A new $Q$-matrix in the eight-vertex model

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Abstract
We construct a $Q$-matrix for the eight-vertex model at roots of unity for crossing parameter $\eta = 2mK/L$ with odd $L$, a case for which the existing constructions do not work. The new $Q$-matrix $\hat{Q}$ depends on the spectral parameter $v$ and also on a free parameter $t$. For $t = 0$, $\hat{Q}$ has the standard properties. For $t \neq 0$, however, it does not commute with the operator $S$ nor with itself for different values of the spectral parameter. We show that the six-vertex limit of $\hat{Q}(v, t = iK'/2)$ exists.

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An essential tool in Baxter’s solution of the eight-vertex model [1–4] is the $Q$-matrix which satisfies the $TQ$ equation

$$T(v)Q(v) = [\rho h(v - \eta)]^N Q(v + 2\eta) + [\rho h(v + \eta)]^N Q(v - 2\eta)$$

and commutes with $T$. Here $T(v)$ is the transfer matrix of the eight-vertex model (A.1). Combined with periodicity properties of $Q(v)$ in the complex $v$-plane equation (1) leads to the derivation of Bethe’s equations and the solution of the model. For generic values of the crossing parameter $\eta$ the transfer matrix $T$ has a non-degenerate spectrum. For rational values of $\eta/K$ however this is not the case. This leads to the existence of different $Q$-matrices which all satisfy equation (1). In [1], Baxter constructs a $Q$-matrix valid for

$$2L\eta = 2m_1K + im_2K'$$

with integers $m_1, m_2, L$. In [2], Baxter derived a $Q$-matrix valid for generic values of $\eta$. As these $Q$-matrices are different we distinguish them by writing $Q_{72}$ and $Q_{73}$ respectively for the constructions in [1, 2]. It turned out, however, that $Q_{72}$ has interesting properties beyond its role in equation (1) because of its restriction to rational values of $\eta/K$.

In [5], it is conjectured that $Q_{72}(v)$ satisfies the following functional relation.

For $N$ even and $\eta = m_1K/L$ where either $L$ even or $L$ and $m_1$ odd

$$e^{-N\pi i v/2K}Q_{72}(v - iK') = A \sum_{l=0}^{L-1} h^N(v - (2l + 1)\eta) \frac{Q_{72}(v)}{Q_{72}(v - 2l\eta)} \frac{Q_{72}(v)}{Q_{72}(v - 2(l + 1)\eta)},$$

where

$$A = \frac{\sinh(\pi i v/2K)}{\sinh(\pi i K'/2K)}.$$
A is a normalizing constant matrix independent of \( v \) that commutes with \( Q_{72} \) and \( h(v) = H(v)\Theta(v) \). There is a proof of this conjecture valid for \( L = 2 \) in [6]. This functional relation is important as it allows the conclusion that the dimension of eigenspaces of degenerate eigenvalues of the \( T \)-matrix is a power of 2, a result also true in the six-vertex model provided the roots of the Drinfeld polynomial of the loop algebra symmetry are distinct [7].

The reason why the case \( L \) odd and \( m_1 \) even is excluded in (3) is that \( Q_{72} \) does not exist in this case [5].

The purpose of this paper is to close this gap\(^1\). We construct for even \( N \) a \( Q \)-matrix which exists for \( \eta = 2mK/L \) for odd \( L \) which satisfies the functional relation (3). Beyond that we shall show that for \( \eta = 2mK/L \) a more general \( Q \)-matrix exists which depends on a free parameter \( t \) and which does not commute with \( R \) and \( S \) where

\[
R = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \quad S = \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3
\]

(4)

and not even with itself for different spectral parameters

\[ [Q(v_1, t), Q(v_2, t)] \neq 0. \]

This phenomenon has also been observed by Bazhanov and Stroganov in [8] for their column-to-column transfer matrix \( T_{\text{col}} \) which acts like a \( Q \)-matrix in the six-vertex model: it satisfies (1) and it commutes with \( T_6 \). But it does not commute with itself for different arguments.

We use the notation of Baxter’s 1972 paper. We denote our new \( \hat{Q}_R \) operator and describe its range of validity. In section 1.3 we introduce the matrix \( Q_L \) and show in section 1.4 that the famous equation \( Q_L(u)Q_R(v) = Q_L(v)Q_R(u) \) which Baxter proved for \( Q_{72} \) and \( Q_{73} \) is also satisfied by \( \hat{Q}(v) \). In section 2 we study the quasiperiodicity properties of \( \hat{Q}(v) \) and show that there exists a link between quasiperiodicity of \( \hat{Q}_R \) with quasiperiod \( iK' \) and non-existence of \( Q_{R}^{-1} \). We summarize in section 3 the properties of \( \hat{Q}(v) \) and describe in section 4 the exotic properties of \( \hat{Q}(v, t) \) for \( t \neq 0 \).

1. Construction of a \( Q \)-matrix for \( \eta = 2mK/L \)

1.1. Baxter’s construction of \( Q_{72} \)

The goal is to find a matrix \( Q_R \) of the form

\[
[Q_R(v)]_{\alpha|\beta} = \text{Tr} S_R(\alpha_1, \beta_1)S_R(\alpha_2, \beta_2) \cdots S_R(\alpha_N, \beta_N),
\]

(5)

where \( \alpha_j \) and \( \beta_j = \pm 1 \) and \( S_R(\alpha, \beta) \) is a matrix of size \( L \times L \) such that \( Q_R \) satisfies

\[
T(v)Q_R(v) = [\rho h(v - \eta)]^N Q_R(v + 2\eta) + [\rho h(v + \eta)]^N Q_R(v - 2\eta).
\]

(6)

The \( Q \)-matrix occurring in equation (1) is then

\[
Q(v) = Q_R(v)Q_R^{-1}(v_0)
\]

(7)

for some constant \( v_0 \). Therefore, it is necessary that \( Q_R(v) \) is a regular matrix. The problem to construct a \( Q_R \) of the form (5) satisfying (6) is posed and solved by Baxter in appendix C

\(^1\) There now exists a related investigation by Roan [11].
of [1]. In order to construct a $Q_R$-matrix which is regular for $\eta = mK/L$ for even $m$ and odd $L$, we shall search for other solutions of Baxter’s fundamental equations.

These equations are (see (C10), (C11) in [1])

\[
\begin{align*}
(a_{mn} - b_{pm})S_R(\alpha, \beta)_{m,n} + (c_{pm} - d_{mn})S_R(-\alpha, -\beta)_{m,n} &= 0 \\
(c_{pm} - d_{mn})S_R(\alpha, \beta)_{m,n} + (b_{pm} - a_{mn})S_R(-\alpha, -\beta)_{m,n} &= 0,
\end{align*}
\]

where $\beta = +, -, m, n = 1, \ldots, L$ and $a, b, c, d$ are defined in (A.2). Equations (8) determine the elements of the local matrices $S_R(\alpha, \beta)$ occurring in (5) provided that the determinant of this system of homogeneous linear equations vanishes:

\[
\begin{align*}
(a^2 + b^2 - c^2 - d^2)p_{mn}p_n &= ab(p^2_m + pn^2_n) - cd(1 + p^2_m p^2_n).
\end{align*}
\]

This determines $p_m$ if $p_m$ is given. Setting

\[
p_m = k^{1/2} \sin(u),
\]

it follows from (A.3) that

\[
p_n = k^{1/2} \sin(u \pm 2\eta).
\]

Baxter selected a solution which has non-vanishing diagonal elements $S_R(\alpha, \beta)_{0,0}$ and $S_R(\alpha, \beta)_{L,L}$. In order to allow $S_R(\alpha, \beta)_{m,n}$ to have non-vanishing diagonal elements $S_R(\alpha, \beta)_{0,0}$ and $S_R(\alpha, \beta)_{L,L}$ equation (9) has to be satisfied for $n = m$. Then

\[
\sin(u) = \sin(u \pm 2\eta).
\]

This fixes the parameter $u$ to become $u = K \pm \eta$ and leads to the restriction to discrete $\eta$:

\[
2L\eta = 2m_1K + i m_2K'.
\]

One obtains from (10) and (11) that

\[
p_n = k^{1/2} \sin(K + (2n - 1)\eta)
\]

and from (8)

\[
S_R(\alpha, \beta)(v)_{k,l} = \delta_{k+1,l}u^\alpha(v + K - 2\eta)\tau_{-\eta,\beta} + \delta_{k,1}u^\alpha(v + K + 2\eta)\tau_{\eta,\beta} + \delta_{k,0}u^\alpha(v + K)\tau_{0,\beta} + \delta_{k,L}u^\alpha(v + K + 2L\eta)\tau_{L,\beta},
\]

for $1 < k \leq L$, $1 < l \leq L$ and where

\[
u^+(v) = H(v) \quad u^-(v) = \Theta(v)
\]

if

\[
\eta = m_1K/L.
\]

$Q_{R,72}$ is the matrix $Q_R$ defined in (5) with $S_R$ given by (15).

It has been shown in [5] that $Q_R$ based on (15) is singular if $m_1$ is even and $L$ is odd. In the following subsection we show that an alternative construction leads for these $\eta$-values to a regular $Q_R$-matrix.

1.2. Another $Q$-matrix

To obtain another solution $\hat{S}_R$ of (8) and (9) we consider the possibility that the elements of $\hat{S}_R(\alpha, \beta)_{m,n}$ form cycles

\[
\hat{S}_R(\alpha, \beta)_{1,2}, \hat{S}_R(\alpha, \beta)_{2,3}, \ldots, \hat{S}_R(\alpha, \beta)_{L-1,L}, \hat{S}_R(\alpha, \beta)_{1,1}
\]

and

\[
\hat{S}_R(\alpha, \beta)_{2,1}, \hat{S}_R(\alpha, \beta)_{3,2}, \ldots, \hat{S}_R(\alpha, \beta)_{L,L-1}, \hat{S}_R(\alpha, \beta)_{1,L}
\]
instead of imposing the condition that $\hat{S}_R(\alpha, \beta)_{m,n}$ has two diagonal elements. In this case a set of functions $p_n$ consistent with (10) and (11) is

$$p_n = k^{1/2} \text{sn}(t + (2n - 1)\eta).$$

From the condition that

$$\hat{S}_R(\alpha, \beta)_{L+1,L+1} = \hat{S}_R(\alpha, \beta)_{L,L+1}$$

it follows that $p_1$ and $p_L$ must have arguments which differ by $2\eta$:

$$\text{sn}(t + (2L - 1)\eta) = \text{sn}(t + \eta - 2\eta).$$

This is satisfied if

$$2L\eta = 4mK + 2m_2K'.$$

This condition differs from (13). The solution of equations (8) with the set of $p_n$-functions (18) as input is

$$\hat{S}_R(\alpha, \beta)_{k,l} = \delta_{k+1}^{l+1}\text{sn}(v + t + 2\eta)\tau_{\beta,l} + \delta_{k,l+1}\text{sn}(v + t + 2\eta)\tau_{\beta,l}$$

with $u^\alpha$ defined in equation (16) and $w^\alpha$ is given by

$$w^+(v) = -H(v) \quad w^-(v) = \Theta(v).$$

Note that the first component of $w^\alpha$ differs from $u^\alpha$ by a minus sign.

We consider only the case $m_2 = 0$ in (21). Then

$$\eta = 2mK/L.$$  

We shall denote the $Q_R$-, $Q_L$- and $Q$-matrices derived from $\hat{S}_R$, $\hat{S}_L$ by $\hat{Q}_R$, $\hat{Q}_L$ and $\hat{Q}$. We distinguish the following cases.

1. If $L$ is odd the resulting $\hat{Q}$-matrices cover exactly the set of discrete $\eta$-values which is missing in the original solution (15)–(16). We note that for $t = K$ this solution becomes identical to case (15)–(16) with singular $\hat{Q}_R$. But for generic $t$ (especially $t = 0$) $\hat{Q}_R$ is regular. It must be stressed, however, that the regularity has not been proved analytically but numerically for sufficiently large systems to allow the occurrence of degenerate eigenvalues of the transfer matrix $T$. See also appendix C of [1, 5].

2. $L$ is even but both $L_1 = L/2$ and $m$ are odd.

    Then $\eta = mK/L_1$ is that set of $\eta$-values for which solution (15)–(16) leads to regular $Q_R$ matrices. It turns out that in this case the $\hat{Q}_R$-matrix resulting from solution (22) is singular.

3. $L$ and $L/2$ are even and $m$ is odd.

    In this case both solutions (15)–(16) and (22) give regular $Q_R$-matrices. But the matrices $\hat{S}_R(\alpha, \beta)$ differ in size by a factor of 2.

The conclusion is that the two sets of $Q$-matrices (15)–(16) and (22) are complementary in the sense that for $\eta = mK/L$ and odd $L$ what is missing in the first set is present in the second and vice versa.

### 1.3. The matrix $\hat{Q}_L$

To get finally a $Q$-matrix which commutes with the transfer matrix $T$ and satisfies equation (1) Baxter introduced a second matrix $\hat{Q}_L$. By transposing equation (6) and replacing $v$ by $-v$ one obtains

$$Q_L(v)T(v) = [\rho h(v - \eta)]^N Q_L(v + 2\eta) + [\rho h(v + \eta)]^N Q_L(v - 2\eta)$$
with

\[ Q_L(v) = Q_R(-v) \]

(26)

and

\[ [Q_L(v)]_{\alpha \beta} = \text{Tr} \ S_L(\alpha_1, \beta_1) S_L(\alpha_2, \beta_2) \cdots S_L(\alpha_N, \beta_N). \]

(27)

We perform this construction for the new \( \hat{Q}_R \) matrix. The local matrices \( \hat{S}_L \) are obtained from (22)

\[ \hat{S}_L(\alpha, \beta)_{k,l}(v) = \hat{S}_R(\beta, \alpha)_{k,l}(-v) \]

(28)

\[ \hat{S}_L(\alpha, \beta)_{k,l} = \delta_{k,l+1} \tau_{\alpha,l} u^\beta(v + t + 2k\eta) + \delta_{k,l+1} \tau_{\alpha,l} u^\beta(v - t - 2l\eta). \]

(29)

1.4. The relation \( Q_L(u) Q_R(v) = Q_L(v) Q_R(u) \)

To prove that the \( Q \)-matrix defined by

\[ Q(v) = QR(v) Q_R^{-1}(v_0) \]

(30)

commutes with the transfer matrix \( T \) Baxter shows in [1] that the relation

\[ Q_L(v) Q_R(u) = Q_L(u) Q_R(v) \]

(31)

holds. Then

\[ Q(v) = Q_L^{-1}(u) Q_L(v) = Q_R(v) Q_R^{-1}(u) \]

(32)

commutes with \( T(v) \). To prove (31) it is shown in [1] that \( S_L(\alpha, \gamma)_{m,n}(v) S_R(\gamma, \beta)_{m',n'}(v) \) and \( S_L(\alpha, \gamma)_{m,n}(u) S_R(\gamma, \beta)_{m',n'}(u) \) are related by a similarity transformation:

\[ S_L(\alpha, \gamma)_{m,n}(u) S_R(\gamma, \beta)_{m',n'}(v) = Y_{m,m';n,n'} S_L(\alpha, \gamma)_{m,n}(v) S_R(\gamma, \beta)_{m',n'}(u) \]

(33)

with diagonal matrix \( Y \),

\[ Y_{m,m';n,n'} = y_{m,n'} \delta_{m,k} \delta_{m',k'}. \]

(34)

To investigate whether the matrices \( \hat{Q}_R \) and \( \hat{Q}_L \), defined in (5), (22) and (27), (29) fulfill such a relation we define a series of abbreviations. According to (22) we write

\[ \hat{S}_R(\alpha, \beta)_{m,n} = \Phi^\alpha_{m,n} \eta^\beta_{m,n}, \]

(35)

where

\[ \Phi^\alpha_{m,n} = \epsilon^\alpha_{m,n} f^\alpha(u_{m,n}) \]

(36)

\[ u_{m,n} = \delta_{m-1,n}(v + t + 2m\eta) + \delta_{m+1,n}(v - t - 2m\eta) \]

(37)

\[ \epsilon^\alpha_{m,n} = \delta_{m-1,n} - \alpha \delta_{m+1,n} \]

(38)

(38)

\[ \eta^\beta_{m,n} = \delta_{m-1,n} \tau_{\beta,n} + \delta_{m+1,n} \tau_{\beta,-n} \]

(39)

and \( f^+(v) = H(v), f^-(v) = \Theta(v), \delta_{m+1,n} = \delta_{m,n} \).

Equivalently, we write following (29):

\[ \hat{S}_L(\alpha, \beta)_{m,n} = \tau^\alpha_{m,n} \chi^\beta_{m,n}, \]

(40)

where

\[ \chi^\beta_{m,n} = \lambda^\beta f^\beta(u_{m,n}) \]

(41)

\[ u_{m,n} = \delta_{m-1,n}(v - t - 2m\eta) + \delta_{m+1,n}(v + t + 2m\eta) \]

(42)
\[ \lambda_{m,n}^{\beta} = -\beta \delta_{m-1,n} + \delta_{m+1,n} \tag{43} \]
\[ \gamma_{m,n}^{\alpha} = \delta_{m-1,n} \gamma_{m,n}^\ast + \delta_{m+1,n} \gamma_{m,n}^{\ast m} \tag{44} \]

It follows then from (35) and (40)
\[ \hat{S}_L(\alpha, \gamma)_{m,n}(u) \hat{S}_R(\gamma, \beta)_{m',n'}(v) = \tau_{m,n}^{\omega} \chi_{m,n}^{\omega}(u) \Phi_{m',n'}^{\omega}(v) \tau_{m',n'}^{\beta} \tag{45} \]
and from (36) and (41) one obtains
\[ \chi_{m,n}^{\omega}(u) \Phi_{m',n'}^{\omega}(v) = (\delta_{m+1,n} \delta_{m'+1,n'} + \delta_{m-1,n} \delta_{m'-1,n'}) (\Theta(u,m,n) \Theta(v,m',n') - H(u,m,n) H(v,m',n')) + (\delta_{m+1,n} \delta_{m'-1,n'} + \delta_{m-1,n} \delta_{m'+1,n'}) (\Theta(u,m,n) \Theta(v,m',n') + H(u,m,n) H(v,m',n')) \tag{46} \]

with non-vanishing elements
\[ \chi_{m,m+1}^{\omega}(u) \Phi_{m',m'+1}^{\omega}(v) = \Theta(u,m,m+1) \Theta(v,m',m'+1) - H(u,m,m+1) H(v,m',m'+1) \tag{47} \]
\[ \chi_{m,m-1}^{\omega}(u) \Phi_{m',m'-1}^{\omega}(v) = \Theta(u,m,m-1) \Theta(v,m',m'-1) - H(u,m,m-1) H(v,m',m'-1) \tag{48} \]
\[ \chi_{m,m+1}^{\omega}(u) \Phi_{m',m'-1}^{\omega}(v) = \Theta(u,m,m+1) \Theta(v,m',m'-1) + H(u,m,m+1) H(v,m',m'-1) \tag{49} \]
\[ \chi_{m,m-1}^{\omega}(u) \Phi_{m',m'+1}^{\omega}(v) = \Theta(u,m,m-1) \Theta(v,m',m'+1) + H(u,m,m-1) H(v,m',m'+1). \tag{50} \]

The arguments are
\[ u_{m,m+1}(u) - v_{m',m'+1}(v) = u - v + 2(m + m') \eta + 2t \]
\[ u_{m,m-1}(u) - v_{m',m'-1}(v) = u - v - 2(n + n') \eta - 2t \]
\[ u_{m,m+1}(u) - v_{m',m'-1}(v) = u - v + 2(m - m') \eta \]
\[ u_{m,m-1}(u) - v_{m',m'+1}(v) = u - v - 2(m - m') \eta \]
\[ u_{m,m+1}(u) + v_{m',m'+1}(v) = u + v + 2(m - m') \eta \]
\[ u_{m,m-1}(u) + v_{m',m'-1}(v) = u + v + 2(-n + n') \eta \]
\[ u_{m,m+1}(u) + v_{m',m'-1}(v) = u + v + 2(m + n') \eta + 2t \]
\[ u_{m,m-1}(u) + v_{m',m'+1}(v) = u + v - 2(n + n') \eta - 2t. \]

To rewrite (47)–(50), we use
\[ \Theta(u) \Theta(v) + H(u) H(v) = cf_+(u + v) g_+(u - v) \tag{52} \]
\[ \Theta(u) \Theta(v) - H(u) H(v) = cf_-(u + v) g_-(u - v) \tag{53} \]
\[ f_+(u) = H((iK' + u)/2) H((iK' - u)/2) g_+(u) = H_1((iK' + u)/2) H_1((iK' - u)/2) \tag{54} \]
\[ f_-(u) = H_1((iK' + u)/2) H_1((iK' - u)/2) g_-(u) = H((iK' + u)/2) H((iK' - u)/2). \tag{55} \]

We need especially the following properties of \( g_{\pm} \):
\[ g_{\pm}(-u) = g_{\pm}(u) \quad g_{\pm}(u + 4K) = g_{\pm}(u). \tag{56} \]

After insertion of (52)–(55) into (47)–(50) we get
\[ \chi_{m,m+1}^{\omega}(u) \Phi_{m',m'+1}^{\omega}(v) = cf_-(u + v + 2(m - m') \eta) g_-(u - v + 2(m + m') \eta + 2t) \tag{57} \]
\[ \chi_{m,m-1}^{\omega}(u) \Phi_{m',m'-1}^{\omega}(v) = cf_-(u + v + 2(m' - m) \eta) g_-(u - v - 2(n + n') \eta - 2t) \tag{58} \]
of equations (64)–(65) is free from contradictions on the torus of size \( y^m \). It now remains to show that a matrix \( L^2 \times L^2 \) matrix \( Y \) exists such that equation (33) is satisfied for \( \hat{S}_R \) and \( \hat{S}_L \). As \( \tau \) and \( \tau' \) occurring in the definition of \( \hat{S}_R \) and \( \hat{S}_L \) are free parameters we obtain from (33)

\[
\chi^\gamma_{m,n}(u) \Phi^\gamma_{m',n'}(v) = y_{m,m'}^\gamma \chi^\gamma(v)_L \Phi^\gamma(u)_{L'} Y_{m,n}^{-1}_{L,L'}.
\]

(61)

Taking tentatively \( Y \) to be diagonal

\[
y_{m,m':k,k'} = y_{m,m'} \delta_{m,k} \delta_{m',k'}.
\]

(62)

we get

\[
\chi^\gamma_{m,n}(u) \Phi^\gamma_{m',n'}(v) = \frac{y_{m,m'} \chi^\gamma(v)_{L}}{y_{m,n} \chi^\gamma_{m',n'}(v)} (u)
\]

(63)

and it follows from (57)–(60)

\[
y_{m+1,m'+1} = y_{m,m'} \frac{g_-(u - v - 2(m + m')\eta - 2t)}{g_-(u - v + 2(m + m')\eta + 2t)}
\]

(64)

\[
y_{m+1,m'-1} = y_{m,m'} \frac{g_+(u - v - 2(m - m' + 1)\eta)}{g_+(u - v + 2(m - m' + 1)\eta)}.
\]

(65)

To prove that a matrix \( Y \) can be found such that (61) is satisfied we have to show that the set of equations (64)–(65) is free from contradictions on the torus of size \( L \times L \) where

\[
y_{m+L,n+L} = y_{m,n}.
\]

(66)

It follows from equation (64) that

\[
y_{m+L,n+L} = \frac{g_-(u - v - 2(m + n)\eta - 4(L - 1)\eta - 2t)}{g_-(u - v + 2(m + n)\eta + 4(L - 1)\eta + 2t)} \times \frac{g_-(u - v - 2(m + n)\eta - 4(L - 2)\eta - 2t)}{g_-(u - v + 2(m + n)\eta + 4(L - 2)\eta + 2t)} \ldots
\]

\[
\frac{g_-(u - v - 2(m + n)\eta - 2t)}{g_-(u - v + 2(m + n)\eta + 2t)} y_{m,n}.
\]

(67)

The factor \( g_-(u - v - 2(m + n)\eta + 4r_2\eta - 2t) \) in the numerator cancels the factor \( g_-(u - v + 2(m + n)\eta + 4r_1\eta + 2t) \) in the denominator if \( t = 0 \) and

\[
-2(m + n)\eta - 4r_2\eta = 2(m + n)\eta + 4r_1\eta + 4kK
\]

(68)

for arbitrary \( k \) and if we set \( k = 2m_1k_1 \) for integer \( k_1 \):

\[
r_2 = k_1 L - m - n - r_1.
\]

(69)

It follows that for each factor in the numerator of equation (67) there is a factor in the denominator against which it cancels. Similarly we derive from equation (65) that

\[
y_{m,n} = y_{m-L,n+L}.
\]

(70)

We have shown that all \( y_{m,n} \) can be determined from a single element (e.g. \( y_{1,1} \)) consistently if \( t = 0 \). This conclusion cannot be drawn for \( t \neq 0 \). A numerical test of (31) shows that it is not satisfied for \( t \neq 0 \), and therefore no similarity transformation (33) exists for \( t \neq 0 \).

We summarize what has been found in this section.
We have attained our goal to construct a \( Q \)-matrix which exists for \( \eta = 2mK/L \) for odd \( L \):

The \( \hat{Q}_R \)-matrix defined in equation (5) with local matrices \( \hat{S}_R \) defined in (22) is regular.

If the parameter \( t \) is set to zero relation (31) is satisfied.

Then \( \hat{Q}(v) = \hat{Q}_R(v) \hat{Q}^{-1}_R(v_0) \) satisfies equation (1) and commutes with the transfer matrix \( T \).

2. Quasiperiodicity properties of \( Q \)

It is easily seen that \( Q_{72,R}(v) \) and \( \hat{Q}_R(v, t) \) satisfy

\[
\hat{Q}_{72,R}(v + 2K) = S \hat{Q}_{72,R}(v) \quad \hat{Q}_R(v + 2K, t) = S \hat{Q}_R(v, t).
\]

This is of great importance to find the quasiperiodicity properties of the \( Q \)-matrices in the complex \( v \)-plane. We do that in this section for \( Q_{72,R}(v) \) and \( \hat{Q}_R(v, t = 0) \). It is well known that the quasiperiod of \( Q_{72} \) is \( iK' \). See [9] for details. We shall present plausibility arguments for the statement that \( Q_{72,R} \) as well as \( \hat{Q} \) are singular matrices if their quasiperiod is \( iK' \).

2.1. Quasiperiodicity properties of \( Q_{72} \)

We get from equations (15), (A.4), (A.5) and from

\[
\eta = m_1 K/L
\]

the relations

\[
S_R(\pm, \beta)_{k,k+1}(v + iK) = f(v) \exp(\pm i\pi\lambda v \partial K) S_R(\mp, \beta)(v)_{k,k+1}
\]

\[
S_R(\pm, \beta)_{k+1,k}(v + iK) = f(v) \exp(-i\pi\lambda v \partial K) S_R(\mp, \beta)(v)_{k+1,k}
\]

\[
S_R(\pm, \beta)_{1,1}(v + iK) = f(v) S_R(\mp, \beta)(v)_{1,1}
\]

\[
S_R(\pm, \beta)_{L,L}(v + iK) = (-1)^{m_1} f(v) S_R(\mp, \beta)(v)_{L,L},
\]

where

\[
f(v) = q^{-1/4} \exp\left(-\frac{i\pi v}{2K}\right).
\]

The similarity transformation

\[
S(\alpha, \beta)_{i,j} = A_{i,j} S(\alpha, \beta)_{j,k} A_{k,l}^{-1},
\]

with

\[
A_{i,j} = \delta_{i,j} \exp\left(\frac{i\pi}{2K}(k-1)\alpha v \partial K\right) a_1
\]

leads to

\[
\hat{S}_R(\pm, \beta)_{k,k+1}(v + iK) = f(v) \hat{S}_R(\mp, \beta)(v)_{k,k+1}
\]

\[
\hat{S}_R(\pm, \beta)_{k+1,k}(v + iK) = f(v) \hat{S}_R(\mp, \beta)(v)_{k+1,k}
\]

\[
\hat{S}_R(\pm, \beta)_{1,1}(v + iK) = f(v) \hat{S}_R(\mp, \beta)(v)_{1,1}
\]

\[
\hat{S}_R(\pm, \beta)_{L,L}(v + iK) = (-1)^{m_1} f(v) \hat{S}_R(\mp, \beta)(v)_{L,L}.
\]

If \( m_1 \) is even it follows that

\[
\hat{S}_R(\alpha, \beta)_{k,k}(v + i\gamma K) = f(v) R(\alpha, \gamma) S_R(\alpha, \gamma, \beta)(v)_{k,k}.
\]
where $R$ is defined in equation (4) and

$$Q_{R,72}(v + iK) = f(v)^N R Q_{R,72}(v).$$

(79)

However, it is well known [5] that for even $m_1$ $Q_{R,72}(v)$ is singular and consequently relation (79) cannot be upgraded from $Q_{R,72}$ to $Q_{72}$. It is shown in [5] that instead of (79) the following relation holds:

$$Q_{R,72}(v + 2iK) = q^{-N} \exp(-iN\pi v/K) Q_{R,72}(v),$$

(80)

which is correct for all $\eta = m_1 K/L$. Provided $m_1$ is odd it follows

$$Q_{R,72}(v + 2iK) = q^{-N} \exp(-iN\pi v/K) Q_{72}(v).$$

(81)

### 2.2. Quasiperiodicity properties of $\hat{Q}$

We obtain from equation (22)

$$\hat{S}_{R}(\pm, \beta)(v + iK)_{k,k+1} = f(v)(-i) \exp(+i\pi k\eta/K) \hat{S}_{R}(\mp, \beta)(v)_{k,k+1}$$

(82)

$$\hat{S}_{R}(\pm, \beta)(v + iK)_{k+1,k} = f(v)(+i) \exp(-i\pi k\eta/K) \hat{S}_{R}(\mp, \beta)(v)_{k,k+1}.$$  

(83)

Perform the similarity transformation

$$\hat{S}(\alpha, \beta)_{i,j} = A_{i,j} \hat{S}(\alpha, \beta)_{j,k} A_{k,l}^{-1},$$

(84)

with

$$A_{k,l} = \delta_{k,l}(-i)^{k-1} \exp\left(\frac{i\pi}{2K}(k - 1)k\eta\right) a_1.$$  

(85)

Then for $k < L$,

$$\hat{S}_{R}(\pm, \beta)(v + iK)_{k,k+1} = f(v) \hat{S}_{R}(\mp, \beta)(v)_{k,k+1}$$

(86)

$$\hat{S}_{R}(\pm, \beta)(v + iK)_{k+1,k} = f(v) \hat{S}_{R}(\mp, \beta)(v)_{k,k+1}.$$  

(87)

and for $k = L$,

$$\hat{S}_{R}(\pm, \beta)(v + iK)_{L,1} = f(v) \exp\left[\frac{+i\pi}{2}((2m_1 - 1)L + 2m_1)\right] \hat{S}_{R}(\mp, \beta)(v)_{L,1}$$

(88)

$$\hat{S}_{R}(\pm, \beta)(v + iK)_{1,L} = f(v) \exp\left[\frac{-i\pi}{2}((2m_1 - 1)L + 2m_1)\right] \hat{S}_{R}(\mp, \beta)(v)_{1,L}.$$  

(89)

We find that if

$$\exp\left[-\frac{i\pi}{2}((2m_1 - 1)L + 2m_1)\right] = 1,$$

(90)

$\hat{Q}_R$ satisfies the relation

$$\hat{Q}_R(v + iK) = q^{-N/4} \exp\left(-\frac{i\pi N v}{2K}\right) R \hat{Q}_R(v)$$

(91)

which is the same as (79) for $Q_{R,72}$. This happens only for

(I) even $m_1$ if $L = 4 \times \text{integer}$

(II) odd $m_1$ if $L = 2 \times \text{odd integer}$.
These are exactly those cases in which $\hat{Q}_R$ is singular. Like equation (79) equation (91) does not give the corresponding relation for the $Q$-matrix $\hat{Q}$.

In the following paragraph $Q$ denotes either $Q_{72}$ or $\hat{Q}$.

We note that if the relation

$$Q(v + iK) = q^{-N/4} \exp \left( -\frac{i\pi N v}{2K} \right) RQ(v)$$

(92)

were correct then it would follow that

$$q(v + iK)|q\rangle = q^{-N/4} \exp \left( -\frac{i\pi N v}{2K} \right) q(v)R|q\rangle$$

(93)

where $|q\rangle$ denotes an arbitrary eigenvector of $Q(v)$ and $q(v)$ is its eigenvalue.

In other words: all eigenvectors of $\hat{Q}(v)$ would be eigenvectors of $R$. It is however well known [5] that the eigenvectors of $Q_{72}(v)$ which are degenerate eigenvectors of the transfer matrix $T$ are generally not eigenvectors of $R$.

Equations (79) and (91) allow a coherent explanation of the fact that $QR$ is singular for one set of $\eta$ values and regular for another. Under the assumption that if $Q$ exists there are eigenstates of $Q$ which are not eigenstates of $R$, $QR$ cannot be regular if in case of $Q_{72}$, $m_1$ is even or in case of $\hat{Q}$ (90) is satisfied. This also explains naturally the observation that for fixed $L$ and sufficiently small $NQ^{-1}$ exists always as then all states are singlets and (79) and (91) do not lead to contradictions when upgraded from $Q_R$ to $Q$.

Using the method used in this section it can easily be shown that always

$$\hat{Q}_R(v + 2iK) = q^{-N} \exp(-iN\pi v/K) \hat{Q}_R(v)$$

(94)

and consequently if $\hat{Q}_R^{-1}$ exists

$$\hat{Q}(v + 2iK) = q^{-N} \exp(-iN\pi v/K) \hat{Q}(v)$$

(95)

3. The properties of $\hat{Q}$ for $t = 0$

It follows from (95) that as shown for $Q_{72}$ in [5], $\hat{Q}(v)$ may be written as

$$\hat{Q}(v) = \hat{K}(q; v_k) \exp(i(n_B - v)\pi v/2K) \prod_{j=1}^{n_B} h(v - v_j^B)$$

$$\times \prod_{j=1}^{n_L} H(v - i\omega_j) H(v - i\omega_j - 2\eta) \cdots H(v - i\omega_j - 2(L-1)\eta)$$

(96)

$$2n_B + Ln_L = N.$$ (97)

$n_B$ is the number of Bethe roots $v_B^B$ and $n_L$ is the number of exact $Q$-strings of length $L$. The sum rules (2.16)–(2.21) of [5] are also true for $\hat{Q}(v)$.

From (71) and

$$\hat{Q}_L(v) \hat{Q}_R(u) = \hat{Q}_L(u) \hat{Q}_R(v)$$

(98)

follows that

$$[S, \hat{Q}(v)] = 0.$$ (99)

Finally, we find numerically that the functional relation (3) which was originally conjectured in [5] is also satisfied for $\hat{Q}(v)$.
4. The matrix $\hat{Q}(v, t)$

We have shown that $\hat{Q}_R(v)$ satisfies the $T - \hat{Q}_R$ relation and found that it is not singular. But the proof of relation (31) failed for parameter $t \neq 0$. One finds numerically that (31) is in fact violated for systems large enough to allow degenerate eigenvalues of the transfer matrix. Therefore, the question arises whether $\hat{Q}_R(v, t)$ is useful at all. Surprisingly, we find numerically that for $\eta = 2mK/L$ and odd $L$

$$\hat{Q}_L^{-1}(v_0, t) \hat{Q}_L(v, t) \neq \hat{Q}_R(v, t) \hat{Q}_R^{-1}(v_0, t)$$

(100)

both matrices

$$\hat{Q}^{(L)}(v, t) = \hat{Q}_L^{-1}(v_0, t) \hat{Q}_L(v, t) \quad \text{and} \quad \hat{Q}^{(R)}(v, t) = \hat{Q}_R(v, t) \hat{Q}_R^{-1}(v_0, t)$$

(101)

commute with the transfer matrix $T$. Furthermore, we find that in the cases studied $\hat{Q}^{(L)}(v, t)$ and $\hat{Q}^{(R)}(v, t)$ have the same eigenvalues. This means that there exists a matrix $A$ with

$$\hat{Q}^{(L)}(v, t) = A \hat{Q}^{(R)}(v, t) A^{-1}$$

and consequently instead of (31)

$$\hat{Q}_L(v) A \hat{Q}_R(u) = \hat{Q}_L(u) A \hat{Q}_R(v)$$

(102)

should hold. A consequence of (100) is that

$$[\hat{Q}^{(R)}(v_1, t), \hat{Q}^{(R)}(v_2, t)] \neq 0$$

(103)

as one needs (31) to prove that $Q$-matrices with different arguments commute (see 9.48.41 in [9]). We find that like $Q_{72}$ the matrix $\hat{Q}$ does not commute with $R$:

$$[R, \hat{Q}(v, t)] \neq 0$$

(104)

but unlike $Q_{72}$ as a consequence of (100) does also not commute with $S$ for $t \neq 0$:

$$[S, \hat{Q}(v, t)] \neq 0$$

(105)

This is possible because the degenerate subspaces of $T$ have elements with both eigenvalues $\nu' = 0, 1$ of $S$ if $\eta = 2mK/L$ and $L$ is odd.

These properties of $\hat{Q}^{(L)}(v, t)$ and $\hat{Q}^{(R)}(v, t)$ imply that they act as non-Abelian symmetry operators in all degenerate subspaces of the set of eigenvectors of $T$. We finally mention that whereas the six-vertex limit of $\hat{Q}_{R,72}$ does not exist it exists for $\hat{Q}_R$. The limit of $\hat{Q}_R(v, t = iK'/2)$ for elliptic nome $q \to 0$ is well defined. Using

$$\lim_{q \to 0} H(u \pm iK'/2) = \exp(\mp i(u - \pi/2)) \lim_{q \to 0} \Theta(u \pm iK'/2) = 1,$$

(106)

one gets a regular limiting $\hat{Q}_R$-matrix. It has been checked numerically that the resulting $\hat{Q}$-matrix commutes with $T_{6v}$.

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**Appendix**

The transfer matrix of the eight-vertex model is

$$T(v)_{\mu,v} = \text{Tr} \ W_8(\mu_1, v_1) W_8(\mu_2, v_2) \cdots W_8(\mu_N, v_N),$$

(A.1)
where in the conventions of (6.2) of [1]
\[ \begin{align*}
W_8(1, 1)_{\uparrow, \uparrow} &= W_8(-1, -1)_{\uparrow, \uparrow} = a = \rho \Theta(2\eta) \Theta(v - \eta) H(v + \eta) \\
W_8(-1, -1)_{\uparrow, \uparrow} &= W_8(1, 1)_{\uparrow, \uparrow} = b = \rho \Theta(2\eta) H(v - \eta) \Theta(v + \eta) \\
W_8(-1, -1)_{\uparrow, \downarrow} &= W_8(1, -1)_{\uparrow, \downarrow} = c = \rho H(2\eta) \Theta(v - \eta) \Theta(v + \eta) \\
W_8(1, -1)_{\uparrow, \downarrow} &= W_8(-1, 1)_{\uparrow, \downarrow} = d = \rho H(2\eta) H(v - \eta) H(v + \eta).
\end{align*} \]

\textbf{A.2)}

Relations used in the text. See e.g. [10]

\[ \begin{align*}
\text{sn}(u - v) &= \frac{\text{sn}(u) \text{cn}(v) \text{dn}(v) - \text{sn}(v) \text{cn}(u) \text{dn}(u)}{1 - k^2 \text{sn}^2(u) \text{sn}^2(v)} \\
H(v + iK') &= iq^{-1/4} \exp\left(\frac{-i\pi v}{2K}\right) \Theta(v) \\
\Theta(v + iK') &= iq^{-1/4} \exp\left(\frac{-i\pi v}{2K}\right) H(v).
\end{align*} \]

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