Lattice Boltzmann Method for Fluid-Structure Interaction with incompressible NeoHookean materials in small perturbations

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This paper deals with the numerical modelling of the interaction between a fluid and an incompressible solid (Neo Hookean) in small perturbations with the lattice Boltzmann method (LBM). In order to use a monolithic formulation and to solve the whole problem with the lattice Boltzmann method, an Eulerian approach is employed for the solid medium. The initial problem is thus transformed into a diphasic problem and a unique LBM solver is used for both phases (fluid and solid). With this approach, the force at the fluid-solid interface does not need to be explicitly computed. It is intrinsic to the method. This new method approach is validated with three academic cases: the deformation of a solid at the bottom of a lid driven cavity, with steady and unsteady boundary conditions at the top wall of the cavity and the deformation and motion of a disk in a lid driven cavity.

I. INTRODUCTION

Fluid Structure Interaction (FSI) deals with the study of physical phenomena involving interactions between a fluid and a solid. A classical example is the study of the flow induced motion and/or deformation of solids, which influences the flow because of a change in boundary conditions. Many examples of FSI can be found in biomechanics (blood flows in arteries, in the heart, ...), aerodynamics (air flows around wind turbines, planes, ...), hydrodynamics (flows around ships, submarines, ...). These complex phenomena occur at various space and time scales. The numerical modelling of such phenomena requires fine grids and small time steps, which leads to a high amount of computing time when classical numerical approaches are used to solve the Navier Stokes equations (finite element method, finite volume method, ...).

An important advantage of the lattice Boltzmann method (LBM) is that it can be very efficiently parallelized on graphic processor units (GPUs) [1,2], which enables a substantial decrease in computational time. It is therefore well suited to FSI computations. Since the 1990s, the lattice Boltzmann method becomes a powerful tool for predicting flows. This method consists in solving the discrete Boltzmann equation at a mesoscopic scale (a scale that is located between the microscopic and the macroscopic scales) [3,4]. This work focuses on LBM for FSI.

Various approaches have been developed to simulate FSI phenomena with LBM. Since Ladd’s works [7], the bounce back method has been widely used to calculate flows around motionless or moving bodies [8,9]. The method was also chosen to calculate the boundary conditions at a deforming obstacle, in the context of a coupling between the LBM and the finite element method [10,12]. Another method, which was developed by Noble and Torczynski [13] consists in modifying the collision term in the lattice Boltzmann equation, to take the boundary condition at the solid interface into account. This method was applied to FSI problems [14]. However, with this method, the solid fraction must be calculated in the vicinity of the boundary, and this is a difficult task when moving boundaries are handled. Furthermore, the immersed boundary method, which enables to compute flows around deformable bodies, was implemented in the lattice Boltzmann method since the 2000s [15,17]. With this method, the fluid flow is calculated with an Eulerian approach on a fixed cartesian grid, and the solid boundary is tracked with a Lagrangian approach, by introducing nodal interaction forces between the fluid and the solid. This method, which is very popular in the LBM community, enables to treat FSI problems, by coupling the lattice Boltzmann solver with the finite element method to solve the structure displacement [18,19]. Moreover, the distributed Lagrange multiplier fictitious domain method [20] was introduced into the lattice Boltzmann method. This method consists in writing the momentum equations and the continuity equation on the whole domain (fluid and solid) which does not move, and a Lagrange multiplier (a force) makes the fluid inside the solid move like the solid. With this method, the expensive remeshing step is avoided. By implementing this method in the lattice Boltzmann method, Shi and Phan Tien [21] and Shi and Piang Lim [22] coupled the LBM with the finite element method.

In previous works, we solved FSI problems by coupling the volume penalization method developed by Angot et al. [23], with LBM. With this method, a volume penalization source term that represents the influence of the solid domain on the flow, is introduced into the lattice Boltzmann equation [24,25]. The boundary conditions are naturally applied at the solid obstacle, and the obtained solver can be easily parallelized. This method enabled us to compute satisfactorily flows around rigid obstacles. In this article, we present an approach which enables to calculate the interaction between a fluid and an incompressible neo Hookean solid with the lattice Boltzmann method. Several authors solved FSI problems with classical approaches (Finite Elements or Finite Volumes) by treating the solid in an Eulerian way [26,28, ...]. In this paper, we propose a method that is based on part of these works : the solid is handled with an Eulerian way, and the LBM is used to solve the equations of fluid and solid media. This allows us to develop a diphasic approach where FSI problems involving small deformations are solved with a unique LBM solver (where some derivatives are calculated with the finite difference method). Moreover, the forces between the fluid and the solid are inherent to this fully Eulerian solver. This new approach is tested for several cases found in the literature. In the following paragraph, the lattice Boltzmann method...
is presented. Then, the Eulerian formulation for incompressible neo Hookean solids, and the LBM implementation of this method are explained. The numerical results obtained with this method are described, and the concluding remarks and future works are given.

II. MATHEMATICAL FORMULATIONS

A. Lattice Boltzmann Method

In this study, the lattice Boltzmann method was used to compute the flow, and to solve the Eulerian equation which describes the behavior of the deformable solid. With this method, the discretized Boltzmann equation, which enables to find at a time \( t \) and at a location \( \mathbf{x} \), the distribution function of particles \( f_i \) with a discrete velocity \( \mathbf{c}_i \), is solved:

\[
f_i(\mathbf{x} + \mathbf{c}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = \Omega_i(\mathbf{x}, t),
\]

where \( \Delta t \) is the time step and \( \Omega_i(\mathbf{x}, t) \) is the collision operator. For this work, which deals with two-dimensional cases, the D2Q9 model with 9 discrete velocities is chosen (Fig. 1).

![Figure 1: Discrete velocities of the D2Q9 model.](image)

The coordinates of the discrete velocities \( \mathbf{c}_i \) are:

\[
\mathbf{c} = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 & 0 & -1 & 1 & 1 & -1
\end{bmatrix}
\]

(2)

where \( \mathbf{c}_i \) is the \( i^{th} \) column of the table.

The collision operator can be modelled with various approaches:

- the BGK model [29] where all distribution functions \( f_i(\mathbf{x}, t) \) reach an equilibrium value \( f_i^{eq}(\mathbf{x}, t) \) with the same relaxation time \( \tau \). The collision operator of the BGK model is written as:

\[
\Omega_i(\mathbf{x}, t) = -\frac{\Delta t}{\tau} (f_i(\mathbf{x}, t) - f_i^{eq}(\mathbf{x}, t)),
\]

(3)

- the multi relaxation time (MRT) model [30] where the moments of the distribution functions reach their equilibrium value with different relaxation times,

- the two relaxation time (TRT) models [31 [32], where the moments of the distribution functions reach their equilibrium value with two different relaxation times.

The MRT and TRT models are more stable than the BGK model. The TRT model, which is easier to implement than the MRT model, is thus chosen. The TRT model is based on the fact that the lattice velocities are symmetrical: each velocity \( \mathbf{c}_i \) has an opposite velocity \( \mathbf{c}_i = -\mathbf{c}_i \). This enables to write the distribution function according to:

\[
f_i^+ = \frac{f_i + f_i}{2}, \quad f_i^- = \frac{f_i - f_i}{2}.
\]

(4)

The same operation can be done with the equilibrium functions:

\[
f_i^{eq} + = \frac{f_i^{eq} + f_i^{eq}}{2}, \quad f_i^{eq} - = \frac{f_i^{eq} - f_i^{eq}}{2},
\]

(5)

where \( f_i^{eq} \) for the D2Q9 model is:

\[
f_i^{eq} = \omega_i \rho \left(1 + \frac{\mathbf{v} \cdot \mathbf{c}_i}{c_s^2} + \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2 c_s^2}\right)^2 - \frac{\mathbf{v} \cdot \mathbf{v}}{c_s^2} \right).
\]

(6)

\( \rho \) and \( \mathbf{v} \) are the fluid density and the fluid velocity, \( c_s = \frac{1}{\sqrt{3}} \) is the speed of sound, \( \omega_i \) are the weights defined by \( \omega_0 = \frac{4}{9}, \omega_1 = \omega_2 = \omega_3 = \omega_4 = \frac{1}{9} \) and \( \omega_5 = \omega_6 = \omega_7 = \omega_8 = \frac{1}{36} \) for the D2Q9 model, and \( \cdot \) is the scalar product. For the TRT model, the lattice Boltzmann equation of a flow submitted to a force density \( \mathbf{F} \) is:

\[
f_i(\mathbf{x} + \mathbf{c}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = -\frac{\Delta t}{\tau^+} (f_i^+(\mathbf{x}, t) - f_i^{eq}+(\mathbf{x}, t))
\]

\[+ \frac{\Delta t}{\tau^-} (f_i^-(\mathbf{x}, t) - f_i^{eq}-(\mathbf{x}, t)) + \left(1 - \frac{\Delta t}{2 \tau^+}\right) F_i,
\]

(7)

where \( \tau^+ \) is the relaxation time linked with the non dimensional viscosity \( \eta \) according to:

\[
\eta = \frac{1}{3} \left(\tau^+ - \frac{1}{2}\right).
\]

(8)

The relaxation time \( \tau^- \) is obtained as follows:

\[
\tau^- = \frac{\Delta t \Lambda}{\tau^+ - \frac{1}{2}} + \frac{1}{2},
\]

(9)

with \( \Lambda = \frac{1}{6} \) [32]. In Eq. (7), \( F_i \) is the forcing term according to the \( i \) direction. In this work, it is modelled thanks to the approach developed by Guo et al. [33]:

\[
F_i = \omega_i \left(\frac{(\mathbf{c}_i - \mathbf{v}) \cdot \mathbf{c}_i}{c_s^4} + \frac{\mathbf{v} \cdot \mathbf{c}_i}{c_s^4}\right), \quad \mathbf{F} >,
\]

(10)

where \( \mathbf{F} \) is the body force term in the momentum conservation equation. The macroscopic variables \( \rho \) and \( \mathbf{v} \) are calculated as follows:

\[
\rho = \sum_i f_i, \quad \rho \mathbf{v} = \sum_i \mathbf{c}_i f_i + \frac{\mathbf{F} \Delta t}{2}.
\]

(11)
B. Eulerian formulation for a fluid structure interaction problem between an incompressible fluid and an incompressible Neo-Hookean solid

Let \( \Omega_f \) be the fluid domain, \( \Omega_s \) the solid domain, and \( \Gamma \) the interface between the fluid and the solid domains. The whole domain is \( \Omega = \Omega_f \cup \Omega_s \cup \Gamma \), its boundary is \( \partial \Omega \), and \( H(x) \) is a characteristic function (which depends on time) of the solid domain:

\[
H(x, t) = \begin{cases} 
1 & \text{if } x \in \Omega_s(t), \\
0 & \text{else}, 
\end{cases}
\] (12)

\( \mathbf{v}_f, \rho_f \) (resp \( \mathbf{v}_s, \rho_s \)) are the fluid (resp solid) velocity and density in \( \Omega_f \) (resp \( \Omega_s \)). The velocity and the density fields in the whole domain are:

\[
\rho = \rho_s H + (1 - H) \rho_f, \\
\mathbf{v} = \mathbf{v}_s H + (1 - H) \mathbf{v}_f.
\] (13)

(14)

If it is not mentioned, all derivative operators are written in Eulerian form in the remainder of the paper.

In this work, we consider an Eulerian formulation for a neo-Hookean incompressible solid domain. Therefore the Poisson ratio of the material is \( \nu = 0.5 \). Let us first recall the Navier-Stokes equation in the fluid domain:

\[
\rho_f \frac{\partial \mathbf{v}_f}{\partial t} + \rho_f \mathbf{v}_f \nabla \mathbf{v}_f = \nabla \cdot \mathbf{\sigma}_f + \mathbf{F}_{\mathbf{v}_f},
\] (15)

\[
\nabla \cdot \mathbf{v}_f = 0,
\] (16)

where \( \mathbf{F}_{\mathbf{v}_f} \) are volumic forces applied to \( \Omega_f \). \( \mathbf{\sigma}_f \) is the viscous stress tensor for the fluid, defined as:

\[
\mathbf{\sigma}_f = -p_f \mathbf{I} + \eta_f \left( (\nabla \mathbf{v}_f)^T + \nabla \mathbf{v}_f \right),
\] (17)

where \( p_f \) is the fluid pressure, \( \mathbf{I} \) is the identity matrix and \( \eta_f \) is the fluid viscosity.

Furthermore, for an incompressible material, the momentum and mass conservation equations in a eulerian framework can be written as:

\[
\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s \mathbf{v}_s \nabla \mathbf{v}_s = \nabla \cdot \mathbf{\sigma}_s + \mathbf{F}_{\mathbf{v}_s},
\] (18)

\[
\nabla \cdot \mathbf{v}_s = 0,
\] (19)

where \( \mathbf{F}_{\mathbf{v}_s} \) are volumic forces applied to \( \Omega_s \), and \( \mathbf{\sigma}_s \) is the stress tensor for the solid. For an incompressible Neo-Hookean material, the following law is satisfied (see \[34, 35, 36\] and \[37\]):

\[
\mathbf{\sigma}_{sL} = -p_{sL} \mathbf{I} + \mu_s \left( \mathbf{FF}^T - \mathbf{I} \right),
\] (20)

where \( \mathbf{\sigma}_{sL} \) is the stress tensor in Lagrangian variables, \( p_{sL} \) is the pressure of the solid in Lagrangian variables and \( \mathbf{F} \) is the deformation gradient and \( \mu_s \) is the Lamé coefficient defined by:

\[
\mu_s = \frac{E}{2(1 + \nu)},
\] (21)

with \( E \) the Young modulus and \( \nu \) the Poisson ratio (here \( \nu = 0.5 \)). In an Eulerian formulation, the stress tensor can be written as (see Appendix \[A\]):

\[
\mathbf{\sigma}_s(x, t) = -p_s(x, t) \mathbf{I} + 2 \mu_s \varepsilon (\mathbf{u}_E(x, t)),
\] (22)

where \( \varepsilon (\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \), and \( \mathbf{u}_E \) is the Eulerian displacement.

Introducing equation (22) into equation (18) leads to:

\[
\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s \mathbf{v}_s \nabla \mathbf{v}_s = -\nabla p_s + 2 \mu_s \nabla \cdot (\varepsilon (\mathbf{u}_E)) + \mathbf{F}_{\mathbf{v}_s},
\] (23)

where the Eulerian displacement is calculated according to:

\[
\frac{D \mathbf{u}_E}{Dt} = \mathbf{v}_s,
\] (24)

with \( \frac{D}{Dt} \) the material derivative:

\[
\frac{\partial \mathbf{u}_E}{\partial t} + \mathbf{v}_s \nabla \mathbf{u}_E = \mathbf{v}_s.
\] (25)

Considering a temporal discretisation scheme of equation (24):

\[
\rho_s \mathbf{v}_s(x, t + \Delta t) - \rho_s \mathbf{v}_s(x, t) \frac{\Delta t}{\Delta t} + \rho_s \mathbf{v}_s(x, t) \nabla \mathbf{v}_s(x, t) = \nabla \left( p_s(x, t) \right) + 2 \mu_s \nabla \cdot \varepsilon (\mathbf{u}_E(x, t + \Delta t)) + \mathbf{F}_{\mathbf{v}_s},
\] (26)

with:

\[
\mathbf{u}_E(x, t + \Delta t) = \mathbf{u}_E(x, t) + \Delta t \left( -\mathbf{v}_s \nabla \mathbf{u}_E(x, t) + \mathbf{v}_s(x, t) \right).
\] (27)

Considering the small deformation hypothesis enables to neglect the term \( \nabla \cdot \varepsilon (\mathbf{v}_s \nabla (\mathbf{u}_E)) \). This leads to:

\[
\nabla \cdot \varepsilon (\mathbf{u}_E(x, t + \Delta t)) = \nabla \cdot \varepsilon (\mathbf{u}_E(x, t)) + \Delta t \nabla \cdot \varepsilon (\mathbf{v}_s(x, t)).
\] (28)

Equation (26) becomes:

\[
\rho_s \mathbf{v}_s(x, t + \Delta t) - \rho_s \mathbf{v}_s(x, t) \frac{\Delta t}{\Delta t} + \rho_s \mathbf{v}_s(x, t) \nabla \mathbf{v}_s(x, t) = \nabla p_s(x, t) + 2 \mu_s \nabla \cdot (\varepsilon (\mathbf{u}_E(x, t)) + \Delta t \varepsilon (\mathbf{v}_s(x, t))) + \mathbf{F}_{\mathbf{v}_s}.
\] (29)

This last equation can be seen as an explicit scheme of the Navier-Stokes-like equation:

\[
\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s \mathbf{v}_s \nabla \mathbf{v}_s = -\nabla p_s + 2 \mu_s \Delta t \nabla \cdot (\varepsilon (\mathbf{v}_s)) + \mathbf{G},
\] (30)

with \( \mu_s \Delta t = \eta_s \) an artificial viscosity and \( \mathbf{G} = \mathbf{F}_{\mathbf{v}_s} + 2 \mu_s \nabla \cdot \varepsilon (\mathbf{u}_E) \).
By combining Eq. [15], Eq. [17] and Eq. [30], the multi-phase fluid-solid formulation can be written on the whole domain $\Omega$ as:

$$\frac{\partial \rho v}{\partial t} + \rho v \nabla v = -\nabla p + 2\eta \nabla \cdot \varepsilon (v) + F_v,$$  \hspace{1cm} (31)

$$\nabla \cdot v = 0,$$  \hspace{1cm} (32)

$$\frac{\partial \mathbf{u}_E}{\partial t} + \mathbf{v} \nabla \mathbf{u}_E = \mathbf{v},$$  \hspace{1cm} (33)

$$v(x, 0) = v_0(x),$$  \hspace{1cm} (34)

with:

$$p = p_s H + (1 - H)p_f,$$

$$\mathbf{F}_v = H (\mathbf{F}_{v_s} + 2H \mu_s \nabla \cdot \varepsilon (\mathbf{u}_E)) + (1 - H)\mathbf{F}_{v_f},$$

$$\eta = \eta_s H + (1 - H)\eta_f.$$  \hspace{1cm} (35)

It can be noticed that these equations enable to solve the fluid flow with the Navier-Stokes equations, and solve the solid problem as a particular fluid with an Eulerian formulation.

Furthermore an advantage of this method is that the forces exerted by the fluid on the solid do not need to be calculated. They are inherent to the proposed formulation.

C. Modification of the lattice Boltzmann method

Eq. (31) and (32) are solved with the two relaxation time lattice Boltzmann method presented in section [15], where the force term in Eq. (10) and (11) becomes:

$$\mathbf{F} = H (\mathbf{F}_{v_s} + 2H \mu_s \nabla \cdot \varepsilon (\mathbf{u}_E)) + (1 - H)\mathbf{F}_{v_f}. $$  \hspace{1cm} (36)

This force is introduced into the lattice Boltzmann equation, following the approach proposed by Guo et al [33] (Eq. (10) and Eq. (11)). With the Chapman Enskog procedure, Guo et al. showed that introducing a forcing term Eq. (10) into the lattice Boltzmann equation, and calculating the velocity field according to Eq. (11), yield the exact Navier Stokes equations. For the computation of the gradient and laplacian operators in Eq. (36), a scheme that provides an isotropic discretization is chosen:

$$\nabla \psi = \sum_{i \neq 0} \omega_i c_i \cdot \nabla \psi / c_i^2,$$  \hspace{1cm} (37)

$$\Delta \psi = \sum_{i \neq 0} \omega_i (c_i \cdot \nabla)^2 \psi / c_i^2,$$  \hspace{1cm} (38)

with $\psi$ a macroscopic variable. The derivatives according to the $c_i$ vectors are approximated with centered finite difference schemes:

$$c_i \cdot \nabla \psi (x) = \frac{\psi (x + c_i \Delta t) - \psi (x - c_i \Delta t)}{2\Delta t}, $$

$$\frac{\psi (x + c_i \Delta t) - 2 \psi (x) + \psi (x - c_i \Delta t)}{\Delta t^2}, $$

$$\nabla \cdot (c_i \cdot \nabla)^2 \psi (x) = \frac{\psi (x + c_i \Delta t) - 2 \psi (x) + \psi (x - c_i \Delta t)}{\Delta t^2}, $$

For the computation of the displacement $u_E$, Eq. (33) is approximated according to time with an explicit scheme, and according to space with a centered scheme:

$$u_E (x, t + \Delta t) = u_E (x, t) + \Delta t (v (x, t) - (\nabla \cdot v u_E) (x, t)).$$

The characteristic function $H$ is calculated with the following evolution equation:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \mathbf{v} \cdot \nabla H = 0, \text{ in } \Omega,$$

$$H (x, 0) = H_0 (x), \text{ in } \Omega,$$

$$H = H_{\Omega}, \text{ on } \partial \Omega.$$  \hspace{1cm} (42)

Since the equations are solved on a fixed mesh, we only seek to update $H$ on the grid nodes. Eq. (42) is solved by integrating the material derivative, and $H$ is obtained at each grid node in the following explicit way:

$$H (x, t) = H_0 (x - u (x, t)), \forall x \in \Omega (t).$$  \hspace{1cm} (43)

The procedure for solving the FSI problem is presented in the temporal algorithm below:

Algorithm 1: Resolution algorithm

1. Initialization $v (x, 0), u_E (x, 0), f_i^{eq} (x, 0), f_i^{eq} (x, 0)$

2. while $t \leq t_{\text{final}}$ do

3. Calculate $\mathbf{F} (t)$ (Eq. (36))

4. Solve $v (t + \Delta t)$ by LBM

5. Calculate $u_E (t + \Delta t)$ (Eq. (33))

6. Update $H (t + \Delta t)$ (Eq. (43))

7. $t = t + \Delta t$

III. SIMULATION RESULTS

In this paragraph, the results obtained with this monolithic formulation for solving FSI problems with LBM are presented. Three classical cases from the NeoHookean literature were simulated : lid driven cavity with a neo hookean incompressible solid at the bottom, with two different boundary conditions for the upper wall, and lid driven cavity with a deformable disk. For each case, the position of the interface was compared with that obtained by the authors.

A. Flow driven by the steady motion of the upper wall

The first case deals with a lid driven cavity with a neo-Hookean incompressible solid at the bottom (see Fig. 2). This
configuration was used for validation purposes by several authors [28, 39, 40]. Our results will be compared with those obtained by Wang et al. [40] with the finite element method. The dimensions of the cavity are \([0,2]\text{cm} \times [0,2]\text{cm}\), and the solid is located at a height \(L_s = L/4\). For that case, the fluid viscosity is \(\eta_f = 20\) Pa\(\cdot\)s, the Lamé coefficient is \(\mu = 0.001\) Pa, the fluid and solid densities are \(\rho_f = \rho_s = 1\) g \(\cdot\) cm\(^{-3}\). For the velocity, homogeneous Dirichlet boundary conditions are applied at \(x = 0\), \(y = 0\) and \(x = L\), and the solid displacement is zero on all solid boundaries, except at the fluid solid interface. At \(y = L\), a horizontal velocity profile is applied:

\[
\mathbf{v} \cdot \mathbf{x} = \begin{cases} 
\nu_0 \sin^2 \left( \frac{\pi x}{0.6} \right), & \text{if } x \in [0, 0.3] \\
\nu_0, & \text{if } x \in [0.3, 1.7], \\
\nu_0 \sin^2 \left( \frac{\pi (x - 2)}{0.6} \right), & \text{if } x \in [1.7, 2], 
\end{cases}
\]  

(44)

with \(x\) in cm and \(\nu_0 = 0.5\) cm \(\cdot\) s\(^{-1}\).

Figure 2: Lid driven cavity with a neoHookean incompressible solid at the bottom: flow driven by the steady motion of the upper wall.

For this case, the Reynolds number based on the velocity \(\nu_0\), the length \(L\) and the kinematic viscosity of the fluid is 5. By matching the Reynolds number in the physical units and in the lattice units, the following parameters are obtained for the LBM simulations:

\[
L_{\text{LBM}} = 800, \quad \nu_{0,\text{LBM}} = 0.18333, \quad \tau_f^+ = 88.5 \quad \tau_s^+ = 0.54, 
\]  

(45)

where \(\tau^+\) is the dimensionless relaxation parameter. The conversion factors are:

\[
C_v = \frac{\nu_0}{\nu_{0,\text{LBM}}}, \quad C_L = \frac{L}{L_{\text{LBM}}}, \quad C_t = \frac{C_v}{C_L}. 
\]  

(46)

To compare the results obtained with LBM, with those found by Wang et al. [40], the LBM results are scaled using the coefficients defined in (46). A comparison of the steady solution obtained by Wang et al. [40] and other authors (41, 42, 39 and 43) with that computed with LBM is shown in Fig.3. The shapes of the interface predicted by both methods are in good agreement. The flow and the solid displacement reach a steady state, and the fluid-solid interface goes down in the right part of the cavity, where the fluid forces are higher.

Since the solid is incompressible, the interface is lifted in the left part of the cavity. The volume of the solid is properly conserved.

In addition, a grid convergence study with four grid resolutions \((100 \times 100, 200 \times 200, 400 \times 400\) and \(800 \times 800)\) was performed. In Fig.4 the mean values of the relative error related to the interface position are plotted, according to the number of meshes. This error is calculated as follows:

\[
e = 100 \frac{|y - y_{\text{author}}|}{y_{\text{author}}},
\]  

(47)

where \(y\) is interface coordinate according to the vertical direction and \(y_{\text{author}}\) is the value obtained by Wang et al [40]. The error decreases when the number of cells is increased, and
it becomes very small when at least a mesh of 400 \times 400 cells is used. That confirms the convergence.

**B. Flow driven by an unsteady velocity profile at \( y = L \)**

![Lid driven cavity with a NeoHookean incompressible solid at bottom: flow driven by the unsteady motion of the upper wall.](image)

The second case is similar to the first one, but the velocity profile at \( y = L \) is unsteady and perpendicular to the boundary (see Fig. 5):

\[
v \cdot y = V_0 (1 - \cos (2\pi t)) \sin (2\pi x), \quad x \in [0, L], \quad t \in \mathbb{R}^+.
\]

Our results are compared with those obtained by Zhao et al. [39] who solved the fluid and solid equations on a fixed Eulerian mesh with the finite difference method. This case deals with a square cavity, whose physical dimensions are \( L = 1 \) m. The solid is located at a height \( L_s = L/2 \). The fluid viscosity is \( \eta_f = 10^{-2} \) Pa.s, the Lamé coefficient is \( \mu = 0.25 \) Pa, the fluid and solid densities are \( \rho_f = \rho_s = 1 \) kg \cdot m\(^{-3}\), and \( V_0 = 1 \) m \cdot s\(^{-1}\). Periodic boundary conditions are applied at \( x = 0 \) and \( x = L \), and homogeneous Dirichlet boundary conditions are applied at \( y = 0 \).

To perform the LBM computations for that case, the following parameters, obtained by matching the Reynolds number \( Re = 200 \) based on the maximum velocity \( V_{\text{max}} = 2V_0 \) and on the cavity height \( L \), were used:

\[
L_{\text{LBM}} = 400, \quad V_{0,\text{LBM}} = 0.1, \quad \tau_f^+ = 1.1, \quad \tau_s^+ = 0.501875.
\]

Fig. 6 displays a comparison of the LBM results and of the results obtained by Zhao et al. [39] at time \( t = 1 \) s. It can be noticed that for an unsteady flow, the interface position computed with LBM is in agreement with the one calculated by Zhao et al. For that case, the LBM properly predicts the physical features of that unsteady problem.

**C. Deformable disk in a lid driven cavity**

![Deformable disk in a lid driven cavity.](image)

The last case deals with a deformable disk in a lid driven square cavity whose side lengths are \( L = 1 \) m (see Fig. 7).

At initial time, a disk of radius \( R = L/5 \), made of a material of density \( \rho_s = 1 \) kg \cdot m\(^{-3}\) and of Lamé coefficient \( \mu = 0.01 \) Pa, is located at \((x_0, y_0) = (3L/5, L/2)\). The upper wall of the cavity, which moves at a velocity \( v(x, L) = 1 \) m \cdot s\(^{-1}\), makes the fluid of density \( \rho_f = 1 \) kg \cdot m\(^{-3}\) and of viscosity \( \eta_f = 10^{-2} \) Pa \cdot s move, and induces the displacement and the deformation of the disk. For the LBM computations, the following parameters were used:

\[
L_{\text{LBM}} = 800, \quad V_{0,\text{LBM}} = 0.078, \quad \tau_f^+ = 2.372, \quad \tau_s^+ = 0.501825
\]

The position of the disk and its deformation at time \( t = 1.17 \) s...
are compared with those obtained by Zhao et al. [39] with the finite difference method in Fig. 8. The interface position and its deformation are also shown in this figure. A good agreement with the numerical results obtained by Zhao et al. [39] can be highlighted. It can be concluded that the lattice Boltzmann method gives satisfactory results even for this more complex case.

Figure 8: Interface obtained with LBM (H) and by Zhao et al [39] (points) and dimensional velocity vectors at time $t = 1.17$ s.

IV. CONCLUSION

In the present work, a new method which enables to solve fluid structure interaction problems for incompressible neo-Hookean solids in small perturbations with the lattice Boltzmann method, is presented. A diphasic approach, where the solid medium is considered as an incompressible fluid, is developed. This enables, at the macroscopic level, to extend the Navier-Stokes equations to the structure, and then to use an Eulerian formalism for NeoHookean materials undergoing small deformations. Thus a multiphasic formulation similar to the Navier-Stokes equations is obtained on the whole domain. One interest of this formulation is that the fluid forces exerted at the fluid-solid interface are intrinsic and do not need to be computed. Next, the LBM is used as a fluid solver for the whole domain, fluid and solid.

This approach is applied to three academic configurations: deformation of a neoHookean incompressible solid in a flow at steady state, deformation of a neoHookean incompressible solid in a flow at unsteady state, deformation and displacement of a neoHookean incompressible solid at unsteady state. For all cases, the results are in good agreement with those found in literature.

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Appendix A: Eulerian formulation for NeoHookean materials in small perturbations

For an incompressible solid, the momentum and the mass conservation equations written in an eulerian framework are:

\[ \rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho \mathbf{v}_s \nabla (\mathbf{v}_s) = \nabla \cdot \mathbf{s} + \mathbf{F} \quad (A1) \]

\[ \nabla \cdot \mathbf{v}_s = 0 \quad (A2) \]

with \( \rho \) the density, \( \mathbf{v} \) the velocity, \( \mathbf{F} \) the volume forces and \( \mathbf{s} \) the stress tensor of the solid material (written in Eulerian variables).

The behavior of an incompressible Neo-Hookean material is described in Lagrangian variables (see [34, 35, 36] and [37]).

\[ \sigma_{sl} = -p_{sl} I_d + \mu_s \left( \tilde{\mathbf{F}} \tilde{\mathbf{F}}^T - I_d \right) \quad (A3) \]

\( p_{sl} \) is the pressure written in Lagrangian variables (we will note \( p \) the pressure expressed in Eulerian variables), \( I_d \) is the identity matrix, \( \mu_s \) is the Lamé coefficient defined by

\[ \mu_s = \frac{E}{2(1 + \nu)} \quad (A4) \]

with \( E \) the Young modulus, \( \nu \) the Poisson ratio (for Neo-Hookean materials, \( \nu = 0.5 \)). \( \tilde{\mathbf{F}} \) is given in equation Eq.\((A12)\).

Let us note \( \mathbf{X} \) the Lagrangian variables and \( \mathbf{x} \) the Eulerian variables. \( \varphi_t \) is the transformation of the solid between the initial time and time \( t \):

\[ \varphi_t : \Omega_0 \rightarrow \Omega_t \]
\[ \mathbf{X} \mapsto \mathbf{x} = \varphi_t(\mathbf{X}) \quad (A5) \]

We note \( \mathbf{u}_L \) the displacement between the point at the initial time and the point at time \( t \), \( \mathbf{x} \) can be written as:

\[ \mathbf{x} = \mathbf{X} + \mathbf{u}_L(\mathbf{X}, t) \quad (A6) \]

Since the material is elastic, \( \varphi_t \) is a diffeomorphism, and there is a reciprocal function

\[ \varphi_t^{-1} : \Omega_t \rightarrow \Omega_0 \]
\[ \mathbf{x} \mapsto \mathbf{X} = \varphi_t^{-1}(\mathbf{x}) \quad (A7) \]

In addition, let us note \( \mathbf{u}_E \) the displacement between the point at the initial time and the point at time \( t \) after a displacement in Eulerian variables. We have:

\[ \mathbf{X} = \mathbf{x} - \mathbf{u}_E(\mathbf{x}, t) \quad (A8) \]

thus

\[ \mathbf{x} = \mathbf{X} + \mathbf{u}_E(\mathbf{x}, t) \quad (A9) \]

Combining Eq.\((A6)\) and Eq.\((A9)\) yields

\[ \mathbf{X} + \mathbf{u}_L(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}_E(\mathbf{x}, t) \quad (A10) \]

and finally:

\[ \mathbf{u}_L(\mathbf{X}, t) = \mathbf{u}_E(\mathbf{x}, t). \quad (A11) \]

Besides, by definition we have

\[ \tilde{\mathbf{F}}(\mathbf{x}, t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad (A12) \]

thus:

\[ \tilde{\mathbf{F}}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial \mathbf{X}} + \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} = I_d + \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \quad (A13) \]

and:

\[ \tilde{\mathbf{F}}^T(\mathbf{x}, t) = I_d + \left( \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \right)^T. \quad (A14) \]

Introducing equations Eq.\((A13)\) and Eq.\((A14)\) into equation Eq.\((A3)\) leads to

\[ \sigma_{sl}(\mathbf{X}, t) = -p_{sl}(\mathbf{X}, t) I_d + \mu_s \left( \tilde{\mathbf{F}} \tilde{\mathbf{F}}^T - I_d \right) \]
\[ + \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \left( I_d + \left( \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \right)^T \right) - I_d \]
\[ = -p_{sl}(\mathbf{X}, t) I_d + \mu_s \left( \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \right)^T \]
\[ + \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \left( \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \right)^T \quad (A15) \]

By considering small perturbations, the quadratic terms can be cancelled in equation \((A15)\) and the Lagrangian and Eulerian tensors of deformation are equivalent:

\[ \frac{1}{2} \left( \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} + \left( \frac{\partial \mathbf{u}_L(\mathbf{X}, t)}{\partial \mathbf{X}} \right)^T \right) = \varepsilon_L(\mathbf{u}_L(\mathbf{X}, t)) \]
\[ \approx \varepsilon(\mathbf{u}_E(\mathbf{x}, t)) \quad (A16) \]

and

\[ \varepsilon(\mathbf{u}_E(\mathbf{x}, t)) = \frac{1}{2} \left( \frac{\partial \mathbf{u}_E(\mathbf{x}, t)}{\partial \mathbf{x}} + \left( \frac{\partial \mathbf{u}_E(\mathbf{x}, t)}{\partial \mathbf{x}} \right)^T \right) \]
\[ = \frac{1}{2} \left( \nabla \mathbf{u}_E(\mathbf{x}, t) + \nabla^T \mathbf{u}_E(\mathbf{x}, t) \right) \quad (A17) \]

At least the stress tensor can thus be written in Eulerian vari-
ables as:

\[ \sigma_s(x, t) = -p_s(x, t)I_d + 2\mu_s\varepsilon(u_E(x, t)) \quad (A18) \]

with \( p_s(x, t) = p_{s_L}(X, t) \) the Eulerian pressure field. By injecting equation [A18] into Eq. [A1] we get:

\[ \rho_s \frac{\partial v_s}{\partial t} + \rho_s v_s \nabla v_s = -\nabla p_s + 2\mu_s \nabla \cdot (\varepsilon(u_E)) + F_{vs} \quad (A19) \]