On rank statistics of \textit{PageRank} and \textit{MarkovRank}

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Abstract

The well-known statistic \textit{PageRank} was created in 1998 by co-founders of Google, Sergey Brin and Larry Page, to optimize the ranking of websites for their search engine outcomes. It is computed using an iterative algorithm, based on the idea that nodes with a larger number of incoming edges are more important. Google's \textit{PageRank} involves some information from “aliens”; the 15\% of information is regarded as the connections from the outside of the network system under consideration.

In this paper, seeking a stable statistic which is “close” to an “intrinsic” version of \textit{PageRank}, we will introduce a new statistic called \textit{MarkovRank}. A special attention will be paid to the comparison of rank statistics among standard-\textit{PageRank}, “intrinsic-\textit{PageRank}” and \textit{MarkovRank}. It is concluded that the rank statistic of \textit{MarkovRank}, which is always well-defined, is identical to that of “intrinsic-\textit{PageRank}”, as far as the latter is well-defined.

1 Introduction

The well-known statistic \textit{PageRank} was created in 1998 by co-founders of Google, Sergey Brin and Larry Page, to optimize the ranking of websites for their search engine outcomes. It is computed using an iterative algorithm, based on the idea that nodes with a larger number of incoming edges are more important.

Google’s \textit{PageRank} involves some information from “aliens”; the 15\% of information is regarded as the connections from the outside of the network system under consideration. Without involving the information from “aliens”, Google’s \textit{PageRank} could not be well-defined.

In this paper, seeking a stable statistic which is “close” to an “intrinsic” version of \textit{PageRank}, we will introduce a new statistic called \textit{MarkovRank}. A special attention will be paid to the comparison of rank statistics among standard-\textit{PageRank}, “intrinsic-\textit{PageRank}” and \textit{MarkovRank}, and it is concluded that the rank statistic of \textit{MarkovRank}, which is always well-defined, is identical to that of “intrinsic-\textit{PageRank}”, as far as the latter is well-defined. Notice that, since the sum of either “Rank” values across all nodes equals 1, the absolute values of the “Rank”s in each of concrete network data are not meaningful, and therefore the rank statistics of them are important. Since it is confirmed by some examples that \textit{MarkovRank} returns as stable values as standard-\textit{PageRank}, it can be said that \textit{MarkovRank} has a potential to play a similar role to the well established standard-\textit{PageRank}, not only from a practical point of view but also with some new theoretical validity. Although there are huge literature dealing with \textit{PageRank} and its variations, including Bianchini \textit{et al.} (2005), Bar-Yossef and Mashiah (2008) and Langville and Meyer (2011), the approach presented in the current paper seems novel.

To close this section, let us load two packages in R.

```
library("MASS")  # for the calculation of generalized inverse
library("igraph") # for the analysis of graphical data
```

2 Preliminary discussions

2.1 An “ideal-\textit{PageRank}” (for the easiest case only)

First let us observe the following network data for an illustration.
An intuitive interpretation of the above network is the following:

- The person 1 follows the tweets of the persons 2 and 4;
- The person 2 does those of the persons 1 and 3;
- The person 3 does those of the person 2;
- The person 4 does those of the person 2.

Such information of the network is contained in the adjacency matrix $A$ given by

$$A = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

where

$$A_{ij} = \begin{cases}
1, & \text{if the person } i \text{ follows the tweets of the person } j, \\
0, & \text{otherwise}.
\end{cases}$$

Looking at each column of the matrix $A$, we find that:

- the person 2 is followed by three persons;
- the others are followed only by one person;
- in particular, the persons 1 and 3 are followed by the person 2, and the person 4 is not.
Thus, we can easily imagine that the Twitter account of the person 2 is the most popular among all, and that those of the persons 1 and 3 are more popular than that of 4.

Now, let \( B \) be the diagonal matrix whose \((i,i)\) entry is the out-degree of the node \( i \); that is,

\[
B = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Then, the matrix

\[
M = A^T B^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1/2 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 1/2 & 0 & 0 \\
1/2 & 0 & 1 & 1 \\
0 & 1/2 & 0 & 0 \\
1/2 & 0 & 0 & 0
\end{pmatrix}
\]

is a transition one; actually, all entries are non-negative and the sum of each column is 1.

Note that, in this easy case:

- \( M \) is (already) the transition matrix of a Markov chain;
- The Markov chain is regular (thus \( M \) has a unique eigenvalue 1).

**Definition 1. [ideal-PageRank]** For given \((n \times n)\) adjacency matrix \( A \), construct the matrix \( M \) with the procedure described above. When this \( M \) becomes the transition matrix of a regular Markov chain, the “ideal-PageRank” is defined as the \((n \times 1)\) vector \( p_\infty \) given by

\[
p_\infty = \lim_{k \to \infty} M^k p_0,
\]

with some (actually, any) initial value \( p_0 = (p_0(1), ..., p_0(n))^T \) such that \( p_0(i) \geq 0 \) for all \( i \) and that \( \sum_{i=1}^n p_0(i) = 1 \).

Although the mathematical calculation of the ideal-PageRank via the diagonalization of \( M \) is possible, a numerical computation based on the recurrence equation is faster especially when \( n \) is large.

```r
MC_limit <- function(M){
  n <- nrow(M)
  p <- matrix(1/n, nrow=n, ncol=1)
  diff <- 1
  while(diff > 0.0000001){
    p.pre <- p
    p <- M %*% p
    diff <- max(abs(p-p.pre))
  }
  return(t(p))
}
```

We are now ready to calculate the ideal-PageRank for the network data given above.

```r
n <- nrow(A)
M <- t(A) %*% diag(1/rowSums(A), nrow=n)
MC_limit(M)
```

```r
## [,1] [,2] [,3] [,4]
## [1,] 0.2222222 0.4444444 0.2222222 0.1111111
```
2.2 An “intrinsic-PageRank” (for the regular case only)

In general, however, since some of the out-degrees of the adjacency matrix $A$ may be zero (i.e., some person may not follow any other person’s tweets), the matrix $B$ may not be invertible; thus we may not be able to introduce the transition matrix “$M = A^T B^{-1}$”.

Unfortunately, the idea just replacing $B^{-1}$ with the generalized inverse $B^-$ is not sufficient, because $A^T B^-$ is still not a transition matrix.

**Definition 2. [The transition matrix $\tilde{M}$]** For any given $(n \times n)$ adjacency matrix $A$, the corresponding transition matrix $\tilde{M}$, which coincides with $M$ if $B$ is invertible, is defined by

$$\tilde{M} = A^T B^- + \frac{1}{n} 1(I - BB^-),$$

where $1$ denotes the $(n \times n)$ matrix whose all entries are $1$.

To get better understanding of the above definition of $\tilde{M}$, let us observe a concrete example. If unfortunately the matrix $B$ is not invertible, like

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

since the matrix $A^T B^-$, such as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix},$$

is not a transition one, then the matrix $\frac{1}{n}1(I - BB^-)$ should be added to $A^T B^-$ to obtain the transition matrix $\tilde{M}$; that is,

$$\begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 & 1/4 \\ 1/2 & 0 & 1 & 1/4 \\ 0 & 1/2 & 0 & 1/4 \\ 1/2 & 0 & 0 & 1/4 \end{pmatrix}.$$

An R-function to produce the matrix $\tilde{M}$ is given as follows:

```r
M_tilde <- function(adj){
  n <- nrow(adj)
  B <- diag(rowSums(adj), n)
  one.mat <- matrix(1, nrow=n, ncol=n)
  C <- diag((rowSums(adj)==0), n)
  Mtilde <- t(adj) %*% ginv(B) + (1/n) * one.mat %*% C
  return(Mtilde)
}
```
We then apply the recurrent equation
\[ p_k = \tilde{M} p_{k-1} = \tilde{M}^k p_0, \quad k = 1, 2, \ldots, \]
with the initial value \( p_0 = (1/n, \ldots, 1/n)^T \), to define and compute an intrinsic version of PageRank, which is still different from the standard-PageRank originally created by Brin and Page (1998), given by
\[ \text{intrinsic-PR} = \lim_{k \to \infty} p_k = \lim_{k \to \infty} \tilde{M}^k p_0. \]

Actually, the intrinsic-PageRank can be calculated in the following way:

```r
intrinsic_PR <- function(adj){
  MC_limit(M_tilde(adj))
}
```

Now that the function `intrinsic_PR()` to compute the intrinsic-PageRank is ready, let us double-check the ideal-PageRank for the particular example given in the previous subsection.

```r
intrinsic_PR(A)
```

```
## [,1] [,2] [,3] [,4]
## [1,] 0.2222222 0.4444444 0.2222222 0.1111111
```

On the other hand, using to the function `page.rank()` included in the package `igraph`, the standard-PageRank is computed as follows:

```r
A.ga <- graph.adjacency(A, mode="directed")
page.rank(A.ga)$vector
```

```
## [1] 0.2199138 0.4292090 0.2199138 0.1309634
```

We have found that the standard-PageRank and the intrinsic-PageRank return completely different values.

### 3 The \( \alpha \)-PageRank and a new statistic called “MarkovRank”

#### 3.1 The \( \alpha \)-PageRank (for the general, non-regular case)

Below we give the definition of \( \alpha \)-PageRank. We announce in advance that:

- 1-PageRank coincides with the intrinsic-PageRank, as far as the multiplicity of the eigenvalue 1 of \( \tilde{M} \) is one;
- 0.85-PageRank coincides with the (so-called) PageRank by Brin and Page (1998). We have already called it “standard-PageRank”, which is always well-defined.

We remark also that the formula used to define the \( \alpha \)-PageRank below is commonly called damping (see, e.g., Bar-Yossef and Mashiach (2008)).

**Definition 3.** [The \( \alpha \)-PageRank] Let \( A \) be any \( (n \times n) \) adjacency matrix.

(i) For \( 0 < \alpha < 1 \), the \( \alpha \)-PageRank is (always) well-defined as

\[ \text{\( \alpha \)-PageRank} = \lim_{k \to \infty} \hat{M}^k p_0, \]

where

\[ \hat{M}_\alpha = \alpha \tilde{M} + (1 - \alpha) \frac{1}{n} \]

and \( p_0 = (p_0^{(1)}, \ldots, p_0^{(n)})^T \) is some (actually, any) initial vector such that \( p_0^{(i)} \geq 0 \) for all \( i \) and that \( \sum_{i=1}^n p_0^{(i)} = 1 \).
(ii) For $\alpha = 1$, the 1-PageRank can be defined by the same formula as above, as far as the matrix $\hat{M}_1 = \hat{M}$ has the eigenvalue 1 of the multiplicity one.

The R-code to calculate the $\alpha$-PageRank is the following:

```r
PageRank <- function(A, alpha) {
  n <- nrow(A)
  for (i in 1:n) { A[i,] <- A[i,] + (all(A[i,] == 0)) }
  M <- alpha * t(A) * diag(1/rowSums(A), n) + (1-alpha)/n
  eigen.one <- Re(eigen(M)$values) > 0.99999
  V <- Re(eigen(M)$vectors[, eigen.one])
  if (sum(eigen.one) > 1) {
    print("Multiplicity of the eigenvalue 1 is not one")
  } else {
    MC_limit(M)[1:n]
  }
}
```

### 3.2 A new statistic “MarkovRank” (for the general, non-regular case)

For any $\varepsilon \in (0, 1]$, let us introduce the $((n + 1) \times (n + 1))$-matrix $\hat{M}_\varepsilon$ in the following way; we will actually use this by setting $\varepsilon = 1/k$ for integers $k$ in our procedure, namely, $\hat{M}_{1/k}$'s.

**Step 1** Given an $(n \times n)$ adjacency matrix $A$, replace each zero row of $A$ by one row, say, $(1, \ldots, 1)$, to create the new matrix $\tilde{A} = (\tilde{a}_{ij})_{(i,j) \in \{1, \ldots, n\}^2}$.

**Step 2** Create the new $((n + 1) \times (n + 1))$ matrix $\hat{A}_\varepsilon = (\hat{a}_{ij})_{(i,j) \in \{1, \ldots, n+1\}^2}$ by:

$$
\hat{a}_{ij} = \tilde{a}_{ij}, \quad \forall (i,j) \in \{1, \ldots, n\}^2;
$$

$$
\hat{a}_{i,n+1} = \varepsilon \sum_{j=1}^{n} \tilde{a}_{ij}, \quad \forall i \in \{1, \ldots, n\};
$$

$$
\hat{a}_{n+1,j} = 1, \quad \forall j \in \{1, \ldots, n\}; \quad \hat{a}_{n+1,n+1} = 0.
$$

**Step 3** Introduce the $((n + 1) \times (n + 1))$ transition matrix

$$
\hat{M}_\varepsilon = \hat{A}_\varepsilon ^T \hat{B}^{-1} \varepsilon ,
$$

where $\hat{B}_{\varepsilon}$ is the $((n + 1) \times (n + 1))$ diagonal matrix whose $(i, i)$ entry is the sum of $i$-th row of $\hat{A}_\varepsilon$.

**Definition 4. [The MarkovRank]** For any given $(n \times n)$ adjacency matrix $A$, the MarkovRank is defined by

$$
\text{MarkovRank} = \frac{1}{\sum_{i=1}^{n} v_{\infty}^{(i)} (v_{\infty}^{(1)}, \ldots, v_{\infty}^{(n)})^T},
$$

where $v_{\infty} = (v_{\infty}^{(1)}, \ldots, v_{\infty}^{(n)}, v_{\infty}^{(n+1)})^T$ is given by

$$
v_{\infty} = \lim_{k \to \infty} \hat{M}_{1/k}^{k} v_0 ,
$$

with some (actually, any) initial value $v_0$ of $(n + 1)$-dimensional vector $v_0 = (v_0^{(1)}, \ldots, v_0^{(n)}, v_0^{(n+1)})^T$ such that $v_0^{(i)} \geq 0$ for all $i$ and that $\sum_{i=1}^{n+1} v_0^{(i)} = 1$.

The above statistic is implemented in R, as follows:
MarkovRank <- function(A){
  n <- nrow(A)
  for(i in 1:n){A[i,] <- A[i,] +(all(A[i,]==0))}
  k <- 1
  MR <- rep(1/n,n)
  diff <- 1
  while(diff > 0.0000001){
    MR.pre <- MR
    Acheck <- rbind(cbind(A,matrix(rowSums(A)/k,n,1)),
                    matrix(c(rep(1,n),0),1,n+1))
    M <- t(Acheck)%*%diag(1/rowSums(Acheck),n+1)
    mr <- rep(1/(n+1),n+1)
    for(i in 1:k){
      mr <- M %*% mr
    }
    MR <- mr[1:n]/sum(mr[1:n])
    diff <- max(abs(MR-MR.pre))
    k <- k+1
  }
  MR
}

4  Toy examples

4.1  Non-regular \( \hat{M} \) with the eigenvalue 1 of multiplicity one

Suppose that the adjacency matrix \( A \) is given by:

```
#  [,1] [,2] [,3] [,4] [,5]
# [1,]  0  1  1  1  1
# [2,]  0  0  1  0  0
# [3,]  0  0  0  0  0
# [4,]  0  0  0  0  1
# [5,]  0  0  0  1  0
```
• The Markov chain generated by the transition matrix $\tilde{M}$ is not regular; indeed, for any $k \geq 1$ some entries of the 4th and 5th columns of $\tilde{M}^k$ are zero.
• However, the multiplicity of the eigenvalue 1 of the matrix $\tilde{M}$ is one. Thus, the intrinsic-PageRank is well-defined.

Actually, the eigenvalues and eigenvectors of $\tilde{M}$ are as follows:

```r
eigen(M_tilde(A))
```

```
## eigen() decomposition
## $values
## [1] -1.0000000+0.0000000i 1.0000000+0.0000000i 0.6777330+0.0000000i
## [4] -0.2388665+0.1292988i -0.2388665-0.1292988i
##
## $vectors
## [,1] [,2] [,3] [,4] 
## [1,] -1.390248e-17+0i 3.431396e-18+0i 0.1798040+0i 0.49993367+0.27061485i
## [2,] -8.890445e-19+0i -5.490233e-17+0i 0.2461296+0i 0.21383811-0.16747727i
## [3,] 5.134079e-17+0i 3.500023e-16+0i 0.6092956+0i -0.77203783+0.00000000i
## [4,] -7.071068e-01+0i 7.071068e-01+0i -0.5176146+0i 0.02913303-0.05156879i
## [5,] 7.071068e-01+0i 7.071068e-01+0i -0.5176146+0i 0.02913303-0.05156879i
##
##
##
##
##
##
## [1,] 0.49993367-0.27061485i
## [2,] 0.21383811+0.16747727i
## [3,] -0.77203783+0.00000000i
## [4,] 0.02913303+0.05156879i
## [5,] 0.02913303+0.05156879i

Let us calculate three Ranks for this data.
PageRank(A, 0.85)  # standard-PageRank

## [1] 0.04849124 0.05879563 0.10877194 0.39197059 0.39197059

PageRank(A, 1)  # intrinsic-PageRank

## [1] 6.017267e-08 8.236898e-08 2.039050e-07 4.999998e-01 4.999998e-01

MarkovRank(A)  # MarkovRank

## [1] 0.0001264742 0.0001580828 0.0003161155 0.4996996637 0.4996996637

Observe that

nrow(A)

## [1] 5

sum( rank(PageRank(A, 1)) == rank(PageRank(A, 0.85)) )

## [1] 5

sum( rank(PageRank(A, 1)) == rank(MarkovRank(A)) )

## [1] 5

We have found that the rank statistics for three Ranks are identical in this particular example.

### 4.2 Non-regular $\tilde{M}$ with the eigenvalue 1 of multiplicity two

Suppose that the adjacency matrix $A$ is given by:

## [,1] [,2] [,3] [,4] [,5] [,6]
## [1,] 0 1 1 1 1 1
## [2,] 0 0 1 1 0 0
## [3,] 0 1 0 1 0 0
## [4,] 0 1 1 0 0 0
## [5,] 0 0 0 0 0 1
## [6,] 0 0 0 0 1 0

9
The multiplicity of the eigenvalue 1 of the matrix \( \tilde{M} = M \) is two. The Markov chain generated by the transition matrix \( \tilde{M} \) is necessarily non-regular.

Actually, the eigenvalues and eigenvectors of \( \tilde{M} \) are as follows:

```
eigen(M_tilde(A))
```

```
## eigen() decomposition
## $values
## [1] 1.0 -1.0 1.0 -0.5 -0.5 0.0
##
## $vectors
## [,1] [,2] [,3] [,4] [,5] [,6]
## [1,] 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.9128709
## [2,] 0.0000000 0.0000000 0.5773503 0.4082483 -0.5661385 -0.1825742
## [3,] 0.0000000 0.0000000 0.5773503 -0.8164966 -0.2264554 -0.1825742
## [4,] 0.0000000 0.0000000 0.5773503 0.4082483 0.7925939 -0.1825742
## [5,] 0.7071068 -0.7071068 0.0000000 0.0000000 0.0000000 -0.1825742
## [6,] 0.7071068 0.7071068 0.0000000 0.0000000 0.0000000 -0.1825742
```

Let us calculate standard-PageRank, 0.9999-PageRank which is very close to “intrinsic-PageRank”, and MarkovRank.

```
PageRank(A, 0.85)  # standard-PageRank
## [1] 0.025 0.195 0.195 0.195 0.195 0.195

PageRank(A, 0.9999)  # very close to intrinsic-PageRank
## [1] 1.666667e-05 1.999967e-01 1.999967e-01 1.999967e-01 1.999967e-01 1.999967e-01
## [6] 1.999967e-01
```
PageRank(A,1)  # intrinsic-PageRank is not well-defined

## 1 "Multiplicity of the eigenvalue 1 is not one"

MarkovRank(A)  # MarkovRank

## [1] 0.000128999 0.199974200 0.199974200 0.199974200 0.199974200 0.199974200

4.3 Non-trivial example for rank statistics

Suppose that the adjacency matrix \( A \) is given by:

## [,1] [,2] [,3] [,4] [,5] [,6]
## [1,] 0 1 0 1 1 1
## [2,] 1 0 0 0 0 0
## [3,] 0 1 0 0 1 0
## [4,] 0 1 0 0 0 0
## [5,] 0 0 1 1 0 0
## [6,] 0 0 0 0 0 0

PageRank(A,0.85)  # standard-PageRank

## [1] 0.26186686 0.26300739 0.09549044 0.15113717 0.13454079 0.09395734

PageRank(A,1)  # intrinsic-PageRank, which is well-defined in this example

## [1] 0.28846150 0.27403846 0.07692306 0.14903846 0.12500002 0.08653847
Observe an interesting result as follows:

```r
rank(PageRank(A, 0.85))
## [1] 5 6 2 4 3 1
rank(PageRank(A, 1))
## [1] 6 5 1 4 3 2
rank(MarkovRank(A))
## [1] 6 5 1 4 3 2
```

Let us count the number of the same ranks:

```r
sum( rank(PageRank(A, 1)) == rank(PageRank(A, 0.85)) )
## [1] 2
sum( rank(PageRank(A, 1)) == rank(MarkovRank(A)) )
## [1] 6
```

We have found that, in this example, although the rank statistics of standard-PageRank and intrinsic-PageRank are different, those of intrinsic-PageRank and MarkovRank are identical. In the Appendix, we shall mathematically prove that the latter half of this observation always holds true.

### 5 A real data analysis

In this section, we analyze Twitter-following data among US senators as directed network data. In this data set, an edge represents an instance of a senator following another senator's Twitter account. The data consist of two files, one listing pairs of the Twitter screen names of the following and followed politicians, twitter-following.csv, and the other containing information about each politician, twitter-senator.csv.

```r
twitter <- read.csv("twitter-following.csv", stringsAsFactors=FALSE)
senator <- read.csv("twitter-senator.csv", stringsAsFactors=FALSE)
```

Let us create the adjacency matrix $A$ for this network.

```r
n <- nrow(senator)
twitter.adj <- matrix(0, nrow=n, ncol=n)
rownames(twitter.adj) <- senator$screen_name
colnames(twitter.adj) <- senator$screen_name
for(i in 1:nrow(twitter)){
  twitter.adj[,twitter$following[i],
    twitter$followed[i]] <- 1
}
```

Let us observe the in-degrees of the nodes:
Let us finish up this paper with observing the results of three Ranks for this network data.

```r
standard_PR.result <- PageRank(twitter.adj, 0.85)
intrinsic_PR.result <- PageRank(twitter.adj, 1)
MR.result <- MarkovRank(twitter.adj)
names(standard_PR.result) <- senator$screen_name
names(intrinsic_PR.result) <- senator$screen_name
names(MR.result) <- senator$screen_name
```

The top six Twitter accounts, by the three Ranks, are given as follows. Although the Rank values for three definitions are somewhat different, the rankings of top and last Twitter accounts are identical.

```r
head(sort(standard_PR.result, decreasing=TRUE))
## SenJohnMcCain JohnCornyn MartinHeinrich lisamurkowski SenToomey SenDanCoats
## 0.02225510 0.01994213 0.01945448 0.01873310 0.01721254

head(sort(intrinsic_PR.result, decreasing=TRUE))
## SenJohnMcCain JohnCornyn MartinHeinrich lisamurkowski SenToomey SenDanCoats
## 0.02441628 0.02196977 0.02149121 0.02031664 0.01846398 0.01654421
```
However, once we observe the rankings for the whole accounts, we again find that the rank statistic of standard-PageRank does not coincide with that of intrinsic-PageRank, but that of MarkovRank does.

**Appendix: Main theorem and its proof**

In the main context of this paper, we have confirmed that:

1. The MarkovRank seems to behave as stably as the standard-PageRank, while the behavior of the intrinsic-PageRank is not so stable;

2. The rank statistic of the standard-PageRank does not always coincide with that of the intrinsic-PageRank, even when the latter is well-defined;

3. The rank statistic of the MarkovRank seems to coincide “always” with that of the intrinsic-PageRank, as far as the latter is well-defined.

Here, we shall mathematically prove the above (loose) claim 3 as our main theorem of the current paper.

**Main Theorem.** If the intrinsic-PageRank corresponding to given adjacency matrix $A$ is well-defined, then the rank statistic of MarkovRank, which is always well-defined, is identical to that of intrinsic-PageRank.

**Proof.** As preliminaries, let us introduce the $((n + 1) \times (n + 1))$ transition matrices $A$, $B$ and $C$ given by:

$$A = \begin{pmatrix} \frac{1}{n} & & & & \\ \vdots & \tilde{M} & & & \\ 1/n & & & & \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{n} & & & & \\ \vdots & O & & & \\ 1/n & & & & \\ 1 & \cdots & 1 & 0 \end{pmatrix},$$

$$B^2 = \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$
Furthermore, for any \( k = 1, 2, \ldots \), put
\[
\alpha_k = \frac{1}{1 + (1/k)} \quad \text{and} \quad \beta_k = \frac{1/k}{1 + (1/k)}.
\]

Then it holds that
\[
\tilde{M}_{1/k} = \alpha_k A + \beta_k B.
\]

Now, fix any initial vector \( \mathbf{v}_0 = (v_0^{(1)}, \ldots, v_0^{(n)})^T \) such that \( v_0^{(i)} \geq 0 \) for all \( i \) and that \( \sum_{i=1}^{n+1} v_0^{(i)} = 1 \), and for every \( k = 1, 2, \ldots \), define the \((n + 1)\)-dimensional vectors \( \mathbf{v}_1^{(1/k)}, \mathbf{v}_2^{(1/k)}, \ldots, \mathbf{v}_k^{(1/k)} \) by the recursive formula
\[
\mathbf{v}_k^{(1/k)} = \tilde{M}_{1/k} \mathbf{v}_{k-1}^{(1/k)} = \cdots = \tilde{M}_{1/k}^{k-1} \mathbf{v}_1^{(1/k)} = \tilde{M}_{1/k}^k \mathbf{v}_0, \quad k = 1, 2, \ldots.
\]

We shall compute \( \tilde{M}_{1/k}^k \) for general integers \( k \). First, notice the following facts:
\[
\mathbf{\_} = \left( \begin{array}{c}
\tilde{M}^k \\
\vdots \\
\tilde{M}^{k-1,(n)} \\
0 \ldots 0
\end{array} \right),
\]
where the vector \( \mathbf{m}^k = (m^{k,(1)}, \ldots, m^{k,(n)})^T \) is given by
\[
m^{k,(i)} = \frac{1}{n} \sum_{l=1}^n \tilde{M}_{l,l}^k \quad \text{where} \quad \tilde{M}_{l,j}^k \text{ is the } (i, j) \text{ entry of the matrix } \tilde{M}^k,
\]
with the convention that
\[
\tilde{M}^0 = \mathbb{M}_n = \left( \begin{array}{ccc}
1/n & \cdots & 1/n \\
\vdots & \ddots & \vdots \\
1/n & \cdots & 1/n
\end{array} \right)
\]
\[
\mathbb{B}^k = \left\{ \begin{array}{l}
\mathbb{B} \quad \text{for odd positive integer } k, \\
\mathbb{B}^2 \quad \text{for even positive integer } k;
\end{array} \right.
\]
\[
\mathbb{B} A = \mathbb{C} = \left( \begin{array}{cccccc}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
1 & \cdots & 1 & 1
\end{array} \right) \quad \text{and} \quad \mathbb{B}^2 A = \mathbb{D} = \left( \begin{array}{cccccc}
1/n & \cdots & 1/n & 1/n \\
\vdots & \ddots & \vdots & \vdots \\
1/n & \cdots & 1/n & 1/n \\
0 & \cdots & 0 & 0
\end{array} \right);
\]
\[
\mathbf{\_}^{k-l} \mathbb{B}^l = \left( \begin{array}{ccccccc}
m^{k-l-1,(1)} & \cdots & m^{k-l-1,(1)} & m^{k-l,(1)} \\
\vdots & \ddots & \vdots & \vdots \\
m^{k-l-1,(n)} & \cdots & m^{k-l-1,(n)} & m^{k-l,(n)} \\
0 & \cdots & 0 & 0
\end{array} \right), \quad \text{for odd } l;
\]
\[
\mathbf{\_}^{k-l} \mathbb{B}^l = \left( \begin{array}{ccccccc}
m^{k-l,(1)} & \cdots & m^{k-l,(1)} & m^{k-l-1,(1)} \\
\vdots & \ddots & \vdots & \vdots \\
m^{k-l,(n)} & \cdots & m^{k-l,(n)} & m^{k-l-1,(n)} \\
0 & \cdots & 0 & 0
\end{array} \right), \quad \text{for even } l;
\]
\[
\mathbb{C} A = \mathbb{B} = \mathbb{C} \quad \text{and} \quad \mathbb{D} A = \mathbb{B} = \mathbb{D}.
\]
Hence it holds for any \( k \geq 3 \) that
\[
\tilde{M}_{1/k}^k = \alpha_k^k \mathbb{A} + \sum_{l=1}^{k-1} \alpha_k^{k-l} \beta_k^l \mathbf{\_}^{k-l} \mathbb{B}^l + \beta_k^k \mathbb{B}^k + \gamma_k \mathbb{C} + \delta_k \mathbb{D},
\]

\[15\]
where $\gamma_k, \delta_k \geq 0$ and $\sum_{l=0}^{k} \alpha_k^{k-l} \beta_k^l + \gamma_k + \delta_k = 1$, with $\delta_k = \alpha_k \gamma_k$.

Notice that $\lim_{k \to \infty} \alpha_k^k = 1/e$. As for the second term on the right-hand side, we have that any entry is non-negative and bounded by

$$\sum_{l=1}^{k-1} \alpha_k^{k-l} \beta_k^l \leq \alpha_k^k \sum_{l=1}^{\infty} (\beta_k/\alpha_k)^l \leq \frac{1/k}{1 - (1/k)},$$

which converges to zero as $k \to \infty$. It is clear that the third term also vanishes as $k \to \infty$. As for the last two terms, since $\lim_{k \to \infty} (\gamma_k + \delta_k) = 1 - (1/e)$ and $\delta_k = \alpha_k \gamma_k$ for $k \geq 3$, we have that $\lim_{k \to \infty} \gamma_k = \lim_{k \to \infty} \delta_k = (1 - (1/e))/2$. Hence, it holds that

$$\mathbb{M} = \lim_{k \to 0} \mathbb{M}_0^k = (1/e) 2^{\infty} + \frac{1 - (1/e)}{2} (\mathbb{C} + \mathbb{D}).$$

Note that the multiplicity of the eigenvalue 1 for this matrix is one. Thus, when the multiplicity of the eigenvalue 1 for the matrix $\mathbb{M}$ is one, the rescaled eigenvector of $(n \times n)$ matrix $\mathbb{M}^\infty$ and the first $n$ entries of the rescaled eigenvector of $((n + 1) \times (n + 1))$ matrix $\mathbb{M}$, both for the eigenvalue 1, have the same rank statistics. The assertion of the theorem has been proved.

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**References and remark**

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**Remark:** The data files twitter-following.csv and twitter-senator.csv treated in Section 5 are available on the accompanying web-site for the book [4]: http://qss.princeton.press/