Semi-BCI Algebras

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Abstract

The notion of semi-BCI algebras is introduced and some of its properties are investigated. This algebra is another generalization for BCI-algebras. It arises from the “intervalization” of BCI algebras. Semi-BCI have a similar structure to Pseudo-BCI algebras however they are not the same. In this paper we also provide an investigation on the similarity between these classes of algebras by showing how they relate to the process of intervalization.

Keywords: Interval-valued Fuzzy Logic, BCI-algebras, Semi-BCI algebras, Pseudo-BCI algebra

1. Introduction

One of the most well known references on the algebraic approach to logics is the book of Rasiowa\textsuperscript{1} which dates to 70s. In this book, at pages 16-17, the notion of implicative algebra, which aim at modelling a simple notion of implication is provided:

An implicative algebra is an algebra \((A, \Rightarrow, \top)\) of type \((2, 0)\) which satisfies the following properties:

\((i-1)\quad a \Rightarrow a = \top,\)
(i-2) If $a \Rightarrow b = \top$ and $b \Rightarrow c = \top$, then $a \Rightarrow c = \top$.

(i-3) If $a \Rightarrow b = \top$ and $b \Rightarrow a = \top$, then $a = b$.

A direct consequence of such definition is the establishment of an order relation
“≤” on $A$, which is known as **Order Property (OP)** of implications:

\[ a \leq b \text{ if and only if } a \Rightarrow b = \top. \]  

(1)

As a consequence, many implications are axiomatized preserving (OP). However, some interesting implications in the field of fuzzy logics do not satisfy such require-
ment, for example (see \[2,3,4\]):

Consider the algebra $[[0,1], \rightarrow_{YG}, 1]$, such that:

\[ x \rightarrow_{YG} y = \begin{cases} 1, & \text{if } x = y = 0 \\ y^\circ, & \text{otherwise} \end{cases}. \]

In this case $0.3 \leq 0.5$, but $0.3 \rightarrow_{YG} 0.5 \approx 0.81225$. However, “$x \rightarrow_{YG} y = 1$ implies $x \leq y$”.

The interval counterpart of Łukasiewicz implication introduced by Bedregal and
Santiago [5] also fails to satisfy (OP). The authors, however, revealed that the resulting
implication satisfy:

1. if $X \ll Y$ \[1\] then $X \rightarrow Y = 1$;
2. if $X \rightarrow Y = 1$, then $X \leq KM Y$.

The relation “\ll” is precisely the way-below relation \[6\] of the usual Kulisch-
Miranker order on intervals “\leq KM”. Way-below relations, “\ll”, are auxiliary relations
\[6\] of partial orders “\leq”; they have the following properties:

1. if $x \ll y$, then $x \leq y$.
2. if $u \leq x \ll y \leq z$, then $u \ll z$.

\[ ^{1}X \ll Y \text{ iff } \overline{X} \ll \overline{Y}.\]
3. if a smallest element 0 exists, then $0 \preceq x$.

Since (OP) connects an implication to the underlying order relation and this is connected to auxiliary relations, this paper proposes to internalize such connection through two implications; one connected to the usual partial order and the other connected to its way-below relation. The resulting algebraic structure is called semi-BCI algebra which abstracts both BCI-algebras and their intervalization.

Another generalization for BCI-algebras which contains two implications is called Pseudo-BCI algebra which was proposed by W. A. Dudek and Y. B. Jun [7]. The connection of such algebras to Semi-BCIs is investigated here.

Since SBCIs encompass both BCIs and its interval counterpart they tend to model logics in which the notion of impreciseness is required.

The paper is organized in the following way: Section 2 provides a brief review of BCI-algebras and their properties. Section 3 provides an overview of the intervalization process. Section 4 shows the intervalization of BCI-algebras and some properties of this interval algebra. Section 5 introduces the notion of semi-BCI algebra and prove some of its properties. Section 6 discusses the relation between Semi-BCI algebras and Pseudo-BCI algebras. Finally, section 7 provides some concluding remarks.

2. BCI-Algebras

BCI-algebras are mathematical structures for modelling fuzzy logics. They were introduced by Iski [8] in the 60’s and since then have been extensively investigated. There are several axiom systems for BCI-algebras. We will present here the axiom systems defined by [9], in which he assures that the BCIs are algebras of the form $(A, \ast, \bot)$ which satisfy the following properties:

BCI-1 $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = \bot$,  

BCI-2 $(x \ast (x \ast y)) \ast y = \bot$,  

BCI-3 $x \ast x = \bot$,  

BCI-4 $x \ast y = \bot$ and $y \ast x = \bot \implies x = y$.  

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A BCI-algebra is called BCK-algebra if it also satisfies:

\[ \text{BCK-1 } \bot \ast x = \bot \]

On any BCI-algebra it is possible to define a partial order “\( \preceq \)” as: “\( x \preceq y \) iff \( x \ast y = \bot \)”. Therefore, a BCI-algebra is BCK if and only if \( \bot \) is its least element.

**Example 2.1.** The following algebras are BCI.

1. \( \langle [0, +\infty), \ast, 0 \rangle \), s.t. \( x \ast y = \max\{0, x - y\} \).

2. \( \langle \mathcal{P}(X), \ominus, \varnothing \rangle \), where \( A \ominus B \) is the set difference between \( A \) and \( B \).

BCI-logics interpret the Curry combinators: (B) \( \lambda xyz.x(yz) \), (C) \( \lambda xyz.xzy \) and (I) \( \lambda x.x \) — see [10]. This set of combinators are functional counterparts for some Fuzzy Implications. They can also be interpreted by algebras: \( C = \langle A, \rightarrow, \top \rangle \) which satisfy:

\[ \begin{align*}
(C-1) \quad & (y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = \top, \\
(C-2) \quad & x \rightarrow ((x \rightarrow y) \rightarrow y) = \top, \\
(C-3) \quad & x \rightarrow x = \top, \\
(C-4) \quad & x \rightarrow y = \top \text{ and } y \rightarrow x = \top \text{ imply } x = y.
\end{align*} \]

On any such structure it is possible to define a partial order “\( \preceq \)” as:

\[ \begin{align*}
(C-5) \quad & x \preceq y \text{ iff } x \rightarrow y = \top.
\end{align*} \]

**Example 2.2.** The following algebra satisfies properties \([C-1],[C-5]\):

1. \( \langle [0, 1], \rightarrow, 1 \rangle \), s.t. \( x \rightarrow y = \min(1, 1 - x + y) \).

There is a way to obtain the above axioms from those of BCI-algebras and vice-versa, the correspondence can be obtained in the following way:

**Proposition 2.1.** Let \( \langle A, \ast, \bot \rangle \) be a BCI-algebra. The algebra \( C = \langle A, \rightarrow, \top \rangle \), where:

\[ x \rightarrow y \overset{\text{def}}{=} y \ast x \text{ and } \top \overset{\text{def}}{=} \bot \text{ satisfies the axioms } \[C-1],[C-4] \]
Proof: \((y \to z) \to ((z \to x) \to (y \to x)) \overset{\text{def}}{=} ((z \to x) \to (y \to x)) \ast (y \to z) \overset{\text{def}}{=} \ldots \overset{\text{def}}{=} ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = \bot \text{ (BCI-1)}.\) But, \(\top \overset{\text{def}}{=} \bot.\) The other axioms are similarly proved. \(\Box\)

 Proposition 2.2. Let \(C = \langle A, \to, \top \rangle\) be an algebra satisfying properties \((C-1)(C-4)\). The algebra \((A, \ast, \bot)\), where: \(x \ast y \overset{\text{def}}{=} y \to x\) and \(\bot \overset{\text{def}}{=} \top\) is a BCI-algebra.

 Proof: Analogous to Proposition 2.1 \(\Box\)

Corollary 2.1. The relation “\(\preceq\)” is the dual partial order of “\(\succeq\)”; namely \(x \preceq y\) if and only if \(y \succeq x\).

Terminology. Since one kind of algebra can be obtained from the other and any result obtained for one can be easily translated, by duality, to the other, both structures are called BCI-algebras. This work consider the second kind of structure for our generalization. In this context, whenever \(x \to \top = \top\), i.e. \(x \preceq \top\), the BCI-algebra \(C = \langle A, \to, \top \rangle\) will be called BCK-algebra. Now, let be some properties of BCI-algebras:

Some Properties of BCI-Algebra. (for more details see [9])

\[\begin{align*}
(A-1) & \quad \top \preceq x \text{ implies } x = \top, \\
(A-2) & \quad x \preceq y \text{ implies } y \to z \preceq x \to z, \quad \text{— (First place antitonicity)} \\
(A-3) & \quad x \preceq y \text{ implies } z \to x \preceq z \to y, \quad \text{— (Second place isotonicity)} \\
(A-4) & \quad x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z, \\
(A-5) & \quad x \to (y \to z) = y \to (x \to z), \quad \text{— (Exchange)} \\
(A-6) & \quad x \preceq y \to z \text{ implies } y \preceq x \to z, \\
(A-7) & \quad x \to y \preceq (z \to x) \to (z \to y), \\
(A-8) & \quad \top \to x = x, \quad \text{— (Left Neutrality)} \\
(A-9) & \quad ((y \to x) \to x) \to x = y \to x,
\end{align*}\]
(A-10) \( x \rightarrow y \preceq (y \rightarrow x) \rightarrow \top \),

(A-11) \( (x \rightarrow y) \rightarrow \top = (x \rightarrow \top) \rightarrow (y \rightarrow \top) \).

Properties (A-7), (A-5) and (C-3) model the combinators B, C and I of Combinatorial Logic [10].

**Proposition 2.3.** Let \( \langle A, \rightarrow, \top \rangle \) be a BCI-algebra. \( \langle A, \rightarrow, \top \rangle \) is a BCK-algebra if and only if for each \( x \in A \) there exists \( y \in A \) such that \( y \preceq x \) and \( y \preceq \top \).

**Proof:** (\( \Rightarrow \)) Straightforward because in BCK-algebras \( \top \) is the greatest element, i.e. \( x \preceq \top \) for each \( x \in A \).

(\( \Leftarrow \)) Suppose that \( \langle A, \rightarrow, \top \rangle \) is not a BCK-algebra. Then, there exists \( a \in A \) such that \( a \not \preceq \top \). By hypothesis there exists \( b \in A \) such that \( b \preceq a \) and \( b \preceq \top \). So, by (A-8) and definition of \( \preceq \), \( (b \rightarrow \top) \rightarrow ((\top \rightarrow a) \rightarrow (b \rightarrow a)) = \top \rightarrow (a \rightarrow \top) = a \rightarrow \top \neq \top \). Therefore, (C-1) fails.

\( \square \)

As stated in the Introduction, this paper shows that the behavior of the process of intervalization does not preserve (OP). In order to precisely define what does it mean, the next section introduces the concept of intervalization over abstract partial orders.

### 3. Intervalization of Structures

The limited capacity of machines to store just a finite set of finitely represented objects constraints the automatic calculation (computation) of structures in which a machine representation of some objects exceeds such capacity. In the case of real numbers, although most programs provide highly accurate results, it can happen that rounding errors built up during each step in the computation produce results which are not even meaningful. For more details see the early Forsythes report [11]. In 1988, Siegfried Rump [12] published the result of a computed function in an IBM S/370 mainframe. The function was:

\[
y = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b}. \tag{2}
\]

He calculated for \( a = 77617.0 \) and \( b = 33096.0 \), and the result was:
1. single precision: \( y = 1.172603 \ldots \);
2. double precision: \( y = 1.1726039400531 \ldots \);
3. extended precision: \( y = 1.172603940053178 \ldots \).

All results lead any user to conclude that IBM S/370 returned the correct result. However this result is WRONG and the correct result lies in the interval: 
\[-0.82739605994682135 \pm 5 \times 10^{-17}. \]
Note that even the sign is wrong!

One of the proposals to overcome this problem is due, almost simultaneously, to Ramon Moore [13, 14] and Teruo Sunaga [15]. They developed the so-called interval arithmetic. Interval arithmetic is a set of operations on the set of all closed intervals \( I(\mathbb{R}) = \{ [a, b] : a, b \in \mathbb{R} \text{ and } a \leq b \} \). The operations are defined in the following way:

1. \([a, b] + [c, d] = [a + c, b + d]\),
2. \([a, b] \cdot [c, d] = [\min P, \max P] \) — where \( P = \{ a \cdot c, a \cdot d, b \cdot c, b \cdot d \} \),
3. \([a, b] - [c, d] = [a - d, b - c]\),
4. \([a, b] / [c, d] = [a, b] \cdot ([1/b, 1/a]) \) ; provided that \( 0 \notin [c, d] \).

Observe that for each operation \( \odot \in \{ +, -, \cdot, / \} \), \([a, b] \odot [c, d] = \{ x \odot y \in \mathbb{R} : x \in [a, b] \wedge y \in [c, d] \} \). This reveals two important properties of this arithmetic (a) Correctness and (b) Optimality.

"Correctness. The criterion for correctness of a definition of interval arithmetic is that the Fundamental Theorem of Interval Arithmetic holds \(^{[10]}\) when an expression is evaluated using intervals, it yields an interval containing all results of pointwise evaluations based on point values that are elements of the argument intervals.

[...]

\(^{[10]}\) Moore [19] Theorem 3.1, p. 21: If \( F \) is an inclusion monotonic interval extension of \( f \), then \( \overrightarrow{f} (X_1, \ldots, X_n) \subseteq F(X_1, \ldots, X_n) \); where \( \overrightarrow{f} (X_1, \ldots, X_n) = \{ f(x_1, \ldots, x_n) : x_i \in X_i \} \).
Optimality. By optimality, we mean that the computed floating-point interval is not wider than necessary."

Hickey et al. [16, p.1040]

The philosophy behind intervals is the following: Enclosure in intervals the values which are not exact by any reason (e.g. the value comes from an imprecise measurement) and apply correct and optimal operations on such intervals in order to obtain the best interval which contains the desired output. This approach will avoid what happened with the Rump’s example. Therefore, the notion of correctness is indispensable for such philosophy.

The property of correctness was investigated in 2006 by Santiago et al [17, 18]. In those papers, instead of correctness the authors used the term representation, since an interval computation could be understood not just as a machine representation of real numbers, but also as a mathematical representation of real numbers (this idea is confirmed by the Representation Theorems of Euclidean continuous functions in [17, 18]). In what follows this notion is shown for binary operations: A binary interval operation, ◦, represents a binary real operation, ◦, whenever:

\[(x, y) \in [a, b] \times [c, d] \Rightarrow x \circ y \in [a, b] \circ [c, d]\] (3)

This can be easily extended to n-ary operations. The authors showed that this notion is more general than what is stated by the Fundamental Theorem of Interval Arithmetic; given that there are representations which are not inclusion monotonic (see [18, p. 238]).

One noteworthy point which will be taken into account in the present paper: There is a difference between the representation of a function \(f\) as an interval function \(F\) and an extension of a function \(f\) to an interval function \(G\). For example, given intervals \(X = [X_-, X_+]\) and \(Y = [Y_-, Y_+]\), the function \(X - Y = [\min(X - Y_-, X - Y_+), \max(X - Y_-, X - Y_+)]\), presented in [20], extends the subtraction on real numbers, however \([2, 3] - [2, 3] = [0, 0], 2.5, 2.1 \in [2, 3], \) but \(2.5 - 2.1 \not\in [2, 3] - [2, 3]\); in other words, this operation is not correct. So, there are interval extensions which are not correct. They are useless for the proposed philosophy.
The process of giving the correct and optimal interval version $F$ for a function $f$ is called: “intervalization”. There are many proposal of intervalization of algebraic structures further than that of real numbers proposed by Moore and Sunaga. In the literature, the reader can find proposals even for the field of Logic, since there are structures which interpret logics that are susceptible to the same situation of $\mathbb{I}(\mathbb{R})$.

For example: The Łukasiewicz implicative algebra $([0,1], \rightarrow_{LK}, 0, 1)$ s.t. $x \rightarrow_{LK} y = \min(1, 1 - x + y)$ interprets some many-valued logics and was “intervalized” by Bedregal and Santiago in [5]. Its MV-algebra counterpart was intervalized by Cabrer et al in [21], also, in order to overcome the same problems already stated for $\mathbb{I}(\mathbb{R})$. In both cases, the interval algebras did not satisfy the same properties that are satisfied by the algebras that they came from. The same happened with $\mathbb{I}(\mathbb{R})$!

The following section a way of “intervalizing” BCI-algebras is provided. Like the case of MVs and Łukasiewicz algebras the resulting structure does not belong to the same category of its starting algebra. This paper we provide an investigation of the resulting structures. In order to achieve that, some required concepts, like the abstract notion of intervals are introduced. The aim, again, is to provide the ability to use intervals to represent the elements of an algebra $(A, \rightarrow)$.

**Definition 3.1 (Abstract Intervals).** Given a poset $(A, \leq)$, the set $[a, b] = \{x \in A : a \leq x \leq b\}$ is called the closed interal with endpoints $a$ and $b$ and $\mathbb{A} = \{(a, b) \in A \times A : a \leq b\}$ is the set of all intervals of elements of $A$. For any, $X \in \mathbb{A}$ its left and right endpoints by $\underline{X}$ and $\overline{X}$, respectively, i.e. if $X = [a, b]$ then $\underline{X} = a$ and $\overline{X} = b$. When $\underline{X} = \overline{X}$ the interval is called degenerate. The embedding $i : A \to \mathbb{A}$, s.t. $i(a) = [a, a]$ is called natural embedding. On the set $\mathbb{A}$ it is canonical to define the partial order: $X \leq_{km} Y$ if and only if $\underline{X} \leq Y$ and $\overline{X} \leq \overline{Y}$. This relation is called pointwise or Kulisch-Miranker order.

Since BCI-algebras are partially ordered systems, $(B, \leq)$, it is possible to apply Definition 3.1 to obtain the partial order $(\bar{B}, \leq_{km})$. The question is about the implications on $\bar{B}$: If an interval operation on $\bar{B}$ satisfies the BCI axioms, is it also correct? The following section will show that the answer is negative. But what does it mean? It means it is not possible to have both: (1) correctness and (2) the known theory of BCI-
algebras for $(\mathbb{B}, \leq_{km})$. So, since correctness is indispensable, a price must be paid: A
new theory for $(\mathbb{B}, \leq_{km})$ must be developed. This is the reason of this paper!

4. Intervalization of BCI-algebras

This section shows that it is not possible to have an interval BCI-algebra with a
correct implication. Proposition 4.1 shows that it is possible to build an interval BCI-
algebra, but with a non-correct implication, and Theorem 4.1 shows that it is an im-
possible task. Finally, we provide the “BCI-algebra intervalization theorem” and some
properties of resulting algebra.

Lemma 4.1. Let $(A, \to, \tau)$ be a BCI-algebra such that $(A, \preceq)$ is a meet-semilattice
$(A, \wedge)$. For each $a, b, c \in A$, $a \to (b \wedge c) = \tau$ iff $a \to b = \tau$ and $a \to c = \tau$. Moreover, if
$a \to c = \tau$ and $b \to c = \tau$ then $(a \wedge b) \to c = \tau$.

Proof: Straightforward.

Lemma 4.2. Let $(A, \to, \tau)$ be a BCI-algebra such that $(A, \preceq)$ is a meet-semilattice
satisfying:

\[ a \preceq b \to c \text{ iff } a \wedge b \preceq c, \quad (4) \]

for every $a, b, c \in A$. For each $a, b, c, d, e \in A$, if $a \to ((b \to c) \wedge (d \to e)) = \tau$ then
$a \to ((b \wedge d) \to (c \wedge e)) = \tau$.

Proof: If $a \to ((b \to c) \wedge (d \to e)) = \tau$ then, by Lemma 4.1 and (C-5) $a \preceq b \to c$ and
$a \preceq d \to e$. By (4), $a \wedge b \preceq c$ and $a \wedge d \preceq e$ and therefore, $b \preceq a \to c$ and $d \preceq a \to e$.
So, $b \wedge d \preceq a \to c$ and $b \wedge d \preceq a \to e$. Thus, applying again (4), $(b \wedge d) \wedge a \preceq c$ and
$(b \wedge d) \wedge a \preceq e$. Hence, $(b \wedge d) \wedge a \preceq c \wedge e$. So, by (4) and (C-5) $a \to ((b \wedge d) \to (c \wedge e)) = \tau$.

Proposition 4.1. Let $(A, \to, \tau)$ be a BCI-Algebra such that $(A, \preceq)$ is a meet-semilattice
satisfying (4). Then $(A, \implies, [\tau, \tau])$, where

\[ X \implies Y = [(X \to Y) \wedge (\overline{X} \to \overline{Y}), \overline{X} \to \overline{Y}], \quad (5) \]

is also a BCI-algebra which satisfies (4).
Proof: Notice that in this case, defining \( X \leq Y \ iff \ X \Rightarrow Y = [T, T] \), then \( X \leq Y \ iff \ X \rightarrow Y = \overline{X} \rightarrow \overline{Y} = T \ iff \ \overline{X} \leq \overline{Y} \) and \( \overline{X} \leq \overline{Y} \).

Thus, clearly, the properties \([C-3]\) to \([C-5]\) are trivially satisfied. In the following, \([C-1]\) is proved.

Since, \( \langle A, \rightarrow, \top \rangle \) is a BCI-Algebra, by \([C-1]\) each \( X, Y, Z \in A \), \((Y \rightarrow Z) \rightarrow ((Z \rightarrow X) \rightarrow (Y \rightarrow X)) = T \) and \((\overline{Y} \rightarrow \overline{Z}) \rightarrow ((\overline{Z} \rightarrow \overline{X}) \rightarrow (\overline{Y} \rightarrow \overline{X})) = T \). Then, \(((Y \rightarrow Z) \land (\overline{Y} \rightarrow \overline{Z})) \rightarrow (((Z \rightarrow X) \rightarrow (Y \rightarrow X)) \land ((\overline{Z} \rightarrow \overline{X}) \rightarrow (\overline{Y} \rightarrow \overline{X}))) = T \) and \(((Y \rightarrow Z) \land (\overline{Y} \rightarrow \overline{Z})) \rightarrow (((Z \rightarrow X) \rightarrow (Y \rightarrow X)) \land ((\overline{Z} \rightarrow \overline{X}) \rightarrow (\overline{Y} \rightarrow \overline{X}))) = T \). So, by Lemma 4.1, \(((Y \rightarrow Z) \land (\overline{Y} \rightarrow \overline{Z})) \rightarrow (((Z \rightarrow X) \rightarrow (Y \rightarrow X)) \land ((\overline{Z} \rightarrow \overline{X}) \rightarrow (\overline{Y} \rightarrow \overline{X}))) = T \). Thus, by Lemma 4.2 and Eq. (5), \((Y \Rightarrow Z) \rightarrow ((Z \Rightarrow X) \land (Z \Rightarrow \overline{X})) \rightarrow ((Y \Rightarrow Z) \land (\overline{Y} \Rightarrow \overline{X})) = T \). Therefore by Eq. (4), \((*)\) \( Y \Rightarrow Z \rightarrow (Z \Rightarrow X \rightarrow Y \Rightarrow X) = T \). On the other hand, by \((C-1)\), \((\overline{Y} \rightarrow \overline{Z}) \rightarrow ((\overline{Z} \rightarrow \overline{X}) \rightarrow (\overline{Y} \rightarrow \overline{X})) = T \) and so, by Eq. (5), \((**\) \( Y \Rightarrow Z \rightarrow (Z \Rightarrow X \rightarrow Y \Rightarrow X) = T \). Thus, from \((*)\) and \((**\) and Lemma 4.1, \((Y \Rightarrow Z \land \overline{Y} \Rightarrow \overline{Z}) \rightarrow ((Z \Rightarrow X \rightarrow Y \Rightarrow X) \land (Z \Rightarrow \overline{X} \rightarrow \overline{Y} \Rightarrow \overline{X})) = T \). Since, by Eq. (5), \((Y \Rightarrow Z \land \overline{Y} \Rightarrow \overline{Z}) = Y \Rightarrow Z \), then \((***)\) \( Y \Rightarrow Z \rightarrow ((Z \Rightarrow X \rightarrow Y \Rightarrow X) \land (Z \Rightarrow \overline{X} \rightarrow \overline{Y} \Rightarrow \overline{X})) = T \). Thus, from \((***)\) and \((**\), \((Y \Rightarrow Z \rightarrow (Z \Rightarrow X) \land (Y \Rightarrow X)) \land (Y \Rightarrow Z \rightarrow (Z \Rightarrow X) \Rightarrow (Y \Rightarrow X)) = T \) and \((Y \Rightarrow Z \rightarrow (Z \Rightarrow X) \Rightarrow (Y \Rightarrow X)) = T \). Therefore, by Eq. (5), \((Y \Rightarrow Z) \rightarrow ((Z \Rightarrow \overline{X}) \Rightarrow (Y \Rightarrow X)) = T \).

Clearly, \( X \rightarrow Y \geq ((X \rightarrow \overline{Y}) \land (\overline{X} \rightarrow \overline{Y})) \) and therefore, by \((A-2)\), \(((X \rightarrow \overline{Y}) \land (\overline{X} \rightarrow \overline{Y})) \rightarrow Y \geq (X \rightarrow \overline{Y}) \rightarrow Y. \) So, by \((A-3)\) and \((C-2)\), \( X \rightarrow ((X \rightarrow \overline{Y}) \land (\overline{X} \rightarrow \overline{Y})) \rightarrow Y \geq X \rightarrow ((X \rightarrow \overline{Y}) \rightarrow Y) = T \). So, by Eq. (5) and \((A-1)\), \( X \rightarrow ((X \Rightarrow Y) \rightarrow Y) = T \). On the other hand, by Eq. (5) and \((C-2)\), \( (\#) \overline{X} \rightarrow ((X \Rightarrow Y) \rightarrow \overline{Y}) = T \). Therefore, \((##) \ (X \rightarrow ((X \Rightarrow Y) \rightarrow \overline{Y})) \land (\overline{X} \rightarrow ((X \Rightarrow Y) \rightarrow \overline{Y})) = T \). Hence, from \((##)\), \((#)\) and Eq. (5), \( X \rightarrow ((X \Rightarrow Y) \Rightarrow Y) = T \) and \( \overline{X} \rightarrow ((X \Rightarrow Y) \Rightarrow Y) = T. \) Consequently, \( \overline{X} \rightarrow ((X \Rightarrow Y) \Rightarrow Y) \Rightarrow Y \land \overline{X} \rightarrow ((X \Rightarrow Y) \Rightarrow Y \Rightarrow \overline{Y}) \in [T, T]. \) Therefore, by Eq. (5), \( X \rightarrow ((X \Rightarrow Y) \Rightarrow Y) = [T, T]. \)

\( \langle A, \Rightarrow, [T, T] \rangle \) is a meet-semilattice. In fact, let \( X, Y, Z \in A \). Then, \( X \Rightarrow Y \land Z = [T, T] iff (X \rightarrow (Y \land Z)) \land (X \rightarrow (Y \land Z)) = T \) and \( X \rightarrow (Y \land Z) = T \) iff \( X \rightarrow Y = T \), \( X \rightarrow Z = T \), \( X \rightarrow Y = T \) and \( X \rightarrow Z = T \) iff \( X \Rightarrow Y = [T, T] \) and \( X \Rightarrow Z = [T, T] \).
In addition, \( X \leq Y \Rightarrow Z \) iff \( X \leq (Y \rightarrow Z) \wedge (\overline{Y} \rightarrow \overline{Z}) \) and \( \overline{X} \leq \overline{Y} \rightarrow \overline{Z} \) iff \( \overline{X} \leq Y \rightarrow Z \) and \( \overline{X} \leq \overline{Y} \rightarrow \overline{Z} \) iff \( \overline{X} \wedge \overline{Y} \leq \overline{Z} \) and \( \overline{X} \wedge \overline{Y} \leq \overline{Z} \) iff \( X \wedge Y \leq Z \). Therefore, \( \langle A, \Rightarrow, \top, \top \rangle \) satisfies (4).

\[ \square \]

If \( A \) has two different elements, say \( a \) and \( b \), such that \( a \rightarrow b = \top \), i.e. \( a \leq b \), then \( \Rightarrow \) is not an interval representation of \( \rightarrow \). In particular, \([a, b] \Rightarrow [a, b] = [\top, \top] \). Nevertheless, by (C-4) \( b \rightarrow a \neq \top \) and so \( b \rightarrow a \notin [a, b] \Rightarrow [a, b] \). This leads us to the following general theorem:

**Theorem 4.1.** Let \( \langle A, \rightarrow, \top \rangle \) be a BCI-Algebra. If there are \( a, b \in A \) such that \( a \neq b \) and \( a \rightarrow b = \top \), then for any interval \( \Pi \in A \) there is no interval representation \( \Rightarrow \) for \( \rightarrow \) such that \( \langle A, \Rightarrow, \Pi \rangle \) is a BCI-algebra.

**Proof:** Case \( \top \notin \Pi \). Then \( a \rightarrow a = \top \notin \Pi = [a, a] \Rightarrow [a, a] \) and therefore \( \Rightarrow \) is not an interval representation of \( \rightarrow \).

Case \( \Pi = [\top, \top] \), then \( [a, b] \Rightarrow [a, b] = \Pi = [\top, \top] \). Nevertheless, by (C-4) \( b \rightarrow a \neq \top \) and so \( b \rightarrow a \notin [a, b] \Rightarrow [a, b] \). Therefore, in this case \( \Rightarrow \) also is not an interval representation of \( \rightarrow \).

Case \( \Pi = [\alpha, \top] \) for some \( \alpha < \top \). Then \( \top \rightarrow a \neq \top \). However, if \( \langle A, \Rightarrow, \Pi \rangle \) is a BCI-algebra then, by (A-8), \( \Pi \Rightarrow [\top, \top] = [\top, \top] \) and therefore, \( \top \rightarrow \alpha \notin [\alpha, \top] \Rightarrow [\top, \top] \) which means that \( \Rightarrow \) again is not an interval representation of \( \rightarrow \).

\[ \square \]

In the following, we propose a process for intervalization of BCI-algebras.

**Theorem 4.2.** Let \( \langle A, \rightarrow, \top \rangle \) be a BCI-algebra, \( \langle A, \leq \rangle \) be a meet semilattice, such that for each \( x, y, z \in A \), \( x \rightarrow (y \wedge z) = x \rightarrow y \wedge z \) and \( \wedge = \{ [\overline{X}, \overline{X}] : X, \overline{X} \in A \} \) and \( X \leq X \). For \( X, Y \in A \), define:

1. \( X \Rightarrow Y = [X \rightarrow Y, X \rightarrow \overline{Y}] \).
2. \( X \Rightarrow Y = [X \rightarrow Y \wedge \overline{X} \rightarrow \overline{Y}, \overline{X} \rightarrow \overline{Y}] \).

Then \( \Rightarrow \) is the best representation of \( \rightarrow \) and the structure \( \langle A, \Rightarrow, \Rightarrow, [\top, \top] \rangle \) satisfies:
(IBC1) \(X \Rightarrow (Y \Rightarrow Z) = Y \Rightarrow (X \Rightarrow Z)\).

(IBC2) \(X \Rightarrow (Y \Rightarrow Z) = Y \Rightarrow (X \Rightarrow Z)\).

(IBC3) \(X \Rightarrow Y \leq (Z \Rightarrow X) \Rightarrow (Z \Rightarrow Y)\).

(IBC4) \([\top, \top] \Rightarrow X = X\).

(IBC5) \(X \ll Y \leq Z \Rightarrow X \ll Z\).

(IBC6) \(X \leq Y \ll Z \Rightarrow X \ll Z\).

(IBC7) \(X \leq Y \Rightarrow Y \leq Z \Rightarrow X = Y\).

where \(X \ll Y \iff X \Rightarrow Y = [\top, \top] \text{ e } X \leq Y \iff X \Rightarrow Y = [\top, \top]\). However, when \(A\) has at least one element different from \(\top\), then \(\langle A, \Rightarrow, [\top, \top]\rangle\) is not a BCI-algebra.

Proof: According to Proposition 4.4 at [5] the operation \(\Rightarrow\) is the best representation of \(\rightarrow\). Note that:

1. \(X \ll Y \iff X \Rightarrow Y = [\top, \top] \iff X \leq Y \iff X \Rightarrow Y = [\top, \top]\) and \(\overline{X} \leq \overline{Y} \iff \overline{X} \leq \overline{Y}\).

2. \(X \leq Y \iff X \Rightarrow Y = [\top, \top] \iff X \leq Y \iff X \Rightarrow Y = [\top, \top]\) and \(\overline{X} \leq \overline{Y} \iff \overline{X} \leq \overline{Y}\), i.e. \(\leq\) is the Kulisch-Miranker order.

(IBC7) is satisfied, since \(\langle A, \leq\rangle\) is a poset and “\(\leq\)” is the Kulisch-Miranker order.

Case of (IBC1): \(X \Rightarrow (Y \Rightarrow Z) = X \Rightarrow [Y \Rightarrow Z, Y \Rightarrow Z] = [X \rightarrow (Y \rightarrow Z), X \rightarrow (Y \rightarrow Z)]\). According to property (A-5) of BCI-algebras this term is also equal to \([Y \rightarrow (X \rightarrow Z), Y \rightarrow (X \rightarrow Z)] = Y \Rightarrow [X \rightarrow Z, X \rightarrow Z] = Y \Rightarrow (X \Rightarrow Z)\).

Case of (IBC2): \(X \Rightarrow (Y \Rightarrow Z) = X \Rightarrow [Y \Rightarrow Z, Y \Rightarrow Z] = [X \rightarrow (Y \rightarrow Z), X \rightarrow (Y \rightarrow Z)]\). On the other hand, \(Y \Rightarrow (X \Rightarrow Z) = Y \Rightarrow [X \rightarrow Z, X \rightarrow Z] = [Y \rightarrow (X \rightarrow Z), Y \rightarrow (X \rightarrow Z)] = [Y \rightarrow (X \rightarrow Z) \land Y \rightarrow (X \rightarrow Z)] \land [Y \rightarrow (X \rightarrow Z) \land Y \rightarrow (X \rightarrow Z)] = [(Y \rightarrow (X \rightarrow Z) \land Y \rightarrow (X \rightarrow Z))] \land [(Y \rightarrow (X \rightarrow Z) \land Y \rightarrow (X \rightarrow Z)) \land (X \rightarrow Z)]\).
By property [A-5] the last term is equal to: \[ ([X \to (Y \to Z) \land X \to (Y \to Z)] \land X \to (\overline{Y} \to \overline{Z}), \overline{X} \to (Y \to \overline{Z})] \] which is equal \(^{iii}\) to \[ ([X \to (Y \to Z) \land X \to (\overline{Y} \to \overline{Z})) \land \overline{X} \to (Y \to \overline{Z}), \overline{X} \to (Y \to \overline{Z})] = [X \to (Y \land Y \to \overline{Z}) \land \overline{X} \to (Y \to \overline{Z}), \overline{X} \to (Y \to \overline{Z})] = X \Rightarrow (Y \Rightarrow Z) \).

Case of **[IBCI1]** By definition, \(X \Rightarrow Y = [\overline{X} \to Y, \overline{X} \to Y] \) and \( (Z \Rightarrow X) \Rightarrow (Z \Rightarrow Y) = [(\overline{Z} \to X) \Rightarrow (\overline{Z} \to Y) \land (Z \to X) \Rightarrow (Z \to Y)]. \)

Since, by [A-2] (A-3) and (A-7) \( X \rightarrow Y \leq \overline{X} \rightarrow Y \leq (\overline{Z} \rightarrow X) \Rightarrow (\overline{Z} \rightarrow Y) \) and \( X \rightarrow Y \leq \overline{X} \rightarrow Y \leq (\overline{Z} \rightarrow X) \Rightarrow (\overline{Z} \rightarrow Y) \), then \( X \rightarrow Y \leq (\overline{Z} \rightarrow X) \Rightarrow (\overline{Z} \rightarrow Y) \). On the other hand, by [A-2] (A-3) and (A-7) \( X \rightarrow Y \leq (\overline{Z} \rightarrow X) \Rightarrow (\overline{Z} \rightarrow Y) \). Therefore, \(X \Rightarrow Y \leq (Z \Rightarrow X) \Rightarrow (Z \Rightarrow Y).\)

Case of **[IBCI4]** By (A-8), \([\top, \top] \Rightarrow [X, \overline{X}] = [\top \rightarrow X, \top \rightarrow \overline{X}] = [X, \overline{X}].\)

Case of **[IBCI5]** and **[IBCI6]** Suppose \(X \ll Y \leq Z\), then \(X \leq Y \leq Z\). Hence, \(X \ll Z, [\text{IBCI6}]\) is analogous.

For each BCI \(A, \rightarrow, \top\) with at least an element \(a \in A\) such that \(a \neq \top\), the algebra \(A, \Rightarrow, [\top, \top]\) is **not a BCI-algebra**. In fact, it fails to satisfy [C-3] since \(a \land \top, \top\) \(\Rightarrow \) \(\Rightarrow [a \land \top, \top] = [\top \rightarrow a \land \top, a \land \top \rightarrow \top] = [(a \land \top, a \land \top \rightarrow \top) \neq [\top, \top].\)

\(\Box\)

Observe that \(\Rightarrow\) is the best interval representation of \(\rightarrow\), but \(\Rightarrow\) is not an interval representation of \(\rightarrow\).

**Definition 4.1.** Given a BCI(K)-algebra \(A, \rightarrow, \top\), the structure \(A, \Rightarrow, [\top, \top]\) obtained by the method used in Theorem 4.2 is called Interval BCI(K)-algebra. IBCI (IBCK).

The process of intervalization destroys some basic properties of BCI-algebras, like (OP), and some properties are generalized.

**Theorem 4.3.** Given an IBCI(K)-algebra, \(A\), and \(X, Y, Z \in A\), the following properties are satisfied:

\[^{iii}\text{By associativity and commutativity of meet.}\]
Proof:

(G-1) \( X \Rightarrow X = [\overline{X} \rightarrow X, \tau] \).

(G-2) \( X \Rightarrow Y = [\tau, \tau] \iff \overline{X} \leq Y \).

\[ \begin{align*}
(G-1) & \quad X \Rightarrow X = [\overline{X} \rightarrow X, \overline{X}] \Rightarrow [\overline{X} \rightarrow X, \tau]. \\
(G-2) & \quad X \Rightarrow Y = [\tau, \tau] \iff \overline{X} \rightarrow Y = \tau \ \text{and} \ \overline{X} \rightarrow Y = \tau \iff \overline{X} \leq Y.
\end{align*} \]

\[ \square \]

\textbf{Corollary 4.1.} \( X \Rightarrow X = [\tau, \tau] \iff X \text{ is degenerate.} \)

We conclude that the relation “\( \ll \)” corresponding to the operator “\( \Rightarrow \)” will be reflexive (and hence a partial order) only if it is restricted to the subset of the degenerate intervals of \( \mathcal{A} \).

The next proposition provides another situation in which an intervalized BCI-algebra behaves like a BCI.

\textbf{Proposition 4.2.} Given an \textit{IBCI}(K)\-algebra, \( \mathcal{A} \), and \( X, Y, Z, X_d = [u, u], Y_d = [v, v] \in \mathcal{A} \), the following properties are satisfied:

(C.d-1) \( Y \Rightarrow Z \leq ((Z \Rightarrow X_d) \Rightarrow (Y \Rightarrow X_d)) \), i.e.,
\[ (Y \Rightarrow Z) \Rightarrow ((Z \Rightarrow X_d) \Rightarrow (Y \Rightarrow X_d)) = [\tau, \tau]; \]

(C.d-2) \( X_d \Rightarrow ((X_d \Rightarrow Y_d) \Rightarrow Y_d) = [\tau, \tau]; \)

(C.d-3) \( X \Rightarrow X_d = X \Rightarrow X_d. \)

\textbf{Proof:}

(C.d-1) \( Y \Rightarrow Z = [Y \rightarrow Z, Y \rightarrow Z] \). Moreover, \( ((Z \Rightarrow X_d) \Rightarrow (Y \Rightarrow X_d)) = [(Z \Rightarrow X_d) \rightarrow (Y \rightarrow X_d), (Z \Rightarrow X_d) \rightarrow (Y \rightarrow X_d)] = [(Z \rightarrow u) \rightarrow (Y \rightarrow u), (Z \rightarrow u) \rightarrow (Y \rightarrow u)] \). By property (C-1), \( Y \rightarrow Z \leq (Z \rightarrow u) \rightarrow (Y \rightarrow u) \) and \( Y \rightarrow Z \leq (Z \rightarrow u) \rightarrow (Y \rightarrow u) \). Therefore, \( (Y \Rightarrow Z) \leq ((Z \Rightarrow X_d) \Rightarrow (Y \Rightarrow X_d)). \)
In what follows a list of properties of IBCI-algebras is provided.

Theorem 4.4. An IBCI-algebra has the following properties: For all $X_d = [u, u], Z_d = [w, w], X, Y, Z \in \mathfrak{A}$,

(B-1) $[\tau, \tau] \preceq X$ implies $X = [\tau, \tau]$.

(B-2) $X \preceq Y$ implies $Y \Rightarrow Z \preceq X \Rightarrow Z$.

(B-3) $X \preceq Y$ implies $Z \Rightarrow X \preceq Z \Rightarrow Y$.

(B-4) $X \preceq Y$ and $Y \preceq Z$ implies $X \preceq Z$.

(B-5) $X \preceq Y \Rightarrow Z_d$ implies $Y \preceq X \Rightarrow Z_d$.

(B-6) $X \Rightarrow Y \preceq (Z_d \Rightarrow X) \Rightarrow (Z_d \Rightarrow Y)$.

(B-7) $((Y \Rightarrow X_d) \Rightarrow X_d) \Rightarrow X_d = Y \Rightarrow X_d$.

(B-8) $X \Rightarrow Y \preceq (Y \Rightarrow X) \Rightarrow [\tau, \tau]$.

(B-9) $(X \Rightarrow Y) \Rightarrow [\tau, \tau] = (X \Rightarrow [\tau, \tau]) \Rightarrow (Y \Rightarrow [\tau, \tau])$.

Proof:

(B-1) Suppose $[\tau, \tau] \preceq X$, then $\tau \preceq X$ and $\tau \preceq X$. By (A-1), $X = \tau$ and $X = \tau$. Therefore, $X = [\tau, \tau]$.

(B-2) Follows from the relation $\preceq$ and (A-2).

(B-3) Follows from the relation $\preceq$ and (A-3).

(B-4) Straightforward.
Theorem 4.5. The properties:

\begin{itemize}
  \item \textbf{(OP}_{M_1}) X \ll Y \text{ if and only if } X \Rightarrow Y = [\top, \top]
  \item \textbf{(OP}_{M_2}) X \preceq Y \text{ if and only if } X \Rightarrow Y = [\top, \top]
\end{itemize}

are not satisfied. However, the following holds:

\begin{itemize}
  \item \textbf{(OP}_a) \text{ If } X \ll Y, \text{ then } X \Rightarrow Y = [\top, \top]
\end{itemize}
If \( X \Rightarrow Y = [\tau, \tau] \), then \( X \preceq Y \).

**Proof:** (OP\(_{M_1}\)) is not satisfied. If fact, if \( X \Rightarrow Y = [\tau, \tau] \) then \( \overline{X} \preceq Y \) and \( \overline{X} \preceq Y \), what does not mean that \( \overline{X} \preceq Y \).

(\( \text{OP}_a \)) Suppose \( \overline{X} \preceq Y \), then \( \overline{X} \preceq \overline{Y} \preceq \overline{Y} \). Therefore, \( X \Rightarrow Y = [\tau, \tau] \).

(\( \text{OP}_b \)) Suppose \( X \Rightarrow Y = [\tau, \tau] \), then \( \overline{X} \preceq Y \), \( \overline{X} \preceq \overline{Y} \) and \( \overline{X} \preceq \overline{Y} = \tau \), so, \( \overline{X} \preceq \overline{Y} \). Therefore \( \overline{X} \preceq \overline{Y} \) and \( \overline{X} \preceq \overline{Y} \).

\( \Box \)

**Proposition 4.4.** The implications map degenerate intervals to degenerate intervals.

**Proof:** Straightforward, since \([u, u] \Rightarrow [v, v] = [u \rightarrow v, u \rightarrow v] = [w, w]\) and by (CD-3) \([u, u] \Rightarrow [v, v] = [w, w]\).

\( \Box \)

Therefore, the mathematical structure that arises from the intervalization of a BCI-algebra is a new mathematical structure which deserves to be developed. This structure will be called here **Semi-BCI algebra** and is what is exposes from now on.

5. Semi-BCI Algebra

This paper showed that some implications do not satisfy the order property (OP) and the correct intervalization of structures leads to relaxed structures. This section proposes a new algebraic structure which aims to capture both situations.

**Definition 5.1 (Semi-BCI (SBCI) algebra).** Given a set \( A \) endowed with two binary operations: \( \rightarrow \) and \( \Rightarrow \), a structure \( (A, \rightarrow, \Rightarrow, \tau) \) is a **Semi-BCI (SBCI) Algebra** whenever for all \( x, y, z \in A \),

- \((\text{SBCI1})\) \( x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z) \),
- \((\text{SBCI2})\) \( x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \),
Example 5.1. Consider the following structure: $A = ([0, 1], \rightarrow_R, \rightarrow_{LK}, 1)$, such that $x \rightarrow_R y = 1 - x + xy$, see Reichenbach [22]. Let $x \rightarrow_{LK} y = \min\{1, 1 - x + y\}$. Thus, since $\rightarrow_{LK}$ is the Łukasiewicz implication, then the corresponding relation $\leq$ is the usual order and $x \ll y$ if and only if $x \rightarrow_R y = 1$ if and only if $x = 0$ or $y = 1$. It is straightforward to check the satisfaction of (SBCI3), (SBCI7), (SBCI1), $x \rightarrow_R (y \rightarrow_R z) = 1 - x y + x y z = y \rightarrow_R (x \rightarrow_R z)$, and $x \rightarrow_{LK} (x \rightarrow_{LK} y) = \min(1, 1 - x + \min(1, 1 - y + z))$, let be the following cases: (1) If $x \rightarrow_{LK} (y \rightarrow_{LK} z) = 1$, then $1 - x + \min(1, 1 - y + z) \geq 1 \iff \min(1, 1 - y + z) \geq x \iff 1 - y + z \geq x \iff 1 - x + z \geq y \iff \min(1, 1 - x + z) \geq y \iff 1 - y + \min(1, 1 - x + z) \geq 1 \iff y \rightarrow_{LK} (x \rightarrow_{LK} z) = 1$. (2) If $x \rightarrow_{LK} (y \rightarrow_{LK} z) \neq 1$, then $1 - x + \min(1, 1 - y + z) < 1 \iff \min(1, 1 - y + z) < x \iff 1 - y + z < y \iff \min(1, 1 - x + z) < y \iff 1 - y + \min(1, 1 - x + z) < 1$. Therefore, $y \rightarrow_{LK} (x \rightarrow_{LK} z) = 1 - y + \min(1, 1 - x + z) = 1 - y + 1 - x + z = 1 - x + 1 - y + z = 1 - x + \min(1, 1 - y + z) = x \rightarrow_{LK} (y \rightarrow_{LK} z)$. Concerning $x \rightarrow_R y = 1 - x + xy$ and $(z \rightarrow_R x) \rightarrow_{LK} (z \rightarrow_R y) = \min\{1, 1 - (1 - z + x z) + (1 - z + z y)\} = \min\{1, 1 - z x + z y\}$ Since, $1 - x + xy \leq 1 - z x + z y$ then $1 - x + xy \leq_{LK} \min\{1, 1 - z x + z y\}$ and therefore $(x \rightarrow_R y) \leq_{LK} ((z \rightarrow_R x) \rightarrow_{LK} (z \rightarrow_R y))$ and therefore $(x \rightarrow_R y) \leq_{LK} (z \rightarrow_R x) \rightarrow_{LK} (z \rightarrow_R y)$ and therefore $(x \rightarrow_R y) \leq_{LK} (z \rightarrow_R x)$. Therefore $A = \{[0, 1], \rightarrow_R, \rightarrow_{LK}, 1\}$ is an SBCI-algebra.
Remark 5.1. Note that any IBCI-algebra is a SBCI-algebra.

Proposition 5.1. In a SBCI-algebra, \( \langle A, \to, \to, \top \rangle \), the following hold:

(SBCI8) If \( x \ll y \) and \( y \ll z \) then \( x \ll z \),

(SBCI9) If \( x \ll y \) and \( y \ll x \) then \( x = y \),

(SBCI10) \( (y \to x) \leq ((z \to x) \to (y \to x)) \),

(SBCI11) If \( \top \leq x \), then \( x = \top \),

(SBCI12) \( x \to x = \top \)

(SBCI13) If \( x \ll y \) then \( x \leq y \),

(SBCI14) \( x \to y \leq x \to y \),

(SBCI15) \( x \to ((x \to y) \to y) = \top \),

(SBCI16) If \( x \ll y \) then \( x \to ((x \to y) \to y) = \top \),

(SBCI17) If \( x \ll y \), then \( z \to x \leq z \to y \),

(SBCI18) If \( x \ll y \), then \( y \to z \leq x \to z \).

Proof: The proof of items (SBCI8)-(SBCI10) and (SBCI14)-(SBCI16) are straightforward. (SBCI11) By (SBCI3) \( (\top \to x) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI12) By (SBCI10) \( (\top \to \top) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI13) By (SBCI10) \( (\top \to \top) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI14) By (SBCI10) \( (\top \to x) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI15) By (SBCI10) \( (\top \to x) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI16) By (SBCI10) \( (\top \to x) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI17) By (SBCI10) \( (\top \to x) \to ((\top \to x) \to (\top \to x)) = \top \). (SBCI18) By (SBCI10) \( (\top \to x) \to ((\top \to x) \to (\top \to x)) = \top \). as \( x \to y = \top \) then by (SBCI11) \( (z \to x) \to (z \to y) = \top \), thus \( (z \to x) \leq (z \to y) \). (SBCI18) By (SBCI10) \( (x \to y) \to ((y \to z) \to (x \to z)) = \top \), as \( x \to y = \top \) then by (SBCI11) \( (y \to z) \to (x \to z) = \top \), thus \( y \to z \leq (x \to z) \). □
We conclude from \([\text{SBCI18}]\) and \([\text{SBCI19}]\) that the relation “\(\preccurlyeq\)" is transitive and anti-symmetric, but it is not necessarily reflexive, whereas the relation “\(\preceq\)" is antisymmetric and reflexive by \([\text{SBCI7}]\) and \([\text{SBCI12}]\) respectively, but it is not necessarily transitive. The following proposition provides a condition for them to be partial orders.

**Proposition 5.2.** The relation “\(\preccurlyeq\)" coincides with “\(\preceq\)" if and only if “\(\preccurlyeq\)" is reflexive.

**Proof:** Suppose \(\preccurlyeq\) is reflexive, then \(x \preccurlyeq y \Rightarrow x \preceq y\), by \([\text{SBCI15}]\). The rest of the proof is trivial.

\(\Box\)

**Example 5.2.** The structure: \(A = (\{0, 1\}, \rightarrow_{GD}, \rightarrow_{FD}, 1)\) is a SBCI, in which:

\[
x \rightarrow_{GD} y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}
\]

and

\[
x \rightarrow_{FD} y = \begin{cases} 1, & \text{if } x \leq y \\ \max(1 - x, y), & \text{if } x > y \end{cases}
\]

and the following relations coincide: (1) \(x \preccurlyeq y \quad \text{if and only if} \quad x \rightarrow_{GD} y = 1\) if and only if \(x \preceq y\) and (2) \(x \preceq y \quad \text{if and only if} \quad x \rightarrow_{FD} y = 1\) if and only if \(x \preceq y\). In fact, “\(\preccurlyeq\)" is reflexive and according to Baczyński et al. \([2]\)[p.10, Table 1.4], \(\rightarrow_{GD}\) satisfies \([\text{SBCI1}]\) and \(\rightarrow_{FD}\) satisfies \([\text{SBCI2, SBCI3}]\). The remaining of SBCI-algebra properties are straightforward.

**Proposition 5.3.** Let \(\langle A, \rightarrow_{1}, \top \rangle\) and \(\langle A, \rightarrow_{2}, \top \rangle\) be BCI-algebras with \(\leq_{1}\) and \(\leq_{2}\) as their respective partial orders. If \(\leq_{2}\) extends \(\leq_{1}\) and \(x \rightarrow_{1} y \leq_{2} x \rightarrow_{2} y\) and

\[
w \leq_{2} x \leq_{1} y \leq_{2} z \Rightarrow w \leq_{1} z
\]

(6)
for each \( w, x, y, z \in A \), then \( (A, \to_{1}, \to_{2}, \top) \) is an SBCI-algebra. The same applies to BCK-algebras and SBCK-algebras.

**Proof:**

(SBCI1) Straightforward from \([A-5]\)

(SBCI2) Straightforward from \([A-5]\)

(SBCI3) By (C-1), \((z \to_{1} x) \to_{1} ((x \to_{1} y) \to_{1} (z \to_{1} y)) = \top\). So, by \([A-5]\), \((x \to_{1} y) \to_{1} ((z \to_{1} x) \to_{1} (z \to_{1} y)) = \top\), i.e. \((x \to_{1} y) \leq_{1} ((z \to_{1} x) \to_{1} (z \to_{1} y))\). Thus, since \( \leq_{2} \) extends \( \leq_{1} \) and \( \to_{1} \leq_{2} \to_{2} \), then \((x \to_{1} y) \leq_{2} ((z \to_{1} x) \to_{2} (z \to_{1} y))\).

(SBCI4) Straightforward from \([A-8]\)

(SBCI5) Straightforward from \([6]\) by taking \( w = x \).

(SBCI6) Straightforward from \([6]\) by taking \( y = z \).

(SBCI7) Straightforward from \([A-4]\).

\[\square\]

**Corollary 5.1.** If \((A, \to, \top)\) is BCI-algebra, then \((A, \to_{1}, \to_{2}, \top)\) is an SBCI-algebra. The same applies to BCK-algebras and SBCK-algebra.

**Proof:** Straightforward from Proposition 5.3 once that \([6]\) holds from \([A-4]\)

\[\square\]

**Proposition 5.4.** Given an SBCI-algebra of the form \((A, \to, \top)\), the structure \((A, \to_{1}, \to_{2}, \top)\) is a BCI-algebra.

**Proof:**

(C-1) By \([SBCI3]\) \( z \to x \leq (y \to z) \to (y \to x) \), i.e. \((z \to x) \to ((y \to z) \to (y \to x)) = \top\). Therefore, by \([SBCI1]\) \((y \to z) \to ((z \to x) \to (y \to x)) = \top\).

(C-2) By \([SBCII]\) and \([SBCII2]\) \( x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = \top\).
Proposition 5.5. There are SBCI-algebras \( A, \to, \to, \tau \) s.t. the reduct \( A, \to, \tau \) are not BCI-algebras.

Proof: Consider the SBCI \( A = ([0,1], \to_R, \to_{LK}, 1) \) seen in Example 5.1. The reduct \( ([0,1], \to_R, 1) \) is not a BCI-algebra, since \( x \to_R x = 1 - x + x^2 \neq 1 \), for all \( x \in (0,1) \). Therefore, \((C-3)\) does not hold.

Proposition 5.6. There are algebras \( A, \to, \tau \) in which the exchange principle (EP) is satisfied, but \( A, \to, \to, \tau \) is not an SBCI-algebra.

Proof: Take the algebra \( ([0,1], \to_{YD}, 1) \), such that:

\[
    x \to_{YD} y = \begin{cases} 
        1, & \text{if } x = y = 0 \\
        y^x, & \text{otherwise}
\end{cases}
\]

The implication “\( \to_{YD} \)” satisfies (EP) — see [2][p.10, Table 1.4]. However, it is easy to check that it fails to satisfy \([\text{SBCI12}]\) — take \( x \in (0,1) \).

Alternatively, consider the algebra \( ([0,1], \to_{WB}, 1) \), such that:

\[
    x \to_{WB} y = \begin{cases} 
        1, & \text{if } x < 1 \\
        y, & \text{if } x = 1
\end{cases}
\]

This implication satisfies \([\text{SBCII}]\) [p.10, Table 1.4]. However, it is easy to check that it fails to satisfy \([\text{SBCI7}]\) — take \( x = 0.5 \) and \( y = 0.3 \).
6. Relating Semi-BCI Algebras and Pseudo-BCI algebras

The generalization of BCI/BCK-algebras is not new. In fact, G. Georgescu and A. Iorgulescu [23] proposed an extension for BCK-algebras and later W. A. Dudek and Y. B. Jun [7] proposed an extension for BCI-algebras. The first was called **Pseudo-BCK algebras** and the second **Pseudo-BCI algebras**. Like SBCI-algebras they propose two operations which generalize the primitive operation of BCK/BCI-algebras. Therefore a natural question arises: Are SBCI-algebras just a rewritten of those algebras? This section shows that the answer to this question is negative, meaning that SBCI/SBCK-algebras are completely new structures which generalize BCI-algebras. This section also shows how both structures are related.

**Definition 6.1 (24).** A pseudo-BCI algebra (PBCIA) is a structure \(\langle A, \leq, \rightarrow, \implies, \top \rangle\) s.t. “\(\leq\)” is a binary relation on the set \(A\), “\(\rightarrow\)” and “\(\implies\)” are binary operations on \(A\), \(\top \in A\) and for all \(x, y, z \in A\):

- (PB-1) \(x \rightarrow y \leq (y \rightarrow z) \implies (x \rightarrow z)\),
- (PB-2) \(x \implies y \leq (y \implies z) \implies (x \implies z)\),
- (PB-3) \(x \leq (x \rightarrow y) \implies y\),
- (PB-4) \(x \leq (x \implies y) \implies y\),
- (PB-5) \(x \leq x\),
- (PB-6) if \(x \leq y\) and \(y \leq x\), then \(x = y\),
- (PB-7) \(x \leq y \iff x \rightarrow y = \top \iff x \implies y = \top\).

**Example 6.1.** The structure \(A = (\mathbb{R}^2, \leq, \rightarrow, \implies, (0,0))\), such that \((x_1, y_1) \rightarrow (x_2, y_2) = (x_2 - x_1, (y_2 - y_1) e^{-x_1})\) and \((x_1, y_1) \implies (x_2, y_2) = (x_2 - x_1, y_2 - y_1 e^{x_2 - x_1})\), is a PBCI-algebra.

The next proposition shows that PBCI-algebras are not suitable to model the intervalization of BCIs.
Proposition 6.1. Let \( \langle A, \rightarrow, \tau \rangle \) be a BCI-algebra, \( \langle A, \preceq \rangle \) be a meet semilattice, such that for each \( x, y, z \in A \), \( x \rightarrow (y \land z) = x \rightarrow y \land x \rightarrow z \), then the structure: \( \langle A, \Rightarrow, \Rightarrow, [\tau, \tau] \rangle \), where \( \Rightarrow \) and \( \Rightarrow \) are defined in Theorem 4.2, is not a PBCI-algebra.

Proof: By \([\text{PB-5}]\) and \([\text{PB-7}]\) for all \( X \in A \), \( X \Rightarrow \Rightarrow X = [\tau, \tau] \) should hold, however by Corollary 4.1, this only applies if \( X \) is degenerate.

Given a PBCI-algebra, if the relations: (a) \( x \preceq 1 y \iff x \rightarrow y = \top \) and (b) \( x \preceq 2 y \iff x \sim y = \top \) are defined, then the axiom \([\text{PB-7}]\) imposes that they must coincide. In the case of SBCI-algebras the relations \( \ll \) and \( \preceq \) does not necessarily coincide. Moreover, even if they coincide there are SBCIs which are not PBCIs — see Proposition 6.2.

Finally, there are SBCIs in which the relation \( \ll \) can be irreflexive refuting the axiom \([\text{PB-5}]\) (see Proposition 5.5). Therefore, this lead us to conclude that SBCI and PBCI-algebras are different structures.

Proposition 6.2. There are SBCI-algebras \( \langle A, \rightarrow, \rightarrow, \tau \rangle \) whose relations \( \ll \) and \( \preceq \) coincide but are not PBCI-algebras.

Proof: The SBCI \( A = \langle [0, 1], \rightarrow_{GD}, \rightarrow_{FD}, 1 \rangle \) provided at Example 5.2 is not a PBCI-algebra. In fact, take \( x = \frac{3}{4}, y = \frac{1}{2} \) and \( z = \frac{1}{5} \), then \( x \rightarrow_{FD} y = \frac{1}{2} \) and \( y \rightarrow_{FD} z \rightarrow_{GD} (x \rightarrow_{FD} z) = \frac{1}{4} \), but since the relations \( \ll \) and \( \preceq \) coincide with the usual order, \([\text{PB-2}]\) is not satisfied.

Proposition 6.3. There are PBCI-algebras \( \langle A, \preceq, \rightarrow, \tau \rangle \) which are not SBCI-algebras.

Proof: The PBCI \( A = \langle \mathbb{R}^2, \preceq, \rightarrow, (0, 0) \rangle \) presented in Example 6.1 is not a SBCI-algebra. In fact, take \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) such that \( y_2(e^{x_1} - 1) \neq y_1(e^{x_2} - 1) \) and any \((x_3, y_3) \in \mathbb{R}^2\). Note that:

\[
(x_1, y_1) \rightarrow ((x_2, y_2) \rightarrow (x_3, y_3)) = \\
= (x_1, y_1) \rightarrow (x_3 - x_2, y_3 - y_2e^{x_3-x_2}) \\
= ((x_3 - x_2) - x_1, (y_3 - y_2e^{x_3-x_2}) - y_1e^{(x_3-x_2)-x_1}) \\
= (x_3 - x_2 - x_1, y_3 - y_2e^{x_3-x_2} - y_1e^{x_3-x_2-x_1})
\]
and

\[(x_2, \ y_2) \mapsto (x_3, y_3) = \]
\[= (x_2, y_2) \mapsto (x_3 - x_1, y_3 - y_1 e^{x_3-x_1}) \]
\[= ((x_3 - x_1) - x_2, (y_3 - y_1 e^{x_3-x_1}) - y_2 e^{(x_3-x_1) - x_2}) \]
\[= (x_3 - x_2 - x_1, y_3 - y_1 e^{x_3-x_1} - y_2 e^{x_3-x_2-x_1}). \]

Since \(y_2(e^{x_1} - 1) \neq y_1(e^{x_2} - 1)\), then \(y_3 - y_2 e^{x_3-x_2} - y_1 e^{x_3-x_2-x_1} \neq y_3 - y_1 e^{x_3-x_1} - y_2 e^{x_3-x_2-x_1}\), and therefore \(A\) does not satisfy the axiom \((\text{SBCI1})\).

The next proposition ensures that the intersection between the PBCIs and SBCIs is formed only by BCI-algebras.

**Proposition 6.4.** Let \(A = \langle A, \rightarrow, \cdot, \top \rangle\) be a SBCI-algebra, s.t. the relations: “\(\ll\)” and “\(\leq\)” correspond to the operations: \(\rightarrow\) and \(\cdot\), respectively. If \(A\) is also a PBCI-algebra, then \(\langle A, \rightarrow, \cdot, \top \rangle\) is a BCI-algebra.

**Proof:** Since \(A\) is also a PBCI-algebra then the relations “\(\leq\)” and “\(\ll\)” coincide. Now, by \((\text{PB-4})\)

\[x \leq (x \rightarrow y) \rightarrow y \quad \Rightarrow \quad x \ll (x \rightarrow y) \rightarrow y \]
\[\Rightarrow \quad x \rightarrow ((x \rightarrow y) \rightarrow y) = \top \]
\[= \quad (x \rightarrow y) \ll (x \rightarrow y) \]
\[\Rightarrow \quad (x \rightarrow y) \ll (x \rightarrow y) \]
\[\Rightarrow \quad (x \rightarrow y) \leq (x \rightarrow y).\]

Therefore, since \(x \rightarrow y \leq x \rightarrow y\) \((\text{SBCI4})\) for all \(x, y \in A\) then, from axiom \((\text{SBCI7})\) follows \(x \rightarrow y = x \rightarrow y\), for all \(x, y \in A\). We conclude by Proposition 5.4 that \(\langle A, \rightarrow, \top \rangle\) is a BCI-algebra.

Figure 1 shows how PBCI and SBCI algebras are classified.
7. Final Remarks

This paper proposes a new algebraic structure which generalizes the notion of BCI-algebra. It is an algebraic structure which captures the most important properties of a Fuzzy Implication after it has been “intervalized” in a correct way. The resulting structures, called Semi-BCI algebras, capture the properties of the structures which arise from the “intervalization” of BCI-algebras. In other words, as BCI-algebras abstract the Łukasiewicz algebra, the SBCI-algebras abstract the respective intervalization of the Łukasiewicz algebra. The paper also provides the connection of such structures with PBCI-algebras.

As further steps, the authors aim to investigate more closely this structure. Some questions like the logical counter-part of such algebras requires a deeper investigation. Entities like filters, ideals, category and others require investigation. In other words, the generalization of BCI-algebras in terms of its intervalization is provided from algebraic viewpoint, but the logical correspondence (The resulting Interval Fuzzy Logic) requires more reflection. Contrary to BL-algebras (BCIs with extra properties) a question is posed:

“Is interval correctness incompatible with logical principles, like the notion of deducibility and its whole connection with implications?”

This question is important, since the answer can limit the term: “Interval Fuzzy Logic” in a broad sense.
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References

[1] H. Rasiowa. (1974). An Algebraic Approach to Non-classical Logics. Studies in logic and the foundations of mathematics. North-Holland Publishing Company.

[2] M. Baczyński and B. Jayaram. (2008). Fuzzy Implications. Springer, Berlin.

[3] J. Pinheiro, B. Bedregal, R. Santiago, and H. Santos. A study of (t,n)-implications and its use to construct a new class of fuzzy subsethood measure. Submitted to International Journal of Approximate Reasoning.

[4] R. R. Yager. (1980). An approach to inference in approximate reasoning. International Journal of Man-Machine Studies, 13(3):323 – 338.

[5] B. C. Bedregal and R. H. N. Santiago. (2013). Interval representations, ukasiewicz implicators and SmetsMagrez axioms. Information Sciences, 221:192 – 200.

[6] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. (2003). Continuous Lattices and Domains. Encyclopedia of Mathematics and its Applications. Cambridge University Press.

[7] W. A. Dudek and Y. B. Jun. (2008). Pseudo-BCI algebras. East Asian Mathematical Journal, 24:187–190.
[8] K. Iski. (1966). An algebra related with a propositional calculus. *Proc. Japan Acad.*, 42(1):26–29.

[9] Y. Huang. (2006). *BCI-Algebra*. Science Press, Beijing.

[10] J. R. Hindley and J. P. Seldin. (1986). *Introduction to Combinators and Lambda-Calculus*. Cambridge University Press.

[11] G. E. Forsythe. (1970). Pitfalls in computation, or why a math book isn’t enough. Technical report, Stanford University, Stanford, CA, USA.

[12] S. M. Rump. (1988). Algorithms for verified inclusions: Theory and practice. In Ramon E. Moore, editor, *Reliability in Computing*, pages 109 – 126. Academic Press.

[13] R. E. Moore. (1959). Automatic error analysis in digital computation. Technical Report Space Div. Report LMSD84821, Lockheed Missiles and Space Co., Sunnyvale, CA, USA.

[14] R. E. Moore. (1962). *Interval Arithmetic and Automatic Error Analysis in Digital Computing*. Ph.D. dissertation, Department of Mathematics, Stanford University, Stanford, CA, USA. Also published as Applied Mathematics and Statistics Laboratories Technical Report No. 25.

[15] T. Sunaga. (2009). Theory of an interval algebra and its application to numerical analysis [reprint of res. assoc. appl. geom. mem. 2 (1958), 2946]. *Japan J. Indust. Appl. Math.*, 26(2-3):125–143.

[16] T. Hickey, Q. Ju, and M. Emdem. (2001). Interval arithmetic: from principles to implementation. *Journal of the ACM*, 48(5):1038–1068.

[17] B. Bedregal and R. Santiago. (2013). Some continuity notions for interval functions and representation. *Computational and Applied Mathematics*, 32(3):435–446.
[18] R. H. N. Santiago, B. C. Bedregal, and B. M. Acióly. (2006). Formal aspects of correctness and optimality in interval computations. *Formal Aspects of Computing*, 18(2):231–243.

[19] R.E. Moore. (1979). *Methods and Applications of Interval Analysis*. SIAM, Philadelphia.

[20] S. Markov. (1977). A non-standard subtraction of intervals. *Serdica*, 3:359–370.

[21] L. M. Cabrer and D. Mundici. (2014). Interval MV-algebras and generalizations. *International Journal of Approximate Reasoning*, 55(8):1623 – 1642.

[22] H. Reichenbach. (1935). Wahrscheinlichkeitslogik. *Erkenntnis*, 5(1):37–43.

[23] G. Georgescu and A. Iorgulescu. (2001). Pseudo-bck algebras: An extension of bck algebras. In C. S. Calude, M. J. Dinneen, and S. Sburlan, editors, *Combinatorics, Computability and Logic: Proceedings of the Third International Conference on Combinatorics, Computability and Logic, (DMTCS’01)*, pages 97–114, London. Springer London.

[24] G. Dymek. (2013). On the category of pseudo-BCI-algebras. *Demonstratio Mathematica. Warsaw Technical University Institute of Mathematics*, Vol. 46, nr 4:631–644.