LINEAR INDEPENDENCE
of Covariant Derivatives and Space-Curvatures

Nenad O. Vesić

Abstract

The considerations about curvature tensors and pseudotensors for a non-symmetric affine connection space (see S. M. Minčić, [12–15], M. Prvanović [16]) are advanced in this paper. It is obtained which kinds of covariant derivatives, and identities of Ricci Type and curvatures of a non-symmetric affine connection space as well, are linearly independent. At the end of this research, curvature tensors of a special generalized Riemannian space are physically interpreted.

Key words: linear independence, affine connection, curvature tensor, energy-momentum tensor

Math. Subj. Classification: 53A55, 53B05, 53C15, 53Z05

1 Introduction

An \(N\)-dimensional manifold \(M_N\) equipped with an affine connection \(\nabla\) (with torsion) is the generalized affine connection space \(G\!\!\!A_N\) (see [6, 12–16, 19]). The affine connection coefficients of the affine connection \(\nabla\) are \(L^i_{jk}\) and they are non-symmetric by indices \(j\) and \(k\).

The symmetric and anti-symmetric parts of the coefficients \(L^i_{jk}\) are

\[
L^i_{2jk} = \frac{1}{2} \left( L^i_{jk} + L^i_{kj} \right) \quad \text{and} \quad L^i_{\tilde{jk}} = \frac{1}{2} \left( L^i_{jk} - L^i_{kj} \right). \tag{1.1}
\]

The symmetric parts \(L^i_{2jk}\) are the affine connection coefficients of the corresponding symmetric affine connection \(\nabla^0\) because they satisfy the corresponding transformation rule \([11, 18]\). The affine connection space \(A_N\) equipped with affine connection \(\nabla^0\) is the associated space (of the space \(G\!\!\!A_N\)). The anti-symmetric part \(L^i_{\tilde{jk}}\) is equal to a half of the torsion tensor for the space \(G\!\!\!A_N\).

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1.1 Symmetric affine connection space

As we said above, the N-dimensional manifold $\mathcal{M}_N$ equipped with a symmetric affine connection $0\nabla$ is the symmetric affine connection space $A_N$ (see [11,15]). One kind of covariant differentiation with respect to the affine connection $0\nabla$ exists

$$a_{j|k}^i = a_{j,k}^i + L_{ak}^i a_j^a - L_{jk}^a a^i_a,$$  

(1.2)

In the last equation, the object $a_j^j$ is a tensor of the type $(1,1)$ and the partial derivative $\partial/\partial x^k$ is denoted by comma.

It is founded one Ricci Type identity with respect to the affine connection $0\nabla$. Moreover, it exists one curvature tensor of the space $A_N$ $R_{jmn}^i = L_{jm,n}^i - L_{jn,m}^i + L_{jm}^\alpha L_{\alpha mn}^i - L_{jn}^\alpha L_{\alpha m}^i$. 

(1.3)

Many authors have promoted the theory of symmetric affine connection spaces. Some of them are J. Mikeš and his research group [11], N. S. Sinyukov [18] and many others.

1.2 Non-symmetric affine connection space

An $N$-dimensional manifold equipped with a non-symmetric affine connection $\nabla$ is the non-symmetric affine connection space $GA_N$ (see [2,12–16,19]). Unlike in the case of a symmetric affine connection space, there are defined four kinds of covariant derivatives $2,12–16,19$

$$a_{j|k}^i = a_{j,k}^i + L_{ak}^i a_j^a - L_{jk}^a a^i_a,$$  

(1.4)

With respect to these kinds of covariant derivatives, it is obtained four curvature tensors, eight derived curvature tensors and fifteen curvature pseudotensors $2,12–15,19$ of the space $GA_N$. In the papers about non-symmetric affine connection spaces, authors have cited and confirmed the following statements $12,13,15$

Statement 1: In the set of twelve curvature tensors of the space $GA_N$ with non-symmetric affine connection $\nabla$, there are five independent ones, while the rest can be expressed in terms of these five tensors and the curvature tensor of the associated space.

Statement 2: All fifteen curvature pseudotensors are given in the terms of five of them, five curvature tensors of the space $GA_N$ and the curvature tensor of the associated space $A_N$.

Statement 3: Curvature pseudotensors have the role and the form of the curvature tensors.

The role of the curvature tensor $1.3$ of the symmetric affine connection space $A_N$ is to uniquely generate the curvature $a_{j|m|n}^i - a_{j|m|n}^i$ of this space. In other words, the geometrical object $a_{j|m|n}^i R_{a|n|m}^i - a_{j|n|m}^i R_{a|n|m}^i$ is the only component of this curvature.

In this paper, we are aimed to find all components for the curvatures $a_{j|m|n}^i - a_{j|n|m}^i$, $p,q,r,s \in \{1,2,3,4\}$, of the space $GA_N$. Moreover, we will obtain and prove what are roles of
the curvature tensors, torsion tensor and curvature pseudotensors with respect to the curvatures of the space $GA_N$ in this study.

1.2.1 Generalized Riemannian space

Special kind of affine connection spaces with torsion are the generalized Riemannian spaces $GR_N$ (in the sense of Eisenhart [7, 8]). These spaces are manifolds $M_N$ equipped with a non-symmetric metric tensor $g_{ij}$.

The symmetric and anti-symmetric part of the tensor $g_{ij}$ are

$$g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) \quad \text{and} \quad g_{ij}^\lor = \frac{1}{2}(g_{ij} - g_{ji}).$$ (1.5)

We assume that the matrix $[g_{ij}]_{N \times N}$ is non-singular. The contravariant symmetric metric tensor $g^{ij}$ is obtained by inverting the matrix $[g_{ij}]$. Hence, it holds the equality $g^{i\alpha}g_{j\alpha} = \delta^i_j$, for the Kronecker delta-symbols $\delta^i_j$.

The affine connection coefficients of the space $GR_N$ are the generalized Christoffel symbols [7, 8]

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\alpha}(g_{j\alpha,k} - g_{jk,\alpha} + g_{\alpha k,j}).$$ (1.6)

The symmetric part of the coefficient $\Gamma^i_{jk}$, $\Gamma^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})$, is the corresponding Christoffel symbol. The anti-symmetric part of this symbol equal to a half of the torsion tensor of the space $GR_N$, i.e. $\Gamma^i_{j\lor} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$, is

$$\Gamma^i_{j\lor} = \frac{1}{2}g^{i\alpha}(g_{j\alpha,k} - g_{j\alpha,k} + g_{\alpha k,j}).$$ (1.7)

1.3 Motivation

A. Einstein [3–5] was the first who applied the complex metrics (equivalent with the non-symmetric metrics) in his research about the Unified Field Theory. The affine connection coefficients in Einstein’s works are not explicit functions of non-symmetric metrics but they satisfy the Einstein Metricity Condition

$$g_{ij,k} - \Gamma^i_{ik}g_{oj} - \Gamma^o_{kj}g_{io} = 0.$$ (1.8)

With respect to Einstein’s work, the symmetric parts $\Gamma^i_{jk}$ of the affine connection coefficients refer to gravity but the anti-symmetric parts $\Gamma^i_{j\lor}$ are important for researching in the theory of electromagnetism.

S. Ivanov and M. Lj. Zlatanović [10] contributed to the theory of differential geometry motivated by Einstein’s work.

Motivated by the Einstein’s considerations [3–5] and Eisenhart’s results [6–8], many authors have developed the theory of non-symmetric affine connection spaces [2, 12, 16, 19] and many others. The identities of Ricci Type in their papers are confirmations of the corresponding
results initially presented in [12, 13]. We are aimed to computationally complete the derivation of the Ricci Type identities in this paper.

The main purposes of the following research are:

a. To examine how many covariant derivatives of the covariant derivatives \((1.2, 1.4)\) may be expressed as the linear combinations of the rest ones,

b. To find linearly independent identities of Ricci type (linearly independent curvatures) for the space \(\mathbb{G}A_N\),

c. To find linearly independent curvatures of the space \(\mathbb{G}A_N\),

d. To interpret the curvature tensors of the space \(\mathbb{G}R_4\) with respect to cosmology.

2 Kinds of covariant derivatives: four plus one but three

By the equations \((1.2, 1.4)\), the covariant derivatives are expressed in the corresponding vector forms. The summands at the right sides of the equalities in these equations are the components of the corresponding covariant derivatives (treated as vectors). Analogously as in the case of curvature tensors [12–15, 19], we are aimed to search how many of these vectors are linearly independent at the start of this section. After that, we will obtain the linearly independent identities of Ricci Type and the curvature tensors from these identities.

With respect to the equations \((1.2, 1.4)\), we get:

\[
\begin{align*}
  a_{j|k}^i &= \frac{1}{2} a_{j|k}^i + \frac{1}{2} a_{j|k}^2, \\
  a_{j|k}^i &= \frac{1}{2} a_{j|k}^3 + \frac{1}{2} a_{j|k}^4, \\
  a_{j|k}^i &= 2 a_{j|k}^1 - a_{j|k}^2, \\
  a_{j|k}^i &= -a_{j|k}^2 + a_{j|k}^3 + a_{j|k}^4, \\
  a_{j|k}^i &= 2 a_{j|k}^2 - a_{j|k}^3, \\
  a_{j|k}^i &= -a_{j|k}^1 + a_{j|k}^2 + a_{j|k}^4, \\
  a_{j|k}^i &= 2 a_{j|k}^3 - a_{j|k}^4, \\
  a_{j|k}^i &= a_{j|k}^1 + a_{j|k}^2 - a_{j|k}^3, \\
  a_{j|k}^i &= 2 a_{j|k}^1 - a_{j|k}^3, \\
  a_{j|k}^i &= a_{j|k}^1 + a_{j|k}^2 - a_{j|k}^3.
\end{align*}
\]

(2.1) for the tensor \(a_{j}^i\) of the type \((1, 1)\).
Based on the equations (2.1 - 2.10), we conclude that the covariant derivatives \( a^i_{jk} \) and \( a^i_{jk} \) may be expressed as the linear combinations of the covariant derivatives \( a^i_{jk}, a^i_{jk}, a^i_{jk} \). As the tensors, the covariant derivatives \( a^i_{jk}, a^i_{jk} \) and \( a^i_{jk} \) are linearly independent what may be checked in the standard way.

For this reason, three kinds of covariant derivatives whose corresponding linear combinations express the rest two kinds of covariant derivatives will be called the linearly independent (kinds of) covariant derivatives.

Therefore, it holds the following theorem:

**Theorem 2.1.** Three of the covariant derivatives \( a^i_{jk}, a^i_{jk}, a^i_{jk}, a^i_{jk}, a^i_{jk} \) are linearly independent.

**Corollary 2.1.** The triples of linearly independent covariant derivatives \( a^i_{jk}, a^i_{jk}, a^i_{jk}, a^i_{jk} \) are

\[
\begin{align*}
  b_1 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_2 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_3 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_4 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_5 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_6 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_7 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\}, \\
  b_8 &= \{a^i_{jk}, a^i_{jk}, a^i_{jk}\},
\end{align*}
\]

unlike the covariant derivatives in the triples \( \{a^i_{jk}, a^i_{jk}, a^i_{jk}\} \) and \( \{a^i_{jk}, a^i_{jk}, a^i_{jk}\} \).

### 2.1 Identities of Ricci Type

Ricci-Type identities with respect to non-symmetric affine connection are obtained in many papers. They are initially founded in (S. M. Minčić, [12 13]). After that, many authors confirmed these identities for different non-symmetric affine connection spaces (see the papers [2 14 15 19]).

In the last cited papers, the authors combined the first and the second kind of covariant derivatives ([14]) together with one identity with respect to the third and the fourth kind of covariant derivatives. Because the covariant derivatives \( \{1,2,3\} \) are linearly independent, we will complete the computations about Ricci-Type identities with respect to the linearly independent covariant derivatives \( \{1,2,3\} \). Moreover, we will simplify the obtained identities of Ricci Type obtained in [2 12 15 19].

The double covariant derivatives of the tensor \( a^i_j \) of the type \((1,1)\) with respect to these three linearly independent ones are

\[
\begin{align*}
  a^i_{jk}\mid_{mn} &= a^i_{jm,n} - L_{\alpha mn} a^\alpha_{j,n} + L_{\alpha mn} a^\alpha_{j,m} + L_{\alpha mn} a^\alpha_{j,a} + L_{\alpha mn} a^\alpha_{j,m} + L_{\alpha mn} a^\alpha_{j,a} \\
  &+ a^i_a \left( L_{\alpha mn} \alpha_{j,n} + L_{\beta mn} \beta_{j,n} - L_{\beta mn} \beta_{j,m} \right) - a^i_{\alpha m,n} - L_{\beta mn} \beta_{j,m} - L_{\beta mn} \beta_{j,a} \\
  &+ a^i_\beta \left( L_{\alpha mn} \beta_{j,m} + L_{\alpha mn} \beta_{j,a} \right),
\end{align*}
\]  

(2.11)
\[ a_{j,m}^i = a_{j,m}^i - L_{nm}^\alpha a_{a,m}^i - L_{jm}^\alpha a_{a,n}^i - L_{ma}^\alpha a_{j,m}^i + L_{na}^\alpha a_{j,n}^i + a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) - a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) \]
\[ - a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) + a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) \]
\[ a_{j,m}^i = a_{j,m}^i - L_{nm}^\alpha a_{a,m}^i - L_{jm}^\alpha a_{a,n}^i - L_{ma}^\alpha a_{j,m}^i + L_{na}^\alpha a_{j,n}^i + a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) - a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) \]
\[ - a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) + a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) \]
\[ a_{j,m}^i = a_{j,m}^i - L_{nm}^\alpha a_{a,m}^i - L_{jm}^\alpha a_{a,n}^i - L_{ma}^\alpha a_{j,m}^i + L_{na}^\alpha a_{j,n}^i + a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) - a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) \]
\[ - a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) + a_{j,m}^i (L_{nm}^\alpha L_{jn}^\beta - L_{nm}^\beta L_{jn}^\alpha) \]

We conclude that there are \( 3 \cdot 3 \cdot 3 \cdot 3 = 81 \) identities of Ricci Type obtained from the differences \( a_{j,m}^i - a_{j,m}^i \) for \( p, q, r, s \in \{1, 2, 3\} \).

After using the equality \( L_{jk}^i = L_{jk}^i + L_{jk}^i \), one gets

\[ L_{jm,n}^i = L_{jm,n}^i + L_{jm,n}^i \quad \text{and} \quad L_{jk}^i L_{bc}^i = L_{jk}^i L_{bc}^i + L_{jk}^i L_{bc}^i + L_{jk}^i L_{bc}^i + L_{jk}^i L_{bc}^i. \]
the next theorem and its corollaries are satisfied. We will prove this theorem for the case of
\( p = r = 1 \) and \( q = s = 2 \) in the subsection 2.3. All other identities of the Ricci Type from
the next theorem may be proved analogously.

**Theorem 2.2.** The family of Ricci-Type identities with respect to the covariant derivatives \( 1 \)
\( 2, 3 \)

\[ a^i_{j|m|n} - a^i_{j|m|n} = 2\hat{c}_1 L^\alpha_{\beta} a^i_{\alpha|n} + 2\hat{c}_2 L^\alpha_{\beta} a^i_{\alpha|n} + 2\hat{c}_3 L^\alpha_{\beta} a^i_{\alpha|n} + 2\hat{c}_4 L^\alpha_{\beta} a^i_{\alpha|n} + 2\hat{c}_5 L^\alpha_{\beta} a^i_{\alpha|n} \\
+ \hat{c}_6 (R^i_{\alpha|m} + \hat{c}_7 L^\beta_{\alpha|n} + \hat{c}_8 L^\beta_{\alpha|n} + \hat{c}_9 L^\beta_{\alpha|n} + \hat{c}_{10} L^\beta_{\alpha|n}) \\
- \hat{c}_a (R^i_{\alpha|m} + \hat{c}_1 L^\beta_{\alpha|n} + \hat{c}_2 L^\beta_{\alpha|n} + \hat{c}_3 L^\beta_{\alpha|n} + \hat{c}_4 L^\beta_{\alpha|n} + \hat{c}_5 L^\beta_{\alpha|n} + \hat{c}_6 L^\beta_{\alpha|n}) \\
- 2\hat{c}_a (c_{16} L^\beta_{\alpha|n} + \hat{c}_1 L^\beta_{\alpha|n}) \tag{2.21} \]

for \( p, q, r, s \in \{1, 2, 3\} \), the covariant derivative with respect to the symmetric affine connection
denoted by \( 1 \) and the corresponding coefficients \( \hat{c}_k \in \{0, 1, -1\} \), \( k = 1, \ldots, 17 \).

**Corollary 2.2.** Seventeen of the identities from the family (2.21) are linearly independent. Some
seventeen of these linearly independent Ricci-Type identities are given by the equations (5.1—
5.17) in the Appendix I. All other Ricci-Type identities are the corresponding linear combinations
of the linearly independent ones.

**Corollary 2.3.** The family (2.21) and the families

\[ a^i_{j|m|n} - a^i_{j|m|n} = 2\hat{c}_1 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_2 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_3 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_4 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_5 d^i_{\alpha|n} a^i_{\alpha|n} \\
+ 2\hat{c}_6 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_7 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_8 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_9 d^i_{\alpha|n} a^i_{\alpha|n} + 2\hat{c}_{10} d^i_{\alpha|n} a^i_{\alpha|n} \\
+ \hat{c}_a (R^i_{\alpha|m} + \hat{c}_7 L^\beta_{\alpha|n} + \hat{c}_8 L^\beta_{\alpha|n} + \hat{c}_9 L^\beta_{\alpha|n} + \hat{c}_{10} L^\beta_{\alpha|n}) \\
- \hat{c}_a (R^i_{\alpha|m} + \hat{c}_1 L^\beta_{\alpha|n} + \hat{c}_2 L^\beta_{\alpha|n} + \hat{c}_3 L^\beta_{\alpha|n} + \hat{c}_4 L^\beta_{\alpha|n} + \hat{c}_5 L^\beta_{\alpha|n} + \hat{c}_6 L^\beta_{\alpha|n}) \\
- 2\hat{c}_a (c_{16} L^\beta_{\alpha|n} + \hat{c}_1 L^\beta_{\alpha|n}) \tag{2.22} \]

of the identities of Ricci Type, for real constants \( d^1_k, d^2_k, d^3_k, d^4_k + d^5_k = 1, d^6_k = \mathbb{R}, k = 1, \ldots, 5, l = 1, 2, 3 \), are equivalent.

With respect to the definition of the curvature tensor for the space \( GA_N (R(X; Y, Z) = \\
\nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X + \nabla_{[Y, Z]} X) \), we conclude that the right sides of the identities of Ricci Type
(2.21) are the curvatures for the space \( GA_N \) expressed in the standard base.
Remark 2.1. The covariant derivatives \((2.1, 2.4)\) are expressed as vectors. We proved that three of these five vectors are linearly independent. For this reason, the double covariant derivatives \((2.11, 2.19)\) are vectors. Finally, the right sides of the corresponding identities of Ricci Type are treated as vectors. For this reason, the left and right sides of the Ricci Type identities are obtained as the linear combinations of the corresponding sides in the equations \((5.1, 5.17)\). Hence, when we say that some Ricci Type identities are linearly (in)dependent we think that for the corresponding equalities.

Note also that the equation \((2.22)\) holds with respect to the equalities

\[
a^i_{jk} = a^i_{jk} + L^i_{\alpha k}a^\alpha_j - L^i_{\alpha k}a^\alpha_j, \quad a^i_{j|k} = a^i_{|jk} - L^i_{\alpha k}a^\alpha_j + L^i_{\alpha j}a^\alpha_k, \quad a^i_{j|k} = a^i_{j|k} + L^i_{\alpha k}a^\alpha_j + L^i_{\alpha j}a^\alpha_k,
\]

substituted into the equation \((2.21)\). In the same manner, we may change the covariant derivatives \(L^i_{jm|n}\) to linear combinations of the covariant derivatives \(L^i_{jm|n}, L^i_{jm|n}, L^i_{jm|n}\), but we preserved the covariant derivative \(L^i_{jm|n}\) in the equations \((2.21, 2.22)\) with respect to the results from the previously published papers about linearly independent curvature tensors. Because linear combinations of tensors are tensors, such as the linear combinations of the covariant derivatives \(L^i_{jm|n}, \ldots, L^i_{jm|n}\) as well, this transformation would not change the tensor characteristics of the curvature tensors for the space \(GA_N\). The curvature pseudotensors are lost through the computation and they composed by the tensor \(a^i_j\) are not components of the curvatures for the space \(GA_N\).

In the rows above, we obtained that curvature tensors for the non-symmetric affine connection space \(GA_N\) together with its torsion tensor are enough to express the family of curvatures for this space.

We will discuss the curvature characteristics of pseudocurvature tensors obtained in \([2, 12, 15, 19]\) in the subsection 2.3.

2.2 Linearly independent curvatures

From the linearly independent Ricci-Type identities, it is obtained fourteen curvature tensors. They are given by the equations \((6.1, 6.14)\) from the Appendix II. As in the previous works \([2, 12, 15, 19]\) and many others, these tensors are elements of the family

\[
\rho^i_{jmn} = R^i_{jmn} + uL^i_{jm|n} + u'L^i_{jm|n} + vL^\alpha_{jmn}L^i_{\alpha\nu} + v'L^\alpha_{jmn}L^i_{\alpha\nu} + wL^\alpha_{jmn}L^i_{\alpha\nu},
\]

for the corresponding coefficients \(u, u', v, v', w\).

It holds the following theorem and its corollary.

**Theorem 2.3.** Six of the curvature tensors from the family \((2.23)\) are linearly independent.

**Corollary 2.4.** The curvature tensors \(\rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}\), given by the equations \((6.1, 6.2, 6.3, 6.4, 6.7, 6.10)\) in the Appendix II, are linearly independent.

The curvature tensors \(R^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}, \rho^i_{jmn}\), given by the equation \((1.3)\) and the equations \((6.2, 6.3, 6.4, 6.7, 6.10)\), in the Appendix II, are linearly independent.
The curvature tensors $\frac{1}{6}j_{mn}, \frac{1}{2}j_{mn}, \frac{1}{3}j_{mn}, \frac{1}{4}j_{mn}, \frac{1}{7}j_{mn}, R_{jmn}$, given by the equations (6.1), (6.2), (6.3), (6.4), (6.7), in the Appendix II, and the equation (1.3), are linearly independent.

2.3 Curvature pseudotensors

It is obtained in [12,13], and confirmed in many papers after (see for example [2,14,15,19]), the existence of the fifteen curvature pseudotensors for the space $G_{A_N}$.

The main characteristic of the fourteen identities of Ricci Type where the curvature pseudotensors are obtained is that they depend of partial derivatives of the tensor $a^i_j$.

If we substitute the definition (1.2) of the covariant derivative with respect to symmetric affine connection into the equation (2.21), we will obtain the equation

\[
\begin{align*}
\frac{a^{i}_{mj\alpha m} - a^{i}_{nj\alpha m}}{p - q} &= \frac{1}{c_1}L_{\alpha m}a_{\alpha, n} + \frac{1}{c_2}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_3}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_4}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_5}L_{\alpha m}a_{\alpha, m} \\
&+ \frac{1}{c_6}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_7}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_8}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_9}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_{10}}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_{11}}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_{12}}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_{13}}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_{14}}L_{\alpha m}a_{\alpha, m} + \frac{1}{c_{15}}L_{\alpha m}a_{\alpha, m}.
\end{align*}
\]

(2.24)

Let us consider the geometrical objects $a^{i}_{j<mn>}, a^{i}_{j<mn>}, a^{i}_{j<mn>}, a^{i}_{j<mn>}, a^{i}_{j<mn>}$, defined in [12,13] and used in later papers. We also need to use the equality $L_{jk} = L_{jk} + L_{jk}$ in the following computations.

Hence, one gets

\[
\begin{align*}
a^{i}_{j<mn>} &= L_{\alpha m}a_{\alpha, m} - L_{\alpha m}a_{\alpha, m}, \\
a^{i}_{j<mn>} &= (L_{\alpha m}L_{\beta j} - L_{\alpha m}L_{\beta j})a_{\beta}, \\
a^{i}_{j<mn>} &= (L_{\alpha m}L_{\beta j} - L_{\alpha m}L_{\beta j})a_{\beta}, \\
a^{i}_{j<mn>} &= (L_{\alpha m}L_{\beta j} - L_{\alpha m}L_{\beta j})a_{\beta}, \\
L_{\alpha mn}^{a^{i}_{j<mn>}} &= L_{\alpha mn}^{a^{i}_{j<mn>}}(a^{i}_{j,\alpha} + L_{\beta \alpha}a^{i}_{j} - L_{\alpha \beta}a^{i}_{j}) + L_{\alpha mn}^{a^{i}_{j<mn>}}(L_{\beta \alpha}a^{i}_{j} - L_{\alpha \beta}a^{i}_{j}), \\
L_{\alpha mn}^{a^{i}_{j<mn>}} &= L_{\alpha mn}^{a^{i}_{j<mn>}}(a^{i}_{j,\alpha} + L_{\beta \alpha}a^{i}_{j} - L_{\alpha \beta}a^{i}_{j}) + L_{\alpha mn}^{a^{i}_{j<mn>}}(L_{\beta \alpha}a^{i}_{j} - L_{\alpha \beta}a^{i}_{j}).
\end{align*}
\]

(2.25)
After recognizing the geometrical objects \(2.25 - 2.29\) in the equation \(2.24\) and adding the left sides of the equations \(2.30, 2.31\) but subtracting the right sides of them from the brackets in the second, third, fourth and fifth row of this equation, one obtains all the curvature pseudotensors searched in the papers \([2, 12–15, 19]\).

The curvatures of the space \(\mathcal{G}A_N\) are the right sides of the equations \(2.21\), for different \(p, q, r, s\). With respect to these equations, we are able to obtain the curvature \(a_{ij|n}^m - a_{ij|m}^n\) and the curvature tensor \(R_{i\;jmn}^\alpha a_{\alpha}^i - R_{i\;jmn}^\alpha a_{\alpha}^i\). For this reason, any of them is the component of curvature for the space \(\mathcal{G}A_N\).

No one of the summands \(L^1_{jk}L_{p\;q}^m a_{v|c}^u\), \(d = 1, \ldots, 5\) from the equation \(2.21\) may be obtained as the corresponding linear combination of the differences \(a_{ij|n}^m - a_{ij|m}^n\) and the curvature \(R_{i\;jmn}^\alpha a_{\alpha}^i - R_{i\;jmn}^\alpha a_{\alpha}^i\). That means that these summands indirectly generate the curvatures of the space \(\mathcal{G}A_N\). The curvature tensors together with the torsion tensor uniquely determine the curvature of the space \(\mathcal{G}A_N\). The curvature pseudotensors do not have the same role as the curvature tensors for the space \(\mathcal{G}A_N\).

The same statement holds for the family \(2.22\).

To point, curvature pseudotensors of the space \(\mathcal{G}A_N\) may anticipate linear combinations of components of curvatures for this space. However, they are linear combinations of one complete and one incomplete component of this curvature. For this reason, we need to solve a system of equations to find the curvature with respect to curvature pseudotensors.

### 3 EXAMPLE: Application of curvature tensors of space \(\mathcal{G}A_N\)

Matthias Blau recalled different published results to gather different findings in the field of cosmology (see [1]). From the other side, I. Shapiro [17] and many other authors have studied cosmology with respect to torsion. We will correlate these results below.

Let us consider the generalized Riemannian spacetime \(\mathbb{G}R_4\) equipped by the metric

\[
(b_{ij}) = \begin{bmatrix}
    s_1(t) & 0 & 0 & 0 \\
    0 & s_2(t) & n(t) & 0 \\
    0 & -n(t) & s_3(t) & 0 \\
    0 & 0 & 0 & s_4(t)
\end{bmatrix},
\]

for the differentiable functions \(s_1(t), \ldots, s_4(t), n(t)\).

The symmetric and anti-symmetric parts of the metric tensor \(b_{ij}\) are

\[
(b_{ij}) = \begin{bmatrix}
    s_1(t) & 0 & 0 & 0 \\
    0 & s_2(t) & 0 & 0 \\
    0 & 0 & s_3(t) & 0 \\
    0 & 0 & 0 & s_4(t)
\end{bmatrix}
\]

and

\[
(b_{ij}) = \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & n(t) & 0 \\
    0 & -n(t) & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}.
\]
The covariant anti-symmetric part of the corresponding generalized Christoffel symbol is
\[ \Gamma_{1.23} = -\Gamma_{1.32} = -\Gamma_{2.13} = \Gamma_{2.31} = \Gamma_{3.12} = -\frac{1}{2}n'(t), \quad (3.3) \]
and \[\Gamma_{i,jk} = 0\] in all other cases.

3.1 Motivation from cosmology

In the Blau’s book [1], it is analyzed the Einstein-Hilbert action
\[ S = \int d^4x \sqrt{|b|} (R + \mathcal{L}_M), \quad (3.4) \]
for the scalar curvature \( R = g^{\alpha\beta}R_{\alpha\beta\gamma} \) of the Riemannian space \( \mathbb{R}_4 \), the metric determinant \( b = \det(b_{ij}) \) and part \( \mathcal{L}_M \) describing any matter fields appearing in the theory.

With respect to the equation (3.4), it is obtained the Einstein’s equations of motion
\[ R_{ij} - \frac{1}{2}Rb_{ij} = T_{ij}, \quad (3.5) \]
for the energy-momentum tensor \( T_{ij} \). The Energy-Momentum Tensors \( T_{ij} \) with respect to different parts \( \mathcal{L}_M \) are recalled in [1].

From the other side, I. Shapiro [17] have studied the cosmology with respect to torsion. The results in Shapiro’s article are equivalent to the results from [1]. The Shapiro’s findings are correlated with torsion unlike the results in [1].

Our purpose is to recognize some of the expressions from the Blau’s and Shapiro’s works with respect to the curvature tensors of the above defined generalized spacetime \( \mathbb{GR}_4 \) in this section.

3.2 Physical interpretation of curvature tensors and torsion

The family \( R = b^{\alpha\beta\gamma\delta}\Gamma_{\alpha.\beta.\gamma}\Gamma_{\beta.\gamma.\delta}, (3.6) \)
with respect to the corresponding coefficients \( v' \) and \( w \).

Based on the considerations from the Shapiro’s work ([17], section 2), we conclude that the family of the Einstein-Hilbert actions with torsion is
\[ \bar{S} = \int d^4x \sqrt{|b|} (R + (v' - w)\frac{\partial R}{\partial b_{ij}}\Gamma_{\alpha.\beta.\gamma}\Gamma_{\beta.\gamma.\delta}, \quad (3.7) \]
for the further work, we need to recall the term of the functional derivative [9].
Consider the functional

\[ J[f] = \int_a^b L[f(t), f'(t)] dt, \quad (3.9) \]

for \( f'(t) = df(t)/dt \). The variational derivative of the functional \( J[f] \) by the function \( f(t) \) is

\[ \frac{\delta J[f]}{\delta f(t)} = \int dt \left( \frac{\partial L[f(t), f'(t)]}{\partial f(t)} - \frac{d}{dt} \frac{\partial L[f(t), f'(t)]}{f'(t)} \right) = \lim_{\varepsilon \to 0} \frac{J[f(t) + \varepsilon \delta(x - t)] - J[f]}{\varepsilon}, \quad (3.10) \]

for the Dirac \( \delta \)-function \( \delta(x) \) and \( x \neq t \). As in [9], the limit \( \varepsilon \to 0 \) has to be taken first, before other limiting operations.

The variation of the functional \( J[f] \) is

\[ \delta J[f] = \int \frac{\delta J[f]}{\delta f(t)} \delta f(t) dt. \quad (3.11) \]

- Note the following equalities

\[ \frac{\delta \{ F[f] \cdot G[f] \}}{\delta f(t)} = \frac{\delta F[f]}{\delta f(t)} \cdot G[f] + \frac{\delta G[f]}{\delta f(t)} \cdot F[f], \]

\[ \frac{\delta F[G[f]]}{\delta f(t)} = \int dt \frac{\delta F[G]}{\delta G(t)} \cdot \frac{\delta G[f]}{\delta f(t)}, \]

\[ \frac{\delta F[f]}{\delta C} = 0, \]

for a constant function \( f(t) = C \).

Based on the equations [3.12, 3.13, 3.18], we get

\[ \mathcal{L}_M = \frac{3(v' - w)}{2} \cdot \frac{(n'(t))^2}{s_1(t)s_2(t)s_3(t)}, \quad (3.15) \]

It is evident that the components

\[ n_{1,2}(t) = \pm \frac{2}{3(v' - w)} \int \sqrt{\mathcal{L}_M \cdot s_1(t)s_2(t)s_3(t)} dt, \quad (3.16) \]

of the anti-symmetric part \( h_{ij} \) for the metric tensor \((3.1)\) correspond to the same operator \( \mathcal{L}_M \).

Geometrically, the operator \( \mathcal{L}_M \) generates two generalized Riemannian spaces \( \text{GR}_4^+ \) and \( \text{GR}_4^- \) in the sense of Eisenhart’s definitions [7, 8]. Moreover, the operator \( \mathcal{L}_M \) generates two opposite torsion tensors \( T_{ijk}^j = 2T_{ijk}^j \) and \( T_{ijk}^j = -2T_{ijk}^j = -T_{ijk}^j \).

If \( n_1(t) = n_2(t) \) in the equation (3.16), we get \( \mathcal{L}_M = 0 \). Equivalently, the equality \( n_1(t) = n_2(t) \) corresponds to the no matter part of the space. The physical considerations about the cases of \( \mathcal{L}_M = 0 \) are geometrically covered by the corresponding Riemannian spaces \( \text{R}_4 \).
After vanishing the variation (3.11) of the Einstein-Hilbert action (3.3) and using the equation (3.8), we get the family of Energy-Momentum Tensors \( T_{ij} = -2\delta L_M/\delta g_{ij} + b_{ij}L_M \), i.e.

\[
T_{ij} = -3(v' - w)\frac{\delta \left( \frac{u'^2(t)}{s_1(t)s_2(t)s_3(t)} \right)}{\delta b_{ij}} + \frac{3(v' - w)}{2} b_{ij} \frac{\delta (u'^2(t))}{\delta s_1(t)s_2(t)s_3(t)},
\]

with respect to the family of scalar curvatures (3.6) of the space \( \mathbb{G}_4 \).

For researches in physics, it is the most common to examine the case of \( v' - w = 1 \) for the coefficients \( v' \) and \( w \) in the equation (3.6).

4 Conclusion

We achieved the aims of this paper above.

In the section 2, we proved that three kinds of covariant derivatives are enough to be defined for the complete analysis of a non-symmetric affine connection. Moreover, we obtained that there are seventeen linearly independent Ricci-Type identities and six linearly independent curvature tensors of the non-symmetric affine connection space \( \mathbb{G}_4 \). In that section, it is explained why the curvature pseudotensors are not components of curvatures for this space.

In the section 3, we physically interpreted the curvature tensors obtained in the section 2. Namely, we founded that the anti-symmetric parts of affine connection coefficients correspond to matter and obtained the corresponding family of Energy-Momentum Tensors. The results obtained in that section motivate the author to find the general formulae for pressure, energy-density and state parameter of a cosmological fluid.

In the future, we are aimed to generalize the results from the previous paper about invariants of geometric mappings. Moreover, we will try to apply differential geometry in physics, specially in cosmology, more detail than in this paper.

5 Appendix I: Linearly independent identities of Ricci Type

\[
a^i_{j m n} - a^i_{j |n|m} = -2L^\alpha_{m n} a^j_{|\alpha|} + a^\alpha_{j} R^i_{\alpha \alpha m n} + L^i_{\alpha \alpha m n} - L^j_{\alpha \alpha m n} + L^j_{\alpha \alpha m n} L^i_{\alpha \alpha m n} - L^\alpha_{\alpha \alpha m n} L^i_{\alpha \alpha m n} - 2L^\alpha_{\alpha \alpha m n} L^i_{\alpha \alpha m n},
\]

\[
(5.1)
\]

\[
a^i_{j m n} - a^i_{j |n|m} = 2L^\alpha_{j \alpha m n} a^j_{|\alpha|} - 2L^\alpha_{j \alpha m n} a^j_{|\alpha|} + a^\alpha_{j} R^i_{\alpha \alpha m n} + L^i_{\alpha \alpha m n} - L^j_{\alpha \alpha m n} + L^j_{\alpha \alpha m n} L^i_{\alpha \alpha m n} - L^\alpha_{\alpha \alpha m n} L^i_{\alpha \alpha m n} - 2L^\alpha_{\alpha \alpha m n} L^i_{\alpha \alpha m n},
\]

\[
(5.2)
\]
\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|m} + a^j_j(R^i_{\alpha mn} + L^i_{\alpha n|m} - L^i_{\alpha n|m} + L^\beta_{\alpha m}L^i_{\beta n} - L^\beta_{\alpha m}L^i_{\beta n}) + a^j_j(R^i_{\alpha jmn} + L^\alpha_{\alpha j|m} - L^\alpha_{\alpha j|m} + L^\beta_{\alpha jn}L^\alpha_{\beta n} + L^\beta_{\alpha jn}L^\alpha_{\beta n} + 2L^\alpha_{\alpha jn}L^\beta_{\alpha jn}, \quad (5.3) \]

\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|n} - 2L^\alpha_{mn}a^i_{\alpha|n} - 2L^\alpha_{vn}a^i_{\alpha|n} + 2a^\alpha_{\alpha jn}L^\beta_{\alpha jn} + L^\alpha_{\alpha jn}L^\beta_{\alpha jn}, \quad (5.4) \]

\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|n} + 2L^\alpha_{jn}a^i_{\alpha|n} - 2L^\alpha_{mn}a^i_{\alpha|n} - 2L^\alpha_{vn}a^i_{\alpha|n} + a^j_j(R^i_{\alpha mn} - L^i_{\alpha n|m} - L^i_{\alpha n|m} + L^\beta_{\alpha m}L^i_{\beta n} - L^\beta_{\alpha m}L^i_{\beta n} - 2L^\alpha_{\alpha jn}L^\beta_{\alpha jn}) \quad (5.5) \]

\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|n} + 2L^\alpha_{jn}a^i_{\alpha|n} - 2L^\alpha_{mn}a^i_{\alpha|n} + a^j_j(R^i_{\alpha mn} - L^i_{\alpha n|m} - L^i_{\alpha n|m} + L^\beta_{\alpha m}L^i_{\beta n} - L^\beta_{\alpha m}L^i_{\beta n} - 2L^\alpha_{\alpha jn}L^\beta_{\alpha jn}) + 2a^\alpha_{\alpha jn}L^\beta_{\alpha jn} \quad (5.6) \]

\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|n} - 2L^\alpha_{mn}a^i_{\alpha|n} + a^j_j(R^i_{\alpha mn} + L^i_{\alpha n|m} - L^i_{\alpha n|m} + L^\beta_{\alpha m}L^i_{\beta n} - L^\beta_{\alpha m}L^i_{\beta n} - 2L^\alpha_{\alpha jn}L^\beta_{\alpha jn}) \quad (5.7) \]

\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|n} + 2L^\alpha_{jn}a^i_{\alpha|n} - 2L^\alpha_{mn}a^i_{\alpha|n} + a^j_j(R^i_{\alpha mn} + L^i_{\alpha n|m} - L^i_{\alpha n|m} + L^\beta_{\alpha m}L^i_{\beta n} - L^\beta_{\alpha m}L^i_{\beta n} - 2L^\alpha_{\alpha jn}L^\beta_{\alpha jn}) + 2a^\alpha_{\alpha jn}L^\beta_{\alpha jn} \quad (5.8) \]

\[ a^i_{j|m} - a^i_{j|n} = 2L^\alpha_{jm}a^i_{\alpha|n} - 2L^\alpha_{mn}a^i_{\alpha|n} + a^j_j(R^i_{\alpha mn} + L^i_{\alpha n|m} - L^i_{\alpha n|m} + L^\beta_{\alpha m}L^i_{\beta n} - L^\beta_{\alpha m}L^i_{\beta n} - 2L^\alpha_{\alpha jn}L^\beta_{\alpha jn}) + 2a^\alpha_{\alpha jn}L^\beta_{\alpha jn}, \quad (5.9) \]
\[ a_{ij}^{m} - a_{ij}^{i} | m = -2L_{jm}^\alpha a_{ij}^{\alpha | m} + 2L_{jm}^\alpha a_{ij}^{i | m} + 2L_{m\alpha}^\alpha a_{ij}^{\alpha | m} - 2L_{m\alpha}^\alpha a_{ij}^{i | m} + 2L_{m\alpha}^\alpha a_{ij}^{i | m} \]

(5.10)

\[ a_{ij}^{m} - a_{ij}^{i} | m = -2L_{jm}^\alpha a_{ij}^{\alpha | m} + 2L_{jm}^\alpha a_{ij}^{i | m} + 2L_{m\alpha}^\alpha a_{ij}^{\alpha | m} + 2L_{m\alpha}^\alpha a_{ij}^{i | m} + 2L_{m\alpha}^\alpha a_{ij}^{i | m} \]

(5.11)

\[ a_{i}^{m} - a_{i}^{i} | m = -2L_{jm}^\alpha a_{i}^{\alpha | m} + 2L_{jm}^\alpha a_{i}^{i | m} + 2L_{m\alpha}^\alpha a_{i}^{\alpha | m} + 2L_{m\alpha}^\alpha a_{i}^{i | m} \]

(5.12)

\[ a_{j}^{m} - a_{j}^{i} | m = 2L_{jm}^\alpha a_{j}^{\alpha | m} - 2L_{jm}^\alpha a_{j}^{i | m} + 2L_{m\alpha}^\alpha a_{j}^{\alpha | m} - 2L_{m\alpha}^\alpha a_{j}^{i | m} \]

(5.13)

\[ a_{i}^{m} - a_{i}^{i} | m = 2L_{jm}^\alpha a_{i}^{\alpha | m} - 2L_{jm}^\alpha a_{i}^{i | m} + 2L_{m\alpha}^\alpha a_{i}^{\alpha | m} - 2L_{m\alpha}^\alpha a_{i}^{i | m} \]

(5.14)

\[ a_{j}^{m} - a_{j}^{i} | m = 2L_{jm}^\alpha a_{j}^{\alpha | m} - 2L_{jm}^\alpha a_{j}^{i | m} + 2L_{m\alpha}^\alpha a_{j}^{\alpha | m} - 2L_{m\alpha}^\alpha a_{j}^{i | m} \]

(5.15)

\[ a_{j}^{m} - a_{j}^{i} | m = 2L_{jm}^\alpha a_{j}^{\alpha | m} - 2L_{jm}^\alpha a_{j}^{i | m} + 2L_{m\alpha}^\alpha a_{j}^{\alpha | m} - 2L_{m\alpha}^\alpha a_{j}^{i | m} \]

(5.16)
\[ a_{j|m}^i - a_{j|n}^i = 2L_{m}^{\alpha}a_{j|\alpha} \]
\[ + a_{j}^i (R_{\alpha\nu}^{i\alpha} + L_{\alpha\nu|m}^{i\alpha} - L_{\alpha\nu|m}^{i\alpha} - L_{\alpha\nu|m}^{i\alpha} - L_{\alpha\nu|m}^{i\alpha} + 2L_{\alpha\beta}^{i\alpha} - L_{\alpha\nu|m}^{i\alpha}) \] (5.17)

- \[ a_{i}^i (R_{\alpha\nu}|m - L_{\alpha\nu|m}^{i\alpha} + L_{\alpha\nu|m}^{i\alpha} - 2L_{\alpha\beta}^{i\alpha} - L_{\alpha\nu|m}^{i\alpha}). \]

6 Appendix II: Curvature tensors obtained from linearly independent identities of Ricci Type

\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.1)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.2)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.3)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.4)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.5)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.6)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.7)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.8)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.9)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.10)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.11)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.12)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}, \] (6.13)
\[ \rho_{jmn}^{i} = R_{jmn}^{i} + L_{jmn}^{i} - L_{jmn}^{i} + L_{jmn}^{i} - 2L_{jmn}^{i}. \] (6.14)

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