NONLINEAR STRUCTURE FORMATION, BACKREACTION AND WEAK GRAVITATIONAL FIELDS

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Abstract. There is an ongoing debate in the literature concerning the effects of averaging out inhomogeneities ("backreaction") in cosmology. In particular, some simple models of structure formation studied in the literature seem to indicate that the backreaction can play a significant role at late times, and it has also been suggested that the standard perturbed FLRW framework is no longer a good approximation during structure formation, when the density contrast becomes nonlinear. In this work we use Zalaletdinov’s covariant averaging scheme (macroscopic gravity or MG) to show that as long as the metric of the Universe can be described by the perturbed FLRW form, the corrections due to averaging remain negligibly small. Further, using a fully relativistic and reasonably generic model of pressureless spherical collapse, we show that as long as matter velocities remain small (which is true in our model), the perturbed FLRW form of the metric can be explicitly recovered. Together, these results imply that the backreaction remains small even during nonlinear structure formation, and we confirm this within the toy model with a numerical calculation.

1 Introduction

The application of general relativity (GR) to the large length scales relevant in cosmology, necessarily requires an averaging operation to be performed on the Einstein equations (Ellis 1984). The nonlinearity of GR then implies that such an averaging will modify the Einstein equations on these large scales. Symbolically, this happens since $E(\langle g \rangle) \neq \langle E[g] \rangle$, where $g$ is the metric and $E$ the Einstein tensor.

In recent times, attention has been focused on two promising candidates for a consistent nonperturbative averaging procedure, namely the spatial averaging of scalars (Buchert 2000) and the covariant approach known as macroscopic gravity (MG)(Zalaletdinov 1992, 1993). The magnitude of the corrections has been

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debated in the literature (see e.g. Kolb et al. 2006; Ishibashi & Wald 2006; Rasanen 2006; Hirata & Seljak 2005). Broadly speaking, due to the structure of the Riemann tensor, the modification terms $C$ in any averaging approach, will have a symbolic structure given by

$$C \sim \langle \tilde{\Gamma}^2 \rangle - \langle \tilde{\Gamma} \rangle^2,$$

where $\Gamma$ denotes the Christoffel connection, and the tilde represents any processing of the Christoffels required by the averaging operation.

In this talk we will work within the MG framework of Zalaletdinov, since this framework allows us to consistently define an averaged metric, which we will assume to be of the flat FLRW form

$$ds^2_{\text{FLRW}} = -d\tau^2 + a(\tau)^2 d\vec{x}^2.$$ (1.2)

Specifically, we will work in the spatial averaging limit of MG, which is relevant to cosmology (Paranjape & Singh 2007). We will also assume, consistently with observations of the cosmic microwave background (CMB), that the early universe was well described by a metric which is a perturbation around the FLRW form, given by

$$ds^2 = -(1 + 2\varphi)d\tau^2 + a(\tau)^2(1 - 2\psi)d\vec{x}^2.$$ (1.3)

The main argument of this talk is that although technically possible, in the real world it is extremely unlikely that backreaction significantly influences the average cosmological expansion. In section 2 we will detail the calculation of the backreaction during the linear regime of perturbation theory, and mention some lessons that one can learn from this calculation. In section 3 we will describe a toy model of fully relativistic and nonlinear structure formation, and demonstrate that the metric for this model can be explicitly brought to the perturbed FLRW form (1.3), even during the nonlinear phase of the evolution. A calculation of the backreaction along similar lines as in the linear case then shows that the backreaction in this toy model remains small even at late times, thus supporting our argument.

2 Building the argument

2.1 The linear regime

The expression in (1.1) can be used to obtain a simple estimate for the backreaction, assuming a perturbed FLRW metric of the form (1.3) with $\varphi = \psi$ in the absence of anisotropic stresses. We then have $C \sim a^{-2} \langle \nabla \varphi \cdot \nabla \psi \rangle$. Assuming a two component flat background consisting of cold dark matter (CDM) and radiation (known as standard CDM or sCDM), and taking the averaging to be an ensemble average over the initial conditions, it is not difficult to show that in the matter dominated era, this estimate leads to $C \sim 10^{-4} H_0^2 / a(\tau)^2$. This indicates...
that at least for epochs around the last scattering epoch, the backreaction due to averaging was negligible.

The real situation is somewhat more complex than this simple calculation indicates. On the one hand, the time evolution of \( a(\tau) \) is needed in order to solve the equations satisfied by the perturbations, as we effectively did above by assuming a form for \( a(\tau) \). On the other hand, the evolution of the perturbations is needed to compute the correction terms \( C \). Until these corrections are known, the evolution of the scale factor cannot be determined; and until we know this evolution, we cannot solve for the perturbations.

To break this circle, we will adopt an iterative procedure. We first compute a “zeroth iteration” estimate for the backreaction, by assuming a fixed standard background \( a^{(0)} \) such as sCDM, evolve the perturbations and compute the time dependence of the objects \( C \), denoted \( C^{(0)} \). Now, using these known functions of time, we form a new estimate for the background \( a^{(1)} \) using the modified equations, and hence calculate the “first iteration” estimate \( C^{(1)} \). This process can then be repeated, and is expected to converge as long as perturbation theory in the metric remains a valid approximation.

With these ideas in mind, we can go ahead and compute the “zeroth iteration” estimate for the backreaction in the Zalaletdinov framework. This is given by the following equations (Paranjape 2008)

\[
\left( \frac{1}{a} \frac{da}{d\tau} \right)^2 = \frac{8\pi G_N}{3} \bar{\rho} - \frac{1}{6} \left[ P^{(1)} + S^{(1)} \right],
\]

\[
\frac{1}{a} \frac{d^2 a}{d\tau^2} = -\frac{4\pi G_N}{3} (\bar{\rho} + 3\bar{p}) + \frac{1}{3} \left[ P^{(1)} + P^{(2)} + S^{(2)} \right],
\]

where we have defined

\[
P^{(1)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) (\Phi_k')^2,
\]

\[
S^{(1)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) (k^2 \Phi_k^2),
\]

\[
P^{(1)} + P^{(2)} = -\frac{8H}{a^2} \int \frac{dk}{2\pi^2} P_{\varphi i}(k) (\Phi_k \Phi_k'),
\]

\[
S^{(2)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) \Phi_k'' \left( \Phi_k - \frac{2H}{k^2} \Phi_k' \right).
\]

Here, the prime denotes a derivative wrt. conformal time \((\prime \equiv \partial_\eta = a\partial_\tau)\), with \( H = a'/a \), \( \Phi_k \) is the Fourier space transfer function defined by \( \varphi(\vec{k},\eta) = \varphi_{\vec{k}} \Phi_k(\eta) \), and \( P_{\varphi i}(k) \) is the initial power spectrum of \( \varphi \). The results of a numerical calculation are shown in figure 1 where all functions are normalised by the Hubble parameter \( H^2(a) \), and confirm that this zeroth iteration estimate in fact gives a negligible contribution. Further, were we to carry out the next iteration, we would essentially obtain no difference between the zeroth and first iteration scale factors up to the accuracy of the calculation, and hence this iteration has effectively converged at the first step itself.
Fig. 1. The backreaction for the sCDM model, normalised by $H^2(a)$. $S^{(1)}$, $P^{(1)}$ and $S^{(2)}$ are negative definite and their magnitudes have been plotted. The vertical line marks the epoch of matter radiation equality $a = a_{eq}$.

2.2 Lessons from linear theory

We see that the dominant contribution to the backreaction at late times, is due to a curvature-like term $\sim a^{-2}$, as expected from our simple estimate above. In order to obtain a correction which grows faster than this, we need a nonstandard evolution of the metric potential $\varphi$, which can only happen if the scale factor evolves very differently from the sCDM model, which in turn would require a significant contribution from the backreaction. The same circle of dependencies as before, now implies that as long as the metric is perturbed FLRW, the backreaction appears to be dynamically suppressed.

Secondly, as figures 2 and 3 show, scales which are approaching nonlinearity, do not contribute significantly to the backreaction, which is a consequence of the suppression of small scale power by the transfer function. We will return to this point when discussing the backreaction during epochs of nonlinear structure formation.

3 The nonlinear regime

Let us now ask whether one can make meaningful statements concerning the backreaction during epochs of nonlinear structure formation, when matter density contrasts become very large and perturbation theory in the matter variables has broken down. We begin by considering some order of magnitude estimates.

3.1 Dimensional arguments, and why they fail

Let us start with the assumption that although the matter perturbations are large, one can still expand the metric as a perturbation around FLRW. We are looking for either self-consistent solutions using this assumption, or any indication that this
**Fig. 2.** The dimensionless integrand of $S^{(1)}$, namely the function $(k/H_0)^2 \Phi^2_k$, at three sample values of the scale factor. The function dies down rapidly for large $k$, with the value at some $k$ being progressively smaller with increasing scale factor. The declining behaviour of the curves for $a = a_{eq}$ and $a = 200a_{eq}$ extrapolates to large $k$.

**Fig. 3.** The dimensionless CDM density contrast. Together with figure 3 this shows that nonlinear scales do not impact the backreaction integrals significantly.

assumption is not valid. Given that the metric has the form (1.3) (and further assuming $\varphi = \psi$ as before), the relevant gravitational equation at late times and at length scales small comparable to $H^{-1}$, is the Poisson equation given by

\[
\frac{1}{a^2} \nabla^2 \varphi = 4\pi G \bar{\rho} \delta,
\]

where $\delta \equiv (\rho(t, \vec{x})/\bar{\rho}(t) - 1)$ is the density contrast of CDM. As before, we can estimate the dominant backreaction component to be $C \sim a^{-2} \langle \nabla \varphi \cdot \nabla \varphi \rangle$.

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We are only worried about the small, sub-Hubble scales, since larger scales are well described by linear theory where we know the form of the backreaction.
Now, for an over/under-density of physical size $R$, treating $a^{-1}\nabla \sim R^{-1}$ and $G\bar{\rho} \sim H^2$ on dimensional grounds, we have

$$|\varphi| \sim (HR)^2 |\delta|.$$  \hfill (3.2)

For voids, we can set $\delta \sim -1$, and then $C \sim H^2 (HR)^2 \ll H^2$, since we have assumed $HR \ll \frac{1}{a}$. This shows that sub-Hubble underdense voids are expected to give a negligible backreaction. For overdense regions we need to be more careful, since here $\delta$ can grow very large. In a typical spherical collapse scenario, the following relations hold,

$$R \sim (1 - \cos u)r; \quad H^{-1} \sim (G\bar{\rho})^{-1/2} \sim t \sim H^{-1}_0 (u - \sin u),$$  \hfill (3.3)

$$G\rho \sim \frac{(H_0 r)^2}{R^2 R'} \sim \frac{H_0^2}{(1 - \cos u)^3}; \quad \delta \sim \frac{(\rho/\bar{\rho})}{(1 - \cos u)^3},$$  \hfill (3.4)

which lead to

$$|\varphi| \sim \frac{(H_0 r)^2}{(1 - \cos u)}; \quad C \sim H^2 \left[\left(\frac{H_0 r}{(1 - \cos u)^3}\right)^2 \frac{(u - \sin u)^2}{1 - \cos u}\right].$$  \hfill (3.5)

It would therefore appear that at late enough times, the perturbative expansion in the metric breaks down with $|\varphi| \sim 1$, and the backreaction grows large $|C| \sim 1$. However, the crucial question one needs to answer is the following: Is this situation actually realised in the universe, or are we simply taking these models too far? We make the claim that perturbation theory in the metric does not break down at late times, since observed peculiar velocities remain small. The spherical collapse model is not a good approximation when model peculiar velocities in the collapsing phase grow large. To support this claim, we will work with an exact toy model of spherical collapse.

### 3.2 Calculations in an exact model

The model we consider was presented by Paranjape & Singh (2008a), and can be summarized as follows. The matter content of the model is spherically symmetric “dust”, and hence the relevant exact solution is the Lemaître-Tolman-Bondi (LTB) metric given by

$$ds^2 = -dt^2 + \frac{R'^2 dr^2}{1 - k(r)r^2} + R^2 d\Omega^2.$$  \hfill (3.6)

Here $t$ is the proper time measured by observers with fixed coordinate $r$, which is comoving with the dust. $R(t, r)$ is the physical area radius of the dust shell labelled by $r$, and satisfies the equation $R^2 = 2GM(r)/R - k(r)r^2$. Here $M(r)$ is the mass contained inside each comoving shell, and a dot denotes a derivative with

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*I thank Karel Van Acoleyen for pointing this out to me.*
Fig. 4. Density contrast in the LTB toy model, at $r = 8.35$ Mpc. $t_0 = 2/3H_0 \approx 9$ Gyr.

respect to the proper time $t$. The energy density of dust measured by an observer comoving with it satisfies the equation $\rho(t, r) = M'(r)/4\pi R^2 R'$, where the prime now denotes a derivative with respect to the LTB radius $r$.

Initial conditions are set at a scale factor value of $a_i = 10^{-3}$, and are chosen such that the initial situation describes an FLRW expansion with a perturbative central overdensity out to radius $r = r_*$, surrounded by a perturbative underdensity out to radius $r = r_v$, with appropriately chosen values for the various parameters in the model (see Table 1 of Paranjape & Singh [2008a]). Figure 4 shows the evolution of the overdensity contrast in the central region. Clearly, at late times the situation is completely nonlinear. Nevertheless, it can be shown that a coordinate transformation in this model can bring its metric to the form (1.3), provided one has $|av| \ll 1$ where $v = \partial_t(R/a)$. Physically $v$ is the “comoving” peculiar velocity. The metric potentials (which are actually equal at the leading order, see Van Acoleyen [2008]) have the expressions $\varphi = -\xi^0 + (1/2)(av)^2$, $\psi = \xi^0 H + \xi$, where $\xi$ and $\xi^0$ are obtained by integrating $\xi' = (1/2)(k(r)r^2 + (av)^2)(R'/R)$ and $\xi'^0 = avR'$. A numerical calculation shows that $av$ and hence the metric potentials do in fact remain small for the entire evolution, for this model. Further, the infall peculiar velocity can only become large if the true infall velocity $\dot{R}$ is large, in which case the specific background chosen to define the peculiar velocity, becomes irrelevant (since $HR \ll 1$). Hence, the fact that relativistic infall velocities are not observed in real clusters etc., leads us to expect very generally that the perturbed FLRW form for the metric should in fact be recoverable even at late times.

Finally, figure 5 shows the dominant contribution to the backreaction in the toy model (Paranjape & Singh [2008b]). There is a significant departure from a curvature-like behaviour, due to evolution of the metric potentials. More importantly, the maximum value of the backreaction here is $\sim 10^{-6}H^2$, as opposed to $\sim 10^{-4}H^2$ as seen in the linear theory. This can be understood by noting that the inhomogeneity of our toy model is only on relatively small, nonlinear scales ($\lesssim 20h^{-1}$ Mpc), and the value of the backreaction is therefore consistent with our
earlier observation that nonlinear scales contribute negligibly to the total backreaction. To conclude, we have seen that as long as the metric has the perturbed FLRW form, the backreaction remains small. Further, there are strong reasons to expect that the metric remains a perturbation around FLRW even at late times during nonlinear structure formation, a claim that is supported by our toy model calculation. It should be possible to test this claim in $N$-body simulations as well.

It appears therefore, that backreaction cannot explain the observed acceleration of the universe.

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