LYAPUNOV CRITERIA FOR THE FELLER-DYNKIN PROPERTY OF MARTINGALE PROBLEMS

DAVID CRIENS

Abstract. We give necessary and sufficient criteria for the Feller-Dynkin property of solutions to martingale problems in terms of Lyapunov functions. Moreover, we derive a Khasminskii-type integral test for the Feller-Dynkin property of multidimensional diffusions with random switching. For one dimensional switching diffusions with finite and state-independent switching, we provide an integral-test, which is equivalent to the Feller-Dynkin property.

1. Introduction

It is a classical question for a Markov process whether its transition semigroup is a self-map on the space of bounded continuous functions and on the space of continuous functions vanishing at infinity, respectively. If the first property holds we call the Markov process a Feller process and when the second property holds we call it a Feller-Dynkin process. Because Markov processes are usually defined by its infinitesimal description, it is particularly interesting to find criteria for these properties in terms of the generalized infinitesimal generator of the Markov process.

In this article we give such criteria for Markov processes defined via abstract martingale problems (MPs). Our contributions are two-fold. First, we show that the Feller-Dynkin property can be described by a Lyapunov-type criterion in the spirit of the classical Lyapunov-type criteria for explosion, recurrence and transience, see, e.g., [21, 31]. More precisely, we prove a sufficient condition for the Feller-Dynkin property, see Theorem 1 below, and a condition to reject the Feller-Dynkin property, see Theorem 2 below. Under an additional assumption on the input data, we extend the sufficient condition for the Feller-Dynkin property to be necessary, see Theorem 3 below. The necessity is for instance useful when one studies coupled processes, i.e. processes whose infinitesimal description is built from the infinitesimal description of other processes. We illustrate this in our applications. Moreover, we provide a technical condition for a reduction or an enlargement of the input data of a MP, see Proposition 4 below. A reduction helps to check the additional assumption of our necessary and sufficient criterion, while an enlargement simplifies finding Lyapunov functions for our sufficient conditions. We apply our criteria to derive conditions for the Feller-Dynkin property of multidimensional diffusions.

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D. Criens - Technical University of Munich, Center for Mathematics, Germany, daniel.criens@tum.de.

In the literature Feller-Dynkin processes are often also called Feller processes, see, e.g., [26, 29, 32]. Our terminology is borrowed from [34].
with random switching. In particular, we derive a Khasminskii-type integral test for the Feller-Dynkin property.

Our second contribution is a systematic study of the Feller-Dynkin property of switching diffusions with finite and state-independent switching. In other words, we consider a process \((Y_t, Z_t)_{t \geq 0}\), where \((Z_t)_{t \geq 0}\) is a continuous-time Markov chain with finite state space and \((Y_t)_{t \geq 0}\) solves the stochastic differential equation (SDE)

\[
    dY_t = b(Y_t, Z_t) \, dt + \sigma(Y_t, Z_t) \, dW_t,
\]

where \((W_t)_{t \geq 0}\) is a Brownian motion. One may think of the process \((Y_t, Z_t)_{t \geq 0}\) as a diffusion in a random environment given by the Markov chain \((Z_t)_{t \geq 0}\). The process \((Y_t)_{t \geq 0}\) has a natural relation to processes with fixed environments, i.e. solutions to the SDEs

\[
    dY^k_t = b(Y^k_t, k) \, dt + \sigma(Y^k_t, k) \, dW_t,
\]

where \(k\) is in the state space of \((Z_t)_{t \geq 0}\). When \((Y_t, Z_t)_{t \geq 0}\) is a Feller process and the SDEs \((1.1)\) have a strong existence property\(^2\), we show that \((Y_t, Z_t)_{t \geq 0}\) is a Feller-Dynkin process if and only if the processes in the fixed environments are Feller-Dynkin processes. Furthermore, using a weak convergence argument, we show that \((Y_t, Z_t)_{t \geq 0}\) is a Feller process whenever it exists uniquely and the coefficients are continuous. We also explain that the uniqueness of \((Y_t, Z_t)_{t \geq 0}\) is implied by the strong existence of the diffusions in the fixed environments. The last two observations are also true when we allow for countably many different environments. For the one dimensional case we deduce an equivalent integral-test for the Feller-Dynkin property of \((Y_t, Z_t)_{t \geq 0}\) and for multidimensional settings we give a Khasminskii-type integral test.

We end this introduction with comments on related literature. To the best of our current knowledge, Lyapunov-type criteria for the Feller-Dynkin property are only used in specific case studies and a systematic study as given in this article does not appear in the literature. For continuous-time Markov chains, explicit conditions for the Feller-Dynkin property can be found in [28, 32]. In [28] also a Lyapunov-type condition appears. Infinitesimal conditions for the Feller-Dynkin property of diffusions are given in [3]. In the context of jump-diffusions, linear growth conditions for the Feller-Dynkin property were recently proven in [24, 25]. The proofs include a Lyapunov-type argument based on Gronwall’s lemma. For switching diffusions the Feller and the strong Feller property\(^3\) are studied profoundly, see, e.g., [30, 36, 39, 40]. We think that our study of the Feller-Dynkin property for switching diffusions is the first of its kind. Also our continuity criterion for the Feller property in the state-independent case seems to be new.

The article is structured as follows. In Section 2 we explain our setup. In particular, in Section 2.2 we recall the different concepts for the Feller properties of

\[ dY_t = \sqrt{1 + |Y_t|^4} \, dW_t, \]

where \((W_t)_{t \geq 0}\) is a one-dimensional Brownian motion; the SDE \((1.2)\) has a unique in law \(\mathbb{R}\)-valued solution (see [20, Theorems 5.5.15, 5.5.29]), which has the strong Feller property (see [37, Corollary 10.1.4]), but is not a Feller-Dynkin process (see [3, Proposition 4.3]).
martingale problems. In Section [3] we discuss Lyapunov-type conditions for the Feller-Dynkin property in a general abstract setting and in Section [11] we discuss the case of switching diffusions.

We added an Appendix including a limit theorem and an existence result for state-independent switching diffusions, whose proofs are close to the proof of the continuity criterion for the Feller property. We think the results deserve a statement of their own. The existence theorem can be seen as a version of Skorokhod’s existence result for usual stochastic differential equations. We stress that it does not require any uniqueness or strong existence assumption.

2. The Feller Properties of Martingale Problems

2.1. The Setup. Let $S$ be a locally compact Hausdorff space with countable base (LCCB space), define $\Omega$ to be the space of all càdlàg functions $\mathbb{R}_+ \to S$ and let $(X_t)_{t \geq 0}$ be the coordinate process on $\Omega$, i.e. the process defined by $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in \mathbb{R}_+$. We set $\mathcal{F} \triangleq \sigma(X_t, t \in \mathbb{R}_+)$ and $\mathcal{F}_t \triangleq \bigcap_{s > t} \mathcal{F}_s^0$, where $\mathcal{F}_s^0 \triangleq \sigma(X_s, s \in [0, t])$. If not stated otherwise, all terms such as local martingale, supermartingale, etc. refer to $(\mathcal{F}_t)_{t \geq 0}$ as the underlying filtration. In general, we equip $\Omega$ with the Skorokhod topology (see [11, 17]). In this case, $\mathcal{F}$ is the Borel $\sigma$-field, see [11, Proposition 3.7.1].

We use standard notation for function spaces, i.e. for example we denote by $M(S)$ the set of Borel functions $S \to \mathbb{R}$, by $B(S)$ the set of bounded Borel functions $S \to \mathbb{R}$, by $C(S)$ the set of continuous functions $S \to \mathbb{R}$ and by $C_0(S)$ the space of continuous functions $S \to \mathbb{R}$ which are vanishing at infinity, etc. We take the following four objects as input data for our abstract MP:

(i) A set $D \subseteq C(S)$ of test functions.
(ii) A candidate $\mathcal{L} : D \to M(S)$ for an extended generator, which satisfies

$$
\int_0^t |\mathcal{L} f(X_s(\omega))| ds < \infty
$$

for all $t \in \mathbb{R}_+, \omega \in \Omega$ and $f \in D$.
(iii) A set $\Sigma \in \mathcal{F}$, which can be seen as the state space for the paths.
(iv) An initial law $\eta$, which is a Borel probability measure on $S$.

**Definition 1.** A probability measure $P$ on $(\Omega, \mathcal{F})$ is called a solution to the MP $(D, \mathcal{L}, \Sigma, \eta)$ if $P(\Sigma) = 1$, $P \circ X_0^{-1} = \eta$ and for all $f \in D$ the process

$$
(2.1) \quad f(X_t) - \int_0^t \mathcal{L} f(X_s) ds, \quad t \in \mathbb{R}_+,
$$

is a local $P$-martingale. When $\eta = \delta_x$ for some $x \in S$, then we write $(D, \mathcal{L}, \Sigma, x)$ instead of $(D, \mathcal{L}, \Sigma, \delta_x)$. We call the martingale problem well-posed if for all $x \in S$ a unique solution with initial law $\delta_x$ exists. Furthermore, we call the martingale problem completely well-posed when for all initial laws a unique solution exists.

**Example 1.** The following MP corresponds to the classical MP of Stroock and Varadhan [37]. Let $S \triangleq \mathbb{R}^d$, $D \triangleq C_0^2(\mathbb{R}^d)$, $\mathcal{L} f(x) \triangleq \langle \nabla f(x), b(x) \rangle + \frac{1}{2} \text{trace} (\nabla^2 f(x) a(x))$,

$$
(2.2) \quad \mathcal{L} f(x) \triangleq \langle \nabla f(x), b(x) \rangle + \frac{1}{2} \text{trace} (\nabla^2 f(x) a(x))
$$

where $\nabla$ denotes the gradient, $\nabla^2$ denotes the Hessian matrix and $b : \mathbb{R}^d \to \mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathbb{S}^d$ are locally bounded Borel functions with $\mathbb{S}^d$ denoting the set of all real symmetric non-negative definite $d \times d$ matrices, and $\Sigma \triangleq \{ \omega \in \Omega : t \mapsto$
£(t) is continuous}. We have Σ ⊆ F, because Σ is a closed subset of Ω, see [11, Problem 3.25].

Before we recall the definitions of the Markov property and the Feller properties of martingale problems, we comment on the relation of martingale problems with degenerated and general initial laws. In the setting of Example 1, it is known that well-posed martingale problems are even completely well-posed, see [18, Proposition 1], and that the existence of solutions for all degenerated initial laws alone implies the existence of a solution for an arbitrary initial law, see [18, Proposition 2]. In many cases these implications allow it to transfer the Markov property and a version of the Feller property from degenerated to arbitrary initial laws. Because of this observation, we provide a formal statement for abstract martingale problems.

**Proposition 1.** Suppose that D is countable, D ⊆ \( C_b(S) \) and that \( \mathcal{L}(D) \subseteq B_{\text{loc}}(S) \). Furthermore, let η be a Borel probability measure on S. If for all y ∈ S the MP \( (D, \mathcal{L}, \Sigma, y) \) has a solution \( P_y \), then also the MP \( (D, \mathcal{L}, \Sigma, \eta) \) has a solution. Moreover, if the family \( (P_y)_{y \in S} \) is unique, then \( y \mapsto P_y(A) \) is Borel for all \( A \in \mathcal{F} \) and \( \int P_y \eta(dy) \) is the unique solution to the MP \( (D, \mathcal{L}, \Sigma, \eta) \).

The proof of this proposition is similar to the diffusion case as discussed in [18] and given in Appendix B. It is often the case that the input data of a martingale problem can be reduced such that the prerequisites of Proposition 1 are met. We will comment on the reduction of the initial data in Proposition 4 below.

In the remaining of this article we impose the following assumption.

**Standing Assumption.** For all \( x \in S \) the MP \( (D, \mathcal{L}, \Sigma, x) \) has a solution \( P_x \).

2.2. The Markov, the Feller and the Feller-Dynkin Property of MPs. The family \( (P_x)_{x \in S} \) is called a Markov family or simply Markov if the map \( x \mapsto P_x(A) \) is Borel for all \( A \in \mathcal{F} \) and for all \( x \in S, t \in \mathbb{R}_+ \) and all \( G \in \mathcal{F} \) we have \( P_x \)-a.s.

\[
P_x(\theta_t^{-1}G|\mathcal{F}_t) = P_{X_t}(G),
\]

where \( \theta_t \omega(s) \triangleq \omega(t + s) \) denotes the shift operator. We call \( (2.3) \) the Markov property. The family \( (P_x)_{x \in S} \) is called a strong Markov family or simply strongly Markov if \( (P_x)_{x \in S} \) is Markov and for all \( x \in S \), all stopping times \( \xi \) and all \( G \in \mathcal{F} \) we have \( P_x \)-a.s. on \( \{ \xi < \infty \} \)

\[
P_x(\theta_\xi^{-1}G|\mathcal{F}_\xi) = P_{X_\xi}(G).
\]

The identity \( (2.4) \) is called the strong Markov property. Often families of solutions to MPs are strong Markov families.

**Proposition 2.** If \( D \) is countable, \( D \subseteq C_b(S), \mathcal{L}(D) \subseteq B_{\text{loc}}(S) \), \( (P_x)_{x \in S} \) is unique and \( \Sigma \subseteq \theta_\xi^{-1}\Sigma \) for all bounded stopping times \( \xi \), then \( (P_x)_{x \in S} \) is strongly Markov.

The proof is close to the diffusion case and given in Appendix B. If \( (P_x)_{x \in S} \) is not unique it might still be possible to pick a Markov family from the set of solutions. For instance, in the setting of Example 1, this is the case when \( a \) and \( b \) are bounded and continuous, see [37, Theorem 12.2.3]. When the family \( (P_x)_{x \in S} \) is (strongly) Markov, the (strong) Markov property transfers to \( P_\eta \triangleq \int P_x \eta(dx) \) for any Borel probability measure \( \eta \), i.e. in the case of the Markov property this means that for all \( t \in \mathbb{R}_+ \) and \( G \in \mathcal{F} \) we have \( P_\eta \)-a.s.

\[
P_\eta(\theta_t^{-1}G|\mathcal{F}_t) = P_{X_t}(G).
\]

[^1]: [11, Problem 3.25]
This follows easily from (2.3) by integration, i.e., for all \( A \in \mathcal{F}_t \) we have
\[
E^{P_\eta}[1_{\xi^{-1}_t 1_A}] = \int E^{P_x}[1_{\xi^{-1}_t 1_A}] \eta(dx) = \int E^{P_x}[P_{X_t}(G) 1_A] \eta(dx) = E^{P_\eta}[P_{X_t}(G) 1_A],
\]
which implies (2.3). In view of the Propositions \( 1 \) and \( 2 \) this means that in many cases where uniqueness holds solutions to martingale problems with arbitrary initial laws satisfy the (strong) Markov property.

In the case where \( (P_x)_{x \in S} \) is Markov, we can define a semigroup \( (T_t)_{t \geq 0} \) of positive contraction operators on \( B(S) \) via
\[
T_t f(x) \triangleq E_x[f(X_t)], \quad f \in B(S).
\]
It is obvious that \( T_t \) is a positive contraction, i.e., if \( f(S) \subseteq [0,1] \) then also \( T_t f(S) \subseteq [0,1] \), and the semigroup property follows easily from the Markov property (2.3).

If \( (P_x)_{x \in S} \) is Markov and
\[
T_t(C_1(S)) \subseteq C_0(S),
\]
we call \( (P_x)_{x \in S} \) a Feller family or simply Feller. The inclusion (2.6) is called the Feller property. The Feller property of the family \( (P_x)_{x \in S} \) has a natural relation to the continuity\(^4\) of \( x \mapsto P_x \) for which many conditions are known, see, e.g., [17, Theorem IX.4.8] for conditions in a jump diffusion setting. In the setup of Example \( 1 \) if \( (P_x)_{x \in S} \) is unique, \( (P_x)_{x \in S} \) is Feller whenever \( b \) and \( a \) are continuous. However, in the same setting, if \( (P_x)_{x \in S} \) is not unique, it might not be possible to choose a Feller family from the set of solutions, even if the coefficients are continuous and bounded, see [33, Exercise 12.4.2]. If \( x \mapsto P_x \) is continuous, then \( \eta \mapsto P_\eta \) is continuous\(^5\) too. This follows immediately from the definition of weak convergence, because the continuity of \( x \mapsto P_x \) implies that \( x \mapsto E^{P_x}[f] \) is bounded and continuous for any bounded and continuous \( f : \Omega \to \mathbb{R} \). In view of Proposition \( 1 \) this means that in many cases where uniqueness holds the solutions to martingale problems are continuous w.r.t. their initial laws.

We call \( (P_x)_{x \in S} \) a Feller-Dynkin family or simply Feller-Dynkin if it is a Feller family and
\[
T_t(C_0(S)) \subseteq C_0(S),
\]
the inclusion (2.7) is called the Feller-Dynkin property. From a semigroup point of view, the definition of a Feller-Dynkin semigroup also includes strong continuity in zero, see, e.g., [33, Definition III.2.1]. In our case, when \( (P_x)_{x \in S} \) is Feller-Dynkin, the semigroup \( (T_t)_{t \geq 0} \) is strongly continuous in zero due to the right-continuous paths of \( (X_t)_{t \geq 0} \), the dominated convergence theorem and [33, Proposition III.2.4]. Any Feller-Dynkin family is also strongly Markov, see, e.g., [19, Theorem 17.17]. Let us also comment on the issue of uniqueness. If \( (P_x)_{x \in S} \) is Feller-Dynkin and \( (L,D(L)) \) is its generator, i.e.,
\[
L f \triangleq \lim_{t \downarrow 0} \frac{T_t f - f}{t}
\]
\(^4\)i.e. \( P_{x_n} \to P_x \) weakly as \( n \to \infty \) whenever \( x_n \to x \) as \( n \to \infty \)
\(^5\)i.e. \( P_{\eta_n} \to P_\eta \) weakly as \( n \to \infty \) whenever \( \eta_n \to \eta \) weakly as \( n \to \infty \)
for $f \in \mathcal{D}(L)$, where
\begin{equation}
\mathcal{D}(L) \triangleq \left\{ f \in C_0(S) : \exists g \in C_0(S) \text{ such that } \lim_{t\searrow 0} \left\| \frac{T_t f - f}{t} - g \right\|_\infty = 0 \right\},
\end{equation}
then $P_x$ is the unique solution to the MP $(D, L, \Sigma, x)$, where $D$ is any subset of $\mathcal{D}(L)$ on which $L$ is uniquely determined, see [23, Theorem 4.10.3]. Consequently, conditions for the Feller-Dynkin property imply in some cases also uniqueness. However, usually the Feller-Dynkin property is established under a uniqueness assumption.

For an overview on different concepts of the Feller property from a semigroup point of view we refer to the first chapter in [7].

Most of the general conditions for the Feller-Dynkin property are formulated in terms of the semigroup $(T_t)_{t \geq 0}$ and therefore are often not easy to check, see, e.g., [7, Theorem 1.10] and the discussion below its proof. In the following section we give a criterion for the Feller-Dynkin property in terms of the existence of Lyapunov functions.

3. Lyapunov Criteria for the Feller-Dynkin Property

Lyapunov-type criteria often appear in the context of explosion, recurrence and transience of a Markov process, see, e.g., [21, 31]. In this section we present such criteria for the Feller-Dynkin property of $(P_x)_{x \in S}$. We start with a sufficient condition.

**Theorem 1.** Fix $t \in \mathbb{R}_+$ and suppose that the map $x \mapsto T_t f(x)$ is continuous for all $f \in C_0(S)$. Assume that for any compact set $K \subseteq S$ there exists a function $V : S \to \mathbb{R}_+$ with the following properties:

(i) $V \in D \cap C_0(S)$.

(ii) $V \triangleq \min_{x \in K} V(x) > 0$.

(iii) $LV \leq cV$ for a constant $c > 0$.

Then, $T_t(C_0(S)) \subseteq C_0(S)$. The function $V$ is called a Lyapunov function.

**Proof:** We first explain that it suffices to show that for all compact sets $K \subseteq S$ and all $\epsilon > 0$ there exists a compact set $O \subseteq S$ such that

$$P_x(X_t \in K) < \epsilon$$

for all $x \not\in O$. To see this, let $f \in C_0(S)$ and $\epsilon > 0$. By the definition of $C_0(S)$, there exists a compact set $K \subseteq S$ such that

$$|f(x)| < \frac{\epsilon}{2}$$

for all $x \not\in K$. By hypothesis, there exists a compact set $O \subseteq S$ such that

$$\sup_{y \in S} |f(y)| P_x(X_t \in K) < \frac{\epsilon}{2}$$

for all $x \not\in O$. Thus, for all $x \not\in O$ we have

$$|E_x[f(X_t)]| \leq E_x[(f(X_t))(1\{X_t \in K\} + 1\{X_t \not\in K\})] \leq \sup_{y \in S} |f(y)| P_x(X_t \in K) + \frac{\epsilon}{2} < \epsilon.$$ 

In other words, $T_t f \in C_0(S)$, i.e. the claim is proven.

Next, we verify that this condition holds under the hypothesis of the theorem. Fix $x \in S$ and a compact set $K \subseteq S$. Let $V$ be as described in the prerequisites.
of the theorem. The following lemma is an easy consequence of the integration by parts formula. For completeness, we give a proof after the proof of Theorem 1 is complete.

**Lemma 1.** If \( f \in C(S) \) is such that the process \((2.1)\) is a local martingale and \( c : \mathbb{R}_+ \to \mathbb{R} \) is an absolutely continuous function with Lebesgue density \( c' \), then the process

\[
(3.1) \quad f(X_t)c(t) - \int_0^t (f(X_s)c'(s) + c(s)\mathcal{L}f(X_s))ds, \quad t \in \mathbb{R}_+,
\]

is a local martingale.

Since \( V \in D \), the definition of the martingale problem and Lemma 1 imply that the process

\[
Y_s \equiv V(X_s)e^{-cs} - \int_0^s e^{-cr}(\mathcal{L}V(X_r) - cV(X_r))dr, \quad s \in \mathbb{R}_+,
\]

is a local \( P_x \)-martingale. Using (iii), we see that \( Y_s \geq V(X_s)e^{-cs} \geq 0 \) for all \( s \in \mathbb{R}_+ \). Thus, since non-negative local martingales are supermartingales due to Fatou’s lemma, \((Y_s)_{s \geq 0}\) is a \( P_x \)-supermartingale. Using Markov’s inequality, we obtain that

\[
P_x(X_t \in K) \leq P_x(V(X_t) \geq V) \leq \frac{1}{V} E_x[V(X_t)] \leq e^{ct} \frac{1}{V} E_x[Y_0] = e^{ct} \frac{1}{V} V(x).
\]

Take an \( \epsilon > 0 \). Since we assume that \( V \in C_0(S) \), there exists a compact set \( O \subseteq S \) such that

\[
V(y) < e^{-ct} V \epsilon
\]

for all \( y \not\in O \). We conclude that

\[
P_x(X_t \in K) \leq e^{ct} \frac{1}{V} V(x) < \epsilon
\]

when \( x \not\in O \). This finishes the proof. \( \square \)

**Proof of Lemma 1.** Denote the local martingale \((2.1)\) by \((M_t)_{t \geq 0}\). Moreover, set

\[
N_t \equiv \int_0^t \mathcal{L}f(X_s)ds, \quad t \in \mathbb{R}_+.
\]

As an absolutely continuous function, \( c \) is of finite variation over finite intervals. Thus, integration by parts yields that

\[
d[Mtc(t)] = c(t)dM_t + f(X_t)c'(t)dt - d[Ntc(t)] + c(t)\mathcal{L}f(X_t)dt.
\]

We see that the process \((3.1)\) equals the local martingale \((\int_0^t c(s)dM_s)_{t \geq 0}\). \( \square \)

Next, we give a condition for rejecting the Feller-Dynkin property.

**Theorem 2.** Suppose that \( S \) is not compact and that there exist compact sets \( K, C \subseteq S \), a constant \( \alpha > 0 \) and a bounded function \( U : S \to \mathbb{R}_+ \) with the following properties:

(i) \( U \in D \).
which implies that the process \((3.4)\) also a non-negative bounded \((3.4)\). In particular, due to \([34, \text{Lemma 67.1} \text{0}\) \((3.2)\), which has a terminal value due to the submartingale convergence theorem (see, e.g., \([19, \text{Lemma 6.8}\) \((3.2)\). We set \(\text{Proposition 3.}\) \((3.2)\) for the equality in \((3.2)\). We set \(\tau \triangleq \inf \{ t \in \mathbb{R}_+: X_t \in K \}, \)

which is well-known to be an \((\mathcal{F}_t^x)_{t \geq 0}\)-stopping time, see \([19, \text{Theorem 6.7}\) \((3.2)\). \(\text{Step 1:}\) The proof of the following observation is given after the proof of Theorem \([2]\) is complete.

**Proposition 3.** Assume that \((P_x)_{x \in S}\) is Feller-Dynkin and denote its generator by \((L, \mathcal{D}(L))\) (see \([2, \text{8} \text{ and 2.3}\) \((3.2)\)). For any compact set \(K \subseteq S\) and any \(\alpha > 0\) there exists a function \(V: S \to \mathbb{R}_+\) with the following properties:

\(\begin{align*}
(i) & \quad V \in \mathcal{D}(L), \\
(ii) & \quad \min_{y \in K} V(y) > 0, \\
(iii) & \quad LV \leq \alpha V.
\end{align*}\)

Let \(V\) be as in Proposition \([3]\) Due to Dynkin’s formula (see \([33, \text{Proposition VII.1.6}\) \([3]\) and Lemma \([1]\) \((3.2)\) the process

\[
Z^x_t \triangleq e^{-\alpha t}V(X_t) + \int_0^t e^{-\alpha s}(\alpha V(X_s) - LV(X_s))ds, \quad t \in \mathbb{R}_+,
\]

is a local \(P_x\)-martingale. In particular, because it is bounded (recall that \(\mathcal{D}(L) \subseteq C_0(S)\) and that \(Lf \in C_0(S)\) for all \(f \in \mathcal{D}(L)\), the process \((Z_t)_{t \geq 0}\) is even a true \(P_x\)-martingale. Consequently, for \(s < t\) we have \(P_x\)-a.s.

\[
E_x[e^{-\alpha t}V(X_t)|\mathcal{F}_s] \leq E_x[Z_t|\mathcal{F}_s] - \int_0^t e^{-\alpha r}(\alpha V(X_r) - LV(X_r))dr
\]

\[
\quad = Z_s - \int_0^s e^{-\alpha r}(\alpha V(X_r) - LV(X_r))dr = e^{-\alpha s}V(X_s),
\]

which implies that the process \((e^{-\alpha t}V(X_t))_{t \geq 0}\) is a non-negative \(P_x\)-supermartingale, which has a terminal value due to the submartingale convergence theorem (see, e.g., \([20, \text{Theorem 1.3.15}\) \([3]\)). In particular, due to \([34, \text{Lemma 67.10}\), \((e^{-\alpha t}V(X_t))_{t \geq 0}\) is also a non-negative bounded \((\mathcal{F}_t^x)_{t \geq 0}, P_x\) -supermartingale. Recalling that \(\tau\) is an

\[\text{We stress that a right-continuous \((\mathcal{F}_t^x)_{t \geq 0}\)-martingale is also an \((\mathcal{F}_t^x)_{t \geq 0}\)-martingale. This follows from the downward theorem (\([34, \text{Theorem II.51.1}\) \([3]\) as in the proof of \([34, \text{Lemma II.67.10}\) \([3]\).}
Because \( V(x) \geq E_x [e^{-\alpha t} V(X_t)] \)
\[
V(x) \geq E_x [e^{-\alpha t} V(X_t) \mathbf{1}\{\tau < \infty\}] \\
\geq E_x [e^{-\alpha t} \min_{y \in K} V(y)].
\] (3.5)

Here, we use the fact that \( X_t \in K \) on \( \{\tau < \infty\} \). This follows from the right-continuity of \( (X_t)_{t \geq 0} \). To see this, fix \( \omega \in \{\tau < \infty\} \). For any \( \epsilon > 0 \) we find a \( t \in [\tau(\omega), \tau(\omega) + \epsilon) \) such that \( X_t(\omega) \in K \). Consequently, we find a sequence \( (t_n)_{n \in \mathbb{N}} \) such that \( t_n \downarrow \tau(\omega) \) as \( n \to \infty \) and \( X_t(\omega) \in K \) for all \( n \in \mathbb{N} \). Because \( K \) is closed, the right-continuity of \( t \mapsto X_t(\omega) \) implies that \( X_\tau(\omega) \in K \).

**Step 2:** In the following all terms such as **local martingale**, **submartingale**, etc. refer to \( (\mathcal{F}_t^\omega)_{t \geq 0} \) as the underlying filtration. Lemma [1] and [34, Lemma 67.10] imply that the stopped process
\[
Y_t \triangleq e^{-\alpha (t \wedge \tau)} U(X_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\alpha s} (\alpha U(X_s) - \mathcal{L} U(X_s)) \, ds, \quad t \in \mathbb{R}_+,
\]
is a local \( P_x \)-martingale. Due to property (iv) of the function \( U \), we have
\[
Y_t \leq e^{-\alpha (t \wedge \tau)} U(X_{t \wedge \tau}) \leq \text{const.}
\]
for all \( t \in \mathbb{R}_+ \). Because local martingales bounded from above are submartingales,\footnote{Let \( (M_t)_{t \geq 0} \) be a local martingale bounded from above by a constant \( c \). Then, \( (c - M_t)_{t \geq 0} \) is a non-negative local martingale and hence a supermartingale by Fatou’s lemma. Consequently, also \( (-M_t)_{t \geq 0} \) is a supermartingale, which implies that \( (M_t)_{t \geq 0} \) is a submartingale.} the process \( (Y_t)_{t \geq 0} \) is a \( P_x \)-submartingale. Thus, it follows similar to (3.4) that the stopped process \( (e^{-\alpha (t \wedge \tau)} U(X_{t \wedge \tau}))_{t \geq 0} \) is a non-negative bounded \( P_x \)-submartingale, which has a terminal value \( e^{-\alpha \tau} U(X_\tau) \) by the submartingale convergence theorem.

We note that on \( \{\tau = \infty\} \) up to a null set we have \( e^{-\alpha \tau} U(X_\tau) = 0 \). To see this, let \( \omega \in \{\tau = \infty\} \) be such that \( \lim_{t \to \infty} e^{-at} U(X_t(\omega)) \) exists and is strictly positive. Then,
\[
\limsup_{t \to \infty} U(X_t(\omega)) = \limsup_{t \to \infty} e^{at} e^{-at} U(X_t(\omega))
\]
\[
= \limsup_{t \to \infty} \lim_{s \to \infty} e^{-\alpha s} U(X_s(\omega)) = \infty,
\]
which is a contradiction to the boundedness of the function \( U \). Another application of the optional stopping theorem yields that
\[
U(x) \leq E_x [e^{-\alpha \tau} U(X_\tau)]
\]
\[
= E_x [e^{-\alpha \tau} U(X_\tau) \mathbf{1}\{\tau < \infty\}]
\]
\[
\leq \max_{y \in K} U(y) E_x [e^{-\alpha \tau}].
\] (3.6)

**Step 3:** We deduce from (3.5) and (3.6) that for all \( x \notin C \)
\[
\inf_{y \in S \setminus C} U(y) \leq \frac{1}{2} \max_{y \in K} U(y) \leq \frac{V(x)}{\min_{y \in K} V(y)}.
\]
Because \( V \in \mathcal{D}(L) \subseteq C_0(S) \), we find a compact set \( G \subset S \) such that for all \( x \notin G \)
\[
V(x) \leq \frac{1}{2} \inf_{y \in S \setminus C} U(y) \min_{y \in K} V(y) > 0,
\]
which implies that for all \( x \not\in C \cup G \neq S \)
\[
0 < \frac{\inf_{y \in S \setminus C} U(y)}{\max_{y \in K} U(y)} \leq E_x [e^{-\alpha \tau}] \leq \frac{1}{2} \frac{\inf_{y \in S \setminus C} U(y)}{\max_{y \in K} U(y)}.
\]
This is a contradiction and the proof of Theorem 2 is complete. \( \square \)

**Proof of Proposition 3.** We construct \( V \) via the \( \alpha \)-potential operator of \((T_t)_{t \geq 0}\), i.e. the operator \( U_\alpha : C_0(S) \to C_0(S) \) defined by
\[
U_\alpha f(x) \triangleq \int_0^\infty e^{-\alpha s} T_s f(x) ds, \quad f \in C_0(S), \ x \in S.
\]
Take a function \( f \in C_0(S) \) with \( 0 \leq f \leq 1 \) and \( f \equiv 1 \) on \( K \). Such a function exists due to Urysohn’s lemma for locally compact spaces, see, e.g., [9, Proposition 7.1.9]. We set \( V \triangleq U_\alpha f \). It is well-known that \( V = U_\alpha f \in D(L) \) and
\[
(3.7) \quad (\alpha 1 - L) V = (\alpha 1 - L) U_\alpha f = f \geq 0,
\]
see, e.g., [20, Proposition 6.12]. Thus, \( V \) has the first and the third property. It remains to show that \( V \) has the second property. Since \( U_\alpha \) is positivity preserving we have \( V \geq 0 \). Assume that \( \min_{y \in K} V(y) = 0 \). Then, there exists an \( x_0 \in K \) such that \( V(x_0) = 0 \) and we obtain
\[
LV(x_0) = \lim_{t \downarrow 0} \frac{1}{t} (T_t V(x_0) - V(x_0)) = \lim_{t \downarrow 0} \frac{1}{t} E_{x_0} [V(X_t)] \geq 0.
\]
Therefore, we conclude from (3.7) that
\[
\alpha V(x_0) = f(x_0) + LV(x_0) = 1 + LV(x_0) \geq 1.
\]
This is a contradiction and it follows that \( V \) has also the second property. \( \square \)

**Remark 1.** The arguments from the proofs of the Theorems 1 and 2 imply a version of [3, Proposition 3.1] beyond a diffusion setting. More precisely, when \((P_x)_{x \in S}\) is Feller, the following are equivalent:

(i) \((P_x)_{x \in S}\) is Feller-Dynkin.

(ii) For all compact sets \( K \subset S \) and all constants \( \alpha > 0 \) the function \( x \mapsto E_x [e^{-\alpha \tau}] \) vanishes at infinity, where \( \tau \) is defined in (3.3).

(iii) For all compact sets \( K \subset S \) and all constants \( \alpha > 0 \) the function \( x \mapsto P_x (\tau \leq \alpha) \) vanishes at infinity, where \( \tau \) is defined in (3.3).

The implication (i) \( \Rightarrow \) (ii) is shown in the proof of Theorem 2. The implication (ii) \( \Rightarrow \) (iii) follows from the inequality
\[
P_x (\tau \leq \alpha) \leq e^{\alpha^2} E_x [e^{-\alpha \tau} 1_{\{\tau \leq \alpha\}}] \leq e^{\alpha^2} E_x [e^{-\alpha \tau}],
\]
and the final implication (iii) \( \Rightarrow \) (i) follows from the fact that
\[
P_x (X_0 \in K) \leq P_x (\tau \leq \alpha)
\]
and the argument from the proof of Theorem 1.

In some cases Theorem 1 and Proposition 3 can be combined to one sufficient and necessary Lyapunov-type condition for the Feller-Dynkin property:

**Example 2.** Suppose that \( S \) is a countable discrete space and let \( Q = (q_{xy})_{x,y \in S} \) be a conservative \( Q \)-matrix, i.e. \( q_{xy} \in \mathbb{R}_+ \) for all \( x \neq y \) and
\[
-q_{xx} = \sum_{y \neq x} q_{xy} < \infty.
\]
Set $\Sigma \triangleq \Omega$, $D \triangleq \{ f \in C_0(S) : Qf \in C_0(S) \}$, and $\mathcal{L} \triangleq Q$, where $Qf$ is defined by
\begin{equation}
Qf(x) = \sum_{y \in S} q_{xy} f(y).
\end{equation}
We stress that the r.h.s. of (3.8) converges absolutely whenever $f \in C_0(S)$. If $(P_x)_{x \in S}$ is Feller-Dynkin, the corresponding generator $(L, D(L))$ is given by $(\mathcal{L}, D)$, see [22, Theorem 5]. Thus, when $(P_x)_{x \in S}$ is Markov (or, equivalently, Feller, because of the discrete topology), Theorem 1 and Proposition 3 imply that the following are equivalent:

(i) $(P_x)_{x \in S}$ is Feller-Dynkin.

(ii) For each $x \in S$ there exists a function $V: S \to \mathbb{R}_+$ such that $V \in D$, $V(x) > 0$, $QV \leq cV$ for a constant $c > 0$.

This observation is also contained in [28, Theorem 3.2].

Under reasonable assumptions on the input data, we can deduce a related equivalence for more general martingale problems. To formulate it we need further terminology. By an extension of the input data $(D, \mathcal{L})$ we mean a pair $(D', \mathcal{L}')$ consisting of $D' \subseteq C(S)$ and $\mathcal{L}' : D' \to M(S)$ such that $D \subseteq D'$, $\mathcal{L}' = \mathcal{L}$ on $D$,
\begin{equation}
\int_0^t |\mathcal{L}' f(X_s(\omega))| ds < \infty
\end{equation}
for all $t \in \mathbb{R}_+, \omega \in \Omega$ and $f \in D'$, and such that for all $x \in S$ the probability measure $P_x$ solves the MP $(D', \mathcal{L}', \Sigma, x)$.

**Theorem 3.** Suppose that for all $f \in D \cap C_0(S)$ we have $\mathcal{L}f \in C_0(S)$ and that $(P_x)_{x \in S}$ is Feller. Then, the following are equivalent:

(i) $(P_x)_{x \in S}$ is Feller-Dynkin.

(ii) The input data $(D, \mathcal{L})$ can be extended such that for any compact subset of $S$ a Lyapunov function in the sense of Theorem 1 exists.

**Proof:** The implication (ii) $\Rightarrow$ (i) is due to Theorem 1. Assume that (i) holds, let $(L, D(L))$ be the generator of $(P_x)_{x \in S}$ and set $D' \triangleq D \cup D(L)$ and
\begin{equation}
\mathcal{L}' f \triangleq \begin{cases}
\mathcal{L} f, & f \in D, \\
L f, & f \in D(L).
\end{cases}
\end{equation}
Of course, we have to explain that $\mathcal{L}'$ is well-defined, i.e. that $Lf = \mathcal{L} f$ for all $f \in D \cap D(L)$. Because $\mathcal{L} f \in C_0(S)$ for any $f \in D \cap D(L)$ by assumption, the process
\begin{equation}
f(X_t) - \int_0^t \mathcal{L} f(X_s) ds, \quad t \in \mathbb{R}_+,
\end{equation}
is a $P_x$-martingale for all $x \in S$, because it is a bounded (on finite time intervals) local $P_x$-martingale. Consequently, [33, Proposition VII.1.7] implies $\mathcal{L} f = L f$. Due to Dynkin’s formula, $P_x$ solves also the MP $(D', \mathcal{L}', \Sigma, x)$ for all $x \in S$. In other words, $(D', \mathcal{L}')$ is an extension of $(D, \mathcal{L})$. Now, (ii) follows from Proposition 3.

Let us comment on the prerequisites of the previous theorem. Even if the coefficients are continuous, in the case of Example 1 it is not always true that $\mathcal{L} f \in C_0(\mathbb{R}^d)$ whenever $f \in D \cap C_0(\mathbb{R}^d) = C^2_b(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. However, if we
could take $D = C_2^2(\mathbb{R}^d)$ instead of $D = C_0^2(\mathbb{R}^d)$, then $\mathcal{L}f \in C_0(\mathbb{R}^d)$ holds for all $f \in D = D \cap C_0(\mathbb{R}^d)$ when the coefficients are continuous. In other words, when we could reduce the input data, we would get an equivalent characterization of the Feller-Dynkin property from Theorem \ref{thm:fdp_characterization}. Next, we explain that a reduction of the input data is often possible.

A sequence $(f_n)_{n \in \mathbb{N}} \subset M(S)$ is said to converge locally bounded pointwise to a function $f \in M(S)$ if

(i) $\sup_{n \in \mathbb{N}} \sup_{y \in K} |f_n(y)| < \infty$ for all compact sets $K \subseteq S$;

(ii) $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in S$.

Moreover, we say that $(f_n)_{n \in \mathbb{N}} \subset B(S)$ converges bounded pointwise to $f \in M(S)$ if $f_n \to f$ as $n \to \infty$ locally bounded pointwise and $\sup_{n \in \mathbb{N}} \|f_n\| \to \infty$.

For a set $A \subseteq C(S) \times M(S)$ we denote by $\text{cl}(A)$ the set of all $(f,g) \in C(S) \times M(S)$ for which there exist sequences $(f_n,g_n)_{n \in \mathbb{N}} \subset A$ such that $f_n \to f$ as $n \to \infty$ bounded pointwise and $g_n \to g$ as $n \to \infty$ locally bounded pointwise. The following proposition can be viewed as an extension of \cite[Proposition 4.3.1]{11}, which allows a local convergence in the second variable.

Proposition 4. Let $D_1, D_2 \subseteq C(S), \mathcal{L}_1: D_1 \to M(S)$ and $\mathcal{L}_2: D_2 \to M(S)$ be such that

$$\int_0^t \left( |\mathcal{L}_1 f(X_s(\omega))| + |\mathcal{L}_2 g(X_s(\omega))| \right) ds < \infty$$

for all $t \in \mathbb{R}_+, \omega \in \Omega, f \in D_1$ and $g \in D_2$. Suppose that

$$(3.9) \quad \{ (f, \mathcal{L}_2 f): f \in D_2 \} \subseteq \text{cl}(\{(f, \mathcal{L}_1 f): f \in D_1\}).$$

If $P$ is a solution to the MP $(D_1, \mathcal{L}_1, \Sigma, x)$, then $P$ is also a solution to the MP $(D_2, \mathcal{L}_2, \Sigma, x)$.

Proof: Due to \cite[Proposition 6.2.10]{15}, there exists a sequence $(K_n)_{n \in \mathbb{N}} \subset S$ of compact sets such that $K_n \subset \text{int}(K_{n+1})$ and $\bigcup_{n \in \mathbb{N}} K_n = S$. Now, define

$$(3.10) \quad \tau_n \equiv \inf \{ t \geq 0: X_t \notin \text{int}(K_n) \text{ or } X_{t-} \notin \text{int}(K_n) \}, \quad n \in \mathbb{N}.$$ 

It is well-known that $\tau_n$ is a stopping time, see \cite[Proposition 2.1.5]{11}, and that $\tau_n \not\to \infty$ as $n \to \infty$, see \cite[Problem 4.27]{11}. Take $f \in D_2$. Due to \cite[Proposition 6.2.10]{15} there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset D_1$ such that $f_n \to f$ as $n \to \infty$ bounded pointwise and $\mathcal{L}_1 f_n \to \mathcal{L}_2 f$ as $n \to \infty$ locally bounded pointwise. For $g \in D^1$ we set

$$M^{g,i}_t \equiv g(X_t) - \int_0^t \mathcal{L}_1 g(X_s) ds, \quad t \in \mathbb{R}_+.$$ 

Since the class of local martingales is stable under stopping, the process $(M^{f_n,1}_{t \wedge \tau_m})_{t \geq 0}$ is a local $P$-martingale. Furthermore,

$$\sup_{s \in [0,t]} \left| M^{f_n,1}_{s \wedge \tau_m} \right| \leq \sup_{k \in \mathbb{N}} \|f_k\| + t \sup_{k \in \mathbb{N}} \sup_{y \in K_m} |\mathcal{L}_1 f_k(y)| < \infty,$$

by the definition of (local) bounded pointwise convergence. Consequently, $(M^{f_n,1}_{t \wedge \tau_m})_{t \geq 0}$ is a $P$-martingale by the dominated convergence theorem. Since

$$\sup_{s \in [0,t \wedge \tau_m]} |\mathcal{L}_1 f_n(X_{s-})| \leq \sup_{k \in \mathbb{N}} \sup_{y \in K_m} |\mathcal{L}_1 f_k(y)| < \infty,$$

the dominated convergence theorem also yields that for any $t \in \mathbb{R}_+$ we have $\omega$-wise $M^{f_n,1}_{t \wedge \tau_m} \to M^{f,2}_{t \wedge \tau_m}$ as $n \to \infty$. Thus, for all $s < t$, applying the dominated
convergence theorem a third time yields that $M_{t∧m}^{f_1,2}$, $M_{t∧m}^{f_2,3} \in L^1(P)$ and that for all $G \in \mathcal{F}_s$
\[E^P[M_{t∧m}^{f_1,2}1_G] = \lim_{n→∞} E^P[M_{t∧m}^{f_n,1}1_G] = \lim_{n→∞} E^P[M_{t∧m}^{f_n,1}1_G] = E^P[M_{t∧m}^{f_2,3}1_G].\]

In other words, the stopped process $(M_{t∧m}^{f_2,3})_{t≥0}$ is a $P$-martingale. Because $\tau_m ∧ n → ∞$ as $m → ∞$, we conclude that the process $(M_{t∧m}^{f_2,3})_{t≥0}$ is a local $P$-martingale, i.e. $P$ solves the MP $(D_2, \mathcal{L}_2, \Sigma, x)$ as claimed.

The previous proposition can also be used to verify the prerequisites of the Propositions [1] and [2].

**Example 1 (continued).** We have
\[(\{ (f, \mathcal{L}f): f \in C^2_b(\mathbb{R}^d) \}) \subseteq cl(\{ (f, \mathcal{L}f): f \in C^2_b(\mathbb{R}^d) \}).\]

To see this, let $g_n \in C^2_b(\mathbb{R}^d)$ be such that $0 ≤ g_n ≤ 1$ and $g_n \equiv 1$ on $\{ x \in \mathbb{R}^d : \|x\| ≤ n \}$. For any $f \in C^2_b(\mathbb{R}^d)$ it is easy to verify that $f_n \defeq f g_n \in C^2_b(\mathbb{R}^d)$, $f_n → f$ as $n → ∞$ bounded pointwise and $\mathcal{L}f_n → \mathcal{L}f$ as $n → ∞$ locally bounded pointwise. Consequently, $P$ solves the MP $(C^2_b(\mathbb{R}^d), \mathcal{L}, \Sigma, x)$ if and only if it solves
\[the\ MP\ (C^2_b(\mathbb{R}^d), \mathcal{L}, \Sigma, x).\] This fact is of course well-known, see, e.g., [20, Proposition 5.4.11].

In summary, if the family $(P_x)_{x \in \mathbb{R}^d}$ is unique and $b$ and $a$ are even continuous, then $(P_x)_{x \in \mathbb{R}^d}$ is Feller (see [37, Corollary 11.1.5]) and Theorem [3] implies that the following are equivalent:

(i) $(P_x)_{x \in \mathbb{R}^d}$ is Feller-Dynkin.
(ii) The input data $(C^2_b(\mathbb{R}^d), \mathcal{L})$ can be extended such that for all compact subsets of $\mathbb{R}^d$ a Lyapunov function (in the sense of Theorem [1]) exists.

The larger the set $D$, the easier it is to find a suitable Lyapunov function and to apply the Theorems [1] and [2]. Thus, when we have applications in mind, we would like to choose $D$ as large as possible. We stress that Proposition [3] also works in this direction, i.e. it gives a condition such that $D$ can be enlarged.

Let us summarize the observation from this section. We have seen a sufficient condition for the Feller-Dynkin property (Theorem [1]) and a sufficient condition to reject the Feller-Dynkin property (Theorem [2]). Moreover, we gave one sufficient and necessary condition under some additional assumptions (Theorem [3]) and discussed its prerequisites (Proposition [4]).

4. The Feller-Dynkin Property of Switching Diffusions

In this section we derive Khasminskii-type integral tests for the Feller-Dynkin property of diffusions with random switching. In particular, for the one dimensional finite state-independent case we present equivalent integral-type conditions for the Feller-Dynkin property.

Before we start our program, we fix some notation. Let $\mathcal{S}_d$ be a countable discrete space and let $S \defeq \mathbb{R}^d × \mathcal{S}_d$ equipped with the product topology. Take the following coefficients:

(i) $b: S → \mathbb{R}^d$ being Borel and locally bounded.
(ii) $a: S → \mathbb{S}^d$ being Borel and locally bounded.
(iii) For each $x ∈ \mathbb{R}^d$, let $Q(x) = (q_{ij}(x))_{i,j ∈ \mathcal{S}_d}$ be a conservative $Q$-matrix, such that the map $x ↦ Q(x)$ is Borel.
4.1. Conditions for the Feller-Dynkin Property. For $i, j \in S_d$, we set

$$q_{ij} \triangleq \begin{cases} \sup_{x \in \mathbb{R}^q} q_{ij}(x), & i \neq j, \\ -\sum_{k \neq i} q_{ik}, & i = j. \end{cases}$$

In this section, we impose the following standing assumption.

**Standing Assumption.** For all $i \in S_d$ we have $|q_{ii}| < \infty$ and

$$\sup_{n \in S_d} \sup_{x \in \mathbb{R}^d} |q_{nn}(x) - \bar{q}_{nn}| < \infty. \tag{4.1}$$

Set $\bar{Q} \triangleq (\bar{q}_{ij})_{i,j \in S_d}$ and note that $\bar{Q}$ is a conservative $Q$-matrix. Denote

$$C \triangleq \{ f \in C_0(S_d) : \bar{Q} f \in C_0(S_d) \}$$

and

$$\Sigma_d \triangleq \{ \omega : \mathbb{R}_+ \to S_d : t \mapsto \omega(t) \text{ is càdlàg} \}.$$

We also impose the following standing assumption.

**Standing Assumption.** For all $x \in S_d$ the MP $(C, \bar{Q}, \Sigma_d, x)$ has a unique solution $P^d_x$ such that the family $(P^d_x)_{x \in S_d}$ is Feller-Dynkin. Here, the state space for the MP is assumed to be $S_d$ (i.e. $\Omega = \Sigma_d$).

If $|S_d| < \infty$ this standing assumption holds. In the following remark we collect also some conditions when the previous standing assumption holds for the case $|S_d| = \infty$.

**Remark 2.**

(i) Conditions for the existence of $(P^d_x)_{x \in S_d}$ can be found in [2, Corollary 2.2.5, Theorem 2.2.27] and [8, Theorem 16]. If, in addition to one of these conditions, we have

$$\forall \lambda > 0, k \in S_d, \quad \{ y \in l_1 : y(\lambda I - \bar{Q}) = 0 \} = \{ 0 \} \text{ and } \bar{q}_{kk} \in C_0(S_d), \tag{4.2}$$

then $(P^d_x)_{x \in S_d}$ is Feller-Dynkin, see [32, Theorem 8].

(ii) If $\sup_{n \in S_d} |\bar{q}_{nn}| < \infty$, then $(P^d_x)_{x \in S_d}$ exists, see [2, Corollary 2.2.5, Proposition 2.2.9], and $\{ y \in l_1 : y(\lambda I - \bar{Q}) = 0 \} = \{ 0 \}$ holds for all $\lambda > 0$, see [32, pp. 273]. In this case, the second part of (4.2) is necessary and sufficient for $(P^d_x)_{x \in S_d}$ to be Feller-Dynkin, see [32, Theorem 9].

(iii) If $S_d = \{ 0, 1, 2, \ldots \}$ and $\bar{q}_{ij} = 0$ for all $i \geq j + 2$, then $\{ y \in l_1 : y(\lambda I - \bar{Q}) = 0 \} = \{ 0 \}$ if and only if $\{ y \in l_1^+ : y(\lambda I - \bar{Q}) = 0 \} = \{ 0 \}$, see [27, Proposition 2]. Latter is necessary for $(P^d_x)_{x \in S_d}$ to be Feller-Dynkin, see [32, Theorem 7].

We suppose that

$$\Sigma \triangleq \{ (\omega^1, \omega^2) \in \Omega : \omega^1 : \mathbb{R}_+ \to \mathbb{R}^d \text{ is continuous} \},$$

and

$$\{ f, fg, g : f \in C^2_0(\mathbb{R}^d), g \in C \} \subseteq D, \tag{4.3}$$

where

$$\mathcal{L} f(x, n) \triangleq \mathcal{K} f(x, n) + \sum_{k \in S_d} q_{nk}(x) f(x, k),$$

and

$$\mathcal{K} f(x, n) \triangleq \langle \nabla x f(x, n), b(x, n) \rangle + \frac{1}{2} \text{trace } (\nabla^2_x f(x, n)a(x, n)).$$
We explain in the proof of Lemma 10 below that $\Sigma$ is a closed. Consequently, we have $\Sigma \in \mathcal{F}$.

By our assumption that $(P^d_x)_{x \in S_d}$ is Feller-Dynkin, there exist Lyapunov functions (in the sense of Theorem 11) for $(P^d_x)_{x \in S_d}$ due to Proposition 3 (see also Example 2). We will combine these Lyapunov functions with Lyapunov functions for the diffusion part, which we can define under each of the following two conditions.

**Condition 1.** There exist two locally Hölder continuous functions $a_d: \left[\frac{1}{2}, \infty\right) \to (0, \infty)$ and $b_d: \left[\frac{1}{2}, \infty\right) \to \mathbb{R}$ such that

$$\langle x, a(x, i)x \rangle \leq a_d \left(\frac{\|x\|^2}{2}\right),$$

trace $a(x, i) + 2\langle x, b(x, i) \rangle \geq b_d \left(\frac{\|x\|^2}{2}\right)$ for all $i \in S_d$ and $x \in \mathbb{R}^d$: $\|x\| \geq 1$. Moreover, either

$$p(r) \triangleq \int_1^r \exp \left( -\int_1^y b_d(z)dz \right) dy, \quad \lim_{r \to \infty} p(r) < \infty,$$

or

$$\lim_{r \to \infty} p(r) = \infty \text{ and } \int_1^\infty p'(y) \int_y^\infty \frac{dz}{a_d(z)p'(z)} dy = \infty.$$

Furthermore, we have

$$\sup_{k \in S_d} \sup_{\|x\| \leq 1} (\|b(x, k)\| + \text{trace } a(x, k)) < \infty.$$

**Condition 2.** There exists a constant $\beta > 0$ such that

$$\|b(x, i)\| \leq \beta(1 + \|x\|), \quad \text{trace } a(x, i) \leq \beta(1 + \|x\|^2),$$

for all $(x, i) \in S$.

**Proposition 5.** If the family $(P_x)_{x \in S}$ is Feller and one of the Conditions 1 and 2 holds, then $(P_x)_{x \in S}$ is also Feller-Dynkin.

**Proof:** We assume that Condition 1 holds. Fix an arbitrary compact set $K \subset S$. Since the projections $\pi_1: S \to \mathbb{R}^d$ and $\pi_2: S \to S_d$ are continuous for the product topology, the sets $\pi_1(K)$ and $\pi_2(K)$ are compact and $K \subseteq \pi_1(K) \times \pi_2(K)$.

Because we assume the family $(P^d_x)_{x \in S_d}$ to be Feller-Dynkin, Proposition 3 (see also Example 2) implies that there exists a function $\zeta: S_d \to \mathbb{R}_+$ such that $\zeta \in C, \zeta > 0$ on $\pi_2(K)$ and $\int_0^\infty \zeta \leq c\zeta$ for a constant $c > 0$. Due to 3. Lemma 4.2, there exists a twice continuously differentiable decreasing solution $u: \left[\frac{1}{2}, \infty\right) \to (0, \infty)$ to the differential equation

$$\frac{1}{2}a_db_du' + \frac{1}{2}a_du'' = u, \quad u\left(\frac{1}{2}\right) = 1,$$

which satisfies $\lim_{x \to \infty} u(x) = 0$. For the last property we require that either 4.4 or 4.5 holds. At the end of the proof we explain the application of 3. Lemma 4.2 in detail. We find a twice continuously differentiable function $\phi: [0, \infty) \to (0, \infty)$ such that $\phi \geq 1$ on $[0, \frac{1}{2}]$ and $\phi = u$ on $(\frac{1}{2}, \infty)$. Now, we define

$$V(x, n) \triangleq \phi \left(\frac{\|x\|^2}{2}\right) \zeta(n).$$

We see that $V \geq 0$, $V \in D$ and that $V > 0$ on $K$. Fix $\epsilon > 0$. Because $x \mapsto \phi(\|x\|^2/2) \in C_0(\mathbb{R}^d)$ there exists a compact set $K_1 \subset \mathbb{R}^d$ such that

$$\phi \left(\frac{\|x\|^2}{2}\right) < \frac{\epsilon}{\|x\|_{\infty}}$$
for all $x \notin K_1$. Similarly, because $\zeta \in C \subseteq C_0(S_d)$, there exists a compact set $K_2 \subseteq S_d$ such that

$$\zeta(n) < \|\varphi\|_\infty$$

for all $n \notin K_2$. Consequently, we have $V(x,n) < \epsilon$ for all $(x,n) \notin K_1 \times K_2$. This shows that $V \in C_0(S)$. It remains to check that $\mathcal{L}V \leq \text{const. } V$. For all $n \in S_d$ and $x \in \mathbb{R}^d$: $\|x\| > 1$ we have

$$\mathcal{K}V(x,n) = \zeta(n) \frac{1}{2} \left( (x,a(x,n)x)u'' \left( \frac{\|x\|^2}{2} \right) + \text{trace } a(x,n) + 2 \langle x, b(x,n) \rangle \right) u' \left( \frac{\|x\|^2}{2} \right)$$

$$\leq \zeta(n) \frac{(x,a(x,n)x)}{2} \left( u'' \left( \frac{\|x\|^2}{2} \right) + b_d \left( \frac{\|x\|^2}{2} \right) u' \left( \frac{\|x\|^2}{2} \right) \right),$$

where we used that $u$ is decreasing, i.e. that $u' \leq 0$. Due to (4.7), we have

$$u'' + b_d u' = \frac{2u}{a_d} \geq 0.$$

Thus, we obtain

$$\mathcal{K}V(x,n) \leq \zeta(n) \frac{1}{2} a_d \left( \frac{\|x\|^2}{2} \right) \left( u'' \left( \frac{\|x\|^2}{2} \right) + b_d \left( \frac{\|x\|^2}{2} \right) u' \left( \frac{\|x\|^2}{2} \right) \right) = V(x,n).$$

Due to (1.6), we find a constant $c^* \geq 1$ such that $\mathcal{K}V(x,n) \leq c^* \zeta(n) \leq c^* V(x,n)$ for all $n \in S_d$ and $x \in \mathbb{R}^d$: $\|x\| \leq 1$. In summary, using (4.1) and (4.8), we obtain

$$\mathcal{L}V(x,n) \leq c^* V(x,n) + \left( \sum_{k \neq n} q_{nk}(x) \zeta(k) + q_{nn}(x) \zeta(n) \right) \phi(x)$$

$$\leq c^* V(x,n) + \left( \sum_{k \in S_d} \overline{q}_{nk} \zeta(k) + (q_{nn}(x) - \overline{q}_{nn}) \zeta(n) \right) \phi(x)$$

$$\leq \left( c^* + c + \sup_{k \in S_d} \sup_{y \in \mathbb{R}^d} |q_{kk}(y) - \overline{q}_{kk}| \right) V(x,n)$$

$$= \text{const. } V(x,n).$$

Consequently, Theorem 1 implies the claim.

For the case where Condition 2 holds, we only have to replace $\phi(x)$ by $(1+2x)^{-1}$. The remaining argument stays unchanged. We omit the details.

Finally, we explain how [3, Lemma 4.2] has to be applied. Let $-\infty \leq l < r \leq +\infty$ and $x_0 \in (l,r)$. Then, [3, Lemma 4.2] states that for a locally H"older function $g: (l,r) \to (0,\infty)$ there exists a decreasing positive function $v \in C([x_0,r)) \cap C^2((x_0,r))$ such that

$$(4.9) \quad \frac{1}{2} g v'' = v \text{ on } (x_0,r), \quad \lim_{x \uparrow r} v(r) = 0,$$

if and only if $r < \infty$ or $r = \infty$ and $\int_{x_0}^{\infty} \int_{z}^{\infty} (g(y))^{-1} dy dz = \infty$. We find two local Hölder functions $b^*: (\frac{1}{2},\infty) \to \mathbb{R}, a^*: (\frac{1}{2},\infty) \to (0,\infty)$ such that $b^* = b_d$ and $a^* = a_d$ on $[\frac{1}{2},\infty)$. Next, set

$$H^*(x) \triangleq \exp \left( - \int_{1}^{x} b^*(z) dz \right), \quad p^*(x) \triangleq \int_{1}^{x} H^*(z) dz$$

for $x \in (\frac{1}{2},\infty)$. It is well-known that $p^*$ has a twice continuously differentiable inverse $q^*$, see, e.g., [20, Section 5.5.B]. Take $l \triangleq \lim_{x \to \frac{1}{2}^+} p^*(x), r \triangleq \lim_{x \to \infty} p^*(x)$
and 
\[ g(x) \triangleq a^*(q^*(x))(H^*(q^*(x)))^2, \quad x \in (l,r). \]

The function \( g \) is positive and, because products and compositions of local Hölder functions are again locally Hölder, it is also locally Hölder. Note that \( r < \infty \) is exactly \([1,4]\). Let \( k \in \left(\frac{1}{r}, \frac{1}{l} \right) \) and set \( x_0 \triangleq p^*(k) \in (l,r) \). If \([1,5]\) holds we have \( r = \infty \) and

\[
\int_{x_0}^{\infty} \int_{z}^{\infty} (g(y))^{-1} dy dz = \int_{x_0}^{\infty} \int_{z}^{\infty} \frac{(q^*)'(y)}{a^*(q^*(y))H^*(q^*(y))} dy dz = \int_{x_0}^{\infty} \frac{(q^*)'(y)}{a^*(y)H^*(y)} dy dz = \int_{1}^{\infty} p'(z) \int_{z}^{\infty} \frac{dy}{a_d(y)p'(y)} dz = \infty.
\]

We conclude that there exists a decreasing positive function \( v \in C((x_0,r)) \cap C^2((x_0,r)) \) with the properties \([1,9]\). Now, we set

\[ u(x) \triangleq \frac{v(p^*(x))}{v(p^*(\frac{1}{l}))}, \quad x \in \left(\frac{1}{l}, \infty\right). \]

Clearly, \( u \) is positive, \( u(\frac{1}{l}) = 1 \) and

\[ \lim_{x \to \infty} u(x) = \frac{\lim_{y \to r} v(y)}{v(p^*(\frac{1}{l}))} = 0. \]

Because \( p^* \) is increasing and \( v \) is decreasing, also \( u \) is decreasing, and, because \( v \in C^2((x_0,r)) \), we have \( u \in C^2((k, \infty)) \), which implies \( u \in C^2((\frac{1}{l}, \infty)) \). We compute that for \( x \in \left(\frac{1}{l}, \infty\right) \)

\[ \frac{1}{2}b^*(x)a^*(x)u'(x) + \frac{1}{2}a^*(x)u''(x) = \left( v(p^*(\frac{1}{l})) \right)^{-1} \frac{1}{2}a^*(x)v''(p^*(x))(H^*(x))^2 = \left( v(p^*(\frac{1}{l})) \right)^{-1} v(p^*(x)) = u(x). \]

In particular, because \( a^* = a_d \) and \( b^* = b_d \) on \( (\frac{1}{l}, \infty) \), we have

\[ \frac{1}{2}b_d a_d u' + \frac{1}{2}a_d u'' = u \text{ on } (\frac{1}{l}, \infty). \]

We conclude that \( u \) has all properties as claimed. \( \square \)

Conditions for the Feller property of \((P_x)_{x \in S}\) can be found in \([30, 36, 39, 40]\).

We collect some of these in the following corollary, where we also assume that

\[ D \equiv \{ f : S \to \mathbb{R} : x \mapsto f(x,e) \in C_b^2(\mathbb{R}^d), e \mapsto f(x,e) \in B(S_d) \text{ for all } (x,e) \in S \}. \]

**Corollary 1.** Suppose the following:

(i) \( S_d = \{0,1,\ldots,N\} \text{ for } 1 \leq N \leq \infty \)

(ii) There exists a constant \( c_1 > 0 \) such that for all \((x,i) \in S\) we have \( q_{ij}(x) = 0 \) for all \( j \in S_d \text{ with } |j-i| > c_1 \).

\(^8\text{Of course, when } N = \infty \text{ we mean } S_d = \mathbb{N}.\)
(iii) There exists a constant $c_2 > 0$ such that for all $i \in S_d$
\[
\sup_{x \in \mathbb{R}^d} \left| q_{ii}(x) \right| \leq c_2(i + 1).
\]

(iv) There exists a constant $c_3 > 0$ such that for all $i \in S_d$ and $x, y \in \mathbb{R}^d$
\[
\sum_{j \neq i} \left| q_{ij}(x) - q_{ij}(y) \right| \leq c_3 \| x - y \|.
\]

(v) Condition (ii) holds and there exists a constant $c_4 > 0$ and a root $a^{1 \over 2}$ of a
such that for all $i \in S_d$ and $x, y \in \mathbb{R}^d$
\[
\| b(x, i) - b(x, i) \| + \| a^{1 \over 2}(x, i) - a^{1 \over 2}(y, i) \| \leq c_4 \| x - y \|.
\]

Then, a Feller-Dynkin family $(P_x)_{x \in S}$ exists.

Proof: The existence of a family $(P_x)_{x \in S}$ follows from [39, Theorem 2.1]. Furthermore, [39, Theorem 3.3] yields that $(P_x)_{x \in S}$ is Feller. Thus, Proposition [5] implies that $(P_x)_{x \in S}$ is Feller-Dynkin, too. \hfill \square

Remark 3. (i) Assumption (ii) in Corollary [1] can be replaced by a weaker,
but less explicit, condition of Lyapunov-type, see [39, Assumption 1.2].
(ii) In general, the conditions from Corollary [1] do not imply the strong Feller
property of $(P_x)_{x \in S}$. For example, it is allowed to take the first coordinate
as linear motion, which gives a process without the strong Feller property.

If, in addition to (i) – (v) in Corollary [1] we assume that there exists a
c constant $c > 0$ such that for all $(x, i) \in S$ and $y \in \mathbb{R}^d$
\[
\langle y, a(x, i)y \rangle \geq c \| y \|^2,
\]
then [36, Theorem 3.1] implies that $(P_x)_{x \in S}$ has the strong Feller property,
too. In this case, $(P_x)_{x \in S}$ has the Feller, the strong Feller and the Feller-
Dynkin property.

The following example illustrates that our results include cases where $Q$ is un-
bounded.

Example 3. Suppose that $Q(x) \equiv Q$ corresponds to a classical birth-death chain,
i.e. $S_d \triangleq \{0, 1, 2, \ldots \}$ and $q_{ij} \triangleq$
\[
\begin{cases}
\lambda_i, & j = i + 1, i \geq 0, \\
\mu_i, & j = i - 1, i \geq 1, \\
-(\lambda_i + \mu_i), & i = j, i \geq 0, \\
0, & \text{otherwise},
\end{cases}
\]
for strictly positive sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$ and $\mu_0 = 0$ and $\lambda_0 > 0$. Set
\[
r \triangleq \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \frac{\mu_n \mu_{n-1}}{\lambda_n \lambda_{n-1} \lambda_{n-2}} + \cdots + \frac{\mu_n \cdots \mu_2}{\lambda_n \cdots \lambda_2 \lambda_1} \right),
\]
\[
s \triangleq \sum_{n=1}^{\infty} \frac{1}{\mu_{n+1}} \left( 1 + \frac{\lambda_n}{\mu_n} + \frac{\lambda_n \lambda_{n-1}}{\mu_n \mu_{n-1}} + \cdots + \frac{\lambda_n \lambda_{n-1} \cdots \lambda_2 \lambda_1}{\mu_n \mu_{n-1} \cdots \mu_2 \mu_1} \right).
\]
It is well-known that a Feller-Dynkin family $(P^s_x)_{x \in S_d}$ exists if $r = s = \infty$, see [2, Theorems 3.2.2, 3.2.3] and Remark [2] (i) and (iii). In this case, if also one of the
Conditions \([\text{1}]\) and \([\text{2}]\) holds, the family \((P_x)_{x \in S}\) is Feller-Dynkin whenever it is Feller. To be more concrete, if we choose 
\[
\lambda_n \triangleq n^\alpha \lambda, \quad \mu_n \triangleq n^\alpha \mu, \quad \alpha \geq 0, \lambda, \mu > 0,
\]
then \(s = r = \infty\) if and only if either \(\alpha \leq 1\) or \([\alpha \in (1, 2] \text{ and } \lambda = \mu\). In other words, we find coefficients \(a, b\) and \(Q\) which satisfy the conditions from Corollary \([\text{4}]\) with an unbounded \(Q\).

4.2. **Conditions not to be Feller-Dynkin.** Next, we give conditions for rejecting the Feller-Dynkin property for two specific situations: First, we assume that \(|S_d| < \infty\) and, second, we assume that \(Q(x) \equiv Q\).

4.2.1. **Finitely Many Environments.** In this section we impose the following:

**Standing Assumption.** We have \(|S_d| < \infty\).

Let \(\Sigma\) and \(L\) be as in Section \([\text{4.1}]\) and define 
\[
D \triangleq \{f, fg, g: f \in C^2_b(\mathbb{R}^d), g: S_d \rightarrow \mathbb{R}\}. 
\]

**Proposition 6.** Assume that there exists an \(r > 0\) and two locally Hölder continuous functions \(b_d: [r, \infty) \rightarrow \mathbb{R}\) and \(a_d: [r, \infty) \rightarrow (0, \infty)\) such that for all \(i \in S_d\) and \(x \in \mathbb{R}^d, \|x\| \geq 2r\)
\[
\langle x, a(x, i)x \rangle \geq a_d \left(\frac{\|x\|^2}{2}\right),
\]
\[
\text{trace } a(x, i) + 2\langle x, b(x, i) \rangle \leq b_d \left(\frac{\|x\|^2}{2}\right) \langle x, a(x, i)x \rangle,
\]

and 
\[
p(t) \triangleq \int_{r+1}^t \exp \left(-\int_{r+1}^y b_d(z)dz\right) dy \rightarrow \infty \text{ as } t \rightarrow \infty,
\]

and 
\[
\int_{r+1}^\infty p'(y) \int_y^\infty \frac{dz}{a_d(z)p'(z)} dy < \infty.
\]

Then \((P_x)_{x \in S}\) is not Feller-Dynkin.

**Proof:** Due to \([\text{3}]\) Lemma 4.2], there exists a twice continuously differentiable decreasing solution \(u: [r, \infty) \rightarrow (0, \infty)\) to the differential equation 
\[
\frac{1}{2}a_d b_d u' + \frac{1}{4}a_d u'' = u, \quad u(r) = 1,
\]
which satisfies \(\lim_{x \nearrow \infty} u(x) > 0\). We find a twice continuously differentiable function \(\phi: [0, \infty) \rightarrow (0, \infty)\) such that \(\phi \geq 1\) on \([0, r]\) and \(\phi = u\) on \((r, \infty)\). It follows similarly to the proof of Proposition \([\text{3}]\) that 
\[
U(x, n) \triangleq \phi \left(\frac{\|x\|^2}{2}\right), \quad x \in \mathbb{R}^d, n \in S_d,
\]
has the properties from Theorem \([\text{2}]\) for the compact sets \(C \equiv K \triangleq \{x \in \mathbb{R}^d: \|x\| \leq \sqrt{2r}\} \times S_d\), which implies the claim. \(\Box\)
4.2.2. State-Independent Switching. In this section we assume the following:

**Standing Assumption.** We have $Q(x) \equiv Q$ and there exists a continuous-time Markov chain $\mathbb{P}$ with $Q$-matrix $Q$. We denote its unique law by $(P^*_t)_{t \in S_d}$, where the subscript indicates the starting value. Furthermore, $(P^*_t)_{t \in S_d}$ is Feller-Dynkin, see Remark 2.

From now on we fix a root $a^{\frac{1}{2}}$ of $a$. Let $\mathcal{L}$ and $\Sigma$ be as in Section 4.1 and set
\[ D \triangleq \{ f, fg, g : f \in C^2_0(\mathbb{R}^d), g \in C \}, \quad C \triangleq \{ g \in C_0(S_d) : Qf \in C_0(S_d) \}. \]

It seems to be known that the family $(P^*_x)_{x \in S}$ has a one-to-one relation to a switching diffusion defined via an SDE, see, for instance, \cite{5} for a partial result in this direction. However, we did not find a complete reference, such that we provide a formal statement and a proof:

**Lemma 2.** Fix $y = (x, i) \in S$. A probability measure $P_y$ solves the MP $(D, \mathcal{L}, \Sigma, y)$ if and only if it is the law of a process $(Y_t, Z_t)_{t \geq 0}$, where $(Z_t)_{t \geq 0}$ be a Markov chain with $Q$-matrix $Q$ and initial value $Z_0 = i$ and $(Y_t)_{t \geq 0}$ solves the SDE
\[ dY_t = b(Y_t, Z_t)dt + a^{\frac{1}{2}}(Y_t, Z_t)dW_t, \quad Y_0 = x, \]
where $(W_t)_{t \geq 0}$ is a Brownian motion such that the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent.

**Proof:** The implication $\Leftarrow$ is a consequence of Itô’s formula, see, e.g., \cite[p. 29]{40}.

It remains to show the implication $\Rightarrow$. We consider the completion of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$ as underlying filtered probability space. Denote $(X_t)_{t \geq 0} = (Y_t, Z_t)_{t \geq 0}$, where $(Y_t)_{t \geq 0}$ is $\mathbb{R}^d$-valued and $(Z_t)_{t \geq 0}$ is $S_d$-valued. It follows from Example 2 and Theorem 3.33 that $P_y \circ (Z_t)_{t \geq 0} = P_t$. This means that $(Z_t)_{t \geq 0}$ is a Markov chain with $Q$-matrix $Q$ and $Z_0 = i$. Furthermore, in view of \cite[Remark 5.4.12]{20}, we can argue as in the proof of \cite[Proposition 5.4.6]{20} to conclude the existence of a Brownian motion $(W_t)_{t \geq 0}$ (possibly defined on a standard extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$; see \cite[Remark 3.4.1]{20}) such that $(Y_t)_{t \geq 0}$ satisfies the SDE \eqref{4.11}. It remains to explain that the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent. With abuse of notation, we denote the standard extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$ again by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_y)$.

Adapt an idea from \cite[Theorem 4.10.1]{20}. Because the martingale property is not affected by a standard extension (see \cite[Proposition 10.46]{13}), for all $f \in C$ the process

\[ M^f_t = f(Z_t) - \int_0^t Qf(Z_s)ds, \quad t \in \mathbb{R}_+, \]

is a $P_y$-martingale. For $g \in C^2_b(\mathbb{R}^d)$ with $\inf_{x \in \mathbb{R}^d} g(x) > 0$ set
\[ K^g_t \triangleq g(W_t) \exp \left( -\frac{1}{2} \int_0^t \frac{\Delta g(W_s)}{g(W_s)} ds \right), \quad t \in \mathbb{R}_+. \]

Of course, here $\Delta$ denotes the Laplacian. Itô’s formula yields that
\[ dK^g_t = \exp \left( -\frac{1}{2} \int_0^t \frac{\Delta g(W_s)}{g(W_s)} ds \right) \langle \nabla g(W_t), dW_t \rangle, \]
which implies that also $(K^g_t)_{t \geq 0}$ a $P_y$-martingale, because it is a bounded (on finite time intervals) local $P_y$-martingale. Because $(Z_t)_{t \geq 0}$ has only finitely many jumps

\footnote{For a Markov chain is always non-explosive.}
in a finite interval, \((M^f_t)_{t \geq 0}\) is of finite variation on finite intervals and we have \(P_y\)-a.s.

\[ [M^f_t, K^g_t] = 0 \text{ for all } t \in \mathbb{R}_+, \]

see [17, Proposition I.4.49]. Consequently, integration by parts yields that \((M^f_t K^g_t)_{t \geq 0}\) is a local \(P_y\)-martingale and a true \(P_y\)-martingale due to its boundedness on finite time intervals. Fix an arbitrary bounded stopping time \(\psi\) and define

\[ Q(G) \equiv \frac{E_y [1_G K^g_{\psi}]}{g(0)}, \quad G \in \mathcal{F}. \]

Due to the optional stopping theorem, for all bounded stopping times \(\phi\) we have

\[ E^Q [M^f_{\phi}] = E^Q [M^f_{\phi \wedge \psi} K^g_{\phi \wedge \psi}] = f(i). \]

We conclude from [33, Proposition II.1.4] that \((M^f_t)_{t \geq 0}\) is a \(Q\)-martingale. Consequently, in view of Example 2, we have

\[ Q(\Gamma) = P_y(\Gamma), \]

where

\[ \Gamma \equiv \{ Z_{t_1} \in F_1, \ldots, Z_{t_n} \in F_n \} \]

for arbitrary \(0 \leq t_1 < \cdots < t_n < \infty\) and \(F_1, \ldots, F_n \in \mathcal{B}(S_d)\). Suppose that \(P_y(\Gamma) > 0\). Then, set

\[ \hat{Q}(G) \equiv \frac{E_y [1_G 1\Gamma]}{P_y(\Gamma)}, \quad G \in \mathcal{F}. \]

We have

\[ E^{\hat{Q}} [K^g_{\psi}] = \frac{E_y [K^g_{\psi} 1\Gamma]}{P_y(\Gamma)} = \frac{Q(\Gamma) g(0)}{P_y(\Gamma)} = g(0). \]

Thus, because \(\psi\) was arbitrary, we deduce from [33, Proposition II.1.4] and [11, Proposition 4.3.3] that \((W_t)_{t \geq 0}\) is a \(\hat{Q}\)-Brownian motion and the uniqueness of the Wiener measure yields that

\[ \hat{Q}(W_{s_1} \in G_1, \ldots, W_{s_k} \in G_k) = P_y(W_{s_1} \in G_1, \ldots, W_{s_k} \in G_k) \]

for arbitrary \(0 \leq s_1 < \cdots < s_k < \infty\) and \(G_1, \ldots, G_k \in \mathcal{B}(\mathbb{R}^d)\). Using the definition of \(\hat{Q}\), we conclude that

\[ P_y(Z_{t_1} \in F_1, \ldots, Z_{t_n} \in F_n, W_{s_1} \in G_1, \ldots, W_{s_k} \in G_k) = P_y(Z_{t_1} \in F_1, \ldots, Z_{t_n} \in F_n) P_y(W_{s_1} \in G_1, \ldots, W_{s_k} \in G_k), \]

which implies the desired independence. \(\square\)

We set

\[ \Sigma_c \equiv \{ \omega: \mathbb{R}_+ \to \mathbb{R}^d: t \mapsto \omega(t) \text{ is continuous} \}, \]

and

\[ \mathcal{K}^k f(x) \equiv \langle \nabla f(x), b(x, k) \rangle + \frac{1}{2} \text{trace } (\nabla^2 f(x)a(x, k)) \]

for \(f \in C^2_b(\mathbb{R}^d)\) and \((x, k) \in S\).
Definition 2. A family \((P^k_x)_{x \in \mathbb{R}^d}\) of solutions to the MP \((C^2_0(\mathbb{R}^d), \mathcal{K}^i, \Sigma_\pi)\) is said to exist strongly, if on any filtered probability space with right-continuous complete filtration \((\mathcal{G}_t)_{t \geq 0}\), which supports a Brownian motion \((W_t)_{t \geq 0}\) and an \(\mathbb{R}^d\)-valued \(\mathcal{G}_0\)-measurable random variable \(\pi\), there exists a unique (up to indistinguishability) continuous adapted process \((Y^k_t)_{t \geq 0}\) which solves the SDE
\[
dY^k_t = b(Y^k_t, k)dt + a^2(Y^k_t, k)dW_t, \quad Y^0_t = \pi,
\]
and a universally adapted\(^{\dagger}\) Borel\(^{\ddagger}\) map \(F^k: \mathbb{R}^d \times \Sigma_\pi \to \Sigma_\pi\) such that \(Y^k = F^k(\pi, W)\) up to a null set. Here, the state space for the MP is \(\mathbb{R}^d\).

Remark 4. We stress that our definition of strong existence includes a version of pathwise uniqueness and that the function \(F^k\) in the previous definition is independent of the law of \(\pi\). A generalization\(^{\ddagger}\) of the classical Yamada-Watanabe theorem yields that \((P^k_x)_{x \in \mathbb{R}^d}\) exists strongly if and only if the SDE (4.13) satisfies weak existence and pathwise uniqueness, see [19, Theorem 18.14].

The following observation can be seen as a version of [36, Theorem 3.2 (ii)] for the Feller-Dynkin property. In the next section, we will also see a version of [36, Theorem 3.2 (i)] under the additional assumption that \(|S_d| < \infty\).

Proposition 7. Suppose that there exists an \(i \in S_d\) such that for all \(x \in \mathbb{R}^d\) the MP \((C^2_0(\mathbb{R}^d), \mathcal{K}^i, \Sigma_\pi, x)\) has a (unique) solution \(P^i_x\) and the family \((P^i_x)_{x \in \mathbb{R}^d}\) exists strongly and is Feller, but not Feller-Dynkin. Then, \((P_x)_{x \in S}\) is not Feller-Dynkin.

Proof: As shown in the proof of Theorem 1, since \((P^i_x)_{x \in \mathbb{R}^d}\) is Feller, if for any compact set \(K \subset \mathbb{R}^d\) and any \(t > 0\) we have
\[
\limsup_{\|x\| \to \infty} P^i_x(X_t \in K) = 0,
\]
then \((P^i_x)_{x \in \mathbb{R}^d}\) is Feller-Dynkin. Consequently, since we assume \((P^i_x)_{x \in \mathbb{R}^d}\) not to be Feller-Dynkin, there exists a sequence \((x_k)_{k \in \mathbb{N}} \subset \mathbb{R}^d\) with \(\|x_k\| \to \infty\) as \(k \to \infty\), a compact set \(K^0 \subset \mathbb{R}^d\) and a \(\rho^0 > 0\) such that
\[
\limsup_{k \to \infty} P^i_{x_k}(X_{\rho^0} \in K^0) > 0.
\]
The set \(G \triangleq K^0 \times \{i\} \subset S\) is compact. If we show that
\[
\limsup_{k \to \infty} P_{(x_k, i)}(X_{\rho^0} \in G) > 0,
\]
then \((P_x)_{x \in S}\) cannot be Feller-Dynkin. To see this, assume for contradiction that \((P_x)_{x \in S}\) is Feller-Dynkin. Due to the locally compact version of Urysohn’s lemma, there exists a function \(f \in C_0(S)\) such that \(0 \leq f \leq 1\) and \(f \equiv 1\) on \(G\). Consequently, we have
\[
P_{(x_k, i)}(X_{\rho^0} \in G) = E_{(x_k, i)}[f(X_{\rho^0})1\{X_{\rho^0} \in G\}] \\
\leq E_{(x_k, i)}[f(X_{\rho^0})] \to 0 \text{ as } k \to \infty,
\]
\(^{\dagger}\)i.e., adapted to the filtration \(\mathcal{G}_t \triangleq \bigcap_{\omega \in \mathcal{P}} \mathcal{G}_t^\omega\), where \(\mathcal{P}\) is the set of all Borel probability measures on \(\mathbb{R}^d\) and \((\mathcal{G}_t^\omega)_{t \geq 0}\) is the completion of the canonical filtration on \(\mathbb{R}^d \times \Sigma_\pi\) w.r.t. the product measure \(\mu \otimes W\), where \(W\) is the Wiener measure; see [18, p. 346]
\(^{\ddagger}\)where \(\Sigma_\pi\) is equipped with the local uniform topology; the Borel \(\sigma\)-field is the \(\sigma\)-field generated by the coordinate process
\(^{11}\)In the usual formulation of the Yamada-Watanabe theorem as given, for instance, in [20], the function \(F^k\) depends on the law of \(\pi\). This dependence was removed in [18].
because \((P_x)_{x \in S}\) is Feller-Dynkin. This, however, is a contradiction and we conclude that \((P_x)_{x \in S}\) cannot be Feller-Dynkin. In summary, it suffices to show (4.15).

For a càdlàg \(S_t\)-valued process \((Z_t)_{t \geq 0}\), we set
\[
\tau(Z) \triangleq \inf \{ t \in \mathbb{R}_+ : Z_t \neq Z_0 \},
\]
which is a stopping time for any right-continuous filtration to which \((Z_t)_{t \geq 0}\) is adapted, see [11, Proposition 2.1.5]. In the following let \((Y_t)_{t \geq 0}, (Z_t)_{t \geq 0}\) and \((W_t)_{t \geq 0}\) be as in Lemma 2 for \(y = (x, i)\). On \(\{t \leq \tau(Z)\}\), we have
\[
Y_t = x + \int_0^t b(Y_s, i)ds + \int_0^t a^i(Y_s, i)dW_s,
\]
which is the SDE corresponding to the MP \((C_b^2(\mathbb{R}^d), \mathcal{K}, \Sigma_c, x)\), see [21, Corollary 5.4.8]. We now need a local version of pathwise uniqueness. The proof of the following lemma is given after the proof of Proposition 7 is complete.

**Lemma 3.** Suppose that the SDE
\[
(4.16) \quad dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t
\]
satisfies weak existence and pathwise uniqueness (see [32, Section IX.1]). Let \((\Theta, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)\) be a filtered probability space with right-continuous complete filtration \((\mathcal{G}_t)_{t \geq 0}\), which supports a Brownian motion \((W_t)_{t \geq 0}\) and an \(\mathbb{R}^d\)-valued \(\mathcal{G}_0\)-measurable random variable \(\psi\). Take a \((\mathcal{G}_t)_{t \geq 0}\)-stopping time \(\tau\) and let \((Y_t)_{t \geq 0}\) be the solution to (4.16) with initial value \(\psi\). Then, all solutions to
\[
dO_t = \mu(O_t)\mathbf{1}_{\{t \leq \tau\}} dt + \sigma(O_t)\mathbf{1}_{\{t \leq \tau\}} dW_t, \quad O_0 = \psi,
\]
are indistinguishable from \((Y_t)_{t \geq 0}\).

Because we assume that \(P^*_x\) exists strongly, the previous lemma implies that
\[
P_{(x,i)}(X_{t^0} G) \geq P(Y_{t^0 \wedge \tau(Z)} \in K^o, Z_{t^0} = i, t^0 < \tau(Z))
\]
\[
= P(F^i(x, W)_{t^0 \wedge \tau(Z)} \in K^o, t^0 < \tau(Z))
\]
\[
= P(F^i(x, W)_{t^0} \in K^o) P(t^0 < \tau(Z))
\]
\[
= P^*_x(X_{t^0} \in K^o) P^*_i(t^0 < \tau),
\]
where \(F^i\) is as in Definition 2. Here, we use that the \(\sigma\)-fields \(\sigma(W_t, t \in \mathbb{R}_+)\) and \(\sigma(Z_t, t \in \mathbb{R}_+)\) are independent, see Lemma 2. It is well-known that under \(P^*_x\) the random variable \(\tau\) is exponentially distributed with parameter \(-q_{ii}\), see, e.g., [19, Lemma 10.18]. Therefore, we have
\[
P_{(x,i)}(X_{t^0} G) \geq P^*_x(X_{t^0} \in K^o) e^{q_{ii}t^0}.
\]
We conclude (4.15) from (4.14). This finishes the proof. \(\square\)

**Proof of Lemma 3:** Due to stopping, we can assume that \(\tau\) is finite. Let \((B_t)_{t \geq 0}\) be defined by
\[
B_t \triangleq W_{t+\tau} - W_t, \quad t \in \mathbb{R}_+.
\]

\[\text{In other words, we assume that the martingale problem corresponding to the SDE (4.16) exists strongly, see Remark 4 and [20, Section 5.4].}\]
\[\text{If } q_{ii} = 0, \text{ then } P^*_i \text{-a.s. } \tau = \infty, \text{ which implies } P^*_i(t^0 < \tau) = 1.\]
By [33, Proposition V.1.5] and Lévy’s characterization (see, e.g., [20, Theorem 3.3.16]), the process \((B_t)_{t \geq 0}\) is a \((\mathcal{G}_{t+})_{t \geq 0}\)-Brownian motion. Due to the strong existence hypothesis, there exists a solution \((U_t)_{t \geq 0}\) to the SDE
\[
dU_t = \mu(U_t) dt + \sigma(U_t) dB_t, \quad U_0 = O_\tau.
\]
Now, we set
\[
V_t \triangleq \begin{cases} 
O_t, & t \leq \tau, \\
U_{t-\tau}, & t > \tau.
\end{cases}
\]
Because \(U_0 = O_\tau\), the process \((V_t)_{t \geq 0}\) has continuous paths. Below, we show that \((V_t)_{t \geq 0}\) is furthermore adapted. On \(\{t \leq \tau\}\) we have
\[
V_t = \psi + \int_0^t \mu(V_s) ds + \int_0^t \sigma(V_s) dW_s.
\]
Classical rules for time-changed stochastic integrals (see, e.g., [33, Propositions V.1.4, V.1.5]; we comment on this below with more details) yield that on \(\{t > \tau\}\)
\[
\begin{align*}
(4.17) & \quad V_t = O_\tau + \int_0^{t-\tau} \mu(U_s) ds + \int_0^{t-\tau} \sigma(U_s) dB_s \\
(4.18) & \quad = V_\tau + \int_\tau^t \mu(U_{s-\tau}) ds + \int_\tau^t \sigma(U_{s-\tau}) dW_s \\
& \quad = \psi + \int_0^t \mu(V_s) ds + \int_0^t \sigma(V_s) dW_s.
\end{align*}
\]
Consequently, \((V_t)_{t \geq 0}\) solves the SDE
\[
dV_t = \mu(V_t) dt + \sigma(V_t) dW_t, \quad V_0 = \psi.
\]
By the strong existence hypothesis, we conclude that a.s. \(V_t = Y_t\) for all \(t \in \mathbb{R}_+\).

The definition of \((V_t)_{t \geq 0}\) implies the claim.

Let us end the proof with a detailed explanation how to come from (1.17) to (1.18). Define \(H_t \triangleq \sigma(U_{t-\tau}) 1_{\{\tau < t\}}\) and \(h_t \triangleq U_{t-\tau} 1_{\{\tau < t\}}\). We claim that \((H_t)_{t \geq 0}\) is progressive. In particular, this implies that \((V_t)_{t \geq 0}\) is adapted. Because \(H_t = \sigma(h_t) 1_{\{\tau < t\}}\), the process \((H_t)_{t \geq 0}\) is progressive whenever \((h_t)_{t \geq 0}\) is progressive. We note that \(t \mapsto h_t\) is left-continuous. Consequently, it suffices to explain that \((h_t)_{t \geq 0}\) is adapted, see, e.g., [20, Proposition 1.1.13]. It is well-known that we can approximate \(\tau\) from above by a sequence \((\tau^n)_{n \in \mathbb{N}}\) of stopping times such that \(\tau^n\) takes values in the countable set \(2^{-n} \mathbb{N}\), see, e.g., [13, Lemma 6.4]. Note that \(s \mapsto U_{t-\tau} 1_{\{s < t\}}\) is right-continuous. Thus, it suffices to show that \(h^n_t \triangleq U_{t-\tau^n} 1_{\{\tau^n < t\}}\) is adapted. Let \(G \in \mathcal{B}(\mathbb{R}^d)\) and set \(N_{t,n} \triangleq 2^n \mathbb{N} \cap [0, t)\). We have
\[
\{h^n_t \in G\} = \left( \bigcup_{k \in N_{t,n}} \{h^n_k \in G\} \cap \{\tau^n = k\} \right) \cup \{h^n_t \in G\} \cap \{\tau^n \geq t\}.
\]
Note that
\[
\{h^n_t \in G\} \cap \{\tau^n \geq t\} = \{0 \in G\} \cap \{\tau^n \geq t\} \in \mathcal{G}_t.
\]
If \(k \in N_{t,n}\) we have
\[
\{h^n_k \in G\} \cap \{\tau^n = k\} = \{U_{t-k} \in G\} \cap \{t-k + \tau^n = t\}.
\]
We have \( \{U_t \}_{t \geq 0} \subseteq G_t \). Recalling that for any stopping time \( \zeta \) we have \( G_{\zeta} \cap \{ \zeta = t \} \subseteq G_t \), see, e.g., [19, Lemma 6.1], we conclude that \( (h^n_t)_{t \geq 0} \) is adapted. Finally, we note that

\[
\int_\tau^{+\tau} H_s dW_s = \int_0^{+\tau} H_s dW_s = \int_0^{+\tau} \sigma(U_s-\tau) dW_s,
\]

and we deduce from [33, Propositions V.1.5] that

\[
(4.19) \quad \int_\tau^{+\tau} H_s dW_s = \int_0^{+\tau} H_s dW_s = \int_0^{+\tau} \sigma(U_s) dW_s.
\]

We note that for any \( t \geq 0 \) the random time \( \zeta \equiv (t-\tau)\mathbf{1}_{\{t>\tau\}} \) is a \( (G_{s+\tau})_{s \geq 0} \)-stopping time and that inserting \( \zeta \) in (4.19) and using [20, Exercise 3.2.30] yields the claimed formula. The \( ds \)-integral can be treated in the same manner. □

Now, we can deduce conditions from Theorem 2 based on similar ideas as in the previous section.

**Corollary 2.** Assume that there exists an \( i \in S_d \) such that the maps \( x \mapsto b(x,i) \) and \( x \mapsto a^2(x,i) \) are locally Lipschitz continuous and that for all \( x \in \mathbb{R}^d \) the MP \((C^2_b(\mathbb{R}^d), K_i, \Sigma_c, x)\) has a solution. Furthermore, suppose there is an \( r > 0 \) and two locally Hölder continuous functions \( b_d: [r, \infty) \to \mathbb{R} \) and \( a_d: [r, \infty) \to (0, \infty) \) such that for all \( x \in \mathbb{R}^d \):

\[
\langle x, a(x,i)x \rangle \geq a_d \left( \frac{\|x\|^2}{2r} \right),
\]

\[
\text{trace } a(x,i) + 2 \langle x, b(x,i) \rangle \leq b_d \left( \frac{\|x\|^2}{2r} \right) \langle x, a(x,i)x \rangle,
\]

and

\[
p(t) \equiv \int_r^{t+1} \exp \left( -\int_y^u b_d(z) dz \right) dy \to \infty \text{ as } t \to \infty,
\]

and

\[
\int_{t+1}^{\infty} p'(y) \int_y^{\infty} \frac{dz}{a_d(z)p'(z)} dy < \infty.
\]

Then, the family \( (P_x)_{x \in S} \) is not Feller-Dynkin.

**Proof:** Due to Remark [20, Theorem 5.2.5, Proposition 5.3.20, Corollary 5.4.9] and [37, Corollary 10.1.4], the family \( (P^i_{x})_{x \in \mathbb{R}^d} \) exists strongly and is Feller. Moreover, as in the proof of Proposition [20] we deduce from Theorem [2] that \( (P^i_{x})_{x \in \mathbb{R}^d} \) is not Feller-Dynkin. Finally, we deduce the claim from Proposition [7]. □

### 4.3. Conditions for the Finite and State-Independent Case

The contribution of this section is twofold. First, under a strong existence hypothesis, we show that a Feller family of switching diffusions with finite state-independent switching is Feller-Dynkin if and only if the diffusions corresponding to the fixed environments are Feller-Dynkin, see Theorem [4] below. Second, using a weak convergence argument as in the classical diffusion case (see [37]), we show that unique families of switching diffusions with state-independent switching (with countably many possible environments) are Feller whenever the coefficients are continuous, see Proposition [9] below. Building on these observations, we give an equivalent integral-type

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15 see, e.g., [37, Chapter 10] for explicit conditions.
condition for the Feller-Dynkin property in dimension one, see Corollary 3 below, and a converse to Corollary 2, see Corollary 4 below. In this section we assume the following standing assumption.

**Standing Assumption.** We have $S_d = \{0, 1, \ldots, N\}$ for $0 \leq N \leq \infty$, $Q(x) \equiv Q$ and there exists a continuous-time Markov chain with $Q$-matrix $Q$. We denote its unique law by $(P^n_x)_{x \in S_d}$, where the subscript indicates the starting value. Furthermore, $(P^n_x)_{x \in S_d}$ is Feller-Dynkin, see Remark 2.

We assume that $\mathcal{L}$ and $\Sigma$ are as in Section 4.1 and that $D$ is given by (1.11).

**Condition 3.** We have $N < \infty$.

**Condition 4.** We have $q_{ii} \neq 0$ for all $i \in S_d$.

**Condition 5.** The family $(P^y_y)_{y \in S}$ is unique and Feller, and for all $k \in S_d$ and $x \in \mathbb{R}^d$ the MP $(C^k_0(\mathbb{R}^d), K^k, \Sigma, x)$, where $K^k$ is given as in (4.12), has a unique solution $P^k_x$. Furthermore, for all $k \in S_d$ the family $(P^k_x)_{x \in \mathbb{R}^d}$ is Feller and exists strongly.

Next, we state the main observation of this section. Below, we will also comment on explicit conditions implying Condition 5.

**Theorem 4.** Suppose that the Conditions 3, 4 and 5 hold. The following are equivalent:

(i) The family $(P^y_y)_{y \in S}$ is Feller-Dynkin.

(ii) For all $k \in S_d$ the family $(P^k_x)_{x \in \mathbb{R}^d}$ is Feller-Dynkin.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows from Proposition 7.

We prove the implication (ii) $\Rightarrow$ (i) using an explicit construction of the family $(P^y_y)_{y \in S}$. Take a filtered probability space $(\Theta, \mathcal{G}, (G_t)_{t \geq 0}, P)$ satisfying the usual hypothesis of a right-continuous and complete filtration, which supports a Brownian motion $(W_t)_{t \geq 0}$ and an $S_d$-valued continuous-time Markov chain $(Z_t)_{t \geq 0}$ with $Q$-matrix $Q$ and $Z_0 = i$. Here, we mean that $(Z_t)_{t \geq 0}$ is a Markov chain for the filtration $(\mathcal{G}_t)_{t \geq 0}$, which also implies that the $\sigma$-fields $\sigma(W_t, t \in \mathbb{R}_+)$ and $\sigma(Z_t, t \in \mathbb{R}_+)$ are independent, see the proof of Lemma 2. Define inductively

\[
\tau_0 \triangleq 0, \quad \tau_n \triangleq \inf \left( t \geq \tau_{n-1} : Z_t \neq Z_{\tau_{n-1}} \right), \quad n \geq 1,
\]

and

\[
\sigma_0 \triangleq 0, \quad \sigma_n \triangleq \tau_n - \tau_{n-1} = \inf \left( t \geq 0 : Z_{t+\tau_{n-1}} \neq Z_{\tau_{n-1}} \right), \quad n \geq 1.
\]

Because no state of $(Z_t)_{t \geq 0}$ is absorbing due to Condition 4, we have a.s. $\tau_n < \infty$ for all $n \in \mathbb{N}$. Furthermore, for all $n \in \mathbb{N}$ the random time $\tau_n$ is a $(G_{\tau_{n-1}})_{t \geq 0}$-stopping time and the random time $\sigma_n$ is a $(G_{\tau_{n-1}})_{t \geq 0}$-stopping time, see [20, Proposition 1.1.12] and [19, Lemma 6.5, Theorem 6.7]. Due to [33, Proposition V.1.5] and Lévy’s characterization, the process $(W^n_t)_{t \geq 0} = (W_{t+\tau_n} - W_{\tau_n})_{t \geq 0}$ is a $(G_{\tau_{n-1}})_{t \geq 0}$-Brownian motion and therefore independent of $G_{\tau_n}$. For all $k \in S_d$ let $F^k : \mathbb{R}^d \times \mathbb{C}_c \to \mathbb{C}_c$ be as in Definition 2 and set $(Y^n_t)_{t \geq 0} \triangleq F^k(x, W)$. By induction, define further

\[
(Y^n_t)_{t \geq 0} \triangleq \sum_{k=0}^{N} F^k(Y^{n-1}_{\sigma_n}, W^n)1 \{ Z_{\tau_n} = k \}, \quad n \in \mathbb{N},
\]

and set

\[
Y_t \triangleq x 1 \{ t = 0 \} + \sum_{n=0}^{\infty} Y^n_{t-\tau_n} 1 \{ \tau_n < t \leq \tau_{n+1} \}.
\]
The process \((Y_t)_{t \geq 0}\) has continuous paths and similar arguments as used in the proof of Lemma 3 show that \((Y_t)_{t \geq 0}\) is adapted, too. Next, four technical lemmata follow.

**Lemma 4.** The law of \((Y_t, Z_t)_{t \geq 0}\) is given by \(P_{(x,i)}\).

**Proof:** The process \((V_t)_{t \geq 0} \triangleq F^c(Y^n_{\sigma_n}^{-1}, W^n)\) has the dynamics
\[
dV_t = b(V_t, e)dt + a^\frac{1}{2}(V_t, e)dW^n_t, \quad V_0 = Y^n_{\sigma_n}^{-1}.
\]
Thus, due to classical rules for time-changed stochastic integrals, for \(t \in [\tau_n, \tau_{n+1}]\) on \(\{Z_{\tau_n} = e\}\) we have
\[
Y^n_{t - \tau_n} = F^c(Y^n_{\sigma_n}^{-1}, W^n)_{t - \tau_n}
\]
\[
= Y^n_{\sigma_n}^{-1} + \int_0^{t-\tau_n} b(Y_s, e)ds + \int_0^{t-\tau_n} a^\frac{1}{2}(Y_s, e)dW^n_s
\]
\[
= Y^n_{\sigma_n}^{-1} + \int_{\tau_n}^t b(Y_{s-\tau_n}, e)ds + \int_{\tau_n}^t a^\frac{1}{2}(Y_{s-\tau_n}, e)dW_s
\]
\[
= Y^n_{\sigma_n}^{-1} + \int_{\tau_n}^t b(Y_s, e)ds + \int_{\tau_n}^t a^\frac{1}{2}(Y_s, e)dW_s
\]
\[
= Y^n_{\sigma_n}^{-1} + \int_{\tau_n}^t b(Y_s, Z_s)ds + \int_{\tau_n}^t a^\frac{1}{2}(Y_s, Z_s)dW_s.
\]
Iterating yields that for \(t \in [\tau_n, \tau_{n+1}]\)
\[
Y^n_{t - \tau_n} = x + \int_0^t b(Y_s, Z_s)ds + \int_0^t a^\frac{1}{2}(Y_s, Z_s)dW_s.
\]
Therefore, the process \((Y_t)_{t \geq 0}\) satisfies the SDE
\[
dY_t = b(Y_t, Z_t)dt + a^\frac{1}{2}(Y_t, Z_t)dW_t, \quad Y_0 = x,
\]
and, consequently, the uniqueness of \(P_{(x,i)}\) and Lemma 2 imply that the law of \((Y_t, Z_t)_{t \geq 0}\) coincides with \(P_{(x,i)}\). \(\square\)

**Lemma 5.** For all Borel sets \(G \subseteq \Sigma_c\) we have a.s.
\[
P\left((W^n_t)_{t \geq 0} \in G | (G_{\sigma_n}, \sigma_{n+1})\right) = P\left((W^n_t)_{t \geq 0} \in G\right).
\]

**Proof:** Let \(\mathcal{W}_x\) be the Wiener measure with starting value \(x \in \mathbb{R}^d\) and \(P^*_e\) be the law of a Markov chain with \(Q\)-matrix \(Q\) and starting value \(e \in S_d\). It follows from Itô's formula that the process \((Z_t, W_t)_{t \geq 0}\) solves a martingale problem with respect to the filtration \((\mathcal{G}_t)_{t \geq 0}\) in the sense of \(\text{[11], Section 4.3.}\) Moreover, due to \(\text{[11], Theorem 4.10.1},\) the product measure \(P^*_e \otimes \mathcal{W}_x\) is the unique solution to same martingale problem on the canonical space with starting value \((e, x)\). It follows from \(\text{[22], Theorem 14.22}\) that for all Borel \(F \subseteq \Sigma_d \times \Sigma_c\) the map \((x, e) \mapsto (P^*_e \otimes \mathcal{W}_x)(F)\) is Borel. Consequently, it follows from \(\text{[11], Proposition 4.1.5, Theorems 4.4.2}\) that the process \((Z_t, W_t)_{t \geq 0}\) is a strong Markov process in the following sense: For all \(F \in \mathcal{F}\) and all a.s. finite \((\mathcal{G}_t)_{t \geq 0}\)-stopping times \(\theta\) a.s.
\[
P\left((Z_{t+\theta}, W_{t+\theta})_{t \geq 0} \in F | \mathcal{G}_\theta\right) = (P^*_e \otimes \mathcal{W}_x)(F).
\]
\(^{16}\)Where we use the Skorokhod topology for \(\Sigma_d\)
Let $F \subseteq \Sigma_d$ be Borel. The strong Markov properties of $(Z_t)_{t \geq 0}, (W_t)_{t \geq 0}$ and $(Z_t, W_t)_{t \geq 0}$ imply that a.s.

$$P((Z_{t+\tau_n})_{t \geq 0} \in F, (W_t)_{t \geq 0} \in G|\mathcal{G}_{\tau_n}) = P_{Z_{\tau_n}}^* (F) W_{\mathcal{G}_{\tau_n}} (G)$$

$$= P((Z_{t+\tau_n})_{t \geq 0} \in F|\mathcal{G}_{\tau_n}) P((W_t)_{t \geq 0} \in G|\mathcal{G}_{\tau_n}).$$

This implies that $\sigma(W^n_t, t \in \mathbb{R}_+)$ and $\sigma(\sigma_n, \sigma_{n+1})$ are independent given $\mathcal{G}_{\tau_n}$. Thus, [19, Proposition 5.6] and the independence of $\sigma(W^n_t, t \in \mathbb{R}_+)$ and $\mathcal{G}_{\tau_n}$ yield that a.s.

$$P((W^n_t)_{t \geq 0} \in G|\sigma(\mathcal{G}_{\tau_n}, \sigma_{n+1})) = P((W^n_t)_{t \geq 0} \in G|\mathcal{G}_{\tau_n}) = P((W^n_t)_{t \geq 0} \in G),$$

which is the claim.

\[\square\]

**Lemma 6.** For all $n \in \mathbb{N}_0$ we have $\|Y^0_{n+1}\| \to \infty$ in probability as $\|x\| \to \infty$.

**Proof:** We use induction. Because the process $(Y^0_t)_{t \geq 0}$ has law $P^1$ (by the uniqueness assumption) and $(Y^0_t)_{t \geq 0}$ is independent of $\sigma_1 = \tau_1$, we can conclude the induction base from the hypothesis (ii) of Theorem 4. More precisely, we have for all $m \in \mathbb{N}$

$$P(|Y^0_{\sigma_1}| \leq m) = \int_0^\infty P^1_{x}(|X_s| \leq m) P(\sigma_1 \in ds) \to 0$$

as $\|x\| \to \infty$, see the proof of Proposition 7. Suppose now that the claim holds for $n \in \mathbb{N}_0$. Using the Lemmata 2 and 5 and [19, Theorem 5.4], we obtain

\begin{align}
P(|Y^0_{\sigma_{n+2}}| \leq m) &= \sum_{k=0}^N P(|F^k(Y^0_{\sigma_{n+1}}, W^n+1)_{\sigma_{n+2}}| \leq m, Z_{\tau_{n+1}} = k) \\
&= \sum_{k=0}^N E[P(|F^k(Y^n_{\sigma_{n+1}}, W^n+1)_{\sigma_{n+2}}| \leq m|\sigma(\mathcal{G}_{\tau_{n+1}}, \sigma_{n+2})) 1 \{Z_{\tau_{n+1}} = k\}] \\
&= \sum_{k=0}^N \int P(|F^k(Y^n_{\sigma_{n+1}}(\omega), W^n+1)_{\sigma_{n+2}}(\omega)| \leq m) 1 \{Z_{\tau_{n+1}}(\omega) = k\} P(d\omega) \\
&= \sum_{k=0}^N \int P_{k}^n_{Y^n_{\sigma_{n+1}}(\omega)}(|X_{\sigma_{n+2}}(\omega)| \leq m) 1 \{Z_{\tau_{n+1}}(\omega) = k\} P(d\omega).
\end{align}

Take $(x_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^d$ such that $\|x_k\| \to \infty$ as $k \to \infty$. A well-known characterization of convergence in probability is the following: A sequence $(Z^k)_{k \in \mathbb{N}}$ converges in probability to a random variable $Z$ if and only if each subsequence of $(Z^k)_{k \in \mathbb{N}}$ contains a further subsequence which converges almost surely to $Z$, see, e.g., [19, Lemma 3.2]. Consequently, $(x_k)_{k \in \mathbb{N}}$ contains a subsequence along which $\|Y^0_{\sigma_{n+1}}\| \to \infty$ almost surely. Due to the dominated convergence theorem, we deduce from (4.21) that along the same subsequence $\|Y^0_{\sigma_{n+1}}\| \to \infty$ in probability. Thus, applying again the subsequence criterion, we can extract a further subsequence such that the convergence holds almost surely. Finally, applying the subsequence criterion a third time (but this time the converse direction), we conclude the claim. \[\square\]
Lemma 7. For all $n \in \mathbb{N}_0$ we have $\|Y^n_{t-\tau_n}\| \to \infty$ on $\{t > \tau_n\}$ in probability as $\|x\| \to \infty$.

Proof: Because $\sigma(W^n_t, t \in \mathbb{R}_+)$ is independent of $\mathcal{G}_{\tau_n}$, we show as in the proof of Lemma 6 that
\[
P(\|Y^n_{t-\tau_n}\| \leq m, t > \tau_n)
= \sum_{k=0}^{N} \int P^{k}_{\sigma_{\tau_n}(\omega)}(\|X_{t-\tau_n(\omega)}\| \leq m)1\{t > \tau_n(\omega)\}1\{Z_{\tau_n(\omega)}(\omega) = k\}P(\omega).
\]
Using Lemma 6 and the argument in its proof, we see that the claim follows. $\square$

Let $f \in C_0(S)$ and $t > 0$. We have
\[
E_{(x,t)}[f(X_t)] = E[f(Y_t, Z_t)]
= \sum_{n=0}^{\infty} \sum_{k=0}^{N} E[f(Y_t, Z_t)1\{\tau_n < t \leq \tau_{n+1}\}1\{Z_{\tau_n} = k\}]
= \sum_{n=0}^{\infty} \sum_{k=0}^{N} E[f(Y^n_{t-\tau_n}, k)1\{\tau_n < t \leq \tau_{n+1}\}1\{Z_{\tau_n} = k\}] \to 0
\]
as $\|x\| \to \infty$, which follows from the Lemmata 4 and 5 and the dominated convergence theorem. This completes the proof. $\square$

In the remaining of this section we will comment on Condition 5 and give deterministic conditions implying it. The main consequence of our proceeding discussion are the following two corollaries. We state them now and postpone their proofs till the end of the section.

Corollary 3. Suppose that $d = 1$, that Condition 4 holds and that for all $k \in S_d$ the map $x \mapsto b(x, k)$ is locally Hölder continuous and the map $x \mapsto a^2(x, k)$ is locally Hölder continuous with exponent larger or equal than $\frac{1}{2}$ and that $a^2(\cdot, k) \neq 0$. Furthermore, for all $k \in S_d$ suppose that
\begin{equation}
\lim_{x \to \pm \infty} \int_0^x \exp \left(-2 \int_0^y \frac{b(k, z)}{a(k, z)}dz\right) \int_y^0 \exp \left(2 \int_0^u \frac{b(k, z)}{a(k, z)}dz\right)du dy = \infty.
\end{equation}
Then, the family $(P_x)_{x \in S}$ exists uniquely, is strongly Markov and Feller. Moreover, if in addition $N < \infty$ the following are equivalent:

(i) $(P_x)_{x \in S}$ is Feller-Dynkin.

(ii) For all $k \in S_d$ one of the conditions (4.23) and (4.24) holds and one of the conditions (4.25) and (4.26) holds:
\begin{equation}
\int_0^{\infty} \exp \left(-2 \int_0^y \frac{b(k, z)}{a(k, z)}dz\right)dy < \infty.
\end{equation}

\begin{equation}
\begin{cases}
\int_0^{\infty} \exp \left(-2 \int_0^y \frac{b(k, z)}{a(k, z)}dz\right)dy = \infty, \\
\int_0^{\infty} \exp \left(-2 \int_0^y \frac{b(k, z)}{a(k, z)}dz\right) dy = \infty,
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\int_0^{\infty} \exp \left(-2 \int_0^y \frac{b(k, z)}{a(k, z)}dz\right) dy = \infty, \\
\int_0^{\infty} \exp \left(-2 \int_0^y \frac{b(k, z)}{a(k, z)}dz\right) \int_y^{\infty} \exp \left(2 \int_0^u \frac{b(k, z)}{a(k, z)}dz\right)du dy = \infty.
\end{cases}
\end{equation}
Assume that Conditions 3 and 4 hold and that for all Corollary 4.

\[ (4.25) \]
\[ \int_{-\infty}^{0} \exp \left( 2 \int_{y}^{0} \frac{b(k, z)}{a(k, z)} dz \right) dy < \infty. \]

\[ (4.26) \]
\[ \left\{ \begin{array}{l}
\int_{-\infty}^{0} \exp \left( 2 \int_{y}^{0} \frac{b(k, z)}{a(k, z)} dz \right) dy = \infty,
\int_{-\infty}^{0} \exp \left( 2 \int_{y}^{0} \frac{b(k, z)}{a(k, z)} dz \right) dy = \infty,
\int_{-\infty}^{0} \exp \left( \int_{y}^{0} \frac{b(k, z)}{a(k, z)} dz \right) dy = \infty.
\end{array} \right. \]

Corollary 4. Assume that Conditions 3 and 4 hold and that for all \( i \in S_{d} \) the maps \( x \mapsto b(x, i) \) and \( x \mapsto a^{2}(x, i) \) are locally Lipschitz continuous and that for all \( (x, i) \in S \) the solution \( (C_{0}^{2}(\mathbb{R}^{d}), \mathcal{K}^{i}, \Sigma, x) \) has a solution. Furthermore, suppose that for each \( i \in S_{d} \) there is an \( r_{i} > 0 \) and two locally Hölder continuous functions \( b_{i} : [r_{i}, \infty) \to \mathbb{R} \) and \( a_{i} : [r_{i}, \infty) \to (0, \infty) \) such that for all \( x \in \mathbb{R}^{d} \colon \|x\| \geq 2r_{i} \):
\[ \langle x, a(x, i)x, \rangle \leq a_{i} \left( \frac{\|x\|^2}{2} \right), \]
\[ \text{trace}(a(x, i)x) + 2\langle x, b(x, i)x \rangle \geq b_{i} \left( \frac{\|x\|^2}{2} \right) \langle x, a(x, i)x \rangle, \]
and either
\[ p_{i}(r) \triangleq \int_{1}^{r} \exp \left( - \int_{1}^{y} b_{i}(z) dz \right) dy, \quad \lim_{r \to \infty} p_{i}(r) < \infty, \]
or
\[ \lim_{r \to \infty} p(r) = \infty \quad \text{and} \quad \int_{1}^{\infty} p'(y) \int_{y}^{\infty} \frac{dz}{a_{i}(z)p(z)} dy = \infty. \]

Then, \( (P_{x})_{x \in S} \) is Feller-Dynkin.

In the one dimensional case without drift the conditions from Corollary 3 are particularly easy to state as the following corollary shows.

Corollary 5. Suppose that \( d = 1 \), that Condition 4 holds, that \( b \equiv 0 \), that the map \( x \mapsto a^{2}(x, k) \) is locally Hölder continuous with exponent larger or equal that \( \frac{1}{2} \) and that \( a^{2}(\cdot, k) \not\equiv 0 \). Then, the family \( (P_{x})_{x \in S} \) exists uniquely, is strongly Markov and Feller. Moreover, if in addition \( N < \infty \) the following are equivalent:

(i) \( (P_{x})_{x \in S} \) is Feller-Dynkin.

(ii) For all \( k \in S_{d} \) the following hold:

\[ (4.27) \]
\[ \int_{0}^{\infty} \frac{u}{a(u, k)} du = \int_{-\infty}^{0} \frac{-u}{a(u, k)} du = \infty. \]

Proof: Due to [20, Problem 5.5.27], (4.27) holds in the case \( b \equiv 0 \). Thus, the existence, uniqueness, the strong Markov and the Feller property of the family \( (P_{x})_{x \in S} \) follow from Corollary 3. Fubini’s theorem yields that
\[ \int_{0}^{\infty} \int_{y}^{\infty} \frac{1}{a(u, k)} du dy = \int_{0}^{\infty} \int_{0}^{u} dy \frac{1}{a(u, k)} du = \int_{0}^{\infty} \frac{u}{a(u, k)} du. \]

In the same manner we obtain that
\[ \int_{-\infty}^{0} \int_{y}^{0} \frac{1}{a(u, k)} du dy = \int_{-\infty}^{0} \int_{0}^{u} dy \frac{1}{a(u, k)} du = \int_{-\infty}^{0} \frac{-u}{a(u, k)} du. \]

\[ \text{see, e.g., [37, Chapter 10] for explicit conditions} \]
Thus, the equivalence of (i) and (ii) follows also from Corollary 8 when \( N < \infty \). □

By [36, Theorem 3.2], the family \((P_x)_{x \in S}\) has the strong Feller property if it is Feller and for all \( k \in S_d \) the families \((P_{x}^{k})_{x \in \mathbb{R}^d}\) have the strong Feller property. Consequently, when \( N < \infty \), the strong Feller property and the Feller-Dynkin property are both inherited from the relative properties of processes in the fixed environments. We give a short example for a switching diffusion which has the strong Feller property, but not the Feller-Dynkin property.

Example 4. Let \( d = 1, S_d = \{1,2\}, b \equiv 0 \) and
\[
a(x,k) \equiv \begin{cases} 
1 + x^4, & k = 1, \\
1, & k = 2.
\end{cases}
\]
We conclude from Corollary 5 that \((P_y)_{y \in S}\) exists uniquely and is Feller. Furthermore, due to [37, Corollary 10.1.4], \((P_{x}^{k})_{x \in \mathbb{R}}\) has the strong Feller property for \( k = 1,2 \). (Of course, \( P_{x}^{0} \) is the Wiener measure and \((P_{x}^{k})_{x \in \mathbb{R}}\) is well-known to be strong Feller.) Therefore, [36, Theorem 3.2] implies that \((P_y)_{y \in S}\) has the strong Feller property, too. However, for \( k = 1 \) the condition (4.27) fails because
\[
\int_{0}^{\infty} \frac{x \, dx}{1 + x^4} = \frac{\pi}{4} < \infty.
\]
Therefore, the family \((P_x)_{x \in S}\) is not Feller-Dynkin due to Corollary 5.

Next, we give a condition for the Feller property of \((P_x)_{x \in S}\) in the spirit of the classical continuity conditions for diffusions. It improves several results known for switching diffusions.

Proposition 8. Suppose that \( b \) and \( a \) are continuous and that \((P_y)_{y \in S}\) is unique, then \((P_y)_{y \in S}\) is strongly Markov and Feller.

Proof: We deduce the strong Markov property from Proposition 2 and use the strategy from the proof of [17, Theorem IX.3.39] to show the Feller property.

We denote
\[
C \triangleq \{ u \in C_0(S_d) : Qu \in C_0(S_d) \}, \quad C_i \triangleq \{ (u,Qu) : u \in C \} \subset C_0(S_d) \times C_0(S_d).
\]
Because \( C_0(S_d) \) endowed with the uniform metric is a separable metric space, the space \( C_0(S_d) \times C_0(S_d) \) endowed with the taxicap uniform metric is a separable metric space. Moreover, since also subspaces of separable metric spaces endowed with the subspace metric are separable metric spaces, the space \( C_i \) is a separable metric space endowed with the metric \( d \) given by
\[
d(x,y) = \| x_1 - y_1 \|_{\infty} + \| x_2 - y_2 \|_{\infty}
\]
for \( x = (x_1,x_2), y = (y_1,y_2) \in C_i \). Consequently, we find a countable set \( C_d \subset C \) such that for any \((x,y) \in C_i \) there exists a sequence \((f_n)_{n \in \mathbb{N}} \subset C_d \) such that
\[
d((f_n, Qf_n),(x,y)) \to 0
\]
as \( n \to \infty \). Due to Proposition 4 this implies that a Borel probability measure on \( \Sigma_d \) solves the MP \((C,Q,\Sigma_d,\varepsilon)\) if and only if it solves the MP \((C_d,Q,\Sigma_d,\varepsilon)\). For \( 1 \leq i,j \leq d \) and \( k \in \mathbb{N} \) let \( f_i(x) = x_i \) and \( f_{ij}(x) = x_ix_j \) and \( g_{ik}^k, g_{ij}^k \in C_c^2(\mathbb{R}^d) \) be
such that \( g_i^k = f_i \) and \( g_j^k = f_{ij} \) on \( \{ x \in \mathbb{R}^d ; ||x|| \leq k \} \). The proof of Lemma \( \text{[8, Remark 5.4.12]} \) reveal that \( D \) can be replaced by the countable set
\[
D \triangleq \left\{ u, g_{ij}^k, g_i^k : 1 \leq i, j \leq d, k \in \mathbb{N}, u \in C_d \right\}.
\]
Now, Proposition \( \text{[2]} \) implies that the family \( (P_x)_{x \in S} \) is strongly Markov.

It remains to prove that \( (P_x)_{x \in S} \) has the Feller property. In the following, we show that \( x \mapsto P_x \) is continuous, i.e. that \( x_n \to x \) implies \( P_{x_n} \to P_x \) weakly as \( n \to \infty \). In this case, for all \( x \in S \) and \( t \in \mathbb{R}_+ \) the map \( \omega \mapsto \omega(t) \) is \( P_x \)-a.s. continuous (see \([\text{11, Proposition 3.5.2}]) and note that \( P_x(\Delta X_t \neq 0) = 0 \), the continuous mapping theorem implies that \( (P_x)_{x \in S} \) has the Feller property. This finishes the proof of our claim.

Before we proceed showing the continuity, let us clarify our terminology: When we say that a sequence of càdlàg processes is tight, we mean that its laws are tight or, equivalently, relatively compact by Prohorov’s theorem (see \([\text{11, Theorem 3.2.2}])). If we speak of an accumulation point of a sequence of processes, we refer to an accumulation point of the corresponding sequence of laws.

Because of the product topology of \( S_d \) and the discrete topology of \( S_d \) it suffices to show that for all \( (x, i) \in S \) we have \( P_{y_n} \to P_{(x, i)} \) weakly as \( n \to \infty \) whenever \( (y_n)_{n \in \mathbb{N}} \subset S \) is such that \( y_n = (x_n, i) \) for \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d \) with \( x_n \to x \) as \( n \to \infty \). For all \( n \in \mathbb{N} \) denote by \( (Y^n_t)_{t \geq 0}, (Z^n_t)_{t \geq 0} \) and \( (W^n_t)_{t \geq 0} \) the processes from Lemma \( \text{[2]} \) corresponding to \( P_{(x_n, i)} \). Let \( \| \cdot \| \) be the Euclidean norm on \( \mathbb{R}^{d+1} \). For \( m \in \mathbb{R}_+ \) we define
\[
\tau_m \triangleq \inf \{ t \in \mathbb{R}_+ : \| X_t \| \geq m \text{ or } \| X_{t-} \| \geq m \}.
\]
We note that \( \tau_m \) is an \( (\mathcal{F}^n_t)_{t \geq 0} \)-stopping time, see \([\text{11, Proposition 2.1.5}]). For \( n \in \mathbb{N} \) and \( m \in \mathbb{R}_+ \) we set
\[
\tau_{n,m} \triangleq \tau_m \circ (Y^n_t, Z^n_t)_{t \geq 0}.
\]
Next, four technical lemmata follow.

**Lemma 8.** For all \( m \in \mathbb{R}_+ \) the sequence \( \{(Y^n_{t \wedge \tau_{n,m}}, Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \) is tight.

**Proof:** The sequence \( \{(Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \) is tight in \( S_d \) when equipped with the Skorokhod topology, because the law of \( (Z^n_t)_{t \geq 0} \) is independent of \( n \in \mathbb{N} \) and all Borel probability measures on Polish spaces are tight (see \([\text{11, Lemma 3.2.1}]) for all \( n \in \mathbb{N} \) the process \( (Y^n_{t \wedge \tau_{n,m}})_{t \geq 0} \) has continuous paths. Below, we show that \( \{(Y^n_{t \wedge \tau_{n,m}})_{t \geq 0}, n \in \mathbb{N}\} \) is tight in \( S_c \) equipped with the local uniform topology. In this case, \([\text{11, Problem 4.25}]) implies that \( \{(Y^n_{t \wedge \tau_{n,m}}), n \in \mathbb{N}\} \) is also tight in \( D(\mathbb{R}_+, \mathbb{R}^d) \), which is the space of càdlàg functions \( \mathbb{R}_+ \to \mathbb{R}^d \) equipped with the Skorokhod topology.

We claim that this already implies the tightness of \( \{(Y^n_{t \wedge \tau_{n,m}}, Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \). To see this, we use the characterization of tightness given in \([\text{11, Corollary 3.7.4}]). Let us recall it as a fact:

**Fact 1.** Let \( (E, r) \) be a Polish space. A sequence \( (\mu^n)_{n \in \mathbb{N}} \) of Borel probability measures on \( D(\mathbb{R}_+, E) \) is tight if and only if the following hold:
\[
(a) \text{ For all } t \in \mathbb{Q}_+ \text{ and } \epsilon > 0 \text{ there exists a compact set } C(t, \epsilon) \subseteq E \text{ such that } \limsup_{n \to \infty} \mu^n(X_t \notin C(t, \epsilon)) \leq \epsilon.
\]
\( \text{[18, see also \([\text{11, Corollary 3.3.2}] \text{ and } \text{[17, Proposition VI.1.17}])} \)
(b) For all $\epsilon > 0$ and $t > 0$ there exists a $\delta > 0$ such that
\[
\limsup_{n \to \infty} \mu^n(w'(X, \delta, t) \geq \epsilon) \leq \epsilon,
\]
where
\[
w'(\alpha, \theta, t) \triangleq \inf_{\{t_i\}} \sup_{u, v \in [t_{i-1}, t_i]} r(\alpha(u), \alpha(v)),
\]
with $\{t_i\}$ ranging over all partitions of the form $0 = t_0 < t_1 < \cdots < t_n < t$ with $\min_{1 \leq i \leq n} (t_i - t_{i-1}) \geq \theta$ and $n \geq 1$.

We equip $S$ with the metric $r((x, i), (y, k)) \triangleq ||x - y|| + 1\{i \neq j\}$, which generates the product topology on $S$. Let us first check that $\{(Y^m_{t_{i,m}}, Z^n_{t_{i,m}})_{i \geq 0}, n \in \mathbb{N}\}$ satisfies Fact 1(a). Fix $t \in \mathbb{Q}_+$ and $\epsilon > 0$. Using Fact 1 the tightness of $\{(Z^n_{s})_{s \geq 0}, n \in \mathbb{N}\}$ in $\Sigma_d$ and $\{(Y^m_{s_{i,m}})_{x \geq 0}, n \in \mathbb{N}\}$ in $D([\mathbb{R}_+], \mathbb{R}^d)$ implies that there exists a compact set $C_1(t, \epsilon) \subset S_d$ and a compact set $C_2(t, \epsilon) \subset \mathbb{R}^d$ such that
\[
\limsup_{n \to \infty} P(Y^m_{t_{i,m}} \not\in C_1(t, \epsilon)) \leq \frac{\epsilon}{2},
\]
\[
\limsup_{n \to \infty} P(Y^m_{t_{i,m}} \not\in C_2(t, \epsilon)) \leq \frac{\epsilon}{2}.
\]
The set $K(t, \epsilon) \triangleq C_1(t, \epsilon) \times C_2(t, \epsilon) \subset S$ is also compact and we have
\[
\limsup_{n \to \infty} P(Y^m_{t_{i,m}}, Z^n_{t_{i,m}} \not\in K(t, \epsilon)) \leq \limsup_{n \to \infty} P(Y^m_{t_{i,m}} \not\in C_2(t, \epsilon)) + \limsup_{n \to \infty} P(Z^n_{t_{i,m}} \not\in C_1(t, \epsilon)) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
In other words, $\{(Y^m_{t_{i,m}}, Z^n_{t_{i,m}})_{i \geq 0}, n \in \mathbb{N}\}$ satisfies Fact 1(a). Next, we explain that it also satisfies Fact 1(b). We claim that the continuous paths of $(Y^m_{t})_{t \geq 0}$ imply that up to a null set
\[
(4.30) \quad w'((Y^m_{s_{i,m}}), Z^n_{s_{i,m}})_{s \geq 0}, \theta, t) \leq 2w'((Y^m_{s_{i,m}}), 0)_{s \geq 0}, 2\theta, t) + w'((0, Z^n_{s_{i,m}})_{s \geq 0}, \theta, t).
\]
To see this, take $(\alpha, \omega) \in \Sigma_c \times \Sigma_d$. Let $\{t_i\}$ be a partition of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n \leq t$ with $\min_{1 \leq i \leq n} (t_i - t_{i-1}) \geq \theta$. By adding points if necessary, we can assume that $\max_{1 \leq i \leq n} (t_i - t_{i-1}) \leq 2\theta$. In this case, we have
\[
\sup_{u, v \in [t_{i-1}, t_i]} r((\alpha(u), 0), (\alpha(v), 0)) \leq \sup \{r((\alpha(u), 0), (\alpha(v), 0)) : 0 \leq u, v \leq t, |u - v| \leq 2\theta\}.
\]
Due to [13, Lemma 15.3], we have
\[
\sup \{r((\alpha(u), 0), (\alpha(v), 0)) : 0 \leq u, v \leq t, |u - v| \leq 2\theta\} \leq 2w'(0, 2\theta, t).
\]
Therefore, we conclude that
\[
w'((\alpha, \omega), \theta, t) \leq 2w'(0, 2\theta, t) + w'((0, \omega), \theta, t),
\]
which implies (4.30). Fix $\epsilon > 0$ and $t > 0$ and let $\delta > 0$ be such that
\[
\limsup_{n \to \infty} P(w'((Y^m_{s_{i,m}}), 0)_{s \geq 0}, 2\delta, t) \geq \frac{\epsilon}{4} \leq \frac{\epsilon}{4},
\]
\[
\limsup_{n \to \infty} P(w'((0, Z^n_{s})_{s \geq 0}, \delta, t) \geq \frac{\epsilon}{2} \leq \frac{\epsilon}{2}.
\]
This δ exists due to Fact 1(b) and the fact that \( w' \) is increasing in δ. Note that for two non-negative random variables \( V \) and \( U \) we have

\[
P(2U + V \geq 2\epsilon) \leq P(U \geq \epsilon) + P(V \geq \epsilon).
\]

Hence, we deduce from (4.30) that

\[
\lim_{n \to \infty} \sup_{n} P(w'(Y_{s+n}^n, Z_{s+n}^n), \delta, t) \geq \epsilon \leq \frac{3\delta}{4} \leq \epsilon.
\]

We conclude from Fact 1 that \( \{(Y_{s+n}^n, Z_{s+n}^n), \delta, t \geq 0, n \in \mathbb{N} \} \) is tight.

It remains to show that \( \{(Y_{s+n}^n, Z_{s+n}^n), t \geq 0, n \in \mathbb{N} \} \) is tight in \( \Sigma_{e} \). Let \( p > 2 \) and recall the inequalities

\[
(v + u)^{p} \leq 2^{p}(v^{p} + u^{p}), \quad v, u \geq 0, \quad \left\| \int_{0}^{t} f(s) ds \right\| \leq \int_{0}^{t} \| f(s) \| ds.
\]

Let \( T \in \mathbb{R}_{+} \) and \( s < t \leq T \). We write \( x \preceq y \) whenever \( x \leq \text{const.} \ y \) where the constant only depends on \( T, p, m, b \) and \( a \). We deduce from the triangle inequality, (4.31) and [21], Remark 3.3.30 (i.e. a multidimensional version of the Burkholder-Davis-Gundy inequality) that

\[
E\left[ \| Y_{s+n}^n - Y_{s+n}^n \|^{p} \right] = E\left[ \left\| \int_{s \wedge \tau_{n}}^{t \wedge \tau_{n}} b(Y_{r}^{n}, Z_{r}^{n}) dr + \int_{s \wedge \tau_{n}}^{t \wedge \tau_{n}} \frac{1}{2} a^{2}(Y_{r}^{n}, Z_{r}^{n}) dW_{r}^{n} \right\|^{p} \right]
\]

\[
\leq 2^{p} E\left[ \left\| \int_{s \wedge \tau_{n}}^{t \wedge \tau_{n}} b(Y_{r}^{n}, Z_{r}^{n}) dr \right\|^{p} \right] + 2^{p} E\left[ \left\| \int_{s \wedge \tau_{n}}^{t \wedge \tau_{n}} \frac{1}{2} a^{2}(Y_{r}^{n}, Z_{r}^{n}) dW_{r}^{n} \right\|^{p} \right]
\]

\[
\leq E\left( \left( \int_{s \wedge \tau_{n}}^{t \wedge \tau_{n}} \| b(Y_{r}^{n}, Z_{r}^{n}) \| dr \right)^{p} \right) + E\left( \left( \int_{s \wedge \tau_{n}}^{t \wedge \tau_{n}} \| a^{2}(Y_{r}^{n}, Z_{r}^{n}) \|^{2} dr \right) \right)^{\frac{p}{2}}
\]

\[
\leq (t - s)^{p} + |t - s|^{\frac{p}{2}}.
\]

Furthermore, we have

\[
\sup_{n \in \mathbb{N}} E[\| Y_{0}^{n} \|] = \sup_{n \in \mathbb{N}} \| x_{n} \| < \infty,
\]

because convergent sequences are bounded. Consequently, [21], Problem 2.4.11, Remark 2.4.13 (i.e. Kolmogorov’s tightness criterion) imply that \( \{(Y_{s+n}^n, Z_{s+n}^n), t \geq 0, n \in \mathbb{N} \} \) is tight in \( \Sigma_{e} \). This completes the proof. \( \square \)

The following lemma is a version of Lemma 3 for uniqueness of martingale problems.

**Lemma 9.** Let \( \rho \) be an \( (\mathcal{F}_{t})_{t \geq 0} \)-stopping time and suppose that \( P \) is a probability measure on \( (\Omega, \mathcal{F}) \) such that \( P(X_{0} = x) = P(\Sigma) = 1 \) and

\[
M_{t}^{\rho} = f(X_{t \wedge \rho}) - f(X_{0}) - \int_{0}^{t \wedge \rho} \mathcal{L} f(X_{s}) ds, \quad t \in \mathbb{R}_{+},
\]

is a local \( P \)-martingale for all \( f \in D \). Then, \( P = P_{x} \) on \( \mathcal{F}_{\rho}^{o} \).
Proof: The claim of this lemma is closely related to the concept of local uniqueness as introduced in [17] and it can be proven with the strategy from [17, Theorem III.2.40]. To each $G \in \mathcal{F}$ we can associate a (not necessarily unique) set $G' \in \mathcal{F}_\rho^0 \otimes \mathcal{F}$ such that

$$G \cap \{\rho < \infty\} = \{\omega \in \Omega: \rho(\omega) < \infty, (\omega, \theta_{\rho(\omega)}\omega) \in G'\},$$

see [17, Lemma III.2.44]. Now, set

$$Q(G) \triangleq P(G \cap \{\rho = \infty\}) + \int P(d\omega)P_{\omega(\rho(\omega))}(d\omega^*)1_{\{\rho(\omega) < \infty\}}1_{G'}(\omega, \omega^*).$$

Due to [17, Lemma III.2.47], $Q$ is a probability measure on $(\Omega, \mathcal{F})$. For $G \in \mathcal{F}_0^\rho$ we can choose $G' = G \times \Omega$. Consequently, we have

$$Q(X_0 = x) = P(X_0 = x) = 1.$$

Set

$$\Sigma^* \triangleq \{\omega \in \Omega: (\omega_t, t)_{t \geq 0} \in \Sigma\} \supseteq \Sigma$$

and note that

$$\Sigma \cap \{\rho < \infty\} = \{\omega \in \Omega: \rho(\omega) < \infty, (\omega, \theta_{\rho(\omega)}\omega) \in \Sigma^* \times \Sigma\}.$$

Consequently, we have

$$Q(\Sigma) = P(\Sigma \cap \{\rho = \infty\}) + \int P(d\omega)P_{\omega(\rho(\omega))}(\Sigma)1_{\{\rho(\omega) < \infty\}}1_{\Sigma^*}(\omega)
= P(\Sigma \cap \{\rho = \infty\}) + P(\Sigma^* \cap \{\rho < \infty\}) \geq P(\Sigma) = 1.$$

Fix $m \in \mathbb{N}$ and a bounded $(\mathcal{F}_t^\rho)_{t \geq 0}$-stopping time $\psi$. For $\omega, \alpha \in \Omega$ and $t \in \mathbb{R}_+$ we set

$$z(\omega, \alpha)(t) \triangleq \begin{cases} \omega(t), & t \leq \rho(\omega), \\ \alpha(t - \rho(\omega)), & t > \rho(\omega), \end{cases}$$

and

$$V(\omega, \alpha) \triangleq \begin{cases} (\psi \wedge \tau_m) \vee \rho - \rho(z(\omega, \alpha)), & \alpha(0) = \omega(\rho(\omega)), \\ 0, & \text{otherwise}. \end{cases}$$

Due to [10, Theorem IV.103] the map $V$ is $\mathcal{F}_\rho^\rho \otimes \mathcal{F}$-measurable and $V(\omega, \cdot)$ is an $(\mathcal{F}_t^\rho)_{t \geq 0}$-stopping time for all $\omega \in \Omega$. Furthermore, it is evident from the definition that

$$(\psi \wedge \tau_m)(\omega) \vee \rho(\omega) = \rho(\omega) + V(\omega, \theta_{\rho(\omega)}\omega)$$

for $\omega \in \Omega$. Because $\omega(t) = z(\omega, \alpha)(t)$ for all $t \leq \rho(\omega)$, Galmarino’s test (see [17, Lemma III.2.43 c)]) yields that for all $G \in \mathcal{F}_\rho^\rho$

$$\omega \in G \iff z(\omega, \alpha) \in G.$$

Thus, we have for $\omega \in \{\rho < \psi \wedge \tau_m\} \in \mathcal{F}_\rho^\rho$ and $\alpha \in \Omega$ starting at $\alpha(0) = \omega(\rho(\omega))$

$$V(\omega, \alpha) = (\psi \wedge \tau_m)(z(\omega, \alpha)) - \rho(\omega) \leq \tau_m(\alpha).$$
We take \( f \in D \) and note that for \( \omega \in \{ \rho < \psi \land \tau_m \} \)
\[
M_{V(\omega, \rho, \omega)} f \theta (\rho(\omega)) \omega = M_{F(\psi \land \tau_m)(\omega)-\rho(\omega)} f (\rho(\omega)) \omega
\]
\[
= f(\omega((\psi \land \tau_m)(\omega))) - f(\omega(\rho(\omega))) - \int_{\rho(\omega)}^{(\psi \land \tau_m)(\omega)} \mathcal{L}f(\omega(s))ds
\]
\[
= M_{F(\psi \land \tau_m)(\omega)} (\omega) - M_{\rho(\omega)} (\omega).
\]
Because \((M_{\rho \land \psi \land \tau_m})_{\geq \geq 0}\) is a \( P \)-martingale, we have
\[
E^Q [M_{\rho \land \psi \land \tau_m}] = E^P [M_{\rho \land \psi \land \tau_m}] = 0,
\]
due to the optional stopping theorem. Therefore, we have
\[
E^Q [M_{\psi \land \tau_m}] = E^Q [M_{\rho \land \psi \land \tau_m} - M_{\rho \land \psi \land \tau_m}]
\]
\[
= E^Q [(M_{\psi \land \tau_m} - M_{\rho \land \psi \land \tau_m}) \mathbf{1}_{\{\rho \geq \psi \land \tau_m\}}]
\]
\[
= 0 + E^Q [(M_{\psi \land \tau_m} - M_{\rho \land \psi \land \tau_m}) \mathbf{1}_{\{\rho < \psi \land \tau_m\}}]
\]
\[
= E^Q [M_{V(\psi, \rho, \omega)} (\theta (\rho)) \mathbf{1}_{\{\rho < \psi \land \tau_m\}}]
\]
\[
= \int P(d\omega)E^P(\rho(\omega)) [M_{V(\psi, \rho, \omega)} (\mathbf{1}_{\{\rho < \psi \land \tau_m\}})] = 0,
\]
again due to the optional stopping theorem (recall that \( V(\omega, \cdot) \) is bounded and that \((M_{\rho \land \psi \land \tau_m})_{\geq \geq 0}\) is a \( P \)-martingale for all \( y \in S \)). We conclude from \([33], \text{Proposition II.1.4}\) and the downwards theorem \((\[34], \text{Theorem II.5.1}\)) that \((M_{\rho \land \psi \land \tau_m})_{\geq \geq 0}\) is a \( Q \)-martingale. Since \( m \in \mathbb{N} \) was arbitrary, this implies that \( Q \) solves the MP \((D, \mathcal{L}, \Sigma, x)\). The uniqueness assumption yields that \( Q = P_x \). Because also for \( G \in \mathcal{F}_\rho^x \) we can choose \( G' = G \times \Omega \), we obtain that
\[
P_x(G) = Q(G) = P(G).
\]
This finishes the proof. \( \square \)

**Lemma 10.** For all \( m \geq 1 \), all accumulation points of \( \{(Y_{\xi \land \tau_m}^n, Z_{\xi}^n)_{\xi \geq 0}, n \in \mathbb{N}\} \) coincide with \( P_{(x, t)} \) on \( \mathcal{F}_{\xi_{\xi-1}} \).

**Proof:** We recall some continuity properties of functions on \( \Omega \). For \( \omega \in \Omega \), define
\[
J(\omega) \triangleq \{ t > 0 : \omega(t) \neq \omega(t-) \},
\]
\[
V(\omega) \triangleq \{ k > 0 : \tau_k(\omega) < \tau_{k+}(\omega) \},
\]
\[
V'(\omega) \triangleq \{ u > 0 : \omega(\tau_u(\omega)) \neq \omega(\tau_u(\omega)-) \text{ and } \| \omega(\tau_u(\omega)-) \| = u \},
\]
which are countable sets, see \([17], \text{Lemma VI.2.10}\). The map \( \omega \mapsto \omega(t) \) is continuous at \( \omega \) whenever \( t \notin J(\omega) \), see \([11], \text{Proposition 3.5.2}\), and the map \( \omega \mapsto \tau_m(\omega) \) is continuous at \( \omega \) whenever \( m \notin V(\omega) \), see \([11], \text{Problem 13, p. 151}\) and \([17], \text{Proposition VI.2.11}\). Furthermore, the map \( \omega \mapsto \omega(\cdot \land \tau_m(\omega)) \) is continuous at \( \omega \) whenever \( m \notin V(\omega) \cup V'(\omega) \), see \([11], \text{Problem 13, p. 151}\) and \([17], \text{Proposition VI.2.12}\).
In the following take an arbitrary \( f \in D \) and let \( Q^m \) be an accumulation point of \( \{(Y^n_{t_{n,m}}, Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \). With abuse of notation we denote the subsequence corresponding to \( Q^m \) also by \( (n)_{n \in \mathbb{N}} \). The set
\[
F \triangleq \{ t > 0 : Q^m(t \in V \cup V') > 0 \}
\]
is countable, see the proof of [17, Proposition IX.1.17]. Thus, we find a \( t_m \in [m-1, m] \) such that \( t_m \notin F \). Set
\[
U \triangleq \{ t \in \mathbb{R}_+ : Q^m(t \in J(X \wedge t_m)) = 0 \}.
\]
By [11, Lemma 3.7.7], the complement of \( U \) in \( \mathbb{R}_+ \) is countable. Thus, \( U \) is dense in \( \mathbb{R}_+ \). Next, we explain that for all \( z \in \mathbb{R}_+ \) the map
\[
\omega \mapsto I_{t \wedge \tau_z(\omega)}(\omega) \triangleq \int_0^{t \wedge \tau_z(\omega)} \mathcal{L}f(\omega(s))ds
\]
is continuous at all continuity points of \( \omega \mapsto \tau_z(\omega) \). Let \( (\omega_n)_{n \in \mathbb{N}} \subset \Omega \) and \( \omega \in \Omega \) be such that \( \omega_n \rightarrow \omega \) and \( \tau_z(\omega_n) \rightarrow \tau_z(\omega) \) as \( n \rightarrow \infty \). Because \( \tau_a \nearrow \infty \) as \( a \rightarrow \infty \) and \( \omega \mapsto \tau_a(\omega) \) is continuous at \( \omega \) for all but countably many \( a \in \mathbb{R}_+ \) (namely for all \( a \notin V(\omega) \)), we find a \( \lambda > 0 \) and an \( N \in \mathbb{N} \) such that \( \tau_\lambda(\omega_n) \geq t \) for all \( n \geq N \).

W.l.o.g. we assume that \( N = 1 \). Now, we have for all \( s \in [0, t \wedge \tau_z(\omega)] \)
\[
|\mathcal{L}f(\omega(s))-\mathcal{L}f(\omega_n(s))| \leq 2 \sup_{\|f\| \leq \lambda} |\mathcal{L}f(y)|.
\]
Thus, [11, Proposition 3.5.2], the fact that \( J(\omega) \) is countable, the dominated convergence theorem and the continuity of \( x \mapsto \mathcal{L}f(x) \), which is due to the hypothesis that \( b \) and \( a \) are continuous, imply that
\[
|I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega)}(\omega_n)| \rightarrow 0
\]
as \( n \rightarrow \infty \). We obtain
\[
|I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega)}(\omega_n)| \leq |I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega)}(\omega_n)| + |I_{t \wedge \tau_z(\omega)}(\omega_n) - I_{t \wedge \tau_z(\omega)}(\omega_n)|
\leq |I_{t \wedge \tau_z(\omega)}(\omega) - I_{t \wedge \tau_z(\omega)}(\omega_n)| + \sup_{\|f\| \leq \lambda} |\mathcal{L}f(y)| |\tau_z(\omega) - \tau_z(\omega_n)| \rightarrow 0
\]
as \( n \rightarrow \infty \), where we use that \( \tau_z(\omega_n) \rightarrow \tau_z(\omega) \) as \( n \rightarrow \infty \). It follows that for each \( t \in U \) there exists a \( Q^m \)-null set \( N_t \) such that the map
\[
\omega \mapsto K_t(\omega) \triangleq f(\omega(t \wedge \tau_m(\omega))) - f(\omega(0)) - \int_0^{t \wedge \tau_m(\omega)} \mathcal{L}f(\omega(s))ds
\]
is continuous at all \( \omega \notin N_t \). Fix \( s < t \). Because \( U \) is dense in \( \mathbb{R}_+ \), we find a sequence \( (z_n)_{n \in \mathbb{N}} \subset U \) such that \( z_n \nearrow t \) as \( n \rightarrow \infty \) and a sequence \( (u_n)_{n \in \mathbb{N}} \subset U \) such that \( u_n \searrow s \) as \( n \rightarrow \infty \). W.l.o.g. we can assume that \( u_n \leq z_n \) for all \( n \in \mathbb{N} \). Let \( v : \Omega \rightarrow \mathbb{R} \) be continuous, bounded and \( \mathcal{F}^m_t \)-measurable. Denote by \( P^m \) the law of \( (Y^n_{t \wedge \tau_n, Z^n_t})_{t \geq 0} \). Using the dominated convergence theorem, the right-continuity of \( (X_t)_{t \geq 0} \) and the continuous mapping theorem, we obtain
\[
(4.34) \quad E^m[\mathcal{L}f(\omega(t \wedge \tau_m(\omega))) - f(\omega(0)) - \int_0^{t \wedge \tau_m(\omega)} \mathcal{L}f(\omega(s))ds]
\]
for all \( \omega \notin N_t \). The process \( \mathcal{L} \) is a \( P^m \)-martingale. To see this, note that
\[
\tau_m \circ (Y^n_{s \wedge \tau_n, Z^n_s})_{s \geq 0} = \tau_n \cdot t_m,
\]
see [17, Lemma III.2.43], and recall that martingales are stable under stopping. Consequently, using again the continuous mapping theorem and the dominated convergence theorem, we conclude from (4.34) that
\[
E^{Q^m}[K_tv] = \lim_{k \to \infty} \lim_{n \to \infty} E^{P_n,m}[K_{s_k}v] = \lim_{k \to \infty} E^{P_n,m}[K_{s_k}v] = E^{Q^m}[K_sv].
\]
Recall that \(s < t\) and \(v\) were arbitrary.

We claim that this already implies that \((K_q)_{q \geq 0}\) is a \(Q^m\)-martingale. To show this, we use an approximation argument. We still keep \(s < t\) fixed and take an arbitrary \(l \in \mathbb{N}\) and \(0 \leq q_1, \ldots, q_l \leq s\). Furthermore, we take \(g_1, \ldots, g_l \in C_b(S)\) and set \(a_k \triangleq (n + k)^{-1}\) and
\[
g_k \triangleq \prod_{i=1}^l g_i^{k_l}(X_{q_i}).
\]
where \(n\) is the smallest natural number such that \(s + a_0 = s + n^{-1} < t\). Finally, set
\[
g^k(\omega) \rightarrow g(\omega) \triangleq \prod_{i=1}^l g_i(\omega(q_i))
\]
as \(k \to \infty\) for all \(\omega \in \Omega\). Thus, it follows from our previous arguments and the dominated convergence theorem that
\[
E^{Q^m}[K_tg] = \lim_{k \to \infty} E^{Q^m}[K_tg^k] = \lim_{k \to \infty} E^{Q^m}[K_{s+a_k}g^k] = E^{Q^m}[K_sg].
\]
It is well-known that for any closed set \(F \subseteq S\) the indicator \(1_F\) can be approximated pointwise by a sequence \((h^k)_{k \in \mathbb{N}} \subseteq C_b(S)\) such that \(\sup_{y \in S} |h^k(y)| \leq 1\) for all \(k \in \mathbb{N}\), see, e.g., the proof of [4, Lemma 30.14]. Consequently, (4.35) and the dominated convergence theorem yield that
\[
E^{Q^m}[K_t1_F(X_{q_i})] = E^{Q^m}[K_s1_F(X_{q_i})]
\]
for arbitrary closed sets \(F_1, \ldots, F_l \subseteq S\). Finally, using monotone class arguments, we conclude that
\[
E^{Q^m}[K_t1_G] = E^{Q^m}[K_s1_G]
\]
for all \(G \in \mathcal{F}_1^\circ\). Together with the downwards theorem, this implies that \((K_t)_{t \geq 0}\) is a \(Q^m\)-martingale.

Because \(\omega \mapsto \omega(0)\) is continuous, we have \(Q^m(X_0 = (x,i)) = 1\) due to the continuous mapping theorem. Due to [11, Problem 4.25] the set \(\Sigma = \Sigma_c \times \Sigma_d\) is a closed set in the product Skorokhod topology on \(\Omega = D(\mathbb{R}^+, \mathbb{R}^d) \times \Sigma_d\), and [11, Proposition 3.5.3] implies that \(\Sigma\) is closed in \(\Omega\), too. Thus, by the Portmanteau theorem, we have \(Q^m(\Sigma) = 1\). It follows from Lemma 9 that \(Q^m\) coincides with \(P_{(x,i)}\) on \(\mathcal{F}_{\tau_m}\) and thus also on \(\mathcal{F}_{\tau_{m-1}}\), because \(t_m \geq m - 1\) implies \(\tau_m \geq \tau_{m-1}\). This completes the proof. \(\square\)
Lemma 11. The sequence \( \{(Y^n_t, Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \) is tight.

Proof: We use again Fact 1. As in the proof of the previous lemma, let \( P^{n,m} \) be the law of \( (Y^n_t, Z^n_t)_{t \geq 0} \) and \( P^n \) be the law of \( (Y^n_t, Z^n_t)_{t \geq 0} \). We fix \( t \in \mathbb{R}_+ \). Due to [11, Problem 13, p. 151] and [13, Lemma 15.20], the set \( \{\tau_m - 1 \leq t\} \) is closed. Moreover, \( \{\tau_m - 1 \leq t\} \in F_{\tau_m - 1} \), because \( \tau_m \) is an \( (F_t)_{t \geq 0} \)-stopping time. We deduce from the Portmanteau theorem and Lemma 10 that

\[
\limsup_{n \to \infty} P^{n,m}(\tau_m - 1 \leq t) \leq P_{(x,i)}(\tau_m - 1 \leq t).
\]

(4.36)

Fix \( \epsilon > 0 \). Since \( P_{(x,i)}(\tau_m - 1 \leq t) \searrow 0 \) as \( m \to \infty \), we find an \( m^o \in \mathbb{N}_2 \) such that

\[
P_{(x,i)}(\tau_m - 1 \leq t) \leq \frac{\epsilon}{2}.
\]

(4.37)

Because \( (P^{n,m-1})_{n \in \mathbb{N}} \) is tight due to Lemma 8 we deduce from Fact 1 that there exists a compact set \( C(t, \epsilon) \subseteq \mathcal{S} \) such that

\[
\limsup_{n \to \infty} P^{n,m-1}(X_t \notin C(t, \epsilon)) \leq \frac{\epsilon}{2}.
\]

(4.38)

In view of [17, Lemma III.2.43] we obtain

\[
P^n(X_t \notin C(t, \epsilon)) = P^n(X_t \notin C(t, \epsilon), \tau_{m^o} - 1 > t) + P^n(X_t \notin C(t, \epsilon), \tau_{m^o} - 1 \leq t)
\]

\[
\leq P^{n,m-1}(X_t \notin C(t, \epsilon)) + P^{n,m^o}(\tau_{m^o} - 1 \leq t).
\]

From this, (4.36), (4.37) and (4.38), we deduce that

\[
\limsup_{n \to \infty} P^n(X_t \notin C(t, \epsilon)) \leq \epsilon.
\]

This proves that the sequence \( (P^n)_{n \in \mathbb{N}} \) satisfies (a) in Fact 1.

Next, we show that \( (P^n)_{n \in \mathbb{N}} \) satisfies (b) in Fact 1. Let \( \epsilon, t \) and \( m^o \) be as before. Because \( (P^{n,m^o-1})_{n \in \mathbb{N}} \) is tight due to Lemma 8 there exists a \( \delta > 0 \) such that

\[
\limsup_{n \to \infty} P^{n,m^o-1}(u'(X_s)_{s \geq 0}, \delta, t) \geq \epsilon \leq \frac{\epsilon}{2}.
\]

(4.39)

Thus, similar as above, using (4.36), (4.37) and (4.39), we obtain

\[
\limsup_{n \to \infty} P^n(u'(X_s)_{s \geq 0}, \delta, t) \geq \epsilon
\]

\[
\leq \limsup_{n \to \infty} P^{n,m^o-1}(u'(X_s)_{s \geq 0}, \delta, t) \geq \epsilon + \limsup_{n \to \infty} P^{n,m^o}(\tau_{m^o} - 1 \leq t)
\]

\[
\leq \epsilon.
\]

In other words, \( (P^n)_{n \in \mathbb{N}} \) satisfies also (b) in Fact 1 and the proof is complete. \( \square \)

We are in the position to complete the proof of Proposition 8. To wit, in view of the classical result [3, Corollary to Theorem 5.1], because \( \{(Y^n_t, Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \) is tight by the previous lemma, for \( P_{(x,i)}(\tau_m) \rightarrow P_{(x,i)} \) weakly as \( n \rightarrow \infty \), it remains to show that any accumulation point \( Q \) of \( \{(Y^n_t, Z^n_t)_{t \geq 0}, n \in \mathbb{N}\} \) coincides with \( P_{(x,i)} \). As in the proof of Lemma 10 we find a sequence \( (t_m)_{n \in \mathbb{N}} \) with \( t_m \in [m - 1, m] \) such that the stopped process \( (M^f_t)_{t \geq 0} \) is a \( Q \)-martingale for all \( f \in D \). Thus, \( (M^f_t)_{t \geq 0} \) is a local \( Q \)-martingale, because \( \tau_m \searrow \infty \) as \( n \rightarrow \infty \). Since \( \omega \rightarrow \omega(0) \) is continuous, we also have \( Q(X_0 = (x,i)) = 1 \) and, because \( \Sigma \) is closed in \( \Omega \), the Portmanteau theorem yields that \( Q(\Sigma) = 1 \). It follows that \( Q \) solves the MP \( (D, L, \Sigma, (x,i)) \). Due to the uniqueness assumption, \( Q = P_{(x,i)} \) and the proof is complete. \( \square \)
Remark 5. A slight modification of the proof of Proposition 8 shows a version of the convergence result [37, Theorem 11.14] and the existence result of Skorokhod (see, e.g., [20, Theorem 5.4.22]) for switching diffusions. Because we think that such results are of independent interest, we provide formal statements in Appendix A.

Next, we also comment on the strong existence assumption on \((P^k_x)_{x \in \mathbb{R}^d}\) and the uniqueness assumption on \((P^y_y)_{y \in S}\).

Proposition 9. Suppose that Condition 2 holds and that for all \(k \in S_d\) the family \((P^k_x)_{x \in \mathbb{R}^d}\) exists strongly, then a unique family \((P^y_y)_{y \in S}\) exists.

Proof: The existence is shown in the proof of Theorem 4. The uniqueness follows from a Yamada-Watanabe argument, which we sketch. Fix \(y = (x, i) \in S\) and suppose that \(P^y_y\) and \(Q^y_y\) solve the MP \((\mathcal{L}, D, \Sigma, y)\). Using similar arguments as in the proof of \([10, \text{Theorem 8.3}]\), we obtain the following: We find a filtered probability space satisfying the usual hypothesis on which we can realize \(P^y_y\) as the law of the process \((Y_t, Z_t)_{t \geq 0}\), where \((Z_t)_{t \geq 0}\) is a Markov chain with \(Q\)-matrix \(Q\) and \(Z_0 = i\) and

\[
dY_t = b(Y_t, Z_t)dt + a^\frac{1}{2}(Y_t, Z_t)dW_t, \quad Y_0 = x,
\]

where \((W_t)_{t \geq 0}\) is a Brownian motion; on the same probability space, we can be realized \(Q^y_y\) as the law of \((V_t, Z_t)_{t \geq 0}\), where

\[
dV_t = b(V_t, Z_t)dt + a^\frac{1}{2}(V_t, Z_t)dW_t, \quad V_0 = x.
\]

We stress that the driving system \((Z, W)\) implies \(Q\), \(Q^y\), and \((Y, V)\) solve the MP \((\mathcal{L}, D, \Sigma, y)\). Using similar arguments as in the proof of \([10, \text{Theorem 8.3}]\), we obtain the following: We find a filtered probability space satisfying the usual hypothesis on which we can realize \(P^y_y\) as the law of the process \((Y_t, Z_t)_{t \geq 0}\), where

\[
dY_t = b(Y_t, Z_t)dt + a^\frac{1}{2}(Y_t, Z_t)dW_t, \quad Y_0 = x,
\]

where \((W_t)_{t \geq 0}\) is a Brownian motion; on the same probability space, we can be realized \(Q^y_y\) as the law of \((V_t, Z_t)_{t \geq 0}\), where

\[
dV_t = b(V_t, Z_t)dt + a^\frac{1}{2}(V_t, Z_t)dW_t, \quad V_0 = x.
\]

The strong existence hypothesis and Lemma 3 imply that \(Y_t = V_t\) for all \(t \leq \tau_1\) up to a null set. Suppose that \(n \in \mathbb{N}\) is such that \(Y_t = V_t\) for all \(t \leq \tau_n\) up to a null set. Using classical rules for time-changed stochastic integrals, we obtain that on \(\{t \leq \tau_{n+1} - \tau_n\} \cap \{Z_{\tau_n} = k\}\)

\[
Y_{t+\tau_n} = Y_{\tau_n} + \int_{\tau_n}^{t+\tau_n} b(Y_s, k)ds + \int_{\tau_n}^{t+\tau_n} a^\frac{1}{2}(Y_s, k)dW_s
\]

\[
= Y_{\tau_n} + \int_{0}^{t} b(Y_{s+\tau_n}, k)ds + \int_{0}^{t} a^\frac{1}{2}(Y_{s+\tau_n}, k)dW_s^n
\]

and

\[
V_{t+\tau_n} = V_{\tau_n} + \int_{0}^{t} b(V_{s+\tau_n}, k)ds + \int_{0}^{t} a^\frac{1}{2}(V_{s+\tau_n}, k)dW_s^n,
\]

where

\[
W_t^n \equiv W_{t+\tau_n} - W_{\tau_n}, \quad t \in \mathbb{R}_+.
\]

We conclude again from the strong existence hypothesis and Lemma 3 that \(Y_{n+1} = V_{n+1}\), for all \(t \leq \tau_{n+1} - \tau_n\) up to a null set. Consequently, \(Y_t = V_t\) for all \(t \leq \tau_{n+1}\) up to a null set and our claim follows. \(\square\)
Proof of Corollary 3. By [20, Theorems 5.5.15, 5.5.29], for all \( k \in S_d \) the family \((P^n_k)_{k \in \mathbb{R}^d}\) exists uniquely. Thus, using the local Hölder condition on the diffusion coefficient, [13, Lemma IX.3.3, Proposition IX.3.2] and [14, Theorem 18.14] imply that \((P^n_k)_{k \in \mathbb{R}^d}\) exists strongly. Consequently, \((P^k_x)_{x \in S}\) exists uniquely due to Proposition 9. Now, \((P^k_x)_{x \in S}\) is strongly Markov and Feller due to Proposition 8 and the equivalence of (i) and (ii) follows from Theorem 4 and [3, Proposition 4.3] when \( N < \infty \).

Proof of Corollary 4. Due to [20, Theorem 5.2.5] and [19, Theorem 18.14], for all \( k \in S_d \) the family \((P^n_k)_{k \in \mathbb{R}^d}\) exists strongly. Consequently, \((P^n_x)_{x \in S}\) exists uniquely due to Proposition 9. Now, \((P^n_x)_{x \in S}\) is strongly Markov and Feller due to Proposition 8 and the claim follows from Theorem 4 and [3, Proposition 5.2]. □

Appendix A. A Limit Theorem and an Existence Theorem for Switching Diffusions

In this appendix we give a limit and an existence theorem for switching diffusions with state-independent switching. We think the results are of independent interest and that they deserve a statement of their own. We pose ourselves in the setting of Section 4.3.

Theorem 5. For all \( n \in \mathbb{N} \) let \( b^n: S \to \mathbb{R}^d \) and \( a^n: S \to \mathbb{S}^d \) be Borel functions such that for all \( m \in \mathbb{R}_+ \)
\[
\sup_{n \in \mathbb{N}} \sup_{|y| \leq m} \left( \|b^n(y)\| + \|a^n(y)\| \right) < \infty.
\]
Assume that \( b: S \to \mathbb{R}^d \) and \( a: S \to \mathbb{S}^d \) are continuous functions and that for all \( m \in \mathbb{R}_+ \)
\[
\sup_{|y| \leq m} \left( \|b(y) - b^n(y)\| + \|a(y) - a^n(y)\| \right) \to 0
\]
as \( n \to \infty \). Furthermore, let \((Q^n)_{n \in \mathbb{N}}\) be a sequence of \( Q \)-matrices on \( S_d \) such that for all \( n \in \mathbb{N} \) and \( k \in S_d \) the MP \((C^n, Q^n, \Sigma_d, k)\), where
\[
C^n \triangleq \{ f \in C_0(S_d): Q^n f \in C_0(S_d) \},
\]
has a unique solution \( P^n_k \) such that \((P^n_k)_{k \in S_d}\) is Feller-Dynkin. Suppose that for all \( f \in C_d \), where \( C_d \) is as in (4.28), there exists a sequence \((f_n)_{n \in \mathbb{N}}\) consisting of \( f \in C^n \) such that
\[
\|f - f_n\|_{\infty} + \|Qf - Q^n f_n\|_{\infty} \to 0
\]
as \( n \to \infty \). Finally, take \((x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \) and \((i_n)_{n \in \mathbb{N}} \subseteq S_d \) such that \( x_n \to x \in \mathbb{R}^d \) and \( i_n \to i \in S_d \) as \( n \to \infty \). Set \( L \) as in (4.3) and \( L^n \) as in (4.3) with \( b \) replaced by \( b^n \), \( a \) replaced by \( a^n \) and \( Q \) replaced by \( Q^n \). If \( P^n \) is a solution to the MP
\[
(D^n; L^n, \Sigma, (x_n, i_n)),
\]
and for all \( y \in S \) the MP \((D, L, \Sigma, y)\) has a unique solution \( P_y \), then \( P^n \to P(x, i) \) weakly as \( n \to \infty \).

19 the sufficiency of the conditions can also be proven with a Lyapunov-type argument as given in Theorem 4 and the necessity follows from Theorem 2; see [3, Lemma 4.2].

20 because there exists a \( m \in \mathbb{N} \) such that \( i_n = i \) for all \( n \geq m \) we could also simply consider sequences \( y_n = (x_n, i) \).
Proof: The proof is almost identical to the proof of Proposition 3. We comment on the necessary changes. For $m \in \mathbb{R}_+$ let $\tau_m$ be as in (4.28) and denote $P^{m,n} \triangleq P^n \circ (X_1^{1,\tau_m}, X_2^{1,\tau_m}) \geq 0$, where $(X_t^{1,\tau_m})_{\geq 0} = (X_t^1, X_t^2)_{\geq 0}$. For each $n \in \mathbb{N}$, let $(Y^n_t)_{\geq 0}$, resp. $(\tilde{Y}_t)_{\geq 0}$, be the process from Lemma 2 corresponding to $P^n$, resp. $P_{(x,i)}$. The Kato-Trotter theorem [11, Theorem 17.25] implies that $(Y^n_t)_{\geq 0}$ is tight. Furthermore, a version of (4.22) holds due to (A.1). Therefore, we can conclude as in the proof of Lemma 8 that the sequence $(P^{m,n})_{n \in \mathbb{N}}$ is tight. Let $Q^m$ be an accumulation point of $(P^{m,n})_{n \in \mathbb{N}}$. Recall that $D$ is defined as in (4.28). For $f \in D$ let $(M^f_t)_{t \geq 0}$ be given as in (4.33). As in the proof of Lemma 11 we find a $t_m \in [m-1,m]$ and a dense set $U \subseteq \mathbb{R}_+$, such that for each $t \in U$ there exists a $Q^m$-null set $N_t \in \mathcal{F}$ such that the map

$$\omega \mapsto M^f_{t \wedge \tau_m}(\omega)$$

is continuous at all $\omega \not\in N_t$. Now, take $f \in D$ which is independent of the $\mathbb{R}^d$-coordinate (i.e. $f \in C_d$) and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions $f_n \in C^n$ such that (A.3) holds. Define $(M^{f,n}_t)_{t \geq 0}$ as in (4.33) with $f$ replaced by $f_n$ and $\mathcal{L}$ replaced by $\mathcal{L}^n$. Furthermore, fix $\omega \not\in N_t$ and let $(\omega_n)_{n \in \mathbb{N}} \subseteq \Omega$ be a sequence such that $\omega_n \to \omega$ as $n \to \infty$. Then, for any bounded continuous function $v : \Omega \to \mathbb{R}$ we have

$$\left| M^f_{t \wedge \tau_m}(\omega)v(\omega) - M^{f,n}_{t \wedge \tau_m}(\omega_n)v(\omega_n) \right| \leq \left| M^f_{t \wedge \tau_m}(\omega)v(\omega) - M^{f}_{t \wedge \tau_m}(\omega_n)v(\omega_n) \right| + \left| M^{f,n}_{t \wedge \tau_m}(\omega_n)v(\omega_n) - M^{f,n}_{t \wedge \tau_m}(\omega_n)v(\omega_n) \right| \to 0,$$

as $n \to \infty$, where the first term converges to zero because the the continuity of $\omega \mapsto M^f_{t \wedge \tau_m}(\omega)v(\omega)$ at $\omega$ and the second term converges to zero because of the boundedness of $v$ and

$$\left| M^f_{t \wedge \tau_m}(\omega_n) - M^{f,n}_{t \wedge \tau_m}(\omega_n) \right| \leq 2\|f - f_n\|_{\infty} + t\|Q f - Q^n f_n\|_{\infty} \to 0,$$

as $n \to \infty$ by (A.3). Similarly, (A.4) holds for all $f \in D$ which only depend on the $\mathbb{R}^d$-coordinate when $(M^{f,n}_t)_{t \geq 0}$ is defined as in (4.33) with $\mathcal{L}$ replaced by $\mathcal{L}^n$. In this case, the second term in (A.4) converges to zero because

$$\left| M^f_{t \wedge \tau_m}(\omega_n) - M^{f,n}_{t \wedge \tau_m}(\omega_n) \right| \leq \text{const. } t \sup_{|m| \leq m} \left( \|b(y) - b^n(y)\| + \|a(y) - a^n(y)\| \right) \to 0,$$

due to (A.2). We conclude from the continuous mapping theorem (see, e.g., [19, Theorem 3.27] for a suitable statement) that for all $f \in D$ and $t \in U$

$$E^{P^{m,n}}\left[ M^{f,n}_{t \wedge \tau_m}v \right] \to E^{Q^m} \left[ M^f_{t \wedge \tau_m}v \right]$$

as $n \to \infty$. By the same arguments as in the proof of Lemma 10 this yields that $(M^f_{t \wedge \tau_m})_{t \geq 0}$ is a $Q^m$-martingale. Arguing as in the proofs of the Lemmata 10 and 11 shows that $(P^n)_{n \in \mathbb{N}}$ is tight.

It remains to explain that any accumulation point $Q$ of $(P^n)_{n \in \mathbb{N}}$ coincides with $P_{(x,i)}$. As above we find a sequence $(t_m)_{n \in \mathbb{N}}$ with $t_m \in [m-1,m]$ such that the
stopped process \((M^f_t)_{t \geq 0}\) is a \(Q\)-martingale for all \(f \in D\). Thus, \((M^f_t)_{t \geq 0}\) is a local \(Q\)-martingale, because \(\tau_{im} \nearrow \infty\) as \(n \to \infty\). We conclude as in the proof of Proposition 38 that \(Q\) solves the MP \((D, \mathcal{L}, \Sigma, (x, i))\). The uniqueness assumption yields that \(Q = P_{(x,i)}\) and the proof is complete. \(\square\)

We now turn to the existence result, which we state for arbitrary initial laws.

**Theorem 6.** Let \(b: S \to \mathbb{R}^d\) and \(a: S \to \mathbb{S}^d\) be continuous functions such that for all \(m \in \mathbb{R}_+\)

\[(A.5) \sup_{\|x\| \leq m} \sup_{k \in S_m} (\|b(x,k)\| + \|a(x,k)\|) < \infty.\]

Let \(K^k\) be given as in (1.12). Suppose that there exists two constants \(c, \lambda > 0\), a function \(v: \mathbb{R}_+ \to (0, \infty)\) and a twice continuously differentiable function \(V : \mathbb{R}^d \to (0, \infty)\) such that \(V(x) \geq v(\|x\|)\) for all \(x \in \mathbb{R}^d: \|x\| \geq \lambda\), \(\limsup_{n \to \infty} v(n) = \infty\) and

\[\mathcal{K}^k V(x) \leq c V(x),\]

for all \((x,k) \in S\). Then, for any Borel probability measure \(\eta\) on \(S\) there exists a solution to the MP \((D, \mathcal{L}, \Sigma, \eta)\).

**Proof:** Due to Proposition 41 it suffices to show the claim for degenerated initial laws, i.e. we assume that \(\eta\{y\} = 1\) for some \(y \in S\).

**Step 1.** We first show the claim under the assumptions that \(b\) and \(a\) are continuous and bounded, i.e.

\[\|b(x,k)\| + \|a(x,k)\| \leq c\]

for all \((x,k) \in S\). Our initial step is a standard mollification argument. Let \(\phi\) be the standard mollifier, i.e.

\[\phi(x) \triangleq \begin{cases} \theta \exp \{- (1 - \|x\|^2)^{-1}\}, & \text{if } \|x\| < 1, \\ 0, & \text{otherwise,} \end{cases}\]

where \(\theta\) is a constant such that \(\int \phi(x)dx = 1\). Let \(\sigma\) be a root of \(a\). We set

\[b^n(x,k) \triangleq n^d \int b(y,k)\phi(n(x - y))dy,\]
\[\sigma^n(x,k) \triangleq n^d \int \sigma(y,k)\phi(n(x - y))dy.\]

It is well-known that \(x \mapsto b^n(x,k)\) and \(x \mapsto \sigma^n(x,k)\) are smooth and that \(b^n \to b\) and \(\sigma^n(\sigma^n)^t \to a\) as \(n \to \infty\) uniformly on compact subsets of \(S\). Furthermore, using that \(\int \phi(x)dx = 1\), we obtain

\[\|b^n(x,k)\| \leq n^d \int \|b(y,k)\|\phi(n(x - y))dy = \int \|b(x - n^{-1}z,k)\|\phi(z)dz \leq c\]

and, in the same manner, \(\|\sigma^n(x,k)\| \leq c\). Because smooth functions are locally Lipschitz continuous, we deduce from [33, Theorem 18.16], [19, Theorem 18.14] and Proposition 39 that for each \(n \in \mathbb{N}\) there exists a solution \(P^n\) to the MP \((D, \mathcal{L}^n, \Sigma, y)\), where \(\mathcal{L}^n\) is defined as in (1.3) with \(b\) replaced by \(b^n\) and \(a\) replaced by \(a^n\). If we show that the sequence \((P^n)_{n \in \mathbb{N}}\) is tight and that any accumulation point of it solves the MP \((D, \mathcal{L}, \Sigma, y)\) the claim of the theorem follows. That any accumulation point of \((P^n)_{n \in \mathbb{N}}\) solves the MP \((D, \mathcal{L}, \Sigma, y)\) can be shown as in the proof of Theorem 4 and that \((P^n)_{n \in \mathbb{N}}\) is tight follows as in the proof of Lemma...
Thus, the claim holds under the assumptions that $b$ and $a$ are continuous and bounded.

**Step 2.** We now tackle the general case. Let $\psi^n : \mathbb{R}^d \to [0, 1]$ be a sequence of cutoff functions, i.e. non-negative smooth functions with compact support such that $\psi^n(x) = 1$ for $x \in \mathbb{R}^d : \|x\| \leq n$. We set

$$b^n(x, k) \triangleq \psi^n(x)b(x, k), \quad a^n(x, k) \triangleq \psi^n(x)a(x, k).$$

The functions $b^n$ and $a^n$ are continuous and bounded and for all $m \in \mathbb{R}_+$ we have

$$\sup_{\|y\| \leq m} (\|b(y) - b^n(y)\| + \|a(y) - a^n(y)\|)$$

$$\leq 2 \sup_{\|y\| \leq m} (\|b(y)\| + \|a(y)\|) \sup_{\|y\| \leq m} |1 - \psi^n(x)| \to 0$$
as $n \to \infty$. Therefore, due to our first step, for each $n \in \mathbb{N}$ there exist a solution $P^n$ to the MP $(D, C^n, \Sigma, y)$. We define

$$\tau_m \triangleq \inf (t \in \mathbb{R}_+ : \|X^n_t\| \geq m \text{ or } \|X^n_{-t}\| \geq m), \quad m \in \mathbb{R}_+.$$  

Furthermore, we denote $P^{n, m} \triangleq P^n \circ (X^{1, n}_{t \wedge \tau_m}, X^{2, n}_{t \wedge \tau_m})_{t \geq 0}$. It follows as in the proof of Lemma 8 that the sequence $(P^{n, m})_{n \in \mathbb{N}}$ is tight for every $m \in \mathbb{R}_+$. Recalling the proof of Lemma 11 and Step 1 reveal that the existence of a solution to the MP $(D, C, \Sigma, y)$ follows once we prove that for each $T > 0$ and $\epsilon > 0$ we find a $m \in \mathbb{R}_+$ such that

$$\lim\sup_{n \to \infty} P^n(\tau_m \leq T) \leq \epsilon.$$  

We show this with a Lyapunov-type argument. Define $K^{k, n}$ as $K^k$ with $b$ and $a$ replaced by $b^n$ and $a^n$. We have

$$K^{k, n}V(x) = \psi^n(x)K^kV(x) \leq c\psi^n(x)V(x) \leq cV(x)$$

for all $(x, k) \in S$ and $n \in \mathbb{N}$. By Lemma 1 the process

$$U_t \triangleq e^{-c(T \wedge \tau_m)}V(X^{1, n}_{t \wedge \tau_m}) + \int_0^{t \wedge \tau_m} e^{-cs} (cV(X^k_s) - K^{X^{2, n}_{t \wedge \tau_m}}V(X^k_s)) ds, \quad t \in \mathbb{R}_+,$$

is a local $P^n$-martingale. Because $U_t \geq e^{-c(T \wedge \tau_m)}V(X^{1, n}_{t \wedge \tau_m}) \geq 0$ for all $t \in \mathbb{R}_+$, the process $(U_t)_{t \geq 0}$ is a non-negative $P^n$-supermartingale. We deduce that for all $m \geq \lambda$

$$P^n(\tau_m \leq T)e^{-cT}v(m) = E^n\left[1_{\{\tau_m \leq T\}}e^{-cT}v(\|X^{1, n}_{\tau_m}\|)\right]$$

$$\leq E^n\left[1_{\{\tau_m \leq T\}}e^{-c(T \wedge \tau_m)}V(X^{1, n}_{T \wedge \tau_m})\right]$$

$$\leq E^n\left[e^{-c(T \wedge \tau_m)}V(X^1_{T \wedge \tau_m})\right]$$

$$\leq E^n\left[U_T\right] \leq V(y),$$

where $y = (y_1, y_2)$. The assumption $\lim\sup_{m \to \infty} v(m) = \infty$ yields that we find a $m \geq \lambda$ such that (A.6) holds. This completes the proof.

**Remark 6.** (i) We stress that the previous existence result does not require any uniqueness or strong existence hypothesis on the SDEs for the fixed environments. Furthermore, no moment condition on the initial law is required.
(ii) Using $V(x) = 1 + ||x||^2$ yields that the growth condition

$$2(x, b(x, k)) + \text{trace } a(x, k) \leq c(1 + ||x||^2), \quad \text{for all } (x, k) \in S,$$

implies the existence of a solution to the MP $(D, \mathcal{L}, \Sigma, \eta)$ whenever the coefficients $b$ and $a$ are continuous and satisfy (A.5).

(iii) Theorem 6 can be seen as a version of the main result of [14] for switching diffusions. The argument used in [14] is based on the compactness method as introduced in [12] and a Lyapunov-type argument. The proof of Theorem 6 uses a different method based on Kolomogrov’s tightness criterion together with a Lyapunov-type argument.

**Appendix B. Proofs of Propositions 1 and 2.**

We follow the proofs of [18, Proposition 2], [19, Theorem 18.10] and [11, Theorem 4.4.2].

First, we assume that the MP $(D, \mathcal{L}, \Sigma, \eta)$ has a solution for all $y \in S$. Let $\eta$ be a Borel probability measure on $S$ and let $\mathcal{P}$ denote the set of all solutions to the MP $(D, \mathcal{L}, \Sigma, \eta)$ for all $y \in S$. We consider $\mathcal{P}$ as a subspace of the Polish space $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ equipped with the topology of convergence in distribution. We note that the space $\mathcal{P}$ is separable and metrizable. Let $(K_n)_{n \in \mathbb{N}} \subset S$ be a sequence of compact sets such that $K_n \subseteq \text{int}(K_{n+1})$ and $\bigcup_{n \in \mathbb{N}} K_n = S$. Define the sequence $(\tau_n)_{n \in \mathbb{N}}$ as in (3.10). We note that an $(\mathcal{F}_t^n)_{t \geq 0}$-martingale process is an $(\mathcal{F}_t)_{t \geq 0}$-martingale if and only if it is an $(\mathcal{F}_t^n)_{t \geq 0}$-martingale. The implication $\Rightarrow$ follows from the downwards theorem and the converse implication is due to the tower rule. Therefore, because we assume that $\mathcal{L}(D) \subset B_{\text{dec}}(S)$, a probability measure $P$ solves the MP $(D, \mathcal{L}, \Sigma, \eta)$ if and only if $P(\Sigma) = 1, P \circ X^{-1}_0 = \eta$ and for all $f \in D$ and $n \in \mathbb{N}$ the stopped process $(M_{t \wedge \tau_n}^f)_{t \geq 0}$ is an $(\mathcal{F}_t^n)_{t \geq 0}$-martingale. Here, $(M_t^n)_{t \geq 0}$ is defined as in (1.38).

**Lemma 12. The set $\mathcal{P}$ is a Borel subset of $\mathcal{P}$.**

**Proof:** Let $I \triangleq \{P \in \mathcal{P}: P \circ X^{-1}_0 \in \{\delta_x, x \in S\}\}$ and $J$ be the set of all $P \in \mathcal{P}$ such that $P(\Sigma) = 1$ and

$$E^P \left[ (M_{t \wedge \tau_n}^f - M_{s \wedge \tau_n}^f) 1_G \right] = 0,$$

for all $f \in D, 0 \leq s < t < \infty, m \in \mathbb{N}$ and $G \in \mathcal{F}_s^n$. In (B.1) we can restrict ourselves to rational $0 \leq s < t < \infty$ because of the right-continuity of $(M_t^f)_{t \geq 0}$. Furthermore, $\mathcal{F}_s^n = \sigma(X_r, r \in [0, s] \cap \mathbb{Q}_+) \cap \mathcal{G}_s$ is countable generated, i.e. contains a countable determining class. Thus, in (B.1) it also suffices to take only countably many sets from $\mathcal{F}_s^n$ into consideration. We conclude that $J$ is Borel due to [11, Theorem 15.13]. Due to [3, Theorem 8.3.7] the set $\{\delta_x, x \in S\}$ is Borel. Thus, since $P \mapsto P \circ X^{-1}_0$ is continuous by the continuous mapping theorem, we also conclude that $I$ is Borel. Finally, it follows that $\mathcal{P} = I \cap J$ is Borel. \hfill \Box

In view of [19, Theorem A.1.6], the previous lemma implies that $\mathcal{P}$ is a Borel space in the sense of [14, p. 456]. Let $\Phi: \mathcal{P} \rightarrow S$ be the map such that $\Phi(P)$ is the starting point associated to $P \in \mathcal{P}$. We claim that $\Phi$ is continuous and therefore Borel. To see this let $(P^n)_{n \in \mathbb{N}}, P \in \mathcal{P}$ such that $P^n \rightarrow P$ weakly as $n \rightarrow \infty$. Denote $\Phi(P^n) = x_n \in S$ and $\Phi(P) = x \in S$. We have to show that $x_n \rightarrow x$ as $n \rightarrow \infty$. For
all \( f \in C_b(S) \) we have

\[
(B.2) \quad f(x_n) = E^{P_{x_n}}[f(X_0)] \to E^P[f(X_0)] = f(x) \quad \text{as } n \to \infty.
\]

This follows from the definition of convergence in distribution because the map \( \omega \mapsto f(\omega(0)) \) is continuous and bounded. Since (B.2) holds for all \( f \in C_b(S) \) the convergence \( x_n \to x \) as \( n \to \infty \) follows from [1], Corollaries 2.57, 2.74. We conclude that \( \Phi \) is continuous. Furthermore, its graph \( G = \{(P, \Phi(P)) : P \in \mathcal{P} \} \) is a Borel subset of \( \mathcal{P} \times S \) due to [3, Proposition 8.1.8]. We have \( \mathcal{B}(\mathcal{P} \times S) = \mathcal{B}(\mathcal{P}) \otimes \mathcal{B}(S) \), see [3, Proposition 8.1.7], and

\[
\bigcup_{P \in \mathcal{P}} \{ s \in S : s = \Psi(P) \} = S,
\]

by the assumption that there exists a solution for all degenerated initial laws. Thus, by the section theorem [19, Theorem A.1.8] there exists a Borel map \( x \mapsto P_x \) and a \( \eta \)-null set \( N \in \mathcal{B}(S) \) such that \( (P_x, x) \in G \) for all \( x \not\in N \). By the definition of \( G \), for all \( x \not\in N \) the probability measure \( P_x \) solves the MP \( (D, \mathcal{L}, \Sigma, x) \). Clearly, the probability measure \( P_\eta \triangleq \int P_x \eta(dx) \) satisfies \( P_\eta \circ X_0^{-1} = \eta \) and \( P_\eta(\Sigma) = 1 \). Furthermore, for all \( x \not\in N \) we have

\[
E^{P_x}\left([M_{s \wedge \tau_n}^f - M_{s \wedge \tau_n}^r]_F\right) = 0,
\]

for all \( 0 \leq s < t < \infty, n \in \mathbb{N}, F \in \mathcal{F}_s \) and \( f \in D \). Consequently, we conclude that \( P_\eta \) solves the MP \( (D, \mathcal{L}, \Sigma, \eta) \) and the proof of the first part of Proposition 10 is complete.

From now on we assume that \( P_x \) is the unique solution to the MP \( (D, \mathcal{L}, \Sigma, x) \) for all \( x \in S \).

**Lemma 13.** The map \( x \mapsto P_x \) is Borel.

**Proof:** By Lemma [12] the set \( \{P_x, x \in S\} \) is Borel. Let \( \Phi \) be the injection which maps \( P_x \) to its initial value \( x \). The map \( \Phi \) is Borel as a composition of the continuous map \( P \mapsto P \circ X_0^{-1} \) and the inverse of \( x \mapsto \delta_x \), which is Borel due to Kuratovski’s theorem (see [3, Proposition 8.3.5, Theorem 8.3.7]). Because the set \( \{\delta_x, x \in S\} \) is Borel, Kuratovski’s theorem also implies that \( \Phi^{-1} \) is Borel.

Due to [1, Theorem 19.7], the Borel measurability of \( x \mapsto P_x \) is equivalent to saying that \( x \mapsto P_x(G) \) is Borel for all \( G \in \mathcal{F} \). For \( P \in \mathcal{P} \) and \( G \subseteq \mathcal{F} \), let \( P(\cdot|G) \) be the regular conditional probability given \( G \). Because \( (\Omega, \mathcal{F}) \) is a Polish space with its Borel \( \sigma \)-field, such a version is well-known to exist, see, e.g., [20, Theorem 5.3.8].

**Lemma 14.** There exists a null set \( N \in \mathcal{F}_0^\omega \) such that \( P(\cdot|\mathcal{F}_0^\omega)(\omega) \) solves the MP \( (D, \mathcal{L}, \Sigma, X_0(\omega)) \) for all \( \omega \not\in N \).

**Proof:** Because \( \mathcal{F}_0^\omega = \sigma(X_0) \) is countably generated, we find a null set \( N \in \mathcal{F}_0^\omega \) such that

\[
P(X_0 = X_0(\omega)|\mathcal{F}_0^\omega)(\omega) = 1\{X_0(\omega) = X_0(\omega)\} = 1
\]

for all \( \omega \not\in N \). Furthermore, we can enlarge \( N \) such that \( P(\Sigma|\mathcal{F}_0^\omega)(\omega) = 1 \) for all \( \omega \not\in N \). We enlarge \( N \) a second time such that for all \( \omega \not\in N \)

\[
(B.3) \quad E^P\left([M_{s \wedge \tau_m}^f - M_{s \wedge \tau_m}^r]_G|\mathcal{F}_0^\omega\right)(\omega) = 0
\]

for all rational \( s < t \), all \( G \) in a countable determining class of \( \mathcal{F}_0^\omega = \sigma(X_r, r \in [0, s] \cap \mathbb{Q}_+) \) and all \( f \in D \). Using the right-continuity of \( (M_u^f)_{u \geq 0} \) and a monotone
class argument yields that the identity (B.3) holds for all \( \omega \notin N \), \( s < t \) and all \( G \in \mathcal{F}_s \). Thus, the claim follows. \( \square \)

We now show the last part of Proposition 1, i.e. that \( P = \int P_x \eta(dx) \) whenever \( P \) solves the MP \((D, \mathcal{L}, \Sigma, \eta)\). The previous lemma and the uniqueness assumption yield that \( P \)-a.s. \( P_{X_0} = P(\cdot | \mathcal{F}_0^0) \). We conclude that for all \( G \in \mathcal{F} \)

\[
P(G) = E^P[P(G|\mathcal{F}_0^0)] = E^P[P_{X_0}(G)] = \int P_x(\eta(dx),
\]
which completes the proof of Proposition 1.

We turn to the proof of Proposition 2, i.e. the strong Markov property.

**Lemma 15.** Let \( \xi \) be a bounded stopping time such that \( \Sigma \subseteq \theta_{\xi}^{-1} \Sigma \) and \( P \) be a solution to the MP \((D, \mathcal{L}, \Sigma, \eta)\). For all \( F \in \mathcal{F}_\xi \) with \( P(F) > 0 \) the probability measures

\[
P_1 \triangleq \frac{E^P[1_F P(\theta_{\xi}^{-1} \cdot \mid \mathcal{F}_\xi)]}{P(F)}, \quad P_2 \triangleq \frac{E^P[1_F P_{X_\xi}(\cdot)]}{P(F)}
\]
both solve the MP \((D, \mathcal{L}, \Sigma, \zeta)\), where

\[
\zeta(G) = \frac{E^P[1_F 1 \{X_\xi \in G\}]}{P(F)}, \quad G \in \mathcal{B}(S).
\]

**Proof:** Obviously, we have

\[
P_1(X_0 \in G) = P_2(X_0 \in G) = \zeta(G), \quad G \in \mathcal{B}(S).
\]

Moreover, we have

\[
P_2(\Sigma) = \frac{E^P[1_F P_{X_\xi}(\Sigma)]}{P(F)} = \frac{P(F)}{P(F)} = 1,
\]
and

\[
P_1(\Sigma) = \frac{E^P[1_F P(\theta_{\xi}^{-1} \Sigma \mid \mathcal{F}_\xi)]}{P(F)} \geq \frac{E^P[1_F P(\Sigma \mid \mathcal{F}_\xi)]}{P(F)} = \frac{E^P[1_F 1_{\Sigma}]}{P(F)} = 1,
\]
due to the assumption that \( \Sigma \subseteq \theta_{\xi}^{-1} \Sigma \). The random time

\[
\sigma_{z,k} \triangleq z \wedge \tau_k \circ \theta_{\xi} + \xi, \quad z, k \in \mathbb{R}_+,
\]
is a stopping time, see \( \Box \), Proposition 7.8. For any \( m \leq k \) on \( \{\xi < \tau_k\} \) we have \( \Box \), Proposition 7.8. For any \( m \leq k \) we have \( P \)-a.s.

\[
E^P\left[\left(\left(M_{s \wedge \tau_m} - M_{r \wedge \tau_m}\right) 1_{G} \circ \theta_{\xi} \ 1 \{\xi < \tau_k\} \mid \mathcal{F}_\xi\right)\right]
\]

\[
= E^P\left[\left(M_{s \wedge \tau_m} - M_{r \wedge \tau_m}\right) 1_{\theta_{\xi}^{-1} G} 1 \{\xi < \tau_k\} \mid \mathcal{F}_\xi\right]
\]

\[
= E^P\left[\left(M_{s \wedge \tau_m} - M_{r \wedge \tau_m}\right) 1_{\theta_{\xi}^{-1} G} 1 \{\xi < \tau_k\} \mid \mathcal{F}_\xi\right]
\]

\[
= E^P\left[E^P\left[\left(M_{s \wedge \tau_m} - M_{r \wedge \tau_m}\right) 1_{\theta_{\xi}^{-1} G} 1 \{\xi < \tau_k\} \mid \mathcal{F}_{\tau + \xi}\right] 1_{\theta_{\xi}^{-1} G} 1 \{\xi < \tau_k\} \mid \mathcal{F}_\xi\right]
\]

\[
= E^P\left[\left(M_{s \wedge \tau_m} - M_{r \wedge \tau_m}\right) 1_{\theta_{\xi}^{-1} G} 1 \{\xi < \tau_k\} \mid \mathcal{F}_\xi\right]
\]

\[
= E^P\left[\left(M_{s \wedge \tau_m} - M_{r \wedge \tau_m}\right) 1_{\theta_{\xi}^{-1} G} 1 \{\xi < \tau_k\} \mid \mathcal{F}_\xi\right] = 0.
\]
For the second equality we used (1.3), for the third equality we used the tower rule and the fact that $\theta^{-1}_\xi F_{r} \subseteq F_{r+\xi}$ and for the fourth equality we used the optional stopping theorem. Using the dominated convergence theorem, we obtain that $P$-a.s.

\[
E^P \left[ (M^f_{\tau \wedge m} - M^f_{\tau \wedge m}) 1_G \right] = \lim_{k \to \infty} E^P \left[ (M^f_{\tau \wedge m} - M^f_{\tau \wedge m}) 1_G \right] \circ \theta_\xi \left( F_\xi \right)
\]

which implies that

\[
E^P \left[ (M^f_{\tau \wedge m} - M^f_{\tau \wedge m}) 1_G \right] = 0.
\]

Using the downwards theorem, this yields that $P_1$ solves the MP $(D, \mathcal{L}, \Sigma, \xi)$. Finally, because for all $\omega \in \Omega, 0 \leq r < s < \infty, m \in \mathbb{N}$ and $G \in \mathcal{F}_r$

\[
E_{X_{\xi(\omega)}(\omega)} \left[ (M^f_{\tau \wedge m} - M^f_{\tau \wedge m}) 1_G \right] = 0,
\]

which follows from the fact that $P_{X_{\xi(\omega)}(\omega)}$ solves the MP $(D, \mathcal{L}, \Sigma, X_{\xi(\omega)}(\omega))$, also $P_2$ solves the MP $(D, \mathcal{L}, \Sigma, \xi)$. \hfill \Box

Let $F, \xi, P, P_1$ and $P_2$ as in the previous lemma. We have already proven that the MP $(D, \mathcal{L}, \Sigma)$ is completely well-posed. Thus, we have $P_1 = P_2$, which implies that

\[
E^P \left[ 1_F P(\theta^{-1}_\xi G | F_\xi) \right] = E^P \left[ 1_F P_{X_\xi} (G) \right], \quad G \in \mathcal{F}.
\]

Because this identity holds trivially when $P(F) = 0$, it holds for all $F \in \mathcal{F}_\xi$ and we conclude that $P$-a.s.

\[
P(\theta^{-1}_\xi G | F_\xi) = P_{X_\xi} (G),
\]

which is the strong Markov property for the bounded stopping time $\xi$. We now deduce the strong Markov property for arbitrary stopping times. Let $\xi$ be a stopping time and fix $G \in \mathcal{F}$. For all $t \in \mathbb{R}_+$ we have $P$-a.s.

\[
P(\theta^{-1}_\xi G | F_{\xi \wedge t}) = P_{X_{\xi \wedge t}} (G).
\]

Because for all stopping times $\tau$ and $\rho$ it holds that $\mathcal{F}_\tau \cap \{ \tau \leq \rho \} \subseteq \mathcal{F}_\rho$, $\{ \tau \leq \rho \} \in \mathcal{F}_\tau$ and $\mathcal{F}_{\tau \wedge \rho} = \mathcal{F}_\tau \cap \mathcal{F}_\rho$, see [21, Lemmata 1.2.15, 1.2.16], we have

\[
\mathcal{F}_\tau \cap \{ \tau \leq \rho \} \subseteq \mathcal{F}_{\tau \wedge \rho}.
\]

Therefore, $P$-a.s. on $\{ \xi \leq t \}$

\[
P(\theta^{-1}_\xi G | F_{\xi \wedge t}) = P(\theta^{-1}_\xi G | F_\xi),
\]

and we conclude that $P$-a.s. on $\{ \xi \leq t \}$

\[
P(\theta^{-1}_\xi G | F_\xi) = P_{X_\xi} (G).
\]

Letting $t \to \infty$ yields the strong Markov property for $\xi$. \hfill \Box

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21This follows from the fact that $X_{s+\xi}$ is $\mathcal{F}_{s+\xi}$-measurable for all $z \leq r$, see, e.g., [19, Lemma 6.5]
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D. Criens - Technical University of Munich, Center for Mathematics, Germany
E-mail address: david.criens@tum.de