String background fields and the Riemann–Cartan geometry

Milovan Vasilić

Institute of Physics, PO Box 57, 11001 Belgrade, Serbia

E-mail: mvasilic@ipb.ac.rs

Received 21 September 2010, in final form 14 February 2011
Published 7 March 2011
Online at stacks.iop.org/CQG/28/075008

Abstract

We study classical dynamics of cylindrical membranes wrapped around the extra compact dimension of a \((D+1)\)-dimensional Riemann–Cartan spacetime. The world-sheet equations and boundary conditions are obtained from the universally valid conservation equations of the stress–energy and spin tensors. Specifically, we consider membranes made of macroscopic matter with maximally symmetric distribution of spin. In the narrow membrane limit, the dimensionally reduced theory is obtained. It describes how effective strings couple to the effective \(D\)-dimensional geometry. The striking coincidence with the string theory \(\sigma\)-model is observed. In this correspondence, the string background fields \(G_{\mu\nu}, B_{\mu\nu}, A_\mu\) and \(\Phi\) are related to the metric and torsion of the Riemann–Cartan spacetime.

PACS numbers: 04.40.-b, 11.27.+d

1. Introduction

The problem of motion of extended objects in curved backgrounds is usually addressed by using the multipole formalism originally proposed in [1, 2]. In [3–5], a manifestly covariant version of this method has been developed, and applied to strings and higher branes in the Riemann–Cartan geometry. One starts with the covariant conservation law of the stress–energy and spin tensors of matter fields, and analyzes it under the assumption that matter is localized to resemble a brane. In the lowest, single-pole approximation, the moving matter is viewed as an infinitely thin brane. In the pole–dipole approximation, its non-zero thickness is taken into account.

The known results concerning higher branes in Riemann–Cartan geometry can be summarized as follows. In the spinless case, the background curvature is coupled to the internal orbital angular momentum of a thick brane. The coupling is shown to disappear in the limit of an infinitely thin brane [3, 4]. The dynamics of spinning branes has been investigated...
in [5]. The spin–torsion coupling has been derived for an arbitrary distribution of spinning matter over the brane. In [6], the general results of [5] have been applied to infinitely thin membranes. Specifically, the membranes with maximally symmetric distribution of stress–energy and spin have been considered. In the spinless case, such membranes are characterized by the tension alone, and are known as the Nambu–Goto membranes [7, 8]. These membranes do not couple to torsion, but their minimal extension characterized by a uniformly distributed spin, does. It has been shown in [6] that these membranes couple to torsion in the same way as string theory membranes couple to the string theory 3-form field [9]. Strings have been introduced by considering cylindrical membranes wrapped around the extra compact dimension of a (D + 1)-dimensional spacetime. After a (D + 1 → D)-dimensional reduction, the effective string dynamics has been obtained. The corresponding action functional has been verified to share many features with the string theory $\sigma$-model action of [10–15]. In particular, the effective string couples to the spacetime metric and torsion the same way as fundamental strings couple to the string background fields $G_{\mu\nu}$ and $B_{\mu\nu}$.

In this paper, we continue investigating the behavior of membrane-like extended objects in Riemann–Cartan spacetime. Our effort is motivated by the partial success of [6] to relate spacetime metric and torsion to the string background fields. In [6], this has been achieved by viewing strings as dimensionally reduced cylindrical membranes made of uniformly distributed spinning matter. Precisely, the distribution of the membrane stress–energy and spin has been attributed the maximal symmetry. As a consequence, the dimensional reduction turns them into strings characterized by two constants only: the tension and spin magnitude. Such strings are minimally extended Nambu–Goto strings, which turn out to couple to the spacetime metric and torsion in the same way as fundamental strings couple to the string background fields $G_{\mu\nu}$ and $B_{\mu\nu}$.

In what follows, we shall continue the analysis of cylindrical membranes by relaxing the condition of the maximal symmetry used in [6]. The reason for this is twofold. First, what we are really interested in is the effective string obtained in the narrow membrane limit, not the membrane itself. This means that the membrane constituent matter does not have to be uniformly distributed along the membrane compact dimension. Indeed, the violation of the maximal symmetry in this direction disappears after the dimensional reduction. The effective string can still be maximally symmetric, possibly determined with more than two free parameters. The second reason is that the effective strings characterized by additional parameters may have new couplings with the dimensionally reduced background. In fact, it is well known that, beside the $D$-dimensional metric, the $D + 1 \rightarrow D$ dimensional reduction leaves us with additional vector and scalar fields. It is our hope that these fields can be related to the string background fields $A_\mu$ and $\Phi$ very much the same as metric and torsion of [6] have been related to $G_{\mu\nu}$ and $B_{\mu\nu}$.

We emphasize that our work is not a part of the mainstream string theory considerations. Our membrane-like objects are made of conventional matter, and are used to probe the Riemann–Cartan geometry. The only connection with string theory is seen in the form of our resulting equations. Namely, we demonstrate in this paper that our effective string indeed couples to the spacetime metric and torsion in the same way as fundamental string couples to the low-energy string fields [10–15]. In this correspondence, the string background fields $G_{\mu\nu}$, $A_\mu$ and $\Phi$ are related to the spacetime metric, while the string axion $B_{\mu\nu}$ is related to torsion. Whether this is just a coincidence, or there is more content in this analogy is not the subject of this paper.

The layout of the paper is as follows. In section 2, we review the basic notions of the multipole formalism developed in [3–5]. The manifestly covariant world-sheet equations and boundary conditions are explicitly displayed. The brane stress–energy and spin tensors appear
in the world-sheet equations as free coefficients. In section 3, the general equations are applied to the membrane case. The membrane spin tensor is chosen maximally symmetric, whereas its stress–energy is left arbitrary. In section 4, a narrow membrane wrapped around the extra compact dimension of a \((D + 1)\)-dimensional spacetime is considered. The membrane stress–energy is chosen maximally symmetric only in large dimensions, while left arbitrary in the compact dimension. In section 5, the obtained string dynamics is compared with the string \(\sigma\)-model of [10–15]. Section 6 is devoted to concluding remarks.

Our conventions are the same as in [4]. Greek indices \(\mu, \nu, \ldots\) are the spacetime indices, and run over \(0, 1, \ldots, D - 1\). Latin indices \(a, b, \ldots\) are the world-sheet indices and run over \(0, 1, \ldots, p\). Latin indices \(i, j, \ldots\) refer to the world-sheet boundary and take values \(0, 1, \ldots, p - 1\). The coordinates of spacetime, world-sheet and world-sheet boundary are denoted by \(x^\mu\), \(\xi^a\) and \(\lambda^i\) respectively. The spacetime metric is denoted by \(g_{\mu\nu}(x)\), and the induced world-sheet metric by \(\gamma_{ab}(\xi)\). The signature convention is defined by \(\text{diag}(-, +, \ldots, +)\), and the indices are raised using the inverse metrics \(g^{\mu\nu}\) and \(\gamma^{ab}\).

2. Multipole formalism

It has been shown in [3, 4] that an exponentially decreasing function can be expanded as a series of \(\delta\)-function derivatives. For example, a tensor-valued function \(F_{\mu\nu}(x)\), well localized around the \(p + 1\)-dimensional surface \(M\) in \(D\)-dimensional spacetime, can be decomposed in a manifestly covariant way as

\[
F_{\mu\nu}(x) = \int_M d^{p+1}\xi \sqrt{-\gamma} \left[ M_{\mu\nu}^{\delta(D)}(x - z) - \nabla_{\rho} \left( M_{\mu\nu\rho}^{\delta(D)}(x - z) \sqrt{-g} \right) + \cdots \right].
\]

(1)

The surface \(M\) is defined by the equation \(x^\mu = z^\mu(\xi)\), where \(\xi^a\) are the surface coordinates, and the coefficients \(M_{\mu\nu}(\xi), M_{\mu\nu\rho}(\xi), \ldots\) are spacetime tensors called multipole coefficients. Here, and throughout the paper, we make use of the surface coordinate vectors

\[
u^\mu_a \equiv \frac{\partial z^\mu}{\partial \xi^a},
\]

and the surface-induced metric tensor

\[
\gamma_{ab} = g_{\mu\nu}u^\mu_a u^\nu_b.
\]

The induced metric is assumed to be nondegenerate, \(\gamma \equiv \det(\gamma_{ab}) \neq 0\), and of the Minkowski signature. The same holds for the target space metric \(g_{\mu\nu}(x)\) and its determinant \(g(x)\). The covariant derivative \(\nabla_{\rho}\) is defined by the Levi-Civita connection.

Decomposition (1) is particularly useful for the description of brane-like objects in spacetimes of the general geometry. In [5], it has been used to model the fundamental matter currents: stress–energy \(\tau_{\mu\nu}\), and spin tensor \(\sigma_{\lambda\mu\nu}\). The brane dynamics in Riemann–Cartan backgrounds is obtained from the covariant conservation equations [16]

\[
\begin{align*}
(D_\nu + T_\lambda^{\nu, \lambda}) \tau^\nu_{\mu} &= \tau^\nu_{\rho} T_\rho^{\nu, \mu} + \frac{1}{2} \sigma^{\nu\rho\sigma} R_{\rho\sigma\mu\nu}, \\
(D_\nu + T_\lambda^{\nu, \lambda}) \sigma^\nu_{\rho\sigma} &= \tau_{\rho\sigma} - \tau_{\sigma\rho}.
\end{align*}
\]

(2)

Here, \(D_\nu\) is the covariant derivative with the nonsymmetric connection \(\Gamma^\lambda_{\mu\nu}\), which acts on a vector \(v^\nu\) according to the rule \(D_\nu v^\mu \equiv \partial_\nu v^\mu + \Gamma^\lambda_{\nu\mu} v^\lambda\). The torsion \(T_\lambda^{\mu\nu}\) and curvature \(R^{\mu}_{\nu\rho\sigma}\) are defined in the standard way:

\[
\begin{align*}
T_\lambda^{\mu\nu} &\equiv \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}, \\
R^{\mu}_{\nu\rho\sigma} &\equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho}.
\end{align*}
\]

3
The covariant derivative $D_\nu$ is assumed to satisfy the metricity condition $D_\nu g_{\mu \nu} = 0$. As a consequence, the connection $\Gamma^\lambda_{\mu \nu}$ is split into the Levi-Civita connection and the contortion:

$$\Gamma^\lambda_{\mu \nu} = \left\{ \frac{\lambda}{\mu \nu} \right\} + K^\lambda_{\mu \nu},$$

$$K^\lambda_{\mu \nu} = -\frac{1}{2} \left( T^\lambda_{\mu \nu} - T^\lambda_{\nu \mu} + T_{\mu \nu} \right).$$

The curvature tensor can then be rewritten in terms of the Riemann curvature $R^\mu_{\nu \rho \sigma} \equiv R^\mu_{\nu \rho \sigma} (\Gamma \rightarrow \{\})$, and the Riemannian covariant derivative $\nabla_\mu \equiv D_\mu (\Gamma \rightarrow \{\})$:

$$R^\mu_{\nu \lambda \rho} = R^\mu_{\nu \lambda \rho} + 2 \nabla_\left[ \lambda \right] K_{\mu \nu \rho} + 2 K^\mu_{\left[ \sigma \right] \lambda \rho} K_{\sigma \mu \nu \rho}.\tag{3}$$

Given the system of conservation equations (4), one finds that the second one has no dynamical content. One can use (2b) to eliminate $\tau_{\mu \nu}$ from (2a), and thus obtain

$$\nabla_\nu (\theta_{\mu \nu} - D_{\mu \nu}) = \frac{1}{2} \frac{T_{\rho \sigma}}{T^{\rho \sigma}} \nabla_\mu K^{\rho \sigma},$$

where $\theta_{\mu \nu}$ stands for the generalized Belinfante tensor

$$\theta_{\mu \nu} \equiv \tau_{\mu \nu} - \nabla_\rho \epsilon_{\mu \nu \rho} - \frac{1}{2} K_{\rho [\mu \sigma] \rho \lambda},$$

and $D_{\mu \nu} \equiv K_{\mu \left[ \rho \nabla \right] \rho \lambda} + \frac{1}{2} K_{\lambda [\mu \sigma] \rho \lambda}$. The conservation law in the form (3) is the starting point for the derivation of brane world-sheet equations. In the particle case, this form of the conservation equations has been used in [17].

In this paper, we are interested in infinitely thin branes, and therefore, restrict our analysis to the single-pole approximation. The multipole expansion of our basic variables then reads

$$\theta_{\mu \nu} = \int_M d^{p+1} \xi \sqrt{-g} \left[ T_{\mu \nu}(\xi - z) \right],\tag{4a}$$

$$\sigma_{\lambda \mu \nu} = \int_M d^{p+1} \xi \sqrt{-g} \left[ S^{\lambda \mu \nu} \delta(D)(\xi - z) \right],\tag{4b}$$

where $T_{\mu \nu}(\xi)$ and $S^{\lambda \mu \nu}(\xi)$ are the corresponding multipole coefficients. Decomposition (4) is used as an ansatz for solving the conservation equations (3). This has already been done in [5], resulting in manifestly covariant $p$-brane world-sheet equations.

### 3. Membrane dynamics

In [6], the general result of [5] has been applied to the membrane case. In particular, the membrane with maximally symmetric distribution of spin has been thoroughly examined. In what follows, the world-sheet equations and boundary conditions of [5] will be used as the starting point of our analysis.

Let us start with the brief recapitulation of the known results. The $p$-brane world-sheet equations in the single-pole approximation are obtained in the following way. We insert ansatz (4) into the conservation equations (3), and solve for the unknown variables $z^\mu(\xi)$, $T_{\mu \nu}(\xi)$ and $S^{\lambda \mu \nu}(\xi)$. The algorithm for solving this type of equation has been discussed in detail in [4, 5], and here we use the ready-made result. According to [5], the single-pole world-sheet equations are given by

$$P_{\rho \mu} P_{\nu \nu} D^{\rho \sigma} = 0,\tag{5a}$$

$$\nabla_\nu \left( u_{\rho \mu} u_{\nu} - 2 u_{\rho \mu} D^{\rho \sigma} + u_{\mu} u_{\rho} u_{\nu} D^{\rho \sigma} \right) = \frac{1}{2} S_{\mu \sigma} \nabla_\mu K^{\rho \sigma \nu},\tag{5b}$$
while the boundary conditions have the form
\[ n_a \left( \tau^{ab} u^b \mu^n D^{\mu^n} u^a \right) \bigg|_{\partial M} = 0. \] (6)

Here, \( P^\nu_{\nu} \equiv \delta^\nu_{\nu} - u^\mu u_\mu \) is the orthogonal world-sheet projector, \( n^a \) is the unit boundary normal, and \( \nabla_a \) stands for the total covariant derivative that acts on both types of indices:
\[ \nabla_a V^{\mu b} \equiv \partial_a V^{\mu b} + \left\{ \frac{\mu}{\lambda} \rho \right\} u_\rho^{\mu} V^{\lambda b} + \left\{ \frac{b}{c a} \right\} V^{\mu c}. \]

The coefficients \( \tau^{ab}(\xi) \) and \( S^{\lambda \mu \nu}(\xi) \) are the residual free parameters of the theory. While \( \tau^{ab} \) represents the effective stress–energy tensor of the brane, the \( S^{\lambda \mu \nu} \) currents are related to its spin density. The shorthand notation
\[ D^{\mu \nu} \equiv K_{[\mu} S^{\nu \lambda \rho]} + \frac{1}{2} K_{\lambda \rho} [\mu S^{\nu \lambda}] \]
is introduced for convenience.

The world-sheet equations (5) and boundary conditions (6) describe the dynamics of an infinitely thin \( p \)-brane in \( D \)-dimensional spacetime with curvature and torsion. By inspecting their form, we realize that only branes made of spinning matter can probe spacetime torsion. Moreover, if the spin tensor \( S^{\lambda \mu \nu} \) is totally antisymmetric, only axial component of torsion survives in the brane equations. In this paper, we are interested in membranes characterized by maximally symmetric distribution of spin. It has already been shown in [6] that such membranes must have the axial spin tensor of the form
\[ S^{\lambda \mu \nu} = s e^{a b c} u^a_{\lambda} u^b_{\mu} u^c_{\nu}, \] (7)
where \( e^{a b c} \) is the covariant Levi-Civita symbol, and \( s \) is a constant. This leads us to restrict our considerations to backgrounds with totally antisymmetric torsion. Indeed, the axial spin tensor exclusively couples to the axial component of torsion. Thus, without loss of generality, we shall assume that \( K^{\mu \nu \rho} \) is totally antisymmetric.

Let us now see how the membrane spin tensor of the form (7) influences equations (5) and (6). The computations are straightforward, and have already been done in [6]. They lead to the world-sheet equations
\[ \nabla_a \left( m^{ab} u^b_{\mu} \right) = s \frac{u^\lambda_{\mu}}{2} K_{\mu \nu \rho}, \] (8a)
and boundary conditions
\[ n_a \left( m^{ab} u^b_{\mu} + \frac{3s}{2} e^{a b c} K_{\mu \nu \rho} \right) \bigg|_{\partial M} = 0. \] (8b)

Here, the antisymmetric tensor \( K_{\mu \nu \rho} \) is defined as
\[ K_{\mu \nu \rho} \equiv \partial_\mu K_{\nu \rho} - \partial_\nu K_{\rho \mu} + \partial_\rho K_{\mu \nu} - \partial_\lambda K_{\lambda \mu \nu}, \]
while \( u^{\mu \nu} \equiv e^{a b c} u^a_{\mu} u^b_{\nu} u^c_{\rho} \), and \( K_{a b \rho} \equiv u^a_{\mu} u^b_{\nu} K_{\mu \nu \rho} \) are introduced for convenience. The residual free coefficients \( m^{ab} \) are related to the original \( \tau^{ab} \) as \( m^{ab} = \tau^{ab} - \frac{s}{2} y^{ab} u^{\mu \nu} K_{\mu \nu \rho} \).

The world-sheet equations (8a), and boundary conditions (8b) describe a membrane with maximally symmetric distribution of spin, and arbitrary distribution of stress–energy. In [6], the maximal symmetry has been attributed to the membrane stress–energy, too. This case is defined by the choice \( m^{ab} = T_{\gamma \mu \nu} \), where \( T \) is a constant commonly interpreted as the membrane tension. With this, the world-sheet equations and boundary conditions have been shown to follow from an action functional. The resemblance of this action to the \( \sigma \)-model action of [9] has been pointed out in [6]. The two actions differ in one instance only: the role of the 3-form field \( B_{\mu \nu \rho} \) in [9] is played by the contortion field \( \frac{1}{T} K_{\mu \nu \rho} \) in [6].
In the next section, we shall demonstrate how effective string dynamics is obtained in the narrow membrane limit. The starting point of our analysis will be equations (8). They differ from those used in [6] by the form of $m^{ab}$ coefficients, which are allowed to violate the condition $m^{ab} = T_{Y^{ab}}$. As a consequence, we shall be able to derive additional string couplings to the background geometry.

4. Dimensional reduction

The results of the preceding section are obtained under very general assumptions concerning the dimensionality and topology of spacetime and world-sheet. In what follows, we shall use this freedom to apply these results to a cylindrical membrane wrapped around the extra compact dimension of a $(D + 1)$-dimensional spacetime. In the limit of a narrow membrane, we expect to obtain the effective string dynamics. This kind of double dimensional reduction has already been considered in [9]. There, the string effective action in ten dimensions has been obtained from the membrane action in 11 dimensions. In what follows, a similar $(D+1 \rightarrow D)$-dimensional reduction will be applied to classical membranes in Riemann–Cartan backgrounds.

Let us consider a $(D + 1)$-dimensional spacetime with one small compact dimension. It is parametrized by the coordinates $X^M(M = 0, 1, \ldots, D)$, which we divide into the ‘observable’ coordinates $x^\mu (\mu = 0, 1, \ldots, D - 1)$, and the extra periodic coordinate $y$. In the limit of the small extra dimension, we use the Kaluca–Klein ansatz

$$
\partial_y K_{MNL} = 0, \quad \partial_y G_{MN} = 0
$$

(9)

to model the contortion and metric. This ansatz violates the $(D + 1)$-dimensional diffeomorphisms, leaving us with the coordinate transformations

$$
x'^\mu = x^\mu (x), \quad y' = y + \varepsilon (x).
$$

(10)

In what follows, we shall use the decomposition

$$
G_{MN} = \left( g_{\mu\nu} + \phi a_\mu a_\nu, \phi a_\mu, \phi a_\nu, \phi \right),
$$

(11)

as it yields the variables that transform as tensors with respect to the residual D-diffeomorphisms. The same kind of argument applies to $K_{MNL}$. We shall use a totally antisymmetric contortion, and decompose it as

$$
K_{MNL} = (K_{\mu\nu\lambda}, K_{\mu\nu y}),
$$

with

$$
K_{\mu\nu} = k_{\mu\nu}, \quad K_{\mu\nu\lambda} = K_{\mu\nu\lambda} + k_{\mu\lambda}a_\nu + k_{\nu\lambda}a_\mu + k_{\nu\lambda}a_\mu.
$$

(12)

With respect to the complete residual transformations (10), the variables $K_{\mu\nu\lambda}$ and $k_{\mu\nu}$, as well as $g_{\mu\nu}$ and $\phi$, transform as tensors, while $a_\mu$ transforms as a connection: $a'_\mu = (a_\mu - \partial_\mu \varepsilon) \partial x^\mu / \partial x'^\mu$.

Now, we consider a membrane wrapped around the extra compact dimension $y$. Its world-sheet $X^M = Z^M(\xi^A)$ is denoted by $\mathcal{M}_3$, and is chosen in the form

$$
x^\mu = z^\mu (\xi^a), \quad y = \xi^2,
$$

(13)

where the world-sheet coordinates $\xi^A$ ($A = 0, 1, 2$) are divided into $\xi^a$ ($a = 0, 1$) and $\xi^2$. This ansatz reduces the reparametrizations $\xi^N = \xi^A (\xi^b)$ to

$$
\xi'^a = \xi'^a (\xi^b), \quad \xi'^2 = \xi^2 + \varepsilon (z^a (\xi)),
$$

(14)
and the world-sheet tangent vectors $U^M_A = \partial Z^M / \partial \xi^A$ to
\[ U^\mu_A = u^\mu_a, \quad U^\nu_A = U^\nu_2 = 0, \quad U^2_A = 1. \]
One can verify that $\nu^a = \partial \xi^\nu / \partial \xi^a$ transforms as a tensor with respect to the residual spacetime and world-sheet diffeomorphisms. The induced metric $\Gamma_{AB} = G_{MN} U^M_A U^N_B$ is shown to satisfy the condition $\partial^2 \Gamma_{AB} = 0$. It is decomposed as
\[ \Gamma_{AB} = \left( \gamma_{ab} + \phi_a \phi b \phi \right), \]
with $\gamma_{ab} = g_{\mu\nu} u^\mu_a u^\nu_b$, and $\phi_a = a^\mu u^\mu_a$. In what follows, we shall refer to $x^\mu = z^\mu(\xi^a)$ as the string world-sheet, and denote it by $M_2$.

The membrane boundary $\partial M_3$ is given by $\xi^A = \zeta^A(\lambda_i)$, where $\lambda_i (i = 0, 1)$ are the boundary coordinates. In accordance with ansatz (13), it is chosen in the form
\[ \xi^a = \zeta^a(\lambda^0), \quad \xi^2 = \lambda^1. \]
The boundary tangent vectors $V^A_i \equiv \partial \zeta^A / \partial \lambda^i$ are then reduced to
\[ V^a_0 = v^a, \quad V^a_1 = V^2_0 = 0, \quad V^2_1 = 1, \]
and the boundary metric $\delta_{ij} = \Gamma_{AB} V^A_i V^B_j$ becomes
\[ \delta_{ij} = \left( v^2 + \phi_a \phi b \phi \right). \]
Here, $v^a = \partial v^a / \partial \lambda^0$, $v^2 = \gamma_{ab} v^a v^b$ and $\alpha = v^a a^a$. The boundary normal $N_A \equiv \varepsilon_{ABC} V^B_0 V^C / \sqrt{-\delta}$ then reduces to $N_A = (n_a, 0)$, with $n_a = \varepsilon_{ab} v^b / \sqrt{-v^2}$. In what follows, we shall refer to $\xi^a = \zeta^a(\lambda^0)$ as the string boundary $\partial M_2$.

In this section, the dimensional reduction is applied to the membrane equations (8) of section 3. In $(D+1)$-dimensional background, the word-sheet equations (8a) are rewritten as
\[ \nabla_A (m_{AB} U^B_M) = \frac{S}{2} U^{NLR} K_{MNL}, \]
while boundary conditions (8b) become
\[ N_A \left( m_{AB} U^B_M + \frac{3S}{2} \varepsilon_{ABC} K_{BCM} \right) \big|_{\partial M_3} = 0. \]
The dimensional reduction of (16) with $m_{AB} = T \Gamma_{AB}$ has already been studied in [6]. Here, we shall consider a more general situation
\[ m_{AB} = T \Gamma_{AB} + \mu_{AB}, \]
with $\mu_{AB} \neq 0$, and $\partial_\xi \mu_{AB} = 0$, in accordance with the adopted Kaluza–Klein ansatz. Note, however, that leaving the stress–energy $\mu_{AB}$ completely arbitrary produces an inhomogeneous effective string after the dimensional reduction. This is not what we want. As we explained in the introduction, our idea is to violate the maximal symmetry of the membrane stress–energy $m_{AB} = T \Gamma_{AB}$ in a way which will preserve the maximal symmetry of the effective string after dimensional reduction. To this end, we are led to retain $m^{ab} = T \gamma^{ab}$ while leaving the other $m^{AB}$ components unconstrained. In terms of $\mu_{AB}$, this means that we must adopt $\mu^{ab} = 0$. In what follows, we shall use the decomposition
\[ \mu_{AB} = \begin{pmatrix} 0 & f^a \\ f^b & \omega - 2 f^c a_c \end{pmatrix}, \]
as it yields parameters that transform as tensors with respect to the residual reparametrizations. Indeed, the remaining free coefficients $f^a(\xi)$ and $\omega(\xi)$ are shown to transform covariantly
under (14). We shall see later that \( j^a \) and \( \omega \) are related to the electric and dilatonic charges of the effective string. While the condition \( \mu^{ab} = 0 \) ensures the maximal symmetry of the string stress–energy, it does not ensure uniform distribution of its electric and dilatonic charges. Nevertheless, in what follows, we shall keep the currents \( j^a \) and \( \omega \) arbitrary.

Let us now solve the world-sheet equations (16a). First, we decompose (16a) into components parallel and orthogonal to the world-sheet. The parallel component yields the conservation equation
\[
\nabla_\alpha A^{\alpha\beta} = 0.
\]
Using the Kaluza–Klein ansatz (9), (13), (15), and the decompositions (11), (12) and (17), it is reduced to
\[
2\phi f_{ab} j^b - \omega \partial_a \phi = 0, \quad 2\phi \nabla_\alpha j^\alpha + 3 j^a \partial_a \phi = 0,
\]
with \( f_{ab} \equiv \partial_b a_a - \partial_a a_b \). The \( M^y \) component of the remaining orthogonal equations is then shown to be identically satisfied, while \( M^\mu \) components become
\[
2T \left( \nabla_\mu u_\mu - \frac{1}{2\phi} P^{\nu \mu \lambda} \partial_\nu \phi \right) - \frac{3s}{\sqrt{\phi}} u^\nu k_{\mu \nu \lambda} + 2\phi f_{\mu \nu} j^\nu - \omega \partial_\mu \phi = 0. \tag{18c}
\]
The totally antisymmetric tensor \( k_{\mu \nu \lambda} \) is defined as
\[
k_{\mu \nu \lambda} \equiv \partial_\mu k_{\nu \lambda} + \partial_\nu k_{\lambda \mu} + \partial_\lambda k_{\mu \nu},
\]
while \( f_{\mu \nu} \equiv \partial_\nu a_\mu - \partial_\mu a_\nu \), and \( j^\mu \equiv u^\mu j^a \).

The world-sheet equations (18) show how effective string couples to the background metric, torsion, electromagnetic and scalar fields. These equations can be simplified by removing (18a), as it coincides with the parallel component of (18c). Also, we are free to make redefinitions of the background fields and free parameters. As it turns out, the world-sheet equations (18) are nicely simplified when expressed in terms of the rescaled metric
\[
\tilde{g}_{\mu \nu} \equiv \sqrt{\phi} g_{\mu \nu},
\]
and the rescaled current
\[
\tilde{j}^a \equiv \phi j^a.
\]
Indeed, in terms of \( \tilde{g}_{\mu \nu} \) and \( \tilde{j}^a \), the string dynamics (18) is rewritten as
\[
\tilde{\nabla}_\mu j_\mu = 0, \quad T \tilde{\nabla}_\mu u_\mu = \frac{3s}{\sqrt{\phi}} \tilde{u}^\nu k_{\mu \nu \lambda} - f_{\mu \nu} j^\nu + \frac{\omega}{2} \partial_\mu \phi. \tag{19b}
\]
The world-sheet equations (19) govern the dynamics of the effective string which carries more charges than the mere tension and spin. As a consequence, the string is coupled not only to the metric and torsion, as described in [6], but also to the electromagnetic and scalar fields.

The dimensional reduction of boundary conditions (16b) is performed in a similar way. In terms of \( \tilde{g}_{\mu \nu} \) and \( \tilde{j}^a \), the resulting equations take the form
\[
\tilde{n}_a j_\mu |_{\beta M_2} = 0, \quad \tilde{n}_\mu \left( T u_\mu + 3s \tilde{e}^{ab} u_\gamma j^b \right) |_{\beta M_2} = 0. \tag{20b}
\]
As we can see, the electromagnetic field \( a_\mu \) does not appear in the boundary conditions. This may lead us to the conclusion that the behavior of our effective strings in Riemann–Cartan spacetime is quite different from that of the fundamental strings in the low-energy string backgrounds. In the next section, we shall compare our effective string equations with the string \( \sigma \)-model of [10–15], and demonstrate that it is not quite so.
As a preparation for this comparison, let us transform our equations (19) and (20) to a more convenient form. We start with the observation that (19b) implies the constraint \( \omega j^a \partial_a \phi = 0 \), which reduces to
\[
j^a \partial_a \phi = 0
\]
in the generic case \( \omega \neq 0 \). The general solution of this constraint has the form
\[
j^a = e e^{ab} \partial_b \phi
\]
where \( e(\xi) \) is a residual coefficient that defines the distribution of the electric charge along the string. In what follows, we shall adopt
\[
e = \text{const}
\]
in accordance with our systematic consideration of strings with maximally symmetric distribution of matter. With this, the conservation equation (19a) is identically satisfied, while the world-sheet equations (19b) become
\[
T \tilde{\nabla}_a u^a_\mu = \frac{3s}{2} \tilde{u}^{\nu \lambda} \left( k_{\mu \nu \lambda} - \frac{2}{3s} f_{\mu \nu \lambda} \right) + \frac{\omega}{2} \partial_\mu \phi.
\]
Now, we are free to make additional redefinitions of the background fields and free parameters. As it turns out, the world-sheet equations are nicely simplified when expressed in terms of the redefined torsion
\[
\tilde{k}_{\mu \nu} \equiv k_{\mu \nu} - \frac{e}{3s} \phi f_{\mu \nu},
\]
and redefined dilatonic charge
\[
\tilde{\omega} \equiv \omega + e \nu^{\mu \nu} f_{\mu \nu}.
\]
Indeed, in terms of \( \tilde{k}_{\mu \nu} \) and \( \tilde{\omega} \), our world-sheet equations are rewritten as
\[
T \tilde{\nabla}_a u^a_\mu = \frac{3s}{2} \tilde{u}^{\nu \lambda} \tilde{k}_{\mu \nu \lambda} + \frac{\tilde{\omega}}{2} \partial_\mu \phi.
\]
As we can see, the resultant world-sheet equations contain no coupling to the electromagnetic field.

Let us now see how the above results influence the boundary conditions (20). First, we note that (21) implies the constraint \( \tilde{\omega} u^a_\mu \partial_\mu \phi = 0 \), which reduces to \( \tilde{\omega} \partial_\mu \phi = 0 \) in the generic case \( \tilde{\omega} \neq 0 \). This means that our effective string is forced to live on the surface \( \phi = \text{const.} \)
As a consequence, the boundary condition (20a), which reduces to \( v^a \partial_a \phi = 0 \), is identically satisfied. The remaining boundary conditions (20b) are then rewritten in terms of \( \tilde{k}_{\mu \nu} \) and \( \tilde{\omega} \) as
\[
\tilde{n}_a \left[ T u^a_\mu + \tilde{e}^{ab} u^b_\mu \left( 3s \tilde{k}_{\mu \nu \lambda} + \tilde{\omega} f_{\mu \nu \lambda} \right) \right] \nmid \partial M_2 = 0.
\]
As we can see, the coupling to the electromagnetic field reappears in the boundary conditions.

To summarize, we have shown that the effective string dynamics is governed by the world-sheet equations (21), and boundary conditions (22). The equations are parametrized by four parameters, \( T, s, \tilde{e}, \) and \( \tilde{\omega} \), which define the string tension, spin, electric charge and dilatonic charge, respectively. The constant parameters \( T, s \) and \( \tilde{e} \) define uniform distribution of stress–energy, spin and electric charge, while dilatonic charge \( \tilde{\omega} \) is left arbitrary. In the next section, this form of string dynamics will be shown to follow from an action functional that coincides with the string theory \( \sigma \)-model discussed in [10–15].
5. Comparison with the string sigma model

In this section, we shall compare our results with the predictions of the string \(\sigma\)-model \([10–15]\).

This model is defined by the action functional

\[
I = T \int d^2 \xi \sqrt{-h} \left[ G_{\mu \nu}(x) u_\mu^a u_\nu^b h^{ab} + B_{\mu \nu}(x) u_\mu^a u_\nu^b e^{ab} + \Phi(x) R^{(2)} \right],
\]

(23)

in which \(x^\mu(\xi)\) and \(h_{ab}(\xi)\) are considered to be independent variables. The string background fields \(G_{\mu \nu}(x)\), \(B_{\mu \nu}(x)\) and \(\Phi(x)\) are commonly referred to as metric, string axion and dilaton, respectively. The 2D curvature \(R^{(2)}\) is constructed out of the auxiliary metric \(h_{ab}\).

By varying action (23) with respect to \(x^\mu\) one finds the world-sheet equations

\[
\nabla_a u_\mu^a = \frac{1}{2} u^{\nu_1} B_{\nu_1 \mu_2} + \frac{1}{2} R^{(2)} \partial_\mu \Phi,
\]

(24a)

and boundary conditions

\[
n_a (u_\mu^a + \epsilon^{ab} u_\nu^b B_{\mu \nu}) \mid_{\partial M^2} = 0, \tag{24b}
\]

with \(\nabla_a = \nabla_a (h)\), \(R^{(2)} = R^{(2)}(h)\), and \(B_{\mu \nu} \equiv \partial_\mu B_{\nu \lambda} + \partial_\nu B_{\lambda \mu} + \partial_\lambda B_{\mu \nu}\). The variation with respect to \(h_{ab}\), on the other hand, gives

\[
\gamma_{ab} - \frac{1}{2} \left( \gamma_{cd} h^{cd} \right) h_{ab} = \nabla_a \nabla_b \Phi - h_{ab} \nabla^2 \Phi,
\]

(24c)

with \(\nabla^2 \equiv h^{ab} \nabla_a \nabla_b\). These equations are nicely simplified by noting that (24a) implies the constraint \(R^{(2)} u_\mu^a \partial_\mu \Phi = 0\). In the generic situation, characterized by the non-vanishing scalar curvature \(R^{(2)}\), it takes the simple form

\[
\partial_\mu \Phi = 0,
\]

telling us that the string is forced to live on the surface \(\Phi = \) const. The equations (24c) are thereby reduced to

\[
\gamma_{ab} = e^\chi h_{ab},
\]

(25)

where \(\chi(\xi)\) is an arbitrary function on the world-sheet. Equation (25) can be used to replace the auxiliary variable \(h_{ab}\) with \(\gamma_{ab}\) in all but the last term of (24a). Indeed, in terms of \(\gamma_{ab}\), the world-sheet equations (24a) are rewritten as

\[
\nabla_a u_\mu^a = \frac{1}{2} u^{\nu_1} B_{\nu_1 \mu_2} + \frac{1}{2} (R^{(2)} - \nabla^2 \chi) \partial_\mu \Phi,
\]

while boundary conditions (24b) are shown to retain the same form. This time, however, \(\nabla_a = \nabla_a (\gamma)\), \(R^{(2)} = R^{(2)}(\gamma)\) etc. As we can see, the conformal factor \(\chi\) of the auxiliary metric \(h_{ab}\) remains in the field equations as a free coefficient. As \(\chi(\xi)\) is an unconstrained arbitrary function, we are free to make the redefinition

\[
\Omega \equiv R^{(2)} - \nabla^2 \chi,
\]

thereby bringing the field equations to the simple form

\[
\nabla_a u_\mu^a = \frac{1}{2} u^{\nu_1} B_{\nu_1 \mu_2} + \frac{\Omega}{2} \partial_\mu \Phi, \tag{26a}
\]

\[
n_a (u_\mu^a + \epsilon^{ab} u_\nu^b B_{\mu \nu}) \mid_{\partial M^2} = 0. \tag{26b}
\]

The new free coefficient \(\Omega(\xi)\) is a remnant of the auxiliary variable \(h_{ab}\).

The string dynamics in the form (26) lacks the interaction with the background electromagnetic field \(A_\mu\). This interaction is conventionally introduced by the replacement

\[
B_{\mu \nu} \to B_{\mu \nu} + F_{\mu \nu},
\]
where $F_{\mu \nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$. Being the gauge transform of the $B_{\mu \nu}$ field, this replacement does not influence the world-sheet equations, as they depend on $B_{\mu \nu}$ only through the gauge invariant field strength $B_{\mu \nu \lambda}$. On the other hand, the boundary conditions (26b) are not gauge invariant. The replacement $B \rightarrow B + F$ brings them to the form

$$n_\mu \left[ u_\mu^a + e^{a b} u_\mu^b \left( B_{\mu \nu} + F_{\mu \nu} \right) \right] \bigg|_{\partial M_2} = 0$$

showing that the string ends have nontrivial coupling to the electromagnetic field.

Now, we are ready to compare our equations (21) and (22) with the string $\sigma$-model equations (26) and (27). The first set of equations describes an effective string obtained by dimensional reduction of a narrow membrane in $(D + 1)$-dimensional Riemann–Cartan spacetime. The second set governs the dynamics of an elementary string coupled to the string background fields $G_{\mu \nu}$, $B_{\mu \nu}$, $A_\mu$ and $\Phi$. By comparing the two sets of equations, we find that they differ in one instance only: the role of the string background fields in (26) and (27) is played by the dimensionally reduced Riemann–Cartan geometry in (21) and (22). Precisely, the identification of external fields

$$\tilde{g}_{\mu \nu}(x) \rightarrow G_{\mu \nu}(x), \quad 3s \tilde{k}_{\mu \nu}(x) \rightarrow T B_{\mu \nu}(x),$$

and free coefficients

$$\tilde{\omega}(\xi) \rightarrow \Omega(\xi)$$

establishes 1–1 correspondence between the two theories. Whether this is just a coincidence, or there is more content in this matching is not the subject of this work. Anyhow, we find it interesting enough to justify our presentation.

In summary, we have derived how effective string, obtained by dimensional reduction of a narrow membrane, behaves in a dimensionally reduced Riemann–Cartan spacetime. We have considered an effective string made of uniformly distributed spinning matter, thus representing a simple generalization of the Nambu–Goto case. As a consequence, novel couplings to the background geometry have been discovered. In particular, the effective string has been shown to couple to the dimensionally reduced Riemann–Cartan geometry the same way as fundamental string couples to the low-energy string fields. We may say that the low-energy string background has geometric origin. Precisely, it is identified with the curved geometry with torsion. This is because the properties of a background are almost exclusively obtained by probing it with material objects. In general relativity, for example, the curvature is detected by probing it with massive particles. In string theory, on the other hand, the low-energy string background is probed by strings themselves. As these are believed to be elementary building blocks of all the existing matter, there is no way to obtain more detailed information on the background than by probing it with fundamental strings. At the same time, the string dynamics obtained in this paper retains its form irrespective of the initial position and shape of the probe string. By the repeated experiments, all the spacetime points are subject to probing, and therefore, the established correspondence holds true for the whole background.

6. Concluding remarks

In this paper, we have analyzed the behavior of a narrow membrane wrapped around the extra compact dimension of a $(D + 1)$-dimensional Riemann–Cartan spacetime. The membrane constituent matter is specified in terms of its stress–energy and spin tensors. A membrane with maximally symmetric distribution of stress–energy and spin has already been considered in [6]. After dimensional reduction, such a membrane has been shown to reduce to a string that
couples to the metric and torsion the same way as fundamental string couples to the low-energy string fields $G_{\mu\nu}$ and $B_{\mu\nu}$. In this paper, we have relaxed the condition of maximal symmetry used in [6] by allowing the membrane stress–energy to have non-uniform distribution in the compact dimension. This way, the effective string retains maximal symmetry, but is characterized by more free parameters than the mere tension and spin. We have demonstrated that such strings carry electric and dilatonic charges, and couple to the electromagnetic and scalar fields the same way as fundamental strings couple to the low-energy string fields $A_\mu$ and $\Phi$. In fact, we have established a 1–1 correspondence between the macroscopic string dynamics in Riemann–Cartan spacetime, and the fundamental string dynamics in the low-energy string backgrounds.

We have started our exposition in section 2 by reviewing the basics of the multipole formalism developed in [3–5]. This method is a generalization of the Mathisson–Papapetrou method for pointlike matter [1, 2]. It has already been used in [3–6] for the study of strings and higher branes in Riemann–Cartan backgrounds. Its essence is the usage of stress–energy and spin tensor conservation laws to specify the brane dynamics. The advantage of this method is that it is model independent as no action functional is specified. In section 3, the basic results of [6] have been reviewed. The single-pole solution of the conservation equations has been applied to the membrane with maximally symmetric distribution of spin. The obtained equations have been marked as a starting point for the main analysis of the paper. This analysis has been done in section 4. We have considered cylindrical membranes wrapped around the extra compact dimension of a $(D+1)$-dimensional spacetime. The effective string dynamics has been obtained in the narrow membrane limit. As compared to [6], our effective string is characterized by two more free parameters, and additionally couples to the electromagnetic and scalar fields. In section 5, we have compared our effective string dynamics with the string theory $\sigma$-model of [10–15]. We have demonstrated that our macroscopic strings couple to the dimensionally reduced Riemann–Cartan geometry the same way as fundamental strings couple to the low-energy string fields.

In conclusion, let us say something about the prospects of our research. We have already established a 1–1 correspondence between the macroscopic string dynamics in Riemann–Cartan spacetime, and the fundamental string dynamics in the low-energy string backgrounds. In this correspondence, the string background fields $G_{\mu\nu}$, $B_{\mu\nu}$, $A_\mu$ and $\Phi$ are related to the metric and torsion of the Riemann–Cartan spacetime. An interesting challenge would be to establish equivalence on the level of background field equations. There have been attempts in the literature to rewrite the low-energy string field action in geometric terms [18, 19]. These have not been very successful though, as they included some unnatural constraints to be imposed on torsion prior to varying the action. Note that establishing equivalence on the level of background field equations is not about comparing the existing equations, but rather about constructing the gravity equations that match those of the low energy string fields. This is because there is no a preferred geometric action with dynamical torsion that could readily be compared with the low energy string field action. Indeed, the Einstein–Hilbert action is torsion free, while the torsion of Einstein–Cartan theory does not propagate. If we stay with membranes, however, the construction of the needed action is quite simple. One should start with the low-energy string field action, and replace the symmetric field with the spacetime metric, and the 3-form field with the axial component of the torsion. It is much more complicated to obtain geometric counterparts of the 2-form field, electromagnetic field and dilaton field. The natural idea is to follow the dimensional reduction procedure used in this paper. One should try and find a $(D+1)$-dimensional geometric action that reduces to the needed string background action after dimensional reduction. This is, however, a difficult task for itself and should be considered in a separate paper.
Acknowledgment

This work is supported by the Serbian Ministry of Science and Technological Development, under contract no 141036.

References

[1] Mathisson M 1937 *Acta. Phys. Pol.* 6 163
[2] Papapetrou A 1951 *Proc. R. Soc.* A 209 248
[3] Vasići M and Vojinović M 2006 *Phys. Rev.* D 73 124013
[4] Vasići M and Vojinović M 2007 *J. High Energy Phys.* JHEP07(2007)028
[5] Vasići M and Vojinović M 2008 *Phys. Rev.* D 78 104002
[6] Vasići M and Vojinović M 2010 *Phys. Rev.* D 81 024025
[7] Nambu Y 1974 *Phys. Rev.* D 10 4262
[8] Goto T 1971 *Prog. Theor. Phys.* 46 1560
[9] Duff M J, Howe P S, Inami T and Stelle K S 1987 *Phys. Lett.* B 191 70
[10] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory* (Cambridge: Cambridge University Press)
[11] Polchinski J 1998 *String Theory* (Cambridge: Cambridge University Press)
[12] Callan C G, Friedan D, Martinec E J and Perry M J 1985 *Nucl. Phys.* B 262 593
[13] Banks T, Nemeschansky D and Sen A 1986 *Nucl. Phys.* B 277 67
[14] Callan C G, Klebanov I R and Perry M J 1986 *Nucl. Phys.* B 278 78
[15] Fradkin E S and Tseytlin A A 1985 *Phys. Lett.* B 158 316
  Fradkin E S and Tseytlin A A 1985 *Nucl. Phys.* B 261 1
[16] Blagojevic M 2002 *Gravitation and Gauge Symmetries* (Bristol: Institute of Physics Publishing)
[17] Yasskin P B and Stoeger W R 1980 *Phys. Rev.* D 21 2081
[18] Nepomechie R I 1985 *Phys. Rev.* D 32 3201
[19] Freund P G O and Nepomechie R I 1982 *Nucl. Phys.* B 199 482