The first law of thermodynamics in Lifshitz black holes revisited

Yongwan Gim,1,* Wontae Kim,1,2,† and Sang-Heon Yi3,‡

1Department of Physics, Sogang University, Seoul 121-742, Republic of Korea
2Research Institute for Basic Science, Sogang University, Seoul, 121-742, Republic of Korea
3Department of Physics, College of Science, Yonsei University, Seoul 120-749, Korea

(Dated: March 20, 2014)

Abstract

We obtain the mass expression of the three- and five-dimensional Lifshitz black holes by employing the recently proposed quasilocal formulation of conserved charges, which is based on the off-shell extension of the ADT formalism. Our result is consistent with the first law of black hole thermodynamics and resolves the reported discrepancy between the ADT formalism and the other conventional methods. The same mass expression of Lifshitz black holes is obtained by using another quasilocal method by Padmanabhan. We also discuss the reported discrepancy in the context of the extended first law of black hole thermodynamics by allowing the pressure term.

Keywords: Lifshitz black hole, conserved charge, thermodynamics

* yongwan89@sogang.ac.kr
† wtkim@sogang.ac.kr
‡ shyi@yonsei.ac.kr
I. INTRODUCTION

Recently, there has been much attention to Lifshitz black holes [1], because they may give rise to a new perspective on a condensed matter physics via the Lif/CFT correspondence which is one of the generalization of the AdS/CFT correspondence [2–5]. In black hole physics, identifying conserved charges of black holes is one of the most important issues. This issue becomes more involved in the case of Lifshitz black holes, because of the anisotropic scaling between time and space. After the Lifshitz black hole of the dynamical exponent $z = 3$ was realized as an analytic solution [6] to the three-dimensional new massive gravity [7], there have been many efforts to identify the mass of the three-dimensional Lifshitz black hole, for instance, by using the Abbott-Deser-Teukin (ADT) method [8], Euclidean action approach [9], boundary stress tensor method [10], and dilaton gravity approach [11]. However, there exists a discrepancy since the results of their methods are not compatible with each other. Concretely, the mass expression from the ADT method is claimed to be given by $M_{DS} = 7r_H^4 / (8 G^\ell^4)$ where $r_H = \sqrt{m}$ [8], while the one from other methods is calculated as $M = r_H^4 / (4 G^\ell^4)$ [9–11]. Moreover, the claimed expression for the mass of the Lifshitz black hole from the ADT method does not satisfy the first law of black hole thermodynamics while the others respect the first law exactly.

In the original ADT method [12–16], which is a covariant generalization of the Arnowit-Deser-Misner(ADM) method [17], the metric is linearized as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ around the asymptotic infinity where $\bar{g}_{\mu\nu}$ is a background metric and $h_{\mu\nu}$ is the rapidly vanishing perturbed metric. However, in the three-dimensional Lifshitz black hole, it is not so clear the above condition on the perturbed metric $h_{\mu\nu}$ is satisfied. Explicitly speaking, the metric of the Lifshitz black hole may be taken in the form of $ds^2 = -g_{tt} dt^2 + g_{rr} dr^2 + r^2 d\phi^2$ with $g_{tt} = r^6 / \ell^6 - mr^4 / \ell^4$ and $g_{rr} = (r^2 / \ell^2 - m)^{-1}$. By taking the background metric to be the case of $m = 0$ in this metric, one can see that, in the metric component $g_{rr}$, the background metric $\bar{g}_{rr} = \ell^2 / r^2$ becomes exclusively dominant term at the asymptotic infinity, while the perturbed metric $h_{rr}$ vanishes sufficiently fast at the asymptotic infinity. On the other hand, the dominant term of the metric component $g_{tt}$ at the asymptotic infinity seems to be the background metric $\bar{g}_{tt} = r^6 / \ell^6$, but the perturbed metric component $h_{tt} = mr^4 / \ell^4$ is also divergent so that it is a little bit subtle to regard $h_{tt}$ as vanishing fast at the asymptotic infinity even though their ratio asymptotically vanishes.
There is a quasilocal generalization of the ADT formalism to obtain conserved charges of black holes [18], which can be used even in the slow fall-off condition on the perturbed metric. Furthermore, this formulation gives us a radial coordinate independent values of conserved charges in the sense of quasilocal charges. Though the off-shell ADT potential as an extension of the original ADT method was used in the higher derivative theory of gravity for computational convenience [19, 20], it was shown to have more interesting aspects; the off-shell ADT potential is equivalent to the linearized off-shell Noether potential up to the surface term [18]. This means that the off-shell ADT method can be related directly to the covariant phase space formalism [21–25] at the off-shell level. By integrating the ADT potential along the one parameter path in the solution space (i.e. the on-shell space) [25–29], quasilocal conserved charges can be calculated [18]. As a matter of fact, there is a practical advantage in this formulation in that one can easily find out the quasilocal conserved charges for the corresponding Killing vectors without resort to the complicated equations of motion.

There exists another convenient method presented by Padmanabhan in order to derive conserved charges of black hole directly from the relationship between gravitational field equations and thermodynamics [30]. The essential ingredient in this method is to rearrange the equation of motion in order to obtain the form of thermodynamic first law of black holes. Since this approach uses the local equation of motion not the integration of a certain potential on the asymptotic space, conserved charges obtained in this approach can be regarded as the quasilocal quantities. In the simplest model of static spherically symmetric black holes in the presence of matters, the equation of motion is decomposed into three parts; the first one corresponds to the mass and the second one does the entropy, and the last one does the pressure, which eventually yield the mass, entropy, and pressure of black holes as long as one can identify the black hole temperature. On the other hand, the pressure can come from sources in classical or quantum-mechanical origins, and particularly in the latter case, the source term, after integrating out classical fields in the action level, can be incorporated in the semi-classical equation of motion through the metric function. If the pressure term in the first law is eliminated, then the mass and entropy can be changed so that they are coincident with the ADM mass and the Wald entropy [31, 32].

In this work, we would like to obtain the quasilocal mass and entropy for the three- and five-dimensional Lifshitz black holes and investigate the validity of the first law of thermodynamics. To this purpose, we obtain the radial-coordinate invariant expression of
the mass of the black holes by using the quasilocal ADT method and then show that their near-horizon value of the mass is complete consistent with the one by the Padmanabhan method. In section II, we recapitulate quasilocal formulation of conserved charges in Ref. [18], which provides a very convenient way to determine quasilocal conserved charges of black hole including the mass. Applying this formula to the Lifshitz black holes in section III, we find the quaislocal mass and entropy of the black hole, and check the validity of the first law of black hole thermodynamics. Eventually it will be shown that they are completely consistent with those of Euclidean action approach [9], boundary stress tensor method [10], and dilaton gravity approach [11]. In section IV, the mass and entropy of the Lifshitz black hole are obtained by the Padmanabhan method and the results turn out to be the same with those in section III. Finally, conclusion and discussion will be given in section V.

II. QUASILOCAL FORMUALTION OF CONSERVED CHARGES

We would like to encapsulate the formulation of quasilocal conserved charges developed in Ref. [18]. Let us consider a variation of action with respect to $g_{\mu\nu}$ for a generally covariant theory of gravity defined in D-dimensional spacetime, which is given as

$$\delta I = \frac{1}{\kappa} \int d^D x [\sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} + \partial_\mu \Theta^\mu (g; \delta g)],$$

where $G^{\mu\nu} = 0$ is the equation of motion for the metric and $\Theta^\mu$ means the surface term. The transformation of the metric, under the diffeomorphism $\zeta$, is $\delta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$, and the corresponding transformation of the Lagrangian density is written as $\delta \zeta (L\sqrt{-g}) = \partial_\mu (\zeta^\mu \sqrt{-g} L)$. We can derive the identically conserved off-shell Noether current $J^\mu$ from Eq. (1) with the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$,

$$J^\mu (g; \zeta) \equiv \nabla_\nu K^{\mu\nu} = 2\sqrt{-g} G^{\mu\nu} (g) \zeta_\nu + \zeta^\mu \sqrt{-g} L (g) - \Theta^\mu (g; \zeta),$$

where $K^{\mu\nu}$ is called as the off-shell Noether potential. On the other hand, the on-shell ADT current is given as $J^\mu = \delta G^{\mu\nu} \xi_\nu$ [12–16], where $\xi_\nu$ is a Killing vector and $\delta G^{\mu\nu}$ denotes the generic variation of the equation of motion. This ADT current can be extended up to the off-shell current [18–20],

$$J^\mu_{\text{ADT}} \equiv \nabla_\nu Q^{\mu\nu}_{\text{ADT}} = \delta G^{\mu\nu} \xi_\nu + G^{\alpha\beta} \delta g_{\alpha\beta} \xi_\nu - \frac{1}{2} \epsilon^{\mu \alpha \beta} g_{\alpha \beta} \delta g_{\mu \nu} + \frac{1}{2} \epsilon^{\alpha \beta \gamma} g_{\alpha \beta} G^{\mu \nu} \xi_\nu,$$
where $Q_{\text{ADT}}^{\mu\nu}$ is the off-shell ADT potential.

Now, it can be shown that the off-shell ADT potential is related to the off-shell Noether potential. To this purpose, the diffeomorphism $\zeta$ is taken as a Killing vector $\xi$ in the Noether potential. Assuming that the Killing vector is preserved as $\delta \xi^\mu = 0$, one can use the following relation on the surface term [21, 22],

$$\mathcal{L}_\xi \Theta^\mu(g; \delta g) - \delta \Theta^\mu(g; \xi) = 0,$$

(4)

where $\mathcal{L}_\xi$ represents a Lie derivative along the Killing vector $\xi$ and the second term denotes the generic variation of the surface term with respect to the metric $g_{\mu\nu}$. This relation combined with the off-shell ADT and Noether potentials yields a key relation for the potentials,

$$\sqrt{-g} Q^{\mu\nu}_{\text{ADT}}(g; \delta g) = \frac{1}{2} \delta K^{\mu\nu}(g; \xi) - \xi^{[\mu} \Theta^{\nu]}(g; \delta g).$$

(5)

Then, one can calculate the linearized quasilocal ADT charge by using the ADT potential as

$$\delta Q(\xi) = \frac{2}{\kappa} \int_B d^{D-2} x_{\mu\nu} \sqrt{-g} Q_{\text{ADT}}^{\mu\nu},$$

(6)

where $B$ does not need to be located at the asymptotic infinity. Note that we can perform the more generic linearization on the black hole metric than the one taken in the conventional on-shell ADT method where it is just the difference between the black hole and the vacuum background. Our linearization is taken along the one-parameter path in the solution space and then the integration is performed along that path as $Q(\xi) = \int_0^1 ds \delta Q(\xi | s\mathcal{M})$ [25–29], where the free parameter $\mathcal{M}$ is parametrized by the variable $s$ such as $0 \leq s\mathcal{M} \leq \mathcal{M}$. By using the relation (5) and the formula (6) with the one-parameter path integral, the quasilocal conserved charge can be finally written as [18]

$$Q(\xi) = \frac{1}{\kappa} \int_B d^{D-2} x_{\mu\nu} \left( \Delta K^{\mu\nu}(\xi) - 2 \xi^{[\mu} \int_0^1 ds \Theta^{\nu]}(\xi | s\mathcal{M}) \right),$$

(7)

where $\Delta K^{\mu\nu}(\xi) \equiv K^{\mu\nu}_{s=1}(\xi) - K^{\mu\nu}_{s=0}(\xi)$ and $K^{\mu\nu}_{s=0}$ means the Noether potential of the vacuum solution. The symmetry in terms of Killing vector $\xi$ will determine the corresponding charge from Eq. (7). This formulation is extended to a theory of gravity with a gravitational Chern-Simons term [33] and to the case of asymptotic Killing vectors [34].

5
III. THERMODYNAMIC FIRST LAW IN THE QUASILOCAL METHOD

We are now in a position to calculate the conserved charges of the three- and five-dimensional Lifshitz black holes by employing the quasilocal formulation introduced in the previous section. The action for a generic quadratic curvature gravity theory is given by

\[ I = \int d^D x \sqrt{-g} \left[ \frac{1}{\kappa} (R + 2\Lambda) + \alpha R^2 + \beta R_{\mu \nu} R^{\mu \nu} + \gamma (R_{\mu \nu \sigma \rho} R^{\mu \nu \sigma \rho} - 4 R_{\mu \nu} R^{\mu \nu} + R^2) \right]. \] (8)

From the action (8), the equation of motion is generally written as

\[ G_{\mu \nu} \equiv \frac{1}{\kappa} G_{\mu \nu} + \alpha A_{\mu \nu} + \beta B_{\mu \nu} + \gamma C_{\mu \nu}, \] (9)

where

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \Lambda g_{\mu \nu}, \]
\[ A_{\mu \nu} = 2 R R_{\mu \nu} - \nabla_{\mu} \nabla_{\nu} R + g_{\mu \nu} (2 \nabla_{\sigma} \nabla^{\sigma} R - \frac{1}{2} R^2), \]
\[ B_{\mu \nu} = 2 R_{\mu \rho \sigma \nu} R^{\rho \sigma} - \nabla_{\mu} \nabla_{\nu} R + \nabla_{\sigma} \nabla^{\sigma} R_{\mu \nu} + \frac{1}{2} g_{\mu \nu} (\nabla_{\sigma} \nabla^{\sigma} R - R_{\rho \sigma} R^{\rho \sigma}), \]
\[ C_{\mu \nu} = 2 R R_{\mu \nu} - 4 R_{\mu \rho \sigma \nu} R^{\rho \sigma} + 2 R_{\mu \lambda \rho \sigma} R_{\nu}^{\lambda \rho \sigma} - 4 R_{\mu \rho} R_{\nu}^{\rho} - \frac{1}{2} g_{\mu \nu} (R_{\lambda \delta \rho \sigma} R^{\lambda \delta \rho \sigma} - 4 R_{\rho \sigma} R^{\rho \sigma} + R^2), \]

where the gravitational constant \( \kappa \), the cosmological constant \( \Lambda \), and the other coupling constants \( \alpha, \beta, \gamma \) will be determined appropriately depending on the models in what follows.

A. Three-dimensional Lifshitz black hole

In three-dimensional black hole, the parameters can be chosen as \( \Lambda = \frac{13}{2\ell^2}, \alpha = -3\ell^2/4\kappa, \beta = 2\ell^2/\kappa, \gamma = 0, \kappa = 16\pi G \). From the equation of motion (9), the metric solution is given as [6]

\[ ds^2 = - \left( \frac{r^2}{\ell^2} \right)^z \left( 1 - \frac{m \ell^2}{r^2} \right) dt^2 + \frac{1}{r^2} \left( 1 - \frac{m \ell^2}{r^2} \right) dr^2 + r^2 d\phi^2, \] (10)

where \( m \) is an integration constant, and the dynamical exponent should be fixed as \( z = 3 \) to satisfy the equation of motion. In this three-dimensional Lifshitz black hole, the location of the horizon is \( r_H = \ell \sqrt{m} \). Using the formulas for higher curvature terms [18, 35],

\[ \Theta^\mu(\delta g) = 2 \sqrt{-g} [P^{\mu(\alpha \beta)}_{\gamma} \nabla_\gamma \delta g_{\alpha \beta} - \delta g_{\alpha \beta} \nabla_\gamma P^{\mu(\alpha \beta)}_{\gamma}], \] (11)
\[ K^{\mu \nu} = \sqrt{-g} [2 P^{\mu \nu \rho \sigma} \nabla_\rho \xi_\sigma - 4 \xi_\sigma \nabla_\rho P^{\mu \nu \rho \sigma}], \] (12)
where $P_{\mu\nu\rho\sigma} = \partial L/\partial R_{\mu
u\rho\sigma}$, and taking the one-parameter path along the integration constant $m$, it is straightforward to obtain the mass of the Lifshitz black hole. By expanding $g_{\mu\nu}$ with respect to an infinitesimal parametrization $dm$, we can not only obtain $\delta g_{\mu\nu}$ but also the value of the surface term $\Theta^r$. The time-like Killing vector is given by $\xi_t = (-1, 0, 0)$ at the horizon where we took the Killing vector with appropriate sign to avoid the negative mass and the negative entropy. Now, we can calculate the Noether potential and the surface term which are integrated along the one free parameter path such as

$$\int_0^m dm \, \Theta^r = -2m^2 + \frac{8m r^2}{\ell^2}, \quad \Delta K^{tr} = \frac{8m r^2}{\ell^2}. \quad (13)$$

It can be shown that the mass $M$ of the Lifshitz black hole is calculated as

$$M \equiv Q(\xi_t) = \frac{1}{16\pi G} \int_0^{2\pi} d\phi \sqrt{h} \left[ 2\epsilon_{tr} K_s^{tr} - 2\epsilon_{tr}\xi_t \int_0^r ds \Theta^s \right]$$

$$= \frac{r_H^4}{4G\ell^4}, \quad (14)$$

where $h$ denotes determinant of induced metric. Note that the Noether potential for the vacuum solution vanishes, i.e., $K_s^{tr} = 0$. We would like to emphasize that this mass expression is valid even in the interior region of the black hole space time not only at the asymptotic infinity. In fact, our mass expression is invariant along the radial coordinate $r$.

The result in Eq. (14) at the asymptotic infinity is coincident with the result which has been obtained from the other methods [9–11] but it is different from the claimed expression $M_{DS} = 7r_H^4/(8G\ell^4)$ in Ref. [8]. Note that we employed the black hole background rather than the vacuum background adopted in Ref. [8]. As was mentioned in the introduction, it is not warranted in the case of Lifshitz black holes to take the linearization around the vacuum solution.

Next, the entropy from our formulation by using Eq. (7) or from the Wald formula [23, 24] can be obtained as

$$S = \frac{2\pi r_H}{G}, \quad (15)$$

and the Hawking temperature becomes $T_H = r_H^3/(2\pi\ell^4)$ from the definition of the surface gravity. The black hole mass (14) and entropy (15) satisfy the first law of black hole thermodynamics as $dM = T_H dS$. In fact, the first law of black hole thermodynamics should always hold in our quasilocal formulation of the ADT method, because it is shown to be equivalent to the covariant phase space formalism [18, 33] and the first law of black hole thermodynamics is proved to hold in that formalism [23].
B. Five-dimensional Lifshitz black hole

In the five-dimensional black hole, the parameters in the action (8) are chosen as \( \Lambda = \frac{2197}{1551 \ell^2} \), \( \alpha = -16 \ell^2/725 \), \( \beta = 1584 \ell^2/13775 \), \( \gamma = 2211 \ell^2/11020 \), and \( \kappa = 1 \). The metric solution is obtained as

\[
 ds^2 = -\left(\frac{r^2}{\ell^2}\right)^z \left(1 - \frac{m \ell^{5/2}}{r^{5/2}}\right) dt^2 + \frac{1}{r^2 \left(1 - \frac{m \ell^{5/2}}{r^{5/2}}\right)} dr^2 + r^2 d\Omega_3^2,
\]

(16)

where \( \Omega_3 \) is the three-dimensional angular part, and \( z = 2 \) is fixed for the five-dimensional Lifshitz black hole, and its horizon is located at \( r_H = \ell m^{2/5} \). The time-like Killing vector is defined by \( \xi_t = (1, 0, 0, 0, 0) \), and by using Eqs. (11) and (12) for higher curvature terms, the surface term and Noether potential are calculated as

\[
 \int_0^m dm \, \Theta^r = \frac{33m}{2755\ell^5} (382m \ell^3 - 933\ell^4 r^2),
\]

(17)

\[
 \Delta K^{tr} = \frac{33}{5510\ell^9} (1072r^5 + 719m^2 \ell^5 - 1866mr^2 \ell^2),
\]

(18)

respectively. In a similar way to the three-dimensional case, we can determine the mass for the five-dimensional Lifshitz black hole by employing Eq. (7),

\[
 M = \frac{297 \ell^3}{1102\ell^3} r_H^3 \Omega_3, \tag{19}
\]

and the entropy can be read off from our quasilocal formulation or the Wald formula [23, 24] as

\[
 S = \frac{396\pi r_H^3}{551} \Omega_3, \tag{20}
\]

where the Hawking temperature is given by calculating the surface gravity as \( T_H = 5r_H^2/(8\pi \ell^3) \). Note that the mass was found to be \( M_{DS} = 536r_H^3 \Omega_3/(2755\ell^6) \) which did not satisfy the first law of thermodynamics in Ref. [8]. It can be easily shown that the mass (19) respects the first law of thermodynamics with the entropy (20) such as \( dM = T_H dS \).

IV. THERMODYNAMIC FIRST LAW IN PADMANABHAN METHOD

In this section, we will derive the conserved charges of Lifshitz black holes and study the first law of thermodynamics by the use of relation between the thermodynamic first law and equations of motion based on the Padmanabhan method [30]. This computation confirms
our claim that our quasilocal mass expression of Lifshitz black hole is valid even near the black hole horizon. In the original work, the mass, entropy, and pressure can be read off from the equation of motion, in particular, the entropy is written as the well-known area law and the pressure depends on classical or quantum-mechanical matter. Note that the action (8) consists of two parts; one is the Einstein-Hilbert action with the cosmological constant and the other is composed of the higher-curvature terms. So, there are largely two options whether these two pieces of action should be treated as a whole, otherwise the higher-curvature terms should be treated as the independent source which is of relevance to the pressure term. Now, we will choose the first option because in the absence of the pressure term it was shown that the mass and entropy were written as the ADM mass and the Wald entropy in Einstein gravity [31].

A. Three-dimensional Lifshitz black hole

Let us rewrite the metric (10) for \( z = 3 \) for convenience as

\[
ds^2 = -\frac{r^4}{\ell^4} f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2,
\]

using the function defined by \( f(r) \equiv r^2/\ell^2 - m \). The Hawking temperature of the black hole (21) is written as

\[
T_H = \frac{r_H^3 f'(r_H)}{4\pi \ell^2}.
\]

Let us consider the equation of motion of \( G^r_r = 0 \), then it is written at the horizon \( r_H \) as

\[
G^r_r = \frac{1}{8r_H^2 \ell^2} \left[ -52r_H^2 + 4a\ell^2 f'(r_H) + 12\ell^4 f'(r_H)^2 \\
- 2a\ell^4 f''(r_H) f'''(r_H) - 2a^2 \ell^4 f'(r_H) f''(r_H) f'''(r_H) + a^2 \ell^4 f'''(r_H)^2 \right] = 0.
\]

Note that \( f''(r_H) = 2/\ell^2 \) and \( f'''(r_H) = 0 \), so that the equation of motion (23) can be factorized as

\[
G^r_r = \frac{8\ell^2}{r_H^2} \left( \frac{f'(r_H)}{2} + \frac{r_H}{\ell^2} \right) \left( \frac{f'(r_H)}{2} - \frac{r_H}{\ell^2} \right) = 0.
\]

The factors such as \( 8\ell^2/r_H^2 \) and \( (f'(r_H)/2 + r_H/\ell^2) \) are always positive, what it means is that

\[
\frac{f'(r_H)}{2} dr_H - \frac{r_H}{\ell^2} dr_H = 0,
\]
after multiplying $dr_H$. We want to rewrite this equation in the form of the first law of black hole thermodynamics. By taking into account a proper factor, we can get

$$0 = \frac{r_H^2 f'(r_H)}{4\pi \ell^2} d\left(\frac{2\pi r_H}{G}\right) - d\left(\frac{r_H^4}{4G\ell^4}\right).\quad (26)$$

For a given Hawking temperature (22), it can be shown that Eq. (26) is manifestly written in the form of the first law of black hole thermodynamics as $0 = T_H dS - dM$. Then, it is natural to identify the mass and entropy of the three-dimensional Lifshitz black hole with $M = r_H^4/(4G\ell^4)$, $S = 2\pi r_H/G$, respectively, where they are exactly coincident with those in section III.

**B. Five-dimensional Lifshitz black hole**

Let us consider the metic (16) for $z = 2$ as

$$ds^2 = -\frac{r^2}{\ell^2} g(r) dt^2 + \frac{1}{g(r)} dr^2 + r^2 d\Omega^2,\quad (27)$$

where the function is defined as $g(r) \equiv r^2/\ell^2 - m(\ell/r)^{1/2}$, and the Hawking temperature is given by

$$T_H = \frac{r_H g'(r_H)}{4\pi \ell}.\quad (28)$$

In a similar way to the three-dimensional case, using $g''(r_H) = 2/\ell^2 - 3m\ell^{1/2}/(4r_H^{5/2})$, $g'''(r_H) = 15m\ell^{1/2}/(7r_H^{1/2})$, the equation of motion $G_r^r = 0$ is written as

$$G_r^r = -\frac{3}{11020m^{5/2}\ell^2}(2909m^{2/5} + 208\ell g'(r_H))(5m^{2/5} - 2\ell g'(r_H))$$

$$= 0.\quad (29)$$

By multiplying $dr_H$ with some constants to the latter part of equation of motion $5m^{2/5} - 2\ell g'(r_H) = 0$, one can get

$$0 = \frac{r_H g'(r_H)}{4\pi \ell} d\left(\frac{396\pi}{551} r_H^2 \Omega_3\right) - d\left(\frac{297}{1102\ell^2} r_H^2 \Omega_3\right).\quad (30)$$

Using the temperature (28), the above equation is written in the form of the first law of thermodynamics of $0 = T_H dS - dM$, and the mass and entropy are easily identified with $M = 297r_H^2\Omega_3/(1102\ell^3)$, $S = 396\pi r_H^2\Omega_3/551$. As expected, they are compatible with the expressions in the previous section III.
V. DISCUSSION

In this work, we have calculated the mass of the three- and five-dimensional Lifshitz black holes by using the quasilocal formulation of the conserved charges and obtained the quasilocal mass consistent with the first law of thermodynamics, which has also been confirmed by the Padmanabhan method which uses the relation between the equations of motion and the first law of black hole thermodynamics. The advantage for these two methods resides in the fact that those do not resort to the background vacuum metric, so that the result is naturally independent of the vacuum metric. We have resolved the discrepancy in the mass expression of Lifshitz black holes between the naive ADT method \[8\] and the other ones by showing that the correct way to incorporate the ADT method is to use one-paramter path in the solution space or in other words to use the nonlinear completion of the linearization. This resolution is completely parallel to the case of warped AdS black holes which also requires such a nonlinear completion of the naive ADT method \[19, 33\].

Though we have resolved the discrepancy in the mass expression of Lifshitz black holes, it might be intriguing to discuss the interpretation of the result in \[8\] in the framework done by Padmanabhan method where the relationship between gravitational field equations and thermodynamics can be found in the simplest context \[30\]. The Lagrangian is assumed to be two parts; one is the Einstein tensor with the cosmological constant and the other consists of the higher curvature terms, for instance, \(L = L_0 + L_1\), where \(L_0 = R + 13/\ell^2\) and \(L_1 = 3\ell^2 R^2/4ipe + 2\ell^2 R_{\mu\nu}R^{\mu\nu}\) especially in three dimensions. Then, the first law of thermodynamics corresponding to the equation of motion can be written by reshuffling Eq. (23) as

\[-d\tilde{M} + T_H d\tilde{S} = \tilde{P} dV,\]

where \(\tilde{M} = 13r_H^4/(48G\ell^4), \tilde{S} = \pi r_H^2/(3G), \tilde{P} = -5r_H f'(r_H)/(24\pi G\ell^2) - r_H^2/(24\pi G\ell^4)\) and \(V = \pi r_H^2\). The right hand side of the pressure term \(\tilde{P}\) comes from the two higher curvature terms in \(L_1\) which play a role of source term in gravitational equations based on the original procedure in Ref. [30]. Note that the mass and entropy are not familiar with the conventional ones. To overcome this problem, the pressure term can be eliminated so that it can be split into two parts and eventually they are absorbed into the mass and entropy, respectively. Then, the resulting equation becomes the desired expressions as

\[dM = T_H dS\]

where \(M = r_H^4/(4G\ell^4)\) and \(S = 2\pi r_H^2/G\), so that the mass and entropy are the same with Eqs. (14) and (15), respectively. In other words, it means that if we allow the pressure term, then the mass and entropy can be changed.
according to the way to separate the action. Conversely speaking, the form of mass can be written in a different way if the pressure term is allowed in the first law. So, one may ask it is possible to accommodate \( M_{DS} \) if we allow the pressure term. If we consider \( M_{DS} \) with the Wald entropy assuming the first law of thermodynamics with a certain pressure term, then it should be written as \( P = -5r_H^2/(4\pi G\ell^4) \) which is unfortunately incompatible with the above pressure \( \tilde{P} \). Therefore, it means that it is very hard to accommodate the mass \( M_{DS} \) as a conserved charge in these frameworks.

**ACKNOWLEDGMENTS**

We would like to thank M. Eune for exciting discussions. WK was supported by the Sogang University Research Grant of (2013)201310022. S.-H.Yi was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MOE) (No. 2012R1A1A2004410).

[1] S. Kachru, X. Liu and M. Mulligan, “Gravity duals of Lifshitz-like fixed points,” Phys. Rev. D 78, 106005 (2008) [arXiv:0808.1725 [hep-th]].
[2] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200].
[3] J. M. Maldacena, AIP Conf. Proc. 484, 51 (1999) [Int. J. Theor. Phys. 38, 1113 (1999)].
[4] S. A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009) [arXiv:0903.3246 [hep-th]].
[5] D. T. Son, Phys. Rev. D 78, 046003 (2008) [arXiv:0804.3972 [hep-th]].
[6] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D 80, 104029 (2009) [arXiv:0909.1347 [hep-th]].
[7] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. 102, 201301 (2009) [arXiv:0901.1766 [hep-th]].
[8] D. O. Devecioglu and O. Sarioglu, Phys. Rev. D 83, 021503 (2011) [arXiv:1010.1711 [hep-th]].
[9] H. A. Gonzalez, D. Tempo and R. Troncoso, JHEP 1111, 066 (2011) [arXiv:1107.3647 [hep-th]].
[10] O. Hohm and E. Tonni, JHEP 1004, 093 (2010) [arXiv:1001.3598 [hep-th]].
[11] Y. S. Myung, Y.-W. Kim and Y.-J. Park, Eur. Phys. J. C 70, 335 (2010) [arXiv:0910.4428 [hep-th]].
[12] L. F. Abbott and S. Deser, Nucl. Phys. B 195, 76 (1982);
[13] L. F. Abbott and S. Deser, Phys. Lett. B 116, 259 (1982).
[14] S. Deser and B. Tekin, Phys. Rev. D 67, 084009 (2003);
[15] S. Deser and B. Tekin, Phys. Rev. Lett. 89, 101101 (2002);
[16] C. Senturk, T. C. Sisman and B. Tekin, Phys. Rev. D 86, 124030 (2012).
[17] R. L. Arnowitt, S. Deser and C. W. Misner, Gen. Rel. Grav. 40, 1997 (2008) [gr-qc/0405109].
[18] W. Kim, S. Kulkarni and S.-H. Yi, Phys. Rev. Lett. 111, 081101 (2013).
[19] A. Bouchareb and G. Clement, Class. Quant. Grav. 24, 5581 (2007);
[20] S. Nam, J.-D. Park and S.-H. Yi, Phys. Rev. D 82, 124049 (2010).
[21] J. Lee and R. M. Wald, J. Math. Phys. 31, 725 (1990).
[22] V. Iyer and R. M. Wald, Phys. Rev. D 50, 846 (1994) [gr-qc/9403028].
[23] R. M. Wald, Phys. Rev. D 48, 3427 (1993) [gr-qc/9307038].
[24] V. Iyer and R. M. Wald, Phys. Rev. D 52, 4430 (1995) [gr-qc/9503052].
[25] R. M. Wald and A. Zoupas, Phys. Rev. D 61, 084027 (2000) [gr-qc/9911095].
[26] G. Barnich, Class. Quant. Grav. 20, 3685 (2003) [hep-th/0301039].
[27] G. Barnich and G. Compere, Phys. Rev. D 71, 044016 (2005) [Erratum-ibid. D 73, 029904 (2006)] [gr-qc/0412029].
[28] G. Barnich and F. Brandt, Nucl. Phys. B 633, 3 (2002) [hep-th/0111246].
[29] G. Barnich and G. Compere, J. Math. Phys. 49, 042901 (2008) [arXiv:0708.2378 [gr-qc]].
[30] T. Padmanabhan, Res. Astron. Astrophys. 12, 891 (2012) [arXiv:1207.0505 [astro-ph.CO]].
[31] E. J. Son and W. Kim, Phys. Rev. D 87, 067502 (2013) [arXiv:1303.0491 [gr-qc]].
[32] Y. Kwon and S. Nam, “Thermodynamics from field equations for black holes with multiple horizons,” arXiv:1310.4933 [gr-qc].
[33] W. Kim, S. Kulkarni and S.-H. Yi, Phys. Rev. D 88, 124004 (2013) [arXiv:1310.1739 [hep-th]].
[34] S. Hyun, S.-Y. Park and S.-H. Yi, arXiv:1403.2196 [hep-th].
[35] G. Lopes Cardoso, B. de Wit and T. Mohaupt, Fortsch. Phys. 48, 49 (2000) [hep-th/9904005].
[36] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, JHEP 1004, 030 (2010) [arXiv:1001.2361 [hep-th]].