On the conformal transformation and duality in gravity

ILYA L. SHAPIRO

Departamento de Fisica Teorica, Universidad de Zaragoza, 50009, Spain
and
Departamento de Fisica – ICE, Universidade Federal de Juiz de Fora – MG, Brazil

Abstract. The theory described by the sum of the Einstein-Hilbert action and the action of conformal scalar field possesses the duality symmetry which includes some special conformal transformation of the metric, and also inversion of scalar field and of the gravitational constant. In the present paper the conformal duality is generalized for an arbitrary space-time dimension $n \neq 2$ and for the general sigma-model type conformal scalar theory. We also consider to apply the conformal duality for the investigation of quantum gravity in the strong curvature regime. The trace of the first coefficient of the Schwinger-DeWitt expansion is derived and it’s dependence on the gauge fixing condition is considered. After that we discuss the way to extract the gauge-fixing independent result and also it’s possible physical applications.

Introduction

It is well known that the study in the field of quantum gravity meets serious difficulties. On classical level General Relativity is in a good accordance with all the known tests, but quantum theory based on GR is nonrenormalizable [1]. Indeed there are a number of approaches to investigate some aspects of quantum gravitational phenomenons. One can mention the idea of inducing GR from the quantum effects of matter fields or from the theory of string (see, for example, [2] and [3]). Recently there were a very interesting attempts to extract the low-energy predictions from the GR-based quantum theory of gravity [4, 5]. However the principal problem with the lack of quantum gravitational experiments doesn’t enable one to choose among the numerous theories and therefore the models for quantum gravity are subjects of wide arbitrariness. This concerns, for instance, the choice of the model for the description of low-energy quantum gravitational phenomena, where the string-inspired action contains, along with the metric, massless dilaton field. In such a situation a lot of attention has been attracted by the models which have more symmetries than GR. In particular, one can mention the supergravity theories, and also the theories with local conformal invariance. The conformal transformation and conformal symmetry in gravity are especially interesting in many respects [6, 7] because of their relation with the cosmological applications [8] and with the

\footnote{On leave from Tomsk Pedagogical University, 634041, Tomsk, Russia
E-mail: shapiro@fisica.ufjf.br}
physically important issues of renormalization group [4], conformal anomaly and anomaly induced gravitational effective action [11, 12, 13, 14, 15].

Another kind of problem concerns the methods of calculations in quantum gravity. The standard perturbative techniques give the effective action in a form of a power series in the curvature tensor and its derivatives (see [16] and [17] for the review). At the same time one of the most important applications of quantum gravity should be the regions of extremely strong gravitational field in the vicinity of cosmological and black hole singularities. One way to study this subject is to develop essentially nonperturbative methods in quantum gravity like dynamical triangulations which have been recently applied to the four dimensional gravity [18]. Another one is to explore a specific models for quantum gravity which have a symmetry linking the regimes of strong and weak gravitational field. In this paper we consider an example of such a symmetry – conformal duality, which has been originally discovered by Bekenstein [7] as a property of dynamical equations for the conformal scalar field coupled to gravity.

Our purpose is to investigate the conformal duality on quantum level, and hence we face the problem of nonrenormalizability of quantum gravity. The unitary second derivative models of quantum gravity are nonrenormalizable already at one loop if they are considered together with the matter fields [1, 19]. In fact the one-loop approximation has a special significance. If considering the asymptotically free theory [2] then the one-loop approximation provides the most of the information about the effective action in the strong gravitational field regime (which is natural UV limit for the nontrivial background metric [17]). One of the possible ways to avoid the problem of divergences is to consider the space-time dimension $n$ different from four. It is well known that in some dimensions the divergences can be kept under control, at least in some approximation. For instance, in any odd dimension the divergences are lacking at the one-loop level [21]. That is why in this work we start with generalizing the conformal duality of [7] for the most general second derivative metric-dilaton model in almost arbitrary space-time dimension $n \neq 2$. For this purpose we use the action instead of dynamical equations [22] that is easier and facilitates the consequent quantum consideration, which is based on the lower orders of the Schwinger-DeWitt expansion of the effective action.

The paper is organized as follows. In the next section we write the actions and symmetry transformations for the conformal metric-scalar theory and for the theory with conformal duality in arbitrary dimension $n$. Section 3 is devoted to the calculation of the first term in the Schwinger-DeWitt expansion of the one-loop effective action. In section 4 we consider the gauge-fixing dependence of the one-loop effective action and also construct the gauge-independent combination of its components. In the last section some conclusions and also discussion of the next reasonable steps in the study of conformal duality are presented. Some information which is aimed to support the main text, is placed in the Appendixes. The possibility of the soft breaking of conformal symmetry is discussed in Appendix A, and the equations of motion for the dual model are written down in Appendix B, where we also solve the equation for the time-dependent dilaton. Both conformal and dual actions are singular in the limit $n \rightarrow 2$, which is explored in some details in Appendix C. Appendix D contains bulky expansion of the dual action which has been used in the one-loop calculations.

2. General theory with conformal symmetry and duality.

We consider the second-derivative metric-dilaton model with the action of sigma-model type in $n$-dimensional space-time.

$$S[g_{\mu \nu}, \phi] = \int d^n x \sqrt{-g} \left\{ A(\phi) g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + B(\phi) R + C(\phi) \right\}$$  (1)

One can suppose that the renormalization group for gravity can be formulated in the nonperturbative way, and so can be applied even for the nonrenormalizable theories [20].
This theory possesses general covariance and, for some special choice of the functions $A, B, C$ (otherwise arbitrary), an extra conformal symmetry. As a first step we describe the local conformal transformation in the theory (1) without any restrictions for these functions. For this purpose we need the relations between geometric quantities related with the metric $g_{\mu\nu}$ and with the transformed metric $\bar{g}_{\mu\nu} = g_{\mu\nu} e^{2\sigma}$, where $\sigma = \sigma(x)$.

$$\sqrt{-\bar{g}} = \sqrt{-g} e^{n\sigma}, \quad \bar{R} = R(\bar{g}_{\mu\nu}) = e^{-2\sigma} \left[ R - 2(n-1)(\Box \sigma) - (n-2)(n-1)(\nabla \sigma)^2 \right]$$

(2)

Substituting (2) into (1) we find

$$S[\bar{g}, \phi] = \int d^n x \sqrt{-\bar{g}} \left\{ e^{(n-2)\sigma} \left[ A(\nabla \phi)^2 + (n-1)(n-2)B(\nabla \sigma)^2 + 2(n-1)B_1(\nabla^\mu \phi)(\nabla_\mu \sigma) + RB \right] + e^{n\sigma} C \right\}$$

(3)

where we have used the notations of [22]: $X_i = d^i X / d\phi^i$, $(\nabla X)^2 = g^{\mu\nu} \partial_\mu X \partial_\nu X$.

If one performs an arbitrary reparametrization of the scalar field $\phi = \phi(\psi)$, then the action becomes $S[\bar{g}_{\mu\nu}, \psi]$. The condition of symmetry for the action (1)

$$S[\bar{g}_{\mu\nu}, \psi] = S[g_{\mu\nu}, \phi]$$

can be rewritten as the equations for the functions $A(\phi), B(\phi), C(\phi)$ and for $\sigma(\phi)$.

$$B(\phi) = e^{(n-2)\sigma} B(\psi(\phi))$$

$$A(\phi) = A(\psi(\phi)) \left( \frac{d\psi}{d\phi} \right)^2 + (n-1)(n-2)B(\psi(\phi)) \left( \frac{d\sigma}{d\phi} \right)^2 + 2(n-1) \left( \frac{d\psi}{d\phi} \right) \left( \frac{d\sigma}{d\phi} \right) \left( \frac{dB(\psi)}{d\psi} \right)_{\psi = \psi(\phi)}$$

$$C(\phi) = e^{\sigma} C(\psi(\phi))$$

(4)

The solution for these equations has the form

$$e^{(n-2)\sigma(\phi)} = \frac{B(\phi)}{B(\psi(\phi))}$$

$$A(\phi) = \frac{n-1}{n-2} B_1^2(\phi), \quad C(\phi) = \lambda \frac{B(\psi(\phi))}{\pi^{\frac{n-2}{2}}} \lambda = const$$

(5)

(6)

Indeed the equations (1), (2) have different sense. (1) shows the relation between arbitrary reparametrization of scalar field and the corresponding conformal transformation for the metric. Therefore the symmetry we found is nothing but the direct generalization of the ordinary one-parameter conformal symmetry. The equations (2) give the constraints on $A(\phi), C(\phi)$ which are caused by conformal symmetry. In the Appendix A it is shown that the dynamical equations for the theory with $A(\phi)$ satisfying (2) and $C(\phi)$ not, do not have any solutions. It is useful to introduce the special notation for the action (1) which satisfies the conformal constraints (3).

Following [22] we denote the action of such theory as $S_{B(\phi), \lambda}$. One can easily see that one of the particular cases of $S_{B(\phi), \lambda}$ is the Einstein gravity with

$$B = \gamma \kappa^{-2}$$

(7)

Here $[\kappa^2]$ is universal dimensional constant, and $\gamma$ is some dimensionless one. We split the Newton constant into two parts for convenience. $C$ term here becomes cosmological constant.

Another particular case of $S_{B, \lambda}$ is the conventional conformal scalar field. One can start with arbitrary $S_{B(\phi), \lambda}$ and put

$$B(\phi) = \frac{n-2}{2(n-1)} \phi^2$$

(8)
where $\psi$ is some other scalar. Then, solving (8) in the form $\phi = \phi(\psi)$ we find the reparametrization which links an arbitrary $S_{B(\phi),\lambda}$ with particular model of conventional conformal scalar. The only one sort of $S_{B(\phi),\lambda}$ which can not be reparametrized in this way is Einstein-Hilbert action (7).

In fact GR is also equivalent to conformal scalar field [3]. To see this we substitute the conditions (6) into the transformed expression (3). It turns out that the last also satisfies (6) so that

$$S_{B(\phi),\lambda}\left[g_{\mu\nu},\phi\right] = S_{K(\phi),\lambda}\left[g_{\mu\nu},\phi\right]$$

for any given function $K(\phi)$ (with only sign restrictions), if only

$$\bar{g}_{\mu\nu} = \left[\frac{K(\phi)}{B(\phi)}\right]^{\frac{2}{n-2}} g_{\mu\nu}$$

In particular one can choose $K = \text{const}$ and demonstrate the conformal equivalence between (7) and other models $S_{B(\phi),\lambda}$ with nonconstant $B$.

The equality (10) can be used as a basis for another interesting symmetry, which holds for some of the nonconformal versions of (1). Let us consider the sum

$$S_{B(\phi),\lambda}\left[g_{\mu\nu},\phi\right] + S_{L(\phi),\tau}\left[g_{\mu\nu},\phi\right]$$

where $B(\phi), L(\phi)$ are some functions and $\lambda, \tau$ are some (arbitrary) constants. Perform the special conformal transformation (10) and find that (11) becomes

$$S_{K(\phi),\lambda}\left[g_{\mu\nu},\phi\right] + S_{M(\phi),\tau}\left[g_{\mu\nu},\phi\right], \quad M(\phi) = \frac{L(\phi)K(\phi)}{B(\phi)}$$

Especially interesting particular case of the above symmetry takes place when one of the components in (11) is GR $L = \gamma \kappa^{-2} = \text{const}$. For the sake of simplicity we choose the second function to be $B(\phi) = \phi \kappa^{-2}$, therefore scalar is dimensionless. Then the symmetry transformation reads

$$\left\{S_{\phi,\frac{1}{\kappa^2},\lambda} + S_{\gamma^{-2},\frac{1}{\gamma^2},\tau}\right\}_{g_{\mu\nu}} = \left\{S_{\frac{1}{\phi^2},\tau} + S_{\frac{1}{\gamma^2},\lambda}\right\}_{\bar{g}_{\mu\nu}}$$

and hence for this specific case we face a conformal duality which exchanges

$$\phi \leftrightarrow \phi^{-1}, \quad \gamma \leftrightarrow \gamma^{-1}, \quad \tau \leftrightarrow \lambda$$

$$\bar{g}_{\mu\nu} \leftrightarrow g_{\mu\nu}, \quad \bar{g}_{\mu\nu} = g_{\mu\nu} \left(\phi \gamma\right)^{-\frac{2}{n-2}}$$

The equations of motion for the theory (13) have some interesting features, which are considered in Appendix B. One can see that the conformal solution (10) and (consequently) the action of the theory with conformal duality become singular when $n$ tends to 2. Some discussion of this limit is given in Appendix C.

The dual symmetry links different metrics, different values of scalar field and different values of coupling constants. Let us, for instance, choose weakly changing $\phi << 1$ and also $\gamma << 1$. Then the curvatures of two metrics $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ are linked by relation (14). If one supposes that the last terms in the brackets in (2) are of the same order or smaller than the scalar curvature, we arrive at the transformation which links the regimes of strong and weak gravitational field. The metric $g_{\mu\nu}$ can be almost flat and it’s curvature very small while the curvature of the metric $\bar{g}_{\mu\nu}$ can have

\footnote{Of course some sign restrictions for the possible form of $B(\phi)$ appears (see second reference in [3]), and for some cases one needs to change the sign in (9), otherwise the correspondence with (7) can not be achieved.}
a big value. Simultaneously one meets a big value of the scalar \(1/\phi\) and a of the dimensionless constant \(1/\gamma\).

This physical interpretation of conformal duality looks different from the one which is common for other types of dual symmetries (see, for example, the review \[23\]). Contrary to the string dualities, the conformal duality does not link the regimes of weak and strong coupling, because the coupling constant \(\kappa\) which is the parameter of the loop expansion in the path integral, does not change under the conformal duality transformation.

However the conformal duality can be important in the study of quantum gravity effects in the regime of strong gravitational field in a given order of the loop expansion. To illustrate this, consider the one-loop effective action in the theory of quantum gravity with classical action \(S[g_{\mu\nu}, \phi]\). The effective action \(\Gamma\) can be derived on the basis of the background field method and Schwinger-DeWitt expansion. The last gives the local representation of \(\Gamma\) as an infinite power series in the proper time parameter \(s\). Since this parameter is dimensional, the local \(\text{Tr} a_k(x,x)\) coefficients have dimensions of \((\text{curvature})^k\), so such an expansion can be efficient for the weak gravitational field only. Below we derive the first coefficient \(\text{Tr} a_1(x,x)\) coefficient for the theory with conformal duality and investigate it’s gauge dependence.

3. One-loop calculation

The purpose of this section is to formulate the one-loop effective action for the theory with conformal duality \(13\) and to evaluate the first nontrivial term in the Schwinger-DeWitt expansion for the effective action in an arbitrary dimension. Since all the theories with conformal duality \(12\) differ from the most simple one \(13\) by the reparametrization of the scalar field only, we shall perform the calculations for this simple case and thus start with the theory

\[
S = \frac{1}{\kappa^2} \int d^n x \sqrt{-g} \left\{ \frac{n-1}{n-2} \phi \frac{1}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (\phi + \gamma) R + V(\phi) \right\}
\]

where \(V(\phi) = \lambda \phi^{n-2} + \tau \gamma^{n-2}\).

For the sake of quantum calculations we apply the background field method (see \[17\] for the introduction). The features of the metric-dilaton theory require the special background gauge, which has been originally introduced in the similar two-dimensional theory \[24\] and recently applied for the calculation of the one-loop divergences in general four-dimensional metric-scalar theory \[1\] in \[25\]. The starting point of the calculations is the usual splitting of the fields into background \(W^a = (g_{\mu\nu}, \phi)\) and quantum \(w^a = (\bar{h}_{\mu\nu}, h, \varphi)\) ones as

\[
\phi \rightarrow \phi' = \varphi + \kappa \phi, \quad g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa h_{\mu\nu}, \quad h_{\mu\nu} = \bar{h}_{\mu\nu} + 1/n g_{\mu\nu} h, \quad h = h^\mu_{\mu}\]

where the trace and the traceless parts of the quantum metric have been separated for convenience. The details of expansion of the action \[15\] into series one can find in the Appendix D. The one-loop effective action is given by the standard general expression

\[
\Gamma^{1-\text{loop}} = \frac{i}{2} \text{Tr} \ln \hat{H} - i \text{Tr} \ln \hat{H}_{\text{ghost}} + \frac{i}{2} \text{Tr} \ln Y^{\mu\nu}
\]

where \(\hat{H}\) is the Hermitean bilinear form of the action \(S + S_{gf}\) with added gauge fixing term

\[
S_{gf} = \int d^n x \sqrt{-g} \chi_\mu Y^{\mu\nu} \chi_\nu,
\]

which contains a weight function \(Y^{\mu\nu}\). \(\hat{H}_{\text{ghost}}\) is the bilinear form of the action of the gauge ghosts. The general form of the gauge fixing condition and weight function are\(
\chi_\mu = \nabla_\mu \tilde{h}_\mu^\lambda + \beta \nabla_\mu h + \rho \nabla_\mu \varphi, \quad Y^{\mu\nu} = -\alpha g^{\mu\nu}
\)

\[4\]We consider the linear covariant background gauges only.

5
where the gauge fixing parameters $\alpha, \beta, \rho$ are some functions of the background dilaton, which can be fine tuned to make the calculations more simple. For instance, if one chooses these functions as

$$\alpha = \frac{1}{2} (\phi + \gamma), \quad \beta = \frac{2 - n}{n}, \quad \rho = -\frac{1}{\phi + \gamma} \quad (20)$$

then the bilinear part of the action $S + S_{gf}$ and the operator $\bar{H}$ have especially simple (minimal) structure

$$(S + S_{gf})^{(2)} = \int d^4 x \sqrt{-g} \left( \bar{h}_{\mu
u}, h, \phi \right) \left( \bar{H} \right) \left( \bar{h}_{\alpha\beta}, h, \phi \right)^T$$

$$\bar{H} = \bar{K} \Box + \bar{L}_\lambda \nabla^\lambda + \bar{M} \quad (21)$$

Here $T$ means transposition, and the matrices in $\bar{L}_\lambda$ have the form (we do not write the projectors to the symmetric traceless states for brevity, but they have to be restored when the matrices are contracted)

$$\bar{K} = \begin{pmatrix} \frac{2n}{\phi} & 0 & 0 \\ 0 & -\frac{2}{n} (\phi + \gamma) & -(1/4) \\ 0 & -(1/4) & \frac{1}{2(\phi + \gamma)} - \frac{n-1}{\phi(n-2)} \end{pmatrix}$$

$$\bar{L}_\lambda = \begin{pmatrix} \frac{1}{2} \delta^{\mu\nu,\alpha\beta} g^{\lambda\tau} + \frac{1}{2} g^\nu (g^{\alpha\lambda} - g^{\beta\lambda}) & -\frac{1}{2} g^\nu (g^{\alpha\lambda} - g^{\beta\lambda}) & -\frac{n-1}{\phi(n-2)} g^{\mu\tau} g^{\nu\lambda} \\ \frac{1}{2} g^\nu (g^{\alpha\lambda} - g^{\beta\lambda}) & -\frac{n-1}{2n\phi} g^{\lambda\tau} & \frac{n-1}{2n\phi} g^{\lambda\tau} \\ \frac{n-1}{(n-2)n} g^{\alpha\lambda} g^{\beta\lambda} & \frac{1-n}{2n\phi} g^{\alpha\lambda} & \left[ \frac{n-1}{\phi(n-2)} - \frac{1}{2(\phi + \gamma)} \right] g^{\lambda\tau} \end{pmatrix}$$

$$\bar{M} = \begin{pmatrix} \frac{1}{2} (\phi + \gamma) \left( R^{\mu\alpha\nu\beta} - R^{\mu\alpha} g^{\nu\beta} \right) + \frac{1}{2} \delta^{\mu\nu,\alpha\beta} \left( \Box \phi \right) - (1/4) (A + V) \delta^{\mu\nu,\alpha\beta} + g^\nu \delta^{\mu\alpha} A^\alpha & \frac{1}{2n} (\Box^\nu \phi - A^\nu \phi) - \frac{n-1}{2(n-2)\phi} \phi^\mu \phi^\nu - \frac{1}{2} R^{\mu\nu} \\ \frac{n-1}{2n} (\Box^\nu \phi - A^\nu \phi) & \frac{n-1}{2n} \frac{1}{(n-2)\phi} \left( \Box \phi + A + V \right) - \frac{n-1}{2n} \frac{1}{(n-2)^2} \phi^\mu \phi^\nu \end{pmatrix}$$

$$A^\mu = \frac{n-1}{n-2} (\nabla^\nu \phi) (\nabla^\mu \phi) + (\phi + \gamma) R^{\mu\nu}, \quad A = A^\mu g_{\mu\nu}, \quad \phi^\mu = \nabla^\mu \phi \quad (22)$$

To apply the Schwinger-DeWitt method we rewrite $\text{Tr} \ln \bar{H}$ in the following way.

$$\text{Tr} \ln \bar{H} = \ln \text{Det} \bar{K} + \text{Tr} \ln \left( \bar{1} \Box + \bar{K}^{-1} \bar{L}_\lambda \nabla^\lambda + \bar{K}^{-1} \bar{M} \right) \quad (23)$$

The first term gives simple contribution to the 1-loop effective action, because it is nothing but an ordinary functional determinant of the c-number matrix $\bar{K}$.

$$\det \bar{K} = \frac{n-1}{2n} \frac{\gamma (\phi + \gamma)}{\phi} \quad (24)$$

$^4$Here we abandon all the additive constants, all evident integrations over the space-time variables and related $\sqrt{-g}$ factor, and also the surface terms in the effective action.
The same concerns the last term in (17) which can be evaluated as
\[
\det Y^\mu\nu = \phi + \gamma
\]  
(25)
The second term in (23) is \(\ln \text{Det}\) of the operator of the standard minimal form and it can be, in principle, evaluated with some accuracy in the framework of the Schwinger-DeWitt method. The bilinear form of the ghost action also has the minimal form
\[
\hat{H}_{\text{ghost}} = \delta_\mu^\nu \Box - \frac{1}{\phi + \gamma} \phi^\nu \nabla_\mu - \frac{1}{\phi + \gamma} (\nabla_\mu \nabla^\nu \phi) + R_\mu^\nu
\]  
(26)
and its contribution can be also, in principle, evaluated with the use of the standard technique.

Here we perform the derivation of the effective action with an accuracy to the first order in curvature and in the corresponding second order in the derivatives of the scalar field. This approximation corresponds to the first coefficient of the Schwinger-Dewitt expansion, which has, for the operator
\[
\hat{H}_{\text{min}} = \hat{1} \Box + \hat{E}^\lambda \nabla_\lambda + \hat{D}
\]  
(27)
the form (see [16, 21, 28] for full details)
\[
\ln \text{Det}(-\hat{H}_{\text{min}}) = -\int_{\epsilon^2}^\infty \frac{ds}{is} \text{Tr} U(x,x';s)
\]
\[
U(x,x';s) = e^s \hat{H}_{\text{min}} = U_0(x,x';s) \sum_{n=0}^\infty A_n (is)^n
\]  
(28)
The trace of the first \(A_1(x,x)\) coefficient is defined, for the operator \(\hat{H}_{\text{min}}\), as
\[
\text{tr} A_1^{\text{min}} = \int d^4x \sqrt{-g} \text{ tr} \left\{ \hat{D} - \frac{1}{4} \hat{E}^\lambda \hat{E}_\lambda + \frac{1}{6} R \right\}
\]  
(29)
Using (17), (23), (27) and (28), after some algebra, we arrive at the following expression
\[
\Gamma_1^{(1- \text{loop})} = \mu^n \int d^n x \sqrt{-g} \{ A(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + B(\phi) R + C(\phi) \}
\]  
(30)
where \(\mu\) is dimensional parameter related to \(\epsilon\) in (28), and
\[
A(\phi) = \frac{1}{8 (n - 2) (n - 1) (\gamma + \phi)} \left\{ \frac{\gamma^2}{\phi^2} (6n - 4 - 2n^2) + \frac{\gamma}{\phi} (16 - 28n + 4n^2 + 12n^3 - 4n^4) - 26n + 49n^2 - 30n^3 + 7n^4 + \frac{\phi}{\gamma} (16 - 28n + 16n^2 - 2n^3) \right\}
\]
\[
B(\phi) = \frac{1}{12} (3n - 5n^2 - 1) + \frac{n \phi}{2\gamma (n - 1)}
\]
\[
C(\phi) = \frac{\lambda n \phi^{\frac{n-2}{n-3}}}{\gamma (n^2 - 3n + 2)} (n \phi + \phi + \gamma) + \frac{n (-\gamma + \gamma n^2 + \phi) \left( \frac{\lambda \phi^{\frac{n-2}{n-3}} + \gamma^{\frac{n-2}{n-3}} \tau}{2\gamma (1 - n) (\gamma + \phi)} \right)}
\]  
(31)
The above result (31) is the lowest (in background dimension) part of the one-loop correction to the classical action. It is easy to see that (31) doesn’t satisfy the conditions imposed by conformal duality, and therefore the quantum corrections violate the symmetry in this approximation. At the
same time, as it will be shown below, the expression (31) contains a big gauge fixing arbitrariness which can be fixed if only one uses the equations of motion.

4. Gauge fixing independent part of quantum corrections

The analysis of (31) can be performed in a reasonable way only after we explore it’s gauge fixing dependence. Such a dependence for the off-shell effective action takes place for any gauge theory. Fortunately it can be investigated in general form [26, 27] (see also [17]). For the one-loop contribution to the effective action one can easily find this dependence explicitly, following the method of [28] (see also it’s development in [29, 30] for the gauge fixing condition with an additional weight operator (18)). From the general expression (which is derived, for instance, in an Appendix of [30]) it follows that if the weight operator $Y_{\mu\nu}$ and the gauge fixing condition $\chi_{\mu}$ depend on arbitrary parameter $t$, then the derivative of (17) on $t$ is

$$\frac{d}{dt}\Gamma^{(1\text{-loop})}_{1} = -\frac{1}{2} G^{nk} \varepsilon_{p} \frac{\delta\nabla_{\alpha}^{n}}{\delta w^{p}} M^{\alpha\mu} \left[ \frac{\delta\chi_{\mu}}{\delta w^{k}} Y_{\mu\nu} + 2 Y_{\mu\nu} \left( \frac{\delta\chi_{\nu}}{\delta w^{k}} \right) \right]$$

(32)

where the touch denotes the derivative on $t$, $M^{\alpha\mu}$ and $G^{nk}$ are the propagators of the gauge ghosts and fields $w^{p} = (\bar{h}_{\mu\nu}, h, \varphi)$ correspondingly, $\nabla_{\mu}^{p}$ are the generators of the field $w^{p}$ and $\varepsilon_{p} = \delta S / \delta w^{p}$ are classical equations of motion (B.1), (B.2). Taking $t$ to be, in turn, the gauge fixing parameters $\alpha, \beta, \rho$ and integrating over these parameters one arrives at the complete explicit form of the gauge fixing dependence of the one-loop effective action.

Thus at one loop the dependence of $\Gamma^{(1\text{-loop})}_{1}$ on the gauge parameters $\alpha, \beta, \rho$ is proportional to the equations of motion (B.1), (B.2). These equations have the same dimension as the integrand of $\Gamma^{(1\text{-loop})}_{1}$, and therefore the gauge fixing dependence is rather strong in this case. In particular all the functions $A(\phi), B(\phi), C(\phi)$ can be essentially changed for different values of $\alpha, \beta, \rho$, and (as will be shown below) only one combination of these functions is gauge independent. One can supposes that this rate can be changed if we take into account the higher orders in the Schwinger-Dewitt expansion, where the integrands of $\text{Tr} a_{\mu}(x, x)$ have higher dimension.

In order to extract some invariant quantity from the expression (30) one can assume that the background fields $g_{\mu\nu}$ and $\phi$ satisfy classical equations of motion (B.1), (B.2). If fact, due to the dimensional reasons, we need only the trace of the equation for the metric, so instead of (B.1), (B.2) one can use (B.3), (B.4). One can directly substitute (B.3) into (30), however in order to use (B.4) one needs first to perform some transformations. From (B.4) it follows that for any function $F(\phi)$ and for any solution of the equations of motion (B.3), (B.4)

$$\int d^{n}x \sqrt{-g} F(\phi) \left\{ \frac{1}{\phi} (\nabla\phi)^{2} - 2 (\Box\phi) - \phi S(\phi) \right\} = 0$$

(33)

Integrating by parts we arrive at

$$\int d^{n}x \sqrt{-g} \left\{ \frac{1}{\phi} F(\phi) + 2F_{1}(\phi) \right\} (\nabla\phi)^{2} = \int d^{n}x \sqrt{-g} F(\phi) \phi S(\phi)$$

(34)

The lhs of (34) is invariant under the change $F(\phi) \rightarrow F(\phi) + C_{1} \phi^{-1/2}$ with $C_{1} = \text{const}$. In fact this means that for any solution of (B.3), (B.4) the function $C_{1} \phi^{-1/2}$ possesses the property

$$\int d^{n}x \sqrt{-g} C_{1} \phi^{-1/2} \phi S(\phi) = 0$$

(35)

and therefore can be always safely omitted. Now we remind (30) and put

$$\frac{1}{\phi} F(\phi) + 2F_{1}(\phi) = A(\phi)$$

(36)
The solution of the last equation has the form

\[ F(\phi) = C_1 \phi^{-1/2} + \phi^{-1/2} \int_{\phi_0}^{\phi} d\phi \phi^{1/2} A(\phi) \]  

(37)

When (37) is substituted into (34) and then to (30), one has to take into account (35), that makes the values of constants \( C_1, \phi_0 \) irrelevant. Finally we arrive at the following expression for the gauge-fixing independent on-shell effective action:

\[
\Gamma^{(1\text{-loop})}_{1, ms} = \mu^n \int d^n x \sqrt{-g} \left\{ n \frac{\tau}{2} \frac{\gamma^2}{n-2} \left( \frac{5n^2 - 3n + 22}{6} - \frac{n}{(n-1)\gamma} \right) + \right.
\]

\[
+ V_1(\phi) \left( \frac{\phi}{\gamma} - \frac{1}{n-1} \right) - \frac{n V(\phi)}{2(n-1)(\phi + \gamma)} \left( n^2 - 1 + \frac{\phi}{\gamma} \right) + 
\]

\[
+ n S(\phi) \left[ 1 - \frac{2(n^3 - 2n^2 - 4n + 6)}{4(n-2)} \frac{\phi}{\gamma} + \frac{2n^3 - 3n^2 - 2n + 2}{(n-1)} \left( \frac{\phi}{\gamma} - \sqrt{\frac{\phi}{\gamma} \arctan \sqrt{\frac{\phi}{\gamma}}} \right) \right.
\]

\[
\left. + \frac{11n^4 - 40n^3 + 27n^2 + 36n - 36}{4(n-1)(n-2)} \left( \frac{\phi}{\phi + \gamma} + \sqrt{\frac{\phi}{\gamma} \arctan \sqrt{\frac{\phi}{\gamma}}} \right) \right\} \]  

(38)

The arctan’s appear because of integration in (37). One can see that all the divergences depend on the potential through \( V(\phi) / (\phi + \gamma) \), on it’s first derivative \( V_1(\phi) \), on function \( S(\phi) \), which is defined in (B.4) and also on the ratio \( (\phi / \gamma) \). The on-shell expression possesses the homogeneity under the simultaneous rescaling of scalar and constant \( \gamma \)

\[
\phi \rightarrow k \phi, \quad \gamma \rightarrow k \gamma, \quad k = \text{const}
\]  

(39)

just as the corresponding on-shell expression for the classical action

\[
S_{ms} = \frac{1}{\kappa^2} \int d^nx \sqrt{-g} \left\{ \frac{\tau}{n-2} \frac{\gamma^2}{n-2} (n\phi - 2\gamma) - \frac{\lambda(n+2)}{n-2} \phi^{n/2} \right\}
\]  

(40)

One can see that the order of homogeneity for (38) is different from the one for (40) (the exception is only singular case \( n = 2 \), see Appendix C) and therefore the on-shell part of \( \Gamma^{(1\text{-loop})}_{1, ms} \) is not invariant under the transformation of conformal duality (44). In fact the violation of the conformal duality in this approximation is not a wonder because the similar thing happens for conformal scalar field. The \( a_1 \) coefficient can not be conformal invariant because of it’s dimension with the only one exception of \( n = 2 \).

5 Conclusion

We have formulated the conformal transformation and conformal duality in the metric-dilaton models in arbitrary dimension, and derived the lower-order one-loop correction to the effective action for the theory with conformal duality. For odd dimensions the one-loop effective action is free of logarithmic divergences and one can consider the above result as an approximation to the effective action in the weak coupling and weak gravitational field limit. The conformal duality can link the strong and weak gravitational field regimes for the metric-dilaton theories. The first approximation which we have explored here, shows the general structure of quantum corrections, but unfortunately it provides too little information because of the strong dependence on the gauge-fixing arbitrariness. The only one part of the one-loop correction to the classical action, is the expression (38) which results from the substitution of the classical equations of motion into the first coefficient of the Schwinger-DeWitt expansion. In the regime of weak gravitational field the parameters of such an
expansion are i) the dimensional coupling constant $\kappa$ and ii) the curvature tensor of the background metric. If we regard both quantities as small parameters, then the quantum correction is small too. In this case one can perform the finite renormalization of the metric and scalar field in such a way that, in terms of the new fields, all but the last (potential) terms in the action have the ”classical” form. If we accept this procedure then the only quantum contribution will be the correction to the potential $V(\phi)$ in (15). This correction is indeed defined by the expression (38) with additional factor of $\kappa$, therefore this correction is gauge fixing and parametrization independent. Thus derived, the effective potential can be used to extract the information about the strong curvature regime, hence it enables one to explore how quantum corrections change the expansion rate of the early Universe, or how they affect the black hole solutions in the vicinity of singularity. We hope to investigate these problems in a close future.

Indeed it should be interesting also to use a more complicated methods of calculations (see for example [31], [32]) to the theories with conformal duality. These methods can help to extract the nonlocal part of the effective action, which can be free of the gauge arbitrariness, and can be directly applicable to cosmology and black hole physics. On the other hand, one can explore the functional properties of the operators resulting from the conformally dual theory, as it has been done for the conformal symmetry itself (see, for example, [33]).

Acknowledgments

Author is grateful to I.G. Avramidi and M.S. Plyushchay for helpful discussions and also thanks for the warm hospitality the Departamento de Fisica Teorica en Universidad de Zaragoza and the Departamento de Fisica en Universidade Federal de Juiz de Fora, where this work has been started and completed, and also CICYT for financial support under grant AEN94-0218 and MEC-DGICYT for fellowship. The work was supported in part by Russian Foundation for Basic Research under the project No.96-02-16017.

Appendix A. Soft breaking of conformal symmetry in $n$ dimensions.

Following [22] one can consider the possibility of the soft breaking of the conformal symmetry of the action $S_{B(\phi), \lambda}$ in $n$ dimensions. For this purpose we need the conformal Noether identity, which has exactly the same form as in $n = 4$

$$B_1(\phi) \, g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} - B(\phi) \, \frac{\delta S}{\delta \phi} = 0 \quad (A.1)$$

where the factor $-\frac{2B}{\delta B_1}$ stands for the conformal weight of the scalar field $\phi$ which depends on the form of the function $B(\phi)$. It is an analog (one can say generalization) of the conventional conformal weight $\frac{2-n}{2}$ of the scalar field in $\mathbb{R}$. The eq. (A.1) is the operator form of the symmetry transformation (8). According to (A.1) the equations of motion for the scalar field and for the conformal factor of the metric are linearly dependent. For the special case of GR one meets $B = const$ and both therms in (A.1) vanish. Consider the nonconstant $B$.

The soft breaking of the conformal symmetry means that the functions $A$ and $B$ satisfy the symmetry condition (8) whereas the restrictions on the potential term $C$ are not imposed. In such a case, however, an arbitrary function $C(\phi)$ satisfies the differential equation, which results from the equations of motion [22]

$$C_1(\phi) B(\phi) = \frac{n}{n-2} C(\phi) B_1(\phi) \quad (A.2)$$

\footnote{The quantum correction leads to the soft breaking of dual symmetry. The source of this breaking is the the loop expansion of the effective action. Consequently if we take into account quantum correction, the procedure of going to dual regime may lead to the nontrivial change of the potential.}
that has the unique nonzero solution \( C(\phi) = \text{constant} \cdot [B(\phi)]^{\frac{1}{n-2}} \). One can easily check that this statement is correct even if we add the action of matter, if this matter does not depend on the field \( \phi \). Thus, just as in \( n = 4 \) case [22], only the symmetric form of \( C(\phi) \) is consistent with the equations of motion. It is interesting to note that the soft breaking of conformal symmetry is possible (at least in the sense we are considering it here) for the both types of the higher derivative conformal metric-dilaton models [30] and [34, 22].

Appendix B. Equations of motion for the dual theory.

Let us consider the equations of motion for the theory with conformal duality. For the sake of simplicity we start with the special simple case (15) which is related with the general one (11) by i) conformal transformation of the metric ii) reparametrization of the scalar field. The dynamical equations for the theory (15) have the form

\[
\frac{\delta S}{\delta g^{\mu\nu}} = \frac{n-1}{n-2} \phi \left( \frac{1}{2} (\nabla \phi)^2 g_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi \right) + (\nabla_\mu \nabla_\nu \phi) - (\Box \phi) g_{\mu\nu} - \\
- (\phi + \gamma) \left( \frac{1}{2} R g_{\mu\nu} - R_{\mu\nu} \right) + \frac{1}{2} V g_{\mu\nu} = 0 \tag{B.1}
\]

\[
\frac{\delta S}{\delta \phi} = \frac{n-1}{n-2} \left[ \frac{(\nabla \phi)^2}{\phi^2} - \frac{2}{\phi^2} (\Box \phi) \right] + \lambda \frac{n}{n-1} \phi^{\frac{2}{n-2}} + R = 0 \tag{B.2}
\]

Taking trace of (B.1) and comparing it with (B.2) we find the constraint

\[
R = \frac{n \tau}{2 - n} \gamma^{\frac{n}{n-2}} \tag{B.3}
\]

Substituting (B.3) back to (B.2) we obtain

\[
\frac{1}{\phi^2} (\nabla \phi)^2 - \frac{2}{\phi^2} (\Box \phi) = \frac{n}{n-1} \left( \tau \gamma^{\frac{n}{n-2}} - \lambda \phi^{\frac{2}{n-2}} \right) = S(\phi) \tag{B.4}
\]

Since the equation for \( \phi \) is factorized it is not very difficult to solve it for some special cases. For instance, if we are looking for the spatially homogeneous solutions and \( \phi \) depends only on time, (B.4) becomes

\[
(\phi')^2 - 2 \phi \phi'' = \frac{n}{n-1} \phi^2 \left( \tau \gamma^{\frac{2}{n-2}} - \lambda \phi^{\frac{2}{n-2}} \right) \tag{B.5}
\]

that can be easily solved. Along with the obvious constant solution

\[
\phi = \gamma \left( \frac{\tau}{\lambda} \right)^{\frac{n-2}{2}} \tag{B.6}
\]

there is a nonconstant one which can be presented in the integral form

\[
t - t_0 = \pm \int_{\phi_0}^{\phi} \frac{d\phi}{\left( C\phi + \phi^2 \left[ \frac{n \tau}{n-1} \gamma^{\frac{n}{n-2}} - \frac{n-2}{n-1} \lambda \phi^{\frac{2}{n-2}} \right] \right)^{1/2}} \tag{B.7}
\]

where \( C \) is integration constant, which should be, in principle, defined on physical backgrounds. The last integral is elementary for the special case \( n = 3, 4; \ C = 0 \). In particular, for \( n = 3 \) we find

\[
\phi = \gamma \left( \frac{\lambda}{3 \tau} \right)^{1/2} \left[ 1 + \tan^2 \left( A + |\gamma| \sqrt{3 \tau} (t - t_0) \right) \right], \quad A = \text{const} \tag{B.8}
\]
In more general case $C \neq 0$ eq. (B.7) can be evaluated within some approximation scheme. For example, if we are interested in the cosmological solution for which the dilaton becomes constant in the limit $t \to +\infty$ then the fixed points method can be applied, and we need to find the values of dilaton which illuminate the denominator of the integrand in (B.6). It is useful to rewrite the condition

$$C\phi + \phi^2 \left[ \frac{n\tau}{n-1} \gamma^{\frac{2}{n-1}} - \frac{n-2}{n-1} \lambda \phi^{\frac{2}{n-1}} \right] = 0 \quad (B.9)$$

in the algebraic form via the new variable $\psi$, where $\phi = \psi^{n-2}$. The resulting equation

$$\psi^{n-2} \left[ C + \frac{n\tau}{n-1} \gamma^{\frac{2}{n-2}} - \frac{n-2}{n-1} \lambda \phi^{n-2} \right] = 0 \quad (B.10)$$

can have a nonzero solutions. It is remarkable that the fixed point $\psi = \phi = 0$ is stable at $t \to +\infty$ only for the even space-time dimensions $n$.

Of course the simple constraint (B.3) and equation (B.4) hold only for the special model (15) while for the general theory with conformal duality (11) scalar curvature is not constant and the equation for scalar is not so simple. However one can always obtain the corresponding quantities in the general (11) with the use of conformal transformation and reparametrization of the special model (15), that is using a version of solution-generating technique [7].

Appendix C. Special case of two-dimensional theory.

The actions for both conformal metric-dilaton gravity (11) and for the theory with conformal duality (12), (13) become singular at $n \to 2$, and therefore two dimension is special case, where those symmetries can not be realized in their literal form. Of course we know that the conformal symmetry can be realized in $n = 2$, and therefore it is interesting to see how the correspondence can be achieved. This may be useful, for instance, in the framework of $2 + \varepsilon$ approach to quantum gravity (see [35] for discussion and references). First we start from (15) and change the scalar variable $\phi$ to another one $\chi$ as

$$\phi = \frac{n-2}{2(n-1)} \chi^2 \quad (C.1)$$

In terms of new variable the action becomes

$$S = \int d^nx \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{n-2}{4(n-1)} \chi^2 R + \gamma R + \lambda \left( \frac{n-2}{n-1} \right)^{\frac{n}{2n-2}} \chi^{\frac{2n}{n-2}} + \tau \gamma^{\frac{2n}{n-2}} \right\} \quad (C.2)$$

The conformal model can be extracted from (C.2) (just as from (15)) when one puts $\tau = \gamma = 0$.

In the limit $n \to 2$ we meet the following behaviour of the model. The only remaining dynamical term is $g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi$, the $\gamma R$ term becomes topological and $\chi^2 R$ one disappears. One can easily check that the potential term tends to zero, while the cosmological constant $\tau \gamma^{\frac{2n}{n-2}}$ becomes infinite. No any dual symmetry is observed in this limit, while the conformal symmetry is restored at $\tau = \gamma = 0$.

One can perform another singular (at $n = 2$) change of variables in the action (15). Let's put

$$\phi^{\frac{n-2}{n-1}} = \psi \quad (C.3)$$

Then the action (15) becomes

$$S = \int d^nx \sqrt{-g} \left\{ \frac{(n-1)(n-2)}{n^2} \psi^{\frac{n+4}{n}} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \psi^{\frac{n-2}{n}} R + \gamma R + \lambda \psi + \tau \gamma^{\frac{2n}{n-2}} \right\} \quad (C.4)$$

In these variables, just as in (C.2), the limit $n \to 2$ is singular in the cosmological term only. However, contrary to (C.2), the dynamical scalar term disappears. Therefore the dynamics of the
theory (15) at $n \to 2$, which we have met in (C.2), is caused by the singular change of variables (C.1), and such a dynamics doesn’t take place for another choice of the variables. Thus we see that one can perform such a limit in a different ways and arrive at the essentially different $2d$ metric-dilaton models. The reason for this is that the original model (15) is singular at $n \to 2$.

**Appendix D. The elements of the background field method.**

Here we collect some intermediate formulas related with the bilinear expansion of the action (13) in the background field method. One can find an expansion of scalar curvature and $\sqrt{-g}$, corresponding to (16), in many papers or in [17], so we do not write them here. For the kinetic term we use

$$\frac{1}{\varphi'} = \frac{1}{\phi (1 + \varphi/\phi)} = \frac{1}{\phi} - \frac{\varphi}{\phi^2} + \frac{\varphi^2}{\phi^3} + ... \quad (D.1)$$

and after some integrations by parts and a little tedious algebra arrive at

$$S^{(2)} = S^{(2)}_{(\varphi,\varphi)} + S^{(2)}_{(\varphi,\bar{h}_{\alpha\beta})} + S^{(2)}_{(h,\varphi)} + S^{(2)}_{(h,h)} \quad (D.2)$$

where

$$S^{(2)}_{(\varphi,\varphi)} = \int d^n x \sqrt{-g} \varphi \left[ -\frac{a}{\phi} \Box + \frac{a}{\phi^2} (\nabla^\mu \phi) \nabla_\mu \frac{a}{\phi} (\nabla^\rho \phi)^2 + \frac{a}{\phi^2} (\Box \phi) + \frac{1}{2} V_2 \right] \varphi$$

$$S^{(2)}_{(\varphi,\bar{h}_{\alpha\beta})} = \int d^n x \sqrt{-g} \varphi \left[ \frac{1}{2} \nabla_\alpha \nabla_\beta + \frac{a}{\phi} \phi_\alpha \nabla_\beta - \frac{a}{2 \phi^2} \phi_\alpha \phi_\beta + \frac{a}{\phi} (\nabla_\alpha \nabla_\beta \phi) - \frac{1}{2} R_{\alpha\beta} \right] \bar{h}^{\alpha\beta}$$

$$S^{(2)}_{(h,\varphi)} = \int d^n x \sqrt{-g} \bar{h}^{\mu\nu} \left[ \frac{1}{2} \nabla_\mu \nabla_\nu - \frac{a}{\phi} \phi_\mu \nabla_\nu + \frac{a}{2 \phi^2} \phi_\mu \phi_\nu - \frac{1}{2} R_{\mu\nu} \right] \varphi$$

$$S^{(2)}_{(h,h)} = \int d^n x \sqrt{-g} \left[ \frac{1}{2} \Box + \frac{n-1}{2 n} \varphi \phi_\lambda \nabla_\lambda + \frac{1}{2} (\nabla^\rho \phi)^2 - (\Box \phi) + \frac{\phi}{2 a} R \right] + \frac{1}{4} V_1 \right] h$$

$$S^{(2)}_{(\bar{h}_{\alpha\beta},\bar{h}_{\alpha\beta})} = \int d^n x \sqrt{-g} \bar{h}^{\mu\nu} \left[ \frac{1}{4} \delta_{\mu\nu,\alpha\beta} \left( (\phi + \gamma) \Box + \phi^\lambda \nabla_\lambda + 2 (\Box \phi) - A - V \right) + \frac{1}{2} g_{\alpha\beta} (\nabla_\alpha \nabla_\mu - \phi_\alpha \nabla_\mu - 2 (\nabla_\alpha \nabla_\mu \phi) + A_{\alpha\mu} \right] \bar{h}^{\alpha\beta}$$

$$S^{(2)}_{(\bar{h}_{\mu\nu},\bar{h}_{\mu\nu})} = \int d^n x \sqrt{-g} \bar{h}^{\mu\nu} \left[ \frac{1}{4} \delta_{\mu\nu,\alpha\beta} \left( (\phi + \gamma) \Box + \phi^\lambda \nabla_\lambda + 2 (\Box \phi) - A - V \right) + \frac{1}{2} g_{\alpha\beta} (\nabla_\alpha \nabla_\mu - \phi_\alpha \nabla_\mu - 2 (\nabla_\alpha \nabla_\mu \phi) + A_{\alpha\mu} \right] \bar{h}^{\alpha\beta}$$

$$S^{(2)}_{(\bar{h}_{\nu\mu},\bar{h}_{\nu\mu})} = \int d^n x \sqrt{-g} \bar{h}^{\mu\nu} \left[ \frac{1}{4} \delta_{\mu\nu,\alpha\beta} \left( (\phi + \gamma) \Box + \phi^\lambda \nabla_\lambda + 2 (\Box \phi) - A - V \right) + \frac{1}{2} g_{\alpha\beta} (\nabla_\alpha \nabla_\mu - \phi_\alpha \nabla_\mu - 2 (\nabla_\alpha \nabla_\mu \phi) + A_{\alpha\mu} \right] \bar{h}^{\alpha\beta}$$

where $a = \frac{\phi}{n}$. Supplemented by the gauge fixing term (18) – (20) the expressions (D.2), (D.3) lead to the minimal matrix form of $S^{(2)} + S_{gf}$ in eq. (21).
References

[1] G. t’Hooft and M. Veltman, Ann.Inst.H.Poincare. A20, 69 (1974).
[2] S.L. Adler, Rev. Mod. Phys. 54, 729 (1982).
[3] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory (Cambridge University Press, Cambridge, 1987).
[4] J.F. Donoghue, Phys.Rev.Lett.72 (1994) 2996; Phys.Rev.D50 3874 (1994).
[5] N.C. Tsamis and R.P. Woodard, Ann.Phys., 238, 1, (1995).
[6] S. Deser, Ann. Phys. 59, 248, (1970); Phys.Lett. 134B, 419 (1984).
[7] J.D. Bekenstein, Ann.Phys. 82,535,(1974).
[8] J.D. Barrow, S. Cotsakis, Phys.Lett. 214B,515, (1988).
[9] J.L. Cardy, Phys.Lett. B215, 749 (1988);
   I. Jack and H. Osborn, Nucl.Phys. B343, 647 (1990);
   G.M. Shore, Phys.Lett. B253, 380 (1991);
   A. Capelli, D. Friedan and J.I. Latorre, Nucl.Phys. B352,616(1991).
[10] S. Deser, M.J. Duff and C Isham, Nucl.Phys.111B, 45, (1976);
    M.J. Duff, Nucl.Phys. 125B 334 (1977);
    M.J. Duff, Class.Quant.Grav. 11, 1387 (1994).
[11] R.Y. Reigert, Phys.Lett. 134B, 56, (1984); E.S. Fradkin and A.A. Tseytlin, Phys.Lett. 134B, 187 (1984).
[12] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, Phys.Lett. B162, 92 (1985).
[13] I. Antoniadis and E. Mottola, Phys. Rev. 45D, 2013 (1992).
[14] I.L. Shapiro and G. Cognola, Phys.Rev. 51D 2775 (1995);
    I.L. Shapiro, Mod. Phys. Lett. 9A, 1985 (1994).
[15] J. Erdmenger and H. Osborn, Preprint DAMTP/96-7, hep-th/9605009 (see also other refer-
    ences from this paper).
[16] B.S. DeWitt, Dynamical Theory of Groups and Fields. (Gordon and Breach, NY, 1965).
[17] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, Effective Action in Quantum Gravity (IOP, Bristol, 1992).
[18] J. Ambjorn and J. Jurkevich, Phys.Lett. B278, 42, (1992); M.E. Aginstein and A.A. Migdal,
    Mod.Phys.Lett. A7, 1039, (1992).
[19] S.Deser and P. van Nieuwenhuisen, Phys. Rev. 10D, 401 (1974); 10D, 411 (1974).
[20] S. Weinberg, in: General Relativity, ed. S.W.Howking and W. Israel, (Cambridge University Press, 1979).
[21] B.S. De Witt, *in: General Relativity, ed. S.W.Hawking and W. Israel*, (Cambridge University Press, 1979).

[22] I.L. Shapiro and H. Takata, Phys. Lett. **B361**, 31 (1995).

[23] J. Polchinski, Preprint NSF-ITP-96-60, [hep-th/9607050](http://arxiv.org/abs/hep-th/9607050).

[24] S.D. Odintsov and I.L. Shapiro, Class. Quant. Grav. **8**, L57 (1991).

[25] I.L. Shapiro and H. Takata, Phys.Rev. **52D**, 2162 (1995).

[26] De Witt B.S., *Phys.Rev. 162D* 1195 (1967).

[27] Voronov B.L., Lavrov P.M., Tyutin I.V., *Sov.J.Nucl.Phys.* **36** 498 (1982).

[28] A.O. Barvinsky and G.A. Vilkovisky, *Phys. Repts.* **119**, 1 (1985).

[29] I.G. Avramidi, *Kand. (PhD) Dissertation*, (1986) – [hep-th/9510144](http://arxiv.org/abs/hep-th/9510144).

[30] I.L. Shapiro and A.G. Jacksenaev, Phys.Lett. **324B**, 284 (1994).

[31] I.G. Avramidi, J. Math. Phys. **37**, 374 (1996); J. Math. Phys. **36**, 5055 (1995).

[32] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*. (Springer, 1995);

E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications*. (World Sci., Singapore, 1994);

A.A. Bytsenko, G.Cognola, L. Vanzo and S. Zerbini, Phys.Repts. **266**, 1 (1996).

[33] T. Branson, P. Gilkey and B. Orsted, Comm. Part. Dif. Eq. 15 (1990) 245

[34] I. Antoniadis, J. Iliopoulos and T.N. Tomaras, *Nucl.Phys.* **B261**157 (1985);

[35] T. Aida, Y. Kitazawa, H. Kawai, M. Ninomiya, Nucl. Phys. **427B**, 159 (1994).