On the evolution of classic charged particles

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Abstract. In this work, we obtain a closed system of equations for the non-probabilistic microscopic distribution functions of charged point particles moving in the electromagnetic field generated by them. The laws of change of both the total energy of a system of charged point particles and its momentum are also derived. All calculations have been performed in the covariant form. The resulting equations describe an irreversible evolution.

1. Introduction

It is known that the total energy of a system of point charges and an electromagnetic field is conserved. The energy of the charged particles themselves changes over time. The nontrivial interaction of particles and the field leads to the irreversible evolution of a particle system. To clarify the full picture of the evolution of a system of point charges, it is reasonable to excluded electromagnetic field variables in the equations of motion. In this way, one can obtain a closed system of equations for the microscopic distribution functions of point particles moving in the electromagnetic field generated by them.

All interparticle interactions in classical physics have an electromagnetic nature. Thus the dynamics of a many-particles system is determined by the interaction between particles and an electromagnetic field. The solutions of the field equations are determined by the retarded and advanced Green functions. Physically significant solutions are associated with a retarded Green function [1, 2]. The inclusion of only retarded effects allows one to obtain irreversible equations of motion for a system of many particles. Thus, molecular chaos hypothesis could be replaced by the retarded effect in the interaction between particles [3, 4].

In a present work we obtained manifestly covariant equations for the non-probabilistic microscopic distribution functions. The resulting equations form exact closed system of equations with excluded field variables. We also showed that the equation for the rate of change of particle energy demonstrates the irreversibility of the dynamics of a system of many particles.

2. The covariant microscopic distribution function

We introduce the microscopic distribution function for particles of type $A$:

$$F_A(x, p) = \sum_a \int \delta^4 \left( x - x_{at} \left( \tau_{at} \right) \right) \delta^4 \left( p - p_{at} \left( \tau_{at} \right) \right) d\tau_{at} \ .$$

(1)
Here $x$ is the space-time four-vector $(x^0, \mathbf{r}) = (ct, \mathbf{r})$, $p$ is the energy-momentum four-vector $(p^0, \mathbf{p}) = \gamma m(c, \mathbf{v})$, $d\tau_a = dt/\gamma_a$, $\gamma = 1/\sqrt{1-(\mathbf{v}/c)^2}$ is the Lorentz factor, $t$ is coordinate time, $\delta^4(x-x(t)) = \delta^4(\mathbf{r} - \mathbf{r}(t))\delta(x^0 - ct)$ is a delta function in four-dimensional Minkowski space-time, $x_{aa}(\tau_{aa})$ is the world line of a charged particle.

The idea of introducing such an invariant construction (1) belongs to R.L. Stratonovich. The relation of $F_a (x, p)$ to the classical Klimontovich microscopic phase density

$$f_A (r, p, t) = \sum_a \delta^4(\mathbf{r} - \mathbf{r}_a (t)) \delta^4(p - \mathbf{p}_a (t))$$

is given by

$$\int p^0 dp^0 F_a (x, p) = m_a f_A (r, p, t).$$

We now write the 4-vector of current density using the microscopic distribution function (1):

$$j^\alpha (x) = \langle e \rho (r, t), J(r, t) \rangle = e \sum_{a=1}^N \langle e u_a^\alpha \delta^4(x - x_a (\tau_a)) \rangle d\tau_a =$$

$$e \sum_{a=1}^N \int d^4 p u_a^\alpha \delta^4(x - x_a (\tau_a)) \delta^4(p - p_a (\tau_a)) d\tau_a = e \sum_{a=1}^N \sum_{m_a} e_{ma} \int d^4 p p^\alpha F_a (x, p).$$

Here $e_a$ is charge of $a$-th particle, $u_a^\alpha$ is a four-velocity, $\rho (r, t) = \sum_{a=1}^N \langle e \delta^3(\mathbf{r} - \mathbf{r}_a) \rangle = j^0 (x)$ is the charge density, $J(r, t) = \sum_{a=1}^N \langle e \mathbf{v}_a \delta^3(\mathbf{r} - \mathbf{r}_a) \rangle$.

Similarly, any additive function $\sum_a g(x_a, p_a)$ can be expressed in terms of a distribution functions $F_a (x, p)$. It is important that the determination of the distribution function (1) does not require any averaging. Nevertheless, it is customary to carry out additional averaging over the Gibbs ensemble [5, 6].

### 3. Evolution of the covariant distribution function

We obtain the evolution equation for microscopic distribution function $F_a (x, p)$. First of all, note that there is an identity

$$\int \sum_{a=1}^N \frac{d}{d\tau_a} \left[ \delta^4(x - x_a (\tau_a)) \delta^4(p - p_a (\tau_a)) \right] d\tau_a = 0. \tag{5}$$

Then we take into account that

$$\frac{d}{d\tau_a} \left[ \delta^4(x - x_a (\tau_a)) \delta^4(p - p_a (\tau_a)) \right] =$$

$$- u_a^\alpha \frac{\partial}{\partial x^\alpha} \delta^4(x - x_a (\tau_a)) \delta^4(p - p_a (\tau_a)) - \frac{dp_a^\alpha}{d\tau_a} \delta^4(x - x_a (\tau_a)) \frac{\partial}{\partial p^\alpha} \delta^4(p - p_a (\tau_a)). \tag{6}$$

Then equation (5) takes the form

$$\frac{\partial}{\partial x^\alpha} \sum_{a=1}^N \frac{p_a^\alpha}{m_a} \delta^4(x - x_{aa} (\tau_{aa})) \delta^4(p - p_{aa} (\tau_{aa})) d\tau_{aa} +$$

$$\frac{\partial}{\partial p^\alpha} \sum_{a=1}^N \frac{dp_a^\alpha}{d\tau_a} \delta^4(x - x_{aa} (\tau_{aa})) \delta^4(p - p_{aa} (\tau_{aa})) d\tau_{aa} = 0. \tag{7}$$

The first term can be represented as
\[
\frac{\partial}{\partial x^\mu} \sum_a \frac{p_a^\mu}{m_a} \delta^4 (x - x_{a \alpha} (\tau_{a \alpha})) \delta^4 (p - p_{a \alpha} (\tau_{a \alpha})) \, d\tau_{a \alpha} = \frac{p^\mu}{m_a} \frac{\partial F_A (x, p)}{\partial x^\mu}.
\] (8)

Taking into account the Lorenz equation
\[
m_a \frac{du^\mu}{d\tau_a} = e_a \frac{F^\mu}{c} u_{a \mu}
\] (9)

we rewrite the second term in (7) in the form
\[
\frac{\partial}{\partial p^\nu} \sum_a \frac{dp_a^\nu}{d\tau_a} \delta^4 (x - x_{a \alpha} (\tau_{a \alpha})) \delta^4 (p - p_{a \alpha} (\tau_{a \alpha})) \, d\tau_{a \alpha} = \frac{e_a}{cm_a} F^{\nu \alpha} (x) p_\mu \frac{\partial F_A (x, p)}{\partial p^\nu}.
\] (10)

Here \( F^{\nu \alpha} (x) = \partial^\nu A^\alpha - \partial^\alpha A^\nu \) is the electromagnetic field tensor, \( A^\mu \) is the 4-vector potential. As a result, the evolution equation for the microscopic distribution function \( F_A (x, p) \) will take the final form
\[
p^\mu \frac{\partial F_A (x, p)}{\partial x^\mu} + \frac{e_a}{c} F^{\nu \alpha} (x) p_\mu \frac{\partial F_A (x, p)}{\partial p^\nu} = 0.
\] (11)

It should be emphasized that the system of equations (10) for particles of type \( A \) contains not averaged distribution functions and therefore the dynamics of a particle system is completely deterministic. Since the electromagnetic field tensor is determined by the evolution of particles, we can write that
\[
F^{\mu \nu} = F^{\nu \alpha} (x; [F_A]_{\alpha}^{\nu \alpha}).
\] (12)

To obtain an expression (12), field variables must be expressed in terms of particle variables.

4. The relationship of electromagnetic fields with microscopic distribution function

Let us consider the Maxwell equations
\[
\partial_a F^{a \beta} = \frac{4\pi}{c} j^\beta.
\] (13)

Equations (13), with the choice of gauge \( \partial_a A^a = 0 \) reduces to
\[
\Box A^a = \frac{4\pi}{c} j^a.
\] (14)

Retarded solution of the equation (14) can be written as
\[
A^a (x) = -\frac{1}{c} \int \frac{d^4 k}{4\pi^2} \int e^{-i k \cdot (x - x')} \frac{1}{k^2 k_a + i\epsilon k} j^a (x') d^4 x'.
\] (15)

Here \( k = (\omega/c,k) \) is wave four-vector. The value \( \epsilon \) determines the rule of bypassing the pole. Let us introduce the Fourier transform of a 4-vector current density
\[
j^a (k) = \int j^a (x') \exp (ik \cdot x') d^4 x'.
\] (16)

Take into account (16) and (15) we rewrite the electromagnetic field tensor in the form
\[
F^{a \beta} (x) = \int \frac{d^4 k}{4\pi^2} \int e^{-i k \cdot x'} \frac{1}{k^a k^\beta + i\epsilon k} [k^a j^\beta (k) - k^\beta j^a (k)].
\] (17)

In view of (4), the Fourier transform of a 4-vector current density (16) can be represented as
\[
j^a (k) = \sum A \frac{e}{m_A} \int d^4 pp' \int \exp (ik \cdot x') F_A (x', p) d^4 x'.
\] (18)

The equation (17) and (18) determines the relationship between the variables of the electromagnetic field \( F^{a \beta} (x) \) and the variables \( F_A (x, p) \) of point charged particles.
5. Complete system of relativistic kinetic equations

Equation (11) together with equations (17) and (18) determine a closed system of covariant kinetic equations for the microscopic distribution functions $F_A(x, p)$:

$$p^{\alpha} \frac{\partial F_A(x, p)}{\partial x^{\alpha}} + \frac{i}{\hbar} \sum_B \sum_{\mu, \nu} \frac{e_B}{m_B} \int d^4x' \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik' \cdot (x' - x)}}{4\pi^3 k'^{\mu} k'^{\nu}} \left[ (-1)^{\mu+\nu} k' k'' - k'^{\mu} k''^{\nu} \right] = 0.$$  

(19)

Given equality (3), equations for the classical microscopic density [4] can be obtained from the system of equations (19).

The resulting system (19) describes the irreversible dynamics of a system of charged particles. Moreover, it is important that irreversibility is not associated with any probabilistic hypothesis. Irreversibility is a consequence of the retardation in interactions between particles. Operation of time reversibility corresponding to replacement

$$r \rightarrow -r, \quad t \rightarrow -t, \quad p \rightarrow -p.$$  

(20)

By taking into account only the retarded Green's function, the system of equations (19) under inversion (20) changes their functional dependence. As a consequence of this fact, the solutions of the system (19) also change.

In our approach, to describe the irreversible behavior of the system, there is no need to introduce additional assumptions that do not follow from equations of the motion (9). The system of equations (19) connects the distribution functions $F_A(x, p)$ at point $(x, p)$ (i.e. at instant of time $t$) with all the distribution functions $F_A(x', p')$ at other points $(x', p')$, i.e. at all earlier instants of times $t' \leq t$. This means that system (19) is nonlocal. The reason for this non-locality lies in the fundamental aspects of classical electrodynamics. There is no need to introduce additional assumptions, such as the principle of equal probability of initial states or averaging of the solution of the equation of motion with over the initial instants of time [7].

6. The law of energy-momentum change of a system of charged particles

Let us find the law of change of both the total energy of the system of particles and its momentum in terms of 4-vectors.

The energy-momentum tensor of a system of point charges

$$T_{\mu\nu}^{\text{part}} = \sum_a m_a c \int d\tau_a u_a^\mu u_a^\nu \delta^4(x - x_a(\tau_a))$$  

(21)

satisfies the equation [1,2]

$$\partial_{\nu} T_{\mu\nu}^{\text{part}} = \frac{1}{c} F^{\mu\nu} j_{\nu}.$$  

(22)

The solution of the system (19) allows one to find the evolution of both the energy

$$T_{\mu\nu}^{\text{part}} = \sum_a \delta^4(x - x_a) \frac{m_a c^2}{\sqrt{1 - (v_a/c)^2}}$$  

(23)

and the momentum

$$T_{\mu\nu}^{\text{part}} = \sum_a \delta^4(x - x_a) m_a u_a^\mu c$$  

(24)

of the system of point charge particles in terms of only particle variables:

$$\partial_{\nu} T_{\mu\nu}^{\text{part}} = F^{\mu\nu}(x, p) \sum_a m_a \int d^4p \sum_{\alpha} \frac{e^{\alpha}}{m_{\alpha}} F_{\alpha}(x, p).$$  

(25)

Whereas convolution can be represented as
we can integrate Eq. (22) over $r$:

$$
\int d^4 \mathbf{r} \partial_\nu T^\nu_{\text{part}} = \frac{i}{c} \sum_{a=1}^N \frac{e_a}{m_a} \int d^4 k e^{-i k \cdot r} \left[ k^\nu j_\nu (k) \right] d^4 r e^{i k \cdot r} j_\nu (x).
$$

(26)

Note that the following relation holds

$$
\int d^4 r e^{i k \cdot r} j_\nu (x) = \frac{1}{2 \pi^3} \sum_{a=0}^{N} \frac{e_a}{m_a} \frac{1}{i k \cdot p_a} e^{i k \cdot p_a}.
$$

We now refine our previous equation (26) to take the form

$$
\int d^4 \mathbf{r} \partial_\nu T^\nu_{\text{part}} = \frac{i}{c} \sum_{a=1}^N \frac{e_a}{m_a} \int d^4 k e^{-i k \cdot r} \left[ k^\nu j_\nu (k) \right] d^4 r e^{i k \cdot r} j_\nu (x).
$$

(27)

According to the Gauss theorem, the terms on the left side of equality (27) containing divergence can be reduced to the flow of the energy-momentum tensor through an infinitely remote surface. Such integrals are equal to zero, since there are no charges and currents at infinity. As a result, we obtain

$$
\frac{d}{dt} \sum_{a=1}^N m_a u_\mu^a = \frac{i}{c} \sum_{a=1}^N \frac{e_a}{m_a} \int d^4 k e^{-i k \cdot r} \left[ k^\mu j_\mu (k) - j^\mu (k) k^\nu j_\nu (k) \right].
$$

(28)

If the index on the left-hand side of (28) runs through the spatial components, then we are dealing with the law of change in the momentum of the system. If the index corresponds to the zero component of the 4-vector, then we come to the formula for the law of change in the total energy of the system of particles. Let us consider the second case in more detail:

$$
\frac{d}{dt} \sum_{a=1}^N m_a u_\mu^a = \frac{i}{c} \sum_{a=1}^N \frac{e_a}{m_a} \int d^4 k e^{-i k \cdot r} \left[ k^0 j^0 (k) - j^0 (k) k^\nu j_\nu (k) \right].
$$

(29)

Taking into account, that

$$
\begin{align*}
\begin{aligned}
\rho (k, \omega) &= \sum_{a=1}^N e_a \int f_A (r, p, t) \exp (i \omega t - kr) d^3 r d^3 p dt, \\
\mathbf{j} (k, \omega) &= \sum_{a=1}^N \frac{e_a}{m_a} \int \mathbf{p} f_A (r, p, t) \exp (i \omega t - kr) d^3 r d^3 p dt.
\end{aligned}
\end{align*}
$$

we can rewrite (29) in terms of Fourier images $\rho (k, \omega)$ and $\mathbf{j} (k, \omega)$:

$$
\frac{d}{dt} \sum_{a=1}^N \frac{m_a c^2}{\sqrt{1 - (v_a / c)^2}} = \sum_{a=1}^N \frac{e_a}{m_a} \int d^3 r d^3 p \int d^3 \mathbf{k} \omega d\omega \frac{e^{-i \mathbf{k} \cdot \mathbf{r}}}{4 \pi^3} \frac{\omega}{\omega^2 - (c \mathbf{k})^2 + i \varepsilon \omega} \left[ c^2 \rho (\mathbf{k}, \omega) \mathbf{k} - \omega \mathbf{j} (\mathbf{k}, \omega) \right].
$$

(30)

Integrating the equation (30) over $\omega$ and $\mathbf{k}$, we can rewrite (30) in the form

$$
\frac{d}{dt} \sum_{a=1}^N \frac{m_a c^2}{\sqrt{1 - (v_a / c)^2}} = \sum_{a=1}^N \frac{e_a}{m_a} \int d^3 r d^3 p \int d^3 p \left[ f_A (r, p, t) \frac{f_A (\mathbf{r} , \mathbf{p} , t)}{\sqrt{m_a^2 c^2 + p^2}} \right] \times
$$

$$
\left[ \frac{c^2}{|\mathbf{r} - \mathbf{r}'|} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{c \partial_t} \right) - \frac{\mathbf{p}}{\sqrt{m_a^2 c^2 + p^2}} \frac{1}{c \partial_t} \right] f_B \left( \mathbf{r}', \mathbf{p}', t - \frac{\mathbf{r} - \mathbf{r}'}{c} \right).
$$

(31)
The equation (31) describes a time-irreversible change in the energy of charged particles. Irreversibility is of the same nature as in the equation (19).

7. Conclusion
The closed system of covariant kinetic equations for the microscopic distribution functions $F_A(x,p)$ is obtained. The system of kinetic equations describes the irreversible evolution of the system of point charged particles.

Irreversibility is caused by a nontrivial interaction between particles and an electromagnetic field generated by them. In mathematical terms, irreversibility is associated with the choice of only the retarded Green’s function. Due to such a physically justified choice of the Green's function, the system (19) during time inversion (20) turns into functionally different equations with other solutions.

Thus, no probabilistic assumptions and hypothesis are required to justify irreversibility.

The law of change in terms of the 4-vectors of both the total energy of a system and its momentum is obtained. The irreversibility of the evolution of the system of charged particles leads to an irreversible change in their total energy.

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