Non-linear maximum rank distance codes in the cyclic model for the field reduction of finite geometries

N. Durante and A. Siciliano

Abstract

In this paper we construct infinite families of non-linear maximum rank distance codes by using the setting of bilinear forms of a finite vector space. We also give a geometric description of such codes by using the cyclic model for the field reduction of finite geometries and we show that these families contain the non-linear maximum rank distance codes recently provided by Cossidente, Marino and Pavese.

1 Introduction

Let $M_{m,m'}(\mathbb{F}_q)$, $m \leq m'$, be the rank metric space of all the $m \times m'$ matrices with entries in the finite field $\mathbb{F}_q$ with $q$ elements, $q = p^h$, $p$ a prime. The distance between two matrices by definition is the rank of their difference. An $(m,m',q;s)$-rank distance code (also rank metric code) is any subset $X$ of $M_{m,m'}(\mathbb{F}_q)$ such that the minimum distance between two of its distinct elements is $s+1$. An $(m,m',q;s)$-rank distance code is said to be linear if it is a linear subspace of $M_{m,m'}(\mathbb{F}_q)$.

It is known [11] that the size of an $(m,m',q;s)$-rank distance code $X$ is bounded by the Singleton-like bound:

$$|X| \leq q^{m'(m-s)}.$$ 

When this bound is achieved, $X$ is called an $(m,m',q;s)$-maximum rank distance code, or $(m,m',q;s)$-MRD code, for short.

Although MRD codes are very interesting by their own and they caught the attention of many researchers in recent years [11] [9] [32], such codes have also applications
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in error-correction for random network coding [18, 22, 37], space-time coding [38] and cryptography [17, 36].

Obviously, investigations of MRD codes can be carried out in any rank metric space isomorphic to $M_{m,m'}(\mathbb{F}_q)$. In his pioneering paper [11], Ph. Delsarte constructed linear MRD codes for all the possible values of the parameters $m, m', q$ and $s$ by using the framework of bilinear forms on two finite-dimensional vector spaces over a finite field (Delsarte used the terminology Singleton systems instead of maximum rank distance codes).

Few years later, Gabidulin [16] independently constructed Delsarte’s linear MRD codes as evaluation codes of linearized polynomials over a finite field [26]. That construction was generalized in [21] and these codes are now known as Generalized Gabidulin codes.

In the case $m' = m$, a different construction of Delsarte’s MRD codes was given by Cooperstein [7] in the framework of the tensor product of a vector space over $\mathbb{F}_q$ by itself. Very recently, Sheekey [35] and Lunardon, Trombetti and Zhou [28] provide some new linear MRD codes by using linearized polynomials over $\mathbb{F}_q^m$.

In finite geometry, $(m,m,q;m-1)$-MRD codes are known as spread sets [12]. To the extent of our knowledge the only non-linear MRD codes that are not spread sets are the $(3,3,q;1)$-MRD codes constructed by Cossidente, Marino and Pavese in [8]. They got such codes by looking at the geometry of certain algebraic curves of the projective plane $\text{PG}(2,q^3)$. Such curves, called $C_{1}^{q}$-sets, were introduced and studied by Donati and Durante in [13]. In this paper, we construct infinite families of non-linear $(m,m,q;m-2)$-MRD codes, for $q \geq 3$ and $m \geq 3$. We also show that the Cossidente, Marino and Pavese non-linear MRD codes belong to these families. Our investigation will carry out in the framework of bilinear forms on a finite dimensional vector space over $\mathbb{F}_q$.

Let $\Omega = \Omega(V,V)$ be the set of all bilinear forms on $V$, where $V = V(m,q)$ denotes an $m$-dimensional vector space over $\mathbb{F}_q$. Clearly, $\Omega$ is an $m^2$-dimensional vector space over $\mathbb{F}_q$.

The left radical $\text{Rad}(f)$ of any $f \in \Omega$ by definition is the subspace of $V$ consisting of all vectors $v$ satisfying $f(v,v') = 0$ for every $v' \in V$. The rank of $f$ is the codimension of $\text{Rad}(f)$, i.e.

$$\text{rk}(f) = m - \dim_{\mathbb{F}_q}(\text{Rad}(f)).$$

Let $u_1, \ldots, u_m$ be a basis of $V$. For a given $f \in \Omega$, the matrix $(f(u_i,u_j))_{i,j=1,\ldots,m}$,
is called the matrix of \( f \) in the basis \( u_1, \ldots, u_m \) and the map

\[
\nu = \nu_{\{u_1, \ldots, u_m\}} : \Omega \to M_{m,m}(\mathbb{F}_q)
\]

\[
f \mapsto (f(u_i, u_j))_{i,j=1,\ldots,m}
\]

is an isomorphism of rank metric spaces giving \( \text{rk}(f) = \text{rk}(\nu(f)) \).

The group \( H = \text{GL}(V) \times \text{GL}(V) \) acts on \( \Omega \) as a subgroup of \( \text{Aut}_{\mathbb{F}_q}(\Omega) \): for every \((g, g') \in H\), the \((g, g')\)-image of any \( f \in \Omega \) is defined to be the bilinear form \( f^{(g, g')} \) given by

\[
f^{(g, g')}(v, v') = f(gv, g'v').
\]

Any \( \theta \in \text{Aut}(\mathbb{F}_q) \) naturally defines a semilinear transformation of \( V \). For any \( f \in \Omega \) and \( \theta \in \text{Aut}(\mathbb{F}_q) \), we can define the bilinear form \( f^\theta \) given by

\[
f^\theta(v, v') = f(v^\theta, v'^\theta).
\]

The involutorial operator \( \top : f \in \Omega \to f^\top \in \Omega \), where \( f^\top \) is given by

\[
f^\top(v, v') = f(v', v),
\]

is an automorphism of \( \Omega \). It turns out that the above automorphisms are all the elements in \( \text{Aut}_{\mathbb{F}_q}(\Omega) \), i.e. \( \text{Aut}_{\mathbb{F}_q}(\Omega) = (\text{GL}(V) \times \text{GL}(V)) \rtimes \langle \top \rangle \rtimes \text{Aut}(\mathbb{F}_q) \).

Two MRD codes \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are said to be equivalent if there exists \( \varphi \in \text{Aut}_{\mathbb{F}_q}(\Omega) \) such that \( \mathcal{X}_2 = \mathcal{X}_1^\varphi \).

This paper is organized as follows. In Section 2 we introduce a cyclic model of \( \Omega \). In this model we construct infinite families of non-linear MRD codes. More precisely, for \( q \geq 3, m \geq 3 \) and \( I \) any subset of \( \mathbb{F}_q \setminus \{0, 1\} \), we provide a subset \( \mathcal{F}_{m,q,I} \) of \( \Omega \) which turns out to be a non-linear \((m, m, q; m-2)\)-MRD code (Theorem 2.19).

In Section 3 we give a geometric description of such codes. If a given rank distance code \( \mathcal{X} \) is considered as a subset of \( V(m^2, q) \), then one can consider the corresponding set of projective points in \( \text{PG}(m^2 - 1, q) \) under the canonical homomorphism \( \psi : \text{GL}(V(m^2, q)) \to \text{PGL}(m^2, q) \). We prove (Theorem 3.5) that the projective set defined by \( \mathcal{F}_{m,q,I} \), with \(|I| = k\), is a subset of a Desarguesian \( m \)-spread of \( \text{PG}(m^2 - 1, q) \) consisting of two spread elements, \( k \) pairwise disjoint Segre varieties \( S_{m,m}(\mathbb{F}_q) \) and \( q - 1 - k \) hyperreguli. Additionally, if one consider the projective space \( \text{PG}(m^2 - 1, q) \) as the field reduction of \( \text{PG}(m - 1, q^m) \) over \( \mathbb{F}_q \), then the projective set defined by \( \mathcal{F}_{m,q,I} \) is, in fact, the field reduction of the union of two projective points, \( k \) mutually disjoint \((m - 1)\)-dimensional \( \mathbb{F}_q \)-subgeometries and \( q - 1 - k \) scattered \( \mathbb{F}_q \)-linear sets of pseudoregulus type of \( \text{PG}(m - 1, q^m) \). The main tool we use to get the above geometric description is the field reduction of \( V(m, q^m) \) over \( \mathbb{F}_q \) in the cyclic model for the tensor product \( \mathbb{F}_q^m \otimes V \) as described in [7].
2 The non-linear MRD codes in the cyclic model of bilinear forms

In the paper [7], the cyclic model of the \( m \)-dimensional vector space \( V = V(m, q) \) over \( \mathbb{F}_q \) was introduced by taking eigenvectors, say \( v_1, \ldots, v_m \), of a given Singer cycle \( \sigma \) of \( V \), where a Singer cycle of \( V \) is an element of \( \text{GL}(V) \) of order \( q^m - 1 \). Since the vectors \( v_1, \ldots, v_m \) have distinct eigenvalues over \( \mathbb{F}_q \), they form a basis of the extension \( \hat{V} = V(m, q^m) \) of \( V \). In this basis the vector space \( V \) is represented by

\[
V = \left\{ \sum_{j=1}^{m} a^{q^j-1} v_j : a \in \mathbb{F}_{q^m} \right\}.
\]

We call \( v_1, \ldots, v_m \) a Singer basis of \( V \) and the above representation is called the cyclic model for \( V \) [19, 15].

The set of all 1-dimensional \( \mathbb{F}_q \)-subspaces of \( \hat{V} \) spanned by vectors in the cyclic model for \( V \) is called the cyclic model for the projective space \( \text{PG}(V) \). Note that the above cyclic model corresponds to the cyclic model of \( \text{PG}(V) \) where the points are identified with the elements of the group \( \mathbb{Z}_{q^m-1+q^{m-2}+\cdots+q+1} \) [19, pp. 95–98] [15]. Very recently, the cyclic model for \( V(3, q) \) has been used to give an alternative model for the triality quadric \( Q^+(7, q) \) [2].

Let \( \hat{V}^* \) be the dual vector space of \( \hat{V} \) with basis \( v_1^*, \ldots, v_m^* \), the dual basis of the Singer basis \( v_1, \ldots, v_m \). Then the dual vector space of \( V \) is

\[
V^* = \left\{ \sum_{i=1}^{m} \alpha^{q^{i-1}} v_i^* : \alpha \in \mathbb{F}_{q^m} \right\}.
\]

A linear transformation from \( V \) to itself is called an endomorphism of \( V \). We will denote the set of all endomorphisms of \( V \) by \( \text{End}(V) \).

An \( m \times m \) Dickson matrix (or \( q \)-circulant matrix) over \( \mathbb{F}_{q^m} \) is a matrix of the form

\[
D_{(a_0, a_1, \ldots, a_{m-1})} = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{m-1} \\
a_0^q & a_0 & \cdots & a_{m-2}^q \\
\vdots & \vdots & \ddots & \vdots \\
a_0^{q^{m-1}} & a_2^{q^{m-1}} & \cdots & a_0^{q^{m-1}}
\end{pmatrix}
\]

with \( a_i \in \mathbb{F}_{q^m} \). We say that the above matrix is generated by the array \( (a_0, a_1, \ldots, a_{m-1}) \).

Let \( D_m(\mathbb{F}_{q^m}) \) denote the Dickson matrix algebra formed by all \( m \times m \) Dickson matrices over \( \mathbb{F}_{q^m} \). The set \( B_m(\mathbb{F}_{q^m}) \) of all invertible Dickson \( m \times m \) matrices is known as the Betti-Mathieu group [6].
Proposition 2.1. \[\text{[39, Lemma 4.1]}\] \(\text{End}(V) \simeq \mathcal{D}_m(\mathbb{F}_{q^m})\) and \(\text{GL}(V) \simeq \mathcal{B}_m(\mathbb{F}_{q^m})\).

A polynomial of the form
\[
L(x) = \sum_{i=0}^{m-1} \alpha_i x^{q^i}, \quad \alpha_i \in \mathbb{F}_{q^m},
\]
is called a linearized polynomial (or \(q\)-polynomial) over \(\mathbb{F}_{q^m}\). It is known that every endomorphism of \(\mathbb{F}_{q^m}\) over \(\mathbb{F}_q\) can be represented by a unique \(q\)-polynomial [33].

Let \(\mathcal{L}_m(\mathbb{F}_{q^m})\) be the set of all \(q\)-polynomials over \(\mathbb{F}_{q^m}\). In the paper [39], it was showed that the map
\[
\varphi : \mathcal{L}_m(\mathbb{F}_{q^m}) \longrightarrow \mathcal{D}_m(\mathbb{F}_{q^m})
\]
\[
\sum_{i=0}^{m-1} \alpha_i x^{q^i} \longmapsto D(\alpha_0, \ldots, \alpha_{m-1})
\]
is an isomorphism between the non-commutative \(\mathbb{F}_q\)-algebras \(\mathcal{L}_m(\mathbb{F}_{q^m})\) and \(\mathcal{D}_m(\mathbb{F}_{q^m})\). From Proposition 2.1 we see that any Singer basis of \(V\) realizes this isomorphism.

Proposition 2.2. Let \(v_1, \ldots, v_n\) be a Singer basis of \(V\). Then the matrix of any \(f \in \Omega\) with respect to \(v_1, \ldots, v_n\) is an \(m \times m\) Dickson matrix. Conversely, every \(m \times m\) Dickson matrix defines a bilinear form on \(V \times V\).

Proof. Let \(D_\mathbf{a}\) be an \(m \times m\) Dickson matrix generated by the \(m\)-ple \(\mathbf{a} = (a_0, a_1, \ldots, a_{m-1})\) over \(\mathbb{F}_{q^m}\). Let \(f_\mathbf{a}\) be the bilinear mapping on \(\widehat{V} \times \widehat{V}\) defined by
\[
f_\mathbf{a}(v_i, v_j) = a_{m-i+j}^{q^i-1} \quad \text{for } i, j = 1, \ldots, m
\]
where subscripts are taken modulo \(m\), and then extended over \(\widehat{V}\) by linearity. Set \(L_\mathbf{a}(x) = \sum_{i=0}^{m-1} a_i x^{q^i}\) and let \(\text{Tr}\) denote the trace function from \(\mathbb{F}_{q^m}\) onto \(\mathbb{F}_q\):
\[
\text{Tr} : y \in \mathbb{F}_{q^m} \rightarrow \text{Tr}(y) = \sum_{j=0}^{m-1} y^{q^j} \in \mathbb{F}_q.
\]

It is easily seen that the action of \(f_\mathbf{a}\) on \(V \times V\) is given by
\[
f_\mathbf{a}(v, v') = f_\mathbf{a}(x, x') = \text{Tr}(L_\mathbf{a}(x')x),
\]
with \(v = \sum_{i=1}^{m} x^{q^i-1}v_i, v' = \sum_{j=1}^{m} x^{q^j-1}v_j\), which is a bilinear form on \(V \times V\). The assertion follows from consideration on the size of \(\mathcal{D}_m(\mathbb{F}_{q^m})\). \(\Box\)
For any $m$-ple $a = (a_0, \ldots, a_{m-1})$ over $\mathbb{F}_{q^m}$, $f_a$ will denote the bilinear form having matrix $D_a$ in the Singer basis $v_1, \ldots, v_m$. For any set $\mathcal{A}$ of $m$-ples over $\mathbb{F}_{q^m}$ we put

$$\mathcal{F}_A = \{ f_a \in \Omega : a \in \mathcal{A} \}.$$

**Corollary 2.3.** Let $a = (a_0, \ldots, a_{m-1})$. Then

$$\nu_{\{v_1, \ldots, v_m\}} : \Omega \rightarrow D_m(\mathbb{F}_{q^m}) \quad f_a \mapsto D(a_0, \ldots, a_{m-1}) \quad (3)$$

is an isomorphism of rank metric spaces giving $\text{rk}(f_a) = \text{rk}(D(a_0, \ldots, a_{m-1}))$.

**Remark 2.4.** By Proposition 2.1, $\text{Aut}_{\mathbb{F}_q}(\Omega)$ is represented by the group $\left( \mathcal{B}_m(\mathbb{F}_{q^m}) \times \mathcal{B}_m(\mathbb{F}_{q^m}) \right) \rtimes (t) \rtimes \text{Aut}(\mathbb{F}_q)$ in the Singer basis $v_1, \ldots, v_m$. Here, $t$ denote transposition in $M_{m,m}(\mathbb{F}_{q^m})$ and it corresponds to the operator $\top$.

**Remark 2.5.** Note that (2) coincides with the bilinear form (6.1) in [11] when $m' = m$.

**Remark 2.6.** Since a change of basis in $\hat{V} \times \hat{V}$ preserves the rank of bilinear forms, for any given $f \in \Omega$ we can consider its matrix representation in the Singer basis $v_1, \ldots, v_m$. Therefore, we can assume $f = f_a$ for some $m$-ple $a$ over $\mathbb{F}_{q^m}$, so that $\text{Rad}(f_a)$ is the set of vectors $v' = x'v_1 + \ldots + x'q^{m-1}v_m \in V$, $x' \in \mathbb{F}_{q^m}$, such that $L_a(x') = 0$.

We are now in position to construct non-linear MRD codes as subsets of $\Omega$.

Let $N$ denote the norm map from $\mathbb{F}_{q^m}$ onto $\mathbb{F}_q$:

$$N : x \in \mathbb{F}_{q^m} \mapsto N(x) = \prod_{j=0}^{m-1} x^{q^j} \in \mathbb{F}_q.$$ 

For every nonzero element $\alpha \in \mathbb{F}_{q^m}$, let

$$\pi_\alpha = \{ (\lambda x, \lambda x^q, \lambda x^{1+q}, x^{q^2}, \ldots, \lambda x^{1+q+\ldots+q^{m-2}} x^{q^{m-1}}) : \lambda, x \in \mathbb{F}_{q^m} \setminus \{0\} \}.$$ 

**Remark 2.7.** The matrix of the Singer cycle $\sigma$ of $V$ in the basis $v_1, \ldots, v_m$ is $\text{diag}(\mu, \mu^q, \ldots, \mu^{q^{m-1}})$, where $\mu$ is a generator of the multiplicative group of $\mathbb{F}_{q^m}$ [2]. If $S$ is the Singer cyclic group generated by $\sigma$, then the set $\mathcal{F}_{\pi_\alpha}$ is the $(S \times S)$-orbit of the bilinear form $f_a$, with $a = (1, \alpha, \alpha^{1+q}, \ldots, \alpha^{1+q+\ldots+q^{m-2}})$. It turns out that the bilinear forms in $\mathcal{F}_{\pi_\alpha}$ have constant rank.
Proposition 2.8. $\pi_\alpha = \pi_\beta$ if and only if $N(\alpha) = N(\beta)$.

Proof. Let $\alpha, \beta \in \mathbb{F}_{q^m} \setminus \{0\}$ such that $N(\alpha) = N(\beta)$. By Remark 2.7 it suffices to show that $(1, \alpha, \alpha^{1+q}, \ldots, \alpha^{1+q+\ldots+q^{m-2}})$ is in $\pi_\beta$.

Since $N(\alpha) = N(\beta)$, then $\alpha = \beta c^{q-1}$ for some $c \in \mathbb{F}_{q^m} \setminus \{0\}$. As $(1 + q + \ldots + q^k)(q-1) = q^{k+1} - 1$, we have

$$\alpha^{1+q+\ldots+q^k} = c^{-1}\beta^{1+q+\ldots+q^k}.$$ 

Conversely, let $\pi_\alpha = \pi_\beta$. Then

$$1 = \lambda x \quad \alpha = \lambda \beta x q \quad \alpha^{1+q+\ldots+q^{m-2}} = \lambda \beta^{1+q+\ldots+q^{m-2}} x^{q^{m-1}}$$

for some $\lambda, x \in \mathbb{F}_{q^m} \setminus \{0\}$. From the last equation we get

$$\alpha^{q+q^2+\ldots+q^{m-1}} = \lambda^q \beta^{q+q^2+\ldots+q^{m-1}} x.$$ 

By taking into account the first and second equation of (4) we get

$$N(\alpha) = \lambda^q \lambda N(\beta) x x^q = N(\beta).$$

We will write $\pi_a$ instead of $\pi_\alpha$, if $\alpha$ is an element of $\mathbb{F}_{q^m} \setminus \{0\}$ with $N(\alpha) = a$.

Lemma 2.9. Every $\pi_a$ has size $(q^m - 1)^2/(q - 1)$.

Proof. Let $\alpha \in \mathbb{F}_{q^m} \setminus \{0\}$ with $N(\alpha) = a$. Clearly, we have

$$(\lambda x, \lambda \alpha x^q, \lambda \alpha^{1+q} x^{q^2}, \ldots, \lambda \alpha^{1+q+\ldots+q^{m-2}} x^{q^{m-1}}) = (\rho y, \rho \alpha y^q, \rho \alpha^{1+q} y^{q^2}, \ldots, \rho \alpha^{1+q+\ldots+q^{m-2}} y^{q^{m-1}})$$

if and only if $\lambda x^{q^i} = \rho y^{q^i}$, for $i = 0, \ldots, m - 1$. If we compare the equalities with $i = 0$ and $i = 1$, we get $x^{q-1} = y^{q-1}$. For every fixed $x \in \mathbb{F}_{q^m}$ there are exactly $q - 1$ elements $y$ in $\mathbb{F}_{q^m}$ such that $y^{q-1} = x^{q-1}$.

Let $\lambda$ and $x$ be fixed elements in $\mathbb{F}_{q^m} \setminus \{0\}$. Then, for each element $y \in \mathbb{F}_{q^m}$ such that $y^{q-1} = x^{q-1}$ we get the unique element $\rho = \lambda x y^{-1}$ and the result is proved.

Lemma 2.10. i) If $a \in \pi_1$, then $\text{rk}(f_a) = 1$. 


ii) If \( a, b \in \mathbb{F}_q \setminus \{0, 1\} \), then \( \text{rk}(f_a - f_b) \geq m - 1 \), for any \( a \in \pi_a \) and \( b \in \pi_b \), with \( b \neq a \) if \( a = b \).

**Proof.** i) Let \( a = (\lambda x, \lambda x^q, \ldots, \lambda x^{q^{m-1}}) \in \pi_1 \). It suffices to note that \( L_a(z) = (\lambda x)z + (\lambda x^q)z^q + \ldots (\lambda x^{q^{m-1}})z^{q^{m-1}} = 0 \) is the equation of a hyperplane in the cyclic model of \( V \).

ii) By Remark 2.7 we can assume \( a = (1, \alpha, \ldots, a^{1+\ldots+q^{m-2}}) \), with \( N(\alpha) = a \neq 1 \).

Let \( b = (\lambda x, \lambda \beta x^q, \ldots, \lambda \beta^{1+q^{m-2}} x^{q^{m-1}}) \), with \( N(\beta) = b \neq 1 \).

Suppose there exist \( z_1, z_2 \in \mathbb{F}_q \) linearly independent over \( \mathbb{F}_q \) such that \( L_{a-b}(z_i) = 0 \). Then we get

\[
(1 - \lambda x)z_i + (\alpha - \lambda \beta x^q)z_i^q + \ldots + (\alpha^{1+\ldots+q^{m-2}} - \lambda \beta^{1+\ldots+q^{m-2}} x^{q^{m-1}})z_i^{q^{m-1}} = 0
\]

and

\[
(\alpha^{q+\ldots+q^{m-1}} - \lambda \beta^{q+\ldots+q^{m-1}} x)z_i + (1 - \lambda^q x^q)z_i^q + \ldots + (\alpha^{q+\ldots+q^{m-2}} - \lambda \beta^{q+\ldots+q^{m-2}} x^{q^{m-1}})z_i^{q^{m-1}} = 0,
\]

for \( i = 1, 2 \).

After subtracting Equation (5) side-by-side from Equation (6) multiplied by \( \alpha \), we get

\[
[a - 1 + (\lambda - \lambda^q \alpha \beta^{q+\ldots+q^{m-1}})x]z_i + (\lambda \beta - \lambda^q \alpha) x^q z_i^q + \ldots + (\lambda \beta - \lambda^q \alpha) \beta^{q+\ldots+q^{m-2}} x^{q^{m-1}} z_i^{q^{m-1}} = 0,
\]

for \( i = 1, 2 \). Then, the \( m \)-ple

\[
(a - 1 + (\lambda - \lambda^q \alpha \beta^{q+\ldots+q^{m-1}})x, (\lambda \beta - \lambda^q \alpha)x^q, \ldots, (\lambda \beta - \lambda^q \alpha) \beta^{q+\ldots+q^{m-2}} x^{q^{m-1}})
\]

is a solution of the linear system

\[
\begin{cases}
  z_1 X_1 + z_2^q X_2 + \ldots + z_1^{q^{m-1}} X_m = 0 \\
  z_2 X_1 + z_2^q X_2 + \ldots + z_2^{q^{m-1}} X_m = 0
\end{cases}
\]

with \( \Delta = \begin{vmatrix} z_1 & z_1^q \\ z_2 & z_2^q \end{vmatrix} \neq 0 \).

The generic solution \((x_1, x_2, \ldots, x_m)\) of (9) has

\[
x_1 = -\Delta^{-1} \begin{pmatrix} z_1^q \\ z_2^q \end{pmatrix} (0, 1, 2, \ldots, m) = -\Delta^{-1} \begin{pmatrix} z_1^q X_3 + z_2^q X_4 + \ldots + z_1^{q^{m-1}} X_m \\ z_2^q X_3 + z_2^{q^{m-1}} X_m \end{pmatrix}
\]

(10)
and
\[ x_2 = -\Delta^{-1} \left( \begin{array}{c|c|c|c|c} z_1 & z_1^q & z_1^{q^2} & \ldots & z_1^{q^{m-1}} \\ \hline z_2 & z_2^q & z_2^{q^2} & \ldots & z_2^{q^{m-1}} \\ \end{array} \right) x_3 + \begin{array}{c|c|c|c|c} z_1 & z_1^q & z_1^{q^2} & \ldots & z_1^{q^{m-1}} \\ \hline z_2 & z_2^q & z_2^{q^2} & \ldots & z_2^{q^{m-1}} \\ \end{array} x_4 + \ldots + \begin{array}{c|c|c|c|c} z_1 & z_1^q & z_1^{q^2} & \ldots & z_1^{q^{m-1}} \\ \hline z_2 & z_2^q & z_2^{q^2} & \ldots & z_2^{q^{m-1}} \\ \end{array} x_m \right). \] (11)

In the expression (11) set
\[ c_i = \begin{array}{c|c|c|c|c} z_1 & z_1^{q^i} & z_1^{q^{i+1}} & \ldots & z_1^{q^{m-1}} \\ \hline z_2 & z_2^{q^i} & z_2^{q^{i+1}} & \ldots & z_2^{q^{m-1}} \\ \end{array}, \quad i = 3, \ldots, m; \]
in particular \( c_m = \begin{array}{c|c|c|c|c} z_1 & z_1^{q^{m-1}} & z_1^{q^m} & \ldots & z_1^{q^{2m-1}} \\ \hline z_2 & z_2^{q^{m-1}} & z_2^{q^m} & \ldots & z_2^{q^{2m-1}} \\ \end{array} \) giving \( c_m^q = -\Delta \). Similarly, in the expression (10) set
\[ d_i = -\begin{array}{c|c|c|c|c} z_1 & z_1^{q^{i-2}}q & z_1^{q^{i-1}}q & \ldots & z_1^{q^{m-1}}q \\ \hline z_2 & z_2^{q^{i-2}}q & z_2^{q^{i-1}}q & \ldots & z_2^{q^{m-1}}q \\ \end{array}, \quad i = 3, \ldots, m. \]

We have,
\[ d_i = -c_i^{q}, \quad \text{for } i = 4, \ldots, m \]
and \( d_3 = -\Delta^q \). We then write
\[
\begin{align*}
  x_1 &= -\Delta^{-1}(-\Delta^q x_3 - c_3^q x_4 - \ldots - c_{m-1}^q x_m) \\
  x_2 &= -\Delta^{-1}(c_3 x_3 + c_4 x_4 + \ldots + c_m x_m)
\end{align*}
\]

By plugging (8) in the right-hands of the above equalities we get
\[-\Delta^q x_3 - c_3^q x_4 - \ldots - c_{m-1}^q x_m = (\lambda \beta - \lambda^q \alpha)(-\Delta^q \beta^q x^q - c_3^q \beta^q x^q - \ldots - c_{m-1}^q \beta^{q + \ldots + q^{m-2}} x^q) \]
and
\[ c_3 x_3 + c_4 x_4 + \ldots + c_m x_m = (\lambda \beta - \lambda^q \alpha)(c_3 \beta^q x^q + c_4 \beta^q x^q + \ldots + c_m \beta^{q + \ldots + q^{m-2}} x^q). \]

Therefore
\[
\beta^q \left( \frac{-\Delta x_2}{\lambda \beta - \lambda^q \alpha} \right)^q + \frac{-\Delta x_1}{\lambda \beta - \lambda^q \alpha} = -\Delta \beta^{q + \ldots + q^{m-1}} x - \Delta^q x^q \beta^q. \] (12)

From (8), we have \( x_2 = (\lambda \beta - \lambda^q \alpha)x^q \) giving
\[
\beta^q \left( \frac{-\Delta x_2}{\lambda \beta - \lambda^q \alpha} \right)^q = (-1)^q \Delta^q x^q \beta^q.
\]
From (12) it turns out that the value of $x_1$ must satisfy
\[- \frac{\Delta x_1}{\lambda \beta - \lambda q \alpha} = -\Delta \beta^{q^2 + \ldots + q^{m-1}} x \]
giving
\[x_1 = (\lambda \beta - \lambda q \alpha) \beta^{q^2 + \ldots + q^{m-1}} x = (\lambda b - \lambda q \alpha \beta^{q^2 + \ldots + q^{m-1}}) x \]
since $\Delta \neq 0$.

From (8), we have
\[x_1 = (a - 1) + (\lambda - \lambda q \alpha \beta^{q+\ldots+q^{m-1}}) x \]
and therefore
\[(b - 1) \lambda x = a - 1 \]
i.e.,
\[\lambda = \frac{a - 1}{b - 1} x^{-1}. \quad (13)\]

By plugging this value in $b$, we get
\[b = \frac{a - 1}{b - 1} \left(1, \beta x^{q-1}, \beta^{1+q} x^{q^2-1}, \ldots, \beta^{1+q+\ldots+q^{m-2}} x^{q^{m-1}-1} \right) \]
Note that if $b = a$, we can assume $\beta = \alpha$ giving $x \notin \mathbb{F}_q$ as $b \neq a$.

We claim that the bilinear form $(f_a - f_b)$ has maximum rank $m$. Indeed, suppose there exists a nonzero $z \in \mathbb{F}_{q^m}$ such that $L_{a-b}(z) = 0$. By plugging (13) in Equation (7) we get
\[a - 1 \left[(\beta - \alpha (x^{-1})^{q-1}) \beta^{q+\ldots+q^{m-1}} z + (\beta x^{q-1} - \alpha) z + \ldots + (\beta x^{q^{m-1}-1} - \alpha x^{q^{m-1}-q}) \beta^{q+\ldots+q^{m-2}} z^{q^{m-1}} \right] = 0 \]
or, equivalently,
\[\left(\frac{\beta}{x} - \frac{\alpha}{x^q} \right) (\beta^{q+\ldots+q^{m-1}} x z + (xz)^q + \beta^q (xz)^{q^2} + \ldots + \beta^{q+\ldots+q^{m-2}} (xz)^{q^{m-1}}) = 0, \]
where $\frac{\beta}{x} - \frac{\alpha}{x^q} \neq 0$ since either $b \neq a$ or $x^q \neq x$ if $b = a$. Therefore, the following equation holds:
\[\beta^{q+\ldots+q^{m-1}} y + y^q + \beta^q y^{q^2} + \beta^{q+q^2} y^{q^3} + \ldots + \beta^{q+\ldots+q^{m-2}} y^{q^{m-1}} = 0 \quad (14)\]
given
\[\beta^{q+\ldots+q^{m-1}} y + \beta^{1+q^2+\ldots+q^{m-1}} y^q + y^{q^2} + \beta^2 y^{q^3} + \ldots + \beta^{2+\ldots+q^{m-2}} y^{q^{m-1}} = 0. \quad (15)\]

By subtracting Equation (14) from (15) multiplied by $\beta^q$ we get $b = 1$, a contradiction. \(\square\)
For every nonzero element \( \alpha \in \mathbb{F}_{q^m} \), let
\[
J_\alpha = \{ (\lambda x, 0, \ldots, 0, -\lambda \alpha x^{q^m-1}) : \lambda, x \in \mathbb{F}_{q^m} \setminus \{0\} \}.
\]

\textbf{Remark 2.11.} Note that the set \( \mathcal{F}_{J_\alpha} \) is the \((S \times S)\)-orbit of the bilinear form \( f_\alpha \), with \( a = (1, 0, \ldots, 0, -\alpha) \). It turns out that the bilinear forms in \( \mathcal{F}_{J_\alpha} \) have constant rank.

By arguing similarly to the proof of Proposition 2.8 and Lemma 2.9, we get the following result.

\textbf{Lemma 2.12.} Each set \( J_\alpha \) has size \( (q^m - 1)^2/(q - 1) \) and \( J_\alpha = J_\beta \) if and only if \( N(\alpha) = N(\beta) \).

We will write \( J_a \) instead of \( J_\alpha \), if \( \alpha \) is an element of \( \mathbb{F}_{q^m} \) with \( N(\alpha) = a \).

\textbf{Lemma 2.13.} For any \( a = (x, 0, \ldots, 0, y) \) with \( x, y \in \mathbb{F}_{q^m} \) not both zero, \( \text{rk}(f_a) \geq m-1 \).

\textit{Proof.} The bilinear form \( f_a \), is equivalent to the bilinear form \( \hat{f}_a \), with \( \hat{a} = (x, y^q, 0, \ldots, 0) \), via the automorphism \( \top \). The result then follows from Remark 2.5 and Theorem 6.3 in [11]. \( \square \)

\textbf{Corollary 2.14.} Let \( a, b \) be nonzero elements in \( \mathbb{F}_q \). Then \( \text{rk}(f_a - f_b) \geq m-1 \), for any \( a \in J_a \) and \( b \in J_b \), with \( a \neq b \) if \( a = b \).

\textbf{Lemma 2.15.} Let \( a, b \) be distinct nonzero elements in \( \mathbb{F}_q \). Then \( \text{rk}(f_a - f_b) \geq m-1 \) for any \( a \in \pi_a \) and \( b \in J_b \).

\textit{Proof.} By Remark 2.7 we can assume \( a = (1, \alpha, \ldots, \alpha^{1+\ldots+q^m-2}) \) with \( N(\alpha) = a \). By arguing as in the proof of Lemma 2.10 we see that the triple
\[
(a - 1 + (\lambda + \alpha \beta^q \lambda^q)x, -\alpha \lambda^q x^q, -\lambda \beta x^{q^{m-1}})
\]
is a solution of the linear system
\[
\begin{align*}
 z_1 X_1 + z_1^q X_2 + z_1^{q^{m-1}} X_3 &= 0 \\
 z_2 X_1 + z_2^q X_2 + z_2^{q^{m-1}} X_3 &= 0
\end{align*}
\]
for some \( z_1, z_2 \in \mathbb{F}_{q^m} \) linearly independent over \( \mathbb{F}_q \) with \( \Delta = \begin{vmatrix} z_1 & z_1^q \\ z_2 & z_2^q \end{vmatrix} \neq 0 \). Any solution \((x_1, x_2, x_3)\) of \( (17) \) satisfies
\[
x_2 = -\frac{\Delta'}{\Delta} x_3
\]
where $\Delta' = \begin{vmatrix} z_1 & z_2^q & z_2^{q^m-1} \\ z_2 & z_2^q & z_2^{q^m-1} \end{vmatrix}$. Since $\Delta'' = \begin{vmatrix} z_1^q & z_1 & z_2 \\ z_2^q & z_2 & z_2 \\ z_2 & z_2 & z_2 \end{vmatrix} = -\Delta$ we get $x_2 = \frac{1}{\Delta''} x_3$ giving $N(x_2) = N(x_3)$. As a solution of (17), the triple (10) must satisfies $a N(\lambda) N(x) = b N(\lambda) N(x)$ giving either $\lambda x = 0$ or $a = b$, a contradiction. \hfill $\square$

Let $A_1 = \{(x,0,\ldots,0) : x \in F_q^n \setminus \{0\}\}$ and $A_2 = \{(0,0,\ldots,x) : x \in F_q^n \setminus \{0\}\}$.

**Lemma 2.16.** $\text{rk}(f_a) = m$, for any $a \in A_i$, $i = 1,2$. Further, $\text{rk}(f_a - f_b) \geq m - 1$, for any $a \in A_1$ and $b \in A_2$.

**Proof.** The first part can be easily proved by taking the Dickson matrix $D_a$ with $a \in A_i$. The second part follows from Lemma 2.13. \hfill $\square$

**Lemma 2.17.** Let $a \in F_q \setminus \{0,1\}$. Then $\text{rk}(f_a - f_b) \geq m - 1$, for any $a \in \pi_a$ and $b \in A_i$, $i = 1,2$.

**Proof.** By Remark 2.7 we can assume $a = (1,\alpha,\ldots,\alpha^{1+\ldots+q^{m-2}})$ with $N(\alpha) = a$. Let $b = (x,0,\ldots,0)$. By proceeding as in the proof of Lemma 2.10 we see the pair $(a - (1-x),-\alpha x^q)$ is a solution of the linear system

$$\begin{cases}
    z_1 X_1 + z_2^q X_2 = 0 \\
    z_2 X_1 + z_2^q X_2 = 0
\end{cases}$$

with $\Delta = \begin{vmatrix} z_1 & z_2^q \\ z_2 & z_2^q \end{vmatrix} \neq 0$. Then the above linear system has the unique solution $(0,0)$ giving $x = 0$ and $a = 1$, a contradiction.

For $i = 2$, similar arguments lead to the same contradiction. \hfill $\square$

**Lemma 2.18.** Let $a \in F_q \setminus \{0\}$. Then $\text{rk}(f_a - f_b) \geq m - 1$, for any $a \in J_a$ and $b \in A_i$, $i = 1,2$.

**Proof.** Use Lemma 2.13. \hfill $\square$

Finally, we have the main theorem.

**Theorem 2.19.** Let $q > 2$ be a prime power and $m \geq 3$ a positive integer. For any subset $I$ of $F_q \setminus \{0,1\}$, put $\Pi_I = \bigcup_{a \in I} \pi_a$, $\Gamma_I = \bigcup_{b \in F_q \setminus \{I \cup \{0\}\}} J_b$ and set

$$\mathcal{A}_{m,q;I} = \Pi_I \cup \Gamma_I \cup A_1 \cup A_2 \cup \{0\}$$

where $0$ is the zero $m$-ple. Then the subset $\mathcal{F}_{m,q;I} = \{f_a : a \in \mathcal{A}_{m,q;I}\}$ of $\Omega$ is a non-linear $(m,m,q;m-2)$-MRD code.
Proof. By Lemmas 2.9, 2.12 we get that $\mathcal{A}_{m,q;I}$ has size $q^{2m}$. By Lemmas 2.10, 2.13, 2.15, 2.16, and Corollary 2.14 we see that $\mathcal{F}_{m,q;I}$ has minimum distance $m - 1$, i.e. it is a $(m, m, q; m - 2)$-MRD code. To show the non-linearity of $\mathcal{F}_{m,q;I}$, it suffices to find two distinct elements in it whose $\mathbb{F}_q$-span is not contained in $\mathcal{F}_{m,q;I}$.

Let $f_a \in \mathcal{F}_{A_a}$ and $f_b \in \mathcal{F}_{\pi_a}$, $a \in I$. By corollary 2.3, we can work with the Dickson matrices $D_a$ and $D_b$, or equivalently, with $m$-plies $a$ and $b$ as arrays in $V(m, q^m)$. Let $a = (0, \ldots, 0, \mu)$ and $b = (\lambda x, \lambda x q, \ldots, \lambda q^{1+q^{m-2}} x q^{m-1})$. Suppose $a + b \in \pi_b$, for some $b \in \mathbb{F}_q$. Then

$$\left(\frac{\lambda \alpha^{1+\ldots+q^{m-3}} x q^{m-2}}{\lambda \alpha^{1+\ldots+q^{m-4}} x q^{m-2}}\right)^q = \alpha^{q^{m-2} x q^{m-1} - q^{m-2}} = \frac{\mu + \lambda \alpha^{1+\ldots+q^{m-2}} x q^{m-1}}{\lambda \alpha^{1+\ldots+q^{m-3}} x q^{m-2}}$$

giving $\mu = 0$. Therefore, the subspace spanned by $a$ and $b$ meets trivially every $\pi_b$ if $b \neq a$, or just in the 1-dimensional subspace spanned by $b$ if $b = a$. The result then follows. \hfill $\square$

3 A geometric description for the non-linear MRD codes

For any $v \in V(t, q^s) \setminus \{0\}$, $[v]$ will denote the point of $\text{PG}(t - 1, q^s)$ defined by $v$ via the canonical homomorphism $\psi : \text{GL}(V(t, q^s)) \rightarrow \text{PGL}(t, q^s)$. For any subset $X$ of $V(t, q^s) \setminus \{0\}$, we set $[X] = \{[v] : v \in X, v \neq 0\}$. The set $[X]$ is said to be an $\mathbb{F}_q$-linear set of rank $r$ if $X$ is an $r$-dimensional $\mathbb{F}_q$-linear subspace of $V(t, q^s)$. An $\mathbb{F}_q$-linear set $[X]$ of rank $r$ is said to be scattered if the size of $[X]$ equals $|\text{PG}(r - 1, q)|$; see [31] for more details on $\mathbb{F}_q$-linear sets and [27] for a relationship between linear MRD-codes and $\mathbb{F}_q$-linear sets.

Consider the set $\mathcal{A}_{m,q;I}$ defined in Theorem 2.19 as a subset of $\hat{V} = V(m, q^m)$, by setting $a_0 v_1 + a_1 v_2 + \ldots + a_{m-1} v_m$, for any $a = (a_0, \ldots, a_{m-1}) \in \mathcal{A}_{m,q;I}$; here, $v_1, \ldots, v_m$ is the Singer basis of $V$ defined in Section 2. Therefore, $[\pi_1] = [V]$ is a scattered $\mathbb{F}_q$-linear set of rank $m$ of $\text{PG}(m - 1, q^m)$ isomorphic to the projective space $\text{PG}(m - 1, q)$.

For any $\alpha \in \mathbb{F}_{q^m} \setminus \{0\}$, the endomorphism

$$\tau_\alpha : \hat{V} \rightarrow \hat{V}$$

$$a_0 v_1 + a_1 v_2 + \ldots + a_{m-1} v_m \mapsto a_0 v_1 + a_1 \alpha v_2 + \ldots + a_{m-1} \alpha^{1+\ldots+q^{m-2}} v_m$$

maps $\pi_1$ into $\pi_a$, with $a = N(\alpha)$, and $J_1$ into $J_b$, with $b = a^{m-1}$.
Let \( W \) be the span of \( v_1 \) and \( v_m \) in \( \hat{V} \). For any \( a \in \mathbb{F}_q \setminus \{0\} \), \([J_a]\) is a scattered \( \mathbb{F}_q \)-linear set of rank \( m \) of \([W]\). In particular \([J_a]\) is a maximum scattered \( \mathbb{F}_q \)-linear set of pseudoregulus type of \([W]\) [24, 29].

Summarizing we have the following result.

**Theorem 3.1.** Let \( q > 2 \) be a prime power and \( m > 2 \) a positive integer. Let \( I \) be any nonempty subset of \( \mathbb{F}_q \setminus \{0,1\} \) with \( k = |I| \). Then, the projective image of \( \mathcal{A}_{m,q,1} \) in \( \text{PG}(m-1,q^m) \) is union of two points \([A_1],[A_2]\), \( k \) mutually disjoint \((m-1)\)-dimensional \( \mathbb{F}_q \)-subgeometries \([\pi_a]\), \( a \in I \), and \( q-1-k \) mutually disjoint \( \mathbb{F}_q \)-linear sets \([J_b]\), \( b \in \mathbb{F}_q \setminus (I \cup \{0\}) \), of pseudoregulus type of rank \( m \) contained in the line spanned by \([A_1]\) and \([A_2]\).

We now investigate the geometry in \( \text{PG}(m^2-1,q) \) of the projective set defined by each MRD code \( F_{m,q,1} \) viewed as a subset of \( V(m^2,q) \).

Let \( V = V(m,q) \) be the \( \mathbb{F}_q \)-span of \( u_1, \ldots, u_m \) and set \( \hat{V} = V(m,q^m) = \mathbb{F}_q^m \otimes V(m,q) \). The rank of a vector \( v = a_1u_1 + a_2u_2 + \ldots + a_m u_m \in \hat{V} \) by definition is the maximum number of linearly independent coordinates \( a_i \) over \( \mathbb{F}_q \).

If we consider \( \mathbb{F}_q^m \) as the \( m \)-dimensional vector space \( V \), then every \( \alpha \in \mathbb{F}_q^m \) can be uniquely written as \( \alpha = x_1u_1 + x_2u_2 + \ldots + x_m u_m \), \( x_i \in \mathbb{F}_q \). Hence, \( \hat{V} \) can be viewed as \( V \otimes V \), the tensor product of \( V \) with itself, with basis \( \{u(i,j) = u_i \otimes u_j : i,j = 1, \ldots, m\} \). Elements of \( V \otimes V \) are called tensors and those of the form \( v \otimes v' \), with \( v, v' \in V \) are called fundamental tensors. In \( \text{PG}(V \otimes V) \), the set of fundamental tensors correspond to the Segre variety \( S_{m,m}(\mathbb{F}_q) \) of \( \text{PG}(V \otimes V) \) [20].

Let \( \phi \) be the map defined by

\[
\phi = \phi(u_1, \ldots, u_m) : \quad \hat{V} \quad \longrightarrow \quad V \otimes V
\]

\[
\alpha_1 u_1 + \ldots + \alpha_m u_m \quad \mapsto \quad \sum_{i=1}^m x_{i1} u(i,1) + \ldots + \sum_{i=1}^m x_{im} u(i,m),
\]

with \( \alpha_k = x_{1k}u_1 + x_{2k}u_2 + \ldots + x_{mk}u_m, x_{ik} \in \mathbb{F}_q \). We call this map the field reduction of \( \hat{V} \) over \( \mathbb{F}_q \) with respect to the basis \( u_1, \ldots, u_m \). The projective space \( \text{PG}(V \otimes V) \) is the field reduction of \( \text{PG}(\hat{V}) \) over \( \mathbb{F}_q \) with respect to the basis \( u_1, \ldots, u_m \).

Under the map \( \phi \), every 1-dimensional subspace \( \langle v \rangle \) of \( \hat{V} \) is mapped to the \( m \)-dimensional \( \mathbb{F}_q \)-subspace \( k_v = \phi(\langle v \rangle) \) of \( V \otimes V \). It turns out that the set \( \mathcal{K} = \{ k_v : v \in \hat{V}, v \neq 0 \} \) is a partition of the nonzero vectors of \( V \otimes V \). In particular \( \mathcal{K} \) is a Desarguesian partition, i.e. the stabilizer of \( \mathcal{K} \) in \( \text{GL}(V \otimes V) \) contains a cyclic subgroup acting regularly on the components of \( \mathcal{K} \) [34, 14].
To any component $k_v$ of $K$ there corresponds a projective $(m - 1)$-dimensional subspace $[k_v]$ of PG($V \otimes V$). The set $S = \{[k_v] : v \in \hat{V}, v \neq 0\}$ is so called a Desarguesian $(m - 1)$-spread of PG($V \otimes V$) \cite{34, 14}.

In addition, the projective set of PG($V \otimes V$) corresponding to the $\phi$-image of the 1-dimensional subspaces spanned by non-zero vectors in $V$ is the Segre variety $S_{m,m}(\mathbb{F}_q)$.

Let $\nu$ be the map defined by
$$
\nu = \nu_{\{u_1, \ldots, u_m\}} : \quad V \otimes V \longrightarrow M_{m,m}(\mathbb{F}_q)
$$
$$
\sum_{i,j} x_{ij} u_{(i,j)} \longrightarrow (x_{ij})_{i,j=1,\ldots,m}.
$$

For every $v = \alpha_1 u_1 + \ldots + \alpha_m u_m \in \hat{V}$, the $k$-th column of the matrix $\nu(\phi(v))$ is the $m$-ple $(x_{1k}, \ldots, x_{mk})$ of the coordinates of $\alpha_k$ with respect to the basis $u_1, \ldots, u_m$ of $\mathbb{F}_q^m$. From \cite{16}, the rank of $v$ equals the rank of $\nu(\phi(v))$, for all $v \in \hat{V}$. In addition, the $\nu$-image of fundamental tensors is precisely the set of rank 1 matrices.

**Remark 3.2.** Evidently, $\nu$ is an isomorphism of rank metric spaces which also provides an isomorphism between the field reduction $V \otimes V$ of $\hat{V}$ with respect to $u_1, \ldots, u_m$ and the metric space $\Omega$ of all bilinear forms on $V = \langle u_1, \ldots, u_m \rangle_{\mathbb{F}_q}$.

Now embed $V \otimes V$ into $\hat{V} \otimes \hat{V}$ by extending the scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^m}$. By taking a Singer basis $v_1, \ldots, v_m$ of $V$ defined by the Singer cycle $\sigma$, Cooperstein \cite{7} defined a cyclic model for $V \otimes V$ within $\hat{V} \otimes \hat{V}$ with basis $v_{(i,j)} = v_i \otimes v_j$, $i, j = 1, \ldots, m$. Let
$$
\Phi(j) = \{\sum_{i=1}^m a^{q^{i-1}} v_{(i,j-1+i)} : a \in \mathbb{F}_{q^m}\},
$$
where the subscript $j-1+i$ is taken modulo $m$. As an $\mathbb{F}_q$-space, $\Phi(j)$ has dimension $m$ and by consideration on dimension we have
$$
V \otimes V = \bigoplus_{j=1}^m \Phi(j);
$$
see \cite{7}. We call this representation the cyclic representation of the tensor product $V \otimes V$.

**Proposition 3.3.** Let $\tilde{\phi}$ be the map defined by
$$
\tilde{\phi} = \phi_{\{v_1, \ldots, v_m\}} : \quad \hat{V} \longrightarrow \hat{V} \otimes \hat{V}
$$
$$
\alpha_1 v_1 + \ldots + \alpha_m v_m \longrightarrow \sum_{i=1}^m \alpha_1^{q^{i-1}} v_{(i,i)} + \ldots + \sum_{i=1}^m \alpha_m^{q^{i-1}} v_{(i,m-1+i)}.
$$
Then $\text{Im}(\tilde{\phi})$ is linearly equivalent to $\text{Im}(\phi)$ in $\hat{V} \otimes \hat{V}$. 
Proof. Let \( v = \sum_{i=1}^{m} \alpha_i v_i \in \hat{V} \) be linear combination of \( k \) vectors of rank 1, \( 1 \leq k \leq m \).

Let \( \tau \) be the change of basis map of \( \hat{V} \) from the basis \( u_1, \ldots, u_m \) to the Singer basis \( v_1, \ldots, v_m \).

Assume \( k = 1 \), i.e. \( v = \lambda(\sum_{i=1}^{m} a^{q^{i-1}} v_i) \), and set \( \lambda = \sum_{i=1}^{m} l_i u_i, a = \sum_{i=1}^{m} x_i u_i \), with \( l_i, x_i \in \mathbb{F}_q \). Therefore, \( v = \lambda \sum_{i=1}^{m} x_i u_i \) and

\[
\tilde{\phi}(v) = (\sum_{i=1}^{m} \lambda^{q^{i-1}} v_i) \otimes (\sum_{i=1}^{m} a^{q^{i-1}} v_i)
\]

\[
= (\sum_{i=1}^{m} l_i u_i) \otimes (\sum_{i=1}^{m} x_i u_i)
\]

\[
= [(\sum_{i=1}^{m} l_i u_i) \otimes (\sum_{i=1}^{m} x_i u_i)]^{(\tau,\tau)}
\]

\[
= [\sum_{i=1}^{m} l_i x_1 u(i_1) + \ldots + \sum_{i=1}^{m} l_i x_m u(i_m)]^{(\tau,\tau)}
\]

\[
= \phi(v)^{(\tau,\tau)}.
\]

Now assume \( v = \lambda_1(\sum_{i=1}^{m} a_1^{q^{i-1}} v_i) + \ldots + \lambda_k(\sum_{i=1}^{m} a_k^{q^{i-1}} v_i), k > 1 \). Set \( \lambda_j = \sum_{i=1}^{m} l_{ij} u_i, a_j = \sum_{i=1}^{m} x_{ij} u_i \), with \( l_{ij}, x_{ij} \in \mathbb{F}_q \). Therefore,

\[
v = \lambda_1(\sum_{i=1}^{m} x_{i1} u_i) + \ldots + \lambda_k(\sum_{i=1}^{m} x_{ik} u_i) = \sum_{i=1}^{m} (\lambda_1 x_{i1} + \ldots + \lambda_k x_{ik}) u_i
\]

giving \( \phi(v) = \sum_{i=1}^{m} (l_{i1} x_{11} + \ldots + l_{ik} x_{1k}) u(i,1) + \ldots + \sum_{i=1}^{m} (l_{i1} x_{m1} + \ldots + l_{ik} x_{mk}) u(i,m) \).

On the other hand we have

\[
\tilde{\phi}(v) = (\sum_{i=1}^{m} \lambda_1^{q^{i-1}} v_i) \otimes (\sum_{i=1}^{m} a_1^{q^{i-1}} v_i) + \ldots + (\sum_{i=1}^{m} \lambda_k^{q^{i-1}} v_i) \otimes (\sum_{i=1}^{m} a_k^{q^{i-1}} v_i)
\]

\[
= (\sum_{i=1}^{m} l_{i1} u_i) \otimes (\sum_{i=1}^{m} x_{i1} u_i) + \ldots + (\sum_{i=1}^{m} l_{ik} u_i) \otimes (\sum_{i=1}^{m} x_{ik} u_i)
\]

\[
= [(\sum_{i=1}^{m} l_{i1} u_i) \otimes (\sum_{i=1}^{m} x_{i1} u_i)]^{(\tau,\tau)} + \ldots + [(\sum_{i=1}^{m} l_{ik} u_i) \otimes (\sum_{i=1}^{m} x_{ik} u_i)]^{(\tau,\tau)}
\]

\[
= [\sum_{i=1}^{m} l_{i1} x_{11} u(i_1) + \ldots + \sum_{i=1}^{m} l_{i1} x_{m1} u(i_m)]^{(\tau,\tau)} + \ldots + [\sum_{i=1}^{m} l_{ik} x_{1k} u(i_1) + \ldots + \sum_{i=1}^{m} l_{ik} x_{mk} u(i_m)]^{(\tau,\tau)}
\]

\[
= \phi(v)^{(\tau,\tau)}.
\]

We call the map \( \tilde{\phi} \) the field reduction of \( \hat{V} \) over \( \mathbb{F}_q \) with respect to the Singer basis \( v_1, \ldots, v_m \) and its image the \textit{cyclic model for the field reduction of} \( \hat{V} \) over \( \mathbb{F}_q \). The projective space whose points are the 1-dimensional \( \mathbb{F}_q \)-subspaces generated by the elements of \( \tilde{\phi}(\hat{V}) \) is the \textit{cyclic model for the field reduction of} \( PG(\hat{V}) \) over \( \mathbb{F}_q \).
3 A geometric description for the non-linear MRD codes

Let \( \tilde{\nu} = \nu_{\{v_1, \ldots, v_m\}} : \hat{V} \otimes \hat{V} \rightarrow M_{m,m}(\mathbb{F}_{q^m}) \) be the map defined by

\[
\sum_{i,j} x_{ij} v_{(i,j)} \rightarrow (x_{ij})_{i,j=1}^{1,m}.
\]

Then, for any \( v = \alpha_1 v_1 + \ldots + \alpha_m v_m \in \hat{V} \), the matrix \( \tilde{\nu}(\phi(v)) \) is the Dickson matrix \( D_{(\alpha_1, \ldots, \alpha_m)} \). Since the cyclic model for the field reduction of \( \hat{V} \) is obtained from the field reduction \( \phi(\hat{V}) \) by changing a basis in \( \hat{V} \otimes \hat{V} \), we get that the rank of \( \tilde{\nu}(\phi(v)) \) equals the rank of \( \nu(\phi(v)) \), for any \( v \in \hat{V} \).

In addition, the element \( k_v = \tilde{\phi}(\langle v \rangle) \) of the \( m \)-partition \( \mathcal{K} \) is

\[
k_v = \left\{ \sum_{i=1}^{m} (\lambda \alpha_1)^{q^{i-1}} v_{(i,i)} + \ldots + \sum_{i=1}^{m} (\lambda \alpha_m)^{q^{i-1}} v_{(i,m-1+i)} : \lambda \in \mathbb{F}_{q^m} \right\}.
\]

In particular, \( \bigcup_{v \in V \setminus \{0\}} \tilde{\nu}(k_v) \) is the set of all rank 1 matrices in \( \mathcal{D}_m(\mathbb{F}_{q^m}) \).

From the arguments above, we see that the set \( \mathcal{F}_{m,q;I} \) can be considered, via the isomorphism (3), as the field reduction of the set \( \mathcal{A}_{m,q;I} \) with respect to the Singer basis \( v_1, \ldots, v_m \).

As \([\pi_1] = [V]\), then the set \( \mathcal{F}_{\pi_1} = \tilde{\phi}(\pi_1) \) defines the Segre variety \( S_{m,m}(\mathbb{F}_q) \) of \( PG(V \otimes V) \) and \( \mathcal{F}_{\pi_a} \) defines a Segre variety projectively equivalent to \( S_{m,m}(\mathbb{F}_q) \) under the element of \( PGL(V \otimes V) \) corresponding to the linear transformation \( \tau_a \) with \( N(\alpha) = a \).

**Remark 3.4.** Note that, whenever \( a \neq 1 \), elements in \( \mathcal{F}_{\pi_a} \) have rank bigger then 1 by Lemma 2.10. This is explained by the fact that the linear transformation of \( V \otimes V = V(m^2, q) \) corresponding to \( \tau_a \) is not in \( Aut_{\mathbb{F}_q}(V \otimes V) \).

Let \( W = (v_1, v_m) \subset \hat{V} \). Then \( \tilde{\phi}(W) \) is a 2\( m \)-dimensional vector subspace of \( V \otimes V \). In \([\tilde{\phi}(W)]\), the set \([\tilde{\phi}(J_1)]\) is the Bruck norm-surface

\[
\mathcal{N} = \mathcal{N}_{(-1)^m} = \{ [\tilde{\phi}(xv_1 + yv_m)] : x, y \in \mathbb{F}_{q^m}, N(y/x) = (-1)^m \}
\]

introduced in [3] and widely investigated in [4, 5] and recently in [10, 23]. For any \( x \in \mathbb{F}_{q^m} \setminus \{0\} \) set \( J_x = \{ \lambda xv_1 - \lambda x v_{m-1} : \lambda \in \mathbb{F}_{q^m} \} \). Then \([\tilde{\phi}(J_x)] \subset \mathcal{N} \) and the set \([\tilde{\phi}(J_x)] : x \in \mathbb{F}_{q^m} \) is a so-called hyper-regulus of \( PG(\hat{V}) \) [30]. It turns out, that under the linear transformation \( \tau_a \) with \( N(\alpha) = a \), also \( J_a \) defines a hyper-regulus of \([\tilde{\phi}(W)]\).

The following result, which summarizes all above arguments, gives a geometric description of the MRD codes \( \mathcal{F}_{m,q;I} \).

Let \( q > 2 \) be a prime power and \( m > 2 \) a positive integer. Let \( I \) be any nonempty subset of \( \mathbb{F}_q \setminus \{0,1\} \) with \( k = |I| \). The projective image of the MRD code \( F_{m,q,I} \) in \( \text{PG}(m^2 - 1, q) \) is a subset of a Desarguesian spread which is union of two spread elements, \( k \) mutually disjoint Segre varieties \( S_{m,m}(\mathbb{F}_q) \) and \( q - 1 - k \) mutually disjoint hyperplanes all contained in the \((2m - 1)\)-dimensional projective subspace generated by the two spread elements.

## 4 The Cossidente-Marino-Pavese non-linear MRD code

Recently, Cossidente, Marino and Pavese constructed non-linear \((3,3,q;1)\)-MRD codes in a totally geometric setting \cite{Theorem 3.6}.

In \( \text{PG}(2,q^3) \), \( q \geq 3 \), let \( C \) be the set of points whose coordinates satisfy the equation \( X_1X_2^3 - X_3^{q+1} = 0 \), that is a \( C^*_1 \)-set of \( \text{PG}(2,q^3) \) as introduced and studied in \cite{13}. The set \( C \) is the projective image of a subset of \( \text{V}(3,q^3) \) which is the union of \( A_1 \), \( A_2' = \{(0,x,0) : x \in \mathbb{F}_{q^3}\setminus\{0\}\} \) and the \( q - 1 \) sets \( \gamma_a = \{(\lambda, \alpha x^q, \lambda x^q) : \lambda, x \in \mathbb{F}_{q^3}\setminus\{0\}, N(x) = a\} \), with \( a \) a nonzero element of \( \mathbb{F}_q \).

For any nonzero \( a \in \mathbb{F}_q \), let \( \alpha \in \mathbb{F}_{q^3} \) with \( N(\alpha) = a \) and set \( Z_a = \{(\lambda x, -\alpha x^q, 0) : \lambda, x \in \mathbb{F}_{q^3}\setminus\{0\}\} \). Let \( I \) be any non-empty subset of \( \mathbb{F}_q \setminus \{0,1\} \) and put

\[
\mathcal{A}'(q; I) = \bigcup_{a \in I} \gamma_a \cup \bigcup_{b \in \mathbb{F}_q \setminus \{I \cup \{0\}\}} Z_b \cup A_1 \cup A_2' \cup \{0\}.
\]

Up to an endomorphism of \( \mathcal{V} \otimes \mathcal{V} \) viewed as the vector space \( \mathcal{V}(9, q) \), the image of set \( \mathcal{A}'(q; I) \) under \( \nu \circ \phi \) is a non-linear \((3,3,q;1)\)-MRD code \cite{8} Proposition 3.8.

**Lemma 4.1.** Let \( \theta \) be the semilinear transformation of \( \mathcal{V}(3,q^3) \) defined by

\[
\theta : \quad v_1 \mapsto v_3 \\
v_2 \mapsto v_1 \\
v_3 \mapsto v_2
\]

with associated automorphism \( x \mapsto x^{q^2} \). Then \( \theta \) maps \( \gamma_a \) into \( \pi_{a^{-1}} \) and \( Z_a \) into \( J_{a^{-1}} \), for any nonzero element \( a \) of \( \mathbb{F}_q \).

**Proof.** Every element \( x \in \mathbb{F}_{q^3} \) with \( N(x) = a \) can be written as \( x = \alpha t^{q^2} \) for some \( t \in \mathbb{F}_{q^3} \) and \( \alpha \) a fixed element in \( \mathbb{F}_{q^3} \) such that \( N(\alpha) = a \). By straightforward calculations, we can write \( \gamma_a = \{(\lambda x, \lambda x^q, \lambda x^{q^2}) : \lambda, x \in \mathbb{F}_{q^3}\} \). Then, we get \( \theta(\gamma_a) = \{(\lambda x, \lambda x^{q^2} \lambda^{q^2} x^{q^2}) : \lambda, x \in \mathbb{F}_{q^3}\} = \pi_{a^{-1}} \) as \( N(\alpha^{-q^2}) = N(\alpha) = a^{-1} \).

The last part of the statement follows from straightforward calculations.

\( \square \)
Corollary 4.2. Let $I$ be any non-empty subset $I$ of $\mathbb{F}_q \setminus \{0, 1\}$ and put $I^{-1} = \{a^{-1} : a \in I\}$. Then, up to the endomorphism $\theta$ of $V(3, q^3)$ and the changing of basis in $V(3, q^3) \otimes V(3, q^3)$ from $u_{(i,j)}$ to $v_{(i,j)}$, the Cossidente-Marino-Pavese family of non-linear MRD codes is the set $F_{3, q, I^{-1}}$.

Let $L$ be any line of $\text{PG}(2, q^3)$ disjoint from a subgeometry $\text{PG}(2, q^3)$. The set of points of $L$ that lie on some proper subspace spanned by points of $\text{PG}(2, q^3)$ is called the exterior splash of $\text{PG}(2, q^3)$ on $L$.

Proposition 4.3. [10] The exterior splash of the subgeometry $[\pi_a]$ on the line $[W]$ is the set $[J_b]$ with $b = a^m - 1$.

Proof. First we note that $[W]$ is disjoint from $[\pi_1]$. The $\mathbb{F}_q^m$-span of some hyperplane in the cyclic model of $V$ is a hyperplane of $\hat{V}$ with equation $\sum_{i=1}^{m} \alpha^{q_i - 1} X_i = 0$, for some nonzero $\alpha \in \mathbb{F}_q^m$. As the Singer cycle $\sigma$ acts on the hyperplanes of $V$ by mapping the hyperplane with equation $\sum_{i=1}^{m} \alpha^{q_i - 1} X_i = 0$ to the hyperplane with equation $\sum_{i=1}^{m} (\mu \alpha)^{q_i - 1} X_i = 0$, then $\sigma$ maps the hyperplane of $\hat{V}$ with equation $\sum_{i=1}^{m} \alpha^{q_i - 1} X_i = 0$ into the hyperplane with equation $\sum_{i=1}^{m} (\mu \alpha)^{q_i - 1} X_i = 0$. Note that $\sigma$ fixes $W$.

The hyperplane $\sum_{i=1}^{m} X_i = 0$ of $\hat{V}$ meets $W$ in the $\mathbb{F}_q^m$-subspace spanned by $v_1 - v_m$. By looking at the action of the Singer cyclic group $S = \langle \sigma \rangle$ on $W$, we see that the exterior splash of $[\pi_1]$ on $[W]$ is the set $[J_1]$. By using he map $\tau_\alpha$ defined above with $N(\alpha) = a$, we get the result. \qed

Remark 4.4. Let $U$ be the $\mathbb{F}_q^m$-span of $v_1$ and $v_2$ in $\hat{V}$. It is evident that the semilinear transformation $\theta$ maps the exterior splash of $[\gamma_a]$ on $[U]$ into the exterior splash of $[\pi_{a^{-1}}]$ on $[W]$.

The exterior splash of $[\gamma_a]$ on $[U]$ is

$$[\gamma_a] = \{(1, x, 0) : x \in \mathbb{F}_q^3, N(x) = -a^2\}.$$ 

In [8], the splash of $[\gamma_a]$ was erroneously given as the set $[Z_a]$. Note that, $[Z_a]$ never coincides with $[\gamma_a]$, unless $a = 1$.

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