On the Complexity of Scheduling Problems With a Fixed Number of Identical Machines

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Abstract
In scheduling, we are given a set of jobs, together with a number of machines and our goal is to decide for every job, when and on which machine(s) it should be scheduled in order to minimize some objective function. Different machine models, job characteristics and objective functions result in a multitude of scheduling problems and many of them are NP-hard, even for a fixed number of identical machines. However, using pseudo-polynomial or approximation algorithms, we can still hope to solve some of these problems efficiently.

In this work, we give conditional running time lower bounds for a large number of scheduling problems, indicating the optimality of some classical algorithms. In particular, we show that the dynamic programming algorithm by Lawler and Moore is probably optimal for \(1||\sum w_j U_j\) and \(Pm||C_{\text{max}}\). Moreover, the FPTAS by Gens and Levner for \(1||\sum w_j U_j\) and the algorithm by Lee and Uzsoy for \(P2||\sum w_j C_j\) are probably optimal as well. There is still small room for improvement for the \(1|\text{Rej}|Q||\sum w_j U_j\) algorithm by Zhang et al. and the algorithm for \(1||\sum T_j\) by Lawler. We also give a lower bound for \(P2|\text{any}|C_{\text{max}}\) and improve the dynamic program by Du and Leung from \(O(nP^2)\) to \(O(nP)\) to match this lower bound. Here, \(P\) is the sum of all processing times. The same idea also improves the algorithm for \(P3|\text{any}|C_{\text{max}}\) by Du and Leung from \(O(nP^3)\) to \(O(nP^2)\).

The lower bounds in this work all either rely on the (Strong) Exponential Time Hypothesis or the \((\min,+)^\dagger\)-conjecture. While our results suggest the optimality of some classical algorithms, they also motivate future research in cases where the best known algorithms do not quite match the lower bounds.

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1 Introduction
Consider the problem of working on multiple research papers. Each paper \(j\) has to go to some specific journal or conference and thus has a given deadline \(d_j\). Some papers might be more important than others, so each one has a weight \(w_j\). In order to not get distracted, we may only work on one paper at a time and this work may not be interrupted. If a paper does not make its deadline, it is not important by how much it missed it; it is either late or on time. This problem is known as \(1||\sum w_j U_j\) and it is one of Karp’s original 21 NP-hard problems [19]. Even when restricted to a fixed number of identical machines, many combinations of job characteristics and objective functions lead to NP-hard problems. For this reason, a lot of effort has been put towards finding either pseudo-polynomial exact or
polynomial approximation algorithms. Sticking to our problem $1||\sum w_j U_j$, where we aim to minimize the weighted number of late jobs on a single machine, there are e.g. an $O(nW)$ algorithm by Lawler and Moore [24] and an FPTAS by Gens and Levner [13]. Here, $W$ is the sum of all weights $w_j$.

In recent years, research regarding scheduling has made its way towards parameterized and fine-grained complexity (see e.g. [2, 21, 29, 30]), where one goal is to identify parameters that make a problem difficult to solve. If those parameters are assumed to be small, parameterized algorithms can be very efficient. Similarly, one may consider parameters like the total processing time $P$ and examine how fast algorithms can be in terms of $n$ (the number of jobs) and $P$. That is our main goal in this work. A few years ago, Abboud et al. [1] gave a beautiful reduction from $k$-Sat to Subset Sum:

▶ Problem 1. Subset Sum

Instance: Items $a_1, \ldots, a_n \in \mathbb{N}$, integer target $T \in \mathbb{N}$.

Task: Decide whether there is a subset $S \subseteq [n]$ such that $\sum_{i \in S} a_i = T$.

Previous results based on the Exponential Time Hypothesis (ETH) excluded $2^{o(n)} T^{o(1)}$-time algorithms [13], while this new result based on the Strong Exponential Time Hypothesis (SETH) suggests that we cannot even achieve $O(2^{\delta n} T^{1-\varepsilon})$:

▶ Theorem 2 (Abboud et al. [1]). For every $\varepsilon > 0$, there is a $\delta > 0$ such that Subset Sum cannot be solved in time $O(2^{\delta n} T^{1-\varepsilon})$, unless the SETH fails.

Intuitively, ETH conjectures that 3-Sat cannot be solved in sub-exponential time, while SETH conjectures that the trivial running time of $O(2^n)$ is optimal for k-Sat, if $k$ tends to infinity. For details, see the original publication by Impagliazzo and Paturi [16]. By revisiting many classical reductions in the context of fine-grained complexity, we transfer this lower bound to scheduling problems like $1||\sum w_j U_j$.

Although lower bounds do not have the immediate practical value of an algorithm, it is clear from the results of this paper how finding new lower bounds can push research into the right direction: Our lower bound for $P2\{\text{any}\}|C_{\text{max}}$ indicated that an $O(nP)$-time algorithm was probably optimal, but the best known algorithm (by Du and Leung [9]) had running time $O(nP^2)$. A modification of this algorithm closes this gap.

It should be noted that all lower bounds in this paper are conditional, that is, they rely on some complexity assumption. However, all of these assumptions are reasonable in the sense that a lot of effort has been put towards refuting them. And in the unlikely case that they are indeed falsified, this would have big complexity theoretical implications.

This paper is organized as follows: We first give an overview on terminology, the related lower bounds by Abboud et al. [2] and our results in Section 2. Then we examine scheduling problems with a single machine in Section 3, problems with two machines in Section 4 and then we look at problems with an arbitrary but fixed number of machines in Section 5. Finally, we give a summary as well as open problems and promising research directions in Section 6. The appendix holds omitted proofs, implications of our reductions for different objective functions and lower bounds for strongly NP-hard problems.

## 2 Preliminaries

In this section, we first introduce the Partition problem, a special case of Subset Sum from which many of our reductions start. Then we recall common terminology from scheduling theory and finally, we give a short overview of related work and our results.
Throughout this paper, log denotes the base 2 logarithm. Moreover, we write \([n]\) for the set of integers from 1 to \(n\), i.e. \([n] := \{1, \ldots, n\}\). If we consider a set of items or jobs \([n]\) and a subset \(S \subseteq [n]\), we use \(\bar{S} = [n] \setminus S\) to denote the complement of \(S\). The \(\tilde{O}\)-notation hides poly-logarithmic factors.

In order to have a unified notation for all the different scheduling parameters, we use e.g. \(p_j\) for the processing times, \(p_{\text{max}}\) for the largest processing time, \(p_{\text{min}}\) for the smallest processing time and \(P\) for the sum of all processing times. We use the same notation for the other parameters, e.g. deadlines, release dates and so on.

### 2.1 Subset Sum and Partition

In this work, we provide lower bounds for several scheduling problems; our main technique are fine-grained reductions, which are like polynomial-time reductions, but with more care for the exact sizes and running times. With these reductions, we can transfer the (supposed) hardness of one problem to another. Our reductions will most of the time start with an instance of \textsc{Subset Sum} or \textsc{Partition} and construct an instance of some scheduling problem. \textsc{Partition} is the special case of \textsc{Subset Sum}, where the sum of all items is exactly twice the target value:

\begin{itemize}
  \item \textbf{Problem 3. Partition} \\
  \textbf{Instance:} Items \(a_1, \ldots, a_n \in \mathbb{N}\). \\
  \textbf{Task:} Decide whether there is a subset \(S \subseteq [n]\) such that \(\sum_{i \in S} a_i = \sum_{i \in \bar{S}} a_i\).
\end{itemize}

For \textsc{Subset Sum} and \textsc{Partition}, we will in the following always denote the total size of all items by \(A := \sum_{i=1}^{n} a_i\). Note that there is a subset of items summing up to \(T\), if and only if there is a subset of items summing up to \(A - T\). Hence, we can and will always assume without loss of generality that \(T \geq \frac{1}{2}A\), and hence \(A = O(T)\). This assumption and Theorem 2 yield:

\begin{itemize}
  \item \textbf{Corollary 4.} For every \(\varepsilon > 0\), there is a \(\delta > 0\) such that \textsc{Subset Sum} cannot be solved in time \(O(2^\delta A^{1-\varepsilon})\), unless the SETH fails.
\end{itemize}

Using a classical reduction from \textsc{Subset Sum} to \textsc{Partition} that only adds two large items, we also get the following lower bound for \textsc{Partition} (for a detailed proof, see Appendix A):

\begin{itemize}
  \item \textbf{Proposition 5.} For every \(\varepsilon > 0\), there is a \(\delta > 0\) such that \textsc{Partition} cannot be solved in time \(O(2^\delta A^{1-\varepsilon})\), unless the SETH fails.
\end{itemize}

### 2.2 Scheduling

In all scheduling problems we consider, we are given a number of machines and a set of \(n\) jobs with processing times \(p_j\), \(j \in [n]\); our goal is to assign each job to (usually) one machine such that the resulting schedule minimizes some objective. So these problems all have a similar structure: A machine model, some (optional) job characteristics and an objective function. This structure motivates the use of the three-field notation introduced by Graham et al. [13]. Hence, we denote a scheduling problem as a triple \(\alpha|\beta|\gamma\), where \(\alpha\) is the machine model, \(\beta\) is a list of (optional) job characteristics and \(\gamma\) is the objective function. As is usual in the literature, we leave out job characteristics like deadlines that are implied by the objective function, e.g. for \(1||\sum w_j U_j\). In this work, we mainly consider the decision variants of scheduling problems (as opposed to the optimization variants). In the decision problems, we are always given a threshold denoted by \(y\) and the task is to decide whether there is a
solution with value at most \( y \). Note that the optimization and the decision problems are - at least in our context - equivalent: An algorithm for the decision problem can be used to find a solution of the optimization problem with a binary search over the possible objective values (which are always integral and bounded, here). Clearly, an algorithm for the optimization problem can also solve the decision problem.

2.3 The Lower Bounds by Abboud et al.

In their work [2], Abboud et al. show similar lower bounds for scheduling problems, e.g. for \( 1\| \sum w_j U_j \). As we will see however, the results by Abboud et al. are not directly comparable to our results.

Standard dynamic programming approaches often give running times like \( O(nP) \); on the other hand, it is usually possible to try out all permutations, subsets or partitions of jobs, yielding an exponential running time like \( O(2^n \text{polylog}(P)) \) (see e.g. the work by Jansen et al. [18]). The intuitive way of thinking about our lower bounds is that we cannot have the best of both worlds, i.e.: 'An algorithm cannot be sub-exponential in \( n \) and sub-linear in \( P \) at the same time.' To be more specific, most of our lower bounds have this form: For every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that the problem cannot be solved in time \( O(2^n P^{1-\varepsilon}) \).

However, note that algorithms with running time \( \tilde{O}(n + P) \) or \( \tilde{O}(n + p_{\max}) \) are not excluded by our bounds, as they are not sub-linear in \( P \). But in a setting where \( n \) and \( P \) (resp. \( p_{\max} \)) are roughly of the same order, such algorithms would be much more efficient than the dynamic programming approaches. In particular, they would be near-linear in \( n \) instead of quadratic. This is where the lower bounds by Abboud et al. [2] come into play, as they have the following form: There is no \( \varepsilon > 0 \) such that the problem can be solved in time \( \tilde{O}(n + p_{\max} n^{1-\varepsilon}) \), unless the \( \forall\exists \)-SETH fails. These lower bounds can successfully exclude algorithms with an additive-type running time \( \tilde{O}(n + p_{\max}) \). Algorithms with running time \( \tilde{O}(n + p_{\max} n) \) may still be possible, but they would only be near-quadratic instead of near-linear in the \( n \approx p_{\max} \) setting. It should be mentioned that the lower bounds by Abboud et al. [2] rely on the \( \forall\exists \)-SETH and as noted by them, this assumption is stronger than the SETH, even strictly stronger, if we assume the NSETH. For the sake of completeness, we give a detailed proof in Appendix A. Moreover, it should be noted that our lower bounds include parameters other than \( p_{\max} \), e.g. the largest deadline \( d_{\max} \) or the threshold for the objective value \( y \).

2.4 Our Results

In this work, we give reductions from \textsc{Subset Sum}, \textsc{Knapsack} and \textsc{Partition} to scheduling problems. While some reductions include new ideas, most of them are classical reductions from works like [19] and [27], described in more detail and with more focus on the size of the parameters. These reductions, together with the result by Abboud et al. [1], exclude algorithms with certain running times under the SETH. Note that our SETH-based lower bounds mainly show that improvements for some pseudo-polynomial algorithms are unlikely. For problems that are strongly NP-hard, pseudo-polynomial algorithms cannot exist, unless \( P=NP \) [5]. We give lower bounds under SETH for some strongly NP-hard scheduling problems nonetheless, since they still exclude algorithms that are sub-exponential but super-polynomial in \( n \). However, since these results are clearly not as strong as those for the weakly NP-hard problems, they can be found in Appendix B.

On the algorithmic side, we show how to solve the problem \( P2\{\text{any}\}|C_{\max} \) in time \( O(nP) \) with a simple dynamic program, which is probably optimal, as it matches our lower bound.
We also give an algorithm for \( P_3 \text{||} \text{any} \text{||} C_{\text{max}} \) that runs in time \( O(n P^2) \), improving upon the previously best \( O(n P^5) \)-algorithm by Du and Leung [9].

Figure 1 displays the SETH-reductions from the main part of this paper. With the help of well-known reductions between objective functions (see e.g. [23]), the lower bounds can be transferred to a wide range of other scheduling problems. A detailed examination of these reductions and their implications, as well as a table containing all the obtained SETH-based lower bounds are given in Appendix C.

3 Single-Machine Problems

In this section, we consider problems on a single machine. For these problems, the main task is to order the jobs. Here, we highlight lower bounds for the problem \( 1\| \sum w_j U_j \) that are based on the SETH and the \((\min, +)\)-conjecture. Both bounds suggest the optimality of state-of-the-art algorithms for solving this problem.

3.1 Jobs With Deadlines

Consider again the problem \( 1\| \sum w_j U_j \), where each job \( j \) has a weight \( w_j \) and a deadline \( d_j \) and we aim to minimize the weighted number of late jobs on a single machine. Here, \( U_j = 1 \) if job \( j \) is late, i.e. \( C_j > d_j \), where \( C_j \) is the completion time of job \( j \) and \( U_j = 0 \) otherwise. With a reduction very similar to the one by Karp [19], we get the following lower bound:

\[ \text{Proposition 6. For every } \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that } 1\| \sum w_j U_j \text{ cannot be solved in time } O(2^{\delta n}(d_{\text{max}} + y + P + W)^{1+\varepsilon}), \text{ unless the SETH fails.} \]

**Proof.** Let \( a_1, \ldots, a_n \) be a PARTITION instance and let \( T = \frac{1}{2} \sum_{i=1}^n a_i \). Construct an instance of \( 1\| \sum w_j U_j \) by setting \( p_j = w_j = a_j, d_j = T \) for each \( j \in [n] \) and \( y = T \). The idea is that the jobs corresponding to items in one of the partitions can be scheduled early (i.e. before the uniform deadline \( T \)).

Formally, assume that there is a solution \( S \) of the given PARTITION instance. We schedule the jobs corresponding to items in \( S \) first, in any order; after that, we schedule the rest of the jobs (also in any order). Now the items in \( S \) sum up to \( T \), so they finish exactly at \( T \) and are all early. The other jobs are all late and have total weight \( T = y \).

\[ \text{Proposition 6. For every } \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that } 1\| \sum w_j U_j \text{ cannot be solved in time } O(2^{\delta n}(d_{\text{max}} + y + P + W)^{1+\varepsilon}), \text{ unless the SETH fails.} \]

**Proof.** Let \( a_1, \ldots, a_n \) be a PARTITION instance and let \( T = \frac{1}{2} \sum_{i=1}^n a_i \). Construct an instance of \( 1\| \sum w_j U_j \) by setting \( p_j = w_j = a_j, d_j = T \) for each \( j \in [n] \) and \( y = T \). The idea is that the jobs corresponding to items in one of the partitions can be scheduled early (i.e. before the uniform deadline \( T \)).

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Formally, assume that there is a solution \( S \) of the given PARTITION instance. We schedule the jobs corresponding to items in \( S \) first, in any order; after that, we schedule the rest of the jobs (also in any order). Now the items in \( S \) sum up to \( T \), so they finish exactly at \( T \) and are all early. The other jobs are all late and have total weight \( T = y \).

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1 It should be noted that some of the parameters in our lower bounds could be omitted, as they are overshadowed by others. For example, we can assume w.l.o.g. that \( d_{\text{max}} \leq P \) for \( 1\| \sum w_j U_j \), since gaps in the schedule make no sense and deadlines larger than \( P \) could be set to \( P \). But having all the parameters in the lower bound makes the comparison with known upper bounds easier.
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For the other direction, indirectly assume that there is no solution for the Partition instance and consider any optimal schedule for the constructed instance. Without loss of generality, there are no gaps in the schedule, as they can only increase the weighted number of late jobs. Since there is no subset of items with total size exactly $T$, there is also no set of jobs with total processing time exactly $T$. Let $S$ be the set of jobs that are scheduled early and note that $\sum_{j \in S} p_j < T$. Now the schedule has total weighted number of late jobs

$$\sum_{j=1}^{n} w_j U_j = \sum_{j=1}^{n} w_j - \sum_{j \in S} w_j = \sum_{j=1}^{n} p_j - \sum_{j \in S} p_j > \sum_{j=1}^{n} p_j - T = T = y,$$

which means that an optimal schedule has value larger than $y$ and hence the instance is negative.

With this reduction, we get $N := n$ jobs. Since $P = \sum_{i=1}^{n} a_i = A$ and hence $K := d_{\text{max}} + y + P + W = T + T + A + A = \mathcal{O}(A)$, we can conclude that there is no algorithm that solves 1|| $\sum w_j U_j$ in time

$$\mathcal{O}(2^{\delta N} K^{1-\varepsilon}) = \mathcal{O}(2^{\delta n} \mathcal{O}(A)^{1-\varepsilon}) = \mathcal{O}(2^{\delta n} c^{1-\varepsilon} A^{1-\varepsilon}) = \mathcal{O}(2^{\delta n} A^{1-\varepsilon})$$

for some $\varepsilon > 0$ and every $\delta > 0$, unless the SETH fails. Otherwise, we could solve Partition faster than Proposition 5 allows. Here, $c$ is large enough to cover the constants in the $\mathcal{O}$-term. It should be noted that the reduction takes time $\mathcal{O}(N)$, but for large enough $N$, we get $\mathcal{O}(N) \leq \mathcal{O}(2^{\delta N})$, and for smaller $N$, we can solve Partition efficiently, anyway (for more details on this, see Lemma 22 in Appendix A).

Using the algorithm by Lawler and Moore [24], 1|| $\sum w_j U_j$ is solvable in time $\mathcal{O}(nW)$ or $\mathcal{O}(n \min\{d_{\text{max}}, P\})$. Our lower bound of $\mathcal{O}(2^{\delta n}(d_{\text{max}} + y + P + W)^{1-\varepsilon})$ suggests the optimality of both variants, as we cannot hope to reduce the linear dependency on $W$, $d_{\text{max}}$ or $P$.

One interesting property of 1|| $\sum w_j U_j$ is that its straightforward formulation as an Integer Linear Program has a triangular structure that collapses to a single constraint when all deadlines are equal (see e.g. Lenstra and Shmoys [23]). This shows that the problem is closely related to the Knapsack problem:

**Problem 7. Knapsack**

*Instance:* Item values $v_1, \ldots, v_n \in \mathbb{N}$, item sizes $a_1, \ldots, a_n \in \mathbb{N}$, knapsack capacity $T \in \mathbb{N}$ and threshold $y$.

*Task:* Decide whether there is a subset $S$ of items with $\sum_{j \in S} a_j \leq T$ and $\sum_{j \in S} v_j \geq y$.

As noted by Mucha et al. [31], the results by Cygan et al. [7] imply the following lower bound for Knapsack:

**Theorem 8** (Corollary 9.6 in [31]). For any constant $\delta > 0$, the existence of an exact algorithm for Knapsack with running time $\mathcal{O}\left(\left(\frac{n}{\log n} + \sum_{j=1}^{n} v_j\right)^{2-\delta}\right)$ refutes the $(\min, +)$-conjecture.

The $(\min, +)$-conjecture states that the $(\min, +)$-Convolution problem cannot be solved in sub-quadratic time (see Cygan et al. [7] for details). We now show that the hardness of Knapsack transfers to 1|| $\sum w_j U_j$:

**Theorem 9.** For any constant $\delta > 0$, the existence of an exact algorithm for 1|| $\sum w_j U_j$ with running time $\mathcal{O}\left((n + W)^{2-\delta}\right)$ refutes the $(\min, +)$-conjecture.
Proof. We give a reduction from Knapsack to $1||\sum w_j U_j$. Consider an instance $v_1, \ldots, v_n, a_1, \ldots, a_n, T, y$ of Knapsack. We construct jobs with $p_j = a_j$, $w_j = v_j$ and $d_j = T$ for every $j \in [n]$. The threshold is set to $y' = \sum_{j=1}^n v_j - y$.

Let $S \subseteq [n]$ be a solution of a given Knapsack instance, i.e. $\sum_{j \in S} a_j \leq T$ and $\sum_{j \in S} v_j \geq y$. In the constructed $1||\sum w_j U_j$ instance, we schedule the jobs corresponding to items in $S$ first (in any order) and afterwards the jobs in $\overline{S}$ (also in any order). Now, since $\sum_{j \in S} a_j \leq T$ and $p_j = a_j$ for every job $j$, we can see that all jobs corresponding to items in $S$ are early. The weighted number of late jobs in the schedule is therefore at most:

$$\sum_{j \in S} w_j = \sum_{j \in S} v_j = \sum_{j=1}^n v_j - \sum_{j \in \overline{S}} v_j \leq \sum_{j=1}^n v_j - y = y'$$

Hence, the constructed instance of $1||\sum w_j U_j$ is positive.

Now, consider a solution of a constructed $1||\sum w_j U_j$ instance, i.e. a schedule with weighted number of late jobs at most $y'$. Let $S \subseteq [n]$ be the set of jobs that are scheduled early. Now, $S$ is a solution of the original Knapsack instance, since $\sum_{j \in S} a_j = \sum_{j \in S} p_j \leq T$ and $\sum_{j \in S} v_j = \sum_{j=1}^n v_j - \sum_{j \in \overline{S}} v_j = \sum_{j=1}^n v_j - \sum_{j \in \overline{S}} w_j \geq \sum_{j=1}^n v_j - y' = \sum_{j=1}^n v_j - \left(\sum_{j=1}^n v_j - y\right) = y$. So the original Knapsack instance is also positive.

Now suppose that there is an algorithm that solves $1||\sum w_j U_j$ in time $O\left((n + W)^{2-\delta}\right)$. Since $W = \sum_{j=1}^n v_j$ in the reduction and the reduction takes time $O(n)$, we could then solve Knapsack in time $O(n) + O\left((n + W)^{2-\delta}\right) = O\left((n + \sum_{j=1}^n v_j)^{2-\delta}\right)$, which is a contradiction to Theorem $\boxed{[8]}$ unless the $(\min, +)$-conjecture fails. ▶

For the optimization version of $1||\sum w_j U_j$, there is a $(1+\epsilon)$-approximation algorithm by Gens and Levner $[13]$ that has running time $O\left(n^2 (\log(n) + \frac{1}{\epsilon})\right)$. The following corollary shows that this algorithm is probably optimal up to logarithmic factors.

**Corollary 10.** For any constant $\delta > 0$, the existence of a $(1 + \epsilon)$-approximation algorithm for the optimization version of $1||\sum w_j U_j$ with running time $O\left((n + \frac{1}{\epsilon})^{2-\delta}\right)$ refutes the $(\min, +)$-conjecture.

**Proof.** Suppose that for some $\delta > 0$, there is an $O\left((n + \frac{1}{\epsilon})^{2-\delta}\right)$-time $(1 + \epsilon)$-approximation algorithm that solves the optimization version of $1||\sum w_j U_j$. Since $W \geq \text{opt}$ for any given instance, setting $\epsilon := \frac{1}{1+W}$ yields a solution with value $z$ such that

$$\text{opt} \leq z \leq (1 + \epsilon) \text{opt} = \text{opt} + \frac{\text{opt}}{1+W} < \text{opt} + 1.$$  

Since all weights are integer, opt is also integer and hence, $z = \text{opt}$. So we just solved the optimization version of $1||\sum w_j U_j$ exactly in time $O\left((n + \frac{1}{\epsilon})^{2-\delta}\right) = O\left((n + W)^{2-\delta}\right)$. With that, we can also solve the decision problem in the same running time for any given threshold $y$ and this is a contradiction to Theorem $\boxed{[8]}$ unless the $(\min, +)$-conjecture fails. ▶

Now, consider the problem $1||\sum T_j$ of minimizing the total tardiness on a single machine. The tardiness $T_j$ of job $j$ is zero if $j$ is early (i.e. if $C_j \leq d_j$) and otherwise it is the time that passes between its deadline $d_j$ and completion time $C_j$. In order to get a lower bound for $1||\sum T_j$, we again revisit reductions from the literature in the context of fine-grained complexity. Since the details of these reductions are quite technical, we refer to Appendix $\boxed{A}$ for details and just give the lower bound, here:
\textbf{Proposition 11.} For every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( 1||\sum T_j \) cannot be solved in time \( \mathcal{O}(2^{\delta n}P^{1-\varepsilon}) \), unless the SETH fails.

For \( 1||\sum T_j \), there is an \( \mathcal{O}(n^4P) \)-time algorithm by Lawler [22]. While we can derive no statement about the dependency on \( n \), our lower bound suggests that an improvement of the linear factor \( P \) is unlikely.

\section*{Rejectable Jobs}

In the problem \( 1|\text{Rej} \leq Q|C_{\text{max}} \), each job has a weight \( w_j \) and we may reject jobs with total weight at most \( Q \), while we aim to minimize the makespan (i.e., the largest completion time among all jobs). In Appendix A, we give a reduction from \textsc{Subset Sum} and get the following lower bound:

\textbf{Proposition 12.} For every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( 1|\text{Rej} \leq Q|C_{\text{max}} \) cannot be solved in time \( \mathcal{O}(2^{\delta n}(y + P + Q + W)^{1-\varepsilon}) \), unless the SETH fails.

Using classical reductions between objective functions, this lower bound can also be shown to hold for \( 1|\text{Rej} \leq Q|\sum w_j U_j \) (see Appendix C for the details). For rejectable jobs, Zhang et al. [34] have given a multitude of algorithmic results. Most notably, \( 1|\text{Rej} \leq Q|\sum w_j U_j \) is solvable in time \( \mathcal{O}(nQP) \), which almost matches our \( \mathcal{O}(2^{\delta n}(y + P + Q + W)^{1-\varepsilon}) \) lower bound.

\section*{4 Two-Machine Problems}

We now turn our attention to problems on two machines. Naturally, the problem of splitting jobs among two machines is quite similar to \textsc{Partition}, so the lower bounds in this section all directly or indirectly follow from Proposition 5.

\subsection*{4.1 Standard Jobs}

It is quite easy to show that we can model \textsc{Partition} using \( P_2||C_{\text{max}} \) by trying to perfectly distribute jobs among two machines. This yields the following lower bound for \( P_2||C_{\text{max}} \) (for a formal proof, see Appendix A):

\textbf{Proposition 13.} For every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( P_2||C_{\text{max}} \) cannot be solved in time \( \mathcal{O}(2^{\delta n}(y + P)^{1-\varepsilon}) \), unless the SETH fails.

Again, using reductions between objective functions, this lower bound can also be transferred to the more difficult objective functions (see Corollary 30 in Appendix C). A dynamic program by Lawler and Moore [24] (which is also sometimes attributed to Rothkopf [33]) solves the problem \( Pm||C_{\text{max}} \) even for the objectives \( \sum w_j C_j \) and \( \sum w_j U_j \). Clearly, the algorithm then also works for the easier objectives. For a comprehensive description of the algorithm, see [25] by Lenstra and Shmoys. They note that the running time can be improved from \( \mathcal{O}(mny^m) \) to \( \mathcal{O}(mny^{m-1}) \) for most objectives, but not for \( \sum w_j U_j \) (see exercise 8.10). Hence, while our lower bounds match the running times \( \mathcal{O}(ny) \) for most objectives on two machines, there is still a gap for \( \sum w_j U_j \).

A variation of this algorithm by Lawler et al. [23] also solves the more general problem \( Q||\sum w_j U_j \) on uniform machines in time \( \mathcal{O}(nmd_{\text{max}}^m) \), which matches our lower bound \( \mathcal{O}(2^{\delta n}(P + d_{\text{max}} + y)^{1-\varepsilon}) \) in the case of two identical machines.
Another variation by Lee and Uzsoy [26] solves $Pm||\sum w_j C_j$ in time $O(\min W^{n-1})$. In order to get a matching lower bound for this objective function and $m = 2$, we give another reduction:

**Proposition 14.** For every $\varepsilon > 0$, there is a $\delta > 0$ such that $P2||\sum w_j C_j$ cannot be solved in time $O(2^n(\sqrt{\gamma} + P + W)^{1-\varepsilon})$, unless the SETH fails.

**Proof.** We now show that the lower bound for PARTITION can be transferred to $P2||\sum w_j C_j$ using the reductions by Lenstra et al. [27] and Bruno et al. [3].

Given a PARTITION instance $a_1, \ldots, a_n$, we construct a $P2||\sum w_j C_j$ instance in the following way: Define $p_j = w_j = a_j$ for all $j \in [n]$ and set the limit $y = \sum_{1 \leq i \leq j \leq n} a_j a_i - \frac{1}{4} A^2$. Note that the execution order of the jobs on one specific machine does not influence the sum of weighted completion times for that machine, since $p_j = w_j$ for all jobs $j$: This follows from the observation that $\sum_{j \in [k]} w_j C_j = \sum_{j \in [k]} p_j (\sum_{i \leq j} p_i) = \sum_{1 \leq i \leq j \leq k} p_j p_i$ holds for any $k$ jobs running on one machine.

Before we prove the correctness of the construction, consider a schedule of jobs $\{1, \ldots, n\} = S \cup \overline{S}$ with $w_j = p_j$, where jobs in $S$ are scheduled on the first machine and jobs in $\overline{S}$ are scheduled on the second machine. We wish to show that the total weighted completion time of this schedule is $\sum_{j=1}^n w_j C_j - \sum_{j \in \overline{S}} w_j \sum_{i \leq j} p_i = \sum_{1 \leq i \leq j \leq n} a_j a_i - \sum_{i \leq j \leq n} a_j p_i - \sum_{j \in \overline{S}} w_j \sum_{i \leq j} p_i$. If all jobs were scheduled on the first machine, the value would be just $\sum_{1 \leq i \leq j \leq n} p_j p_i$, as argued above. Again, this does not depend on the order of the jobs, but let us assume that the jobs in $\overline{S}$ are scheduled first. Moving them to the second machine reduces the total weighted completion time by $\sum_{j \in \overline{S}} w_j \sum_{i \leq j} p_i$, as the completion time of each job in $S$ is reduced by $\sum_{j \in \overline{S}} p_j$ and the completion time of the jobs in $\overline{S}$ stays the same.

Now, consider a solution $S \cup \overline{S}$ of PARTITION. We schedule the jobs corresponding to items in $S$ on the first machine and the rest on the second machine, in arbitrary order. By the previous observations, this schedule has total weighted completion time:

$$\sum_{1 \leq i \leq j \leq n} p_j p_i - \sum_{j \in \overline{S}} w_j \sum_{i \leq j} p_i = \sum_{1 \leq i \leq j \leq n} a_j a_i - \sum_{j \in S} a_j \sum_{i \leq j} a_i = \sum_{1 \leq i \leq j \leq n} a_j a_i - \frac{1}{2} A^2$$

So this schedule meets the threshold $y = \sum_{1 \leq i \leq j \leq n} a_j a_i - \frac{1}{4} A^2$.

For the other direction, it is only important to see that we have the term $\sum_{1 \leq i \leq j \leq n} a_j a_i$ in the total weighted completion time no matter what and that the selection of jobs $S$ we put on the second machine determines the second part $\sum_{j \in S} a_j \sum_{i \leq j} a_i$ that is subtracted from the first term. Since the total sum $A = \sum_{i=1}^n a_i = \sum_{j \in \overline{S}} a_j + \sum_{j \in S} a_j$ is fixed, the maximum of the product $\sum_{j \in S} a_j \sum_{i \leq j} a_i$ is attained when $\sum_{j \in S} a_j = \sum_{j \in \overline{S}} a_j$. This is also exactly the case where the value is equal to $\frac{1}{4} A^2$. In all other cases, the product is smaller and hence subtracting less from $\sum_{1 \leq i \leq j \leq n} a_j a_i$ gives us an objective value larger than $y$. With this observation, clearly the jobs must be split such that the jobs on machine 1 have the same total processing time as the jobs on machine 2, which is only possible if the corresponding PARTITION instance is positive.

Now, assume that there is an algorithm that solves an instance of $P2||\sum w_j C_j$ in time $O(2^n K^{1-\varepsilon})$ for some $\varepsilon > 0$ and every $\delta > 0$, where $N := n$ and $K := \sqrt{\gamma} + P + W$. By the choice of $y$, we can see that

$$y = \sum_{1 \leq i \leq j \leq n} a_j a_i - \frac{1}{4} A^2 \leq \left( \sum_{j \in [n]} a_j \right)^2 - \frac{1}{4} A^2 = \frac{3}{4} A^2 = O(A^2).$$
Since \( w_j = p_j = a_j \), we also have \( P = W = A \). Hence, we have \( K = \sqrt{y} + P + W = O(A + A + A) = O(A) \) and an algorithm with running time

\[
O\left(2^{\delta n} K^{1-\varepsilon}\right) = O\left(2^{\delta n} O(A)^{1-\varepsilon}\right) = O\left(2^{\delta n} c^{1-\varepsilon} A^{1-\varepsilon}\right) = O\left(2^{\delta n} A^{1-\varepsilon}\right)
\]

would contradict the lower bound for Partition from \[\text{Proposition 5}\]. Again, \( c \) is large enough to cover the constants in the \( O \)-term and we can assume \( N \) to be large enough such that the running time \( O(N) \) of the reduction vanishes.

So the \( O(nW) \)-time algorithm by Lee and Uzsoy [20] for \( P2||\sum w_jC_j \) is probably optimal.

### 4.2 Rigid Jobs

Rigid jobs have a predetermined size that indicates on how many machines they have to be scheduled. By reducing from \( P2||C_{\text{max}} \) and setting the size (not processing time!) of each job to 1, we can easily get the following result (for a formal proof, see Appendix A):

\[\text{Proposition 15. For every } \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that } P2|\text{size}|C_{\text{max}} \text{ cannot be solved in time } O\left(2^{\delta n}(y + P)^{1-\varepsilon}\right), \text{ unless the SETH fails.}\]

Since for the \( C_{\text{max}} \)-objective, the order of the jobs does not matter, we can schedule all two-machine jobs at the beginning and then use the algorithm by Lawler and Moore [24] to schedule the one-machine jobs. This gives an \( O(ny) \) algorithm for \( P2|\text{size}|C_{\text{max}} \), which matches our lower bound. For other objectives, the problem quickly becomes more difficult: Already \( P2|\text{size}|L_{\text{max}} \) is strongly NP-hard, as well as \( P2|\text{size}|\sum w_jC_j \) (for both results, see Lee and Cai [25]). It is still open whether the unweighted version \( P2|\text{size}|\sum C_j \) is also strongly NP-hard or whether there is a pseudo-polynomial time algorithm; this question has already been asked by Lee and Cai [25], more than 20 years ago. If there is indeed a pseudo-polynomial algorithm for this problem, it would be interesting to also find a matching lower bound.

### 4.3 Moldable Jobs

In the variant \( P2|\text{any}|C_{\text{max}} \), the scheduler may decide on how many machines a job is executed, but a different number of machines can cause a different processing time. So we have processing times \( p_j(k) \), indicating how long it takes to run job \( j \) on \( k \) machines. By setting \( p_j(2) = p_j(1) = p_j \), there is no benefit from scheduling any job on two machines, so we can reduce \( P2||C_{\text{max}} \) to \( P2|\text{any}|C_{\text{max}} \) and get the following result (for a formal proof, see Appendix A):

\[\text{Proposition 16. For every } \varepsilon > 0, \text{ there is a } \delta > 0 \text{ such that } P2|\text{any}|C_{\text{max}} \text{ cannot be solved in time } O\left(2^{\delta n}(y + P)^{1-\varepsilon}\right), \text{ unless the SETH fails.}\]

Note that \( \text{any} \) is a more difficult constraint than \( \text{size} \): We can reduce a problem with the \( \text{size} \) constraint to the corresponding \( \text{any} \)-problem by essentially forbidding \( i \)-machine jobs to be scheduled on \( k \neq i \) machines. To achieve this, we just need to set \( p_j(i) = p_j \) and \( p_j(k) = M \) for every job \( j \), every \( k \neq i \) and some large enough value \( M \) that depends on the objective function (for \( C_{\text{max}} \), we could choose \( M = y + 1 \), for example). Using this idea, the strong NP-hardness results for \( \text{size} \) transfer to \( \text{any} \).

However, the problems \( P2|\text{any}|C_{\text{max}} \) and \( P3|\text{any}|C_{\text{max}} \) can be solved via dynamic programming, as shown by Du and Leung [9]. A nice summary is given by Drozdowski [8]. We
show that these programs can be improved to match our new lower bound, at least in the
case of two machines.

In the problem $P2|\text{any}|C_{\text{max}}$, the main difficulty is to decide whether a job is to be
processed on one or on two machines. Our dynamic program fills out a table $F(j, t)$ for every
$j \in [n]$ and $t \in [y]$, where the entry $F(j, t)$ is the minimum load we can achieve on machine 2,
while we schedule all the jobs in $[j]$ and machine 1 has load $t$. To fill the table, we use the
following recurrence formula:

$$F(j, t) = \min \left\{ F(j - 1, t - p_j(1)), F(j - 1, t + p_j(1)), F(j - 1, t - p_j(2)) + p_j(2) \right\}$$

Intuitively speaking, job $j$ is executed on machine 1 in the first case, on machine 2 in the
second case and on both machines in the third case. The initial entries of the table are
$F(0, 0) = 0$ and $F(0, t) = \infty$ for every $t \in [y]$.

There are $ny \leq n \sum_{j=1}^{n} \max\{p_j(1), p_j(2)\} = O(nP)$ entries we have to compute. Then,
we can check for every $t \in [y]$ whether $F(n, t) \leq y$. If we find such an entry, this directly
corresponds to a schedule with makespan at most $y$, so we can accept. Otherwise, there
is no such schedule and we can reject. The actual schedule can be obtained by traversing
backwards through the table; alternatively, we can store the important bits of information
while filling the table (this works exactly like in the standard knapsack algorithm). Note
that we might have to reorder the jobs such that the jobs executed on two machines are run
in parallel. But it can be easily seen that all two-machine jobs can be executed at the begin
of the schedule. Computing the solution and reordering does not change the running time in
$O$-notation, so we get an $O(nP)$ algorithm, which matches our lower bound.

For three machines, the idea stays the same; the recurrence formula just becomes a bit
more complicated. We now have a field $F(j, t_1, t_2)$, which tells us the minimum load we can
get on machine 3, if we schedule all the jobs in $[j]$ such that machine 1 and 2 have load $t_1$
and $t_2$, respectively. In analogy to the dynamic program above, we define the recurrence
formula:

$$F(j, t_1, t_2) = \min \left\{ T(j - 1, t_1 - p_j(1), t_2), T(j - 1, t_1, t_2 - p_j(1)), T(j - 1, t_1, t_2) + p_j(1), T(j - 1, t_1 - p_j(2), t_2 - p_j(2)), T(j - 1, t_1 - p_j(2), t_2) + p_j(2), T(j - 1, t_1, t_2 - p_j(2)) + p_j(2), T(j - 1, t_1 - p_j(3), t_2 - p_j(3)) + p_j(3) \right\}$$

The cases correspond to scheduling job $j$ on machine 1, on machine 2, on machine 3, on
machine 1 and 2, on machine 1 and 3, on machine 2 and 3 and finally on all three machines.
Again, the initial entries are $F(0, 0, 0) = 0$ and $F(0, t_1, t_2) = \infty$ for every $t_1, t_2 \in [y]$.

This time, we need to compute $O(nP^2)$ entries; the actual distribution of our jobs
among the machines can be obtained in the standard way, i.e. by remembering how we
obtained every entry or by going backwards through the table. However, in this case, it is
not directly clear that this distribution of jobs yields a feasible schedule with makespan at
most $y$. Fortunately, as Du and Leung [9] observed, there is also a canonical schedule in the
case of three machines: The jobs are swapped such that there are only two-machine jobs
on machines 1 and 2 and on machines 2 and 3. Then, the three-machine jobs are moved to the beginning of the schedule, followed by the two-machine jobs on machine 1 and 2. Finally, the two-machine jobs on machine 2 and 3 are executed at the end of the schedule and the one-machine jobs are executed in between. Using this canonical schedule and the distribution of the jobs to machines, we can obtain the actual schedule (i.e., with starting times). This dynamic programming algorithm has running time $O(n^p_2)$, improving upon the $O(n^p_k)$-algorithm by Du and Leung [9].

Clearly, this approach could be expanded and filling out a similar table could easily be done for an arbitrary number of machines $m$ in time $O(nm^m - 1)$. However, the strong NP-hardness of $Pm|\text{any}|C_{\text{max}}$ for $m \geq 4$ shows that we cannot find a canonical schedule in those cases, unless P=NP (see Henning et al. [15] as well as Du and Leung [9]).

### 5 Makespan Minimization With a Fixed Number of Machines

As noted before, the algorithm by Lawler and Moore [24] solves $Pm|\text{any}|C_{\text{max}}$ in a running time of $O(nm^m - 1) \leq O(nm^{m-1})$, since the exponent can be improved to $m - 1$ in this case. Our matching lower bound $O(n^p)$ for $m = 2$ gives rise to the question whether the algorithm is optimal for general $m > 1$. In the following, we will assume that $m > 1$, as the problem is trivial on a single machine.

In [6], Chen et al. show that the known approximation schemes for $Pm|\text{any}|C_{\text{max}}$ are essentially optimal. They also show that there is no $O\left(\left(\frac{m^{\frac{1}{2}+\varepsilon}}{\sqrt{m}}\right)^{\frac{1}{3}}\right)$-time exact algorithm for any $\varepsilon > 0$, unless the ETH fails. This is done by a reduction from 3-Sat via 3-Dimensional-Matching to $Pm|\text{any}|C_{\text{max}}$.

For us, the crucial part about these reductions is that we can choose $m$ arbitrarily and if the original 3-Sat formula has $n'$ variables, the $Pm|\text{any}|C_{\text{max}}$ instance has $O(n' + m)$ jobs and total processing time bounded by $P \leq n'm\frac{O(n' \log(m))}{m}$.

This can be seen in the paper by Chen et al. [6] on page 666, where a job is constructed for each of the $O(n')$ matches and elements in addition to at most $m$ dummy jobs and one huge job. The processing time of the huge job is set to $6m^4(m + 1)\sum_{i=1}^{\frac{n'q + \tau}{m}}\alpha^i$ minus the total processing time of the other constructed jobs, where $q = \frac{n'}{m}$, $\tau = O\left(\frac{n' \log(m)}{m}\right)$ and $\alpha = 6m^4 + 6m^3 + 6m^2$. Note that in the reduction it is assumed that $q$ is integer, so $n' \geq m$. This is achieved by adding dummy elements to the 3-Dimensional-Matching instance. Hence, the total processing time is equal to $6m^3(m + 1)\sum_{i=1}^{\frac{n'q + \tau}{m}}\alpha^i$, which can be bounded by

$$n'm\frac{O\left(n' \log(m)\right)}{m} = 2^{\log(n')}\frac{2^{\left(\frac{n' \log^2(m)}{m}\right)}}{2^{\left(\log(n') + \frac{n' \log^2(m)}{m}\right)}} = 2\left(\frac{n' \log^2(m)}{m}\right),$$

using

$$m \leq n' \Rightarrow \frac{m}{\log(m)} \leq \frac{n'}{\log(n')} \Rightarrow \frac{m}{\log^2(m)} \leq \frac{n'}{\log(n')} \iff \log(n') \leq \frac{n' \log^2(m)}{m},$$

where the first implication follows because the function $\frac{k}{\log(k)}$ is monotone for values $k \geq 2$ (and we assumed $m$ - and hence also $n'$ - to be larger than 1).

Now, suppose that we have an algorithm for $Pm|\text{any}|C_{\text{max}}$ that runs in time $O\left(nm^{P_{\log^2(m)}}\right)$. Using the reduction by Chen et al. [6], which has a running time poly($n$) = $n^c$, we could...
then solve 3-Sat in time:

\[
O\left(n^e+nmP^{\frac{m}{\log(m)}}\right) \leq 2O(\log(n')) \left(\frac{2^{\frac{1}{2}\log^2(m)}}{\log^2(m)}\right)^{\frac{m}{\log^2(m)}} \leq 2O(\log(n'))2O\left(n'\log\left(n'\right)\right)
\]

This is equal to \(2O(\log(n')+\frac{m}{\log(m)})\) and by setting \(n'\) large enough such that \(\log(n')\log(m) \leq \log^2(n') \leq n'\), we can assume \(\log(n') \leq \frac{n'}{\log(m)}\). So we get the term \(2O\left(\frac{m}{\log(m)}\right)\), which can be written as \(2\frac{c}{\log(m)}\), where \(c\) covers the constants in the \(O\)-term. Now, for large enough \(m\), we get \(\delta := \frac{c}{\log(m)}\) to be smaller than 1. Setting \(\varepsilon := 1 - \delta > 0\) then gives a \(2(1-\varepsilon)n'\)-time algorithm for 3-Sat, which contradicts the ETH, proving the following theorem:

\[\text{Theorem 17. There is no algorithm for } Pm\|C_{\text{max}} \text{ with running time } O\left(nmP^{\frac{m}{\log(m)}}\right), \text{ unless the ETH fails.}\]

So the algorithm by Lawler and Moore \[24\] with running time \(O\left(nmP^{m-1}\right)\) is almost optimal, as we can at best hope to shave off logarithmic factors in the exponent (assuming ETH). Since the algorithm not only works for \(C_{\text{max}}\), one might ask whether we can find similar lower bounds for other objectives as well. For most common objective functions, we answer this question positively in Appendix C, but it remains open for \(\sum C_j\) and \(\sum w_jC_j\).

6 Conclusion

In this work, we examined the complexity of scheduling problems with a fixed number of machines. Our conditional lower bounds indicate the optimality of multiple well-known classical algorithms. For the problems \(P2|\text{any}|C_{\text{max}}\) and \(P3|\text{any}|C_{\text{max}}\), we managed to improve the currently best known algorithm, closing the gap for two machines.

Aside from \(1||\sum w_jU_j\), where we showed the near-optimality of an FPTAS under the \((\min,+)\)-conjecture, we only considered exact algorithms. However, we strongly believe that similar techniques can be used to either show tightness results for other approximation schemes or to indicate room for improvement.

For exact algorithms, there is a number of open problems motivated by our results: First of all, our lower bound for \(Pm||C_{\text{max}}\) directly extends to many other objectives, but not to \(\sum C_j\). It would be interesting to see whether a similar reduction also works for \(Pm||\sum C_j\), since the algorithm by Lawler and Moore \[24\] solves that problem as well (and even the weighted version). Another interesting question is where the *true* complexity lies between \(m-1\) and \(\frac{m}{\log(m)}\) in the exponent. Zhang et al. also give an \(O(n(r_{\text{max}}+P))\)-time algorithm for \(1|r_j, Rej|C_{\text{max}}\) in their work \[34\]. Since \(r_{\text{max}}+P \geq y\) w.l.o.g., it would be interesting to find an \(O\left(2^{5h}(r_{\text{max}}+P)^{1-\varepsilon}\right)\) or \(O\left(2^{5h}y^{1-\varepsilon}\right)\) lower bound for this problem. As noted by Lenstra and Shmoys \[28\], the algorithm by Lawler and Moore \[24\] cannot be improved to \(O\left(mny^{m-1}\right)\) for the objective \(\sum w_jU_j\). So this algorithm would be quadratic in \(y\) for two machines, while our lower bound excludes anything better than linear. Hence, it would be interesting to see whether there is a different algorithm with running time \(O\left(ny\right)\). Similarly, there is an algorithm for \(1|Rej|\gamma|\) with running time \(O\left(nQP\right)\), while our lower bound suggests that an \(O\left(n(Q+P)\right)\)-time algorithm could be possible.

On another note, it would be interesting to extend the sub-quadratic equivalences by Cygan et al. \[7\] and Klein \[20\] to scheduling problems. Finally, the question by Lee and Cai \[25\] whether \(P2|\text{size}|\sum C_j\) is strongly NP-hard or not is still open since 1999.
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A Omitted Proofs

In this section, we show the proofs that were omitted from the main part. These are mostly reductions, but we also show that the $\forall \exists$-SETH is probably a strictly stronger assumption than SETH.

A.1 SETH and $\forall \exists$-SETH

Before we prove Proposition 21, we restate the SETH by Impagliazzo and Paturi [16] and the $\forall \exists$-SETH by Abboud et al. [2]:

▶ Conjecture 18 (Strong Exponential Time Hypothesis [16]). For every $\varepsilon > 0$, there is some $k \geq 3$ such that $k$-Sat cannot be solved in time $O\left(2^{(1-\varepsilon)n}\right)$.

▶ Conjecture 19 ($\forall \exists$ Strong Exponential Time Hypothesis [2]). For every $\alpha \in (0, 1)$, $\varepsilon > 0$ there is some $k \geq 3$ such that the problem of deciding whether

$$\forall x_1, \ldots, x_{\lceil \alpha n \rceil}, \exists x_{\lceil \alpha n \rceil + 1}, \ldots, x_n : \phi(x_1, \ldots, x_n) = \text{true}$$

cannot be solved in time $O\left(2^{(1-\varepsilon)n}\right)$ for any $n$-variable formula $\phi$ in conjunctive normal form with $k$ variables per clause.

We show the "strictly stronger" part of the claim using the Non-Deterministic Strong Exponential Time Hypothesis (NSETH):

▶ Conjecture 20 (Non-Deterministic Strong Exponential Time Hypothesis [4]). For every $\varepsilon > 0$, there exists a $k$ such that there is no non-deterministic algorithm solving the complement of $k$-Sat in time $O\left(2^{(1-\varepsilon)n}\right)$.

This conjecture is particularly useful for proving non-reducibility results: If there are non-deterministic algorithms for a problem $A$ and its complement $\overline{A}$, both with running time bounded by $T$, then we cannot prove a SETH-based lower bound for $A$ that is higher than $T$, assuming NSETH (see Corollary 2 in [4]).

▶ Proposition 21. $\forall \exists$-SETH implies SETH. But SETH does not imply $\forall \exists$-SETH, unless NSETH fails.

Proof. We prove the first part indirectly by showing how a faster-than-SETH algorithm for $k$-Sat would imply a faster-than-$\forall \exists$-SETH algorithm for $\forall \exists$-$k$-SAT. The second part is then shown by providing non-deterministic algorithms for $\forall \exists$-k-SAT and $\forall \exists$-k-SAT, which under NSETH rules out a corresponding reduction from k-SAT.

Now, assume that SETH does not hold (i.e. there is an $\varepsilon > 0$ such that $k$-Sat can be solved in time $O\left(2^{(1-\varepsilon)n}\right)$ for every $k$) and consider an instance of $\forall \exists$-$k$-SAT, consisting of an $\alpha \in (0, 1)$, a number $k \in \mathbb{N}$ and a $k$-CNF formula $\phi(x_1, \ldots, x_n)$ depending on $n = n_1 + n_2$ variables, where $n_1 = \lfloor \alpha n \rfloor \leq \alpha n + 1$ and $n_2 = n - n_1 \leq (1 - \alpha)n$.

Our first step is to use the Sparsification Lemma by Impagliazzo et al. [17] in order to transform $\phi$ into $2^{k\alpha} k$-CNF formulas, with $O(n)$ clauses each, in time $\text{poly}(n)2^{k\alpha}$, where $\delta > 0$ will be specified later and $\phi$ is feasible if and only if at least one of the $2^{k\alpha}$ constructed formulas is feasible. Hence, we do the rest of the procedure for each of the $2^{k\alpha}$ formulas; let $\phi_i$ be the $i$-th such formula for each $i \in [2^{k\alpha}]$. Now, given a $\phi_i$, we go through all $2^{n_1}$ assignments of the $n_1$ $\forall$-quantified variables and for each of them we fix the corresponding variables in $\phi_i$, i.e. for every appearance of a variable $x_j$ in some clause, we either remove
that clause (since the clause is already satisfied) or we remove the literal from the clause (since the literal is false). Going through all the clauses takes time $\text{poly}(n)$, as we have $n$ variables and $O(n)$ clauses after the sparsification. By fixing the $n_1$ variables, we get a formula with $n_2$ variables. Now, as we assumed SETH to be false, we can solve this formula in time $O(2^{\frac{1}{1-\alpha}n_2})$ for some $\varepsilon > 0$ and large enough $k$.

In total, we need the following running time to solve the $\forall \exists\text{-k-SAT}$-problem, where $c$ is some constant:

$$\text{poly}(n)2^{\delta n} + O\left(2^{\delta n}2^{n_1}n2^{(1-\varepsilon)n_2}\right) \leq O\left(n^c2^{\delta n}2^{\alpha n}2^{(1-\varepsilon)(n-\lceil\alpha n\rceil)}\right)$$

$$\leq O\left(2^{c\log(n)}2^{\delta n}2^{\alpha n}+\log(2^{(1-\alpha) n})\right)$$

$$= O\left(2^{c\log(n)}2^{\delta n}2^{\alpha n}+(1-\varepsilon)(1-\alpha) n\right)$$

$$= O\left(2^{\alpha n}\frac{\log(n)}{\varepsilon}\right)$$

$$= O\left(2^{\alpha n}\frac{\log(n)}{\varepsilon}\right)$$

$$= O\left(2^{\alpha n}\frac{\log(n)}{\varepsilon}\right)$$

So if we can assure that $\varepsilon' := \frac{\log(n)}{\varepsilon} < \frac{\log(n)}{\varepsilon} > \varepsilon > \varepsilon \alpha$, we have contradicted the $\forall \exists$-SETH. This is exactly the case if $\varepsilon - \varepsilon \alpha - \frac{\log(n)}{\varepsilon} > \varepsilon$. But we have to set $\delta$ to be a positive value as well, so we also need to assure that $\varepsilon - \varepsilon \alpha - \frac{\log(n)}{\varepsilon} > \varepsilon$. This is true if and only if $\frac{\log(n)}{\varepsilon} < \varepsilon - \varepsilon \alpha$, which is equivalent to $\frac{\log(n)}{\varepsilon - \varepsilon \alpha} < n$. So as long as this inequality holds, we are fine by setting $\delta$ to be some value slightly smaller than $\varepsilon - \varepsilon \alpha - \frac{\log(n)}{\varepsilon}$. But what if $\frac{\log(n)}{\varepsilon - \varepsilon \alpha} > n$? In that case, $n$ has to be bounded by some constant and we can solve the $\forall \exists\text{-k-SAT}$-problem efficiently, anyway. Hence, if SETH fails, $\forall \exists$-SETH fails as well and we have shown the first part of the claim.

For the second part, consider an instance of $\forall \exists\text{-k-SAT}$, consisting of $\alpha \in (0,1)$, $k \in \mathbb{N}$ and $\phi(x_1, \ldots, x_n)$ depending on $n = n_1 + n_2$ variables, where $n_1 = \lceil\alpha n\rceil \leq \alpha n + 1$ and $n_2 = n - n_1 \leq (1 - \alpha)n$.

To define a non-deterministic algorithm for $\forall \exists\text{-k-SAT}$, we proceed similar to the above reduction: First, we use the Sparsification Lemma to get only linear in $n$ many clauses in time $\text{poly}(n)2^{\delta n}$. This generates $2^{\delta n}$ many formulas and this time we can non-deterministically guess a satisfiable one in time $O(\delta n)$ (if one exists). We then again try out all $2^{n_1}$ assignments for the $\forall$-quantified variables and fix the corresponding variables in the given formula in time $\text{poly}(n)$, as we need to go through all clauses. We proceed to guess a satisfying assignment of the remaining $n_2$ variables in time $O(n_2)$. In total, for some constant $c$, this yields a non-deterministic algorithm for $\forall \exists\text{-k-SAT}$ with running time

$$\text{poly}(n)2^{\delta n} + O(\delta n) + O(2^{n_1}\text{poly}(n)n_2) \leq O\left(n^c2^{\delta n}2^{n_1}\right) \leq O\left(2^{c\log(n)+\alpha n+1+\delta n}\right)$$

$$= O\left(2^{n\frac{\log(n)}{\varepsilon}+\alpha+\delta}\right)$$

$$= O\left(2^{n\left(1-\frac{\log(n)}{\varepsilon}\right)+\alpha+\delta}\right)$$

and setting $\varepsilon := 1 - \frac{\log(n)}{n} - \alpha - \delta$ gives us the desired running time of $O\left(2^{(1-\varepsilon)n}\right)$, but we have to again assure that $\varepsilon > 0$. This holds if and only if $1 - \frac{\log(n)}{n} - \alpha > \delta$. Again, we have to set $\delta$ such that it is positive, and hence we need $\frac{\log(n)}{\varepsilon} + \alpha < 1$. This is equivalent to $\frac{\log(n)}{1-\alpha} < n$. And as above, if the opposite is the case, we can solve the problem even more efficiently, as $n$ is then bounded by some constant.
In a non-deterministic algorithm for $\forall 3$-k-SAT, we need to decide whether there exists an assignment for the first $n_1$ variables such that for every assignment of the remaining $n_2$ variables, the formula $\phi$ evaluates to false. Up to changes in the order of the steps, the algorithm works almost identical to the above one: We again use the Sparsification Lemma to construct $2^{\delta n}$ formulas in time $\text{poly}(n)2^{\delta n}$ and guess a feasible one in time $O(\delta n)$. Then, we first guess a feasible assignment for the $n_1$ variables in time $O(n_1)$ and finally, we try out all $2^{n_2}$ assignments for the $n_2$ variables and evaluate the resulting formulas in time $\text{poly}(n)$.

So for some constant $c$, we get the following running time:

$$\text{poly}(n)2^{\delta n} + O(\delta n) + O(n_1) + O(2^{n_2}\text{poly}(n)) \leq O(n^c2^{n_2}2^{\delta n})$$

$$= O\left(2^{n\left(-\frac{\log(n)}{n}\right) + 1 - \alpha + \beta}\right)$$

$$= O\left(2^n(1 - \frac{\log(n)}{n} + \alpha - \beta)\right)$$

Again, setting $\varepsilon := -\frac{\log(n)}{n} + \alpha - \delta$ yields a running time of $O(2^{(1-\varepsilon)n})$. We can assure that $\varepsilon > 0$ by setting $\delta$ to be smaller than $-\frac{\log(n)}{n} + \alpha$. Again, we can do this if $-\frac{\log(n)}{n} + \alpha > 0$, which is equivalent to $n > \frac{\log(n)}{n}$. Otherwise, $n$ is bounded by a constant and we can solve the problem efficiently, anyway.

Hence, there is an $\varepsilon > 0$ such that both $\forall 3$-k-SAT and $\exists 3$-k-SAT can be solved in time $O(2^{(1-\varepsilon)n})$, which implies that there is no $O(2^{(1-\varepsilon)n})$ lower bound via SETH, unless NSETH fails (see Corollary 2 in [4]).

### A.2 SETH-Based Lower Bounds

We now give the reductions that did not make it into the main part of this paper. To avoid repetitions, we first show a useful lemma that encapsulates the technical parts in the computations of our lower bounds:

**Lemma 22.** Suppose there is an $O(\text{poly}(n))$-time reduction from SUBSET SUM (or PARTITION) with $n$ items and $\sum_{i=1}^{n} a_i = A = O(T)$ to some scheduling problem $\alpha|\beta|\gamma$ with $N = O(n)$ jobs and parameter $K = O(\text{poly}(n)A)$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\alpha|\beta|\gamma$ cannot be solved in time $O(2^{\delta N K^{1-\varepsilon}})$, unless the SETH fails.

**Proof.** For the sake of contradiction, assume that there exists an $\varepsilon > 0$ such that for every $\delta > 0$, $\alpha|\beta|\gamma$ can be solved in time $O(2^{\delta N K^{1-\varepsilon}})$. Now, consider an instance of SUBSET SUM (or PARTITION) with $n$ items and $\sum_{i=1}^{n} a_i = A = O(T)$. Using the reduction, we construct an instance of $\alpha|\beta|\gamma$ with $N = O(n) = c_1 n$ jobs and parameter $K = O(\text{poly}(n)A) = c_2 n^{c_3} A$ in time $O(\text{poly}(n)) = n^{c_4}$.

In order to contradict the lower bound for SUBSET SUM (or PARTITION), we set $\varepsilon' := \varepsilon$ and consider some arbitrary but fixed $\delta' > 0$. Since we can - by assumption - solve $\alpha|\beta|\gamma$ in time $O(2^{\delta N K^{1-\varepsilon}})$ for every $\delta > 0$, we can also do so for $\delta = (\delta' - \frac{\log^2(n)}{n} + \frac{\log(n)}{n} \epsilon^{c_1 n})$, as long
as this is larger than 0. For this, we need that $n$ is large enough so we get:

\[
\begin{aligned}
\delta' - \left(\frac{(c_3 + c_4)}{n}\right) &> 0 \\
\delta' - \left(\frac{(c_3 + c_4)}{n}\right) &> 0 \\
\frac{n > (c_3 + c_4)
\end{aligned}
\]

Note that for smaller $n$, the inequality $n \leq \frac{(c_3 + c_4)}{\delta'}$ means that $n$ has to be bounded by some function in $\delta'$ and hence (since $\delta'$ is fixed), we could solve \textsc{Subset Sum} (or \textsc{Partition}) in polynomial time. So let us now assume that $n$ is large enough so that $\delta > 0$ and we can use the supposed algorithm for $\alpha/\beta'\gamma$. Using the reduction and this algorithm, we can then solve the \textsc{Subset Sum} (or \textsc{Partition}) instance in time:

\[
\begin{aligned}
n^{\varepsilon} + \mathcal{O}(2^{\delta N} K^{1-\varepsilon}) &\leq \mathcal{O}(2^{\delta N} c_2 c_3 + c_4 A^{1-\varepsilon}) \\
&\leq \mathcal{O}(2^{\delta N} c_2 c_3 + c_4 A^{1-\varepsilon}) \\
&\leq \mathcal{O}(2^{\delta N} n (c_3 + c_4) \log(n) A^{1-\varepsilon}) \\
&= \mathcal{O}(2^{\delta N} (n + (c_3 + c_4) \log(n) A^{1-\varepsilon}) \\
&= \mathcal{O}(2^{\delta N} A^{1-\varepsilon})
\end{aligned}
\]

So there exists a fixed $\varepsilon' > 0$ such that for every fixed $\delta' > 0$, we can solve \textsc{Subset Sum} (or \textsc{Partition}) in time $\mathcal{O}(2^{\delta n} A^{1-\varepsilon'})$, which contradicts the corresponding lower bound under SETH and concludes the proof.

\begin{proposition}
For every $\varepsilon > 0$, there is a $\delta > 0$ such that \textsc{Partition} cannot be solved in time $\mathcal{O}(2^{\delta n} A^{1-\varepsilon})$, unless the SETH fails.
\end{proposition}

\begin{proof}
We use a simple reduction from \textsc{Subset Sum} to \textsc{Partition} (a similar reduction from \textsc{Knapsack} to \textsc{Partition} has been given by Karp [19].

Consider a \textsc{Subset Sum} instance with items $a_1, \ldots, a_n$ and target $T$. We construct a \textsc{Partition} instance by copying all items $a'_i = a_i$ for all $i \in [n]$ and then adding the two items $a'_{n+1} = T + 1$ and $a'_{n+2} = A + 1 - T$ to the instance. Let $N = n + 2$.

Given a solution $S$ of the \textsc{Subset Sum} instance, we get the partitions $S \cup \{a'_{n+2}\}$ and $\overline{S} \cup \{a'_{n+1}\}$, which both sum up to $A + 1$. For the other direction, note that the sum of all items is equal to $2A + 2$ and hence the items $a'_{n+1}$ and $a'_{n+2}$ cannot be in the same partition, as they sum up to $T + 1 + A + 1 - T = A + 2$. So given a solution $S \cup \overline{S}$ of the \textsc{Partition} instance, assume w.l.o.g. that $a'_{n+1}$ is in $\overline{S}$ and $a'_{n+2}$ is in $S$. Then in order for the items in $S$ to have a total sum of $A + 1$, the other items in $\overline{S}$ need to have a total sum that is exactly $T$. Hence, those items give us a solution of the original \textsc{Subset Sum} instance.

With $K := \sum_{i \in [N]} a'_i = A + (T + 1) + (A + 1 - T) = \mathcal{O}(A)$ as parameter and $N := n + 2$ jobs, \textbf{Lemma 22} yields the claim, since the reduction takes time $\mathcal{O}(n)$ \footnote{Note that we only use the \textsc{Subset Sum}-part of \textbf{Lemma 22}. This way, we do not actually use a lemma to prove a result that is used by the lemma itself.}.
\end{proof}
Proposition 11. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $1\| \sum T_j$ cannot be solved in time $O(2^n n^{1+\varepsilon})$, unless the SETH fails.

The NP-hardness of $1\| \sum T_j$ is shown by Du and Leung [10], who reduce from the NP-hard problem EVEN-ODD-PARTITION (or EO-PARTITION for short) via a restricted version thereof (REO-PARTITION). While the hardness of EO-PARTITION is usually attributed to Garey and Johnson [11], the first reduction in the literature (to the best of our knowledge) is due to Garey, Tarjan and Wilfong [12]. We revisit the reductions in [12] and [10] to prove Proposition 11.

Problem 23. EO-PARTITION

Instance: Integers $b_1, \ldots, b_{2n} \in \mathbb{N}$ with $b_i > b_{i+1}$ for each $i \in [2n - 1]$.

Task: Decide whether there is a subset $S \subseteq [2n]$ such that $\sum_{i \in S} b_i = \sum_{i \notin S} b_i$ and $|S \cap \{b_{2i-1}, b_{2i}\}| = 1$ for each $i \in [n]$.

In other words, the items $b_i$ are strictly decreasing and consist of $n$ pairs of items $(b_{2i-1}, b_{2i})$, where the items of a pair may not be in the same partition.

The reduction from PARTITION to EO-PARTITION by to Garey, Tarjan and Wilfong [12] is as follows: Given a PARTITION instance $a_1, \ldots, a_n$, we may assume that $A = \sum_{i \in [n]} a_i$ is even because otherwise the instance is trivial. We set $b_{2n} = 1$, $b_{2i-1} = b_{2i} + a_i$ for each $i \in [n]$, and $b_{2i} = b_{2i+1} + 1$ for each $i \in [n-1]$. In other words, we start at the smallest item (which also has the largest index) and set it to 1. Then we recursively define the other items, step by step: If we stay in the same pair $i$, we add $a_i$ and if we go from one pair to the next, we add only 1. Hence, the items increase throughout the construction and we get $b_i > b_{i+1}$ for each $i \in [2n - 1]$. Moreover, the difference between the larger item of a pair and the smaller one is $b_{2i-1} - b_{2i} = a_i$ for every pair $i \in [n]$.

Suppose that the PARTITION instance is positive, i.e. we have a set $S \subseteq [n]$ such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$. Consider the set $T = \{j \in [2n] \mid (i \in S \land j = 2i - 1) \lor (i \in S \land j = 2i)\}$, where we take all the odd-indexed (i.e. larger) items corresponding to items in $S$ and the even-indexed (i.e. smaller) items corresponding to items in $\bar{S}$. It is not quite clear how large $\sum_{j \in T} b_j$ is, but using the fact that $b_{2i-1} - b_{2i} = a_i$, we can see that

$$
\sum_{j \in T} b_j - \sum_{j \notin T} b_j = \left(\sum_{i \in S} b_{2i-1} + \sum_{i \notin S} b_{2i}\right) - \left(\sum_{i \in S} b_{2i} + \sum_{i \notin S} b_{2i-1}\right)
$$

$$
= \sum_{i \in S} (b_{2i-1} - b_{2i}) + \sum_{i \in S} (b_{2i} - b_{2i-1})
$$

$$
= \sum_{i \in S} a_i - \sum_{i \notin S} a_i
$$

$$
= 0
$$

and hence, $T$ and $\bar{T}$ are a valid partition.

For the other direction, suppose we have a solution $T$, $\bar{T}$ of the EO-PARTITION instance. Define a solution of the corresponding PARTITION instance as follows: $S = \{i \in [n] \mid b_{2i-1} \in T\}$. Using essentially the same transformations as above, it follows that $\sum_{i \in S} a_i - \sum_{i \notin S} a_i = \sum_{j \in \bar{T}} b_j - \sum_{j \in T} b_j$, which is equal to zero, by assumption. So $S$ is indeed a solution of the PARTITION instance.

Note that $\sum_{j \in [2n]} b_j = O(n^2 + nA)$, since the largest item $b_1$ is bounded by $O(n + A)$. Furthermore, note that in the resulting EO-PARTITION instance we have $\sum_{i \in [n]} (b_{2i-1} - b_{2i}) = \sum_{i \in [n]} a_i$ and therefore may assume that this number is even in the following.
The REO-Partition problem was introduced by Du and Leung \[10\] and is very similar to EO-Partition. However, in this version of the problem the input consist of integers \(c_1, \ldots, c_{2n} \in \mathbb{N}\) with \(c_i > c_{i+1}\) for each \(i \in \{2n-1\}\), \(c_{2i} > c_{2i+1} + \delta\) for each \(i \in [n-1]\), and \(c_i > n(4n + 1)\delta + 5n(c_{2i} - c_{2n})\) for each \(i \in [2n]\), where \(\delta = \frac{1}{2} \sum_{i \in [n]} (c_{2i} - c_{2i+1})\). So in this restricted variant, the items of subsequent pairs have difference \(> \delta\) and each item is larger than some value depending on \(n\), \(\delta\) and the difference between the largest and smallest item.

The reduction by Du and Leung \[10\] from EO-Partition to REO-Partition is as follows: Let \(\Delta = \frac{1}{2} \sum_{i \in [n]} (b_{2i+1} - b_{2i+1}) = \frac{1}{2} A\) and

\[c_{2i-1} = b_{2i-1} + (9n^2 + 3n - (i - 1))\Delta + 5n(b_1 - b_{2n})\]
\[c_{2i} = b_{2i} + (9n^2 + 3n - (i - 1))\Delta + 5n(b_1 - b_{2n})\]

for every \(i \in [n]\). We know \(\Delta \in \mathbb{N}\) as the sum is even, by a previous assumption. Note that \(\delta = \Delta\) since \(c_{2i-1} - c_{2i} = b_{2i-1} - b_{2i}\) holds for each \(i \in [n]\). Moreover, \(c_i > c_{i+1}\) for each \(i \in [2n-1]\) since \(b_i > b_{i+1}\). We also have \(c_{2i} > c_{2i+1} + \delta\) for each \(i \in [n-1]\), since we get an additional \(\Delta = \delta\) between subsequent pairs and since already \(b_{2i} > b_{2i+1}\). Finally, we also get \(c_i > n(4n + 1)\delta + 5n(c_{2i} - c_{2n})\) for each \(i \in [2n]\), as we will show. First note that

\[c_i - c_{2n} = b_1 + (9n^2 + 3n - (1 - 1))\Delta + 5n(b_1 - b_{2n})\]
\[= (b_{2n} + (9n^2 + 3n - (n - 1))\Delta + 5n(b_1 - b_{2n}))\]
\[= b_1 - b_{2n} + (n - 1)\Delta\]

and hence:

\[n(4n + 1)\delta + 5n(c_i - c_{2n}) = n(4n + 1)\Delta + 5n(b_1 - b_{2n} + (n - 1)\Delta)\]
\[= n(4n + 1)\Delta + 5n(n - 1)\Delta + 5n(b_1 - b_{2n})\]
\[= n(4n + 1) + 5n(n - 1)\Delta + 5n(b_1 - b_{2n})\]
\[= (4n^2 + n + 5n^2 - 5n)\Delta + 5n(b_1 - b_{2n})\]
\[= (9n^2 - 4n)\Delta + 5n(b_1 - b_{2n})\]
\[= b_{2n} + (9n^2 + 2n + 1)\Delta + 5n(b_1 - b_{2n})\]
\[= b_{2n} + (9n^2 + 3n - (n - 1))\Delta + 5n(b_1 - b_{2n})\]
\[= c_{2n}\]

So the inequality holds for the smallest item, which means that it holds for all items. We can conclude that the constructed instance of REO-Partition is valid.

It is not hard to verify that a solution of the EO-Partition instance can be transformed to a solution of the REO-Partition instance and vice-versa by just selecting the corresponding \(c_j\) (respectively \(b_j\)).

If we consider the two reductions one after another and use previously observed bounds for \(b_1\), \(\sum_{j \in [2n]} b_j\) and \(\Delta\), we get:

\[\sum_{j \in [2n]} c_i = \sum_{j \in [2n]} (b_j + (9n^2 + 3n - (j - 1))\Delta + 5n(b_1 - b_{2n}))\]
\[\leq 2n((9n^2 + 3n)\Delta + 5n(b_1 - b_{2n})) + \sum_{j \in [2n]} b_j\]
\[= O(n^3\Delta) + O(n^2b_1) + \sum_{j \in [2n]} b_j\]
\[= O(n^3) + O(n^3 + n^2A) + O(n^2 + nA)\]
\[= O(n^3A)\]
In the last step, Du and Leung construct a $1|| \sum T_j$ instance from the REO-PARTITION instance. For the details, see the paper [10]. What is important for our lower bound are the parameter sizes of the $1|| \sum T_j$ instance, namely the number of jobs and the total processing time. The due dates are trivially bounded by the total processing time, which makes them less interesting in a lower bound. An examination of the construction (page 487) gives the following parameters:

- $N := 3n + 1$ jobs and total processing time $K := \sum_{i \in [2n]} c_i + (n + 1)(4n + 1)\delta = \mathcal{O}(n^3 A) = \mathcal{O}(\text{poly}(n)A)$.

Since the reduction is polynomial in $n$, we can use Lemma 22 to prove the claim.

**Proposition 12.** For every $\varepsilon > 0$, there is a $\delta > 0$ such that $1|\text{Rej} \leq Q|C_{\text{max}}$ cannot be solved in time $\mathcal{O}(2^{\delta n}(y + P + Q + W)^{1-\varepsilon})$, unless the SETH fails.

**Proof.** Consider an instance $a_1, \ldots, a_n, T$ of SUBSET SUM. We create a $1|\text{Rej} \leq Q|C_{\text{max}}$ instance with $n$ jobs, $p_j = w_j = a_j$ for every $j \in [n]$, $y = T$ and $Q = A - T$. We can now see that there is a subset of items that sums up to $T$, if and only if there is a subset of jobs that is scheduled in time $T$, the rest of the jobs is rejected and total weight of rejected jobs is at most $A - T$.

In the constructed instance, we have $N := n$ jobs and using $K := y + P + Q + W = T + A + A - T + A = \mathcal{O}(A)$ as parameter with Lemma 22 proves the claim, since the reduction takes time $\mathcal{O}(n)$. \hfill ◀

**Proposition 13.** For every $\varepsilon > 0$, there is a $\delta > 0$ such that $P2||C_{\text{max}}$ cannot be solved in time $\mathcal{O}(2^{\delta n}A^{1-\varepsilon})$, unless the SETH fails.

**Proof.** We now show that the lower bound $\mathcal{O}(2^{\delta n}A^{1-\varepsilon})$ for PARTITION can be transferred to $P2||C_{\text{max}}$.

Let $a_1, \ldots, a_n$ be a PARTITION instance. Construct the $P2||C_{\text{max}}$ instance by setting $p_j = a_j$ for every $j \in [n]$ and $y = \frac{1}{2}A$. It is easy to see that there is a partition of the items into two subsets of equal sum, if and only if the jobs can be split among the two machines such that each one gets assigned jobs with total processing time $y$.

Since the constructed instance has $N := n$ jobs and takes time linear in $n$, we can prove the claim by using $K := y + P = \frac{1}{2}A + A = \mathcal{O}(A)$ as parameter in Lemma 22. \hfill ◀

**Proposition 15.** For every $\varepsilon > 0$, there is a $\delta > 0$ such that $P2|\text{size}|C_{\text{max}}$ cannot be solved in time $\mathcal{O}(2^{\delta n}(y + P)^{1-\varepsilon})$, unless the SETH fails.

**Proof.** We now show that the lower bound $\mathcal{O}(2^{\delta n}(y + P)^{1-\varepsilon})$ for $P2|\text{size}|C_{\text{max}}$ can be transferred to $P2|\text{size}|C_{\text{max}}$.

Construct the $P2|\text{size}|C_{\text{max}}$ instance from the $P2||C_{\text{max}}$ instance by setting the size to 1 for each job. The correctness for this reduction is trivial and neither the number of jobs nor the total processing time or threshold changes, so the lower bound for $P2||C_{\text{max}}$ directly applies to $P2|\text{size}|C_{\text{max}}$ and we can conclude that there is no algorithm that solves $P2|\text{size}|C_{\text{max}}$ in $\mathcal{O}(2^{\delta n}(y + P)^{1-\varepsilon})$, unless the SETH fails. \hfill ◀

**Proposition 16.** For every $\varepsilon > 0$, there is a $\delta > 0$ such that $P2|\text{any}|C_{\text{max}}$ cannot be solved in time $\mathcal{O}(2^{\delta n}(y + P)^{1-\varepsilon})$, unless the SETH fails.

**Proof.** We now show that the lower bound $\mathcal{O}(2^{\delta n}(y + P)^{1-\varepsilon})$ for $P2||C_{\text{max}}$ can be transferred to $P2|\text{any}|C_{\text{max}}$.

Construct the $P2|\text{any}|C_{\text{max}}$ instance from the $P2||C_{\text{max}}$ instance by setting $p_{j,1} = p_{j,2} = p_j$ for every $j \in [n]$. Any schedule for $P2||C_{\text{max}}$ also represents a schedule for $P2|\text{any}|C_{\text{max}}$. \hfill ◀
and since no advantage can be achieved by scheduling any job on two machines, a feasible schedule for \( P2|\text{any}| C_{max} \) is also feasible for \( P2|| C_{max} \).

Again, this does not change the size of the instance and hence the lower bound for \( P2|| C_{max} \) directly applies to \( P2|\text{any}| C_{max} \), so we can conclude that there is no algorithm that solves the problem in time \( \mathcal{O}(2^{\delta n}(y+P)^{1-\varepsilon}) \), unless the SETH fails.

\section*{B Strongly NP-Hard Problems}

In this section, we show all our SETH-based lower bounds for strongly NP-hard problems. It is important to note that unless \( \text{P=NP} \), these problems cannot have pseudo-polynomial algorithms \cite{5}. However, our lower bounds do not only exclude pseudo-polynomial algorithms; algorithms with a super-polynomial but sub-exponential dependency on \( n \) (and a linear dependency on the other parameters) are also impossible under SETH. So even though these results are not as strong as those for weakly NP-hard problems, they might still be of interest for parameterized or approximation algorithms.

\subsection*{Jobs With Deadlines}

Our lower bound from Proposition \ref{prop: subset sum} also implies a lower bound for \( 1|| \sum_{j} T_j \) (see Corollary \ref{corollary: subset sum}). But with a more elaborate reduction that actually uses (though only two) different weights, we get a different lower bound for \( 1|| \sum_{j} w_j T_j \); the problem of minimizing the total weighted tardiness on a single machine.

\begin{proposition}
For every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( 1|| \sum_{j} w_j T_j \) cannot be solved in time \( \mathcal{O}(2^{\delta n}(d_{max} + P + W + \sqrt{y})^{1-\varepsilon}) \), unless the SETH fails.
\end{proposition}

\begin{proof}
We revisit the reduction by Lenstra et al. \cite{27} from \textsc{Subset Sum} to \( 1|| \sum_{j} w_j T_j \).

Consider a \textsc{Subset Sum} instance \( a_1, \ldots, a_n, T \). We set \( N = n + 1, p_j = w_j = a_j, d_j = 0 \) for each \( j \in [n] \), \( p_N = 1, w_N = 2, d_N = T + 1, \) and \( y = \sum_{1 \leq i < j \leq n} a_i a_j + A - T \). Again, the idea is that the newly added job \( n \) acts as a barrier at time \( T \) and the solution to the \textsc{Subset Sum} instance mapped to the \( 1|| \sum_{j} w_j T_j \) instance and job \( N \) have to be scheduled before \( T + 1 \).

Suppose the \textsc{Subset Sum} instance is positive, i.e. there is a subset \( S \) of items summing up to exactly \( T \). Schedule the jobs corresponding to \( S \) first, then job \( N \) and finally the rest of the jobs. If we ignore job \( N \) for a moment, we have only the \( n \) jobs with \( p_j = w_j = a_j \), so their total weighted completion time is equal to \( \sum_{1 \leq i < j \leq n} a_i a_j \), which is also the total weighted tardiness, since the deadlines of these jobs are all zero. Now, adding job \( N \) to the schedule at time \( T \) increases the tardiness of every job in \( S \) by 1. This increase by 1 is multiplied by the weight of each job in \( S \) and we get a total increase of \( A - T \), since that is the sum of the weights of the jobs in \( S \). So the total weighted tardiness of the constructed schedule is equal to \( \sum_{1 \leq i < j \leq n} a_i a_j + A - T = y \) and hence the instance is positive.

For the other direction, assume that we are given a schedule with total weighted tardiness at most \( y \). Consider two cases: If job \( N \) is scheduled after \( T \), say at time \( T + k \) with \( k > 0 \), then the minimum total weighted tardiness is achieved by having no gaps in the schedule, i.e. there are jobs with total processing time \( T + k \) scheduled before job \( N \) and jobs with total processing time \( A - T - k \) are scheduled after. Now, the total weighted tardiness of this schedule is \( \sum_{1 \leq i < j \leq n} a_i a_j + (A - T - k) + 2k = \sum_{1 \leq i < j \leq n} a_i a_j + A - T + k > y \), since the jobs after job \( N \) are delayed by one time unit and have total weight \( A - T - k \) and job \( N \) is late by \( k \) time units and has weight 2. This is a contradiction, so job \( N \) has to be scheduled
at time $T$. Now we know from the observations in the proof of the first direction that a gap-less schedule with job $N$ scheduled at time $T$ has total weighted tardiness exactly $y$. And if there is a gap, the total weighted tardiness strictly increases. Hence, we can conclude that there can be no gap and that the jobs scheduled before job $N$ have a total processing time exactly $T$ and the corresponding items form a solution of the Subset Sum instance.

Since the constructed instance has $N = n + 1$ jobs and moreover $d_{\text{max}} = T + 1$, $P = A + 1$, $W = A + 2$ and

$$y = \sum_{1 \leq i \leq j \leq n} a_i a_j + A - T \leq \sum_{i=1}^{n} \left( a_i \sum_{j=1}^{n} a_j \right) + O(T) = \sum_{i=1}^{n} (a_i A) + O(T) \leq O(A^2) + O(T) \leq O(T^2)$$

setting $K := d_{\text{max}} + P + W + \sqrt{y} = T + 1 + A + 1 + A + 2 + O(T) = O(A)$ and using Lemma 22 finishes the proof. Note that the reduction is polynomial in $n$.  

\textbf{Jobs With Release Dates}

We consider the problem of scheduling jobs with release dates $r_j$ for every job. With a classical reduction from Subset Sum, we get the following lower bound:

\textbf{Proposition 25.} For every $\varepsilon > 0$, there is a $\delta > 0$ such that $1|\text{rjob}|\sum w_j C_j$ cannot be solved in time $O\left(2^{\delta n} (r_{\text{max}} + P + W + \sqrt{y})^{1-\varepsilon}\right)$, unless the SETH fails.

\textbf{Proof.} We revisit the reduction by Rimoo, who reduces from Subset Sum. Let $a_1, \ldots, a_n, T$ be a Subset Sum instance. Construct a $1|\text{rjob}|\sum w_j C_j$ instance by setting $N = n + 1$, $p_j = w_j = a_j$, $r_j = 0$ for each $j \in [N - 1]$ and $p_N = 1$, $w_N = 2$, $r_N = T$, $y = \sum_{1 \leq i \leq j \leq n} a_i a_j + A + T + 2$. The idea is that the split job $N$ has to be scheduled at its release date $T$ and there cannot be any gaps in the schedule. The job $N$ then acts as a barrier between jobs corresponding to items from the Subset Sum solution and the rest.

For the first direction, assume that there is a subset $S$ that is a solution of our Subset Sum instance. Schedule the jobs corresponding to items in $S$ before $T$, then schedule the split job $N$ and finally the rest of the jobs. If we ignore the split job for a second, the total weighted completion time of this schedule is $\sum_{1 \leq i \leq j \leq n} a_i a_j$, regardless of the order of jobs (this follows again from the fact that $p_j = w_j$ for all the jobs). Now, if we add the split job, we get its completion time $T + 1$, multiplied with its weight 2; moreover, all jobs scheduled after it are delayed by 1. These delays are in turn multiplied by the weights (which are equal to their processing times). Hence, the total weighted completion time becomes:

$$\sum_{i=1}^{N} w_i C_j = \sum_{1 \leq i \leq j \leq n} a_i a_j + 2(T + 1) + A - T = \sum_{1 \leq i \leq j \leq n} a_i a_j + A + T + 2 = y$$

So our constructed schedule meets the target $y$ and is therefore feasible.

For the other direction, consider any schedule for the constructed instance with total weighted completion time at most $y$ and distinguish two cases:

If the split job $N$ is scheduled directly at time $T$, there cannot be a gap before job $N$. Otherwise, the weighted completion time is

$$\sum_{i=1}^{N} w_i C_j \geq \sum_{1 \leq i \leq j \leq n} a_i a_j + 2(T + 1) + A - T + 1 > y,$$
since the load of the jobs scheduled after job $N$ is at least $A - T + 1$ because of the gap. So this sub-case leads to a contradiction. If there is no gap, the jobs scheduled before job $N$ have total processing time exactly $T$ and the corresponding items are a solution of the original Subset Sum instance.

For the second case, assume that job $N$ starts after its release date $T$, say at time $T + k$, where $k > 0$. Without loss of generality, we can also assume that there is no gap in the schedule before the execution of job $N$, since such a gap would only increase the weighted completion time. So there are jobs with processing time $T + k$ scheduled before $N$ and jobs with processing time $A - (T + k)$ scheduled after job $N$. Thus, the total weighted completion time of the schedule is

$$\sum_{i=1}^{N} w_i C_j \geq \sum_{1 \leq i \leq j \leq n} a_i a_j + 2(T + k + 1) + A - (T + k) = \sum_{1 \leq i \leq j \leq n} a_i a_j + A + T + 2 + k > y,$$

which is a contradiction.

As job $N$ cannot be scheduled before $T$ because of its release date, we have to end up in the first case, where we find a solution of the original Subset Sum instance.

By construction, we have $r_{\max} = T$, $P = A + 1$ and $W = A + 2$. Moreover, we show that $y \leq O(T^2)$:

$$y = \sum_{1 \leq i \leq j \leq n} a_i a_j + A + T + 2 \leq \sum_{i=1}^{n} \left( a_i \sum_{j=1}^{n} a_j \right) + O(T) = \sum_{i=1}^{n} a_i A + O(T)$$

$$\leq O(A^2 + T) = O(T^2)$$

Now, since the constructed instance has $N = n + 1$ jobs and the reduction is polynomial in $n$, setting $K := r_{\max} + P + W + \sqrt{5} = T + A + 1 + A + 2 + O(T) = O(A)$ and using Lemma 22 proves the claim. ▶

A similar idea also works for $1|\tau_j|T_{\max}$, where we aim to minimize the maximum tardiness and have additional release dates. Again, a classical reduction from Subset Sum gives us a lower bound:

\begin{proposition}
For every $\varepsilon > 0$, there is a $\delta > 0$ such that $1|\tau_j|T_{\max}$ cannot be solved in time $O\left(2^{\Theta(a_{\max} + r_{\max} + P + y)^{1-\varepsilon}}\right)$, unless the SETH fails.
\end{proposition}

\begin{proof}
Lenstra et al. [27] show the NP-hardness of $1|\tau_j|L_{\max}$ by a reduction from Knapsack (which is a generalization of Subset Sum: If for each item weight and profit are the same, the problems are equivalent). We revisit this reduction to prove Proposition 26. Note that the reduction by Lenstra et al. is supposedly for the $L_{\max}$-version, but the same reduction also works for $T_{\max}$. This is because $T_{\max}$ and $L_{\max}$ are the same in the constructed instance, since there is a job with $d_j = r_j + p_j$. Hence, the maximum lateness cannot be negative and has to be equal to the maximum tardiness.

Consider a Subset Sum instance $a_1, \ldots, a_n, T$. We set $N = n + 1$, $r_j = 0$, $p_j = a_j$, $d_j = A + 1$ for each $j \in [n]$, $r_N = T$, $p_N = 1$, $d_N = T + 1$, and $y = 0$. Once more, the idea is that the newly added job $N$ acts as a barrier at time $T$ and the solution to the Subset Sum instance mapped to the $1|\tau_j|T_{\max}$ instance has to be scheduled before $T$.

For the first direction, assume we have a subset $S$ of items summing up to $T$. Then we can schedule all the jobs corresponding to the items in $S$ before $T$ (with processing time $T$),
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then the job $N$ (with processing time 1) and then the rest of the jobs (with total processing time $A - T$). So we get a schedule in which each job meets its deadline and release date, i.e. one with objective value $y = 0$.

For the other direction, assume we are given a schedule with objective value $y \leq 0$. This means that no job can be late, which has two consequences: Job $N$ with processing time 1 has to be scheduled exactly at time $T$ to meet its release date $T$ and its deadline $T + 1$. Moreover, all other jobs have to be finished before their uniform deadline $A + 1$. Since their total processing time is equal to $A$ and the remaining space is also equal to $A$, there can be no gap in the schedule. Instead, the jobs are perfectly divided into a set of jobs $S$ that are scheduled before time $T$ and the rest of the jobs. So the jobs in $S$ have total processing time $T$ and correspond to a subset of the items summing up to $T$ in the original Subset Sum instance.

Since $N = n + 1$, $r_{\text{max}} = T$, $y = 0$ and $P = d_{\text{max}} = A + 1$, using parameter $K := d_{\text{max}} + r_{\text{max}} + P + y = A + 1 + T + A + 1 + 0 = O(A)$ with Lemma 22 proves the claim. Note that the reduction is linear in $n$.

Jobs With Hard Deadlines

In the problem $1|C_j \leq d_j| \sum w_j C_j$, we aim to minimize the total weighted completion time subject to deadlines $d_j$. With a classical reduction from Subset Sum, we get the following result:

**Proposition 27.** For every $\varepsilon > 0$, there is a $\delta > 0$ such that $1|C_j \leq d_j| \sum w_j C_j$ cannot be solved in time $O\left(2^{\delta n} (d_{\text{max}} + P + W + \sqrt{y})^{1-\varepsilon}\right)$, unless the SETH fails.

**Proof.** We revisit the reduction by Lenstra et al. 27 from Subset Sum to $1|C_j \leq d_j| \sum w_j C_j$.

Consider a Subset Sum instance $a_1, \ldots, a_n, T$. We set $N = n + 1, P_j = w_j = a_j, d_j = A + 1$ for each $j \in [n], p_n = 1, w_N = 0, d_N = T + 1$, and $y = \sum_{1 \leq i \leq n} a_i a_j + A - T$. Once again, the idea is that the newly added job $N$ acts as a barrier at time $T$ and the solution to the Subset Sum instance mapped to the $1|C_j \leq d_j| \sum w_j C_j$ instance and job $N$ have to be scheduled before $T + 1$.

Suppose we are given a subset $S$ of items summing up to exactly $T$. Then we can schedule the corresponding jobs first, then job $N$ and finally the rest of the jobs. All jobs meet their deadline, the total weighted completion time without job $N$ is $\sum_{1 \leq i \leq n} a_i a_j$ and adding job $N$ delays the later jobs by one time unit, resulting in an increase of $A - T$ in the total weighted completion time, since that is that is the total weight of these jobs. Hence, we have a feasible schedule with total weighted completion time $\sum_{1 \leq i \leq n} a_i a_j + A - T = y$.

For the other direction, suppose we are given a schedule with total weighted completion time at most $y$ and consider two cases: If job $N$ is scheduled at time $T - k$ with $k > 0$, even a gap-less schedule has total weighted completion time $\sum_{1 \leq i \leq n} a_i a_j + (A - T + k) > y$, since now jobs with total weight $A - T + k$ are delayed by the job $N$. If - on the other hand - job $N$ is scheduled at time $T$, a gap-less schedule has total weighted completion time $y$ (see above) and only a gap-less one. Hence, we get a subset of jobs with total completion time exactly $T$, which corresponds to a solution of the Subset Sum instance.

Since the reduction is polynomial in $n$ and we have $N = n + 1, d_{\text{max}} = T + 1, P = A + 1, W = A$ and $\sqrt{y} = O(T)$ (see the proof of Proposition 24), using $K := d_{\text{max}} + P + W + \sqrt{y} = T + 1 + A + 1 + A + O(T) = O(A)$ as parameter with Lemma 22 proves the claim. ▶
C Implications for Other Objective Functions

In this section, we make use of the fact that the usual objective functions are somewhat ordered in difficulty. Using classical reductions, we can transfer lower bounds to a wide range of other scheduling problems.

But let us first introduce the objective functions that were not considered in previous sections: We define $F_j := C_j - r_j$ to be the flow of job $j$, which is simply the amount of time that passes between $j$’s release and completion. Naturally, we might then consider to minimize the maximum flow $F_{max}$, the total flow $\sum F_j$ or the total weighted flow $\sum w_j F_j$. Note that for a given instance, the release dates $r_j$ are fixed values in the objective function and can be removed. So in our context of exact algorithms and decision problems, $1|r_j|\sum C_j$ is equivalent to $1||\sum F_j$, for example. Also note that in the three-field notation, the release dates are implicitly included by the objective function, like the deadlines are implicitly included for objectives like $T_{max}$ or $\sum w_j U_j$. The other concept is the late work $V_j := \min\{p_j, T_j\}$, where $T_j = \max\{0, C_j - d_j\}$ is the tardiness as usual. Intuitively - as the name suggests - the late work is the duration for which the job is executed after its deadline. The late work most commonly appears in the objectives of minimizing the total late work $\sum V_j$ or the total weighted late work $\sum w_j V_j$.

We now revisit classical reductions between the usual objective functions in the context of fine-grained complexity. Reductions like these can e.g. be found in the work by Lawler et al. [23]. The content of the following lemma is also visualized in Figure 2. Moreover, all SETH-based lower bounds - including those from this section - are summarized in Table 1.

Lemma 28. Consider machine model $\alpha$ and additional constraints $\beta$. We have:

1. If $\gamma = C_{max}$ and $\gamma' = T_{max}$ or $\gamma = \sum C_j$ and $\gamma' = \sum T_j$ or $\gamma = \sum w_j C_j$ and $\gamma' = \sum w_j T_j$, there exists a reduction from $\alpha|\beta|\gamma$ to $\alpha|\beta|\gamma'$, where we only introduce zero-deadlines, i.e. $d'_j = 0$ for every $j \in [n]$.

2. If $\gamma = C_{max}$ and $\gamma' = F_{max}$ or $\gamma = \sum C_j$ and $\gamma' = \sum F_j$ or $\gamma = \sum w_j C_j$ and $\gamma' = \sum w_j F_j$, there exists a reduction from $\alpha|\beta|\gamma$ to $\alpha|\beta|\gamma'$, where we only introduce zero-release-dates, i.e. $r'_j = 0$ for every $j \in [n]$.

3. If $\gamma = \sum C_j$ and $\gamma' = \sum w_j C_j$ or $\gamma = \sum T_j$ and $\gamma' = \sum w_j T_j$ or $\gamma = \sum U_j$ and $\gamma' = \sum w_j U_j$ or $\gamma = \sum V_j$ and $\gamma' = \sum w_j V_j$ or $\gamma = \sum F_j$ and $\gamma' = \sum w_j F_j$, there exists a reduction from $\alpha|\beta|\gamma$ to $\alpha|\beta|\gamma'$, where we only introduce unit-weights, i.e. $w'_j = 1$ for every $j \in [n]$.

4. If $\gamma = T_{max}$ and $\gamma' = L_{max}$, there exists a reduction from $\alpha|\beta|\gamma$ to $\alpha|\beta|\gamma'$, where we do not change anything about the instance.

5. If $\gamma = L_{max}$ and $\gamma' = \sum T_j$, $\gamma' = \sum U_j$ or $\gamma' = \sum V_j$, there exists a reduction from $\alpha|\beta|\gamma$ to $\alpha|\beta|\gamma'$, where we only increase the deadlines by $y$, i.e. $d'_j = d_j + y$ for every $j \in [n]$ and set the target value to zero, i.e. $y' = 0$.

Note that in all of these reductions, only the mentioned parameters are modified. Everything else about the instance (e.g. the number of jobs $n$) stays the same. The running time of each reduction is polynomial in $n$.

Proof. For part 1, we observe that setting the deadline of every job $j \in [n]$ to zero means that its completion time $C_j$ is identical to its tardiness $T_j$:

$$T_j = \max\{C_j - d_j, 0\} = \max\{C_j, 0\} = C_j$$

Hence, minimizing $T_{max}$, $\sum T_j$ and $\sum w_j T_j$ is equivalent to minimizing $C_{max}$, $\sum C_j$ and $\sum w_j C_j$, respectively.
Part 2 is similar: Setting a release date $r_j$ to zero means that the flow time $F_j$ of job $j$ is equal to its completion time $C_j$.

Part 3 is obvious: Weighting all jobs equally is equivalent to having no weights at all. Having the weights all set to 1 also means that the objective value does not change.

For part 4, observe that any schedule that minimizes the maximum lateness $L_{\text{max}}$ also minimizes the maximum tardiness and that we only have non-negative $y$-values in the $T_{\text{max}}$-problem.

For part 5, observe that the original instance of $\alpha|\beta|\gamma$ has a schedule with maximum lateness $L_{\text{max}} \leq y$ if and only if in the instance with delayed deadlines $d'_j = d_j + y$ no job is late. No job being late is equivalent to $\sum T_j$, $\sum U_j$ and $\sum V_j$ all being zero.

All of the reductions are polynomial in $n$, since we only need to construct new jobs with not too large parameters.

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| $\alpha$ | $\beta$ | $\gamma$ | Lower Bound | Reference |
|---------|---------|---------|-------------|----------|
| P2      | –       | $C_{\text{max}}$ | $O\left(2^n(y + P)^{1-\varepsilon}\right)$ | Proposition 13 |
| P2      | size    | $C_{\text{max}}$ | $O\left(2^n(y + P)^{1-\varepsilon}\right)$ | Proposition 15 |
| P2      | any     | $C_{\text{max}}$ | $O\left(2^n(y + P)^{1-\varepsilon}\right)$ | Proposition 16 |
| P2      | –       | $L_{\text{max}}$ | $O\left(2^n(y + P + f(D))^{1-\varepsilon}\right)$ | Corollary 30 |
|        | size    | $T_{\text{max}}$ | $O\left(2^n(y + P + f(D))^{1-\varepsilon}\right)$ | |

3 The possibility of negative $y$-values in the $L_{\text{max}}$-problem prevents the same ‘reduction’ from working in the other direction.
Table 1 Overview of our SETH-based lower bounds. Throughout, \( f \) is some arbitrary computable function.

In Proposition 12 we showed a lower bound for \( 1 | \text{Rej} \leq Q | C_{\text{max}} \). Using Lemma 28 we now transfer this result to other rejection problems with more difficult objective functions:

**Corollary 29.** Let \( f \) be a computable function. Moreover, let \( \gamma_1 \in \{ L_{\text{max}}, T_{\text{max}} \} \), \( \gamma_2 \in \{ \sum U_j, \sum V_j, \sum T_j \} \) and \( \gamma_3 \in \{ \sum w_j U_j, \sum w_j V_j, \sum w_j T_j \} \) \[^4\] Then unless SETH fails, for

\[^4\] Note that in the case of weighted objective functions we already have weights for each job (i.e. rejection
every $\varepsilon > 0$ there exists a $\delta > 0$ such that

1. $|\text{Rej} \leq Q|\gamma_1$ cannot be solved in time $O(2^{5n}(y + P + Q + W + f(D))^{1-\varepsilon})$,
2. $|\text{Rej} \leq Q|\gamma_2$ cannot be solved in time $O(2^{5n}(d_{\text{max}} + P + Q + W + f(y))^{1-\varepsilon})$,
3. $|\text{Rej} \leq Q|\gamma_3$ cannot be solved in time $O(2^{5n}(d_{\text{max}} + P + Q + W + f(y, w_{\text{max}}))^{1-\varepsilon})$ and
4. $|\text{Rej} \leq Q|F_{\text{max}}$ cannot be solved in time $O(2^{5n}(y + P + Q + W + f(R))^{1-\varepsilon})$.

**Proof.** The lower bound for $|\text{Rej} \leq Q|C_{\text{max}}$ from Proposition 12 for $|\text{Rej} \leq Q|C_{\text{max}}$ is $O(2^{5n}(y + P + Q + W)^{1-\varepsilon})$. We use the reductions from Lemma 28 to obtain lower bounds for the more difficult objective functions.

For part 1, the reductions only introduce zero-deadlines. Hence, $D = 0$ and we get the lower bound $O(2^{5n}(y + P + Q + W + f(D))^{1-\varepsilon})$, where $f$ is some computable function that only depends on $D$ and is hence constant for $D = 0$.

For part 2, applying the reductions in sequence gives us deadlines that are equal to the original target, together with a new target $y = 0$. Hence, we get the lower bound $O(2^{5n}(d_{\text{max}} + P + Q + W + f(y))^{1-\varepsilon})$.

For the third part, we get $O(2^{5n}(d_{\text{max}} + P + Q + W + f(y, w_{\text{max}}))^{1-\varepsilon})$ as our lower bound, since the reduction only adds unit-weights.

In the case of $F_{\text{max}}$, we simply introduce zero-release-dates and get the lower bound $O(2^{5n}(y + P + Q + W + f(R))^{1-\varepsilon})$.

Similarly, the lower bounds from Proposition 13, Proposition 15 and Proposition 16 for the two-machine problems with $C_{\text{max}}$-objective can also be transferred to problems with more difficult objective functions:

**Corollary 30.** Let $\beta \in \{\text{size, any, -}\}$, $\gamma_1 \in \{L_{\text{max}}, T_{\text{max}}\}$, $\gamma_2 \in \{\sum U_j, \sum V_j, \sum T_j\}$ and $\gamma_3 \in \{\sum w_jU_j, \sum w_jV_j, \sum w_jT_j\}$. Moreover, let $f$ be any computable function. Then unless SETH fails, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

1. $P2|\beta|\gamma_1$ cannot be solved in time $O(2^{5n}(y + P + f(D))^{1-\varepsilon})$,
2. $P2|\beta|\gamma_2$ cannot be solved in time $O(2^{5n}(d_{\text{max}} + P + f(y))^{1-\varepsilon})$,
3. $P2|\beta|\gamma_3$ cannot be solved in time $O(2^{5n}(d_{\text{max}} + P + f(w_{\text{max}}, y))^{1-\varepsilon})$ and
4. $P2|\beta|F_{\text{max}}$ cannot be solved in time $O(2^{5n}(y + P + f(R))^{1-\varepsilon})$.

**Proof.** The lower bounds for $P2|\beta|C_{\text{max}}$ are all $O(2^{5n}(y + P)^{1-\varepsilon})$. We apply Lemma 28 and add the additional parameters to the lower bound.

In case 1 and 4, the reduction from $P2|\beta|C_{\text{max}}$ to $P2|\beta|\gamma_1$ (resp. $P2|\beta|F_{\text{max}}$) involves only adding zero-deadlines (resp. zero-release-dates). Hence, $D = R = 0$ in the constructed instances and the lower bounds follow.

In case 2, the instance is constructed by adding deadlines equal to the original target value and the new target value is set to zero. So we get $O(2^{5n}(d_{\text{max}} + P + f(y))^{1-\varepsilon})$ as our new lower bound.

In case 3, in addition to deadlines, we get unit-weights. So we get the lower bound $O(2^{5n}(d_{\text{max}} + P + f(w_{\text{max}}, y))^{1-\varepsilon})$, concluding the proof.

The lower bound for $P2||\sum w_jC_j$ from Proposition 14 together with Lemma 28 yields the following implications:

**Corollary 31.** Let $f$ be any computable function. Then unless SETH fails, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
1. $P2|\sum w_jT_j$ cannot be solved in time $O\left(2^{dn}(\sqrt{y} + P + W + f(D))^{1-\varepsilon}\right)$ and
2. $P2|\sum w_jF_j$ cannot be solved in time $O\left(2^{dn}(\sqrt{y} + P + W + f(R))^{1-\varepsilon}\right)$.

**Proof.** The lower bound for $P2|\sum w_jC_j$ from Proposition 14 is $O\left(2^{dn}(\sqrt{y} + P + W)^{1-\varepsilon}\right)$ and we use the reductions from Lemma 28.

In both cases, we only introduce zero-deadlines/zero-release-dates and have $D = R = 0$; so the lower bounds follow.

Using the lower bound for $1||\sum T_j$ from Proposition 11 and Lemma 28, we get the following result by adding unit-weights:

**Corollary 32.** Let $f$ be any computable function. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $1||\sum w_jT_j$ cannot be solved in time $O\left(2^{dn}(P + f(w_{max}))^{1-\varepsilon}\right)$, unless the SETH fails.

**Proof.** The lower bound for $1||\sum T_j$ from Proposition 11 is $O\left(2^{dn}P^{1-\varepsilon}\right)$ and using the unit-weight reduction from Lemma 28, we get the claimed lower bound, since $w_{max} = 1$.

As noted before, $1|r_j|\sum w_jC_j$ is equivalent to $1||\sum w_jF_j$. Hence, we get the same lower bound for that problem. By introducing zero-deadlines, we also get a lower bound for $1|r_j|\sum w_jT_j$ using Lemma 28.

**Corollary 33.** Let $f$ be any computable function. Then unless the SETH fails, for every $\varepsilon > 0$, there is a $\delta > 0$ such that:

1. $1|r_j|\sum w_jT_j$ cannot be solved in time $O\left(2^{dn}(r_{max} + P + W + \sqrt{y} + f(D))^{1-\varepsilon}\right)$ and
2. $1||\sum w_jF_j$ cannot be solved in time $O\left(2^{dn}(r_{max} + P + W + \sqrt{y})^{1-\varepsilon}\right)$.

**Proof.** From Proposition 26, we have $O\left(2^{dn}(r_{max} + P + W + \sqrt{y})^{1-\varepsilon}\right)$ as lower bound for $1|r_j|\sum w_jC_j$, so using the zero-deadlines reduction from Lemma 28, we get $D = 0$ and the lower bound for $1|r_j|\sum w_jT_j$ follows.

Using our lower bound for $1|r_j|T_{max}$ from Proposition 26 and Lemma 28, we get the following implications for other single-machine release date problems:

**Corollary 34.** Let $\gamma_1 \in \{\sum U_j, \sum V_j, \sum T_j\}$, $\gamma_2 \in \{\sum w_jU_j, \sum w_jV_j, \sum w_jT_j\}$ and let $f$ be any computable function. Then unless SETH fails, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that:

1. $1|r_j|L_{max}$ cannot be solved in time $O\left(2^{dn}(d_{max} + r_{max} + P + y)^{1-\varepsilon}\right)$,
2. $1|r_j|\gamma_1$ cannot be solved in time $O\left(2^{dn}(d_{max} + r_{max} + P + f(y))^{1-\varepsilon}\right)$ and
3. $1|r_j|\gamma_2$ cannot be solved in time $O\left(2^{dn}(d_{max} + r_{max} + P + f(y, w_{max}))^{1-\varepsilon}\right)$.

**Proof.** From Proposition 26, we have the lower bound $O\left(2^{dn}(d_{max} + r_{max} + P + y)^{1-\varepsilon}\right)$ for $1|r_j|T_{max}$ and using the reductions from Lemma 28, we get the following results.

The lower bound for $1|r_j|L_{max}$ directly follows, since the reduction does not change the instance in any way.

For the second part, the reduction increases the deadlines by the original target value and sets the new target value to zero. The lower bound $O\left(2^{dn}(d_{max} + r_{max} + P + f(y))^{1-\varepsilon}\right)$ follows.

The same lower bound holds for the third case, but we also introduce unit-weights, so the bound becomes $O\left(2^{dn}(d_{max} + r_{max} + P + f(y, w_{max}))^{1-\varepsilon}\right)$. 

\[\square\]
Using Lemma 28, we can also transfer the lower bound for $1|C_j \leq d_j| \sum w_jC_j$ from Proposition 27 to flow minimization. Note that we already have deadlines in the original problem, so we cannot introduce zero-deadlines to transfer the result to $1|C_j \leq d_j| \sum w_jT_j$.

**Corollary 35.** Let $f$ be any computable function. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $1|C_j \leq d_j| \sum w_jF_j$ cannot be solved in time $O \left(2^{\delta n} \left(d_{\text{max}} + P + W + \sqrt{y} + f(R)\right)^{1-\varepsilon}\right)$, unless the SETH fails.

**Proof.** From Proposition 27, we have the lower bound $O \left(2^{\delta n} \left(d_{\text{max}} + P + W + \sqrt{y}\right)^{1-\varepsilon}\right)$ for $1|C_j \leq d_j| \sum w_jC_j$. Using the zero-release-dates reduction from Lemma 28, the lower bound for $1|C_j \leq d_j| \sum w_jF_j$ follows. ▶

Finally, we can transfer the result for $Pm||C_{\text{max}}$ from Theorem 17 to other objectives, using the reductions from Lemma 28.

**Corollary 36.** Let $\gamma \in \{F_{\text{max}}, T_{\text{max}}, L_{\text{max}}, \sum T_j, \sum U_j, \sum V_j, \sum w_jT_j, \sum w_jU_j, \sum w_jV_j\}$. There is no algorithm for $Pm||\gamma$ with running time $O \left(nmP^{\frac{1}{\text{max}(\gamma)}}\right)$, unless the ETH fails.

**Proof.** Here, we use simply that there is a reduction from $Pm||C_{\text{max}}$ to each of these problems, where neither the number of jobs, nor the number of machines or the processing times change. The lower bound then directly holds for the more difficult problems. ▶