String Theory on $G_2$ Manifolds Based on Gepner Construction

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Abstract

We study the type II string theories compactified on manifolds of $G_2$ holonomy of the type $(\text{Calabi-Yau 3-fold } \times S^1)/\mathbb{Z}_2$ where CY$_3$ sectors realized by the Gepner models. We construct modular invariant partition functions for $G_2$ manifold for arbitrary Gepner models of the Calabi-Yau sector. We note that the conformal blocks contain the tricritical Ising model and find extra massless states in the twisted sectors of the theory when all the levels $k_i$ of minimal models in Gepner constructions are even.
1 Introduction

Recently a number of papers have appeared investigating the dynamics of M theory and string theory compactified on manifolds with $G_2$ holonomy \[1\]-\[20\]. M theory, when compactified on 7-dimensional $G_2$ manifolds, provides $\mathcal{N} = 1$ 4-dimensional supersymmetric gauge theories which are of basic phenomenological importance. It has also been pointed out that some interesting duality between gauge and gravity systems in type IIA theory may be interpreted as geometrical transitions when lifted to M-theory compactified on certain $G_2$ manifolds \[8\] \[9\] \[10\]. In the case of string theory compactified on $G_2$ manifold, on the other hand, worldsheet description is expected to possess some exotic features, i.e. existence of tricritical Ising model and extended conformal symmetry \[21\], \[22\], \[23\]. Construction of Gepner type soluble models for strings on $G_2$ manifolds has been an challenging problem.

In a previous communication we have constructed candidate partition functions for strings propagating on non-compact $G_2$ manifolds associated with $A - D - E$ singularities \[24\]. In this paper we instead would like to consider the case of compact $G_2$ manifolds $(CY_3 \times S^1)/Z_2$ constructed by taking a Calabi-Yau 3-fold $CY_3$ times a circle $S^1$ and then by dividing by $Z_2$ which acts as anti-holomorphic involution on $CY_3$ \[25\]. We will use the Gepner construction for the $CY_3$ sector of the theory based on the tensor products of $\mathcal{N} = 2$ minimal models \[26\]. $Z_2$ orbifolding of $\mathcal{N} = 2$ minimal models is somewhat non-trivial. It turns out, however, the representation theory and character formulas developed for the "twisted (orbifoldized) $\mathcal{N} = 2$ superconformal algebra" of refs. \[27\], \[28\], \[29\] provide the necessary information and we can carry out the orbifoldization procedure in a straightforward manner.

In the following section we will construct a modular invariant partition function on $G_2$ manifold $(CY_3 \times S^1)/Z_2$ for an arbitrary Gepner model describing $CY_3$. While the amplitude in untwisted sector contains massless states of $b_2 + b_3 = h^{1,1} + h^{2,1} + 1$ chiral multiplets in 3 dimensions in agreement with geometry \[27\], \[1\], \[4\], there appear new massless states (2 chiral multiplets) in the twisted sector if and only if all the levels $k_i$ of the minimal models of the tensor product are even. When all levels are even, anti-holomorphic involution acts on CY manifolds without fixed points and classically we do not expect new massless states. Appearance of these states seem to be stringy quantum effects. Section 3 is devoted to the case of $G_2$ manifolds constructed from singular CY 3-folds and ALE spaces. Discussions and conclusions are given in section 4.
While this paper was in preparation a preprint \cite{30} has appeared which discusses the construction of string theory on \((CY_3 \times S^1)/Z_2\) for 3 special cases of Calabi-Yau 3 folds. Results of \cite{30} are completely consistent with ours.

2 SCFT on \(G_2\) Manifold \((CY_3 \times S^1)/Z_2\)

Let us construct the partition function of a string compactified on a \(G_2\) manifold \((CY_3 \times S^1)/Z_2\) with the \(CY_3\) sector being realized by a Gepner model. Gepner model is given by a tensor product of \(N = 2\) minimal models

\[
\left[ \mathcal{M}^N_{k_1} \times \cdots \times \mathcal{M}^N_{k_r} \right]_{U(1)}\text{-projected} \equiv (k_1, \cdots, k_r),
\]

where \(\mathcal{M}^N_{k}\) denotes the level \(k\) \(N = 2\) minimal model with central charge \(c = \frac{3k}{k+2}\). The criticality condition is given by

\[
\sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 9.
\]

When \(r = 5\), the condition (2.2) becomes equivalent to the Calabi-Yau condition for the hypersurface

\[
Z_{1}^{k_1+2} + \cdots + Z_{5}^{k_5+2} = 0
\]

in the weighted projective space \(WCP^4\left(\frac{1}{k_1+2}, \ldots, \frac{1}{k_5+2}\right)\).

The sector of the circle \(S^1\) is described by a free boson and fermion \(X, \psi\). Orbifoldization along this direction is simply given by

\[
X \rightarrow -X, \quad \psi \rightarrow -\psi.
\]

It is somewhat non-trivial to perform the orbifoldization on the sector of the Gepner model. Geometrically the \(Z_2\)-action (we shall denote it as \(\sigma\)) is an anti-holomorphic involution on \(CY_3\) and have the properties \(\sigma^*(K) = -K\), \(\sigma^*(\Omega) = e^{i\theta} \bar{\Omega}\), where \(K\) denotes the Kähler form and \(\Omega\) is the holomorphic 3-form. Therefore, it is natural to assume that \(\sigma\) acts on each sub-theory \(\mathcal{M}^N_{k}\) as an automorphism of \(N = 2\) superconformal algebra \(\{T, J, G^+, G^-\}\);

\[
\sigma : T \rightarrow T, \quad J \rightarrow -J, \quad G^\pm \rightarrow G^\mp.
\]

In the computation of toroidal partition functions \(Z_2\)-orbifoldization is enforced by \(\sigma\)-twisting along the “space” and “time” directions. (2.5) implies, in the NS sector for instance, when
the $\sigma$-twisting is applied in the spatial direction the moding of the $G_1$ remains half-integral while that of $G_2$ is switched to integral values ($G^\pm = G_1 \pm iG_2$). We next introduce characters of the $\mathcal{N} = 2$ minimal model in various $\sigma$-twisted sectors.

### 2.1 Twisted Characters in $\mathcal{N} = 2$ Minimal Model

Characters of the untwisted sector of $\mathcal{N} = 2$ theories are well-known. For various spin structures they are given by

\[
\begin{align*}
\text{ch}_{t,m}^{k,(\text{NS})}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{t,m}(\text{NS})} q^{L_0-\zeta/8} y^{J_0} = \chi_{m}^{1,0}(\tau, z) + \chi_{m}^{1,2}(\tau, z), \\
\text{ch}_{t,m}^{k,(\text{R})}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{t,m}(\text{R})} q^{L_0-\zeta/8} y^{J_0} = \chi_{m}^{1,1}(\tau, z) + \chi_{m}^{1,3}(\tau, z), \\
\text{ch}_{t,m}^{k,(\bar{\text{R}})}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{t,m}(\bar{\text{R}})} q^{L_0-\zeta/8} y^{J_0} = \chi_{m}^{1,1}(\tau, z) - \chi_{m}^{1,3}(\tau, z), \\
\end{align*}
\]

(2.6)

Here we set

\[
\chi_{m}^{l,n}(\tau, z) = \sum_{r \in \mathbb{Z}} c_{t,m}^{(k)}(\tau) \Theta_{2m+(k+2)(-s+4r),2k(k+2)/2}^{(2m+(k+2)(-s+4r),2k(k+2)/2)}(\tau, z/(k + 2)),
\]

(2.7)

and $c_{t,m}^{(k)}$ denotes the string function associated to the affine $SU(2)$ algebra at level $k$.

Let us now consider sectors with $\sigma$-twisting. We denote the twisted characters as $\text{ch}_{t,m}^{k,(I)}(\tau)$ where $I$ runs over the spin structures NS, $\bar{\text{NS}}$, $\text{R}$, $\bar{\text{R}}$, and $S, T = \pm$ describes the spatial and temporal boundary conditions of the $\sigma$-twist. Since the twisting $\sigma : J \to -J$ leaves only the states with vanishing $U(1)$-charge, it is obvious that the twisted characters are labeled only by the “$t$-index”. Recall that the usual twisting by $(-1)^F$ insertion acts as

\[
(-1)^F : T \to T, \quad J \to -J, \quad G^\pm \to -G^\mp.
\]

(2.8)

Thus under the combined twist $\sigma \cdot (-1)^F (\equiv (-1)^F \cdot \sigma)$ we have

\[
\sigma \cdot (-1)^F : T \to T, \quad J \to -J, \quad G^\pm \to -G^\mp.
\]

(2.9)

(2.9) differs from (2.3) only in the exchange of $G_1$ and $G_2$ and thus leads to the same character formulas. These facts imply the following relations among the twisted characters:

\[
\begin{align*}
\text{ch}_{t,(+,-)}^{k,(\text{NS})}(\tau) &= \text{ch}_{t,(+,-)}^{k,(\bar{\text{NS}})}(\tau), \quad \text{ch}_{t,(+,-)}^{k,(\text{NS})}(\tau) = \text{ch}_{t,(+,-)}^{k,(\text{R})}(\tau), \quad \text{ch}_{t,(+,-)}^{k,(\bar{\text{NS}})}(\tau) = \text{ch}_{t,(+,-)}^{k,(\bar{\text{R}})}(\tau), \\
\text{ch}_{t,(-,+)}^{k,(\text{R})}(\tau) &= \text{ch}_{t,(-,+)}^{k,(\bar{\text{R}})}(\tau), \quad \text{ch}_{t,(-,+)}^{k,(\text{NS})}(\tau) = \text{ch}_{t,(-,+)}^{k,(\text{R})}(\tau), \quad \text{ch}_{t,(-,+)}^{k,(\bar{\text{NS}})}(\tau) = \text{ch}_{t,(-,+)}^{k,(\bar{\text{R}})}(\tau).
\end{align*}
\]

(2.10)
Characters in the 2nd line above actually all vanish due to a fermion zero mode and we are left with 3 independent characters which are related to each other by the modular transformations;

\begin{align}
\text{ch}^k_{l(+,-)}(\tau) & \xleftarrow{S} \text{ch}^k_{l(-,+)}(\tau) \xleftarrow{T} \text{ch}^k_{l(-,-)}(\tau), \\
\text{ch}^k_{l(+,+)}(\tau) & \xleftarrow{S} \text{ch}^k_{l(-,+)}(\tau) \xleftarrow{T} \text{ch}^k_{l(-,-)}(\tau).
\end{align}

(2.12) (2.13)

Fortunately we can make use of the results given in [27, 28, 29] to calculate these character functions. As in [28], we first consider the sector \( I = \text{NS}, (S, T) = (-, +) \), which is known as the “twisted \( \mathcal{N} = 2 \) minimal model”. In this sector \( J \) and \( G_1 \) have half-integer modes and \( G_2 \) has integer modes. Making use of the well-known decomposition [27, 31]

\[ \mathcal{M}_{k}^{\mathcal{N}=2} \cong \frac{[\mathbb{Z}_k\text{-parafermion theory}] \times U(1)}{\mathbb{Z}_k}, \]

(2.14)

the primary fields (in NS sector) in the twisted minimal model are constructed as

\[ \Phi_l(z) = \varphi_l(z) \sigma(z), \quad (l = 0, 1, \ldots, k), \]

(2.15)

where \( \varphi_l(z) \) are “C-disorder fields” [27] in the \( \mathbb{Z}_k \)-parafermion theory [32] and \( \sigma(z) \) is the twist field of the \( U(1) \) sector. \( \Phi_l(z) \) has the conformal weight

\[ h_l^t(\equiv h(\Phi_l)) = h(\varphi_l) + h(\sigma) = \frac{k - 2 + (k - 2l)^2}{16(k + 2)} + \frac{1}{16}. \]

(2.16)

Since we have the field identification \( \Phi_l = \Phi_{k-l} \), we can assume the range \( l = 0, 1, \ldots, \left[ \frac{k}{2} \right] \).

The character of the representation associated to the field \( \Phi_l \) has been calculated in [33] (See also [28, 23]);

\[ \chi^k_{l(-,+)}(\tau)(\equiv \text{ch}^k_{l(-,+)}(\tau)) = \frac{1}{\theta_4(\tau)} \left( \Theta_1 \tau^{l_1 - \frac{k + 2}{2}k + 2} - \Theta_{-(l+1) - \frac{k + 2}{2}k + 2} \right) \]

\[ = \frac{1}{\theta_4(\tau)} \left( \Theta_{2(l+1)-(k+2),4(k+2)} + \Theta_{2(l+1)+3(k+2),4(k+2)} \right) \]

\[ - \Theta_{2(l+1)-(k+2),4(k+2)} - \Theta_{2(l+1)+3(k+2),4(k+2)} \]

(2.17)

Characters in other sectors are found from the modular transformation

\[ \chi^k_{l(-,+)}(\tau + 1) = e^{2\pi i \left( \frac{k}{16(k + 2)} \right)} \chi^k_{l(-,+)}(\tau), \]

(2.18)

\[ \chi^k_{l(-,+)} \left( \frac{-1}{\tau} \right) = \sum_{l' = 0}^{k} S_{l,l'}^{(k)} \left( -1 \right)^{l'/2} \chi^k_{l',(+,-)}(\tau), \]

(2.19)
where we have

\[
\chi^k_{l(\pm\pm)}(\tau) = \frac{1}{\theta_3(\tau)} \left( \Theta_{2(l+1)-(k+2),4(k+2)}(\tau) + (-1)^k \Theta_{2(l+1)+3(k+2),4(k+2)}(\tau) \right) + (-1)^l \theta_{2(\pm)}(\tau) \Theta_{2(l+1)-(k+2),4(k+2)}(\tau) + (-1)^{k+l} \theta_{2(l+1)+3(k+2),4(k+2)}(\tau) \right), \\
\chi^k_{l(\pm\mp)}(\tau) = \begin{cases} \\
\frac{2}{\theta_2(\tau)} \left( \Theta_{2(l+1),4(k+2)}(\tau) + (-1)^k \Theta_{2(l+1)+4(k+2),4(k+2)}(\tau) \right) & (l : \text{even}), \\
0 & (l : \text{odd}). 
\end{cases} 
\] (2.20)

In (2.19) \( S_{l,l'}^{(k)} \equiv \sqrt{\frac{2}{k+2}} \sin \left( \frac{(l+1)(l'+1)}{k+2} \right) \) is the coefficient of the S-matrix of the \( SU(2) \) WZW model at level \( k \).

Modular properties of \( \chi^k_{l(\pm\pm)}(\tau) \), \( \chi^k_{l(\pm\mp)}(\tau) \) are similarly obtained as

\[
\chi^k_{l(\pm\pm)}(\tau + 1) = e^{2\pi i (h_l - \frac{k}{4(k+2)})} \chi^k_{l(\pm\pm)}(\tau), \quad \chi^k_{l(\pm\mp)}(-\frac{1}{\tau}) = \sum_{l' = 0}^{k} (-1)^{l/2} S_{l,l'}^{(k)} \chi^k_{l'(-,\pm)}(\tau), \\
\chi^k_{l(-\cdash)}(\tau + 1) = e^{2\pi i (h_l - \frac{k}{4(k+2)})} \chi^k_{l(-\cdash)}(\tau), \quad \chi^k_{l'(-\cdash)}(-\frac{1}{\tau}) = (-i) \sum_{l'}^{k} \tilde{S}_{l,l'}^{(k)} \chi^k_{l'(-\cdash)}(\tau). 
\] (2.22)

Here \( h_l \equiv \frac{(l+2)}{4(k+2)} \) and we set \( \tilde{S}_{l,l'}^{(k)} = e^{\frac{\pi i}{2} (l+l'+2-k+2)} S_{l,l'}^{(k)} \) in the above. In summary

\[
\chi^k_{l(\pm\pm)}(\tau) = \text{ch}^k_{l(\pm\pm)}(\tau) = \text{ch}^k_{l(\pm\pm)}(\tau), \\
\chi^k_{l(\pm\mp)}(\tau) = \text{ch}^k_{l(\pm\mp)}(\tau) = \text{ch}^k_{l(\pm\mp)}(\tau), \\
\chi^k_{l(-\cdash)}(\tau) = \text{ch}^k_{l(-\cdash)}(\tau) = \text{ch}^k_{l(-\cdash)}(\tau). 
\] (2.24)

Remaining characters all vanish since they contain a free Majorana fermion with the \( (P,P) \) boundary condition. A few remarks are in order:

1. It is easy to see \( \chi^k_{l(-\cdash)}(\tau) = \chi^k_{l(\pm\pm)}(\tau) \) from the definitions (2.17), (2.20). This is consistent with the field identification \( \Phi_{k-l} \cong \Phi_l \).

2. \( \chi^k_{l(\pm\mp)}(\tau) \) is identified with the trace \( \text{Tr}_{\mathcal{H}_l} (\sigma q^{L_0 - \frac{c}{24}}) \), where \( \mathcal{H}_l \) is the representation space of (untwisted) \( \mathcal{N} = 2 \) superconformal algebra over the primary state with \( h = \frac{l(l+2)}{4(k+2)}, q = 0 \) in the NS sector. \( \chi^k_{l(\pm\mp)} = 0 \) for \( l \) = odd is consistent with the fact that \( q \neq 0 \) states do not contribute to the trace \( \text{Tr}_{\mathcal{H}_l} (\sigma q^{L_0 - \frac{c}{24}}) \).

3. We present a few examples of the twisted character formulas;
\( k = 1 \) \((c = 1)\):

\[
\begin{align*}
\chi^1_{0(+,+)}(\tau) &= \chi^1_{1(+,+)}(\tau) = \sqrt{\frac{\eta}{\theta_4}}, \\
\chi^1_{0(+,-)}(\tau) &= \chi^1_{1(+,-)}(\tau) = \sqrt{\frac{\eta}{\theta_3}}, \\
&= \sqrt{\frac{2\eta}{\theta_2}}.
\end{align*}
\] (2.25)

\( k = 2 \) \((c = 3/2)\):

\[
\begin{align*}
\chi^2_{0(+,+)}(\tau) &= \chi^2_{2(+,+)}(\tau) = \sqrt{\frac{\eta}{\theta_4}} \sqrt{\frac{\theta_3}{2\eta}}, \chi^2_{1(+,+)}(\tau) = \sqrt{\frac{\eta}{\theta_3}} \sqrt{\frac{\theta_3}{\eta}}, \\
\chi^2_{0(+,-)}(\tau) &= \chi^2_{2(+,-)}(\tau) = \sqrt{\frac{\eta}{\theta_3}} \sqrt{\frac{\theta_2}{2\eta}}, \chi^2_{1(+,-)}(\tau) = \sqrt{\frac{\eta}{\theta_2}} \sqrt{\frac{\theta_4}{\eta}}, \\
&= \sqrt{\frac{2\eta}{\theta_2}} \cdot \frac{1}{2} \left( \sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}} \right), \ \chi^2_{2(+,-)}(\tau) = \sqrt{\frac{2\eta}{\theta_2}} \cdot \frac{1}{2} \left( \sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}} \right).
\end{align*}
\] (2.26)

In deriving these formulas some theta-function identities are used. (2.25) (2.26) is consistent with the fact that the \( k = 1 \) \((k = 2)\) model is described by a free boson \((\text{a free boson and fermion})\).

### 2.2 Partition Function of SCFT for \( G_2 \) Orbifold \((CY_3 \times S^1)/\mathbb{Z}_2\)

Now we are ready to discuss the construction of toroidal partition functions of string theory on the orbifold \((CY_3 \times S^1)/\mathbb{Z}_2\) where the \( CY_3 \) sector is described by an arbitrary Gepner model \((k_1, k_2, \ldots, k_r)\). We first consider the partition function of the \( \mathcal{N} = 1 \) non-linear \( \sigma \)-model on this orbifold \((\text{using the standard diagonal modular invariant})\), and then go on to the construction of the partition function of type II string theory on \( R^{2,1} \times (CY_3 \times S^1)/\mathbb{Z}_2\).

According to the standard argument of \( \mathbb{Z}_2 \)-orbifold, partition function of \( \sigma \)-model has the following form

\[
Z_{\sigma} = \frac{1}{4} \sum_I \sum_{S,T} Z^{(I)}_{S,T},
\] (2.27)

where \( I \) runs over spin structures NS, \( \overline{\text{NS}} \), R, \( \overline{\text{R}} \) and \( S, T = \pm \) characterize the boundary conditions for the \( \sigma \)-twist. The overall factor \( 1/4 \) comes from the \( \mathbb{Z}_2 \)-orbifolding and the GSO projection.

The partition function in the untwisted sector is quite simple. If the partition function of the Gepner model \((k_1, \ldots, k_r)\) is given by

\[
Z_{CY_3} = \frac{1}{2} \sum_I Z^{(I)}_{CY_3},
\] (2.28)
then the partition function for the orbifold is given by

\[ Z^{(I)}_{+,+} = Z^{(I)}_{\text{CY}_{3}} \cdot Z^{(I)}_{S^1}. \]  

(2.29)

Amplitudes of the \( S^1 \) sector \( Z^{(I)}_{S^1} \) are given by the standard expressions

\[
Z^{(\text{NS})}_{S^1} = \left| \frac{\theta_4}{\eta} \right| Z_{S^1}(R), \quad Z^{(\overline{\text{NS}})}_{S^1} = \left| \frac{\eta}{\theta_4} \right| Z_{S^1}(R), \\
Z^{(R)}_{S^1} = \left| \frac{\theta_2}{\eta} \right| Z_{S^1}(R), \quad Z^{(\overline{R})}_{S^1} = \left| \frac{\eta}{\theta_2} \right| Z_{S^1}(R) (\equiv 0),
\]

(2.30)

where \( Z_{S^1}(R) \) denotes the partition function of a compact free boson \( X \) (\( R \) is the radius of \( S^1 \)). We later discuss the general structure of the partition function \( Z_{\text{CY}_{3}} \) in the Gepner model.

Now let us turn to the twisted sectors. Since twisted characters include only states with vanishing \( U(1) \)-charge, the orbifoldization enforcing the integrality of total \( U(1) \)-charge acts trivially in these sectors. We combine the conformal blocks in, say, the NS \((+,-)\) sector as;

\[
Z^{(\text{NS})}_{+,,-} = \sum_{k_i} \prod_{l_i,i_i=0}^{r} \mathcal{N}_{l_i,i_i}^{k_i} ch_{l_i, (+,-)}^{k_i} \left[ \frac{2\eta}{\theta_2} \right] \left| \frac{\theta_4}{\eta} \right| \sum_{l_i,i_i=0}^{r} \prod_{i=1}^{k_i} \mathcal{N}_{l_i,i_i}^{k_i} \chi_{l_i, (+,-)}^{k_i} \chi_{l_i, (+,-)}^{k_i*} \left( \frac{\theta_3 \theta_4^2}{\eta^3} \right). 
\]

(2.31)

Here \( \mathcal{N}_{l_i,i_i}^{k_i} \) denotes the coefficient matrix for the modular invariants of the sub-theory of level \( k_i \). Summing over spin structures we obtain

\[
\sum_{I} Z^{(I)}_{+,+} = \sum_{l_i,i_i=0}^{r} \prod_{i=1}^{k_i} \mathcal{M}_{l_i,i_i}^{(k_i)} \left[ \frac{\theta_2 \theta_3^2}{\eta^3} \right] \left[ \frac{\theta_3 \theta_4^2}{\eta^3} \right].
\]

(2.32)

As it turns out, in the case of general Gepner model describing CY 3-fold we have to be careful in choosing the coefficient matrices \( \mathcal{N}_{l_i,i_i}^{(k_i)} \) in order to ensure a suitable projection onto \( \mathbb{Z}_2 \) invariant states. When not all levels \( k_i (i = 1, \ldots, r) \) are even, we can use the diagonal invariant for all sub-theories. On the other hand, in the special case with all \( k_i \) even, a particular mixture of A-type and D-type invariants has to be used as we discuss below.

Other twisted sectors are obtained from the \((+,-)\) sector by modular transformations. Partition functions are given by

\[
\sum_{I} Z^{(I)}_{-,+} = \sum_{l_i,i_i=0}^{r} \prod_{i=1}^{k_i} \mathcal{M}_{l_i,i_i}^{(k_i)} \left[ \frac{\theta_2 \theta_3^2}{\eta^3} \right] \left[ \frac{\theta_3 \theta_4^2}{\eta^3} \right],
\]

\[
\sum_{I} Z^{(I)}_{-,-} = \sum_{l_i,i_i=0}^{r} \prod_{i=1}^{k_i} \mathcal{M}_{l_i,i_i}^{(k_i)} \left[ \frac{\theta_2 \theta_3^2}{\eta^3} \right] \left[ \frac{\theta_3 \theta_4^2}{\eta^3} \right].
\]

(2.33)
The matrix $M_{\{k_i\}_{i=1}^l,\{l_i\}_{i=1}^l}$ is obtained from the matrix $\prod_i \Lambda_{l_i,l_i}^{k_i}$ by modular transformations.

Let us now consider the partition function of type II string theory on $R^{2,1} \times (CY_3 \times S^1)/\mathbb{Z}_2$.

Our remaining task is to:
1. Fix the coefficient matrix of modular invariant.
2. Incorporate the contribution from the space-time $R^{2,1}$ (we only consider the transversal degrees of freedom).
3. Take account of the GSO projection as the type II theory. Namely, we sum over the spin structures of left and right movers independently, while the $\sigma$-twist acts in a diagonal manner.

Due to 2 and 3, the partition function should have the following form:

$$Z_{\text{string}} = \frac{1}{4} \cdot 2 \frac{1}{\sqrt{\tau_2|\eta|^2}} \sum_{I_L,I_R} \sum_{S,T} Z_{S,T}^{(I_L,I_R)}(\tau_2 = \Im \tau).$$

(2.34)

where we factored out the contribution from the transverse boson of $R^{2,1}$ while that of the fermion is incorporated in $Z_{S,T}^{(I_L,I_R)}$ to take account of the GSO projection. The overall factor 1/4 is due to GSO projection while an additional 1/2 is due to $\mathbb{Z}_2$-orbifolding.

Let us now introduce some formulas obtained in [34] which are convenient for the discussion of the general structure of Gepner models (see Appendix B). Contributions of the tensor product of minimal models are organized into orbits $F_i$ generated by the spectral flow

$$F_i(\tau) \equiv \frac{\theta_3}{\eta} \text{NS}_i(\tau) - \frac{\theta_4}{\eta} \tilde{\text{NS}}_i(\tau) - \frac{\theta_2}{\eta} R_i(\tau) - \frac{\theta_1}{\eta} \tilde{R}_i(\tau)$$

(2.35)

where $\text{NS}_i(\tau), \tilde{\text{NS}}_i(\tau), R_i(\tau), \tilde{R}_i(\tau)$ are the conformal blocks of the $CY_3$ sector defined by

$$Z_{CY_3} = \frac{1}{2} \sum_i D_i \left( |\text{NS}_i(\tau)|^2 + |\tilde{\text{NS}}_i(\tau)|^2 + |R_i(\tau)|^2 + |\tilde{R}_i(\tau)|^2 \right),$$

(2.36)

$D_i$ are non-negative integers with the properties

$$D_i S_{ij} = D_j S_{ij} \quad \text{(no sum on i, j)}$$

(2.37)

where $S_{ij}$ is the $S$-transformation matrix of the conformal blocks $F_i$. Blocks $F_i$ actually all vanish identically $F_i \equiv 0$ due to some theta-function identity reflecting the space-time SUSY in Calabi-Yau compactification. After a little algebra, we obtain string theory amplitude in the untwisted sector

$$\sum_{I_L,I_R} Z_{S^1}(I_L,I_R) = Z_{S^1}(R) \sum_i D_i |F_i(\tau)|^2.$$
Now we consider the twisted sector \((+, -)\) and discuss a suitable projection onto \(\mathbb{Z}_2\) invariant states when combined with the untwisted sector. Under the action of \(\mathbb{Z}_2\) symmetry \(U(1)\) charge flips sign and thus in the twisted sector we should consider only neutral states. Let us consider a state, for instance, \(\text{ch}^{k, (\text{NS})}_{l_i, m=0}\) in the NS sector of a sub-theory. In the orbit of this state generated by the spectral flow, there appears another neutral state \(\text{ch}^{k, (\text{NS})}_{k-l_i, m=0}\) if \(k\) is even. These two representations of spin \(l/2\) and \((k - l)/2\) are paired and they contribute an off-diagonal term to the partition function. Therefore when the level \(k\) is even, we have to adopt an analogue of D-type modular invariant. On the other hand when \(k\) is odd, we use the standard A-type modular invariant.

In the case of a general tensor product of minimal models additional neutral states appear when all the levels \(k_i\) of sub-theories are even. In this case an additional neutral state in the orbit of \(\prod_i \text{ch}^{k_i, (\text{NS})}_{l_i, m=0}\) has a form \(\prod_{i \in S_1} \text{ch}^{k_i, (\text{NS})}_{k_i-l_i, m=0} \prod_{j \in S_2} \text{ch}^{k_j, (\text{NS})}_{l_j, m=0}\). Here the two sets \(S_1, S_2\) are defined as

\[
\begin{align*}
  i & \in S_1 \text{ if } \frac{D}{k_i + 2} = \text{odd}, \\
  j & \in S_2 \text{ if } \frac{D}{k_j + 2} = \text{even}.
\end{align*}
\]

and

\[
D = \text{Least Common Multiple of } \{k_i + 2 (i = 1, \cdots, r)\}.
\]

We then see that the D-type pairing has to be used for the sub-theories in the set \(S_1\) while A-type invariant is used for sub-theories in \(S_2\). Thus we introduce

\[
\mathcal{N}^{(k_i)}_{\{l_i\}, \{\bar{l}_i\}} = \prod_{i \in S_2} \delta_{l_i, \bar{l}_i} \prod_{j \in S_1} \left(\delta_{l_j, \bar{l}_j} + \delta_{l_j, k_j - \bar{l}_j}\right)
\]

Then the amplitude in the twisted sector \((+, -)\) is given by

\[
\sum_{I_L, I_R} Z^{(I_L, I_R)}_{+, -} = \sum_{l_i, \bar{l}_i} \left(\mathcal{N}^{(k_i)}_{\{l_i\}, \{\bar{l}_i\}} \prod_i x^{k_i}_{I_i, (+)} x^{k_i}_{I_i, (-)} \right)^2 \left| \sqrt{\frac{\theta_3}{\eta}} \sqrt{\frac{\theta_3 \theta_4^2}{\eta^3}} - \sqrt{\frac{\theta_4}{\eta}} \sqrt{\frac{\theta_3 \theta_4^2}{\eta^3}} \right|^2
\]

when all level are even. When an odd level is contained in the tensor product, coefficient \(\mathcal{N}^{(k_i)}_{\{l_i\}, \{\bar{l}_i\}}\) is replaced by the product of Kronecker delta’s.

Amplitudes in other twisted sectors are now obtained by modular transformations. When all levels are even, we obtain

\[
\sum_{I_L, I_R} Z^{(I_L, I_R)}_{+, +} = \sum_{l_i} \left(1 + (-1)^{\sum_{i \in S_1} l_i}\right) \prod_i |x^{k_i}_{l_i, (+)}|^2 \left| \sqrt{\frac{\theta_3}{\eta}} \sqrt{\frac{\theta_3 \theta_4^2}{\eta^3}} - \sqrt{\frac{\theta_4}{\eta}} \sqrt{\frac{\theta_3 \theta_4^2}{\eta^3}} \right|^2
\]

and

\[
\sum_{I_L, I_R} Z^{(I_L, I_R)}_{-, -} = \sum_{l_i} \left(1 + (-1)^{\sum_{i \in S_1} l_i}\right) \prod_i |x^{k_i}_{l_i, (-)}|^2 \left| \sqrt{\frac{\theta_3}{\eta}} \sqrt{\frac{\theta_3 \theta_4^2}{\eta^3}} - \sqrt{\frac{\theta_4}{\eta}} \sqrt{\frac{\theta_3 \theta_4^2}{\eta^3}} \right|^2
\]

when all level are even. When an odd level is contained in the tensor product, coefficient \(\mathcal{N}^{(k_i)}_{\{l_i\}, \{\bar{l}_i\}}\) is replaced by the product of Kronecker delta’s.
\[
\sum_{I_{L,R}} Z^{(I_{L,R})} = \sum_{I_i} \left( 1 + (-1)^{\sum_{i \in S_1} l_i} \right) \prod \left| \chi_{i_i(-,-)}^{k_i} \right|^2 \left\{ \sqrt{\frac{\theta_4}{\eta}} \sqrt{\frac{\theta_4 \theta_2^2}{\eta^3}} - \sqrt{\frac{\theta_2}{\eta}} \sqrt{\frac{\theta_2 \theta_4^2}{\eta^3}} \right\}. \tag{2.45}
\]

In checking modular invariance of these formulas we have to cancel some unwanted sign factors by using

1) \(D\) is an integer divisible by a factor 4.

2) The set \(S_1\) is not empty and its number of elements is even.

3) When a sub-theory of level \(k_i\) belongs to \(S_1\), \(k_i \in 4\mathbb{Z} + 2\).

These facts are easily derived by using the criticality condition (2.2).

When an odd level is contained in the tensor product, a factor \((-1)^{\sum_{i \in S_1} l_i}\) is absent in the above formulas (2.44), (2.45). Note that all the twisted amplitudes vanish identically, which is consistent with the existence of SUSY in our orbifold construction.

Space-time SUSY charges are constructed as vertex operators in the untwisted sector and hence are the \(\mathbb{Z}_2\)-invariant combinations of SUSY charges of the \(CY_3\) compactification. Since the \(\sigma\)-twisting commutes with \((-1)^{F_L}, (-1)^{F_R}\), such SUSY charges consistently act on the Hilbert space of twisted sectors also, and give rise to the manifest cancelation of amplitudes in twisted sectors. Thus our string vacuum possesses the space-time SUSY charges which are half as many as those of Calabi-Yau compactification \(\frac{1}{2} \times 8 = 4\). This is of course the expected number of SUSY charges in the compactification on a \(G_2\) manifold.

Let us next check the consistency of our results with the general argument by Shatashvili and Vafa [21] of string compactification on \(G_2\) manifold and in particular the existence of tricritical Ising model. As is shown in Appendix B, conformal blocks \(\mathcal{F}_i\) of \(CY_3\) compactification are expanded in terms of functions \(g_1, g_2\) defined by

\[
g_1(\tau) \equiv \frac{\theta_3 \Theta_{0,3/2}}{\eta \eta} - \frac{\theta_4 \Theta_{0,3/2}}{\eta \eta} - \frac{\theta_2 \Theta_{3/2,3/2}}{\eta \eta}, \tag{2.46}
\]

\[
g_2(\tau) \equiv \frac{\theta_3 \Theta_{1,3/2}}{\eta \eta} + \frac{\theta_4 \Theta_{1,3/2}}{\eta \eta} - \frac{\theta_2 \Theta_{1/2,3/2}}{\eta \eta}. \tag{2.47}
\]

We then use the following identities and reexpress \(g_1, g_2\) in terms of functions \(F_1, F_2\) which involve tricritical Ising models

\[
g_1(\tau) = \eta c_{0,0}^{(3)}(\tau) F_1(\tau) + \eta c_{2,0}^{(3)}(\tau) F_2(\tau), \tag{2.48}
\]

\[
g_2(\tau) = \eta c_{0,2}^{(3)}(\tau) F_1(\tau) + \eta c_{2,2}^{(3)}(\tau) F_2(\tau), \tag{2.49}
\]

\[
F_1(\tau) = \sqrt{\frac{\theta_3}{\eta}} \chi_0^{\text{tri}} - \sqrt{\frac{\theta_4}{\eta}} \chi_0^{\text{tri}} - \sqrt{2} \frac{\theta_2}{\eta} \sqrt{\theta_7/16}, \tag{2.50}
\]
\[
F_2(\tau) = \sqrt{\frac{\theta_3}{\eta}} \chi_{1/10}^{\text{tri}} + \sqrt{\frac{\theta_4}{\eta}} \chi_{1/10}^{\text{tri}} - \sqrt{2} \sqrt{\frac{\theta_2}{\eta}} \chi_{3/80}^{\text{tri}}.
\] (2.51)

Here \(\chi_h^{\text{tri}}, \bar{\chi}_h^{\text{tri}}\) denote the \((\mathcal{N} = 1)\) characters of tricritical Ising model of conformal dimension \(h\) and \(\mathcal{c}_{l,m}^{(3)}\) is the level 3 string function of affine SU(2) algebra. The above relations (2.48) (2.49) can be derived by comparing two ways of rewriting Jacobi’s identity [24, 35]

\[
0 = \frac{1}{\eta^4} \left( \theta_3^4 - \theta_1^4 - \theta_4^4 \right) = g_1(\tau) \chi_{b}^{SU(3)}(\tau) + g_2(\tau) \left( \chi_{f}^{SU(3)}(\tau) + \chi_{\bar{f}}^{SU(3)}(\tau) \right)
= F_1(\tau) \chi_{b}^{G_2}(\tau) + F_2(\tau) \chi_{f}^{G_2}(\tau).
\] (2.52)

(2.53)

Here \(\chi_i^{SU(3)}\) \(i = b, f, \bar{f}\) denote the level 1 \(SU(3)\) characters of the basic, fundamental and anti-fundamental representations, and \(\chi_i^{G_2}\) \(i = b, f\) denotes the level 1 \(G_2\) character of the basic and fundamental representations. We also remark that

\[
(G_2)_{1/SU(3)} \cong SU(2)_3/U(1)_3 \cong \mathbb{Z}_3-\text{Parafermion},
\] (2.54)

as pointed out in [33].

Above formulas (2.52), (2.53) show that in CY compactification of SU(3) holonomy branching functions \(g_1, g_2\) should necessarily appear in the CFT description while functions \(F_1, F_2\) should appear in compactification on \(G_2\) manifold. In fact \(F_1, F_2\) contain tricritical Ising model as claimed by Shatashvili and Vafa. All these functions \(g_i, F_i\) vanish due to the Jacobi identity.

Let us next look at the massless spectrum contained in our amplitudes. It is easy to identify the massless states in the untwisted sector; they are nothing but the \(\mathbb{Z}_2\)-invariant combinations of the massless states in the string theory on \(\mathbb{R}^{2,1} \times S^1 \times CY_3\). It is straightforward to count these states and it is known that in addition to the gravity multiplet there exist \(b_2 + b_3 = h^{1,1} + h^{2,1} + 1\) massless chiral fields where \(h^{1,1}, h^{2,1}\) are the Hodge numbers of Calabi-Yau 3-fold [24, 1, 4].

The extra massless states originating from the twisted sectors are somewhat non-trivial. We first recall the formula for conformal weights of primary fields in the twisted \(\mathcal{N} = 2\) minimal model

\[
h_t^i(\equiv h(\Phi_i)) = \frac{k - 2 + (k - 2l)^2}{16(k + 2)} + \frac{1}{16}.
\] (2.55)

Thus we find

\[
h_t^i - \frac{k}{8(k + 2)} = \frac{1}{4(k + 2)} \left( l + 1 - \frac{k + 2}{2} \right)^2 \geq 0.
\] (2.56)
Therefore, when \( k \) is even, the inequality (2.56) is saturated at \( l = k/2 \), while when \( k \) is odd, there is no saturation. This leads to the following rules on the existence of extra massless states:

1. In the case when at least one of \( k_i \) is odd in the tensor product of minimal models \((k_1, \ldots, k_r)\), there are no massless states in the twisted sector.

2. In the case when all the levels \( k_i \) are even, we have \( 2 \times 2 = 4 \) massless bosonic states in the twisted sector as is read off from the above partition function (a factor 2 corresponds to the choice of NS-NS, R-R sectors). These form 2 massless chiral multiplets.

3 **G\(_2\) Manifolds associated to Singular Calabi-Yau and K\(_3\) Spaces**

Finally we construct the partition functions of G\(_2\)-orbifolds based on the Calabi-Yau spaces with isolated singularities. We here focus on the case of A\(_{k+1}\)-type singularity. The conformal system describing such Calabi-Yau space is given by the Gepner model like construction;

\[
\left[ \mathcal{M}^{N=2}_k \times (\mathcal{N} = 2 \text{ Liouville}) \right]_{U(1)\text{-projected}}. \tag{3.1}
\]

In addition to the minimal model we have an \( \mathcal{N} = 2 \) Liouville system which is necessary to describe the non-compact geometry of target manifold. The partition function of this system is studied in detail in [36].

The \( \sigma \)-twist acts on the minimal sector \( \mathcal{M}^{N=2}_k \) in the same way as before, and acts on the \( \mathcal{N} = 2 \) Liouville fields \( \phi, Y, \psi^\phi, \psi^Y \) as follows;

\[
\sigma : \phi \to \phi, \quad Y \to -Y, \quad \psi^\phi \to \psi^\phi, \quad \psi^Y \to -\psi^Y \tag{3.2}
\]

These fields contribute extra theta functions to the twisted sectors. For example, in the (NS, (+, −))-sector, we obtain an extra factor

\[
\left| \frac{2\eta}{\theta_2} \right| \cdot \left| \frac{\theta_3 \theta_4}{\eta^2} \right| = \left| \frac{\theta_3^2 \theta_4^2}{\eta^4} \right|. \tag{3.3}
\]

String theory partition functions are then given by

\[
Z_{\text{string}} = \frac{1}{\tau_2 |\eta|^4} \sum_{I_L, I_R} \sum_{S,T} Z_{S,T}^{(I_L, I_R)} = \frac{1}{\tau_2 |\eta|^4} \left( Z_{\text{string}}^u + Z_{\text{string}}^d \right). \tag{3.4}
\]
The factor $\frac{1}{r_2|\eta|^r}$ comes from the contribution of the Liouville field $\phi$ as well as the transverse boson. The conformal blocks in the untwisted sector has essentially the same form as given in [36].

$$Z_{\text{string}}^u = \sum_{l,r} Z_{S_1}(R) |\mathcal{F}_{l,r}^{(k)}|^2,$$

(3.5)

$$\mathcal{F}_{l,r}^{(k)}(\tau) = \frac{1}{2} \sum_{m \in \mathbb{Z}_{4(k+2)}} \frac{1}{\eta} \Theta_{(k+4)m+(k+2)r,2(k+2)(k+4)}(\tau)$$

\begin{align*}
\times \left\{ \left( \frac{\theta_3}{\eta} \right)^2 \text{ch}_{l,m}^{k,(NS)} - \frac{1}{2^2} \left( \frac{\theta_4}{\eta} \right)^2 \text{ch}_{l,m}^{(\tilde{NS})} - \left( \frac{\theta_2}{\eta} \right)^2 \text{ch}_{l,m}^{(R)} \right\}.
\end{align*}

(3.6)

The conformal blocks $\mathcal{F}_{l,r}^{(k)}(\tau)$ identically vanish for arbitrary $l, r$ as discussed in [36] and we can further prove that these functions are expanded by $g_1(\tau), g_2(\tau)$ as in the case of usual Gepner models.

The contribution from the twisted sectors are again written in a form where manifest cancelation takes place

$$Z_{\text{string}}^t = \left( \sum_{l=0}^k |\lambda_{l,+}^{k}|^2 \right) \left\{ \sqrt{\frac{\theta_3}{\eta}} \sqrt{\frac{\theta_3^3 \theta_4^4}{\eta^7}} - \sqrt{\frac{\theta_4}{\eta}} \sqrt{\frac{\theta_3^3 \theta_4^4}{\eta^7}} \right\}^2$$

\begin{align*}
+ \left( \sum_{l=0}^k |\lambda_{l,-}^{k}|^2 \right) \left\{ \sqrt{\frac{\theta_4}{\eta}} \sqrt{\frac{\theta_3^3 \theta_4^4}{\eta^7}} - \sqrt{\frac{\theta_2}{\eta}} \sqrt{\frac{\theta_3^3 \theta_4^4}{\eta^7}} \right\}^2

+ \left( \sum_{l=0}^k |\lambda_{l,-}^{k}|^2 \right) \left\{ \sqrt{\frac{\theta_2}{\eta}} \sqrt{\frac{\theta_3^3 \theta_4^4}{\eta^7}} - \sqrt{\frac{\theta_2}{\eta}} \sqrt{\frac{\theta_3^3 \theta_4^4}{\eta^7}} \right\}^2.
\end{align*}

(3.8)

In the case of singular manifolds theory has a mass gap due to the Liouville background charge and there exist no massless normalizable states in the spectrum of the untwisted sector. However, somewhat surprisingly, massless states do appear in the twisted sector. By a similar argument as before it is easy to see that there appear 4 massless chiral multiplets in the case of even $k$ in the twisted sector while no massless states exist at odd $k$.

The case of orbifold $(ALE \times T^3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is also interesting. We focus on the case of $A_{k+1}$-singularity and assume that $T^3$ is a rectangular torus with the radii $R_1, R_2, R_3$ for simplicity. Let $X^1, X^2, X^3 (\psi^1, \psi^2, \psi^3)$ be the bosonic (fermionic) coordinates of $T^3$. Discussion here becomes slightly involved, since we must consider $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold instead of $\mathbb{Z}_2$ orbifolding in order to realize the right amount of supersymmetry.
In this case the system is described by the $SU(2)_k$ WZW model together with the Liouville field $\phi$, and 4 free fermions $\chi^a$ ($a = 0, 1, 2, 3$), which form an $\mathcal{N} = 4$ superconformal field theory with $c = 6$ \[^{37}\].

We should consider the following twists $\sigma_i$ ($i = 1, 2, 3$)

$$
\begin{align*}
\sigma_i & : \phi \rightarrow \phi, \quad K^i \rightarrow K^i, \quad K^j \rightarrow -K^j, \quad (j \neq i), \\
\chi^0 & \rightarrow \chi^0, \quad \chi^i \rightarrow \chi^i, \quad \chi^j \rightarrow -\chi^j, \quad (j \neq 0, i).
\end{align*}
$$

where $K^i$ denotes the $SU(2)_k$ currents. The above action is identified as an automorphism in $\mathcal{N} = 4$ SCA

$$
\begin{align*}
\sigma_i & : \ G^0 \rightarrow G^0, \quad G^i \rightarrow G^i, \quad G^j \rightarrow -G^j \quad (j \neq i), \\
J^i & \rightarrow J^i, \quad J^j \rightarrow -J^j \quad (j \neq i).
\end{align*}
$$

We also assume that $\sigma_i$ act on the $T^3$ sector as

$$
\begin{align*}
\sigma_i & : \begin{cases} 
X^j \rightarrow -X^j & (j \neq i), \\
\psi^j \rightarrow -\psi^j & (j \neq i), \\
\psi^i \rightarrow \psi^i.
\end{cases}
\end{align*}
$$

Since only the two of $\sigma_1, \sigma_2, \sigma_3$ are independent ($\sigma_1\sigma_2 = \sigma_3$), such transformations define a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $ALE \times T^3$.

These $\sigma_i$-twists give rise to various twisted sectors. At first glance, it appears hard to treat the sectors with different twists in different directions. However, it is easy to check that such twisted sectors always include at least one free fermion with the (P,P) boundary condition and thus do not contribute to the amplitude.

Therefore, the remaining twisted sectors are given by the same twist (or absence of twist) in both space and time directions. Only the non-trivial point is to see how the twists act on the $SU(2)_k$ currents $K^a$. Fortunately, this problem has been already discussed in \[^{29}\] and it has been shown that the characters of “twisted $SU(2)_k$” are the same as those of twisted $\mathcal{M}^\mathcal{N}=2_k$, that is, \[(2.17), (2.20), (2.21)\] which we have studied in section 2.

Now the results of the computation are summarized as follows.

\[
Z_{\text{string}} = \frac{1}{\tau_2 |\eta|^4} \cdot \frac{1}{16} \left( Z^u_{\text{string}} + Z^t_{\text{string}} \right),
\]

\[
Z^u_{\text{string}} = Z_{T^3}(R_1, R_2, R_3) \sum_{l=0}^{k} |F_l(\tau)|^2,
\]

\[
Z^t_{\text{string}} = \frac{1}{2} \left( Z^u_{\text{string}} \right).
\]
\[ \mathcal{F}_i(\tau) = \chi_i^{(k)}(\tau) \left\{ \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right\}, \tag{3.15} \]

\[ Z^*_\text{string} = \left( \sum_{j=1}^{3} Z_{S^1}(R_j) \right) \left\{ \left( \sum_{l=0}^{k} |\chi_{i(\pm,-)}|^2 \right) \sqrt{\frac{\theta_3}{\eta} \frac{\theta_3^4}{\eta^7} - \sqrt{\frac{\theta_2}{\eta} \frac{\theta_2^4}{\eta^7}}^2} + \left( \sum_{l=0}^{k} |\chi_{i(\pm,+)}|^2 \right) \sqrt{\frac{\theta_3^4}{\eta^7} - \sqrt{\frac{\theta_2}{\eta} \frac{\theta_2^4}{\eta^7}}^2} \right\}. \tag{3.16} \]

Again all the normalizable states in the untwisted sector are massive. In the twisted sectors we find 6 massless chiral multiplets in the case of even \( k \) while no massless states exist when \( k \) is odd.

### 4 Discussions and Conclusions

In this paper we have constructed partition functions of type II string theory compactified on general \( G_2 \) manifolds of the type \((CY_3 \times S^1)/\mathbb{Z}_2\) by making use of Gepner construction for the CY sector. It turned out there appear extra massless states in the twisted sector if and only if all the levels \( k_i \) of the minimal sub-theories are even. This seems somewhat problematic since in these cases the corresponding hypersurface

\[ \sum_i z_i^{k_i+2} = 0 \tag{4.1} \]

do not have fixed points under anti-holomorphic involution and we do not expect new massless states to emerge. In [30] a possible resolution of this problem is suggested based on the behavior of the NS \( B \) field taking discrete values in \( G_2 \) manifolds. Related problem exists in the \( G_2 \) manifolds with \( A - D - E \) singularities fibered over \( S^3 \) which feature in gauge/gravity duality [3, 5]. Since the moduli of the metric preserving \( G_2 \) structure is given by \( b_3 \) there exists no smooth resolution of \( A - D - E \) singularities. These are interesting issues which require further study.

*Note added:*
Very recently a new preprint has appeared [38] where it is also pointed out that there exist extra massless states in \((CY_3 \times S^1)/(\mathbb{Z}_2)\) theory when all levels are even in the Gepner construction.

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**Appendix A: Conventions for Theta Functions**

Set \( q := e^{2\pi i \tau}, y := e^{2\pi i z}; \)

\[ \begin{align*}
\theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m), \\
\theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m), \\
\theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{-m/2})(1 + y^{-1}q^{-m/2}), \\
\theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{-m/2})(1 - y^{-1}q^{-m/2}).
\end{align*} \]  

(A.1)

\[ \begin{align*}
\Theta_{m,k}(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2})^2} y^{k(n+\frac{m}{2})}, \\
\tilde{\Theta}_{m,k}(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{k(n+\frac{m}{2})^2} y^{k(n+\frac{m}{2})}.
\end{align*} \]  

(A.2, A.3)

We use abbreviations; \( \theta_1 \equiv \theta_1(\tau, 0) \) \((\theta_1 \equiv 0)\), \( \Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0) \), \( \tilde{\Theta}_{m,k}(\tau) \equiv \tilde{\Theta}_{m,k}(\tau, 0) \). We also set

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \]  

(A.4)

The character of \( SU(2)_k \) with spin \( l/2 \) \((0 \leq l \leq k)\) is given by

\[ \chi_l^{(k)}(\tau, z) = \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}}, \]  

(A.5)

and the string function \( c_{l,m}^{(k)}(\tau) \) is defined by

\[ \chi_l^{(k)}(\tau, z) = \sum_{m \in \mathbb{Z}_{2k}} c_{l,m}^{(k)}(\tau) \Theta_{m,k}(\tau, z). \]  

(A.6)
Appendix B: Review on the Partition Functions of SCFT for CY$_3$ $\sigma$-Model

In this appendix we summarize the structure of the partition function of non-linear $\sigma$-model on CY$_3$ following [34]. The partition function for CY$_3$ has the following form

$$Z_{CY_3}(\tau, \tilde{\tau}, z, \bar{z}) = \frac{1}{2} \sum_{i=1}^{2d+d'+d''} D_i \left( |\text{NS}_i(\tau, z)|^2 + |\overline{\text{NS}}_i(\tau, z)|^2 + |\text{R}_i(\tau, z)|^2 + |\overline{\text{R}}_i(\tau, z)|^2 \right), \quad (B.1)$$

where the index $i$ runs over orbits generated by the spectral flow and $D_i$ are non-negative integers with the property

$$D_i S_{ij} = D_j S_{ji}, \quad \text{(no sum on } i, j). \quad (B.2)$$

In this expression, $S_{ij}$ denotes the modular $S$-matrix of the conformal blocks NS$_i(\tau)$. There appear various orbits in Gepner construction. In the NS-sector we have

1. Graviton orbit: ($i = 1$)

$$\text{NS}_1(\tau, z) = G_1(\tau) \left( f_1(z) + f_{-1}(z) \right) + H_1(\tau) f_0(z), \quad (B.3)$$

$$G_1(\tau) = \sum_{n=1}^{\infty} g_n^{(1)} q^n, \quad H_1(\tau) = q^{-\frac{1}{3}} \left( 1 + \sum_{n=1}^{\infty} h_n^{(1)} q^n \right) \quad (B.4)$$

2. Massless matter orbits: ($i = 2, \ldots, d - 1$, $i^* = i + d - 1 = d, \ldots, 2d - 1$)

$$\text{NS}_i(\tau, z) = f_1(z) + G_i(\tau) \left( f_1(z) + f_{-1}(z) \right) + H_i(\tau) f_0(z), \quad (B.5)$$

$$\text{NS}_i^*(\tau, z) \equiv \text{NS}_i(\tau, -z) = f_{-1}(z) + G_i(\tau) \left( f_1(z) + f_{-1}(z) \right) + H_i(\tau) f_0(z), \quad (B.6)$$

$$G_i(\tau) = \sum_{n=1}^{\infty} g_n^{(i)} q^n, \quad H_i(\tau) = \sum_{n=1}^{\infty} h_n^{(i)} q^{n-\frac{1}{3}}. \quad (B.7)$$

We assume $D_i = D_i^*$. 

3. Self-conjugate matter orbits: ($j = 2d, \ldots, 2d + d' - 1$)

$$\text{NS}_j(\tau, z) = G_j(\tau) \left( f_1(z) + f_{-1}(z) \right) + H_j(\tau) f_0(z), \quad (B.8)$$

$$G_j(\tau) = 1 + \sum_{n=1}^{\infty} g_n^{(j)} q^n, \quad H_j(\tau) = \sum_{n=1}^{\infty} h_n^{(j)} q^{n-\frac{1}{3}}. \quad (B.9)$$
4. Massive orbits: \((m = 2d + d', \ldots, 2d + d' + d'')\)

\[
\text{NS}_m(\tau, z) = G_m(\tau) (f_1(z) + f_{-1}(z)) + H_m(\tau) f_0(z) ,
\]

(B.10)

\[
G_m(\tau) = \sum_{n=1}^{\infty} g_n^{(m)} q^{n + r_m}, \quad H_m(\tau) = \sum_{n=1}^{\infty} h_n^{(m)} q^{n + r'_m - \frac{1}{4}},
\]

(B.11)

where \(r_m, r'_m \in \mathbb{Q}, r_m, r'_m > 0\).

Here \(f_Q\) denotes the level \(3/2\) theta function

\[
f_Q(z) \equiv \frac{1}{\eta(\tau)} \Theta_{Q,3/2}(\tau, 2z) ,
\]

(B.12)

characteristic of the \(c = 9\) system of \(\mathcal{N} = 2\) SCFT.

Conformal blocks in other spin structures are given by spectral flow

\[
\widetilde{\text{NS}}_i(\tau, z) \equiv \text{NS}_i(\tau, z + \frac{1}{2}), \quad \text{R}_i(\tau, z) \equiv q^{3/4} y^{3/2} \text{NS}_i(\tau, z + \frac{\tau}{2}), \quad \widetilde{\text{R}}_i(\tau, z) \equiv \text{R}_i(\tau, z + \frac{1}{2}).
\]

(B.13)

These blocks are compatible with the space-time SUSY. Namely, the following relation holds

\[
\left( \frac{\theta_2}{\eta} \right) \text{NS}_i(\tau, z) - \left( \frac{\theta_4}{\eta} \right) \widetilde{\text{NS}}_i(\tau, z) - \left( \frac{\theta_2}{\eta} \right) \text{R}_i(\tau, z) - \left( \frac{\theta_1}{\eta} \right) \widetilde{\text{R}}_i(\tau, z) \equiv 0,
\]

(B.14)

We also note that

\[
\lim_{z \to 0} \widetilde{\text{R}}_i(\tau, z) = I_i \quad (\text{Witten index})
\]

\[
= \begin{cases} 
1 & (i : \text{massless matter orbits}) \\
-1 & (i : \text{conjugate massless matter orbits}) \\
0 & (i : \text{others})
\end{cases}
\]

(B.15)

Hence we obtain

\[
\lim_{z \to 0} \sum_i D_i |\widetilde{\text{R}}_i(\tau, z)|^2 = 2 \sum_{i=2}^{d} D_i = -2 \chi(CY_3) .
\]

(B.16)
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