Hölder continuous solutions to quaternionic Monge-Ampère equations

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Abstract. We prove the Hölder continuity of the unique solution to quaternionic Monge-Ampère equation with densities in $L^p$, $p > 2$, on a bounded strictly pseudoconvex domains.

Introduction

Recently, people are interested in developing Quaternion analysis, which has become an important branch of mathematics, has many application in mathematical physics. The quaternionic Monge-Ampère operator is defined as the Moore determinant of the quaternionic Hessian of $u$:

$$\det(u) = \det\left[\frac{\partial^2 u}{\partial q_j \partial q_k}(q)\right].$$

The following Dirichlet problem for the quaternionic Monge-Ampère equation in $\Omega \subset \mathbb{H}^n$:

$$\begin{cases}
\det\left[\frac{\partial^2 u}{\partial q_j \partial q_k}(q)\right] = f, \\
\lim_{q^i \to q} u(q^i) = \varphi(q), \quad \forall q \in \partial \Omega, \varphi \in C(\partial \Omega)
\end{cases}$$

(1)

It has been shown by Alesker [A3, Theorem 1.3] that (1) is solvable when $\Omega$ is a strictly pseudoconvex domain and $f \in C(\Omega)$, $f \geq 0$, $\varphi \in C(\partial \Omega)$ and the solution is continuous on $\overline{\Omega}$. For the smooth case, in [A3, Theorem 1.4] S.Alesker proved a result on existence and uniqueness of the smooth solution of (1) when the domain $\Omega$ is the Euclidean ball $B$ in $\mathbb{H}^n$ and $f \in C^\infty(\Omega)$, $f > 0$, $\varphi \in C^\infty(\partial \Omega)$. He said the reason why he failed to solve (1) on general strictly pseudoconvex bounded domains is the fact that the class of diffeomorphisms preserving the class of quaternionic plurisubharmonic (psh) functions must be affine transformations. Relating to this problem, the Dirichlet problem for quaternionic Monge-Ampère equations on arbitrary strictly pseudoconvex bounded domains was an open problem. For solving this issue, Zhu proved in [Z] the existence of a subsolution to the Dirichlet problem in quaternionic strictly pseudoconvex bounded domain. By this end and the fact that the subsolutions lead the solutions [Z, Theorem 1.1] Zhu proved that (1) is solvable when $\Omega$ is a strictly pseudoconvex bounded domain and $f \in C^\infty(\Omega)$, $f > 0$, $\varphi \in C^\infty(\partial \Omega)$ and the solution is in $C^\infty(\overline{\Omega})$.

Sroka in [SM] found a continuous solution of this problem (1) under the much milder assumption $f \in L^p(\Omega)$, $p > 2$.

To develop the quaternionic pluripotential theory, Alesker defined the quaternionic Monge-Ampère operator on general quaternionic manifolds, he introduced in [A2] an operator in terms of the Baston operator $\Delta$, which is the first operator of the quaternionic complex on quaternionic manifolds. The n-th power of this operator is exactly the quaternionic Monge-Ampère operator when the manifold is flat. On the flat space $\mathbb{H}^n$, the Baston operator $\Delta$ is the first operator of 0-Cauchy-Fueter complex:

$$0 \to C^\infty(\Omega, \mathbb{C}) \to^\Delta C^\infty(\Omega, \wedge^2 \mathbb{C}^{2n}) \to^D C^\infty(\Omega, \wedge^3 \mathbb{C}^{2n}) \to \ldots$$

(2)

Wang [W1] wrote down explicity each operator of the $k$-Cauchy-Fueter complex in terms of real variables. Motivated by this, D.Wan and W.Wang introduced in [WW] two first-order differential operators $d_0$ and $d_1$ acting on the quaternionic version of differential forms. The second operator $D$ in (2) can be written
as \( D := \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} \). The behavior of \( d_0, d_1, \) and \( \Delta = d_0 d_1 \) is very similar to \( \partial, \bar{\partial}, \) and \( \partial \bar{\partial} \) in several complex variables. The quaternionic Monge-Ampère operator can be defined as \( (\Delta u)^n = (d_0 d_1 u)^n \) and has a simple explicit expression, which is much more convenient than the definition by using Moore determinant. Based on this observation, some authors established and developed the quaternionic versions of several results in complex pluripotential theory (for more informations see [WW, WZ, WK]).

Motivated by this, we consider the following Dirichlet problem for the quaternionic Monge-Ampère equation in a given strictly pseudoconvex domain \( \Omega \subset \mathbb{H}^n \):

\[
\begin{cases}
  u \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega}), \\
  (\Delta u)^n = d\nu, \\
  \lim_{q' \to q} u(q') = \psi(q), \quad \forall q \in \partial \Omega, \psi \in C(\partial \Omega)
\end{cases}
\]  

The purpose of this paper is to study the regularity of solutions to this problem. To begin with, we describe the background. The Hölder continuous solutions to complex Monge-Ampère equations was proved by [GKZ]. In particular, it is proved that the solution is Hölder continuous if \( d\nu = f dV, \) \( 0 \leq f \in L^p, p > 1, \) and \( \varphi \) is Hölder continuous. Then we are going to follow the method of [GKZ] to prove our main result, which is the following Theorem.

**Theorem.** Let \( \Omega \) be a bounded strongly pseudoconvex domain of \( \mathbb{H}^n \) with smooth boundary. Assume that \( \psi \in C^{1,1}(\partial \Omega) \) and \( 0 \leq f \in L^p(\Omega), \) for some \( p > 2. \) Then the unique solution \( u \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega}) \) to the problem 3 for \( d\nu = f dV, \) belongs to \( C^{0,\alpha}(\Omega) \) for any \( 0 < \alpha < \frac{2}{qn+1-\frac{2}{q}-1}, \) where \( \frac{1}{p} + \frac{1}{q} = 1. \)

The paper is organized as follows. In section 1, we recall basic facts about plurisubharmonic functions, and the quaternionic Monge-Ampère operator. In section 2, we give an estimate of the modulus of continuity of the solution to the Dirichlet problem for the quaternionic Monge-Ampère equation, and prove its useful consequence (Corollary 2.5) which plays key role in the rest. In section 3, we prove our main tool which is the stability estimate. In section 4, we show that the unique solution to the quaternionic Monge-Ampère equation with densities in \( L^p, p > 2, \) is Hölder continuous if the boundary data \( \psi \) is so.

## 1 Preliminaries

### 1.1 Plurisubharmonic functions of quaternionic variables

In this part, let us remind few standard notions by [A].

**Definition 1.1.** A real valued function \( f : \Omega \subset \mathbb{H}^n \to \mathbb{R} \) is called quaternionic plurisubharmonic if it is upper semi-continuous and its restriction to any right quaternionic line is subharmonic.

**Remarks 1.2.** On \( \mathbb{H}^1 \) the class of plurisubharmonic functions coincides with the class of subharmonic functions in \( \mathbb{R}^4. \)

**Definition 1.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{H}^n. \) Then \( \Omega \) is called strictly pseudoconvex if there exists a strictly plurisubharmonic defining function \( \varphi, \) i.e., \( \Omega = \{ q \in \mathbb{H}^n; \varphi(q) < 0 \}. \)

The analogous classical results for subharmonic functions also holds for the quaternionic plurisubharmonic functions. We list these properties here without proofs; all of them can be derived from the subharmonic case (see [K, Chapter 2]).
Proposition 1.4. 1. If $u \in C^2$ then $u$ is plurisubharmonic if and only if the form $\Delta u$ is positive in $\Omega$.

2. If $u, v \in PSH(\Omega)$ then $\lambda u + \mu v \in PSH(\Omega)$, $\forall \lambda, \mu > 0$

3. If $u$ is plurisubharmonic in $\Omega$ then the standard regularization $u * \chi_\varepsilon$ are also plurisubharmonic in $\Omega_\varepsilon := \{ q \in \Omega / d(q, \partial \Omega) > \varepsilon \}$.

4. If $(u_i) \subset PSH(\Omega)$ is locally uniformly bounded from above then $(\sup u_i)^* \in PSH(\Omega)$, where $v^*$ is the upper semi continuous regularization of $v$

5. $PSH(\Omega) \subset SH(\Omega)$.

6. Let $\emptyset \neq U \subset \Omega$ be a proper open subset such that $\partial U \cap \Omega$ is relativenment compact in $\Omega$. If $u \in PSH(\Omega)$, $v \in PSH(\Omega)$ and $\limsup_{q \to q'} v(q) \leq u(q')$ for each $q' \in \partial U \cap \Omega$ then the function $w$, defined by

$$ w(q) = \begin{cases} u(q), & q \in \Omega \setminus U; \\ \max(u(q), v(q)), & q \in U. \end{cases} $$

is plurisubharmonic in $\Omega$.

Denote by $PSH$ the class of all quaternionic plurisubharmonic functions (cf. [A, A1, A2]) for more information about plurisubharmonic functions).

For the complex case, it is well known that the psh functions are locally integrable with any exponent, but in the quaternionic case we have this following result for local integrability of psh functions.

Proposition 1.5. (Proposition 2 in [SM])
Suppose $u \in PSH(\Omega)$ is such that $u \neq -\infty$. Then $u \in L^p_{loc}(\Omega)$ for any $p < 2$ and the bound on $p$ is optimal.
What is more if $u_j \neq -\infty$ is a sequence of psh functions converging in $L^1_{loc}(\Omega)$ to some $u$, neccessarily belonging to $PSH(\Omega)$, then convergence holds in $L^p_{loc}(\Omega)$ for any $p < 2$.

1.2 The operators $d_0$, $d_1$ and the Baston operator $\Delta$

We use the well-known embedding of the quaternionic algebra $\mathbb{H}$ into $End(\mathbb{C}^2)$ defined by

$$ \tau : \mathbb{H} \rightarrow \mathbb{C}^{2 \times 2} $$

$$ q = x_0 + ix_1 + jx_2 + kx_3 \rightarrow \begin{pmatrix} x_0 - ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 + ix_1 \end{pmatrix} $$

Actually we use the conjugate embedding

$$ \tau : \mathbb{H}^n \cong \mathbb{R}^{4n} \rightarrow \mathbb{C}^{2n \times 2} $$

$$ (q_0, q_1, ... q_{n-1}) \rightarrow z = (z^0) \in \mathbb{C}^{2n \times 2} $$

with $q_j = x_{4j} + ix_{4j+1} + jx_{4j+2} + kx_{4j+3}$, $j = 0, 1, ..., 2n - 1$, $\alpha = 0, 1$, with

$$ \begin{pmatrix} z^0 \end{pmatrix}^T \begin{pmatrix} z^0 & z^0 \\ z^1 & z^1 \\ \vdots & \vdots \\ z^{(2n-2)} & z^{(2n-2)} \\ z^{(2n-1)} & z^{(2n-1)} \end{pmatrix} = \begin{pmatrix} x_0 - ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 + ix_1 \end{pmatrix} $$

$$ \begin{pmatrix} x_0 - ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 + ix_1 \end{pmatrix} = \begin{pmatrix} x^{4l} - ix_{4l+1} & -x^{4l+2} + ix_{4l+3} \\ x^{4l+2} + ix_{4l+3} & x^{4l} + ix_{4l+1} \end{pmatrix} $$

(4)
Pulling back to the quaternionic space $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by the embedding (4), we define on $\mathbb{R}^{4n}$ first-order differential operators $\nabla_{j\alpha}$ as following

$$
\begin{pmatrix}
\nabla_{00} & \nabla_{01} \\
\nabla_{10} & \nabla_{11} \\
\vdots & \vdots \\
\nabla_{(2l)0} & \nabla_{(2l)1} \\
\nabla_{(2l+1)0} & \nabla_{(2l+1)1} \\
\vdots & \vdots \\
\nabla_{(2n-2)0} & \nabla_{(2n-2)1} \\
\nabla_{(2n-1)0} & \nabla_{(2n-1)1}
\end{pmatrix}
= 
\begin{pmatrix}
\partial x_0 + i\partial x_1 & -\partial x_2 - i\partial x_3 \\
\partial x_2 - i\partial x_3 & \partial x_0 - i\partial x_3 \\
\vdots & \vdots \\
\partial x_{4l} + i\partial x_{4l+1} & -\partial x_{4l+2} - i\partial x_{4l+3} \\
\partial x_{4l+2} - i\partial x_{4l+3} & \partial x_{4l} - i\partial x_{4l+1} \\
\vdots & \vdots \\
\partial x_{4n-4} + i\partial x_{4n-3} & -\partial x_{4n-2} - i\partial x_{4n-1} \\
\partial x_{4n-2} - i\partial x_{4n-1} & \partial x_{4n-4} - i\partial x_{4n-3}
\end{pmatrix}
$$

(5)

$\omega^{k\beta}$ can be viewed as independent variables and $\nabla_{j\alpha}$’s are derivatives with respect to these variables. The operators $\nabla_{j\alpha}$’s play very important roles in the investigating of regular functions in several quaternionic variables.

Let $\wedge^k \mathbb{C}^{2n}$ be the complex exterior algebra generated by $\mathbb{C}^{2n}$, with $0 \leq k \leq n$. Fix a basis $\{\omega^0, \omega^1, \ldots, \omega^{2n-1}\}$ of $\mathbb{C}^{2n}$. Let $\Omega$ be a domain in $\mathbb{R}^{4n}$. We define $d_0, d_1 : C_0^\infty(\Omega, \wedge^p \mathbb{C}^{2n}) \to C_0^\infty(\Omega, \wedge^{p+1} \mathbb{C}^{2n})$ by:

$$d_0 F = \sum_{k,l} \nabla_{k0} f_i \omega^k \wedge \omega^l$$

$$d_1 F = \sum_{k,l} \nabla_{k1} f_i \omega^k \wedge \omega^l$$

$$\Delta F = d_0 d_1 F$$

for $F = \sum_I f_I \omega^I \in C_0^\infty(\Omega, \wedge^p \mathbb{C}^{2n})$, where the multi-index $I = (i_1, \ldots, i_p)$ and $\omega^I = \omega^{i_1} \wedge \ldots \wedge \omega^{i_p}$. The operators $d_0, d_1$ depend on the choice of coordinates $x_i$’s and the basis $\{\omega^i\}$. It is known (cf. [WW]) that the second operator $D$ in the 0-Cauchy-Fueter complex can be written as $DF := \begin{pmatrix} d_0 F \\ d_1 F \end{pmatrix}$.

Although $d_0, d_1$ are not exterior differential, their behavior is similar to exterior differential: $d_0 d_1 = -d_1 d_0$, $d_0^2 = d_1^2 = 0$; for $F \in C_0^\infty(\Omega, \wedge^p \mathbb{C}^{2n})$, $G \in C_0^\infty(\Omega, \wedge^q \mathbb{C}^{2n})$, we have

$$d_\alpha (F \wedge G) = d_\alpha F \wedge G + (-1)^p F \wedge d_\alpha G, \quad \alpha = 0, 1, \quad d_\alpha \Delta = d_\alpha = 0$$

(6)

We say $F$ is closed if $d_0 F = d_1 F = 0$, ie, $DF = 0$. For $u_1, u_2, \ldots, u_n \in C^2$, $\Delta u_1 \wedge \ldots \wedge \Delta u_k$ is closed, $k = 1, \ldots, n$. Moreover, it follows easily from (6) that $\Delta u_1 \wedge \ldots \wedge \Delta u_n$ satisfies the following remarkable identities:

$$\Delta u_1 \wedge \ldots \wedge \Delta u_n = d_0(d_1 u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n) = -d_1(d_0 u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n)$$

$$= d_0 d_1(u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n) = \Delta(u_1 \wedge \Delta u_2 \wedge \ldots \wedge \Delta u_n).$$

To write down the explicit expression, we define for a function $u \in C^2$,

$$\Delta_{ij} u := \frac{1}{2} (\nabla_{i0} \nabla_{j1} u - \nabla_{i1} \nabla_{j0} u).$$

$2\Delta_{ij}$ is the determinant of $(2 \times 2)$- submatrix of $i$-th row in (5). Then we can write

$$\Delta u = \sum_{i,j=0}^{2n-1} \Delta_{ij} u \omega^i \wedge \omega^j.$$
and for \( u_1, \ldots, u_n \in C^2 \),

\[
\Delta u_1 \wedge \ldots \wedge \Delta u_n = \sum_{i_1, j_1, \ldots} \Delta_{i_1, j_1} u_{i_1} \ldots \Delta_{i_n, j_n} u_{i_n} \omega^{i_1} \wedge \omega^{j_1} \wedge \ldots \wedge \omega^{i_n} \wedge \omega^{j_n}
\]

\[
= \sum_{i_1, j_1, \ldots} \delta^i_{j_1, -1} \delta_{01, (2n-1)} \Delta_{i_1, j_1} u_{i_1} \ldots \Delta_{i_n, j_n} u_{i_n} \Omega_{2n},
\]

where \( \Omega_{2n} \) is defined as

\[
\Omega_{2n} := \omega^0 \wedge \omega^1 \wedge \ldots \omega^{2n-2} \wedge \omega^{2n-1},
\]

and \( \delta^i_{j_1, -1} = 1 \) is the sign of the permutation from \((i_1, j_1, \ldots, j_n)\) to \((0, 1, \ldots, 2n-1)\), if \( \{i_1, j_1, \ldots, i_n, j_n\} = \{0, 1, \ldots, 2n-1\} \); otherwise, \( \delta^i_{j_1, -1} = 0 \). Note that \( \Delta u_1 \wedge \ldots \wedge \Delta u_n \) is symmetric with respect to the permutation of \( u_1, \ldots, u_n \). In particular, when \( u_1 = \ldots = u_n = u \), \( \Delta u_1 \wedge \ldots \wedge \Delta u_n \) coincides with \( (\Delta u)^n = \wedge^n \Delta u \). We denote by \( \Delta_n(u_1, \ldots, u_n) \) the coefficient of the form \( \Delta u_1 \wedge \ldots \wedge \Delta u_n \), i.e., \( \Delta u_1 \wedge \ldots \wedge \Delta u_n = \Delta_n(u_1, \ldots, u_n) \Omega_{2n} \). Then \( \Delta_n(u_1, \ldots, u_n) \) coincides with the mixed Monge-Ampère operator \( \det(u_1, \ldots, u_n) \) while \( \Delta_n u \) coincides with the quaternionic Monge-Ampère operator \( \det(u) \), we gave an elementary and simpler proof in Appendix A of [WW].

Denote by \( \wedge^{2k}_{\mathbb{R}} \mathbb{C}^{2n} \) the subspace of all real elements in \( \wedge^{2k} \mathbb{C}^{2n} \) following Alesker [A2]. They are counterparts of \((k, k)\) forms in several complex variables. In the space \( \wedge^{2k}_{\mathbb{R}} \mathbb{C}^{2n} \) and \( \wedge^{2k}_{\mathbb{R}} \mathbb{C}^{2n} \) and \( \mathbb{P}^{2k} \mathbb{C}^{2n} \) of positive and strongly positive elements, respectively. Denoted by \( \mathcal{D}^{2k}(\Omega) \) the set of all \( C^0(\Omega) \) functions valued in \( \wedge^{2k} \mathbb{C}^{2n} \). \( \eta \in \mathcal{D}^{2k}(\Omega) \) is called a positive form (respectively, strongly positive form) if for any \( \Omega \), \( \eta(\Omega) \) is positive (respectively, strongly positive) element. Such forms are the same as the sections of certain line bundle introduced by Alesker [A2] when the manifold is flat. We proved that for \( u \in \mathcal{P}SH \cap C^2(\Omega) \), \( \Delta u \) is a closed strongly positive 2-form.

An element of the dual space \( (\mathcal{D}^{2n-p}(\Omega))' \) is called a \( p \)-current. Denoted by \( \mathcal{D}^p_0(\Omega) \) the set of all \( C^0(\Omega) \) functions valued in \( \wedge^p \mathbb{C}^{2n} \). The elements of the dual space \( (\mathcal{D}^{2n-p}_0(\Omega))' \) are called \( p \)-currents of order zero. Obviously, the \( 2n \)-currents are just the distributions on \( \Omega \), whereas the \( 2n \)-currents of order zero are Radon measures on \( \Omega \).

A \( 2k \)-current \( T \) is said to be positive if we have \( T(\eta) \geq 0 \) for any strongly positive form \( \eta \in \mathcal{D}^{2n-2k}(\Omega) \). Although a \( 2n \)-form is not an authentic differentiel form and we cannot integrate it, we can define

\[
\int_{\Omega} F := \int_{\Omega} f dV,
\]

if we write \( F = f \Omega_{2n} \in L^1(\Omega, \wedge^{2n} \mathbb{C}^{2n}) \), where \( dV \) is the Lebesgue measure.

In particular, if \( F \) is positive \( 2n \)-form, then \( \int_{\Omega} F \geq 0 \). For a \( 2n \)-current \( F = \mu \Omega_{2n} \) with coefficient to be measure \( \mu \), define

\[
\int_{\Omega} F := \int_{\Omega} \mu.
\]

Any positive \( 2k \)-current \( T \) on \( \Omega \) has measure coefficients (i.e.is of order zero)(cf [WW] for more details). For a positive \( 2k \)-current \( T \) and a strongly positive test form \( \varphi \), we can write \( T \wedge \varphi = \mu \Omega_{2n} \) for some Radon measure \( \mu \). We have

\[
T(\varphi) = \int_{\Omega} T \wedge \varphi.
\]

Now for the \( p \)-current, \( F \), we define \( d_\alpha F \) as \( (d_\alpha F)(\eta) := -F(d_\alpha \eta), \alpha = 0, 1 \), for any test \( (2n-p-1) \)-form \( \eta \).

We say a current \( F \) is closed if \( d_0 F = d_1 F = 0 \), i.e, \( DF = 0 \). Wan and Wang proved \( \Delta u \) is closed positive \( 2 \)-current for any \( u \in \mathcal{P}SH(\Omega) \).
Lemma 1.6. (Stokes-type formula, [WW, Lemma 3.2]).
Assume that \( T \) is a smooth \((2n - 1)\)-form in \( \Omega \), and \( h \) is a smooth function with \( h = 0 \) on \( \partial \Omega \). Then we have
\[
\int_{\Omega} hd_{\alpha}T = -\int_{\Omega} d_{\alpha}h \wedge T, \quad \alpha = 0, 1,
\]

Bedford-Taylor theory [BT] in complex analysis can be generalized to the quaternionic case. Let \( u \) be a locally bounded PSH function and let \( T \) be a closed positive \(2k\)-current. Define
\[
\Delta u \wedge T := \Delta(uT),
\]
i.e., \((\Delta u \wedge T)(\eta) := uT(\Delta \eta)\) for test form \( \eta \). \( \Delta u \wedge T \) is also a closed positive current. Inductively,
\[
\Delta u_1 \wedge \ldots \wedge \Delta u_p := \Delta(u_1 \Delta u_2 \wedge \ldots \wedge \Delta u_p)
\]
is closed positive \(2p\)-current. In particular, for \( u_1, \ldots, u_n \in PSH \cap L_{\text{loc}}^{\infty}(\Omega) \), Wan and Wang showed that
\[
\Delta u_1 \wedge \ldots \wedge \Delta u_n = \mu_{\Omega_{2n}}
\]
for a well-defined positive Radon measure \( \mu \).

For any test \((2n - 2p)\)-form \( \psi \) on \( \Omega \), we have
\[
\int_{\Omega} \Delta u_1 \wedge \ldots \wedge \Delta u_p \wedge \psi = \int_{\Omega} u_1 \Delta u_2 \wedge \ldots \wedge \Delta u_p \wedge \Delta \psi,
\]
where \( u_1, \ldots, u_p \in PSH \cap L_{\text{loc}}^{\infty}(\Omega) \).

Given a bounded plurisubharmonic function \( u \) one can define the quaternionic Monge-Ampère measure
\[
(\Delta u)^n = \Delta u \wedge \Delta u \wedge \ldots \wedge \Delta u.
\]
This is a nonnegative Borel measure.

The following capacity was introduced in [WZ] for Borel sets \( E \subset \Omega \):
\[
cap(E, \Omega) = \sup\{ \int_{E} (\Delta u)^n : u \in PSH(\Omega), -1 \leq u < 0 \}.
\]
It is closely related to the relative extremal function of the given compact set \( K \):
\[
u_K(q) = \sup\{ u(q) : u \in PSH \cap L^{\infty}(\Omega), u < 0 \text{ in } \Omega, u \leq -1 \text{ on } K \},
\]
Its upper semicontinuous regularization \( u_K^*(q) := \lim_{\zeta \to q} u_K(\zeta) \) is a plurisubharmonic function and by [WK] we have
\[
cap(K, \Omega) = \int_{K} (\Delta u_K^*)^n = \int_{\Omega} (\Delta u_K^*)^n.
\]
W.Wang introduced in [W] the operator
\[
\Delta_a v := \frac{1}{2} Re \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 v}{\partial q_j \partial q_k} = \frac{1}{2} Re Tr(a(\frac{\partial^2 v}{\partial q_j \partial q_k}))
\]
for \( a = (a_{jk}) \in \mathcal{H}_n \), with \( \mathcal{H}_n \) the set of all positive quaternionic hyperhermitian \((n \times n)\) matrices, and a \( C^2 \) real function \( v \). This is an elliptic operator of constant coefficients. This operator is the
quaternionic counterpart of complex Kähler operator, which plays key role in the viscosity approach for the complex case. For more details see [W] and [WW1]. With the help of this operator, we can prove this following result, by applying the same ideas from the proof of proposition 3.1 in [WW1] and Proposition 5.9 in [GZ17].

We set

\[ \mathcal{H}_n' := \{ a \in \mathcal{H}_n / \det a \geq 1 \}. \]

**Proposition 1.7.** Let \( u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega) \) and \( 0 \leq f \in C(\overline{\Omega}) \). The following conditions are equivalent:

1. \( \Delta_au \geq a_n f^\frac{1}{n} \) for all \( a \in \mathcal{H}_n' \).

2. \( (\Delta u)^n \geq fdV \) in \( \Omega \)

where \( a_n = \frac{n}{2(n!)^\frac{1}{n}} \).

**Proof.** \( 2 \implies 1 \). Fix \( q_0 \in \Omega \) and \( \varphi \in C^2 \) in neighborhood \( B \in \Omega \) of \( q_0 \) such that \( u \leq \varphi \) in \( B \) and \( u(q_0) = \varphi(q_0) \). We will prove that \( (\Delta \varphi)^n_{q_0} \geq f(q_0)dV \). Suppose by contradiction that \( (\Delta \varphi)^n_{q_0} < f(q_0)dV \), by choosing \( \epsilon > 0 \) small enough and letting \( \varphi_\epsilon := \varphi + \epsilon |q - q_0|^2 \), we have \( 0 < (\Delta \varphi_\epsilon)^n_{q_0} < f(q_0)dV \) in \( B \) by the continuity of \( f \). It follows from the proof of proposition 3.1 in [WW1] that \( \varphi_\epsilon \) is plurisubharmonic in \( B \). Now for \( \delta > 0 \) small enough, we have \( \varphi_\epsilon - \delta \geq u \) near \( \partial B \) and \( (\Delta \varphi_\epsilon)^n_{q_0} \leq (\Delta u)^n_{q_0} \). The pluripotential comparison principle yields \( \varphi_\epsilon - \delta \geq u \) on \( B \). But \( \varphi_\epsilon(q_0) = \varphi(q_0) = u(q_0) \), a contradiction. Hence \( (\Delta \varphi)^n_{q_0} \geq f(q_0)dV \). Then the hyperhermitian matrix \( Q = [\frac{\partial^2 \varphi}{\partial q_j \partial q_k}(q_0)] \) satisfies \( \det(Q) \geq f \) at \( q_0 \).

By lemma 3.4 in [WW1], we have

\[ (n! \det Q)^\frac{1}{n} = (n!)^\frac{1}{n} \frac{2}{\pi} \inf_a \Delta_a \varphi \geq f^\frac{1}{n} \]

for every \( a \in \mathcal{H}_n' \). Hence \( \Delta_a \varphi \geq a_n f^\frac{1}{n} \).

If \( f > 0 \) is smooth function, there exists \( g \in C^\infty(\overline{\Omega}) \) such that \( \Delta_ag = a_n f^\frac{1}{n} \). Thus \( h = u - g \)

is subharmonic respect to \( \Delta_a \) by Proposition 3.2.10 in [H], and satisfies \( \Delta_h \geq 0 \) in the sense of distributions. Hence \( \Delta_au \geq a_n f^\frac{1}{n} \).

If \( f > 0 \) is only continuous, we observe that

\[ f = \sup \{ W, W \in C^\infty(\overline{\Omega}), f \geq W > 0 \} \]

Since \( (\Delta u)^n \geq fdV \), we get \( (\Delta u)^n \geq WdV \). By the proof above, we can see that \( (\Delta_au) \geq a_n W^\frac{1}{n} \), therefore \( \Delta_au \geq a_n f^\frac{1}{n} \).

Now let \( f \geq 0 \) be continuous. We observe that \( u_\epsilon(q) = u(q) + \epsilon \|q\|^2 \) satisfies \( (\Delta u_\epsilon)^n \geq (f + 8^n e^n)dV \), since \( (\Delta \|q\|^2)^n = 8^n \beta_n^n \). By the last part above, we have \( \Delta_au_\epsilon \geq a_n (f + 8^n e^n)^\frac{1}{n} \). The result follows by letting \( \epsilon \to 0 \).

1 \( \implies \) 2. Suppose that \( u \in C^2(\Omega) \) then by lemma 3.4 in [WW1], we have \( \Delta_au \geq a_n f^\frac{1}{n} \) is equivalent to \( (\det(\frac{\partial^2 u}{\partial q_j \partial q_k})))^{\frac{1}{n}} \geq f^\frac{1}{n} \), which it itself equivalent to \( (\Delta u)^n \geq fdV \) in \( \Omega \).

If \( u \) is not smooth, we consider the standart regularisation \( u_\epsilon \) of \( u \) by convolution with a smoothing kernel. The function \( u_\epsilon := u * \chi_\epsilon \) are plurisubharmonic in \( \Omega_\epsilon \) and decrease to \( u \) as \( \epsilon \) decrease to 0. We have \( \Delta_au_\epsilon \geq (a_n f^\frac{1}{n})_\epsilon \), since \( u_\epsilon \) is smooth, we have

\[ (\Delta u_\epsilon)^n \geq ((f^\frac{1}{n})_\epsilon)^n dV. \]
Letting $\epsilon \to 0$, and applying the convergence theorem for the quaternionic Monge-Ampère operator, we get $(\Delta u)^n \geq fdV$ in $\Omega$.

Consider

$$U = U(\Omega, \psi, f) = \{ u \in PSH \cap C(\overline{\Omega}), u|_{\partial \Omega} \leq \psi \text{ and } \Delta_a u \geq a_n f^\frac{1}{n}, \forall a \in \mathcal{H}_n' \}$$

It is easy to show that $U$ is non empty. Then by proposition 1.7, we can describe the solution as the following

$$U = \sup \{ u \in U(\Omega, \psi, f) \}.$$

## 2 The Modulus of continuity of The solution

With the help of [C], we can use in this part the modulus of continuity of the solution to Dirichlet problem for quaternionic Monge-Ampère equation (3).

Recall that a real function $\theta$ on $[0, r]$, $0 < r < \infty$ is called a modulus of continuity if $\theta$ is continuous, subadditive, nondecreasing and $\theta(0) = 0$. In general, $\theta$ fails to be concave, we denote $\overline{\theta}$ to be the minimal concave majorant of $\theta$. We denote $\theta_\varphi$ the optimal modulus of continuity of the continuous function $\varphi$ which is defined by

$$\theta_\varphi(t) = \sup_{|x-y| \leq t} |\varphi(x) - \varphi(y)|.$$

Now, we will prove the following result which is the one of the useful properties of $\overline{\theta}$

**Lemma 2.1.** Let $\theta$ be a modulus of continuity on $[0, r]$ and $\overline{\theta}$ be the minimal concave majorant of $\theta$. Then $\theta(\lambda t) < \overline{\theta}(\lambda t) < (1 + \lambda)\theta(t)$ for any $t > 0$ and $\lambda > 0$.

**Proof.** The same proof of Lemma 3.1 in [C].

In the following result, we establish a barrier to the problem (3) and give an estimate of its modulus of continuity, which will be used in the proof of Theorem 2.4

**Proposition 2.2.** Let $\Omega \subset \mathbb{H}^n$ be a bounded strongly pseudoconvex domain with smooth boundary, assume that $\theta_\psi$ is the modulus of continuity of $\psi \in C(\partial \Omega)$ and $0 \leq f \in C(\overline{\Omega})$. Then there exists a subsolution $u \in U(\Omega, \psi, f)$ such that $u = \psi$ on $\partial \Omega$ and the modulus of continuity of $u$ satisfies the following inequality

$$\theta_u(t) \leq \eta \max \{ \theta_\psi(t^\frac{1}{n}), t^\frac{1}{n} \},$$

where $\eta = (1 + a_n\|f\|_{L_\infty(\overline{\Omega})})$ and $\lambda \geq 1$ is a constant depending on $\Omega$.

**Proof.** Fix $\xi \in \partial \Omega$. We will prove that there exists $u_\xi \in U(\Omega, \psi, f)$ such that $u_\xi(\xi) = \psi(\xi)$.

As in the proof of proposition 3.2 in [C], and by using Lemma 2.1 we prove that there exists a constant $C > 0$ depending only on $\Omega$ such that every point $\xi \in \partial \Omega$ and $\psi \in C(\partial \Omega)$, there is a function $v_\xi \in PSH(\Omega) \cap C(\overline{\Omega})$ such that

1. $v_\xi(q) \leq \psi(q)$ for all $q \in \partial \Omega$
2. $v_\xi(\xi) = \psi(\xi)$
3. $\theta_{v_\xi}(t) \leq C\theta_\psi(t^{1\over 2})$.

Fix a point $q_0 \in \Omega$ and choose $K_1 \geq 0$ such that $K_1 = a_n \sup_{\text{harmonic} f} f^{1\over n}$. Then

$$\Delta_a(K_1|q - q_0|^2) = K_1 \Delta_a|q - q_0|^2 \geq a_n f^{1\over n}(q),$$

for all $a \in \mathcal{H}'_n$. Set $K_2 = K_1|\xi - q_0|^2$. Then for the continuous function $\tilde{\psi}(q) := \psi(q) - K_1|q - q_0|^2 + K_2$ we have $v = v_\xi$ such that 1, 2 and 3 hold. Then $u_\xi \in \mathcal{U}(\Omega, \psi, f)$ is given by

$$u_\xi(q) := v(q) + K_1|q - q_0|^2 - K_2$$

Indeed, $u_\xi \in PSH(\Omega) \cap C(\overline{\Omega})$ and we have $v(q) \leq \tilde{\psi}(q) = \psi(q) - K_1|q - q_0|^2 + K_2$ on $\partial \Omega$. So that $u_\xi(q) \leq \psi(q)$ on $\partial \Omega$ and $u_\xi(\xi) = \psi(\xi)$. We have

$$\Delta_a u_\xi = \Delta_a v + K_1 \Delta_a|q - q_0|^2 \geq a_n f^{1\over n} \text{ in } \Omega.$$ 

Then, by the hypothesis, we can get an estimate for the modulus of continuity of $u_\xi$

$$\theta_{u_\xi}(t) = \sup_{|u(q) - u(q')| \leq t} \left| u(q) - u(q') \right| \leq \theta_v(t) + K_1 \theta_{|q - q_0|^2}(t) \leq C\theta_\psi(t^{1\over 2}) + 4d^{1\over 2}K_1 t^{1\over 2} \leq C\theta_\psi(t^{1\over 2}) + 2dK_1(C + 2d^{1\over 2})t^{1\over 2} \leq (C + 2d^{1\over 2})(1 + 2dK_1) \max\{\theta_\psi(t^{1\over 2}), t^{1\over 2}\}$$

Then, we choose $\lambda$ so that $\theta_{u_\xi}(t) \leq \lambda(1 + a_n\|f\|^{1\over L_\infty(\Omega)}) \max\{\theta_\psi(t^{1\over 2}), t^{1\over 2}\}$. Hence the desired result follows. 

Corollary 2.3. Taking the same assumption of Proposition 2.2. There exists a plurisuperharmonic function $\tilde{u} \in C(\overline{\Omega})$ such that $\tilde{u} = \psi$ on $\partial \Omega$ and

$$\theta_{\tilde{u}}(t) \leq \eta \max\{\theta_\psi(t^{1\over 2}), t^{1\over 2}\},$$

where $\eta = \lambda(1 + a_n\|f\|^{1\over L_\infty(\Omega)})$ and $\lambda \geq 1$ is a constant depending on $\Omega$.

Proof. We can use the same construction as in the proof of Proposition 2.2 for $\psi_1 = -\psi \in C(\partial \Omega)$, then there exists $u_1 \in \mathcal{U}(\Omega, \psi_1, f)$ such that $u_1 = \psi_1$ on $\partial \Omega$ and $\theta_{u_1}(t) \leq \eta \max\{\theta_{\psi_1}(t^{1\over 2}), t^{1\over 2}\}$. Then, we set $\tilde{u} = -u_1$ which is plurisuperharmonic function on $\Omega$, continuous on $\overline{\Omega}$ and satisfies $\tilde{u} = \psi$ on $\partial \Omega$ and $\theta_{\tilde{u}}(t) \leq \eta \max\{\theta_{\psi}(t^{1\over 2}), t^{1\over 2}\}$. 

Now, we are in position to prove an estimate for the modulus of continuity of the solution to Dirichlet problem for quaternionic Monge-Ampère equation.

Theorem 2.4. Let $\Omega$ be a smoothly bounded strongly pseudoconvex domain in $\mathbb{H}^n$, suppose that $0 \leq f \in C(\overline{\Omega})$ and $\psi \in C(\partial \Omega)$. Then the modulus of continuity $\theta_u$ of the solution $u$ satisfies the following estimate

$$\theta_u(t) \leq \gamma(1 + a_n\|f\|^{1\over L_\infty(\Omega)}) \max\{\theta_\psi(t^{1\over 2}), a_n t^{1\over n}(t), t^{1\over 2}\}$$

where $\gamma \geq 1$ is a constant depending only on $\Omega$. 

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Proof. Thanks to Proposition 2.2, Corollary 2.3 and the comparaison principle (Corollary 1.1 in [WZ]), we can follow the same proof of Theorem 1.1 in [C], with setting \( g(t) = \max\{\eta \max(\theta_\psi(t_1^{\frac{1}{2}}, t_2^{\frac{1}{2}}), a_n\theta f_\psi(t)\} \) and we get the desired result. \( \square \)

Now, it is easy to check that this previous theorem has the following consequence.

**Corollary 2.5.** Let \( \Omega \) be a smoothly bounded strongly pseudoconvex domain in \( \mathbb{H}^n \). Let \( \psi \in \text{Lip}_{2\alpha}(\partial \Omega) \) and \( 0 \leq f < 0 \in \text{Lip}_\alpha(\Omega) \), \( 0 < \alpha \leq \frac{1}{2} \). Then the unique solution of Dirichlet problem \( u \) is \( \alpha \)-Hölder continuous on \( \overline{\Omega} \).

### 3 The stability estimate

In this section, the main goal is to prove the stability estimate, Theorem 3.5. For this end, we need some results which are the following:

**Lemma 3.1.** Let \( u, v \in PSH \cap L^\infty(\Omega) \) such that \( \lim_{\zeta \to \partial \Omega} (u - v)(\zeta) > 0 \). Then for all \( t, s > 0 \),

\[
s^n \text{cap}(\{u - v < -t - s\}) \leq \int_{\{u - v < -t\}} (\Delta u)^n.
\]

**Proof.** Take \(-1 \leq \varphi \leq 0\) a psh function in \( \Omega \). We have \( \{u - v < -t - s\} \subset \{u < v - t + sv\} \subset \{u < v - t\} \subset \Omega \). By the comparaison principle [WZ, Theorem 1.2] we find

\[
s^n \int_{\{u - v < -t - s\}} (\Delta \varphi)^n \leq \int_{\{u - v < -t + sv\}} (\Delta(-t + v + s\varphi))^n \leq \int_{\{u - v < -t\}} (\Delta u)^n.
\]

Taking the supremum and the lemma follows. \( \square \)

Now, we are going to prove the following estimate which play an important role in the rest.

**Lemma 3.2.** Assume that \( 0 \leq f \in L^p(\Omega), p > 2 \), and a fixed \( \alpha \in (1, 2) \). Then there exists a constant \( D = D(\alpha, \|f\|_{L^p}) > 0 \) such that for every \( E \subset \Omega \)

\[
0 \leq \int_E f dV \leq D[\text{cap}(E)]^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** By Holder inequality and using Lemma 3 in [SM], we have

\[
\int_E f dV \leq \|f\|_{L^p(\Omega)} V(E)^{\frac{1}{q}} \leq C(\alpha)\|f\|_{L^p(\Omega)}[\text{cap}(E)]^{\frac{1}{q}} \leq D(\alpha, \|f\|_{L^p(\Omega)})[\text{cap}(E)]^{\frac{1}{q}}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \), hence the lemma follows. \( \square \)

We will also need the following result, which its proof is similar to Lemma 2.4 in [EGZ].
Lemma 3.3. Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a decreasing right-continuous function such that \( \lim_{t \to \infty} f = 0 \). Assume there exists \( \tau > 1, B > 0 \) such that \( f \) satisfies

\[
\tau f(s + t) \leq B f(t)^\tau, \forall t, s > 0.
\]

Then, there exists \( S_\infty := \frac{2Bf(0)^{r-1}}{1-2^{1-r}} \) such that \( f(s) = 0 \) for all \( s \geq S_\infty \).

Proposition 3.4. Let \( u, v \in \text{PSH} \cap L^\infty(\Omega) \) be such that \( \lim_{r \to 0} (u-v)(\zeta) \geq 0 \) and \( 0 \leq f \in L^p(\Omega), p > 2 \). Suppose that \( (\Delta u)^n = f dV \), then for any \( 0 < \beta < \frac{1}{n}(\frac{2}{q} - 1), \frac{1}{p} + \frac{1}{q} = 1 \), there exists a constant \( C = C(\alpha, \| f \|_{L^p(\Omega)}) \) such that for all \( \epsilon > 0 \)

\[
\sup_{\Omega} (v - u) \leq \epsilon + C[\text{cap}(\{u - v < -\epsilon\})]^\beta.
\]

Proof. By Lemmas 3.1 et 3.2, the function \( g(s) := \text{cap}(\{u - v < -\epsilon - s\}) \) satisfies the conditions of Lemma 3.3, we obtain \( \text{cap}(\{u - v < -s_\infty - \epsilon\}) = 0 \) which means that \( v - u \leq \epsilon + s_\infty \) almost everywhere on \( \Omega \). Finally, if we choose \( \tau := 1 + \beta n \) we obtain \( \sup_{\Omega} (v - u) \leq \epsilon + C[\text{cap}(\{u - v < -\epsilon\})]^\beta \) where \( C := 2B/(1 - 2^{-\beta n}) \).

We are now in the position to prove the main stability estimate, which is similar to Theorem 1.1 in [GKZ] for the complex case.

Theorem 3.5. Let \( u_1, u_2 \in \text{PSH} \cap L^\infty(\Omega) \) be such that \( u_1 \geq u_2 \) on \( \partial \Omega \), and \( 0 \leq f \in L^p(\Omega), p > 2 \). Suppose that \( (\Delta u_1)^n = f dV \) in \( \Omega \). Fix \( r \geq 1 \) and \( 0 < \gamma < \gamma_r \), with \( \gamma_n := \frac{r}{nq + r + \frac{q}{q - 1}}, \frac{1}{p} + \frac{1}{q} = 1 \). Then there exists a constant \( C = C(\gamma; \| f \|_{L^p(\Omega)}) > 0 \) such that

\[
\sup_{\Omega} (u_2 - u_1) \leq C[\| (u_2 - u_1)_+ \|_{L^r(\Omega)}]^\gamma,
\]

where \( (u_2 - u_1)_+ := \max(u_2 - u_1, 0) \).

Proof. Using Lemma 3.1 with \( s = t = \epsilon > 0 \) and by Hölder inequality, we obtain

\[
\text{Cap}(\{u_1 - u_2 < -2\epsilon\}) \leq \epsilon^{-n} \int_{\{u_1 - u_2 < -\epsilon\}} f dV \\
\quad \leq \epsilon^{-n - \frac{\gamma}{q}} \int_{\Omega} (u_2 - u_1)^\frac{\gamma}{q} f dV \\
\quad \leq \epsilon^{-n - \frac{\gamma}{q}} \| (u_2 - u_1)_+ \|_{L^r(\Omega)} \| f \|_{L^p(\Omega)}^\gamma.
\]

By Proposition 3.4, we get

\[
\sup_{\Omega} (u_2 - u_1) \leq 2\epsilon + C \epsilon^{-\beta(n + \frac{\gamma}{q})} \| (u_2 - u_1)_+ \|_{L^r(\Omega)}^\gamma \| f \|_{L^p(\Omega)}^\beta.
\]

Fix \( \gamma \) and set \( \epsilon := \| (u_2 - u_1)_+ \|_{L^r(\Omega)}^\gamma \), we get

\[
\sup_{\Omega} (u_2 - u_1) \leq 2\| (u_2 - u_1)_+ \|_{L^r(\Omega)}^\gamma + C \| (u_2 - u_1)_+ \|_{L^r(\Omega)}^{\frac{-\gamma \beta}{q}} \| f \|_{L^p(\Omega)}^\beta.
\]

If we choose \( \beta = \frac{\gamma q}{r - \gamma(r + nq)} \), we easily obtain the estimate of this Theorem. \( \square \)
Hölder continuous solutions to quaternionic Monge-Ampère equations

For a fixed $\delta > 0$, we set $\Omega_\delta := \{q \in \Omega / \text{dist}(q, \partial \Omega) > \delta\}$;

$$u_\delta(q) := \sup_{\|\zeta\|<\delta} u(q + \zeta), \ q \in \Omega_\delta;$$

and

$$\hat{u}_\delta(q) := \frac{1}{\tau_{4n}\delta^{4n}} \int_{\|\zeta - q\|\leq \delta} u(\zeta) dV_{4n}(\zeta), \ q \in \Omega_\delta,$$

where $\tau_{4n}$ is the volume of the unit ball in $\mathbb{H}^n$.

In the following result, we show the link between $u_\delta$ and $\hat{u}_\delta$.

Lemma 4.1. Given $0 < \beta < 1$, the following two conditions are equivalent:

1. There exist $\eta_1, A_1 > 0$ such that for any $0 < \delta \leq \eta_1$

   $$u_\delta - u \leq A_1\delta^\beta, \ \text{on} \ \Omega_\delta.$$

2. There exist $\eta_2, A_2 > 0$ such that for any $0 < \delta \leq \eta_2$

   $$\hat{u}_\delta - u \leq A_2\delta^\beta, \ \text{on} \ \Omega_\delta.$$

Proof. This result is proved in [GKZ] for the complex case, we will follow the same proof of Lemma 4.2 in [GKZ].

The content of our next result (Lemma 4.3) is to control the growth of $\|u_\delta - u\|_{L^2(\Omega_\delta)}$ and $\|\hat{u}_\delta - u\|_{L^1(\Omega_\delta)}$, but before we are in need of this following lemma.

Lemma 4.2. Suppose that $\Omega$ is a domain, $a \in \Omega$, $B(a, r) \Subset \Omega$, and $u$ is a psh function. Then for $r > 0$, $q \in \mathbb{H}^n$,

$$\int_{B(a, r)} \Delta u \wedge \Delta\left(\frac{-1}{\|q - a\|^2}\right)^{n-1} = \frac{1}{\tau_{4n}^{4n-4}} \int_{B(a, r)} \Delta u \wedge \beta_n^{n-1}.$$ 

Proof. First, we are going to prove that

$$\int_{\{a\}} \Delta u \wedge \Delta\left(\frac{-1}{\|q - a\|^2}\right)^{n-1} = 0.$$ 

It follows from the proof of Proposition 4.1 in [WW], and by lemma 4.1 in [WW] for $\frac{-1}{\|q - a\|^2 + \epsilon}$, we get

$$\Delta u \wedge \Delta\left(\frac{-1}{\|q - a\|^2 + \epsilon}\right)^{n-1} = \sum_{i_1j_1...i_nj_n} \delta_{01...(2n-1)}^{i_1j_1...i_nj_n} \Delta_{i_1j_1} u \Delta_{i_2j_2} \left(\frac{-1}{\|q - a\|^2 + \epsilon}\right) \ldots \Delta_{i_nj_n} \left(\frac{-1}{\|q - a\|^2 + \epsilon}\right) \Omega_{2n}$$

$$= \left(\frac{-4}{\|q - a\|^2 + \epsilon}\right)^{n-1} \sum_{i_1j_1...i_nj_n} \delta_{01...(2n-1)}^{i_1j_1...i_nj_n} \Delta_{i_1j_1} u \left(\mathbb{M}_{i_2j_2} - \sum_{k_2} \delta_{i_2j_2}^{(2k_2)(2k_2+1)} (\|q - a\|^2 + \epsilon)\right) \ldots$$
\[
\cdots (M_{i,j} - \sum_{k_n} \delta_{(2k_n)(2k_n+1)} (\|q-a\|^2 + \epsilon))\Omega_{2n} = \left(\frac{-4}{\|q-a\|^2 + \epsilon}\right)^n [\sum_{k_1 \ldots k_n} 2^n \delta_{(2k_1)(2k_1+1)} \ldots (2k_{n}) \Delta_{(2k_1)(2k_1+1)} u. (-\|q-a\|^2 - \epsilon)^{n-1} + \sum_{i_1 j_1 i_2 j_2 k_1 \ldots k_n} 2^{n-2} \delta_{(2k_1)(2k_1+1)} \ldots (2k_{n}) \Delta_{i_1 j_1} u. (M_{i_2 j_2} \ldots M_{i_n j_n}) \Omega_{2n}.
\]

Note that in the right hand side above, except for the first two sums, all other sums vanish by simple computation, (for more details see proof of proposition 4.1 in [WW]).

\[u\] is a locally bounded psh function on \(\Omega\), so there exists \(C > 0\) such that \(\|\Delta_{ij} u\|_{L^\infty(\Omega)} \leq C\) for all \(i, j\), and there exists \(C' > 0\) such that \(\|\Delta_{(2k)(2k+1)} u\|_{L^\infty(\Omega)} \leq C'\) for all \(k\). Then, by straightforward computation we get

\[
\int_{B(a,s)} 2^n \delta_{(2k_1)(2k_1+1)} \ldots (2k_{n}) \Delta_{(2k_1)(2k_1+1)} u. (-\|q-a\|^2 - \epsilon)^{n-1} dV \leq 2^n! C' \int_{B(a,s)} (-\|q-a\|^2 - \epsilon)^{n-1} dV,
\]

and for \(C > 0\) large enough, we have

\[
\int_{B(a,s)} \sum_{i_1 j_1 i_2 j_2 k_1 \ldots k_n} 2^{n-2} \delta_{(2k_1)(2k_1+1)} \ldots (2k_{n}) \Delta_{i_1 j_1} u. (M_{i_2 j_2} \ldots M_{i_n j_n} \Omega_{2n}.
\]

by the fact that \(\|q-a\|^2 = \sum_{k=0}^n M_{(2k)}\). So by simple computation, we get

\[
\int_{B(a,s)} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2 + \epsilon}))^{n-1} dV \leq C' \int_{B(a,s)} \frac{8^n n!}{\|q-a\|^2 + \epsilon}^{2n-2} dV.
\]

Then

\[
\int_{\|q-a\| < \epsilon} (\Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2 + \epsilon}))^{n-1}) dV \leq \lim_{\epsilon \to 0} \int_{\|q-a\| < \epsilon} (\Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2 + \epsilon}))^{n-1}) dV
\]

\[
\leq \lim_{\epsilon \to 0} s_{4n} C' \int_0^s \frac{8^n n! 4^{n-1}}{(\epsilon + t^2)^{2n-2}} dt
\]

\[
= s_{4n} C' \int_0^s \frac{8^n n! t^{4n-1}}{t^{4n-4}} dt
\]

\[
= s_{4n} C' s^n n! \int_0^s t^3 dt = s_{4n} C' s^n n! \frac{s^4}{4}
\]

So

\[
\int_{\{a\}} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2}))^{n-1} \leq \lim_{s \to 0} s_{4n} C' \frac{8^n n!}{4} s^4 = 0.
\]
then

$$\int_{\{a\}} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2}))^{n-1} = 0.$$  

On the other hand, by proposition 4.2 in [WW], we have for $0 < s < r$,

$$\int_{B(a,r) \setminus B(a,s)} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2 + \epsilon}))^{n-1} = \frac{1}{r^{4n-4}} \int_{B(a,r)} \Delta u \wedge \beta_n^{n-1} - \frac{1}{s^{4n-4}} \int_{B(a,s)} \Delta u \wedge \beta_n^{n-1}$$

tend $s$ to 0, we get

$$\int_{B(a,r) \setminus \{a\}} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2 + \epsilon}))^{n-1} = \frac{1}{r^{4n-4}} \int_{B(a,r)} \Delta u \wedge \beta_n^{n-1} - \nu_u(a),$$

where $\nu_u(a)$ is the Lelong number of $u$ at point $a$. Since $u$ is bounded function, $\nu_u(a) = 0$. So by the first part of this proof, we have

$$\int_{B(a,r)} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2}))^{n-1} = \int_{B(a,r) \setminus \{a\}} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2}))^{n-1} + \int_{\{a\}} \Delta u \wedge (\Delta (\frac{-1}{\|q-a\|^2}))^{n-1}$$

$$= \frac{1}{r^{4n-4}} \int_{B(a,r)} \Delta u \wedge \beta_n^{n-1}.$$ 

\[ \square \]

**Lemma 4.3.**

1. Assume that $\nabla u \in L^2(\Omega)$. Then for $\delta > 0$ small enough, we have

$$\int_{\Omega_\delta} |u_\delta(q) - u(q)|^2 dV_{4n}(q) \leq C_n \|\nabla u\|_{L^2(\Omega)}^2,$$

2. Assume that $\|\Delta u\|_{\Omega} < +\infty$. Then for $\delta > 0$ small enough, we have

$$\int_{\Omega_\delta} |\tilde{u}_\delta(q) - u(q)|^2 dV_{4n}(q) \leq C_n \|\Delta u\|_{\Omega} \delta^2,$$

where $\Delta u \wedge \beta_n^{n-1} = \Delta u_{\tilde{u}_\delta} u_{\Omega_2n}$, and $C_n > 0$ is a constant depends only on $n$.

**Proof.** For 1) see the last part in the proof of Theorem 3.1 in [GKZ].

2) It follows from Lelong-Jensen type formula (Theorem 5.1 in [WW]) and lemma 4.2, that for $q \in \Omega_\delta$, $0 < r < \delta$, $r' = \frac{1}{r^2}$, $\varphi(\xi) = \frac{-A}{\|\xi-q\|^2}$ where $A = (\frac{(2n)!}{4^{2n} n!^2})^\frac{1}{2}$, and $B_\varphi(r') = \{\xi \in \Omega, \varphi(\xi) \leq r'\}$.

$$\frac{1}{\sigma_{4n-1}} \int_{|\xi|=1} u(q + r\xi) dS_{4n-1} = u(q) + \int_{-\infty}^{r'} t^{2n-2} \int_{\|\xi-q\|^2 \leq t} \Delta u \wedge \beta_n^{n-1} dt$$

Using polar coordinates we get, for $q \in \Omega_\delta$

$$\tilde{u}(q) - u(q) = \frac{1}{\sigma_{4n-1} \delta^{4n}} \int_{0}^{\delta} r^{4n-1} dr \int_{0}^{(\frac{\delta}{r})^\frac{1}{2}} s^{1-4n} \int_{\|\xi-q\| \leq s} \Delta u \wedge \beta_n^{n-1} ds.$$

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So, by Fubini’s theorem we have
\[
\int_{\Omega_{\delta}} (\hat{u} - u) dV \leq a_n c_{\Omega}^{-4n} \int_{0}^{s} r^{n-1} dr \int_{0}^{r} s^{1-4n} \max_{|\xi - q| \leq s} (\int_{\Omega} \Delta_{H^n} u) ds
\]
\[
\leq C_n c_{\Omega}^{2} \| \Delta_{H^n} u \|.
\]

For giving us the Hölder norm estimate in $\overline{\Omega}$ of the solution $u$, we need to apply the stability estimate with $u_2 := u_{\delta}$. And in order to do that, we have to extend $u_{\delta}$ to $\Omega$, since it is only defined on $\Omega_{\delta}$.

**Proposition 4.4.** Let $u \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $u = \psi \in Lip_{2\beta}(\partial \Omega)$ on $\partial \Omega$. Then there exist a constant $c_0 = c_0(u) > 0$ and $\delta_0$ small enough such that for any $0 < \delta < \delta_0$ the function
\[
\tilde{u}_{\delta} = \begin{cases} 
\max\{u_{\delta}, u + c_0 \delta^\beta\} & \text{in } \Omega_{\delta}, \\
u + c_0 \delta^\beta, & \text{in } \Omega \setminus \Omega_{\delta}.
\end{cases}
\]
is a bounded plurisubharmonic function on $\Omega$ and $(\tilde{u}_{\delta})$ decreases to $u$ as $\delta$ decrease to $0$.

For the proof we need the following result.

**Lemma 4.5.** Fix $\psi \in Lip_{2\alpha}(\partial \Omega), f \in L^p(\Omega), p > 2$ and set $u := u(\Omega, \psi, f)$. Then there exist $\varphi, \phi \in PSH(\Omega) \cap C^\alpha(\Omega)$ such that
1. $\varphi(\xi) = \psi(\xi) = -\phi(\xi), \forall \xi \in \partial \Omega.$
2. $\varphi(q) \leq u(q) \leq -\phi(q) \forall q \in \Omega.$

**Proof.** We are going to construct a weak barrier $b_f \in PSH(\Omega) \cap Lip_{1}(\Omega)$ for the Dirichlet problem $MA(\Omega, 0, f)$ such that
- $b_f(\xi) = 0 \forall \xi \in \partial \Omega$
- $b_f \leq u(\Omega, 0, f) \text{ in } \Omega$
- $|b_f(q) - b_f(\zeta)| \leq C \|q - \zeta\| \forall q \in \Omega \forall \zeta \in \Omega$

for some uniform constant $C > 0$. First, assume $f$ is bounded near $\partial \Omega$, so $\exists K \subset \Omega 0 \leq f \leq M$ on $\Omega \setminus K$, where $K$ is a compact subset in $\Omega$.

Set $b_f := A \rho$, where $\rho$ be a $C^2$ strictly psh defining function for $\Omega$, by taking $A > 0$ large enough so that
\[
(\Delta b_f)^n \geq MdV \geq f dV \text{ on } \Omega \setminus K \text{ and } b_f \leq m \leq u(\Omega, 0, f) \text{ near } K.
\]
where $m := \min_{\Omega} u(\Omega, 0, f)$. Then $(\Delta b_f)^n \geq (\Delta u(\Omega, 0, f))$ on $\Omega \setminus K$, and $b_f \leq u(\Omega, 0, f)$ on $\partial(\Omega \setminus K)$.

This implies, by the comparaison principle ( Corollary 1.1 in [WZ]) that $b_f \leq u(\Omega, 0, f)$ in $\Omega$.

For the general case, $f$ is not bounded near $\partial \Omega$. Fix a large ball $B \subset H^n$ so that $\Omega \subset B \subset H^n$.

Set $\tilde{f} := f$ in $\Omega$ and $\tilde{f} = 0$ in $B \setminus \Omega$. By the first part of this proof, we can find a barrier function $b_{\tilde{f}} \in PSH(B) \cap C^2(B)$ for the Dirichlet problem $MA(B, 0, \tilde{f})$. Set $h := u(\Omega, -b_{\tilde{f}}, 0)$.

Since $-b_{\tilde{f}} \in C^2(\partial \Omega)$, by Corollary 2.5 $h$ is Lipshitz on $\Omega$. Set $b_f := h + b_{\tilde{f}} \in PSH(\Omega) \cap Lip_{1}(\Omega)$ is a barrier function for $MA(\Omega, 0, f)$. Moreover, by corollary 2.5 we have $u(\Omega, \pm \psi, 0)$ is Hölder continuous of order $\alpha$, where $\psi \in C^{2\alpha}(\partial \Omega)$. Then, the functions $\varphi := u(\Omega, \psi, 0) + b_f$ and $\phi := u(\Omega, -\psi, 0) + b_f$ belong to $PSH(\Omega) \cap Lip_{\alpha}(\Omega)$ and satisfies 1) and 2).
Using Lemma 4.5, and follow the same proof of Proposition 2.1 in [GKZ].

Now, we are in position to prove our main tool, which is the following.

**Theorem 4.6.** Let \( \Omega \) be a bounded pseudoconvex domain of \( \mathbb{H}^n \). Assume that \( \psi \in \text{Lip}_p(\partial \Omega) \) and fix \( f \in L^p(\Omega) \) for some \( p > 2 \). Let \( u \) be the unique solution to \( \beta \) for \( d\nu = f dV \).

1. If \( \nabla u \) belongs to \( L^2(\Omega) \), then \( u \in \text{Lip}_{\beta'}(\overline{\Omega}) \) for all \( \beta' < \min(\beta, \gamma_2) \).

2. If the total mass of \( \Delta_{\Omega^\infty} u \) is finite, then \( u \in \text{Lip}_{\beta''}(\overline{\Omega}) \) for all \( \beta'' < \min(\beta, 2\gamma_1) \).

where \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( \gamma_1, \gamma_2 \) are defined in Theorem 3.5.

**Proof.** 1) We have \( f \in L^p(\Omega) \), \( p > 2 \). By [SM], we have the solution \( u \in \text{PSH}(\Omega) \cap C(\Omega) \) is a continuous plurisubharmonic function. Then, we have to show that \( u \) is Hölder continuous on \( \Omega \).

Given \( 0 < \gamma < \frac{2}{qn+2+\frac{2}{\beta-1}} \). Applying the stability estimate Theorem 3.5 with \( r = 2 \), \( u_2 = \tilde{u}_\delta \) and \( u_1 := u + c_0 \delta^\beta \) we get

\[
\sup_{\Omega}[\tilde{u}_\delta - (u + c_0 \delta^\beta)] \leq C\|u - c_0 \delta^\beta\|_{L^2(\Omega)},
\]

Since \( \tilde{u}_\delta = u + c_0 \delta^\beta \) in \( \Omega \setminus \delta \), we have

\[
\sup_{\delta}[u - u - c_0 \delta^\beta] \leq C\|u - c_0 \delta^\beta\|_{L^2(\Omega)}.
\]

Since \((u - u - c_0 \delta^\beta)_+ \leq u - u \) and by Lemma 4.3, we have

\[
\sup_{\delta}(u - u) \leq c_0 \delta^\beta + C\|u - u\|_{L^2(\Omega)} \leq c_0 \delta^\beta + CC_n\|\nabla u\|_{L^2(\Omega)}\delta^\gamma.
\]

Then,

\[
\sup_{\delta}(u - u) \leq A\delta^{\min\{\beta, \gamma\}},
\]

for \( \delta \) small enough, where \( A = c_0 + CC_n\|\nabla u\|_{L^2(\Omega)} \). This proves the first part of this result.

2) Given \( 0 < \gamma < \gamma_1 \).

\[
u_\delta(q) \leq u(q) + c_0 \delta^\beta \implies \tilde{u}_\delta \leq u < u + c_0 \delta^\beta \text{ on } \partial \delta \Omega.
\]

The function

\[
u_\delta' = \begin{cases} \max\{\tilde{u}_\delta, u + c_0 \delta^\beta\} & \text{in } \Omega; \\ u + c_0 \delta^\beta, & \text{in } \Omega \setminus \delta \Omega.
\end{cases}
\]

is a bounded plurisubharmonic function on \( \Omega \), continuous in \( \overline{\Omega} \). Using Theorem 3.5 with \( u_1 := u + c_0 \delta^\beta \), \( u_2 := u_\delta' \) and \( r = 1 \), we get

\[
\sup_{\Omega}[u_\delta' - u - c_0 \delta^\beta] \leq C\|(u_\delta' - u - c_0 \delta^\beta)\|_{L^1(\Omega)}.
\]
We have $u'_\delta = u + c_0 \delta^\beta$ in $\Omega \setminus \Omega_\delta$, hence
\[
\sup_{\Omega_\delta} [(\hat{u}_\delta - u - c_0 \delta^\beta)] \leq C \|(\hat{u}_\delta - u - c_0 \delta^\beta)\|_{L^1(\Omega_\delta)}.
\]
Also, we have $(\hat{u}_\delta - u - c_0 \delta^\beta)_+ \leq \hat{u}_\delta - u$, so we get
\[
\sup_{\Omega_\delta} (\hat{u}_\delta - u) \leq c_0 \delta^\beta + C \|\hat{u}_\delta - u\|_{L^1(\Omega_\delta)} 
\leq c_0 \delta^\beta + CC_n \|\Delta_{\Omega_\delta} u\|_{\Omega_\delta} \delta^{2\gamma}.
\]
Then, $\sup_{\Omega_\delta} (\hat{u}_\delta - u) \leq M_1 \delta^{\min\{\beta, 2\gamma\}}$, for $\delta$ small enough, and $M_1 = c_0 + CC_n \|\Delta_{\Omega_\delta} u\|_{\Omega_\delta}$.

By Lemma 4.1, we have
\[
\sup_{\Omega_\delta} (u_\delta - u) \leq M_2 \delta^{\min\{\beta, 2\gamma\}},
\]
for $\delta$ small enough, and some uniform constant $M_2 > 0$. This finishes the last part of this result. \[\square\]

Now, we are in the last part of this paper. We are going to prove the main Theorem, using these following results.

**Lemma 4.7.** Let $u, v$ be continuous functions on $\overline{\Omega}$ and be plurisubharmonic functions in $\Omega$, such that $u \geq v$ in $\Omega$ and $u = v$ on $\partial \Omega$. Then
\[
\int_{\Omega} \Delta u \wedge \beta_n^{n-1} \leq \int_{\Omega} \Delta v \wedge \beta_n^{n-1}
\]
\[
\int_{\Omega} d_0 u \wedge d_1 u \wedge \beta_n^{n-1} \leq 2 \int_{\Omega} \gamma(v, u) \wedge \beta_n^{n-1} + \int_{\Omega} d_0 v \wedge d_1 v \wedge \beta_n^{n-1},
\]
where $\gamma(u, v) := \frac{1}{2}(d_0 u \wedge d_1 v - d_1 u \wedge d_0 v)$, $\beta_n := \frac{1}{8} \Delta(\|q\|^2)$. Furthermore, if $\int_{\Omega} d_0 v \wedge d_1 v \wedge \beta_n^{n-1} < +\infty$ then $\int_{\Omega} d_0 u \wedge d_1 u \wedge \beta_n^{n-1} < +\infty$.

**Proof.** First, we set $u_\epsilon = \max\{u - \epsilon, \nu\}$ for $\epsilon > 0$. We have $u, v$ are continuous and $u = v$ on $\partial \Omega$, so $u_\epsilon = v$ in a neighborhood of $\partial \Omega$. Let $\{\Omega_j\}$ a hyperconvex open in $\Omega$ such that $\{u_\epsilon \neq v\} \subset \subset \Omega_1 \subset \subset \ldots \Omega_j \subset \subset \ldots \subset \Omega$ and $\{\chi_j\} \subset C_0^\infty(\Omega)$ such that $\chi_j \equiv 1$ in a neighborhood of $\overline{\Omega}_j$ and $\chi_j \not\equiv 1$. By Lemma 1.6, we have
\[
\int_{\Omega} \chi_j \Delta u_\epsilon \wedge \beta_n^{n-1} = - \int_{\Omega} d_0 \chi_j \wedge d_1 u_\epsilon \wedge \beta_n^{n-1}
\]
\[
= - \int_{\Omega \setminus \Omega_j} d_0 \chi_j \wedge d_1 u_\epsilon \wedge \beta_n^{n-1}
\]
\[
= - \int_{\Omega \setminus \Omega_j} d_0 \chi_j \wedge d_1 v \wedge \beta_n^{n-1}
\]
\[
= \int_{\Omega} \chi_j \Delta v \wedge \beta_n^{n-1}
\]
Letting $j$ tend to $+\infty$, we get
\[
\int_{\Omega} \Delta u_\epsilon \wedge \beta_n^{n-1} = \int_{\Omega} \Delta v \wedge \beta_n^{n-1}.
\]
Since \( u \geq v \) in \( \Omega \), we have \( u_\epsilon \nrightarrow u \) in \( \Omega \). By the monotone convergence theorem, we get
\[
\Delta u_\epsilon \wedge \beta_n^{n-1} \longrightarrow \Delta u \wedge \beta_n^{n-1}.
\]
So
\[
\int_\Omega \Delta u \wedge \beta_n^{n-1} \leq \liminf_{\epsilon \to 0} \int_\Omega \Delta u_\epsilon \wedge \beta_n^{n-1} = \int_\Omega \Delta v \wedge \beta_n^{n-1}.
\]
The first one follows. For the second one, we have also \( u, v \) are continuous and \( u = v \) on \( \partial \Omega \), so we set \( u_\epsilon := \max\{u - \epsilon, v\} = v \) in a neighborhood of \( \partial \Omega \) and \( u_\epsilon \geq v \) on \( \Omega \). We have
\[
\int_\Omega d_0v \wedge d_1v \wedge \beta_n^{n-1} - \int_\Omega d_0u_\epsilon \wedge d_1u_\epsilon \wedge \beta_n^{n-1} + 2 \int_\Omega \gamma(v,u_\epsilon) \wedge \beta_n^{n-1} = \int_\Omega d_0(v - u_\epsilon) \wedge d_1(v + u_\epsilon) \wedge \beta_n^{n-1} = \int_\Omega (u_\epsilon - v) \wedge \Delta(v + u_\epsilon) \wedge \beta_n^{n-1} \geq 0.
\]
Then,
\[
\int_\Omega d_0v \wedge d_1v \wedge \beta_n^{n-1} \geq \int_\Omega d_0u_\epsilon \wedge d_1u_\epsilon \wedge \beta_n^{n-1} - 2 \int_\Omega \gamma(v,u_\epsilon) \wedge \beta_n^{n-1}
\]
By convergence Theorem, we have
\[
\int_\Omega d_0u_\epsilon \wedge d_1u_\epsilon \wedge \beta_n^{n-1} - 2 \int_\Omega \gamma(v,u_\epsilon) \wedge \beta_n^{n-1} \longrightarrow \int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1} - 2 \int_\Omega \gamma(v,u) \wedge \beta_n^{n-1} \text{ as } \epsilon \searrow 0.
\]
Thus
\[
\int_\Omega d_0v \wedge d_1v \wedge \beta_n^{n-1} + 2 \int_\Omega \gamma(v,u) \wedge \beta_n^{n-1} \geq \int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1}.
\]
By Corollary 3.1 in [WZ], we have
\[
\int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1} \leq \int_\Omega d_0b_\gamma \wedge d_1b_\gamma \wedge \beta_n^{n-1} + 2 \int_\Omega \gamma(b_\gamma, u) \wedge \beta_n^{n-1} \leq \int_\Omega d_0b_\gamma \wedge d_1b_\gamma \wedge \beta_n^{n-1} + 2 \sqrt{\int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1}} \sqrt{\int_\Omega d_0b_\gamma \wedge d_1b_\gamma \wedge \beta_n^{n-1}}
\]
and we obtain
\[
\sqrt{\int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1}} \leq 2 \sqrt{\int_\Omega d_0b_\gamma \wedge d_1b_\gamma \wedge \beta_n^{n-1}} + \frac{\int_\Omega d_0b_\gamma \wedge d_1b_\gamma \wedge \beta_n^{n-1}}{\sqrt{\int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1}}}.
\]
So necessary we have \( \int_\Omega d_0u \wedge d_1u \wedge \beta_n^{n-1} < +\infty \). This finishes the lemma.

**Proposition 4.8.** Fix \( 0 \leq f \in L^p(\Omega) \ (p > 2) \). If \( \psi \in C^{1,1}(\partial \Omega) \), Then \( \Delta_{\Omega^*} u(\Omega, \psi, 0) \) has finite mass in \( \Omega \). Moreover \( \Delta_{\Omega^*} u(\Omega, \psi, f) \) also has finite mass in \( \Omega \).
Proof. We fix a defining function \( \rho \) of \( \Omega \). Setting \( \Omega = \{ \rho < 0 \}, \rho \in C^2(\overline{\Omega}) \).
First, we claim that
\[
h(q) = \sup\{v(q) : v \in PSH(\Omega) \cap C(\overline{\Omega}), v \leq \psi \text{ on } \partial\Omega\}
\]
is psh function in \( \Omega \) and is Lipschitz continuous in \( \overline{\Omega} \), it satisfies \( h = \psi \) on \( \partial\Omega \). Moreover
\[
\int_{\Omega} \Delta h \wedge \beta_n^{n-1} < +\infty.
\]
Assume \( f = 0 \), set \( u := u(\Omega, \psi, 0) \), we may choose \( A > 0 \) big enough such that \( A\rho + h \leq u \) in a neighborhood of \( F \in \Omega \), as \( \rho < -\epsilon \) in \( F \) for some \( \epsilon > 0 \), and \( (\Delta(A\rho + h))^n \geq (\Delta A\rho)^n \geq 0 \) in \( \Omega \setminus F \),
by comparaison principle (Corollary 1.1 in [WZ]), we have \( A\rho + h \leq u \) in \( \Omega \setminus F \).
Therefore, \( b := A\rho + h \leq u \) in \( \Omega \) and \( b \) is Lipschitz continuous in \( \overline{\Omega} \). By Lemma 4.7 and the fact that \( \rho \) is \( C^2 \) smooth in a neighborhood of \( \overline{\Omega} \), by using the claim above we get
\[
\int_{\Omega} \Delta u \wedge \beta_n^{n-1} \leq \int_{\Omega} \Delta b \wedge \beta_n^{n-1} < +\infty.
\]
For the last part of this proposition, we are going to prove that \( \Delta_{\mathbb{H}^n} u(\Omega, 0, f) \) has finite mass in \( \Omega \).
Let \( \tilde{f} \) be the trivial extension of \( f \) to a large ball \( B \) containing \( \Omega \). Let \( b_\tilde{f} \in C^2(\mathbb{B}) \) be a psh barrier for \( MA(\mathbb{B}, 0, \tilde{f}) \) (see the proof of Lemma 4.5 ). Then \( b_\tilde{f} := u(\Omega, -b_\tilde{f}, 0) + b_\tilde{f} \) is psh barrier for \( MA(\Omega, 0, f) \). Since \( b_\tilde{f} \) is smooth, we have \( \Delta_{\mathbb{H}^n} b_\tilde{f} \) has finite mass in \( \Omega \).
On the other hand, we have \( (\Delta b_\tilde{f})^n \geq fdV \) in \( \Omega \), so \( b_\tilde{f} \leq u(\Omega, 0, f) \) in \( \Omega \) by comparaison principle (Corollary 1.1 in [WZ]). Using Lemma 4.7, we get
\[
\int_{\Omega} \Delta u(\Omega, 0, f) \wedge \beta_n^{n-1} \leq \int_{\Omega} \Delta b_\tilde{f} \wedge \beta_n^{n-1} < +\infty.
\]
Now set \( v := u(\Omega, 0, f) + u(\Omega, \psi, 0) \), \( v \) is a psh function in \( \Omega \) such that \( v = \psi \) on \( \partial\Omega \) and \( (\Delta v)^n \geq fdV \) in \( \Omega \).
Since \( \psi \) is \( C_{1,1} \) in \( \partial\Omega \), we have \( \Delta_{\mathbb{H}^n} u(\Omega, \psi, 0) \) has finite mass in \( \Omega \), and \( \Delta_{\mathbb{H}^n} u(\Omega, 0, f) \) is so.
Then \( \Delta_{\mathbb{H}^n} v \) has finite mass in \( \Omega \), with \( v \leq u(\Omega, \psi, f) \) in \( \Omega \). So by Lemma 4.7, we get
\[
\int_{\Omega} \Delta u(\Omega, \psi, f) \wedge \beta_n^{n-1} \leq \int_{\Omega} \Delta v \wedge \beta_n^{n-1} < +\infty.
\]
For the proof of the claim, we let the reader to see the proof of Lemma 3.5 in [N]. \[\square\]

**Proposition 4.9.** Fix \( 0 \leq f \in L^p(\Omega) \) \( (p > 2) \). If \( \psi \in C^{1,1}(\partial\Omega) \), Then \( \nabla u(\Omega, \psi, f) \in L^2(\Omega) \).

**Proof.** First, we claim that : for \( 0 \leq \gamma < \frac{1}{2} \), the function \( \rho_\gamma = -|\rho|^{1-\gamma}, \rho \) as in the proof of proposition 4.8, setting \( (\Delta \rho)^n \geq g\beta^n \) on \( \overline{\Omega} \), with \( g > 0 \) \( \rho_\gamma \in PSH(\Omega) \cap Lip_{1-\gamma}(\overline{\Omega}) \) and satisfies
\[
\int_{\Omega} d_0 \rho_\gamma \wedge d_1 \rho_\gamma \wedge \beta_n^{n-1} < +\infty.
\]
Now, assume that \( f(q) \leq C|\rho(q)|^{-n\gamma} \) near \( \partial\Omega \) for some \( C > 0 \). So there is a compact subset \( E \in \Omega \) such that \( f(q) \leq C|\rho(q)|^{-n\gamma} \) in \( \Omega \setminus E \).
Then, we have
\[(\Delta \rho_\gamma)^n = (d_0 d_1 (-\rho)^{1-\gamma})^n = ((1 - \gamma)\rho^{-\gamma}\Delta \rho + \gamma(1 - \gamma)|\rho|^{-1-\gamma}d_0 \rho \wedge d_1 \rho)^n \geq (1 - \gamma)^n|\rho|^{-n\gamma}g \beta_n^\gamma \geq \frac{g(1 - \gamma)^n}{C}f \beta_n^\gamma \text{ in } \Omega \setminus E.\]

Therefore, we may choose \(A > 0\) big enough such that \(b_\gamma := A \rho_\gamma + h \leq u\) in a neighborhood of \(E\), and
\[(\Delta b_\gamma)^n \geq (\Delta A \rho_\gamma)^n \geq f \beta_n^\gamma \text{ in } \Omega \setminus E,\]
where \(h\) as in the proof of proposition 4.8. Then, by the comparison principle (Corollary 1.1 in [WZ]), we obtain \(b_\gamma \leq u\) in \(\Omega \setminus E\). So \(b_\gamma \leq u\) in \(\Omega\) and \(b_\gamma \in Lip_{1-\gamma}(\Omega)\). By Lemma 4.7, we have
\[\int_\Omega d_0 u \wedge d_1 u \wedge \beta_n^{n-1} \leq \int_\Omega d_0 b_\gamma \wedge d_1 b_\gamma \wedge \beta_n^{n-1} + 2 \int_\Omega \gamma(b_\gamma, u) \wedge \beta_n^{n-1},\]
and by the claim above, we get \(\int_\Omega d_0 u \wedge d_1 u \wedge \beta_n^{n-1} < +\infty\).

For the general case, we set \(f = 0\), we obtain \(\int_\Omega d_0 u \wedge d_1 u \wedge \beta_n^{n-1} < +\infty\), by the first part of this proof. Now, for \(f \neq 0\). Set \(v := u(\Omega, \psi, 0) + b_f\), where \(b_f\) is the plurisubharmonic barrier constructed in the proof of Lemma 4.5. We have \(v = \psi + 0 = u\) on \(\partial \Omega\), \((\Delta v)^n \geq (\Delta b_f)^n \geq f dV\) in \(\Omega\), so \(v \leq u\) in \(\Omega\). Moreover, we have \(\nabla u(\Omega, \psi, 0) \in L^2(\Omega)\) and \(\nabla b_f \in L^2(\Omega)\) hence \(\nabla v \in L^2(\Omega)\). By Lemma 4.7, we get
\[\int_\Omega d_0 u \wedge d_1 u \wedge \beta_n^{n-1} < +\infty.\]

Now, we prove the claim. we have
\[\Delta \rho_\gamma = d_0 d_1 (-(-\rho)^{1-\gamma}) = (1 - \gamma)|\rho|^{-\gamma}\Delta \rho + \gamma(1 - \gamma)|\rho|^{-1-\gamma}d_0 \rho \wedge d_1 \rho,\]
and
\[d_0 \rho_\gamma \wedge d_1 \rho_\gamma \wedge \beta_n^{n-1} = (1 - \gamma)^2|\rho|^{-2\gamma}d_0 \rho \wedge d_1 \rho \wedge \beta_n^{n-1}\]

Since \(-2\gamma > -1\), we have \(\int_\Omega d_0 \rho_\gamma \wedge d_1 \rho_\gamma \wedge \beta_n^{n-1} < +\infty\). Then, the proof is finished. \(\square\)

**Proof of main Theorem** According to Proposition 4.8 and Proposition 4.9, the assumptions of Theorem 4.6 are satisfied. Thus, the main Theorem follows.

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