DEGENERATION OF INTERMEDIATE JACOBIANS AND THE TORELLI THEOREM

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Abstract. Mumford and Newstead generalised the classical Torelli theorem to higher rank i.e., a smooth, projective curve $X$ is uniquely determined by the second intermediate Jacobian of the moduli space of stable rank 2 bundles on $X$, with fixed odd degree determinant. In this article we prove the analogous result in the case $X$ is an irreducible nodal curve with one node. As a byproduct, we obtain the degeneration of the intermediate Jacobians of a family of such moduli spaces.

1. Introduction

Throughout this article the underlying field will be $\mathbb{C}$. Given a smooth, projective variety $Y$, the $k$-th intermediate Jacobian of $Y$, denoted $J^k(Y)$ is defined as:

$$J^k(Y) := \frac{H^{2k-1}(Y, \mathbb{C})}{F^k H^{2k-1}(Y, \mathbb{C}) + H^{2k-1}(Y, \mathbb{Z})},$$

where $F^\cdot$ denotes Hodge filtration. The classical Torelli theorem states that a smooth, projective curve $X$ is uniquely determined by its Jacobian variety $J^1(X)$, along with the polarisation on $J^1(X)$ induced by a non-degenerate, integer valued symplectic pairing on $H^1(X, \mathbb{Z})$. In [Nam73] Namikawa extended the Torelli theorem to irreducible nodal curves. In [MN68], Mumford and Newstead generalised the classical Torelli to higher rank. More precisely they proved that the Jacobian $J^1(X)$ of the curve $X$ is isomorphic to the (second) intermediate Jacobian $J^2(M_X(2, L))$ of the moduli space $M_X(2, L)$ of stable rank 2 vector bundles with odd degree determinant $L$ over $X$.

In this article, we prove the analogous higher rank version of the Torelli theorem in the case $X$ is an irreducible nodal curve:

**Theorem 1.1** (See Theorem 4.4). Let $X_0$ and $X_1$ be irreducible nodal curves of genus $g \geq 4$ with exactly one node such that the normalizations $\tilde{X}_0$ and $\tilde{X}_1$ are not hyper-elliptic. Let $L_0$ and $L_1$ be invertible sheaves of odd degree on $X_0$ and $X_1$, respectively. Denote by $G_{X_0}(2, L_0)$ (resp. $G_{X_1}(2, L_1)$) the Gieseker moduli space of rank 2 semi-stable sheaves with determinant $L_0$ (resp. $L_1$) on curves semi-stably equivalent to $X_0$ (resp. $X_1$). If $G_{X_0}(2, L_0) \cong G_{X_1}(2, L_1)$ then $X_0 \cong X_1$.

We now discuss the strategy of the proof of the theorem. Since the Hodge structure of a smooth, projective variety is pure, Mumford and Newstead use the theory of correspondences to give an isomorphism of pure Hodge structures between $H^1(X, \mathbb{C})$ (in the case $X$ is smooth) and $H^2(M_X(2, L), \mathbb{C})$ (if $X$ is smooth, so is $M_X(2, L)$). In the case $X$ is a reducible curve, Basu in [Bas16] proves an analogous result. Moreover, he also shows that in this case, the Hodge
structure on \( H^1(X, \mathbb{C}) \) and \( H^3(M_X(2, L)) \) are again pure (see Bas16 Lemma 4.1 and 4.3). However, in the case when \( X \) is an irreducible nodal curve, these cohomology groups no longer have a pure Hodge structure (see Theorem 2.9). Therefore, the techniques of Mumford-Newstead as well as of Basu do not generalise to this case.

In this article, we prove a relative version of Mumford-Newstead on a regular family of projective curves,

\[
\pi_1 : \mathcal{X} \to \Delta
\]
of genus \( g \geq 2 \), smooth over the punctured disc \( \Delta^\ast \) and central fiber \( X_0 \) an irreducible nodal curve with exactly one node. Fix an invertible sheaf \( L \) on \( \mathcal{X} \) of odd degree and let \( L_0 := L|_{X_0} \). Denote by \( \pi_2 : \mathcal{G}(2, L) \to \Delta \) (resp. \( \mathcal{G}_X(2, L_0) \)) the relative Gieseker moduli space of rank 2 semi-stable sheaves on \( \mathcal{X} \) (resp. \( X_0 \)) with determinant \( L \) (resp. \( L_0 \)) (see Notation A.5). Define the second intermediate Jacobian of \( \mathcal{G}_X(2, L_0) \) to be

\[
J^2(\mathcal{G}_X(2, L_0)) := \frac{H^3(\mathcal{G}_X(2, L_0), \mathbb{C})}{F^2H^3(\mathcal{G}_X(2, L_0), \mathbb{C}) + H^3(\mathcal{G}_X(2, L_0), \mathbb{Z})}.
\]

Note that \( J^2(\mathcal{G}_X(2, L_0)) \) is not an abelian variety. Since the families \( \mathcal{X} \) and \( \mathcal{G}(2, L) \) are smooth over the punctured disc \( \Delta^\ast \), there exists a family of (intermediate) Jacobians \( J^1_{\Delta^\ast} \) (resp. \( J^2_{\mathcal{G}(2, L)_{\Delta^\ast}} \)) over \( \Delta^\ast \) associated to the family of curves \( \mathcal{X} \) (resp. family of Gieseker moduli spaces \( \mathcal{G}(2, L) \)) restricted to \( \Delta^\ast \) i.e., for all \( t \in \Delta^\ast \), the fibers

\[
(J^1_{\Delta^\ast})_t \cong J^1(\mathcal{X}_t) \quad \text{and} \quad (J^2_{\mathcal{G}(2, L)_{\Delta^\ast}})_t \cong J^2(\mathcal{G}(2, L)_t).
\]

Using the isomorphism obtained by Mumford-Newstead, we have an isomorphism of families of intermediate Jacobians over \( \Delta^\ast \):

\[
\Phi : J^1_{\Delta^\ast} \to J^2_{\mathcal{G}(2, L)_{\Delta^\ast}}.
\]

Our goal is to extend the morphism \( \Phi \) to the entire disc \( \Delta \). Clemens in Cle83 and Zucker in Zac76 show that under certain conditions there exist holomorphic, canonical Néron models \( J^1_{\mathcal{X}} \) and \( J^2_{\mathcal{G}(2, L)} \) extending the families of intermediate Jacobians \( J^1_{\mathcal{X}_{\Delta^\ast}} \) and \( J^2_{\mathcal{G}(2, L)_{\Delta^\ast}} \) respectively, to \( \Delta \). We prove in Theorem 2.9 that these conditions are satisfied. Although the construction of the Néron model by Zucker differs from that by Clemens, we prove in Theorem 3.2 that they coincide in our setup. Moreover, we prove:

**Theorem 1.2.** Notations as above. Denote by

\[
J^1(X_0) := \frac{H^1(X_0, \mathbb{C})}{F^1H^1(X_0, \mathbb{C}) + H^1(X_0, \mathbb{Z})}
\]

the generalised Jacobian of \( X_0 \). Then,

(1) there exist complex manifolds \( J^1_{\mathcal{X}} \) and \( J^2_{\mathcal{G}(2, L)} \) over \( \Delta \) extending \( J^1_{\mathcal{X}_{\Delta^\ast}} \) and \( J^2_{\mathcal{G}(2, L)_{\Delta^\ast}} \), respectively. Furthermore, the extension is canonical,

(2) the central fiber of \( J^1_{\mathcal{X}} \) (resp. \( J^2_{\mathcal{G}(2, L)} \)) is isomorphic to \( J^1(X_0) \) (resp. \( J^2(\mathcal{G}_X(2, L_0)) \)). Furthermore, \( J^1(X_0) \) and \( J^2(\mathcal{G}_X(2, L_0)) \) are semi-abelian varieties,

(3) the isomorphism \( \Phi \) extends holomorphically to an isomorphism

\[
\overline{\Phi} : J^1_{\mathcal{X}} \to J^2_{\mathcal{G}(2, L)},
\]

over the entire disc \( \Delta \). Furthermore, the induced isomorphism on the central fiber is an isomorphism of semi-abelian varieties i.e., the abelian part of \( J^1(X_0) \) maps isomorphism to that of \( J^2(\mathcal{G}_X(2, L_0)) \).
See Theorem 3.2, Remark 3.3 and Corollaries 3.5 and 4.3 for the proof.

Finally, we use the Torelli theorem on generalised Jacobian of irreducible nodal curves by Namikawa, to obtain the higher rank analogue as mentioned in Theorem 1.1.

Note that in this article, we use the Gieseker’s relative moduli space of semi-stable sheaves with fixed determinant, since the central fiber of the moduli space is a simple normal crossings divisor (this is not the case for Simpson’s relative moduli space). This is needed for computing the limit mixed Hodge structure using Steenbrink spectral sequences.

We remark that since the work of Namikawa in [Nam73] on extending the classical Torelli map to the moduli of stable curves, there has been pioneering work by Alexeev in [Ale02, Ale04], where he gives a different compactification of the Torelli map, using principally polarized “semi-abelic pairs”. In [CV11], Caparaso and Viviani give an analogue of the Torelli theorem for stable curves, using the compactified Torelli map. We expect to obtain an analogous higher rank Torelli theorem for stable curves using this theory, in a future project.

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We use the following notations

Notations

\begin{itemize}
  \item $k$ algebraically closed field of characteristic 0
  \item $X_0, g, x_0$ an irreducible nodal $k$-curve of genus $g \geq 2$ with exactly one node, say at the point $x_0$
  \item $\pi : \tilde{X}_0 \to X_0$ the normalization of $X_0$
  \item $\{x_1, x_2\}$ the preimage $\pi^{-1}(x_0)$
  \item $Y_t$ fiber over $t \in T$ of a family $Y \to T$
  \item $S, s_0$ a smooth $k$-curve $S$ and $s_0 \in S$ a closed point
  \item $\delta : X \to S$ flat family of projective curves with $X$ regular, $X_{s_0} \cong X_0$ and smooth over $S \setminus \{s_0\}$
  \item $O_X(1)$ an $S$-ample line bundle on $X$ of relative degree 1
  \item $\Delta$ unit disc in $S$ containing $s_0$ as the origin
\end{itemize}

2. Limit mixed Hodge structure on the relative moduli space

In this section we compute the limit mixed Hodge structures and monodromy associated to degeneration of curves and the corresponding Gieseker moduli space with fixed determinant, defined in Appendix A (see Theorem 2.9). We assume familiarity with basic results on limit mixed Hodge structures. See [PS08] for reference.

**Notation 2.1.** Denote by $\pi_1 : X \to \Delta$ a family of projective curves of genus $g \geq 2$, smooth over the punctured disc $\Delta^*$ and central fiber isomorphic to an irreducible nodal curve $X_0$ with exactly one node, say at $x_0$. Assume further that $X$ is regular. To compute the limit mixed Hodge structure, we need the central fiber to be a simple normal crossings divisor. For this purpose, we blow-up $X$ at the point $x_0$. Denote by $\tilde{X} := \text{Bl}_{x_0} X$ and by $\tilde{\pi}_1 : \tilde{X} \to X$ $\tilde{\pi}_1 : \tilde{X} \to X$ $\frac{\text{Bl}_{x_0} X}{x_0}$. Note that for $t \neq 0$, $\tilde{\pi}_1^{-1}(t) = \pi_1^{-1}(t)$. The central fiber of $\tilde{\pi}_1$ is the union of two irreducible components, the normalization $\tilde{X}_0$ of $X_0$ and the exceptional divisor $F \cong \mathbb{P}^{1}_{x_0}$. 
Fix an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ of relative odd degree, say $d$. Set $\mathcal{L}_0 := \mathcal{L}|_{\Delta_0}$, the restriction of $\mathcal{L}$ to the central fiber. Let $\tilde{\mathcal{X}}_0$ be the pullback of $\mathcal{L}_0$ by the normalisation $\tilde{X}_0 \to X_0$. Denote by $\pi_2 : \mathcal{G}(2, \mathcal{L}) \to \Delta$ the relative Gieseker moduli spaces of rank 2 semi-stable sheaves on $\mathcal{X}$ with determinant $\mathcal{L}$ as defined in Notation A.3. Let $\mathcal{G}_X(2, \mathcal{L}_0)$ denote the central fiber of the family $\pi_2$. Recall, $\mathcal{G}(2, \mathcal{L})$ is regular, smooth over $\Delta$, with reduced simple normal crossings divisor $\mathcal{G}_X(2, \mathcal{L}_0)$ (Theorem A.7). Denote by $M_{\tilde{X}_0}(2, \mathcal{L}_0)$ the fine moduli space of semi-stable sheaves of rank 2 and with determinant $\mathcal{L}_0$ over $\tilde{X}_0$ (see [HL10, Theorem 4.3.7 and 4.6.6]). By Theorem A.7 $\mathcal{G}_X(2, \mathcal{L}_0)$ can be written as the union of two irreducible components, say $\mathcal{G}_0$ and $\mathcal{G}_1$, where $\mathcal{G}_1$ (resp. $\mathcal{G}_0 \cap \mathcal{G}_1$) is isomorphic to a $\mathbb{P}^3$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$)-bundle over $M_{\tilde{X}_0}(2, \mathcal{L}_0)$.

2.2. We recall the construction of the limit mixed Hodge structure due to Schmid for the families $\tilde{\pi}_1$ and $\tilde{\pi}_2$. Let us first introduce the relevant Hodge bundles. Consider the restriction of the families $\tilde{\pi}_1$ and $\tilde{\pi}_2$ to the punctured unit disc:

$$\tilde{\pi}_1' : \tilde{X}_{\Delta^*} \to \Delta^*$$ and $$\tilde{\pi}_2' : \tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*} \to \Delta^*,$$

where $\tilde{X}_{\Delta^*} := \tilde{\pi}_1^{-1}(\Delta^*)$ and $\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*} := \tilde{\pi}_2^{-1}(\Delta^*)$. Using Ehresmann’s theorem (see [Vo02, Theorem 9.3]), we have

$$H^m_{\tilde{\mathcal{X}}_{\Delta^*}} := R^m \tilde{\pi}_1^* \mathbb{Z}$$ and $$H^m_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}} := R^m \tilde{\pi}_2^* \mathbb{Z}$$

are the local systems over $\Delta^*$ with fibers $H^1(\mathcal{X}_t, \mathbb{Z})$ and $H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{Z})$ respectively, for $t \in \Delta^*$.

One can canonically associate to these local systems, the holomorphic vector bundles

$$\mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}} := H^1_{\tilde{\mathcal{X}}_{\Delta^*}} \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*}$$ and $$\mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}} := H^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}} \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*},$$

called the Hodge bundles. There exist holomorphic sub-bundles $F^p \mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}} \subset \mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}}$, and $F^p \mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}} \subset \mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}}$ defined by the condition: for any $t \in \Delta^*$, the fibers

$$(F^p \mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}})_t \subset \left( \mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}} \right)_t$$ and $$(F^p \mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}})_t \subset \left( \mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}} \right)_t$$

can be identified respectively with

$$F^p \mathcal{H}^1(\mathcal{X}_t, \mathbb{C}) \subset H^1(\mathcal{X}_t, \mathbb{C})$$ and $$F^p \mathcal{H}^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C}) \subset H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C}),$$

where $F^p$ denotes the Hodge filtration (see [Vo02 §10.2.1]).

The Hodge bundles and their holomorphic sub-bundles defined above can be extended to the entire disc. In particular, there exist canonical extensions, $\overline{\mathcal{H}}^1_{\mathcal{X}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})}$ of $\mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}}$ and $\mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}}$ respectively, to $\Delta$ (see [PS08, Definition 11.4]). Note that, $\overline{\mathcal{H}}^1_{\mathcal{X}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})}$ are locally-free over $\Delta$. Denote by $j : \Delta^* \to \Delta$ the inclusion morphism,

$$F^p \overline{\mathcal{H}}^1_{\mathcal{X}} := j_* \left( F^p \mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}} \right) \cap \overline{\mathcal{H}}^1_{\mathcal{X}}$$ and $$F^p \overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})} := j_* \left( F^p \mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}} \right) \cap \overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})}.$$ Note that, $F^p \overline{\mathcal{H}}^1_{\mathcal{X}}$ (resp. $F^p \overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})}$) is the unique largest locally-free sub-sheaf of $\overline{\mathcal{H}}^1_{\mathcal{X}}$ (resp. $\overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})}$) which extends $F^p \mathcal{H}^1_{\tilde{\mathcal{X}}_{\Delta^*}}$ (resp. $F^p \mathcal{H}^3_{\tilde{\mathcal{G}}(2, \mathcal{L})_{\Delta^*}}$).

Consider the universal cover $\tilde{\eta} \to \Delta^*$ of the punctured unit disc. Denote by $e : \tilde{\eta} \to \Delta^*$ the composed morphism and $\tilde{X}_\infty$ (resp. $\mathcal{G}(2, \mathcal{L})_\infty$) the base change of the family $\mathcal{X}$ (resp. $\mathcal{G}(2, \mathcal{L})$) over $\Delta$ to $\tilde{\eta}$, by the morphism $e$. There is an explicit identification of the central fiber of the canonical extensions $\overline{\mathcal{H}}^1_{\mathcal{X}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2, \mathcal{L})}$ and the cohomology groups $H^1(\tilde{X}_\infty, \mathbb{C})$ and
$H^3(G(2,L)_\infty, \mathbb{C})$ respectively, depending on the choice of the parameter $t$ on $\Delta$ (see [PS08 XI-8]):

$$g_{\tilde{X},t} : H^1(\tilde{X}_\infty, \mathbb{C}) \xrightarrow{\sim} \left(\tilde{\mathcal{H}}^1_X\right)_0$$

and $g_{G(G(2,L)_\infty, \mathbb{C})} : H^3(G(2,L)_\infty, \mathbb{C}) \xrightarrow{\sim} \left(\tilde{\mathcal{H}}^3_{G(2,L)}\right)_0$.

This induce (Hodge) filtrations on $H^1(\tilde{X}_\infty, \mathbb{C})$ and $H^3(G(2,L)_\infty, \mathbb{C})$ as:

$$F^pH^1(\tilde{X}_\infty, \mathbb{C}) := (g_{\tilde{X},t})^{-1}\left(F^p\tilde{\mathcal{H}}_X^1\right)_0$$

and $F^pH^3(G(2,L)_\infty, \mathbb{C}) := (g_{G(G(2,L)_\infty, \mathbb{C}))}^{-1}\left(F^p\tilde{\mathcal{H}}^3_{G(2,L)}\right)_0$.

Next, we define the limit weight filtration on $H^1(\tilde{X}_\infty, \mathbb{Q})$ and $H^3(G(2,L)_\infty, \mathbb{C})$. Let $T_{\tilde{X},\Delta^*}$ and $T_{\tilde{G}(2,L)_{\Delta^*}}$ denote the monodromy automorphisms:

$$T_{\tilde{X},\Delta^*} : \mathbb{H}^1_{\tilde{X},\Delta^*} \to \mathbb{H}^1_{\tilde{X},\Delta^*}$$

and

$$T_{\tilde{G}(2,L)_{\Delta^*}} : \mathbb{H}^3_{\tilde{G}(2,L)_{\Delta^*}} \to \mathbb{H}^3_{\tilde{G}(2,L)_{\Delta^*}}$$

defined by parallel transport along a counterclockwise loop about $0 \in \Delta$ (see [PS08 §11.1.1]). By [PS08 Proposition 11.2] the automorphisms extend to automorphisms of $\tilde{\mathcal{H}}^1_X$ and $\tilde{\mathcal{H}}^3_{G(2,L)}$, respectively. Since the monodromy automorphisms $T_{\tilde{X},\Delta^*}$ and $T_{\tilde{G}(2,L)_{\Delta^*}}$ were defined on the integral local systems, the induced automorphisms on $\tilde{\mathcal{H}}^1_X$ and $\tilde{\mathcal{H}}^3_{G(2,L)}$ after restriction to the central fiber gives the following automorphisms:

$$T_{\tilde{X}} : H^1(\tilde{X}_\infty, \mathbb{Q}) \to H^1(\tilde{X}_\infty, \mathbb{Q})$$

and

$$T_{\tilde{G}(2,L)} : H^3(G(2,L)_\infty, \mathbb{Q}) \to H^3(G(2,L)_\infty, \mathbb{Q}).$$

**Remark 2.3.** There exists an unique increasing monodromy weight filtration $W_\bullet$ (see [PS08 Lemma-Definition 11.9]) on $H^1(\tilde{X}_\infty, \mathbb{Q})$ (resp. $H^3(G(2,L)_\infty, \mathbb{Q})$) such that,

1. for $i \geq 2$, $\log T_{\tilde{X}}(W_iH^1(\tilde{X}_\infty, \mathbb{Q})) \subset W_{i-2}H^1(\tilde{X}_\infty, \mathbb{Q})$

   (resp. $\log T_{\tilde{G}(2,L)}(W_iH^3(G(2,L)_\infty, \mathbb{Q})) \subset W_{i-2}H^3(G(2,L)_\infty, \mathbb{Q})$),

2. the map $(\log T_{\tilde{X}})_l : \text{Gr}^W_{1+i}H^1(\tilde{X}_\infty, \mathbb{Q}) \to \text{Gr}^W_{1+i}H^1(\tilde{X}_\infty, \mathbb{Q})$

   (resp. $(\log T_{\tilde{G}(2,L)})_l : \text{Gr}^W_{3+i}H^3(G(2,L)_\infty, \mathbb{Q}) \to \text{Gr}^W_{3+i}H^3(G(2,L)_\infty, \mathbb{Q})$)

   is an isomorphism for all $l \geq 0$.

By [Sch73 Theorem 6.16], the induced filtrations on $H^1(\tilde{X}_\infty, \mathbb{C})$ (resp. $H^3(G(2,L)_\infty, \mathbb{C})$) defines a mixed Hodge structure $(H^1(\tilde{X}_\infty, \mathbb{Z}), W_\bullet, F^\bullet)$ (resp. $(H^3(G(2,L)_\infty, \mathbb{Z}), W_\bullet, F^\bullet)$).

2.4. We now recall the construction of limit mixed Hodge structure via Steenbrink spectral sequences.

Let $\rho : \mathcal{Y} \to \Delta$ be a flat, family of projective varieties, smooth over $\Delta \setminus \{0\}$ such that the central fiber $\mathcal{Y}_0$ is a simple normal crossings divisor in $\mathcal{Y}$ consisting of exactly two irreducible components, say $Y_1$ and $Y_2$. Assume further that $\mathcal{Y}$ is regular. Denote by $\mathcal{Y}_\infty$ the fiber product

$$\mathcal{Y}_\infty \ar{r} & \mathcal{Y}$$

$$\rho_\infty \ar{r} & \square$$

where $\mathfrak{h}$ is the universal cover of $\Delta^*$. As $\mathcal{Y}_0$ consists of exactly two irreducible components, we have the following terms of the (limit) weight spectral sequence:

**Proposition 2.5** ([PS08 Corollary 11.23] and [Ste76 Example 3.5]). The limit weight spectral sequence $\bigwedge^\infty E^1_{p,q} \Rightarrow H^{p+q}(\mathcal{Y}_\infty, \mathbb{Q})$ consists of the following terms:
(1) if \( |p| \geq 2 \), then \( \varepsilon_{w} E_{1}^{0,q} = 0 \),
(2) \( \varepsilon_{w} E_{1}^{0,q} = H^{q}(Y_{1} \cap Y_{2}, \mathbb{Q})(0), \) \( \varepsilon_{w} E_{1}^{0,q} = H^{q}(Y_{1}, \mathbb{Q})(0) \oplus H^{q}(Y_{2}, \mathbb{Q})(0) \) and \( \varepsilon_{w} E_{1}^{0,q} = H^{q}(Y_{1} \cap Y_{2}, \mathbb{Q})(-1) \),
(3) the differential map \( d_{1} : \varepsilon_{w} E_{1}^{0,q} \to \varepsilon_{w} E_{1}^{1,q} \) is the restriction morphism and
\[
d_{1} : \varepsilon_{w} E_{1}^{0,q} \to \varepsilon_{w} E_{1}^{1,q}
\]
is the Gysin morphism.

The limit weight spectral sequence \( \varepsilon_{w} E_{1}^{p,q} \) degenerates at \( E_{2} \). Similarly, the weight spectral sequence \( \varepsilon_{w} E_{1}^{p,q} \) on \( Y_{0} \) consists of the following terms:

(1) for \( p \geq 2 \) or \( p < 0 \), we have \( \varepsilon_{w} E_{1}^{p,q} = 0 \),
(2) \( \varepsilon_{w} E_{1}^{0,q} = H^{q}(Y_{1} \cap Y_{2}, \mathbb{Q})(0) \) and \( \varepsilon_{w} E_{1}^{0,q} = H^{q}(Y_{1}, \mathbb{Q})(0) \oplus H^{q}(Y_{2}, \mathbb{Q})(0) \),
(3) the differential map \( d_{1} : \varepsilon_{w} E_{1}^{0,q} \to \varepsilon_{w} E_{1}^{1,q} \) is the restriction morphism.

The spectral sequence \( \varepsilon_{w} E_{1}^{p,q} \) degenerates at \( E_{2} \).

**Notation 2.6.** To avoid confusions arising from the underlying family, we denote by \( \varepsilon_{w} E_{1}^{p,q}(\rho) \) and \( \varepsilon_{w} E_{1}^{p,q}(\rho) \), the limit weight spectral sequence on \( Y_{\infty} \) and the weight spectral sequence on \( Y_{0} \) respectively, associated to the family \( \rho \), defined above.

**Remark 2.7.** Recall, the central fiber of \( \tilde{X} \) (resp. \( G(2, \mathcal{L}) \)) consists of two non-singular, irreducible components \( \tilde{X}_{0} \) and \( F \) (resp. \( G_{0} \) and \( G_{1} \), see Notation 2.1). The weight filtration on \( H^{1}(\tilde{X}_{\infty}, \mathbb{C}) \) and \( H^{3}(G(2, \mathcal{L}), \mathbb{C}) \) induced by the limit weight spectral sequences \( \varepsilon_{w} E_{1}^{p,q}(\tilde{\pi}_{1}) \) and \( \varepsilon_{w} E_{1}^{p,q}(\tilde{\pi}_{2}) \) respectively, coincide with the monodromy weight filtration defined in Remark 2.3 (see [PS08, Corollary 11.41]).

**2.8.** In the remainder of this section we compute the mixed Hodge structures on \( H^{1}(\tilde{X}_{\infty}, \mathbb{C}) \) and \( H^{3}(G(2, \mathcal{L}), \mathbb{C}) \) using Steenbrink spectral sequence. By the local invariant cycle theorem [PS08, Theorem 11.43], we have the following exact sequences:

\[ H^{1}(\tilde{X}_{0}, \mathbb{Q}) \xrightarrow{sp_{1}} H^{1}(\tilde{X}_{\infty}, \mathbb{Q}) \xrightarrow{T_{\tilde{X}} - \text{Id}} H^{1}(\tilde{X}_{\infty}, \mathbb{Q}) \quad (2.4) \]

\[ H^{3}(G_{X_{0}}, G_{L_{0}}, \mathbb{Q}) \xrightarrow{sp_{2}} H^{3}(G(2, \mathcal{L}), \mathbb{Q}) \xrightarrow{T_{G(2, \mathcal{L})} - \text{Id}} H^{3}(G(2, \mathcal{L}), \mathbb{Q}), \quad (2.5) \]

where \( sp_{i} \) denotes the specialization morphisms for \( i = 1, 2, \) which are morphisms of mixed Hodge structures (see [PS08, Theorem 11.29]). See [Ste76, Example 3.5] for the description of the Hodge filtration \( F^{\bullet} \) on \( H^{1}(\tilde{X}_{0}, \mathbb{C}) \) and \( H^{3}(G_{X_{0}}, G_{L_{0}}, \mathbb{C}) \).

**Theorem 2.9.** Under the natural specialization morphisms \( sp_{i} \) for \( i = 1, 2 \), we have

(1) the natural morphism \( H^{1}(\tilde{X}_{0}, \mathbb{C})/F^{1}H^{1}(\tilde{X}_{0}, \mathbb{C}) \xrightarrow{sp_{1}} H^{1}(\tilde{X}_{\infty}, \mathbb{C})/F^{1}H^{1}(\tilde{X}_{\infty}, \mathbb{C}) \) is pure of type \((1,1)\) in the sense that \( Gr_{1}^{T_{\tilde{X}}}H^{1}(\tilde{X}_{\infty}, \mathbb{C}) = Gr_{1}^{H^{1}(\tilde{X}_{\infty}, \mathbb{C})} \) and \((T_{\tilde{X}} - \text{Id})^{2} = 0, \)
(2) \( H^{3}(G_{X_{0}}, G_{L_{0}}, \mathbb{C})/F^{2}H^{3}(G_{X_{0}}, G_{L_{0}}, \mathbb{C}) \xrightarrow{sp_{2}} H^{3}(G(2, \mathcal{L}), \mathbb{C})/F^{2}H^{3}(G(2, \mathcal{L}), \mathbb{C}) \) is pure of type \((2,2)\) i.e., \( Gr_{2}^{H^{3}(G(2, \mathcal{L}), \mathbb{C})} = Gr_{2}^{F^{2}H^{3}(G(2, \mathcal{L}), \mathbb{C})} \) and \((T_{G(2, \mathcal{L})} - \text{Id})^{2} = 0. \)

The proof of the theorem requires the cohomology computations given in rest of this section. We give the proof of the theorem at the end of the section.

**Notation 2.10.** For the rest of this section we use Notation 2.1. Denote by \( Y_{b} := M_{\tilde{X}_{0}}(2, \tilde{L}_{0}) \) and \( \rho_{12} : G_{0} \cap G_{1} \to Y_{b} \) the natural bundle morphism. For any \( y \in Y_{b} \), denote by \( (G_{0} \cap G_{1})_{y} := \rho_{12}^{-1}(y) \) and \( \alpha_{y} : (G_{0} \cap G_{1})_{y} \hookrightarrow G_{0} \cap G_{1} \) the natural inclusion.
Lemma 2.11. The cohomology group $H^2(G_0 \cap G_1, \mathbb{Q})$ sits in the following short exact sequence of vector spaces:

$$0 \to H^2(Y_b, \mathbb{Q}) \xrightarrow{\rho_1^\ast} H^2(G_0 \cap G_1, \mathbb{Q}) \xrightarrow{\alpha_y^\ast} H^2((G_0 \cap G_1)_y, \mathbb{Q}) \to 0$$

for a general $y \in Y_b$.

Proof. Since $G_0 \cap G_1$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $Y_b$, there exists an open subset $U \subset Y_b$ such that $\rho_1^\ast(U) \cong U \times (G_0 \cap G_1)_y$ and $H^1(O(G_0 \cap G_1)_y) = 0$ for any $y \in U$. By [Har77, Ex. III.12.6], this implies

$$\text{Pic}(\rho_1^\ast(U)) \cong \text{Pic}(U \times (G_0 \cap G_1)_y) \cong \text{Pic}(U) \times \text{Pic}((G_0 \cap G_1)_y).$$

We then have the following composition of surjective morphisms,

$$\alpha_y^\ast : \text{Pic}(G_0 \cap G_1) \to \text{Pic}(\rho_1^\ast(U)) \to \text{Pic}((G_0 \cap G_1)_y)$$

where the first surjection follows from [Har77, Proposition II.6.5] and the second is simply projection. By [Har77, Ex. III.12.4], for a general $y \in Y_b$, we have $\ker \alpha_y^\ast \cong \rho_1^\ast \text{Pic}(Y_b)$. Using the projection formula and the Zariski main theorem, observe that for any invertible sheaf $\mathcal{L}$ on $Y_b$, we have $\rho_1^\ast, \rho_2^\ast \mathcal{L} \cong \mathcal{L}$. In particular, the morphism $\rho_1^\ast$ is injective and we have the following short exact sequence for a general $y \in Y_b$:

$$0 \to \text{Pic}(Y_b) \xrightarrow{\rho_1^\ast} \text{Pic}(G_0 \cap G_1) \xrightarrow{\alpha_y^\ast} \text{Pic}((G_0 \cap G_1)_y) \to 0.$$

As $Y_b$ and $(G_0 \cap G_1)_y$ are rationally connected for any $y \in Y_b$, so is $G_0 \cap G_1$. Using the exponential short exact sequence associated to each of these varieties we conclude that the corresponding first chern class maps:

$$c_1 : \text{Pic}(Y_b) \to H^2(Y_b, \mathbb{Z}), \quad c_1 : \text{Pic}(G_0 \cap G_1) \to H^2(G_0 \cap G_1, \mathbb{Z}) \quad \text{and} \quad c_1 : \text{Pic}((G_0 \cap G_1)_y) \to H^2((G_0 \cap G_1)_y, \mathbb{Z})$$

are isomorphisms. This gives us the short exact sequence in the lemma, thereby proving it. \[
\square
\]

We now study the Gysin morphism relevant to the limit weight spectral sequence.

Proposition 2.12. Denote by $i : G_0 \cap G_1 \to G_1$ the natural inclusion. Then,

$$\ker(i_\ast : H^2(G_0 \cap G_1, \mathbb{Q}) \to H^4(G_1, \mathbb{Q})) \cong \mathbb{Q},$$

where $i_\ast$ denotes the Gysin morphism.

Proof. By Proposition [A.6], $G_1$ is a $\mathbb{P}^3$-bundle over $Y_b$. Denote by $\rho_1 : G_1 \to Y_b$ the corresponding bundle morphism. For any $y \in Y_b$, denote by $G_{1,y} := \rho_1^{-1}(y)$ and $\alpha_{1,y} : G_{1,y} \to G_1$ the natural inclusion. Note that, $G_0 \cap G_1$ is a divisor in $G_1$. Thus, the cohomology class $[G_0 \cap G_1]$ is an element of $H^2(G_1, \mathbb{Q})$. Moreover, for every $y \in Y_b$, $([G_0 \cap G_1]_y)$ generate $H^2(G_{1,y}, \mathbb{Q})$ ($([G_0 \cap G_1]_y)$ is a hypersurface in $P_{1,y}$). By the Leray-Hirsch theorem [Voisin, Theorem 7.33] we have

$$H^4(G_1, \mathbb{Q}) \cong \rho_1^\ast H^4(Y_b, \mathbb{Q}) \oplus \rho_1^\ast H^2(Y_b, \mathbb{Q}) \oplus [G_0 \cap G_1] \oplus \mathbb{Q}[G_0 \cap G_1]^2,$$

where $\cup$ denotes cup-product. Since $H^4(G_{1,y}, \mathbb{Q}) = H^4(\mathbb{P}^3, \mathbb{Q}) \cong \mathbb{Q}$ generated by $([G_0 \cap G_1]_y)^2$, we have the following short exact sequence:

$$0 \to H^4(Y_b, \mathbb{Q}) \oplus H^2(Y_b, \mathbb{Q}) \xrightarrow{\alpha_{1,y}} H^4(G_1, \mathbb{Q}) \xrightarrow{\alpha_{1,y}} H^4(G_{1,y}, \mathbb{Q}) \to 0,$$
where $\alpha(\xi, \gamma) = \rho_1^*(\xi) + \rho_1^*(\gamma) \cup [G_0 \cap G_1]$. We claim that the following diagram of short exact sequences is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(Y_b, \mathbb{Q}) & \stackrel{\rho_{12}^*}{\longrightarrow} & H^2(G_0 \cap G_1, \mathbb{Q}) & \stackrel{\alpha_g^*}{\longrightarrow} & H^2((G_0 \cap G_1)_y, \mathbb{Q}) & \longrightarrow & 0 \\
\vphantom{0} & \beta \downarrow & \bigcirc \downarrow i_* & \bigcirc \downarrow i_\gamma & & & & \\
0 & \longrightarrow & H^4(Y_b, \mathbb{Q}) \oplus H^2(Y_b, \mathbb{Q}) & \stackrel{\alpha}{\longrightarrow} & H^4(G_1, \mathbb{Q}) & \stackrel{\alpha_{1,y}^*}{\longrightarrow} & H^4(G_{1,y}, \mathbb{Q}) & \longrightarrow & 0
\end{array}
$$

(2.6)

where $\beta(\xi) = (0, \xi)$, $i_{y*}$ is the Gysin morphism induced by the inclusion $(G_0 \cap G_1)_y \hookrightarrow G_{1,y}$ and the top horizontal short exact sequence follows from Lemma 2.11. Indeed, we observed in the proof of Lemma 2.11 that $H^2(Y_b, \mathbb{Q}), H^2(G_0 \cap G_1, \mathbb{Q})$ and $H^2((G_0 \cap G_1)_y, \mathbb{Q})$ are generated by the cohomology class of divisors (the corresponding varieties are rationally connected). The commutativity of the left square then follows from the observation that for any invertible sheaf $L$ on $Y_b$, we have

$$i_*\rho_{12}^*([L]) = i_*\rho_1^*([L]) = \rho_1^*([L]) \cup [G_0 \cap G_1],$$

where the first equality follows from $\rho_1 \circ i = \rho_{12}$ and the second by the projection formula. The commutativity of the right hand square follows from the definition of pull-back and push-forward of cycles (see [FM13, Theorem 6.2(a)] for the general statement). This proves the claim.

Clearly, $\beta$ is injective. Since the entries in the diagram are vector spaces, the two rows are in fact split short exact sequences. Then, the diagram (2.6) implies that $\ker(i_*) \cong \ker(i_{y*})$. Since

$$i_\gamma = i_*|(G_0 \cap G_1)_y : (G_0 \cap G_1)_y \hookrightarrow G_{1,y}$$

can be identified with the inclusion $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, under the Segre embedding, we have $\ker(i_{y*}) \cong \mathbb{Q}$ (use $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}, H^4(\mathbb{P}^3, \mathbb{Q}) \cong \mathbb{Q}$ and $i_{y*}$ is surjective). This implies $\ker i_* \cong \mathbb{Q}$. This proves the proposition. 

By [PS08, Theorem 5.39], note that $H^1(\bar{X}_0, \mathbb{Q}) = W_1H^1(\bar{X}_0, \mathbb{Q})$ and $H^3(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) = W_3H^3(\mathcal{G}_{X_0}(2, \mathcal{L}_0), \mathbb{Q})$. Since the specialization morphisms $s_{p_1}$ are morphisms of mixed Hodge structures, we conclude:

**Proposition 2.13.** The specialization morphism $s_{p_1}$ (resp. $s_{p_2}$) factors through $W_1H^1(\bar{X}_\infty, \mathbb{Q})$ (resp. $W_3H^3(\mathcal{G}(2, \mathcal{L}_\infty), \mathbb{Q})$) and the induced morphisms

$$s_{p_1} : H^1(\bar{X}_0, \mathbb{Q}) \rightarrow W_1H^1(\bar{X}_\infty, \mathbb{Q})$$

and

$$s_{p_2} : H^3(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \rightarrow W_3H^3(G(2, \mathcal{L}_\infty), \mathbb{Q})$$

are isomorphisms. Moreover, the natural inclusions

$$W_2H^1(\bar{X}_\infty, \mathbb{Q}) \hookrightarrow H^1(\bar{X}_\infty, \mathbb{Q})$$

and

$$W_4H^3(G(2, \mathcal{L}_\infty), \mathbb{Q}) \hookrightarrow H^3(G(2, \mathcal{L}_\infty), \mathbb{Q})$$

are isomorphisms and the Q-vector spaces $Gr^W_2H^1(\bar{X}_\infty, \mathbb{Q})$ and $Gr^W_4H^3(G(2, \mathcal{L}_\infty), \mathbb{Q})$ are of dimension at most one.

**Proof.** Using the (limit) weight spectral sequence (Proposition 2.9 and Remark 2.7) we have the following (limit) weight decompositions:

$$H^1(\bar{X}_0, \mathbb{Q}) \cong Gr^W_0H^1(\bar{X}_0, \mathbb{Q}) \oplus Gr^W_1H^1(\bar{X}_0, \mathbb{Q})$$

(2.7)

$$H^1(\bar{X}_\infty, \mathbb{Q}) \cong Gr^W_0H^1(\bar{X}_\infty, \mathbb{Q}) \oplus Gr^W_1H^1(\bar{X}_\infty, \mathbb{Q})$$

(2.8)

$$H^3(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \cong Gr^W_2H^3(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \oplus Gr^W_3H^3(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q})$$

(2.9)

$$H^3(G(2, \mathcal{L}_\infty), \mathbb{Q}) \cong Gr^W_2H^3(G(2, \mathcal{L}_\infty), \mathbb{Q}) \oplus Gr^W_3H^3(G(2, \mathcal{L}_\infty), \mathbb{Q})$$

(2.10)

$$Gr^W_{p+q}(\bar{X}_0, \mathbb{Q}) = \bigoplus_{p} E^p_q(\bar{X}_0), Gr^W_{p+q}(\bar{X}_\infty, \mathbb{Q}) = \bigoplus_{q} E^p_q(\bar{X}_\infty), \text{ for all } p, q$$

(2.11)
Similarly, \( \text{Gr}_p^W H^{p+q}(\mathcal{G}_X(2, \mathcal{L}), \mathbb{Q}) = w E_2^{p,q}(\pi_2) \) and \( \text{Gr}_q^W H^{p+q}(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q}) = \infty E_2^{p,q}(\pi_2) \) for all \( p, q \). As observed earlier, we have \( w E_1^{p,q}(\pi_1) = \infty E_1^{p,q}(\pi_1) \) and \( w E_1^{p,q}(\pi_2) = \infty E_2^{p,q}(\pi_2) \) for all \( p \geq 0 \) and the differential maps,

\[
d_1 : w E_1^{p,q}(\pi_1) \rightarrow w E_1^{p+1,q}(\pi_1) \text{ and } d_1 : \infty E_1^{p,q}(\pi_1) \rightarrow \infty E_1^{p+1,q}(\pi_1) \text{ coincide for all } p \geq 0.
\]

Similarly for all \( p \geq 0 \), the differential maps

\[
d_1 : w E_1^{p,q}(\pi_2) \rightarrow w E_1^{p+1,q}(\pi_2) \text{ and } d_1 : \infty E_1^{p,q}(\pi_2) \rightarrow \infty E_1^{p+1,q}(\pi_2) \text{ coincide.}
\]

Note that, \( \infty E_1^{1,1}(\pi_1) = 0 \) (by definition) and \( \infty E_1^{1,3}(\pi_2) = H^1(G_0 \cap G_1, \mathbb{Q}) = 0 \) (\( G_0 \cap G_1 \) is rationally connected, see Proposition \ref{A.6}). This implies

\[
w E_2^{p,1-p}(\pi_1) = \infty E_2^{p,1-p}(\pi_1) \text{ and } w E_2^{p,3-p}(\pi_2) = \infty E_2^{p,3-p}(\pi_2) \text{ for all } p \geq 0.
\]

Using (2.8) and (2.10), we conclude that

\[
sp_1 : H^1(\tilde{X}_0, \mathbb{Q}) \rightarrow W_1 H^1(\tilde{X}_\infty, \mathbb{Q}) \text{ and } sp_2 : H^3(G_X(2, \mathcal{L}), \mathbb{Q}) \rightarrow W_3 H^3(G(2, \mathcal{L})_\infty, \mathbb{Q})
\]

are isomorphisms. Furthermore, the natural inclusions

\[
W_2 H^1(\tilde{X}_\infty, \mathbb{Q}) \hookrightarrow H^1(\tilde{X}_\infty, \mathbb{Q}) \text{ and } W_4 H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q}) \hookrightarrow H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q})
\]

are isomorphisms. It remains to prove that \( \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q}) \) and \( \text{Gr}_4^W H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q}) \) are of dimension at most one.

Using Proposition \ref{2.5} we observe that

\[
\infty E_2^{1,2}(\pi_1) = \ker(\infty E_2^{1,2}(\pi_1) \xrightarrow{d_1} \infty E_1^{0,2}(\pi_1)) = \ker(H^0(\tilde{X}_0 \cap F, \mathbb{Q}) \xrightarrow{(i_1, i_2)} H^2(\tilde{X}_0, \mathbb{Q}) \oplus H^2(F, \mathbb{Q}))
\]

where \( i_1, (\text{resp. } i_2) \) is the Gysin morphism from \( H^0(\tilde{X}_0 \cap F, \mathbb{Q}) \) to \( H^2(\tilde{X}_0, \mathbb{Q}) \) (resp. \( H^2(F, \mathbb{Q}) \)) induced by the inclusion maps

\[
i_1 : \tilde{X}_0 \cap F \hookrightarrow \tilde{X}_0 \text{ (resp. } i_2 : \tilde{X}_0 \cap F \hookrightarrow F).
\]

Since \( \tilde{X}_0 \cap F \) consists of 2 points and the morphism \((i_1, i_2)\) is non-zero, one can check that \( \ker(i_1, i_2) \) is isomorphic to either 0 or \( \mathbb{Q} \). Therefore, \( \infty E_2^{1,2}(\pi_1) = \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q}) \) is of dimension at most 1.

Similarly, using Proposition \ref{2.5} we have,

\[
\infty E_2^{1,4}(\pi_2) = \ker(\infty E_2^{1,4}(\pi_2) \xrightarrow{d_1} \infty E_2^{0,4}(\pi_2)) = \ker(H^2(G_0 \cap G_1, \mathbb{Q}) \xrightarrow{(j_1, j_2)} H^4(G_0, \mathbb{Q}) \oplus H^4(G_1, \mathbb{Q}))
\]

where \( j_1 \) and \( j_2 \) are the Gysin morphisms associated to the natural inclusions

\[
j_1 : G_0 \cap G_1 \hookrightarrow G_0 \text{ and } j_2 : G_0 \cap G_1 \hookrightarrow G_1.
\]

Proposition \ref{2.12} then implies that \( \ker(j_1, j_2) \) is isomorphic to either 0 or \( \mathbb{Q} \). Thus, \( \infty E_2^{1,4}(\pi_2) = \text{Gr}_4^W H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q}) \) is of dimension at most 1. This completes the proof of the proposition. 

We are now ready to prove the main theorem of this section.

**Proof of Theorem 2.11** Since \( \tilde{X}_t \) is smooth (resp. \( \mathcal{G}(2, \mathcal{L})_t \) is rationally connected) for all \( t \in \Delta^t \), we have \( \text{Gr}_p^W H^1(\tilde{X}_t, \mathbb{C}) = 0 \) (resp. \( \text{Gr}_p^W H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C}) = 0 \)) for all \( p \geq 2 \) (resp. \( p \geq 3 \)). Then Corollary 11.24 implies that \( \text{Gr}_p^W H^1(\tilde{X}_\infty, \mathbb{C}) = 0 \) (resp. \( \text{Gr}_p^W H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{C}) = 0 \)) for all \( p \geq 2 \) (resp. \( p \geq 3 \)). Thus, \( F^3 H^1(\tilde{X}_\infty, \mathbb{C}) = 0 = F^3 H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{C}) = 0 \), which means

\[
\text{Gr}_p^W H^1(\tilde{X}_\infty, \mathbb{Q}) = F^1 \text{Gr}_p^W H^1(\tilde{X}_\infty, \mathbb{Q}) \text{ and } \text{Gr}_p^W H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q}) = F^2 \text{Gr}_p^W H^3(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Q}).
\]

(2.12)
Consider now the following diagram of short exact sequences:

\[
\begin{array}{ccc}
F^m H^p(A, \mathbb{C}) \cap W_p H^p(A, \mathbb{C}) & \hookrightarrow & F^m H^p(A, \mathbb{C}) \cap W_{p+1} H^p(A, \mathbb{C}) \\
\downarrow & & \downarrow \\
W_p H^p(A, \mathbb{C}) & \hookrightarrow & W_{p+1} H^p(A, \mathbb{C}) \\
\circ & & \circ \\
& & f_0
\end{array}
\]

for the two cases \( \{ A = \tilde{X}_\infty, m = 1 = p \} \) and \( \{ A = G(2, \mathcal{L})_\infty, m = 2, p = 3 \} \). Since \( \text{Gr}_W^W H^1(\tilde{X}_\infty, \mathbb{Q}) \) and \( \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \) are pure Hodge structures of dimension at most one (Proposition 2.13), we have

\[
\text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{C}) = \text{Gr}_3^W H^1(\tilde{X}_\infty, \mathbb{C}) \quad \text{and} \quad \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{C}) = \text{Gr}_3^W \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{C}).
\]

Using (2.12), this implies \( f_0 \) is an isomorphism in both the cases. Applying Snake lemma to the diagram (2.13) we conclude that in both cases

\[
\frac{W_p H^p(A, \mathbb{C})}{F^m H^p(A, \mathbb{C}) \cap W_p H^p(A, \mathbb{C})} \cong \frac{W_{p+1} H^p(A, \mathbb{C})}{F^m H^p(A, \mathbb{C}) \cap W_{p+1} H^p(A, \mathbb{C})} \cong \frac{H^p(A, \mathbb{C})}{F^m H^p(A, \mathbb{C})}
\]

where the last isomorphism follows from \( W_{p+1} H^p(A, \mathbb{C}) \cong H^p(A, \mathbb{C}) \) (Proposition 2.13). Proposition 2.13 further implies

\[
\begin{align*}
\frac{H^1(\tilde{X}_0, \mathbb{C})}{F^1 H^1(\tilde{X}_0, \mathbb{C})} & \cong \frac{W_1 H^1(\tilde{X}_\infty, \mathbb{C})}{F^1 H^1(\tilde{X}_\infty, \mathbb{C})} \cong \frac{H^1(\tilde{X}_\infty, \mathbb{C})}{F^1 H^1(\tilde{X}_\infty, \mathbb{C})} \\
\frac{H^3(G(2, \mathcal{L})_\infty, \mathbb{C})}{F^2 H^3(G(2, \mathcal{L})_\infty, \mathbb{C})} & \cong \frac{W_3 H^3(G(2, \mathcal{L})_\infty, \mathbb{C})}{F^2 H^3(G(2, \mathcal{L})_\infty, \mathbb{C})} \cong \frac{H^3(G(2, \mathcal{L})_\infty, \mathbb{C})}{F^2 H^3(G(2, \mathcal{L})_\infty, \mathbb{C})}
\end{align*}
\]

It now remains to check that \((T_{\tilde{X}} - \text{Id})^2 = 0 = (T_{G(2, \mathcal{L})} - \text{Id})^2\). Using Proposition 2.13 and the exact sequences (2.4) and (2.5), we have

\[
\ker(T_{\tilde{X}} - \text{Id}) = \text{Im} \text{sp}_1 = W_1 H^1(\tilde{X}_\infty, \mathbb{Q}) \quad \text{and} \quad \ker(T_{G(2, \mathcal{L})} - \text{Id}) = \text{Im} \text{sp}_2 = W_3 H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}).
\]

Hence, \( T_{\tilde{X}} - \text{Id} \) (resp. \( T_{G(2, \mathcal{L})} - \text{Id} \)) factors through \( \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q}) \) (resp. \( \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \)). Recall, \( \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q}) \) (resp. \( \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \)) is either trivial or isomorphic to \( \mathbb{Q} \) (Proposition 2.13). Now, consider the composed morphisms

\[
\begin{align*}
T_1 : & \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q}) \xrightarrow{T_{\tilde{X}} - \text{Id}} H^1(\tilde{X}_\infty, \mathbb{Q}) \xrightarrow{\text{pr}_1} \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q}) \\
T_2 : & \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \xrightarrow{T_{G(2, \mathcal{L})} - \text{Id}} H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \xrightarrow{\text{pr}_2} \text{Gr}_4^W H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}),
\end{align*}
\]

where \( \text{pr}_1 \) and \( \text{pr}_2 \) are natural projections. Since \( T_1 \) (resp. \( T_2 \)) is a morphism of \( \mathbb{Q} \)-vector spaces of dimension at most one, \( T_1^N = 0 \) (resp. \( T_2^N = 0 \)) for some \( N \) if and only if \( T_1 = 0 \) (resp. \( T_2 = 0 \)). As the monodromy operators \( T_{\tilde{X}} \) and \( T_{G(2, \mathcal{L})} \) are unipotent, there exists \( N \) such that

\[
(T_{\tilde{X}} - \text{Id})^N = 0 = (T_{G(2, \mathcal{L})} - \text{Id})^N
\]

which implies \( T_1^N = 0 = T_2^N \). Thus, \( T_1 = 0 = T_2 \). This implies,

\[
\text{Im}(T_{\tilde{X}} - \text{Id}) \subset W_1 H^1(\tilde{X}_\infty, \mathbb{Q}) \quad \text{and} \quad \text{Im}(T_{G(2, \mathcal{L})} - \text{Id}) \subset W_3 H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}).
\]

In other words, \( \text{Im}(T_{\tilde{X}} - \text{Id}) \subset \ker(T_{\tilde{X}} - \text{Id}) \) and \( \text{Im}(T_{G(2, \mathcal{L})} - \text{Id}) \subset \ker(T_{G(2, \mathcal{L})} - \text{Id}) \). Therefore, \((T_{\tilde{X}} - \text{Id})^2 = 0 = (T_{G(2, \mathcal{L})} - \text{Id})^2\). This proves the theorem. \( \square \)
3. Néron model for families of intermediate Jacobians

In this section, we compare the various Néron models of the families of intermediate Jacobians. The Néron model of a family of intermediate Jacobians over a punctured disc $\Delta^*$, is its extension to the entire disc $\Delta$. We use the same notations as in [22].

3.1. We now define the Hodge bundles and the sub-bundles on the extension $\Delta$. Denote by $\tilde{H}^1_{\Delta^*} := j_* \tilde{H}^1_{\Delta^*}$ and $\tilde{H}^3_{G(2,\mathcal{L})} := j_* \tilde{H}^3_{G(2,\mathcal{L})_{\Delta^*}}$, where $j : \Delta^* \to \Delta$. By the Hodge decomposition for all $t \in \Delta^*$, we have

$$H^1(\mathcal{X}_t, \mathbb{Z}) \cap F^1 H^1(\mathcal{X}_t, \mathbb{C}) = 0 \quad \text{and} \quad H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{Z}) \cap F^2 H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C}) = 0.$$ 

Thus the natural morphisms,

$$H^1(\mathcal{X}_t, \mathbb{Z}) \to H^1(\mathcal{X}_t, \mathbb{C})/F^1 H^1(\mathcal{X}_t, \mathbb{C}) \quad \text{and} \quad H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{Z}) \to H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C})/F^2 H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C})$$

are injective. This induces natural injective morphisms,

$$\Phi_1 : \tilde{H}^1_{\Delta^*} \to H^1_{\Delta^*}/F^1 H^1_{\Delta^*} \quad \text{and} \quad \Phi_2 : \tilde{H}^3_{G(2,\mathcal{L})_{\Delta^*}} \to H^3_{G(2,\mathcal{L})_{\Delta^*}}/F^2 H^3_{G(2,\mathcal{L})_{\Delta^*}}.$$

Since $F^p \tilde{H}^1_{\Delta^*} = j_* \left( F^p H^1_{\Delta^*} \right) \cap \tilde{H}^1_{\Delta^*}$ and $F^p \tilde{H}^3_{G(2,\mathcal{L})} := j_* \left( F^p H^3_{G(2,\mathcal{L})_{\Delta^*}} \right) \cap \tilde{H}^3_{G(2,\mathcal{L})}$ are vector bundles, one can immediately check that the natural morphisms,

$$\overline{\Phi}_1 : \tilde{H}^1_{\Delta^*} \to \tilde{H}^1_{\Delta^*}/F^1 \tilde{H}^1_{\Delta^*} \quad \text{and} \quad \overline{\Phi}_2 : \tilde{H}^3_{G(2,\mathcal{L})} \to \tilde{H}^3_{G(2,\mathcal{L})}/F^2 \tilde{H}^3_{G(2,\mathcal{L})}$$

extending $\Phi_1$ and $\Phi_2$ respectively, are injective. Denote by

$$\mathcal{J}^1_{\Delta^*} := \text{coker}(\Phi_1), \mathcal{J}^2_{G(2,\mathcal{L})_{\Delta^*}} := \text{coker}(\Phi_2), \mathcal{J}^1_{\Delta^*} := \text{coker}(\overline{\Phi}_1) \quad \text{and} \quad \mathcal{J}^2_{G(2,\mathcal{L})} := \text{coker}(\overline{\Phi}_2).$$

Note that for any $t \in \Delta^*$, we have the following fibers

$$\mathcal{J}^1_{\Delta^*} \otimes k(t) = J^1(\mathcal{X}_t) = \frac{H^1(\mathcal{X}_t, \mathbb{C})}{F^1 H^1(\mathcal{X}_t, \mathbb{C}) + H^1(\mathcal{X}_t, \mathbb{Z})} \quad \text{and} \quad \mathcal{J}^2_{G(2,\mathcal{L})} \otimes k(t) = J^2(\mathcal{G}(2, \mathcal{L})_t) = \frac{H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C})}{F^2 H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{C}) + H^3(\mathcal{G}(2, \mathcal{L})_t, \mathbb{Z})}.$$ 

Then, $J^1_{\Delta^*} := \bigcup_{t \in \Delta^*} J^1(\mathcal{X}_t)$ and $J^2_{G(2,\mathcal{L})} := \bigcup_{t \in \Delta^*} J^2(\mathcal{G}(2, \mathcal{L})_t)$ has naturally the structure of a complex manifold such that

$$J^1_{\Delta^*} \to \Delta^* \quad \text{and} \quad J^2_{G(2,\mathcal{L})} \to \Delta^*$$

are analytic fibre spaces of complex Lie groups with $\mathcal{O}_J^1 \cong J^1_{\Delta^*}$ and $\mathcal{O}_J^2 \cong J^2_{G(2,\mathcal{L})_{\Delta^*}}$ as sheaves of abelian groups.

**Theorem 3.2.** The family of intermediate Jacobians $J^1_{\Delta^*}$ and $J^2_{G(2,\mathcal{L})}$ over $\Delta^*$, extend holomorphically and canonically to $J^1_{\Delta}$ and $J^2_{G(2,\mathcal{L})}$ over $\Delta$. Furthermore, $J^1_{\Delta^*}$ and $J^2_{G(2,\mathcal{L})}$ has the structure of smooth, complex Lie groups over $\Delta$ with a natural isomorphism

$$\mathcal{O}_{J^1_{\Delta}} \cong J^1_{\Delta^*} \quad \text{and} \quad \mathcal{O}_{J^2_{G(2,\mathcal{L})}} \cong J^2_{G(2,\mathcal{L})}$$

of sheaves of abelian groups.

**Proof.** Fix an $s \in \Delta^*$. Denote by

$$T_{\mathcal{X}_s} : H^1(\mathcal{X}_s, \mathbb{Z}) \rightarrow H^1(\mathcal{X}_s, \mathbb{Z}) \quad \text{and} \quad T_{\mathcal{G}(2,\mathcal{L})_s} : H^3(\mathcal{G}(2, \mathcal{L})_s, \mathbb{Z}) \rightarrow H^3(\mathcal{G}(2, \mathcal{L})_s, \mathbb{Z}).$$
the restriction of the monodromy operators \(T_{\tilde{X}_s} \) and \(T_{G(2,\mathcal{L})_s} \) as in [2.3], respectively, to the fiber over \(s\). Denote by \(T'_{\tilde{X}_s,\mathbb{Q}} := (T_{\tilde{X}_s} - \text{Id})H^1(\mathcal{X}_s, \mathbb{Q}) \) and \(T'_{G(2,\mathcal{L})_s,\mathbb{Q}} := (T_{G(2,\mathcal{L})_s} - \text{Id})H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{Q}) \).

Consider the following finite groups:

\[
G_{\mathcal{X}_s} := \frac{T'_{\tilde{X}_s,\mathbb{Q}} \cap H^1(\mathcal{X}_s, \mathbb{Z})}{(T_{\tilde{X}_s} - \text{Id})H^1(\mathcal{X}_s, \mathbb{Z})} \quad \text{and} \quad G_{G(2,\mathcal{L})_s} := \frac{T'_{G(2,\mathcal{L})_s,\mathbb{Q}} \cap H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{Z})}{(T_{G(2,\mathcal{L})_s} - \text{Id})H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{Z})}.
\]

Recall by Theorem [2.3] we have

1. \((T_{\tilde{X}_s} - \text{Id})^2 = 0 = (T_{G(2,\mathcal{L})} - \text{Id})^2\),
2. \(G^W_1 H^1(\tilde{X}_s, \mathbb{C})\) (resp. \(G^W_1 H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{C})\)) is pure of type \((1,1)\) (resp. of type \((2,2)\)).

Using [GGK10] Proposition II.A.8 and Theorem II.B.9 (see also [Sai96, Propositions 2 and 2.7]) we conclude that if \(G_{\mathcal{X}_s} = 0 = G_{G(2,\mathcal{L})_s}\), then the family of intermediate Jacobians \(\bar{J}^1_{\mathcal{X}}\) and \(\bar{J}^2_{G(2,\mathcal{L})}\) extend holomorphically and canonically to \(\bar{J}^1_{\tilde{X}}\) and \(\bar{J}^2_{G(2,\mathcal{L})}\) over \(\Delta\), which have the structure of smooth, complex Lie groups over \(\Delta\). Furthermore, we have a natural isomorphism of sheaves of abelian groups

\[
\mathcal{O}_{\bar{J}^1_{\tilde{X}}} \cong \bar{J}^1_{\tilde{X}} \quad \text{and} \quad \mathcal{O}_{\bar{J}^2_{G(2,\mathcal{L})}} \cong \bar{J}^2_{G(2,\mathcal{L})}.
\]

It therefore suffices to check that \(G_{\mathcal{X}_s} = 0 = G_{G(2,\mathcal{L})_s}\).

Denote by \(\delta \in H_1(\mathcal{X}_s, \mathbb{Z})\) the vanishing cycle associated to the degeneration of curves defined by \(\pi_1\) (see [Voi03 §3.2.1]). Note that \(\delta\) is the generator of the kernel of the natural morphism

\[
H_1(\mathcal{X}_s, \mathbb{Z}) \xrightarrow{\iota} H_1(\mathcal{X}, \mathbb{Z}) \xrightarrow{r_0} H_1(X_0, \mathbb{Z}),
\]

where \(\iota : \mathcal{X}_s \to \mathcal{X}\) is the natural inclusion of fiber and \(r_0 : \mathcal{X} \xrightarrow{\sim} X_0\) is the retraction to the central fiber. Since \(X_0\) is an irreducible nodal curve, the homology group \(H_1(X_0, \mathbb{Z})\) is torsion-free. Therefore, \(\delta\) is non-divisible i.e., there does not exist \(\delta' \in H_1(\mathcal{X}_s, \mathbb{Z})\) such that \(n\delta' = \delta\) for some integer \(n \neq 1\). Denote by \((-,-)\) the intersection form on \(H_1(\mathcal{X}_s, \mathbb{Z})\), defined using cup-product (see [Voi02 §7.1.2]). Since the intersection form \((-,-)\) induces a perfect pairing on \(H_1(\mathcal{X}_s, \mathbb{Z})\), the non-divisibility of \(\delta\) implies that there exists \(\gamma \in H_1(\mathcal{X}_s, \mathbb{Z})\) such that \((\gamma, \delta) = 1\). Recall the Picard-Lefschetz formula,

\[
T_{\tilde{X}_s}(\eta) = \eta + (\delta, \eta)\delta \quad \text{for any} \quad \eta \in H_1(\mathcal{X}_s, \mathbb{Z}).
\]

This implies \(T'_{\tilde{X}_s,\mathbb{Q}} \cap H^1(\mathcal{X}_s, \mathbb{Z}) = \mathbb{Z}\delta^c = (T_{\tilde{X}_s} - \text{Id})H^1(\mathcal{X}_s, \mathbb{Z})\), where \(\delta^c\) is the Poincaré dual to the vanishing cycle \(\delta\). Therefore, \(G_{\mathcal{X}_s} = 0\).

Now, there exists an isomorphism of local system \(\Phi_{\Delta^* : \mathbb{H}^1_{\Delta^*} \to \mathbb{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}\) which commutes with the respective monodromy operators (see the diagram (4.2) below). Denote by \(\Phi_s : H^1(\mathcal{X}_s, \mathbb{Z}) \xrightarrow{\sim} H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{Z})\) the restriction of \(\Phi_{\Delta^*}\) to the fiber over \(s\). Note that,

\[
T'_{G(2,\mathcal{L})_s,\mathbb{Q}} \cap H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{Z}) = \mathbb{Z}\Phi_s(\delta^c) = (T_{G(2,\mathcal{L})_s} - \text{Id})H^3(\mathcal{G}(2,\mathcal{L}), \mathbb{Z}).
\]

This implies \(G_{G(2,\mathcal{L})_s} = 0\), thereby proving the theorem. \(\square\)

**Remark 3.3.** The extensions \(\bar{J}^1_{\tilde{X}}\) and \(\bar{J}^2_{G(2,\mathcal{L})}\) are called Néron models for \(J^1_{\tilde{X}}\) and \(J^2_{G(2,\mathcal{L})}\) (see [GGK10] for this terminology). The construction of the Néron model by Zucker in [Zuc76] differs from that by Clemens in [Cle83] by the group \(G_{\mathcal{X}_s}\) and \(G_{G(2,\mathcal{L})_s}\) mentioned in the proof of Theorem [3.2] above (see [Sai96 Proposition 2.7]). As we observe in the proof above that \(G_{\mathcal{X}_s}\) and \(G_{G(2,\mathcal{L})_s}\) vanish in our setup, thus the two Néron models coincide in our case.

We now describe the central fiber of the Néron model of the intermediate Jacobians.
**Notation 3.4.** Recall, \( H^1(X_0, \C) \) and \( H^3(G_{X_0}(2, L_0)) \) are equipped with mixed Hodge structures. Define the generalised intermediate Jacobian of \( X_0 \) and \( G_{X_0}(2, L_0) \) as

\[
J^1(X_0) := \frac{H^1(X_0, \C)}{F^1H^1(X_0, \C) + H^1(X_0, \Z)} \quad \text{and} \quad J^2(G_{X_0}(2, L_0)) := \frac{H^3(G_{X_0}(2, L_0), \C)}{F^2H^3(G_{X_0}(2, L_0), \C) + H^3(G_{X_0}(2, L_0), \Z)}.
\]

Denote by \( W_i J^1(X_0) \) (resp. \( W_i J^2(G_{X_0}(2, L_0)) \)) the image of the natural morphism

\[
W_i J^1(X_0) \to J^1(X_0) \quad \text{(resp.} \quad W_i J^2(G_{X_0}(2, L_0)) \to J^2(G_{X_0}(2, L_0)).
\]

By Theorem 3.2 we have,

\[
\left( \mathcal{T}_{X_0}^1 \right)_0 \cong \left( \mathcal{T}_{\tilde{X}_0}^1 \right) \otimes k(0) = \frac{H^1(\tilde{X}_\infty, \C)}{F^1H^1(\tilde{X}_\infty, \C)} \quad \text{and}
\]

\[
\left( \mathcal{T}_{G(2,L)}^1 \right)_0 \cong \left( \mathcal{T}_{G(2,L)}^1 \right) \otimes k(0) = \frac{H^3(\tilde{G}(2, L, \infty), \C)}{F^2H^3(\tilde{G}(2, L, \infty), \C)}.
\]

Denote by \( W_i \left( \mathcal{T}_{X_0}^1 \right)_0 \) (resp. \( W_i \left( \mathcal{T}_{G(2,L)}^1 \right)_0 \)) the image of the natural morphism

\[
W_i J^1(\tilde{X}_\infty, \C) \to \left( \mathcal{T}_{X_0}^1 \right)_0 \quad \text{(resp.} \quad W_i J^2(\tilde{G}(2, L, \infty), \C) \to \left( \mathcal{T}_{G(2,L)}^1 \right)_0
\]

**Corollary 3.5.** The central fiber of \( \mathcal{T}_{X_0}^1 \) and \( \mathcal{T}_{G(2,L)}^1 \) satisfies the following:

1. denote by \( i_0 : \tilde{X}_0 \to X_0 \) the natural morphism contracting the rational curve \( F \) to the nodal point. The natural morphism of mixed Hodge structures

\[
(sp_1 \circ i_0^*) : H^1(X_0, \C) \to H^1(\tilde{X}_\infty, \C),
\]

induces an isomorphism from \( J^1(X_0) \) to the central fiber \( \left( \mathcal{T}_{X_0}^1 \right)_0 \) such that \( W_i J^1(X_0) \cong W_i \left( \mathcal{T}_{X_0}^1 \right)_0 \) for all \( i \),

2. the specialization morphism \( sp_2 \) induces an isomorphism from \( J^3(G_{X_0}(2, L_0)) \) to the central fiber \( \left( \mathcal{T}_{G(2,L)}^1 \right)_0 \) such that \( W_i J^3(G_{X_0}(2, L_0)) \cong W_i \left( \mathcal{T}_{G(2,L)}^1 \right)_0 \) for all \( i \).

**Proof.** Consider the restriction of the morphisms \( g_{\tilde{X},t} \) and \( g_{G(2,L),t} \) as in (2.1) to \( H^1(\tilde{X}_\infty, \Z) \) and\( H^1(\tilde{G}(2, L, \infty), \Z) \), respectively:

\[
g_0 : H^1(\tilde{X}_\infty, \Z) \to H^1(\tilde{X}_\infty, \C) \xrightarrow{g_{\tilde{X},t}} \left( \mathcal{H}_{\tilde{X}_\infty}^1 \right)_0 \quad \text{and}
\]

\[
g_1 : H^3(\tilde{G}(2, L, \infty), \Z) \to H^3(\tilde{G}(2, L, \infty), \C) \xrightarrow{g_{G(2,L),t}} \left( \mathcal{H}_{G(2,L)}^3 \right)_0.
\]

Using the explicit description of \( g_{\tilde{X},t} \) and \( g_{G(2,L),t} \) as in [PS08, XI-6], observe that

\[
\ker(T_{\tilde{X}} - Id) \cap H^1(\tilde{X}_\infty, \Z) \cong \Im g_0 \cap \left( \mathcal{H}_{\tilde{X}_\infty}^1 \right)_0 \quad \text{and} \quad \ker(T_{G(2,L)} - Id) \cap H^3(\tilde{G}(2, L, \infty), \Z) \cong \Im g_1 \cap \left( \mathcal{H}_{G(2,L)}^3 \right)_0.
\]

By the local invariant cycle theorem (2.4) and (2.5), we have

\[
\ker(T_{\tilde{X}} - Id) \cap H^1(\tilde{X}_\infty, \Z) = \Im sp_1 \cap H^1(\tilde{X}_\infty, \Z) = sp_1(H^1(\tilde{X}_0, \Z)) \quad \text{and}
\]

\[
\ker(T_{G(2,L)} - Id) \cap H^3(\tilde{G}(2, L, \infty), \Z) = \Im sp_2 \cap H^3(\tilde{G}(2, L, \infty), \Z) = sp_2(H^3(G_{X_0}(2, L_0), \Z)).
\]
Using Theorem 2.9 we can then conclude
\[
\frac{H^1(\overline{X}_0, \mathbb{C})}{F^1H^1(X_0, \mathbb{C}) + H^1(X_0, \mathbb{Z})} \cong \left( \mathcal{J}^1 \overline{X}_0 \right)_0 \quad \text{and} \quad \frac{H^1(\overline{X}_\infty, \mathbb{C})}{F^1H^1(X_\infty, \mathbb{C}) + sp_1(H^1(X_0, \mathbb{Z}))} \cong \left( \mathcal{J}^1 \overline{X}_\infty \right)_0
\]

which is isomorphic to \((\mathfrak{J}^2_{\mathcal{G}(2, \mathcal{L})})_0^0\). Let \(f : x_0 \to X_0\) be the natural inclusion. Applying [PS08, Corollary-Definition 5.37] to the proper modification \(i_0 : \overline{X}_0 \to X_0\), we obtain the following exact sequence of mixed Hodge structures:
\[
H^0(X_0, \mathbb{Z}) \to H^0(\overline{X}_0, \mathbb{Z}) \to H^0(F, \mathbb{Z}) \to H^1(X_0, \mathbb{Z}) \to H^1(\overline{X}_0, \mathbb{Z}) \to H^1(F, \mathbb{Z}).
\]

Since \(F \cong \mathbb{P}^1\) and \(x_0\) is a point, we have \(H^1(F, \mathbb{Z}) = 0 = H^1(x_0, \mathbb{Z})\). Moreover, \(H^0(X_0, \mathbb{Z}) = H^0(\overline{X}_0, \mathbb{Z}) = H^0(F, \mathbb{Z}) = \mathbb{C}\). This implies \(i_0^* : H^1(X_0, \mathbb{Z}) \to H^1(\overline{X}_0, \mathbb{Z})\) is an isomorphism of mixed Hodge structures. Therefore,
\[
J^1(X_0) = \frac{H^1(X_0, \mathbb{C})}{F^1H^1(X_0, \mathbb{C}) + H^1(X_0, \mathbb{Z})} \cong \left( \mathcal{J}^1 \overline{X}_0 \right)_0
\]

and
\[
J^2(\mathcal{G}(X_0, (2, \mathcal{L}))) \cong \left( \mathfrak{J}^2_{\mathcal{G}(2, \mathcal{L})} \right)_0.
\]

Since the specialization morphisms and \(i_0^*\) are morphisms of mixed Hodge structures, we have \(W_iJ^1(X_0) \cong W_i \left( \mathcal{J}^1 \overline{X}_0 \right)_0\) and \(W_iJ^2(\mathcal{G}(X_0, (2, \mathcal{L}))) \cong W_i \left( \mathfrak{J}^2_{\mathcal{G}(2, \mathcal{L})} \right)_0\) for all \(i\). This proves the corollary.

4. Relative Mumford-Newstead and the Torelli theorem

We use notations of [2 and 3. In this section, we prove that the relative version of [MN68, Theorem p. 1201] (Proposition 4.11). We use this to show that the generalised intermediate Jacobians \(J^1(X_0)\) and \(J^2(\mathcal{G}(X_0, (2, \mathcal{L})))\) defined in Notation 4.3 are semi-abelian varieties and are isomorphic (Corollary 4.3). As an application, we prove the Torelli theorem for irreducible nodal curves (Theorem 4.4).

We first consider the relative version of the construction in [MN68]. Denote by \(\mathcal{W} := \mathcal{X}_\Delta \times \Delta - \Delta \times \mathcal{G}(2, \mathcal{L})_{\Delta}^\times\) and \(\pi_3 : \mathcal{W} \to \Delta^\times\) the natural morphism. Recall, for all \(t \in \Delta^\times\), the fiber \(\mathcal{W}_t := \pi_3^{-1}(t)\), is \(\mathcal{X}_t \times \mathcal{G}(2, \mathcal{L})_t \cong \mathcal{X}_t \times M_{\mathcal{X}_t}(2, \mathcal{L}_t)\) (Theorem 4.3). There exists a (relative) universal bundle \(\mathcal{U}\) over \(\mathcal{W}\) associated to the (relative) moduli space \(\mathcal{G}(2, \mathcal{L})_{\Delta^\times}\) i.e., \(\mathcal{U}\) is a vector bundle over \(\mathcal{W}\) such that for each \(t \in \Delta^\times\), \(\mathcal{U}|_{\mathcal{W}_t}\) is the universal bundle over \(\mathcal{X}_t \times M_{\mathcal{X}_t}(2, \mathcal{L}_t)\) associated to fine moduli space \(M_{\mathcal{X}_t}(2, \mathcal{L}_t)\) (use [Pan96, Theorem 9.1.11]). Denote by \(\mathbb{H}^i_{\mathcal{W}} := R^i\pi_3^*Z_{\mathcal{W}}\) the local system associated to \(\mathcal{W}\). Using K"unneth decomposition, we have
\[
\mathbb{H}^i_{\mathcal{W}} := \bigoplus_i \left( \mathbb{H}^i_{\mathcal{X}_\Delta} \otimes \mathbb{H}^{4-i}_{\mathcal{G}(2, \mathcal{L})_{\Delta}^\times} \right).
\]

Denote by \(c_2(\mathcal{U})^{1,3} \in \Gamma \left( \mathbb{H}^1_{\mathcal{X}_\Delta} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta}^\times} \right)\) the image of the second Chern class \(c_2(\mathcal{U}) \in \Gamma (\mathbb{H}^4_{\mathcal{W}})\) under the natural projection from \(\mathbb{H}^4_{\mathcal{W}}\) to \(\mathbb{H}^1_{\mathcal{X}_\Delta} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta}^\times}\). Using Poincaré duality applied to the local system \(\mathbb{H}^1_{\mathcal{X}_\Delta}\), we have
\[
\mathbb{H}^1_{\mathcal{X}_\Delta} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta}^\times} \overset{PD}{\cong} \left( \mathbb{H}^1_{\mathcal{X}_\Delta} \right)^\vee \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta}^\times} \cong \mathcal{H}_{\hom} \left( \mathbb{H}^1_{\mathcal{X}_\Delta}, \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta}^\times} \right).
\]
Therefore, $c_2(U)^{1,3}$ induces a homomorphism
\[ \Phi_{\Delta^*} : \mathbb{H}^1_{\Delta^*} \to \mathbb{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}. \]

By [MN68 Lemma 1 and Proposition 1], we conclude that the homomorphism $\Phi_{\Delta^*}$ is an isomorphism such that the induced isomorphism on the associated vector bundles:
\[ \Phi_{\Delta^*} : \mathcal{H}^1_{\Delta^*} \xrightarrow{\sim} \mathcal{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}} \text{ satisfies } \Phi_{\Delta^*}(F^p\mathcal{H}^1_{\Delta^*}) = F^{p+1}\mathcal{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}} \text{ for all } p \geq 0. \]

Therefore, the morphism $\Phi_{\Delta^*}$ induces an isomorphism $\Phi : J^1_{\Delta^*} \xrightarrow{\sim} J^2_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$.

Since $\overline{\mathcal{H}}^1_{\Delta^*}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$ are canonical extensions of $\mathcal{H}^1_{\Delta^*}$ and $\mathcal{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$, respectively, the morphism $\Phi_{\Delta^*}$ extend to the entire disc:
\[ \overline{\Phi} : \overline{\mathcal{H}}^1_{\Delta^*} \xrightarrow{\sim} \overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}. \]

Using the identification (2.1) and restricting $\overline{\Phi}$ to the central fiber, we have an isomorphism:
\[ \overline{\Phi}_0 : H^1(\tilde{X}_\infty, \mathbb{C}) \xrightarrow{\sim} H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}). \]

We can then conclude:

**Proposition 4.1.** For the extended morphism $\overline{\Phi}$, we have $\overline{\Phi}(F^p\overline{\mathcal{H}}^1_{\tilde{X}}) = F^{p+1}\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$ for $p = 0, 1$ and $\overline{\Phi}(\overline{\mathcal{H}}^1_{\tilde{X}}) = \overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$. Moreover, $\overline{\Phi}_0(W_iH^1(\tilde{X}_\infty, \mathbb{C})) = W_{i+2}H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C})$ for all $i \geq 0$.

**Proof.** The proof of $\overline{\Phi}(F^p\overline{\mathcal{H}}^1_{\tilde{X}}) = F^{p+1}\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$ for $p = 0, 1$ and $\overline{\Phi}(\overline{\mathcal{H}}^1_{\tilde{X}}) = \overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$, follows directly from construction. We now prove the second statement.

Since $c_2(U)^{1,3}$ is a (single-valued) global section of $\mathbb{H}^1_{\Delta^*} \otimes \mathbb{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$, it is monodromy invariant. In other words, using (4.1), we have the following commutative diagram:
\[ \begin{array}{ccc}
\mathbb{H}^1_{\tilde{X}_{\Delta^*}} & \xrightarrow{\Phi_{\Delta^*}} & \mathbb{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}} \\
T_{\tilde{X}_{\Delta^*}} & \circ & T_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}} \\
\mathbb{H}^1_{\tilde{X}_{\Delta^*}} & \xrightarrow{\Phi_{\Delta^*}} & \mathbb{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}
\end{array} \quad (4.2) \]

Note that the monodromy operators extend to the canonical extensions $\overline{\mathcal{H}}^1_{\tilde{X}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$ ([PS08 Proposition 11.2]). This gives rise to a commutative diagram similar to (1.2), after substituting $\overline{\mathcal{H}}^1_{\tilde{X}_{\Delta^*}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$ by $\overline{\mathcal{H}}^1_{\tilde{X}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$, respectively. Restricting this diagram to the central fiber, we obtain the following commutative diagram (use the identification (2.1)):
\[ \begin{array}{ccc}
H^1(\tilde{X}_\infty, \mathbb{C}) & \xrightarrow{\overline{\Phi}_0} & H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}) \\
T_{\tilde{X}} & \circ & T_{\mathcal{G}(2,\mathcal{L})} \\
H^1(\tilde{X}_\infty, \mathbb{C}) & \xrightarrow{\overline{\Phi}_0} & H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C})
\end{array} \quad (4.3) \]

The exact sequences (2.4) and (2.5) combined with Proposition 2.13 then implies:
\[ W_1H^1(\tilde{X}_\infty, \mathbb{C}) \cong \text{Im } sp_1 \cong \ker(T_{\tilde{X}} - \text{Id}) \cong \ker(T_{\mathcal{G}(2,\mathcal{L})} - \text{Id}) \cong \text{Im } sp_2 \cong W_3H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}). \]
This implies, \( \text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{C}) \cong \text{Im}(T_{\tilde{X}} - \text{Id}) \cong \text{Im}(T_{g(2, \mathcal{L})} - \text{Id}) \cong \text{Gr}_4^W H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}). \) By (13), it is easy to check that for all \( k \geq 0 \) and \( \alpha \in H^1(\tilde{X}_\infty, \mathbb{C}), \) we have

\[ \tilde{\Phi}_0(T_{\tilde{X}} - \text{Id})^k(\alpha) = (T_{g(2, \mathcal{L})} - \text{Id})^k(\tilde{\Phi}_0(\alpha)). \]

Using Remark 2.13, we then conclude

\[ \text{Gr}_0^W H^1(\tilde{X}_\infty, \mathbb{C}) \cong \text{log} T_{\tilde{X}}(\text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{C})) \cong \text{log} T_{g(2, \mathcal{L})}(\text{Gr}_4^W H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C})) \cong \text{Gr}_2^W H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}). \]

By Proposition 2.13, it is easy to check that

\[ \text{Gr}_0^W H^1(\tilde{X}_\infty, \mathbb{C}) \cong W_0 H^1(\tilde{X}_\infty, \mathbb{C}) \quad \text{and} \quad \text{Gr}_2^W H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}) \cong W_2 H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}). \]

Moreover, \( W_2 H^3(\tilde{X}_\infty, \mathbb{C}) \cong H^1(\tilde{X}_\infty, \mathbb{C}) \) and \( W_4 H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}) \cong H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}) \) (Proposition 2.13). Therefore for all \( i, \) we have \( \tilde{\Phi}_0(W_i H^1(\tilde{X}_\infty, \mathbb{C})) = W_{i+2} H^3(\mathcal{G}(2, \mathcal{L})_{\infty}, \mathbb{C}). \) This proves the proposition.

\[ \square \]

**Remark 4.2.** Using Proposition 4.1, observe that the isomorphism \( \Phi \) between the families of intermediate Jacobians extend to an isomorphism

\[ \Phi : \mathcal{J}_0^1(\tilde{X}) \cong \mathcal{J}^2_{g(2, \mathcal{L})}. \]

By Corollary 3.5 and Proposition 4.1, we conclude that the composition

\[ \Psi : J^1(X_0) \xrightarrow{\text{spin}(Z)} \left( \mathcal{J}_0^1 \right)_0 \cong \mathcal{J}^1 \otimes k(0) \xrightarrow{\tilde{\Phi}} \mathcal{J}^2_{g(2, \mathcal{L})} \otimes k(0) \cong \left( \mathcal{J}^2_{g(2, \mathcal{L})} \right)_0 \xrightarrow{\text{spin}^{-1}} J^2(\mathcal{G}_{X_0}(2, \mathcal{L})) \]

satisfying \( \Psi(W_i J^1(X_0)) = W_{i+2} J^2(\mathcal{G}_{X_0}(2, \mathcal{L})). \)

We prove that \( \Psi \) is an isomorphism of semi-abelian varieties which induces an isomorphism of the associated abelian varieties.

**Corollary 4.3.** Denote by \( \text{Gr}_1^W J^1(X_0) := W_i J^1(X_0)/W_{i-1} J^1(X_0) \) and \( \text{Gr}_1^W J^2(\mathcal{G}_{X_0}(2, \mathcal{L})) := W_i J^2(\mathcal{G}_{X_0}(2, \mathcal{L}))/W_{i-1} J^2(\mathcal{G}_{X_0}(2, \mathcal{L})). \) Then,

1. \( \text{Gr}_1^W J^1(X_0) \) and \( \text{Gr}_3^W J^2(\mathcal{G}_{X_0}(2, \mathcal{L})) \) are principally polarized abelian varieties,
2. \( J^1(X_0) \) and \( J^2(\mathcal{G}_{X_0}(2, \mathcal{L}))) \) are semi-abelian varieties. In particular, \( J^1(X_0) \) (resp. \( J^2(\mathcal{G}_{X_0}(2, \mathcal{L})) \)) is an extension of \( \text{Gr}_1^W J^1(X_0) \) (resp. \( \text{Gr}_3^W J^2(\mathcal{G}_{X_0}(2, \mathcal{L})) \)) by the complex torus \( W_0 J^1(X_0) \) (resp. \( W_1 J^2(\mathcal{G}_{X_0}(2, \mathcal{L})) \)),
3. \( \Psi \) is an isomorphism of semi-abelian varieties sending \( \text{Gr}_1^W J^1(X_0) \) to \( \text{Gr}_3^W J^2(\mathcal{G}_{X_0}(2, \mathcal{L})). \)

**Proof.** Applying [PS08, Corollary-Definition 5.37] to the proper modification \( \pi : \tilde{X}_0 \to X_0, \) we obtain the following exact sequence of mixed Hodge structures:

\[ 0 \to H^0(X_0, \mathbb{Z}) \to H^0(\tilde{X}_0, \mathbb{Z}) \oplus H^0(x_0, \mathbb{Z}) \to H^0(x_1 \oplus x_2, \mathbb{Z}) \to H^1(X_0, \mathbb{Z}) \to H^1(\tilde{X}_0, \mathbb{Z}) \oplus H^1(x_0, \mathbb{Z}) \to 0, \]

where surjection on the right follows from \( H^1(x_1 \oplus x_2, \mathbb{Z}) = 0. \) Since \( Z = H^0(\tilde{X}_0, \mathbb{Z}) = H^0(X_0, \mathbb{Z}) = H^0(x_i, \mathbb{Z}) \) for \( i = 0, 1, 2, \) we obtain the short exact sequence

\[ 0 \to Z \xrightarrow{\pi^*} H^1(X_0, \mathbb{Z}) \xrightarrow{\pi^*} H^1(\tilde{X}_0, \mathbb{Z}) \to 0 \]

with \( F^1 H^1(X_0, \mathbb{C}) \cong F^1 H^1(\tilde{X}_0, \mathbb{C}) \cong H^1(\tilde{X}_0, \mathbb{C}), \) \( W_0 H^1(X_0, \mathbb{Z}) \cong Z \) and \( \text{Gr}_1^W H^1(X_0, \mathbb{Z}) \cong \text{Gr}_1^W H^1(\tilde{X}_0, \mathbb{C}) = H^1(\tilde{X}_0, \mathbb{C}) \) (as \( Z \subset H^0(x_1 \oplus x_2, \mathbb{Z}) \) is pure of weight 0 and \( H^1(\tilde{X}_0, \mathbb{Z}) \) is pure of weight 1). This induces an isomorphism

\[ \text{Gr}_1^W J^1(X_0) \cong \frac{\text{Gr}_1^W H^1(X_0, \mathbb{C})}{F^1 \text{Gr}_1^W H^1(X_0, \mathbb{C}) + \text{Gr}_1^W H^1(X_0, \mathbb{Z})} \cong \frac{H^1(\tilde{X}_0, \mathbb{C})}{F^1 H^1(\tilde{X}_0, \mathbb{C}) + H^1(\tilde{X}_0, \mathbb{Z})} \cong J^1(\tilde{X}_0). \]

Hence, \( \text{Gr}_1^W J^1(X_0) \) is a principally polarised abelian variety.
By Proposition 4.1, we have $\Psi(Gr^W_1 J^1(X_0)) = Gr^W_2 J^2(\mathcal{G}_{X_0}(2, L_0))$ (see Remark 4.2). Therefore, $Gr^W_3 J^2(\mathcal{G}_{X_0}(2, L_0))$ is also a principally polarised abelian variety.

Note that $F^1 Gr^W_0 H^1(X_0, \mathbb{C}) = 0$ as $Gr^W_0 H^1(X_0, \mathbb{Z}) \cong \mathbb{Z}$ ($Gr^W_0 H^1(X_0, \mathbb{Z})$ must be of Hodge type $(0, 0)$). Therefore,

$$Gr^W_0 J^1(X_0) \cong \frac{Gr^W_0 H^1(X_0, \mathbb{C})}{F^1 Gr^W_0 H^1(X_0, \mathbb{C}) + Gr^W_0 H^1(X_0, \mathbb{Z})} \cong \mathbb{C}^{\delta_0} \mathbb{C}^{\exp} \mathbb{C}^*.$$  

Similarly as before, $\Psi(Gr^W_0 J^1(X_0)) = Gr^W_2 J^2(\mathcal{G}_{X_0}(2, L_0)) \cong \mathbb{C}^*$ (as $\Psi$ is $\mathbb{C}$-linear). By [PS08, Theorem 5.39], $H^1(X_0, \mathbb{C})$ (resp. $H^3(\mathcal{G}_{X_0}(2, L_0))$) is of weight at most 1 (resp. 3). Therefore, $J^1(X_0)$ and $J^2(\mathcal{G}_{X_0}(2, L_0))$ are isomorphic (via $\Psi$) semi-abelian varieties, inducing an isomorphism between the associated abelian varieties $Gr^W_1 J^1(X_0)$ and $Gr^W_3 J^2(\mathcal{G}_{X_0}(2, L_0))$. This proves the corollary.

We now prove the Torelli theorem for irreducible nodal curves.

**Theorem 4.4.** Let $X_0$ and $X_1$ be irreducible nodal curves of genus $g \geq 4$ with exactly one node such that the normalizations $\tilde{X}_0$ and $\tilde{X}_1$ are not hyper-elliptic. Let $L_0$ and $L_1$ be invertible sheaves of odd degree on $X_0$ and $X_1$, respectively. If $\mathcal{G}_{X_0}(2, L_0) \cong \mathcal{G}_{X_1}(2, L_1)$ then $X_0 \cong X_1$.

**Proof.** Since the genus of $X_0$ and $X_1$ is $g \geq 4$, the curves $X_0$ and $X_1$ are stable. As the moduli space of stable curves is complete, we get algebraic families

$$f_1 : X_1 \to \Delta \text{ and } f_2 : X_2 \to \Delta$$  

of curves with $X_1$ and $X_2$ regular, $f_1, f_2$ smooth over the punctured disc $\Delta^*$, $f_1^{-1}(0) = X_0$ and $f_2^{-1}(0) = X_1$. Recall, the obstruction to extending invertible sheaves from the central fiber to the entire family lies in $H^2(\mathcal{O}_{\tilde{X}_0})$ and $H^2(\mathcal{O}_{\tilde{X}_1})$, respectively (see [Har10, Theorem 6.4]). Since $X_0$ and $X_1$ are curves the obstruction vanishes. Thus, there exist invertible sheaves $\mathcal{M}_0$ and $\mathcal{M}_1$ on $X_0$ and $X_1$, respectively, such that $\mathcal{M}_0|_{X_0} \cong L_0$ and $\mathcal{M}_1|_{X_1} \cong L_1$. Since $\mathcal{G}_{X_0}(2, L_0) \cong \mathcal{G}_{X_1}(2, L_1)$, Theorem 4.3 implies that

$$J^1(X_0) \cong J^2(\mathcal{G}_{X_0}(2, L_0)) \cong J^2(\mathcal{G}_{X_1}(2, L_1)) \cong J^1(X_1) \text{ with } Gr^W_1 J^1(X_0) \cong Gr^W_1 J^1(X_1).$$

By [Nam73, Proposition 9], this implies $X_0 \cong X_1$. This proves the theorem.

**Appendix A. Gieseker moduli space of stable sheaves**

In this section, we recall the (relative) Gieseker’s moduli space as defined in [NS99] and with fixed determinant as in [Tah92]. We observe that the later moduli space is regular, with central fiber a reduced simple normal crossings divisor.

**Notation A.1.** Denote by $X_r$ the curve “semi-stably equivalent to $X_0$”, which is obtained by gluing a chain of rational curves of length $r$ to the normalization $\tilde{X}_0$ of $X_0$ through $x_1$ and $x_2$ (see [NS99, Definition-Notation 2] for the precise definition). Let $\pi_r : X_r \to X_0$ be the natural morphism from $X_r$ to $X_0$ contracting the chain of rational curves. Fix a positive integer $n$ and an integer $d$ coprime to $n$.

**Definition A.2.** By [HL10, Theorem 3.3.7], there exists an integer $l \gg 0$ such that any semi-stable torsion-free sheaf $F$ on $X_0$ of rank $n$ and degree $d$ is $l$-regular (in the sense of Castelnuovo-Mumford regularity). Fix such an integer $l$ and let $m := H^0(F(l))$. Denote by $Gr(m, n)$ the Grassmannian variety parametrizing $n$-dimensional quotients of a (fixed) $m$-dimensional vector space. Let $Gr(m, n)_g = Gr(m, n) \times_k S$.

Denote by $\tilde{\mathcal{G}}_{X_0}(n, d)$ (resp. $\tilde{\mathcal{G}}(n, d)$) the Gieseker (relative Gieseker) moduli functor from $(\text{Sch}/k)$ (resp. $(\text{Sch}/S)$) to (sets) such that for any $k$-scheme (resp. $S$-scheme) $T$, $\tilde{\mathcal{G}}_{X_0}(n, d)(T)$
(resp. \( \tilde{G}(n,d)(T) \)) consists of pairs \( (Y_T, i_{Y_T}) \), where \( i_{Y_T} : Y_T \hookrightarrow X_0 \times_k T \times_k \text{Gr}(m,n) \) (resp. \( i_{Y_T} : Y_T \hookrightarrow X \times S T \times S \text{Gr}(m,n)_S \)) is a closed immersion satisfying the following properties:

1. the natural morphism \( p_2 : Y_T \to T \) is a flat morphism of relative dimension 1 such that for all \( t \in T \), \( Y_{T,t} \cong X_t \) for some \( r \) (resp. \( Y_{T,t} \cong X_r \) for some \( r \) if \( t \) is over the closed point \( s_0 \) of \( S \) and \( Y_{T,s_0} \cong X_t := X \times S \{ t \} \) if \( t \) does not lie over \( s_0 \)) and the natural morphism \( Y_{T,t} \to X_0 \times \{ t \} \) (resp. \( Y_{T,t} \to X_r \times \{ t \} \)) is the canonical morphism contracting the chain of rational curves \( \pi_r : X_r \cong Y_{T,t} \to X_0 \) if \( t \) maps to \( s_0 \) and \( Y_{T,t} \to X_t \) is smooth if \( t \) does not lie over \( s_0 \),
2. the projection morphism \( \text{pr}_{23} \) from \( Y_T \) to \( T \times_k \text{Gr}(m,n) \) (resp. \( T \times S \text{Gr}(m,n)_S \)) is a closed immersion,
3. Denote by \( O^{\text{iem}}_{Y_T} \to E_{Y_T} \) the pullback to \( Y_T \) of the tautological rank \( n \) quotient bundle on \( \text{Gr}(m,n) \) (resp. \( \text{Gr}(m,n)_S \)), via the morphism \( \text{pr}_{23} \). Then, the vector bundle \( E_{Y_T} \) is of degree \( e \) and rank \( n \) with \( e = m + n(g - 1) \),
4. the natural quotient morphism \( O^{\text{iem}}_{Y_{T,t}} \to E_{Y_T}|_{Y_{T,t}} \) induces an isomorphism from \( H^0(O^{\text{iem}}_{Y_{T,t}}) \) to \( H^0(E_{Y_T}|_{Y_{T,t}}) \).

See [NS99, Definition 5 and 7] for further details.

Recall, there exists a fine moduli space \( M_X(n,d) \) of semi-stable sheaves of rank \( n \) and degree \( d \) on \( X_t \) for \( t \neq s_0 \) (see [HL10, Theorem 4.3.7 and Corollary 4.6.6]). We know that the moduli functor \( \tilde{G}(n,d) \) is representable by a scheme \( G(n,d) \) with every fiber \( G(n,d)_t \) isomorphic to \( M_X(n,d) \) for \( t \neq s_0 \). More precisely,

**Theorem A.3.** The functor \( \tilde{G}(n,d) \) (resp. \( \tilde{G}_{X_0}(n,d) \)) is representable by an open subscheme \( G(n,d) \) (resp. \( G_{X_0}(n,d) \)) of the \( S \)-scheme (resp. \( k \)-scheme) \( \text{Hilb}^P(X \times S \text{Gr}(m,n)_S) \) (resp. \( \text{Hilb}^P(X_0 \times_k \text{Gr}(m,n)) \)), for some Hilbert polynomial \( P \). Furthermore,

1. the closed fiber \( G(n,d)_{s_0} \) of \( G(n,d) \) over \( s_0 \) is irreducible, isomorphic to \( G_{X_0}(n,d) \) and is a (analytic) normal crossings divisor in \( G(n,d) \),
2. for all \( t \neq s_0 \), the fiber \( G(n,d)_{S,t} \) is smooth and isomorphic to \( M_{X_t}(n,d) \),
3. as a scheme over \( k \), \( G(n,d) \) is regular.

**Proof.** See [NS99, Proposition 8] (or [Gie84, pp. 179] for the case \( n = 2 \)). □

We now briefly recall the construction of the moduli space \( G_{X_0}(2,d) \). Let \( M_{X_0}(2,d) \) be the fine moduli space of rank 2, degree \( d \) stable bundles over \( X_0 \). Consider the universal bundle \( E \) over \( X_0 \times M_{X_0}(2,d) \). Let \( E_{x_1} := E|_{x_1 \times M_{X_0}(2,d)} \) and \( E_{x_2} := E|_{x_2 \times M_{X_0}(2,d)} \). Consider the projective bundle \( S_1 := \mathbb{P}(\text{Hom}(E_{x_1}, E_{x_2}) \oplus \mathcal{O}_{M_{X_0}(2,d)}) \) over \( M_{X_0}(2,d) \). Denote by

\[
0_s := \bigcup_{t \in M_{X_0}(2,d)} 0_{s,t}.
\]

the zero section of the projective bundle \( S_1 \) over \( M_{X_0}(2,d) \), where \( 0_{s,t} := \{ 0, \lambda \} \subseteq S_{1,t} \), for \( \lambda \in \mathcal{O}_{M_{X_0}(2,d)} \otimes k(t) \). Consider the two sub-bundles of \( S_1 \),

\[
H_2 := \bigcup_{t \in M_{X_0}(2,d)} H_{2,t} \quad \text{and} \quad D_1 := \bigcup_{t \in M_{X_0}(2,d)} D_{1,t}, \quad \text{where}
\]

\[
H_{2,t} := \{ [\phi, 0] \in S_{1,t} \mid \phi \in \text{Hom}(E_{x_1}, E_{x_2}) \otimes k(t), \phi \neq 0 \} \quad \text{and}
\]

\[
D_{1,t} := \{ [\phi, \lambda] \mid \phi \in \text{Hom}(E_{x_1}, E_{x_2}) \otimes k(t), \lambda \in \mathcal{O}_{M_{X_0}(2,d)} \otimes k(t) \text{ and } \ker(\phi) \neq 0 \}.
\]
For any \( t \in M_{X_0}(2, d) \), the fiber \( S_{1,t} \) of \( S_1 \) over \( t \) is isomorphic to \( \mathbb{P}(\text{End}(\mathbb{C}^2) \oplus \mathbb{C}) \), after identifying \( \mathcal{E}|_{x_1 \times \{t\}} \cong \mathbb{C}^2 \cong \mathcal{E}^*|_{x_2 \times \{t\}} \). Under this identification, we have \( H_{2,t} \cong \mathbb{P}(\text{End}(\mathbb{C}^2) \setminus \{0\}) \cong \mathbb{P}^3 \) and \( H_{2,t} \cap D_{1,t} = \{(M, 0) \in \mathbb{P}(\text{End}(\mathbb{C}^2) \oplus \mathbb{C}) | M \in \text{End}(\mathbb{C}^2), \det(M) = 0\} \). It is easy to check that \( H_{2,t} \cap D_{1,t} \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) (use the Segre embedding into \( H_{2,t} \cong \mathbb{P}^3 \)). Denote by \( Q = H_2 \cap D_1 \) and \( S_2 := \text{Bl}_Q(S_1) \). Let \( \tilde{Q} \) be the strict transform of \( Q \) in \( S_2 \) and \( S_3 := \text{Bl}_{\tilde{Q}}S_2 \). Denote by \( H_1 \) (resp. \( D_2 \)) the exceptional divisor of the blow-up \( S_2 \to S_1 \) (resp. \( S_3 \to S_2 \)). Replace \( H_2 \) and \( D_1 \) by their strict transforms in \( S_3 \). There is a natural (open) embedding

\[ \text{GL}_2 \hookrightarrow \mathbb{P}(\text{End}(\mathbb{C}^2) \oplus \mathbb{C}) \cong S_{1,t} \text{ defined by End}(\mathbb{C}^2) \ni M \mapsto [M, 1]. \]

Then, \( S_{3,t} = \text{Bl}_{\tilde{Q}_t}(\text{Bl}_{0,s_t}S_{1,t}) \) is called the wonderful compactification of \( \text{GL}_2 \) (see \cite{Pez10} Definition 3.3.1). Denote by \( \overline{\text{SL}}_2 \subset S_{1,t} \) the closure of \( \text{SL}_2 \) in \( S_{1,t} \) under the above mentioned embedding of \( \text{GL}_2 \hookrightarrow S_{1,t} \). Then, \( \overline{\text{SL}}_2 \cong \{(M, \lambda) \in \mathbb{P}(\text{End}(\mathbb{C}^2) \oplus \mathbb{C}) | \det(M) = \lambda^2\} \). Note that \( \overline{\text{SL}}_2 \) is regular, \( 0_{s,t} \not\in \overline{\text{SL}}_2 \) and \( Q_t = H_{2,t} \cap D_{1,t} \) is a divisor in \( \overline{\text{SL}}_2 \). Thus the strict transform of \( \overline{\text{SL}}_2 \) in \( S_{3,t} \) is isomorphic to itself.

**Proposition A.4** \cite{Gie84}. Denote by \( \mathcal{G}'_{X_0}(2, d) \) the normalization of \( \mathcal{G}_{X_0}(2, d) \). There exist closed subschemes \( Z \subset D_1 \cap D_2 \) and \( Z' \subset \mathcal{G}'_{X_0}(2, d) \) of codimension at least \( g - 1 \) such that \( S_3 \setminus Z \cong \mathcal{G}'_{X_0}(2, d) \setminus Z' \).

**Proof.** See \cite{Gie84} §8, 9 and §10 for a concrete description of \( Z, Z' \) and proof. Also see \cite{Tha92} Chapter 1, §5. \qed

By Theorem A.3 there exists an universal closed immersion \( \mathcal{Y} \hookrightarrow \mathcal{X} \times_S \mathcal{G}(2, d) \times_S \text{Gr}(m, 2)_S \) corresponding to the flat family of curves \( \mathcal{Y} \hookrightarrow \mathcal{G}(2, d) \). Denote by \( \mathcal{U} \) the vector bundle on \( \mathcal{Y} \) obtained as the pull-back of the tautological quotient bundle of rank 2 on \( \text{Gr}(m, 2)_S \).

**Notation A.5.** Let \( \mathcal{L} \) be an invertible sheaf on \( \mathcal{X} \) of relative odd degree \( d \) i.e., for every \( t \in S \), we have \( \deg(\mathcal{L}|_{x_0}) = d \). Denote by \( \mathcal{L}_0 := \mathcal{L}|_{x_0} \). Consider the reduced family \( \mathcal{G}(2, \mathcal{L}) \subset \mathcal{G}(2, d) \) such that the fiber over \( s \in S \) consists of all points \( z_s \in \mathcal{G}(2, d)_s \) such that the corresponding vector bundle \( \mathcal{U}_{z_s} \) satisfies the property \( H^0(\det \mathcal{U}_{z_s} \otimes \mathcal{L}_s) \neq 0 \). By upper semi-continuity, \( \mathcal{G}(2, \mathcal{L}) \) is a closed subvariety of \( \mathcal{G}(2, d) \). Denote by \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) := \mathcal{G}(2, \mathcal{L})_{s_0} \) the fiber over \( s_0 \).

We now study the geometry of the subvariety \( \mathcal{G}(2, \mathcal{L}) \). Denote by \( \mathcal{Y}'_0 \) the base-change to \( S_3 \setminus Z \) of the universal family of curves \( \mathcal{Y} \) (over \( \mathcal{G}(2, d) \) ) via the composition

\[ S_3 \setminus Z \cong \mathcal{G}'_{X_0}(2, d) \setminus Z' \to \mathcal{G}'_{X_0}(2, d) \to \mathcal{G}_{X_0}(2, d) \to \mathcal{G}(2, d). \]

Denote by \( \mathcal{U}'_0 \) the pullback to \( \mathcal{Y}'_0 \) the universal bundle \( \mathcal{U} \) on \( \mathcal{G}(2, d) \). Denote by \( \mathcal{E}_{S_3} \) the pullback of the universal bundle \( \mathcal{E} \) on \( X_0 \times M_{\overline{X}_0}(2, d) \), by the natural morphism from \( S_3 \) to \( M_{\overline{X}_0}(2, d) \). Define,

\[ P_0 := \{ s \in S_3 \mid (\mathcal{Y}'_0)_s \cong X_0 \text{ and } \det \mathcal{U}'_{0,s} \simeq \mathcal{L}_0 \} \cup \{ s \in D_1 \cap D_2 \mid \det \mathcal{E}_{S_3,s} \simeq \pi^\ast \mathcal{L}_0 \}, \]

\[ P_1 := \{ s \in H_1 \mid \det \mathcal{E}_{S_3,s} \simeq \pi^\ast \mathcal{L}_0(x_2 - x_1) \}, \]

\[ P_2 := \{ s \in H_2 \mid \det \mathcal{E}_{S_3,s} \simeq \pi^\ast \mathcal{L}_0(x_1 - x_2) \}. \]

Recall, there exists a fine moduli space \( M_X(2, \mathcal{L}_t) \) of semi-stable sheaves of rank 2 and determinant \( \mathcal{L}_t \) on \( X_t \) for \( t \neq s_0 \) (see \cite{Lan96} §3.3). Observe that

**Proposition A.6.** The variety \( P_0 \) (resp. \( P_1, P_2 \)) is a \( \mathbb{P}^3 \)-bundle over \( M_{\overline{X}_0}(2, d) \). Each \( P_1 \) contains a natural \( \mathbb{P}^1 \times \mathbb{P}^1 \)-bundle over \( M_{\overline{X}_0}(2, d) \), namely \( P_0 \cap D_1 \cap D_2 \), \( P_1 \cap D_1 \) and \( P_2 \cap D_2 \). The subvariety \( Z \subset S_3 \) (Proposition A.4) does not intersect \( P_1 \) or \( P_2 \).
Proof. See [Tha92, §6].

Using this we have the following description of $G(2, L)$:

**Theorem A.7.** The variety $G(2, L)$ is regular and for all $t \neq s_0$, the fiber $G(2, L)_t$ is isomorphic to $M_X(2, \mathcal{L}_t)$. The fiber $G_{X_0}(2, \mathcal{L}_0)$ is a reduced simple normal crossings divisor in $G(2, L)$, consisting of two irreducible components, say $G_0$ and $G_1$, with one of the irreducible components isomorphic to $P_1$. Moreover, the intersection $G_0 \cap G_1$ is isomorphic to $P_1 \cap D_1$, which is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $M_{X_0}(2, \mathcal{L}_0)$.

**Proof.** For proof see [Tha92, §6] or [Abe09, §5 and §6].

**Remark A.8.** Note that, for any $t \neq s_0$, $G(2, L)_t \cong M_X(2, \mathcal{L}_t)$ is non-singular ([HL10, Corollary 4.5.5]). Therefore, $G(2, L)$ is smooth over the punctured disc $\Delta^*$.

**References**

[Abe09] T. Abe. The moduli stack of rank-two Gieseker bundles with fixed determinant on a nodal curve ii. *International Journal of Mathematics*, 20(07):859–882, 2009.

[Ale02] V. Alexeev. Complete moduli in the presence of semiabelian group action. *Annals of mathematics*, pages 611–708, 2002.

[Ale04] V. Alexeev. Compactified Jacobians and Torelli map. *Publications of the Research Institute for Mathematical Sciences*, 40(4):1215–1265, 2004.

[Bas16] S. Basu. On a relative Mumford–Newstead theorem. *Bulletin des Sciences Mathématiques*, 140(8):953–989, 2016.

[Cle83] H. Clemens. The nérond model for families of intermediate Jacobians acquiring algebraic singularities. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 58(1):5–18, 1983.

[CV11] L. Caporaso and F. Viviani. Torelli theorem for stable curves. *Journal of the European Mathematical Society*, 13(5):1289–1329, 2011.

[Ful13] W. Fulton. *Intersection theory*, volume 2. Springer Science & Business Media, 2013.

[GGK10] M. Green, P. Griffiths, and M. Kerr. Néron models and limits of Abel–Jacobi mappings. *Compositio Mathematica*, 146(2):288–366, 2010.

[Gie84] D. Gieseker. A degeneration of the moduli space of stable bundles. *Journal of Differential Geometry*, 19(1):173–206, 1984.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate text in Mathematics-52. Springer-Verlag, 1977.

[Har70] R. Hartshorne. *Deformation Theory*. Graduate text in Mathematics. Springer-Verlag, 2010.

[HL10] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Springer, 2010.

[Lan06] A. Langer. Moduli spaces and Castelnuovo-Mumford regularity of sheaves on surfaces. *American journal of mathematics*, pages 373–417, 2006.

[MN68] D Mumford and P Newstead. Periods of a moduli space of bundles on curves. *American Journal of Mathematics*, 90(4):1200–1208, 1968.

[Nam73] Y. Namikawa. On the canonical holomorphic map from the moduli space of stable curves to the Igusa monoidal transform. *Nagoya Mathematical Journal*, 52:197–259, 1973.

[NS99] D. S. Nagaraj and C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves II (Generalized Gieseker moduli spaces). In *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, volume 109, pages 165–201. Springer, 1999.

[Pan96] R. Pandharipande. A compactification over $M_g$ of the Universal Moduli Space of Slope-Semistable vector bundles. *Journal of the American Mathematical Society*, 9(2):425–471, 1996.

[Pez10] G. Pezzini. Lectures on spherical and wonderful varieties. *Les cours du CIRM*, 1(1):33–53, 2010.

[PS08] C. Peters and J. H. M. Steenbrink. *Mixed Hodge structures*, volume 52. Springer Science & Business Media, 2008.

[Sai96] M. Saito. Admissible normal functions. *J. Algebraic Geom.*, 5(2):235–276, 1996.

[Sch73] W. Schmid. Variation of Hodge structure: the singularities of the period mapping. *Inventiones mathematicae*, 22(3-4):211–319, 1973.

[Ste76] J. Steenbrink. Limits of Hodge structures. *Inventiones mathematicae*, 31:229–257, 1976.

[Tha92] M. Thaddeus. *Algebraic geometry and the Verlinde formula*. PhD thesis, University of Oxford, 1992.

[Vo12] C. Voisin. *Hodge Theory and Complex Algebraic Geometry-I*. Cambridge studies in advanced mathematics-76. Cambridge University press, 2002.
[Voi03] C. Voisin. *Hodge Theory and Complex Algebraic Geometry-II*. Cambridge studies in advanced mathematics-77. Cambridge University press, 2003.

[Zuc76] S. Zucker. Generalized intermediate Jacobians and the theorem on normal functions. *Inventiones mathematicae*, 33(3):185–222, 1976.

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