On the Fiber Characters of $\mathbb{F}_{p^m}^*$ and related Polynomial Algebras

Michele Elia *

February 16, 2021

Abstract

Let $p$ be a prime, $m$ be a positive integer ($m \geq 1$, and $m \geq 2$ if $p = 2$), and $\chi_n$ be a multiplicative complex character on $\mathbb{F}_{p^m}^*$ with order $n | (p^m - 1)$. We show that a partition $A_1 \cup A_2 \cup \cdots \cup A_n$ of $\mathbb{F}_{p^m}^*$ is the partition by fibers of $\chi_n$ if and only if these fibers satisfy certain additive properties. This is equivalent to showing that the set of multivariate characteristic polynomials of these fibers, completed with the constant polynomial 1, is the basis of an $(n + 1)$-dimensional commutative algebra with identity in the ring $\mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^{p^m - 1}, \ldots, x_n^{p^m - 1} \rangle$.

Mathematics Subject Classification (2000): 11A15, 11N69, 11R32

Key words: $n$th power residue, cyclotomic coset, character, polynomial ring.

1 Introduction

In 1952, Perron gave some additive properties of the fibers of the quadratic character on $\mathbb{F}_p$. Specifically in [12], he showed that if $\mathcal{A}, \mathcal{B} \subset \mathbb{F}_p$ are the subsets of quadratic residues and non-residues, respectively, and letting $d_p = \frac{p-1}{4}$ if $p \equiv 1 \pmod{4}$, and $d_p = \frac{p+1}{4}$ if $p \equiv 3 \pmod{4}$, then

1. Every element of $\mathcal{A}$ [respectively $\mathcal{B}$] can be written as the sum of two elements of $\mathcal{A}$ [respectively $\mathcal{B}$] in exactly $d_p - 1$ ways.

2. Every element of $\mathcal{A}$ [respectively $\mathcal{B}$] can be written as the sum of two elements of $\mathcal{B}$ [respectively $\mathcal{A}$] in exactly $d_p$ ways.

It was natural to inquire just how strong this result is, and to what extent it may hold for any character $\chi_m$ other than $\chi_2$. In [10] it is shown that these additive properties uniquely characterize the even partition of $\mathbb{F}_p$ into quadratic residues and non-residues. In [11], the even restriction is removed, and the result is generalized to fibers of arbitrary multiplicative character $\chi$ on $\mathbb{F}_p$ ($n$ being a divisor of $(p - 1)$), with suitable cyclotomic numbers in place of the constants $d_p$ above. Lastly, in [5], the generalization of the even partition (i.e. by the quadratic character $\chi_2$) to every finite field of odd characteristic, that is, the partition of $\mathbb{F}_{p^m}$ into squares and non-squares, is discussed and settled. Perron’s view is attractive, but the formulation of the problem purely in terms of characteristic polynomials and their algebras permits a full description and proof of facts that occur in every finite field. The purpose of this paper is to prove this definitive result.

*Politecnico di Torino Corso Duca degli Abruzzi 24, I - 10129 Torino – Italy; e-mail: elia@polito.it
2 Preliminary results

Let $\mathbb{F}_{p^m}$ be a finite field with $p^m$ elements generated by a root $\gamma$ of a primitive irreducible polynomial

$$p(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_0 \text{ over } \mathbb{F}_p.$$ 

Let $\mathcal{B} = \{1, \gamma, \ldots, \gamma^{m-1}\}$ be a basis of $\mathbb{F}_{p^m}$, any non-zero element $\beta \in \mathbb{F}_{p^m}$ is represented either as a power $\gamma^h$ or in the basis $\mathcal{B}$ as $\sum_{i=1}^m b_i \gamma^{i-1}$ with $b_i \in \mathbb{F}_p$. In the following, $\beta$ will be interchangeably indicated with the $m$-dimensional vector $b = [b_1, b_2, \ldots, b_m] \in \mathbb{F}_p^m$, whenever necessary.

A multiplicative complex character is an isomorphism $\chi: \mathbb{F}_{p^m}^* \rightarrow \mathbb{C}_{p^m-1}$ between the multiplicative cyclic group $\mathbb{F}_{p^m}^*$ and the complex multiplicative group $\mathbb{C}_{p^m-1}$ of the units of order $p^m - 1$ in the complex field $\mathbb{C}$. Let $n > 1$ be a non-trivial positive divisor of $p^m - 1$, that is $n \cdot s = p^m - 1$ (if $p = 2$ then $m$ must be greater than $1$), then the subset consisting of the powers of $\rho = \gamma^n$ is a cyclic subgroup of order $s$ of $\mathbb{F}_{p^m}^*$.

Let $\zeta_n$ be a primitive $n$th complex root of unity, i.e. $\zeta_n$ satisfies the $n$th cyclotomic polynomial. A character of order $n$ is explicitly defined as the mapping $\chi_n: \gamma \mapsto e^{2\pi i/n} = \zeta_n$, that is

$$\chi_n(\gamma^{n\ell+h}) = e^{2\pi i(n\ell+h)/n} = e^{2\pi i h/n} = \zeta_n^h \quad \forall \ell \in \mathbb{Z}, \quad \text{and } h \in \{0, 1, \ldots, n-1\}.$$ 

For each integer $0 \leq k \leq (n-1)$ let $\mathcal{A}_k$ be the fiber $\chi^{-1}(\zeta_n^{k-1})$, then the fiber $\mathcal{A}_1 = \chi^{-1}(1)$ is the subgroup of $\mathbb{F}_{p^m}^*$ consisting of the $n$th powers of $\gamma$, and the fiber $\mathcal{A}_k = \chi^{-1}(\zeta_n^{k-1})$, with $k > 1$, is clearly the coset $\gamma^{k-1}\mathcal{A}_1$. We have $|\mathcal{A}_k| = \frac{p^{m-1}}{n} = s$, and for each $1 \leq k \leq n$, the corresponding multivariate characteristic polynomial is

$$q_k(x) = q_k(x_1, \ldots, x_n) = \sum_{\beta \in \mathcal{A}_k} \prod_{i=1}^m x_i^{b_i} \in \mathbb{Z}[x_1, \ldots, x_n].$$ 

The set of fibers $\mathcal{A}_1, \ldots, \mathcal{A}_n$ form a partition of $\mathbb{F}_{p^m}^* = \{1, \gamma, \gamma^2, \ldots, \gamma^{p^m-2}\}$, thus, defining the polynomial $q_0(x) = 1$ which is the characteristic polynomial of the set $\{0\}$, we have

$$\sum_{k=0}^n q_k(x) = \prod_{i=1}^m \frac{x_i^{p} - 1}{x_i - 1}.\] 

The following lemmas and theorem show that the set of these $n+1$ multivariate polynomials is the basis of an algebra of dimension $n+1$ in the polynomial ring $\mathbb{R}_n[x] = \mathbb{Q}[x]/(x_1^p - 1, \ldots, x_n^p - 1)$, where $(x_1^p - 1, \ldots, x_n^p - 1)$ denotes the ideal generated by the polynomials included in brackets.

Since the fiber $\mathcal{A}_1$ is a sub-group of order $s = \frac{p^m-1}{n}$ of $\mathbb{F}_{p^m}^*$, and the remaining fibers are its cosets, which form a partition of $\mathbb{F}_{p^m}^*$, the following proposition easily follows

**Proposition 1.** The set $\{q_0(x), q_1(x), \ldots, q_n(x)\}$ of $n+1$ multivariate polynomials is a basis of a $\mathbb{Q}$-subspace $V_{n+1}$ of dimension $n+1$ in the $p^m$-dimensional vector space $\mathbb{Q}[x]/(x_1^p - 1, \ldots, x_n^p - 1)$ of multivariate polynomials of degree at most $p-1$ in each variable $x_i$.

The elements of $\mathcal{A}_1$ have the following properties:

**Lemma 1.** Let $p$ be an odd prime, and assuming the above hypotheses, we have

1. If $s$ is even, for any $\beta \in \mathcal{A}_1$ there exists a $\alpha \in \mathcal{A}_1$ such that $\beta + \alpha = 0$.

2. If $s$ is odd, there exists a coset $\mathcal{E} = \eta \mathcal{A}_1$ such that for any $\beta \in \mathcal{A}_1$ there is a $\alpha \in \mathcal{E}$ such that $\beta + \alpha = 0$.  

2
Let $p = 2$, then

3. In $\mathbb{F}_{2^m}$, any element $\beta$ is the opposite of itself, i.e. $\beta + \beta = 0$.

**Proof.** Consider the primitive element $\gamma$ of $\mathbb{F}_{p^m}$, then

1. If $s$ is even, the elements of $A_1$ are all the roots of $X^s - 1$, which splits as $(X^{s/2} - 1)(X^{s/2} + 1)$. Let $\eta = \gamma^n$ denote a root of $X^{s/2} + 1$, and $\beta = \gamma^{2nt}$ be any root of $X^{s/2} - 1$. Since $\eta^{s/2} = -1$, we have

$$\beta \eta^{s/2} = -\beta = \gamma^{2nt} \eta^{ns/2} = \gamma^{(2t+s/2)n} \in A_1,$$

therefore $\beta + \gamma^{(2t+s/2)n} = 0$, i.e. $\alpha = \gamma^{(2t+s/2)n}$.

2. If $s$ is odd, no power of any element in $A_1$ is equal to $\gamma$. However, let $\theta = \gamma^n$ be a generator of the cyclic group $A_1$, then an $\eta = \gamma^t \in \mathbb{F}_{p^m}$ certainly exists such that $\theta + \eta = 0$. Consider the coset $\eta A_1$, therefore for any $\beta = \theta^u \in A_1$, the element $\zeta = \eta \theta^{u-1}$ is such that $\beta + \zeta = 0$ because we have

$$\beta + \zeta = \theta^u + \eta \theta^{u-1} = \theta^{u-1}(\theta + \eta) = 0,$$

i.e. $\alpha = \gamma^t \theta^{u-1} = \gamma^t \gamma^{n(u-1)} = \gamma^{t+n(u-1)}$.

3. If $p = 2$, then we trivially have $\beta + \beta = 0$, thus in any fiber $A_k$ in $\mathbb{F}_{2^m}$, the sum of every element with itself is $0$, and the sum of two elements that are not in the same fiber is always different from zero.

□

The immediate goal is to show that $V_{n+1}$ is actually a $\mathbb{Q}$-sub-algebra of $\mathbb{Q}[x]/(x_1^p - 1, x_2^p - 1, \ldots, x_m^p - 1)$.

**Lemma 2.** The following properties hold for the sums of elements of cosets in $\mathbb{F}_{p^m}$ with odd $p$:

1. If a fixed $u \in \mathbb{F}_{p^m}$ can be expressed as the sum $\alpha_1 + \alpha_2 = u$, with $\alpha_1 \in A_i$ and $\alpha_2 \in A_j$, then every element of the coset $A_{k(u)} = u A_1$ can be expressed as the sum of two elements, one from $A_i$, and one from $A_j$.

2. As a direct consequence of the previous point, the product $q_i(x) q_j(x)$ is a linear combination of the basis elements of $V_{n+1}$.

**Proof.** The proof of claim 1 is immediate, assuming $\alpha_1 + \alpha_2 = u$, we have

$$\alpha(\alpha_1 + \alpha_2) = \alpha \alpha_1 + \alpha \alpha_2 = \alpha u \quad \forall \alpha \in A_1,$$

and the conclusion follows from the definition of the coset $A_{k(u)} = u A_1$, and group closure.

The proof of claim 2 is a little more elaborate. Due to the definition of the monomials $m(x)$ that form part of the definition of the polynomials $q_i(x)$, and the correspondence $m(x) \leftrightarrow \eta \in \mathbb{F}_{p^m}$, the product of two monomials in the ring $\mathbb{Q}[x]/(x_1^p - 1, x_2^p - 1, \ldots, x_m^p - 1)$ corresponds to the sum of the corresponding elements in $\mathbb{F}_{p^m}$. Now the product $q_i(x) q_j(x)$ consists of $s^2$ distinct monomials, which can be partitioned into groups of $s$ monomials, each group corresponding to some polynomial $q_{k(i,j)}(x)$ by the previous claim 1; the conclusion follows, by linearity. □
\textbf{Theorem 1.} Let $2 \leq n|\!(p^m - 1)$, $p$ prime, $m$ positive integer ($m \geq 2$ if $p = 2$), and $s = \frac{p^m - 1}{n}$. The $\mathbb{Q}$-vector space $V_{n+1}$ of Proposition 1 is a $\mathbb{Q}$-sub-algebra of the residue ring $\mathbb{Q}[x]/\langle x_1^p - 1, x_2^p - 1, \ldots, x_m^p - 1 \rangle$. In particular, for every $1 \leq i, j \leq n$ there exist integers $c_{ijk}$ such that

$$q_i(x)q_j(x) \mod \langle x_1^p - 1, x_2^p - 1, \ldots, x_m^p - 1 \rangle = c_{ij0} + \sum_{k=1}^{n} c_{ijk} q_k(x) \, .$$

The coefficients $c_{ij0}$ can be explicitly expressed considering $p$ odd and $p = 2$ separately:

\begin{enumerate}
\item $p$ odd
  \begin{enumerate}
  \item $c_{i0} = s$ and $c_{ij0} = 0$ for every $j \neq i$ if $s$ is even;
  \item $c_{i0} = 0$ and $c_{ij0} = s$ for a suitable pair $j \neq i$, if $s$ is odd.
  \end{enumerate}
\item $p = 2$, in this case $s$ is always odd, and we have
  \begin{enumerate}
  \item $c_{i0} = s$, and $c_{ij0} = 0$ for every $j \neq i$.
  \end{enumerate}
\end{enumerate}

\textbf{Proof.} The $\mathbb{Q}$-vector space $V_{n+1}$ is a sub-algebra of $\mathbb{Q}[x]/\langle x_1^p - 1, x_2^p - 1, \ldots, x_m^p - 1 \rangle$ by Lemma 2.

In general it does not seem possible to obtain a closed form for all constants $c_{ijk}$ holding for every $p$ and every $m$, except for the following exceptions.

Let $1$ be the all-ones $m$-dimensional vector, then we have

$$q_i(1)q_j(1) = c_{ij0} + \sum_{k=1}^{n} c_{ijk} q_k(1) \, ,$$

since $q_i(1) = s$, we have $s^2 = c_{ij0} + s \sum_{k=1}^{n} c_{ijk}$; this equation implies that $s|c_{ij0}$; since the integer $c_{ij0} \leq s$, it follows that $c_{ij0}$ is either 0 or $s$. If $p$ is odd, by Lemma 2 it follows that

\begin{enumerate}
\item $c_{i0} = s$ and $c_{ij0} = 0$ for every $j \neq i$ if $s$ is even;
\item $c_{i0} = 0$ and $c_{ij0} = s$ for a suitable pair $j \neq i$, if $s$ is odd.
\end{enumerate}

If $p = 2$, then $s$ is necessarily odd, however in $\mathbb{F}_{p^m}$ every element is the opposite of itself, then letting $m_\beta(x)$ be the monomial associated to $\beta$, it follows that $m_\beta(x)^2 \mod \langle x_1^2 - 1, x_2^2 - 1, \ldots, x_m^2 - 1 \rangle$ is the monomial associated to $\beta + \beta = 0$, that is the monomial 1; it follows that

\begin{enumerate}
\item $c_{i0} = s$ and $c_{ij0} = 0$ for every $j \neq i$.
\end{enumerate}

\hfill \square

Theorem 1 shows that the vector space $V_{n+1}$ is a commutative sub-algebra with identity of the ring of residue polynomials $\mathbb{Q}[x]/\langle x_1^p - 1, x_2^p - 1, \ldots, x_m^p - 1 \rangle$. As observed in the proof of Theorem 1 in general it seems that the structure constants cannot be given in closed form for every prime $p$, extension degree $m$, and power residue exponent $n$, thus the computational aspects for obtaining numerical values of every $c_{ijk}$ may be of interest.
3 Computation of the structure constants

The structure constants $c_{ijk}$ are easily found in closed form for $n = 2$, $m = 1$, and any odd $p$; however, for every $m \geq 2$ and $n > 2$, in general these constants must be numerically computed by means of convenient algorithms. We briefly describe two different computational methods.

3.1 Direct method

For fixed $i, j$, equation (1) can be directly used to compute the structure constants. A consistent linear system of $n$ equations in the $n$ unknowns $c_{ijk}$, $k = 1, \ldots, n$, can be obtained by comparing the coefficients of equal multivariate monomials on the two sides of (1); actually, we would obtain a consistent linear system of $n^2$ linear equations in $n$ unknowns. The search for the solution could present some difficulty because the product

$$q_1(x_1, \ldots, x_2)q_1(x_1, \ldots, x_2) \mod (x_i^n - 1, \ldots, x_n^n - 1)$$

consists of $n^2$ monomials in some order that, a priori, we do not know. They must all be computed, but only $n$ are used. When $n$ is small, as in the following examples, the method is very efficient, but when $n$ is large, $n^2$ multivariate monomials must be sorted according to some ordering criterion: this computational issue is left as an open problem.

**Example 1.** Let $p = 3$ and $m = 2$, thus $p^m - 1 = 8$ and $n$ may be 2 or 4, which are the only proper divisors of 8. Let $m(z) = z^2 + z + 2$ be a primitive quadratic polynomial over $\mathbb{F}_3$. Let $\alpha$ be a root of $m(z)$, the 9 elements of $\mathbb{F}_9$ are

| $\alpha$ | $\beta$ |
|--------|--------|
| $\alpha^0$ | $0$ |
| $\alpha^1$ | $1$ |
| $\alpha^2$ | $\alpha + 1$ |
| $\alpha^3$ | $1 + 2\alpha$ |
| $\alpha^4$ | $2 + 2\alpha$ |
| $\alpha^5$ | $2\alpha$ |
| $\alpha^6$ | $2\alpha + 1$ |
| $\alpha^7$ | $1 + \alpha$ |

**Case 1: $n = 2, s = 4$ ; we have two fibers (cosets)**

$$A_1 = \{1, 1 + 2\alpha, 2 + \alpha\}, \quad A_2 = \{\alpha, 2 + 2\alpha, 2\alpha, 1 + \alpha\},$$

and the corresponding characteristic multivariate polynomials are

$$q_1(x_1, x_2) = x_1 + x_1x_2^2 + x_2^2 + x_1x_2^2, \quad q_2(x_1, x_2) = x_2 + x_1^2x_2^2 + x_2^2 + x_1x_2^2.$$ 

The structure constants of the polynomial algebra of $V_3$, with basis $\{1, q_1(x_1, x_2), q_2(x_1, x_2)\}$, are identified by the system

$$\begin{cases}
q_1(x_1, x_2)q_1(x_1, x_2) \mod (x_1^2 - 1, x_2^3 - 1) = c_{110} + c_{111}q_1(x_1, x_2) + c_{112}q_2(x_1, x_2) \\
q_1(x_1, x_2)q_2(x_1, x_2) \mod (x_1^2 - 1, x_2^3 - 1) = c_{120} + c_{121}q_1(x_1, x_2) + c_{122}q_2(x_1, x_2) \\
q_2(x_1, x_2)q_2(x_1, x_2) \mod (x_1^2 - 1, x_2^3 - 1) = c_{220} + c_{221}q_1(x_1, x_2) + c_{222}q_2(x_1, x_2)
\end{cases}$$
where the constants with the third index equal to 0 are known by Theorem \( \Pi \). 

\( c_{110} = 4, c_{120} = 0, \) and \( c_{220} = 4 \).

To find the remaining 6 constants with the direct method we compute \( q_i(x_1, x_2)q_j(x_1, x_2) \mod (x_1^3 - 1, x_2^3 - 1) \) and subtract \( c_{ij1}q_1(x_1, x_2) + c_{ij2}q_2(x_1, x_2) \) + \( c_{ij0} \), obtaining three multivariate polynomials which must be identically zero

\[
\begin{align*}
4 + (2 - c_{112})x_1^2 x_2^2 + (1 - c_{111})x_1^2 x_2 + (1 - c_{111})x_1 x_2^2 + (2 - c_{112})x_1 x_2 & = 0 \\
(2 - c_{122})x_1^2 x_2^2 + (2 - c_{121})x_1^2 x_2 + (2 - c_{121})x_1 x_2^2 + (2 - c_{122})x_1 x_2 & = 0 \\
4 + (1 - c_{222})x_1^2 x_2^2 + (2 - c_{221})x_1^2 x_2 + (2 - c_{221})x_1 x_2^2 + (1 - c_{222})x_1 x_2 & = 0
\end{align*}
\]

From the first equation we obtain \( c_{111} = 1, c_{112} = 2 \), from the second equation we obtain \( c_{121} = 2, c_{122} = 2 \), and from the third equation \( c_{221} = 2, c_{222} = 1 \), which allows us to write the multiplication table with the coefficients of the linear combinations (the trivial multiplications by \( q_0(x_1, x_2) = 1 \) are not reported)

|            | \( q_0(x_1, x_2) \) | \( q_1(x_1, x_2) \) | \( q_2(x_1, x_2) \) |
|------------|---------------------|---------------------|---------------------|
| \( q_1(x_1, x_2)q_1(x_1, x_2) \) | 0                   | 2                   | 2                   |
| \( q_1(x_1, x_2)q_2(x_1, x_2) \) | 0                   | 2                   | 2                   |
| \( q_2(x_1, x_2)q_2(x_1, x_2) \) | 0                   | 2                   | 1                   |

Case 2: \( n = 4, s = 2 \); we have four cosets

\( A_1 = \{1, 2\} , A_2 = \{\alpha, 2\alpha\} , A_3 = \{2 + \alpha, 1 + 2\alpha\} , A_4 = \{2 + 2\alpha, 1 + \alpha\} , \)

and, correspondingly, the characteristic multivariate polynomials are

\( q_1(x_1, x_2) = x_1 + x_1^2 , q_2(x_1, x_2) = x_2 + x_2^2 , q_3(x_1, x_2) = x_1 x_2 + x_1^2 x_2 , q_4(x_1, x_2) = x_1^2 x_2^2 + x_1 x_2 . \)

The multiplication table can be conveniently written as a \( 4 \times 4 \) table, where rows and columns are orderly indexed by the polynomials \( q_i(x) \), and the entries are five-tuples of integers which are the five coefficients of the linear combinations

|            | \( q_1 \)       | \( q_2 \)       | \( q_3 \)       | \( q_4 \)       |
|------------|----------------|----------------|----------------|----------------|
| \( q_1 \)   | [2, 1, 0, 1, 0] | [0, 0, 1, 1, 0] | [0, 0, 1, 1]   | [0, 0, 1, 1]   |
| \( q_2 \)   | [0, 0, 0, 1, 1] | [2, 0, 1, 0]   | [0, 0, 1, 1]   | [0, 0, 1, 1]   |
| \( q_3 \)   | [0, 0, 1, 0, 1] | [0, 0, 1, 1]   | [2, 0, 0, 1]   | [0, 0, 1, 1]   |
| \( q_4 \)   | [0, 0, 1, 1, 0] | [0, 0, 1, 1]   | [0, 0, 1, 0]   | [2, 0, 0, 0, 1] |

For instance we have

\[
\begin{align*}
\{ q_1(x_1, x_2)q_1(x_1, x_2) \mod (x_1^3 - 1, x_2^3 - 1) & = 2 + q_1(x_1, x_2) \Rightarrow [2, 1, 0, 0, 0] \\
q_1(x_1, x_2)q_2(x_1, x_2) \mod (x_1^3 - 1, x_2^3 - 1) & = q_3(x_1, x_2) + q_4(x_1, x_2) \Rightarrow [0, 0, 0, 1, 1]
\end{align*}
\]
Example 2. Let \( m(z) = z^4 + z + 1 \) be a 4-degree primitive polynomial over \( \mathbb{F}_2 \). Let \( \alpha \) be a root of \( m(z) \), the 16 elements of \( \mathbb{F}_{16} \) are

\[
\begin{array}{c|c}
0 & 0 \\
1 & 1 \\
\alpha & \alpha \\
\alpha^2 & \alpha^2 \\
\alpha^3 & \alpha^3 \\
\alpha^4 & 1 + \alpha \\
\alpha^5 & \alpha + \alpha^2 \\
\alpha^6 & \alpha^2 + \alpha^3 \\
\alpha^7 & 1 + \alpha + \alpha^3 \\
\alpha^8 & 1 + \alpha^2 \\
\alpha^9 & \alpha + \alpha^3 \\
\alpha^{10} & 1 + \alpha + \alpha^2 \\
\alpha^{11} & \alpha + \alpha^2 + \alpha^3 \\
\alpha^{12} & 1 + \alpha + \alpha^2 + \alpha^3 \\
\alpha^{13} & 1 + \alpha^2 + \alpha^3 \\
\alpha^{14} & 1 + \alpha^3 \\
\end{array}
\]

In this case \( n \) may be 3 or 5; only \( n = 3 \) is considered, being fully illustrative.

Case: \( n = 3, s = 5 \); we have three cosets

\[
\begin{align*}
\mathcal{A}_1 &= \{1, \alpha^3, \alpha^2 + \alpha^3, \alpha + \alpha^3, 1 + \alpha + \alpha^2 + \alpha^3\} \\
\mathcal{A}_2 &= \{\alpha, 1 + \alpha, 1 + \alpha + \alpha^3, 1 + \alpha + \alpha^2, 1 + \alpha^2 + \alpha^3\} \\
\mathcal{A}_3 &= \{\alpha^2, \alpha + \alpha^2, 1 + \alpha^2, \alpha + \alpha^2 + \alpha^3, 1 + \alpha^3\}
\end{align*}
\]

and correspondingly three characteristic multivariate polynomials

\[
\begin{align*}
q_1(x_1, x_2, x_3, x_4) &= x_1 + x_4 + x_3x_4 + x_2x_4 + x_1x_2x_3x_4 \\
q_2(x_1, x_2, x_3, x_4) &= x_2 + x_1x_2 + x_1x_2x_4 + x_1x_2x_3 + x_1x_3x_4 \\
q_3(x_1, x_2, x_3, x_4) &= x_3 + x_2x_3 + x_1x_3 + x_2x_3x_4 + x_1x_4
\end{align*}
\]

Let \( x = [x_1, x_2, x_3, x_4] \), a basis of \( \mathcal{V}_4 \) is \( \{1, q_1(x), q_2(x), q_3(x)\} \), and the structure constants of the polynomial algebra can be computed from the following system of six equations

\[
\begin{align*}
q_1(x)q_1(x) &= c_{110} + c_{111}q_1(x) + c_{112}q_2(x) + c_{113}q_3(x) \\
q_1(x)q_2(x) &= c_{120} + c_{121}q_1(x) + c_{122}q_2(x) + c_{123}q_3(x) \\
q_1(x)q_3(x) &= c_{130} + c_{131}q_1(x) + c_{132}q_2(x) + c_{133}q_3(x) \\
q_2(x)q_2(x) &= c_{220} + c_{221}q_1(x) + c_{222}q_2(x) + c_{223}q_3(x) \\
q_2(x)q_3(x) &= c_{230} + c_{231}q_1(x) + c_{232}q_2(x) + c_{233}q_3(x) \\
q_3(x)q_3(x) &= c_{330} + c_{331}q_1(x) + c_{332}q_2(x) + c_{333}q_3(x)
\end{align*}
\]

Now \( c_{110} = c_{220} = c_{330} = 5 \), and \( c_{120} = c_{130} = c_{230} = 0 \), then we have to compute only 18 constants instead of 24. Proceeding as in the previous example we obtain all structure constants \( c_{ijk} \) and write the multiplication table where the coefficients of the linear combinations for \( q_i(x)q_j(x) \) are reported in the corresponding row (the trivial
multiplications by $q_0(x_1, x_2) = 0$ are not reported)

|     | $q_0(x)$ | $q_1(x)$ | $q_2(x)$ | $q_3(x)$ |
|-----|----------|----------|----------|----------|
| $q_1(x)q_1(x)$ | 5 | 0 | 2 | 2 |
| $q_1(x)q_2(x)$ | 0 | 2 | 2 | 1 |
| $q_1(x)q_3(x)$ | 0 | 2 | 1 | 2 |
| $q_2(x)q_2(x)$ | 5 | 2 | 0 | 2 |
| $q_2(x)q_3(x)$ | 0 | 1 | 2 | 2 |
| $q_3(x)q_3(x)$ | 5 | 2 | 2 | 0 |

3.2 A numerical method based on cyclotomic fields

Let $\mathbb{Q}(\zeta_p)$ be the cyclotomic field of $p$-th roots of unity, with $\zeta_p$ denoting a primitive root of unity, that is a root of the cyclotomic polynomial of degree $p - 1$. Thus $\mathbb{Q}(\zeta_p)$ is an extension of degree $p - 1$ of $\mathbb{Q}$. Let $\mathfrak{G}_p$ denote the multiplicative cyclic group generated by $\zeta_p$. Let $u = (\zeta_i^1, \zeta_i^2, \ldots, \zeta_i^m)$ denote an $n$-tuple of elements of $\mathfrak{G}_p$, thus from the evaluation of equation (1) for $x = u$, we get a polynomial in $\zeta$ that is equal to 0

$$c_{ij0} + \sum_{k=1}^{n} c_{ijk} q_k(u) - q_i(u)q_j(u) = 0 .$$

(2)

We thus obtain a system of $p$ linear equations with integral coefficients in $n$ unknowns. If $n \leq p$ a solution is easily obtained, since it certainly exists by Theorem 1. If $n > p$ we need more linear equations, then we consider the equations obtained using $\ell$ different vectors $u$, with the aim of getting $n$ linearly independent equations.

**Example 3.** Reconsider the problem of example 1. Its solutions by this second method are obtained working in $\mathbb{Q}(\zeta_3)$ with $\zeta_3$ a primitive complex cubic root of unity.

Take $u_0 = (1, 1)$, and $u_1 = (1, \zeta_3)$; in this case we obtain two equations using (2), considering that $c_{110} = 4$, $c_{120} = 0$, and $c_{220} = 4$, $q_1(1, 1) = q_2(1, 1) = 4$, $q_1(1, \zeta_3) = 1$, and $q_2(1, \zeta_3) = -2$. Thus we can write the system

$$\begin{cases} 4 + 4c_{111} + 4c_{112} &= 16 \\ 4 + c_{111} - 2c_{112} &= 1 \end{cases}$$

Solving for $c_{111}, c_{112}$ we obtain $c_{111} = 1, c_{112} = 2$.

Similarly, we obtain all structure constants summarized in the following table

|     | $q_1(x_1, x_2)$ | $q_2(x_1, x_2)$ |
|-----|----------------|----------------|
| $q_1(x_1, x_2)q_1(x_1, x_2)$ | 4 | 1 |
| $q_1(x_1, x_2)q_2(x_1, x_2)$ | 0 | 2 |
| $q_2(x_1, x_2)q_2(x_1, x_2)$ | 4 | 2 |

3.3 A new proof of Perron’s original observations

The history of the $\mathbb{F}_{p^m}^*$ partition by the fibers of a given character began with Perron’s characterization of the sets or quadratic residues and non-residues in prime fields, and several independent proofs have
since been given. A “new” proof is obtained by specializing the general results given above, and holds for every finite field of odd characteristic.
Consider the prime field \( F_p \), \( p \) odd, and the character \( \chi_2 \) of order 2 defined over \( F_p^* \). Let \( R \) and \( \mathcal{N} \) be the subsets of \( F_p^* \) of squares and non-squares, respectively, that is \( R = \chi^{-1}(1) \) and \( \mathcal{N} = \chi^{-1}(-1) \). The corresponding characteristic polynomials are
\[
q_R(x) = \sum_{\beta \in F_p^*} \frac{1 + \chi_2(\beta)}{2} \prod_{i=1}^{m} x_i^{b_i}, \quad q_{\mathcal{N}}(x) = \sum_{\beta \in F_p^*} \frac{1 - \chi_2(\beta)}{2} \prod_{i=1}^{m} x_i^{b_i},
\]
depending on whether \( \frac{p^{m-1}}{2} \) is odd or even, we have
\[
\begin{align*}
\frac{p^{m-1}}{2} \text{ odd } & \quad \begin{cases} q_R(x)q_R(x) \mod (x_1^p - 1, \ldots, x_m^p - 1) = 0 + a_{11}q_R(x) + b_{11}q_{\mathcal{N}}(x) \\
q_R(x)q_{\mathcal{N}}(x) \mod (x_1^p - 1, \ldots, x_m^p - 1) = \frac{p-1}{2} + a_{12}q_R(x) + b_{12}q_{\mathcal{N}}(x) \\
q_{\mathcal{N}}(x)q_{\mathcal{N}}(x) \mod (x_1^p - 1, \ldots, x_m^p - 1) = 0 + a_{22}q_R(x) + b_{22}q_{\mathcal{N}}(x)
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\frac{p^{m-1}}{2} \text{ even } & \quad \begin{cases} q_R(x)q_R(x) \mod (x_1^p - 1, \ldots, x_m^p - 1) = \frac{p-1}{2} + a_{11}q_R(x) + b_{11}q_{\mathcal{N}}(x) \\
q_R(x)q_{\mathcal{N}}(x) \mod (x_1^p - 1, \ldots, x_m^p - 1) = 0 + a_{12}q_R(x) + b_{12}q_{\mathcal{N}}(x) \\
q_{\mathcal{N}}(x)q_{\mathcal{N}}(x) \mod (x_1^p - 1, \ldots, x_m^p - 1) = \frac{p-1}{2} + a_{22}q_R(x) + b_{22}q_{\mathcal{N}}(x)
\end{cases}
\end{align*}
\]
Let \( u_o \) be the vector of all ones, then we have
\[
q_R(u_o) = \sum_{\beta \in F_p^*} \frac{1 + \chi_2(\beta)}{2} = \frac{p^{m-1}}{2}, \quad q_{\mathcal{N}}(u_o) = \sum_{\beta \in F_p^*} \frac{1 - \chi_2(\beta)}{2} = \frac{p^{m-1}}{2}
\]
\[
q_R(-u_o) = \sum_{\beta \in F_p^*} \frac{1 + \chi_2(\beta)}{2} (-1)^{\sum_{i=1}^{m} b_i} = t, \quad q_{\mathcal{N}}(-u_o) = \sum_{\beta \in F_p^*} \frac{1 - \chi_2(\beta)}{2} (-1)^{\sum_{i=1}^{m} b_i} = -t
\]
If \( q_R(-u_o) = 0 \), it is necessary to use a vector \( u \) different from \( -u_o \): there are \( 2^m - 2 \) possible choices for \( u \neq -u_o \), and one of them certainly works because of Theorem [3]

References

[1] E. Bach, J. Shallit, Algorithmic Number Theory, vol.1, Cambridge: MIT Press, 1996.
[2] B.C. Berndt, R.J. Evans, K.S. Williams, Gauss and Jacobi Sums, Wiley, New York, 1998.
[3] D.A. Cox, Galois Theory, Wiley, Hoboken, 2004.
[4] L. E. Dickson, Algebras and their Arithmetics, Dover, New York, NY, 1960.
[5] M. Elia, On a Problem of Perron, JPANTA, Volume 42, Number 2, 2019, pp. 255-266, ISSN: 0972-5555 http://dx.doi.org/10.17654/NT042020255
[6] A Frölich, M. Taylor, Algebraic Number Theory, Cambridge University Press, 1994.
[7] C.F. Gauss, Disquisitiones Arithmeticae, New York: Springer-Verlag, 1986.
[8] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford at the Clarendon Press, 1971.

[9] Hua Loo Keng, *Introduction to Number Theory*, New York: Springer, 1981.

[10] C. Monico, M. Elia, Note on an additive characterization of quadratic residues modulo $p$, *J. Comb. Inf. Syst. Sci.*, 31 (2006), pp. 209-215.

[11] C. Monico, M. Elia, An Additive Characterization of Fibers of Characters on $\mathbb{F}_p^*$, *Int. J. Algebra* 4 (2010), pp. 109-117.

[12] O. Perron, Bemerkungen über die Verteilung der quadratischen Reste, *Math. Z.* 56 (1952), pp. 122-130.

[13] B.L. van der Waerden, *Modern Algebra*, 2 vol., New York: Ungar, 1966.

[14] H. Weyl, *Algebraic Theory of Numbers*, Princeton University Press, Princeton, 1998.

[15] A. Winterhof, On the distribution of powers in finite fields, *Finite Fields Appl.* 4 (1998), pp. 43-54.