THE MAXIMIZATION OF THE P-LAPLACIAN ENERGY
FOR A TWO-PHASE MATERIAL

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Abstract. We consider the optimal arrangement of two diffusion materials in a bounded open set \( \Omega \subset \mathbb{R}^N \) in order to maximize the energy. The diffusion problem is modeled by the \( p \)-Laplacian operator. It is well known that this type of problems has no solution in general and then that it is necessary to work with a relaxed formulation. In the present paper we obtain such relaxed formulation using the homogenization theory, i.e. we replace both materials by microscopic mixtures of them. Then we get some uniqueness results and a system of optimality conditions. As a consequence we prove some regularity properties for the optimal solutions of the relaxed problem. Namely, we show that the flux is in the Sobolev space \( H^1(\Omega)^N \) and that the optimal proportion of the materials is derivable in the orthogonal direction to the flux. This will imply that the unrelaxed problem has no solution in general. Our results extend those obtained by the first author for the Laplace operator.

1. Introduction

The present paper is devoted to study an optimal design problem for a diffusion process in a two-phase material modeled by the \( p \)-Laplacian operator. Namely, we are interested in the control problem

\[
\begin{aligned}
\max_{\omega} & \int_{\Omega} \left( \alpha \chi_{\omega} + \beta (1 - \chi_{\omega}) \right) |\nabla u|^p dx \\
- \nabla \cdot \left( (\alpha \chi_{\omega} + \beta (1 - \chi_{\omega})) |\nabla u|^{p-2} \nabla u \right) &= f & \text{in } \Omega \\
u &\in W_0^{1,p}(\Omega), \quad \omega \subset \Omega \text{ mesurable , } |\omega| \leq \kappa,
\end{aligned}
\]

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with $\Omega$ a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $p \in (1, \infty)$, $\alpha, \beta, \kappa > 0$, $\alpha < \beta$, $\chi_\omega$ the characteristic function of the set $\omega$, and $f \in W^{-1,p'}(\Omega)$, with $p'$ is the Holder conjugate of $p \left( p' = \frac{p}{p-1} \right)$.

In (1.1) the equation is understood to hold in the sense of distributions, combined with $u \in W^{1,p}_0(\Omega)$, denoting by $u^\alpha$ and $u^\beta$ the values of $u$ in $\omega$ and $\Omega \setminus \omega$ respectively and assuming $\omega$ smooth enough, this means that the interphase conditions on $\partial \omega$ are given by

$$u^\alpha = u^\beta, \quad \alpha |\nabla u^\alpha|^{p-2} \nabla u^\alpha \cdot \nu = \beta |\nabla u^\beta|^{p-2} \nabla u^\beta \cdot \nu \quad \text{on } \partial \omega \cap \Omega$$

in the sense of the traces in $W^{1/p',p}(\partial \omega)$ and $W^{-1/p',p'}(\partial \omega)$ respectively. Here $\nu$ denotes a unitary normal vector on $\partial \omega$.

Physically the constants $\alpha$ and $\beta$ represent two diffusion materials that we are mixing in order to maximize the corresponding functional, which in (1.1) represent the potential energy. The control variable is the set $\omega$ where we place the material $\alpha$. If we do not impose any restriction on the amount of this material, it is simple to check that the solution of (1.1) is the trivial one given by $\omega = \Omega$. Thus, the interesting problem corresponds to $\kappa < |\Omega|$, i.e. the material $\alpha$ is better than $\beta$ but it is also more expensive and therefore, we do not want to use a large amount of it in the mixture. The case corresponding to $p = 2$ has been studied in several papers (see e.g. [5], [15], [26]) where some classical applications are the optimal mixture of two materials in the cross-section of a beam in order to minimize the torsion, and the optimal arrangement of two viscous fluids in a pipe. For $p \in (1,2) \cup (2,\infty)$ the $p$-Laplacian operator models the torsional creep in the cross-section of a beam [16] and therefore problem (1.1) corresponds to find the material which minimizes the torsion for the mixture of two homogeneous materials in non-linear elasticity.

It is well known that a control problem in the coefficients like (1.1) has no solution in general ([24], [25]). In fact, some counterexamples to the existence of solution for (1.1) with $p = 2$ can be found in [5] and [26]. Thus, it is necessary to work with a relaxed formulation. One way to obtain this formulation is to use the homogenization theory ([2], [26], [30]). The idea is to replace the material $\alpha \chi_\omega + \beta (1 - \chi_\omega)$ in (1.1) by microscopic mixtures of $\alpha, \beta$ with a certain proportion $\theta = \theta(x) \in [0,1], \, x \in \Omega$. The new materials do not only depend on the proportion of each original material but also on their microscopical distribution. In the case $p = 2$, this relaxed formulation has been obtained in [26]. Here we show that a relaxed formulation for (1.1) is given by

$$\max_\theta \left\{ \frac{1}{p} \int_\Omega \left( \theta \alpha^{\frac{1}{p-1}} + (1 - \theta) \beta^{\frac{1}{p-1}} \right)^{1-p} |\nabla u|^p dx \right\}$$

$$-\text{div} \left( \left( \theta \alpha^{\frac{1}{p-1}} + (1 - \theta) \beta^{\frac{1}{p-1}} \right)^{1-p} |\nabla u|^{p-2} \nabla u \right) = f \quad \text{in } \Omega$$

$$u \in W^{1,p}_0(\Omega), \quad \theta \in L^\infty(\Omega; [0,1]), \quad \int_\Omega \theta(x) \, dx \leq \kappa, \quad \text{for } \theta \in L^\infty(\Omega; [0,1]),$$

(1.2)
which is equivalent to the Calculus of Variations problem

\[
\begin{array}{l}
\min_{\theta} \left\{ \frac{1}{p} \int_{\Omega} \left( \theta \alpha |\nabla u|^p + (1 - \theta) \beta |\nabla u|^p \right)^{1-p} |\nabla u|^p \right\} \\
u \in W^{1,p}_0(\Omega), \quad \theta \in L^\infty(\Omega; [0,1]), \quad \int_{\Omega} \theta(x) \, dx \leq \kappa,
\end{array}
\]

(1.3)

where here and in what follows, \((f,u)\) denotes the duality product of \(f\) and \(u\) as elements of \(W^{-1,p'}(\Omega)\) and \(W^{1,p}_0(\Omega)\) respectively.

Our main results extend those obtained in [5] (see also [26]) for \(p = 2\) relative to the uniqueness and regularity of a solution for (1.2). Namely, we prove that although it is not clear that (1.3) has a unique solution \((u,\theta)\), the flux

\[\sigma = \left( \frac{\theta}{\alpha^{1-p}} + \frac{1 - \theta}{\beta^{1-p}} \right)^{1-p} |\nabla u|^{p-2} \nabla u\]

is unique. Moreover, assuming \(\Omega \in C^{1,1}\) and \(f \in L^q(\Omega) \cap W^{1,1}(\Omega)\), with \(q > N\), we have that \(\sigma\) belongs to \(H^1(\Omega)^N \cap L^\infty(\Omega)\). This is related to some regularity results for the \(p\)-Laplacian operator obtained in [20]. We also prove that every solution \((u,\theta)\) of (1.3) satisfies

\[u \in W^{1,\infty}(\Omega), \quad \partial_i \theta \sigma_j - \partial_j \theta \sigma_i \in L^2(\Omega), \quad 1 \leq i, j \leq N,\]

(1.4)

where \(\sigma_i\) denotes the \(i\)-th component of the vector function \(\sigma\), i.e. \(\theta\) is derivable in the orthogonal subspace to \(\sigma\). The existence of first derivatives for \(\sigma\) and \(\theta\) will imply that we cannot hope in general an existence result for the unrelaxed problem (1.1). Namely, the existence of a solution for (1.1) is equivalent to the existence of a solution for (1.3) where \(\theta\) only takes the values zero and one, but then the derivatives of \(\theta\) in (1.4) vanish. Assuming \(\Omega\) simply connected with connected boundary, we show that this implies \(\sigma = |\nabla w|^{p-2} \nabla w\), with \(w\) the unique solution of

\[
\begin{cases}
-\text{div} (|\nabla w|^{p-2} \nabla w) = f & \text{in } \Omega \\
w \in W^{1,p}_0(\Omega).
\end{cases}
\]

Similarly to the result obtained in ([5], [26]), we prove that this is only possible if \(\Omega\) is a ball.

We finish this introduction remembering that the results obtained in the present paper are also related to those given in [11] where, for \(p = 2\), it is considered the minimization in (1.1) instead of the maximization. Problem (1.1) is also related to the minimization of the first eigenvalue for the \(p\)-Laplacian operator (see [5], [9], [10], [22] for \(p = 2\)), problem which we hope to study in a later work.
2. Position of the problem. Relaxation and equivalent formulations

For a bounded open set \( \Omega \subset \mathbb{R}^N \), three positive constants \( \alpha, \beta, \kappa \) with \( 0 < \alpha < \beta, \kappa < |\Omega| \), and a distribution \( f \in W^{-1,p'}(\Omega), p > 1 \), we are interested in the control problem

\[
\begin{aligned}
\max_{\omega} & \int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u_\omega|^p dx \\
\omega & \subset \Omega \text{ measurable, } |\omega| \leq \kappa
\end{aligned}
\]  

(2.1)

Here \( \alpha \) and \( \beta \) represent the diffusion coefficients of two materials, where the diffusion process is modeled by the \( p \)-Laplacian operator. The problem consists in maximizing the potential energy.

Using \( u_\omega \) as test function in the state equation we have

\[
\int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u_\omega|^p dx = \langle f, u_\omega \rangle,
\]

By the above equality and since \( p' = \frac{p}{p-1} \) we have

\[
\begin{aligned}
&\int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u_\omega|^p dx \\
&= -p' \left( \frac{1}{p} \int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u_\omega|^p dx - \int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u_\omega|^p dx \right) \\
&= -p' \left( \frac{1}{p} \int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u_\omega|^p dx - \langle f, u_\omega \rangle \right)
\end{aligned}
\]

which combined with \( u_\omega \), unique solution of the minimization problem

\[
\min_{u \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u|^p dx - \langle f, u \rangle \right\},
\]

gives the equivalent formulation for problem (2.1):

\[
\begin{aligned}
\min_{\omega, u} & \left\{ \frac{1}{p} \int_{\Omega} (\alpha \mathcal{X}_\omega + \beta \mathcal{X}_{\Omega \setminus \omega}) |\nabla u|^p dx - \langle f, u \rangle \right\} \\
u & \in W^{1,p}_0(\Omega), \quad \omega \subset \Omega \text{ measurable, } |\omega| \leq \kappa.
\end{aligned}
\]  

(2.2)

It is known that the maximum in (2.1) or the minimum in (2.2) are not achieved, i.e., that (2.1) (or (2.2)) has no solution in general. Namely, for \( p = 2 \) and \( f = 1 \), it has been proved in [5] and [26] that if \( \Omega \) is smooth, with connected smooth boundary, and (2.1) has a solution, then \( \Omega \) is a ball. Some other classical counterexamples to the existence of solution for problems related to (2.1) can be found in [24] and [25]. Due to this difficulty it is then necessary to find a relaxed formulation for (2.1). This is done by the following theorem
Theorem 2.1. A relaxed formulation of problem \((2.2)\) is given by
\[
\begin{align*}
\min_{\theta,u} & \left\{ \frac{1}{p} \int_{\Omega} \left( \theta \alpha^{1-p} + (1 - \theta) \beta^{1-p} \right)^{1-p} |\nabla u|^p dx - \langle f, u \rangle \right\} \\
u & \in W^{1,p}_0(\Omega), \quad \theta \in L^\infty(\Omega; [0,1]), \quad \int_\Omega \theta dx \leq \kappa,
\end{align*}
\] (2.3)
in the following sense:

1. Problem \((2.3)\) has a solution.
2. The infimum for problem \((2.2)\) agrees with the minimum for \((2.3)\).
3. Every minimizing sequence \((u_n, \omega_n)\) for \((2.2)\) has a subsequence still denoted by \((u_n, \omega_n)\) such that
\[
u_n \rightharpoonup \nu \quad \text{in} \quad W^{1,p}_0(\Omega), \quad \omega_n \rightharpoonup \omega \quad \text{weak-star in} \quad L^\infty(\Omega),
\] (2.4)
with \((\nu, \omega)\) solution of \((2.3)\).
4. For every pair \((\nu, \omega)\) \in \(W^{1,p}_0(\Omega) \times L^\infty(\Omega; [0,1])\) there exist \(u_n \in W^{1,p}_0(\Omega), \omega_n \subset \Omega\) measurable, \(|\omega_n| \leq \kappa\) such that \((2.4)\) holds and such that
\[
\lim_{n \to \infty} \int_{\Omega} (\alpha \omega + \beta \Omega \setminus \omega_n) |\nabla u_n|^p dx = \int_{\Omega} \left( \theta \alpha^{1-p} + (1 - \theta) \beta^{1-p} \right)^{1-p} |\nabla \nu|^p dx.
\] (2.5)

Remark 2.1. Such as we will see in the proof of Theorem 2.1, the relaxed materials in \((2.3)\) are obtained as a simple lamination in a parallel direction to \(\nabla \nu\). In this context, a laminated material corresponds to a particular distribution of two materials, which depends exclusively on one direction, say \(\xi \in \mathbb{R}^N\), which is represented by a function \(\varphi \in L^\infty(\Omega; [0,1])\) with a generic form as follows:
\[
\varphi(x) = g(\xi \cdot x) \quad \forall x \in \Omega,
\]
where \(g\) is a real-valued function. (see sections 2.3.5 and 2.2.1 in [2] for more details on laminated materials).

Proof of Theorem 2.1. Using that the function \(J: \mathbb{R}^N \times (0, \infty) \to \mathbb{R}\) defined by
\[
J(\xi, t) = \frac{|\xi|^p}{t^{p-1}}, \quad \forall (\xi, t) \in \mathbb{R}^N \times (0, \infty),
\] (2.6)
is convex, and the sequential compactness of the bounded sets in \(W^{1,p}_0(\Omega) \times L^\infty(\Omega)\) with respect to the weak-* topology, it is immediate to show that \((2.3)\) has at least a solution and that every minimizing sequence \((u_n, \theta_n)\) for \((2.3)\) has a subsequence which converges in \(W^{1,p}_0(\Omega) \times L^\infty(\Omega)\) weak-* to a minimum.

Since problem \((2.2)\) consists in minimizing the same functional than the one in \((2.3)\), but on the smaller set
\[
\left\{ (u, \chi_\omega) \in W^{1,p}_0(\Omega) \times L^\infty(\Omega; [0,1]) : \chi_\omega \subset \Omega, \int_\Omega \chi_\omega dx \leq \kappa \right\},
\]
it is clear that the infimum in (2.2) is bigger or equal than the minimum in (2.3). Thus, taking into account that the convergence of the minimizing sequences stated above will imply statement (3), we deduce that it is enough to prove statement (4) to complete the proof of Theorem 2.1. For this purpose, we introduce the functions (the index ♯ means periodicity) \( H \in L^\infty((0,1) \times \mathbb{R}) \cap C^0([0,1]; L^1_\#(0,1)) \), \( G \in W^{1,\infty}((0,1) \times \mathbb{R}) \cap C^0([0,1]; W^{1,1}_\#(0,1)) \), by

\[
H(q, r) = \sum_{k=-\infty}^{\infty} \mathcal{X}_{k,k+q}(r), \quad G(q, r) = qr - \int_{0}^{r} H(q, s) \, ds, \quad \forall \, q, r \in [0,1] \times \mathbb{R}. \tag{2.7}
\]

Now, for a pair \((u, \theta) \in C^1_c(\Omega) \times C^0(\Omega)\) with \(\int_{\Omega} \theta \, dx < \kappa\), and \(\delta > 0\), we consider a family of cubes \(Q_i, 1 \leq i \leq n_\delta\), of side \(\delta\) such that \(\Omega \subset \bigcup_{i=1}^{n_\delta} Q_i\), \(|Q_i \cap Q_j| = 0, \text{ if } i \neq j|, and a partition of the unity in \(\overline{\Omega}\) by functions \(\psi_i \in C^\infty_c(\mathbb{R}^N), \psi_i(x) \geq 0, 1 \leq i \leq n_\delta\) and \(\sum_{i=1}^{n_\delta} \psi_i(x) = 1, \forall x \in \Omega\).

Then, we take

\[
q_i = \frac{1}{\delta N} \int_{Q_i} \theta \, dx, \quad \xi_i = \frac{1}{\delta N} \int_{Q_i} \nabla u \, dx, \quad \zeta_i = \begin{cases} \xi_i & \text{if } \xi_i \neq 0 \\ e & \text{if } \xi_i = 0 \end{cases}
\]

with \(e \in \mathbb{R}^N \setminus \{0\}\) fixed, and we introduce, for every \(\varepsilon > 0\), the sets \(\omega_{\delta,\varepsilon} \subset \Omega\) and the functions \(u_{\delta,\varepsilon} \in W^{1,\infty}(\Omega)\), with compact support by

\[
\mathcal{X}_{\omega_{\delta,\varepsilon}} = \sum_{i=1}^{n_\delta} H\left(q_i, \frac{\xi_i \cdot x}{\varepsilon}\right) \mathcal{X}_{Q_i}, \quad u_{\delta,\varepsilon} = u + \varepsilon \sum_{i=1}^{n_\delta} \psi_i \frac{G(q_i, \frac{\xi_i \cdot x}{\varepsilon})}{\alpha^\frac{\gamma}{\gamma^*}} (\beta^\frac{1}{\gamma^*} - \alpha^\frac{\gamma}{\gamma^*}) q_i + \beta^\frac{1}{\gamma^*} (1 - q_i).
\]

Using the result (see e.g. \(\mathbb{P}\))

\[
\Phi\left(x, \frac{x \cdot \xi}{\varepsilon}\right) \overset{\ast}{\rightarrow} \int_{0}^{1} \Phi(x, s) \, ds \in L^\infty(\Omega), \tag{2.8}
\]

for every \(\Phi \in C^0(\overline{\Omega}; L^1_\#(0,1)) \cap L^\infty(\Omega \times \mathbb{R})\) and every \(\xi \in \mathbb{R}^N \setminus \{0\}\), we have that \(\omega_{\delta,\varepsilon}\) satisfies

\[
\mathcal{X}_{\omega_{\delta,\varepsilon}} \overset{\ast}{\rightarrow} \theta_\delta := \sum_{i=1}^{n_\delta} q_i \mathcal{X}_{Q_i} \in L^\infty(\Omega), \text{ when } \varepsilon \to 0, \tag{2.9}
\]
where thanks to $\theta$ uniformly continuous, we also have

$$\theta_{\delta} \to \theta \text{ in } L^\infty(\Omega; [0, 1]), \text{ when } \delta \to 0. \quad (2.10)$$

In particular, since the integral of $\theta$ is strictly smaller than $\kappa$, we deduce that for every $\delta > 0$ small enough, there exists $\varepsilon_{\delta} > 0$ such that

$$|\omega_{\delta, \varepsilon}| < \kappa, \quad \forall \ 0 < \varepsilon < \varepsilon_{\delta}. \quad (2.11)$$

Since $q(q - 1) \leq G(q, r) \leq 0$, for every $q \in [0, 1]$ and every $r \in \mathbb{R}$, we also have the existence of $C > 0$ such that

$$\|u_{\delta, \varepsilon} - u\|_{C^0(\Omega)} \leq C\varepsilon, \quad \forall \varepsilon, \delta > 0 \quad (2.12)$$

and taking into account that $u$ has compact support and that $G(q, 0) = 0$, we deduce that, for $\delta$ small enough, $u_{\delta, \varepsilon}$ has compact support and thus belongs to $W_0^{1,p}(\Omega)$. Moreover, thanks to (2.8) (observe that there is not problem if $\xi_i = 0$ because then $G(q_i, \frac{\xi_i - x}{\varepsilon}) = 0$ for every $x \in \mathbb{R}^N$)

$$\nabla u_{\delta, \varepsilon} = \nabla u + \sum_{i=1}^{n_\delta} \left(\frac{\beta \frac{1}{1-p} - \alpha \frac{1}{1-p}}{\alpha \frac{1}{1-p} q_i + \beta \frac{1}{1-p} (1 - q_i)}\right) \left(\varepsilon \nabla \psi_i G(q_i, \frac{\xi_i \cdot x}{\varepsilon}) + \psi_i (q_i - H(q_i, \xi_i \cdot x))\right) \xi_i$$

$$\nabla u \text{ in } L^\infty(\Omega) \text{ when } \varepsilon \to 0, \ \forall \delta > 0.$$

Therefore

$$u_{\delta, \varepsilon} \nabla u \text{ in } W^{1,\infty}(\Omega) \cap W_0^{1,p}(\Omega) \text{ when } \varepsilon \to 0, \ \forall \delta > 0 \text{ small enough.} \quad (2.13)$$

On the other hand, using the above expression of $\nabla u_{\delta, \varepsilon}$, and denoting $H_i(s) = H(q_i, s)$, we can use (2.8) combined with $H(q, s) = 1$ if $s \in (0, q)$, $H(q, s) = 0$ if $s \in (q, 1)$, and $\xi_i = 0$ is $\zeta_i = \xi_i$ to deduce

$$\lim_{\varepsilon \to 0} \int_{\Omega} \left(\alpha \lambda_{\omega_{\delta, \varepsilon}} + \beta (1 - \lambda_{\omega_{\delta, \varepsilon}})\right) |\nabla u_{\delta, \varepsilon}|^p dx$$

$$= \sum_{i=1}^{n_\delta} \int_{Q_i} \int_0^1 (\alpha H_i(s) + \beta (1 - H_i(s))) \left|\nabla u + \sum_{i=1}^{n_\delta} \psi_i \left(\frac{q_i - H_i(s)}{\alpha \frac{1}{1-p} q_i + \beta \frac{1}{1-p} (1 - q_i)}\right) \xi_i\right|^p ds dx$$

$$= \sum_{i=1}^{n_\delta} \int_{Q_i} \alpha q_i \left|\nabla u + \frac{(q_i - 1) (\beta \frac{1}{1-p} - \alpha \frac{1}{1-p})}{\alpha \frac{1}{1-p} q_i + \beta \frac{1}{1-p} (1 - q_i)} \xi_i\right|^p dx$$

$$+ \sum_{i=1}^{n_\delta} \int_{Q_i} \beta (1 - q_i) \left|\nabla u + \frac{q_i (\beta \frac{1}{1-p} - \alpha \frac{1}{1-p})}{\alpha \frac{1}{1-p} q_i + \beta \frac{1}{1-p} (1 - q_i)} \xi_i\right|^p dx.$$
Thanks to the uniform continuity of $\theta$ and $\nabla u$, we can also take the limit when $\delta$ tends to zero in the right-hand side of the above equality to get

$$
\lim_{\delta \to 0} \left( \sum_{i=1}^{n^3} \int_{Q_i} \alpha q_i \left| \nabla u + \frac{(q_i - 1)(\beta_{1/p} - \alpha_{1/p})}{\alpha_{1/p} q_i + \beta_{1/p} (1 - q_i)} \xi_i \right|^p dx \right)
+ \sum_{i=1}^{n^3} \int_{Q_i} \beta (1 - q_i) \left| \nabla u + \frac{q_i (\beta_{1/p} - \alpha_{1/p})}{\alpha_{1/p} q_i + \beta_{1/p} (1 - q_i)} \xi_i \right|^p dx
= \int_{\Omega} \left( \alpha \theta \left| \nabla u + \frac{(\theta - 1) (\beta_{1/p} - \alpha_{1/p})}{\alpha_{1/p} \theta + \beta_{1/p} (1 - \theta)} \right|^p \right) dx
+ \beta (1 - \theta) \left| \nabla u \right|^p dx
= \int_{\Omega} \left( \theta \alpha_{1/p} + (1 - \theta) \beta_{1/p} \right)^{1-p} \left| \nabla u \right|^p dx.
$$

Let us now use that for $\varepsilon < 1$, $\nabla u$, $\varepsilon$, and $\chi_{\omega_{\delta,e}} \in \{0, 1\}$. Thus, there exists $C \geq 1$ such that

$$
\|\chi_{\omega_{\delta,e}}\|_{L^\infty(\Omega)} \leq 1, \quad \|\partial_j u_{\delta,e}\|_{L^\infty(\Omega)} \leq C, \quad 1 \leq j \leq N, \quad \forall \varepsilon, \delta > 0, \quad 0 < \varepsilon < 1.
$$

Here, we recall that the closed ball $\overline{B}_C$ of center 0 and radius $C$ in $L^\infty(\Omega)$, endowed with the weak-* topology is metrizable. Taking $d$ a suitable distance, and using (2.9), (2.11) and (2.13), we can choose for every $\delta > 0$, $\varepsilon(\delta) > 0$ such that

$$
d(\chi_{\omega_{\delta,e}}(\theta_{\delta}), \delta) < \delta, \quad |\omega_{\delta,e}| < \kappa, \quad d(\partial_j u_{\delta,e}(\theta_{\delta}), \partial_j u) < \delta, \quad 1 \leq j \leq N;
$$

Then, taking into account (2.10) and (2.14), we get

$$
\chi_{\omega_{\delta,e}} \xrightarrow{\ast} \theta \text{ in } L^\infty(\Omega), \quad |\omega_{\delta,e}| \leq \kappa, \quad u_{\delta,e} \xrightarrow{\ast} u \text{ in } W^{1,\infty}(\Omega) \cap W^{1,p}_0(\Omega),
$$

$$
\lim_{\delta \to 0} \int_{\Omega} \left( \alpha \chi_{\omega_{\delta,e}}(\theta_{\delta}) + \beta (1 - \chi_{\omega_{\delta,e}}(\theta_{\delta})) \right) \left| \nabla u_{\delta,e}(\theta_{\delta}) \right|^p dx = \int_{\Omega} \left( \theta \alpha_{1/p} + (1 - \theta) \beta_{1/p} \right)^{1-p} \left| \nabla u \right|^p dx.
$$

This proves assertion (4) for $u$, $\theta$ smooth and $\int_\Omega \theta \, dx < \kappa$. The general result follows by density. \qed
Remark 2.2. We can express problem (2.3) in a simpler way defining
\[ c := \left( \frac{\beta}{\alpha} \right)^{\frac{1}{p-1}} - 1 > 0, \quad \tilde{f} := \frac{f}{\beta}, \] (2.16)
which provides
\[
\begin{aligned}
\min_{\theta, u} \left\{ \frac{1}{p} \int_{\Omega} \frac{|\nabla u|^p}{(1 + c \theta)^{p-1}} dx - <\tilde{f}, u> \right\} \\
u \in W^{1, p}_0(\Omega), \quad \theta \in L^\infty(\Omega; [0, 1]), \quad \int_{\Omega} \theta dx \leq \kappa.
\end{aligned}
\] (2.17)

For simplicity, in the following we will redefine \( f \) as \( \tilde{f} \).

3. **Uniqueness results and optimality conditions for the relaxed problem**

Since in problem (2.17) the cost functional is not strictly convex, the uniqueness of solution is not clear. However, let us prove in Proposition 3.1 that the flux
\[ \hat{\sigma} := \frac{|\nabla \hat{u}|^{p-2}}{(1 + c \hat{\theta})^{p-1}} \nabla \hat{u}, \] (3.1)
with \((\hat{u}, \hat{\theta})\) a solution of (2.17) is uniquely defined. The result follows from a dual formulation of (2.17) as a min-max problem. In the case \( p = 2 \), a similar result has been obtained in [26].

**Proposition 3.1.** For every solution \((\hat{u}, \hat{\theta}) \in W^{1, p}_0(\Omega) \times L^\infty(\Omega; [0, 1])\) of (2.17), the flux \( \hat{\sigma} \) defined by (3.1) is the unique solution of
\[
\begin{aligned}
\min_{-\text{div} \sigma = f} \quad \max_{\sigma \in L^p(\Omega)^N} \quad \int_{\Omega} (1 + c \theta) |\sigma|^p dx.
\end{aligned}
\] (3.2)

The function \( \hat{\theta} \) solves the problem
\[
\begin{aligned}
\max_{\theta \in L^\infty(\Omega; [0, 1])} \quad \min_{-\text{div} \sigma = f} \quad \int_{\Omega} (1 + c \theta) |\sigma|^p dx,
\end{aligned}
\] (3.3)
and the minimum value in (3.2) agrees with the maximum in (3.3).

**Proof.** For \( \theta \in L^\infty(\Omega; [0, 1]) \), we define \( \sigma_\theta \in L^p(\Omega)^N \) as the unique solution of
\[
\begin{aligned}
\min_{-\text{div} \sigma = f} \quad \int_{\Omega} (1 + c \theta) |\sigma|^p dx.
\end{aligned}
\]
The uniqueness of $\sigma_\theta$ is ensured by the strictly convexity of the problem. Then, taking into account that $\sigma_\theta$ satisfies
\[ p' \int_\Omega (1 + c\theta)|\sigma_\theta|^{p'-2}\sigma_\theta \cdot \eta dx = 0, \quad \forall \eta \in L^{p'}(\Omega), \text{ with } \text{div} \eta = 0, \]
we deduce the existence of $u_\theta \in W^{1,p}_0(\Omega)$ such that $(1 + c\theta)|\sigma_\theta|^{p'-2}\sigma_\theta = \nabla u_\theta$ in $\Omega$. Using also that $-\text{div} \sigma_\theta = f$ in $\Omega$, we get that $u_\theta$ is the unique solution of
\[ -\text{div} \left( \frac{|\nabla u_\theta|^{p-2}}{(1 + c\theta)^{p-1}} \nabla u_\theta \right) = f \quad \text{in } \Omega, \quad u_\theta \in W^{1,p}_0(\Omega), \]
or equivalently, of the minimization problem
\[ \min_{u \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} \int_\Omega \frac{|\nabla u|^p}{(1 + c\theta)^{p-1}} dx - \langle f, u \rangle \right\}, \]
which combined with
\[ \frac{1}{p} \int_\Omega \frac{|\nabla u_\theta|^p}{(1 + c\theta)^{p-1}} dx - \langle f, u_\theta \rangle = - \frac{1}{p'} \int_\Omega (1 + c\theta)|\sigma_\theta|^{p'} dx, \]
proves that $(\hat{u}, \hat{\theta})$ is a solution of (2.17) if and only if $\hat{\sigma}$ is a solution of the max-min problem (3.3), and $(\hat{\theta}, \hat{\sigma})$, with $\hat{\sigma}$ defined by (3.1), is a saddle point. From the von Neumann Min-Max Theorem [31, Theorem 2.G and Proposition 1 in Chapter 2], we get that the minimum in (3.2) agrees with the maximum in (3.3), and that $\hat{\sigma}$ is a solution of (3.2). Taking into account that the functional
\[ \sigma \in L^{p'}(\Omega)^N \mapsto \max_{\theta \in L^{\infty}(\Omega; [0,1])} \int_\Omega (1 + c\theta)|\sigma|^{p'} dx \]
is strictly convex, as a maximum of a family of strictly convex functions, we deduce the uniqueness of $\hat{\sigma}$. \qed

The following theorem provides a system of optimality conditions for the convex problem (2.3). It proves in particular that $\hat{u}$ is the solution of a nonlinear Calculus of Variations problem which does not contain the proportion $\hat{\theta}$. We refer to Section 4 in [15] for a related result in the case $p = 2$.

**Theorem 3.1.** A pair $(\hat{u}, \hat{\theta}) \in W^{1,p}_0(\Omega) \times L^{\infty}(\Omega; [0,1])$ is a solution of (2.17) if and only if there exists $\hat{\mu} \geq 0$ such that $\hat{u}$ is a solution of
\[ \min_{u \in W^{1,p}_0(\Omega)} \left( \int_\Omega F(|\nabla u|) dx - \langle f, u \rangle \right), \quad (3.4) \]
with $F \in C^{1}((0, \infty)) \cap W^{1, \infty}_{loc}(0, \infty)$, the convex function defined by

$$F(0) = 0, \quad F'(s) = \begin{cases} s^{p-1} & \text{if } 0 \leq s < \hat{\mu} \\ \hat{\mu}^{p-1} & \text{if } \hat{\mu} \leq s \leq (1 + c)\hat{\mu} \\ s^{p-1} & \text{if } (1 + c)\hat{\mu} < s, \end{cases}$$

(3.5)

and $\hat{\mu}, \hat{\theta}$ are related by

- If $\hat{\mu} = 0$ then

$$\hat{\theta} = 1 \text{ a.e. in } \{|\nabla \hat{u}| > 0\}, \quad \int_{\Omega} \hat{\theta} \, dx \leq \kappa. \quad (3.6)$$

- If $\hat{\mu} > 0$, then

$$\hat{\theta} = \begin{cases} 0 & \text{if } 0 \leq |\nabla \hat{u}| < \hat{\mu} \\ \frac{1}{c} \left( \frac{|\nabla \hat{u}|}{\hat{\mu}} - 1 \right) & \text{if } \hat{\mu} \leq |\nabla \hat{u}| < (1 + c)\hat{\mu} \\ 1 & \text{if } (1 + c)\hat{\mu} < |\nabla \hat{u}|, \end{cases} \quad \int_{\Omega} \hat{\theta} \, dx = \kappa. \quad (3.7)$$

Proof. Applying Kuhn-Tucker’s theorem to the convex problem (2.3), we get that $(\hat{u}, \hat{\theta})$ is a solution if and only if there exists $\hat{\mu} \geq 0$ such that $(\hat{u}, \hat{\theta})$ solves

$$\min_{u \in W^{1, p}_{0}(\Omega)} \left\{ \int_{\Omega} \left( \frac{|\nabla u|^{p}}{p \,(1 + c \theta)^{p-1}} + \frac{c \hat{\mu}^{p}}{p} \theta \right) \, dx - \langle f, u \rangle \right\}, \quad (3.8)$$

and

$$\int_{\Omega} \hat{\theta} \, dx \leq \kappa, \quad \hat{\mu} \left( \int_{\Omega} \hat{\theta} \, dx - \kappa \right) = 0. \quad (3.9)$$

Differentiating in (3.8) we have that $(\hat{u}, \hat{\theta})$ is a solution of (3.8) if and only if

$$\int_{\Omega} \frac{|\nabla \hat{u}|^{p-2} \nabla \hat{u} \cdot \nabla \hat{v}}{(1 + c \theta)^{p-1}} \, dx = \langle f, v \rangle, \quad \forall v \in W^{1, p}_{0}(\Omega), \quad (3.10)$$

$$\int_{\Omega} \left( \hat{\mu}^{p} - \frac{|\nabla \hat{u}|^{p}}{(1 + c \theta)^{p}} \right) (\theta - \hat{\theta}) \, dx \geq 0, \quad \forall \theta \in L^{\infty}(\Omega; [0, 1]). \quad (3.11)$$

Condition (3.10) is equivalent to $\hat{u}$ solution of the minimum problem

$$\min_{u \in W^{1, p}_{0}(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} \frac{|\nabla u|^{p}}{(1 + c \theta)^{p-1}} \, dx - \langle f, u \rangle \right\}, \quad (3.12)$$
while (3.11) is equivalent to \( \hat{\theta} \) satisfying (3.6) or (3.7) depending on whether \( \hat{\mu} = 0 \) or \( \hat{\mu} > 0 \). Replacing this value of \( \hat{\theta} \) in (3.8) we have the equivalence between (3.12) and (3.4).

\[ \square \]

Remark 3.1. Using (3.6) or (3.7) and expression (3.1) of \( \hat{\sigma} \), we have that \( \hat{\theta} \) satisfies

\[
\hat{\theta}(x) = \begin{cases} 
1 & \text{if } |\hat{\sigma}| > \hat{\mu} \\
0 & \text{if } |\hat{\sigma}| < \hat{\mu}.
\end{cases}
\]  

(3.13)

Moreover, Theorem 3.1 implies \( \hat{\mu} = 0 \) if and only if the unique solution \( \tilde{u} \) of

\[
\min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} \frac{|
abla u|^p}{(1 + c)^{p-1}} \, dx - \langle f, u \rangle \right\},
\]

satisfies

\[ |\{ x \in \Omega : |
\nabla \tilde{u}| > 0 \} | \leq \kappa, \]

where in this case \( \tilde{u} = \hat{u} \).

4. Regularity for the relaxed problem

In the present section we study the regularity of the solutions of problem (2.17). As a consequence we show that the unrelaxed problem (2.2) has no solution in general. We begin by stating the main results. The corresponding proofs are given later.

Theorem 4.1. Let \( \Omega \subset \mathbb{R}^N \) be a \( C^{1,1} \) bounded open set and \((\hat{u}, \hat{\theta}) \in W_0^{1,p}(\Omega) \times L^\infty(\Omega; [0, 1])\) be a solution of (2.17), then, for \( \hat{\sigma} \) defined by (3.1) and \( \hat{\mu} \) given by Theorem 3.1 we have:

1. If \( f \in W^{-1,q}(\Omega) \), \( p' \leq q < \infty \), then \( \nabla \hat{u} \in L^q(\Omega)^N \) and there exists \( C > 0 \), which only depends on \( p, q, N, \hat{\mu} \) and \( \Omega \) such that

\[
\|\nabla \hat{u}\|_{L^q(\Omega)^N} \leq C \left( \|f\|_{W^{-1,q}(\Omega)}^{\frac{1}{p'q}} + \hat{\mu} \right).
\]  

(4.1)

2. If \( f \in L^q(\Omega) \) with \( q > N \), then there exists \( C > 0 \) which only depends on \( p, q, N \) and \( \Omega \) such that

\[
\|\nabla \hat{u}\|_{L^\infty(\Omega)^N} \leq C \left( \|f\|_{L^q(\Omega)}^{\frac{1}{p}q} + \hat{\mu} \right).
\]  

(4.2)

3. If \( f \in W^{1,1}(\Omega) \cap L^{2(1+r)}(\Omega) \), with \( r \geq 0 \) or \( f \in W^{1,2(1+r)}(\Omega) \) with \( r \in (-1/2, 0) \), then the function \( |\hat{\sigma}|^r \hat{\sigma} \) is in \( H^1(\Omega)^N \) and there exists \( C > 0 \), which only depends on \( p, q, N, \hat{\mu} \) and \( \Omega \) such that

\[
\|\hat{\sigma}|^r \sigma\|_{H^1(\Omega)^N} \leq \begin{cases} 
C \left( 1 + \|f\|_{W^{1,1}(\Omega)} + \|f\|_{L^{2(1+r)}(\Omega)}^{2(1+r)} \right) & \text{if } r \geq 0 \\
C \left( 1 + \|f\|_{W^{1,2(1+r)}(\Omega)} \right) & \text{if } -\frac{1}{2} < r < 0.
\end{cases}
\]  

(4.3)

Moreover

\[ \hat{\sigma} \text{ is parallel to } \nu \text{ on } \partial \Omega, \]

(4.4)
with \( \nu \) the unitary outside normal to \( \partial \Omega \).

(4) For \( 1 \leq i, j \leq N \) and \( f \in W^{1,1}(\Omega) \cap L^2(\Omega) \)

\[
\partial_i \hat{\sigma}_j - \partial_j \hat{\sigma}_i = (1 + \hat{\theta})(\partial_j \hat{\sigma}_i - \partial_i \hat{\sigma}_j) \chi_{|\sigma| = \hat{\mu}} \in L^2(\Omega). \tag{4.5}
\]

Moreover, if \( \hat{\theta} \) only takes a finite number of values a.e. in \( \Omega \), then

\[
\partial_i \hat{\sigma}_j - \partial_j \hat{\sigma}_i = 0, \quad 1 \leq i, j \leq N,
\]

\[
\text{curl}(|\hat{\sigma}|^{p' - 2} \hat{\sigma}) = 0 \quad \text{in} \quad \Omega. \tag{4.6}
\]

where, for a distribution from \( \Omega \) into \( \mathbb{R}^N \), the curl operator is defined as

\[
\text{curl}(\Phi) := \frac{1}{2} \left( \nabla \Phi - \nabla \Phi^\top \right).
\]

**Remark 4.1.** As in [5] we can also obtain some local regularity results for \( \hat{u}, \hat{\theta} \) and \( \hat{\sigma} \) but, for the sake of simplicity, we have preferred to only state and prove the global regularity result.

**Remark 4.2.** If we assume that \( f \) belongs to \( W^{1,1}(\Omega) \cap L^2(\Omega) \), that the unrelaxed problem \( (2.2) \) has a solution \( (\hat{u}, \hat{\theta}) \), and that \( \Omega \) is simply connected, then \( (4.6) \) proves the existence of \( w \in W^{1,p}(\Omega) \) such that \( \hat{\sigma} = |\nabla w|^{p - 2} \nabla w \) a.e. in \( \Omega \). By \( (4.4) \), we must also have \( \hat{u} \) constant in each connected component of \( \partial \Omega \). Assuming then that \( \partial \Omega \) has only a connected component and taking into account that \( w \) is defined up to an additive constant, we get

\[
\hat{\sigma} = |\nabla w|^{p - 2} \nabla w, \quad w \text{ solution of } \begin{cases} 
- \text{div} \left( |\nabla w|^{p - 2} \nabla w \right) = f \quad \text{in} \ \Omega \\
 w = 0 \quad \text{on} \ \partial \Omega. 
\end{cases} \tag{4.7}
\]

We will show that this implies that the unrelaxed problem has no solution in general.

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^N \) be a connected open set of class \( C^{1,1} \) with connected boundary and \( f = 1 \). If there exists a solution of \( (1.1) \), then \( \Omega \) is a ball.

**Remark 4.3.** In the case \( p = 2 \), Theorem 4.2 has been proved in [26] assuming that \( (1.1) \) has a smooth solution and in [5] in the general case.

The proof of Theorem 4.1 will follow from the following Lemma.

**Lemma 4.1.** Let \( \Omega \subset \mathbb{R}^N \) be a \( C^2 \) bounded open set and \( G : [0, \infty) \to [0, \infty) \) be a \( C^1 \) function such that there exist \( \lambda, \mu > 0 \) and \( p > 1 \) satisfying

\[
G(s) = s^{p - 2}, \quad \forall s \geq \mu, \tag{4.8}
\]

\[
0 \leq G(s) + G'(s)s, \quad G(s) \leq \lambda s^{p - 2}, \quad \forall s \geq 0. \tag{4.9}
\]

Let \( u \in C^2(\overline{\Omega}) \) be such that there exists \( f \in C^{1,1}(\overline{\Omega}) \) satisfying

\[
- \text{div} \left( G(|\nabla u|) \nabla u \right) = f \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega. \tag{4.10}
\]

Then, the following estimates hold:
(1) For every \( q \in (p', \infty) \), there exists \( C > 0 \) depending only on \( p, q \) and \( \Omega \), such that
\[
\| \nabla u \|_{L^q(p^{-1}, \Omega)^N} \leq C \left( \| f \|_{W^{1, p}((\Omega)}^{\frac{1}{p-1}} + \mu \right).
\] (4.11)

(2) For every \( q > N \) there exists \( C > 0 \) depending only on \( p, q \) and \( \Omega \) such that
\[
\| \nabla u \|_{L^\infty(\Omega)^N} \leq C \left( \| f \|_{W^{1, p}((\Omega)}^{\frac{1}{p-1}} + \mu \right).
\] (4.12)

(3) For every \( \gamma > -1 \), there exists \( C > 0 \) depending only on \( p, N, \lambda, \gamma \) and \( \Omega \) such that
\[
\int_{\Omega} |\nabla u|^\gamma \left( \frac{G'(|\nabla u|)}{|\nabla u|} |\nabla^2 u| \right) dx \leq C \mu^{p+\gamma} + C \mu^{1+\gamma} \| f \|_{W^{1, p}((\Omega)}^{\frac{p+\gamma}{p}} \| f \|_{L^\infty(\Omega)^N}^{\gamma},
\] if \( \gamma \geq p - 2 \), (4.13)

\[
\int_{\Omega} |\nabla u|^\gamma \left( \frac{G'(|\nabla u|)}{|\nabla u|} |\nabla^2 u| \right) dx \leq C \mu^{p+\gamma} + C \| f \|_{W^{1, p}}^{\frac{p+\gamma}{p}} \| f \|_{W^{1, p}}^{\gamma},
\] if \( -1 < \gamma < p - 2 \). (4.14)

Proof. In order to prove (4.11), we write (4.10) as
\[
- \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f - \text{div} \left( |\nabla u|^{p-2} \nabla u - G(|\nabla u|) \nabla u \right) \quad \text{in} \ \Omega,
\]
where the last term in the right-hand side is bounded in \( W^{-1, \infty}(\Omega) \) by \( C \mu^{p-1} \). Then the result follows from Theorem 2.3 in [21].

For the rest of the proof let us differentiate equation (4.10) with respect to \( x_i \). This gives
\[
- \text{div} \left( L \nabla \partial_i u \right) = \partial_i f \quad \text{in} \ \Omega,
\] (4.15)
with
\[
L = \frac{G'(|\nabla u|)}{|\nabla u|} \nabla u \otimes \nabla u + G(|\nabla u|) I.
\] (4.16)

Observe that \( L \) is non-negative thanks to (4.9).

In order to estimate \( \partial_i u \) from (4.15), we also need to add some boundary conditions. For this purpose, fixed \( \bar{x} \in \partial \Omega \), we use that there exist \( \delta > 0 \) and functions \( \tau^1, \ldots, \tau^N \in C^1(B(\bar{x}, \delta))^N \) such that for every \( x \in B(\bar{x}, \delta) \)
\[
\{ \tau^1(x), \ldots, \tau^N(x) \} \quad \text{is an orthonormal basis of} \ \mathbb{R}^N,
\]
\[
\tau^N(x) \quad \text{agrees with the unitary outside normal vector to} \ \Omega \ \text{on} \ \partial \Omega \cap B(\bar{x}, \delta).
\] (4.17)

Using that
\[
\nabla u = \sum_{i=1}^N (\nabla u \cdot \tau^i) \tau^i \quad \text{a.e. in} \ B(\bar{x}, \delta),
\]
and \((4.10)\), we get
\[
- \sum_{i=1}^{N} \text{div}(G(|\nabla u|^{i}) \nabla u \cdot \tau^{i}) - \sum_{i=1}^{N} \nabla(\nabla u \cdot \tau^{i}) \cdot \tau^{i} G(|\nabla u|) = f \quad \text{in } \Omega, \tag{4.18}
\]
where thanks to \(u\) vanishing on \(\partial \Omega\), we have
\[
\nabla u = (\nabla u \cdot \tau^{N}) \tau^{N}, \quad \nabla u \cdot \tau^{i} = 0, \quad \nabla(\nabla u \cdot \tau^{i}) \cdot \tau^{i} = 0 \quad \text{on } \partial \Omega, \quad 1 \leq i \leq N - 1.
\]
Thus, developing (4.18), we get
\[
-L \nabla^{2} u \tau^{N} \cdot \tau^{N} = f + G(|\nabla u|) \left( \text{div } \tau^{N} I + \left( \nabla \tau^{N} \right)^{t} \right) \tau^{N} \cdot \nabla u \quad \text{on } \partial \Omega \cap B(\bar{x}, \delta).
\]
By the arbitrariness of \(\bar{x}\), we then deduce the existence of a vector function \(h \in L^{\infty}(\partial \Omega)^{N}\), which only depends on \(\Omega\), such that \(\nabla u\) satisfies the boundary conditions
\[
\left\{
\begin{array}{l}
\nabla u = |\nabla u| s \nu, \quad s \in \{0, 1\} \ \text{a.e. on } \partial \Omega, \\
-L \nabla^{2} u \nu \cdot \nu = f + G(|\nabla u|) h \cdot \nabla u \quad \text{on } \partial \Omega,
\end{array}
\right. \tag{4.19}
\]
with \(\nu\) the unitary outside normal on \(\partial \Omega\).

Let us now prove (4.11). We reason similarly to [12]. For
\[
w = |\nabla u|^{2}, \tag{4.20}
\]
and \(k > \mu^{p}\), we multiply (4.15) by \((w^{\frac{p}{2}} - k)^{+} \partial_{i} u \in H^{1}(\Omega)\) and integrate by parts. Adding in \(i\) and taking into account (4.19), we get
\[
\frac{p}{4} \int_{\{w^{\frac{p}{2}} \geq k\}} w^{\frac{p-2}{2}} L \nabla w \cdot \nabla w \ dx + \sum_{i=1}^{N} \int_{\Omega} (w^{\frac{p}{2}} - k)^{+} L \nabla \partial_{i} u \cdot \nabla \partial_{i} u \ dx
\]
\[
= - \int_{\partial \Omega} s |\nabla u| (f + G(|\nabla u|) h \cdot \nabla u) (w^{\frac{p}{2}} - k)^{+} \ ds(x) + \int_{\Omega} \nabla f \cdot \nabla u (w^{\frac{p}{2}} - k)^{+} \ dx
\]
\[
= - \int_{\partial \Omega} s |\nabla u| G(|\nabla u|) h \cdot \nabla u (w^{\frac{p}{2}} - k)^{+} \ ds(x) - \int_{\Omega} f \Delta u (w^{\frac{p}{2}} - k)^{+} \ dx
\]
\[
- \frac{p}{2} \int_{\{w^{\frac{p}{2}} \geq k\}} w^{\frac{p-2}{2}} f \nabla u \cdot \nabla w \ dx,
\]
which thanks to \(k > \mu\), (4.8) and (4.16) proves
\[
\int_{\{w^{\frac{p}{2}} \geq k\}} w^{p-2} |\nabla w|^{2} \ dx + \int_{\Omega} (w^{\frac{p}{2}} - k)^{+} w^{\frac{p-2}{2}} |\nabla^{2} u|^{2} \ dx
\]
\[
\leq C \int_{\partial \Omega} (w^{\frac{p}{2}} - k)^{+} \ dx + C \int_{\Omega} |f| |\nabla^{2} u| (w^{\frac{p}{2}} - k)^{+} \ dx + C \int_{\{w^{\frac{p}{2}} \geq k\}} w^{\frac{p-1}{2}} |f| |\nabla w| \ dx,
\]
and then, using Young’s inequality
\[
\int_{\{w^p \geq k\}} w^{p-2} |\nabla w|^2 \, dx + \int_{\Omega} (w^p - k)^+ w^{\frac{p-2}{p}} |\nabla u|^2 \, dx \\
\leq C \int_{\partial \Omega} w^p (w^p - k)^+ \, ds(x) + C \int_{\{w^p \geq k\}} |f|^2 w \, dx.
\] (4.21)

In the first term on the right-hand side we use that, thanks to the compact embedding of \( W^{1,1}(\Omega) \) into \( L^1(\partial \Omega) \), for every \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
\int_{\partial \Omega} |v| ds(x) \leq C_\varepsilon \int_{\Omega} |v| \, dx + \varepsilon \int_{\Omega} |\nabla v| \, dx, \quad \forall v \in W^{1,1}(\Omega).
\]
Therefore there exists a constant \( C \) depending on \( p \) and \( \epsilon \) such that
\[
\int_{\partial \Omega} w^p (w^p - k)^+ \, ds(x) \leq C \int_{\Omega} w^p (w^p - k)^+ \, dx + \varepsilon \int_{\{w^p \geq k\}} w^{p-1} |\nabla w| \, dx.
\]
Replacing this inequality in (4.21), taking \( \varepsilon \) small enough, and using Young’s inequality, we get
\[
\int_{\{w^p \geq k\}} w^{p-2} |\nabla w|^2 \, dx \leq C \int_{\{w^p \geq k\}} w^p \, dx + C \int_{\{w^p \geq k\}} |f|^2 \, dx,
\]
which by Sobolev’s inequality and \( f \) in \( L^q(\Omega) \) provides
\[
\left( \int_{\Omega} |(w^p - k)^+|^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C \int_{\{w^p \geq k\}} w^p \, dx + C \|f\|_{L^q(\Omega)}^2 \left( \int_{\{w^p \geq k\}} w^{\frac{q}{2}} \, dx \right)^{\frac{4-2}{q}},
\] (4.22)
with
\[
2^* = \frac{2N}{N-2} \quad \text{if} \quad N > 2, \quad 2^* \in (2, \infty) \quad \text{if} \quad N = 2.
\]
Now, we use that \( q > N \) allows us to take \( r > 1 \) large enough to have
\[
\frac{2^*}{2} \left( \frac{q-2}{q} - \frac{1}{r} \right) > 1, \quad \frac{2^*}{2} \left( 1 - \frac{p}{r} \right) > 1.
\]
For such \( r \), we use Hölder’s inequality in (4.22) to get
\[
\left( \int_{\Omega} |(w^p - k)^+|^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq C \left( \int_{\Omega} w^r \, dx \right)^{\frac{2}{r}} \left\{ w^p \geq k \right\}^{1-\frac{2}{r}} + C \|f\|_{L^q(\Omega)}^2 \left( \int_{\Omega} w^r \, dx \right)^{\frac{1}{r}} \left\{ w^p \geq k \right\}^{\frac{q-2}{q}-\frac{1}{r}}
\]
which by (4.11) with \( q = 2r/(p-1) \) and
\[
\|f\|_{W^{1,\frac{2r}{p-1}}(\Omega)} \leq C \|f\|_{L^q(\Omega)},
\]
implies
\[
\left( \int_{\Omega} \left( |w^p - k|^2 + k^2 \right)^{2^*} dx \right)^{\frac{2}{2^*}} \leq C \left( \left\| f \right\|_{L^p(\Omega)}^{\frac{1}{p}} + \mu \right)^{2p} \min \left( \left\{ w^p \right\}_{k} \right)^{\min \left( -\frac{2}{q}, -\frac{1}{r} \right)}.
\]

Taking \( h > k \) and defining \( \varphi \) by
\[
\varphi(k) = \left\{ w^p \geq k \right\},
\]
we have then proved
\[
\varphi(h)^{\frac{2}{2^*}} \leq C \left( \left\| f \right\|_{L^p(\Omega)}^{\frac{1}{p}} + \mu \right)^{2p} \varphi(k)^{\min \left( -\frac{2}{q}, -\frac{1}{r} \right)}, \quad \text{for } h > k \geq \mu^p,
\]
where \( C \) only depends on \( p, N \), and \( \Omega \). Lemma 4.1 in [29] then proves (4.12).

Let us now prove (4.13). Defining \( w \) by (4.20), we take \( (w + \varepsilon)^{\frac{2}{2^*}} \partial_i u \), with \( \varepsilon > 0 \), \( \gamma > -1 \), as test function in (4.10). Using (4.19), we get
\[
\frac{\gamma}{4} \int_{\Omega} \left( w + \varepsilon \right)^{\frac{2-2}{2^*}} L \nabla w \cdot \nabla w dx + \sum_{i=1}^{N} \int_{\Omega} \left( w + \varepsilon \right)^{\frac{2}{2^*}} L \nabla \partial_i u \cdot \nabla \partial_i u dx = - \int_{\partial \Omega} s |\nabla u| (f + G(|\nabla u|)h \cdot \nabla u) (w + \varepsilon)^{\frac{2}{2^*}} ds(x) + \int_{\Omega} \nabla f \cdot \nabla w \varepsilon^{\frac{2}{2^*}} dx.
\]
In this inequality, we observe that the integrand in the left-hand side is nonnegative due to
\[
2w \sum_{i=1}^{N} L \nabla \partial_i u \cdot \nabla \partial_i u - L \nabla w \cdot \nabla w = 2|\nabla u|^2 \sum_{i=1}^{N} L \nabla \partial_i u \cdot \nabla \partial_i u - 2L(\nabla^2 u \nabla u) \cdot (\nabla^2 u \nabla u) \geq 0 \quad \text{a.e. in } \Omega,
\]
and \( \gamma > -1 \). This allows us to use the Fatou Lemma on the left-hand side and the dominated convergence theorem on the right-hand side, when \( \varepsilon \) tends to zero, to deduce
\[
\frac{\gamma}{4} \int_{\Omega} \left( w + \varepsilon \right)^{\frac{2-2}{2^*}} L \nabla w \cdot \nabla w dx + \sum_{i=1}^{N} \int_{\Omega} \left( w + \varepsilon \right)^{\frac{2}{2^*}} L \nabla \partial_i u \cdot \nabla \partial_i u dx \leq - \int_{\partial \Omega} s |\nabla u| (f + G(|\nabla u|)h \cdot \nabla u) w^{\frac{2}{2^*}} ds(x) + \int_{\Omega} \nabla f \cdot \nabla w w^{\frac{2}{2^*}} dx.
\]
Let us first consider the case $\gamma \geq p - 2$. Defining $T \in W^{1,\infty}(0, \infty)$ by

$$T(s) = \begin{cases} 
0 & \text{if } 0 \leq s \leq \mu^2 \\
\frac{s}{\mu^2} - 1 & \text{if } \mu^2 \leq s \leq 2\mu^2 \\
1 & \text{if } s \geq 2\mu^2,
\end{cases}$$

we decompose the last term in (4.25) as

$$\int_\Omega \nabla f \cdot \nabla u \, \frac{\partial h}{\partial t} \, dx = \int_\Omega \nabla f \cdot (1 - T(w)) \nabla u \, \frac{\partial h}{\partial t} \, dx + \int_\Omega \nabla f \cdot T(w) \nabla u \, \frac{\partial h}{\partial t} \, dx.$$

Integrating by parts the last term, replacing in (4.25) and using Young’s inequality, $h \in L^\infty(\partial \Omega)$, and (4.8), we deduce

$$\int_{\partial \Omega} w^{2-2} \, dx + \sum_{i=1}^N \int_\Omega w^2 \nabla \partial_i u \cdot \nabla \partial_i \gamma u \, dx \leq \mu^{1+\gamma} \int_{\partial \Omega} |f| \, ds(x)$$

$$+ C \int_{\partial \Omega} w^{\frac{p-2}{2}} \, ds(x) + \mu^{1+\gamma} \int_\Omega |\nabla f| \, dx + C \int_{\partial \Omega} w^{\frac{p+2}{2}} \, dx + C \mu^{1+\gamma} \int_\Omega |f| \, dx.$$  \hspace{1cm} (4.26)

For the second term on the right-hand side we use the continuous embedding of $W^{1,1}(\Omega)$ into $L^1(\partial \Omega)$ and Young’s inequality to get

$$\int_{\partial \Omega} w^{\frac{p-2}{2}} \, dx \leq C \mu^{p+\gamma} + \int_{\partial \Omega} |(w - \mu^2)^{\frac{p+\gamma}{2}}| \, dx$$

$$\leq C \mu^{p+\gamma} + C \int_\Omega w^{\frac{p-2}{2}} \, dx + C \int_{\{w > \mu^2\}} w^{\frac{p+2}{2}} \, dx$$

$$\leq C \mu^{p+\gamma} + C \left(1 + \frac{1}{\delta}\right) \int_\Omega w^{\frac{p-2}{2}} \, dx + C \delta \int_{\{w > \mu^2\}} w^{\frac{p+2}{2}} \, dx,$$  \hspace{1cm} (4.27)

with $\delta > 0$ arbitrary. Taking $\delta$ small enough, replacing in (4.26) and using Hölder’s inequality we have

$$\int_{\partial \Omega} w^{\frac{p-2}{2}} \, dx + \sum_{i=1}^N \int_\Omega w^2 \nabla \partial_i u \cdot \nabla \partial_i \gamma u \, dx \leq \mu^{1+\gamma} \int_{\partial \Omega} |f| \, ds(x)$$

$$+ C \mu^{p+\gamma} + C \int_\Omega w^{\frac{p-2}{2}} \, dx + \mu^{1+\gamma} \int_\Omega |\nabla f| \, dx + C \int_\Omega |f| \, dx.$$  \hspace{1cm} (4.28)

Using (4.11) with $q = \frac{p+2}{p-1}$ and the continuous imbedding of $L^q(\Omega)$ into $W^{-1,q}(\Omega)$, combined with (4.24) and

$$\sum_{i=1}^N L \nabla \partial_i u \cdot \nabla \partial_i u = \frac{G'(|\nabla u|)}{|\nabla u|} |\nabla^2 u \nabla u|^2 + G(|\nabla u|) |D^2 u|^2, \text{ a.e. in } \Omega,$$  \hspace{1cm} (4.29)
we conclude \((4.13)\).

We now assume \(-1 < \gamma < p - 2\). In this case we estimate the right-hand side in \((4.25)\) as follows:

For the first term, using \((4.27)\), we have for \(\delta < 1\)

\[
\left| \int_{\partial \Omega} s|\nabla u| (f + G(|\nabla u|) h \cdot \nabla u) w^2 \, ds(x) \right| \leq C \int_{\partial \Omega} (|f| w^{\frac{\gamma + 1}{p}} + w^{\frac{\gamma + 2}{p}}) \, ds(x) 
\]

\[
\leq C \int_{\partial \Omega} |f| w^{\frac{\gamma + 2}{p}} \, ds(x) + C \int_{\partial \Omega} w^{\frac{\gamma + 2}{p}} \, ds(x) \tag{4.29}
\]

\[
\leq C \int_{\partial \Omega} |f| w^{\frac{\gamma + 2}{p}} \, ds(x) + C \mu^{p+\gamma} + C \delta \int_{\{w \geq \mu^2\}} w^{\frac{\gamma + 2}{p} - 4} |\nabla w|^2 \, dx.
\]

For the second term on the right-hand side of \((4.25)\), we just use Hölder’s inequality to get

\[
\left| \int_{\Omega} \nabla f \cdot \nabla u w^2 \, dx \right| \leq C \int_{\Omega} |\nabla f|^{\frac{p-1}{p}} \, dx + C \int_{\Omega} w^{\frac{\gamma + 2}{p}} \, dx. \tag{4.30}
\]

Using \((4.29)\) with \(\delta\) small enough, and \((4.30)\) in \((4.25)\), and then using \((4.11)\) with \(q = \frac{p+\gamma}{p-1}\), we conclude \((4.14)\). \(\square\)

**Remark 4.4.** Since the constant in the previous theorem only depends on the norm in \(L^\infty\) of the first derivative of the functions \(\{r^i\}_{i=1}^N\) defined in \((4.17)\), we can relax the conditions \(u \in C^2(\Omega)\) and \(f \in C^{1,1}(\Omega)\) and \(\Omega\) of class \(C^{1,1}\) by a density argument.

**Remark 4.5.** As a simple case, Lemma \((4.7)\) can be applied to the \(p\)-Laplacian operator, \(G(s) = |s|^{p-2} s\). Indeed, since here \(\mu = 0\) it is simple to check that the proof above does not use the assumption \(f \in W^{1,1}(\Omega)\) in \((4.13)\). Thus, it shows that for \(f \in W^{-1,p'}(\Omega) \cap L^{\frac{2}{p-1}}(\Omega)\), if \(\gamma \geq p - 2\) or \(f \in W^{-1,p'}(\Omega) \cap W^{1,\frac{2}{p-1}}(\Omega)\) if \(-1 < \gamma < p - 2\), there exists a solution \(u\) of \((4.10)\) such that

\[
|\nabla u|^{\frac{p+\gamma}{2}} \, |\nabla^2 u| \text{ belongs to } L^2(\Omega),
\]

i.e. \(|\nabla u|^{\frac{p+\gamma}{2}}\) belongs to \(H^1(\Omega)\). In particular, it proves that \(u\) belongs to \(H^2(\Omega)\) if \(p < 3\) and \(f\) belongs to \(W^{1,\frac{2}{p-1}}(\Omega)\). This is a known result which can be found in \([11]\). It also proves that for \(f \in L^{2(1+r)}(\Omega)\) if \(r \geq 0\), or \(f \in W^{1,2(1+r)}(\Omega)\) if \(-1/2 < r < 0\) the flux \(\sigma = |\nabla u|^{p-2} \nabla u\) satisfies that \(|\sigma|^r D\sigma\) belongs to \(L^2(\Omega)^{N \times N}\), or equivalently, that \(|\sigma|^r \sigma\) belongs to \(H^1(\Omega)^N\). The case \(r = 0\) has been proved in \([20]\).

**Proof of Theorem 4.7.** Let us assume the right-hand side \(f\) in \((2.17)\) smooth enough, which by \(\hat{u}\) solution of \((3.4)\) implies that \(\hat{u} \in C^{0,\alpha}(\Omega)\) for some \(\alpha > 0\) (see e.g. \([12]\)) and satisfies

\[
- \text{div} \left( \frac{F'(|\nabla \bar{u}|) |\nabla \bar{u}|}{|\nabla \bar{u}|} \nabla \bar{u} \right) = f \text{ in } \Omega, \quad \bar{u} \in W_0^{1,p}(\Omega). \tag{4.31}
\]
For $\varepsilon > 0$ small and $F$ defined by (3.5), we take $F_\varepsilon : [0, \infty) \to [0, \infty)$ of class $C^2([0, \infty))$ such that for some $k > 0$, it satisfies

$$
\begin{cases}
F_\varepsilon(0) = 0, & F_\varepsilon'(s) \geq \frac{s^{p-1}}{2(1 + c)p - 1}, \quad \varepsilon \leq F_\varepsilon''(s) \leq \varepsilon + ks^{p-2}, \quad \forall s \geq 0, \\
F_\varepsilon(s) = F(s), \quad \forall s \geq (1 + c)\hat{\mu}, \quad \lim_{\varepsilon \to 0} \|F_\varepsilon - F\|_{L^\infty([0, \infty))} = 0.
\end{cases}
$$

(4.32)

The existence of this approximation is ensured by Theorem 2.1 and Remark 3.1 in [13]. Then, we define $u_\varepsilon$ as the unique solution of

$$
\min_{u \in W^{1,p}_0(\Omega) \cap L^2(\Omega)} \left\{ \int_{\Omega} F_\varepsilon(|\nabla u|) dx + \frac{1}{2} \int_{\Omega} |u - \hat{u}|^2 dx - \int_{\Omega} fu dx \right\}.
$$

(4.33)

and therefore

$$
- \text{div} \left( \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon \right) + u_\varepsilon - \hat{u} = f \quad \text{in} \ \Omega.
$$

(4.34)

Since

$$\int_{\Omega} F_\varepsilon(|\nabla u_\varepsilon|) dx + \frac{1}{2} \int_{\Omega} |u_\varepsilon - \hat{u}|^2 dx - \int_{\Omega} fu_\varepsilon dx \leq \int_{\Omega} F_\varepsilon(|\nabla \hat{u}|) dx - \int_{\Omega} f \hat{u} dx,$$

we have that $u_\varepsilon$ is bounded in $W^{1,p}_0(\Omega) \cap L^2(\Omega)$ and thus, up to a subsequence, it converges weakly in $W^{1,p}_0(\Omega) \cap L^2(\Omega)$ to a certain function $u_0$. Taking into account the uniform convergence of $F_\varepsilon$ to $F$, and $F$ convex, we can pass to the limit in the above inequality to deduce

$$\int_{\Omega} F(|\nabla u_0|) dx + \frac{1}{2} \int_{\Omega} |u_0 - \hat{u}|^2 dx - \int_{\Omega} fu_0 dx \leq \liminf_{\varepsilon \to 0} \left( \int_{\Omega} F_\varepsilon(|\nabla u_\varepsilon|) dx + \frac{1}{2} \int_{\Omega} |u_\varepsilon - \hat{u}|^2 dx - \int_{\Omega} fu_\varepsilon dx \right)$$

$$\leq \int_{\Omega} F(|\nabla \hat{u}|) dx - \int_{\Omega} f \hat{u} dx,$$

which combined with $\hat{u}$ solution of (3.4) shows $u_0 = \hat{u}$ and

$$\lim_{\varepsilon \to 0} \int_{\Omega} F(|\nabla u_\varepsilon|) dx = \lim_{\varepsilon \to 0} \int_{\Omega} F_\varepsilon(|\nabla u_\varepsilon|) dx = \int_{\Omega} F(|\nabla \hat{u}|) dx.
$$

(4.35)

On the other hand, the assumptions of $F_\varepsilon$ imply that

$$\sigma_\varepsilon := \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon$$

is bounded in $L^p(\Omega)^N$, and then by (4.34), for a subsequence, there exists $\sigma_0 \in L^p(\Omega)^N$ such that

$$\sigma_\varepsilon \rightharpoonup \sigma_0 \quad \text{in} \ L^p(\Omega)^N, \quad -\text{div}(\sigma_0) = f \quad \text{in} \ \Omega.
$$

(4.36)
Taking $V \in L^p(\Omega)^N$ and using the convexity of $F_\varepsilon$, we have
\[
\int_{\Omega} \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon \cdot (V - \nabla u_\varepsilon) \, dx \leq \int_{\Omega} \left( F_\varepsilon(|V|) - F_\varepsilon(|\nabla u_\varepsilon|) \right) \, dx,
\]
which can also be written as
\[
\int_{\Omega} \left( \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon - \frac{F'_\varepsilon(|\nabla \hat{u}|)}{|\nabla \hat{u}|} \nabla (\hat{u} - u_\varepsilon) \right) \cdot \nabla (\hat{u} - u_\varepsilon) \, dx
+ \int_{\Omega} \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon \cdot (V - \nabla \hat{u}) \, dx
\leq \int_{\Omega} \left( F_\varepsilon(|V|) - F_\varepsilon(|\nabla u_\varepsilon|) \right) \, dx.
\]
From (4.31), (4.35) and (4.36) we can pass to the limit in this inequality to deduce
\[
\int_{\Omega} \sigma_0 \cdot (V - \nabla \hat{u}) \, dx \leq \int_{\Omega} \left( F(|V|) - F(|\nabla \hat{u}|) \right) \, dx, \quad \forall V \in L^p(\Omega)^N.
\]
Taking $V = \nabla \hat{u} + tW$, with $W \in L^p(\Omega)^N$, $t > 0$, dividing by $t$ and passing to the limit when $t$ tends to zero, we get
\[
\int_{\Omega} \sigma_0 \cdot W \, dx \leq \int_{\Omega} \frac{F'(|\nabla \hat{u}|)}{|\nabla \hat{u}|} \nabla \hat{u} \cdot W \, dx, \quad \forall W \in L^p(\Omega)^N,
\]
which shows
\[
\sigma_0 = \frac{F'(|\nabla \hat{u}|)}{|\nabla \hat{u}|} \text{ a.e. in } \Omega.
\]
We have thus proved
\[
u_\varepsilon \to \hat{u} \text{ in } W^{1,p}_0(\Omega), \quad \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon \to \frac{F'(|\nabla \hat{u}|)}{|\nabla \hat{u}|} \nabla \hat{u} \text{ in } L^{p'}(\Omega)^N.
\]
Assuming $\Omega \in C^{2,\alpha}$ we can apply for example Theorem 15.12 in [14] to deduce that $u_\varepsilon$ belongs to $C^{2,\alpha}(\Omega)$. On the other hand, we have that $G_\varepsilon \in C^1([0, \infty))$ defined by
\[
G_\varepsilon(s) = \frac{F'_\varepsilon(s)}{s} \text{ if } s > 0, \quad G_\varepsilon(0) = 0,
\]
satisfies
\[
\frac{G'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} |\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2 + G_\varepsilon(|\nabla u_\varepsilon|) |\nabla^2 u_\varepsilon|^2
= \frac{F'_\varepsilon(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \left( |\nabla u_\varepsilon|^2 - \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2} \right)
+ F''_\varepsilon(|\nabla u_\varepsilon|) \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2},
\]
while
\[
|D\sigma_\varepsilon|^2 = \frac{F'_\varepsilon(|\nabla u_\varepsilon|)^2}{|\nabla u_\varepsilon|^2} \left( |\nabla u_\varepsilon|^2 - \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2} \right)
+ F''_\varepsilon(|\nabla u_\varepsilon|)^2 \frac{|\nabla^2 u_\varepsilon \nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2},
\]
Then, the assumptions of $F_\varepsilon$ imply the existence of a constant $C > 0$, which only depends on the constant $k$ in (4.32) such that

$$|D\sigma_\varepsilon|^2 \leq C(\varepsilon + |\nabla u_\varepsilon|^{p-2}) \left( \frac{G_\varepsilon(\nabla u_\varepsilon)}{|\nabla u_\varepsilon|} |\nabla^2 u_\varepsilon|^2 + G_\varepsilon(|\nabla u_\varepsilon|) |\nabla^2 u_\varepsilon|^2 \right).$$

Using Lemma 4.1 and

$$|\nabla u_\varepsilon| \leq 2^{\frac{1}{p-1}} (1 + c)|\sigma_\varepsilon|^\frac{1}{p-1},$$

we conclude (4.1), (4.2) and (4.3) for $f$ and $\Omega$ smooth. The general case follows by an approximation argument.

Let us now show (4.5). First, we recall that since we are assuming $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$, we have $\sigma$ in $H^1(\Omega)^N$. Using that (3.1) implies

$$\nabla \hat{u} = (1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\hat{\sigma} \text{ a.e. in } \Omega,$$

and taking $i, j \in \{1, \ldots, N\}$, and $\Phi \in C_0^\infty(0, \infty)$, such that $\Phi = 1$ in a neighborhood of $\mu$, we get in the distributional sense

$$\partial_j \hat{u} \partial_i [\Phi(|\hat{\sigma}|)] - \partial_i \hat{u} \partial_j [\Phi(|\hat{\sigma}|)] = \partial_i \left( \partial_j \hat{u} \Phi(|\hat{\sigma}|) \right) - \partial_j \left( \partial_i \hat{u} \Phi(|\hat{\sigma}|) \right)$$

$$= \partial_i \left( (1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)|\hat{\sigma}| \hat{\sigma}_j \right) - \partial_j \left( (1 + c\hat{\theta})|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|)|\hat{\sigma}| \hat{\sigma}_i \right)$$

$$= c\partial_i \hat{\theta} |\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \hat{\sigma}_j - c\partial_j \hat{\theta} |\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \hat{\sigma}_i$$

$$+ (1 + c\hat{\theta}) \left( \partial_i \left( \Phi(|\hat{\sigma}|)|\hat{\sigma}|^{p'-2}\hat{\sigma}_j \right) - \partial_j \left( \Phi(|\hat{\sigma}|)|\hat{\sigma}|^{p'-2}\hat{\sigma}_i \right) \right),$$

which using that the support of $\Phi$ is compact and that $\sigma$ belongs to $H^1(\Omega)^N$ shows

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i \in L^2(\Omega).$$

Now we recall that

$$\hat{\theta} = 0 \text{ in } \{|\hat{\sigma}| < \hat{\mu}\}, \quad \hat{\theta} = 1 \text{ in } \{|\hat{\sigma}| > \hat{\mu}\}.$$

This implies that for every $\Psi \in C_0^\infty((0, \infty) \setminus \{\hat{\mu}\})$ we have

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \left( \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i \right) = |\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \left( \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i \right) (1 - \Psi(|\hat{\sigma}|)).$$

By (4.38) we can take $\Psi = \hat{\Psi}_s$ with

$$0 \leq \hat{\Psi}_s \leq 1, \quad \hat{\Psi}_s(\hat{\mu}) = 0, \quad \hat{\Psi}_s(s) \rightarrow 1, \forall s \neq \hat{\mu},$$

to deduce that

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \left( \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i \right)$$

vanishes a.e. in $\{|\hat{\sigma}| \neq \hat{\mu}\}$ and then that

$$|\hat{\sigma}|^{p'-2}\Phi(|\hat{\sigma}|) \left( \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i \right) = \hat{\mu}^{p'-2}\Phi(\hat{\mu}) \left( \partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i \right) X_{\{|\hat{\sigma}| = \hat{\mu}\}}.$$

On the other hand, recalling that $\nabla |\hat{\sigma}| = 0$ a.e. in $\{|\hat{\sigma}| = \hat{\mu}\}$, we can return to (4.37) to conclude (4.5).
Assertion (4.6) now follows from Proposition 2.1 in [3], which shows that

$$\partial_i \hat{\sigma}_j - \partial_j \hat{\sigma}_i \in L^2(\Omega),$$

implies

$$\partial_i \hat{\theta} \hat{\sigma}_j - \partial_j \hat{\theta} \hat{\sigma}_i = 0 \text{ a.e. in } \{\hat{\theta} = c\}, \quad \forall c \in [0, 1].$$

□

**Proof of Theorem 4.2.** Let $\hat{\omega}$ a measurable subset of $\Omega$, and $\hat{u} \in W^{1,p}_0(\Omega)$ be such that $(\chi_{\hat{\omega}}, \hat{u})$ is a solution of (2.17) with $\hat{f} = f$. By Remark 4.2, we have

$$(\alpha \chi_{\hat{\omega}} + \beta \chi_{\Omega \setminus \hat{\omega}}) \nabla \hat{u} = \nabla w,$$

with $w$ the unique solution of

$$\begin{cases}
-\text{div} \left(|\nabla w|^{p-2} \nabla w \right) = 1 \text{ in } \Omega \\
w \in W^{1,p}_0(\Omega).
\end{cases}$$

(4.39)

Thanks to Theorem 1.1 in [19] and the first corollary in [12], we know that $w$ is in $C^{1,\beta}(\Omega)$ for some $\beta \in (0, 1)$, and (see [23]) that it is analytic in $\{|\nabla w| > 0\}$. Using Theorem 1.1 in [20] (or Theorem 4.1) we also have that $\hat{\sigma} = |\nabla w|^{p-2} \nabla w$ is in $H^1(\Omega)^N$. Thus, $-\text{div} \hat{\sigma} = 0$ a.e. in $\{\hat{\sigma} = 0\}$, which combined with $w$ solution of (4.39) implies that $\nabla w \neq 0$ a.e. in $\Omega$. Analogously, let us prove that for every $\lambda > 0$, the set $\{|\nabla w| = \lambda\}$ has zero measure. For this purpose we observe that a.e. in $\{|\nabla w| = \lambda\},$ we have

$$0 = \nabla \text{div}(|\nabla w|^{p-2} \nabla w) = \lambda^{p-2} \nabla \Delta w = \lambda^{p-2} \Delta \nabla w.$$

Therefore $\nabla^2 w = 0$ a.e. in $\{|\nabla w| = \lambda\}$, which combined with

$$-\lambda^{p-2} \Delta w = -\text{div}(|\nabla w|^{p-2} \nabla w) = 1 \text{ a.e. in } \{|\nabla w| = \lambda\},$$

implies that the set $\{|\nabla w| = \lambda\}$ has zero measure. Now, we recall that thanks to (3.13), the constant $\hat{\mu}$ in Theorem 3.1 satisfies

$$\{x \in \Omega : |\nabla w| > \hat{\mu}\} \subset \hat{\omega} \subset \{x \in \Omega : |\nabla w| \geq \hat{\mu}\},$$

while Theorem 3.1 implies $|\hat{\omega}| = \kappa$. So, using that $|\{|\nabla w| = \hat{\mu}| = 0$, we get (up to a set of null measure)

$$\omega = \{x \in \Omega : |\nabla w| < \hat{\mu}\},$$

(4.40)

and $|\hat{\omega}| < |\Omega|$. Then, taking a connected component $O$ of the open set $\{x \in \Omega : |\nabla w| > \hat{\mu}\},$ we can repeat the argument in [6] to deduce that $O \subset \Omega$ is an analytic manifold with connected boundary such that

$$\begin{cases}
-\text{div} \left(|\nabla w|^{p-2} \nabla w \right) = 1 \text{ in } O \\
w, \frac{\partial w}{\partial \nu} \text{ are constant on } \partial O.
\end{cases}$$

(4.41)
From Serrin’s Theorem (\cite{27}), this proves that $O$ is an open ball and that $w$ is a radial function in $O$ with respect to its center. Taking into account the analyticity of $w$ in $\{|\nabla w| \neq 0\}$, the unique continuation principle shows that $\Omega$ is a ball.

5. Conclusion Section

In the present paper we have studied the optimal design of a two-phase material modeled by the $p$-Laplacian operator posed in a bounded open set $\Omega \subset \mathbb{R}^N$. The goal is to maximize the potential energy (problem (1.1)) when we only dispose of a limited amount of the best material. Since the problem has not solution in general, we have obtained a relaxed formulation (problems (1.2) and (1.3)) where instead of taking in every point of $\Omega$ one of both materials, we use a microscopic mixture where the proportion $\theta$ of the best material takes values in the whole interval $[0,1]$. This new formulation is obtained using homogenization theory. Reasoning by duality, we have also obtained a new formulation of the minimization problem as a min-max problem (problems (3.2) and (3.3)). As a consequence we show that although the relaxed problem has not uniqueness in general, the flux $\hat{\sigma}$ is unique.

The optimal conditions for the relaxed problem show that the state function $\hat{u}$ is the solution of a nonlinear Calculus of Variation problem (3.4). Since the second derivative of the function $F$ in this problem is not uniformly elliptic, the corresponding Euler-Lagrange equation does not provide in general the existence of second derivatives for $\hat{u}$. However it allows us to show that if the data es smooth enough then, for every $r > -1/2$, the function $|\hat{\sigma}|^r \hat{\sigma}$ is in the Sobolev space $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Moreover, the optimal proportion $\hat{\theta}$ is derivable in the orthogonal directions to $\nabla \hat{u}$. As an application of these results, we show that the original problem has a solution in a smooth open set $\Omega$ with a connected boundary if and only if $\Omega$ is a ball.

The results obtained in the present paper extend those obtained by other authors in the case of the Laplacian operator (see e.g. \cite{3, 8, 15, 26}).
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