Interior of Schwarzschild Black-Hole: A model of relativistic free particle

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Abstract: Using the standard ADM model [4] for the geometry of a Schwarzschild Black-hole, it can be shown that the interior behaves as a relativistic free particle which can be quantized in great detail [5],[6],[8],[9],[10]. And here, our present endeavour is to provide a quantum picture of the interior of the Schwarzschild’s Black-hole. Exploiting the standard tools and techniques of the Hamiltonian approach to general relativity [5], it can be established that the metric components of invariant length element, act as dynamical quantities in field theoretic viewpoint. This enables the theory to be tested in the frame of gauge theory. This is a novel observation in the context of gauge theoretic approach to gravity. The implication of the relativistic free particle approach [9],[10] to provide a complete quantum mechanical description of the interior of the Schwarzschild Black-hole is the key theme of our endeavour.

PACS numbers: 04.60.-m; 04.70.Dy; 11.10.Ef.

Keywords: Schwarzschild Black-hole, Relativistic free particle, First class constraints, Gauge theory, Metric, Gauge invariant action, BRST and anti-BRST symmetries
1 Introduction

The most familiar interaction, gravity is described by Einstein’s general relativity theory which is a classical theory and deals with the underlying spacetime geometry. On the other hand, the gauge theory which describes the rest of three fundamental interactions of nature, is a quantum theory. This has been a long awaited outstanding problem to provide a quantum description of gravity or a spacetime geometric version of gauge theories. There have been many attempts to provide a quantum picture of the gravity e.g. string theory, loop quantum gravity etc. To quantize the gravity one needs to rewrite the Einstein equations in Hamilton or Lagrangian formulation, best suited for quantum theories. The Hamilton’s formulation of gravity, was proposed firstly by P. A. M. Dirac [5] which flourished in time. Based on that, many attempts to quantize the gravity have been done. Since the gravity is a large scale interaction while the gauge interactions are small scale interactions; therefore, a theory which incorporates the gravity and quantum mechanics both, should meet the extreme conditions of gravity. A black-hole is such an object where the spacetime is so warped that a quantum description of gravity becomes essential for underlying spacetime.

Black-hole physics has been one of the leading area in theoretical physics since the derivation of first non-trivial solution of Einstein’s equation for a homogeneous and isotropic spacetime, by Karl Schwarzschild in his famous paper [2]. This attracted the attention of many leading physicists of the time. Since then it has been an active area of research. The most fascinating point of the Schwarzschild’s Black-hole, is the concept of event-horizon; a hypersurface around the Black-hole which acts as a boundary to it. Inside the boundary, even the fastest moving light, can not escape once entered in its interior. There are many such bizarre things associated with the event-horizon of the Black-holes. One of the well known result for the event-horizon of a Schwarzschild’s Black-hole, is the interchange of behaviour of space and time. The time behaves as if it were the radial coordinate while the radial coordinate behaves as if it were a time coordiante, a monotonically increasing sequence of real numbers [12],[13].

2 Developments

Inside the event-horizon of a Schwarzschild’s Black-Hole, the nature of radial and time coordinate interchange [12],[13]. In the famous ADM (Arnowitt, Deser and Misner) approach [3],[4],[17] the general isotropic and static, Black- Hole metric is given as [3],[4],[17].

\[ ds^2 = -\frac{N^2(t)}{B(t)}dt^2 + B(t)dr^2 + H^2(t)d\Omega^2 \]  

(1)

where \(N(t)\) is the lapse function and

\[ d\Omega^2 = r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \]  

(2)

Substituting this metric solution in Hilbert-Einstein action,

\[ A = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + A_{YGH} \]  

(3)
where $R$ is Ricci scalar and $g$ is the determinant of metric under consideration. Here $A_{YGH}$ stands for York-Gibbon-Hawking boundary term. Integrating over spatial dimensions, we get the Lagrangian density as following [1],[2].

\[
L = -\frac{V_0}{8\pi G} \left[ \frac{1}{N}(H\dot{H}\dot{B} + \dot{H}^2B) - N \right]
\]

(4)

where $V_0 = 4\pi \int dr$

Now substituting,

\[
x - y = H, \quad x + y = HB,
\]

\[
\frac{dH}{dt} = \dot{H} = \dot{x} - \dot{y}
\]

(5)

And also,

\[
\frac{d}{dt} (HB) = \frac{d}{dt} (x + y) = \dot{x} + \dot{y},
\]

\[
\dot{H}B + H\dot{B} = \dot{x} + \dot{y}.
\]

(6)

Multiplying equation (5) with equation (6), we obtain,

\[
\dot{H}^2B + H\dot{H}\dot{B} = (\dot{x} - \dot{y})(\dot{x} + \dot{y}),
\]

\[
\dot{H}^2B + H\dot{H}\dot{B} = \dot{x}^2 - \dot{y}^2.
\]

(7)

Substitution of equation (7) in equation (4), we obtain,

\[
\mathcal{L} = - \left[ \frac{1}{N}(\dot{x}^2 - \dot{y}^2) - N \right]
\]

(8)

where we have replaced $L$ by $\mathcal{L} = 8\pi G \frac{L}{V_0}$.

Here one thing is to be noted that absorbing the factor $8\pi G \frac{L}{V_0}$ in the expression of $\mathcal{L}$ does not alter the dynamics of the system.

### 3 Intereior: a model of relativistic free particle

Now, in order to make the expression (8) more simplified, let

\[
\sqrt{x - y} = \frac{1}{2}(a - b),
\]

\[
\sqrt{x + y} = \frac{1}{2}(a + b).
\]

(9)

Or

\[
x - y = \frac{1}{4}(a - b)^2,
\]

\[
x + y = \frac{1}{4}(a + b)^2.
\]

(10)
Therefore,

\[ (\dot{x} - \dot{y})(\dot{x} + \dot{y}) = \frac{1}{16} \frac{d}{dt} (a - b)^2 \frac{d}{dt} (a + b)^2, \]

\[ \Rightarrow \dot{x}^2 - \dot{y}^2 = \frac{1}{4} (a^2 - b^2)(\dot{a}^2 + \dot{b}^2). \]  

(11)

Now, let us take further,

\[ N = \sqrt{\dot{x}^2 - \dot{y}^2} \tilde{N} \]  

(12)

With substitution from equation (11) and equation (12), we obtain,

\[ \mathcal{L} = -\left[ \frac{1}{4\sqrt{x^2 - y^2} \tilde{N}} (a^2 - b^2)(\dot{a}^2 - \dot{b}^2) - 4\sqrt{x^2 - y^2} \tilde{N} \right], \]

\[ \Rightarrow \mathcal{L} = -\left[ \frac{(\dot{a}^2 - \dot{b}^2)}{\tilde{N}} - (a^2 - b^2) \tilde{N} \right]. \]  

(13)

The canonical momenta corresponding to the various variables, can be calculated from above Lagrangian density as follows,

\[ \Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{a}}, \]

\[ \Rightarrow \Pi_a = -2 \frac{\dot{a}}{\tilde{N}}. \]  

(14)

And similarly,

\[ \Pi_b = \frac{\partial \mathcal{L}}{\partial \dot{b}}, \]

\[ \Rightarrow \Pi_b = 2 \frac{\dot{b}}{\tilde{N}}. \]  

(15)

And

\[ \Pi_{\tilde{N}} \approx 0 = \Omega_1 \]  

(16)

is the primary constraint on the theory [15],[16].

In order to calculate the secondary constraint on the theory, we calculate the Hamiltonian density first. The Hamiltonian density can be calculated as,

\[ \mathcal{H} = \dot{a}\Pi_a + \dot{b}\Pi_b - \mathcal{L}, \]

\[ \Rightarrow \mathcal{H} = -\frac{\tilde{N}}{4} (\Pi_a^2 - \Pi_b^2) - \tilde{N} (a^2 - b^2). \]  

(17)
The primary Hamiltonian density can be constructed as follows,

\[ H_p = H + \xi \Pi \tilde{N} \]

\[ \Rightarrow H_p = -\frac{\tilde{N}}{4}(\Pi_a^2 - \Pi_b^2) - \kappa \tilde{N}(a^2 - b^2) + \xi \Pi \tilde{N} \]  \hspace{1cm} (18)

where \( \xi \) is the Lagrange’s multiplier.

Now, with help of equation (18), we can calculate the secondary constraint as follows,

\[ \Omega_2 = \frac{d}{dt} \Omega_1 = \frac{d}{dt} \Pi \tilde{N} = \{\Pi \tilde{N}, H_p\} \approx 0 \]

\[ \Rightarrow \Omega_2 = \frac{\Pi_a^2 - \Pi_b^2}{4} + (a^2 - b^2) \approx 0. \]  \hspace{1cm} (19)

One can check that there is no further constraint on the theory by applying the same procedure. The two constraint on the theory are,

\[ \Omega_1 = \Pi \tilde{N} \approx 0, \]

\[ \Omega_2 = \frac{\Pi_a^2 - \Pi_b^2}{4} + (a^2 - b^2) \approx 0. \]  \hspace{1cm} (20)

Since the Poission bracket of these two constraints vanishes therefore in Dirac’s prescription of constraints [15], these two appear to be first-class constraints on the theory.

And therefore, the theory is a gauge theory. The generator for the gauge transformations can be obtained, as follows,

\[ G = \int dr \left( \dot{\xi} \Omega_1 - \xi \Omega_2 \right) \]  \hspace{1cm} (21)

The various gauge transformation can be obtained using the general procedure of transformations as,

\[ \delta \Phi = \{G, \Phi\} \]  \hspace{1cm} (22)

where \( \Phi = a, b, \tilde{N} \). Using this definition, one can calculate the transformations for above mentioned fields as,

\[ \delta a = -\frac{\xi}{\tilde{N}} \dot{a}, \quad \delta b = -\frac{\xi}{\tilde{N}} \dot{b}, \quad \delta \tilde{N} = -\dot{\xi} \]  \hspace{1cm} (23)

The Lagrangian density (15) varies under the gauge symmetry transformations as (23),

\[ \delta L = \delta \left[ \left( \frac{-\dot{a}^2 - \dot{b}^2}{\tilde{N}} \right) + \tilde{N}(a^2 - b^2) \right], \]

\[ \Rightarrow \delta L = \frac{d}{dt} \left[ \xi \left( \frac{\dot{a}^2 - \dot{b}^2}{\tilde{N}^2} - \tilde{N}(a^2 - b^2) \right) \right]. \]  \hspace{1cm} (24)
A total derivative. This ensures that the Hilbert- Einstein action remains quasi-invariant under the gauge symmetry transformations. We can make the Lagrangian density further more simplified with use of equations of motion, which are as follows,

\[
\frac{d}{dt} \left( \frac{\dot{a}}{N} \right) = -a \ddot{N},
\]

(25)

\[
\frac{d}{dt} \left( \frac{\dot{b}}{N} \right) = -b \ddot{N},
\]

(26)

and

\[
\ddot{N}^2 = -\frac{(\dot{a}^2 - \dot{b}^2)}{a^2 - b^2}.
\]

(27)

If we further substitute,

\[
a = \cosh \varphi, \quad b = \sinh \varphi
\]

(28)

The Lagrangian density (13) reduces to the form,

\[
L = \left( \frac{\dot{\varphi}^2}{N} + \ddot{N} \right)
\]

(29)

And with help of set of transformations (23), the transformations for \( \varphi \) and \( \ddot{N} \) turn out to be

\[
\delta \varphi = -\frac{\xi}{N} \dot{\varphi}, \quad \delta \ddot{N} = -\dot{\xi}
\]

(30)

under this symmetry transformations the Lagrangian density(29) varies as,

\[
\delta L = d \left[ - \left( \frac{\dot{\varphi}^2}{N^2} \xi + \xi \right) \right]
\]

(31)

A total time derivative. This again ensures that the action remains semi-invariant under the symmetry transformations given by equation (30). According to Noether’s theorem, there must be a conserved current for this set of symmetry transformations. The conserved current turns out to be

\[
\begin{align*}
    j &= (\delta \varphi) \frac{\partial L}{\partial \dot{\varphi}} + (\delta \ddot{N}) \frac{\partial L}{\partial \ddot{N}} + \left( \frac{\dot{\varphi}^2}{N^2} \xi + \xi \right), \\
    \Rightarrow j &= \xi \left( 1 - \frac{\dot{\varphi}^2}{N^2} \right) - \dot{\xi} \Pi \ddot{N}
\end{align*}
\]

(32)

For one dimensional case the conserved charge \( Q \) is current itself. Therefore, the conserved charge \( Q \) is none the less but current \( j \) itself, i.e.

\[
Q = \xi \left( 1 - \frac{\dot{\varphi}^2}{N^2} \right) - \dot{\xi} \Pi \ddot{N}
\]

(33)

Under the application of equations of motion,
The charge $Q$ can be easily shown to be conserved. This conserved charge itself is able to produce all the symmetry transformations, given by equation (23), with use of formula,

$$
\delta \Phi = \{ \Phi, Q \}.
$$

where $\Phi = \varphi, \tilde{N}$.

4 Nilpotent (Anti-)BRST Symmetries in Lagrangian Formulation

The Lagrangian density (29) is gauge invariant. It can be shown that with a gauge fixing term $-\frac{1}{2} \frac{\dot{\tilde{N}}^2}{\tilde{N}^2}$, one can form a modified Lagrangian density, free from the constraints and therefore is suitable for the canonical quantization. But this comes at the cost of gauge invariance of the theory. The gauge invariance is gone. To restore the gauge invariance of the Lagrangian density, we add an extra Faddeev-Popov term $-i \frac{\dot{\tilde{N}}}{\tilde{N}} \dot{\bar{C}} \dot{C}$ to the modified Lagrangian density [11].

With these modifications the Lagrangian density (29) becomes

$$
\mathcal{L} = \frac{\dot{\varphi}^2}{\tilde{N}} + \tilde{N} - \frac{1}{2} \frac{\tilde{N}^2}{\tilde{N}^2} - i \frac{\dot{\tilde{N}}}{\tilde{N}} \dot{\bar{C}} \dot{C}.
$$

The BRST symmetry transformations for this Lagrangian density, are as follows,

$$
s_b \varphi = -C \frac{\dot{\varphi}}{\tilde{N}}, \quad s_b \tilde{N} = -\dot{\bar{C}}, \quad s_b C = 0, \quad s_b \bar{C} = -i \frac{\dot{\tilde{N}}}{\tilde{N}}.
$$

The Lagrangian density (37), under the set of BRST transformations (38) varies as

$$
s_b \mathcal{L} = \frac{d}{dt} \left( \frac{\dot{\tilde{N}} \dot{C}}{\tilde{N}^2} - \frac{\dot{\varphi}^2 C}{\tilde{N}^2} - C \right)
$$

where we have used the equation of motion for $\varphi$ as,

$$
\frac{d}{dt} \left( 2 \frac{\dot{\varphi}}{\tilde{N}} \right) = 0
$$

In a similar fashion it can be shown that the Lagrangian density (37) varies under the set of anti-BRST transformations

$$
s_{ab} = -\bar{C} \frac{\dot{\varphi}}{\tilde{N}}, \quad s_{ab} \tilde{N} = -\dot{\bar{C}}, \quad s_{ab} C = 0, \quad s_{ab} \bar{C} = -i \frac{\dot{\tilde{N}}}{\tilde{N}}.
$$
\[ s_{ab} \mathcal{L} = \frac{d}{dt} \left( \frac{\dot{N} \dot{C}}{N^2} - \frac{\dot{\varphi}^2 C}{N^2} - \dot{C} \right) \]  \hspace{1cm} (42)

It is easy to check that the on-shell nilpotency of order -2 and the on-shell anti-commutativity are the essential features of these set of (anti-)BRST transformations i.e.

\[ s_b^2 = 0 = s_{ab}^2, \quad s_b s_{ab} + s_{ab} s_b = 0 \]  \hspace{1cm} (43)

Corresponding to these symmetry transformations, there exist conserved Noether charges which are nothing but the conserved currents themselves. In order to calculate the conserved charges of the BRST and anti-BRST symmetries as well as, we need to calculate the canonical momenta first, which are

\[ \Pi_\varphi = 2 \frac{\dot{\varphi}}{N}, \quad \Pi_{\dot{N}} = -\frac{\dot{N}}{N^2}, \quad \Pi_c = i \frac{\dot{C}}{N}, \quad \Pi_{\bar{C}} = -i \frac{\dot{\bar{C}}}{N} \]  \hspace{1cm} (44)

The various equations of motion for the Lagrangian density (38) are following:

\[ \frac{d}{dt} \left( \frac{2\varphi}{N} \right) = 0, \]

\[ \ddot{N} + \dot{N}^2 + i \dot{C} \dot{\bar{C}} = \dot{\varphi}^2 + \frac{\dot{N}^2}{N}, \]

\[ \frac{d}{dt} \left( i \frac{\dot{C}}{N} \right) = 0, \]

\[ \frac{d}{dt} \left( -i \frac{\dot{\bar{C}}}{N} \right) = 0. \]  \hspace{1cm} (45)

The Noether conserved current and hence the conserved charges corresponding to BRST and anti-BRST symmetry transformations can be calculated and are found to be

\[ Q_b = \frac{\dot{N}}{N^2} \dot{C} - \frac{\dot{\varphi}^2 C}{N^2} + C, \]

\[ Q_{ab} = \frac{\dot{N}}{N^2} \dot{C} - \frac{\dot{\varphi}^2 \bar{C}}{N^2} + \bar{C}. \]  \hspace{1cm} (46)

Further, using the various equations of motion (45), this can be shown that the both the charges are invariant in time. Now, the Lagrangian density (38) under canonical quantization scheme produces the following (anti-)commutators,

\[ [\varphi, \Pi_\varphi] = i, \quad [\tilde{N}, \Pi_{\tilde{N}}] = i, \quad \{C, \Pi_C\} = i, \quad \{\bar{C}, \Pi_{\bar{C}}\} = i. \]  \hspace{1cm} (47)
We have taken the natural units $c = 1$ and $\hbar = 1$ everywhere. The (anti-)commutators (48) can be obtained in terms of our usual field variables as,

\[ [\varphi, \dot{\varphi}] = \frac{i}{2} \tilde{N}, \quad [\tilde{N}, \dot{\tilde{N}}] = -i \tilde{N}^2, \quad \{C, \dot{C}\} = \tilde{N}, \quad \{\bar{C}, \dot{\bar{C}}\} = -\tilde{N} \quad (48) \]

On the basis of above set of (anti-)commutators (48), it is easy to establish the connections between various (anti-)commutators as,

\[
\{C, \dot{\bar{C}}\} = -2i [\varphi, \dot{\varphi}], \quad \{\bar{C}, \dot{C}\} = 2i [\varphi, \dot{\varphi}], \quad \{C, \dot{C}\} = -\{\bar{C}, \dot{\bar{C}}\},
\]

\[ [\tilde{N}, \dot{\tilde{N}}] = -4i [\varphi, \dot{\varphi}][\varphi, \dot{\varphi}], \quad [\tilde{N}, \dot{\tilde{N}}] = i\{\bar{C}, \dot{\bar{C}}\}\{C, \dot{C}\}. \quad (49) \]

With help of various momenta (44), the expression for the (anti-)BRST charges can be given as,

\[
Q_b = -i \Pi_{\tilde{N}} \tilde{N} \Pi_C - \frac{1}{4} \Pi^2_{\varphi} C + C,
\]

\[
Q_{ab} = i \Pi_{\tilde{N}} \tilde{N} \Pi_{\bar{C}} - \frac{1}{4} \Pi^2_{\varphi} \bar{C} + \bar{C}. \quad (50) \]

On the basis of (anti-)commutation relations (48) the absolute anti-commutativity and nilpotency of order-2 of the (anti-)BRST charges can be established. i.e.

\[
Q^2_b = 0 = Q^2_{ab},
\]

\[
\{Q_b, Q_{ab}\} = Q_bQ_{ab} + Q_{ab}Q_b = 0. \quad (51) \]

With all these treatise one can easily check that the charges $Q_b$ and $Q_{ab}$ are able to generate all the BRST and ant-BRST transformations respectively, using the formula

\[
s_b \chi = \pm i \{\chi, Q_b\}, \quad s_{ab} \chi = \pm i \{\chi, Q_{ab}\}. \quad (52) \]

where $\chi = \varphi, \tilde{N}, C, \bar{C}$ and $[A, B]_+ = \{A, B\}$ and $[A, B]_- = [A, B]$.

Now, substituting back the values of $\varphi$ and $\tilde{N}$ in the commutators given by equation (49), we obtain the commutation relation between the metric components of the ADM metric as follows,

\[
\left[ \ln B, \frac{d}{dt}(\ln B) \right] = \frac{8iN}{\sqrt{\dot{H}(\dot{H}B + H\dot{B})}}, \quad [N, \dot{\tilde{N}}] = -i N^2. \quad (53) \]
5 Conclusions and Discussions

The central theme of our endeavour has been accomplished. We have been able to provide a complete BRST and usual gauge theoretic quantization of the interior of the simplest case of Black-Hole, the Schwarzschild’s Black-hole. This is really a novel achievement for us to see the whole spacetime interior of the Schwarzschild Black-Hole, behaving as a relativistic free particle. We conclude that the metric inside the event-horizon of a Black-hole behaves analogous to the relativistic free particle, moving in the plain Minkowskian spacetime. This is a duality between gauge sector and gravitation sector, in which we are able to replace the whole curved geometry without any dynamical force to a well known and easily solvable problem of relativistic free particle in plain Minkowskian space. Thus, we have been able to establish a one to one correspondence between the gravitational sector(describing the interior of black-hole) and the gauge sector(describing the relativistic free particle theory, a gauge theory). Further, we have also established the relations among the various components of the metric of the interior of spacetime of the Schwarzschild’s Black-hole. And this way, we are able to understand the dynamics of the interior of Schwarzschild’s Black-hole.

My next task is to generalize this method for spinning Black-hole(Kerr Black-hole). I want to investigate the effect of the spin angular momentum of spinning Black-hole on its underlying spacetime. My attempt will be focused on the relativistic spinning free particle and to explore if there exists any connection between two spins; namely, spin of Black-hole and the spin of relativistic spinning free particle.

Acknowledgements:

I am very grateful to Prof. Loriano Bonora from Trieste, Italy (who visited our group at the physics department for a couple of week in December, 2013) for his valuable suggestions and discussions in the problem. I am also very thankful to my supervisor Prof. R. P. Malik for his generous support and encouragement for working on this problem. I would like to thank UGC, Government of India, New Delhi, for financial support through RFSMS scheme.

References

[1] Shahram Jalalzadeh, Babak Vakili, Int. J. Theor. Phys. (2012) 51:263275
[2] Evangelos Melas, Journal of Physics:Conference Series 442 (2013) 012037
[3] R. Arnowitt, S. Deser, C. Misner, Physical Review 116 (5): 13221330D
[4] R. Arnowitt, S. Deser, C. Misner, Physical Review 117 (6): 15951602X
[5] P. A. M. Dirac, Proceedings of the Royal Society of London A 246 (1246): 326332
[6] L. Brink, P. Di Vecchia, P. Howe, Nucl. Phys. B 118, 76 (1977)
[7] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, P. Howe, *Phys. Lett. B* 64, 435 (1976)

[8] D. Namshansky, C. Preitschopf, M. Weinstein, *Ann. Phys. (N. Y.)* 183, 226 (1988)

[9] R. P. Malik, *Eur.Phys.J.C*45:513-524 (2006)

[10] S. Krishna, D. Shukla, R. P. Malik, *Int. J. Mod. Phys. A* 28: 1350108 (p01-p14) (2013)

[11] L. Bonora, M. Tonin, *Phys. Lett. B* 98, 48 (1991)

[12] Ray D’Inverno, *Introducing Einstein’s Relativity*, Oxford University Press, 1992, 1993, 1995, 1996, 1998

[13] James B. Hartle, *Gravity: An Introduction to Einstein’s General Relativity*, Addison Wesley Publication, Pearson Education India, 01-Sep-2003

[14] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler *Gravitation*, W. H. Freeman and Company, 1973

[15] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University Press, New York, 1964

[16] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, New Jersey, Princeton, 1992

[17] M. Chaichian, M. Oksanen, Anca Tureanu *Eur.Phys.J.C*71:1657, 2011