THE SLICE SPECTRAL SEQUENCE FOR THE $C_4$ ANALOG OF REAL $K$-THEORY

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Abstract. We describe the slice spectral sequence of a 32-periodic $C_4$-spectrum $K_H$ related to the $C_4$ norm $NC_2 MU_{\mathbb{R}}$ of the real cobordism spectrum $MU_{\mathbb{R}}$. We will give it as a spectral sequence of Mackey functors converging to the graded Mackey functor $\mathbb{Z}.K_H$, complete with differentials and exotic extensions in the Mackey functor structure.

The slice spectral sequence for the 8-periodic real $K$-theory spectrum $K_R$ was first analyzed by Dugger. The $C_8$ analog of $K_H$ is 256-periodic and detects the Kervaire invariant classes $\theta_j$. A partial analysis of its slice spectral sequence led to the solution to the Kervaire invariant problem, namely the theorem that $\theta_j$ does not exist for $j \geq 7$.

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We will study the slice spectral sequence of Mackey functors for a 32-periodic $C_4$-spectrum $K_{H}$ (to be defined below in §6, specifically Definition 6.3) related to the $C_4$ norm (defined in [HHRa, §2.2.3])

$$MU^{((C_4))} = N_{C_2}^{C_4}MU_{R}$$

of the real cobordism spectrum $MU_{R}$. We will rely extensively on the results, methods and terminology of [HHRa].

Part of this spectral sequence is illustrated in an unpublished poster produced in late 2008 and shown in Figure 1. It shows the spectral sequence converging to the homotopy of the fixed point spectrum $K^{C_4}_{H}$. The corresponding spectral sequence of Mackey functors converges to the graded Mackey functor $\pi_{-}\bar{K}_{H}$. Our analog of the poster is Figure 16. The Mackey functors appearing in it are denoted by symbols listed in Table 2.

In both illustrations some patterns of $d_3$s and families of elements in low filtration are excluded to avoid clutter. In the poster, representative examples of these are shown in the second and fourth quadrants, the spectral sequence itself being...
concentrated in the first and third quadrants. In this paper those patterns are spelled out in §10 and §11.

The $C_2$ analog is the 8-periodic real $K$-theory spectrum $K_R$ first defined and studied by Atiyah in [Ati66]. Its slice spectral sequence was first analyzed by Dugger [Dug05] and is described in §7. The $C_8$ analog of $K_H$ is 256-periodic and detects the Kervaire invariant classes $\theta_j$.

In more detail, §1 collects some notions from equivariant stable homotopy theory with an emphasis on Mackey functors. Definition 1.6 introduces new notation that we will occasionally need.

§2 concerns the equivariant analog of the homology of a point namely, the $RO(G)$-graded homotopy of the integer Eilenberg-Mac Lane spectrum $HZ$. In particular Lemma 2.6 describes some relations among certain elements in it including the “gold relation” between $av$ and $uv$.

§3 describes some general properties of spectral sequences of Mackey functors. These include Theorem 3.3 about the relation between differential and exotic extensions in the Mackey functor structure and Theorem 3.5 on the norm of a differential.

§4 lists some concise symbols for various specific Mackey functors for the groups $C_2$ and $C_4$ that we will need. Such functors can be spelled out explicitly by means

Figure 1. The 2008 poster. The first and third quadrants show $E_4(G/G)$ of the slice spectral sequence for $K_H$ with the elements of Prop. 10.3 excluded. The second quadrant indicates $d_3$s as in Figures 8 and 9. The fourth quadrant indicates comparable $d_3$s in the third quadrant of the slice spectral sequence as in Figures 10 and 11.
of Lewis diagrams \[4.1\], which we usually abbreviate by symbols shown in Tables 1 and 2.

In \[5\] we study some chain complexes of Mackey functors that arise as cellular chain complexes for \(G\)-CW complexes of the form \(S^V\).

In \[6\] we introduce the \(C_4\)-spectra of interest in this paper, \(k_{\mathbf{H}}\) and \(K_{\mathbf{H}}\) (Definition 6.3) and various elements in their homotopy groups. The latter are collected in Table 3, which spans several pages.

In \[7\] we describe the slice spectral sequence for an easier case, the \(C_2\)-spectrum \(K_R\), real \(K\)-theory. This is due to Dugger [Dug05] and serves as a warmup exercise for us. It turns out that everything in the SS is formally determined by the structure of its \(E_2\)-term and Bott periodicity.

In \[8\] we determine the \(E_4\)-term of the slice SS for \(k_{\mathbf{H}}\) and \(K_{\mathbf{H}}\).

In \[9\] we use the Slice Differentials Theorem of [HHRa] to determine some differentials in our spectral sequence.

In \[10\] we determine the \(E_4\)-term of our spectral sequence. It is far smaller than \(E_2\) and the results of \[11\] enable us to ignore most of it. What is left is small enough to be shown legibly in the SS charts of Figures 14 and 16.

In \[11\] we examine the \(C_4\)-spectrum \(k_{\mathbf{H}}\) as a \(C_2\)-spectrum. This leads to a calculation only slightly more complicated than Dugger’s. It gives a way to remove a lot of clutter from the \(C_4\) calculation.

We finish the calculation in \[12\] by dealing with the remaining differentials and exotic Mackey functor extensions. It turns out that they are all formal consequences of \(C_2\) differentials of the previous section along with the results of \[3\].

1. Recollections about equivariant stable homotopy theory

We first discuss some structure on the equivariant homotopy groups of a \(G\)-spectrum \(X\). We will assume throughout that \(G\) is a finite cyclic \(p\)-group. This means that its subgroups are well ordered by inclusion and each is uniquely determined by its order. The results of this section hold for any prime \(p\), but the rest of the paper concerns only the case \(p = 2\). We will define several maps indexed by pairs of subgroups of \(G\). We will often replace these indices by the orders of the subgroups, sometimes denoting \(|H|\) by \(h\).

The homotopy groups can be defined in terms of finite \(G\)-sets \(T\) Let

\[ \Xi^G_0 X(T) = [T_+, X]^G, \]

be the set of homotopy classes of equivariant maps from \(T_+\), the suspension spectrum of the union of \(T\) with a disjoint base point, to the spectrum \(X\). We will often omit \(G\) from the notation when it is clear from the context. For an orthogonal representation \(V\) of \(G\), we define

\[ \Xi^V X(T) = [S^V \wedge T_+, X]^G. \]

As an \(RO(G)\)-graded contravariant abelian group valued functor of \(T\), this converts disjoint unions to direct sums. This means it is determined by its values on the sets \(G/H\) for subgroups \(H \subseteq G\).

Since \(G\) is abelian, \(H\) is normal and \(\Xi^V X(G/H)\) is a \(\mathbb{Z}[G/H]\)-module.
Given subgroups $K \subseteq H \subseteq G$, one has pinch and fold maps between the $H$-spectra $H/H_+$ and $H/K_+$. This leads to a diagram

$$
\begin{array}{ccc}
H/H_+ & \xrightarrow{\text{pinch}} & H/K_+ \\
\downarrow^{G_+\wedge H(\cdot)} & & \\
G/H_+ & \xrightarrow{\text{fold}} & G/K_+ \\
\end{array}
$$

(1.1)

Note that while the fold map is induced by a map of $H$-sets, the pinch map is not. It only exists in the stable category.

**Definition 1.2. The Mackey functor structure maps in $\pi^G_V X$.** The fixed point transfer and restriction maps

$$
\begin{array}{ccc}
\pi_V X(G/H) & \xrightarrow{\text{tr}^H_K} & \pi_V X(G/K) \\
\downarrow^{\text{res}^H_K} & & \\
\pi_V X(G/H) & \xrightarrow{\text{pinch}} & \pi_V X(G/K) \\
\end{array}
$$

are the ones induced by the composite maps in the bottom row of (1.1).

These satisfy the formal properties needed to make $\pi_V X$ into a Mackey functor; see [HHRa, Def. 3.1]. They are usually referred to simply as the transfer and restriction maps. We use the words “fixed point” to distinguish them from another similar pair of maps specified below in Definition 1.10.

When $X$ is a ring spectrum, we have the fixed point Frobenius relation

$$
\text{tr}^H_K(\text{res}^H_K(a)b) = a(\text{tr}^H_K(b)) \quad \text{for } a \in \pi_* X(G/H) \text{ and } b \in \pi_* X(G/K).
$$

In particular this means that

$$
\text{tr}^H_K(\text{res}^H_K(a)b) = 0 \quad \text{when } \text{res}^H_K(a) = 0.
$$

For a representation $V$ of $G$, the group

$$
\pi^G_V X(G/H) = \pi^H_V X = [S^V, X]^H
$$

is isomorphic to

$$
[S^0, S^{-V} \wedge X]^H = \pi_0(S^{-V} \wedge X)^H.
$$

However fixed points do not respect smash products, so we cannot equate this group with

$$
\pi_0(S^{-V} \wedge X)^H = [S^V, X]^H = \pi_0[V, H]X^H = \pi_0^G[V, H]X(G/H).
$$

Conversely a $G$-equivariant map $S^V \to X$ represents an element in

$$
[S^V, X]^G = \pi^G_V X = \pi^G_V X(G/G).
$$

The following notion is useful.

**Definition 1.5. Mackey functor induction and restriction.** For groups $H \subseteq G$ and an $H$-Mackey functor $M$, the induced $G$-Mackey functor $\uparrow^G_H M$ is given by

$$
\uparrow^G_H M(T) = M(i^*_H T)
$$

for each finite $G$-set $T$, where $i^*_H$ denotes the forgetful functor from $G$-sets (or spaces or spectra) to $H$-sets.
For a $G$-Mackey functor $\underline{N}$, the restricted $H$-Mackey functor $\downarrow^G_H \underline{N}$ is given by

$$\downarrow^G_H \underline{N}(S) = \underline{N}(G \times_H S)$$

for each finite $H$-set $S$.

This notation is due Thévenaz-Webb [TW95]. They put the arrow on the right and denote $G \times_H S$ by $S \uparrow^G_H$ and $i_H^* T$ by $T \downarrow^G_H$. We also need notation for $X$ as an $H$-spectrum for subgroups $H \subseteq G$. For this purpose we will enlarge the orthogonal representation ring of $G$, $RO(G)$, to the representation ring Mackey functor $RO(G)$ defined by $RO(G)(G/H) = RO(H)$. This was the motivating example for the definition of a Mackey functor in the first place. In it the transfer map on a representation $V$ of $H$ is the induced representation of a supergroup $K \supseteq H$, and its restriction to a subgroup is defined in the obvious way. In particular the restriction of the transfer of $V$ is $|K/H|\cdot V$.

More generally for a finite $G$-set $T$, $RO(G)(T)$ is the ring (under pointwise direct sum and tensor product) of functors to the category of finite dimensional orthogonal real vector spaces from $BG T$, the split groupoid (see [Rav86, A1.1.22]) whose objects are the elements of $T$ with morphisms defined by the action of $G$.

**Definition 1.6.** $RO(G)$-graded homotopy groups. For each $G$-spectrum $X$ and each pair $(H, V)$ consisting of a subgroup $H \subseteq G$ and a virtual orthogonal representation $V$ of $H$, let the $G$-Mackey functor $\pi_{H,V}(X)$ be defined by

$$\pi_{H,V}(X)(T) := [(G_+ \wedge^H S^V) \wedge T_+ \wedge X]^G \cong [S^V \wedge i_H^* T_+, i_H^* X]^H = \underline{Z}_v(i_H^* X)/(i_H^* T),$$

for each finite $G$-set $T$. Equivalently, $\pi_{H,V}(X) = \uparrow^G_H \pi_{V}(i_H^* X)$ (see 1.5) as Mackey functors. We will often denote $\pi_{H,V}$ by $\pi_H$ or $\underline{Z}_v$.

We will be studying the $RO(G)$-graded slice $SS \{E^s_{r,*}\}$ of Mackey functors with $r, s \in \mathbb{Z}$ and $* \in RO(G)$. We will use the notation $E^s_{r,*}(H, V)$ for such Mackey functors, abbreviating to $E^s_{r,*}$ when the subgroup is $G$. Most of our spectral sequence charts will display the values of $E^s_{r,*}$ for integral values of $t$ only.

The following definition should be compared with [Ada84, (2.3)].

**Definition 1.7.** An equivariant homeomorphism. Let $X$ be a $G$-space and $Y$ an $H$-space for a subgroup $H \subseteq G$. We define the equivariant homeomorphism

$$\tilde{u}_H^G(Y, X) : G \times_H (Y \times i_H^* X) \to (G \times_H Y) \times X$$

by $(g, y, x) \mapsto (g, y, g(x))$ for $g \in G$, $y \in Y$ and $x \in X$. We will use the same notation for a similarly defined homeomorphism

$$\tilde{u}_H^G(Y, X) : G_+ \wedge^H (Y \wedge i_H^* X) \to (G_+ \wedge^H Y) \wedge X$$

for a $G$-spectrum $X$ and $H$-spectrum $Y$. We will abbreviate

$$\tilde{u}_H^G(S^0, X) : G_+ \wedge^H i_H^* X \to G/H_+ \wedge X$$

by $\tilde{u}_H^G(X)$.

For representations $V$ and $V'$ of $G$ both restricting to $W$ on $H$, but having distinct restrictions to all larger subgroups, we define $u^G_{V - V'} = u^G_H(S^V)u^G_H(S^{V'})^{-1}$.
so the following diagram of equivariant homeomorphisms commutes:

\[
\begin{array}{ccc}
G_+ \wedge_H S^W & \xrightarrow{\tilde{u}_{G}(S^V)} & G/H \wedge S^V \\
\downarrow & & \downarrow \\
G/H \wedge S^{V'} & \xleftarrow{\tilde{u}_{V-V'}} & G/H \wedge S^{V''}.
\end{array}
\]

(1.8)

When \( V' = |V| \) (meaning that \( H = G_V \) acts trivially on \( W \)), then we abbreviate \( \tilde{u}_{V-V'} \) by \( \tilde{u}_V \).

If \( V \) is a representation of \( H \) restricting to \( W \) on \( K \), we can smash the diagram (1.1) with \( S^V \) and get

\[
\begin{array}{ccc}
S^V & \xrightarrow{\text{pinch}} & H/K_+ \wedge S^V \\
& \downarrow \quad \text{fold} & \\
& & \quad G_+ \wedge (H/K_+ \wedge S^V) \xrightarrow{\cong} G_+ \wedge (H/K_+ \wedge S^W) \equiv G_+ \wedge S^W,
\end{array}
\]

(1.9)

where the homeomorphism is induced by that of Definition 1.7.

**Definition 1.10. The group action restriction and transfer maps.** For subgroups \( K \subseteq H \subseteq G \), let \( V \) be a representation of \( H \) restricting to \( W \) on \( K \). The group action transfer and restriction maps

\[
\begin{array}{cccc}
\uparrow^{G}_{H} \cong_{V} (i_{H}^{*} X) & \xrightarrow{\cong} & \cong_{H,V} X & \xrightarrow{\cong_{K,W}^{H}} & \cong^{K}_{K} \cong^{K}_{W}(i_{K}^{*} X)
\end{array}
\]

(see (1.5)) are the ones induced by the composite maps in the bottom row of (1.9). The symbols \( t \) and \( r \) here are underlined because they are maps of Mackey functors rather than maps within Mackey functors.

We include \( V \) as an index for the group action transfer \( \uparrow^{H,V}_{K} \) because its target is not determined by its source.

Thus we have abelian groups \( \cong_{H,V}(X)(G/H'') \) for all subgroups \( H', H'' \subseteq G \) and representations \( V \) of \( H' \). Most of them are redundant in view of Theorem 1.12 below. In what follows, we will use the notation \( H_{\cup} = H' \cup H'' \) and \( H_{\cap} = H' \cap H'' \).

**Lemma 1.11. An equivariant module structure.** For a \( G \)-spectrum \( X \) and \( H' \)-spectrum \( Y \),

\[
[G_+ \wedge_{H'} Y, X]^{H''} = \mathbb{Z}[G/H_{\cup}] \otimes [H_{\cup} \wedge_{H'} Y, X]^{H''}
\]

as \( \mathbb{Z}[G/H''] \)-modules.

**Proof.** As abelian groups,

\[
[G_+ \wedge_{H'} Y, X]^{H''} = [i_{H''}^{*}(G_+ \wedge_{H'} Y), X]^{H''}
\]
Theorem 1.12. The module structure for RO(G)-graded homotopy groups. For subgroups \( H', H'' \subseteq G \) with \( H_\cup = H' \cup H'' \) and \( H_\cap = H' \cap H'' \), and a representation \( V \) of \( G' \) restricting to \( W \) on \( H_\cap \),

\[
\pi_{H',V}(G/G'') = [G_+ \wedge_{H' \cap} S^V, X]^{H''} = G_+ \wedge_{H' \cap} [S^W, X]^{H''} = \bigoplus_{[G/H_\cup]} [H_\cap, X]^{H''} = \bigoplus_{[G/H_\cup]} [H_\cup, X]^{H''}
\]

and \( G/H'' \) permutes the wedge summands of \( \bigvee_{[G/H_\cup]} H_\cup \wedge X \) as it permutes the elements of \( G/H_\cup \).

Proof. We start with the definition and use the homeomorphism of Definition 1.7 and the module structure of Lemma 1.11.

\[
\begin{align*}
\pi_{H',V}(G/G'') &= [(G_+ \wedge_{H'} S^V) \wedge G/H', X]^G \\
&= [G_+ \wedge_{H'} (G_+ \wedge_{H'} S^V), X]^G \\
&= [G_+ \wedge_{H'} S^V, X]^{H''} = [G_+ \wedge_{H' \cap} S^W, X]^{H''}
\end{align*}
\]

and

\[
[H_\cup \wedge_{H'} S^V, X]^{H''} = [S^W, X]^{H''} = [G_+ \wedge_{H' \cap} S^W, X]^{H''} = \pi_{H',W}(G/H_\cap) \sim \pi_{H',W}(G/G).
\]

For the statement about nonoriented \( V \), we have

\[
\pi_{H',V}(G/G'') = [G/H'] \otimes [S^W \wedge_{H' \cap} X]^{H''} = [G/H'] \otimes [S^W, X]^{H''}.
\]

Then \( \gamma \) induces a map of degree \( \pm 1 \) on the sphere depending on the orientability of \( V \).

Theorem 1.12 means that we need only consider the groups

\[
\pi_{H',V}(G/G) = \pi_{H',W}(G/H).
\]

When \( H \subseteq G \) and \( V \) is a representation of \( G \) restricting to \( W \) on \( H \), we have

\[
\pi_{V}(G/H) \cong \pi_{H,W}(G/G).
\]

This isomorphism makes the following diagram commute for \( K \subseteq H \).

\[
\begin{array}{ccc}
\pi_{V}(G/H) & \cong & \pi_{H,W}(G/G) \\
\xrightarrow{\text{res}_K^H} & \xrightarrow{\text{res}_K^H} & \xrightarrow{\text{res}_K^H} \\
\pi_{V}(G/K) & \cong & \pi_{K,W}(G/G)
\end{array}
\]
We will use these two groups of \([1,3]\) interchangeably as convenient. Note that the group on the left is indexed by \(RO(G)\) while the one the right is indexed by \(RO(H)\). This means that if \(V\) and \(V'\) are representations of \(G\) each restricting to \(W\) on \(H\), then \(\overline{\pi}_V X(G/H)\) and \(\overline{\pi}_{V'} X(G/H)\) are canonically isomorphic. The first of these is

\[
[G/H_+ \wedge S^V, X]^G \cong [G_+ \wedge H S^W, X]^G \cong [S^W, i^H_H X]^H
\]

where the first isomorphism is induced by the homeomorphism \(\overline{u}_H^G(X)\) of Definition \([1,7]\) and the second is the fact that \(G_+ \wedge H(\cdot)\) is the left adjoint of the forgetful functor \(i^H_H\).

For a ring spectrum \(X\), such as the one we are studying in this paper, an indecomposable element in \(\overline{\pi}_j X(G/H)\) may map to a product in \(\overline{\pi}_j X(G/G)\) of elements in groups indexed by representations of \(H\) that are not restrictions of representations of \(G\). This factoring can make some computations easier.

2. THE \(RO(G)\)-GRADED HOMOTOPY OF \(HZ\)

We describe part of the \(RO(G)\)-graded Green functor \(\overline{\pi}_j (HZ)\), where \(HZ\) is the integer Eilenberg-Mac Lane spectrum \(HZ\) in the \(G\)-equivariant category, for some finite cyclic 2-group \(G\). For each actual (as opposed to virtual) \(G\)-representation \(V\) we have an equivariant reduced cellular chain complex \(C_V^j\) for the space \(S^V\). It is a complex of \(Z[G]\)-modules with \(H_j(C^V) = H_*(S^V)\).

One can convert such a chain complex \(C^j_{\ast}\) of \(Z[G]\)-modules to one of Mackey functors as follows. Given a \(Z[G]\)-module \(M\), we get a Mackey functor \(M\) defined by

\[
M(G/H) = M^H \quad \text{for each subgroup} \ H \subseteq G.
\]

We call this a fixed point Mackey functor. In it each restriction map \(res^K_H\) (for \(K \subseteq H \subseteq G\) is one to one. When \(M\) is a permutation module, meaning the free abelian group on a \(G\)-set \(B\), we call \(M\) a permutation Mackey functor \([HHRa,3.2]\).

In particular the \(Z[G]\)-module \(Z\) with trivial group action (the free abelian group on the \(G\)-set \(G\)) leads to a Mackey functor \(Z\) in which each restriction map is an isomorphism and the transfer map \(tr^K_H\) is multiplication by \(|H/K|\). For each Mackey functor \(M\) there is an Eilenberg-Mac Lane spectrum \(HZM\) \([GM55,5]\), and \(HZ\) is the same as \(HZ\) with trivial group action.

Given a finite \(G\)-CW spectrum \(X\), meaning one built out of cells of the form \(G_+ \wedge H e^n\), we get a reduced cellular chain complex of \(Z[G]\)-modules \(C_*X\), leading to a chain complex of fixed point Mackey functors \(C_*X\). Its homology is a graded Mackey functor \(H_*X\) with

\[
H_*X(G/H) = \pi_* (X \wedge HZ)(G/H) = \pi_* (X \wedge HZ)^H.
\]

In particular \(H_*X(G/\{\epsilon\}) = H_*X\), the underlying homology of \(X\). In general \(H_*X(G/H)\) is not the same as \(H_*X^H\) because fixed points do not commute with smash products.

For a finite cyclic 2-group \(G = C_{2^k}\), the irreducible representations are the 2-dimensional ones \(\lambda(m)\) corresponding to rotation through an angle of \(2\pi m/2^k\) for \(0 < m < 2^{k-1}\), the sign representation \(\sigma\) and the trivial one of degree one, which we denote by \(1\). The 2-local equivariant homotopy type of \(S^{\lambda(m)}\) depends only on the 2-adic valuation of \(m\), so we will only consider \(\lambda(2^j)\) for \(0 \leq j \leq k-2\) and denote
it by $λ_j$. The planar rotation $λ_{k-1}$ though angle $π$ is the same representation as $2σ$. We will denote $λ(1) = λ_0$ simply by $λ$.

We will describe the chain complex $C^V$ for

$$V = a + bσ + \sum_{2 \leq j \leq k} c_j λ_{k-j}.$$ 

for nonnegative integers $a$, $b$ and $c_j$. The isotropy group of $V$ (the largest subgroup fixing all of $V$) is

$$G_V = \begin{cases} \mathbb{Z} & \text{for } n = d_0 \\ \mathbb{Z}[G/G'] & \text{for } d_0 < n \leq d_1 \\ \mathbb{Z}[G/C_{2k-1}] & \text{for } d_{j-1} < n \leq d_j \text{ and } 2 \leq j \leq \ell \\ 0 & \text{otherwise.} \end{cases}$$ 

where

$$d_j = \begin{cases} a & \text{for } j = 0 \\ a + b & \text{for } j = 1 \\ a + b + 2c_2 + \cdots + 2c_j & \text{for } 2 \leq j \leq \ell, \end{cases}$$

so $d_\ell = |V|$.

The boundary map $∂_n : C^V_n \to C^V_{n-1}$ is determined by the fact that $H_\ast(C^V) = H_\ast(S^{|V|})$. More explicitly, let $γ$ be a generator of $G$ and

$$ζ_j = \sum_{0 \leq t < 2j} γ^t$$

for $1 \leq j \leq k$.

Then we have

$$∂_n = \begin{cases} \nabla & \text{for } n = 1 + d_0 \\ (1 - γ)x_n & \text{for } n - d_0 \text{ even and } 2 + d_0 \leq n \leq d_n \\ x_n & \text{for } n - d_0 \text{ odd and } 2 + d_0 \leq n \leq d_n \\ 0 & \text{otherwise,} \end{cases}$$

where $\nabla$ is the fold map sending $γ \mapsto 1$, and $x_n$ denotes multiplication by an element in $\mathbb{Z}[G]$ to be named below. We will use the same symbol below for the quotient map $\mathbb{Z}[G/H] \to \mathbb{Z}[G/K]$ for $H \subseteq K \subseteq G$. The elements $x_n \in \mathbb{Z}[G]$ for $2 + d_0 \leq n \leq |V|$ are determined recursively by $x_{2+d_0} = 1$ and

$$x_nx_{n-1} = ζ_j \quad \text{for } 2 + d_{j-1} < n \leq 2 + d_j.$$ 

It follows that $H_{|V|}^V = \mathbb{Z}$ generated by either $x_{1+|V|}$ or its product with $1 - γ$, depending on the parity of $b$.

This complex is

$$C^V = \Sigma_{|V_0|} C^{V/V_0}$$

where $V_0 = V^G$. This means we can assume without loss of generality that $V_0 = 0$.

An element

$$x \in H_n(C^V)(G/H) = H_nS^V(G/H)$$

corresponds to an element $x \in π_{n-V}HZ(G/H)$. 
We will denote the dual complex \( \mathrm{Hom}_\mathbb{Z}(C^V, \mathbb{Z}) \) by \( C^{-V} \). Its chains lie in dimensions \(-n\) for \( 0 \leq n \leq |V| \). An element \( x \in H_{-n}(S^{-V})(G/H) \) corresponds to an element \( x \in \pi_{-n} H\mathbb{Z}(G/H) \).

The method we have just described determines only a portion of the \( RO(G) \)-graded Mackey functor \( \pi (G, *) \mathbb{Z} \), namely the groups in which the index differs by an integer from an actual representation \( V \) or its negative. For example it does not give us \( \pi_{-\lambda} H\mathbb{Z} \) for \( |G| \geq 4 \).

We leave the proof of the following as an exercise for the reader.

**Proposition 2.3.** The top (bottom) homology groups for \( S^V \) (\( S^{-V} \)). Let \( G \) be a finite cyclic 2-group and \( V \) a nontrivial representation of \( G \) of degree \( d \) with \( V^G = 0 \) and isotropy group \( G_V \). Then \( C^V_d = C^{-V}_d = \mathbb{Z}[G/G_V] \) and

(i) If \( V \) is oriented then \( H_d S^V = \mathbb{Z} \), the constant \( \mathbb{Z} \)-valued Mackey functor in which each restriction map is an isomorphism and each transfer \( tr^K_H \) is multiplication by \( |K/H| \).

(ii) \( H_{-d} S^{-V} = \mathbb{Z}(G, G_V) \), the constant \( \mathbb{Z} \)-valued Mackey functor in which

\[
\begin{align*}
\text{res}^K_H &= \begin{cases} 
1 & \text{for } K \subseteq G_V \\
|K/H| & \text{for } G_V \subseteq H 
\end{cases} \\
\text{tr}^K_H &= \begin{cases} 
|K/H| & \text{for } K \subseteq G_V \\
1 & \text{for } G_V \subseteq H 
\end{cases}
\end{align*}
\]

(The above completely describes the cases where \( |K/H| = 2 \), and they determine all other restrictions and transfers.) The functor \( \mathbb{Z}(G, e) \) is also known as the dual \( \mathbb{Z}^* \). These isomorphisms are induced by the maps

\[
\begin{array}{ccc}
H_d S^V & \xrightarrow{\Delta} & H_{-d} S^{-V} \\
\mathbb{Z} & \xrightarrow{\nabla} & \mathbb{Z}(G, G_V) \\
\end{array}
\]

(iii) If \( V \) is not oriented then \( H_d S^V = \mathbb{Z}_- \), where

\[
\mathbb{Z}_-(G/H) = \begin{cases} 
0 & \text{for } H = G \\
\mathbb{Z}_- := \mathbb{Z}[G]/(1 + \gamma) & \text{otherwise}
\end{cases}
\]

where each restriction map \( \text{res}^K_H \) is an isomorphism and each transfer \( tr^K_H \) is multiplication by \( |K/H| \) for each proper subgroup \( K \).

(iv) We also have \( H_{-d} S^{-V} = \mathbb{Z}(G, G_V)_- \), where

\[
\mathbb{Z}(G, G_V)_-(G/H) = \begin{cases} 
0 & \text{for } H = G \text{ and } V = \sigma \\
\mathbb{Z}/2 & \text{for } H = G \text{ and } V \neq \sigma \\
\mathbb{Z}_- & \text{otherwise}
\end{cases}
\]

with the same restrictions and transfers as \( \mathbb{Z}(G, G_V) \). These isomorphisms are induced by the maps

\[
\begin{array}{ccc}
H_d S^V & \xrightarrow{\Delta_-} & H_{-d} S^{-V} \\
\mathbb{Z}_- & \xrightarrow{\nabla_-} & \mathbb{Z}(G, G_V)_- \\
\end{array}
\]

The Mackey functor \( \mathbb{Z}(G, G_V) \) is one of those defined (with different notation) in \cite[Def. 2.1]{HHRS}. 

\[\text{THE SLICE SPECTRAL SEQUENCE FOR THE } C_4 \text{ ANALOG OF REAL } K\text{-THEORY \hfill 11}\]
Definition 2.4. Three elements in $\pi^G_*(HZ)$. Let $V$ be an actual (as opposed to virtual) representation of the finite cyclic 2-group $G$ with $V^G = 0$ and isotropy group $G_V$.

(i) The equivariant inclusion $S^0 \to S^V$ defines an element in $\pi_{-V} S^0(G/G)$ via the isomorphisms

$$\pi_{-V} S^0(G/G) = \pi_0 S^V(G/G) = \pi_0 S^V = \pi_0 S^0 = \mathbb{Z},$$

and we will use the symbol $\alpha_v$ to denote its image in $\pi_{-V} H\mathbb{Z}(G/G)$.

(ii) The underlying equivalence $S^V \to S^{[V]}$ defines an element in

$$\pi_{V} S^V(G/G_V) = \pi_{[V]} S^0(G/G_V)$$

and we will use the symbol $\epsilon_v$ to denote its image in $\pi_{V} H\mathbb{Z}(G/G_V)$.

(iii) If $W$ is an oriented representation of $G$ (we do not require that $W^G = 0$), there is a map

$$\Delta : \mathbb{Z} \to C_{[W]}^W = \mathbb{Z}[G/G_W]$$

as in Proposition 2.3 giving an element

$$u_W \in \mathcal{H}_{[W]} S^W(G/G) = \pi_{[W]} H\mathbb{Z}(G/G).$$

For nonoriented $W$, Proposition 2.3 gives a map

$$\Delta : \mathbb{Z} \to C_{[W]}^W$$

and an element

$$u_W \in \mathcal{H}_{[W]} S^W(G/G') = \pi_{[W]} H\mathbb{Z}(G/G').$$

The element $u_W$ above is related to the element $\tilde{u}_V$ of (1.8) as follows.

Lemma 2.5. The restriction of $u_W$ to a unit and permanent cycle. Let $W$ be a nontrivial representation of $G$ with $H = G_W$. Then the homeomorphism

$$\Sigma^{-W} \tilde{u}_W : G/H_+ \land S^{[W]} \to G/H_+$$

of (1.8) induces an isomorphism $\pi_0 H\mathbb{Z}(G/H) \to \pi_{[W]} H\mathbb{Z}(G/H)$ sending the unit to $\text{res}_H^K(u_W)$ for $u_W$ as defined in (iii) above and $K = G$ or $G'$ depending on the orientability of $W$.

The product

$$\text{res}_H^K(u_W) e_W \in \pi_0 H\mathbb{Z}(G/H) = \mathbb{Z}$$

is a generator, so $e_W$ and $\text{res}_H^K(u_W)$ are units in the ring $\pi_* H\mathbb{Z}(G/H)$, and $\text{res}_H^K(u_W)$ is in the Hurewicz image of $\pi_* S^0(G/H)$.

Proof. The diagram

$$G/K_+ \land S^{[W]} \xrightarrow{\text{fold}} G/H_+ \land S^{[W]} \xrightarrow{\tilde{u}_W} G/H_+$$

induces (via the functor $[\cdot, H\mathbb{Z}]^G$)

$$\pi_{[W]} H\mathbb{Z}(G/K) \xrightarrow{\text{res}_H^K} \pi_{[W]} H\mathbb{Z}(G/H) \xrightarrow{\sim} \pi_0 H\mathbb{Z}(G/H) \xrightarrow{\sim} \mathbb{Z}$$

The restriction map is an isomorphism by Proposition 2.3 and the group on the left is generated by $u_W$. 
The product is the composite of $H$-maps

$$S^W \longrightarrow S^{|W|} \longrightarrow \Sigma^W \mathbb{H}_Z,$$

which is the standard inclusion. \hfill $\square$

Note that $a_V$ and $e_V$ are induced by maps to equivariant spheres while $u_W$ is not. This means that in any spectral sequence based on a filtration where the subquotients are equivariant $\mathbb{HZ}$-modules, elements defined in terms of $a_V$ and $e_V$ will be permanent cycles, while multiples and powers of $u_W$ can support nontrivial differentials. Lemma 2.5 says a certain restriction of $u_W$ will be permanent cycles, while multiples and powers of $u_W$ can support nontrivial differentials. Lemma 2.6 says a certain restriction of $u_W$ is a permanent cycle.

Each nonoriented $V$ has the form $W + \sigma$ where $\sigma$ is the sign representation and $W$ is oriented. It follows that

$$u_V = u_0 r e_{e_0}^G (u_W) \in \pi_{|V| - V} \mathbb{H}_Z(G/G').$$

Note also that $a_0 = e_0 = u_0 = 1$. The trivial representations contribute nothing to $\pi_*(\mathbb{H}_Z)$. We can limit our attention to representations $V$ with $V^G = 0$. Among such representations of cyclic 2-groups, the oriented ones are precisely the ones of even degree.

**Lemma 2.6. Properties of $a_V$, $e_V$ and $u_W$.** The elements $a_V \in \pi_{-V} \mathbb{H}_Z(G/G)$, $e_V \in \pi_{-|V|} \mathbb{H}_Z(G/G_V)$ and $u_W \in \pi_{|W|-W} \mathbb{H}_Z(G/G)$ for $W$ oriented of Definition 2.4 satisfy the following.

(i) This follows from the existence of the pairing $C^V \otimes C^W \rightarrow C^{V+W}$. It induces an isomorphism in $H_0$ and (when both $V$ and $W$ are oriented) in $H_{|V+W|}$.

(ii) This holds because $H_0(V)$ is killed by $|G/G_V|$.

(iii) This follows from Proposition 2.3.

(iv) Using the Frobenius relation we have

$$tr_G^G (e_V) u_V = tr_G^G (e_V r e_{e_0}^G (u_V)) = tr_G^G (1) \quad \text{by 2.5}$$

$$= |G/G_V|$$

$$tr_G^G (e_{V+\sigma}) u_{V+\sigma} = tr_G^G (e_{V+\sigma} r e_{e_0}^G (u_{V+\sigma})) = tr_G^G (1) = |G'/G_V|.$$
(v) We have
\[ a_{V+W}^G \cdot tr_{G_{V+W}}^G (e_{V+W}) : S^{[V]-[U]} \to S^{W-U}. \]
It is null because the bottom cell of \( S^{W-U} \) is in dimension \(-[U]\).

(vi) Since \( V \) is oriented, then we are computing in a torsion free group so we can tensor with the rationals. It follows from (iv) that
\[ \text{tr}_{G_{V+W}}^G (e_{V+W}) = \frac{|G/G_{V+W}|}{u_V u_W} \]
and
\[ \text{tr}_{G_V}^G (e_V) = \frac{|G/G_V|}{u_V} \]
so
\[ u_W \cdot \text{tr}_{G_{V+W}}^G (e_{V+W}) = \frac{|G/G_{V+W}|}{u_V} = |G_V/G_{V+W}| \cdot \text{tr}_{G_V}^G (e_V). \]

(vii) For \( G = C_2^m \), each oriented representation of degree 2 is 2-locally equivalent to a \( \lambda_j \) for \( 0 \leq j < n \). The isotropy group is \( G_{\lambda_j} = C_2^j \). Hence the assumption that \( G_V \subset G_W \) can be replaced with \( V = \lambda_j \) and \( W = \lambda_k \) with \( 0 \leq j < k < n \).

We wish to prove is
\[ a_{\lambda_k} a_{\lambda_j} = 2^{k-j} a_{\lambda_k} a_{\lambda_j}. \]

One has a map \( S^{\lambda_j} \to S^{\lambda_k} \) which is the suspension of the \( 2^{k-j} \)-th power map on the equatorial circle. Hence its underlying degree is \( 2^{k-j} \). We will denote it by \( a_{\lambda_k}/a_{\lambda_j} \) since there is a diagram

\[
\begin{array}{ccc}
S^0 & \xrightarrow{a_{\lambda_k}} & S^{\lambda_j} \\
\downarrow{a_{\lambda_j}} & & \downarrow{a_{\lambda_k}/a_{\lambda_j}} \\
S^{\lambda_k} & \xrightarrow{a_{\lambda_j}} & S^{\lambda_k} 
\end{array}
\]

We claim there is a similar diagram

\[
\begin{array}{ccc}
S^2 & \xrightarrow{u_{\lambda_k}} & S^{\lambda_k} \wedge H\mathbb{Z} \\
\downarrow{u_{\lambda_j}} & & \downarrow{u_{\lambda_k}/u_{\lambda_j}} \\
S^{\lambda_k} & \xrightarrow{a_{\lambda_j}} & S^{\lambda_k} \wedge H\mathbb{Z}. 
\end{array}
\]

in which the underlying degree of the vertical map is one.

Smashing \( a_{\lambda_k}/a_{\lambda_j} \) with \( H\mathbb{Z} \) and composing with \( u_{\lambda_j}/u_{\lambda_k} \) gives a factorization of the degree \( 2^{k-j} \) map on \( S^{\lambda_j} \wedge H\mathbb{Z} \). Thus we have
\[ \frac{u_{\lambda_j} a_{\lambda_k}}{u_{\lambda_k} a_{\lambda_j}} = 2^{k-j} \]
\[ u_{\lambda_j} a_{\lambda_k} = 2^{k-j} u_{\lambda_k} a_{\lambda_j} \]
as desired.

The vertical map in (2.7) would follow from a map
\[ S^{\lambda_k-\lambda_j} \to H\mathbb{Z} \]
with underlying degree one. Let \( G = C_2^m \) and \( G \supset H = C_2^j \). Then \( S^{-\lambda_j} \) has a cellular structure of the form
\[ G/H_+ \wedge S^{-2} \cup G/H_+ \wedge e^{-1} \cup e^0. \]
We need to smash this with $S^{\lambda_k}$. Since $\lambda_k$ restricts trivially to $H$,

$$G/H_+ \wedge S^{\lambda_k} = G/H_+ \wedge S^2.$$  

This means

$$S^{\lambda_k - \lambda_j} = S^{\lambda_k} \wedge S^{-\lambda_j} = G/H_+ \wedge S^0 \cup G/H_+ \wedge e^1 \cup e^0 \wedge S^{\lambda_k}.$$  

Thus its cellular chain complex has the form

$$
\begin{array}{ccc}
2 & Z[G/K] & \ \downarrow^{1-\gamma} \\
1 & Z[G/K] & \Delta \ \\
0 & Z & \ \downarrow^{1-\gamma} \\
\end{array}
$$

where $K = G/C_{p^k}$ and the left column is the chain complex for $S^{\lambda_k}$.

There is a corresponding chain complex of fixed point Mackey functors. Its value on the $G$-set $G/L$ for an arbitrary subgroup $L$ is

$$
\begin{array}{ccc}
2 & Z[G/\text{max}(K, L)] & \ \downarrow^{1-\gamma} \\
1 & Z[G/\text{max}(K, L)] & \Delta \ \\
0 & Z & \ \downarrow^{1-\gamma} \\
\end{array}
$$

For each $L$ the map $\Delta$ is injective and maps the kernel of the first $1-\gamma$ isomorphically to the kernel of the second one. This means we can replace the above by a diagram of the form

$$
\begin{array}{ccc}
1 & \text{coker } (1-\gamma) & \ \downarrow^{1-\gamma} \\
0 & Z & \Delta \ \\
\end{array}
$$

where each cokernel is isomorphic to $Z$ and each map is injective.

This means that $H_* S^{\lambda_k - \lambda_j}$ is concentrated in degree 0 where it is the pushout of the diagram above, meaning a Mackey functor whose value on each subgroup is $Z$. Any such Mackey functor admits a map to $Z$ with underlying degree one. This proves the claim of (2.7).  

The $Z$-valued Mackey functor $H_0 S^{\lambda_k - \lambda_j}$ is discussed in more detail in [HHR5], where it is denoted by $Z(k, j)$.

3. Generalities on differentials and Mackey functor extensions

First we make some observations about the relation between exotic transfers and restriction with certain differentials in the slice spectral sequence. By “exotic” we mean in a higher filtration. In a spectral sequence of Mackey functors converging to $\pi_* X$, it can happen that an element $x \in \pi_* X(G/H)$ has filtration $s$, but its restriction or transfer has a higher filtration. In the spectral sequence charts in this paper, exotic transfers and restrictions will be indicated by blue and green lines respectively.
Lemma 3.1. Restriction kills \( a_\sigma \) and \( a_\sigma \) kills transfers. Let \( G \) be a finite cyclic 2-group with sign representation \( \sigma \) and index 2 subgroup \( G' \), and let \( X \) be a \( G \)-spectrum. Then in \( \pi_* X(G/G) \) the image of \( \text{tr}_{G'}^G \) is the kernel of multiplication by \( a_\sigma \), and the kernel of \( \text{res}_G^{G'} \) is the image of multiplication by \( a_\sigma \).

Proof. Consider the cofiber sequence obtained by smashing \( X \) with

\[
S^{-1} \stackrel{a_\sigma}{\longrightarrow} S^{\sigma - 1} \longrightarrow G_+ \wedge_{G'} S^0 \longrightarrow S^0 \stackrel{a_\sigma}{\longrightarrow} S^\sigma
\]

Since \((G_+ \wedge_s X)^G\) is equivalent to \(X^{G'}\), passage to fixed point spectra gives

\[
\Sigma^{-1}X^G \longrightarrow (\Sigma^{-1}X)^G \longrightarrow X^{G'} \longrightarrow X^G \longrightarrow (\Sigma G)X^G
\]

so the exact sequence of homotopy groups is

\[
\mathbb{E}_{k+1}(G/G) \longrightarrow \mathbb{E}_{k+1-\sigma}X(G/G) \longrightarrow \mathbb{E}_k \left( G_+ \wedge_{G'} X \right) (G/G) \longrightarrow \mathbb{E}_k X(G/G) \left( G/G' \right) \longrightarrow \mathbb{E}_k \left( X \right) (G/G') \longrightarrow \mathbb{E}_k \left( G\right) (G/G').
\]

Note that the isomorphism \( u_\sigma \) is invertible. This gives the exactness required by both statements.

This implies that when \( a_\sigma x \) is killed by a differential but \( x \in E_* (G/G) \) is not, then \( x \) represents an element that is \( \text{tr}_{G'}^G(y) \) for some \( y \) in lower filtration. Similarly if \( x \) supports a nontrivial differential but \( a_\sigma x \) is a nontrivial permanent cycle, then the latter represents an element with a nontrivial restriction to \( G' \) of higher filtration. In both cases the converse also holds.

Theorem 3.3. Exotic transfers and restrictions in the \( RO(G) \)-graded slice spectral sequence. Let \( G \) be a finite cyclic 2-group with index 2 subgroup \( G' \) and sign representation \( \sigma \), and let \( X \) be a \( G \)-equivariant spectrum with \( x \in E_*^{r,V}(X(G/G)) \) (for \( V \in RO(G) \)) in the slice spectral sequence for \( X \). Then

(i) Suppose there is a permanent cycle \( y' \in E_*^{r+V+r-1}(X(G/G')). \) Then there is a nontrivial differential \( d_r(x) = \text{tr}_{G'}^G(y') \) iff \( a_\sigma x \) is a permanent cycle with \( \text{res}_G^{G'}(a_\sigma x) = u_\sigma y' \). In this case \( a_\sigma x \) represents the Toda bracket \( \langle a_\sigma, \text{tr}_{G'}^G, y' \rangle \).

(ii) Suppose there is a permanent cycle \( y \in E_*^{r-V-r-2}(X(G/G)). \) Then there is a nontrivial differential \( d_r(x) = a_\sigma y \) iff \( \text{res}_G^{G'}(x) \) is a permanent cycle with \( \text{tr}_{G'}^G(u_\sigma^{-1}\text{res}_G^{G'}(x)) = y \). In this case \( \text{res}_G^{G'}(x) \) represents the Toda bracket \( \langle \text{res}_G^{G'}, a_\sigma, y \rangle \).

In each case a nontrivial \( d_r \) is equivalent to a Mackey functor extension raising filtration by \( r - 1 \). In (i) the permanent cycle \( a_\sigma x \) is not divisible in \( \pi_* X \) by \( a_\sigma \) and therefore could have a nontrivial restriction in a higher filtration. Similarly in (ii) the element denoted by \( \text{res}_G^{G'}(x) \) is not a restriction in \( \pi_* X \), so we cannot use the Frobenius relation to equate \( \text{tr}_{G'}^G(u_\sigma^{-1}\text{res}_G^{G'}(x)) \) with \( \text{tr}_{G'}^G(u_\sigma^{-1})x \).
We remark that the proof below makes no use of any properties specific to the slice filtration. The result holds for any equivariant filtration with suitable formal properties.

**Proof.** First we make a formal observation. Suppose we have a commutative diagram

\[
\begin{array}{cccc}
A_{0,0} & \xrightarrow{a_{0,0}} & A_{0,1} & \xrightarrow{a_{0,1}} A_{0,2} & \xrightarrow{a_{0,2}} \Sigma A_{0,0} \\
\downarrow{b_{0,0}} & & \downarrow{b_{0,1}} & & \downarrow{b_{0,2}} & & \downarrow{b_{0,0}} \\
A_{1,0} & \xrightarrow{a_{1,0}} & A_{1,1} & \xrightarrow{a_{1,1}} A_{1,2} & \xrightarrow{a_{1,2}} \Sigma A_{1,0} \\
\downarrow{b_{1,0}} & & \downarrow{b_{1,1}} & & \downarrow{b_{1,2}} & & \downarrow{b_{1,0}} \\
A_{2,0} & \xrightarrow{a_{2,0}} & A_{2,1} & \xrightarrow{a_{2,1}} A_{2,2} & \xrightarrow{a_{2,2}} \Sigma A_{2,0} \\
\downarrow{b_{2,0}} & & \downarrow{b_{2,1}} & & \downarrow{b_{2,2}} & & \downarrow{b_{2,0}} \\
\Sigma A_{0,0} & \xrightarrow{a_{0,0}} & \Sigma A_{0,1} & \xrightarrow{a_{0,1}} \Sigma A_{0,2} & \xrightarrow{a_{0,2}} \Sigma^2 A_{0,0}
\end{array}
\]

in which each row and column is a cofiber sequence. Then suppose that from some spectrum \(W\) we have hypothetical maps \(f_i\) making the following diagram commute, where \(c_{i,j} = b_{i,j+1} + a_{i,j} = a_{i+1,j} b_{i,j}\).

\[
\begin{array}{cccc}
W & & & \text{f}_3 \\
\downarrow{\text{f}_1} & & & \downarrow{\text{f}_2} \\
A_{2,1} & \xrightarrow{a_{2,1}} & A_{2,2} & \xrightarrow{a_{2,2}} \Sigma A_{2,0} \\
\downarrow{b_{2,1}} & & \downarrow{b_{2,2}} & & \downarrow{b_{2,0}} \\
\Sigma A_{0,0} & \xrightarrow{a_{0,0}} & \Sigma A_{0,1} & \xrightarrow{a_{0,1}} \Sigma A_{0,2} \\
\downarrow{c_{0,0}} & & \downarrow{b_{0,1}} & & \downarrow{b_{0,2}} \\
\Sigma A_{1,1} & \xrightarrow{a_{1,1}} \Sigma A_{1,2}
\end{array}
\]

(3.4)

Then the existence of any two of the three maps \(f_i\) implies that of the third. In particular, if one exists then the existences of the other two are equivalent to each other. Moreover when all three exist, \(c_{0,0} f_3\) is null homotopic and we have Toda brackets

\[
(a_{1,1}, c_{0,0}, f_3) \ni f_2 \quad \text{and} \quad (b_{1,1}, c_{0,0}, f_3) \ni f_1.
\]

Next observe that for a \(G\)-spectrum \(X\) and integers \(a < b < c \leq \infty\) there is a cofiber sequence

\[
P^c_{b+1}X \xrightarrow{i} P^c_a X \xrightarrow{j} P^b_a X \xrightarrow{k} \Sigma P^c_{b+1}X.
\]

When \(c = \infty\), we omit it from the notation. We will combine this and the one of (3.2) to get a diagram similar to (3.4) with \(W = S^V\) to prove our two statements.

For (i) note that \(x \in E^{s+1}_1 X(G/G)\) is by definition an element in \(\bigoplus_{r>0} P^s_r X(G/G)\). We will assume for simplicity that \(s = 0\), so \(x\) is represented by a map from some \(S^V\) to \((P^0_0 X)^G\). Its survival to \(E_r\) and supporting a nontrivial differential means that it lifts to \((P^{r-2}_0 X)^G\) but not to \((P^{r-1}_0 X)^G\). The value of \(d_r(x)\) is represented...
by the composite $kx$ in the diagram below.

\[ \begin{array}{ccc}
S^{V-1} & \xrightarrow{y'} & (P_{r-1}X)^{G'} \\
\downarrow w & & \downarrow i \\
\Sigma^{-1}P_0X & \xrightarrow{u_{\alpha^{-1}}res^G_{G'}} & (P_0X)^{G'} \\
\downarrow j & & \downarrow j \\
\Sigma^{-1}P_0^{-2}X & \xrightarrow{a_{\alpha}} & (\Sigma^{-1}P_0^{-2}X)^{G'} \\
\downarrow k & & \downarrow k \\
(P_{r-1}X)^{G'} & \xrightarrow{tr_{G'}} & (P_{r-1}X)^G \\
\end{array} \]

The commutativity of the lower left trapezoid is the differential of (i), $d_r(x) = tr_{G'}^G(y')$. The existence of the map $w$ making the diagram commute follows from that of $x$ and $y'$. It is the representative of $a_{\alpha}x$ as a permanent cycle, which represents the indicated Toda bracket. The commutativity of the upper right trapezoid identifies $y'$ as $u_{\alpha^{-1}}res^G_{G'}(x)$ as claimed. For the converse we have the existence of $y'$ and $w$ and hence that of $x$.

The second statement follows by a similar argument based on the diagram

\[ \begin{array}{ccc}
S^{V+\sigma-1} & \xrightarrow{y} & (P_{r-1}X)^G \\
\downarrow w & & \downarrow i \\
\Sigma^{-1}P_0X & \xrightarrow{tr_{G'}} & (P_0X)^{G'} \\
\downarrow j & & \downarrow j \\
\Sigma^{-1}P_0^{-2}X & \xrightarrow{a_{\alpha}} & (\Sigma^{-1}P_0^{-2}X)^{G'} \\
\downarrow k & & \downarrow k \\
(P_{r-1}X)^G & \xrightarrow{a_{\alpha}} & (\Sigma P_{r-1}X)^G \\
\end{array} \]

Here $w$ represents $u_{\alpha^{-1}}res^G_{G'}(x)$ as a permanent cycle, so we get a Toda bracket containing $res^G_{G'}(x)$ as indicated.

Next we study the way differentials interact with the norm. Suppose we have a subgroup $H \subset G$ and an $H$-equivariant ring spectrum $X$ with $Y = N^G_H X$. Suppose we have spectral sequence $s$ converging to $\pi_*X$ and $\pi_*Y$ based on towers

\[ \ldots \rightarrow P^H_nX \rightarrow P^H_{n-1}X \rightarrow \cdots \text{ and } \ldots \rightarrow P^G_nY \rightarrow P^G_{n-1}Y \rightarrow \cdots \]

for functors $P^H_n$ and $P^G_n$ equipped with suitable maps

\[ P^H_mX \wedge P^H_nX \rightarrow P^H_{m+n}X, \quad P^G_mY \wedge P^G_nX \rightarrow P^G_{m+n}Y \quad \text{and} \quad N^G_H P^H_mX \rightarrow P^G_m[G/H]Y. \]

Our slice $SS$ for each of the spectra studied in this paper fits this description.

**Theorem 3.5. The norm of a differential.** Suppose we have spectral sequences as described above and a differential $d_r(x) = y$ for $x \in E^{r+s}_{\infty}X(H/H)$. Let $\rho = Ind^G_H 1$ and suppose that $a_{\rho}$ has filtration $\lceil G/H \rceil - 1$. Then in the $SS$ for $Y = N^G_H X$,

\[ d_{[G/H](r-1)+1}(a_{\rho}N^G_H x) = N^G_H y \in E^{[G/H](s+1)}_{[G/H](r-1)+1}Y(G/G). \]
Proof. The differential can be represented by a diagram

\[
\begin{array}{ccc}
S^V & \rightarrow & S(1 + V) \\
\downarrow \scriptstyle{\rho} & & \downarrow \scriptstyle{\rho + W} \\
\rightarrow & \rightarrow & \rightarrow \\
SPH_s X & \rightarrow & SPH_s X \\
\downarrow \scriptstyle{\rho} & & \downarrow \scriptstyle{\rho + W} \\
\rightarrow & \rightarrow & \rightarrow \\
P_{s+r}^H X & \rightarrow & P_{s+r}^H X \\
\rightarrow & \rightarrow & \rightarrow \\
\end{array}
\]

for some orthogonal representation \(V\) of \(H\), where each row is a cofiber sequence. We want to apply the norm functor \(N_H^G\) to it. Let \(W = \text{Ind}_H^G V\). Then we get

\[
\begin{array}{ccc}
S^W & \rightarrow & N_H^G S(1 + V) \\
\downarrow \scriptstyle{\rho} & & \downarrow \scriptstyle{\rho + W} \\
\rightarrow & \rightarrow & \rightarrow \\
N_H^G \rho + W & \rightarrow & N_H^G S^{\rho + W} \\
\downarrow \scriptstyle{\rho} & & \downarrow \scriptstyle{\rho + W} \\
\rightarrow & \rightarrow & \rightarrow \\
N_H^G P_{s+r}^H X & \rightarrow & N_H^G P_s^H X \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
N_H^G (P_s^H X / P_{s+r}^H X) & \rightarrow & N_H^G (P_{s+r}^H X / P_{s+r}^H X).
\end{array}
\]

Neither row of this diagram is a cofiber sequence, but we can enlarge it to one where the top and bottom rows are, namely

\[
\begin{array}{ccc}
S^W & \rightarrow & D(1 + W) \\
\downarrow \scriptstyle{\rho} & & \downarrow \scriptstyle{\rho + W} \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
N_H^G P_{s+r}^H X & \rightarrow & N_H^G P_s^H X \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
N_H^G (P_s^H X / P_{s+r}^H X) & \rightarrow & N_H^G (P_{s+r}^H X / P_{s+r}^H X).
\end{array}
\]

Here the first two bottom vertical maps are part of the multiplicative structure the SS is assumed to have. Composing the maps in the three columns gives us the diagram for the desired differential. \(\square\)

Given a \(G\)-equivariant ring spectrum \(X\), let \(X'\) denote its restriction as an \(H\)-spectrum. Then \(N_H^G X' = X'^{(G/H)}\) and the multiplication on \(X\) gives us a map from this smash product to \(X\). This gives us a map \(\pi_*X' \rightarrow \pi_*X\) called the internal norm, which we denote abusively by \(N_H^G\). The argument above yields the following.

Corollary 3.6. The internal norm of a differential. With notation as above, suppose we have a differential \(d_r(x) = y\) for \(x \in E_\epsilon^r X^{(H/H)}\). Then

\[
d_{(G/H)((r-1)+1)}(a_r N_H^G x) = N_H^G y \in E_\epsilon^{(G/H((r-1)+1)} Y(G/H).
\]

4. Some Mackey functors for \(C_4\) and \(C_2\)

We need some notation for Mackey functors to be used in spectral sequence charts. In this paper, when a subgroup appears as an index, we will often replace it by its order. We can specify Mackey functors \(M\) for the group \(C_2\) and \(N\) for \(C_4\) by means of Lewis diagrams (first introduced in \[\text{LewSS}\]),

\[
\begin{array}{c}
M(C_2/C_2) \\
\text{res}^2 \blash \downarrow \uparrow \downarrow \text{tr}_1^2 \\
M(C_2/e) \\
\text{res}^2 \blash \downarrow \uparrow \downarrow \text{tr}_1^2 \\
M(C_2/C_2) \\
\text{res}^2 \blash \downarrow \uparrow \downarrow \text{tr}_1^2 \\
N(C_4/C_4) \\
\text{res}^2 \blash \downarrow \uparrow \downarrow \text{tr}_1^2 \\
N(C_4/C_2) \\
\text{res}^2 \blash \downarrow \uparrow \downarrow \text{tr}_1^2 \\
N(C_4/e).
\end{array}
\]

\(\blash\)
We omit Lewis’ looped arrow indicating the Weyl group action on \( M(G/H) \) for proper subgroups \( H \). This notation is prohibitively cumbersome in SS charts, so we will abbreviate specific examples by more concise symbols. These are shown in Tables 1 and 2. Admittedly some of them are arbitrary and take some getting used to, but we have to do start somewhere. Lewis denotes the fixed point Mackey functor for a \( \mathbb{Z}G \)-module \( M \) by \( R(M) \). He abbreviates \( R(\mathbb{Z}) \) and \( R(\mathbb{Z}^-) \) by \( R \) and \( R^- \). He also defines (with similar abbreviations) the orbit group Mackey functor \( L(M) \) by

\[
L(M)(G/H) = M/H.
\]

In this case each transfer map is the surjection of the orbit space for a smaller subgroup onto that of a larger one. The functors \( R \) and \( L \) are the left and right adjoints of the forgetful functor \( M \mapsto M(G/e) \) from Mackey functors to \( \mathbb{Z}G \)-modules.

Over \( C_2 \) we have short exact sequences

\[
\begin{align*}
0 & \longrightarrow \square \longrightarrow \square \longrightarrow \bullet \longrightarrow 0 \\
0 & \longrightarrow \bullet \longrightarrow \square \longrightarrow \square \longrightarrow 0 \\
0 & \longrightarrow \square \longrightarrow \widehat{\square} \longrightarrow \square \longrightarrow 0
\end{align*}
\]

We can apply the induction functor to each of them to get a short exact sequence of Mackey functors over \( C_4 \).

Five of the Mackey functors in Table 2 are fixed point Mackey functors (2.1), meaning they are fixed points of an underlying \( Z[G] \)-module \( M \), such as \( Z[G] \) or

\[
\begin{align*}
\mathbb{Z}^- = \mathbb{Z}[G]/(\gamma - 1) & \quad \mathbb{Z}[G/G'] = \mathbb{Z}[G]/(\gamma^2 - 1) \\
\mathbb{Z}_- = \mathbb{Z}[G]/(\gamma + 1) & \quad \mathbb{Z}[G/G']_- = \mathbb{Z}[G]/(\gamma^2 + 1)
\end{align*}
\]

We will use the following notational conventions for \( C_4 \)-Mackey functors.

- Given a \( C_2 \)-Mackey functor \( M \) with Lewis diagram

\[
A \\
\alpha \begin{array}{c} \begin{array}{c} \beta \end{array} \\
B
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
\text{Symbol} & \square & \circ & \bullet & \square & \circ & \square \\
\text{Lewis} & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\text{diagram} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\text{Lewis symbol} & R & R^- & \langle \mathbb{Z}/2 \rangle & L & L^- & R(\mathbb{Z}^2)
\end{array}
\]
Table 2. Some $C_4$-Mackey functors, where $G = C_4$ and $G'$ is its index 2 subgroup. The notation $Z(G, H)$ is defined in [2,3;i].

| $\square = \mathbb{Z}$ | $\tilde{\square} = \mathbb{Z}[G/G']$ | $\tilde{\square} = \mathbb{Z}_-$ | $\circ$ | $\tilde{\square} = \mathbb{Z}[G]$ | $\tilde{\square}$ |
|-------------------------|---------------------------------|-------------------|--------|-----------------|-----------------|
| $\mathbb{Z}$            | $\mathbb{Z}$                   | 0                 | $\mathbb{Z}/4$ | $\mathbb{Z}$   | $\mathbb{Z}/2$  |
| $1\{\}^2$              | $\Delta\{\} \vee$             | 1                 | $1\{\}^2$    | $\Delta\{\} \vee$ | $\Delta\{\} \vee$ |
| $\mathbb{Z}$            | $\mathbb{Z}[G/G']$             | $\mathbb{Z}_-$    | $\mathbb{Z}/2$ | $\mathbb{Z}[G/G']$ | $\mathbb{Z}/2[G/G']$ |
| $1\{\}^2$              | $1\{\}^2$                     | $\mathbb{Z}_-$    | 0             | $\mathbb{Z}[G]$ | $\mathbb{Z}[G/G']$ |
| $\mathbb{Z}$            | $\mathbb{Z}[G/G']$             | $\mathbb{Z}_-$    | 0             | $\mathbb{Z}[G/G']$ | $\mathbb{Z}[G/G']$ |
| $\mathbb{Z}/2$          | $\mathbb{Z}$                   | $\mathbb{Z}/2$    | 0             | 0               | 0               |
| 0$\{\}^1$              | $2\{\}^1$                     | 0$\{\}^1$        | 1$\{\}^0$    | 0               | 0               |
| $\mathbb{Z}_-$          | $\mathbb{Z}$                   | $\mathbb{Z}/2$    | 0             | $\mathbb{Z}/2$  | $\mathbb{Z}/2[G/G']$ |
| 1$\{\}^2$              | $2\{\}^1$                     | 0                 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$  | $\mathbb{Z}/2[G/G']$ |
| $\mathbb{Z}_-$          | $\mathbb{Z}$                   | 0                 | 0             | $\mathbb{Z}[G/G']$ | $\mathbb{Z}[G/G']$ |
| $\mathbb{Z}[G/G']$     | $\mathbb{Z}$                   | 0                 | 0             | $\mathbb{Z}[G/G']$ | $\mathbb{Z}[G/G']$ |

$\square = \mathbb{Z}(G, e)$ | $\tilde{\square} = \mathbb{Z}(G, G')$ | $\tilde{\square} = \mathbb{Z}[G/G']$ | $\circ$ | $\tilde{\square} = \mathbb{Z}[G]$ | $\tilde{\square}$ |
|-------------------------|---------------------------------|-------------------|--------|-----------------|-----------------|
| $\mathbb{Z}/2$          | $\mathbb{Z}$                   | $\mathbb{Z}/2$    | 0     | $\mathbb{Z}/2$  | $\mathbb{Z}/2$  |
| 0$\{\}^1$              | $2\{\}^1$                     | $\mathbb{Z}/2$    | 0$\{\}^1$| 1$\{\}^0$    | 0               |
| $\mathbb{Z}/2$          | $\mathbb{Z}$                   | 0$\{\}^1$        | $\mathbb{Z}/2$ | 0               | 0               |
| 0$\{\}^1$              | $\mathbb{Z}_-$                 | $\mathbb{Z}/2$    | 0$\{\}^1$| 0               | 0               |
| $\mathbb{Z}[G/G']$     | $\mathbb{Z}_-$                 | 0                 | $\mathbb{Z}/2$ | $\mathbb{Z}/2$  | $\mathbb{Z}/2[G/G']$ |

$\square = \mathbb{Z}_-$ | $\tilde{\square} = \mathbb{Z}_-$ | $\tilde{\square} = \mathbb{Z}_-$ | $\circ$ | $\tilde{\square} = \mathbb{Z}_-$ | $\tilde{\square}$ |
|-------------------------|---------------------------------|-------------------|--------|-----------------|-----------------|
| 0$\{\}^1$              | $\mathbb{Z}$                   | $\mathbb{Z}$      | $\mathbb{Z}/2$ | $\mathbb{Z}/4$  | $\mathbb{Z}/2$  |
| $\mathbb{Z}_-$          | $\mathbb{Z}[G/G']$             | $\mathbb{Z}_-$    | $\mathbb{Z}/2$ | $\mathbb{Z}/2$  | $\mathbb{Z}/2[G/G']$ |
| 2$\{\}^1$              | $2\{\}^1$                     | $\mathbb{Z}_-$    | 0$\{\}^1$    | 0               | 0               |
| $\mathbb{Z}_-$          | $\mathbb{Z}[G/G']$             | $\mathbb{Z}_-$    | 0$\{\}^1$    | 0               | 0               |
| $\mathbb{Z}/2$          | $\mathbb{Z}/2 \oplus \mathbb{Z}_-$ | $\mathbb{Z}/2 \oplus \mathbb{Z}_-$ | $\mathbb{Z}_-$ | $\mathbb{Z}_4$ | $\mathbb{Z}/4$ |
| $[1 \ 0]$               | $[1 \ 0]$                     | $[1 \ 0]$        | $[1 \ 0]$    | $[2 \ 2]$      | $[0 \ 2]$      |
| $\mathbb{Z}/2 \oplus \mathbb{Z}_{-}$ | $\mathbb{Z}/2 \oplus \mathbb{Z}_{-}$ | $\mathbb{Z}_-$ | $\mathbb{Z}_-$ | $\mathbb{Z}_-$ | $\mathbb{Z}_-$ |
| $[0 \ 2]$               | $[0 \ 2]$                     | $[0 \ 2]$        | $[0 \ 2]$    | $[0 \ 2]$      | $[0 \ 2]$      |
with $A$ and $B$ cyclic, we will use the symbols $\underline{M}$, $\underline{M}$ and $\dot{M}$ for the $C_4$-Mackey functors with Lewis diagrams

\[
\begin{array}{ccc}
A & 0 & \text{and} & \mathbb{Z}/2 \\
\alpha \downarrow \beta & 0 \downarrow \tau & & \\
B & A_+ & & \\
1 \downarrow 2 & \alpha \downarrow \beta & \alpha \downarrow \beta & \\
B_- & & & B_-
\end{array}
\]

where a generator $\gamma \in C_4$ acts via multiplication by $-1$ on $A$ and $B$ in the second two, and the transfer $\tau$ is nontrivial.

• For a $C_2$-Mackey functor $\underline{M}$ we will denote $\uparrow C_2^2 \underline{M}$ (see 1.5) by $\widehat{\underline{M}}$. For a Mackey functor $\underline{M}$ defined over the trivial group, we will denote $\uparrow C_2^1 \underline{M}$ and $\uparrow C_2^1 \underline{M}$ by $\hat{\underline{M}}$ and $\breve{\underline{M}}$.

Over $C_4$, in addition to the short exact sequences induced up from $C_2$, we have

\[
\begin{array}{ccccccc}
0 & \rightarrow & \bullet & \rightarrow & \square & \rightarrow & 0 \\
0 & \rightarrow & \nabla & \rightarrow & \circ & \rightarrow & \bullet & \rightarrow & 0 \\
0 & \rightarrow & \nabla & \rightarrow & \square & \rightarrow & \hat{\square} & \rightarrow & 0 \\
0 & \rightarrow & \bullet & \rightarrow & \circ & \rightarrow & \blacktriangle & \rightarrow & 0 \\
0 & \rightarrow & \square & \rightarrow & \diamond & \rightarrow & \circ & \rightarrow & 0 \\
0 & \rightarrow & \nabla & \rightarrow & \square & \rightarrow & \bullet & \rightarrow & 0
\end{array}
\]

(4.2)

**Definition 4.3. A $C_4$-enriched $C_2$-Mackey functor.** For a $C_2$-Mackey functor $\underline{M}$ as above, $\underline{M}$ will denote the $C_2$-Mackey functor enriched over $\mathbb{Z}[C_4]$ defined by

\[
\underline{M}(S) = \mathbb{Z}[C_4] \otimes_{\mathbb{Z}[C_2]} \underline{M}(S)
\]

for a finite $C_2$-set $S$. Equivalently, in the notation of Definition 1.5, $\underline{M} = \downarrow C_2^1 \uparrow C_2^1 \underline{M}$.

5. Some chain complexes of Mackey functors

As noted above, a $G$-CW complex $X$, meaning one built out of cells of the form $G_+ \wedge e^n$, has a reduced cellular chain complex of $\mathbb{Z}[G]$-modules $C_* X$, leading to a chain complex of fixed point Mackey functors (see (2.1)) $C_* X$. When $X = S^V$ for a representation $V$, we will denote this complex by $C_*^V$; see (2.2). Its homology is the graded Mackey functor $H_* X$. Here we will apply the methods of (2) to three examples.

(i) Let $G = C_2$ with generator $\gamma$, and $X = S^{n \rho}$ for $n > 0$, where $\rho$ denotes the regular representation. We have seen before [HHRa, Ex. 3.7] that it has a reduced cellular chain complex $C$ with

\[
C_i^{n,\rho} = \begin{cases} 
\mathbb{Z}[G]/(\gamma - 1) & \text{for } i = n \\
\mathbb{Z}[G] & \text{for } n < i \leq 2n \\
0 & \text{otherwise}
\end{cases}
\]

(5.1)
Let $c_i^{(n)}$ denote a generator of $C_i^{n\rho_2}$. The boundary operator $d$ is given by

$$d(c_i^{(n+1)}) = \begin{cases} c_i^{(n)} & \text{for } i = n \\ \gamma_{i+1-n}(c_i^{(n)}) & \text{for } n < i \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

where $\gamma_i = 1 - (-1)^i$. For future reference, let

$$\epsilon_i = 1 - (-1)^i = \begin{cases} 0 & \text{for } i \text{ even} \\ 2 & \text{for } i \text{ odd} \end{cases}$$

This chain complex has the form

$$\begin{array}{cccccccc}
\bigcirc & \rho & \bigcirc & \gamma_2 & \bigcirc & \gamma_3 & \bigcirc & \cdots & \gamma_n & \bigcirc \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
Z & Z & Z & Z & \cdots & Z \\
\gamma_2 & \Delta & \gamma_3 & \Delta & \gamma_4 & \Delta & \cdots & \gamma_n & \Delta \\
Z & Z/G & Z/G & Z/G & \cdots & Z/G \\
\Delta & \Delta & \Delta & \Delta & \cdots & \Delta \\
Z/G & Z/G & Z/G & Z/G & \cdots & Z/G \\
\epsilon_1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_n \\
\end{array}$$

Passing to homology we get

$$\begin{array}{cccccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \cdots & \bigcirc \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \cdots & \mathbb{Z}/2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\end{array}$$

$$H_{2n}$$

$$H_{2n}(G/G) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$H_{2n} = \begin{cases} \bigcirc & \text{for } n \text{ even} \\ \bigcirc & \text{for } n \text{ odd} \end{cases}$$

where $\mathbb{Z}$ and $\mathbb{Z}/2$ are fixed point Mackey functors but $\bullet$ is not.

Similar calculations can be made for $S_i^{n\rho_2}$ for $n < 0$. The results are indicated in Figure 2. This is originally due to unpublished work of Stong and is reported in [Lew88, Theorem 2.1 and Table 2.2]. This information will be used in §7.

In other words the $RO(G)$-graded Mackey functor valued homotopy of $HZ$ is as follows. For $n \geq -1$ we have

$$\pi_* \Sigma^{n\rho_2} HZ = \pi_{-n\rho_2} HZ = \begin{cases} \bigcirc & \text{for } n \text{ even and } i = 2n \\ \bigcirc & \text{for } n \text{ odd and } i = 2n \\ \bullet & \text{for } n \leq i < 2n \text{ and } i + n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

For $n \leq -2$ we have

$$\pi_* \Sigma^{n\rho_2} HZ = \pi_{-n\rho_2} HZ = \begin{cases} \bigcirc & \text{for } n \text{ even and } i = 2n \\ \bigcirc & \text{for } n \text{ odd and } i = 2n \\ \bullet & \text{for } 2n < i \leq n - 3 \text{ and } i + n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

We can use Definition 2.4 to name some elements of these groups.
Figure 2. The (collapsing) Mackey functor slice spectral sequence for $\bigvee_{n \in \mathbb{Z}} \Sigma^n \mathbb{Z}$. The symbols are defined in Table 1. When the Mackey functor $\mathbb{M}(2-\rho_2)n-s \mathbb{H} \mathbb{Z} = \mathbb{H}(2n-s)\Sigma^{s+n\rho_2}$ is nontrivial, it is shown at $(2n-s, s)$ in the chart. Compare with Figure 7.

Note that $\mathbb{H} \mathbb{Z}$ is a commutative ring spectrum, so there is a commutative multiplication in $\pi_\ast \mathbb{H} \mathbb{Z}$, making it a commutative $RO(G)$-graded Green functor. For such a functor $M$ on a general group $G$, the restriction maps are a ring homomorphisms while the transfer maps satisfy the Frobenius relations (1.3).

Then the generators of various groups in $\pi_\ast \mathbb{H} \mathbb{Z}$ are

$$(4m-2)\text{-slices for } m > 0 :$$

$$a^{2m-1-2i}u^i = a_{(2m-1-2i), s}u_{2i}$$

$$\in \pi_{2m-1+2i}(2m-1)\Sigma^{2m-1} \mathbb{H} \mathbb{Z}(G/G)$$

$$= \pi_{2i-(2m-1)s} \mathbb{H} \mathbb{Z}(G/G)$$

for $0 \leq i < m$

$$x^{2m-1} = u_{(2m-1)s} \in \pi_{4m-2}(2m-1)\Sigma^{2m-1} \mathbb{H} \mathbb{Z}(G/\{e\})$$

$$= \pi_{(2m-1)(1-s)} \mathbb{H} \mathbb{Z}(G/\{e\})$$

with $\gamma(x) = -x$

$4m\text{-slices for } m > 0 :$

$$a^{2m-2i}u^i = a_{(2m-2i), s}u_{2i}$$

$$\in \pi_{2m-1+2i}(2m-1)\Sigma^{2m-1} \mathbb{H} \mathbb{Z}(G/G)$$

$$= \pi_{2i-(2m-1)s} \mathbb{H} \mathbb{Z}(G/G)$$

for $0 \leq i \leq m$

and with $res(u) = x^2$

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$z_n = e_{2n\rho_2} \in \pi_{-4n} \Sigma^{-2n\rho_2} H\mathbb{Z}(G/\{e\})$
$= \pi_{2n(\sigma-1)} H\mathbb{Z}(G/\{e\})$ for $n > 0$

$a^{-i} tr(x^{-2n-1}) \in \pi_{-4n-2-i} \Sigma^{-(2n+1+i)\rho_2} H\mathbb{Z}(G/G)$
$= \pi_{(2n+1)(\sigma-1)+\sigma} H\mathbb{Z}(G/G)$
for $n > 0$ and $i \geq 0$.

We have relations

$2a = 0 \quad res(a) = 0$

$z_n = x^{-2n} \quad tr(x^n) = \begin{cases} 2n^{n/2} & \text{for } n \text{ even and } n \geq 0 \\ tr(z_{-n/2}) & \text{for } n \text{ even and } n < 0 \\ 0 & \text{for } n \text{ odd and } n > -3.\end{cases}$

(ii) Let $G = C_4$ with generator $\gamma$, $G' = C_2 \subseteq G$, the subgroup generated by $\gamma^2$, and $\hat{S}(n, G') = G_+ \wedge S^{n\rho_2}$. Thus we have

$C_*(\hat{S}(n, G')) = \mathbb{Z}[G] \otimes \mathbb{Z}[G'] C_+^{n\rho_2}$

with $C_+^{n\rho_2}$ as in (5.1). The calculations of the previous example carry over verbatim by the exactness of Mackey functor induction of Definition 1.5.

(iii) Let $G = C_4$ and $X = S^{n\rho_4}$. Then the reduced cellular chain complex (2.2) has the form

$C_i^{n\rho_4} = \begin{cases} \mathbb{Z} & \text{for } i = n \\ \mathbb{Z}[G/G'] & \text{for } n < i \leq 2n \\ \mathbb{Z}[G] & \text{for } 2n < i \leq 4n \\ 0 & \text{otherwise} \end{cases}$

in which generators $c_i^{(n)} \in C_i^{n\rho_4}$ satisfy

$d(c_i^{(n)}) = \begin{cases} c_i^{(n)} & \text{for } i = n \\ \gamma_{i+1-n} c_i^{(n)} & \text{for } n < i \leq 2n \\ \mu_{i+1-n} c_i^{(n)} & \text{for } 2n < i < 4n \text{ and } i \text{ even} \\ \gamma_{i+1-n} c_i^{(n)} & \text{for } 2n < i < 4n \text{ and } i \text{ odd} \\ 0 & \text{otherwise} \end{cases}$

where

$\mu_i = \gamma_i(1 + \gamma^2) = (1 - (-1)^i \gamma)(1 + \gamma^2).$

The values of $H_* S^{n\rho_4}$ are illustrated in Figure 3. The Mackey functors in filtration 0 (the horizontal axis) are the ones described in Proposition 2.3.

As in (i), we name some of these elements. Let $G = C_4$ and $G' = C_2 \subseteq G$. Recall that the regular representation $\rho_2$ is $1 + \sigma + \lambda$ where $\sigma$ is the sign representation and $\lambda$ is the 2-dimensional representation given by a rotation of order 4.

Note that while Figure 2 shows all of $\pi_* H\mathbb{Z}$ for $G = C_2$, Figure 3 shows only a bigraded portion of this trigraded Mackey functor for $G = C_4$, namely the groups

$\pi_{-4n} \Sigma^{-2n\rho_2} H\mathbb{Z}(G/\{e\})$ for $n > 0$ and $i \geq 0$.
Figure 3. The Mackey functor slice spectral sequence for 
\( \bigvee_{n \in \mathbb{Z}} \Sigma^{\rho_4} H\mathbb{Z} \). The symbols are defined in Table 2. The Mackey functor at position \((4n - s, s)\) is 
\[ \pi_{n(4-\rho_4)} H\mathbb{Z} = H_{4n-s} S^{\rho_4} \]
for which the index differs by an integer from a multiple of \( \rho_4 \). We will need to refer to some elements not shown in the latter chart, namely

\[
\begin{align*}
\mathcal{a}_\sigma &\in H_0 S^\sigma (G/G) & a_\lambda &\in H_0 S^\lambda (G/G) & \mathcal{a}_\lambda &= \text{res}_2^4 (a_\lambda) \\
\mathcal{u}_{2\sigma} &\in H_2 S^{2\sigma} (G/G) & u_\sigma &\in H_2 S^\sigma (G/G') & \mathcal{u}_\sigma &= \text{res}_1^2 (u_\sigma) \\
\mathcal{u}_\lambda &\in H_2 S^\lambda (G/G) & \mathcal{u}_\lambda &= \text{res}_2^2 (u_\lambda) & \mathcal{u}_\lambda &= \text{res}_1^1 (u_\lambda)
\end{align*}
\]

subject to the relations

\[
\begin{align*}
2a_\sigma &= 0 & \text{res}_2^4 (a_\sigma) &= 0 \\
4a_\lambda &= 0 & 2\mathcal{a}_\lambda &= 0 & \text{res}_1^2 (a_\lambda) &= 0 \\
\text{res}_1^1 (u_{2\sigma}) &= u_\sigma^2 & a_\sigma^2 u_\lambda &= 2a_\lambda u_{2\sigma} & \text{(gold relation)}
\end{align*}
\]

see Definition 2.4 and Lemma 2.6.

We will denote the generator of \( E_2^{s,t} (G/H) \) (when it is nontrivial) by \( x_{l-s,s}, y_{l-s,s} \) and \( z_{l-s,s} \) for \( H = G, G' \) and \( \{e\} \) respectively. Then the generators for the groups in the 4-slice are

\[
\begin{align*}
y_{4,0} &= u_{\rho_4} = u_\sigma \text{res}_2^4 (u_\lambda) \in \pi_4 \Sigma^{\rho_4} H\mathbb{Z}(G/G') = \pi_{3-\sigma-\lambda} H\mathbb{Z}(G/G') \\
&\text{with } \gamma(x_{4,0}) = -x_{4,0} \\
x_{3,1} &= a_\sigma u_\lambda \in \pi_3 \Sigma^{\rho_4} H\mathbb{Z}(G/G) = \pi_{2-\sigma-\lambda} H\mathbb{Z}(G/G) \\
y_{2,2} &= \text{res}_2^4 (a_\lambda) u_\sigma \in \pi_2 \Sigma^{\rho_4} H\mathbb{Z}(G/G') = \pi_{1-\sigma-\lambda} H\mathbb{Z}(G/G') \\
x_{1,3} &= a_{\rho_4} = a_\sigma a_\lambda \in \pi_1 \Sigma^{\rho_4} H\mathbb{Z}(G/G) = \pi_{-\sigma-\lambda} H\mathbb{Z}(G/G)
\end{align*}
\]
and the ones for the 8-slice are
\[ x_{8,0} = u_{2\lambda+2\sigma} = u_{2\rho_4} \in \Sigma^{2\rho_4}H\mathbb{Z}(G/G) = \Sigma^{6-2\sigma-2\lambda}H\mathbb{Z}(G/G) \]
with \( y_{8,0}^2 = y_{8,0} = \text{res}_1^4(x_{8,0}) \)
\[ x_{6,2} = a_{\lambda}u_{\lambda+2\sigma} \in \Sigma^{6\rho_4}H\mathbb{Z}(G/G) = \Sigma^{1-2\sigma-2\lambda}H\mathbb{Z}(G/G) \]
with \( x_{3,1}^2 = 2x_{6,2} \)
and \( y_{4,0}y_{2,2} = y_{6,2} = \text{res}_2^4(x_{6,2}) \)
\[ x_{4,4} = a_{\lambda}u_{2\sigma} \in \Pi^{4\rho_4}H\mathbb{Z}(G/G) = \Sigma^{3-2\sigma-2\lambda}H\mathbb{Z}(G/G) \]
with \( y_{2,2}^2 = y_{4,4} = \text{res}_1^4(x_{4,4}) \)
and \( x_{1,3}x_{3,1} = 2x_{4,4} \)
\[ x_{2,6} = x_{1,3}^2 \in \Sigma^{2\rho_4}H\mathbb{Z}(G/G) = \Sigma^{3-2\sigma-2\lambda}H\mathbb{Z}(G/G) \].

These elements and their restrictions generate \( \mathbb{Z}^{\Sigma^{m\rho_4}}H\mathbb{Z} \) for \( m = 1 \) and \( 2 \). For \( m > 2 \) the groups are generated by products of these elements.

The element
\[ z_{4,0} = \text{res}_2^4(y_{4,0}) = \text{res}_2^4(u_{\rho_4}) \in \Pi^{4\rho_4}H\mathbb{Z}(G/\{e\}) \]
is invertible with \( \gamma(y_{4,0}) = -y_{4,0}, z_{4,0}^2 = z_{8,0} = \text{res}_1^4(x_{8,0}) \) and
\[ z_{-4m,0} := z_{4,0}^{-m} = e_{m\rho_4} \in \Pi^{-4m\rho_4}H\mathbb{Z}(G/\{e\}) \]
for \( m > 0 \), where \( e_{m\rho_4} \) is as in Definition 2.4. These elements and their transfers generate the groups in
\[ \Pi^{-4m\Sigma^{-m\rho_4}}H\mathbb{Z} \]
for \( m > 0 \).

**Theorem 5.5.** Divisibilities in the negative regular slices for \( C_4 \). There are the following infinite divisibilities in the third quadrant of the spectral sequence in Figure 3:

1. \( x_{-4,0} = tr_1^4(z_{-4,0}) \) is divisible by any monomial in \( x_{1,3} \) and \( x_{4,4} \), meaning that
\[ x_{1,3}^i x_{4,4}^j x_{-4-4j-i-4j-3k} = x_{-4,0} \quad \text{for } i, j \geq 0. \]
Moreover, no other basis element killed by \( x_{3,1} \) and \( x_{4,4} \) has this property.

2. \( x_{-4,0} \) and \( x_{-7,-1} \) are divisible by any monomial in \( x_{4,4}, x_{6,2} \) and \( x_{8,0} \), subject to the relation \( x_{6,2}^2 = x_{8,0}x_{4,4} \). Note here that \( x_{2,3,1} = 2x_{6,2} \).
Moreover, no other basis element killed by \( x_{4,4}, x_{6,2} \) and \( x_{8,0} \) has this property.

3. \( y_{-7,-1} = \text{res}_2^4(x_{-7,-1}) \) is divisible by any monomial in \( y_{2,2} \) and \( y_{4,0} \), meaning that
\[ y_{2,2}^k y_{4,0}^j y_{-7-2j-4k} = y_{-7,-1} \quad \text{for } j, k \geq 0. \]
Moreover, no other basis element killed by \( y_{2,2} \) and \( y_{4,0} \) has this property.

We will prove Theorem 5.5 as a corollary of a more general statement (Lemma 5.11 and Corollary 5.13) in which we consider all representations of the form \( m\lambda+n\sigma \) for \( m, n \geq 0 \). Let
\[ R = \bigoplus_{m,n \geq 0} H_\ast S^{m\lambda+n\sigma}. \]
It is generated by the elements of Figure 3 subject to the relations of Figure 4.
In the larger ring
\[ \tilde{R} = \bigoplus_{m,n \in \mathbb{Z}} H_* \mathbb{S}^{m\lambda+n\sigma}, \]
the elements \( u_\sigma, \pi_\sigma \) and \( \pi_\lambda \) are invertible with
\[ e_\sigma = u_\sigma^{-1} \in H_{-1} S^{-\sigma}(G/G') \]
\[ e_\lambda = \pi_\lambda^{-1} \in H_{-2} S^{-\lambda}(G/e). \]

Define spectra \( L_m \) and \( K_n \) to be the cofibers of \( a_{m\lambda} \) and \( a_{n\sigma} \). Thus we have cofiber sequences
\[ \Sigma^{-1} L_m \xrightarrow{c_{m\lambda}} S^0 \xrightarrow{a_{m\lambda}} \mathbb{S}^{m\lambda} \xrightarrow{b_{m\lambda}} L_m \]
\[ \Sigma^{-1} K_n \xrightarrow{c_{n\sigma}} S^0 \xrightarrow{a_{n\sigma}} \mathbb{S}^{n\sigma} \xrightarrow{b_{n\sigma}} K_n \]

Dualizing gives
\[ DL_m \xrightarrow{D b_{m\lambda}} S^{-m\lambda} \xrightarrow{D a_{m\lambda}} S^0 \xrightarrow{D c_{m\lambda}} \Sigma DL_m \]
\[ DK_n \xrightarrow{D b_{n\sigma}} S^{-n\sigma} \xrightarrow{D a_{n\sigma}} S^0 \xrightarrow{D c_{n\sigma}} \Sigma DK_n \]
The maps \( D a_{m\lambda} \) and \( D a_{n\sigma} \) are the same as desuspensions of \( a_{m\lambda} \) and \( a_{n\sigma} \), which implies that
\[ DL_m = \Sigma^{-1-m\lambda} L_m \quad \text{and} \quad DK_n = \Sigma^{-1-n\sigma} K_n. \]

Inspection of the cellular chain complexes for \( L_m \) and \( K_n \) and certain of their suspensions reveals that
\[ \Sigma^{2-\lambda} L_m \wedge H\mathbb{Z} = L_m \wedge H\mathbb{Z} = \Sigma^{2-2\sigma} L_m \wedge H\mathbb{Z} \]
and
\[ \Sigma^{2-2\sigma} K_n \wedge H\mathbb{Z} = K_n \wedge H\mathbb{Z}, \]
while \( \Sigma^{1-\sigma} \) alters both \( L_m \wedge H\mathbb{Z} \) and \( K_n \wedge H\mathbb{Z} \). We will denote \( \Sigma^{k(1-\sigma)} L_m \wedge H\mathbb{Z} \) by \( L_m^{(k)} \wedge H\mathbb{Z} \) and similarly for \( K_n \).

The homology groups of \( L_m^\pm \) and \( K_n^\pm \) for \( m,n > 0 \) are indicated in Figures 4 and 5, and those for \( \mathbb{S}^{m\lambda} \) and \( \mathbb{S}^{n\sigma} \) are shown in Figure 6.

In the following diagrams we will use the same notation for a map and its smash product with any identity map. Let \( V = m\lambda+n\sigma \) with \( m,n > 0 \), and let \( R_V \) denote

\[ \begin{array}{cccccccccccc}
1 & 2 & 4 & 6 & 8 & 10 & 12 & 0 & 2 & 4 & 6 & 8 \\
1 & 2 & 4 & 6 & 8 & 10 & 12 & 0 & 2 & 4 & 6 & 8 \\
\end{array} \]

**Figure 4.** Charts for \( H_i L_m^\pm \). The horizontal coordinate is \( i \) and the vertical one is \( m \). \( L_m \) is on the left and \( L_m^- \) is on the right.
the fiber of $a_V$. Since $a_V$ is self-dual up to suspension, we have $DR_V = \Sigma^{-1-V} R_V$.

In the following each row and column is a cofiber sequence.

\[
\Sigma^{-1} K_n \xrightarrow{c_{n\sigma}} S^0 \xrightarrow{a_{n\sigma}} S^{n\sigma} \xrightarrow{b_{n\sigma}} K_n \\
\Sigma^{-1} R_V \xrightarrow{c_{V}} S^0 \xrightarrow{a_{V}} S^V \xrightarrow{b_{V}} R_V \\
\Sigma^{n\sigma} L_m \xrightarrow{c_{m\lambda}} S^{m\lambda} \xrightarrow{a_{m\lambda}} S^V \xrightarrow{b_{m\lambda}} \Sigma^{n\sigma} L_m
\]

The homology sequence for the third column is the easiest way to compute $H_* S^V$.

That column is

\[
\Sigma^{n\sigma-1} L_m \xrightarrow{c_{m\lambda}} S^{n\sigma} \xrightarrow{a_{m\lambda}} S^V \xrightarrow{b_{m\lambda}} \Sigma^{n\sigma} L_m,
\]
which dualizes to
\[
\Sigma^{1-n} DL_m \xrightarrow{c_{m\lambda}} S^{-n} \xrightarrow{a_{m\lambda}} S^{-V} \xrightarrow{c_{m\lambda}} \Sigma^{-n} DL_m \xrightarrow{\cong} \Sigma^{-1-V} L_m.
\]

or
\[
(5.8) \quad \Sigma^{-1-V} L_m \xrightarrow{c_{m\lambda}} S^{-V} \xrightarrow{a_{m\lambda}} S^{-n} \xrightarrow{b_{m\lambda}} \Sigma^{-V} L_m.
\]

For \(5.7\), the long exact sequence in homology includes
\[
H_{i+1-n} L_m(-1)^n \xrightarrow{c_{m\lambda}} H_i S^n \xrightarrow{a_{m\lambda}} H_i S^V \xrightarrow{b_{m\lambda}} H_{i-1-n} L_m(-1)^n \xrightarrow{c_{m\lambda}} H_{i-1} S^n.
\]

**Divisibility by** \(a_\lambda\). Multiplication by \(a_\lambda\) leads to
\[
H_{i+1-n} L_m(-1)^n \xrightarrow{c_{m\lambda}} H_i S^n \xrightarrow{a_{m\lambda}} H_i S^V \xrightarrow{b_{m\lambda}} H_{i-1-n} L_m(-1)^n \xrightarrow{c_{m\lambda}} H_{i-1} S^n.
\]

where \(m' = m + 1\) and \(a'_\lambda\) is induced by the inclusion \(L_m \to L_{m'}\).

In the dual case we get
\[
(5.9) \quad H_{i+1} S^{-n} \xrightarrow{b} H_{i+1+|V|} L_m(-1)^n \xrightarrow{c} H_i S^{-V} \xrightarrow{a} H_i S^{-n} \xrightarrow{b} H_{i+1+|V|} L_m(-1)^n \xrightarrow{Da'_\lambda}.
\]

Here the subscripts on the horizontal maps (\(m\lambda\) in the top row and \(m'\lambda\) in the bottom row) have been omitted to save space. The five lemma implies that the middle vertical map is onto when the left hand \(Da'_\lambda\) is onto and the right hand one is one to one. The left version of \(Da'_\lambda\) is onto in every case except \(i = |V|\) and the right version of it is one to one in all cases except \(i = |V|\) and \(i = -1 - |V|\). This is illustrated for small \(m\) in the following diagram in which trivial Mackey functors are indicated by blank spaces.

It follows that the map \(a_\lambda\) in \(5.9\) is onto for all \(i\) except \(-|V|\). This is a divisibility result. Note that \(a_\lambda\) is trivial on \(H_* X(G/e)\) for any \(X\) since \(res^i_1(a_\lambda) = 0\).
Divisibility by $u_\lambda$. For $u_\lambda$ multiplication we use the diagram

$$
\begin{align*}
H_{i+1}S^{-n_\sigma} & \xrightarrow{b} H_{i+1}L_m^{(-1)^n} & \xrightarrow{c} H_iS^{-V} & \xrightarrow{a} H_iS^{-n_\sigma} & \xrightarrow{b} H_iL_m^{(-1)^n} \\
\downarrow u_\lambda & & \downarrow u_\lambda & & \downarrow u_\lambda \\
H_{i-1}S^{-n_\sigma-\lambda} & \xrightarrow{b} H_{i+1}L_m^{(-1)^n} & \xrightarrow{c} H_{i-2}S^{-V-\lambda} & \xrightarrow{a} H_{i-2}S^{-n_\sigma-\lambda} & \xrightarrow{b} H_{i-2}L_m^{(-1)^n}
\end{align*}
$$

The rightmost $u_\lambda$ is onto in all cases except $i = -n$ and $n$ even. This is illustrated for $n = 6$ and 7 in the following diagram.

\[\begin{array}{cccccccc}
 & j & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \\
H_jS^{-6_\sigma} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
H_jS^{-6_\sigma-\lambda} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
H_jS^{-7_\sigma} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
H_jS^{-7_\sigma-\lambda} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}\]

Thus the central $u_\lambda$ in (5.10) fails to be onto only in when $i = -n$ and $n$ is even.

Divisibility by $a_\sigma$. The corresponding diagram is

$$
\begin{align*}
H_{i+1}S^{-n_\sigma} & \xrightarrow{b} H_{i+1+|V|}L_m^{(-1)^n} & \xrightarrow{c} H_iS^{-V} & \xrightarrow{a} H_iS^{-n_\sigma} & \xrightarrow{b} H_i+|V|L_m^{(-1)^n} \\
\downarrow a_\sigma & & \downarrow a_\sigma & & \downarrow a_\sigma \\
H_{i+1}S^{-n_\sigma} & \xrightarrow{b} H_{i+2+|V|}L_m^{(-1)^n} & \xrightarrow{c} H_iS^{-V-\sigma} & \xrightarrow{a} H_iS^{-n_\sigma} & \xrightarrow{b} H_i+1+|V|L_m^{(-1)^n}
\end{align*}
$$

Here we have abbreviated $n + 1$ by $n^\prime$. Since $res_2^1(a_\sigma) = 0$, the map $a_\sigma$ must vanish on $H_\ast X(G/G')$ and $H_\ast X(G/e)$. It can be nontrivial only on $G/G$.

By Lemma 3.1, the image of $a_\sigma$ is the kernel of the restriction map $u_\sigma^{-1}res_2^1$ and the kernel of $u_\sigma$ is the image of the transfer $tr_2^1$. From Figure 6 we see that $res_2^1(a_\sigma)$ kills $H_iS^{-n_\sigma}(G/G)$ except the case $i = -n$ for even $n$. From Figure 4 we see that it kills $H_jL_m(G/G)$ for all $j$ and $H_jL_m(G/G)$ for odd $j > 1$, but not the generators for $j = 1$ nor the ones for even values of $j$ from 2 to 2$m$. The transfer has nontrivial image in $H_jL_m$ only for $j = 1$ and in $H_jL_m$ only for $j = 1$ and for even $j$ from 2 to 2$m$.

It follows that for odd $n$, each element of $H_iS^{-V}(G/G)$ is divisible by $u_\sigma$, except when $i = -|V| = -2m-n$. For even $n$ it is onto except when $i = -n$, $i = -n-2m$, and $i$ odd from $1 - n - 2m$ to $-1 - n$.

Divisibility by $u_{2\sigma}$. For $u_{2\sigma}$ multiplication, the diagram is

$$
\begin{align*}
H_{i+1}S^{-n_\sigma} & \xrightarrow{b} H_{i+1}L_m^{(-1)^n} & \xrightarrow{c} H_iS^{-V} & \xrightarrow{a} H_iS^{-n_\sigma} & \xrightarrow{b} H_iL_m^{(-1)^n} \\
\downarrow u_{2\sigma} & & \downarrow u_{2\sigma} & & \downarrow u_{2\sigma} \\
H_{i-1}S^{-(n+2)_\sigma} & \xrightarrow{b} H_{i+1}L_m^{(-1)^n} & \xrightarrow{c} H_{i-2}S^{-V-2_\sigma} & \xrightarrow{a} H_{i-2}S^{-(n+2)_\sigma} & \xrightarrow{b} H_{i-2}L_m^{(-1)^n}
\end{align*}
$$

The rightmost $u_{2\sigma}$ is onto in all cases, so every element in $H_iS^{-V}$ is divisible by $u_{2\sigma}$.

The arguments above prove the following.

Lemma 5.11. $RO(G)$-graded divisibility. Let $G = C_4$ and $V = m\lambda + n\sigma$ for $m, n \geq 0$. 
(i) Each element in $H_*S^V(G/G)$ or $H_*S^V(G/G')$ is divisible by $a_\lambda$ or $\overline{a_\lambda}$ except when $i = -|V|$.  
(ii) Each element in $H_*S^V(G/H)$ is divisible by a suitable restriction of $u_\lambda$ except when $i = -n$ for even $n$.  
(iii) Each element in $H_*S^V(G/G)$ for odd $n$ is divisible by $u_\sigma$ except when $i = -|V|$.  For even $n$ it is divisible by $u_\sigma$ except when $i = -n$, $i = -|V|$ and $i$ is odd from $i = 1 - |V|$ to $-1 - n$. 
(iv) Each element in $H_*S^V(G/H)$ is divisible by a $u_{2\sigma}$, $u_\sigma$ or $\overline{u_\sigma}$. 

In Theorem 5.5 we are looking for divisibility by 

\[
\begin{align*}
\begin{cases} 
\begin{aligned}
x_{1,3} &= a_\sigma a_\lambda \\
x_{4,4} &= a_\lambda^2 u_{2\sigma} \\
y_{2,2} &= \overline{a_\lambda} u_\sigma \\
x_{6,2} &= a_\lambda u_{2\sigma} u_\lambda \\
x_{8,0} &= u_{2\sigma} u_\lambda^2 \\
y_{4,0} &= u_\sigma \overline{a_\lambda}
\end{aligned}
\end{cases}
\end{align*}
\]

(5.12) 

In view of Lemma 5.11(iv), we can ignore the factors $u_{2\sigma}$ and $u_\sigma$ when analyzing such divisibility.

**Corollary 5.13.** Infinite divisibility by the divisors of (5.12). Let 

\[ V = m\lambda + n\sigma \quad \text{for} \quad m, n \geq 0. \]

Then 

- Each element of $H_*S^V(G/G)$ is infinitely divisible by $x_{1,3} = a_\sigma a_\lambda$ for $i > -n$ when $n$ is even and for $i \geq -n$ when $n$ is odd. 
- Each element of $H_*S^V(G/G)$ is infinitely divisible by $x_{4,4} = a_\lambda^2 u_{2\sigma}$ for $i > -|V|$. 
- Each element of $H_*S^V(G/G')$ is infinitely divisible by $y_{2,2} = \overline{a_\lambda} u_\sigma$ for $i > -|V|$. 
- Each element of $H_*S^V(G/G)$ is infinitely divisible by $x_{6,2} = a_\lambda u_{2\sigma} u_\lambda$ for $i > -|V|$ when $n$ is odd and for $-|V| < i < -n$ when $n$ is even. 
- Each element of $H_*S^V(G/G')$ is infinitely divisible by $x_{8,0} = u_{2\sigma} u_\lambda^2$ for $i < -n$ when $n$ is even and for all $i$ when $n$ is odd. 
- Each element of $H_*S^V(G/G')$ is infinitely divisible by $y_{4,0} = u_\sigma \overline{a_\lambda}$ for $i < -n$ when $n$ is even and for all $i$ when $n$ is odd.

This implies Theorem 5.5.

6. THE SPECTRA $k_R$ AND $k_H$

Before defining our spectrum we need to recall some definitions and formulas from \[\text{HHRa}\]. Let $H \subset G$ be finite groups. In \[\text{HHRa} \]§2.2.3 we define a norm functor $N^G_H$ from the category of $H$-spectra to that of $G$-spectra. Roughly speaking, for an $H$-spectrum $X$, $N^G_H X$ is the $G$-spectrum underlain by the smash power $X_{(i(G/H))}$ with $G$ permuting the factors and $H$ leaving each one invariant. When $G$ is cyclic, we will denote the orders of $G$ and $H$ by $g$ and $h$, and the norm functor by $N^g_h$.

There is a $C_2$-spectrum $MU_R$ underlain by the complex cobordism spectrum $MU$ with group action given by complex conjugation. For a finite cyclic 2-group $G$
we define

$$MU^{((G))} = N_2^g MU_R.$$

Choose a generator $\gamma$ of $G$. In [HHRa, (5.47)] we defined generators

\begin{equation}
\tau_k = \tau_k^G \in \pi^{C_2}_{k,p_2} MU^{((G))}(C_2/C_2) = \pi_{C_2,k,p_2} MU^{((G))}(G/G)
\end{equation}

(note that this group is a module over $G/C_2$) and

$$r_k = \frac{1}{2}(\tau_k) \in \pi_{\{e\},2k} MU^{((G))}(G/G) = \pi_{2k}^{\{e\}} MU^{((G))}(\{e\} / \{e\}) = \pi_{2k}^u MU^{((G))}.$$

The Hurewicz images of the $\tau_k$ (for which we use the same notation) are defined in terms of the coefficients (see Definition 1.6)

$$m_k \in \pi_{k,p_2}^{C_2} H\mathbb{Z}(2) \wedge MU^{((G))}(C_2/C_2) = \pi_{C_2,k,p_2} H\mathbb{Z}(2) \wedge MU^{((G))}(G/G)$$

of the logarithm of the formal group law $F$ associated with the left unit map from $MU$ to $MU^{((G))}$. The formula is

$$\sum_{k \geq 0} \tau_k x^{k+1} = \left( x + \sum_{l > 0} \gamma(l) x^{2l} \right)^{-1} \circ \log F(x)$$

where

$$\log F(x) = x + \sum_{k > 0} m_k x^{k+1}.$$

For small $k$ we have

$$\tau_1 = (1 - \gamma)(m_1)$$
$$\tau_2 = m_2 - 2\gamma(m_1)(1 - \gamma)(m_1)$$
$$\tau_3 = (1 - \gamma)(m_3) - \gamma(m_1)(m_1^2 + 2m_1\gamma(m_1) - 3\gamma(m_1)^2 - 2m_2)$$

Now let $G = C_2$ or $C_4$ and, in the latter case $G' = C_2 \subseteq G$. The generators $\tau_k^{G'}$ are the $\tau_k$ defined above. We also have elements $\tau_k^{G'}$ defined by similar formulas with $\gamma$ replaced by $\gamma^2$; recall that $\gamma^2(m_k) = (-1)^k m_k$. They are the images of similar generators of

$$\pi^{C_2}_{k,p_2} MU^{((G'))}(C_2/C_2) = \pi_{C_2,k,p_2} MU^{((G'))}(G'/G')$$

under the left unit map

$$MU^{((G'))} \to MU^{((G'))} \wedge MU^{((G'))} = MU^{((G))}.$$

Thus we have

$$\tau_1^{G'} = 2m_1$$
$$\tau_2^{G'} = m_2 + 4m_1^2$$
$$\tau_3^{G'} = 2m_3 - 2m_1m_2 - 4m_1^3$$
If we set $r_2 = 0$ and $r_3 = 0$, we get

\[
\begin{align*}
\pi'^{G'}_1 & = (1+\gamma)(\pi_1) \\
\pi'^{G'}_2 & = 3\pi_1\gamma(\pi_1) + \gamma(\pi_1)^2 \\
\pi'^{G'}_3 & = 5\pi_1^2\gamma(\pi_1) + 5\pi_1\gamma(\pi_1)^2 + \gamma(\pi_1)^3 \\
\pi'^{G'}_3(\pi'^{G'}_3) & = -\pi_1\gamma(\pi_1) (5\gamma(\pi_1)^2 - 5\pi_1\gamma(\pi_1) + \pi_1^2) \\
& \quad \left(\gamma(\pi_1)^2 + 5\pi_1\gamma(\pi_1) + 5\pi_1^2\right) \\
& \quad -5\pi_1^2\gamma(\pi_1) + 20\pi_1^3\gamma(\pi_1)^2 - 2\pi_1^3\gamma(\pi_1)^3 \\
& \quad -20\pi_1^4\gamma(\pi_1)^4 - 5\pi_1\gamma(\pi_1)^5
\end{align*}
\]

(6.2)

**Definition 6.3.** $k_R$, $K_R$ $k_H$ and $K_H$. The $C_2$-spectrum $k_R$ (connective real $K$-theory), is the spectrum obtained from $MU_R$ by killing the $r_n$s for $n \geq 2$. Its periodic counterpart $K_R$ is the telescope obtained from $k_R$ by inverting $\gamma_1 \in \pi_{n+2}k_R(C_2/C_2)$.

The $C_4$-spectrum $k_H$ is obtained from $MU^{(C_4)}$ by killing the $r_n$s and their conjugates for $n \geq 2$. Its periodic counterpart $K_H$ is the telescope obtained from $k_H$ by inverting a certain element $D \in \pi_{4p+1}k_H(C_4/C_4)$ defined below in (6.6) and Table 3.

We remark that while $MU^{(C_4)}$ is $MU_R \wedge MU_R$ as a $C_2$-spectrum, $k_H$ is not $k_R \wedge k_R$ as $C_2$-spectrum. The former has torsion free underlying homotopy but the latter does not.

For $G = C_4$ we will often use a (second) subscript $\epsilon$ on elements such as $r_n$ to indicate the action of a generator $\gamma$ of $G = C_4$, so $\gamma(x_{\epsilon}) = x_{1+\epsilon}$ and $x_{2+\epsilon} = \pm x_{\epsilon}$. Then we have

\[
\pi'^{G'}_n k_H = \pi_{n+1}k_H(G/\{\epsilon\}) = \pi_{2+3n}k_H(G/G) = \mathbb{Z}[r_1, \gamma(r_1)] = \mathbb{Z}[r_{1,0}, r_{1,1}]
\]

where $\gamma^2(r_{1,\epsilon}) = -r_{1,\epsilon}$. Here we use $r_{1,\epsilon}$ and $\gamma_{1,\epsilon}$ to denote the images of elements of the same name in the homotopy of $MU^{(G)}$.

\[
\begin{array}{|c|c|c|c|c|}
\hline
s \backslash t & 0 & 1 & 2 & 3 \\
\hline
0 & 2 & \bar{a}_1 & a_\sigma & u_\sigma & \bar{u}_\sigma \\
1 & a_\sigma & a_{\sigma_2} & \eta & \eta' & \theta_1, \theta_2 \\
2 & u_\sigma & u_{\sigma_2} & \bar{u}_\sigma & u_{\sigma_2} & \bar{u}_\sigma \\
\hline
\end{array}
\]

(6.5)

Here the vertical coordinate is $s$, the horizontal coordinate is $|t| - s$. More information about these elements can be found in Table 3 below.

We are using the following notational convention. When $x = tr_2^k(y)$ for some element $y \in \pi_{2k}k_H(G/G')$, we will write $x' = tr_2^k(u_{\sigma}y)$. Examples above include the cases $x = \eta$ and $x = \bar{t}_2$. The primes could be iterated, i.e., we might write $x^{(k)} = tr_2^k(u_{\sigma}^{k}y)$, but this turns out to be unnecessary.
The group action (by $G'$ on $\tau_{1,\epsilon}$, $a_{\sigma_2}$, and $u_{\sigma_2}$, and by $G$ on all the others) fixes each generator but $u_{\sigma}$, and $u_{\sigma_2}$. For them the action is given by
\[
\begin{align*}
    u_{\sigma} &\xrightarrow{\gamma} -u_{\sigma} \quad \text{and} \quad u_{\sigma_2} \xrightarrow{\gamma^2} -u_{\sigma_2}
\end{align*}
\]
by Theorem 1.12. This is compatible with the following $G$-action:
\[
\begin{array}{c}
\begin{array}{c}
r_{1,0} \\
1
\end{array} \\
\end{array}\xrightarrow{\gamma} \begin{array}{c}
\begin{array}{c}
r_{1,1} \\
1
\end{array} \\
\end{array}
\begin{array}{c}
\begin{array}{c}
-r_{1,1} \\
1
\end{array} \\
\end{array}\xrightarrow{\gamma} \begin{array}{c}
\begin{array}{c}
r_{1,0} \\
1
\end{array} \\
\end{array}
\end{array}
\]
where $r_{1,\epsilon} = 2^2(\tau_{1,\epsilon}) \in \pi_{(\epsilon,2)}k_H(G/G)$.

We will see below (Theorem 9.10) that $d_5(u_{2\sigma}) = a_3^2a_1\bar{d}_1$ and $u_{2\sigma}^2$ is a permanent cycle. Since all transfers are killed by $a_{\sigma}$, multiplication (Lemma 3.1), this implies that $u_{2\sigma}x$ is a permanent cycle representing the Toda bracket
\[
u_{2\sigma}x = u_{2\sigma}tr^4(y) = (x, a_{\sigma}, a_{\sigma}^2a_1\bar{d}_1).
\]
This element is $x''$ since in $E_2$ we have (using the Frobenius relation (1.3))
\[
x'' = tr^4(u_{2\sigma}y) = tr^4(res^4(u_{2\sigma})y) = u_{2\sigma}tr^4(y) = u_{2\sigma}x.
\]
Similarly $x''' = u_{2\sigma}x'$. For $k \geq 4$, $x^{(k)} = u_{2\sigma}x^{(k-4)}$ in $\pi_{\epsilon}$ as well as $E_2$.

The Periodicity Theorem [HHRa] Thm. 9.19] states that inverting a class in $\pi_{4p_k}k_H(G/G)$ whose image under $\pi_{2}res^4_2$ is divisible by $\pi_{3,0}^{G'}\pi_{3,1}^{G'}$ (see (6.2) and $\tilde{r}_{1,0}\tilde{r}_{1,1} = \pi_{1,0}\pi_{1,1}$ makes $u_{8\rho_4}$ a permanent cycle. One such class is
\[
D = N^2_2(\bar{d}_1^2)^2\bar{d}_2^2 = u_{2\sigma}^2(res^4_2(u_{2\sigma})y) = u_{2\sigma}tr^4_2(y)
\]
where $\tilde{r}_2 = tr^4(u_{\sigma}^{-1}\pi_{1,0})$ and $\bar{d}_1$ is as in (6.8) below, and $K_H = D^{-1}k_H$. Then we know that $\Sigma^{32}K_H$ is equivalent to $K_H$.

The Slice and Reduction Theorems [HHRa] Thms. 6.1 and 6.5] imply that the 2th slice of $k_H$ is the 2th wedge summand of
\[
\begin{align*}
H\mathbb{Z} \wedge N^2_2 \left( \bigvee_{i \geq 0} S^{i\rho_2} \right).
\end{align*}
\]
It follows that over $G'$ the 2th slice is a wedge of $k+1$ copies of $H\mathbb{Z} \wedge S^{k\rho_2}$. Over $G$ we get the wedge of the appropriate number of copies of $G_+ \wedge H\mathbb{Z} \wedge S^{k\rho_2}$, wedged with a single copy of $H\mathbb{Z} \wedge S^{(k/2)\rho_4}$ for even $k$. This is spelled out in Theorem 8.2 below.

The group $\pi_{4p_k}^Gk_H(G'/\{e\})$ is not in the image of the group action restriction $\pi_2^G$ because $\rho_2$ is not the restriction of a representation of $G$. However, $\pi_2^{k_H}$ is refined (in the sense of [HHRa], Def. 5.28]) by a map from
\[
(6.7) \quad S_{\rho_2} := G_+ \wedge_{\rho_2} S^{\rho_2} \xrightarrow{\pi_{\rho_2}} k_H.
\]
The Reduction Theorem implies that the 2-slice $P^2_2k_H$ is $S_{\rho_2} \wedge H\mathbb{Z}$. We know that
\[
\pi_2(S_{\rho_2} \wedge H\mathbb{Z}) = \mathbb{Z}.
\]
We use the symbols $r_1$ and $\gamma(r_1)$ to denote the generators of the underlying abelian group of $\hat{\pi}(G/\{e\}) = \mathbb{Z}[G/G']$. These elements have trivial fixed point transfers and

$$\hat{\pi}_2(S_{p_2} \wedge H\mathbb{Z})(G/G') = 0.$$  

Table 3 describes some elements in the slice spectral sequence for $k_H$ in low dimensions, which we now discuss.

Given an element in $\pi_{\ast}MU((G))$, we will often use the same symbol to denote its image in $\pi_{\ast}k_H$. For example, in [HHRa, §9.1]

(6.8) $\delta_n \in \pi_{r}^{G}(2^{n}-1)_{\rho_{4}}MU((G)) = \pi_{r}^{G}(2^{n}-1)_{\rho_{4}}MU((G))$ was defined to be the composite

$$S(2^{n}-1)_{\rho_{4}} \xrightarrow{N_{2}^{2}} S(2^{n}-1)_{\rho_{2}} \xrightarrow{N_{2}^{1}r_{2n-1}} N_{2}^{1}MU((G)) \xrightarrow{MU((G))}. $$

We will use the same symbol to denote its image in $\pi_{1}k_{H}(G/G')$. The element $\eta \in \pi_{1}S^{0}$ (coming from the Hopf map $S^{3} \rightarrow S^{2}$) has image $a_{\sigma}r_{1} \in \pi_{1}^{G}k_{R}(G'/G')$. There are two corresponding elements

$$\eta_{\epsilon} \in \pi_{1}^{G}k_{H}(G'/G') $$

for $\epsilon = 0, 1$. We use the same symbol for their preimages under $r_{2}$ in $\pi_{1}^{G}k_{H}(G'/G')$, and there we have

$$\eta_{\epsilon} = a_{\sigma_{2}}r_{1, \epsilon}. $$

We denote by $\eta$ again the image of either under the transfer $tr_{2}^{1}$, so

$$res_{2}^{1}(\eta) = \eta_{0} + \eta_{1}. $$

Its cube is killed by a $d_{3}$ in the slice spectral sequence, as is the sum of any two monomials of degree 3 in the $\eta$. It follows that in $E_{4}$ each such monomial is equal to $r_{0}^{3}$. It has a nontrivial transfer, which we denote by $x_{3}$.

In [HHRa, Def. 5.51] we defined

(6.9) $f_{k} = a_{p_{2}}^{k}N_{2}^{2}(r_{k}) \in \pi_{k}MU((G))((G/G))$

for a finite cyclic 2-group $G$. In particular, $f_{2^{n}-1} = a_{p_{2}}^{2^{n}-1}\delta_{n}$ for $\delta_{n}$ as in (6.8). The slice filtration of $f_{k}$ is $k(g-1)$ and we will see below (Lemma 3.1 and, for $G = C_{4}$, Theorem 9.10) that

(6.10) $tr_{G}^{C}(u_{\sigma}) = a_{\sigma}f_{1}$. 

Note that $u_{\sigma} \in E_{2}^{0,1-\sigma}(G/G')$ since the maximal subgroup for which the sign representation $\sigma$ is oriented is $G'$, on which it restricts to the trivial representation of degree 1. This group depends only on the restriction of the $RO(G)$-grading to $G'$, and the isomorphism extends to differentials as well. This means that $u_{\sigma}$ is a place holder corresponding to the permanent cycle $1 \in E_{2}^{0,0}(G/G')$.

For $G = C_{4}$ (6.10) implies

$$tr_{2}^{4}(u_{\sigma}) = a_{\sigma}f_{1} = a_{\sigma}a_{\lambda}\delta_{1}. $$
For example
\[ tr_2^4(\eta_0^1) = tr_2^4(a_{\sigma_2}^2 \tau_{1,0} \tau_{1,1}) = tr_2^4(u_\sigma res_2^4(a_\lambda \delta_1)) = tr_2^4(a_\sigma a_\lambda \delta_1) = a_\sigma f_1 a_\lambda \delta_1 = f_1^2 \]

The Hopf element \( \nu \in \pi_3 S^0 \) has image
\[ a_\sigma u_\lambda \delta_1 \in \pi_3 k H(G/G), \]
so we also denote the latter by \( \nu \). (We will see below in (9.5) that \( u_\lambda \) is not a permanent cycle, but \( \nu := a_\sigma u_\lambda \) is (9.6).) It has an exotic restriction \( \eta_0^3 \) (filtration jump two), which implies that
\[ 2\nu = tr_2^4(res_2^4(\nu)) = tr_2^4(\eta_0^3) = x_3. \]

One way to see this is to use the Periodicity Theorem to equate \( \pi_3 k H \) with \( \pi_{-29} k H \), which can be shown to be the Mackey functor \( \phi \) in slice filtration \(-32\). Another argument not relying on periodicity is given below in Theorem 9.10.

The exotic restriction on \( \nu \) implies
\[ res_2^4(\nu^3) = \eta_0^6, \]
with filtration jump 4.

**Theorem 6.11. The Hurewicz image.** The elements \( \nu \in \pi_3 k H(G/G), \epsilon \in \pi_3 k H(G/G), \kappa \in \pi_{14} k H(G/G), \) and \( \pi \in \pi_{20} k H(G/G) \) are the elements of the same names in \( \pi_3 S^0 \). The image of the Hopf map \( \eta \in \pi_1 S^0 \) is either \( \eta = tr_2^4(\eta_\epsilon) \) or its sum with \( f_1 \).

We refer the reader to [Rav86, Table A3.3] for more information about these elements.

**Proof.** Suppose we know this for \( \nu \) and \( \pi \). Then \( \Delta_1^{-4} \nu \) is represented by an element of filtration \(-3\) whose product with \( \nu^2 \) is nontrivial. This implies that \( \nu^3 \) has nontrivial image in \( \pi_3 k H(G/G) \). This is a nontrivial multiplicative extension in the first quadrant, but not in the third. The spectral sequence representative of \( \nu^3 \) has filtration 11 instead of 3. We will see later that \( \nu^3 = 2n \) where \( n \) has filtration 1, and \( \nu^3 \) is the transfer of an element in filtration 1.

Since \( \nu^3 = \nu \epsilon \) in \( \pi_3 S^0 \), this implies that \( \eta \) and \( \epsilon \) are both detected and have the images stated in Table 3. It follows that \( \sigma \kappa \) has nontrivial image here. Since \( \kappa^2 = \epsilon \kappa \) in \( \pi_3 S^0 \), \( \kappa \) must also be detected. Its only possible image is the one indicated.

Both \( \nu \) and \( \pi \) have images of order 8 in \( \pi_1 TMF \) and its \( K(2) \) localization. The latter is the homotopy fixed point set of an action of the binary tetrahedral group \( G_{24} \) acting on \( E_2 \). This in turn is a retract of the homotopy fixed point set of the quaternion group \( Q_8 \). A restriction and transfer argument shows that both elements have order at least 4 in the homotopy fixed point set of \( C_4 \subset Q_8 \).

There is an orientation map \( MU \rightarrow E_2 \), which extends to a \( C_2 \)-equivariant map \( MU_R \rightarrow E_2 \). Norming up and multiplying on the right gives us a \( C_4 \)-equivariant map \( N^3_2 MU_R \rightarrow E_2 \). This \( C_4 \)-action on the target is compatible with the \( G_{24} \)-action leading to \( L_{K(2)} TMF \).

The image of \( \eta \in \pi_1 S^0 \) must restrict to \( \eta_0 + \eta_1 \), so modulo the kernel of \( res_2^4 \) it is the element \( tr_2^4(\eta_\kappa) \), which we are calling \( \eta \). The kernel of \( res_2^4 \) is generated by \( f_1 \).
Table 3. Some elements in the slice spectral sequence for $k_H$.

| Element | Description |
|---------|-------------|
| $\tau_{1,c} \in \overline{\pi}_{d,c}^G \iota_{c,c}^{-1} k_H (G'/G') $ | Images from [6.1] defined in [HHRA] (5.47) |
| with $\tau_{1,2} = - \tau_{1,0}$ | |
| $r_{1,c} \in \overline{\pi}_{(c),2} k_H (G/G)$ | $\tau_{c-1}^2 (\tau_{1,c})$, generating $\pi_0^G k_H / torsion = \widehat{\pi}$ |
| $\cong \overline{\pi}_{G;2} k_H (G/\{c\}) \cong \pi_0^G k_H$ | |
| $u_{2\sigma} \in \overline{\pi}_{a,2} \overline{\pi}_{G;2} k_H (G/G)$ with | |
| $d_5 (u_{2\sigma}) = a_3^2 a_\lambda \overline{d}_1$ | Element corresponding to $u_{2\sigma} \in \overline{\pi}_{2-2\sigma} HZ(G/G)$ |
| $2 u_{2\sigma} = \langle 2, a_\sigma, a^2_\lambda a_\lambda \overline{d}_1 \rangle$ | Slice differential of (9.3) |
| $u_{2\sigma} = \langle a_3^2 a_\lambda \overline{d}_1, a_3^2 a_\lambda \overline{d}_1 \rangle$ | |
| $u_{2\sigma} \in \overline{\pi}_{1-2\sigma} k_H (G/G')$ | Isomorphic image of $1 \in \overline{\pi}_0 k_H (G/G') \cong \overline{\pi}_{G';0} k_H (G/G)$ |
| $\cong \overline{\pi}_{G';0} k_H (G/G)$ with | |
| res$_2^4 (u_{2\sigma}) = a_2^2$ | |
| $\gamma (u_{2\sigma}) = - u_{2\sigma}$ | Follows from Theorem 3.3 and $d_5 (u_{2\sigma})$ in [9.3] |
| $tr_2^2 (u_{2\rangle}^{k+1}) = a_\sigma f_1 u_{2\rangle}^{k}$ | |
| (exotic transfer) | |
| $tr_2^2 (u_{2\rangle}^k) = 2 u_{2\sigma}$ | |
| $tr_2^2 (u_{2\rangle}^{k+3}) = 0$ | |
| $u_{\lambda} \in \overline{\pi}_{a,2} \overline{\pi}_{G;2} k_H (G/G)$ with | |
| $2 u_{\lambda} \in \overline{\pi}_{2-\lambda} k_H (G/G)$ | |
| $a_3^2 u_{\lambda} = 0$ | Element corresponding to $u_{\lambda} \in \overline{\pi}_{2-\lambda} HZ(G/G)$ |
| $d_3 (u_{\lambda}) = \eta u_{\lambda} = tr_2^2 (a_3^2 \overline{d}_1) + \overline{d}_1$ | follows from the gold relation, Lemma 2.6vii |
| $d_5 (u_{\lambda}) = \pi a_\lambda \overline{d}_1$ | Slice differential of Theorem 9.10 |
| $d_7 (2 u_{\lambda}) = \eta' a_3^2 \overline{d}_1$ | Slice differential of Theorem 9.10 |
| $d_3 (u_{\lambda}) = \eta u_{\lambda} = tr_2^2 (a_3^2 \overline{d}_1) + \overline{d}_1$ | |
| $4 u_{\lambda} \in \overline{\pi}_{4-2\lambda} k_H (G/G)$ | |
| $2 a_\sigma u_{\lambda} \in \overline{\pi}_{4-\sigma-2\lambda} k_H (G/G)$ | |
| $d_7 (u_{\lambda}) = \langle \eta', \overline{d}_1, a_3^2 \overline{d}_1, a_3^2 \overline{d}_1 \rangle$ | |
| $2 u_{\lambda} \in \overline{\pi}_{8-4\lambda} k_H (G/G)$ | |
| $\overline{\pi}_{\lambda} \in \overline{\pi}_{a,2} \overline{\pi}_{G;2} (G/G')$ with | |
| $d_3 (\overline{\pi}_{\lambda}) = a_3^2 (\overline{\tau}_{1,0} + \overline{d}_1)$ | |
| $2 \overline{\pi}_{\lambda} \in \overline{\pi}_{2-\lambda} \overline{\pi}_{2-\lambda} k_H (G/G')$ | |
| $d_7 (\overline{\pi}_{\lambda}) = a_\lambda \overline{d}_1$ | |
| $2 \overline{\pi}_{\lambda} \in \overline{\pi}_{4-2\lambda} k_H (G/G')$ | |
| $\overline{\pi}_{\lambda} \in \overline{\pi}_{4-4\lambda} k_H (G/G')$ | |
| $u_{\sigma} \in \overline{\pi}_{a} (G;1-\sigma) k_H (G/e)$ with | |
| $res_2^3 (\overline{\pi}_{\lambda}) = u_{\sigma}^2$ | |
| res$_2^2 (d_3 (u_{\lambda}))$ | Isomorphic image of $1 \in \overline{\pi}_0 k_H (G/e)$ |
| $\gamma^2 (u_{\sigma}) = - u_{\sigma}$ and | |
| $tr_2^2 (u_{\sigma}) = a_{\sigma}^2 (\overline{\tau}_{1,0} + \overline{d}_1)$ | |
| (exotic transfer) | |
| $\overline{\pi}_{2,e} \in \overline{\pi}_{a,e} k_H (G/G')$ | $u_{\sigma} \overline{\tau}_{1,e}$ | |

Continued on next page
Table 3. Some elements in the slice spectral sequence for $k\mathbb{H}$, continued.

| Element | Description |
|---------|-------------|
| $d_1 \in \pi_{n-1}^G k_{\mathbb{H}}(G/G)$ with $res_1^3(d_1) = u_\sigma^3 \pi_{1,0}^1 \bar{\pi}_{1,1}$ | Image from [6,8] defined in [HHRa] §9.1 |
| $\bar{t}_2 \in \pi_{n+1}^G k_{\mathbb{H}}(G/G)$ with $res_2^3(\bar{t}_2) = \bar{\pi}_{2,0} + \bar{\pi}_{2,1}$ | $tr_2^3(\bar{\pi}_{2,\epsilon})$ |
| $\bar{t}_2 \in \pi_{n+1}^G k_{\mathbb{H}}(G/G)$ with $res_2^3(\bar{t}_2) = u_\sigma(\bar{\pi}_{2,0} - \bar{\pi}_{2,1})$ | $(-1)^{\epsilon} tr_2^3(u_\sigma \bar{\pi}_{2,\epsilon})$ |
| $D \in \pi_{4p+1}^G k_{\mathbb{H}}(G/G)$, the periodicity element | $\bar{\delta}_1^2(-5 \bar{\pi}_2^2 + 20 \bar{t}_2 \bar{\pi}_1 + 9 \bar{\delta}_1^2)$ |
| $\Sigma_{2,\epsilon} \in E_2^{3,4} k_{\mathbb{H}}(G/G'')$ with $\Sigma_{2,2} = \Sigma_{2,0}$ and $d_3(\Sigma_{2,\epsilon}) = \pi_2^1(h_0 + \eta_1)$ | $(-1)^{\epsilon} u_\rho u_\sigma \bar{\pi}_{2,\epsilon} = (-1)^{\epsilon} \pi_2 \bar{\pi}_1^1,\epsilon$ |
| $T_2 \in E_2^{3,4} k_{\mathbb{H}}(G/G'')$ with $res_3^3(T_2) = \Sigma_{2,0} + \Sigma_{2,1}$ and $d_3(T_2) = \eta_1^2$ | $tr_3^2(\Sigma_{2,\epsilon}) = (-1)^{\epsilon} u_\lambda tr_3^2(\bar{\pi}_1^1,\epsilon)$ |
| $T_4 \in E_2^{3,4} k_{\mathbb{H}}(G/G'')$ with $T_4 = \Delta_1(T_2^2 - 4 \Delta_1)$, $res_3^4(T_4) = (\Sigma_{2,0} - \Sigma_{2,1}) \delta_1$ and $d_3(T_4) = 0$ | $(-1)^{\epsilon} tr_3^2(\Sigma_{2,\epsilon}, \delta_1) = u_2 u_1^2 \bar{t}_2^2 \bar{\pi}_1$ |
| $\bar{\delta}_1 \in E_2^{3,4} k_{\mathbb{H}}(G/G'')$ with $\gamma(\delta_1) = - \delta_1$, $tr_3^2(\bar{\delta}_1) = 0$ and $d_3(\delta_1) = \eta_0 \eta_1 (h_0 + \eta_1)$ | $u_\rho u_\sigma res_3^2(\bar{\delta}_1) = \pi_\lambda \bar{\pi}_{1,0} \bar{\pi}_{1,1}$ |
| $\Delta_1 \in E_2^{3,4} k_{\mathbb{H}}(G/G'')$ with $res_3^3(\Delta_1) = \delta_1^2$, $res_4^4(\Delta_1) = \pi_1^2 \bar{\pi}_{1,1}$ and $d_5(\Delta_1) = \nu \pi_4$ | $u_2 u_1^2 \bar{\pi}_1^2 = u_2 u_1^2 \bar{\pi}_1^2$ |

| Filtration 1 |
|-----------------|
| $a_{\sigma_2} \in \pi_{2g-2} k_{\mathbb{H}}(G/G)$ |
| $\pi_{g}^1 k_{\mathbb{H}}(G'/G')$ with $2a_{\sigma_2} = 0$ | See Definition 2.4 |
| $\eta \in \pi_{g}^1 k_{\mathbb{H}}(G/G')$ with $2\eta_0 = 0$ | $a_{\sigma_2} \bar{\pi}_{1,\epsilon}$ |
| $\eta \in \pi_{g}^1 k_{\mathbb{H}}(G/G')$ with $2\eta_0 = 0$ | $tr_2^3(\eta_\epsilon) = tr_2^3(a_{\sigma_2} \bar{\pi}_{1,0}) = tr_2^3(a_{\sigma_2} \bar{\pi}_{1,1})$ |
| $\eta' \in \pi_{g}^1 k_{\mathbb{H}}(G/G')$ with $2\eta_0 = 0$ | $tr_2^3(\eta_0 a_{\lambda}) = tr_2^3(a_{\sigma_2} u_{\sigma} \bar{\pi}_{1,0}) = tr_2^3(a_{\sigma_2} u_{\sigma} \bar{\pi}_{1,1})$ |
| $\text{Filtration } 1$ | $\bar{\eta}^2 = a_{\sigma_2} \eta_{1,0}^2$ |

Continued on next page
Table 3. Some elements in the slice spectral sequence for $k_\mathbb{H}$, continued.

| Element | Description |
|---------|-------------|
| $\eta_2^2 = 0$ | |
| $\xi \in \pi_{1-3\sigma-\lambda}k_{\mathbb{H}}(G/G)$ with $\text{res}_2^2(\xi) = a_\sigma a_\sigma^{-1} u_{\sigma^{-1}_0} \pi_{1,0}$ | |
| $2\xi = a_\lambda \langle \eta' \rangle, a_\sigma^2, f_1 \rangle$ | |
| $d_0(u_\sigma, u_{\sigma}^2) = \xi u_\lambda^2 \delta_1$ | |
| $\eta \xi = 2a_\lambda u_{\sigma}^2 \delta_1$ (exotic multiplication) | |
| $\eta' \xi = a_\sigma^2 a_\sigma^{-1} u_{\sigma}^2 \delta_1^2$ (exotic multiplication) | $\tau u_{2\sigma} = \langle \eta', a_\sigma^2, f_1 \rangle$ Follows from value of $\text{res}_2^4(\eta')$ Transfer of the above |
| $\nu \in \pi_{x}k_{\mathbb{H}}(G/G)$ with $\text{res}_2^4(\nu) = \eta_0^3$ and $2\nu = x_3$ (exotic restriction and group extension) | $a_\sigma u_\lambda \delta_1 = \nu_0 \delta_1$, generating $\circ = \pi_{x}k_{\mathbb{H}}$ Follows from those on $\tau$ |
| $\eta_2^3, \eta_0 \eta_1 \in \pi_2^G k_{\mathbb{H}}(G/G')$ with $\text{tr}_2^3(\eta_2^3) = (-1)^a a_\lambda \tau_2$ and $\text{tr}_2^3(\eta_0 \eta_1) = f_1^2$ (exotic transfer) | $u_\sigma a_\sigma^2 \pi_2, \epsilon$ and $u_\sigma a_\sigma^2 \text{res}_2^4(\delta_1)$, generating the torsion $\bullet \oplus \bigtriangleup$ in $\pi_2^G k_{\mathbb{H}}$ |
| $\nu' \in \pi_{1,0}k_{\mathbb{H}}(G/G)$ | $2a_\lambda u_\lambda u_{\sigma} \delta_1^2 = (2, \eta, f_1, f_1^2)$ |
| $\kappa \in \pi_{1,3}k_{\mathbb{H}}(G/G)$ | $2a_\lambda u_{\sigma}^2 u_{\lambda}^2 \delta_1^2$ |
| $f_1 \in \pi_{1}k_{\mathbb{H}}(G/G)$ | $a_\sigma a_\lambda \delta_1$, generating the summand $\bullet$ of $\pi_{1}k_{\mathbb{H}}$ |
| $\eta_0^3 = \eta_0 \eta_1 = \eta_0 \eta_1^3 = \eta_1^3 \in \pi_2^G k_{\mathbb{H}}(G/G')$ | $\eta_0 u_\sigma a_\sigma^2 \text{res}_2^4(\delta_1) = \eta_0 u_\sigma a_\sigma^2 \pi_2, \epsilon$ |
| $x_3 \in \pi_{4}k_{\mathbb{H}}(G/G)$ with $\text{res}_2^4(x_3) = 0$ | $\text{tr}_2^3(\eta_0 \eta_1) = \text{tr}_2^3(a_\lambda^3 \pi_2, \epsilon^{-1}) = a_\lambda^3 \eta \delta_1$ |
| $x_4 \in E_2^{5,4}(G/G)$ with $d_5(x_4) = f_1^3$, $\text{res}_2^4(x_4) = (\eta_0 \eta_1)^2 = \eta_0^4$ and $2x_4 = f_1 \nu$ | $a_\sigma a_\sigma^{-1} u_{\sigma}^2 \delta_1^2$ |
| $\pi \in \pi_{4}k_{\mathbb{H}}(G/G)$ | $2\pi = \text{tr}_2^3(u_\sigma \text{res}_2^4(u_{\sigma}^2 u_{\lambda}^2 \delta_1^2))$ (exotic transfer) $a_\lambda u_{\sigma}^2 u_{\lambda}^2 \delta_1^2$ |
| $\epsilon \in \pi_6k_{\mathbb{H}}(G/G)$ | $x_4^2 = (f_1, f_1^3, f_1, f_1^3) \in E_6^{8,4}(G/G)$ |
| $\nu^4 = \eta \in \pi_{9}k_{\mathbb{H}}(G/G)$ | Represents $f_1 x_4^2 \in E_2^{11,20}(G/G)$ |
7. The slice spectral sequence for $K_R$

In this section we describe the slice spectral sequence for $K_R$. These results are originally due to Dugger [Dug05], to which we refer for many of the proofs. This case is far simpler than that of $K_H$, but it is very instructive.

**Theorem 7.1. The slice $E_2$-terms for $K_R$ and $k_R$.** The slices of $K_R$ are

\[ P^t K_R = \begin{cases} \Sigma^{(t/2)\rho^2} H\mathbb{Z} & \text{for } t \text{ even} \\ \ast & \text{otherwise} \end{cases} \]

For $k_R$ they are the same in nonnegative dimensions, and contractible below dimension 0.

Hence we know the homotopy groups of these slices by the results of §5 and they are shown in Figure 2. It shows the $E_2$-term for the wedge of all of the slices of $K_R$, and $K_R$ itself has the same $E_2$-term. It turns out that the differentials and Mackey functor extensions are determined by the fact that $\pi_* K_R$ is 8-periodic, while the $E_2$-term is far from it. This explanation is admittedly circular in that the proof of the Periodicity Theorem itself of [HHRa, §9] relies on the existence of certain differentials described below in (9.2).

**Theorem 7.2. The slice spectral sequence for $K_R$.** The differentials and extensions in the spectral sequence are as indicated in Figure 7.

**Proof.** There are four phenomena we need to establish:

(i) The differentials in the first quadrant, which are indicated by red lines.

(ii) The differentials in the third quadrant.

(iii) The exotic transfers in the first quadrant, which are indicated by blue lines.

(iv) The exotic restrictions in the third quadrant, which are indicated by green lines.

For (i), note that there is a nontrivial element in $E_3^{3,6}(G/G)$, which is part of the 3-stem, but nothing in the $(-5)$-stem. This means the former element must be killed by a differential, and the only possibility is the one indicated. The other differentials in the first quadrant follow from this one and the multiplicative structure.

For (ii), we know know that $\pi_7 K_R = 0$, so the same must be true of $\pi_{-9}$. Hence the element in $E_3^{-3,-12}$ cannot survive, leading to the indicated third quadrant differentials.

For (iii), note that $\pi_2$ and $\pi_{-6}$ must be the same as Mackey functors. This forces the indicated exotic transfers. For each $m \geq 0$ one has a nonsplit short exact sequence of $C_2$ Mackey functors

\[ 0 \longrightarrow E_2^{2,8m+4} \longrightarrow \pi_{8m+2} K_R \longrightarrow E_2^{0,8m+2} \longrightarrow 0 \]

For (iv), note that $\pi_{-8}$ and $\pi_0$ must also agree. This forces the indicated exotic restrictions. For each $m < 0$ one has a nonsplit short exact sequence

\[ 0 \longrightarrow E_2^{8,8m} \longrightarrow \pi_{8m} K_R \longrightarrow E_2^{-2,8m-2} \longrightarrow 0 \]
Figure 7. The slice spectral sequence for $K_R$. Compare with Figure 2. Exotic transfers and restrictions are indicated respectively by solid blue and dashed green lines. Differentials are in red.

8. Slices for $k_H$ and $K_H$

In this section we will identify the slices for $k_H$ and $K_H$ and the generators of their integrally graded homotopy groups. For the latter we will use the notation of Table 3. Let

$$X_{m,n} = \begin{cases} \sum_{\rho_4}^m H\mathbb{Z} & \text{for } m = n \\ G_+ \wedge \sum_{G'}^{(m+n)\rho_2} H\mathbb{Z} & \text{for } m < n. \end{cases}$$

The slices of $k_H$ are certain finite wedges of these, and those of $K_H$ are a certain infinite wedges. Fortunately we can analyze these slices by considering just one value of $m$ at a time, this index being preserved by the first differential $d_3$. These are illustrated below in Figures 8–11. They show both $E_2$ and $E_4$ in four cases depending on the sign and parity of $m$.

**Theorem 8.2. The slice $E_2$-term for $k_H$.** The slices of $k_H$ are

$$P_t^t k_H = \begin{cases} \bigvee_{0 \leq m \leq t/4} X_{m,t/2-m} & \text{for } t \text{ even and } t \geq 0 \\ \ast & \text{otherwise} \end{cases}$$

where $X_{m,n}$ is as in (8.1).

The structure of $\pi^* k_H$ as a $\mathbb{Z}[G]$-module (see (6.4)) leads to four types of orbits and slice summands:

1. $\{(r_1,0)^{2\ell}\}$ leading to $X_{2\ell,2\ell}$ for $\ell \geq 0$; see the leftmost diagonal in Figure 8. On the $0$-line we have a copy of $\ast$ (defined in Table 3) generated under restrictions by $\Delta_1^\ell = u_{2\ell,\rho_4}^\ell = u_{2\ell,0}^\ell u_\lambda^\ell \in E_2^{0,8\ell}(G/G)$. 


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In positive filtrations we have

\[ \circ \subseteq E^{2 \lambda, 8 \ell}_2 \] generated by

\[ a^{\lambda}_1 u^{\lambda}_{2 \lambda} u^{2 \lambda - j \lambda}_{1 \lambda} \delta^{1}_{1 \lambda} \in E^{2 \lambda, 8 \ell}_2 (G/G) \] for $0 < j \leq 2 \lambda$ and

\[ \bullet \subseteq E^{2k + 4 \lambda, 8 \ell}_2 \] generated by

\[ a^{\lambda}_2 a^{\lambda}_{2 \lambda} u^{\lambda}_{2 \lambda} - k \lambda \delta^{1}_{1 \lambda} \in E^{2k + 4 \lambda, 8 \ell}_2 (G/G) \] for $0 < k \leq \ell$.

(2) \{$(r_{1,0} r_{1,1})^{2 \ell+1}$\} leading to $X_{2 \ell+1,2 \ell+1}$ for $\ell \geq 0$; see the leftmost diagonal in Figure 8. On the $0$-line we have a copy of $\square$ generated under restrictions by

\[ \delta^{2 \ell+1} = u^{2 \ell+1}_\sigma r^{4}_s (u^{\lambda}_\lambda) \delta^{1}_{1 \lambda} \in E^{0, 8 \ell+4}_2 (G/G') \].

In positive filtrations we have

\[ \bullet \subseteq E^{2 \lambda, 8 \ell+4}_2 \] generated by

\[ u^{2 \ell+1}_\sigma r^{4}_s (a^{\lambda}_1 u^{2 \lambda - 1 - j \lambda}_{1 \lambda} \delta^{1}_{1 \lambda}) \in E^{2 \lambda, 8 \ell+4}_2 (G/G') \] for $0 < j \leq 2 \ell + 1$,

\[ \bullet \subseteq E^{2j + 1, 8 \ell+4}_2 \] generated by

\[ a^{\lambda}_1 a^{\lambda}_j u^{\lambda}_{2 \lambda} u^{2 \lambda - 1 - j \lambda}_{1 \lambda} \delta^{1}_{1 \lambda} \in E^{2j + 1, 8 \ell+4}_2 (G/G') \] for $0 \leq j \leq 2 \ell + 1$ and

\[ \bullet \subseteq E^{2k + 4 \lambda + 3, 8 \ell+4}_2 \] generated by

\[ a^{\lambda}_2 a^{\lambda}_{2 \lambda} u^{\lambda}_{2 \lambda} - k \lambda \delta^{1}_{1 \lambda} \in E^{2k + 4 \lambda + 3, 8 \ell+4}_2 (G/G) \] for $0 < k \leq \ell$.

(3) \{$(r_{1,0} r_{1,1})^{2 \ell-i}$\} leading to $X_{i,2 \ell-i}$ for $0 \leq i < \ell$; see other diagonals in Figure 8. On the $0$-line we have a copy of $\square$ generated (under $tr^{4}_2$, $res^{2}_1$ and the group action) by

\[ u^{\ell}_\sigma \pi^{\ell-i}_2 r^{4}_s (u^{\lambda}_\lambda) \delta^{1}_{1 \lambda} \in E^{0, 4 \ell}_2 (G/G') \].

In positive filtrations we have

\[ \circ \subseteq E^{2 \lambda, 4 \ell}_2 \] generated by

\[ u^{\ell}_\sigma \pi^{\ell-i}_2 r^{4}_s (a^{\lambda}_1 u^{2 \lambda - 1 - j \lambda}_{1 \lambda} \delta^{1}_{1 \lambda}) \in E^{2 \lambda, 4 \ell}_2 (G/G') \] for $0 < j \leq \ell$

= $n^{2 \lambda}_\sigma u^{\ell-i}_\sigma j \pi^{\ell-i-i}_2 r^{4}_s (u^{\lambda}_\lambda) \delta^{1}_{1 \lambda}$ for $0 < j < \ell - i$.

(4) \{$(r_{1,0} r_{1,1})^{2 \ell+1-i}$\} leading to $X_{i,2 \ell+1-i}$ for $0 \leq i \leq \ell$; see other diagonals in Figure 9. On the $0$-line we have a copy of $\square$ generated (under transfers and the group action) by

\[ r^{i}_1 0 r^{1}_2 (u^{\ell-i}_\sigma \pi^{\ell-i}_2) r^{4}_s (u^{\lambda}_\lambda) \delta^{1}_{1 \lambda} \in E^{0, 4 \ell+2}_2 (G/ \{e\}) \].

In positive filtrations we have

\[ \circ \subseteq E^{2j + 1, 4 \ell+2}_2 \] generated by

\[ \eta^{2 \lambda}_\sigma u^{\ell-i}_\sigma j \pi^{\ell-i-i}_2 r^{4}_s (a^{\lambda}_1 u^{\lambda}_\lambda) \delta^{1}_{1 \lambda} \in E^{2j + 1, 4 \ell+2}_2 (G/G') \] for $0 \leq j \leq \ell$

= $\eta^{2 \lambda+1}_\sigma u^{\ell-i}_\sigma j \pi^{\ell-i-i}_2 r^{4}_s (u^{\lambda}_\lambda) \delta^{1}_{1 \lambda}$ for $0 < j < \ell - i$. 
Corollary 8.3. A subring of the slice $E_2$-term. The ring $E_2 k^k_H(G/G')$ contains $\mathbb{Z}[\delta_1, \Sigma_2, \Sigma_3, \eta_1, \epsilon, \delta_1^2 - \Sigma_2, \Sigma_2, \epsilon + \eta_1 \delta_1] / (2\eta_1, \eta_1^2 - \Sigma_2, \Sigma_2, \epsilon + \eta_1 \delta_1)$; see Table 3 for the definitions of its generators. In particular the elements $\eta_0$ and $\eta_1$ are algebraically independent mod 2 with
\[
\gamma'((\eta_0^m \eta_1^n) \in \mathbb{Z}/m \mathbb{Z} \times \mathbb{Z}/n \mathbb{Z} \quad \text{for} \ m \leq n.
\]
The element $(\eta_0 \eta_1)^2$ is the fixed point restriction of $u_2 a_2^2 \delta_1^2 \in E_2^2 k^k_H(G/G')$, which has order 4, and the transfer of the former is twice the latter. The element $\eta_0 \eta_1$ is not in the image of $\text{res}_2^4$ and has trivial transfer in $E_2$.

Proof. We detect this subring with the monomorphism
\[
E_2 k^k_H(G/G') \xrightarrow{\gamma'} E_2 k^k_H(G'/G')
\]
\[
\eta_1 \mapsto a_\sigma \Sigma_1, \epsilon
\]
\[
\Sigma_2, \epsilon \mapsto u_2 a_2^2 \delta_1^2, \epsilon
\]
\[
\delta_1 \mapsto u_2 a_2^2 \delta_1, \epsilon
\]
in which all the relations are transparent.

Corollary 8.4. Slices for $K_H$. The slices of $K_H$ are
\[
P_s K_H = \begin{cases} \bigvee_{m \leq s/4} X_{m, s/2-m} & \text{for } s \text{ even} \\ \bigvee_{m \leq s/4} X_{m, s/2-m} & \text{otherwise} \end{cases}
\]
where $X_{m, n}$ is as in Theorem 8.2. Here $m$ can be any integer, and we still require that $m \leq n$.

Proof. Recall that $K_H$ is obtained from $k^k_H$ by inverting a certain element $D \in \mathbb{Z}/4p_4 k^k_H(G/G)$ described in Table 3. Thus $K_H$ is the homotopy colimit of the diagram
\[
k^k_H \xrightarrow{D} \Sigma^{-4p_4} k^k_H \xrightarrow{D} \Sigma^{-8p_4} k^k_H \xrightarrow{D} \ldots
\]
Desuspending by $4p_4$ converts slices to slices, so for even $s$ we have
\[
P_s K_H = \lim_{k \to \infty} \Sigma^{-4p_4} \bigvee_{m \leq s/4} X_{m, s/2-8k-m}
\]
\[
= \lim_{k \to \infty} \Sigma^{-4p_4} \bigvee_{0 \leq m \leq s/4+4k} X_{m, s/2+8k-m}
\]
\[
= \lim_{k \to \infty} \bigvee_{0 \leq m \leq s/4+4k} X_{m, s/2-4k-m}
\]
\[
= \bigvee_{-4k \leq m \leq s/4} X_{m, s/2-4k-m}
\]
\[
= \bigvee_{m \leq s/4} X_{m, s/2-m}.
\]
Corollary 8.5. A filtration of $k_H$. Consider the diagram

$$
\begin{array}{cccccc}
\quad & k_H & \xleftarrow{\delta_1} & \Sigma^p_k H & \xleftarrow{\delta_1} & \Sigma^{2p} k_H & \xleftarrow{\delta_1} & \cdots \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
Y_0 & Y_1 = \Sigma^p Y_0 & Y_2 = \Sigma^{2p} Y_0 & \quad & \quad & \quad & \quad 
\end{array}
$$

where $Y_0$ is the cofiber of the map induced by $\delta_1$. Then the slices of $Y_m$ are

$$
P_t^s Y_m = \begin{cases} 
X_{m,t/2-m} & \text{for } t \text{ even and } t \geq 4m \\
* & \text{otherwise.}
\end{cases}
$$

9. Some differentials in the slice spectral sequence for $k_H$

Now we turn to differentials. The only generators in (6.5) that are not permanent cycles are the $u$’s. We will see that it is easy to account for the elements in $E_2^{0,|V|-V}(G/H)$ for proper subgroups $H$ of $G = C_4$. From (6.5) we see that

$$
E_2^{s,t} = 0 \quad \text{for } |t| \text{ odd.}
$$

This sparseness condition implies that $d_r$ can be nontrivial only for odd values of $r$.

Our starting point is the Slice Differentials Theorem of [HHRa, Thm. 9.9], which is derived from the fact that the geometric fixed point spectrum of $MU^{((G))}$ is $MO$. It says that in the slice spectral sequence for $MU^{((G))}$ for an arbitrary finite cyclic 2-group $G$ of order $g$, the first nontrivial differential on various powers of $u_{2r}$ is

$$
d_r(u_{2r}^{2k-1}) = a^k_r a^{k-1}_r N_2^g(2^r - 1) \in E_2^{r,r+2^k(1-\sigma)-1} MU^{((G))}(G/G),
$$

where $r = 1 + (2^k - 1)g$ and $p$ is the reduced regular representation of $G$. In particular

$$
\begin{align*}
\begin{cases}
  d_5(u_{2r}) &= a^3_\sigma a^4_\lambda \bar{d}_1 \in E_5^{5,6-2\sigma} MU^{((G))}(G/G) & \text{for } G = C_4 \\
  d_3(u_{2r}) &= a^2_\sigma T_1 \in E_3^{3,4-2\sigma} MU_R(G/G) & \text{for } G = C_2 \\
  d_3(u_{2r}^2) &= a^2_\sigma a^3_\lambda \bar{d}_2 \in E_3^{1,16-4\sigma} MU^{((G))}(G/G) & \text{for } G = C_4 \\
  d_7(u_{2r}^2) &= a^2_\sigma T_3 \in E_3^{7,10-4\sigma} MU_R(G/G) & \text{for } G = C_2.
\end{cases}
\end{align*}
$$

These lead to Massey products which are permanent cycles,

$$
\begin{align*}
\langle 2, a^2_\sigma, f_1 \rangle &= 2u_{2r} = tr^3_2(u_{2r}^2) \in \begin{cases} 
E_6^{0,2-2\sigma} MU^{((G))}(G/G) & \text{for } G = C_4 \\
E_6^{2,2-2\sigma} MU_R(G/G) & \text{for } G = C_2
\end{cases} \\
\langle 2, a^2_\sigma, f_3 \rangle &= 2u_{2r}^2 = tr^3_2(u_{2r}^4) \in \begin{cases} 
E_8^{0,4-4\sigma} MU^{((G))}(G/G) & \text{for } G = C_4 \\
E_8^{2,4-4\sigma} MU_R(G/G) & \text{for } G = C_2
\end{cases}
\end{align*}
$$

and (by Theorem 3.3) to exotic transfers

\[
a_{\sigma}f_1 = \begin{cases} 
tr_4^2(u_\sigma) & \in E^{1,5-\sigma}_\infty MU^{((G))}(G/G) \\
\text{filtration jump 4} \\
tr_4^2(u_\sigma) & \in E^{2,3-\sigma}_\infty MU_R(G/G) \\
\text{filtration jump 2} \\
tr_4^2(u_\sigma) & \in E^{12,15-3\sigma}_\infty MU_R(G/G) \\
\text{filtration jump 12} \\
tr_4^2(u_\sigma) & \in E^{6,9-3\sigma}_\infty MU_R(G/G) \\
\text{filtration jump 6} 
\end{cases}
\]

(9.4)

\[
a_{\sigma}^3f_3 = \begin{cases} 
\in C_3^{4}(G/G) \\
\text{filtration jump 4} \\
\in C_3^{4}(G/G) \\
\text{filtration jump 12} \\
\in C_3^{4}(G/G) \\
\text{filtration jump 6} 
\end{cases}
\]

Now, as before, let \(G = C_4\) and \(G' = C_2 \subseteq G\). We need to translate the \(d_3\) above in the slice spectral sequence for \(MU_R\) into a statement about the one for \(k_H\) as a \(G'\)-spectrum. We have an equivariant multiplication map \(m\) of \(G'\)-spectra

\[
MU^{((G))} \xrightarrow{\eta} MU_R \land MU_R \xrightarrow{m} MU_R
\]

\[
\xrightarrow{a_{\sigma}^3(\tau_1^{G'} + \tau_1^{G'})} \xrightarrow{a_{\sigma}^3\tau_1^{G'}} \xrightarrow{a_{\sigma}^3\tau_1^{G'}}
\]

\[
\xrightarrow{(5\tau_1^{G'} + \tau_3^{G'}) \mod(\tau_2^{G'}, \tau_3^{G'})} \xrightarrow{\tau_3^{G'}}
\]

where the elements lie in \(E^{G'}_{m+1}(\cdot)(G'/G')\) and \(E^{G'}_{m+1}(\cdot)(G'/G')\). In the slice spectral sequence for \(MU^{((G))}\) as a \(G'\)-spectrum, \(d_3(u_{2\sigma})\) and \(d_7(u_{2\sigma}^2)\) must be \(G\)-invariant since \(u_{2\sigma}\) is, and they must map respectively to \(a_{\sigma}^3\tau_1^{G'}\) and \(a_{\sigma}^3\tau_3^{G'}\), so we have

\[
d_3(u_{2\sigma}) = d_3(\pi_\lambda) = a_{\sigma}(\tau_1^{G'} + \tau_3^{G'}) = a_{\sigma}(\eta_0 + \eta_1)
\]

and

\[
d_7(u_{2\sigma}^2) = d_7(\pi_\lambda^2) = a_{\sigma}(5\tau_1^{G'} + \tau_3^{G'}) + (\tau_1^{G'})^3 + \cdots
\]

We get similar differentials in the slice spectral sequence for \(k_H\) as a \(C_2\)-spectrum in which the missing terms in \(d_7(\pi_\lambda^2)\) vanish.

Pulling back along the isomorphism \(\tau_3^{G'}\) gives

\[
d_3(res_{1}^{2}(u_\lambda)) = d_3(\pi_\lambda) = a_{\sigma}(\eta_0 + \eta_1) = res_2^{2}(a_{\lambda}\eta)
\]

and

\[
d_7(res_{1}^{2}(u_\lambda^2)) = d_7(\pi_\lambda^2) = a_{\sigma}^2\tau_1^{G'} = res_2^{2}(a_{\lambda}^2\eta_0) = res_2^{2}(a_{\lambda}^2\nu)
\]

These imply that

\[
d_3(u_\lambda) = a_{\lambda}\eta \quad \text{and} \quad d_5(u_\lambda^2) = a_{\lambda}^2\nu.
\]

The differential on \(u_\lambda\) leads to the following Massey products, which are permanent cycles.

\[
2u_\lambda = \langle 2, \eta, a_{\lambda} \rangle \in E^{0,2-\lambda}_4(G/G)
\]

and

\[
\nu := a_{\sigma}u_\lambda = \langle a_{\sigma}, \eta, a_{\lambda} \rangle \in E^{1,3-\sigma-\lambda}_4(G/G)
\]
where $\nu$ satisfies
\[
\nu^2 = (\nu, \eta, a\lambda) = a\nu^3 u\lambda = a\nu 2a\lambda u_{2\sigma} = 0
\]
\[
\text{res}^4_2(\nu) = a\nu^3 u\lambda \tau_{1,0} \in E^3,5-3\sigma-\lambda(G/G')
\]
(exotic restriction with filtration jump 2)
\[
2\nu = tr^4_2(\text{res}^4_2(\nu)) = tr^4_2(u\nu a\nu^3 \tau_{1,0})
= \eta' a\lambda \in E^4,5-3\sigma-\lambda(G/G')
\]
(exotic group extension with jump 2)
\[
tr^4_2(x)\nu = tr^4_2(x \cdot \text{res}^4_2(\nu)) = tr^4_2(xa\nu^3 u\nu \tau_{1,0})
\]
\[
= tr^4_2(a\nu^2 u\nu \tau_{1,1})\nu = tr^4_2(a\nu^2 u\nu \tau_{1,1} a\nu^3 u\nu \tau_{1,0})
\]
\[
= tr^4_2(a\nu^2 u\nu \tau_{1,0} \tau_{1,1}) = a\nu^2 \tilde{d}_1 tr^4_2(u\nu^2)
\]
\[
= a\lambda \tilde{d}_1 (2, a\nu, a\nu f_1) = \langle 2, a\nu, f_1^2 \rangle
\]
\[
\eta' \nu = a\nu^2 \tilde{d}_1 \eta' tr^4_2(u\nu^2) = 0
\]
\[
d^7(\tilde{\pi}_1^3) = a\nu^2 \tilde{d}_1 (u\nu a\nu_2 \tau_{1,0}) \cdot (u\nu a\nu^2 \tau_{1,0} a\nu \tau_{1,1}) \cdot (u\nu a\nu^3 \tau_{1,2}) \in E^4
\]
\[
= \text{res}^4_2(\nu) \text{res}^4_2(a\nu^2 \tilde{d}_1) = \text{res}^4_2(\tilde{\nu} a\nu^3 \tilde{d}_1) = \text{res}^4_2(\tilde{\nu} \lambda \lambda a\nu^2 \tilde{d}_1)
\]
\[
d_7(2u\lambda^2) = (\nu^2 a\nu^2 \tilde{d}_1) = a\nu^2 \eta' \tilde{d}_1.
\]
Note that $\nu = \tilde{\nu} \tilde{d}_1$, with the exotic restriction and group extension on $\nu$ being consistent with those on $\nu$.

The differential on $u\lambda^2$ yields
\[
a\nu^2 u\lambda^2 = (a\nu^2, \nu, a\nu^2 \tilde{d}_1)
\]
and
\[
\eta' u\lambda^2 = (\eta', \nu, a\nu^2 \tilde{d}_1).
\]

We can use this to find the differential on $u\lambda^3$. We have
\[
d(u\lambda^3) = 2u\lambda^3 d(u\lambda^3) = 2u\lambda^3 \nu a\lambda^3 \tilde{d}_1 = (2\nu) a\lambda^3 u\lambda^3 \tilde{d}_1
\]
\[
= \eta' a\lambda^3 u\lambda^3 \tilde{d}_1 = \eta' a\lambda^3 u\lambda^3 \tilde{d}_1 = \langle \eta', \nu, a\lambda^2 \tilde{d}_1, a\nu^3 \tilde{d}_1 \rangle.
\]

The differential on $u_{2\sigma}$ yields
\[
xu_{2\sigma} = \langle x, a\nu^2, f_1 \rangle
\]
for any permanent cycle $x$ killed by $a\nu^2$. Possible values of $x$ include 2, $\eta$, $\eta'$ (each of which is killed by $a\nu$ as well) and $\bar{\nu}$. For the last of these we write
\[
\xi := \bar{\nu} u_{2\sigma} = \langle \bar{\nu}, a\nu^2, f_1 \rangle = \langle a\nu^2 u\lambda, a\nu^2, f_1 \rangle \in E^1,5-3\sigma-\lambda(G/G'),
\]
which satisfies
\[
\text{res}^4_2(\xi) = a\nu^2 u\lambda \tau_{1,0} \in E^3,7-3\sigma-\lambda(G/G')
\]
(exotic restriction with jump 2)
\[
2\xi = tr^4_2(\text{res}^4_2(\xi)) = \eta' a\lambda u_{2\sigma} \in E^3,7-3\sigma-\lambda(G/G')
\]
(exotic group extension with jump 2)
Theorem 9.10. The differentials on powers of $u_\lambda$ and $u_{2\sigma}$. The following differentials occur in the slice spectral sequence for $k_\text{H}$. Here $\bar{\pi}_\lambda$ denotes $\text{res}^2_2(u_\lambda)$.

$$
\begin{align*}
&d_5(u_{2\sigma} u_\lambda^3) = a_3^2 a_\lambda u_\lambda^2 \bar{d}_1 + \nu a_3^2 u_{2\sigma} \bar{d}_1 = (a_3^2 u_\lambda^2 + \nu a_{2\sigma}) a_\lambda^2 \bar{d}_1 \\
&\quad = (2 a_\sigma a_\lambda u_\lambda u_{2\sigma} + \xi) a_\lambda^2 \bar{d}_1 = \xi a_\lambda^2 \bar{d}_1 \\
&d_7(2 u_{2\sigma} u_\lambda^2) = 2 \xi \cdot a_\lambda^2 \bar{d}_1 = \eta' a_3^2 u_{2\sigma} \bar{d}_1 \\
&\text{res}^2_2(\text{d}_5(2 u_{2\sigma} u_\lambda^2)) = u_\lambda^3 a_3^2 \bar{\pi}_{1.1}, \text{res}^2_2(2 a_\lambda^2 \bar{d}_1) = u_\lambda^2 a_\sigma^7 \bar{\pi}_{1,0} = u_\lambda^2 d_7(\bar{\pi}_\lambda^2).
\end{align*}
$$

The elements

$$
\begin{align*}
u_{\sigma}, \\
2 u_{2\sigma} &= (2, a_\sigma^2, f_1) = tr^4_2(u_{2\sigma}), \\
u_{\sigma}^2 &= (a_\sigma^2, f_1, a_\sigma^2, f_1) = 2 \bar{\pi}_\lambda^2 = (2, a_\sigma^6, a_\sigma^2 \bar{\pi}_{1,0}) = tr^4_1(u_{2\sigma}), \\
nu_{2\sigma} u_{2\sigma} &= (2 u_{2\sigma}, \eta, a_\lambda), \\
u_{2\sigma}^4 &= (2, \nu', \nu, a_\lambda^2 \bar{d}_1) = tr^4_2(\bar{\pi}_\lambda^4), \\
\bar{\pi}_\lambda^4 &= (a_\sigma^7, \bar{\pi}_{1,0}, a_\sigma^7, \bar{\pi}_{1,0})
\end{align*}
$$

are permanent cycles.

We also have the following exotic restriction and transfers.

$$
\text{res}^4_2(a_\sigma u_{2\sigma} u_\lambda) = u_{2\sigma} \text{res}^4_2(a_\lambda) = u_{2\sigma} a_\sigma^3 \bar{\pi}_{1,1} \quad \text{(filtration jump 2)}
$$

$$
\begin{align*}
tr^4_2(2 u_{2\sigma}^k) &= \begin{cases} 
2 a_\sigma^2 a_\lambda u_\lambda \bar{d}_1 (u_{2\sigma}^{k-1})/2 = a_\sigma f_1 u_{2\sigma}^{k-1}/2 & \text{for } k \equiv 1 \mod 4 \\
0 & \text{for } k \text{ even}
\end{cases} \\
&\quad \text{ (filtration jump 4)}
\end{align*}
$$

$$
\begin{align*}
tr^4_2(u_{2\sigma}^k) &= \begin{cases} 
2 a_\lambda^2 \bar{\pi}_{1,0} (u_{2\sigma}^{k-1})/2 = a_\lambda (\eta_0 + \eta_1) \bar{\pi}_{1,0}^{(k-1)/2} & \text{for } k \equiv 1 \mod 4 \\
0 & \text{for } k \equiv 3 \mod 4
\end{cases} \\
&\quad \text{ (filtration jump 2)}
\end{align*}
$$

$$
\begin{align*}
tr^4_1(2 u_{2\sigma}^k) &= \begin{cases} 
2 a_\sigma^2 \bar{\pi}_{1,0} (u_{2\sigma}^{k-1})/2 = a_\sigma (\eta_0 + \eta_1) \bar{\pi}_{1,0}^{(k-1)/2} & \text{for } k \equiv 1 \mod 4 \\
0 & \text{for } k \equiv 3 \mod 4
\end{cases} \\
&\quad \text{ (filtration jump 6)}
\end{align*}
$$

Proof. All differentials were established above.
The differential on \( u_3^2 \) does not lead to an exotic transfer because neither \( \eta \alpha \) nor \( u_\lambda a_\lambda^2 \delta_1 \) is a permanent cycle as required by Theorem 3.3.

We need to discuss the element \( 2u_{2\sigma} \eta u_\lambda = (2u_{2\sigma}, \eta, a_\lambda) \). To see that this Toda bracket is defined, we need to verify that \( 2u_{2\sigma} \eta = 0 \). For this we have

\[
2u_{2\sigma} \cdot \eta = 2u_{2\sigma} tr_2^1(\eta_0) = tr_2^1(2u_{2\sigma} \eta_0) = tr_2^1(0) = 0.
\]

The exotic restriction and transfers are applications of Theorem 3.3 to the differentials on \( u_\lambda \) and on \( u_{2\sigma}^{(k+1)/2} \) and \( \eta \alpha^{(k+1)/2} \) for odd \( k \). For even \( k \) we have

\[
tr_2^1(u_k^k) = tr_2^1(res_2^1(u_{2\sigma}^{k/2})) = 2u_{2\sigma}^{k/2} \quad \text{since} \quad tr_2^1(res_2^1(x)) = (1 + \gamma)x,
\]

and similarly for even powers of \( u_{2\sigma} \).

As remarked above, we lose no information by inverting the class \( D \), which is divisible by \( \delta_1 \). It is shown in [Hir] Thm. 9.16 that inverting the latter makes \( u_{2\sigma}^2 \) a permanent cycle. One can also see this from [9.3]. Since \( d_5(u_{2\sigma}) = a_\lambda^2 \alpha \delta_1 \), \( d_5(u_{2\sigma} \delta_1) = a_\lambda^2 a_\lambda \). This means that \( d_{13}(u_{2\sigma}^2) = a_\lambda^2 a_\lambda \delta_3 \) is trivial in \( E_6(G/G) \). It turns out that there is no possible target for a higher differential. \( \square \)

10. The effect of the first differentials over \( C_4 \)

Theorem 8.2 lists elements in the slice spectral sequence for \( kH \) over \( C_4 \) in terms of

\[
r_1, \bar{r}_2, \delta_1; \quad \eta, \alpha, a_\lambda; \quad u_\lambda, u_{2\sigma}, \text{ and } u_{2\sigma}.
\]

All but the \( u \)'s are permanent cycles, and the action of \( d_3 \) on \( u_\lambda, u_{\sigma} \) and \( u_{2\sigma} \) is described above in Theorem 9.10.

**Proposition 10.1.** \( d_3 \) on elements in Theorem 8.2. We have the following \( d_3 \)'s, subject to the conditions on \( i, j, k \) and \( \ell \) of Theorem 8.3:

- **On** \( X_{2\ell+2,2\ell} \):

  \[
  d_3(a_\lambda^i a_\sigma^j a_\lambda^2 u_{2\sigma}^{2\ell - j} \delta_1^2) = \begin{cases} 
    a_\lambda^{i+1} \eta a_\sigma^j u_\lambda^{2\ell - j - 1} \delta_1^{2\ell} & \text{for } i \text{ odd} \\
    0 & \text{for } i \text{ even}
  \end{cases}
  \]

  \[
  d_3(a_\lambda^i a_\sigma^j a_\lambda^2 u_{2\sigma}^{2\ell + j} \delta_1^2) = 0
  \]

- **On** \( X_{2\ell+1,2\ell+1} \):

  \[
  d_3(a_\lambda^i a_\sigma^j a_\lambda^2 u_{2\sigma}^{2\ell + j} \delta_1^{2\ell+1}) = \eta a_\lambda^{2\ell + 1} res_2^4(a_\lambda a_\lambda^2 \delta_1^{2\ell+1})
  \]

  \[
  = \begin{cases} 
    \eta a_\lambda^{2\ell + 1} res_2^4(a_\lambda^2 u_{2\sigma}^{2\ell - j} \delta_1^{2\ell+1}) & \text{for } j \text{ even} \\
    0 & \text{for } j \text{ odd}
  \end{cases}
  \]

  \[
  d_3(a_\lambda^i a_\sigma^j a_\lambda^2 u_{2\sigma}^{2\ell + j - 1} \delta_1^{2\ell+1}) = \eta a_\lambda^{2\ell + 1} res_2^4(a_\lambda a_\lambda^2 u_{2\sigma}^{2\ell - j} \delta_1^{2\ell+1})
  \]

  \[
  = \begin{cases} 
    \eta a_\lambda^{2\ell + 1} res_2^4(a_\lambda^2 u_{2\sigma}^{2\ell - j} \delta_1^{2\ell+1}) & \text{for } j \text{ odd} \\
    0 & \text{for } j \text{ even}
  \end{cases}
  \]

  \[
  d_3(a_\lambda^i a_\sigma^j a_\lambda^2 u_{2\sigma}^{2\ell + j - 1} \delta_1^{2\ell+1}) = 0
  \]
Theorem 10.2. The slice $E_{2\ell}$-term for $k_H$. The elements of Theorem 8.2 surviving to $E_4$, which live in the appropriate subquotients of $\pi_* X_{m,n}$, are as follows.

(i) In $\pi_* X_{2\ell,2\ell}$ (see the leftmost diagonal in Figure 8), on the 0-line we still have a copy of $\Box$ generated under fixed point restrictions by $\Delta^f_1 \in E^{0,8\ell}_4$. In positive filtrations we have

\[ \circ \subset E^{2j,8\ell}_4 \text{ generated by} \]

\[ a^j_{\lambda} u^j_{2\sigma} u^{2\ell-j}\bar{\delta}_1 \in E^{2j,8\ell}_4(G/G) \]

\[ 2a^{j+1}_{\lambda} u^{j+1}_{2\sigma} u^{2\ell-j-1}\bar{\delta}_1 \in E^{2j+1,8\ell}_4(G/G) \]

for $j$ even and $0 < j \leq 2\ell$.

(ii) In $\pi_* X_{2\ell+1,2\ell}$ (see the rightmost diagonal in Figure 8), on the 0-line we have a copy of $\Box$ generated under fixed point restrictions by $\Delta^f_1 \in E^{0,8\ell}_4$. In positive filtrations we have

\[ \bullet \subset E^{2k+2\ell,8\ell}_4 \text{ generated by} \]

\[ 2a_{\lambda} u^j_{2\sigma} u^{2\ell-j}\bar{\delta}_1 = a^j_{\lambda} u^j_{2\sigma} u^{2\ell-j-1}\bar{\delta}_1 \in E^{2j+1,8\ell}_4(G/G) \]

for $j$ odd and $0 < j < 2\ell$ and $\ell + 1$.
\[ a_{2^k} \sigma^2 u_{2^k} \delta_1^{1} \in E_4^{2^j+2k, 8\ell}(G/G) \quad \text{for} \quad 0 < k \leq \ell. \]

(ii) In \( \pi_\ast X_{2\ell, 2\ell+1} \) (see the second leftmost diagonal in Figure 8), in filtration 0 we have \( \bullet \), generated (under transfers and the group action) by

\[ r_1 \res_1^4(\mu_2 \res_1^4 \mu_3 \res_1^4) \in E_4^{0, 8\ell+2}(G/\{e\}). \]

In positive filtrations we have

- \( \bullet \subseteq E_4^{1, 8\ell+2} \) generated (under transfers and the group action) by

\[ \eta \mu_2 \res_2 \mu_3 \res_2 \mu_3 \in E_4^{1, 8\ell+2}(G/G'). \]

- \( \bigcirc \subseteq E_4^{4k+1, 8\ell+2} \) for \( 0 < k \leq \ell \) generated by

\[ x = \eta \mu_2 \res_2 \mu_3 \res_2 \mu_3 \in E_4^{4k+1, 8\ell+2}(G/G') \]

with \( (1 - \gamma)x = tr_2(x) = 0. \)

(iii) In \( \pi_\ast X_{2\ell+1, 2\ell+1} \) (see the leftmost diagonal in Figure 8'), on the 0-line we have a copy of \( \bigcirc \) generated under fixed point \( \Delta_1(2^{\ell+1})/2 \in E_4^{0, 8\ell+4} \). In
positive filtrations we have

\[ \bullet \subseteq \mathbb{E}_2^{2j,8\ell+4} \quad \text{generated by} \quad u_\sigma^{2\ell+1}r_1s_1^4(a_\lambda^{2\ell+1-j}d_1^{2\ell+1}) \in \mathbb{E}_2^{2j,8\ell+4}(G/G') \quad \text{for} \quad 0 < j \leq 2\ell + 1, \]

\[ \bullet \subseteq \mathbb{E}_2^{2j+1,8\ell+4} \quad \text{generated by} \quad a_\sigma a_\lambda^{2\ell,j}u_\lambda^{2\ell+1-j}d_1^{2\ell+1} \in \mathbb{E}_2^{2j+2k,8\ell+4}(G/G) \quad \text{for} \quad 0 \leq j < 2\ell + 1 \quad \text{and} \]

\[ \bullet \subseteq \mathbb{E}_2^{2k+4\ell+3,8\ell+4} \quad \text{generated by} \quad a_\sigma^{2k+1}a_\lambda^{2\ell+1}u_\lambda^{-k}d_1^{2\ell+1} \in \mathbb{E}_2^{2k+4\ell+2,8\ell+4}(G/G) \quad \text{for} \quad 0 < k \leq 2\ell + 1. \]

(iv) In \( \mathbb{E}_4 X_{2\ell+1,2\ell+2} \) (see the second leftmost diagonal in Figure 9), in filtration 0 we have \( \square \), generated (under transfers and the group action) by

\[ r_1r_1s_1^2u_\sigma^{2\ell+1}r_1s_1^4(u_\lambda^{2\ell+1}d_1^{2\ell+1}) \in \mathbb{E}_4^{6,8\ell+6}(G/\{e\}). \]
In positive filtrations we have

\[ \bigwedge \subseteq E^{4k+3,3\ell+6}_4 \] for \(0 \leq k \leq \ell \) generated under transfer by

\[ x = \eta^{k+3}\Delta^\ell_{1} \in E^{4k+3,3\ell+6}_4 \]

with \((1-\gamma)x = 0\).

The generator of \( E^{4k+3,3\ell+6}_4 (G/G') \) is the exotic restriction of the one in \( E^{4k+1,3\ell+4}_4 (G/G) \).

(v) In \( \pi_*X_{m,m+i} \) for \( i \geq 2 \) (see the rest of Figures 8 and 9), in filtration 0 we have

\[ \bigcirc \subseteq E^{0,4m+4j+2}_4 \]
\[ \bigcirc \subseteq E^{0,8\ell+4}_4 \]
\[ \bigcirc \subseteq E^{0,8\ell}_4 \]
\[ \bigcirc \subseteq E^{4,4m+j+2}_4 \]
\[ \bigcirc \subseteq E^{4,8\ell+2}_4 \]

\[ \bigcirc \subseteq E^{2,8\ell+4}_4 \]
\[ \bigcirc \subseteq E^{8,8\ell+2s}_4 \]

\[ \eta^s x_{8\ell,m} \in E^{8,8\ell+2s}_4 (G/G') \]

for \( s = 1, 2 \) and \( 0 \leq m \leq 2\ell - 1 \).

Each generator of \( E^{2,8\ell+4}_4 (G/G') \) is an exotic transfer of one in \( E^{4,8\ell+2s}_4 (G/e) \).

Proposition 10.3. Some nontrivial permanent cycles. The elements listed in Theorem 10.2(v) other than \( \eta^2\Delta^2_{1} \) are all nontrivial permanent cycles.

Proof. Each such element is either in the image of \( E^{0,*}_4 (G/\{e\}) \) under the transfer and therefore a nontrivial permanent cycle, or it is one of the ones listed in Corollary 11.2.

In subsequent discussions and charts, starting with Figure 14, we will omit the elements in Proposition 10.3. These elements all occur in \( E^{s}_4 \) for \( 0 \leq s \leq 2 \).

Analogous statements can be made about the slice spectral sequence for \( K_H \). Each of its slices is a certain infinite wedge spelled out in Corollary 8.4. Their homotopy groups are determined by the chain complex calculations of Section 6 and illustrated in Figures 2 (with Mackey functor induction \( \lambda^1_2 \) applied) and 3. Analogous Figures 8, 9 are shown in Figures 10, 11. In each figure, exotic transfers and restrictions are indicated by blue and green lines respectively. As in the \( k_H \) case, most of the elements shown in this chart can be ignored for the purpose of...
calculating higher differentials. In the third quadrant the elements we are ignoring all occur in $E^s_{4} \times f$ for $-2 \leq s \leq 0$.

The resulting reduced $E^4_4$ for $K_{H}$ is shown in Figure [16]. The information shown there is very useful for computing differentials and extensions. The periodicity theorem tells us that $\pi_n K_{H}$ and $\pi_{n-32} K_{H}$ are isomorphic. For $0 \leq n < 32$ these groups appear in the first and third quadrants respectively, and the information visible in the spectral sequence can be quite different.

For example, we see that $\pi_{0} K_{H}$ has summand of the form $\Box$, while $\pi_{-32} K_{H}$ has a subgroup isomorphic to $\text{fourbox}$. The quotient $\Box/fourbox$ is isomorphic to $\circ$. This leads to the exotic restrictions and transfer in dimension $-32$ shown in Figure [16]. Information that is transparent in dimension 0 implies subtle information in dimension $-32$. Conversely, we see easily that $\pi_{-4} K_{H} = \Box$ while $\pi_{-32} K_{H}$ has a quotient isomorphic to $\Box$. This leads to the “long transfer” (which raises filtration by 12) in dimension 28.

11. $k_{H}$ AS A $C_2$-SPECTRUM

Before studying the slice spectral sequence for $k_{H}$ further, it is helpful to explore its restriction to $G'$, for which the $\mathbb{Z}$-bigraded portion

$$E^2_{s,t} k_{H}(G'/G') = E^2_{s,t}(G'/G) k_{H}(G/G) = E^2_{s,t} k_{H}(G/G)$$
(see Theorem 1.12 for these isomorphisms) is the isomorphic image of the subring of Corollary 8.3. In the following we identify $\Sigma_{2, e}$, $\delta_1$ and $\tilde{r}_{1, e}$ with their images under $L_2^3$. From the differentials of (9.3) we get

$$d_3(\Sigma_{2, e}) = \eta_0^2(\eta_0 + \eta_1) = a_3^2\tilde{r}_{1, e}^2(\tilde{r}_{1, 0} + \tilde{r}_{1, 1})$$

$$d_3(\delta_1) = \eta_0^2\eta_1 + \eta_0\eta_1^2 = a_3^3\tilde{r}_{1, 0}\tilde{r}_{1, 1}^2(\tilde{r}_{1, 0} + \tilde{r}_{1, 1})$$

$$d_7(\delta_1^2) = d_7(a_3^2\tilde{r}_{1, 0}^2\tilde{r}_{1, 1}^2) = a_7^2\tilde{r}_{3, 0}\tilde{r}_{1, 0}\tilde{r}_{1, 1}^2$$

$$= a_7^2(5\tilde{r}_{1, 1}^2\tilde{r}_{1, 0} + 5\tilde{r}_{1, 0}\tilde{r}_{1, 1}^2 + \tilde{r}_{1, 1}^3)\tilde{r}_{1, 0}\tilde{r}_{1, 1}^2.$$

The $d_3$s above make all monomials in $\eta_0$ and $\eta_1$ of any given degree $\geq 3$ the same in $E_4(G/G')$ and $E_4(G'/G')$, so $d_7(\delta_1^2) = \eta_0^2$. Similar calculations show that

$$d_7(\Sigma_{2, e}) = \eta_0^2 = a_7^2\tilde{r}_{1, 0}.$$

This leads to the following, for which Figure 12 is a visual aid.

**Theorem 11.1.** The slice spectral sequence for $k_{\mathbb{H}}$ as a $C_2$-spectrum. Using the notation of Table 1 and Definition 4.3, we have

$$E_2^{s, s}(G'/\{e\}) = \mathbb{Z}[r_{1, 0}, r_{1, 1}]$$

with $r_{1, e} \in E_2^{0, 2}(G'/\{e\})$

$$E_2^{s, s}(G'/G') = \mathbb{Z}[\delta_1, \Sigma_{2, e}, \eta_{e}: e = 0, 1]$$
Figure 12. The slice spectral sequence for $k_\mathbb{H}$ as a $C_2$-spectrum. The Mackey functor symbols are defined in Table 1. The $C_4$-structure of the Mackey functors is not indicated here. In each bidegree we have a direct sum of the indicated number of copies of the indicated Mackey functor. Each $d_3$ has maximal rank, leaving a cokernel of rank 1, and each $d_7$ has rank 1. Blue lines indicate exotic transfers. The ones raising filtration by 2 have maximal rank while the ones raising it by 6 have rank 1. The resulting $E_8 = E_\infty$-term is shown below.
The first set of differentials and determined by 
\[ d_3(\Sigma_{2,e}) = \eta_0^2(\eta_0 + \eta_1) \quad \text{and} \quad d_5(\delta_1) = \eta_0\eta_1(\eta_0 + \eta_1) \]
and there is a second set of differentials determined by 
\[ d_7(\Sigma_{2,e}^2) = d_7(\delta_1^2) = \eta_0^7 \]

**Corollary 11.2. Some nontrivial permanent cycles.** The elements listed below in \( E_2^{s,8i+2s} k_H(G/G') \) are nontrivial permanent cycles. Their transfers in \( E_2^{s,8i+2s} k_H(G/G') \) are also permanent cycles.

- \( \Sigma_{2,e}^{2i-1}\delta_1^i \) for \( 0 \leq j \leq 2i \) (4i + 1 elements of infinite order including \( \delta_1^2i \)), i even and \( s = 0 \).
- \( \eta_0^i\Sigma_{2,e}^{2i-1}\delta_1^i \) for \( 0 \leq j < 2i \) and \( \eta_0^i\delta_1^2i \) (4i + 2 elements or order 2) for i even and \( s = 1 \).
- \( \eta_0^i\Sigma_{2,e}^{2i-1}\delta_1^i \) for \( 0 \leq j < 2i \) and \( \eta_0^i\delta_1^2i \) \{\( \eta_0^i, \eta_0^i\eta_1, \eta_0^i\eta_2 \} \) (4i + 3 elements or order 2) for i even and \( s = 2 \).
- \( \eta_0^i\delta_1^3i \) for \( 3 \leq s \leq 6 \) (4 elements or order 2) \( i \) and even.
- \( \Sigma_{2,e}^{2i-1}\delta_1^i + \delta_1^2i \) for \( 0 \leq j \leq 2i \) (4i + 1 elements of infinite order including \( 2\delta_1^2i \)), i odd and \( s = 0 \).
- \( \eta_0^i\Sigma_{2,e}^{2i-1}\delta_1^i + \delta_1^2i \) for \( 0 \leq j \leq 2i-1 \) and \( \eta_0^i\delta_1^2i-1(\Sigma_{2,1}+\delta_1) = \eta_0^i\delta_1^2i-1(\Sigma_{2,0}+\delta_1) \) (4i + 1 elements of order 2), i odd and \( s = 1 \).
- \( \eta_0^i\Sigma_{2,e}^{2i-1}\delta_1^i + \delta_1^2i \) for \( 0 \leq j \leq 2i-1 \), \( \eta_0^i\delta_1^2i-1(\Sigma_{2,1}+\delta_1) = \eta_0^i\eta_1^i\delta_1^2i-1(\Sigma_{2,0}+\delta_1) \) and \( \eta_0^i\eta_1^i\delta_1^2i-1(\Sigma_{2,1}+\delta_1) = \eta_0^i\eta_1^i\delta_1^2i-1(\Sigma_{2,0}+\delta_1) \) (4i + 2 elements of order 2) for i odd and \( s = 2 \).

In \( E_2^{0,8i+4} k_H(G/G') \) we have \( 2\Sigma_{2,e}^{2i+1-j}\delta_1^i \) for \( 0 \leq j \leq 2i \) and \( 2\delta_1^2i, 4i + 3 \) elements of infinite order, each in the image of the transfer \( tr_1^2 \).

### 12. Higher differentials and exotic Mackey functor extensions

We can use the results of the previous section to study higher differentials and extensions. The \( E_7 \)-term implied by them is illustrated in Figure 13. For each \( \ell, s \geq 0 \) there is a generator 
\[ y_{8\ell+s,s} := \eta_0^i\delta_1^{2\ell} \in E_7^{s,8\ell+2s}(G/G') \]
with 
\[ d_7(y_{16k+s+8,s}) = y_{16k+s+7,s+7}. \]

If the source has the form \( res_7^{j}(x_{16k+s+8,s}) \), then such an \( x \) must support a nontrivial \( d_r \) for \( r \leq 7 \). If it has a nontrivial transfer \( x' \) then \( x' \) cannot support
an earlier differential, and we must have
\[ d_r(x'_{16k+s+8,s}) = tr_2^3(d_7(y_{16k+s+8,s})) = tr_2^3(y_{16k+s+7,s+7}) \quad \text{for some } r \geq 7. \]

We could get a higher differential (meaning \( r > 7 \)) if \( y_{16k+s+7,s+7} \) supports an exotic transfer.

We have seen (Figure 14 and Theorem 10.2) that for \( s \geq 3 \) and \( k \geq 0 \),
\[
E_5^{s,16k+8+2s} = \begin{cases} 
\circ & \text{for } s \equiv 0 \mod 4 \\
\blacksquare & \text{for } s \equiv 1, 2 \mod 4 \\
\blacktriangle & \text{for } s \equiv 3 \mod 4.
\end{cases}
\]

For \( s = 1, 2 \), \( E_5^{s,16k+8+2s} \) has \( \blacksquare \) as a direct summand. For \( s = 0 \) it has \( \square \) as a summand, and the differentials on it factor through its quotient \( \circ \); see (4.2).

The corresponding statement in the third quadrant is
\[
E_5^{s,-16k-2s-24} = \begin{cases} 
\circ & \text{for } s \equiv 0 \mod 4 \\
\blacksquare & \text{for } s \equiv 1, 2 \mod 4 \\
\blacktriangle & \text{for } s \equiv 0 \mod 4.
\end{cases}
\]

for \( s \geq 3 \) and \( k \geq 0 \). For \( s = 1, 2 \) the groups have similar summands, and for \( s = 0 \) there is a summand of the form \( \blacksquare \), which has \( \blacktriangle \) as a subgroup; again see (4.2). This is illustrated in Figure 16.

**Theorem 12.2.** Differentials for \( C_4 \) related to the \( d_7 \)s for \( C_2 \). The differential
\[ d_7(y_{16k+s+8,s}) = y_{16k+s+7,s+7} \quad \text{with } s \geq 3 \]
has the following implications for the four congruence classes of \( s \).

(i) For \( s \equiv 0 \), \( E_7^{s,16k+8+2s} = \circ \) and \( E_7^{s+7,16k+14+2s} = \blacktriangle \). Hence \( y_{16k+s+8,s} \) is a restriction with a nontrivial transfer, and
\[ d_5(x_{16k+s+8,s}) = x_{16k+s+7,s+5} \]
and \( d_7(2x_{16k+s+8,s}) = d_7(tr_4(y_{16k+s+8,s})) \)
\[ = tr_2(y_{16k+s+7,s+7}) = x_{16k+s+7,s+7}. \]

(ii) For \( s \equiv 1, \)
\( d_7(y_{16k+s+8,s}) = y_{16k+s+7,s+7} \)
and \( d_5(x_{16k+s+8,s+2}) = tr_2(y_{16k+s+7,s+7}) = 2x_{16k+s+7,s+7} \)
This leaves the fate of \( x_{16k+s+7,s+7} \) undecided; see below.

(iii) For \( s \equiv 2, E_7^{s,16k+8+2s} = \bar{u} \) and \( E_7^{s+7,16k+14+2s} = \bar{u}. \) Neither the source nor target is a restriction or has a nontrivial transfer, so no additional differentials are implied.

(iv) For \( s \equiv 3, E_7^{s,16k+8+2s} = \nu \) and \( E_7^{s+7,16k+14+2s} = \bar{u}. \) In this case the source is an exotic restriction; again see Figure 9. Thus we have
\[ d_7(y_{16k+s+8,s}) = y_{16k+s+7,s+7} \]
and \( d_5(x_{16k+s+8,s+2}) = x_{16k+s+7,s+3} \)
with \( \text{res}_2(x_{16k+s+7,s+3}) = y_{16k+s+7,s+7}. \)
Moreover, \( tr_2(y_{16k+s+8,s}) \) is nontrivial and it supports a nontrivial transfer \( d_1 \) when \( 4k+s \equiv 3 \text{ mod } 8. \) The other case, \( 4k+s \equiv 7, \) will be discussed below.

Proof. (i) The target Mackey functor is \( \nu \) and \( y_{16k+s+7,s+7} \) is the exotic restriction of \( x_{16k+s+7,s+5}; \) see Figure 9 and Theorem 10.2. The indicated \( d_5 \) and \( d_7 \) follow.

(ii) The differential is nontrivial on the \( G/G' \) component of
\[ \bar{u} = E_7^{s,16k+8+2s} \xrightarrow{d_7} E_7^{s+7,16k+14+2s} = 0 \]
Thus the target has a nontrivial transfer, so the source must have an exotic transfer. The only option is \( x_{16k+s+8,s+2}, \) and the result follows.

(iv) We prove the statement about \( d_{11} \) by showing that
\[ y_{16k+s+7,s+7} = \eta_0 \delta_1^k \]
supports an exotic transfer that raises filtration by 4. First note that
\[ tr_2(y_8) = tr_2(a_2^2 r_{10} r_{10}) = tr_2(u_{a_2} res_2(a_{10} \bar{d}_1)) = tr_2(u_{a_0} a_{10} \bar{d}_1) = a_0 a_{10} \bar{d}_1 a_{10} \bar{d}_1 \text{ by } (9.4). \]
Next note that the three elements
\[ y_{8,8} = \eta_0^8 = res_2^4(\epsilon), \quad y_{20,4} = \eta_0^{14} = res_2^4(\bar{r}) \] and \( y_{32,0} = \delta_1^{14} = res_2^4(\Delta) \)
are all permanent cycles, so the same is true of all
\[ y_{16m+4l,4l} = \eta_0^{4l} \delta_1^{4m} \text{ for } m, l \geq 0 \text{ and } m + l \text{ even}. \]
It follows that for such \( \ell \) and \( m, \)
\[ \eta_0 \eta_1 y_{16m+4l,4l} = \eta_0 \eta_1 \eta_0^{4l} \delta_1^{4m} = \eta_0^{4l+2} \delta_1^{4m} = y_{16m+4l+2,4l+2} = \eta_0 \eta_1 \text{res}_2^4(x_{16m+4l+2,4l+2}), \]
so
\[ tr_2(y_{16m+4l+2,4l+2}) = tr_2(y_8) x_{16m+4l,4l} = f_1 x_{16m+4l,4l}. \]
This is the desired exotic transfer. \( \square \)
We now turn to the unsettled part of [12.2 iv].

**Theorem 12.3. The fate of \(x_{16k+s,8,s}\) for \(4k + s \equiv 7 \mod 8\) and \(s \geq 7\).** Each of these elements is the target of a \(d_7\) and hence a permanent cycle.

**Proof.** Consider the element \(\Delta^3 \in E^{0,16}_2(G/G)\). We will show that

\[
d_7(\Delta^3) = x_{15,7} = tr^s_2(y_{15,7}).
\]

This is the case \(k = 0\) and \(s = 7\). The remaining cases will follow via repeated multiplication by \(e, \pi\) and \(\Delta^3\).

We begin by looking at

\[
\Delta_1 = u_2\sigma u_\lambda \bar{\delta}_1^2.
\]

From Theorem [9.10] we have

\[
d_5(u_{2\sigma}) = a^3_\lambda a_\lambda \bar{\delta}_1 \quad \text{and} \quad d_5(u^2) = a_\sigma a^3_\lambda u_\lambda \bar{\delta}_1
\]

Using the gold relation \(a^2_\sigma u_\lambda = 2a_\lambda u_{2\sigma}\), we have

\[
d_5(\Delta_1) = d_5(u_{2\sigma}u^2)\bar{\delta}_1 = (a^3_\sigma a_\lambda u^2_\lambda \bar{\delta}_1 + a_\sigma a^2_\lambda u_\lambda u_{2\sigma} \bar{\delta}_1)\bar{\delta}_1
\]

\[
= a_\sigma a_\lambda u_\lambda (a^2_\sigma u_\lambda + a_\lambda u_{2\sigma}) \bar{\delta}_1^2
\]

\[
= a_\sigma a^3_\lambda u_\lambda u_{2\sigma} \bar{\delta}_1^2 \quad \text{since } 2a_\sigma = 0
\]

\[
= \nu x_4.
\]

Since \(\nu\) supports an exotic group extension, \(2\nu = x_3\), we have

\[
2d_5(\Delta_1) = d_7(2\Delta_1) = x_3 x_4.
\]

From this it follows that

\[
d_7(\Delta^3) = \Delta_1 d_7(2\Delta_1) = x_{15,7}
\]

as claimed. \(\square\)

The resulting reduced \(E_{12}\)-term is shown in Figure [13]. It is sparse enough that the only possible remaining differentials are the indicated \(d_{13}\). In order to establish them we need the following.

**Theorem 12.4. Normed up slice differentials for \(k_H\) and \(K_H\).** In the slice spectral sequence \(s\) for \(k_H\) and \(K_H\),

\[
d_5(a_\sigma u^3_\lambda) = 0
\]

and

\[
d_{13}(a_\sigma u^4_\lambda) = a^7_\lambda u^2_{2\sigma} \bar{\delta}_1^3.
\]

**Proof.** The two slice differentials over \(G'\) are

\[
d_3(u_{2\sigma}) = a^3_\sigma a^3_\sigma' r_{11} = a^3_\sigma (r_{1,0} + r_{1,1})
\]

and

\[
d_7(u^2_{2\sigma}) = a^7_\sigma a^7_\sigma' r_{11} = a^7_\sigma (5r_{1,0}r_{1,1} + 5r_{1,0}r_{1,1} + r_{1,1})
\]

We need to find the norms of both sources and targets. [HHKa] Lemma 3.13] tells us that

\[
N^3_2(a^k_\sigma) = a^k_\lambda
\]

and

\[
N^4_2(u^4_{2\sigma}) = u^{2k}_{\lambda} / u^{k}_{2\sigma}
\]
Figure 14. The $E_4$-term of the slice spectral sequence for $k\mathbb{H}$ with elements of Proposition 10.3 removed. Differentials are shown in red. Exotic transfers and restrictions are shown as solid blue and dashed green lines respectively. The Mackey functor symbols are as in Table 2.

Figure 15. The $E_{12}$-term of the slice spectral sequence for $k\mathbb{H}$ with elements of Proposition 10.3 removed. Differentials are shown in red. Exotic transfers and restrictions are shown as solid blue and dashed green lines respectively. The Mackey functor symbols are as in Table 2.
The surviving class in $\kappa H$.

There are differentials

$$d_{13}(f_1 x_4^m \Delta_1^{2m}) = f_1^{m-1} x_4^{m+4} \Delta_1^{2(n-1)}$$

for $\epsilon = 1, 2$, $m + n$ odd, $n \geq 1$ and $m \geq 1 - \epsilon$. The SS collapses from $E_{14}$.}

To finish the calculation we have

**Theorem 12.6. Exotic transfers from and restrictions to the 0-line.** In $\pi_* k H$, for $i \geq 0$ we have

$$tr^2_1(r_{1,1}^i r_{1,0}^{i+4} r_{1,1}^{4i}) = \eta_6 r_{1,0}^{4i} x_{1,1} \in \mathbb{P}_{8i+2}$$

$$tr^4_1((r_{1,0}^{8i+1} r_{1,1}^{8i+1}) = 2x_4 \Delta_1^{4i} \in \mathbb{P}_{32i+4}$$

$$tr^4_1(r_{1,0}^3 + r_{1,1}^3) r_{1,0}^{i+4} r_{1,1}^{4i} = \eta_6 \Delta_1^{2i} \in \mathbb{P}_{32i+6}$$

$$tr^4_1(r_{1,0}^{8i+5} r_{1,1}^{8i+5}) = 2x_4 \Delta_1^{4i+2} \in \mathbb{P}_{32i+20}$$
\[ \text{tr}_4^5((r_{1,0}^3 + r_{1,1}^3)r_{4,1}^3r_{4,1}^3) = \eta_0^3 \Delta_{4}^{24} \in \mathbb{Z}_{32+22} \]

and

\[ \text{tr}_4^5(2\delta_{8+7}^3) = \chi_4^3 \Delta_{4}^{4+2} \in \mathbb{Z}_{32+28} \text{ (the long transfer)}. \]

Let \( M_k \) denote the reduced value of \( \mathbb{Z}_k \), meaning the one obtained by removing the elements of Proposition 10.3. Its values are shown in purple in Figure 17, and each has at most two summands. For even \( k \) one of them contains torsion free elements, and we denote it by \( M_k' \). Its values depend on \( k \) mod 32 and are as follows, with symbols as in Table 2.

| \( k \) | 0   | 2   | 4   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  | 22  | 24  | 26  | 28  | 30  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( M_k' \) | ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ H |

**Proof.** We have two tools at our disposal: the periodicity theorem and Theorem 3.3, which relates exotic transfers to differentials.

Figure 16 shows that \( M_k' \) has the indicated value for \(-8 \leq k \leq 0\) because the same is true of \( E_{4,k}^{0,k} \) and there is no room for any exotic extensions. On the other hand \( E_{4,k}^{0,k+32} \) does not have the same value for \( k = -8, k = -6 \) and \( k = -4 \). This comparison via periodicity forces

- the indicated \( d_5 \) and \( d_7 \) in dimension 24, which together convert \( \square \) to \( \blacklozenge \). These were also established in Theorem 12.2.
- the short transfer in dimension 26, which converts \( \hat{\square} \) to \( \hat{\blacklozenge} \). It also follows from the the results of Section 11.
- the long transfer in dimension 28, which converts \( \hat{\blacklozenge} \) to \( \hat{\bullet} \).

The differential corresponding to the long transfer is

\[ d_{13}(2u^3_\lambda) = a_\sigma a_\lambda^6 u_{2\sigma} u_{3\lambda}^4 \delta_1^3, \]

so

\[ d_{13}(a_\sigma \cdot 2u^3_\lambda) = a_\sigma^2 a_\lambda^6 u_{2\sigma} u_{3\lambda}^4 \delta_1^3 = 2a_\lambda^7 u_{2\sigma}^2 v_2^3 \delta_1^3. \]

This compares well with the \( d_{13} \) of Theorem 12.4, namely

\[ d_{13}(a_\sigma u^3_\lambda) = a_\lambda^7 u_{2\sigma}^2 \delta_1^3. \]

The statements in dimensions 4 and 20 have similar proofs, and we will only give the details for the former. It is based on comparing the \( E_{4}\)-term for \( K_H \) in dimensions \(-28 \) and 4. They must converge to the same thing by periodicity. From the slice \( E_{4}\)-term in dimension 4 we see there is a short exact sequence

\[
\begin{array}{c}
0 \rightarrow \nabla \rightarrow M'_4 \rightarrow M' \rightarrow 0
\end{array}
\]

\[ (12.7) \]

- \( \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}_- \rightarrow \mathbb{Z}_- \]

- \( \mathbb{Z}/2 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}_- \rightarrow \mathbb{Z}_- \]

\[ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} b \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \mathbb{Z}_-, \]

\[ \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}_- \rightarrow \mathbb{Z}_- \rightarrow \mathbb{Z}_- \]
while the \((-28)\)-stem gives

\[ \begin{array}{c}
0 \rightarrow \mathbb{Z}/2 \rightarrow M'_{4} \rightarrow 0 \\
\mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow 0 \\
\mathbb{Z} \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z} \rightarrow \mathbb{Z}/2 \\
\mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2
\end{array} \]

The commutativity of the second diagram requires that \(a+b = c = 1\) and \(b+d = c + d = 0\), giving \((a, b, c, d) = (0, 1, 1, 1)\). The diagram for \(M_{4}\) is that of \(\mathfrak{a}\) in Table 2.

In dimension 20 the short exact sequence of (12.7) is replaced by

\[ \begin{array}{c}
0 \rightarrow \mathbb{Z}/2^{k} \rightarrow M'_{20} \rightarrow 0
\end{array} \]

and the resulting diagram for \(M'_{20}\) is that of \(\mathfrak{a}\).

Similar arguments can be made in dimensions 6 and 22.

We could prove a similar statement about exotic restrictions hitting the 0-line in the third quadrant in dimensions congruent to 0, 4, 6, 14, 16, 20 (where there is an exotic transfer) and 22. The problem is naming the elements involved.

In Table 4 we show short or 4-term exact sequences in the 16 even dimensional congruence classes. In each case the value of \(M'_{k}\) is the symbol appearing in both rows of the diagram. For even \(k\) with \(0 \leq k < 32\), we typically have short exact sequences

\[ \begin{array}{c}
0 \rightarrow E_{4}^{0,k-32} \rightarrow M'_{k} \rightarrow \text{quotient} \rightarrow 0 \\
0 \rightarrow \text{subgroup} \rightarrow M'_{k} \rightarrow E_{4}^{0,k} \rightarrow 0,
\end{array} \]

where the quotient or subgroup is finite and may be spread over several filtrations. This happens for the quotient in dimensions \(-32, -16\) and \(-12\), and for the subgroup in dimensions 6 and 22.

This is the situation in dimensions where no differential hits [originates on] the 0-line in the third [first] quadrant. When such a differential occurs, we may need a 4-term sequence, such as the one in dimension -22.

In dimensions 8 and 24 there is more than one such differential, the targets being a quotient and subgroup of the Mackey functor \(\circ = \square/\mathbb{Z}\).

In dimension \(-18\) we have a \(d_{7}\) hitting the 0-line. Its source is written as \(\circ \subseteq E_{7}^{-7,-24}\) in Figure 16. Its generator supports a \(d_{5}\), leaving a copy of \(\nabla\) in \(E_{7}^{-7,-24}\).

There is no case in which we have such differentials in both the first and third quadrants.

**Corollary 12.8.** The \(E_{\infty}\)-term of the slice spectral sequence for \(K_{H}\). The surviving elements in the spectral sequence for \(K_{H}\) are shown in Figure 17.
Table 4. Infinite Mackey functors in the reduced $E_\infty$-term for $K_H$. In each even degree there is an infinite Mackey functor on the 0-line that is related to a summand of $\pi_{2k}K_H$ in the manner indicated. The rows in each diagram are short or 4-term exact sequences with the summand appearing in both rows.

| Dimension mod 32 | Third quadrant | Dimension mod 32 | Third quadrant |
|------------------|----------------|------------------|----------------|
|                  | First quadrant |                  | First quadrant |
| 0                | $\bullet \rightarrow \square \rightarrow \circ$ | 16              | $\bullet \rightarrow \square \rightarrow \nabla$ |
|                  | $\circ \rightarrow \square \rightarrow \square$ | 18, 26          | $\triangle \rightarrow \square \rightarrow \circ$ |
| 2, 10            | $\hat{\bullet} \rightarrow \hat{\square} \rightarrow \hat{\circ}$ | 20              | $\hat{\triangle} \rightarrow \hat{\square} \rightarrow \hat{\circ}$ |
| 4                | $\nabla \rightarrow \hat{\square} \rightarrow \hat{\circ}$ | 22              | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ |
|                  | $\circ \rightarrow \hat{\square} \rightarrow \hat{\circ}$ | 24              | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ |
| 8                | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ | 28              | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ |
|                  | $\circ \rightarrow \hat{\square} \rightarrow \hat{\circ}$ | 30              | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ |
| 12               | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ | 30              | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ |
|                  | $\circ \rightarrow \hat{\square} \rightarrow \hat{\circ}$ | 30              | $\bullet \rightarrow \hat{\square} \rightarrow \circ$ |
Figure 16. The reduced $E_4$-term of the slice spectral sequence for the periodic spectrum $K_H$. Differentials are shown in red. Exotic transfers and restrictions are shown in solid blue and dashed green vertical lines respectively. The Mackey functor symbols are indicated in the table below Figure 17.
Figure 17. The reduced $E_{14} = E_{\infty}$-term of the slice spectral sequence for $K_H$. The exotic Mackey functor extensions lead to the Mackey functors shown in violet in the second and fourth quadrants. The Mackey functor symbols are indicated below.
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