Local calibration of mass and systolic geometry

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Abstract: We prove the simultaneous \((k, n-k)\)-systolic freedom, for a pair of adjacent integers \(k < \frac{n}{2}\), of a simply connected \(n\)-manifold \(X\). Our construction, related to recent results of I. Babenko, is concentrated in a neighborhood of suitable \(k\)-dimensional submanifolds of \(X\). We employ calibration by differential forms supported in such neighborhoods, to provide lower bounds for the \((n-k)\)-systoles. Meanwhile, the \(k\)-systoles are controlled from below by the monotonicity formula combined with the bounded geometry of the construction in a neighborhood of suitable \((n-k+1)\)-dimensional submanifolds, in spite of the vanishing of the global injectivity radius. The construction is geometric, with the algebraic topology ingredient reduced to Poincare duality and Thom’s theorem on representing multiples of homology classes by submanifolds. The present result is different from the proof, in collaboration with A. Suciu, and relying on rational homotopy theory, of the \(k\)-systolic freedom of \(X\). Our results concerning systolic freedom contrast with the existence of stable systolic inequalities, studied in joint work with V. Bangert.

1. Introduction

Let \(X\) be an orientable manifold of dimension \(n\). A choice of a Riemannian metric \(g\) on \(X\) allows us to compute the total volume \(\text{vol}_n(g)\), as well as the \(k\)-volumes of \(k\)-dimensional submanifolds of \(X\). Given an integer homology class \(\alpha \in H_k(X, \mathbb{Z})\), let \(\text{vol}_k(\alpha) = \inf_{x \in \alpha} \text{vol}(x)\), where the infimum is taken over Lipschitz cycles. Here the volume

* Supported by The Israel Science Foundation (grant no. 620/00-10.0). Partially supported by the Emmy Noether Research Institute and the Minerva Foundation of Germany. Paper visible at the site [http://www.cs.biu.ac.il/~katzmik/publications.html](http://www.cs.biu.ac.il/~katzmik/publications.html) and will appear in *Geometric and Functional Analysis* 12, issue 3 (2002).
of an integer $k$-cycle $x = \sum_i r_i \sigma_i$ is $vol(x) = \sum_i |r_i|vol(\sigma_i)$, while the volume of a Lipschitz singular chain $\sigma_i : \Delta \rightarrow X$ is the integral over the standard $k$-simplex $\Delta$, of the “volume form” of the pullback $\sigma_i^*(g)$.

We define the $k$-systole of $(X, g)$, denoted $sys_k(g)$, as the infimum of volumes of nonzero integer homology classes:

$$sys_k(g) = \inf_{\alpha \neq 0} vol_k(\alpha).$$

Note that the $n$-systole coincides with the total volume: $sys_n(g) = vol(g)$. An alternative definition involves enlarging the class of $k$-dimensional submanifolds to rectifiable $k$-currents, so as to enable the solution of extremal problems, and extending the $k$-volume to $k$-mass (cf. [Mo]). Then the $k$-systole is the least mass of a rectifiable $k$-current representing a nonzero integer $k$-dimensional homology class. For a leisurely historical introduction to $k$-systoles, and for further references, see M. Berger’s survey article [Be2], or [KS1], sections 2 and 3.

M. Gromov asks the following question in his recent book [G3], p. 268. Let $\mathcal{C}_l$ be the space of all left-invariant metrics on the unitary group $U(d)$. Let $S_l : \mathcal{C}_l \rightarrow \mathbb{R}^{d^2}$ be the ‘systolic’ map, i.e. the map which associates to every metric $g$, its vector of systolic invariants $sys_k(g)$, $k = 1, \ldots, \dim(U(d)) = d^2$. Gromov asked:

**Question 1.** What is the image of the map $S_l$?

More generally, let $\mathcal{C}_g(X)$ denote the space of arbitrary Riemannian metrics $g$ on a given smooth $n$-manifold $X$. Let $S_g : \mathcal{C}_g \rightarrow \mathbb{R}^n$ be the analogous ‘systolic’ map.

**Question 2.** What is the image of the map $S_g$?

Typically, the constraints on the image of $S$ have been expressed in terms of inequalities satisfied by the various systoles or their “stable” analogues, cf. [G1], [H], [BanK], [K2]. The absence of such inequalities, termed “systolic freedom”, has been exhibited in a number of recent works, starting with Gromov’s example of 1993.

Systolic freedom involving a single systole with regard to the total volume has by now been well understood, at least in the case where the relevant homology group is torsionfree. Thus, even-dimensional manifolds with middle-dimensional systole growing faster than the volume were constructed in [BKS], generalizing [K1]. Manifolds whose $k$-systole, for any given $2 \leq k < n$, grows faster than the volume, were constructed in [KS1], [KS2].

Systolic freedom of manifolds involving a pair of systoles of complementary dimensions was studied in [BeK], [P], [BK]. Metrics $g$ on general $n$-dimensional polyhedra with the product, $sys_k(g)sys_{n-k}(g)$, growing faster than the total volume, were constructed in [B2, B3], vastly generalizing the results of [BK] (cf. section 2).

A shortcoming of such constructions is the fact that, as a by-product of the construction, the injectivity radius tends to vanish, and similarly for the systoles below dimension $k$. 

\[2\]
Remark 2.2 for more details). Thus, such constructions are limited to inequalities involving only 2 systoles.

In the present work we attempt to tackle Question 2 above in a situation where a larger number of systoles is involved, in the following sense.

**Definition 1.0.** Let $X$ be an $n$-dimensional smooth manifold, and let $k$ be an integer satisfying $1 \leq k < \frac{n}{2}$. Then $X$ is called *simultaneously $(k, n-k)$- and $(k-1, n-k+1)$-systolically free* if

$$\inf_g \frac{1}{\text{vol}(g)} \left( \frac{1}{\text{sys}_k(g) \text{sys}_{n-k}(g)} + \frac{1}{\text{sys}_{k-1}(g) \text{sys}_{n-k+1}(g)} \right) = 0,$$

where the infimum is over all smooth metrics $g$ on $X$.

In other words, we show that the product of systoles of complementary dimensions can be made large, compared to the volume, simultaneously for two pairs of complementary dimensions, modulo mild restrictions on the topology of the manifold.

We will adopt the convention that a systolic invariant defined over a trivial homology group, is infinite.

**Theorem 1.1.** Let $X$ be a simply connected closed $n$-dimensional manifold. Let $k < \frac{n}{2}$ be an integer. Assume that the group $H_{k-1}(X)$ is torsionfree. Then $X$ admits submanifolds $C_{k-1}$ and $C_k$ with $\dim(C_i) = i$, and a sequence of metrics $(g_j)$ as $j \to \infty$, fixed outside a neighborhood of the union $C_{k-1} \cup C_k \subset X$, which satisfy the following three conditions:

1. We have a uniform lower bound for the $(k-1)$-systole and the $k$-systole: $\text{sys}_{k-1}(g_j) \geq 1$ and $\text{sys}_k(g_j) \geq 1$;
2. The total volume of $X$ is linear in $j$, namely $\text{vol}(g_j) \leq \mu(X) \sum_i b_i(X) j$, where $b_i$ are the Betti numbers, and $\mu(X)$ is a constant depending only the topology of $X$ (cf. 5.1 and 5.3);
3. We have a quadratic lower bound for the complementary stable systoles: $\text{stsys}_{n-k}(g_j) \geq j^2$ and $\text{stsys}_{n-k+1}(g_j) \geq j^2$.

Therefore, $X$ is simultaneously $(k, n-k)$- and $(k-1, n-k+1)$-systolically free.

Note that our method has the usual consequence in terms of systolic freedom in a (single) pair of complementary dimensions, as in Corollary 1.2 below. This corollary follows from more general results for arbitrary polyhedra proved in [B3], cf. section 2 below.

**Corollary 1.2.** Let $X^n$ be orientable, let $k < \frac{n}{2}$, and assume that the group $H_{n-k}(X)$ is torsion free. Then $X$ is $(k, n-k)$-systolically free.

Indeed, the estimates above imply that the ratio

$$\frac{\text{vol}_n(g_j)}{\text{sys}_k(g_j) \text{sys}_{n-k}(g_j)} \leq \frac{\text{vol}_n(g_j)}{\text{sys}_k(g_j) \text{stsys}_{n-k}(g_j)} \leq \frac{\mu(X) \sum_i b_i(X) j}{1 \cdot j^2} = \text{Const} \to 0 \quad (2)$$
tends to zero. Here we do not need the assumption of simple connectivity of Theorem 1.1, which is only required for the simultaneous construction, cf. section 6.

Our theorem has the drawback of eliminating the interesting case of dimension and codimension one. The result below partly compensates for it.

**Theorem 1.3.** (cf. 7.2 and 7.4) Let $X$ be an orientable $n$-manifold, $n \geq 5$. Then $X$ admits metrics which are simultaneously $(1, n-1)$-free and $(2, n-2)$-free in the following two cases: (a) the fundamental group $\pi_1(X)$ is free abelian; (b) $H_1(X) = \mathbb{Z}$.

We will place the present work in its context among other systolic results in section 2. The local construction is described in section 3. Bounded geometry is discussed in section 4. Calibration by differential forms in a neighborhood of a submanifold and the use of the monotonicity formula for minimizing rectifiable currents appear in section 5, which describes the main construction using triangulations and pullback. Theorems 1.1 is proved in section 6. We use the classification of surfaces and an analysis of systems of disjoint loops on surfaces to prove Theorem 1.3 in section 7.

### 2. Historical remarks and motivation

An early result on the $(k, n-k)$-freedom from [BK] (cf. Theorem 2.1 below) contained a modulo 4 restriction on $k$, as well as an additional restriction of high connectivity.

I. Babenko [B2, B3] proved the systolic freedom in a pair of complementary dimensions, $(k, n-k)$, of an arbitrary $n$-dimensional polyhedron. Babenko’s proof [B3] involves polyhedral constructions using products of spheres. Babenko’s result, which is valid in particular for all manifolds regardless of their orientability, shows that systolic freedom in a pair of complementary dimensions has nothing to do with Poincare duality.

As compared to [B3], our theorem shows that such $(k, n-k)$-freedom can be achieved simultaneously for a pair of distinct $k$, modulo the stated assumptions on $X$.

More precisely, the following result follows from [BK], Lemma 5.1.

**Theorem 2.1** [BK]. Let $X$ be an orientable $n$-dimensional manifold, and let $k < \frac{n}{2}$. Assume that a multiple of every $k$-dimensional homology class can be represented by a submanifold $A \subset X$ with trivial normal bundle, which is either a sphere $S^k$ or a product $A = B \times C$ where $C$ is a circle. Then $X$ admits metrics $g_j$ with injectivity radius $\text{InjRad}(g_j) \geq 1$ uniformly bounded from below, sectional curvature uniformly bounded in absolute value, while the total volume grows at most linearly: $\text{vol}(g_j) \leq j$, and the stable $(n-k)$-systole at least quadratically: $\text{stsys}_{n-k}(g_j) \geq \text{Const} j^2$. Thus,

$$\text{InjRad}(g_j)^k \text{stsys}_{n-k}(g_j) \geq \text{Const} \text{vol}(g_j) j$$

where the constant, Const, is independent of the metric.
Remark 2.1.A: The monotonicity formula. The theorem implies, in particular, the $(k, n-k)$-systolic freedom modulo suitable torsion hypotheses. This follows either from the coarea formula, or, alternatively, from the monotonicity formula [Mo] applied to a minimizer for $k$-dimensional homology, which gives a lower bound for $\text{sys}_k$ in terms of $(\text{InjRad})^k$ (cf. equation (13) below).

A key ingredient in the proof of (3) along the lines of [BK] is the map

$$f : U_\epsilon A \to A \times D^{n-k},$$

which identifies a neighborhood of a suitable representative, $A$, of a homology class, with the product with the $(n-k)$-disk.

In contrast, the starting point of the argument in [B3] is the CW complex $X/X^{(k-1)}$ obtained by collapsing the $(k-1)$-skeleton of $X$ to a point. Thus, one immediately loses control of the injectivity radius and the systoles below dimension $k$.

The origin of the present paper was an attempt to understand the argument of [B3]. It turned out that one can interpret this argument as a local construction using a calibration, generalizing the map (4) from the construction of [BK] by introducing the degree 1 map $f : U_\epsilon \to S^k \times D^{n-k}$ of formula (7) below, where $U_\epsilon = U_k$ is a tubular neighborhood of a submanifold $C_k \subset X$ representing a $k$-dimensional homology class.

Remark 2.2: Vanishing injectivity radius. The inclusion of $C_k$ in $X$ may have nonzero image in the lower dimensional homology groups. Our construction relies on the pullback of metrics by the map $f$ of (7), which annihilates the lower dimensional homology of $C_k$. In particular, the resulting metrics may have a vanishing $(k-1)$-systole. The author is grateful to I. Babenko for pointing out this drop in the lower systoles, analogous to the collapse of the $(k-1)$-skeleton in the construction of [B3].

Note that $(k, n-k)$-systolically free metrics on $X$ are obtained in [B3] by pullback by a map $f : X \to W$, where $W$ a polyhedron obtained from $X/X^{(k-1)}$ by attaching copies of the product of spheres, $S^k \times S^{n-k}$. Here [B3] uses $(k, n-k)$-free metrics from [BK] on the product of spheres. Meanwhile, the construction of the $(n-k)$-free metrics on the product of spheres of [BK] is local in a neighborhood of the factor $S^k$. Thus, ultimately, the construction of [B3] relies on the construction of $(k, n-k)$-freedom in a neighborhood of a specific $k$-cycle in $X$. In the case when $X$ is a manifold, such a specific cycle is the inverse image of a suitable copy of $S^k$, or, more precisely, the inverse image of a regular value of the composition of $f$ with the projection to the second factor $S^{n-k}$. It is the latter observation that originally motivated the author to undertake the present work.

Placing the local nature of the construction in the forefront allows us to prove simultaneous freedom in two adjacent pairs of complementary dimensions, while remaining in the category of manifolds. The input from algebraic topology is reduced to Poincare duality and Thom’s theorem, as follows.
To protect the \((k - 1)\)-systole from collapse, we sharpen the construction of [BK] to include control over \((n - k + 1)\)-dimensional submanifolds \(B_{n-k+1} \subset X\). More precisely, the geometry in a neighborhood of \(B_{n-k+1}\) remains uniformly bounded. This provides a uniform lower bound for the volume of a minimizer for an infinite order integer \((k - 1)\)-dimensional homology class, by Poincare duality and the monotonicity formula applied in a neighborhood of \(B_{n-k+1}\).

**Remark 2.3: Relation to \(k\)-systolic freedom.** If we rescale the metric \(g_j\) of Theorem 1.1 to unit total volume, then the \((n-k)\)-systoles above the middle dimension become arbitrarily large, while the \(k\)-systoles below the middle dimension tend to 0. In other words, these metrics are “\((n-k)\)-systolically free”, but not “\(k\)-systolically free”.

Can the systoles below the middle dimension also be made arbitrarily large, compared to the total volume of \(X\)? The answer is typically negative for \(k = 1\). Namely, M. Gromov proved the following result in [G1]. Let \(\text{sys}_{\pi_1}(g)\) be the homotopy 1-systole of \((X,g)\), i.e. the length of the shortest noncontractible loop. Assume \(X\) is aspherical, or, more generally, “essential”, cf. [G1]. Then we have the inequality

\[
\text{sys}_{\pi_1}(g) \leq C_n \text{vol}_n(g)^{\frac{1}{n}},
\]

where the constant \(C_n\) depends only on the dimension.

On the other hand, the answer is affirmative for \(k \geq 2\), for an arbitrary manifold \(X\) with torsion free \(k\)-dimensional homology, as shown by the author in collaboration with A. Suciu [KS1, KS2]. Compared to the \((k,n-k)\)-freedom, such results seem to be harder to obtain in a purely geometric way. Thus, they seem to require “classifying space” type arguments from algebraic topology. The proof of [KS1, KS2] uses rational homotopy theory. In contrast, the present paper uses direct geometric constructions.

**Question 2.4.** Can \(k\)-systolic freedom be attained simultaneously for more than a single dimension \(k \leq \frac{n}{2}\)?

Our theorem can be usefully compared to the following result of J. Hebda [H], cf. [BanK].

Let \(H_k(X,\mathbb{Z})_{\mathbb{R}} \subset H_k(X,\mathbb{R})\) be the maximal lattice obtained as the image of integer homology. Denote by \(\alpha_{\mathbb{R}} \in H_k(X,\mathbb{Z})_{\mathbb{R}}\) the image of the class \(\alpha \in H_k(X,\mathbb{Z})\). We set

\[
\text{mass}_k(\alpha_{\mathbb{R}}) = \lim_{i \to \infty} \frac{1}{i} \text{vol}(i\alpha).
\]

We define the stable \(k\)-systole, denoted \(\text{stsys}_k(g)\), by minimizing mass over nonzero elements in the integer lattice:

\[
\text{stsys}_k(g) = \inf_{\alpha_{\mathbb{R}} \neq 0 \in H_k(X,\mathbb{Z})_{\mathbb{R}}} \text{mass}_k(\alpha_{\mathbb{R}}).
\]
Alternatively, the stable $k$-systole is the least mass of a minimizing normal $k$-current representing a nonzero element of the lattice $H_k(X, \mathbb{Z})_{\mathbb{R}}$.

**Theorem 2.5.** [H] Let $X$ be a compact orientable manifold of dimension $n$. Let $k < n$ and assume that the $k$-th Betti number is positive, $b_k(X) > 0$. Then every metric $g$ on $X$ satisfies the inequality

$$\text{stsys}_k(g) \leq C(n) b_k(X) \text{vol}(g),$$

where $C(n)$ depends only on the dimension $n$ of $X$.

Gromov’s paper [G1, section 7.4] contains general results of this type; see also [BanK]. The combination of our Theorem 1.1 with Hebda’s Theorem 2.5 shows that the ratio

$$\frac{\text{sys}_k(g_j)}{\text{stsys}_k(g_j)} \geq O(j) \quad (5)$$

tends to infinity. Thus our theorem can be viewed as a systematic way of constructing minimizing rectifiable currents which are far from minimizing as normal currents.

The subject of systolic freedom has received renewed attention recently in connection with the theoretical work in the context of quantum computers. Thus, M. Freedman [Fr] proved that the 3-manifold $S^1 \times S^2$ admits (1,2)-systolically free metrics even when we allow nonorientable surfaces to compete in the definition of the 2-systole. Note that the case of orientable surfaces is easier and was proved in [BeK] and generalized in [P] and [BK]. The general study of systoles was pioneered by M. Berger [Be1]; see Gromov’s 1999 book [G3] for an overview.

### 3. The local “two circle” construction

Our construction is local in a neighborhood $U_k$ of a $k$-dimensional submanifold $C_k \subset X$, specified in the lemma below. We will refer to it as the “two circle” construction, because it involves the splitting off of a pair of circles, denoted $C$ and $S^1$. The circles are split off, respectively, of the $k$-dimensional class and of the $(n - k)$-dimensional class, as in Proposition 3.3 below.

Let $(c_i)$ be an integer basis for a maximal lattice in $H_k(X, \mathbb{Z})$. The following lemma is well known [T] (cf. Theorem 6.2 below).

**Lemma 3.1.** Let $k < \frac{n}{2}$. There exists a submanifold $C_k \subset X$, and an integer $\lambda \in \mathbb{Z}$, satisfying the following two conditions:

(a) the connected components $C_{k,i}$ of $C_k$ represent a fixed integer multiple of the basis $c_i$, namely $[C_{k,i}] = \lambda c_i$ for all $i$;

(b) the normal bundle of $C_k$ in $X$ is trivial.
In view of condition (b), a neighborhood $U_k$ of $C_k$ is diffeomorphic to a product $C_k \times D^{n-k}$ with a ball $D^{n-k}$. Let $(m_i)$ be a basis for $H_{n-k}(X)$ modulo torsion, dual to the basis $(c_i)$. Representative fibers $D^{n-k}$ from the connected components of the fibration $U_k \to C_k$ may be “closed up”, by Poincare duality and hypothesis (a), to $(n-k)$-cycles $M_{n-k,i} \subset X$ representing the elements of the basis $(m_i)$. Thus the algebraic intersection numbers with the components of the $k$-cycle $C_k$ satisfy the relation

$$C_k,i \cdot M_{n-k,\ell} = \lambda \delta_{i\ell}.$$  

We have the excision isomorphism $r : H_{n-k}(U_{k,i}, \partial U_{k,i}) \to H_{n-k}(X, X \setminus U)$ relating the classes $[M_i] = r([D^{n-k}])$, where

$$[D^{n-k}] \in H_{n-k}(U_{k,i}, \partial U_{k,i})$$

is a relative homology class.

**Definition 3.2.** Consider the map

$$f_k : U_k \to S^k \times D^{n-k},$$

where $S^k$ is the $k$-sphere, defined by sending each connected component of $C_k$ to $S^k$ by a degree 1 map (or any nonzero degree), where $f_k$ is the identity on the second factor $D^{n-k}$. If $\alpha \subset U_k$ is a $k$-cycle representing a nonzero homology class, then the class of $f(\alpha) \in H_k(S^k \times D^{n-k})$ is also nonzero. Indeed, consider the generator $[C_k] \in H_k(U_k) = \mathbb{Z}$, and let $[\alpha] = a[C_k]$, where $a \neq 0$. Then the image under $f = f_k$ is

$$[f(\alpha)] = f_*([\alpha]) = f_*(a[C_k]) = af_*([C_k]) = a \deg(f)[S^k] \neq 0$$

in the group $H_k(S^k \times D^{n-k}) = \mathbb{Z}$.

Our target space, $S^k \times D^{n-k}$, of the map $f$ plays a key role in the construction. Namely, we will pull back “systole-rich” metrics from the target to the source, using a simplicial approximation of $f$ with respect to suitable triangulations of source and target. For the purpose of proving “simultaneous” freedom, we will need a simplicial approximation which remains a diffeomorphism in certain neighborhoods where $f$ itself is a diffeomorphism.

**Proposition 3.3 (The two-circle construction).** Let $C \subset S^k$ be a distinguished circle. Then the manifold $Y = S^k \times D^{n-k}$ admits a submanifold $S^{k-1} \times C$, where the distinguished circle occurs as a copy of the second factor, with the following property. A neighborhood $Y' = S^{k-1} \times C \times D^{n-k}$ of $S^{k-1} \times C$ contains a hypersurface

$$\Sigma = S^{k-1} \times C \times S^1 \times K,$$
with a neighborhood \( Y'' = \Sigma \times I \), where \( S^1 \) is a circle, while \( \dim(K) = n - k - 2 \), and furthermore:

(i) we have \( h([S^{k-1} \times C]) = [S^k] \) for the inclusion homomorphism \( h : H_k(\Sigma) \to H_k(Y) \);

(ii) we have \( h'([S^1 \times K \times I]) = [D^{n-k}] \) for the excision isomorphism

\[
h' : H_{n-k}(Y'', \partial Y'') \to H_{n-k}(Y, Y \setminus \text{int}(Y'')),
\]

where \( \text{int} \) denotes interior.

**Proof.** Let \( C \subset S^k \) be a circle, and consider the class \( [S^k] \in H_k(Y) \). It can clearly be represented by an imbedded submanifold \( S^{k-1} \times C \subset Y \) with trivial normal bundle. A tubular neighborhood \( Y' \) of \( S^{k-1} \times C \) in \( Y \) is diffeomorphic to the product

\[
Y' = S^{k-1} \times C \times D^{n-k}.
\]

All the \( j \)-dependent constructions will take place inside \( Y' \). Now consider a codimension 2 submanifold \( K \subset D^{n-k} \) (for example, an \((n - k - 2)\)-sphere), still with trivial normal bundle. A neighborhood of \( K \) in \( D^{n-k} \) is diffeomorphic to \( D^2 \times K \), with boundary \( S^1 \times K \). Thus a neighborhood of \( S^{k-1} \times C \times K \) in \( Y' \) is diffeomorphic to \( S^{k-1} \times C \times D^2 \times K \), with boundary denoted

\[
\Sigma = S^{k-1} \times C \times S^1 \times K.
\]

Now a tubular neighborhood \( Y'' \) of \( \Sigma \) in \( Y' \) is diffeomorphic to the product

\[
Y'' = \Sigma \times I,
\]

where \( I \) is an interval. All the \( j \)-dependent constructions will take place inside \( Y'' = \Sigma \times I \).

**4. Bounding geometry near a submanifold while \( \text{InjRad} = 0 \)**

We continue with the notation of the previous section. Our goal is to construct systolically free metrics while retaining a uniform bound on the geometry in a neighborhood of a suitable submanifold \( Z \subset Y = S^k \times D^{n-k} \).

**Definition 4.1.** Denote by \( Z \subset Y'' \subset Y \) the submanifold \( Z = C \times S^1 \times K \times I \) with boundary, so that \( Y'' = S^{k-1} \times Z \).

**Definition 4.2.** We will say that a family \( g_j \) of metrics on a manifold with boundary is of “bounded geometry” if it has the following three properties:

(a) the sectional curvature \( K \) is bounded, \(|K| \leq 1\), uniformly in \( j \);

(b) the metric is constant (independent of \( j \)) in a unit neighborhood of the boundary;

(c) the injectivity radius \( \iota \) satisfies \( \iota_x(g_j) \geq 1 \), for all points \( x \) at least a unit distance away from the boundary.
**Definition 4.4.** We will say that a $p$-form $w$ is *calibrating* if $w(X_1, \ldots, X_p) \leq 1$ for all $p$-tuples of unit vectors $X_i$ (cf. [G2], 4.4).

By the natural pairing $\langle w, \rangle$ between homology and cohomology, an $(n-k)$-form $w$ with compact support can be integrated over $D^{n-k} \subset Y$, viewed as a relative cycle defining a class in the group $H_{n-k}(Y, Y' \setminus Y'') = H_{n-k}(Y'', \partial Y'')$.

**Proposition 4.5.** The manifold $Y'' \subset Y = S^k \times D^{n-k}$, where

$$Y'' = \Sigma \times I = S^{k-1} \times C \times S^1 \times K \times I$$

admits metrics $\tilde{g}_j$, together with closed $(n-k)$-forms $w_j$ compactly supported in the interior of $Y''$, with the following four properties:

(i) the metrics have “bounded geometry” in the sense of 4.2;

(ii) the total volume is linear in $j$, i.e. $\text{vol}(\tilde{g}_j) \leq j$;

(iii) we have the “area” lower bound $\langle w_j, D^{n-k} \rangle \geq j^2$;

(iv) the $(n-k+1)$-volume of the submanifold $Z = C \times S^1 \times K \times I \subset Y''$ is linear in $j$.

**Proof.** The starting point of the construction is the fundamental domain for the manifold

$H/G(j),$

where $H \subset SL(3, \mathbb{R})$ is the Heisenberg group of unipotent matrices, while $G(j) \subset SL(3, \mathbb{Z})$ is the subgroup consisting of matrices

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}; \; x, y, z \in \mathbb{Z}$$

such that $x$ is congruent to 0 modulo $j$ (cf. [G3, p. 88] and [G2, section 3.6]).

Suitable metrics $g_j$, and calibrating 2-forms $\psi_j$, on the 3-manifold with boundary $T^2 \times I = C \times S^1 \times I$ were constructed in [BK] (see also [KS1, Appendix A]). These are related to F. Almgren’s example on $S^1 \times S^2$ (cf. [Fe2], p. 397).

Here one may equip the manifold $Y = T^2 \times I$ with metrics $g_j$ defined by

$$g_j(x, y, z) = h(\hat{x})(y, z) + dx^2,$$

where $x \in I = [0, 2j]$, $\hat{x} = \min(x, 2j - x)$, while $T^2$ is the quotient of the $(y, z)$-plane by the integer lattice, and the formula

$$h(x)(y, z) = (dz - xdy)^2 + dy^2$$
defines a metric on the 2-torus \( T^2 \times \{ x \} \). Here the z-axis parametrizes the circle \( C \) which has unit length, while the y-axis parametrizes the circle \( S^1 \) of length \( \sqrt{1 + (\hat{x})^2} \). Note that the two circles, \( C \) and \( S^1 \), play very different roles in this seminal “two-circle” construction.

The calibrating form is \( \psi_j = (1 + x^2)^{-\frac{1}{2}} \ast dz \), where \( \ast \) is the Hodge star operator of the metric \( g_j \). For details, see [BK] and [KS1], Appendix A.

The 1-systole \( \text{sys}_1(g_j) \) is uniformly bounded from below, while the stable 1-systole tends to zero at the rate of \( \frac{1}{j} \). The volume grows linearly in \( j \).

Meanwhile, the area of the surface \( S^1 \times I \subset T^2 \times I \) grows quadratically in \( j \) by calibration by \( \psi_j \). Such growth is reflected in the fact that the minimizing surface \( S^1 \times I \) accumulates on itself, in the sense that distinct sheets of the surface squeeze together to within distance on the order of \( \frac{1}{j} \), comparable to the stable 1-systole. In contrast, the injectivity radius remains uniformly bounded from below.

Here a basic building block is a fundamental domain for the standard compact nilmanifold of the Heisenberg group with its left invariant metric.

We imbed \( (T^2 \times I, g_j) \) in \( T^2 \times K \times I \), by incorporating a fixed (independent of \( j \)) metric on \( K \) as a direct summand, and let \( w_j = \psi_j \wedge d_{vol_K} \) be the \( (n - k) \)-form obtained by exterior product with the volume form of \( K \). Finally, we pull the form \( w_j \) back to \( Y'' = \Sigma \times I \) by the coordinate projection \( S^{k-1} \times T^2 \times K \times I \to T^2 \times K \times I \), to obtain the desired form on \( Y'' \). Here the metric of the factor \( S^{k-1} \) is a direct summand of the metric on \( Y'' = S^{k-1} \times (T^2 \times K \times I) \).

The quadratic lower bound for an \( (n - k) \)-cycle \( \alpha \) follows, as in [BK] and [KS1], by integrating the calibrating form \( w_j \) over \( \alpha \) in the connected component of \( \hat{U}_k \) containing a \( C_{k,i} \) which has nonzero algebraic intersection with \( \alpha \).

Finally, the linear upper bound for the total volume as well as the \( (n - k + 1) \)-volume of the manifold \( Z = C \times S^1 \times K \times I \) follows from the linear upper bound for the volume of the metrics on the 3-manifold \( T^2 \times I \), together with the fact that the metrics \( \tilde{g}_j \) on \( Y'' \) are a direct sum with a fixed metric on the factors \( K \) and \( S^{k-1} \).

**Lemma 4.6.** The \( k \)-systole of the metrics \( \tilde{g}_j \) on \( Y'' \) is uniformly bounded from below.

**Proof.** Let \( \alpha \) be a \( k \)-cycle in \( Y'' \) representing a nonzero multiple of the class \( [S^{k-1} \times C] \). Recall that \( Y'' = S^{k-1} \times Z \) where \( Z = C \times S^1 \times K \times I \). Consider the projection to the first factor \( p : Y'' \to S^{k-1} \). Then \( p \) is a Riemannian submersion by construction. By the coarea (Eilenberg’s) formula [Mo], p. 31, we have

\[
\text{vol}_k(\alpha) \geq \int_{S^{k-1}} \alpha \cap p^{-1}(x) dx, \tag{9}
\]

where the intersection \( \alpha \cap p^{-1}(x) \) with a typical fiber \( Z \) is a loop in \( Z \) which is not nullhomologous. Hence \( \text{sys}_k(Y'') \geq \text{vol}_k(\alpha) \geq \text{sys}_1(Z) \text{vol}_{k-1}(S^{k-1}) \), providing the desired lower bound in view of the bounded geometry of \( Y'' \) and \( Z \). Note that the stable 1-systole of \( Z \) tends to zero as \( j \) increases.
However, we will provide an alternative argument, as well, which will work even in a situation where the projection \( p \) to \( S^{k-1} \) is not available, and the geometry far away from a specific copy of \( Z \) may not be bounded. This is due to the collapse which occurs under the map \( f_k \) of formula (7) above. Therefore we would like to argue in a fixed neighborhood of the submanifold \( Z = C \times S^1 \times K \times I \). Indeed, the bounded geometry in a neighborhood of \( Z \) allows us to apply the monotonicity formula, centered at a point of \( Z \), to a minimizer in the class \([\alpha]\), immediately yielding the desired lower bound.

5. Triangulations, pullback metrics, and calibrations

In this section we will prove the \((k, n-k)\)-systolic freedom for a single \( k \), preparing the ground for simultaneity in the next section.

The idea is to construct first the metrics \( \tilde{g}_j \) on the product \( S^k \times D^{n-k} \), pull them back to \( X \) by the map \( f_k \) of formula (7), and then apply calibration by the pullbacks of the form \( w_j \) of Proposition 4.5 to obtain the quadratic lower bound for the \((n-k)\)-systole.

By the work of I. Babenko [B1], the map \( f_k : U_k \to S^k \times D^{n-k} \) of formula (7) may be replaced by a map \( \tilde{f} \) which has the following property with respect to suitable triangulations of its domain and target: on each simplex, \( \tilde{f} \) is either a diffeomorphism onto its image, or the collapse onto a wall of positive codimension. Moreover, we will use the following theorem of A. H. Wright (already exploited in [B1], Theorem 8.1). Recall that a map is called \emph{monotone} if the inverse image of every point in the range is a compact connected subset of the domain.

Theorem 5.1. ([W], Theorem 7.3.) Let \( M^n \) and \( N^n \) be closed piecewise linear manifolds. Let \( f : M^n \to N^n \) be a continuous map of absolute degree one. Then \( f \) is homotopic to a piecewise linear monotone map.

Note that if \( M^n \) and \( N^n \) are orientable, and if \( f \) has degree 1, then \( f \) has absolute degree one. In the terminology of section 3, the map from the connected component \( C_{k,i} \) to \( S^k \) may be chosen to be monotone by Wright’s theorem. Hence there is exactly one \( k \)-dimensional simplex mapping diffeomorphically to a \( k \)-dimensional simplex in the target \( S^k \). By Cartesian product with the disk, we may assume that the map \( \tilde{f} \), defined on each connected component of the neighborhood of the submanifold \( C_k \) of Lemma 3.1, has the property that each top dimensional simplex in the range has a unique simplex as its inverse image.

The map \( \tilde{f} \) can be slightly perturbed to \( \tilde{f}^{*} \) to make it smooth, so the pullbacks by \( \tilde{f}^{*} \) are well-defined, while not significantly affecting the volumes. Consider the pulled-back positive (symmetric) 2-forms \( \tilde{f}^{*} (\tilde{g}_j) \). By construction, its total volume is nearly equal to that of \( \tilde{g}_j \). The form \( \tilde{f}^{*} (\tilde{g}_j) \) may not be definite, but a small multiple of a fixed metric on \( X \) can always be added on to make it positive definite, in the end of the construction,
without significantly increasing the total volume, and certainly not decreasing the systolic invariants.

**Proposition 5.2.** Let $X$ be an orientable $n$-manifold (not necessarily simply connected). Let $k < \frac{n}{2}$. Then the metrics $g_j$ on the neighborhoods $U_k$ of Lemma 3.1, constructed by pullback, can be extended from $U_k$ to $X$ so as to satisfy a uniform lower bound for the $k$-systole, a quadratic (in $j$) lower bound for the stable $(n-k)$-systole, and a linear (in $j$) upper bound for the total volume.

**Proof.** Outside the neighborhood $U_k$, the metric $g = g_j$ on $X$ is chosen fixed. To patch together the metric $\tilde{f}^*(\tilde{g}_j)$ on $U_k$ and the metric $g$ on $X \setminus U_k$, we consider the boundary $\partial U_k = C_k \times S^{n-k-1} \subset X$. Its tubular neighborhood in $X$ is of the form $C_k \times S^{n-k-1} \times I \subset X$.

We use a partition of unity along the “cylinder” $C_k \times S^{n-k-1} \times I$ (all choices independent of $j$). Let $\alpha$ be an integer $k$-cycle, which represents a nontrivial homology class. We replace $\alpha$ by the minimizing rectifiable current in its homology class.

**Remark 5.3.** If $\alpha$ lies in $U_k$, then a uniform lower bound

$$\text{vol}_k(\alpha) \geq \mu(X)$$

for $\text{vol}_k(\alpha)$ follows from the second argument in the proof of Lemma 4.6. If $\alpha$ ventures outside $U_k$, then such a lower bound for its volume follows from the monotonicity formula centered at a point of $\alpha \cap (X \setminus U_k)$, together with the fact that the metric on $X \setminus U_k$ is fixed.

**Remark 5.4.** Note that there is no lower bound for the injectivity radius of $\tilde{f}^*(\tilde{g}_j)$, since the map $\tilde{f}$ may collapse top-dimensional cells.

Next, the total volume of $(X, g_j)$ is dominated by $j$ times the number of connected components of $U_k$ by Theorem 5.1 and property (ii) of Proposition 4.5. This number is controlled by the Betti number $b_k$ in view of Lemma 3.1(a), proving the upper bound for the volume.

Now the pullback form $\tilde{f}^*(w_j)$ on $U_k$ may be extended by zero to give a calibrating form on all of $X$. Let $M$ be a Lipschitz $(n-k)$-cycle representing an infinite order class $[M] \in H_{n-k}(X)$. We decompose it with respect to the basis $(m_i)$:

$$[M] = T + \sum_i \alpha_i m_i,$$

where $T \in H_{n-k}(X)$ is torsion and $\alpha_i \in \mathbb{Z}$. Choose a nonzero index $\alpha_{i_0} \neq 0$. Let $w_{j,i_0}$ be the pullback of $w_j$ supported in the connected component $U_{k,i_0}$. Thus it is the extension
by zero of the pullback of \( w_j \) by the restriction of the map \( f \) to this component. Then by Stokes’ theorem,

\[
\int_M w_{j,i_0} = \int_{M \cap U_{n-k,i_0}} w_{j,i_0} = \alpha_{i_0} \int_{M_{n-k,i_0}} \tilde{f}^*(w_j) = \alpha_{i_0} \int_{D^{n-k}} w_j,
\]

where the \( M_{n-k,i} \) are the cycles from formula (6). Since \( w_j \) is a calibrating form and \( \tilde{f} \) is an isometry, we have

\[
\text{vol}_{n-k}(M) \geq |\alpha_{i_0}| \int_{D^{n-k}} w_j \geq j^2
\]

by property (iii) of Proposition 4.5. Since the lower bound is obtained by integration of differential forms, it holds for an arbitrary normal current, proving the proposition.

6. Proof of simultaneous freedom

**Theorem 6.1.** Let \( X \) be a simply connected closed manifold of dimension \( n \). Let \( k < \frac{n}{2} \) and assume that \( H_{k-1}(X) \) is torsion free. Then \( X \) is simultaneously \((k, n-k)\)-free and \((k-1, n-k+1)\)-free, in the precise sense stated in Theorem 1.1.

**Proof.** The proof breaks up into 5 steps. Simple connectivity is used in step 3, while torsion freeness is used in step 4.

**Step 1: Background metric and injectivity radius.** Consider a fixed smooth background Riemannian metric \( g(0) = g_j(0) \) on \( X \), with positive fixed injectivity radius.

**Step 2: The \((k, n-k)\)-freedom.** Choose a rational basis for \( H_k(X) \), representable by manifolds \( C_{k,i} \) with trivial normal bundles. Modify the background metric in a neighborhood \( U_k \) of each \( C_{k,i} \), to construct \((k, n-k)\)-free metrics \( g_j(1) \) on \( X \) as in Proposition 5.2. Recall that we have the submanifold \( Z = C \times S^1 \times K \times I \) of \( Y'' = S^{k-1} \times Z \) which occurred in the construction of the metrics. The metrics \( g_j(1) \) have the following four properties:

(a) the \( k \)-systole is uniformly bounded from below: \( \text{sys}_k(g_j(1)) > 1 \);

(b) the \((n-k)\)-systole grows quadratically: \( \text{sys}_{n-k}(g_j(1)) = j^2 \);

(c) the volume grows at most linearly: \( \text{vol}_{n-k}(g_j(1)) < j \).

(d) the volume of the submanifold \( Z \) grows linearly in \( j \).

Note that the construction of the \((k, n-k)\)-free metrics results in metrics \( g_j(1) \) with zero injectivity radius. Notwithstanding, the uniform lower bound for the \( k \)-systole is ensured by Proposition 5.2, in view of the bounded geometry in a neighborhood of \( Z \).

**Step 3: Choice of the \((n-k+1)\)-dimensional submanifold \( B \).** Unlike the uniform lower bound for the \( k \)-systole, in the wake of step 2, we have no control over either the
(\(k-1\))-systole or the injectivity radius, which may both be zero if we work with positive, possibly nondefinite, quadratic forms \(g_j(1)\). To repair this and ensure a lower bound for the \((k-1)\)-systole, we need to modify the construction as follows.

Choose a rational basis for \(H_{n-k+1}(X)\) which can be represented by \(b_{k-1}\) imbedded submanifolds \(B = B_{(n-k+1),i}\). See Sh. Weinberger’s comments below in Theorem 6.2, concerning the existence of the submanifolds \(B\).

We will modify the construction so as to have precise control over the metric in a neighborhood of \(B\). By Poincare duality, every \((k-1)\)-cycle \(\alpha\) representing a class of infinite order must meet one of the submanifolds \(B\). This will yield a lower bound for the \((k-1)\)-volume of \(\alpha\).

If \(B\) lies outside the neighborhood \(U_k\) of \(C_k\), where the \(j\)-dependent construction took place in step 2, then, by step 1, there is a lower bound for the injectivity radius in a fixed neighborhood of \(B\), yielding the desired lower bound for a \((k-1)\)-cycle with nonzero algebraic intersection with \(B\).

Now assume that the connected imbedded submanifold \(B_{n-k+1,i}\) meets \(C_k\). We place them in transverse position. Then

\[
\dim(B_{n-k+1,i} \cap C_k) = \dim(B_{n-k+1,i}) + \dim(C_k) - n = 1,
\]

hence the intersection \(B_{n-k+1,i} \cap C_k\) is a disjoint union of circles.

If \(k = 2\), then the \((k-1, n-k+1) = (1, n-1)\)-freedom is not an issue, since \(X\) is assumed simply connected (cf. section 7 for a result on non-simply connected manifolds in the case \(k = 2\)). Therefore we may assume that \(k \geq 3\). Then by transversality we may assume that the triple intersections

\[
B_{n-k+1,i} \cap B_{n-k+1,j} \cap C_k = \emptyset
\]

are empty (cf. formula (14) of section 7). Thus the full intersection \((\cup_i B_{n-k+1,i}) \cap C_k\) is also a disjoint collection of circles, denoted \(C_\alpha\):

\[
(\cup_i B_{n-k+1,i}) \cap C_k = \cup_{\alpha} C_\alpha.
\]

A tubular neighborhood of \(C_\alpha \subset B\) is of the form

\[
C_\alpha \times D^{n-k} \subset B.
\]

Recall that \(X\) is assumed simply connected. Therefore, after a suitable surgery, we can assume that the manifold \(C_k\) is simply connected, as well. Hence, the circle \(C_\alpha\) is contractible in \(C_k\). Choose a neighborhood \(A_\alpha\) of an imbedded disk which bounds \(C_\alpha \subset C_k\), such that \(A_\alpha\) is diffeomorphic to a \(k\)-ball. We choose such neighborhoods disjoint for distinct circles \(C_\alpha\).
Now choose a fixed circle $C \subset S^k$. Let $f : C_k \to S^k$ be a map with the following three properties.

1) $f$ maps the interior of each $A_\alpha$ diffeomorphically onto the complement of a fixed basepoint in $S^k$;

2) $f$ collapses the complement $C_k \setminus (\cup_\alpha A_\alpha)$ to the basepoint;

3) Each circle $C_\alpha$ is mapped diffeomorphically to the fixed circle $C \subset S^k$.

Note that the degree of $f$ is the total number of circles $C_\alpha$. We may use our map $f$ to define the map $f_k : U_k \to Y = S^k \times D^{n-k}$, in place of the degree one map of formula (7). The diffeomorphism property 1) above allows us to argue as if a neighborhood of a circle $C_\alpha \subset B$ is actually contained in the standard manifold $Y$, and apply transversality techniques there to achieve the identification of formula (12) below.

We can assume that the image $f(C_\alpha)$ of the circle $C_\alpha \subset C_k$ coincides with a circle $C$ which is a copy of the second factor in the submanifold $S^{k-1} \times C \subset Y$ homologous to $S^k$, and that the product structure (11) in $B$ coincides with the last two components of decomposition (8) of $Y'$ = $S^{k-1} \times C \times D^{n-k}$. Therefore we can assume that

$$f_k(B) \cap Y'' = Z = C \times S^1 \times K \times I.$$ (12)

Note that the volume of $B$ for our metrics is at most linear in $j$ by property (d) of Step 2 above. Denote by $g_j(2)$ the resulting modification of the metric $g_j(1)$.

**Step 4: Uniform lower bound for the $(k-1)$-systole.** Let $\alpha$ be a $(k-1)$-cycle in $X$ representing a class of infinite order. We replace $\alpha$ by a minimizing rectifiable current.

By Poincaré duality, the cycle $\alpha$ must meet one of the manifolds $B$ of step 3 (it is here that we need the assumption of Theorem 1.1 that $H_{k-1}(X) = \text{torsion free}$). By construction, the metrics $g_j(2)$ of $X$ have bounded geometry (in the sense of 4.2) in a neighborhood of $B$ of a fixed size. Here the geometry near $B$ is bounded inside $U_k$ by step 3, while the geometry near $B \cap (X \setminus U_k)$ is bounded by step 1. A lower bound for the volume of $\alpha$ now follows from the monotonicity formula applied at a point of the intersection $(\alpha \cap B) \subset X$.

**Step 5: The $(k-1, n-k+1)$-freedom.** Now choose a basis $C_{k-1,i}$ for $H_{k-1}(X)$, as in Lemma 3.1. Let $U_{k-1} \subset X$ be a neighborhood, disjoint from $U_k$, of the $C_{k-1}$. We modify the metrics $g_j(2)$ in the neighborhood $U_{k-1}$, to construct $(k-1, n-k+1)$-free metrics $g_j(3)$ with $\text{sys}_{n-k+1}(g_j(3)) = j^2$, similarly to step 2. Note that the volume of the manifold $B_{n-k+1}$ is increased to quadratic growth in $j$.

The metrics $g_j(3)$ cannot be chosen to dominate the metrics $g_j(2)$. Therefore we need to explain why our construction of $(k-1, n-k+1)$-free metrics in the neighborhood $U_{k-1} \subset X$ respects the lower bound of step 4 for the $(k-1)$-systole.

We may assume that the manifold $B_{n-k+1}$ meets $C_{k-1}$ transversely in a finite number of points $x_1, \ldots, x_N$. We choose a neighborhood $O \subset C_{k-1,i}$, such that $x_i \in O$ and $O$ is diffeomorphic to a $(k-1)$-ball.
We choose a degree 1 map $C_{k-1,i} : S^{k-1}$ to be a diffeomorphism from $O$ onto the complement of a point of the sphere, and define the map $f_{k-1} : U_{k-1} \to S^{k-1} \times D^{n-k+1}$ as in formula (7). Then we can ensure that, for each connected component $(B_{n-k+1} \cap U_{k-1})_0$ of the intersection $B_{n-k+1} \cap U_{k-1}$, we have

$$f_{k-1}((B_{n-k+1} \cap U_{k-1})_0) = D^{n-k+1},$$

so that again the metric near $B$ is bounded in all three regions, $U_k, U_{n-k+1}$, and their complement in $X$. The lower bound for the $(k-1)$-systole now results from the monotonicity formula as before. This completes the proof of Theorem 6.1.

The comments below were kindly provided by Shmuel Weinberger, for the benefit of the reader who is a Riemannian geometer and is not necessarily familiar with the details of the techniques of [T].

**Theorem 6.2** [T] Let $X$ be a compact manifold, $A$ a homology class. Then a multiple of $A$ can be represented by an imbedded submanifold.

**Idea of proof.** If the codimension of $A$ in $X$ is odd, then a multiple of $A$ can always be represented by a submanifold with trivial normal bundle. Indeed, the Poincare dual cohomology class is a map to $K(\mathbb{Z}, odd)$, which is, at least rationally, $S^{odd}$. Thus a multiple is represented by a map to the sphere. To construct the desired submanifold, simply take the transverse inverse image of a point of the sphere $S^{odd}$.

For even codimension $2n$ the proof is slightly more involved. We consider first an example in codimension 2. Let $z \in H^2(X, \mathbb{Z})$ be the dual 2-class to the homology class of codimension 2 that we want to represent by a submanifold. Let $f_z : X \to \mathbb{C}P^N$ be the associated map to a skeleton of the classifying space, for a sufficiently high $N$. Then the transverse inverse image of a hyperplane,

$$f_z^{-1}(f_z(X) \cap \mathbb{C}P^{N-1}),$$

is the desired submanifold.

Let $\zeta \to K(\mathbb{Z}, 2)$ be the universal 2-bundle. Its Euler class is the generator of cohomology. The Thom space $MG$ of $\zeta$ is again the Eilenberg Maclane space: $MG = K(\mathbb{Z}, 2)$. Namely, $CP^N$ is the one point compactification of $\zeta$ over $CP^{N-1}$.

Returning to the general case of even codimension $2n$, recall that the cohomology of the Eilenberg Maclane space

$$H^*(K(Z, 2n)) \otimes Q$$

is a polynomial algebra on a $2n$-dimensional generator. Now $BU(n)$ admits a map to a product

$$BU(n) \to K(Z, 2) \times K(Z, 4) \times \cdots \times K(Z, 2n),$$
which is defined (integrally) via Chern classes, and induces isomorphism in rational cohomology.

Now if \( X \) and \( Y \) are rationally equivalent, there may not in general exist maps from \( X \) to \( Y \) and from \( Y \) to \( X \). But for any fixed \( k \), one can map the \( k \)-skeleton of \( X \) to \( Y \) in such a way that the map will induce an isomorphism in rational homology up to dimension \( n - 1 \). This applies, for example, if \( X \) is a finite cell complex, and allows one to replace rational complexes by finite integral ones in the argument that follows.

Thus there is an \( n \)-dimensional complex bundle over \( K(\mathbb{Z}, 2n) \) whose \( n \)-th Chern class (i.e. its Euler class) is a nonzero multiple of the generator of

\[
H^{2n}(K(\mathbb{Z}, 2n)).
\]

A Gysin sequence argument then shows that the Thom space \( T \) of this bundle is rationally \( K(\mathbb{Z}, 2n) \). Thus one can (after multiplying) lift the map \( X \to K(\mathbb{Z}, 2n) \) to \( T \). The desired submanifold of \( X \) is then the transverse inverse image of the 0-section of \( \zeta \subset T \).

Note that this argument really only works on skeleta, while \( K(\mathbb{Z}, 2n) \) doesn’t have the bundle, but rather, it rationally has a bundle. It integrally gets a bundle on its skeleta.

7. Curves and surfaces in \( n \)-manifolds

Our goal is to prove \((k, n-k)\) systolic freedom simultaneously for \( k = 1 \) and \( k = 2 \), for a manifold with \( H_1(X) = \mathbb{Z} \) (cf. Theorem 7.4 below). However, we start with the simpler case of abelian fundamental group.

**Proposition 7.2.** Let \( X \) be an orientable \( n \)-manifold with free abelian fundamental group. Then \( X \) admits metrics which are simultaneously \((1, n-1)\)-free and \((2, n-2)\)-free.

**Proof.** We will prove a stronger statement, namely, that \( X \) admits metrics \( g_j \) satisfying the inequality

\[
\text{vol}(g_j) \left( \frac{1}{\text{InjRad}(g_j) \text{sys}_{n-1}(g_j)} + \frac{1}{\text{InjRad}(g_j)^2 \text{sys}_{n-2}(g_j)} \right) \leq \frac{\text{Const}}{j}
\]

(13)

where InjRad is the injectivity radius (cf. equations (1) and (3) above).

Since the fundamental group is abelian, every incompressible orientable surface in \( X \) is either a sphere or a torus. We choose a rational basis \( C_2 = \cup_i C_{2,i} \) for \( H_2(X) \), where each surface \( C_{2,i} \) is a torus \( T^2 \) with trivial normal bundle (spherical classes may be represented by tori by adding a handle). Then a neighborhood \( U_2 \) of \( C_2 \) has the form \( U_2 = C_2 \times D^{n-2} \).

We write \( T^2 = S^{2-1} \times C \). Here the circle \( S^{2-1} \) will play a role similar to that of the sphere \( S^{k-1} \) in the simply connected case (Theorem 6.1). The circle \( C \) is similar to the circle \( C \) in the simply connected case, namely the second factor of the submanifold \( S^{k-1} \times C \) representing the class \([S^k] \in H_k(Y) \) (cf. Proposition 3.3). The difference from
the simply connected case is that now $C$ may not be contractible in $X$. Here the analogue of the map $f_k$ of formula (7) is the map

$$f_2 : U_2 \to S^{2-1} \times C \times D^{n-k}$$

which is a diffeomorphism on each connected component of $U_2 \subset X$.

We choose a fixed smooth background metric $g$ on $X$. We modify $g$ in $U_2$ to build $(2, n-2)$-free metrics $g_j$ on $X$ as in the proof of Theorem 6.1, with quadratic growth of the $(n-2)$-systole: $\text{sys}_{n-2}(g_j) \sim j^2$, and linear volume growth: $\text{vol}_n(g_j) \sim j$. The metrics $g_j$ have injectivity radius and 1-systole which are uniformly bounded from below, since $f_2$ is a diffeomorphism.

We continue by modifying the metrics $g_j$ in a neighborhood $U_1 \subset X$ of a system of loops $C_1 = \cup_i C_{1,i}$ generating $\pi_1(X) = H_1(X)$. Here each connected component of $U_1$ is diffeomorphic to $C \times D^{n-1}$. We choose a neighborhood $Y'' \subset U_1$ of the form $Y'' = C \times S^1 \times K \times I$ and construct $(1, n-1)$-free metrics as in section 4. The resulting metrics satisfy quadratic growth in the $(n-1)$-systole, and obey a uniform lower bound for the injectivity radius. This concludes the proof of Theorem 7.2.

**Lemma 7.3.** Let $R_g$ be a closed orientable surface. Let $C_\alpha \subset R_g$ be a finite family of disjoint imbedded loops, where $\alpha \in$ some index set. Then there exists a smooth map $f : R_g \to S^2$ to the sphere $S^2$, of nonzero degree, which maps each loop $C_\alpha$ onto the equator of $S^2$, while a neighborhood of $C_\alpha$ maps diffeomorphically onto a neighborhood of the equator.

**Proof.** Denote by $NP \in S^2$ and $SP \in S^2$ respectively the north and south poles of the sphere. It is obvious that there exists such a map to the sphere with two points identified, $S^2/(SP \sim NP)$.

To construct the map to the sphere itself, note that we may assume that the family of loops is “maximal” in the sense that the complement of $\cup_\alpha C_\alpha$ in $R_g$ is a collection of open surfaces of one of two types: either a “pair of pants” or a cylinder (some of the loops may be isotopic). We choose disjoint tubular neighborhoods, called annuli, $A_\alpha \subset R_g$ of the $C_\alpha$. We identify the boundary of the annulus $\partial A_\alpha = C_\alpha \times S^0$ with the product of the loop with the pair of points $S^0 = \{s, n\}$. Let

$$f : A_\alpha \to S^2 \setminus \{SP, NP\}$$

be an orientation-preserving diffeomorphism onto the complement of the poles, while $f(C_\alpha)$ is the equator, so that $f$ extends by continuity to the two boundary loops as follows:

$$f(C_\alpha \times \{s\}) = SP \in S^2, \ f(C_\alpha \times \{n\}) = NP \in S^2.$$

The complement $R_g \setminus (\cup_\alpha A_\alpha)$ of the annuli is a disjoint union of connected “complementary regions” each of which is either a pair of pants or a cylinder. Each boundary loop of a
complementary region has a marking $\epsilon = SP$ or $\epsilon = NP$, depending on whether it is the boundary loop of the adjacent annulus $A_\alpha$ which is sent to $SP \in S^2$ or to $NP \in S^2$. If the marking $\epsilon$ is the same for all boundary loops of such a complementary region, we collapse the entire region to the corresponding pole $\epsilon \in S^2$. If the boundary loops do not all have the same markings, there are three possibilities:

1. The complementary region is a cylinder whose boundary loops are marked $SP$ and $NP$. Then we map it to $S^2$ by an orientation-preserving diffeomorphism in the interior of the cylinder, while the boundary loops are sent respectively to $SP \in S^2$ and $NP \in S^2$;

2. The complementary region is a pair of pants and its triple of boundary loops are marked $SP$, $SP$, and $NP$. Then we subdivide the pair of pants into two regions, by choosing a new circle parallel to the boundary loop which is marked $NP$, and cutting along it. We label this new circle $SP$ (see figure below). Now, one of the regions is a cylinder, which can be handled in case 1. The other region is a pair of pants with the identical marking, $SP$, on all three boundary loops, and we collapse it to the pole $SP \in S^2$;

3. The complementary region is a pair of pants and its boundary loops are marked $NP$, $NP$, and $SP$. This case is similar to the previous one.

Since $f$ preserves orientation on each cylinder and pair of pants, its degree is greater than or equal to the total number of circles $C_\alpha$ at the outset, and is therefore nonzero (cf. calculation following formula (7)).

**Theorem 7.4.** Let $X$ be an orientable $n$-manifold. Assume that $H_1(X) = \mathbb{Z}$. Then $X$ admits metrics which are simultaneously $(1, n - 1)$-free and $(2, n - 2)$-free.
Proof. Let $C_2 \subset X$ be a surface whose connected components $C_{2,i}$ define a rational basis for 2-dimensional homology. Let $B = B_{n-1}$ be a hypersurface representing a generator of the group $H_{n-1}(X) = \mathbb{Z}$. We place $C_2$ and $B$ in transverse position. The intersection $C_2 \cap B$ is a disjoint union of imbedded circles $C_\alpha \subset C_2$.

The reason for requiring a unit first Betti number is that, for $b_1 \geq 2$, distinct components of $B_{n-1}$ may have a nonempty triple intersection

$$B_{n-1,i} \cap B_{n-1,j} \cap C_2 \neq \emptyset,$$

which by transversality is a finite set. Thus, the collection of loops on $C_2$ is in general not disjoint when $b_2 \geq 2$, and the construction of $(2,n-2)$-freedom cannot be accomplished with bounded geometry near $B$.

Let $f : C_2 \to S^2$ be the map of Lemma 7.3. We now proceed as in the proof of Theorem 6.1. We first construct $(2,n-2)$-free metrics, as in Proposition 5.2, by a modification in a neighborhood $U_2 \subset X$ of the surface $C_2$, with bounded geometry in a neighborhood of the loops $C_\alpha$. Here the map of formula (7) is replaced by the map $f_2 : C_2 \times D^{n-2} \to S^2 \times D^{n-2}$, which is the map $f$ of Lemma 7.3 times the identity on the disk. Let $Y'' \subset U_{2,i}$ denote the submanifold $Y'' = S^{2-1} \times C \times S^1 \times K \times I$ constructed as in section 3. Then the hypersurface $B$ may be assumed to satisfy

$$f_2(B) \cap Y'' = \bigcup_\alpha C_\alpha \times S^1 \times K \times I,$$

so as to guarantee bounded geometry in a neighborhood of the hypersurface $B$.

The resulting metrics may have zero injectivity radius, since the map $f_2$ is no longer regular (as it was in Proposition 7.2). However, a loop representing a homology class of infinite order must meet the hypersurface $B$ by Poincare duality. Hence its length is bounded below by the injectivity radius in a neighborhood of $B$. Finally, we modify the metric in a neighborhood of a loop $C_1$ representing a generator of $H_1(X)$, as in Proposition 5.2, to attain $(1,n-1)$-freedom.

Acknowledgment. The author is grateful to I. Babenko for catching an error in an earlier version of the paper, for signaling the reference [W] (cf. 5.1 above), and for a number of helpful suggestions. Sh. Weinberger contributed the comments following Theorem 6.2, for which I thank him warmly. The author expresses appreciation to L. Ambrosio, V. Bangert, F. Morgan, and B. White for insightful comments, and to D. Garber for an expert drawing.
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