Self-similar growth-fragmentation processes with types

by

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Summary. In this paper, we are interested in self-similar growth-fragmentation processes with types. It roughly consists of two parts. In the first part, we investigate multitype versions of the self-similar growth-fragmentation processes introduced in [Ber17], therefore extending the signed case of [Sil21] to finitely many types. Our main result in this direction describes the law of the spine in the multitype setting. We stress that our arguments only rely on the structure of the underlying Markov additive processes, and hence is more general than [Sil21]. In the second part, we study \(\mathbb{R}^d\)-valued self-similar growth-fragmentation processes driven by an isotropic process. These can be seen as multitype growth-fragmentation processes, where the set of types is the sphere \(S^{d-1}\). We give the spinal description in this setting. Finally, we prove that such a family of processes shows up when slicing half-space excursions with hyperplanes.

Keywords. Growth-fragmentation process, self-similar Markov process, Markov additive process, spinal decomposition, excursion theory.

1 Introduction

Self-similar growth-fragmentation processes first appeared in [Ber17] to describe the evolution of a cloud of atoms which may grow and dislocate in a binary way. More precisely, these atoms are assumed to have a specific one-dimensional trait of interest, which we can think of as its mass or size. Initially, the cloud starts from one particle (the common ancestor of all future particles) whose size is a positive quantity evolving in time in a Markovian way. This size will have jumps, and at each negative jump \(y < 0\), we wish to add to the cloud a new particle whose size at birth will be given by \(-y\), at the time when the jump occurs. This creates children of the original ancestor in such a way that the divisions are conservative, that is summing the size of the child and the size of the parent just after division exactly gives the size of the parent before division. Then, the newborn particles evolve independently of the parent, and independently of one another, in the same Markovian way as the parent. We proceed similarly creating the offspring of those particles, thereby introducing the grandchildren, great grandchildren, and so on, of the original ancestor.

Such growth-fragmentation models have been given a striking geometric flavour, in the context of random planar maps. This originated from [BCK18] and [BBCK18], where a remarkable class of self-similar growth-fragmentations shows up in the scaling limit of perimeter processes (see [Bud16]) in Markovian explorations of Boltzmann planar maps. These growth-fragmentation processes are closely related to stable Lévy processes with stability parameter \(\theta \in (\frac{1}{2}, \frac{3}{2})\). Since then, the same

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growth-fragmentation processes were directly constructed in the continuum \([MSW20]\) for \(1 < \theta \leq \frac{3}{2}\) by drawing a CLE exploration on a quantum disc. Moreover, the boundary case \(\theta = \frac{3}{2}\), corresponding to the random triangulations in \([BCK18]\), already appeared in \([LGR20]\) as the collection of perimeters obtained when slicing a Brownian disc at heights. The critical Cauchy case \(\theta = 1\), in turn, corresponds to slicing a Brownian half-plane excursion at heights \([AS20]\). This approach was recently extended to \(\frac{1}{2} < \theta < 1\) \([Sil21]\) by considering other half-plane excursions.

Let us point out that in \([AS20]\) (and subsequently in \([Sil21]\)), negative mass is taken into account in the system, whereas the aforementioned construction of growth-fragmentation processes deals with positive mass only. In those examples, the sign depends on the time orientation of the excursions. In particular, slicing a half-plane Brownian excursion only yields the critical case \(\theta = 1\) in \([BBCK18]\) provided one discards the negative cells. On a related note, the driving cell processes in the distinguished family of growth-fragmentations in \([BBCK18]\) also have positive jumps (except for \(\theta = \frac{3}{2}\)). This has a geometric meaning: Boltzmann planar maps correspond to the gasket of a loop \(O(n)\)-model, and the positive jumps occur when discovering a loop, which could then be explored. In the continuum, positive jumps also arise in \([MSW20]\) when hitting a CLE loop for the first time. This prompted \([Sil21]\) to provide a framework for self-similar signed growth-fragmentations.

Adding negative mass to the system presents some technical issues. The analysis of the positive case carried out in \([Ber17]\) and \([BBCK18]\) relies heavily on the Lamperti representation \([Lam72]\) for positive self-similar Markov processes, allowing for a large toolbox of Lévy techniques. This breaks down if one is willing to deal with signed processes, in other words the effect of introducing a sign is to move from the class of Lévy processes to the one of Markov additive processes, see for instance \([CPR13]\), \([KKPW14]\), \([KP21]\) and \([PR13]\). Part of this paper aims at extending the framework to a general set of types. This has a counterpart in the pure fragmentation setting, see for instance \([Ste18]\). In this case, we show that natural martingales arise, in connection to the additive martingales appearing in the context of multitype branching random walks (Section 3). These martingales have the same form as in \([Ber17]\), except that they are weighted by the types. Following the same lines as \([BBCK18, Theorem 4.2]\), our main theorem in the multitype setting (Theorem 4.3) describes the cell system under the change of measures with respect to these martingales (Section 4).

We stress that, although the framework developed here includes the signed case which was already treated in \([Sil21]\), our methodology is completely different. Indeed, \([Sil21]\) hinges upon a change of driving cell process to reduce to the positive case, whereas in this paper we directly work with Markov additive processes.

Next, we were interested in extending the growth-fragmentation framework to \(\mathbb{R}^d\)-valued Markov processes (Section 5). In this case, we take advantage of the interplay between self-similar Markov processes and Markov additive processes on the sphere \(S^{d-1}\), see \([KP21]\). This can be considered as a multitype growth-fragmentation model as described in the previous paragraph, where the types are the angles, so that they live in the (uncountable) set \(S^{d-1}\). Because of the complexity of the Markov additive process structure on uncountable state space, we mainly restrict to the isotropic setting, which forms a nicer subclass of self-similar Markov processes in \(\mathbb{R}^d\). In this case, we prove martingales, and prove that the corresponding exponents can be found as the roots of a convex function reminiscent of the cumulant function defined in \([Ber17]\). We prove that a similar spine decomposition holds under the associated change of measures (Section 6).

This lays the groundwork for the construction in Section 7 of a distinguished family of spatial growth-fragmentations. In light of \([AS20]\) and \([Sil21]\), we consider some excursions in the half-space \(\{(x_1, \ldots, x_d) \in \mathbb{R}^d, x_d > 0\}\). Slicing these excursions at heights along the hyperplanes \(\{x_d = a\}\), we obtain a collection of excursions which exhibit a branching structure. We define the size of such an excursion as the difference between the endpoint and the starting point (this is a vector in \(\mathbb{R}^{d-1}\)). We show that considering the collection of these sizes at varying heights constructs a special
growth-fragmentation in $\mathbb{R}^{d-1}$. Finally, we specify the spine obtained in this context.

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2 Self-similar Markov processes with types

We start by presenting some shared features of the Markov processes we will be interested in, revolving around the notion of self-similarity. We explain how to deal with types for self-similar processes, in the cases when the set of types is finite or the sphere $S^{d-1}$. As a common thread and a key ingredient of our analysis, we will use the Lamperti-Kiu representation, which gives a bijection between these self-similar processes and a class of Markov additive processes. We refer to [KP21] for a detailed treatment of these questions.

Markov additive processes. Let $E$ be a finite set or more generally a locally compact, complete and separable metric space, endowed with a cemetery state $\dagger$. We also let $(\xi(t), \Theta(t), t \geq 0)$ be a regular Feller process in $\mathbb{R} \times E$ with probabilities $P_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, on $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $(\mathcal{G}_t)_{t \geq 0}$ the natural standard filtration associated with $(\xi, \Theta)$. We say that $(\xi, \Theta)$ is a Markov
additive process (MAP for short) if for every bounded measurable \( f : \mathbb{R} \times E \to \mathbb{R} \), \( s, t \geq 0 \) and \((x, \theta) \in \mathbb{R} \times E\),
\[
E_{x,\theta} \left[ f(\xi(t + s) - \xi(t), \Theta(t + s)) \mathbb{1}_{\{t + s < \varsigma\}} \big| \mathcal{G}_t \right] = E_{0,\Theta(t)} \left[ f(\xi(s), \Theta(s)) \mathbb{1}_{\{s < \varsigma\}} \right],
\]
where \( \varsigma := \inf\{t > 0, \Theta(t) = 1\} \). Observe that the process \( \Theta \) is itself a regular Feller process in \( E \).

We call \( \xi \) the *ordinate* and \( \Theta \) the *modulator* of the MAP. The notation
\[
P_\theta := P(\cdot \mid \xi(0) = 0 \text{ and } \Theta(0) = \theta) \quad \text{for} \quad \theta \in E,
\]
will be in force throughout the paper. Whilst MAPs have found a prominent role in e.g. classical applied probability models for queues and dams when \( \Theta \) is a Markov chain with a finite state space (see for instance [Asm08] and [Iva11]), the case that \( \Theta \) is a general Markov process has received somewhat less attention. However, this case has been treated in the literature before, see for instance [Çin75] and references therein.

Informally, one should think of a MAP as a natural extension of a Lévy process in the sense that \( \Theta \) is an arbitrary well-behaved Markov process and \((\xi(t), \Theta(t))_{t \geq 0}, \mathbb{P}_{x,\theta}\) is equal in law to \((\xi(t) + x, \Theta(t))_{t \geq 0}, \mathbb{P}_\theta\). Moreover when \( \Theta \) is a Markov chain with a finite state space a more natural description can be given for the ordinate process \( \xi \). Indeed it can be thought as the concatenation of Lévy processes which depend on the current type in \( E \) given by \( \Theta \). Here we are interested in two specific cases which will be developed below: the case when \( E = \mathcal{I} \) is finite and the case when \( E = \mathbb{S}^{d-1} \) which describes the angles of a process in \( \mathbb{R}^d \).

**Markov additive processes with finite type set \( \mathcal{I} \).** One particularly important situation is when the set of types \( E = \mathcal{I} \) is finite, in which case \( \Theta \) is a continuous-time Markov chain with values in \( \mathcal{I} \). As we mentioned before, this case has been deeply studied, see for instance [Asm08], [Iva11] and the references therein.

An important property in this case, which in particular describes the structure of MAPs, is given by the following proposition, see [Iva11], [KKPW14], [KP21] or the survey [PR13].

**Proposition 2.1.** The process \((\xi, \Theta)\) is a Markov additive process if and only if there exist independent sequences \((\xi_i^{(n)}, n \geq 0)_{i \in \mathcal{I}}\) and \((U_{i,j}^{(n)}, n \geq 0)_{i,j \in \mathcal{I}}\), all independent of \( \Theta \), such that:

- for \( i \in \mathcal{I}, (\xi_i^{(n)}, n \geq 0) \) is a sequence of i.i.d. Lévy processes,
- for \( i, j \in \mathcal{I}, (U_{i,j}^{(n)}, n \geq 0) \) are i.i.d. random variables,
- if \((T_n)_{n \geq 0}\) denotes the sequence of jump times of the chain \( \Theta \) (with the convention \( T_0 = 0 \)), then for all \( n \geq 0 \),

\[
\xi(t) = \left( \xi(T_n^{-}) + U_{\Theta(T_n),\Theta(T_n)}^{(n)} \right) \mathbb{1}_{\{n \geq 1\}} + \xi_{\Theta(T_n)}^{(n)}(t - T_n), \quad T_n \leq t < T_{n+1}. \tag{2.1}
\]

We now turn to defining the *matrix exponent* of a MAP, which is the analogue of the Laplace exponent in the setting of Lévy processes. Without loss of generality, we assume that \( \mathcal{I} = \{1, \ldots, N\} \) where \( N \in \mathbb{N} \), and that \( \Theta \) is irreducible. We write \( Q = (q_{i,j})_{i,j \leq N} \) for its intensity matrix, and \( \rho_i \), \( i \in \mathcal{I} \) for the exponential time \( \Theta \) takes to jump from state \( i \) to some other state. Also, we denote for all \( i, j \in \mathcal{I} \), all on the same probability space, by \( \xi_i \) a Lévy process distributed as the \( \xi_i^{(n)} \)'s, and by \( U_{i,j} \) a random variable distributed as the \( U_{i,j}^{(n)} \)'s, with the convention \( U_{i,i} = 0 \) and \( U_{i,j} = 0 \) if \( q_{i,j} = 0 \). Finally, we introduce the Laplace exponent \( \psi_i \) of \( \xi_i \) and the Laplace transform \( G_{i,j}(z) := \mathbb{E} \left[ e^{z U_{i,j}} \right] \)
We write with additional clarification from [ACGZ17], building on the original work of [Lam72].

Then, for $R$ for our purposes, that is self-similar Markov processes in $\mathbb{R}^d$ for our purposes, that is self-similar with index $\alpha$ in the sense of (2.5), then one can find a MAP such that (2.4) holds, with the time change (2.3). This construction is reminiscent of the Lamperti or Lamperti-Kiu representations [Lam72,CPR13, KKPW14] for positive or real-valued self-similar Markov processes respectively. In the latter case, the type $J$ is the sign, see [CPR13, KKPW14]. We call this process $(X, J)$, or sometimes just $X$, a self-similar Markov process with types.

Self-similar Markov processes in $\mathbb{R}^d$ and isotropy. We now describe the other important case for our purposes, that is $E = \mathbb{S}^{d-1}$. Let $\alpha \in \mathbb{R}$. The Lamperti representation of $\alpha$-self-similar $\mathbb{R}^d$-valued Markov processes is the content of the following proposition which is attributed to [Kiu80] with additional clarification from [ACGZ17], building on the original work of [Lam72].
Proposition 2.2. Let $X$ be a self-similar $\mathbb{R}^d$-valued Markov process with index $\alpha$. Then there exists a Markov additive process $(\xi, \Theta)$ in $\mathbb{R} \times \mathbb{S}^{d-1}$ such that

$$X(t) = e^{\xi(\varphi(t))} \Theta(\varphi(t)), \quad t \leq I_\zeta := \int_0^\zeta e^{\alpha \xi(s)} ds,$$

where

$$\varphi(t) := \inf \left\{ s > 0, \int_0^s e^{\alpha \xi(u)} du > t \right\},$$

and $I_\zeta$ is the lifetime of $X$. Conversely, any process $X$ satisfying (2.6) is a self-similar Markov process in $\mathbb{R}^d$ with index $\alpha$.

In the previous statement we implicitly took the convention that $0 \times \uparrow = 0$. The integral $\zeta = I_\zeta$ is the lifetime of $X$ until it eventually hits 0, which acts as an absorbing state. For $x \in \mathbb{R}^d \setminus \{0\}$, we denote by $\mathbb{P}_x$ for the law of $X$ issued from $x$.

The analysis of MAPs with uncountable state space is much more intricate. One way to capture their properties is using the celebrated compensation formula. It was proved in [Çin75] that any MAP $(\xi, \Theta)$ on $\mathbb{R} \times \mathbb{S}^{d-1}$ is associated with a so-called Lévy system $(H, L)$, made up of an increasing additive functional $t \mapsto H_t$ of $\Theta$ and a transition kernel $L$ from $\mathbb{S}^{d-1}$ to $\mathbb{R}^* \times \mathbb{S}^{d-1}$ such that, for all $\theta \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{R}^*} \left( 1 \wedge |x|^2 \right) L_\theta(dx \times \{\theta\}) < \infty.$$  

More importantly, this Lévy system satisfies the following compensation formula for all bounded measurable $F : \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{R}$, and all $(x, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}$,

$$\mathbb{E}_{x, \theta} \left[ \sum_{s \geq 0} F(s, \xi(s^-), \Delta \xi(s), \Theta(s^\cdot), \Theta(s)) 1_{\{\xi(s^-) \neq \xi(s) \text{ or } \Theta(s^\cdot) \neq \Theta(s)\}} \right] = \mathbb{E}_{x, \theta} \left[ \int_0^\infty dH_s \int_{\mathbb{R}_+^* \times \mathbb{S}^{d-1}} L\Theta(s)(dx, d\Phi) F(s, \xi(s), x, \Theta(s), \Phi) \right].$$  

For the remainder of the paper we restrict ourselves to the usual setting $H_t = t$. Because of the bijection in Proposition 2.2, this naturally puts us in a restricted class of self-similar Markov processes through the underlying driving MAP. Observe how (2.7) compares with the compensation formula for Lévy processes: $L$ essentially plays the role of a Lévy measure, albeit now depending on the current angle from which the process jumps.

A nice subclass of MAPs is provided by isotropic self-similar Markov processes, and we shall mainly restrict ourselves to this setting. We say that a self-similar Markov process $X$ is isotropic if, for all isometry $U$, and all $x \in \mathbb{R}^d \setminus \{0\}$, the law of $(U \cdot X(t), \mathbb{P}_x)$ is $\mathbb{P}_{U \cdot x}$. Equivalently, this means [KP21, Theorem 11.14] that for all $(x, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}$, the law of $((\xi, U \cdot \Theta), \mathbb{P}_{x, \theta})$ is $\mathbb{P}_{x, U \cdot \theta}$. The key advantage of restricting to isotropic processes is the following proposition, which is [KP21, Corollary 11.15].

Proposition 2.3. If $X$ is an isotropic self-similar Markov process, then the underlying ordinate $\xi$ is a Lévy process.

Let us briefly mention that the proof of Proposition 2.3 relies on the fact that by isotropy, $|X|$ is a positive self-similar Markov process, for which we can apply the classical Lamperti theory. This result opens the way to many useful Lévy tools, such as the Lévy-Itô description of $\xi$, the
compensation or exponential formulas, or the existence of an exponential martingale and the corresponding change of measures. We will make heavy use of these additional properties when describing growth-fragmentations driven by isotropic processes in Section 5. Note that this notion of isotropy in particular covers the \( \alpha \)-stable isotropic Lévy case [Kyp18, Theorem 3.13], for which the Lévy system is given by \( H_t = t \) and

\[
L_\theta(dx, d\Phi) = \frac{c(\alpha)e^{dx}}{|e^{\theta} - \theta|^\alpha + d} \frac{dx}{d\Phi} \sigma_{d-1}(d\Phi),
\]

where \( c(\alpha) = 2^{\alpha-1} - d \Gamma((d+\alpha)/2)\Gamma(d/2) \left| \Gamma(-\alpha/2) \right| \), and \( \sigma_{d-1}(d\Phi) \) is the surface measure on the sphere \( S^{d-1} \).

See also [BW96] for the planar case. Numerous applications of Lévy systems can be found in [KRS18, KRŞY20] to name but a few.

3 Multitype growth-fragmentation processes

In this section, we present an extension of the growth-fragmentation framework [Ber17] to particles with finitely many types in \( \mathcal{I} \). We point out that the approach here presented is completely different than the treatment [Sil21] of the signed case, which relies on a change of Eve cell to go back to the positive setting. We shall describe the martingales appearing in this context, and how they can be found in the roots of multitype cumulants.

3.1 Construction of the multitype growth-fragmentation cell system

Following section 2, we will consider either a càdlàg self-similar Markov process with types \((X, J)\). For technical reasons, we further assume that \((X, J)\) is either absorbed at the cemetery state \( \partial \) after a finite time \( \zeta \), or that \( X \) converges to 0 under \( \mathbb{P}_{x,i} \) for all \( x \in \mathbb{R}^*_+ \), \( i \in \mathcal{I} \). We write \( \Delta X(s) := (X(s) - X(s^-))\mathbb{I}_{\{X(s) < X(s^-)\}} \) for the jump of \( X \), when it is negative.

We now construct a cell system whose building block is the self-similar Markov process with types \((X, J)\). This cell system will start from a single particle whose size and type are given by the process \((X, J)\), that will split in a binary way whenever \( X \) has a negative jump. This will create new particles with initial size given by the intensity of the jump, and which will then evolve as \((X, J)\) independently of the mother cell, and independently of one another, conditionally on their sizes at birth. This construction takes the viewpoint presented in [Ber17], but note that in our context we need to clarify the types of the offspring. To this end, we introduce some preliminary notation. Call \((\xi, \Theta)\) the MAP underlying \((X, J)\) via the Lamperti-Kiu transform (2.4). We assume throughout the paper that for all \( i \in \mathcal{I} \), the Lévy measure \( \Lambda_i \) of \( \xi_i \) can be decomposed as a sum of Lévy measures

\[
\Lambda_i(dx) := \sum_{k \in \mathcal{I}} \Lambda_{i}^{(k)}(dx), \tag{3.1}
\]

satisfying the following integrability condition

\[
\int_{\mathbb{R}} (1 \wedge |x|^2) \Lambda_i(dx) < \infty.
\]

Likewise, for \( i, j \in \mathcal{I} \), we give ourselves a decomposition of the law \( \Lambda_{U_{i,j}} \) of \( U_{i,j} \) as

\[
\Lambda_{U_{i,j}}(dx) := \sum_{k \in \mathcal{I}} \Lambda_{U_{i,j}}^{(k)}(dx). \tag{3.2}
\]
Equations (3.1) and (3.2) can be interpreted as a thinning of $\xi_i$ and $U_{i,j}$ respectively: the jumps of $\xi_i$ and $U_{i,j}$ should be understood as the result of a superimposition of jumps coming with a type $k \in \mathcal{I}$. Through the Lamperti time change (2.3), we see that any jump $\Delta X(s)$ of $X$ now also comes with some type, that we denote $J(\Delta(s))$.

We may now construct the cell system associated with $(X,J)$ and indexed by the tree $\mathbb{U} := \bigsqcup_{i \geq 0} \mathbb{N}^i$, with $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}^0 := \{\emptyset\}$ is the label of the $\text{Eve cell}$. For $u := (u_1,\ldots,u_i) \in \mathbb{U}$, we denote by $|u| = i$, the generation of $u$. In this tree, the offspring of $u$ will be labelled by the lists $(u_1,\ldots,u_i,k)$, with $k \in \mathbb{N}$.

We then define the law $\mathcal{P}_{x,i}$, $x > 0$, $i \in \mathcal{I}$, of the cell system $((\mathcal{X}_u(t),\mathcal{J}_u(t)), u \in \mathbb{U})$ driven by $X$ similarly to [Ber17]. Let $b_\emptyset = 0$ and take a copy $(\mathcal{X}_\emptyset,\mathcal{J}_\emptyset)$ of $(X,J)$ started from $(x,i)$. We can rank the sequence of positive jumps and times $(x_1,\beta_1),(x_2,\beta_2),\ldots$ and $-\mathcal{X}_\emptyset$ by descending lexicographical order of the $x_k$’s (this is possible because in any case $X$ is either absorbed at the cemetery state $\partial$ or converges to 0). We write $j_1,j_2,\ldots$ for the corresponding types. Given this sequence $(x_k,j_k,\beta_k,k \in \mathbb{N})$, we define the first generation $(\mathcal{X}_k,\mathcal{J}_k),k \in \mathbb{N}$, of our cell system as

independent processes with respective law $\mathbb{P}_{x_k,j_k}$, and we set $b_k = b_\emptyset + \beta_k$ for the birth time of $k$ and $\zeta_k$ for its lifetime. Likewise, we define the $n$-th generation given generations $1,\ldots,n-1$. A cell $u = (u_1,\ldots,u_{n-1}) \in \mathbb{N}^{n-1}$ gives birth to the cell $u = (u_1,\ldots,u_{n-1},k)$, with lifetime $\zeta_u$, at time $b_u = b_u' + \beta_k$ where $\beta_k$ is the $k$-th jump of $\mathcal{X}_{u'}$ (in terms of the same ranking as before). Moreover, conditionally on the jump sizes, types and times of $\mathcal{X}_{u'}$, $(\mathcal{X}_u,\mathcal{J}_u)$ has law $\mathbb{P}_{y,j}$ and is independent of the other daughter cells at generation $n$, where $-y = \Delta \mathcal{X}_{u'}(\beta_k)$ comes with type $j$. Note that division events are conservative in the sense that the sum of the size of a particle born at time $t$ and of its mother cell at time $t$ exactly equals the size of the mother cell before dislocation.

Although by construction the cells are not labelled chronologically, this uniquely defines the law $\mathcal{P}_{x,i}$ of the cell system driven by $(X,J)$ and started from $(x,i)$. The cell system $((\mathcal{X}_u(t),\mathcal{J}_u(t)),u \in \mathbb{U})$ is meant to describe the evolution of a population of atoms $u$ with size $\mathcal{X}_u(t)$ and type $\mathcal{J}_u(t)$ evolving with its age $t$ and fragmenting in a binary way.

Finally, we define the multitype growth-fragmentation process

$$X(t) := \{((\mathcal{X}_u(t-b_u),\mathcal{J}_u(t-b_u)), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}, \ t \geq 0,$$

where the double brackets denote multisets: $X(t)$ is the collection of all the particles alive in the system at time $t$. We denote by $\mathbb{P}_{x,i}$ the law of $X$ started from $(x,i)$ and $(\mathcal{F}_t, t \geq 0)$ the natural filtration of $X$.

**Remark 3.1.** We emphasize that only the negative jumps of $X$ give birth to new cells. One could also be willing to create new particles at the positive jump times, corresponding to cells with negative mass, so that the conservation rule still holds at splittings, similarly as in [Sil21]. This will simply result in doubling the number of types of the chain $J$, by considering the sign itself as a type. Hence we can restrict without loss of generality to considering only positive cells, i.e. negative jumps.

It is clear from the definition of growth-fragmentation processes that the cell system enjoys a genealogical branching structure. Under mild assumptions, this extends to a temporal branching property. Construct

$$\mathcal{X}(t) := \{((\mathcal{X}_u(t-b_u),\mathcal{J}_u(t-b_u),|u|), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}, \ t \geq 0,$$

by adjunction of the generations to the growth-fragmentation process, and consider its associated filtration $(\mathcal{F}_t, t \geq 0)$. A measurable function $f : \mathbb{R}_+ \to [0,\infty)$ is called excessive for $X$ if
If such an excessive function exists, then one can rank the elements \((X_t(1), J_t(1)), (X_t(2), J_t(2)), \ldots\) of \(X(t)\) by descending order of their size for any fixed \(t\).

**Proposition 3.2.** Assume that \(X\) has an excessive function. Then for any \(t \geq 0\), conditionally on \(X(t) = \{((x_i, j_i, n_i))\}\), the process \((\Xi(t+s), s \geq 0)\) is independent of \(\mathcal{F}_t\) and distributed as

\[
\bigcup_{i \geq 1} \Xi_i(s) \circ \theta_{n_i},
\]

where the \(\Xi_i, i \geq 1\), are independent processes distributed as \(\Xi\) under \(\mathbb{P}_{x_i,j_i}\), \(\theta_n\) is the shift operator, i.e. \(\{(z_i, y_i, k_i), i \geq 1\} \circ \theta_n := \{((z_i, y_i, k_i + n), i \geq 1)\}\), and \(\cup\) denotes union of multisets.

**Proof.** We refer to [Ber17, Proposition 2] for a proof of the statement in the classical framework, which is then easily extended to the multitype case. \(\square\)

### 3.2 Martingales in multitype growth-fragmentation processes

We continue the study of martingales for multitype growth-fragmentation processes initiated in [Sil21] in the signed case. The fact that \((-\log X_u(0), J_u(0))_{u \in \mathbb{U}}\) forms a multitype branching random walk provides several tools, including genealogical martingales for the growth-fragmentation cell system. The key feature is the following matrix \(m(q)\) indexed by the type set \(\mathcal{I}\), with entries

\[
m_{i,j}(q) := \mathbb{E}_i \left[ \sum_{0 < s < \zeta} |\Delta X(s)|^q \mathbb{I}_{\{J_\Delta(s) = j\}} \right], \quad q \in \mathbb{R}.
\]

This matrix has only nonnegative entries. We make the following two assumptions throughout the paper.

**Assumption A :** For \(q \in \mathbb{R}\) such that \(m(q)\) has finite entries, the matrix \(m(q)\) is irreducible.

In other words, Assumption A means that all the types communicate in the growth-fragmentation cell system (this is not too restrictive, since we could restrict to communication classes otherwise). Since \(\mathcal{I}\) is finite, this enables us to consider the Perron-Frobenius eigenvalue \(e^{\lambda(q)}\) and an associated positive eigenvector.

**Assumption B :** There exists \(\omega \in \mathbb{R}\) such that \(\lambda(\omega) = 0\).

We shall give a criterion for Assumption B later on in section 3.3. If \((v_i)_{i \in \mathcal{I}}\) has positive entries and \(\omega \geq 0\), we say that \((- \log X_u(0), J_u(0))_{u \in \mathbb{U}}\) is admissible for \(X\) if \(\lambda(\omega) = 0\), and \((v_i)_{i \in \mathcal{I}}\) is an associated eigenvector of \(m(\omega)\). In other words, \((- \log X_u(0), J_u(0))\) is admissible for \(X\) if, and only if,

\[
\forall i \in \mathcal{I}, \quad \mathbb{E}_i \left[ \sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^{\omega} \right] = v_i.
\]

This invariance property extends to a genealogical martingale as follows. Define

\[
\mathcal{G}_n := \sigma(X_u, |u| \leq n),
\]

noting that by definition, if \(u \in \mathbb{U}\) is such that \(|u| = n + 1\), then \(X_u(0)\) is \(\mathcal{G}_n\)-measurable.
Proposition 3.3. For all \((x, i) \in \mathbb{R}_+^* \times \mathcal{I}\), the process

\[ M(n) := \sum_{|u| = n+1} v_{\mathcal{J}_u(0)} |X_u(0)|^\omega, \quad n \geq 0, \]

defines a \((\mathcal{G}_n, n \geq 0)\)-martingale under \(P_{x,i}\).

Proof. The process \(M\) is obtained as the genealogical martingale of the multitype branching random walk \((- \log X_u(0), \mathcal{J}_u(0))_{u \in U}\), see [Sil21, Theorem 3.3].

Moreover, the following martingale for \(X\) will turn out useful in the next section. In particular, it implies the existence of an excessive function by extending [Ber17, Theorem 1] to the multitype case.

Proposition 3.4. For all \((x, i) \in \mathbb{R}_+^* \times \mathcal{I}\), under \(P_{x,i}\) the process

\[ M(t) := v_{J(t)} |X(t)|^\omega + \sum_{0 < s \leq t \wedge \zeta} v_{J(s)} |\Delta X(s)|^\omega, \quad t \geq 0, \]

is a uniformly integrable martingale for the filtration \((F^X_t, t \geq 0)\) of \(X\), with terminal value \(\sum_{0 < s < \zeta} v_{J(s)} |\Delta X(s)|^\omega\).

Proof. We omit the proof as it essentially follows from [Sil21, Proposition 3.5].

3.3 Multitype cumulant functions

For any sequence \(((v_i)_{i \in \mathcal{I}}, \omega)\), define

\[ M(t) := v_{J(t)} |X(t)|^\omega + \sum_{0 < s \leq t \wedge \zeta} v_{J(s)} |\Delta X(s)|^\omega, \quad s \geq 0, \]

where we omit the dependence on \(\omega\) and \((v_i)_{i \in \mathcal{I}}\) in the notation of \(M\) for simplicity. Proposition 3.4 states that when the underlying sequence is admissible, \(M\) is a martingale under \(P_i\) for all \(i \in \mathcal{I}\) (see the signed case). A converse statement also holds, providing a more tractable characterisation of admissibility.

Proposition 3.5. Let \(H\) be the first jump time of \(J\). Then \(((v_i)_{i \in \mathcal{I}}, \omega)\) is admissible for \(X\) if and only if, for all \(i \in \mathcal{I}\),

\[ \mathbb{E}_i[M(H)] = v_i. \]

Proof. The implication \(\Rightarrow\) follows easily from the optional stopping theorem applied to the martingale \(M\) in Proposition 3.4. Conversely, if we denote \((H_k, k \geq 0)\), the successive jump times of \(J\) (with \(H_0 = 0\)), then for any \(i \in \mathcal{I}\), by the Markov property of \((X, J)\) and self-similarity of \(X\),

\[
\mathbb{E}_i \left[ \sum_{0 < s < \zeta} v_{J(s)} |\Delta X(s)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_i \left[ \sum_{H_k < s \leq H_{k+1}} v_{J(s)} |\Delta X(s)|^\omega \right]
= \sum_{k \geq 0} \mathbb{E}_i \left[ |X(H_k)|^\omega \mathbb{E}_{J(H_k)} \left[ \sum_{s \leq H_k} v_{J(s)} |\Delta X(s)|^\omega \right] \right].
\]
Because we have assumed \( \mathbb{E}_j[M(H)] = v_j \) for all \( j \in \mathcal{I} \), this is

\[
\mathbb{E}_i \left[ \sum_{0 < s < \zeta} v_{\Delta(s)} |\Delta X(s)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_i \left[ X(H_k)^\omega \left( v_{J(H_k)} - \mathbb{E}_{J(H_k)} \left[ v_{J(H)}|X(H)|^\omega \right] \right) \right].
\]

Hence, using again the Markov property and self-similarity of \( X \) backwards, we find ourselves with

\[
\mathbb{E}_i \left[ \sum_{0 < s < \zeta} v_{\Delta(s)} |\Delta X(s)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_i \left[ v_{J(H_k)}|X(H_k)|^\omega \right] - \sum_{k \geq 0} \mathbb{E}_i \left[ v_{J(H_{k+1})}|X(H_{k+1})|^\omega \right],
\]

which ultimately cancels out, leaving

\[
\mathbb{E}_i \left[ \sum_{0 < s < \zeta} v_{\Delta(s)} |\Delta X(s)|^\omega \right] = v_i. \,
\]

Next, we identify multitype cumulant functions \( K_i, i \in \mathcal{I} \), whose common roots correspond to the admissible exponents \( \omega \). To do so, we compute \( \mathbb{E}_i[M(H)] \) in terms of the underlying MAP characteristics, for any (not necessarily admissible) sequence \((v_i)_{i \in \mathcal{I}}, \omega\). The expectation can be written as \( \mathbb{E}_i[M(H)] = A + B \), where

\[
A := \mathbb{E}_i \left[ \sum_{0 < s \leq H \land \zeta} v_{\Delta(s)} |\Delta X(s)|^\omega \right] \quad \text{and} \quad B := \mathbb{E}_i \left[ v_{J(H)}|X(H)|^\omega \right].
\]

Let us start with the term \( A \). For \( s > 0 \), we write as in (2.4), \( X(\varphi^{-1}(s)) = e^{\xi(s)} \) and \( J(\varphi^{-1}(s)) = \Theta(s) \) under \( \mathbb{P}_i \), where \( \varphi \) is the usual time-change (2.3). From this standpoint,

\[
A = \mathbb{E}_i \left[ \sum_{0 < s < \rho_i} v_{\Delta(s)} e^{\omega \xi_i(s^-)} \left( 1 - e^{\Delta \xi_i(s)} \right)^\omega \right] + \mathbb{E}_i \left[ v_{\Delta(\rho_i)} e^{\omega \xi_i(\rho_i^-)} \left( 1 - e^{U_{i,\omega}(\rho_i)} \right)^\omega \right], \tag{3.4}
\]

where \( \Delta(s) \) stands for the type corresponding to the jump of \( \xi \) at time \( s \). By independence and the compensation formula for \( \xi_i \), the first term of (3.4) is

\[
\mathbb{E}_i \left[ \sum_{s < \rho_i} v_{\Delta(s)} e^{\omega \xi_i(s^-)} \left( 1 - e^{\Delta \xi_i(s)} \right)^\omega \right] = \int_0^\infty dt \left( -q_{i,i} \right) e^{q_{i,i} t} \sum_{k \in \mathcal{I}} v_k \mathbb{E}_i \left[ \int_0^t ds e^{\omega \xi_i(s)} \right] \int_{(-\infty,0)} \Lambda_i^{(k)}(dx)(1 - e^x)^\omega
\]

\[
= \sum_{k \in \mathcal{I}} v_k \int_{(-\infty,0)} \Lambda_i^{(k)}(dx)(1 - e^x)^\omega \cdot \frac{1}{\psi_i(\omega)} \int_0^\infty dt \left( -q_{i,i} \right) e^{q_{i,i} t} (e^{\psi_i(\omega)t} - 1)
\]

\[
= -\frac{1}{\psi_i(\omega) + q_{i,i}} \sum_{k \in \mathcal{I}} v_k \int_{(-\infty,0)} \Lambda_i^{(k)}(dx)(1 - e^x)^\omega,
\]

provided \( \psi_i(\omega) + q_{i,i} < 0 \) (otherwise the expectation blows up). Now, let \( \iota^* = \Theta(\rho_i) \) be the type to which the Markov chain jumps at time \( \rho_i \). Then \( \iota^* \) is independent of \( \rho_i \), and for all \( j \in \mathcal{I} \setminus \{i\} \),
\( \nu^* = j \) with probability \(-\frac{q_{i,j}}{q_{i,i}}\). By conditioning on \( \rho_i \), we obtain that the second term of (3.4) is

\[
E_i \left[ v_{i,\Delta}(\rho_i) e^{\omega \xi_i(\rho_i)} \left( 1 - e^{U_{i,\nu^*}} \right) \right]
\]

\[
= \int_0^\infty dt \left( -q_{i,i} \right) e^{q_{i,i}t} \sum_{j \in I \setminus \{i\}} \frac{q_{i,j}}{(-q_{i,i})} E_i \left[ e^{\omega \xi_j(t)} \right] \sum_{k \in I} v_k \int_{(-\infty,0)} \Lambda_{U_{i,j}}^{(k)}(dx)(1 - e^x)^\omega
\]

\[
= \int_0^\infty dt \frac{e^{\psi_i(\omega) + q_{i,i}t}}{\psi_i(\omega) + q_{i,i}} \sum_{j \in I \setminus \{i\}} \frac{q_{i,j}}{(-q_{i,i})} \sum_{k \in I} v_k \int_{(-\infty,0)} \Lambda_{U_{i,j}}^{(k)}(dx)(1 - e^x)^\omega
\]

\[
= - \frac{1}{\psi_i(\omega) + q_{i,i}} \sum_{j \in I \setminus \{i\}} \frac{v_k}{v_{i,j}} q_{i,j} \int_{(-\infty,0)} \Pi_{i,k}(dx)(1 - e^x)^\omega,
\]

provided again that \( \psi_i(\omega) + q_{i,i} < 0 \). Therefore, we end up with

\[
A = - \frac{1}{\psi_i(\omega) + q_{i,i}} \sum_{k \in I} v_k \int_{(-\infty,0)} \Pi_{i,k}(dx)(1 - e^x)^\omega,
\]

with \( \Pi_{i,k}(dx) := \Lambda_{U_{i,j}}^{(k)}(dx) + \sum_{j \in I \setminus \{i\}} q_{i,j} \Lambda_{U_{i,j}}^{(k)}(dx) \).

We now compute

\[
B = E_i \left[ v_{i,*} e^{\omega (\xi_i(\rho_i) + U_{i,\nu^*})} \right].
\]

As before, we condition on \( \rho_i \) and decompose over the possible values \( j \in I \setminus \{i\} \) for \( \nu^* \):

\[
B = \sum_{j \in I \setminus \{i\}} \int_0^\infty ds \left( -q_{i,i} \right) e^{q_{i,i}s} \frac{q_{i,j}}{(-q_{i,i})} v_{j} E_i \left[ e^{\omega (\xi_j(s) + U_{i,j})} \right]
\]

\[
= \sum_{j \in I \setminus \{i\}} q_{i,j} v_j \int_0^\infty ds e^{q_{i,i}s} e^{\psi_j(\omega)s} G_{i,j}(\omega)
\]

\[
= - \frac{1}{\psi_i(\omega) + q_{i,i}} \sum_{j \in I \setminus \{i\}} q_{i,j} v_j G_{i,j}(\omega),
\]

as long as \( \psi_i(\omega) + q_{i,i} < 0 \). We come to the conclusion that

\[
E_i[M(H)] = - \frac{1}{\psi_i(\omega) + q_{i,i}} \cdot \left( \sum_{k \in I} v_k \int_{(-\infty,0)} \Pi_{i,k}(dx)(1 - e^x)^\omega + \sum_{j \in I \setminus \{i\}} q_{i,j} v_j G_{i,j}(\omega) \right).
\]

This is equal to \( v_i \) if and only if,

\[
K_i(\omega) := (\psi_i(\omega) + q_{i,i}) + \sum_{k \in I} \frac{v_k}{v_i} \int_{(-\infty,0)} \Pi_{i,k}(dx)(1 - e^x)^\omega + \sum_{j \in I \setminus \{i\}} \frac{v_j}{v_i} q_{i,j} G_{i,j}(\omega) = 0,
\]

and, thanks to Proposition 3.5, Assumption A in Section 3.2 boils to the existence of \( \omega \in \mathbb{R} \) and a sequence \( (v_i)_{i \in I} \) of positive numbers such that, for all \( i \in I \), \( K_i(\omega) = 0 \). We will call the family \( (K_i, i \in I) \) the multitype cumulant functions. We also write

\[
\kappa_i(q) := (\psi_i(q) + q_{i,i}) + \int_{(-\infty,0)} \Pi_{i,i}(dx)(1 - e^x)^q, \quad q \geq 0,
\]

(3.5) \{eq: kappa\}

for the cumulant function corresponding to type \( i \), so that for \( q \geq 0 \),

\[
K_i(q) := \kappa_i(q) + \sum_{j \in I \setminus \{i\}} \frac{v_j}{v_i} \left( \int_{(-\infty,0)} \Pi_{i,j}(dx)(1 - e^x)^q + q_{i,j} G_{i,j}(q) \right).
\]

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3.4 Example of multitype self-similar growth-fragmentation processes

We now briefly present an example of growth-fragmentation processes with finitely many types. This example may seem artificial, because it exhibits some symmetries, so that one may treat them without resorting to the general theory of multitype growth-fragmentation processes that we just described. Nonetheless, it is natural to view it as part of this larger family of processes, and we believe that other examples may be found in this context.

CLE explorations of a quantum disc. This example is based on the work of Miller, Sheffield and Werner [MSW20]. We review their results and reformulate them in the multitype setting. Let \( h \) be an instance of the Gaussian free field in a simply connected planar domain \( D \), and \( \gamma \in (0, 2) \). Loosely speaking, the \( \gamma \)-Liouville quantum gravity (in short, \( \gamma \)-LQG) surface \((D, h)\) is the surface parametrised by \( D \) in a conformally covariant way, with the random area measure \( \mu_h(dz) := e^{\gamma h(z)}dz \).

The issue is that \( h \) is not a random function but a random distribution, hence making sense of \( e^{\gamma h} \) requires some clarification, going through some mollification procedure. We will not discuss this here, and refer to [DS11,RV14] for details. Furthermore, by conformally covariant, we mean that a quantum surface is actually an equivalence class of pairs \((D, h)\), where we consider that \((D, h)\) and \((D', h')\) are equivalent if

\[
h = h' \circ \phi + Q \log |\phi'|, \tag{3.6}
\]

where \( Q := \frac{\gamma}{2} + \frac{2}{\gamma} \) and \( \phi \) is a conformal transformation mapping \( D \) to \( D' \). We stress that the relationship (3.6) is natural, since it comes from the condition that \( \mu_{h'} = \mu_h \circ \phi^{-1} \). Finally, one can equally make sense of the \( \gamma \)-LQG boundary length measure \( \nu_h \) of \( \partial D \) or several curves in \( D \), including SLE\( \kappa \)-type curves for \( \kappa \in \{\gamma^2, 16/\gamma^2\} \) [She16]. We will focus on one particular quantum surface stemming from a specific instance of the Gaussian free field, namely the unit-boundary \( \gamma \)-quantum disc, which was first introduced in [DMS14]. It has the property that its quantum boundary length is 1 and that its quantum area is finite almost surely.

![Figure 1: Drawing of the CPI on the CLE\( \kappa \) carpet in the \( \gamma \)-quantum disc. The CLE\( \kappa \) loops are shown in purple, and the CPI in blue (the target point is implicitly fixed).](Image)

\[{\text{fig: CLE on LQG}}\]
Quantum surfaces enjoy a rich interplay with some specific SLE$_\kappa$–type of curves, which can be seen as a continuum analogue to decorating a random planar map with some finely tuned statistical mechanics model. From now on, we fix $\kappa := \gamma^2$ and $\kappa' := 16/\gamma^2$. We assume that $\gamma \in (\sqrt{8}/3, 2)$. First, we consider on top of the quantum disc an independent conformal loop ensemble CLE$_\kappa$ [She09, SW12], which is a random collection of non-crossing loops in the disc. The other key player involves the conformal percolation interface (CPI) in the CLE$_\kappa$ carpet between boundary points $x$ and $y$ [MSW17], as represented in Figure 1. Roughly speaking, this is the only curve staying in the CLE$_\kappa$ carpet, enjoying conformal invariance and locality properties, such that anytime it hits a CLE$_\kappa$ loop, it leaves it to its right. It is also possible to make sense simultaneously of all the CPI branches towards any point in the disc.

The paper [MSW20] constructs through these CPI branches a growth-fragmentation process, which is the continuum analogue of the one arising from peeling explorations of Boltzmann planar maps [BBCK18]. Consider some CPI branch towards some point $z$ in the CLE$_\kappa$ carpet, and parametrise it by its quantum length. The branch in the growth-fragmentation towards $z$ corresponds to recording the quantum boundary length of the connected component containing $z$ as the CPI evolves. To see how this growth-fragmentation process comes into play, it is important to understand when jumps occur for the boundary length of the domain containing $z$. There are two types of jumps, corresponding to the following events (see Figure 2 and 3):

(i) The CPI hits a new CLE$_\kappa$ loop. Denote by $\ell$ the quantum boundary length of the loop. Then at this time the quantum boundary length of the domain containing $z$ goes from $L$ to $L + \ell$. Such times are therefore associated to positive jumps for the quantum boundary length of the component containing $z$.

![Figure 2: The CPI discovers a new CLE$_\kappa$ loop (event (i)). The to-be-explored domain swallows the loop at once and its total boundary length has a positive jump.](fig: CLE on LQG event (i))

(ii) The CPI splits the remaining-to-be-explored domain into two smaller subdomains. In this case, the quantum boundary length of the domain containing $z$ drops from $L$ to $L - \ell$, where $\ell$ is the quantum boundary length of the other subdomain. Hence these times correspond to negative jumps for the quantum boundary length of the component containing $z$.

Moreover, if $x$ and $y$ are any two points in the CLE$_\kappa$ carpet, the branches targeting $x$ and $y$ respectively will coincide up to some time when they will get disconnected by the CPI.
The main result of [MSW20] (Theorem 1.1) describes this branching structure in terms of the growth-fragmentation processes introduced in [BBCK18]. Call $X^\gamma$ this growth-fragmentation process. Note that in $X^\gamma$, only negative jumps are considered as fragmentation events. Thus, to recover a genuine growth-fragmentation structure, one should strictly speaking either discard the times when the CPI hits a CLE loop for the first time (case (i)) from birth events, or use the setting of signed growth-fragmentation processes [Sil21]. But it is also natural to consider another multitype growth-fragmentation process stemming from this construction. Indeed, one could in principle make a distinction between different events that may occur when the domain is split into smaller domains (case (ii) above). When the CPI hits the boundary of the remaining-to-be-explored region, the surface cut out may lie to its left or to its right. One could view these two events as different fragmentation events, by assigning a type (left or right) to the cut out surfaces. One should then obtain a multitype growth-fragmentation process $\tilde{X}^\gamma$. However, a consequence of [MSW20, Theorem 1.1] is that these surfaces have the same law (see also [MSW20, Theorem 1.4] for an analogous statement in the quantum half-plane case). This means that, although $\tilde{X}^\gamma$ contains more information than $X^\gamma$, the features of $\tilde{X}^\gamma$ can be deduced from those of $X^\gamma$. Such is for example the case of the martingales involved in Section 3.2, which are the same as those appearing in [MSW20].

4 The spine decomposition of multitype growth-fragmentation processes

4.1 Description of the spine under the change of measure

A change of measure. The martingale $\langle M(n), n \geq 0 \rangle$ in Proposition 3.3 enables us to introduce a new probability measure $\tilde{P}_{x,i}$ for $x > 0, i \in I$. Under this change of measure, the cell system has a spine decomposition that we aim to describe (see [BBCK18, Section 4.1]). The measure $\tilde{P}_{x,i}$ singles out a particular leaf $L \in \partial U = \mathbb{N}^N$. On $\mathcal{G}_n$, for $n \geq 0$, it has Radon-Nikodym derivative $M(n)$ with respect to $P_{x,i}$, up to normalisation, viz. for all $G_n \in \mathcal{G}_n$,

$$\tilde{P}_{x,i}(G_n) := \frac{1}{v_{x,i}^\omega} \varepsilon_{x,i}[M(n)1_{G_n}].$$

Moreover, conditionally on $\mathcal{G}_n$, the parent of the particular leaf $L$ at generation $n + 1$ is chosen under $\tilde{P}_{x,i}$ proportionally to its weight in the martingale $M(n)$. More precisely, let $\ell(n)$ denote the

Figure 3: Different situations when event (ii) may occur: (a) the CPI hits the boundary of the disc (b) the CPI hits itself (c) the CPI hits a previously visited CLE loop.
ancestor of the leaf $\ell \in \partial \mathcal{U}$ at generation $n$. Then for all $n \geq 0$ and all $u \in \mathcal{U}$ such that $|u| = n + 1$,
\[
\hat{\mathcal{P}}_{x,i}(\mathcal{L}(n+1) = u | \mathcal{G}_n) := \frac{v(J_u(0)|\mathcal{X}_u(0)|^\omega}{\mathcal{M}(n)}.
\]  
(4.1) \{eq: generation leaf\}

The consistency of formula (4.1) stems from the martingale property of $(\mathcal{M}(n), n \geq 0)$ and the branching structure of the system, thus defining a unique probability measure by an application of the Kolmogorov extension theorem.

One key player is provided by the so called tagged cell or spine, which consists in following the evolution of the cell associated with the leaf $\mathcal{L}$. The tagged cell will have the role of a backbone in the spine decomposition of the cell system under $\hat{\mathcal{P}}_{x,i}$. Let $b_\ell = \lim t \uparrow b_\ell(n)$ for any leaf $\ell \in \partial \mathcal{U}$. Then, we define $\hat{X}$ by $(\hat{X}(t), \hat{J}(t)) := \partial$ if $t \geq b_\mathcal{L}$ and
\[
\hat{X}(t) := \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \quad \text{and} \quad \hat{J}(t) := \mathcal{J}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \quad \text{if} \quad t < b_\mathcal{L},
\]
where $n_t$ is the unique integer $n$ such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$. From the very definition of $\hat{\mathcal{P}}_{x,i}$, for all nonnegative measurable function $f$ and all $\mathcal{G}_n$–measurable nonnegative random variable $B_n$,
\[
v_i|\omega|\hat{\mathcal{E}}_{x,i}\left[f(\mathcal{X}_{\mathcal{L}(n+1)}(0), \mathcal{J}_{\mathcal{L}(n+1)}(0))B_n\right] = \mathcal{E}_{x,i}\left[\sum_{|u|=n+1} v(J_u(0)|\mathcal{X}_u(0)|^\omega f(\mathcal{X}_u(0), \mathcal{J}_u(0))B_n\right].
\]

This extends to a temporal identity in the following way. Recall that $\mathcal{X}(t) = \{(X_k(t), J_k(t)), k \geq 1\}$, for $t \geq 0$, have been enumerated by descending order of the $X_k(t)$.

**Proposition 4.1.** For every $t \geq 0$, every nonnegative measurable function $f$ vanishing at $\partial$, and every $\mathcal{F}_t$–measurable nonnegative random variable $B_t$, we have
\[
v_i|\omega|\hat{\mathcal{E}}_{x,i}\left[f(\hat{X}(t), \hat{J}(t))B_t\right] = \mathcal{E}_{x,i}\left[\sum_{k \geq 1} v(J_k(t)|X_k(t)|^\omega f(X_k(t), J_k(t))B_t\right].
\]

**Proof.** The proof essentially follows from the arguments presented in the proof of [BBCK18, Proposition 4.1].

Let $t \geq 0$. Consider the case when $B_t$ is $\mathcal{F}_t \cap \mathcal{G}_k$–measurable for some $k \in \mathbb{N}$ (the result would then be readily extended by a monotone class argument). Since $f(\partial) = 0$, almost surely,
\[
f(\hat{X}(t), \hat{J}(t))B_t1_{\{b_{\mathcal{L}(n+1)}>t\}} \xrightarrow{n \to \infty} f(\hat{X}(t), \hat{J}(t))B_t.
\]
Therefore, by monotone convergence,
\[
\hat{\mathcal{E}}_{x,i}\left[f(\hat{X}(t), \hat{J}(t))B_t\right] = \lim_{n \to \infty} \hat{\mathcal{E}}_{x,i}\left[f(\hat{X}(t), \hat{J}(t))B_t1_{\{b_{\mathcal{L}(n+1)}>t\}}\right].
\]
Now, we want to condition on $\mathcal{G}_n$ and decompose $\mathcal{L}(n+1)$ over the cells at generation $n+1$, provided $n > k$ so that $B_t$ is $\mathcal{G}_n$–measurable. For $u$ such that $b_u > t$, write $u(t)$ for the most recent ancestor of $u$ at time $t$. Then
\[
\hat{\mathcal{E}}_{x,i}\left[f(\hat{X}(t), \hat{J}(t))B_t1_{\{b_{\mathcal{L}(n+1)}>t\}}\right]
\]
\[=
\frac{1}{v_i(\omega)|\omega|\hat{\mathcal{E}}_{x,i}\left[\sum_{|u|=n+1} v(J_u(0)|\mathcal{X}_u(0)|^\omega 1_{\{b_u>t\}}f(\mathcal{X}_u(t)(t-b_u(t)), \mathcal{J}_u(t)(t-b_u(t)))B_t\right].
\]

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Splitting over the value of \( u(t) \) yields

\[
\mathcal{E}_{x,i} \left[ \sum_{|u|=n+1} v_{\mathcal{J}_u(0)}|X_u(0)|^{\omega} \mathbb{1}_{\{b_u>t\}} f(\mathcal{X}_u(t) - b_u(t), \mathcal{J}_u(t) - b_u(t)) B_t \right] \\
= \mathcal{E}_{x,i} \left[ \sum_{|u'| \leq n} \sum_{|u|=n+1} v_{\mathcal{J}_u(0)}|X_u(0)|^{\omega} \mathbb{1}_{\{b_u>t\}} f(\mathcal{X}_u(t) - b_u(t), \mathcal{J}_u(t) - b_u(t)) B_t \mathbb{1}_{\{u(t)=u'\}} \right] \tag{4.2} \tag{eq:decompose}
\]

and by conditioning on \( \mathcal{F}_t \) and applying the temporal branching property stated in Proposition 3.2,

\[
\mathcal{E}_{x,i} \left[ \sum_{|u|=n+1} v_{\mathcal{J}_u(0)}|X_u(0)|^{\omega} \mathbb{1}_{\{b_u>t\}} f(\mathcal{X}_u(t) - b_u(t), \mathcal{J}_u(t) - b_u(t)) B_t \right] \\
= \mathcal{E}_{x,i} \left[ \sum_{|u'| \leq n} f(\mathcal{X}_{u'}(t) - b_{u'}, \mathcal{J}_{u'}(t) - b_{u'}) B_t \right] \\
\times \mathcal{E}_{\mathcal{X}_{u'}(t-b_{u'}), \mathcal{J}_{u'}(t-b_{u'})} \left[ \sum_{|u|=n+1-|u'|} v_{u'u} |X_{u'u}(0)|^{\omega} \mathbb{1}_{\{b_{u'} \leq t - b_{u'} + \zeta_{u'}\}} \right] \\
= \mathcal{E}_{x,i} \left[ \sum_{|u'| \leq n} f(\mathcal{X}_{u'}(t) - b_{u'}, \mathcal{J}_{u'}(t) - b_{u'}) B_t \mathbb{1}_{\{b_{u'} \leq t - b_{u'} + \zeta_{u'}\}} v_{\mathcal{J}_{u'}(t-b_{u'})}^{\omega} |X_{u'}(t) - b_{u'}|^{\omega} \right].
\]

Finally, taking \( n \to \infty \) and using monotone convergence, we obtain the desired result. \( \square \)

**Remark 4.2.** Proposition 4.1 applied with \( f := \mathbb{1}_{\{x \neq \emptyset\}} \) yields that the temporal analogue of \( \mathcal{M}(n) \) in Proposition 3.3,

\[
\mathcal{M}_t := \sum_{i=1}^{\infty} v_{i,t} |X_i(t)|^{\omega}, \quad t \geq 0,
\]

is a supermartingale with respect to \( (\mathcal{F}_t)_{t \geq 0} \).

The law of the growth-fragmentation under \( \hat{\mathbb{P}}_{x,i} \). We now describe the law of \( \hat{X} \) under \( \hat{\mathbb{P}}_{x,i} \). Loosely speaking, the tagged cell will serve as a backbone evolving as some explicit self-similar multitype Markov process, to which we attach independent copies of the original growth-fragmentation process. We must first reconstruct the whole cell system from the spine by recording the negative jumps of \( \hat{X} \), as detailed in [BBCK18, Section 4.1]. We will label these by couples \( (n,j) \), where \( n \geq 0 \) is the generation of the tagged cell immediately before the jump, and \( j \geq 1 \) is the rank (for the usual ranking) of the jump among those of the tagged cell at generation \( n \) (including the final jump when the generation changes from \( n \) to \( n+1 \)). To each such \( (n,j) \) corresponds a growth-fragmentation \( \hat{X}_{n,j} \) stemming from the corresponding jump: if the generation does not change during the \((n,j)\)-jump, then we set

\[
\hat{X}_{n,j}(t) := \{(\mathcal{X}_{uw}(t - b_{uw} + b_u), \mathcal{J}_{uw}(t - b_{uw} + b_u)), \; u \in \mathbb{U} \text{ and } b_{uw} \leq t + b_u < b_{uw} + \zeta_{uw}\},
\]

where \( u \) is the label of the cell born at the \((n,j)\)-jump. Otherwise, the \((n,j)\)-jump corresponds to a jump for the generation of the tagged cell and the tagged cell jumps from label \( u \) to label \( uk \) say, in which case

\[
\hat{X}_{n,j}(t) := \{\{(\mathcal{X}_{uw}(t - b_{uw} + b_{uk}), \mathcal{J}_{uw}(t - b_{uw} + b_{uk})), \; w \in \mathbb{U} \setminus \{k\} \text{ and } b_{uw} \leq t + b_{uk} < b_{uw} + \zeta_{uw}\}\}.
\]
We agree that $\hat{X}_{n,j} := \partial$ when the $(n,j)$-jump does not exist, and this completely defines $\hat{X}_{n,j}$ for all $n \geq 0$ and all $j \geq 1$.

Let $\hat{F}(q) := (\hat{F}_{i,j}(q))_{i,j \in \mathcal{I}}$ be the matrix with entries

$$
\hat{F}_{i,j}(q) = \begin{cases} 
\frac{v_j}{v_i} \int_{\mathbb{R}^+} \Pi_{i,j}(dx) [e^x - 1] q^{+\omega} + q_{i,j} G_{i,j}(q + \omega) & \text{if } i \neq j, \\
\kappa_i(\omega + q) & \text{if } i = j.
\end{cases}
$$

(4.3) \{eq: Fhat spine\}

\textbf{Theorem 4.3.} Under $\hat{\mathcal{P}}_{x,i}$, $(\hat{X}(t), \hat{J}(t), 0 \leq t < b_{\mathcal{L}})$ is a self-similar Markov process with types in $\mathcal{I}$, whose underlying Markov additive process has the matrix exponent $\hat{F}$ in (4.3).

Moreover, conditionally on $(\hat{X}(t), \hat{J}(t))_{0 \leq t < b_{\mathcal{L}}}$, the processes $\hat{X}_{n,j}$, $n \geq 0$, $j \geq 1$, are independent and each $\hat{X}_{n,j}$ has law $P_{x(n,j)}$ where $-x(n,j)$ is the size of the $(n,j)$-th jump.

\textbf{Remark 4.4.} 1. The law of the generation $n_i$ of the spine at time $t$ is not so explicit as in [BBCK18] or [Sil21] in the constant sign case. In fact, $b_{\mathcal{L}(1)}$ may not be exponential because of the current type of the spine before it jumps.

2. The proof of Theorem 4.3 goes through determining all three components $\hat{\psi}_i$, $\hat{q}_{i,j}$, and $\hat{G}_{i,j}$. of the MAP in (2.2). This sheds light on the structure of the MAP under (4.3).

\textbf{4.2 Proof of Theorem 4.3}

It is plain that the spine $(\hat{X}, \hat{J})$ inherits the Markov property and self-similarity of $(X, J)$, and therefore it can be described in terms of a MAP \textit{via} the representation (2.4). Without loss of generality, we may restrict to the homogeneous case when $\alpha = 0$. The result is then easily extended thanks to Lamperti time-change. We aim at finding the characteristics $(\hat{\psi}_i, \hat{q}_{i,j}, \hat{G}_{i,j})$ of the matrix exponent of this MAP.

\textbf{Description of the spine.} Let $\hat{H}$ be the first time when the type of the spine changes, and $\hat{J}(\hat{H})$ denote the corresponding type. Fix $q \geq 0$, and $i, j \in \mathcal{I}$.

\textbf{Determining the Laplace exponent $\hat{\psi}_i$.} This part is similar to the proof of [BBCK18, Theorem 4.2]. We denote by $\hat{\xi}$ the first component of the MAP corresponding to $\hat{X}$, that is

$$
\hat{\xi}(s) = \log \hat{X}(s), \quad s \geq 0,
$$

and $\hat{\xi}_k$ the underlying Lévy processes depending on type $k \in \mathcal{I}$. We want to show that the Lévy process $\hat{\xi}_i$ has Laplace exponent $\hat{\psi}_i(q) = \kappa_i(q + \omega) - \kappa_i(\omega)$. Notice that a process $\eta$ with Laplace exponent $\hat{\psi}_i$ can be written as the superposition $\eta = \eta^{(1)} + \eta^{(2)}$ of independent Lévy processes $\eta^{(1)}$ and $\eta^{(2)}$, with respective Laplace exponents $\psi^{(1)}(q) := \psi_i(q + \omega) - \psi_i(\omega)$ and

$$
\psi^{(2)}(q) := \int_{-\infty}^{0} \left( (1 - e^x)^{q+\omega} - (1 - e^x)^{\omega} \right) \Pi_{i,i}(dx).
$$

In particular, $\eta^{(2)}$ is a compound Poisson process with Lévy measure $e^{\omega x} \tilde{\Pi}_{i,i}(dx)$, where $\tilde{\Pi}_{i,i}(dx)$ is the image measure of $\Pi_{i,i}(dx) \mathbf{1}_{x < 0}$ through $x \mapsto \log(1 - e^x)$. Let $T$ be the first time when $\eta^{(2)}$ has a jump. The branching property of the cell system and the Markov property of $\eta$ ensures that the result will hold if we manage to prove that the distribution of $(\hat{\xi}_i(t), t \leq b_{\mathcal{L}(1)})$ is the same as that of $(\eta(t), t \leq T)$. Let $f, g$ be two nonnegative measurable functions defined respectively on the space of càdlàg trajectories and on $(-\infty, 0)$. Let

$$
\Delta\hat{\xi}(s) = \log \frac{\hat{X}(s)}{\hat{X}(s-)}, \quad s \geq 0,
$$

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We first split over the possible current generations for this special jump to occur:

\[
\mathbb{E}_i \left[ f(\xi(s), s < b_{L(1)}) g(\Delta \xi(b_{L(1)})) \mathbb{I}_{\{b_{L(1)} < \hat{T}\}} \right] = \mathbb{E}_i \left[ \sum_{t > 0} \frac{\nu_{\Delta(t)}}{b_t} e^{\omega \xi(t^-)} \left(1 - e^{\Delta \xi(t)}\right)^\omega \mathbb{I}_{\{t_{\Delta(t)} = i, t \leq \rho_i\}} f(\xi(s), s < t) g(\log(1 - e^{\Delta \xi(t)}) \right] = \mathbb{E}_i \left[ \sum_{0 < t < \rho_i} e^{\omega \xi(t^-)} \left(1 - e^{\Delta \xi(t)}\right)^\omega \mathbb{I}_{\{t_{\Delta(t)} = i\}} f(\xi(s), s < t) g(\log(1 - e^{\Delta \xi(t)}) \right] + \mathbb{E}_i \left[ e^{\omega \xi(\rho_i^-)} \left(1 - e^{U_i(\xi(\rho_i^-))}\right)^\omega \mathbb{I}_{\{t_{\Delta(\rho_i)} = i\}} f(\xi(s), s < \rho_i) g(\log(1 - e^{U_i(\xi(\rho_i^-))} \right].
\]

The compensation formula for $\xi_i$ entails that the first term is

\[
\mathbb{E}_i \left[ \sum_{0 < t < \rho_i} e^{\omega \xi(t^-)} \left(1 - e^{\Delta \xi(t)}\right)^\omega \mathbb{I}_{\{t_{\Delta(t)} = i\}} f(\xi(s), s < t) g(\log(1 - e^{\Delta \xi(t)}) \right] = \int_0^\infty dt \mathbb{E}_i \left[ f(\xi(s), s < t) e^{\omega \xi(t)} \right] \int_{-\infty}^0 g(\log(1 - e^x))(1 - e^x)^\omega \Lambda_i^{(1)}(dx). \tag{4.4} \]

The second term can be computed as follows:

\[
\mathbb{E}_i \left[ e^{\omega \xi(\rho_i^-)} \left(1 - e^{U_i(\xi(\rho_i^-))}\right)^\omega \mathbb{I}_{\{t_{\Delta(\rho_i)} = i\}} f(\xi(s), s < \rho_i) g(\log(1 - e^{U_i(\xi(\rho_i^-))} \right] = \int_0^\infty dt \mathbb{E}_i \left[ f(\xi(s), s < t) e^{\omega \xi(t)} \right] \sum_{k \in \mathbb{Z} \setminus \{i\}} q_{i,k} \int_{-\infty}^0 g(\log(1 - e^x))(1 - e^x)^\omega \Lambda_i^{(1)}(dx). \tag{4.5} \]

Combining (4.4) and (4.5), we finally obtain

\[
\mathbb{E}_i \left[ f(\xi(s), s < b_{L(1)}) g(\Delta \xi(b_{L(1)})) \mathbb{I}_{\{b_{L(1)} < \hat{T}\}} \right] = \int_0^\infty dt \mathbb{E}_i \left[ f(\xi(s), s < t) e^{\omega \xi(t)} \right] \int_{-\infty}^0 g(y) e^{y \tilde{\Pi}_i}(dy).
\]

This proves that $(\xi_i(s), s < b_{L(1)})$ and $\Delta \xi(b_{L(1)})$ are independent. The latter is distributed as $-\mathbb{I}_{\{y < 0\}} \frac{1}{q_{i,i} + \psi_i(\omega)} e^{y \tilde{\Pi}_i}(dy)$, which is the law of $\Delta \eta^{(2)}(T)$. The former is $\xi_i$ biased by the exponential martingale, and killed at an independent exponential time with parameter $-q_{i,i} - \psi_i(\omega)$, hence has Laplace exponent $\psi_i(q + \omega) + q_{i,i}$. We retrieve the Laplace exponent of $\eta^{(1)}$ killed at $T$, and conclude that $(\xi_i(t), t \leq b_{L(1)})$ evolves as $(\eta(t), t \leq T)$.

\begin{itemize}
  \item [\textbf{Determining the Laplace Transform $\hat{G}_{i,j}$ of the Special Jumps}]
  We want to compute
  \[
  \mathbb{E}_i \left[ \hat{\lambda}(\hat{T}) \mathbb{I}_{\{\hat{T} = j\}} \right] = \int_0^\infty dt \mathbb{E}_i \left[ f(\xi(s), s < t) e^{\omega \xi(t)} \right] \int_{-\infty}^0 g(y) e^{y \tilde{\Pi}_i}(dy).
  \]
  \end{itemize}

We first split over the possible current generations for this special jump to occur:

\[
\mathbb{E}_i \left[ \hat{\lambda}(\hat{T}) \mathbb{I}_{\{\hat{T} = j\}} \right] = \sum_{k = 0}^\infty \mathbb{E}_i \left[ \hat{\lambda}(\hat{T}) \mathbb{I}_{\{\hat{T} = j\}} \mathbb{I}_{\{b_{L(k)} < \hat{T} \leq b_{L(k+1)}\}} \right]. \tag{4.6} \]

\[}\]
For $k \geq 1$, the definition of $\hat{\rho}_i$ and the Markov property at time $\hat{H}$ yield

$$a_k = \mathbb{E}_i \left[ \sum_{0 < s < \zeta} \frac{v_{J(s)}}{v_i} \left| \Delta X(s) \right| \mathbb{1}_{\{H \geq s\}} \mathbb{1}_{\{J(s) = i\}} \right] \cdot a_{k-1},$$

where recall that $H$ denotes the first jump time of $J$. Therefore $a_k = \mu_{i,i}(\omega) \cdot a_{k-1} = \mu_{i,i}(\omega)^k \cdot a_0$, with

$$\mu_{i,i}(\omega) := \mathbb{E}_i \left[ \sum_{0 < s < \zeta} \left| \Delta X(s) \right| \mathbb{1}_{\{H \geq s\}} \mathbb{1}_{\{J(s) = i\}} \right].$$

Then, provided $\mu_{i,i}(\omega) < 1$, identity (4.6) triggers

$$\mathcal{E}_i \left[ \frac{\hat{\mathcal{X}}(\hat{H})}{\mathcal{X}(\hat{H}^-)} \right]^{q} \mathbb{1}_{\{\hat{J}(\hat{H}) = j\}} = \frac{a_0}{1 - \mu_{i,i}(\omega)}. \quad (4.7)$$

It remains to compute $a_0$ and $\mu_{i,i}(\omega)$. We begin with the latter:

$$\mu_{i,i}(\omega) = \mathbb{E}_i \left[ \sum_{0 < s < \rho_i} e^{\omega \xi(s^-)} \left( 1 - e^{\Delta \xi(s)} \right) e^{\omega \mathbb{1}_{\{t \Delta(s) = i\}}} \right] = \mathbb{E}_i \left[ \sum_{0 < s < \rho_i} e^{\omega \xi(s^-)} \left( 1 - e^{\Delta \xi(s)} \right) e^{\omega \mathbb{1}_{\{t \Delta(s) = i\}}} \right] + \mathbb{E}_i \left[ e^{\omega \xi_i(\rho_i^-)} \left( 1 - e^{\Delta \xi_i(\rho_i)} \right) e^{\omega \mathbb{1}_{\{t \Delta(\rho_i) = i\}}} \right].$$

A computation similar to (3.4) gives

$$\mu_{i,i}(\omega) = -\frac{1}{q_{i,i} + \psi_i(\omega)} \int_{-\infty}^{0} \Pi_{i,i}(dx) (1 - e^x)^\omega,$$

provided $\psi_i(\omega) + q_{i,i} < 0$. On the other hand, in $a_0$ the type of the spine can either change because $J$ jumps to $j$ (i.e. $\hat{H} < b_{E(1)}$), or because one picks a jump of type $j$ at time $b_{E(1)}$ (i.e. $\hat{H} = b_{E(1)}$). This writes

$$a_0 = A + B,$$

with

$$A = \mathcal{E}_i \left[ \frac{\hat{\mathcal{X}}(\hat{H})}{\mathcal{X}(\hat{H}^-)} \right]^{q} \mathbb{1}_{\{\hat{J}(\hat{H}) = j\}} \mathbb{1}_{\{\hat{H} < b_{E(1)}\}} \quad \text{and} \quad B = \mathcal{E}_i \left[ \frac{\hat{\mathcal{X}}(\hat{H})}{\mathcal{X}(\hat{H}^-)} \right]^{q} \mathbb{1}_{\{\hat{J}(\hat{H}) = j\}} \mathbb{1}_{\{\hat{H} = b_{E(1)}\}}.$$
By admissibility of \((v_i)_{i \in I, \omega}\),
\[ E_i \left[ \sum_{0 < t < \zeta} v_{J_{\Delta}(t)}|\Delta X(t)|^\omega \right] = v_j. \]
Hence,
\[
A = \frac{v_j}{v_i} E_i \left[ \left| \frac{X(H)}{X(H^-)} \right|^q \frac{|\Delta X(t)|^\omega 1_{\{J(t)=j\}}}{1_{\{J(t)=j\}}} \right] = \frac{v_j}{v_i} E_i \left[ e^{(q+\omega)U_{i,j}} e^{\omega \xi_i(\rho_i)} 1_{\{\Theta(\rho_i) = j\}} \right],
\]
and by independence we obtain
\[
A = \frac{v_j}{v_i} \sum_{q_i,j} q_i,j G_{i,j}(q + \omega) \int_0^\infty ds (-q_i,i) e^{q_i,i \xi_i(s)} E_i[e^{\omega \xi_i(s)}] = -\frac{v_j}{v_i} q_i,j G_{i,j}(q + \omega) \psi_i(\omega) + q_i,i.
\]
Besides,
\[
B = E_i \left[ \sum_{0 < t < \zeta} 1_{\{t \leq H\}} 1_{\{J_{\Delta}(t)=j\}} \frac{v_{J_{\Delta}(t)}}{v_i} |\Delta X(t)|^\omega \frac{\Delta X(t)|^q}{X(t^-)} \right]
\]
\[
= \frac{v_j}{v_i} E_i \left[ \sum_{0 < t < \rho_i} e^{\omega \xi_i(t^-)} \left( 1 - e^{\Delta \xi_i(t)} \right)^{q + \omega} 1_{\{i,j\}} \right] + \frac{v_j}{v_i} E_i \left[ e^{\omega \xi_i(\rho_i)} \left( 1 - e^{U_i,\Theta(\rho_i)} \right)^\omega 1_{\{i,j\}} \right].
\]
By the compensation formula as in (3.4), we finally get
\[
B = -\frac{v_j}{v_i} \frac{1}{q_i,i + \psi_i(\omega)} \int_{-\infty}^0 \Pi_{i,j}(dx)(1 - e^x)^{q + \omega}.
\]
We can now come back to (4.7) and deduce that
\[
\tilde{E}_i \left[ \left| \frac{\hat{X}(H)}{\hat{X}(H^-)} \right|^q 1_{\{\hat{J}(\hat{H})=j\}} \right] = -\frac{v_j}{v_i} \left( \int_{-\infty}^0 \Pi_{i,j}(dx)(1 - e^x)^{q + \omega} + q_i,j G_{i,j}(q + \omega) \right) \frac{1}{\psi_i(\omega) + q_i,i + \int_{-\infty}^0 \Pi_{i,i}(dx)(1 - e^x)^\omega}.
\]
Recalling (3.5), we are left with
\[
\tilde{E}_i \left[ \left| \frac{\hat{X}(H)}{\hat{X}(H^-)} \right|^q 1_{\{\hat{J}(\hat{H})=j\}} \right] = -\frac{v_j}{v_i} \left( \int_{-\infty}^0 \Pi_{i,j}(dx)(1 - e^x)^{q + \omega} + q_i,j G_{i,j}(q + \omega) \right) \frac{1}{\kappa_i(\omega)}.
\]
Note that, because \(K_i(\omega) = 0\), we get
\[
\kappa_i(\omega) = -\sum_{j \in I \setminus \{i\}} \frac{v_j}{v_i} \left( \int_{-\infty}^0 \Pi_{i,j}(dx)|e^x - 1|^\omega + q_i,j G_{i,j}(\omega) \right),
\]
which upon taking \(q = \omega\) already gives \(\hat{q}_{i,j}\) up to a multiplicative constant.
Determining the exponential parameters $\tilde{q}_{i,j}$. Recall that we have assumed homogeneity. Thus, for $q \geq 0$, we wish to compute,

$$
\mathcal{E}_i \left[ e^{q\tilde{R}} \mathbb{1}_{\{\tilde{J}(\tilde{R})=j\}} \right] = \sum_{k=0}^{\infty} \mathcal{E}_i \left[ e^{q\tilde{R}} \mathbb{1}_{\{\tilde{J}(\tilde{R})=j\}} \mathbb{1}_{\{b_{\mathcal{L}(k)} < \tilde{R} \leq b_{\mathcal{L}(k+1)}\}} \right].
$$

Again, using the definition of $\tilde{P}_i$ and the Markov property just as we did with $a_k$, we end up with

$$
a'_k = ra'_{k-1}, \quad k \geq 1,
$$

where

$$
r = \mathbb{E}_i \left[ \sum_{0<s<\rho_i} e^{q\xi_i(s^-)} \left( 1 - e^{\Delta \xi(s)} \right) e^{\omega_{i(\Delta(s))=i}} \right]
= \mathbb{E}_i \left[ \sum_{0<s<\rho_i} e^{q\xi_i(s^-)} \left( 1 - e^{\Delta \xi(s)} \right) \mathbb{1}_{\{\Delta(s)=i\}} \right] + \mathbb{E}_i \left[ e^{q\rho_i \xi_i(\rho_i^-)} \left( 1 - e^{U_i,\omega(\rho_i)} \right) \mathbb{1}_{\{\Delta(\rho_i)=i\}} \right].
$$

Then, we use the compensation formula and we obtain that the first term is

$$
\mathbb{E}_i \left[ \sum_{0<s<\rho_i} e^{q\xi_i(s^-)} \left( 1 - e^{\Delta \xi(s)} \right) e^{\omega_{i(\Delta(s))=i}} \right] = \int_0^\infty ds e^{(q+\Theta_i)s} e^{\psi_i(s)} e^{\omega \Delta \xi(s)} \int_{-\infty}^0 \Lambda_i^{(i)}(dx)(1-e^x)^\omega
= -\frac{1}{q+q_i+\psi_i(\omega)} \int_{-\infty}^0 \Lambda_i^{(i)}(dx)(1-e^x)^\omega.
$$

By independence, the second term of (4.8) is

$$
\mathbb{E}_i \left[ e^{q\rho_i \xi_i(\rho_i^-)} \left( 1 - e^{U_i,\omega(\rho_i)} \right) e^{\omega_{i(\Delta(\rho_i))=i}} \right]
= \sum_{k \in \mathcal{I}_\{i\}} \frac{q_{i,k}}{-q_{i,i}} \int_0^\infty ds (-q_{i,i})e^{(q+\Theta_i)s} e^{\psi_i(s)} e^{\omega \Delta \xi(s)} \int_{-\infty}^0 \Lambda_i^{(i)}(dx)(1-e^x)^\omega
= -\frac{1}{q+q_i+\psi_i(\omega)} \sum_{k \in \mathcal{I}_\{i\}} q_{i,k} \int_{-\infty}^0 \Lambda_i^{(i)}(dx)(1-e^x)^\omega.
$$

Thanks to (4.9) and (4.10), equation (4.8) boils down to

$$
r = -\frac{1}{q+q_i+\psi_i(\omega)} \int_{-\infty}^0 \Pi_{i,i}(dx)(1-e^x)^\omega.
$$

On the other hand,

$$
a'_0 = \mathcal{E}_i \left[ e^{q\tilde{R}} \mathbb{1}_{\{\tilde{J}(\tilde{R})=j\}} \mathbb{1}_{\{\tilde{R} \leq b_{\mathcal{L}(i)}\}} \right],
$$

and we may split the indicator over $\{\tilde{R} < b_{\mathcal{L}(1)}\}$ and $\{\tilde{R} = b_{\mathcal{L}(1)}\}$. We therefore get $a'_0 = A' + B'$, where

$$
A' = \mathbb{E}_i \left[ \sum_{0<t<\xi} e^{q\tilde{R}} \mathbb{1}_{\{J(\tilde{R})=j\}} \mathbb{1}_{\{\tilde{R} \leq b_{\mathcal{L}(i)}\}} \frac{v_j \Lambda(t)}{v_i} |\Delta X(t)|^\omega \right],
$$
and

\[ B' = E_i \left[ \sum_{0 < t \leq \rho_i} e^{\eta \xi(t^-)} \left( 1 - e^{\Delta \xi(t)} \right) \mathbf{1}_{\{i, \Delta(t) = j\}} \right]. \]

First of all, \( B' \) can be rewritten as follows

\[ B' = \frac{v_j}{v_i} E_i \left[ \sum_{0 < t < \rho_i} e^{\eta \xi(t^-)} \left( 1 - e^{\Delta \xi(t)} \right) \mathbf{1}_{\{i, \Delta(t) = j\}} \right] + \frac{v_j}{v_i} E_i \left[ e^{q \xi \omega_i(r^i)} \left( 1 - e^{U_i(\omega_i)} \right) \mathbf{1}_{\{i, \Delta(r^i) = j\}} \right]. \]

Continuing along the lines of (4.9), (4.10), we eventually get to

\[ B' = -\frac{v_j}{v_i} \frac{1}{q + q_{i,i} + \psi_i(\omega)} \int_{-\infty}^{0} \Pi_{i,j}(dx)(1 - e^x)^\omega. \]

Moreover, by using the Markov property at time \( H \), self-similarity of \( X \), and by admissibility of \( \{(v_i)_{i \in \mathcal{I}}, \omega\} \), we have

\[
A' = E_i \left[ e^{\eta H} \mathbf{1}_{\{J(H) = j\}} \mathbb{E}_H \left[ \sum_{0 < t < \zeta} \frac{v_j \Delta(t)}{v_i} |\Delta X(t)|^\omega \right] \right] \\
= E_i \left[ e^{\eta H} \mathbf{1}_{\{J(H) = j\}} |X(H)|^\omega \mathbb{E}_H \left[ \sum_{0 < t < \zeta} \frac{v_j \Delta(t)}{v_i} |\Delta X(t)|^\omega \right] \right] \\
= \frac{v_j}{v_i} E_i \left[ e^{\eta H} \mathbf{1}_{\{J(H) = j\}} |X(H)|^\omega \right].
\]

Now, on the event that \( J(H) = j \), we have \( X(H) = e^{\xi(r^i)+U_{i,j}} \) under \( \mathbb{P}_i \). This entails

\[
A' = -\frac{v_j}{v_i} \frac{q_{i,j} G_{i,j}(\omega)}{q + q_{i,i} + \psi_i(\omega)}.
\]

Therefore,

\[
a_0 = -\frac{\frac{v_j}{v_i} q_{i,j} G_{i,j}(\omega) + \frac{v_j}{v_i} \int_{-\infty}^{0} \Pi_{i,j}(dx)(1 - e^x)^\omega}{q + q_{i,i} + \psi_i(\omega)}.
\]

We finally conclude that

\[
\hat{a}_i \left[ e^{\eta H} \mathbf{1}_{\{\hat{J}(H) = j\}} \right] = \frac{a_0}{1 - r} = -\frac{\frac{v_j}{v_i} q_{i,j} G_{i,j}(\omega) + \frac{v_j}{v_i} \int_{-\infty}^{0} \Pi_{i,j}(dx)(1 - e^x)^\omega}{q + q_{i,i} + \psi_i(\omega) + \int_{-\infty}^{0} \Pi_{i,i}(dx)(1 - e^x)^\omega}.
\]

This shows that, for all \( i, j \in \mathcal{I} \), with \( j \neq i \), the jump time of the chain \( \hat{J} \) from state \( i \) to state \( j \) is an exponential random variable, with parameter

\[
\hat{q}_{i,j} = \frac{v_j}{v_i} \left( q_{i,j} G_{i,j}(\omega) + \int_{-\infty}^{0} \Pi_{i,j}(dx)|1 - e^x|^\omega \right).
\]
The matrix exponent. The previous calculations determine \( \hat{F}(q) = (\hat{F}_{i,j}(q))_{i,j \in \mathcal{I}} \), the matrix exponent of the spine as the matrix with entries:

\[
\forall i \in \mathcal{I}, \quad \hat{F}_{i,i}(q) = \kappa_i(\omega + q)
\]

and

\[
\forall i, j \in \mathcal{I}, i \neq j, \quad \hat{F}_{i,j}(q) = \frac{v_j}{v_i} \left( \int_{-\infty}^{0} \Pi_{i,j}(dx)(1 - e^x)^q + q_i G_{i,j}(q + \omega) \right).
\]

5 Spatial isotropic growth-fragmentation processes

Now, we extend the framework of [Ber17] to isotropic \( \mathbb{R}^d \)-valued Markov processes for \( d \geq 2 \). We exclude the case \( d = 1 \), since it can be deduced from the previous construction by considering a symmetric self-similar Markov process or from [Sil21]. It is important to note that the construction in the previous sections does not consider the isotropy assumption as well as in [Sil21].

In what follows \( X \) will be an isotropic \( \mathbb{R}^d \)-valued self-similar Markov process with index \( \alpha \), as defined in the last paragraph of Section 2, which under \( \mathbb{P}_x \), \( x \in \mathbb{R}^d \setminus \{0\} \), starts from \( x \). For technical reasons, we shall assume that \( X \) is either absorbed after time \( \zeta \) at some cemetery state \( \partial \), or that \( X \) converges to \( 0 \) at infinity, for all starting points. We also recall that \( (\xi, \Theta) \) denotes the MAP associated with \( X \).

5.1 Construction of spatial growth-fragmentation processes

The construction of the cell system in this case is similar to (and simpler than) the multitype case, so that we will only outline the construction. This construction actually holds without the self-similarity or isotropy assumptions. We alter a bit the previous notation by now letting \( \Delta X(t) := X(t) - X(t^-) \), for \( t \geq 0 \), denote the possible jump of \( X \) at time \( t \). At any jump time \( t \) of \( X \), one places a new particle in the system and, conditionally given their size \( -\Delta X(t) \) at birth, each of these newborn particles evolves independently as \( P_{-\Delta X(t)} \). Then, one repeats this construction for any such child, thus creating the second generation, and so on. As in the multitype case (Section 3.1), a more formal construction goes through iteratively defining variables \( X_u, u \in \mathcal{U} \), modelling the evolution of particles indexed by the Ulam tree.

In this construction, the cells are not labelled chronologically. However, it still uniquely defines the law \( \mathbb{P}_x \) of the cell system \( (X_u(t), u \in \mathcal{U}, t \geq 0) \) started from \( x \). Finally, we introduce the (spatial) growth-fragmentation process

\[
X(t) := \{ (X_u(t - b_u), u \in \mathcal{U} \text{ and } b_u \leq t < b_u + \zeta_u) \}, \quad t \geq 0,
\]

describing the collection of cells alive at time \( t \geq 0 \) (the double brackets here denote multisets). We define \( \mathbb{P}_x \) to be the law of the growth-fragmentation \( X \) started at \( x \).

We point out that one can view this construction as a multitype growth-fragmentation process, where the types correspond to the directions (in the \( d = 1 \) case, it is the sign). The set of types is therefore the sphere \( \mathbb{S}^{d-1} \), which is uncountable, so that the construction does not quite fall into the framework developed in Section 3. From this standpoint, note that the type corresponding to the daughter cell created by the jump \( \Delta X(t) \) is, up to time-change,

\[
\Theta_{\Delta}(t) := \frac{\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)}{|\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)|}.
\]
Similarly as in the multitype case, we have a temporal version of the branching property, see Proposition 3.2. Let

\[ \mathcal{X}(t) := \{(X_u(t - b_u), |u|), \ u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}, \quad t \geq 0. \]

We shall denote by \((\mathcal{F}_t, t \geq 0)\) the natural filtration associated with \(X\), and \((\overline{\mathcal{F}}_t, t \geq 0)\) the one associated with \(\overline{X}\). Under the existence of an excessive function (see (3.3) for its definition) for \(X\), one can rank the elements \(X_1(t), X_2(t), \ldots\) of \(X(t)\) by descending order of their norm for any fixed \(t\). Under the same assumption, we have the following.

**Proposition 5.1.** Assume that \(X\) has an excessive function. Then for any \(t \geq 0\), conditionally on \(\overline{X}(t) = \{(x_i, n_i)\}\), the process \((\overline{X}(t + s), s \geq 0)\) is independent of \(\mathcal{F}_t\) and distributed as

\[ \bigcup_{i \geq 1} \overline{X}_i(s) \circ \theta_{n_i}, \]

where the \(X_i, i \geq 1\), are independent processes distributed as \(\overline{X}\) under \(\mathcal{P}_{x_i}\), \(\theta_n\) is the shift operator \(\{(z_i, k_i), i \geq 1\} \circ \ theta := \{(z_i, k_i + n), i \geq 1\}\), and \(\sqcup\) denotes union of multisets.

### 5.2 The isotropic cumulant function and genealogical martingales

We are first of all interested in pointing out martingales as in Section 3.2 in the spatial growth-fragmentation setting. It turns out that the exponents \(\omega\) corresponding to these martingales will be found as the roots of an *isotropic cumulant function* which generalises the cumulant function \(\kappa\) in [Ber17, BBCK18]. Recall that, as readily seen from the rotational invariance property, the radial part of \(X\) is a positive self-similar Markov process, so that the ordinate \(\xi\) is in fact a Lévy process. We will extensively make use of this argument and its consequences.

Let us start with a simple but typical calculation: for \(q \geq 0\) and \(\theta \in \mathbb{S}^{d-1}\), we aim at computing the quantity \(E_\theta \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^q \right]\) in terms of the MAP characteristics of \(X\). We will now consider the Lévy system \((L, H)\) of \((\xi, \Theta)\) (see Section 2), and we take as usual \(H_t = t\) to avoid notational clutter. Since we want to sum over all \(t\)'s, we can omit the Lamperti-Kiu time-change between \(X\) and \((\xi, \Theta)\), so that

\[ E_\theta \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = E_{0,\theta} \left[ \sum_{0 < t < \zeta} e^{q\xi(t^-)} |\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)|^q \right]. \]

The compensation formula (2.7) for Markov additive processes then yields

\[ E_{0,\theta} \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = E_{0,\theta} \left[ \int_0^\infty dt \ e^{q\xi(t)} \int_{R^+ \times \mathbb{S}^{d-1}} L_{\Theta(t)}(dx, d\Phi) |\Theta(t) - e^{x\Phi}|^q \right]. \quad (5.1) \]

Remark that the integral

\[ \int_{R^+ \times \mathbb{S}^{d-1}} L_{\theta}(dx, d\Phi) |\theta - e^{x\Phi}|^q, \]

does not depend on the angle \(\theta\), since isotropy entails that if \(\theta, \theta' \in \mathbb{S}^{d-1}\), and \(U\) is an isometry mapping \(\theta\) to \(\theta'\), then \(L_{\theta'}(dx, d\Phi) = L_{\theta}(dx, U^{-1}d\Phi)\). More generally, the image measures \(L_{\theta'}(dy, d\phi)\)
of $L_\theta(dx, d\Phi)$ through the mapping $(y, \phi) = (\log |\theta - e^y\Phi|, \frac{\theta - e^y\Phi}{\theta - e^x\Phi})$ satisfy the same relationship. Indeed, for any nonnegative measurable function $F$, since $|U| = 1$, we have

$$
\int_{\mathbb{R}^* \times S^{d-1}} L_{\theta'}(dx, d\Phi) F \left( \log |\theta' - e^x\Phi|, \frac{\theta' - e^x\Phi}{\theta' - e^y\Phi} \right) = \int_{\mathbb{R}^* \times S^{d-1}} L_\theta(dx, U^{-1}d\Phi) F \left( \log |U\theta - e^x\Phi|, \frac{U\theta - e^x\Phi}{U\theta - e^y\Phi} \right)
$$

(5.2) \{eq: Ltilde isotropy\}

$$
= \int_{\mathbb{R}^* \times S^{d-1}} L_\theta(dx, d\phi) F \left( \log |U\theta - e^xU\varphi|, \frac{U\theta - e^xU\varphi}{U\theta - e^yU\varphi} \right).
$$

(5.3) \{eq:kappa\}

Hence $\tilde{L}_\theta'(dx, d\Phi) = \tilde{L}_\theta(dx, U^{-1}d\Phi)$. Singling out the image measure $\tilde{L}(dy, d\phi)$ of $L_\theta(dx, d\Phi)$ when $\theta = (1, 0, \ldots, 0)$ say, (5.1) boils down to

$$
\mathbb{E}_\theta \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = \mathbb{E}_{0, \theta} \left[ \int_0^\infty dt \, e^{\theta \xi(t)} \right] \int_{\mathbb{R}^* \times S^{d-1}} \tilde{L}(dy, d\phi)e^{\eta y}.
$$

Recall that $\xi$ is a Lévy process and assume that its Laplace exponent $\psi$ satisfies $\psi(q) < 0$, otherwise the first integral blows up. Then we are left with

$$
\mathbb{E}_\theta \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = 1 - \frac{\kappa(q)}{\psi(q)},
$$

where we have set

$$
\kappa(q) = \psi(q) + \int_{\mathbb{R}^* \times S^{d-1}} \tilde{L}(dy, d\phi)e^{\eta y}.
$$

(5.4) \{eq: sum kappa\}

We stress once more that $\kappa$ can be calculated using any of the measures $\tilde{L}_\theta$ in place of $\tilde{L}$. The previous calculations finally show that

$$
\mathbb{E}_\theta \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = \begin{cases} 
1 - \frac{\kappa(q)}{\psi(q)} & \text{if } \kappa(q) < \infty \text{ and } \psi(q) < 0, \\
+\infty & \text{otherwise}.
\end{cases}
$$

(5.5) \{eq: omega generalised\}

We call the function $\kappa$ the isotropic cumulant function. Its roots will lead to martingales for the growth-fragmentation cell system through the identity (5.4). Thus, throughout the paper we make the following assumption

**H** \hspace{1em} There exists $\omega \geq 0$ such that $\kappa(\omega) = 0$.

Notice that, as readily seen from (5.3), $\kappa$ is a convex function, so that there exist at most two such roots. For such a root $\omega$, we obtain by self-similarity and (5.4) that for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$
\mathbb{E}_x \left[ \sum_{0 < t < \zeta} |\Delta X(t)|^\omega \right] = |x|^\omega.
$$

Following the strategy of Section 3.2, we now show that the roots of $\kappa$ pave the way for remarkable martingales. The proof of the following result follows exactly from the same arguments as those used in Proposition 3.4.
Proposition 5.2. Under $\mathbb{P}_x$, for all $x \in \mathbb{R}^d \setminus \{0\}$, the process

$$M(t) := |X(t)|^\omega + \sum_{0 < s \leq t \wedge \zeta} |\Delta X(s)|^\omega,$$

is a martingale for the filtration $(F^X_t, t \geq 0)$ associated with $X$.

Moreover, the definition of $\omega$ and the branching structure of growth-fragmentation processes entail the existence of the following genealogical martingale, which will be crucial for the spine decomposition.

Let $G := \sigma (X_u, |u| \leq n), n \geq 0$.

Theorem 5.3. The process

$$\mathcal{M}(n) := \sum_{|u| = n+1} |X_u(0)|^\omega, \quad n \geq 0,$$

is a $(\mathcal{G}_n, n \geq 0)$-martingale under $\mathbb{P}_x$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

The arguments used to deduce the previous result are the same as those presented in Proposition 3.3.

5.3 A change of measures

We introduce a new probability measure $\hat{\mathbb{P}}_x$ for $x \in \mathbb{R}^d \setminus \{0\}$ using the martingale $(\mathcal{M}(n))_{n \geq 0}$ in Theorem 5.3. This is the analogue of [BBCK18, Section 4.1] in the positive case or [Sil21, Section 3.3] in the $d = 1$ case. It describes the law of a new cell system $(X_u)_{u \in \mathcal{U}}$ together with an infinite distinguished ray $\mathcal{L}$ or leaf, $\mathcal{L} \in \partial \mathcal{U} = \mathbb{N}^\infty$. On $\mathcal{G}_n$, for $n \geq 0$, it has Radon-Nikodym derivative with respect to $\mathbb{P}_x$ given by $\mathcal{M}(n)$, normalized to be a probability measure, i.e. for all $G_n \in \mathcal{G}_n$,

$$\hat{\mathbb{P}}_x(G_n) := |x|^{-\omega} \mathbb{E}_x [\mathcal{M}(n) \mathbb{1}_{G_n}].$$

The law of the particular leaf $\mathcal{L}$ under $\hat{\mathbb{P}}_x$ is chosen so that, for all $n \geq 0$ and all $u \in \mathcal{U}$ such that $|u| = n + 1$

$$\hat{\mathbb{P}}_x (\mathcal{L}(n+1) = u | \mathcal{G}_n) := \frac{|X_u(0)|^\omega}{\mathcal{M}(n)}, \quad (5.6)$$

where for any $\ell \in \partial \mathcal{U}$, $\ell(n)$ denotes the ancestor of $\ell$ at generation $n$. In words, to define the next generation of the spine, we select one of its jumps proportionally to its size to the power $\omega$ (the spine at generation 0 being the Eve cell). By an application of the Kolmogorov extension theorem, the martingale property and the branching structure of the system ensure that these definitions are compatible, and therefore this uniquely defines the probability measure $\hat{\mathbb{P}}_x$.

We will be interested in the evolution of the tagged cell, which is the cell associated with the distinguished leaf $\mathcal{L}$. More precisely, set $b_\ell = \lim \uparrow b_{\ell(n)}$ for any leaf $\ell \in \partial \mathcal{U}$. Then, define $\hat{\mathcal{L}}$ by $\hat{\mathcal{L}}(t) := \partial$ if $t \geq b_\mathcal{L}$ and

$$\hat{\mathcal{L}}(t) := \mathcal{L}_{n_t}(t - b_{\mathcal{L}(n_t)}), \quad t < b_\mathcal{L}, \quad (5.7)$$

where $n_t$ is the unique integer $n$ such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$.

By construction of $\hat{\mathbb{P}}_x$, we have the following genealogical many-to-one formula: for all nonnegative measurable function $f$ and all $\mathcal{G}_n$-measurable nonnegative random variable $B_n$,

$$|x|^{-\omega} \hat{\mathbb{E}}_x [f(\mathcal{L}_{n+1}(0))B_n] = \mathbb{E}_x \left[ \sum_{|u| = n+1} |X_u(0)|^\omega f(X_u(0))B_n \right].$$
This may be extended to a temporal many-to-one formula. The existence of \((v, \omega)\) ensures that we may rank the elements in \(X(t) = \{\{X_i(t), i \geq 1\}\}, \ t \geq 0\), by decreasing order of the norms.

**Proposition 5.4.** For every \(t \geq 0\), every nonnegative measurable function \(f\) vanishing at \(\partial\), and every \(\mathcal{F}_t\)-measurable nonnegative random variable \(B_t\), we have

\[
|\omega|^w \tilde{\mathcal{E}}_x[f(\tilde{X}(t))B_t] = \mathcal{E}_x \left[ \sum_{i \geq 1} |X_i(t)|^w f(X_i(t))B_t \right].
\]

**Proof.** See Proposition 4.1 for the multitype case, which is easily extended. \(\square\)

6 The spine decomposition of spatial isotropic growth-fragmentation processes

6.1 The spine decomposition for isotropic growth-fragmentation processes

In this section, we describe the law of the growth-fragmentation process under the change of measures \(\tilde{\mathcal{P}}_x, x \in \mathbb{R}^d\), and in particular the law of the tagged cell \(\tilde{X}\) (5.7). In order to make sense of this, we need to rebuild the growth-fragmentation along the spine, and so we must first label the jumps of \(\tilde{X}\). In general, one cannot rank those in lexicographical order. Instead, they will be labelled by couples \((n, j)\), where \(n \geq 0\) stands for the generation of the tagged cell immediately before the jump, and \(j \geq 1\) is the rank (in the usual lexicographical sense) of the jump among those of the tagged cell at generation \(n\) (including the final jump, when the generation changes to \(n + 1\)). For each such \((n, j)\), we define the growth-fragmentation \(\hat{X}_{n,j}\) induced by the corresponding jump. More precisely, if the generation stays constant during the \((n, j)\)-jump, then we set

\[
\hat{X}_{n,j}(t) := \{\{X_{uw}(t - b_{uw} + b_u), w \in U \text{ and } b_{uw} \leq t + b_u < b_{uw} + \zeta_{uw}\}\},
\]

where \(u\) is the label of the cell born at the \((n, j)\)-jump. Otherwise the \((n, j)\)-jump corresponds to a jump for the generation of the tagged cell so that the tagged cell jumps from label \(u\) to label \(uk\) say.

In this case, we set

\[
\hat{X}_{n,j}(t) := \{\{X_{uw}(t - b_{uw} + b_{uk}), w \in U \setminus \{k\} \text{ and } b_{uw} \leq t + b_{uk} < b_{uw} + \zeta_{uw}\}\}.
\]

Finally, we agree that \(\hat{X}_{n,j} := \partial\) when the \((n, j)\)-jump does not exist, and this sets \(\hat{X}_{n,j}\) for all \(n \geq 0\) and all \(j \geq 1\).

Recall also that \(n_t\) was defined in (5.7) and stands for the generation of the spine at time \(t\). We can now state our main theorem describing the law of the growth-fragmentation under \(\tilde{\mathcal{P}}_x\).

**Theorem 6.1.** Under \(\tilde{\mathcal{P}}_x\), \(\tilde{X}\) is a self-similar Markov process with values in \(\mathbb{R}^d\) and index \(\alpha\). The Lévy system of the underlying Markov additive process \((\xi, \Theta)\) is given by \((\tilde{H}, \tilde{L})\) where \(\tilde{H}_t = t\) and

\[
\tilde{L}_\theta(dy, d\phi) = e^{w\theta} \left( L_\theta(dy, d\phi) + \tilde{L}_\theta(dy, d\phi) \right).
\]

(6.1) **(eq: Lhat)**

Besides, \(\tilde{X}\) is isotropic, and the ordinate \(\tilde{\xi}\) is a Lévy process with Laplace exponent \(\tilde{\psi}(q) = \kappa(\omega + q)\). Moreover, conditionally on \((\tilde{X}(t), n_t)_{0 \leq t < b_{\tilde{X}}},\) the processes \(\hat{X}_{n,j}, n \geq 0, j \geq 1,\) are independent and each \(\hat{X}_{n,j}\) has law \(P_{x(n,j)}\) where \(-x(n,j)\) is the size of the \((n, j)\)-th jump.
Remark 6.2. 1. Observe that we have the following description of the MAP \((\hat{\xi}, \hat{\Theta})\). Let \((\eta^0, \Phi^0)\) be a MAP with Lévy system \((H^0, L^0)\) given by \(H^0_t := t\) and \(L^0_t(dy, d\phi) := \psi^y L_\theta(dy, d\phi)\). Consider an independent compound Poisson process \(D = (D_1, D_2)\) on \(\mathbb{R}^+ \times S^{d-1}\) with intensity measure \(e^{\psi y} L(dy, d\psi)\). This definition makes sense because, since \(\kappa(\omega) = 0\),

\[
\int_{\mathbb{R}^+ \times S^{d-1}} \tilde{L}(dy, d\varphi) e^{\varphi y} = -\psi(\omega) < \infty.
\]

Then \((\hat{\xi}, \hat{\Theta})\) is the superimposition of \((\eta^0, \Phi^0)\) and \(D\), in the following sense. Let \(T_1\) the first jump time of \(D\), which is exponential with parameter \(-\psi(\omega)\). Then \((\hat{\xi}(s), \hat{\Theta}(s), s < b_{L(1)})\) evolves as \((\eta^0(s), \Phi^0(s), s < T_1)\), and \((\hat{\xi}(b_{L(1)}), \hat{\Theta}(b_{L(1)}))\) is distributed as

\[
(\eta^0(T_1) + D_1(T_1), U_{\Phi^0(T_1)} \cdot D_2(T_1)),
\]

where \(U_\theta\) is an isometry mapping \((1, 0, \ldots, 0)\) to \(\theta\).

2. The proof actually provides a more precise statement describing the law of \((\hat{\lambda}(t), n_t, t \geq 0)\). The process \(n_t\) is then the Poisson process counting the jumps arising in \(D\) up to the usual Lamperti time change.

3. The MAP \((\xi^0, \eta^0)\) is exactly the so-called Esscher transform \((\xi^{\omega}, \Theta^{\omega})\) of \((\xi, \Theta)\). More precisely, recall that in the isotropic setting, \(\xi\) is itself a Lévy process, so that we can consider the usual exponential martingale \((e^{\omega(t)} - \psi(\omega), t \geq 0)\). Then the law of \((\xi, \Theta)\) under the exponential change of measures is \((\xi^{\omega}, \Theta^{\omega})\). This will appear in the proof.

4. Combining these two remarks casts light on equation (6.1). Loosely speaking, it is a decomposition of \(\hat{L}\) in terms of the jumps of the Esscher transform of \((\xi, \Theta)\) and the special jumps when the spine picks one of the jumps according to (5.6).

5. We deduce from Theorem 6.1 that the temporal version of \((\mathcal{M}(n), n \geq 0)\), namely

\[
\mathcal{M}_t := \sum_{i=1}^{\infty} |X_i(t)|^\omega, \quad t \geq 0,
\]

is a \((\mathcal{F}_t)\)-martingale if, and only if, \(\alpha\kappa'(\omega) < 0\). Indeed, by taking \(f = 1_\beta\) the many-to-one formula (Proposition 5.4) yields that \((\mathcal{M}_t, t \geq 0)\) is a supermartingale, and that it is a martingale if, and only if, \(X\) has infinite lifetime. From the Lamperti representation of \(|X|\), and the expression \(\hat{\psi}(q) = \kappa(\omega + q)\) of the Laplace exponent of \(\hat{\xi}\), this happens exactly when \(\alpha\kappa'(\omega) < 0\).

6.2 Proof of Theorem 6.1

The proof will roughly follow the same lines as the one of Theorem 4.3, although the structure of the modulator is more involved.

The law of the spine \(\hat{\lambda}\). The definition of \(\hat{\lambda}\) readily shows that \(\hat{\lambda}\) is an \(\alpha\)-self-similar Markov process. By Lamperti time change, we may place ourselves in the homogeneous case \(\alpha = 0\). In this case, note that there is no time change between \(\hat{\lambda}\) and \((\hat{\xi}, \hat{\Theta})\). For this reason, and to avoid notational clutter, we will sometimes make an abuse of notation by considering them on the same probability space. Likewise, we will use expressions involving both \(X\) and its MAP \((\xi, \Theta)\) as a
shorthand. Moreover, the Markov property implies that we only need to check the compensation formula up to the first time $b_{\mathcal{L}(1)}$ when the spine selects another generation. More precisely, we want to show that

$$\hat{\mathcal{E}}_\theta \left[ \sum_{s>0} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \mathbb{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right]$$

$$= \hat{\mathcal{E}}_\theta \left[ \int_0^\infty \mathrm{d}s e^{\psi(s)} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \L_{\hat{\Theta}(s)}(\mathrm{d}x, \mathrm{d}\varphi) F(s, \hat{\xi}(s), x, \hat{\Theta}(s), \varphi) \right]. \quad (6.2)$$

We may split the sum into two parts:

$$\hat{\mathcal{E}}_\theta \left[ \sum_{s>0} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \mathbb{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] =$$

$$\hat{\mathcal{E}}_\theta \left[ \sum_{s<b_{\mathcal{L}(1)}} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \right] + \hat{\mathcal{E}}_\theta \left[ F(b_{\mathcal{L}(1)}), \hat{\xi}(b_{\mathcal{L}(1)}), \Delta \hat{\xi}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}) \right]. \quad (6.3) \quad \text{[eq: sum decomposed b(1)]}$$

We compute the first term of (6.3). By definition of $b_{\mathcal{L}(1)}$,

$$(\hat{\xi}(s), \hat{\Theta}(s), s < b_{\mathcal{L}(1)}) = (\xi(s), \Theta(s), s < b_{\mathcal{L}(1)}).$$

Applying the change of measure (5.6), and recalling that we are in the homogeneous case, we get

$$\hat{\mathcal{E}}_\theta \left[ \sum_{s<b_{\mathcal{L}(1)}} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \right]$$

$$= \mathbb{E}_\theta \left[ \sum_{s>0} \sum_{t>s} F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) |\Delta X(t)|^\omega \right]. \quad (6.4) \quad \text{[eq: split b1]}$$

Now, the Markov property of $X$ at fixed time $s > 0$ yields that

$$\mathbb{E}_\theta \left[ \sum_{t>s} F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) |\Delta X(t)|^\omega \right]$$

$$= \mathbb{E}_\theta \left[ F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) \mathbb{E}_X(s) \left[ \sum_{t>0} |\Delta X(t)|^\omega \right] \right],$$

and using the definition of $\omega$ in identity (5.5),

$$\mathbb{E}_\theta \left[ \sum_{t>s} F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) |\Delta X(t)|^\omega \right] = \mathbb{E}_{0,\theta} \left[ F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) e^{\omega \xi(s)} \right].$$

Coming back to (6.4), this means

$$\hat{\mathcal{E}}_\theta \left[ \sum_{s<b_{\mathcal{L}(1)}} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \right] = \mathbb{E}_{0,\theta} \left[ \sum_{s>0} F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) e^{\omega \xi(s)} \right].$$
Using the compensation formula entails

\[
\hat{\mathbb{E}}_\theta \left[ \sum_{s < b_{\mathbb{L}(1)}} F(s, \tilde{\xi}(s), \Delta \tilde{\xi}(s), \tilde{\Theta}(s)) \right] = \mathbb{E}_{0,\theta} \left[ \int_0^\infty ds e^{\omega \xi(s)} \int_{\mathbb{R}^+ \times \mathbb{S}^{d-1}} L_{\Theta(s)}(dx, d\varphi) e^{\omega x} F(s, \xi(s), x, \Theta(s), \varphi) \right]. \tag{6.5} \]

We now tilt the measure using the classical Esscher transform (see for example [KP21]). Recall from Remark 6.2 that the process obtained has the law \(\mathbb{P}_{0,\theta}^0\) of \((\eta^0, \Phi^0)\). Thus equation (6.5) rewrites

\[
\hat{\mathbb{E}}_\theta \left[ \sum_{s < b_{\mathbb{L}(1)}} F(s, \tilde{\xi}(s), \Delta \tilde{\xi}(s), \tilde{\Theta}(s)) \right] = \mathbb{E}_{0,\theta} \left[ \int_0^\infty ds e^{\psi(\omega)s} \int_{\mathbb{R}^+ \times \mathbb{S}^{d-1}} L_{\Phi^0(s)}(dx, d\varphi) e^{\omega x} F(s, \eta^0(s), x, \Phi^0(s), \varphi) \right]. \tag{6.6} \]

Note that, since \(L_\theta(dx, d\varphi)e^{\omega x}\) is the jump measure of the Lévy system associated with \((\eta^0, \Phi^0)\), this shows that \((\tilde{\xi}(s), \tilde{\Theta}(s), s < b_{\mathbb{L}(1)})\) behaves as \((\eta^0(s), \Phi^0(s), s < T_1)\), where \(T_1\) is an independent exponential time with parameter \(-\psi(\omega)\), a fact that could have been derived directly.

Let us now compute the second term of (6.3). Changing the measure according to (5.6) again, one obtains

\[
\hat{\mathbb{E}}_\theta \left[ F(b_{\mathbb{L}(1)}), \tilde{\xi}(b_{\mathbb{L}(1)}), \Delta \tilde{\xi}(b_{\mathbb{L}(1)}), \tilde{\Theta}(b_{\mathbb{L}(1)}), \tilde{\Theta}(b_{\mathbb{L}(1)}) \right] = \mathbb{E}_\theta \left[ \sum_{s > 0} |\Delta X(s)|^{\omega} F \left( s, \xi(s), \log |\Delta X(s)| - \xi(s), \Theta(s) - \frac{\Delta X(s)}{|\Delta X(s)|} \right) \right]
\]

\[
= \mathbb{E}_{0,\theta} \left[ \sum_{s > 0} e^{\omega \xi(s)} |\Theta(s) - e^{\Delta \xi(s)} \Theta(s)|^{\omega} F(s, \xi(s), \log |\Theta(s) - e^{\Delta \xi(s)} \Theta(s)|, \Theta(s), \Theta(s), \Theta(s) \Delta(s)) \right],
\]

where as usual

\[
\Theta(s) \Delta(s) = \frac{\Theta(s) - e^{\Delta \xi(s)} \Theta(s)}{|\Theta(s) - e^{\Delta \xi(s)} \Theta(s)|}.
\]

Using the compensation formula for \((\xi, \Theta)\), this is

\[
\hat{\mathbb{E}}_\theta \left[ F(b_{\mathbb{L}(1)}), \tilde{\xi}(b_{\mathbb{L}(1)}), \Delta \tilde{\xi}(b_{\mathbb{L}(1)}), \tilde{\Theta}(b_{\mathbb{L}(1)}), \tilde{\Theta}(b_{\mathbb{L}(1)}) \right] = \mathbb{E}_{0,\theta} \left[ \int_0^\infty ds e^{\omega \xi(s)} \int_{\mathbb{R}^+ \times \mathbb{S}^{d-1}} L_{\Theta(s)}(dx, d\varphi) |\Theta(s) - e^{\varphi}|^{\omega}
\]

\[
\times F \left( s, \xi(s), \log |\Theta(s) - e^{\varphi}|, \Theta(s), \frac{\Theta(s) - e^{\varphi}}{|\Theta(s) - e^{\varphi}|} \right)
\]

We want to perform the change of variables \((y, \phi) = (\log |\theta - e^\varphi|, \frac{\theta - e^\varphi}{|\theta - e^\varphi|})\) for fixed \(\theta\) in the second integral. Recall that we have defined \(\tilde{L}_\theta\) as the image measure of \(L_\theta\) through this mapping, and that these measures satisfy the isotropy relationship (5.2). Therefore,

\[
\hat{\mathbb{E}}_\theta \left[ F(b_{\mathbb{L}(1)}), \tilde{\xi}(b_{\mathbb{L}(1)}), \Delta \tilde{\xi}(b_{\mathbb{L}(1)}), \tilde{\Theta}(b_{\mathbb{L}(1)}), \tilde{\Theta}(b_{\mathbb{L}(1)}) \right] = \mathbb{E}_{0,\theta} \left[ \int_0^\infty ds e^{\omega \xi(s)} \int_{\mathbb{R}^+ \times \mathbb{S}^{d-1}} \tilde{L}_{\Theta(s)}(dy, d\phi) e^{\omega y} F(s, \xi(s), y, \Theta(s), \phi) \right].
\]
Tilting with the exponential martingale of $\xi$ finally provides
\[
\tilde{E}_\theta \left[ F(b_{\mathcal{L}(1)}, \xi(b_{\mathcal{L}(1)}), \Delta \tilde{\xi}(b_{\mathcal{L}(1)}), \tilde{\Theta}(b_{\mathcal{L}(1)}), \tilde{\Theta}(b_{\mathcal{L}(1)})) \right] 
= \mathbb{E}_{0,\theta}^0 \left[ \int_0^\infty ds e^{\psi(s)} \int_{\mathbb{R}^+ \times S^{d-1}} \tilde{L}_{\Phi_0(s)}(dy, d\phi) e^{\omega y} F(s, \eta^0(s), y, \Phi^0(s), \phi) \right]. \tag{6.7} \]

Putting together (6.3), (6.6) and (6.7), we end up with
\[
\tilde{E}_\theta \left[ \sum_{s > 0} F(s, \xi(s^-), \Delta \tilde{\xi}(s), \tilde{\Theta}(s^-)) \mathbb{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] 
= \mathbb{E}_{0,\theta}^0 \left[ \int_0^\infty ds e^{\psi(s)} \int_{\mathbb{R}^+ \times S^{d-1}} \tilde{L}_{\Phi_0(s)}(dx, d\varphi) F(s, \eta^0(s), x, \Phi^0(s), \varphi) \right],
\]
and since $(\eta^0(s), \Phi^0(s), s < T_1)$ has the same law as $(\xi(s), \tilde{\Theta}(s), s < b_{\mathcal{L}(1)})$, we can rewrite this as
\[
\tilde{E}_\theta \left[ \sum_{s > 0} F(s, \xi(s^-), \Delta \tilde{\xi}(s), \tilde{\Theta}(s^-)) \mathbb{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] 
= \tilde{E}_\theta \left[ \int_0^\infty ds e^{\psi(s)} \int_{\mathbb{R}^+ \times S^{d-1}} \tilde{L}_{\Theta(s)}(dx, d\varphi) F(s, \tilde{\xi}(s), x, \tilde{\Theta}(s), \varphi) \right]. \tag{6.8} \]

This completes the proof of (6.1).

The second assertion of the theorem is then a straightforward consequence. First, it is clear that since $X$ is isotropic, so is $\tilde{X}$ by construction. Hence, by Proposition 2.3, $\tilde{X}$ must be a Lévy process. The expression for $\tilde{\psi}$ can be found using a particular case of the compensation formula (6.8). Alternatively, for any nonnegative measurable functionals $F$ and $G$ defined respectively on the space of finite càdlàg paths and on $\mathbb{R}$, we may compute
\[
\tilde{E}_\theta \left[ F(\tilde{\xi}(s), s < b_{\mathcal{L}(1)}) G(\Delta \tilde{\xi}(b_{\mathcal{L}(1)})) \right] 
= \mathbb{E}_\theta \left[ \sum_{t > 0} |\Delta X(t)|^\varphi F(\log |X(s)|, s < t) G \left( \log \frac{|\Delta X(t)|}{|X(t^-)|} \right) \right] 
= \mathbb{E}_{0,\theta} \left[ \sum_{t > 0} e^{\omega \xi(t^-)} \Theta(t^-) - e^{\Delta t(\xi)} \Theta(t) |\psi F(\xi(s), s < t) G \left( \log |\Theta(t^-) - e^{\Delta t(\xi)} \Theta(t)| \right) \right] 
= \mathbb{E}_{0,\theta} \left[ \int_0^\infty dt e^{\omega \xi(t)} F(\xi(s), s < t) \int_{\mathbb{R}^+ \times S^{d-1}} L_{\Theta(t)}(dx, d\varphi) |\Theta(t) - e^{\varphi} G(\log |\Theta(t)| - e^{\varphi}) | \right].
\]

By isotropy of $X$, the second integral does not depend on the angle $\Theta(t)$ (see (5.2)). Hence by applying the change of variables $(y, \phi) = \left( \log |\Theta(t)| - e^{\varphi}, \frac{\Theta(t) - e^{\varphi}}{|\Theta(t)| - e^{\varphi}} \right)$, we end up with
\[
\tilde{E}_\theta \left[ F(\tilde{\xi}(s), s < b_{\mathcal{L}(1)}) G(\Delta \tilde{\xi}(b_{\mathcal{L}(1)})) \right] 
= \mathbb{E}_{0,\theta} \left[ \int_0^\infty dt e^{\omega \xi(t)} F(\xi(s), s < t) \right] \int_{\mathbb{R}^+ \times S^{d-1}} \tilde{L}(dy, d\phi) e^{\omega y} G(y).
\]

In words, this proves that $(\tilde{\xi}(s), s < b_{\mathcal{L}(1)})$ and $\Delta \tilde{\xi}(b_{\mathcal{L}(1)})$ are independent. The former has the law of $\xi$ killed according to its exponential martingale, leading to a Lévy process with Laplace exponent
$q \mapsto \psi(\omega + q)$. On the other hand, the latter is distributed as $(\psi(\omega))^{-1} \int_{\phi \in S^{d-1}} \mathcal{L}(dy, d\phi)e^{\omega y}$, which is the law of the first jump of a compound Poisson process with intensity measure $\int_{\phi \in S^{d-1}} \mathcal{L}(dy, d\phi)e^{\omega y}$.

By removing the killing, this entails that $\xi$ has Laplace exponent

$$\hat{\psi}(q) = \psi(\omega + q) - \psi(\omega) + \int_{\mathbb{R}^* \times S^{d-1}} \mathcal{L}(dy, d\phi)e^{\omega y}(e^{\omega y} - 1), \quad q \geq 0.$$  

Using that $\kappa(\omega) = 0$, this is

$$\hat{\psi}(q) = \psi(\omega + q) + \int_{\mathbb{R}^* \times S^{d-1}} \mathcal{L}(dy, d\phi)e^{(\omega + q)y}, \quad q \geq 0,$$

whence $\hat{\psi}(q) = \kappa(\omega + q)$.

**The law of the growth-fragmentations $\hat{X}_n,j$.** We prove the last assertion of Theorem 6.1. It actually follows from the same arguments as in [BBCK18], but we provide the proof for the sake of completeness. To avoid cumbersome notation, we will restrict to proving the statement for the first generation. This is then easily extended thanks to the branching property. Let $F$ be a nonnegative measurable functional on the space of càdlàg trajectories, and $G_j$, $j \geq 1$, be nonnegative measurable functionals on the space of multiset-valued paths. For $t > 0$, denote by $(\Delta_j(t), j \geq 1)$ the sequence consisting of all the jumps of $\mathcal{X}_\varnothing$ that happened strictly before time $t$, and the extra value of $\mathcal{X}_\varnothing(t)$, all ranked in descending order of their absolute value. We are after the identity

$$\mathcal{E}_1 \left[ \sum_{t>0} |\Delta \mathcal{X}_\varnothing(t)|^\omega F(\mathcal{X}_\varnothing(s), 0 \leq s \leq t) \prod_{j \geq 1} G_j(\hat{\mathcal{X}}_{0,j}) \right].$$

We start from the left-hand side, and apply the change of measure (5.6):

$$\mathcal{E}_1 \left[ F(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_L(1)) \prod_{j \geq 1} G_j(\hat{\mathcal{X}}_{0,j}) \right] = \mathcal{E}_1 \left[ F(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_L(1)) \prod_{j \geq 1} \mathbb{E}_{\Delta_j(b_L(1))} [G_j(X)] \right].$$

Using the definition of the $\hat{X}_{0,j}$ together with the branching property under $\mathcal{P}_1$ give

$$\mathcal{E}_1 \left[ F(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_L(1)) \prod_{j \geq 1} G_j(\hat{\mathcal{X}}_{0,j}) \right] = \mathcal{E}_1 \left[ \sum_{t>0} |\Delta \mathcal{X}_\varnothing(t)|^\omega F(\mathcal{X}_\varnothing(s), 0 \leq s \leq t) \prod_{j \geq 1} \mathbb{E}_{\Delta_j(t)} [G_j(X)] \right].$$

Applying the change of measure backwards, we get the desired identity. This concludes the proof of Theorem 6.1.
6.3 Comments on the isotropy assumption

The previous analysis of $\mathbb{R}^d$-valued growth-fragmentations relies heavily on the isotropy assumption. Because of the complications caused by the underlying MAP structure, describing growth-fragmentations driven by anisotropic processes is a much more challenging task. We stress the importance of the isotropy assumption and comment on possible extensions to anisotropic growth-fragmentation processes.

First, we expect that in the anisotropic case, there should be an angular component in all the (super-)martingales, appearing in particular in Theorem 5.3. This already takes place in the $d = 1$ case [Sil21], for asymmetric signed growth-fragmentation processes, where the angular component is nothing but the sign. Remember in addition that, in analogy with the discrete multitype case (Section 3), the types in the spatial framework are the angles, and that the martingales in the multitype setting also involve the types (Section 3.2). If $X$ is a $\mathbb{R}^d \setminus \{0\}$-valued self-similar Markov process, this actually prompts us to define, for $q \geq 0$, the linear operator

$$T_q : f \in C \mapsto \left( \theta \in S^{d-1} \mapsto \mathbb{E}_\theta \left[ \sum_{t>0} f(\Theta_\Delta(\varphi(t)))|\Delta X(t)|^q \right] \right),$$

where $\varphi$ is the Lamperti-Kiu time-change. This is the analogue of the matrix $m$ appearing in the multitype case. Assume that $X$ has jumps (otherwise the construction is irrelevant), and that $M_q := \sup_{\theta \in S^{d-1}} \mathbb{E}_\theta \left[ \sum_{t>0} |\Delta X(t)|^q \right] < \infty$. Then $T_q(f)$ is well-defined for all $f \in C$, and for $f \in C$,

$$||T_q(f)||_\infty \leq M_q ||f||_\infty,$$

whence $T_q$ is a continuous operator. Note also that, at least under the assumption that $X$ jumps with positive probability to any open set $D \subset S^{d-1}$ of directions, $T_q$ is strongly positive, in the sense that for all nonnegative $f \neq 0$, $T_q(f) > 0$. Assume moreover that $T_q$ takes values in $C$, and that it is a compact operator. Then, by the Krein-Rutman theorem [Dei10], it must have positive spectral radius $r(q)$, which is moreover a simple eigenvalue associated to a positive eigenfunction $v$. In the spirit of Assumption $(H)$, Section 5.2, one could impose the additional assumption

$$(H') \quad \text{There exists } \omega \geq 0 \text{ such that } r(\omega) = 1.$$

Then by definition, we have

$$\forall \theta \in S^{d-1}, \quad \mathbb{E}_\theta \left[ \sum_{t>0} v(\Theta_\Delta(\varphi(t)))|\Delta X(t)|^\omega \right] = v(\theta).$$

This generalises to vectors in $\mathbb{R}^d$ by self-similarity of $X$:

$$\forall (r, \theta) \in \mathbb{R}_+ \times S^{d-1}, \quad \mathbb{E}_{r\theta} \left[ \sum_{t>0} v(\Theta_\Delta(\varphi(t)))|\Delta X(t)|^\omega \right] = v(\theta)r^\omega. \quad (6.9)$$

**Remark 6.3.** When $X$ is isotropic in the sense of Section 2, one can show that $v(\theta) = 1$ for all $\theta \in S^{d-1}$ up to normalisation, and one therefore retrieves the cumulant approach presented in Section 5.2. Indeed, isotropy entails that if $v$ is an eigenfunction associated with $r(q)$, then for all isometries $U$, $v(U\cdot)$ is also an eigenfunction associated with $r(q)$, and we conclude by simplicity of the eigenvalue that $v(U\cdot) = v$, so that $v$ is constant.
Once (6.9) holds for some positive function $v$, then modulo these adjustments one can carry through the arguments for the genealogical martingale (Theorem 5.3) and the many-to-one formula (Proposition 5.4). However, the description of the spine in Theorem 6.1 is more involved. This is mainly due to the fact that the jump intensity at time $b_{\mathcal{L}(1)}$ depends on the current angle of the spine. In the isotropic case, one can more or less get rid of this dependency. The proof of Theorem 6.1 hinges upon the existence of an Esscher transform. In the isotropic case, this readily comes from the fact that the ordinate $\xi$ of $X$ is a Lévy process, which does not hold anymore for anisotropic processes. This in particular yielded that $b_{\mathcal{L}(1)}$ (up to Lamperti time change) is an exponential random variable. This last feature should not hold in general, as already indicated by the discrete multitype case.

7 The growth-fragmentation embedded in Brownian excursions from hyperplanes

7.1 The excursion measure

Construction of the excursion measure $n$. We fix $N \geq 3$ and recall from [Bur86] how one may define the Brownian excursion measure from hyperplanes in $\mathbb{R}^N$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a complete filtered probability space, on which is defined a $N$–dimensional Brownian motion $B^N$. We single out the last coordinate and write $B^N = (B^N_{-1}, Z)$. Introduce the set $\mathcal{X}$ of càdlàg functions $x$ defined on some finite time interval $[0, R(x)]$, and the set $\mathcal{X}_0$ of such functions $x$ in $\mathcal{X}$ that are continuous and vanish at $R(x)$. Moreover, we define $U := \{ u := (x_1, \ldots, x_{N-1}, z) \in \mathcal{X}^{N-1} \times \mathcal{X}_0, R(x_1) = \ldots = R(x_{N-1}) = R(z) \text{ and } u(0) = 0 \}.$

For $u \in U$, we shall write $R(u)$ for the common value of the lifetimes. All these sets are equipped with their usual $\sigma$–fields. Finally, in order to study the excursions of $B^N$ from the hyperplane $\mathcal{H} = \{x_N = 0\}$, we introduce the local time $(\ell_s, s \geq 0)$ at 0 of the Brownian motion $Z$, as well as its inverse $(\tau_s, s \geq 0)$.

The excursion process $(e_s, s > 0)$ of our interest is easily defined following the one-dimensional case (see [RY99], Chapter XII), by

(i) if $\tau_s - \tau_{s-} > 0$, then

$$e_s : r \mapsto \left( B^{N-1}_{r+\tau_{s-}} - B^{N-1}_{\tau_s}, Z_{r+\tau_{s-}} \right), \quad r \leq \tau_s - \tau_{s-},$$

(ii) if $\tau_s - \tau_{s-} = 0$, then $e_s = \partial$,

where $\partial$ is some cemetery state. The following proposition directly stems from the one-dimensional case.

Proposition 7.1. The excursion process $(e_s, s > 0)$ is a $(\mathcal{F}_t)_{t \geq 0}$–Poisson point process of excursions in $U$. Its intensity measure is

$$n(du', dz) := n(dz)\mathbb{P}\left( (B^{N-1}_{R(z)}) \in du' \right),$$

where $n$ denotes the one-dimensional Itô measure on $\mathcal{X}_0$, and for any process $X$, and any time $T$, $X^T := (X_t, t \in [0, T])$. 35
We shall denote by $n_+$ and $n_-$ the restrictions of $n$ to $U^+ := \{(u', z) \in U, z \geq 0\}$ and $U^- := \{(u', z) \in U, z \leq 0\}$ respectively. In [Bur86], excursion measures from hyperplanes in $\mathbb{R}^N$ are rather constructed using Bessel processes. More precisely, one first samples the duration of the excursion with density $r \mapsto \{(2\pi r^3)^{-1/2} \mathbb{1}_{\{t \geq 0\}}\}$ with respect to Lebesgue measure, and then for the last coordinate, one samples a $3$–dimensional Bessel bridge from $0$ to $0$ over $[0, r]$. This is equivalent to $n_+$ in our representation (up to a multiplicative factor) thanks to Itô’s description of $\gamma$, for which we refer again to [RY99]. We conclude this paragraph with the following Markov property under $n_+$.

**Proposition 7.2.** On the event that $T_a := \inf\{0 \leq t \leq R(u), z(t) = a\} < \infty$, the process $(u(T_a + t) − u(T_a), 0 \leq t \leq R(u) − T_a)$ is independent of $(u(t), 0 \leq t \leq T_a)$ and is a $d$-dimensional Brownian motion stopped when hitting $\{x_N = −a\}$.

**Disintegration of $n_+$.** We now construct measures $\gamma_x, x \in \mathbb{R}^{N−1}$, for Brownian excursions from the hyperplane $\{x_N = 0\}$ conditioned on ending at $(v, 0)$, by disintegrating $n_+$ over its endpoint. Whenever $r \geq 0$ and $x \in \mathbb{R}^{N−1}$, we write $\Pi_r$ for the law of a Bessel bridge from $0$ to $0$ over $[0, r]$, and $\mathbb{P}_r^{0 \to x}$ for the law of a $(N−1)$-dimensional Brownian bridge from $0$ to $x$ with duration $r$. See [AS20] for the case $N = 2$.

**Proposition 7.3.** The following disintegration formula holds:

$$n_+ = \int_{\mathbb{R}^{N−1} \setminus \{0\}} dx \frac{\Gamma(N)}{2\pi^{N/2}|x|^{N}} \cdot \gamma_x,$$

where $\gamma_x, x \in \mathbb{R}^{N−1} \setminus \{0\}$, are probability measures. In addition, for all $x \in \mathbb{R}^{N−1} \setminus \{0\}$,

$$\gamma_x = \int_0^\infty dr \frac{e^{\frac{−r}{2}}}{\Gamma(N)} \frac{N}{r^{N+1}} \mathbb{P}_r^{0 \to x} \wedge \Pi_r|x|^2.$$

**Proof.** The proposition follows from Theorem 3.3 in [Bur86], but we rephrase it in our framework for completeness. Let $f : \mathcal{X}^{N−1} \rightarrow \mathbb{R}_+$ and $g : \mathcal{X}_0 \rightarrow \mathbb{R}_+$ be two nonnegative measurable functions. Then by Proposition 7.1,

$$\int_{U^+} f(u')g(z)n_+(du', dz) = \int_{U^+} f(u')g(z)n_+(dz)\mathbb{P}\left((B^{N−1})^{R(z)} \in du'\right).$$

Then by Itô’s description of $n_+$ (see Chap. XII, Theorem 4.2 in [RY99]), we may split this integral over the duration $R(z)$:

$$\int_{U^+} f(u')g(z)n_+(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^3} \Pi_r[g]E[f((B^{N−1})^r)].$$

We now condition on $B^{N−1}_r$, and we obtain

$$\int_{U^+} f(u')g(z)n_+(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi}r^3} \int_{\mathbb{R}^{N−1}} dx \frac{e^{\frac{−|x|^2}{2r}}}{(2\pi r)^{N+1}} \Pi_r[g]E^{0 \to x}[f].$$

Finally, we perform the change of variables $r \mapsto t = r/|x|^2$:

$$\int_{U^+} f(u')g(z)n_+(du', dz) = \int_{\mathbb{R}^{N−1}} dx \frac{dt}{|x|^N} \int_0^\infty dt \frac{e^{\frac{−t}{2t}}}{(2\pi t)^{N+1}} \Pi_t|x|^2[g]E^{0 \to x}[f].$$
Since
\[ \int_0^\infty dt \frac{e^{-\frac{t^2}{2}}}{{(2\pi)}^{\frac{N}{2}} t^{\frac{N+1}{2}}} = \frac{1}{2} \pi^{-\frac{N}{2}} \Gamma \left( \frac{N}{2} \right), \]
this gives that \( \gamma_x \), for \( x \in \mathbb{R}^{N-1} \setminus \{0\} \), are probability measures, and the disintegration claim holds.

**Bismut’s description of \( n_+ \).** The following decomposition of \( n_+ \) describes the left and right parts of the trajectory seen from a point chosen uniformly at random on the Brownian excursion weighted by its lifetime.

**Proposition 7.4. (Bismut’s description of \( n_+ \))**

Let \( \pi_+ \) be the measure defined on \( \mathbb{R}_+ \times U^+ \) by

\[ \pi_+(dt, du) = \mathbb{1}_{\{0 \leq t \leq R(u)\}} dt n_+(du). \]

Then under \( \pi_+ \) the "law" of \((t, (u', z)) \mapsto z(t)\) is the Lebesgue measure \( dA \) on \( \mathbb{R}_+ \), and conditionally on \( z(t) = A \), \( u^{t \leftarrow} = (u(t-s) - u(t))_{0 \leq s \leq t} \) and \( u^{t \rightarrow} = (u(t+s) - u(t))_{0 \leq s \leq R(u)-t} \) are independent Brownian motions killed when reaching the hyperplane \( \{x_N = -A\} \).

Proposition 7.4 is a straightforward consequence of Bismut’s description of the one-dimensional Itô measure \( n \). The following picture illustrates how the excursion splits when seen from a uniform point.

\[ u^{t \leftarrow}(s) := u(t-s) - u(t) \quad u^{t \rightarrow}(s) := u(t+s) - u(t) \]

**Figure 4:** Bismut’s description of \( n_+ \) in dimension \( N = 3 \). The height of a uniformly chosen point \( t \) on the excursion weighted by its duration is distributed according to the Lebesgue measure \( dA \). Moreover, conditionally on the height, the excursion splits into two independent trajectories depicted in blue and red. Both are distributed as Brownian motion killed when hitting the bottom half-plane (in grey).
7.2 Slicing excursions with hyperplanes

This section is an easy extension of the framework introduced in [AS20]. Let \( u \in U^+ \), and \( a \geq 0 \).

We may write \( u := (u', z) \) with \( u' \in \mathcal{X}^{N-1} \) and \( z \in \mathcal{X}_0, z \geq 0 \).

Notation and setup. Define the superlevel set
\[
I(a) = \{ s \in [0, R(u)], \ z(s) > a \}. \tag{7.1}
\]
This is a countable (possibly empty) union of disjoint open intervals, and for any such interval \( I = (i_-, i_+), \) we write \( u_I(s) := u(i_- + s) - u(i_-), 0 \leq s \leq i_+ - i_- \), for the restriction of \( u \) to \( I \), and \( \Delta u_I := x(i_+) - x(i_-) \). Remark that \( \Delta u_I \) is a vector in the hyperplane \( \mathcal{H}_a := \{ x_N = a \} \), which we call the size or length of the excursion \( u_I \), see Figure 5. If \( 0 \leq t \leq R(u) \), we denote by \( e_a^{(t)} \) the excursion \( u_I \) corresponding to the unique such interval \( I \) which contains \( t \). Moreover, we define \( \mathcal{H}_a^+ \) as the set of excursions above \( \mathcal{H}_a \) corresponding to the previous partition of \( I(a) \).

\[ \text{Figure 5: Slicing at height } a \text{ of an excursion } u \text{ away from } \mathcal{H}. \] The blue trajectory represents an excursion in the half-space \( \{ x_N > 0 \} \), \( N = 3 \). For some fixed height \( a > 0 \) we draw the hyperplane \( \mathcal{H}_a \) and record the sub-excursions above \( \mathcal{H}_a \). The four largest of them are represented in dark blue (the reader should imagine many infinitesimal excursions). The red arrows indicate the size of the sub-excursions, counted with respect to the orientation of \( u \).

We may now present an application of Proposition 7.4, which is similar to Proposition 2.7 in [AS20]. We show that, almost surely, excursions cut at heights do not make bubbles above any hyperplane. More precisely, we set
\[
\mathcal{L} := \{ u \in U^+, \exists 0 \leq t \leq R(u), \exists 0 \leq a < z(t), \ \Delta e_a^{(t)}(u) = 0 \}. \tag{7.2}
\]
This is the set of \( u \in U^+ \) making above some level an excursion which comes back to itself. Then

Proposition 7.5.
\[
n_+ (\mathcal{L}) = 0.
\]
Proof. We first notice that if \( u \in \mathcal{L} \), then the set of \( t \)'s such that \( \Delta e_a^{(t)}(u) = 0 \) for some \( 0 \leq a < z(t) \) has positive Lebesgue measure. Therefore
\[
\mathcal{L} \subset \left\{ u \in U^+; \int_0^{R(u)} 1_{\{30 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\}} \, dt > 0 \right\}. \tag{7.3}
\]
Now using the notation in Proposition 7.4, and defining
\[
T_a^{t,\leftarrow} := \inf\{s > 0, z(t-s) = a\} \quad \text{and} \quad T_a^{t,\rightarrow} := \inf\{s > 0, z(t+s) = a\},
\]
we get
\[
n_+ \left( \int_0^{R(u)} 1_{\{30 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\}} \, dt \right) = \mathbb{P} \left( \exists 0 < a \leq A, B_1^{N-1}(T_a^{t,\leftarrow}) = B_2^{N-1}(T_a^{t,\rightarrow}) \right),
\]
where \( B_1^{N-1}, B_2^{N-1} \) are independent \((N-1)\)-dimensional Brownian motions, and \( T_a^{1}, T_a^{2} \) are independent Brownian hitting times. It is now well-known that the entries of \( B_1^{N-1}(T_a^{1}) \) and \( B_2^{N-1}(T_a^{2}) \) are symmetric Cauchy processes in \( a \). By independence, the entries of \( B_1^{N-1}(T_a^{1}) - B_2^{N-1}(T_a^{2}) \) are also Cauchy processes, for which points are polar (see [Ber96], Chap. II, Section 5). Hence
\[
n_+ \left( \int_0^{R(u)} 1_{\{30 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\}} \, dt \right) = 0.
\]
This yields that for \( n_+ \)-almost every excursion \( u \),
\[
\int_0^{R(u)} 1_{\{30 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\}} \, dt = 0,
\]
and given the inclusion (7.3), we infer that \( n_+ (\mathcal{L}) = 0 \).

The branching property of excursions in \( \mathcal{H}_a^{+} \). When cutting excursions with the hyperplanes \( \mathcal{H}_a \), the natural filtration is the one carrying the information below these hyperplanes. We call \((\mathcal{G}_a, a \geq 0)\) this filtration, completed with the \( n_+ \)-negligible sets. Recall that we have set \( T_a := \inf\{0 \leq t \leq R(u), z(t) = a\} \). Finally, we let \( a > 0 \) and rank the excursions \( (\epsilon_i^{a,\leftarrow}, i \geq 1) \) in \( \mathcal{H}_a^{+} \) by descending order of the norm of their sizes \( (x_i^{a,\leftarrow}, i \geq 1) \). Then the following branching property holds.

Proposition 7.6. For all \( A \in \mathcal{G}_a \), and all nonnegative measurable functions \( F_1, \ldots, F_k : U^+ \rightarrow \mathbb{R}_+ \), \( k \geq 1 \),
\[
n_+ \left( \mathbb{1}_{\{T_a < \infty\}} \prod_{i=1}^k F_i(\epsilon_i^{a,\leftarrow}) \right) = n_+ \left( \mathbb{1}_{\{T_a < \infty\}} \prod_{i=1}^k \mathbb{1}_{x_i^{a,\leftarrow} = 0} F_i \right),
\]
and the same also holds under \( \gamma_x \) for all \( x \in \mathbb{R}^{N-1} \setminus \{0\} \).

Proof. We refer to [AS20] for the proof in the planar case, which is easily extended to higher dimensions. \qed
7.3 A temporal martingale

Recall from (7.1) the notation $\mathcal{H}_a^+$ for the set of excursions above $\mathcal{H}_a$.

**Theorem 7.7.** Under $\gamma_x$ for all $x \in \mathbb{R}^{N-1} \setminus \{0\}$, the process

$$\mathcal{M}_a := \mathbb{1}_{\{T_a < \infty\}} \sum_{e \in \mathcal{H}_a^+} |\Delta e|^N, \quad a \geq 0,$$

is a martingale with respect to $(\mathcal{G}_a, a \geq 0)$.

The proof is again an adaptation of [AS20], Proposition 3.7, which is the planar case $N = 2$.

**Proof.** By the branching property (Proposition 7.6), we may restrict ourselves to proving that $\gamma_x[\mathcal{M}_a] = |x|^N$ for all $x \in \mathbb{R}^{N-1} \setminus \{0\}$ and all $a \geq 0$. Let $f : \mathbb{R}^{N-1} \setminus \{0\} \to \mathbb{R}_{+}$ a nonnegative measurable function. We aim at computing $\mathbb{E}(\mathcal{M}_a f(u'(R(u))))$, where as usual we write $u = (u', z) \in \mathcal{X}^{N-1} \times \mathcal{Z}_0$ for $u \in \mathbb{R}_{+}^\infty$. By the Markov property (Proposition 7.2) under $\mathbb{n}_+$, we have

$$n_+ (\mathcal{M}_a f(u'(R(u)))) = n_+ \left( \mathbb{1}_{\{T_a < \infty\}} \mathbb{E} \left[ \sum_{s \leq \ell_{T-a}} \mathbb{1}_{\ell_s \in U^+} |\Delta e|^N f(B^{N-1}(T-a)) \right] \right). \quad (7.4)$$

Using the Master formula for the excursion process $(\ell_s, s > 0)$ and the density of the endpoint under $\mathbb{n}_+$ given by Proposition 7.3 yields

$$\mathbb{E} \left[ \sum_{s \leq \ell_{T-a}} \mathbb{1}_{\ell_s \in U^+} |\Delta e|^N f(B^{N-1}(T-a)) \right]$$

$$= \mathbb{E} \left[ \int_0^{T-a} d\ell_s \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{\Gamma(N/2)}{2\pi^{N/2}|x|^N} \mathbb{E}[f(\mathbf{x}' + B^{N-1}(T-a))]\bigg|_{\mathbf{x}'=B^{N-1}(s)+\mathbf{x}} \right].$$

A change of variables then gives

$$\mathbb{E} \left[ \sum_{s \leq \ell_{T-a}} \mathbb{1}_{\ell_s \in U^+} |\Delta e|^N f(B^{N-1}(T-a)) \right]$$

$$= \mathbb{E} \left[ \int_0^{T-a} d\ell_s \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{\Gamma(N/2)}{2\pi^{N/2}} \mathbb{E}[f(\mathbf{x} + B^{N-1}(T-a))] \right],$$

and since the Lebesgue measure is an invariant measure for Brownian motion,

$$\mathbb{E} \left[ \sum_{s \leq \ell_{T-a}} \mathbb{1}_{\ell_s \in U^+} |\Delta e|^N f(B^{N-1}(T-a)) \right] = \mathbb{E}[\ell_{T-a}] \times \left( \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x})d\mathbf{x} \right).$$

Recall from [RY99], Chap. VI, Section 4, that $\ell_{T-a}$ is an exponential random variable with mean $2a$, so that we end up with

$$\mathbb{E} \left[ \sum_{s \leq \ell_{T-a}} \mathbb{1}_{\ell_s \in U^+} |\Delta e|^N f(B^{N-1}(T-a)) \right] = 2a \times \left( \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x})d\mathbf{x} \right).$$

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We then disintegrate.

We now use Proposition 3.6, Chapter XII, of [RY99] to get that

\[ n_+ (\mathcal{M}_a f(u'(R(u)))) = 2a \times \left( \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1}\setminus\{0\}} f(x)dx \right) \times n_+(T_a < \infty). \]

We then disintegrate \( n_+ \) thanks to Proposition 7.3:

\[ \int_{\mathbb{R}^{N-1}\setminus\{0\}} \frac{\Gamma(N/2)}{2\pi^{N/2}|x|^N} f(x) \gamma_x (\mathcal{M}_a) \, dx = \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1}\setminus\{0\}} f(x)dx. \]

This holds for all nonnegative measurable functions \( f \), and so we may deduce by using a continuity argument that for all \( x \in \mathbb{R}^{N-1} \setminus \{0\} \), \( \gamma_x (\mathcal{M}_a) = |x|^N \).

### 7.4 A change of measures

We fix \( x \in \mathbb{R}^{N-1} \setminus \{0\} \). To the martingale in Theorem 7.7, we can associate the change of measure \( \mu_x \) such that

\[ \frac{d\mu_x}{d\gamma_x} \bigg|_{\mathcal{G}_a} := \frac{\mathcal{M}_a}{|x|^N}, \quad a \geq 0. \]

The martingale property of \( \mathcal{M} \) makes this definition consistent, and the existence of \( \mu_x \) then follows by Kolmogorov’s extension theorem. Performing this change of measure will result in splitting the excursion into two independent excursions in the half-space \( \mathcal{H}^+ := \{ x_N > 0 \} \) going to infinity, as in Figure 6. To describe the law \( \mu_x \), we call \( \mathcal{H}^+ \)-excursions a process in \( \mathbb{R}^N \) whose first \( (N-1) \) entries are independent Brownian motions, and whose last entry is an independent 3-dimensional Bessel process starting at 0 (so that this process actually remains in \( \mathcal{H}^+ \)). We recall that \( T_a := \inf \{ 0 < t \leq R(u), z(t) = a \} \), and set \( S_a := \inf \{ 0 < t < T_a, z(R(u) - t) = a \} \).

**Theorem 7.8.** Under \( \mu_x \), for all \( a > 0 \), the processes \( (u(s), s \leq T_a) \) and \( (u(R(u) - s), s \leq S_a) \) are independent \( \mathcal{H}^+ \)-excursions started respectively from 0 and \( (x, 0) \) and stopped when hitting \( \mathcal{H}_a \).

**Proof.** The proof is taken almost verbatim from [AS20]. Let \( f, g \) two nonnegative measurable functions defined on \( \mathcal{X}^N \). By the Markov property (Proposition 7.2),

\[ n_+ \left( f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a)\mathcal{M}_a \right) \]

\[ = n_+ \left( 1_{\{T_a < \infty\}} f(u(s), 0 \leq s \leq T_a) \right) \times \mathbb{E} \left[ \sum_{r<\ell_{T-a}} 1_{t_r \in U^+} |\Delta t_r|^N g(B^N(T-a-s), 0 \leq s \leq S) \right], \]

where we have set \( S := \inf \{ 0 \leq t \leq T-a, Z(R(u) - t) = 0 \} \). Yet another application of the Master formula triggers that the second expectation is

\[ \mathbb{E} \left[ \sum_{r<\ell_{T-a}} 1_{t_r \in U^+} |\Delta t_r|^N g(B^N(T-a-s), 0 \leq s \leq S) \right] \]

\[ = \mathbb{E} \left[ \int_0^{T-a} dt_r \int_{\mathbb{R}^{N-1}\setminus\{0\}} dx \frac{\Gamma(N/2)}{2\pi^{N/2}|x|^N} |x|^N \mathbb{E} \left[ g((x', a) + B^N(T-a-s), s \leq S) \right] \bigg|_{x'=B^{N-1}(r)+x} \right]. \]
Figure 6: Splitting of the excursion under \( \mu_X \). Under the change of measure, the excursion splits into two independent \( \mathcal{H}^+ \)-excursions (blue and red). For fixed height \( a > 0 \), the sub-excursion above \( a \) straddling the point at infinity is obtained by running two independent \( N \)-dimensional Brownian motions started from infinity and stopped when hitting the hyperplane \( \mathcal{H}_a \).

Therefore, by a change of variables, and using that \( \mathbb{E}[\ell(T_{-a})] = 2a \), we are left with

\[
\mathbb{E} \left[ \sum_{r < \ell_{T_{-a}}} \mathbb{1}_{\mathcal{E}_r \in U^+} |\Delta \mathbf{e}_r| \right]^{N} g(B^{N}(T_{-a} - s), 0 \leq s \leq S) \]

\[= 2a \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} \mathbf{x} \mathbb{E}[g((\mathbf{x}, a) + B^{N}(T_{-a} - s), s \leq S)].\]

As usual, we write \( B^{N} = (B^{N-1}, Z) \). Given \( Z \), \( (B^{N-1}(s), s \leq T_{-a}) \) is a \((N - 1)\)-dimensional Brownian motion, stopped at \( T_{-a} \). Since the Lebesgue measure is reversible for Brownian motion, we get

\[
\mathbb{E} \left[ \sum_{r < \ell_{T_{-a}}} \mathbb{1}_{\mathcal{E}_r \in U^+} |\Delta \mathbf{e}_r| \right]^{N} g(B^{N}(T_{-a} - s), 0 \leq s \leq S) \]

\[= 2a \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} \mathbf{x} \mathbb{E}[g((\mathbf{x} + B^{N-1}(s), a + Z(T_{-a} - s)), s \leq S)].\]

Finally, recall from [RY99], Chap. VII, Corollary 4.6, that \((a + Z(T_{-a} - s), s \leq S)\) has the law of a 3-dimensional Bessel process \( V \) run until time \( T_{a}^{V} \) when it first hits \( a \). Thus

\[
n_+(f(u(s), 0 \leq s \leq T_{a})g(u(R(u) - s), 0 \leq S_{a})\mathcal{M}_{a})
\]

\[= n_+(\mathbb{1}_{T_{a} < \infty} f(u(s), 0 \leq s \leq T_{a}) \times 2a \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} \mathbf{x} \mathbb{E}[g((\mathbf{x} + B^{N-1}(s), V(s)), s \leq T_{a}^{V})],\]
and since \( n_+(T_a < \infty) = \frac{1}{2a} \), we obtain

\[
n_+ \left( f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a)M_a \right) = n_+ \left( f(u(s), 0 \leq s \leq T_a) \right| T_a < \infty \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1}\{0\}} d\mathbb{E}[g((x + B^{N-1}(s), V(s)), s \leq T_a^V)].
\]

On the other hand, one can prove using Williams’ description of \( n_+ \) that conditionally on \( T_a < \infty \), \((u(s), 0 \leq s \leq T_a)\) is distributed as \(( (B^{N-1}(s), V(s)), 0 \leq s \leq T_a^V ) \). Hence,

\[
n_+ \left( f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a)M_a \right) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1}\{0\}} d\mathbb{E}[g((x + B^{N-1}(s), V(s)), s \leq T_a^V)] \times \mathbb{E}[f((B^{N-1}(s), V(s)), 0 \leq s \leq T_a^V)].
\]

Disintegrating over the endpoint as in Proposition 7.3, and using a continuity argument, this concludes the proof of Theorem 7.8 

\textbf{Remark 7.9.} We reformulate the previous results in the parlance of Section 5. Setting \( Z(a) := \{ \{ \Delta e, e \in \mathcal{H}_a^+ \} \}, \ a \geq 0, \) it follows from Proposition 7.6 that \( Z \) enjoys a branching property akin to Proposition 5.1. We could have pointed out an Eve cell in the spirit of [AS20, Theorem 3.3] by considering the locally largest excursion. Together with an avatar of [AS20, Theorem 3.6], this proves that under \( \gamma_X \), \( Z \) is a spatial growth-fragmentation process. Actually, one should first check that the evolution of the Eve cell generates all the excursions, but this is a simple consequence of the arguments presented in [AS20, Theorem 4.1]. In the previous exposition, we chose to rather dwell on the spine description. More specifically, the martingale in Theorem 7.7 is a temporal version of the martingale in Theorem 5.3. Then, Theorem 7.8 determines the law of the spine without reference to Theorem 6.1. The spine is described as the Brownian motion \( B^{N-1} \) taken at the hitting times of another independent linear Brownian motion, and hence is a \((N - 1)\)–dimensional isotropic Cauchy process.

### 7.5 Extension to isotropic stable Lévy processes

As in [Sil21], we can extend the previous construction to stable processes with index \( \alpha \in (0, 2) \). We recall that we have set \( N \geq 3 \), and that the case \( N = 2 \) was already treated in [Sil21]. We will not provide all the details of the proofs since the arguments are similar to the Brownian case.

**The excursion measure \( n^\alpha \).** We shall consider the following excursions, which consist in replacing the first \((N - 1)\) entries of the previous setting by an isotropic \( \alpha \)-stable Lévy process in \( \mathbb{R}^{N-1} \). We keep the notation in Section 7.1, except that now is defined on the probability space a \((N - 1)\)–dimensional isotropic stable Lévy process \( X^{N-1} \), and we consider the process \( Z^N := (X^{N-1}, Z) \) with Brownian last coordinate. Then, we introduce the excursion process \((e^\alpha_s, s > 0)\) as

(i) if \( \tau_s - \tau_s^- > 0 \), then

\[
e^\alpha_s : r \mapsto \left( X_{r+\tau_s^-}^{N-1} - X_{\tau_s^-}^{N-1}, Z_{r+\tau_s^-} \right), \ r \leq \tau_s - \tau_s^-;
\]

(ii) if \( \tau_s - \tau_s^- = 0 \), then \( e^\alpha_s = \partial \).
As in Proposition 7.1, this defines a Poisson point process with intensity measure

$$n^\alpha(du', dz) := n(dz)\mathbb{P}\left((X^{N-1})^{R(z)} \in du'\right).$$

Let $n^\alpha_+$ be the restriction of $n^\alpha$ to positive excursions. We now want to condition $n^\alpha_+$ on the endpoint of the excursion. For $x \in \mathbb{R}^{N-1}$ and $r > 0$, let $\mathbb{P}^r\alpha,0 \rightarrow x$ denote the law of an $\alpha$–stable bridge from 0 to $x$ over $[0,r]$. In addition, we write $(p^\alpha_r, r \geq 0)$ for the transition densities of $X^{N-1}$. Throughout this section, we fix $\omega_N := N - 1 + \frac{\alpha}{2}$.

**Proposition 7.10.** The following disintegration formula holds:

$$n^\alpha_+ = \int_{\mathbb{R}^{N-1} \setminus \{0\}} dx \frac{C_N}{|x|^\omega_N} \cdot \gamma^\alpha_x,$$

where $\gamma^\alpha_x, x \in \mathbb{R}^{N-1} \setminus \{0\}$, are probability measures, and

$$C_N = \frac{\alpha}{2\sqrt{2\pi}} \int_{\mathbb{R}^+} dv p^\alpha_1(v) v^{\omega_N - 1}.$$ 

In addition, for all $x \in \mathbb{R}^{N-1} \setminus \{0\}$,

$$\gamma^\alpha_x = \int_0^\infty dr \frac{p^\alpha_1(r^{-1/\alpha})}{2\sqrt{2\pi} r^{1+\frac{\omega_N}{\alpha}}} \mathbb{P}^{\alpha,0 \rightarrow x}_r \otimes \Pi_r| |x|^2.$$ 

**Proof.** Let $f$ and $g$ be two nonnegative measurable functions, respectively defined on $\mathcal{F}^{N-1}$ and $\mathcal{F}_0$. Following the proof of Proposition 7.3, we end up with

$$
\int_{U^+} f(u')g(z)n^\alpha_+(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi} r^3} \mathbb{E}_r[f((X^{N-1})^r)]
\mathbb{E}_r[g]\Pi_r| |x|^2 \Pi_r| |x|^2 \frac{dr}{2\sqrt{2\pi} r^3} \int_{\mathbb{R}^{N-1}} dx p^\alpha_1(z) \Pi_r| |x|^2 \mathbb{E}_r[g](X^{N-1})^r).
$$

Note that, by self-similarity, for all $r > 0$ and $x \in \mathbb{R}^{N-1}$,

$$p^\alpha_r(x) = r^{-\frac{N-1}{\alpha}} p^\alpha_1(r^{-1/\alpha} x).$$

Hence

$$
\int_{U^+} f(u')g(z)n^\alpha_+(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi} r^3} \int_{\mathbb{R}^{N-1}} dx \frac{r^{-N+1}}{\alpha} p^\alpha_1(r^{-1/\alpha} x) \Pi_r[g]\mathbb{E}_r^{\alpha,0 \rightarrow x}(f(X^{N-1}))
$$

and by the change of variables $u(r) := \frac{r}{|x|^\alpha}$, this is

$$
\int_{U^+} f(u')g(z)n^\alpha_+(du', dz) = \int_{\mathbb{R}^{N-1}} dx \frac{u^{-N+1}}{\alpha} p^\alpha_1(u^{-1/\alpha} x) \Pi_u| |x|^\alpha \mathbb{E}_u^{\alpha,0 \rightarrow x}(f(X^{N-1})).
$$

Observe that the isotropy of $X^{N-1}$ yields the relationship $p^\alpha_1(u^{-1/\alpha} x) = p^\alpha_1(u^{-1/\alpha})$, so that

$$
\int_{U^+} f(u')g(z)n^\alpha_+(du', dz) = \int_{\mathbb{R}^{N-1}} dx \frac{u^{-N+1}}{\alpha} \Pi_u| |x|^\alpha \mathbb{E}_u^{\alpha,0 \rightarrow x}(f(X^{N-1})).
$$

The proposition follows.
**Remark 7.11.** We emphasize that the proof of Proposition 7.10 uses the isotropy assumption on $X^{N-1}$, and indeed formula (7.5) shows that the excursion measure $n^\alpha_+$ assigns a weight to the endpoint $x$ which only depends on its radial part $|x|$. If $X^{N-1}$ were not isotropic, then one would have to deal with the angular part of $x$ in the disintegration.

The following proposition is a Bismut description of $n^\alpha_+$, which is easily extended from Proposition 7.4. The picture looks roughly the same as in Figure 4, albeit the two trajectories have their first $(N-1)$ entries distributed as an isotropic stable process in $\mathbb{R}^{N-1}$.

**Proposition 7.12.** *(Bismut’s description of $n^\alpha_+$)*

Let $\overline{n}^\alpha_+$ be the measure defined on $\mathbb{R}_+ \times U^+$ by

$$\overline{n}^\alpha_+(dt, du) = \mathbb{I}_{\{0 \leq t \leq R(u)\}} dt \ n^\alpha_+(du).$$

Then under $\overline{n}^\alpha_+$ the "law" of $(t, (u', z)) \mapsto z(t)$ is the Lebesgue measure $dA$ on $\mathbb{R}_+$, and conditionally on $z(t) = A$, $u^{s,t} = (u(t) - u(s))_{0 \leq s \leq t}$ and $u^{s,t} = (u(t) - u(s))_{0 \leq s \leq R(u(t))}$ are independent and evolve as $Z^N$ killed when reaching the hyperplane $\{x_N = -A\}$.

One of the consequences of this decomposition is that for $n^\alpha_+$—almost every excursion, there is no loop above any level. More precisely, recall the definition of $L$ in (7.2). Then $n^\alpha_+(L) = 0$. The proof can be taken verbatim from Proposition 7.5, using that a stable process in dimension $N - 1 \geq 2$ does not hit points (see [Ber96, II, Corollary 17]).

![Figure 7: Slicing of an excursion in $\mathcal{H}_+$ with stable first two coordinates, in dimension $N = 3$. The excursion is drawn in blue. The trajectory is càdlàg but jumps never occur for the height. We record the length (in red) of the sub-excursions (in dark blue) made above $\mathcal{H}_a$.](fig:slicing)

**The branching property under $n^\alpha_+$.** We will be interested in cutting excursions with hyperplanes at varying heights, and study the length of the subexcursions above these hyperplanes (Figure 7). As in Proposition 7.6, this exhibits a branching structure that we summarise in the next result, in the language introduced in Section 7.2.
Proposition 7.13. For all $A \in \mathcal{G}_a$, and all nonnegative measurable functions $F_1, \ldots, F_k : U^+ \to \mathbb{R}_+$, $k \geq 1$,
\[
\begin{aligned}
\mathbb{n}_+^a \left( \mathbf{1}_{\{T_a<\infty\}} \mathbf{1}_A \prod_{i=1}^k F_i(e_i^{a,+}) \right) &= \mathbb{n}_+^a \left( \mathbf{1}_{\{T_a<\infty\}} \mathbf{1}_A \prod_{i=1}^k \gamma_{X_i^{a,+}}^a[F_i] \right),
\end{aligned}
\]
and the same also holds under $\gamma_X^a$ for all $x \in \mathbb{R}^{N-1} \setminus \{0\}$.

Martingale and spine decomposition under $\gamma_X^a$. In line with Theorem 7.7 and Theorem 7.8, we reveal the martingale in the stable setting and describe the law after the change of measure. The notation is implicitly taken from the Brownian case. All the proofs are omitted because they are simple extensions of their Brownian analogues. Recall that $\omega_N = N - 1 + \frac{\alpha}{2}$.

Theorem 7.14. Under $\gamma_X^a$ for all $x \in \mathbb{R}^{N-1} \setminus \{0\}$, the process
\[
\begin{aligned}
\mathcal{M}_a^\alpha := \mathbf{1}_{\{T_a<\infty\}} \cdot \sum_{e \in \mathcal{H}_a} |\Delta e|^\omega_N,
\end{aligned}
\]
is a martingale with respect to $(\mathcal{G}_a, a \geq 0)$.

Figure 8: The excursion $u$ seen under $\mu_X^\alpha$. Under the change of measure, $u$ splits into two independent $(\alpha, \mathcal{H}_+)$-excursions (blue and red), which are the analogues of the Brownian half-space excursions appearing in Figure 6, when the first $(N - 1)$ coordinates are replaced with an isotropic stable process. The length of the sub-excursion above some height $a$ straddling the point at infinity is obtained by subordinating the isotropic process at the Brownian hitting time of level $a$. Let us stress once more that the last coordinate is continuous, so that this length is well defined for all positive height $a > 0$.  

\{fig:change measures stable\}
Let \( x \in \mathbb{R}^{N-1} \setminus \{0\} \). Consider the change of measure \( \mu^a_x \) such that
\[
\frac{d\mu^a_x}{d\gamma^\alpha_x} \bigg| \bigg|_a := \frac{M^a_x}{|x|^\alpha N}, \quad a \geq 0.
\]

We now come to the description of the excursion under \( \mu^a_x \). Call \((\alpha, \mathcal{H}^+)-\)excursion a process in \( \mathbb{R}^N \) whose first \((N-1)\) entries form an isotropic \( \alpha \)-stable Lévy process, and whose last entry is an independent 3-dimensional Bessel process starting at 0 (so that this process actually remains in \( \mathcal{H}^+ \)). We set
\[
T_a := \inf\{0 \leq t \leq R(u), z(t) = a\}, \quad S_a := \inf\{0 \leq t \leq R(u), z(R(u) - t) = a\}.
\]

**Theorem 7.15.** Under \( \mu^a_x \), for all \( a > 0 \), the processes \((u(s), s \leq T_a)\) and \((u(R(u) - s), s \leq S_a)\) are independent \((\alpha, \mathcal{H}^+)-\)excursions started respectively from 0 and \((x, 0)\) and stopped when hitting \( \mathcal{H}_a \).

Figure 8 illustrates the theorem.

**Remark 7.16.** Let us notice that, similarly to Remark 7.9, the process
\[
Z(a) := \left\{ \{\Delta e, \ e \in \mathcal{H}^+_a\} \right\}, \quad a \geq 0,
\]
is a spatial growth-fragmentation process under \( \gamma^\alpha_x \). One could fiddle with the ideas of [Sil21, Theorem 6.8] in order to define an Eve cell process driving \( Z \), but beware that the (signed) growth-fragmentation process described therein is not isotropic as such (one needs to adjust the constants \( c_+ \) and \( c_- \) to recover an isotropic process). Theorem 7.15 provides the law of the spine as an isotropic \((N-1)\)-dimensional \( \alpha^{2/3} \)-stable process.

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