Spherical model and quantum phase transitions

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Abstract. The spherical Berlin-Katz model is considered in the framework of the epsilon expansion in one-dimensional and two-dimensional space. For the two-dimensional and three-dimensional cases in this model, an exact solution was previously obtained in the presence of a field, and for the two-dimensional case the critical temperature is zero, that is, a "quantum" phase transition is observed. On the other hand, the epsilon expansion of critical exponents with an arbitrary number of order parameter components is known. This approach is consistent with the scaling paradigm. Some critical exponents are found for the spherical model in one- and two-dimensional space in accordance with the generalized scaling paradigm and the ideas of quantum phase transitions. A new formula is proposed for the critical heat capacity exponent, which depends on the dynamic index $z$, at a critical temperature equal to zero. An expression is proposed for the order of phase transition with a change in temperature (developing the approach of R. Baxter), which also depends on the $z$ index. An interpolation formula is presented for the effective dimension of space, which is valid for both a positive critical temperature and a critical temperature equal to zero. This formula is general. Transitions with a change in the field in a spherical model at absolute zero are also considered.

1. Introduction
Phase transitions (PT) have been studied for over 60 years within the framework of the concept of critical exponents and the scaling paradigm [1-12]. For three-dimensional systems (and even two-dimensional systems in the field) no exact solutions have been found [1-3]. There is only one exception to this rule - the so-called Berlin-Katz spherical model [1, 2], for which exact solutions for equilibrium states have been found for two-dimensional and three-dimensional cases in the presence of a field. It is interesting that in the two-dimensional case (in contrast to the usual Ising model) the critical temperature in this model is zero [1].

The purpose of the work is to consider the spherical model at zero critical temperature, taking into account the ideas of quantum phase transitions and the generalized scaling paradigm [8, 11-13].

2. Spherical model
The spherical model in a sense is a generalization of the Ising model or the Heisenberg model [1, 2]. An extensive literature is devoted to the spherical model (see references in [1, 2]). The Hamiltonian of the model in zero field has the form [1, 2]

\[ H^{(D)} = -J \sum_{<i,j>} \langle^{(D)} \rangle \cdot s_i \cdot s_j \]  

(1)
Here, the summation is carried out over the pairs of nearest neighbours, $i, j$ are the numbers of the lattice sites, the dot denotes the scalar product, and $J$ is a positive constant. However, if in the Ising and Heisenberg models $s^{(D)}$ – are $D$-dimensional unit vectors, then for the spherical model there is only one condition on the length of the vectors [1] ($N$ is the total number of lattice sites)

$$\sum_{i=1}^{N} s_i^2 = N,$$

which defines the surface of the hypersphere in $N$-dimensional space, over which the averaging is carried out. This condition introduces into consideration the continuum of states even for a model of finite size, in contrast to a countable set of states for the Ising model [1], so summation is replaced by integration. This leads to the fact that for a spherical model the order parameter has an infinite number of components [1], which corresponds to the limit

$$D \rightarrow +\infty.$$

For one, two and three dimensions for the spherical model, exact solutions were obtained [1], and for the three-dimensional case this is the only obtained exact solution (for equilibrium states). This determines the interest in this model; besides, the static scaling relations are satisfied for the spherical model [1, 2]. Exact solutions are obtained in the thermodynamic limit (for a system of infinite size). Unfortunately, no physical or other system corresponding to the spherical model has been found yet [1, 2].

For the two-dimensional case for the spherical model, the critical temperature is zero (as for the one-dimensional Ising model) [1, 2], that is, a "quantum" phase transition (PT) occurs in it [4-8].

3. Quantum phase transitions

Quantum PT are usually called phase transitions at a temperature of absolute zero [5, 6], when the behaviour of the system is determined by quantum fluctuations. However, quantum phase transitions also manifest themselves at finite temperatures (pretransition phenomena), sometimes even close to room temperatures [6, 12]. It turns out that in this case the statics cannot be separated from the dynamics and the properties depend on the dynamic critical exponent $z$ [5, 6]

$$\tau \propto \xi^z,$$

where $\tau$ is the relaxation time of the order parameter (for example, for a ferromagnet, the order parameter is spontaneous magnetization), $\xi$ is the correlation length [1-9]. All these quantities are measured experimentally or can be calculated, including by computer simulation methods [1, 2, 9-13]. In a zero field, the correlation length depends on temperature [1-3]

$$\xi \propto \tau^{-\nu}, \tau = |T - T_c|, \tau \rightarrow 0.$$

Here $\nu > 0$ is the critical exponent of the correlation length [1-3].

When approaching the quantum critical point with respect to critical properties (critical exponents), the system acquires an effective spatial dimension [5, 6] ($d$ is the dimension of space)

$$d_{\text{eff}} = d + z.$$

Here the dynamic exponent $z$ plays the role of additional spatial dimensions due to nonequilibrium processes near the quantum critical point at zero PT temperature.
4. Epsilon expansion and quantum phase transitions

Within the framework of the so-called ε-expansion, expansions of critical exponents are obtained for an arbitrary number of $D$ order parameter components [10]. For example, for the Fisher index we have [10]

$$\eta = \frac{\varepsilon^2 (D+2)}{2(D+8)^2} \left\{ 1 + \frac{\varepsilon}{4(D+8)^2} \left[ -D^2 + \ldots \right] + \frac{\varepsilon^2}{16(D+8)^4} \left[ -5D^2 + \ldots \right] + \ldots \right\}. $$

(6)

Here [9, 10]

$$\varepsilon = 4 - d. $$

(7)

The dots in (6) denote degrees $D$ less than the previous ones. The spherical model corresponds to the limit (3); in this limit, the curly brace in (6) remains finite. The factor in front of the bracket at $D \to +\infty$ tends to zero, therefore, the Fisher exponent for the spherical model is zero, and for any dimension of the space, this is consistent with [1]. Fisher’s index $\eta$ determines the behaviour of the pair correlation function $G \propto \phi(x)\phi(x') > - \phi(x) << \phi(x') > [2, 3, 10]$

$$G \propto r^{-2-\eta} f(\frac{r}{\varepsilon}), r = |x - x'|. $$

Here $\phi$ is the fluctuating order parameter depending on the coordinates $x$.

More interesting results are obtained for other exponents. For the critical exponent of the correlation length $\nu$ we have [10]

$$\frac{1}{\nu} = 2 - \frac{\varepsilon(D+2)}{(D+8)} \left\{ 1 + \frac{\varepsilon}{2(D+8)^2} \left[ 13D^2 + 44 \right] + \frac{\varepsilon^2}{8(D+8)^4} \left[ -3D^2 + \ldots \right] + O(\varepsilon^3) \right\}. $$

(8)

In limit (3) for the spherical model, we obtain

$$\frac{1}{\nu} = 2 - \varepsilon = 2 - (4 - d), $$

whence

$$\nu = \frac{1}{d-2}, $$

(9)

which agrees with [1] for three-dimensional space and with [10]. (10) is true for a positive critical temperature, if the temperature of the phase transition $T_c = 0$, then for the quantum PT the formula takes the form (see (5))

$$\nu = \frac{1}{d_{\text{eff}}-2}. $$

(11)

For a spherical model in two-dimensional space ($T_c = 0$) we find that the correlation length exponent is related to the dynamic critical index

$$\nu = \frac{1}{z} (d = 2, T_c = 0, D = +\infty). $$

(12)

For the spherical model in the one-dimensional case (11) takes the form

$$\nu = \frac{1}{z-1} (d = 1, T_c = 0, D = +\infty). $$

(13)
Since the exponent of the correlation length is positive by definition, in the latter case the dynamic index $z$ must be greater than one.

Even more interesting results are obtained for the critical heat capacity exponent $\alpha$. According to [1-12], we have for the usual case $T_c > 0$ the classical scaling relation

$$\alpha = 2 - d \nu \ (T_c > 0) \ .$$

(14)

Substituting (10) here, we find

$$\alpha = \frac{d - 4}{d - 2} \ (T_c > 0, D = +\infty ) ,$$

(15)

which agrees with [10]. However, for zero critical temperature, PT (14) changes its form [8, 11, 12]

$$\alpha = 1 - d \nu \ (T_c = 0) \ .$$

(16)

Using (10), we obtain a new formula

$$\alpha = \frac{2}{2 - d} = -2 \nu \ (T_c = 0, D = +\infty ) \ .$$

(17)

For quantum PT, the expression takes the form

$$\alpha = \frac{2}{2 - d_{\text{eff}}} = -2 \nu \ (T_c = 0, D = +\infty ) \ .$$

(18)

For a two-dimensional spherical model near absolute zero, the heat capacity exponent is (see (5))

$$\alpha = \frac{2}{1 - z} < 0 \ (T_c = 0, d = 2, D = +\infty ) \ .$$

(19)

For the one-dimensional case, the result will change

$$\alpha = \frac{2}{1 - z} < 0 \ (T_c = 0, d = 1, D = +\infty ) \ .$$

(20)

5. The order of the phase transition and the index $z$

For a positive critical temperature, the order of the PT according to R. Baxter [2, 11, 13] is (it is assumed that the phase transition occurs with a change in temperature)

$$r = 2 - \alpha \ (T_c > 0) .$$

(21)

However, this expression changes if the critical temperature is zero [8, 11, 13], which will be for quantum PT with decreasing temperature

$$r = 1 - \alpha \ (T_c = 0) .$$

(22)

Substitute here (5) and (18)

$$r = \frac{d_{\text{eff}}}{d_{\text{eff}} - 2} = \frac{1}{1 - \frac{2}{d_{\text{eff}}}} = \frac{1}{1 - \frac{2}{(d + z)}} \ (T_c = 0, D = +\infty ) \ .$$

(23)

This expression makes sense for a spherical model for one-dimensional and two-dimensional cases (in the three-dimensional case $T_c > 0 \ [1, 2]$).

First, let's find out the possible values of the kinetic exponent $z \ [10]$. The calculation of this index is associated with the modelling of nonequilibrium (kinetic) processes and is a much more complicated problem than the calculation of static indices (for example, the correlation length exponent $\nu$). This is
all the more difficult for quantum PTs, for which not everything is clear, so we will restrict ourselves to preliminary considerations.

There can be many variants of dynamics, let’s consider two variants. According to the first option, for any number of order parameter components $D$ the following formula is valid [10]

$$ z = 2 + R \eta. $$ (24)

Here $\eta$ is the Fisher critical exponent (see Section 4), and $R$ is defined as follows [10]

$$ R = 0.76 (1 - 0.19 (4 - d) + ...) . $$ (25)

For one-dimensional and two-dimensional space, the coefficient $R$ remains finite, and the Fisher index for the spherical model is zero (see Section 4), therefore, for the spherical model in the first version, the dynamic exponent $z$ is

$$ z_1 = 2. $$ (26)

In the second variant of the dynamics [10]

$$ z_2 = 4 - \eta = 4. $$ (27)

Now let us find the kind of quantum PT for the spherical model. For a one-dimensional space in the first version of the dynamics, the genus of the phase transition is (see (23))

$$ r = 3. $$ (28)

that is, there is a third-order PT with a zero jump in heat capacity and entropy. In the second variant of the dynamics for a one-dimensional spherical model, we obtain

$$ r = \frac{5}{3} > 1 . $$ (29)

This corresponds to a continuous phase transition with a zero jump in heat capacity and entropy [13].

For two-dimensional space in the first version of the dynamics for a spherical model, the genus of the phase transition is equal to two

$$ r = 2. $$ (30)

however, the jump in heat capacity is equal to zero in this case. In the second variant of the dynamics for a two-dimensional spherical model, the kind of the phase transition is

$$ r = \frac{3}{2} > 1 . $$ (31)

A continuous PT with a zero jump in the specific heat is also observed here [13]. All these phase transitions are continuous (the state jump at the phase transition is zero), one-sided, and occur with decreasing temperature at a constant zero field and other parameters (for example, pressure). The low-temperature phase, which exists theoretically at absolute zero, is not experimentally observed. In all considered cases, the critical heat capacity index is negative, which is consistent with the third law of thermodynamics [3]. It is desirable, of course, to directly calculate the kinetic exponent $z$ for a spherical model, including one-dimensional and two-dimensional space.

Speaking about other quantum PTs (for example, about the transition to the superconducting state), we note the following. In such cases, there is a PT line, which begins at finite temperatures, and then, with a change in the parameters, the PT temperature smoothly decreases and becomes zero [1, 9]. In this case, expression (5) for the effective dimension of the space must be modified. The following interpolation formula can be suggested

$$ d_{\text{eff}} = d + z (1 - S) . $$ (32)
\( S_i \)-function is equal to [13]

\[
S_i = (1 + A_1 |t|^n) \left( \frac{T_c}{T} \right)^x, \quad T_c \geq 0
\]

(33)

where \( t = T - T_c, T_c \geq 0 \), \( A_1 \) is a constant of any sign; \( m, n \) are positive constants. The constants \( A_1, m, n \) can be found by computer processing of exact experimental data for a certain PT or by computer simulation methods (based on experimental data). Then, when approaching a positive critical temperature, the \( S_i \)-function tends to unity and the effective dimension of space tends to the usual dimension of space. If the critical temperature is zero (as for a spherical model in one and two dimensions or for superconductivity in a magnetic field), then the \( S_i \)-function is zero and we obtain formula (5) for the quantum phase transition.

6. PT with a change in the field

Let us consider a PT with a change in the field at a critical temperature (equal to zero), which corresponds to the usual formulation of the problem of a quantum PT [5, 6]. First, we introduce some concepts for the spherical model [2]. The free energy of the model depends on the function [2]

\[
\Phi(z) = Kz + Kd - (g(z)) / 2 + \frac{\hbar^2}{4Kz^2}
\]

(34)

Here \( z \) is a complex variable (not to be confused with a dynamic exponent!);

\[
K = J \langle(T); \quad h = H / T
\]

(35)

\( H \) - the projection of the magnetic field strength on the selected axis; \( g(z) \) - defined function [2], temperature is measured in energy units. If \( z \) is a real and a positive number, then \( \Phi(z) \) has one minimum at a positive value \( z_0 > 0 \) ( \( h \neq 0, K > 0 \) ) [2], at the point of which the derivative is equal to zero (corresponds to equilibrium)

\[
\Phi'(z) = K - \frac{g'(z_0)}{2} - \frac{\hbar^2}{4Kz_0^2} = 0
\]

(36)

(36) can be reduced to the form

\[
J \left( J - \frac{Tg'(z_0)}{2} \right) = \frac{H^2}{4z_0^2}
\]

(37)

Under some assumptions for the spherical model [2]

\[
g'(z) = c_z^{(\epsilon-2)/3}
\]

(38)

It is known that the magnetization for this model is [2]

\[
M = \frac{H}{2Jz_0}
\]

(39)

At a critical temperature, the formula [1-3] is correct

\[
M \propto H^{1/\delta}, \quad H \to 0, T = T_c
\]

(40)

\( \delta \) is another critical exponent. Using (39), we find for the spherical model

\[
M \propto H^{1/\delta} \propto \frac{H}{z_0}
\]

(41)
hence
\[ z_0 \propto H^{1-1/d}. \] (42)

On the other hand, for a one-dimensional space (see (38))
\[ g'(z) = c_1 z^{-1/2} \] (43)
and (37) takes the form
\[ J (J - c_1 T) = \frac{H^2}{4 z_0^2}. \] (44)

But for one-dimensional space the critical temperature is zero for the spherical model [1, 2], therefore, (44) gives
\[ \frac{H}{2 z_0} = J, T = T_c = 0 \] (45)
or, using (42)
\[ H \propto z_0 \propto H^{1-1/d}. \] (46)

As an equality, we obtain \( c_1 \) is some constant
\[ H = c_1 H^{1-1/d}, H \rightarrow 0, T = T_c = 0. \] (47)

Hence, it can be shown that for a one-dimensional spherical model without taking into account quantum fluctuations
\[ \delta = \infty, d = 1, 2. \] (48)

Let us show that the same result is also true for a two-dimensional space, in this case (see (38))
\[ g'(z) = c_1 z^{1/2} \rightarrow 1 = c_1 > 0. \] (49)

(37) will take the form
\[ J (J - c_1 T) = \frac{H^2}{4 z_0^2}, d = 2. \] (50)

In the case \( d = 2 \) for the spherical model, the critical temperature is equal to zero [1, 2] and we again obtain (45), therefore, again we come to (48). The same result is obtained from the general formula for the index \( \delta \) [2, p. 77] (although Baxter believes that the formula is true for \( 2 < d < 4 \))
\[ \delta = \frac{d + 2}{d - 2}. \] (51)

If \( d \rightarrow 2 \), then from this we obtain (48). However, since in one-dimensional and two-dimensional space for a spherical model the critical temperature is zero [1, 2], it is necessary to use formula (5), which leads to the result
\[ \delta = \frac{d_{\text{eff}} + 2}{d_{\text{eff}} - 2}. \] (52)

For two-dimensional space we get
\[ \delta = \frac{4 + z}{z}; d = 2, T = T_c = 0. \] (53)
Moreover, in one of the kinetics variants \( z = 2 \) (see Section 5) and
\[
\delta = 3; \ d = 2, d_{\sigma} = 4, T = T_c = 0.
\] (54)

This agrees with (51) for four-dimensional space. Thus, taking into account quantum effects and kinetics leads to the fact that the critical exponent \( \delta \) remains finite, at least for a two-dimensional spherical model. The same quantum effect can also manifest itself in the one-dimensional case.

7. Conclusions
A new approach to the consideration of a spherical model for one-dimensional and two-dimensional space, when the critical temperature is zero, is proposed. In this case, the dynamic critical exponent \( z \) must be taken into account. The possible order of phase transition in the spherical model for one-dimensional and two-dimensional cases with a change in temperature is determined for the first time. Direct calculation of the dynamic exponent \( z \) for the spherical model is desirable.

An expression is proposed for the effective dimension of space when approaching the quantum phase transition point, when the critical temperature is zero or tends to zero. This expression has a general character, that is, it is true for any phase transition.

The phase transitions are also considered when the field changes at a critical temperature equal to absolute zero. The critical exponent \( \delta \) is calculated for a two-dimensional spherical model, which depends on the dynamic critical exponent \( z \).

The approach considered can be applied to other quantum phase transitions [5, 6], for example, to a transition to a superconducting or superfluid state (in a mixture of helium isotopes) near absolute zero.

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