Adaptive Decision Making via Entropy Minimization

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An agent choosing between various actions tends to take the one with the lowest cost. But this choice is arguably too rigid (not adaptive) to be useful in complex situations, e.g., where exploration-exploitation trade-off is relevant in creative task solving or when stated preferences differ from revealed ones. Here we study an agent who is willing to sacrifice a fixed amount of expected utility for adaptation. How can/ought our agent choose an optimal (in a technical sense) mixed action? We explore consequences of making this choice via entropy minimization, which is argued to be a specific example of risk-aversion. This recovers the $\epsilon$-greedy probabilities known in reinforcement learning. We show that the entropy minimization leads to rudimentary forms of intelligent behavior: (i) the agent assigns a non-negligible probability to costly events; but (ii) chooses with a sizable probability the action related to less cost (lesser of two evils) when confronted with two actions with comparable costs; (iii) the agent is subject to effects similar to cognitive dissonance and frustration.

Neither of these features are shown by entropy maximization.

Keywords: prior probability, risk, entropy maximization/minimization, exploration-exploitation

I. INTRODUCTION

Consider an agent who has to choose between a number of different actions $A_1, \ldots, A_n$. Before taking action, consequences of each action $A_k$ are subjectively estimated to have the cost $\epsilon_k$ (or the utility $u_k = -\epsilon_k$). The basic tenet of decision theory is that the agent ought to choose the action that minimizes the cost (or maximizes the utility) \cite{1}. In terms of probabilities $p_k$ for various actions ($p_k \geq 0, \sum_{k=1}^{n} p_k = 1$), this amounts to taking the action $\ell$ related to the least cost (if it exists and is available):

$$ p_\ell = 1, \quad p_{k \neq \ell} = 0, \quad \epsilon_\ell < \epsilon_{k \neq \ell}, \quad k = 1, \ldots, n. \quad (1) $$

There are, however, situations where $\epsilon_k$ may change after actions are taken, also as a result of those actions \textsuperscript{1}. Here is an example that points against choosing (1), and illustrates our problem. You got 100 eggs and several baskets, which seem to have different durabilities (i.e. utilities). The probability with which an action, i.e. a basket, is taken refers to the fraction of eggs in it. Even if the most durable basket may appear to support all the eggs, it is not wise to put everything in one basket. First the durability of a basket can change due to very eggs inside of it. Second, the durability can change unexpectedly due to hindrances. Third, you loose the possibility to explore other baskets that may turn out to be more durable than you thought. Let us now mention several more, broadly defined situations, where “putting all eggs into one basket” is not good.

– In reinforcement learning the preliminary costs $\epsilon_k$ do change due to the actions taken \textsuperscript{3}. Even if these changes are assumed to be predictable, the agent still needs to make several actions before enough experience is accumulated.

– The exploration-exploitation dilemma is known in adaptive (biological, organizational, social) systems; see \textsuperscript{4, 5} for reviews. Exploring possibilities that seem inferior from a local viewpoint may provide advantages in the long run. Exploitation (in the narrow sense) makes the choice that does seem optimal at the moment of choice. Broader exploitation scenarios do account for adaptivity, but still concentrate on the most useful possibilities \textsuperscript{5}.

– In creative problem solving there are conceptually simple tasks which are nevertheless not easy to solve in practice because solving them via the least cost (implied by the statement of the problem and/or the previous experience of the solver) is a dead end \textsuperscript{6–8}. This \textit{Einstellung} effect is one of the main hindrances of human creativity \textsuperscript{6–8}. Creative tasks can be solved only if (subjectively) less probable ways are looked at \textsuperscript{6, 7}.

\textsuperscript{1} See section II for more details. We stress that we do not mean the delayed reward situation, where the utility is constant, but is discounted by some known factor, because the action is performed now, while its reward will come in future.
How to assign prior probabilities to avoid the strictly deterministic (1)? Such probabilities should hold a natural constraint that actions related to higher cost are getting smaller probabilities. Two ad hoc solutions are especially simple: one can take into account only the second-best action, or take all non-best actions with the same (small) probability. In reinforcement learning the latter prior probability is known as the $\epsilon$-greedy [3]. It is preferable to have a regular method of choosing non-deterministic probabilities, which will reflect people’s attitudes towards the decision making in an uncertain situation, and which will include the above ad hoc solutions as particular cases.

Here we explore the possibility of defining the prior probabilities via risk minimization (or maximization); see [9, 10] for reviews on the notion of risk and its various interpretations. We assume that the agent first decides how much average utility $E - \min_k [\varepsilon_k]$ he invests into exploration by going into nonoptimal—in the sense of not holding (1)—behavior. We employ the notion of risk in a specific context, namely when comparing the behavior of agents having the same utilities for various actions and the same value of $E$. We argue below that maximizing (minimizing) risk in this specific situation can be done via maximizing (minimizing) the entropy $- \sum_{k=1}^n p_k \ln p_k$. People demonstrate both risk minimization (aversion) and maximization (seeking) [12, 25], though the risk in those situations is a less specific (and more difficult to describe) notion—first because it involves agents having different utilities for same actions, and second because it involves a difference between the monetary value (gain or loss) and its utility.

Our results show that there are important behavioral differences between entropy-minimizing and entropy-maximizing agents. They are seen for at least three different actions (and the same $E$). The entropy-minimizing agent implements risk-aversion by weighting the least-cost action more, but he also assigns a non-negligible probability for the high-cost action—whereas the entropy-maximizing agent ignores it. The extent to which the high-cost action is accounted for by the entropy-minimizing agent depends on the amount of utility invested into exploration: investing more utility leads to assigning less probability. As we argue below, this closely relates with the notion of cognitive dissonance [60, 61]. Another feature is frustration: due to competing local minima of entropy, the entropy-minimizing agent can abruptly change the action probabilities as a result of a small change of $E$. Also, when confronted with two actions with different, but comparable costs, the entropy-minimizing agent tends to select the one with a smaller cost (chooses the lesser of two evils), while the entropy-maximizing agent simply does not distinguish between them. The important point is that for a risk-minimizing agent (which does a constrained minimization of a concave function in a convex domain) choosing the probabilities of actions means selecting between several local minima. In contrast, the risk-seeking agent always has a unique and well-defined probabilistic solution that results from minimizing a convex function [13]. We relate the above features of the entropy-minimizing agent with a rudimentary form of intelligence (see Section VII).

The remainder of this paper is organized as follows. Section II explains the statement of our problem. Section III discusses stochastic dominance, risk, majorization and its relation with entropy. In particular, Section III D provides general remarks and references on entropy optimization. The reader who agrees from the outset with entropy as a measure of risk (and uncertainty) can consult Sections II and III very briefly. The next two sections, IV and V, present details of (resp.) entropy minimization and maximization. The latter may seem to be standard, but Section V still contains salient points that are frequently overlooked. Section VI compares entropy maximization and minimization scenarios from the viewpoint of the agent’s behavior. We summarize in Section VII.

II. STATEMENT OF THE PROBLEM

A. Costs and constraints

Let us explain the above problem. An agent faces different actions $\{A_k\}_{k=1}^n$ with (resp.) costs $\{\varepsilon_k\}_{k=1}^n$. These costs are subjective estimates of future consequences of actions made by the agent before deciding on those actions. After taking several actions, the agent can change his estimates also as a result of the actions taken. However, before taking actions he does not know in which specific way the costs will change. In such an agnostic situation, the agent faces two normative demands—he should behave according to $\{\varepsilon_k\}_{k=1}^n$, but he also should also explore all actions. Hence he decides to act via probabilities and implements two constraints. First, he decides to invest in the exploration of the average utility $E - \min_k [\varepsilon_k]$, where

$$E \equiv \sum_{k=1}^n p_k \varepsilon_k, \quad p_k \geq 0, \quad \sum_{k=1}^n p_k = 1,$$  \hspace{1cm} (2)$$

and where $p_k$ are probabilities to be chosen. Within this formulation of the problem, we can regard $E$ to be under control of the agent.

Second, actions related to a lesser cost get higher probability:

$$(p_k - p_l)(\varepsilon_k - \varepsilon_l) \leq 0. \hspace{1cm} (3)$$
Recall that costs are negative utilities. We work with costs, since this makes explicit the analogy with statistical physics, where the cost corresponds to energy, and the natural tendency is to minimize the energy.\(^2\)

We order the costs and probabilities as

\[
\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n, \quad (4)
\]

\[
p_1 \geq p_2 \geq \ldots \geq p_n. \quad (5)
\]

Note that the inequalities in (4) between costs are strict, since within the present study coinciding costs mean coinciding actions. Hence once the actions are different, so should be the costs. The reason for insisting on non-equal costs in (4) will be seen below.

**B. Stated versus revealed preferences**

We can look at this problem from a different angle. In human decision-making, it is known that preferences with respect to different actions can be determined in laboratory conditions (e.g., via surveys). This then defines their costs (negative utilities) (4). However, in reality (e.g., in complex conditions of a market) people generally do not strictly follow these preferences due to lack of cognitive abilities, insufficient attention, lack of a decision time, the need of an exploration behavior etc.. The difference between experimental and real-world choices is known as the problem of stated versus revealed preferences; see [57] for a recent review. It is assumed in the psychology of choice that such cases can be described by assigning probabilities to each costs [57]. Now, the above formulation does correspond to this situation, where (5) is a natural constraint on probabilities, and where \(E - \min_k [\varepsilon_k] > 0\) is not simply chosen by the agent, but is also due to objective problems described above. For \(E - \min_k [\varepsilon_k] \to 0\), the agent is back to choosing the least-cost solution only.

**C. Differences with respect to other models of decision theory**

Above we described an agent who is uncertain about future consequences of his actions. Decision theory has models for that situation. The classic model (by Savage and others) [1, 2, 12] assumes that at the moment of action-taking there is an uncertain state of nature (environment) \(S_\alpha\) to be realized from \(\{S_\alpha\}_{\alpha=1}^m\) with probabilities \(\{\pi_\alpha\}_{\alpha=1}^m\), which can be the agent’s subjective degrees of belief. These are called states of nature, since their future realization is independent from the action taken, but an action \(A_i\) in a state \(S_\alpha\) leads to consequences with costs \(c_\alpha\) [1, 2, 12]. The agent does know \(\{\pi_\alpha\}_{\alpha=1}^m\) and \(\{c_\alpha\}_{\alpha=1}^m\). Now one criterion for looking for the best action is to choose \(i\) such that the expected cost \(\sum_{\alpha=1}^m \pi_\alpha c_\alpha\) is minimized over \(i\) (i.e., the expected utility is maximized) [1, 2, 12].

The classic model has limitations—e.g., that environmental states \(\{S_\alpha\}_{\alpha=1}^m\) are realized independently from actions. There are many cases where this simplistic assumption does not hold. Causal decision theory [14–18] attempts to remedy this problem, but creates its own issues—e.g., replacing \(\pi_\alpha\) by the conditional probability \(\pi_{\alpha|i}\) of the state \(S_\alpha\) given the action \(A_i\), leads to paradoxical (if not unacceptable) conclusions nicknamed “voodoo” decisions [16]. In response, the expected utility principle was changed to the concept of ratifiability [2, 17], but this proposal has its own problems [17], e.g. it demands an unusual environment.\(^3\) Another response was to change the conditional probability \(\pi_{\alpha|i}\) to the probability of conditional \(\pi_{\alpha\rightarrow i}\), which, in contrast to \(\pi_{\alpha|i}\), is supposed to describe the causal influence of the action \(A_i\) on \(S_\alpha\) [14, 15]. But \(\pi_{\alpha\rightarrow i}\) does not have a general definition [14], while some of its particular definitions are flawed [18]. Recent reviews show that all non-classical models of causal decision theory are problematic in one way or another [14, 17, 18].

Our statement of the problem refers to an agnostic agent that (as yet) does not know about possible consequences of his actions in a complex and uncertain world. Even if he is going to learn about them, he still needs to perform several actions with certain prior probabilities. We do not assume any specific mechanism by which uncertain states of the world are realized and changed in response to actions. Hence, our model can be at best descriptive, but it starts from two normative demands: behave according to your own estimated costs (4) and explore all actions.

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\(^2\) Note that (3) is the standard condition of physical stability [19]. Relations between statistical physics and decision theory are mentioned in [20]. Ref. [21] discusses the maximum entropy method for elicitation of costs (utilities) from an incomplete data.

\(^3\) Ratifiability advises to take action \(A_i\) if for all \(j\) [2, 16, 17]: \(C(A_j|A_i) \geq C(A_i|A_i)\), where \(C(A_j|A_i) \equiv \sum_{\alpha} \pi_{\alpha|j} c_{\alpha}\). (Minimization of the expected cost reads in this notation \(C(A_j|A_i) \geq C(A_i|A_i)\)) This assumes an unusual environment: \(C(A_j|A_i)\) describes an agent who commits to act \(A_i\), conditionally constrains the environment by this commitment, but still keeps the freedom of acting \(A_j\). The ratifiability then means consistency between the commitment and the actual action.
III. RISK AND ENTROPY

A. Stochastic dominance, risk and majorization

Together with probabilities \( p_1, ..., p_n \) in (2–5), we define another candidate probability with the same costs (4) and holding the same constraints (2, 3) with the same value of \( E \):

\[
q_1 \geq q_2 \geq \ldots \geq q_n \geq 0, \quad \sum_{k=1}^n q_k = 1, \quad E = \sum_{k=1}^n q_k \varepsilon_k. \quad (6)
\]

We ask for criteria that should make \( \{p_k\}_{k=1}^n \) more preferable than \( \{q_k\}_{k=1}^n \). The well-known criterion of choosing for a smaller average cost obviously does not apply here. However, one can ask whether \( \{p_k\}_{k=1}^n \) is less risky than \( \{q_k\}_{k=1}^n \).

We emphasize that risk is generally a complex notion that raises conceptual and practical (resp. normative and descriptive) issues; see [9–12, 23–27, 54–56] for reviews from various viewpoints. Our application of risk will be to a large extent free of these issues. There are two reasons for this. First, we do not focus on the difference between the commodity and its utility (cost); hence we are not concerned with the problem of convex/concave utility functions [12, 25]. Second, we always compare between lotteries (i.e., probabilities and their outcome costs) that are ordered in the same way [cf. (6, 5)] and refer to the same costs (4). Then a stringent definition of risk goes via the stochastic dominance condition \(^4\); see [1, 22] for reviews. A risk-averse agent can start with the largest possible cost \( \varepsilon_n \) and asks for its \( p \)-probability \( p_n \) to be not larger than the \( q \)-probability \( q_n \). Next, the \( p \)-probability \( p_n + p_{n-1} \) of the cost to be larger or equal to \( \varepsilon_{n-1} \) is compared with the corresponding \( q \)-probability \( q_n + q_{n-1} \) etc. We end up with \( n-1 \) conditions \( \sum_{i=n}^k p_i \leq \sum_{i=n}^k q_i \), or

\[
\sum_{i=1}^k p_i \geq \sum_{i=1}^k q_i, \quad k = 1, \ldots, n-1, \quad (7)
\]

so that if one of them holds strictly, then \( \{p_k\}_{k=1}^n \) is less risky than \( \{q_k\}_{k=1}^n \). We emphasize that the same \( n-1 \) conditions (7) emerge with an alternative definition of risk, when the agent maximizes the probability of the best outcome \( \varepsilon_1 \) etc. So within the present definition it does not matter whether a risk-averse agent selects for higher probabilities of lower-cost options, or for lower probabilities of higher-cost actions.

Due to (4, 5, 6), the stochastic dominance condition (7) coincides with the majorization condition \([69]\).\(^5\)

\[
\{p_k\}_{k=1}^n \succ \{q_k\}_{k=1}^n. \quad (8)
\]

In particular, for any \( \{p_k\}_{k=1}^n \) we get \( \{p_k\}_{k=1}^n \succ \{q_k = \frac{1}{n}\}_{k=1}^n \) (once \( q_k \) holds the first condition in (6)), while \( (1, 0, \ldots, 0) \succ \{p_k\}_{k=1}^n \). Thus, in the present situation we have a natural correspondence between risk and majorization.

B. Measure of risk and Schur-concave functions

According to (7), we have \( n-1 \) different conditions; i.e., it is not a single number as a measure for risk.\(^6\) Hence even if (7) holds, i.e., \( \{q_k\}_{k=1}^n \) is more risky than \( \{p_k\}_{k=1}^n \), we do not have any obvious way of quantifying this relation via a single number or determine its approximate validity.

A more serious drawback of (7) is that for many interesting cases the risk is simply not defined, since neither \( \{p_k\}_{k=1}^n \succ \{q_k\}_{k=1}^n \) holds nor \( \{q_k\}_{k=1}^n \succ \{p_k\}_{k=1}^n \):

\[
\{p_k\}_{k=1}^n \nprec \{q_k\}_{k=1}^n \quad \text{and} \quad \{q_k\}_{k=1}^n \nprec \{p_k\}_{k=1}^n. \quad (9)
\]

\(^4\) This is the first-order stochastic dominance condition [1, 22]. Second and higher-order conditions refer to the situation, where there are two different concepts related to the cost (or negative utility): money and its proper utility. We do not employ them here.

\(^5\) We stress the difference between majorization and stochastic dominance: in the latter the probabilities are not ordered, only the costs are ordered as in (4). For majorization the probabilities should be ordered as in (5). Given this difference in orderings, both majorization and stochastic dominance are defined by (7). Generally, the stochastic dominance does differ from majorization [1, 22].

\(^6\) The literature on mathematical economics suggests certain single-number measures of risk; see e.g., [23]. They do not apply for our situation since they demand that \( E \leq 0 \) (the average gain), but there are certainly some indices \( i \) for which \( \varepsilon_i > 0 \). This condition is too restrictive for us. Ref. [24] presents another interesting attempt to go beyond the stochastic dominance conditions.
For $n = 3$, (9) is equivalent to

$$(p_1 - q_1)(p_3 - q_3) > 0.$$  

(10)

These drawbacks motivate us to take (7) as a sufficient condition for risk, but still look for a single-function measure of risk $S[p_1, ..., p_n]$ such that if (7) holds, then

$$S[p_1, ..., p_n] \leq S[q_1, ..., q_n].$$  

(11)

While $S[p_1, ..., p_n]$ is initially defined for ordered probabilities (5), it is natural to extend this definition by demanding that $S[p_1, ..., p_n]$ is invariant with respect to any permutation of its $n$ arguments. Functions that hold these two conditions—permutation invariance and that (8) implies (11)—are called Schur-concave\(^7\) \cite{69}. For a differentiable function to be Schur-concave it is necessary and sufficient that \cite{69}

$$(p_i - p_j) \left( \frac{\partial S}{\partial p_i} - \frac{\partial S}{\partial p_j} \right) \leq 0.$$  

(12)

Naturally, the choice of a Schur-concave function as a measure of risk is not unique; hence certain additional conditions are needed to fix it. Below we discuss one set of such conditions that allows us to fix entropy as a measure of risk \cite{70}. But before doing so, let us stress that a candidate Schur-concave function will in fact define the risk for a composite set of actions $A$ such that according to (12) there cannot be a differentiable $S[p_1, ..., p_n]$ for which, e.g., $S[p_1, ..., p_n] < S[q_1, ..., q_n]$ whenever $p_3 = \min[p_1, p_2, p_3] < q_3 = \min[q_1, q_2, q_3]$.

\section{Entropy}

Following \cite{70, 72}, we now outline several additional conditions that lead to choosing entropy among other Schur-concave functions.

A. $S[p_1, ..., p_n]$ is additive with respect to the index $k$, i.e.,

$$S[p_1, ..., p_n] = \sum_{k=1}^{n} \psi(p_k),$$  

(13)

with a smooth function $\psi(x)$; cf. (14). Condition (13) can be motivated from demanding that if we compare $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ with each other we want to draw conclusions from non-equal probabilities only. Likewise, if $p_i$ and $p_j$ ($1 \leq i < j \leq n$) change as $p_i \rightarrow p_i + \delta$ and $p_j \rightarrow p_j - \delta$ with a suitable (but not necessarily small) $\delta$, we want the change of $S[p_1, ..., p_n]$ to depend only on $p_i$, $p_j$ and $\delta$, but not on other probabilities $p_l$ with $l \neq i$ and $l \neq j$; see \cite{32, 72} for further discussion of (13).

B. The function $\psi(x)$ is concave: $\frac{d^2 \psi}{dx^2} \leq 0$. This makes $S[p_1, ..., p_n] = \sum_{k=1}^{n} \psi(p_k)$ consistent with (7, 12), because for any concave function $\psi(x)$ relations (7) and (12) imply \cite{69}

$$\sum_{k=1}^{n} \psi(p_k) \leq \sum_{k=1}^{n} \psi(q_k).$$  

(14)

C. Instead of actions $A_1, ..., A_n$ that refer (resp.) to probabilities $p_1, ..., p_n$, consider a composite set of actions $\{A_k, A'_k\}_{k=1}^{n} \times \{A''_k\}_{k=1}^{n''}$ with probabilities (resp.) $\{p_k p'_k\}_{k=1}^{n'} \times \{q_k \}$. Hence $\{A_k\}_{k=1}^{n}$ is independent from $\{A'_k\}_{k=1}^{n'}$; e.g., $\{A_1\}_{k=1}^{n}$ and $\{A'_1\}_{k=1}^{n'}$ may refer to the same set of actions performed independently at different times. It is natural to demand from a risk measure $S[...]$ that it adds up for independent actions \cite{32, 72}:

$$S[p_1 p'_1, ..., p_n p'_n] = S[p_1, ..., p_n] + S[p'_1, ..., p'_n].$$  

(15)

\(^7\) Multiplying by minus one makes such a function Schur convex.
Conditions (A–C) are natural for a measure of uncertainty—hence for a risk measure—and they lead to [70]:
$$\psi(p) = -p \ln p,$$
up to a positive multiplicative constant that we fixed to 1, and an additive constant that we fixed to 0 [70]. Thus the sought-after quantity amounts to the entropy
$$S[p] = -\sum_{k=1}^{n} p_k \ln p_k.$$  \hspace{1cm} (16)

Eq. (14) means that entropy is larger for more risky probability. The expression (16) for the entropy can be recovered via other axiomatic schemes; see [73, 74] for a review.

Thus, instead of conditions (7) we shall employ the entropy (16) as a measure of risk.8 Now the risk-averse agent chooses to minimize entropy (16) under constraints (2, 3):
$$\min \left[-\sum_{k=1}^{n} p_k \ln p_k; \quad p_1 \geq \ldots \geq p_n \geq 0, \quad \sum_{k} p_k \varepsilon_k = E\right],$$  \hspace{1cm} (17)

while the risk-seeking agent chooses in (17) max instead of min.

The maximization or minimization can be done via the Lagrange function
$$\mathcal{L} = -\sum_{k=1}^{n} p_k \ln p_k + \beta \sum_{k=1}^{n} p_k \varepsilon_k,$$  \hspace{1cm} (18)

where $\hat{\beta}$ is the Lagrange multiplier that corresponds with $E$ in (2). It is important to stress that (18) emerged as a measure of risk and uncertainty in several alternative axiomatic schemes [52, 54–56]; see Appendix A for a discussion.

D. A general discussion on entropy maximization versus its minimization

The entropy maximization is a well-known method for determination of prior probabilities [31–37]. This method was developed within statistical physics [38], and it reflects the second law of thermodynamics, i.e., the natural tendency of the entropy to increase in closed systems [38]. The method was also motivated from within the probabilistic inference theory [32–34], and applied to Bayesian decision making [35, 36], social group decision making [42], game theory [43], and learning algorithms [44], where it describes bounded rational agents, etc. The entropy maximization was also discussed from the viewpoint of approximate (and causal) reasoning [45–48]. The maximum entropy method is fundamental for statistics, e.g., it was recently motivated from within the objective Bayesianism program [36, 37]. In social sciences the outcome of the entropy maximization method is known as the logit distribution; see [43, 44] for reviews. Several basic features of entropy were related to the utility maximization in economics [49–51]. Mathematical psychology and choice theory also provide interesting situations, where the agent behavior is determined by entropy maximization for a fixed average utility; see [57] for a thorough review.

In contrast, the entropy minimization for a risk-averse agent is compared with a general trend of social and biological systems that create order, i.e., decrease the entropy locally [58, 59]. Formalizations of such processes within statistical physics are less known, but do exist [28–30]. Below we shall employ results from [30]. Occasionally, aspects of entropy minimization are also discussed in probabilistic inference [39–41], e.g., for the feature extraction problem [40].

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8 The use of entropy as a measure of risk was criticized in literature (see e.g., [23]) on the grounds that it depends only on probability and not (also) on the costs $\varepsilon_k$. However, we note that the criticisms of [23] are not general, and it certainly does not apply to the situation we consider, where both probabilities $\{p_k\}_{k=1}^{n}$ and $\{q_k\}_{k=1}^{n}$ refer to the same costs $\{\varepsilon_k\}_{k=1}^{n}$. Indeed, the argument of [23] against using the entropy as a measure of risk refers (say) to comparing two situations, where within the first situation the agent gets 10$ and $-10$ with probabilities $p_1$ and $p_2$. Within the second situation the agent gets 1000$ and $-1000$ with the same probabilities $p_1$ and $p_2$. The entropies here will be the same, $-p_1 \ln p_1 - p_2 \ln p_2$, but the risks are obviously different.
IV. PRIOR PROBABILITIES VIA ENTROPY MINIMIZATION

A. Parametrization of probabilities

Here we discuss the minimization of entropy (16) under constraints (2, 3); see also [30] in this context. First, we note that (3) and (2) are compatible only for

\[
\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \geq E = \sum_{k=1}^{n} p_k \varepsilon_k.
\]

Indeed, we apply the summation-by-parts formula

\[
\sum_{k=1}^{n} p_k \varepsilon_k = \varepsilon_n \sum_{k=1}^{n} p_k - \sum_{m=1}^{n-1} [\varepsilon_{m+1} - \varepsilon_m] \sum_{k=1}^{m} p_k,
\]

to both sides of (19) and obtain:

\[
\sum_{k=1}^{n} \frac{1}{n} \varepsilon_k - \sum_{k=1}^{n} p_k \varepsilon_k = \sum_{m=1}^{n-1} [\varepsilon_{m+1} - \varepsilon_m] \left[ \sum_{k=1}^{m} p_k - \sum_{k=1}^{m} \frac{1}{n} \right],
\]

Now (19) follows, because the first (second) term inside square brackets in (21) is non-negative due to (4) (due to (5)). Using (4, 5) we parametrize the sought-after probabilities \((p_1, \ldots, p_n)\) as [30]

\[
(p_1, \ldots, p_n) = \sum_{\alpha=1}^{n} \lambda_\alpha \pi_\alpha, \quad \pi_\alpha \equiv \frac{1}{\alpha} (1, \ldots, 1, 0, \ldots, 0),
\]

\[
\lambda_\alpha \geq 0, \quad \sum_{\alpha=1}^{n} \lambda_\alpha = 1,
\]

\[
p_k = \sum_{\alpha=k}^{n} \lambda_\alpha, \quad k = 1, \ldots, n.
\]

It is easy to show that (22) is necessary and sufficient for holding (3); see also (4). The advantage of using (22) is that the probabilities \(\lambda_\alpha\) do not have any other constraints besides (23) and (2) that in terms of \(\lambda_\alpha\) are written as

\[
E = \sum_{\alpha=1}^{n} \lambda_\alpha \xi_\alpha,
\]

\[
\xi_\alpha \equiv \frac{1}{\alpha} \sum_{i=1}^{\alpha} \varepsilon_i.
\]

B. Minimization of a concave function on a convex set

Note that constraints (5) and (2) define a convex set [13], which we denote by \(\Omega\). The same convex set, but in different coordinates, is defined via (23) and (25). We now recall that the entropy \(S[p]\) in (16) is a concave function of \(p = (p_1, \ldots, p_n)\) on \(\Omega\) [31–34] [cf. (14)]:

\[
S[\chi p + (1 - \chi) q] \geq \chi S[p] + (1 - \chi) S[q], \quad 0 \leq \chi \leq 1,
\]

where \(q = (q_1, \ldots, q_n)\). Eq. (27) is well-known, but it can be verified by looking at an unconstrained Hessian matrix (i.e., without accounting for constraints (5), (2)):

\[
\frac{\partial^2 S}{\partial p_k \partial p_l} = -\frac{\delta_{kl}}{p_k}
\]

where \(\delta_{kl}\) is a unit matrix. Once \(S\) is concave without the constraints, it is also concave on \(\Omega\).
Now representation (22) implies that the entropy $S$ is also a concave function of $\lambda_\alpha$, e.g., takes derivatives over $\lambda_\alpha$ and $\lambda_\beta$ as in (28). The non-constant concave function $S[\lambda]$ (entropy in variables $\lambda_\alpha$) on the convex set $\Omega$ can reach its local minima only on vertices of $\Omega$—i.e., those elements of $\Omega$ that cannot be represented as a convex sum of other elements of $\Omega$ [13]. For those vertices $\lambda_\alpha = 0$ for as many as possible indices $\alpha$. Now for each vertex of $\Omega$, generically only two $\lambda$’s are non-zero; otherwise (25) cannot hold for a given $E$. (For particular values of $E$ there can be one non-zero $\lambda_\alpha$). Let us take all $(n(n-1)/2)$ vectors (23) with only two non-zero $\lambda_\alpha$’s that are determined from the normalization $\sum_{\alpha=1}^n \lambda_\alpha = 1$ in (24) and from (25). Now when minimizing the entropy on those vectors one finds the global entropy minimum [30].

Note that the above concavity argument does not imply that all solutions are local minima with respect to an infinitesimal perturbation that holds (2, 3). Nevertheless, it is the case that for the present problem all above $n(n-1)/2$ solutions are local minima; see Appendix B for details. The local minimality is important because it makes each solution meaningful even if it does not provide the global minimum of entropy. Note that choosing the global minimum demands $O(n(n-1)/2)$ operations. This number scales polynomially with $n$.

Denote by $\{\alpha \beta\}$ $\alpha < \beta$ the solution, where only $\lambda_\alpha \equiv \lambda_\alpha$ and $\lambda_\beta = 1 - \lambda_\alpha$ are non-zero. For we get (25, 26):

$$E = \lambda_\alpha \xi_\alpha + (1 - \lambda_\alpha) \xi_\beta, \quad \lambda_\alpha = \frac{\xi_\beta - E}{\xi_\beta - \xi_\alpha},$$

(29)

$$p_1 = ... = p_\alpha = \frac{\lambda_\alpha}{\alpha} + \frac{\lambda_\beta}{\beta}, \quad p_{\alpha+1} = ... = p_\beta = \frac{\lambda_\beta}{\beta}, \quad p_{\beta+1} = ... = p_n = 0.$$  

(30)

we get the minimized entropy from (16, 22, 23, 29, 30)

$$S_{\{\alpha \beta\}}(E) = -\nu_{\alpha \beta} \ln \left[ \frac{\nu_{\alpha \beta}}{\alpha} \right] - (1 - \nu_{\alpha \beta}) \ln \left[ \frac{1 - \nu_{\alpha \beta}}{\beta - \alpha} \right],$$

(31)

$$\nu_{\alpha \beta} \equiv \lambda_\alpha (1 - \frac{\alpha}{\beta}) + \frac{\alpha}{\beta} = \frac{\xi_\beta - E}{\xi_\beta - \xi_\alpha} (1 - \frac{\alpha}{\beta}) + \frac{\alpha}{\beta}.$$  

(32)

Now note from (4, 26) that [for $\alpha < \beta$]

$$\xi_\alpha - \xi_\beta = \frac{\beta - \alpha}{\alpha \beta} \sum_{i=1}^\beta \xi_i - \frac{1}{\beta} \sum_{i=1}^\beta \xi_i \leq \frac{\beta - \alpha}{\alpha \beta} \sum_{i=1}^\alpha \xi_i - \frac{\beta - \alpha}{\beta} \xi_{\alpha+1} = \frac{\beta - \alpha}{\alpha \beta} \sum_{i=1}^\alpha \xi_i - \xi_{\alpha+1} \leq 0.$$  

(33)

Hence, within solution $\{\alpha \beta\}$ $\alpha < \beta$ we get [see (29)]

$$\xi_\alpha \leq E \leq \xi_\beta.$$  

(34)

Recall from (19) that the allowed range of $E$ is $\xi_1 = \xi_1 \leq E \leq \xi_n$.

**C. Features of the minimized entropy**

We obtain from (26–33)

$$\frac{dS_{\{\alpha \beta\}}}{dE} = \frac{\partial S_{\{\alpha \beta\}}}{\partial E} \frac{\partial S_{\{\alpha \beta\}}}{\partial \nu_{\alpha \beta}} = \frac{1 - \frac{\alpha}{\beta}}{\xi_\beta - \xi_\alpha} \ln \left[ \frac{\nu_{\alpha \beta}}{1 - \nu_{\alpha \beta}} \right] \frac{\beta - \alpha}{\alpha \beta} \ln \left[ \frac{1 + \frac{\beta}{\alpha} \frac{\xi_\beta - E}{\xi_\beta - \xi_\alpha}}{1 + \frac{\beta}{\alpha} \frac{\xi_\beta - E}{\xi_\beta - \xi_\alpha}} \right] \geq 0,$$  

(35)

---

9 This fact follows from the negativity of the Hessian matrix as well directly: let $\lambda_\alpha$ be a local minimum of $S[\lambda]$ that is not a vertex of $S$. Then there exist $\lambda_1$ and $\lambda_2$, both from $S$, and both close to $\lambda_0$, so that $\lambda_\alpha = \chi \lambda_1 + (1 - \chi) \lambda_2$, and $0 \leq \chi \leq 1$. Now $S[\lambda_\alpha] \geq \chi S[\lambda_1] + (1 - \chi) S[\lambda_2]$ from concavity of $S[\lambda]$, and $S[\lambda_1] > S[\lambda_0]$, $S[\lambda_2] > S[\lambda_0]$, because $\lambda_0$ is a local minimum. These lead to $S[\lambda_\alpha] > S[\lambda_0]$, which is contradictory. Likewise, one can show that for a concave function $S[\lambda]$ on a convex domain $S$, any local maximum coincides with the global one. Let $\lambda_1$ and $\lambda_2$ be, respectively, the local and global maxima of $S[\lambda]$. Consider $S[\chi \lambda_0 + (1 - \chi) \lambda_1]$, where $0 < \chi < 1$ is sufficiently close to 0. Then $\chi \lambda_0 + (1 - \chi) \lambda_1$ is close to the local maximum $\lambda_1$, but $S[\chi \lambda_0 + (1 - \chi) \lambda_1] \geq S[\lambda_1]$, again is a contradiction.

10 As an example consider a sphere intersected by a horizontal plane. The part of the plane that is inside of the sphere defines a convex set, and the surface of the sphere is a concave function. Vertices of the set are intersection points of the sphere with the plane, but they are not local minima.
where (35) follows from (34). According to (35), $S_{\{\alpha\beta\}}$ is an increasing function of $E$ within each solution $\{\alpha\beta\}$, hence also for the global minimum. Likewise, one can show from (35) that
\[
\frac{d^2 S_{\{\alpha\beta\}}}{dE^2} < 0,
\] (36)
i.e., $S_{\{\alpha\beta\}}(E)$ is concave within each solution. Hence the global minimum $\min_{1 \leq \alpha < \beta \leq n}[S_{\{\alpha\beta\}}(E)]$ is also a concave function of $E$.

Eq. (34) shows that different local solutions exist for different values of $E$. Now solutions $\{\alpha\beta\}$ ($\alpha > \beta$) and $\{\beta\gamma\}$ ($\beta < \gamma$) can be glued together at $E = \xi_\beta$ so that the probabilities (30) behave continuously at $E = \xi_\beta$; cf. (29). Indeed, at $E = \xi_\beta$ we get $p_1 = \ldots = p_\beta = \frac{1}{\beta}$, $p_{\beta+1} = \ldots = p_n = 0$, both for $\{\alpha\beta\}$ and $\{\beta\gamma\}$. The gluing operation is denoted by $\vee$. The purpose of gluing is that local solutions are combined to cover the whole possible range $\varepsilon_1 = \xi_1 \leq E \leq \xi_n$ (of $E$) via continuous probabilities. For $n = 3$ there are two global solutions: $\{12\} \vee \{23\}$ and $\{13\}$. The entropies are given as
\[
S_{\{12\} \vee \{23\}}(E) = \begin{cases} S_{\{12\}}(E) & \text{for} \quad \xi_1 \leq E \leq \xi_2, \\ S_{\{23\}}(E) & \text{for} \quad \xi_2 \leq E \leq \xi_3, \end{cases}
\] (37)
and where $S_{\{\alpha\beta\}}$ is given by (31). For $n = 4$ we have: $\{12\} \vee \{23\} \vee \{34\}$, $\{12\} \vee \{24\}$, $\{13\} \vee \{34\}$, and $\{14\}$.

Within solution $\{1n\}$ (e.g., $\{13\}$ for $n = 3$) the action with the lowest cost $\varepsilon_1$ is assigned the largest probability, while all other actions have the same probability; see (22). This is indeed the simplest possible prescription that provides non-zero probabilities for all utilities and holds for all possible values of $E$. The local minimum solution $\{1n\}$ coincides with the $\varepsilon$-greedy scheme from reinforcement learning [3]. It is seen to correspond to (at least) a local minimum of entropy. The candidate solution $\{12\}$ refers to the situation, where only the best and the second-best actions are assigned non-zero probabilities. The validity range for this local minimum is restricted by $\xi_1 = \varepsilon_1 \leq E \leq \xi_2 = \frac{\varepsilon_1 + \varepsilon_2}{2}$.

Thus within the present set-up these two simple schemes of assigning non-deterministic prior probability appear naturally. But in contrast to imposing them ad hoc, these schemes now have their validity conditions.

V. ENTROPY MAXIMIZATION FOR RISK-SEEKING AGENTS

Within the maximum entropy scheme, the risk-seeking agent will maximize over probabilities $p_k$ the entropy (16) under constraint (25) [31–34]. This maximization starts from (18), and it produces the Gibbs-Boltzmann probabilities [31–37] [cf. Footnote 9]:
\[
\hat{p}_k = \frac{1}{Z} e^{-\hat{\beta}\varepsilon_k}, \quad Z = \sum_{l=1}^n e^{-\hat{\beta}\varepsilon_l},
\] (38)
where $\hat{\beta}$ is the Lagrange factor determined from $E = \sum_{k=1}^n \hat{p}_k \varepsilon_k$; cf. (2, 18). The sign of $\hat{\beta}$ coincides with the sign of $\frac{1}{\beta} \sum_{k=1}^n \varepsilon_k - E$; see (19). Hence $\hat{\beta} \geq 0$, since we assume the validity of (19). In statistical mechanics $\hat{\beta}$ refers to inverse (absolute) temperature, and its positivity is a well-known fact [38].

It is not widely known in non-physical communities that the Gibbs-Boltzmann probabilities (38) have specific responses to parameters, which were uncovered before statistical mechanics [38]. Below we recall some features of (38) and compare them with the entropy minimization scheme.\footnote{\textsuperscript{11}}

Note that the probabilities at the local entropy minimum $\{\alpha\beta\}$ are not susceptible to changes of $\varepsilon_\gamma$ (with $\gamma > \beta$) that take place under a constant $E$; see (29–32). This differs from the Gibbs-Boltzmann probabilities (16) for the risk-seeking agent, where changing any cost $\varepsilon_k$ under a fixed $E$ will change $\hat{\beta}$:
\[
\left. \frac{\partial \hat{\beta}}{\partial \varepsilon_k} \right|_E = \frac{\hat{p}_k [1 + \hat{\beta}(E - \varepsilon_k)]}{\sum_{l=1}^n \hat{p}_l (\varepsilon_l - E)^2},
\] (39)

\textsuperscript{11} Probabilities in statistical mechanics can have an objective meaning; e.g., the Gibbs-Boltzmann probabilities (38) can refer to energy occupations of a system in contact with a much larger thermal bath at temperature $1/\beta$ [38]. But they can also have a subjective meaning—e.g., when making predictions about a complex, closed system [38].

\textsuperscript{12} An alternative way of obtaining (38) is to fix the entropy (16) to a non-zero value $\hat{S}$, and then minimize the average cost $\sum_{k=1}^n p_k \varepsilon_k$. Now $\hat{\beta}$ in (38) is defined from a fixed value of entropy. In the context of our problem, this way of looking at (38) is not adequate, since for us it is important that $E$ is an independent variable that can be decided by the agent. However, in a different setting this way of looking at (38) led to a definition of risk [26].
where the derivative is taken under a constant $E$. Hence all the probabilities \( \{\hat{p}_i\}_{i=1}^n \) will generally change including when changing $\varepsilon_k$. Ratios $\hat{p}_l/\hat{p}_m = e^{-\beta(\varepsilon_l - \varepsilon_m)}$, where $l \neq k$ and $m \neq k$, will change as well. This corresponds with the fact that there is a single maximum entropy solution (38), while the entropy minimization produces several competing local minima.

Another difference is that in the maximum entropy situation, as seen from (38)

\[
\varepsilon_k \approx \varepsilon_l \quad \text{implies} \quad \hat{p}_k \approx \hat{p}_l.
\]

For the entropy minimizing situation, (40) is not necessarily the case, as implied by (30), and as seen more explicitly below in (51–53). Due to this feature the entropy minimizing solution is capable of detecting even small differences between the costs. This is also one reason we insisted on strict inequalities in (4).

The maximized entropy is expressed from (38) as

\[
\hat{S} = -\sum_{k=1}^n \hat{p}_k \ln \hat{p}_k = \hat{\beta} E + \ln Z.
\]

Differentiating (41) and employing

\[
\frac{d\hat{S}}{d\beta} = -\frac{1}{\sum_{k=1}^n \hat{p}_k (\varepsilon_k - E)^2},
\]

obtained from $\sum_{k=1}^n \hat{p}_k \varepsilon_k = E$, we get

\[
\frac{d\hat{S}}{dE} = \hat{\beta}.
\]

Eq. (43) shows that for $\hat{\beta} > 0$ (which holds due to (19)), $\hat{S}$ is an increasing function of $E$; cf. (35). Likewise, one shows that $\hat{S}(E)$ is a concave function of $E$: $\frac{d^2 \hat{S}}{dE^2} \leq 0$; cf. (36). Now using (41, 39) we find

\[
\left. \frac{\partial \hat{S}}{\partial \varepsilon_i} \right|_E = -\hat{\beta} \hat{p}_i.
\]

It is seen that (44) is negative for $\beta > 0$; cf. (19). Hence the uncertainty $\hat{S}$ decreases if one of the costs increases for a fixed $E$. A similar feature holds as well for the minimized entropy in (31):

\[
\left. \frac{\partial S_{[\alpha \beta]}}{\partial \varepsilon_\gamma} \right|_E = \left. \frac{\partial S_{[\alpha \beta]}}{\partial \varepsilon_\gamma} \right|_E = (1 - \frac{\alpha}{\beta}) \left. \frac{\partial \lambda_\alpha}{\partial \varepsilon_\gamma} \right|_E \left. \frac{\partial S_{[\alpha \beta]}}{\partial \varepsilon_\alpha} \right|_E \leq 0.
\]

Inequality (45) follows from $\alpha < \beta$ (by definition of $S_{[\alpha \beta]}$), from $\left. \frac{\partial \lambda_\alpha}{\partial \varepsilon_\gamma} \right|_E \geq 0$ (which can be shown easily from (29)), and from $\left. \frac{\partial S_{[\alpha \beta]}}{\partial \varepsilon_\alpha} \right|_E \leq 0$, which is seen from (35). Eqs. (44, 45) imply an interesting feature of entropy optimization that we explore below in more detail. A large cost $\varepsilon_\gamma$ tends to be relevant (irrelevant) for entropy minimization (maximization), since it decreases the entropy.

Using (38, 42) one can show that

\[
\left. \frac{\partial \hat{p}_k}{\partial E} \right|_{\{\varepsilon_i\}_{i=1}^n} = -\frac{d\hat{\beta}}{dE} \hat{p}_k (\varepsilon_k - E).
\]

Due to $\frac{d\hat{\beta}}{dE} \leq 0$, we get from (46) that the probabilities of sufficiently high ($\varepsilon_k > E$) costs increase upon increasing $E$. This feature is both natural and expected: the more utility one is ready to invest (into adaptation), the more higher-costs actions he explores. We shall see below that it need not hold for the entropy-minimizing agent.

Finally, we recall that there is a well-known freedom associated with the definition of costs $\varepsilon_k$ [1]:

\[
\varepsilon_k \to a \varepsilon_k + b,
\]

where $a > 0$ and $b$ are arbitrary. In particular, (47) means that probabilities assigned to a given set of actions should not change, if all utilities (of those actions) are multiplied by a common positive factor $a$, and/or are shifted by a common arbitrary number $b$. It is clear that the probabilities for both entropy minimization and entropy maximization do hold (47); see in this context (29, 31, 32) and (38) and note that under (47), $E$ also changes as $E \to aE + b$. 
VI. EXAMPLES

A. Probabilities for three actions (n = 3)

Let us now study in detail the n = 3 situation (37): this is the simplest case that illustrates the difference between maximizing entropy and minimizing it. Indeed, for n = 2 the probabilities p₁ and p₂ = 1 − p₁ are uniquely determined by the contraint (2). Eqs. (22, 29) imply

\[
\{12\} : (p₁, p₂, p₃) = \left( \frac{1}{2} + \frac{\lambda_{12}}{2}, \frac{1}{2} - \frac{\lambda_{12}}{2}, 0 \right), \quad \lambda_{12} = \frac{\xi₂ - E}{\xi₂ - \xi₁},
\]

\[
\{23\} : (p₁, p₂, p₃) = \left( \frac{1}{3} + \frac{\lambda_{23}}{6}, \frac{1}{3} - \frac{\lambda_{23}}{6}, 1 - \frac{\lambda_{23}}{3} \right), \quad \lambda_{23} = \frac{\xi₃ - E}{\xi₃ - \xi₂},
\]

\[
\{13\} : (p₁, p₂, p₃) = \left( \frac{1}{3} + \frac{2\lambda_{13}}{3}, \frac{1}{3} - \frac{\lambda_{13}}{3}, 1 - \frac{\lambda_{13}}{3} \right), \quad \lambda_{13} = \frac{\xi₃ - E}{\xi₃ - \xi₁}.
\]

Recall that within \{12\} only the two smallest costs get non-zero probabilities; \{23\} prescribes equal probabilities to those two smallest costs, while \{13\} gives equal probabilities to the two highest cost actions.

Using the freedom provided by (47), we fix \( \epsilon₁ = 0 \) and \( \epsilon₃ = 1 \); hence \( 0 < \epsilon₂ < 1 \). Now (48–50) simplify as follows:

\[
\{12\} : (p₁, p₂, p₃) = \frac{1}{\epsilon₂} (\epsilon₂ - E, E, 0), \quad 0 \leq E \leq \frac{\epsilon₂}{2},
\]

\[
\{23\} : (p₁, p₂, p₃) = \frac{1}{2 - \epsilon₂} (1 - E, 1 - E, 2E - \epsilon₂), \quad \frac{\epsilon₂}{2} \leq E \leq \frac{1 + \epsilon₂}{3},
\]

\[
\{13\} : (p₁, p₂, p₃) = \frac{1}{1 + \epsilon₂} (1 + \epsilon₂ - 2E, E, E), \quad 0 \leq E \leq \frac{1 + \epsilon₂}{3}.
\]

It is seen that (52) and (53) do not coincide with each other for \( \epsilon₂ \rightarrow \epsilon₁ + 0 \) (\( \epsilon₁ = 0 \)); cf. the discussion around (40).

Note that within each solution (51–53) increasing \( E \) leads to larger probabilities for costly actions 2 and 3. We shall show below that this intuitively expected feature does not hold for transitions between different solutions (i.e., local minima of entropy).

For various parameter regimes we shall now compare the behavior of the risk-averse (entropy-minimizing) agent with the risk-seeking (entropy-maximizing) one. Everywhere, such comparisons are made for the same values of \( \{\epsilon_k\}_{k=1}^n \) and the same value of \( E \) (where \( E \) is the utility invested into exploration). As seen below, entropy-minimizing probabilities are neither majorized nor majorized by the entropy-maximizing ones, i.e., (9) and (10) are valid. This fact makes the situation interesting, in particular because (for \( n = 3 \)) there are two strategies for risk-aversion: putting a larger weight on the lowest-cost action or a lesser weight on the highest-cost action; cf. our discussion after (12).

B. Two low-cost actions

In this regime two lower costs are approximately equal: \( \epsilon₁ \approx \epsilon₂ \). Fig. 1 displays the entropies \( S_{\{12\}}(E) \) and \( S_{\{13\}}(E) \) for a representative set of parameters; see (37). It is seen that for \( \epsilon₁ \approx \epsilon₂ \) the global minimum is always \{13\}; cf. Fig. 1. Hence the corresponding probabilities are given by (53), where two actions with non-minimal costs \( \epsilon₂ \) and \( \epsilon₃ \) are given the same weight.

In the regime \{13\} the entropy minimizing probabilities (53) hold [cf. Fig. 2]

\[
\{13\} : \quad p₁ > \hat{p₁}, \quad p₃ > \hat{p₃},
\]

where \( 0 < E < \frac{1 + \epsilon₂}{3} \), and \( \hat{p_k} \) are the Gibbs-Boltzmann probabilities (38). Hence \( (p₁, p₂, p₃) \) neither majorizes \( (\hat{p₁}, \hat{p₂}, \hat{p₃}) \) nor is majorized by that, i.e., (54, 55) amount to a particular case of (10).

Now (54) means that the entropy-minimizing agent invests more probability on the lowest-cost action than the entropy-maximizing one. This is one strategy for risk-aversion.\(^{13}\) Inequality (55) means that the entropy-minimizing

\(^{13}\text{We can look at (54) from a different viewpoint. Recall that for } E \rightarrow \epsilon₁ = 0 \text{ both entropy-minimization and entropy-maximization}\)
agent assigns to costly actions more probability than the entropy-maximizing agent. Note however that in the considered regime $\varepsilon_1 \approx \varepsilon_2$, where the solution \{13\} is the global minimum of energy we have (in addition to (55))

$$p_1 - \hat{p}_1 \gg p_3 - \hat{p}_3,$$

except $E = 0$ and $E = \frac{1 + \varepsilon_2}{2}$, where $p_1 - \hat{p}_1 = p_3 - \hat{p}_3 = 0$; cf. Fig. 2. However, if $\varepsilon_2$ is sufficiently larger than $\varepsilon_1 = 0$, relation (56) need not hold—e.g., for $0.51 < \varepsilon_2 < 0.68$ and sufficiently small $E$, the global minimum of entropy is still given by \{13\}. There we can have (together with (54, 55)): $p_3 - \hat{p}_3 - (p_1 - \hat{p}_1) \approx 0.01 > 0$; i.e., though the difference is relatively small it is still positive.

Another difference is that the entropy-minimizing (maximizing) agent tends to underweight (overweight) the middle-cost action:

The entropy-maximizing agent focuses on this middle-cost action and ignores the most costly action since it has $\hat{p}_1 \simeq \hat{p}_2 \gg \hat{p}_3$; cf. Fig. 2.

Thus, the feature of \{13\} is that it puts more weight on the lowest-cost action—which is one possibility of risk-aversion—and does account for highest-cost actions in the sense of (55, 57).

C. Two high-cost actions: distinguishing between actions with approximately equal costs

Let us now study the opposite case $\varepsilon_1 \ll \varepsilon_2 \approx \varepsilon_3 = 1$. For a sufficiently small $E$ the global minimum of entropy is \{12\}; see Fig. 1. For the solution \{12\} ∨ \{23\} the inequalities (54, 55) are inverted:

$$\{12\} ∨ \{23\} : p_1 < \hat{p}_1,$$

$$p_3 < \hat{p}_3.$$  

In this regime another strategy of risk-aversion is realized—the entropy-minimizing agent puts less weight on the highest-cost action. In particular, for $0 \leq E \leq \frac{2}{3}$ this action gets no probability at all: $p_3 = 0$; see (48). Hence when having two possibilities $\varepsilon_2$ and $\varepsilon_3$ with comparable high costs $\varepsilon_2 \lesssim \varepsilon_3$, the agent prefers the lesser of two evils. In contrast, the entropy-maximizing agent will take these options with nearly equal probabilities. For larger values of $E$, the global minimum is \{23\}, where all probabilities are non-zero, but the probability for the action with the (intermediate) cost $\varepsilon_2$ is as large as for the least-cost action: $p_2 = p_1$; see (49) and Figs. 1 and 3.

We see here an effect whose traces were also observed in the previous scenario: when choosing between two actions of comparable cost $\varepsilon_2 \approx \varepsilon_3$, the entropy-minimizing agent is able to distinguish between them despite a small difference. In contrast to this, the entropy-maximizing agent will just take them with (approximately) equal probability, neglecting that small difference.

D. Transitions from one regime to another, re-entrance, cognitive dissonance and frustration

1. Transition between \{13\} and \{12\} upon increasing $E$

So far we studied cases $\varepsilon_1 \approx \varepsilon_2 \approx \varepsilon_3 = 1$ and $\varepsilon_1 \ll \varepsilon_2 \approx \varepsilon_3 = 1$, where one (global) solution—respectively, \{13\} and \{12\} ∨ \{23\}—provides the global entropy minimum for all values of $E$. Now we turn to studying cases, where $\varepsilon_1 \neq \varepsilon_2$ and $\varepsilon_3 \neq \varepsilon_2$. Here it is possible to have transitions between different local minima upon changing $E$; see Fig. 1 with the case $\varepsilon_2 = 0.33$.

For a sufficiently small $E$ the global minimum is \{13\}. But for $E > 0.15$, the global minimum becomes \{12\}. This transition from one regime to another is continuous in terms of entropy, but discontinuous in terms of probabilities produce: $p_1 \to 1$ and $\hat{p}_1 \to 1$, i.e., they converge (as they should) to taking the least-cost action. Now one can ask to which extent this least-cost action is stable ($\hat{p}_1, \hat{p}_2, \hat{p}_3) = (1, 0, 0)$ with respect to a small, but non-zero $E$. This question can be addressed by looking at one of standard distances between probabilities, e.g., the variational distance $\Delta_1[p(E), \tilde{p}] = \frac{1}{2} \sum_{k=1}^3 |p_k(E) - \tilde{p}_k|$ or the Hellinger distance $\Delta_2[p(E), \tilde{p}] = 1 - \sum_{k=1}^3 \sqrt{p_k(E) \tilde{p}_k}$. It is clear that $\Delta_1[p(E), \tilde{p}] = \Delta_1[p(E), \hat{p}] = p_1 - \hat{p}_1$ and $\Delta_2[p(E), \tilde{p}] = \Delta_2[p(E), \hat{p}] = \sqrt{p_1} - \sqrt{\hat{p}_1}$. Now (54) and Fig. 1 show that for a small but non-zero $E$ we have $\Delta_k[p(E), \hat{p}] > \Delta_k[p(E), \tilde{p}]$ ($k = 1, 2$), i.e. the least-cost action is more stable for the entropy minimizing agent.
for various actions, as seen from (48–50). This is an analogue of the first-order phase transition in statistical physics systems (e.g., the liquid-vapor transition) [38], where the role of \( E \) is played by the (physical) temperature. There the thermodynamic potential whose minimization determines stability (for our case this is entropy) changes continuously, but the order parameter (the difference between densities of liquid and vapor) suffers a discontinuous change [38].

The transition at \( E = 0.15 \) is against naive intuition, because with a larger average cost \( E \) the agent neglects the action \( \varepsilon_3 \), which is related to the highest cost. Here the agent who invests less into adaptation assigns more probability to costly actions. For the entropy-maximizing agent it is impossible that the probability of the most costly action decreases upon increasing \( E \); cf. our discussion around (46). For the entropy-minimizing agent such an effect does take place—the probability \( p_3 \) of the most costly state changes from a positive value to zero—due to transition from one local minimum to another. For changes within the local minimum this cannot happen, as seen from (51–53). This effect can be related to cognitive dissonance [60–62] 14.

2. Re-entrance

However, the above transition from \{13\} to \{12\} upon increasing \( E \) is not the end of the story: at \( E = 0.165 \) (and \( \varepsilon_2 = 0.33 \)) the solution \{12\} continuously (both in terms of entropy and probabilities) changes to \{23\}. This is a natural transition: for \( E > 0.165 \), the solution \{12\} does not exist anymore, since it cannot hold the constraint \( E = p_2\varepsilon_2 \); cf. (34). This is an analogue of a second-order phase transition in statistical physical systems (e.g., paramagnetic-ferromagnetic transition in magnets) [38], where one solution ceases to exist and is replaced by another via a continuous change of the order parameter (magnetization, in our case probabilities of actions).15

But eventually, for \( E > 0.18 \), the global minimum is back from \{23\} to \{13\} (re-entrance). The probabilities of the solution \{13\} stay almost constant in the whole interval \( 0.15 \leq E \leq 0.18 \). Note that the re-entrance effect persists up to \( \varepsilon_2 \leq 0.45 \). For \( 0.68 > \varepsilon_2 > 0.45 \) (not shown on figures) the re-entrance behavior is absent; there is only a single transition from \{13\} to \{12\} upon increasing \( E \). Eventually, for \( 0.68 < \varepsilon_2 \) the global entropy minimum is always \{12\} \( \lor \) \{23\}, and we revert to the studied regime \( \varepsilon_1 \ll \varepsilon_2 \approx \varepsilon_3 = 1 \).

The transitions from \{13\} to \{12\} \( \lor \) \{23\} and back mean that there are three local minima with comparable values of entropy but different values for probabilities of actions; cf. Fig.1. They compete with each other upon relatively small changes of \( E \). This effect resembles frustration, whose psychological content is that there are two (or more) different (incommensurate) goals or motivations that compete with one other. The concept of frustration is also well-known in statistical physics of complex systems (see [71] for a recent review), where its meaning is very close to the above, since it relates to competing local minima of the thermodynamic potential (entropy in our case).

We stress that neither of the above effects is seen for the risk-seeking agent, because in this situation probabilities (38) are unique and depend smoothly on the parameters involved. This has to do with the fact that (38) was obtained via maximization of a concave function in a convex set, which generically produces unique and well-behaved results [13]. Possible non-uniqueness of the Gibbs-Boltzmann probabilities (38) can show up only in the thermodynamic limit \( n \to \infty \) for specific systems that can be subject to phase-transitions. (We however refrain from considering the \( n \to \infty \) case, since it is far from an agent facing a limited amount of different choices.)

E. Four actions

We turn to the entropy-minimizing scenarios for \( n = 4 \). We do so briefly, because though \( n = 4 \) is richer then \( n = 3 \), it does not offer conceptual novelties. Here we also fix \( \varepsilon_1 = 0 \) and \( \varepsilon_4 = 1 \) [cf. (47)], and we are left with two parameters: \( 0 < \varepsilon_2 < \varepsilon_3 < 1 \). For \( \varepsilon_2 \approx \varepsilon_3 \approx 1 \) the global minimum of entropy is provided by the solution

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14 It is characterized by the following feature: the more energy and/or effort people invest into some situation, the narrower the set of their actions or intentions tends to become. (This narrowing was described theoretically within probabilistic opinion formation [63].) E.g., the more (possessions and/or time) people invest into a sectarian movement, the more vigorously they tend to support it [60–62].

15 Another example: once people already buy something, they tend to have fewer doubt about its value and relevance. The above behavior of the entropy-minimizing agent agrees with cognitive dissonance—the more utility the agent decides to invest into adaptation (i.e., \( E \) increases), and the narrower his action set tends to become. The underlying cause of this effect in our situation also relates to one of the phenomenological features of cognitive dissonance [60, 61]: that is, people tend to minimize the uncertainty in their action (belief or intention) set, which for our situation refers to minimizing entropy (uncertainty). Within the cognitive dissonance theory these two aspects—namely that investing more means believing more and uncertainty minimization—are related to each other on the grounds of a general plausibility [60, 61]. Here we see that they are related to each directly, and risk minimization explains this relation.
It has the same meaning as above: among actions of comparable cost the risk-minimizing agent chooses the one with the lowest cost (i.e., $p_2 > 0$, but $p_3 = p_4 = 0$) for $\xi_1 \leq E \leq \xi_2$. Now this solution does not exist for $\xi_2 \leq E \leq \xi_3$. In this regime the agent takes the solution $\{23\}$, where $p_2 > 0$, $p_3 > 0$, but $p_4 = 0$. For $\varepsilon_2 \approx \varepsilon_3 \ll 1$ the global minimum is $\{14\}$, as expected. Now $p_2 = p_3 = p_4 > 0$. These two regimes are similar to those for $n = 3$. For $\varepsilon_1 \approx \varepsilon_2 \ll \varepsilon_3 \approx \varepsilon_4$ the global minimum $\{14\}$ for a relatively low $E$ changes to $\{13\} \lor \{34\}$ for a large $E$. Finally, when both $\varepsilon_2$ and $\varepsilon_3$ are close to 0.5, there is a sequence of transitions from one solution to another, such that every solution becomes a global minimum for a certain $E$; e.g., for $\varepsilon_2 = 0.4$ and $\varepsilon_3 = 0.5$ we observe the following transitions upon increasing $E$: $\{14\} \rightarrow \{12\} \rightarrow \{24\} \rightarrow \{23\} \rightarrow \{34\}$.

VII. SUMMARY AND DISCUSSION

An agent who wants to be adaptive in choosing between several actions in a varying and complex environment cannot exclusively focus on the least-cost action. The agent should also explore actions with non-minimal costs, and not only exploit the action with the minimal cost. This is the known exploration-exploitation trade-off, which exists in various forms and fields. We here worked out a set-up that allows us to study the trade-off within decision-making.

The agent faces several actions, whose initial costs are known, but it is not known how they will change given the time and actions of the agent. Now the agent needs to perform several actions so that if there is regular information about costs, then this information can be gathered. With which probabilities should he choose initial actions in such an agnostic situation?

We worked out one possibility, where the exploration goes via risk-minimization—a heuristic rule that people frequently apply in uncertain situations. Risk is a wide notion that appears in various situations. In particular, both risk-minimization (aversion) and risk-maximization are seen in experiments with people gambling on uncertain monetary outcomes [12].

In our situation the treatment of risk is easier than usual (cf. [22–24]) because we compare agents that have the same costs for their actions and decide to invest the same amount of average utility into exploration, i.e., into not taking the least-cost action only. We start with the stochastic dominance (or majorization) condition, which constraints specific risk measures. Given certain standard constraints, we show that the entropy can be employed as a measure of risk. This result is obtained via several different axiomatization schemes; see section III. Thus the risk-averse (risk-seeking) agent will minimize (maximize) the entropy given the average utility invested into adaptation, and also the constraint that more costly actions should get less probability. There are two different strategies of risk aversion: putting more probability on low-cost vs. high-cost actions. These strategies are different, and they relate to a rich behavior spectrum even in the simplest case of three actions.

While the entropy maximization is a well-known rule in probabilistic inference [31–40], the entropy minimization is an under-explored idea; cf. [28–30, 39, 40, 58, 59]. We show that this method can lead to useful predictions, e.g., it recovers (under definite conditions) the $\varepsilon$-greedy probability known in reinforcement learning theory [3]. Our main result is that the entropy-minimizing agent (in contrast to the entropy-maximizing one) shows certain aspects of intelligent behavior: (i) taking into account costly actions; (ii) choosing the best alternative among two comparable ones; (iii) cognitive dissonance.

Within (i) the entropy-minimizing (risk-averse) agent puts more probability into the least-cost action than the risk-seeking (entropy-maximizing) agent, and distributes the remainder such that the largest-cost action gets more probability than the risk-seeking agent; see (54, 55). Empirically, this scenario coincides with the $\varepsilon$-greedy strategy known in reinforcement learning [3], but now it is derived—together with its validity conditions—from a more general principle of entropy minimization. Overall, this scenario resembles the behavior of a scientist who follows the incremental character of any science—hence devotes most of his time to traditional, low-cost subjects but is still open-minded enough to venture on alternatives that have higher costs of implementation and recognition.

When choosing between two actions with comparable cost, the entropy-minimizing agent chooses to put a higher probability on the lower-cost action; cf. (ii). We find this to be a (rudimentary) scientific attitude. The progress of science does not only relate to noting small differences in experiments and/or in theory; e.g., successful scientists are prone to see potential contradictions that are ignored by others. Also, modern theories of physics (e.g., quantum mechanics) are based on small experimental differences between their predictions and those of classical physics. In contrast, the entropy-maximizing agent gives approximately the same probability for actions with close costs.

The entropy-minimizing agent also demonstrates cognitive dissonance: by increasing the amount of utility $E$ invested into adaptation, this agent tends to nullify probabilities of high-cost actions. This effect relates to the fact that the minimization of entropy produces several local minima that compete with each other (frustration). Within the cognitive dissonance theory this effect is taken as a general sign of a loosely defined cognitive consistency. We relate it with risk-minimization.

We mention that maximizing the entropy of probability paths was proposed as a scheme for the emergence of
intelligent behavior [64]. Though the mathematical details of the original proposal are unclear [65], the proposal was generalized to social collective systems [66], and formalized within the convex analysis [67]. We stress however that within the set-up studied here, the entropy maximization did not show features of intelligent behavior. Further work is needed to connect the presented research with the ones reported in [64, 66, 67]. There are also proposals of employing entropy minimization for improving the performance of machine learning algorithms [68]. Future research may clarify relations of this proposal with the presented results.

Another open problem is how to modify/continue the presented theory for applying it to problems of creativity modeling. One difference here is that in creative task solving it is the action space that has to be conceived and understood, while in various types of decision theories the action space is fixed. As examples show, even with this serious difference there are clear analogies between creative task solving and the exploration-exploitation dilemma.

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16 Two friends approach a river and want to pass it. They ask a fisherman who has a boat to help them. The fisherman has two conditions: only one person can be in the boat; the boat should be brought back from where it is taken. People normally start solving this task by over-concentrating on one specific (subjectively most likely) option: The two friends together approach the same side of the river. This would be a usual scenario for friends, but this makes the problem unsolvable. Insisting on this option, people come up with rather artificial constructions for solving the problem. But it is nowhere said that the friends approached the same side of the river. If they approached the different sides, the problem has a trivial solution, which would be found if people devoted some time to this (subjectively less likely) possibility.
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FIG. 1: (Color online) The locally minimal entropies $S_{\{12\} \vee \{23\}}(E)$ (red curves) and $S_{\{13\}}(E)$ (blue curves) as functions of $E$ for $n = 3$; see (37). Using (47) we fixed for costs: $\varepsilon_1 = 0$, $0 < \varepsilon_2 < 1$, and $\varepsilon_3 = 1$.
Dotted curves: $\varepsilon_2 = 0.1$. The blue curve is always lower: $S_{\{12\} \vee \{23\}}(E) > S_{\{13\}}(E)$; hence the local minimum $\{13\}$ is the global entropy minimum.

Full curves: $\varepsilon_2 = 0.33$. Now the blue and red curves intersect two times: the local minimum $\{13\}$ is not the global one for $0.15 \leq E \leq 0.18$. The transition from $\{12\}$ to $\{23\}$ takes place at $E = 0.165$.
Dashed curves: $\varepsilon_2 = 0.8$. Now the red curve is always lower, $S_{\{12\} \vee \{23\}}(E) < S_{\{13\}}(E)$, meaning that the solution $\{12\} \vee \{23\}$ is the global minimum.

FIG. 2: (Color online) Probabilities as a function of $E$ for the $n = 3$ situation with $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$ and $\varepsilon_3 = 1$. Full curves: $\hat{p}_1$ (black), $\hat{p}_2$ (magenta), $\hat{p}_3$ (green). Dotted curves: $p_1$ (black), $p_2 = p_3$ (green).
The Gibbs-Boltzmann probabilities $\hat{p}_k$ for the risk-seeking agent are calculated from (38). The probabilities $\{p_k\}_{k=1}^3$ refer to $\{13\}$ [see (53)], which is the global minimum for the present values of $\varepsilon_k$; cf. Fig. 1.

FIG. 3: (Color online) The same as in Fig. 2, but now $\varepsilon_1 = 0$, $\varepsilon_2 = 0.8$ and $\varepsilon_3 = 1$. Dotted curves refer to the solution $\{12\} \vee \{23\}$ [see (52)], which is now the global minimum: $p_1$ (black), $p_2$ (magenta; note that $p_2 = p_3$ for $E \geq \xi_2 = \frac{\varepsilon_2}{2} = 0.4$), and $p_3$ (green; we have $p_3 = 0$ for $E \leq \xi_2 = \frac{\varepsilon_2}{2} = 0.4$).
Appendix A: Alternative route to constrained entropy optimization

The purpose of this Appendix is to outline an alternative method for obtaining entropy as a measure of risk and uncertainty; see (18). Recall that optimization (i.e., minimization or maximization) of entropy (16) under constraint (2) can be done via the optimization of the Lagrange function

$$L = - \sum_{k=1}^{n} p_k \ln p_k + \beta \sum_{k=1}^{n} p_k \varepsilon_k,$$  \hspace{1cm} (A1)

where $\beta$ is the Lagrange multiplier that corresponds with $E$ in (2). Following [52] we shall mention an approach that allows us to recover directly (A1) via few reasonable axioms. This is useful as an alternative (and more direct) route to optimizing entropy under (2).

Let us reinterpret actions $A_1, ..., A_n$ in (1) as events of a classical probability space. One seeks a measure $L_n(A_1, p_1; ..., A_n, p_n)$ of uncertainty (or risk) that depends on both the probabilities and the corresponding events and that hold the following axioms [52]:

(a) $L_n(A_1, p_1; ..., A_n, p_n)$ is symmetric with respect to any permutation of $n$ elements $(z_1, ..., z_n)$, where $z_k = (A_k, p_k)$.

(b) $L_n(A_1, p_1; ..., A_n, p_n)$ holds the branching feature

$$L_n(A_1, p_1; ..., A_n, p_n) = L_{n-1}(A_1 \cup A_2, p_1 + p_2; A_3, p_3; ..., A_n, p_n) + (p_1 + p_2) L_2(A_1, \frac{p_1}{p_1 + p_2}; A_2, \frac{p_2}{p_1 + p_2}).$$  \hspace{1cm} (A2)

This is a natural feature for an uncertainty, where joining to events $A_1$ and $A_2$ (thus $p_1 + p_2$ is the joint probability) leaves the residual uncertainty $L_2(A_1, \frac{p_1}{p_1 + p_2}; A_2, \frac{p_2}{p_1 + p_2})$ with conditional probabilities $\frac{p_1}{p_1 + p_2}$ and $\frac{p_2}{p_1 + p_2}$ for $A_1$ and $A_2$, respectively.

(c) $L_n(A_1, p_1; ..., A_n, p_n)$ is a continuous function of $p_1, p_2, ..., p_n$.

The three axioms lead to [52]:

$$L_n(A_1, p_1; ..., A_n, p_n) = - \sum_{k=1}^{n} p_k \ln p_k + \hat{\gamma} \sum_{k=1}^{n} p_k \varepsilon_k(A_k),$$  \hspace{1cm} (A3)

where $\varepsilon_k(A_k)$ is an arbitrary function of $A_k$, and $\hat{\gamma}$ is an arbitrary constant. (An irrelevant additive constant was fixed to zero). Interpreting $\varepsilon_k(A_k)$ as the cost related to $A_k$, and equating $\hat{\gamma} = \hat{\beta}$ we revert from (A3) to (A1). Eq. (A3) and many related results can be proved via the functional equations methods reviewed in [53].

Note, that expressions similar to (A3), i.e., a convex combination of entropy and expected cost (negative utility) were proposed in [54, 55] as a measure of risk. Refs. [54, 55] employ this measure for elucidating several controversies in the decision theory. The same measure was axiomatically deduced and studied in [56]. Taking into account the axiomatic development, one can say that the measure of risk (A3) expressed by a linear combination of entropy and expected cost does have normative features.

Appendix B: Local minimality of solutions (29).

The aim of this Appendix is to show that the solutions for entropy minimization given by (29) do provide local minima of entropy. This is an important point, because once the local minimality is established, the solutions become meaningful even if they do not provide the global entropy minimum. To illustrate ideas, we start with the simplest non-trivial situation.
1. \( n = 3 \)

To check the local minimality we represent the probabilities of different actions as [see (22)]

\[
\begin{align*}
p_1 &= \mu_1 + \frac{\mu_2}{2} + \frac{\mu_3}{3} + \lambda_1 + \frac{\lambda_2}{2} + \frac{\lambda_3}{3}, \\
p_2 &= \frac{\mu_2}{2} + \frac{\mu_3}{3} + \frac{\lambda_2}{2} + \frac{\lambda_3}{3}, \\
p_3 &= \frac{\mu_3}{3} + \frac{\lambda_3}{3},
\end{align*}
\]

(B1)

(B2)

(B3)

(B4)

where the unperturbed probabilities are [see (25, 26)]

\[
\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 &= 1, \\
\lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 &= E,
\end{align*}
\]

(B5)

(B6)

and where small perturbations \( \mu_k \) hold

\[
\begin{align*}
\mu_1 + \mu_2 + \mu_3 &= 0, \\
\mu_1 \xi_1 + \mu_2 \xi_2 + \mu_3 \xi_3 &= 0.
\end{align*}
\]

(B7)

(B8)

We recall from (33) that

\[
\xi_1 \leq \xi_2 \leq \xi_3.
\]

(B9)

To check the local minimality of the first solution we take \( \lambda_2 = 0 \) in (B1–B3). Now (B4) demands [in addition to (B7, B8)]

\[
\mu_2 > 0.
\]

(B10)

Now we have for the entropy changes due to the perturbation:

\[
\Delta S = -\sum_{k=1}^{3} p_k \ln p_k + \sum_{k=1}^{3} p_k |_{\mu_i=0} \ln p_k |_{\mu_i=0}
\]

(B11)

\[
= -\sum_{k=1}^{3} (p_k - p_k |_{\mu_i=0}) \ln p_k |_{\mu_i=0}
\]

(B12)

\[
= -(\mu_1 + \frac{\mu_2}{2} + \frac{\mu_3}{3}) \ln \left[ \frac{p_1 |_{\mu_i=0}}{p_2 |_{\mu_i=0}} \right]
\]

(B13)

\[
= \frac{2\mu_2}{3} \left[ \frac{\xi_3 - \xi_2}{\xi_3 - \xi_1} \right] \ln \left[ \frac{p_1 |_{\mu_i=0}}{p_2 |_{\mu_i=0}} \right],
\]

(B14)

where in (B12) we kept only the linear order over \( \mu_i \), and where we employed (B7, B8) in (B13) and in (B14). Due to (B10) and to \( \ln \left[ \frac{p_1 |_{\mu_i=0}}{p_2 |_{\mu_i=0}} \right] \geq 0 \), we get that \( \Delta S > 0 \) (we are looking for the local minimum of entropy) is achieved for

\[
\frac{\xi_3 - \xi_2}{\xi_3 - \xi_1} > \frac{1}{4}.
\]

(B15)

This inequality always holds, once one recalls the definition of \( \xi_k \); see (26, 4).

To check the local minimality of the second solution we take \( \lambda_1 = 0 \) in (B1–B3). Now (B4) demands [in addition to (B7, B8)]

\[
\mu_1 > 0.
\]

(B16)

Repeating the same steps as in (B11–B14), we get

\[
\Delta S = \frac{\mu_1}{3} \frac{\xi_2 - \xi_1}{\xi_3 - \xi_2} \ln \left[ \frac{p_2 |_{\mu_i=0}}{p_3 |_{\mu_i=0}} \right] > 0,
\]

(B17)
i.e. this solution is local minimum (without additional conditions) due to (B16), \( \ln \left[ \frac{p_2|_{\mu_l=0}}{p_3|_{\mu_l=0}} \right] > 0 \) and (B9).

For the third solution we take \( \lambda_3 = 0 \) in (B1–B3). Now (B4) demands [in addition to (B7, B8)]

\[
\mu_3 > 0.
\]  

(B18)

We get instead of (B17)

\[
\Delta S = O(\mu_3) + \frac{\mu_3}{3} \ln \left[ \frac{3 e p_2|_{\mu_l=0}}{\mu_3} \right] > 0.
\]  

(B19)

This expression is always non-negative, whenever \( \mu_3 > 0 \) is sufficiently small. Thus all solutions are always local minima.

2. \( n > 3 \)

We now turn to the more general situation and write probabilities as

\[
p_k = \sum_{l=k}^{n} \frac{\mu_l}{T} + \sum_{l=k}^{n} \frac{\lambda_l}{T}, \quad k = 1, \ldots, n,
\]  

(B20)

\[
p_1 \geq p_2 \geq \ldots \geq p_n,
\]  

(B21)

\[
\sum_{l=k}^{n} \mu_l = 0,
\]  

(B22)

\[
\sum_{l=k}^{n} \xi_l \mu_l = 0,
\]  

(B23)

where \( \mu_l \) are perturbations. Now the unperturbed solution is defined by only two non-zero elements in \( \{\lambda_\alpha\}_{\alpha=1}^{n} \geq 0: \lambda_\alpha \text{ and } \lambda_\beta. \) Eq. (B21) then implies that besides \( \mu_\alpha \) and \( \mu_\beta \) all other \( \mu_k \) are necessarily non-negative:

\[
\mu_k \geq 0, \quad k \neq \alpha, \quad k \neq \beta.
\]  

(B24)

Using (B22, B23), \( \mu_\alpha \) and \( \mu_\beta \) are expressed as

\[
\mu_\alpha = \sum_{k=1, k\neq \alpha, k\neq \beta}^{n} \mu_k \frac{\xi_k - \xi_\beta}{\xi_\beta - \xi_\alpha}, \quad \mu_\beta = \sum_{k=1, k\neq \alpha, k\neq \beta}^{n} \mu_k \frac{\xi_k - \xi_\alpha}{\xi_\alpha - \xi_\beta}.
\]  

(B25)

Eqs. (B25) imply that if at least one \( p_k|_{\mu_l=0} \) equals to zero, the corresponding solution is locally stable via the same mechanism as in (B19).

Solutions for which \( p_k|_{\mu_l=0} > 0 \) can be studied on the case-by-case basis. For the solution with \( \lambda_1 > 0 \) and \( \lambda_n > 0 \) we obtain from (B25) [cf. (B11)]:

\[
\Delta S = - \sum_{k=1}^{n} (p_k - p_k|_{\mu_l=0}) \ln p_k|_{\mu_l=0}
\]  

(B26)

\[
= \ln \left[ \frac{p_1|_{\mu_l=0}}{p_2|_{\mu_l=0}} \right] \frac{n-1}{n} \sum_{k=2}^{n-1} \mu_k \left\{ \frac{\xi_\alpha - \xi_k}{\xi_\alpha - \xi_1} - \frac{n-k}{k(n-1)} \right\}
\]  

(B27)

Now using (26, 4) for \( \xi_k \) one can show directly that all the curly brackets in (B27) are non-negative, which together with (B24) and (B21) implies \( \Delta S \geq 0 \), i.e., this solution is a local minimum of entropy. Generalizing this argument we converge to a conclusion that all the solutions are local minima.