A Multifront Problem of Freezing-Thawing Moist Soil

B G Aksenov¹, Y E Karyakin², S V Karyakina³

¹Department of Industrial thermal Power Engineering, Industrial University of Tyumen, Volodarskogo str. 38, Tyumen, 625001, Russia
²Department of Information Systems, Tyumen State University, Perekopskaya str. 15a, Tyumen, 625013, Russia
³Department of business informatics and mathematics, Industrial University of Tyumen, Volodarskogo str. 38, Tyumen, 625001, Russia

E-mail: aksenovbg@tyuiu.ru

Abstract. Construction activities in the North of Western Siberia is carried out in specific conditions of the cryolithozone. The study of heat transfer processes in freezing soils near buildings is necessary to ensure their successful operation. Heat exchange with phase transition is traditionally described by Stefan problem, which is a system of differential equations of parabolic type with standard boundary conditions and a subsidiary condition on the phase transition boundary. It is possible to formally pass on to one equation of the conductivity type, but in that case the delta function appears in one of the coefficients. The delta function reflects the Joule heat release at the phase transition temperature. A widespread "pass-through counting" method reduces the Stefan problem to a boundary problem for a nonlinear heat conductivity equation. With this approach, the calculation results in a temperature field. But it is difficult to identify the position of the phase transition boundary on the temperature field. A large number of methods for solving the Stefan problem are developed, in which the required value is the phase transition front coordinate. The common disadvantage of these methods is their unsuitability for situations with several fronts, when these fronts appear and disappear, change the movement direction, merge with each other. The article presents a method for solving the Stefan problem, which allows to obtain the front coordinate as a zero isotherm. As an example this method is used to solve the problem of freezing-thawing of moist soil under the influence of seasonal surface temperature fluctuations. The method eliminates the need to control the evolution of each front. The Stefan problem is considered as a limiting case of the general phase transition problem in a certain temperature range. Standard transformations and application of Green's function allow to write down a problem in the form of an integral equation. The approximate solution is obtained in the form of a recurrent formula.

1. Introduction

When designing, constructing and operating buildings in cryolithozone specific conditions it is necessary to study the heat transfer processes in freezing soils near the buildings. Of particular importance is the forecast of the freezing-thawing front evolution during the operation life. According to the obtained results measures to prevent the catastrophic degradation of permafrost can be developed.

Heat exchange processes with phase transitions, like seasonal thermophysical processes in moist soils, are usually described by the Stefan problem [1, 2]. Exact solutions of the Stefan problem are
either obtained under restrictive conditions [1] or are completely unsuitable for practical calculations [3]. In engineering practice, various approximate methods [4, 11-21] are used, and if the rigorous approach is necessary then numerical methods are used. The use of standard mathematical methods of differential equations numerical modeling, such as the transition to finite differences, is hampered by the presence of the phase transition front. The front "smearing", which consist in replacing the Delta function by a Delta-shaped function, removes these problems, since the nonlinear heat equation is solved and the front as such is out of the consideration. With this approach, the calculations result is a temperature field, due to which it is difficult to identify the phase transition boundary position. In case of the "smeared" front a zero isotherm is not a specific line on which the phase transition heat is released/absorbed. And since the "smeared" front should cover several sampling discretization points on the spatial coordinate, the determination of the front position is possible only with a large error, although the error in determining the temperature is small. Meanwhile, the temperature field calculation is often made mainly to determine the phase transformations dynamics. The temperature itself is less important. Therefore, a large number of methods for solving the Stefan problem are developed, in which the required value is the front coordinate [5, 6]. The common disadvantage of these methods is that they are not suitable to situations with several fronts, when these fronts appear and disappear, change the movement direction, merge with each other. In this case, it is difficult to monitor the movement of each front.

This article presents a method for solving the Stefan problem, which allows to obtain the coordinate of the front as a zero isotherm as illustrated by the problem of freezing-thawing of moist soil under the influence of seasonal fluctuations in surface temperature. In this method there is no need to specially monitor each front evolution.

2. Method for solving the multiphont problem of moist soil freezing-thawing

The Stefan problem is considered as a limiting case of a general phase transition problem in a certain temperature range. In [8] this problem is phrased as follows:

$$c(t) \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(t) \frac{\partial t}{\partial x}\right) - \kappa \frac{\partial W(t(x, \tau))}{\partial \tau}$$

$$\tau > 0, x > 0, t(0, \tau) = F(\tau), t(x, 0) = t_u = \text{const}, |t(x, \tau)| < M .$$

Where \(t, c, \lambda, \tau, x, \kappa\) are temperature, heat capacity, coefficient of thermal conductivity, time, spatial coordinate, latent heat of phase transition of water multiplied by the density of soil respectively; \(W\) is the unfrozen moisture content; \(F(\tau)\) is arbitrary, for example periodic, function with limited variation; \(M\) is a positive constant.

Using the Kirchhoff substitution and Golant equation (1) is reduced to form

$$\frac{\partial u}{\partial \tau} = K \frac{\partial^2 t}{\partial x^2} - \frac{\partial V}{\partial \tau} ,$$

where \(u, V\) are some functions of \(t\); \(K\) is a constant.

By taking

$$c(t) = c = \text{const}, \lambda(t) = \lambda = \text{const}, a = \frac{\lambda}{c}$$

equation (1) is reduced to form

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} - \frac{\kappa}{c} \frac{\partial W(t(x, \tau))}{\partial \tau}$$
with boundary conditions (2).

As shown in [9-10] the solution of this problem can be phrased in the form of an integral equation

$$
t(x, \tau) = t_0(x, \tau) - \frac{\kappa}{c} \int_{00}^{\infty} G(x, \xi, \tau - y) \frac{\partial W(t(\xi, y))}{\partial \tau} d\xi dy, \tag{4}
$$

where Green function is

$$
G(x, \xi, \tau - y) = \frac{\exp\left(-\frac{(x - \xi)^2}{4a(\tau - y)}\right) - \exp\left(-\frac{(x + \xi)^2}{4a(\tau - y)}\right)}{2(\pi a(\tau - y))^{1/2}},
$$

$$
t_0(x, \tau) \text{ is solution of the equation } \frac{\partial t_0}{\partial \tau} = a \frac{\partial^2 t_0}{\partial x^2}, \text{ with the conditions of the form (2).}
$$

From (2), (3) we obtain the Stefan problem, assuming that \( \frac{\partial W}{\partial \tau} \) on the moving boundary of the transition is equal to the humidity \( W_0 \) multiplied by the Delta function and is equal to zero at all other points. We define the function \( p(x) \) as the solution of the equation \( t(x, \tau) = t_p \) solvable for \( \tau \) where \( t_p \) is the temperature of the phase transition. Hereafter we take \( t_p = 0 \).

Under the conditions (2) \( p(x) \) is non-unique, but it can be expanded into a finite number of unique branches: \( p_i(x), i = 1, 2, ..., n(x) \), where \( p_i(x^*) \) is the time points in which the fronts of the phase transitions pass a point \( x^* \).

To take into account the direction of the phase transition, we also define the functions

$$
z(p_i) = \text{sign}\left(\frac{\partial t}{\partial \tau}_{x = p_i}\right), \ i = 1, ..., n(x).
$$

Then the equation (3) is reduced to the form

$$
\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} - \frac{\kappa W_0}{c} \cdot z(p(x)) \delta(p(x)), \tag{5}
$$

where \( \delta(x) \) is the Dirac Delta function, and the equation (4) can be reduced to the form

$$
t(x, \tau) = t_0(x, \tau) - \frac{\kappa W_0}{c} \int_0^{x_m} \left( \sum_{i=1}^{n} z(p_i(\xi))G(x, \xi, \tau - p_i(\xi)) \right) d\xi, \tag{6}
$$

where \( x_m \) is the maximum value of the front coordinate.

From (6) we obtain a method for approximate solution of the problem using a recursive formula

$$
t_{j+1}(x, \tau) = t_0(x, \tau) - \frac{\kappa W_0}{c} \int_0^{x_m} \left( \sum_{i=1}^{n} z(p_{ji}(\xi))G(x, \xi, \tau - p_{ji}(\xi)) \right) d\xi, \tag{7}
$$

where \( i = 1, ..., n(x), \ j = 1, 2, ... \).
For the initial approximation $t_0(x, \tau)$ is taken, $p_{0i}(x), i = 1, 2, ..., n_0(x)$ according to which $p_{0i}(x), i = 1, 2, ..., n_0(x)$ are defined as unique branches of the solution of the equation $t_0(x, \tau) = 0$, as well as $z(p_{0i})$ by the formula

$$z(p_{0i}) = \text{sign} \left( \frac{\partial t_0}{\partial \tau} |_{\tau = p_{0i}} \right).$$

From successive approximations $p_{ji}, z(p_{ji})$ are found in a similar way.

To find $p_{ji}$ it is possible to use one of interpolation methods on the known temperatures field. The procedure (7) gives good results and is easily software-implemented on the computer. However, the need to store in memory a two-dimensional temperatures array $t_j(x, \tau)$ used for the calculation of $t_{j+1}(x, \tau)$ is a significant obstacle to its use for the calculation of long processes with great accuracy. Therefore, it is advisable to apply the iterative formula (7) at each time step, and to consider the result as an initial condition for the next time step.

So, we consider the moments $0, \tau_1, \tau_2, ..., \tau_k, ..., \tau_\infty$, where $\tau_k = \Delta \tau \cdot k; \Delta \tau$ is the sample spacing. Let us assume that the temperature field at the instant of time $\tau_k$ is known. Then to determine $t(x, \tau_{k+1})$ the following integral equation can be phrased

$$t(x, \tau_{k+1}) = t_k(x, \tau_{k+1}) - \frac{\kappa W_0}{c} x_m \int_0^1 \left( n(\xi) \sum \frac{z(p_i(\xi))G(x, \xi, \tau - p_i(\xi))}{i=1} \right) d\xi,$$  

(8)

where $t_k(x, \tau)$ is the solution of the problem

$$\frac{\partial t_k}{\partial \tau} = a \frac{\partial^2 t_k}{\partial x^2}, \tau > \tau_k, x > 0;$$

(9)

$$t_k(0, \tau) = F(\tau); t_k(x, \tau_k) = t(x, \tau_k); |t_k(x, \tau)| < M.$$

The solution of the problem (9) is well known [1].

Equation (8) is solved by successive approximations when the initial approximation is $t_k(x, \tau)$. The recurrent procedure begins with $t(x, 0) = t_u$. Following the method [7] it is easy to show that the iterative process (8) converges on a sufficiently small segment $\tau_k \leq \tau \leq \tau^*$. Therefore, it is always possible to take $\Delta \tau < \tau^*$ which provides a solution. Using formula (8) instead of (7) significantly increases the count rate.

3. Results of solving the multiphront problem of freezing-thawing of moist soil

Table 1 shows the calculation results for the following initial data values:
\[ F(\tau) = t_u + t_c \sin(\omega \tau + \varepsilon); \quad t_u = 0.7^\circ C; \quad t_c = 15^\circ C; \]
\[ \omega = 1.9722 \cdot 10^{-7} \, \text{1/s}; \quad \varepsilon = 3.1416; \quad a = 0.56 \cdot 10^{-6} \, \text{m}^2/\text{s}; \]
\[ c = 2093 \frac{kJ}{\text{m}^3 \cdot K}; \quad \kappa = 1339776 \frac{kJ}{\text{m}^3}; \quad W_0 = 0.4. \]

**Table 1.** Results of the numeric simulation/

| \( \tau \) (hr) | \( x \) (m) |
|----------------|------------|
| 0.3            | 0.6        | 0.9        | 1.2        | 1.5        | 1.8        | 2.1        | 2.4        | 2.7        | 3.0        |
| 730            | -2.0       | 0.3        | 0.4        | 0.4        | 0.5        | 0.5        | 0.6        | 0.6        | 0.6        |
| 2190           | -9.6       | -7.1       | -4.1       | -0.3       | 0.3        | 0.3        | 0.4        | 0.4        | 0.4        | 0.5        |
| 3650           | -5.6       | -4.8       | -3.8       | -2.7       | -1.5       | -0.1       | 0.3        | 0.3        | 0.4        | 0.4        |
| 5110           | 2.6        | -0.3       | -0.3       | -0.2       | -0.2       | -0.1       | 0.2        | 0.3        | 0.3        | 0.3        |
| 6570           | 11.2       | 7.4        | 3.7        | 0.1        | -0.2       | -0.1       | 0.1        | 0.3        | 0.3        | 0.3        |
| 8030           | 6.9        | 5.6        | 4.2        | 2.7        | 1.1        | 0.0        | 0.2        | 0.3        | 0.3        | 0.3        |
| 9490           | -2.0       | 0.3        | 0.3        | 0.3        | 0.3        | 0.2        | 0.2        | 0.3        | 0.3        | 0.3        |
| 10950          | -9.6       | -7.1       | -4.1       | -0.3       | 0.3        | 0.2        | 0.2        | 0.3        | 0.3        | 0.3        |
| 12410          | -5.6       | -4.8       | -3.9       | -2.8       | -1.6       | -0.2       | 0.2        | 0.3        | 0.3        | 0.3        |
| 13870          | 2.6        | -0.3       | -0.3       | -0.3       | -0.2       | 0.1        | 0.2        | 0.3        | 0.3        | 0.3        |
| 15330          | 11.1       | 7.3        | 3.6        | 0.0        | -0.2       | -0.2       | -0.1       | 0.1        | 0.3        | 0.3        |
| The solution according to Samarskii method |
| 5110           | 2.6        | 0.0        | -0.1       | -0.3       | -0.2       | 0.0        | 0.1        | 0.2        | 0.3        | 0.3        |
| 13870          | 2.6        | 0.1        | -0.1       | -0.3       | -0.2       | -0.1       | 0.0        | 0.2        | 0.3        | 0.3        |

Table 1 shows the temperature values for the different \( x \) and \( \tau \). The results of parallel calculation by Samarskii method are given to compare with two available values of \( \tau \).

**4. Conclusion**

The obtained results are in good agreement with the calculations by the method of A. Samarskii when finding the temperature. If in the Samarskii method zero isotherm is considered to be the interphase boundary, the divergence turns out to be expectedly significant.

It should be noted that the formulas (7), (8) are obtained as a generalization to the Stefan problem of the methods used in [8-10] to obtain estimates of the freezing–thawing problems in the form of a functions system that alternately majorize the required solution from above and from below. For the monotonous Stefan problems building estimations and their successive refinement provokes no principal objections. The approximate solutions, for example, obtained by the Leibzenzon method, possessing the properties of estimates, should be taken as the first approximations, and then (7) or (8). However, it is not possible to apply techniques [9], [10] directly to non-monotinous Stefan problems, so that additional research is required to obtain estimates of such problems.

Numerical experiments show that the method described in this article is a correct and convenient way of computer simulation of Stefan multi-frontal problems.
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