SPECTRAL ANALYSIS OF THE KRONIG-PENNEY MODEL WITH WIGNER-VON NEUMANN PERTURBATIONS

VLADIMIR LOTOREICHIK AND SERGEY SIMONOV

Abstract. The spectrum of the self-adjoint Schrödinger operator associated with the Kronig-Penney model on the half-line has a band-gap structure: its absolutely continuous spectrum consists of intervals (bands) separated by gaps. We show that if one changes strengths of interactions or locations of interaction centers by adding an oscillating and slowly decaying sequence which resembles the classical Wigner-von Neumann potential, then this structure of the absolutely continuous spectrum is preserved. At the same time in each spectral band precisely two critical points appear. At these points “instable” embedded eigenvalues may exist. We obtain locations of the critical points and discuss for each of them the possibility of an embedded eigenvalue to appear. We also show that the spectrum in gaps remains pure point.

1. Introduction

In the classical paper [vNW29] von Neumann and Wigner studied the one-dimensional Schrödinger operator with the potential of the form \( e^{\frac{c}{x}\sin(2\omega x)} \), and discovered that such an operator may have an eigenvalue at the point of the continuous spectrum \( \lambda = \omega^2 \). Since then such potentials permanently attracted the interest of different authors [Al72, Ma73, RS78, Be91, Be94, HKS91, KN07, N07, L10, JS10]. Potentials of the type \( e^{\frac{c}{x}\sin(2\omega x)} \) with \( \gamma \in (0,1) \) also possess such a property and they are often called Wigner-von Neumann potentials.

Consider the operator \( H_\kappa \) on \( \mathbb{R}_+ \) that formally corresponds to the differential expression

\[-\frac{d^2}{dx^2} + \sum_{n \in \mathbb{N}} \alpha_0 \delta_{nd}(\delta_{nd}, \cdot)\]

with the boundary condition

\[\psi(0) \cos \kappa - \psi'(0) \sin \kappa = 0,\]

at the origin, where \( \alpha_0 \in \mathbb{R} \), \( d > 0 \), \( \kappa \in [0, \pi) \) and \( \delta_x \) is the delta-distribution supported on the point \( x \in \mathbb{R}_+ \) (for the mathematically rigorous definition of the operator see Section 2.1, cf. [AGHH05, Chapter III]). The operator \( H_\kappa \) is self-adjoint in \( L^2(\mathbb{R}_+) \). It describes the behavior of a free non-relativistic charged quantum particle interacting with the lattice \( d\mathbb{N} \). The constant \( \alpha_0 \) characterizes the strength of interaction between the free charged quantum particle and each interaction center in the lattice. The spectrum of the operator \( H_\kappa \) has a band-gap structure: it consists of infinitely many bands of the purely absolutely continuous spectrum and outside these bands the spectrum of \( H_\kappa \) is discrete. The investigation of such operators was initiated in the classical paper [KP31] by Kronig and Penney.

In the present paper we study what happens with the spectrum of the Kronig-Penney model in the case of perturbation of strengths or positions of interactions by a slowly decaying oscillating sequence resembling the Wigner-von Neumann potential. Let constants \( d, \alpha_0, c, \omega, \gamma \) and a real-valued sequence \( \{q_n\}_{n \in \mathbb{N}} \) be such...
that
\begin{equation}
    d > 0, \alpha_0 \in \mathbb{R}, \omega \in \left(0, \frac{\pi}{2}\right), \gamma \in \left(\frac{1}{2}, 1\right), \{q_n\}_{n \in \mathbb{N}} \in \ell^1(\mathbb{N}).
\end{equation}

\textbf{Model I: Wigner-von Neumann amplitude perturbation.} We add a discrete Wigner-von Neumann potential to the constant sequence of interaction strengths. Let the discrete set \( \tilde{X} = \{\tilde{x}_n : n \in \mathbb{N}\} \) supporting the interaction centers be defined as
\begin{equation}
    \tilde{x}_n := nd, \ n \in \mathbb{N},
\end{equation}
and let the sequence of interaction strengths \( \tilde{\alpha} = \{\tilde{\alpha}_n\}_{n \in \mathbb{N}} \) be as below:
\begin{equation}
    \tilde{\alpha}_n := \alpha_0 + \frac{c \sin(\omega n)}{n^\gamma} + q_n, \ n \in \mathbb{N}.
\end{equation}
We study the self-adjoint operator \( H_{\kappa, \tilde{X}, \tilde{\alpha}} \) which formally corresponds to the differential expression
\[- \frac{d^2}{dx^2} + \sum_{n \in \mathbb{N}} \tilde{\alpha}_n \delta_{\tilde{x}_n}(\delta_{\tilde{x}_n} \cdot)\]
with the boundary condition (1.1). This operator reflects an amplitude perturbation of the Kronig-Penney model.

\textbf{Model II: Wigner-von Neumann positional perturbation.} We change the distances between interaction centers in a “Wigner-von Neumann” way, i.e. we add a sequence of the form of Wigner-von Neumann potential to the coordinates of interaction centers leaving the strengths constant. Let the discrete set \( \hat{X} = \{\hat{x}_n : n \in \mathbb{N}\} \) be as below:
\begin{equation}
    \hat{x}_n := nd + \frac{c \sin(2 \omega n)}{n^\gamma} + q_n, \ n \in \mathbb{N},
\end{equation}
and the sequence of strengths be defined as
\begin{equation}
    \hat{\alpha}_n := \alpha_0, \ n \in \mathbb{N}.
\end{equation}
We study the operator \( H_{\kappa, \hat{X}, \hat{\alpha}} \) which formally corresponds to the differential expression
\[- \frac{d^2}{dx^2} + \sum_{n \in \mathbb{N}} \hat{\alpha}_n \delta_{\hat{x}_n}(\delta_{\hat{x}_n} \cdot)\]
with the boundary condition (1.1). This operator reflects a positional perturbation of the Kronig-Penney model and describes properties of one-dimensional crystals with global defects. We also mention that local defects in the Kronig-Penney are discussed in [AGHH05, §III.2.6]; situations of random perturbations of positions were recently considered in [HIT10].

The essential spectrum of the operator \( H_\kappa \) has a band-gap structure similar to the case of Schrödinger operator with regular periodic potential:
\[ \sigma_{\text{ess}}(H_\kappa) = \sigma_{\text{ac}}(H_\kappa) = \bigcup_{n=1}^\infty \left( [\lambda_{2n-1}^+, \lambda_{2n-1}^-] \cup [\lambda_{2n}^+, \lambda_{2n}^-] \right), \]
where \( \lambda_1^+ < \lambda_1^- < \lambda_2^+ < \lambda_2^- < \cdots < \lambda_{2n-1}^+ < \lambda_{2n-1}^- < \lambda_{2n}^- < \lambda_{2n}^+ < \cdots \).

The locations of boundary points of the spectral bands are determined by the parameters \( \alpha_0 \) and \( d \). Namely, the values \( \lambda_n^\pm \) are the \( n \)-th roots of the corresponding Kronig-Penney equations
\[ L_\delta \left( \sqrt{\lambda} \right) = \pm 1, \]
where
\begin{equation}
    L_\delta(k) = \cos(\delta d) + \alpha_0 \frac{\sin(\delta d)}{2k}.
\end{equation}
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\[ L_\delta(k) = \alpha_0 \frac{\sin(kd)}{2k} + \cos(kd) \]

Figure 1. The curve is the graph of \( L_\delta \); bold intervals are bands of the absolutely continuous spectrum; crosses denote the critical points \( \{ \lambda_{n,cr}^\pm \}_{n \in \mathbb{N}} \).

For the details the reader is referred to the monograph [AGHH05, Chapter III].

In the present paper we show that the absolutely continuous spectra of the operators \( H_{\kappa,\tilde{X}}, \tilde{\alpha} \) and \( H_{\kappa,\tilde{X}}, \tilde{\alpha} \) coincide with the absolutely continuous spectra of the non-perturbed operator \( \tilde{H}_{\kappa} \). However the spectrum in bands may not remain purely absolutely continuous. Namely, at certain (critical) points embedded eigenvalues may appear. In each band there are two such points. The critical points \( \lambda_{n,cr}^\pm \) in the \( n \)-th spectral band are the \( n \)-th roots of the equations

\[ L_\delta \left( \sqrt{\lambda} \right) = \pm \cos \omega. \]

The illustration is given on the Figure I.

For the operators considered we give exact conditions which ensure that a given critical point is indeed an embedded eigenvalue for some \( \kappa \in [0, \pi) \). This can occur only for one value of \( \kappa \in [0, \pi) \). The possibility of the appearance of an embedded eigenvalue at certain critical point depends on the rate of the decay of the subordinate generalized eigenvector. We calculate the asymptotics of generalized eigenvectors for all values of the spectral parameter \( \lambda \) including the critical points, except the endpoints of the bands. We also show that the spectrum in gaps remains pure point.

Our results are close to the results for one-dimensional Schrödinger operator with the Wigner-von Neumann potential and a periodic background potential. Such operators were considered very recently in [KN07, NS11, KS11].

To study spectra in bands we make a discretization of the spectral equations and further we perform an asymptotic integration of obtained discrete linear system using Benzaid-Lutz-type theorems [BL87]. As the next step we apply a modification of Gilbert-Pearson subordinacy theory [GP87]. To study spectra in gaps we apply compact perturbation argument in Weyl function method, see, e.g., [DM95, BGP08].

The reader can trace some analogies of our case with Jacobi matrices. The coefficient matrix of the discrete linear system that appears in our analysis has a form similar to the transfer-matrix for some Jacobi matrix and the Weyl function that appears in our analysis takes Jacobi matrices as its values.
The body of the paper contains two parts: the preliminary part — which consists of mostly known material — and the main part, where we obtain new results. In the preliminary part we give a rigorous definition of one-dimensional Schrödinger operators with δ-interactions (Section 2.1), show how to reduce the spectral equations for these operators to discrete linear systems in $C^2$ (Section 2.2), provide a formulation of the subordinacy theory analogue for the operators considered (Section 2.3). Further we formulate few results from asymptotic integration theory for discrete linear systems (Section 2.4) and provide a characterization of the essential spectra of considered operators using the Weyl function associated with certain ordinary boundary triple (Section 2.5). In the main part, in Section 3.1 we study a special class of discrete linear systems in $C^2$ and find asymptotics of solutions of these systems. After certain technical preliminary calculations in Section 3.2, we proceed to Section 3.3, where we obtain asymptotics of generalized eigenvectors of these systems. After certain technical preliminary calculations in Section 3.2, we pass to the conclusions about the spectra in bands putting an emphasis on critical points. Section 3.4 is devoted to the analysis of spectra in gaps by means of the Weyl function.

Notations. By small letters with integer subindices, e.g., $\xi_n$, we denote sequences of complex numbers. By small letters with integer subindices and arrows above, e.g. $\overleftarrow{\xi}_n$, we denote sequences of $C^2$-vectors. By capital letters with integer indices, e.g., $R_n$, we denote sequences of $2 \times 2$ matrices with complex entries. We use notations $\mathcal{L}(\mathbb{N})$, $\mathcal{L}(\mathbb{N}, C^2)$ and $\mathcal{L}(\mathbb{N}, C^{2\times 2})$ for spaces of summable $(p = 1)$, square-summable $(p = 2)$ and bounded $(p = \infty)$ sequences of complex values, complex two-dimensional vectors and complex $2 \times 2$ matrices, respectively. We denote for a self-adjoint operator its pure point, absolutely continuous, singular continuous, essential and discrete spectra by $\sigma_{pp}$, $\sigma_{ac}$, $\sigma_{sc}$, $\sigma_{ess}$ and $\sigma_d$, respectively. We write $T \in \mathcal{S}_\infty$ in the case $T$ is a compact operator. Other notations are assumed to be clear for the reader.

2. Preliminaries

2.1. Definition of operators with point interactions. In this section we give a rigorous definition of operators with δ-interactions, see, e.g., [GK85, Ko89]. Let $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers and let $X = \{x_n; n \in \mathbb{N}\}$ be a discrete set on $\mathbb{R}_+$ ordered as $0 < x_1 < x_2 < \ldots$. Assume that the set $X$ satisfies
\begin{equation}
\inf_{n \in \mathbb{N}} |x_{n+1} - x_n| > 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} |x_{n+1} - x_n| < \infty.
\end{equation}

Denote also $x_0 := 0$. In order to define the operator corresponding to the expression
\[\tau_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{N}} \alpha_n \delta_{x_n}(\delta_{x_n}, \cdot)\]
and the boundary condition
\[\psi(0) \cos \kappa - \psi'(0) \sin \kappa = 0,
\]
consider the following set of functions:
\[S_{X,\alpha} := \left\{ \psi : \psi, \psi' \in AC_{loc}(\mathbb{R}_+ \setminus X), \psi(x_{n+}) = \psi(x_{n-}) = \psi(x_n), \psi'(x_{n+}) - \psi'(x_{n-}) = \alpha_n \psi(x_n), n \in \mathbb{N} \right\},
\]
and let the operator $H_{\kappa,X,\alpha}$ be defined by its action
\[H_{\kappa,X,\alpha}\psi := -\psi''\]
on the domain
\[\text{dom} \ H_{\kappa,X,\alpha} := \{ \psi \in L^2(\mathbb{R}_+) \cap S_{X,\alpha} : \psi'' \in L^2(\mathbb{R}_+), \psi(0) \cos \kappa = \psi'(0) \sin \kappa \}.
\]
The spectral equation $\tau_{X,\alpha} \psi = \lambda \psi$ is understood as the equation $-\psi''(x) = \lambda \psi(x)$ for $\psi \in S_{X,\alpha}$. The latter is equivalent to the following system:

$$
-\psi''(x) = \lambda \psi(x), \quad x \in \mathbb{R}_+ \setminus X,
$$

$$
\psi(x_n^+) = \psi(x_n^-), \quad \psi'(x_n^+) = \psi'(x_n^-) + \alpha_n \psi(x_n^-), \quad n \in \mathbb{N}.\tag{2.2}
$$

The equation (2.2) has two linear independent solutions which are called generalized eigenvectors. If $\psi \in L^2(\mathbb{R}_+)$ satisfies (2.2) and the boundary condition, then $\psi$ is an eigenfunction of $H_{\kappa,X,\alpha}$.

2.2. Reduction of the eigenfunction equation to a discrete linear system.

In this subsection we recall rather well-known way of reduction of the spectral equation (2.2) to a discrete linear system. Let the discrete set $X$ and the sequence of strengths $\alpha$ be as in the previous section. To make our formulas more compact we introduce the following notations

$$
s_n(k) := \sin(k(x_{n+1} - x_n)) \quad \text{and} \quad c_n(k) := \cos(k(x_{n+1} - x_n)), \quad n \in \mathbb{N}_0.
$$

For a solution $\psi$ of (2.2) corresponding to $\lambda = k^2$ we introduce the sequence below

$$
\xi_n := \psi(x_n), \quad n \in \mathbb{N}_0.
$$

If the condition (2.1) is satisfied, then by [AGHH05, Chapter III, Theorem 2.1.5] for $k \in \mathbb{C}_+ \cup \mathbb{R}$ and $n \in \mathbb{N}$ such that $s_{n-1}(k), s_n(k) \neq 0$ it holds that

$$
-k \left( \frac{\xi_{n-1}}{s_{n-1}(k)} + \frac{\xi_{n+1}}{s_n(k)} \right) + \left( \alpha_n + k \left( \frac{c_{n-1}(k)}{s_{n-1}(k)} + \frac{c_n(k)}{s_n(k)} \right) \right) \xi_n = 0.\tag{2.3}
$$

Inversely, solutions of the eigenfunction equation on each of the intervals $(x_n, x_{n+1})$ can be recovered from their values at the endpoints $x_n$ and $x_{n+1}$:

$$
\psi(x) = \frac{\xi_n \sin(k(x_{n+1} - x)) + \xi_{n+1} \sin(k(x - x_n))}{\sin(k(x_{n+1} - x_n))}, \quad x \in [x_n; x_{n+1}].\tag{2.4}
$$

The reader may confer with [E97], where a more general case of a quantum graph is considered.

Instead of working with recurrence relation (2.3) we will consider a discrete linear system in $\mathbb{C}^2$. Define

$$
\overrightarrow{u}_n := \left( \frac{\xi_{n-1}}{\xi_n} \right).
$$

Then (2.3) is equivalent to

$$
\overrightarrow{u}_{n+1} = T_n(k) \overrightarrow{u}_n \tag{2.5}
$$

with

$$
T_n(k) := \begin{pmatrix}
0 & \frac{1}{s_n(k)} - \frac{s_{n-1}(k)}{s_n(k)} \\
-\frac{s_{n-1}(k)}{s_n(k)} & \frac{\alpha_n}{s_{n-1}(k)}
\end{pmatrix}\tag{2.6}
$$

The coefficient matrix of this system $T_n(k)$ is called the transfer-matrix.

2.3. Subordinacy. The subordinacy theory as suggested in [GP87] by D. Gilbert and D. Pearson produced a strong influence on the spectral theory of one-dimensional Schrödinger operators. Later on the subordinacy theory was translated to difference equations [KP92]. For Schrödinger operators with $\delta$-interactions there exists a modification of the subordinacy theory, see, e.g., [SCS94] which relates the spectral properties of the operator $H_{\kappa,X,\alpha}$ with the asymptotic behavior of the solutions of the spectral equation (2.2). Analogously, to the classical definition of the subordinacy [GP87] we say that a solution $\psi_1$ of the equation $\tau_{X,\alpha} \psi_1 = \lambda \psi$ is subordinate
Assume that for all Proposition 2.2. \( \Delta \lambda \) corresponding to the discrete set Proposition 2.1.

Proof. Let \( \psi \) be an arbitrary solution of \( \tau X, \alpha \psi = \lambda \psi \). Then \([a, b] \subset \sigma(H_{\alpha, X, \alpha})\) and \(\sigma(H_{\alpha, X, \alpha})\) is purely absolutely continuous in \((a, b)\).

Proposition 2.2. Let the discrete set \( X = \{x_n : n \in \mathbb{N}\} \) on \( \mathbb{R}_+ \) be ordered as \( 0 < x_1 < x_2 < \ldots \). Assume that it satisfies conditions \( \Delta x_n := \inf_{n \in \mathbb{N}} |x_{n+1} - x_n| > 0 \) and \( \Delta x^* := \sup_{n \in \mathbb{N}} |x_{n+1} - x_n| < \infty \). Let \( \alpha = \{\alpha_n\}_{n \in \mathbb{N}} \) be a sequence of real numbers and let \( \lambda \in \mathbb{R} \). Assume that \( \lim \inf_{n \to \infty} |\alpha_n(\sqrt{\lambda})| > 0 \). If every solution of the equation \( \tau X, \alpha \psi = \lambda \psi \) (see (2.2)) is bounded, then for such \( \lambda \) there exists no subordinate solution.

Proof. Let \( \psi_1 \) be an arbitrary solution of \( \tau X, \alpha \psi = \lambda \psi \). Differentiating (2.4) one gets in the case \( s_n(k) \neq 0 \)

\[
|\psi_1'(x)| \leq \frac{|k|(\xi_n + \xi_n+1)}{|s_n(k)|} \quad \text{for } x \in [x_n, x_{n+1}].
\]

There exists \( N(k) \) such that \( \inf_{n \geq N(k)} |s_n(k)| > 0 \). Since the sequence \( \{\xi_n\}_{n \in \mathbb{N}} \) is bounded, one has that \( \psi_1'(x) \) is also bounded for \( x \geq x_{N(k)} \). Obviously for \( x \leq x_{N(k)} \) it is bounded too, since it is piecewise continuous with finite jumps at the points of discontinuity. Let \( \psi_2 \) be any other solution of \( \tau X, \alpha \psi = \lambda \psi \), which is linear independent with \( \psi_1 \). It follows that there exists a constant \( C > 0 \) such that

\[
(2.7) \quad \|\psi_1\|_\infty, \|\psi_2\|_\infty, \|\psi_1'\|_\infty, \|\psi_2'\|_\infty \leq C.
\]

The Wronskian of the solutions \( \psi_1 \) and \( \psi_2 \) is independent of \( x \):

\[
(2.8) \quad W\{\psi_1, \psi_2\} := \psi_1(x)\psi_2'(x) - \psi_1'(x)\psi_2(x) \quad \text{for all } x \in \mathbb{R}_+ \setminus X.
\]

This is easy to check: it is constant on every interval \([x_n, x_{n+1}]\) and at the points \( \{x_n\}_{n \in \mathbb{N}} \) one has \( W\{\psi_1, \psi_2\}(x_{n+}) = W\{\psi_1, \psi_2\}(x_{n-}) \) from (2.2). The Wronskian is non-zero since the solutions are linear independent. From (2.8) one has:

\[
(2.9) \quad 0 < C_* \leq |\psi_1(x)| + |\psi_1'(x)| \leq C^*, \quad \text{for all } x \in \mathbb{R}_+ \setminus X.
\]

Now we apply the trick used in the proof of [592, Lemma 4]. We consider for an arbitrary \( n \in \mathbb{N} \) the interval \([x_n, x_{n+1}]\). Since the function \( \psi_1 \) is continuous on \([x_n, x_{n+1}]\), the formula

\[
(2.10) \quad \psi_1(x_{n+1}) - \psi_1(x_n) = \int_{x_n}^{x_{n+1}} \psi_1'(t)dt
\]

holds. Set

\[
p_* := \frac{\Delta x_*}{\Delta x_* + 2}.
\]

Next we show that there exists a point \( x_n^* \in [x_n, x_{n+1}] \) such that \( |\psi_1(x_n^*)| \geq p_*C_* \). Let us suppose that such a point does not exist, i.e. \( |\psi_1(x)| < p_*C_* \) for all \( x \in [x_n, x_{n+1}] \). We get from (2.9) that \( |\psi_1'(x)| > (1 - p_*)C_* \) for every \( x \in [x_n, x_{n+1}] \). In
particular $\psi'_1$ is sign-definite in $[x_n, x_{n+1}]$, so using (2.10) and $\Delta x_n \geq \Delta x_\ast$ we get a contradiction

$$2p_\ast C_\ast > |\psi_1(x_{n+1})| + |\psi_1(x_n)| \geq |\psi_1(x_{n+1}) - \psi_1(x_n)| =$$

$$= \int_{x_n}^{x_{n+1}} |\psi'_1(t)|dt > \Delta x_n (1 - p_\ast)C_\ast \geq \Delta x_\ast (1 - p_\ast)C_\ast = 2p_\ast C_\ast.$$ 

Thus the point $x_\ast^\ast$ with required properties exists.

Since $|\psi'_1(x)| \leq C$, for every $x \in [x_n, x_{n+1}]$ such that $|x - x_\ast^\ast| \leq \frac{p_\ast C_\ast}{2n}$ one has $|\psi_1(x)| \geq \frac{p_\ast C_\ast}{2n}$. We have shown that every interval $[x_n, x_{n+1}]$ contains a subinterval of length $l := \min (\Delta x_\ast, \frac{p_\ast C_\ast}{2C_\ast})$, where $|\psi_1(x)| \geq \frac{p_\ast C_\ast}{2n}$. Therefore

$$\int_{0}^{x_n} |\psi_1(t)|^2dt \geq \frac{h_0^2 C^2_\ast}{4} n.$$ 

On the other hand,

$$\int_{0}^{x_n} |\psi_1(t)|^2dt \leq \Delta x_\ast C^2 n.$$ 

Summing up, for every solution $\psi$ the integral $\int_{0}^{x_n} |\psi(t)|^2dt$ has two-sided linear estimate. Thus no subordinate solution exists. \(\square\)

2.4. Benzaid-Lutz theorems for discrete linear systems in $C^2$. The results of [BL87] translate classical theorems due to N. Levinson [L85] and W. Harris and D. Lutz [HL75] on the asymptotic integration of ordinary differential linear systems to the case of discrete linear systems. The major advance of these methods that allow to reduce under certain assumptions the asymptotic integration of general discrete linear systems to almost trivial asymptotic integration of diagonal discrete linear systems. For our applications it is sufficient to formulate Benzaid-Lutz theorems only for discrete linear systems in $C^2$. The first lemma of this subsection is a direct consequence of [BL87] Theorem 3.3.

Lemma 2.3. Let $\mu_\pm \in \mathbb{C} \setminus \{0\}$ be such that $|\mu_+| \neq |\mu_-|$ and let $\{V_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, C^{2 \times 2})$. If the coefficient matrix of the discrete linear system

$$\overrightarrow{u}_{n+1} = \begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix} + V_n \overrightarrow{u}_n$$ 

is non-degenerate for every $n \in \mathbb{N}$, then this system has a basis of solutions $\overrightarrow{u}_n^\pm$ with the following asymptotics:

$$\overrightarrow{u}_n^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1) \prod_{k=1}^{n}(\mu_+ + (V_k)_{11}) \quad \text{as } n \to \infty,$$

$$\overrightarrow{u}_n^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1) \prod_{k=1}^{n}(\mu_- + (V_k)_{22}) \quad \text{as } n \to \infty,$$

where by $(V_k)_{11}$ and $(V_k)_{22}$ we denote the diagonal entries of the matrices $V_k$, and the factors $(\mu_+ + (V_k)_{11})$ and $(\mu_- + (V_k)_{22})$ should be replaced by 1 for those values of the index $k$ for which they vanish.

The following lemma is a simplification of [BL87] Theorem 3.2.

Lemma 2.4. Let $\{t_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ and $\{V_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, C^{2 \times 2})$ satisfy the following conditions.

(i) Levinson condition: there exists $M_\ast$ such that for every $n_1 \geq n_2$

$$\prod_{n=n_1}^{n_2} |1 - t_n| < M_\ast$$
and either there exists $M_+$ such that for every $n_1 \geq n_2$

$$\prod_{n=n_1}^{n_2} |1 + t_n| > M_+$$

or

$$\prod_{k=1}^{n} |1 + t_k| \to \infty \text{ as } n \to \infty.$$ 

(ii) The sum $\sum_{n=1}^{\infty} V_n$ is conditionally convergent and

$$\left\{ \sum_{k=n}^{\infty} V_k \right\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{C}^{2 \times 2}).$$

If for every $n \in \mathbb{N}$

$$\det \left[ \begin{pmatrix} 1 + t_n & 0 \\ 0 & 1 - t_n \end{pmatrix} + V_n \right] \not= 0,$$

then the discrete linear system

$$\overrightarrow{u}_{n+1} = \left[ \begin{pmatrix} 1 + t_n & 0 \\ 0 & 1 - t_n \end{pmatrix} + V_n \right] \overrightarrow{u}_n$$

has a basis of solutions $\overrightarrow{u}_n^\pm$ with the following asymptotics:

$$\overrightarrow{u}_n^+ = \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \right] \prod_{k=1}^{n} (1 + t_k) \text{ as } n \to \infty,$$

$$\overrightarrow{u}_n^- = \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \right] \prod_{k=1}^{n} (1 - t_k) \text{ as } n \to \infty,$$

where the factor $\left(1 - t_k\right)$ should be replaced by $1$ for those values of the index $k$ for which it vanishes.

2.5. Characterization of spectra for operators with point interactions via Weyl function. In this section we view the operator $H_{\kappa, X, \alpha}$ as a self-adjoint extension of a certain symmetric operator and we parametrize this self-adjoint extension via suitable ordinary boundary triple [Ko75, Br76]. As a benefit of this parametrization we reduce the spectral analysis of the self-adjoint operator $H_{\kappa, X, \alpha}$ to the spectral analysis of an operator-valued function in the auxiliary Hilbert space. The reader is referred to [DM95, BGP08] for an extensive discussion of ordinary boundary triples and their applications. The Weyl function method was applied to Schrödinger operators with $\delta$-interactions by many authors [Ko89, M95, KM10, GO10, AKM10].

Let $X = \{x_n; n \in \mathbb{N}\}$ be a discrete set satisfying the condition (2.1) and let $\kappa$ be a parameter in the interval $[0, \pi)$. Define the minimal symmetric operator via its action and domain

$$H_{\kappa, X} \psi = -\psi'',$$

$$\text{dom } H_{\kappa, X} = \left\{ \psi; \psi, \psi'' \in L^2(\mathbb{R}^+), \psi, \psi' \in AC_{\text{loc}}(\mathbb{R}^+ \setminus X), \psi(x_n-)=\psi(x_n+)=0 \iff n \in \mathbb{N}, \psi''(x_n-)\neq\psi''(x_n+), n \in \mathbb{N} \right\}.$$ 

It can be easily verified that $H_{\kappa, X}$ is a closed symmetric densely defined operator with the adjoint (the maximal operator)

$$H_{\kappa, X}^* \psi = -\psi'',$$

$$\text{dom } H_{\kappa, X}^* = \left\{ \psi; \psi, \psi'' \in L^2(\mathbb{R}^+), \psi, \psi' \in AC_{\text{loc}}(\mathbb{R}^+ \setminus X), \psi(x_n-)=\psi(x_n+)=0 \iff n \in \mathbb{N} \right\}.$$
Definition 2.5. For the discrete set $X$ and the parameter $\kappa$ as above we define $\Gamma_{0,\kappa,X}$ and $\Gamma_{1,\kappa,X}$ as

$$
\Gamma_{0,\kappa,X} : \text{dom } H^*_{\kappa,X} \to \mathbb{L}^2(\mathbb{N}), \quad \Gamma_{0,\kappa,X}\psi := \{ -\psi(x_n) \}_{n \in \mathbb{N}},
$$

$$
\Gamma_{1,\kappa,X} : \text{dom } H^*_{\kappa,X} \to \mathbb{L}^2(\mathbb{N}), \quad \Gamma_{1,\kappa,X}\psi := \{ \psi'(x_n) - \psi'(x_n - 1) \}_{n \in \mathbb{N}}.
$$

By [BGP08, Proposition 3.17] the triple $\Pi_{\kappa,X} := \{ \mathbb{L}^2(\mathbb{N}), \Gamma_{0,\kappa,X}, \Gamma_{1,\kappa,X} \}$ is an ordinary boundary triple for $H^*_\kappa$ in the following sense:

(i) $\text{ran} \left( \frac{\Gamma_0}{\Gamma_1} \right) = \mathbb{L}^2(\mathbb{N}) \times \mathbb{L}^2(\mathbb{N})$;

(ii) for all $f, g \in \text{dom } H^*_\kappa$

$$
(\mathcal{H}^*_\kappa f, g)_{\mathcal{L}^2} - (f, \mathcal{H}^*_\kappa g)_{\mathcal{L}^2} = (\Gamma_{1,\kappa,X} f, \Gamma_{0,\kappa,X} g)_{\mathcal{L}^2} - (\Gamma_{0,\kappa,X} f, \Gamma_{1,\kappa,X} g)_{\mathcal{L}^2}.
$$

Remark 2.6. The assumption $\inf_{n \in \mathbb{N}} |x_{n+1} - x_n| > 0$ is essential. In a recent work [KM10] an ordinary boundary triple was constructed for a class of sets $X$ not satisfying this condition.

For further purposes we define the set $\sigma_{\kappa,X} := \sigma(H^*_\kappa \mid \ker \Gamma_{0,\kappa,X})$. We associate the Weyl function with the triple $\Pi_{\kappa,X}$

$$
M_{\kappa,X}(\lambda) := \Gamma_{1,\kappa,X} \left( \Gamma_{0,\kappa,X} \ker(H^*_\kappa - \lambda I) \right)^{-1}
$$

which is defined for all $\lambda \in \mathbb{C} \setminus \sigma_{\kappa,X}$.

Recall that $s_n(k) = \sin(k(x_{n+1} - x_n))$ and $c_n(k) = \cos(k(x_{n+1} - x_n))$. Given a vector $\xi \in \mathbb{L}^2(\mathbb{N})$ we can find a solution $\psi \in \ker(H^*_\kappa - \lambda I)$ such that $\psi(x_n) = -\xi_n$. The operator $M_{\kappa,X}(\lambda)$ maps $\xi$ to the vector $\{ \psi'(x_n) - \psi'(x_n - 1) \}_{n \in \mathbb{N}} \in \mathbb{L}^2(\mathbb{N})$.

A straightforward calculation based on the formula (2.4) gives us the values of all entries of the matrix $M_{\kappa,X}(\lambda)$ except the first row. To find the values of the first row entries one needs to know $\psi'(x)$ on the interval $[0, x_1]$, i.e., to solve the non-homogenious Sturm-Liouville problem $-\psi''(x) = \lambda \psi(x)$, $\psi(0) \cos \kappa - \psi'(0) \sin \kappa = 0$, $\psi(x_1) = -\xi_1$. The result of calculations is the following:

$$
M_{\kappa,X}(k^2) = \begin{pmatrix}
    b_1(k) & a_1(k) & 0 & \ldots \\
    a_1(k) & b_2(k) & a_2(k) & \ldots \\
    0 & a_2(k) & b_3(k) & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

with

$$
b_1(k) := k \frac{c_0(k) \cos \kappa - ks_0(k) \sin \kappa}{s_0(k) \cos \kappa + ks_0(k) \sin \kappa} + k \frac{c_1(k)}{s_1(k)},
$$

$$
b_n(k) := k \frac{c_n(k)}{s_n(k)} + k \frac{c_{n-1}(k)}{s_{n-1}(k)}, \quad n \geq 2,
$$

$$
a_n(k) := -k \frac{c_n(k)}{s_n(k)}.
$$

Remark 2.7. Note that the set $\sigma_{\kappa,X}$ consists of squares of all zeroes of denominators in the formulas above:

$$
\sigma_{\kappa,X} = \{ k^2 \in \mathbb{R} : s_n(k) = 0 \text{ for some } n \in \mathbb{N} \text{ or } s_0(k) \cos \kappa + ks_0(k) \sin \kappa = 0 \}.
$$

Let $\alpha = \{ \alpha_n \}_{n \in \mathbb{N}}$ be a bounded sequence of real numbers. Define an operator in $\mathbb{L}^2(\mathbb{N})$

$$
B_\alpha \xi = \{ -\alpha_n \xi_n \}_{n \in \mathbb{N}}, \quad \xi \in \mathbb{L}^2(\mathbb{N}),
$$
which is bounded and self-adjoint. The operator $H_{\kappa,X,\alpha}$ is a restriction of the operator $H'_{\kappa,X}$ corresponding to $B_\alpha$ in the following sense

$$H_{\kappa,X,\alpha} = H'_{\kappa,X} \mid \ker(\Gamma_{\kappa,X} - B_\alpha \Gamma_{0,\kappa,X}).$$

**Proposition 2.8.** [BGPS05, Theorem 3.3] Let the discrete set $X$, the sequence of strengths $\alpha$ and the self-adjoint operator $H_{\kappa,X,\alpha}$ be defined as in Section 2.1. Let $\sigma_{\kappa,X}$ be the set defined in (2.1) and $B_\alpha$ be given by (2.13). Let $M_{\kappa,X}$ be the Weyl function (2.11) associated with the ordinary boundary triple $\Pi_{\kappa,X}$. For $\lambda \in \mathbb{R} \setminus \sigma_{\kappa,X}$ the following holds:

$$\lambda \in \sigma_{\text{ess}}(H_{\kappa,X,\alpha}) \quad \text{if and only if} \quad 0 \in \sigma_{\text{ess}}(B_\alpha - M_{\kappa,X}(\lambda)).$$

3. **Spectral and asymptotic analysis**

3.1. **Asymptotic analysis of a special class of discrete linear systems.** In this section we study a special class of discrete linear systems that encapsulates system (2.5) corresponding to $X = \{x_n : n \in \mathbb{N}\}$ and $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ as in Models I and II described in the introduction.

Let the parameters $l$, $a$, $b$, $\gamma$ and $\omega$ satisfy conditions

$$l \in \mathbb{R}_+, \quad a, b \in \mathbb{C}, \quad \gamma \in \left(\frac{1}{2}, 1\right) \quad \text{and} \quad \omega \in \left(0, \frac{\pi}{2}\right).$$

For further purposes we define

$$\mu_{\pm} := l \pm \sqrt{l^2 - 1},$$

$$z := \frac{ae^{-i\omega} + be^{-2i\omega}}{2i \sin \omega}, \quad \beta := |z|, \quad \varphi := \arg z,$

and

$$f_n^+(\beta) := \left\{ \begin{array}{ll}
\exp\left(\pm \frac{2n^{1-\gamma}}{1-\gamma}\right), & \text{if } \gamma < 1, \\
n^{1-\gamma}, & \text{if } \gamma = 1.
\end{array} \right.$$  

The following has technical nature and helps to simplify the analysis of cases (Model I and Model II).

**Lemma 3.1.** Let the parameters $l$, $a$, $b$, $\gamma$ and $\omega$ be as in (3.1). Let $\mu_{\pm}$, $\beta$, $\varphi$ and $f_n^+(\cdot)$ be as in (3.2), (3.3) and (3.4), respectively. Let the sequence of matrices $\{R_n\}_{n \in \mathbb{N}} \in l^1(\mathbb{N}, \mathbb{C}^{2 \times 2})$. If the coefficient matrix of the discrete linear system

$$\overrightarrow{u}_{n+1} = \begin{bmatrix} 0 & 1 \\ -1 & 2l \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \frac{e^{2in\omega}}{n^\gamma} + \begin{bmatrix} 0 & 0 \\ \frac{\omega}{\pi} & -b \end{bmatrix} \frac{e^{-2in\omega}}{n^\gamma} + R_n \overrightarrow{u}_n$$

is non-degenerate for every $n \in \mathbb{N}$, then this system has a basis of solutions $\overrightarrow{u}_n^\pm$ with the following asymptotics.

(i) If $l \in \mathbb{R}_+ \setminus \{\cos \omega, 1\}$, then

$$\overrightarrow{u}_n^\pm = \left[ \begin{array}{c}
1 \\
\mu_{\pm}
\end{array} \right] + o(1) \quad \mu_{\pm}^n \quad \text{as } n \to \infty.$$

(ii) If $l = \cos \omega$, then

$$\overrightarrow{u}_n^+ = \left[ \begin{array}{c}
\cos(\omega(n + \varphi/2)) \\
\cos(\omega(n + 1) + \varphi/2)
\end{array} \right] + o(1) \quad f_n^+(\beta) \quad \text{as } n \to \infty,$$

and

$$\overrightarrow{u}_n^- = \left[ \begin{array}{c}
\sin(\omega(n + \varphi/2)) \\
\sin(\omega(n + 1) + \varphi/2)
\end{array} \right] + o(1) \quad f_n^-(\beta) \quad \text{as } n \to \infty.$$

Before providing the proof we make several remarks.
Remark 3.2. We do not consider the (double-root) case $l = 1$. The analysis in this special case is technically involved and we refer the reader to [J06, JNSh07].

Remark 3.3. The case of negative values of the parameter $l$ can be treated by the same lemma if one makes the substitution

$$\overrightarrow{u}_n \rightarrow (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overrightarrow{u}_n.$$ 

This leads to the same answer in the case $l = -\cos \omega$:

$$\overrightarrow{u}_n^+ = (-1)^n \left[ \begin{pmatrix} \cos(\omega n + \varphi_1/2) \\ -\cos(\omega(n+1) + \varphi_1/2) \end{pmatrix} + o(1) \right] f_n^+(\beta_1) \quad \text{as} \ n \to \infty,$$

and

$$\overrightarrow{u}_n^- = (-1)^n \left[ \begin{pmatrix} \sin(\omega n + \varphi_1/2) \\ -\sin(\omega(n+1) + \varphi_1/2) \end{pmatrix} + o(1) \right] f_n^-(\beta_1) \quad \text{as} \ n \to \infty,$$

where

$$z_1 := \frac{ae^{-i\omega} - be^{-2i\omega}}{2i\sin \omega}, \quad \beta_1 := |z_1|, \quad \varphi_1 := \arg z_1.$$ 

The (double-root) case $l = -1$ is out of our considerations.

Proof of Lemma 3.1. Since $l \in \mathbb{R}_+ \setminus \{1\}$, the constant term in the coefficient matrix in (3.5) can be diagonalized as follows

$$\begin{pmatrix} 1 & 1 \\ \mu_+ & \mu_- \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 2l \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \mu_+ & \mu_- \end{pmatrix} = \begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix}.$$ 

In view of the identity

$$\begin{pmatrix} 1 & 1 \\ \mu_+ & \mu_- \end{pmatrix}^{-1} = \frac{1}{\mu_- - \mu_+} \begin{pmatrix} \mu_- & -1 \\ -\mu_+ & 1 \end{pmatrix},$$

the substitution

$$\overrightarrow{u}_n = \begin{pmatrix} 1 \\ \mu_+ \end{pmatrix} \overrightarrow{v}_n$$

transforms the system (3.5) on $\overrightarrow{u}_n$ into the system on $\overrightarrow{v}_n$ given below

$$\overrightarrow{v}_{n+1} = \left[ \begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix} + \frac{1}{\mu_- - \mu_+} \begin{pmatrix} -(a + \mu_+ b) & -(a + \mu_- b) \\ (a + \mu_+ b) & (a + \mu_- b) \end{pmatrix} e^{2i\omega n} \overrightarrow{v}_n \right.$$

$$\left. + \frac{1}{\mu_- - \mu_+} \begin{pmatrix} -(a + \mu_+ \beta) & -(a + \mu_- \beta) \\ (a + \mu_+ \beta) & (a + \mu_- \beta) \end{pmatrix} e^{-2i\omega n} + R_n \right] \overrightarrow{v}_n.$$ 

(3.8)

(i) The case $l \in \mathbb{R}_+ \setminus \{\cos \omega, 1\}$ splits into two subcases: $l > 1$ and $l \in [0,1) \setminus \{\cos \omega\}$. The condition $l > 1$ implies $0 < \mu_- < 1 < \mu_+$, thus Lemma 2.3 is applicable to the system (3.8) and it gives us a basis. Reverting the substitution (3.7) we get the statement.

The condition $l \in (0,1) \setminus \{\cos \omega\}$ implies $|\mu_+| = |\mu_-| = 1$. We reduce this situation to Lemma 2.4 by the following substitution:

$$\overrightarrow{v}_n = \begin{pmatrix} \mu_+^\alpha & 0 \\ 0 & \mu_-^\beta \end{pmatrix} \overrightarrow{w}_n.$$ 

(3.9)
The system on \( \overrightarrow{w}_n \) has the following form
\[
\overrightarrow{w}_{n+1} = \left[ I + \frac{1}{\mu_- - \mu_+} \left( -\mu_- (a + \mu_+ b) \mu_+^2 - \mu_- (a + \mu_- b) \mu_-^2 \right) \rho \right] \overrightarrow{w}_n + R_n.
\]

Now Lemma 2.4 is applicable and reverting substitutions (3.7) and (3.9) we get the statement.

(ii) We start with the system (3.10). In the case \( l = \cos \omega \) Lemma 2.4 could not be applied immediately. In order to bring the system into the form in which Lemma 2.4 is applicable we make further transformations. Since \( l = \cos \omega \), we, clearly, have \( \mu_{\pm} = \pm i \omega \) and thus, grouping summands in a right way, we can rewrite our system in the form
\[
\overrightarrow{w}_{n+1} = \left[ I + \frac{1}{\mu_- - \mu_+} \left( -\mu_- (a + \mu_+ b) \mu_+^2 - \mu_- (a + \mu_- b) \mu_-^2 \right) \rho \right] \overrightarrow{w}_n + R_n.
\]
with
\[
z = \frac{ae^{-i\omega} + be^{-2i\omega}}{2i\sin \omega}
\]
and with
\[
V_n := \frac{1}{(\mu_- - \mu_+)^{-1}} \left[ -\mu_- (a + \mu_+ b) \mu_+^2 - \mu_- (a + \mu_- b) \mu_-^2 \right] \rho + \left[ -\mu_- (a + \mu_+ b) \mu_+^2 - \mu_- (a + \mu_- b) \mu_-^2 \right] \rho + R_n.
\]

The sequence \( \{V_n\}_{n \in \mathbb{N}} \) satisfies the condition (ii) in Lemma 2.4. It can checked by a straightforward calculation that \( |z| = \beta \) and \( \arg(z) = \varphi \) with \( \beta \) and \( \varphi \) defined in (3.3).

Since
\[
\begin{pmatrix}
e^{i\varphi} & -e^{i\varphi} \\
e^{-i\varphi} & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
0 & \beta e^{i\varphi} \\
\beta e^{-i\varphi} & 0
\end{pmatrix}
\begin{pmatrix}
e^{i\varphi} & -e^{i\varphi} \\
e^{-i\varphi} & 1
\end{pmatrix} =
\begin{pmatrix}
\beta & 0 \\
0 & -\beta
\end{pmatrix},
\]
the substitution
\[
\overrightarrow{w}_n = \begin{pmatrix} e^{i\varphi} & -e^{i\varphi} \\ 1 & 1 \end{pmatrix} \overrightarrow{x}_n
\]
leads to the system on \( \overrightarrow{x}_n \) of the form
\[
\overrightarrow{x}_{n+1} = \left[ I + \frac{1}{\mu_- - \mu_+} \left( -\mu_- (a + \mu_+ b) \mu_+^2 - \mu_- (a + \mu_- b) \mu_-^2 \right) \rho \right] \overrightarrow{x}_n + R_n.
\]
with \( \{V_n\}_{n \in \mathbb{N}} \) satisfying the condition (ii) in Lemma 2.4.

For sufficiently large \( n_0 \)
\[
\prod_{k=n_0}^{n} \left( 1 \pm \frac{\beta}{k^2} \right) \sim \text{const} \cdot f^+_{n}(\beta) \quad \text{as} \; n \to \infty,
\]
with $f_±(\cdot)$ defined in (3.3). Now we apply Lemma 2.4 to the system (3.12) and using (3.13) we get a basis of solutions of that system of the form

$$
\mathbf{x}^+_n = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + o(1) \right) f_+^n(\beta) \quad \text{as } n \to \infty,
$$

(3.14)

$$
\mathbf{x}^-_n = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] + o(1) \right) f_-^n(\beta) \quad \text{as } n \to \infty.
$$

The asymptotics above imply the statement immediately after reverting substitutions (3.7), (3.9), (3.11) and coming back to the sequence $\tilde{u}_n$. □

### 3.2. Decomposition of the transfer matrices.
Recall that $T_n(k)$ denotes the transfer matrix (2.6). In this section we decompose this transfer matrix subject to Model I and Model II in the form

$$
T_n(k) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 2 \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ a & b \end{array} \right) \frac{e^{i\omega n}}{n^{\gamma}} + \left( \begin{array}{cc} 0 & 0 \\ -a & -b \end{array} \right) \frac{e^{-i\omega n}}{n^{\gamma}} + R_n,
$$

where $R_n \in \ell^1(N; \mathbb{C}^{2 \times 2})$. This decomposition allows to apply later Lemma 3.1.

**Model I: Amplitude perturbation.** The sequence of strengths $\tilde{\alpha} = \{\tilde{\alpha}_n\}_{n \in \mathbb{N}}$ and the discrete set $\tilde{X} = \{\tilde{x}_n: n \in \mathbb{N}\}$ are defined as follows

$$
\tilde{\alpha}_n := \alpha_0 + \frac{c \sin(2\omega n)}{n^{\gamma}} + q_n, \quad \tilde{x}_n := nd \quad \text{and} \quad \tilde{x}_{n+1} - \tilde{x}_n \equiv d.
$$

Substitution of these expression into (2.6) and tedious calculation show that $T_n(k)$ for $k \notin \pi \mathbb{Z}/d$ has the representation of the type (3.15) with the following values of the parameters:

$$
l = \cos(kd) + \frac{\alpha_0 \sin(kd)}{2k} = \delta(k),
$$

$$
a = 0, \quad b = \frac{c \sin(kd)}{2ik}.
$$

**Model II: Positional perturbation.** The sequence of strengths $\hat{\alpha} = \{\hat{\alpha}_n\}_{n \in \mathbb{N}}$ and the discrete set $\hat{X} = \{\hat{x}_n: n \in \mathbb{N}\}$ are defined as follows

$$
\hat{\alpha}_n \equiv \alpha_0, \quad \hat{x}_n := nd + \frac{c \sin(2\omega n)}{n^{\gamma}} + q_n.
$$

In this case substitution into (2.6) and calculation give for $k \notin \pi \mathbb{Z}/d$ the representation of the type (3.15) with the following values of the parameters:

$$
l = \cos(kd) + \frac{\alpha_0 \sin(kd)}{2k} = \delta(k),
$$

$$
a = -2ikc \cot(kd) \sin^2(\omega), \quad b = c \cos(kd) \sin(\omega) \left[ 4k \cos(\omega) \cot(2kd) - 2k \cot(kd) e^{-i\omega} + \alpha_0 e^{i\omega} \right].
$$

**Remark 3.4.** We do not provide here long calculations for the decompositions of the transfer matrices. In these calculations we in fact expand the transfer matrix in powers of $\frac{1}{n}$ as $n \to \infty$ up to a summable term.

### 3.3. Asymptotics of generalized eigenvectors for the spectral bands.
In this section we obtain using Lemma 3.1 asymptotics of solutions $\xi$ of the difference equation (2.3) in the cases of Model I and Model II. Using the statements from Section 2.3 we come from the asymptotics of these solutions to the conclusions about the spectral bands and critical points inside them.
In what follows we use the inverse Joukowsky mapping $\Phi: \mathbb{C} \mapsto \mathbb{C}$ defined as
\begin{equation}
\Phi(w) := w + \sqrt{w^2 - 1}, \quad w \in \mathbb{C},
\end{equation}
where the branch of the square root is chosen so that $|\Phi(w)| > 1$ for $w \in \mathbb{C}_+$. In the table below we list certain functions playing a role in the main theorem of this section.

| Function | Perturb. | Amplitude | Positional |
|----------|----------|-----------|------------|
| $\beta(k) :=$ | $\frac{\sin(kd)}{kd \sin \omega}$ | $\frac{\cos \omega}{2}$ |
| $\vartheta_{\pm}(k) :=$ | $\frac{1}{2} \arg \left( \mp \frac{\sin(kd)}{k} \right)$ | $\frac{1}{2} \arg (\cos \omega)$ |

**Theorem 3.5.** Let the parameters $d, a_0, c, \omega, \gamma$ and $\{q_n\}_{n \in \mathbb{N}}$ be as in (1.2). Let the function $L_0$ be as in (1.4). Assume either that $X = \bar{X}$, $\alpha = \bar{\alpha}$ as in Model I or that $X = \bar{X}$, $\alpha = \bar{\alpha}$ as in Model II. Then for $\lambda \in \mathbb{R}$ the finite difference equation (3.8) with $k = \sqrt{k}$ has a basis of solutions $\xi_{\pm}(k) = \{\xi_{\pm}(k)\}_{n \in \mathbb{N}}$ with the following asymptotics:

(i) If $|L_0(k)| \in \mathbb{R}_+ \setminus \{\cos \omega, 1\}$, then
\begin{equation}
\xi_{\pm}^n(k) = (\Phi(L_0(k)))^{\pm n} \left( 1 + o(1) \right) \quad \text{as} \quad n \to \infty,
\end{equation}
where $\Phi$ is defined in (3.18).

(ii) If $L_0(k) = \cos \omega$, then
\begin{align*}
\xi_+^n(k) &= \left( \cos(\omega n + \vartheta_+(k)) + o(1) \right) f_+^n(\beta(k)) \quad \text{as} \quad n \to \infty, \\
\xi_-^n(k) &= \left( \sin(\omega n + \vartheta_-(k)) + o(1) \right) f_-^n(\beta(k)) \quad \text{as} \quad n \to \infty.
\end{align*}
If $L_0(k) = -\cos \omega$, then
\begin{align*}
\xi_+^n(k) &= (-1)^n \left( \cos(\omega n + \vartheta_-(k)) + o(1) \right) f_+^n(\beta(k)) \quad \text{as} \quad n \to \infty, \\
\xi_-^n(k) &= (-1)^n \left( \sin(\omega n + \vartheta_+(k)) + o(1) \right) f_-^n(\beta(k)) \quad \text{as} \quad n \to \infty.
\end{align*}
Here $f_{\pm}^n(\cdot)$ are defined in (3.4) and the functions $\beta(\cdot)$ and $\vartheta_{\pm}(\cdot)$ are given in the 1st column of the Table 1 in the case of Model I and in the 2nd column in the case of Model II, respectively.

Proof. Let us fix $k = \sqrt{k}$ such that $|L_0(k)| \neq 1$. For such $k$ the limit property $\lim_{n \to +\infty} |\xi_n(k)| > 0$ holds, and the transfer matrix $T_n(k)$ of the discrete linear system (2.5) decomposes into the form (3.3) with the parameters $a, b$ and $l$ given in (3.10) for the Model I and given in (3.11) for the Model II, respectively.

In the case $|L_0(k)| \in \mathbb{R}_+ \setminus \{\cos \omega, 1\}$ Lemma 3.1(i) could be applied and it automatically implies the item (i). Assume now $L_0(k) = \pm \cos \omega$. In the case of Model I we plug into formulas (3.3) and (3.10) the values $a$ and $b$ from (3.10) and we easily get the following
\begin{equation}
z(k) = -z_1(k) = -\frac{c \sin(kd)e^{-2i\omega}}{4k \sin \omega}.
\end{equation}
In the case of Model II we plug into formulas (3.3) and (3.6) the values $a$ and $b$ from (3.11) and using that $L_\delta(k) = \pm \cos \omega$ after a long and tedious calculation we get

$$z(k) = z_1(k) = \frac{ic_0 e^{-2i\omega}}{2}.$$ 

Now Lemma 3.1(ii) could be applied. It gives us the asymptotics for the sequences of $\mathbb{C}^2$-vectors

$$\tilde{\varphi}_n^{\pm}(k) = \left(\xi_{n-1}^{\pm}(k), \xi_n^{\pm}(k)\right).$$

Extracting the second components from these asymptotics we obtain the statement. 

Given the asymptotics, we can now come to the conclusions about the structure of the purely absolutely continuous spectrum using the subordinacy theory, see Section 2.3. Let the parameters $d, \alpha_0, c, \omega$ and $\gamma$ be as in (1.2). Recall that the self-adjoint operator $H_\kappa$ corresponds to the Kronig-Penney without perturbation, where the distance between interaction is the constant $d$ and the strength of interactions is the constant $\alpha_0$. The spectral properties of this operator are discussed in the introduction. The absolutely continuous spectrum of the operator $H_\kappa$ is the set

$$\sigma_{ac}(H_\kappa) := \left\{ \lambda \in \mathbb{R} : L_\delta \left(\sqrt{\lambda}\right) \in [-1, 1] \right\}.$$

Denote the set of all critical points by

$$\mathcal{g}_{cr} := \left\{ \lambda \in \mathbb{R} : L_\delta \left(\sqrt{\lambda}\right) = \pm \cos \omega \right\} \subset \text{Int}(\sigma_{ac}(H_\kappa)) = \left\{ \lambda \in \mathbb{R} : L_\delta \left(\sqrt{\lambda}\right) \in (-1, 1) \right\}.$$

**Corollary 3.6.** Let the parameters $d, \alpha_0, c, \omega, \gamma$ and $\{q_n\}$ be as in (1.2) and let the sequences $\tilde{X}, \tilde{\alpha}$ and $\hat{X}, \hat{\alpha}$ be as in (1.3), (1.4) and in (1.5), (1.6), respectively. Then the following statements hold.

(i) The spectrum of the operators $H_{\kappa, \tilde{X}, \tilde{\alpha}}$ and $H_{\kappa, \hat{X}, \hat{\alpha}}$ is purely absolutely continuous on the set

$$\text{Int}(\sigma_{ac}(H_\kappa)) \setminus \mathcal{g}_{cr}.$$

(ii) Let $\lambda = k^2 \in \mathcal{g}_{cr}$. If either $\gamma < 1$ or

$$\gamma = 1 \quad \text{and} \quad \frac{c \sin(kd)}{2k \sin \omega} > 1,$$

then there exists a unique value $\kappa$ such that $\lambda$ is an embedded eigenvalue of $H_{\kappa, \tilde{X}, \tilde{\alpha}}$.

(iii) Let $\lambda = k^2 \in \mathcal{g}_{cr}$. If either $\gamma < 1$ or

$$\gamma = 1 \quad \text{and} \quad |\cos| > 1,$$

then there exists a unique value $\kappa$ such that $\lambda$ is an embedded eigenvalue of $H_{\kappa, \hat{X}, \hat{\alpha}}$.

**Proof.** For $k^2 \in \text{Int}(\sigma_{ac}(H_\kappa))$ one has that $\lim_{n \to \infty} |s_n(k)| > 0$. From the asymptotics of $\xi_n^\pm(k)$ and the formula (2.14) it follows that all the solutions of the spectral equations for both operators $H_{\kappa, \tilde{X}, \tilde{\alpha}}$ and $H_{\kappa, \hat{X}, \hat{\alpha}}$ are bounded. The assertion of the item (i) follows now from Proposition 2.1. Proposition 2.2 and boundedness of all solutions.

Conditions of the items (ii) and (iii) guarantee the existence of the square-integrable solution of the spectral equations at the point $\lambda$ for the operators $H_{\kappa, \tilde{X}, \tilde{\alpha}}$ and $H_{\kappa, \hat{X}, \hat{\alpha}}$, respectively. This fact means that there is a unique value of the boundary parameter $\kappa$ such that this solution satisfies the boundary condition, and thus in this case $\lambda$ is an eigenvalue.
3.4. Spectral gaps. In this section we show that the essential spectra of perturbed operators coincide with the essential spectrum of the non-perturbed Kronig-Penney model. Let the sets \( \sigma_{\alpha,\hat{X}} \) and \( \sigma_{\alpha,\hat{X}} \) be defined by \( \hat{X} \) and \( \hat{X} \), respectively, by the formula (2.12). Recall that the Weyl functions \( M_{\alpha,\hat{X}} \) and \( M_{\alpha,\hat{X}} \) are defined in (2.11) and operators \( B_{\alpha} \) and \( B_{\alpha} \) are defined in (2.13).

Theorem 3.7. Let the parameters \( d, \alpha_0, c, \omega, \gamma \) and \( \{q_n\} \) be as in (1.2). Let the sets \( \hat{X}, \hat{X} \) and the sequences \( \hat{\alpha}, \hat{\alpha} \) be as in (1.3), (1.5) and (1.3), (1.6), respectively.

Let the self-adjoint operator \( H_{\alpha} \) correspond to the non-perturbed Kronig-Penney model. Let the operators \( H_{\alpha,\hat{X},\hat{\alpha}} \) and \( H_{\alpha,\hat{X},\hat{\alpha}} \) correspond to \( \hat{X}, \hat{\alpha} \) and \( \hat{X}, \hat{\alpha} \) as in Section 2.1, respectively. Then the spectrum of both operators \( H_{\alpha,\hat{X},\hat{\alpha}} \) and \( H_{\alpha,\hat{X},\hat{\alpha}} \) on the set \( \mathbb{R} \setminus \sigma_{\text{ess}}(H_{\alpha}) \) is pure point.

Proof. (i) Define \( B_{\alpha_0} := -\alpha_0 I \), a diagonal operator in \( L^2(\mathbb{N}) \). One has: \( \hat{\alpha}_n \to \alpha_0 \) as \( n \to \infty \). Thus

\[ B_{\alpha} - B_{\alpha_0} \in \mathcal{S}_\infty. \]

Consider \( \lambda \in \mathbb{R} \setminus (\sigma_{\text{ess}}(H_{\alpha}) \cup \sigma_{\alpha,\hat{X}}) \). By Proposition 2.6, \( 0 \notin \sigma_{\text{ess}}(B_{\alpha_0} - M_{\alpha,\hat{X}}(\lambda)) \). Therefore by compact perturbation argument \( 0 \notin \sigma_{\text{ess}}(B_{\alpha} - M_{\alpha,\hat{X}}(\lambda)) \) and again by Proposition 2.6 \( \lambda \notin \sigma_{\text{ess}}(H_{\alpha,\hat{X},\hat{\alpha}}) \). Thus the spectrum of the operator \( H_{\alpha,\hat{X},\hat{\alpha}} \) on the set \( \mathbb{R} \setminus (\sigma_{\text{ess}}(H_{\alpha}) \cup \sigma_{\alpha,\hat{X}}) \) is discrete. Since the set \( \sigma_{\alpha,\hat{X}} \cap (\mathbb{R} \setminus \sigma_{\text{ess}}(H_{\alpha})) \) is at most countable, it cannot support absolutely continuous or singular continuous spectral measure.

(ii) One has \( B_{\alpha} = B_{\alpha_0} \). Consider \( \lambda \in \mathbb{R} \setminus (\sigma_{\text{ess}}(H_{\alpha}) \cup \sigma_{\alpha,\hat{X}} \cup \sigma_{\alpha,\hat{X}}) \). Let \( \hat{b}_n(k) \) be diagonal and \( \hat{a}_n(k) \) be anti-diagonal entries of the Jacobi matrix \( M_{\alpha,\hat{X}}(\lambda) \) (where \( \lambda = k^2 \)). Clearly \( \hat{a}_n(k) \to -\frac{k}{\sin(kd)} \) and \( \hat{b}_n(k) \to 2k \cot(kd) \) as \( n \to \infty \). It follows that

\[ M_{\alpha,\hat{X}}(k^2) - M_{\alpha,\hat{X}}(k^2) \in \mathcal{S}_\infty. \]

Again, Proposition 2.6 implies \( \lambda \notin \sigma_{\text{ess}}(H_{\alpha,\hat{X},\hat{\alpha}}) \). Hence the spectrum of \( H_{\alpha,\hat{X},\hat{\alpha}} \) on \( \mathbb{R} \setminus (\sigma_{\text{ess}}(H_{\alpha}) \cup \sigma_{\alpha,\hat{X}} \cup \sigma_{\alpha,\hat{X}}) \) is discrete. The set \( (\sigma_{\alpha,\hat{X}} \cup \sigma_{\alpha,\hat{X}}) \cap (\mathbb{R} \setminus \sigma_{\text{ess}}(H_{\alpha})) \) is at most countable, so the spectrum of \( H_{\alpha,\hat{X},\hat{\alpha}} \) on the set \( \mathbb{R} \setminus \sigma_{\text{ess}}(H_{\alpha}) \) is pure point.

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References

[AGHH05] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, Solvable models in quantum mechanics. With an appendix by Pavel Exner., AMS Chelsea Publishing, 2005.

[Al72] S. Albeverio, On bound states in the continuum of N-body systems and the virial theorem, Ann. Physics 71 (1972), 167–276.

[AKM10] S. Albeverio, A. Kostenko and M. Malamud, Spectral theory of semibounded Sturm-Liouville operators with local interactions on a discrete set, J. Math. Phys. 51 (2010), 102102.
[Be91] H. Behncke, Absolute continuity of hamiltonians with von Neumann-Wigner Potentials I, Proc. AMS 111 (1991), 373–384.

[Be94] H. Behncke, The m-function for Hamiltonians with Wigner-von Neumann potentials, J. Math. Phys. 35 (1994), 1445–1462.

[BL87] Z. Benzaid and D. Lutz, Asymptotic representation of solutions of perturbed systems of linear difference equations, Stud. Appl. Math. 77 (1987), 195–221.

[Br76] V. M. Bruk, A certain class of boundary value problems with a spectral parameter in the boundary condition, Mat. Sb. (N.S.) 100 (142) (1976), 210–216 (in Russian); translated in: Math. USSR-Sb. 29 (1976), 186–192.

[BGP08] J. Brüning, V. Geyler and K. Pankrashkin, Spectra of self-adjoint extensions and applications to solvable Schrödinger operators, Rev. Math. Phys. 20 (2008), 1–70.

[DM95] V. A. Derkach and M. M. Malamud, The extension theory of Hermitian operators and the moment problem, J. Math. Sci. (New York) 73 (1995), 141–242.

[E97] P. Exner, A duality between Schrödinger operators on graphs and certain Jacobi matrices, Ann. Inst. Henri Poincare, Phys. Theor. 66 (1997), 359–371.

[GO10] N. Goloschapova and L. Oridoroga, On the negative spectrum of one-dimensional Schrödinger operators with point interactions, Integral Equations Operator Theory 67 (2010), 1–14.

[GP87] D. Gilbert and D. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, J. Math. Anal. Appl. 128 (1987), 30–56.

[HT10] J. Hernandez-Herrejon, F. Izrailev and L. Tessner, Anomalous localization in the aperiodic Kronig-Penney model, J. Phys. A, Math. Theor. 43 (2010), 425004.

[HL75] W. A. jun. Harris and D. A. Lutz, Asymptotic integration of adiabatic oscillators, J. Math. Anal. Appl. 51 (1975), 76–93.

[HKS91] D. Hinton, M. Klaus and J. Shaw, Embedded halfbound states for potentials of Wigner-von Neumann type, Proc. Lond. Math. Soc. 62 (1991), 607–646.

[J06] J. Janas, The asymptotic analysis of generalized eigenvectors of some Jacobi operators. Jordan box case, J. Difference Equ. Appl., 12 (2006), 597–618.

[JS10] J. Janas and S. Simonov, Weyl-Titchmarsh type formula for discrete Schroedinger operator with Wigner-von Neumann potential, Studia Math. 201 (2010), 167–189.

[KP92] S. Khan and D. Pearson, Subordinacy and spectral theory for infinite matrices, Helv. Phys. Acta 65 (1992), 905–927.

[Ko75] A. Kochubei, Extensions of symmetric operators and symmetric binary relations, Mat. Zametki 17 (1975), 41–48 (in Russian); translated in: Math. Notes 17 (1975), 25–28.

[Ko89] A. Kochubei, One-dimensional point interactions, Ukrain. Math. J. 41 (1989), 1301–1395.

[KM10] A. Kostenko and M. Malamud, 1-D Schrödinger operators with local point interactions on a discrete set, J. Differential Equations 249 (2010), 253–304.

[KN07] P. Kurasov and S. Naboko, Wigner-von Neumann perturbations of a periodic potential: spectral singularities in bands, Math. Proc. Camb. Philos. Soc. 142 (2007), 161–183.

[KP31] R. de L. Kronig and W. G. Penney, Quantum mechanics of electrons in crystal lattices, Proc. Roy. Soc. Lond. 130 (A) (1931), 499–513.

[KN10] P. Kurasov and S. Naboko, Wigner-von Neumann perturbations of a periodic potential: spectral singularities in bands, Math. Proc. Camb. Philos. Soc. 142 (2007), 161–183.

[KS11] P. Kurasov and S. Simonov, Weyl-Titchmarsh type formula for periodic Schrödinger operator with Wigner-von Neumann potential, submitted, arXiv:1102.5213.

[L48] N. Levinson, The asymptotic nature of solutions of linear systems of differential equations, Duke Math. J. 15 (1948), 111–126.

[L10] M. Luik, Orthogonal polynomials with recursion coefficients of generalized bounded variation, Comm. Math. Phys. 306 (2011), 485–509.

[Ma73] V. B. Matveev, Wave operators and positive eigenvalues for a Schrödinger equation with oscillating potential, Theor. Math. Phys. 15 (1973), 574–583.

[M95] V. A. Mikhailets, Spectral properties of the one-dimensional Schrödinger operator with point interactions, Rep. Math. Phys. 36 (1995), 495–500.

[NS11] S. Naboko and S. Simonov, Zeros of the spectral density of the periodic Schrödinger operator with Wigner-von Neumann potential, submitted, arXiv:1102.5207.
[N07] P. Nesterov, Averaging method in the asymptotic integration problem for systems with oscillatory-decreasing coefficients, *Differ. Equ.* **43** (2007), 745–756.

[vNW29] J. von Neumann and E. Wigner, Über das Verhalten von Eigenwerten bei adiabatischen Prozessen, *Physikal. Z.* **30** (1929), 467–470.

[RS78] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, vol. 4*, New York: Academic Press, 1978.

[SCS94] C. Shubin Christ and G. Stolz, Spectral theory of one-dimensional Schrödinger operators with point interactions, *J. Math. Anal. Appl.*, **184** (1994), 491–516.

[S92] G. Stolz, Bounded solutions and absolute continuity of Sturm-Liouville operators, *J. Math. Anal. Appl.* **169** (1992), 210–228.

Department of Mathematics, Saint-Petersburg State University of Information Technologies, Mechanics and Optics, Kronverkskiy, 49, Saint-Petersburg, Russia

Department of Computational Mathematics, Graz University of Technology, Steyrergasse 30, 8010, Graz, Austria

E-mail address: vladimir.lotoreichik@gmail.com

Chebyshev Laboratory, Department of Mathematics and Mechanics, Saint-Petersburg State University, 14th Line, 29b, Saint-Petersburg, 199178, Russia

Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse, 8–10, Wien, A–1040, Austria

E-mail address: sergey.a.simonov@gmail.com