Gamma function method for the nonlinear cubic-quintic Duffing oscillators

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Abstract
In this article, the gamma function method, for the first time ever, is used to solve the nonlinear cubic-quintic Duffing oscillators. The nonlinear cubic-quintic Duffing oscillators with and without the damped and quadratic terms are considered respectively. By the gamma function method, it only needs one-step to get the approximate solution. The comparisons with the existing solutions reveal that the proposed method is simple but effective in solving the small amplitude oscillation.

Keywords
Gamma function, small amplitude oscillation, Duffing oscillators, damped and quadratic terms

Introduction
Nonlinear oscillation can be seen everywhere in our daily life. The research on its vibration characteristics has always been a research hot spot. However, the explicit analytical solutions of nonlinear oscillator equations are few, and either numerical solutions or approximate analytical techniques are frequently used. Many scholars have made outstanding contributions and many different methods are obtained such as homotopy perturbation method,1–3 variational approach,4–9 variational iteration method,10–16 He’s frequency formulation,17–19 Hamiltonian approach,20 Taylor series method,21 and so on.22–24 The well-known Duffing oscillator equation was named after a German electrical engineer Georg Duffing who first proposed the equation in 1918,25 and then, it is developed into different forms to describe many physical, mechanical engineering, circuits and biological processes in various areas of science.26–28 Thus, the study of the Duffing oscillator equation is important. In this article, we consider the nonlinear small amplitude cubic-quintic Duffing oscillators, which is

\[ v'' + av + \beta v^3 + \gamma v^5 = 0 \]  

with the initial condition as

\[ v(0) = \Lambda < 1, \quad v'(0) = 0 \]  

Inspired by the recent study on the special functions and nonlinear oscillators,29,30 we will use a new method so called the gamma function method to seek its frequency–amplitude formulation of equation (1.1).

The overall structure of this article is arranged as follows. The gamma function and its properties are presented in the section The Gamma Function. In the section The Gamma Function Method, the gamma function method is proposed and used to solve the nonlinear cubic-quintic Duffing oscillators without the damped and the quadratic terms. In the section Considering the Damped and Quadratic Terms, the nonlinear cubic-quintic Duffing oscillators considering the damped and the quadratic terms are studied. And the conclusion is presented in the Conclusion section.
The Gamma function

The well-known gamma function is defined as

$$\Gamma(x) = \int_0^\infty e^{-t}t^{-x}dt$$  \hspace{1cm} (2.1)$$

And there are the following properties\textsuperscript{30}

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$  \hspace{1cm} (2.2)$$

$$\Gamma(1 + x) = x\Gamma(x)$$  \hspace{1cm} (2.3)$$

$$\int_0^\frac{\pi}{2} \cos^m\theta \sin^n\theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1 + \frac{m+n}{2}\right)} \hspace{1cm} m, n \geq 0$$  \hspace{1cm} (2.4)$$

The Gamma function method

For obtaining the solution of equation (1.1), we first linearize equation (1.1) as\textsuperscript{30}

$$v'' + \Omega^2 v = 0, \quad v(0) = \Lambda \ll 1, \quad v'(0) = 0$$  \hspace{1cm} (3.1)$$

And its solution is

$$v(t) = \Lambda \cos(\Omega t)$$  \hspace{1cm} (3.2)$$

The Galerkin technology is a general approximation method for seeking the solution via introducing the trial functions and weight function.\textsuperscript{31} Here, we select the trial function as $v(t) = \Lambda \cos(\Omega t)$ and the weight function as $\cos(\Omega t)$. By the condition of vanishing residual integrated over a quarter period implies that\textsuperscript{31}

$$\int_0^{\frac{T}{4}} \{av + \beta v^3 + \gamma v^5 - \Omega^2 v\} \cos(\Omega t)dt = 0$$  \hspace{1cm} (3.3)$$

Substituting equation (3.2) into above equation, we have

$$\int_0^{\frac{T}{4}} \{ (a - \Omega^2)\Lambda \cos(\Omega t) + \beta \Lambda^3 \cos^3(\Omega t) + \gamma \Lambda^5 \cos^5(\Omega t) \} \cos(\Omega t)dt = 0$$  \hspace{1cm} (3.4)$$

According to the properties of the gamma function in equation (2.4), equation (3.4) can result in

$$\frac{1}{2} (a - \Omega^2)\Lambda \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} + \frac{1}{2} \beta \Lambda^3 \frac{\Gamma\left(\frac{4}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} + \frac{1}{2} \gamma \Lambda^5 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = 0$$  \hspace{1cm} (3.5)$$

That is

$$\left(\frac{3}{4} a - \Omega^2\right) + \frac{3}{8} \beta \Lambda^2 + \frac{5}{8} \gamma \Lambda^4 = 0$$  \hspace{1cm} (3.6)$$

So the frequency–amplitude formulation of equation (1.1) can be obtained as

$$\Omega^2 = a + \frac{3}{4} \beta \Lambda^2 + \frac{5}{8} \gamma \Lambda^4, \quad \Lambda \ll 1$$  \hspace{1cm} (3.7)$$

With this, the solution of equation (1.1) can be written as

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By using He’s frequency formulation, we can get the solution of equation (1.1) as
\[ \Omega^2 = \alpha + \frac{3}{4}\beta \Lambda^2 + \frac{5}{16}\gamma \Lambda^4 \tag{3.9} \]
which has a good agreement with the solution given in equation (3.7) for \( \Lambda \ll 1 \).

By equation (3.9), we obtain the solution of equation (1.1) as
\[ v(t) = \Lambda \cos \left( \sqrt{\alpha + \frac{3}{4}\beta \Lambda^2 + \frac{5}{16}\gamma \Lambda^4} \, t \right) \tag{3.10} \]

For \( \alpha = 1, \beta = 1, \) and \( \gamma = 1 \), the comparison between the gamma function method in equation (3.8) and Ref. 32 with different \( \Lambda \) is plotted in Figure 1. It shows that the different methods agree well when \( \Lambda \ll 1 \). The comparison of the approximate periods with the exact one is given in Table 1.

**Considering the damped and quadratic terms**

In this section, we use the gamma function method to solve the cubic-quintic Duffing oscillators with the damped and the quadratic terms as
\[ v'' + av + \varepsilon_1 v' + \varepsilon_2 v^2 + \beta v^3 + \gamma v^5 = 0 \tag{4.1} \]
with the initial condition as
\[ v(0) = \Lambda \ll 1, \quad v'(0) = 0 \tag{4.2} \]

Here, we assume its solution is
\[ v(t) = \Lambda \cos(\Omega t) \tag{4.3} \]

Similarly, we aim to seek the frequency as
\[ \int_0^\pi \{ av + \varepsilon_1 v' + \varepsilon_2 v^2 + \beta v^3 + \gamma v^5 - \Omega^2 v \} \cos(\Omega t) \, dt = 0 \tag{4.4} \]
which is
\[ \int_0^\pi \{ (a - \Omega^2) \Lambda \cos(\Omega t) - \varepsilon_1 \Omega \Lambda \sin(\Omega t) + \varepsilon_2 \Lambda^2 \cos^2(\Omega t) + \beta \Lambda^3 \cos^3(\Omega t) + \gamma \Lambda^5 \cos^5(\Omega t) \} \cos(\Omega t) \, dt = 0 \tag{4.5} \]

According to the properties of the gamma function, equation (4.5) can result in
\[ \frac{1}{2}(a - \Omega^2) \Lambda \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \frac{1}{2} \varepsilon_1 \Omega \Lambda \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} + \frac{1}{2} \varepsilon_2 \Lambda^2 \frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} \]
\[ + \frac{1}{2} \beta \Lambda^3 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} + \frac{1}{2} \gamma \Lambda^5 \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = 0 \tag{4.6} \]
That is
\[ (a - \Omega^2) \pi - 2\varepsilon_1 \Omega + \frac{8}{3} \varepsilon_2 \Lambda + \frac{3}{4} \beta \Lambda^2 \pi + \frac{5}{8} \gamma \Lambda^4 \pi = 0 \tag{4.7} \]
which leads to
\[ \Omega = \sqrt{\frac{\varepsilon_1^2}{\pi^2} + \frac{64\alpha + 2424 + 18\pi^2 + 15\pi^2 + 15\pi^2 + 15\pi^2}{24\pi} - \frac{\varepsilon_1}{\pi}} \]  (4.8)

With this, the solution of equation (4.1) can be obtained as

Table 1. Comparison of the approximate frequency with different \( \Lambda \).

| \( \Lambda \) | 0.1 | 0.2 | 0.4 | 0.6 | 0.8 | 1   | 1.2   |
|--------------|-----|-----|-----|-----|-----|-----|-------|
| Equation (3.7) | 1.003774 | 1.015382 | 1.065833 | 1.162325 | 1.317574 | 1.541104 | 1.837389 |
| Ref. 27      | 1.003759 | 1.015135 | 1.062073 | 1.147711 | 1.268069 | 1.436141 | 1.651666 |
| Relative error % | 0.0015 | 0.0243 | 0.3540 | 1.5334 | 3.9040 | 7.3087 | 11.2445 |
\[ v(t) = \Lambda \cos \left( \sqrt{\frac{\epsilon_1^2}{2} + \frac{64\Lambda\epsilon_2 + 24\alpha\pi + 18\beta\Lambda^2\pi + 15\gamma\Lambda^4\pi}{24\pi}} \frac{\epsilon_1}{\pi} \right) t \]  \hspace{1cm} (4.9)

For \( \epsilon_1 = \epsilon_2 = 0 \), the solution becomes the solution of equation (1.1).

The solution of equation (4.1) can be obtained by the ancient Chinese algorithm (ACG), which is

\[ \Omega = \sqrt{-\frac{32\epsilon_1 + 64\Lambda\epsilon_2 + 24\alpha\pi + 18\beta\Lambda^2\pi + 15\gamma\Lambda^4\pi}{8(2\epsilon_1 + 3\pi)}} \]  \hspace{1cm} (4.10)

**Figure 2.** The comparison between the gamma function method and the ACG with different \( \Lambda \). (a) \( \Lambda = 0.2 \). (b) \( \Lambda = 0.4 \). (c) \( \Lambda = 0.6 \). (d) \( \Lambda = 0.8 \).

**Table 2.** Comparison of the approximate frequency with different \( \Lambda \).

| \( \Lambda \) | 0.1  | 0.2  | 0.4  | 0.6  | 0.8  | 1   | 1.2  |
|-------------|------|------|------|------|------|-----|------|
| Equation (4.8) | 0.774286 | 0.82278 | 0.937418 | 1.08227 | 1.268000 | 1.505189 | 1.80205 |
| Equation (4.10) | 0.742353 | 0.80028 | 0.931188 | 1.08836 | 1.281471 | 1.519656 | 1.80974 |
| Relative error % | 4.301 | 2.811 | 0.6690 | 0.5595 | 1.5012 | 0.95199 | 0.42492 |
For $\alpha = 1, \beta = 1, \gamma = 1$, and $\varepsilon_1 = \varepsilon_2 = 1$, we compare the two different methods in Figure 2. Obviously, the two methods match well with each other. The comparison of the approximate frequency is shown in Table 2. These obtained results all strongly prove the correctness and effectiveness of our proposed method.

Conclusion

In this work, the gamma function method is used to solve the nonlinear cubic-quintic Duffing oscillators. It only takes one-step to obtain the amplitude–frequency relationship. Compared with the existing solution, it reveals that the gamma function method is remarkably accurate for the small amplitude oscillation. The obtained results in this article are expected to open up new horizons for the study of the small amplitude oscillation theory.

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