THE OPTIMAL SOLUTION TO A PRINCIPAL-AGENT PROBLEM WITH UNKNOWN AGENT ABILITY

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ABSTRACT. We investigate a principal-agent model featured with unknown agent ability. Under the exponential utilities, the necessary and sufficient conditions of the incentive contract are derived by utilizing the martingale and variational methods, and the solutions of the optimal contracts are obtained by using the stochastic maximum principle. The ability uncertainty reduces the principal’s ability of incentive provision. It is shown that as time goes by, the information about the ability accumulates, giving the agent less space for belief manipulation, and incentive provision will become easier. Namely, as the contractual time tends to infinity (long-term), the agent ability is revealed completely, the ability uncertainty disappears, and the optimal contracts under known and unknown ability become identical.

1. Introduction and literature review. In a principal-agent problem, the principal (investor) hires an agent (manager) to manage a firm or projects. In the case of moral hazard, the agent’s effort or action is unobservable. Therefore, in order to motivate the agent to take the recommended effort, the principal has to provide the agent with incentives. In addition to the unobservable effort process, an unknown agent ability makes this problem more complicated. To deal with this issue, the principal needs to update the information about the agent ability through time. In

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existing literature, the principal-agent problem features learning process (see Adrian and Westerfield[1], Bergemann and Hege[2], etc.), which makes solving the optimal contract become difficult. In this work, we aim to find solutions to a principal-agent problem with known and unknown agent ability under certain assumptions.

In our dynamic environment, we assume that both the contractual parties have the same common prior about the project, which means that they have the same information on the equilibrium path. On the non-equilibrium path, only the agent knows his actual effort level so that he has certain information advantage over the principal. Specifically, if the agent follows the recommended effort, the principal would have the accurate belief about agent ability and the project profitability. If the agent is currently shirking, the lower level effort leads to a reduction on the project output. Following Bayesian learning process, the principal would attribute the low output to low profitability by mistake. By shirking, the agent can shift the principal’s inference about the profitability or the agent ability downward. When the future output is higher than that inferred by the principal, the agent gets more reward, which is regarded as the information rent caused by private information. This is called a belief manipulation. In a salesforce contract, Chen et al. (2017)[5] show that ability uncertainty helps the agent to get higher income while putting forth less effort. There’s no doubt that belief manipulation is beneficial to the agent. To avoid belief manipulation when the agent ability is unknown, the principal should provide the agent with a contract in which the payment scheme is closely linked to the agent’s performance to lower his welfare. Wang et al. (2017)[21] study the wage contract design in moral hazard and show that private information lowers the principal’s profits. Bergemann and Hege (1998)[2] and Hörner and Samuelson (2013)[13] investigate the principal-agent problem with long-last belief manipulation. In Bergemann and Hege (1998)[2], the agent manages a project which may succeed with a probability depending on the agent ability. The agent’s effort and the agent ability affect the success rate in a multiplicative way. In our model, we apply an additive output function where the agent’s effort is independent of the agent ability.

Unknown agent ability (or project quality) in dynamic contractual problems has drawn much attention from many researchers. For example, Hopenhayn and Jarque (2007)[12] study the problem with persistent unknown ability and a single effort decision at the beginning. Fudenburg and Rayo (2019)[7] consider the optimal contract for a principal with commitment power to train a cash-constrained agent and find that the optimal contract for the principal is inefficient because the agent receives slow training and will work inefficiently hard to compensate the principal for this training. Adrian and Westerfield (2009)[1] investigate a dynamic contracting model in which there exists a disagreement between the contractual parties about the resolution of uncertainty, and illustrate the contract design in an application with Bayesian learning. A closed form solution to the optimal contract with the disagreement and several meaningful conclusions are found in Adrian and Westerfield (2009)[1]. As the principal and the agent disagree about the belief of the agent ability, the belief manipulation vanishes. Our model mainly focuses on discussing how the known and unknown abilities affect the optimal contract.

Giat et al. (2009)[9] build a structural model based on that in Holmstrom and Milgrom (1987)[11] and analyze the effects of asymmetric beliefs. Prat and Jovanovic (2014)[16] consider a repeated principal-agent problem with unknown agent ability, where a risk-averse agent receives payments periodically. He et al.
focus on stationary learning and show that, on average, the recommended effort in the optimal contract decreases over time. Uğurlu (2018)\cite{19} investigates the principal-agent problem with risk-averse principal and agent, and uses non-stationary learning process to characterize the information of the agent ability. Wang and Yang (2019)\cite{20} show that when the agent has limited commitment, stochastic outside opportunities and random demands for his ability, both good and bad outside offers could lead to termination of the contract. Wong (2019)\cite{24} investigates a principal-agent model in which a multitasking agent engages in unobserved risk-taking and points out that completely downside risk hedging induces the agent to gamble and optimal contract incentivizes excessive risk-taking when the agent has insufficient skin in the game. In the case of hidden savings, Mitchell and Zhang (2010)\cite{15} consider the contract design of unemployment insurance by using exponential utility.

In this paper, we assume that the principal cannot observe the agent’s actual effort, but he can observe the agent’s wealth process. Thus, without loss of generality, we let the agent’s wealth remain zero(see \cite{23}). Following Holmstrom and Milgrom (1987)\cite{11}, in order to obtain the solutions of incentive contract, we assume that the principal and agent possess exponential utilities. When the model is embedded with unknown agent ability, for simplicity, we solve the contract with three steps. Firstly, as suggested in Sannikov (2008)\cite{17}, we cast our model in continuous time so that we are able to use the optimization techniques originally introduced by Schättler and Sung (1993)\cite{18}. Secondly, we assume both the ability and noise are normally distributed so that the ability is derived from the conjugate priors of likelihood function. Thirdly, we apply linearity to the output process. Williams (2009, 2015)\cite{22, 23} studies the contractual problem under the assumption of linear technologies. This makes the value of private information independent of the posteriors, and as a result, we only need to capture the information of the cumulative output.

As the implementable contract relies on the promised utility, there is history-dependence. To deal with this problem, we apply a change of variables introduced by Bismut (1978)\cite{4}. The optimal contract in our model is derived via the first-order approach. Using the martingale method and the variational method, we derive the necessary and sufficient conditions for the incentive contracts. When the agent ability is known, we can easily obtain the optimal contracts. By contrast, when the agent ability is unknown, a learning process is added. As time goes by, the agent ability is progressively revealed and the principal’s power to provide incentives becomes stronger. Thus the payment received by the agent is front-loaded. When the contractual time tends to infinity (long-term contract), the unknown ability case would become the known one.

The novelty of our work has the following folds. Different from the constant productivity of effort in Prat and Jovanovic (2014)\cite{16} and Williams (2015)\cite{23}, we discuss a general form of the effort process. That is, the function of efforts can take many forms instead of a linear function (an example is given in Section 4). Williams (2015)\cite{23} investigates a continuous time dynamic moral model with hidden actions and hidden states, we consider ability uncertainty in the output process for the principal. Comparing with the works in Adrian and Westerfield (2009)\cite{1}, in which the principal and agent would have different beliefs, our model only contains the
known or the unknown ability in the output process of the principal. The backward stochastic differential equation of the agent’s promised utility differs from the equation expressing the agent’s beliefs in Adrian and Westerfield (2009)[1].

The rest of this paper proceeds as follows. In Section 2, we give the general settings of the model and cast a change of variables. In Section 3, we derive the incentive compatible conditions including the necessary condition and the sufficient condition for the recommended effort. Section 4 solves the optimal contract under both known and unknown ability with an example of a specific form of the effort function. The numerical analysis is provided in Section 5. A conclusion remark is given in Section 6.

2. Model and methodology. In this section, we introduce the basic settings of our model and approaches about the change of variable.

2.1. The model. \(\{W_t\}_{t \geq 0}\) is a standard Brownian motion on the probability space \((\Omega, \mathcal{F}, P)\), where \(\mathcal{F}_t\) denotes the filtration generated by Brownian motion \(W_t\). In time interval \([0, t]\), we assume that the cumulative output \(Y_t\) satisfies the following stochastic integral equation

\[
Y_t = \int_0^t (f(e_s) + \eta) \, ds + \int_0^t \sigma dW_s,
\]

where \(e_t\) is the agent’s effort level. Function \(f(e_t)\) satisfies \(f(e) \geq 0\), its first order derivative \(f'(e) > 0\) and second order derivative \(f''(e) \leq 0\). For \(0 \leq t \leq T\), \(e_t \in [0, M]\), in which \(M > 0\) is the maximum effort that the agent can exert. The time-invariant agent ability is denoted by \(\eta\). Constant \(\sigma\) is the volatility. In our model, the agent’s effort choice affects the output without affecting its volatility. The differential form of the output process evolves by

\[
dY_t = (f(e_t) + \eta) \, dt + \sigma dW_t.
\]

At time 0, the agent ability \(\eta\) is unknown. Both the principal and the agent have common priors about the ability which are normal with precision \(h_0\) and mean \(m_0\). The posteriors are denoted by \(\hat{\eta}\) which rely on the output \(Y_t\) and the cumulative effort \(\alpha_t \equiv \int_0^t f(e_s) \, ds\). Thus, the mean of the normal posteriors \(\hat{\eta}\) follows

\[
\hat{\eta}(Y_t - \alpha_t, t) = E_t[\eta | Y_t, \alpha_t] = \frac{h_0 m_0 + \sigma^{-2} (Y_t - \alpha_t)}{h_t}
\]

and the precision evolves by

\[
h_t = h_0 + \sigma^{-2} t.
\]

At time 0, we have

\[
\hat{\eta}(0, 0) = m_0.
\]

If the priors over the mean of the normal distribution process are given, all the statistical information is reflected by cumulative output \(Y_t\), cumulative effort \(\alpha_t\) and time \(t\). The fact that the beliefs depend on the history of agent’s effort through \(\alpha_t\) alone is useful to characterize the incentive contract. Therefore, we only need to keep track of the information provided by the cumulative effort instead of by the whole effort path in solving the optimal contract.

In a moral hazard model, with the unobservable effort, the principal has to assume that the agent exerts the equilibrium effort \(e^*\). The principal’s beliefs satisfy
Brownian motion to the filtering theorem in Fujisaki et al. (1972)[8], we have the following standard

$$Y_t$$ believes that the output is

$$F_0 = \{ \mathcal{F}_t \}_{t \geq 0}$$ be the P-augmentation filtration of $$\mathcal{F}$$.

When the agent provides the effort sequence $$\{e_t; 0 \leq t \leq T\}$$, the principal believes that the output is $$Y_t$$ with the cumulative external shocks $$W_t$$. According to the filtering theorem in Fujisaki et al. (1972)[8], we have the following standard Brownian motion

$$dW_t = \frac{1}{\sigma} [dY_t - (f(e_t) + \eta(Y_t - \alpha_t, t)) dt]$$

on the probability space $$(\Omega, \mathcal{F}_t, \mathbb{P})$$. Moreover, $$\eta$$, which is a P-martingale, evolves by

$$d\eta(Y_t - \alpha_t, t) = \sigma^{-1} \frac{\partial f}{\partial y} dt + \sigma dW_t$$

with decreasing variance. In our settings, we denote the principal’s assets by $$y_t$$, which grows at the risk-free rate $$r$$. Furthermore, with the agent’s payment $$s_t \in S$$ and the principal’s consumption (or dividend) $$d_t$$, the principal’s assets $$y_t$$ evolve by

$$dy_t = (r_y + f(e_t) + \eta - s_t - d_t) dt + \sigma dW_t.$$  

As the assets $$y_t$$ carry the same information as output $$Y_t$$, we regard assets $$y_t$$ as “output”.

The agent’s effort $$e$$ belongs to the class of control processes $$A \triangleq \{ e : [0, T] \times \Omega \rightarrow [0, M] \}$$ which are $$\mathcal{F}_t$$-predictable. Although the principal does not observe the agent’s actual effort, he observes the output of the project. Thus the principal’s available information corresponds to the filtration $$\mathcal{F}_t^\mathcal{F} \triangleq \sigma \{ y_t; 0 \leq s \leq t \}$$ which is generated by output $$y$$. We define the $$\mathcal{F}_t$$-augmentation filtration as $$\mathbb{P}^y = \{ \mathcal{F}_t^y \}_{t \geq 0}$$. Let $$C$$ be the space of continuous functions mapping from $$[0, T]$$ into $$\mathbb{R}$$. The time path of output $$\bar{y} = \{ y_t; t \in [0, T] \}$$ is a random element in $$C$$, which defines the output path observed by the principal. We let $$W_t^0 = \omega_t$$ be the family of coordinate functions with the filtration $$\mathcal{F}_t^0 = \sigma \{ W_s^0; s \leq t \}$$. Let $$P^0$$ be the corresponding Wiener measure on $$\{ \Omega, \mathcal{F}_t^0 \}$$ and $$\mathcal{F}_t$$ be the completion of $$\mathcal{F}_t^0$$ with the null sets of $$\mathcal{F}_t^0$$. In the space $$(\Omega, \mathcal{F}_t, \mathbb{P})$$, we define $$W_t^0$$ as the Brownian motion in (1). For any time $$t$$, both the agent’s payment and the recommended effort $$(\hat{e}, \hat{s})$$ depend on the whole history of output $$\bar{y}_t$$. To deal with moral hazard, we assume that the agent’s consumption, the recommended consumption, and the agent’s payment are identical, i.e., $$\hat{e}_t = c_t = s_t$$ (see Williams 2015[23]). For a given contract, the agent chooses his own level of effort. Consequently, the admissible set of agent’s actions is a $$\mathcal{F}_t$$-measurable function $$(\hat{e}, \hat{c}) : [0, T] \times C \rightarrow [0, M] \times S$$. After accepting the contract, the agent remains in the contract until the termination date $$T$$. If the agent accepts the contract at time zero and chooses the principal’s recommended actions $$(\hat{e}, \hat{c})$$ during the contract period, i.e., $$(e, c) = (\hat{e}, \hat{c})$$, this contract is called an implementable contract.

In the model, both the principal and the agent are risk-averse and have exponential utilities, which are second-order continuous differential functions. Specifically, we assume that the agent’s utility takes the following form

$$u(c, c) = -\exp \left( -\lambda (c - \mu e) \right),$$

where $$\lambda$$ is the risk aversion coefficient, $$\mu$$ is a parameter related to endogenous technology and can be expressed as a function of volatility. To ensure that $$e = M$$
is the agent’s optimal strategy, we restrict \( \mu \) to the interval \([0, C_0]\). \( \mu \) is the agent’s cost associated with endogenous technology and effort. Taking advantage of the terminal condition of the agent’s expected utility in Williams (2015)[23], we have

\[
v(s_T, m_T) = V_T(s_T + m_T),
\]

\[
V_T(x) = -\frac{1}{r} \exp\left(\frac{U - \rho}{r} - \lambda rx\right).
\]

In moral hazard cases, we usually assume that the agent keeps the wealth \( m_t \) at zero and does not borrow or save from outside. That is, \( m_t \equiv 0 \) and \( c_t = s_t \). Therefore, the agent’s expected utility at terminal date can be written as \( \upsilon(s_T) = V_T(s_T) \).

For a given process \((\bar{e}, \bar{c})\), the agent’s expected discounted utility at \( t = 0 \) evolves by

\[
V(\bar{e}, \bar{c}) = \mathbb{E}\left[\int_0^T e^{-\rho t} u(c_t, e_t) dt + e^{-\rho T} \upsilon(s_T)\right], \tag{9}
\]

where \( \rho \) is the discount rate.

Similarly, the principal’s utility evolves by

\[
u(d) = -\exp(-\lambda d), \tag{10}\]

where \( d \) is the principal’s consumption or dividend. The terminal condition of the principal’s expected utility is \( L(y_T, s_T) = V_T(y_T - s_T) \). For a given process \( \bar{d} \), the principal’s expected discounted utility at \( t = 0 \) evolves by

\[
U(\bar{d}) = \mathbb{E}\left[\int_0^T e^{-\rho t} u(d_t) dt + e^{-\rho T} L(y_T, s_T)\right]. \tag{11}
\]

During the entire contract period, maximizing \( U(\bar{d}) \) is the principal’s goal.

2.2. The change of variables. As mentioned before, since the whole history path of output \( \bar{y} \) is involved in contract as a state variable, we cannot deal with the agent’s problem directly. In order to overcome this problem, we replace the output process with the density of the output and use it as a key variable. This change of variable method is also applied in Bismut (1973, 1978)[3, 4], Cvitanić et al. (2009)[6] and Williams (2015)[23]. Let \( W_t^0 \) be a Wiener process on space \( C \). The agent’s different choice of effort changes the distribution of output. Hence, the agent’s choice of effort is the choice of a probability measure over output. We regard the relative density \( \Gamma_t(\bar{e}) \) as a key state variable. Next, we show the details of the change of variables.

We denote the drift term of output by \( g \), namely,

\[
g(t, \bar{y}, e) = ry_t + f(e_t) + \bar{q}(Y_t - \alpha_t, t) - s_t - d_t, \tag{12}\]

in which \( \bar{y} \) is the history path of output. The relative density \( \Gamma_t(\bar{e}) \), which depends on the effort \( \bar{e} \), is a \( \mathcal{F}_t \)-measurable process and has the following form

\[
\Gamma_t(\bar{e}) = \exp\left(\int_0^t \sigma^{-1} g dW_s^0 - \frac{1}{2} \int_0^t |\sigma^{-1} g|^2 ds\right). \tag{13}
\]

Since the settings of \( g \) guarantee the Novikov’s condition, for all \( \bar{e} \in \mathcal{A} \), \( \Gamma_t \) is a \( \mathcal{F}_t \)-measurable martingale with \( \mathbb{E}[\Gamma_T(\bar{e})] = 1 \). Using Girsanov theorem, we have a new measure \( P^{\bar{e}} \)

\[
\frac{dP^{\bar{e}}}{dP^0} = \Gamma_T(\bar{e}).
\]

and consequently, the Brownian motion \( W_t^{\bar{e}} \) evolves by

\[
W_t^{\bar{e}} = W_t^0 - \int_0^t \sigma^{-1} g(s, \bar{y}, e_s) ds. \tag{14}
\]
In addition, we have the following stochastic differential equation (SDE)

\[ dy_t = \sigma dW_t^0 \] (15)

\[ = \sigma \left[ dW_t^e + \sigma^{-1} g(t, \tilde{y}, e_t) dt \right] \] (16)

\[ = g(t, \tilde{y}, e_t) dt + \sigma dW_t^e. \] (17)

For a given \( \sigma > 0 \), (15) indicates that the output is a \( P^0 \)-martingale. When the agent exerts \( \bar{e} \), the drift term of the output (15) equals zero for \( t \in [0, T] \). When the agent chooses \( \bar{e} \), we have (17). Differentiating \( \Gamma_t \), we obtain

\[ d\Gamma_t = \Gamma_t \sigma^{-1} \left( r y_t + f(e_t) + \hat{\eta}(Y_t - \alpha_t, t) - s_t - d_t \right) dW_t^0 \] (18)

with the initial condition \( \Gamma_0 = 1 \). (18) is a SDE with random coefficient. Different from the key state variable which depends directly on the whole history of output, the transformed state variable relies on a fixed path of output \( \tilde{y} \).

As we can see, different choice of effort corresponds to different Brownian motion. By Ito’s lemma, the agent ability \( \hat{\eta} \) under \( P^\bar{e} \) has the following form

\[ d\hat{\eta}(Y_t - \alpha_t, t) = \sigma^{-1} h_t dW_t^e. \]

Under the relative density process \( \Gamma_t \), the agent’s problem can be written as

\[ V(\bar{e}, \bar{c}) = E^0 \left[ \int_T^0 \Gamma_s e^{-\rho(s-t)} u(c_s, e_s) ds + e^{-\rho(T-t)} v(s_T, 0) | F_t \right], \] (19)

where \( E^0 \) is the expectation under \( P^0 \).

3. Incentive compatible conditions.

3.1. The agent’s promised utility. For a given effort level, we first derive the necessary condition for the optimal contract. Subsequently, we impose restrictions on the necessary condition and obtain the sufficient condition of the optimal contract. At time 0, the only constraint of the contract is the participation constraint, which ensures the agent’s participation. We assume that the agent’s outside reservation utility is \( V_0 \). Therefore, the optimal contract at time 0 must satisfy the participation constraint, i.e., \( V(\bar{e}, \bar{c}) \geq V_0 \).

Since the participation constraint only needs to be satisfied at time 0, it can be obtained by standard Lagrangian method. As the agent’s effort is unobservable, the contracts should depend on the agent’s promised utility. Thus, we introduce the agent’s promised utility which is crucial for characterizing the contracts with private information. Specifically, the agent’s promised utility \( q_t \) is the remaining expected discounted utility in the contract from time \( t \) onward, which takes the following form

\[ q_t = E \left[ \int_t^T e^{-\rho(s-t)} u(c_s, e_s) ds + e^{-\rho(T-t)} v(s_T, 0) | F_t \right]. \] (20)

Through calculating the expectation under \( F_t \), we obtain the level of the agent’s promised utility which varies with the cumulative effort. In addition, with unobservable effort, the principal should keep track of all the levels of the promised utility. In continuous time, the first-order approach simplifies the problem of solving the contracts. We consider the agent’s promised utility on the equilibrium path, and then show that the solution to the agent’s problem is globally optimal so as
to establish the incentive compatible conditions. By the martingale representation theorem, the backward stochastic differential form of the promised utility reads
\[
\begin{aligned}
dq_t &= [\rho q_t - u(c_t, e_t)]dt + \gamma_t \sigma dW_t, \\
q_T &= \upsilon(s_T, 0),
\end{aligned}
\] (21)
where \(\gamma_t\) is the sensitivity of the agent’s promised utility to external shocks. In moral hazard environment, \(\gamma_t\) is closely linked to incentive provision and is generally referred as the incentive compatible parameter.

3.2. Necessary condition. To find the solutions of the optimal contract, we solve the agent’s problem by maximizing the agent’s expected utility. As the objective function (20) depends on the agent’s payment, which is non-Markovian because of the history dependence, we cannot use the standard method to analyze the optimization problem. For a given contract \((c, e)\), the agent controls the distribution of his payment by changing his effort strategies. The agent also chooses the appropriate probability measure according to his actual payment \(s_t\). Because the Radon-Nikodym derivative associated with any effort path is a Markovian process, we apply the martingale method to solve the agent’s optimization problem. The idea of using the distribution function as a state variable to solve the principal-agent problem dates back to Mirlees (1976)\[14\]. In our model, the agent ability is unknown and the principal gradually captures the information of it over time. The learning process complicates the optimal contract problem. Following Cvitanić et al. (2009)\[6\] and Prat and Jovanovic (2014) \[16\], we give the necessary condition for agent’s problem. Before doing so, we define the following Hamiltonian
\[
H(t, y, e, \alpha, \gamma) = u(c_t, e_t) + (\rho y_t + f(e_t) + \hat{\eta}(\alpha_t, Y_t) - s_t - d_t)\gamma_t.
\] (22)
The value function of the agent’s problem is denoted by \(V(t, e)\).

**Proposition 1.** Maximizing the Hamiltonian defined by (22) is sufficient for maximizing the agent’s value function \(V(t, e)\). For any \(e_t \in [0, M]\), the necessary condition for the agent’s optimal effort \(e_t^*\) is that the incentive compatible parameter \(\gamma_t\) satisfies
\[
\left[ (\gamma_t + \frac{\sigma^2}{h_t} p_t) f'(e_t) + u_e(c_t, e_t) \right] (e_t - e_t^*) \leq 0,
\] (23)
where
\[
p_t = h_t E\left[ - \int_t^T e^{-\rho(s-t)} \frac{\gamma_s}{h_s} ds | \mathcal{F}_t \right].
\] (24)

**Proof.** We denote the agent’s optimal effort by \(\hat{e}\). In the time interval \([t, T]\), integrating both sides of (21), we have
\[
e^{-\rho t} q_T = e^{-\rho t} v(s_T)
\]
\[
= e^{-\rho t} V(t, \hat{e}) - \int_t^T e^{-\rho s} u(c_s, \hat{e}_s) ds + \int_t^T \hat{\zeta}_s \sigma dW_s^{\hat{e}},
\] (25)
where \(\hat{\zeta}_s = e^{-\rho s} \gamma_s\). We denote any other effort level by \(\bar{e}\). In the time interval \([t, T]\), integrating both sides of (21) yields
\[
e^{-\rho t} q_T = e^{-\rho t} v(s_T)
\]
\[
= e^{-\rho t} V(t, \bar{e}) - \int_t^T e^{-\rho s} u(c_s, \bar{e}_s) ds + \int_t^T \bar{\zeta}_s \sigma dW_s^{\bar{e}},
\] (26)
where \( \zeta_s = e^{-\rho_s \gamma_s} \). According to the derivation process of the change of variables, from (14)-(17), we obtain
\[
dt y_t = \sigma dW^0_t,
\]
\[
dt W^\epsilon_t = dW^0_t - \frac{1}{\sigma} (r y_t + f(\bar{e}_t) + \bar{\eta}(Y_t - \bar{\alpha}_t, t) - s_t - d_t) dt,
\]
\[
dt W^\bar{\epsilon}_t = dW^0_t - \frac{1}{\sigma} (r y_t + f(\bar{e}_t) + \bar{\eta}(Y_t - \bar{\alpha}_t, t) - s_t - d_t) dt.
\]
From (27) and (28), we have
\[
dt W^\epsilon_t = dt + \frac{1}{\sigma} [f(\bar{e}_t) - f(\bar{e}_t) + (\bar{\eta}(Y_t - \bar{\alpha}_t, t) - \bar{\eta}(Y_t - \bar{\alpha}_t, t))] dt,
\]
from which we derive
\[
V(t, \bar{\epsilon}) - V(t, \bar{\epsilon}) = e^{\rho t} E_t \left[ \int_t^T e^{-\rho s} \left[ u(c_s, \bar{e}_s) - u(c_s, \bar{e}_s) \right] ds + \int_t^T e^{-\rho s} (\bar{\gamma}_s - \bar{\gamma}_s) \sigma dW^\bar{\epsilon}_s \right. \\
+ \int_t^T e^{-\rho s} \bar{\gamma}_s \left[ f(\bar{e}_s) - f(\bar{e}_s) + (\bar{\eta}(Y_s - \bar{\alpha}_s, s) - \bar{\eta}(Y_s - \bar{\alpha}_s, s)) \right] ds \\
= e^{\rho t} E_t \left[ \int_t^T e^{-\rho s} \left[ H(\bar{e}, \bar{\gamma}) - H(\bar{e}, \bar{\gamma}) \right] ds + \int_t^T e^{-\rho s} \bar{\gamma}_s \sigma dW^\bar{\epsilon}_s \right]
\]
\[
\leq e^{\rho t} E_t \left[ \int_t^T e^{-\rho s} \bar{\gamma}_s \sigma dW^\bar{\epsilon}_s \right] = 0.
\]
According to (25), (26), (27) and \( E_T \left[ \int_t^T e^{-\rho s} \bar{\gamma}_s \sigma dW^\bar{\epsilon}_s \right] = 0 \), equation (30) holds. Using the definition of the Hamiltonian, equation (31) is established. Since \( \bar{\epsilon} \) is the optimal effort and \( H(t, y, \bar{\epsilon}, \alpha, \gamma) \) is the maximum value of the Hamiltonian, we can obtain the inequality (32). As \( \bar{\gamma}_t \) is square-integrable and \( \epsilon \in [0, M] \), the last term is a martingale and its expectation is 0. Therefore, the sufficient condition for maximizing the agent’s expected utility is obtained.

Next, we show the necessary condition of agent’s optimal effort. We define a control perturbation
\[
\tilde{e}_t = e_t + \epsilon \Delta e_t,
\]
where \( \epsilon \in [0, \epsilon_0] \) and \( \epsilon_0 \) is positive. Define
\[
\nabla V_t(e) = \lim_{\epsilon \to 0} \frac{V(t, \tilde{\epsilon}) - V(t, \epsilon)}{\epsilon}.
\]
By small perturbation \( \epsilon \), we have
\[
e^{-\rho t} \nabla V_t(e) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} E_t \left[ \int_t^T e^{-\rho s} \left[ u(c_s, \tilde{e}_s) - u(c_s, e_s) \right] ds + \int_t^T \zeta_s \sigma dW^\bar{\epsilon}_s \right. \\
+ \int_t^T \zeta_s \left[ f(\tilde{e}_s) - f(e_s) + (\bar{\eta}(Y_s - \bar{\alpha}_s, s) - \bar{\eta}(Y_s - \bar{\alpha}_s, s)) \right] ds \right].
\]
On the basis of the proof of the sufficient condition, we have
\[
E_t \left[ \int_t^T \zeta_s \left[ f(\tilde{e}_s) - f(e_s) + (\bar{\eta}(Y_s - \bar{\alpha}_s, s) - \bar{\eta}(Y_s - \bar{\alpha}_s, s)) \right] ds \\
+ \int_t^T e^{-\rho s} \left[ u(c_s, \tilde{e}_s) - u(c_s, e_s) \right] ds \right] \leq 0.
\]
As $\epsilon \to 0$, for any $\Delta e_s$, the following inequality holds

$$E_t \left[ \int_t^T \left( e^{-\rho s} u_e e_s + \zeta_s(f'(e_s) \Delta e_s - \frac{\sigma^{-2}}{h_s} \int_t^s f'(e_\tau) \Delta e_\tau d\tau) \right) ds \right] \leq 0,$$

where $u_e$ is the partial derivative of the agent’s utility $u(c_t, e_t)$ with respect to $e_t$ and $f'(e_s) = \frac{df(e_s)}{de_s}$. Changing the order of integral, we get

$$E_t \left[ \int_t^T \left( e^{-\rho s} u_e + \zeta_s f'(e_s) - \int_s^T \frac{\sigma^{-2}}{h_\tau} \zeta_\tau d\tau \right) f'(e_s) \Delta e_s ds \right] \leq 0. \tag{35}$$

At time $t$, we denote the agent’s optimal effort by $e_t^*$ and any other effort level by $e_t$. As $\Delta e_s$ is arbitrary, we obtain

$$\left[ E_t \left[ - \int_t^T \frac{\sigma^{-2}}{h_s} \zeta_s ds \right] + \zeta_t \right] f'(e_t) + e^{-\rho t} u_e \right] (e_t - e_t^*) \leq 0. \tag{36}$$

Multiplying both sides of (36) by $e^{\rho t}$, noticing $\zeta_t = e^{-\rho t} \gamma_t$ and letting

$$p_t = h_t E_t \left[ - \int_t^T e^{-\rho (s-t)} \frac{\gamma_s}{h_s} ds \right],$$

we get

$$\left[ (\gamma_t + \frac{\sigma^{-2}}{h_t} p_t) f'(e_t) + u_e(c_t, e_t) \right] (e_t - e_t^*) \leq 0. \tag{37}$$

Thus, the necessary condition for agent’s optimal effort is proved.

An increase of agent’s current effort has two effects: (1) it raises the promised utility, (2) it increases the cumulative effort. The amount of the first effect is proportional to the sensitivity coefficient $\gamma$. The second effect is described by the expectation term in (24). When $\eta$ is known, this expectation term disappears, i.e., $p_t = 0$. Consequently, we obtain

$$(\gamma_t f'(e_t) + u_e(c_t, e_t)) (e_t - e_t^*) \leq 0. \tag{38}$$

The Hamiltonian corresponding to the first-order condition (38) is

$$H(t, y, e, \gamma) = u(c_t, e_t) + (ry_t + f(e_t) - s_t - d_t) \gamma_t. \tag{39}$$

If we choose $f(e) = Be$ (constant $B > 0$) and $\eta = 0$, our model is turned into the moral hazard model in Williams (2015) [23]. The uncertainty of agent ability leads to an additional expectation term in (23). Using (3) and differentiating $\hat{\eta}$ with respect to $a_\tau$, we have

$$\frac{\partial \hat{\eta}}{\partial a_\tau} (Y_\tau - a_\tau, \tau) = - \frac{\sigma^{-2}}{h_\tau} < 0.$$

Thus an increase in the current cumulative effort reduces the value of the posteriors $\hat{\eta}$. Namely, for all future time $\tau > t$, an upward deviation from the recommended effort generates a negative output shock by $\sigma^{-2}/h_\tau$. The expectation that $\sigma^{-2}/h_\tau$ multiplies the sensitivity $\gamma_\tau$ measures the effects induced by the uncertainty of agent ability. Through summing and discounting all these marginal effects, the expected marginal returns of manipulating beliefs are acquired.

$p_t$ defined by (24) is a stochastic process and measures the private information value. As $p_t$ is negative and satisfies (23), for any recommended effort $e_t^*$ and any given payment $s_t$, unknown agent ability leads to a higher volatility $\gamma_t$. In other words, with ability uncertainty, it is more difficult for the principal to provide incentives. This conclusion does not rely on any specific form of the utility.
The necessary condition (23) involves two stochastic variables: $\gamma$ and $p$. This is a general result of the dynamic contract with private information. We use the promised utility to characterize the past history. According to the incentive constraint (23) and the assumption $f'(e_t) > 0$, we obtain

$$\gamma_t \geq -\frac{u_e(c_t, e_t)}{f'(e_t)} - \frac{\sigma^2}{h_t} p_t.$$  \hspace{1cm} (40)

As the agent is risk-averse, the principal would like to minimize $\gamma$. Hence, for any time $\tau \in [t, T]$ and $e_t > 0$, the necessary condition (40) holds with equality almost everywhere on the equilibrium path. That is

$$\gamma_t = -\frac{u_e(c_t, e_t)}{f'(e_t)} - \frac{\sigma^2}{h_t} p_t.$$  \hspace{1cm} (41)

Taking advantage of the assumption of exponential utility, we know that (41) indeed holds. Using the definition of $p_t$ and equation (41), we obtain the expression of $p_t$.

We show the details of derivation process below.

Letting $\tilde{p}_t = (\sigma^2/h_t)p_t$ and using the definition of $p_t$ in (24), we get

$$\tilde{p}_t = E \left[ -\int_t^T e^{-\rho(s-t)} \frac{\sigma^2}{h_s} \gamma_s ds \right].$$  \hspace{1cm} (42)

Differentiating both sides of (42) with respect to $t$, we obtain the following equation

$$\frac{d\tilde{p}_t}{dt} = \rho E \left[ -\int_t^T e^{-\rho(s-t)} \frac{\sigma^2}{h_s} \gamma_s ds \right] + \frac{\sigma^2}{h_t} \gamma_t$$

$$= \rho \tilde{p}_t + \frac{\sigma^2}{h_t} \gamma_t.$$  \hspace{1cm} (43)

Substituting (41) into (43) gives rise to

$$\frac{d\tilde{p}_t}{dt} = \left(\rho - \frac{\sigma^2}{h_t}\right) \tilde{p}_t - \frac{\sigma^2}{h_t} \frac{u_e(c_t, e_t)}{f'(e_s)}.$$  \hspace{1cm} (44)

Integrating both sides of (44) and noticing $\tilde{p}_T = 0$ yield

$$\tilde{p}_t = E \left[ \int_t^T e^{-\rho(s-t)} \frac{\sigma^2}{h_s} \frac{u_e(c_s, e_s)}{f'(e_s)} ds \right].$$  \hspace{1cm} (45)

As $h_\tau = h_0 + \sigma^{-2} \tau$, we have

$$\frac{\sigma^{-2}}{h_\tau} = \frac{\sigma^{-2}}{h_0 + \sigma^{-2} \tau} = \frac{d(ln h_\tau)}{d\tau}. $$  \hspace{1cm} (46)

Thus, we obtain

$$\exp \left[ \int_t^s \frac{\sigma^{-2}}{h_\tau} d\tau \right] = e^{(ln h_s - ln h_t)} = \frac{h_s}{h_t}. $$  \hspace{1cm} (47)

Substituting (47) into (45), the expression of $\tilde{p}_t$ becomes

$$\tilde{p}_t = \left(\rho - \frac{\sigma^2}{h_t}\right) \tilde{p}_t + \frac{\sigma^2}{h_t} E \left[ \int_t^T e^{-\rho(s-t)} \frac{u_e(c_s, e_s)}{f'(e_s)} ds \right].$$  \hspace{1cm} (48)

Combining the definition of $\tilde{p}_t$ and (48), the expression of $p_t$ is turned into

$$p_t = E \left[ \int_t^T e^{-\rho(s-t)} \frac{u_e(c_s, e_s)}{f'(e_s)} ds \right].$$  \hspace{1cm} (49)
When the agent’s optimal effort \( e^*_t > 0 \), for any time \( s \in [t, T] \), the differential of \( p_t \) becomes

\[
\begin{align*}
dp_t &= \left[ \rho p_t - \frac{u_s(e_s, e^*_s)}{f_t(e^*_s)} \right] dt + \beta_t \sigma dW_t, \\
p_T &= 0.
\end{align*}
\]

(50)

Similarly, the volatility \( \beta \) is determined by the principal, which maximizes his own expected return. According to (49), \( p_t \) is proportional to the expectation of the discounted value of the marginal cost of future efforts. Multiplying \( p_t \) by \( \sigma^{-2}/h_t \) measures the effect of cumulative effort on promised utility. As time goes by, the principal gets to know the agent ability \( \eta \) more precisely and \(- (\sigma^{-2}/h_t) p_t \) decreases, meaning that the principal’s power to provide incentives for the agent becomes stronger over time.

### 3.3. Sufficient condition

The global concavity of the agent’s objective function is crucial in characterizing the first-order conditions. However, the existence of persistent private information makes it difficult to ensure the concavity of agent’s objective function. Because any agent’s deviation from the recommended effort leads to a permanent gap of beliefs between the contracting parties, we should verify the sufficiency of the optimal strategy.

Compared to discrete-time models, continuous-time models make it possible to verify the sufficiency of the optimal strategy and incentive compatibility by the concavity of the agent’s Hamiltonian function. Theorem 3.5.2 in Yong and Zhou (1999)[25] summarizes this general mathematical result. Proposition 2 gives the sufficient conditions for the agent’s optimal effort.

The efforts on the equilibrium and arbitrary path are denoted by \( e^*_t \) and \( e_t \), respectively. By comparing \( e^*_t \) and \( e_t \), we obtain the sufficient conditions of agent’s optimal effort. We define the current effort difference between the arbitrary and recommended path as \( \delta_t = f(e_t) - f(e^*_t) \). For the cumulative efforts, we let

\[
\begin{align*}
\alpha_t &= \int_0^t f(e_s)ds, \\
\alpha^*_t &= \int_0^t f(e^*_s)ds, \\
\Delta_t &= \int_0^t \delta s ds = \alpha_t - \alpha^*_t.
\end{align*}
\]

**Proposition 2.** For \( t \in [0, T] \), if the matrix

\[
\begin{pmatrix}
u_{ee}(c, e^*_t) - \frac{u_s(c, e^*_t)}{f_t(e^*_t)} f''(e^*_t), & e^{\rho t} \xi_t f'(e^*_t) \\
e^{\rho t} \xi_t f'(e^*_t), & -2 e^{\rho t} \xi_t \frac{\sigma^2}{h_t}
\end{pmatrix}
\]

is negative semidefinite and (23) holds, then the control \( e^*_t \) is incentive compatible. \( \xi_t \) is the predictable process defined by

\[
E \left[ - \int_0^T e^{-\rho s} \frac{\sigma^2}{h_s} ds \right] \mathcal{F}_t^e = \mathcal{F}_0^e
\]

\[
= \int_0^t \xi_s \sigma dW_s.
\]

**Proof.** For simplicity, we abbreviate \((e_t, e^*_t)\) to \((e, e^*_t)\). For an arbitrary and recommended effort, the corresponding output processes become

\[
\begin{align*}
dy_t &= (ry_t + f(e_t) + \dot{\eta}(Y_t - \alpha_t, t) - s_t - d_t) dt + \sigma dW^e_t, \\
dy_t &= (ry_t + f(e^*_t) + \dot{\eta}(Y_t - \alpha^*_t, t) - s_t - d_t) dt + \sigma dW^e_t.
\end{align*}
\]
Thus, we get
\[ \sigma dW^e_t = \sigma dW^e_t + \left( \delta_t - \frac{\sigma^2}{h_t} \Delta_t \right) dt. \]  
(52)
We denote agent’s reward by
\[ I^e_t = E_t \left[ \int_0^T e^{-\rho t} u(c_t, e_t) dt + e^{-\rho T} v(s_T, 0) \right] = \int_0^T e^{-\rho s} u(c_s, e_s) ds + e^{-\rho T} V^e_T(t, e_t). \]  
(53)
According to the extended martingale representation theorem in Fujisaki et al. (1972) [8], there is a process \( \zeta \) in \( L^2 \) such that
\[ I^e_T = I^e_t + \int_t^T \zeta_s \sigma dW^e_s. \]  
(54)
By (53), the agent’s total reward on the equilibrium path is
\[ I^e_T = \int_0^T e^{-\rho t} u(c_t, e_t^*) dt + e^{-\rho T} V^e_T(0, e_t^*) = \int_0^T \left( \delta_t - \frac{\sigma^2}{h_t} \Delta_t \right) \zeta_t^* dt + \int_0^T \zeta_t^* \sigma dW^e_t, \]  
(55)
where \( V^e_T(t, e_t^*) \) represents the agent’s value function at time \( t \) and the superscript indicates the agent’s effort path. Similarly, the agent’s total return of arbitrary path is
\[ I^e_T = \int_0^T e^{-\rho t} \left[ u(c_t, e_t) - u(c_t, e_t^*) \right] dt + I^e_T = \int_0^T e^{-\rho t} \left( u(c_t, e_t) - u(c_t, e_t^*) \right) dt + V^e_T(0, e_t) \]
\[ + \int_0^T \left( \delta_t - \frac{\sigma^2}{h_t} \Delta_t \right) \zeta_t^* dt + \int_0^T \zeta_t^* \sigma dW^e_t. \]  
(56)
Calculating the third term of the second equality in (56), we get
\[ - \int_0^T \frac{\sigma^2}{h_t} \Delta_t \delta_t^* dt = - \int_0^T \frac{\sigma^2}{h_t} \zeta_t^* \left( \int_0^t \delta_s \right) dt \]
\[ = \int_0^T \delta_t \left( - \int_0^T \frac{\sigma^2}{h_t} \zeta_s^* ds \right) dt \]
\[ = \int_0^T \delta_t \left( - \int_t^T e^{-\rho s} \frac{\sigma^2}{h_t} \gamma_s^* ds \right) dt. \]  
(57)
Substituting
\[ E \left[ - \int_t^T e^{-\rho s} \gamma_s^* ds \right] = \frac{1}{\rho} \gamma_t e^{-\rho T} \]  
and (51) into (57), we obtain
\[ - \int_0^T \frac{\sigma^2}{h_t} \Delta_t \zeta_t^* dt = \int_0^T \delta_t \left( \frac{\sigma^2}{h_t} \gamma_t e^{-\rho T} + \int_t^T \xi_t^* \sigma dW^e_t \right) dt. \]  
(58)
Changing the Brownian motion and taking the expectation, we have
\[ V^c(0, e) - V^{e^*}(0, e^*) \]
\[ = E_0^c [I^e(T)] - V^{e^*}(0, e^*) \]
\[ = E_0^c \left[ \int_0^T e^{-pt} (u(c_t, e_t) - u(c_t, e_t^*)) dt + \int_0^T \delta_t \xi_t^* dt \right. \]
\[ + \left. \int_0^T \delta_t \left( \frac{\sigma^{-2}}{h_t} e^{-pt} p_t^* + \int_t^T \xi_t^* \sigma dW_t^e \right) dt \right] \]
\[ = E_0^c \left\{ \int_0^T e^{-pt} \left[ (u(c_t, e_t) - u(c_t, e_t^*)) + \delta_t \left( \gamma_t^* + \frac{\sigma^{-2}}{h_t} p_t^* \right) \right] dt \right\} \]
\[ + E_0^c \left[ \xi_t^* \Delta_t \int_0^T \left( \delta_t - \frac{\sigma^{-2}}{h_t} \Delta_t \right) \xi_t^* \Delta_t dt \right]. \] (59)

Since \( e_t^* \) is the effort on the equilibrium path, the first term of the third equality in (59) is non-positive with a maximum of zero. However, we are not clear about the sign of the second term of the third equality. To check its sign, we introduce a predictable process \( \chi^* \) which satisfies
\[ \chi^* = \gamma_t^* - e^{pt} \xi_t^* \alpha_t^*. \] (60)

In addition, we define a Hamiltonian function \( H \) by
\[ H(t, e, \alpha, \chi^*, \xi^*, p^*) = u(c, e) + (\chi^* + e^{pt} \xi^* \alpha_t) f(e) - e^{pt} \xi^* \frac{\sigma^{-2}}{h_t} \alpha^2 \]
\[ + \frac{\sigma^{-2}}{h_t} p^* f(e). \] (61)

In fact, (61) is the Hamiltonian function corresponding to the agent’s optimal control problem. Taking a linear approximation of \( H \) around \( \alpha^* \), we have
\[ H_t(e_t, \alpha_t) - H_t(e_t^*, \alpha_t^*) - \frac{\partial H_t(e_t^*, \alpha_t^*)}{\partial \alpha} \Delta_t \]
\[ = u(c, e) - u(c, e^*) + \delta_t (\chi^* + e^{pt} \xi_t^* \alpha_t^* + \frac{\sigma^{-2}}{h_t} p_t^*) \]
\[ + e^{pt} \xi^* \Delta_t (\delta_t - \frac{\sigma^{-2}}{h_t} \Delta_t). \] (62)

Combining (59) and (62) yields
\[ V^c(0, e) - V^{e^*}(0, e^*) \]
\[ = E_0^c \left[ \int_0^T e^{-pt} \left( H_t(e_t, \alpha_t) - H_t(e_t^*, \alpha_t^*) - \frac{\partial H_t(e_t^*, \alpha_t^*)}{\partial \alpha} \Delta_t \right) \right]. \] (63)

If (61) is concave, the value of (63) is negative, which turns out to be the sufficient condition. We obtain that the Hessian matrix of Hamiltonian function \( H(\cdot) \) is
\[ H(t, e, \alpha) = \begin{pmatrix} u_{ee}(c_t, e_t) - \frac{u_e(c_t, e_t)}{f_t(e_t)} f_t''(e_t), & e^{pt} \xi_t f_t'(e_t) \\ e^{pt} \xi_t f_t'(e_t), & -2e^{pt} \xi_t \frac{\sigma^{-2}}{h_t} \end{pmatrix}. \] (64)

The concavity of (61) is ensured if the Hessian matrix \( H(t, e, \alpha) \) is negative semi-definite. \( \square \)
According to Prat and Jovanovic (2014)[16], the process $\xi_t$ measures the random fluctuation of the sum of the discounted marginal utility accumulated from time 0. Proposition 2 imposes rigorous restrictions on $\xi_t$. If a control process satisfies Proposition 2, it is incentive compatible. In fact, for a given contract, the sufficient condition stated in Proposition 2 should be verified ex-post for $(c_t, \gamma_t)$ being endogenous.

4. Incentive compatible contracts.

4.1. Value function. In this section, incentive provision is considered. We discuss the optimal contracts under known and unknown agent ability, respectively. It is shown that when contract time tends to infinity, the case with unknown ability is turned into the case with known ability. In order to maximize the expected utility, the principal chooses an appropriate strategies $(c_t, d_t, e_t, \gamma_t)$. From (10) and (11), we denote the principal’s value function $J(t,y,q)$ at time $t$ by

$$J(t,y,q) = \max_{\{c_t, d_t, e_t, \gamma_t\}} E\left[\int_t^T e^{-\rho t}u(d_t)dt + e^{-\rho T}L(y_T, s_T)\right].$$

(65)

The principal’s constraints are (7), (21), (41) and (50). The reason why the principal’s value function does not contain the variable $\hat{\eta}$ will be illustrated later. According to the terminal condition, we obtain

$$L(y_T, s_T) = V_T(y_T - s_T) = -\frac{1}{r} \exp\left[\frac{r - \rho}{r} - \lambda r(y_T - s_T)\right].$$

(66)

Using (21), we have

$$q_T = V_T(s_T) = v(s_T) = -\frac{1}{r} \exp\left(\frac{r - \rho}{r} - \lambda r s_T\right).$$

(67)

Combining (66) and (67), we get the principal’s value function at time $T$, i.e.,

$$J(T, y_T, q_T) = L(y_T, s_T) = \frac{1}{r^2} \exp^2\left(\frac{r - \rho}{r}\right) \exp(-\lambda r y_T).$$

(68)

From (68), we conjecture that the expression of the principal’s value function at time $t$ is

$$J(t, y, q) = \frac{e^{K(t)}}{q} \exp(-\lambda r y),$$

(69)

where $K(t)$ is a function of time $t$.

In this section, we study the optimal contracts with a specific effort function.

**Assumption 1.** Let $f(e) = a_0 + a_1 e + a_2 e^2, e \in [0, M]$, where $a_0, a_1, a_2$ and $M > 0$ are constants, $f'(e) = a_1 + 2a_2 e > 0$, $f''(e) = 2a_2 \leq 0$ and $|a_2|$ is sufficiently small.

In fact, the linear function $f(e) = B_0 e$ (constant $B_0 > 0$) is used in Williams (2015)[23] to find the optimal contract in the cases of full information, hidden action and hidden savings. Our assumption on $f(e)$ is assumption 1. The techniques to find optimal contracts are similar to that in Williams (2015)[23].
4.2. Optimal contract under known ability. To intuitively understand the influence of agent ability on the optimal contract, we first analyze the optimal contract when the agent ability is known. Apparently, when the agent ability is known, belief manipulation does not exist. Thus the private information \( p_t = 0 \) and the necessary condition (41) is converted into

\[
\gamma_t = \frac{-u_e(c_t, e_t)}{f'(e_t)}. \tag{70}
\]

Under this assumption, the principal’s problem is similar to those in Sannikov (2008)[17] and Williams (2015)[23]. As the model does not contain persistent private information, the necessary condition is also sufficient.

As the state variable \( q \) is a Markovian process, we solve the principal’s problem via Hamilton-Jacobi-Bellman (HJB) equation. According to the dynamic programming principle, we derive the HJB equation for the principal

\[
\rho J(t, y, q) - J_t(t, y, q) = \max_{c, d, e} \left\{ -\exp(-\lambda d) + J_y(t, y, q)(ry + f(e) + \eta - c - d) + J_q(t, y, q)\left[\rho q + \exp(-\lambda(c - \mu e))\right] + \frac{1}{2}J_{yy}(t, y, q)\sigma^2 + \frac{1}{2}J_{qq}(t, y, q)\sigma^2 \gamma^2 \right\}. \tag{71}
\]

Taking advantage of exponential utility, the solutions of (71) can be obtained.

**Proposition 3.** Denote the optimal contract by \((c^N, d^N, e^N, \gamma^N)\). Assume that the contract expires at time \( T \) and the agent ability is known, i.e., for any time \( t \in [0, T] \), \( h_t = \infty \) holds. If assumption 1 holds and \( a_1 > \mu \). Then the optimal strategies at time \( t \) are

\[
\begin{cases}
  c^N = M, \\
  \gamma^N = -\lambda \mu k q, \\
  e^N = \mu M - \frac{1}{2} \ln k + \ln(-q), \\
  d^N = ry - \frac{1}{\lambda} \left[ K(t) + \ln r - \ln(-q) \right],
\end{cases}
\]

in which \( k \) is the root of the following equation

\[
2\lambda^2 \mu^2 \sigma^2 k^2 + \left( \left[f'(M)\right]^2 - r \mu \lambda^2 \sigma^2 f'(M) \right) k - r \left[f'(M)\right]^2 = 0 \tag{72}
\]

and \( K(t) \) is expressed by

\[
K(t) = e^{-r(T-t)} \left( B + \frac{2(r - \rho)}{r} - 2 \ln r \right) - \frac{B}{r},
\]

where

\[
B = 2 \rho - r - k - \frac{\lambda^2 \mu^2 \sigma^2}{2} + \frac{r \lambda}{2} \left[f(M) + \eta - M \mu + \frac{1}{\lambda} \ln(rk)\right] + \frac{r \mu k \sigma^2}{a_1 + 2a_2 M} - \frac{\lambda^2 \mu^2 k^2 \sigma^2}{(a_1 + 2a_2 M)^2}.
\]

**Proof.** For conciseness, we drop out all the subscripts in the proof of this proposition. Differentiating the right hand side of (71) with respect to \((c, e, d)\), we get the
first-order conditions for \((c, e, d)\)

\[- J_y - \lambda J_y e^{-\lambda(c-\mu e)} + J_{qq} \sigma^2 \gamma \gamma_c + J_{yq} \sigma^2 \gamma_c = 0, \quad (73)\]

\[f'(e)J_y + \lambda \mu J_y e^{-\lambda(c-\mu e)} + J_{qq} \sigma^2 \gamma_c + J_{yq} \sigma^2 \gamma_c \geq 0, \quad (74)\]

\[\lambda e^{-\lambda d} - J_y = 0. \quad (75)\]

To obtain the optimal contracts, we assume that the agent’s utility satisfies

\[u(c, e) = kq. \quad (76)\]

From (70), \(\gamma\) satisfies

\[\gamma = - \frac{\lambda \mu u(c, e)}{f'(e)} = - \frac{\lambda \mu kq}{f'(e)}. \quad (76)\]

Differentiating \(\gamma\) with respect to \((c, e)\), we have the first-order derivatives as follows

\[\frac{\partial \gamma}{\partial c} = -\lambda \gamma, \quad (77)\]

\[\frac{\partial \gamma}{\partial e} = \left(\frac{\lambda \mu}{a_1 + 2a_2 e}\right) \gamma. \quad (78)\]

Differentiating \(J(t, y, q)\) with respect to \((t, y, q)\), we have

\[J_t(t, y, q) = K'(t)J(t, y, q), \quad J_y(t, y, q) = -\lambda r J(t, y, q), \quad J_y(t, y, q) = (\lambda r)^2 J(t, y, q), \quad J_{qq}(t, y, q) = \frac{\lambda r}{q} J(t, y, q), \quad J_{qq}(t, y, q) = \frac{2}{q^2} J(t, y, q). \quad (79)\]

If (73) holds, from (74), we derive that

\[
\left(a_1 + 2a_2 e - \mu + \frac{2a_2}{\lambda(a_1 + 2a_2 e)}\right)(-\lambda r) - \frac{2a_2}{q(a_1 + 2a_2 e)} e^{-\lambda(c-\mu e)} \leq 0.
\]

If Assumption 1 holds and \(a_1 > \mu\), we can choose a very small \(|a_2|\) such that

\[a_1 + 2a_2 e - \mu + \frac{2a_2}{\lambda(a_1 + 2a_2 e)} > 0,
\]

then (74) holds. Therefore, we derive that the agent’s optimal effort can only be obtained at the endpoint. If the agent puts forth zero effort, incentive provision is unnecessary. Since the right derivative of the necessary condition (70) with respect to \(e\) at \(e = 0\) is positive, the value of \(\gamma_t\) increases with the effort level. As a result, the necessary condition (70) cannot be optimal at \(e = 0\). Similarly, the left derivative of (70) with respect to \(e\) at \(e^N = e^*\) is positive (see (78)) and the agent’s effort satisfies \(e^* \in [0, M]\), so the optimal effort \(e^N = e^* = M\). Thus, the agent’s consumption is

\[e^N = \mu M - \frac{1}{\lambda} \left[\ln k + \ln(-q)\right]. \quad (79)\]

Substituting \(J_y(t, y, q)\) into (75), we get the principal’s optimal consumption (dividend)

\[d^N = ry - \frac{1}{\lambda} [K(t) + \ln r - \ln(-q)]. \quad (80)\]
In order to find \( k \), we substitute the partial derivatives of \( J(t, y, q) \) into the first-order condition (73) and obtain

\[
2\lambda^2 \mu^2 \sigma^2 k^2 + \left( [f'(M)]^2 - r\mu \lambda^2 \sigma^2 f'(M) \right) k - r[f'(M)]^2 = 0. \quad (81)
\]

Letting

\[
\Delta_1 = \left( [f'(M)]^2 - r\mu \lambda^2 \sigma^2 f'(M) \right) + 8r\lambda^2 \mu^2 \sigma^2 [f'(M)]^2,
\]

we obtain

\[
k = \frac{r\mu \lambda^2 \sigma^2 f'(M) - [f'(M)]^2 + \sqrt{\Delta_1}}{4\lambda^2 \mu^2 \sigma^2} > 0. \quad (82)
\]

Substituting \((c^N, d^N, e^N, \gamma^N)\) and (69) into (71), we derive the first-order ODE about \( K(t) \), which satisfies

\[
-K'(t) = -rK(t) + r - 2\rho + k + \frac{r^2 \lambda^2 \sigma^2}{2} - r\mu[f(M) + \eta - M\mu + \frac{1}{\lambda} \ln(rk)]
\]

\[
+ \frac{k\lambda^2 \mu^2 \sigma^2}{(a_1 + 2a_2 M)^2} - \frac{r\mu k\lambda^2 \sigma^2}{a_1 + 2a_2 M}.
\]

Setting

\[
B = 2\rho - r - k - \frac{\lambda^2 \sigma^2 r^2}{2} + r\lambda \left[ f(M) + \eta - M\mu + \frac{1}{\lambda} \ln(rk) \right]
\]

\[
+ \frac{r\mu k\sigma^2 \lambda^2}{a_1 + 2a_2 M} - \frac{\lambda^2 \mu^2 k^2 \sigma^2}{(a_1 + 2a_2 M)^2},
\]

in which \( k \) satisfies (82), thus (83) is simplified to

\[
K'(t) - rK(t) = B. \quad (84)
\]

Using the terminal condition (68), we get the terminal condition of \( K(t) \)

\[
K(T) = \frac{2(r - \rho)}{r} - 2\ln r. \quad (85)
\]

Employing (85) and solving the first-order ODE, \( K(t) \) becomes

\[
K(t) = e^{-r(T-t)} \left( \frac{B}{r} + \frac{2(r - \rho)}{r} - 2\ln r \right) - \frac{B}{r}. \quad (86)
\]

The proof is finished. \( \square \)

From Proposition 3, we see that the agent’s effort is a constant. For any time, the agent exerts his maximum effort. From (79), the agent’s consumption is also a constant. \( k \) is interpreted as the effective rate of return which varies with volatility. The agent’s consumption is linear with the logarithm of both \( k \) and the promised utility. When the parameters of the model are given and \( k \) is fixed, the promised utility characterizes the dynamics of the agent’s consumption. In the expression of \( d^N \), we see that the principal’s dividend contains the risk-free return and is proportional to the logarithm of \( r \) and the promised utility. Compared with the agent’s consumption, the principal’s consumption varies over time.
4.3. Optimal contract under unknown ability. In this subsection, we discuss the optimal contract in the case where the agent ability is unknown. The uncertainty of agent ability makes the principal’s problem complex. To find the solutions of optimal contract, we simplify the optimal control problem by eliminating two state variables. The first state variable is \( \hat{\eta} \), and the second state variable is the private information \( p \).

Since the constraints (21), (41) and (50) are not directly affected by the posterior mean \( \hat{\eta} \), we can use the precision of beliefs to replace the posterior mean \( \hat{\eta} \). Moreover, as \( h_t \) is a function of time \( t \), \( t \) could be used to describe the precision of beliefs. In fact, for the principal, the expectation of \( \eta \) is of no significance, which means that the purpose of incentive provision is to reward the agent’s effort instead of his ability.

When the agent’s utility satisfies (8), we get \( u_e(c, e) = \lambda \mu u(c, e) \), which simplifies the principal’s problem substantially. For all time \( \tau \in [t, T] \), the incentive constraint (41) always holds. By substituting \( u_e(c, e) = \lambda \mu u(c, e) \) into (20) and (21), the private information can be expressed as

\[
p_t = \frac{\lambda \mu}{f'(e_t)} \left( q_t - e^{-\rho(T-t)} E_t[q_T] \right).
\] (87)

When the expiration of the contract tends to infinity and the transversality condition \( \lim_{T \to \infty} e^{-\rho T} E_t[q_T] = 0 \) holds, we have \( p_t = \frac{\lambda \mu}{f'(e_t)} q_t \). The fact that \( p_t \) is proportional to \( q_t \) implies that these variables carry the same information. To find the quantitative relationship between \( p_t \) and \( q_t \), we suppose that

\[
p_t = \frac{\lambda \mu \varphi^T(t)}{f'(e_t)} q_t,
\] (88)

where \( \varphi^T(t) \) is a function of time \( t \) and its superscript \( T \) means the expiration date of contract. Simultaneously, we assume that

\[
u(c, e) = k^T(t) q,
\] (89)

where \( k^T(t) \) is a function of time \( t \). Substituting (88) and (89) into the necessary condition (41), we have

\[
g_t = -\frac{\lambda \mu q}{f'(e_t)} \left( k^T(t) + \frac{\sigma^2}{h_t} \varphi^T(t) \right) = \Gamma^T_t q_t
\] (90)

where

\[
\Gamma^T_t = -\frac{\lambda \mu}{f'(e_t)} \left( k^T(t) + \frac{\sigma^2}{h_t} \varphi^T(t) \right).
\]

Substituting (90) into (21), we denote the promised utility by

\[d\mu_t = q_t \left[ (\rho - k^T(t)) dt + \Gamma^T_t \sigma dW_t \right].\]

When \( \Gamma^T_t \) is bounded, we obtain

\[E_t[q_T] = q_t e^{\int_t^T (\rho - k^T(\tau)) d\tau}.
\] (91)

From (87) and (91), \( p_t \) satisfies the following equality

\[
p_t = \frac{\lambda \mu q_t}{f'(e_t)} \left( 1 - e^{-\int_t^T k^T(\tau) d\tau} \right).
\]
Therefore, the evolution of \( \varphi^T(t) \) becomes
\[
\varphi^T(t) = 1 - e^{-\int_t^T k^T(r) \, dr}.
\]
(92)

From (92), we see that \( \varphi^T(t) \) is a function of \( k^T(t) \). Thus, the key to the principal’s problem is to find the solution of \( k^T(t) \). Because \( k^T(t) \) is positive for all \( t < T \), we have \( \varphi^T(t) \in (0, 1) \). Moreover, \( k^T(t) \) is the effective rate of return and \( \varphi^T(t) \in (0, k) \). Therefore, \( \Gamma_T^T \) is indeed bounded. At time \( T \), we have \( p_T = \varphi(T) = 0 \), which indicates that at time \( T \), the agent ability is fully revealed and there is no longer information rent caused by manipulation beliefs. In other words, as time goes by, the principal’s ability of providing incentives becomes better.

Applying the dynamic programming principle, we get the HJB equation corresponding to the principal’s problem as
\[
\rho J(t, y, q) - J_t = \max_{c, d, e} \left\{ -\exp(-\lambda d) + J_y(t, y, q)(ry + f(e) + \eta - c - d)
+ J_q(t, y, q)[pq + \exp(-\lambda(c - \mu e))] + \frac{1}{2} J_{yy}(t, y, q)\sigma^2 + \frac{1}{2} J_{qq}(t, y, q)\sigma^2 \gamma^2
+ J_{qq}(t, y, q)\sigma^2 \right\}.
\]
(93)

Although (71) and (93) have the same form, their sensitivity coefficients are different. The following Proposition 4 gives the optimal strategies of the incentive contract.

**Proposition 4.** Denote the optimal contract by \((c^{un}, d^{un}, e^{un}, \gamma^{un})\). Let the Assumption 1 hold and \( a_1 > \mu \). Assume that the expiration of the contract is \( T \) and the agent ability \( \eta \) is unknown. Then the optimal policy at time \( t \) is
\[
\begin{align*}
e^{un} & = M, \\
\gamma^{un} & = -\frac{\lambda \mu q}{f'(M)} \left( k^T(t) + \frac{\sigma^2}{h_t} \varphi^T(t) \right), \\
c^{un} & = \mu M - \frac{1}{\lambda} \left[ \ln k^T(t) + \ln(-q) \right], \\
d^{un} & = ry - \frac{1}{\lambda} \left[ K_1(t) + \ln r - \ln(-q) \right],
\end{align*}
\]
in which \( k^T(t) \) is the root of the following equation
\[
2(\lambda \mu \sigma k^T(t))^2 + \left( \frac{2\lambda^2 \mu^2 \sigma^2 \varphi^T(t)}{h_t} - r \mu \lambda^2 \sigma^2 f'(M) + [f'(M)]^2 \right) k^T(t) = [f'(M)]^2 r
\]
(94)
and \( K_1(t) \) satisfies
\[
K'_1(t) - rK_1(t) = B_1(t),
\]
where the corresponding terminal condition is (85) and
\[
B_1(t) = 2\rho - r - \frac{1}{2} \lambda^2 r^2 \sigma^2 + r \lambda[f(M) - M\mu + \eta] + r \ln(rk^T(t))
+ \frac{\lambda^2 \sigma^2 \mu r}{f'(M)} \left( k^T(t) + \frac{\sigma^2}{h_t} \varphi^T(t) \right) - \frac{2\lambda^2 \mu^2 \sigma^2}{h_t^2} \left( k^T(t) + \frac{\sigma^2}{h_t} \varphi^T(t) \right)^2.
\]

**Proof.** The first-order conditions of (93) for \((c, c, d)\) can be expressed as
\[
-J_y - \lambda J_q e^{-\lambda(c - \mu e)} + J_{qq} \sigma^2 \gamma_c + J_{qq} \sigma^2 \gamma_c = 0,
\]
(95)
\[
f(e^*)J_y + \lambda \mu J_q e^{-\lambda(c - \mu e)} + J_{qq} \sigma^2 \gamma_c + J_{qq} \sigma^2 \gamma_c \geq 0,
\]
(96)
\[
\lambda e^{-\lambda d} - J_y = 0.
\]
(97)
We have
\[
\frac{\partial \gamma}{\partial e} = -\lambda \mu q \left[ \frac{\lambda \mu k^T(t)}{a_1 + 2a_2e} - \frac{2a_2k^T(t)}{(a_1 + 2a_2e)^2} \right] > 0
\]
and
\[
\gamma_e = \frac{\partial \gamma}{\partial e} = \frac{\lambda^2 \mu k^T(t)q}{f'(e^*)} < 0.
\]
Similar to the analysis of Proposition 3, if the first-order condition (95) holds, (96) is valid. Hence the agent’s optimal effort is \(e^* = M\), meaning that the agent provides his maximum effort. Since the agent’s utility satisfies \(u(c, e) = k^T(t)q\), the optimal consumption of agent is
\[
e^{un} = \mu M - \frac{1}{\lambda} \left[ \ln k^T(t) + \ln(-q) \right].
\]
Substituting \(J(t, y, q)\) into (97), we have the principal’s consumption
\[
d^{un} = ry - \frac{1}{\lambda} \left[ K_1(t) + \ln r - \ln(-q) \right].
\]
To find \(k^T(t)\) and \(K_1(t)\), substituting \(\gamma, \gamma_e\) and \(J(t, y, q)\) into (95), we know that \(k^T(t)\) satisfies
\[
2 \left[ \lambda \mu \sigma k^T(t) \right]^2 + \left( \frac{2\lambda^2 \mu^2 \sigma^2 \sigma^T(t)}{h_\tau} - r \mu \lambda^2 \sigma^2 \left( \frac{1}{k^T(t)} - \frac{\sigma^{-2}}{h_\tau} \phi^T(t) \right)^2 \right)
- \left[ f(M) + \eta - M \mu + \frac{1}{\lambda} \ln(k^T(t)r) + \frac{1}{\lambda} K_1(t) \right] \lambda r
- \frac{\mu r \lambda^2 \sigma^2}{f'(M)} \left( \frac{1}{k^T(t)} - \frac{\sigma^{-2}}{h_\tau} \phi^T(t) \right) = 0.
\]
Let \(B_1(t) = 2\rho r - \frac{1}{2} \lambda^2 \sigma^2 r^2 + r \lambda [f(M) - M \mu + \eta] + r \ln(rk^T(t)) + \lambda^2 \mu^2 \sigma^2 \left( \frac{1}{k^T(t)} - \frac{\sigma^{-2}}{h_\tau} \phi^T(t) \right)^2 \]
\[
+ \frac{\mu r \lambda^2 \sigma^2}{f'(M)} \left( \frac{1}{k^T(t)} - \frac{\sigma^{-2}}{h_\tau} \phi^T(t) \right) - \frac{\lambda^2 \mu^2 \sigma^2}{f'(M)^2} \left( \frac{1}{k^T(t)} - \frac{\sigma^{-2}}{h_\tau} \phi^T(t) \right)^2.
\]
Substituting \(B_1(t)\) into (98), \(K_1(t)\) satisfies the following first-order ODE
\[
K_1'(t) - r K_1(t) = B_1(t),
\]
with terminal condition (85). Then
\[
K_1(t) = e^{\sigma(t-T)} \left( \frac{2(r - \rho)}{r} - 2 \ln r - \int_t^T B_1(\tau)e^{-(\tau-T)}d\tau \right).
\]

From Proposition 3, we know that only \(d^N\) depends on time \(t\). According to Proposition 4, we know that when the agent ability is unknown, the agent’s optimal effort is still \(e^{un} = M\), and \(\gamma^{un}, e^{un}, d^{un}\) rely on time \(t\). Compared with known ability case, the agent’s optimal effort does not change because the posteriors \(\tilde{\eta}\) has no effect on the first-order condition of \(e\). The agent’s optimal consumption is linear with the logarithm of \(k^T(t)\) and \((-q\)). Since \(k^T(t)\) is an increasing function of time
When we substitute \( \phi_k \) into (94), we obtain (72). That is, when the contract expires and the agent ability is positive, this implies that the agent requires more incentives and can benefit from belief manipulation when the principal cannot observe the agent ability.

Now we show that ability uncertainty will eventually disappear in a long-term contract. Specifically, when we substitute \( \varphi^T(T) = 0 \) into (94) at \( t = T \), equation (94) is turned into (72). That is, when the contract expires and the agent ability is revealed completely, the two cases become identical. We show the features of the optimal contract when the expiration of the contract tends to infinity.

Using the terminal condition \( \varphi^T(T) = 0 \) and \( \varphi^T(t) \) are found by backward induction. As the terminal date \( T \to \infty \), we discuss the convergence of \( k^T(t) \) and \( \varphi^T(t) \). Let

\[
\Delta_2 = \left( \frac{2\lambda^2 \mu^2 \sigma^2 \varphi^T(t)}{h_t} - \mu r \lambda^2 \sigma^2 f'(M) + [f'(M)]^2 \right)^2 + 8\lambda^2 \mu^2 \sigma^2 B^2 r [f'(M)]^2,
\]

then the solution of \( k^T(t) \) can be expressed as

\[
k^T(t) = -\frac{\left( \frac{2\lambda^2 \mu^2 \sigma^2 \varphi^T(t)}{h_t} - \mu r \lambda^2 \sigma^2 f'(M) + [f'(M)]^2 \right) + \sqrt{\Delta_2}}{4\lambda^2 \mu^2 \sigma^2}.
\]

Differentiating \( k^T(t) \) with respect to \( \varphi^T(t) \), we obtain

\[
\frac{dk^T(t)}{d\varphi^T(t)} = -\left( \frac{2\lambda^2 \mu^2 \sigma^2 \varphi^T(t)}{h_t} - \mu r \lambda^2 \sigma^2 f'(M) + [f'(M)]^2 \right) - 1 \cdot \frac{1}{2h_t} < 0.
\]

For any time \( t < T \), \( k^T(t) > 0 \) and \( \varphi^T(t) \in (0, 1) \). Therefore, \( k^T(t) \) is bounded. When \( \varphi^T(t) = 1 \), \( k^T(t) \) becomes

\[
k(t) = \sqrt{\Delta_3} - \left( \frac{2\lambda^2 \mu^2 \sigma^2}{h_t} - \frac{\lambda^2 \sigma^2}{h_t} f'(M) + [f'(M)]^2 \right)
\]

with

\[
\Delta_3 = \left( \frac{2\lambda^2 \mu^2 \sigma^2}{h_t} - \mu r \lambda^2 \sigma^2 f'(M) + [f'(M)]^2 \right)^2 + 8\lambda^2 \mu^2 \sigma^2 [f'(M)]^2 r.
\]

Furthermore, differentiating \( k(t) \) with respect to \( t \), we have

\[
\frac{dk(t)}{dt} = \frac{1}{2} \left[ 1 - \left( \frac{2\lambda^2 \mu^2 \sigma^2}{h_t} - \frac{\lambda^2 \sigma^2 \mu r f(e^*) + [f(e^*)]^2}{\sqrt{\Delta_3}} \right) \left( \frac{\sigma^2}{h_t} \right)^2 \right] > 0,
\]

in which we have used

\[
\left| \frac{2\lambda^2 \mu^2 \sigma^2}{h_t} - \mu r \lambda^2 \sigma^2 f'(M) + [f'(M)]^2 \right| < \sqrt{\Delta_3}.
\]

As a result, \( k(t) \) is an increasing function of time \( t \) and

\[
\int_t^T k^T(s) ds > \int_t^T k(s) ds > k(t)(T - t).
\]
According to (99), we have \( \lim_{T \to \infty} \int_t^T k^T(s) ds = \infty \). Then we obtain

\[
\varphi(t) = \lim_{T \to \infty} \varphi^T(t) = \lim_{T \to \infty} \left( 1 - e^{-\int_t^T k^T(s) ds} \right) = 1.
\]  
(100)

Substituting (100) into (94), we have \( \lim_{T \to \infty} k^T(t) = k(t) \) and

\[
2(\lambda \mu \sigma k(t))^2 + \left( \frac{2\lambda^2 \mu^2 \sigma^2}{h_t} - \mu \rho \mu^2 \sigma^2 f'(M) + [f'(M)]^2 \right) k(t) = 0.
\]  
(101)

When \( t \to \infty, \lim h(t) = \infty \). Then (101) can be written as

\[
2(\lambda \mu \sigma k(t))^2 + \left( -\mu \rho \mu^2 \sigma^2 f'(M) + [f'(M)]^2 \right) k(t) = r [f(M)]^2.
\]  
From the above derivation, we know that when the expiration of the contract tends to infinity, (101) is turned into (72). In other words, when the principal and the agent commit to a long-term contract, as time goes by, the principal gradually understands the agent ability \( \eta \) and the information rent caused by belief manipulation will eventually disappear. As \( t \to \infty \), the two cases would become identical. We can also infer that long-term contracts make the principal better in providing incentives than short-term contracts does.

Now we verify that as \( t \to \infty \) the optimal contract is indeed an incentive compatible contract, i.e., we should ensure whether the contract satisfies the conditions given by Proposition 2. The following Corollary 1 gives the sufficient conditions for the incentive contract. As suggested in Prat and Jovanovic (2014)[16], this sufficient conditions can also be obtained directly in the settings of parameters in the model.

**Corollary 1.** Let \( a_1 > 0, a_2 = 0, \) Assumption 1 hold and

\[
\frac{(2 - f'(M)) \left[ (\lambda^2 \sigma^2 \mu \rho - f'(M)) f'(M) + \sqrt{\Delta_1} \right]}{4 \lambda^2 \mu^2} > \frac{1}{h_t},
\]  
(102)

where

\[
\Delta_1 = (f'(M) - \lambda^2 \sigma^2 \mu \rho)^2 + 8 r \lambda^2 \mu^2 \sigma^2 [f'(M)]^2,
\]
then the agent’s optimal effort \( e = M \) is incentive compatible, i.e., \( e^* = M \) satisfies (23) and Proposition 2. Moreover, since the precision \( h_t \) is an increasing function of time \( t \), for any time \( s \geq t \), (102) always holds.

**Proof.** When the expiration \( T \) tends to infinity, we can use a standard SDE instead of the BSDE used before. In order to derive the state equation of \( p_t \), we introduce an auxiliary variable, i.e., for any time \( t \in [0, T] \),

\[
b_t = E \left[ - \int_0^T e^{-\rho s} \frac{\gamma_s}{h_s} ds | \mathcal{F}_t^x \right] = b_0 + \int_0^t \xi_s \sigma dW_s,
\]
where the second equality is obtained from (51). Consequently, \( p_t \) defined by (24) is turned into

\[
p_t = e^{\rho t} \sigma^2 h_t \left[ b_t + \int_0^t e^{-\rho s} \gamma_s \frac{\sigma^2}{h_s} ds \right].
\]
When \( t \) is sufficiently large, \( p_t \) satisfies
\[
d p_t = \left[ \rho p_t + \frac{d \sigma^2 h_t}{h_t} \right] dt + e^{\rho t} \sigma^2 h_t dB_t
\]
\[
= \left[ \left( \rho + \frac{\sigma^2}{h_t} \right) p_t + \gamma_t \right] dt + \beta_t \sigma dW_t,
\]
where \( \beta_t = e^{\rho t} \sigma^2 h_t \xi_t \).

When the agent’s optimal effort \( e^* = M \), we have
\[
\beta^*_t = \lambda \mu \gamma^*_t = -\frac{\lambda^2 \mu^2 q}{f'(M)} \left( k(t) + \frac{\sigma^2}{h_t} \right).
\]

Therefore, the sufficient condition of Proposition 2 becomes
\[
-2k(t)q + f'(M)q \left( k(t) + \frac{\sigma^2}{h_t} \right) > 0.
\]
(103)

Since the solution of \( k(t) \) is (82), we obtain the sufficient condition (102) by substituting \( k(t) \) into (103). Because \( \frac{dk(t)}{dt} > 0 \) and \( \frac{\sigma^2}{h_r} \) decreases with time, for any time \( \tau > t \), the condition (103) holds.

If both the principal and agent are very patient, the sufficient condition (102) holds almost surely. Especially, when the time lasts long enough, the sufficient condition holds. As the agent ability is revealed completely, the problem of asymmetric information disappears. Note that (102) is a sufficient condition, but may not be a necessary condition.

5. **Numerical comparison.** In Propositions 3 and 4, the sensitivity coefficients \( \gamma \) are different. Thus we only analyze the optimal consumption and dividend. Table 1 gives the optimal consumption and dividend under known and unknown ability. As we can see, the optimal consumption and dividend have strong resemblance. The only differences come from \( k \) and \( k^T(t) \) affecting the agent’s consumption, and \( K(t) \) and \( K_1(t) \) affecting the principal’s dividend. To illustrate these effects, we cast the following numerical analysis. Throughout the analysis, unless otherwise stated, the basic numbers of the parameters are given by: \( M = 1, \rho = 0.1, \lambda = 2, \mu = 0.5, q = -1, r = 0.15, \sigma = 1, \eta = 1, h_0 = 1, a_0 = 0.01, a_1 = 2, a_2 = -0.001 \).

Fig.1(a) gives the evolution of the agent’s consumption over time under known and unknown ability. As we have seen, when there is no ability uncertainty, the optimal consumption is a constant over time. In contrast, when there exist ability uncertainty, the agent’s consumption \( c^u_n \) is larger than that under known ability, decreasing over time and eventually tending to \( c^N \) as \( t \to \infty \). This can be explained by that the agent has more bargaining power and can benefit from belief manipulation before his ability is completely revealed. As time goes by, the information rent

| Known ability              | Unknown ability              |
|----------------------------|------------------------------|
| Consumption \( c^N \)      | \( c^u_n = \mu M - \frac{1}{\lambda} \left[ \ln k + \ln(-q) \right] \) |
| Dividend \( d^N \)         | \( d^u_n = \mu M - \frac{1}{\lambda} \left[ \ln k^T(t) + \ln(-q) \right] \) |
|                            |                              |

Table 1. Comparison of the optimal consumption and dividend under known and unknown ability.
Figure 1. (a) The evolution of the agent’s consumption over time $t$ (b) Reduction in the principal’s dividend over time $t$.

decreases as the principal gradually learns more about the agent ability and when the principal gets all the information, the two cases will coincide.
According to Table 1, the difference between \( d^N \) and \( d^un \) comes from the time-dependent functions \( K(t) \) and \( K_1(t) \) for each level of output \( y \), promised utility \( q \), and rate of return \( r \). Similar to Williams (2015)[23], we regard \( K(t)/\lambda \) and \( K_1(t)/\lambda \) as the reductions in dividend. Fig.1(b) compares the reduction in the principal’s dividend under known and unknown ability. With unknown ability, the reduction in dividend is larger than that under known ability because the principal has to pay the information rent. However, as we can see, in the infinite horizon limit, the two lines will eventually coincide, implying that the difference will disappear. In other words, in the long term, the adverse effect of the ability uncertainty on the principal’s dividend vanishes when the agent ability is no longer private information.

6. Conclusion. In a continuous-time principal-agent model, we investigate a contracting problem with the agent ability unknown. We derive the necessary and the sufficient conditions for the incentive contracts when both the priors and the posteriors about the agent ability are normally distributed. We investigate a general form of the effort function which could be implemented in the model. Applying exponential utility and a specific effort function, we obtain the optimal contracts and analyze the influence of the uncertainty of agent ability on the incentive contract.

In our principal-agent problem featured with learning process, the uncertainty of the agent ability reduces the principal’s ability to provide incentives. When the agent ability is unknown, the agent’s optimal consumption is not only affected by the promised utility but also by the degree of his ability. Ability uncertainty leads to belief manipulation and benefits the agent. For long-term contract, the principal gradually understands the agent ability and thus, his ability of incentive provision is improved as time goes by. As a result, the agent’s payment or wage is front-loaded. When time tends to infinity, the agent ability is completely revealed and the known and unknown ability cases become identical.

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