Choice of the best geometry to explain physics

José B. Almeida

Universidade do Minho, Physics Department, Campus de Gualtar, 4710-057 Braga, Portugal

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Choosing the appropriate geometry in which to express the equations of fundamental physics can have a determinant effect on the simplicity of those equations and on the way they are perceived. The point of departure in this paper is the geometry of 5-dimensional spacetime, where monogenic functions are studied. Monogenic functions verify a very simple first order differential equation and the paper demonstrates how they generate the line interval of special relativity, as well as the Dirac equation of quantum mechanics. Monogenic functions act as a unifying principle between those two areas of physics, which is in itself very significant for the perception one has of them. Another consequence is the possibility of studying the same phenomena in Euclidean 4-dimensional space, providing a different point of view to physics, from which one has an unusual and enriching perspective.

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I. INTRODUCTION

There is a general consensus among physicists about what is a physical theory. The point of departure is a set of principles or axioms which are unproven statements, whose validity is sustained on the consistency of the whole theory and its ability to make correct predictions. Using accepted rules of mathematics and logic it is possible to derive consequences from the set of principles, some of which are observables and can be confronted with experiment and or observation of the physical world. In general every theory has its own application domain, that is, a set of conditions where it is capable of providing verifiable predictions.

No physical theory has yet been formulated whose application domain is universal and the search for a unified theory of physics is a strong motivation for many researchers. The goal is to establish a reduced number of principles from which one could derive a formalism applicable to physics of all scales, from particles to the cosmos, and to all times, from the origin of the Universe, through the present time, allowing predictions for the Universe’s future.

In this paper we will have a brief look at how a carefully chosen point of departure in geometry can lead us to several important equations of fundamental physics, using only geometrical reasoning. We will derive equations with significance both for General Relativity and Quantum Mechanics, in order to emphasize the potential unifying power of this approach to physics.

Before we begin our exposition a notation issue must be resolved. We will be dealing with 5-dimensional space but we are also interested in two of its 4-dimensional subspaces and one 3-dimensional subspace; ideally our choice of indices should clearly identify their ranges in order to avoid the need to specify the latter in every equation. The diagram in Fig. 1 shows the index naming convention used in this paper; Einstein’s summation convention will be adopted as well as the compact notation for partial derivatives \( \partial_\alpha = \partial / \partial x^\alpha \).

II. CHOICE OF GEOMETRY

Our purpose being to establish geometrical relationships which can be read as equations of physics, the choice of an appropriate geometry becomes crucial, no less than an educated assignment of coordinates to physical entities. Here we will postulate the geometry as first principle and in doing so we will not try to find any special justification for that particular choice. The history of trials leading to the present postulates can be traced through various papers by the author; the interested reader can find guidance for such search in the webpage http://bda.planetaclix.pt although our recommendation for complementary reading is centred in three recent papers.

In the introduction we gave a hint that we would be using 5-dimensional space and this is actually true, because we have found that 5 is the smallest number of dimensions that one needs in order to find the topology and symmetries that can produce equations applicable to physics. In the scope of this paper we need 5 dimensions in order to establish a unifying principle from which both special relativity and quantum mechanics can be derived; however there are other reasons for this choice, which have to do with the incorporation of the standard model gauge group symmetries and a possible hyperspherical
symmetry of the Universe. This 5-dimensional space can be designated as 5-dimensional spacetime because one of its dimensions is associated with a frame vector with negative norm. We can now add that dimension 0 will be associated with physical or Compton time \( t \), the time measured by the Compton frequency of elementary particles, while coordinate 4 will be associated with proper or cosmological time \( \tau \); these associations will become clearer further down (see Appendix A). For the moment we will consider only geometry without any physical implications.

We will need a reference frame associated with the set of coordinates and we will thus assume an orthonormed frame of vectors, designated by \( \sigma \alpha \); these vectors are such that \( \sigma \alpha \cdot \sigma \beta = 0 \), where the dot represents the inner product and \( \alpha \neq \beta \). Furthermore, \( \sigma_0 \) has norm minus unity, given by \( (\sigma_0)^2 = \sigma_0 \cdot \sigma_0 = -1 \) and all the others have unit norm.

An elementary displacement \( dx \) is given by
\[
\mathbf{dx} = \sigma_\alpha dx^\alpha. \tag{1}
\]
As usual the square of a vector equals the square of its length, which we apply to the elementary displacement
\[
(\mathbf{dx})^2 = - (dx^0)^2 + \sum_i (dx^i)^2. \tag{2}
\]
Because we have chosen this space to have one negative norm frame vector, the length of a vector is not necessarily positive and it can even be zero; we will explore this possibility at great length.

### III. MONOGENIC FUNCTIONS AND WAVES

We will now introduce the vector derivative defined by
\[
\nabla = \sigma^\alpha \partial_\alpha; \tag{3}
\]
here we used \( \sigma^\alpha \) to represent the reciprocal frame such that \( \sigma^\alpha \cdot \sigma_\beta = \delta^\alpha_\beta \), where \( \delta^\alpha_\beta \) is the Kronecker delta. One sees easily that \( \sigma^0 = -\sigma_0 \) and \( \sigma^i = \sigma_i \). It turns out that there is a class of functions of great importance, called monogenic functions, characterized by having null vector derivative; a function \( \psi \) is monogenic if and only if
\[
\nabla \psi = 0. \tag{4}
\]
These functions are not usually scalars and we will say a bit more about them later on but for now let us define the scalar Laplacian operator \( \nabla^2 = \nabla \cdot \nabla \). The Laplacian is just the sum of second order partial derivatives with respect to all the coordinates, the term corresponding to coordinate 0 having a negative sign \( \nabla^2 = - \partial_{00} + \sum \partial_\alpha^2 \).

A monogenic function has by necessity null Laplacian, as can be seen by dotting Eq. (4) with \( \nabla \) on the left. We are then allowed to write
\[
\sum_i \partial_i \psi = \partial_{00} \psi. \tag{5}
\]
This can be recognized as a wave equation in the 4-dimensional space spanned by \( \sigma \), which will accept plane wave type solutions of the general form
\[
\psi = \psi_0 e^{i(\mathbf{p} \cdot \mathbf{x}^0 + \delta)}, \tag{6}
\]
where \( \psi_0 \) is an amplitude whose characteristics we shall not discuss for now, \( \delta \) is a phase angle and \( p_\alpha \) are constants such that
\[
\sum_i (p_i)^2 - (p_0)^2 = 0. \tag{7}
\]
By setting the argument of \( \psi \) constant in Eq. (6) and differentiating we can get the differential equation
\[
p_\alpha dx^\alpha = 0. \tag{8}
\]
The first member can equivalently be written as the inner product of the two vectors \( p \cdot dx = 0 \), where \( p = \sigma^\alpha p_\alpha \). In 5D hyperbolic space the inner product of two vectors can be null when the vectors are perpendicular but also when the two vectors are null; since we have established that \( p \) is a null vector, Eq. (8) can be satisfied either by \( dx \) normal to \( p \) or by \( (dx)^2 = 0 \). In the former case the condition describes a 3-volume called wavefront and in the latter case it describes the wave motion. Notice that the wavefronts are not surfaces but volumes, because we are working with 4-dimensional waves.

The condition describing wave motion can be expanded as
\[
- (dx^0)^2 + \sum_i (dx^i)^2 = 0. \tag{9}
\]
This is a purely scalar equation and can be manipulated as such, which means we are allowed to rewrite it with any chosen terms in the second member; some of those manipulations are particularly significant. Suppose we decide to isolate \( (dx^4)^2 \) in the first member: \( (dx^4)^2 = (dx^0)^2 - \sum_i (dx^i)^2 \). We can then rename coordinate \( x^4 \) as \( \tau \), to get the interval squared of special relativity for space-like displacements
\[
d\tau^2 = (dx^0)^2 - \sum_i (dx^i)^2. \tag{10}
\]
We have thus derived the space-like part of special relativity as a consequence of monogeneity in 5D hyperbolic space and simultaneously found a physical interpretation for coordinates \( x^3 \) and \( x^4 \) as time and proper time, respectively.

A different manipulation of Eq. (9) has great significance because it leads to the concept of 4-dimensional optics (4DO). If we isolate \( (dx^0)^2 \) and replace \( x^0 \) by the letter \( t \), we see that time becomes the interval in Euclidean 4D space
\[
dt^2 = \sum (dx^i)^2. \tag{11}
\]
From this we conclude that the monogenic condition produces plane waves whose wavefronts are 3D volumes but
can be represented by wavefront normals, just as it happens in standard optics with electromagnetic waves.

Several readers may be worried with the fact that proper time is a line integral and not a coordinate in special relativity; to this we will argue that the manipulations we have done, collapsing 5D spacetime into 4 dimensions through a null displacement condition and then promoting one of the coordinates into interval, is exactly equivalent to the process of defining a light cone in Minkowski spacetime and then applying Fermat’s principle to define an Euclidean 3D metric on the light cone; we have just upgraded the procedure by including one extra dimension.

IV. QUANTUM MECHANICS AND THE DIRAC EQUATION

The Dirac equation can also be derived from the monogenic condition but since it appears formulated in terms of matrices in all textbooks we will have to rewrite Eq. 4 also in terms of matrices, so that it can then be further manipulated. This is easily achieved if we assign our frame vectors to Dirac matrices that square to the identity matrix or minus the identity matrix as appropriate; the following list of assignments can be used but others would be equally effective.  

\[ \sigma^0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \sigma^2 \equiv \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \sigma^4 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}. \]  

(12)

There is no need to adopt different notations to refer to the frame vectors or to their matrix counterparts because the context will usually be sufficient to determine what is meant.

We can check that matrices \( \sigma^\alpha \) form an orthonormal basis of 5D space by defining the inner product of square matrices as

\[ A \cdot B = \frac{AB + BA}{2}. \]  

(13)

It will then be possible to verify that the inner product of any two different \( \sigma \)-matrices is null, \( (\sigma^0)^2 = -I \) and \( (\sigma^1)^2 = I \); these are the conditions defining an orthonormal basis expressed in matrix form.  

It will now be convenient to expand the monogenic condition as \( (\sigma^\mu \partial_\mu + \sigma^4 \partial_4) \psi = 0 \). If this is applied to the solution \( \psi_0 \) and the derivative with respect to \( x^4 \) is evaluated we get

\[ (\sigma^\mu \partial_\mu + \sigma^4 ip_4) \psi = 0. \]  

(14)

Let us now multiply both sides of the equation on the left by \( \sigma^4 \) and note that matrix \( \sigma^4 \sigma^\mu \) squares to the identity while the 3 matrices \( \sigma^4 \sigma^\mu \) square to minus identity; we rename these products as \( \gamma \)-matrices in the form \( \gamma^\mu = \sigma^4 \sigma^\mu \). Rewriting the equation in this form we get

\[ (\gamma^\mu \partial_\mu + ip_4) \psi = 0. \]  

(15)

The only thing this equation needs to be recognized as Dirac’s is the replacement of \( p_4 \) by the particle’s mass \( m \).

We turn now our attention to the amplitude \( \psi_0 \) in Eq. 4 because we know that the Dirac equation accepts solutions which are spinors and we want to find out their equivalents in our formulation. Applying the monogenic condition to Eq. 4 we see that the following equation must be verified

\[ \psi_0 (\sigma^\alpha p_\alpha) = 0. \]  

(16)

If the \( \sigma \)s are interpreted as matrices, remembering that \( p \) is null, the only way the equation can be verified is by \( \psi_0 \) being some constant multiplied by the matrix in parenthesis, which is a matrix representation of \( p \). We can set the multiplying constant to unity and \( \psi_0 \) becomes equal to \( p \); the wavefunction \( \psi \) can then be interpreted as a Dirac spinor.

In order to separate left and right spinor components we use a technique adapted from Ref. 4. We choose an arbitrary \( 4 \times 4 \) matrix which squares to identity, for instance \( \sigma_1 \), with which we form the two idempotent matrices \( (I + \sigma_1)/2 \) and \( (I - \sigma_1)/2 \). These matrices are called idempotents because they reproduce themselves when squared. These idempotents absorb any \( \sigma_4 \) factor; as can be easily checked \( (I + \sigma_1)\sigma_4 = (I + \sigma_4) \) and \( (I - \sigma_4)\sigma_4 = -I \).

Obviously we can decompose the wavefunction \( \psi \) as

\[ \psi = \psi_1 + \psi_2 = \frac{I + \sigma_4}{2} \psi_1 + \frac{I - \sigma_4}{2} \psi_2 = \psi_+ + \psi_. \]  

(17)

This apparently trivial decomposition produces some surprising results due to the following relations

\[ e^{i\theta}(I + \sigma_4) = (\cos \theta + i \sin \theta)(I + \sigma_4) = (I \cos \theta + i \sigma_4 \sin \theta)(I + \sigma_4) = e^{i\sigma_4 \theta}(I + \sigma_4), \]  

and similarly

\[ e^{i\theta}(I - \sigma_4) = e^{-i\sigma_4 \theta}(I - \sigma_4). \]  

(18)

The different idempotents produce similar results and it has been argued that they may be related to different elementary particles.
V. CONCLUSION

The choice of an adequate geometry to write the fundamental equations of physics is important in order to make the equations as simple as possible. The perception one has of those equations is also greatly dependent on the geometry and on the assignment between coordinates and physical entities. This paper makes use of 5-dimensional spacetime and studies a special class of functions, called monogenic functions, demonstrating that this is sufficient for the derivation of equations in special relativity and quantum mechanics and can thus be seen as a unifying principle between those two areas of physics.

Monogenic functions in 5D spacetime produce consequences to 4D Euclidean space and Minkowski spacetime simultaneously, which provides two different points of view from which to perceive the physical meaning of the solutions. For instance, solutions that can be interpreted as Dirac particles in Minkowski spacetime are also 4D "plane like" waves in Euclidean space; although the paper does not explore this latter point of view, other works by the author have shown that it provides a different and interesting perception of quantum mechanics.

APPENDIX A: NON-DIMENSIONAL UNITS

The interpretation of $t$ and $\tau$ as time coordinates implies the use of a scale parameter which is naturally chosen as the vacuum speed of light $c$. We don’t need to include this constant in our equations because we can always recover time intervals, if needed, introducing the speed of light at a later stage. We can even go a step further and eliminate all units from our equations so that they become pure number equations; in this way we will avoid cumbersome constants whenever coordinates have to appear as arguments of exponentials or trigonometric functions. We note that, at least for the macroscopic world, physical units can all be reduced to four fundamental ones: we can, for instance, choose length, time, mass and electric charge as fundamental, as we could just as well have chosen others. Measurements are then made by comparison with standards; of course we need four standards, one for each fundamental unit. But now note that there are four fundamental constants: Planck constant ($\hbar$), gravitational constant ($G$), speed of light in vacuum ($c$) and proton electric charge ($e$), with which we can build four standards for the fundamental units. Table I lists the standards of this units’ system, frequently called Planck units, which the author prefers to designate by non-dimensional units. In this system all the fundamental constants, $\hbar$, $G$, $c$, $e$, become unity, a particle’s Compton frequency, defined by $\nu = mc^2/\hbar$, becomes equal to the particle’s mass and the frequent term $GM/(c^2r)$ is simplified to $M/r$. We can, in fact, take all measures to be non-dimensional, since the standards are defined with recourse to universal constants; this will be our posture. Geometry and physics become relations between pure numbers, vectors, bivectors, etc. and the geometric concept of distance is needed only for graphical representation.

TABLE I: Standards for non-dimensional units’ system

| Length | Time | Mass | Charge |
|--------|------|------|--------|
| $\sqrt{\hbar c}$ | $\sqrt{\hbar c}$ | $\sqrt{\hbar c}$ | $e$ |

* Electronic address: bda@fisica.uminho.pt

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9. For any space it is possible to define a set of orthogonal frame vectors in a number equal to the the number of spatial dimensions; if one of these frame vectors has negative length, the associated dimension is called a time dimension. The frame is orthonormed when all its vectors have length $\pm 1$. Alternatively one can also have only one positive length frame vector, in which case this corresponds to the time dimension.
10. There are 16 possible $4 \times 4$ Dirac matrices of which we must choose 5 such that $(\sigma_0)^2 = -I$, $(\sigma_i)^2 = I$ and $\sigma_i \sigma_\beta = -\sigma_\beta \sigma_i$, for $\alpha \neq \beta$.
11. A more formal approach to this subject would lead us to invoke the isomorphism between the complex algebra of $4 \times 4$ matrices and Clifford algebra $\mathcal{C}_{4,1}$, the geometric algebra of 5D spacetime.