The Effect of Slope in the Casimir Rack Gear

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Abstract

The effect of slope for the rack gear in the massless scalar field model is considered. It appears, that the slope of profile surfaces can essentially change the value of normal Casimir force, whereas average value of tangential force remains almost unchanged. At the same time we observe essential shift of maximum and minimum tangential force positions.

1 Introduction

In recent years the Casimir effect [1] has been a subject of extremely active consideration. Experimental [2, 3, 4, 5], and theoretical results [4, 5] deal with the attractive and repulsive Casimir force, with the Casimir force between separated bodies (including anisotropic atoms) and the Casimir effect for one isolated body, Casimir effect for various topology, geometry, type of boundary condition and curvature of surfaces. For the purposes of construction of nanomechanical devices there are two the most important aspects of the Casimir effect. The first aspect is the positive sign of energy (repulsion [6]) without intermediate dielectric fluid, and the second one is the value of tangential force. We will concentrate on the second task.

In this paper we consider the effect of slope for the rack gear. For the sake of simplicity we study the case of scalar field, so one numerical calculation

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Figure 1: System parameters: $a$ — period, $u$ — length of first profile edge, $v$ — length of second profile edge, $s$ — shift, $l$ — width of the gap.
permits us to obtain results for both two-dimensional and three dimensional geometry. The standard rack gear consists of two profiled plates with identical period $a$. We assume, that the plates are parallel, whereas the surface of profiles has a slope (see Fig 1. for geometry details and parameters). Three-dimensional model implies additional dimension $z$, orthogonal to $x$ and $y$, with the translational invariance along $z$ axis. We assume the simplest zero boundary condition on the plate surface. We start with the zero slope ($u=v$), and study the dependence of both normal and tangential Casimir forces on the parameter $v$ for fixed $l$ and $u$. It is obvious, that both forces will decrease for decreasing $v$ (increasing slope), due to the enlargement of average distance between plates. But the type of the dependence for normal and tangential forces can be different. Indeed, the density of normal force (normal force per unit length along the $y$ axis for two-dimensional case and normal force corresponding to unit area at $y-z$ plane for three-dimensional case) depends essentially on the distance between plates for all $y$ positions. At the same time the tangential force isn’t equal to zero only when this distance is not constant, so for rack gear it is mostly side effect. For the geometry assumed the distance between profile edges with the length $u$ remains unchanged (it is equal to $l$), so the effect of slope for tangential force should be less significant, then for normal force. We perform the direct numerical calculations to verify all these speculations.

2 Calculation method

As is well known, the simplest method to estimate the Casimir force between two isolated bodies for the scalar field is to calculate numerically the Euclidean Green function for the surface conditions imposed [7, 8, 9]. So we’ll find the solution to the equation

$$\triangle G(\mu, x, y) - \mu^2 G(\mu, x, y) = \delta(x - y)$$

with the boundary condition

$$G(\mu, \xi, x) = 0,$$

where $\xi$ lies on the plate surface.

For any point $x$ between plates the vacuum expectation value of energy-
momentum tensor density can be calculated as

\[ \langle 0 | T_{11}(x) | 0 \rangle = \frac{1}{\pi} \int_0^\infty dq \left( \frac{1}{2} \partial_{x_1} \partial_{y_1} - \frac{1}{2} \partial_{x_2} \partial_{y_2} - \frac{1}{2} (q^2 + m^2) \right) \times \]

\[ \times G(\sqrt{q^2 + m^2}, x, y) \bigg\vert_{y=x} \] (3)

\[ \langle 0 | T_{12}(x) | 0 \rangle = \frac{1}{\pi} \int_0^\infty dq \left( \frac{1}{2} \partial_{x_1} \partial_{y_2} \right) G(\sqrt{q^2 + m^2}, x, y) \bigg\vert_{y=x} \] (4)

for two-dimensional case. Here \( m \) is the mass of the scalar field.

Both expressions appear to be singular at \( y = x \) and should be renormalized. For two distant bodies the renormalization procedure is quite trivial — it is sufficient to subtract the same expression for Minkowski space.

The expression for three-dimensional case is almost identical due to translational invariance along \( z \) axis. One can perform Fourier transformation along \( z \) axis and obtain two-dimensional task for each Fourier component. So the three-dimensional expressions take the form:

\[ \langle 0 | T_{11}(x) | 0 \rangle = \frac{1}{2\pi} \int_0^\infty dq \sqrt{q^2 + m^2} \left( \frac{1}{2} \partial_{x_1} \partial_{y_1} - \frac{1}{2} \partial_{x_2} \partial_{y_2} - \frac{1}{2} (q^2 + m^2) \right) \times \]

\[ \times G(\sqrt{q^2 + m^2}, x, y) \bigg\vert_{y=x} \] (5)

\[ \langle 0 | T_{12}(x) | 0 \rangle = \frac{1}{2\pi} \int_0^\infty dq \sqrt{q^2 + m^2} \left( \frac{1}{2} \partial_{x_1} \partial_{y_2} \right) G(\sqrt{q^2 + m^2}, x, y) \bigg\vert_{y=x} \] (6)

Finally we should integrate renormalized expressions (3)-(6) along the \( y \) axis from point \( x_1 = (x_0, y_0) \) to point \( x_2 = (x_0, y_0 + a) \), where \( x_1 = (x_0, y_0) \) — arbitrary point, that is placed in the gap between plates. For instance, for (3) (two-dimensional case, normal force):

\[ F_n = \int_0^a \langle 0 | T^{\text{ren}}_{11}(x_0, y_0 + w) | 0 \rangle \, dw. \] (7)
It is not the force density, but the force for one period; to obtain the force
density one should divide this result by the period length $a$.

To calculate the difference between Green function and Green function for
Minkowski space $G^{\text{ren}}(\mu, x, y) = G(\mu, x, y) - G(0)(\mu, x, y)$ we use a slightly
modified boundary-element method [10, 11, 12]. This difference is equal
to the solution of the homogeneous Helmholtz equation with the boundary
condition $G^{\text{ren}}(\mu, x, \xi) = -G(0)(\mu, x, \xi)$, where $\xi$ is positioned on the plate
surface. Instead of the standard spline approach for the given boundary
element we use polynomial approximation, based on surrounding elements.
It is similar, but not precisely equal to spline.

We estimate calculation errors by increasing points density and subse-
quent comparison of results. The relative error appears to be about $10^{-2}$.

### 3 Results and discussion

We use the natural system of units $\hbar = c = 1$ and choose the arbitrary
geometry parameters $a = 2$, $u = 0.5$, $l = 1$ and three values for $v = 0.5$ (zero
slope), $v = 0.4$ (medium slope), $v = 0.3$ (large slope). We should calculate
Green functions for all $\mu$ values, so we can obtain results for arbitrary mass
of the scalar field. But we restrict ourselves to the massless case $m = 0$.

From equations (3)-(6) one can easily conclude, that the mass dependence
can’t drastically modify results, so the type of dependence should remain
unchanged. Direct calculations confirm this assumption.

From Figs. 2 and 3 one can easily find, that the value of normal force
essentially decreases for increasing slope for both two- and three-dimensional
cases. This result seems quite reasonable, if we take into account, that we
change slope for fixed absolute gap width, whereas average gap width in-
creases. Moreover, this effect appears to be almost homogeneous: for differ-
cent values of shift $s$ we observe almost identical slope dependence. It also
should be noted, that there is no ”phase shift” in the $s$-dependence, i.e. the
position of maximum and minimum force values remains almost unchanged,
at $s = 0$ and $s = a/2$ correspondingly.

On the contrary, for the tangential force (Figs. 4 and 5) we observe
moderate decrease for increasing slope. Indeed, in two-dimensional case the
normal force maximum for large slope ($v = 0.3$) is about 45% of the maximum
for zero slope ($v = 0.5$), whereas the tangential force maximum for large
slope is about 85% of the maximum for zero slope. For three-dimensional
Figure 2: Normal force for one period as the function of shift $s$ for two-dimensional case. Solid line — $v = 0.5$, dashed line — $v = 0.4$, dotted line — $v = 0.3$. 
Figure 3: Normal force for one period as the function of shift $s$ for three-dimensional case. Solid line $v = 0.5$, dashed line $v = 0.4$, dotted line $v = 0.3$. 
Figure 4: Tangential force per one period as the function of shift s for two-dimensional case. Solid line — $v = 0.5$, dashed line — $v = 0.4$, dotted line — $v = 0.3$. 
Figure 5: Tangential force per one period as the function of shift $s$ for three-dimensional case. Solid line — $v = 0.5$, dashed line — $v = 0.4$, dotted line — $v = 0.3$. 
In the three-dimensional case we obtain almost the same behavior, because the expressions for three-dimensional case differ from two-dimensional case only by one additional multiplier in the integral.

It also should be noted, that there is an essential "phase shift" in the $s$-dependence, i.e. the shift of the position for maximum and minimum force values at various slopes. For instance, in the case of zero shift ($s = 0$) we get nonzero tangential force for nonzero slopes, whereas for the zero slope this force is absent due to obvious symmetry reasons. For the large slope this "phase shift" is almost equal to $a/2$ (about 40% of the period $a$). In some sense the tangential force for the zero shift and nonzero slope is partly "normal" — it can be interpreted as the projection of attracting force between parallel sloped surfaces (this force is orthogonal to the surfaces) on the $y$ axis.

4 Conclusions

Direct numerical computations lead us to the conclusion, that (at least for the geometry considered) the slope of profile surfaces can essentially change the value of normal Casimir force for the rack gear, whereas tangential force remains almost unchanged, but exhibits essential "phase shift" — shift of maximum and minimum positions. Both results are quite reasonable, because normal force is defined mostly by the width of the gap, whereas tangential force depends essentially on the change of this width. For the geometry considered the derivative of the gap width along $y$ axis depends on the slope, but average value of this derivative remains almost unchanged.

It should be mentioned, that we consider the trivial case of zero boundary condition. For Neumann or Robin boundary conditions results may change drastically, because Green functions at the presence of sharp edges (for instance, $\pi/2$ angles between flat surfaces) essentially depend on the type of boundary condition considered.

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