AN IMPROVED UPPER BOUND FOR THE RIGHT-SIDE TAIL OF THE CROSSOVER DISTRIBUTION AT THE EDGE OF THE RAREFACTION FAN

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This note is a refinement of a calculation done in Sections 5.3 and 5.4 of [2] by Corwin and Quastel. By improving some of the estimates, we were able to obtain the following result:

**Corollary 1.** Let $T_0 > 0$. Then there exist $c_1, c_2, c_3 < \infty$ depending only on $T_0$ such that for all $T \geq T_0$,

$$1 - F_{T_0}^{\text{edge}}(s) \leq c_1(e^{-c_2T^{1/3}s}e^{-c_3s^{3/2}}).$$

(1)

We will need the following useful inequality related to Stirling’s approximation. For $x > 0$,

$$1 < (2\pi)^{-1/2}x^{1/2}e^{-x}\Gamma(x) < e^{1/(12x)}.$$  

(2)

We will also use the fact from Lemma 49 of [2] that there exists a constant $C > 0$ such that for all $\text{Re}(z) > 0$,

$$|1/\Gamma(z)| \leq Ce^{2|z|}.$$  

(3)

The essential improvement of this note over [2] is the following lemma, which gives improved bounds for the Airy Upper and Airy Lower Gamma functions.

**Lemma 2.** Fix a constant $T_0 > 0$, and let $\kappa_T = 2^{-1/3}T^{1/3}$. Then there exists a constant $C > 0$ depending only on $T_0$ such that the following inequalities hold:

$$(1) \quad |\text{Ai}^{\Gamma}(x, \kappa_T^{-1}, 0)| \leq CT^{1/3} \text{ for all } x \in \mathbb{R}$$

(4)

$$(2) \quad |\text{Ai}_R(x, \kappa_T^{-1}, 0)| \leq CT^{-1/3}e^{-2\kappa_T^{-3/2}x^{3/2}} \text{ for all } x \geq 0$$

(5)

$$(3) \quad |\text{Ai}_L(x, \kappa_T^{-1}, 0)| \leq CT^{-1/3}e^{2\kappa_T^{-1}|x|^{1/2}} \text{ for all } x < 0$$

(6)

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Proof of Lemma 2:

In this proof, \( C > 0 \) is a constant that can change from line to line, but only depends on \( T_0 \).

**Ai^F Bounds, \( x \geq 0 \).** We begin by proving (4). We begin by proving the \( x \geq 0 \) case.

We deform the contour \( \tilde{\Gamma}_\zeta \) to the vertical line \( s_0 + it \), where \( s_0 = -\kappa T_0 / 2 \), \( t \in (-\infty, \infty) \). On this new contour,

\[
|\Gamma(\zeta^{-1}(s_0 + it))| = \left| \frac{\Gamma(\zeta^{-1}(s_0 + it) + 1)}{\zeta^{-1}(s_0 + it)} \right| \leq \left| \frac{\Gamma(\text{Re}(\zeta^{-1}(s_0 + it) + 1))}{\zeta^{-1}(s_0 + it)} \right| = \left| \frac{\Gamma(\zeta^{-1}s_0 + 1)}{\zeta^{-1}(s_0 + it)} \right|.
\]

In the first equation, we use the functional equation \( \Gamma(z) = \Gamma(z + 1)/z \), and in the second line, we use the simple fact that \(|\Gamma(z)| \leq \Gamma(\text{Re}(z)) \). Now, \( 3/2 \geq \zeta^{-1}s_0 + 1 \geq 1/2 \), by choice of \( s_0 \). Therefore, we conclude that the following is true:

\[
|\Gamma(\zeta^{-1}(s_0 + it))| \leq \frac{\Gamma(3/2)}{\zeta^{-1}(s_0 + it)} \leq \frac{\Gamma(3/2)2^{-1/3}T^{1/3}}{\sqrt{s_0^2 + t^2}} \leq CT^{1/3}.
\]

By deforming the contour, we pick up only the residue at \( z = 0 \), since the poles of \( \Gamma(\zeta^{-1}z) \) occur at \( \zeta^{-1}z = -n, n \in \mathbb{N} \cup \{0\} \), and

\[
s_0 = -\frac{T_0}{2} \geq -\frac{\kappa T}{2} > \kappa T.
\]

Now, it is easy to verify that

\[
\text{Res}(e^{-1/3z^3 + x^2} \Gamma(\zeta^{-1}z), z = 0) = \frac{1}{\kappa_T} = 2^{-1/3}T^{1/3}.
\]
Therefore, the following is true:
\[
|Ai^T(x, \kappa_T^{-1}, 0)| \leq 2^{-1/3}T^{1/3} + \left| \int_{-\infty}^{\infty} e^{-\frac{1}{3}(s_0+it)^3+x(s_0+it)} \Gamma(\kappa_T^{-1}(s_0+it)) dt \right|
\]
\[
\leq 2^{-1/3}T^{1/3} + CT^{1/3} \int_{-\infty}^{\infty} \left| e^{-\frac{1}{3}(s_0^3+3s_0^2it-3s_02-it^3)+x(s_0+it)} \right| dt
\]
\[
= 2^{-1/3}T^{1/3} + CT^{1/3} \int_{-\infty}^{\infty} e^{-\frac{1}{3}t^3+s_0^3+s_0^2t^2+x_s} dt
\]
\[
\leq 2^{-1/3}T^{1/3} + CT^{1/3} \int_{-\infty}^{\infty} e^{-s_0|t^2|} dt
\]
\[
= CT^{1/3}.
\]

In the second-to-last line, we use the fact that \(x > 0\) and \(s_0 < 0\).

\textbf{Ai^T Bound, } x < 0.\textbf{ Now, we move on to the } x < 0 \textbf{ case. Let } \tilde{x} = -x. \textbf{ We break this proof into three cases.}

\textit{Case 1: } 0 < \tilde{x}^{1/2} \leq \kappa_{T_0}. \textbf{ We deform the contour } \tilde{\Gamma}_z \textbf{ to the vertical line } s_0 + it, \textbf{ where } s_0 = \frac{-\kappa_{T_0}}{2}, \textbf{ and } t \in (-\infty, \infty). \textbf{ Just as in the } x \geq 0 \textbf{ case, } |\Gamma(\kappa_T^{-1}(s_0+it))| \leq CT^{1/3}. \textbf{ Furthermore, just as in the } x \geq 0 \textbf{ case, the only residue picked up when deforming the contour is at } z = 0. \textbf{ Therefore,}

\[
|Ai^T(x, \kappa_T^{-1}, 0)| \leq 2^{-1/3}T^{1/3} + CT^{1/3} \int_{-\infty}^{\infty} e^{\text{Re}(\frac{1}{3}(s_0+it)^3+x(s_0+it))} dt
\]
\[
= 2^{-1/3}T^{1/3} + CT^{1/3} \int_{-\infty}^{\infty} e^{-\frac{1}{3}s_0^3+s_0^2t^2+x_s} dt
\]
\[
\leq 2^{-1/3}T^{1/3}CT^{1/3} \int_{-\infty}^{\infty} e^{\frac{1}{24}s_0^3 - \frac{1}{2} \kappa_{T_0} t^2 + \frac{3}{2}t^2 s_0^2} dt
\]
\[
\leq CT^{1/3}.
\]

In the second-to-last line, we used the assumption that \(\tilde{x} < \kappa_{T_0}^2\).

\textit{Case 2: } \kappa_{T_0} < \tilde{x}^{1/2} < \kappa_T + 1. \textbf{ In the contour integral formula for } Ai^T, \textbf{ make the change of variables } z = s\tilde{x}^{1/2}, \textbf{ to find that}

\[
Ai^T(x, \kappa_T^{-1}, 0) = \int_{\tilde{\Gamma}_z'} e^{-\tilde{x}(3/2)(\frac{1}{3}s^3+s)} \Gamma(\kappa_T^{-1}\tilde{x}^{1/2}s)\tilde{x}^{1/2} ds,
\]

where \(\tilde{\Gamma}_z'\) is the contour obtained by dividing each point on the contour \(\tilde{\Gamma}_z\) by \(\tilde{x}^{1/2}\).

Deform the contour \(\tilde{\Gamma}_z'\) to the following contour: \textbf{ a straight line passing from } -\infty e^{-i\frac{3\pi}{4}} \textbf{ to } -i, \textbf{ a semicircular arc passing from } -i \textbf{ to } -i, \textbf{ and a straight line passing from } i \textbf{ to } \infty e^{i\frac{3\pi}{4}}. \textbf{ Notice that the contour does not pass over any singularities when being deformed. We deal with the arc first.}
Parameterize the arc by \( s = e^{i\theta}, \theta \in [-\pi/2, \pi/2] \). First, we bound \( \Gamma(\kappa_T^{-1} \tilde{x}^{1/2} s) \) on this arc. By (2),

\[
|\Gamma(\kappa_T^{-1} \tilde{x}^{1/2} e^{i\theta} + 1)| \leq \exp((\tilde{x}^{1/2} \kappa_T^{-1} \cos(\theta) + 1) \log(\tilde{x}^{1/2} \kappa_T^{-1} \cos(\theta) + 1))
\]

\[
\leq \exp(((\kappa_T + 1) \kappa_T^{-1} \cos(\theta) + 1) \log((\kappa_T + 1) \kappa_T^{-1} \cos(\theta) + 1))
\]

\[
\leq \exp(((1 + \kappa_T^{-1}) \cos(\theta) + 1) \log((1 + \kappa_T^{-1}) + 1))
\]

\[
= C.
\]

In the last inequality, we used the fact that \( \kappa_T \geq \kappa_T^0 \) and \( 1 \geq \cos(\theta) \geq 0 \) for all \( \theta \in [-\pi/2, \pi/2] \).

By the functional equation for the gamma function,

\[
|\Gamma(\kappa_T^{-1} \tilde{x}^{1/2} e^{i\theta})| = \frac{|\Gamma(\kappa_T^{-1} \tilde{x}^{1/2} e^{i\theta} + 1)|}{|\kappa_T^{-1} \tilde{x}^{1/2} e^{i\theta}|} \leq \frac{C T^{1/3}}{\tilde{x}^{1/2}},
\]

where \( C' \) only depends on \( \kappa_T^0 \).

Now, we bound the exponential part of the integrand:

\[
\left| \exp \left( -\tilde{x}^{3/2} \left( \frac{1}{3} (e^{i\theta})^3 + (e^{i\theta}) \right) \right) \right| = \exp \left( -\tilde{x}^{3/2} \left( \frac{1}{3} \cos(3\theta) + \cos(\theta) \right) \right) \leq 1,
\]

since \( \frac{1}{3} \cos(3\theta) + \cos(\theta) \leq 0 \) for all \( \theta \in [-\pi/2, \pi/2] \), as is easily verified by basic calculus.

The integrand is therefore bounded by \( \frac{C T^{1/3}}{\tilde{x}^{1/2}} \tilde{x}^{1/2} = C T^{1/3} \), and since the arc has length \( \pi \), the integral along the arc is bounded by \( C T^{1/3} \).

Now, we move on to the rays. By symmetry, it is enough to prove the \( C T^{1/3} \) bound on the upper ray, parameterized by \( s = i + re^{i\frac{3\pi}{4}}, r \in [0, \infty) \). We split the argument into two parts.

The first part is when \( r \leq \frac{2}{\sqrt{2}} \left( \frac{1}{\kappa_T^{-1} \tilde{x}^{1/2}} - 1 \right) \). Let \( z = \kappa_T^{-1} \tilde{x}^{1/2} s \). Then by choice of \( r \), the following is true:

\[
0 \geq Re(z) = \kappa_T^{-1} \tilde{x}^{1/2} (r \cos \left( \frac{3\pi}{4} \right)) \geq -1 + \kappa_T^{-1} \tilde{x}^{1/2}.
\]
Thus, \( 1 \geq \text{Re}(z + 1) \geq \kappa_T^{-1} \tilde{x}^{1/2} \). Therefore,

\[
|\Gamma(z)| = \frac{|\Gamma(z + 1)|}{|z|} \leq \frac{2^{-1/3} T^{1/3} |\Gamma(\text{Re}(z + 1))|}{\tilde{x}^{1/2} |i + re^{\frac{3\pi}{4}}|} \leq 2^{-1/3} T^{1/3} \Gamma(\kappa_T^{-1} \tilde{x}^{1/2}) \leq \frac{2^{-1/3} T^{1/3}}{\tilde{x}^{1/2}} \exp(\kappa_T^{-1} \tilde{x}^{1/2} \log(\kappa_T^{-1} \tilde{x}^{1/2})) \leq \frac{2^{-1/3} T^{1/3}}{\tilde{x}^{1/2}} \exp(\kappa_T^{-1} (\kappa_T + 1) \log(\kappa_T^{-1} (\kappa_T + 1))) \leq CT^{1/3} \tilde{x}^{1/2}.
\]

The second part is when \( r \geq \frac{2}{\sqrt{2}} (\frac{1}{\kappa_T - 3/2} - 1) \). Choose a natural number \( k \) such that \( 1/2 \leq \text{Re}(z + k) \leq 3/2 \). Note that \( k \geq 2 \), since by choice of \( r \), \( \text{Re}(z) \leq -1 \). Also, note that for all \( j \geq 1 \), \( |z + j| \geq |\text{Im}(z)| \geq 1 \), by choice of \( r \). Therefore, by repeatedly applying the functional equation of the gamma function, we see that the following is true:

\[
|\Gamma(z)| = \frac{|\Gamma(z + k)|}{|z||z + 1| \cdots |z + k - 1|} \leq \frac{|\Gamma(\text{Re}(z + k))|}{|z|} \leq \frac{2^{-1/3} T^{1/3} \Gamma(3/2)}{\tilde{x}^{1/2} |i + re^{\frac{3\pi}{4}}|} \leq CT^{1/3} \tilde{x}^{1/2}.
\]

Now, we address the exponential part of the integrand:

\[
|\exp(-\tilde{x}^{3/2}(1/3s^3 + s))| = \exp(|\text{Re}(-\tilde{x}^{3/2}(1/3s^3 + s))|) = \exp \left( -\tilde{x}^{3/2}r^2 - \tilde{x}^{3/2}r^3 \sqrt{\frac{3}{6}} \right).
\]

Therefore, the integral on the ray is bounded above by the following:

\[
\int_0^\infty \exp(-\tilde{x}^{3/2}r^2) \frac{CT^{1/3}}{\tilde{x}^{1/2}} \tilde{x}^{1/2} dr \leq \frac{CT^{1/3}}{\tilde{x}^{3/4}}.
\]

Case 3: \( \kappa_T + 1 < \tilde{x}^{1/2} \). In this argument, we use the same contour as in Case 2. We prove the \( CT^{1/3} \) bound on the semicircular arc first. We start with bounding the gamma function. By the functional equation for the gamma function, and [2],

\[
|\Gamma(\kappa_T^{-1} \tilde{x}^{1/2} e^{i\theta})| \leq \frac{\exp((\kappa_T^{-1} \tilde{x}^{1/2} \cos(\theta) + 1) \log(\kappa_T^{-1} \tilde{x}^{1/2} \cos(\theta) + 1))}{\kappa_T^{-1} \tilde{x}^{1/2}}.
\]

As for the exponential part, it is easy to verify using calculus that as a function of \( \theta \),

\[
\text{Re}(-\tilde{x}^{3/2}(\frac{1}{3}s^3 + s)) = -\tilde{x}^{3/2}(1/3 \cos(3\theta) + \cos(\theta)).
\]
decreases on \([-\pi/2,0]\) and increases on \([0,\pi/2]\). The same is clearly true for \((\kappa_T^{-1}\tilde{x}^{1/2}\cos(\theta)+1)\log(\kappa_T^{-1}\tilde{x}^{1/2}\cos(\theta)+1)\). Therefore, the following is true for \(s\) on the arc:

\[
\left|e^{-x^{3/2}(\frac{3}{2}s^3+s)}\Gamma(\kappa_T^{-1}\tilde{x}^{3/2}s)^{1/2}\right| \leq e^{-\tilde{x}^{3/2}(\frac{3}{2}\cos(\frac{3\pi}{2})+\cos(\frac{3\pi}{2}))}e^{(\kappa_T^{-1}\tilde{x}^{1/2}\cos(\frac{\pi}{2})+1)\log(\kappa_T^{-1}\tilde{x}^{1/2}\cos(\frac{\pi}{2})+1)}
\]

\[
= \frac{1}{\kappa_T^{-1}}
\]

\[
= 2^{-1/3}T^{1/3}.
\]

Since the arc has length \(\pi\), we conclude that the integral along the arc is bounded above by \(CT^{1/3}\).

Now, we check the \(CT^{1/3}\) bound on the rays. By symmetry, it is necessary only to check the bound on the upper ray, \(s = 1 + re^{i\theta}\). Choose a natural number \(k\) such that \(1 \leq Re(z + k) \leq 2\). Note that \(|z \pm j| \geq \Im(z + j) \geq \kappa_T^{-1}\tilde{x}^{1/2} > 1\) for all \(j \geq 0\). By again repeatedly applying the functional equation for the gamma function, the following holds:

\[
|\Gamma(\kappa_T^{-1}\tilde{x}^{1/2}(1 + re^{\frac{3\pi}{2}}))| \leq \frac{|\Gamma(z + k)|}{|z||z + 1| \cdots |z + k - 1|} \leq \frac{\Gamma(2)}{|\kappa_T^{-1}\tilde{x}^{1/2}(1 + re^{\frac{3\pi}{2}})|} \leq \frac{CT^{1/3}}{\tilde{x}^{1/2}}.
\]

The exponential part is easily bounded, as follows:

\[
\exp(-\tilde{x}^{3/2}(1/3(i + re^{\frac{3\pi}{2}})^3 + (i + re^{\frac{3\pi}{2}}))) \leq \exp(-r^2\tilde{x}^{3/2}/r^3\tilde{x}^{3/2}\sqrt{2}/6).
\]

Therefore, the integral along the ray is bounded by the following:

\[
\int_0^\infty \exp(-r^2\tilde{x}^{3/2})\frac{CT^{1/3}}{\tilde{x}^{1/2}}\tilde{x}^{1/2}dr \leq \frac{C'T^{1/3}}{\tilde{x}^{3/4}}.
\]

**Ai\_r Bound, \(x \geq 0\).**

**Case 1:** \(0 \leq x^{1/2} \leq \kappa_T\). In this case, we deform the contour \(\tilde{\Gamma}_e\) to the contour \(z = 1 + ri\), \(r \in (-\infty, \infty)\). The function \(1/\Gamma(\kappa_T^{-1}z)\) has no singularities, so we do not pick up any residues. By (8) and the functional equation for the gamma function, it now holds that

\[
\frac{1}{|\Gamma(\kappa_T^{-1}(1 + ri))|} = \frac{|\kappa_T^{-1}(1 + ri)|}{|\Gamma(\kappa_T^{-1}(1 + ri) + 1)|} \leq CT^{-1/3}(1 + r)e^{2\kappa_T^{-1}(2 + r^{1/2})}.
\]

Therefore, the integral is bounded as follows, with suitable \(C, C'\):

\[
\int_{-\infty}^\infty \exp(1/3(1 + ri)^3 - x(1 + ri))CT^{-1/3}(1 + r)dr \leq CT^{-1/3} \int_{-\infty}^\infty \exp(1/3 - r^2 - x)(1 + r)e^{2\kappa_T^{-1}(2 + r^{1/2})}dr
\]

\[
\leq C'T^{-1/3}e^{-2/3x^{3/2}}.
\]
Case 2: \( \kappa T_0 < x^{1/2} \).
In this case, we make the usual change of variables \( z = sx^{1/2} \). We are free to deform the contour \( \tilde{\Gamma}' \) to a steepest descent contour that passes along a straight line from \( \infty e^{-\frac{i \pi}{4}} \) to 1, and then on a straight line from 1 to \( \infty e^{\frac{i \pi}{4}} \). By Stirling’s approximation, it is easy to see that \( 1/\Gamma(z) \) is bounded above by an absolute constant on both rays. Therefore, again by the functional equation for the gamma function,

\[
\left| \frac{1}{\Gamma(\kappa_T^{-1} z^{1/2} s)} \right| \leq CT^{-1/3}(1 + r)x^{1/2}.
\]

The integral along the upper ray is bounded by the following:

\[
\int_0^\infty e^{3/2\left( \frac{1}{4}(1+re^{i\pi})^3 + (1+re^{i\pi^2}) \right)} CT^{-1/3}(1 + r)xdr = CT^{-1/3}e^{-\frac{2}{3}x^{3/2}} \int_0^\infty e^{-\frac{1}{2}r^2 x^{3/2}} dr
\]

\[
\leq CT^{-1/3}e^{-\frac{2}{3}x^{3/2}} \int_0^\infty e^{-\frac{1}{2}r^2 x^{3/2}} dr
\]

\[
\leq \frac{CT^{-1/3}e^{-\frac{2}{3}x^{3/2}}}{x^{3/4}}
\]

**Airy Bound, \( x < 0 \).** Again, let \( \tilde{x} = -x \).

Case 1: \( 0 \leq \tilde{x} \leq \kappa T_0 \).
We deform the contour \( \tilde{\Gamma}' \) to the vertical line \( 1 + ri \), \( r \in (-\infty, \infty) \). We have already shown that

\[
\left| \frac{1}{\Gamma(\kappa_T^{-1} z)} \right| \leq CT^{-1/3}(1 + |r|)e^{4\kappa_T^{-1}(1+|r|^{1/2})}
\]

on this contour. Therefore, the integral is bounded above by the following:

\[
\int_{-\infty}^\infty e^{Re(1/3(1+ri)^3 - x(1+ri))} CT^{-1/3}(1 + r)e^{4\kappa_T^{-1}(1+|r|^{1/2})} dr = CT^{-1/3}e^{1/3-x} \int_{-\infty}^\infty e^{-r^2(1 + |r|)}e^{4\kappa_T^{-1}(1+|r|^{1/2})} dr
\]

\[
\leq CT^{-1/3}e^{1/3 - \kappa T_0} \int_{-\infty}^\infty e^{-r^2(1 + |r|)}e^{4\kappa_T^{-1}(1+|r|^{1/2})} dr,
\]

which is bounded by \( CT^{-1/3} \) because \( \tilde{x} < \kappa T_0 \).

Case 2: \( \kappa T_0 < \tilde{x}^{1/2} \).
In this case, we first make the change of variables \( z = sx^{1/2} \). Then, we deform the contour to the contour (in the s-plane) made up of a straight line passing from \( \infty e^{-i\frac{\pi}{4}} \) to \(-i\), then a straight line from \(-i\) to \(i\), and finally a straight line from \(i\) to \( \infty e^{i\frac{\pi}{4}} \).

We deal with the vertical line segment from \(-i\) to \(i\) first. We parameterize the vertical line segment by \( s = -i + ti, \ t \in (0, 2) \), so by \( \Box \)
On the vertical line segment, the exponential part is bounded as follows:

\[|\exp(\tilde{x}^{3/2}/(1/3(-i + ti)^3) + (-i + ti)))| = 1.\]

Since the length of the line segment is 2, we conclude that the integral on the line segment is bounded by \(CT^{-1/3}\tilde{x}^{1/2}e^{\kappa_T^{-1}\tilde{x}^{1/2}}\).

Now, we consider the integral along the ray. By symmetry, we only consider the upper ray. We parameterize the upper ray by \(s = i + re^{i\pi/4}\), on which it holds that

\[\left|\frac{1}{\Gamma(\kappa_T^{-1}\tilde{x}^{1/2}(i + re^{i\pi/4}))}\right| \leq CT^{-1/3}\tilde{x}^{1/2}(1 + r).\]

The exponential part of the integrand is bounded by

\[|\exp(\tilde{x}^{3/2}(1/3(i + re^{i\pi}))^3 + (i + re^{i\pi}))| \leq \exp(-\tilde{x}^{3/2}r^2)dr,\]

and hence, the integral along the ray satisfies

\[\int_0^\infty \exp(\tilde{x}^{3/2}(1/3s + s))CT^{-1/3}\tilde{x}^{1/2}(1 + r)\tilde{x}^{1/2}dr \leq CT^{-1/3}\tilde{x} \int_0^\infty \exp(-\tilde{x}^{3/2}r^2)dr = CT^{-1/3}\tilde{x}^{-1/4}.\]

1. Upper Tail of Crossover Distribution

We give upper bounds for the upper tail of the following [2]:

\[1 - P_{\text{edge}}^{\text{edge}}(s) = -\int C e^{-\tilde{u} \frac{d\tilde{u}}{\tilde{\mu}}} \det(I - \tilde{K}_{T,\tilde{\mu}}) - \det I,\]

where \(\tilde{K}_{T,\tilde{\mu}}\) is evaluated on \(L^2(s, \infty)\) and

\[\tilde{K}_{T,\tilde{\mu}}(x, y) = \int_{-\infty}^{\infty} e^{-2^{-1/3T^{1/3}t - \tilde{\mu}}} \text{Ai}_r(x + t, \kappa_T^{-1}, 0) \text{Ai}_r(x + t, \kappa_T^{-1}, 0).\]
In order to prove our upper bounds, we follow [2] directly, but use the new bounds for the \( \text{Ai}^\Gamma \) and \( \text{Ai}_1^\Gamma \) we proved in the last section. We factor the operator \( \tilde{K}_{T,\tilde{\mu}} \) into a product of two Hilbert-Schmidt operators, bound them, and then use the continuity of the Fredholm determinant. We will be consistent with the notation and structure of [2] in order to make the argument easy to follow.

By [3],

\[
\det(I - \tilde{K}_{T,\tilde{\mu}}) = \det(I - A),
\]

where \( A = U^{-1}\tilde{K}_{T,\tilde{\mu}}U \), and

\[
Uf(x) = (x^4 + 1)^{1/2}f(x).
\]

We factor \( A = A_1A_2 \), where \( A_1 : L^2(\mathbb{R}) \rightarrow L^2(s, \infty) \), \( A_2 : L^2(s, \infty) \rightarrow L^2(\mathbb{R}) \), and \( A_1 \) and \( A_2 \) have the following kernels:

\[
A_1(x, t) = \text{Ai}^\Gamma(x + t, \kappa_T^{-1}, 0)(x^4 + 1)^{-1/2}(t^4 + 1)^{-1/2}
\] (10)

\[
A_2(t, y) = \frac{\tilde{\mu}}{e^{-\kappa_T t} - \tilde{\mu}} \text{Ai}(y + t, \kappa_T^{-1}, 0)(x^4 + 1)^{1/2}(t^4 + 1)^{1/2}.
\] (11)

We will use the fact that since \( A_1 \) and \( A_2 \) are Hilbert Schmidt and \( A_1A_2 = A \),

\[
|\det(I + A) - \det I| \leq \|A\|_1e^{\|A\|_2+1} \leq \|A_1\|_2\|A_2\|_2 e^{\|A_1\|_2\|A_2\|_2+1}. \] (12)

By (4) and the definition of the Hilbert-Schmidt norm, the following is true:

\[
\|A_1\|_2^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{Ai}^\Gamma(x + t, \kappa_T^{-1}, 0)(x^4 + 1)^{-1/2}(t^4 + 1)^{-1/2})^2 dxdt
\] (13)

\[
\leq CT^{2/3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^4 + 1)^{-1/2}(t^4 + 1)^{-1/2}^2 dxdt
\] (14)

\[
= CT^{2/3}.
\] (15)

In order to bound \( A_2 \), we will use the following bound, along with the bound we proved for \( \text{Ai}^\Gamma \):

\[
\left| \frac{\tilde{\mu}}{e^{-\kappa_T t} - \tilde{\mu}} \right| \leq C|\tilde{\mu}|(e^{2\kappa_T t} \wedge 1),
\] (16)

which is formula (116) in [2].
By our upper bounds for $A_1^F$, the following is true:

$$
\|A_2\|_2^2 \leq C T^{-2/3} |\tilde{\mu}|^2 \left( \int_s^\infty dx \int_{-x}^\infty dt (e^{2\kappa_T t} \wedge 1) e^{-\frac{1}{2}t^2} (x^4 + 1) (t^4 + 1) \right)
+ \int_s^\infty dx \int_{-\infty}^{-x} dt (e^{2\kappa_T t} \wedge 1) |x + t| e^{-\kappa_T |x+t|/2} (x^4 + 1) (t^4 + 1) \right) C |\tilde{\mu}|^2 (I_1 + I_2).
$$

First of all, notice that since $s > 1$, $t < 0$ over the range of the inner integral. Thus, $e^{2\kappa_T t} \wedge 1 = e^{2\kappa_T t}$ in the inner integral. Make the change of variables $u = t/x$ in the inner integral. Then the integral now has the following form:

$$
\int_{-\infty}^{-1} e^{2\kappa_T u x} |x + xu| e^{2\kappa_T^{-1} x^{1/2} |1 + xu|} (x^4 + 1) (t^4 + 1) x du.
$$

Now, choose $s > 64\kappa_T^4$. Then since $x \geq s$, it is easy to verify that

$$
-2\kappa_T x + 8\kappa_T^{-1} x^{1/2} < -\kappa_T x
$$

This implies the following inequality:

$$
I_2 \leq \int_s^\infty dx \int_{-\infty}^{-1} x du e^{-\kappa_T x u} |x + xu| (x^4 + 1) ((xu)^4 + 1) \leq C e^{\kappa_T s} s^9.
$$

Now, we bound $I_1$. We write $I_1 = I_3 + I_4$, where the $t$-integration runs from $-x$ to $-x/2$ in $I_3$, and the $t$-integration runs from $-x/2$ to $\infty$ in $I_4$. Now,

$$
I_3 \leq \int_s^\infty dx \int_{-x}^{-x/2} dt e^{2\kappa_T t} (x^4 + 1) (t^4 + 1) \leq Cs^8 e^{\kappa_T s},
$$

and

$$
I_4 \leq \int_s^\infty dx \int_{-x/2}^\infty dt e^{-4/3 |x + t|^{3/2}} (x^4 + 1) (t^4 + 1) \leq Cs^8 e^{-cs^{3/2}}.
$$

Therefore,

$$
\|A_2\|_2^2 \leq C |\tilde{\mu}|^2 s^9 (e^{-\kappa_T s} + e^{-cs^{3/2}}).
$$

Now, we combine our estimates for $A_1$ and $A_2$, use \[12\] and conclude the following:

$$
|\det(I - \tilde{K}_{T,\tilde{\mu}})| \leq C \|\tilde{\mu}\| s^{9/2} (e^{-\kappa_T s} + e^{-cs^{3/2}}) e^{\frac{1}{2} s |\tilde{\mu}|},
$$

where we have chosen $s$ large enough that $Cs^{9/2} (e^{-\kappa_T s} + e^{-cs^{3/2}}) \leq 1/2$.

Now, we insert this estimate into the formula for the probability distribution:

$$
|1 - F_{T,0}^{\text{edge}}(s)| \leq \int C |e^{-\tilde{\mu}} \frac{d|\tilde{\mu}|}{|\tilde{\mu}|} \|\tilde{\mu}\| s^{9/2} (e^{-\kappa_T s} + e^{-cs^{3/2}}) e^{\frac{1}{2} s |\tilde{\mu}|} \leq Cs^{9/2} (e^{-cT^{1/3} s} + e^{-cs^{3/2}}).
$$

In the second inequality, we have used the fact that $|\int C e^{-\tilde{\mu}} |\tilde{\mu}| \frac{d|\tilde{\mu}|}{|\tilde{\mu}|} e^{\frac{1}{2} s |\tilde{\mu}|}$ converges.
Now, $T \geq T_0$, so for all sufficiently large $s$, there exists $c' > 0$ such that the following is true:
\[
s^{9/2} \left( e^{-cT^{1/3}s} + e^{-cs^{3/2}} \right) \leq e^{-cT_0^{1/3}s + \log s^{9/2}} + e^{-cT_0^{1/3}s + \log s^{9/2}}
\]
\[
\leq e^{-cT_0^{1/3}s} + e^{-cT_0^{1/3}s}.
\]
Therefore, we conclude the following:
\[
|1 - F_{T,0}^{\text{edge}}(s)| \leq C(e^{-cT_0^{1/3}s} + e^{-cT_0^{1/3}s}).
\]

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