CLOSED GENERALIZED EINSTEIN MANIFOLDS WITH RADIIALLY FLAT RICCI CURVATURE

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Abstract. In this paper, we show that a closed $n$-dimensional generalized $(\lambda, n+m)$-Einstein manifold of constant scalar curvature with weakly radially zero Ricci curvature is isometric to either a sphere $S^n$, or a product $S^1 \times S^{n-1}$ of a circle with an $(n-1)$-dimensional Einstein manifold of positive Ricci curvature, up to finite cover and rescaling. Furthermore, if we assume $(M, g)$ has positive isotropic curvature, $M$ must be isometric to either a sphere $S^n$, or a product $S^1 \times S^{n-1}$ of a circle with an $(n-1)$-sphere.

1. Introduction

A closed generalized $(\lambda, n+m)$-Einstein manifold is a triple $(M^n, g, f)$, where $(M^n, g)$ is a closed $n$-dimensional Riemannian manifold and $f$ is a smooth function on $M$ satisfying

$$Ddf = \frac{f}{m}(\text{Ric} - \lambda g).$$

Furthermore, $\lambda$ is a smooth function on $M$ and $m$ is a positive real number. In particular, $(M, g, f)$ will be called a $(\lambda, n+m)$-Einstein manifold if $\lambda$ is constant.

The motivation to approach this type of manifolds is studying Einstein manifolds that have a structure of warped product. In fact, if $m > 1$ is an integer and $f > 0$, it is known [13] that $(M^n, g, f)$ is a $(\lambda, n+m)$-Einstein manifold if and only if there is a smooth $(n+m)$-dimensional warped product Einstein manifold having $M$ as the base space. We also observe that if we define the function $\phi$ by $e^{-\frac{\lambda}{m}} = f$, the equation (1.1) becomes

$$\text{Ric}^m_{\phi} := \text{Ric} + Dd\phi - \frac{d\phi \otimes d\phi}{m} = \lambda g.$$

Here, the tensor $\text{Ric}^m_{\phi}$ is the well-known $m$-Bakry-Émery Ricci tensor. Taking $m \to \infty$, we get the gradient (generalized) Ricci soliton equation

$$\text{Ric} + Dd\phi = \lambda g.$$

A generalized $(\lambda, n+m)$-Einstein manifold $(M^n, g, f)$ is called trivial if the potential function $f$ is constant, and in this case $(M, g)$ is Einstein. We point out that, if a closed $(\lambda, n+m)$-Einstein manifold has constant scalar curvature, $f$ does not change sign on $M$ and $\lambda$ is constant, it follows from the maximum principle that $(M^n, g, f)$ is trivial. On the other hand, if $\lambda$ is non-constant, a closed generalized $(\lambda, n+m)$-Einstein manifold with constant scalar curvature and $f > 0$ is Einstein and isometric to a sphere, see [1]. Motivated by these results, from now on, we always assume that

$$\min_M f := a < 0 < b := \max_M f$$

so that $f^{-1}(0)$ is a non-empty set whenever we consider closed generalized $(\lambda, n+m)$-Einstein manifolds.

In this paper, we consider closed generalized $(\lambda, n+m)$-Einstein manifolds satisfying a 2-form $\omega = df \wedge i\nabla_f z_g = 0$, where $z_g$ is the traceless Ricci curvature tensor defined by $z_g = \text{Ric} - \frac{\text{Scal}}{n} g$ and $i\nabla_f z$ denotes the interior...
product. We can see that the vanishing of $\omega$ is equivalent to $z_\theta(\nabla f, X) = 0$ for any vector field $X$ orthogonal to $\nabla f$, or $\text{Ric}_g(\nabla f, X) = 0$ since vacuum static space have constant scalar curvature.

Our main result is the following.

**Theorem 1.1.** Let $(M^n, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Suppose that $(M, g)$ has constant scalar curvature and $\omega = df \wedge \imath_{\nabla f} z = 0$. Then, up to finite cover and rescaling, we have the following.

1. If $f^{-1}(0)$ is connected, then $(M, g)$ is isometric to a sphere $\mathbb{S}^n$.
2. If $f^{-1}(0)$ is disconnected, then $f^{-1}(0)$ has exactly two connected components and $(M, g)$ is isometric to the product $\mathbb{S}^1 \times \Sigma^{n-1}$. Here $\Sigma$ is an $(n-1)$-dimensional Einstein manifold of positive Ricci curvature.

We point out that, when $(M^n, g, f)$ is a closed $(\lambda, n + m)$-Einstein manifold, that is, $\lambda$ is a constant function, the condition $\omega = 0$ is equivalent to constant scalar curvature, see Lemma 6.1. W can also easily show that there are no critical points on the set $f^{-1}(0)$ and each connected component of $f^{-1}(0)$ is totally geodesic (see Section 2). Moreover, motivated by [10], we show that the vanishing of $\omega = 0$ implies that the potential function $f$ does not have critical points except at minimum and maximum points. Finally, supposing that $(M, g)$ has constant scalar curvature, we are able to provide a proof of the above result.

Furthermore, if we assume that $(M, g)$ has positive isotropic curvature (PIC in short), then $\Sigma$ can be shown to be a sphere $\mathbb{S}^{n-1}$.

**Theorem 1.2.** Let $(M^n, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Suppose that $(M, g)$ has PIC, constant scalar curvature and $\omega = df \wedge \imath_{\nabla f} z = 0$. If $f^{-1}(0)$ is disconnected, then, up to finite cover and rescaling, $(M, g)$ is isometric to the product $\mathbb{S}^1 \times \Sigma^{n-1}$.

We recall that a Riemannian manifold $M$ has positive isotropic curvature if and only if, for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, we have that

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0.$$  

The positive isotropic curvature was first introduced by Micalleff and Moore [15] in consideration of the second variation of energy of maps from surfaces into $M$. It is easy to see that a standard sphere $\mathbb{S}^n$ and a product $\mathbb{S}^{n-1} \times \mathbb{S}^1$ or $\mathbb{S}^{n-1} \times \mathbb{R}$ have PIC. It is also well-known that if the sectional curvature of a Riemannian manifold is pointwise strictly quarter-pinched, then it has PIC, and the connected sum of manifolds with PIC admits a PIC metric (see [15] and [16]). This condition was also used by Brendle and Schoen to prove the celebrated differentiable pointwise 1/4-pinching sphere theorem, see [4].

**Convention and Notations:** Basically, we follow curvature conventions and operator conventions in [2] except only one the Laplace operator. Hereafter, for convenience and simplicity, we denote curvatures $\text{Ric}_g, z_\theta, \text{Scal}_g,$ and the Hessian and Laplacian of $f, D_g df, \Delta_g$ by $r, z, s,$ and $Ddf, \Delta,$ respectively, if there is no ambiguity. We also use the notation $\langle \cdot, \cdot \rangle$ for metric $g$ or inner product induced by $g$ on tensor spaces.

## 2. Basic Properties

In this section, we give some basis properties on closed generalized $(\lambda, n + m)$-Einstein manifolds. Before doing this, we first enumerate equivalent equations to (1.1) and basic identities which are used later.

**Basic Identities:**

1. Taking the trace of (1.1), we have

$$\Delta f = -\frac{n}{m} \left( \lambda - \frac{s}{n} \right) f \quad \text{or} \quad \Delta f = \frac{f}{m} (s - n \lambda). \quad (2.1)$$
Denoting $Ddf = Ddf - \frac{\Delta f}{n} g$, (1.1) is reduced to

$$fz = mDdf.$$

Since $\frac{\lambda f}{m} = \frac{1}{n} \left( sf - \Delta f \right)$ in (2.1), by substituting this into (1.1), we obtain

$$Ddf = \frac{f}{m} r - \frac{1}{n} \left( sf - n g \right).$$

Using $z = r - \frac{s}{n} g$, from (1.1), we have

$$fz = mDdf + \left( \lambda - \frac{s}{n} \right) fg = mDdf - \frac{m}{n} (\Delta f) g.$$

Letting $\mu := \lambda f - \frac{s}{n-1} f$, from (2.2), we have

$$fz = mDdf - \frac{sf}{n} g + \left( \mu + \frac{s}{n-1} \right) g.$$

Taking the trace of (2.3), we obtain

$$\Delta f = -\frac{s}{m(n-1)} f - \frac{n}{m} \mu.$$

**Lemma 2.1.** Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Then, the set \( \{ f > 0 \} \cap \{ \lambda \geq \frac{s}{n} \} \) is nonempty.

**Proof.** Suppose, on the contrary, that \( \{ f > 0 \} \cap \{ \lambda \geq \frac{s}{n} \} = \emptyset \). Multiplying (2.1) by $f$ and integrating it over the set $f \geq 0$, we obtain

$$\int_{f \geq 0} |\nabla f|^2 = \frac{n}{m} \int_M \left( \lambda - \frac{s}{n} \right) f^2 \leq 0,$$

which shows that $\nabla f = 0$ and so $f = 0$ on the set \( \{ f > 0 \} \), a contradiction. \qed

The same argument in the proof of Lemma 2.1 shows that

$$\{ f < 0 \} \cap \{ \lambda \geq \frac{s}{n} \} \neq \emptyset.$$

**Lemma 2.2.** Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Then, there are no critical points on the set $f^{-1}(0)$.

**Proof.** Suppose that there is a critical point $p \in f^{-1}(0)$ of $f$. Let $\gamma$ be a unit-speed geodesic starting at $p$ and define $h(t) = f(\gamma(t))$. From (2.2), we have

$$h''(t) = \frac{1}{m} \left[ z(\gamma'(t), \gamma'(t)) - \left( \lambda - \frac{s}{n} \right) h(t) \right]$$

with $h(0) = 0$ and $h'(0) = 0$. So, it follows from the uniqueness of ODE solution that $f$ vanishes identically, which is a contradiction. \qed

We can easily deduce from Lemma 2.2 that any connected component of the set $f^{-1}(0)$ is a hypersurface of $M$. Moreover, we can easily see that $|\nabla f|^2$ is a positive constant on each connected component of $f^{-1}(0)$. In fact, if $X$ is tangent to (a connected component of) $f^{-1}(0)$, by (2.2), we have $X(|\nabla f|^2) = 0$. Furthermore, we have the following.

**Lemma 2.3.** Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Then, any connected component of $f^{-1}(0)$ is a totally geodesic hypersurface in $M$. 

Proof. We can take \( N = \frac{\nabla f}{|\nabla f|} \) as a unit normal vector field on (a component of) \( f^{-1}(0) \) by Lemma 2.2. Choosing a local frame \( \{N, e_2, e_3, \ldots, e_n\} \) so that \( \{e_2, e_3, \ldots, e_n\} \) are tangent to \( f^{-1}(0) \), we have \( e_i(\nabla f) = e_i \left( \frac{\nabla f}{|\nabla f|} \right) = 0 \) on the set \( f^{-1}(0) \), which implies \( D_{e_i}N = 0 \). \( \square \)

It is well-known that the following identities hold for a Riemannian manifold in general.
\[
\delta r = -\frac{1}{2} ds \quad \text{and} \quad \delta z = -\frac{n-2}{2n} ds.
\] (2.4)

Here \( \delta = -\text{div} \) denotes the negative divergence operator.

**Lemma 2.4.** Let \((M, g, f)\) be a generalized \((\lambda, n+m)\)-Einstein manifold. Let \( \mu := \lambda f - \frac{s}{n-1} f \). Then we have
\[
(m-1)i\nabla f z + \frac{m-1}{n} sdf = \frac{1}{2} f ds + (n-1)d\mu.
\] (2.5)

**Proof.** Taking the divergence operator \( \delta \) of (2.3), we obtain
\[
-i\nabla f z + f\delta z = m(-i\nabla f r - d\Delta f) - \frac{s}{n(n-1)} df - \frac{f}{n(n-1)} ds - d\mu.
\] (2.6)

Note that, by (2.4) and definition, \( \delta z = -\frac{n-2}{2n} ds \), \( i\nabla f r = i\nabla f z + \frac{s}{n} df \) and \( \Delta f = -\frac{s}{m(n-1)} f - \frac{n}{m} \mu \). Substituting these into (2.6), we obtain
\[
(m-1)i\nabla f z + \frac{m-1}{n} sdf = \frac{1}{2} f ds + (n-1)d\mu.
\] \( \square \)

Recall that, from a result in [1], we always assume that \( f^{-1}(0) \) is nonempty whenever we consider generalized \((\lambda, n+m)\)-Einstein manifolds. When \( m = 1 \), we have the following.

**Corollary 2.5.** Let \( m = 1 \). Let \((M, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold with \( f^{-1}(0) \neq \emptyset \). If the scalar curvature \( s \) is constant, then \( s \) should be positive and \( \lambda \) is also a positive constant. In fact, we have
\[
\lambda = \frac{s}{n-1}.
\]

**Proof.** Suppose that \( s \) is constant. Letting \( \mu = \lambda f - \frac{s}{n-1} f \), it follows from Lemma 2.4 that \( \mu \) is constant. Since \( f^{-1}(0) \neq \emptyset \), we have \( \mu = 0 \) and so \( \lambda = \frac{s}{n-1} \), which shows that \( \lambda \) is constant. By (2.1), we have
\[
\Delta f = sf - n\lambda f = -\frac{s}{n-1} f.
\]

Hence \( s \) should be positive by maximum principle. \( \square \)

**Theorem 2.6.** Let \( m = 1 \). Let \((M, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold with \( f^{-1}(0) \neq \emptyset \). If \((M, g)\) has constant scalar curvature and \( \omega = df \wedge i\nabla f z = 0 \), then, up to finite cover and rescaling, \( M \) is isometric to a sphere \( S^n \) or the product \( S^1 \times \Sigma^{n-1} \) of a circle and an \((n-1)\)-dimensional Einstein manifold \( \Sigma \) of positive Ricci curvature.

**Proof.** From Corollary 2.5, we have \( \lambda = \frac{s}{n-1} \) and so the equation (2.2) is reduced to
\[
f z = Ddf + \frac{s}{n(n-1)}.
\]
which is exactly a vacuum static space. Thus, applying the main result in [12], we obtain the conclusion. \( \square \)

In case of positive isotropic curvature, we have the following.

**Corollary 2.7.** Let \( m = 1 \). Let \((M, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold of constant scalar curvature with \( f^{-1}(0) \neq \emptyset \). Suppose that \((M, g)\) has PIC and \( \omega = 0 \). Then, up to finite cover and rescaling, \( M \) is isometric to a sphere \( S^n \) or the product \( S^1 \times S^{n-1} \).
From now on, we assume $m > 1$ for generalized $(\lambda, n + m)$-Einstein manifolds $(M^n, g, f)$ throughout the paper. For the case $m > 1$, we can also show that the scalar curvature is positive as Corollary 2.5 if it is constant.

**Proposition 2.8.** Let $m > 1$ be an integer. Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. If the scalar curvature $s$ is constant, then $s$ should be positive.

**Proof.** Letting $\mu = \lambda f - \frac{m}{n-1} f$, it follows from Lemma 2.4 that

$$(m-1)i_{\nabla f}z + \frac{m-1}{n} sdf = (n-1)d\mu.$$ 

Taking the divergence operator $\delta$ of this equation, we have

$$(m-1)\delta i_{\nabla f}z - \frac{m-1}{n} s\Delta f = -(n-1)\Delta \mu.$$ 

Since $\delta i_{\nabla f}z = -\frac{n-2}{2n}(\nabla s, \nabla f) - \frac{1}{m} |z|^2 = -\frac{1}{m} |z|^2$, we obtain

$$\Delta \left[(n-1)\mu - \frac{m-1}{n} sf\right] = \frac{m-1}{m} f|z|^2.$$ 

Applying the maximal principle to $(n-1)\mu - \frac{m-1}{n} sf$ on the set $f \geq 0$, from $(n-1)\mu - \frac{m-1}{n} sf = 0$ on the set $f^{-1}(0)$, we have

$$(n-1)\mu - \frac{m-1}{n} sf \leq 0$$

on the set $f \geq 0$. Substituting $\mu = \lambda f - \frac{m}{n-1} f$ into this inequality, we have

$$n\lambda \leq \frac{m+n-1}{n-1} s = s + \frac{m}{n-1} s$$

on the set $f \geq 0$. Since $\{f > 0\} \cap \{n\lambda > s\} \neq \emptyset$ by Lemma 2.1 and $s$ is constant, we must have $s > 0$. \hfill $\Box$

### 3. Tensorial Properties

In this section, we investigate relations of generalized $(\lambda, n + m)$-Einstein manifolds to the Bach tensor, Cotton tensor, Weyl tensor and a structural tensor $T$ which will be defined later.

Let $(M^n, g)$ be a Riemannian manifold of dimension $n$ with the Levi-Civita connection $D$, and let $h$ be a symmetric 2-tensor on $M$. The differential $d^D h$ is defined by

$$d^D h(X, Y, Z) = D_X h(Y, Z) - D_Y h(X, Z)$$

for any vectors $X, Y$ and $Z$.

**Definition 3.1.** The Cotton tensor $C \in \Gamma(\Lambda^2 M \otimes T^* M)$ is defined by

$$C = d^D \left( \text{Ric} - \frac{s}{2(n-1)} g \right) = d^D r - \frac{1}{2(n-1)} ds \wedge g,$$

where $ds \wedge g$ is defined by $ds \wedge g(X, Y, Z) = ds(X)g(Y, Z) - ds(Y)g(X, Z)$ for vectors $X, Y, Z$.

Related to the Cotton tensor $C$, the followings are well-known.

- The Weyl tensor $w$ satisfies

$$\delta w = -\frac{n-3}{n-2} d^D \left( \text{Ric} - \frac{s}{2(n-1)} g \right) = -\frac{n-3}{n-2} C$$

under the following identification

$$\Gamma(T^* M \otimes \Lambda^2 M) \cong \Gamma(\Lambda^2 M \otimes T^* M).$$

- Using $r = z + \frac{z}{n} g$, the Cotton tensor can be written as

$$C = d^D z + \frac{n-2}{2n(n-1)} ds \wedge g.$$ \hfill (3.1)
• Introducing a local orthonormal frame \( \{e_i\} \), and denoting \( C_{ijk} = C(e_i, e_j, e_k) \), we have

\[
\langle \delta C, z \rangle = -C_{ijk;i} z_{ijk} = -(C_{ijk} z_{jk};i) + C_{ijk} z_{jk;i}
\]

\[
= -(C_{ijk} z_{jk};i) + \frac{1}{2} |C|^2,
\]

where the semi-colon denotes covariant derivative.

• The cyclic summation of indices in \( C \) vanishes: \( C_{ijk} + C_{jki} + C_{kij} = 0 \), and trace of \( C \) in any two summands also vanishes.

For a generalized \((\lambda, n + m)\)-Einstein manifold, \((M, g, f)\), define a 3-tensor \( T \) as

\[
T = \frac{1}{n - 2} df \wedge z + \frac{1}{(n - 1)(n - 2)} i_{\nabla f z} \wedge g,
\]

where \( i_{\nabla f z} \) denotes the interior product given by \( i_{\nabla f z}(X) = z(\nabla f, X) \) for any vector field \( X \).

Then we have the following identity.

**Lemma 3.2.** Let \((M, g, f)\) be a generalized \((\lambda, n + m)\)-Einstein manifold. Then

\[
f C = m i_{\nabla f} W - (m + n - 2) T.
\]

Here \( i_{\nabla f} W \) is defined by \( i_{\nabla f} W(X, Y, Z) = W(X, Y, Z, \nabla f) \).

**Proof.** Taking \( d^D \) in (2.2), we obtain

\[
d f \wedge z + f d^D z = m \tilde{i}_{\nabla f} R - \frac{m}{n} d \Delta f \wedge g,
\]

where \( R \) denotes the Riemannian curvature tensor. From the following curvature decomposition

\[
R = W + \frac{s}{2n(n - 1)} g \otimes g + \frac{1}{n - 2} z \otimes g,
\]

we have

\[
\tilde{i}_{\nabla f} R = \tilde{i}_{\nabla f} W - \frac{s}{n(n - 1)} d f \wedge g - \frac{1}{n - 2} d f \wedge z - \frac{1}{n - 2} i_{\nabla f z} \wedge g.
\]

Recall that, from (3.1), the Cotton tensor can be written as

\[
C = d^D z + \frac{n - 2}{2n(n - 1)} d s \wedge g.
\]

Thus,

\[
f C = f d^D z + \frac{n - 2}{2n(n - 1)} f d s \wedge g
\]

\[
= m \tilde{i}_{\nabla f} R - \frac{m}{n} d \Delta f \wedge g - d f \wedge z + \frac{n - 2}{2n(n - 1)} f d s \wedge g
\]

\[
= m \tilde{i}_{\nabla f} W - \frac{m + n - 2}{n - 2} d f \wedge z - \frac{m}{n - 2} i_{\nabla f z} \wedge g
\]

\[
+ \frac{1}{n - 1} \left( \frac{n - 2}{2n} f d s \wedge g - \frac{m}{n} d f \wedge g \right) - \frac{m}{n} d \Delta f \wedge g.
\]

Next, by taking the divergence operator \( \delta \) in (2.2), we have

\[
- i_{\nabla f z} + f \delta z = m (-i_{\nabla f r} - d \Delta f) + \frac{m}{n} d \Delta f.
\]

Since \( \delta z = -\frac{n - 2}{2n} d s \) and \( i_{\nabla f r} = i_{\nabla f z} + \frac{s}{n} d f \), we obtain

\[
(m - 1) i_{\nabla f z} = \frac{n - 2}{2n} f d s \wedge g - \frac{m}{n} d f \wedge g - \frac{m(n - 1)}{n} d \Delta f \wedge g.
\]

Substituting this into (3.3), we have

\[
f C = m \tilde{i}_{\nabla f} W - \frac{m + n - 2}{n - 2} d f \wedge z - \frac{m + n - 2}{(n - 1)(n - 2)} i_{\nabla f z} \wedge g
\]

\[
= \tilde{i}_{\nabla f} W - (m + n - 2) T.
\]
For the Weyl tensor \( W \) and the traceless Ricci tensor \( z \), the symmetric 2-tensor \( \tilde{W}z \) is defined by
\[
\tilde{W}z(X, Y) = z(W(X, e_i)Y, e_i)
\]
for a local frame \( \{e_i\} \).

**Lemma 3.3.** Let \((M, g, f)\) be a generalized \((\lambda, n + m)\)-Einstein manifold. Then
\[
\delta(i\nabla_f W) = -\frac{n-3}{n-2} \tilde{C} + \frac{f}{m} \tilde{W}z,
\]
where \( \tilde{C} \) is a 2-tensor defined as
\[
\tilde{C}(X, Y) = C(Y, \nabla f, X)
\]
for any vectors \( X, Y \).

**Proof.** From definition together with (2.2), we have
\[
\delta(i\nabla_f W)(X, Y) = -D_e(i\nabla_f W)(e_i, X, Y) = -e_i(W(e_i, X, Y, \nabla f))
\]
\[
= -D_eW(e_i, X, Y, \nabla f) - W(e_i, X, Y, D_e \nabla f)
\]
\[
= \delta W(X, Y, \nabla f) - Df(e_i, e_j)W(e_i, X, Y, e_j)
\]
\[
= \delta W(X, Y, \nabla f) - \frac{f}{m}z(e_i, e_j)W(e_i, X, Y, e_j) \quad (\because \text{tr}_{14}W = 0)
\]
\[
= -\frac{n-3}{n-2}C(Y, \nabla f, X) - \frac{f}{m}z(e_i, e_j)W(e_i, X, Y, e_j)
\]
\[
= \frac{n-3}{n-2}C(Y, \nabla f, X) + \frac{f}{m} \tilde{W}z(X, Y).
\]

For a closed generalized \((\lambda, n + m)\)-Einstein manifold \((M, g, f)\), the vanishing of the tensor \( T \) shows that \((M, g)\) satisfies some constrained geometric structures. We say that \((M, g)\) has harmonic Weyl curvature if \( \delta W = 0 \), and is Bach-flat if the Bach tensor \( B \) defined by
\[
B = \frac{1}{n-3} \delta D \delta W + \frac{1}{n-2} \tilde{W}z
\]
vanishes. It is easy to see, from definition and (3.1), that \((n-2)B = -\delta C + \tilde{W}z \).

**Theorem 3.4.** Let \((M, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold. If \( T = 0 \), then \((M, g)\) has harmonic Weyl curvature and is Bach-flat.

**Proof.** By the definition of \( T \), for an orthonormal frame \( \{e_i\}_{1 \leq i \leq n} \) with \( e_1 = N = \frac{\nabla_f}{|\nabla_f|} \), we have
\[
(n-2)i\nabla_f T(e_i, e_j) = |\nabla f|^2 z(e_i, e_j) + \frac{1}{n-1} z(\nabla f, \nabla f) \delta_{ij}
\]
\[
- \frac{1}{n-1} z(\nabla f, e_i) df(e_j) - z(\nabla f, e_j) df(e_i).
\]
Since \( T = 0 \) by assumption, for \( 2 \leq i, j \leq n \)
\[
z_{ij} = z(e_i, e_j) = -\frac{1}{n-1} \alpha \delta_{ij}.
\]
Also, for \( 2 \leq i \leq n \)
\[
0 = (n-2)i\nabla_f T(e_i, N) = \frac{n-2}{n-1} z(e_i, N) |\nabla f|^2,
\]
implying that
\[
z(e_i, N) = \frac{\alpha}{n-1} C(\nabla f, e_i, e_i) = \frac{\alpha}{n-1} C(\nabla f, N, N) = 0.
\]
for \( 2 \leq i \leq n \). By (3.4) and (3.5), we have
\[
\langle i\nabla_f C, z \rangle = -\frac{\alpha}{n-1} \sum_{i=1}^{n-1} C(\nabla f, e_i, e_i) = \frac{\alpha}{n-1} C(\nabla f, N, N) = 0.
\]
On the other hand, it follows from Lemma 3.2 together with $T = 0$ that
\[ fC = m\tilde{i}_{\nabla f}W. \]  
(3.7)

In particular, for any vector fields $X$ and $Y$, we have $C(X, Y, \nabla f) = 0$, and since the cyclic summation of $C$ is vanishing, this implies
\[ C(Y, \nabla f, X) + C(\nabla f, X, Y) = 0. \]

So,
\[ \tilde{C} = -i_{\nabla f}C. \]

By taking the divergence $\delta$ of (3.7), we have the following
\[ -i_{\nabla f}C + f\delta C = m\delta(i_{\nabla f}W) = -\frac{m(n - 3)}{n - 2}\tilde{C} + f\dot{W}z. \]

That is,
\[ f\delta C = i_{\nabla f}C + \frac{m(n - 3)}{n - 2}i_{\nabla f}C + f\dot{W}z \]
\[ = \frac{n - 2 + m(n - 3)}{n - 2}i_{\nabla f}C + f\dot{W}z. \]  
(3.8)

It follows from the definition of $\dot{W}z$ together with (3.7) that
\[ \dot{W}z(\nabla f, X) = -\frac{f}{m}(i_{X}C, z) \]
for any vector field $X$. In particular, by (3.6), we have
\[ \dot{W}z(\nabla f, \nabla f) = -\frac{f}{m}(i_{\nabla f}C, z) = 0. \]

Consequently, by (3.8) and (3.9)
\[ \delta C(N, N) = \dot{W}z(N, N) = 0. \]

Therefore, by (3.6) and (3.9) again,
\[ f\langle \delta C, z \rangle = \frac{n - 2 + m(n - 3)}{n - 2}f(i_{\nabla f}C, z) + f(\dot{W}z, z) = f \sum_{2\leq i,j\leq n} \dot{W}z(e_i, e_j)z_{ij} \]
\[ = -\frac{\alpha f}{n - 1} \sum_{2\leq i\leq n} \dot{W}z(e_i, e_i) = \frac{\alpha f}{n - 1} \dot{W}z(N, N) = 0, \]
implying that
\[ \langle \delta C, z \rangle = 0. \]

Hence, from
\[ 0 = \int_{M} \langle \delta C, z \rangle = \frac{1}{2} \int_{M} |C|^2, \]
we have $C = 0$, and hence $i_{\nabla f}W = 0 = \dot{W}z$. Therefore
\[ \delta W = -\frac{n - 3}{n - 2}C = 0 \quad \text{and} \quad (n - 2)B = -\delta C + \dot{W}z = 0. \]

Now, we define a 2-form $\omega$ by
\[ \omega := df \wedge i_{\nabla f}z. \]  
(3.10)

By the definition of $T$, for any vector fields $X$ and $Y$
\[ T(X, Y, \nabla f) = \frac{1}{n - 1}df \wedge i_{\nabla f}z(X, Y) = \frac{1}{n - 1}\omega(X, Y) = -\frac{f}{m + n - 2}i_{\nabla f}C(X, Y). \]

Here, the last equality follows from (3.2). Thus, we have
\[ \omega = (n - 1)i_{\nabla f}T = -\frac{n - 1}{m + n - 2}f i_{\nabla f}C. \]  
(3.11)
Let \( \{e_i\}_{i=1}^n \) be a local orthonormal frame with \( e_1 = N = \nabla f /|\nabla f| \). It is clear that
\[
\omega(e_j, e_k) = 0
\]
for \( 2 \leq j, k \leq n \) by definition of \( \omega \). Thus, if \( \tilde{i}_f C(N, e_i) = 0 \) for \( 2 \leq i \leq n \), then \( \omega = 0 \).

**Lemma 3.5.** As a 2-form, we have
\[
\tilde{i}_f C = \text{div}_f z - \frac{n-2}{2n(n-1)} df \wedge ds.
\]
In particular, if \((M, g)\) has constant scalar curvature, then \( \tilde{i}_f C \) is an exact form.

**Proof.** Choose a local orthonormal frame \( \{e_i\} \) which is normal at a point \( p \in M \), and let \( \{\theta^i\} \) be its dual coframe so that \( d\theta^i|_p = 0 \). Since \( i_f z = \sum_{i,k=1}^n f_iz_k\theta^k \) with \( e_i(f) = f_i, e_i(s) = s_i \) and \( z(e_i, e_k) = z_{ik} \), by (2.2) and (3.1), we have
\[
d\text{div}_f z = \sum_{j,k} \sum_l (f_{ij}z_{lk} + f_{ij}\delta_{lk}) \theta^j \wedge \theta^k
= \sum_{j<k} \sum_l (f_{ij}z_{lk} - f_{ik}z_{lj} + f_{ij}(z_{lk} - z_{lj})) \theta^j \wedge \theta^k
= \sum_{j<k} \sum_l \left[ \left( \frac{f}{m} z_{ij} + \frac{1}{n} \delta_{ij} \right) z_{lk} - \left( \frac{f}{m} z_{ik} + \frac{1}{n} \delta_{ik} \right) z_{lj} \right] \theta^j \wedge \theta^k
+ \sum_{j<k} \sum_l f_i \left[ C_{ijkl} - \frac{n-2}{2n(n-1)} (s_j \delta_{lk} - s_k \delta_{lj}) \right] \theta^j \wedge \theta^k
= \sum_{j<k} \sum_l f_i C_{ijkl} \theta^j \wedge \theta^k - \frac{n-2}{2n(n-1)} \sum_{j<k} (f_k s_j - f_j s_k) \theta^j \wedge \theta^k
= \tilde{i}_f C - \frac{n-2}{2n(n-1)} \sum_{j<k} (f_k s_j - f_j s_k) \theta^j \wedge \theta^k.
\]

**Lemma 3.6.** \( \omega \) is a closed 2-form, i.e., \( d\omega = 0 \).

**Proof.** Choose a local orthonormal frame \( \{e_i\} \) with \( e_1 = N = \nabla f /|\nabla f| \), and let \( \{\theta^i\} \) be its dual coframe. Then, by Lemma 3.5
\[
d\text{div}_f z = \sum_{j<k} \sum_l f_i C_{ijkl} \theta^j \wedge \theta^k - \frac{n-2}{2n(n-1)} \sum_{j<k} (f_k s_j - f_j s_k) \theta^j \wedge \theta^k
= \sum_{j<k} |\nabla f| C_{ijkl} \theta^j \wedge \theta^k + \frac{n-2}{2n(n-1)} |\nabla f| \sum_{k=2}^n s_k \theta^1 \wedge \theta^k
= |\nabla f| \sum_{k=2}^n C_{1kl} \theta^1 \wedge \theta^k + \frac{n-2}{2n(n-1)} |\nabla f| \sum_{k=2}^n s_k \theta^1 \wedge \theta^k.
\]
Thus, by taking the exterior derivative of \( \omega \) in (3.10), we have
\[
d\omega = -df \wedge d\text{div}_f z = -|\nabla f| \theta^1 \wedge \left( |\nabla f| \sum_{k=2}^n C_{1kl} \theta^1 \wedge \theta^k \right)
- |\nabla f| \theta^1 \wedge \frac{n-2}{2n(n-1)} |\nabla f| \sum_{k=2}^n s_k \theta^1 \wedge \theta^k
= 0.
\]
In this section we will prove that there are no critical points of \( f \) in \( M \) for any tangent vector \( X \). First of all, recall that both totally geodesic, or each set contains only a single point.

**Lemma 4.1.** Let \( (M,g,f) \) be a closed generalized \((\lambda,n+m)\)-Einstein manifold. Assume that \( \omega = 0 \). Then \( D_{\nabla f} N = 0 \) and the function \( \alpha \) is constant along each level hypersurface \( f^{-1}(t) \) of \( f \).

**Proof.** Note that \( N(\nabla f) = Ddf(N,N) = \frac{af}{m} + \frac{\Delta f}{n} \) and
\[
N \left( \frac{1}{|\nabla f|^2} \right) = - \frac{1}{|\nabla f|^2} \left( \frac{af}{m} + \frac{\Delta f}{n} \right).
\]

So,
\[
D_{\nabla f} N = N \left( \frac{1}{|\nabla f|^2} \right) \nabla f + \frac{1}{|\nabla f|^2} D_N df
\]
\[
= - \frac{1}{|\nabla f|^2} \left( \frac{af}{m} + \frac{\Delta f}{n} \right) \nabla f + \frac{1}{|\nabla f|^2} \left( \frac{f}{m} z(N,N) + \frac{\Delta f}{n} N \right)
\]
\[
= - \frac{1}{|\nabla f|^2} \left( \frac{af}{m} + \frac{\Delta f}{n} \right) \nabla f + \frac{1}{|\nabla f|^2} \left( \frac{af}{m} + \frac{\Delta f}{n} \right) \nabla f
\]
\[
= 0.
\]

Now, let \( X \) be a vector field orthogonal to \( \nabla f \). Since \( D_{\nabla f} N = 0 \), we have \( g(D_{\nabla f} X,N) = -g(X,D_{\nabla f} N) = 0 \) and so
\[
D_{\nabla f} z(X,N) = - z(D_{\nabla f} X,N) - z(X,D_{\nabla f} N) = 0.
\]

Since \( i_{\nabla f} C = 0 \) by (3.11), we have
\[
0 = C(X,N,\nabla f) = D_X z(N,\nabla f) - D_N z(X,\nabla f) = D_X z(N,\nabla f) = |\nabla f| X(\alpha),
\]

implying that \( \alpha \) is a constant on \( f^{-1}(t) \).

**Remark 4.2.** The property \( D_{\nabla f} N = 0 \) also implies that \( [X,N] \) is orthogonal to \( \nabla f \). Using this, one can show that \( |z|^2 \) is also constant along each level hypersurface \( f^{-1}(t) \) of \( f \).
Lemma 4.3. Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Assume that $\omega = 0$. Then $\alpha$ is constant and, in particular, $\langle \nabla s, \nabla f \rangle = 0$ on the set $f^{-1}(0)$.

Proof. To show $\alpha$ is constant, it suffices to prove $\langle \nabla \alpha, \nabla f \rangle = 0$ on $M$ from Lemma 4.1. Since $i_{\nabla f}C = 0$ by (3.11) and $div_f = \alpha d f$, we can see, in the proof of Lemma 3.5 and Lemma 3.6, that

$$da = \frac{n-2}{2n(n-1)} \sum_{k=2}^{n} s_k \theta^k,$$

which implies that $\langle \nabla \alpha, \nabla f \rangle = 0$. (Note that by taking $N = e_1$, we have $f_1 = |\nabla f|, f_k = 0$ for $k \geq 2$.)

Now since $\alpha$ is constant and $i_{\nabla f} = \alpha d f$, we have $\delta i_{\nabla f} = -\alpha \Delta f$. Also, from the definition of divergence, we have

$$\delta i_{\nabla f} = -\frac{n-2}{2n} \langle \nabla s, \nabla f \rangle - \frac{f}{m} |z|^2,$$

and hence

$$\frac{n-2}{2n} \langle \nabla s, \nabla f \rangle = \alpha \Delta f - \frac{f}{m} |z|^2. \quad (4.1)$$

In particular, we have $\langle \nabla s, \nabla f \rangle = 0$ on the set $f^{-1}(0)$ since $\Delta f = 0$ on $f^{-1}(0)$. \qed

Corollary 4.4. Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Assume that $\omega = 0$. If $\langle \nabla s, \nabla f \rangle \geq 0$ on $M$, then $\alpha$ is nonpositive constant. Furthermore, if $\alpha = 0$, then $M$ is Einstein, and so isometric to a sphere $S^n(r)$.

Proof. From (4.1), we have the following inequality

$$\frac{n-2}{2n} \int_{f>0} \langle \nabla s, \nabla f \rangle = -\alpha \int_{f=0} \langle \nabla f - \frac{1}{m} \int_{f>0} f |z|^2 \rangle,$$

which shows that $\alpha$ is nonpositive on the set $f^{-1}(0)$. Since $\alpha$ is constant, $\alpha \leq 0$ on the whole $M$. If $\alpha = 0$, then we have $z = 0$ from (4.1) on the set $f > 0$, and so $\langle \nabla s, \nabla f \rangle = 0$ on the set $f > 0$. By elliptic theory, we have $z = 0$ on the whole $M$. The argument on existence of a conformal vector field in [8] (Proposition 1) also does work in our case and we can deduce the conclusion. \qed

Lemma 4.5. Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Assume that $\omega = 0$. Then $\nabla s$ is parallel to $\nabla f$. In particular $\nabla s = 0$ on the set $f^{-1}(0)$.

Proof. Since $i_{\nabla f} = \alpha d f$ and $\alpha$ is constant, by taking the differential operator $d$ of (2.5), we obtain

$$\left( \frac{m-1}{n} + \frac{1}{2} \right) ds \wedge df = 0,$$

which shows $\nabla s$ is parallel to $\nabla f$. In particular, since $\langle \nabla s, \nabla f \rangle = 0$ on $f^{-1}(0)$, by Lemma 4.3, we have $\nabla s = 0$ on the set $f^{-1}(0)$ because $\nabla f \neq 0$ on $f^{-1}(0)$. \qed

Lemma 4.6. Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold. Assume that $\omega = 0$ and $s$ is a (positive) constant. Then we have

$$\lambda = \frac{m+n-1}{n(n-1)} s + \frac{m-1}{n-1} \alpha. \quad (4.2)$$

In particular, $\lambda$ is a positive constant and $s < n \lambda$ on $M$.

Proof. Since $i_{\nabla f} = \alpha d f$ and $\alpha$ is constant, it follows from Lemma 2.4 that

$$(m-1) \alpha df + \frac{m-1}{n} s df = \frac{1}{2} fds + (n-1) d \mu,$$

which shows that

$$(m-1) \alpha f + \frac{m-1}{n} sf - (n-1) \mu = 0$$
on the whole $M$. Substituting $\mu = \lambda f - \frac{s}{m} f$, we obtain

$$\lambda = \frac{m + n - 1}{n(n - 1)} s + \frac{m - 1}{n - 1} \alpha.$$ 

In particular, $\lambda$ is constant. If $\lambda \leq 0$, from $\Delta f = \frac{f}{m} (s - n \lambda)$, the function $f$ is subharmonic on the set $f \geq 0$, which is a contradiction to the maximum principle. \hfill $\square$

**Lemma 4.7.** Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. If the scalar curvature $s$ is constant and $\omega = 0$, we have

$$\text{Ric}(N, N) \geq 0$$
on the set $M$.

*Proof.* Suppose that $\text{Ric}(N, N) < 0$ at a point $x \in M$. Since $\text{Ric}(N, N) = \alpha + \frac{s}{n}$ is constant, $\text{Ric}(N, N) < 0$ on the whole $M$. Considering a connected component $\Gamma$ of $f^{-1}(0)$ which is totally geodesic by Lemma 2.3, we have

$$\int_{\Gamma} \left[ |\nabla^\Gamma \varphi|^2 - \text{Ric}(N, N) \varphi^2 \right] \geq 0$$

for any smooth function $\varphi$ defined on $\Gamma$. By Fredholm alternative (cf. [7], Theorem 1), there exists a positive $\varphi > 0$ on $\Gamma$ satisfying

$$\Delta^\Gamma \varphi + \text{Ric}(N, N) \varphi = 0.$$ 

However, it follows from the maximum principle $\varphi$ must be a constant which is impossible. \hfill $\square$

**Lemma 4.8.** Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. If the scalar curvature $s$ is constant and $\omega = 0$, then

$$\text{Ric}(N, N) = \frac{(n - 1) \lambda - s}{m - 1}.$$ 

In particular, we have $(n - 1) \lambda \geq s$.

*Proof.* Following [9], we define a tensor $P := \text{Ric} - \rho g$ with $\rho = \frac{(n - 1) \lambda - s}{m - 1}$. By taking the divergence of (1.1), we can obtain (cf. [8])

$$\frac{f \nabla s}{2(m - 1)} - \frac{(n - 1) f \nabla \lambda}{m - 1} + P(\nabla f, \cdot) = 0.$$ 

Since $\omega = 0$ and the scalar curvature $s$ is constant, $\lambda$ is also constant by Lemma 4.6. Thus, we have

$$P(\nabla f, \cdot) = 0.$$ 

In particular, we have

$$\text{Ric}(N, N) = \rho = \frac{(n - 1) \lambda - s}{m - 1}.$$ 

The last inequality follows from Lemma 4.7. \hfill $\square$

**Lemma 4.9.** Let $(M, g, f)$ be a closed generalized $(\lambda, n + m)$-Einstein manifold with $f^{-1}(0) \neq \emptyset$. Assume that $\omega = 0$ and $s$ is a (positive) constant. Then there are no critical points of $f$ except at the minimum and maximum points of $f$.

*Proof.* We have $s > 0$, $\alpha \leq 0$, $s - n \lambda < 0$ and these are all constants. If $\alpha = 0$, then $M$ is Einstein and so is isometric to a sphere by Corollary 4.4. So, we may assume $\alpha < 0$. Moreover, we have

$$\Delta f = \frac{s - n \lambda}{m} f \quad \text{and} \quad \alpha \Delta f = \frac{f}{m} |z|^2 \quad \text{(4.3)}$$ 

and so $|z|^2 = \alpha (s - n \lambda)$ is also a positive constant. It follows from the Bochner-Weitzenb"{o}ck formula that

$$\frac{1}{2} \Delta |\nabla f|^2 = |Dd f|^2 + \langle \nabla \Delta f, \nabla f \rangle + z(\nabla f, \nabla f) + \frac{s}{n} |\nabla f|^2.$$
From (2.2), we have
\[ z(\nabla f, \nabla f) = \frac{m}{f} Ddf(\nabla f, \nabla f) - \frac{m}{n} \frac{\Delta f}{f} |\nabla f|^2 = \frac{m}{2f} \nabla f(|\nabla f|^2) - \frac{s-n\lambda}{n} |\nabla f|^2. \]

Thus,
\[ \frac{1}{2} \Delta |\nabla f|^2 - \frac{m}{2f} \nabla f(|\nabla f|^2) = |Ddf|^2 + \left( \frac{s-n\lambda}{m} + \lambda \right) |\nabla f|^2. \tag{4.4} \]

Note that (4.4) is valid only in \( f > 0 \) and \( f < 0 \).

**Assertion:** We claim \( \frac{s-n\lambda}{m} + \lambda > 0 \).

From Lemma 4.6, we have
\[ \frac{s-n\lambda}{m} = - \frac{s}{n-1} - \frac{n(m-1)}{m(n-1)} \alpha. \]

So, by Lemma 4.8,
\[ \frac{s-n\lambda}{m} + \lambda = \lambda - \frac{s}{n-1} - \frac{n(m-1)}{m(n-1)} \alpha > 0 \]

Now applying the maximum principle to (4.4) on the set \( f > 0 \), the function \( |\nabla f|^2 \) cannot have its local maximum in \( f > 0 \).

Now, suppose that there is a critical point \( p \) of \( f \) with \( 0 < f(p) = c < b = \max f \). Since \( |\nabla f| \) is constant on each level set of \( f \), \( |\nabla f| = 0 \) on \( f^{-1}(c) \). However, this implies that there should be a local maximum of \( |\nabla f|^2 \) in the set \( \{ x \in M \mid c < f(x) < b \} \), which is impossible by the above maximum principle.

Since \( |\nabla f|^2 \) cannot have its local maximum in the set \( f > 0 \), the exactly same argument as above shows that \( f \) cannot have its critical points in the set \( f < 0 \).

Let \((M, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold with \( f^{-1}(0) \neq \emptyset \). Assume that \( \omega = 0 \) and \( s \) is a (positive) constant. Let \( \min_M f = a \) and \( \max_M f = b \) with \( a < b < b \). Lemma 4.9 shows that the sets \( f^{-1}(a) \) and \( f^{-1}(b) \) are both connected, and either each set is a single point or a hypersurface by the Isotopy lemma. In fact, if \( f^{-1}(a) \) has at least two components, \( f \) may have a critical point other than \( f^{-1}(a) \).

Lemma 4.9 also shows that if \( f^{-1}(0) \) is disconnected, then it has only two connected components. Furthermore, if \( f^{-1}(0) \) is connected, then both \( f^{-1}(a) \) and \( f^{-1}(b) \) consist of only a single point, and in this case each level hypersurface including \( f^{-1}(0) \) is homotopically an \((n-1)\)-sphere \( \mathbb{S}^{n-1} \).

The following result shows that if \( f^{-1}(a) \) contains a single point, then so does \( f^{-1}(b) \), and vice versa. Moreover in case of hypersurface, it should be a totally geodesic stable minimal hypersurface.

**Lemma 4.10.** Let \((M^n, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold with \( f^{-1}(0) \neq \emptyset \). Assume that \( \omega = 0 \) and \( s \) is a (positive) constant. Let \( \min_M f = a \) and \( \max_M f = b \) with \( a < b < b \). If \( f^{-1}(a) \) contains only a single point, then so does \( f^{-1}(b) \), and vice versa. Furthermore, if \( f^{-1}(a) \) is a single point, then every level set \( f^{-1}(t) \) except \( t = a \) and \( t = b \) is a hypersurface and is homotopically a sphere \( \mathbb{S}^{n-1} \).

**Proof.** Suppose that \( f^{-1}(a) \) is a hypersurface, but \( f^{-1}(b) \) is a single point. It follows from Lemma 4.9 together with Isotopy lemma that the set \( f^{-1}(b) \) consists of only two points, and the set \( M - f^{-1}(a) \) has two connected components.

If \( \alpha = 0 \), then \((M, g)\) is Einstein by Corollary 4.4. We may assume that \( \alpha < 0 \) and so, from (4.1),
\[ \Delta f = \frac{f}{m\alpha} |z|^2. \tag{4.5} \]

Applying the maximum principle to (4.5) on each connected component of \( M - f^{-1}(a) \), \( f \) attains its maximum on the boundary \( f^{-1}(a) \), which is a contradiction.
Lemma 4.11. Let \((M, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). Assume that \(\omega = 0\) and \(s\) is a (positive) constant. Suppose that \(\Sigma := f^{-1}(b) [\Sigma = f^{-1}(a)]\) is a hypersurface for \(b = \max_M f\) \([a = \min_M f]\) and assume \(\nu\) is a unit normal vector field on \(\Sigma\). Then we have the following.

1. \(\alpha = z(\nu, \nu) = \frac{n - 1}{m} (n\lambda - s) < 0\) and \(Dd f_p(X, X) = 0\) for a vector \(X\) orthogonal to \(\nu\) at any point \(p \in \Sigma\).

2. \(\Sigma\) is totally geodesic.

Proof. First, we will show that \(Dd f|_\Sigma = 0\). For a sufficiently small \(\epsilon > 0\), \(f^{-1}(a - \epsilon)\) has two connected components \(\Sigma^+\) and \(\Sigma^-\). Let \(\nu\) be a unit normal vector field on \(\Sigma = f^{-1}(a)\). On a tubular neighborhood of \(\Sigma\), \(\nu\) can be extended smoothly to a vector field \(\tilde{\nu}\) such that \(\tilde{\nu}|_{\Sigma} = \nu\) with \(\tilde{\nu}|_{\Sigma^+} = \frac{\nabla}{\nabla \nu|_{\Sigma^+}}\) and \(\tilde{\nu}|_{\Sigma^-} = -\frac{\nabla}{\nabla \nu|_{\Sigma^-}}\).

The Laplacian of \(f\) on \(f^{-1}(a - \epsilon) = \Sigma^- \cup \Sigma^+\) is given by

\[
\Delta f = \Delta' f + Dd f(\tilde{\nu}, \tilde{\nu}) + H(\tilde{\nu}, \nabla f),
\]

where \(\Delta'\) and \(H\) denote the intrinsic Laplacian and the mean curvature of \(f^{-1}(a - \epsilon)\), respectively. Since \(\nabla f = 0\) at \(\Sigma\) and \(\Delta' f = 0\) in (4.6), by letting \(\epsilon \to 0\), we have

\[
\Delta f = Dd f(\nu, \nu)
\]

on \(\Sigma\). It is clear that \(H\) is bounded around \(\Sigma\). Thus,

\[
\frac{n\lambda - s}{m} a = \Delta f = Dd f(\nu, \nu) = \frac{a}{m} z(\nu, \nu) - \frac{n\lambda - s}{mn} a,
\]

implying that

\[
zm(\nu, \nu) = -\frac{n - 1}{n} (n\lambda - s) < 0.
\]

Now let \(p \in \Sigma\) be a point and choose an orthonormal basis \(\{e_i\}_{i=1}^n\) at \(p\) with \(e_1 = \nu\). Since \(\Sigma\) is the maximum set of \(f\), we have

\[
Dd f_p(e_i, e_i) = \frac{a}{m} z_p(e_i, e_i) - \frac{n\lambda - s}{mn} a \leq 0
\]

for \(2 \leq i \leq n\), and so

\[
z_p(e_i, e_i) \leq \frac{n\lambda - s}{n} = \frac{1}{n - 1} z_p(\nu, \nu).
\]

Since \(\sum_{i=2}^n z_p(e_i, e_i) = -z_p(\nu, \nu)\), this implies that

\[
z_p(e_i, e_i) = \frac{n\lambda - s}{n} > 0
\]

and so

\[
Dd f_p(e_i, e_i) = 0
\]

on \(\Sigma\) for each \(i, 2 \leq i \leq n\). In case of minimum set \(\Sigma = f^{-1}(a)\) with \(a = \min_M f\), the inequalities are just reversed above and we have the same conclusion.

Second, we will show that \(\Sigma\) is totally geodesic. Since \(z(\nu, X) = 0\) for \(X\) orthogonal to \(\nu\) at \(p \in \Sigma\), we may take the previously mentioned orthonormal basis \(\{e_i\}_{i=1}^n\) so that \(\{e_i\}_{i=2}^n\) are tangent to \(\Sigma\).

For \(e_2\), let \(\gamma : [0, l) \to M\) be a unit speed geodesic such that \(\gamma(0) = p, \gamma'(0) = e_2\) for some \(l > 0\). Defining \(\varphi(t) = f \circ \gamma(t)\), we have \(\varphi'(0) = df_p(\gamma'(0)) = 0\), and by (4.8)

\[
\varphi''(0) = Dd f(\gamma'(0), \gamma'(0)) = Dd f_p(e_2, e_2) = 0.
\]

Note that

\[
\varphi''(t) = Dd f(\gamma'(t), \gamma'(t)) = \left[ z(\gamma'(t), \gamma'(t)) + \frac{(s - n\lambda)}{m} \right] \varphi(t).
\]

So, it follows from the uniqueness of ODE solution that \(\varphi\) vanishes identically, which implies that \(\gamma(t)\) stays in \(\Sigma\). Since \(e_2\) is an arbitrary tangent vector, \(\Sigma\) is totally geodesic. 

\[\square\]
Let \((M^n, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). Assume that \(\omega = 0\) and \(s\) is a (positive) constant. By Lemma 4.8, we have

\[
\text{Ric}(N, N) = \rho = \frac{(n-1)\lambda - s}{m-1} \geq 0 \tag{4.8}
\]

and so

\[
\alpha = z(N, N) = \text{Ric}(N, N) - \frac{s}{n} = \frac{(n-1)\lambda - s}{m-1} - \frac{s}{n} \tag{4.9}
\]

Furthermore, from (4.7), we have

\[
\frac{(n-1)\lambda - s}{m-1} - \frac{s}{n} = \alpha = -\frac{n-1}{n}(n\lambda - s)
\]

So, we obtain

\[
(n-1)\lambda - s = 0
\]

and hence

\[
\text{Ric}(N, N) = 0.
\]

By continuity, this equality hold on the whole \(M\) including the set \(\Sigma = f^{-1}(a)\) for \(a = \min_M f\) or \(\Sigma = f^{-1}(b)\) for \(b = \max_M f\) if we assume \(\Sigma\) is a hypersurface. Finally, since \(n\lambda - s = \lambda\), it follows from (4.7) again that

\[
z(N, N) = \alpha = -\frac{n-1}{n}\lambda = -\frac{s}{n}.
\]

Since \(\Sigma\) is totally geodesic, we have

\[
s\Sigma = s - 2\text{Ric}(\nu, \nu) = s.
\]

**Lemma 4.12.** Let \((M, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). Assume that \(\omega = 0\) and \(s\) is a (positive) constant. Suppose that \(\Sigma := f^{-1}(a) [\Sigma = f^{-1}(b)]\) is a hypersurface for \(a = \min_M f\) \([b = \max_M f]\) and assume \(\nu\) is a unit normal vector field on \(\Sigma\). Then we have

\[
\lambda = \frac{s}{n-1}, \quad \alpha = -\frac{s}{n} \quad \text{and} \quad \text{Ric}(N, N) = 0.
\]

Moreover, \(\Sigma\) is stable.

**Proof.** The stability operator becomes

\[
\int_{\Sigma} \left\{ |\nabla \varphi|^2 - (|A|^2 + \text{Ric}(\nu, \nu))\varphi^2 \right\} = \int_{\Sigma} |\nabla \varphi|^2 \geq 0
\]

for any function \(\varphi\) on \(\Sigma\). Here \(A\) denotes the second fundamental form of \(\Sigma\). \(\square\)

**Theorem 4.13.** Let \((M, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). Assume that \(\omega = 0\) and \(s\) is a (positive) constant. If \(f^{-1}(a)\) is a hypersurface for \(a = \max_M f\) or \(a = \min_M f\), then \((M, g)\) has harmonic Weyl curvature and Bach-flat.

**Proof.** Note that

\[
|T|^2 = \frac{2}{(n-2)^2} |\nabla f|^2 \left( |z|^2 - \frac{n}{n-1} \alpha^2 \right).
\]

By (4.3), we have

\[
|z|^2 = \alpha(s - n\lambda) = -\alpha \lambda.
\]

So,

\[
|z|^2 - \frac{n}{n-1} \alpha^2 = \frac{\alpha}{n-1} (n-1)\lambda + n \alpha
\]

\[
= \frac{n}{n-1} ((n-1)\lambda - s) = 0,
\]

which means \(T = 0\). The conclusion follows from Theorem 3.4. \(\square\)
Let \((M^n, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). Assume that \(\omega = 0\) and \(s\) is a (positive) constant. If \(f^{-1}(a)\) is a hypersurface for \(a = \max_M f\) or \(a = \min_M f\) so that \(f^{-1}(0)\) has two connected components which are both totally geodesics, then \(T = 0\) and so we have

\[ z_{ij} = -\frac{\alpha}{n - 1} \delta_{ij} = \frac{s}{n(n - 1)} \delta_{ij} \quad \text{for} \quad 2 \leq i, j \leq n \]

and

\[ z_{1k} = \alpha \delta_{1k} = -\frac{s}{n} \delta_{1k} \quad \text{for} \quad 1 \leq k \leq n. \]

That is,

\[ \text{Ric}_{ij} = \frac{s}{n - 1} \delta_{ij} \quad \text{for} \quad 2 \leq i, j \leq n \]

and

\[ \text{Ric}_{1k} = 0. \]

In particular, on each level hypersurface \(f^{-1}(t)\), we have

\[ \frac{s}{n(n - 1)} g - z = 0 \]

and from \(\frac{s}{n(n - 1)} g(N, N) - z(N, N) = \frac{s}{n - 1}\), we obtain

\[ \frac{s}{n(n - 1)} g = z + \frac{s}{n - 1} \frac{df}{|df|} \otimes \frac{df}{|df|}. \]

On the other hand, for the curvature tensor \(R\) with \(N = \nabla f / |\nabla f|\), \(R_N\) is defined as follows

\[ R_N(X, Y) = R(X, N, Y, N) \]

for any vector fields \(X\) and \(Y\). From \(C = 0\) and Lemma 3.2, we have

\[ i_{\nabla f} \mathcal{W} = 0. \]

So, from the curvature decomposition

\[ R = \frac{s}{2n(n - 1)} g \otimes g + \frac{1}{n - 2} z \otimes g + \mathcal{W} \]

we can obtain

\[ R_N(X, Y) = \frac{s}{n(n - 1)} g(X, Y) + \frac{1}{n - 2} z(X, Y) + \frac{\alpha}{n - 2} g(X, Y) + \mathcal{W}_N(X, Y) \]

\[ = \left( \frac{s}{n(n - 1)} + \frac{s}{n(n - 1)(n - 2)} - \frac{s}{n(n - 2)} \right) g(X, Y) \]

\[ = 0 \]

for any vector fields \(X\) and \(Y\) orthogonal to \(\nabla f\).

Since \(\Sigma = f^{-1}(a)\) with \(a = \max_M f\) or \(a = \min_M f\) is totally geodesic by Lemma 4.11, it follows from the Gauss equation that

\[ \text{Ric}^\Sigma(X, X) = \lambda - R(X, \nu, X, \nu) = \lambda, \]

which shows that \(\Sigma\) is an Einstein manifold with positive Ricci curvature. From the same reason, any connected component of the set \(f^{-1}(0)\) is also Einstein by Lemma 3.2. In fact, since \((M, g)\) has harmonic Weyl curvature and \(i_{\nabla f} \mathcal{W} = 0\), from a result in [6] \(g\) is locally a warped product of an interval with an Einstein manifold around any regular point of \(f\). In our case, from (4.10), we can see that \(g\) is, in fact, a product of a circle \(S^1\) with an Einstein manifold \(\Sigma_0\) which is totally geodesic and Einstein with positive Ricci curvature.

Therefore, we have the following.
Lemma 4.15. Let \((M^n, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold of constant scalar curvature with \(f^{-1}(0) \neq \emptyset\) and \(\omega = 0\). Assume that \((M, g)\) has PIC and \(f^{-1}(0)\) is disconnected. Then \(f^{-1}(0)\) has only two connected components and \(M\) is isometric to \(S^1 \times \Sigma^{n-1}\) up to finite cover and rescaling.

Theorem 4.15. Let \((M^n, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold of constant scalar curvature with \(f^{-1}(0) \neq \emptyset\) and \(\omega = 0\). Assume that \((M, g)\) has PIC and \(f^{-1}(0)\) is disconnected. Then \(f^{-1}(0)\) has only two connected components and \(M\) is isometric to \(S^1 \times \Sigma^{n-1}\), up to finite cover and rescaling.

5. Compact Generalized Manifold with Connected Zero Set

Throughout this section, we assume that \((M^n, g, f)\) is a closed generalized \((\lambda, n+m)\)-Einstein manifold with constant scalar curvature \(s\). We also assume that \((M, g)\) has \(\omega = df \wedge \mathcal{L}_f z = 0\) and the zero set \(f^{-1}(0)\) is connected. Under these hypotheses, we show that \(M\) is isometric to a sphere \(S^n\), up to finite cover and rescaling.

Recall that \(\alpha \leq 0\), and if \(\alpha = 0\), it follows from Lemma 4.9 and Lemma 4.10 that the minimum set \(f^{-1}(a)\) with \(a = \min_M f\) and the maximum set \(f^{-1}(b)\) with \(b = \max_M f\) consist of a single point, respectively, and \(f^{-1}(0)\) must be connected. In particular, we have \(\text{Ric}(N, N) > 0\). The following shows that the converse is also true.

Lemma 5.1. Let \((M, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold with constant scalar curvature \(s\). If \(\omega = 0\) and \(f^{-1}(0)\) is connected, then we have \(\text{Ric}(N, N) > 0\) on the set \(M\).

Proof. By Lemma 4.7, we have \(\text{Ric}(N, N) \geq 0\). Suppose that \(\text{Ric}(N, N) = 0\) on \(M\) so that \(\alpha = -\frac{s}{n}\) and \((n-1)\lambda - s = 0\) by (4.9). Then as in the proof of Theorem 4.13 and the argument just below it, we have \(T = 0\) in this case, and so the metric \(g\) has the same form as (4.10):

\[
\frac{s}{n(n-1)} g = z + \frac{s}{n-1} \frac{df}{|df|} \otimes \frac{df}{|df|}
\]

which shows that \(g\) is, in fact, a product metric. However, since \(f^{-1}(0)\) is connected, both minimum set \(f^{-1}(a)\) and maximum set \(f^{-1}(b)\) consist of a single point, respectively, the metric \(g\) cannot be a product, a contradiction. \(\square\)

Let \((M^n, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold. Assume the scalar curvature \(s\) is constant. If \((M, g)\) has PIC and \(f^{-1}(0)\) is connected, then, from Lemma 4.10 and [15], \(M\) is homeomorphic to a sphere.

Now, we introduce a warped product metric involving \(\frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}\) as a fiber metric on each level set \(f^{-1}(c)\). Consider a warped product metric \(\tilde{g}\) on \(M\) by

\[
\tilde{g} = \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + |\nabla f|^2 g_r,
\]

where \(g_r\) is the restriction of \(g\) to \(\Gamma := f^{-1}(0)\). Note that, from Lemma 4.9, the metric \(\tilde{g}\) is smooth on \(M\) except, possibly at two points, the maximum and minimum points of \(f\).

The following lemma shows that \(\nabla f\) is a conformal vector field with respect to the metric \(\tilde{g}\).
Lemma 5.2. Let \((M, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with constant scalar curvature \(s\) and \(\omega = 0\). Then,
\[
\frac{1}{2} \mathcal{L}_f \bar{g} = N((\nabla f)) \bar{g} = \frac{1}{n}(\bar{\Delta} f) \bar{g}.
\]
Here, \(\mathcal{L}\) denotes the Lie derivative.

Proof. Note that, by (1.1) we have,
\[
\frac{1}{2} \mathcal{L}_f g = D_g df = \frac{f}{m} (\text{Ric} - \lambda g).
\]
By the definition of Lie derivative,
\[
\frac{1}{2} \mathcal{L}_f (df \otimes df)(X, Y) = Ddf(X, \nabla f) df(Y) + df(X) Ddf(Y, \nabla f)
\]
\[= \frac{2f}{m} (\text{Ric}(N, N) - \lambda) df \otimes df(X, Y).\]
Therefore,
\[
\frac{1}{2} \mathcal{L}_f \left( \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \right) = N(|\nabla f|) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}. \tag{5.1}
\]
Since
\[
\frac{1}{2} \mathcal{L}_f(|\nabla f|^2 g_\Sigma) = \frac{1}{2} \nabla f(|\nabla f|^2) g_\Sigma = Ddf(\nabla f, \nabla f) g_\Sigma = N(|\nabla f|)|\nabla f|^2 g_\Sigma,
\]
we conclude that
\[
\frac{1}{2} \mathcal{L}_f \bar{g} = \bar{D} df = N(|\nabla f|) \bar{g}.
\]
In particular, we have \(\bar{\Delta} f = nN(|\nabla f|)\). \qed

Lemma 5.3. Let \((M^n, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with constant scalar curvature \(s\) and \(\omega = 0\). If \(f^{-1}(0)\) is connected, then \(T = 0\) on \(M\).

Proof. Let \(p, q \in M\) be the only two points such that \(f(p) = \min_M f\) and \(f(q) = \max_M f\), respectively, and let \(\bar{M} = M \setminus \{p, q\}\). Due to Lemma 4.9 together with our assumption that \(f^{-1}(0)\) is connected and Lemma 5.2, we can apply Tashiro’s result [17] and can see that \((\bar{M}, \bar{g})\) is conformally equivalent to \(S^n \setminus \{\bar{p}, \bar{q}\}\), where \(\bar{p}\) and \(\bar{q}\) are the points in \(S^n\) corresponding to \(p\) and \(q\), respectively. In particular, by Theorem 1 in [5], the fiber space \((\Gamma, g|_r)\) is a space of constant curvature. Thus,
\[
(\Gamma, g|_r) \equiv (S^{n-1}, r \cdot g_{S^{n-1}}),
\]
where \(r > 0\) is a positive constant and \(g_{S^{n-1}}\) is a round metric.

Now, replacing \(\Gamma = f^{-1}(0)\) by \(\Gamma_t := f^{-1}(t)\) in (5.1), it can be easily concluded that the warped product metric \(\bar{g}_t\) also satisfies Lemma 5.2, and hence, the same argument mentioned above shows that, for any level hypersurface \(\Gamma_t := f^{-1}(t)\),
\[
(\Gamma_t, g|_{r_t}) \equiv (S^{n-1}, r(t) \cdot g_{S^{n-1}}).
\]
Therefore, the original metric \(g\) can also be written as
\[
g = \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + b(f)^2 g_\Gamma, \tag{5.2}
\]
where \(b(f) > 0\) is a positive function depending only on \(f\). From (5.1) and the following identity
\[
\frac{1}{2} \mathcal{L}_f (b^2 g_\Gamma) = b(\nabla f, \nabla b) g_\Gamma = b|\nabla f|^2 \frac{db}{df} g_\Gamma,
\]
we obtain
\[
\frac{1}{2} \mathcal{L}_f g = N(|\nabla f|) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + b|\nabla f|^2 \frac{db}{df} g_\Gamma. \tag{5.3}
\]
On the other hand, from (1.1) and (5.2), we have
\[
\frac{1}{2} \mathcal{L}_f g = \frac{f}{m} (\text{Ric} - \lambda g) = N (|\nabla f| \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}) + \frac{f}{m} \text{Ric} - \frac{f}{m} \text{Ric}(N, N) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} - \frac{\lambda f}{m} b^2 g. 
\]
Comparing this to (5.3), we obtain
\[
(\frac{b^2}{n-1} \frac{db}{df} + \frac{\lambda f}{m} b^2) g_T = \frac{f}{m} \left( \text{Ric} - \text{Ric}(N, N) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \right). 
\]
Now, let \{e_1, e_2, \ldots, e_n\} be a local frame with \(e_1 = N\). Then, by applying \((e_j, e_j)\) for each \(2 \leq j \leq n\) to (5.4), we have
\[
\frac{b^2}{n-1} \frac{db}{df} = \frac{f}{m} \text{Ric}(e_j, e_j) - \frac{\lambda f}{m} b^2 
\]
for \(2 \leq j \leq n\). Summing up these, we obtain
\[
(n-1) \frac{b^2}{n-1} \frac{db}{df} = \frac{f}{m} [s - \text{Ric}(N, N)] - \frac{(n-1) \lambda f}{m} b^2. 
\]
Substituting this into (5.4), we get
\[
\frac{1}{n-1} [s - \text{Ric}(N, N)] g_T = \left( \frac{\text{Ric} - \text{Ric}(N, N)}{|\nabla f|} \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \right). 
\]
Replacing \((\Gamma, g_T)\) by \((\Gamma_t, g_{T_t})\), we can see that the argument mentioned above is also valid. Thus, (5.5) shows that, on each level hypersurface \(f^{-1}(t)\), we have
\[
\text{Ric}(e_j, e_j) = \frac{1}{n-1} [s - \text{Ric}(N, N)] 
\]
for \(2 \leq j \leq n\), which is equivalent to
\[
z(e_i, e_i) = -\frac{\alpha}{n-1}. 
\]
Hence,
\[
|z|^2 = \alpha^2 + \frac{n-1}{n} \alpha^2 = \frac{n}{n-1} |z_N|^2, 
\]
since \(z(N, e_i) = 0\) for \(i \geq 2\). As a result, we have \(T = 0\).

\[\square\]

**Theorem 5.4.** Let \((M, g, f)\) be a closed generalized \((\lambda, n+m)\)-Einstein manifold. Assume the scalar curvature \(s\) is constant and \(\omega = 0\) If \(f^{-1}(0)\) is connected, then \(M\) is isometric to a sphere \(\mathbb{S}^n\).

**Proof.** By Lemma 5.1 together with (4.8) and (4.9), we have
\[
\text{Ric}(N, N) = \frac{(n-1) \lambda - s}{m-1} > 0 \quad \text{and} \quad \alpha = \frac{(n-1) \lambda - s}{m-1} - \frac{s}{n} \leq 0. 
\]
Suppose that \(\alpha < 0\). From Lemma 5.3 and (4.3), we have
\[
\frac{n}{n-1} \alpha^2 = |z|^2 = \alpha (s - n \lambda) 
\]
and so
\[
\alpha = \frac{n-1}{n} s - (n-1) \lambda = [s - (n-1) \lambda] - \frac{s}{n}. 
\]
Comparing this to (5.6), we have \((n-1) \lambda - s = 0\), which contradicts \(\text{Ric}(N, N) > 0\). Hence we have \(\alpha = 0\) and consequently \((M, g)\) is Einstein. Finally, using an argument in [8], we can show that \((M, g)\) is isometric to a sphere.

\[\square\]
6. Final Remarks

Let \((M^n, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). If \(\omega = 0\) and the scalar curvature \(s\) is constant, by Lemma 4.6, the function \(\lambda\) is also a (positive) constant (see also Corollary 2.5 for \(m = 1\) without vanishing of \(\omega\)). In case of \(m \geq 1\), we can show that the converse is also true. Namely, we have the following.

**Lemma 6.1.** Let \((M, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold with \(f^{-1}(0) \neq \emptyset\). If \(\omega = 0\) and the function \(\lambda\) is constant, then the scalar curvature \(s\) is also a (positive) constant.

**Proof.** Suppose that \(\lambda\) is constant. We recall that, from Lemma 4.3, the function \(\alpha\) is constant. Since \(i_{\xi} f z = o df\), it follows from Lemma 2.4 that

\[
ds + \frac{2(m + n - 1)}{n} s df = 2 [(n - 1)\lambda - (m - 1)\alpha] df.
\]

Now let \(k_1 = \frac{m+n-1}{n}\) and \(k_2 = (n - 1)\lambda - (m - 1)\alpha\). Considering the set \(M^0 = \{ x \in M : f(x) > 0 \}\) and multiplying by \(\frac{1}{f}\), we obtain, on the set \(M^0\),

\[
\nabla s + 2k_1 s \nabla \ln f = 2k_2 \nabla \ln f. \quad (6.1)
\]

Defining \(\varphi = 2k_1 \ln f\), we can rewrite (6.1) as

\[
\nabla s + s \nabla \varphi = \frac{k_2}{k_1} \nabla \varphi,
\]

or equivalently

\[
\nabla (se^{\varphi}) = \frac{k_2}{k_1} \nabla e^{\varphi}.
\]

Since \(k_1\) and \(k_2\) are constants, we conclude that

\[
s = \frac{k_2}{k_1} + c_0 e^{-\varphi} = \frac{k_2}{k_1} + \frac{c_0}{f^{2k_1}}.
\]

Consequently, we obtain

\[
s f^{2k_1} = \frac{k_2}{k_1} f^{2k_1} + c_0
\]

on the set \(M^0 = \{ x \in M : f(x) > 0 \}\). Taking a sequence \(p_l \in M^0\) tending to a point \(p \in f^{-1}(0)\) as \(l \to \infty\), we have \(c_0 = 0\) and hence

\[
s = \frac{k_2}{k_1} = \frac{n [(n - 1)\lambda - (m - 1)\alpha]}{m + n - 1} \quad (6.2)
\]

which shows that \(s\) is constant on the set \(f > 0\). The same argument works on the set \(M_0 := \{ x \in M : f(x) < 0 \}\) and the proof is complete. \(\square\)

Note that (6.2) is exactly the same as (4.2) in Lemma 4.6. Applying Theorem 4.15 and Theorem 5.4, we have the following result.

**Theorem 6.2.** Let \((M^n, g, f)\) be a closed generalized \((\lambda, n + m)\)-Einstein manifold. Assume \((M, g)\) has PIC and the function \(\lambda\) is constant. Then

1. if \(f^{-1}(0)\) is connected, then \(M\) is isometric to a sphere \(S^n\).
2. if \(f^{-1}(0)\) is disconnected, then, it has only two connected components and \(M\) is isometric to \(S^1 \times S^{n-1}\), up to finite cover and rescaling.

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