Univoque bases of real numbers: local dimension, Devil’s staircase and isolated points

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Motivation

Given $q \in (1, 2]$, for each $x \in I_q := [0, \frac{1}{q-1}]$ there exists a sequence $(d_i) \in \{0, 1\}^\mathbb{N}$ such that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i} =: \pi_q((d_i)).$$

The sequence $(d_i) = d_1d_2\ldots$ is called a $q$-expansion of $x$. 
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- For each $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1, 2]$ and $x \in I_q$ such that $x$ has $k$ different $q$-expansions (Erdős, Joó and Komornik 1990; Erdős, Horváth and Joó 1991; Erdős and Joó 1992).

- Let $q \in (1, 2)$. Then Lebesgue a.e. $x \in I_q$ has a continuum of $q$-expansions (Sidorov, 2003).
There is a great interest in unique $q$-expansions, due to their close connections with open dynamical systems.

Figure: The overlapping graphs of $T_0 : x \mapsto qx$ and $T_1 : x \mapsto qx - 1$. 
Univoque set

Let
\[ \mathcal{U} := \{(x, q) : x \text{ has a unique } q \text{ expansion}\} . \]

Then for each \( q \in (1, 2] \) the horizontal slice
\[ \mathcal{U}_q := \{x \in I_q : (x, q) \in \mathcal{U}\} \]

is the set of \( x \) having a unique \( q \)-expansion.
Critical values $q_G = \frac{1+\sqrt{5}}{2}$ and $q_{KL} \approx 1.78723$ (Erdős, Joó and Komornik 1990; Glendinning and Sidorov 2001);
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What about the vertical slice of $\mathbb{U}$?
Univoque bases

For \( x \geq 0 \) let

\[
\mathcal{U}(x) := \{ q \in (1,2] : (x,q) \in \mathbb{U} \}.
\]

- If \( x = 0 \), then \( \mathcal{U}(0) = (1,2] \) (trivial!).
- If \( x \in (0,1] \), then \( q_x = 2 \).
- If \( x \in (1,\infty) \), then \( q_x = 1 + \frac{1}{x} \), and in this case, \( x = \infty \sum_{i=1}^{\infty} \frac{1}{q_i} \).
Univoque bases

For $x \geq 0$ let

$$\mathcal{U}(x) := \{q \in (1, 2] : (x, q) \in \mathcal{U}\}.$$  

- If $x = 0$, then $\mathcal{U}(0) = (1, 2]$ (trivial!).
- If $x > 0$, then the largest element of $\mathcal{U}(x)$ is

$$q_x := \min \left\{ 2, 1 + \frac{1}{x} \right\}.$$  

- If $x \in (0, 1]$, then $q_x = 2$.
- If $x \in (1, \infty)$, then $q_x = 1 + \frac{1}{x}$, and in this case,

$$x = \sum_{i=1}^{\infty} \frac{1}{q_x^i}.$$
For $x = 1$ the set $\mathcal{U} = \mathcal{U}(1)$ was well-studied:

- $\mathcal{L}(\mathcal{U}) = 0$ (Erdős, Joó and Komornik 1990) and $\dim_H \mathcal{U} = 1$ (Daróczy and Kátai 1995);
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- $\mathcal{U}$ has a smallest member $q_{KL} \approx 1.78723$ (Komornik and Loreti, 1998), and is transcendental (Allouche and Cosnard 2000);
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- Local dimension (Allaart and K. 2020).
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- For $x \in (0, 1)$ we have $\mathcal{L}(U(x)) = 0$ and $\dim_H U(x) = 1$ (Lü, Tan and Wu 2014);
- For $x \in (0, 1]$ the algebraic difference $U(x) - U(x)$ contains an interval (Dajani, Komornik, K. and Li 2018);
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- The smallest element of $\mathcal{U}(x)$ (K. 2016; Allaart and K. 2020).
Variation principle

Theorem (K., Li, Lü, Wang and Xu, 2020)

For any $x > 0$ and for any $q \in (1, q_x] \setminus \overline{U}$ we have

$$\lim_{\delta \to 0} \dim H(U(x) \cap (q - \delta, q + \delta)) = \lim_{\delta \to 0} \dim H(U_q \cap (x - \delta, x + \delta)).$$
Proof

The proof is based on the local bi-Hölder continuity of the map

$$\Phi_x : \mathcal{U}(x) \to \mathcal{U}(x); \quad q \mapsto x_1(q)x_2(q)\ldots,$$

where $\mathcal{U}(x)$ is the set of all unique expansions of $x$ for some $q \in \mathcal{U}(x)$. 

We also need the local bi-Hölder continuity of the projection map

$$\pi_q : \mathcal{U}_q \to \mathcal{U}_q; \quad (d_i) \mapsto \sum_{i=1}^{\infty} d_i q^i,$$

where $\mathcal{U}_q$ is the set of all unique $q$-expansions.
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\[ \Phi_x(U(x) \cap (q - \delta, q + \delta)) \to \text{cylinder set } U_q[x, n] \]

\[ U(x) \cap (q - \delta, q + \delta) \to U_q \cap (x - \eta, x + \zeta) \]
Proof conti

Let \( q \in (1, q_x] \setminus \overline{\mathcal{U}} \) and \( x = \pi_q(\Phi_x(q)) \). Then \( \exists \, \delta > 0 \) such that \( (q - \delta, q + \delta) \cap \overline{\mathcal{U}} = \emptyset \).
Let \( q \in (1, q_x] \setminus \overline{U} \) and \( x = \pi_q(\Phi_x(q)) \). Then \( \exists \delta > 0 \) such that \((q - \delta, q + \delta) \cap \overline{U} = \emptyset \). This defines a nearly bijective map

\[
\phi : \quad U(x) \cap (q - \delta, q + \delta) \quad \rightarrow \quad U_q \cap (x - \eta, x + \zeta)
\]

\[
p \quad \mapsto \quad \pi_q(\Phi_x(p)).
\]

Note that \( \delta \rightarrow 0 \) implies \( \eta, \zeta \rightarrow 0 \).
Proof conti

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$$

Note that $\delta \rightarrow 0$ implies $\eta, \zeta \rightarrow 0$. It is also nearly bi-Lipschitz:

$$
C_1 |p_1 - p_2|^{1+\varepsilon} \leq |\phi(p_1) - \phi(p_2)| \leq C_2 |p_1 - p_2|^{1-\varepsilon}.
$$
Proof conti

Let $q \in (1, q_x] \setminus \overline{U}$ and $x = \pi_q(\Phi_x(q))$. Then $\exists \delta > 0$ such that $(q - \delta, q + \delta) \cap \overline{U} = \emptyset$. This defines a nearly bijective map

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This implies

$$
\lim_{\delta \rightarrow 0} \dim_H(U(x) \cap (q - \delta, q + \delta)) = \lim_{\eta \rightarrow 0} \dim_H(U_q \cap (x - \eta, x + \eta)).
$$
Devil’s staircase

Recall that

\( U(x) = \{(d_i) : (d_i) \text{ is the unique expansion of } x \text{ in some base} \} , \)

\( U_q = \{(d_i) : (d_i) \text{ is a unique } q \text{ expansion of some point} \} . \)

**Theorem (K., Li, Lü, Wang and Xu, 2020)**

For any \( x > 0 \) we have

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\dim_H U(x) = \dim_H U_{q_x},
\]

where \( q_x = \max U(x) = \min \{2, 1 + \frac{1}{x} \} . \)

Therefore, \( D : x \mapsto \dim_H U(x) \) is a non-increasing Devil’s staircase on \((0, \infty) .\)
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*Therefore,* \( D : x \mapsto \dim_H U(x) \) *is a non-increasing Devil’s staircase on* \((0, \infty)\).

In general, we are not able to calculate \( \dim_H U(x) \).
Figure: The graph of $D(x) = \dim_H U(x)$. 

$\phi(x) = \dim_H U(x)$
Proof

\[ q_x = \min \{ 2, 1 + \frac{1}{x} \} \]

- \( U(x) \subseteq U_{q_x} \), and then \( \dim_H U(x) \leq \dim_H U_{q_x} \);
Proof

- $U(x) \subseteq U_{q_x}$, and then $\dim_H U(x) \leq \dim_H U_{q_x}$;
- For any $s < \dim_H U_{q_x}$ we can construct a subset $\Gamma \subset U(x)$ close to $\Phi_x(q_x)$ such that $\dim_H \Gamma \geq s$. 
Critical values

Theorem (K., Li, Lü, Wang and Xu, 2020)

The set $\mathcal{U}(x)$ has zero Lebesgue measure for any $x > 0$.

(i) If $x \in (0, 1]$, then $\dim_H \mathcal{U}(x) = 1$;
(ii) If $x \in (1, x_{KL})$, then $0 < \dim_H \mathcal{U}(x) < 1$;
(iii) If $x \in [x_{KL}, x_G)$, then $|\mathcal{U}(x)| = \aleph_0$;
(iv) If $x \geq x_G$, then $\mathcal{U}(x) = \{q_x\}$.

\[ q_x = \min\left\{2, 1 + \frac{1}{x}\right\} \]

\[ x_{KL} = \frac{1}{q_{KL} - 1} \approx 1.27 \]

\[ x_G = q_G = \frac{1 + \sqrt{5}}{2} \]
Recall that $\mathcal{U} = \mathcal{U}(1)$ has no isolated points and $\overline{\mathcal{U}}$ is a Cantor set.

Theorem (K., Li, Lü, Wang and Xu, 2020) $X_{iso}$ is dense in $(0, \infty)$. Furthermore, $\mathcal{U}(x)$ contains isolated points for any $x > 1$. 

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Let

$$X_{iso} := \{ x \in (0, \infty) : \mathcal{U}(x) \text{ contains isolated points} \}.$$  

**Theorem (K., Li, Lü, Wang and Xu, 2020)**

$X_{iso}$ is dense in $(0, \infty)$. Furthermore, $\mathcal{U}(x)$ contains isolated points for any $x > 1$.  


proof

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Then $U \subset V$ and $\#(V \setminus U) = \aleph_0$. 

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Then $U \subset V$ and $\#(V \setminus U) = \aleph_0$.

Observe that

$$(1, 2] \setminus \overline{U} = \bigcup (q_0, q_0^*).$$

For each $(q_0, q_0^*)$ we have $V \cap (q_0, q_0^*) = \{q_n\}_{n=1}^{\infty}$ such that

$q_0 < q_1 < q_2 < \cdots < q_n < q_{n+1} < \cdots$, and $q_n \uparrow q_0^*$.

So the map $q \mapsto U_q$ is constant on each interval $(q_n, q_{n+1}]$. 

proof
Proof conti

Set $U_{q_{n+1}}^* := U_{q_{n+1}} \setminus U_{q_n}$. Then $U_{q_{n+1}}^*$ is dense in $U_{q_{n+1}}$.

Lemma

For any

$$x \in \bigcup_{n=1}^{\infty} \bigcup_{p \in (q_n, q_{n+1})} \pi_p(U_{q_{n+1}}^*)$$

the set $\mathcal{U}(x)$ contains at least one isolated point.
Proof conti

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- Using this lemma we can show that the union covers a dense subset of $(0,1)$;
Proof conti

Set $U^*_{q_{n+1}} := U_{q_{n+1}} \setminus U_{q_n}$. Then $U^*_{q_{n+1}}$ is dense in $U_{q_{n+1}}$.

**Lemma**

For any $x \in \bigcup_{n=1}^{\infty} \bigcup_{p \in (q_n, q_{n+1})} \pi_p(U^*_{q_{n+1}})$ the set $U(x)$ contains at least one isolated point.

- Using this lemma we can show that the union covers a dense subset of $(0, 1)$;
- Furthermore, the union covers the whole interval $(1, \infty)$ (techniques from combinatorics on words).
Proof conti

Set $U^*_{n+1} := U_{n+1} \setminus U_n$. Then $U^*_{n+1}$ is dense in $U_{n+1}$.

Lemma

For any

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Conjecture

$U(x)$ contains isolated points $\iff x \neq 1$. 
Open questions

1. When is $\mathcal{U}(x)$ a closed set for $x \in (0, x_G)$?

2. What is the Hausdorff dimension of $\mathcal{U}(x)$ for $x \in (1, x_{KL})$?
Thank you!
And welcome to Chongqing!