On inverse problems for uncoupled space-time fractional operators involving time-dependent coefficients

Li Li

Institute for Pure and Applied Mathematics, University of California, Los Angeles, CA 90095, USA

ABSTRACT. We study the uncoupled space-time fractional operators involving time-dependent coefficients and formulate the corresponding inverse problems. Our goal is to determine the variable coefficients from the exterior partial measurements of the Dirichlet-to-Neumann map. We exploit the integration by parts formula for Riemann-Liouville and Caputo derivatives to derive the Runge approximation property for our space-time fractional operator based on the unique continuation property of the fractional Laplacian. This enables us to extend early unique determination results for space-fractional but time-local operators to the space-time fractional case.

1 Introduction

The basic uncoupled space-time fractional diffusion equation
\[ \partial_t^\alpha u + (-\Delta)^s u = 0, \quad 0 < \alpha < 1, \quad 0 < s < 1 \] (1)
models anomalous diffusion. Here the Caputo derivative \( \partial_t^\alpha \) (time-fractional derivative) describes particle trapping phenomena; The fractional Laplacian \( (-\Delta)^s \) (space-fractional derivative) describes long particle jumps. See [24] for more background information on (1). See [2] for a probabilistic interpretation for (1).

In this paper, we consider uncoupled space-time fractional operators involving time-dependent coefficients which generalize \( \partial_t^\alpha + (-\Delta)^s \) and formulate the corresponding inverse problems.

1.1 Main results

We will first study the initial exterior problem
\[
\begin{cases}
\partial_t^\alpha u + R_{A(t)}^s u + q(t)u = 0 & \Omega \times (0, T) \\
u = g & \Omega_e \times (0, T) \\
u = 0 & \Omega \times \{0\}
\end{cases}
\] (2)

where \( \Omega \) is a bounded domain and \( \Omega_e := \mathbb{R}^n \setminus \Omega \). Here the time-dependent fractional operator \( R_{A(t)}^s \) is formally defined by
\[
R_{A(t)}^s u(x) := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (u(x) - R_{A(t)}(x, y)u(y)) K(x, y) \, dy
\] (3)
where

\[ K(x, y) := c_{n,s} / |x - y|^{n+2s}, \]  

(4)

\( A(\cdot, t) \) is a time-dependent real vector-valued magnetic potential and

\[ R_{A(t)}(x, y) := \cos((x - y) \cdot A(x + y / 2, t)). \]  

(5)

The operator \( R_{A(t)}^* \) coincides with the fractional Laplacian \((-\Delta)^s\) when \( A \equiv 0\).

Under appropriate regularity and support assumptions on magnetic potential \(A\), electric potential \(q\) and the exterior data \(g\), the above theorem is the first main result in this paper, which can be viewed as a space-time fractional analogue of Theorem 1.1 in [21].

Theorem 1.1. Suppose \( \Omega \subset B_r (0) \) for some constant \( r > 0 \), \( \text{supp} A_j(t) \subset \Omega \) for \( t \in [0, T] \), \( A_j \in \mathcal{C}([0, T]; L^\infty(\mathbb{R}^n)) \), \( q_j \in \mathcal{C}([0, T]; L^\infty(\Omega)) \), \( W_j \) are open sets s.t. \( W_j \cap B_{3r}(0) = \emptyset \) (\( j = 1, 2 \)). Let

\[ W^{(1,2)} = \left\{ \frac{x + y}{2} : x \in W_1, y \in W_2 \right\}. \]

Also assume \( W^{(1,2)} \setminus \Omega \neq \emptyset \). If

\[ \Lambda_{A_1,q_1} g \big|_{W_2 \times (0,T)} = \Lambda_{A_2,q_2} g \big|_{W_2 \times (0,T)} \]  

(7)

for any \( g \in C_c^\infty(W_1 \times (0, T)) \), then \( A_1(t) = \pm A_2(t) \) and \( q_1 = q_2 \) in \( \Omega \times (0, T) \).

We remark that the seemingly unnatural assumptions on \( W_j \) in the statement are necessary (see the remark after Theorem 1.1 in [24]). However, if we replace \( R_{A(t)}^* \) by \((-\Delta)^s\) in [2], only interested in the determination of \( q \), then the assumptions on \( W_j \) in the statement can be simply replaced by \( W_j \subset \Omega_c \) (see Subsection 3.2 for details).

We will next study the semilinear problem

\[ \begin{cases} 
\partial_t^s u + (-\Delta)^s u + a(x, t, u) = 0 & \Omega \times (0, T) \\
u = g & \Omega_c \times (0, T) \\
u = 0 & \Omega \times \{0\} 
\end{cases} \]  

(8)

where the nonlinearity satisfies

\[ a(x, t, z) = \sum_{k=1}^{m} a_k(x, t)|z|^{b_k}z, \]  

(9)

\( a_k \geq 0 \) are smooth in \( \bar{\Omega} \times [0, T] \), \( b_1 = 0 \) and the powers \( 0 < b_2 < \cdots < b_m \) are not necessarily integers. We will see that the Dirichlet-to-Neumann map

\[ \Lambda_a : g \mapsto (-\Delta)^s u_g|_{\Omega_c \times (0,T)} \]  

(10)

is well-defined at least for \( g \in C_c^\infty(\Omega_c \times (0,T)) \).

Our goal here is to determine the nonlinearity \( a \) from the exterior partial measurements of \( \Lambda_a \). The following theorem is the second main result in this paper, which can be viewed as a space-time fractional analogue of Theorem 1.1 in [22].
Theorem 1.2. Let $W_1, W_2 \subset \Omega$ be nonempty and open. If

$$\Lambda_{a^{(1)}} g|_{W_2 \times (0,T)} = \Lambda_{a^{(2)}} g|_{W_2 \times (0,T)}, \quad g \in C^\infty_c(W_1 \times (0,T)), \tag{11}$$

then $a_k^{(1)} = a_k^{(2)}$ in $\Omega \times (0,T)$, $k = 1, 2, \cdots, m$.

1.2 Connection with earlier literature

So far there have been many contributions in the study of inverse problems for time-fractional but space-local operators. See, for instance, [14] for a work in this direction where the authors determined various time-independent smooth coefficients appearing in the equation from the knowledge of the associated Dirichlet-to-Neumann map based on the inverse spectral theory.

The study of (Calderón type) inverse problems for space-fractional operators dates back to [11]. See [9, 10, 12, 4, 20, 23] for further results in this direction. The proof of unique determination results in all these works heavily relies on exploiting the unique continuation property of the fractional Laplacian, which makes inverse problems for space-fractional operators often more manageable than their space-local counterparts. We adopt this framework here as well. Our approaches will be mainly based on the ones introduced in [21, 22].

To the best knowledge of the author, there are very few existing rigorous works on inverse problems for space-time fractional operators. Calderón type problems for the coupled space-time fractional operator $(\partial_t - \Delta)^s$ has been studied in [18, 1]. This fractional evolutionary operator actually behaves more like a fractional elliptic operator, which distinguishes it from our uncoupled space-time fractional operators. Another work in this direction can be found in [13] where the authors determined the Riemannian metric appearing in the uncoupled space-time fractional equation up to an isometry from the knowledge of the associated source-to-solution map in the setting of closed manifolds (also see [5, 26] for such geometric inverse problems for other fractional operators).

We also mention that our first inverse problem involving a magnetic potential can be viewed as a space-time fractional analogue of the classical magnetic Calderón problem studied in [25, 6, 17]. See [3, 19] for the study of Calderón type problems for a different kind of fractional magnetic operators.

1.3 Organization

The rest of this paper is organized in the following way. In Section 2, we will summarize the background knowledge. In Section 3, we will first show the well-posedness of (2); Then we will derive the integral identity for the Dirichlet-to-Neumann maps and the Runge approximation property of the space-time fractional operator to prove Theorem 1.1. In Section 4, we will first prove an a priori $L^\infty$ estimate and the well-posedness of (5); Then we will derive a linearization result to prove Theorem 1.2.

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2 Preliminaries

Throughout this paper we use the following notations.

- Fix the space dimension $n \geq 2$. 
• Fix the fractional powers $0 < \alpha < 1$ and $0 < s < 1$.
• $\Omega$ denotes a bounded domain with smooth boundary and $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$.
• $B_r(0)$ denotes the open ball centered at the origin with radius $r > 0$ in $\mathbb{R}^n$.
• $c, C, C', C_{1}, \ldots$ denote positive constants.
• $X$ denotes a Banach space and $X^*$ denotes the continuous dual space of $X$.
• $\langle f, u \rangle$ denotes the standard $L^2$ distributional pairing between $f$ and $u$ if $u$ (resp., $f$) is a (spatial) $n$-variable function (resp., functional).
• For functions $f(t), g(t)$ $(t > 0)$, we use the standard convolution notation

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau.$$ 

2.1 Function spaces
Throughout this paper we refer all function spaces to real-valued function spaces. We use $H^r$ to denote the standard Sobolev space $W^{r,2}$. We use $C^\alpha$ to denote the standard Hölder space $C^{0,\alpha}$.

Let $U$ be an open set in $\mathbb{R}^n$. Let $F$ be a closed set in $\mathbb{R}^n$. Then

$$H^r(U) := \{u|_{U} : u \in H^r(\mathbb{R}^n)\}, \quad H^r_F(\mathbb{R}^n) := \{u \in H^r(\mathbb{R}^n) : \text{supp} u \subset F\},$$

and

$$\tilde{H}^r(U) := \text{the closure of } C^\infty_c(U) \text{ in } H^r(\mathbb{R}^n).$$

For $r \in \mathbb{R}$, we have the natural identifications

$$H^{-r}(\mathbb{R}^n) = H^r(\mathbb{R}^n)^*, \quad \tilde{H}^r(\Omega) = H^r_{\Omega}(\mathbb{R}^n), \quad H^{-r}(\Omega) = \tilde{H}^r(\Omega)^*.$$ 

We use $C([0, T]; X)$ (resp., $C^\alpha([0, T]; X)$) to denote the space consisting of the corresponding Banach space-valued continuous (resp., $C^\alpha$-continuous) functions on $[0, T]$. $L^2(0, T; X)$ (resp., $H^1(0, T; X)$) denotes the space consisting of the corresponding Banach space-valued $L^2$-functions (resp., $H^1$-functions).

2.2 Riemann-Liouville and Caputo derivatives
Throughout this paper we use the notation

$$\phi_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0$$

where $\Gamma$ is the standard Gamma function. It is straightforward to verify that

$$\phi_\alpha \ast \phi_{1-\alpha} = 1, \quad t > 0. \quad (12)$$

The left Riemann-Liouville fractional integral of order $\alpha$ is defined by

$$I^\alpha_{0,t} u := \phi_\alpha \ast u.$$
and the left Riemann-Liouville fractional derivative of order $\alpha$ is defined by
\[
D_0^\alpha u := \partial_t (I_{0,t}^{1-\alpha} u) = \partial_t (\phi_{1-\alpha} \ast u).
\]
Correspondingly, the right Riemann-Liouville fractional integral of order $\alpha$ is defined by
\[
I_0^\alpha u := \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{\alpha-1} u(\tau) \, d\tau
\]
and the right Riemann-Liouville fractional derivative of order $\alpha$ is defined by
\[
D_0^\alpha u := -\partial_t (I_{0,t}^{1-\alpha} u).
\]

The (left) Caputo fractional derivative of order $\alpha$ is defined by
\[
\partial_t^\alpha u(t) := D_0^\alpha (u(t) - u(0)).
\]
It is straightforward to verify that $\partial_t^\alpha u = I_{0,t}^{1-\alpha} (\partial_t u)$ if $u$ is sufficiently regular.

For (scalar-valued) functions $f(t), g(t)$, we have the following integration by parts formula (see, for instance, (3.6) in [28])
\[
\int_0^T g \partial_t^\alpha f = \int_0^T f D_0^\alpha T g + (f I_{0,T}^{1-\alpha} g)|_{t=0}^{t=T}.
\]
(13)

Here the value of $I_{0,T}^{1-\alpha} g$ at $t = T$ should be interpreted in the limit sense. We also have the following inequality (see, for instance, Lemma 2.1 in [27])
\[
\partial_t^\alpha (H(f(t))) \leq H'(f(t)) \partial_t^\alpha f(t)
\]
(14)
where $H \in C^1(\mathbb{R})$ is convex. We note that both (13) and (14) can be generalized for Banach space-valued $f(t), g(t)$ if we replace the pointwise product by the distributional pairing.

Let $L$ be a sectorial operator (i.e. a closed, densely defined linear operator which generates an analytic semigroup) in $X$. For sufficiently regular $f$, we can take Laplace transform to show that
\[
u(t) = S_\alpha(t) u_0 + \int_0^t P_\alpha(t-\tau) f(\tau) \, d\tau
\]
(15)
satisfies the continuity at $t = 0$, and this formula gives the solution of
\[
\partial_t^\alpha u = Lu + f, \quad u(0) = u_0;
\]
(16)
We can also take Laplace transform to show that
\[
u(t) = P_\alpha(t) h_0 + \int_0^t P_\alpha(t-\tau) f(\tau) \, d\tau
\]
(17)
satisfies that $I_{0,t}^\alpha u$ is continuous at $t = 0$, and this formula gives the solution of
\[
D_0^\alpha u = Lu + f, \quad I_{0,t}^\alpha u(0) = h_0.
\]
(18)
Here $\{S_\alpha(t)\}, \{P_\alpha(t)\}$ are the resolvent families associated with $L$, which can be expressed by integrals involving the semigroup $\{S(t)\}$ generated by $L$ and the Wright type function $\Phi_\alpha(t)$. Since we are not going to use the explicit expressions of $S_\alpha(t), P_\alpha(t)$, we refer interested readers to (2.1.9) in [8] for details.
2.3 Space-fractional operators

It is well-known that the fractional Laplacian \((-\Delta)^s\) can be defined by

\[
(-\Delta)^s u(x) := c_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy
\]

and for each \(r \in \mathbb{R}\), we have

\((-\Delta)^s : H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)\).

We will see that many properties of the fractional Laplacian are preserved under the perturbation by the magnetic potential \(A\). Throughout this paper we assume

\(A \in C([0,T]; L^\infty(\mathbb{R}^n)), \quad \text{supp} \ A(t) \subset \Omega\)

for each \(t \in [0,T]\) and \(q \in C([0,T]; L^\infty(\Omega))\).

It has been shown that (see (1) and (6) in [21]) for \(u, v \in H^s(\mathbb{R}^n)\),

\[
\langle R_A^s u, v \rangle := 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - R_A(t)(x,y)u(y))v(x)K(x,y) \, dx \, dy,
\]

and \(R_A^s \) is symmetric:

\[
\langle R_A^s u, v \rangle = \langle R_A^s v, u \rangle.
\]

We define the bilinear form \(B_t\) associated with \(A, q\) by

\[
B_t[u,v] := \langle R_A^s u, v \rangle + \int_{\Omega} q(t)uv, \quad t \in [0,T].
\]

**Lemma 2.1.** \(B_t\) is bounded:

\[
|B_t[u,v]| \leq C_0 ||u||_{H^r} ||v||_{H^r}, \quad u, v \in H^s(\mathbb{R}^n).
\]

**Proposition 2.2.** Suppose \(\Omega \subset B_{r}(0)\) for some \(r > 0\), \(W\) is a nonempty open set s.t. 

\(W \cap B_{3r}(0) = \emptyset\).

(i) For \(g \in C^\infty_c(W \times (0,T))\), we have

\[R_A^s g|_{\Omega \times (0,T)} = (-\Delta)^s g|_{\Omega \times (0,T)};\]

(ii) (Proposition 2.4 in [21]) If 

\[u \in \tilde{H}^s(\Omega), \quad R_A^s u|_W = 0,\]

then \(u = 0\) in \(\mathbb{R}^n\).

It is straightforward to verify (i) based on the support assumptions on \(A\) and \(g\). (ii) is based on the following unique continuation property of \((-\Delta)^s\) (see Theorem 1.2 in [11]).

**Proposition 2.3.** Let \(u \in H^r(\mathbb{R}^n)\) for some \(r \in \mathbb{R}\). Let \(W \subset \mathbb{R}^n\) be nonempty and open. If 

\[(-\Delta)^s u = u = 0 \quad \text{in } W,\]

then \(u = 0\) in \(\mathbb{R}^n\).
3 Linear problem

3.1 Forward problem

We will study (2) as well as the related problem

\[
\begin{aligned}
D_{0,t}u + R_{A(t)}^s u + q(t)u &= 0 \quad \Omega \times (0, T) \\
u &= g \quad \Omega_e \times (0, T) \\
I_{0,t}^0 u &= 0 \quad \Omega \times \{0\}
\end{aligned}
\]  

(25)

for \( g \in C_c^\infty(\Omega_e \times (0, T)) \).

By using the substitution \( u = w + g \), (2) can be converted into the initial value problem

\[
\begin{aligned}
\partial_t w + R_{A(t)}^s w + q(t)w &= f \quad \Omega \times (0, T) \\
w &= 0 \quad \Omega \times \{0\}
\end{aligned}
\]  

(26)

and (25) can be converted into the (integral) initial value problem

\[
\begin{aligned}
D_{0,t}w + R_{A(t)}^s w + q(t)w &= f \quad \Omega \times (0, T) \\
I_{0,t}^0 w &= 0 \quad \Omega \times \{0\}
\end{aligned}
\]  

(27)

Based on Lemma 2.1 in Subsection 2.3 and Theorem 3.1 in [29], we have the following well-posedness result.

**Proposition 3.1.** Suppose \( f \in L^2(0, T; H^{-s}(\Omega)) \). Then (27) has a unique solution satisfying

\[
w \in L^2(0, T; \tilde{H}^s(\Omega)), \quad I_{0,t}^0 w \in H^1(0, T; H^{-s}(\Omega)).
\]

This actually implies \( I_{0,t}^0 w \in C([0, T]; L^2(\Omega)) \) (see Theorem 2.1 in [27]).

We note that

\[
0 \leq 1 - R_{A(t)}(x, y) = 2 \sin^2\left(\frac{1}{2}(x - y) \cdot A\left(\frac{x+y}{2}, t\right)\right) \leq C_A \min\{1, |x - y|^2\}
\]

and (based on (19) and (3)) we have

\[
\|((-\Delta)^s - R_{A(t)}^s)u(x)\| \leq \int (1 - R_{A(t)}(x, y))K(x, y)||u(y)|dy.
\]

Since we have

\[
\int (1 - R_{A(t)}(x, y))K(x, y)dy \leq C, \quad x \in \mathbb{R}^n,
\]

by the generalized Young’s inequality (see, for instance, Proposition 0.10 in [7]), we get

\[
\|((-\Delta)^s - R_{A(t)}^s)u\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}. \quad (28)
\]

Now we know that the operator \( R_{A(t)}^s + q(t) \) can be written as

\[(-\Delta)^s + F(t, \cdot)\]
where the linear operator
\[ F(t, \cdot) := \mathcal{R}_A^s(t) - (-\Delta)^s + q(t) \]
is \( L^2 \) bounded (uniformly in \( t \)).

We also note that for sufficiently regular \( f \), both \eqref{eq:26} and \eqref{eq:27} can be written as the integral equation (see \eqref{eq:15} and \eqref{eq:17})
\[
w(t) = \int_0^t P_\alpha(t - \tau) (f(\tau) - F(\tau, w(\tau))) \tau \, d\tau.
\]
Here \( \{P_\alpha(t)\} \) is the resolvent family associated with the sectorial operator
\[ L : w \to -(-\Delta)^s w|_\Omega, \quad D(L) = \{w \in \dot{H}^s(\Omega) : (-\Delta)^s w|_\Omega \in L^2(\Omega)\}. \]

Based on Theorem 1.1 in \cite{15}, we have the following well-posedness result.

**Proposition 3.2.** Let \( f \in C^\alpha([0, T]; L^2(\Omega)) \) Then \eqref{eq:26} (or \eqref{eq:27}) has a unique solution \( w \in C([0, T]; L^2(\Omega)) \).

We remark that the well-posedness result above actually holds true for the general Lipschitz type nonlinear map \( F(t, \cdot) \) and the regularity assumption on \( f \) can be weakened. We refer readers to \cite{15} for more details.

In the rest of this section, we will assume \( \Omega \subset B_r(0) \) for some constant \( r > 0 \) and \( W \) is a nonempty open set in \( \mathbb{R}^n \) s.t. \( W \cap B_{3r}(0) = \emptyset \) unless otherwise stated. We note that for \( g \in C^\infty_c(W \times (0, T)) \), by Proposition 2.2 (i) we have
\[
f := -\mathcal{R}_A^s g|_{\Omega \times (0, T)} = -(-\Delta)^s g|_{\Omega \times (0, T)},
\]
which is smooth over \( \bar{\Omega} \times [0, T] \).

Based on Proposition 3.1 and 3.2, we have the following well-posedness result

**Proposition 3.3.** Let \( g \in C^\infty_c(W \times (0, T)) \) Then \( \eqref{eq:3} \) (or \eqref{eq:27}) has a unique solution \( u \) with \( w := u - g \) satisfying
\[
w \in L^2(0, T; \dot{H}^s(\Omega)) \cap C([0, T]; L^2(\Omega)), \quad I_{\alpha}^\top w \in H^1(0, T; H^{-s}(\Omega)) \cap C([0, T]; L^2(\Omega)).
\]

We denote the solution operator \( g \to u_g \) associated with \eqref{eq:2} by \( P_{A,q} \). We note that we have the same well-posedness result for the dual problem
\[
\begin{cases}
D_{t,T}^\alpha u + \mathcal{R}_A^s(t) u + q(t) u = 0 & \Omega \times (0, T) \\
u = g & \Omega_\epsilon \times (0, T) \\
I_{t,T}^\alpha u = 0 & \Omega \times \{T\}
\end{cases}
\]
and we denote the associated solution operator by \( P^*_{A,q} \).
3.2 Inverse problem

3.2.1 DN map

Recall that we defined the Dirichlet-to-Neumann map \( \Lambda_{A,q} \) associated with (2) in (1). We also define the Dirichlet-to-Neumann map \( \Lambda^*_{A,q} \) associated with (29) by

\[
\Lambda^*_{A,q}g := \mathcal{R}^*(P_{A,q}^*)|_{\Omega_t \times (0,T)}.
\]

The well-posed results ensure that \( \Lambda_{A,q} \) and \( \Lambda^*_{A,q} \) are well-defined at least for \( g \in C^\infty_c(W \times (0,T)) \).

Let \( g \in C^\infty_c(W_1 \times (0,T)) \) and \( h \in C^\infty_c(W_2 \times (0,T)) \).

By the definition of \( B_t, P_{A,q} \) and \( \Lambda_{A,q} \) we have

\[
\int_0^T \langle \Lambda_{A,q}g(t), h(t) \rangle dt = \int_0^T \langle \mathcal{R}^*_{\Lambda(t)}u(t), \tilde{h}(t) \rangle dt - \int_0^T \langle \mathcal{R}^*_{\Lambda(t)}u(t), \tilde{h}(t) - h(t) \rangle dt
\]

\[
= \int_0^T \langle \mathcal{R}^*_{\Lambda(t)}u(t), \tilde{h}(t) \rangle dt + \int_0^T \langle \partial_t u(t) + q(t)u(t), \tilde{h}(t) - h(t) \rangle dt
\]

\[
= \int_0^T B_t[u(t), \tilde{h}(t)] dt + \int_0^T \langle \partial_t u(t), \tilde{h}(t) \rangle_{\Omega_t} dt
\]

for any \( \tilde{h} \) satisfying \( \tilde{h} - h \in L^2(0,T; \tilde{H}^*(\Omega)) \). Here \( u := P_{A,q}g, w := u - g \) and

\[
\langle \partial_t u(t), \tilde{h}(t) \rangle_{\Omega_t} = \langle \partial_t w(t), \tilde{h}(t) - h(t) \rangle.
\]

Similarly we have

\[
\int_0^T \langle \Lambda^*_{A,q}h(t), g(t) \rangle dt = \int_0^T B_t[u^*(t), \tilde{g}(t)] dt + \int_0^T \langle D^*_{I_{t,T}}u^*(t), \tilde{g}(t) \rangle_{\Omega_t} dt
\]

where \( u^* := P_{A,q}^*h \) for any \( \tilde{g} \) satisfying \( \tilde{g} - g \in L^2(0,T; \tilde{H}^*(\Omega)) \).

**Proposition 3.4.** For \( g \in C^\infty_c(W_1 \times (0,T)) \) and \( h \in C^\infty_c(W_2 \times (0,T)) \), we have

\[
\int_0^T \langle \Lambda_{A,q}g(t), h(t) \rangle dt = \int_0^T \langle \Lambda_{A,q}^*h(t), g(t) \rangle dt.
\]

**Proof.** Let \( \tilde{h} = u^* \) in (31) and let \( \tilde{g} = u \) in (32). Since \( u(0) = I^*_t u^*(T) = 0 \), we have

\[
\int_0^T \langle \Lambda_{A,q}g(t), h(t) \rangle dt - \int_0^T \langle \Lambda_{A,q}^*h(t), g(t) \rangle dt
\]

\[
= \int_0^T \langle \partial_t u(t), u^*(t) \rangle_{\Omega_t} - \langle D^*_{I_{t,T}}u^*(t), u(t) \rangle_{\Omega_t} dt = \langle u(t), I^*_t u^*(t) \rangle_{\Omega_t} |^{t=T}_{t=0} = 0
\]

based on the symmetry of \( B_t \) and the integration by parts formula (13). \( \square \)
Now we are ready to derive the integral identities for Dirichlet-to-Neumann maps. For \( g_j \in C_c^\infty(W_j \times (0, T)) \) \((j = 1, 2)\), let \( u_1 = P_{A_1,q_1}(g_1) \) and \( u_2^* = P_{A_2,q_2}^*(g_2) \), i.e. \( u_1 \) is the solution of

\[
\begin{aligned}
\begin{cases}
\partial_t^\alpha u + \mathcal{R}_{A_1(t)}^* u + q_1(t) u = 0 & \text{in } \Omega \times (0, T) \\
u = g_1 & \text{in } \Omega_e \times (0, T) \\
u = 0 & \text{in } \Omega \times \{0\}
\end{cases}
\end{aligned}
\] (34)

and \( u_2^* \) is the solution of

\[
\begin{aligned}
\begin{cases}
D_{t,T}^\alpha u + \mathcal{R}_{A_1(t)}^* u + q_2(t) u = 0 & \text{in } \Omega \times (0, T) \\
u = g_2 & \text{in } \Omega_e \times (0, T) \\
I_{t,T}^\alpha u = 0 & \text{in } \Omega \times \{T\}.
\end{cases}
\end{aligned}
\] (35)

Then by \( \text{(31), (32), (13)} \) and Proposition 3.4 we have

\[
\begin{aligned}
\int_0^T \langle \Lambda_{A_1,q_1} g_1(t), g_2(t) \rangle - \langle \Lambda_{A_2,q_2} g_1(t), g_2(t) \rangle \, dt \\
= \int_0^T \langle \Lambda_{A_1,q_1} g_1(t), g_2(t) \rangle \, dt - \int_0^T \langle \Lambda_{A_2,q_2}^* g_2(t), g_1(t) \rangle \, dt \\
= \int_0^T B_t^{(1)}[u_1(t), u_2^*(t)] + (\partial_t^\alpha u_1(t) + u_2^*(t))_\Omega \, dt - \int_0^T B_t^{(2)}[u_2^*(t), u_1(t)] + (D_{t,T}^\alpha u_2^*(t), u_1(t))_\Omega \, dt \\
= \int_0^T B_t^{(1)}[u_1(t), u_2^*(t)] \, dt - \int_0^T B_t^{(2)}[u_1(t), u_2^*(t)] \, dt \\
= \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G(x,y,t) u_1(y,t) u_2^*(x,t) - \int_0^T \int_{\Omega} (q_2 - q_1) u_1 u_2^* 
\end{aligned}
\] (36)

where

\[
G(x,y,t) := 2(R_{A_2}(x,y) - R_{A_1(t)}(x,y)) K(x,y)
\]

3.2.2 Runge approximation

**Proposition 3.5.** Suppose \( \Omega \subset B_r(0) \) for some constant \( r > 0 \) and \( W \) is a nonempty open set in \( \mathbb{R}^n \) s.t. \( W \cap B_{3r}(0) = \emptyset \). Then

\[
S := \{ P_{A,q} g |_{\Omega \times (0,T)} : g \in C_c^\infty(W \times (0,T)) \},
\]

\[
S^* := \{ P_{A,q}^* g |_{\Omega \times (0,T)} : g \in C_c^\infty(W \times (0,T)) \}
\]

are dense in \( L^2(\Omega \times (0,T)) \).

**Proof.** By the Hahn-Banach Theorem, it suffices to show that:

If \( f \in L^2(\Omega \times (0,T)) \) and \( \int_0^T \int_\Omega w f = 0 \) for all \( w \in S \), then \( f = 0 \) in \( \Omega \times (0,T) \).
In fact, for an arbitrary given \( f \in L^2(\Omega \times (0, T)) \), by (the dual version of) Proposition 3.1 we know that the solution of

\[
\begin{cases}
D_{t,T}^\alpha v + \mathcal{R}_{A(t)}^s v + q(t) v = f & \text{in } \Omega \times (0, T) \\
I_{t,T}^\alpha v = 0 & \text{in } \Omega \times \{T\}.
\end{cases}
\]

(37)

satisfies

\[
v \in L^2(0, T; \tilde{H}^s(\Omega)), \quad I_{t,T}^\alpha v \in H^1(0, T; H^{-s}(\Omega)) \cap C([0, T]; L^2(\Omega)).
\]

For \( g \in C_c^\infty(W \times (0, T)) \), write \( u_g := P_{A,q} g \), then we have

\[
\int_0^T \int_\Omega u_g f = \int_0^T \langle D_{t,T}^\alpha v(t) + \mathcal{R}_{A(t)}^s v(t) + q(t) v(t), u_g(t) - g(t) \rangle \, dt
\]

\[
= \int_0^T \langle \partial_t^\alpha u_g(t), v(t) \rangle + B_t[u_g(t), v(t)] \, dt - \int_0^T \langle \mathcal{R}_{A(t)}^s g(t), v(t) \rangle \, dt
\]

\[
= - \int_0^T \langle \mathcal{R}_{A(t)}^s v(t), g(t) \rangle \, dt.
\]

(38)

The first equality holds since \( u_g - g \in L^2(0, T; \tilde{H}^s(\Omega)) \); The second equality holds since \( u_g(0) = I_{t,T}^\alpha v(T) = 0 \) ensures

\[
\int_0^T \langle D_{t,T}^\alpha v(t), u_g(t) \rangle = \int_0^T \langle \partial_t^\alpha u_g(t), v(t) \rangle;
\]

The last equality holds since \( v \in L^2(0, T; \tilde{H}^s(\Omega)) \) and \( u_g \) is the solution of (2).

Hence, if \( \int_0^T \int_\Omega w f = 0 \) for all \( w \in S \), then (38) yields

\[
\int_0^T \langle \mathcal{R}_{A(t)}^s v(t), g(t) \rangle \, dt = 0, \quad g \in C_c^\infty(W \times (0, T))
\]

so for each \( t \) we have

\[
v(t) \in \tilde{H}^s(\Omega), \quad \mathcal{R}_{A(t)}^s v(t)|_{W} = 0,
\]

which implies \( v(t) = 0 \) in \( \mathbb{R}^n \) for each \( t \) by Proposition 2.2 (ii) and thus \( f = 0 \) in \( \Omega \times (0, T) \).

Similarly we can show \( S^* \) is dense in \( L^2(\Omega \times (0, T)) \).

We remark that a similar argument has been used to prove the approximate controllability result for \( \partial_t^\alpha + (-\Delta)^s \) (see Theorem 2.6 in [28]).

### 3.2.3 Proof of Theorem 1.1

We note that time-fractional derivatives do not appear in the integral identity (36) so we can use the same argument as the one in [21] to prove Theorem 1.1 based on the Runge approximation property (Proposition 3.5). To avoid redundancy, we will skip inessential steps in the proof. We refer readers to Subsection 4.3 in [21] for full details.
Proof. We write \( u_1 = P_{A_1,q_1}(g_1) \) and \( u_2^* = P_{A_2,q_2}(g_2) \) for \( g_j \in C_0^\infty(W_j \times (0,T)) \).

Based on (30) and the assumptions stated in Theorem 1.1, we have
\[
\int_0^T \int_\Omega G(x,y,t)u_1(y,t)u_2^*(x,t) = \int_0^T \int_\Omega (q_2 - q_1)u_1u_2^*.
\] (39)

**Determine A:** We fix open sets \( \Omega_j \subset \Omega \) s.t. \( \Omega_1 \cap \Omega_2 = \emptyset \). We fix \( \phi_j \in C_0^\infty(\Omega_j) \) and the constants \( a, b \in (0,T) \). We define
\[
\tilde{\phi}_j(x,t) := 1_{[a,b]}(t)\phi_j(x).
\]

By Proposition 3.5, we can choose \( g_1 \in C_0^\infty(W_1 \times (0,T)) \) (resp. \( g_2 \in C_0^\infty(W_2 \times (0,T)) \)) s.t. \( u_1 \) (resp. \( u_2^* \)) approximates \( \tilde{\phi}_1 \) (resp. \( \tilde{\phi}_2 \)) arbitrarily in \( L^2(\Omega \times (0,T)) \).

The key observation is that \( \phi_1 \phi_2 = 0 \) while \( \phi_1 \otimes \phi_2 \neq 0 \) in general. This enables us to take the limit on both sides of (39) to get
\[
\int_a^b \int_{\Omega_1} \int_{\Omega_2} G(x,y,t)\phi_1(y)\phi_2(x) \, dx \, dy \, dt = 0.
\]

Since the choices of \( \phi_1, \phi_2 \) and \([a,b]\) are arbitrary, we can conclude that \( G(x,y,t) = 0 \) for \( x, y \in \Omega \) whenever \( x \neq y \). Thus we know that
\[
R_{A_1(t)}(x,y) = R_{A_2(t)}(x,y), \quad x, y \in \Omega
\] (40) for each \( t \). Then we can show that \( A_1(t) = \pm A_2(t) \).

**Determine q:** Now (30) becomes
\[
\int_0^T \int_\Omega (q_2 - q_1)u_1u_2^* = 0.
\]

For \( f \in L^2(\Omega \times (0,T)) \), again by Proposition 3.5, we can choose \( g_1 \in C_0^\infty(W_1 \times (0,T)) \) (resp. \( g_2 \in C_0^\infty(W_2 \times (0,T)) \)) s.t. \( u_1 \) (resp. \( u_2^* \)) approximates \( f \) (resp. constant function 1) arbitrarily in \( L^2(\Omega \times (0,T)) \). Then we take the limit to get
\[
\int_0^T \int_\Omega (q_2 - q_1)f = 0.
\]

We conclude that \( q_1 = q_2 \) since the choice of \( f \) is arbitrary.

We remark that if we are only interested in the problem
\[
\left\{ \begin{array}{ll}
\partial_t^\alpha u + (-\Delta)^\alpha u + q(t)u &= 0 & \Omega \times (0,T) \\
u &= g & \Omega \times (0,T) \\
u &= 0 & \Omega \times \{0\}
\end{array} \right.
\] (41)

instead of the more general (2), then the inverse problem will be reduced to the determination of \( q \) from partial knowledge of \( \Lambda_q \) (i.e. \( \Lambda_{0,q} \)), and in this case the assumptions on \( W_j \) in the statement of Theorem 1.1 are not necessary for the unique determination.

In fact, based on the unique continuation property of fractional Laplacian (Proposition 2.3) instead of Proposition 2.2, we can use the same argument as the one in the proof of Proposition 3.5 to prove the following Runge approximation property.
Proposition 3.6. Let \( W \subset \Omega_e \) be nonempty and open. Then
\[
S := \{ P_q g |_{\Omega \times (0, T)} : g \in C_c^\infty (W \times (0, T)) \}, \quad S^* := \{ P_q^* g |_{\Omega \times (0, T)} : g \in C_c^\infty (W \times (0, T)) \}
\]
are dense in \( L^2 (\Omega \times (0, T)) \). (Here \( P_q := P_{0,q} \) and \( P_q^* := P_{0,q}^* \).)

Then we can use the argument in the second half of the proof of Theorem 1.1 to prove the following unique determination theorem.

Proposition 3.7. Let \( W_1, W_2 \subset \Omega_e \) be nonempty and open. If
\[
\Lambda_{q_1} g |_{W_2 \times (0, T)} = \Lambda_{q_2} g |_{W_2 \times (0, T)}, \quad g \in C_c^\infty (W_1 \times (0, T)),
\]
then \( q_1 = q_2 \) in \( \Omega \times (0, T) \).

4 Semilinear problem

4.1 Forward problem

Now we turn to the semilinear problem (8). We first prove the following a priori \( L^\infty \) estimate.

Proposition 4.1. Let \( g \in C_c^\infty (\Omega_e \times (0, T)) \). Suppose \( u \in C([0, T]; L^2 (\mathbb{R}^n)) \) is a solution of
\[
\begin{aligned}
\partial_t^\alpha u + (-\Delta)^s u + a(x, t, u) &= f \quad \Omega \times (0, T) \\
\partial_t^\alpha u &= g \quad \Omega_e \times (0, T) \\
\partial_t^\alpha u &= 0 \quad \Omega \times \{0\}.
\end{aligned}
\]

Then we have
\[
\| u \|_{L^\infty} \leq \frac{T^{\alpha}}{\Gamma (\alpha + 1)} (\| f \|_{L^\infty (\Omega \times (0, T))} + \| g \|_{L^\infty (\Omega_e \times (0, T))}).
\]

Proof. We fix \( \varphi \in C_c^\infty (\mathbb{R}^n) \) s.t. \( 0 \leq \varphi \leq 1 \) and \( \varphi = 1 \) on \( \overline{\Omega} \cup \text{supp}\_x g \). We define
\[
\tilde{\varphi}(x, t) := (\| f \|_{L^\infty (\Omega \times (0, T))}) \frac{t^{\alpha}}{\Gamma (\alpha + 1)} + \| g \|_{L^\infty (\Omega_e \times (0, T))} \varphi(x).
\]
Clearly \( (-\Delta)^s \varphi \geq 0 \) in \( \Omega \) from the pointwise definition (19) of \( (-\Delta)^s \). Also note that
\[
\partial_t^\alpha (t^{\alpha}) = \Gamma (\alpha + 1), \quad \partial_t^\alpha (c) = 0
\]
so we have
\[
\partial_t^\alpha \tilde{\varphi} + (-\Delta)^s \tilde{\varphi} + a(x, t, \tilde{\varphi}) \geq \| f \|_{L^\infty (\Omega \times (0, T))}
\]
in \( \Omega \times (0, T) \). Now we consider \( \tilde{u} := u - \tilde{\varphi} \). Note that \( \tilde{u} \leq 0 \) in \( \Omega_e \times (0, T) \), \( \tilde{u} \leq 0 \) at \( t = 0 \) and
\[
\partial_t^\alpha \tilde{u} + (-\Delta)^s \tilde{u} + a(x, t, \tilde{u}) - a(x, t, \tilde{\varphi}) \leq 0
\]
in \( \Omega \times (0, T) \). We write \( \tilde{u} = \tilde{u}^+ - \tilde{u}^- \) where \( \tilde{u}^\pm = \max \{ \pm \tilde{u}, 0 \} \). Then \( \tilde{u}^+ = 0 \) in \( \Omega_e \times (0, T) \) and \( \tilde{u}^+ = 0 \) at \( t = 0 \).
If we choose $H(y) = \frac{1}{2}(y^+)^2$ in (43), then we have
\[
\frac{1}{2} \partial^\alpha_t (||\tilde{u}^+(t)||^2_{L^2(\Omega)}) \leq \langle \partial^\alpha_t \tilde{u}(t), \tilde{u}^+(t) \rangle.
\] (44)

Note that by the definition of the Caputo derivative, we have
\[
\frac{1}{2} \partial^\alpha_t (||\tilde{u}^+(t)||^2_{L^2(\Omega)}) = \frac{1}{2} \partial_t (\phi_{1-\alpha} * (||\tilde{u}^+(t)||^2_{L^2(\Omega)} - ||\tilde{u}^+(0)||^2_{L^2(\Omega)})) = \frac{1}{2} \partial_t (\phi_{1-\alpha} * ||\tilde{u}^+(t)||^2_{L^2(\Omega)})
\]
since $\tilde{u}^+ = 0$ at $t = 0$. Also note that $u - g \in C([0, T]; L^2(\Omega))$ implies
\[
\phi_{1-\alpha} * ||\tilde{u}^+(t)||^2_{L^2(\Omega)} = 0
\]
at $t = 0$, by (12) we have
\[
\phi_{1-\alpha} * \partial_t (\phi_{1-\alpha} * ||\tilde{u}^+(t)||^2_{L^2(\Omega)}) = \partial_t (\phi_{1-\alpha} * ||\tilde{u}^+(t)||^2_{L^2(\Omega)}).
\] (45)

Now we let both sides of (43) act on $\tilde{u}^+$. Based on (44) we can apply the convolution operation $\phi_{1-\alpha}$ and (45) to get
\[
\frac{1}{2} ||\tilde{u}^+(t)||^2_{L^2(\Omega)} + \phi_{1-\alpha} \langle \langle (-\Delta)^\alpha \tilde{u}(t), \tilde{u}^+(t) \rangle \rangle + \phi_{1-\alpha} \langle (a(\cdot, t, \tilde{u}(\cdot, t)) - a(\cdot, t, \tilde{u}(\cdot, t)), \tilde{u}^+(\cdot, t) \rangle \rangle \leq 0
\]
since $\phi_{1-\alpha}$ is a positive function. Note that $a(x, t, u) - a(x, t, \tilde{u})$ has the same sign as $\tilde{u}$ and
\[
\langle \langle (-\Delta)^\alpha \tilde{u}(t), \tilde{u}^+(t) \rangle \rangle \geq 0
\] (46)

Hence we have $||\tilde{u}^+(t)||^2_{L^2(\Omega)} \leq 0$ and thus the only possibility is $\tilde{u}^+ = 0$, i.e. $u \leq \tilde{u}$ in $\Omega \times (0, T)$.

Also note that $a(x, t, \tilde{u}) + a(x, t, u)$ has the same sign as $\tilde{u} + u$ so similarly we can consider
\[
\tilde{u} := -u - \tilde{u}
\]
and show $-\tilde{u} \leq u$ in $\Omega \times (0, T)$. Hence we have $|u| \leq \tilde{u}$ in $\Omega \times (0, T)$.

We remark that we have
\[
\langle \langle (-\Delta)^\alpha \tilde{u}(t), \tilde{u}^+(t) \rangle \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\tilde{u}(x, t) - \tilde{u}(y, t))(\tilde{u}^+(x, t) - \tilde{u}^+(y, t))}{|x - y|^{n+2\alpha}} \, dxdy.
\]

Since $(\tilde{u}^+(x, t) - \tilde{u}^+(y, t))(\tilde{u}^-(x, t) - \tilde{u}^-(y, t)) \leq 0$, we have
\[
\langle \langle (-\Delta)^\alpha \tilde{u}(t), \tilde{u}^+(t) \rangle \rangle \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\tilde{u}^+(x, t) - \tilde{u}^+(y, t))^2}{|x - y|^{n+2\alpha}} \, dxdy \geq 0.
\]

Hence (46) holds true.

To study the well-posedness of (5), we again use the substitutions $u = w + g$ and $f = -(\Delta)^\alpha g|_{\Omega \times (0, T)}$ so (5) can be converted into the initial value problem
\[
\begin{cases}
\partial^\alpha_t w + (\Delta)^\alpha w + a(x, t, w) = f & \Omega \times (0, T) \\
w = 0 & \Omega \times \{0\}.
\end{cases}
\] (47)

Our assumptions on $g$ and $a$ ensure that $f - a$ satisfies the conditions (F1), (F2) (and thus (F3), (F4)) on Page 66 and the condition (F5) on Page 81 in [5]. Hence we can apply Theorem 3.2.2, Corollary 3.2.3 and (3.2.32) in [5] to obtain the following well-posedness result.
Proposition 4.2. There exists $0 < T_{\text{max}} \leq \infty$ s.t. for $0 < T < T_{\text{max}}$, $w$ has a unique solution

$$w \in C([0, T]; L^\infty(\Omega)).$$

Furthermore, We have either (i) $T_{\text{max}} = \infty$ or (ii) $0 < T < \infty$ and $\lim_{t \to T_{\text{max}}} ||u(\cdot, t)||_{L^\infty} = \infty$.

We remark that the solution $w$ actually belongs to certain Hölder type function spaces. We refer readers to Subsection 3.2 in [8] for more details.

By the previous $L^\infty$ estimate, we have $T_{\text{max}} = \infty$ for $g \in C_0^\infty(\Omega \times (0, T))$ under our assumptions on the nonlinearity $a$. Hence we do not need any smallness assumptions on the constant $T$.

Now we prove a linearization result. For $g \in C_0^\infty(\Omega \times (0, T))$, we use $u_g$ to denote the solution of the linear problem

$$\begin{cases}
\partial_t^\alpha u + (-\Delta)^s u + a_1(x, t)u = 0 & \Omega \times (0, T) \\
u = g & \Omega \times (0, T) \\
u = 0 & \Omega \times \{0\}
\end{cases}$$

and we use $u_{\lambda, g}$ to denote the solution of the semilinear problem

$$\begin{cases}
\partial_t^\alpha u + (-\Delta)^s u + a(x, t, u) = 0 & \Omega \times (0, T) \\
u = \lambda g & \Omega \times (0, T) \\
u = 0 & \Omega \times \{0\}
\end{cases}$$

in the rest of this section.

Proposition 4.3. Let $w_{\lambda, g} := u_g - \frac{u_{\lambda, g}}{\lambda}$. Then $\lim_{\lambda \to 0} w_{\lambda, g} = 0$ in $C([0, T]; L^\infty(\Omega))$.

Proof. Note that $w_{\lambda, g} = 0$ in $\Omega \times (0, T)$ and

$$\partial_t w_{\lambda, g} + (-\Delta)^s w_{\lambda, g} + a_1(x, t)w_{\lambda, g} = \frac{1}{\lambda} \sum_{k=2}^m a_k(x, t)|u_{\lambda, g}|^{b_k} u_{\lambda, g}$$

in $\Omega \times (0, T)$. By the $L^\infty$ estimate (Proposition 4.1), we have

$$||w_{\lambda, g}||_{L^\infty} \leq \frac{T^\alpha}{\lambda \Gamma(\alpha + 1)} ||\sum_{k=2}^m a_k(x, t)|u_{\lambda, g}|^{b_k} u_{\lambda, g}||_{L^\infty(\Omega \times (0, T))}$$

and $||u_{\lambda, g}||_{L^\infty} \leq \lambda ||g||_{L^\infty(\Omega \times (0, T))}$ so

$$||w_{\lambda, g}||_{L^\infty} \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \sum_{k=2}^m \lambda^{b_k} ||a_k(x, t)||_{L^\infty(\Omega \times (0, T))} ||g||_{L^\infty(\Omega \times (0, T))}^{b_k+1},$$

which implies $||w_{\lambda, g}||_{L^\infty} \to 0$ as $\lambda \to 0$. \qed
4.2 Inverse problem

We will use the same argument as the one in [21] to prove Theorem 1.2 based on the unique continuation property of the fractional Laplacian (Proposition 2.3), the Runge approximation property (Proposition 3.6), the $L^\infty$ estimate (Proposition 4.1) and the linearization result (Proposition 4.3). We remark that this kind of regimes work for solving Calderón type inverse problems for a broad class of fractional evolutionary operators involving the fractional Laplacian (see Theorem 2.18 in [16] for more details). To avoid redundancy, we will skip inessential steps in the proof. We refer readers to Subsection 4.2 in [22] for full details.

Proof. Based on Proposition 2.3, the assumption (11) implies that

$$u_{\lambda,g}^{(1)} = u_{\lambda,g}^{(2)} =: u_{\lambda,g}$$

in $\mathbb{R}^n \times (0,T)$. Hence we have

$$(a_1^{(1)} - a_1^{(2)}) u_{\lambda,g} = R_1^{(2)}(x,t,u_{\lambda,g}) - R_1^{(1)}(x,t,u_{\lambda,g})$$

in $\Omega \times (0,T)$ where

$$R_j^{(i)}(x,t,z) := \sum_{k=j+1}^m a_k^{(i)}(x,t)|z|^{b_k}z.$$ 

Now we note that

$$||a_1^{(1)} - a_1^{(2)}||_{L^2(\Omega \times (0,T))} \leq ||a_1^{(1)} - a_1^{(2)}||_{L^\infty} ||1 - \frac{u_{\lambda,g}}{\lambda}||_{L^2(\Omega \times (0,T))}$$

$$+ \frac{1}{\lambda} ||(a_1^{(1)} - a_1^{(2)}) u_{\lambda,g}||_{L^2(\Omega \times (0,T))}.$$  

For given $\delta > 0$, by Proposition 3.6 we can choose $g \in C_c^{\infty}(W_1 \times (0,T))$ s.t.

$$||1 - u_g||_{L^2(\Omega \times (0,T))} \leq \delta$$

and for this chosen $g$, we have

$$||1 - \frac{u_{\lambda,g}}{\lambda}||_{L^2(\Omega \times (0,T))} \leq 2\delta$$

for small $\lambda$ by Proposition 4.3. Since Proposition 4.1 implies that

$$||u_{\lambda,g}||_{L^\infty} \leq \lambda ||g||_{L^\infty(\Omega \times (0,T))},$$

by (50) we have

$$\frac{1}{\lambda} ||(a_1^{(1)}(\cdot,t) - a_1^{(2)}(\cdot,t)) u_{\lambda,g}(\cdot,t)||_{L^2(\Omega)}$$

$$\leq C' \left( \sum_{k=2}^m \lambda^{b_k} (||a_k^{(1)}||_{L^\infty} + ||a_k^{(2)}||_{L^\infty}) ||g||_{L^\infty}^{b_k+1} \right) \to 0$$

as $\lambda \to 0$. Then by (51) and (52) we get

$$||a_1^{(1)} - a_1^{(2)}||_{L^2(\Omega \times (0,T))} \leq 2\delta ||a_1^{(1)} - a_1^{(2)}||_{L^\infty(\Omega \times (0,T))}.$$
Now we conclude that $a_1^{(1)} = a_1^{(2)}$ since $\delta$ is arbitrary.

Iteratively, once we have shown $a_j^{(1)} = a_j^{(2)}$ ($1 \leq j \leq m' - 1$), we have

$$(a_m^{(1)}(x, t) - a_m^{(2)}(x, t))|_{u_{\lambda,g}}^{b_{m'}} = R_m^{(2)}(x, t, u_{\lambda,g}) - R_m^{(1)}(x, t, u_{\lambda,g})$$

in $\Omega \times (0, T)$. Then we can repeat the procedure above to conclude that $a_m^{(1)} = a_m^{(2)}$. $\square$

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