SIGNER ENUMERATION OF RIBBON TABLEAUX

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ABSTRACT. We give an extension of the classical Schensted correspondence to the case of ribbon tableaux, where ribbons are allowed to be of different sizes. This is done by extending Fomin’s growth diagram approach of the classical correspondence between permutations and pairs of standard tableaux of the same shape, in particular by allowing signs in the enumeration. As an application we give a combinatorial proof for the column sums of the character table of the symmetric group.

1. INTRODUCTION

The Schensted correspondence [22] is a bijection between permutations and pairs of standard Young tableaux of the same shape. It has been extended in numerous ways, the most famous being certainly the Robinson-Schensted-Knuth correspondence between matrices of integers and pairs of semi-standard tableaux of the same shape. Other extensions exist, for instance oscillating tableaux [29, 2, 3, 4], skew tableaux [20], shifted Young tableaux [21], and k-ribbon tableaux [23, 28].

Sergey Fomin developed a general theory of such correspondences, cf. [5, 6, 7, 8, 9]. It unifies the correspondences listed above by interpreting these tableaux as paths in so-called graded graphs. For instance, a Young tableau in this context is viewed as a particular kind of path in the graph whose vertices are integer partitions, and where \((\lambda, \mu)\) is an edge if \(\mu\) is a partition obtained after adding 1 to a part of \(\lambda\). This graph is usually called the Young graph (or Young lattice). Other combinatorial objects can be then represented by considering other paths, for instance by modifying the extreme points of the path, or the edges that one can use. This is a way of looking at oscillating or skew tableaux inside the Young graph for instance. Then the local properties of the graph will give rise to various bijective correspondences, all consequences of one elementary bijection.

Furthermore, Fomin gives in parallel a linear algebraic approach to his results, which is directly inspired by the work of Stanley [25]. As a matter of fact, most of Fomin’s results have both a bijective and an algebraic proof.

In [31], Dennis White describes another extension of the Schensted correspondence for ribbon tableaux where ribbons are allowed to have different sizes; his goal was in fact to give a combinatorial proof of the second orthogonality relation for characters of the symmetric group. The algorithm describing his correspondence is a complicated insertion mechanism, along the lines of the original Schensted correspondence; this forces him moreover to put some restrictions on the ribbon tableaux he considers.

Key words and phrases. tableau, ribbon, involution principle, growth diagrams, graded graphs, Schensted correspondence.
We will show here how the approaches of Fomin—both bijective and algebraic—can be adapted to apply to the correspondence of White. There will be two benefits: first, this will extend the original work of White, by getting rid of his restrictions in particular. Second, in the process we will have to extend Fomin’s setting to graphs which are more general than the ones considered in [6, 7].

Note that we will deal here with signed enumerations, meaning that we will count objects with weights plus or minus one. It will appear that, for the bijective approach, we will have to appeal to the famous involution principle of Garsia and Milne [12].

The paper is organized as follows: in Section 2 we give the main definitions about ribbons and ribbon tableaux, as well as some elementary operations on these objects. In Section 3 we define different notions about signed enumeration, and we recall the famous Involution Principle of Garsia and Milne. Section 4 introduces hook permutations and involutions, which are the objects in correspondence with ribbon tableaux in our main results. Section 5 states these results, which are Theorems 10 and 12: there exists a signed bijection between hook permutations and pairs of ribbon tableaux of the same shape. The description of the bijections is given in the two following sections: in Section 6 we recall some local rules that depend on the operations introduced in Section 2 and Section 7 shows how to define a global correspondence from these local rules. The technical parts of the proofs are given in Appendix B. In Section 8 an algebraic version of Theorem 10 is given.

Section 9 contains an application of Theorem 12 to the column sums of the character table of the symmetric group. Finally, Section 10 explains how the methods developed can be used for other enumerations, and details in what ways this bijective and algebraic setting is a generalization of Sergey’s Fomin graded graphs in duality.

2. Ribbons

2.1. Definitions. A partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \) is a nonincreasing finite sequence of positive integers; these integers are the parts of the partition, the size of the partitions being their sum \( |\lambda| := \sum \lambda_i \). A composition \( c \) is as finite sequence of positive integers; we can associate to \( c \) a partition \( \tilde{\lambda} \) by rearranging the integers in nonincreasing order. A partition \( \lambda \) of size \( n \) can be described with the exponential notation \( \lambda = [1^{j_1}, 2^{j_2} \cdots n^{j_n}] \), where \( j_i \) is the number of parts of size \( i \). If \( j_i = 0 \), then \( i^{j_i} \) is not written and if \( j_i = 1 \), \( j_i \) is not written.

We will identify a partition \( (\lambda_1, \ldots, \lambda_m) \) with its Ferrers diagram, which is the left justified set of cells (i.e. unit squares of \( \mathbb{Z}^2 \)) such that the \( i \)th row from the top contains \( \lambda_i \) cells; the diagram on the left of Figure 1 represents the partition \( (8, 6, 5, 2, 2, 1, 1) \).

Let \( Y \) be the set of integer partitions, and \( Y_n \) the subset of partitions of size \( n \). Two partitions \( \lambda \preceq \mu \) (in the sense of inclusion of Ferrers diagrams) define a skew shape \( \mu/\lambda \). We will identify here a skew shape with the set of cells \( \mu \setminus \lambda \), whenever \( \mu \) or \( \lambda \) is clear from the context; though, in general, two distinct skew shapes may define the same set, this will not create any ambiguity here. The size of \( \mu/\lambda \) is its number of cells \( |\mu| - |\lambda| \) and will be noted \( |\mu/\lambda| \). The skew shape \( (9, 8, 7, 4, 4, 1, 1)/(8, 6, 5, 2, 2, 1, 1) \) represented on the right of Figure 1 has size 9.
Let us say that a subset $S$ of cells of $\mathbb{Z}^2$ is connected if, for every two cells $c, c'$ in $S$, there exists a sequence of cells $c = c_0, c_1, \ldots, c_t = c'$ in $S$ such that $c_i, c_{i+1}$ have a common side for all $i$. We can then define the notion of ribbons:

**Definition 1.** A ribbon is a connected skew shape that does not contain a $2$ by $2$ square of cells.

Let $r = \mu/\lambda$ be a non empty ribbon. Such a ribbon is said to be $\mu$-addable and $\lambda$-removable. Its height $h(r)$ is defined as the number of lines it occupies minus one, and its sign is then $\varepsilon(r) := (-1)^{h(r)}$. By convention we will set $\varepsilon(\lambda/\lambda) := 1$. The bottom left cell of $r$ is its tail, and the top right one is its head. Given a partition $\lambda$, the ribbons that can be removed or added to $\lambda$ are entirely determined by the coordinates of their heads and tails. On Figure 2 are two ribbons: the left one has size 4, height 2 and sign +1, and the right one has size 6, height 1 and thus sign $-1$.

A hook is a non empty ribbon of shape $\lambda/\emptyset$, which is equivalently a partition of the kind $(k, 1, \ldots, 1)$. Note that a hook is characterized by the data of its size $s$ and height $h \in [0, s - 1]$.

For $i$ a positive integer, we note $\text{Rib}_i$ the set of ribbons of size $i$, and $\text{Rib} := \bigcup_{i > 0} \text{Rib}_i$. One should think of $Y$ as the vertices of a graph $\text{GR}$ whose edges are the elements of $\text{Rib}$; each edge carries in addition a sign which is simply the sign of the corresponding ribbon. $\text{GR}$ is structured in different levels given by the partitions of a given size. Figure 3 shows the first levels of $\text{GR}$, where the dotted edges correspond to negative ribbons. Adding (respectively removing) a ribbon corresponds simply to making a step up (resp. down) in $\text{GR}$.
Definition 2. A ribbon tableau of shape $\lambda \in Y$ and length $\ell$ is a sequence of partitions $\lambda^0 = \emptyset \subset \lambda^1 \subset \ldots \lambda^{\ell-1} \subset \lambda^\ell = \lambda$ such that $r_i := \lambda^{i+1}/\lambda^i$ is a nonempty ribbon for every $i < \ell$.

We will often represent a ribbon tableau by a labeling of the cells of $\lambda$, in which the cells labeled $i$ coincide with the ribbon $r_i$. Note that a ribbon tableau is equivalently a path of length $\ell$ in the graph $GR$, going up from $\emptyset$ to $\lambda$; this interpretation is the key to the extensions described in Section 10.

We need some more definitions. The sign $\varepsilon(P)$ of a tableau $P$ is the product of the signs of the ribbons $r^{(i)}$. The content $c(P)$ is the composition of $|\lambda|$ in $\ell$ parts formed by the sequence of sizes $|r^{(i)}|$. We will note $RT_{\lambda,c}$ the tableaux of shape $\lambda$ and content $c$, where $c$ is a given composition of $|\lambda|$ and $RT_{\lambda,l}$ the set of all the ribbon tableaux of shape $\lambda$ and length $l$.

Figure 3 shows a tableau of shape $(8, 6, 6, 2, 1)$, content $(1, 6, 6, 3, 7)$ and sign $(-1)^0(-1)^2(-1)^2(-1)^1(-1)^2 = -1$.

2.2. Operations on ribbons. We will now introduce some classical operations on ribbons which are necessary for the definition of the local rules of Section 6.

bumpin, bumpout: let $\lambda$ be a partition, and $r$, $r'$ be two ribbons that are $\lambda$-addable, such that $r$ and $r'$ have different heads and different tails. Then $bumpout(r, r')$ is the set of cells $(r \setminus r') \cup (r' \setminus r) \cup (r' \cap r) \setminus$ where $A \setminus$ is the translated of $A$ by the vector $(1, -1)$. The operation bumpin$(r, r')$ is similarly defined for two $\xi$-removable ribbons, by translating the common cells between $r$ and $r'$ by the vector $(-1, 1)$ (these definitions vary slightly from the ones commonly used). Figure 4 shows the
way this operation acts, according to the relative positions of $r$ and $r'$: they can be disjoint, or partially overlap, or one can be included in the other.

\textbf{prev, next, first}: Let $\lambda$ be a partition, $k$ a positive integer, and $h$ a nonnegative integer. A result of Shimozono and White \cite{23} is that if $(r_i)_{i=0 \ldots t}$ (respectively $(r'_i)_{i=1 \ldots t'}$) are all ribbons of size $k$ and height $h$ that are $\lambda$-addable (resp. $\lambda$-removable), then (1) $t=t'$, and (2) the enumeration order of the ribbons can be chosen so that $r_0 < r'_1 < r_1 < \ldots < r'_t < r_t$ where $\text{rib}_1 < \text{rib}_2$ means that the head of $\text{rib}_1$ is north east of the head of $\text{rib}_2$. 

Figure 4: A ribbon tableau: two equivalent representations.

Figure 5: Operations \textit{bumpin} and \textit{bumpout}
Figure 6: Addable and removable ribbons of height 1 and size 3.

Figure 6 illustrates Shimozono and White’s result. This allows to define certain operations first, next and prev on ribbons:

- if $hk$ is a hook of size $k$ and height $h$, we define $\text{first}(\lambda, hk)$ as the ribbon $r_0$ above.
- If $r'$ is a $\lambda$-removable ribbon, i.e. $r' = r'_i$ for a certain $i \in [1, t]$, then $\text{next}(\lambda, r') := r_i$.
- Reciprocally, if $r = r_i$ for any $i \in [1, t]$ is a $\lambda$-addable ribbon, we define $\text{prev}(\lambda, r_i) := r'_i$ for $i \geq 1$, and $\text{prev}(\lambda, r_0) := \emptyset$.

slideout, switchout, slidein, switchin: let $\lambda$ be a partition, and $r, r'$ two $\lambda$-addable ribbons, with identical tails but different heads; we assume without loss of generality that $|r| > |r'|$. The external band of $\lambda$ consists of all cells between the infinite south east boundary of $\lambda$ and its translated by $(1, -1)$, i.e. the cells enclosed by the dotted line on Figure 7. Let $\tau$ be the subset of the external band formed by the $|r'|$ connected cells, north west of $r$ and adjacent to it. Then two cases can occur:

- if $\tau \cup r$ forms a $\lambda$-addable ribbon, we define the partition $\text{slideout}(\lambda, r, r') = \lambda \cup r \cup \tau$.
- otherwise, we define $\text{switchout}(\lambda, r, r') = (\lambda \cup r') \setminus \tau \setminus$, where $\tau \setminus$ is the translated of $\tau$ by the vector $(-1, 1)$.

If $r$ and $r'$ have the same head but different tails, one performs the same operations on the transposed partitions. The operations switchin and slidein are defined similarly on $\lambda$-removable ribbons: see White [31] for supplementary explanations.

3. Signed sets and signed bijections

In this work we have to deal with signed enumerations, so we need some definitions and notations to explain what we mean by a bijection in this context. All sets are assumed to be finite.
and $B$ they give a combinatorial explanation of the equality of signed cardinals.

explains why signed bijections are the correct generalizations of bijections, in that for $i$ a bijection proving this equality (i.e. a bijection $\delta(i)$ is the data of 3 functions on $A$ between $\pm a$ and $a$).

Definition 3 (Signed Sets). A signed set is a set $A$ together with a decomposition $A = A^+ \cup A^-$ where $A^+ \cap A^- = \emptyset$. The members of $A^+$ are positive elements, those of $A^-$ are negative.

Such a decomposition is equivalent to a function $\delta : A \to \{1,-1\}$, with the obvious correspondence $A^+ = \delta^{-1}(\{1\})$ and $A^- = \delta^{-1}(\{-1\})$. Our objects of study here are the sets $RT_{\lambda,\mu}$, the sign being given by the function $\varepsilon$. Note also that unless explicitly stated, usual sets are considered as positive sets.

A function $f$ between two signed sets is sign preserving (resp. sign reversing) if $a$ and $f(a)$ have the same sign (resp. different signs) for all $a$. Fixed points of a function $i$ form the set $Fix(i)$.

Definition 4 (Signed bijections). A signed bijection between the signed sets $A$ and $B$ is the data of 3 functions $i_A, i_B$ and $\phi$ such that $i_A$ (resp. $i_B$) is an involution on $A$ (resp. $B$) which is sign reversing outside of its fixed point set, and $\phi$ is a sign preserving bijection between $Fix(i_A)$ and $Fix(i_B)$.

The signed cardinal (or signed sum) of a signed set $A$ is $|A|_\pm = |A^+| - |A^-|$; if the sign is given by a function $\delta$, then we have $|A|_\pm = \sum_{a \in A} \delta(a)$. A signed bijection between $A$ and $B$ proves that $|A|_\pm = |B|_\pm$; indeed we have

$$|A|_\pm = |Fix(i_A)|_\pm = |Fix(i_B)|_\pm = |B|_\pm.$$

The central equality comes from the sign preserving bijection $\phi$, the other ones from the fact that $i_A$ and $i_B$ are sign reversing, so the pairs $\{a, i_A(a)\}$ such that $a \neq i_A(a)$ have a zero contribution to the signed cardinal of $A$, the same being true for $B$ and $i_B$. Now $|A|_\pm = |B|_\pm$ is equivalent to $|A^+| + |B^-| = |B^+| + |A^-|$, and a bijection proving this equality (i.e. a bijection $\psi$ between the usual sets $A^+ \cup B^-$ and $B^+ \cup A^-$) is clearly equivalent to a signed bijection between $A$ and $B$. This explains why signed bijections are the correct generalizations of bijections, in that they give a combinatorial explanation of the equality of signed cardinals.

![Figure 7: Operations switchout and slideout.](image-url)
The Involition Principle of Garsia and Milne

Garsia and Milne gave the first fully bijective proof of a combinatorial version of a famous identity of Rogers-Ramanujan [11, 12]: this states that the number of partitions \((\lambda_1, \ldots, \lambda_k)\) of \(n\) verifying \(\lambda_i - \lambda_{i+1} \geq 2\) for all \(i < k\) is the same as those verifying \(\lambda_i \equiv 1\) or \(4\) modulo \(5\) for all \(i \leq k\). To achieve this, they defined and used a general principle, that we recall now.

Let \(A, B\) be two finite signed sets. Let also \(i_A, i_B\) be two involutions on \(A\) and \(B\) respectively, and \(\phi\) a bijection between \(A\) and \(B\). We suppose that \(\phi\) preserves signs, whereas \(i_A\) and \(i_B\) reverse signs outside their fixed point sets.

Under those assumptions one has clearly \(|Fix(i_A)|_\pm = |Fix(i_B)|_\pm\), but not necessarily a signed bijection proving this equality. The principle of Garsia and Milne is the construction of such a signed bijection \((\psi, j_A, j_B)\) between \(Fix(i_A)\) and \(Fix(i_B)\) in the following manner: let \(a \in A\). We apply to it the function \(\phi : A \to B\), then alternatively, the functions \(\phi^{-1} \circ i_B : B \to A\) and \(\phi \circ i_A : A \to B\), stopping as soon as the element \(x\) obtained is:

- either in \(Fix(i_A)\), in which case one sets \(j_A(a) := x\);
- or in \(Fix(i_B)\), in which case one sets \(\psi(a) := x\) (and \(j_A(a) := a\)).

To define \(j_B\) (and \(\psi^{-1}\)), one uses the symmetric procedure starting from \(b \in B\). These procedures terminate and give the sought signed bijection: see for instance [13, p.76] for a proof.

4. Hook permutations and hook involutions

We now introduce the notions of hook permutations and hook involutions, which play the role of the ordinary permutations and involutions of the Schensted correspondence in our main results stated in Section 5.

4.1. Hook Permutations. If \(H = (H_1, \ldots, H_\ell)\) is an ordered sequence of \(\ell\) hooks, its content \(c(H)\) is the composition \((|H_1|, \ldots, |H_\ell|)\).

**Definition 5.** A hook permutation \((H, \sigma)\) is an ordered sequence \(H = (H_1, \ldots, H_\ell)\) of \(\ell\) hooks, together with a permutation \(\sigma\) of \([\ell]\). The length of a hook permutation is \(\ell\), its size is \(\sum |H_i|\), and its content \(c(H, \sigma)\) is the composition \((|H_{\sigma(1)}|, \ldots, |H_{\sigma(\ell)}|)\).

We will write \(\mathcal{HP}\) for the set of hook permutations, its elements of content \(\mu\) forming \(\mathcal{HP}(\mu)\) (where \(\mu\) is any composition). Hook permutations can be represented by the list \(H\) where the cells of hook \(H_i\) are numbered by \(\sigma(i)\), or by square arrays of size \(\ell\) such that entry \((i, j)\) is empty unless \(j = \sigma(i)\) in which case it is occupied by the \(i\)th hook \(H_i\). Illustrations are given Figure 8.

We then have the following proposition:

**Proposition 6.** The number of hook sequences of length \(\ell\) and total size \(n\) is equal to \(\binom{n+\ell-1}{2\ell-1}\).

**Proof:** Given such a hook sequence, we can associate to it \(2\ell - 1\) integers in \([n+\ell-1]\), as illustrated graphically on Figure 9 in which \(n = 23\), \(\ell = 5\), so \(n+\ell-1 = 27\) and \(2\ell - 1 = 9\), and the subset of integers is \(\{4, 7, 9, 12, 13, 19, 21, 22, 26\}\).

Conversely, if we have \(2\ell - 1\) integers \(1 \leq i_1 < \ldots < i_{2\ell-1} \leq n + \ell - 1\), and we set by convention \(i_0 = 0\) and \(i_{2\ell} = n + \ell\), then the list of hooks \((H_1, \ldots, H_\ell)\) in bijection is characterized by the fact that \(H_i\) is the hook of size \(i_{2i} - i_{2i-2} - 1\) and height \(i_{2i-1} - i_{2i-2} - 1\). This is clearly bijective. \(\square\)
Figure 8: Two representations of the same hook permutation of length 5, size 23 and content $(6, 4, 6, 2, 5)$.

Figure 9: The bijection in Proposition 6.

4.2. Hook Involutions.

Definition 7. Hook involutions are hook permutations such that their array representation is symmetric with respect to the diagonal $i = j$.

In other words, these are hook permutations $(H, \sigma)$ such that $\sigma$ is an involution, and $H_i = H_j$ if $j = \sigma(i)$. For a hook involution $I = (H, \sigma)$, we define its sign as $\varepsilon(I) = \prod_{i/\sigma(i)=i} \varepsilon(H_i)$. It is the product of the signs of the hooks associated to fixed points.

We note $\mathcal{HI}$ the signed set of hook involutions, $\mathcal{HI}(\mu)$ the signed subset of those hook involutions with content $\mu$, and finally $\mathcal{HI}_{\text{spec}}(\mu)$ the elements of $\mathcal{HI}(\mu)$ all of whose fixed points are hooks of odd size and of height 0. Note that all elements of $\mathcal{HI}_{\text{spec}}(\mu)$ are then positive.

Lemma 8. There is a sign reversing involution on $\mathcal{HI}(\mu) \setminus \mathcal{HI}_{\text{spec}}(\mu)$.

Proof: If $I = (H, \sigma) \notin \mathcal{HI}_{\text{spec}}(\mu)$ then let $i = \sigma(i)$ be its smallest fixed point contradicting the definition of $\mathcal{HI}_{\text{spec}}(\mu)$. Let $h$ be the height of the hook $H_i$. If $H_i$ is of even size, then we let $H_i'$ be the hook of the same size and of height $h + 1$ (resp. $h - 1$) if $h$ is even (resp. odd). If $H_i$ is of odd size, so that necessarily $h \neq 0$
by the definition of $H_i$, then we let $H'_i$ be the hook of the same size and of height $h+1$ (resp. $h-1$) if $h$ is odd (resp. even).

Let $H'$ be the hook list equal to $H$ except in position $i$ where $H'_i$ replaces $H_i$. If we define $f(I) = (H', \sigma)$, then we have the desired sign reversing involution on $\mathcal{HI}(\mu) \setminus \mathcal{H}_{\text{spec}}(\mu)$. $\square$

**Corollary 9.** For any composition $\mu$, $|\mathcal{HI}(\mu)|_{\pm} = |\mathcal{H}_{\text{spec}}(\mu)|$.

We will give some consequences of this result in Section 9.

5. Main results

The first result is a generalization of the Schensted correspondence. We note by $Id$ the identity function on hook permutations.

**Theorem 10.** Let $n, \ell$ be two positive integers. There exists an explicit signed bijection $(Id, i, \phi)$ between hook permutations of size $n$ and length $\ell$, and pairs of ribbon tableaux of size $n$ and length $\ell$ with the same shape.

This bijection preserves contents, which means: if $i(P, Q) = (P_1, Q_1)$, then $c(P) = c(P_1)$ and $c(Q) = c(Q_1)$; and if $\phi(\sigma, H) = (P, Q)$, then $c(H) = c(Q)$ and $c(H, \sigma) = c(P)$.

We will define the signed bijection is given in Section 7 based on local rules given in Section 6; the proof of the correctness of the bijection is in Appendix B. In the special case where contents are partitions with certain constraints, then the preceding theorem is equivalent to the main result of White in [31]. The idea here is to use Fomin’s techniques [6, 7] in the proof of this result: this sheds a new light on White’s result, and lends itself to generalization in a more straightforward fashion.

The theorem has the following consequences concerning the signed enumeration of ribbon tableaux:

**Corollary 11.** Let $\mu, \nu$ be two compositions of $n$ with $\ell$ parts, and write $\bar{\mu} = [1^{j_1}, 2^{j_2}, \ldots]$. Then

$$
\sum_{P \in RT_{\lambda, \mu}, Q \in RT_{\lambda, \nu}} \varepsilon(P) \varepsilon(Q) = \delta_{\mu \bar{\mu}} \cdot 1^{j_1}(j_1!)2^{j_2}(j_2!)\ldots;
$$

$$
\sum_{P, Q \in RT_{\lambda, \ell}} \varepsilon(P) \varepsilon(Q) = \left(\frac{n + \ell - 1}{2\ell - 1}\right) \cdot \ell!
$$

We will show that this corollary can be proved by techniques of linear algebra in Section 8. Finally, the Schensted correspondence has the property that it restricts to a bijection between involutions of $S_n$ and standard tableaux of size $n$. We will prove a version of this result for ribbon tableaux:

**Theorem 12.** Let $n, \ell$ be two positive integers. There exists an explicit signed bijection between hook involutions of size $n$ and length $\ell$, and ribbon tableaux of size $n$ and length $\ell$.

As explained in Section 6, this cannot be deduced from White’s correspondence for pairs of tableaux. We will prove this Theorem in Section 7 and Appendix B.
6. Local Rules

We wish to extend local the rules used by Shimozono and White \cite{ShimozonoWhite2019} to deal with ribbons of all possible sizes; this will be done simply by reformulating White’s insertion rules of \cite{White2002} as local rules. In all this section $\mu, \nu$ are two partitions of respective sizes $m$ and $n$.

For $i$ a nonnegative integer, we define $\mathcal{U}_i(\mu, \nu)$ as the set of partitions of size $\max(m, n) + i$ such that $\xi/\mu$ and $\xi/\nu$ are ribbons. $\mathcal{U}_i(\mu, \nu)$ is a signed set through

$$\text{sgn}(\xi) := \varepsilon(\xi/\mu) \cdot \varepsilon(\xi/\nu).$$

Similarly, we define $\mathcal{D}_i(\mu, \nu)$ the set of partitions of size $\min(m, n) - i$ such that $\lambda/\mu$ and $\lambda/\nu$ are ribbons; $\mathcal{D}_i(\mu, \nu)$ is a signed set through

$$\text{sgn}(\lambda) := \varepsilon(\mu/\lambda) \cdot \varepsilon(\nu/\lambda).$$

( Note that, as signed sets, $\mathcal{U}_i(\mu, \mu)$ and $\mathcal{D}_i(\mu, \mu)$ contain only positive elements.)

We draw a square where 2 corners are labeled by $\mu$ and $\nu$ as shown on the right. The bottom left corner will be labeled by partitions $\lambda$ from a set $\mathcal{D}_i(\mu, \nu)$, the top right one by partitions $\xi$ from a set $\mathcal{U}_i(\mu, \nu)$. In the case $\lambda = \mu = \nu$, the interior $C$ may be marked by a nonempty hook, or be left empty; in all other cases it is left empty.

To define local rules, it is necessary to use the operations on ribbons and partitions defined in Section 2.

Let $((\lambda, C), \mu, \nu)$ be given in a square, as above. What we mean by applying a local rule is the following: first find out which case applies in the list below, then erase $\lambda$ and $C$ from the square, and finally write the outcome of the rule in the appropriate corner: the top right one for the rules D1 to D6, and the bottom left one for the rule S.

**Direct rules:** in the following rules $\lambda$ is an element of a certain $\mathcal{D}_i(\mu, \nu)$, $C$ is empty unless possibly when $\lambda = \mu = \nu$ in which case it may be filled by a hook. Let $r$, $r'$ be the ribbons $\mu/\lambda$ and $\nu/\lambda$ (allowing empty ribbons).

- If $\lambda = \mu = \nu$ and $C$ is empty, then $\xi := \lambda$. (D1)
- If $\lambda = \mu = \nu$ and $C$ is a nonempty hook eq, then $\xi := \lambda \cup \text{first}(\lambda, \text{eq})$. (D2)
- If $\lambda \neq \mu = \nu$, then $\xi := \mu \cup \text{next}(\mu, \mu/\lambda)$. (D3)
- If $\lambda = \mu \neq \nu$ (resp. $\lambda = \nu \neq \mu$), then $\xi := \nu$ (resp. $\xi := \mu$). (D4)
- If $\lambda \neq \mu \neq \nu$, then:
  - if $r$ and $r'$ have neither the same tail nor the same head, then $\xi := \lambda \cup \text{bumpout}(r, r')$. (D5)
  - if $r$ and $r'$ have the same head but different tails, or the same tail but different heads, then:
    * if $\text{slideout}(\lambda, r, r')$ is well defined, then $\xi := \text{slideout}(\lambda, r, r')$. (D6)
    * otherwise, we set $\hat{\lambda} := \text{switchout}(\lambda, r, r') \in D_i(\mu, \nu)$. (S)

**Inverse rules:** Now $\xi$ belongs to a certain set $U_i(\mu, \nu)$. We write $r$, $r'$ for the ribbons $\xi/\mu$ and $\xi/\nu$ (possibly empty). $C$ is left empty except in rule (I2).

- If $\xi = \mu = \nu$, then $\lambda := \xi$. (I1)
- If $\xi \neq \mu = \nu$, then
  - if $\text{prev}(\xi, r) = \emptyset$, we define $\lambda := \mu$ and $C$ is filled with the unique hook of size $|r|$ and height $h(r)$; (I2)
  - otherwise $\lambda := \mu \setminus \text{prev}(\xi, r)$. (I3)
- If $\xi = \mu \neq \nu$ (resp. $\xi = \nu \neq \mu$), then $\lambda := \nu$ (resp. $\lambda := \mu$). (I4)
• If $\xi \neq \mu \neq \nu$, then:
  – if $r$ and $r'$ have neither the same tail nor the same head, then $\lambda := \xi \backslash \text{bumpin}(r, r')$. (I5)
  – if $r$ and $r'$ have the same head but different tails, or the same tail but different heads, then:
    * if $\text{slidein}(\xi, r, r')$ is defined, then $\lambda := \text{slidein}(\xi, r, r')$; (I6)
    * otherwise we set $\hat{\xi} := \text{switchin}(\xi, r, r') \in U_i(\mu, \nu)$. (T)

Proposition 13. The rules D1 to D6 are the respective inverses of I1 to I6; S and T are involutions. Furthermore, D1-D6 and I1-I6 preserve signs between $D_i(\mu, \nu)$ and $U_i(\mu, \nu)$, whereas S and T are sign reversing on $D_i(\mu, \nu)$ and $U_i(\mu, \nu)$ respectively.

Proof: All these properties are already proved elsewhere, albeit sometimes in a different form. For D2, D3 and I2, I3 this was proved by Shimozono and White [23]. For the rules D5, D6, I5, I6, S and T, the result can be found in White [31]. We will anyway give the proof for the rule S in Appendix A using the encoding of partitions by infinite sequences $\delta(\lambda)$: we wish to show how this encoding is particularly suited to the study of ribbons.

Let us sum up the local rules in terms of signed bijections, since this will be useful in particular in the algebraic approach of Section 8:

Proposition 14. Let $\mu, \nu$ be two partitions and $i$ a positive integer.
  (a) There exists an explicit bijection $\phi_1$ between $U_i(\mu, \mu)$ and $D_i(\mu, \mu) \sqcup [0, i-1]$.
  (b) For $\mu \neq \nu$, there exists a signed bijection $(i_D, i_U, \phi_2)$ between $D_i(\mu, \nu)$ and $U_i(\mu, \nu)$.

Proof: (a) is given by rules D2 and D3; for (b), the involutions $i_D$ and $i_U$ are given by rules S and T respectively, and the bijection $\phi_2$ consists of rules D4, D5 and D6.

This proposition has to be interpreted as a local property of the graph $GR$: given $\mu$ and $\nu$ of size $m$ and $n$, it gives a combinatorial link between vertices adjacent to both $\mu$ and $\nu$ at the level $\min(m, n) - i$, and at the level $\max(m, n) - i$. Sections 7 and 8 will use this local structure to deduce global results on ribbon tableaux.

7. Bijective approach

We want to use growth diagram techniques (see [7]) to carry out a signed correspondence proving Theorems 10 and 12. This will be done in this section, but requires more work than a simple application of Fomin’s setting. In order to prove Theorem 10, we will actually need to make some back and forth moves in a growth diagram: the correction of the correspondence will rely on the Involution Principle.

7.1. Bijection for hook permutations. Fix a positive integer $\ell$, and let $G_{\ell}$ be the grid of size $\ell \times \ell$, made of $\ell^2$ squares (and $(\ell + 1)^2$ vertices). Square $(i, j)$ is at the intersection of the $i$th column from the left and the $j$th row from the bottom of the grid. We wish now to label some of the vertices by partitions and then apply the local rules of Section 6 in the squares: for this, we order the squares (partially) by $(i, j) \preceq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$.

We fix from now on a total order $\mathcal{O}$ extending this partial order. Every square $sq \neq (1, 1)$ has then a predecessor $\text{pred}(sq)$, and every square $sq \neq (\ell, \ell)$ has a
successor $\text{succ}(sq)$. For $\text{dir} = \pm 1$, we define also $\text{next}(sq, \text{dir})$ to be $\text{succ}(sq)$ if $\text{dir} = 1$, $\text{pred}(sq)$ if $\text{dir} = -1$, and to return “undefined” when $\text{pred}$ or $\text{succ}$ is not defined.

Given $\text{dir} \in \{+1, -1\}$ and a square $sq$ of $G_\ell$:
- if $\text{dir} = 1$, and $\mu, \nu, \lambda, C$ label $sq$ as in the definition of direct rules, we apply the corresponding rule;
- if $\text{dir} = -1$, and $\mu, \nu, \xi$ label $sq$ as in the definition of inverse rules, we apply the corresponding rule.

Let us call this procedure $\text{Apply\_local\_rule}$, and write $\text{loc} := \text{Apply\_local\_rule}(\text{dir}, sq)$ for the local rule that applies.

Now we want to go from local rules to a correspondence on the entire grid. Let us be given a hook permutation drawn $G_\ell$, in which we also label by $\emptyset$ (the empty partition) all vertices of $G_\ell$ on the bottom and left sides (see Configuration A on Figure 10). We may now describe the bijection $\phi$ of Theorem 10, which we do in an algorithmic fashion:

**Algorithm $\phi$**:

Input: a hook permutation $(H, \sigma)$.
Output: A pair $(P, Q)$ of ribbon tableaux of the same shape.

Begin

$sq := (1, 1); \text{dir} := 1$;

repeat

loc := $\text{Apply\_local\_rule}(\text{dir}, sq)$;

If $(\text{loc} \in \{S, T\})$ then $\text{dir} := -\text{dir}$; end if;

sq := $\text{next}(sq, \text{dir})$;

until (sq = “undefined”);

End

We will show that this algorithm is well defined and does not loop indefinitely: it ends when $\text{succ}(\ell, \ell)$ is not defined, in which case the vertices on the top and right side of $G_\ell$ are labeled by partitions forming two ribbon tableaux of the same shape.

**Example**: we illustrate the algorithm on the example of Figure 10. We choose the total order $(i, j) < (i', j')$ if $j < j'$, or $j = j'$ and $i < i'$. A square $c'$ is thus larger than a square $c$ if it is above in the same column, or if $c'$ is in a column to the right of $c$.

We start with the hook permutation given on $G_3$ (configuration A). We apply direct local rules to reach configuration B, the rules being successively D2, D4, D4, D4, D1, D2, D4 and D2. Now the rule $S$ applies, and the direction changes (configuration C). Note that we deleted all contents of visited squares as well as the label of their bottom left corner; we did not show this for direct rules in order to keep the number of pictures reasonable. From this configuration C, we apply inverse rules I5,I2 and I5 successively to reach configuration D. There rule $T$ applies, and the direction changes. We finally reach configuration F with the rules D3,D2,D3 and D6. We can finally read off the ribbon tableaux $P$ and $Q$ respectively on the right and top sides of the grid.

We will prove the correctness of the bijection in Appendix B; we can already give the proof of its corollary:
Figure 10: The algorithm $\phi$. 
Proof of Corollary 11: For the first formula, the signed bijection of Theorem 10 implies that the left hand side is equal to the number of hook permutations of type \((c, c')\); this number is zero unless \(c\) can be obtained from \(c'\) by permuting some of its parts. When \(\tilde{c} = \tilde{c}'\), we have to compute, for a part \(i\) appearing \(j_i\) times, how many hook permutations of length \(j_i\) exist so that all hooks have size \(i\); this number is clearly \(i^{j_i} j_i!\). We obtain finally the right hand side by multiplying such terms for all sizes \(i\). The second equality of Corollary 11 is an immediate consequence of Theorem 10 and of Proposition 6.

7.2. Bijection for hook involutions. The Schensted correspondence is known to restrict to a bijection between involutions and standard tableaux, and we wish to extend this to hook involutions and ribbon tableaux. But whereas this restriction is immediate with Fomin’s version of the Schensted correspondence, some extra work has to be done here: first, the bijection defined by the algorithmic procedure depends on the fixed total order specified on the squares of the grid, so the procedure can not be done symmetrically in general. Then, notice that hook involutions have signs, whereas hook permutations are defined as positive: we must prove that these signs are preserved by the bijections. We will deal with these two extra difficulties and give the proof of Theorem 12 in Appendix B.

To end this section, we formulate in terms of signed sets the properties which are crucial in the proof of Theorem 12. Define \(U_i(\mu)\) as the set \(U_i(\mu, \mu)\) but with sign \(\varepsilon(\lambda/\mu)\) for \(\lambda \in U_i(\mu)\); similarly, \(D_i(\mu)\) is \(D_i(\mu, \mu)\) with sign \(\varepsilon(\mu/\lambda)\).

Proposition 15 (Shimozono and White [23]). Consider \([0, i - 1]\) as a signed set with \(\text{sgn}(h) = (-1)^h\). Then there is a sign preserving bijection between \(U_i(\mu)\) and \(D_i(\mu) \sqcup [0, i - 1]\).

7.3. Relation with previous works. As already mentioned, Theorem 10 was partly demonstrated by White [31] (in this article, White evokes the possibility of extending his ideas to obtain the form in which we gave it). In [28], the authors notice that if one only considers hook permutations with all hooks of size \(k\), and ribbon tableaux with all ribbons of size \(k\), then rules S and T can never be applied: so in this case we have a (sign-preserving) bijection between ribbon tableaux and \(k\)-colored permutations.

As a matter of fact, rules D2 and D3 are not used in these two articles, but alternative rules that do not preserve signs in the sense of Proposition 15. Theorem 12 cannot thus be a consequence of White’s original work, but is based on the work of Shimozono and White [23] (for ribbons and hooks of fixed size), in which the authors introduce the operations \(\text{prev, next, first}\) that are used to define rules D2, D3 and I2, I3.

8. Algebraic approach

The previous section generalized the bijection of Robinson-Schensted; we will give an algebraic proof of the enumerative counterpart of these results, that is Corollary 11 in the spirit of Stanley [25] and Fomin [9].

Let \(K\) be a field of characteristic zero. We consider \(K\mathcal{Y} = \bigoplus_n K\mathcal{Y}_n\), the vector space of all linear combinations of partitions with coefficients in \(K\). For \(i\) a positive integer we define two linear operators \(U_i\) and \(D_i\) via their action on the basis of partitions:
Definition 16. For \( \lambda \in \mathcal{Y} \),

\[
U_i \lambda = \sum_{r = \lambda / \mu \in \text{Rib}_i} \varepsilon(r) \mu ; \quad D_i \lambda = \sum_{r = \lambda / \mu \in \text{Rib}_i} \varepsilon(r) \mu.
\]

\( U_i \) and \( D_i \) are endomorphisms of \( \mathbb{K} \mathcal{Y} \), that send \( \mathbb{K} \mathcal{Y}_n \) in \( \mathbb{K} \mathcal{Y}_{n+i} \) and \( \mathbb{K} \mathcal{Y}_{n-i} \) respectively. We note that these two operators were already defined by Stanley [25, 26] and which are the basis of [6] (let us remark that our operators involve signs, which is not the case in the classical case). For \( \lambda, \mu \) two partitions, we set \( \langle \lambda, \mu \rangle = 1 \) if \( \lambda = \mu \) and 0 otherwise. We may then extend \( \langle \cdot, \cdot \rangle \) to \( \mathbb{K} \mathcal{Y} \times \mathbb{K} \mathcal{Y} \) by bilinearity, and we notice that \( U_i \) and \( D_i \) are dual endomorphisms for this bilinear form; indeed \( \langle U_i \lambda, \mu \rangle = \langle \lambda, D_i \mu \rangle \) since each member in the equality is equal to \( \varepsilon(r) \) if \( \lambda \subseteq \mu \) and \( r = \mu / \lambda \) is a ribbon, and to 0 otherwise.

The fundamental relation between these endomorphisms is the following (we note \( AB = A \circ B \) for composition):

**Proposition 17.** For nonnegative integers \( i, j \), we have

\[
\begin{align*}
U_i D_j &= U_i D_j + i \cdot \text{Id} \quad (3) \\
D_i U_j &= D_j D_i \quad \text{if } i \neq j \quad (4)
\end{align*}
\]

**Proof:** The first equality can be rewritten

\[
\langle D_i U_i \mu, \nu \rangle = \langle U_i D_i + i \cdot \text{Id} \mu, \nu \rangle
\]

i.e. \( \langle D_i U_i \mu, \nu \rangle = \langle U_i D_i \mu, \nu \rangle + i \cdot \delta_{\mu, \nu} \)

i.e. \( \langle U_i \mu, U_i \nu \rangle = \langle D_i \mu, D_i \nu \rangle + i \cdot \delta_{\mu, \nu} \)

for \( \mu \) and \( \nu \) any two partitions, while the second equality is equivalent to

\[
\langle D_i U_j \mu, \nu \rangle = \langle U_j D_i \mu, \nu \rangle.
\]

Those two equalities can be rephrased as \( |U_i(\mu, \mu)| = |D_i(\mu, \mu)| + i \) for all \( \mu \), and \( |U_i(\mu, \nu)|_\pm = |D_i(\mu, \nu)|_\pm \) for \( \mu \neq \nu \). But this is exactly the enumerative signification of Proposition 14 which ends the proof. \( \square \)

The first of these two relations is characteristic of \textit{i-differential posets}, defined by Stanley [23, 26], and which are the basis of [6] (let us remark that our operators involve signs, which is not the case in the classical case.).

Let us now specialize \( \mathbb{K} = \mathbb{Q}(\!(q)\!) \), the field of formal Laurent series in \( q \) (We could in fact limit ourselves to formal power series \( \mathbb{Q}(\![q]\!) \), and in general work with commutative rings of characteristic zero).

Let us define \( \mathbb{K} \mathcal{Y} = \prod \mathbb{K} \mathcal{Y}_n \) the vector space of functions from \( \mathcal{Y} \) to \( \mathbb{K} \), which we will write as infinite linear combinations of partitions, with coefficients in \( \mathbb{K} \). It is then possible to extend \( \langle \cdot, \cdot \rangle \) to \( \mathbb{K} \mathcal{Y} \times \mathbb{K} \mathcal{Y} \) without difficulty, and one checks that \( D_i \) and \( U_i \) extend also to endomorphisms of \( \mathbb{K} \mathcal{Y} \). Let us now define operators \( U \) and \( D \):

**Definition 18.**

\[
U = \sum_i q^i U_i ; \quad D = \sum_j q^j D_j
\]

\( U \) and \( D \) are themselves endomorphism of \( \mathbb{K} \mathcal{Y} \). \( U(\lambda) \) (respectively \( D(\lambda) \)) is by definition the sum of all partitions \( \mu \) such that \( \mu / \lambda \) (resp. \( \lambda / \mu \)) is a ribbon, with a coefficient \( \varepsilon q^k \) for a ribbon of size \( k \) and sign \( \varepsilon \) (note that \( U \) does not have its
Theorem 19 (Stanley [24]). Suppose that two endomorphisms of a vector space $E$ verify $DU = UD + rI$. Then for all positive integers $\ell$ we have

$$D^{\ell}U^{\ell} = (UD + rI)(UD + 2rI) \cdots (UD + \ell rI)$$  \hfill (7)

As a consequence, if $\hat{O}$ is an element of $E$ such that $D\hat{O} = 0$, we have $\langle \hat{O}, D^{\ell}U^{\ell}\hat{O} \rangle = r^{\ell} \ell!$.

We have such a relation for $U$ and $D$ with $\hat{O} = \emptyset$ and $r = q^{2}/(1 - q^{2})^{2}$; it’s just equation (6). So the second formula of (11) is the consequence of the following computation:

$$r^{\ell} \ell! = \ell! \cdot q^{2\ell} \cdot \frac{1}{(1 - q^{2})^{2\ell}} = \sum_{n \geq 2\ell} \left[ \binom{n + \ell - 1}{2\ell - 1} \ell! \right] q^{2n}.$$

Let us now turn to the first formula of Corollary 11. $U_{\mu} \cdots U_{\mu_{i}}\emptyset$ is the linear combination of ribbon tableaux of content $\mu$, with the sign of the tableau as coefficient. So the left hand side of (11) is $\langle \emptyset, D_{\nu_{1}} \cdots D_{\nu_{i}} U_{\mu_{i}} \cdots U_{\mu_{1}} \emptyset \rangle$, which by duality is $\langle \emptyset, D_{\nu_{1}} \cdots D_{\nu_{i}} U_{\mu_{i}} \cdots U_{\mu_{1}} \emptyset \rangle$.

Lemma 20. Let $\mu, \nu$ be two compositions with $\ell$ parts. Then

$$\langle \emptyset, D_{\nu_{1}} \cdots D_{\nu_{i}} U_{\mu_{i}} \cdots U_{\mu_{1}} \emptyset \rangle = \nu_{\ell} \times \sum_{\rho} \langle \emptyset, D_{\nu_{1}} \cdots D_{\nu_{i-1}} U_{\rho} \emptyset \rangle,$$

where $\rho$ goes through the multiset of all compositions of length $\ell - 1$ obtained by deleting a part of $\mu$ of size $\nu_{i}$.

Proof of the lemma: It is just an iterated use of Proposition 17 with $D_{\nu_{i}}$ and $U_{\mu_{i}}$, for $i$ equals $\mu_{1}, \ldots, \mu_{i}$ successively. Two cases can occur: if $\mu_{i} \neq \nu_{i}$, then $D_{\nu_{i}}$ and $U_{\mu_{i}}$ commute; otherwise, for each index $i$ such that $\mu_{i} = \nu_{i}$, a term $\nu_{i} \langle \emptyset, D_{\nu_{1}} \cdots D_{\nu_{i-1}} U_{\rho} \emptyset \rangle$ has to be added, where $\rho$ is the composition with the part $\mu_{i}$ deleted in $\mu$. Then when $D_{\mu_{i}} U_{\mu_{i}}$, the scalar product is zero since $D_{\nu_{i}} \emptyset = 0$. □

The proof of the first part of the Corollary 11 is now a simple induction on $\ell$ based on the preceding Lemma.
9. Columns of the character table of $\mathfrak{S}_n$

Ribbon tableaux are known to have a strong connection with the representation theory of the symmetric group, which we will now recall briefly. Let $n$ be a positive integer, $\lambda$ a partition of size $n$; we note $\chi^\lambda$ the irreducible character of $\mathfrak{S}_n$ indexed by $\lambda$ (for more information on these topics, see for instance [10, 19] that have a combinatorial approach). Let also $\chi^\mu_\lambda$ be the value of this character on a permutation of cycle type $\mu$: this means that the permutation has $m_i$ cycles of length $i$ for each $i$ if $\mu = (1^{m_1}2^{m_2}\cdots)$. The Murnaghan-Nakayama rule [16, 17] states that:

**Theorem 21.** Let $\mu, \nu$ be two partitions of the same size $n$. Then

$$\chi^\lambda_\mu = |RT^\lambda_{\mu}| \pm$$

This rule gives a combinatorial interpretation of $\chi^\lambda_\mu$, and shows in particular that it is an integer. We will now show that Theorem 12 is adapted to study the column sums of the character table.

9.1. A formula for $\sum_\lambda \chi^\lambda_\mu$. Define $C(\mu) = \sum_\lambda \chi^\lambda_\mu$, the sum of all entries of column $\mu$ in the character table of $\mathfrak{S}_n$. By the Murnaghan-Nakayama rule, $C(\mu)$ is equal to the signed sum of all ribbon tableaux of content $\mu$. By Theorem 12 this last quantity is itself equal to the signed sum of hook involutions of content $\mu$. The preceding result can thus be summed up by $C(\mu) = |RT^\lambda_\mu| = |HI| \pm$. Using Corollary 9 we finally obtain

$$C(\mu) = |HI_{\text{spec}}(\mu)|. \ (8)$$

This shows in particular that $C(\mu)$ is a nonnegative integer; the following theorem gives a formula for the exact value of this integer.

**Theorem 22.** Let $\mu = (1^{m_1}2^{m_2}\cdots)$ be a partition. Then $C(\mu) = \prod_{i>0} c_{i,m_i}$ with:

$$c_{i,m_i} = \begin{cases} 0 & \text{if } i \text{ is even and } m_i \text{ is odd;} \\ (m_i - 1)! \cdot i^{m_i/2} & \text{if } i \text{ is even and } m_i \text{ is even;} \\ \frac{1}{i} \sum_{k=0}^{\frac{m_i}{2k}} \binom{m_i}{m_i-2k} (2k-1)! \cdot i^k & \text{if } i \text{ is odd.} \end{cases}$$

We will exhibit two proofs: first a bijective one, and the second algebraic, using the tools of Section 8.

**First Proof:** The computation of $|HI_{\text{spec}}(\mu)|$ for general $\mu$ reduces clearly to the case $\mu = [i^{m_i}]$ where $\mu$ has only one part size.

In this case an element of $HI_{\text{spec}}(\mu)$ is an involution on $[1,a_k]$ with a choice of a hook of size $a_k$ for each cycle of length 2. Remembering that elements of $HI_{\text{spec}}(\mu)$ have no fixed points corresponding to even parts, we obtain easily the above expression for the coefficient $c_{i,m_i}$, and the proof is complete.

**Second Proof:** We now give a proof that does not use Theorem 12 (and thus also not the Equality (8)). For this we need the following algebraic consequence of Proposition 15 in terms of the operators $D_i$ and $U_i$ (considered as endomorphisms of $\mathbb{K}^Y$); we note $Y$ the vector $\sum_{\lambda \in Y} \lambda \in \mathbb{K}^Y$, and $o_i$ is 1 when $i$ is odd and 0 otherwise.

**Proposition 23.** For all $i \geq 1$, we have $D_iY = U_iY + o_i \cdot Y$. 
Proof: Take the scalar product of each member of the equality with a partition \( \lambda \), and remembering that \( U_i \) and \( D_i \) are dual operators, the result is equivalent to

\[
\sum_{\mu \in \mathcal{M}((\lambda))} \varepsilon(\mu/\lambda) = \sum_{\mu \in \mathcal{D}(\lambda)} \varepsilon(\lambda/\mu) + o_i.
\]

This is an immediate corollary to Proposition 17.

By the Murnaghan-Nakayama rule, we have \( C(\mu) = \langle D_\mu Y, \emptyset \rangle \). We will use the relations of Propositions 17 and 23 to compute this scalar product.

Lemma 24. We have the following formulas:

1. For \( m \geq 2 \) and \( i \geq 1 \), \( D_m^i Y = a_i \cdot D_m^{i-1} Y + (m-1)i \cdot D_m^{i-2} Y + U_i D_m^{i-1} Y \).
2. For \( m \geq 1 \) and \( i \geq 1 \), \( D_m^i Y = c_{i,m} Y + U_i A_{i,m} Y \), where \( c_{i,m} \) is defined in Theorem 22, and \( A_{i,m} \) is an endomorphism of \( \mathbb{K} Y \).
3. For \( m \geq 1 \) and \( i \geq 1 \), \( \langle D_\mu D_m^i Y, \emptyset \rangle = c_{i,m}(D_\mu Y, \emptyset) \) if all parts of \( \mu \) are greater than \( i \).

Proof of the lemma: We have \( D_m^i Y = a_i \cdot D_m^{i-1} Y + D_m^{i-1} U_i Y \) thanks to Proposition 23. Using \( m - 1 \) times the relation \( D_i U_i = U_i D_i + i \cdot I \), point 1. of the lemma is proved.

By an immediate induction on 1., we can write for all \( m \geq 2 \) that \( D_m^m Y = b_{i,m} Y + U_i B Y \) for a certain endomorphism \( B \) and an integer \( b_{i,m} \), necessarily equal to \( (D_m^m Y, \emptyset) \). Substituting in 1., and taking the coefficient of \( \emptyset \) in each member, we obtain \( b_{i,m} = a_i b_{i,m-1} + (m-1)ib_{i,m-2} \). The numbers \( c_{i,m} \) verify the same recurrence relation, as can be easily seen, directly or by the combinatorial interpretation given in the first proof. Since we have in addition \( b_{i,0} = c_{i,0} = 1 \) and \( b_{i,1} = c_{i,1} = a_i \), it follows \( b_{i,m} = c_{i,m} \) for all \( i, m \) and point 2. is proved.

Finally, thanks to point 2., the left member of 3. is equal to

\[
c_{i,m}(D_\mu Y, \emptyset) + \langle D_\mu U_i A_{i,m} Y, \emptyset \rangle,
\]

Since \( D_\mu \) commutes with \( U_i \) by Equation 4, the second term is then 0 because the image of \( U_i \) has null intersection with \( \mathbb{K} \emptyset \), and the lemma is proved.

The proof of Theorem 22 is now immediate by induction on the number of different part sizes of \( \mu \), using formula 3. in the previous lemma.

9.2. Other enumerations of \( C(\mu) \). The formula of Theorem 22 is not new, but the proof above is (to the best of our knowledge) the first fully bijective proof of it based on the Murnaghan Nakayama rule. Let us mention two other places in the literature where this result is shown, and show the equivalence to our formulation.

The computation of \( C(\mu) \) is an exercise in Macdonald’s book [13, p.122, ex.11], and relies on symmetric function techniques. It is proved that \( C(\mu) \) is equal to the product \( \prod_{i \geq 1} a_i^{(m)} \), where \( a_i^{(m)} \) is the coefficient of \( t^m/(m!) \) in \( \exp(t + \frac{1}{2} it^2) \) (respectively \( \exp(-\frac{1}{2} it^2) \)) if \( i \) is odd (resp. even). Through an expansion of the series, one checks easily that \( a_i^{(m)} \) is indeed equal to the coefficient \( c_{i,m} \) of Theorem 22.

Another proof can be found in Exercise 7.69 of Stanley’s book [27]; the proof is based on a general result in character theory, whose specialization to the symmetric group is the following theorem:

Theorem 25 ([13, 27]). Let \( \sigma \) be a permutation of \([1,n]\) with cycle type \( \mu \). Then \( C(\mu) \) is equal to the number of square roots of \( \sigma \) in \( S_n \), i.e. to the number of permutations \( \tau \in S_n \) such that \( \tau^2 = \sigma \).
Proof: Thanks to the formula (8), we can prove this result by exhibiting a bijection $HIToRoot$ between $\mathcal{H}_{\text{spec}}(\mu)$ and $\{\tau \mid \tau^2 = \sigma\}$.

We consider each cycle of $\sigma$ as a word $x\sigma(x)\sigma^2(x)\ldots$ where $x$ is minimal in its orbit, and decompose $\sigma$ canonically in the form $[c_1^{(1)} \cdots c_1^{(m_1)}][c_2^{(1)} \cdots c_2^{(m_2)}] \cdots$, where $c_i^{(k)}$, $k = 1 \ldots m_k$, are the cycles of length $i$ written in increasing order of their minimal elements. For instance, the permutation 57432896 has cycle type $\mu = (3, 2^3)$ and will be written $[(27)(34)(69)][(158)]$.

Lemma 26. Let $c_1, c_2$ be two disjoint cycles of length $m$ in $S_n$.

- There exist exactly $m$ cycles $c$ of length $2m$ in $S_n$ such that $c^2 = c_1c_2$.
- If $m$ is odd, there exist a unique cycle $c$ of length $m$ in $S_n$ such that $c^2 = c_1$.

The proof of this lemma is immediate. The $m$ cycles of the first part of the Lemma will be denoted $\text{root}(c_1, c_2, j)$, $j = 0 \ldots m - 1$, and the unique cycle of length $j$ determined by the second part is $\text{root}(c_1)$.

Let us now fix an element $I \in \mathcal{H}_{\text{spec}}(\mu)$; it is equivalent to a sequence of hook involutions $I_j, j = 1 \ldots k$ where $I_j$ is element of $\mathcal{H}_{\text{spec}}(j^{m_j})$. Write $t_j$ (resp. $f_j$) for the number of transpositions (resp. fixed points) in the involution $I_j$. Let us also write $(x_s, y_s), s = 1 \ldots t_j$ these transpositions and $z_s, t = 1 \ldots f_j$ for the fixed points. Finally let $h_s \in [0, j - 1]$ be the height of the hook associated to the transposition $(x_s, y_s)$.

We now associate to $I_j$ the $t_j$ cycles of length $2j$ defined by $\text{root}(c_j^{(r)}(c_j^{(s)}), h_s), s = 1 \ldots t_j$, as well as the cycles of length $j$ $\text{root}(c_j^{(z_t)}), t = 1 \ldots f_j$. The product of all these disjoint cycles for all indices $j$ form a permutation, which is the desired root $HIToRoot(I)$.

We can also use the formula to answer the question: for a given integer $k$, what are the partitions $\mu$ such that the column sum $C(\mu)$ is equal to $k$? Let $\mathcal{OD}$ be the set of partitions with odd distinct parts. The answers for the first integers are:

- $C(\mu) = 0$ if and only if $\mu$ has at least an even part with odd multiplicity;
- $C(\mu) = 1$ if and only if $\mu \in \mathcal{OD}$;
- $C(\mu) = 2$ if and only if $1$ has multiplicity 2 and $\mu - 1^2 \in \mathcal{OD}$, or 2 has multiplicity 2 and $\mu - 2^2 \in \mathcal{OD}$;
- $C(\mu) = 3$ has no solution.
- $C(\mu) = 4$ if and only if 3 has multiplicity 2 and $\mu - 3^2 \in \mathcal{OD}$, or 4 has multiplicity 1 and $\mu - 4^1 \in \mathcal{OD}$, or 2 and 1 have multiplicity 2 and $\mu - 1^2 2^2 \in \mathcal{OD}$.

The number of solutions to $C(\mu) = 0$ is sequence A085642 in Sloane’s Online Encyclopedia [24]. The article [1] proves that another family of partitions is in bijection with $\mathcal{OD}$, namely the partitions with at least one part congruent to 2 modulo 4.

10. Extensions

In this last section we sketch three different directions for which the ideas of this work can be applied.

10.1. Combinatorial proof that characters are class functions. Stanton and White [28] show combinatorially that, if $c$ and $c'$ are 2 compositions verifying $\bar{c} = \bar{c}'$, then
then
\[ |RT_{\lambda,c}|_\pm = |RT_{\lambda,c'}|_\pm \] (9)

This expresses in fact that the value of the character \( \chi^\lambda \) on a permutation \( \sigma \) depends only on the conjugacy class of \( \sigma \): it is a class function. In this Section we give local rules that realize Stanton and White’s result, building on Fomin’s version of jeu de taquin explained for instance in his appendix to Stanley’s book [27]. We have the following proposition, whose proof can be easily done by using the encoding of partitions by infinite sequences explained in Appendix A.

**Proposition 27.** Let \( \lambda, \mu, \xi \) be three partitions, such that \( \mu/\lambda \) and \( \xi/\mu \) are nonempty ribbons. Then exactly one of the two following cases occur:

1. either there exists \( \nu \) such that \( \nu/\lambda \) and \( \xi/\nu \) are ribbons of respective sizes \( |\nu/\lambda| = |\xi/\nu| = |\mu/\lambda| \), or
2. there exists \( \hat{\mu} \) such that \( \hat{\mu}/\lambda \) and \( \xi/\hat{\mu} \) are ribbons of respective size \( |\hat{\mu}/\lambda| = |\mu/\lambda| \) and \( |\xi/\hat{\mu}| = |\xi/\mu| \).

Furthermore, we have the sign relations \( \varepsilon(\nu/\lambda)\varepsilon(\xi/\nu) = \varepsilon(\mu/\lambda)\varepsilon(\xi/\mu) \) in the first case, and \( \varepsilon(\hat{\mu}/\lambda)\varepsilon(\xi/\hat{\mu}) = -\varepsilon(\mu/\lambda)\varepsilon(\xi/\mu) \) in the second case.

We can now define the local rules: given \( \lambda, \mu, \xi \) as in the proposition, we draw them on a lozenge as on the left of Figure 11. If the first case of the proposition occurs, we erase \( \mu \), and write \( \nu \) on the right vertex of the lozenge; otherwise we replace \( \mu \) by \( \hat{\mu} \). We also define inverse local rules by a simple vertical symmetry. Finally, we define the trivial local rule which consists of simply moving the partition \( \mu \) from one side to the other.

Now we go from the local to the global as we did in Section 7. Consider the grid \( P_\ell \) on the right of Figure 11 made of lozenges. We attach to each lozenge coordinates \( (i, j) \), with \( 1 \leq i \leq \ell - 1 \) and \( i \leq j \leq \ell - 1 \) in the manner shown in the example. We fix a total order on lozenges, such that each lozenge \( (i, j) \) has to be bigger than the two lozenges on its top left and bottom left, namely \( (i-1, j) \) and \( (i, j-1) \) when they are defined. For the examples, we will use the following linear order: \( (i, j) \) is greater than \( (i', j') \) if \( i > i' \), or \( i = i' \) and \( j > j' \).

![Figure 11: Lozenge for local rules and associated grid for the global correspondence.](image-url)
Let us fix two compositions $c, c'$ of length $\ell$, such that $\tilde{c} = \tilde{c}'$. We will represent elements of $RT_{\lambda, c}$ on the left side of $P_\ell$, as chains of partitions labeling the vertices from bottom to top, and elements of $RT_{\lambda, c'}$ on the right side in the same fashion.

Now we want to select a subset of the lozenges so that when local rules are applied in the grid, we obtain indeed a correspondence between $RT_{\lambda, c}$ and $RT_{\lambda, c'}$. Fix $\sigma$ a permutation such that $\sigma(c) = c'$, and mark certain lozenges of $P_\ell$, in the following way: for $1 < j \leq \ell$, if $\sigma_j = k$, then mark the lozenges $(i, j - 1)$, $i \leq \ell - k$.

For instance, consider $c = (1, 3, 2, 1)$ and $c' = (3, 1, 1, 2)$, and fix the permutation $w = 3142$ which indeed verifies $w(c) = c'$; the corresponding marking is represented on the left hand side of Figure 12.

Now we start from a ribbon tableau represented on the left hand side, and go on performing local rules as in the beginning of Section 7, the rule being that we perform nontrivial local rules in the marked lozenges, while in the other (unmarked) ones we will use the trivial local rule. An example is given on Figure 13, the definition of the marked lozenges in the previous paragraph is thus made so that the size of ribbons “match” between $c$ and $c'$, thanks to Proposition 27, this can be visualized on the right hand side of Figure 12.

We remark that in the case when all lozenges are marked with a cross, then this corresponds to a generalization of Schützenberger’s involution which is the case where all ribbons are of size 1.
10.2. Other correspondences based on the graph $GR$. In the correspondence of Theorem 10, we considered pairs of ribbon tableaux of the same shape. As already noticed, these are very special paths in the ribbon graph $GR$: they start and end at $\emptyset$, going up $\ell$ steps and then down $\ell$ steps. The same ideas work to build correspondences for other kinds of paths, and we give an example that is well known in the standard case.

We consider the paths in $GR$ of length $2\ell$, that start and end at $\emptyset$, and which possess $\ell$ steps up and $\ell$ steps down (but in no imposed order). These are called oscillating tableaux (of shape $\emptyset$) in the case where all ribbons are of size 1, and we will thus call these paths oscillating ribbon tableaux.

The size of such a path is the half sum of the sizes of the $2\ell$ edges (i.e. ribbons), and the sign is the product of the signs of those ribbons. The oscillating tableau above has size 5, sign $+1$ and length 6. Let us note $\text{Osc}_{n,\ell}$ this signed set; we can then prove the following formula:

$$|\text{Osc}_{n,\ell}| = (2\ell - 1)! \binom{n + \ell - 1}{\ell - 1}$$

This can be done in two ways, algebraic and bijective, following the steps of what was done in the case of pairs of ribbon tableaux of the same shape.

The algebraic way to prove the identity is to notice that the quantity $|\text{Osc}_{n,\ell}|$ can be expressed as the coefficient of $q^{2n}$ in the series $\langle \emptyset, (D + U)^{2\ell} \emptyset \rangle$. Now this series is $(2\ell - 1)!\sum iq^{2i}$ (this a consequence of Corollary 2.6 (a) of [25]); then the end of the proof goes as for the second equality of Corollary [11].

The bijective way consists in constructing a signed bijection between $\text{Osc}_{n,\ell}$ and hook matchings of $[1, 2\ell]$ with size $n$: these are perfect matchings on $[1, 2\ell]$ such that to each pair $\{i, j\}$ of the matching is associated a hook $H_{\{i, j\}}$, such that the sum of the sizes of the $\ell$ hooks is $n$. Since there are $(2\ell - 1)!$ perfect matchings, Proposition [8] shows that there are $(2\ell - 1)!\binom{n + \ell - 1}{\ell - 1}$ such hook perfect matchings.

The bijective correspondence between $\text{Osc}_{n,\ell}$ and the hook matchings is done as in Roby [18]. We illustrate this on an example on Figure [14] instead of the grid $G_\ell$, we perform the bijection on a grid $T_\ell$ illustrated on the Figure by dashed lines for $\ell = 3$.

Hook matchings can be represented by labeling by $\emptyset$ the bottom and left side, and for each pair $\{i, j\}$ of the matching, the corresponding hook is drawn in the square of column $i$ from the left and row $j$ from the top. In the example, the matching is then $\{\{1, 3\}, \{2, 6\}, \{4, 5\}\}$. Oscillating ribbon tableaux $(\lambda_0 = \emptyset, \lambda_1, \ldots, \lambda_{2\ell}, \lambda_{2\ell} = \emptyset)$ are represented on the outside corners of the north east border, from top left to bottom right (moreover, in each of the corresponding inside corners, one draws the smallest shape between $\lambda_i$ and $\lambda_{i+1}$).

Now a signed correspondence goes along the exact same lines as what we did in Section [7] for pairs of ribbon tableaux of the same shape: we fix a total order on the squares of $T_\ell$, and apply local rules in these squares, changing directions when we encounter a rule $S$ or $T$. The example of Figure [14] does not present any occurrence of those last rules for the sake of simplicity.
10.3. **Layered graphs in duality.** The techniques used in this article are a generalization of Fomin’s framework developed in [6, 7]; we now present our theoretical setting in this paragraph.

Consider a graph \( G = (V, E) \) with a sign function on the edges \( \varepsilon : E \to \{+1, -1\} \). Suppose that \( V \) is the disjoint union of finite sets \( V_i, i \in \mathbb{N} \) where \( V_0 \) is a singleton \( \{O\} \). We will say that \( G \) is *layered* graph (with zero).

Let now \( U_i, D_j \) be the endomorphisms of \( K^V \) defined for \( v \in V_k \) by \( U_i(v) = \sum_e \varepsilon(e) v' \) where \( e \) goes through all edges from \( v \) to \( v' \in V_{k+i} \), and \( D_j \) is defined dually from \( K^V \) to \( K^V \) for \( i \neq j \).

Let us say that the layered graph \( G \) is *self dual* if there exist nonnegative integers \( \alpha_i \) such that:

\[
D_i U_i = U_i D_i + \alpha_i \cdot \text{Id} \\
D_i U_j = U_j D_i \quad \text{if } i \neq j
\]

Fomin’s framework of self dual graded graphs is the case where edges exist only between consecutive levels \( V_i \) and \( V_{i+1} \), and when the sign function is constant equal to 1.

It is then possible to use the algebraic techniques of this work to study the enumeration of paths in such graphs. The relations above correspond to certain equalities of signed cardinals, as in the graph \( GR \). If signed bijections proving these equalities are fixed, then we can determine global correspondences in the same way. But it obviously remains to see if there exists interesting examples to which this theory can be applied.
An important remark is that this generalizes only Theorem 10 and its Corollary; to give an extension of Theorem 12, one needs additional local properties on the self-dual layered graph, analogous to Proposition 23 in the case of $GR$.

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This appendix will show how the encoding of partitions by words is well suited to the study of the operations on ribbons of Section 2.2. We follow van Leeuwen [30] for notations. Let \( \lambda \) be a partition, and \( \delta(\lambda) \in \{0,1\}^\mathbb{Z} \) the sequence defined by the following procedure: we extend the top and left borders of a Ferrers diagram to infinity, and read the lower right boundary from bottom to top, recording 1 for every vertical edge encountered, and 0 for the horizontal edges.

For instance, the partition \((4,2,2,1)\) has for coding word \((\cdots 1110101|1001000 \cdots)\), cf. Figure 15: Infinite word encoding a partition. The sign “|” separates the parts of the border below and above the diagonal of the diagram, and we consider that nonnegative indices of \( \delta(\lambda) \) are those on the right of |. Notice that the encoding sequences have the following characteristic properties (see [30]):

1. they differ from \((\cdots 1111|0000 \cdots)\) (corresponding to the empty partition) in a finite number of positions;
2. the number of 0s to the left of | is equal to the number of 1s to its right.

Now, ribbons addable to \( \lambda \) (respectively removable from \( \lambda \)) are in bijection with pairs of indices \((i,j)\) in \( \mathbb{Z} \) where \( i < j \) such that \( \delta_i(\lambda) = 1 \) and \( \delta_j(\lambda) = 0 \) (resp.
Lemma 28. Let $\lambda$ be a partition with associated sequence $\delta(\lambda)$, and $i < j$ indices of $\delta(\lambda)$ corresponding to a ribbon $r$ addable to (or removable from) $\lambda$: this just means that $\{\delta_i(\lambda), \delta_j(\lambda)\} = \{0, 1\}$. Then

1. the size of $r$ is $|r| = j - i$.
2. the height $h$ of $r$ is the number of 1 in $\delta(\lambda)$ between the indices $i$ and $j$, i.e. $h = \{| k \in \mathbb{Z} \mid i < k < j \text{ and } \delta_k(\lambda) = 1 \}$. 

Proof: It goes simply by using the fact that the 1s correspond to vertical steps on the boundary of $\lambda$, and the 0s to horizontal ones; so $j - i$ is equal to the number of cells occupied by $r$, and each 1 between $i$ and $j$ corresponds to going up from one row to another in the ribbon. \hfill $\square$

Now the data of $\lambda, \mu, \nu$ (when $\mu, \nu \neq \lambda$) in a direct rule is equivalent to the data of $\delta(\lambda)$ and integers $i_1 < j_1, i_2 < j_2$ (corresponding to $\mu/\lambda$ and $\nu/\lambda$), where $\delta_{i_1}(\lambda) = 1, \delta_{j_1}(\lambda) = 0$ and $\delta_{i_2}(\lambda) = 1, \delta_{j_2}(\lambda) = 0$. Every operation of Section 2.2 can be in fact easily explicated given this representation; we shall do it for the switchout operation of rule $S$.

The rule $S$ applies precisely when one of the following two cases occur:

1. $i_1 = i_2, j_1 \neq j_2$ and $\delta_{j_1 + j_2 - i_1}(\lambda) = 1$, or
2. $j_1 = j_2, i_1 \neq i_2$ and $\delta_{i_1 + i_2 - j_1}(\lambda) = 0$

Let $i$ be the common value of $i_1$ and $i_2$ in the first case, and $j$ the common value of $j_1$ and $j_2$ in the second case. Then applying rule $S$ consists simply in defining $\tilde{\lambda}$ as the partition whose code is obtained from $\delta(\lambda)$ by exchanging 0 and 1 at positions $i, j_1, j_2$ and $j_1 + j_2 - i$ in the first case, and at positions $i_1, i_2, j$ and $i_1 + i_2 - j$ in the second case. On Figure 17 the first case is illustrated: $\lambda$ is shown above with the ribbons, and the applications of rule $S$ gives the partition below. (The symbols ’x’ represent indifferently 1s or 0s).

Figure 16: Addition of a ribbon on $\delta(\lambda)$. 

$\delta_i(\lambda) = 0$ and $\delta_j(\lambda) = 1$: $i$ indicates the position of the head, and $j$ the position of the tail. The partition $\mu$ obtained after addition or removal of the ribbon is the result of the exchange of 0 and 1 at positions $i$ and $j$.

One may think of $\delta(\lambda)$ as a configuration of particles on the infinite discrete line: the 1s represent particles, and 0s represent empty positions. So moving a particle in an empty position to its left (respectively right) corresponds to removing (resp. adding) a ribbon. Figure 16 shows a ribbon $\lambda/\mu$ by an arrow between its tail and head, and the codes of $\lambda$ and $\mu$ are given on the right.

One has then the following result:

One may think of the 1s as corresponding to vertical steps on the boundary of $\lambda$, and the 0s to horizontal ones; so $j - i$ is equal to the number of cells occupied by $r$, and each 1 between $i$ and $j$ corresponds to going up from one row to another in the ribbon. \hfill $\square$
One notices that the partition $\hat{\lambda}$ is element of $D_{j_1-i}(\mu,\nu)$, as was $\lambda$: indeed, one gets the same partition by moving a particle following the long arrow (resp. the short arrow) in both partitions of Figure 17. The rule is clearly an involution, and in fact exchanges cases 1. and 2. defined above.

Now we will check finally that it exchanges signs, which is just a matter of counting particles, thanks to Lemma 28. Let $a$ (respectively $b$, $c$) be the number of 1 in $\lambda$ that are strictly between the indices $i$ and $j_1$ (resp. $j_1$ and $j_2$, resp. $j_2$ and $j_1 + j_2 - i$); these numbers are the same in $\lambda$ and $\hat{\lambda}$. As elements of the signed set $D_{j_1-i}(\mu,\nu)$, $\lambda$ and $\hat{\lambda}$ have signs $(-1)^x$ and $(-1)^y$, with $x = 2a + b$ and $y = b + 2c + 1$; this gives opposite signs since $x$ and $y$ have opposite parity. 

**Appendix B. Proof of Theorems [10] and [12]**

In this Section we will give the proofs of Theorems [10] and [12], the signed bijections having been defined in Section [7]. The proof is directly inspired by Fomin’s constructions [7], but some extra technicalities are needed in both proofs.

**B.1. Proof Of Theorem [10]** We will show in particular that the construction $\phi$ defined algorithmically in [7.1] verifies indeed all properties stated in the theorem. We will demonstrate that this algorithm is in fact a consequence of Garsia and Milne’s involution principle: therefore, we have to construct signed sets $A, B$ and adequate functions. For this, a certain number of concepts have to be defined.

We call *border* of the grid $G_\ell$ a path from the top left vertex to the bottom right one, with South and East steps. We call *inside* of the border $F$ the squares of $G_\ell$ to the south west of $F$, and *outside* the rest of the squares.

A *good labeling* of a border $F$ is the labeling of each of its vertices by a partition, such that:

- the vertices at the top left and bottom right are labeled by the empty partition $\emptyset$.
- for every horizontal edge of $F$, the labels $\lambda$ and $\mu$ at the left and right end respectively form a ribbon $\mu/\lambda$. 

![Figure 17: The rule S.](image)
for every vertical edge of $F$, the labels $\lambda$ and $\mu$ at the bottom and top end respectively form a ribbon $\mu/\lambda$.

Remember that we fixed a total order $O$ on the squares of $G_\ell$. Let a $O$-border be a border $F$ when the squares inside $F$ are smaller than the squares outside. Such a border defines two squares in general: $sq_Q(F)$ which is the largest square inside $F$, and $sq_P(F)$ the smallest one outside; $sq_Q(F)$ is not defined when $F$ consists of the left and bottom side of the grid, and $sq_P(F)$ is not defined when $F$ consists of the top and right side of the grid.

Let $F$ be a border with a good labeling label: note that label induces a labeling by ribbons on the edges of $F$. Suppose that certain squares of the grid are filled (we will also say colored) by nonempty hooks. Such a coloring col is compatible with $(F,label)$ if:

- The squares inside $F$ are not filled.
- for every horizontal edge $h$ of $F$ labeled by $r$, there is exactly one square filled by a hook in the column above $h$ when $r$ is empty, and none if $r$ is non empty.
- for every vertical edge $v$ of $F$ labeled by $r$, there is exactly one square filled by a hook in the row right of $v$ when $r$ is empty, and none if $r$ is non empty.

**Configurations.** We can now introduce configurations, which are the main objects we will consider for the rest of the proof

**Definition 29.** A configuration is a 3-tuple $(F,label,col)$ where $F$ is a $O$-border, which is well labeled by label, and col is a coloring of $G_\ell$ compatible with $(F,label)$.

Let $(F,label,col)$ be a configuration on $G_\ell$. For each $i \in [1,\ell]$ we note $c_i > 0$ the size of the hook in column $i$, or the size of the ribbon labeling the edge of $F$ appearing in column $i$: by compatibility of label and col, exactly one of these two cases occur. Likewise, we note $c'_i$ the size of the hook in row $i$, or the size of the ribbon labeling the edge of $F$ appearing row $i$. The content of a configuration is then defined as the two compositions $c, c'$, where $c = (c_1, \ldots, c_\ell)$ and $c' = (c'_1, \ldots, c'_\ell)$.

Let us define the sign of a configuration $(F,label,col)$ as the product of all $2\ell$ ribbons labeling the edges of $F$ (recall that the empty ribbon has sign +1). For instance, the steps C and E of Figure 10 show two configurations: C and E have both content $((2,1,2),(2,2,1))$, and C has negative sign, whereas E has positive sign.

A hook permutation $(\sigma,H)$ is a positive configuration: the border is the left and bottom side of $G_\ell$, all vertices are labeled by $\emptyset$, and the coloring is just the representation of $(\sigma,H)$ in the grid. The content of such a configuration is $(c(H),c(H,\sigma^{-1}))$. A pair $(P,Q)$ of ribbon tableaux of length $\ell$ with the same shape is also a configuration: the border is the top and right side, the right side being labeled by $P$ and the top side by $Q$; and all squares of the grid are empty. The sign of the configuration is $\varepsilon(P)\varepsilon(Q)$, and the content is $c(Q),c(Q)$. We will from now on identify hook permutations and pairs of ribbon tableaux to such configurations.

We now notice that in the algorithm describing our correspondence, the transformation ApplyLocalRule entails a change from one configuration to another. Indeed, an inspection of each local rule shows that the compatibility conditions in the definition are indeed respected by this transformation.
Application of the Involution principle. A configuration \((F, label, col)\) is of type \(A\) if rule \(T\) has to be applied in \(sq_<(F)\), or if it is a permutation (which is when \(sq_<(F)\) is not defined). It is of type \(B\) if rule \(S\) has to be applied in \(sq_>(F)\), or if it is a pair of ribbon tableaux (which is when \(sq_>(F)\) is not defined).

We fix two compositions \(c_1\) and \(c_2\) of length \(\ell\) and size \(n\). Let \(A\) (respectively \(B\)) be the configurations of type \(A\) (resp. \(B\)) and content \((c_1, c_2)\). For instance, the configurations \(C\) and \(E\) of Figure 10 are of type \(B\) and \(A\) respectively, for the content \((2, 1, 2), (2, 2, 1)\).

Let us define an involution on \(A\) by applying the rule \(T\) in \(sq_<(F)\) when it is defined, and letting the hook permutations unchanged. Likewise, we have an involution on \(B\) by applying the rule \(S\) in \(sq_>(F)\) when it is defined, and letting the pairs of tableaux unchanged. Notice that these two involutions are sign reversing by Proposition 13. We define also a bijection \(A \rightarrow B\) by applying direct rules \(sq_>(F)\) on a type \(A\) configuration, until we reach a type \(B\) configuration. Since in this case we only apply rules of the form \(D_i\), the sign is preserved, thanks to Proposition 13 again.

We now have all the functions verifying the involution principle of Garsia and Milne: this gives us a signed bijection between hook permutations and pairs of ribbon tableaux that verifies exactly the properties of Theorem 10, which completes the proof.

\[\square\]

B.2. Proof of Theorem 12. We will use here the definitions used in the proof of Theorem 10 and adapt them to the case of involutions. We consider a total order \(\mathcal{HO}\) on \(HG_\ell\), the half grid made of the squares \((i, j)\) of \(G_\ell\) with \(i \geq j\), and suppose that this order extends the order \(\preceq\), defined by \((i, j) \preceq (i', j')\) if \(i \leq i'\) and \(j \leq j'\) as seen before. This induces a partial order on the squares of \(G_\ell\), given by \((i, j)\) is smaller than \((i', j')\) if and only if \((\min(i, j), \max(i, j))\) is smaller than \((\min(i', j'), \max(i', j'))\) for \(\mathcal{HO}\). We will still note \(\mathcal{HO}\) this partial order.

We consider configurations as in Definition 29, with the exception that the order \(\mathcal{HO}\) is now used. A configuration \((F, label, col)\) on \(G_\ell\) is called symmetric if \(F, label\) and \(col\) are all symmetric with respect to the diagonal \(i = j\), and we name half-configuration the restriction to \(HG_\ell\) of a symmetric configuration. Note that the border \(f\) of a half-configuration consists of a path form the top left corner of \(HG_\ell\) to any vertex of the diagonal \(i = j\). Figure 18 shows an example of half configuration.

For a half configuration, applying a local rule in the square \((i, j)\) \in \(HG_\ell\) means applying it in both \((i, j)\) and \((j, i)\) in the associated symmetric configuration, and restrict the result to \(HG_\ell\). Note that local rules in \((i, j)\) and \((j, i)\) will give the same outputs since all our local rules are symmetric in \(\mu\) and \(\nu\).

We represent ribbon tableaux by a chain of partitions on the top side of \(HG_\ell\), and hook involutions by the restriction of their matrix representation to \(HG_\ell\) with \(\emptyset\) labeling the vertices on the left. The content \(c\) of a symmetric configuration is of the form \((c, c)\), so we define the content of the associated half configuration as \(c\). The sign of a half configuration \((f, label, col)\) is the product of all the signs labeling \(f\), multiplied by the product of the signs of hooks appearing in the square \((i, i)\). Note that this gives the desired sign on \(\mathcal{HI}\) and on ribbon tableaux, so that we are indeed in the setting of Theorem 12.

Let us note \(\mathcal{HA}\) and \(\mathcal{HB}\) the sets of half configurations with associated symmetric configurations in \(A\) and \(B\) respectively, with the partial order \(\mathcal{HO}\). We define the
involutions on $\mathcal{HA}$ and $\mathcal{HB}$ in the same fashion as for $A$ and $B$, as well as the bijection between $\mathcal{HA}$ and $\mathcal{HB}$.

Now we wish to apply once again the involution principle: we have here to check the sign modifications, since the definition of the sign of a configuration has been modified.

For the non diagonal squares $(i,j)$ with $i > j$, everything works as before: the application of a rule $S$ or $T$ changes the sign, whereas the other rules preserve it.

Now there remains the application of a rule on a diagonal square $sq$. First we notice that only rules D1-D3 and I1-I3 can be applied there, since they are the only rules with $\mu = \nu$. One needs to prove that the sign of the ribbon on the left side of $sq$, times the sign of the hook in $sq$ in the case of D2, is equal to the sign of the ribbon on the top side of $sq$.

This is trivial for the rule D1. For the rules D2 and D3, one has to look at the definition of $\text{prev}$ and $\text{first}$ (Section 2.2): $\text{first}(\lambda, eq)$ is of the same height as $eq$ by definition, and thus of the same sign: this implies that rule D2 will indeed have the sign preserving property. For the rule D3, $\text{next}(\mu, \mu/\lambda)$is a ribbon of the same height as $\mu/\lambda$, and so of the same sign, which implies here also that the rule preserves the sign. The involution principle can thus be applied, and this achieves the proof of Theorem 12.
