CENTRAL LIMIT THEOREMS WITH A RATE OF CONVERGENCE FOR SEQUENCES OF TRANSFORMATIONS

OLLI HELLA

Abstract. Using Stein’s method, we prove an abstract result that yields multivariate central limit theorems with a rate of convergence for time-dependent dynamical systems. As examples we study a model of expanding circle maps and a quasistatic model. In both models we prove multivariate central limit theorems with a rate of convergence.

Acknowledgements. I thank my Ph.D. advisor Mikko Stenlund for many helpful advices and suggestions he gave me during the research and writing process of this paper. I also thank the Jane and Aatos Erkko Foundation, and Emil Aaltosen Säätiö for their financial support.

1. Introduction

Time-dependent dynamical systems have gathered a lot of interest recently, see for example [1–5,11–14,19,20,23–25,29,30,34–36,38,45–50] and for older papers [6,27,28]. In this paper we approach time-dependent systems by first providing abstract results estimating the distributions of sums of random vectors and variables. The sum of random variables (vectors) is nearly (multi)normally distributed, when certain decay of correlations properties are satisfied. These conditions are specifically designed so that they can be applied to time-dependent dynamical systems yielding CLTs with a rate of convergence.

To be more precise, the setting we study in this paper is the following: Let \((X, \mathcal{B}, \mu)\) be a probability space and \(f : X \to \mathbb{R}^d\) a function, where \(d \geq 1\), and let \(T_1, T_2, \ldots\) be measurable transformations on \((X, \mathcal{B})\). We denote \(\bar{T}_k = T_k \circ \ldots \circ T_1\) and define \(\bar{T}_0 = \text{Id}\). We study the problem of approximating the distribution of normalized and centered Birkhoff sum

\[
W(N) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f \circ \bar{T}_k - \mu(f \circ \bar{T}_k)
\]

by a normal distribution of \(d\) variables. The transformations \(T_i, i = 1, 2, \ldots\), do not have to preserve the measure \(\mu\). This restricts the methods that can be used to prove CLTs for the system in question. Denoting \(\bar{f}^i = f \circ \bar{T}_i - \mu(f \circ \bar{T}_i)\) we may view \(W\) as a normalized sum of random variables \(\bar{f}^i\). We show that a method in probability theory, introduced by Stein in [44], can be adapted to this setting.

Stein’s method has been researched a lot in probability theory, see [8–10,15,16,18,32,33,37,40–42], but has mostly been neglected in theory of dynamical systems. Without obtaining convergence rates, the method is applied to some special cases in [17] and [26], but to our knowledge the first systematic treatment of Stein’s method in the context of dynamical systems was not done until recently in [21]. The results of [21] were then

Key words and phrases. Stein’s method, multivariate normal approximation, time-dependent dynamical systems, quasistatic dynamical systems.

2010 Mathematics Subject Classification. 60F05; 37A50.
applied in [31] for non-uniformly expanding maps. In this paper the results of [21] are
generalized to be applicable for time-dependent systems.

In Section 2 we state two theorems; one concerning random vectors \( W \) and second
concerning random variables. The proofs of these results are given in Section 7. These
are then applied in two models demonstrating the usefulness of this method. Applications
to time-dependent expanding circle maps and in a quasistatic model introduced in [12] are
stated in Sections 3 and 4, respectively. We also give a quenched CLT result concerning
randomly selected circle expanding maps in Section 3. The results for the applications
are proved in Sections 5 and 6.

This paper uses some of the results and proofs in aforementioned papers [21] and [12].
We also make certain improvements to those results. The main theorems have some
similarity to Pène’s results in [39], which uses an adapted Rio’s method. However we
work in a time-dependent setting, while in [39] the system in question is assumed to be
stationary.

Notations and conventions. Through the paper we reserve the letter \( Z \) for a random
variable with the standard normal distribution. We write \( C = C(x_1, \ldots, x_n) \), when \( C \) is a
constant whose numerical value can be calculated from the variables \( x_1, \ldots, x_n \).

Various norms are used through the paper. For a vector \( v \in \mathbb{R}^d \) with components \( v_\alpha \),
\( \alpha = 1, \ldots, d \), we denote
\[
|v| = \max\{|v_\alpha| : \alpha = 1, \ldots, d\}
\]
and for vector valued functions \( \|f\|_\infty = \max\{|f_\alpha| : \alpha = 1, \ldots, d\} \). For a function
\( B : \mathbb{R}^d \to \mathbb{R}^{d'} \), we write \( D^kB \) for the \( k \)th derivative. We define
\[
\|D^kB\|_\infty = \max\{|\partial^{t_1} \cdots \partial^{t_d} B_\alpha| : t_1 + \cdots + t_d = k, 1 \leq \alpha \leq d'\}.
\]
Here \( B_\alpha, 1 \leq \alpha \leq d' \) are the coordinate functions of \( B \).

2. Results in the abstract setting

Let \((X, \mathcal{B}, \mu)\) be a probability space and \( (f^i)_{i=0}^\infty \) a sequence of random vectors. We also
assume that every \( \|f^i\|_\infty, i \in \mathbb{N}_0 \), have a common upper bound denoted by \( \|f\|_\infty \). We
write \( f^i = f^i - \mu(f^i) \) and given an \( N \in \mathbb{N}_0 \)
\[
W = W(N) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} f^i.
\]
The covariance matrix of \( W(N) \) is denoted by \( \Sigma_N \), i.e.,
\[
\Sigma_N = \mu(W^2) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mu(f^i \otimes f^j).
\]

Let \( K \in \mathbb{N}_0 \cap [0, N-1] \). Then we define
\[
[n]_K = \{i \in \mathbb{N} : 0 \leq i \leq N - 1, |n - i| \leq K\}.
\]
and
\[
W_n = \frac{1}{\sqrt{N}} \sum_{i=0,|i|\notin[n]_K}^{N-1} f^i.
\]
We denote by $\Phi_\Sigma(h)$ the expectation of a function $h : \mathbb{R}^d \to \mathbb{R}$ with respect to the $d$-dimensional centered normal distribution $\mathcal{N}(0, \Sigma)$ with positive definite covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$, i.e.,

$$
\Phi_\Sigma(h) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}w \cdot \Sigma^{-1}w} h(w) \, dw.
$$

The next theorem concerns approximating the distribution of the sum of random vectors by a normal distribution. It is formulated in such a way that it can easily be applied to time-dependent dynamical systems: Let $X$ be a state space and $f : X \to \mathbb{R}^d$ a function, and let $(T_i)_{i=1}^\infty$, $T_i : X \to X$, be a sequence of transformations. Denote $T_0 = T_1 \circ T_0 \circ \ldots \circ T_1$, when $i \geq 1$, and $T_0 = \text{Id}$. Then simply substituting $f^i$ in the theorem by $f \circ T_i$ yields a result for the centered Birkhoff sums $W = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} (f \circ T_i - \mu(f \circ T_i))$ on the probability space $(X, \mathcal{B}, \mu)$.

**Theorem 2.1.** Let $(X, \mathcal{B}, \mu)$ be a probability space and $(f^i)_{i=0}^\infty$ a sequence of random vectors with common upper bound $\|f\|_\infty \geq \|f^i\|_\infty$ for every $i \in \mathbb{N}_0$. Let $h : \mathbb{R}^d \to \mathbb{R}$ be three times differentiable with $\|D^k h\|_\infty < \infty$ for $1 \leq k \leq 3$. Fix integers $N > 0$ and $0 \leq K < N$. Suppose that the following conditions are satisfied:

(A1) There exist constants $C_2 > 0$ and $C_4 > 0$, and a non-increasing function $\rho : \mathbb{N}_0 \to \mathbb{R}_+$ with $\rho(0) = 1$ and $\sum_{i=1}^\infty i \rho(i) < \infty$, such that for all $0 \leq i \leq j \leq k \leq l \leq N-1$,

$$
|\mu(\tilde{f}_a \tilde{f}_b)| \leq C_2 \rho(j-i),
$$

$$
|\mu(\tilde{f}_a \tilde{f}_d \tilde{f}_e | \leq C_4 \rho(\max\{j-i, l-k\}),
$$

$$
|\mu(\tilde{f}_a \tilde{f}_b \tilde{f}_c \tilde{f}_d) - \mu(\tilde{f}_a \tilde{f}_b \tilde{f}_c)) \mu(\tilde{f}_d \tilde{f}_a)| \leq C_4 \rho(k-j)
$$

hold whenever $k \geq 0$; $0 \leq i \leq j \leq k \leq n < N$; $\alpha, \beta, \gamma, \delta \in \{\alpha', \beta'\}$ and $\alpha', \beta' \in \{1, \ldots, d\}$.

(A2) There exists a function $\tilde{\rho} : \mathbb{N}_0 \to \mathbb{R}_+$ such that

$$
|\mu(\tilde{f}_n \cdot \nabla h(v + W_n t))| \leq \tilde{\rho}(K)
$$

holds for all $0 \leq n \leq N-1$, $0 \leq t \leq 1$ and $v \in \mathbb{R}^d$.

(A3) $\Sigma_N$ is positive-definite $d \times d$ matrix.

Then

$$
|\mu(h(W)) - \Phi_{\Sigma_N}(h)| \leq C_* \left( \frac{K + 1}{\sqrt{N}} + \sum_{i=K+1}^\infty \rho(i) \right) \sqrt{N} \tilde{\rho}(K),
$$

where

$$
C_* = 6d^3 \max\{C_2, \sqrt{C_4}\} \left( \|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty \right) \sqrt{\sum_{i=0}^\infty (i+1) \rho(i)}
$$

is independent of $N$ and $K$.

This theorem is similar to Theorem 2.1 in [21]. The theorem above can be applied to dynamical systems where transformations are time-dependent. As a side note, the constant $C_*$ in (2) is better than in [21].
Let $f^i, i \in \mathbb{N}_0$, be random variables. Then we denote the variance of $W(N)$ by
\[
\sigma_N^2 = \mu(W^2) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mu(f^i f^j).
\]

For univariate $f^i$ we can improve the result of the previous theorem. Instead of three times differentiable functions $h$, we can assume that $h$ is Lipschitz and still get an upper bound result for $\mu(h(W)) - \Phi_{\sigma_N^2}(h)$. A downside is that the bound obtained is inversely proportional to the variance $\sigma_N^2$. To state the result rigorously, we introduce the concept of Wasserstein distance.

Let $X_1$ and $X_2$ be two random variables in $(X, \mathcal{B}, \mu)$. Then the Wasserstein distance between $X_1$ and $X_2$ is defined as
\[
d_W(X_1, X_2) = \sup_{h \in \mathcal{H}} |\mu(h(X_1)) - \mu(h(X_2))|,
\]
where
\[
\mathcal{H} = \{h : \mathbb{R} \to \mathbb{R} : |h(x) - h(y)| \leq |x - y|\}
\]
is the class of all 1-Lipschitz functions.

**Theorem 2.2.** Let $(X, \mathcal{B}, \mu)$ be a probability space and $(f^i)_{i=0}^{\infty}$ a sequence of random vectors with common upper bound $\|f\|_{\infty}$. Fix integers $N > 0$ and $0 \leq K < N$. Suppose that the following conditions are satisfied.

1. **(B1)** There exist constants $C_2, C_4$ and a non-increasing function $\rho : \mathbb{N} \to \mathbb{R}$ with $\rho(0) = 1$, such that for all $0 \leq i \leq j \leq k \leq l \leq N - 1$,
   \[
   |\mu(f^i f^j)| \leq C_2 \rho(j - i),
   \]
   \[
   |\mu(f^i f^j f^k f^l)| \leq C_4 \rho(\max\{j - i, l - k\}),
   \]
   \[
   |\mu(f^i f^j f^k f^l) - \mu(f^i f^j) \mu(f^k f^l)| \leq C_4 \rho(k - j).
   \]
2. **(B2)** There exists a function $\tilde{\rho} : \mathbb{N}_0 \to \mathbb{R}_+$ such that, given a differentiable $A : \mathbb{R} \to \mathbb{R}$ with $A'$ absolutely continuous and $\max_{0 \leq k \leq 2} \|A^{(k)}\|_{\infty} \leq 1$,
   \[
   |\mu(f^n A(W_n))| \leq \tilde{\rho}(K)
   \]
   holds for all $0 \leq n < N$.
3. **(B3)** $\sigma_N^2 > 0$.

Then the Wasserstein distance $d_W(W, \sigma_N Z)$ is bounded from above by
\[
C_\# \left( \frac{K + 1}{\sqrt{N}} + \sum_{i=K+1}^{\infty} \rho(i) \right) + C'_\# \sqrt{N} \tilde{\rho}(K),
\]
where
\[
C_\# = 12 \max\{\sigma_N^{-1}, \sigma_N^{-2}\} \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_{\infty}) \sqrt{\sum_{i=0}^{\infty} (i + 1) \rho(i)}
\]
and
\[
C'_\# = 2 \max\{1, \sigma_N^{-2}\}
\]
are independent of $N$ and $K$.

Note that if $\sigma_N = 0$, then trivially $d_W(W, \sigma_N Z) = 0$. 

3. Application I: time-dependent expanding maps

In this section we present some CLTs in a concrete model of expanding circle maps. They are proved in Section 5 by applying Theorems 2.1 and 2.2. We also give a result in the case, where transformations are chosen randomly.

3.1. The model. Let \( S^1 \) be the state space equipped with Borel sigma-algebra \( \mathcal{B} \) and an initial probability measure \( \mu \). Let \( \mathcal{M} \) denote the set of \( C^2 \) expanding circle maps \( T : S^1 \to S^1 \) with the following bounds:

\[
\inf T' = \lambda > 1, \quad \|T''\|_\infty \leq A_+.
\]

Let \( f : S^1 \to \mathbb{R}^d \). We write \( \text{Lip}(f) = \max\{\text{Lip}(f_\alpha) : \alpha \in 1, \ldots, d\} \) and \( \|f\|_{\text{Lip}} = \|f\|_\infty + \text{Lip}(f) \). From now on we assume that all transformations belong to \( \mathcal{M} \) and that \( f \) is Lipschitz continuous, i.e., all the coordinate functions are Lipschitz continuous. Furthermore we assume that the initial probability measure \( \mu \) with density \( \varrho \) with respect to Lebesgue measure \( m \) on \( S^1 \) is such that \( \log \varrho \) is Lipschitz continuous with constant \( L_0 = \text{Lip}(\log \varrho) \). Notice that this implies that \( \varrho = e^{\log \varrho} \) is also Lipschitz continuous and \( \varrho \geq c > 0 \) with some \( c \in \mathbb{R}_+ \). \( W \) is defined as in the abstract setting in the previous section, as are \( \Sigma_N \) and \( \sigma_N^2 \).

The results in this section contain constants \( \vartheta, C_2, C_4 \) and \( B_0 \). Some exact bounds to their values could be calculated by using the results of section 5 of [12], but it is omitted here. Instead we just state here the most important features of those constants. First of all \( \vartheta \in [0, 1[ \) measures the decorrelation speed of the system and depends only on the model constants \( \lambda \) and \( A_+ \). It is defined as in Lemma 5.6 of [12]. In particular \( \vartheta \geq \lambda^{-1} \). Constants \( C_2 > 0 \) and \( C_4 > 0 \) depend on \( \lambda, A_+ \), \( \|f\|_{\text{Lip}} \) and Lipschitz constant of \( \varrho \), and are introduced in Lemma 5.2. The last constant \( B_0 = B_0(L_0, \lambda, A_+) > 0 \) is defined after Lemma 5.6.

Now we are ready to present the first theorem concerning expanding circle maps.

**Theorem 3.1.** Let \((T_i)_{i=1}^\infty \subset \mathcal{M}\) be a sequence of transformations in the model \( \mathcal{M} \). Let \( h : \mathbb{R}^d \to \mathbb{R} \) be three times differentiable with \( \|D^kh\|_\infty < \infty, \ k = 1, 2, 3 \). Suppose that \( N \geq \max\{3, 16/(1-\vartheta)^2\} \) is such that the matrix \( \Sigma_N \) is positive definite. Then

\[
|\mu(h(W)) - \Phi_{\Sigma_N}(h)| \leq C N^{-\frac{1}{2}} \log N,
\]

where

\[
C = \frac{30d^3 \max\{C_2, \sqrt{C_4}\} \left(\|f\|_\infty \|D^3h\|_\infty + \|D^2h\|_\infty\right)}{(1-\vartheta)^2} + 2d^2 \|D^2h\|_\infty \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-\frac{3}{2}} - \vartheta^2} + 3dB_0 \|f\|_{\text{Lip}} \|Dh\|_\infty + \frac{2d \|Dh\|_\infty \|f\|_{\text{Lip}}}{\vartheta^\frac{3}{2}}.
\]

In addition to the previous theorem, for univariate \( f \), the following theorem also holds:

**Theorem 3.2.** Let \((T_i)_{i=1}^\infty \subset \mathcal{M}\) be a sequence of transformations in the model \( \mathcal{M} \). Let \( N \geq \max\{3, 16/(1-\vartheta)^2\} \) and \( \sigma_N \geq C_0N^{-p} \), where \( C_0 > 0 \), \( p \geq 0 \). Then

\[
d_{\vartheta}(W, \sigma_N^2Z) \leq \tilde{C} \max\{1, C_0^{-2}\} N^{-\frac{1}{2}+2p} \log N,
\]

where
\[ \hat{C} = \frac{60 \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_\infty)}{(1 - \vartheta)^2} + \frac{4\|f\|_{\text{Lip}}^2}{\vartheta^{-\frac{3}{2}} - \vartheta^{\frac{1}{2}}} + 6B_0\|f\|_{\text{Lip}} + \frac{4\|f\|_{\text{Lip}}^2}{\vartheta^{\frac{1}{2}}} \]  

is independent of \( N \).

In particular, if \( \sigma_N > C_0 \) (case \( p = 0 \)) for \( N \geq 3 \), the upper bound becomes

\[ \hat{C} \max\{1, C_0^{-\frac{3}{2}}\} N^{-\frac{1}{2}} \log N. \]

If the variance \( \sigma_N \) decreases fast towards zero, then Theorem 3.2 is not useful. However, \( d_{\mathcal{W}}(W, \sigma_N Z) \leq 2\sigma_N \) as is proven in Section 5. Since this second estimate is stronger, when \( \sigma_N \) is smaller, we are able to provide the following CLT result which is independent of variance.

**Corollary 3.3.** Let \( (T_i)_{i=1}^{\infty}, T_i : S^1 \to S^1, i \in \mathbb{N} \) be a sequence of transformations in the model \( \mathcal{M} \). Then

\[ d_{\mathcal{W}}(W, \sigma_N Z) \leq \max\{\hat{C}, 2\} N^{-\frac{1}{2}} \log N, \]

for all \( N \geq \max\{3, 16/(1 - \vartheta)^2\} \), where \( \hat{C} \) is as in (4).

Finally, in the last result of this subsection we consider the self-normalized version of \( W \).

For this purpose define

\[ S_N = \sum_{i=0}^{N-1} \tilde{f}^i = \sqrt{N} W(N) = \sqrt{N} W \]  

and

\[ s_N^2 = \text{Var}(S_N) = \text{Var}(\sqrt{N} W) = N \sigma_N^2, \]

i.e., \( S_N \) is the Birkhoff sum with variance \( s_N^2 \). Notice that if \( s_N > 0 \), then \( S_N/s_N \) has a variance 1 and it is thus \( W \) after self-normalization. With these definitions, we have the following corollary to Theorem 3.2:

**Corollary 3.4.** Let \( (T_i)_{i=1}^{\infty} \subset \mathcal{M} \) be a sequence of transformations in the model \( \mathcal{M} \). Let \( N \geq \max\{3, 16/(1 - \vartheta)^2\} \) and \( s_N^2 \geq C_0 N^p \), where \( C_0 > 0 \), \( p \geq 0 \). Then

\[ d_{\mathcal{W}}\left( \frac{S_N}{s_N}, Z \right) = \hat{C} \max\{C_0^{-\frac{3}{2}}, C_0^{-\frac{1}{2}}\} N^{1-\frac{p}{2}} \log N. \]

We make two final remarks. First, if the growth of \( s_N^2 \) is linear \( (p = 1) \), then the upper bound of Wasserstein distance is of the form \( CN^{-1/2} \log N \). Second, if \( p > 2/3 \), then \( d_{\mathcal{W}}(S_N/s_N, Z) \to 0 \), when \( N \to \infty \).

### 3.2. Random dynamical system.

In this subsection we study a setup, where expanding circle maps are picked at random from the set \( \mathcal{M} \). We show that under some assumptions there exists a limit variance for \( W \) and it is the same for almost every random sequence of transformations.

Let \( (T_\omega)_{i=1}^{\infty} \) be a sequence of transformations on \( S^1 \) such that each index \( \omega_i \) is drawn randomly from a probability space \((\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_0^{Z_+}, \mathcal{E}^{Z_+}, \mathbb{P})\). Here \((\Omega_0, \mathcal{E})\) is a measurable space and \( Z_+ = \{1, 2, \ldots\} \). We assume the following about the random dynamical system in question:

**Assumption (RDS)**

i) Each \( T_\omega \in \mathcal{M} \).
ii) The law $\mathbb{P}$ is stationary, i.e., the shift $\tau: \Omega \to \Omega: (\tau(\omega))_i = \omega_{i+1}$ preserves $\mathbb{P}$.

iii) The random selection process is strong mixing satisfying

$$\sup_{i \geq 1} \sup_{A \in \mathcal{F}_i, B \in \mathcal{F}_i^\infty} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| \leq Cn^{-\gamma}$$

for each $n \geq 1$, where $\gamma > 0$ and $\mathcal{F}_i^k$ is a sigma-algebra generated by projections $\pi_1, \ldots, \pi_i, \pi_k(\omega) = \omega_k$ and $\mathcal{F}_i^\infty$ generated by $\pi_i, \pi_i+1, \ldots$.

iv) The map

$$(\omega, x) \mapsto T_{\omega_n} \circ \cdots \circ T_{\omega_1}(x)$$

is measurable from $\mathcal{F} \otimes \mathcal{B}$ to $\mathcal{B}$ for every $n \in \mathbb{N} = \{0, 1, \ldots\}$.

Define $\sigma_N^2(\omega) = \sigma_N^2 = \text{Var}_\omega W(N)$ and $\sigma^2 = \lim_{N \to \infty} \mathbb{E}\sigma_N^2$, when the limit exists. Here $W$ is defined as in the abstract setting except that it now also has $\omega$-dependence. The next theorem gives a quenched convergence result for $W$ that holds for almost every sequence of transformations.

**Theorem 3.5.** Assume that (RDS) is satisfied. Then

$$\sigma^2 = \sum_{k=0}^{\infty} (2 - \delta_k) \lim_{i \to \infty} \mathbb{E}[(\mu(f_i f_{i+k}) - \mu(f_i)\mu(f_{i+k}))]$$

is well-defined and non-negative. We have $\sigma > 0$ if and only if

$$\sup_{N \geq 1} N\mathbb{E}\mu(W^2) = \infty.$$ 

Furthermore if $\sigma > 0$ holds, then for arbitrary $\delta > 0$ and almost every $\omega$

$$d_\mathcal{W}(W(N), \sigma Z) = \begin{cases} O(N^{-\frac{1}{2}}\log^{\frac{3}{2}+\delta} N), & \gamma > 1, \\ O(N^{-\frac{1}{2}+\delta}), & \gamma = 1, \\ O(N^{-\frac{1}{2}+\delta} N), & 0 < \gamma < 1. \end{cases}$$

**Sketch of proof.** First of all, Assumption (RDS) together with Lemmas 5.2 and 5.6 are applied to show that Assumptions (SA1)–(SA4) in [22] are satisfied. The condition for $\sigma > 0$ is shown by verifying that Assumption (SA5') in [22] holds for the given system and then using Lemma B.1 (iv)(b) & (v)(b) in that paper.

Theorem 4.1 in the same paper is then applied, giving the limit variance and bounds for $|\sigma_N^2(\omega) - \sigma^2|$. Lemma 6.4 and Theorem 3.2 are applied to complete the proof of the latter part in the above theorem.

4. **Application II: A Quasistatic Dynamical System**

The model that we present in this section is introduced in [12]. First we present the following definition from [12]:

**Definition 4.1.** Let $X$ be a set and $\mathcal{M}$ a collection of self-maps $T: X \to X$ equipped with a topology. Consider a triangular array

$$T = \{T_{n,k} \in \mathcal{M}: 0 \leq k \leq n, n \geq 1\}$$

of elements of $\mathcal{M}$. If there exists a piecewise continuous curve $\gamma: [0, 1] \to \mathcal{M}$ such that

$$\lim_{n \to \infty} T_{n,\lfloor nt \rfloor} = \gamma_t, \quad t \in [0, 1],$$

we say that $(T, \gamma)$ is a quasistatic dynamical system (QDS). The set $X$ is called the state phase and $\mathcal{M}$ the system phase of the QDS.
4.1. The model. We define a following QDS, also introduced in [12]. The state space is \( S^1 \) and the system space \( M \) is the same set of transformations on \( S^1 \) as in the model of Section 3. We define a metric \( d_{C^1} \) to the set \( M \) by

\[
d_{C^1}(T_1, T_2) = \sup_{x \in S^1} d(T_1 x, T_2 x) + \|T_1' - T_2'\|_\infty
\]

for \( T_1, T_2 \in M \). Here \( d \) is the natural metric on \( S^1 = \mathbb{R} \setminus \mathbb{Z} \). We assume that \( \gamma : [0, 1] \to M \) is a Hölder continuous curve with exponent \( \eta \in ]0, 1[ \) and constant \( C_H \geq 0 \). Let \( T \) be a triangular array of maps

\[
T = \{ T_{n,k} \in M : 0 \leq k \leq n, n \geq 1 \},
\]

which satisfies

\[
\sup_{0 \leq t < 1} d_{C^1}(T_{nt}, \gamma_t) \leq C_H n^{-\eta}.
\]

It is known that for every \( T \in M \) there exists a unique invariant probability measure that is equivalent to the Lebesgue measure \( m \) on \( S^1 \). For \( \gamma_t \) we denote this measure by \( \hat{\mu}_t \). Furthermore we write \( \hat{f}_t = f - \hat{\mu}_t(f) \).

If \( f \) is univariate we define

\[
\hat{\sigma}_t^2(f) = \lim_{m \to \infty} \hat{\mu}_t \left[ \left( \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} \hat{f}_t \circ \gamma_k^t \right)^2 \right]
\]

and

\[
\sigma_t^2(f) = \int_0^t \hat{\sigma}_s^2(f) ds.
\]

We may write \( \sigma_t^2 \) instead of \( \sigma_t^2(f) \) if \( f \) is known from the context.

Analogously if \( f \) is multivariate we define

\[
\hat{\Sigma}_t(f) = \lim_{m \to \infty} \hat{\mu}_t \left[ \left( \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} \hat{f}_t \circ \gamma_k^t \right) \otimes \left( \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} \hat{f}_t \circ \gamma_k^t \right) \right]
\]

and

\[
\Sigma_t(f) = \int_0^t \hat{\Sigma}_s(f) ds.
\]

Let us introduce some notations. We denote \( \mathcal{T}_{n,i} = T_{n,i} \circ T_{n,i-1} \circ ... \circ T_{n,1} \) and \( \mathcal{T}_{n,i,j} = T_{n,i} \circ T_{n,i-1} \circ ... \circ T_{n,j} \). Furthermore we denote \( \tilde{f}_{n,i} = f \circ \mathcal{T}_{n,i} \) and \( \bar{f}_{n,i} = f_{n,i} - \mu(f_{n,i}) \), and define

\[
\xi_n(t) = \xi(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]-1} \tilde{f}_{n,i} + \frac{\{nt\}}{\sqrt{n}} \bar{f}_{n,[nt]}, \quad (6)
\]

where \( \{nt\} = nt - [nt] \). Note that \( \xi_n(t) = n^{1/2} \int_0^t \bar{f}_{n,[ns]} ds \).

We denote the covariance matrix of \( \xi_n(t) \) (with respect to \( \mu \)) by \( \Sigma_{n,t} \). If \( f \) is univariate, then the variance of \( \xi_n(t) \) is denoted by \( \sigma_{n,t}^2 \). We aim to prove an upper bound on \( |\sigma_{n,t}^2 - \sigma_t^2| \) as a function of \( n \). This in turn is used to prove an upper bound to Wasserstein distance between \( \xi_n(t) \) and \( \sigma_t Z \).
4.2. Results. The next theorem concerns approximating the distribution of $ξ_n(t)$ by the multivariate normal distribution $N(0, Σ_t)$. By the definition of $ξ_n(t)$ and Theorem 3.1 it is not surprising that for large $nt$ the distribution of $ξ_n(t)$ is close to $N(0, Σ_{nt})$. Thus the essential new content of this theorem is that $Σ_{nt} ≈ Σ_t$ for large $n$. We also see that the more regular the curve $γ$ is, the better is the speed of convergence.

**Theorem 4.2.** Let $t_0 ∈ [0, 1]$ be such that $Σ_{t_0}$ is positive definite and let $h : ℝ^d → ℝ$ be three times differentiable with $‖D^kh‖_∞ < ∞$ for $1 ≤ k ≤ 3$. Then for all $η' < η$ there exists constant $C$ independent of $t$ such that for every $t ≥ t_0$ and $n ≥ 1$

$$|μ(h(ξ_n(t))) − Φ_{Σ_t}(h)| ≤ Cn^{-η'} + Cn^{-\frac{1}{2}} \log n.$$  

It is actually true that if $Σ_0$ is positive definite, then $Σ_{t_0}$ is positive definite with some $t_0 ≥ 0$. However the constant $C$ depends on the choice of $t_0$, which explains the formulation of the previous theorem.

If $f$ is univariate we can again use the Wasserstein distance. As in the previous theorem, regularity of $γ$ effects the speed of convergence. A simple assumption that $\hat{σ}^2_t$ is non-zero somewhere is also required for providing the speed of convergence given in the theorem.

**Theorem 4.3.** Let $t_0 ∈ [0, 1]$ be such that $\hat{σ}^2_{t_0} > 0$. Then for all $η' < η$ there exist constants $C = C(λ, A, η', η, C_H, ‖f‖_{lip}, L_0, t_0, \hat{σ}^2_{t_0})$ such that for every $t ≥ t_0$ and $n ≥ 1$

$$d_W(ξ_n(t), σ_tZ) ≤ Cn^{-η'} + Cn^{-\frac{1}{2}} \log n.$$  

We point out that $\hat{σ}^2_t(f) = 0$ only in the very special case that $f = g − g ∘ γ_t$ for some Hölder continuous $g$.

The last result we present in this section is analogous to Corollary 3.3 in Section 3. It holds without any restriction on the behaviour of the variance $\hat{σ}^2_t$.

**Theorem 4.4.** Let $η' < η$. Then there exists a constant $C = C(λ, A, η', η, C_H, ‖f‖_{lip}, L_0)$ such that the following holds for every $t ∈ [0, 1]$ and $n ≥ 1$:

$$d_W(ξ_n(t), σ_tZ) ≤ Cn^{-\frac{η}{2}} + Cn^{-\frac{1}{2}} \log n.$$  

5. Proofs for Application I

In this section we study the model described in Section 3.

5.1. Upper bounds for $ρ$ and $\hat{ρ}$. In this subsection we calculate upper bounds for $ρ(K)$ in Assumptions (A1) and (B1), and $\hat{ρ}(K)$ in Assumptions (A2) and (B2) of Theorems 2.1 and 2.2, respectively. First we introduce the following definition from [12]:

**Definition 5.1.** We define a class $D_L$, $L ∈ ℝ^+$, of probability densities $ψ : S → ℝ$ in the following way: $ψ ∈ D_L$ if

i) $ψ > 0$

ii) there exists $z ∈ S^1$ such that $log ψ$ is Lipschitz continuous on $J_z = S \setminus \{z\}$ with constant $L$.

Thus $L$ describes the regularity of probability densities in that class, smaller $L$ meaning smoother density. By Remark 4.3ii) in [12] every Lipschitz continuous probability density $ψ > 0$ belongs to $D_L$ with some value of $L$.  


Given a transformation $T \in \mathcal{M}$, the transfer operator $\mathcal{L}_T : L^1(m) \rightarrow L^1(m)$ is defined by

$$\mathcal{L}_T g(x) = \sum_{y \in T^{-1}\{x\}} \frac{g(y)}{FP(y)}.$$  

(7)

It satisfies the following rule: For every $g \in L^1(m)$ and $f \in L^\infty(m)$

$$\int_{\mathcal{G}} f \mathcal{L}_T g dm = \int_{\mathcal{G}} g f \circ T dm.$$  

(8)

Furthermore, we introduce the new notation $\mathcal{T}_{k,j} = T_k \circ \ldots \circ T_j$.

Applying (8) repeatedly gives $\mathcal{L}_{T_{k,j}} = \mathcal{L}_{T_k} \ldots \mathcal{L}_{T_j}$. We write $\mathcal{L}_{k,j} = \mathcal{L}_{T_k} \ldots \mathcal{L}_{T_j}$ and $\mathcal{L}_k = \mathcal{L}_{T_k} \ldots \mathcal{L}_{T_1}$.

In general transfer operators tend to smooth probability densities; see, e.g., Lemma 5.2 in [12]. Concerning this paper, the most important content of that lemma is that there exists a constant $L_\ast = L_\ast(\lambda, A_\ast)$ with the property that for every $L > L_\ast$ there exists $k$ such that

$$\mathcal{L}_k \mathcal{D}_L \subset \mathcal{D}_{L_\ast}.$$  

(9)

Actually we can choose $L_\ast = A_\ast \lambda (1 - \lambda^{-1})^{-2}$, as the reader may verify by going through the proofs of Lemmas 5.1 and 5.2 in that paper.

Throughout the paper $\vartheta = \vartheta(\lambda, A) \in ]0,1[$, mentioned in the next lemma, is the same constant as in Lemma 6 of [12].

The next lemma is implied by Lemma 5.10 of [12]. Recall that $\vartheta \in \mathcal{D}_{L_0}$, where $L_0 = \text{Lip}(\log \vartheta)$. By Remark 4.4(ii) of [12] $L_0$ determines some upper bound to $\text{Lip}(\vartheta)$, which is why we can replace $\text{Lip}(\vartheta)$ by $L_0$ in the following lemma:

**Lemma 5.2.** There exist constants

$$C_2 = C_2(\lambda, A_\ast, \|f\|_{\text{Lip}}(L_0)) > 0 \quad \text{and} \quad C_4 = C_4(\lambda, A_\ast, \|f\|_{\text{Lip}}(L_0)) > 0$$

such that by choosing $\rho(i) = \vartheta^i$ the system satisfies the condition $(A1)$ of Theorem 2.1 and the condition $(B1)$ of Theorem 2.2.

Assume that $0 \leq n \leq N - 1$. The following two theorems determine some upper bounds on $|\mu(f^n \cdot \nabla h(v + W_n t))|$ and $|\mu [f^n A(W_n)]|$ in the case of multivariate and univariate $f$, respectively. The proof of Theorem 5.3 is given after Lemmas 5.5–5.9. Proof of Theorem 5.4 is omitted since it follows exactly the same steps as the proof of Theorem 5.3.

There is some $N$-dependence in the formulations of these theorems which will be removed later to bound $\hat{\rho}(K)$. Therefore only $K$-dependence is left in the formulation of Assumptions (A2) and (B2) for the sake of simplicity.

**Theorem 5.3.** Given a two times differentiable $h : \mathbb{R}^d \rightarrow \mathbb{R}$ with $\|Dh\|_{\infty}, \|D^2 h\|_{\infty} \leq \infty$, $\|D^n f\|_{\infty}, \|D^2 f\|_{\infty} \leq \infty$,

$$|\mu(f^n \cdot \nabla h(v + W_n t))| \leq 2d^2 \|f\|_{\infty} \|D^2 h\|_{\infty} \text{Lip}(f) \frac{\lambda^{-\frac{K-1}{2}}}{(\lambda - 1)\sqrt{N}}$$

$$+ 3dB_0 \|f\|_{\infty} \|Dh\|_{\infty} \vartheta^\frac{K}{2} + 2d \|Dh\|_{\infty} \text{Lip}(f) \frac{\lambda^{-\frac{K-1}{2}}}{(\lambda - 1)\sqrt{N}}$$

holds for all $0 \leq n \leq N - 1$, where $B_0 = C(\lambda, A_\ast, L_0)$ is the constant that will be introduced after Lemma 5.6.
Theorem 5.4. Given a differentiable $A : \mathbb{R} \to \mathbb{R}$ with $A'$ absolutely continuous and $\max_{0 \leq k \leq 2} \|A^{(k)}\|_\infty \leq 1,$

$$|\mu [\bar{f}_n A(W_n)]| \leq 2\|f\|_\infty \text{Lip}(f)\lambda^{-\frac{K-1}{2}} (\lambda - 1)\sqrt{N} + 3\|f\|_\infty B_0 \vartheta^{\frac{K}{2}} + 2\text{Lip}(f)\lambda^{-\frac{K-1}{2}},$$

holds for all $0 \leq n \leq N - 1,$ where $B_0 = B_0(\lambda, A_*, L_0).$

By Theorem 5.3 and inequalities $\lambda^{-1} \leq \vartheta,$ $\sqrt{N} \geq 1$ and $\|f\|_\infty, \text{Lip}(f) \leq \|f\|_{\text{Lip}},$ we may deduce the following

$$|\mu(\bar{f}_n \cdot \nabla h(v + W_n t))|$$

\[
\leq 2d^2\|f\|_\infty \|D^2 h\|_\infty \frac{\text{Lip}(f)\lambda^{-\frac{K-1}{2}}}{\sqrt{N}(\lambda - 1)\sqrt{N}} + 3dB_0\|f\|_\infty \|Dh\|_\infty \vartheta^{\frac{K}{2}} + 2d\|Dh\|_\infty \text{Lip}(f)\lambda^{-\frac{K-1}{2}}
\]

\[
\leq 2d^2\|D^2 h\|_\infty \frac{\|f\|_{Lip}^2 \vartheta^{\frac{K-1}{2}}}{\vartheta^{-1} - 1} + 3dB_0\|f\|_{\text{Lip}} \|Dh\|_\infty \vartheta^{\frac{K}{2}} + 2d\|Dh\|_\infty \|f\|_{\text{Lip}} \vartheta^{\frac{K-1}{2}}.
\]

Thus when we apply Theorem 2.1 in the model of expanding circle maps introduced in Section 3 we may choose in Assumption (A2) that

$$\tilde{\rho}(K) = 2d^2\|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2 \vartheta^{\frac{K-1}{2}}}{\vartheta^{-1} - 1} + 3dB_0\|f\|_{\text{Lip}} \|Dh\|_\infty \vartheta^{\frac{K}{2}} + 2d\|Dh\|_\infty \|f\|_{\text{Lip}} \vartheta^{\frac{K-1}{2}}. \quad (10)
$$

By similar computations for an univariate $f$ we may choose in Assumption (B2) that

$$\tilde{\rho}(K) = 2\frac{\|f\|_{\text{Lip}}^2 \vartheta^{\frac{K-1}{2}}}{\vartheta^{-1} - 1} + 3dB_0\|f\|_{\text{Lip}} \|Dh\|_\infty \vartheta^{\frac{K}{2}} + 2\|f\|_{\text{Lip}} \vartheta^{\frac{K-1}{2}}. \quad (11)
$$

For the rest of the section, we assume that $n$ is fixed and define $B = \max\{\lfloor n - K/2 \rfloor, 0\}.$

Each $T$ induces a finite partition of $\mathbb{S}^1$ into intervals $I_i,$ $i \in J$ such that $T$ maps int $I_i$ diffeomorphically on $\mathbb{S}^1 \setminus \{0\}.$ We call $\{I_i : i \in J\}$ the partition induced by $T.$ The next lemma shows that when $K$ is large, then $\sum_{i=0}^{n-1} \bar{f}_i$ is almost a constant in elements $I_i \in \mathbb{S}^1$ of the partition induced by $T_B.$

Lemma 5.5. Let $I_i$ be an element of the partition induced by $T_B.$ There exists $C_i \in \mathbb{R}^d$ such that

$$|v + W_n(x) t| + \left( C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j(x) \right) \leq \frac{\text{Lip}(f)\lambda^{-\frac{K}{2}}}{\sqrt{N}(\lambda - 1)}$$

for every $x \in I_i.$

Proof. Assume first that $n \leq K.$ Then $W_n = \sum_{j=n+K+1}^{N-1} \bar{f}_j.$ Thus choosing $C_i = v - \sum_{j=n+K+1}^{N-1} \mu(f^j)$ yields

$$|v + W_n(x) t| + \left( C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j(x) \right) = 0.$$

Assume then that $n > K.$ Let $x, y \in I_i$ and $j \leq B = n - \lfloor K/2 \rfloor.$ Then $|T_j(x) - T_j(y)| \leq \lambda^{j-n+\lfloor K/2 \rfloor},$ which implies $|f_\alpha \circ T_j(x) - f_\alpha \circ T_j(y)| \leq \text{Lip}(f_\alpha)\lambda^{j-n+\lfloor K/2 \rfloor}.$
Thus
\[
\left| \frac{t}{\sqrt{N}} \sum_{j=0}^{n-K-1} \bar{f}^j_\alpha(x) - \frac{t}{\sqrt{N}} \sum_{j=0}^{n-K-1} \bar{f}^j_\beta(y) \right| \leq \frac{t}{\sqrt{N}} \sum_{j=0}^{n-K-1} \lambda^{j-n+\lfloor \frac{j}{\lambda} \rfloor} \lambda^{-j} \leq \lambda^{n-K-1} \left[ n-\frac{n}{\lambda} \right] \frac{\text{Lip}(f_\alpha)}{\sqrt{N}} \sum_{j=0}^{\infty} \lambda^{-j} = \frac{\text{Lip}(f_\alpha) \lambda^{-\lfloor \frac{n}{\lambda} \rfloor}}{\sqrt{N}(\lambda - 1)}.
\]

Therefore there exists \( \bar{C}_i \in \mathbb{R}^d \) such that
\[
\left| v + \frac{t}{\sqrt{N}} \sum_{i=0}^{n-K-1} \bar{f}^i(x) - \bar{C}_i \right| \leq \frac{\text{Lip}(f) \lambda^{-\lfloor \frac{n}{\lambda} \rfloor}}{\sqrt{N}(\lambda - 1)}
\]
for every \( x \in I_i \). Thus there exists a constant \( C_i = \bar{C}_i - \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} \mu(f^j) \) such that
\[
\left| (v + W_n(x))t - \left( C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j(x) \right) \right|
\leq \frac{\text{Lip}(f) \lambda^{-\lfloor \frac{n}{\lambda} \rfloor}}{\sqrt{N}(\lambda - 1)}
\]
for every \( x \in I_i \). \( \square \)

The next standard lemma shows that the transfer operator decreases the distance of two probability measures in the \( L^1 \) norm.

**Lemma 5.6.** Let \( T_{c,a}, T_{c,b} \) be two compositions of any maps in \( \mathcal{M} \), where \( a \leq b \leq c \), and let \( \varrho_1, \varrho_2 \in \mathcal{D}_L \), where \( L \geq L_* \). Then there exist constants \( D_0 = D_0(L, \lambda, A_*) \) and \( \vartheta = \vartheta(\lambda, A_*) \in ]0, 1[ \) such that
\[
\| \mathcal{L}_{c,a}(\varrho_1) - \mathcal{L}_{c,b}(\varrho_2) \|_{L^1} \leq D_0 \vartheta^{c-b+1}.
\]

**Proof.** Lemma 5.2 (i) in [12] gives that \( T_{b-1,a}(\varrho_1) \in \mathcal{D}_L \). Thus applying Lemma 5.6 of the same article gives
\[
\| \mathcal{L}_{c,a}(\varrho_1) - \mathcal{L}_{c,b}(\varrho_2) \|_{L^1} = \| \mathcal{L}_{c,b}(\mathcal{L}_{b-1,a}(\varrho_1) - \varrho_2) \|_{L^1} \leq D_0 \vartheta^{c-b+1}.
\]
That \( D_0 = D_0(L, \lambda, A_*) \) and \( \vartheta = \vartheta(\lambda, A_*) \in ]0, 1[ \) follows from Section 5 of [12]. \( \square \)

We define the new constant \( L_1 = \max \{ L_*, L_0 \} \). Lemma 5.6 now implies that there exists a constant \( B_0 = B_0(\lambda, A_*, L_1) = B_0(\lambda, A_*, L_0) \) such that
\[
\| \mathcal{L}_{c,a}(\varrho) - \mathcal{L}_{c,b}(\varrho) \|_{L^1} \leq B_0 \vartheta^{c-b+1}. \tag{12}
\]

The following result is Lemma 5.2(iii) in [12].

**Lemma 5.7.** Let \( L \geq L_* \), \( \varrho_0 \in \mathcal{D}_L \), \( m \geq 1 \) and \( T_m \) be a composition of \( m \) maps in \( \mathcal{M} \). Then for every \( I_i, i \in J \) it holds that
\[
\mathcal{L}_m \left( \frac{\varrho_0 1_{I_i}}{\mu_0(I_i)} \right) \in \mathcal{D}_L.
\]
Lemmas 5.6 and 5.7 yield the following corollary.

**Corollary 5.8.** Let \( q_1, q_2 \in \mathcal{D}_L, L \geq L_s, 1 < m + 1 < n \), \( \mathcal{T}_m \) composition of \( m \) maps in \( \mathcal{M} \). Then for every \( I_i, i \in J \) it holds that

\[
\left\| \mathcal{L}_n(q_1) - \mathcal{L}_{n,m+1} \left( \mathcal{L}_m \left( \frac{q_0 1_{I_i}}{\mu_0(I_i)} \right) \right) \right\|_{L^1} \leq D_0 \vartheta^{n-m},
\]

where \( D_0 = D_0(L, \lambda, A_s) \) is the same constant as in Lemma 5.6.

Similarly to Lemma 5.6, the previous corollary holds for two probability densities \( q_1, q_2 \) in the class \( \mathcal{D}_{L_0} \subset \mathcal{D}_{L_1} \). The constant \( D_0 \) is \( B_0(L_0, \lambda, A_s) \), where \( B_0 \) is the same constant as in (12).

The content of the next lemma is exponential decay of pair correlations when any sequence of transformations in \( \mathcal{M} \) is applied.

**Lemma 5.9.** Let \( g, h : \mathbb{S}^1 \to \mathbb{R} \), where \( g \) is Lipschitz, \( h \) bounded and \( \mathcal{T}_m = T_m \circ \ldots \circ T_1 \) a composition of maps in \( \mathcal{M} \) and \( q_0 \in \mathcal{D}_L, L \geq L_s \). Then

\[
\left| \int_{\mathbb{S}^1} g h \circ \mathcal{T}_m q_0 dm - \int_{\mathbb{S}^1} g q_0 dm \int_{\mathbb{S}^1} h \circ \mathcal{T}_m q_0 dm \right| \leq \|h\|_{\infty} \left( 2 \text{Lip}(g) \lambda^{-\lfloor \frac{m}{2} \rfloor} + \|g\|_{\infty} D_0 \vartheta^{\lfloor \frac{m}{2} \rfloor} \right),
\]

where \( D_0 = D_0(L, \lambda, A_s) \) is the same constant as in Lemma 5.6.

**Proof.** Let \( \{I_i : i \in J\} \) be the partition induced by \( \mathcal{T}_{\lfloor m/2 \rfloor} \). Thus \( |I_i| \leq \lambda^{-\lfloor m/2 \rfloor} \). Define \( g_i = \frac{\int_{I_i} g q_0 dm}{\int_{I_i} q_0 dm} \). We have \( g_i = g(x_i) \) for some \( x_i \in I_i \). Thus

\[
\int_{\mathbb{S}^1} g h \circ \mathcal{T}_m q_0 dm = \sum_{i} \int_{I_i} g h \circ \mathcal{T}_m q_0 dm
\]

\[
= \sum_{i} \left( g_i \int_{I_i} h \circ \mathcal{T}_m q_0 dm + \int_{I_i} (g - g_i) h \circ \mathcal{T}_m q_0 dm \right) \quad (13)
\]

\[
= \sum_{i} g_i \int_{I_i} h \circ \mathcal{T}_m q_0 dm + E_1,
\]

where \( |E_1| \leq \sum_{i} \text{Lip}(g) \lambda^{-\lfloor m/2 \rfloor} \|h\|_{\infty} \int_{I_i} |q_0| dm = \text{Lip}(g) \lambda^{-\lfloor m/2 \rfloor} \|h\|_{\infty} \). Furthermore by the properties of the transfer operator:

\[
\int_{I_i} h \circ \mathcal{T}_m q_0 dm = \int_{\mathbb{S}^1} h \circ \mathcal{T}_{\lfloor m/2 \rfloor + 1} \circ \mathcal{T}_{\lfloor m/2 \rfloor} q_0 1_{I_i} dm
\]

\[
= \mu_0(I_i) \int_{\mathbb{S}^1} h \mathcal{L}_{\lfloor m/2 \rfloor + 1} \left( \mathcal{L}_{\lfloor m/2 \rfloor} \left( \frac{q_0 1_{I_i}}{\mu_0 1_{I_i}} \right) \right) dm
\]

\[
= \mu_0(I_i) \left( \int_{\mathbb{S}^1} h \mathcal{L}_m(q_0) dm + E_i \right),
\]
where \(|E_i| \leq \|h\|_\infty D_0 \hat{g}^{m-[m/2]} = \|h\|_\infty D_0 \hat{g}^{m/2}\) by Corollary 5.8. Thus (13) is
\[
\sum_{i} \left( g_i \mu_0(I_i) \left( \int_{\mathcal{S}^1} h \mathcal{L}_m \right) dm + E_i \right) + E_1
\]
\[
= \sum_{i} g_i \mu_0(I_i) \int_{\mathcal{S}^1} h \mathcal{L}_m dm + \sum_{i} (\mu_0(I_i) g_i E_i) + E_1
\]
\[
= \sum_{i} \int_{\mathcal{S}^1} g_i 1_{I_i} \varrho_0 dm \int \int_{\mathcal{S}^1} h \circ T_m \varrho_0 dm + E_2 + E_1,
\]
where \(|E_2| \leq \|g\|_\infty \|h\|_\infty D_0 \hat{g}^{m/2}\). Furthermore we have
\[
\left| \sum_{i} \int_{\mathcal{S}^1} g_i 1_{I_i} \varrho_0 dm - \int_{\mathcal{S}^1} g \varrho_0 dm \right| = \int \left( \sum_{i} g_i 1_{I_i} - g \right) \varrho_0 dm \leq \text{Lip}(g) \lambda^{- \left[ \frac{m}{2} \right]}.
\]
From which it follows that
\[
\left| \sum_{i} \left( \int_{\mathcal{S}^1} g_i 1_{I_i} \varrho_0 dm \int \int_{\mathcal{S}^1} h \circ T_m \varrho_0 dm \right) - \int_{\mathcal{S}^1} g \varrho_0 dm \int \int_{\mathcal{S}^1} h \circ T_m \varrho_0 dm \right| \leq \text{Lip}(g) \lambda^{- \left[ \frac{m}{2} \right]} \|h\|_\infty.
\]
By (13), (14) and (15), we have
\[
\int_{\mathcal{S}^1} g h \circ T_m \varrho_0 dm = \int_{\mathcal{S}^1} g \varrho_0 dm \int \int_{\mathcal{S}^1} h \circ T_m \varrho_0 dm + E_3 + E_2 + E_1,
\]
where \(|E_3| \leq \text{Lip}(g) \lambda^{- \left[ \frac{m}{2} \right]} \|h\|_\infty\). Thus
\[
\left| \int_{\mathcal{S}^1} g h \circ T_m \varrho_0 dm - \sum_{i} \int_{\mathcal{S}^1} g \varrho_0 dm \int \int_{\mathcal{S}^1} h \circ T_m \varrho_0 dm \right|
\]
\[
\leq 2 \text{Lip}(g) \lambda^{- \left[ \frac{m}{2} \right]} \|h\|_\infty + \|g\|_\infty \|h\|_\infty D_0 \hat{g}^{\left[ \frac{m}{2} \right]}.
\]
\]

5.2. The proof of Theorem 5.3. The overall strategy used in the following proof is described in section 7 of [21].

Proof. Step 1. In Step 1 we split the measure \(\mu\) to a sum of conditional measures on small intervals \(I_i\). In these intervals \(W_n = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f^j\) can be approximated by \(C_i + \sum_{j=n+K+1}^{N-1} f^j\) with only small error. Here \(C_i \in \mathbb{R}^d\) depends on the interval. The choice of intervals \(I_i\) is delicate. The smaller the intervals are, the smaller is the error made in Step 1. However, for the purposes of computations in Steps 2 and 3 only very specific choices produce small errors.

Recall that \(B = \max\{\lfloor n-K/2 \rfloor, 0\}\). Let \(\mathcal{I} = \{I_i : i \in J\}\) be the partition of \(\mathbb{S}^1\) induced by \(\mathcal{T}_B\). We may represent the measure \(\mu\) as \(\sum_{i} \mu_i\), where \(\mu_i(U) = \mu(U \cap I_i), U \subset \mathbb{S}^1\). Thus
\[
\mu(\bar{f}_n \cdot \nabla h(v + W_n t)) = \sum_{i} \int_{I_i} \sum_{\alpha} \left( f^n_{\alpha} - \mu(f^n_{\alpha}) \right) \cdot \partial_\alpha h(v + W_n t) \varrho dm.
\]
By Lemma 5.5 we may write \(v + W_n(x) = C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j(x) + E(x), \) where
\[
\|E\|_\infty \leq \frac{\text{Lip}(f) \lambda^{- \left[ \frac{m}{2} \right]}}{\sqrt{N} (\lambda - 1)}.\]
Now the right side of (16) equals
\[
\sum_{\alpha} \sum_{i} \int_{I_i} \left((f^n_{\alpha} - \mu(f^n_{\alpha})) \partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j + E\right)\right) \ dm
\]

\[
= \sum_{\alpha} \sum_{i} \int_{I_i} \left((f^n_{\alpha} - \mu(f^n_{\alpha})) \partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j\right)\right) \ dm
\]

\[
+ \sum_{\alpha} \sum_{i} \int_{I_i} \left(f^n_{\alpha} - \mu(f^n_{\alpha})\right) \cdot \left(\partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j + E\right) - \partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j\right)\right) \ dm
\]

\[
= \sum_{\alpha} \sum_{i} \int_{I_i} \left((f^n_{\alpha} - \mu(f^n_{\alpha})) \partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j\right)\right) \ dm + E',
\]

where

\[
|E'| = |(17)| \leq \sum_{\alpha} \|f^n_{\alpha} - \mu(f^n_{\alpha})\|_{\infty}
\]

\[
\cdot \int_{S^1} \left|\partial_{\beta} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j + E\right) - \partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j\right)\right| \ dm
\]

\[
\leq 2\|f\|_{\infty} \sum_{\alpha} \sum_{\beta} \|\partial_{\beta}(\partial_{\alpha} h)\|_{\infty} \|E_{\beta}\|_{\infty}
\]

\[
\leq 2d^2\|f\|_{\infty} D^2 h_{\infty} \frac{\text{Lip}(f)\lambda^{-|\frac{K}{2}|}}{\sqrt{N}(\lambda - 1)}.
\]

Thus we now have

\[
\left|\mu(f^n \cdot \nabla h(v + W_n t)) - \sum_{\alpha} \sum_{i} \int_{I_i} \left((f^n_{\alpha} - \mu(f^n_{\alpha})) \partial_{\alpha} h \left(C_i + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j\right)\right) \ dm\right|
\]

\[
\leq 2d^2\|f\|_{\infty} D^2 h_{\infty} \frac{\text{Lip}(f)\lambda^{-|\frac{K}{2}|}}{\sqrt{N}(\lambda - 1)}.
\]

(18)

**Step 2.** In Step 2 we modify the integral in the previous equation. The trick done in Step 1 enables to write the integral in (18) as

\[
\int_{S^1} G \circ T_n \theta 1_{I}, dm = \mu(I_1) \int_{S^1} G \circ T_{n+1} L_B \left(\frac{\partial I_1}{\mu(I_1)}\right) \ dm
\]

\[
= \mu(I_1) \int_{S^1} G L_{n+1} \left(L_B \left(\frac{\partial I_1}{\mu(I_1)}\right)\right) \ dm,
\]

where \(G\) is some function on \(S^1\). By Corollary 5.8, \(L_n(\varphi) \approx L_{n+1} \left(L_B \left(\frac{\partial I_1}{\mu(I_1)}\right)\right)\). Thus in the Step 3 we are only left to evaluate \(\int_{S^1} G L_n(\varphi) dm\).
Beginning Step 2, notice that we can write
\[
\sum_{\alpha} \sum_{i} \int_{I_{i}} \left( (f_{a}^{n} - \mu(f_{a}^{n})) \partial_{\alpha} h \left( C_{i} + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^{j} \right) \right) q dm
\]
\[
= \sum_{\alpha} \sum_{i} \int_{I_{i}} \left( (f_{a} \circ \mathcal{T}_{n,B+1} - \mu(f_{a}^{n})) \partial_{\alpha} h \left( C_{i} + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f \circ \mathcal{T}_{j,B+1} \right) \right) \circ \mathcal{T}_{B} q dm.
\]
\]
(19)

We introduce the notation \( \hat{W}_{l,i} = C_{i} + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f \circ \mathcal{T}_{j,l} \).

Let \( 0 \leq l_{1} \leq l_{2} \leq n + K \). Then we see that
\[
\hat{W}_{l_{1},i} = \left( C_{i} + \frac{1}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f \circ \mathcal{T}_{j,l_{2}+1} \right) \circ \mathcal{T}_{l_{2}l_{1}} = \hat{W}_{l_{1}+1,i} \circ \mathcal{T}_{l_{2}l_{1}}.
\]
\]
(20)

Now by using the properties of the transfer operator and (20)
\[
\sum_{\alpha} \sum_{i} \int_{I_{i}} \left( (f_{a} \circ \mathcal{T}_{n,B+1} - \mu(f_{a}^{n})) \partial_{\alpha} h \left( C_{i} + \frac{t}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f \circ \mathcal{T}_{j,B+1} \right) \right) \circ \mathcal{T}_{B} q dm
\]
\[
= \sum_{\alpha} \sum_{i} \int_{I_{i}} \left( (f_{a} \circ \mathcal{T}_{n,B+1} - \mu(f_{a}^{n})) \partial_{\alpha} h(\hat{W}_{B+1,i}) \right) \mathcal{L}_{B}(\frac{q_{1,i}}{\mu_{0}(I_{i})}) dm
\]
\[
= \sum_{\alpha} \sum_{i} \mu(I_{i}) \int_{I_{i}} \left( (f_{a} - \mu(f_{a}^{n})) \partial_{\alpha} h(\hat{W}_{B+1,i}) \right) \mathcal{L}_{B}(\frac{q_{1,i}}{\mu_{0}(I_{i})}) dm
\]
\[
= \sum_{\alpha} \sum_{i} \mu(I_{i}) \int_{I_{i}} \left( (f_{a} - \mu(f_{a}^{n})) \partial_{\alpha} h(\hat{W}_{B+1,i}) \right) \mathcal{L}_{n,B+1}(\frac{q_{1,i}}{\mu_{0}(I_{i})}) dm.
\]
\]
(21)

Since \( q \in \mathcal{D}_{L_{0}} \), Corollary 5.8 yields
\[
\left\| \mathcal{L}_{n,B+1} \left( \mathcal{L}_{B} \left( \frac{q_{1,i}}{\mu_{0}(I_{i})} \right) \right) - \mathcal{L}_{n}(q) \right\|_{L_{1}} \leq B_{0} \vartheta^{n-B},
\]
\]
(22)

where \( B_{0} = B_{0}(\lambda, A_{s}, L_{0}), \vartheta = \vartheta(A_{s}, \lambda) \). If \( |n - K/2| \leq 0 \), then \( B = 0 \) and
\[
\mathcal{L}_{n,B+1} \left( \mathcal{L}_{B} \left( \frac{q_{1,i}}{\mu_{0}(I_{i})} \right) \right) = \mathcal{L}_{n}(q).
\]
\]
(23)

From (22) and (23) it follows that
\[
\left\| \mathcal{L}_{n,B+1} \left( \mathcal{L}_{B} \left( \frac{q_{1,i}}{\mu_{0}(I_{i})} \right) \right) - \mathcal{L}_{n}(q) \right\|_{L_{1}} \leq B_{0} \vartheta^{\lceil \frac{n}{2} \rceil},
\]
\]
(24)

Now \( \| (f_{a} - \mu(f_{a}^{n})) \partial_{\alpha} h(\hat{W}_{n+1,i}) \|_{\infty} \leq 2 \| f \|_{\infty} \| Dh \|_{\infty} \) for every \( \alpha \), and thus (19), (21) and (24) give
\[ \left| \sum_{\alpha} \sum_{i} \int_{S^1} \left( (f_\alpha^n - \mu(f_\alpha^n)) \partial_\alpha h \left( C_\alpha + \frac{l}{\sqrt{N}} \sum_{j=n+K+1}^{N-1} f^j \right) \right) \varrho \, dm \right. \\
\left. - \sum_{\alpha} \sum_{i} \mu(I_i) \int_{S^1} \left( (f_\alpha - \mu(f_\alpha^n)) \partial_\alpha h(\tilde{W}_{n+1,i}) \right) \mathcal{L}_n(\varrho) \, dm \right| \\
\leq 2d \| f \|_\infty \| Dh \|_\infty B_0 \vartheta \left( \frac{N}{2} \right). \tag{25} \]

**Step 3.** In Step 3 we use Lemma 5.9 to show that \( f_\alpha - \mu(f_\alpha^n) \) and \( \partial_\alpha h(\tilde{W}_{n+1,i}) \) are nearly uncorrelated. Furthermore \( \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \mathcal{L}_n(\varrho) \, dm = 0 \). These facts then yield that

\[ \sum_{\alpha} \sum_{i} \mu(I_i) \int_{S^1} \left( (f_\alpha - \mu(f_\alpha^n)) \partial_\alpha h(\tilde{W}_{n+1,i}) \right) \mathcal{L}_n(\varrho) \, dm \approx 0. \]

More precisely, first (20) gives

\[ \sum_{\alpha} \sum_{i} \mu(I_i) \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \partial_\alpha h(\tilde{W}_{n+1,i}) \mathcal{L}_n(\varrho) \, dm = \sum_{\alpha} \sum_{i} \mu(I_i) \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \partial_\alpha h(\tilde{W}_{n+1,i}) \circ T_{n+K,n+1} \mathcal{L}_n(\varrho) \, dm. \]

Lemma 5.9 then yields

\[ \left| \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \partial_\alpha h(\tilde{W}_{n+K+1,i}) \circ T_{n+K,n+1} \mathcal{L}_n(\varrho) \, dm \right. \\
\left. - \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \mathcal{L}_n(\varrho) \, dm \int_{S^1} \partial_\alpha h(\tilde{W}_{n+K+1,i}) \circ T_{n+K,n+1} \mathcal{L}_n(\varrho) \, dm \right| \\
\leq \| Dh \|_\infty \left( 2 \operatorname{Lip}(f) \lambda^{-\left\lceil \frac{N}{2} \right\rceil} + 2 \| f \|_\infty B_0 \vartheta \left( \frac{N}{2} \right) \right). \]

Hence \( \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \mathcal{L}_n(\varrho) \, dm = 0 \) and \( \sum_i \mu(I_i) = 1 \), and we deduce

\[ \left| \sum_{\alpha} \sum_{i} \mu(I_i) \int_{S^1} (f_\alpha - \mu(f_\alpha^n)) \partial_\alpha h(\tilde{W}_{n+1,i}) \mathcal{L}_n(\varrho) \, dm \right| \leq d \| Dh \|_\infty \left( 2 \operatorname{Lip}(f) \lambda^{-\left\lceil \frac{N}{2} \right\rceil} + \| f \|_\infty B_0 \vartheta \left( \frac{N}{2} \right) \right). \tag{26} \]

**Step 4.** Using the triangle inequality and the estimates collected in (18),(25) and (26) we get

\[ |\mu(\bar{f}_n \cdot \nabla h(v + W_n t))| \leq 2d^2 \| f \|_\infty \| D^2 h \|_\infty \frac{\operatorname{Lip}(f) \lambda^{-\left\lceil \frac{N}{2} \right\rceil}}{\sqrt{N} (\lambda - 1)} \]
\[ + 2d \| f \|_\infty \| Dh \|_\infty B_0 \vartheta \left( \frac{N}{2} \right) \]
\[ + d \| Dh \|_\infty \left( 2 \operatorname{Lip}(f) \lambda^{-\left\lceil \frac{N}{2} \right\rceil} + \| f \|_\infty B_0 \vartheta \left( \frac{N}{2} \right) \right) \]
\[ \leq 2d^2 \| f \|_\infty \| D^2 h \|_\infty \frac{\operatorname{Lip}(f) \lambda^{-\frac{N-1}{2}}}{\sqrt{N} (\lambda - 1)} \]
\[ + 3dB_0 \| f \|_\infty \| Dh \|_\infty \vartheta \left( \frac{N}{2} \right) + 2d \| Dh \|_\infty \operatorname{Lip}(f) \lambda^{-\frac{N-1}{2}}. \]
This finishes the proof of Theorem 5.3.

Theorem 5.4 is proved with exactly same steps, by replacing $\nabla h$ by $A$, $v$ by 0 and $t$ by 1.

5.3. **Finishing the proofs of Theorems in Section 3.** After calculating upper bounds for $\rho$ and $\tilde{\rho}$ we are now ready to prove the theorems and corollaries in Section 3.

We use Theorems 2.1 and 2.2 to prove the results in Section 3. Using those results requires choosing values of $N$ and $K$ such that $N > K$. It turns out that to minimize the upper bounds in results of Section 3 we need to choose $K = C \log N$, where $C$ is some constant. For small values of $N$ it might be that $C \log N < N$, therefore we have formulated the results in such way that they hold only for large enough $N$.

We are going to choose $K = \left\lceil \frac{2 \log N}{-\log \vartheta} \right\rceil$ and the purpose of next lemma is to guarantee that this choice works in the proof as meant.

**Lemma 5.10.** Let $\vartheta \in ]0, 1[$. Then

i) If $x \geq 3$, then $\left\lceil \frac{2 \log x}{-\log \vartheta} \right\rceil + 1 \leq \frac{4 \log x}{1 - \vartheta}$.

ii) If $x \geq \max\{3, 16/(1 - \vartheta)^2\}$, then $x > \left\lceil \frac{2 \log x}{-\log \vartheta} \right\rceil$.

**Proof.** i). First we introduce a following fact: If $a \geq 1$ and $b > 0$, then

\[ \left\lceil \frac{2a}{b} \right\rceil \leq \frac{3a}{\min\{1, b\}}. \]  \hspace{1cm} (27)

This can be seen by studying the cases $b \leq 1$ and $b > 1$ separately. Thus it holds that

\[ \left\lceil \frac{2 \log x}{-\log \vartheta} \right\rceil + 1 \leq \frac{3 \log x}{\min\{1, -\log \vartheta\}} + 1 \leq \frac{4 \log x}{\min\{1, -\log \vartheta\}} \leq \frac{4 \log x}{1 - \vartheta}, \]

which completes the proof of i).

ii). Assume first that $3 \geq 16/(1 - \vartheta)^2$. Then using i) yields

\[ \left\lceil \frac{2 \log x}{-\log \vartheta} \right\rceil + 1 \leq \frac{4 \log x}{1 - \vartheta} \leq \sqrt{3} \log x < x. \]

Assume then that $16/(1 - \vartheta)^2 > 3$, and let $x_0 = 16/(1 - \vartheta)^2$. Then by i)

\[ \left\lceil \frac{2 \log x_0}{-\log \vartheta} \right\rceil \leq \frac{4 \log x_0}{1 - \vartheta} - 1. \]

Since for all $y > 0$ it holds that $\log y^2 = 2 \log y < y$, we have

\[ \frac{4 \log x_0}{1 - \vartheta} - 1 = \frac{4 \log \left( \frac{4}{(1 - \vartheta)} \right)^2}{1 - \vartheta} - 1 \leq \left( \frac{4}{1 - \vartheta} \right)^2 - 1 < x_0. \]

Let $x \geq x_0$. The derivative of $4 \log x/(1 - \vartheta)$ with respect to $x$ is $4/x(1 - \vartheta)$, which is at most $(1 - \vartheta)/4$, when $x \geq 16/(1 - \vartheta)^2$. By the assumption $16/(1 - \vartheta)^2 \geq 3$ we have that $(1 - \vartheta)/4 \leq 1/\sqrt{3} \leq 1$. Thus

\[ \left\lceil \frac{2 \log x}{-\log \vartheta} \right\rceil < \frac{4 \log x}{1 - \vartheta} \leq x_0 + \int_{x_0}^{x} \frac{4}{t(1 - \vartheta)} dt \leq x_0 + \int_{x_0}^{\infty} \frac{1 - \vartheta}{4} dt \leq x_0 + (x - x_0) = x, \]

for every $x \geq 16/(1 - \vartheta)^2$. 

\[ \Box \]
5.3.1. Proof of Theorem 3.1.

Proof. The proof of Theorem 3.1 is based on applying Theorem 2.1 to the model introduced in Section 3. First we verify that the assumptions of Theorem 2.1 are satisfied.

Clearly the transformations $T_i$, and the functions $f$ and $h$ in Theorem 3.1 are such that the corresponding assumptions in Theorem 2.1 hold. Assumption (A3) is also explicitly stated in Theorem 3.1.

Let then

$$N \geq \max \left\{ 3, \frac{16}{(1-\vartheta)^2} \right\} \quad \text{and} \quad K = \left\lfloor \frac{2\log N}{-\log \vartheta} \right\rfloor$$

be fixed. By Lemma 5.10.ii), we have $K < N$. We choose the functions $\rho(K)$ and $\tilde{\rho}(K)$ to be as in Lemma 5.2 and (10), respectively. As was proven in the previous section, with those choices, the Assumptions (A1) and (A2) hold.

It is also crucial to notice that the constants $C_2, C_4$ and $B_0$ in those definitions do not depend on $N$ or $K$. Therefore in the forthcoming computation, every dependence on $N$ and $K$ is explicit.

We have thus checked that the Theorem 2.1 is applicable under the setting described in Theorem 3.1 with the choices described above. It yields

$$|\mu(h(W)) - \Phi_{\Sigma_N}(h)| \leq 6d^3 \max\{C_2, \sqrt{C_4}\} \left( \|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty \right) \sqrt{\sum_{i=0}^\infty (i+1)\rho(i) \left( \frac{K+1}{\sqrt{N}} + \sum_{i=K+1}^\infty \rho(i) \right)}$$

$$+ \left( 2d^2 \|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-1} - 1} \vartheta^{K+1} + 3dB_0 \|f\|_{\text{Lip}} \|Dh\|_\infty \vartheta^{K} + 2d \|Dh\|_\infty \|f\|_{\text{Lip}} \vartheta^{K+1} \right) \sqrt{N}. \quad (28)$$

Since $\rho(i) = \vartheta^i$ we have

$$\sum_{i=K+1}^\infty \rho(i) = \sum_{i=K+1}^\infty \vartheta^i = \frac{\vartheta^{K+1}}{1-\vartheta} \quad (29)$$

and, by some calculations omitted here,

$$\sqrt{\sum_{i=0}^\infty (i+1)\rho(i)} \leq \frac{1}{1-\vartheta}. \quad (30)$$

Thus by (29) and (30):

$$(28) \leq 6d^3 \max\{C_2, \sqrt{C_4}\} \left( \|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty \right) \frac{1}{1-\vartheta} \left( \frac{K+1}{\sqrt{N}} + \frac{\vartheta^{K+1}}{1-\vartheta} \right)$$

$$+ \left( 2d^2 \|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-1} - 1} \vartheta^{K+1} + 3dB_0 \|f\|_{\text{Lip}} \|Dh\|_\infty \vartheta^{K} + 2d \|Dh\|_\infty \|f\|_{\text{Lip}} \vartheta^{K+1} \right) \sqrt{N}. \quad (31)$$

We now make the substitution $K = \left\lfloor \frac{2\log N}{-\log \vartheta} \right\rfloor$ to (31). Then $\vartheta^K \leq \vartheta^{-\frac{2\log N}{\log \vartheta}} = N^{-2}$, and by Lemma 5.10.i)

$$K + 1 \leq \frac{4\log N}{1-\vartheta}. \quad (32)$$
Thus
\[(31) \leq 6d^3 \max\{C_2, \sqrt{C_4}\} (\|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty) \frac{1}{1 - \vartheta} \left( \frac{4 \log N}{1 - \vartheta} + \frac{N^{-2}}{(1 - \vartheta)\sqrt{N}} \right)
+ \left(2d^2 \|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2 N^{-1}}{\vartheta^{-\frac{1}{2}} - \vartheta^{\frac{1}{2}}} + 3dB_0 \|f\|_{\text{Lip}} \|D h\|_\infty N^{-1} + \frac{2d\|D h\|_\infty \|f\|_{\text{Lip}}^{-1}}{\vartheta^{-\frac{1}{2}}} \right) \sqrt{N}.
\]
\[
\leq \left(6d^3 \max\{C_2, \sqrt{C_4}\} (\|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty) \frac{4 \log N}{1 - \vartheta} + \frac{1}{1 - \vartheta} \right) \left(1 - \vartheta\right)^2
+ \left(2d^2 \|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-\frac{1}{2}} - \vartheta^{\frac{1}{2}}} + 3dB_0 \|f\|_{\text{Lip}} \|D h\|_\infty + \frac{2d\|D h\|_\infty \|f\|_{\text{Lip}}^{-1}}{\vartheta^{-\frac{1}{2}}} \right) \left(1 - \vartheta\right) N^{-\frac{1}{2}} \log N.
\]
\[
\leq \left(30d^3 \max\{C_2, \sqrt{C_4}\} \left(\|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty\right) \frac{4 \log N}{1 - \vartheta} + \frac{1}{1 - \vartheta}\right) \left(1 - \vartheta\right)^2
+ 2d^2 \|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-\frac{1}{2}} - \vartheta^{\frac{1}{2}}} + 3dB_0 \|f\|_{\text{Lip}} \|D h\|_\infty + \frac{2d\|D h\|_\infty \|f\|_{\text{Lip}}^{-1}}{\vartheta^{-\frac{1}{2}}} \right) \left(1 - \vartheta\right) N^{-\frac{1}{2}} \log N.
\]
\[
(32)
\]
Since we assumed $N \geq 3$ we have $\log N \geq 1$ which finally yields
\[
|\mu(h(W)) - \Phi_{\Sigma_N}(h)|
\leq \left(30d^3 \max\{C_2, \sqrt{C_4}\} \left(\|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty\right) \frac{4 \log N}{1 - \vartheta} + \frac{1}{1 - \vartheta}\right) \left(1 - \vartheta\right)^2
+ 2d^2 \|D^2 h\|_\infty \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-\frac{1}{2}} - \vartheta^{\frac{1}{2}}} + 3dB_0 \|f\|_{\text{Lip}} \|D h\|_\infty + \frac{2d\|D h\|_\infty \|f\|_{\text{Lip}}^{-1}}{\vartheta^{-\frac{1}{2}}} \right) \left(1 - \vartheta\right) N^{-\frac{1}{2}} \log N.
\]
Since (32) holds for all $N \geq \max\left\{3, \frac{16}{(1 - \vartheta)^2}\right\}$, we have now completed the proof of Theorem 3.1.

5.3.2. Proofs of Theorem 3.2 and Corollary 3.3. The proof of Theorem 3.2 proceeds similarly to the last one. As in the previous proof let

\[
N \geq \max\left\{3, \frac{16}{(1 - \vartheta)^2}\right\} \quad \text{and} \quad K = \left\lfloor \frac{2 \log N}{-\log \vartheta} \right\rfloor
\]

be fixed, and let $p \geq 0$ and $C_0 > 0$ be such that $\sigma_N \geq C_0 N^{-p}$. Define functions $\rho(K)$ and $\tilde{\rho}(K)$ as in Lemma 5.2 and (11), respectively. The assumptions of Theorem 2.2 are now satisfied as the reader may verify. By reusing the results in the proof of Theorem 3.1, it yields
$$d_\#(W, \sigma_N Z)$$

\[
\leq 12 \max\{\sigma_N^{-1}, \sigma_N^{-2}\} \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_\infty) \left( \sum_{i=0}^{\infty} (i + 1) \rho(i) \left( \frac{K + 1}{\sqrt{N}} + \sum_{i=K+1}^{\infty} \rho(i) \right) \right.
\]

\[
+ 2 \max\{1, \sigma_N^{-2}\} \sqrt{N} \rho(K)
\]

\[
\leq \frac{12 \max\{\sigma_N^{-1}, \sigma_N^{-2}\} \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_\infty)}{1 - \vartheta} \left( \frac{4 \log N}{(1 - \vartheta) \sqrt{N}} + \frac{N^{-2}}{1 - \vartheta} \right)
\]

\[
+ 2 \max\{1, \sigma_N^{-2}\} \sqrt{N} \left( 2 \frac{\|f\|_{\text{Lip}}^2 \vartheta^{-2} \left( 1 + \frac{1}{K} \right)}{\vartheta^{-1} - 1} + 3 B_0 \|f\|_{\text{Lip}} \vartheta^{-2} + 2 \|f\|_{\text{Lip}} \vartheta^{-2} \right)
\]

\[
\leq \frac{60 \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_\infty)}{(1 - \vartheta)^2} \max\{\sigma_N^{-1}, \sigma_N^{-2}\} N^{-\frac{1}{2}} \log N
\]

\[
+ \left( 4 \frac{\|f\|_{\text{Lip}}^2}{\vartheta^{-2} - \vartheta^{-2}} + 6 B_0 \|f\|_{\text{Lip}} + 4 \frac{\|f\|_{\text{Lip}}}{\vartheta^{-2}} \right) \max\{1, \sigma_N^{-2}\} N^{-\frac{1}{2}}.
\]

We have \(\max\{\sigma_N^{-1}, \sigma_N^{-2}\} \leq \max\{1, \sigma_N^{-2}\}\) and thus

$$d_\#(W, \sigma_N Z) \leq \tilde{C} \max\{1, \sigma_N^{-2}\} N^{-\frac{1}{2}} \log N,$$

(33)

where

$$\tilde{C} = \frac{60 \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_\infty)}{(1 - \vartheta)^2} + \frac{4 \|f\|_{\text{Lip}}^2}{\vartheta^{-2} - \vartheta^{-2}} + 6 B_0 \|f\|_{\text{Lip}} + 4 \|f\|_{\text{Lip}} \vartheta^{-2}.$$ 

Theorem 3.2 now follows: Since it was assumed that \(\sigma_N \geq C_0 N^{-p}\) it holds that \(\max\{1, \sigma_N^{-2}\} \leq \max\{1, C_0^{-2}\} N^{2p}\) and by (33) we have

$$d_\#(W, \sigma_N Z) \leq \tilde{C} \max\{1, C_0^{-2}\} N^{-\frac{1}{2} + 2p} \log N.$$ 

Furthermore, notice that \(\tilde{C}\) does not depend on \(N\).

The idea behind the proof of Corollary 3.3 is as follows:

If the variance of \(W\) is large, then Theorem 3.2 gives good upper bound to Wasserstein distance \(d_\#(W, \sigma_N Z)\). On the other hand if \(\sigma_N\) is close to zero, then both the distribution of \(W\) and \(\sigma_N Z\) are close to the Dirac delta distribution \(\delta_0\) in the sense of Wasserstein distance. This gives us two distinct ways to find upper bound to \(d_\#(W, \sigma_N Z)\). It turns out that the worst-case scenario happens when variance behaves like \(CN^{-\frac{1}{2}}\).

To handle the small values of \(\sigma_N\), we introduce the following fact:

Let \(X\) and \(Y\) be two random variables with means 0 and variances \(\sigma_X^2, \sigma_Y^2\), respectively. Then

$$d_\#(X, Y) \leq \sigma_X + \sigma_Y.$$ 

(34)

To see this, assume that \(X_0\) is a random variable such that \(P(X_0 = 0) = 1\). Then

$$d_\#(X_0, X) = \sup_{h \in \mathcal{H}} \left| \int h(x) dF_{X_0}(x) - \int h(x) dF_X(x) \right|$$

$$= \sup_{h \in \mathcal{H}} |h(0) - \mathbb{E}[h(X)]| \leq \mathbb{E}[|X|] \leq \sqrt{\mathbb{E}[X^2]} = \sigma_X.$$

Now (34) follows from triangle inequality, since \(d_\#\) is a metric.
Assume that $\sigma_N \geq CN^{-p}$, where $p$ and $C$ are some constants. Then the Wasserstein distance $d_W(W, Z)$ has an upper bound of type $CN^{-\frac{1}{2}+2p} \log N$ by Theorem 3.2. On the contrary, if $\sigma_N < CN^{-p}$ then formula (34) gives an upper bound of type $CN^{-p}$. Since the equation $-\frac{1}{2} + 2p = -p$ is solved by $p = \frac{1}{6}$, we make the following choices:

Let $p = \frac{1}{6}$. Choose $C_p = 1$ in Theorem 3.2. Then

$$d_W(W, \sigma_N Z) \leq \tilde{C} N^{-\frac{1}{6}} \log N,$$

(35)

when $\sigma_N \geq N^{-\frac{1}{4}}$. If $\sigma_N < N^{-\frac{1}{4}}$, then by (34)

$$d_W(W, \sigma_N Z) \leq 2N^{-\frac{1}{6}} \leq 2N^{-\frac{1}{4}} \log N.$$

(36)

Since either (35) or (36) holds, we have

$$d_W(W, \sigma_N Z) \leq \max \left\{ \tilde{C}, 2 \right\} N^{-\frac{1}{6}} \log N.$$

(37)

This proves Corollary 3.3.

5.3.3. Proof of Corollary 3.4. Let $C'_0 > 0$, $r \geq 0$ and $s^2_N = N\sigma_N^2 > C'_0 N^r$, which implies that $\sigma_N = \frac{s^2_N}{\sqrt{N}} > \sqrt{C'_0} N^{-\frac{r}{m-1}}$. Then using properties of Wasserstein distance gives

$$d_W\left( \frac{S_N}{s_N}, Z \right) = d_W\left( \frac{W}{\sigma_N}, Z \right) = \sigma_N^{-1} d_W(W, \sigma_N Z) = C'_0^{-\frac{1}{2}} N^{-\frac{1}{2}} d_W(W, \sigma_N Z).$$

We may now apply Theorem 3.2 to $d_W(W, \sigma_N Z)$, with values $p = (1-r)/2$ and $C_0 = \sqrt{C'_0}$ which yields

$$d_W\left( \frac{S_N}{s_N}, Z \right) = C'_0^{-\frac{1}{2}} N^{-\frac{1}{2}} d_W(W, \sigma_N Z)$$

$$\leq C'_0^{-\frac{1}{2}} N^{-\frac{1}{2}} \left( \tilde{C} \max\{1, C'_0^{-1}\} N^{-\frac{1}{2}+1-r} \log N \right)$$

$$\leq \tilde{C} \max\{C'_0^{-\frac{1}{2}}, C'_0^{-\frac{3}{2}}\} N^{1-\frac{3r}{2}} \log N.$$

This completes the proof of Corollary 3.4.

6. Proofs for Application II

In this section we use the notation defined in Section 4. The reader should recall the definitions of $T_{n,i}, T_{n,i,j}$ and $f_{n,i}$ from that section to avoid confusion with the notations used on Sections 3 and 5. The pushforward measure $(T_{n,k})_* \mu$ is denoted by $\mu_{n,k}$ and the corresponding density by $\vartheta_{n,k}$.

The density $\hat{\vartheta}_t$ of the SRB measure $\hat{\mu}_t$ is Lipschitz continuous by Remark 4.1 of [12]. By the same remark $L^k_t$ converges to $\hat{\vartheta}_t$ in the supremum norm and thus $\hat{\vartheta}_t > 0$. Furthermore by Remark 4.4. (iii) of [12] $\hat{\vartheta}_t \in \mathcal{D}_L$ for some $L \geq 0$. Since $L^k_t \hat{\vartheta}_t = \hat{\vartheta}_t$ for all $k \geq 0$, by (9), we have $\hat{\vartheta}_t \in \mathcal{D}_L^*.$

6.1. Preliminary results. By duality

$$\hat{\sigma}_s^2(f) = \hat{\mu}_s[\hat{f}_s^2] + 2 \sum_{k=1}^{\infty} m[\hat{f}_s L^k_t(\hat{\vartheta}_t \hat{f}_s)] = \hat{\mu}_s[f^2] - \hat{\mu}_s[f]^2 + 2 \sum_{k=1}^{\infty} \left\{ \hat{\mu}_s[f f \varnothing \gamma^k_s] - \hat{\mu}_s[f]^2 \right\}.$$

(38)
For later use, notice that $\hat{\sigma}_s^2(f)$ can be represented in the integral form
\[ n \int_{-\infty}^{\infty} \hat{\mu}_s(f f \circ \gamma_s^{[n s]-[n(s+r)]}) - \hat{\mu}_s(f)^2 \, dr, \tag{39} \]
where $n = 1, 2, \ldots$

In Lemma 6.1 of [12] it is proven that $t \to \hat{\sigma}_s^2(f)$ is uniformly continuous. We improve the proof to show that it is even Hölder continuous.

**Lemma 6.1.** $t \to \hat{\sigma}_s^2(f)$ is Hölder continuous with every exponent $\eta'' < \eta$. An upper bound for the corresponding Hölder constant $C$ can be given as a function of $\lambda, A_s, \eta, \eta''$, $C_H$ and $\|f\|_{\text{Lip}}$.

**Proof.** Let $k \geq 0$. We have $m(\hat{f}_t \mathcal{L}^k(\hat{\sigma}_s \hat{f}_t)) = m(\mathcal{L}^k(\hat{\sigma}_s f)) - m(\hat{\sigma}_s f)^2$. The computation given in the proof of Lemma 6.1 in [12] yields
\[ |m(\mathcal{L}^k(\hat{\sigma}_s f)) - m(\mathcal{L}^k(\hat{\sigma}_s f))| \leq \|f\|_{\text{Lip}} \left( \|\mathcal{L}^k - \mathcal{L}^k_{\text{Lip}}\| + \|\hat{\sigma}_s - \hat{\sigma}_s\|_{\text{Lip}} \right). \]

Let $\eta'' < \eta$ and $k \geq 1$. By (19), (8) and (22) of [12] the right side can be approximated from above by
\[ C(k \delta c_0(\gamma_t, \gamma_s) + |t - s|^{\eta''}) \leq C(k \delta c_0(\gamma_t, \gamma_s) + |t - s|^{\eta''}) \leq C k \delta c_0(\gamma_t, \gamma_s) + |t - s|^{\eta''}, \]

where $C = C(\lambda, A_s, C_H, \|f\|_{\text{Lip}}, \eta'')$. Using the same results for $k = 0$ it also follows that
\[ |\mu_k(\hat{\sigma}_s^2) - \mu_k(\hat{\sigma}_s^2)| = \left| m(\hat{\sigma}_s^2) - m(\hat{\sigma}_s f)^2 + m(\hat{\sigma}_s f)^2 - m(\hat{\sigma}_s f)^2 \right| \leq C |t - s|^{\eta''}. \]

Furthermore we have that for every $M \in \mathbb{N}$
\[ \sum_{k = M}^{\infty} m(\hat{f}_t \mathcal{L}^k(\hat{\sigma}_s \hat{f}_t)) = \sum_{k = M}^{\infty} m(\mathcal{L}^k(\hat{\sigma}_s f)) - m(\hat{\sigma}_s f)^2 \leq C \vartheta^M \]

by Lemma 5.9. Combining all these observations, formula (38) yields
\[ |\hat{\sigma}_t^2(f) - \hat{\sigma}_s^2(f)| \leq \sum_{k = 0}^{M-1} M \vartheta \delta^M \leq C |t - s|^{\eta''} + \vartheta^M \]

for all $M = 1, 2, \ldots$, where $C = C(\lambda, A_s, C_H, \|f\|_{\text{Lip}}, \eta'')$. Choosing $C$ large enough,
\[ |\hat{\sigma}_t^2(f) - \hat{\sigma}_s^2(f)| \leq C |t - s|^{\eta''} + \vartheta^M \]

holds also for all real numbers $M \geq 0$. To prove Hölder-continuity, we choose $M$ depending on $|t - s| > 0$ in the following way:
\[ |t - s|^{\eta''} = \vartheta^M \Rightarrow M = \frac{\log |t - s|^{\eta''}}{\log \vartheta} = \frac{\eta''}{\log \vartheta} \log |t - s| > 0. \]

Using the well known fact that for all $x \in [0, 1]$ and $\alpha \in [0, 1]$ there exists a constant $C = C(\alpha)$ such that
\[ x |\log x| \leq C x^\alpha, \]
we deduce that
\[ M^2 = \frac{\eta''^2}{\log^2 \vartheta} \log^2 |t - s| \leq \frac{\eta''^2}{\log^2 \vartheta} C |t - s|^{2\alpha - 2}, \]

where $C = C(\lambda, A_s, \eta'', \alpha)$. Let $0 < \eta'' < \eta'$. Choose $\alpha = 1 - (\eta' - \eta'')/2$. Then
\[ |\hat{\sigma}_t^2(f) - \hat{\sigma}_s^2(f)| \leq C (M^2 |t - s|^{\eta''} + \vartheta^M) \leq C \left( \frac{\eta''^2}{\log^2 \vartheta} C |t - s|^{\eta'' + 2\alpha - 2} + |t - s|^{\eta''} \right) \leq C |t - s|^{\eta''}, \]

where, in the rightmost expression, $C = C(\lambda, A_s, C_H, \|f\|_{\text{Lip}}, \eta', \eta'')$. 
Since \( \eta'' \) can be arbitrarily close to \( \eta' \) we see that \( t \mapsto \hat{\sigma}^2_t(f) \) is Hölder continuous in \([0,1]\) for any Hölder exponent \( \eta'' \leq \eta \). The result now follows by choosing for example \( \eta' = (\eta + \eta'')/2 \).

The next lemma follows from Lemma 5.9 in [12]

**Lemma 6.2.** There exists a constant \( b > 0 \) such that the following holds. Given \( \eta' < \eta \) there exists \( C = C(\lambda, A_s, C_H, \eta', L_0) \) such that

\[
\| \varrho_{n,[nt]} - \hat{\sigma}_s \|_{L^1} \leq C(n^{-\eta'} + |t - s|^{\eta'}),
\]

when \( t \geq bn^{-1} \log n \).

Recall that the variance of \( \xi_n(t) \) with respect to \( \mu \) is denoted by \( \sigma^2_{n,t} \). Since \( \xi_n(t) \) is a sum of random variables with mean 0, we have \( \mu(\xi_n(t)) = 0 \) and that \( \sigma^2_{n,t} = \mu(\xi_n(t)^2) \).

Next we approximate

\[
|\sigma^2_{n,t} - \sigma^2_t|.
\]

The proof of the following lemma follows that of Lemma 6.2 of [12]. We need a more explicit version in this paper.

**Lemma 6.3.** Let \( \eta'' < \eta \). Then there exists a constant \( C = C(\lambda, A_s, C_H, \eta, \eta'', L_0, \|f\|_{\text{lip}}) \) such that

\[
|\sigma^2_{n,t} - \sigma^2_t| \leq C n^{-\eta''}
\]

for every \( t \in [0,1] \).

**Proof.** Let \( \eta' < \eta \). We have

\[
\sigma^2_{n,t} = \mu((\xi_n(t))^2) = n \int_0^t \int_0^t \mu(f_{n,[ns]} f_{n,[nr]}) dr ds.
\]

Let \( \kappa \in \left]0, \frac{1}{4}\right[ \) satisfy \( 2\kappa < \eta'(1 - \kappa) \) and define \( a_n = n^{-1+\kappa} \). Then \( a_n > bn^{-1} \log n \) for big enough \( n \), where \( b \) is the same constant as in Lemma 6.2. Define the sets

\[
P_n = \{(s,r) \in [0,t]^2 : 2a_n \leq s \leq t - a_n \text{ and } |r - s| \leq a_n\},
\]

\[
Q_n = \{(s,r) \in [0,t]^2 : |r - s| \leq a_n \text{ and either } s < 2a_n \text{ or } s > t - a_n\}
\]

and

\[
R_n = \{(s,r) \in [0,t]^2 : |r - s| > a_n\}.
\]

Notice that \( P_n \cup R_n \cup Q_n = [0,t]^2 \). The area of \( Q_n \) is at most \( 6a_n^2 \) and \( \|f_{n,[ns]} f_{n,[nr]}\| \leq \|f\|_{\text{lip}}^2 \). Thus

\[
\left| n \int_{Q_n} \mu(f_{n,[ns]} f_{n,[nr]}) dr ds \leq 6\|f\|_{\text{lip}}^2 a_n^2 = 6\|f\|_{\text{lip}}^2 n^{-1+2\kappa}\right.
\]

From now on \( E \) denotes a real valued function such that there exists a constant \( C = C(\lambda, A_s, C_H, \eta', L_0, \kappa, \|f\|_{\text{lip}}) > 0 \) such that \( |E| \leq C \). The specific formulas for values of \( C \) might change from line to line in the computation.

By Lemma 5.10 in [12] we know that

\[
|\mu(f_{n,[ns]} f_{n,[nr]})| \leq E \varrho^{n|r-s|}.
\]

By (40) we see that

\[
\left| n \int_{R_n} \mu(f_{n,[ns]} f_{n,[nr]}) dr ds \leq E \varrho^{na_n} = E \varrho^n \right.
\]
For large enough \( n \), we have \( E\hat{\vartheta}^n \leq 6\|f\|_{\infty}^2 n^{-1+2\varepsilon} \). Thus
\[
\left| n \int_{Q_{n}\cup R_n} \mu(f_{n,\lfloor ns \rfloor}, f_{n,\lfloor nr \rfloor}) dr ds \right| \leq 6\|f\|_{\infty}^2 n a_n^2 = Ena_n^2.
\]

The only major contribution to the integral now comes from \( P_n \), i.e.
\[
n \int_{-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}, f_{n,\lfloor nr \rfloor}) dr ds = n \int_{-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}, f_{n,\lfloor nr \rfloor}) dr ds + Ena_n^2, \tag{41}
\]

Next we will show that \( n \int_{-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}, f_{n,\lfloor nr \rfloor}) dr \approx \hat{\sigma}_s^2 \).

By Lemma 6.2 we have
\[
\|\varrho_{n,\lfloor nr \rfloor} - \hat{\varrho}_s\|_{L^1} = E(n^{-\eta'} + |r - s|^\eta'),
\]
when \( r > bn^{-1} \log n \). Thus
\[
\sup_{r \in (s-a_n, s+a_n)} \|\varrho_{n,\lfloor nr \rfloor} - \hat{\varrho}_s\|_{L^1} = E(n^{-\eta'} + a_n^{1+\eta'}) = Ea_n^{\eta'}. \]

From this it follows that
\[
n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}, f_{n,\lfloor nr \rfloor}) dr
= n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor} - f_{n,\lfloor nr \rfloor}) dr
= n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}) dr - n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor nr \rfloor}) dr
= n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}) dr - n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor nr \rfloor}) dr + E(na_n^{1+\eta'}) \tag{42}
\]

Define \( b_n = \frac{1}{n}(1 - \{ns\}) \). We have
\[
n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor}) dr = n \int_{0}^{a_n} \mu(f_{n,\lfloor ns \rfloor}) dr
= b_n n \mu_{\lfloor ns \rfloor}(f^2) + n \int_{0}^{a_n} \mu_{\lfloor ns \rfloor} (ff \circ T_{n,\lfloor n(s+r) \rfloor} \circ \cdots \circ T_{n,\lfloor ns \rfloor+1}) dr
= b_n \hat{\mu}_{\lfloor ns \rfloor}(f^2) + n \int_{0}^{a_n} \hat{\mu}_{\lfloor ns \rfloor} (ff \circ T_{n,\lfloor n(s+r) \rfloor} \circ \cdots \circ T_{n,\lfloor ns \rfloor+1}) dr + E(a_n^{1+\eta'})
= n \int_{0}^{b_n} m(f \hat{\varrho}_s f) dr + n \int_{b_n}^{a_n} m(f \mathcal{L}_{n,\lfloor n(s+r) \rfloor} \cdots \mathcal{L}_{n,\lfloor ns \rfloor+1} (\hat{\varrho}_s f)) dr + E(a_n^{1+\eta'}). \]

We want to replace \( \mathcal{L}_{n,\lfloor n(s+r) \rfloor} \cdots \mathcal{L}_{n,\lfloor ns \rfloor+1} \) by \( \mathcal{L}_{\gamma_{\lfloor n(s+r) \rfloor}^{-\lfloor ns \rfloor}} \). For this purpose notice that for every \( j \in \{\lfloor ns \rfloor, \ldots, \lfloor n(s+r) \rfloor\} \) and \( r \leq a_n \), we have \( d_{C^1}(\gamma_s, T_{n,j}) \leq d_{C^1}(\gamma_s, \gamma_j) + d_{C^1}(\gamma_j, T_{n,j}) \leq E r^\eta + E n^{-\eta} \leq E a_n^\eta \). We have
\[
\|\mathcal{L}_{n,\lfloor n(s+r) \rfloor} \cdots \mathcal{L}_{n,\lfloor ns \rfloor+1} (\hat{\varrho}_s f) - \mathcal{L}_{\gamma_{\lfloor n(s+r) \rfloor}^{-\lfloor ns \rfloor}} (\hat{\varrho}_s f)\|_{L^1} = Ena_n^\eta = Ea_n^{\eta'+1}. \]

Hence,
\[
n \int_{s-a_n}^{s+a_n} \mu(f_{n,\lfloor ns \rfloor} f_{n,\lfloor nr \rfloor}) dr = n \int_{0}^{a_n} m(f \mathcal{L}_{\gamma_{\lfloor n(s+r) \rfloor}^{-\lfloor ns \rfloor}} (\hat{\varrho}_s f)) dr + E(a_n^{2+\eta'}). \tag{43}
\]

By a similar computation
\[
n \int_{s-a_n}^{s} \mu(f_{n,\lfloor ns \rfloor} f_{n,\lfloor nr \rfloor}) dr = n \int_{-a_n}^{0} m(f \mathcal{L}_{\gamma_{\lfloor n(s+r) \rfloor}^{-\lfloor ns \rfloor}} (\hat{\varrho}_s f)) dr + E(a_n^{2+\eta'}). \tag{44}
\]
Thus by (42), (43),(44) and using the formula (39) for the variance, we have
\[
\begin{align*}
& n \int_{s-a_n}^{s+a_n} \mu(f_n, [ns]) \mu(f_n, [nr]) dr \\
& = n \int_{s-a_n}^{s+a_n} \mu(f_n, [ns]) - \hat{\mu}_s(f)^2 dr + Ena_n^{2+\eta'} \\
& = n \int_{s-a_n}^{s+\eta} m(f, L_s([ns]-[n(s+s)])(\hat{g}_s f))) - \hat{\mu}_s(f)^2 dr + Ena_n^{1+\eta'} + En^2a_n^{2+\eta'} \\
& = n \int_{s-a_n}^{s-a_n} \hat{\mu}_s(f f \circ \gamma_s([ns]-[n(s+s)])) - \hat{\mu}_s(f)^2 dr + Ena_n^{2+\eta'} \\
& = n \int_{-\infty}^{\infty} \hat{\mu}_s(f f \circ \gamma_s([ns]-[n(s+s)])) - \hat{\mu}_s(f)^2 dr + Ena_n^{2+\eta'} + E \vartheta^{\eta a_n} \\
& = \sigma_s^2(f) + En^2a_n^{2+\eta'}. 
\end{align*}
\]

Note that we can choose an upper bound for $|E|$ that is independent of $s$. This is because $\hat{g}_s \in D_{L_s}$.

Therefore by (41) and (45)
\[
\begin{align*}
\mu((\xi_n(t))^2) &= n \int_{2a_n}^{t-a_n} \int_{s-a_n}^{s+a_n} \mu(f_n, [ns]) \mu(f_n, [nr]) dr ds + Ena_n^2 \\
& = \int_{2a_n}^{t-a_n} \hat{\sigma}_s^2(f) ds + Ena_n^2 + En^2a_n^{2+\eta'} \\
& = \int_{2a_n}^{t-a_n} \hat{\sigma}_s^2(f) ds + Ena_n^2 + En^2a_n^{2+\eta'} \\
& = \int_{0}^{t} \hat{\sigma}_s^2(f) ds + Ena_n^2 + En^2a_n^{2+\eta'} + E a_n \\
& = \sigma_t^2 + Ena_n^2 + En^2a_n^{2+\eta'} + E a_n \\
& = \sigma_t^2 + E n^{-1+2\kappa} + En^{2-\eta'(1-\kappa)}. 
\end{align*}
\]

Let $0 < \eta'' < \eta$. Recall that we have assumed that $\eta' \in [0, \frac{1}{4}]$, $2\kappa < \eta'(1-\kappa)$ and $\eta' < \eta$. By choosing $\eta' = (\eta + \eta'')/2$ and $\kappa = (\eta - \eta'')/(4(1+\eta))$ these assumptions are satisfied as the reader may check, and we have
\[
n^{-1+2\kappa} = n^{-1+\frac{2-\eta''}{2(1+\eta)}} = n^{-\frac{2-2\eta''}{4(1+\eta)}} \leq n^{-\eta''} = n^{-\eta''} \\
\]
and
\[
n^{2(1-\kappa)} = n^{\frac{1-2}{2(1+\eta)}} + \frac{n^{\eta''}}{2} \left( \frac{n^{\eta''}}{2(1+\eta)} - 1 \right) = n^{\frac{\eta'' - n^{\eta''} + n^{\eta''} - (4-3\eta' - \eta'')}{8(1+\eta)}} = n^{\frac{\eta'' - 4\eta'' - (4-3\eta' - \eta'')^2}{8(1+\eta)}} \leq n^{-\eta''}. 
\]

Thus it follows that $\sigma_{n,t}^2 = \mu((\xi_n(t))^2) = \sigma_t^2 + En^{-\eta''}$, where
\[
|E| < C = C(\lambda, A, C_H, \eta, \eta'', L_0, \|f\|_{Lip}). 
\]

6.2. Proofs of Theorems 4.3 and 4.4. An upper bound on the Wasserstein distance of two normal distributions is given by the next lemma.

Lemma 6.4. Let $a, b \geq 0$ and $Z \sim N(0, 1)$. Then $d_{\mathcal{W}}(aZ, bZ) \leq \frac{\sqrt{2}|a - b|}{\sqrt{\pi}}$. 

Proof. Let $h$ be 1-Lipschitz and $a, b \geq 0$. Then
\[
|E[h(aZ)] - E[h(aZ)]| \leq |a - b|E|Z| = \frac{\sqrt{2}|a - b|}{\sqrt{\pi}}.
\]

Next theorem proves Theorem 4.3 for large values of $n$. For small $n$ Theorem 4.3 holds trivially by choosing large enough $C$.

**Theorem 6.5.** Let $t_0 \in [0, 1]$ and $\gamma$ be such that $\sigma^2_{t_0}(f) > 0$. Then for all $\eta' \leq \eta$ there exists a constant $C = C(\lambda, A_s, C_H, \eta, \eta', L_0, \|f\|_{Lip}, t_0, \sigma^2_{t_0}) > 0$ and a constant $n_0 > 0$ such that for every $t \geq t_0$ and $n \geq n_0$
\[
d_W(\xi_n(t), \sigma_tZ) \leq C(n^{-\frac{1}{2}} \log n + n^{-\eta'}).
\]

**Proof.** This proof is divided in three steps. The Wasserstein distances
\[
d_W(\xi_n(t), \xi_n([nt]/n)), \quad d_W(\xi_n([nt]/n), \sigma_{[nt]/n}Z) \quad \text{and} \quad d_W(\sigma_{[nt]/n}Z, \sigma_tZ)
\]
are estimated in the corresponding order. The final result then follows by triangle inequality.

Before computing upper bounds on the Wasserstein distances in (46) we need to guarantee that for every $t \geq t_0$ and large enough $n$ the variances $\sigma_t$ and $\sigma_{n,t}$ are greater than some constant.

Since $t \mapsto \sigma^2_t$ is Hölder continuous by Lemma 6.1 it follows that there exists $t_1 = t_1(\lambda, A_s, C_H, \|f\|_{Lip}, \sigma^2_{t_0}, t_0) \leq t_0$ such that for every $t \in [t_1, t_0]$ it holds that $\sigma^2_t \geq \frac{\sigma^2_{t_0}}{2}$.

For $t \geq t_0$ this implies $\sigma^2_t \geq \frac{(t_0 - t_1)\sigma^2_{t_0}}{2}$ and by Lemma 6.3 for every $\eta' < \eta$ there exists $C = C(\lambda, A_s, C_H, \eta, \eta', L_0, \|f\|_{Lip}, \sigma^2_{t_0})$ such that
\[
|\sigma^2_{n,t} - \sigma^2_t| \leq Cn^{-\eta'} \quad \Rightarrow \quad \sigma^2_{n,t} \geq \sigma^2_t - Cn^{-\eta'} \geq \frac{(t_0 - t_1)\sigma^2_{t_0}}{2} - Cn^{-\eta'}.
\]

Thus there exists $n_0 = n_0(\lambda, A_s, C_H, \eta, \eta', L_0, \|f\|_{Lip}, \sigma^2_{t_0})$ such that
\[
\sigma^2_t \geq \frac{(t_0 - t_1)\sigma^2_{t_0}}{2} \quad \text{and} \quad \sigma^2_{n,t} \geq \frac{(t_0 - t_1)\sigma^2_{t_0}}{4},
\]
when $n \geq n_0$ and $t \geq t_0$.

To be able to apply Theorem 3.2 we also assume that $n_0t_0 \geq \max\{3, 16/(1 - \vartheta)^2\}$.

**Step 1.** Notice that $\xi_n([nt]/n) = \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]-1} \tilde{f}_{n,i}$. Thus
\[
d_W(\xi_n(t), \xi_n([nt]/n)) \leq \left\| \xi_n([nt]/n) - \xi_n(t) \right\|_\infty = \left\| \frac{1 - \{nt\}}{\sqrt{n}} \tilde{f}_{n,[nt]} \right\|_\infty \leq \frac{\|f\|_\infty}{\sqrt{n}},
\]
where the first inequality follows easily from the definition of Wasserstein distance.

**Step 2.** Let $t \geq t_0$ and $n \geq n_0$. We have by definition
\[
\xi_n([nt]/n) = \frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} \left( \frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{i=0}^{[nt]-1} \tilde{f}_{n,i} \right).
\]
Denote
\[ V = V([nt]) = \frac{1}{\sqrt{|nt|}} \sum_{i=0}^{[nt]-1} f_n,i = \frac{\xi_n \left( \frac{[nt]}{n} \right) \sqrt{n}}{\sqrt{|nt|}}. \]

Denote the variance of \( V([nt]) \) by \( v_n^2 = \frac{n}{|nt|} \sigma_n^2 [nt]/n \). Since \( [nt] \geq \max\{3, 16/(1-\vartheta)^2\} \), we can apply Theorem 3.2 to \( V([nt]) \) and it yields
\[
d_w \left( \xi_n \left( \frac{[nt]}{n} \right), \sigma_{n,[nt]}/n, Z \right) = d_w \left( \frac{\sqrt{|nt|}}{\sqrt{n}} V, \sigma_{n,[nt]}/n, Z \right)
\[ = \frac{\sqrt{|nt|}}{\sqrt{n}} d_w \left( V, v_{[nt]} Z \right) \]
\[ \leq C [nt]^{-\frac{1}{2}} \log [nt] \leq C t_0^{-\frac{1}{2}} n^{-\frac{1}{2}} \log n = C n^{-\frac{1}{4}} \log n, \]
where \( C = C(\lambda, A, \|f\|_{\text{Lip}}, L_0) \).

We have \( [nt]/n \geq t \) and thus \( \sigma_{[nt]/n} \geq ((t_0 - t_1)\hat{\sigma}_{t_0}^2/2)^{\frac{1}{2}} \). Therefore by Lemma 6.3
\[
|\sigma_{n,[nt]/n} - \sigma_{[nt]/n}| = \frac{|\sigma_n^2 - \sigma_{[nt]/n}^2|}{\sigma_{n,[nt]/n} + \sigma_{[nt]/n}} \leq C n^{-\eta'} \left( \frac{(t_0 - t_1)\hat{\sigma}_{t_0}^2}{2} \right)^{-\frac{1}{2}} = C n^{-\eta'}, \]
where in the last expression \( C = C(\lambda, A, C_H, \eta, \eta', L_0, \|f\|_{\text{Lip}}, t_0, \hat{\sigma}_{t_0}^2) \). Thus by (49) and (50), Lemma 6.4 yields
\[
d_w \left( \xi_n \left( \frac{[nt]}{n} \right), \frac{\sigma_{[nt]/n}}{n}, Z \right) \leq d_w \left( \xi_n \left( \frac{[nt]}{n} \right), \frac{\sigma_{n,[nt]/n}}{n}, Z \right) + d_w \left( \frac{\sigma_{n,[nt]/n}}{n}, \frac{\sigma_{[nt]/n}}{n}, Z \right)
\[ \leq C(n^{-\frac{1}{2}} \log n + n^{-\eta'}), \]
where \( C = C(\lambda, A, C_H, \eta, \eta', L_0, \|f\|_{\text{Lip}}, t_0, \hat{\sigma}_{t_0}^2) \).

**Step 3.** By Lemma 6.1 \( t \mapsto \hat{\sigma}_t^2 \) is Hölder continuous and thus \( \|\hat{\sigma}_t^2\|_{\infty} \leq C \), for every \( t \in [0, 1] \), where \( C = C(\lambda, A, \|f\|_{\text{Lip}}, \eta) \). Therefore \( |\sigma_{n,[nt]/n}^2 - \sigma_t^2| \leq C n^{-1} \). Let \( t \geq t_0 \) and \( n \geq n_0 \). Now by (47)
\[
|\sigma_{[nt]/n} - \sigma_t| \leq \frac{|\sigma_n^2 - \sigma_t^2|}{\sigma_{n,[nt]/n} + \sigma_t} \leq C n^{-1} \left( \frac{(t_0 - t_1)\hat{\sigma}_{t_0}^2}{2} \right)^{-\frac{1}{2}} \leq C n^{-1},
\]
where in the last expression \( C = C(\lambda, A, C_H, \eta, \|f\|_{\text{Lip}}, t_0, \hat{\sigma}_{t_0}^2) \). Thus by Lemma 6.4
\[
d_w \left( \sigma_{[nt]/n}, Z, \sigma_t Z \right) \leq C n^{-1}. \]

Collecting the estimates from (48), (51) and (52), we see that for \( n \geq n_0 \) and \( t \geq t_0 \)
\[
d_w \left( \xi_n(t), \sigma_t Z \right) \leq C(n^{-\eta'} + n^{-\frac{1}{4}} \log n),
\]
where \( C = C(\lambda, A, C_H, \eta, \|f\|_{\text{Lip}}, L_0, t_0, \hat{\sigma}_{t_0}^2) \). \( \Box \)

Next we give the proof of Theorem 4.4:

Let \( 0 < \eta' < \eta \). Let \( n \geq 1 \) and \( t \in [0, 1] \). Then at least one of the following cases holds: **Case 1:** \( nt \geq \max\{3, 16/(1-\vartheta)^2\} \); **Case 2:** \( t \leq n^{-\eta'} \); **Case 3:** \( n \leq \max\{3, 16/(1-\vartheta)^2\} \). We show that in each of these cases, there exists \( C = C(\lambda, A, \eta', \eta, C_H, \|f\|_{\text{Lip}}, L_0) \) such that
\[
d_w \left( \xi_n(t), \sigma_t Z \right) \leq C n^{-\frac{\eta}{2}} + C n^{-\frac{1}{4}} \log n. \]
In Case 1, we follow the proof of Theorem 6.5 making the following changes. First, we do not define any $t_0$ or $t_1$. Second, in (49) instead of Theorem 3.2, we apply Corollary 3.3, which yields the bound $Cn^{-1/6} \log n$ on $d_\infty (\xi_n ([nt]/n), \sigma_{n,[nt]/n} Z)$. Third, in estimating $|\sigma_{n,[nt]/n} - \sigma_{[nt]/n}|$ and $|\sigma_{[nt]/n} - \sigma_t|$ we use that for $x_1, x_2 \geq 0$ we have $|x_1 - x_2| \leq \sqrt{|x_1^2 - x_2^2|}$. This yields the estimates $|\sigma_{n,[nt]/n} Z - \sigma_{[nt]/n} Z| \leq Cn^{-\frac{\eta}{2}}$ and $|\sigma_{[nt]/n} Z - \sigma_t Z| \leq Cn^{-\frac{\eta}{2}}$. By (48) we have $d_\infty (\xi_n (t), \sigma_n ([nt]/n)) \leq Cn^{-1/2}$. Overall, collecting these estimates, we have that (53) holds in Case 1.

In Case 2 we see that $\sigma_{n,nt} \leq \|f\|_\infty \leq Cn^{-\eta'/2}$. Furthermore since $\hat{\sigma}_n^2$ is bounded we also have $\sigma_t \leq Ct^{1/2} \leq Cn^{-\eta'/2}$. Now (34) yields $d_\infty (\xi_n (t), \sigma_t Z) \leq Cn^{-\eta'/2}$. Clearly in Case 3 we can choose large enough $C$ such that (53) holds.

We can now choose $C$ in Theorem 4.4 to be the maximum of the corresponding constants in Cases 1–3. This completes the proof of Theorem 4.4.

6.3. Proof of Theorem 4.2. In this subsection we present the proof of Theorem 4.2.

Let $M$ be a $d \times d$ matrix. We introduce the following norm that is used through this subsection:

$$|M| = \max\{|M_{\alpha\beta}| : \alpha, \beta = 1, ..., d\}.$$ 

The following two lemmas generalize Lemmas 6.1 and 6.3. The proofs are similar and thus omitted.

**Lemma 6.6.** $t \to (\hat{\Sigma}_t^2)_{\alpha\beta}(f)$ is Hölder continuous with every exponent $\eta'' < \eta$ and entries $\alpha, \beta \in \{1, ..., d\}$. An upper bound for the corresponding Hölder constant $C$ can be given as a function of $\lambda, A_*, \eta, \eta'', C_H$ and $\|f\|_{\text{Lip}}$.

**Lemma 6.7.** Let $\eta'' < \eta$. Then there exists a constant $C = C(\lambda, A_*, C_H, \eta, \eta'', L_0, \|f\|_{\text{Lip}})$ such that

$$|\Sigma_{n,t} - \Sigma_t| \leq Cn^{-\eta''}$$

for every $t \in [0, 1]$.

The upper bound on $|\mu(h(\xi_n(t)) - \Phi_{\Sigma_t}(h))|$ is found by bounding the following four terms:

$$|\mu [h(\xi_n(t)) - h(\xi_n ([nt]/n))]|, \tag{54}$$

$$|\mu [h(\xi_n ([nt]/n)) - \Phi_{\Sigma_{n,[nt]/n}}(h)]|, \tag{55}$$

$$|\Phi_{\Sigma_{n,[nt]/n}}(h) - \Phi_{\Sigma_{[nt]/n}}(h)|, \tag{56}$$

and

$$|\Phi_{\Sigma_{[nt]/n}}(h) - \Phi_{\Sigma_t}(h)|. \tag{57}$$

The proof is analogous to the proof of Theorem 6.5. Bounding (54) corresponds to Step 1, (55) and (56) to Step 2, and (57) to Step 3. However, computing with matrices introduces some complications that need to be dealt with more closely. We therefore first introduce some matrix-related notations.

If a matrix $M$ is positive semi-definite, we denote $M \geq 0$, and if it is positive definite, $M > 0$. The minimal eigenvalue of $M$ is denoted by $\lambda_1(M)$. Recall that $\lambda_1(M) \geq 0$ or $\lambda_1(M) > 0$ if and only if $M \geq 0$ or $M > 0$, respectively.
Let $v \in \mathbb{R}^d$. We denote its Euclidean norm by $|v|_d$ and for a $d \times d$ matrix $M$, we write
\[
\|M\| = \sup_{v \in \mathbb{R}^d \setminus \{0\}} \frac{|Mv|_d}{|v|_d} \leq d|M|
\]
and call $\|M\|$ the spectral norm of $M$.

**Matrix fact.** Let $M$ be a positive definite matrix satisfying $\lambda_1(M) \geq C_l$ and $\|M\| \leq C_u$ for some $0 < C_l < C_u$. Then there exists $\delta = \delta(C_l, C_u) > 0$ such that if $\tilde{M}$ is positive semi-definite and $|M - \tilde{M}| < \delta$, then $\lambda_1(\tilde{M}) \geq C_l/2$ and especially $\tilde{M}$ is positive definite.

Let now $t_0 \in [0,1]$ be such that $\Sigma_{t_0} > 0$. By Lemma 6.6, the entries of $\hat{\Sigma}_t$ vary Hölder continuously. Thus there exists a neighbourhood of $t_0$ such that if $t$ is in that neighbourhood, then $\hat{\Sigma}_t > 0$. For all $t \in [0,1]$, we also have $\hat{\Sigma}_t \geq 0$, since covariance matrices are positive semi-definite. This guarantees that $\Sigma_t = \int_0^t \hat{\Sigma}_s ds > 0$ for every $t \geq t_0$. To be more precise
\[
\lambda_1(\Sigma_t) = \lambda_1(\Sigma_{t_0} + \int_{t_0}^t \hat{\Sigma}_s ds) \geq \lambda_1(\Sigma_{t_0}) + \lambda_1(\int_{t_0}^t \hat{\Sigma}_s ds) \geq \lambda_1(\Sigma_{t_0}),
\]
when $t \geq t_0$.

**Bound on (54).** As in the Step 1 of the proof of Theorem 6.5, we have that (54) is bounded by $C n^{-\frac{1}{2}}$.

**Bound on (55).** We are going to apply Theorem 3.1 as in Step 2 of Theorem 6.5. To this end $\Sigma_{n,t}$ must be positive definite. As will be apparent later it is crucial that we can choose $n_0$ independent of $t$ such that for every $n \geq n_0$ and $t \geq t_0$ we have $\Sigma_{n,t} > 0$.

By (59), in the set $\{\Sigma_t : t \geq t_0\}$ there exists an uniform bound to $\lambda_1(\Sigma_t)$ namely $\lambda_1(\Sigma_{t_0})$. By Lemma 6.6 the entries of $\Sigma_t$ are also uniformly bounded. Thus by Matrix fact and Lemma 6.7 there exists $n_0$ independent of $t$ such that $\Sigma_{n,t} > 0$ for every $n \geq n_0$, $t \geq t_0$. Choose $n_0$ such that also $n_0 t_0 \geq \max\{3, 16/(1 - \vartheta)^2\}$. We will bound (55) by first applying Theorem 3.1 to $\frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{i=0}^{\lfloor nt \rfloor - 1} \tilde{f}_{n,i}$. Denote
\[
V = V(\lfloor nt \rfloor) = \frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{i=0}^{\lfloor nt \rfloor - 1} \tilde{f}_{n,i} = \frac{\sqrt{n}}{\sqrt{\lfloor nt \rfloor}} \xi_n \left( \frac{\lfloor nt \rfloor}{n} \right).
\]
Now the covariance matrix $\Sigma_V$ of $V(\lfloor nt \rfloor)$ is $\frac{n}{\lfloor nt \rfloor} \Sigma_{n,\lfloor nt \rfloor/n}$ and thus positive definite, when $n \geq n_0$ and $t \geq t_0$.

Let $h : \mathbb{R}^d \to \mathbb{R}$ be as in the theorem. Define $h^* : \mathbb{R}^d \to \mathbb{R}$, $w \mapsto h \left( \frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} w \right)$.

Thus we have $\mu(h^*(V)) = \mu(h(\xi_n(\lfloor nt \rfloor/n)))$ and $\|D^k h^*\|_\infty \leq \|D^k h\|_\infty < \infty$. We have $\Phi_{\Sigma_{n,\lfloor nt \rfloor/n}}(h) = \Phi_{\Sigma_V}(h^*)$. Since $\lfloor nt \rfloor \geq \max\{3, 16/(1 - \vartheta)^2\}$, we can apply Theorem 3.1 to $V(\lfloor nt \rfloor)$ and it yields
\[
|\mu(h(\xi_n(\lfloor nt \rfloor/n))) - \Phi_{\Sigma_{n,\lfloor nt \rfloor/n}}(h)| = |\mu(h^*(V)) - \Phi_{\Sigma_V}(h^*)| \leq C n^{-\frac{1}{2}} \log n.
\]

**Bound on (56).** Let $Z \sim \mathcal{N}(0, I_{d \times d})$, where $\mathcal{N}(0, I_{d \times d})$ is a standard $d$-dimensional normal distribution. If $Z$ is a positive definite $d \times d$ matrix, then it has unique positive definite square root matrix $\Sigma^{1/2}$. Furthermore
\[
\Phi_{\Sigma}(h) = \mathbb{E}[h(\Sigma^{1/2} Z)].
\]
We define
\[
\text{Lip}_d(h) = \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|h(x) - h(y)|}{|x - y|_d}.
\]

With these definitions
\[
\left| \mathbb{E} \left[ h \left( \Sigma_{n,[nt]/n}^{1/2} Z \right) \right] - \mathbb{E} \left[ h \left( \Sigma_{[nt]/n}^{1/2} Z \right) \right] \right| \leq \text{Lip}_d(h) \mathbb{E} \left[ \left| \Sigma_{n,[nt]/n}^{1/2} Z - \Sigma_{[nt]/n}^{1/2} Z \right|_d \right] 
\leq \text{Lip}_d(h) \mathbb{E} |Z|_d \left\| \Sigma_{n,[nt]/n}^{1/2} - \Sigma_{[nt]/n}^{1/2} \right\|.
\]

The following bound holds for the spectral norm of the difference of two square root matrices (see [43])
\[
\left\| \Sigma_{n,[nt]/n}^{1/2} - \Sigma_{[nt]/n}^{1/2} \right\| \leq \frac{\left\| \Sigma_{n,[nt]/n} - \Sigma_{[nt]/n} \right\|}{\sqrt{\lambda_1 (\Sigma_{n,[nt]/n}) + \sqrt{\lambda_1 (\Sigma_{[nt]/n})}}}. \tag{62}
\]

Now (60), (61), (62) and (58) yield
\[
\left| \Phi_{\Sigma_{n,[nt]/n}}(h) - \Phi_{\Sigma_{[nt]/n}}(h) \right| \leq \text{Lip}_d(h) \mathbb{E} |Z|_d \frac{d \left\| \Sigma_{n,[nt]/n} - \Sigma_{[nt]/n} \right\|}{\sqrt{\lambda_1 (\Sigma_{n,[nt]/n}) + \sqrt{\lambda_1 (\Sigma_{[nt]/n})}}}. \tag{63}
\]

Let \( \eta'' < \eta \). By Lemma 6.7 we have \( \left| \Sigma_{n,[nt]} - \Sigma_{[nt]} \right| \leq C n^{-\eta''} \). The other terms on the right side of inequality (63) are uniformly bounded. Thus
\[
\left| \Phi_{\Sigma_{n,[nt]/n}}(h) - \Phi_{\Sigma_{[nt]/n}}(h) \right| \leq C n^{-\eta''}.
\]

**Bound on (57).** Following the same steps as in the previous calculation, we have
\[
\left| \Phi_{\Sigma_{[nt]/n}}(h) - \Phi_{\Sigma_{t}}(h) \right| \leq \text{Lip}_d(h) \mathbb{E} |Z|_d \frac{d \left\| \Sigma_{[nt]/n} - \Sigma_{t} \right\|}{\sqrt{\lambda_1 (\Sigma_{[nt]/n}) + \sqrt{\lambda_1 (\Sigma_{t})}}}. \tag{64}
\]

Since \( |\hat{\Sigma}_{t}| \) is clearly uniformly bounded, we have the uniform estimate
\[
\left| \Sigma_{[nt]/n} - \Sigma_{t} \right| = \left| \int_{t}^{[nt]/n} \hat{\Sigma}_{s} ds \right| \leq C n^{-1}.
\]

This and (64) yields the bound
\[
\left| \Phi_{\Sigma_{[nt]/n}}(h) - \Phi_{\Sigma_{t}}(h) \right| \leq C n^{-1}.
\]

**Bound on \( |\mu(h(\xi_n(t))) - \Phi_{\Sigma_{t}}(h)| \).** There exist \( n_0 \) such that for all \( n \geq n_0 \) the bounds of (54), (55), (56) and (57) computed above hold and thus we have
\[
|\mu(h(\xi_n(t))) - \Phi_{\Sigma_{t}}(h)| \leq C n^{-\frac{3}{2}} + C n^{-\frac{1}{2}} \log n + C n^{-\eta''} + C n^{-1} \leq C n^{-\frac{1}{2}} \log n + C n^{-\eta''} \tag{65}
\]
for every \( n \geq n_0 \). It is easy to choose large enough \( C \) so that (65) holds also when \( 1 \leq n \leq n_0 \). This completes the proof of Theorem 4.2. \( \square \)
7. Proofs of abstract results

This section assumes familiarity with [21]. However, accepting certain results given, we have made an effort to provide a comprehensible proof.

Recall that the goal of Theorem 2.1 is to control the term \( |\mu(h(W)) - \Phi_{\Sigma_N}(h)| \).

First, assuming that the matrix \( \Sigma_N \in \mathbb{R}^{d \times d} \) is positive definite, the normal distribution \( \mathcal{N}(0, \Sigma_N) \) has a density function \( \phi_{\Sigma_N} \), and we define

\[
A(w) = -\int_0^\infty \left\{ \int_{\mathbb{R}^d} h(e^{-s}w + \sqrt{1 - e^{-2s}} z) \phi_{\Sigma_N}(z) \, dz - \Phi_{\Sigma_N}(h) \right\} \, ds,
\]

where \( \Phi_{\Sigma_N}(h) \) stands for the expectation of \( h \) with respect to the same normal distribution. Furthermore, if \( h : \mathbb{R}^d \to \mathbb{R} \) is three times differentiable with \( \|D^k h\|_\infty < \infty \) for \( 1 \leq k \leq 3 \), then \( A \in C^3(\mathbb{R}^d, \mathbb{R}) \), and it solves the Stein equation

\[
\text{tr} \Sigma D^2 A(w) - w \cdot \nabla A(w) = h(w) - \Phi_{\Sigma}(h) \tag{66}
\]

We refer to [7,15,16,18] or Lemma 3.3 in [21]. Moreover \( \|\partial_t \cdots \partial_{d} \cdot \partial_{d} h\|_\infty \leq k^{-1} \|\partial_t \cdots \partial_{d} \cdot \partial_{d} h\|_\infty \) whenever \( t_1 + \cdots + t_d = k, 1 \leq k \leq 3 \).

Thus in (66), taking the expected value with respect to \( \mu \), we have

\[
|\mu[h(W)] - \Phi_{\Sigma}(h)| = |\mu[\text{tr} \Sigma D^2 A(W) - W \cdot \nabla A(W)]|.
\]

It turns out that the expression on the right side is easier to bound than the left, which is the core of Stein’s method.

Instead of proving Theorem 2.1 directly, we prove the following preliminary result.

**Theorem 7.1.** Let \((X, \mathcal{B}, \mu)\) be a probability space and \((f_i)_{i=0}^\infty\) a sequence of random vectors with common upper bound \( \|f\|_\infty \). Let \( A \in C^3(\mathbb{R}^d, \mathbb{R}) \) be a given function satisfying \( \|D^k A\|_\infty < \infty \) for \( 1 \leq k \leq 3 \). Fix integers \( N > 0 \) and \( 0 \leq K < N \). Suppose that the following conditions are satisfied:

(A1) There exist constants \( C_2 > 0 \) and \( C_4 > 0 \), and a non-increasing function \( \rho : \mathbb{N}_0 \to \mathbb{R}_+ \) with \( \rho(0) = 1 \) and \( \sum_{i=1}^\infty i \rho(i) < \infty \), such that for all \( 0 \leq i \leq j \leq k \leq l \leq N-1 \),

\[
|\mu(\tilde{f}_\alpha^i \tilde{f}_\beta^j)| \leq C_2 \rho(j - i),
\]

\[
|\mu(\tilde{f}_\alpha^i \tilde{f}_\beta^j \tilde{f}_\gamma^k)| \leq C_4 \rho(\max\{j - i, l - k\}),
\]

\[
|\mu(\tilde{f}_\alpha^i \tilde{f}_\beta^j \tilde{f}_\gamma^k) - \mu(\tilde{f}_\alpha^i \tilde{f}_\beta^j) \mu(\tilde{f}_\gamma^k)| \leq C_4 \rho(k - j).
\]

hold whenever \( k \geq 0 \); \( 0 \leq i \leq j \leq k \leq l < N \); \( \alpha, \beta, \gamma, \delta \in \{\alpha', \beta'\} \) and \( \alpha', \beta' \in \{1, \ldots, d\} \).

(A2’) There exists a function \( \eta : \mathbb{N}_0^2 \to \mathbb{R}_+ \) such that

\[
\left| \mu\left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \tilde{f}_\alpha^i \cdot \nabla A(W_n) \right) \right| \leq \eta(N, K).
\]
Then
\[
\left| \mu(\text{tr} \Sigma_N D^2 A(W) - W \cdot \nabla A(W)) \right| \\
\leq 2d^3 C_2 \| f \|_{\infty} \| D^3 A \|_{\infty} \frac{2K + 1}{\sqrt{N}} \left( \rho(0) + 2 \sum_{i=1}^{2K} \rho(i) \right) \\
+ 2d^2 C_2 \| D^2 A \|_{\infty} \sum_{i=K+1}^{N-1} \rho(i) \\
+ 11d^2 \max\{C_2, \sqrt{C_4}\} \| D^2 A \|_{\infty} \frac{\sqrt{K + 1}}{\sqrt{N}} \left( \sum_{i=0}^{N-1} (i + 1) \rho(i) \right) \\
+ \eta(N, K).
\]

7.1. **Proof of Theorem 7.1.** Let the Assumptions of Theorem 7.1 hold. By basic matrix computations we see that \( \mu(\text{tr} \Sigma_N D^2 A(W) - W \cdot \nabla A(W)) \) can be represented as a sum

\[
\mu \left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} D^2 A(W)(W - W_n) - \bar{f}^n \cdot (\nabla A(W) - \nabla A(W_n)) \right) 
\]

(67)

\[
+ \mu \left( \text{tr} \left( \left( \Sigma_N - \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m \in [n]_K} \bar{f}^n \otimes \bar{f}^m \right) D^2 A(W) \right) \right) 
\]

(68)

\[
+ \mu \left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} - \bar{f}^n \cdot \nabla A(W_n) \right). 
\]

(69)

By Assumption (A2'), the absolute value of (69) is bounded by \( \eta(N, K) \). Bounds for the absolute values of (67) and (68) are stated in Propositions 7.2 and 7.3, respectively.

Proposition 7.2 is proved exactly as Proposition 4.3 in [21] and the proof is thus omitted. The only difference is that every \( \| f^i \|_{\infty} \) is bounded above by \( 2\| f \|_{\infty} \) which explains the coefficient 2 in the bound.

**Proposition 7.2.** The absolute value of expression in (67) is bounded by

\[
2d^3 C_2 \| f \|_{\infty} \| D^3 A \|_{\infty} \frac{2K + 1}{\sqrt{N}} \rho(0) + 2 \sum_{i=1}^{2K} \rho(i) .
\]

The next proposition gives a bound on the absolute value of expression (68).

**Proposition 7.3.** The absolute value of the expression in (68) is bounded by

\[
2d^2 C_2 \| D^2 A \|_{\infty} \sum_{i=K+1}^{N-1} \rho(i) \\
+ 11d^2 \max\{C_2, \sqrt{C_4}\} \| D^2 A \|_{\infty} \frac{\sqrt{K + 1}}{\sqrt{N}} \sqrt{\sum_{i=0}^{N-1} (i + 1) } \rho(i) .
\]
To prove Proposition 7.3, we first define \( \tilde{\Sigma} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m \in [n]_K} \mu(\bar{f}_n \otimes \bar{f}_m) \). Using this definition, we have the following upper bound on the absolute value of (68)

\[
\mu \left( \left\| \text{tr} \left( \left( \tilde{\Sigma} - \frac{1}{N} \sum_{n=0}^{N-1} \bar{f}_n \otimes \bar{f}_m \right) D^2 A(W) \right) \right\| \right)
\]

(70)

\[
+ \| \text{tr}((\Sigma_N - \tilde{\Sigma})D^2 A)\|_{\infty}.
\]

(71)

Bounds on (70) and (71) are given in the Lemmas 7.4 and 7.5, respectively.

The next lemma is proven exactly as Lemma 4.6 in [12].

**Lemma 7.4.** The expression (70) satisfies the bound

\[
\mu \left( \left\| \text{tr} \left( \left( \tilde{\Sigma} - \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m \in [n]_K} \mu(\bar{f}_n \otimes \bar{f}_m) \right) D^2 A(W) \right) \right\| \right)
\]

\[
\leq 11d^2 \max\{C_2, \sqrt{C_4}\} \|D^2 A\|_{\infty} \frac{\sqrt{K} + 1}{\sqrt{N}} \sqrt{\sum_{i=0}^{N-1} (i + 1)\rho(i)}.
\]

**Lemma 7.5.** The expression in (71) satisfies the bound

\[
\| \text{tr}((\Sigma_N - \tilde{\Sigma})D^2 A)\|_{\infty} \leq 2d^2 C_2 \|D^2 A\|_{\infty} \sum_{i=K+1}^{N-1} \rho(i).
\]

**Proof.** By definitions we have

\[
\Sigma_N - \tilde{\Sigma} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mu(\bar{f}_n \otimes \bar{f}_m) - \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m \in [n]_K} \mu(\bar{f}_n \otimes \bar{f}_m)
\]

\[
= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{0 \leq m \leq N-1, m \notin [n]_K} \mu(\bar{f}_n \otimes \bar{f}_m).
\]

Assumption (A1) yields

\[
|((\Sigma_N - \tilde{\Sigma})_{\alpha\beta}| = \frac{1}{N} \left| \sum_{n=0}^{N-1} \sum_{0 \leq m \leq N-1, m \notin [n]_K} \mu(\bar{f}_n \bar{f}_m) \right|
\]

\[
\leq \frac{1}{N} \sum_{n=0}^{N-1} \sum_{0 \leq m \leq N-1, m \notin [n]_K} C_2 \rho(|n - m|)
\]

\[
\leq 2C_2 \sum_{i=K+1}^{N-1} \rho(i).
\]
Thus
\[
\|\text{tr}((\Sigma_N - \tilde{\Sigma})D^2A)\|_\infty \leq \|D^2A\|_\infty \sum_{1 \leq \alpha, \beta \leq d} |(\Sigma - \tilde{\Sigma})_{\alpha\beta}|
\]
\[
\leq \|D^2A\|_\infty \sum_{1 \leq \alpha, \beta \leq d} 2C_2 \sum_{i=K+1}^{N-1} \rho(i)
\]
\[
\leq 2d^2C_2\|D^2A\|_\infty \sum_{i=K+1}^{N-1} \rho(i).
\]

This finishes the proof of the lemma. \(\square\)

Observe that Proposition 7.3 follows from Lemmas 7.4 and 7.5. This completes the proof of Proposition 7.3.

Proposition 7.2, Proposition 7.3 and Assumption (A2') now yield the bounds on (67), (68) and (69), respectively. This completes the proof of Theorem 7.1. \(\square\)

7.2. **Proof of Theorem 2.1.** To be able to use Theorem 7.1 in proving Theorem 2.1, we need to check that Assumption (A2') is implied by the assumptions of Theorem 2.1. This follows from the next lemma, which is proven with exactly the same steps as Lemma 5.1 in [21].

**Lemma 7.6.** Under the conditions of Theorem 2.1,
\[
\left| \mu \left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{f}^n \cdot \nabla A(W^n) \right) \right| \leq \sqrt{N\check{\rho}(K)}.
\]

Now Assumption (A2') of Theorem 7.1 is satisfied by defining
\[
\eta(N, K) = \sqrt{N\check{\rho}(K)}.
\]

For the purpose of computing the constant \(C_*\) in the error bound of Theorem 2.1 we introduce the following lemma.

**Lemma 7.7.** Let \(\rho: \mathbb{N} \to \mathbb{R}\) be a function such that \(0 \leq \rho(i) \leq 1\) for all \(i \in \mathbb{N}\) and \(\sum_{i=0}^{\infty} (i + 1)\rho(i) < \infty\) Then \((\sum_{i=0}^{\infty} \rho(i))^2 \leq 2 \sum_{i=0}^{\infty} (i + 1)\rho(i)\)

**Proof.** We have
\[
\left( \sum_{i=0}^{\infty} \rho(i) \right)^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho(i)\rho(j).
\]
Let \(n \in \mathbb{N}\). In the sum on the right side of (72) there exist exactly \(2n + 1\) terms \(\rho(i)\rho(j)\) such that \(\max\{i, j\} = n\). The result now follows easily from rearranging the terms according to \(\max\{i, j\}\) and noticing that \(\rho(i)\rho(j) \leq \rho(\max\{i, j\})\). \(\square\)

Taking square roots of the result in Lemma 7.2, we have
\[
\sum_{i=0}^{\infty} \rho(i) \leq \sqrt{2 \sum_{i=0}^{\infty} (i + 1)\rho(i)}. \quad (73)
\]
Lemma 3.3 in [21] and Theorem 7.1, followed by elementary estimates and using (73), now yield
\[
|\mu(h(W)) - \Phi_{\Sigma_N}(h)| = |\mu(\text{tr} \Sigma_N D^2 A(W) - W \cdot \nabla A(W))| \\
\leq 2d^3 C_2 \|f\|_\infty \|D^3 A\|_\infty \frac{2K + 1}{\sqrt{N}} \left( \rho(0) + 2 \sum_{i=1}^{2K} \rho(i) \right) \\
+ 2d^2 C_2 \|D^2 A\|_\infty \sum_{i=K+1}^{N-1} \rho(i) \\
+ 11d^2 \max\{C_2, \sqrt{C_4}\} \|D^2 A\|_\infty \frac{K + 1}{\sqrt{N}} \sqrt{\sum_{i=0}^{N-1} (i + 1)\rho(i)} \\
+ \eta(N, K). \\
\leq \frac{8}{3} d^3 C_2 \|f\|_\infty \|D^3 h\|_\infty \frac{K + 1}{\sqrt{N}} \sum_{i=0}^{\infty} \rho(i) + d^2 C_2 \|D^2 h\|_\infty \sum_{i=K+1}^{\infty} \rho(i) \\
+ \frac{11}{2} d^2 \max\{C_2, \sqrt{C_4}\} \|D^2 h\|_\infty \frac{K + 1}{\sqrt{N}} \sqrt{\sum_{i=0}^{\infty} (i + 1)\rho(i)} \\
+ \sqrt{N}\hat{\rho}(K)
\]
\[
\leq \left( \frac{8\sqrt{2}}{3} d^3 C_2 \|f\|_\infty \|D^3 h\|_\infty + 6d^2 \max\{C_2, \sqrt{C_4}\} \|D^2 h\|_\infty \right) \frac{K + 1}{\sqrt{N}} \sqrt{\sum_{i=0}^{\infty} (i + 1)\rho(i)} \\
+ d^2 C_2 \|D^2 h\|_\infty \sum_{i=K+1}^{\infty} \rho(i) + \sqrt{N}\hat{\rho}(K)
\]
\[
\leq C_* \left( \frac{K + 1}{\sqrt{N}} + \sum_{i=K+1}^{\infty} \rho(i) \right) + \sqrt{N}\hat{\rho}(K),
\]
where
\[
C_* = 6d^3 \max\{C_2, \sqrt{C_4}\} \left( \|f\|_\infty \|D^3 h\|_\infty + \|D^2 h\|_\infty \right) \sqrt{\sum_{i=0}^{\infty} (i + 1)\rho(i)}.
\]
The proof of Theorem 2.1 is now complete. \(\blacksquare\)

7.3. Proof of Theorem 2.2. The Stein equation in the univariate case is
\[
\sigma^2 A'(w) - w A(w) = h(w) - \Phi_\sigma^2(h).
\]
The following theorem is proven as Theorem 4.2 in [21] with the same modifications as in the proof of Theorem 7.1.

**Theorem 7.8.** Let \((X, \mathcal{B}, \mu)\) be a probability space and \((f^i)_{i=0}^{\infty}\) a sequence of random variables with common upper bound \(\|f\|_\infty\). Let \(A \in C^1(\mathbb{R}, \mathbb{R})\) be a given function with absolutely continuous \(A'\), satisfying \(\|A^{(k)}\|_\infty < \infty\) for \(0 \leq k \leq 2\). Fix integers \(N > 0\) and \(0 \leq K < N\). Suppose that the following conditions are satisfied:
(B1) There exist constants $C_2, C_4$ and a decreasing function $\rho : \mathbb{N} \to \mathbb{R}$ with $\rho(0) = 1$, such that for all $i \leq j \leq k \leq l$,

$$|\mu(\bar{f}_i \bar{f}_j)| \leq C_2 \rho(j - i),$$

$$|\mu(\bar{f}_i \bar{f}_j \bar{f}_k \bar{f}_i)| \leq C_4 \rho(\max\{j - i, l - k\}),$$

$$|\mu(\bar{f}_i \bar{f}_j \bar{f}_k \bar{f}_i) - \mu(\bar{f}_i \bar{f}_j)\mu(\bar{f}_k \bar{f}_i)| \leq C_4 \rho(k - j).$$

(B2') There exists a function $\eta : \mathbb{N}_0^2 \to \mathbb{R}_+$ such that

$$\left| \mu \left[ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \bar{f}^n A(W_n) \right] \right| \leq \eta(N, K).$$

Then

$$|\mu(\sigma_N^2 A'(W) - WA(W))| \leq C_2 \|f\|_\infty \|A''\|_\infty \frac{2K + 1}{\sqrt{N}} \left( \rho(0) + 2 \sum_{i=1}^{2K} \rho(i) \right)$$

$$+ 2C_2 \|A'\|_\infty \sum_{i=K+1}^{N-1} \rho(i)$$

$$+ 11 \max\{C_2, \sqrt{C_4}\} \|A''\|_\infty \frac{\sqrt{K + 1}}{\sqrt{N}} \sqrt{\sum_{i=0}^{N-1} (i + 1) \rho(i)}$$

$$+ \eta(N, K).$$

Let $\mathcal{F}_{\sigma_N^2}$ be the class of differentiable functions $A : \mathbb{R} \to \mathbb{R}$ with absolutely continuous derivative and satisfying the following bounds

$$\|A\|_\infty \leq 2, \quad \|A'\|_\infty \leq \sqrt{2/\pi} \sigma^{-1} \quad \text{and} \quad \|A''\|_\infty \leq 2\sigma^{-2}.$$

If $h : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, then the corresponding solution $A$ for the Stein equation (74) belongs to $\mathcal{F}_{\sigma_N^2}$.

Following the proof of Theorem 2.3 in [21], we see that if assumptions in Theorem 2.2 are satisfied, then for every $A \in \mathcal{F}_{\sigma_N^2}$ the assumptions of Theorem 7.8 are satisfied with the choice

$$\eta(N, K) = 2 \max\{1, \sigma_N^{-2}\} \sqrt{N} \rho(K).$$

Therefore using the same methods as in the proof of Theorem 2.1, we have

$$d_{\mathcal{W}}(W, \sigma_N Z) \leq |\mu(\sigma_N^2 A'(W) - WA(W))|$$

$$\leq C_2 \|f\|_\infty 2\sigma_N^{-2} \frac{2K + 1}{\sqrt{N}} \left( \rho(0) + 2 \sum_{i=1}^{2K} \rho(i) \right)$$

$$+ 2C_2 \sqrt{2/\pi} \sigma_N^{-1} \sum_{i=K+1}^{N-1} \rho(i)$$

$$+ 11 \max\{C_2, \sqrt{C_4}\} \sqrt{2/\pi} \sigma_N^{-1} \frac{\sqrt{K + 1}}{\sqrt{N}} \sqrt{\sum_{i=0}^{N-1} (i + 1) \rho(i)} + \eta(N, K)$$
\[ \leq 8C_2 \|f\|_\infty \sigma_N^{-2} \sum_{i=0}^\infty \rho(i) + 2C_2 \sqrt{2/\pi \sigma_N^{-1}} \sum_{i=K+1}^\infty \rho(i) \]

\[ + 9 \max\{C_2, \sqrt{C_4}\} \sigma_N^{-1} \frac{K + 1}{\sqrt{N}} \sqrt{\sum_{i=0}^\infty (i + 1)\rho(i) + 2 \max\{1, \sigma_N^{-2}\} \sqrt{N} \bar{\rho}(k)} \]

\[ \leq \left( 8\sqrt{2}C_2 \|f\|_\infty \sigma_N^{-2} + 9 \max\{C_2, \sqrt{C_4}\} \sigma_N^{-1} \right) \frac{K + 1}{\sqrt{N}} \sqrt{\sum_{i=0}^\infty (i + 1)\rho(i)} \]

\[ + 2C_2 \sqrt{2/\pi \sigma_N^{-1}} \sum_{i=K+1}^\infty \rho(i) + 2 \max\{1, \sigma_N^{-2}\} \sqrt{N} \bar{\rho}(k) \]

\[ \leq C_\# \left( \frac{K + 1}{\sqrt{N}} + \sum_{i=K+1}^\infty \rho(i) \right) + C'_{\#} \sqrt{N} \bar{\rho}(k), \]

where

\[ C_\# = 12 \max\{\sigma_N^{-1}, \sigma_N^{-2}\} \max\{C_2, \sqrt{C_4}\} (1 + \|f\|_\infty) \sqrt{\sum_{i=0}^\infty (i + 1)\rho(i)} \]

and

\[ C'_{\#} = 2 \max\{1, \sigma_N^{-2}\}. \]

This completes the proof of Theorem 2.2. \qed
REFERENCES

[1] Romain Aimino, Huyi Hu, Matthew Nicol, Andrei Török, and Sandro Vaienti. Polynomial loss of memory for maps of the interval with a neutral fixed point. *Discrete Contin. Dyn. Syst.*, 35(3):793–806, 2015. doi:10.3934/dcds.2015.35.793.

[2] Romain Aimino and Jérôme Rousseau. Concentration inequalities for sequential dynamical systems of the unit interval. *Ergodic Theory Dynam. Systems*, 36(6):2384–2407, 2016. doi:10.1017/etds.2015.19.

[3] Pierre Arnoux and Albert M. Fisher. Anosov families, renormalization and non-stationary subshifts. *Ergodic Theory Dynam. Systems*, 36(8):2384–2407, 2016. doi:10.1017/etds.2015.19.

[4] Arvind Ayyer, Carlangelo Liverani, and Mikko Stenlund. Quenched CLT for random toral automorphisms. *Ergodic Theory Dynam. Systems*, 35(3):793–806, 2015. doi:10.1017/S0143385704000641.

[5] Arvind Ayyer and Mikko Stenlund. Exponential decay of correlations for randomly chosen hyperbolic toral automorphisms. *Chaos*, 17(4):043116, 7, 2007. doi:10.1063/1.2785145.

[6] V. I. Bakhtin. Random processes generated by a hyperbolic sequence of mappings. I. *Izv. Ross. Akad. Nauk Ser. Mat.*, 58(2):40–72, 1994. doi:10.1070/IM1995v044n02ABEH001596.

[7] Andrew Barbour. Stein’s method for diffusion approximations. *Probab. Theory Related Fields*, 84(3):297–322, 1990. doi:10.1007/BF01197887.

[8] B. Berckmoes, R. Lowen, and J. Van Casteren. Stein’s method and a quantitative Lindeberg CLT for the Fourier transforms of random vectors. *J. Math. Anal. Appl.*, 433(2):1441–1458, 2016. doi:10.1016/j.jmaa.2015.08.040.

[9] Sourav Chatterjee and Elizabeth Meckes. Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat.*, 4:257–283, 2008. URL: http://alea.impa.br/articles/v4/04-13.pdf.

[10] Louis H. Y. Chen, Larry Goldstein, and Qi-Man Shao. *Normal approximation by Stein’s method*. Probability and its Applications (New York). Springer, Heidelberg, 2011. doi:10.1007/978-3-642-15007-4.

[11] Jean-Pierre Conze and Albert Raugi. Limit theorems for sequential expanding dynamical systems on [0,1]. In *Ergodic theory and related fields*, volume 430 of *Contemp. Math.*, pages 89–121. Amer. Math. Soc., Providence, RI, 2007. doi:10.1090/conm/430/08253.

[12] Neil Dobbs and Mikko Stenlund. Quasistatic dynamical systems. *Ergodic Theory and Dynamical Systems*, 2016. doi:10.1017/etds.2016.9.

[13] Ana Cristina Freitas, Jorge Freitas, and Sandro Vaienti. Extreme value laws for sequences of intermittent maps. *Proceedings of the American Mathematical Society*, 2018. doi:10.1090/proc/13892.

[14] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Sandro Vaienti. Extreme value laws for non stationary processes generated by sequential and random dynamical systems. *Ann. Inst. H. Poincaré Probab. Statist.*, 53(3):1341–1370, 08 2017. doi:10.1214/16-AIHP757.

[15] Robert E. Gaunt. Rates of convergence in normal approximation under moment conditions via new bounds on solutions of the Stein equation. *J. Theoret. Probab.*, 29(1):231–247, 2016. doi:10.1007/s10959-014-0562-z.

[16] Larry Goldstein and Yosef Rinott. Multivariate normal approximations by stein’s method and size bias couplings. *Journal of Applied Probability*, 33(1):1–17, 1996. doi:10.2307/3215259.

[17] Mikhail Gordin. A homoclinic version of the central limit theorem. *M. J Math Sci*, 68(4):451–458, 1994. doi:10.1007/BF01254269.

[18] Friedrich Götze. On the rate of convergence in the multivariate clt. *The Annals of Probability*, pages 724–739, 1991. doi:10.1214/aop/1176990788.

[19] Chinmaya Gupta, William Ott, and Andrei Török. Memory loss for time-dependent piecewise expanding systems in higher dimension. *Math. Res. Lett.*, 20(1):141–161, 2013. doi:10.4310/MRL.2013.v20.n1.a12.

[20] Nicolai Haydn, Matthew Nicol, Andrew Török, and Sandro Vaienti. Almost sure invariance principle for sequential and non-stationary dynamical systems. *Transactions of the American Mathematical Society*, 2017. doi:10.1090/tran/6812.

[21] Olli Hella, Juho Leppänen, and Mikko Stenlund. Stein’s method for dynamical systems. 2016. Preprint. arXiv:1701.02966.

[22] Olli Hella and Mikko Stenlund. Quenched normal approximation for random sequences of transformations. Preprint. URL: https://arxiv.org/abs/1810.10760.

[23] Christoph Kawan. Metric entropy of nonautonomous dynamical systems. *Nonauton. Dyn. Syst.*, 1:26–52, 2014. doi:10.2478/mds-2013-0003.
[24] Christoph Kawan. Expanding and expansive time-dependent dynamics. *Nonlinearity*, 28(3):669–695, 2015. doi:10.1088/0951-7715/28/3/669.

[25] Christoph Kawan and Yuri Latushkin. Some results on the entropy of nonautonomous dynamical systems. *Dynamical Systems: An International Journal*, 2015. doi:10.1080/14689367.2015.1112199.

[26] Jonathan L. King. On M. Gordin’s homoclinic question. *Internat. Math. Res. Notices*, (5):203–212, 1997. doi:10.1155/S1073792897000147.

[27] Sergii Kolyada, Michał Misurowsicz, and Lubomír Snoha. Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval. *Fund. Math.*, 160(2):161–181, 1999. URL: http://eudml.org/doc/212386.

[28] Sergii Kolyada and Lubomír Snoha. Topological entropy of nonautonomous dynamical systems. *Random Comput. Dynam.*, 4(2-3):205–233, 1996.

[29] A Korepanov, Z Kosloff, and I Melbourne. Martingale–coboundary decomposition for families of dynamical systems. In *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*. Elsevier, 2017. doi:10.1016/j.anihpc.2017.08.005.

[30] Juho Leppänen and Mikko Stenlund. Quasistatic dynamics with intermittency. *Mathematical Physics, Analysis and Geometry*, 19(2):8, 2016. doi:10.1007/s11040-016-9212-2.

[31] Juho Leppänen. Functional correlation decay and multivariate normal approximation for non-uniformly expanding maps. *Nonlinearity*, 30(11):4239, 2017. URL: http://stacks.iop.org/0951-7715/30/i=11/a=4239.

[32] Elizabeth Meckes. On Stein’s method for multivariate normal approximation. In *High dimensional probability V: the Luminy volume*, volume 5 of *Inst. Math. Stat. Collect.*, pages 153–178. Inst. Math. Statist., Beachwood, OH, 2009. doi:10.1214/09-IMSCOLL511.

[33] Elizabeth Meckes. Approximation of projections of random vectors. *J. Theoret. Probab.*, 25(2):333–352, 2012. doi:10.1007/s10959-010-0299-2.

[34] Anushaya Mohapatra and William Ott. Memory loss for nonequilibrium open dynamical systems. *Discrete Contin. Dyn. Syst.*, 34(9):3747–3759, 2014. doi:10.3934/dcds.2014.34.3747.

[35] Péter Nándori, Domokos Szász, and T amás Varjú. A Central Limit Theorem for Time-Dependent Dynamical Systems. *Journal of Statistical Physics*, 146(6):1213–1220, MAR 2012. doi:10.1007/s10955-012-0451-8.

[36] Matthew Nicol, Andrew Török, and Sandro Vaienti. Central limit theorems for sequential and random intermitent dynamical systems. *Ergodic Theory and Dynamical Systems*, page 1–27, 2016. doi:10.1017/etds.2016.69.

[37] Ivan Nourdin, Giovanni Peccati, and Anthony Réveillac. Multivariate normal approximation using Stein’s method of exchangeable pairs under a general linearity condition. *Ann. Probab.*, 37(6):2150–2173, 2009. doi:10.1214/09-AOP467.

[38] Adrian Röllin. Stein’s method in high dimensions with applications. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(2):529–549, 2013. doi:10.1214/11-AIHP473.
[45] Mikko Stenlund. Non-stationary compositions of Anosov diffeomorphisms. *Nonlinearity*, 24:2991–3018, 2011. doi:10.1088/0951-7715/24/10/016.

[46] Mikko Stenlund. A vector-valued almost sure invariance principle for Sinai billiards with random scatterers. *Commun. Math. Phys.*, 325:879–916, 2014. doi:10.1007/s00220-013-1870-3.

[47] Mikko Stenlund. An almost sure ergodic theorem for quasistatic dynamical systems. *Mathematical Physics, Geometry and Analysis*, 19:14, 2016. doi:10.1007/s11040-016-9217-x.

[48] Mikko Stenlund and Henri Sulku. A coupling approach to random circle maps expanding on the average. *Stochastics and Dynamics*, 14(4):1450008 (29 pages), 2014. doi:10.1142/S0219493714500087.

[49] Mikko Stenlund, Lai-Sang Young, and Hongkun Zhang. Dispersing billiards with moving scatterers. *Commun. Math. Phys.*, 322(3):909–955, 2013. doi:10.1007/s00220-013-1746-6.

[50] Yujun Zhu, Zhaofeng Liu, Xueli Xu, and Wenda Zhang. Entropy of nonautonomous dynamical systems. *J. Korean Math. Soc.*, 49(1):165–185, 2012. doi:10.4134/JKMS.2012.49.1.165.

(Olli Hella) Department of Mathematics and Statistics, P.O. Box 68, Fin-00014 University of Helsinki, Finland.

E-mail address: olli.hella@helsinki.fi