The trapping effect on degenerate horizons

Y. Angelopoulos\textsuperscript{1}, S. Aretakis\textsuperscript{1}, and D. Gajic\textsuperscript{2}

\textsuperscript{1}Princeton University, Department of Mathematics, Fine Hall, NJ 08544, USA
\textsuperscript{2}University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

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Abstract

We show that degenerate horizons exhibit a new trapping effect. Specifically, we obtain a non-degenerate Morawetz estimate for the wave equation in the domain of outer communications of extremal Reissner–Nordström up to and including the future event horizon. We show that such an estimate requires 1) a higher degree of regularity for the initial data, reminiscent of the regularity loss in the high-frequency trapping estimates on the photon sphere, and 2) the vanishing of an explicit quantity that depends on the restriction of the initial data on the horizon. The latter condition demonstrates that degenerate horizons exhibit a global trapping effect (in the sense that this effect is not due to individual underlying null geodesics as in the case of the photon sphere). We moreover uncover a new stable higher-order trapping effect; we show that higher-order estimates do not hold regardless of the degree of regularity and the support of the initial data. We connect our findings to the spectrum of the stability operator in the theory of marginally outer trapped surfaces (MOTS). Our methods and results play a crucial role in our upcoming works on linear and non-linear wave equations on extremal black hole backgrounds.

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1 Introduction

1.1 Overview

Black holes are one of the most celebrated predictions of general relativity and as such their stability properties are of fundamental importance. The first step in resolving the non-linear stability problem for black holes is to establish quantitative decay estimates for the wave equation

$$\square_g \psi = 0$$

on fixed black hole backgrounds. One of the main difficulties in the analysis of the wave equation on such backgrounds is the so-called trapping effect due to the existence of a family of trapped null geodesics in the domain of outer communications whose limit point is the future timelike infinity. In the well-known Kerr–Newman family of black holes, there is a family of trapped null geodesics with constant area radius $r$. In the special subfamily of Schwarzschild backgrounds these trapped null geodesics span the hypersurface $r = 3M$ known as the photon sphere. Here $M$ is the mass parameter. From every other point in the Schwarzschild exterior region there is a codimension-one subset of future-directed null directions whose corresponding geodesics approach the photon sphere, and all other null geodesics either cross the event horizon $\mathcal{H} = \{ r = r_{\text{hor}} \}$ or terminate at null infinity.

The trapped null geodesics pose a well-known high-frequency obstruction, known as the trapping effect, for the existence of a non-degenerate Morawetz estimate for the wave equation of the form

$$\int_0^T \int_{\Sigma_t \cap \{ r_{\text{hor}} < R_1 \leq r \leq R_2 \}} |\partial_\tau \psi|^2 \, dg_{\Sigma_t} \, dt \leq C \int_{\Sigma_0} |\partial_\nu \psi|^2 \, dg_{\Sigma_0},$$

(1.2)
where the trapped null geodesics exist in the region \( r_{\text{hor}} < R_1 \leq r \leq R_2 \). The obstruction for (1.2) originates from the existence of high-frequency solutions to the wave equation with finite initial energy which are supported in a neighborhood of trapped null geodesics for arbitrarily long times. This phenomenon has long been studied in the context of the obstacle problem for the wave equation in Minkowski space, where the analogue of trapped null geodesics are null lines which reflect off the obstacle’s boundary. Recently, Sbierski [31], building on previous work of Ralston [29] on the Gaussian beam approximation, showed that, on general Lorentzian manifolds, the energy at time \( t \) of the localized high frequency solutions is comparable to the energy of the underlying null geodesic at time \( t \). The fact that the energy of the trapped null geodesics in the Kerr–Newman family is constant immediately contradicts estimate (1.2). On the other hand, it can be shown (see, for instance, [13, 14, 16, 35] and references there-in) that on sub-extremal backgrounds estimate (1.2) holds if the right hand side loses derivatives (i.e includes higher order energies).

Having introduced the trapping effect on the photon sphere we next consider the event horizon. The event horizon is a null hypersurface ruled by null geodesics, known as the null generators. In the sub-extremal case, however, one can show a local integrated decay Morawetz estimate in a neighborhood of the event horizon without any loss of differentiation (see [13]). This is possible because the energy of the null generators decays exponentially in time \( t \), in view of the of the so-called redshift effect which in turn is based on the positivity of the surface gravity (see Section 2 for an introduction to these notions). Hence, no trapping takes place on the event horizon of sub-extremal black holes, even though the latter is ruled by null geodesics.

Nonetheless, the situation is drastically different for degenerate horizons, which are null hypersurfaces with vanishing surface gravity. This is the case for the event horizon of extremal black holes. In this case the null generators have energy that is constant in time. Hence, Sbierski’s result implies that loss of regularity is a necessary condition for the existence of a non-degenerate Morawetz estimate up to and including the event horizon. However, until now there had not been any works providing sufficient conditions.

In a series of papers [3, 4, 5] the second author initiated the mathematical study of the wave equation on extremal black holes and obtained a mixture of stability and instability results. Specifically, it was shown that solutions to the wave equation decay in time towards the future, first-order derivatives remain bounded but do not decay along the event horizon whereas higher order derivatives asymptotically blow up along the event horizon. We remark that these stability and instability results do not suffice to determine whether a non-degenerate Morawetz estimate up to and including the event horizon holds for general smooth solutions to the wave equation on extremal black holes (see Section 1.3 for more details).

1.2 Summary of results and techniques

In this paper we derive necessary and sufficient conditions for the existence of a non-degenerate Morawetz estimate in the domain of outer communications of extremal
Reissner–Nordström backgrounds up to and including the event horizon. Such estimates play a fundamental role in the analysis of non-linear wave equations and hence necessary and sufficient conditions for their existence are relevant for the study of the black hole stability problem.

Summary of results

We obtain a complete characterization of the trapping effect on the event horizon of extremal Reissner–Nordström. Specifically, we obtain a non-degenerate Morawetz estimate (see Theorem 3.1 in Section 3) in the domain of outer communications of extremal Reissner–Nordström up to and including the future event horizon. We show that such an estimate requires

1. a higher degree of regularity for the initial data,
2. the vanishing of the quantity $H[\psi]$ given by (3.7). We remark that $H[\psi]$ depends only on the restriction of the initial data on the horizon.

Note that the first condition is reminiscent of the regularity loss in the high-frequency trapping estimates on the photon sphere. In fact, we will show a result in the converse direction (see Theorem 3.3 in Section 3):

- if a weighted higher-order norm of the initial data is infinite then no non-degenerate Morawetz estimate holds.

This implies that the loss of regularity of our result is optimal. We also prove that the vanishing of the quantity $H[\psi]$ is necessary in the following sense (see Theorem 3.2 in Section 3):

- if the quantity $H[\psi]$ given by (3.7) is initially non-zero then no non-degenerate Morawetz estimate holds, regardless of the degree of regularity of the initial data.

The above result demonstrates that degenerate horizons exhibit a new global trapping effect, in the sense that this effect is not due to individual underlying null geodesics as in the case of the photon sphere. This global trapping effect seems to be closely related to the spectrum of the stability operator for the sections of the event horizon (see Section 6).

Furthermore, a new stable higher order trapping effect is uncovered (see Theorem 3.4 in Section 3), in the sense that

- higher order estimates up and including the event horizon do not hold regardless of the degree of regularity and the support of the initial data.

Summary of techniques

We decompose $\psi$ into its spherical mean $\psi_0$ and its projection $\psi_{\geq 1}$ on angular frequencies $\ell \geq 1$ (see Section 4.1) and for each part we use novel physical space vector field multipliers.

\footnote{No decomposition in time frequencies is needed.}
For the spherical mean we use the singular vector field
\[ S = \frac{1}{r - r_{\text{hor}}} \cdot Y \]
as a multiplier vector field (see Section 2.4 for an introduction to the vector field method). Here \( r_{\text{hor}} \) is the radius of the event horizon and \( Y \) is a regular translation-invariant transversal to the event horizon vector field (see Section 2). Clearly, \( S \) is singular on the event horizon. We apply the divergence identity in regions \( A_{r_0} \) (see Section 2, figure 2.1) which do not include the event horizon if \( r_0 > r_{\text{hor}} \) and study the limiting behavior of the resulting equations as \( r_0 \to r_{\text{hor}} \). We use a special structure of the geometry of degenerate horizons to show that all the resulting singular terms can be estimated by singular norms of the initial data. The boundedness of these singular norms (which is an assumption on the initial data only) implies a non-degenerate Morawetz estimate.

One of the most critical terms is the integral
\[ \int_{A_{r_0}} \frac{1}{r - r_{\text{hor}}} T\psi \cdot Y\psi, \]
where \( T \) is the stationary Killing field (see (4.22)). Clearly the integral of the integrand quantity \( \frac{1}{r - r_{\text{hor}}} T\psi \cdot Y\psi \) over a spacial slice \( \Sigma_\tau \) is infinite. However, we were able to show that if we first integrate \( \frac{1}{r - r_{\text{hor}}} T\psi \cdot Y\psi \) over time and then over space then the resulting expression has a finite limit as \( r \to r_{\text{hor}} \). In Section 4.6 it is shown that in the sub-extremal case the corresponding limit is infinite demonstrating thus a new distinctive feature of degenerate horizons.

For the projection \( \psi_{\geq 1} \), we use the vector field \( Y \) as a commutator vector field (see Section 2.4) and the degenerate vector field \( \bar{\partial} = -(r - r_{\text{hor}})Y \) as multiplier vector field. Our estimates follow by appropriate use of Hardy and Poincaré inequalities and the special structure of the wave equation in a neighborhood of the degenerate event horizon which yield various cancellations of the most dangerous terms. It is worth contrasting this with the trapping estimate at the photon sphere where one is required to commute with either the Killing field \( T \) or the standard Killing fields \( \Omega_i, i = 1, 2, 3 \) of the sphere. All these vector fields are tangential to the photon sphere.

We note that our methods and results play a crucial role in our upcoming works on linear and non-linear wave equations on extremal black hole backgrounds.

### 1.3 Previous results

To put our results into context, we briefly summarize previous work on the wave equation on black hole backgrounds. The study of the wave equation (1.1) on black hole backgrounds has a long history, starting in 1957 with the pioneering work of Regge–Wheeler [30] on the mode stability of (1.1) on Schwarzschild (\( a = 0 \)). The first quantitative result in Schwarzschild was obtained in 1987 by Kay–Wald [19], who proved uniform boundedness of solutions to the wave equation. In the last two decades there have been many (partial) results on the asymptotic behaviour of linear waves in the domain of outer communication of sub-extremal Kerr backgrounds, for which \( |a| < M \), culminating in the proof of polynomial decay in time for solutions
to (1.1) on the full sub-extremal range $|a| < M$ of Kerr backgrounds by Dafermos–Rodnianski–Shlapentokh-Rothman [13]; see also [14, 13, 11, 33] for a comprehensive list of references to earlier works. We also refer the reader to the inverse logarithmic decay results for solutions to the wave equation on spacetimes exhibiting stable trapping [18, 25, 20].

The rigorous study of mathematical properties of the wave equation on extremal black holes was initiated in a series of papers [4, 5, 6, 11] where a mixture of stability and instability results was presented. Subsequent works of Reall, Murata, Lucietti et al [27, 26, 22, 23] studied in a series of papers these instability properties on more general linear and non-linear settings. The authors of [27] studied numerically spherically symmetric perturbations of extremal Reissner–Nordström in the context of the Cauchy problem for the Einstein–Maxwell-scalar field system and discovered that derivatives of the scalar field grow, even for arbitrarily small initial perturbations, in complete agreement with the work in the linear case. Ori [28] and Sela [32] have numerically investigated the relation of long time dynamics of scalar fields and the conservation laws. Rigorous non-linear results have appeared in [10, 2]. Further applications and extensions have been presented in [18, 34, 15, 8]. For work in the interior of extremal black holes we refer to [17].

2 Background on the geometry of extremal R–N

2.1 The extremal R–N spacetime

We define the extremal Reissner–Nordström spacetime as the Lorentzian manifold $(\mathcal{M}, g)$, where $\mathcal{M} = \mathbb{R} \times \mathbb{R}_+ \times S^2$. We introduce the *ingoing Eddington–Finkelstein coordinate chart* $(v, r, \theta, \varphi)$, where $v \in \mathbb{R}$, $r \in \mathbb{R}_+$ and $(\theta, \varphi)$ are the standard polar coordinates on the round sphere $S^2$. In these coordinates, the metric can be expressed as:

$$g = -Ddv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $D := r^{-2}(M-r)^2$. This metric is a solution to the Einstein–Maxwell equations.

The null hypersurface $\mathcal{H} := \{r = M\}$ is called the *future event horizon*. The region $\mathcal{M}_{\text{ext}} := \mathcal{M} \cap \{r > M\}$ is known as the *exterior region* or *domain of outer communications* of extremal Reissner–Nordström. The *interior region* $\mathcal{M}_{\text{int}} := \mathcal{M} \cap \{0 < r < M\}$ will not feature in the remainder of this paper.

We will refer to the coordinate $v$ as the *advanced null coordinate*. We can introduce a *retarded null coordinate* $u$, given by $u = v - 2r_*$, where $r_*$ is defined as a solution to $\frac{dr_*}{dr} = D^{-1}$ and is given explicitly by

$$r_*(r) = \frac{M^2}{M - r} + 2M \log(M - r) + r.$$

The tuple $(u, v, \theta, \varphi)$ constitutes a coordinate chart on $\mathcal{M}_{\text{ext}}$, commonly referred to as *Eddington–Finkelstein double-null coordinates*. In these coordinates, the metric
is given by
\[ g = -Ddudv + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]

Figure 1: The Penrose diagram of extremal Reissner–Nordström

Let \( \Sigma \) be an asymptotically flat spacelike hypersurface in \( \mathcal{M} \) such that \( \Sigma \cap \mathcal{H}^+ \neq \emptyset \). Let \( M < R < R_2 \). We define \( \Sigma_0 \) as:

\[ \Sigma_0 = (\Sigma \cap \{ R \leq r \leq R_2 \}) \cup \{ v = v_\Sigma(R), M \leq r \leq R \} \cup \{ u = u_\Sigma(R_2), v_\Sigma(R_2) \leq v < \infty \}. \]

Here, \( v_\Sigma \) and \( u_\Sigma \) denote the restrictions to \( \Sigma \) of the functions \( v \) and \( u \) to, respectively. See Figure 2.1. In this paper, we will only consider the region \( J^+(\Sigma_0) \), which is foliated by the hypersurfaces \( \Sigma_r \) obtained by time-translating \( \Sigma_0 \). We next define several regions and hypersurfaces that will be very important in our analysis. Consider the

Figure 2: Representation of spacetime regions
outgoing hypersurfaces $H_{r_0}$ which are defined as follows:

$$H_{r_0} = J^+ (\Sigma_0) \cap \{ u = u_{\Sigma_0} (r_0), \ r \leq R \}$$

and the following timelike hypersurface:

$$B_R = \{ r = R \} \cap J^+ (\Sigma_0).$$

We will frequently restrict to the region $A_{r_0} \subset J^+ (\Sigma_0)$, where

$$A_{r_0} = J^+ (\Sigma_0) \cap \{ M \leq r \leq R \}.$$

### 2.2 Photon sphere

By the causal geometry of $M_{\text{ext}}$, one can easily infer the existence of geodesics that do not cross $M$ or approach future null infinity $I^+$. A class of these geodesics $\gamma : \mathbb{R} \rightarrow M_{\text{ext}}$ can be parametrized as follows:

$$\gamma (\tau) = (v(\tau), 2M, \frac{\pi}{2}, \varphi (\tau)),$$

where $v(\tau)$ and $\varphi (\tau)$ depend linearly on $\tau$. The timelike hypersurface $\{ r = 2M \}$ is called the photon sphere. As was mentioned before, the existence of these geodesics gives rise to the trapping effect of the photon sphere. This is manifested in the loss of regularity to Morawetz estimates for the wave equation (1.1) (see estimate 3.2).

### 2.3 The red-shift effect

The vector field $T = \partial_u$ is a causal Killing vector field that is timelike in $M_{\text{ext}}$ and null on $\mathcal{H}$. Correspondingly, we can define the surface gravity of $\mathcal{H}$ with respect to $T$ as the function $\kappa : \mathcal{H} \rightarrow \mathbb{R}$, such that

$$\nabla_T T|_{\mathcal{H}} = \kappa \cdot T|_{\mathcal{H}}.$$
the event horizon to another observer crossing the event horizon at a later time is red-shifted. This redshift effect in the context of decay results for the wave equation was first used by Dafermos and Rodnianski in [12]. The degeneracy of the redshift effect in the extremal case introduces an additional difficulty in the analysis of the wave equation on such backgrounds.

2.4 Energy currents and the vector field method

The energy-momentum tensor $T$ corresponding to the wave equation (1.1) is a symmetric 2-tensor with components

$$T_{\alpha\beta}[f] = \partial_\alpha f \partial_\beta f - \frac{1}{2} g_{\alpha\beta}(g^{-1})^{\gamma\delta}\partial_\gamma f \partial_\delta f.$$  

Moreover,

$$\nabla^\beta T_{\alpha\beta}[f] = \partial_\alpha f \Box g f,$$

so $T[\psi]$ is divergence-free for solutions $\psi$ to (1.1).

Let $V$ be a vector field. Then we denote the energy current with respect to $V$ by $J^V$, where $J^V = T(V, \cdot)$. Let $W$ be another vector field, then we denote

$$J^V[f] \cdot W = T[f](V, W).$$

An immediate calculation yields

$$\operatorname{div} J^V[f] = K^V[f] + \mathcal{E}^V[f],$$

where

$$K^V[f] = T^{\alpha\beta}\nabla_\alpha V_\beta,$$
$$\mathcal{E}^V[f] = V^\alpha \nabla^\beta T_{\alpha\beta}[f] = V(f) \Box g f.$$

Note that $\mathcal{E}^V[\psi] = 0$ for solutions $\psi$ to (1.1). Furthermore, $K^V[f] = 0$ if $V$ is a Killing vector field. The vector field method (see also [21]) comprises a careful choice of the vector field $V$ (the vector field multiplier) and $W$, where $f = W \psi$ (the commutation vector field), so as to obtain suitable energy estimates after applying Stokes’ theorem in appropriate spacetime regions.

Consider the vector fields

$$T := \partial_v, \quad Y := \partial_r,$$

with respect to the $(v, r, \theta, \phi)$ coordinate system. The vector field $T$ is timelike in $\mathcal{M}_{\text{ext}}$ and null along $\mathcal{H}$. The vector field $Y$ is regular and transversal to $\mathcal{H}$.

We introduce the vector fields $P, N$ (see also [4, 5]), which close to the event horizon satisfy the following

$$P \sim T - (r - M) \cdot Y,$$
$$N \sim T - Y,$$
Observe that $N$ is future-directed and \textit{strictly} timelike in $J^+(\Sigma_0)$. On the other hand, the vector field $P$ is timelike in $\mathcal{M}_{\text{ext}}$ and null at $\mathcal{H}^+$.

One can easily check that close to the event horizon it holds

\begin{align*}
J_T[f] \cdot N &\sim (Tf)^2 + (r - M)^2 \cdot (Yf)^2 + |\nabla f|^2, \\
J_P[f] \cdot N &\sim (Tf)^2 + (r - M) \cdot (Yf)^2 + |\nabla f|^2, \\
J_N[f] \cdot N &\sim (Tf)^2 + (Yf)^2 + |\nabla f|^2,
\end{align*}

where $\nabla$ denotes the covariant derivative restricted to the spheres of constant $v$ and $r$.

\section{The main theorems}

For the definition of relevant notions and notation and, in particular, for the definition of specific regions, hypersurfaces, vector fields and their fluxes, see section 2.

We study the Cauchy problem for the linear wave equation (1.1) on the exterior $\mathcal{M}$ of the extreme Reissner–Nordström spacetime up to and including the event horizon $\mathcal{H}$ with initial data

$$\psi|_{\Sigma_0} \in H^s_{\text{loc}}(\Sigma_0), \quad D_{\text{tran}}\psi|_{\Sigma_0} \in H^{s-1}_{\text{loc}}(\Sigma_0),$$

where $D_{\text{tran}}$ is a transversal to $\Sigma_0$ vector field. The hypersurface $\Sigma_0$ is as defined in Section 2. We assume that $s$ is sufficiently large so that all the weighted norms of our estimates are finite.

\subsection{Summary of previous results}

For the reader’s convenience we recall some of the key results of [4, 5] for solutions $\psi$ to (1.1) on extremal Reissner–Nordström:

(1) \textbf{Energy boundedness.}

$$\int_{\Sigma_r} (J^V[\psi] \cdot n_{\Sigma_r}) \, dg_{\Sigma_r} \leq C \int_{\Sigma_0} (J^V[\psi] \cdot n_{\Sigma_0}) \, dg_{\Sigma_0}, \quad (3.1)$$

where $V \in \{T, P, N\}$.

(2) \textbf{Integrated local energy decay.}

\begin{align*}
&\int_{\{M \leq r \leq r_0\}} ((T\psi)^2 + |\nabla \psi|^2 + (r - M) \cdot (Y\psi)^2) \\
&\quad + \int_{\{r_0 \leq r \leq r_1\}} ((\partial_r \psi)^2 + (r - 2M)^2 \cdot (T\psi)^2 + |\nabla \psi|^2) \\
&\leq C \int_{\Sigma_0} (J^N[\psi] \cdot n_{\Sigma_0}) \, dg_{\Sigma_0}, \quad (3.2)
\end{align*}
where $M < r_0 < 2M < r_1$. Note that this estimate degenerates both on the photon sphere and on the horizon. Specifically, the degeneracy applies to

- the *tangential* derivatives to the photon sphere,
- the *transversal* derivative to the event horizon.

Removing the degeneracy at the photon sphere comes at the expense commuting with the Killing field $T$. Theorem 3.1 below provides an estimate which does not degenerate at the horizon.

(3) Energy decay

$$\int_{\Sigma_{\tau} \cap \mathcal{A}_0} (J^T[\psi] \cdot n_{\Sigma_{\tau}}) \, dg_{\Sigma_{\tau}} \leq \frac{C}{r^2} \cdot I[\psi],$$

(3.3)

where $I[\psi]$ is a suitable weighted norm of the initial data.

(4) Conserved quantities and instabilities on the horizon.

a) The quantity

$$\int_{\Sigma_{\tau} \cap \mathcal{H}} \left( Y \psi + \frac{1}{M} \psi \right)$$

is conserved along the event horizon (i.e. independent of $\tau$). In fact, in [5] a hierarchy of conservation laws was established for each angular frequency.

b) For generic solutions we have the following blow-up result

$$\max_{\Sigma_{\tau} \cap \mathcal{H}} |Y^k T^m \psi| \to \infty,$$

asymptotically along $\mathcal{H}^+$ as $\tau \to \infty$, for $k \geq m + 2$.

3.2 The statements of the main theorems

The main results of the present paper concern non-degenerate Morawetz estimates up to and including the event horizon $\mathcal{H}$.

**Theorem 3.1 (The trapping estimate).** Consider the following weighted norm on the Cauchy hypersurface $\Sigma_0$

$$D^w_{\Sigma_0}[\psi] := \int_{\Sigma_0 \cap \{r \leq R\}} \left[ \frac{1}{r-M} \cdot \left( Y \psi + \frac{1}{r} \psi \right)^2 \right] \, dr \, d\omega$$

$$+ \int_{\Sigma_0} (J^P[\psi] \cdot n_{\Sigma_0}) \, dg_{\Sigma_0} + \int_{\Sigma_0} (J^P[T\psi] \cdot n_{\Sigma_0}) \, dg_{\Sigma_0}$$

$$+ \int_{\Sigma_0 \cap \{r \leq R\}} (J^P[Y \psi] \cdot n_{\Sigma_0}) \, dg_{\Sigma_0},$$

(3.5)

where the $J^P$ flux and the constant $R$ are as defined in Section 2.
Then, there is a constant $C > 0$ that depends only on the mass parameter $M$ such that for all solutions $\psi$ to the wave equation on extremal Reissner–Nordström which arise from initial data with bounded $D^v_{\Sigma_0}$ norm the following non-degenerate Morawetz estimate holds
\[
\int_A \left[ \psi^2 + (Y\psi)^2 + (T\psi)^2 + |\nabla\psi|^2 \right] \, dg_A \leq C \cdot D^v_{\Sigma_0}[\psi]. \tag{3.6}
\]

Here $A$ is the spacetime region $\{ M \leq r \leq R \}$ which includes the future event horizon $H = \{ r = M \}$.

**Remark 3.1.** Observe that the right hand side of (3.6) requires higher regularity than the left hand side. Furthermore, the boundedness of the $D^v_{\Sigma_0}$ norm forces the conserved charge
\[
H[\psi] = \int_{\Sigma_0 \cap H} \left( Y\psi + \frac{1}{M} \psi \right) \, d\omega
\]
to vanish. Note that if the data are in $C^2$ and the conserved charge vanishes, then the integral
\[
\int_{\Sigma_0 \cap \{ r \leq R \}} \left[ \frac{1}{(r-M)^2} \cdot \left( Y\psi + \frac{1}{r} \psi \right)^2 \right] \, dr \, d\omega
\]
is bounded by the remaining three integrals in (3.5).

Hence not only Theorem 3.1 requires high regularity for the initial data but also requires the vanishing of the conserved charge $H[\psi]$. The next two theorems show that this result is optimal by providing results in the converse direction.

**Theorem 3.2 (Trapping and conserved charges).** If $\psi$ is a solution to the wave equation on extremal Reissner–Nordström with smooth compactly supported initial data for which the memory charge
\[
H[\psi] := \int_{\Sigma_0 \cap H} \left( Y\psi + \frac{1}{M} \psi \right) \, d\omega \neq 0 \tag{3.7}
\]
then
\[
\int_A (Y\psi)^2 \, dg_A = \infty.
\]
Hence, no non-degenerate Morawetz estimate holds for such solutions $\psi$ to the wave equation.

**Remark 3.2.** Theorem 3.2 shows that trapping takes place on degenerate horizons even for smooth solutions as long as their conserved charge is non-vanishing. This implies that the trapping effect on degenerate horizon is not due to a high frequency obstruction that requires loss of regularity but also due to global properties of the horizon which are independent of the degree of regularity of the initial data. Hence, the degenerate horizon $H$ should be thought of as being trapped as a whole. This is in stark contrast with the trapping effect at the photon sphere where trapping is due to high frequency solutions which are supported for arbitrarily long times in small neighborhoods of individual null geodesics on the photon sphere.
The next theorem shows that the regularity required in Theorem 3.1 for the non-degenerate estimate (3.6) to hold is in fact optimal. Specifically, we will show that the non-degenerate spacetime integral $\int_A |\partial \psi|^2$ is not bounded if we assume that the initial data are less regular than required for the boundedness of $D^w_{\Sigma_0}$. We have the following

**Theorem 3.3 (Trapping and optimal loss of regularity).** Consider initial data for the wave equation on extremal Reissner–Nordström such that the conserved charge is vanishing

$$H[\psi] = 0$$  \hspace{1cm} (3.8)

and

$$\int_{\Sigma_0 \cap \{r \leq R\}} \left[ \frac{1}{(r - M)} \cdot \left( Y \psi + \frac{1}{r} \psi \right) \right]^2 \, dr \, d\omega = \infty.$$  \hspace{1cm} (3.9)

Clearly condition (3.9) implies that

$$D^w_{\Sigma_0}[\psi] = \infty.$$  

Then, no non-degenerate Morawetz estimate holds for $\psi$, that is

$$\int_A (Y\psi)^2 \, dg_A = \infty.$$  

The last theorem concerns higher order trapping estimates. Specifically, we show that higher order stable trapping takes place on degenerate horizons. We have the following

**Theorem 3.4 (Stable higher-order trapping).** Generic solutions to the wave equation on extremal Reissner–Nordström with generic smooth initial data which are supported in $\{M < R_1 \leq r \leq R_2\}$ satisfy

$$\int_A (Y^k \psi)^2 \, dg_A = \infty, \text{ for } k \geq 2$$

and hence no non-degenerate higher order Morawetz estimate holds.

Hence, no non-degenerate higher order Morawetz estimate holds even for initial data which are compactly supported and supported away from the horizon and as such any charge on the event horizon initially vanishes and any weighted higher order norm is finite. This implies that stable higher order trapping takes place on degenerate horizons.
4 The non-degenerate Morawetz estimate

4.1 Elliptic theory and Poincaré’s inequality

We recall briefly some basic facts of the spectral theory of a standard sphere \( S^2(r) \) of radius \( r > 0 \). The space \( L^2(S^2(r)) \) admits the following orthogonal decomposition:

\[
L^2(S^2(r)) = \bigoplus_{l=0}^{\infty} E_l,
\]

where the eigenspaces \( E_l \) are of dimension \( 2l + 1 \), and their corresponding eigenfunctions are denoted by \( Y^{m,\ell} \) for \( m \in \mathbb{Z} \cap [-\ell, \ell] \) (the functions \( Y^{m,\ell} \) are usually referred to as spherical harmonics). The eigenvalues of the spherical Laplacian \( \Delta \) are equal to \( -\ell(\ell+1)r^2 \).

Hence, any function \( f \in S^2(r) \) can be written as:

\[
f = \sum_{l=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{m,\ell}(r) \cdot Y^{m,\ell} = \sum_{l=0}^{K-1} f_l + f_{\geq K},
\]

for any \( K \geq 1 \), where we denote by \( f_l \) the projection of \( f \) on the eigenspace \( E_l \).

In view of the spherical symmetry of the extremal Reissner–Nordström spacetime \( M \), if decompose any solution \( \psi \) of the linear wave equation on \( M \):

\[
\psi = \sum_{\ell=0}^{\infty} \psi_\ell,
\]

then every projection \( \psi_\ell \) will also satisfy the wave equation. For example, \( \psi \) can be uniquely written as

\[
\psi = \psi_0 + \psi_{\geq 1},
\]

where \( \psi_0 = \int_{S^2} \psi \, d\omega \) is the spherical mean of \( \psi \).

From now on, we will say that \( \psi \) is supported on angular frequencies \( \ell \geq K \) for some \( K \geq 0 \) if initially we have that \( \psi_k = 0 \) for \( k \in \mathbb{N} \cap [0, K-1] \), and that \( \psi \) is supported on angular frequency \( K \) if \( \psi \in E_K \).

Finally we record here Poincaré’s inequality (a proof of it can be found in [4]):

**Proposition 4.1.1.** Let \( f \in S^2(r) \) for some \( r > 0 \), and let \( f_\ell = 0 \) for \( \ell \in \mathbb{N} \cap [0, K-1] \) for some \( K \in \mathbb{N}, K \geq 1 \). Then we have that:

\[
\int_{S^2(r)} f^2 \, d\omega \leq \frac{r^2}{K(K+1)} \int_{S^2(r)} |\nabla f|^2 \, d\omega. \tag{4.3}
\]

Additionally we note that equality in (4.3) holds if and only if \( f_\ell = 0 \) for all \( \ell \neq K \).

4.2 Hardy inequalities

We here list a few Hardy-type inequalities for functions defined on the exterior \( M \) of the extremal Reissner–Nordström spacetime.

The regions \( A, A_{\ell_0} \) and the hypersurfaces \( B_R, H_{\ell_0} \) are as defined in Section 2.
**Proposition 4.2.1** (First Hardy inequality). Let \( f : \mathcal{M} \to \mathbb{R} \) be a \( C^1 \) function. Let \( p \in \mathbb{R} \setminus \{-1\} \) and suppose that \( \lim_{r \to \mathcal{M}} (r - M)^{p+1} f^2 = 0 \). Then
\[
\int_A (r - M)^p f^2 \, d\omega dv \leq \frac{4}{(p+1)^2} \int_A (r - M)^{p+2} (\partial_r f)^2 \, d\omega dv
+ \frac{2}{p+1} \int_{B_R} (r - M)^{p+1} f^2 \, d\omega dv. \tag{4.4}
\]
In particular, if \( p < -1 \), then we have that
\[
\int_A (r - M)^p f^2 \, d\omega dv \leq \frac{4}{(p+1)^2} \int_A (r - M)^{p+2} (\partial_r f)^2 \, d\omega dv. \tag{4.5}
\]

**Proof.** Integrate \( \partial_r ((r - M)^{p+1} f^2) \) on \( A \) with \( p \neq -1 \) and use that \( \lim_{r \to \mathcal{M}} (r - M)^{p+1} f^2 = 0 \) to obtain:
\[
\int_A (p+1)(r - M)^p f^2 + 2(r - M)^p f \partial_r f \, d\omega dv = (p + 1)^{-1} \int_{B_R} (r - M)^{p+1} f^2 \, d\omega dv. \tag{4.6}
\]
We rearrange the terms above and multiply both sides by \( (p+1)^{-1} \):
\[
\int_A (r - M)^p f^2 \, d\omega dv = (p + 1)^{-1} \int_{B_R} (r - M)^{p+1} f^2 \, d\omega dv - 2(p + 1)^{-1} \int_A (r - M)^{p+1} f \partial_r f \, d\omega dv. \tag{4.7}
\]
We apply a weighted Cauchy–Schwarz inequality to estimate
\[
2(p + 1)^{-1} \int_A (r - M)^{p+1} |f| \, |\partial_r f| \, d\omega dv \leq \alpha \int_A (r - M)^p f^2 \, d\omega dv + \alpha^{-1} (p + 1)^{-2} \int_A (r - M)^{p+2} (\partial_r f)^2 \, d\omega dv,
\]
where \( 0 < \alpha < 1 \). We use the above inequality together with (4.6) to obtain:
\[
\int_A (r - M)^p f^2 \, d\omega dv \leq \alpha^{-1} (1 - \alpha)^{-1} (p + 1)^{-2} \int_A (r - M)^{p+2} (\partial_r f)^2 \, d\omega dv + (1 - \alpha)^{-1} (p + 1)^{-1} \int_{B_R} (r - M)^{p+1} f^2 \, d\omega dv.
\]
The function \( \alpha^{-1} (1 - \alpha)^{-1} \) attains its minimum at \( \alpha = \frac{1}{2} \). By taking \( \alpha = \frac{1}{2} \) in the above inequality, we arrive at (4.4). \( \square \)

**Proposition 4.2.2** (Second Hardy inequality). Let \( f : \mathcal{M} \to \mathbb{R} \) be a \( C^1 \) function. Let \( r_1 > r_0 \). Then, for any \( \epsilon > 0 \) we can estimate
\[
(r_1 - r_0) \int_{H_{r_0}} f^2 \, d\omega dv \leq \epsilon \int_{A_{r_0}} (\partial_r f)^2 \, d\omega dv + (1 + \epsilon^{-1}) \int_{A_{r_0}} f^2 \, d\omega dv + (r_1 - R) \int_{B_R} f^2 \, d\omega dv. \tag{4.8}
\]
In particular, if we take \( r_1 = R \) then there is no boundary integral over \( B_R \) in (4.8).
Proof. Integrate $\partial_r ((r - r_1)f^2)$ over $A_{r_0}$ to obtain:

$$(r_1 - r_0) \int_{H_{r_0}} f^2 d\omega dv = \int_{A_{r_0}} f^2 d\omega dv + 2 \int_{A_{r_0}} (r - r_1) f \partial_r f d\omega dv + (r_1 - R) \int_{B_R} f^2 d\omega dv.$$  

The inequality (4.8) follows immediately after applying a weighted Cauchy–Schwarz inequality on the second term.

4.3 The estimate for the spherical mean

Let $\psi_0$ denote the spherical mean of $\psi$, that is

$$\psi_0(v, r) = \int_{S^2} \psi(v, r, \omega) d\omega,$$

where $\omega = (\theta, \phi)$ and $d\omega = \sin \theta d\theta d\phi$. We will prove the following proposition for $\psi_0$.

Proposition 4.3.1. There is a constant $C > 0$ that depends only on the mass parameter $M$ such that for spherically symmetric solutions $\psi_0$ to the wave equation on extremal Reissner–Nordström which arise from initial data with bounded norm $D_{\Sigma_0}[\psi_0] := \int_{\Sigma_0 \cap \{r \leq R\}} \left[ \frac{1}{(r - M)} \cdot \left( \partial_r (r \psi_0) \right)^2 \right] dr d\omega + \int_{\Sigma_0} (J^T[\psi_0] \cdot n_{\Sigma_0}) dg_{\Sigma_0}$ (4.9)

the following estimate holds

$$\int_{A} \left[ \psi_0^2 + (\partial_r \psi_0)^2 + (\partial_v \psi_0)^2 \right] dg_A \leq C \cdot D_{\Sigma_0}[\psi].$$ (4.10)

Here the $J^T$ flux is as defined in Section 2.

Proof. We apply the singular vector field

$$S = -\frac{1}{(r - M)} \cdot \partial_r$$ (4.11)

as our multiplier vector field in the spacetime region $A_{r_0}$ bounded by the hypersurfaces $\Sigma_0 = \{v = 0\} \cap \{r \leq R\}$, $B_R = \{r = R\}$, for some large $R > M$, and $H_{r_0} = \{u = u_0\}$, where $u_0 = v(p)$ and the point $p$ is on the hypersurface $\Sigma_0$ such that $R > r(p) = r_0 > M$. Here $u$ is the retarded null coordinate and $v$ is the advanced null coordinate (see Section 2). Clearly, we have that $H_{r_0} \to \mathcal{H}^+$ as $r_0 \to M$. We therefore use that

$$\int_{A_{r_0}} \left( \frac{1}{(r - M)} \cdot \partial_r \psi_0 \cdot \Box_g \psi_0 \right) \cdot r^2 dr dv d\omega = 0$$ (4.12)
where $d\omega = \sin \theta d\theta \, d\phi$.

Since $\psi_0$ is spherically symmetric we have

$$\Box \psi_0 = D \cdot \partial_r \partial_r \psi_0 + 2 \partial_r \left( H[\psi_0] \right) + R \cdot \partial_r \psi_0 = 0$$

(4.13)

where the partial derivatives are taken with respect to the ingoing Eddington–Finkelstein coordinates $(v, r, \theta, \phi)$ and

$$D = \left( \frac{r - M}{r} \right)^2, \quad R = \frac{2}{r^2} (r - M), \quad H[\psi_0] = \partial_r \psi_0 + \frac{1}{r} \psi_0.$$  

(4.14)

We therefore obtain that

$$I_1 + I_2 + I_3 = 0,$$  

(4.15)

where

$$I_1 = \int_{A_{r_0}} \frac{(r - M) \cdot \partial_r \psi_0 \cdot \partial_r \psi_0}{\partial_r \psi_0} \, dr \, dv \, d\omega,$$  

(4.16)

$$I_2 = \int_{A_{r_0}} \frac{2r^2}{(r - M)} \cdot \partial_r \psi_0 \cdot \partial_r \left( H[\psi_0] \right) \, dr \, dv \, d\omega,$$  

(4.17)

$$I_3 = \int_{A_{r_0}} 2(\partial_r \psi_0)^2 \, dr \, dv \, d\omega.$$  

(4.18)

By integrating by parts with respect to $\partial_r$ in $A_{r_0}$\footnote{Note that $\partial_r$ is tangential to $\Sigma_0$.} we obtain

$$I_1 = \int_{A_{r_0}} \left( \partial_r \left( \frac{1}{2} (r - M) \cdot (\partial_r \psi_0)^2 \right)^2 - \frac{1}{2} (\partial_r \psi_0)^2 \right) \, dr \, dv \, d\omega$$

$$= - \frac{1}{2} \int_{A_{r_0}} (\partial_r \psi_0)^2 \, dr \, dv \, d\omega + \int_{B_R} \frac{1}{2} (r - M) \cdot (\partial_r \psi_0)^2 \, dv \, d\omega$$

$$- \int_{H_{r_0}} \frac{1}{2} (r - M) \cdot (\partial_r \psi_0)^2 \, dv \, d\omega.$$  

(4.19)

Note that the coefficient of the spacetime integral on the right hand side has the wrong sign and hence its precise value plays a fundamental role in our analysis. Specifically, it is crucial that the coefficient of the spacetime integral is strictly less than 2 and hence this integral can be absorbed by the integral $I_3$ (see equation (4.18)).

By integrating by parts with respect to $\partial_v$ in $A_{r_0}$\footnote{Note that $\partial_v$ is tangential to $B_R$.} and using equations (4.14), (4.18) and that

$$dr = \frac{(r - M)^2}{2r^2} \, dv$$

along $H_{r_0}$

(4.20)
we obtain

\[ I_2 = I_4 + \int_{A_{r_0}} \partial_v \left( \frac{r^2}{(r-M)} \cdot (\partial_r \psi_0)^2 \right) \, dv \, dr \, dv \]

\[ = I_4 + \int_{H_{r_0}} \frac{r^2}{(r-M)} \cdot (\partial_r \psi_0)^2 \, dr \, dv - \int_{\Sigma \cap \{ r_0 \leq r \leq R \}} \frac{r^2}{(r-M)} \cdot (\partial_r \psi_0)^2 \, dr \, dv \]

where

\[ I_4 = \int_{A_{r_0}} \frac{2r}{(r-M)} \cdot (\partial_v \psi_0) \cdot (\partial_r \psi_0) \, dv \, dr \, dv. \]  

(4.21)

Clearly, all the terms are regular apart from the term \( I_4 \) which is singular when we take the limit \( r_0 \to M \). We will show that in view of the special structure of the geometry of degenerate horizons we are able to bound this integral in terms of a weighted norm of the initial data on \( \Sigma_0 \) only (see also the discussion in Section 4.6). Indeed, by integrating by parts with respect to \( \partial_r \) we obtain

\[ I_4 = I_5 + \int_{A_{r_0}} \partial_v \left( \frac{2r}{(r-M)} \cdot \psi_0 \cdot \partial_r \psi_0 \right) \, dv \, dr \, dv \]

\[ = I_5 + \int_{H_{r_0}} \frac{2r}{(r-M)} \cdot \psi_0 \cdot (\partial_r \psi_0) \, dr \, dv - \int_{\Sigma \cap \{ r_0 \leq r \leq R \}} \frac{2r}{(r-M)} \cdot \psi_0 \cdot \partial_r \psi_0 \, dr \, dv \]

where

\[ I_5 = \int_{A_{r_0}} \frac{r}{(r-M)} \cdot \psi_0 \cdot (-2 \partial_v \partial_r \psi_0) \, dv \, dr \, dv. \]  

(4.24)

The wave equation (4.13) and the expression (4.14) for \( H[\psi_0] \) yield

\[ I_5 = \int_{A_{r_0}} \frac{r}{(r-M)} \cdot \psi_0 \left[ \frac{(r-M)^2}{r^2} \cdot \partial_r \partial_r \psi_0 + \frac{2}{r} \cdot \partial_v \psi_0 + \frac{2}{r^2} \cdot (r-M) \cdot \partial_r \psi_0 \right] \, dv \, dr \, dv \]

\[ = I_6 + I_7 + I_8, \]  

(4.25)

where

\[ I_6 = \int_{A_{r_0}} \frac{2}{(r-M)} \cdot \psi_0 \cdot \partial_v \psi_0 \, dv \, dr \, dv; \]

\[ I_7 = \int_{A_{r_0}} \frac{2}{r} \cdot \psi_0 \cdot \partial_r \psi_0 \, dv \, dr \, dv; \]

\[ I_8 = \int_{A_{r_0}} \frac{(r-M)}{r} \cdot \psi_0 \cdot \partial_\nu \partial_r \psi_0 \, dv \, dr \, dv. \]  

(4.26)
Furthermore,

\[
I_6 = \int_{A_{r_0}} \partial_v \left( \frac{1}{r-M} \cdot \psi_0^2 \right) dr dv d\omega
= \int_{H_{r_0}} \frac{1}{r-M} \cdot \psi_0^2 dr d\omega - \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \frac{1}{r-M} \cdot \psi_0^2 dr d\omega
\tag{4.29}
\]

\[
\int_{H_{r_0}} \frac{(r-M)}{2r^2} \cdot \psi_0^2 dv d\omega - \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \frac{1}{r-M} \cdot \psi_0^2 dr d\omega.
\]

Similarly, we obtain

\[
I_7 = \int_{A_{r_0}} \frac{1}{r} \cdot \partial_r \psi_0^2 dr dv d\omega = \int_{A_{r_0}} \left[ \partial_r \left( \frac{\psi_0^2}{r} \right) + \frac{1}{r^2} \cdot \psi_0^2 \right] dr dv d\omega
= \int_{B_{R}} \frac{\psi_0^2}{r} dv d\omega - \int_{H_{r_0}} \frac{\psi_0^2}{r} dv d\omega + \int_{A_{r_0}} \frac{1}{r^2} \cdot \psi_0^2 dr d\omega
\tag{4.30}
\]

\[
\int_{B_{R}} \frac{\psi_0^2}{r} dv d\omega - I_9 + \int_{A_{r_0}} \frac{1}{r^2} \cdot \psi_0^2 dr dv d\omega.
\]

where

\[
I_9 = - \int_{H_{r_0}} \frac{\psi_0^2}{r} dv d\omega.
\tag{4.31}
\]

Observe that the middle integral \( I_9 \) above has the wrong sign in the expression for \( I_7 \) in (4.30). This will be later remedied using an appropriate Hardy inequality.

Regarding the integral \( I_8 \) we obtain the following

\[
I_8 = \int_{A_{r_0}} \left[ \partial_r \left( \frac{(r-M)}{r} \cdot \psi_0 \cdot \partial_r \psi_0 \right) - \frac{(r-M)}{r} \cdot (\partial_r \psi_0)^2 - \frac{M}{r^3} \cdot \psi_0 \cdot \partial_r \psi_0 \right] dr dv d\omega
= \int_{B_{R}} \frac{(r-M)}{r} \cdot \psi_0 \cdot \partial_r \psi_0 dv d\omega - \int_{H_{r_0}} \frac{(r-M)}{r} \cdot \psi_0 \cdot \partial_r \psi_0 dv d\omega
- \int_{A_{r_0}} \frac{(r-M)}{r} \cdot (\partial_r \psi_0)^2 dr dv d\omega - I_{10},
\tag{4.32}
\]

where

\[
I_{10} = \int_{A_{r_0}} \frac{M}{2r^2} \cdot \partial_r \psi_0^2 dr dv d\omega = \int_{A_{r_0}} \left[ \partial_r \left( \frac{M}{2r^2} \cdot \psi_0^2 \right) + \frac{M}{r^3} \cdot \psi_0^2 \right] dr dv d\omega
= \int_{B_{R}} \frac{M}{2r^2} \cdot \psi_0^2 dv d\omega - \int_{H_{r_0}} \frac{M}{2r^2} \cdot \psi_0^2 dv d\omega + \int_{A_{r_0}} \frac{M}{r^3} \cdot \psi_0^2 dr dv d\omega.
\tag{4.33}
\]

Therefore, by using equations (4.15), (4.18), (4.19), (4.21), (4.23), (4.25), (4.29),
(4.30), (4.32), (4.33) and grouping all the integral terms in $I_1, I_2, I_3$ we obtain

$$0 = \int_{A_0} \left[ -\frac{1}{2} \cdot (\partial_r \psi_0)^2 + \frac{1}{r^2} \cdot \psi_0^2 - \frac{(r - M)}{r} \cdot (\partial_r \psi_0)^2 - \frac{M}{r^3} \cdot \psi^2 + 2(\partial_r \psi_0)^2 \right] \, dr \, dv \, d\omega$$

$$+ \int_{H_0} \left[ -\frac{1}{2}(r - M) \cdot (\partial_r \psi_0)^2 + \frac{(r - M)}{r} \cdot \psi_0 \cdot \partial_r \psi_0 + \frac{(r - M)}{2r^2} \cdot \psi_0 \right] \, dv \, d\omega$$

$$+ \int_{H_0} \left[ -\frac{1}{r} \cdot \psi_0^2 + \frac{1}{2}(r - M) \cdot (\partial_r \psi_0)^2 + \frac{M}{2r^2} \cdot \psi_0^2 - \frac{(r - M)}{r} \cdot \psi_0 \cdot \partial_r \psi_0 \right] \, dv \, d\omega$$

$$+ \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \left[ -\frac{r^2}{(r - M)} \cdot (\partial_r \psi_0)^2 - \frac{2r}{(r - M)} \cdot \psi_0 \cdot \partial_r \psi_0 - \frac{1}{(r - M)} \cdot \psi_0 \right] \, dr \, d\omega$$

$$+ \int_{B_R} \left[ K_R[\psi] \right] \, dv \, d\omega,$$

where

$$K_R[\psi_0] = \frac{1}{2} \cdot (r - M) \cdot (\partial_r \psi_0)^2 + \left( \frac{2r - M}{2r^2} \right) \cdot \psi_0^2 + \frac{(r - M)}{r} \cdot \psi_0 \cdot \partial_r \psi_0. \quad (4.35)$$

Hence, by noting all terms that cancel out, we have established that

$$\int_{A_0} \left[ \left( \frac{3}{2} - \frac{(r - M)}{r} \right) \cdot (\partial_r \psi_0)^2 + \left( \frac{r - M}{r^3} \right) \cdot \psi_0^2 \right] \, dr \, dv \, d\omega$$

$$= \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \left[ \frac{r^2}{(r - M)} \cdot (\partial_r \psi_0 + \frac{1}{r} \psi_0)^2 \right] \, dr \, d\omega \quad (4.36)$$

$$+ \int_{H_0} \left[ -\frac{1}{2r^2} \cdot \psi_0^2 \right] \, dv \, d\omega - \int_{B_R} \left[ K_R[\psi_0] \right] \, dv \, d\omega.$$
where
$$E_2[\psi_0] \sim \psi^2 + (\partial_r \psi_0)^2$$ (4.40)

where the constants in $\sim$ depend only on $M$. Noting that for all $r \geq M$ we have
$$\frac{3}{2} - \frac{(r - M)}{r} > \frac{1}{2}$$

and using $\epsilon$ in (4.37), (4.39) we obtain

$$\int_{A_{r_0}} \frac{1}{2} \cdot (\partial_r \psi_0)^2 dr \, dv \, d\omega \leq \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \left[ \frac{1}{(r - M)^2} \cdot \left( \partial_r (r \psi_0) \right)^2 \right] dr \, d\omega$$

$$+ \int_{A_0} \left[ \epsilon + \frac{(r - M)^2}{\epsilon^2} \right] (\partial_r \psi_0)^2 dr \, dv \, d\omega$$

$$+ \int_{B_R} E_3[\psi_0] \, dv \, d\omega,$$

where
$$E_3[\psi_0] = E_1[\psi_0] + E_2[\psi_0] - K_R[\psi_0].$$ (4.42)

We now choose $\epsilon$ such that
$$\epsilon = \min \left\{ \frac{1}{16}, \epsilon_1, \epsilon_2 \right\},$$ (4.43)

where $\epsilon_1, \epsilon_2$ are the constants of the Hardy inequalities (4.37), (4.39), respectively. Clearly, with this choice $\epsilon$ depends only on $M$. Recalling that in region $A_{r_0}$ we have $r \leq R$, we impose on $R$ the condition
$$\frac{(R - M)^2}{\epsilon^2} \leq \frac{1}{16}$$

which implies
$$R \leq M + \frac{\epsilon}{4},$$ (4.44)

where $\epsilon$ is given by (4.43). With these conditions for $\epsilon, R$, estimate (4.41) yields the following

$$\int_{A_{r_0}} \frac{1}{4} \cdot (\partial_r \psi_0)^2 dr \, dv \, d\omega \leq \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \left[ \frac{1}{(r - M)^2} \cdot \left( \partial_r (r \psi_0) \right)^2 \right] dr \, d\omega$$

$$+ \int_{B_R} E_3[\psi_0] \, dv \, d\omega,$$

(4.45)

We finally need to bound the boundary integral over $B_R$. In view of the degenerate Morawetz estimate of [4] we have that

$$\int_{\{M + \frac{\epsilon}{4} \leq r \leq M + \frac{\epsilon}{4}\}} \left[ (\partial_r \psi_0)^2 + \psi_0^2 \right] dr \, dv \, d\omega \leq C_\epsilon \int_{\Sigma_0} \left( J^T[\psi_0] \cdot \eta_{\Sigma_0} \right) \, dg_{\Sigma_0}.$$
Hence, by the averaging principle, there is a value 
\[ \tilde{R} \in \left[ M + \frac{\epsilon}{8}, M + \frac{\epsilon}{4} \right] \] (4.46)
such that
\[ \int_{\{r=\tilde{R}\}} \left[ (\partial_r \psi_0)^2 + \psi_0^2 \right] dv d\omega \leq C \int_{\Sigma_0} \left( J^T[\psi_0] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0}. \] (4.47)

Note that \( C \) depends only on \( M \) since \( \epsilon \) has already been chosen in (4.43). Therefore, if we define
\[ R := \tilde{R} \]
then (4.45) becomes
\[ \int_{A_{r_0}} \frac{1}{4} \left( (\partial_r \psi_0)^2 \right) dv d\omega \leq \int_{\Sigma_0 \cap (r_0 \leq r \leq R)} \left[ \frac{1}{(r - M)} \cdot (\partial_r (r \psi_0))^2 \right] dr d\omega \]
\[ + C \int_{\Sigma_0} \left( J^T[\psi_0] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0}. \] (4.48)

Clearly, all the constants are independent of the constant \( r_0 \) in the definition of the spacetime region \( A_{r_0} \). Therefore, by taking \( r_0 \to M \) in (4.48) we obtain Proposition (4.3.1). The bound on the zeroth order term follow from the first Hardy inequality (4.4).

\[ \square \]

4.4 The estimate for angular frequencies \( \ell \geq 1 \)

For the projection on angular frequencies \( \ell \geq 1 \) we apply regular multiplier and commutator vector fields in the spacetime region \( \mathcal{R}(0, \tau) \) bounded by the hypersurfaces \( \Sigma_0 \) and \( \Sigma_{\tau} \).

**Proposition 4.4.1.** Let \( \psi \) be a solution of the linear wave equation \( \Box_g \psi = 0 \). Then for any \( \tau > 0 \) we have that for the part of \( \psi \) that is localized in angular frequencies \( \ell \geq 1 \) the following estimate holds true:
\[ \int_{\Sigma_0 \cap \mathcal{A}_0} \left( J^P[\partial_\ell \psi_{\geq 1}] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0} + \] (4.49)
\[ + \int_{\mathcal{A}_0^\ell} \left( (\partial_\ell \partial_r, \psi_{\geq 1})^2 + (r - M)^2 \cdot (\partial_\ell \partial_r, \psi_{\geq 1})^2 + |\nabla \partial_\ell \psi_{\geq 1}|^2 \right) dg_{\mathcal{A}_0^\ell} \lesssim \]
\[ \lesssim \sum_{l=0,1} \int_{\Sigma_0} \left( J^P[\partial_\ell \psi_{\geq 1}] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0} + \int_{\Sigma_0 \cap \mathcal{A}_0^\ell} \left( J^P[\partial_\ell \psi_{\geq 1}] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0}, \]
where \( \mathcal{A}_0^\ell = \mathcal{R}(0, \tau) \cap \mathcal{A} \),
where \( \mathcal{A} \) is as defined in Section 2.
Proof. We consider the equation for $\partial_r \psi_{\geq 1}$, we have that:

$$\Box_g(\partial_r \psi_{\geq 1}) = D^r \partial_r \psi_{\geq 1} + \frac{2}{r^2} \partial_r \psi_{\geq 1} - R' \partial_r \psi_{\geq 1} + \frac{2}{r} \Delta \psi_{\geq 1}. \quad (4.50)$$

Now consider the vector field:

$$L_p = f^v(r) \partial_v + f^r(r) \partial_r,$$

where $f^v$ and $f^r$ are smooth functions satisfying

$$f^v \approx 1, \quad \partial_r f^v \approx \frac{1}{\sigma}, \quad f^r = -M \sqrt{D}, \quad \partial_r f^r = -\frac{M^2}{r^2},$$

close to the horizon (in the region $A$ where $r_0$ is chosen to be very close to $M$), with $f^v \equiv 1$, $f^r \equiv 0$ in $r \geq r_1$ for some $r_0 < r_1 < 2M$, and where $\sigma > 0$ is chosen to be small.

Applying Stokes’ Theorem for $J^{LP}[\partial_r \psi_{\geq 1}]$ we have that the following bulk terms:

$$K^{LP}[\partial_r \psi_{\geq 1}] + \mathcal{E}^{LP}[\partial_r \psi_{\geq 1}] = H_1(\partial_v \partial_r \psi_{\geq 1})^2 + H_2(\partial_r \partial_r \psi_{\geq 1})^2 + H_3|\nabla \psi_{\geq 1}|^2 +$$

$$+ H_4(\partial_v \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) + H_5(\partial_v \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) + H_6(\partial_r \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) + H_7(\partial_r \partial_r \psi_{\geq 1}) \cdot (\Delta \psi_{\geq 1}) +$$

$$+ H_8(\partial_r \partial_r \psi_{\geq 1}) \cdot (\Delta \psi_{\geq 1}) + H_9(\partial_v \partial_r \psi_{\geq 1}) \cdot (\partial_v \partial_r \psi_{\geq 1}) + H_{10}(\partial_v \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}),$$

where close to the horizon

$$H_1 = (\partial_r f^v) \approx \frac{1}{\sigma}, \quad H_2 = \frac{D(\partial_r f^r)}{2} - \frac{D f^r}{r} - \frac{3D f^r}{2} = \frac{5M^2 D}{2r^2} + \frac{M D^{3/2}}{r},$$

$$H_3 = -\frac{1}{2} (\partial_r f^r) = \frac{M^2}{2r^2}, \quad H_4 = \frac{2 f^v}{r^2} \approx \frac{2}{r^2}, \quad H_5 = -f^v R' \approx -R',$$

$$H_6 = \frac{2 f^r}{r^2} = \frac{2M \sqrt{D}}{r^2}, \quad H_7 = \frac{2 f^v}{r} \approx \frac{2}{r}, \quad H_8 = \frac{2 f^r}{r} = -\frac{2M \sqrt{D}}{r},$$

$$H_9 = D(\partial_r f^v) - D' f^v - \frac{2 f^r}{r} \approx \frac{D}{\sigma} - D' + \frac{2M \sqrt{D}}{r}, \quad H_{10} = M \sqrt{D} R'.$$

Here the functions $D(r), R(r)$ are given by (4.14). We will not deal with the terms away from the horizon since they can be bounded by a degenerate Morawetz estimate away from the photon sphere for $J^T[\partial_r \psi_{\geq 1}]$.

We deal first with the terms $H_4 - H_{10}$.

$H_4$: We have that

$$\int_{A_5} H_4(\partial_v \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) d\sigma \lesssim \int_{A_5} \frac{2}{r^2} (\partial_v \partial_r \psi_{\geq 1}) \cdot (\partial_v \psi_{\geq 1}) d\sigma \lesssim$$

$$\lesssim \beta \int_{A_5} (\partial_v \partial_r \psi_{\geq 1})^2 d\sigma + \frac{1}{\beta} \int_{A_5} (\partial_r \psi_{\geq 1})^2 d\sigma,$$
and now we absorb the first term in the right hand by a choice of a $\beta = \beta(M)$ that is small enough, and we bound the second term by the Morawetz estimate.

Note that the $H_4$ term introduced only a $\beta$ loss (for $\beta$ very small) from the $H_1$ term.

$H_5$: We have that

$$\int_{A_0^*} H_5(\partial_r \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) \, dg_A \simeq \int_{A_0^*} -R'(\partial_r \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) \, dg_A \lesssim (4.51)$$

$$\lesssim \frac{1}{\beta} \int_{A_0^*} (\partial_r \partial_r \psi_{\geq 1})^2 \, dg_A + \beta \int_{A_0^*} (\partial_r \psi_{\geq 1})^2 \, dg_A,$$

where $\beta = \beta(M)$ since in $A$ we have that

$$R' = D'' + \frac{2D'}{r} - \frac{\sqrt{D}}{r^2} = \frac{MD'}{\sqrt{D}r^2} - \frac{2M\sqrt{D}}{r^2} + \frac{2D'}{r} - \frac{2D}{r^2} \simeq \frac{2M^2}{r^4},$$

and it is chosen to be small enough but much bigger than $\sigma$, so that the first term of (4.51) can be absorbed in the right hand side, while for the second one we have that:

$$\beta \int_{A_0^*} (\partial_r \psi_{\geq 1})^2 \, dg_A$$

$$\leq C\beta \int_{R(0,\tau) \cap \{r_0 \leq r_1 < 2M\}} (\partial_r \psi_{\geq 1})^2 \, dg_R + C\beta \int_{A_0^*} D (\partial_r \partial_r \psi_{\geq 1})^2 + (\partial_r \partial_r \psi_{\geq 1})^2 \, dg_A +$$

$$+ C\beta \int_{R(0,\tau) \cap \{r_0 \leq r_1 < 2M\}} D (\partial_r \partial_r \psi_{\geq 1})^2 + (\partial_r \partial_r \psi_{\geq 1})^2 \, dg_R,$$

where the first and the third term of the above estimate can be bounded by the Morawetz estimate for $\psi_{\geq 1}$ and $\partial_r \psi_{\geq 1}$ respectively, while the second one can be absorbed in the right hand side as

$$C\beta D \ll H_1 \text{ and } C\beta D \ll H_2,$$

due to the higher degeneracy of $D$ on the horizon compared to the other terms for the first inequality, and due to the smallness of $\beta$ for the second one.

Note that the $H_5$ term introduced only a $\beta$ loss (for $\beta$ very small) from the $H_1$ and $H_2$ terms.

$H_6$: We have that

$$\int_{A_0^*} H_6(\partial_r \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) \, dg_A = - \int_{A_0^*} \frac{2M\sqrt{D}}{r^2}(\partial_r \partial_r \psi_{\geq 1}) \cdot (\partial_r \psi_{\geq 1}) \, dg_A \lesssim$$

$$\lesssim \beta \int_{A_0^*} D(\partial_r \partial_r \psi_{\geq 1})^2 \, dg_A + \frac{1}{\beta} \int_{A_0^*} (\partial_r \psi_{\geq 1})^2 \, dg_A,$$

for $\beta = \beta(M)$ small enough, and the first term can be absorbed in the right hand side, while the second one is bounded by the Morawetz estimate.
Note that $H_6$ term introduced only a $\beta$ loss (for $\beta$ very small) from the $H_1$ term. 

$H_7$: We have that
\[
\int_{A_0^*} H_7(\partial_r, \partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \, dg_A \sim \int_{A_0^*} \frac{2}{r} (\partial_r, \partial_r \psi_{>1}) \cdot \Delta \psi_{>1} \, dg_A = \quad (4.52)
\]
\[
= \int_{\Sigma_0} \frac{2}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma - \int_{\Sigma_0} \frac{2}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma - \int_{A_0^*} \frac{2}{r} (\partial_r \psi_{>1}) \cdot (\partial_r \Delta \psi_{>1}) \, dg_A,
\]
by an application of Stokes’ Theorem.

For the last term of (4.52) we have after integrating by parts on the sphere that
\[
- \int_{A_0^*} \frac{2}{r} (\partial_r \psi_{>1}) \cdot (\partial_r \Delta \psi_{>1}) \, dg_A = \int_{A_0^*} \frac{2}{r} (\nabla \partial_r \psi_{>1}, (\nabla \partial_r \psi_{>1})) \, dg_A \leq \beta_1 \int_{A_0^*} |\nabla \partial_r \psi_{>1}|^2 \, dg_A + \frac{1}{\beta_1} \int_{A_0^*} |\nabla \partial_r \psi_{>1}|^2 \, dg_A,
\]
for some $\beta_1 = \beta_1(M)$ that is chosen to be small enough so that the first term can be absorbed from the right hand side (from $H_3$), while the second term can be bounded by the $T$-flux of $\partial_r \psi_{>1}$.

For the second term of (4.52) we have after integrating by parts on the sphere that
\[
- \int_{\Sigma_r} \frac{2}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma = \int_{\Sigma_r} \frac{2}{r} (\nabla \partial_r \psi_{>1}, \nabla \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma \leq \beta_2 \int_{\Sigma_r} |\nabla \partial_r \psi_{>1}|^2 \, dg_\Sigma + \frac{1}{\beta_2} \int_{\Sigma_r} |\nabla \psi_{>1}|^2 \, dg_\Sigma,
\]
where $\beta_2 = \beta_2(M)$ is chosen to be small enough so that the first term can be absorbed from the right hand side (from $H_3$), while the second term can be bounded by the $T$-flux of $\psi_{>1}$. The first term of (4.52) can be treated in a similar manner.

Note that the $H_7$ term introduced only a $\beta$ loss (for $\beta$ very small) from the $H_3$ term.

$H_8$: We have that
\[
\int_{A_0^*} H_8(\partial_r, \partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \, dg_A = - \int_{A_0^*} \frac{2M \sqrt{D}}{r} (\partial_r, \partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \, dg_A \Rightarrow \nabla \psi_{>1} \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma \Rightarrow - \int_{A_0^*} \frac{2M \sqrt{D}}{r} (\partial_r, \partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \, dg_A = \quad (4.53)
\]
\[
= - \int_{\Sigma_0} \frac{2M \sqrt{D}}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma + \int_{\Sigma_0} \frac{2M \sqrt{D}}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma + 25
\]
\[ + \int_{A^5} \partial_r \left( \frac{2M \sqrt{D}}{r} \Delta \psi_{>1} \right) \cdot (\partial_r \psi_{>1}) \, dg_A + \int_{A^5} \frac{2M \sqrt{D}}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \, dg_A, \]

by an application of Stokes’ Theorem.

For the last two spacetime terms of (4.53) we have by using Stokes’ theorem on the sphere

\[ \int_{A^5} \partial_r \left( \frac{2M \sqrt{D}}{r} \Delta \psi_{>1} \right) \cdot (\partial_r \psi_{>1}) \, dg_A + \int_{A^5} \frac{2M \sqrt{D}}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \, dg_A = (4.54) \]

\[ = - \int_{A^5} \frac{2M \sqrt{D}}{r} |\nabla \partial_r \psi_{>1}|^2 \, dg_A - \int_{A^5} \partial_r \left( \frac{2M \sqrt{D}}{r} \right) (\nabla \psi_{>1}, \nabla \partial_r \psi_{>1}) \, dg_A. \]

Since

\[ \partial_r \left( \frac{2M \sqrt{D}}{r} \right) = - \frac{2M \sqrt{D}}{r^2} + \frac{2M^2}{r^3} \simeq \frac{2}{M} \text{ in } A, \]

the second term on the right hand side of (4.54) can be treated as follows:

\[ - \int_{A^5} \partial_r \left( \frac{2M \sqrt{D}}{r} \right) (\nabla \psi_{>1}, \nabla \partial_r \psi_{>1}) \, dg_A \leq \beta_1 \int_{A^5} |\nabla \partial_r \psi_{>1}|^2 \, dg_A + \frac{1}{\beta_1} \int_{A^5} |\nabla \psi_{>1}|^2 \, dg_A, \]

where \( \beta_1 = \beta_1(M) \) is chosen to be small enough so that the first term can be absorbed from the right hand side (from \( H_3 \)), while the second term can be bounded by the Morawetz estimate.

The first term on the right hand side of (4.54) can be absorbed as well from \( H_3 \) since \( \frac{2M \sqrt{D}}{r} \ll H_3 \) close to the horizon.

Finally for the second term on the right hand side of (4.53) we have after integrating by parts on the sphere that

\[ \int_{\Sigma_r} \frac{2M \sqrt{D}}{r} (\partial_r \psi_{>1}) \cdot (\Delta \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma = \]

\[ = - \int_{\Sigma_r} \frac{2M \sqrt{D}}{r} (\nabla \partial_r \psi_{>1}), (\nabla \psi_{>1}) \cdot (\partial_r \mathbf{n}_\Sigma) \, dg_\Sigma \leq \]

\[ \leq \int_{\Sigma_r} \frac{4M^2 D}{r^2} |\nabla \partial_r \psi_{>1}|^2 \, dg_\Sigma + \int_{\Sigma_r} |\nabla \psi_{>1}|^2 \, dg_\Sigma, \]

where we can absorb the first term by \( H_3 \) (since \( \frac{4M^2 D}{r^2} \ll H_3 \) in \( A \)), and the second one is bounded by the \( T \)-flux of \( \psi_{>1} \). We can treat the first term of the right hand side of (4.53) similarly.

Note that the \( H_8 \) term introduced only a \( \beta \) loss (for \( \beta \) very small) from the \( H_3 \) term.
$H_9$: We have that

$$
\int_{A_0^+} H_9(\partial_v \partial_r \psi \geq 1) \cdot (\partial_r \partial_r \psi \geq 1) dg_A \leq \int_{A_0^+} \sqrt{D} \left( \frac{\sqrt{D}}{\sigma} + \frac{2M}{r^2} + \frac{2}{r} \right) (\partial_v \partial_r \psi \geq 1) \cdot (\partial_r \partial_r \psi \geq 1) dg_A \leq \beta \int_{A_0^+} D(\partial_r \partial_r \psi \geq 1)^2 dg_A + \frac{1}{\beta} \int_{A_0^+} (\partial_r \partial_r \psi \geq 1)^2 dg_A,
$$

where $\beta = \beta(M)$ is chosen to be very small so that the first term of the last inequality can be absorbed by $H_1$, but also to satisfy $\frac{1}{\beta} \ll \frac{1}{\sigma}$ (which is possible by choosing $\sigma$ to be extremely small from the beginning) so that the second term can be absorbed by $H_2$ as well.

Note that the $H_9$ term introduced only a $\beta$ loss (for $\beta$ very small) from the $H_1$ and $H_2$ terms.

$H_{10}$: We have that

$$
\int_{A_0^+} H_{10}(\partial_r \partial_r \psi \geq 1) \cdot (\partial_r \psi \geq 1) dg_A \leq 4.55 \leq \frac{1}{2\beta} \int_{A_0^+} (H_{10})^2 \frac{M^2}{r^2} (\partial_r \psi \geq 1)^2 dg_A + \frac{\beta}{2} \int_{A_0^+} \frac{2}{M^2} (\partial_r \psi \geq 1)^2 dg_A,
$$

where we just applied the Cauchy–Schwarz inequality, for some $\beta$ that will be chosen later.

We deal first with the second term for which we apply Poincaré’s inequality:

$$
\frac{\beta}{2} \int_{A_0^+} \frac{2}{M^2} (\partial_r \psi \geq 1)^2 dg_A \leq \frac{\beta (M + \beta')^2}{2} \int_{A_0^+} |\nabla \partial_r \psi \geq 1|^2 dg_A,
$$

where we denoted the $r_0$ given in the definition of $A$ by $r_0 = M + \beta'$ for some very small $\beta' > 0$. Now we note that for an appropriate choice of $\beta$ close to 1 we can have that:

$$
H_3 = \frac{M^2}{2r^2} > \frac{\beta (M + \beta')^2}{2} \Rightarrow H_3 - \frac{\beta (M + \beta')^2}{2} > c > 0,
$$

so in the end the second term of the right hand side of (4.55) can be absorbed by the $H_3$ term. Note that this is possible also because of the fact that from all the previous terms $H$ terms that we examined, we only had $\beta''$ loss in $H_3$ for $\beta'' > 0$ being extremely small.

We now look at the first term of the right hand side of (4.55). We have that:

$$
(H_{10})^2 = M^2 D(R')^2 = M^2 D \left( \frac{6M^2}{r^4} - \frac{4M}{r^3} + \frac{2M\sqrt{D}}{r^3} - \frac{2MD}{r^2} \right)^2.
$$
In $\mathcal{A}$, the first two terms of $R'$ are the important ones, since they are much bigger than the last two. We recall the following precise estimate:

$$\frac{6M^2}{r^4} - \frac{4M}{r^3} + \frac{2M\sqrt{D}}{r^3} - \frac{2MD}{r^2} \sim \frac{2M^2}{r^4} \text{ in } \mathcal{A},$$

that we also used in the estimate for $H_4$.

We would like to show that:

$$H_2 = \frac{5M^2D}{2r^2} + \frac{MD^{3/2}}{r} > \frac{1}{2\beta} \cdot \frac{M^2}{2} \cdot \frac{4M^6D}{r^8} = \frac{M^8D}{\beta r^8}.$$

Indeed, by our choice of $\beta$ (which as we mentioned it is chosen to be close to 1), we can have that:

$$\frac{5M^2}{2r^2} > \frac{M^8D}{\beta r^8},$$

and this proves the required estimate (note that the term $\frac{MD^{3/2}}{r}$ is of lower order in $(r - M)$ and hence cannot be used in the proof of the estimates).

\[\square\]

**Remark 4.1.** It should be noted that for the last term in the proof of the Proposition 4.4.1 (the $H_{10}$ term) we used the actual form of the coefficients and some smallness condition coming from a Cauchy–Schwarz inequality.

### 4.5 Finishing the proofs of the theorems

We now have all the tools necessary to complete the proofs of Theorems 3.1, 3.2 and 3.3.

**Proof of Theorem 3.1.** Clearly, in view of the degenerate Morawetz estimate (3.2) and the Hardy inequality (4.4) (that controls the zeroth order term) it suffices to show that

$$\int_{\mathcal{A}} (\partial_r \psi)^2 \, dg_{\mathcal{A}} \leq C \cdot D^w_{\Sigma_0}[\psi],$$

where the norm $D^w_{\Sigma_0}[\psi]$ is defined by (3.5). We decompose

$$\psi = \psi_0 + \psi_{\geq 1}$$

as in Section 4.1. In view of the orthogonality of $\psi_0$ and $\psi_{\geq 1}$ in $L^2(\mathbb{S}^2)$ we have

$$\int_{\mathcal{A}} (\partial_r \psi)^2 \, dg_{\mathcal{A}} = \int_{\mathcal{A}} (\partial_r \psi_0)^2 \, dg_{\mathcal{A}} + \int_{\mathcal{A}} (\partial_r \psi_{\geq 1})^2 \, dg_{\mathcal{A}}.$$

(4.57)

For the spherically symmetric term we apply Proposition 4.3.1. For the term $\psi_{\geq 1}$ we apply the Hardy inequality (4.4) for $p = 0$ combined with the degenerate Morawetz estimate to get

$$\int_{\mathcal{A}} (\partial_r \psi_{\geq 1})^2 \, dg_{\mathcal{A}} \leq C \int_{\mathcal{A}} (r - M)^2 (\partial_r \partial_r \psi_{\geq 1})^2 \, dg_{\mathcal{A}} + C \int_{\Sigma_0} \left(J^T[\psi_{\geq 1}] \cdot n_{\Sigma_0}\right) \, dg_{\Sigma_0}. \tag{4.58}$$
The first integral on the right hand side can be bounded by Proposition 4.4.1. By adding the two estimates and using (4.57) we obtain the desired results.

**Proof of Theorem 3.2.** Let \( \psi \) be a spherically symmetric solution to the wave equation with smooth compactly supported initial data such that the conserved charge

\[
H[\psi] = \int_{\Sigma_0 \cap \mathcal{H}} \left( \partial_r \psi + \frac{1}{M} \psi \right) \, d\omega = 1.
\]  
(4.59)

Then we clearly have that

\[
\int_{\Sigma_0 \cap \{r \leq R\}} \left[ \frac{1}{(r-M)} \cdot \left( \partial_r \psi + \frac{1}{r} \psi \right) \right]^2 \, dr \, d\omega = \infty.
\]  
(4.60)

On the other hand, we have established that for the spherically symmetric solution \( \psi \) the exact identity (4.36) holds, which can be re-written as

\[
\int_{A_{r_0}} \left[ \left( \frac{3}{2} - \frac{(r-M)}{r} \right) \cdot (\partial_r \psi_0)^2 + \left( \frac{r-M}{r^3} \right) \cdot \psi_0^2 \right] \, dr \, dv \, d\omega \\
+ \int_{H_{r_0}} \left[ \frac{1}{2r^2} \cdot \psi_0^2 \right] \, dv \, d\omega + \int_{B_R} \left[ K_R[\psi_0] \right] \, dv \, d\omega \\
= \int_{\Sigma_0 \cap \{r_0 \leq r \leq R\}} \left[ \frac{r^2}{(r-M)} \cdot \left( \partial_r \psi_0 + \frac{1}{r} \psi_0 \right) \right]^2 \, dr \, d\omega
\]  
(4.61)

where \( K_R[\psi] \) is given by (4.35). In view of (4.60), the right hand side of (4.61) is tends to infinity as \( r_0 \to M \). Since the initial data of \( \psi \) are assumed to be smooth and compactly supported with have that the \( T \)-flux of \( \psi \) through \( \Sigma_0 \) is finite. Hence, the integrals in (4.61) over the hypersurface \( B_R \) is uniformly (in \( r_0 \)) bounded using the averaging principle and the degenerate Morawetz estimate (3.2). On the other hand, if we assume that

\[
\int_{A} (\partial_r \psi)^2 \, dg_A < \infty
\]  
(4.62)

then the integral in (4.61) over the hypersurface \( H_{r_0} \) is uniformly (in \( r_0 \)) bounded using the Hardy inequality (4.8) and and the zeroth order term in the integral over \( A_{r_0} \) is uniformly (in \( r_0 \)) bounded using the Hardy inequality (4.4). Hence, if we assume (4.62) then the limit as \( r_0 \to M \) of left hand side of (4.61) is finite, which, since \( A_{r_0} \) tends to \( A \) as \( r_0 \to M \), contradicts (4.62)! This completes the proof of the theorem.

**Proof of Theorem 3.3.** We consider initial data for \( \psi \) with vanishing conserved charge \( H[\psi] = 0 \) but which are singular in the sense that (3.9) holds. Clearly such the data are not \( C^2 \).

The proof in this case mimics that of the proof of Theorem 3.2. Indeed we use the identity (4.61) for the spherical mean of \( \psi \) and argue by contradiction. Assuming
that (4.62) holds we obtain that left hand side of (4.61) is uniformly bounded in $r_0$. This however contradicts the fact that the right hand side of (4.61) blows up as $r_0 \to M$.

\[ \square \]

### 4.6 Remarks about the singular multiplier $S$

The proof of Theorem 3.1 heavily relies on the use of the singular vector field

$$ S = -\frac{1}{r-r_{\text{hor}}} \cdot \partial_r $$

as a multiplier vector field. Here $r_{\text{hor}}$ is the radius of the event horizon and hence this vector field is singular on the event horizon.

As was noted in Section 4.3, the most critical term in the analysis is the integral $I_4$ given by (4.22). Note that the boundeness of the limit as $r_0 \to r_{\text{hor}}$ (where the event horizon is located at $r = r_{\text{hor}}$) of this integral in the extremal case is a new feature of the geometry of degenerate horizons. In other words, the product $\partial_r \psi \cdot \partial_v \psi$ oscillates in time faster and faster as we approach the event horizon forcing thus the singular integral $I_4$ to have a finite limit. Such an oscillation cease to hold in the sub-extremal case. Indeed, consider solutions to the wave equation with initial data in the class $C_{\text{data}}$ given by

$$ C_{\text{data}} = \left\{ \text{smooth data supported on } \Sigma_0 \cap \{r_{\text{hor}} < R_1 \leq r \leq R_2 \} \right\}, $$

for some constants $R_1, R_2 > r_{\text{hor}}$. We will next show that if we consider generic solutions to the wave equation on sub-extremal black holes with data in the class $C_{\text{data}}$ then

$$ \lim_{r_0 \to r_{\text{hor}}} \int_{A_{r_0}} \frac{1}{r - r_{\text{hor}}} \partial_r \psi \cdot \partial_v \psi \, dg_{A_{r_0}} = \infty. \quad (4.63) $$

Note that in the sub-extremal case we schematically have

$$ \square \psi = (r - r_{\text{hor}}) \cdot \partial_r \partial_r \psi + \partial_r \partial_v \psi + \partial_v \partial_v \psi + \partial_v \psi + \Delta \psi = 0 \quad (4.64) $$

and

$$ dr = (r - r_{\text{hor}}) \cdot dv \text{ on } H_{r_0}. \quad (4.65) $$

We next restrict our attention to spherically symmetric solutions to the wave equation on subextremal backgrounds. Let us assume that for all such solutions with initial data in the class $C_{\text{data}}$ the following integral is in fact bounded:

$$ \lim_{r_0 \to r_{\text{hor}}} \int_{A_{r_0}} \frac{1}{r - r_{\text{hor}}} \partial_r \psi \cdot \partial_v \psi \, dg_{A_{r_0}} < \infty, \quad (4.66) $$

Then, we clearly have that if $\psi$ has data in the class $C_{\text{data}}$ then

$$ \varphi = \partial_v \psi $$
also has data in $\mathcal{D}_{\text{data}}$. Hence,

$$
\lim_{r_0 \to r_{\text{hor}}} \int_{\mathcal{A}_{r_0}} \frac{1}{(r - r_{\text{hor}})} \partial_r \varphi \cdot \partial_r \varphi \, dg_{\mathcal{A}_{r_0}} < \infty. \tag{4.67}
$$

By integrating with respect to $\partial_\nu$ we obtain

$$
\lim_{r_0 \to r_{\text{hor}}} \int_{\mathcal{A}_{r_0}} \frac{1}{(r - r_{\text{hor}})} \partial_r \varphi \cdot \partial_\nu \varphi \, dr \, d\omega = \int_{\mathcal{H}_{r_0}} \frac{1}{r - r_{\text{hor}}} \varphi \cdot \partial_r \varphi \, dr \, d\omega - \int_{\mathcal{H}_{r_0}} \frac{1}{r - r_{\text{hor}}} \varphi \cdot \partial_r \varphi \, dr \, d\omega
$$

$$
- \int_{\mathcal{A}_{r_0}} \frac{1}{(r - r_{\text{hor}})} \partial_\tau \partial_\nu \varphi \cdot \varphi \, dr \, d\omega. \tag{4.68}
$$

In view of (4.65) we have

$$
\int_{\mathcal{H}_{r_0}} \frac{1}{r - r_{\text{hor}}} \varphi \cdot \partial_r \varphi \, dr \, d\omega = \int_{\mathcal{H}_{r_0}} \varphi \cdot \partial_r \varphi \, dv \, d\omega
$$

$$
\leq \int_{\mathcal{H}_{r_0}} \left[ \varphi^2 + (\partial_r \varphi)^2 \right] dv \, d\omega \tag{4.69}
$$

$$
\leq C \int_{\Sigma_0} \left( \sum_{k=1,2} J^N \left[ N^k \varphi \right] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0} \leq \infty,
$$

where in the last step we used the redshift estimate of Dafermos and Rodnianski [?]. Note that the bound is uniform in $r_0$.

The integral over $\Sigma_0$ in (4.68) is finite since the integrand quantity depends only on the initial data of $\varphi$ and by assumption these data are supported away from the horizon. Furthermore, in view of (4.64), we schematically obtain

$$
\int_{\mathcal{A}_{r_0}} (r - r_{\text{hor}}) \partial_r \partial_\nu \varphi \cdot \varphi \, dr \, dv \, d\omega
$$

$$
= \int_{\mathcal{A}_{r_0}} \frac{1}{(r - r_{\text{hor}})} \left[ (r - r_{\text{hor}}) \partial_r \partial_\nu \varphi + \partial_\nu \partial_r \varphi + \partial_r \partial_r \varphi \right] \cdot \varphi \, dr \, dv \, d\omega \tag{4.70}
$$

$$
= \int_{\mathcal{A}_{r_0}} \left[ \partial_\nu \partial_r \varphi + \frac{1}{(r - r_{\text{hor}})} \partial_\nu \partial_\nu \varphi + \frac{1}{(r - r_{\text{hor}})} \partial_r \partial_r \varphi \right] \cdot \varphi \, dr \, dv \, d\omega
$$

Similarly as above we have

$$
\int_{\mathcal{A}_{r_0}} \partial_r \partial_\nu \varphi \cdot \varphi \, dv \, d\omega
$$

$$
\leq \int_{\mathcal{A}_{r_0}} \left[ \varphi^2 + (\partial_r \varphi)^2 \right] dv \, d\omega \tag{4.71}
$$

$$
\leq C \int_{\Sigma_0} \left( \sum_{k=1,2} J^N \left[ N^k \varphi \right] \cdot n_{\Sigma_0} \right) dg_{\Sigma_0} \leq \infty,
$$

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where we used once again the redshift estimate. Furthermore, in view of (4.65)
\[
\int_{A_{r_0}} \frac{1}{(r - r_{\text{hor}})} (\partial_v \varphi^2) \, dv \, d\omega
\]
\[
= \int_{H_{r_0}} \frac{1}{(r - r_{\text{hor}})} \varphi^2 \, d\omega - \int_{\Sigma_0} \frac{1}{(r - r_{\text{hor}})} \varphi^2 \, d\omega
\]
\[
= \int_{H_{r_0}} \varphi^2 \, dv \, d\omega - \int_{\Sigma_0} \frac{1}{(r - r_{\text{hor}})} \varphi^2 \, d\omega
\]
\[
\leq C \int_{\Sigma_0} \left( J^N[\varphi] \cdot n_{\Sigma_0} \right) \, dg_{\Sigma_0} - \int_{\Sigma_0} \frac{1}{(r - r_{\text{hor}})} \varphi^2 \, d\omega < \infty
\]  
(4.72)
since \( \varphi \) is initially supported away from the horizon. Regarding the remaining term we schematically have the following
\[
\int_{A_{r_0}} \frac{1}{(r - r_{\text{hor}})} \cdot \partial_v \varphi \cdot \varphi \, dv \, d\omega
\]
\[
= \int_{A_{r_0}} \frac{1}{(r - r_{\text{hor}})} \cdot \partial_r \partial_v \psi \cdot \partial_v \psi \, dv \, d\omega
\]
\[
= \int_{A_{r_0}} \frac{1}{(r - r_{\text{hor}})} \cdot \left[ (r - r_{\text{hor}}) \cdot \partial_r \partial_v \psi + \partial_v \psi + \partial_r \psi \right] \cdot \partial_v \psi \, dv \, d\omega
\]
\[
= \int_{A_{r_0}} \left[ \partial_r \partial_v \psi + \frac{1}{(r - r_{\text{hor}})} \cdot \partial_v \psi + \frac{1}{(r - r_{\text{hor}})} \cdot \partial_r \psi \right] \cdot \partial_v \psi \, dv \, d\omega.
\]  
(4.73)
As before, by Cauchy–Schwarz and the redshift estimate we have
\[
\int_{A_{r_0}} \partial_r \partial_v \psi \cdot \partial_v \psi \, dv \, d\omega \leq C \int_{\Sigma_0} \left( \sum_{k=1,2} J^N[N^k \varphi] \cdot n_{\Sigma_0} \right) \, dg_{\Sigma_0} < \infty.
\]
By our assumption (4.66) we also have
\[
\lim_{r_0 \to r_{\text{hor}}} \int_{A_{r_0}} \frac{1}{(r - r_{\text{hor}})} \partial_r \psi \cdot \partial_v \psi \, dv \, d\omega \leq \infty.
\]  
(4.74)
Hence, in view of (4.67), (4.68), (4.69), (4.70), (4.71), (4.72), (4.73) and (4.74), finally obtain that
\[
\lim_{r_0 \to r_{\text{hor}}} \int_{A_{r_0}} \frac{1}{(r - r_{\text{hor}})} \cdot \left( \partial_v \psi \right)^2 \, dv \, d\omega < \infty.
\]  
(4.75)
However, the integrand quantity is non-negative definite and moreover for generic \( \psi \) we have that there is a \( v_1 > 0 \) such that
\[
(\partial_v \psi)(v_1, r = r_{\text{hor}}) \neq 0.
\]  
(4.76)
Indeed, we can impose data on the event horizon such that \( \partial_v \psi \) is a bump function with \( (\partial_v \psi)(v_1, r = r_{\text{hor}}) = 1 \) and \( (\partial_v \psi)(v, r = r_{\text{hor}}) = 0 \) for all \( 0 \leq v \leq v_0 \) and for all
$v_2 \leq v$, for some $0 < v_0 < v_1 < v_2$. Since $\psi$ is spherically symmetric we may also impose (ill-posed) trivial data on the hypersurface $v = v_2$. By solving backwards we obtain a solution $\psi$ to the wave equation which is compactly supported (in fact it is zero in the region where $v \geq v_2$) and also $(\partial_v^k \psi)(v = 0, r = r_{\text{hor}}) = 0$ at $\Sigma_0 \cap \mathcal{H}$, where $\mathcal{H}$ denotes the event horizon. Since the function $\frac{1}{(r-r_{\text{hor}})}$ is not in $L_{\text{loc}}^1$ in $\mathcal{A}$, the condition (4.76) immediately forces the limit on the left hand side of (4.75) to be infinite, which is a contradiction.

## 5 Higher order estimates and stable trapping

In this section we prove Theorem 3.4. We show that for generic smooth initial data there is no non-degenerate higher order Morawetz estimate for solutions arising from smooth initial data on a Cauchy hypersurface $\Sigma_0$ supported in a compact region $\{ M < \tilde{R}_1 \leq r \leq \tilde{R}_2 \}$ away from $\Sigma_0 \cap \mathcal{H}$, where $\mathcal{H} = \{ r = M \}$ denotes the event horizon.

**Proof of Theorem 3.4.** First divide $\Sigma_0$ in the following regions:

$$\Sigma_0 = \Sigma_0^1 \cup \Sigma_0^2 \cup \Sigma_0^3$$

where

$$\Sigma_0^1 = \Sigma_0 \cap \{ M \leq r \leq \tilde{R}_1 \}, \quad \Sigma_0^2 = \Sigma_0 \cap \{ \tilde{R}_1 \leq r \leq \tilde{R}_2 \}, \quad \Sigma_0^3 = \Sigma_0 \cap \{ r \geq \tilde{R}_2 \},$$

where $M < \tilde{R}_1 < \tilde{R}_2 < \infty$.

We consider the following data on $\Sigma_0$:

$$\phi|_{\Sigma_0^1} = 1, \quad \phi|_{\Sigma_0^3} = 0, \quad \partial_v \phi|_{\Sigma_0^1} = 0, \quad \partial_v \phi|_{\Sigma_0^3} = 0, \quad (5.1)$$
and $\phi|_{\Sigma_2}$ and $\partial_v \phi|_{\Sigma_3}$ are smooth and spherically symmetric. These data give rise to a spherically symmetric solution $\phi$ that is smooth in the domain of outer communications. Note also that

$$H[\phi] = \frac{1}{M} \neq 0.$$  \hfill (5.2)

We now consider:

$$\psi := \partial_v \phi$$  \hfill (5.3)

which is also a solution to the wave equation by the fact that $[\partial_v, \Box_g] = 0$. By Proposition 1 of [8] we have that the initial data of $\psi$ on $\Sigma_0$ are smooth and supported on a compact region $\left\{ M < \tilde{R}_1 \leq r \leq \tilde{R}_2 \right\}$. Hence, it follows that $H[\psi] = 0$.

More specifically, we have that:

$$\psi|_{\Sigma_1^0} = 0, \quad \psi|_{\Sigma_2^0} = 0,$$

$$\partial_v \psi|_{\Sigma_1^0} = 0, \quad \partial_v \psi|_{\Sigma_2^0} = 0,$$

and $\psi|_{\Sigma_2}$ and $\partial_v \psi|_{\Sigma_3}$ are smooth and spherically symmetric.

We will next show that for $\psi$ given by (5.3) we have

$$\int_A (\partial_r \partial_r \psi)^2 \, dg_A = \infty.$$  \hfill (5.4)

In view of the wave equation for $\phi$ we have

$$-2 \partial_r \partial_v \phi = D \cdot \partial_r \partial_v \phi + \frac{2}{r} \partial_v \phi + R \cdot \partial_r \phi$$  \hfill (5.5)

and hence commuting with the vector field $\partial_r$ we obtain

$$-2 \partial_r \partial_r \partial_v \phi = D \cdot \partial_r \partial_r \phi + \left( \partial_r D + R \right) \cdot \partial_r \partial_v \phi + \partial_r R \cdot \partial_r \phi + \partial_r \left( \frac{2}{r} \partial_v \phi \right).$$  \hfill (5.6)

By virtue of (4.14) we have

$$r^2 \cdot (\partial_r D + R) = 2M \left( \frac{r - M}{r} \right) + 2(r - M) = 4(r - M) - \frac{2}{r} \cdot (r - M)^2$$

and

$$r^2 \cdot \partial_r R = 2 - \frac{4}{r} \cdot (r - M).$$

Hence, if we define

$$\mathcal{E}_1[\phi] := 2r \cdot \partial_r \partial_v \phi - 2\partial_v \phi - \frac{2}{r} (r - M)^2 \cdot \partial_r \partial_v \phi - \frac{4}{r} (r - M) \cdot \partial_r \phi,$$  \hfill (5.7)

then we obtain

$$4r^2 \cdot |\partial_r \partial_r \partial_r \phi| = (r - M)^2 \cdot \partial_r \partial_r \partial_r \phi + 4(r - M) \cdot \partial_r \partial_r \phi + 2 \cdot \partial_r \phi + \mathcal{E}_1[\phi]$$  \hfill (5.8)
and, therefore,

$$4r^2 \cdot |\partial_r \partial_r \partial_v \phi| = \partial_r \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) + \mathcal{E}_1[\phi]. \quad (5.9)$$

By the Cauchy–Schwarz inequality we obtain

$$4r^4 \cdot \left( |\partial_r \partial_r \partial_v \phi| \right)^2 \geq (1 - \epsilon) \cdot \partial_r \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right)^2 + \left( 1 - \frac{1}{\epsilon} \right) \cdot \left( \mathcal{E}_1[\phi] \right)^2 \quad (5.10)$$

for sufficiently small $\frac{1}{2} > \epsilon > 0$. Therefore, after choosing such $\epsilon$ and using that $r$ is bounded in $\mathcal{A}$, there exists a positive constant $C$ that depends only on $M$ such that

$$\int_{\mathcal{A}} \left( \partial_r \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) \right)^2 \, dg_{\mathcal{A}} \leq C \int_{\mathcal{A}} r^4 \cdot \left( |\partial_r \partial_r \partial_v \phi| \right)^2 \, dg_{\mathcal{A}} + C \int_{\mathcal{A}} \left( \mathcal{E}_1[\phi] \right)^2 \, dg_{\mathcal{A}} \quad (5.11)$$

We will next show that

$$\int_{\mathcal{A}} \left( \mathcal{E}_1[\phi] \right)^2 \, dg_{\mathcal{A}} < \infty. \quad (5.12)$$

First we observe that in view of the wave equation (4.13) for $\phi$ we can write

$$- \frac{2}{r}(r - M)^2 \cdot \partial_r \partial_v \phi = 4r \cdot \partial_r \partial_v \phi + 4 \partial_v \phi + \frac{4}{r}(r - M) \cdot \partial_r \phi$$

and hence $\mathcal{E}_1[\phi]$ becomes

$$\mathcal{E}_1[\phi] = 6r \cdot \partial_r \partial_v \phi + 2 \partial_v \phi. \quad (5.13)$$

Therefore,

$$\int_{\mathcal{A}} \left( \mathcal{E}_1[\phi] \right)^2 \, dg_{\mathcal{A}} \leq C \int_{\mathcal{A}} \left( (\partial_v \phi)^2 + (\partial_r \partial_v \phi) \right)^2 \, dg_{\mathcal{A}} \leq C \int_{\mathcal{A}} \left( (\partial_v \phi)^2 + (\partial_r \psi) \right)^2 \, dg_{\mathcal{A}}. \quad (5.14)$$

In view of the degenerate Morawetz theorem established in [5] we have

$$\int_{\mathcal{A}} (\partial_v \phi)^2 \, dg_{\mathcal{A}} \leq C \int_{\Sigma_0} \left( J^N[\phi] \cdot n_{\Sigma_0} \right) \, dg_{\Sigma_0} < \infty \quad (5.15)$$

since $\phi$ is a smooth on $\Sigma_0$. Furthermore, in view of the Proposition 4.3.1 we have

$$\int_{\mathcal{A}} (\partial_r \psi)^2 \, dg_{\mathcal{A}} \leq C \cdot D_{\Sigma_0}[\psi] < \infty \quad (5.16)$$

since

$$D_{\Sigma_0}[\psi] = \int_{\Sigma_0 \cap (r \leq R)} \left[ \frac{1}{r - M} \cdot \left( \partial_r (r \psi) \right)^2 \right] \, dr \, d\omega + \int_{\Sigma_0} \left( J^T[\psi] \cdot n_{\Sigma_0} \right) \, dg_{\Sigma_0} < \infty \quad (5.17)$$
since $\psi$ is a smooth function on $\Sigma_0$ supported on the set \( \{ M < \tilde{R}_1 \leq r \leq \tilde{R}_2 \} \).

Clearly, (5.12) follows from (5.15) and (5.16).

Arguing by contradiction, let us assume that we in fact have
\[
\int_{A} (\partial_r \partial_r \psi)^2 \, dg_A < \infty. \tag{5.17}
\]
Then, in view of (5.11), (5.12) and (5.17), we have
\[
\int_{A} \left( \partial_r \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) \right)^2 \, dg_A < \infty. \tag{5.18}
\]
Since
\[
\partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) \sim (r - M)
\]
in $A$, we can apply the Hardy inequality (4.5) for $p = -2$ to get
\[
\int_{A} \frac{1}{(r - M)^2} \left( \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) \right)^2 \, dg_A \leq C \int_{A} \left( \partial_r \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) \right)^2 \, dg_A \tag{5.19}
\]
Applying the Hardy inequality (4.5) one more time for $p = -4$ we obtain
\[
\int_{A} \frac{1}{(r - M)^4} \cdot \left( (r - M)^2 \cdot \partial_r \phi \right)^2 \, dg_A \leq C \int_{A} \frac{1}{(r - M)^2} \cdot \left( \partial_r \left( (r - M)^2 \cdot \partial_r \phi \right) \right)^2 \, dg_A \tag{5.20}
\]
Hence, in view of the estimates (5.17), (5.19) and (5.20) we get
\[
\int_{A} (\partial_r \phi)^2 < \infty \tag{5.21}
\]
On the other hand, by virtue of (5.2) we have that the conserved charge $H[\phi] \neq 0$ and hence by Theorem 3.2 it follows that
\[
\int_{A} (\partial_r \phi)^2 = \infty \tag{5.22}
\]
which of course contradicts (5.21). Hence, our assumption that the integral (5.17) is finite is wrong and this completes the proof for $k = 2$.

For $k \geq 3$ we simply argue by repeatedly using the Hardy inequality (4.4) to obtain
\[
\int_{A} (\partial_r \partial_r \psi)^2 \, dg_A \leq C_k \int_{A} (\partial_r^k \psi)^2 \, dg_A
\]
which implies that the integral on the left hand side must also be infinite.

Clearly, (5.4) holds generically. Indeed, if we consider a solutions $\Psi$ to the wave equation such that
\[
\int_{A} (\partial_r \partial_r \Psi)^2 \, dg_A < \infty
\]
then (5.4) holds for $\Psi + \epsilon \psi$ for arbitrarily small $\epsilon$ and $\psi$ given by (5.3) above. This completes the proof.

\[\square\]
6 Relation with the stability theory of MOTS

In this section we shall attempt to provide a connection of our findings to the stability theory of marginally outer trapped surfaces (MOTS).

Our analysis shows a necessary condition for the existence of a non-degenerate Morawetz estimate up to and including the event horizon for solutions $\psi$ to (1.1) is the vanishing of the conserved charge

$$H[\psi] = \int_{S_\tau} (Y\psi + \frac{1}{M}\psi),$$

(6.1)

where $S_\tau = \Sigma_\tau \cap \mathcal{H}$. If the conserved charge does not vanish then $\psi$ fails to satisfy a Morawetz estimate regardless of its degree of regularity. This is in stark contrast with the trapping effect on the photon sphere where Morawetz estimates hold as long as loss of regularity is allowed. The necessity of the vanishing of the charge (6.1) on each section $S_\tau$ implies that a global trapping effect takes place on degenerate horizons. We shall interpret this in terms of the stability theory of the sections $S_v$, the latter seen as marginally outer trapped surfaces on extremal Reissner–Nordström.

It was shown in [7] that the conserved charge (6.1) arises from the kernel of the elliptic operator

$$O_{S_\tau} \psi = \triangle \psi + 2\nabla \cdot \psi + \left[2d\nabla \zeta^2 + \partial_v(tr\chi) + \frac{1}{2}(tr\chi)(tr\chi) \right] \cdot \psi,$$

$$O_{S_\tau} \Psi = \triangle \Psi + d\nabla(2\Psi \cdot \zeta) + tr\chi \cdot \kappa \cdot \Psi,$$

(6.2)

where $\zeta$ denotes the torsion of the section $S_\tau$ and $tr\chi, tr\chi$ the outgoing and ingoing null mean expansions, respectively. This operator was introduced in [9] where it was shown that in the case of Killing horizons it reduces to

$$O_{S_\tau} \psi = \triangle \psi + 2d\nabla \zeta^2 + \partial_v(tr\chi) + \frac{1}{2}(tr\chi)(tr\chi) \cdot \psi,$$

$$O_{S_\tau} \Psi = \triangle \Psi + d\nabla(2\Psi \cdot \zeta) + tr\chi \cdot \kappa \cdot \Psi,$$

where $\kappa$ is the surface gravity.

In view of work of Mars[4][23] the operator (6.2) coincides with the stability operator on the marginally outer trapped surface $S_\tau$. In the case of sub-extremal black holes we have that $tr\chi < 0$ and $\kappa > 0$ and hence the principal eigenvalue of the operator $O_{S_\tau}$ must be strictly negative. Hence the whole spectrum of the operator $O_{S_\tau}$ is strictly negative. This implies that all sections $S_\tau$ of sub-extremal event horizons are stable as MOTS.

On the other hand, in the extremal case we have $\kappa = 0$ which implies that the principal eigenvalue of $O_{S_\tau}$ is zero. Hence, there is a unique function (up to a constant) that belongs in the kernel of $O_{S_\tau}$ and all the other eigenfunctions correspond to strictly negative eigenvalues. Hence, the sections $S_\tau$ of are stable as MOTS in all but exactly one transversal perturbation with respect to which they are marginally stable. That is, there is transversal perturbation of the degenerate horizon with respect to which the outgoing null mean expansion is stationary (i.e. has a critical point). In other words, there is a unique perturbation of the degenerate horizon with respect

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4We acknowledge private communication with Marc Mars.
to which its sections do not expand to second order. Recalling that the conserved charge (6.1) arises from the same perturbation (the kernel of $O_{S_+}$), we conclude the failure of the expansion of the horizon in this direction induces the global trapping effect.

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