A Polymer Expansion for the Quantum Heisenberg Ferromagnet Wave Function

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Abstract

A polymer expansion is given for the Quantum Heisenberg Ferromagnet wave function. Working on a finite lattice, one is dealing entirely with algebraic identities; there is no question of convergence. The conjecture to be pursued in further work is that effects of large polymers are small. This is relevant to the question of the utility of the expansion and its possible extension to the infinite volume. In themselves the constructions of the present paper are neat and elegant and have surprising simplicity.
This paper assumes the fundamentals of the Heisenberg model but is basically self-contained; it arises from the work in [1], [2], [3], but these references need not be referred to. We intend to continue the work in the present paper, to obtain bounds on polymer contributions enabling extension to the infinite lattice. We also have some hope of using this expansion in a proof of the phase transition.

We work with a finite rectangular lattice, \( V \), in \( d \)-dimensions, \( V \) the set of its vertices. The Hamiltonian is taken as

\[
H = -\sum_{i \sim j} \frac{1}{2} (\vec{\sigma}_i \cdot \vec{\sigma}_j - 1) = -\sum_{i \sim j} (I_{ij} - 1)
\]

where \( I_{ij} \) interchanges the spins at nearest neighbor sites \( i \) and \( j \). The Hilbert space \( \mathcal{H} \) is constructed from basis elements \( \vec{i}_S \), basis elements in 1–1 correspondence with subsets \( S \) of \( V \), used for their labeling. In a spin-up spin-down representation

\[
\vec{i}_S = \bigotimes_{i \in S} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{i} \bigotimes_{j \notin S} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{j}.
\]

A vector \( \vec{f} \) in \( \mathcal{H} \) may be expanded as

\[
\vec{f} = \sum_S f(S) \vec{i}_S.
\]

For two sets \( S \) and \( S' \) we write \( S \sim S' \) if \( S' \) is constructed from \( S \) by replacing some single element of \( S \) by one of its nearest neighbors. That is, \( S \sim S' \) if there is a set \( F \) and elements of \( V \), \( i \) and \( j \), so that

\[
S = F \cup i
\]

\[
S' = F \cup j
\]
where \( i \sim j \) and the unions in (4) are disjoint. If we write
\[
\tilde{f}(t) = e^{-\mathcal{H}t} \tilde{f} = \sum_S f(S, t) \vec{i}_S. \tag{5}
\]
It is easy to see that the \( f(S, t) \) satisfy the differential equations
\[
\frac{\partial}{\partial t} f(S, t) = \sum_{S' \sim S} (f(S', t) - f(S, t)). \tag{6}
\]
This is the graph heat equation, corresponding to a graph with vertices the subsets of \( V \), and with an edge connecting vertices \( S_1 \) and \( S_2 \) if and only if \( S_1 \sim S_2 \).

We now write \( \mathcal{H} \) as direct sum
\[
\mathcal{H} = \bigoplus_{n=0}^{\#(V)} \mathcal{H}^n \tag{7}
\]
where as indicated \( n \) ranges from 0 to \( \#(V) \). \( \mathcal{H}^n \) is spanned by the basis elements \( \vec{i}_S \) where \( \#(S) = n \). \( \mathcal{H}^n \) is the \( n \) spin-wave sector of the Hilbert space \( \mathcal{H} \). The \( \mathcal{H}^n \) are invariant subspaces of \( \mathcal{H} \). We write \( \mathcal{H}^n \) for \( \mathcal{H} \) restricted to \( \mathcal{H}^n \).

We introduce operators \( T^{r,s} \), where \( T^{r,s} \) is a linear mapping from \( \mathcal{H}^r \) to \( \mathcal{H}^s \). They are defined as follows:

1) \( T^{r,r} \) is the identity on \( \mathcal{H}^r \)

2) If \( s > r \),
\[
T^{r,s} = 0
\]

3) If \( s < r \) let \( \vec{g} \) be in \( \mathcal{H}^r \)
\[
\vec{g} = \sum_S g(S) \vec{i}_S \tag{8}
\]
where \( g(S) \) is non-zero only if \( \#(S) = r \). Let
\[
\vec{h} = T^{r,s} \vec{g} = \sum_S h(S) \vec{i}_S. \tag{9}
\]
Then $h(S) = 0$ unless $\#(S) = s$, and

$$h(S) = \sum_{S' \supset S} g(S') \text{ if } \#(S) = s. \quad (10)$$

We note that if $r > s > k$ then

$$T^{s,k} T^{r,s} = \frac{(r-k)!}{(s-k)!(r-s)!} T^{r,k} \quad (11)$$

This is easy counting.

A nice result is that $T^{r,s}$ intertwines $H^r$ and $H^s$. That is

$$T^{r,s} H^r = H^s T^{r,s} \quad (12)$$

where both sides of (12) are viewed as mappings from $\mathcal{H}^r$ to $\mathcal{H}^s$. This is treated in Appendix A. The formalism is from Section II of [1]. A similar more complex parallel theory is given in [3] for random walks on the permutation group, instead of subspaces of a lattice.

We start presenting the polymer expansion for $\vec{f}(t)$ of equation (5). We assume $\vec{f}(t)$ is normalized so that

$$\sum_{S} f(S, t) = 1. \quad (13)$$

We note that if at any time this equation holds, the heat equation, equation (6), preserves the identity. We do not consider the possibility that the sum on the left side of (13) be zero, so no such normalization is possible.

We let $\mathcal{P}$ be a partition of $\mathcal{V}$. We write $S_\alpha < \mathcal{P}$ for a subset $S_\alpha$ of the partition $\mathcal{P}$. One has

$$S_\alpha \cap S_\beta = \emptyset, \alpha \neq \beta \quad (14)$$

$$\bigcup_{\alpha \in \mathcal{P}} S_\alpha = \mathcal{V}. \quad (15)$$
We will have
\[ \vec{f}(t) = \sum_{\mathcal{P}} \otimes_{\mathcal{S}_\alpha < \mathcal{P}} \vec{u}(\mathcal{S}_\alpha, t) \]  
(16)

where
\[ \vec{u}(\mathcal{S}_\alpha, t) = \left( \begin{array}{c} \phi_i(t) \\ 1 - \phi_i(t) \end{array} \right) \]  
(17)

if \( \mathcal{S}_\alpha = \{ i \} \).

If \( \#(\mathcal{S}_\alpha) = r > 1 \)
\[ \vec{u}(\mathcal{S}_\alpha, t) = u^r(\mathcal{S}_\alpha, t) \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \]  
(18)

We also write
\[ u(\mathcal{S}, t) = \begin{cases} \phi_i(t) & \text{if } \mathcal{S} = \{ i \} \\ u^r(\mathcal{S}, t) & \text{if } \#(\mathcal{S}) = r > 1 \end{cases} \]  
(19)

We write \( \vec{f}(t) \) as a sum of its different spin-wave number components
\[ \vec{f}(t) = \sum_{n=0}^{\#(\mathcal{V})} \vec{f}_n(t) \]  
(20)
\[ \vec{f}_n(t) \in \mathcal{H}^n \]  
(21)

We set
\[ \vec{c}_r(t) = \sum_{n=0}^{\#(\mathcal{V})} T^{n,r} \vec{f}_n(t) \]  
(22)

and
\[ \vec{c}_r(t) = \sum_{\mathcal{S}} c^r(\mathcal{S}, t) \vec{1}_\mathcal{S} \]  
(23)

(Do notice that the \( c^r, s \) satisfy the graph heat equation, (6).) Then we find that equation (16) is satisfied if the \( u(\mathcal{S}, t) \) are chosen to satisfy:
\[ c^r(\mathcal{S}, t) = u^r(\mathcal{S}, t) + \sum_{\mathcal{P}} \otimes_{\mathcal{S}_\beta < \mathcal{P}} u(\mathcal{S}_\beta, t) \]  
(24)

Where here \( \mathcal{P} \) is a proper partition of \( \mathcal{S} \) and \( \#(\mathcal{S}) = r \). \( r \) will range from 1 to \( \#(\mathcal{V}) \).

Equations (16) and (24) are prototype cluster-expansion/polymer-expansion equations.
But the form of equation (18) is perhaps surprising? Appendix B treats the consistency of the formalism; that there is a unique solution for the $u's$ from (24), and they yield equation (16).
Appendix A. Intertwining Result

In virtue of equation (11) it is enough to show $T_{r,r}^{-1}$ intertwines. We choose to show equivalently that $T_{r,r}^{-1}$ carries a solution of the heat equation into a solution of the heat equation. Let $f(S, t)$ satisfy the heat equation, and be zero unless $(S) = r$. We define

$$g(s, t) = \sum_j f(s \cup j, t), \#(s) = r - 1 \quad (A.1)$$

We wish to show $g$ satisfies the heat equation. Writing the heat equation for $f$:

$$\frac{\partial f}{\partial t}(s \cup i, t) = \sum_{S' \sim (s \cup i)} (f(S', t) - f(s \cup i, t)) \quad (A.2)$$

We sum the two sides of (A.2) over $i$.

$$\frac{\partial}{\partial t} g(s, t) = \sum_i \sum_{S' \sim (s \cup i)} (f(S', t) - f(s \cup i, t)) \quad (A.3)$$

The right side splits into two terms $I_1$ and $I_2$

$$I_1 = \sum_i \sum_{s' \sim s} (f(s' \cup i, t) - f(s \cup i, t)) \quad (A.4)$$

and

$$I_2 = \sum_i \sum_{j \sim i} (f(s \cup j, t) - f(s \cup i, t)) \quad (A.5)$$

It is easy to see

$$I_1 = \sum_{s' \sim s} (g(s', t) - g(s, t)) \quad (A.6)$$

and just a little harder to see

$$I_2 = 0$$

and the result is proved.

Appendix B. In Partes Tres.

We divide the demonstration of consistency into three parts.
I) We first note that equation (24) has a unique solution for the $u^r$ (these are the unknowns). One solves inductively over $r$, the $r$th equation uniquely determining $u^r$.

II) Once the $u$'s are determined from equation (24), we substitute them in the right side of equation (16) which we call $\vec{X}(t)$, so equation (16) becomes

$$ \vec{f}(t) = \vec{X}(t). \quad (B.1) $$

(Of course we do not know whether (B.1) is true, that is what we're trying to show.) We decompose $\vec{X}(t)$

$$ \vec{X}(t) = \sum_{n=0}^{\#(\mathcal{V})} \vec{X}_n(t) \quad (B.2) $$

$$ \vec{X}_n(t) \in \mathcal{H}^n \quad (B.3) $$

and define

$$ \vec{d}_r(t) = \sum_{n=0}^{\#(\mathcal{V})} T^{n,r} \vec{X}_n(t) \quad (B.4) $$

$$ \vec{d}_r(t) = \sum_S d^r(S, t) \vec{i}_S. \quad (B.5) $$

The result we seek to now show is the following: If $d^r(S, t) = c^r(S, t) \text{ all } S, r$ then $\vec{f}(t) = \vec{X}(t)$.

This we also show by induction over $r$, but in the opposite direction, from $r = \#(\mathcal{V})$ down to $r = 0$. At the step $r = r$ we clearly get

$$ \vec{f}_r(t) = \vec{X}_r(t). \quad (B.6) $$

(One only needs $T^{r,r} = I$, and $T^{r,s} = 0$ if $s > r$.)

III) We are left with the task of showing

$$ d^r(S, t) = c^r(S, t). \quad (B.7) $$

We first do a preliminary investigation.

Let

$$ \vec{h}(t) = \sum h(S, t) \vec{i}_S \quad (B.8) $$
\[
\sum_{n=0}^{\#(V)} \vec{h}_n(t) = \sum_{n=0}^{\#(V)} \vec{h}_n(t) \quad (B.9)
\]

\[
\vec{h}_n(t) \in \mathcal{H}^n \quad (B.10)
\]

and define

\[
\tilde{g}_r(t) = \sum_n T^{n,r} \vec{h}_n(t) \quad (B.11)
\]

\[
= \sum_S g^r(S, t) \vec{1}_S \quad (B.12)
\]

We then find the following expression for \( g^r(S, t) \)

\[
g^r(S, t) = \sum_{S'} h(S \cup S', t) \quad (B.13)
\]

where \( \#(S) = r \).

Now when we compute \( d^r(S, t) \) using the expression (B.13) with \( X \) replacing \( h \) \((X(t) = \sum_S X(S, t) \vec{1}_S)\), the only terms in the expression for \( \tilde{X}(t) \) from (16) which will contribute are of the form

\[
\left( \vec{u}'(S, t) + \sum_{P} \bigotimes_{S_{\beta}<P} \vec{u}(S_{\beta}, t) \right) \bigotimes_{i \notin S} \left( \phi_i(t) \begin{pmatrix} \phi_i(t) \\ 1 - \phi_i(t) \end{pmatrix} \right)_i \quad (B.14)
\]

using the notation from equation (24). That is because the sum over \( \mathcal{S}' \) in (B.13) may be written as an iterated sum, summing for each vertex not in \( \mathcal{S} \), whether the vertex is in \( \mathcal{S}' \) or not. This amounts to summing over spin-up and spin-down at that vertex. At vertex \( k \) this sum applied to the term in the tensor product

\[
\begin{pmatrix} \phi_k(t) \\ 1 - \phi_k(t) \end{pmatrix}_k
\]

yields 1, and applied to

\[
\begin{pmatrix} 1 \\ -1 \end{pmatrix}_k
\]
yields 0. We get from the terms in $\vec{X}(t)$ in (B.14) that

$$d^{r}(S, t) = c^{r}(S, t) \quad (B.15)$$

Quod erat demonstrandum.

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References

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