Can non-private channels transmit quantum information?

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We study the power of quantum channels with little or no capacity for private communication. Because privacy is a necessary condition for quantum communication, one might expect that such channels would be of little use for transmitting quantum states. Nevertheless, we find strong evidence that there are pairs of such channels that, when used together, can transmit far more quantum information than the sum of their individual private capacities. Because quantum transmissions are necessarily private, this would imply a large violation of additivity for the private capacity. Specifically, we present channels which display either (1) A large joint quantum capacity but very small individual private capacities or (2) a severe violation of additivity for the Holevo information.

Shannon’s information theory, which mathematically formalizes the problem of communication in the presence of noise, underlies the reliability of all modern communications technologies \cite{Shannon}. The cornerstone of Shannon’s theory is his capacity formula, which gives an elegant expression quantifying the capability of a communication channel for noiseless transmission. Capacities quantify the ultimate limits on communication with a physical channel, and provide essential insight for the design of practical error correction and mitigation schemes \cite{Cove}. The starting point of information theory is to model the noise in a communication link probabilistically. In many physical systems this is a reasonable approximation, as evidenced by the engineering success of the theory. However, the physical systems underlying all communication are fundamentally quantum mechanical and when quantum effects become prominent, classical probabilistic modeling will provide a poor approximation. One of the first quantitative investigations of this was the work of Holevo \cite{Holevo}, who gave an upper bound on the capacity of a noisy quantum channel for classical communication.

In these early investigations, quantum effects were generally considered to be a nuisance—quantum mechanics was a fundamental source of noise that had to be dealt with to enable faithful communication. In contrast, in 1984 Bennett and Brassard suggested \cite{BB84} that quantum effects might be useful for carrying out communication and cryptographic tasks that are impossible in a classical theory. Specifically, they proposed a quantum method for unconditionally secure key distribution and classical communication. These ideas spawned a broad array of work on both the theory and experiment of quantum key distribution, and there are many groups worldwide working on practical implementations. Indeed, quantum key distribution is, and probably will remain for some time, the only practical quantum information based technology.

Much as the classical capacity of a channel characterizes its capability for noiseless classical communication, the private capacity of a quantum channel tells us about a channel’s capability for communication that is secret from an eavesdropper. More formally, the classical capacity of a quantum channel $\mathcal{N}$ is denoted by $C(\mathcal{N})$, and is defined as the maximal number of bits per channel use that can be sent with transmission errors vanishing in the asymptotic limit. The private capacity has the additional constraint that an eavesdropper with access to the environments of the channels used \cite{H09} can learn arbitrarily little about the key.

Unfortunately, unlike the classical capacity of a classical channel, no simple characterization is known for either the classical or private capacity of a quantum channel. For example, the classical capacity of a quantum channel is known \cite{MLT,smolin} to satisfy

\[ C(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \chi(\mathcal{N}^\otimes n), \]

where the Holevo information is defined as

\[ \chi(\mathcal{N}) = \max_\mathcal{E} \chi(\mathcal{N}, \mathcal{E}) \]

with

\[ \chi(\mathcal{N}, \mathcal{E}) = S \left( \sum_i p_i N(\rho_i) \right) - \sum_i p_i S(N(\rho_i)) \]

for $\mathcal{E}$ an ensemble $\{p_i, \rho_i\}$ of probabilities $p_i$ and quantum states $\rho_i$ and $S(\rho) = -\text{Tr}\rho \log \rho$ is the von Neumann entropy. For some special channels it is known that $C(\mathcal{N}) = \chi(\mathcal{N})$—in other words, the limit is unnecessary. It has, however, recently been reported that there exist channels for which this is not true \cite{H09}, though the violation is extremely small. Similarly, the private capacity satisfies

\[ P(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} P^{(1)}(\mathcal{N}^\otimes n), \]

where the private information is defined as

\[ P^{(1)}(\mathcal{N}) = \max_\mathcal{E} \left( \chi(\mathcal{N}, \mathcal{E}) - \chi(\tilde{\mathcal{N}}, \mathcal{E}) \right) \]

where the complementary channel $\tilde{\mathcal{N}}$ is defined below \cite{H09}. In this case, it is known that the limit in Eq. (4)
cannot be removed in general \[8\], even for some very natural qubit channels.

These difficulties in evaluating capacities are closely related to the family of problems known as additivity problems. A real function, \( f \), on the set of quantum channels is said to be additive if \( f(N \otimes M) = f(N) + f(M) \). Determining whether a given function is additive is a problem that arises constantly in quantum information science in a variety of very natural settings \[9\]. For example, if it were possible to show that \( \chi \) is additive, we would immediately be able to conclude that \( C(N) = \chi(N) \). Similarly, the fact that the regularization in Eq. (4) cannot be removed is a consequence of the fact that \( P^{(1)} \) is not additive. In the context of quantum Shannon theory, the importance of additivity questions is twofold: first, showing additivity of some entropic quantity may often lead to a simple capacity formula; and second, when a capacity is additive it uniquely specifies the channel’s communication capabilities independent of what other channels may be available.

The quantum capacity of a channel is the maximal rate, in qubits per channel use, at which a sender can reliably transmit quantum information in the asymptotic limit. The essential feature of the quantum capacity is that transmission must be reliable not only on a set of orthogonal states, but also on arbitrary superpositions. The quantum capacity of a channel \( N \) is denoted \( Q(N) \).

It was recently shown that the capacity of a quantum channel for quantum communication is not additive \[10\]. In fact, the quantum capacity is very strongly non-additive: there are pairs of quantum channels \( N \) and \( A \), both with a quantum capacity of zero, that nevertheless can be combined to achieve a positive capacity: \( Q(N) = Q(A) = 0 \) but \( Q(N \otimes A) > 0 \), where \( Q \) is the quantum capacity. This superactivation is not yet completely understood, but from \[10\] it seemed to be related to the existence of channels, termed “private Horodecki channels”, with zero quantum capacity but positive private capacity \[11, 12\]. Indeed, at the heart of \[10\] is an argument showing there is an \( A \) with \( Q(A) = 0 \) such that if \( N \) has \( Q(N) = 0 \) but \( P(N) > 0 \) the joint capacity \( Q(N \otimes A) \geq (1/2)P(N) \). One interpretation of this effect is that, while neither \( N \) nor \( A \) is capable of transmitting noiseless quantum information, the two channels have complementary capabilities for communication which can be combined for sending quantum information. Naturally, one would expect \( N \)’s capability is somehow related to its private capacity.

In this work we connect the additivity questions for the Holevo information and the private capacity by showing that either one or the other is highly nonadditive. Specifically, we show that for every \( \epsilon > 0 \) there is a family of channels \( R^d \) with increasing input dimension \( d \) and a channel \( A \) with \( P(A) = 0 \) such that either (1) \( P(R^d_\epsilon \otimes A) \) is \( O(\log d) \) larger than \( P(R^d_\epsilon) \) or (2) \( C(R^d_\epsilon) \) is \( O(\log d) \) larger than \( \chi(R^d_\epsilon) \). Assuming the additivity of \( \chi \) for this channel, which we regard as more likely, allows us to conclude that \( P(R^d_\epsilon) \leq \epsilon \) but \( P(R^d_\epsilon \otimes A) \gtrsim 1/2 \log d \).

Thus, while it was natural to conjecture that “privacy” is the feature that the private Horodecki channel contributes allowing the superactivation effect, our results suggest the situation cannot be quite as simple as that. Indeed, it appears that two channels with little or no private capacity can be combined to send an arbitrarily large amount of private and even quantum data.

Our main building block in what follows will be the retro- or echo-correctable channels of \[13\] (see FIG. 1). The standard echo-correctable channel \( R^d_\epsilon \) has a \( d \)-dimensional data input and a corresponding output; a control input of dimension \( c = (K/\epsilon^2)d\log(d)\)\[19\] with \( K \) a constant; and an infinite-dimensional classical control output. The channel internally selects a random basis \( b \), for \( H_c \), and a set of \( c \) random unitaries \( \{U\} = U_1...U_c \) on \( H_d \). The channel measures the control input in the basis \( b \), yielding result \( j \in \{1...c\} \) and according to that result applies one of the unitaries \( U_i \) to the data input, which is then emitted as the data output \( B_1 \). The channel also emits a classical control output \( B_2 \) consisting of the random basis \( b \) and the set of random unitaries \( \{U\} = U_1...U_c \). It does not, however, emit the measurement result \( i \) but keeps it hidden.

It can be shown \[13\] that for any \( \epsilon > 0 \) and sufficiently large \( d \) that \( \chi(R^d_\epsilon) \leq \epsilon \). Thus, if \( C = \chi \) such a channel has almost no classical capacity and since the classical capacity upper bounds \( P \), it too becomes small.

However, when used in combination with an erasure channel \( A^p \) which takes a \( c \)-dimensional input and with probability \( 1-p \) transmits the input to the output perfectly, but with probability \( p \) outputs only an erasure flag.
then the combination has a great deal of both quantum and private capacity: $\mathcal{P} \geq \mathcal{Q} \simeq (1 - p) \log d$. This is most striking, of course, when $p \geq 1/2$ since it is then that the erasure channel has no private or quantum capacity at all [14].

The way to use the two channels together is shown in FIG. 2. Alice prepares a maximally-entangled state of $d \times d$-dimensions on Hilbert space $AA'$ and another of $c \times c$-dimensions on space $FF'$. She feeds the $A'$ and $F'$ systems into the data and control inputs of $\mathcal{R}_d$, respectively, and she also puts the $F$ system into the erasure channel, whose output we will call $B_3$.

The coherent information of the resulting bipartite state $\varrho_{AB_1B_2B_3}$ is a lower bound on $\mathcal{Q}$ and $\mathcal{P}$ [13, 10, 17]:

$$I_{coh} = S(B_1B_3|B_2) - S(AB_1B_3|B_2).$$

(6)

Here, since $B_2$ is classical, the conditional entropies are given by averages over $b_2$, the possible values of $B_2$:

$$S(B_1B_3|B_2) = \int db_2 S(\varrho_{B_1B_3}^{b_2})$$

and

$$S(AB_1B_3|B_2) = \int db_2 S(\varrho_{AB_1B_3}^{b_2})$$

where $\varrho_{AB_1B_3}^{b_2}$ and $\varrho_{B_1B_3}^{b_2}$ are conditional states given $b_2$. Using the slightly nonstandard expression in Eq. (6) allows us to avoid any complications due to the fact that $B_2$ is infinite dimensional.

The coherent information is straightforward to calculate since the erasure channel's flag breaks the quantity into the sum of two terms:

$$I_{coh} = (1 - p)I_{coh}^{not \, erased} + pI_{coh}^{erased}. \quad (7)$$

In the unerased case, since Bob knows what basis to measure in he can measure the $F$ system which he has received through the successful use of the erasure channel and determine exactly which $U_i$ has occurred. Thus

$$I_{coh}^{not \, erased} = \log d. \quad (8)$$

When the $F$ system is erased, the the conditional entropy of the $AB_1$ given $B_2$ is at most $\log c = \log d + 4 \log \log d + \log(K/\epsilon^2)$, while the conditional entropy of $B_1$ given $B_2$ is $\log d$. So, we have

$$I_{coh}^{erased} \geq -4 \log \log d - \log(K/\epsilon^2) \quad (9)$$

and

$$I_{coh} \geq (1 - p) \log d - p(4 \log \log d + \log(K/\epsilon^2)), \quad (10)$$

which is positive as $d \to \infty$ for

$$\frac{1 - p}{p} > 4 \log \log d + \log(K/\epsilon^2) \quad (11)$$

A consequence of the above argument is that at least one of $\mathcal{P}$ and $\chi$ violates additivity severely. In particular, let $p = 1/2$, if $\mathcal{P}(\mathcal{R}_d^i \otimes \mathcal{A}_e) - \mathcal{P}(\mathcal{R}_d^i) = O(\log d)$ we have $\mathcal{P}(\mathcal{R}_d^i) = O(\log d)$. Since $C \geq \mathcal{P}$, this implies that $C(\mathcal{R}_d^i) = O(\log d)$ while $\chi(\mathcal{R}_d^i) \leq \epsilon$. Otherwise, if $\mathcal{P}(\mathcal{R}_d^i \otimes \mathcal{A}_e) - \mathcal{P}(\mathcal{R}_d^i) = O(\log d)$, we have a large violation of additivity for $\mathcal{P}$, since then $\mathcal{P}(\mathcal{A}_e) = 0$, but $\mathcal{P}(\mathcal{R}_d^i \otimes \mathcal{A}_e) \gg \mathcal{P}(\mathcal{R}_d^i)$. We have plotted the joint and individual capacities or $\mathcal{R}_d^i$ (assuming it has additive $\chi$) and $\mathcal{A}_e^i$ in FIG. 3.

In summary, we have explored violations of additivity arising from two channels, $\mathcal{R}_d^i$ and $\mathcal{A}_e^i$. $\mathcal{R}_d^i$ is a retro-correctable channel, described in FIG 1, and satisfies $\chi(\mathcal{R}_d^i) \leq \epsilon$. $\mathcal{A}_e^i$ is a 50% quantum erasure channel, whose private capacity is zero. Our main result, illustrated in FIG 2, is that these two channels can be used together to transmit large amounts of quantum information. This leaves only two possibilities: either (1) $\mathcal{R}_d^i$ has a large classical capacity, which would imply severe
nonadditivity of $\chi$ or (2) a large violation of additivity for the private capacity.

As we have mentioned above, we consider the extreme nonadditivity of $\chi$ for the retro-correctable channel to be rather unlikely, and tend to believe instead that it is $P$ which is nonadditive. We believe this despite the recent results of Hastings [12] since his results are a tiny effect for a very specially designed family of channels. It is nevertheless an important open problem to find an argument that shows nonadditivity of the private capacity without additivity assumptions on $\chi$. While the $p \geq 1/2$ erasure channels we have used have quantum and private capacities exactly equal to zero, we have only been able to show that the retro-correctable channels, $R_d^c$, have capacity less than $\epsilon$ (even this is conditional on the additivity of $\chi$). One would hope for the stronger result of pairs of channels with strictly zero private capacity that can jointly allow nonzero private capacity. This would be parallel to the quantum capacity findings in [10]. In the quantum setting, there are two distinct types of zero capacity channels—PPT channels and channels whose environment can simulate the channel output (sometimes called “antidegradable”). Unfortunately, the only type of channels known to have zero private capacity are the antidegradable ones (which include symmetric channels as a special case). Because the product of antidegradable channels is itself antidegradable, and therefore has zero private capacity, a necessary step for finding genuine superactivation would be identifying a class of channels with zero private capacity that are not antidegradable. Finding such channels is an intriguing open problem.

We do know that the classical capacity is often a very weak bound on the private capacity (for example, a classical channel always has exactly zero private capacity, regardless of its classical capacity). It is then plausible that the private capacity of $R_d^c$ or if not that $R_d^c$ with some small additional noise that would leave the joint capacity essentially unchanged, may actually be zero. If this is so, then the superactivation effect of [10] requires no privacy and we are left to wonder just what it is that $R_d^c$ provides.

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[18] Any channel $N$ can be expressed as an isometry followed by a partial trace: $N(\rho) = \text{Tr}_E U \rho U^\dagger$, with $U: A \to BE$ satisfying $UU^\dagger = I$. $E$ is referred to as the environment of the channel, and the eavesdropper is given access to the channel $\hat{N}(\rho) = \text{Tr}_B U \rho U^\dagger$.
[19] The alert reader will note the additional power of log $d$ beyond that needed in [10]. This is the result of a slightly more conservative approach we have taken and, being a sublinear factor, doesn’t change any capacity quantity or qualitative result.