QUASILINEAR ITERATIVE METHOD FOR THE BOUNDARY VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, the existence and uniqueness of solution for a class of boundary value problems of nonlinear fractional order differential equations involving the Caputo fractional derivative are studied. The estimation of error between the approximate solution and the solution for such equation is presented by employing the quasilinear iterative method, and an example is given to demonstrate the application of our main result.

1. Introduction. In this paper, we consider the following boundary value problems of fractional order differential equation

\begin{align*}
& cD^\alpha_a x(t) = f(t, x(t)), \quad t \in J = [a, b], \quad n-1 < \alpha \leq n, \\
& x^{(k)}(a) = 0, \quad k = 0, 1, 2, \ldots, n-2; \quad x^{(n-1)}(b) = 0,
\end{align*}

where \( cD^\alpha_a \) is the Caputo fractional derivative, \( f : J \times \mathbb{R} \to \mathbb{R} \) is continuous function.

Fractional order differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. We can find some applications in control, porous media, electromagnetic, etc., in reference to the monographs of Oldham and Spanier[17], Miller and Ross[16], Podlubny [18], Hilfer [9] and the papers of Stynes et al [19], El-Sayed [7], He [8], Lyu and Vong et al [15], Li and Yi et al [13], Benchohra et al [3, 4], Agarwal et al [1] and the others[22, 23, 20, 2, 6, 5, 14] of the references therein.

Recently some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator \( 0 < \alpha < 1 \) has been

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discussed by Lakshmikantham et al [10, 11, 12]. In a series of papers (see [3, 1]) the authors considered some classes of boundary value problems for differential equations involving Riemann-Liouville and Caputo fractional derivatives of order \(0 < \alpha < 1\) and \(2 < \alpha < 3\).

In this paper, we propose a quasilinear iterative method for the the estimations of error between the approximate solution and the solution of BVP (1), and an example is given to demonstrate the application of our main result.

2. Preliminaries. In order to state our result, some notions and lemmas are given in this section. We denote by \(C(J, \mathbb{R})\) the Banach space of all continuous functions from \(J\) into \(\mathbb{R}\) with the norm

\[
\|x\|_\infty := \sup_{t \in J}\{|x(t)|\}, \quad J = [a, b].
\]

**Definition 2.1.** [3, 1] The fractional order integral of the function \(h(t) \in L^1([a, b], \mathbb{R}_+)\) is defined by

\[
I^\alpha_a h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,
\]

where \(\Gamma\) is the gamma function.

**Definition 2.2.** [3, 1] For a function \(h\) given on the interval \([a, b]\), the \(\alpha\)th Caputo fractional-order derivative of \(h\) is defined by

\[
(cD^\alpha_a h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,
\]

where \(n=[\alpha]+1\) and \([\alpha]\) denotes the integer part of \(\alpha\).

**Definition 2.3.** By a solution of (1) we mean a function \(x(t) \in C^{n-1}(J, \mathbb{R})\), which satisfies (1).

**Lemma 2.4.** [21] Let \(\alpha > 0\). Then

\[
I^\alpha_a cD^\alpha a h(t) = h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}.
\]

In particular, if \(a = 0\), we have

\[
I^\alpha_0 cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},
\]

where \(c_i = -\frac{h^{(i)}(a)}{i!} \in \mathbb{R}, \quad i = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1\).

As a consequence of Lemma 2.4, we have the following result.

**Lemma 2.5.** Let \(n-1 < \alpha < n\), \(n = [\alpha] + 1\), and let \(h : J \to \mathbb{R}\) be continuous. A function \(x(t)\) is a solution of the fractional BVP

\[
\begin{cases}
(cD^\alpha a x(t) = h(t), \quad t \in J, \\
x^{(k)}(a) = 0, \quad k = 0, 1, 2, \ldots, n-2; \quad x^{(n-1)}(b) = 0,
\end{cases}
\]

if and only if \(x(t)\) is a solution of the fractional integral equation

\[
x(t) = -\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} h(s)ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds.
\]
Proof. Assume \( x(t) \) satisfies (4), then Lemma 2.4 implies that

\[
x(t) = -c_0 - c_1(t-a) - c_2(t-a)^2 - \cdots - c_{n-1}(t-a)^{n-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds.
\]

And the following simple calculation can be obtained by (4)

\[
c_k = 0, \quad k = 0, 1, 2, \ldots, n - 2;
\]

\[
c_{n-1} = \frac{1}{(n-1)! \Gamma(\alpha - n + 1)} \int_a^b (b-s)^{\alpha-n} h(s) ds.
\]

Hence we get equation (5). Conversely, it is clear that if \( x(t) \) satisfies equation (5), then equations (4) hold. \( \square \)

3. Main result. We consider the BVP (1). As a consequence of Lemma 2.5, it is easy to verify the following results.

**Lemma 3.1.** A function \( x(t) \) is a solution of the fractional BVP (1), if and only if \( x(t) \) is a solution of the fractional integral equation

\[
x(t) = \int_a^b G(t, s) f(s, x(s)) ds,
\]

(6)

where

\[
G(t, s) = \begin{cases} 
\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-a)^{n-1}(b-s)^{\alpha-n}}{(n-1)! \Gamma(\alpha - n + 1)}, & a \leq s \leq t \leq b, \\
- \frac{(t-a)^{n-1}(b-s)^{\alpha-n}}{(n-1)! \Gamma(\alpha - n + 1)}, & a \leq t \leq s \leq b.
\end{cases}
\]

(7)

\( G(t, s) \) is Green’s function of BVP (1).

**Lemma 3.2.** With respect to \( G(t, s) \), the following inequality is established

\[
\int_a^b |\partial^k G(t, s)| ds \leq C_{\alpha, k} (b-a)^{\alpha-k},
\]

where \( C_{\alpha, k} := \left( \frac{1}{\Gamma(\alpha-k+1)} + \frac{1}{(n-k-1)! \Gamma(\alpha - n + 2)} \right) \), \( k = 0, 1, 2, \ldots, n-1 \).

**Definition 3.3.** A function \( z(t) \) is called an approximate solution of the BVP (1), if there exists a constant \( \varepsilon > 0 \) such that

\[
\max_{a \leq t \leq b} |^c D^\alpha_a z(t) - f(t, z(t))| < \varepsilon,
\]

and

\[
z^{(k)}(a) = 0, \quad k = 0, 1, 2, \ldots, n-2; \quad z^{(n-1)}(b) = 0.
\]

In fact, the approximate solution \( z(t) \) can be expressed as

\[
z(t) = \int_a^b G(t, s) \{ f(s, z(s)) + \eta(s) \} ds,
\]

(8)

in which

\[
\eta(t) = ^c D^\alpha_a z(t) - f(t, z(t)), \quad \max_{a \leq t \leq b} |\eta(t)| \leq \varepsilon.
\]

Denote: \( R_\alpha = \{ x(t) : x(t) \in C^{(n-1)}(J, \mathbb{R}); \quad x^{(k)}(a) = 0, \quad k = 0, 1, \ldots n-2; \quad x^{(n-1)}(b) = 0; \quad ^c D^\alpha_a x(t) \text{ on } [a, b] \text{ has only some limited discontinuous points of the first kind} \} \).
For $x(t) \in R_\alpha$, we define the norm
\[
\|x\| = \max_{0 \leq k \leq n-1} \left\{ \frac{C_{\alpha,0}(b-a)^k}{C_{\alpha,k}} \max_{a \leq t \leq b} |x^{(k)}(t)| \right\}.
\]

It is easy to verify $R_\alpha$ is a Banach space. Let $D = \{x(t) : |x(t) - z(t)| \leq N\} \subset R_\alpha$, $N$ is a nonnegative constant. Consider the sequence $\{x_m(t)\}$ generated by the quasilinear iterative scheme
\[
\begin{align*}
^{(k)} D_{\alpha} x_{m+1}(t) &= f(t, x_m(t)) + \beta(t)(x_{m+1}(t) - x_m(t)) \frac{\partial f(t, x_m(t))}{\partial x_m(t)}, \\
\end{align*}
\]
with $x_{0}(t) = z(t)$ (approximate solution), such that (9),(10) converges to the unique solution $z^*(t)$ of the BVP (1), where $\beta(t) \in C([a, b])$.

**Theorem 3.4.** With respect to the BVP (1), we assume that there exists an approximate solution $z(t)$, and

(H1) There exists a nonnegative constants $L_1$ such that
\[|f(t, u) - f(t, v)| \leq L_1|u - v|, \quad \forall \ t \in J; \ u, v \in C^{n-1}(J, \mathbb{R}).\]

(H2) The function $f(t, x(t))$ is a continuous differentiable with respect to $x(t)$ and such that for all $(t, x(t)) \in [a, b] \times D$, \[\left| \frac{\partial f(t, x(t))}{\partial x(t)} \right| \leq L_2, \text{ in which } L_2 \text{ is a nonnegative constants.}\]

(H3) $\beta = \max_{a \leq t \leq b} |\beta(t)|$, $\vartheta = L_2 C_{\alpha,0}(b-a)^\alpha$, $\vartheta_\beta = (L_1 + L_2 \beta) C_{\alpha,0}(b-a)^\alpha$, $\beta \vartheta + \vartheta_\beta < 1$.

(H4) $N_1 = \varepsilon (1 - \beta \vartheta - \vartheta_\beta)^{-1} C_{\alpha,0}(b-a)^\alpha \leq N$, $S(z, N_1) = \{x(t) \in R_\alpha : \|x - z\| < N_1, \ N_1 > 0\}$.

Then the following conclusions hold

(i) The sequence $\{x_m(t)\}$ generated by the process (9),(10) remains in $S(z, N_1)$.

(ii) The sequence $\{x_m(t)\}$ converges to the unique solution $z^*(t)$ of the BVP (1).

(iii) A bound on the error is given by
\[\|x_m - z^*\| \leq \left( \frac{\vartheta_\beta}{1 - \beta \vartheta} \right)^m \left( \frac{1 - \beta \vartheta - \vartheta_\beta}{1 - \beta \vartheta} \right)^{-1} \|x_1 - x_0\|\]
\[\leq \left( \frac{\vartheta_\beta}{1 - \beta \vartheta} \right)^m \left( 1 - \beta \vartheta - \vartheta_\beta \right)^{-1} \varepsilon C_{\alpha,0}(b-a)^\alpha.\]

**Proof.** Firstly, we shall prove that $\{x_m(t)\} \subseteq S(z, N_1)$. From Lemma 3.1, defining an implicit operator $T$ maps $R_\alpha$ into itself, such that $x_{m+1} = Tx_m$, obviously, by norm of $R_\alpha$, $\forall x(t) \in D$, we have (see [14]):
\[
(Tx)(t) = \int_a^b G(t, s) \left\{ f(s, x(s)) + \beta(s)((Tx)(s) - x(s)) \frac{\partial f(s, x(s))}{\partial x(s)} \right\} ds,
\]
whose form is patterned on the integral equation representation of (8),(9).

Since $x_0(t) = z(t) \in S(z, N_1)$, we shall show that if $x_m(t) \in S(z, N_1)$, then $(Tx_m)(t) \in S(z, N_1)$, Let $x_m(t) \in S(z, N_1)$, implies that $x_m(t) \in D$, and hence
from (8) and (13), we have

\[(Tx_m)(t) - z(t) = \int_{a}^{b} G(t, s) \left\{ f(s, x_m(s)) + \beta(s) [(Tx_m)(s) - x_m(s)] - f(s, z(s)) - \eta(s) \right\} ds.\]

Thus, from Lemma 3.2, (H1), (H2), (H3) that

\[|(Tx_m)^{(k)}(t) - z^{(k)}(t)| \leq \max_{a \leq t \leq b} |\eta(t)| + |f(t, x_m(t)) - f(t, z(t))| + L_2 |\beta(t)||x_m(t) - z(t)| + L_2 |\beta|(Tx_m(t) - x_m(t))\]

\[\times C_{\alpha, k}(b - a)^{\alpha - k}.\]

And hence

\[\|(Tx_m) - z\| \leq (\varepsilon + L_1 \|x_m - z\| + L_2 \beta \|(Tx_m) - x_m\|)C_{\alpha, 0}(b - a)^{\alpha}\]

\[\leq (\varepsilon + L_1 \|x_m - z\| + L_2 \beta \|(Tx_m) - z\| + \|x_m - z\|)C_{\alpha, 0}(b - a)^{\alpha}.\]

Furthermore, we have

\[\|(Tx_m) - z\| \leq (1 - \beta \theta)^{-1}(\varepsilon C_{\alpha, 0}(b - a)^{\alpha} + \theta \beta \|x_m - z\|).\]

Therefore, \(\|(Tx_m) - z\| \leq N_1\) from the definition of \(N_1\) and \(\|x_m - z\| < N_1\), i.e. \(x_{m+1}(t) = (Tx_m)(t) \in S(z, N_1)\). Known by the mathematical induction, the part (i) follows.

Next to prove part (ii). From (13) and \(x_{m+1} = Tx_m\), we have

\[x_{m+1}(t) - x_m(t) = \int_{a}^{b} G(t, s) \left\{ f(s, x_m(s)) - f(s, x_{m-1}(s)) + \beta(s) \left[ (x_{m+1}(s) - x_m(s)) - \frac{\partial f(s, x_m(s))}{\partial x_m(s)} \right] - (x_m(s) - x_{m-1}(s)) \frac{\partial f(s, x_{m-1}(s))}{\partial x_{m-1}(s)} \right\} ds.\]

Thus, from Lemma 3.2, (H1) and part (i), we get

\[|x_{m+1}^{(k)}(t) - x_m^{(k)}(t)| \leq C_{\alpha, k}(b - a)^{\alpha - k} \max_{a \leq t \leq b} \left\{ L_1 |x_m(t) - x_{m-1}(t)| + |\beta(t)||L_2 \left[ |x_{m+1}(t) - x_m(t)| + |x_m(t) - x_{m-1}(t)| \right] \right\}.\]

Hence

\[\|x_{m+1} - x_m\| \leq C_{\alpha, 0}(b - a)^{\alpha} \left\{ (L_1 + L_2 \beta) \|x_m - x_{m-1}\| + L_2 \beta \|x_{m+1} - x_m\| \right\}\]

\[\leq \theta \|x_m - x_{m-1}\| + \beta \theta \|x_{m+1} - x_m\|.\]

Furthermore, we obtain

\[\|x_{m+1} - x_m\| \leq \frac{\theta \beta}{1 - \beta \theta} \|x_m - x_{m-1}\|.\]

By induction, we have

\[\|x_{m+1} - x_m\| \leq \left( \frac{\theta \beta}{1 - \beta \theta} \right)^m \|x_1 - x_0\|. \quad (14)\]
Because of $\vartheta + \beta \vartheta < 1$, the $\{x_m(t)\}$ is a Cauchy sequence, and converges to $z^*(t) \in S(z, N_1)$. It can be easily verified that $z^*(t)$ is the unique solution of (1).

To prove (11), we can follow from (14) and the triangle inequality

$$\|x_{m+p} - x_m\| \leq \|x_{m+p} - x_{m+p-1}\| + \cdots + \|x_{m+1} - x_m\|$$

$$\leq \left\{ \left( \frac{\vartheta}{1 - \beta \vartheta} \right)^{m+p-1} + \cdots + \left( \frac{\vartheta}{1 - \beta \vartheta} \right)^m \right\} \|x_1 - x_0\|$$

$$\leq \left( \frac{\vartheta}{1 - \beta \vartheta} \right)^m \left( \frac{1 - (\frac{\vartheta}{1 - \beta \vartheta})^p}{1 - \frac{\vartheta}{1 - \beta \vartheta}} \right) \|x_1 - x_0\|,$$

and now letting $p \to \infty$, then, $\lim_{p \to \infty} x_{m+p}(t) = z^*(t)$, hence we get (11)

$$\|x_m - z^*\| \leq \left( \frac{\vartheta}{1 - \beta \vartheta} \right)^m \left( \frac{1 - \beta \vartheta - \vartheta}{1 - \beta \vartheta} \right)^{-1} \|x_1 - x_0\|.$$

Next, from (8), (9) and (10), we have

$$x_1(t) - x_0(t) = \int_a^b G(t, s)\left\{-\eta(s) + \beta(s)(x_1(s) - x_0(s)) \frac{\partial f(s, x_0(s))}{\partial x_0(s)} \right\} ds,$$

and hence, proceeding as above, we get

$$\|x_1 - x_0\| \leq (1 - \beta \vartheta)^{-1} \varepsilon C_{\alpha,0}(b - a)^\alpha. \quad (15)$$

Using (15) in (11), we can obtain the inequality (12). Theorem 3.4 is completed. \(\square\)

4. Example. Consider the following boundary value problem:

$$\begin{cases}
D_0^a x(t) = \frac{1}{10} t + \frac{1}{5} x(t), & 0 \leq t \leq 1, \quad n - 1 < \alpha \leq n \\
x^{(k)}(0) = 0, & k = 0, 1, 2, \cdots, n - 2; \quad x^{(n-1)}(1) = 0.
\end{cases} \quad (16)$$

Taking $z(t) \equiv 0$ so that $\varepsilon = \max_{0 \leq t \leq 1} | - \frac{1}{10} | = \frac{1}{10}$. Therefore $z(t) \equiv 0$ is the approximate solution of BVP (16). Defining $D = \{x(t) : |x(t)| \leq N\}$, $f(t, x(t))$ is continuously differentiable with respect to $x(t)$, and there exist nonnegative constants $l_1 = \frac{1}{5}, l_2 = \frac{1}{5}$, such that for all $(t, x(t)) \in [0, 1] \times D$, $\left| \frac{\partial f(t, x(t))}{\partial x(t)} \right| \leq l_2 = \frac{1}{5}$.

Taking $\beta(t) = 0$ then $\beta = \max_{0 \leq t \leq 1} ||\beta(t)|| = 0$. Let $L_1 = \frac{1}{5} N$, $L_2 = \frac{1}{5} N$. Thus, the conditions of Theorem 3.4 are satisfied provided that

$$\frac{1}{5} C_{\alpha,0} N < 1 \quad \text{and} \quad N_1 = \frac{1}{10} \left( 1 - \frac{1}{5} C_{\alpha,0} N \right)^{-1} C_{\alpha,0} \leq N, \quad (17)$$

where $C_{\alpha,0} = \left( \frac{1}{\Gamma(n+1)} + \frac{1}{(n-1)! \Gamma(\alpha - n + 2)} \right)$. Taking $\alpha = \frac{3}{2}$, then $n = 2$, $\alpha \in (1, 2]$, and $0.2036646499... \leq N \leq 2.4550161264...$.

From (9) and (10), we have

$$\varepsilon D_0^a x_{m+1}(t) = \frac{1}{10} t + \frac{1}{5} x_m(t), \quad 0 \leq t \leq 1, \quad \alpha = \frac{3}{2} \quad (18)$$

$$x_{m+1}(0) = 0, \quad x_{m+1}'(1) = 0, \quad m = 0, 1, 2, \cdots. \quad (19)$$
Since $x_0(t) = z(t) \equiv 0$, then

$$cD^\alpha_{0+} x_1(t) = \frac{1}{10} t, \quad 0 \leq t \leq 1, \quad \alpha = \frac{3}{2}$$

$$x_1(0) = 0, \quad x'_1(1) = 0.$$ 

From Lemma 3.1 and (7), we have

$$x_1(t) = \frac{1}{10\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{10\Gamma(\alpha+1)} t^2 - \frac{2}{15\sqrt{\pi}} t.$$ 

Similarly, we have $x_2(t), x_3(t), \ldots, x_{10}(t)$, if we take $N = 0.5000000000$, then $N_1 = 0.2316229456 < N$. Thus, Theorem 3.4 ensures the BVP (16) has a unique solution $z^*(t)$, and the iterative sequence $\{x_m(t)\}$ generated by the (18) and (19) remains in $S(z, N_1)$. Meanwhile, a bound on the error is given by

$$\|x_m - z^*\| \leq (0.1880631945)^m \times (0.2316229456),$$

and

$$z^*(t) = \lim_{m \to \infty} x_m(t).$$

The numerical results of $x_1(t), x_2(t), x_3(t), \ldots, x_{10}(t)$ for (18) and $z^*(t)$ for (21) by using the quasilinear iterates methods of Theorem 3.4 are presented in Table 1.

Table 1: The numerical results

| $t$    | $x_1(t)$ | $x_2(t)$ | $x_3(t)$ | $x_4(t)$ | ... | $x_9(t)$ | $x_{10}(t)$ | $z^*(t)$ |
|--------|----------|----------|----------|----------|-----|----------|-------------|----------|
| 0.0000 | 0.0000   | 0.0000   | 0.0000   | 0.0000   | ... | 0.0000   | 0.0000      | 0.0000   |
| 0.1111 | -0.0083  | -0.0072  | -0.0075  | -0.0074  | ... | -0.0074 | -0.0074     | -0.0074 |
| 0.2222 | -0.0164  | -0.0146  | -0.0145  | -0.0145  | ... | -0.0145 | -0.0145     | -0.0145 |
| 0.3333 | -0.0241  | -0.0211  | -0.0210  | -0.0210  | ... | -0.0210 | -0.0210     | -0.0210 |
| 0.4444 | -0.0315  | -0.0269  | -0.0267  | -0.0267  | ... | -0.0267 | -0.0267     | -0.0267 |
| 0.5556 | -0.0383  | -0.0318  | -0.0317  | -0.0317  | ... | -0.0317 | -0.0317     | -0.0317 |
| 0.6667 | -0.0447  | -0.0344  | -0.0359  | -0.0357  | ... | -0.0357 | -0.0357     | -0.0357 |
| 0.7778 | -0.0505  | -0.0322  | -0.0389  | -0.0387  | ... | -0.0387 | -0.0387     | -0.0387 |
| 0.8889 | -0.0557  | -0.0390  | -0.0407  | -0.0406  | ... | -0.0406 | -0.0406     | -0.0406 |
| 1.0000 | -0.0602  | -0.0396  | -0.0414  | -0.0412  | ... | -0.0412 | -0.0412     | -0.0412 |

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REFERENCES

[1] R. P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Georgian Mathe. J.*, 16 (2009), 401–411.
[2] R. P. Agarwal, Y. Zhou and Y.-Y. He, Existence of fractional neutral functional differential equations, *Comput. Math. Appl.*, 59 (2010), 1095–1100.
[3] M. Benchohra, S. Hamani and S. K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.*, 3 (2008), 1–12.
[4] M. Benchohra, S. Hamani and S. K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.*, 71 (2009), 2391–2396.
[5] J. Brzdek and N. Eghbali, On approximate solutions of some delayed fractional differential equations, *Appl. Math. Lett.*, 54 (2016), 31–35.
[6] J.-W. Deng, L.-J. Zhao and Y.-J. Wu, Efficient algorithms for solving the fractional ordinary differential equations, *Appl. Math. Comput.*, 269 (2015), 196–216.
[7] A. M. A. El-Sayed, Fractional differential equations, *Kyungpook Math. J.*, 28 (1988), 119–122.
[8] J.-H. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol., 15 (1999), 86–90.
[9] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[10] V. Lakshmikantham and A. S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Appl. Math. Lett., 21 (2008), 828-834.
[11] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal., 69 (2008), 2677–2682.
[12] V. Lakshmikantham and J. V. Devi, Theory of fractional differential equations in a Banach space, Eur. J. Pure Appl. Math., 1 (2008), 38–45.
[13] C. Li, Q. Yi and A. Chen, Finite difference methods with non-uniform meshes for nonlinear fractional differential equations, J. Comput. Phys., 316 (2016), 614–631.
[14] H. Liang and M. Stynes, Collocation methods for general Caputo two-point boundary value problems, J. Sci. Comput., 76 (2018), 390–425.
[15] P. Lyu, S. Vong and Z. Wang, A finite difference method for boundary value problems of a Caputo fractional differential equation, East. Asia. J. Appl. Math., 7 (2017), 752–766.
[16] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, INC., New York, 1993.
[17] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, London, 1974.
[18] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, vol. 198, Academic Press, New York, 1999.
[19] M. Stynes and J.-L. Gracia, A finite difference method for a two-point boundary value problem with a Caputo fractional derivative, IMA J. Numer. Anal., 35 (2015), 698–721.
[20] Y.-F. Sun and P.-G. Wang, Quasilinear iterative scheme for a fourth-order differential equation with retardation and anticipation, Appl. Math. Comput., 217 (2010), 3442–3452.
[21] Y.-F. Sun, Z. Zeng and J. Song, Existence and uniqueness for the boundary value problems of nonlinear fractional differential equation, Appl. Math., 8 (2017), 312–323.
[22] P.-G. Wang, S.-H. Tian and Y.-H. Wu, Monotone iterative method for first-order functional difference equations with nonlinear boundary value conditions, Appl. Math. Comput., 203 (2008), 266–272.
[23] P.-G. Wang, H.-X. Wu and Y.-H. Wa, Higher even-order convergence and coupled solutions for second-order boundary value problems on time scales, Comput. Math. Appl., 55 (2008), 1693–1705.

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