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A Warp Factor
in the Nonsymmetric
Kaluza–Klein (Jordan–Thiry) Theory

Abstract. We consider in the paper some consequences of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory with spontaneous symmetry breaking connecting to the existence of the warp factor.

We develop in the paper some applications and consequences of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory extensively presented in the first point of Ref. [1]. We refer for all the details to Ref. [1], especially to the book on nonsymmetric fields theory and its applications.

Let us give a short description of the theory.

We develop a unification of the Nonsymmetric Gravitational Theory and gauge fields (Yang–Mills’ fields) including spontaneous symmetry breaking and the Higgs’ mechanism with scalar forces connected to the gravitational constant. The theory is geometric and unifies tensor-scalar gravity with massive gauge theory using a multidimensional manifold in a Jordan–Thiry manner. We use a nonsymmetric version of this theory. The general scheme is the following. We introduce the principal fibre bundle over the base $V = E \times G/G_0$ with the structural group $H$, where $E$ is a space-time, $G$ is a compact semisimple Lie group, $G_0$ is its compact subgroup and $H$ is a semisimple compact group. The manifold $M = G/G_0$ has an interpretation as a “vacuum states manifold” if $G$ is broken to $G_0$ (classical vacuum states). We define on the space-time $E$, the nonsymmetric tensor $g_{\alpha\beta}$ from N.G.T., which is equivalent to the existence of two geometrical objects

$$ g = g^{(\alpha\beta)} \theta^\alpha \otimes \theta^\beta $$

$$ g = g_{[\alpha\beta]} \theta^\alpha \wedge \theta^\beta $$

the symmetric tensor $g$ and the 2-form $g$. Simultaneously we introduce on $E$ two connections from N.G.T. $\nabla^{\alpha}_{\beta\gamma}$ and $\nabla_{\beta\gamma}^\alpha$. On the homogeneous space $M$ we define the nonsymmetric metric tensor

$$ g_{\tilde{a}\tilde{b}} = h_{\tilde{a}\tilde{b}}^0 + \zeta k_{\tilde{a}\tilde{b}}^0 $$

where $\zeta$ is the dimensionless constant, in a geometric way. Thus we really have the nonsymmetric metric tensor on or $V = E \times G/G_0$.

$$ \gamma_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & r^2 g_{\tilde{a}\tilde{b}} \end{pmatrix} $$
r is a parameter which characterizes the size of the manifold $M = G/G_0$. Now on the principal bundle $P$ we define the connection $\omega$, which is the 1-form with values in the Lie algebra of $H$.

After this we introduce the nonsymmetric metric on $P$ right-invariant with respect to the action of the group $H$, introducing scalar field $\rho$ in a Jordan–Thiry manner. The only difference is that now our base space has more dimensions than four. It is $(n_1 + 4)$-dimensional, where $n_1 = \dim(M) = \dim(G) - \dim(G_0)$. In other words, we combine the nonsymmetric tensor $\gamma_{AB}$ on $V$ with the right-invariant nonsymmetric tensor on the group $H$ using the connection $\omega$ and the scalar field $\rho$. We suppose that the factor $\rho$ depends on a space-time point only. This condition can be abandoned and we consider a more general case where $\rho = \rho(x,y)$, $x \in E$, $y \in M$ resulting in a tower of massive scalar field $\rho_k, k = 1, 2 \ldots$. This is really the Jordan–Thiry theory in the nonsymmetric version but with $(n_1 + 4)$-dimensional “space-time”. After this we act in the classical manner. We introduce the linear connection which is compatible with this nonsymmetric metric. This connection is the multidimensional analogue of the connection $\tilde{\Gamma}_{\beta\gamma}^\alpha$ on the space-time $E$. Simultaneously we introduce the second connection $W$. The connection $W$ is the multidimensional analogue of the $\tilde{W}$-connection from N.G.T. and Einstein’s Unified Field Theory. Now we calculate the Moffat–Ricci curvature scalar $R(W)$ for the connection $W$ and we get the following result. $R(W)$ is equal to the sum of the Moffat–Ricci curvature on the space-time $E$ (the gravitational lagrangian in Moffat’s theory of gravitation), plus $(n_1 + 4)$-dimensional lagrangian for the Yang–Mills' field from the Nonsymmetric Kaluza–Klein Theory plus the Moffat–Ricci curvature scalar on the homogeneous space $G/G_0$ and the Moffat–Ricci curvature scalar on the group $H$ plus the lagrangian for the scalar field $\rho$. The only difference is that our Yang–Mills’ field is defined on $(n_1 + 4)$-dimensional “space-time” and the existence of the Moffat–Ricci curvature scalar of the connection on the homogeneous space $G/G_0$. All of these terms (including $R(W)$) are multiplied by some factors depending on the scalar field $\rho$.

This lagrangian depends on the point of $V = E \times G/G_0$ i.e. on the point of the space-time $E$ and on the point of $M = G/G_0$. The curvature scalar on $G/G_0$ also depends on the point of $M$.

We now go to the group structure of our theory. We assume $G$ invariance of the connection $\omega$ on the principal fibre bundle $P$, the so called Wang-condition. According to the Wang-theorem the connection $\omega$ decomposes into the connection $\tilde{\omega}_E$ on the principal bundle $Q$ over space-time $E$ with structural group $G$ and the multiplet of scalar fields $\Phi$. Due to this decomposition the multidimensional Yang–Mills’ lagrangian decomposes into: a 4-dimensional Yang–Mills’ lagrangian with the gauge group $G$ from the Nonsymmetric Kaluza–Klein Theory, plus a polynomial of 4th order with respect to the fields $\Phi$, plus a term which is quadratic with respect to the gauge derivative of $\Phi$ (the gauge derivative with respect to the connection $\tilde{\omega}_E$ on a space-time $E$) plus a new term which is of 2nd order in the $\Phi$, and is linear with respect to the Yang–Mills’ field strength. After this we perform the dimensional reduction procedure for
the Moffat–Ricci scalar curvature on the manifold $\mathcal{P}$. We average $R(W)$ with respect to the homogeneous space $M = G/G_0$. In this way we get the lagrangian of our theory. It is the sum of the Moffat–Ricci curvature scalar on $E$ (gravitational lagrangian) plus a Yang–Mills’ lagrangian with gauge group $G$ from the Nonsymmetric Kaluza–Klein Theory plus a kinetic term for the scalar field $\Phi$, plus a potential $V(\Phi)$ which is of 4th order with respect to $\Phi$, plus $\mathcal{L}_{\text{int}}$ which describes a nonminimal interaction between the scalar field $\Phi$ and the Yang–Mills’ field, plus cosmological terms, plus lagrangian for scalar field $\rho$. All of these terms (including $R(W)$) are multiplied of course by some factors depending on the scalar field $\rho$. We redefine tensor $g_{\mu\nu}$ and $\rho$ and pass from scalar field $\rho$ to $\Psi^\rho = e^{-\Psi}$.

After this we get lagrangian which is the sum of gravitational lagrangian, Yang–Mills’ lagrangian, Higgs’ field lagrangian, interaction term $\mathcal{L}_{\text{int}}$ and lagrangian for scalar field $\Psi$ plus cosmological terms. These terms depend now on the scalar field $\Psi$. In this way we have in our theory a multiplet of scalar fields $(\Psi, \Phi)$. As in the Nonsymmetric-Nonabelian Kaluza–Klein Theory we get a polarization tensor of the Yang–Mills’ field induced by the skewsymmetric part of the metric on the space-time and on the group $G$. We get an additional term in the Yang–Mills’ lagrangian induced by the skewsymmetric part of the metric $g_{\alpha\beta}$. We get also $\mathcal{L}_{\text{int}}$, which is absent in the dimensional reduction procedure known up to now. Simultaneously, our potential for the scalar–Higgs’ field really differs from the analogous potential. Due to the skewsymmetric part of the metric on $G/G_0$ and on $H$ it has a more complicated structure. This structure offers two kinds of critical points for the minimum of this potential: $\Phi_{\text{crt}}^0$ and $\Phi_{\text{crt}}^1$. The first is known in the classical, symmetric dimensional reduction procedure and corresponds to the trivial Higgs’ field (“pure gauge”). This is the “true” vacuum state of the theory. The second, $\Phi_{\text{crt}}^1$, corresponds to a more complex configuration. This is only a local (no absolute) minimum of $V$. It is a “false” vacuum. The Higgs’ field is not a “pure” gauge here. In the first case the unbroken group is always $G_0$. In the second case, it is in general different and strongly depends on the details of the theory: groups $G_0$, $G$, $H$, tensors $\ell_{ab}$, $g_{\bar{a}\bar{b}}$ and the constants $\zeta$, $\xi$. It results in a different spectrum of mass for intermediate bosons. However, the scale of the mass is the same and it is fixed by a constant $r$ (“radius” of the manifold $M = G/G_0$). In the first case $V(\Phi_{\text{crt}}^0) = 0$, in the second case it is, in general, not zero $V(\Phi_{\text{crt}}^1) \neq 0$. Thus, in the first case, the cosmological constant is a sum of the scalar curvature on $H$ and $G/G_0$, and in the second case, we should add the value $V(\Phi_{\text{crt}}^1)$. We proved that using the constant $\xi$ we are able in some cases to make the cosmological constant as small as we want (it is almost zero, maybe exactly zero, from the observational data point of view). Here we can perform the same procedure for the second term in the cosmological constant using the constant $\zeta$. In the first case we are able to make the cosmological constant sufficiently small but this is not possible in general for the second case.

The transition from “false” to “true” vacuum occurs as a second order phase transition. We discuss this transition in context of the first order phase transition
in models of the Universe. In this paper the interesting point is that there exists an effective scale of masses, which depends on the scalar field $\Psi$. Using Palatini variational principle we get an equation for fields in our theory. We find a gravitational equation from N.G.T. with Yang–Mills’, Higgs’ and scalar sources (for scalar field $\Psi$) with cosmological terms. This gives us an interpretation of the scalar field $\Psi$ as an effective gravitational constant

$$G_{\text{eff}} = G_N e^{-(n+2)\Psi}$$

We get an equation for this scalar field $\Psi$. Simultaneously we get equations for Yang–Mills’ and Higgs’ field. We also discuss the change of the effective scale of mass, $m_{\text{eff}}$ with a relation to the change of the gravitational constant $G_{\text{eff}}$.

In the “true” vacuum case we get that the scalar field $\Psi$ is massive and has Yukawa-type behaviour. In this way the weak equivalence principle is satisfied. In the “false” vacuum case the situation is more complex. It seems that there are possible some scalar forces with infinite range. Thus the two worlds constructed over the “true” vacuum and the “false” vacuum seem to be completely different: with different unbroken groups, different mass spectrum for the broken gauge and Higgs’ bosons, different cosmological constants and with different behaviour for the scalar field $\Psi$. The last point means that in the “false” vacuum case the weak equivalence principle could be violated and the gravitational constant (Newton’s constant) would increase in distance between bodies.

We explore Einstein $\lambda$-transformation for a connection $W^A_B$ ($(m + 4)$-dimensional) in order to get an interpretation of $\mathbb{R}_+$ gauge invariance for a field $\widetilde{W}_\mu$. We discuss $\mathbb{R}_+$ and $U(1)_F$ invariance. We decide that $U(1)_F$ invariance from G.U.T. is a local invariance. Due to a geometrical construction we are able to identify $\widetilde{\mathcal{A}}_\mu$ from Moffat’s theory of gravitation with the four-potential $\mathcal{A}_\mu^F$ corresponding to the $U(1)_F$ group (internal rotations connected to fermion charge). In this way, the fermion number is conserved and plays the role of the second gravitational charge. Due to the Higgs’ mechanism $\mathcal{A}_\mu^F$ is massive and its strength, $\mathcal{H}_{\mu\nu}^F$, is of short range with Yukawa-type behaviour. This has important consequences. The Lorentz-like force term (or Coriolis-like force term) in the equation of motion for a test particle is of short range with Yukawa-type behaviour. The range of this force is smaller than the range of the weak interactions. Thus it is negligible in the equation of motion for a test particle. We discuss the possibility of the cosmological origin of the mass of the scalar field $\Psi$ and geodetic equations on $\overline{P}$. We consider an infinite tower of scalar fields $\Psi_k(x)$ coming from the expansion of the field $\Psi(x,y)$ on the manifold $M = G/G_0$ into harmonics of the Beltrami–Laplace operator. Due to Friedrichs’ theory we can diagonalize an infinite matrix of masses for $\Psi_k$ transforming them into new fields $\Psi'_k$. The truncation procedure means here to take a zero mass mode $\Psi_0$ and equal it to $\Psi$.

We consider a possibility to take seriously additional dimensions in our theory in a framework similar to a Randall–Sundrum scenario. In our theory the additional dimensions connecting to the manifold $M$ (a vacuum manifold) could be considered in such a way. They are not directly observable because the size of the manifold...
is very small. In order to see them it is necessary to excite massive modes of the scalar field (a tower of those fields). We find an interesting toy model (a 5-dimensional model) which can describe a possibility to travel with speed higher than a speed of light using the fifth dimension. This dimension has nothing to do with the fifth dimension in Kaluza–Klein theory. We do some analysis on an energy to excite a scalar field $\Psi$ to get this special solution to the theory. We discuss also some quantitative relations involving travelling signals in the model via fifth dimension. We consider various possibilities to excite a warp factor due to fluctuations of a tower of scalar fields finding a density of an excitation energy. Eventually we find a zero energy (or almost zero) condition for such an excitation. We consider also a simple solution for hierarchy problem in the framework of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory.

Let us consider the Eqs (5.3.17–19) of the first point of Ref. [1], p. 329, and let us release the condition that $\rho$ is independent of $y$. Thus $\rho = \rho(x, y)$, $y \in G/G_0$. In Eq. (5.3.34) of the first point of Ref. [1], p. 333, we get in a place of $\lambda^2$ the formula

$$\lambda^2 \rho^{n-2} \left( M\gamma^{(\mu\nu)} p_{\gamma\rho} p_{\mu\rho} + n^2 g^{[\mu\nu]} g_{\rho\mu} \gamma^{(\delta\gamma)} \right) p_{\nu\rho} \cdot \rho_{p}$$

(1)

the formula

$$\lambda^2 \rho^{-2} \left( M\gamma^{(CM)} p_{\rho,C} p_{\rho,M} + \frac{n}{2} \gamma^{[MN]} \gamma^{(DC)} p_{N} \cdot \rho_{C} \right) \cdot$$

(2)

This $\rho$ has nothing to do with a density of energy considered below.

Now let us repeat the procedure from Section 5.5 of the first point of Ref. [1], i.e. the redefinition of $g_{\mu\nu}$ and $\rho$. We get the formula (5.5.5) of the first point of Ref. [1], p. 355, but in a place of $\mathcal{L}_{\text{scal}}(\Psi)$ we get the formula (5.14.3). Simultaneously $\Psi = \Psi(x, y)$, $x \in E$, $y \in G/G_0$ and the metric on a space-time $E$ depends on $y \in G/G_0$, i.e.

$$g_{\mu\nu} = g_{\mu\nu}(x, y), \quad x \in E, \quad y \in G/G_0.$$  

(3)

Thus $g_{\mu\nu}$ is parametrized by a point of $G/G_0$. Simultaneously we can interpret a dependence on higher dimensions as an existence of a tower of scalar fields $\Psi_K$ (see Eqs (5.14.4–8), Eqs (5.14.11–14) from the first point of Ref. [1], p. 434–436, and a discussion below).

The interesting point will be to find physical consequences of this dependence for $g_{\mu\nu}$. This can be achieved by considering cosmological solutions of the theory. Thus let us come to Eq. (5.5.5) of the first point of Ref. [1], p. 355, supposing the lagrangian of matter fields is written as $L_{\text{matter}}$, $g_{\mu\nu} = g_{\nu\mu}$ and depends on $y \in G/G_0$. We get

$$L\sqrt{-g}\sqrt{\tilde{\gamma}} = \sqrt{-g}\sqrt{\tilde{\gamma}} \left( \mathcal{R}(\tilde{\gamma}) + \frac{\lambda^2}{4} e^{-(n+2)\Psi} L_{\text{matter}} \right. + \left. \frac{\lambda^2}{4} \mathcal{L}_{\text{scal}}(\Psi) + \frac{1}{\lambda^2} e^{(n+2)\Psi} \mathcal{R}(\tilde{\gamma}) + \frac{1}{r^2} e^{n\Psi} \tilde{P} \right).$$

(4)
According to the standard interpretation of the constant $\lambda$ we have

$$\frac{\lambda^2}{4} \sim \frac{1}{m_{pl}^2} \quad (5)$$

and

$$\frac{1}{r^2} \sim m_{\tilde{A}}^2 \quad (6)$$

where $m_{\tilde{A}}$ is a scale of a mass of broken gauge bosons in our theory and $m_{pl}$ is a Planck mass, $c = 1$, $\hbar = 1$.

Thus one gets form variation principle with respect to $g_{\mu\nu}$ and $\Psi$:

$$\bar{R}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = e^{-(n+2)\psi} T_{\mu\nu}^{\text{matter}} + \frac{1}{m_{pl}^2} T_{\mu\nu}^{\text{scal}}$$

$$+ \left( \frac{m_{pl}^2}{8} e^{(n+2)\psi} \bar{R} + m_{\tilde{A}}^2 e^{n\psi} \bar{\Gamma} \right) g_{\mu\nu} \quad (7)$$

$$M g^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \Psi + \frac{1}{r^2} \tilde{L} \Psi + \frac{m_{\tilde{A}}^4}{4} (n + 2) e^{(n+2)\psi} \bar{R}$$

$$+ (m_{\tilde{A}} m_{pl})^2 n e^{n\psi} \bar{\Gamma} - (n + 2) T e^{-(n+2)\psi} = 0 \quad (8)$$

where $\bar{R}_{\mu\nu}$ and $\bar{R}$ are a Ricci tensor and a scalar curvature of a Riemannian geometry induced by a metric $g_{\mu\nu}$ (which can depend on $y \in G/G_0$).

$\tilde{L}$ is an operator on $G/G_0$ defined by Eq. (5.14.39) of the first point of Ref. [1], p. 442, $\bar{\nabla}_\mu$ a covariant derivative with respect to a connection induced by $g_{\mu\nu}$ (a Riemannian one).

$$T^{\mu\nu} = (p + \varrho) u^\mu u^\nu - pg^{\mu\nu}. \quad (9)$$

Let us consider a simple model with a metric

$$ds^2 = e^{2v(y)} dt^2 - dr^2 \quad (10)$$

and let us suppose that $G/G_0 = S^2 = SO(3)/SO(2)$. Thus

$$ds^2 = r^2 \left( d\chi^2 + \sin^2 \chi d\lambda^2 \right), \quad (\chi, \lambda) = y, \quad x \in \left[ 0, \frac{\pi}{2} \right], \quad \lambda \in (0, 2\pi). \quad (11)$$

In this case $v = v(\chi, \lambda)$ and

$$g_{[\hat{a}, \hat{b}]} = g_{[56]} = -\zeta \sin \chi, \quad (12)$$
$\tilde{L}$ is defined by Eq. (5.14.41) of the first point of Ref. [1], p. 443,

$$
\Psi = \Psi(\vec{r}, t, \chi, \lambda), \quad \rho = \rho(\vec{r}, t, \chi, \lambda), \quad p = p(\vec{r}, t, \chi, \lambda).
$$

(13)

Finally let us come to the toy model for which we suppose $\chi = \frac{\pi}{2}$ and we get effectively a 5-dimensional world $E \times S^1$, i.e.

$$
ds_5^2 = e^{2v(\lambda)} dt^2 - d\vec{r}^2 - r^2 d\lambda^2.
$$

(14)

One easily gets

$$
R_{44} = \left( v'' + (v')^2 \right) e^v
$$

(15)

and

$$
R_{55} = r^2 \left( v'' + (v')^2 \right) e^{-v}
$$

(16)

where $'$ means a derivative with respect to $\lambda$. For simplicity we suppose $\rho = p = 0$ and $\Psi = \Psi(\lambda)$ (does not depend on $\vec{r}$ and $t$). In this way

$$
\tilde{L} = \left( \frac{\partial^2}{\partial \lambda^2} + \frac{n^2 \zeta^2}{\zeta^2 + 1} \right) \frac{\partial^2}{\partial \lambda^2}.
$$

(17)

From Eq. (4) one obtains

$$
\left( \frac{d^2}{d\lambda^2} e^v \right) e^{-v} = \left( \frac{m_{pl}^4}{8} e^{(n+2)\psi} \tilde{R}(\tilde{F}) + m_A^2 e^{n\psi} \tilde{P} \right).
$$

(18)

$$
m_A^2 \left( \frac{\partial^2}{\partial \lambda^2} + \frac{n^2 \zeta^2}{\zeta^2 + 1} \right) \frac{d^2}{d\lambda^2} \Psi + \frac{m_{pl}^4}{4} (n + 2) e^{(n+2)\psi} \tilde{R}(\tilde{F})
$$

$$
+ (m_{pl}^2 m) n e^{n\psi} \tilde{P} = 0.
$$

(19)

The last equation can be transformed into

$$
\frac{d^2\Psi}{d\lambda^2} = A e^{(n+2)\psi} + B e^{n\psi}
$$

(20)

where

$$
A = -\frac{m_{pl}^4 (n + 2)}{4m_A^2 (\tilde{M} + \frac{n^2 \zeta^2}{\zeta^2 + 1})} \tilde{R}(\tilde{F})
$$

(21)

$$
B = -\frac{m_{pl}^2 n}{(\tilde{M} + \frac{n^2 \zeta^2}{\zeta^2 + 1})} \tilde{P}.
$$

(22)
Supposing $\Psi = \Psi_0 = \text{const.}$, one gets

$$A e^{2\Psi_0} + B = 0$$

(23)

$$e^{\Psi_0} = \sqrt{-\frac{B}{A}} = \frac{4m_{\tilde{A}}}{m_{\text{pl}}} \sqrt{\frac{n|\tilde{P}|}{(n+2)\tilde{R}(\tilde{\Gamma})}}.$$  

(24)

Thus from Eq. (18) one gets ($z = e^v$)

$$\frac{d^2z}{d\lambda^2} = \tilde{A} z$$

(25)

where

$$\tilde{A} = \frac{m_{\text{pl}}^2}{8} e^{(n+2)\Psi_0} \tilde{R}(\tilde{\Gamma}) + m_{\tilde{A}}^2 e^n\Psi_0 \tilde{P}$$

$$= m_{\tilde{A}}^2 \left( \frac{4m_{\tilde{A}}}{\alpha_s m_{\text{pl}}} \right)^n \left( \frac{n}{n+2} \right) \left| \frac{\tilde{P}}{n+2} \frac{|\tilde{R}(\tilde{\Gamma})|}{\tilde{\Gamma}} \right|^2 \left( \frac{n-2}{n+2} \right) |\tilde{P}| > 0.$$  

(26)

And eventually one gets

$$z = z_0 e^{\sqrt{\tilde{A}}\lambda} + z'_0 e^{-\sqrt{\tilde{A}}\lambda}.\quad (27)$$

Taking $z'_0 = 0$ we get

$$e^{2v} = z_0^2 e^{2\sqrt{\tilde{A}}\lambda}.\quad (28)$$

Simply rescaling a time in the metric (14) we finally get

$$ds_5^2 = e^{2\sqrt{\tilde{A}}\lambda} dt^2 - d\tilde{r}^2 - r^2 d\lambda^2.\quad (29)$$

In this way we get a funny toy model. We are confined on 3-dimensional brane in a 4-dimensional euclidean space for a $\lambda = 0$. Moreover $\lambda$ is changing from 0 to $2\pi$ resulting in some interesting possibilities of communication and travel in extra dimensions. This is due to a fact that an effective speed of light depends on $\lambda$ in an exponential way:

$$c_{\text{eff}} = e^{\sqrt{\tilde{A}}\lambda}.\quad (30)$$

If we move in $\lambda$ direction (even a little, remember $r = \frac{1}{m_{\tilde{A}}}$) from 0 to $\lambda_0 < 2\pi$ and after this move in space direction from $\tilde{r}_0$ to $\tilde{r}_1$ and again from $\lambda_0$ to 0 (we effectively travel from $\tilde{r}_0$ to $\tilde{r}_1$), then we can be in a point $\tilde{r}_1$ even very distant from $\tilde{r}_0$ ($L = |\tilde{r}_1 - \tilde{r}_0|$) in a much shorter time than $\frac{L}{c}$ (where $c$ is the velocity of light taken to be equal 1).

In some sense we travel in hyperspace from *Star Wars* or *Wing Commander*.

Thus we consider a metric (see Ref. [2])

$$ds_5^2 = e^{2\sqrt{\tilde{A}}\lambda} dt^2 - d\tilde{r}^2 - r^2 d\lambda^2.$$  

(31)
We get the following geodetic equations for a signal travelling in $\vec{r}$, $\lambda$ direction with initial velocity at $\lambda = 0$, $\frac{d\lambda}{dt}(0) = u$, $\frac{d\vec{r}}{dt}(0) = \vec{v}$,

$$\frac{d\vec{r}}{dt} = \vec{v} e^{\sqrt{A\lambda}}$$  \hspace{1cm} (32)

$$\left(\frac{d\lambda}{dt}\right)^2 = e^{\sqrt{A\lambda}} - (1 - u^2)e^{2\sqrt{A\lambda}}.$$  \hspace{1cm} (33)

One integrates

$$\vec{r}(\lambda) = \vec{r}_0 - \vec{v}_0 \cdot 2(1 - u^2)^{3/2} \frac{\arctg}{\sqrt{A}} \left( \frac{1}{\sqrt{1 - \frac{1}{(1 - u^2)^{1/2}}e^{-\sqrt{A\lambda}}}} \right),$$  \hspace{1cm} (34)

$$\lambda = \frac{1}{\sqrt{A}} \ln \left( \frac{4(1 - u^2)^{5/2}}{A (t - t_0)^2 + 4(1 - u^3)} \right)$$  \hspace{1cm} (35)

and

$$\vec{r}(t) = \vec{r}_0 - \vec{v}_0 \frac{(1 - u^2)^{3/2} \arctg}{\sqrt{A}} \left( \frac{4(1 - u^2)^2}{A (t - t_0)^2 + 4(1 - u^3)} \right),$$  \hspace{1cm} (36)

$$0 < \lambda < \min \left[ 2\pi, \frac{1}{2\sqrt{A}} \ln \left( \frac{1}{1 - u^2} \right) \right].$$  \hspace{1cm} (37)

Let us consider a signal travelling along an axis $x$ from 0 to $x_0$. The time of this travel is simply $x_0$. But now we can consider a travel from zero to $\lambda_0$ (in $\lambda$ direction) and after this from $(0, \lambda_0)$ to $(x_0, \lambda_0)$ and after to $(x_0, 0)$. The time of this travel is as follows:

$$t = 2t_1 + t_2$$  \hspace{1cm} (38)

where

$$t_1 = r \int_{0}^{\lambda_0} \frac{d\lambda}{e^{\sqrt{A\lambda}}} = \frac{r}{\sqrt{A}} \left( 1 - e^{-\sqrt{A\lambda_0}} \right)$$  \hspace{1cm} (39)

and

$$t_2 = x_0 e^{-\sqrt{A\lambda_0}}.$$  \hspace{1cm} (40)

For $r$ is of order of a scale of G.U.T. the first term is negligible and

$$t = x_0 e^{-\sqrt{A\lambda_0}}.$$  \hspace{1cm} (41)

Taking maximal value of $\lambda_0 = 2\pi$ we get

$$t = x_0 e^{-2\pi \sqrt{A}}.$$  \hspace{1cm} (42)
Thus we achieve really shorter time of a travelling signal through the fifth dimension.

Let us consider the more general situation when we are travelling along a time-like curve via fifth dimension from \( \vec{r} = \vec{r}_0 \) to \( \vec{r} = \vec{r}_1 \). Let the parametric equations of the curve have the following shape:

\[
\vec{r} = r(\xi), \\
t = t(\xi), \\
\lambda = \lambda(\xi), \\
0 \leq \xi \leq 1,
\]
\[
\lambda(0) = 0, \quad \lambda(1) = \lambda_0, \\
\vec{r}(0) = \vec{r}_0 \\
\vec{r}(1) = \vec{r}_1.
\]

Let a tangent vector to the curve be \((\dot{r}, \dot{\vec{r}}(\xi), \dot{\lambda}(\xi))\), where dot “\( \dot{} \)” means a derivative with respect to \( \xi \). For a total time measured by a local clock of an observer one gets

\[
\Delta t = \int_0^1 d\xi \left( e^{2\sqrt{\Lambda \lambda(\xi)}} \left( \frac{dt}{d\xi} \right)^2 - \left( \frac{d\vec{r}}{d\xi} \right)^2 - r^2 \left( \frac{d\lambda}{d\xi} \right)^2 \right)^{\frac{1}{2}}.
\]

Let us consider a three segment curve such that

1) a straight line from \( \vec{r} = 0, \lambda = 0 \) to \( \vec{r} = 0, \lambda = \lambda_0 \),
2) a straight line from \( \vec{r} = 0, \lambda = \lambda_0 \) to \( \vec{r} = \vec{r}_0, \lambda = \lambda_0 \),
3) a straight line from \( \vec{r} = \vec{r}_0, \lambda = \lambda_0 \) to \( \vec{r} = \vec{r}_0, \lambda = 0 \).

The additional assumptions are such that the traveller travels with a constant velocity \( v_\lambda \) on the first segment and on the third segment and with a velocity \( \vec{v} \) on the second one.

Thus we have a parametrization:

1) On the first segment

\[
\vec{r} = 0, \\
t = \frac{\lambda_0}{v_\lambda} \xi, \quad 0 \leq \xi \leq 1, \\
\lambda = \xi \lambda_0.
\]

2) On the second segment

\[
\vec{r}(\xi) = \vec{r}_0 \xi, \\
t = \frac{\|\vec{r}_0\|}{\|\vec{v}\|} \xi, \quad 0 \leq \xi \leq 1, \\
\lambda = \lambda_0.
\]
3) On the third segment

\[ \vec{r} = \vec{r}_0 \]
\[ t = \frac{\lambda_0}{|\vec{v}|} (1 - \xi), \quad 0 \leq \xi \leq 1 \]
\[ \lambda = (1 - \xi) \lambda_0. \]  

From Eq. (45) one easily gets

\[ \Delta t = \Delta t_1 + \Delta t_2 \]  
\[ \Delta t_1 = \frac{2r}{\sqrt{A}} \left( \frac{1}{\sqrt{v_\lambda \lambda_0}} \left( \sqrt{e^{2\sqrt{\tilde{A}\lambda_0}} - v_\lambda \lambda_0} - \sqrt{1 - v_\lambda \lambda_0} \right) \right. \]
\[ + \ \arctg \left( \frac{\sqrt{e^{2\sqrt{\tilde{A}\lambda_0}} - v_\lambda \lambda_0}}{v_\lambda \lambda_0} \right) - \arctg \left( \frac{1 - v_\lambda \lambda_0}{v_\lambda \lambda_0} \right) \right) \]
\[ \Delta t_2 = \frac{|\vec{r}_0|}{|\vec{v}|} e^{2\sqrt{\tilde{A}\lambda_0}} - |\vec{v}|^2 \]  

For \( r \) is of order of an inverse scale of G.U.T., the first term in Eq. (49) is negligible and

\[ \Delta t \approx \frac{|\vec{r}_0|}{|\vec{v}|} e^{2\sqrt{\tilde{A}\lambda_0}} - |\vec{v}|^2. \]  

The traveller passing from \( \vec{r} = 0 \) to \( \vec{r} = \vec{r}_0 \) (having \( \lambda = 0 \) during his travel) reached \( \vec{r} = \vec{r}_0 \) at time

\[ \Delta t' = \frac{|\vec{r}_0|}{|\vec{v}|} (1 - |\vec{v}|^2)^{1/2}. \]  

The first traveller has the following condition for \( \vec{v} \):

\[ |\vec{v}| \leq e^{\sqrt{\tilde{A}\lambda_0}}, \quad 0 \leq \lambda_0 \leq 2\pi. \]  

The second

\[ |\vec{v}| \leq 1. \]  

Thus we see that we can travel quicker taking a bigger \( \vec{v} \).

For an unmoving observer the time needed for a travel in the first case is

\[ \Delta t = \frac{2r \lambda_0}{v_\lambda} + \frac{|\vec{r}_0|}{|\vec{v}|} \approx \frac{|\vec{r}_0|}{|\vec{v}|}, \]
\[ |\vec{v}| \leq e^{\sqrt{\tilde{A}\lambda_0}}, \quad 0 \leq \lambda_0 \leq 2\pi. \]
In the second case

\[ \Delta t' = \frac{|\vec{r}_0|}{|\vec{v}|}, \quad |\vec{v}| \leq 1 \]  

(52b*)

Thus in some sense we get a dilatation of time:

\[ \Delta t' = \left( e^{2\sqrt{A\lambda}} - \vec{v}^2 \right)^{1/2} \Delta t \]  

(52a**)

in the first case,

\[ \Delta t' = (1 - \vec{v}^2)^{1/2} \Delta t \]  

(52b**)

in the second case.

Let us come back to the Eqs (35–37). We see that a particle coming along a geodesic is confined in a shell surrounding a brane in the fifth dimension

\[ 0 \leq \lambda \leq \frac{r}{\sqrt{A}} \ln \left( \frac{1}{(1 - u^2)^{1/2}} \right). \]  

(54)

The particle oscillates in the shell. Moreover \( 0 \leq \lambda \leq 2\pi \), thus if \( \lambda \) exceeds \( 2\pi \) (i.e. modulo \( 2\pi \)). Usually for \( u \) not close to 1 this is a really thin shell of order \( r \) (a length of a unification scale).

Let us come back to our toy model for a cone with \( \tilde{A} = 0 \). In this case an equation of cone in 5-dimensional Minkowski space is

\[ x^2 + y^2 + z^2 + r^2 \lambda^2 = t^2. \]  

(55)

An interesting question is what is an analog for (55) if \( \tilde{A} \neq 0 \). One easily gets that

\[ x^2 + y^2 + z^2 + r^2 \lambda^2 = t^2 r^2 \lambda^2 A^2 e^{2\sqrt{\lambda}}. \]  

(56)

There is no simple way to get Eq. (55) from Eq. (56). Moreover if we change a coordinate \( t \) into \( rt\lambda\tilde{A} \) one gets

\[ x^2 + y^2 + z^2 + r^2 \lambda^2 = t^2 e^{2\sqrt{\lambda}}. \]  

(57)

and for \( \tilde{A} = 0 \) we get Eq. (55).

Let us notice that Eq. (57) is necessary to be considered for \( \lambda \) modulo \( 2\pi \). It means even \( \lambda \) could be any real number in the equation we should put in a place of \( \lambda \), \( [\lambda] \), where \( [\lambda] \) is defined as follows

\[ \lambda = [\lambda] + 2\pi n, \quad [\lambda] \in (0, 2\pi), \]
$n$ is an integer and $\lambda \in (-\infty, +\infty)$. In this way in a place of Eq. (57) we get
\[
x^2 + y^2 + z^2 + r^2|\lambda|^2 = t^2 e^{2\sqrt{A}|\lambda|}.
\] (57*)
The last equation defines an interesting 4-dimensional hypersurface in 5-dimensional space ($\mathbb{R}^5$).

For $\lambda = 0$ we get a light cone on a brane (4-dimensional Minkowski space). For branes with $0 < \lambda = \lambda_0 < 2\pi$ we have a “cone”
\[
x^2 + y^2 + z^2 = t^2 e^{2\sqrt{A}\lambda_0} - \lambda_0^2
\] (58)
which is a 3-dimensional hyperboloid.

In our model we get in a quite natural way a warp factor known from Ref. [3]. Moreover we have a different reason to get it. The reason of this warp factor is a many dimensional dependence of a scalar field $\Psi = \Psi(x, \lambda)$. In particular higher-dimensional excitations of $\Psi$ forming a tower of massive scalar fields. In this model $\Psi$ can be developed in a Fourier series
\[
\Psi(x, \lambda) = \sum_{m=0}^{\infty} \psi_1^m(x) \cos(m\lambda) + \sum_{m=1}^{\infty} \psi_2^m(x) \sin(m\lambda)
\] (59)
where
\[
\psi_1^m(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi(x, \lambda) \cos(m\lambda) \, d\lambda
\] (60)
\[
\psi_2^2(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi(x, \lambda) \sin(m\lambda) \, d\lambda
\] (61)
\[
\psi_0^1(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi(x, \lambda) \, d\lambda.
\] (62)
Thus an interesting point will be to find a spectrum of mass for this tower. This is simply
\[
m_m^2 = m_A^2 \left( \frac{M}{\zeta^2 + 1} \right) m^2, \quad m = 0, 1, 2, \ldots
\] (63)
Now we come to the calculation of a content of a warp factor (a content in terms of a
tower of scalar fields). Thus we calculate
\[ \psi_m^1 = \frac{1}{2\pi} \int_0^{2\pi} e^{2\sqrt{A}\lambda} \sin(m\lambda) d\lambda, \quad m = 1, 2, 3, \ldots \] (64)
\[ \psi_m^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{2\sqrt{A}\lambda} \cos(m\lambda) d\lambda, \quad m = 1, 2, 3, \ldots \] (65)
\[ \frac{1}{2} \psi_0^1 = \frac{1}{2\pi} \int_0^{2\pi} e^{2\sqrt{A}\lambda} d\lambda. \] (66)

One gets
\[ \frac{1}{2} \psi_0^1 = \frac{1}{4\pi \sqrt{A}} \left( e^{4\pi \sqrt{A}} - 1 \right) \] (67)
\[ \psi_m^1 = -\frac{m}{2\pi} \frac{e^{4\pi \sqrt{A}} - 1}{4A + m^2} \] (68)
\[ \psi_m^2 = \frac{\sqrt{A} (e^{4\pi \sqrt{A}} - 1)}{\pi (4A + m^2)}. \] (69)

The simple question is what is the energy to excite the warp factor in terms of a tower of scalar fields. The answer is as follows:
\[ E_{\text{wf}} = \int_0^{2\pi} T_{44}(e^{2\sqrt{A}\lambda}) d\lambda \] (70)
where \( T_{44} \) is a time component of an energy-momentum tensor for a scalar field calculated for \( \Psi = e^{2\sqrt{A}\lambda} \).

One gets
\[ \frac{\text{scal}}{2\pi} T_{44} = \frac{1}{2} \left( \frac{d\Psi}{d\lambda} \right)^2 + \frac{1}{2} \lambda_{\text{co}}(\Psi), \quad \lambda_{\text{co}}(\Psi) = \frac{1}{2} |\gamma| e^{n\Psi} - \frac{\beta}{2} e^{(n+2)\Psi} \] (71)
\[ \int_0^{2\pi} \frac{1}{2} \left( \frac{d}{d\lambda} e^{2\sqrt{A}\lambda} \right)^2 d\lambda = \frac{\sqrt{A}}{2} \left( e^{8\pi \sqrt{A}} - 1 \right) \] (72)
and
\[ \frac{1}{2} \lambda_{\text{co}}(e^{2\sqrt{A}\lambda}) = -\frac{1}{2} e^{n(e^{2\sqrt{A}\lambda})} \left( \beta e^{2(e^{2\sqrt{A}\lambda})} - |\gamma| \right), \] (73)
where
\[
\gamma = \frac{m^2}{\alpha_s^2} \bar{P} = \frac{1}{r^2 \bar{P}},
\]
\[
\beta = \frac{\alpha_s^2}{\bar{P} P} \tilde{R}(\bar{F}) = \alpha_s^2 m_{pl}^2 \tilde{R}(\bar{F}).
\]

One easily gets
\[
E_{wf} = \sqrt{\bar{A}} \left( e^{8\pi \sqrt{\bar{A}}} - 1 \right) - \frac{1}{4 \sqrt{\bar{A}}} \left[ |\gamma| \left( \text{Li} \left( \exp \left( \frac{e^{4\pi \sqrt{\bar{A}}}}{n} \right) \right) - \text{Li} \left( \sqrt{\bar{A}} \right) \right) \right.
\]
\[
- \beta \left( \text{Li} \left( \exp \left( \frac{e^{4\pi \sqrt{\bar{A}}}}{n+2} \right) \right) - \text{Li} \left( \sqrt{\bar{A}} \right) \right) \right]
\]
\[
(74)
\]
where
\[
\text{Li}(x) = \int_0^x \frac{dz}{\ln z}, \quad x > 1,
\]
\[
(75)
\]
is an integral logarithm. The integral is considered in the sense of a principal value.

Let us consider Eq. (74) using asymptotic formula for \(\text{Li}(x)\) for large \(x\):
\[
\text{Li}(x) \sim x \left[ \frac{1}{\ln x} + \sum_{m=1}^{\infty} \frac{m!}{\ln^{m+1} x} \right].
\]

One gets
\[
E_{wf} = \sqrt{\bar{A}} \left( e^{8\pi \sqrt{\bar{A}}} - 1 \right) - \frac{e^{-4\pi \sqrt{\bar{A}}}}{4 \sqrt{\bar{A}}} \left( n|\gamma| \exp \left( \frac{e^{4\pi \sqrt{\bar{A}}}}{n} \right) \right)
\]
\[
- (n+2)\beta \exp \left( \frac{e^{4\pi \sqrt{\bar{A}}}}{n+2} \right).
\]

However, let us think quantum-mechanically in a following direction. Let us create (maybe in an accelerator as some kind of resonances) scalar particles of mass \(m_n\) (Eq. (63)) with an amount of \(|\psi^1_m|^2 + |\psi^2_m|^2\). In this way we create the full warp factor (for they due to interference will create it as a Fourier series). The question is what is an energy needed to create those particles. The answer is simply
\[
E = \sum_{m=1}^{\infty} \left( |\psi^1_m|^2 + |\psi^2_m|^2 \right) m_m.
\]
\[
(76)
\]
One easily gets
\[
E = \frac{m_A^2}{4\pi^2} \left( e^{4\pi \sqrt{\bar{A}}} - 1 \right)^2 \left( \frac{\bar{M}}{n^2 \zeta^2 + 1} \right)^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{m}{4A + m^2}.
\]
\[
(77)
\]
The series \( \sum_{m=1}^{\infty} \frac{m}{4A + m^2} \) is divergent for

\[
\frac{m}{4A + m^2} \approx \frac{1}{m}.
\] (78)

Thus we need a regularization technique to sum it. We use \( \zeta \)-function regularization technique similar as in Casimir-effect theory. Let us introduce a \( \tilde{\zeta}_A \)-function on a complex plane

\[
\tilde{\zeta}_A = \sum_{m=1}^{\infty} \left( \frac{m}{4A + m^2} \right)^s
\] (79)

for \( \text{Re}(s) \geq 1 + \delta, \delta > 0 \).

This function can be extended analytically on a whole complex plane. Obviously \( \tilde{\zeta}_A \) has a pole for \( s = 1 \) (as a Riemann \( \zeta \) function). Thus we should regularize it at \( s = 1 \). One gets

\[
\tilde{\zeta}_A(1) = \tilde{\zeta}_R(1) + 4\tilde{A} \sum_{m=1}^{\infty} \frac{1}{m(4A + m^2)}
\] (80)

where \( \tilde{\zeta}_R \) means a regularized \( \tilde{\zeta}_A \).

Let us introduce a \( \zeta \)-Riemann function

\[
\zeta_R(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \text{Re}(s) \geq 1 + \delta, \delta > 0.
\] (81)

One can write

\[
\text{“} \tilde{\zeta}_A(1) \text{”} = \text{“} \zeta_R(1) \text{”} + 4\tilde{A} \sum_{m=1}^{\infty} \frac{1}{m(4A + m^2)}
\] (82)

or

\[
\text{“} \zeta_A(1) \text{”} = \text{“} \zeta_R(1) \text{”} + 4\tilde{A} \zeta_R(3) - 16\tilde{A}^2 \sum_{m=1}^{\infty} \frac{1}{m^3(4A^2 + m^2)}.
\] (83)

The series in Eqs (82–83) are convergent. Moreover \( \zeta_R(s) \) has a pole at \( s = 1 \) and

\[
\lim_{s \to 1} \left[ \zeta_R(s) - \frac{1}{s-1} \right] = \gamma_E
\] (84)

where \( \gamma_E \) is an Euler constant.

Thus we can regularize \( \tilde{\zeta}_A(s) \) in such a way that

\[
\tilde{\zeta}_A(s) = \zeta_R(s) - \frac{1}{s-1} + 4\tilde{A} \sum_{m=1}^{\infty} \frac{1}{m(4A + m^2)}.
\] (85)
Thus
\[ \zeta^r_A(1) = \gamma_E + 4A \sum_{m=1}^{\infty} \frac{1}{m(4A + m^2)}. \] (86)

The series
\[ \sum_{m=1}^{\infty} \frac{1}{m(4A + m^2)} \] (87)
is convergent for every \( z^2 = -4A \) and defines an analytic function
\[ f(z) = \sum_{m=1}^{\infty} \frac{1}{m(m^2 - z^2)}. \] (88)

Using some properties of an expansion in simple fraction for
\[ \psi(x) = \frac{d}{dx} \log \Gamma(x) \] (89)
where \( \Gamma(x) \) is an Euler \( \Gamma \)-function, one gets
\[ \psi(x) = -\gamma_E - \sum_{m=0}^{\infty} \left( \frac{1}{m + x} - \frac{1}{m + 1} \right) \] (90)
\[ \zeta^r_A(1) = -\frac{1}{2} \left( \psi(2\sqrt{A}) + \psi(-2\sqrt{A}) \right) \] (91)
or
\[ \zeta^r_A(1) = \frac{1}{4\sqrt{A}} + \frac{\pi}{2} \ctg 2\sqrt{A}\pi \] (92)
(where we use \( \Gamma(z)\Gamma(-z) = \frac{\pi}{\sin \pi z} \)).

And finally
\[ E = \frac{m_A}{8\pi^2} \left( \frac{4n \sqrt{n + 2} m_A}{\alpha_s \sqrt{n + 2} m_{pl}} \right)^\frac{\hat{m}}{2} \left( \frac{|\vec{P}|}{R(\Gamma)} \right)^\frac{\hat{n}}{2} \left( \frac{n - 2}{n + 2} |\vec{P}| \right)^{\frac{\hat{n}}{2}}. \] (93)

It is easy to see that \( E = 0 \) for \( A = 0 \),
\[ \sqrt{A} = m_A \left( \frac{4\sqrt{n} m_A}{\alpha_s \sqrt{n + 2} m_{pl}} \right)^\frac{\hat{m}}{2} \left( \frac{|\vec{P}|}{R(\Gamma)} \right)^\frac{\hat{n}}{2} \left( \frac{n - 2}{n + 2} |\vec{P}| \right)^{\frac{\hat{n}}{2}}. \] (94)

For we use a dimensionless coordinate \( \lambda \) (not \( \tilde{\lambda} = r \cdot \lambda = \frac{\lambda}{m_A} \)), we can omit a factor \( m_A \) in front of the right-hand side in the formula (94).
It is interesting to notice that for $\tilde{A} = 0$, $e^{2\sqrt{\tilde{A}}} = 1$ and we have to do with a Fourier expansion of a constant mode only $(m = 0)$. This mode is a massless mode. However, a massless mode can obtain a mass from some different mechanism. Thus in some sense we have to do with an energy

$$\frac{1}{4} |\psi_0^1(\tilde{A} = 0)|^2 m_\psi = m_\psi.$$  \hfill (95)

One gets the following formula

$$m_\psi = m_{pl} \left[ \frac{|\tilde{P}|(n + 2)}{\alpha_s \sqrt{n + 2} \sqrt{R(\tilde{A})} m_{pl}} \right]^2.$$  \hfill (96)

Thus it seems that in order to create a warp factor $e^{2\sqrt{\tilde{A}}} \lambda$ in a front of $dt^2$ we should deposite an energy of a tower of scalar particles

$$\tilde{E} = E + m_\psi.$$  \hfill (97)

Even a factor equal to 1 is not free. This is of course supported by a classical field formula (Eq. (75)). For $\tilde{A} = 0$ we get

$$E_{wf} = -\pi e^n (\beta e^2 - |\gamma|)$$  \hfill (98)

or

$$E_{wf} = -\frac{\pi e^n}{\alpha_s^2} \left( e^2 \alpha_s^4 m_{pl}^2 \tilde{R}(\tilde{A}) - m_{pl}^2 |\tilde{P}| \right).$$  \hfill (99)

To be honest we should multiply $E_{wf}$ and $E$ by $V_3$, where $V_3$ is a volume of a space and a result will be divergent. Moreover, if we want to excite a warp factor only locally, $V_3$ can be finite and the result also finite.

Let us come back to the Eq. (4). The variational principle based on (4) lagrangian is really in $(n_1 + 4)$-dimensional space $E \times G/G_0$, where a size of a compact space $M = G/G_0$ is $r$. In this way the gravity lives on $(n_1 + 4)$-dimensional manifold, where $n_1$ dimensions are curled into a compact space. The scalar field $\Psi$ lives also on $(n_1 + 4)$-dimensional manifold. The matter is 4-dimensional. Thus we can repeat some conclusions from Randall-Sundrum ([4]) and Arkani-Hamed, Savas Dimopoulos and Dvali ([5]) theory. Similarly as in their case we have gravity for two regions ($V(R)$ is a Newtonian gravitational potential).

1) $R \ll r$

$$V(R) \approx \frac{m_1 \cdot m_2}{m_{pl}^2(n_1+4)} \frac{1}{R^{n_1+1}},$$  \hfill (100)
2) $R \gg r$

$$V(R) \cong \frac{m_1 \cdot m_2}{m_{pl(n_1+4)}^{n_1+2}} \frac{1}{R},$$

where $m_{pl(n_1+4)}$ is a $(n_1 + 4)$-dimensional Planck’s mass. So

$$m^2_{pl} = m_{pl(n_1+4)}^{n_1+2} \cdot m_{pl(n_1+4)}^{-n_1} = \left( \frac{m_{pl(n_1+4)}}{m_{A}} \right)^{n_1} \cdot m^2_{pl(n_1+4)}.$$  \hspace{1cm} (102)

Thus

$$m_{pl(n_1+4)} = \left( m_{pl} \right)^{\frac{n_1}{n_1+2}} \cdot \left( m_{A} \right)^{\frac{n_1}{n_1+2}}.$$  \hspace{1cm} (103)

Eqs (102–103) give us a constraint on parameters in our theory and establish a real strength of gravitational interactions given by $m_{pl(n_1+4)}$.

However, in our case we have to do with a scalar-tensor theory of gravity. Thus our gravitational constant is effective

$$G_{\text{eff}} = G_N e^{-(n+2)\Psi} = G_N \rho^{n+2}.$$  \hspace{1cm} (104)

In this case we have to do with effective “Planck’s masses” in $n_1 + 4$ and 4 dimensions

$$m^2_{pl(n_1+4)} \rightarrow m^2_{pl(n_1+4)} \cdot e^{(n+2)\Psi_{(n_1+4)}}$$  \hspace{1cm} (105)

$$m^2_{pl} \rightarrow m^2_{pl} \cdot e^{(n+2)\Psi}$$  \hspace{1cm} (106)

where $\Psi_{(n_1+4)}$ is a scalar field $\Psi$ for $R \ll r$ and $\Psi$ is for $R \gg r$.

In this way Eq. (102) reads

$$m^2_{pl} e^{(n+2)\Psi} = \left( \frac{m_{pl(n_1+4)} e^{\frac{(n+2)}{2} \Psi_{(n_1+4)}}}{m_{A}} \right)^{n_1} \cdot m^2_{pl(n_1+4)} \cdot e^{(n+2)\Psi_{(n_1+4)}}.$$  \hspace{1cm} (107)

For our contemporary epoch we can take $e^{(n+2)\Psi} \simeq 1$.

If we take as in the case of Ref. [5]

$$m_{pl(n_1+4)} \simeq m_{EW},$$  \hspace{1cm} (108)

where $m_{EW}$ is an electro-weak energy scale. We get

$$\left( \frac{m_{pl}}{m_{EW}} \right)^2 = \left( \frac{m_{EW}}{m_{A}} \right)^{n_1} \cdot e^{\frac{1}{2} (n+2)(n+2) \Psi_{(n_1+4)}}.$$  \hspace{1cm} (109)

For $R \ll r$ the field $\Psi_{(n_1+4)}$ has the following behaviour

$$\Psi_{(n_1+4)} \simeq \Psi_0 + \frac{\alpha}{R^{n_1+1}}.$$  \hspace{1cm} (110)
Thus we can approximate $\psi_{(n_1+4)}$ by $\psi_0$ where $\psi_0$ is a critical point of a selfinteracting potential for $\psi$. In this way one gets

$$\left(\frac{m_{pl}}{m_{EW}}\right) \left(\frac{m_A}{m_{EW}}\right)^{n_1} \exp\left(\frac{(n+2)(n_1+2)}{2} \psi_0\right)$$

(111)

where

$$e^{\psi_0} = \sqrt{\frac{n|\gamma|}{(n+2)\beta}} = \frac{1}{\alpha_s} \sqrt{\frac{m_{pl}}{m_{pl}} \frac{n\bar{P}}{(n+2)R(\bar{\Gamma})}}.$$ 

Thus one gets:

$$\left(\frac{m_{pl}}{m_{EW}}\right) \left(\frac{m_A}{m_{EW}}\right)^{n_1} \left(\frac{m_{pl}}{m_A}\right)^{\frac{(n+2)(n_1+2)}{2}} \left(\frac{n|\bar{P}|}{(n+2)|\bar{R}(\bar{\Gamma})|}\right)^{\frac{(n+2)(n_1+2)}{4}} = \left(\frac{1}{\alpha_s}\right)^{\frac{(n+2)(n_1+2)}{2}} \left(\frac{n|\bar{P}|}{(n+2)|\bar{R}(\bar{\Gamma})|}\right)^{\frac{(n+2)(n_1+2)}{4}}.$$ 

(112)

It is easy to see that we can achieve a solution for a hierarchy problem if $|\bar{P}/\bar{R}(\bar{\Gamma})|$ is sufficiently big, e.g. taking $\bar{R}(\bar{\Gamma})$ sufficiently small and/or $\bar{P}$ sufficiently big playing with constants $\xi$ and $\zeta$. In this way a real strength of gravity will be the same as electro-weak interactions and the hierarchy problem has been reduced to the problem of smallness of a cosmological constant (i.e. $\bar{R}(\bar{\Gamma}) \simeq 0$).

Let us consider Eq. (93) in order to find such a value of $\bar{A}$ for which $\bar{E}$ is minimal. One finds that

$$\bar{E} = 0$$ 

(113)

for

$$\bar{A} = \frac{x^2}{4\pi^2}$$ 

(114)

where $x$ satisfies an equation

$$x = -\tan x, \quad x > 0.$$ 

(115)

Eq. (115) has an infinite number of roots. One can find some roots of Eq. (115) for $x > 0$. We get

$$x_1 = 4.913 \ldots$$
$$x_2 = 7.978 \ldots$$
$$x_3 = 11.086 \ldots$$
$$x_4 = 14.207 \ldots$$

(116)

We are interested in large roots (if we want to have $c_{\text{eff}}$ considerably big).
In this case one finds

$$x = \varepsilon + \frac{\pi}{2}(2l + 1)$$

(117)

where $\varepsilon > 0$ is small and $l = 1, 2, \ldots$. Moreover, to be in line in our approximation, $l$ should be large. One gets

$$\frac{\pi}{2}(2l + 1) + \varepsilon = \text{ctg} \varepsilon.$$  

(118)

For $\varepsilon$ is small, one gets

$$\text{ctg} \varepsilon \simeq \frac{1}{\varepsilon}.$$  

(119)

Finally one gets

$$\varepsilon^2 + (2l + 1)\frac{\pi}{2} - 1 = 0$$

(120)

and

$$\varepsilon \simeq \frac{2}{(2l + 1)\pi}$$  

(121)

$$x = \frac{\pi}{2}(2l + 1) + \frac{2}{(2l + 1)\pi}$$  

(121a)

$$\tilde{A} = \frac{1}{4\pi^2} \left( \frac{\pi}{2}(2l + 1) + \frac{2}{(2l + 1)\pi} \right)^2.$$  

(122)

It is easy to notice that our approximation (121a) is very good even for small $l$, e.g. for $l = 4$ we get $x_4$ (Eq. (116)).

In this way an effective velocity of light can be arbitrarily large,

$$c_{\text{eff}}^{\text{max}} = c \exp \left( \frac{1}{2} \left( \frac{\pi}{2}(2l + 1) + \frac{2}{(2l + 1)\pi} \right) \right)$$

(123)

(i.e. for $\lambda = 2\pi$). However, in order to get such large $c_{\text{eff}}$ we should match Eq. (94) with (122). One gets

$$\frac{\pi}{2}(2l + 1) + \frac{2}{(2l + 1)\pi} = 2\pi \left( \frac{4\sqrt{n} m_{\tilde{A}}}{\alpha_s \sqrt{n + 2} m_{\text{pl}}} \right)^{\frac{3}{4}} \left( \frac{\overline{|\tilde{P}|}}{\overline{R}(\Gamma)} \right)^{\frac{1}{2n + 1}} \left( \frac{n - 2}{n + 2} \frac{|\tilde{P}|}{\overline{P}} \right)^{\frac{1}{2}}.$$  

(124)

In this way we get some interesting relations between parameters in our theory. To have a large $c_{\text{eff}}$ we need to make $|\tilde{P}|$ large and $\overline{R}(\Gamma)$ small. In this case only the first term of the left-hand side of (124) is important and we should play with the ratio

$$\left( \frac{m_{\tilde{A}}}{\alpha_s m_{\text{pl}}} \right)$$

in order to get the relation

$$(2l + 1) = 4 \left( \frac{4\sqrt{n} m_{\tilde{A}}}{\alpha_s \sqrt{n + 2} m_{\text{pl}}} \right)^{\frac{3}{4}} \left( \frac{\overline{|\tilde{P}|}}{\overline{R}(\Gamma)} \right)^{\frac{1}{2n + 1}} \left( \frac{n - 2}{n + 2} \frac{|\tilde{P}|}{\overline{P}} \right)^{\frac{1}{2}}.$$  

(125)
where \( l \) is a big natural number. In this way we can travel in hyperspace (through extra dimension) almost for free. The energy to excite the warp factor is zero (or almost zero). Thus it could be created even from a fluctuation in our Universe.

Let us notice that Eq. (115) is in some sense universal for a warp factor. It means that it is the same for any realization of the Nonsymmetric Jordan-Thiry (Kaluza-Klein) Theory.

However, if we take Eq. (74) and if we demand
\[
E_{\text{wf}} = 0,
\]
we get some roots (they exist) depending on \( n, \beta \) and \( |\gamma| \). Thus the roots, some values of \( A > 0 \) depend on details of the theory, they are not universal as in the case of Eqs (113–115). In this case the Eq. (94) is in some sense a self-consistency condition for a theory and more restrictive. The fluctuations of a tower of scalar fields in the case of \( \sqrt{A} = \frac{1}{4}(2l + 1) \) are as follows
\[
\frac{1}{2}\psi_0 = \frac{e^{\pi(2l+1)} - 1}{2l + 1} \tag{127}
\]
\[
\psi_m^1 = -\frac{m}{\pi} \cdot \frac{e^{\pi(2l+1)} - 1}{(2l + 1)^2 + 4m^2} \tag{128}
\]
\[
\psi_m^2 = \frac{(2l + 1)(e^{\pi(2l+1)} - 1)}{\pi((2l + 1)^2 + 4m^2)} \tag{129}
\]

In the case of very large \( l \) (which is really considered) one gets
\[
\frac{1}{2}\psi_0 = \frac{e^{\pi(2l+1)}}{2l} \tag{127a}
\]
\[
\psi_m^1 = -\frac{m}{2\pi} \cdot \frac{e^{\pi(2l+1)}}{4(l^2 + m^2)} \tag{128a}
\]
\[
\psi_m^2 = \frac{l \cdot e^{\pi(2l+1)}}{2\pi(l^2 + m^2)} \tag{129a}
\]

\( m = 1, 2, \ldots \)

According to our calculations the energy of the fluctuations (excitations) (127a–129a) is zero.

Thus the only difficult problem to get a warp factor is to excite a tower of scalar fields coherently in a shape of (127–129a) or to wait for such a fluctuation. In the first case it is possible to use some techniques from quantum optics applied to the scalar fields \( \psi_m \). In our case we get
\[
\psi_0 = \frac{e^{\pi(2l+1)}}{2l} \tag{130}
\]
\[
\psi_m = \frac{e^{\pi(2l+1)}\sqrt{16l^2 + m^2}}{8\pi(l^2 + m^2)} \sin(m\lambda + \delta_m)e^{im\lambda t} \tag{131}
\]
\[ \tan \delta_m = -\frac{m}{4l} \]  
(132)

and \( m_m \) is given by the formula (63).

For \( \psi_0 \) is constant in time and does not contribute to the total energy of a warp factor (only to the energy of a warp factor equal to 1), we consider only

\[ \psi(\lambda, t) = \frac{e^{i(2l+1)\lambda}}{8\pi} \left[ \sum_{m=1}^{\infty} \sqrt{16l^2 + m^2} \sin(m\lambda + \delta_m) e^{im_m t} + \frac{4\pi}{l} \right]. \]  
(133)

For \( t = 0 \) we get

\[ \psi(\lambda, 0) = e^{i(2l+1)\lambda} \]  
(134)

and the wave packet (133) will disperse. Thus we should keep the wave packet to not decohere in such a way that

\[ \psi(\lambda, t) = \frac{e^{i(2l+1)\lambda}}{8\pi} \left[ \sum_{m=1}^{\infty} \sqrt{16l^2 + m^2} \sin(m\lambda + \delta_m) + \frac{4\pi}{l} \right]. \]  
(135)

It means \( \psi(\lambda, t) \) should be a pure zero mode for all the time \( t \geq 0 \).

Let us come back to the Eq. (125) (earlier to Eq. (94)) taking under consideration that in a real model \( \hat{A} = \alpha_s \sqrt{2} \). One can rewrite Eq. (125) in the following way:

\[ \left( \frac{2l+1}{4\alpha_s} \right)^{4/n} \left( \frac{n+2}{(n-2)|\hat{P}|} \right)^{2/n} \left( \frac{\alpha_s \sqrt{n + 2} m_{pl}}{4\sqrt{n} m_{\hat{A}}} \right)^2 = \frac{|\hat{P}|}{R(\Gamma)} \]  
(136)

or

\[ \hat{R}(\Gamma) = \left( \frac{4\alpha_s}{2l+1} \right)^{4/n} \left( \frac{(n-2)|\hat{P}|}{n+2} \right)^{2/n} \left( \frac{4\sqrt{n} m_{\hat{A}}}{\alpha_s \sqrt{n + 2} m_{pl}} \right)^2 |\hat{P}| = 2F(\zeta). \]  
(137)

We want to find some conditions for \( \mu \) and \( \zeta \) (or \( \xi \) and \( \zeta \)) in order to satisfy Eq. (94) for large \( l \) in a special model with \( H = G2 \), \( \dim G2 = n = 14 \) for \( M = S^2 \) and \( \hat{R}(\Gamma) \) calculated for \( G = SO(3) \). In this case one finds

\[ F(\zeta) = 3^{2/7} \cdot 7 \cdot \left( \frac{\alpha_s}{4} \right)^{4/7} \left( \frac{m_{\hat{A}}}{\alpha_s m_{pl}} \right)^2 |\hat{P}|^{15/7} \]  
(138)

where \( \hat{P} \) is given by the formula

\[ \hat{P}(\zeta) = \left\{ \frac{16\zeta^3(\zeta^2+1)}{3(2\zeta^2+1)(1+\zeta^2)^{3/2}} \left( \zeta^2 E \left( \frac{|\zeta|}{\sqrt{\zeta^2+1}} \right) - (2\zeta^2+1)K \left( \frac{|\zeta|}{\sqrt{\zeta^2+1}} \right) \right) + 8 \ln(|\zeta|) + 4(1+9\zeta^2 - 8\zeta^4)|\zeta|^3 \right\} / (\ln(|\zeta| + \sqrt{\zeta^2+1} + 2\zeta^2 + 1)). \]
We calculate \( \widetilde{R}(\widetilde{\Gamma}) \) for a group \( SO(3) \) in the first point of Ref. [1] and we get

\[
\widetilde{R}(\widetilde{\Gamma}) = \frac{2(2\mu^3 + 7\mu^2 + 5\mu + 20)}{(\mu^2 + 4)^2}.
\] (139)

The polynomial

\[
W(\mu) = 2\mu^3 + 7\mu^2 + 5\mu + 20
\] (140)

possesses only one real root

\[
\mu_0 = \frac{-3}{6} \left( \sqrt[3]{1108 + 3\sqrt{135645}} \right) - \frac{7}{6} - \frac{19}{6 \cdot \sqrt[3]{1108 + 3\sqrt{135645}}} = -3.581552661 \ldots \quad (141)
\]

It is interesting to notice that \( W(-3.581552661) = 2.5 \cdot 10^{-9} \) and for 70-digit approximation of \( \mu_0, \bar{\mu} \) equal to 

\[-3.581552661076733712599740215045436907383569800816123632201827285932446,\]

we have

\[W(\bar{\mu}) = 0.1 \cdot 10^{-67}.\]

The interesting point in \( \bar{\mu} \) is that any truncation of \( \bar{\mu} \) (it means, removing some digits from the end of this number) results in growing \( W \), i.e.,

\[0 < W(\bar{\mu}_n) < W(\bar{\mu}_{n-1}),\]

where \( \bar{\mu}_n \) means a truncation of \( \bar{\mu} \) with only \( n \) digits, \( n \leq 70, \bar{\mu}_{70} = \bar{\mu} \). This can help us in some approximation for \( \widetilde{R}(\widetilde{\Gamma}) > 0 \), close to zero.

Substituting (139) into Eq. (137) we arrive to the following equation:

\[-F(\zeta)\mu^4 + 2\mu^3 + \mu^2(7 - 4F(\zeta)) + 5\mu + 4(-2F(\zeta) + 5) = 0. \quad (142)\]

Considering \( F(\zeta) \) small we can write a root of this polynomial as

\[\mu = \mu_0 + \varepsilon \quad (143)\]

where \( \mu_0 \) is the root of the polynomial (140) and \( \varepsilon \) is a small correction. In this way one gets the solution

\[
\mu = -3.581552661 \ldots + 47 \left( \frac{\alpha_s}{\ell} \right)^{4/7} \left( \frac{m_\Delta}{m_{\text{pl}}} \right)^2 q(\zeta) \quad (144)
\]

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where \( q(\zeta) \) is given by the formula

\[
q(\zeta) = \frac{16|\zeta|^3(\zeta^2 + 1)}{3(2\zeta^2 + 1)(1 + \zeta^2)^{5/2}} \left( \zeta^2 E \left( \frac{|\zeta|}{\sqrt{\zeta^2 + 1}} \right) - (2\zeta^2 + 1)K \left( \frac{|\zeta|}{\sqrt{\zeta^2 + 1}} \right) \right)
+ 8 \ln \left( |\zeta|\sqrt{\zeta^2 + 1} \right) + \frac{4(1 + 9\zeta^2 - 8\zeta^4)|\zeta|^3}{3(1 + \zeta^2)^{3/2}}
\times \left[ \ln \left( |\zeta| + \sqrt{\zeta^2 + 1} \right) + 2\zeta^2 + 1 \right]^{1/3},
\]

\(|\zeta| > |\zeta_0| = 1.36\) (see Ref. [1]), and \( K \) and \( E \) are given by the formulae

\[
K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta, \quad 0 \leq k^2 \leq 1
\]

In this way taking sufficiently large \( l \) we can choose \( \mu \) such that \( \tilde{R}(\tilde{\Gamma}) > 0 \) and quite arbitrary \( \zeta \) \((\tilde{P} < 0)\) to satisfy a zero energy condition for an excitation of a warp factor.

Taking \( m_{\tilde{\Delta}} = m_{\text{EW}} \simeq 80 \text{ GeV}, m_{\text{pl}} \simeq 2.4 \times 10^{18} \text{ GeV}, \alpha_s = \sqrt{\alpha_{\text{em}}} = \frac{1}{\sqrt{137}} \), one gets from Eq. (144)

\[
\mu = -3.581552661\ldots + 4.1 \times 10^{-35} q(\zeta) \frac{1}{147}
\]

which justifies our approximation for \( \varepsilon \) and simultaneously gives an account for a smallness of a cosmological constant \( \tilde{R}(\tilde{\Gamma}(\mu)) \simeq 0 \). Thus it seems that in this simple toy model we can achieve a large \( c_{\text{eff}} \) without considering large cosmological constant.

Let us give some numerical estimation for a time needed to travel a distance of 200 Mps

\[
t_{\text{travel}} = \frac{L}{c_{\text{max}}} = \frac{L}{c} \cdot e^{-(2l+1)\pi/4}.
\]

One gets for \( L = 200 \) Mps, \( t_{\text{travel}} = 2 \times 10^{13} \text{ s} \cdot e^{-(11/2)} \) and for \( l = 100, t_{\text{travel}} \simeq 12 \text{ ns} \).

Let us notice that for \( l = 100, \mu \) from Eq. (146) is equal to

\[
\mu = -3.581552661\ldots + 5.94 \times 10^{-44} q(\zeta)\frac{1}{147}
\]

\( q(\zeta) \) cannot be too large (it is a part of cosmological term \( q(\zeta) = |\tilde{P}|^{15/7} \)). If \( \tilde{P} \) is of order 0.1, \( q \) is of order \( 10^{-16} \); if \( \tilde{P} \) is of order 1.1, \( q \) is of order 0.6. Thus \( \mu \) is very close to \( \mu_0 \) for some reasonable values of \( |\tilde{P}| \). It is interesting to ask how to develop this model to more dimensional case. It means, to the manifold \( M \) which is not a circle. It is easy to see that in this case we should consider a warp factor which depends on more
coordinates taking under consideration some ansatzes. For example we can consider a warp factor in a shape
\[\exp\left(\sum_{i=1}^{m} \sqrt{A_i} f_i(y)\right)\] (148)
where \(f_i(y) = x_i, \, i = 1, 2, \ldots, m = \frac{1}{2}n_1(n_1 + 1)\) are parametric equations of the manifold \(M\) in \(m\)-dimensional euclidean space and
\[ds^2_M = \left(\sum_{k=1}^{m} \left(\frac{\partial f^k}{\partial y^i}\right) \left(\frac{\partial f^k}{\partial y^j}\right)\right)dy^i \otimes dy^j\] (149)
is a line element of the manifold \(M\). In the case of the manifold \(S^2\) it could be
\[
\begin{align*}
  f_1 &= \cos \theta \sin \lambda \\
  f_2 &= \cos \theta \cos \lambda \\
  f_3 &= \sin \theta \\
  m &= 3
\end{align*}
\] (150)
and a warp factor takes the form
\[
\exp \left(\sqrt{A_1} \cos \theta \sin \lambda + \sqrt{A_2} \cos \theta \cos \lambda + \sqrt{A_3} \sin \theta\right),
\] (151)
\(\theta \in (0, \pi), \, \lambda \in (0, 2\pi)\).

Afterwards we should develop a warp factor in a series of spherical harmonics on the manifold \(M\) and calculate an energy of a tower of scalar particles connected with this development. We should of course regularize the series (divergent) using \(\zeta\)-functions techniques. In this case we should use \(\zeta\)-functions connected with the Beltrami-Laplace operator on the manifold \(M\). Afterwards we should find a zero energy condition for special types of constants \(A_i, \, i = 1, 2, \ldots, m\). We can of course calculate the energy of an excitation of the warp factor using some formulae from classical field theory. In the case of \(M = S^2\) we have to do with an ordinary Laplace operator on \(S^2\) and with ordinary spherical harmonics with spectrum given by
\[
\mathfrak{m}(\ell, m) = \frac{1}{r} \sqrt{\left(\frac{1 + \frac{n^2 \zeta^2}{M( \zeta^2 + 1)}\right) \ell(\ell + 1)}}, \quad \ell = 1, 2, 3.
\]
The Fourier analysis of (151) can be proceeded on \(S^2\).

Moreover a toy model with \(S^1\) can be extended to the cosmological background in such a way that
\[
ds_5^2 = e^{\sqrt{A} \lambda} dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) - r^2d\lambda^2.
\] (152)
It seems that from practical (calculational) point of view it is easier to consider a metric of the form
\[ ds_5^2 = dt^2 - e^{-\sqrt{A} R(t)}(dx^2 + dy^2 + dz^2) - r^2 d\lambda^2 \] (153)
where \( R(t) \) is a scale factor of the Universe. For further investigations we should also consider more general metrics, i.e.
\[ ds_6^2 = dt^2 - \exp \left( -\sqrt{A} (\cos \theta (\sin \lambda + \cos \lambda) + \sin \theta) \right) \times R^2(t)(dx^2 + dy^2 + dz^2) - r^2 (d\theta^2 + \sin^2 \theta d\lambda^2) \] (154)
or even
\[ ds_{4+n_1}^2 = dt^2 - \exp \left( -\sum_{i=1}^{m} \sqrt{A_1 f_i(y)} \right) \cdot R^2(t) \frac{dx^2 + dy^2 + dz^2}{1 - k(x^2 + y^2 + z^2)^2} - r^2 \left( \sum_{k=1}^{m} \left( \frac{\partial f_k}{\partial y^i} \right) \left( \frac{\partial f_k}{\partial y^j} \right) \right) \cdot dy^i \otimes dy^j \] (155)
where
\[ m = \frac{1}{2} n_1 (n_1 + 1). \] (156)

Let us notice that our toy model with a travel in a hyperspace (this is really a travel along dimensions of a vacuum state manifold) can go to some acausal properties. This is evident if we consider two signals: one sent in an ordinary way with a speed of light and a second via the fifth dimension to the same point. The second signal will appear earlier than the first one at the point. Thus it will affect this point earlier. Effectively it means a superluminal propagation of signals in a Minkowski space (or in Friedmann-Robertson-Walker Universe). In this way we can construct a time-machine using this solution (even if it does not introduce tachyons in a 4-dimensional space-time), i.e. a space-time where there are closed time-like (or null) curves.

Let us notice that given a superluminal signal we can always find a reference frame in which it is travelling backwards in time. Suppose we send a signal from \( A \) to \( B \) (at time \( t = 0 \)) at an effective speed \( u_{\text{eff}} > 1 \) in frame \( S_1 \) (it means we send a signal through a path described here earlier and an effective speed \( u_{\text{eff}} \) with coordinates \((t, x)\) (we consider only one space dimension). In a frame \( S_2 \) moving with respect to \( S_1 \) with velocity \( v > \frac{1}{u_{\text{eff}}} \), the signal travels backwards in \( t' \) time. This follows from the Lorentz transformation, i.e.
\[ t' = \frac{t_B (1 - vu_{\text{eff}})}{\sqrt{1 - v^2}} \] (157)
\[ x' = \frac{t_B (1 - v)}{\sqrt{1 - v^2}}. \] (158)
We require both \( x'_B > 0, \ t'_B < 0 \), that is \( \frac{1}{u_{\text{eff}}} < v < u_{\text{eff}} \) which is possible only if \( u_{\text{eff}} > 1 \).

However this by itself does not mean a violation of a causality. For that we require that the signal can be returned from \( B \) to a point in the past of light cone of \( A \). Moreover, if we return the signal from \( B \) to \( C \) with the same effective speed (it means we send a signal through the fifth dimension with the same effective speed using the warp factor) in frame \( S_1 \), then it arrives at \( C \) in the future cone of \( B \). The situation is physically equivalent in the Lorentz boosted frame \( S_2 \) — the return signal travels forward in time \( t' \) and arrives at \( C \) in the future cone of \( A \). This is a frame-independent statement. If a backwards-in-time signal \( AC \) is possible in frame \( S_1 \), then a return signal sent with the same speed \( u_{\text{eff}} \) will arrive at a point \( D \) in the past light cone of \( A \) creating a closed time-like curve \( ACDA \). In a Minkowski space (a space for \( \lambda = 0 \), or in general for \( \lambda = \lambda_0 = \text{const.} \)), local Lorentz invariance results in that if a superluminal signal such as \( AB \) is possible, then so is one of \( AC \) for it is just given by an appropriate Lorentz boost. The existence of a global inertial frames in a Minkowski space guarantees the existence of the return signal \( CD \).

Thus we get unacceptable closed time loops. Moreover the theory for the five-dimensional world is still causal, as in five-dimensional General Relativity. Let us notice that in a cosmological background described by a metric (153) the situation will be very similar. We have here also global inertial frame for \( \lambda = 0 \), and similar possibility to send a superluminal signal through fifth dimension. However, in this case we should change a shape of Lorentz transformation to include a scale factor \( R(t) \) which slightly complicates considerations. Moreover in order to create a warp factor we need a zero energy condition. If this condition is not satisfied we need an infinite amount of an energy to create it. The zero energy condition imposes some restriction on parameters which are involved in our unified theory. Maybe it is impossible to satisfy these conditions in our world. In this way it would be an example of Hawkins' chronology protection conjecture: “the laws of physics do not allow the appearance of closed timelike curves”. However, if a zero energy condition is satisfied in a realistic Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory (it means in such a theory which is consistent with a phenomenology of a contemporary elementary particle physics) then we meet possibility of effective closed time-like loops (even the higher dimensions are involved). In this case we should look for some weaker conjectures (especially in an expanding Universe, not just in a Minkowski space). Such weaker condition protects us from some different causal paradoxes as the following paradox: a highly energetic signal can destroy a laboratory before it was created. In some sense it means some kind of selfconsistency conditions.

Thus what is a prescription to construct a time-machine in our theory? It is enough to have a 5-dimensional (or \((n_1 + 4)\)-dimensional) space-time with a warp factor described here earlier. Afterwards we need two frames \( S_1 \) and \( S_2 \) in relative motion with relative velocity \( v > \frac{1}{u_{\text{eff}}} \) where \( u_{\text{eff}} \) is an effective superluminal velocity \( u_{\text{eff}} > 1 \). This could be arranged e.g. as two spaceships with relative motion (\( A \) and \( C \)). Sending a signal with effective velocity \( u_{\text{eff}} \) from \( A \) to \( C \) we get the signal in a past
of \( C \). Afterwards we send a signal (through the fifth dimension) from \( C \) to \( A \). In this way we arrive in a past of \( A \).

If we want to travel into a future we should reverse a procedure with

\[ -1 < v < 0. \]

In both cases we use an effective time-like loop, constructed due to a warp factor in a five-dimensional space-time. Using this loop several times we can go to the future or to the past as far as we want. An interesting point which can be raised consists in an effectiveness of such a travel. It means how much time we need to get e.g. 1 s to past or 1 s to future. It depends on a relative velocity \( v \) and \( u_{\text{eff}} \). The effectiveness can be easily calculated.

\[
\eta = \frac{u_{\text{eff}} v}{\sqrt{1 - v^2}}. \tag{159}
\]

If we use our estimation for \( c_{\text{eff}} \) (see Eq. (123)), we get

\[
\eta = \frac{v}{\sqrt{1 - v^2}} e^{(2l+1)\pi/4} \tag{160}
\]

for \( l \) large. It is easy to see that \( \eta \to \infty \) if \( v \to 1 \) and if \( l \to \infty \).

Let us take \( v = \frac{1}{10} \) and \( l = 100 \). In this case one gets

\[
\eta \approx 10^{67}. \tag{161}
\]

What does this number mean? It simply means that in order to go 1 s into the past or into the future we loose \( 10^{-67} \) s of our life.

However, to be honest, we should use as \( u_{\text{eff}} \) a velocity different from \( c_{\text{eff}} \). In this case a time needed to travel through the fifth dimension really matters. Let a speed along the fifth dimension be equal to the velocity of light (or close to it). In this case one gets

\[
u_{\text{eff}} = \frac{c_{\text{eff}}}{1 + 2(l/2)c_{\text{eff}}} \tag{162}\]

where \( r_0 \simeq 10^{-16} \) cm, \( L \) is the distance between \( A \) and \( B \), e.g. 1000 km. In this case \( u_{\text{eff}} \) is smaller,

\[
u_{\text{eff}} = 10^{26} \tag{163}\]

and

\[
\eta \simeq 10^{25}. \tag{164}\]

In this case we can estimate how much time we loose to go 2000 years back or forward. It is \( 6.3 \cdot 10^{-15} \) s. However, if we want to see dinos we should go back 100 mln yr and we loose \( 10^{-10} \) s. Of course in the formula for \( u_{\text{eff}} \) we can use smaller \( L \) and a velocity in the place \( c_{\text{eff}} \) can be smaller too. We can travel more comfortably changing smoothly a velocity. Any time a time to loose to travel back or forward in time for our time machine is very small. In some sense it is very effective (if it is possible to construct).
Let us notice that the warp factor (or \(\tilde{A}\)) is fixed by the constants of the theory (especially by a cosmological constant \(\lambda_{c0}\)). In this way only in some special cases we get a zero energy condition. However, in the case of the warp factor existence we have to do with a tower of scalar fields (not only with a quintessence). The quintessence field is a zero mode of this tower. Thus in the case of an existing of a tower of scalar fields we will have to do with a corrected cosmological constant, which is really an averaged cosmological term over \(S^1\) (in general over \(M\))

\[
\lambda_{c0}^{\text{corrected}} = \frac{1}{2\pi} \int_0^{2\pi} \left( |\gamma| - \beta e^{2\Psi(\lambda)} \right) e^{n\Psi(\lambda)} d\lambda
\]

where \(\Psi(\lambda)\) is given by Eq. (59). In this way according to our considerations concerning an energy of excitation

\[
\lambda_{c0}^{\text{corrected}} = E_{\text{wf}}.
\]

However, in the formula for \(E_{\text{wf}}\) we dropped an energy for a zero mode for a tower of scalar fields (i.e., for a quintessence). Thus

\[
\lambda_{c0}^{\text{tot}} = \lambda_{c0}(\Psi_0) + \lambda_{c0}^{\text{corrected}}.
\]

If the zero energy condition for a warp factor is satisfied, we get

\[
\lambda_{c0}^{\text{tot}} = \lambda_{c0}
\]

and we do not need any tuning of parameters in the theory. However, now we should write down full 5-dimensional equations in our theory \((n_1 + 4\) in general) to look for more general solutions mentioned here earlier. This will be a subject of a future work.

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