THE ZARISKI TOPOLOGY-GRAPH ON THE MAXIMAL SPECTRUM OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let \( M \) be a module over a commutative ring and let \( \text{Spec}(M) \) (resp. \( \text{Max}(M) \)) be the collection of all prime (resp. maximal) submodules of \( M \). We topologize \( \text{Spec}(M) \) with Zariski topology, which is analogous to that for \( \text{Spec}(R) \), and consider \( \text{Max}(M) \) as the induced subspace topology. For any non-empty subset \( T \) of \( \text{Max}(M) \), we introduce a new graph \( G(\tau^M_T) \), called the Zariski topology-graph on the maximal spectrum of \( M \). This graph helps us to study the algebraic (resp. topological) properties of \( M \) (resp. \( \text{Max}(M) \)) by using the graph theoretical tools.

1. Introduction

Throughout this paper \( R \) is a commutative ring with a non-zero identity and \( M \) is a unital \( R \)-module. By \( N \leq M \) (resp. \( N < M \)) we mean that \( N \) is a submodule (resp. proper submodule) of \( M \).

There are many papers on assigning graphs to rings or modules. Annihilating-ideal graphs of rings, first introduced and studied in [11], provide an excellent setting for studying the ideal structure of a ring. \( AG(R) \), the Annihilating-ideal graph of \( R \), to be a graph whose vertices are ideals of \( R \) with non-zero annihilators and in which two vertices \( I \) and \( J \) are adjacent if and only if \( IJ = 0 \).

In [4], we generalized the above idea to submodules of \( M \) and define the (undirected) graph \( AG(M) \), called the annihilating submodule graph, with vertices \( V(AG(M)) = \{ N \leq M : \text{there exists } \{0\} \neq K < M \text{ with } NK = 0 \} \), where \( NK \), the product of \( N \) and \( K \), is defined by \( (N : M)(K : M)M \) (see [3]). In this graph, distinct vertices \( N, L \in AG(M) \) are adjacent if and only if \( NL \neq 0 \). In section two of this article, we collect some fundamental properties of the annihilating-submodule graph of a module which will be used in this work.

A prime submodule (or a \( p \)-prime submodule) of \( M \) is a proper submodule \( P \) of \( M \) such that whenever \( re \in P \) for some \( r \in R \) and \( e \in M \), either \( e \in P \) or \( r \in p \) [12].

The prime spectrum (or simply, the spectrum) of \( M \) is the set of all prime submodules of \( M \) and denoted by \( \text{Spec}(M) \).

The Zariski topology on \( \text{Spec}(M) \) is the topology \( \tau_M \) described by taking the set \( Z(M) = \{ V(N) : N \leq M \} \) as the set of closed sets of \( \text{Spec}(M) \), where \( V(N) = \{ P \in \text{Spec}(M) : (P : M) \supseteq (N : M) \} \) [15].

The closed subset \( V(N) \), where \( N \) is a submodule of \( M \), plays an important role in the Zariski topology on \( \text{Spec}(M) \). In [4], We employed these sets and defined...
a new graph $G(\tau_T)$, called the Zariski topology-graph. This graph helps us to study algebraic (resp. topological) properties of modules (resp. $\text{Spec}(M)$) by using the graphs theoretical tools. $G(\tau_T)$ is an undirected graph with vertices $V(G(\tau_T)) = \{N < M : \text{there exists } K < M \text{ such that } V(N) \cup V(K) = T \text{ and } V(N), V(K) \neq T\}$, where $T$ is a non-empty subset of $\text{Spec}(M)$ and distinct vertices $N$ and $L$ are adjacent if and only if $V(N) \cup V(L) = T$.

There exists a topology on $\text{Max}(M)$ having $Z^m(M) = V^m(N) : N \leq M$ as the set of closed sets of $\text{Max}(M)$, where $V^m(N) = \{Q \in \text{Max}(M) : (Q : M) \supseteq (N : M)\}$. We denote this topology by $\tau^m_M$. In fact $\tau^m_M$ is the same as the subspace topology induced by $\tau_M$ on $\text{Max}(M)$.

In this paper, we define a new graph $G(\tau^m_M)$, called the Zariski topology-graph on the maximal spectrum of $M$, where $T$ is a non-empty subset of $\text{Max}(M)$, and by using this graph, we study algebraic (resp. topological) properties of $M$ (resp. $\text{Max}(M)$). $G(\tau^m_M)$ is an undirected graph with vertices $V(G(\tau^m_M)) = \{N < M : \text{there exists a non-zero proper submodule } L \text{ of } M \text{ such that } V^m(N) \cup V^m(L) = T \text{ and } V^m(N), V^m(L) \neq T\}$, where $T$ is a non-empty subset of $\text{Max}(M)$ and distinct vertices $N$ and $L$ are adjacent if and only if $V^m(N) \cup V^m(L) = T$.

Let $T$ be a non-empty subset of $\text{Max}(M)$. As $\tau^m_M$ is the subspace topology induced by $\tau_M$ on $\text{Max}(M)$, one may think that $G(\tau_T)$ and $G(\tau^m_M)$ have the identical nature. But the Example 3.3 (case (1)) shows that this is not true and these graphs are different. Also the Example 3.3 (case (2)) shows that for a non-empty subset $T'$ of $\text{Spec}(M)$, under a condition, $G(\tau_T')$ can be regarded as a subgraph of $G(\tau^m_M)$, where $T = T' \cap \text{Max}(M)$. Besides, this case denotes that $G(\tau_T')$ is not a subgraph of $G(\tau^m_M)$, necessarily. Moreover, it is shown that $G(\tau^m_M)$ can not appear as a subgraph of $G(\tau_T')$, where $T \subseteq T' \subseteq \text{Spec}(M)$, in general (see Example 3.3 (case (3))). So, the results related to $G(\tau^m_M)$, where $T$ is a non-empty subset of $\text{Max}(M)$, do not go parallel to those of $G(\tau_T)$, where $T'$ is a non-empty subset of $\text{Spec}(M)$, necessarily. Based on the above remarks, it is worth to study Max-graphs separately.

For any pair of submodules $N \subseteq L$ of $M$ and any element $m$ of $M$, we denote $L/N$ and the residue class of $m$ modulo $N$ in $M/N$ by $\overline{L}$ and $\overline{m}$ respectively.

For a submodule $N$ of $M$, the colon ideal of $M$ into $N$ is defined by $(N : M) = \{r \in R : rM \subseteq N\} = \text{Ann}(M/N)$. Further if $I$ is an ideal of $R$, the submodule $(N : M) I$ is defined by $\{m \in M : IM \subseteq N\}$. Moreover, $\mathbb{Z}$ (resp. $\mathbb{Q}$) denotes the ring of integers (resp. the field of rational numbers).

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of $M$ containing $N$, and in case $N$ is not contained in any prime submodule, $\sqrt{N}$ is defined to be $M$. Note that the intersection of all prime submodule $M$ is denoted by $\text{rad}(M)$.

If $\text{Max}(M) \neq \emptyset$, the mapping $\psi : \text{Max}(M) \rightarrow \text{Max}(R/\text{Ann}(M)) = \text{Max}(\overline{R})$ such that $\psi(Q) = (Q : M)/\text{Ann}(M) = (\overline{Q} : \overline{M})$ for every $Q \in \text{Max}(M)$, is called the natural map of $\text{Max}(M)$ [9].

$M$ is said to be $\text{Max}$-surjective if either $M = (0)$ or $M \neq (0)$ and the natural map of $\text{Max}(M)$ is surjective [2].

For a proper ideal $I$ of $R$, we recall that the $J$ – radical $I$, denoted by $J^m(I)$, is the intersection of all maximal ideals containing $I$.

The $J$ – radical of a submodule $N$ of $M$, denoted by $J^m(N)$, is the intersection of all members of $V^m(N)$. In case that $V^m(N) = \emptyset$, we define $J^m(N) = M$ [5].
A topological space $X$ is said to be connected if there doesn’t exist a pair $U$, $V$ of disjoint non-empty open sets of $X$ whose union is $X$. A topological space $X$ is irreducible if for any decomposition $X = X_1 \cup X_2$ with closed subsets $X_i$ of $X$ with $i = 1, 2$, we have $X = X_1$ or $X = X_2$. A subset $X’$ of $X$ is connected (resp. irreducible) if it is connected (resp. irreducible) as a subspace of $X$.

In section 2, we briefly review some fundamental properties of the annihilating-submodule graph of a module needed later.

In section 3, among other results, it is shown that the Zariski topology-graph on maximal Spectrum of $M$, $G(\tau_M^n)$, is connected and $diam(\tau_M^n) \leq 3$. Moreover, if $G(\tau_M^n)$ containing a cycle satisfies $gr(\tau_M^n) \leq 4$ (see Theorem 3.6). Also we consider some conditions under which $G(\tau_M^n)$ is a non-empty graph.

In section 4, the relationship between $G(\tau_M^n)$ and $AG(M/\mathcal{T}(T))$ is investigated, where $\mathcal{T}(T)$ is the intersection of all members of $T$. It is proved that if $M$ is a Max-surjective $R$-module and $N$ and $L$ are proper submodules of $M$ which are adjacent in $G(\tau_M^n)$, then $J^n(N)/\mathcal{T}(T)$ and $J^n(L)/\mathcal{T}(T)$ are adjacent in $AG(M/\mathcal{T}(T))$ (see Corollary 4.2). Also it is shown, under some conditions, that $AG(M/\mathcal{T}(T))$ is isomorphic to a subgraph of $G(\tau_M^n)$ and $AG(M/\mathcal{T}(T))$ is non-empty if and only if $G(\tau_M^n)$ is non-empty, and any two proper submodules $N$ and $L$ of $M$ are adjacent in $G(\tau_M^n)$ if $N/\mathcal{T}(T)$ and $L/\mathcal{T}(T)$ are adjacent in $M/\mathcal{T}(T)$ (see Proposition 4.3).

For completeness, we now mention some graph theoretic notions and notations that are used in this article. A graph $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non-empty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function $\psi_G$ that associates an unordered pair of distinct vertices with each edge. The edge $e$ joins $x$ and $y$ if $\psi_G(e) = \{x, y\}$, and we say $x$ and $y$ are adjacent. The degree $d_G(x)$ of a vertex $x$ is the number of edges incident with $x$. A path in graph $G$ is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where $x_{i-1}$ and $x_i$ are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between $x_{i-1}$ and $x_i$. The number of edges crossed to get from $x$ to $y$ in a path is called the length of the path, where $x, y \in V(G)$. A graph $G$ is connected if a path exists between any two distinct vertices. For distinct vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of the shortest path from $x$ to $y$ and if there is no such path $d(x, y) = \infty$. The diameter of $G$ is $diam(G) = \sup\{d(x, y) : x, y \in V(G)\}$. The girth of $G$, denoted by $gr(G)$, is the length of a shortest cycle in $G$ ($gr(G) = \infty$ if $G$ contains no cycle)(see [11]).

A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. Two graphs $G$ and $G'$ are said to be isomorphic if there exists a one-to-one and onto function $\phi : V(G) \rightarrow V(H)$ such that if $x, y \in V(G)$, then $x - y$ if and only if $\phi(x) - \phi(y)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$; that is, $U$ and $V$ are each independent sets and complete bipartite graph on $n$ and $m$ vertices, denoted by $K_{n,m}$, where $V$ and $U$ are of size $n$ and $m$ respectively, and $E(G)$ connects every vertex in $V$ with all vertices in $U$ (see [18]).

In the rest of this article, $T$ denotes a non empty subset of $Max(M)$ and $\mathcal{T}(T))$ is the intersection of all members of $T$.

2. Previous results

As we mentioned before, $AG(M)$ is a graph with vertices $V(AG(M)) = \{N \leq M : NL = 0 \text{ for some } \{0\} \neq L < M\}$, where distinct vertices $N$ and $L$ are adjacent
if and only if $NL = 0$ (here we recall that the product of $N$ and $L$ is defined by $(N : M)(L : M)M$ (see [11, Definition 3.1]).

The following results reflect some basic properties of the annihilating-submodule graph of a module.

**Proposition A** ([4, Proposition 3.2]). Let $N$ be a non-zero proper submodule of $M$.

(a) $N$ is a vertex in $AG(M)$ if $Ann(N) \neq Ann(M)$ or $(0 : M (N : M)) \neq 0$.

(b) $N$ is a vertex in $AG(M)$, where $M$ is a multiplication module, if and only if $(0 : M (N : M)) \neq 0$.

**Remark 2.1.** In the annihilating-submodule graph $AG(M)$, $M$ itself can be a vertex. In fact $M$ is a vertex if and only if every non-zero submodule is a vertex if and only if there exists a non-zero proper submodule $N$ of $M$ such that $(N : M) = Ann(M)$. For example, for every submodule $N$ of $Q$ (as $\mathbb{Z}$-module), $(N : Q) = 0$. Hence $\mathbb{Q}$ is a vertex in $AG(\mathbb{Q})$.

**Theorem B** ([4, Theorem 3.3]). Assume that $M$ is not a vertex. Then the following hold.

(a) $AG(M)$ is empty if and only if $M$ is a prime module.

(b) A non-zero submodule $N$ of $M$ is a vertex if and only if $(0 : M (N : M)) \neq 0$.

**Theorem C** ([4, Theorem 3.4]). The annihilating-submodule graph $AG(M)$ is connected and $diam(AG(M)) \leq 3$. Moreover, if $AG(M)$ contains a cycle, then $gr(AG(M)) \leq 4$.

**Proposition D** ([4, Proposition 3.5]). Let $R$ be an Artinian ring and $M$ a finitely generated $R$-module. Then every nonzero proper submodule $N$ of $M$ is a vertex in $AG(M)$.

**Theorem E** ([4, Theorem 3.6]). Suppose $M$ is not a prime module. Then $AG(M)$ has acc (resp. dcc) on vertices if and only if $M$ is a Noetherian (resp. an Artinian) module.

**Theorem F** ([4, Theorem 3.7]). Let $R$ be a reduced ring and let $M$ be a faithful module which is not prime. Then the following statements are equivalent.

(a) $AG(M)$ is a finite graph.

(b) $M$ has only finitely many submodules.

(c) Every vertex of $AG(M)$ has finite degree. Moreover, $AG(M)$ has $n$ ($n \geq 1$) vertices if and only if $M$ has only $n$ nonzero proper submodules.

**Proposition G** ([4, Proposition 3.9]). We have exactly one of the following assertions.

(a) Every non-zero submodule $M$ is a vertex in $AG(M)$.

(b) There exists maximal ideal $m$ of $R$ such that $mM \in V(AG(M))$ if and only if $Soc(M) \neq 0$. 


3. The Zariski topology-graph on the maximal spectrum module

**Definition 3.1.** We define \(G(\tau_T^m)\), a Zariski topology-graph on the maximal spectrum of \(M\), with vertices \(V(G(\tau_T^m)) = \{N < M : \text{there exists } L < M \text{ such that } V^m(N) \cup V^m(L) = T \text{ and } V^m(L) \neq T\}\), where \(T\) is a non-empty subset of \(\text{Max}(M)\) and distinct vertices \(N\) and \(L\) are adjacent if and only if \(V^m(N) \cup V^m(L) = T\).

**Lemma 3.2.** \(G(\tau_T^m) \neq \emptyset\) if and only if \(T\) is closed and is not irreducible subset of \(\text{Max}(M)\).

*Proof.\* Let \(N\) and \(L\) be submodules of \(M\). Since \(V^m(N) = \text{Max}(M) \cup V(N)\), by [5, Result 3], we have \(V^m(N) \cap V^m(L) = V^m(N \cup L) = V^m(NL) (= V^m(NL))\). Note that \(V^m(N) = \{P \in \text{Max}(M) : P \supseteq N\}\). Then for every submodules \(N\) and \(L\) of \(M\), we have \(V^m(N) \cup V^m(L) \subseteq V^m(N \cup L)\). Hence, the proof is straightforward. \(\Box\)

**Remark 3.3.** By [5, Lemma 3.5], \(T\) is closed if and only if \(T = V^m(\exists T)\). Hence \(G(\tau_T^m) \neq \emptyset\) if and only if \(T = V^m(\exists T)\) and \(T\) is not irreducible.

**Remark 3.4.** By [5, Theorem 3.13(a)], if \(M\) is a Max-surjective \(R\)-module, then \(G(\tau_T^m) \neq \emptyset\) if and only if \(T = V^m(\exists T)\) and \(\exists T : M\) is not a \(J\)-radical prime ideal of \(R\). If \(\text{Spec}(M) = \text{Max}(M)\) and \(G(\tau_T^m) \neq \emptyset\), then \(T = V^m(\exists T)\) is not a prime submodule of \(M\) by [15, Proposition 5.4].

**Example 3.5.** Consider the following examples.

- case (1): Set \(R := \mathbb{Z} \oplus \mathbb{Z}\). Then \(\text{Max}(R) = \{\mathbb{Z} \oplus p_i \mathbb{Z}, p_i \mathbb{Z} \oplus \mathbb{Z} : i \in \mathbb{N}\}\) and \(\text{Spec}(R) = \text{Max}(R) \cup \{(0) \oplus \mathbb{Z}, \mathbb{Z} \oplus (0)\}\).

  In this example, we see that \(G(\tau_{\text{Max}(R)}^m)\) and \(G(\tau_{\text{Max}(R)}^m)\) are different. Because \(I = (0) \oplus \mathbb{Z}\) and \(J = \mathbb{Z} \oplus (0)\) are not adjacent in \(G(\tau_{\text{Max}(R)}^m)\) but they are adjacent in \(G(\tau_{\text{Max}(R)}^m)\). In fact, \(G(\tau_{\text{Spec}(R)}^m)\) and \(G(\tau_{\text{Max}(R)}^m)\) are complete bipartite graphs with two parts \(U = \{I \oplus (0)\}\) and \(V = \{(0) \oplus J\}\), where \(I\) and \(J\) are nonzero proper ideals of \(\mathbb{Z}\).

- case (2): Set \(R := \mathbb{Q} \oplus \mathbb{Z}\). Then \(\text{Max}(R) = \{\mathbb{Q} \oplus p_i \mathbb{Z}, i \in \mathbb{N}\}\) and \(\text{Spec}(R) = \text{Max}(R) \cup \{(0) \oplus \mathbb{Z}, \mathbb{Q} \oplus (0)\}\).

  In this example, we see that \(G(\tau_{\text{Spec}(R)}^m)\) is not a subgraph of \(G(\tau_{\text{Max}(R)}^m)\). Because \(G(\tau_{\text{Spec}(R)}^m)\) is a complete bipartite graph with two parts \(U = \{I \oplus (0)\}\) and \(V = \{(0) \oplus J\}\), where \(I\) and \(J\) are non-zero proper ideals of \(\mathbb{Q}\) and \(\mathbb{Z}\), respectively and \(G(\tau_{\text{Max}(R)}^m)\) is an empty graph. In fact, for every non-empty subset \(T\) of \(\text{Spec}(M)\), \(G(\tau_T^m)\) is a subgraph of \(G(\tau_T^m)\) (i.e., \(V(G(\tau_T^m)) \subseteq V(G(\tau_T^m))\) and \(E(G(\tau_T^m)) \subseteq E(G(\tau_T^m))\) iff for every vertex \(N\) of \(G(\tau_T^m)\), \(V^m(N) \neq T\), where \(T' = T \cap \text{Max}(M)\).

- case (3): Set \(R := \mathbb{Q} \oplus \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/p_i \mathbb{Z}\). Then \(\text{Max}(R) = \{p_i R, i \in \mathbb{N}\}\) and \(\text{Spec}(R) = \text{Max}(R) \cup \{(0) \oplus \bigoplus_{i \in \mathbb{N}} \mathbb{Z}/p_i \mathbb{Z}\}\).

  For every \(i, j \in \mathbb{N}, i \neq j, I_i = \mathbb{Q} \oplus \mathbb{Z}/p_i \mathbb{Z}\) and \(I_j = \mathbb{Q} \oplus \mathbb{Z}/p_j \mathbb{Z}\) are not adjacent in \(G(\tau_{\text{Spec}(R)}^m)\), but they are adjacent in \(G(\tau_{\text{Max}(R)}^m)\). This example shows that \(G(\tau_{\text{Max}(R)}^m)\) is not a subgraph of \(G(\tau_{\text{Spec}(R)}^m)\).

  The following theorem illustrates some graphical parameters.

**Theorem 3.6.** The Zariski topology-graph \(G(\tau_T^m)\) is connected and \(\text{diam}(G(\tau_T^m)) \leq 3\). Moreover if \(G(\tau_T^m)\) containing a cycle satisfies \(\text{gr}(G(\tau_T^m)) \leq 4\).
Proposition 3.7. Let $M$ be an $R$-module and let $\psi : \text{Max}(M) \to \text{Max}(\overline{R})$ be the natural map. Suppose $\text{Max}(M)$ is homeomorphic to $\text{Max}(\overline{R})$ under $\psi$. Let $N$ and $L$ be adjacent in $G(\tau^n_T)$ and let $T' = \{P : M : P \in T\}$. Then $N : M$ and $L : M$ are adjacent in $G(\tau^n_T)$. Conversely, if $T$ and $\overline{T}$ are adjacent in $G(\tau^n_T)$, then $IM$ and $JM$ are adjacent in $G(\tau^n_T)$.

Proof. The proof is similar to [4, Proposition 2.11]. □

Lemma 3.8. Let $G(\tau^n_T) \neq \emptyset$ and let $P \in T$. Then $P$ is a vertex if each of the following statements holds.

(a) There exists a subset $T'$ of $T$ such that $P \in T'$, $V^m(\bigcap_{Q \in T'} Q) = T$, and $V^m(\bigcap_{Q \in T', Q \notin P} Q) \neq T$. In particular, this holds when $T$ is a finite set and every element of $T$ is adjacent to a semi maximal submodule of $M$.

(b) For a submodule $N$ of $M$, $N \in V(G(\tau^n_T))$ and $N \cup P \notin V(G(\tau^n_T))$.

Proof. Straightforward. □

Definition 3.9. We define a subgraph $G_d(\tau^n_T)$ of $G(\tau^n_T)$ with vertices $V((G_d(\tau^n_T))) = \{N < M : \text{there exists } L < M \text{ such that } V^m(N) \cup V^m(L) = T \text{ and } V^m(N), V^m(L) \neq T \text{ and } V^m(N) \cup V^m(L) = \emptyset\}$, where distinct vertices $N$ and $L$ are adjacent if and only if $V^m(N) \cup V^m(L) = T$ and $V^m(N) \cup V^m(L) = \emptyset$. It is clear that the degree of each $N \in V((G_d(\tau^n_T)))$ is the number of submodules $K$ of $M$ such that $V^m(K) = V^m(L)$, where $N$ and $L$ are adjacent.

Lemma 3.10. (a) $G_d(\tau_T) \neq \emptyset$ if and only if $T = V^m(\exists(T))$ and $T$ is disconnected.

(b) Suppose $\text{Spec}(M) = \text{Max}(M)$ and $M$ is a Max-surjective module and $T$ is closed. Then $G_d(T^n_T) = \emptyset$ if and only if $R/(\exists(T) : M)$ contains no idempotent other than $\overline{0}$ and $\overline{T}$.

Proof. (a) is straightforward and (b) follows from [6, Proposition 2.9] and [14, Corollary 3.8]. □

Theorem 3.11. $G_d(\tau^n_T)$ is a bipartite graph.

Proof. Use the technique of [4, Theorem 2.17]. □

Corollary 3.12. By Theorem 3.11, if $G_d(\tau^n_T)$ contains a cycle, then $gr(G_d(\tau^n_T)) = 4$.

Example 3.13. Set $M := \mathbb{Z}/12\mathbb{Z}$. Then $\text{Max}(M) = \{2\mathbb{Z}/12\mathbb{Z}, 3\mathbb{Z}/12\mathbb{Z}\}$. Set $T = \text{Max}(M)$. Clearly, $G(\tau^n_T) = G_d(\tau^n_T)$ is a bipartite graph and $\mathbb{Z}/(\bigcup_{P \in T} P : M) \cong \mathbb{Z}/6\mathbb{Z}$ contains idempotents other than $\overline{0}$ and $\overline{T}$.

Example 3.14. Set $M := \mathbb{Z}/30\mathbb{Z}$. Then $\text{Max}(M) = \{2\mathbb{Z}/30\mathbb{Z}, 3\mathbb{Z}/30\mathbb{Z}, 5\mathbb{Z}/30\mathbb{Z}\}$. Set $T = \text{Max}(M)$. Clearly, $G_d(\tau_T)$ is a bipartite graph and $\mathbb{Z}/(\bigcap_{P \in T} P : M) \cong \mathbb{Z}/30\mathbb{Z}$ contains idempotents other than $\overline{0}$ and $\overline{T}$.

The Example 3.14 shows that $G_d(\tau^n_T)$ is not necessarily connected. The following proposition provides some useful characterization for this case.

Proposition 3.15. (a) $G_d(\tau^n_T)$ with two parts $U$ and $V$ is a complete bipartite graph if and only if for every $N, L \in U$ (resp. in $V$), $V^m(N) = V^m(L)$.

(b) $G_d(\tau^n_T)$ is connected if and only if it is a complete bipartite graph.
4. THE RELATIONSHIP BETWEEN $G(T_m^m)$ AND $AG(M)$

The purpose of this section is to illustrate the connection between the Zariski topology-graph on the maximal spectrum of a module and the annihilating-submodule graph.

**Theorem 4.1.** Let $M$ be a Max-surjective module and suppose $N$ and $L$ are adjacent in $G(T_m^m)$. Then $J^m((N : M)M)/\mathfrak{3}(T)$ and $J^m((L : M)M)/\mathfrak{3}(T)$ are adjacent in $AG(M/\mathfrak{3}(T))$.

**Proof.** Assume that $N$ and $L$ are adjacent in $G(T_m^m)$ so that $V^m(N) \cup V^m(L) = T$. Then we have $V^m((N : M)M)((L : M)M)) = T$. It implies that $J^m(((N : M)M)((L : M)M)) = \mathfrak{3}(T)$. Hence we have $\mathfrak{3}(T) \subseteq J^m((N : M)M)$ and $\mathfrak{3}(T) \subseteq J^m((L : M)M)$. Now it is enough to show that $J^m((N : M)M)J^m((L : M)M) = J^m((N : M)M)(J^m((L : M)M) : M)M \subseteq \mathfrak{3}(T)$. Since $M$ is Max-surjective, by Lemma 3.11(a), we have $(J^m((N : M)M) : M) = J^m(((N : M)M : M)) = J^m((N : M)) (J^m((L : M)M) : M) = J^m((L : M))$. By using Proposition 2, it turns out that $(J^m((N : M)M) : M)(J^m((L : M)M) : M)M = J^m((N : M))J^m((L : M)))M \subseteq J^m((N : M)(L : M))M = J^m((N : M)(L : M)M) = J^m(NL) = \mathfrak{3}(T)$. Note that if $J^m((N : M)M) = \mathfrak{3}(T)$ or $J^m((N : M)M) = J^m((L : M)M)$, then we have $(N : M)M \subseteq \mathfrak{3}(T)$. This implies that $V^m(N) = T$, a contradiction. This completes the proof. 

**Corollary 4.2.** Assume that the hypothesis hold as in Theorem 4.1. Then $J^m(N)/\mathfrak{3}(T)$ and $J^m(L)/\mathfrak{3}(T)$ are adjacent in $AG(M/\mathfrak{3}(T))$.

**Proof.** By meeting the above theorem again, we see that $V^m(NL) = T$, $J^m(NL) = \mathfrak{3}(T)$, and

$$J^m(N)J^m(L) = J^m(N : M)J^m(L : M)M \subseteq J^m(NL) = \mathfrak{3}(T).$$

**Proposition 4.3.** Suppose $M/\mathfrak{3}(T)$ is not a vertex in $AG(M/\mathfrak{3}(T))$. Then we have the following.

(a) The annihilating-submodule graph $AG(M/\mathfrak{3}(T))$ is isomorphic to a subgraph of $G(T_m^m)$.

(b) If $M$ is a Max-surjective module or $Spec(M) = Max(M)$, then $AG(M/\mathfrak{3}(T)) = \emptyset$ if and only if $G(T_m^m) = \emptyset$.

(c) If $R$ is an Artinian ring and $M/\mathfrak{3}(T)$ is a finitely generated module, then for every non-zero proper submodule $N/\mathfrak{3}(T)$ of $M/\mathfrak{3}(T)$, $N/\mathfrak{3}(T)$ and $N$ are vertices in $AG(M/\mathfrak{3}(T))$ and $G(T_m^m)$, respectively.

**Proof.** Let us begin our proof by noting that $M/\mathfrak{3}(T)$ is a vertex in $AG(M/\mathfrak{3}(T))$ if and only if there exists $N < M$ containing $\mathfrak{3}(T)$ properly with $V^m(N) = T$.

(a) Let $N/\mathfrak{3}(T) \in V(AG(M/\mathfrak{3}(T)))$. Then there exists a nonzero submodule $L/\mathfrak{3}(T)$ of $M/\mathfrak{3}(T)$ such that it is adjacent to $N/\mathfrak{3}(T)$ ($N \neq L$, because $M/\mathfrak{3}(T)$
Lemma 4.6. is not a vertex. So we have $NL \subseteq \Im(T)$. Hence $V^m(NL) = T$. If $V^m(N) = T$, then $(N : M) = (\Im(T) : M)$. It follows that $M/\Im(T)$ is a vertex, a contradiction. Hence $N$ is a vertex in $G(\tau_T^M)$ which is adjacent to $L$. In particular, if $M = R$ and $\Im(T) = 0$, then $AG(R)$ is a subgraph of $G(\tau_{\text{Max}}(M))$.

(b) To see the forward implication, let $AG(M/\Im(T)) = \emptyset$. Then $M/\Im(T)$ is a prime $R$-module so that $\Im(T)$ is a prime submodule of $M$. Thus we have $G(\tau_T^M) = \emptyset$. Conversely, suppose that $AG(M/\Im(T)) \neq \emptyset$. Then by part (a), $AG(M/\Im(T))$ is isomorphic to a subgraph of $G(\tau_T^M)$. Hence $G(\tau_T^M) \neq \emptyset$, as desired.

(c) This follows from Proposition D and part (a).

We need the following theorem from [17, Theorem 3.3].

Theorem 4.4. Let $M$ be a finitely generated module and let $N$ be a submodule of $M$ such that $(N : M) \subseteq P$, where $P$ is a prime ideal of $R$. Then there exists a prime submodule $K$ of $M$ such that $N \subseteq K$ and $(K : N) = P$.

Theorem 4.5. Suppose $N/\Im(T)$ and $L/\Im(T)$ are adjacent in $M/\Im(T)$. Then $N$ and $L$ are adjacent in $G(\tau_T^M)$ if one of the following conditions holds.

(a) $M/\Im(T)$ is not a vertex in $AG(M/\Im(T))$. In particular, this holds when $M/\Im(T)$ is finitely generated and contains no semi maximal submodule $S \neq \Im(T)$ with $V(S) = T$.

(b) $M/N$ and $M/L$ are Max-surjective and they contain no semi maximal submodule $S \neq \Im(T)$ with $V(S) = T$.

Proof. (a) This follows from Proposition 4.3 (a). For the proof of the second assertion, suppose $M/\Im(T)$ is a vertex. So there exists a non-zero proper submodule $N'/\Im(T)$ of $M/\Im(T)$, where $N' < M$, such that $V^m(N') = T$. It is clear that $M/\Im(T)$ has a structure of $R/(\Im(T) : M)$-module. Let $Q$ be an arbitrary element of $T$. Then we have $(N'/\Im(T) : M/\Im(T)) \subseteq (Q : M)/\Im(T) : M)$. Now by Theorem 4.3 there exists a prime submodule $K/\Im(T)$ such that $N' \subseteq K$ and $(K : M) = (Q : M)$. It follows that $N' \subseteq \Im(T)$, a contradiction.

(c) Clearly, $V^m(N) \cup V^m(L) = T$. Now let $V^m(N) = T$. Then we have $(N : M) \subseteq (Q : M)$ for every $Q \in T$. Since $M/N$ is Max-surjective, there exists a maximal submodule $K$ of $M$ such that $N \subseteq K$ and $(K : M) = (Q : M)$. Hence $N \subseteq \Im(T)$, contrary to our assumption. So $V^m(N) \neq T$ and the proof is completed.

Now we put the following lemma which is needed later. Its proof is easy and is omitted.

Lemma 4.6. Let $N \subseteq M$ and let $\dim(R) = 0$. Then $\text{rad}(N) = N$ if and only if $\text{Nil}(R)M = 0$ (we recall that $\text{Nil}(R) = \sqrt{0}$ is the ideal consisting of the nilpotent elements of $R$).

Proposition 4.7. Suppose $\dim(R) = 0$, $\text{Nil}(R)M = 0$, and $M/\Im(T)$ is not a vertex in $AG(M/\Im(T))$. Then the Zariski topology-graph $G(\tau_T^M)$ and the annihilating-submodule graph $AG(M/\Im(T))$ are isomorphic. In particular, when $M/\Im(T)$ is not a prime module, $G(\tau_T^M)$ has acc (resp. dcc) on vertices if and only if $M/\Im(T)$ is a Noetherian (resp. an Artinian) module.

Proof. By Proposition 4.3 (a), It thus remains to show that $G(\tau_T^M)$ is isomorphic with a subgraph of $AG(M/\Im(T))$. Suppose $N$ and $L$ are adjacent in $G(\tau_T^M)$. So we
have $V^m(NL) = T$. From [10] Theorem 2.3, we have $\text{rad}(N) = N$ if and only if $J^m(N) = N$. Now by Lemma 1.6 $J^m(NL) = NL$. Hence $N/\mathfrak{m}(T)$ and $L/\mathfrak{m}(T)$ are adjacent in $AG(M/\mathfrak{m}(T))$ as required. The second assertion follows from Theorem E.

**Lemma 4.8.** Assume that $M/\mathfrak{m}(T)$ is not a vertex in $AG(M/\mathfrak{m}(T))$. Suppose that for every $Q \in T \cap V(G(\tau^m_T))$, there exists a semi maximal submodule of $M$ such that it is adjacent to $Q$. Then $\text{Max}(M) \cap V(G(\tau^m_T)) \neq \emptyset$ if and only if $\text{Max}(M/\mathfrak{m}(T)) \cap V(AG(M/\mathfrak{m}(T))) \neq \emptyset$.

**Proof.** Choose $Q \in \text{Max}(M)$ and $Q \in V(G(\tau^m_T))$. By assumption,

$$V^m(Q) \bigcup V^m(\bigcap_{Q \in T'} Q) = T,$$

where $T'$ is a subset of $T$. Then $Q/\mathfrak{m}(T)$ and $\bigcap_{Q \in T'} Q/\mathfrak{m}(T)$ are adjacent in $AG(M/\mathfrak{m}(T))$. To see the backward implication, suppose $Q/\mathfrak{m}(T)$ is a vertex in $AG(M/\mathfrak{m}(T))$. So there exists a submodule $N/\mathfrak{m}(T)$ of $M/\mathfrak{m}(T)$ which is adjacent to it. It follows that $Q$ and $N$ are adjacent in $G(\tau^m_T)$ and the proof is completed.

**Proposition 4.9.** Suppose that for every $mM \in T \cap V(G(\tau^m_T))$, there exists a semi maximal submodule of $M$ such that it is adjacent to $mM$, where $m \in \text{Max}(R)$. Then we have exactly one of the following assertions.

(a) There exists a non-zero submodule $K$ of $M$ with $K \neq \mathfrak{m}(T)$ and $V^m(K) = T$.

(b) There exists a maximal ideal $m$ of $R$ such that $mM \in T \cap V(G(\tau^m_T))$ if and only if $\text{Soc}(M/\mathfrak{m}(T)) \neq 0$.

**Proof.** An obvious way to do this is to suppose (a) doesn’t hold and we have $mM \in T \cap V(G(\tau^m_T))$, where $m$ is a maximal ideal of $R$. Then by Lemma 1.8 $mM/\mathfrak{m}(T)$ is a vertex in $AG(M/\mathfrak{m}(T))$. It is clear that $M/\mathfrak{m}(T)$ is not a vertex in $AG(M/\mathfrak{m}(T))$. Thus by Proposition G, $\text{Soc}(M/\mathfrak{m}(T)) \neq 0$. Conversely, let $\text{Soc}(M/\mathfrak{m}(T)) \neq 0$. Then by Proposition G, $mM/\mathfrak{m}(T) \in V(AG(M/\mathfrak{m}(T)))$, where $m$ is a maximal ideal of $R$. Hence by Lemma 1.8 $mM \in V(G(\tau^m_T))$ and the proof is completed.

**Theorem 4.10.** Assume that $M/\mathfrak{m}(T)$ is a faithful module which is not a vertex in $AG(M/\mathfrak{m}(T))$. Then the following assertions are equivalent.

(a) $G(\tau^m_T)$ is a finite graph.

(b) $AG(M/\mathfrak{m}(T))$ is a finite graph

(c) $M/\mathfrak{m}(T)$ has finitely many submodules. Moreover, $G(\tau^m_T)$ has $n$ ($n \geq 1$) vertices if and only if $M/\mathfrak{m}(T)$ has only $n$ nonzero proper submodules.

**Proof.** (a) $\Rightarrow$ (b) This follows from Proposition 1.8 (a).

(b) $\Rightarrow$ (c) Suppose $AG(M/\mathfrak{m}(T))$ is a finite graph with $n$ vertices ($n \geq 1$). By Theorem E, $M/\mathfrak{m}(T)$ has finite length. Now the claim follows from Proposition D.

(c) $\Rightarrow$ (a) $M/\mathfrak{m}(T)$ has finite length so that $\text{dim}(R) = 0$. Also, it follows from the hypothesis that $\text{Nil}(R) = 0$. Therefore, $G(\tau^m_T)$ is a finite graph by Proposition 4.7. The second assertion follows easily from the above arguments.
The following example shows that the hypotheses "$M/\mathfrak{I}(T)$ is a faithful module" is needed in the above theorem.

**Example 4.11.** Consider Example 3.5 (case (1)). When $|T| \geq 2$ and $T \subseteq \{p_1\mathbb{Z} \oplus \mathbb{Z}, \ldots, p_n\mathbb{Z} \oplus \mathbb{Z}\}$.

Then every element $T$ is a vertex and $G(\tau^n_T)$ is an infinite graph because $p_i^k\mathbb{Z} \oplus \mathbb{Z}$ is a vertex for every positive integer $k$ and $1 \leq i \leq n$. Now we have

$$R/\mathfrak{I}(T) = \mathbb{Z} \oplus \mathbb{Z}/(p_1p_2\ldots p_k\mathbb{Z}) \oplus \mathbb{Z}.$$ 

It follows that $AG(R/\mathfrak{I}(T))$ is a finite graph.

We end this work with the following question:

**Question 4.12.** Let $G(\tau^n_T) \neq \emptyset$, where $T$ be an infinite subset of $\text{Max}(M)$. Is $T \cap V(G(\tau^n_T)) \neq \emptyset$?

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THE ZARISKI TOPOLOGY-GRAPH ON THE MAXIMAL SPECTRUM

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