Localized energy equalities for the Navier-Stokes and the Euler equations

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Abstract

Let \((v, p)\) be a smooth solution pair of the velocity and the pressure for the Navier-Stokes (Euler) equations on \(\mathbb{R}^N \times (0, T)\), \(N \geq 3\). We set the Bernoulli function \(Q = \frac{1}{2}|v|^2 + p\). Under suitable decay conditions at infinity for \((v, p)\) we prove that for almost all \(\alpha(t)\) and \(\beta(t)\) defined on \((0, T)\) there holds

\[
\int_{\{\alpha(t) < Q(x, t) < \beta(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx = \nu \int_{\{Q(x, t) = \beta(t)\}} |\nabla Q| dS
- \nu \int_{\{Q(x, t) = \alpha(t)\}} |\nabla Q| dS,
\]

where \(\omega = \text{curl } v\) is the vorticity. This shows that, in each region squeezed between two levels of the Bernoulli function, besides the energy dissipation due to the enstrophy, the energy flows into the region through the level hypersurface having the higher level, and the energy flows out of the region through the level hypersurface with the lower level. Passing \(\alpha(t) \downarrow \inf_{x \in \mathbb{R}^N} Q(x, t)\) and \(\beta(t) \uparrow \sup_{x \in \mathbb{R}^N} Q(x, t)\), we recover the well-known energy equality, \(\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |v|^2 = -\nu \int_{\mathbb{R}^N} |\omega|^2 dx\). A weaker version of the above equality under the weaker decay assumption of the solution at spatial infinity is also derived. The stationary version of the equality implies the previous Liouville type results on the Navier-Stokes equations.
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1 Introduction

We are concerned here on the incompressible Navier-Stokes(Euler) equations on $\mathbb{R}^N$, $N \geq 2$.

\[
\begin{aligned}
&v_t + (v \cdot \nabla)v + \nabla p = \nu \Delta v, \\
&\text{div } v = 0, \\
v(x,0) = v_0(x),
\end{aligned}
\]

where $v = v(x,t) = (v_1(x,t), \cdots, v_N(x,t))$ is the velocity, and $p = p(x,t)$ is the pressure. We assume the viscosity satisfies $\nu \geq 0$. In the case $\nu > 0$ the system (NS,E) becomes the Navier-Stokes equations, while for $\nu = 0$ the system (NS,E) is the Euler equations. By the system (NS,E) we represent both of the cases of the Navier-Stokes and the Euler equations. Below we denote the vorticity of the vector field $v$ in $\mathbb{R}^N$ defined by

\[
\omega = \{\partial_j v_k - \partial_k v_j\}_{j,k=1,j>k},
\]

the magnitude of which is given by

\[
|\omega| = \sqrt{\frac{1}{2} \sum_{j,k=1}^{N} (\partial_j v_k - \partial_k v_j)^2}.
\]

As is well-known in most of the fluid mechanics text books, the smooth solution $v$ of the system (NS,E) with sufficiently fast decays at spatial infinity satisfies the energy equality,

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |v|^2 dx = -\nu \int_{\mathbb{R}^N} |\omega|^2 dx.
\] (1.1)

Our aim in this paper is to localize the domain of the integration in (1.1). The domains are characterized by their boundaries, which are the level hypersurfaces of the Bernoulli function,

\[
Q(x, t) := \frac{1}{2} |v(x, t)|^2 + p(x, t).
\] (1.2)

Our localized integral equalities refine the classical equality (1.1), in the sense that under suitable integrability conditions for the solutions and particular
choice of the levels of the Bernoulli function we recover (1.1). The first theorem below concerns on these localized energy equalities under milder conditions on the asymptotic behavior for the solution \((v, p)\) at spatial infinity.

**Theorem 1.1** Let \(N \geq 2\), and \((v, p)\) be a smooth solution of \((NS,E)\) on \(\mathbb{R}^N \times (0, T)\). Let \(Q(x, t)\) be defined as in (1.2). Suppose there exists \(Q_0 = Q_0(t)\) such that

\[
\lim_{|x| \to \infty} Q(x, t) = Q_0(t) \quad (1.3)
\]

uniformly for each \(t \in (0, T)\). Then, for almost every \(\alpha(t), \beta(t)\) defined on \((0, T)\) such that either \(\beta(t) > \alpha(t) > 0\) or \(0 > \beta(t) > \alpha(t)\) we have the following equality.

\[
\int_{\{\alpha(t) < Q(x, t) < \beta(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx = \nu \int_{\{Q(x, t) = \beta(t)\}} |\nabla Q| dS - \nu \int_{\{Q(x, t) = \alpha(t)\}} |\nabla Q| dS.
\]

(1.4)

**Remark 1.1** Physically the equality (1.4) shows new energy balance in each region squeezed between two levels of the Bernoulli function. Besides the well-known energy dissipation due to the enstrophy the energy flows into the region through the level hypersurface having the higher level, and the energy flows out of the region through the level hypersurface with the lower level. This tendency of flow of the energy into lower values of the Bernoulli function is proportional to the viscosity and the magnitude of the gradient of the Bernoulli function, and there exists no such phenomena in the inviscid case of the Euler equations, where energy is just conserved on the average in each strip between the two levels of the Bernoulli function.

**Remark 1.2** If we approach \(\beta(t) \uparrow \sup_{x \in \mathbb{R}^N} Q(x, t) - Q_0(t)\), then, since \(\int_{\{Q(x, t) - Q_0(t) = \beta(t)\}} |\nabla Q| dS \to 0\), we obtain from (1.4) that

\[
\int_{\alpha(t) + Q_0(t) < Q(x, t)} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx = -\nu \int_{\{Q(x, t) = \alpha(t) + Q_0(t)\}} |\nabla Q| dS \leq 0.
\]

(1.5)
Similarly, if we approach \( \alpha(t) \downarrow \inf_{x \in \mathbb{R}^N} Q(x, t) - Q_0(t) \), then we obtain from (1.4) that
\[
\int \{Q(x, t) < \beta(t) + Q_0(t)\} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) \, dx = \nu \int \{Q(x, t) = \beta(t) + Q_0(t)\} |\nabla Q| \, dS \geq 0.
\] (1.6)

**Remark 1.3** If we let \( \alpha(t) \) and \( \beta(t) \) approach respectively to the critical values of \( Q(x, t) - Q_0(t) \) with \( \alpha(t) \beta(t) > 0 \), then the righthand side of (1.4) vanishes, and the energy balance holds in the region of the set \( \{x \in \mathbb{R}^N \mid \alpha(t) < Q(x, t) - Q_0(t) < \beta(t)\} \).

The following theorem derives stronger version of equalities than (1.4) in the sense that there is no sign condition on the functions \( \alpha(t), \beta(t) \), but under the stronger assumptions on the behaviors of the solutions at spatial infinity represented by the integrability of the solutions.

**Theorem 1.2** Let \( N \geq 3 \), and \((v, p)\) be a smooth solution of (NS,E) on \( \mathbb{R}^N \times (0, T) \). Suppose \((v, p)\) satisfies the following conditions: there exists \( p_0(t) \) defined on \((0, T)\) such that
\[
\lim_{|x| \to \infty} (|v(x, t)| + |p(x, t) - p_0(t)|) = 0,
\] (1.7)
uniformly for each \( t \in (0, T) \),
\[
v \in L^\infty(0, T; L^{\frac{3N}{N-1}}(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)),
\] (1.8)
and
\[
\frac{\partial}{\partial t} |v|^2 \in L^\infty(0, T; L^1(\mathbb{R}^N)).
\] (1.9)
Then, for almost all real valued functions \( \alpha(t) < \beta(t) \) defined on \((0, T)\) we have the following equalities.
\[
\int \{\alpha(t)+p_0(t)<Q(x, t)<\beta(t)+p_0(t)\} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) \, dx
\]
\[
= \nu \int \{Q(x, t) = \beta(t) + p_0(t)\} |\nabla Q| \, dS - \nu \int \{Q(x, t) = \alpha(t) + p_0(t)\} |\nabla Q| \, dS.
\] (1.10)
Remark 1.4 The same remark holds as Remark 1.1 for the physical interpretation of (1.15). If we approach
\[ \alpha(t) \downarrow \inf_{x \in \mathbb{R}^N} Q(x, t) - p_0(t), \quad \beta(t) \uparrow \sup_{x \in \mathbb{R}^N} Q(x, t) - p_0(t), \]
in particular, then from (1.15) we obtain
\[ \int_{\mathbb{R}^N} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx = 0, \quad (1.11) \]
which is equivalent to the classical energy equality (1.1) for the Navier-Stokes equations.

For the stationary solutions of the Navier-Stokes equations with \( v(x, t) = v(x), p(x, t) = p(x), \) and \( \omega(x, t) = \omega(x), \) as an immediate corollary of Theorem 1.2 we obtain the following result, which is previously obtained by Galdi(3).

Corollary 1.1 Let \( N \geq 3. \) Suppose \( (v, p) \) is a stationary smooth solution of the Navier-Stokes equations satisfying: there exists a constant \( p_0 \) such that
\[ \lim_{|x| \to \infty} (|v(x)| + |p(x) - p_0|) = 0, \quad (1.12) \]
uniformly, and
\[ v \in \begin{cases} L^2_\infty(\mathbb{R}^3) \cap L^6(\mathbb{R}^3) & \text{if } N = 3, \\ L^{\frac{N}{N-1}}(\mathbb{R}^N) & \text{if } N \geq 4. \end{cases} \quad (1.13) \]
Then, \( v = 0. \)

Indeed, we first note that \( \frac{3N}{N-1} \geq \frac{2N}{N-2} \) for \( N \geq 4. \) Thus, if \( N \geq 4, \) then \( L^{\frac{2N}{N-1}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) by the standard \( L^p \)-interpolation, and the fact \( v \in L^\infty(\mathbb{R}^N) \) is guaranteed by the hypothesis (1.12). Therefore, the stationary version of (1.11) together with the assumptions (1.12) and (1.13) imply \( \omega = 0. \) Combining this with \( \text{div} \ v = 0, \) we have \( v = \nabla h \) for some harmonic function \( h. \) Therefore the condition (1.12) implies \( v = 0. \)

Remark 1.5 For \( N = 3, \) if we assume (1.12), and replace the assumption (1.13) by \( \int_{\mathbb{R}^3} |\omega|^2 dx < \infty, \) it is still an open question whether \( v = 0 \) or not. In this case we know by the maximum principle from (2.1) implies that
$Q(x) \leq p_0$ for all $x \in \mathbb{R}^3$. Therefore, by choosing $\alpha \downarrow \inf_{x \in \mathbb{R}^3} Q(x) - p_0$, and $\beta \uparrow 0$ in the stationary version of Theorem 1.1, we obtain

$$\int_{\mathbb{R}^3} |\omega|^2 \, dx = \lim_{\beta \uparrow 0} \int_{\{Q(x) = p_0 + \beta\}} |\nabla Q| \, dS.$$  \hfill (1.14)

Therefore, we find that the desired Liouville type theorem holds if

$$\lim_{\beta \uparrow 0} \int_{\{Q(x) = p_0 + \beta\}} |\nabla Q| \, dS < \int_{\mathbb{R}^3} |\omega|^2 \, dx.$$  

In the case of smooth solutions to (NS,E) in a periodic domain (1.12) and (1.13) are not necessary in view of the proof of Theorem 1.2. Therefore, as another corollary of Theorem 1.2 and its proof we obtain the following:

**Corollary 1.2** Let $N \geq 2$. Suppose $(v, p)$ is a smooth solution of (NS,E) on $\mathbb{T}^N \times (0, T)$, where $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$ is the periodic domain in $\mathbb{R}^N$. We normalize the pressure by assuming $\int_{\mathbb{T}^N} p \, dx = 0$. Define the Bernoulli function $Q$ as in (1.2). Then, for all $t \in (0, T)$ and for almost all real valued functions $\alpha(t), \beta(t)$ we have

$$\int_{\{\alpha(t) < Q(x,t) < \beta(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) \, dx$$

$$= \nu \int_{\{Q(x,t) = \alpha(t)\}} |\nabla Q| \, dS - \nu \int_{\{Q(x,t) = \beta(t)\}} |\nabla Q| \, dS.$$  \hfill (1.15)

**Remark 1.6** Similarly to Remark 1.4 the usual energy equality in $\mathbb{T}^N$ follows immediately by approaching

$$\alpha(t) \downarrow \inf_{x \in \mathbb{T}^N} Q(x,t), \quad \beta(t) \uparrow \sup_{x \in \mathbb{T}^N} Q(x,t).$$

We emphasize that we consider the energy equalities only for smooth solutions of the Navier-Stokes and the Euler equations. For the studies of the energy for the weak solutions of the Euler equations we refer to the results, among others, by De Lellis-Székelyhidi(21), Shnirelman(41) and the references therein.
2 Proof of the Main Theorems

Although the lemma below is well-known previously for the stationary solutions of the Navier-Stokes equations, we present its proof for reader’s convenience.

Lemma 2.1 If \((v, p)\) solves \((NS, E)\), then \(Q = p + \frac{1}{2}|v|^2\) satisfies

\[
\frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 = \nu \Delta Q - v \cdot \nabla Q.
\]  
(2.1)

Proof Multiplying \((NS, E)\) by \(v\), we obtain

\[
\frac{1}{2} \frac{\partial}{\partial t} |v|^2 + v \cdot \nabla \left( p + \frac{1}{2} |v|^2 \right) = \nu v \cdot \Delta v = \nu \Delta \left( \frac{1}{2} |v|^2 \right) - \nu \sum_{j,k=1}^{N} (\partial_j v_k)^2. \tag{2.2}
\]

We compute

\[
\sum_{j,k=1}^{N} (\partial_j v_k)^2 = \sum_{j,k=1}^{N} \partial_j v_k (\partial_j v_k - \partial_k v_j) + \sum_{j,k=1}^{N} \partial_j v_k \partial_k v_j = \frac{1}{2} \sum_{j,k=1}^{N} (\partial_j v_k - \partial_k v_j)^2 - \Delta p = |\omega|^2 - \Delta p, \tag{2.3}
\]

where we used the well-known formula \(\Delta p = -\sum_{j,k=1}^{N} \partial_j v_k \partial_k v_j\), which is obtained from \((NS, E)\) by taking \(\text{div}(\cdot)\). Plugging (2.3) into (2.2), we have (2.1). □

Proof of Theorem 1.1 We use argument similar to the one used in [1]. We define \(Q(x, t) := Q(x, t) - Q_0(t)\). Then,

\[
\lim_{|x| \to \infty} Q(x, t) = 0 \quad \forall t \in (0, T) \tag{2.4}
\]

uniformly. From (2.1) we have

\[
\nu \Delta Q - v \cdot \nabla Q = \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2. \tag{2.5}
\]

We assume below

\[
\sup_{x \in \mathbb{R}^N} Q(x, t) \geq \beta(t) > \alpha(t) > 0.
\]
The other case, $0 > \beta(t) > \alpha(t) \geq \inf_{x \in \mathbb{R}^N} Q(x, t)$, is similar. Let us set $\mathbb{D} = \{x \in \mathbb{R}^N | \alpha(t) < Q(x, t) < \beta(t)\}$. Thanks to the Sard theorem and the implicit function theorem $\partial \mathbb{D}$ consists of smooth level surfaces in $\mathbb{R}^N$ except the values of $\alpha(t)$ and $\beta(t)$, having the zero Lebesgue measure, which corresponds to the critical values of the function $z = Q(x, t)$. It is understood that our values of $\alpha(t), \beta(t)$ below avoid these exceptional ones. We integrate (2.5) on $\mathbb{D}$, and use the divergence theorem to obtain

$$\int_{\mathbb{D}} \left( \frac{1}{2} \frac{\partial |v|^2}{\partial t} + \nu |\omega|^2 \right) dx = \nu \int_{\mathbb{D}} \Delta Q dx - \int_{\mathbb{D}} v \cdot \nabla Q dx$$

$$= \nu \int_{\partial \mathbb{D}} \hat{n} \cdot \nabla Q dS - \int_{\partial \mathbb{D}} v \cdot \nabla Q dS$$

$$:= I_1 + I_2, \quad (2.6)$$

where $\hat{n}$ is the outward unit normal vector on $\partial \mathbb{D}$.

$$I_1 = \nu \int_{\{Q(x,t) = \alpha(t)\}} \frac{\nabla Q}{|\nabla Q|} \cdot \nabla Q dS - \nu \int_{\{Q(x,t) = \beta(t)\}} \frac{\nabla Q}{|\nabla Q|} \cdot \nabla Q dS$$

$$= \nu \int_{\{Q(x,t) = \beta(t)\}} |\nabla Q| dS - \nu \int_{\{Q(x,t) = \alpha(t)\}} |\nabla Q| dS. \quad (2.7)$$

Next, by the condition $\text{div} \ v = 0$ we obtain

$$I_2 = - \int_{\{Q(x,t) = \beta(t)\}} \hat{n} \cdot v Q dS - \int_{\{Q(x,t) = \alpha(t)\}} \hat{n} \cdot v dS$$

$$= \beta(t) \int_{\{Q(x,t) = \beta(t)\}} (-\hat{n}) \cdot v dS - \alpha(t) \int_{\{Q(x,t) = \alpha(t)\}} \hat{n} \cdot v dS$$

$$= \beta(t) \int_{\{Q(x,t) > \beta(t)\}} \text{div} \ v dx - \alpha(t) \int_{\{Q(x,t) > \alpha(t)\}} \text{div} \ v dx = 0. \quad (2.8)$$

Combining (2.6)-(2.8), we obtain (1.4). □

**Proof of Theorem 1.2** We set $Q(x,t) := Q(x,t) - p_0(t)$ here. Let $\alpha(t) < \beta(t)$ be real valued functions on $(0, T)$ with $\alpha(t) < Q(x,t) < \beta(t)$. The case $\alpha(t) \beta(t) > 0$ is already covered by Theorem 1.1. Here we assume that

$$m(t) := \inf_{x \in \mathbb{R}^N} Q(x, t) \leq \alpha(t) < 0 < \beta(t) \leq \sup_{x \in \mathbb{R}^N} Q(x, t) := M(t).$$
Since
\[
\int_{\{\alpha(t) < Q(x,t) < \beta(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx
\]
\[
= \int_{\{\alpha(t) < Q(x,t) < M(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx
\]
\[= \int_{\{\alpha(t) < Q(x,t) < M(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx + \int_{\{Q(x,t) = \beta(t)\}} |\nabla Q|dS
\]
by application of Theorem 1.1, it suffices to show that
\[
\int_{\{\alpha(t) < Q(x,t) < \beta(t)\}} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) dx = -\int_{\{Q(x,t) = \alpha(t)\}} |\nabla Q|dS \quad (2.9)
\]
for \(m(t) < \alpha(t) < 0\). The set \(\mathbb{D}^+_\alpha := \{x \in \mathbb{R}^N \mid \alpha(t) < Q(x,t)\}\) is unbounded, while \(\mathbb{D}^-_\alpha := \{x \in \mathbb{R}^N \mid \alpha(t) > Q(x,t)\}\) is a bounded set. Moreover,
\[
\partial \mathbb{D}^+_\alpha = \partial \mathbb{D}^-_\alpha \cup \{\infty\}.
\]
Let \(R > 0\) be large enough so that \(\partial \mathbb{D}^-_\alpha \subset B_R(0) := \{x \in \mathbb{R}^N \mid |x| < R\}\). We set \(Q^+_\alpha := [Q + \alpha]_+ = \max\{Q + \alpha, 0\}\) below. We introduce the radial cut-off function \(\sigma \in C_0^\infty(\mathbb{R}^N)\) such that
\[
\sigma(|x|) = \begin{cases} 
1 & \text{if } |x| < 1, \\
0 & \text{if } |x| > 2,
\end{cases}
\]
and \(0 \leq \sigma(x) \leq 1\) for \(1 < |x| < 2\). Then, for each \(R > 0\), we define
\[
\sigma \left( \frac{|x|}{R} \right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).
\]
Multiplying (2.5) by $\sigma_R$, and integrating on $\mathbb{D}_+^\alpha$, we obtain by the divergence theorem,

$$
\int_{\mathbb{D}_+^\alpha} \left( \frac{1}{2} \frac{\partial}{\partial t} |v|^2 + \nu |\omega|^2 \right) \sigma_R dx = \nu \int_{\mathbb{D}_+^\alpha} \sigma_R \Delta Q_\alpha^+ dx - \int_{\mathbb{D}_+^\alpha} \sigma_R v \cdot Q_\alpha^+ dx
$$

$$
= \nu \int_{\partial \mathbb{D}_+^\alpha} \hat{n} \cdot \nabla Q_\alpha^+ \sigma_R dS - \nu \int_{\mathbb{D}_+^\alpha} \nabla Q_\alpha^+ \cdot \nabla \sigma_R dx - \int_{\partial \mathbb{D}_+^\alpha} v \cdot \hat{n} Q_\alpha^+ \sigma_R dx
$$

$$
+ \int_{\mathbb{D}_+^\alpha} Q_\alpha^+ v \cdot \nabla \sigma_R dx
$$

$$
:= I_1 + I_2 + I_3 + I_4, \tag{2.10}
$$

where $\hat{n} = -\frac{\nabla Q_\alpha^+}{|\nabla Q_\alpha^+|}$ is the outward unit normal vector to $\partial \mathbb{D}_-^\alpha$. As in the proof of Theorem 1.1, we have

$$I_1 = -\nu \int_{\partial \mathbb{D}_-^\alpha} \frac{\nabla Q_\alpha^+}{|\nabla Q_\alpha^+|} \cdot \nabla Q_\alpha^+ \sigma_R dS
$$

$$= -\nu \int_{\{Q(x,t) = \alpha(t)\}} |\nabla Q| \sigma_R dS
$$

$$= -\nu \int_{\{Q(x,t) = \alpha(t)\}} |\nabla Q| dS, \tag{2.11}
$$

since $\sigma_R = 1$ on $B_R(0) \supset \partial \mathbb{D}_-^\alpha = \{x \in \mathbb{R}^N \mid Q(x,t) = \alpha(t)\}$. By the fact $Q_\alpha^+ = 0$ on $\partial \mathbb{D}_+^\alpha$, we have

$$I_3 = 0. \tag{2.12}
$$

Applying the divergence theorem again, we have

$$I_2 = -\nu \int_{\mathbb{D}_+^\alpha} \nabla Q \cdot \nabla \sigma_R dx
$$

$$= -\nu \int_{\partial \mathbb{D}_-^\alpha} Q \hat{n} \cdot \nabla \sigma_R dS + \nu \int_{\mathbb{D}_+^\alpha} Q \Delta \sigma_R dx
$$

since $\nabla \sigma_R = 0$ on $B_R(0) \supset \partial \mathbb{D}_-^\alpha$. Hence, we have the estimates,

$$|I_2| \leq \nu \int_{\mathbb{R}^N} |Q| |\Delta \sigma_R| dx \leq \frac{\|D^2\sigma\|_{L^\infty}}{R^2} \|Q\|_{L^{N/2}(R \leq |x| \leq 2R)} \left( \int_{\{R \leq |x| \leq 2R\}} dx \right)^{\frac{2}{N}}
$$

$$\leq C \nu \|D^2\sigma\|_{L^\infty} \left( \int_{\{R \leq |x| \leq 2R\}} (|p - p_0| + \frac{1}{2} |v|^2)^{\frac{N}{N-2}} dx \right)^{\frac{N-2}{N}}
$$

$$\leq C \nu \left( \|p - p_0\|_{L^{\frac{N}{N-2}}(R \leq |x| \leq 2R)} + \|v\|^2_{L^{2N/(N+2)}(R \leq |x| \leq 2R)} \right) \to 0
$$
as $R \to \infty$, which follows from the assumption $v \in L^\infty(0,T;L^{\frac{2N}{N-2}}(\mathbb{R}^N))$, and the Calderon-Zygmund inequality, $\|p - p_0\|_{L^{\frac{2N}{N-2}}} \leq C\|v\|_{L^{\frac{2N}{N-2}}}^2$, where we used the well-known velocity-pressure relation, $p - p_0 = \sum_{j,k} R_j R_k(v_j v_k)$ with the Riesz transforms $R_j$, $j = 1, \cdots, N$ in $\mathbb{R}^N$. For the estimate of $I_4$, since $\partial D^\alpha \subset B_R(0)$, we have

$$
|I_4| = \left| \int_{\{R \leq |x| \leq 2R\}} (Q + \alpha) v \cdot \nabla \sigma_R dx \right| = \left| \int_{\{R \leq |x| \leq 2R\}} Qv \cdot \nabla \sigma_R dx \right|
$$

$$
\leq \frac{1}{R} \|D\sigma\|_{L^\infty} \left( \int_{\{R \leq |x| \leq 2R\}} dx \right)^\frac{1}{N} \left\{ \int_{\{R \leq |x| \leq 2R\}} \left( |v||p - p_0| + \frac{1}{2}|v|^3 \right)^{\frac{N}{N-1}} dx \right\}^{\frac{N-1}{N}}
$$

$$
\leq C \left( \|v\|_{L^{\frac{2N}{N-2}}(R \leq |x| \leq 2R)} \|p - p_0\|_{L^{\frac{2N}{N-2}}(R \leq |x| \leq 2R)} + \|v\|^3_{L^{\frac{2N}{N-2}}(R \leq |x| \leq 2R)} \right)
$$

$$
\leq C \left( \|v\|_{L^{\frac{2N}{N-2}}(R \leq |x| \leq 2R)} \|v\|^2_{L^{\frac{2N}{N-2}}(R \leq |x| \leq 2R)} + \|v\|^3_{L^{\frac{2N}{N-2}}(R \leq |x| \leq 2R)} \right) \to 0
$$

(2.13)

as $R \to \infty$ by the hypothesis (1.8), where we used the fact $\alpha \int_{\{R \leq |x| \leq 2R\}} v \cdot \nabla \sigma_R dx = 0$ in the first line. On the other hand, if we pass $R \to \infty$ in the left hand side of (2.10), we note that $\int_{D_+} \frac{1}{2|\nabla|^2} |v|^2 \sigma_R dx \to \int_{D_+} \frac{1}{2|\nabla|^2} |v|^2 dx$ by the dominated convergence theorem, while $\int_{D_+} |\omega|^2 \sigma_R dx \to \int_{D_+} |\omega|^2 dx$ by the monotone convergence theorem. Therefore (2.9) follows from (2.10) by passing $R \to \infty$ in both sides of the equality. □

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