Differential Renormalization of QCD in the Background-Field Method†

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A short outline is given on the application of differential regularization to QCD in the background-field method. The derivation of the propagators in the background gluon field as short-distance expansions is described and the renormalization of the theory is mentioned.

I. Introduction

Loop calculations in quantum field theories are generally plagued by infinities, and hence a regularization procedure is needed. The most commonly-used method is dimensional regularization, which respects gauge and Lorentz symmetries, but it is not suitable for chiral gauge theories like the electroweak theory or for calculations involving polarized particles.

In this talk I will give a short outline on the application of a regularization method known as differential regularization (DR) [1] to QCD. This method was extensively studied in recent years and was demonstrated to be applicable to chiral theories. It enjoys the property that no counterterms are needed in the renormalization procedure, and hence the number of Feynman diagrams needed for a given loop calculation in DR is generally less than that in any other known regularization method.

Here I will consider DR for QCD in the background-field method [2-7], which is a useful computational tool that allows one to compute radiative corrections while maintaining manifestly the symmetries of the theory under consideration. However, we should note that the calculational procedure in [1] using DR exhibits some inconveniences, such as that fixing the renormalization scale in gauge theories requires the implementation of Ward identities. I will present a calculational background-field approach with DR to study QCD, so as to eliminate the cumbersome step of implementing Ward identities in the calculations of Green functions. It is hoped that the background-field approach with DR could provide a more systematic method for calculating higher-loop orders in QCD. This talk is thus organized as follows. In section II, I will give a derivation of a short-distance expansion for the gluon propagator in the background gluon field, and then obtain the one-loop effective action. In section III, I will briefly mention the use of the method of differential regularization for loop calculations. Finally, a discussion section is also included.

II. Propagators in the background gluon field

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Consider the action for the gluon field in Euclidean space

\[
S[A] = \int d^4x \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu},
\]

\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu.
\]

In the background-field method \([2-7]\), we let \(A^a_\mu = B^a_\mu + Q^a_\mu(x)\) with \(B^a_\mu(x)\) being the background gluon field and \(Q^a_\mu(x)\) the quantum fluctuation gluon field. To quantize system (1), we use the background-field gauge \(F^a_{\mu\nu}[Q] = D^{ab}_{\mu\nu}[B]Q^b_\mu(x) = 0\) and consider adding to (1) the gauge-fixing term \(S_{gf}[Q] = \frac{1}{2} \int d^4x F^a_{\mu\nu}[Q] F^a_{\mu\nu}[Q]\), which corresponds to the Feynman gauge. Then we have

\[
S[B + Q] + S_{gf}[Q] = S[B] + \int d^4x \left( \frac{1}{2} (D^{ab}_{\mu\nu} Q^b_\mu)^2 + gf^{abc} G^a_{\mu\nu} Q^b_\mu Q^c_\nu \right)
\]

\[
+ gf^{abc} (D^{ad}_{\mu\nu} Q^d_\mu) Q^b_\mu Q^c_\nu
\]

\[
+ \frac{1}{4} g^2 f^{abc} f^{ade} Q^b_\mu Q^c_\nu Q^d_\mu Q^e_\nu,
\]

where \(G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc} B^b_\mu B^c_\nu\) is the field strength for the background gluon field. The corresponding Faddeev-Popov ghost term reads

\[
S_{gh}[B, Q, \bar{\eta}, \eta] = \int d^4x \bar{\eta}^a(x) D^{ac}_{\mu\nu}[B] D_{\mu\nu}[B + Q] \eta^c(x)
\]

\[
= \int d^4x \bar{\eta}^a(x) (D^2[B])^{ab} \eta^b(x)
\]

\[
- gf^{cbe} \int d^4x \bar{\eta}^a(x) D^{ac}_{\mu\nu}[B] Q^c_\mu(x) \eta^b(x),
\]

where \(\bar{\eta}^a(x)\) and \(\eta^a(x)\) are the Faddeev-Popov ghosts. From (2), the gluon propagator in the background gluon field reads

\[
\langle Q^a_\mu(x) Q^b_\nu(y) \rangle = \langle x \left| \left( \frac{1}{P^2 \delta_{\mu\nu} - 2G_{\mu\nu}} \right)^{ab} \right| y \rangle \equiv D^{ab}_{\mu\nu}(x, y),
\]

\[
P^{ab}_\mu = -i D^{ab}_\mu = -i \left( \partial_\mu \delta^{ab} + B^{ab}_\mu \right),
\]

where \(G^{ab}_{\mu\nu} = gf^{acb} G^c_{\mu\nu}\) and \(B^{ab}_\mu = gf^{acb} B^c_\mu\); while from (3), the ghost propagator reads

\[
\langle \bar{\eta}^a(x) \eta^b(y) \rangle = \langle x \left| \left( \frac{1}{P^2} \right)^{ab} \right| y \rangle.
\]

For perturbative calculations, we need the gluon propagator (4), the ghost propagator (5) and the Feynman rules for the vertices, which can be derived from (2) and (3).
Since we are interested in the ultra-violet divergences, we look for the short-distance expansions for propagators (4) and (5). We consider an expansion for the gluon propagator (4) in the following form \((z \equiv x - y)\) \[4\]:

\[
D_{\mu\nu}^{ab}(x,y) = \frac{1}{4\pi^2} \left( \frac{1}{z^2} U_{\mu\nu}^{ab}(x,y) + V_{\mu\nu}^{ab}(x,y) \ln(M^2 z^2) + W_{\mu\nu}^{ab}(x,y) \right),
\]

so that it can be used in conjunction with DR method for loop calculations. In (6), \(U(x,y), V(x,y)\) and \(W(x,y)\) are analytic functions, which will be expanded in series of \(x\) and \(y\), and the parameter \(M\) plays the role of a subtraction point.

The gluon propagator satisfies

\[
\Delta_{\alpha\beta}^{ab}(x)D_{\beta\sigma}^{bc}(x,y) = \delta^4(x - y)\delta_{\alpha\sigma}\delta_{\beta\epsilon},
\]

where

\[
\Delta_{\alpha\beta}^{ab}(x) = [P^2\delta_{\alpha\beta} - 2G_{\alpha\beta}]^{ab}(x)
\]

\[
= -\Box \delta_{\alpha\beta} \delta_{ab} - \left[ \nabla_\mu B_\mu(x) + 2B_\mu(x)\nabla_\mu + B_\mu^2(x) \right]^{ab}(x)
\]

To determine \(U(x,y), V(x,y)\) and \(W(x,y)\), we employ the Fock-Schwinger gauge \(x_\mu B_\mu(x) = 0\) to express \(B_\mu(x)\) in terms of \(G_{\mu\nu}(x)\). We can then rewrite (8), in matrix notation, as

\[
\Delta(x) = -\Box - X_\mu(x)\nabla_\mu - Y(x),
\]

where \(X_\mu(x)\) and \(Y(x)\) are expanded about \(x = 0\) as follows:

\[
X_\mu(x) = x_\alpha G_{\alpha\mu} + \frac{2}{3} x_\alpha x_\beta D_\alpha G_{\beta\mu} + \frac{1}{4} x_\alpha x_\beta x_\rho D_\alpha D_\beta G_{\rho\mu}
\]

\[
+ \frac{1}{15} x_\alpha x_\beta x_\rho x_\sigma D_\alpha D_\beta D_\rho G_{\sigma\mu} + \mathcal{O}(x^5),
\]

\[
Y(x) = 2G + 2x_\alpha D_\alpha G + x_\alpha x_\beta a_{\alpha\beta} + x_\alpha x_\beta x_\rho b_{\alpha\beta\rho}
\]

\[
+ x_\alpha x_\beta x_\rho x_\sigma c_{\alpha\beta\rho\sigma} + \mathcal{O}(x^5);
\]

\[
a_{\alpha\beta} = \frac{1}{8} D_\gamma D_\alpha G_{\beta\gamma} + \frac{1}{4} G_{\alpha\gamma} G_{\beta\gamma} + D_\alpha D_\beta G,
\]

\[
b_{\alpha\beta\rho} = \frac{1}{30} [D_\gamma D_\alpha + D_\alpha D_\gamma] D_\beta G_{\rho\gamma} + \frac{1}{6} [G_{\alpha\gamma} D_\rho + (D_\rho G_{\alpha\gamma})] G_{\beta\gamma} + \frac{1}{3} D_\alpha D_\beta D_\rho G,
\]

\[
c_{\alpha\beta\rho\sigma} = \frac{1}{144} [D_\gamma D_\alpha D_\beta + D_\alpha D_\beta D_\gamma + D_\alpha D_\beta D_\gamma] D_\rho G_{\sigma\gamma} + \frac{1}{9} (D_\rho G_{\alpha\gamma}) (D_\sigma G_{\beta\gamma})
\]

\[
+ \frac{1}{16} [G_{\alpha\gamma} D_\rho D_\sigma + (D_\rho D_\sigma G_{\alpha\gamma})] G_{\beta\gamma} + \frac{1}{12} D_\alpha D_\beta D_\rho D_\sigma G.
\]
Here $G$ is a matrix with elements $g^{a b}G^a_{\mu \nu}$. In obtaining (10) and (11), we have used the equations of motion $D_\mu G_{\mu \nu} = 0$.

Solving (7), we get

$$U(x, y) = I + \frac{1}{2} x_\alpha y_\beta G_{\alpha \beta}(0) + \frac{1}{6} x_\alpha (x_\beta + y_\beta) y_\mu D_\alpha G_{\beta \rho}(0) + \cdots,$$

$$V(x, y) = -\frac{1}{2} G(0) - \frac{1}{4} (x_\alpha + y_\alpha) D_\alpha G(0) + x_\alpha x_\beta V_{20\alpha \beta} + y_\alpha y_\beta V_{02\alpha \beta} + x_\alpha y_\beta V_{11\alpha \beta} + \cdots,$$  \hspace{1em} (12)

where

$$V_{20\alpha \beta} = V_{02\alpha \beta} = \frac{1}{8} \delta_{\alpha \beta} G(0) G(0),$$

$$V_{11\alpha \beta} = \frac{1}{8} G(0) G(0) \delta_{\alpha \beta} + \frac{1}{192} G_{\alpha \rho}(0) G_{\beta \rho}(0)$$

$$- \frac{\delta_{\alpha \beta}}{4} \left( \frac{3}{2} G(0) G(0) + \frac{1}{16} G_{\rho \sigma}(0) G_{\rho \sigma}(0) + \frac{1}{4} D_\rho D_\rho G(0) \right).$$

It can be seen that the function $W(x, y)$ is intimately related to $U(x, y)$ and $V(x, y)$. For example, by rescaling the parameter $M$, $W(x, y)$ can change by an amount proportional to $V(x, y)$. Here $W(x, y)$ is not needed.

The short-distance expansion for the ghost propagator can be obtained simply from the gluon propagator with $G(x) = 0$.

We can now use the short-distance expansions for the propagators to obtain the effective action up to one-loop order,

$$\Gamma[B] \approx S[B] + \Gamma^{(1)}[B],$$

where $\Gamma^{(1)}[B]$ is the one-loop effective action \cite{3,5}

$$\Gamma^{(1)}[B] = \frac{1}{2} \ln \text{Det} D^{-1}_{\text{gluon}} - \ln \text{Det} D^{-1}_{\text{ghost}},$$

with $D^{-1}$ denoting the inverse of the propagator $D$. From the logarithmic terms in the propagators, we obtain the derivative of the renormalized one-loop effective action with respect to $\ln M^2$:

$$\frac{d \Gamma^{(1)}[B]}{d \ln M^2} = \frac{1}{2} \frac{11}{96\pi^2} C_A \int d^4 x G^a_{\rho \sigma}(x) G^a_{\rho \sigma}(x) - \frac{11}{96\pi^2} C_A \int d^4 x G^a_{\rho \sigma}(x) G^a_{\rho \sigma}(x)$$

$$= -\frac{11}{48\pi^2} C_A \frac{1}{4} \int d^4 x G^a_{\rho \sigma}(x) G^a_{\rho \sigma}(x).$$\hspace{1em} (16)
Finally, the one-loop $\beta$ function can be easily obtained by considering the renormalization group equation for $\Gamma[B]$, 

$$M \frac{\partial \Gamma[B]}{\partial M} + \beta(g) \frac{\partial \Gamma[B]}{\partial g} = 0,$$

which gives

$$\beta(g) = -\frac{11}{48\pi^2} C_A g^3.$$  

Next, for simplicity, we briefly consider the massless quark Lagrangian density

$$\mathcal{L}[\bar{\psi}, \psi, A^a_\mu] = i\bar{\psi}\gamma^\mu D^\mu(A^a_\mu)\psi.$$  

In the background-field method, we write

$$\mathcal{L}[\bar{\psi} + \bar{q}, \psi + q, B^a_\mu + Q^a_\mu] = \mathcal{L}[\bar{\psi}, \psi, B^a_\mu] + i\bar{q}\gamma_\mu D_\mu(B^a_\mu)q$$

$$+ g(\bar{\psi}\gamma_\mu Q^a_\mu T^a q + \bar{q}\gamma_\mu Q^a_\mu T^a \psi)$$

$$+ g\bar{q}\gamma_\mu Q^a_\mu T^a q,$$

where $\bar{\psi}$ and $\psi$ are the background quark fields, while $\bar{q}$ and $q$ are the quantum fields. $D_\mu(B^a_\mu) = \partial_\mu - igB^a_\mu T^a$ is the covariant derivative in the background gluon field $B^a_\mu$ in the fundamental representation with $T^a$ being the generators.

The massless quark propagator in the background gluon field is given by

$$S(x, y) = \langle x|\frac{1}{\gamma_\mu P_\mu}\mid y\rangle, P_\mu = -iD_\mu(B^a_\mu).$$  

The short-distance expansion for the quark propagator is more complicated than that for the gluon's but can be found in many references [2-4,7]. I shall refrain from listing its expression and proceed to the next section on the use of DR method for higher-loop calculations or for operator renormalization.

**III. The method of differential regularization**

We saw that the one-loop effective action can be obtained by taking the determinants of the propagators in the background field. However, calculations of the effective action to higher-loop orders and of the evolution of operators contain highly-singular terms of the form

$$\frac{1}{(z^2)^n} \ln^m(M^2 z^2), n \geq 2, m \geq 0,$$

where $M$ is a mass parameter, which may be that in the expansions for the propagators. The essential idea of DR method is to define the highly-singular terms by

$$\frac{1}{(z^2)^n} \ln^m(M^2 z^2) = \underbrace{\square \cdots \square}_{n-1} G(z^2), z^2 \neq 0,$$  

where $\square$ is the infinite propagator.
and to solve for the function \( G(z^2) \), which has a well-defined Fourier transform and depends on \( 2(n - 1) \) integration constants.

We list below the regularized expressions that are used in the loop calculations:

\[
\frac{1}{z^4} = -\frac{1}{4} \Box \ln(z^2 M^2), \\
(24)
\]

\[
\frac{1}{z^6} = \Box \Box H(z^2),
(25)
\]

\[
H(z^2) = -\frac{1}{32} \ln \left( \frac{z^2 M_1^2}{z^2} \right) + M_2^2 \ln(z^2 M_3^2) + M_4 z^2,
\]

\[
\frac{1}{z^4} \ln(z^2 M^2) = \Box I(z^2),
(26)
\]

\[
I(z^2) = -\frac{1}{8} \left[ \ln(z^2 M^2) \right]^2 + 2 \ln(z^2 M_1^2) + M_2,
\]

where \( M_1, M_2, M_3 \) and \( M_4 \) are arbitrary integration constants, which will be fixed by imposing gauge invariance. In the present background-field method, to fix a scale from the many integration constants, we only need to consider the gauge invariance of the background field, and not to use Ward identities as mentioned in [1]. The renormalized effective action at a given loop order is then obtained by considering its derivative with respect to \( \ln M^2 \) (cf. (16)).

**VI. Discussion**

In this talk I have briefly outlined the procedure of using differential regularization for loop calculations in the background-field method with the aim of providing a more systematic approach. It amounts to first writing the propagators in the background gluon field as short-distance expansions in Euclidean space. From the expansions, the one-loop effective action can be immediately obtained. The propagators are expanded in such a way that they can be used in conjunction with DR for loop calculations. We note that the present approach enjoys an advantage of avoiding the use of Ward identities for maintaining the gauge invariance of any loop calculations. The approach is not limited to one-loop calculations, and the number of Feynman diagrams needed for a given loop calculation in DR method is generally less than that in any other regularization method, since DR needs no counterterms.

We should note that for evaluating the effective action to one-loop order, there is no need to use DR, and the short-distance expansion procedure applies in either Euclidean or Minkowski space. Only when we calculate the two-loop or higher-loop orders that we need to regularize the singular terms, and the use of DR requires working in Euclidean space.

The approach mentioned in this talk can naturally be used for studying operator renormalization to higher-loop orders, for which one can easily extend the method in [2-4,6,7].
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