Fermion Determinants and Effective Actions

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Abstract

Configuration space heat-kernel methods are used to evaluate the determinant and hence the effective action for an SU(2) doublet of fermions in interaction with a covariantly constant SU(2) background field. Exact results are exhibited which are applicable to any Abelian background on which the only restriction is that $(B^2 - E^2)$ and $E \cdot B$ are constant. Such fields include the uniform field and the plane wave field. The fermion propagator is also given in terms of gauge covariant objects. An extension to include finite temperature effects is given and the probability for creation of fermions from the vacuum at finite temperature in the presence of an electric field is discussed.

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1 Introduction

Effective actions are the one tool that allow us to probe for quantum effects in a manner that mimics classical studies. It has grown into an important tool since its most famous and perhaps one of the first examples - that of the Euler-Heisenberg Lagrangian obtained by “integrating out ”the fermions in QED.

However, the impetus to study effective actions did not come until the work of Schwinger and DeWitt. Schwinger in his inimitable style constructed the effective action for QED using a powerful and elegant technique, frequently referred to as the proper-time method. While Schwinger worked in flat space, DeWitt generalized his techniques to curved spaces and related the method to the heat equation. For the first time Schwinger had not only rederived the Euler-Heisenberg Lagrangian but went on to find the effective action for QED in the presence of a plane-wave background field.

More recently, the method was revived by Shore who formalized the ideas of Brown and Duff with the techniques of Schwinger to enable the calculation of effective actions for non-Abelian gauge theories with covariantly constant background fields. We shall have more to say about such fields in the paper. Shore’s work has also been generalized to include the effects of finite temperature. It has also been shown how the method enables one to find the effective actions for the electroweak theory SU(2)_L × U(1)_Y. However all these have dealt with only the bosonic sectors of the gauge theories and fermionic contributions have been neglected. The sole exception being where it was shown that the method could be applied to fermions both at zero and at finite temperatures equally well. However dealt with fermions transforming under an adjoint representation of SU(2) with an eye to applicability to supersymmetric theories.

More recently, there has been a resurgence of interest in the study of quantum fields in external backgrounds both at zero and finite temperatures. In this paper, we consider a doublet of fermions transforming under the fundamental representation of SU(2) and interacting with an SU(2) Yang-Mills background gauge field. The heat kernel method is applied to find the exact propagator and effective action contributions coming from vacuum polarization effects of the fermions à la Euler-Heisenberg for the SU(2) gauge fields satisfying a covariant constancy condition.

2 Propagators, Effective Actions and Heat Kernels

Given a differential operator \( D(x,y) \), the corresponding heat equation is given by

\[
\int dz D(x,z) G(z,y; s) = -\frac{\partial G}{\partial s}
\]

where the Kernel, \( G(x,y,s) \) is required to satisfy the following boundary condition

\[
\lim_{s \to 0^+} G(x,y; s) = \delta(x,y)
\]

\( \delta(x,y) \) being the Dirac delta function. The propagator or Green’s function \( G(x,y) \) for the operator \( D(x,y) \) is obtained by a simple proper time \( (s) \) integration

\[
G(x,y) = \int_0^\infty ds G(x,y; s)
\]

Of course, it is understood that the kernel \( G(x,y; s) \) and the Green’s function \( G(x,y) \) will carry all the indices that the operator \( D(x,y) \) carries.

Effective actions up to one loop quantum effects are related to functional determinants of operators obtained by considering quadratic fluctuations in the classical fields. More precisely, the one loop effective action is obtained by a Legendre transformation of the generating functional for connected Green’s function The standard formula for fermions is

\[
\Gamma(A) = \ln det D
\]
Where $\Gamma[A]$ is the effective action functional and $A$, the background field(s). This dependence on the background field(s) comes, of course, from the operator $D(x,y)$ which contains the background field(s) $A$.

Formal manipulations on $\text{Indet} D$ can be carried out to yield

$$\Gamma[A] = \text{Indet} D = -Tr \int_0^\infty \frac{ds}{s} G(x,y;s). \quad (5)$$

This formula holds as long as the operator $D$ has no zero eigenvalues. Traditionally, one excludes such zero modes from consideration to evaluate the effective action and later includes their effects through collective mode methods. For the purposes of this paper, we assume $D$ has no zero eigenvalues. Equations (3) and (5) tell us that to evaluate $G(x,y)$ and $\Gamma[A]$, all we require is a solution to the heat equation (1). This is where the power of the proper-time or heat kernel method is manifest. Before we proceed to find $G(x,y;s)$, we have to note that in equation (5), $Tr$ denotes a composite trace à la DeWitt; it is a trace over both discrete labels carried by $G(x,y;s)$ as well as its continuous indices; in particular,

$$tr_{x,y} \int_0^\infty \frac{ds}{s} G(x,y;s) = \int d^n x \int d^n y \delta^n(x,y) \int_0^\infty \frac{ds}{s} G(x,y;s) \quad (6)$$

### 3 Determinants for Dirac Operators.

In this section, we shall examine the particular Dirac operator that concerns us. Notations and conventions can be found in the appendix. The particular form of the Dirac equation we consider is

$$\left(\gamma^\mu D_\mu + m\right) \psi(x) = 0 \quad (7)$$

The Dirac propagator $S(x,y)$, satisfies

$$\left(\gamma^\mu D_\mu + m\right) S(x,y) = \delta(x,y) \quad (8)$$

Since the spinor $\psi(x)$ is in reality a doublet, $\psi_i(x); i = 1, 2$, we must have

$$\left(\gamma^\mu D_\mu\right)_{ij} + m\delta_{ij}) S_{jk}(x,y) = \delta_{ik} \delta(x,y) \quad (9)$$

having made the “internal” $\text{SU}(2)$ indices explicit. For the rest of the paper, all indices will be suppressed unless otherwise required. The Dirac operator and the Laplacian have been studied extensively in the mathematical literature. The spectrum of the Laplacian has been used to get information on the geometrical properties of manifolds (“can one hear the shape of a drum?”) while the Dirac operator on manifolds has given topological information through the Atiyah-Singer index theorem and its generalizations \cite{10}. Here we shall solve our problem for the determinant of the Dirac operator by converting it into a study of the covariant Laplacian. From equation (8) we can write

$$S(x,y) = \left(\gamma^\mu D_\mu + m\right)^{-1} \delta(x,y) \quad (10)$$

implying

$$S(x,y) = -\left(\gamma^\mu D_\mu - m\right) \left[-(\gamma \cdot D)^2 + m^2\right]^{-1} \delta(x,y) \quad (11)$$

The Green’s function $G(x,y)$, for the covariant Laplacian

$$D = -(\gamma \cdot D)^2 + m^2 \quad (12)$$

is defined by

$$-(\gamma \cdot D)^2 + m^2) G(x,y) = \delta(x,y) \quad (13)$$

By using the properties of the $\gamma$-matrices and

$$[D_\mu, D_\nu] = -igF_{\mu\nu} \quad (14)$$
Given the $D_\mu = \partial_\mu - igA_\mu$, $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$, we find

$$D = (-D)^2 + \frac{1}{2}g(\sigma \cdot F) + m^2$$

where

$$\sigma_{\mu \nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

and $\sigma \cdot F = \sigma_{\mu \nu}F_{\mu \nu}$. Equation (13) now takes the following form:

$$(-D)^2 + \frac{1}{2}g(\sigma \cdot F) + m^2)G(x, y) = \delta(x, y)$$

The associated heat equation may be written down as

$$(-D)^2 + \frac{1}{2}g(\sigma \cdot F) + m^2)G(x, y; s) = -\frac{1}{\partial s}G(x, y; s)$$

From equations (3) and (5), it is clear that a solution of (18) will give us both the effective action as well as the Green’s function and hence the Dirac propagator through equation (10). While it is easy to exponentiate the constant mass term in (18), the “magnetic moment” term $(\sigma \cdot F)$, requires a different approach. A perturbative solution is possible through the trace of the heat kernel. The coefficients of the series are known to be invariants of the underlying manifold. In our case, these invariants include invariants of the gauge field. This particular method has been used extensively for studying gravitational interactions \[2, 11\]. Following the ideas of \[12\] and \[3\], we impose a condition on the background field. Since we do not wish to break the gauge symmetry of the background field, we shall impose a gauge covariant condition:

$$D_\mu F_{\nu \rho} = 0$$

Equations (18) and (19) admit the following solution:

$$G(x, y; s) = \frac{e^{-n/2}}{(4\pi)^{n/2}} \Phi(x, y)$$

$$\times \exp \left\{ -\frac{1}{4} (x - y)_\mu (gFcot(gsF))_{\mu \nu} (x - y)_\nu \right\}$$

$$\times \exp \left\{ -\frac{1}{2} tr\ln[(gsF)^{-1}Sin(gsF)] \right\}$$

$$\times \exp \left\{ -\frac{1}{2}gs(\sigma \cdot F) - m^2s \right\}$$

where

$$\Phi(x, y) = \exp \left\{ ig \int_x^y A_\mu(z)dz_\mu \right\}$$

is a path dependent phase factor with the line integral being taken over the straight line path from $x$ to $y$. It ensures the correct gauge properties for the kernel $G(x, y; s)$. From (5) we see that $\Gamma[A]$ is then given by:

$$\Gamma[A] = -\frac{1}{2}Tr \int_0^\infty ds \frac{s^{-n/2}}{(4\pi)^{n/2}} \Phi(x, y)$$

$$\times \exp \left\{ -\frac{1}{4} (x - y)_\mu (gFcot(gsF))_{\mu \nu} (x - y)_\nu \right\}$$

$$\times \exp \left\{ -\frac{1}{2} tr\ln[(gsF)^{-1}Sin(gsF)] \right\}$$

$$\times \exp \left\{ -\frac{1}{2}gs(\sigma \cdot F) - m^2s \right\}$$
From equation (20) it is clear that
\[
\lim_{s \to 0} G(x, y; s) = \lim_{s \to 0} \frac{s^{-n/2}}{(4\pi)^{n/2}} \exp \left\{ -\frac{1}{4s} (x - y)^2 \right\}
\] (23)
which shows that indeed \( G(x, y; s) \) satisfies the boundary condition as required since the expression on the RHS in (23) is just the Gaussian representation of the Dirac delta function. It is also clear that the asymptotic series one usually adopts for the heat kernel when dealing with arbitrary background fields has been “summed up” [13]. Such a summing up has been made possible by the covariant constancy condition given in (19) implying that all derivative terms in the asymptotic series are put to zero. In the next section we shall consider the trace over Dirac indices in (22).

4 Dirac Traces.

Equation (22) has three sets of indices that need to be traced over; they are, Lorentz, internal gauge group and Dirac spinor indices. As far as the spinor indices are concerned, only one term contains them. The calculation is much easier if we carry out the traces over the spinor indices first. Let us write
\[
D(s) = tr_D \exp \left\{ -\frac{1}{2} g s (\sigma \cdot F) \right\}
\] (24)
where \( tr_D \) denotes a trace over the Dirac spinor indices. Noting that
\[
\frac{1}{2} \{ \sigma_{\mu\nu}, \sigma_{\alpha\beta} \} = \delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha} + i \epsilon_{\mu\nu\alpha\beta} \gamma_5
\] (25)
where \( \{ , \} \) denotes the anti-commutator. We can therefore write
\[
\left( \frac{1}{2} \sigma \cdot F \right)^2_{ij} = \frac{1}{2} (F^a_{\mu\nu} F^a_{\mu\nu}) + \frac{1}{2} \gamma_5 (F^a_{\mu\nu})_{ij}
\] (26)
where \( F^2 = F_{\mu\nu} F_{\mu\nu}; F^a F = \frac{1}{4} F_{\mu\nu} \epsilon_{\mu\alpha\beta} F_{\alpha\beta} \). Since \( F_{\mu\nu} \) is a non-Abelian gauge field, making the group indices explicit, we have \( F_{\mu\nu} = F_{\mu\nu}^a T^a \) where \((T^a)_{ij}\) are the fundamental representation matrices of SU(2) satisfying
\[
[T^a, T^b] = i \epsilon^{abc} T^c
\] (27)
and we choose \( T^a = \frac{1}{2} \sigma^a \) where \( \sigma^a, a = 1, 2, 3, \) are the Pauli matrices (see appendix). Equation(26) reduces to
\[
\left( \frac{1}{2} \sigma \cdot F \right)^2_{ij} = \frac{1}{2} \left\{ (F^a_{\mu\nu} F^a_{\mu\nu}) + \gamma_5 (F^a_{\mu\nu} F^b_{\mu\nu}) \right\} \frac{1}{4} (\sigma^a \sigma^b)_{ij}
\] (28)
Using obvious symmetry properties of this expression, we have
\[
\left( \frac{1}{2} \sigma \cdot F \right)^2_{ij} = - (F^2 + \gamma_5 F^2)_{\mu\nu} \delta_{ij}
\] (29)
Where the following gauge invariant quantities
\[
(\mathcal{F}^2)_{\mu\nu} = \frac{1}{4} (F^a_{\mu\sigma} F^a_{\sigma\nu})
\] (30)
and
\[
(\mathcal{F}^2)_{\mu\nu} = \frac{1}{4} (F^a_{\mu\sigma} F^b_{\sigma\nu})
\] (31)
have been defined. Finally, since the eigenvalues of \( \gamma_5 \) are \( \pm i \), we can write the eigenvalues of the operator \( \left( \frac{1}{2} \sigma \cdot F \right) \) as
\[
\left( \frac{1}{2} \sigma \cdot F \right)'_{ij} = \pm i \text{tr} (\mathcal{F}^2 \pm i \mathcal{F}^2)^{1/2} \delta_{ij}
\] (32)
and

\[ D(s) = \text{tr}_D \exp \left\{ -\frac{1}{2} gs \sigma \cdot F \right\} \]

\[ = \delta_{ij} 4 \Re \cosh \left\{ i gs [\text{tr}(F^2 \pm i F)]^{1/2} \right\} \]  

where \( \Re \) stands for “real part of”. This completes the purpose of this section—that of evaluating the trace over the Dirac spinor indices. In the next section we shall go on to consider the other two sets of discrete indices—Lorentz and group.

5 Lorentz and group index traces.

To carry out traces over the Lorentz and group indices, we notice first that objects such as \( F^\mu_\alpha \) carry both Lorentz and group indices. A method has thus to be devised allowing us to disentangle these two sets of indices. This is in general made possible through the definition of projection operators. Two projection operators have already been introduced for precisely such a purpose, for the group that we are dealing with: \( \text{SU}(2) \). The first is

\[ R^{ij}_{\mu\nu} = (\delta^i_j \delta_{\mu\nu} - (F^2)^{-1}_\mu\sigma F^{i}_\sigma \lambda F^{j}_\lambda \nu) \]  

The second is

\[ Q^{ij}_{\mu\nu} = (\delta^i_j \delta_{\mu\nu} - (F^2 \pm iF^2)^{-1}_\mu\sigma (F^{i}_\sigma \lambda F^{j}_\lambda \nu \pm i* F^{i}_\sigma \lambda F^{j}_\lambda \nu)) \]

It is easy to verify that both of these are projection operators:

\[ R^{ij}_{\mu\sigma} R^{jk}_{\sigma\nu} = R^{ik}_{\mu\nu} \]  

and

\[ Q^{ij}_{\mu\sigma} Q^{jk}_{\sigma\nu} = Q^{ik}_{\mu\nu} \]

where two simply consequences of the covariant constancy condition are utilized:

\[ [F_{\mu\nu}, F_{\alpha\beta}] = 0 = [F_{\mu\nu}, *F_{\alpha\beta}] \]

However, these two projection operators rely on the fact that the representation matrices are adjoint representation matrices. We have to consider what happens when the matrices are \( \text{SU}(2) \) fundamental representation matrices. Since \( D_\mu F_{\alpha\beta} = 0 \) implies that \([F_{\mu\nu}, F_{\alpha\beta}] = 0\), we have that

\[ F^{\alpha}_\mu T^{b}_\nu \]  

and therefore, consider

\[ (F^2)_{\mu\nu ij} = F^{a}_\mu T^{a}_{ik} F^{b}_\lambda T^{b}_{kj} \]

Since \( T^{a}_{ij} \) are \( \text{SU}(2) \) fundamental representation matrices,

\[ T^{a}_{ij} = \frac{1}{2} \sigma^a_{ij} \]

it is easy to see that

\[ (F^2)_{\mu\nu ij} = \frac{1}{4} F^{a}_{\mu\lambda} F^{b}_{\lambda\nu} \delta_{ij} + \frac{1}{8} F^{a}_{\mu\lambda} F^{b}_{\lambda\nu} [\sigma^a, \sigma^b]_{ij} \]

From equation (39) we see that the second term on the RHS is zero. Therefore we write

\[ (F^2)_{\mu\nu ij} = (F^2)_{\mu\nu} \delta_{ij} \]

we notice that the group indices have been separated rather trivially from the Lorentz indices. There is clearly no need to define any projection operator here. We may, for convenience sake write

\[ (F^2)_{\mu\nu ij} = (F^2)_{\mu\nu} F^{ij}_{\sigma\nu} \]
where
\[ P^{ij}_{\mu\nu} = \delta_{ij} \delta_{\mu\nu} \] (45)
and clearly, it follows that
\[ (F^{2n})_{\mu
u ij} = ((F^2)^n)_{\mu\sigma} P^{ij}_{\sigma\nu}; n \geq 1. \] (46)
The required simplifications now follow:
\[
\exp \left\{ -\frac{1}{2} tr \ln [(gF)^{-1} \sin(gF)] \right\}_{ij} = \exp \left\{ -\frac{1}{2} tr \ln [(gF)^{-1} \sin(gF)] \right\} \delta_{ij}
\] (47)
and
\[
(gF \cot(gF))^{\mu\nu}_{ij} = (gF \cot(gF))^{\mu\nu} \delta_{ij}
\] (48)
Therefore,
\[
\exp \left\{ -\frac{1}{4} (x - y)_\mu (gF \cot(gF))^{\mu\nu}(x - y)_\nu \right\} = \delta_{ij} \exp \left\{ -\frac{1}{4} (x - y)_\mu (gF \cot(gF))^{\mu\nu}(x - y)_\nu \right\}
\] (49)
where \( F \) is the square root of the matrix \( F^2 \). We should note here that this particular exponent; \((gF \cot(gF))\) appears multiplied by factors of \((x - y)\) implying that its contribution to the effective action or the propagator vanishes upon tracing over the \( x, y \) indices. However, at finite temperature, its contribution would be non-zero and the result in equations (48) & (49) would be useful then. This completes our discussion on the traces over the Lorentz and group indices. In the next section we shall make explicit, the effective action \( \Gamma[A] \).

### 6 Effective Actions, etc.

We have seen that the effective action for our SU(2) fundamental fermions in interaction with a covariantly constant SU(2) gauge field can be written as
\[
\Gamma[A] = -\frac{1}{2} Tr \int_0^\infty ds \frac{s^{n/2}}{(4\pi)^{n/2}} \Phi(x, y) \times \exp \left\{ -\frac{1}{4} (x - y)_\mu (gF \cot(gF))^{\mu\nu}(x - y)_\nu \right\} \times \exp \left\{ -\frac{1}{2} tr \ln [(gF)^{-1} \sin(gF)] \right\} \delta_{ij} \times \exp \left\{ -\frac{1}{2} gs (\sigma \cdot F) - m^2 s \right\}
\] (50)
Using the results of the last two sections on Dirac, Lorentz and group traces along with equation (6), we find
\[
\Gamma[A] = -\frac{1}{2} tr_{\text{group}} \int d^n x \int_0^\infty ds \frac{s^{n/2}}{(4\pi)^{n/2}} \Re \cosh \left\{ i gs [tr(F^2 + i F^3)]^{1/2} \right\} \times \exp \left\{ -\frac{1}{2} tr \ln [(gF)^{-1} \sin(gF)] \right\} \exp \left\{ -m^2 s \right\}
\] (51)
Before proceeding, we must remember that any quantum calculation involves the appearance of infinities and therefore a process of regularization and renormalization has to be carried out. We know that the proper-time method is an “invariant regularization” method \[\square\] and so we need only worry about subtracting the infinities in (51). It is well known that in the proper-time integration, the traditional UV
(UltraViolet) infinities arise in the limit of $s \to 0$ while IR (InfraRed) infinities arise at the upper limit of the integration: $s \to \infty$. I.e; the short distance behaviour is given by $s \to 0$ and the behaviour at large scales is given by $s \to \infty$. In this paper, we shall not worry about the infinities or their structure. They have been well studied in the literature and while they are useful for the study of renormalization properties, we are more interested in the finite structure of the effective actions. To this end, we simply extract the infinities in the proper-time integration of (51) and subtract the terms from the integrand, leaving a finite expression for $\Gamma[A]$. Note that the mass term in the exponent in (51) tells us that there will be no divergent terms in the integrand for $s \to \infty$, hence no IR divergences. While in four dimensions ($n = 4$), in the limit of $s \to 0$, divergences will arise from terms which are at most quadratic in $s$. Therefore, expanding the integrand in powers of $s$ and subtracting terms up to $s^2$, we find a finite $\Gamma[A]$; we shall carry out these subtractions at the end. Using (47) we can write (51) in a form where the group indices are explicit:

$$\Gamma[A] = -\frac{1}{2} \text{tr}_{\text{group}} \int_{0}^{\infty} \frac{ds}{s} \int d^n x \frac{e^{-s/2}}{(4\pi)^{n/2}}$$

$$4\text{RCosh} \left\{ i gs [tr(F^2 + i F')]^{1/2} \right\} \delta_{ij}$$

$$\times \text{Exp} \left\{ -\frac{1}{2} tr ln [(gsF)^{-1} \text{Sin}(gsF)] \right\} \exp(-m^2 s)$$

Therefore,

$$\Gamma[A] = -\int_{0}^{\infty} \frac{ds}{s} \int d^n x \frac{e^{-s/2}}{(4\pi)^{n/2}}$$

$$4\text{RCosh} \left\{ i gs [tr(F^2 + i F')^{1/2} \right\}$$

$$\times \text{Exp} \left\{ -\frac{1}{2} tr ln [(gsF)^{-1} \text{Sin}(gsF)] \right\} \exp(-m^2 s)$$

Now, we require to find the eigenvalues of $F^2$ in order to express $\Gamma[A]$ in terms of the two Abelian gauge invariants

$$F_1 = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

and

$$F_2 = \frac{1}{4} F^a_{\mu\nu} * F^a_{\mu\nu}$$

Since an Abelian group admits only these two invariants, the fact that our non-Abelian field satisfying the covariant constancy condition (19) and hence (38) tells us that we shall also have only these two invariant Lorentz scalars appearing. The eigenvalues of $F^2$ (and hence of $F^2$) are

$$f^\pm = -F_1 \pm \sqrt{F_1^2 + F_2^2}$$

with each eigenvalue occurring with a degeneracy factor of two. It follows then that:

$$\text{Exp} \left\{ -\frac{1}{2} tr ln [(gsF)^{-1} \text{Sin}(gsF)] \right\} = \frac{g^2 s^2 \sqrt{f^+ \sqrt{f^-}}}{\text{Sin}(gs \sqrt{f^+}) \text{Sin}(gs \sqrt{f^-})}$$

where

$$\sqrt{f^\pm} = \frac{i}{\sqrt{2}} [(F_1 + i F_2)^{1/2} \pm (F_1 - i F_2)^{1/2}]$$

or,

$$\text{Exp} \left\{ -\frac{1}{2} tr ln [(gsF)^{-1} \text{Sin}(gsF)] \right\} = \frac{g^2 s^2 F_2}{3 \text{Cosh}(gsX)}$$

with

$$X^2 = 2(F_1 + i F_2)$$
and ℑ denoting “imaginary part of”.

Noting that \( \text{tr}F^2 = -F_1 \), and \( i\gamma_5 \sqrt{\text{tr}(F_2 + i\theta)} \) \( = -i\theta X \), we can finally write

\[
\Gamma_A = -\frac{1}{2} \int d^n x \int_0^\infty \frac{s^{-n/2}}{(4\pi)^{n/2}} 4 \left\{ \Re \cosh(\gamma_5 \theta X) \left( \frac{2\gamma^2 s^2 \theta_2}{3 \cosh(\gamma_5 \theta X)} \right) - 1 \right\} \exp(-m^2 s)
\]  

In writing the final expression, we have cheated just a little. Because we have the dual of \( F_{\mu\nu} \), \( *F_{\mu\nu} \) appearing in \( \Gamma_A \), the spacetime dimension must be four so that \( *F \) is also a two form along with \( F \). Also, as early as equation (25), because of the appearance of \( \gamma_5 \) and \( \epsilon_{\mu\nu\alpha\beta} \), we should have put \( n = 4 \). The reason we have done so at that stage is that for the case of a purely magnetic background field, \( F_2 = 0 \) and so \( \Gamma_A \) can live in spacetime dimensions \( \geq 4 \)—after the Dirac traces are done. However, notice that we do not really need to go to \( n \) dimensions (usually done for regularizing purposes) because regularization has been effected by the proper-time itself as long as the proper-time integration is carried out last \[1\]. For zero-field, the fermions are free and therefore, the effective action must go to zero. Thus, in four dimensions,

\[
\Gamma_A = -\frac{1}{4\pi^2} \int_0^\infty \int_0^\infty d^4 x \left\{ \gamma^2 \frac{s}{s^3} \frac{\theta_2}{3 \cosh(\gamma_5 \theta X)} - 1 \right\} \exp(-m^2 s)
\]  

Comparing this with the expression obtained by Schwinger (equation (3.44) of [1]) shows the changes brought about by going to a non-Abelian group, SU(2) and its fundamental representation, from the QED result. One may say that equation (62) is for SU(2) QCD, what Schwinger’s result is for U(1) QED and we see that for covariantly constant fields, the contribution from the doublet is simply twice the QED result. In some sense, a decoupling has taken place. To conclude this section, let us consider the UV infinities of \( \Gamma_A \): Expanding the integrand around \( s = 0 \) and retaining terms that lead to divergences, we find the following:

\[
-\frac{1}{6\pi^2} \int_0^\infty \gamma^2 \frac{ds}{s} \left( \frac{1}{s} \right) \exp(-m^2 s)
\]  

Adding the classical Lagrangian to the one-loop effective Lagrangian, we write the final result in the following form:

\[
\Gamma_A = -\left\{ 1 + \frac{\gamma^2}{6\pi^2} \int_0^\infty \frac{ds}{s} e^{-m^2 s} \right\} \theta_1 
- \frac{1}{4\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left\{ \gamma^2 \frac{s}{s^3} \frac{\theta_2}{3 \cosh(\gamma_5 \theta X)} - 1 - \frac{2}{3} \gamma^2 \frac{s}{s^3} \theta_1 \right\}
\]  

In the next section we shall find the propagators for the fermions using our results for the heat kernel before proceeding to consider finite temperature effects.

7 Propagators.

In this section we shall exhibit the propagator for our SU(2) doublet fermions using the relationship between the Green’s function for the Laplacian and the propagator exemplified by equations (3),(10) and (11).

\[
S(x,y) = -(\gamma_\mu D_\mu - m) \int_0^\infty ds G(x,y; s)
\]  

Since \( D_\mu F_{\alpha\beta} = 0 \), it is easy to show [3] that \( D_\mu \Phi(x,y) \) is given by

\[
D_\mu \Phi(x,y) = \frac{i}{2} F_{\mu\lambda}(x) \Phi(x,y)(x - y)_\lambda
\]  

\[
= \frac{i}{2} \Phi(x,y) F_{\mu\lambda}(y)(x - y)_\lambda
\]
Therefore the propagator is simply

\[ S(x, y) = \int_0^\infty \frac{s^{-n/2}}{4\pi^{n/2}} \left[ m - \frac{i}{2} \gamma_\mu F_\mu(x)(x - y)_\nu \right. \]
\[ - \frac{1}{2} (gF \cot g s \lambda F)_{\mu\nu}(x - y)_\nu \Phi(x, y) \]
\[ \times \exp \left\{ -\frac{1}{4} (x - y)_\mu (gF \cot g s \lambda F)_{\mu\nu}(x - y)_\nu \right\} \]
\[ \times \exp \left\{ -\frac{1}{2} \text{tr} \ln [(gF)^{-1} \text{Sin}(gF)] \right\} \]
\[ \times \exp \left\{ -\frac{1}{2} g(\sigma \cdot F)s - m^2 s \right\} \]

Note that this is the exact propagator for SU(2) fundamental fermions in an external SU(2) gauge field satisfying the covariant constancy condition (19). In addition, since we have seen that there is a simple separation of the group and Lorentz indices, \( S(x, y) \) can be reduced to:

\[ S(x, y) = \int_0^\infty \frac{s^{-n/2}}{4\pi^{n/2}} \left[ m - \frac{i}{2} \gamma_\mu F_\mu(x)(x - y)_\nu \right. \]
\[ - \frac{1}{2} (gF \cot g s \lambda F)_{\mu\nu}(x - y)_\nu \Phi(x, y) \]
\[ \times \exp \left\{ -\frac{1}{4} (x - y)_\mu (gF \cot g s \lambda F)_{\mu\nu}(x - y)_\nu \right\} \]
\[ \times \exp \left\{ -\frac{1}{2} g(\sigma \cdot F)s - m^2 s \right\} \]

This is as much of a simplification as one can achieve without further specialization of the background fields (such as to a purely magnetic field). In the next section, we shall extend our considerations to include finite temperature effects using the imaginary time formalism.

### 8 Finite temperature effects.

After the pioneering work on finite temperature effects in gauge theories \([14]\), there has been a resurgence of interest in the subject in recent years. On the one hand, non-equilibrium phenomena are being tackled while equilibrium thermodynamics is being applied (through both the imaginary and real time formalisms) to the study of QED and QCD plasma formations. This has become very important as the progress in the study of heavy-ion collisions holds promise of new developments leading to a better understanding of the structure of matter in extreme environments.

In this paper, we shall apply the imaginary time formalism as modified for applicability through the heat kernel \([7, 5]\), to find the finite temperature corrections to the effective action. In configuration space, the heat kernel method can be generalized to include finite temperature effects through the method of images and the finite temperature heat kernel, \( G^\beta(x, y; s) \) is constructed from the zero temperature kernel, \( G(x, y; s) \) as follows \([5]\):

\[ G^\beta(x, y; s) = \sum_{p = -\infty}^\infty (-1)^p G(x - p\lambda \beta, y; s) \]

where \((-1)^p\) ensures the correct boundary conditions for the fermions, \( \beta = 1/kT \), with \( k \) being Boltzmann’s constant and \( \lambda \) is a unit vector in the time direction. The generalization of the \( \text{Indet} \) is straightforward and is given by:

\[ \text{Indet}_\beta \mathcal{D} = -\text{Tr} \int_0^\infty \frac{ds}{s} G^\beta(x, y; s) \]
\[ \begin{align*}
&= -\text{Tr} \sum_{p=-\infty}^{\infty} (-1)^p \int_0^\infty \frac{ds}{s} \mathcal{G}(x - p\lambda\beta, y; s)
\end{align*} \]

while the Green’s function is given as:

\[ \mathcal{G}^\beta(x, y) = \int_0^\infty ds \mathcal{G}^\beta(x, y; s) \]

From equation (20) it is easy to see that

\[ \mathcal{G}^\beta(x, y; s) = \frac{s^{-n/2}}{(4\pi)^{n/2}} \sum_{p=-\infty}^{\infty} \Phi(x - p\lambda\beta, y) \]

\[ \times \text{Exp} \left\{ -\frac{p^2 \beta^2}{4} \lambda_\mu (gF\cot(gsF))_{\mu\nu} \lambda_\nu \right\} \]

\[ \times \text{Exp} \left\{ \frac{1}{2} \text{tr} \ln \left[ (gsF)^{-1} \text{Sin}(gsF) \right] \right\} \]

\[ \times \text{Exp} \left\{ \frac{1}{2} gs(\sigma \cdot F) - m^2 s \right\} . \]

In this expression, the phase factor \( \Phi(x - p\lambda\beta, y) \) is the only term that needs to be dealt with a little care as it leads to non-equilibrium situations along with a loss of gauge covariance. For the special case of purely magnetic fields however, it reduces to unity. At this stage, for simplicity, we shall impose a second condition on the background field by requiring

\[ \Phi(x - p\lambda\beta, y) = 1 \]

A brief discussion of the consequences of such a restriction has been given in [10] while a more detailed analysis may be found in [12] where they argue (for the case of QCD) that periodic configurations for which this condition does not hold, contribute negligibly to the effective action. For our purposes, it is sufficient to assume this additional condition holds and we note that the only term that is affected by the finite temperature corrections is the term containing \( (gF\cot(gsF)) \), the term which drops out of the zero temperature effective action. In particular, the Dirac spinor index traces will be identical to the zero temperature case. Thus, as the finite temperature effective action, \( \Gamma^\beta[A] \) is given by

\[ \begin{align*}
\Gamma^\beta[A] &= \frac{1}{2} \text{Tr} \int_0^\infty \frac{ds}{s} s^{-n/2} \sum_{p=-\infty}^{\infty} \Phi(x - p\lambda\beta, y) \]

\[ \times \text{Exp} \left\{ -\frac{p^2 \beta^2}{4} \lambda_\mu (gF\cot(gsF))_{\mu\nu} \lambda_\nu \right\} \]

\[ \times \text{Exp} \left\{ \frac{1}{2} \text{tr} \ln \left[ (gsF)^{-1} \text{Sin}(gsF) \right] \right\} \]

\[ \times \text{Exp} \left\{ \frac{1}{2} gs(\sigma \cdot F) - m^2 s \right\} , \]

we can carry out the Dirac traces and simplifying the expression, we find

\[ \begin{align*}
\Gamma^\beta[A] &= -\frac{1}{2} \text{Tr} \int_0^\infty \frac{ds}{s} s^{-n/2} \sum_{p=-\infty}^{\infty} 4\text{ReCosh} \left\{ igs\text{tr}[F^2 + iF^3]^{1/2} \right\} \\
&\quad \times \text{Exp} \left\{ -\frac{1}{2} \text{tr} \ln \left[ (gsF)^{-1} \text{Sin}(gsF) \right] - m^2 s \right\} . \end{align*} \]
For the group index traces, we see from equation (48) that
\[
[\lambda_\mu (gF\cot(gsF))_{\nu\nu}]^{ij} = \delta^{ij} \lambda_\mu (gF\cot(gsF))_{\nu\nu} = \delta^{ij} (gF\cot(gsF))_{00}
\] (77)

Hence,
\[
\Gamma^\beta[A] = -\frac{1}{4\pi^2} \int_0^\infty \frac{ds}{s^3} \int d^4x
\sum_{p=\infty}^{\infty} \text{Exp} \left\{ -\frac{p^2\beta^2}{4} (gF\cot(gsF))_{00} \right\}
\times (\Re \text{Cosh} \left\{ igs[\text{tr}(F^2 + iF^2)]^{1/2} \right\}) \left( \frac{g^2 s^2 \sqrt{f^+ \sqrt{f^+}}}{\text{Sin}(gs\sqrt{f^+})\text{Sin}(gs\sqrt{f^-})} \right) e^{-m^2 s}
\] (78)

The evaluation of \((g\sqrt{F^2\cot(gs\sqrt{F^2})})_{00}\) can be carried out either by constructing the diagonalizing matrix or by noting that \(F, F^2\) and \((g\sqrt{F^2\cot(gs\sqrt{F^2})})_{00}\) are all diagonalized by the same diagonalizing matrix. Simple matrix algebra then gives:
\[
(gF\cot(gsF))_{00} = (g\sqrt{F^2\cot(gs\sqrt{F^2})})_{00} = E^2 \left( \frac{\text{Cot}(gs\sqrt{f^+}) - \text{Cot}(gs\sqrt{f^-})}{\sqrt{f^+} - \sqrt{f^-}} \right)
\] (79)

where we notice that the expected loss of covariance has taken place with an appearance of the electric field. Unfortunately, due to the appearance of singular structures, it is not possible to deduce the result in the case of a purely electric or purely magnetic field from that given in equation (78) for the general case. We can return to the expression in equation (49) along with the result from (58) to find that for either a pure magnetic or pure electric field, we can write
\[
(gF\cot(gsF))_{00} = g\sqrt{f^+\cot(gs\sqrt{f^+})}
\] (80)

and of course, in the limit of zero field, this reduces to \((1/s)\) as it should. Let us conclude this section by examining the singular structure of the integrand in \(\Gamma^\beta[A]\) for the case of a purely electric background field. For the case of a purely electric background field, the integrand in the effective action, \(\Gamma^\beta[A]\) has a term \(qE\cot(gsE)\) as in the case of \([1]\). The singularities therefore lie at properties \(s_n = n\pi/gE\). Normally (as at zero temperature), this would lead to an imaginary contribution to \(\Gamma[A]\). But for non-zero temperatures, we notice that we have an exponent with a non-zero contribution. Therefore, we have the interesting result that for a purely electric background field, the imaginary part of the effective action gets an additional exponential factor. There are some fermionic systems in 2+1 dimensions which seem to show signals of a phase transition from zero to non-zero temperatures\([13]\). A more detailed analysis of the exponent in equation (78) with its temperature dependence is needed for a better understanding of the behaviour of the doublet in the presence of a general covariantly constant background field. Further work on this aspect of our paper is under progress. For the purposes of this paper, this completes the study of finite temperature corrections. In the next section we shall present our conclusions along with prospects for future work and associated applications.

9 Conclusions.

In this paper we have shown how the Euclidean configuration space heat kernel can be used to find the effective Lagrangian for SU(2) doublet fermions interacting with a covariantly constant SU(2) Yang-Mills field. The result is exact and mimics that of Schwinger. Since both a uniform field and a plane-wave
field satisfy the covariant constancy condition, our result holds for both these background configurations. We have also been able to write down the exact propagator in such backgrounds by virtue of having a closed form solution of the heat equation for finite temperature through the imaginary time formalism and the method of images, has yielded a result which suggests a phase transition. For a purely electric background field, we have a situation where the probability for pair creation from the vacuum while being non-zero at zero temperature, vanishes for non-zero temperatures. This warrants further investigation and a more detailed study is in progress. The most important future work we envisage is to utilize the SU(2) propagators to study SU(2) QCD plasma formations. Since much experimental work is being focussed on high energy collisions and the quark-gluon plasma. Such studies would be most relevant. Moreover, from a gauge theoretic point of view, the calculations presented in this paper point towards a more complete analysis of effective actions for unified gauge theories such as $SU(2)_L \times U(1)_Y$. 
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A Appendix

In this appendix, we give a set of consistent notations and conventions which have been used in the text. The Dirac matrix algebra satisfies:

\[ \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu}; \quad \mu\nu = 0, 1, 2, 3. \]  

(A.1)

braces signify an anticommutator: \( \{ a, b \} = ab + ba \). While,

\[ \gamma_\mu^2 = 1; \quad \gamma_\mu^\dagger = \gamma_\mu; \quad \mu = 0, 1, 2, 3. \]  

(A.2)

and

\[ \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}; \quad i = 1, 2, 3. \]  

(A.3)

where

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

(A.4)

are the Pauli matrices. We also define

\[ \gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}; \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

(A.5)

and

\[ \gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3; \quad \gamma_5^\dagger = -\gamma_5; \quad \gamma_5^2 = -1. \]  

(A.6)

This implies that \( \gamma_5 \) has \((\pm i)\) as eigenvalues with multiplicity two. Defining

\[ \sigma_{\mu\nu} = \frac{i}{2} \{ \gamma_\mu, \gamma_\nu \}; \]  

(A.7)

it is easy to establish that

\[ \frac{1}{2} \{ \sigma_{\mu\nu}, \sigma_{\alpha\beta} \} = \delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha} + i \epsilon_{\mu\nu\alpha\beta} \gamma_5. \]  

(A.8)

Lastly, the SU(2) algebra is chosen to satisfy the following Lie bracket:

\[ [T^a, T^b] = i \epsilon^{abc} T^c; \quad a, b, c = 1, 2, 3. \]  

(A.9)

While the generators of the fundamental representation are chosen to be:

\[ T^a = \frac{1}{2} \sigma^a; \quad a = 1, 2, 3. \]  

(A.10)