Abstract

Let $G = (V(G), E(G))$ be an $(n, m)$-graph and $X$ a nonempty proper subset of $V(G)$. Let $X^c = V(G) \setminus X$. The edge density of $X$ in $G$ is given by

$$\rho_G (X) = \frac{n |E_X (G)|}{|X||X^c|},$$

where $E_X (G)$ is the set of edges in $G$ with one end in $X$ and the other in $X^c$. The Laplacian spread of a graph is the difference between the greatest Laplacian eigenvalue and the algebraic connectivity. In this paper, we use the edge density of some nonempty proper subsets of vertices in $G$ to establish new lower bounds for the Laplacian spread. Also, using some known numerical inequalities some lower bounds for the Laplacian spread of a graph with a prescribed degree sequence are presented.

Keywords: Spectral graph theory, matrix spread, Laplacian spread.

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1. Introduction and Motivation

By an \((n,m)\)-graph \(G\) we mean an undirected simple graph with a vertex set \(V(G)\) of cardinality \(n\) and an edge set \(E(G)\) of cardinality \(m\). If \(e \in E(G)\) has end vertices \(u\) and \(v\) then they are neighbors and the edge is denoted by \(uv\). Sometimes, if we label the vertices of \(G\), we denote an edge by the labels of its end vertices, that is, if \(e = v_iv_j\), then it can be simply denoted as \(e = ij\). A graph with no edges (but at least one vertex) is called empty graph.

For \(u \in V(G)\), the number of vertices adjacent to \(u\) is denoted by \(d(u)\), and it is called the vertex degree of \(u\). A pendent vertex is a vertex with degree one. The minimum and maximum vertex degree of \(G\) are denoted by \(\delta(G)\) and \(\Delta(G)\), respectively (or simply \(\delta\) and \(\Delta\), respectively). A \(q\)-regular graph \(G\) is a graph where every vertex has degree \(q\). The complete graph of order \(n\) is an \((n-1)\)-regular graph with \(n\) vertices and it is denoted by \(K_n\). The cycle with \(n\) vertices is denoted by \(C_n\).

Now, some interesting subgraphs of a graph are defined. By a non trivial subset of vertices of \(G\) we mean a nonempty proper subset of \(V(G)\) and we will denote the induced subgraph with a non trivial vertex set \(S \subset V(G)\) by \(\langle S \rangle_G\). If \(S \subseteq V(G)\) then \(S^c\) is the set \(V(G) \setminus S\). A set of vertices that induces an empty subgraph is called an independent set. A matching \(N\) in a graph \(G\) is a nonempty set of edges such that no two have a vertex in common. The size of a matching is the number of edges in it. A matching \(N\) of \(G\) is a dominating induced matching (DIM) of \(G\) if every edge of \(G\) is either in \(N\) or has a common end vertex with exactly one edge in \(N\). A DIM is also called an efficient edge domination set (see for instance [13]). It is important to notice that if \(N\) is a DIM of \(G\), then there is a partition of \(V(G)\) into two disjoint subsets \(V(N)\) and \(R\), where \(R\) is an independent set. Some literature concerning dominating induced matchings can be seen for instance in [3, 5, 18, 19].

The complement, \(\overline{G}\) of a graph \(G\) has the same vertex set as \(G\), where vertices \(u\) and \(v\) are adjacent in \(\overline{G}\) if and only if they are non adjacent in \(G\). On the other hand, a non trivial vertex set \(S \subseteq V(G)\) is \((\kappa, \tau)\)-regular set if \(\langle S \rangle_G\) is a \(\kappa\)-regular subgraph in \(G\) such that every vertex in \(S^c\) has \(\tau\) neighbors in \(S\). This definition appeared at first place in [28] under the designation of eigengraphs and in [26], in the context of strongly regular graphs and designs. More literature concerning this concept can be found for instance in [3, 7].

Considering two vertex disjoint graphs \(G_1\) and \(G_2\), the join of \(G_1\) and \(G_2\)
is the graph \( G_1 \cup G_2 \) such that \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{ uv : u \in V(G_1) \text{ and } v \in V(G_2) \} \).

For a real symmetric matrix \( M \), let us denote by \( \theta_i(M) \) the \( i \)-th largest eigenvalue of \( M \). For the spectrum of a matrix \( M \) (the multiset of the eigenvalues of \( M \)) we use the notation \( \sigma(M) \). If \( \theta \) is an eigenvalue of \( M \) and \( x \) one of its eigenvectors, then the pair \((\theta, x)\) is an eigenpair of \( M \).

As usual we denote the adjacency matrix of \( G \) by \( A(G) \) and the vertex degree matrix, \( D(G) \) is the diagonal matrix where its \( v \)-th diagonal entry is the degree of vertex \( v \), for \( v \in V(G) \).

From Geršgorin Theorem, see [29], the matrix \( L(G) = D(G) - A(G) \) is positive semidefinite. This matrix is called the Laplacian matrix of \( G \) and its spectrum, \( \sigma(L(G)) \) is called Laplacian spectrum of \( G \). Let \( e \) be the all ones vector, then \( L(G)e \) equals the null vector, thus 0 is always a Laplacian eigenvalue and its multiplicity is the number of components of \( G \).

The Laplacian eigenvalues, \( \mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0 \) of \( G \) and \( \overline{\mu}_1 \geq \cdots \geq \overline{\mu}_{n-1} \geq \overline{\mu}_n = 0 \) of \( \overline{G} \) are related by

\[
\overline{\mu}_j = n - \mu_{n-j}, \tag{1}
\]

for \( 1 \leq j \leq n - 1 \). An important result in graph theory, see [22], states that if \( G \) is a graph with \( n \geq 2 \) vertices and has at least one edge, \( \Delta + 1 \leq \mu_1 \), and for a connected graph the equality occurs if and only if \( \Delta = n - 1 \). The algebraic connectivity of a graph \( G \) is defined as the second smallest Laplacian eigenvalue \( \mu_{n-1} \), [12], and it is sometimes denoted by \( a(G) \). Some important properties of this Laplacian eigenvalue can be found in [12]. For instance, it is worth to recall that \( G \) is connected if and only if \( \mu_{n-1} > 0 \).

There are some written surveys about algebraic connectivity, see [22, 23, 24] and for applications see for instance, [10, 14].

It is known, see [12], that if \( G \) is a non complete graph, then \( a(G) \leq \kappa_0(G) \), where \( \kappa_0(G) \) denotes the vertex connectivity of \( G \) (that is, the minimum number of vertices whose removal yields a disconnected graph). Since, \( \kappa_0(G) \leq \delta(G) \), it follows that \( a(G) \leq \delta(G) \). The graphs for which the algebraic connectivity attains the vertex connectivity are characterized in [20].

We introduce now the edge density of a nonempty proper subset of vertices in \( G \). This concept appeared in [9] although there was a previous definition given by Mohar in [24] that did not include the factor \( n \) in his definition but used in [9] for author's purposes.

Let \( G \) be an \((n, m)\)-graph and \( X \) a non trivial subset of vertices of \( G \). The
edge density of $X$ in $G$ is given by

$$
\rho_G(X) = \frac{n |E_X(G)|}{|X||X^c|}
$$

where $E_X(G)$ is the set of edges of $G$ with one end in $X$ and the other in $X^c$. Note that $\rho_G(X) = \rho_G(X^c)$.

The next two concepts are also important in this work. The spread of an $n \times n$ complex Hermitian matrix $M$ with eigenvalues $\theta_1, \ldots, \theta_n$ is defined by

$$
S(M) = \max_{i,j} |\theta_i - \theta_j|
$$

where the maximum is taken over all pairs of eigenvalues of $M$. There are several papers devoted to this matrix parameter, see for instance [17, 21].

Attending to previous concept and in view that the smallest Laplacian eigenvalue is zero, a nontrivial definition for the Laplacian spread of a graph $G$ is the difference among the largest and the second smallest Laplacian eigenvalue of $G$, see [11]. So, the Laplacian spread, denoted by $S_L(G)$, is given by

$$
S_L(G) = \mu_1 - a(G).
$$

Note that if $G$ is not connected then $S_L(G) = \mu_1$. Moreover, since $\kappa_0(G) \leq \delta(G)$, it follows that $a(G) \leq \delta(G)$ and then

$$
S_L(G) \geq \Delta(G) + 1 - \delta(G),
$$

(2)

see e. g. [30]. It is also immediate from [11] that the Laplacian spread of $G$ and $\overline{G}$ coincide. In consequence, if $\overline{G}$ is not connected, then $S_L(G) = S_L(\overline{G}) = n - a(G)$.

Recently, in [1], by combining old techniques of interlacing eigenvalues and rank 1 perturbation matrices, some lower bounds on the Laplacian spread were given. Some of these bounds involve invariant parameters of graphs, as it is the case of the bandwidth, independence number and vertex connectivity.

In this work, considering the referred work in [1] as motivation, we search for new lower bounds on the Laplacian spread using the edge density of some nonempty proper subsets of vertices in $G$. Firstly, we are concerned with those graphs which contain special subgraphs such as empty subgraphs, DIM subgraphs or special subset of vertices such as $(\kappa, \tau)$-regular subset of vertices. Then, using some known numerical inequalities we present some
lower bounds for the Laplacian spread of a graph with a prescribed degree sequence.

The paper is organized in 4 sections. After Introduction, at Section 2, we present a known inequality due to Mohar, [24], that relates the algebraic connectivity and the edge density. This result gives a tool to establish some lower bounds for the Laplacian spread of a connected graph that are related with the edge density. Then, using these results we study lower bounds for the Laplacian spread of graphs that have a particular nontrivial subset of vertices, namely for graphs that have an independent nontrivial subset of vertices and a $(\kappa, \tau)$-regular subset of vertices. At Section 3, using some known numerical inequalities, additional lower bounds for the Laplacian spread of graphs are presented, this time using the degree sequence and another invariant parameter of the graph which depends of its Laplacian spectrum. Finally, in the last section we study an example in order to approach the Laplacian spread of a given graph, using different parameters, such as algebraic connectivity, edge density of a set $X$ and minimum vertex degree. Some relations are observed.

2. Edge density of a subset: relations with Laplacian spread

In [9], the following inequality (previously presented in [24]), that relates the algebraic connectivity and the edge density was proved. The authors also studied the graphs $G$ for which the equality holds.

**Proposition 1.** [9] Let $G$ be an $(n, m)$-graph. For a non trivial subset $X$ of $V$, the edge density of $X$ satisfies

$$a(G) \leq \rho_G(X) \leq \mu_1(G).$$

Moreover, if a graph $G$ satisfies one of the equalities for some cut $E_X(G)$ (set of edges in $G$ whose removal yields a disconnected graph), then there are integers $s$ and $t$ such that the following conditions must hold:

1. Each vertex in $X$ is adjacent to $s$ vertices in $X^c$ and each vertex in $X^c$ is adjacent to $t$ vertices in $X$, and
2. $s|X| = t|X^c|$.

Note that, the condition 2. results from the fact that in $A(G)$, after labeling the vertices of $X$, the matrix $A(G)$ is partitioned into blocks and the blocks corresponding to the positions $(1, 2)$ and $(2, 1)$ have the same number of entries equal to 1.
Remark 2. There are graphs $G$ with subset of vertices $X$ such that $\rho_G(X) = a(G)$. Related with this fact, in [4] the following example was given: For a graph $G = (V, E)$ where $V = X \cup X^c$, with $X$ and $X^c$ having $n_1$ and $n_2$ vertices, respectively and satisfying 1. and 2. of Proposition 1 and such that the induced subgraph $\langle X \rangle |_G = K_{n_1}$ and $\langle X^c \rangle |_G = K_{n_2}$, then $a(G) = \rho_G(X).

As an immediate consequence of Proposition 1, the next result gives a lower bound for $S_L(G)$ in function of edge density of a non trivial subset of the vertex set of $G$.

Corollary 3. Let $G = (V, E)$ be a connected graph of order $n$. Let $X$ a non trivial subset of $V$ and consider $\phi$ such that $a(G) \leq \phi \leq \mu_1$. Then

$$S_L(G) \geq |\phi - \rho_G(X)|.$$  

If the equality holds then the conditions 1. and 2. of Proposition 1 hold for $X$. Moreover, $\{\rho_G(X), \phi\} = \{\mu_1, a(G)\}$. 

Remark 4. Let $G = G_1 \lor G_2$, let us denote by $n_i$ and $\mu_j (G_i)$ the order and the $j$-th largest eigenvalue of $G_i$, $i \in \{1, 2\}$, respectively. Then, taking into account [3],

$$\sigma_L(G) = \{0, n_1+n_2, n_2+\mu_j (G_1), n_1+\mu_i (G_2): 1 \leq j \leq n_1-1, 1 \leq i \leq n_2-1\},$$

Taking $\phi = a(G) = \min \{n_2 + a(G_1), n_1 + a(G_2)\}$ and as $\rho_G(V(G)) = n_1+n_2$, with this chosen $\phi$, one can see that the graph $G$ satisfies the equality in [3].

Given a graph $G$ and $u \in V(G)$, the set of neighbors of $u$ is denoted by $N(u)$. For a non trivial subset of vertices $X$, denote $N_X(u) := X \cap N(u)$ and its cardinality by $d_X(u)$.

Remark 5. Let $G = (V, E)$ be a connected graph of order $n$. Let $X$ be a non trivial subset of $V$ with $n_1$ vertices. We set $n_2 = n - n_1$. Set $\alpha_X = \frac{1}{n_1} \sum_{v \in X} d(v)$ and $m_X = \frac{1}{2} \sum_{v \in X} d_X(v)$. Remark that

$$\left(\alpha_X - \frac{2m_X}{n_1}\right) \left(1 + \frac{n_1}{n_2}\right) = \frac{n_1}{n_2} \left(\alpha_X - \frac{2m_X}{n_1}\right) = \frac{n_1}{n_1n_2} \sum_{v \in X} d_X(v) = \rho_G(X).$$

By direct calculation, it results

$$\rho_G(X) = \left(\alpha_X - \frac{2m_X}{n_1}\right) + \left(\alpha_{X^c} - \frac{2m_{X^c}}{n-n_1}\right).$$
Corollary 6. Let $G = (V, E)$ be a connected graph of order $n$. Let $X$ be a non trivial subset of vertices of $G$ and let us consider $\alpha_X$ as in Remark 5. Then

$$S_L(G) \geq \max \{ |\rho_G(X) - \alpha_X|, |\rho_G(X) - \alpha_X^c| \}. \quad (4)$$

Proof. We set $|X| = n_1$. Note that $a(G) \leq \frac{1}{n_1} n_1 \delta \leq \alpha_X \leq \frac{1}{n_1} n_1 \Delta \leq \mu_1$ and with the same argument we have $a(G) \leq \alpha_X \leq \mu_1$. Thus, the inequality in (4) follows from Corollary 3.

2.1. Lower bound for the Laplacian spread of graphs with an independent nontrivial subset of vertices

The following results and their applications use Corollary 3 in some special cases. In this section we present a lower bound for the Laplacian spread of graphs that have an independent nontrivial subset of vertices.

Corollary 7. Let $G$ be an $(n, m)$-connected graph and $T$ an independent set of $G$ on $n_1$ vertices. Then

$$S_L(G) \geq \frac{n \sum_{v \in T} d(v)}{n_1 (n - n_1)} - \delta(G). \quad (5)$$

If the equality holds, then there exist $s$ and $t$ such that $d_{T_c}(v) = d(v) = s$, for all $v \in T$ and $d_{T}(v) = t$, for all $v \in T_c$, with $sn_1 = t(n - n_1)$. Moreover $\mu_1(G) = \frac{n \sum_{v \in T} d(v)}{n_1 (n - n_1)}$ and $a(G) = \delta(G)$.

Proof. Note that if $T \subset V(G)$ is an independent set, $|E_T(G)| = \sum_{v \in T} d(v) \geq n_1 \delta(G)$ and so $\rho_G(T) \geq \frac{n \delta(G)}{n - n_1} > \delta(G)$. The result follows from Corollary 3.

Corollary 8. Let $G$ be an $(n, m)$-connected graph and $X$ a nontrivial subset of vertices such that $\langle X \rangle|_G = \overline{K}_{n_1}$. Then

$$S_L(G) \geq \frac{n_1 (n - 1 - \Delta(G))}{n - n_1}.$$

Proof. Throughout the proof $\Delta = \Delta(G)$. It is clear that $\langle X \rangle|_{\overline{G}} = \overline{K}_{n_1}$ and therefore it is an independent set in $\overline{G}$. We now apply Corollary 3 to $\overline{G}$ in order to obtain

$$S_L(\overline{G}) \geq \rho_{\overline{G}}(X) - \delta(\overline{G}). \quad (6)$$
Note that $|E_X(G)| = n_1(n - n_1) - |E_X(G)|$ and so $\rho_{\overline{G}}(X) = n - \rho_G(X)$.

Since $\delta(G) = n - 1 - \Delta$, $|E_X(G)| \leq n_1(\Delta - (n_1 - 1))$, for $\langle X \rangle |_G = K_{n_1}$, and recalling that $S_L(\overline{G}) = S_L(G)$ we obtain

$$S_L(G) \geq \Delta + 1 - \rho_{\overline{G}}(X) \geq \Delta + 1 - \frac{n_1(\Delta + 1 - n_1)}{n_1(n - n_1)} = \frac{n_1(n - 1 - \Delta)}{n - n_1}.$$
Proof. Since \( N \) is a \( \text{DIM} \), \( R = V(N)^c \) is an independent set of \( G \) with \( n - 2k \) vertices and \( |E_R(G)| = \sum_{v \in R} d(v) = m - k \). The result follows from Corollary 7. \( \square \)

Remark 12. Since the Laplacian eigenvalues are either integers or irrational numbers, see e.g. in [9], the equality case in Corollary 7 implies that either \( n, m - k \) or both must be even numbers.

Corollary 13. Let \( G \) be a graph with \( n \) vertices. Let \( N = \{i_1j_1, \ldots, i_kj_k\} \) be a \( \text{DIM} \) of \( G \), where \( 2k \leq n - 1 \). Let \( X = V(N) \). Then
\[
S_L(G) \geq \alpha_X - 1.
\]
If equality holds then the conditions 1. and 2. of Proposition 7 hold for \( X = V(N) \). Moreover, \( \rho_G(V(N)) = \mu_1 \) and \( \frac{1}{n - 2k} \sum_{v \in V(N)^c} d(v) = a(G) \).

Proof. Let \( X = V(N) \) and \( n_1 = 2k \). Taking into account Remark 5,
\[
\alpha_X = \frac{1}{2k} \sum_{v \in V(N)} d(v),
\]
\[
\alpha_{X^c} = \frac{1}{n - 2k} \sum_{v \in V(N)^c} d(v),
\]
\[
2m_X = \sum_{v \in V(N)} d_{V(N)}(v) = 2k, \text{ and}
\]
\[
2m_{X^c} = \sum_{v \in V(N)^c} d_{V(N)^c}(v) = 0.
\]
Therefore, \( \rho_G(V(N)) = \alpha_X - \frac{2k}{2k} + \alpha_{X^c} \). We now apply Corollary 6 to obtain
\[
S_L(G) \geq |\rho_G(X) - \alpha_{X^c}| = \alpha_X - 1.
\]
By Corollary 8 if equality holds there exists an integer \( s \) such that \( d_{V(N)^c}(v) = s \) for all \( v \in V(N) \), \( d_{V(N)}(v) = t \) for all \( v \in V(N)^c \), and \( s(2k) = t(n - 2k) \). Additionally, \( \rho_G(V(N)) = \alpha_X - 1 + \alpha_{X^c} = \mu_1 \), and \( a(G) = \alpha_{X^c} \). \( \square \)
2.2. Lower bounds for the Laplacian spread of graphs with \((\kappa, \tau)\)-regular subset of vertices

Now we search for lower bounds for \(S_L(G)\) in terms of the order of \(S\), where \(S\) is a \((\kappa, \tau)\)-regular subset of \(V\).

**Corollary 14.** Let \(G = (V, E)\) be a graph with \(n\) vertices and \(S\) a \((\kappa, \tau)\)-regular subset of \(V\) with cardinality \(n_1\), and suppose that there exists a vertex \(v \in S^c\) with no neighbors in \(S^c\). Then

\[
S_L(G) \geq \frac{\tau(n-n_1)}{n_1}. \tag{9}
\]

If the equality holds there exists an integer \(s\) such that \(d_{S^c}(v) = s\), for all \(v \in S\), \(sn_1 = \tau(n-n_1)\), with \(\mu_1 = \frac{\tau n}{n_1}\) and \(a(G) = \tau\).

**Proof.** Let \(S\) be a \((\kappa, \tau)\)-regular subset of \(V\) with cardinality \(n_1\), \(|E_{S^c}(G)| = (n-n_1)\tau\) and

\[
\rho_G(S) = \rho_G(S^c) = \frac{n(n-n_1)\tau}{n_1(n-n_1)} = \frac{\tau n}{n_1}.
\]

Let \(v \in S^c\) be a vertex with no neighbors in \(S^c\). Then, by removing the \(\tau\) edges connecting \(v\) with its \(\tau\) neighbors in \(S\), the graph \(G\) becomes disconnected. In consequence, \(a(G) \leq \tau \leq \frac{\tau n}{n_1} \leq \mu_1\) and

\[
S_L(G) \geq \frac{\tau n}{n_1} - \tau = \frac{\tau(n-n_1)}{n_1}.
\]

The equality case follows from the equality case in Corollary 3.

**Corollary 15.** Let \(G = (V, E)\) be a graph with \(n\) vertices, \(S\) a \((\kappa, \tau)\)-regular subset of \(V\) with cardinality \(n_1\) and \(\kappa \leq \tau\). Suppose that there exists a vertex \(v \in S\) with no neighbors in \(S^c\). Then

\[
S_L(G) > \frac{\tau n}{n_1} - \kappa. \tag{10}
\]

**Proof.** Let \(v \in S\) in the condition of the statement then, removing \(\kappa\) edges connecting \(v\) with its \(\kappa\) neighbors in \(S\) the graph \(G\) becomes disconnected. Therefore \(a(G) \leq \kappa \leq \tau \leq \frac{\tau n}{n_1} \leq \mu_1\). Taking \(\phi = \kappa\) in Corollary 3 we obtain

\[
S_L(G) \geq \frac{\tau n}{n_1} - \kappa,
\]

and, as \(d_{S^c}(v) = 0\), there is no equality. Then, the result follows.
Corollary 16. Let $G = (V, E)$ be a graph with $n$ vertices and $S$ a $(\kappa, \tau)$-regular subset of $V$ with cardinality $n_1$. If $\tau - \kappa \leq 1$, then

$$S_L(G) \geq \Delta + 1 - \frac{\tau n}{n_1}. \quad (11)$$

If the equality holds there exists an integer $s$ such that $d_{S^c}(v) = s$, for all $v \in S$ and $sn_1 = \tau(n - n_1)$. Moreover, $\mu_1 = \Delta + 1$ and $a(G) = \frac{\tau n}{n_1}$.

Proof. Since $S$ is a $(\kappa, \tau)$-regular subset of $V$ with cardinality $n_1$ and $\tau - \kappa \leq 1$, $|E_S(G)| = \sum_{v \in S} d(v) - \sum_{v \in S} d_S(v) \leq n_1(\Delta - \kappa) \leq n_1(\Delta + 1 - \tau)$.

It follows from Proposition 1 that $a(G) \leq \frac{\tau n}{n_1} = \tau - \frac{(n - n_1)\tau}{n_1} = \tau + \frac{|E_S(G)|}{n_1} \leq \Delta + 1 \leq \mu_1$.

Taking $\phi = \Delta + 1$ in Corollary 3, the result follows. \qed

Corollary 17. Let $G = (V, E)$ be a graph with $n$ vertices and $S$ a $(\kappa, \tau)$-regular subset of $V$ with cardinality $n_1$. Then

$$S_L(G) \geq |\kappa - \tau|. \quad (12)$$

If equality holds there exists an integer $s$, $d_{S^c}(v) = s$ for all $v \in S$ and $sn_1 = \tau(n - n_1)$.

Proof. As $S$ is a $(\kappa, \tau)$-regular subset of $V$ with cardinality $n_1$, $\alpha_S = \frac{1}{n_1} \sum_{v \in S} d(v) = \frac{1}{n_1} \left( \sum_{v \in S} d_S(v) + |E_S(G)| \right)$

$$= \frac{1}{n_1} \left( n_1\kappa + (n - n_1)\tau \right) = \kappa + \frac{n - n_1}{n_1}\tau$$

$$= \kappa - \tau + \frac{n\tau}{n_1} = \kappa - \tau + \rho_G(S).$$

Applying Corollary 6 we obtain

$$S_L(G) \geq |\rho_G(S) - \alpha_S| = |\kappa - \tau|. $$

Moreover, by Corollary 3 if equality holds there exists an integer $s$, $d_{S^c}(v) = s$ for all $v \in S$, and $sn_1 = \tau(n - n_1)$. \qed

Example 18. If $S = V(C_4)$ in $G = C_4 \lor C_4$ then $S$ is a $(2, 4) = (\kappa, \tau)$-regular set of $V(G)$. Moreover, $\Delta = \delta = 6$, and therefore, $\Delta + 1 - \delta = 1 < |\kappa + \tau| = 2 = S_L(G)$. Thus, in this case, the lower bound in Corollary 17 is sharp and it is an improvement of (2).
3. Lower bound for Laplacian spread of graphs with prescribed degree sequences

In this section, using some known numerical inequalities and an appropriate parameter, we present lower bounds for the Laplacian spread of a graph with a prescribed degree sequence.

We start recalling the following numerical inequality, \[25\].

**Lemma 19.** Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be two positive \( n \)-vectors with \( 0 < m_1 \leq a_i \leq M_1 \) and \( 0 < m_2 \leq b_i \leq M_2 \), \( i \in \{1, \ldots, n\} \), and constants \( m_1, m_2, M_1, M_2 \). The following complement of Cauchy’s inequality holds

\[
\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i b_i} - \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^2} \leq \left( \left( \frac{M_1}{m_1} \right)^{\frac{1}{2}} - \left( \frac{m_1}{M_1} \right)^{\frac{1}{2}} \right)^2.
\]  

Recall that the first Zagreb index of a graph is defined as \( Z_g(G) = \sum_{i=1}^{n} d_i^2 \). So, using this invariant we have the next lemma.

**Lemma 20.** Let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0 \) be the Laplacian eigenvalues of a graph \( G \) with \( m \) edges and degrees sequence \( d_1, d_2, \ldots, d_n \). For \( i \neq j \) let us denote by \( \kappa_{ij} \) the number of vertices at the intersection of the neighborhoods of \( v_i \) and \( v_j \). Then,

1. \( \sum_{i=1}^{n-1} \mu_i^2 = Z_g(G) + 2m \);
2. \( \sum_{i=1}^{n-1} \mu_i^4 = \Upsilon \)

where

\[
\Upsilon = \sum_{i=1}^{n} \left( d_i^4 + 3d_i^3 \right) + Z_g(G) + \sum_{i=1}^{n} \left( \sum_{s=1}^{n} \kappa_{is}^2 + \sum_{v_s \sim v_i} 2d_s (d_s - \kappa_{si}) + d_s (d_s - 2\kappa_{si}) \right).
\]
Proof. The equality in 1. is obtained from the Frobenius matrix norm computation of \( L(G) \). For the second equality note that \( F = \sum_{i=1}^{n-1} \mu_i^4 = \text{trace} \ (L(G)^4) \). Let \( e_i \) be the \( i \)-th canonical vector of \( \mathbb{R}^n \). Therefore,

\[
F = \sum_{i=1}^{n} e_i^T L(G)^4 e_i = \sum_{i=1}^{n} \| L(G)^2 e_i \|^2,
\]

where \( \| \cdot \| \) stands for the Euclidean vector norm. Setting \( L(G) = (\delta_{ij}) \), then

\[
\delta_{ij} = \begin{cases} 
  d_i & \text{if } i = j \\
  -1 & \text{if } v_i \sim v_j
\end{cases}.
\]

Since,

\[
L(G)e_i = (\delta_{1i}, \ldots, \delta_{ni})^t,
\]

we have

\[
L(G)^2 e_i = \left( \sum_{i=1}^{n} \delta_{1i} \delta_{1i}, \ldots, \sum_{i=1}^{n} \delta_{ti} \delta_{ti}, \ldots, \sum_{i=1}^{n} \delta_{ni} \delta_{i1} \right)^t
\]

\[
= (d_1 + d_i) \delta_{1i} + \sum_{l=2}^{n} \delta_{li} \delta_{ti}, \ldots, (d_n + d_i) \delta_{ni} + \sum_{l=1}^{n} \delta_{li} \delta_{il}
\]

\[
= ((d_1 + d_i) \delta_{1i} + \kappa_{1i}, \ldots, d_i + d_i^2, \ldots, (d_n + d_i) \delta_{ni} + \kappa_{ni})^t.
\]

Therefore,

\[
\| L(G)^2 e_i \|^2 = (d_1 + d_i^2)^2 + \sum_{s=1}^{n} \sum_{s \neq i} (d_s + d_i) \delta_{si} + \kappa_{si}
\]

\[
= (d_1 + d_i^2)^2 + \sum_{s=1}^{n} \sum_{s \neq i} (x_{is}^2 + \delta_{si}^2 (d_i + d_s)^2 + 2 (d_s + d_i) \delta_{si} \kappa_{si})
\]

\[
= (d_1 + d_i^2)^2 + \sum_{s=1}^{n} \sum_{s \neq i} (x_{is}^2 + \delta_{si}^2 (d_i^2 + 2d_id_s + d_s^2) + 2 \sum_{s=1}^{n} (d_s + d_i) \delta_{si} \kappa_{si}
\]

\[
= (d_1 + d_i^2)^2 + \sum_{s=1}^{n} \sum_{s \neq i} x_{is}^2 + \delta_{si}^2 (d_i^2 + 2d_id_s + d_s^2) + 2 \sum_{s=1}^{n} (d_s + d_i) \delta_{si} \kappa_{si}
\]

\[
= (d_1 + d_i^2)^2 + \sum_{s=1}^{n} \sum_{s \neq i} x_{is}^2 + 2d_i \sum_{v_s \sim v_i} d_s + \sum_{v_s \sim v_i} d_s^2 - 2 \sum_{v_s \sim v_i} d_s \kappa_{si} - 2d_i \sum_{v_s \sim v_i} \kappa_{si}
\]

\[
= d_1^2 + 3d_i^3 + d_i^4 + \sum_{s=1}^{n} \sum_{s \neq i} x_{is}^2 + 2d_i \sum_{v_s \sim v_i} (d_s - \kappa_{si}) + \sum_{v_s \sim v_i} d_s^2 - 2 \sum_{v_s \sim v_i} d_s \kappa_{si}.
\]

As a consequence of the previous two lemmas we have the next result.
Theorem 21. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues of a graph $G$ with $m$ edges and degrees sequence $d_1, d_2, \ldots, d_n$. For $i \neq j$, let $\kappa_{ji}$ the number of vertices at the intersection of the neighborhoods of $v_i$ and $v_j$. Then

$$\left( \frac{\Upsilon(n-1) - (Z_g(G) + 2m)^2}{(n-1)(Z_g(G) + 2m)} \right)^{\frac{1}{2}} \leq S_L(G).$$

Proof. In this proof we use Lemma 19 replacing $a_i = \mu_i^2$ and $b_i = 1$, for $i = 1, \ldots, n$. Note that,

$$\mu_{n-1} \leq \mu_i^2 \leq \mu_1^2 \quad (i = 1, \ldots, n-1).$$

Moreover, $m_1 = \mu_{n-1}^2$, $M_1 = \mu_1^2$, and $m_2 = 1 = M_2$. Replacing in the inequality (13) we obtain

$$\sum_{i=1}^{n-1} \mu_i^4 - \frac{1}{n-1} \left( \sum_{i=1}^{n-1} \mu_i^2 \right)^2 \leq \left( (\mu_1^2)^{\frac{1}{2}} - (\mu_{n-1}^2)^{\frac{1}{2}} \right)^2.$$

Using identities 1. and 2. in Lemma 20 the result follows. ■

Another interesting numerical inequality was presented in [27].

Lemma 22. [27] Let $a \leq a_i \leq A$ and $b \leq b_i \leq B$, $(i = 1, \ldots, n)$. Moreover consider, $t_i \geq 0$, $(i = 1, \ldots, n)$ and $T = \sum t_i$. Then

$$\left| \frac{1}{T} \sum_{i=1}^{n} t_i a_i b_i - \frac{1}{T^2} \sum_{i=1}^{n} t_i a_i \sum_{i=1}^{n} t_i b_i \right| \leq \frac{1}{4} (A - a) (B - b). \quad (14)$$

Then, applying this result we can obtain the next lower bound for the Laplacian spread of a graph with a prescribed degree sequence.

Theorem 23. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues of a graph $G$ with $m$ edges and degrees sequence $d_1, d_2, \ldots, d_n$. For $i \neq j$, let us denote by $\kappa_{ji}$ the number of vertices at the intersection of the neighborhoods of $v_i$ and $v_j$. Then

$$\frac{2}{n-1} \left( (n-1)(Z_g(G) + 2m) - 4m^2 \right)^{\frac{1}{2}} \leq S_L(G).$$
Proof. In this proof we use Lemma 22 replacing \( a_i = b_i = \mu_i \), \( i = 1, \ldots, n-1 \). Moreover, \( t_i = \frac{1}{n-1}, \; (i = 1, \ldots, n-1) \). Since, \\
\[ \mu_{n-1} \leq \mu_i \leq \mu_1, \; (i = 1, \ldots, n-1), \]
we have, \( a = b = \mu_{n-1} \) and \( A = B = \mu_1 \). Therefore, from inequality (14) we obtain \\
\[ \left| \frac{1}{n-1} \sum_{i=1}^{n-1} \mu_i^2 - \frac{1}{(n-1)^2} \left( \sum_{i=1}^{n-1} \mu_i \right)^2 \right| \leq \frac{1}{4} (\mu_1 - \mu_{n-1})^2. \]

From identity 1. in Lemma 20 \\
\[ \frac{2}{n-1} \left( (n-1) (Z_g(G) + 2m) - 4m^2 \right)^{\frac{1}{2}} \leq \mu_1 - \mu_{n-1} \]
and the result follows. \( \blacksquare \)

We observe that the above result was also obtained in [30].

Let us denote by \( J_p \) the all ones matrix of order \( p \).

The next theorem, due to Brauer, relates the eigenvalues of an arbitrary matrix and the matrix resulting from it after a rank one additive perturbation.

**Theorem 24.** [2] Let \( M \) be an arbitrary \( n \times n \) matrix with eigenvalues \( \theta_1, \ldots, \theta_n \). Let \( x_k \) be an eigenvector of \( M \) associated with the eigenvalue \( \theta_k \), and let \( q \) be an arbitrary \( n \)-dimensional vector. Then the matrix \( M + x_k q^T \) has eigenvalues \( \theta_1, \ldots, \theta_{k-1}, \theta_k + x_k^T q, \theta_{k+1}, \ldots, \theta_n \).

Let \( \phi \) be a real number and consider the matrix \\
\[ M = L(G) + \frac{\phi}{n} J_n, \]
obtained after applying Theorem 24 replacing \( M \) by \( L(G) \) and the eigenpair \((\theta_k, q)\) of \( M \) with the eigenpair \((0, e)\) of \( L(G) \). Thus, the matrix \( L(G) + \frac{\phi}{n} J_n \) has eigenvalues \( \mu_1, \ldots, \mu_{k-1}, 0 + \frac{\phi}{n} e^T e, \mu_{k+1}, \ldots, \mu_{n-1} \). Then, if \( \mu_1 \geq \phi \geq \mu_{n-1} \) the matricial spread of \( M \), \( S(M) \) and the Laplacian spread of \( G \), coincide.

Applying Lemma 19 to the spectrum of the matrix \( M \) in (15) we get the following result.
Theorem 25. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues of a graph $G$ with $m$ edges and degrees sequence $d_1, d_2, \ldots, d_n$. Let $\mu_{n-1} \leq \phi \leq \mu_1$. For $i \neq j$, let us denote by $x_{ji}$ the number of vertices at the intersection of the neighborhoods of $v_i$ and $v_j$. Then

\[
\left(\frac{n(\Upsilon + \phi^4) - (Z_g(G) + 2m + \phi^2)^2}{n(Z_g(G) + 2m + \phi^2)} \right)^{\frac{1}{2}} \leq S_L(G).
\]

Proof. In this proof we use Lemma 19 replacing $a_i = \mu_i^2$, $b_i = 1$, for $i = 1, \ldots, n-1$ and setting $a_n = \phi$, as $\mu_{n-1} \leq \phi \leq \mu_1$. Note that, $\mu_{n-1}^2 \leq \phi^2 \leq \mu_1^2$ ($i = 1, \ldots, n$).

Moreover, $m_1 = \mu_{n-1}^2$, $M_1 = \mu_1^2$, and $m_2 = 1 = M_2$. Replacing these values in inequality (13) we have

\[
\frac{\phi^4 + \sum_{i=1}^{n-1} \mu_i^4}{\phi^2 + \sum_{i=1}^{n-1} \mu_i^2} - \frac{\phi^2 + \sum_{i=1}^{n-1} \mu_i^2}{n} \leq \left(\frac{\mu_1}{\mu_{n-1}}\right)^{\frac{1}{2}} - \left(\frac{\mu_{n-1}}{\mu_1}\right)^{\frac{1}{2}})^2.
\]

Using identities 1. and 2. in Lemma 20 the result follows. ■

Applying Lemma 22 to the matrix $M$ in (15) we get the following inequality.

Theorem 26. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues of a graph $G$ with $m$ edges and degrees sequence $d_1, d_2, \ldots, d_n$. Let $\mu_{n-1} \leq \phi \leq \mu_1$. Then

\[
\frac{2}{n} \left| nZ_g(G) + 2mn + (n-1)\phi^2 - 4m(m + \phi) \right|^{\frac{1}{2}} \leq S_L(G).
\]

Proof. In this proof we use Lemma 22 replacing $a_i = b_i = \mu_i$, ($i = 1, \ldots, n-1$) and $a_n = b_n = \phi$. Moreover, $t_i = \frac{1}{n}$, ($i = 1, \ldots, n$). Since,

$\mu_{n-1} \leq a_i \leq \mu_1$ ($i = 1, \ldots, n$)

we have, $a = b = \mu_{n-1}$ and $A = B = \mu_1$. Replacing in the inequality (14) we obtain

\[
\frac{1}{n} \left( \sum_{i=1}^{n-1} \mu_i^2 + \phi^2 \right) - \frac{1}{n^2} \left( \sum_{i=1}^{n-1} \mu_i + \phi \right)^2 \leq \frac{1}{4} (\mu_1 - \mu_{n-1})^2.
\]
Using identity 1. in Lemma 20

\[ \frac{4}{n^2} \left| n \left( 2m + Z_g(G) + \phi^2 \right) - (2m + \phi)^2 \right| \leq (\mu_1 - \mu_{n-1})^2 \]

\[ \Rightarrow \quad \frac{4}{n^2} \left| n \left( 2m + Z_g(G) + \phi^2 \right) - 4m^2 - 4m\phi - \phi^2 \right| \leq (\mu_1 - \mu_{n-1})^2 \]

\[ \Rightarrow \quad \frac{4}{n^2} \left| nZ_g(G) + 2mn + (n - 1) \phi^2 - 4m (m + \phi) \right| \leq (\mu_1 - \mu_{n-1})^2. \]

\[ \blacksquare \]

4. An Example

In the next example, a computational experiment is presented in order to approach the Laplacian spread of $G$. In each column, we use different parameters, namely:

1. algebraic connectivity;
2. edge density of a set $X$;
3. minimum vertex degree,

and we test the lower bounds presented at Corollary 8, Theorem 21, Theorem 23, Theorem 25 and Theorem 26.

In this example, it can be seen that the edge density gives a better approach to algebraic connectivity compared with the minimum vertex degree $\delta$. In consequence, better results are obtained in the final lower bounds for the Laplacian spread when we replace the parameter $\phi$ by the edge density rather than the minimum degree.

Example 27. Let $X = \{1, 2, 3\} \subseteq V(G)$, where $G$ is the graph at Figure 1. For $G$ we have $a(G) = 1.75$, $\delta(G) = 3$, $\rho_G(X) = 2$, $S_L(G) = 5.8$.

| $\phi$ | $a(G)$ | $\rho_G(X)$ | $\delta(G)$ |
|---|---|---|---|
| Corollary 8 | 2 | 2 | 2 |
| $\tau_1$ | 3.2313 | 3.2313 | 3.2313 |
| $\tau_2$ | 3.6636 | 3.6316 | 3.4762 |
| $\tau_3$ | 3.8730 | 3.8730 | 3.8730 |
| $\tau_4$ | 4.3477 | 4.2630 | 3.9752 |
Figure 1: Graph $G$

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