Dynamics and the Emergence of Geometry in an Information Mesh

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The idea of a graph theoretical approach to modeling the emergence of a quantized geometry and consequently spacetime, has been proposed previously, but not well studied. In most approaches the focus has been upon how to generate a spacetime that possesses properties that would be desirable at the continuum limit, and the question of how to model matter and its dynamics has not been directly addressed. Recent advances in network science have yielded new approaches to the mechanism by which spacetime can emerge as the ground state of a simple Hamiltonian, based upon a multi-dimensional Ising model with one dimensionless coupling constant. In this paper we extend that approach and propose a new Hamiltonian, which has the added advantage of producing a spacetime exhibiting enhanced locality. Moreover we indicate that the role of matter and subsequently its dynamics can be investigated as excitations above the ground state of this Hamiltonian.

I. INTRODUCTION

A. Background

Reconciling General Relativity (GR) with Quantum Mechanics (QM) has not yielded a consistent and finite theory [1]. In part, this is due to the fundamentally different role that spacetime, and its geometry, plays in the two theories. In QM spacetime exists external to the theory and its geometry is input, whereas GR is essentially a theory of the geometry of spacetime. As such a quantum theory of gravity entails quantizing the geometry and a fully quantized theory of Gravity, therefore, would have to explain what it means to quantize spacetime. This is not a simple task to undertake.

The original observation that quantum theory must inevitably lead to a discrete spacetime was made by Matvei Bronstein [2], but the first concrete proposal for how this could be reconciled with Lorentz invariance has its origin in the work of Hartland Snyder [3]. He proposed a framework to directly consider the implications of discretized space with a minimum length, originally as an attempt to rationalize the presence of ultra-violet cut-offs in Quantum Field Theory (QFT). The existence of a global fundamental length would seem at odds with the principle of Lorentz invariance, as observers in inertial frames moving relative to each other would disagree about the magnitude of this length due to relativistic length contraction. The approach in this work was subsequently extended to prove that such inconsistencies are reconcilable [4, 5], provided one admits additional dimensions in which to accommodate such a universal minimum distance scale. In these modifications, referred to as Doubly or Deformed Special Relativity (DSR), a discrete spacetime can avoid the presence of problematic preferred observers, by the modification of the invariant interval $d s^2$. We subscribe to the opinion that a discrete microscopic structure, at the scale of the Planck Length $l_p$, must form part of a consistent quantum theory of gravity and the properties of quantized spacetime are important to consider. In fact it is the starting assumption of our model is that at the fundamental level spacetime is discrete.

A direct consequence of this assumption leads to the concept of a quantized spatial mesh (or more mathematically a graph), which can be used to model and explore the quantum nature of spacetime. Most of the studies of such a quantized spacetime have taken place in the context of emergent theories of gravity and geometry. The particular focus of this work is to build upon models that represent the emergent spacetime as a graph. As outlined by Antonsen some years ago [6] the inevitable reduction of the nature of geometry to points and relations leads naturally to the idea of a spacetime graph, capturing the spirit of Wheeler’s “it from bit” hypothesis [7]. A direct application of this concept, ‘Quantum Graphity’ (QG) [8, 9], provides an intriguing pathway to emergent geometry that can be shown to naturally produce a four dimensional universe. The analysis proposes a Hamiltonian, from which the graph emerges as a ground state as the universe ‘cools’ from a hypothetical ‘hot’ early universe. The argument however works backwards from desired properties of the emerged graph to the necessary form of the Hamiltonian which would produce that graph, rather than being derived from an underlying physical model. In an elegant fashion Trugenberger [10] combined some recent advances in Network Science to propose such a mechanism, and outlined an approach he terms the Dynamical Graph Model (DGM). In this work he shows how the emergence of a stable graph can be explained as a transition that would have occurred as the universe cooled. Further the emerged spacetime graph has a convergence of certain measures of dimensionality that would indicate a preference for a four dimensional universe. However the universe ‘graph’ that emerges from this model contains a significant amount of clustering (presence of triangles in the emerged spacetime graph), which has the unfortunate effect of introducing non-locality, whereby common conceptions of the neighborhood of a point in space are violated. Further, in both

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models the emerged spacetime does not make any clear proposition for the role of time, a scheme for how matter could be included, and as a consequence for the inclusion of dynamics in the emerged geometry.

Resolving these issues is the objective of the work presented in this paper, and the contribution can be viewed as an evolution of both models. We propose and describe a model that sits between QG and DGM, with a specific mechanism to include time, and describes how matter can be introduced into the spacetime graph as excited state ‘defects’ in the graph. These defects are free to propagate in the graph, and considering a possible extension of the Hamiltonian that is idempotent on the ground state, we are able to connect these excitations to non-relativistic quantum dynamics. What is obtained is an intriguing model that could explain how geometry evolves as the ground state of an informational graph.

This is of course short of a firm proposal for a quantum theory of gravity, but the presence of a graph is intriguing, as graphs possess measures of informational entropy and therefore a way to connect to thermodynamics. This entropy results from the encoding of information in the structure of the graph, i.e. the number of ‘bits’ needed to precisely specify it. The graph, in the case of our proposal, emerges directly from the entanglement of ‘spins’ between adjacent vertices. There is a deep and interesting correspondence between entanglement across causal boundaries and the emergence of spacetime curvature and hence GR, first detailed in the work of Bekenstein and later Jacobson [11, 12]. A striking result of Jacobson’s work is the connection between the requirement for finite entropy and the need for a fundamental length scale, which he determines to be the Planck Length \( l_p \). Further, this observation directly leads to a derivation of the thermodynamic statement of GR established by Bekenstein and also an ‘area law’ for entropy well understood from Hawking’s laws of black hole thermodynamics [13]. More recently, rekindled interest in this approach has been outlined by Verlinde [14, 15] in his emergent gravity theories.

It seems likely that there is a fundamental connection between the quantized, discreet graph of spacetime, a fundamental length scale, and gravity. To investigate this link it would seem natural to start with a concrete proposal for the structure of this discrete spacetime at the Planck Length, and this forms the basic motivation for the work presented in this paper.

B. Outline of the paper

The principal contribution of this work is an evolution and hybridization of the QG model of Konopka et al and the DGM work of Trugenberger, with a proposal for how the emerged geometry can accommodate dynamics. As such we begin in Section II with a brief introduction to the idea of a discrete spacetime and present an overview of the two prior models of interest, highlighting how they treat the emergence of the graphical structure of spacetime. In the same section we present a comparison of the two models and point out where there is an opportunity to improve upon them. Building upon these two approaches, in Section III we describe an alternative model which preserves the more physical approach of DGM with a ground state graph that has the desirable properties of locality present in QG. We present numerical simulations that show that the obtained spacetime geometries are of a more local nature, whilst preserving the attractive feature of possessing a low dimension where the difference between the extrinsic and intrinsic measures of dimensionality are minimized. Given the computational constraints involved in the numerical simulations, these calculations are practically bounded to spacetime graphs up to 150 nodes, and consequently the error involved in computing dimensionality measures prevent a precise prediction of the ‘natural’ dimension of the graph. It is however likely that the value lies between 3 and 5.

Once we have an emerged geometry, it is natural to ask how matter and dynamics can be included. We consider this problem in Section IV beginning by considering how matter may be modeled as excitations above the ground state. These manifest themselves as defects or ‘holes’ in the mesh, which are bounded by an area of reduced dimensionality, that is a surface. I show in Section V how the entropy of this excitation is dictated by the size of the boundary, an intriguing feature of the model as this reproduces an ‘area’ law for such excitations.

In Section V B we ask what minimal extension can be made to the Hamiltonian that can introduce dynamics into the model. Our choice is guided by considerations of the stability of the ground state, conservation of the number of excitations used to model matter, and dimensional analysis. We then arrive at the final Hamiltonian of the model. Intriguingly we sketch out in Section V C how, in the continuum limit, this extension can lead to the familiar form of the wave equation for a non-relativistic free particle. As a reasonableness check on the model, it is reassuring that it produces the normal quantum behavior for matter in the continuum limit.

I conclude in Section VI with a brief discussion of possible future directions.

II. SPACETIME AS A QUANTUM MESH

A. The Nature of Discrete Space

Starting with the idea that spacetime is quantized, one is naturally led to the concept of a fundamental length, below which it is impossible to measure the separation of two points [3, 4, 6]. Elementary considerations on the limit of localization of a quantum particle using the uncertainty principle, led to the proposal of a minimum distance called the ‘Planck Length’, which using S.I. units is defined to be \( l_p = \sqrt{\frac{\hbar G}{c^3}} \), and has the value...
of $1.6 \times 10^{-35} m$. This assumption can be justified from many different directions, either as a practical limitation of measurement, or as a consequence of the fundamental nature of spacetime at very small separations. We refer the reader to the comprehensive review by Hossenfelder [5], but in this work we take the view that spacetime is fundamentally discrete. Hartland Snyder originally proposed the concept of a quantized spacetime in 1947 [3] as a mechanism to explain UV cutoffs in QFT. In much the same way that the requirement that all observers agree upon the velocity of light requires the mixing of space and time into the familiar Lorentz invariant Minkowski interval, forcing all observers to agree as well on a minimal length can be accomplished by the mixing of a fifth dimension into a new invariant interval. In the ‘Snyder basis’ he proposed a 5 dimensional interval which is invariant under the Lorentz group $SO(4,1)$ using the coordinates $\eta_a$, with $a = 0,1,2,3,4$, and a diagonal metric $g_{\alpha \beta} = \text{diag}(+,-,-,-,-)$. In this definition the interval is $d\sigma^2 = d\eta_0^2 - d\eta_1^2 - d\eta_2^2 - d\eta_3^2 - d\eta_4^2$. To recover the Lorentz invariant subgroup $SO(3,1)$ he defined non-commuting physical coordinate operators $\hat{x}_\alpha$, with $\alpha = 0,1,2,3$ in terms of the $\eta_a$ as

$$
\hat{x}_\alpha = i\hbar \left( \eta_\alpha \frac{\partial}{\partial \eta_\alpha} - \eta_\alpha \frac{\partial}{\partial \eta_\alpha} \right), \quad \alpha = 0,1,2,3, \quad (1)
$$

$$
\hat{x}_0 = \frac{i\hbar}{c} \left( \eta_0 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_0} \right). \quad (2)
$$

Using this definition he is able to prove that Lorentz invariance of the $\hat{x}_\alpha$ is guaranteed and that as operators the spectra of the position coordinates are discretized in units of $\hbar$. The approach outlined by Snyder was significantly built upon in the framework of DSR in the work of Amelino-Camelia [4], which recast the additional coordinates as being those in a de Sitter momentum space where the Planck Length plays the role of the scalar curvature. Fundamentally these attempts show that there is hope of reconciling the existence of a fundamental length with Lorentz invariance, but our goal is a physical model of how such a space could arise. In the following sections we will survey two approaches upon which the proposed model builds.

### B. Graph Theoretical Preliminaries

In this paper we will rely upon graph theoretical terminology, which may not be familiar to the reader, and we follow closely the terminology and definitions used in the standard text by Bollobas [16]. We will summarize the key concepts here. A graph $G(V,E)$, is defined as a collection of $N$ vertices defined in a set $V$, and a collection of pairs of vertices that define the edges in the graph $E \subset V \times V$. An individual edge $e_{ij}$ connects the vertices $v_i$ and $v_j$. There are a number of special graphs, and in particular we will refer to the perfect or fully connected graph on $N$ vertices, $K_N$. This graph $K_N$, has all possible $\binom{N}{2} = \frac{1}{2}N(N-1)$ edges present, that is each pair of vertices is connected by an edge and each vertex is therefore connected to every other vertex. A graph is termed ‘simple’ if it contains no edges beginning and terminating at the same vertex, and there is only one edge between any two vertices. Further we say that it is undirected if $e_{ij}$ is indistinguishable from $e_{ji}$. For the rest of this paper we will only concern ourselves with simple, undirected graphs. We denote by $k_i$ the degree of a vertex $v_i$, defined as the number of edges that begin or terminate at an individual vertex, and we note the basic result that $\sum_i k_i = 2|E|$.

An important property of graphs is the degree of clustering present, referred to as the clustering coefficient. This property of a graph measures how fully connected it is, and is directly dependent on how many of the possible edges in the graph are present. Fundamental to this notion is the presence of closed triples, or triangles, in the graph. In Fig. 1 we denote the two possible ways in which three nodes of a graph can be connected by two or more edges. An open triple has two edges, and a closed triple, or triangle, has three. The ratio of open to closed triples is used to compute the clustering coefficient of the graph (see for example Bollobas [16], or Barabasi [17]). If $\tau_G$ is the number of closed triples, and $\lambda_G$ the number of open triples, with strictly $\tau_G < \lambda_G$, then the clustering coefficient $C_G$ is defined as

$$
C_G = \frac{\tau_G}{\lambda_G}. \quad (3)
$$

It is possible to construct graphs where every node has the same degree in many ways. In general these graphs are referred to as $k$-regular, and in particular a $k$-regular graph with little to no clustering we term a ‘mesh’ and with significant clustering a lattice.

We can also associate a number of important matrices with a graph. Firstly we can define the adjacency matrix $A_{ij}$ such that an element $A_{ij} = 1$ if there is an edge between the vertices $v_i$ and $v_j$, zero if there is no edge, and by convention $A_{ii} = 0$. This matrix has a number of important and useful properties. In particular when raised to the $n^{th}$ power, the value of $A_{ij}$ indicates the number of paths, defined as the non-unique sequence of traversed edges, that exist between vertices $v_i$ and $v_j$. It is important to note that edges in a path may be traversed more than once, so a path is not necessarily the shortest path, in terms of number of hops or edge traversals, between two vertices. Associated with the adjacency matrix is the

![Figure 1: Open and closed triples used to define the degree of clustering of a graph.](image)
degree matrix $\Delta_{ij}$, which is defined as $\Delta_{ij} = k_i \delta_{ij}$, that is the diagonal values are set to the degree of the corresponding node. Finally, constructed from the degree and adjacency matrices is the Laplacian matrix, defined as $L_{ij} = \Delta_{ij} - A_{ij}$. The origin of the name of this matrix is that for certain dynamical processes on a graph (say for example heat diffusion) the Laplacian matrix plays the same role as that of the negative Laplacian operator in the differential equations of the continuous model analog of the same physical process [18].

C. Quantum ‘Graphity’

In a series of papers by Konopka *et al.* [8, 9] the authors proposed a graph theoretical origin for emergent spacetime. The model for QG proceeds by defining a structure where the ‘atoms’ of spacetime are quantum bits, whose permissible values of 0 and 1 define an ‘on’ and an ‘off’ state for each edge. These states define a point and edge structure, *i.e.* a graph, with a finite number of vertices. The presence of an edge implies the nodes that it connects are ‘local’, and, from this construct the traditional notions of geometry arise. The model starts with a perfect graph $K_N$, of fixed size $N$, and a Hamiltonian which maps the precise graph structure to an energy computed from the edges, vertices, loops and paths present in the graph. The precise configuration of edges against a fixed set of vertices, is obtained as the ground state of a proposed Hamiltonian, and generates a very large, but finite, undirected graph of size $|V| = N \sim 10^{100-1000}$ nodes. The very large size of the graph is postulated to reflect the spatial extent of the universe, and also to encompass the estimated information content of the Universe. Evidently the particular configuration of the produced spacetime graph is entirely dependent upon the Hamiltonian, and the authors propose a multi-term Hamiltonian to yield a $k$-regular mesh with minimal clustering. This latter point regarding the absence of clustering is important to guarantee locality, which we discuss in detail in Section II E. Each term in the Hamiltonian is crafted to favor a particular feature of the graph (regularity, locality and so on), and each term requires a new dimensionless coupling constant. The quantum nature of the graph is described by the Hilbert space of the graph defined as the tensor product space of the individual vertex and edge Hilbert spaces. It is proposed that the vertices are separated by the Planck Length, and the graph is postulated to represent the actual structure of spacetime at the smallest scales. In this way the total Hilbert space for the whole graph is defined as the tensor product of the edge and vertex spaces in the following way

$$\mathcal{H}_{\text{total}} = \bigotimes_{e_{ij} \in E} \mathcal{H}_{\text{edge}} \bigotimes_{N} \mathcal{H}_{\text{node}}. \quad (4)$$

In their model, the $\bigotimes_{N} \mathcal{H}_{\text{node}}$ vertex Hilbert spaces are ignored, with the focus being upon the edge spaces. The Hilbert space for each edge can be minimal, *i.e.*

$$\mathcal{H}_{\text{edge}} = \text{span}\{|0\rangle, |1\rangle\},$$

with the $|0\rangle$ basis vector being associated with the absence of an edge, and $|1\rangle$ the presence of an edge. An edge in their model is equated to the presence of a fermionic ‘particle’, the nature of which is left unspecified. Standard annihilation and creation operators are defined having the normal algebra $\hat{a}_{ij} |0\rangle = 0$ and $\hat{a}_{ij}^\dagger |1\rangle = 0$, and anti-commutation relations $\{\hat{a}_{ij}, \hat{a}_{kl}^\dagger\} = \delta_{ik} \delta_{jl}$, all other anti-commutators being zero. One can extract a number operator $\hat{N}_{ij} = \hat{a}_{ij}^\dagger \hat{a}_{ij}$, which has the property of returning the number of particles in the edge ‘state’, which can either be 1 or 0. It is convenient to express the graph’s adjacency matrix, in terms of these operators as follows, $A_{ij} = \hat{a}_{ij}^\dagger \hat{a}_{ij}$, or equivalently define the adjacency operator for an edge and its eigenvalue equation as $A_{ij} |l\rangle = a_{ij} |l\rangle = a_{ij}^\dagger \hat{a}_{ij} |l\rangle$, with $l = \{0, 1\}$.

As stated the physics of the model is dictated by the choice of Hamiltonian for the graph. There is no physical basis of this choice, other than to consciously construct one such that the ground state of the model has the desired properties. The ground state of the QG model is a mesh, and importantly the pre-geometry, high temperature regime of the model is the initial perfect graph with no notion of locality, that is every point in space is local to every other point. The authors propose that a phase transition occurs as the universe cools, whereby edges are deleted from the graph until locality emerges suddenly at the transition point. It is an intriguing proposition but no rigorous proof of such a transition is given.

D. The Trugenberger Model of Emergent Spacetime

In [10] Trugenberger proposed an alternative approach for the emergence of spacetime. The basis of the approach utilizes competing pairs of Ising spin-spin interactions, a ferromagnetic interaction between spins on the vertices of a graph, and an anti-ferromagnetic interaction between spins on the edges of the graph. In this way the ferromagnetic interaction seeks to create links in the graph, and the anti-ferromagnetic term acts to suppress links by introducing an energy penalty for triples of vertices sharing two or three edges. The Hilbert space formulation is identical to basic QG, with the crucial distinction that a physical Hamiltonian is proposed, requiring only one dimensionless coupling constant. The basis for the formulation is that if two adjacent spins are aligned, there is a preference for an edge between them. If we define the Hilbert space for the vertex $v_i$ as $\mathcal{H}_i = \text{span}\{|0\rangle, |1\rangle\}$, such that there is an operator $S_i |s\rangle = s_i |s\rangle$, we can complete the Hilbert Space using the edge space of QG. This yields $\mathcal{H}_{\text{total}} = \bigotimes_{e_{ij} \in E} \mathcal{H}_{\text{edge}} \bigotimes_{N} \mathcal{H}_i$, and
the following Hamiltonian is proposed
\[ H_i = \frac{J}{2} \sum_{j \neq i} A_{ij} A_{kj} - \frac{1}{2} \sum_{j \neq i} s_i A_{ij} s_j, \tag{5} \]
where in the second term the Ising coupling has been absorbed into the dimensionless coupling constant \( J \). This can be converted into a quantum Hamiltonian operator by replacing the terms in \( A_{ij}, s_i \) by their appropriate operators on the total Hilbert space. As noted, the two terms act in opposite directions, in that there is an energy reduction from the second term when edges are created between nodes of the same spin, and from the second term energy increases as a node acquires links. This first term is the ‘link frustration’ term, and when the Hamiltonian is minimized to extract the ground state, the result is a constraint to the node degree, and the emergence of locality. The balance between the two terms in the ground state of Eq. \( \text{(5)} \) leads to a well defined minimum for a value of \( k > 0 \). The action of the link frustration term also reduces the number of connected triples in the graph. Given the dynamical nature of the emergence of the graph structure, he terms these universe graphs Dynamical Graphs, and we refer to this approach as the Dynamical Graph Model (DGM).

By analyzing the graphs produced as the ground state of Eq. \( \text{(5)} \), many attractive properties of this model emerge. In particular, it is possible to prove that a valid ground state is a \( k \)-regular graph with \( k = 2d \), where \( d \) is the (approximate) Euclidean dimension of the emerged regular lattice. Indeed if the assumption is made that the ground state is \( k \)-regular, it is possible to compute that this minimum energy occurs at precisely
\[ k = \frac{1}{2} + \frac{1}{2J}. \tag{6} \]
For a lattice (or mesh) of spatial dimension \( d \), \( k = 2d \), and it is possible to rearrange Eq. \( \text{(6)} \) to establish that for such a mesh this minimum occurs at a value of
\[ J = \frac{1}{4d - 1}. \]
For details we refer the reader to the original work of Trugenberger. Further, as the link frustration term disfavors triples in the graph, clustering is lower than in most random models of networks, although not entirely absent, or indeed very different in magnitude to random graphs obtained using the Erdős-Rényi process \cite{17}.

An intriguing feature of the model, however, is the transition to the regular lattice from a hot ‘disordered information soup’. This transition is marked by a divergence in measures of graph dimensionality (see Section \ref{sec:VII}B) at or around \( d = 4 \). Intriguingly the emerged graph can also exhibit stable deformations in the lattice, referred to as topological black holes. We will make use of this concept to model matter later in this work.

![Figure 2: A small patch of a \( d = 2 \) emerged mesh. The presence of the links \((v_2, v_5)\) and \((v_4, v_5)\) violate conventional locality. Without these links \( v_5 \) is not in the local neighborhood of \( v_2 \) or in the neighborhood of any of \( v_2 \)'s neighbors. With the links established \( v_2 \) is now 'local' to a node that is distant to all of the points in the node's neighborhood.](image)

### E. Comparison and Limitations

The two models both seek to propose a model of emergent geometry and locality, and DGM mostly differs from QG in that it proposes a physical basis for the graph Hamiltonian. The QG model has a ground state that approximates a Euclidean vacuum, in that the graph produced is free of clustering and is locally flat. It is worthwhile pointing out that the assertion of a flat geometry is not substantiated in the QG model. The principal difference in the DGM model is that the Hamiltonian as presented does not uniquely lead to a \( k \)-regular mesh, as it is possible to have triangles in the graph as evidenced by a non-zero clustering coefficient in the simulation results presented in the paper and reproduced in Table \ref{table}. The consequence of any clustering in graph is problematic for the interpretation of the ground state as a Euclidean vacuum. To illustrate this point, assume that all of the links in the mesh are assumed to be of equal length \( l_p \). The presence of a triangle in the graph can link nodes in violation of the normal conceptions of distance, and introduce non-locality. In Fig. \ref{fig:2} we depict a subgraph of a \( d = 2 \) emerged graph, with the presence of the triangle \( \{v_2, v_4, v_5\} \). It is clear that \( v_5 \) is not local to all of the points local to \( v_2 \), and that without the triangular links between \( v_2, v_4 \) and \( v_5 \) the shortest hop distance to \( v_5 \) would be 2 hops from \( v_4 \), and 3 from \( v_2 \). The inclusion of the triangle, and therefore clustering, brings \( v_5 \) into the neighborhood of \( v_2 \) and creates a ‘shortcut’ through the graph compromising conventional ideas of distance and locality.

It is proven in both the DGM and QG model that \( k \)-regularity is a minimum of the Hamiltonian, but it is not the only possible minimum, and not necessarily guaranteed to be the global minimum. The underlying reason for that is the Hamiltonian does not have a term that introduces an additional energy penalty when a link completes a triangle. This feature is in the data reported in \cite{10} and we reproduce that result in Section \ref{sec:VII}.

The QG model does have such a term, but it is intro-
duced in an arbitrary fashion with the requirement for an additional dimensionless coupling constant. Further, and unlike the DGM model, neither simulation or analysis is used to investigate the geometry of the emerged spacetime graph. Instead it is argued that a regular lattice, is an acceptable ground state of the Hamiltonian.

Neither model has a well defined causal structure, nor a rigorous framing of the concept of time and temperature. Temperature in the thermodynamic sense is problematic as the fundamental definition of thermodynamic temperature is a measure of equilibrium between bodies in thermal contact. Equilibrium in turn depends upon ‘time’ for such an equilibrium to be established, and time is not concretely defined. In Section III B we attempt place for such an equilibrium to be established, and time is not discretized. This evolution in time will proceed according to the graph Hamiltonian, with time being used as the iteration label used to control the convergence of the graph to its ground state. In DGM, temperature is proposed to arise from the statistical fluctuations of the graph around the ground state. In broad terms, connectivity as measured by average node degree, decreases as temperature decreases, to a stable value, thereby yielding our current familiar spacetime geometry. In technical terms, the coupling constant, as introduced in the Hamiltonian, runs with temperature such that the coupling increases with temperature.

In this work we propose a hybrid Hamiltonian that combines elements of both models. We show that the graph obtained by minimizing this Hamiltonian approaches k-regularity, and is effectively triangle free without having to introduce additional coupling constants. It is this absence of additional constants that will be particularly advantageous, as we speculate how matter can be modeled as persistent defects in the mesh, and investigate the dynamics of such defects.

III. THE QUANTUM MESH

A. A Triangle Free Hamiltonian

To overcome the presence of clustering in the DGM model we propose an extension to the Hamiltonian to further disfavor links forming between nodes that would ‘close’ a triangle and form a triangle. We can do this by exploiting the third power of the adjacency matrix, which computes the number of paths of length 3 between any two nodes. In particular it is a well known result that $A^3_{ij}$ computes the number of triangles that the vertex $v_i$ is a member of. The starting assumption is a finite graph of $N$ nodes, which is composed of $N$ spins, with each node $v_i$ associated with a Hilbert space $H_i = \text{span}\{|i, 0\rangle, |i, 1\rangle\}$, and a spin operator obeying $\hat{S}_i |i, s\rangle = s_i |i, s\rangle$ on this space. If we were to more closely model a fermionic spin 1/2 particle, the spin operator would have eigenvalues of $\pm \hbar/2$. For simplicity we omit this detail and we bear in mind that in our model $\hat{S} = \hat{S}_z$. Associated with these states are the usual ladder operators and anti-commutators of spin 1/2 fermions

$$\{\hat{s}^+_i, \hat{s}^-_j\}, \{\hat{s}^-_i, \hat{s}^+_j\} = 0, \{\hat{s}^+_i, \hat{s}^+_j\} = \delta_{ij},$$

$$\hat{s}^+_i |i, 0\rangle = |i, 1\rangle,$$

$$\hat{s}^-_i |i, 1\rangle = |i, 0\rangle, \hat{s}^-_i |i, 0\rangle = 0.$$  

Later when we model matter in the mesh the action of a spin `flip’ directly leads to the creation of a matter quantum in the mesh.

Similarly between each of the pairs of spins, there are $\frac{1}{2} N(N - 1)$ Hilbert spaces $H_{ij} = \text{span}\{|i, j, 0\rangle, |i, j, 1\rangle\}$, with the state $|i, j, 1\rangle$ indicating the presence of an edge between the the vertices $v_i, v_j$, and $|i, j, 0\rangle$ its absence.

For each edge, $e_{ij}$ we can define an adjacency operator $A_{ij}$ with eigenvalue $a_{ij}$, such that the adjacency matrix $A_{ij}$, is the collection of these eigenvalues. One can then define the edge annihilation and creation operators as follows

$$\{\hat{a}^+_{ij}, \hat{a}^-_{kl}\}, \{\hat{a}^-_{ij}, \hat{a}^+_{kl}\} = 0, \{\hat{a}^-_{ij}, \hat{a}^+_{kl}\} = \delta_{ik}\delta_{jl},$$

$$\hat{a}^+_{ij} |i, j, 1\rangle = 0, \hat{a}^+_{ij} |i, j, 0\rangle = |i, j, 1\rangle,$$

$$\hat{a}^-_{ij} |i, j, 1\rangle = |i, j, 0\rangle, \hat{a}^-_{ij} |i, j, 0\rangle = 0.$$  

It will become important later to express the adjacency matrix in terms of these annihilation and creation operators. To do so we establish a convention for the ordering of these operators, and we choose the conventional normal ordering such that annihilation operators are always placed to the right of creation operators. This convention will be applied whenever we have an expression involving these operators. With this convention the adjacency matrix is expressed in terms of annihilation and creation as follows

$$A_{ij} = \hat{a}^+_i \hat{a}^-_j.$$  

Our strategy is to introduce a further term into the Hamiltonian to penalize the formation of closed triples by adding a term in the cube of the adjacency matrix. Using $A^3_{ij}$ to represent the cube of the adjacency matrix, the value at $A^3_{ij}$ represents the number of triangles that involve the vertex $v_i$. Using these definitions we propose a new Hamiltonian as follows

$$H = \frac{g^2}{2} \left( \text{Tr}(A^3_{ij}) + \sum_{i<j} \sum_{k \neq i, j} A_{ik} A_{kj} \right) - \frac{g}{2} \sum_{i,j} s_i A_{ij} s_j.$$  

This Hamiltonian is an extension of Eq. 5, using the notation $g$ for the coupling constant, and this model is referred to as Quantum Mesh Dynamics (QMD), for reasons that will be justified in Section IV B but the central assertion is that its ground state is a cluster free mesh.
The significant addition to the DGM model is the term in $A_{ij}^2$, responsible for creating a positive energy penalty for any closed triangles. It is worth noting that the quadratic term in the adjacency matrix penalizes open triples as described in Fig. [1]. In a similar way to the DGM model, but taken together these terms will preferentially remove open triples that participate in clustering. We can establish with simple calculations the relevant minima of Eq. (14) by assuming that the graph is approximately a uniform $k$ regular mesh, with a small but non-zero clustering. In a strict $k$-regular graph it is possible for clustering to arise (in fact in the small world model of Watts and Strogatz [19] they start with a high clustering, however such graphs are non-local in the sense that you can have paths that connect nodes that do not share any other neighbors (the so called small world property). In terms of the paths that can be taken from two such nodes the ‘hop distance’, that is the number of links traversed in a path from one node to the other, is on average much greater than the one hop that now connects these normally ‘distant’ nodes. As discussed in the previous section, as a model of spacetime this is highly undesirable as it would entail distances between points being dependent upon how they are measured.

For our emerged spacetime mesh to be a useful model, we require both $k$-regularity and extremely low or zero clustering, in short a mesh. With zero clustering the triangle penalty term is zero, and the Hamiltonian reduces to the DGM model, but without the additional term significant clustering cannot be avoided in the ground state of the graph that minimizes Eq. (14). Using the definition of clustering coefficient $C_G$ from Eq. (3) as the ratio of closed to open triples, it is possible to place some bounds on ground state of this graph, by considering the energy change upon addition and deletion of links, in the presence of clustering. The calculation relies upon the observation that the clustering coefficient also acts as a probability that any three connected nodes participate in a triangle versus being a connected triple [17], and so it is possible to compute the contribution from $\text{Tr}(A_{ij}^3)$ in Eq. (14).

Firstly we note that a typical node in a $k$-regular graph participates in $k(k-1)$ open triples for $k > 2$, as any triple requires at least one node to have degree at least 2. As links connect spins of the same value, the energy of a typical link from the Hamiltonian can be computed as $E = g^2(k-1)(C_G + 1) - g/2$, and therefore the energy cost of removing a link is $\Delta E_- = g/2 - g^2(k-1)(C_G + 1)$. Using the same argument adding a link contributes $\Delta E_+ = g^2k(C_G + 1) - g/2$. At the ground state $\Delta E_{+/-} > 0$, that is the addition or removal of a link increases the energy. After some manipulation can be expressed as the following limit on $k$

$$\frac{1}{2g(C_G + 1)} < k < \frac{1}{2g(C_G + 1)} + 1. \quad (15)$$

Finally we note that the total energy of the graph on $N$ nodes can be computed as

$$H(N, k, g, C_G) = \frac{Ng^2}{2} k(k-1)(C_G + 1) - \frac{gN}{2}. \quad (16)$$

This is minimized for values of $k_{\text{min}}$ obeying

$$k_{\text{min}} = \frac{1}{2g(C_G + 1)} + \frac{1}{2}, \quad (17)$$

satisfying Eq. (15). All of these results reproduce DGM in the limit $C_G \to 0$.

### B. The role of time in the mesh

In the QMD model, and indeed in both QG and the DGM model, no specific proposal has been made for the role of time. There are essentially two alternatives to include a time like dimension to the mesh. The first possibility is to choose, using some scheme, one of the dimensions of the graph, which would define the causal structure. That is time is intrinsic to the emerged geometry. In fact this approach would mirror the approach taken in Causal Dynamical Triangulation (CDT) and Loop Quantum Gravity [20–22]. This approach requires the identification of one of the emerged dimensions as a time like coordinate, but there is no structure intrinsic to the QMD model which would identify such a dimension.

Further, the dimensionality of the emerged mesh is dependent upon the value of the coupling constant $g$, which we will see in Section IV can be thought to ‘run’ with temperature and decrease as the universe cools. To identify and preserve one of the dimensions as a time coordinate would need the mechanism to survive the running of this coupling constant, and would be an external and arbitrary input to the model.

In the QMD model the second alternative approach is taken, which considers time as defining the dynamic evolution of the mesh geometry. In this approach, time exists as a label for each of the sequence of configurations of the spacetime mesh. It does not matter whether time is continuous or discrete, as there is no fundamental quantization of this degree of freedom of the mesh.

So, in the QMD model the fundamental role of time is to label the evolution of the graph model, and the question of whether time itself is ‘quantized’ is not necessary to answer for it to be well defined. If we assume that the dynamical variables in the graph are simply the spins $s_i$ and the local correlations represented by the edges $A_{ij}$, we can represent the time evolution of the graph as being the operation of the Hamiltonian of the graph on the time independent states of these spins and edges at a particular instant in time, that is their quantum states are in the ‘Heisenberg picture’. It is then possible to add in the time dependence of these states in the usual way for a Hamiltonian that is not time dependent, such as Eq. (14). Denoting by $|A_{ij}, s_i\rangle$ as the state of the entire graph, the time-dependent state is given by
where \( \hat{H} \) is the operator equivalent of Eq. (14). This relationship will be the starting point for our investigation of the dynamical behavior of the mesh analyzed in Section V.

This approach does not define the full causal structure for a covariant description of the emerged mesh. This is an open question for future work.

IV. NUMERICAL SIMULATIONS AND PROPERTIES OF THE EMERGED MESH.

A. Clustering and Degree

Using numerical simulations, it is possible to illustrate the proposal that the emerged graphs of QMD have extremely low to zero clustering, and are true \( k \)-regular meshes. This property is important, as described before, as this is central to our proposal that the QMD Hamiltonian has a ground state corresponding to a Euclidean vacuum. In addition, it is also possible to investigate the dimensionality of the ground state graphs, and in particular how this depends upon the coupling constant \( g \).

What emerges is evidence that like the DGM model there is a preferred dimension, indicated by a convergence of extrinsic and intrinsic measures of dimension.

To compute the ground states, we apply to Eq. (14) techniques taken from the field of neural networks \cite{23, 24} to minimize the Hamiltonian for a given coupling constant. Implicitly we are making the assumption of the absence of random thermal fluctuations which would manifest themselves as departures from the stable configuration of the vacuum. The approach is stochastic, performing random spin and link manipulations, favoring those that reduce the total value of the Hamiltonian for the graph.

To compare the two models, we applied to both the Hamiltonian from Eq. (14) and that of the DGM model defined by Eq. (5). In the simulations the dynamical variables are \( A_{ij}, s_i \) for a fixed value of the coupling constant \( g \), which we iteratively adjust to minimize \( \hat{H} \). We describe that process in detail below.

The simulation is initialized to a fully randomized graph, with the spins of the vertices chosen with even probability to be either \( \pm 1 \). The nodes are then also connected randomly, with each pair of nodes connected with even probability. The simulation proceeds at each unit time step to alter the values of \( A_{ij} \) and \( s_i \) in such a way as to minimize the Hamiltonian, effectively treating it as a `cost function'. The algorithm used is slightly modified from the approach used for solving Hopfield networks \cite{24} as the presence of the triangle suppression term complicates the energy calculation of a single spin flip. At each time step we perform the following steps until we achieve a stationary minimum of \( \hat{H} \):

1. Compute the value of graph Hamiltonian from Eq. (14), at time \( t \) to be \( H_t \).

2. For each vertex, selected in a random sequence, compute

\[
h_i = \sum_{j \in V} A_{ij} s_j.
\]

If \( h_i \geq 0 \) set \( s_i = +1 \), otherwise \( s_i = -1 \).

3. For every pair of vertices, with probability 0.5 change the value of \( A_{ij} \). The direction of the change is recorded in the value of \( \delta \), such that if initially \( A_{ij} = 1 \), then set \( A_{ij} = 0, \delta = -1 \); and conversely for \( A_{ij} = 0 \) set \( A_{ij} = 1, \delta = +1 \). We assert the symmetric nature of \( A_{ij} \) by setting \( A_{ji} = A_{ij} \). Then for each vertex pair compute

\[
h_{ij} = \delta \frac{g^2}{2} \left[ (k_i + k_j - 2) + \sum_{k=1}^{N} A_{ik} A_{kj} \right] - \delta \frac{g}{2} s_i s_j,
\]

representing the contribution to the Hamiltonian of the addition \((\delta = 1)\) or deletion of a link \((\delta = -1)\). The last term computes the contribution from the addition, or deletion of a link that would complete or remove a triangle in the graph, as it calculates the number of open triples that the nodes \( i, k \) participate in. For the DGM model the last term can be ignored. If the value of \( h_{ij} \leq 0 \) from Eq. (20) the random link change is preserved, otherwise the previous value of \( A_{ij} \), and \( A_{ji} \) are restored.

4. After every link has been considered, the new value of the Hamiltonian Eq. (14) at step \( t+1 \), \( H_{t+1} \) is computed and if \( H_{t+1} \geq H_t \), all changes are discarded.

5. Steps 1 to 4 are repeated until we reach a stationary value of \( H_{t+n} \), that is \( H_{t+n} = H_{t+n-1} \) for a configurable number of iterations, being no less than 4. The stationarity condition was experimented with at values much higher than 4, but once the minimum emerged it was observed to be stable.

Due to the computational expense of solving for the minimum value of the Hamiltonian we calculate the ground states for graphs up to \( N = 150 \). With all such minimization techniques, there is the risk that the obtained configuration is a local and not a global minimum. To avert that we repeat the simulations 10 times and average all of the extracted results, each run starting from a different randomized configuration to mitigate the risk of a false or non-global minimum.

Plotted in Figs. 3 and 4 for the DGM and QMD Hamiltonian, are the clustering coefficient and average node degree for both models. It is notable that the QMD
model produces ground states with considerably less clustering than the DGM model, across a range of values of $N$ and $g$. For the clustering coefficient values extracted from the QMD simulation, it is also worth noting that the margin of error in the calculation indicates that the non-zero values may in fact not be significant. Using Eq. [17] we can define the degree dimension of the graph $d_k$ in terms of the minimum calculated degree of the graph $k_{\text{min}}$ as $2d_k = k_{\text{min}}$, reflecting the $k$-regularity of the emerged ground state. It is the most basic measure of dimensionality in a graph that is only exact when the $k$-regular mesh is lattice like. We can however use this to check the accuracy of the simulation against the model by computing the ratio of $d_k$ to $\frac{1}{2}(k)$. For the QMD model this ratio is $0.997 \pm 0.039$, and for DGM $0.999 \pm 0.041$, indicating that the simulations are reproducing the expected and computed minimum energy state with the anticipated accuracy check to verify that the reduction in clustering is only exact when the $k$-regular mesh is lattice like. We can however use this to check the average node degree. We can also investigate the degree distributions of the graphs. For a run of 10 graphs of 100 nodes, in both models close to 100% of the nodes have degree exactly equal to $\langle k \rangle$. For QMD, for a sample of 1000 nodes all but 12 have degree $\langle k \rangle - 1$, for DGM the values are 996 and 4 respectively. These results are within the margin of experimental error, providing validation of the algorithm outlined in this section. Additionally, present in these results is further evidence of the proposition that the QMD model produces a ground state with significantly improved locality through reduced clustering.

In Fig. 4 we plot against a range of coupling constant $g$ the value of $\langle k \rangle$ for both models. This is an important check to verify that the reduction in clustering in the QMD model is not simply a function of the ground state being significantly less connected. Average clustering constant will reduce for smaller average node degree, as the number of edges in the graph is simply $N(\langle k \rangle)$. What we can see in the plots is that the produced graphs have identical average node degree. We can also investigate the degree distributions of the graphs. For a run of 10 graphs of 100 nodes, in both models close to 100% of the nodes have degree exactly equal to $\langle k \rangle$. For QMD, for a sample of 1000 nodes all but 12 have degree $\langle k \rangle - 1$, for DGM the values are 996 and 4 respectively. These results are within the margin of error of being able to claim that both models produce $k$-regular graphs, but only in the case of QMD can it be claimed that the graph is a cluster free mesh. It is tempting to speculate that in the case of QMD, the slightly larger number of $(k) - 1$ degree nodes may indicate the presence of some form of lower dimensional boundary to the emerged geometry, as a closed $k$-regular graph would naturally have zero nodes of differing degree. It is however beyond the scope of this work to rigorously prove this, and it remains an open question.

| $d_k$ | QMD $NC_G$ |  |  | DGM $NC_G$ |  | DGM $NC_G$ |  |
|-------|------------|  |  |  |  |  |  |  |
|       | 0.187 ± 0.23 |  |  | 2.30 ± 0.60 |  | 8.74 ± 0.41 |
|       | 0.465 ± 0.34 |  |  | 4.33 ± 0.54 |  | 6.68 ± 0.55 |
|       | 0.610 ± 0.12 |  |  | 4.52 ± 0.09 |  | 6.27 ± 0.22 |
|       | 0.680 ± 0.20 |  |  | 8.39 ± 0.17 |

Table I: Values of clustering coefficient, multiplied by network size for $N = 100$ for both the QMD and DGM.

The significant reduction in clustering in the QMD model is evident. Also reproduced from [10] are the published values for clustering, which are in line with our results. For QMD we also present the absolute clustering.

B. Dimensionality

A critical feature of the emerged mesh is the dimensionality of the spacetime geometry that it represents. The dimension of a graph has many definitions, categorized as intrinsic or extrinsic. An intrinsic measure of dimensionality does not depend upon any embedding of the graph in a space of the same or higher dimensions, and corresponds to that which would be experienced by an observer in the mesh. These measures of dimension are well understood and I direct the reader to the standard text by Ambjørn et al [25].

The most common of the intrinsic measures is the spectral dimension $d_S$, which measures the return probability of a random walk on the graph. The walk is defined by starting at a random node, and at each time step moving to another connected vertex by randomly selecting an edge and traversing it. As described in Ambjørn et al [25], for $t$ time steps, the probability of returning to the initial vertex $p(t) = t^{-d_S/2}$. Unfortunately this is only well defined on an infinite graph, as the relation to the ‘spectral’ dimension $d_S$ is only valid in the limit of $t \to \infty$, and for a finite graph $p(\infty) = 1$. It is possible to estimate the value of the spectral dimension by restricting the length of the random walk to time steps $t < N$, across a sample of graphs for different values of $N$. When this is done, using a fit to the scaling law it is possible to extract $d_S$ for each value of $N$, and obtain the function for the dependence of $d_S$ upon $N$, $d_S(N)$. As $N \to \infty$ the value of $d_S(N)$ will tend to the final value. Figures 5 and 6 are presented as a representative sample of this scaling against increasing network size $N$, and also a fit of the curve to a test function. To extract a value for the limit $N \to \infty$, a number of test functions were tested, and the best least squares fit to the data was obtained using

$$d_S(N) = d_S(1 - be^{-aN}).$$

With this scaling function it is possible to extrapolate to obtain the value of the spectral dimension. For a selection of graphs with $N$ ranging from 40 to 110, the Pearson correlation coefficient $\rho$ of the fit to Eq. [21] was generally in the range of 0.96 to 0.98.

Turning to extrinsic measures, the standard measure,
(a) Averaged clustering coefficient, QMD and DGM, $N = 50$.

(b) Averaged clustering coefficient, QMD and DGM, $N = 75$.

(c) Averaged clustering coefficient, QMD and DGM, $N = 100$.

(d) Averaged clustering coefficient, QMD and DGM, $N = 150$.

Figure 3: Comparison of both QMD and DGM clustering coefficients across a range of graph sizes. Each datapoint represents the average value across 10 graph samples. Against each of the data points the error bars represent the 1st standard deviation from the mean value. It is evident that the QMD ground state at each value of $N$ has a significantly lower degree of clustering.

as described in Ambjørn et al is the Hausdorff dimension, $d_H$. This is defined on any sized graph, and so does not suffer from the scaling issues of the spectral dimension. It measures how volume and area scale as the size of the graph increases. Standard convention is that volume $V$ in this instance is the average distance between all pairs of nodes in the graph, which can be readily obtained from the adjacency matrix, or the use of a standard algorithm such as a breadth-first search [17]. Using this measure of volume, it is a standard result that this scales with the relation $V \propto N^{1/d_H}$ [26], and by sampling the average distance from the same sample graphs used to calculate the spectral dimension it is possible to compute $d_H$. In addition to the extrinsic measure, the graph dimension $d_k$ is also computed using the Eq. (17).

Figure 5 display the results of this analysis. In particular in Figs. 5a. and 5b. it is evident that the graph and spectral dimensions converge at around $d_k = 3.5$ to 4.0 for both models. In Figs. 5c. and 5d. we plot the differences between $d_H$ and $d_k$. For these measures the convergence is best for the range $d_k = 4.0$ to 5.0 for both models. It is worthwhile remarking that in Fig. 5b. that the obtained values of both the spectral and Hausdorff dimensions behave differently to QMD. In particular $d_H$ is somewhat lower in DGM and $d_S$ is somewhat higher. This can be explained by the presence of clustering in the graphs produced by DGM. Essentially clustering introduces the ‘small world’ property into the graphs [19], which in turn leads to average distances in the graph being smaller than a graph with no clustering (lower $d_H$), and a higher probability that a random walk will transport the walker to a distant location in the graph (higher $d_S$). Although the value of the preferred dimension, where the different measures of dimension converge, is not as precise as produced by Trugenberger, it is nevertheless supportive of a hypothesis that there is a pre-
V. DYNAMICS

A. Modeling Matter in the Mesh, and the Temperature dependence of $g$

In Section II D we noted that in the DGM treatment stable deformations of the mesh were able to form. These correspond to an isolated spin in the mesh, that is a point at vertex $v_i$ with a value $s_i \neq s_j$, where $j$ ranges over all of the neighbors of $v_i$. Unless $A_{ij} = 0$, that is there are no links between $v_i$ and its neighbors, this spin would contribute positively to the Hamiltonian, and so the links would be energetically disfavored. Removing these links after a vertex spin is flipped would reduce the overall energy of the graph and result in a new stable minimum energy. With the neighboring links removed the point is topologically isolated from the mesh. In the DGM model these defects are proposed to be ‘topological black holes’ and an argument is offered for their stability across a wide range of temperatures (using a statistical mechanical treatment of the ground state of the mesh). The basis for the stability argument relies upon treating Equations (19) and (20) as energies in a Fermi-Dirac distribution. It is then possible to postulate that there are temperature dependent fluctuations of both the vertex and link spins subject to a thermal probability of a new link forming being defined as

$$p(a_{ij} : 0 \rightarrow 1) = \frac{1}{1 + e^{-2h_{ij}/kT}},$$

with the factor of 2 arising as links are undirected. Without reproducing the details of the argument in [10], it is possible to show that the average degree of the mesh...
(a) Extrinsic, Intrinsic and Graph dimension for QMD across a range of $d_k$, computed using Eq. 17.

(b) Extrinsic, Intrinsic and Graph dimension for DGM across a range of $d_k$, computed using Eq. 6.

(c) Difference between $d_H$ and $d_S$ for QMD across a range of $d_k$, computed using Eq. 17.

(d) Difference between $d_H$ and $d_S$ for DGM across a range of $d_k$, computed using Eq. 6.

(e) Evolution of spectral dimension $d_S$ with increasing network size $N$, for QMD, and $g = 0.055$.

(f) Evolution of spectral dimension $d_S$ with increasing network size $N$, for DGM, and $g = 0.055$.

Figure 5: Dimensional analysis of the graphs obtained as the ground state of QMD and DGM. Displayed are the three measures of dimensionality against graph dimension, the divergence of the extrinsic and graph measures and the evolution of spectral dimension with $N$. 
\( \langle k \rangle \) increases with increasing temperature, and its minimum value at \( T = 0 \) is the ground state value that emerges from minimizing the Hamiltonian. The additional triangle suppressing term in QMD does not affect the argument, as for the large part the resultant ground state graphs have very little clustering. From Eq. (17), it is clear that \( g \propto 1/\langle k \rangle \), and assuming \( k \)-regularity \( k_{\text{min}} = k \). Given that \( \langle k \rangle \propto T \), it is possible to conclude \( g \propto 1/T \).

Consider a defect in the mesh, as described above. To remove the defect it is necessary to flip one spin and connect \( k \) links to the node. From the earlier discussion it was established that the total energy cost of one new link to be \( \Delta E = g^2/2(\langle k \rangle - 1)(C_g + 1) - g/2 \), assuming that \( C_g \approx 0 \), stability is assured when \( \Delta E > 0 \). The ground state at \( T = 0 \) has average degree \( \langle k_0 \rangle = \frac{1}{2g} + 1/2 \) and with some algebra, stability is guaranteed for \( \langle k \rangle > \langle k_0 \rangle + 1/2 \). We have already indicated that \( \langle k \rangle \) is proportional to temperature and so this defect is stable until temperature drops close to \( T = 0 \) where \( \langle k \rangle = \langle k_0 \rangle \). From Fig. 3 we note that average node degree jumps decreases discontinuously by an integer amount as \( g \) increases and therefore \( T \) decreases. It is also possible to conclude from the simulations that as \( \langle k \rangle \) decreases its value hold for increasing ranges of \( g \). The stability equality will therefore hold until \( \langle k \rangle = \langle k_0 \rangle + 1 \), requiring a very low value of \( T \).

B. Minimal Extensions to the Hamiltonian

Having established the Hilbert Space associated with every edge and vertex in the graph, it is natural to ask how the state of the graph evolves with time. For a vertex \( v_i \), we can define in the standard way a general state vector \( |v_i\rangle \) as an expansion in the basis states
\[
|v_i\rangle = c_0|0\rangle + c_1|1\rangle,
\]
where the \( c_0, c_1 \) are complex coefficients. Clearly the same can be done for edges. For the whole mesh we can of course construct the universe state vector \( |\Psi_G\rangle \) as a an expansion in the \( D = \frac{1}{2}N^2(N-1) \) dimensional tensor product basis vectors that span the total Hilbert space \( \mathcal{H}_{\text{total}} \). For simplicity of exposition we will only consider small segments of the whole graph, whilst remembering that wave functions and operators are properly defined in the whole tensor product Hilbert space.

As time is not present in this definition we can consider the state vector to be in the Heisenberg, or time independent picture. To convert to the Schrödinger picture we can write
\[
|v_i, t\rangle = e^{-i\hat{H}t/h} |v_i\rangle,
\]
as the Hamiltonian \( \hat{H} \), by definition, does not depend upon time. If we now imagine the state of the mesh at some time \( t_{\text{in}} \), evolving under the effect of this Hamiltonian a short time \( t_{\text{out}} = t_{\text{in}} + \tau \), the evolution of the state vector can be written as
\[
|v_i, t_{\text{out}}\rangle = e^{-i\hat{H}\tau/h} |v_i, t_{\text{in}}\rangle.
\]

As \( \tau \) is very small we can expand the exponential, remembering that as \( |v_i, t\rangle \) is an element of the entire tensor product space, and the operators are similarly defined to act on elements of that space, we obtain
\[
|v_i, t_{\text{out}}\rangle = \mathbb{I} |v_i, t_{\text{in}}\rangle - \frac{i\tau}{h} \hat{H} |v_i, t_{\text{in}}\rangle + O(\tau^2) \ldots,
\]
where \( \mathbb{I} \) is the identity operator on \( \mathcal{H}_{\text{total}} \).

We first verify that the Hamiltonian for QMD cannot create dynamics for a defect introduced into the mesh, which we are using to model matter. Let us first recast Eq. (14) in an operator format, using the annihilation and creation operators for links and the spin operator, using the normal ordering introduced in Section III. Taking each term by term we have three components of the Hamiltonian, which using our normal ordering convention are
\[
g^2/2 \text{Tr}(A^3) = \sum_{i,k,l \in [1,N]} \hat{a}^\dagger_{ik} \hat{a}^\dagger_{kl} \hat{a}_{li} \hat{a}_{kl} \hat{a}_{li},
\]
\[
g^2/2 \sum_{i \neq j} \sum_{k \neq i,j} A_{ik} A_{kj} = \sum_{i \neq j} \sum_{k \neq i,j} \hat{a}^\dagger_{ik} \hat{a}^\dagger_{kj} \hat{a}_{ik} \hat{a}_{kj},
\]
\[
g/2 \sum_{i,j} s_i A_{ij} s_j = \sum_{i,j} \hat{S}_i \hat{a}^\dagger_{ij} \hat{a}_{ij} \hat{S}_j,
\]
where the spin operators \( \hat{S}_i \) are as defined in the definition of the Hilbert space for the vertices in Section III.

Figure 6: A section of the emerged quantum mesh with \( \langle k \rangle = 2d = 4 \). Depicted is an isolated defect surrounding vertex \( v_i \), with vertices \( v_1, v_2, v_3, v_4 \) labelled. For the purposes of discussion assume \( s_i = -1 \), all other spins \( s_j = +1 \).

Figure 6 depicts a single isolated defect in a \( d_k = 2 \) dimensional emerged mesh. It is possible to extend the following argument to higher dimensions, but for clarity the argument is outlined in 2 dimensions. We can consider the Hamiltonian as a sum over all possible edges \( e \in E \) in the mesh. For a given configuration not all of these edges are present, so let us denote by \( E_G \subset E \) those
edges that are present and the complement \( E_G \subset E \) as the set of edges that are not present in a given state of the mesh. Formally \( E = E_G \oplus E_{\tilde{G}} \), and we can correspondingly write

\[
\hat{H} = \hat{H}_{E_G} + \hat{H}_{E_{\tilde{G}}}. \tag{30}
\]

By considering the action of Eq. (30) on the total state vector \( |\Psi_G\rangle \) for the mesh we seek to show that the mesh with the defect is an eigenstate of \( \hat{H} \) with an energy strictly above the ground state, and therefore that a defect is unchanged by the operation of \( \hat{H} \) on \( |\Psi_G\rangle \). Let us first consider the action of the second term on \( |\Psi_G\rangle \). As this will apply only operator expressions from Equations [27], [28], [29] that refer to edges not present in \( |\Psi_G\rangle \), the annihilation operator \( \hat{a}_{ij} \), which by our ordering convention will act first on \( |\Psi_G\rangle \), will always return a zero state vector. Our operator expression therefore only contains terms arising from \( \hat{H}_{E_G} \), and as every edge annihilation operator is paired with a corresponding edge creation operator it must return a state vector proportional to \( |\Psi_G\rangle \).

As such we arrive at the result

\[
\hat{H} |\Psi_G\rangle = E_d |\Psi_G\rangle, \tag{31}
\]

where \( E_d \) is the energy eigenvalue for the mesh with one defect. This calculation satisfies the assertion that the defect is preserved under the operation of \( \hat{H} \), but we can go further and consider \( E_d \) as compared to \( E_0 \) the eigenvalue of the mesh ground state with no defects present. We already know that the removal of edges requires energy, so any operator that added back in an edge to a general state vector in the total Hilbert space, would contribute a negative value to the total energy eigenvalue. As such we conclude that \( E_d > E_0 \), and conclude that the operator representation of Eq. (14) as applied in the exponential expansion of Eq. (26) leaves a defect static and unchanged if one is present in \( |v_i, t\rangle \).

In order for the matter defect to propagate through the mesh, a new term in the Hamiltonian is needed that will result in the movement of the defect in the mesh. What is required is a term that flips spins in the mesh in such a way that it favors spins at lattice positions that are disconnected but nearby, in other words the term must favor locality in the mesh. At the same time, it would be desirable for this term not to require additional coupling constants, and be minimally constructed from the same operators used to construct the QMD Hamiltonian. Further, the new term must not interfere with the energization of the ground state. This can be accomplished by a term of the form \(-\hat{s}_i^{\pm}(1 - A_{ij})\hat{s}_j^{\pm}\), with the minus sign inserted to guarantee energetic favorability for a move. For the purposes of the following analysis, noting that the eigenvalues of \( \hat{s}_i^{\pm} \) are \( \hbar/2 \), the convention of setting \( c = \hbar = 1 \) is temporarily dropped. The middle term is zero for all nodes in the network that do not have a link to the node \( v_i \) at position \( i \), and as noted this is guaranteed to be energetically favorable. It remains to prefer local over distant interactions, and this can be done by inserting an inverse proportionality to the square of the hop distance \( l_{ij}^2 \), as measured by the smallest number of links to a neighbor of \( v_i \). The effect of this term on the ground state is negligible as the interaction is specifically between nodes that are not adjacent, and the effect of the distance \( 1/l_{ij}^2 \) will quickly reduce to zero the interaction between spins that are distant. The choice of an inverse square dependency on \( l_{ij} \) is of course arbitrary, but is motivated by dimensional considerations described below. To propose the dynamical Hamiltonian, we use the Laplacian of the graph, noting that the equivalent of \((1 - A_{ij}) \) is \((1 + L_{ij})\) as this is only non-zero when no edge is present between vertices \( v_i \) and \( v_j \), or when \( i = j \). In order for the dynamic Hamiltonian to use the same dimensionless coupling constant, we recall that each spin operator \( \hat{s}_i^\pm \) has eigenvalues \( \hbar/2 \). To include the hop distance \( l_{ij} \) and maintain the measurement of the Hamiltonian in units of energy, one can first introduce the energy to create or destroy a defect \( \epsilon_m \), and identify it as the mass-energy represented by this defect. Using these definitions the full proposal for the dynamic term of the Hamiltonian is

\[
\hat{H}_d = -\frac{gc^2}{2\epsilon_m l_{ij}^2} \hat{s}_i^{\pm}(1 + L_{ij})\hat{s}_j^{-}, \tag{32}
\]

C. Quantum Mechanics and the Continuum Limit

As it is possible to assume that the other terms in the Hamiltonian will have no effect on a matter defect in the mesh, it is possible to use Eq. (32) to illustrate how this formulation could be viewed as equivalent to the non-relativistic formulation of quantum mechanics for a defect of mass \( m \). Begin by noting that for a small (infinitesimal) increment in time \( \tau \), the time evolution of the state vector \( |v_i, t\rangle \) is

\[
|v_i, t + \tau\rangle = e^{-i\hat{H}_d\tau/\hbar} |v_i, t\rangle, \tag{33}
\]

whilst expanding as a Taylor series in \( \tau \) we have,

\[
|v_i, t + \tau\rangle = |v_i, t\rangle + \tau \frac{\partial |v_i, t\rangle}{\partial t} + O(\tau^2) \ldots
\]

Expanding the exponential to \( O(\tau) \), and gathering terms in the first power of \( \tau \) yields

\[
-\frac{\hbar}{i} \frac{\partial |v_i, t\rangle}{\partial t} = \frac{gc^2\hbar^2}{8\epsilon_m l_{ij}^2} |v_i, t\rangle + \frac{gc^2\hbar^2}{8\epsilon_m l_{ij}^2} L_{ij} |v_i, t\rangle, \tag{33}
\]

by noting that \( L_{ij} \) is simply a number and therefore the state vector \( |v_i, t\rangle \), is operated upon by the combination \( \hat{s}_i^+\hat{s}_j^- \). In the continuum limit, as the vertices \( v_j \) and \( v_i \) are neighboring points in the graph, they can be approximated as operating on the same vertex, and so each spin operator contributes the eigenvalue \( \hbar/2 \) to the terms on the right hand side. Further, at the continuum limit,
each vertex of the mesh becomes identified as a point in the $d$ dimensional space $\mathbf{x}$, where $d$ is the dimension of the mesh. At this limit we propose the correspondence between the state vectors on the vertices with a position state vector be as follows $|v_i, t\rangle \rightarrow |\mathbf{x}, t\rangle$.

It is well known [18] that for functions defined upon the vertex of a graph (such as in our case the state vector), the continuum limit of the Laplacian matrix is $-\nabla^2$. However as we shrink the edge length in our matrix to zero, then $l_{ij} \rightarrow 0$, which will cause infinities. The freedom exists to claim that the coupling constant $g$ is a 'bare' coupling constant, valid only when $l_p > 0$, and should not be expected to hold in the continuum limit. Instead the $l_{ij}^2$ factor can be absorbed into $g$, now denoted as the bare coupling constant $g_0$. Additionally as $\epsilon_{m} = mc^2$, one can replace the defect energy with its mass $m$, by absorbing the $c^2$ in the numerators on the right hand side. One can then redefine the physical coupling constant $g_0 = g_0/4l_{ij}^2$, and choose it to be numerically one with dimension length squared.

Bringing this together and making the substitutions $|v_i, t\rangle \rightarrow |\mathbf{x}, t\rangle$ and $L_{ij} \rightarrow -\nabla^2$ gives the following expression

$$\frac{\hbar}{i} \frac{\partial |\mathbf{x}, t\rangle}{\partial t} = g_0 \frac{\hbar^2}{2m} |\mathbf{x}, t\rangle + \frac{g_0}{2m} \left( \frac{\hbar}{i} \nabla \right)^2 |\mathbf{x}, t\rangle$$  \hspace{1cm} (34)

The first term on the right hand side, $g_0 \frac{\hbar^2}{2m} |\mathbf{x}, t\rangle$, is a constant multiplier times the state vector, and one can write this as $V(\mathbf{x}) |\mathbf{x}, t\rangle$, for a constant potential, $V(\mathbf{x}) = \text{const}$, at all points in the space $\mathbf{x}$. Inserting this back generates the final result

$$\frac{\hbar}{i} \frac{\partial |\mathbf{x}, t\rangle}{\partial t} = V(\mathbf{x}) |\mathbf{x}, t\rangle + \frac{1}{2m} \left( \frac{\hbar}{i} \nabla \right)^2 |\mathbf{x}, t\rangle .$$  \hspace{1cm} (35)

This is of course the non-relativistic Schrödinger equation for a free particle of mass $m$, moving in a constant potential. The result confirms that the proposed Lagrangian for the dynamics of the defect $\hat{H}_d$, reproduces the behavior of a free quantum particle in the continuum limit $l_p \rightarrow 0$, in a self consistent way, and hints that it is possible to treat the energy of a defect, the excitation needed to flip a spin, as a quantum ‘particle’ of mass $m = \epsilon_{m}/c^2$.

D. Evidence for an Entropy ‘Area’ law

The entropy of the mesh, arising from its graph structure, is most naturally considered using information theory to quantify the information in ‘bits’ required to describe the structure of the graph. Mathematically this is a well understood measure, originating from the work of Janos Körner in 1973 [20,27]. These measures though are perhaps less well suited to the consideration of a quantum mesh, and a more appropriate measure, the ‘Von Neumann’ (VN) entropy of a graph, has been proposed and explored by many, notably Passerini et al [28, 29] and lately by Bianconi et al [30]. In what follows, when we refer to the entropy of the graph we shall use the Von Neumann form.

One constructs the entropy by solving the eigenvalue problem for $L_{ij}$, obtaining the set of $N$ eigenvalues $\lambda_i$. These are then used to define the dimensionless entropy

$$S_G = - \sum_{i=1}^{N} \lambda_i \log_2 \lambda_i.$$  \hspace{1cm} (36)

In the paper by Passerini [28] it is proved that this quantity is maximized by both the complete graph and also $k$-regular graphs. Therefore any defect in the graph, which causes a departure from $k$-regularity will cause a drop in the VN entropy of the mesh, but the amount will depend upon the precise configuration of the defect. As we create matter defects in the mesh, by definition we create isolated vertices. It is similarly well known that the multiplicity of the eigenvalue $\lambda_i = 0$ yields the number of disconnected subgraphs [16,18,31]. As such the addition of a new defect will create an additional zero eigenvalue of the Laplacian. By convention the contribution of $\lambda = 0$ to Eq. (36) is zero, so the only change to the spacetime graph that can affect the spectrum and therefore the entropy of the graph are the nodes on the boundary of the defect. For a $d$ dimensional graph this boundary is $d - 1$ dimensional, and so in the mesh corresponding to our universe, the boundary has spatial dimension 2, that is it is an area. At the boundary the vertices will lose a link, previously connected to the opposite spin node, and so the total number of links will reduce. From elementary linear algebra $\text{Tr}(L_{ij}) = \sum k_i = \sum \lambda_i$, and by the convexity of Eq. (36), the VN entropy increases as the sum of the eigenvalues decreases. In practice this establishes that the entropy of the zero defect mesh (i.e. the vacuum) is lower than a mesh with defects, and that the change is proportional to the defect areas. This is an intriguing result, providing a direct connection between the size of the boundary and the entropy of the contained matter defects.

VI. CONCLUSION AND FUTURE DIRECTIONS

Using numerical simulations we illustrated how a workable discrete geometry for spacetime can emerge from a disordered and random collection of spins interacting using a similar mechanism to the Ising model of ferromagnetism. The specific proposal extends the foundation in QG and the DGM model of Trugenberger. It has been possible to show that the extension produces an emerged spacetime which has enhanced locality, through the absence of extensive clustering, whilst retaining the attractive features of a low valued preferred dimension. It is
possible that with larger simulations this preferred dimension could approach the three or four dimensions of classical physics.

Further, a model for matter is proposed using a minimal extension to the Hamiltonian defined in Eq. (14). Intriguingly, when this is introduced using a term that acts locally, it is possible in the continuum limit to recover the Schrödinger equation for a non-relativistic quantum particle in a vacuum. It is also possible to argue that the intrinsic informational entropy of such a defect is proportional to its boundary and not its ‘bulk’.

The work presented is far from a concrete proposal for the quantum nature of spacetime in the extreme ultra violet regime, but it does represent a physical ‘toy’ model of how such a spacetime could be manifested, and geometry could occur as an emergent phenomenon in a cooling universe. The focus of future work will be to further refine the critical behavior of the graphs as the coupling constant is varied, and to consider in the presence of matter, how non-trivial geometrical features such as curvature could be represented.

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