Enumerating Isotopy Classes of Tilings guided by the symmetry of Triply-Periodic Minimal Surfaces

Benedikt Kolbe† and Myfanwy E. Evans‡

Abstract. We present a technique for the enumeration of all isotopically distinct ways of tiling a hyperbolic surface of finite genus, possibly nonorientable and with punctures and boundary. This provides a generalization of the enumeration of Delaney-Dress combinatorial tiling theory on the basis of isotopic tiling theory. To accomplish this, we derive representations of the mapping class group of the orbifold associated to the symmetry group in the group of outer automorphisms of the symmetry group of a tiling. We explicitly give descriptions of certain subgroups of mapping class groups and of tilings as decorations on orbifolds, namely those that are commensurate with the Primitive, Diamond and Gyroid triply-periodic minimal surfaces. We use this explicit description to give an array of examples of isotopically distinct tilings of the hyperbolic plane with symmetries generated by rotations, outlining how the approach yields an unambiguous enumeration.

Key words. Isotopic tiling theory, mapping class group, orbifolds, triply-periodic minimal surface, Delaney-Dress tiling theory, hyperbolic tilings

AMS subject classifications. 05B45, 05C30, 52C20, 57M07, 68U05, 82D25

1. Introduction. Tesselations of space from repeating motifs have a long and involved history in mathematics, engineering, art and sciences. Most of the mathematical literature has focussed on patterns in Euclidean spaces but the role of hyperbolic geometry in Euclidean tilings and more generally for the natural sciences is increasingly recognized. Chemists have long been using 3-dimensional symmetries for the analysis of chemical frameworks. From this point of view, one of the focuses is on how many of the intrinsic symmetries of the connected components of the graph are realized as symmetries of the ambient 3-space. However, assemblies of atoms (and molecules) in crystalline arrangements that are energetically favourable involve (intrinsic) curvature [23], and some real Zeolite frameworks were found to reticulate triply-periodic minimal surfaces (TPMS) [21, 22, 6].

These observations and ideas have led to a novel investigation of 3-dimensional Euclidean networks, where TPMS, are used as a convenient route to the enumeration of crystallographic nets and polyhedra in $\mathbb{R}^3$ [46, 36, 44, 24, 41, 5]. The ambient Euclidean symmetries of prominent TPMSs manifest as hyperbolic in-surface symmetries [40], so that symmetric tilings of the TPMSs give rise to symmetric graph embeddings in $\mathbb{R}^3$. Structures such as bounded hy-

∗ Funding: This work was funded by the Emmy Noether Programme of the Deutsche Forschungsgemeinschaft
† Department of Mathematics, Technische Universität Berlin.
‡ Department of Mathematics, Technische Universität Berlin (evans@math.tu-berlin.de).
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(a) A tiling of the hyperbolic plane with symmetry group $\mathbb{22222}$ is represented by the green and red edges, shown on the Poincaré disc model of $\mathbb{H}^2$. The lighter blue tiling shown is the $\ast 246$, which represents the symmetry of the Diamond surface.

(b) The decoration from (a) and a fundamental domain shown as a decoration of the Diamond triply-periodic minimal surface.

(c) The resulting net in $\mathbb{R}^3$ when the tile boundaries are taken as trajectories in Euclidean 3-space.

Figure 1.1: The progression from a tiling of the hyperbolic plane to a 3-dimensional net via decoration of the Diamond triply-periodic minimal surface.

perbolic tilings with kaleidoscopic symmetry [27, 41], hyperbolic tilings with more complicated symmetries [39, 38], simple unbounded tiles with a network-like structure [25, 26, 11, 10], and unbounded tiles with totally geodesic boundaries [12] have been explored, and resulting structures have been used in analysis in real physical systems [28]. Figure 1.1 shows a hyperbolic tiling that corresponds to a decorated orbifold fundamental domain, where the tiling respects the symmetry of the covering map onto the diamond $D$ surface. When the tile boundaries are considered as trajectories in 3-dimensional Euclidean space rather than geodesics in the surface, we obtain a net in $\mathbb{R}^3$.

Delaney-Dress combinatorial tiling theory [8] is an essential tool of the enumerative process. It deals with the classification of combinatorial classes of equivariant tilings, i.e. tilings with a specified symmetry group, of simply connected spaces in which every tile is a bounded disc. The 2-dimensional case lends itself particularly well to computational approaches, which are well-explored [20]. The combinatorial structure does not encapsulate all information about the isotopy class of the embedded tiling. In the context of stellate symmetry groups, tilings of distinct isotopy classes have been briefly explored [38, 11, 12, 10, 25]. We will rigorously examine the related ideas in a more general setting.

Here we present the enumeration of embedded, symmetric isotopy classes of combinatorial structures. Using Delaney-Dress tiling theory and its generalization to the classification of isotopically distinct equivariant tilings of surfaces [29], we will derive ways to completely enumerate all isotopy classes of equivariant tilings with symmetry group commensurate with
some fixed hyperbolic structure on a Riemann surface. To illustrate the approach, we will show how to enumerate all isotopy classes of equivariant tilings with rotational symmetry groups commensurate with a genus-3 Riemann surface with automorphism group equal to $\star 246$. This surface plays the role of unit cell for the target TPMS of the enumeration.

In this paper, we will use the notion of orbifolds and mapping class groups (MCGs). We interpret tilings as decorations on orbifolds, from whose point of view Delaney-Dress symbols, which encode the combinatorics of a tiling, represent triangulations of orbifolds. This gives rise to algorithms that enumerate all equivariant tilings on a hyperbolic Riemann surface in its uniformized metric. The mathematical foundations of this approach are given in [29]. In practice, any enumerative approach requires the conversion of a theoretical framework into a practical setting, and we describe this non-trivial implementation in detail, alongside limitations. Effectiveness of our enumerative process depends not only on the physical limitations of computers but also on what is known about MCGs in general. As far as we know, this represents the first application of MCGs outside of pure mathematics.

This paper is structured into five sections which cumulatively build towards the enumeration process of isotopy classes of tilings, culminating in a collection of examples that illustrate the approach. Sections 2 and 3 will be a recollection of orbifolds, combinatorial tiling theory and the new framework for isotopic tiling theory [29]. In section 4, we relate the geometry of orbifolds to the sets of generators of its group. Section 5 then uses the connection of the semi-pure braid group on the sphere to the mapping class group to move to a enumerative setting for tilings with stellate symmetry.

2. Orbifolds. Orbifolds are topological spaces with some extra structure. The diffeomorphic structure of a developable orbifold, which is the only important examples to us, is encoded in its symmetry group. In this section, we will explain the standard presentation of an orbifolds symmetry group [7, 48, 1]. A developable orbifold $\mathcal{O}$ is given by $\mathbb{H}^2/\Gamma$, where $\Gamma \subset \text{Iso}(\mathbb{H}^2)$ is discrete, i.e. a NEC group (non-Euclidean crystallographic group). The difference between $\mathbb{H}^2/\Gamma$ as a topological space and as an orbifold is that for the orbifold structure, one retains set of data relating to $\Gamma$ and can reconstruct $\mathbb{H}^2$ from $\mathbb{H}^2/\Gamma$. The group $\Gamma$ is called the fundamental group of the orbifold $\mathcal{O}$. The general theory for isotopic tilings developed in [29] also works for the more general case of a codomain with finite area, but in this paper we will focus on the compact codomain case.

The hyperbolic case is the only case that admits infinitely many isomorphism types of NEC groups $\Gamma$, which means that there are infinitely many non-diffeomorphic hyperbolic orbifolds.

Let $\mathcal{O}$ be a 2D orbifold. Conway’s orbifold symbol, as described below and in [7], provides a convenient notation for identifying its fundamental group up to isomorphism. This in turn specifies the diffeomorphic structure of $\mathcal{O}$. The Conway symbol for its fundamental group has the form $\Gamma := A \cdots \star abc \cdots \times \cdots \circ$. The generators have an interpretation as special kinds of curves [7, 42]. There are generators for the translations associated to the handles, given by $X$ and $Y$, and there is an oriented curve going around a handle that traces the commutator $\alpha := [X,Y] = XYX^{-1}Y^{-1}$. There are also generators for each gyration point of order $A$, and for a curve $\gamma$ in $\mathcal{O}$ going around the gyration point once we have $\gamma^A = 1$, where we again interpret the curve as a deck transformation. For each mirror we have the usual Coxeter group relations, which depend on the angles of the intersecting mirrors. However, in the case
where the interior of the orbifold contains nontrivial features, we actually need to choose one mirror per boundary component that we give two generators $P$ and $Q$, ordered in positive orientation corresponding to its two mirror halves and one generator $\lambda$ for the curve that goes around this boundary component once in positive orientation. We then add the relation $P = \lambda^{-1} Q \lambda$. Next, going around a cross-cap corresponds to a generator $\omega$ with $\mathbb{Z}^2 = \omega$, where $Z$ corresponds to the curve entering the cross-cap once. There is one global relation for an orbifold, namely, the product of all Greek letters in the above has to be trivial, i.e.

$$\gamma \ldots \lambda \ldots \omega \ldots \alpha \ldots = 1. \quad \text{(2.1)}$$

In the sequel, we shall refer to this presentation as the standard presentation of the fundamental group of $\mathcal{O}$. For convenience, we can also assume that in the presence of a crosscap, all handles are replaced by two crosscaps each [15].

3. Enumerative aspects of isotopic tiling theory of Riemann surfaces. The ideas in this paper are centered around tessellations of $\mathbb{H}^2$, for which we employ the framework of isotopic tiling theory. The theory makes heavy use of combinatorial tiling theory, which was developed for simply connected spaces in [8] in general and worked out in detail for the 2-dimensional case in [20]. We will only work with the 2-dimensional case and make use of its generalization to isotopic tiling theory in non-simply connected spaces in [29].

Combinatorial tiling theory classifies all possible equivariant types of combinatorial tilings on space forms $\mathbb{X}$. It deals with the case that each tile is a closed and bounded disc. Furthermore, the symmetry group of the tiling is assumed to act cocompactly on $\mathbb{H}^2$. A set $\mathcal{T}$ of such discs in $\mathbb{X}$ is called a tiling if every point $x \in \mathbb{X}$ belongs to some disc (tile) $T \in \mathcal{T}$ and if for every two tiles $T_1$ and $T_2$ of $\mathcal{T}$, $T_1 \cap T_2 = \emptyset$. Here, $S^0$ denotes the interior of a set $S$. This paper assumes all tilings to be locally finite, meaning that any compact disc in $\mathbb{X}$ meets only a finite number of tiles.

We call a point that is contained in at least 3 tiles or is located at a two-fold gyration point a vertex, and the closures of connected components of the boundary of a tile with the vertices removed edges. Let $\mathcal{T}$ be a tiling of $\mathbb{X}$ and $\Gamma$ be a discrete subgroup of $\text{Iso}(\mathbb{X})$. If $\mathcal{T} = \gamma \mathcal{T} := \{ \gamma T \mid t \in \mathcal{T} \}$ for all $\gamma \in \Gamma$, then we call the pair $(\mathcal{T}, \Gamma)$ an equivariant tiling. Note that $\Gamma$, its symmetry group, is not required to be the full group of isometries of the tiling. We call two tiles $T_1, T_2 \in \mathcal{T}$ in a tiling equivalent or symmetry-related if $\gamma T_1 = T_2$ for some $\gamma \in \Gamma$. We call the subgroup of $\Gamma$ that leaves invariant a particular tile $T \in \mathcal{T}$ the stabilizer subgroup $\Gamma_T$ of $T$. A particular tile is called fundamental if $\Gamma_T$ is trivial and we call the whole tiling fundamental if this is true for all tiles. We will call tilings with only one symmetry class of tiles tile transitive. When building such tilings, the whole space can be tesselated starting from one chosen tile, which we will also frequently call the fundamental tile.

One can interpret any fundamental domain for the action of a group on $\mathbb{X}$ as a tile of a fundamental equivariant tiling that is tile transitive. One of the central results in combinatorial tiling theory is that there are only finitely many combinatorially distinct fundamental tile transitive tilings. Starting from these fundamental tilings, all other equivariant tilings with the same symmetry group can be obtained by applying the GLUE and SPLIT operations [20].

There is a direct way from the D-symbol encoding an equivariant tiling to a decoration of a specified realization of an orbifold as a fundamental domain in $\mathbb{H}^2$ (with appropriate edge
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identifications) [29]. In this paper, we focus exclusively on decorations given by geodesics on the orbifold in its induced metric from $\mathbb{H}^2$.

Given a Riemann surface $S$ such as the genus-3 Riemannian surface that gives rise to the diamond surface, to enumerate all symmetric embeddings of graphs, one first has to identify all of its hyperbolic symmetry groups. For the diamond surface, the 131 subgroups of the smallest (area-wise) symmetry group $\Gamma_S = \star 246$ have been listed [43] and the fundamental tilings for the Coxeter groups have been enumerated and projected onto the Diamond TPMS [41]. Note that when two different hyperbolic symmetry groups of $S$ are abstractly isomorphic, in isotopic tiling theory they nevertheless require independent treatment [29].

For a general hyperbolic symmetry group $G$, isotopic combinatorial tiling theory states that there are only a finite number of such fundamental tiles w.r.t. to a given set of generators, i.e. given the generators of a hyperbolic group in $\text{Iso}(\mathbb{H}^2)$, there are only finitely many different fundamental tiles with a different combinatorial structure such that the generators act on the boundary of the tile. Each of these can be enumerated by using Delaney-Dress symbols for tilings [9]. The placements of generators of interest are naturally in bijection to the group of type preserving outer automorphisms $\text{Out}_t(G)$ of $G$ [29, Theorem 5.3]. We will derive ways to enumerate the possible placements for (the invariant subsets associated to the) generators for $G$ in $\mathbb{H}^2$ in section 4 starting from a given set. In this section, we will concentrate on finding a nice starting set of generators.

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Let

$$T \subset G \subset \Gamma_S \subset \text{Iso}(\mathbb{H}^2)$$

be NEC groups, with $T$ the realization of the fundamental group of the Riemann surface $M$ and $\Gamma_S$ its smallest (area-wise) symmetry supergroup. Then there is a fundamental domain $F_G$ for $G$ produced by gluing together some fundamental domains of $\Gamma_S$ such that there is a set of geometric generators for $G$ that act on the boundary of $F_G$. For $T$, this can be done so that the fundamental domain for $T$ is a geodesical polygon $p$ in $\mathbb{H}^2$ assembled from copies of such a domain for $F_{\Gamma_S}$. The group $T$ is generated by the pairwise edge identifications of $p$. Note that there might be different but isomorphic versions of $T$ satisfying (3.1), related to different presentations of $T$ as a subgroup of $\text{Iso}(\mathbb{H}^2)$. We can assume that we fix one that is normal in $\Gamma_S$. We similarly obtain a geometric placement of a set of generators for any $G$ such that $F_G$ lies within $p$. One such set of generators will be our starting point and serve as a reference frame. See figures 6.7 and 6.2 below for an example where this set is not unique. In more complicated cases, with many such choices, we choose a starting set of generators according to the length of the circumference of a fixed fundamental tile in its induced quotient metric, with generators acting on its boundary, where we take the generators with minimal circumference. Note that these two different methods of finding a starting set agree where they overlap. The idea is simply to use a least-sheared version of the tiling as a starting point.

A phenomenon related to the ambiguity of the starting set of generators occurs when some tiling of $\mathbb{H}^2$ exhibits symmetries that do not inherently belong to the symmetries of the surface, for example, when a surface doesn’t have reflectional symmetries but the stellate fundamental domain is obtained by mirroring the fundamental domain of the underlying Coxeter supergroup. The emergence of accidental symmetries is a phenomenon that occurs regularly and is a result of the combinatorial tiling theory being cast in terms of equivariant
tilings, where some symmetry group of a tiling must be specified, possibly without being the full group of symmetries. These accidental symmetries can easily be broken by adding decorations to the tiles, but because of their existence, some tilings are invariant w.r.t. a change of generators and associated decoration of the orbifold. We will see examples of this in section 6.

For two isomorphic subgroups \( G_1, G_2 \subset \Gamma_S \), there is a homeomorphism \( \varphi : \mathbb{H}^2 \to \mathbb{H}^2 \) that takes a set of generators for \( G_1 \) in \( p \) to such a set for \( G_2 \) [33, Theorem 3]. It can be very helpful to study which such \( \varphi \) lift to a homeomorphism of \( M \) (or any translational surface) because homeomorphisms of the surface \( M \) can be assembled from a combination of Dehn twists, whose entangling characteristics in the embedding of \( M \) into \( \mathbb{R}^3 \) are much easier to understand. In order for \( \varphi \) to lift to a homeomorphism of \( M \) we must have \( \varphi_* T \subset T \). Here, each version of \( T = \pi_1(M) \) is understood to be contained in its respective realization as a subgroup of \( \pi_1(O) \).

For example, consider the abstractly isomorphic groups 77 and 54 in [43] in their given realizations. The translations of \( M \) in either group are the result of composing two 2-fold rotations. It is easy to see that no liftable \( \varphi \) exists that maps the given generators of 77 to those given for 54. Any such map realizes the isomorphism that maps the given set of generators that border the fundamental domain to similar generators in the other group. However, such a map would change the side identifications of the underlying genus-3 dodecagon. This implies, in particular, that the tilings with the same D-symbol associated to these generators are topologically distinct, in the sense that they are not related by a homeomorphism of the surface. Note that this does not mean that there are no isomorphisms from 77 to 54 that lift. In view of the proof of proposition 5.5 in [29], there will only be finitely many isomorphism classes of maps from 54 to 77 modulo liftable ones.

We now have a way of obtaining all isotopically distinct tilings and therefore symmetric drawings on the surface \( M \) associated to a given set of generators of the symmetry group as isometries of the hyperbolic plane. So, the question we are interested in investigating asks about all ways that we can choose minimal generating sets for a given symmetry group that is given abstractly. We then look at all of the realizations of this group as a group of symmetries for \( M \). Any decoration on the abstract orbifold can then be interpreted as a tiling of the surface \( M \), possibly with unbounded tiles.

4. Relating the Geometry of Orbifolds to the Sets of Generators of its Group. The set of sets of generators that is relevant for producing isotopically distinct tilings of a Riemann surface \( M \) is the orbifold MCG [29, theorem 5.3]; we refer to this isomorphism as the MCG isomorphism. Just knowing that such a 1 : 1 correspondence exists is insufficient for applications though, hence we begin by making the definition of the MCG isomorphism more suitable in this setting. In order to explain the isomorphism concretely, we use the notation for the generators and group relations of the standard presentation of an orbifold given in section 2.

We will describe the MCG isomorphism by relating the action of certain generators of the MCG on curves to the resulting group elements in the orbifold fundamental group \( \Gamma \), using the interpretation of group elements in \( \Gamma \) as homotopy classes of curves in \( O \). On this note, we have the following classical result for covering spaces, which we will revisit to discuss the orbifold case.
Let \( f \) be a homeomorphism of a surface \( S \) with base point \( p \), which, in case of a regular cover lifts to a homeomorphism \( \tilde{f} \) of the (unbranched) universal covering space, where the group of deck transformations acts transitively on fibres. Therefore, the choice of base point \( \tilde{p} \) in the fiber above \( p \) is arbitrary. Denote the deck transformation corresponding to \( \alpha \), with \( [\alpha] \in \pi_1(S, p) \), by \( \delta_{[\alpha], p} \) and the homomorphism \( f \) induces on the fundamental group by \( f_* \).

We then have the relation

\[
\tilde{f} \circ \delta_{[\alpha], p} \circ \tilde{f}^{-1} = \delta_{f_*([\alpha]), f(p)}.
\]

Indeed, first recall that a deck transformation of a classical covering space is determined uniquely by where it sends a single point of the path-connected covering space. Now, because of how \( \tilde{f} \) is constructed from \( f \), \( \delta_{f_*([\alpha]), f(p)} \) sends \( \tilde{f}(\tilde{p}) \) to \( \tilde{f}(\delta_{[\alpha], p}(\tilde{p})) \). This is exactly what the left hand side of (4.1) does. This also explains the relation of the MCG isomorphism in its form in [29] to the isomorphism given in the formulation of the classical Dehn-Nielsen-Baer theorem in [14]. The statement (4.1) is also true for the more general case of regular branched covering spaces that we consider, since in the above we only used that deck transformations are uniquely determined by where they map a single point. Away from the branch point set, this is also true for our surfaces and their orbifold covering spaces.

This suggests a method to construct the automorphisms of \( \Gamma \) that are the images of homeomorphisms of \( \mathcal{O} \) explicitly. The generators of \( \Gamma \) correspond to homotopy classes of closed orbipaths in the orbifold based at a point not in the singular set (see [42] for details, which also includes the noncompact codomain case). One can now simply draw pictures of these generators as curves in \( \mathcal{O} \), look at how these are changed by some set of geometrical generators of the MCG, and read off the new word representing the resulting path. Doing this for all generators defines an induced automorphism of \( \Gamma \). It is essential to get the loops and their orientations right, so that after cutting \( \mathcal{O} \) along these one obtains the standard group relation \( \Omega \). See [19] for an illustration of such curves on an orientable surface. Different presentations of the fundamental group correspond to a different set of curves in the orbifold and different representations of automorphisms. We will only work out the standard case. For applications, the results have to be translated to the presentation of interest via some appropriate isomorphism. See [50] for an account of similar work in related fields.

One could also use the method described in [35] to find an algebraic representation of the MCG. However, we are interested in a presentation that captures our intuition of what a complexity ordering for decorations on surfaces might look like, so we will look for geometric generators. Also, the method in [35] is very involved and most probably results in a presentation that is too complicated for most computational algorithms to handle. Moreover, the geometric generators feature a relation to the twisting of the tiles around handles of the Riemann surface covering. Note that computational group theory packages in computer algebra programs available today such as GAP can only solve relatively simple problems. For example, even the world problem for the classical MCG for genus-3 surfaces is computationally too involved to comprehensively solve for the Knuth-Bendix program for GAP, despite it being well-known that the problem is solvable.

We can now finally present a concrete form of the MCG isomorphism. We start with an important special case, which will turn out to be the backbone of all our examples. This is the case where the orbifold is stellate, the only features it contains being gyration points. The
restriction means that topologically, $\mathcal{O}$ is a sphere with some singular points. Note, however, that fixing a presentation of the symmetry group $\Gamma$ gives $\mathcal{O}$ more structure than what its topology implies. In particular, the order of gyration points is fixed and gyration points have neighbors, corresponding to the neighboring factors in the global group relation (2.1) of $\Gamma$.

Consider in the stellate case the group relation (2.1) and denote its left hand side by $\Pi$. For any instance of $x_i x_k$ in $\Pi$ define $\Phi(x_k) := x_k x_i x_k^{-1}$, $\Phi(x_j) := x_j$ for $j \neq k$ and $\Pi_\Theta$ as the word where instead of $x_i x_k$ we have $\Phi(x_k)\Phi(x_i)$ and everywhere else $\Phi(x_j)$ instead of $x_j$. Successive application of this operation and subsequent renaming of the two elements involved does not change the global group relation of the $\Phi(x_i)$, as long as the involved gyration points are of the same order, so represents an automorphism. It is well-known that this automorphism corresponds directly to the action of the generators of the Braid group $B_n$ on the word $\Pi$, which also serves as a definition for $B_n$ [31]. The braid group is thus of fundamental interest in the study of such orbifolds.

Figure 4.1 yields a first geometric interpretation of an element of the MCG, namely, a Dehn half-twist of two neighboring gyration points, which is a prominent generator of the MCG. However, because the braid group acts on points on the sphere, we have to add some relations to the braid group to obtain the MCG of the sphere with $n$ marked points. This presentation is classical and given in [14] as

\[
Mod(S_{0,n}) = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i, \sigma_j = 1 \quad |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
(\sigma_1 \cdots \sigma_{n-1})^n = 1, \\
(\sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 = 1) \rangle.
\]

Assume now that not all gyration points are of the same order. We must instead consider appropriate subgroups of the braid group that only account for allowed permutations, and this corresponds to a subgroup of the MCG of finite index. Note that there is a well-known short exact sequence

\[
1 \to \text{PMod}^\pm(\mathcal{O}) \to \text{Mod}^\pm(\mathcal{O}) \to \Sigma_n \to 1,
\]

where $\Sigma_n$ is the symmetric group on $n$ elements. The problem could be solved geometrically by using that Dehn twists around essential closed loops generate the MCG and use a decomposition of permutations into transpositions of only neighboring points to obtain half twists that exchange two arbitrary marked points. In cyclic notation, one readily verifies that $(a, b) = (a, a + 1)^{-1}(a + 1, b)(a, a + 1)$ gives a procedure to decompose permutations in this way. Squaring then yields the Dehn twist along a simple closed loop around these two points. Having found the generators, we could use the Reidemeister-Schreier process to find a presentation for the subgroups of interest. We will use a slightly different approach. Artin proved that the pure braid group $PB_n$, which we can define here as the subgroup of $B_n$ consisting of those elements that maps every generator in $\Pi$ to a conjugate of itself, is generated by the elements

\[
a_{i,j} = (\sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{j-1} \cdots \sigma_{i+1})^{-1}, \quad 1 \leq i < j \leq n.
\]
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We must then simply add those elements \( \sigma_i \) corresponding to the allowed permutations of generators. We will analyse this situation in more detail below.

Let us consider a twist around an essential closed curve \( c \) on \( O \), the underlying topological space of \( O \), that goes around some feature(s), corresponding to an element \( g \in \pi_1(O) \). Let \( \gamma \) be an orbipath corresponding to a generator of \( \Gamma \). If \( \tilde{\gamma} \) denotes the image of \( \gamma \) under the twist associated to \( c \) and the MCG isomorphism, then we see that we have two cases to consider, according to whether or not \( \gamma \) is inside the curve \( c \) or not. If it is inside, then the twist around \( c \) corresponds to the deck transformation \( g \) and therefore, interpreted as deck transformations of \( H_2 \to O \), \( \tilde{\gamma}(g(x)) = g(\gamma(x)) \) for all \( x \in H_2 \), or, equivalently, \( \tilde{\gamma} = g\gamma g^{-1} \). If not, then \( \tilde{\gamma} = \gamma \). This defines the resulting automorphism in a very concrete way. However, there is some leeway here in the choice of which side is the inside of a given curve, which should be made consistently for all automorphisms and corresponds to the choice of a base point for curves in \( \Gamma \).

For example, consider the pure mapping class group \( \text{PMod}(S_5) \subset \text{Mod}(S_5) \) that fixes all punctures. It is well known that \( \text{PMod}(S_5) \) is generated by Dehn twists \( \{ t_i \}_{i=1}^5 \). After stereographic projection, imagine all punctures lying in a (cyclically) ordered row. We find that we can take the curves \( c_i \) to be given by those that enclose exactly two consecutive points. If \( c_i \) encloses the punctures \( \{ p_i \} \subset \{ 1, \ldots, 5 \} \), then \( c_i \) can be represented in \( \pi_1(O) \) by \( \Pi_{j \in K_i} e_j : = t \). We can then define \( \varphi : \text{PMod}(S_5) \to \text{Out}(\pi_1(2222a)) \) by taking \( \varphi(t_i) \) to be the automorphism that fixes each \( r_j \) for \( j \in K_i \) and sends \( r_j \) to \( tr_j t^{-1} \) for \( j \notin K_i \).

We are now in a position to give representatives for the most common elements and generators in MCGs. It is well-known that the pure mapping class group that fixes all marked points individually is generated by Dehn twists in the orientable case and by what are called boundary or crosscap slides in the nonorientable case [32, 30]. Half-twists are generators that interchange the positions of marked points.

Dehn twists along simple closed curves are the most straightforward to handle, but there are also subtleties one has to be wary of. First and foremost, we consider right Dehn twists. As the above example illustrates, we will treat features that are adjacent in (2.1) differently than ones that are not. We then have a natural geometric interpretation of Dehn twists along simple curves that go around a chain of adjacent elements. A Dehn twist \( t_c \) along a curve \( c \) that encloses a chain of features given by Greek symbols, say \( \beta_1, ..., \beta_k \), corresponds to the automorphism of \( \Gamma \) given by

\[
\gamma \mapsto \begin{cases} 
(\beta_1 \cdots \beta_k)\gamma(\beta_1 \cdots \beta_k)^{-1} & \text{if } \gamma \text{ is inside } c, \\
\gamma & \text{otherwise}.
\end{cases}
\]

We derive the representation in \( \text{Out}(\pi_1(O)) \) for half-twists in figure 4.1, with the result of the twist indicated in figure 4.1a given in figure 4.1b. Note that ordering of the marked points, which comes from laying the surface down flat after cutting along any possible curves. Refer to [19, p. 5] for a picture of the situation. The result itself reproduces the well-known presentation of the braid group above, but deriving it using our methods illustrates our formalism.

It is easy to see that figure 4.1 translates to the half twists around \( S_1 \) and \( S_2 \) taking the
\begin{equation}
\begin{aligned}
S_1 &\mapsto S_1 S_2 S_1^{-1}, \\
S_2 &\mapsto S_1.
\end{aligned}
\end{equation}

One can readily check that this transformation squares to the right Dehn twist around these two marked points. Figure 4.2 shows the effect of a half twist on the starting set of generators with a fixed D-symbol for the tiling.

The rest of the section deals with more complicated generators for more complicated orbifolds. We start with the generators of the MCG in the group of type preserving outer automorphisms $\text{Out}_t(\pi_1(\mathcal{O}))$ for a nonorientable closed surface. For a general reference on the MCG of nonorientable surfaces, see [30] and [47] and the references contained therein. It was proved in [30] that additionally to Dehn twists along two-sided curves (Dehn-twists are only well-defined for such curves), cross-cap slides and boundary slides are needed to generate the MCG of a nonorientable surface. Although there is as far as we are aware as-
yet no known presentation for the MCG of a general nonorientable surface in terms of these generators, these generators are very natural and cannot be deconstructed any further into constituent geometric transformations of other types. Along with these, we will also look at crosscap transpositions. The derivation of the form these take in $\text{Out}_1(O)$ are derived from the geometric picture of the surface cut open to form a polygon with pairwise identification of edges to produce the standard presentation of the corresponding fundamental group. Starting from any presentation one can cut and glue the surface suitably to obtain the standard presentation.

Let $\zeta$ be a two sided loop that goes through two neighboring crosscaps represented by their deck transformations in $\mathbb{H}^2$, $A$ and $B$, and encircles no other features. Let $\alpha$ be the curve associated with $A$. A crosscap transposition $U_{\alpha,\zeta}$ is supported in a neighborhood of $\zeta \cup \alpha$, which is a Klein bottle $K$ with one boundary component, see figure 4.3a, and has the following representation in $\text{Out}_1(O)$, where all other generators are constant.

$$\begin{align*}(A, B) &\mapsto (B, (B^2)^{-1}AB^2) \quad (4.6)\end{align*}$$

For an illustration of this and the following transformations, refer to [37]. The Dehn twist $T_{\zeta}$ has the representation

$$\begin{align*}(A, B) &\mapsto (AB^{-1}A^{-1}, AB^2). \quad (4.7)\end{align*}$$

The crosscap slide $Y_{\alpha,\zeta} = U_{\alpha,\zeta}T_{\zeta}$ is represented by

$$\begin{align*}(A, B) &\mapsto (AB^2, (AB^2)^{-1}B^{-1}(AB^2)). \quad (4.8)\end{align*}$$

It is straightforward to check that these transformation yield automorphisms of $\Gamma$ and that they satisfy

$$\begin{align*}Y_{\alpha,\zeta}^2 = U_{\alpha,\zeta}^2 = T_{\partial K},\end{align*}$$

which serves as an algebraic proof of this well-known geometric relation [32]. Now, a boundary slide $S_A$ in a nonorientable surface that has a boundary or puncture $P$ neighboring $A$ is represented by

$$\begin{align*}(P, A) &\mapsto ((PA)P^{-1}(PA)^{-1}, PA), \quad (4.9)\end{align*}$$

which again is readily seen to induce an automorphism and again comes from drawing a picture with curves. Note that our definition of homotopies of surfaces and the MCG means that boundaries behave exactly like punctures. We exemplify the derivations of the above equations by drawing the picture for (4.7) in figure 4.3, with figure 4.3b showing the result of twisting, from which (4.7) can be read off. Note that outside the shown neighborhood, the transformation is the identity.

In the orientable case, following [45] and [3], we use the same approach of trying to find representations of generators corresponding to sliding singular points or boundaries around the fixed point free features, i.e. translations associated with handles.

The pictures one has to draw quickly become rather involved and we know of no viable systematic approach to finding the feasible representations of arbitrary elements of the MCG.
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Figure 4.3: Dehn twists around two sided curves in nonorientable surfaces

by hand, as many pictures are very subtle. We will see and example of this in figure 4.4. The standard relation (2.1) for the orbifold groups fixes a way of choosing the curves that represent the orbifold group elements, but only up to a change of basepoint.

Just like in the nonorientable case, we will focus on building representations of more general geometric generators from simpler transformations that are supported in smaller subsurfaces. Assume that $\Omega$ has a subword of the form $S_n[X,Y]$. Then, moving $S_n$ around $Y$ corresponds to the situation of figure 4.4. Figure 4.4b shows the effect on the blue curve of pushing $S_n$ around the red curve.

This picture shows the difficulty in choosing which picture to draw. While the picture suggests that $S_n \mapsto Y^{-1}S_nY$, the picture is only defined up to conjugation of group elements because the choice of base point is arbitrary. Note also that the outer automorphisms resulting from the isomorphism are also only defined up to conjugation. We thus find that we can write the point push around of $S_n$ around $Y$ as

\[
\begin{align*}
S_n &\mapsto S_n Y^{-1} S_n Y S_n^{-1} \\
X &\mapsto S_n Y^{-1} S_n X S_n^{-1} \\
Y &\mapsto S_n Y S_n^{-1}.
\end{align*}
\]

(4.10)

Note that $XS_n^{-1}$ is a hyperbolic transformation if no fixed point of $S_n$ lies on the axis of $X$. We therefore see that $Y$ and thus $X$ are mapped to hyperbolic transformations. A similar picture for a twist around $X$ yields

\[
\begin{align*}
S_n &\mapsto X^{-1} S_n X \\
X &\mapsto X^{-1} S_n^{-1} X S_n X \\
Y &\mapsto Y X^{-1} S_n X.
\end{align*}
\]

(4.11)

Note that in both (4.10) and (4.11) forgetting that $S$ is a feature represents the trivial automorphism.

Assume now that $\Omega$ has a subword of the form $[a_1,b_1][a_2,b_2]$. A similar argument using Lickorish’s generators for the MCG of a closed surface without features yields that there are two twists per handle.
that leave each commutator relation invariant on its own. These are given by

\begin{align}
\begin{cases}
a_i \mapsto a_i \\
b_i \mapsto b_i a_i,
\end{cases}
\end{align}

and

\begin{align}
\begin{cases}
a_i \mapsto a_i b_i^{-1} \\
b_i \mapsto b_i.
\end{cases}
\end{align}

On the other hand, there is a more complicated twist amongst the Lickorish generators that mingles two adjacent handles, which after drawing another picture takes the following form.

\begin{align}
\begin{cases}
a_1 \mapsto b_2 a_1 b_2^{-1}, \\
b_1 \mapsto b_2 a_1^{-1} b_2^{-1} a_1 b_1 (b_2 a_1^{-1})^{-1}, \\
a_2 \mapsto (b_2 a_1^{-1}) b_2^{-1} a_1 a_2 (b_2 a_1^{-1})^{-1}, \\
b_2 \mapsto (b_2 a_1^{-1}) b_2 (b_2 a_1^{-1})^{-1}.
\end{cases}
\end{align}

All twists are defined through orientation preserving homeomorphisms of a regular neighborhood of some curve to the annulus with standard orientation of the plane. The orientations of curves themselves do not play a role when defining what a right Dehn twist is. Changing the orientation of $\mathcal{O}$ amounts to changing the orientation of every curve on the orbifold as well as the direction of twists. Last but not least, we need a way to represent a point push around a handle that is not neighboring the pushed point. Half-twists can be applied to change the ordering of the singular points and boundary components, cross-cap transpositions achieve the same for cross-caps, so we are only missing a 'transposition of
handles.' For two neighboring handles \([a_1, b_1][a_2, b_2]\) like above, such a transformation can easily be verified to be given by

\[
\begin{align*}
    a_1 &\mapsto a_2, \\
    b_1 &\mapsto b_2, \\
    a_2 &\mapsto [a_2, b_2]^{-1}a_1[a_2, b_2], \\
    b_2 &\mapsto [a_2, b_2]^{-1}b_1[a_2, b_2].
\end{align*}
\]

With these generators, the presentation of most MCGs can be readily translated into the form they take in \(\text{Out}_t(\pi_1(O))\). We will encounter an example of a MCG that needs more work in the next section.

5. The Semi-Pure Braid Group on the Sphere. In this section we will derive presentations for the stellate MCGs to produce the tilings in the next section and look at braid groups \(B_n\) and their connection to MCGs. We will employ slightly non-standard presentations of braid groups and certain subgroups. See [2] and [4] for the mathematical foundations of braid groups. We will assume in the following that \(n \geq 3\). Note that virtually all references for the braid group in the literature are algebraically influenced and as such do not follow functional notation and if interpreted as maps need to be read from left to right. This paper follows that tradition, so implementations of the results in this paper must be translated to functional notation.

Definition 5.1. Given a partition \(\mathcal{P}\) of \(n \in \mathbb{N}\), we define the semi-pure braid group \(SPB_{\mathcal{P}}\) of type \(\mathcal{P}\) to be the subgroup of the braid group \(B_n\) that under the canonical morphism to the symmetric group \(\Sigma_n\) yields only permutations that respect the partition of \(n\), i.e. only permute elements within each set in the partition. We similarly define the semi-pure mapping class group.

While seemingly not well-known, the semi-pure braid group of the plane was studied in [34]. We will examine concrete presentations of these groups in section 6.

The braid group is important to us for the following reasons. There is a well-known short exact sequence that relates the braid group of a surface \(S\) to the MCG of \(S\) and an \(n\)-times punctured version of \(S\), which we denote with \(S^*\). This sequence is known as the Birman exact sequence [14, Theorem 9.1] and, as long as \(\pi_1(\text{Hom}^+(S)) = 1\), reads

\[
1 \to \pi_1(C(S, n)) \to \text{Mod}(S^*) \to \text{Mod}(S) \to 1,
\]

where \(C(S, n)\) is the configuration space of \(n\) distinct, unordered points in \(S\) and the MCGs involved stem from orientable mappings. Surfaces with negative Euler characteristic satisfy \(\pi_1(\text{Hom}^+(S)) = 1\) [16, 17, 18]. Note that the 'simpler' case of stellate orbifolds that we are working with does not obey the above sequence.

The standard presentation of \(B_n\) [2] is phrased in terms of the generators \(\sigma_i\). Each \(\sigma_i\) crosses the strand in position \(i\) in front of the strand in position \(i+1\). In order to find a presentation of the spherical braid group \(B_n(S)\), we need to append the relation \(X := \sigma_1 \cdots \sigma_{n-1} \sigma_n \cdots \sigma_1 = \text{id}\) to the presentation of \(B_n\) in terms of the \(\sigma_i\) [13, 49].

Consider the element \(z := (\sigma_1 \cdots \sigma_{n-1})^n \in B_n\). The infinite cyclic center of the pure braid group \(PB_n\) and of \(B_n\) is generated by \(z\) [2]. Taking the interpretation of \(B_n\) as the MCG of the \(n\)-times punctured disc with the additional restriction that the classes of homeomorphisms fix the boundary pointwise, one sees that geometrically, \(z\) corresponds to the Dehn twist around the boundary. Following equation (9.1) and figure 9.6 of [14], adding the relation \(z = \text{id}\) to the relations of \(B_n(S)\) turns this group into the corresponding MCG of the sphere \(\text{Mod}(S_n)\) with \(n\) identical conical singularities.

The following uses both the notion of the braid group as the fundamental group of a configuration space as well as that on the MCG of a disc with boundary and marked points. For mixed Braid groups on the sphere, it is important to find an expression of the generators of the nontrivial extra
relations resulting from the topology of the sphere in terms of generators of the pure Braid group. This is possible because the generators of the extra relations belong to the subgroup of pure braids. We start with how to express \( z \) in terms of the \( a_{i,j} \). Using the interpretation of the braid group as the fundamental group of a configuration space on \( n \) points [14, Chapter 9], we have

\[
(5.2) \quad z = (a_{1,2}a_{1,3} \cdots a_{1,n}) \cdots (a_{n-2,n-1}a_{n-2,n})(a_{n-1,n}).
\]

We will pursue the same line of reasoning for the relation \( X = \text{id} \). The original proof in [13] that \( X = \text{id} \) suffices as an extra relation for the full braid group on the sphere hinges on the existence of twists that transforms the analogous relations for the other braids into this one. In terms of the pure braid group, however, we must add back in the missing relations and express them in terms of the pure generators. Each relation we need to add has an interpretation as the braid where the \( i \)-th strand passes over all \( n - i \) strands to its right, turning back to pass underneath all strands until the first one, and back over the first \( i - 1 \) back to where it started. The first factor in the expression will be \( a_{i,i+1} \cdots a_{i,n} \). The second factor, for similar reasons, is given by \( a_{i-1,i}a_{i-2,i} \cdots a_{1,i} \). This explains how to write the relation corresponding to \( X \) in terms of \( n \) relations on the generators of the pure braid group.

We shall denote the MCGs thus obtained from a partition \( \mathcal{P} \) by \( \text{Mod}(S_\mathcal{P}) \).

6. Tilings with Stellate Symmetry Groups in \(*246*\). This section will be dedicated to the explicit construction of tilings of \( \mathbb{H}^2 \) that are commensurate with the candidate TPMS. We start with all stellate groups that are subgroups of \(*246* [43]. For producing tilings, we are only interested in the orientation preserving MCGs \([29]\). Another simplification comes from the fact that the three-punctured sphere \( S_{0,3} \) trivial MCG, so the only nontrivial elements stem from permutations of the points. We therefore have

\[
(6.1) \quad \text{Mod}(S_{0,3}) \cong \Sigma_3.
\]

We obtain the following list of relevant MCGs. Note that the generator of \( \mathbb{Z}_2 \) is in both cases a half-twist about the two gyration points of the same order.

In principle, the ordering of the gyration points does not matter in determining the isomorphism class of the orbifold group. However, the way that the group ‘sits inside’ the genus three surface representing a translational unit of the TPMS does pin down one particular presentation, which determines the ordering of the gyration points up to cyclic permutations. In table 6.1, this is only important for 2323 and 6262. The representation of a half twist involving two non-adjacent points is more complicated than (4.5), but can be readily computed using the decomposition of a permutation into adjacent transpositions from section 4.

A presentation of the subgroup of \( B_n \) consisting of exactly those elements that leave invariant a partition of \( n \) of the form \((1 \cdots h_1)(h_1 + 1 \cdots h_2)(h_2 + 1 \cdots h_m)\) was derived in [34], following an application of the Reidemeister-Schreier method to glean presentations for subgroups from a presentation of the group. After reexamining theorem 4, the most general theorem in the paper, we find that there, \( i \) can equal \( j \) for the generators \( A_{h_i,h_{i+1}} \). Also, the braid relations in equation (2.2) have to be replaced by the usual ones as applicable. Lastly, in the indexing of (2.2), \( i \) can equal \( h_{j+1} - 2 \).

A technical but very important detail concerns the algebraic braid group given in terms of abstract group elements, whose multiplication is read from left to right, whereas the MCG elements act from right to left. This leads to \((\sigma_1 \cdots \sigma_j)^{-1} \) acting as \( T_{\sigma_j}^{-1} \cdots T_{\sigma_1}^{-1} \), where the \( T_{\sigma_j} \) are the associated twists in the MCG of a disc corresponding to the twists \( \sigma_j \) in the braid group. Therefore, seeing as \( A_{i,j} \) corresponds to the Dehn twist around the simple curve that encircles the marked points \( i \) and \( j \), we observe that \( A_{i,j} = \sigma_{j-1} \cdots \sigma_{x} A_{x,i} \sigma_{x+1}^{-1} \cdots \sigma_{j-1}^{-1} \) for \( x < j \). This also follows directly from the definition of the Artin generators \( A_{i,j} \). This observation also lets us similarly establish \( A_{x,j} = \sigma_{x}^{-1} \cdots \sigma_{z+1}^{-1} A_{z,j} \sigma_{z-1} \cdots \sigma_{x} \) for \( j > z > x \).
Lastly, to find a presentation for a mixed Braid group it is useful to assume that the elements that can be exchanged are grouped together as in the presentation given in [34]. If this is not the case, then we must conjugate all elements of the mixed braid group with a braid that maps to an element in the symmetric group realizing the appropriate partition from the one at hand. The ordering of the points within a part in the partition is arbitrary.

Using the computer programming language GAP and in particular the Knuth-Bendix package, we have implemented these results to yield a list of MCG elements ordered by word length from a presentation of the MCG. We present here a sequence of pictures of tilings that were produced using MATLAB and are part of an exhaustive enumeration of isotopy classes of tilings. We concentrate on the most challenging cases and highlight some of the subtleties one encounters on the way.

Figure 6.1 shows isotopically distinct fundamental tilings with the same realization of the symmetry group 3232 and the same D-symbol. The tilings are drawn in their lifted versions on the hyperbolic plane and fit onto the genus-3 Riemann surface $S$ obtained by identifying opposite edges of the dodecagon in green, with fundamental group $\pi_1(S)$ a subgroup of $\ast 246$, such that $\pi_1(S)$ is normal. Figure 6.1 represents the most complicated case as the presentation of the MCG uses most of the theoretical results introduced in the earlier sections. On the other hand, groups like 22222 turn out to be particularly simple, using only (4.2), (4.5) and the arrangements of the group elements in a fundamental tiling. Figure 6.2 shows isotopically distinct fundamental tilings with symmetry group 2224 and the same combinatorial structure. Figures 6.3, 6.4 and 6.5 shows isomorphic symmetry groups (in this case 22222) that produce isotopically distinct tilings. Figures 6.3 and 6.4 show isotopically different sets of tilings with the same D-symbol, whereas figure 6.5 shows a further set of combinatorially equivalent tilings with a different D-symbol.

Figure 6.6a shows that there are 6 symmetry elements in each tile in the associated combinatorial class of tilings. The situation is somewhat special because one has different choices of generators acting on the fundamental tile that all lead to the same group presentation. The result of this can be seen when comparing figures 6.7a and 6.6b, which show the same tiling resulting from a different set of nonconjugate generators that correspond to different elements of the MCG. Comparing the numbering

| Orbifold $\mathcal{O}$ | Mapping Class Group $\text{Mod}(\mathcal{O})$ |
|------------------------|---------------------------------------------|
| 246                    | Trivial                                     |
| 266                    | $\mathbb{Z}_2$                              |
| 344                    | $\mathbb{Z}_2$                              |
| 2223                   | $\text{Mod}(S_{3,1})$                       |
| 2224                   | $\text{Mod}(S_{3,1})$                       |
| 2226                   | $\text{Mod}(S_{3,1})$                       |
| 2323                   | $\text{Mod}(S_{2,2})$                       |
| 2244                   | $\text{Mod}(S_{2,2})$                       |
| 6262                   | $\text{Mod}(S_{2,2})$                       |
| 4444                   | $\text{Mod}(S_4)$                          |
| 22222                  | $\text{Mod}(S_5)$                          |
| 22223                  | $\text{Mod}(S_{1,1})$                      |
| 222222                 | $\text{Mod}(S_6)$                          |
| 2222222                | $\text{Mod}(S_8)$                          |

Table 6.1: Mapping class groups of stellate orbifolds
Figure 6.1: Isotopically distinct fundamental tilings with symmetry group $3232$ and the same D-symbol. All tilings fit onto the genus-3 surface obtained by identifying opposite edges of the dodecagon in green in figure 6.1a. Starting from the left, which shows a simplest tiling, the tilings to the right are a result of 'twisting' the tiling, and are the 20th and 40th tilings in our enumeration, respectively. Figure 6.1a shows the placements of the generators, figure 6.1b shows the boundary of the fundamental tile on which the generators act in green.

Figure 6.2: Isotopically distinct fundamental tilings with symmetry group $2224$ and equivalent combinatorial structure. They are numbers 1, 20 and 40 in our enumeration, respectively. All tilings are commensurate with the genus-3 dodecagon from figure 6.1a.

of symmetry elements in the same tiles in figures 6.6b and 6.7b, one sees that each set belongs to a different set of generators for $4444$ acting on the fundamental tile in such a way that the resulting presentation of the group, from Poincaré’s theorem, is the same. The tilings only appear to be the same because the decoration of the orbifold, the embedded graph, is not complicated enough. A marking of, say, point 1 would result in a different tiling. This is in line with the general theory developed in
Figure 6.3: Isotopically distinct fundamental tilings of the symmetry group 22222 in one realization and the same D-symbol. (a) shows the placements of the starting generators, with increasingly complicated shearing (nos 20 and 40 in our enumeration) to the right. All tilings are commensurate with the genus-3 dodecagon from figure 6.1a.

Figure 6.4: Isotopically distinct fundamental tilings of the symmetry group 22222 and the same D-symbol in another realization than figure 6.3. (a) shows the placements of the starting generators, with increasingly complicated shearing (nos 20 and 40 in our enumeration) to the right. All tilings are commensurate with the genus-3 dodecagon from figure 6.1a.

[29], which states that any MCG element changes some decoration of the orbifold. In fact, one readily verifies that the transformation corresponds to interchanging generators 1 and 4, and 2 and 3.

The situation of the last paragraph is related to the situation we encounter for orbifolds such as 266. The MCG is isomorphic to $\mathbb{Z}_2$ and generated by a half-twist around the two 6-fold rotation centers which corresponds to simply permuting the two rotation centers. For example, the effect of this on an embedded graph with a marking of the first of two neighboring 6-fold rotation centers would transfer this marking (and any others) onto the second, while leaving the rest of the decoration invariant. As
Figure 6.5: Isotopically distinct fundamental tilings of the symmetry group 22222 and the same D-symbol in a different realization than both figures 6.3 and 6.4. (a) shows the placements of the starting generators, with increasingly complicated shearing (nos 20 and 40 in our enumeration) to the right. All tilings are commensurate with the genus-3 dodecagon from figure 6.1a.

Figure 6.6: Isotopically distinct fundamental tilings of the symmetry group 44444 and the same D-symbol. (a) shows the placements of the starting generators, with increasingly complicated shearing (nos 20 and 41 in our enumeration) to the right. All tilings are commensurate with the genus-3 dodecagon from figure 6.1a.

such, a tiling of 266 that encodes the same types of edges for both 6-fold rotational centers is invariant under such a transformation. This problem can be remedied by considering the pure MCG, where the rotational centers are never interchanged and treated as different. There is a noteworthy potential drawback to this solution, which we illustrate with a concrete example of a tiling with 2224 symmetry in figure 6.8. Both figure 6.8a and 6.8b show a fundamental tiling that can either be interpreted as
Figure 6.7: Tiling no. 40 with symmetry 4444. Image (a) shows a different placement of generators than Figure 6.6b with the same D-symbol but nevertheless identical tiling, where the placements of the generators here are a permutation of the generators in the other tiling. The numbering of symmetry elements on the same tile is shown in (b), after conjugation by rotation around the indicated symmetry point. The decoration is too simple to detect the differences, so the tilings show up as the same.

the same abstract decoration of the orbifold with different placements for the generators in $\mathbb{H}^2$ or as different decorations of the orbifold with the same placements for the generators. Exchanging the positions of generators 1 and 3 takes the first tiling into the second. Restricting to the pure MCG means that one cannot go from the tiling in figure 6.8a to that in figure 6.8b. Instead, one would have to expand the number of fundamental tilings from given generators. Notice that the combinatorics of the tilings are the same and they therefore share the same D-symbol.

In practice, a close examination of the combinatorial description of the curves on the orbifold reveals when we are in such a situation of ambiguity. In figure 6.7, the decorations are a result of connecting the generators with geodesics according to their ordering. This decoration on an orbifold with only same order gyration points will always lead to the same situation of having multiple sets of generators around one fundamental tile that lead to the same group presentation and therefore an ambiguity for the MCG acting on tilings. Notice that this situation always arises when we obtain the decoration by doubling a Coxeter fundamental domain with an ordering of angles such that the vector of values of the angles in that order is palindromic, for the Kaleidoscopic supergroup generated only by reflections in which the original stellate group has index 2.

For practical purposes, all of the above situations where ambiguities can arise can be read off the simplest fundamental tiling from which one starts as it is generally the case that ambiguities can be detected through the symmetries of the combinatorial description of the decoration. For example, figure 6.4a shows another example of a fundamental tiling where we can predict that the same kind of ambiguity will happen.

7. Summary and Outlook. In this paper, we described a general way to produce all isotopy classes of tilings of a hyperbolic surface starting from a set of generators for the symmetry group of a tiling. Fundamental tile transitive tilings provide a natural starting point for an enumeration. The combinatorial types of such tilings are encoded using D-symbols, which in turn yield decorations of the orbifold associated to the hyperbolic symmetry group starting from the positions of the generators in
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Figure 6.8: Isotopically distinct fundamental tilings with symmetry group 2224 and same D-symbol. The two images show the interchanging of generators 1 and 3, which yields an isotopically distinct tilings with the same combinatorial structure.

\[ \mathbb{H}^2, \] i.e. their manifestation as isometries of \( \mathbb{H}^2 \), or representations of their symmetry groups in \( \text{Iso}(\mathbb{H}^2) \). Prominent generators of all possible MCGs that can be used to enumerate those representations that ultimately yield the same subgroup of \( \text{Iso}(\mathbb{H}^2) \) were translated to their respective forms under the correspondence of outer automorphisms of the symmetry group and the MCG.

These methods were used in section 6 to illustrate the general approach to a systematic enumeration of isotopy classes of tilings by producing an enumeration of such tilings related to symmetry groups generated by rotational symmetries. The importance of the class of examples comes from the fact that the associated tilings all fit onto the genus-3 Riemann surface with hyperbolic structure induced by the hyperbolic orbifold group \( \star 246 \) that constitutes the diamond, primitive, and gyroid minimal surfaces.

In a future paper, we plan to produce and further investigate the entangling and topological types of the resulting structures in \( \mathbb{R}^3 \) from the embedding of the genus-3 hyperbolic surface as the unit cell of a triply-periodic minimal surfaces. We will moreover use the results of this paper to study the related question for more general orbifold groups. In such a systematic enumeration, the search for specific physical properties of the resulting structures provides a natural framework for the synthesis of novel materials. Such physical properties are often a result of entanglement, motivating a more detailed scrutiny of the relation between the action of the MCG of the symmetry group on decorations of orbifolds and the action of the MCG of the Riemann surface on the resulting tiling of the surface.

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