On the role of shear in cosmological averaging II: large voids, non-empty voids and a network of different voids

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Abstract. We study the effect of shear on the cosmological backreaction in the context of matching voids and walls together using the exact inhomogeneous Lemaître-Tolman-Bondi solution. Generalizing JCAP 1010 (2010) 021, we allow the size of the voids to be arbitrary and the densities of the voids and walls to vary in the range $0 \leq \Omega_v \leq \Omega_w \leq 1$. We derive the exact analytic result for the backreaction and consider its series expansion in powers of the ratio of the void size to the horizon size, $r_0/t_0$. In addition, we deduce a very simple fitting formula for the backreaction with error less than 1% for voids up to sizes $r_0 \gtrsim t_0$. We also construct an exact solution for a network of voids with different sizes and densities, leading to a non-zero relative variance of the expansion rate between the voids. While the leading order term of the backreaction for a single void-wall pair is of order $(r_0/t_0)^2$, the relative variance between the different voids in the network is found to be of order $(r_0/t_0)^4$ and thus very small for voids of the observed size. Furthermore, we show that even for very large voids, the backreaction is suppressed by an order of magnitude relative to the estimate obtained by treating the walls and voids as disjoint Friedmann solutions. Whether the suppression of the backreaction due to the shear is just a consequence of the restrictions of the used exact models, or a generic feature, has to be addressed with more sophisticated solutions.

Keywords: gravity, cosmic web, dark energy theory

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1 Introduction

The non-commutativity of time evolution and spatial averaging in general relativity implies that the average expansion of a universe with structures does not evolve in time like the uniform Hubble expansion in a homogeneous Friedmann model \([1-3]\). This effect can be quantified by a backreaction term in generalized Friedmann equations and understood as a consequence of the nonlinearity of gravity \([4]\). Although the cosmological backreaction is conceptually well-understood, the complexity of the structure formation at the nonlinear level means its magnitude in the real universe is difficult to evaluate and is hence widely debated \([5]\): some studies have found that the backreaction is small \([6-11]\), while other studies suggest it can have a significant effect on the cosmological dynamics and observations \([12-16]\), even to the extent of accounting for the observed cosmic acceleration entirely without additional effects \([12-14]\). As many of the current standard values for the cosmological parameters — not just the cosmological constant — rely on the hypothesis of negligible backreaction, its evaluation has become a central issue in cosmology today \([17, 18]\).

A widely used scheme to estimate the backreaction is to average the scalar parts of the Einstein equation on spacelike hypersurfaces, defined by constant proper time of the freely-falling dust particles. In this so-called Buchert approach, the backreaction term is given by the (positive) variance of the expansion rate minus the (positive) average shear \([4]\). In studies that have estimated the backreaction to be significant \([12-15]\), the shear on the boundaries between regions characterized by different expansion rates has been neglected or, equivalently, the boundary regions or matching conditions have been ignored. On the other hand, in perturbative studies that do not neglect the shear the backreaction has been found to be small \([19-21]\); however, see \([22]\) for a discussion on the possible shortcomings of the perturbation theory in modelling the structure formation. Considering that on physical grounds, shear is expected to occur on the boundaries between regions of different expansion
rates, evaluating the effect of the shear on the backreaction is evidently one of the key issues in the problem.

This work is a continuation to our previous work [23], hereafter paper I, in which the effect of shear on the cosmological backreaction was studied in the context of matching voids (with \( \Omega_v = 0 \) and \( Ht = 1 \)) and walls (with \( \Omega_w = 1 \) and \( Ht = 2/3 \)) together using the exact inhomogeneous Lemaître-Tolman-Bondi or LTB solution. Whereas neglecting the exact matching can lead to significant backreaction, the main conclusion of paper I was that the shear arising from the exact matching suppresses the backreaction by the squared ratio of the void size to the horizon size, \( (r_0/t_0)^2 \), thus making it small for voids of the observed size \( r_0/t_0 \lesssim 10^{-2} \) [24]. Here we generalize the study of paper I by considering:

1. The backreaction as a function of the void and wall density parameters in the range \( 0 \leq \Omega_v \leq \Omega_w \leq 1 \), thus relaxing the priors \( \Omega_v = 0 \) and \( \Omega_w = 1 \) of paper I.
2. The backreaction for voids of arbitrary size \( r_0 \), thus relaxing the condition \( r_0 \ll t_0 \) assumed in paper I.
3. A network of voids with different densities \( \Omega_v \) and radii \( r_0 \), thus giving rise to relative variance of the expansion rate between the different void-wall pairs.

Note that since the backreaction of the quasi-spherical Szekeres model reduces to the LTB model, the results of this work apply to the quasi-spherical Szekeres solution as well [25, 26].

Although we focus here on the effect of the structure formation on the average dynamics of the universe, the ultimate goal of the research is to evaluate the total effect of the structure formation on the cosmological observations. In addition to the dynamical backreaction considered in this work, known effects of the structure formation include modifications on the propagation of light not directly determined by the volume-averaged expansion [8, 16, 27–41] and effects due to our non-average location in the universe [15, 42–44]. These effects may be important but we do not try to evaluate them in this work.

The paper is organized as follows. The necessary background of the Buchert averaging method and the LTB solution are introduced in sections 2 and 3, respectively. Section 4 provides the definition of the void-wall LTB model, which we apply to study the effect of shear on the backreaction in section 5: the general analytic expression of the backreaction for the void-wall LTB model is derived in section 5.1 and its expansion as a power series and a simple but accurate fitting formula are considered in section 5.2. The result for the backreaction is then applied to a network of different voids in section 5.3 and to large voids in section 5.4. In section 5.5, we discuss uncompensated voids. Finally, the results are summarized in section 6 and the conclusions are given in section 7.

2 Scalar averaging

The spatial volume-average of a scalar \( S(x,t) \) is defined as

\[
\langle S \rangle_D(t) \equiv \frac{\int_D S(x,t) \epsilon(x,t) \, d^3x}{\int_D \epsilon(x,t) \, d^3x},
\]

where the volume element is determined by the determinant of the spatial part of the metric as \( \epsilon = \sqrt{\det g_{ij}} \, d^3x \) and \( D \) is the averaging domain or a region of the spatial sections. Unless otherwise noted, we take the averaging domain to be an origin-centered ball of coordinate radius \( R \), i.e. \( D = B(R) \), and simply write \( \langle S \rangle \equiv \langle S \rangle_{B(R)} \).
In comoving and synchronous coordinates, the volume expansion scalar \( \theta \equiv \nabla_\mu u^\mu \) is given by the temporal change of the volume,

\[
\theta(x, t) = \partial_t \ln(\epsilon(x, t)),
\]

and is related to the generalized scale factor \( a(t) \) as

\[
\langle \theta \rangle(t) = 3 \frac{\dot{a}(t)}{a(t)},
\]

where

\[
a(t) \equiv \left( \frac{\int \epsilon(x, t) \, dx}{\int \epsilon(x, t_0) \, dx} \right)^{1/3}.
\]

By taking the time derivative of the average (2.1) and writing the expression in terms of the expansion scalar (2.2), we obtain the commutator of time evolution and averaging

\[
\partial_t \langle S \rangle = \langle \partial_t S \rangle + \langle S \theta \rangle - \langle S \rangle \langle \theta \rangle,
\]

which, when applied to the expansion scalar itself, yields

\[
\partial_t \langle \theta \rangle = \langle \partial_t \theta \rangle + \langle (\theta - \langle \theta \rangle)^2 \rangle \geq 0.
\]

The result (2.6) shows that the average expansion always decreases less (or increases more) in time than the average of the time derivative of the local expansion.

By applying the averaging (2.1) to the scalar parts of the Einstein equation in an irrotational dust universe, and using the results (2.3) and (2.5), we obtain the generalized Friedmann equations [4]:

\[
3 \frac{\ddot{a}(t)}{a(t)} = -4\pi G \langle \rho \rangle(t) + Q(t)
\]

\[
3 \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = 8\pi G \langle \rho \rangle(t) - \frac{1}{2} \langle \mathcal{R} \rangle(t) - \frac{1}{2} Q(t)
\]

\[
\partial_t \langle \rho \rangle(t) = -3 \frac{\dot{a}(t)}{a(t)} \langle \rho \rangle(t),
\]

where \( \mathcal{R} \) is the Ricci curvature scalar of the spatial sections and the backreaction \( Q \) is given by the variance of the expansion rate \( \theta \) minus the average shear,

\[
Q(t) \equiv \frac{2}{3} \left( \langle \theta^2 \rangle - \langle \theta \rangle^2 \right) - \langle \sigma^2 \rangle,
\]

where the shear scalar is given by the contraction of the shear tensor as \( \sigma^2 \equiv \sigma^{\mu\nu} \sigma_{\mu\nu} \).

The backreaction (2.10) quantifies the difference of the time evolution of the averages relative to the uniform quantities in a homogeneous Friedmann dust universe.

By partitioning the averaging domain \( \mathcal{D} \) into a set of \( N \) mutually disjoint subregions \( \mathcal{D}_i \) that satisfy \( \bigcup_{i=1}^N \mathcal{D}_i = \mathcal{D} \), the backreaction (2.10) can be written as

\[
Q_D = \sum_{i=1}^N f_i Q_{\mathcal{D}_i} + \frac{1}{3} \sum_{i=1}^N \sum_{j=1}^N f_i f_j \left( \langle \theta \rangle_{\mathcal{D}_i} - \langle \theta \rangle_{\mathcal{D}_j} \right)^2,
\]

where \( f_i \) is the volume fraction of the subregion \( \mathcal{D}_i \), \( f_i \equiv \text{Vol}(\mathcal{D}_i) / \text{Vol}(\mathcal{D}) \). The expression (2.11) makes it explicit that the backreaction is an intensive, rather than extensive, quantity and becomes useful in section 5.3, where we consider a network of voids constructed by matching together different LTB solutions.
3 LTB solution

The exact spherically symmetric dust solution of general relativity was discovered by Lemaître in 1933 [45] and is now commonly referred to as the LTB metric:

\[
 ds^2 = -dt^2 + \frac{[A'(r,t)]^2}{1-k(r)}dr^2 + A^2(r,t)(d\theta^2 + \sin^2 \theta d\varphi^2),
\]

where the prime stands for the radial derivative \(A'(r,t) \equiv \partial_r A(r,t)\), \(k(r)\) is related to the spatial Ricci curvature scalar as

\[
 R = \frac{2}{A^2(r,t)} \frac{\partial_r (A(r,t) k(r))}{A'(r,t)},
\]

and \(A(r,t)\) is determined by the Friedmann-like evolution equation, which, following the notation and parametrization introduced in ref. [46], reads as

\[
 H(r,t) = H_0(r) \left[ \Omega_0(r) \left( \frac{A_0(r)}{A(r,t)} \right)^3 + (1 - \Omega_0(r)) \left( \frac{A_0(r)}{A(r,t)} \right)^2 \right]^{1/2},
\]

where \(H(r,t) \equiv \partial_t A(r,t)/A(r,t) \equiv \dot{A}(r,t)/A(r,t)\), \(H_0(r) \equiv H(r,t_0)\) and \(\Omega_0(r)\) are boundary condition functions specified on a spatial hypersurface \(t = t_0\) that determine the radial inhomogeneity profile, while the freedom to choose the function \(A_0(r) \equiv A(r,t_0)\) corresponds to the scaling of the \(r\)-coordinate. The coordinate freedom is used here to set

\[
 A_0(r) = r.
\]

The curvature function \(k(r)\) in the metric (3.1) is related to these by

\[
 k(r) = H_0^2(r) A_0^3(r)(\Omega_0(r) - 1)
\]

and the boundary condition function \(\Omega_0(r)\) is related to the physical matter density \(\rho(r,t)\) on the \(t = t_0\) hypersurface as

\[
 \Omega_0(r) = \frac{8\pi G}{3H_0^2(r)} \frac{\int_{R(r)} \rho_0(r) d^3x}{\int_{R(r)} d^3x},
\]

where \(\rho_0(r) \equiv \rho(r,t_0)\) and \(d^3x \equiv r^2 \sin \theta \ dr \ d\theta \ d\varphi\).

For general functions \(\Omega_0(r)\) and \(H_0(r)\) that are independent of each other, the LTB solution contains both decaying and growing inhomogeneities [47]. However, given the observed near isotropy of the CMB, the models with close to homogeneous early universe form perhaps the most relevant subcase of the LTB solutions. This corresponds to solutions where growing modes dominate so we only consider inhomogeneity profiles that obey the constraint

\[
 H_0(r) = \frac{1}{t_0} \left( 1 - \frac{\sqrt{\Omega_0(r)}}{3} \right),
\]

which is an accurate approximation for the exact condition; see eq. (2.8) in ref. [23].

To calculate the backreaction (2.10) for the LTB solution in section 5, we need the following quantities: the shear scalar

\[
 \sigma^2(r,t) = \frac{2}{3} \left( \frac{\dot{A}(r,t)}{A(r,t)} - \frac{A'(r,t)}{A(r,t)} \right)^2,
\]
the volume expansion scalar (2.2)

\[
\theta(r, t) = \frac{2}{A(r, t)} \frac{\dot{A}(r, t)}{A'(r, t)} + \frac{\partial_r (A^2(r, t) \dot{A}(r, t))}{A^2(r, t) A'(r, t)},
\]
and their expressions on the \(t = t_0\) hypersurface:

\[
\sigma^2(r, t_0) = \frac{2}{3} \left( r H_0'(r) \right)^2, \quad (3.10)
\]

\[
\theta(r, t_0) = 3 H_0(r) + r H_0'(r) = \frac{1}{r^2} \partial_r \left( r^3 H_0(r) \right). \quad (3.11)
\]

For volume averaging, we also need the LTB volume element

\[
\epsilon(x, t) = \sqrt{\det g_{ij}} \, d\mathbf{r} \, d\theta \, d\varphi = \frac{A'(r, t) A^2(r, t) \sin \theta}{\sqrt{1 - k(r)}} \, dr \, d\theta \, d\varphi. \quad (3.12)
\]

4 The void-wall LTB model

We consider an LTB solution consisting of two different regions: a void with density parameter \(\Omega_v\) and a wall with density parameter \(\Omega_w\), such that the boundary condition function (3.6) has the form

\[
\Omega_0(r) = \left( \sqrt{\Omega_v} + (\sqrt{\Omega_w} - \sqrt{\Omega_v}) \Theta(r - r_0) \right)^2, \quad (4.1)
\]

where \(\Theta\) stands for the Heaviside step function and \(r_0\) determines the size of the void. Because of the constraint (3.7), the density profile (4.1) implies the expansion profile

\[
H_0(r) = t_0^{-1} \left( 3 - \sqrt{\Omega_v} + (\sqrt{\Omega_v} - \sqrt{\Omega_w}) \Theta(r - r_0) \right). \quad (4.2)
\]

To calculate the backreaction (2.10) for the void-wall model defined by eqs. (4.1) and (4.2) in section 5, we also need the first derivative of the expansion profile:

\[
H_0'(r) = -\frac{1}{3} t_0^{-1} (\sqrt{\Omega_w} - \sqrt{\Omega_v}) \delta(r - r_0), \quad (4.3)
\]

where \(\delta\) stands for the Dirac delta function.

Note that in the case \(\Omega_v = 0\) and \(\Omega_w = 1\) the void-wall profile reduces to the step-function limit \(n \to \infty\) of the LTB model considered in paper I. The reason to consider only the step-function profile is that, as already verified in paper I, the dependence of the backreaction on the sharpness of the transition between the void and the wall is weak. The step-function transition has also the advantage of allowing us to analytically perform calculations that would otherwise call for approximations or numerical methods.

5 Backreaction in the void-wall LTB model

In this section, we calculate the backreaction (2.10) for the void-wall LTB model defined by the density profile (4.1) and the expansion profile (4.2). Apart from the profile, the systematic study of the role of shear makes our approach different from the previous studies on the backreaction in the LTB model which have focused on finding profiles that exhibit
acceleration of the average expansion [48–52], on scale-dependence of the averages [53] or on general properties of the backreaction [54, 55].

When considering numerical values for the backreaction, we give the results in units of the backreaction obtained by taking the variance of the expansion rate over two disjoint, shear-free Friedmann models: the empty Milne solution for the void and the spatially flat Einstein-de-Sitter solution for the wall, yielding the result found in paper I:

\[ Q_{\text{FRW}} = \frac{1}{6} t_0^{-2}, \]  

(5.1)

where the regions are chosen to have equal volumes. These units are convenient as they directly address the role of shear by telling us how much the backreaction is suppressed relative to the Friedmann estimate (5.1) where the shear is identically zero.

### 5.1 General expression

With the help of eqs. (3.8) and (3.9), the backreaction (2.10) of the LTB solution simplifies to

\[ Q(t) = 2 \left( \frac{\dot{A}^2(r, t)}{A^2(r, t)} \right) + 4 \left( \frac{\dot{A}(r, t)A' (r, t)}{A( r, t)A' (r, t)} \right) - \frac{2}{3} \left( \frac{2}{A( r, t)} + \frac{A'(r, t)}{A'(r, t)} \right)^2, \]  

(5.2)

which, when evaluated on the \( t = t_0 \) hypersurface using eqs. (3.9) and (3.11), reduces to

\[ Q(t_0) = 6 \left( \langle H_0^2 \rangle - \langle H_0 \rangle^2 \right) + 4 \left( \langle rH'_0 \rangle H_0 \rangle - \langle rH_0' \rangle \langle H_0 \rangle \right) - \frac{2}{3} \langle rH_0' \rangle^2, \]  

(5.3)

where \( H_0 \equiv H_0(r) \).

For the LTB solution, we write the volume average (2.1) as:

\[ \langle S \rangle = \frac{\int_{B(R)} S(r, t) \epsilon(r, t)}{\int_{B(R)} \epsilon(r, t)} = \frac{\int_0^R r^2 dr}{\int_0^R \frac{r^2 dr}{\sqrt{1 - k(r)}}} = \frac{[S]}{v t_0^3}, \]  

(5.4)

where we have defined the reduced dimensionless volume \( v \), which, after the change of variables \( r \equiv t_0 y \), reads as:

\[ v = \int_0^{t_0 x} \frac{y^2 dy}{\sqrt{1 + y^2 \alpha(\Theta)}}, \]  

(5.5)

where \( \varepsilon \equiv r_0/t_0, x \equiv R/r_0 > 1, \Theta \equiv \Theta(r - r_0) \) and

\[ \alpha(\Theta) \equiv \left( 1 - \frac{1}{3} \left[ \sqrt{\Omega_v} + (\sqrt{\Omega_w} - \sqrt{\Omega_v})\Theta \right] \right)^2 \left( 1 - \left[ \sqrt{\Omega_v} + (\sqrt{\Omega_w} - \sqrt{\Omega_v})\Theta \right] \right)^2. \]  

(5.6)

To calculate the backreaction (5.3), we thus need to evaluate \( v, [H_0], [H_0^2], [rH_0'], \) and \([rH_0'H_0] \).

An integral appearing often in these quantities is

\[ I(\mu, \nu, \alpha) \equiv \int_0^{\nu} \frac{y^2 dy}{\sqrt{1 + y^2 \alpha}}, \]  

(5.7)
which can be calculated analytically to yield
\[ I(\mu, \nu, \alpha) = \frac{1}{2\alpha^{3/2}} \left[ \sqrt{\alpha} (\nu \sqrt{1 + \alpha^2} - \mu \sqrt{1 + \alpha^2}) + \ln \left(\frac{\sqrt{\alpha^2 + 1 + \alpha^2 \nu^2}}{\sqrt{\alpha^2 + 1 + \alpha^2 \mu^2}}\right) \right]. \] (5.8)

For example the volume (5.5) can be written in terms of the function (5.7) as
\[ v = \int_0^\varepsilon y^2 dy + \int_{\varepsilon}^{\varepsilon x} y^2 dy = I(0, \varepsilon, \alpha(0)) + I(\varepsilon, \varepsilon x, \alpha(1)), \] (5.9)
where \( \alpha \) is defined in eq. (5.6). Using the definitions (5.7) and (5.9), \([H_0]\) and \([H_0^2]\) can be integrated to yield
\[ [H_0]_0 = t_0^2 \left\{ \left(1 - \frac{\sqrt{\Omega_w}}{3}\right) I_0 + \left(1 - \frac{\sqrt{\Omega_v}}{3}\right) I_1 \right\}, \] (5.10)
\[ [H_0^2]_0 = t_0^3 \left\{ \left(1 - \frac{\sqrt{\Omega_w}}{3}\right)^2 I_0 + \left(1 - \frac{\sqrt{\Omega_v}}{3}\right)^2 I_1 \right\}, \] (5.11)
which combine to give
\[ \langle H_0^2 \rangle - \langle H_0 \rangle^2 = \frac{1}{9} \left(\frac{\sqrt{\Omega_w} - \sqrt{\Omega_v}}{v^2}\right) I_0 I_1. \] (5.12)

Similarly, using eq. (4.3), we have for the derivative terms
\[ [rH_0']_0 = -\frac{t_0^2}{3} \left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right) \int_0^{\varepsilon x} y^3 \delta(t_0 y - r_0) dy \] (5.13)
\[ [rH_0 H_0']_0 = -\frac{t_0}{3} \varepsilon^3 \left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right) \left\{ \left(1 - \frac{\sqrt{\Omega_w}}{3}\right) \int_0^{\varepsilon x} y^3 \delta(t_0 y - r_0) dy \right\} + \right. \]
\[ -\frac{1}{3} \left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right) \int_0^{\varepsilon x} y^3 \Theta(t_0 y - r_0) \delta(t_0 y - r_0) dy \right\}\right\}. \] (5.14)

The integrals in eqs. (5.13) and (5.14) can be recognized as Stieltjes integrals: using
\[ \frac{d\Theta(t_0 y - r_0)}{dy} = t_0 \delta(t_0 y - r_0), \] (5.15)
they can be written as the following ordinary integrals
\[ \mathcal{A} = \int_0^1 \frac{d\vartheta}{\sqrt{1 + \varepsilon^2 \alpha(\vartheta)}} \] (5.16)
\[ \mathcal{B} = \int_0^1 \frac{\vartheta d\vartheta}{\sqrt{1 + \varepsilon^2 \alpha(\vartheta)}}, \] (5.17)
so that eqs. (5.13) and (5.14) become
\[ [rH_0']_0 = -\frac{t_0^2}{3} \left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right) \mathcal{A} \] (5.18)
\[ [rH_0 H_0']_0 = -\frac{t_0}{3} \varepsilon^3 \left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right) \left\{ \left(1 - \frac{\sqrt{\Omega_w}}{3}\right) \mathcal{A} - \frac{1}{3} \left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right) \mathcal{B} \right\}. \] (5.19)
By recalling the definition (5.4) and substituting the expressions (5.10), (5.12), (5.18) and (5.19) in eq. (5.3), we finally obtain:

$$Q = t_0^{-2} \frac{\sqrt{\Omega_w} - \sqrt{\Omega_v}}{t^2} \left\{ \frac{2}{3} T_0 T_1 + \frac{4}{9} [ (T_0 + T_1) B - T_1 A ] \epsilon^3 - \frac{2}{27} A^2 \epsilon^6 \right\},$$

(5.20)

where the integrals (5.16) and (5.17) can be calculated numerically or as an expansion in powers of $\epsilon$ or $\epsilon^{-1}$.

### 5.2 Power expansions and a fitting formula

As the observations suggest that voids in the cosmological matter distribution are much smaller than the horizon [24, 56–59], typically $r_0 \lesssim 10^{-2} t_0$, one of the most useful methods to investigate the backreaction is the series expansion in powers of $\epsilon = r_0/t_0$. By expanding the exact result (5.20) in powers of $\epsilon$ to second order, we obtain:

$$Q = \frac{1}{405} t_0^{-2} \frac{r_0^3}{R} \left( \Omega_w^{1/2} - \Omega_v^{1/2} \right)^2 \left( 54 - \Omega_w^2 - 6 \Omega_v^2 - 15 \Omega_w \Omega_v - 6 \Omega_w^{3/2} \Omega_v^{1/2} + 
- 20 \Omega_w^{3/2} \Omega_v^{1/2} + 45 \Omega_w \Omega_v^{1/2} + 90 \Omega_v \Omega_w^{1/2} - 80 \Omega_v^{1/2} \Omega_w^{1/2} + 36 \Omega_v^{3/2} + 
+ 9 \Omega_w^{3/2} - 20 \Omega_w - 48 \Omega_v - 30 \Omega_w^{1/2} - 36 \Omega_v^{1/2} \left( \frac{r_0}{t_0} \right)^2 \right) + \mathcal{O} \left( \frac{r_0}{t_0} \right)^4,$$

(5.21)

while for the values $\Omega_v = 0$ and $\Omega_w = 1$ the fourth order expansion reads as

$$Q = t_0^{-2} \frac{r_0^3}{R} \left\{ \frac{4}{135} \left( \frac{r_0}{t_0} \right)^2 - \left[ \frac{2677}{102060} - \frac{3359}{437400} \left( \frac{r_0}{R} \right)^3 \right] \left( \frac{r_0}{t_0} \right)^4 \right\} + \mathcal{O} \left( \frac{r_0}{t_0} \right)^6,$$

(5.22)

where the first term in the expansion agrees with the result given by eq. (5.11) in paper I. For the values $\Omega_v = 0$, $\Omega_w = 1$ and $(r_0/R)^3 = 1/2$, the sixth order expansion of eq. (5.20) yields:

$$Q = \frac{2}{135} \left( \frac{r_0}{t_0} \right)^2 - \frac{137107}{12247200} \left( \frac{r_0}{t_0} \right)^4 + \frac{33336241}{394053600} \left( \frac{r_0}{t_0} \right)^6 - \mathcal{O} \left( \frac{r_0}{t_0} \right)^8.$$

(5.23)

By inspecting the power series (5.23), we see that a very simple but accurate fitting formula for the backreaction in terms of elementary functions is given by

$$Q = \frac{2}{135} t_0^{-2} \left( \frac{r_0}{t_0} \right)^2 \left[ \frac{1}{1 + \frac{3}{4} \left( \frac{r_0}{t_0} \right)^2} \right].$$

(5.24)

Testing the fitting formula (5.24) numerically shows that it is very accurate up to horizon sized voids $r_0 \sim t_0$ (for $r_0 < t_0$ we have $|\text{error}| < 1\%$) and an excellent approximation even beyond. To illustrate this, we have plotted the fitting formula (5.24) against the exact backreaction (5.20) and the leading order term of the expansion (5.23) in figure 1. The figure also shows that the mere leading order term gives an accurate approximation for the backreaction even up to voids of size $r_0 \sim t_0/3$.

By solving for the density parameters $\Omega_v$ and $\Omega_w$ that yield the maximum value for the backreaction of small voids (5.21), we obtain $\Omega_v = 0$ and $\Omega_w = 0.7$. This can be seen in figure 2, where the backreaction is plotted as a function of $\Omega_w$ for three values of $r_0/t_0$ and fixed $\Omega_v = 0$ and $(r_0/R)^3 = 1/2$. The figure also shows that the backreaction is only slightly larger for $\Omega_w = 0.7$ than for $\Omega_w = 1$. 


Figure 1. The backreaction as a function of $r_0/t_0$ for the profile (4.1) with $\Omega_v = 0$, $\Omega_w = 1$ and $(r_0/R)^3 = 1/2$ calculated using the exact result (5.20) (red solid curve), the leading order term of the expansion (5.23) (blue dash dotted curve) and the fitting formula (5.24) (black dashed curve).

Figure 2. The backreaction as a function of $\Omega_w$ for the profile (4.1) with $r_0/t_0 = 0.01$ (blue dash dotted curve), $r_0/t_0 = 0.02$ (red dashed curve) and $r_0/t_0 = 0.03$ (black solid curve). All profiles in this figure have $\Omega_v = 0$ and $(r_0/R)^3 = 1/2$.

5.3 Network of voids

Using LTB solutions with the profile defined by eqs. (4.1) and (4.2), we can join together many void-wall pairs to construct a model for a network of voids with different $\Omega_v$ and $r_0$, presuming the following matching conditions are met: the different void-wall pairs must have the same wall-density $\Omega_w$ and the same age of the universe $t_0$. This provides an exact
solution, because, outside the void at $R > r_0$, the void-wall LTB profiles are identical to the homogeneous Friedmann dust solution with $\Omega_w$ as the density parameter and $t_0$ as the age of the universe and thus naturally match together.

For a network of voids, the total backreaction $Q_{\text{tot}}$ is given by the formula (2.11): the volume-weighted average of the backreactions $Q_i$ of the individual void-wall pairs $i$ plus a sum over the relative variances of the expansion rates between all the different pairs. The relative variance term makes the configuration particularly interesting as it seems to offer a way to increase the variance of the expansion rate without having to introduce any counterbalancing extra shear.

Let us determine an upper limit to the relative variance term for a network of voids. As the relative variance is given by the double sum term in the expression (2.11), we need to calculate the volume-average expansion $\langle \theta \rangle$ for a void-wall LTB model as a function of the parameters $\Omega_v, \Omega_w, r_0, t_0$ and $R$:

$$\langle \theta \rangle = (3 - \sqrt{\Omega_w})t_0^{-1} + \frac{1}{540}t_0^{-1}\left(\frac{r_0}{R}\right)^3\left(\sqrt{\Omega_w} - \sqrt{\Omega_v}\right)\left(108 + 12\Omega_v^2 - 60\Omega_v^{3/2}\sqrt{\Omega_w} + 18\sqrt{\Omega_v} - 90\Omega_v\Omega_w + 45\sqrt{\Omega_v}\Omega_w - 80\Omega_w - 80\sqrt{\Omega_v}\Omega_w + 64\Omega_v + \Omega_v^{3/2} - 6\sqrt{\Omega_v}\Omega_w^{3/2} - 6\Omega_w\Omega_v + 45\Omega_w^{3/2} - 6\Omega_w^2\right)\left(\frac{r_0}{t_0}\right)^2.$$  \hspace{1cm} (5.25)

Given the matching condition $\Omega_w(i) = \Omega_w(j)$ between different void-wall LTB regions $i$ and $j$, the leading order terms of the volume expansion (5.25) cancel in the difference $\langle \theta \rangle_i - \langle \theta \rangle_j$. Consequently, the relative variance term in eq. (2.11) gets the maximum value when $\Omega_v(i) = 0$ and $\Omega_v(j) = 1$, yielding the upper limit:

$$\langle \theta \rangle_i - \langle \theta \rangle_j \leq \left[\frac{23}{540}t_0^{-1}\left(\frac{r_0}{t_0}\right)^2\right]^2.$$  \hspace{1cm} (5.26)

Therefore we have

$$\frac{1}{3} \sum_{i=1}^{N} \sum_{j=1}^{N} f_i f_j \left(\langle \theta \rangle_{D_i} - \langle \theta \rangle_{D_j}\right)^2 \leq \frac{1}{3} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N^2} \frac{529}{291600}t_0^{-2}\left(\frac{r_0}{t_0}\right)^4.$$  \hspace{1cm} (5.27)

or

$$\frac{1}{3} \sum_{i=1}^{N} \sum_{j=1}^{N} f_i f_j \left(\langle \theta \rangle_{D_i} - \langle \theta \rangle_{D_j}\right)^2 \leq 10^{-3}t_0^{-2}\left(\frac{r_0}{t_0}\right)^4.$$  \hspace{1cm} (5.28)

where $r_0$ is the radius of the largest void in the network. Eq. (5.28) shows that the relative variance term is at most of the order $(r_0/t_0)^4$, so the backreaction for a network of LTB voids with profile defined by eqs. (4.1) and (4.2) is essentially no greater than the backreaction for a single void-wall pair.

### 5.4 Large voids

Let us then study the backreaction of large voids, i.e. voids that do not satisfy $r_0 \ll t_0$. For voids of the horizon size $r_0 \sim t_0$, the exact backreaction (5.20) has to be evaluated numerically.
Figure 3. The backreaction as a function of $\Omega_w$ for the profile (4.1) with $r_0/t_0 = 1$ (blue dash dotted curve), $r_0/t_0 = 3$ (red dashed curve) and $r_0/t_0 \to \infty$ (black solid curve). All profiles in this figure have $\Omega_v = 0$ and $(r_0/R)^3 = 1/2$.

or using the fitting formula (5.24) but for superhorizon voids with $r_0 \gg t_0$, we can consider the limit $r_0/t_0 \to \infty$ of eq. (5.20), which, after some manipulations, takes the form:

$$Q_{\infty} = t_0^{-2} \alpha_v \alpha_w \left( \sqrt{\Omega_w} - \sqrt{\Omega_v} \right)^2 \left[ \frac{2 (x^2 - 1)}{3 \sqrt{\alpha_v \alpha_w}} - \frac{8}{27} \int_0^{\psi} \frac{d\vartheta}{\sqrt{\alpha(\vartheta)}} \right]^2 + 

+ \frac{8 \sqrt{\alpha_w} + \sqrt{\alpha_v} (x^2 - 1)}{9 \sqrt{\alpha_v \alpha_w}} \int_0^{\psi} \frac{d\vartheta}{\sqrt{\alpha(\vartheta)}} - \frac{8 (x^2 - 1)}{9 \sqrt{\alpha_w} \int_0^{\psi} \frac{d\vartheta}{\sqrt{\alpha(\vartheta)}}},$$

(5.29)

where $\alpha(\vartheta)$ is defined in eq. (5.6), $\alpha_v \equiv \alpha(0)$, $\alpha_w \equiv \alpha(1)$ and the integrals can be evaluated in terms of elementary functions:

$$\int_0^{\psi} \frac{d\vartheta}{\sqrt{\alpha(\vartheta)}} = \frac{3 \sqrt{2}}{4 (\sqrt{\Omega_w} - \sqrt{\Omega_v})} \arctan \left( \frac{3 \sqrt{2} (\vartheta - \frac{1}{3})}{\sqrt{\Omega_v - \Omega_w}} \right),$$

(5.30)

$$\int_0^{\psi} \frac{d\vartheta}{\sqrt{\alpha(\vartheta)}} = \frac{1}{(\sqrt{\Omega_w} - \sqrt{\Omega_v})^2} \left\{ 3 \left( \arcsin \sqrt{\Omega_v} - \arcsin \sqrt{\Omega_w} \right) 

+ \frac{3 \sqrt{2}}{4 (3 - \sqrt{\Omega_v})} \arctan \left( \frac{3 \sqrt{2} (\vartheta - \frac{1}{3})}{4 \sqrt{1 - \vartheta^2}} \right) \right\}. $$

(5.31)

Eq. (5.29) can be used to solve the values of the density parameters $\Omega_v$ and $\Omega_w$ that maximize the backreaction for the superhorizon voids: numerically, we find that the maximum is located at $\Omega_v = 0$ and $\Omega_w = 0.625$ and has the value

$$Q_{\text{max}} = 0.038 t_0^{-2} = 0.23 Q_{\text{FRW}},$$

(5.32)

which can also be read off from figure 3. The location of the maximum differs only slightly from the point $\Omega_v = 0$, $\Omega_w = 0.7$ that gives the maximum backreaction for small $r_0 \lesssim 0.3 t_0$
voids, although the maximum value of the backreaction relative to the value at $\Omega_w = 1$ is in this case noticeably larger (cf. section 5.2). However, even with the superhorizon sized voids, the maximum backreaction (5.32) is still only $\sim 20\%$ of the FRW value (5.1), implying that in the void-wall LTB models with the profile defined by eqs. (4.1) and (4.2) the shear suppresses the backreaction in all cases at least by about an order of magnitude.

In figure 4, we have plotted the backreaction as a function of the size of the void $r_0/t_0$ for different values of the wall density $\Omega_w$ but keeping the void density and the volume fraction fixed: $\Omega_v = 0$ and $(r_0/R)^3 = 1/2$. The figure illustrates how the rapid growth of the backreaction as a function of the void size $r_0$ stops once the horizon size $t_0$ is exceeded.

5.5 Uncompensated voids

As the density function $\Omega_0(r)$ is determined by the integral of the physical matter density (3.6), the profile (4.1) implies an overdense shell in $\rho_0(r)$ between the void and the wall (see section 5.4 of paper I for a more detailed explanation). To avoid the overdense or collapsing shell, we consider the profile defined by

$$\theta(r, t_0) = t_0^{-1}(3 - \Theta(r - r_0)),$$

which, by virtue of eqs. (3.7) and (3.11), implies

$$\Omega_0(r) = \left(1 - \left(\frac{r_0}{r}\right)^3\right)^2 \Theta(r - r_0)$$

and

$$H_0(r) = t_0^{-1}\left[1 - \frac{1}{3}\left(1 - \left(\frac{r_0}{r}\right)^3\right)\Theta(r - r_0)\right].$$
6 Summary of the results

We have generalized the analytic approach of paper I in studying the role of shear in the cosmological backreaction problem. As in paper I, we constructed a void-wall model from the exact inhomogeneous LTB dust solution, but instead of assuming the voids to be small $r_0 \ll t_0$ and fixing the densities in the voids and walls to $\Omega_v = 0$ and $\Omega_w = 1$, we let the void size $r_0$ be arbitrary and the densities vary within the range $0 \leq \Omega_v \leq \Omega_w \leq 1$. Moreover, we constructed an exact solution for a network of voids with different densities $\Omega_v$ and radii $r_0$ to study how the emerging relative variance of the expansion rate between the different void-wall pairs affects the backreaction.

We calculated the exact analytic result for the backreaction of a void-wall pair with profile defined by eqs. (4.1) and (4.2) in section 5.1. To study its general behavior for subhorizon voids, we considered the series expansion in powers of $r_0/t_0$ in section 5.2. The leading order term of order $(r_0/t_0)^2$ in the expansion was verified to agree with the result obtained in paper I for the case $\Omega_v = 0$ and $\Omega_w = 1$. From the leading order term we found that the values $\Omega_v = 0$ and $\Omega_w = 0.7$ yield the maximum value for the backreaction for subhorizon voids of size $r_0 \lesssim t_0/3$. However, the increase in the backreaction was showed to be only $\sim 10\%$ relative to the case where $\Omega_v = 0$ and $\Omega_w = 1$. Furthermore, by inspecting the series expansion with the values $\Omega_v = 0$, $\Omega_w = 1$ and $(r_0/R)^3 = 1/2$, we were able to deduce...
the simple fitting formula \((5.24)\) for the backreaction in terms of elementary functions, with error less than 1\% up to horizon sized voids.

In section 5.3, we considered a network of subhorizon voids and showed that the relative variance of the expansion rate between the different voids is at most of order \((r_0/t_0)^4\). The total backreaction for the network is then essentially given just by the volume-weighted average of the backreactions of the individual void-wall pairs and thus remains of order \((r_0/t_0)^2\) or small for voids of the observed size.

We investigated the behavior of the backreaction for large voids in section 5.4, in particular superhorizon \(r_0 \gg t_0\) voids by considering the limit \(r_0/t_0 \to \infty\). We found that the backreaction approaches a constant value as the size of a superhorizon void is increased, the limiting value depending on the parameters \(\Omega_v, \Omega_w\) and \(R/r_0\). The ultimate maximum value for the backreaction was found to be \(Q_{\text{max}} = 0.038 t_0^{-2}\) at \(\Omega_v = 0\) and \(\Omega_w = 0.62\) in the limit \(r_0/t_0 \to \infty\), keeping the void-to-wall volume ratio fixed to \((r_0/R)^3 = 1/2\). This is still only \(\sim 20\%\) of the maximum backreaction obtained when the shear is neglected by treating the voids and walls as separate Friedmann solutions. Furthermore, as opposed to the \(\sim 10\%\) increase in the case of small voids, the backreaction for large voids is a few times (depending on the ratio \(r_0/t_0\)) greater at the maximum point \(\Omega_v = 0\) and \(\Omega_w = 0.62\) than at \(\Omega_v = 0\) and \(\Omega_w = 1\). Finally, in section 5.5 we demonstrated that the suppression of the backreaction is not due to an overdense or collapsing region between the void and the wall but truly an effect of the shear.

As a final comment, we note that since \(Q < 0.04 t_0^{-2}\) and \(4\pi G \langle \rho \rangle > t_0^{-2}/3\) the right hand side of eq. (2.7) is always negative so none of the models studied in this work have average acceleration.

7 Conclusion

By considering exact solutions of the Einstein equation consisting of one or more LTB solutions, we have pinpointed the issue of small versus large cosmological backreaction to the question of matching conditions: while the variance of the expansion rate alone can induce significant backreaction, the shear arising from matching together the regions with different expansion rates seems to bring down the backreaction by at least five orders of magnitude for voids of the observed size. The crucial question is whether the suppression of the backreaction due to the shear is a general property of all realistic cosmological solutions of general relativity or just a special property of the matching in the considered particular solutions. This issue has to be addressed with solutions more sophisticated than the LTB-based models employed here.

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