Field Theoretic Calculation of the Universal Amplitude Ratio of Correlation Lengths in 3D-IIsing Systems

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Abstract

In three-dimensional systems of the Ising universality class the ratio of correlation length amplitudes for the high- and low-temperature phases is a universal quantity. Its field theoretic determination apart from the $\epsilon$-expansion represents a gap in the existing literature. In this article we present a method, which allows to calculate this ratio by renormalized perturbation theory in the phases with unbroken and broken symmetry of a one-component $\phi^4$-theory in fixed dimensions $D = 3$. The results can be expressed as power series in the renormalized coupling constant of either of the two phases, and with the knowledge of their fixed point values numerical estimates are obtainable. These are given for the case of a two-loop calculation.

1 Introduction

The application of field theoretic methods to critical phenomena and second order phase transitions is one of the main tools for their quantitative theoretical investigation. Based on general theoretical arguments it is believed that physical systems, which undergo a second order phase transition, fall into distinct universality classes characterized by the dimensionality $D$ of space, the number $n$ of components of the order parameter and the underlying symmetry group. This manifests itself in the fact that within these universality classes many interesting quantities such as critical exponents and amplitude ratios have the same values [1, 2].
As a consequence of the long range nature of the effective interaction (for example between the spin variables of a ferromagnetic system showing a collective behaviour in the vicinity of the critical temperature $T_c$) the microscopic structure of the given system can be neglected by introducing a real valued order parameter field $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$, $x \in \mathbb{R}^D$, which is assumed to describe the phase transition. In this way one is led to a field theory with $\phi^4$-self-interaction with spontaneous breaking of its $O(n)$-symmetry, which is one of the standard topics in quantum field theory.

In the past many perturbative calculations have been made to estimate the values for the above mentioned critical quantities. To this end the $\epsilon$-expansion, initiated by M.E. Fisher and K.G. Wilson [3] and subsequently elaborated by E. Brézin, J.C. Le Guillou, J. Zinn-Justin [4] and others for the massless $\phi^4$-theory in $D = 4 - \epsilon$ dimensions, has been applied and yielded very good results.

A quantity whose numerical value is important in comparing various experimentally determined universal quantities with theoretical predictions (see e.g. [5]) is the universal ratio of correlation length amplitudes $f_+/f_-$. It is defined by the behaviour of the correlation length $\xi$ as a function of the temperature $T$ near $T = T_c$ through

$$\xi \sim \begin{cases} f_+ t^{-\nu}, & t > 0 \\ f_-(\,\,\,-t\,\,\,-)\,\,-\nu, & t < 0 \end{cases}, \quad t := \frac{T - T_c}{T_c},$$

(1)

where the symbol $\sim$ denotes the critical behaviour as defined in the usual way [4, 2]. The $\epsilon$-expansion of $f_+/f_-$ is known up to second order for the second-moment correlation length and up to first order for the ‘true’ correlation length [3, 4], which limits the accuracy of numerical estimates.

In this article we follow an idea owing to G. Parisi [7], who suggested the use of renormalized massive perturbation theory in fixed dimensions $D = 3$. For the symmetric phase there exist extensive calculations by G.A. Baker, B.G. Nickel, D.I. Meiron et al. [8, 9, 10], who give the renormalization group functions and the critical exponents respectively up to six- and seven-loop order. Later C. Bagnuls, C. Bervillier et al. [11, 12] have developed a renormalization scheme, which among others makes it possible to determine the amplitude ratios of the susceptibility and the specific heat from these series. In general the quality of results from this method appears to be better than from the $\epsilon$-expansion.

Up to now an extension of this method to the universal amplitude ratio $f_+/f_-$ is not found in the literature since this requires explicit calculations in the phase of broken symmetry, too.

In order to fill this gap we consider Euclidean $\phi^4$-theory with $D = 3$ and $n = 1$, which is a super-renormalizable field theory, that is believed to lie in the same universality class as the three-dimensional Ising model. Its Lagrangian density in the symmetric phase ($t > 0$) is given by

$$\mathcal{L}(\phi_0) = \frac{1}{2} (\partial \phi_0(x))^2 + V(\phi_0)$$
\[ V(\phi_0^+) = \frac{1}{2} m^2_{0^+} \phi^2_{0^+}(x) + \frac{1}{4!} g_0 \phi^4_{0^+}(x) \] (2)

with \( g_0 > 0 \) and \( m^2_{0^+} > 0 \). The quantities in different phases are distinguished by indices + and −, if necessary. In the broken phase \((t < 0)\) the \( Z_2\)-symmetry \( \phi \to -\phi \) is lost for the Lagrangian

\[
L(\phi) = \frac{1}{2} (\partial \phi(x))^2 + V(\phi) \\
V(\phi) = -\frac{1}{4} m^2_{0^-} \phi^2(x) + \frac{1}{4!} g_0 \phi^4(x) + \frac{3}{8} \frac{m^4_{0^-}}{g_0} = \frac{1}{4!} g_0 (\phi^2(x) - v^2_0)^2
\] (3)

with \( m^2_{0^-} > 0 \) and the classical potential minima at \( \phi = \pm v_0 := \pm \sqrt{3m^2_{0^-}/g_0} \). After an expansion of the potential around the positive minimum, which is equivalent to a shift in the field variable via

\[
\phi_0^-(x) := \phi(x) - v_0,
\] (4)

we arrive at

\[
L(\phi_0^-) = \frac{1}{2} (\partial \phi_0^-(x))^2 + V(\phi_0^-) \\
V(\phi_0^-) = \frac{1}{2} m^2_{0^-} \phi^2_{0^-}(x) + \frac{1}{3!} \sqrt{3g_0 m_{0^-} \phi^3_{0^-}(x)} + \frac{1}{4!} g_0 \phi^4_{0^-}(x).
\] (5)

These Lagrangian densities are the starting points of our perturbative calculations in both phases. It should be noted that the additional \( \phi^3\)-self-interaction of the field \( \phi_0^- \) in the broken phase gives rise to tadpole-diagrams as well as to a non-vanishing one-point function, which is the vacuum expectation value of the field \( \phi_0^- \).

The Feynman rules in momentum space follow from the Lagrangians (2) and (3) as usual. In particular each four-vertex is associated with a factor \(-g_0\), each three-vertex gets a factor \(-\sqrt{3g_0 m_{0^-}}\) and the Feynman propagator is

\[
\tilde{\Delta}(k) := \frac{1}{k^2 + m^2_{0^-}}.
\] (6)

The divergences of the theory, appearing in the form of the only two primitively divergent graphs of the two-point function, are isolated by dimensional regularization.

## 2 Renormalized perturbation theory

Before introducing the renormalization schemes for the phases with unbroken and broken symmetry let us briefly outline our further strategy.

To begin with we define the correlation length for general \( D \) as the second moment

\[
\xi^2 := \frac{1}{2D} \left[ \frac{\int d^D x x^2 G_c^{(2,0)}(x)}{\int d^D x G_c^{(2,0)}(x)} \right] = \frac{\partial}{\partial p^2} \frac{G_c^{(2,0)}(p)}{G_c^{(2,0)}(p)} \bigg|_{p^2 = 0}.
\] (7)
of the connected two-point correlation function

\[ G_c^{(2,0)}(x) := \langle \phi_0(x) \phi_0(y) \rangle - \langle \phi_0(x) \rangle \langle \phi_0(y) \rangle. \] (8)

Because of its relation

\[ - \Gamma_0^{(2,0)}(p) = \left( G_c^{(2,0)}(p) \right)^{-1} \] (9)

to the two-point vertex function \( \Gamma_0^{(2,0)} \) in momentum space one verifies the following identity between the correlation length and the renormalized mass \( m_R \):

\[ m_R^2 := \frac{\Gamma_0^{(2,0)}(p)}{\partial^2 \Gamma_0^{(2,0)}(p) \bigg|_{p^2=0}} = \frac{1}{\xi^2}. \] (10)

As \( \Gamma_0^{(2,0)}(p) \) is given by

\[ - \Gamma_0^{(2,0)}(p) = \tilde{\Delta}^{-1}(p) - \Sigma(p), \] (11)

where \( \Sigma(p) \) is the sum of all one-particle irreducible two-point graphs with amputated external legs, equation (10) shows how \( \xi \) can be determined perturbatively. Alternatively one defines the ‘true’ correlation length by the exponential decay of the two-point function. Then \( \xi \) equals the inverse physical mass (see e.g. [13]).

From renormalization group theory the critical behaviour of the model is known to be controlled by the dimensionless renormalized coupling constant \( u_R \) and its non-trivial, infrared-stable fixed point \( u^*_R \). The calculation of renormalized quantities sketched in this section enables us to find analytic functions in \( u_R \), in terms of which the desired ratio \( f_+/f_- \) can be expressed, as will be shown in the next section.

The renormalization scheme is established in terms of the bare \((n,l)\)-vertex functions \( \Gamma_0^{(n,l)}(\{p, q\}; m_0, g_0) \), with \( \{p, q\} = \{p_1, \ldots, p_n; q_1, \ldots, q_l\} \). They emerge from the expectation values \( \langle \phi_0(x_1) \cdots \phi_0(x_n) \frac{1}{2} \phi_0^2(y_1) \cdots \frac{1}{2} \phi_0^2(y_l) \rangle_c \) after Legendre and Fourier transformation. The renormalized mass (14) is written as

\[ m_R^2 = -Z_3 \Gamma_0^{(2,0)}(0; m_0, g_0), \quad \frac{1}{Z_3} := -\frac{\partial \Gamma_0^{(2,0)}(p; m_0, g_0)}{\partial p^2} \bigg|_{p^2=0}, \] (12)

and in addition we define

\[ \frac{1}{Z_2} := -\Gamma_0^{(2,1)}(\{0, 0\}; m_0, g_0) = -\frac{\partial \Gamma_0^{(2,0)}(0; m_0, g_0)}{\partial m_0^2}. \] (13)

The renormalization constants \( Z_3 \) and \( Z_2 \) play a prominent rôle in the theory because they stand in close connection to renormalization group functions and critical exponents. For the renormalized coupling we use different settings in each phase.

**Symmetric phase**

As usual [1, 13] we define a renormalized coupling constant \( g_R^{(4)} \) by the value of the four-point function for vanishing external momenta:

\[ g_R^{(4)} := -Z_3^2 \Gamma_0^{(4,0)}(\{0\}; m_0, g_0). \] (14)
Together with (12) one can invert the relations between \( m_R, g_R^{(4)} \) on the one hand and \( m_0, g_0 \) on the other hand so that the (in \( D = 3 \)) dimensionless renormalized coupling

\[
u_R := \frac{g_R^{(4)}}{m_R}
\]

becomes the natural expansion variable of our perturbation series. Moreover with

\[
\frac{1}{Z_1} := -\frac{1}{g_0} \Gamma_0^{(4,0)}(\{0\}; m_0, g_0)
\]

and (14) we define for later purposes

\[
u := \frac{g_0}{m_R} = \frac{Z_1(u_R)}{Z_3(u_R)}
\]

where it is indicated that the dimensionless coupling \( u \) is to be read as a function of \( u_R \).

To summarize, the renormalization is fixed by the conditions

\[
\begin{align*}
\Gamma_R^{(2,0)}(0; m_R, u_R) &= -m_R^2 \\
\left. \frac{\partial}{\partial p^2} \Gamma_R^{(2,0)}(p; m_R, u_R) \right|_{p^2=0} &= -1 \\
\Gamma_R^{(4,0)}(\{0\}; m_R, u_R) &= -m_R^{4-D} u_R = -g_R^{(4)} \\
\Gamma_R^{(2,1)}(\{0; 0\}; m_R, u_R) &= -1
\end{align*}
\]

for the renormalized \( (n, l) \)-vertex functions

\[
\Gamma_R^{(n,l)}(\{p; q\}; m_R, u_R) = [Z_3(u_R)]^{\frac{n}{2} - l} [Z_2(u_R)]^l \Gamma_0^{(n,l)}(\{p; q\}; m_0, g_0).
\]

**Broken symmetry phase**

In the phase of broken symmetry we follow [14, 5] and define the renormalized coupling constant \( g_R \) by the vacuum expectation value \( v \) of the field \( \phi \) in (3). If \( G_c^{(1,0)} \) stands for the non-vanishing one-point function of the field \( \phi_0 \) in (3) one has

\[
v = v_0 + G_c^{(1,0)}; \quad v_0 = \sqrt{3m_0^2/g_0}; \quad v_R := \frac{1}{\sqrt{Z_3}} v
\]

with a renormalized vacuum expectation value \( v_R \). We set

\[
g_R := \frac{3m_R^2}{v_R^2},
\]

and form a (in \( D = 3 \)) dimensionless renormalized coupling according to

\[
u_R := \frac{g_R}{m_R}.
\]
Proceeding as in the previous subsection we find with

\[ Z_4 := \frac{m_0}{m_R} \left(1 + \sqrt{\frac{g_0}{3m_0^2}v_0}\right) \] (23)

and (21) an expression for the dimensionless quantity

\[ u := \frac{g_0}{m_R} = u_R \frac{Z_4(u_R)}{Z_3(u_R)} \] (24)

in terms of renormalization constants, which again are assumed to be (analytic) functions of \( u_R \) alone.

The renormalization conditions fixing this scheme are:

\[ \Gamma_R^{(2,0)}(0; m_R, u_R) = -m_R^2 \] (25a)

\[ \frac{\partial}{\partial p^2} \Gamma_R^{(2,0)}(p; m_R, u_R) \bigg|_{p^2 = 0} = -1 \] (25b)

\[ \frac{3m_R^2}{u_R^2} = m_R^{4-D} u_R = g_R \] (25c)

\[ \Gamma_R^{(2,1)}(\{0; 0\}; m_R, u_R) = -1. \] (25d)

3 Determination of \( f_+/f_- \)

Now we derive an expression for the amplitude ratio \( f_+/f_- \) in terms of analytic and dimensionless functions calculable by renormalized perturbation theory.

First of all we have to realize the temperature dependence in the parameters of this theory. At first sight the occurrence of spontaneous symmetry breaking in the Lagrangian (3) suggests \( t \propto m_0^2 \), where according to (2) and (3)

\[ m_0^2 + = m_0^2 \] (26a)

\[ m_0^2 - = -2m_0^2. \] (26b)

But when including the perturbative corrections one observes a mass shift by an amount \( m_{0c}^2 \) in the temperature variable, which is perturbatively not calculable \([11, 12]\). (This corresponds to the fact that for \( D < 4 \) mean field theory is no longer valid.) Distinguishing the different phases we have

\[ \frac{1}{m_R^+} = \xi_+ \sim f_+ t_+^{-\nu}, \quad t_+ = (m_0^2 - m_{0c}^2)_{T>T_c} > 0 \] (27a)

\[ \frac{1}{m_R^-} = \xi_- \sim f_- t_-^{-\nu}, \quad t_- = -(m_0^2 - m_{0c}^2)_{T<T_c} > 0 \] (27b)

with \( t_+ = t \) for \( t > 0 \) and \( t_- = -t \) for \( t < 0 \).
In order to circumvent this problem we eliminate the bare mass \( m_0 \) by introducing the functions

\[
F_{\pm}(u_{R\pm}) := \left. \frac{\partial m_{R\pm}^2}{\partial m_{0\pm}^2} \right|_{g_0},
\]

which are partial derivatives with respect to \( m_0^2 \) at fixed \( g_0 \) and depend in renormalized perturbation theory on \( u_{R\pm} \) only. With the identity

\[
\left. \frac{\partial}{\partial t} \right|_{g_0} = \left. \frac{\partial}{\partial m_0^2} \right|_{g_0}
\]

one finds

\[
\begin{align*}
\left. \frac{\partial m_{R+}^2}{\partial t_+} \right|_{g_0} &= \left. \frac{\partial m_{R+}^2}{\partial m_0^2} \right|_{g_0}, \quad \left. \frac{\partial m_{R-}^2}{\partial t_-} \right|_{g_0} = 2 \left. \frac{\partial m_{R-}^2}{\partial m_0^2} \right|_{g_0} = 2F_-(u_{R-}). \tag{30a}
\end{align*}
\]

Then the differentiation of (27a) and (27b) yields

\[2 \frac{F_- (u_{R-})}{F_+ (u_{R+})} = \left. \frac{\partial m_{R+}^2}{\partial m_{R+}^2 / \partial t_+} \right|_{g_0} \sim \left( \frac{f_+}{f_-} \right)^2 \left( \frac{t_-}{t_+} \right)^{2\nu-1}. \tag{31}\]

On the other hand the definition of the correlation length as the inverse renormalized mass implies

\[\left( \frac{f_+}{f_-} \right)^2 \left( \frac{t_-}{t_+} \right)^{2\nu} \sim \left( \frac{m_{R-}}{m_{R+}} \right)^2, \tag{32}\]

and one concludes

\[2 \frac{F_- (u_{R-})}{F_+ (u_{R+})} \sim \left( \frac{m_{R-}}{m_{R+}} \right)^2 \frac{t_+}{t_-}. \tag{33}\]

So far \( t_+ \) and \( t_- \), as well as \( m_{R+} \) and \( m_{R-} \), are independent of each other. In order to specify the approach to the critical point from both sides we choose pairs of points \( (t_+, t_-) \) in the phase diagram such that

\[m_{R+} = m_{R-}. \tag{34}\]

This leads to a (yet unknown) dependence between \( u_{R+} \) and \( u_{R-} \). Consequently (27a), (27b) and (33) read

\[f_+ t_+^{-\nu} \sim f_- t_-^{-\nu}, \quad \frac{t_+}{t_-} \sim \frac{2}{F_+ (u_{R+})} \frac{F_- (u_{R-})}{F_+ (u_{R+})}, \tag{35}\]

and we combine these equations into the formula

\[f_+ \sim \left[ \frac{2}{F_+ (u_{R+})} \right]^{\nu} F_- (u_{R-}) \tag{36}.\]

The crucial point is now that one can express (36) as a function of a single dimensionless renormalized coupling constant \( \bar{u}_R \), which is related to \( u_{R\pm} \) in a definite way.
As mentioned earlier, the critical theory is characterized by the non-trivial fixed point $u^*_R$ of $u_R$. This is equal to the non-vanishing zero of the renormalization group $\beta$-function

$$\beta(u_R) := m_R \frac{\partial}{\partial m_R} \bigg|_{g_0} u_R = - \left( \frac{\partial}{\partial u_R} \bigg|_{m_R} \ln(u) \right)^{-1}, \quad D = 3, \quad (37)$$

according to $u = u(u_R) = g_0/m_R$. Thus, if we are looking for a new coupling $\bar{u}_R$, which is to be used as a renormalized coupling in both phases, we have to ensure that this coupling leads to the same $\beta$-function in both phases. Consequently the functional relation between $m_{R+}$ and $\bar{u}_R$ in the symmetric phase must be the same as the one between $m_{R-}$ and $\bar{u}_R$ in the phase of broken symmetry. In view of (37) one can comply with this condition by using the relations (17) and (24), which represent perturbative expansions of $u_\pm$ in terms of $u_{R\pm}$:

\begin{align}
  u_- &= h_-(u_{R-}) \quad (38a) \\
  u_+ &= h_+(u_{R+}) \quad (38b)
\end{align}

One possibility is to identify $\bar{u}_R$ with $u_{R+}$ in the symmetric phase and to define it in the phase with broken symmetry by means of

$$u_- = h_+(\bar{u}_R). \quad (39)$$

The second possibility considered in this work is to identify $\bar{u}_R$ with $u_{R-}$ in the phase with broken symmetry and to define it in the symmetric phase by means of

$$u_+ = h_-(\bar{u}_R). \quad (40)$$

In both cases the $\beta$-functions relevant for the high- and low-temperature phases

$$\bar{\beta}_\pm(\bar{u}_R) = - \left[ \frac{\partial}{\partial \bar{u}_R} \bigg|_{m_{R\pm}} \ln \left( u_\pm(\bar{u}_R) \right) \right]^{-1} \quad (41)$$

are equal to each other,

$$\bar{\beta}_+(\bar{u}_R) = \bar{\beta}_-(\bar{u}_R). \quad (42)$$

In the first case their common fixed point value is

$$\bar{u}^*_R = u^*_R, \quad (43)$$

for the second choice it is

$$\bar{u}^*_R = u^*_{R-}. \quad (44)$$

With the definition

$$\Phi(\bar{u}_R) := \frac{F_-(u_{R-}(\bar{u}_R))}{F_+(u_{R+}(\bar{u}_R))}, \quad (45)$$

we can evaluate (36) at the critical point,

$$\frac{f_+}{f_-} = \left[ 2 \Phi(\bar{u}^*_R) \right]^{\nu}, \quad (46)$$

to get numerical values for the amplitude ratio $f_+/f_-$ up to a given order in perturbation theory.
4 Results

We have calculated the renormalization functions necessary for the procedure outlined in the previous section in perturbation theory up to the order of two loops. In the symmetric phase 7 massive Feynman graphs and in the phase with broken symmetry 24 additional graphs have been evaluated analytically in $D = 3$ dimensions with dimensional regularization. We omit the details of our calculations and present only the main results.

In the symmetric phase we obtain the bare expansions

$$ m_{R+}^2 = m_{0+}^2 \left\{ 1 - \frac{1}{8\pi} u_{0+} + \frac{1}{64\pi^2} u_{0+}^2 \left[ \frac{79}{162} - \frac{1}{3} B_{+}^{(\text{div})} \right] + O(u_{0+}^3) \right\} \tag{47a} $$

$$ g_R^{(4)} = g_0 \left\{ 1 - \frac{3}{16\pi} u_{0+} + \frac{5}{162\pi^2} u_{0+}^2 + O(u_{0+}^3) \right\} \tag{47b} $$

$$ u_{R+} = u_{0+} \left\{ 1 - \frac{1}{8\pi} u_{0+} + \frac{1}{64\pi^2} u_{0+}^2 \left[ \frac{293}{216} + \frac{1}{6} B_{+}^{(\text{div})} \right] + O(u_{0+}^3) \right\} \tag{47c} $$

which are inverted to

$$ u_{0+} = u_{R+} \left\{ 1 + \frac{1}{8\pi} u_{R+} + \frac{1}{64\pi^2} u_{R+}^2 \left[ \frac{139}{216} - \frac{1}{6} B_{R+}^{(\text{div})} \right] + O(u_{R+}^3) \right\} \tag{48a} $$

$$ m_{0+}^2 = m_{R+}^2 \left\{ 1 + \frac{3}{64\pi} u_{R+} + \frac{1}{64\pi^2} u_{R+}^2 \left[ \frac{19525}{5184} + \frac{2}{3} B_{R+}^{(\text{div})} \right] + O(u_{R+}^3) \right\} \tag{48b} $$

$$ u_{+} = u_{R+} \left\{ 1 + \frac{3}{16\pi} u_{R+} + \frac{575}{20736\pi^2} u_{R+}^2 + O(u_{R+}^3) \right\} \tag{48c} $$

In the broken symmetry phase one has

$$ m_{R-}^2 = m_{0-}^2 \left\{ 1 + \frac{3}{64\pi} u_{0-} + \frac{1}{64\pi^2} u_{0-}^2 \left[ \frac{19525}{5184} - \frac{2}{3} B_{-}^{(\text{div})} \right] + O(u_{0-}^3) \right\} \tag{49a} $$

$$ g_R = g_0 \left\{ 1 - \frac{7}{32\pi} u_{0-} + \frac{37835}{331776\pi^2} u_{0-}^2 + O(u_{0-}^3) \right\} \tag{49b} $$

$$ u_{R-} = u_{0-} \left\{ 1 - \frac{31}{128\pi} u_{0-} + \frac{1}{64\pi^2} u_{0-}^2 \left[ \frac{80125}{13824} - \frac{1}{3} B_{-}^{(\text{div})} \right] + O(u_{0-}^3) \right\} \tag{49c} $$

and

$$ u_{0-} = u_{R-} \left\{ 1 + \frac{31}{128\pi} u_{R-} + \frac{1}{64\pi^2} u_{R-}^2 \left[ \frac{23663}{13824} + \frac{1}{3} B_{R-}^{(\text{div})} \right] + O(u_{R-}^3) \right\} \tag{50a} $$

$$ m_{0-}^2 = m_{R-}^2 \left\{ 1 + \frac{3}{64\pi} u_{R-} + \frac{1}{64\pi^2} u_{R-}^2 \left[ \frac{45125}{10368} + \frac{2}{3} B_{R-}^{(\text{div})} \right] + O(u_{R-}^3) \right\} \tag{50b} $$

$$ u_{-} = u_{R-} \left\{ 1 + \frac{7}{32\pi} u_{R-} + \frac{2191}{165888\pi^2} u_{R-}^2 + O(u_{R-}^3) \right\} \tag{50c} $$

The divergent pieces

$$ B_{\pm}^{(\text{div})} := \frac{1}{\epsilon} - \ln \left( \frac{m_{0\pm}^2}{4\pi} \right) - \gamma + \frac{\text{const.}}{4\pi} + O(\epsilon) \tag{51a} $$

$$ B_{R\pm}^{(\text{div})} := \frac{1}{\epsilon} - \ln \left( \frac{m_{R\pm}^2}{4\pi} \right) - \gamma + \frac{\text{const.}}{4\pi} + O(\epsilon) \tag{51b} $$
vanish after differentiation with respect to $m_{0\pm}^2$. So we get from (28) and the above equations the finite functions

$$F_+(u_{R+}) = 1 - \frac{1}{16\pi} u_{R+} - \frac{1}{384\pi^2} u_{R+}^2 + O(u_{R+}^3)$$  \hspace{1cm} (52)

$$F_-(u_{R-}) = 1 + \frac{3}{128\pi} u_{R-} - \frac{233}{49152\pi^2} u_{R-}^2 + O(u_{R-}^3).$$  \hspace{1cm} (53)

As discussed in the previous section the new coupling constant $\bar{u}_R$ is to be introduced now. We consider both choices explained above.

1. Expansion in $u_{R+}$:

We determine $u_{R-}$ as a function of $\bar{u}_R$ with the help of equation (39). From (48c) and (50c) the result is

$$u_{R-} = \bar{u}_R \left( 1 - \frac{1}{32\pi} \bar{u}_R + \frac{9059}{165888\pi^2} \bar{u}_R^2 + O(\bar{u}_R^3) \right).$$  \hspace{1cm} (54)

Since $\bar{u}_R = u_{R+}^*$ we identify $\bar{u}_R$ with $u_{R+}$. Then

$$\Phi_+(u_{R+}) = 1 + \frac{11}{128\pi} u_{R+} + \frac{41}{16384\pi^2} u_{R+}^2 + O(u_{R+}^3).$$  \hspace{1cm} (55)

2. Expansion in $u_{R-}$:

We adjust $\bar{u}_R$ to the low temperature coupling $u_{R-}$. The equations (10), (48c) and (50c) show that in this case $u_{R+}$ depends on $\bar{u}_R$ via

$$u_{R+} = \bar{u}_R \left( 1 + \frac{1}{32\pi} \bar{u}_R - \frac{8735}{165888\pi^2} \bar{u}_R^2 + O(\bar{u}_R^3) \right),$$  \hspace{1cm} (56)

and $\bar{u}_R^* = u_{R-}^*$ allows the identification $\bar{u}_R = u_{R-}$. Consequently

$$\Phi_-(u_{R-}) = 1 + \frac{11}{128\pi} u_{R-} + \frac{85}{16384\pi^2} u_{R-}^2 + O(u_{R-}^3).$$  \hspace{1cm} (57)

In order to get numerical results for the amplitude ratio $f_+/f_-$ according to (16) we need the values of $u_{R+}^*$ and $\nu$. The high temperature fixed point $u_{R+}^*$ and the index $\nu$ have been calculated in the framework of renormalized perturbation theory in $D = 3$ dimensions in [15, 11]. Therefore the whole analysis only requires the present methods. To improve the quality of the results we prefer, however, to make use of additional information available about $u_{R+}^*$ and $\nu$. For the exponent $\nu$ there are recent Monte Carlo result in [16]. The fixed points $u_{R+}^*$ have been estimated from the longest presently available high- and low-temperature series in [17], and for $u_{R-}^*$ we also use an estimate cited in [3]. To estimate the sensitivity of $f_+/f_-$ to these parameters we list the values resulting from different choices in tables 1 and 2.

The functions $\Phi_\pm(u_{R+})$, given in equations (55) and (57), have been evaluated at the fixed points with Padé approximants. By this we do not mean that the perturbative
expansion is analytic, which it is presumably not, but just take the different results as a measure for the error due to the shortness of the series. The application of Padé-Borel methods, which we also tried, does not yield any improvement.

We summarize the contents of the tables by extracting mean values of \( f_+ / f_- \) for the high and low temperature fixed points:

\[
\left( \frac{f_+}{f_-} \right)_{ht} = 2.18(12), \quad \left( \frac{f_+}{f_-} \right)_{lt} = 2.03(4). \tag{58}
\]

The number given in brackets is the maximal variation owing to the inputs of \( u_{R^+}^*, \nu \) and the different Padé approximants.

The fact that the variation between the different Padé approximations in table 2 is much smaller than in table 1 indicates that the low temperature renormalized coupling constant \( u_{R^-} \) is the better expansion variable for our series. An examination of (55) and (57) confirms this impression since the first and second order perturbative corrections to the leading terms in the coupling \( u_{R^-} \) are smaller than those in the high temperature coupling \( u_{R^+} \) (41% and 8% against 66% and 9%). Therefore we tend to have more faith in the estimate resulting from the expansion in \( u_{R^-} \).

For the sake of completeness we add a remark about the perturbative determination of the critical exponent \( \nu \). It is related to the renormalization group functions \( \eta_i \), for
\[ i = 1, 2 \text{ defined by} \]
\[ \eta_i^\pm(u_{R^\pm}) := \beta_i^\pm(u_{R^\pm}) \frac{\partial}{\partial u_{R^\pm}} \ln \left( Z_i^\pm(u_{R^\pm}) \right), \tag{59} \]

through the equations

\[ \frac{1}{\nu_\pm(u_{R^\pm})} := 2 - \eta_3^\pm(u_{R^\pm}) + \eta_2^\pm(u_{R^\pm}), \quad \nu = \nu_\pm(u_{R^\pm}). \tag{60} \]

The renormalization constants \( Z_i, i = 1, 2 \), are calculated straightforwardly from (47a)-(50c). The expansions of \( \nu_\pm \), which are of the form

\[ \nu_\pm = \frac{1}{2} + O(u_{R^\pm}), \tag{61} \]

can be inserted into (H) and evaluated at the fixed points \( u_{R^\pm}^* \) in the same way as before. The expansion has the form \( f_+/f_- = \sqrt{2} + O(u_{R^\pm}) \). This is the typical structure for renormalized perturbation theory because the higher order corrections in \( u_{R^\pm} \) represent the in \( D < 4 \) non-negligible deviations from the known mean field value \( (f_+/f_-)_{mf} = \sqrt{2} \).

With this method we find \( (f_+/f_-)_{ht} = 2.22(6) \) and \( (f_+/f_-)_{lt} = 1.86(6) \). We must emphasize, however, that these results are strongly influenced by the behaviour of the short series for \( \nu_\pm \) and therefore are certainly less reliable than the results above, where more precise information about \( \nu \) has been employed.

5 Conclusion

Our results for the universal ratio of correlation length amplitudes indicate a value of

\[ f_+/f_- = 2.04(4). \tag{62} \]

The theoretical estimates in the literature are 1.91 from the \( \epsilon \)-expansion \[4, 8\], and 1.96(1) \[18\], 1.94(3) \[17\] from high- and low-temperature expansions. They are definitely below our number. On the other hand, experimental values from binary fluids are 2.05(22), 2.22(5) \[19\] and 1.9(2), 2.0(4) \[20\].

In conclusion it appears quite reasonable to us that the correct value of the universal amplitude ratio of correlation lengths can lie above 2. The calculation of further orders and a detailed investigation of the resulting (non convergent, but likely asymptotic) perturbation series may lead to more accurate estimates.

Finally let us mention that with an appropriate modification our methods are also qualified to determine the amplitude ratios \( (f_+/f_-)_{ph} \) of the ‘true’ correlation length defined via the physical mass, and \( C_+/C_- \) of the susceptibility. We have performed the calculation of \( C_+/C_- \) in the two-loop approximation, too. The result is consistent with the estimate in \[11\]; their series from Feynman graphs in the symmetric phase is, however,
longer. The calculation of the ‘true’ correlation length in the one-loop approximation reveals that the amplitude ratio is only negligibly different from the one considered here.

As a consistency check we have also reproduced the $\epsilon$-expansions of the quantities considered in this work.

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