The Long dimodules category and nonlinear equations

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Abstract

Let $H$ be a bialgebra and $H\mathcal{L}^H$ be the category of Long $H$-dimodules defined, for a commutative and cocommutative $H$, by F. W. Long in [10] and studied in connection to the Brauer group of a so called $H$-dimodule algebra. For a commutative and co-commutative $H$, $H\mathcal{L}^H = H\mathcal{YD}^H$ (the category of Yetter-Drinfel’d modules), but for an arbitrary $H$ the categories $H\mathcal{L}^H$ and $H\mathcal{YD}^H$ are basically different. Keeping in mind that the category $H\mathcal{YD}^H$ is deeply involved in solving the quantum Yang-Baxter equation, we shall study the category $H\mathcal{L}^H$ of $H$-dimodules in connection with what we have called the $\mathcal{D}$-equation: $R_{12}R_{23} = R_{23}R_{12}$, where $R \in \mathrm{End}_k(M \otimes M)$ for a vector space $M$ over a field $k$. The main result is a FRT type theorem: if $M$ is finite dimensional, then any solution $R$ of the $\mathcal{D}$-equation has the form $R = R(M, \cdot, \rho)$, where $(M, \cdot, \rho)$ is a Long $D(R)$-dimodule over a bialgebra $D(R)$ and $R(M, \cdot, \rho)$ is the special map $R(M, \cdot, \rho)(m \otimes n) := \sum_{n<1> \cdot m \otimes n<0>}$. In the last section, if $C$ is a coalgebra and $I$ is a coideal of $C$, we shall introduce the notion of $\mathcal{D}$-map on $C$, that is a $k$-bilinear map $\sigma : C \otimes C/I \rightarrow k$ satisfying a condition which ensures on one hand that for any right $C$-comodule, the special map $R_\sigma$ is a solution of the $\mathcal{D}$-equation, and on the other that, in the finite case, any solution of the $\mathcal{D}$-equation has this form.

0 Introduction

Let $k$ be a field, $M$ a $k$-vector space, $R \in \mathrm{End}_k(M \otimes M)$ such that a certain equation ($\mathcal{E}$) holds in $\mathrm{End}_k(M \otimes M \otimes M)$. The starting point of the present paper is the following general question:

"Does the Hopf algebra theory offer an effective technique for solving the equation ($\mathcal{E}$)"?

Using as a source of inspiration the quantum Yang-Baxter equation, in the solution of which the instruments offered by Hopf algebra theory have proven to be very efficient (see [4], [5], [6], [7], [8]), we can identify two ways of approaching the above problem:
1. Let $H$ be a bialgebra. The first idea is to define a new category $\mathcal{HE}^H$: the objects in this category are threetuples $(M, \cdot, \rho)$, where $(M, \cdot)$ is a left $H$-module and $(M, \rho)$ is a right $H$-comodule such that $(M, \cdot, \rho)$ satisfies a compatibility relation which ensures that the special map

$$R_{(M, \cdot, \rho)} : M \otimes M \to M \otimes M, \quad R_{(M, \cdot, \rho)}(m \otimes n) := \sum n_{<1>} \cdot m \otimes n_{<0>}$$

for all $m, n \in M$, is a solution of the equation $(\mathcal{E})$. The deal is now the converse: if $M$ is a finite dimensional vector space and $R$ is a solution of the equation $(\mathcal{E})$, can we construct a bialgebra $E(R)$ such that $(M, \cdot, \rho)$ has a structure of object in $E(R)\mathcal{E}^E(R)$ and $R = R_{(M, \cdot, \rho)}$? Roughly speaking, an affirmative answer to this FRT type theorem tells us that the maps $R_{(M, \cdot, \rho)}$, where $(M, \cdot, \rho)$ is an object in $\mathcal{HE}^H$ for some bialgebra $H$, “count all solutions” of the equation $(\mathcal{E})$. Two questions arise in a natural way: first, which is the bialgebra $E(R)$? Suppose that $M$ has dimension $n$. Let $\mathcal{M}^n(k)$ be the comatrix coalgebra of order $n$ and $(T(\mathcal{M}^n(k)), m, 1, \Delta, \varepsilon)$ the unique bialgebra structure of the tensor algebra $(T(\mathcal{M}^n(k)), m, 1)$ whose comultiplication $\Delta$ and counit $\varepsilon$ extend the comultiplication and the counit from $\mathcal{M}^n(k)$. As a general rule, $E(R) = T(\mathcal{M}^n(k))/I$, where $I$ is a bi-ideal of $T(\mathcal{M}^n(k))$ generated by some “obstructions”. The second question arising here regards the potential benefits of such an approach to the nonlinear equation $(\mathcal{E})$. The answer to this question is easy: following the technique evidenced by Radford in [13] for the quantum Yang-Baxter equation, reduced to the pointed case, we can classify the solutions of the equation $(\mathcal{E})$, at least for low dimensions.

2. The second approach to our problem is the following: let $H$ be a bialgebra and $\sigma : H \otimes H \to k$ a $k$-bilinear map. We require for the map $\sigma$ to satisfy certain properties $(P)$ which ensure that for any right $H$-comodule $M$, the natural map

$$R_\sigma : M \otimes M \to M \otimes M, \quad R_\sigma(m \otimes n) = \sum \sigma(m_{<1>} \otimes n_{<1>})m_{<0>} \otimes n_{<0>}$$

for all $m, n \in M$, is a solution for the equation $(\mathcal{E})$. Similar to case 1, the converse is interesting: in the finite dimensional case, any solution arises in this way.

In the case of the quantum Yang-Baxter equation the role of the category $\mathcal{HE}^H$ is played by $\mathcal{HYD}^H$, the category of Yetter-Drinfel’d modules (or crossed modules) defined by Yetter in [10]. Of course, here we have in mind the version of the famous FRT theorem given by Radford in [13]. The role of the maps $\sigma$ which satisfy the properties $(P)$ is played by the co-quasitriangular (or braided) bialgebras. For more information regarding the Yetter-Drinfel’d modules or co-quasitriangular bialgebras we refer to [8], [13], [14], or to the more recent [3], [4].

Recently, in [11] we evidence the fact that the clasical category $\mathcal{HM}^H$ of $H$-Hopf modules ([11]) is also deeply involved in solving a certain non-linear equation. We called it the Hopf equation, and it is:

$$R^{12}R^{23} = R^{23}R^{13}R^{12}$$

The Hopf equation is equivalent, taking $W = \tau R\tau$, where $\tau$ is the flip map, with the pentagonal equation

$$W^{12}W^{13}W^{23} = W^{23}W^{12}$$
which plays a fundamental role in the duality theory of operator algebras (see [2] and the references indicated here). We remember that the unitary fundamental operator defined by Takesaki for a Hopf von Neumann algebra is a solution of the Hopf equation (see lemma 4.9 of [13]).

Although at first sight the categories $\mathcal{H}M^H$ and $\mathcal{H}YD^H$ are completely different, in fact they are particular cases of the same general category $\mathcal{A}\mathcal{M}(H)^C$ of Doi-Hopf modules defined by Doi in [3]: if $A$ is a right $H$-comodule algebra and $C$ is a left $H$-module coalgebra then a threetuple $(M, \cdot, \rho)$ is an object in $\mathcal{A}\mathcal{M}(H)^C$ if $(M, \cdot)$ is a left $A$-module, $(M, \rho)$ is a right $C$-comodule and the following compatibility condition holds

$$\rho(a \cdot m) = \sum a_{<0>} \cdot m_{<0>} \otimes a_{<1>} m_{<1>}$$

for all $a \in A$, $m \in M$. Taking $A = C = H$, then $\mathcal{H}M(H)^H = \mathcal{H}M^H$, the category of classical $H$-Hopf modules; on the other hand $\mathcal{H}YD^H = \mathcal{H}M(H^{op} \s H)^H$, where $H$ can be viewed (see [3]) as an $H^{op} \s H$-module (comodule) coalgebra (algebra).

Now, let $A = H$ with the right $H$-comodule structure via $\Delta$ and $C = H$ with the trivial $H$-module structure: $h \cdot k := \varepsilon(h)k$ for all $h, k \in H$. Then the compatibility condition (1) takes the form

$$\rho(h \cdot m) = \sum h \cdot m_{<0>} \otimes m_{<1>}$$

for all $h \in H$, $m \in M$. We shall denote this category with $\mathcal{H}L^H$, which is also a special case of the Doi-Hopf module category. This category $\mathcal{H}L^H$ was defined first by F.W. Long in [10] for a commutative and cocommutative $H$ and was studied in connection to the construction of the Brauer group of an $H$-dimodule algebra (an object in $\mathcal{H}L^H$ was called in [10] an $H$-dimodule). It is interesting to note that for a commutative and cocommutative $H$ the category $\mathcal{H}YD^H$ of Yetter-Drinfel’d modules is precisely $\mathcal{H}L^H$, the category of Long $H$-dimodules (see, for example, [3]). Of course, for an arbitrary $H$, the categories $\mathcal{H}YD^H$ and $\mathcal{H}L^H$ are basically different. Keeping in mind that the category $\mathcal{H}YD^H$ of Yetter-Drinfel’d modules plays a determinant role in describing the solutions of the quantum Yang-Baxter equation, the following question is natural:

"In which equation will the category $\mathcal{H}L^H$ of Long $H$-dimodules play a key role?"

Answering this question would complete the image of the way in which the three well known categories from Hopf algebras theory, $\mathcal{H}M^H$, $\mathcal{H}YD^H$ and $\mathcal{H}L^H$, participate in solving nonlinear equations.

In this paper we shall study in detail what we have called the $\mathcal{D}$-equation: $R \in \text{End}_k(M \otimes M)$ is called a solution of the $\mathcal{D}$-equation if

$$R^{12} R^{23} = R^{23} R^{12}.$$  

In the study of this equation, we shall apply the general treatment presented above. In theorem 3.5 we prove that in the finite dimensional case any solution $R$ of the $\mathcal{D}$-equation has the form $R = R_{(M, \cdot, \rho)}$, where $(M, \cdot, \rho) \in \mathcal{D}(R)\mathcal{L}^{D(R)}$, for some bialgebra $D(R)$. The bialgebra $D(R)$ is a quotient of $T(M^\alpha(k))$ by a bi-ideal generated by some obstruction
elements $o(i,j,k,l)$ which have degree one in the graded algebra $T(\mathcal{M}^n(k))$, i.e. $o(i,j,k,l)$ lie in the coalgebra $\mathcal{M}^n(k)$. This observation led us in the last section to define the concept of $D$-map in a more general case, relative only to a coalgebra. The $D$-equation is obtained from the quantum Yang-Baxter equation by deleting the middle term from both sides. This operation destroys a certain symmetry which exists in the case of the quantum Yang-Baxter equation, and it is reflected in the fact that the $D$-maps are defined as $k$-bilinear maps from $C \otimes C/I$ to $k$, where $I$ is a coideal of a coalgebra $C$. This “asymmetry” of the $D$-equation is also underlined in the second item of the main theorem of this section. As applications, we present several examples of such constructions.

Following Doi’s philosophy ([6]), we can define a more general category $A\mathcal{L}^C$, where $A$ is an algebra and $C$ is a colagebra. Any decomposable left $A$-module can be viewed as an object of $A\mathcal{L}^C$, where $C = k[X]$, for some set $X$. We can thus view the category $A\mathcal{L}^C$ as a generalisation of the decomposable modules category.

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1 Preliminaries

Throughout this paper, $k$ will be a field. All vector spaces, algebras, coalgebras and bialgebras that we consider are over $k$. $\otimes$ and Hom will mean $\otimes_k$ and Hom$_k$. For a coalgebra $C$, we will use Sweedler’s $\Sigma$-notation, that is, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc. We will also use Sweedler’s notation for right $C$-comodules: $\rho_M(m) = \sum m_{<0>} \otimes m_{<1>}$, for any $m \in M$ if $(M, \rho_M)$ is a right $C$-comodule. $\mathcal{M}^C$ will be the category of right $C$-comodules and $\mathcal{C}$-comodules and $A\mathcal{M}$ will be the category of left $A$-modules and $A$-linear maps, if $A$ is a $k$-algebra. An important role in the present paper will be played by the $\mathcal{M}^n(k)$, the comatrix coalgebra of order $n$, i.e. $\mathcal{M}^n(k)$ is the $n^2$-dimensional vector space with $\{c_{ij} \mid i,j = 1,\ldots,n\}$ a $k$-basis such that

$$\Delta(c_{jk}) = \sum_{u=1}^{n} c_{ju} \otimes c_{uk}, \quad \varepsilon(c_{jk}) = \delta_{jk}$$

(2)

for all $j,k = 1,\ldots,n$. We view $T(\mathcal{M}^n(k))$ with the unique bialgebra structure which can be defined on the tensor algebra $T(\mathcal{M}^n(k))$ which extend the comultiplication $\Delta$ and the counity $\varepsilon$ of $\mathcal{M}^n(k)$.

For a vector space $M$, $\tau : M \otimes M \to M \otimes M$ will denote the flip map, that is, $\tau(m \otimes n) = n \otimes m$ for all $m, n \in M$. $\tau^{(123)}$ will be the automorphism of $M \otimes M \otimes M$ given by $\tau^{(123)}(l \otimes m \otimes n) = n \otimes l \otimes m$, for all $l, m, n \in M$. If $R : M \otimes M \to M \otimes M$ is a linear map we denote by $R^{12}$, $R^{13}$, $R^{23}$ the maps of End$_k(M \otimes M \otimes M)$ given by

$$R^{12} = R \otimes I, \quad R^{23} = I \otimes R, \quad R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau).$$
2 The $\mathcal{D}$-equation

We will introduce the following

**Definition 2.1** Let $M$ be a vector space and $R \in \text{End}_k(M \otimes M)$. We shall say that $R$ is a solution for the $\mathcal{D}$-equation if

$$R^{12} R^{23} = R^{23} R^{12}$$

in $\text{End}_k(M \otimes M \otimes M)$.

Solving the $\mathcal{D}$-equation is not an easy task. Even for the two-dimensional case, solving the $\mathcal{D}$-equation is equivalent to solving a homogenous system of 64 nonlinear equations (see equation (3)).

**Proposition 2.2** Let $M$ be a finite dimensional vector space and $\{m_1, \ldots , m_n\}$ a basis of $M$. Let $R, S \in \text{End}_k(M \otimes M)$ given by

$$R(m_v \otimes m_u) = \sum_{i,j} x_{uv}^{ji} m_i \otimes m_j, \quad S(m_v \otimes m_u) = \sum_{i,j} y_{uv}^{ji} m_i \otimes m_j,$$

for all $u, v = 1, \ldots , n$, where $(x_{uv}^{ji})_{i,j,u,v}$, $(y_{uv}^{ji})_{i,j,u,v}$ are two families of scalars of $k$. Then

$$R^{23} S^{12} = S^{12} R^{23}$$

if and only if

$$\sum_v x_{kv}^{ji} y_{lq}^{vp} = \sum_\alpha x_{kl}^{ji} y_{oq}^{ip}$$

(4)

for all $i, j, k, l, p, q = 1, \ldots , n$. In particular, $R$ is a solution for the $\mathcal{C}$-equation if and only if

$$\sum_v x_{kv}^{ji} x_{lq}^{vp} = \sum_\alpha x_{kl}^{ji} x_{oq}^{ip}$$

(5)

for all $i, j, k, l, p, q = 1, \ldots , n$.

**Proof** For $k, l, q = 1, \ldots , n$ we have:

$$R^{23} S^{12}(m_q \otimes m_l \otimes m_k) = R^{23}\left(\sum_{v,p} y_{lp}^{jq} m_p \otimes m_v \otimes m_k\right)$$

$$= \sum_{v,p,i,j} x_{kv}^{ji} y_{lp}^{vp} m_p \otimes m_i \otimes m_j$$

$$= \sum_{i,j,p} \left(\sum_v x_{kv}^{ji} y_{lp}^{vp}\right) m_p \otimes m_i \otimes m_j$$

and

$$S^{12} R^{23}(m_q \otimes m_l \otimes m_k) = S^{12}\left(\sum_{j,\alpha} x_{kl}^{j\alpha} m_q \otimes m_\alpha \otimes m_j\right)$$

$$= \sum_{j,\alpha,p,i} x_{kl}^{j\alpha} y_{lp}^{\alpha q} m_p \otimes m_i \otimes m_j$$

$$= \sum_{i,j,p} \left(\sum_\alpha x_{kl}^{j\alpha} y_{lp}^{\alpha q}\right) m_p \otimes m_i \otimes m_j.$$
Hence, the conclusion follows. □

In the next proposition we shall evidence a few equations which are equivalent to the $\mathcal{D}$-equation.

**Proposition 2.3** Let $M$ be a vector space and $R \in \text{End}_k(M \otimes M)$. The following statements are equivalent:

1. $R$ is a solution of the $\mathcal{D}$-equation.
2. $T := R\tau$ is a solution of the equation:
   \[ T^{12}T^{13} = T^{23}T^{13}\tau^{(123)} \]
3. $U := \tau R$ is a solution of the equation:
   \[ U^{13}U^{23} = \tau^{(123)}U^{13}U^{12} \]
4. $W := \tau R\tau$ is a solution of the equation:
   \[ \tau^{(123)}W^{23}W^{13} = W^{12}W^{13}\tau^{(123)} \]

**Proof** 1 $\iff$ 2. As $R = T\tau$, $R$ is a solution of the $\mathcal{D}$-equation if and only if
\[ T^{12}T^{12}T^{23} = T^{23}T^{23}T^{12}T^{12}. \] (6)
But
\[ \tau^{12}T^{23}\tau^{23} = T^{13}\tau^{13}\tau^{12}, \quad \text{and} \quad \tau^{23}T^{12}\tau^{12} = T^{13}\tau^{12}\tau^{13}. \]
Hence, the equation (6) is equivalent to
\[ T^{12}T^{13}\tau^{13}\tau^{12} = T^{23}T^{13}\tau^{12}\tau^{13}. \]
Now, the conclusion follows because $\tau^{12}\tau^{13}\tau^{12}\tau^{13} = \tau^{(123)}$.

1 $\iff$ 3. $R = \tau U$. Hence, $R$ is a solution of the $\mathcal{D}$-equation if and only if
\[ \tau^{12}U^{12}U^{23} = \tau^{23}U^{23}\tau^{12}U^{12}. \] (7)
But
\[ \tau^{12}U^{12}\tau^{23} = \tau^{23}\tau^{13}U^{13}, \quad \text{and} \quad \tau^{23}U^{23}\tau^{12} = \tau^{23}\tau^{12}U^{13}. \]
Hence, the equation (7) is equivalent to
\[ U^{13}U^{23} = \tau^{13}\tau^{12}U^{13}U^{12} \]
and we are done as $\tau^{13}\tau^{12} = \tau^{(123)}$. 6
1 ⇔ 4. \( R = \tau W \tau \). It follows that \( R \) is a solution of the \( D \)-equation if and only if
\[
\tau^{12} W^{12} \tau^{23} W^{23} \tau^{12} = \tau^{23} W^{23} \tau^{12} W^{12} \tau^{12}
\] (8)

Using the formulas
\[
\begin{align*}
\tau^{12} \tau^{23} &= \tau^{13} \tau^{12}, & \tau^{23} \tau^{12} &= \tau^{13} \tau^{23}, \\
\tau^{12} W^{12} \tau^{13} &= \tau^{12} \tau^{13} W^{23}, & \tau^{12} W^{23} \tau^{12} &= W^{13} \tau^{12} \tau^{23}
\end{align*}
\]
the equation (8) is equivalent to
\[
\tau^{12} \tau^{13} W^{23} \tau^{12} \tau^{23} = \tau^{23} \tau^{13} W^{12} \tau^{12} \tau^{23} \tau^{12}
\]
The prove is done as
\[
\tau^{12} \tau^{13} \tau^{12} \tau^{13} = \tau^{12} \tau^{13}
\]
\( \blacksquare \)

Examples 2.4

1. Suppose that \( R \in \text{End}_k(M \otimes M) \) is bijective. Then, \( R \) is a solution of the \( D \)-equation if and only if \( R^{-1} \) is.

2. Let \( (m_i)_{i \in I} \) be a basis of \( M \) and \( (a_{ij})_{i,j \in I} \) be a family of scalars of \( k \). Then, \( R : M \otimes M \to M \otimes M, R(m_i \otimes m_j) = a_{ij} m_i \otimes m_j \), for all \( i, j \in I \), is a solution of the \( D \)-equation. In particular, the identity map \( \text{Id}_{M \otimes M} \) is a solution of the \( D \)-equation.

3. Let \( M \) be a finite dimensional vector space and \( u \) an automorphism of \( M \). If \( R \) is a solution of the \( D \)-equation then \( u R := (u \otimes u) R (u \otimes u)^{-1} \) is also a solution of the \( D \)-equation.

4. Let \( f, g \in \text{End}_k(M) \) and \( R = f \otimes g \). Then \( R \) is a solution of the \( D \)-equation if and only if \( fg = gf \).

Indeed, a direct computation shows that
\[
R^{23} R^{12} = f \otimes fg \otimes g, \quad R^{12} R^{23} = f \otimes gf \otimes g
\]
i.e. the conclusion follows.

In particular, we consider the example considered in [9] (pg. 339). Let \( M \) be a two-dimensional vector space with \( \{m_1, m_2\} \) a basis. Let \( f, g \in \text{End}_k(M) \) such that with respect to the given basis are:
\[
\begin{align*}
f &= \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, & g &= \begin{pmatrix} b & c \\ 0 & b \end{pmatrix}
\end{align*}
\]
where \( a, b, c \) are scalars of \( k \). Then, \( R = f \otimes g \), with respect to the ordonate basis \( \{m_1 \otimes m_1, m_1 \otimes m_2, m_2 \otimes m_1, m_2 \otimes m_2\} \) of \( M \otimes M \), is given by
\[
R = \begin{pmatrix} ab & ac & b & c \\ 0 & ab & 0 & b \\ 0 & 0 & ab & ac \\ 0 & 0 & 0 & ab \end{pmatrix}
\] (9)
Then \( R \) is a solution for both the quantum Yang-Baxter equation and the \( D \)-equation.

5. Let \( R \in M_4(k) \) given by

\[
R = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & d & e & 0 \\
0 & 0 & 0 & f
\end{pmatrix}
\]

It can be proven, by a direct computation, that \( R \) is a solution of the \( D \)-equation if and only if \( c = d = 0 \). In particular, if \( q \in k, q \neq 0, q \neq 1 \), the two dimensional Yang-Baxter operator

\[
R = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & q^{-1} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}
\]

is a solution of the quantum Yang-Baxter equation and is not a solution for the \( D \)-equation.

6. Let \( G \) be a group and \( M \) be a left \( k[G] \)-module. Suppose that there exists \( \{ M_\sigma \mid \sigma \in G \} \) a family of \( k[G] \)-submodules of \( M \) such that \( M = \bigoplus_{\sigma \in G} M_\sigma \). If \( m \in M \), then \( m \) is a finite sum of homogenous elements \( m = \sum m_\sigma \). The map

\[
R : M \otimes M \to M \otimes M, \quad R(n \otimes m) = \sum_\sigma \sigma \cdot n \otimes m_\sigma, \quad \forall n, m \in M
\]

is a solution of the \( D \)-equation. Furthermore, if \( G \) is non-abelian, \( R \) is not a solution of the quantum Yang-Baxter equation.

Indeed, it is enough to prove that \((3)\) holds only for homogenous elements. Let \( m_\sigma \in M_\sigma, m_\tau \in M_\tau \) and \( m_\theta \in M_\theta \). Then,

\[
R^{23}R^{12}(m_\sigma \otimes m_\tau \otimes m_\theta) = R^{23}(\tau \cdot m_\sigma \otimes m_\tau \otimes m_\theta) = \tau \cdot m_\sigma \otimes \theta \cdot m_\tau \otimes m_\theta
\]

and

\[
R^{12}R^{23}(m_\sigma \otimes m_\tau \otimes m_\theta) = R^{12}(m_\sigma \otimes \theta \cdot m_\tau \otimes m_\theta)
\]

\( (\theta \cdot m_\tau \in M_\tau) = \tau \cdot m_\sigma \otimes \theta \cdot m_\tau \otimes m_\theta \)

Hence \( R \) is a solution of the \( D \)-equation. On the other hand

\[
R^{12}R^{13}R^{23}(m_\sigma \otimes m_\tau \otimes m_\theta) = \tau \theta \cdot m_\sigma \otimes \theta \cdot m_\tau \otimes m_\theta
\]

and

\[
R^{23}R^{12}R^{13}(m_\sigma \otimes m_\tau \otimes m_\theta) = \theta \tau \cdot m_\sigma \otimes \theta \cdot m_\tau \otimes m_\theta
\]

Hence, if \( G \) is a non-abelian group, then \( R \) is not a solution of the quantum Yang-Baxter equation.
3 Categories of Long dimodules and the $D$-equation

We shall start this section by adapting a definition given by Long in [10] for a commutative and cocommutative bialgebra to the case of an arbitrary bialgebra.

**Definition 3.1** Let $H$ be a bialgebra. A Long $H$-dimodule is a threetuple $(M, \cdot, \rho)$, where $(M, \cdot)$ is a left $H$-module, $(M, \rho)$ is a right $H$-comodule such that the following compatibility condition holds:

$$\rho(h \cdot m) = \sum h \cdot m_{<0>} \otimes m_{<1>}$$

(11)

for all $h \in H$ and $m \in M$.

The category of $H$-dimodules over $H$ and $H$-linear $H$-colinear maps will be denoted by $\mathcal{L}^H_H$.

**Examples 3.2**

1. Let $G$ be a group and $H = k[G]$ be the groupal Hopf algebra. Let $(M, \cdot)$ be a left $k[G]$-module. Then, $(M, \cdot, \rho)$ is an object in $k[G]\mathcal{L}^{k[G]}$ if and only if there exists $\{M_\sigma \mid \sigma \in G\}$ a family of $k[G]$-submodules of $M$ such that $M = \bigoplus_{\sigma \in G} M_\sigma$.

Indeed, $(M, \rho)$ is a right $k[G]$-comodule if and only if there exists $\{M_\sigma \mid \sigma \in G\}$ a family of $k$-subspaces of $M$ such that $M = \bigoplus_{\sigma \in G} M_\sigma$ (see [12]). Recall that, for $\sigma \in G$,

$$m_\sigma \in M_\sigma \quad \text{if and only if} \quad \rho(m_\sigma) = m_\sigma \otimes \sigma.$$ 

Now, let $g \in G$ and $m_\sigma \in M_\sigma$. The compatibility condition (11) turn into $\rho(g \cdot m_\sigma) = g \cdot m_\sigma \otimes \sigma$ which is equivalent to $g \cdot M_\sigma \subseteq M_\sigma$ for any $g \in G$, i.e. $M_\sigma$ is a left $k[G]$-submodule of $M$ for all $\sigma \in G$.

In fact, we have proven that the categories $k[G]\mathcal{L}^{k[G]}$ and $(k[G]M)^{(G)}$ are equivalent.

Let us now suppose that $M$ is a decomposable representation on $G$ with the Long length smaller or equal to the cardinal of $G$. Let $X$ be a subset of $G$ and $\{M_x \mid x \in X\}$ a family of indecomposable $k[G]$-submodules of $M$ such that $M = \bigoplus_{x \in X} M_x$. Then $M$ is a right comodule over $k[X]$, and as $X \subseteq G$, $M$ can be viewed as a right $k[G]$-comodule. Hence, $M \in k[G]\mathcal{L}^{k[G]}$.

We obtain that the category of representations on $G$ with the Long length smaller or equal to the cardinal of $G$ can be viewed as a subcategory of $k[G]\mathcal{L}^{k[G]}$.

2. Let $(N, \cdot)$ be a left $H$-module. Then $N \otimes H$ is an object in $\mathcal{L}^H_H$ with the structures:

$$h \bullet (n \otimes l) := h \cdot n \otimes l, \quad \rho(n \otimes l) = \sum n \otimes l_{(1)} \otimes l_{(2)}$$

for all $h, l \in H$, $n \in N$. Hence, we have constructed a functor

$$\bullet \otimes H : \mathcal{M} \rightarrow \mathcal{L}^H.$$

3. Let $(M, \rho)$ be a right $H$-comodule. Then $H \otimes M$ is an object in $\mathcal{L}^H_H$ with the structures:

$$h \bullet (l \otimes m) := hl \otimes m, \quad \rho_{H \otimes M}(l \otimes m) := \sum l \otimes m_{<0>} \otimes m_{<1>}$$
for all \( h, l \in H, m \in M \). Hence, we obtain a functor
\[
H \otimes \bullet \colon \mathcal{M}^H \to H\mathcal{L}^H.
\]

4. Let \((N, \cdot)\) be a left \(H\)-module. Then, with the trivial structure of right \(H\)-comodule, \(\rho : N \to N \otimes H\), \(\rho(n) := n \otimes 1\), for all \(n \in N\), \((N, \cdot, \rho)\) is an object in \(H\mathcal{L}^H\).

5. Let \((M, \rho)\) be a right \(H\)-comodule. Then, with the trivial structure of left \(H\)-module, \(h \cdot m := \varepsilon(h)m\), for all \(h \in H, m \in M\), \((M, \cdot, \rho)\) is an object in \(H\mathcal{L}^H\).

Remarks 3.3

1. In the study of the category \(H\mathcal{L}^H\) an important role is played by the forgetful functors:
\[
F^H : H\mathcal{L}^H \to \mathcal{M} \quad \text{and} \quad F_H : H\mathcal{L}^H \to \mathcal{M}^H
\]
where \(F^H\) (resp. \(F_H\)) is the functor forgetting the \(H\)-comodule (resp. \(H\)-module) structure.

The principle according to which forgetful type functors have adjoints is also valid for the category \(H\mathcal{L}^H\). We have:

1. The functor \(H \otimes \bullet : \mathcal{M}^H \to H\mathcal{L}^H\) is a left adjoint of the functor \(F_H : H\mathcal{L}^H \to \mathcal{M}^H\).

2. The functor \(\bullet \otimes H : \mathcal{M} \to H\mathcal{L}^H\) is a right adjoint of the functor \(F^H : H\mathcal{L}^H \to \mathcal{M}\).

The adjoint pair \((H \otimes \bullet, F_H)\) is given by the following natural transformation:
\[
\eta : \text{Id}_{\mathcal{M}^H} \to F_H \circ (H \otimes \bullet), \quad \eta_N : N \to H \otimes N, \quad \eta_N(n) := 1 \otimes n, \ \forall n \in N
\]
\[
\theta : (H \otimes \bullet) \circ F_H \to \text{Id}_{\mathcal{M}^H}, \quad \theta_M : H \otimes M \to M, \quad \theta_M(h \otimes m) := hm, \ \forall h \in H, m \in M
\]
where \(N \in \mathcal{M}^H, M \in H\mathcal{L}^H\).

The adjoint pair \((F^H, \bullet \otimes H)\) is given by the following natural transformation:
\[
\eta : \text{Id}_{H\mathcal{L}^H} \to (\bullet \otimes H) \circ F^H, \quad \eta_M : M \to M \otimes H, \quad \eta_M(m) := \sum m_{<0>} \otimes m_{<1>}, \ \forall m \in M
\]
\[
\theta : F^H \circ (\bullet \otimes H) \to \text{Id}_{H\mathcal{L}^H}, \quad \theta_N : N \otimes H \to N, \quad \theta_N(n \otimes h) := \varepsilon(h)n, \ \forall h \in H, n \in N
\]
where \(M \in H\mathcal{L}^H, N \in \mathcal{M}\).

Now, using the general properties of pairs of adjoint functors we obtain (see Corollary 2.7 of [3] for a similar results regarding the Yetter-Drinfel’d category \(H\mathcal{YD}^H\)):

1. the functor \(\bullet \otimes H : \mathcal{M} \to H\mathcal{L}^H\) preserves the injective cogenerators. In particular, \(\text{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z}) \otimes H\) is an injective cogenerator of \(H\mathcal{L}^H\).

2. the functor \(H \otimes \bullet : \mathcal{M}^H \to H\mathcal{L}^H\) preserves generators. In particular, if \(H\) is cofrobenius as a coalgebra, then \(H \otimes H\) is a projective generator of \(H\mathcal{L}^H\).
2. If \( M, N \in H \mathcal{L}^H \), then \( M \otimes N \) is also an object in \( H \mathcal{L}^H \) with the natural structures
\[
h \cdot (m \otimes n) = \sum h_{(1)} \cdot m \otimes h_{(2)} \cdot n, \quad \rho(m \otimes n) = \sum m_{<0>} \otimes n_{<0>} \otimes m_{<1>} n_{<1>}
\]
for all \( h \in H, m \in M, n \in N \). \( k \) can be viewed as an object in \( H \mathcal{L}^H \) with the trivial structures
\[
h \cdot a = \varepsilon(h)a, \quad \rho(a) = a \otimes 1
\]
for all \( h \in H, a \in k \). It is easy to see that \((H \mathcal{L}^H, \otimes, k)\) is a monoidal category. This can also be obtained from Proposition 2.1 of [1], keeping in mind that the category \( H \mathcal{L}^H \) is a special case of the Doi-Hopf modules category.

3. If \( M \in H \mathcal{L}^H \), then \( M \) is a left \( H \otimes H^* \)-module with the structure
\[
(h \otimes h^*) \cdot m := \sum < h^*, m_{<1>}> hm_{<0>}
\]
for all \( h \in H, h^* \in H^* \) and \( m \in M \). Furthermore, if \( H \) is finite dimensional the categories \( H \mathcal{L}^H \) and \( H \mathcal{L}^H \mathcal{L}^H \mathcal{M} \) are equivalent.

4. The category \( H \mathcal{L}^H \) can be generalized to a category \( A \mathcal{L}^C \). Let \( A \) be an algebra and \( C \) be a coalgebra over \( k \). Then an object in \( A \mathcal{L}^C \) is a threetuple \((M, \cdot, \rho)\), where \((M, \cdot)\) is a left \( A \)-module, \((M, \rho)\) is a right \( C \)-comodule such that the following compatibility condition holds:
\[
\rho(a \cdot m) = \sum a \cdot m_{<0>} \otimes m_{<1>}
\tag{12}
\]
for all \( a \in A \) and \( m \in M \).

Let \( X \) be a set and \( C = k[\mathcal{X}] \) be the groupal coalgebra. Then \((M, \cdot, \rho)\) is an object in \( A \mathcal{L}^k[\mathcal{X}] \) if and only if there exists \( \{M_x \mid x \in \mathcal{X}\} \) a family of \( A \)-submodules of \( M \) such that \( M = \oplus_{x \in \mathcal{X}} \mathcal{M}_x \). Hence, the category \( A \mathcal{L}^C \) can be seen as a generalization of the category of decomposable left \( A \)-modules.

The next proposition evidences the role which can be played by the category \( H \mathcal{L}^H \) in solving the \( \mathcal{D} \)-equation.

**Proposition 3.4** Let \( H \) be a bialgebra and \((M, \cdot, \rho)\) be a Long \( H \)-dimodule. Then the natural map
\[
R_{(M, \cdot, \rho)}(m \otimes n) = \sum n_{<1>} \cdot m \otimes n_{<0>}
\]
is a solution of the \( \mathcal{D} \)-equation.

**Proof** Let \( R = R_{(M, \cdot, \rho)} \). For \( l, m, n \in M \) we have
\[
R^{12}R^{23}(l \otimes m \otimes n) = R^{12}\left(\sum l \otimes n_{<1>} \cdot m \otimes n_{<0>}\right)
= \sum (n_{<1>} \cdot m)_{<1>} \cdot l \otimes (n_{<1>} \cdot m)_{<0>} \otimes n_{<0>}
\tag{using (11)}
= \sum m_{<1>} \cdot l \otimes n_{<1>} \cdot m_{<0>} \otimes n_{<0>}
= R^{23}\left(\sum m_{<1>} \cdot l \otimes m_{<0>} \otimes n\right)
= R^{23}R^{12}(l \otimes m \otimes n)
\]
Lemma 3.5 Let $H$ be a bialgebra, $(M, \cdot)$ a left $H$-module and $(M, \rho)$ a right $H$-comodule. Then the set
\[
\{ h \in H \mid \rho(h \cdot m) = \sum h \cdot m_{<0>} \otimes m_{<1>}, \forall m \in M \}
\]
is a subalgebra of $H$.

Proof Straightforward. \qed

We obtain from this lemma that if a left $H$-module and right $H$-comodule $M$ satisfies the condition of compatibility \[\text{[□]}\] for a set of generators as an algebra of $H$ and for a basis of $M$, then $M$ is a Long $H$-dimodule.

Now we shall prove the main results of this section. By a FRT type theorem, we shall prove that in the finite dimensional case, any solution of the $D$-equation has the form $R(M, \cdot, \rho)$, where $(M, \cdot, \rho)$ is a Long $D(R)$-dimodule over a special bialgebra $D(R)$.

Theorem 3.6 Let $M$ be a finite dimensional vector space and $R \in \text{End}_k(M \otimes M)$ be a solution of the $D$-equation. Then:

1. There exists a bialgebra $D(R)$ such that $(M, \cdot, \rho)$ has a structure of object in $D(R) \mathcal{L}^{D(R)}$ and $R = R(M, \cdot, \rho)$.

2. The bialgebra $D(R)$ is a universal object with this property: if $H$ is a bialgebra such that $(M, \cdot, \rho) \in H \mathcal{L}^H$ and $R = R(M, \cdot, \rho)$ then there exists a unique bialgebra map $f : D(R) \to H$ such that $\rho' = (I \otimes f)\rho$. Furthermore, $a \cdot m = f(a) \cdot m$, for all $a \in D(R)$, $m \in M$.

Proof 1. Let $\{m_1, \cdots, m_n\}$ be a basis for $M$ and $(x^j_{uv})_{i,j,u,v}$ a family of scalars of $k$ such that
\[
R(m_v \otimes m_u) = \sum_{i,j} x^j_{uv} m_i \otimes m_j
\]
for all $u, v = 1, \cdots, n$.

Let $(C, \Delta, \varepsilon) = \mathcal{M}^n(k)$, be the comatrix coalgebra of order $n$. Let $\rho : M \to M \otimes C$ given by
\[
\rho(m_l) = \sum_{v=1}^n m_v \otimes c_{vl}
\]
for all $l = 1, \cdots, n$. Then, $M$ is a right $C$-comodule. Let $T(C)$ be the unique bialgebra structure on the tensor algebra $T(C)$ which extends $\Delta$ and $\varepsilon$. As the inclusion $i : C \to T(C)$ is a coalgebra map, $M$ has a right $T(C)$-comodule structure via
\[
M \xrightarrow{\rho} M \otimes C \xrightarrow{I \otimes i} M \otimes T(C)
\]
There will be no confusion if we also denote the right $T(C)$-comodule structure on $M$ with $\rho$.

Now, we will put a left $T(C)$-module structure on $M$ in such a way that $R = R_{(M, \cdot, \rho)}$. First we define the $k$-bilinear map

$$\mu : C \otimes M \to M, \quad \mu(c_{ju} \otimes m_v) := \sum_i x_{uv}^{ji} m_i$$

for all $j, u, v = 1, \cdots n$. Using the universal property of the tensor algebra $T(C)$, there exists a unique left $T(C)$-module structure on $(M, \cdot)$ such that

$$c_{ju} \cdot m_v = \mu(c_{ju} \otimes m_v) = \sum_i x_{uv}^{ji} m_i$$

for all $j, u, v = 1, \cdots, n$. For $m_v, m_u$ the elements of the given basis, we have:

$$R_{(M, \cdot, \rho)}(m_v \otimes m_u) = \sum_j c_{ju} \cdot m_v \otimes m_j$$

$$= \sum_{i,j} x_{uv}^{ji} m_i \otimes m_j$$

$$= R(m_v \otimes m_u)$$

Hence, $(M, \cdot, \rho)$ has a structure of left $T(C)$-module and right $T(C)$-comodule such that $R = R_{(M, \cdot, \rho)}$.

Now, we define the obstructions $o(i, j, k, l)$ which measure how far away $M$ is from a Long $T(C)$-dimodule. Keeping in mind that $T(C)$ is generated as an algebra by $(c_{ij})$ and using lemma 3.5 we compute

$$\sum h \cdot m_{<0>} \otimes m_{<1>} - \rho(h \cdot m)$$

only for $h = c_{jk}$, and $m = m_l$, for $j, k, l = 1, \cdots, n$. We have:

$$\sum h \cdot m_{<0>} \otimes m_{<1>} = \sum c_{jk} \cdot (m_l)_{<0>} \otimes (m_l)_{<1>}$$

$$= \sum_v c_{jk} \cdot m_v \otimes c_{vl}$$

$$= \sum_{v, i} x_{ki}^{ji} m_i \otimes c_{vl}$$

$$= \sum_i m_i \otimes \left( \sum_v x_{ki}^{ji} c_{vl} \right)$$

and

$$\rho(h \cdot m) = \rho(c_{jk} \cdot m_l)$$

$$= \sum \alpha x_{kl}^{\alpha} (m_{\alpha})_{<0>} \otimes (m_{\alpha})_{<1>}$$

$$= \sum_{i, \alpha} x_{kl}^{\alpha} m_i \otimes c_{i\alpha}$$

$$= \sum_i m_i \otimes \left( \sum_{\alpha} x_{kl}^{\alpha} c_{i\alpha} \right)$$
Let
\[ o(i, j, k, l) := \sum_v x_{ki}^{ji} c_{vi} - \sum_{\alpha} x_{kl}^{jv} c_{i\alpha} \] (15)
for all \( i, j, k, l = 1, \cdots, n \). Then
\[ \sum h \cdot m_{<0>} \otimes m_{<1>} - \rho(h \cdot m) = \sum_i m_i \otimes o(i, j, k, l) \] (16)

Let \( I \) be the two-sided ideal of \( T(C) \) generated by all \( o(i, j, k, l) \), \( i, j, k, l = 1, \cdots, n \). Then:
\( I \) is a bi-ideal of \( T(C) \) and \( I \cdot M = 0 \).

We first prove that \( I \) is also a coideal and this will result from the following formula:
\[ \Delta(o(i, j, k, l)) = \sum_u \left( o(i, j, k, u) \otimes c_{ul} + c_{iu} \otimes o(u, j, k, l) \right) \] (17)

First, we denote that the formula (15) can be written
\[ o(i, j, k, l) := \sum_v \left( x_{ki}^{ji} c_{vi} - x_{kl}^{jv} c_{iv} \right) \]

We have:
\[
\Delta(o(i, j, k, l)) = \sum_{u,v} \left( x_{ku}^{ji} c_{vu} \otimes c_{ul} - x_{kl}^{jv} c_{iu} \otimes c_{uv} \right) \\
= \sum_u \left( \sum_v x_{ku}^{ji} c_{vu} \right) \otimes c_{ul} - \sum_u c_{iu} \otimes \left( \sum_v x_{kl}^{jv} c_{uv} \right) \\
= \sum_u \left( o(i, j, k, u) + \sum_v x_{ku}^{ji} c_{iv} \right) \otimes c_{ul} \\
- \sum_u c_{iu} \otimes \left( -o(u, j, k, l) + \sum_v x_{ku}^{jv} c_{vi} \right) \\
= \sum_u \left( o(i, j, k, u) \otimes c_{ul} + c_{iu} \otimes u(u, j, k, l) \right)
\]

where in the last equality we use the fact that
\[ \sum_{u,v} x_{ku}^{ji} c_{iv} \otimes c_{ul} = \sum_{u,v} x_{ku}^{ji} c_{iu} \otimes c_{vl} \]

Hence, the formula (17) holds. On the other hand
\[ \varepsilon(o(i, j, k, l)) = x_{ki}^{ji} - x_{kl}^{jv} = 0 \]
so we proved that \( I \) is a coideal of \( T(C) \).

Now, we shall prove that \( I \cdot M = 0 \) using the fact that \( R \) is a solution of the \( D \)-equation. Let \( n \in M \) and \( j, k = 1, \cdots, n \). We have
\[ \left( R^{23} R^{12} - R^{12} R^{23} \right) \left( n \otimes m_k \otimes m_j \right) = \sum_{r,s} o(r, s, j, k) \cdot n \otimes m_r \otimes m_s \] (18)
Let us compute

\[
\left(R^{23} R^{12}\right)(n \otimes m_k \otimes m_j) = R^{23}\left(\sum_\alpha c_{ak} \cdot n \otimes m_\alpha \otimes m_j\right)
\]

\[
= \sum_{\alpha, s} c_{ak} \cdot n \otimes c_{sj} \cdot m_\alpha \otimes m_s
\]

\[
= \sum_{\alpha, r, s} c_{ak} \cdot n \otimes x^{sr}_{ja} m_r \otimes m_s
\]

\[
= \sum_{r, s} \left(\sum_\alpha x^{sr}_{ja} c_{ak}\right) n \otimes m_r \otimes m_s
\]

On the other hand

\[
\left(R^{12} R^{23}\right)(n \otimes m_k \otimes m_j) = R^{12}\left(\sum_s n \otimes c_{sj} \cdot m_k \otimes m_s\right)
\]

\[
= R^{12}\left(\sum_s n \otimes x^{sa}_{jk} m_\alpha \otimes m_s\right)
\]

\[
= \sum_{r, s, \alpha} x^{sa}_{jk} c_{ra} \cdot n \otimes m_r \otimes m_s
\]

It follows that

\[
\left(R^{23} R^{12} - R^{12} R^{23}\right)(n \otimes m_k \otimes m_j) = \sum_{r, s} \left(\sum_\alpha x^{sr}_{ja} c_{ak} - \sum_\alpha x^{sa}_{jk} c_{ra}\right) \cdot n \otimes m_r \otimes m_s
\]

\[
= \sum_{r, s} o(r, s, j, k) \cdot n \otimes m_r \otimes m_s
\]

i.e. the formula (18) holds. But \(R\) is a solution of the \(D\)-equation, hence \(o(r, s, j, k) \cdot n = 0\), for all \(n \in M, j, k, r, s = 1, \ldots, n\). We conclude that \(I\) is a bi-ideal of \(T(C)\) and \(I \cdot M = 0\).

Define now

\[D(R) = T(C)/I\]

\(M\) has a right \(D(R)\)-comodule structure via the canonical projection \(T(C) \to D(R)\) and a left \(D(R)\)-module structure as \(I \cdot M = 0\). As \((c_{ij})\) generate \(D(R)\) and in \(D(R)\), \(o(i, j, k, l) = 0\), for all \(i, j, k, l = 1, \ldots, n\), using (16) we get that \((M, \cdot', \rho) \in D(R)\mathcal{L}^{D(R)}\) and \(R = R_{(M, \cdot', \rho)}\).

2. Let \(H\) be a bialgebra and suppose that \((M, \cdot', \rho') \in H\mathcal{L}^H\) and \(R = R_{(M, \cdot', \rho')}\). Let \((\ell'_{ij})_{i,j=1,\ldots,n}\) be a family of elements of \(H\) such that

\[\rho'(m_i) = \sum_v m_v \otimes \ell'_{vl}\]

Then

\[R(m_v \otimes m_u) = \sum_j \ell'_{ju} \cdot m_v \otimes m_j\]

and

\[\ell'_{ju} \cdot m_v = \sum_i x^{ji}_{uv} m_i = c_{ju} \cdot m_v\]

Let

\[o'(i, j, k, l) = \sum_v x^{ji}_{kv} \ell'_{vl} - \sum_\alpha x^{ja}_{kl} \ell'_{\alpha v}\]
From the universal property of the tensor algebra $T(C)$, there exists a unique algebra map $f_1 : T(C) \to H$ such that $f_1(c_{ij}) = c_{ij}'$, for all $i, j = 1, \ldots, n$. As $(M, \rho', \rho') \in H \mathcal{L}^H$ we get that $o(i, j, k, l) = 0$, and hence $f_1(o(i, j, k, l)) = 0$, for all $i, j, k, l = 1, \ldots, n$. So the map $f_1$ factorizes to the map

$$f : D(R) \to H, \quad f(c_{ij}) = c_{ij}'$$

Of course, for $m_i$ an arbitrary element of the given basis of $M$, we have

$$(I \otimes f)\rho(m_i) = \sum_v m_v \otimes f(c_{vd}) = \sum_v m_v \otimes c_{vd}' = \rho'(m_i)$$

Conversely, the relation $(I \otimes f)\rho = \rho'$ necessarily implies $f(c_{ij}) = c_{ij}'$, which proves the uniqueness of $f$. This completes the proof of the theorem. □

**Remark 3.7** The obstruction elements $o(i, j, k, l)$ are different from the homogenous elements $d(i, j, k, l)$ defined in [13] which correspond to the quantum Yang-Baxter equation, and are also different from the obstruction elements $\chi(i, j, k, l)$ which appear in [14] in connexion with the Hopf equation. The main difference consists in the fact that in the graded algebra $T(\mathcal{M}^n(k))$ the elements $o(i, j, k, l)$ are of degree one, i.e. are elements of the comatrix coalgebra $\mathcal{M}^n(k)$. This will lead us in the next section to the study of some special functions defined only for a coalgebra, which will also play an important role in solving the $D$-equation.

**Examples 3.8**

1. Let $a, b, c \in k$ and $R \in \mathcal{M}_4(k)$ given by equation (19), which is a solution for both the quantum Yang-Baxter equation and the $D$-equation. In [9], if $ab \neq 0, ac+b \neq 0$ a labour-intensive computation will give a description of the bialgebra $A(R)$, obtained looking at $R$ as a solution of the quantum Yang-Baxter equation.

Below we shall describe the bialgebra $D(R)$, which is obtained considering $R$ as a solution for the $D$-equation. If $(b, c) = (0, 0)$ then $R = 0$ i.e. $D(R) = T(\mathcal{M}^4(k))$. Suppose now that $(b, c) \neq (0, 0)$. If we write

$$R(m_v \otimes m_u) = \sum_{i,j=1}^{2} x^{ji}_{uv} m_i \otimes m_j$$

we get that among the elements $(x^{ji}_{uv})$, the only nonzero elements are:

$$x^{11}_{11} = ab, \quad x^{11}_{21} = ac, \quad x^{21}_{21} = ab, \quad x^{11}_{12} = b,$$

$$x^{12}_{12} = ab, \quad x^{11}_{21} = c, \quad x^{21}_{22} = b, \quad x^{12}_{22} = ac, \quad x^{22}_{22} = ab$$

The sixteen relation $o(i, j, k, l) = 0$, written in the lexicographical order acording to $(i, j, k, l)$, starting with $(1, 1, 1, 1)$ are

$$abc_{11} + bc_{21} = abc_{11}, \quad abc_{12} + bc_{22} = bc_{11} + abc_{12}$$

$$acc_{11} + cc_{21} = acc_{11}, \quad acc_{12} + cc_{22} = cc_{11} + acc_{12}$$

$0 = 0, \quad 0 = 0, \quad abc_{11} + bc_{21} = abc_{11}, \quad abc_{12} + bc_{22} = bc_{11} + abc_{12}$

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\[abc_{21} = abc_{21}, \quad abc_{22} = abc_{22}, \quad acc_{21} = acc_{21}, \quad acc_{22} = cc_{21} + acc_{22}\]
\[0 = 0, \quad 0 = 0, \quad abc_{21} = abc_{21}, \quad abc_{22} = abc_{22} + bc_{21}\]

It remains only the four relation
\[bc_{21} = 0, \quad bc_{22} = bc_{11}, \quad cc_{21} = 0, \quad cc_{22} = cc_{11}\]

As, \((b, c) \neq (0, 0)\), there are only two linear independent relation:
\[c_{21} = 0, \quad c_{22} = c_{11}\]

Now, if we denote \(c_{11} = x, c_{12} = y\) we obtain that \(D(R)\) can be described as follows:
\[\bullet\] as an algebra \(D(R) = k < x, y >\), the free algebra generated by \(x\) and \(y\).
\[\bullet\] The comultiplication \(\Delta\) and the counity \(\varepsilon\) are given by
\[\Delta(x) = x \otimes x, \quad \Delta(y) = x \otimes y + y \otimes x, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.\]

We observe that the bialgebra \(D(R)\) does not depend on the parameters \(a, b, c\).

2. Let \(q \in k\) be a scalar and \(R_q \in \mathcal{M}_4(k)\) given by
\[R_q = \begin{pmatrix}
0 & -q & 0 & -q^2 \\
0 & 1 & 0 & q \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}\]

Then \(R_q\) is a solution of the Hopf equation (cf. [11]), and of the \(D\)-equation, as \(R_q\) has the form \(f \otimes g\) with \(fg = gf\). In [11] we have described the bialgebra \(B(R_q)\) which arises thinking of \(R_q\) as a solution of the Hopf equation. The bialgebra \(D(R_q)\) obtained by viewing \(R_q\) as a solution of the \(D\)-equation has a much simpler description and is independent of \(q\).

\[\bullet\] as an algebra \(D(R_q) = k < x, y >\), the free algebra generated by \(x\) and \(y\).
\[\bullet\] The comultiplication \(\Delta\) and the counity \(\varepsilon\) are given in such a way \(x\) and \(y\) are groupal elements
\[\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y \quad \varepsilon(x) = \varepsilon(y) = 1.\]

The above description can be obtained through a computation similar to the one from the previous example. Among the sixteen relations \(o(i, j, k, l) = 0\), the only linear independent ones are:
\[c_{21} = 0, \quad c_{12} = q(c_{11} - c_{22})\]

If we denote \(c_{11} = x, c_{22} = y\) the conclusion follows.

3. We shall now present an example which has a geometric flavour. Let \(f \in \text{End}_k(k^2), f((x, y)) = (x, 0)\) for all \((x, y) \in k^2\), i.e. \(f\) is the projection of the plane \(k^2\) on the \(Ox\) axis. With respect to the canonical basis \(\{e_1, e_2\}\) of \(k^2\), \(f\) has the form
\[f = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}\]
\( g \in \text{End}_k(k^2) \) commute with \( f \) if and only if with respect to the canonical basis, \( g \) has the form
\[
g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}
\]
where \( a, b \in k \). Then \( R = f \otimes g \) with respect to the ordonate basis \( \{ e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2 \} \) is given by
\[
R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

We shall describe the bialgebra \( D(R) \). Suppose \( (a, b) \neq 0 \) (otherwise \( R = 0 \) and \( D(R) = T(M^4(k)) \)). Among the sixteen relations \( o(i, j, k, l) = 0 \), the only linear independent ones are:
\[
ac_{12} = ac_{21} = bc_{12} = bc_{21} = 0
\]
As \( (a, b) \neq 0 \) we obtain \( c_{12} = c_{21} = 0 \). Now, if we denote \( c_{11} = x, c_{22} = y \) we get that the bialgebra \( D(R) \) is exactly the bialgebra described in the previous example.

4 Special functions on a coalgebra and the \( D \)-equation

In this section we shall introduce the concept of \( D \)-map on a coalgebra: if \( C \) is a coalgebra and \( I \) a coideal of \( C \), then a \( D \)-map is a \( k \)-bilinear map \( \sigma : C \otimes C/I \rightarrow k \) which satisfies the equation (20) presented below. This condition ensures that, for any right \( C \)-comodule \( M \), the natural map \( R_\sigma \) is a solution for the \( D \)-equation. Conversely, in the finite dimensional case, any solution for the \( D \)-equation arises in this way.

If \( C \) is a coalgebra and \( I \) is a coideal of \( C \) then the elements of the quotient \( C/I \) will be denoted by \( \overline{c} \). If \( (M, \rho) \) is a right \( C \)-comodule, then \( (M, \overline{\rho}) \) is a right \( C/I \)-comodule via \( \overline{\rho}(m) = \sum m_{<0>} \otimes m_{<1>} \), for all \( m \in M \).

**Definition 4.1** Let \( C \) be a coalgebra and \( I \) be a coideal of \( C \). A \( k \)-bilinear map \( \sigma : C \otimes C/I \rightarrow k \) is called a \( D \)-map if
\[
\sum \sigma(c_{(1)} \otimes \overline{d}) \overline{c_{(2)}} = \sum \sigma(c_{(2)} \otimes \overline{d}) \overline{c_{(1)}}
\]
for all \( c \in C, \overline{d} \in C/I \). If \( I = 0 \), then \( \sigma \) is called strongly \( D \)-map.

**Examples 4.2** 1. If \( C \) is cocommutative then any \( k \)-bilinear map \( \sigma : C \otimes C/I \rightarrow k \) is a \( D \)-map.

2. Let \( C \) be a coalgebra, \( I \) be a coideal of \( C \) and \( f \in \text{Hom}_k(C/I, k) \). Then
\[
\sigma_f : C \otimes C/I \rightarrow k, \quad \sigma_f(c \otimes \overline{d}) := \varepsilon(c)f(\overline{d})
\]
for all \( c \in C, \overline{d} \in C/I \), is a \( D \)-map. In particular, \( \sigma : C \otimes C/I \rightarrow k, \sigma(c \otimes \overline{d}) := \varepsilon(c)\varepsilon(\overline{d}) \), for all \( c \in C, \overline{d} \in C/I \), is a \( D \)-map.
3. Let $C := \mathcal{M}^n(k)$ be the comatrix coalgebra of order $n$. Let $a \in k$ be a scalar of $k$. Then the map
\[ \sigma : C \otimes C \to k, \quad \sigma(c_{ij} \otimes c_{pq}) := \delta_{ij}a \]
for all $i, j, p, q = 1, \ldots, n$ is a strongly $\mathcal{D}$-map.

Indeed, for $c = c_{ij}$, $d = c_{pq}$ we have
\[ \sum \sigma(c_{(1)} \otimes d)c_{(2)} = \sum_t \sigma(c_{it} \otimes c_{pq})c_{tj} = ac_{ij}, \]
and
\[ \sum \sigma(c_{(2)} \otimes d)c_{(1)} = \sum_t \sigma(c_{tj} \otimes c_{pq})c_{it} = ac_{ij}, \]
for all $i, j, p, q = 1, \ldots, n$.

**Proposition 4.3** Let $C$ be a coalgebra, $I$ be a coideal of $C$ and $\sigma : C \otimes C/I \to k$ a $\mathcal{D}$-map. Let $(M, \rho)$ be a right $C$-comodule. Then, the special map
\[ R_\sigma : M \otimes M \to M \otimes M, \quad R_\sigma(m \otimes n) = \sum \sigma(m_{<1>} \otimes n_{<1>})m_{<0>} \otimes n_{<0>} \]
is a solution of the $\mathcal{D}$-equation.

**Proof** Let $l, m, n \in M$ and put $R = R_\sigma$. We have
\[
R^{12}R^{23}(l \otimes m \otimes n) = R^{12}\left(\sum \sigma(m_{<1>} \otimes n_{<1>})l \otimes m_{<0>} \otimes n_{<0>}\right) \\
= \sum \sigma(m_{<1>} \otimes n_{<1>})\sigma(l_{<1>} \otimes m_{<0><1>})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
= \sum \sigma(m_{<1>}(2) \otimes n_{<1>})\sigma(l_{<1>} \otimes m_{<0>(1)})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
= \sum \sigma(l_{<1>} \otimes \sigma(m_{<1>}(2) \otimes n_{<1>})m_{<1>(1)})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
\text{(using (2))} = \sum \sigma(l_{<1>} \otimes \sigma(m_{<1>(1)} \otimes n_{<1>})m_{<1>(2)})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
= \sum \sigma(l_{<1>} \otimes m_{<2>})\sigma(m_{<1>} \otimes n_{<1>})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
\text{and} \\
R^{23}R^{12}(l \otimes m \otimes n) = R^{23}\left(\sum \sigma(l_{<1>} \otimes m_{<1>})l_{<0>} \otimes m_{<0><0>} \otimes n\right) \\
= \sum \sigma(l_{<1>} \otimes m_{<1>})\sigma(m_{<0><1>} \otimes n_{<1>})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
= \sum \sigma(l_{<1>} \otimes m_{<2>})\sigma(m_{<1>} \otimes n_{<1>})l_{<0>} \otimes m_{<0><0>} \otimes n_{<0>} \\
i.e. R_\sigma \text{ is a solution of the } \mathcal{D}-\text{equation.} \qed

**Theorem 4.4** Let $n$ be a positive integer number, $M$ be a $n$ dimensional vector space and $R \in \text{End}_k(M \otimes M)$ a solution of the $\mathcal{D}$-equation. Then:
1. There exist a coideal $I(R)$ of the comatrix coalgebra $\mathcal{M}^n(k)$ and a unique $\mathcal{D}$-map

$$\sigma : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k$$

such that $R = R_\sigma$. Furthermore, if $R$ is bijective, $\sigma$ is invertible in the convolution algebra $\text{Hom}_k(\mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R), k)$.

2. If $R\tau = \tau R$, then there exists a coalgebra $C(R)$ and a unique strongly $\mathcal{D}$-map

$$\tilde{\sigma} : C(R) \otimes C(R) \to k$$

such that $M$ has a structure of right $C(R)$-comodule and $R = R_\sigma$.

**Proof** 1. Let $\{m_1, \cdots, m_n\}$ be a basis for $M$ and $(x_{iv}^{ji})_{i,j,u,v}$ a family of scalars of $k$ such that

$$R(m_v \otimes m_u) = \sum_{i,j} x_{iv}^{ji} m_i \otimes m_j \quad \text{(21)}$$

for all $u, v = 1, \cdots, n$. Let $\mathcal{M}^n(k)$ be the comatrix coalgebra of order $n$. $M$ has a right $\mathcal{M}^n(k)$-comodule structure given by

$$\rho(m_l) = \sum_{v=1}^n m_v \otimes c_{vl}$$

for all $l = 1, \cdots, n$. Let $I(R)$ be the $k$-subspace of $\mathcal{M}^n(k)$ generated by all $o(i, j, k, l), i, j, k, l = 1, \cdots, n$. From equation (21), $I(R)$ is a coideal of $\mathcal{M}^n(k)$.

First we shall prove the uniqueness. Let $\sigma : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k$ be a $\mathcal{D}$-map such that $R = R_\sigma$. Let $u, v = 1, \cdots, n$. Then

$$R_\sigma(m_v \otimes m_u) = \sum_{i,j} \sigma((m_v)_{<1>} \otimes (m_u)_{<1>}) (m_v)_{<0>} \otimes (m_u)_{<0>} = \sum_{i,j} \sigma(c_{iv} \otimes \overline{c_{ju}}) m_i \otimes m_j$$

Hence $R_\sigma(m_v \otimes m_u) = R(m_v \otimes m_u)$ gives us

$$\sigma(c_{iv} \otimes \overline{c_{ju}}) = x_{iv}^{ji} \quad \text{(22)}$$

for all $i, j, u, v = 1, \cdots, n$. Hence, the equation (22) ensure the uniqueness of $\sigma$.

Now we shall prove the existence of $\sigma$. First we define

$$\sigma_0 : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k) \to k, \quad \sigma_0(c_{iv} \otimes c_{ju}) = x_{uv}^{ji}$$

for all $i, j, u, v = 1, \cdots, n$. In order to prove that $\sigma_0$ factorizes to a map $\sigma : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k$, we have to show that $\sigma_0(\mathcal{M}^n(k) \otimes I(R)) = 0$. For $i, j, k, l, p, q = 1, \cdots, n$, we have:

$$\sigma_0(c_{pq} \otimes o(i, j, k, l)) = \sum_v x_{kv}^{ji} \sigma_0(c_{pq} \otimes c_{vl}) - \sum_\alpha x_{kl}^{j\alpha} \sigma_0(c_{pq} \otimes c_{\alpha v})$$

$$= \sum_v x_{kv}^{ji} x_{lv}^{pq} - \sum_\alpha x_{kl}^{j\alpha} x_{\alpha v}^{pq}$$

(from (5)) = 0
Hence we have constructed \( \sigma : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k \) such that \( R = R_\sigma \). It remains to prove that \( \sigma \) is a \( \mathbb{D} \)-map. Let \( c = c_{ij}, \ d = c_{pq} \). We have:

\[
\sum \sigma(c_{(1)} \otimes d) c_{(2)} = \sum_v \sigma(c_{iv} \otimes c_{pq}) c_{vjad} = \sum_v x^{pi}_{vq} c_{vij}
\]

and

\[
\sum \sigma(c_{(2)} \otimes d) c_{(1)} = \sum_\alpha \sigma(c_{\alpha j} \otimes c_{pq}) c_{\alpha i} = \sum_\alpha x^{\alpha o}_{j} c_{\alpha i}.
\]

Hence,

\[
\sum \sigma(c_{(1)} \otimes d) c_{(2)} - \sum \sigma(c_{(2)} \otimes d) c_{(1)} = \sigma(i, p, q, j) = 0,
\]

i.e. \( \sigma \) is a \( \mathbb{D} \)-map.

Suppose now that \( R \) is bijective and let \( S = R^{-1} \). Let \( (y_{ij}^{uv}) \) be a family of scalars of \( k \) such that

\[
S(m_v \otimes m_u) = \sum_{i,j} y_{nu}^{ij} m_i \otimes m_j,
\]

for all \( u, v = 1, \ldots, n \). As \( RS = SR = \text{Id}_{\mathcal{M} \otimes \mathcal{M}} \) we have

\[
\sum_{\alpha, \beta} x^{ip}_{\alpha \beta} y^{\beta \alpha}_{jq} = \delta_{ij} \delta_{pq}, \quad \sum_{\alpha, \beta} y^{ip}_{\beta \alpha} x_{jq}^{\beta \alpha} = \delta_{ij} \delta_{pq}
\]

for all \( i, j, p, q = 1, \ldots, n \). We define

\[
\sigma_0' : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k) \to k, \quad \sigma_0'(c_{iv} \otimes c_{ju}) := y_{uv}^{ji}
\]

for all \( i, j, u, v = 1, \ldots, n \). First we prove that \( \sigma_0' \) is an inverse in the convolution algebra \( \text{Hom}_k(\mathcal{M}^n(k) \otimes \mathcal{M}^n(k), k) \) of \( \sigma_0 \). Let \( p, q, i, j = 1, \ldots, n \). We have:

\[
\sum \sigma_0((c_{pq}(1) \otimes (c_{ij}(1)) \sigma_0'(c_{pq}(2) \otimes (c_{ij}(2)) = \sum_{\alpha, \beta} \sigma_0(c_{pa} \otimes c_{\beta j}) \sigma_0'(c_{aq} \otimes c_{\beta j})
\]

\[
= \sum_{\alpha, \beta} x^{ip}_{\beta \alpha} y^{\beta \alpha}_{pq} = \delta_{ij} \delta_{pq} = \varepsilon(c_{ij}) \varepsilon(c_{pq})
\]

and

\[
\sum \sigma_0'(c_{pq}(1) \otimes (c_{ij}(1)) \sigma_0((c_{pq}(2) \otimes (c_{ij}(2)) = \sum_{\alpha, \beta} \sigma_0'(c_{pa} \otimes c_{\beta j}) \sigma_0(c_{aq} \otimes c_{\beta j})
\]

\[
= \sum_{\alpha, \beta} y^{ip}_{\beta \alpha} x^{\beta \alpha}_{pq} = \delta_{ij} \delta_{pq} = \varepsilon(c_{ij}) \varepsilon(c_{pq}).
\]

Hence, \( \sigma_0 \in \text{Hom}_k(\mathcal{M}^n(k) \otimes \mathcal{M}^n(k), k) \) is invertible in convolution. In order to prove that \( \sigma \in \text{Hom}_k(\mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R), k) \) remains invertible in the convolution, it is enough to prove that \( \sigma_0' \) factorizes to a map \( \sigma' : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k \). For \( i, j, k, l, p, q = 1, \ldots, n \) we have

\[
\sigma_0'(c_{pq} \otimes o(i, j, k, l)) = \sum_v x^{ji}_{kl} y^{vp}_{iq} - \sum_\alpha x^{ji}_{kl} y^{\alpha p}_{aq}
\]
As $S = R^{-1}$ and $R$ is a solution of the $\mathcal{D}$-equation we get that $R^{23} S^{12} = S^{12} R^{23}$. Using the equation (14) we obtain that $\sigma'(c_{pq} \otimes o(i,j,k,l)) = 0$. Hence, $\sigma'$ factorizes to a map $\sigma' : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k$.

2. Suppose now that $R \tau = \tau R$. It follows that

$$x^{ij}_{uv} = x^{ij}_{vu} \quad (23)$$

for all $i, j, u, v = 1, \ldots, n$. Let

$$C(R) : = \mathcal{M}^n(k)/I(R)$$

The rest of the proof is similar to the one in item 1. The only thing we have to prove is that the map $\sigma : \mathcal{M}^n(k) \otimes \mathcal{M}^n(k)/I(R) \to k$ factorizes to a map $\tilde{\sigma} : C(R) \otimes C(R) \to k$. We have

$$\sigma(o(i,j,k,l) \otimes \varpi_{pq}) = \sum_v x^{ij}_{kv} \sigma(c_{il} \otimes \varpi_{pq}) - \sum_{\alpha} x^{j\alpha}_{kl} \sigma(c_{\alpha o} \otimes \varpi_{pq})$$

$$= \sum_v x^{ij}_{kv} x_{ql}^{pq} - \sum_{\alpha} x^{j\alpha}_{kl} x_{\alpha q}^{pq}$$

(using (23) )

$$= \sum_v x^{ij}_{kv} x_{lq}^{pq} - \sum_{\alpha} x^{j\alpha}_{kl} x_{\alpha q}^{pq} = 0$$

for all $i, j, k, l, p, q = 1, \ldots, n$. Hence, $\sigma(I(R) \otimes \mathcal{M}^n(k)/I(R)) = 0$, i.e. $\sigma$ factorizes to a map $\tilde{\sigma} : C(R) \otimes C(R) \to k$. \hfill \Box

Applying the above theorem for the solution of the $\mathcal{D}$-equation mentioned in the previous section, we shall construct the corresponding $\mathcal{D}$-map defined for the coalgebra $C = \mathcal{M}^4(k)$.

**Examples 4.5** 1. Let $C = \mathcal{M}^4(k)$ and $I$ be the two dimensional $k$-subspace of $C$ with \{c_{21}, c_{22} - c_{11}\} a $k$-basis. Then $I$ is a coideal of $C$. Let $a$, $b$, $c$ scalars of $k$ and $(x^{uv}_{ij})$ given by the formulas (19). Then

$$\sigma : \mathcal{M}^4(k) \otimes \mathcal{M}^4(k)/I \to k, \quad \sigma(c_{iv} \otimes c_{ju}) = x^{ij}_{uv}$$

for all $i, j, u, v = 1, \ldots, 4$ is a $\mathcal{D}$-map.

2. Let $C = \mathcal{M}^4(k)$, $q \in k$ and $I$ be the two dimensional coideal of $C$ with \{c_{21}, c_{12} + qc_{22} - qc_{11}\} a $k$-basis. Let $(x^{uv}_{ij})$ be the scalars of $k$ of which

$$x^{11}_{21} = -q, \quad x^{21}_{21} = 1, \quad x^{11}_{22} = -q^2, \quad x^{21}_{22} = q.$$

and all others are zero. Then

$$\sigma : \mathcal{M}^4(k) \otimes \mathcal{M}^4(k)/I \to k, \quad \sigma(c_{iv} \otimes c_{ju}) = x^{ij}_{uv}$$

for all $i, j, u, v = 1, \ldots, 4$ is a $\mathcal{D}$-map.

3. Let $C = \mathcal{M}^4(k)$ and $I$ be the two dimensional coideal of $C$ with \{c_{12}, c_{21}\} a $k$-basis. Let $a$, $b$ $\in k$ and

$$\sigma : \mathcal{M}^4(k) \otimes \mathcal{M}^4(k)/I \to k$$

such that

$$\sigma(c_{11} \otimes c_{11}) = a, \quad \sigma(c_{11} \otimes c_{22}) = b$$

and all others are zero. Then $\sigma$ is a $\mathcal{D}$-map.
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