Geometric field theory and weak Euler-Lagrange equation for classical relativistic particle-field systems

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Abstract

A manifestly covariant, or geometric, field theory for relativistic classical particle-field system is developed. The connection between space-time symmetry and energy-momentum conservation laws for the system is established geometrically without splitting the space and time coordinates, i.e., space-time is treated as one identity without choosing a coordinate system. To achieve this goal, we need to overcome two difficulties. The first difficulty arises from the fact that particles and field reside on different manifolds. As a result, the geometric Lagrangian density of the system is a function of the 4-potential of electromagnetic fields and also a functional of particles’ world-lines. The other difficulty associated with the geometric setting is due to the mass-shell condition. The standard Euler-Lagrange (EL) equation for a particle is generalized into the geometric EL equation when the mass-shell condition is imposed. For the particle-field system, the geometric EL equation is further generalized into a weak geometric EL equation for particles. With the EL equation for field and the geometric weak EL equation for particles, symmetries and conservation laws can be established geometrically. A geometric expression for the energy-momentum tensor for particles is derived for the first time, which recovers the non-geometric form in the existing literature for a chosen coordinate system.

PACS numbers: 52.35.Hr, 52.35.-g, 52.35.We, 42.50.Tx, 52.50.Sw

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I. INTRODUCTION

Energy-momentum conservation is a fundamental law of physics. It applies to both quantum systems and classical systems. From the field-theoretical point of view, energy-momentum conservation is fundamentally due to the space-time symmetry of the Lagrangian (or Lagrangian density) that the system admits \cite{1,2,3}. For example, the Lagrangian density for the electromagnetic field is

\[ L_F = \frac{1}{8\pi} \left( \left( -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \varphi \right)^2 - (\nabla \times A)^2 \right). \tag{1} \]

The energy and momentum conservation laws of the system,

\[ \frac{\partial}{\partial t} \left[ \frac{E^2 + B^2}{8\pi} \right] + \nabla \cdot \left[ \frac{c}{4\pi} E \times B \right] = 0, \tag{2} \]
\[ \frac{\partial}{\partial t} \left[ \frac{E \times B}{4\pi c} \right] + \nabla \cdot \left[ \frac{E^2 + B^2}{8\pi} I - \frac{EE + BB}{4\pi} \right] = 0, \tag{3} \]
can be derived from EL equations,

\[ \frac{\partial L_F}{\partial \varphi} - \frac{D}{Dx} \cdot \frac{\partial L_F}{\partial \dot{\varphi}} = 0, \tag{4} \]
\[ \frac{\partial L_F}{\partial A} - \frac{D}{Dx} \cdot \frac{\partial L_F}{\partial \nabla A} - \frac{D}{Dt} \frac{\partial L_F}{\partial A_t} = 0, \tag{5} \]
and the symmetry conditions,

\[ \frac{\partial L_F}{\partial t} = 0, \tag{6} \]
\[ \frac{\partial L_F}{\partial x} = 0. \tag{7} \]

Here $D/Dx$ and $D/Dt$ denote partial derivative with respect to $x$ and $t$, respectively, when they operate on a field defined on the space time.

Geometrically, the symmetries of time and space of the Lagrangian density are components of the space-time symmetry,

\[ \frac{\partial L_F}{\partial \chi^\mu} = 0, \quad (\mu = 0, 1, 2, 3), \tag{8} \]

where $\chi^\mu$ is an arbitrary world point in 4 dimensional Minkowski space-time, i.e., $\chi^0 \equiv t$ and $\chi^i \equiv x^i$. And the Lagrangian density in Eq. (8) can be equivalently written as
where the 4-potential $A_\nu = (\varphi, -\mathbf{A})$ is defined on the space-time. In this paper, we assume that the space-time is endowed with a Lorentzian metric and the signature of the metric is $(+ − − −)$. Equation (9) is a manifestly covariant form of Eqs. (6) and (7). In this paper, the phrase “manifestly covariant” will be substituted by “geometric”, which indicates that covariance is the intrinsic coordinate-independent property of the physical system [4].

Similarly, the EL equations (4) and (5) or the Maxwell equations can be written in the geometric form as

$$\frac{\partial \mathcal{L}_F}{\partial A_\mu} - \frac{D}{D\chi^\nu} \left[ \frac{\partial \mathcal{L}_F}{\partial (\partial_\nu A_\mu)} \right] = 0.$$  (10)

Here, the operator $D/D\chi^\nu$ denotes partial derivative with respect to $\chi^\nu$ while keeping $\chi^\mu (\mu \neq \nu)$ fixed, when it is operated on a field on space-time. The geometric energy-momentum conservation law is

$$\partial_\nu T^{\mu \nu}_F = 0, \quad (\mu, \nu = 0, 1, 2, 3),$$  (11)

where $T^{\mu \nu}_F$ is the energy-momentum tensor of electromagnetic fields and could be written in an explicit form as

$$T^{\mu \nu}_F = \frac{1}{4\pi} \left( -F^{\mu \sigma} F^\nu_\sigma + \frac{1}{4\pi} \eta^{\mu \nu} F^\sigma_\rho F^\rho_\sigma \right), \quad (\mu, \nu, \sigma, \rho = 0, 1, 2, 3)$$  (12)

where $F$ is electromagnetic tensor and $\eta$ is the Lorentzian metric. To briefly summarize, Eq. (9) is the geometric form of Eq. (1), Eqs. (11) is that of Eqs. (2) and (3). Similarly, Eq. (10) is the geometric form of Eqs. (4) and (5), and Eq. (8) is that of Eqs. (6) and (7). These, of course, are well-known [5].

Classical particle-field systems, where many charged particles evolve under the electromagnetic field generated self-consistently by the particles, are often encountered in astrophysics, accelerator physics, and plasma physics [6, 7]. For these systems, the relations between symmetries and conservation laws have only been established recently by Qin et al [8]. It turns out that the standard EL equation for particles doesn’t hold anymore, because the dynamics of the particles and fields are defined on different manifolds which have different dimensions. The electromagnetic fields are defined on the space-time domain, whereas the particle trajectories as fields are only defined on the time-axis. For particles, a weak EL
equation is established to replace the standard EL equation. It was discovered that the weak EL equation can also link symmetries with conservation laws as in standard field theory. This field theory is non-relativistic, but it can be easily extended to relativistic cases, which will be given in Sec. III. However, this approach is based on the split form of space and time. In another word, it is not geometric.

In this paper, we will geometrically reformulate the field theory for classical particle-field system established in Ref. [8]. A geometric weak EL equation will be derived, and the energy-momentum conservation will be geometrically derived from the space-time symmetry. To achieve this goal, we need to overcome two difficulties. The first difficulty arises from the fact that particles and field reside on different manifolds as noticed in Ref. [8]. In the geometric setting, this difference is more prominent. Particles’ dynamics are characterized by world-lines on the space-time. They are defined on $R^4$ as a field valued in the space-time, and the domain of the particle fields can be the proper time or any other parameterization for the world-lines. The world-line of a particle is uniquely defined, independent of how it is parameterized. The electromagnetic field, on the other hand, are defined on the space-time. The geometric Lagrangian density of the system will be a function of the 4-potential of electromagnetic fields and also a functional of particles’ world-lines. This is qualitatively different from the standard field theory, where the Lagrangian density is a local function of the fields. The other difficulty associated with the geometric setting is due to the mass-shell condition, which exists even for the geometric variational principle for a single particle [9]. The standard EL equation will be generalized into a geometric EL equation when the mass-shell condition is imposed. For the particle-field system, the geometric EL equation is further generalized into a weak geometric EL equation.

We emphasize that it is of significant theoretical and practical value to put the physical laws governing the classical particle-field system into geometric forms. The compact geometric forms, or manifestly Lorentz invariant forms, are especially suitable for analysis in statistical mechanics for relativistic plasmas [7, 10–18]. They also serve as the theoretical foundations for developing Lorentz covariant algorithms [19] for numerical simulations. In quantum field theory, both geometric and non-geometric forms are used. The path integral approach is based on the geometric form of the Lagrangian density, whiles the canonical quantization method is not manifestly covariant because the space and time dimensions are split. The path integral form of the quantum electrodynamics has been recently adopted to
We note that energy-momentum conservation laws are well-known results, and can be found, for example, in Ref. [5]. However, in the existing literature these conservation laws are not derived from the underpinning symmetries. Often one can establish a conservation law without knowing the underpinning symmetry. As in the case studied here, establishing the connection between a symmetry and a conservation law sometimes can be a difficult but rewarding task.

This paper is organized as follows. In Sec. II, we will discuss how to extend the work by Qin, Burby and Davidson to the relativistic case in a non-geometric way. The geometric Lagrangian for a single particle and the corresponding geometric EL equation are discussed in Sec. III. In Sec. IV the geometric Lagrangian density for particle-field system and its properties are discussed. The geometric weak EL equation and the link it provides between conservation laws and space-time symmetries are given in Sec. V.

II. NON-GEOMETRIC FIELD THEORY AND WEAK EULER-LAGRANGE EQUATION FOR RELATIVISTIC PARTICLE-FIELD SYSTEMS

The classical relativistic particle-field system in flat space is governed by the following Newton-Maxwell equations,

\[\frac{d}{dt} (\gamma_{sp} m_s \dot{X}_{sp}) = q_s (E + \frac{1}{c} \dot{X}_{sp} \times B),\]

\[\nabla \cdot E = 4\pi \sum_{s,p} q_s \delta (X_{sp} - x),\]

\[\nabla \times B = \frac{4\pi}{c} \sum_{s,p} q_s \dot{X}_{sp} \delta (X_{sp} - x) + \frac{1}{c} \frac{\partial E}{\partial t},\]

\[\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t},\]

\[\nabla \cdot B = 0,\]

where \(\gamma_{sp} \equiv 1/\sqrt{1 - \dot{X}_{sp}^2/c^2}\) is the relativistic factor for \(p\)-th particle of \(s\)-species, \(m_s\) is the mass for any particle of \(s\)-species, \(X_{sp}\) is the trajectory, \(x\) is a point in the configuration space, and \(\delta (X_{sp} - x)\) is the Dirac delta function on 3D configuration space. The basic
Newton-Maxwell equations (13)-(15) can be equivalently written as the Vlasov-Maxwell equations

\[
\frac{\partial F_s}{\partial t} + \mathbf{v} \cdot \frac{\partial F_s}{\partial \mathbf{x}} + q_s (\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \cdot \frac{\partial F_s}{\partial \mathbf{p}} = 0
\]

(18)

\[
\nabla \cdot \mathbf{E} = 4\pi \sum_s q_s \int F_s d^3 \mathbf{p},
\]

(19)

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_s q_s \int F_s v d^3 \mathbf{p} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t},
\]

(20)

by defining the Klimontovich distribution function in phase space as

\[
F_s = \sum_p \delta(X_{sp} - x) \delta(p_{sp} - p).
\]

(21)

Here, \( p_{sp} = \gamma_{sp} m_s \dot{X}_{sp} \) is the relativistic momentum of \( sp \)-particle. The electric field \( \mathbf{E}(t, \mathbf{x}) \) and magnetic field \( \mathbf{B}(t, \mathbf{x}) \) are functions of space and time. For this system, the action \( \mathcal{A} \) is

\[
\mathcal{A}[\varphi, \mathbf{A}, X_{sp}] = \int L_{PF} dt d^3 x,
\]

(22)

where

\[
L_{PF} = -\sum_{s,p} \gamma_{sp}^{-1} m_s c^2 \delta(X_{sp} - x) + \frac{q_s}{c} \mathbf{A} \cdot \dot{X}_{sp} \delta(X_{sp} - x) - q_s \varphi \delta(X_{sp} - x)
\]

\[
+ \frac{1}{8\pi} \left[ \left( -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \right)^2 - \nabla \times \mathbf{A} \right]
\]

(23)

is the Lagrangian density of this system \[5\]. The variation of \( \mathcal{A} \) induced by \( \delta X_{sp}, \delta \mathbf{A} \) and \( \delta \varphi \) is

\[
\delta \mathcal{A} = \sum_{s,p} \int dt \delta X_{sp} \cdot \int E_{X_{sp}}(L_{PF}) d^3 x + \int E_{\varphi}(L_{PF}) \delta \varphi dt d^3 x + \int E_{\mathbf{A}}(L_{PF}) \cdot \delta \mathbf{A} dt d^3 x,
\]

(24)

where

\[
E_{X_{sp}}(L_{PF}) = \frac{\partial L_{PF}}{\partial \dot{X}_{sp}} - \frac{D}{Dt} \cdot \frac{\partial L_{PF}}{\partial X_{sp}},
\]

(25)

\[
E_{\varphi}(L_{PF}) = \frac{\partial L_{PF}}{\partial \varphi} - \frac{D}{Dx} \cdot \frac{\partial L_{PF}}{\partial \varphi},
\]

(26)

\[
E_{\mathbf{A}}(L_{PF}) = \frac{\partial L_{PF}}{\partial \mathbf{A}} - \frac{D}{Dx} \cdot \nabla \mathbf{A} - \frac{D}{Dt} \cdot \frac{\partial L_{PF}}{\partial \mathbf{A}_t}.
\]

(27)

For \( \delta \mathcal{A} = 0 \), we have

\[
\int E_{X_{sp}}(L_{PF}) d^3 x = 0,
\]

(28)
\[ E_\varphi(\mathcal{L}_{PF}) = 0, \quad (29) \]
\[ E_A(\mathcal{L}_{PF}) = 0 \quad (30) \]
due to the arbitrarinesses of \( \delta X_{sp} \), \( \delta \varphi \) and \( \delta A \) in Eq. (24). Equation (28) will be called sub-manifold EL equation because it is defined only on the time-axis after the integrating over spatial variable. Moreover, substituting Eq. (23) into Eqs. (25)-(30), we will recover Eqs. (13)-(17) by defining \( E = -\partial A / \partial t - \nabla \varphi \) and \( B = \nabla \times A \).

For Eq. (28), in general, we expect that \( E_{X_{sp}}(\mathcal{L}_{PF}) \neq 0 \) although the integral of this term vanishes. We can derive an explicit expression for \( E_{X_{sp}}(\mathcal{L}_{PF}) \) as

\[ E_{X_{sp}}(\mathcal{L}_{PF}) = \frac{\partial \mathcal{L}_{PF}}{\partial X_{sp}} - \frac{D}{Dt} \cdot \frac{\partial \mathcal{L}_{PF}}{\partial \dot{X}_{sp}} = \frac{\partial}{\partial x} \left( H_{sp} - P_{sp} \cdot \dot{X}_{sp} \right) + \frac{\partial}{\partial \dot{x}} \cdot \left( \dot{X}_{sp} P_{sp} \right), \quad (31) \]

where

\[ H_{sp} = \left( \gamma_{sp} m_s c^2 + q_s \varphi \right) \delta (X_{sp} - x), \quad (32) \]
\[ P_{sp} = \left( \gamma_{sp} m_s \dot{X}_{sp} + \frac{q_s}{c} A \right) \delta (X_{sp} - x). \quad (33) \]

Equation (31) is called weak EL equation [8], and the qualifier “weak” is used to indicate the fact that only the spatial integral of \( E_{X_{sp}}(\mathcal{L}_{PF}) \) is zero (see Eq. (28)).

Next, we define the symmetry of the action \( \mathcal{A}[\varphi, A, X_{sp}] \) to be a group of transformation

\[ (t, x, \varphi, A, X_{sp}) \mapsto (\tilde{t}, \tilde{x}, \tilde{\varphi}, \tilde{A}, \tilde{X}_{sp}), \quad (34) \]

such that

\[ \int \mathcal{L}_{PF}(t, x, \varphi, A, X_{sp}) dtd^3x = \int \tilde{\mathcal{L}}_{PF}(\tilde{t}, \tilde{x}, \tilde{\varphi}, \tilde{A}, \tilde{X}_{sp}) dtd^3\tilde{x}. \quad (35) \]

For our Lagrangian density (see Eq. (23)), if the group transformation is the time-translation,

\[ (\tilde{t}, \tilde{x}, \tilde{\varphi}, \tilde{A}, \tilde{X}_{sp}) = (t + \epsilon, x, \varphi, A, X_{sp}), \quad \epsilon \in \mathbb{R} \quad (36) \]

the condition (35) will be satisfied because

\[ \frac{\partial \mathcal{L}_{PF}}{\partial \tilde{t}} = 0. \quad (37) \]

Using weak EL equations (31) for particles, the EL equations for fields (see Eqs. (26), (27), (29), (30)), we obtain the energy conservation law

\[ \frac{\partial}{\partial t} \left[ \sum_{s,p} \gamma_{sp} m_s c^2 \delta (X_{sp} - x) + \frac{E^2 + B^2}{8\pi} \right] + \nabla \cdot \left[ \sum_{s,p} \gamma_{sp} m_s c^2 \dot{X}_{sp} \delta (X_{sp} - x) + \frac{c}{4\pi} E \times B \right] = 0. \quad (38) \]
Equation (35) holds for spatial-translation as well

$$
\left( \tilde{t}, \tilde{x}, \tilde{\varphi}, \tilde{A}, \tilde{X}_{sp} \right) = \left( t, x + \epsilon X, \varphi, A, X_{sp} + \epsilon X \right), \quad \epsilon \in \mathbb{R}
$$

(39)

because $L_{PF}$ satisfies

$$
\frac{\partial L_{PF}}{\partial \dot{x}} + \sum_{s,p} \frac{\partial L_{PF}}{\partial \dot{X}_{sp}} = 0.
$$

(40)

As a consequence, the momentum conservation law due to this symmetry can be written as

$$
\frac{\partial}{\partial t} \left[ \sum_{s,p} \gamma_{sp} m_s \dot{X}_{sp} + \frac{E \times B}{4\pi c} \right] + \nabla \cdot \left[ \sum_{s,p} \gamma_{sp} m_s \dot{X}_{sp} \dot{X}_{sp} + \frac{E^2 + B^2}{8\pi} I - \frac{EE + BB}{4\pi} \right] = 0.
$$

(41)

Details of the derivation can be found in Ref. [8]. However, in relativistic cases, this split form of space and time is not elegant. The space and time should be treated as one identity in the most fundamental and geometric approach. In the following sections, we will explore the geometric way to establish the relations between symmetries and conservation laws.

Now, let’s discuss the necessity of the weak Euler-Lagrange equation introduced in the present study. We note that one can construct a velocity field $v(x, t)$ on spacetime using particles’ trajectories $X_{sp}(t)$ as $v(x, t) = \sum_{s,p} v_{sp}(x, t) \delta(x - X_{sp}(t))$. However, in the variation procedure, we cannot treat the velocity field $v(x, t)$ defined this way on spacetime as an independent field that can be varied freely by an arbitrary $\delta v(x, t)$. In the variation procedure, the quantities that can be independently varied are the 4-potential $A(x, t)$ and particles’ trajectories $X_{sp}(t)$. Obviously, $A(x, t)$ and $X_{sp}(t)$ are defined on different domain. This is the reason that we need to introduce the weak Euler-Lagrange equation to overcome this difficulty.

Having said that, there is indeed an alternative approach, if we insist on varying the velocity field $v(x, t)$, instead of $X_{sp}(t)$. In this case, the velocity field $v(x, t)$ cannot be varied arbitrarily. The variation $\delta v(x, t)$ needs to satisfy certain constraints. A systematic approach for this kind of constrained variation has been developed in the context of Euler-Poincare reduction [21–23]. The dynamic equation resulting from the constrained variation will be very different from the standard Euler-Lagrange equation, and will complicate the symmetry analysis. In the present study, we will not pursue along this route. Note that the standard Euler-Lagrange equation needs to be amended in either approach.
III. GEOMETRIC LAGRANGIAN AND GEOMETRIC EULER-LAGRANGIAN EQUATION FOR A SINGLE PARTICLE

For a classical particle, the action can be expressed as

\[ A = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt, \]  

(42)

where \( L(x, \dot{x}, t) \) is the Lagrangian. Apply the principle of least action,

\[ 0 = \delta A = \int_{t_1}^{t_2} \delta L(x, \dot{x}, t) dt = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right] \cdot \delta x dt + \left[ \frac{\partial L}{\partial \dot{x}} \cdot \delta x \right]_{t_1}^{t_2}. \]  

(43)

Because of the fact that \( \delta x(t_1) = \delta x(t_2) = 0 \) and the arbitrariness of \( \delta x \), we obtain the EL equation for the dynamics of the particle

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0. \]  

(44)

For relativistic particles, the variational principle is more involved. In special relativity, the Lagrangian should be Lorentz invariant. For this reason, the integral variable of the action should be the proper time \( \tau \) and the action is

\[ A = \int_{\tau_1}^{\tau_2} \tilde{L}(\chi, \dot{\chi}, \tau) d\tau, \]  

(45)

where \( \chi \) and \( \dot{\chi} \equiv d\chi/d\tau \) are the world-line and 4-velocity of the particle, respectively. The Lagrangian \( \tilde{L} \) is manifestly covariant and is called geometric Lagrangian in the present study. The integral in Eq. (45) is along any possible world-lines passing \( p_1 \) and \( p_2 \) in space-time. If we parameterize the world-line by the proper time \( \tau \), then the Eq. (45) can be written as

\[ A = \int_{\tau_1}^{\tau_2} \tilde{L}(\chi, \dot{\chi}, \tau) d\tau, \]  

(46)

where \( \tau_1 \) is the proper time at the beginning and \( \tau_2 \) is the proper time at the end. We can choose the parameter \( \tau_1 \) to be same for any possible world-line. But, the parameter \( \tau_2 \) are generally different because of the different length for different world-lines passing the given space-time point \( p_2 \), which means \( \delta(d\tau) \neq 0 \). This is different from the non-relativistic case,
which implicitly assumes $\delta(dt) = 0$. The Hamiltonian principle is

$$0 = \delta A = \int_{\tau_1}^{\tau_2} \delta \tilde{L}(\chi, \dot{\chi}, \tau) d\tau + \int_{\tau_1}^{\tau_2} \tilde{L}(\chi, \dot{\chi}, \tau) \delta(d\tau), \quad (47)$$

where

$$\delta(d\tau) = \frac{1}{c^2} \delta \left( \frac{d\chi^\mu d\chi_{\mu}}{d\tau} \right) = \frac{1}{c^2} \delta \left[ \frac{d\chi^\mu d\chi_{\mu}}{d\tau} \right] = \frac{1}{c^2} \delta [d\chi^\mu \dot{\chi}_{\mu}]$$

$$= \frac{1}{c^2} \delta (d\chi^\mu) \dot{\chi}_{\mu} + \frac{1}{c^2} d\chi^\mu \delta (\dot{\chi}_{\mu}) = \frac{1}{c^2} \delta (d\chi^\mu) \dot{\chi}_{\mu} + \frac{1}{c^2} \dot{\chi}^\mu \delta (\dot{\chi}_{\mu}) d\tau. \quad (48)$$

Here, $\chi^\mu$ and $\dot{\chi}^\mu$ ($\mu = 0, 1, 2, 3$) are the components of the world-line and 4-velocity of the particle. From the mass-shell condition

$$P^\mu P_{\mu} = m_0^2 c^2 \text{ or } \dot{\chi}^\mu \dot{\chi}_{\mu} = c^2, \quad (49)$$

where $P^\mu = m_0 \dot{\chi}^\mu$ are the components of 4-momentum and $m_0$ is the rest mass. We have

$$0 = \delta (c^2) = \delta (\dot{\chi}^\mu \dot{\chi}_{\mu}) = \delta (\dot{\chi}^\mu) \dot{\chi}_{\mu} + \dot{\chi}^\mu \delta (\dot{\chi}_{\mu}) = 2 \dot{\chi}^\mu \delta (\dot{\chi}_{\mu}), \quad (50)$$

which means that Eq. (48) can be reduced to

$$\delta(d\tau) = \frac{1}{c^2} \dot{\chi}_{\mu} \delta (d\chi^\mu). \quad (51)$$

The second term on the right-hand side of Eq. (47) is

$$\frac{1}{c^2} \int_{\tau_1}^{\tau_2} \tilde{L}\dot{\chi}_{\mu} \delta (d\chi^\mu) = \frac{1}{c^2} \int_{\tau_1}^{\tau_2} \tilde{L}\dot{\chi}_{\mu} \frac{d(\delta\chi^\mu)}{d\tau} d\tau$$

$$= -\frac{1}{c^2} \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} \left[ \tilde{L}\dot{\chi}_{\mu} \right] (\delta\chi^\mu) d\tau + \left[ \tilde{L}\dot{\chi}_{\mu} (\delta\chi^\mu) \right]^{\tau_2}_{\tau_1}$$

$$= -\frac{1}{c^2} \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} \left[ \tilde{L}\dot{\chi}_{\mu} \right] \delta\chi^\mu d\tau. \quad (52)$$

The first term on the right-hand side of Eq. (47) is
\[
\int_{\tau_1}^{\tau_2} \left[ \frac{\partial \tilde{L}}{\partial \dot{\chi}^\mu} \delta \chi^\mu + \frac{\partial \tilde{L}}{\partial \ddot{\chi}^\mu} \delta \dot{\chi}^\mu \right] d\tau \\
= \int_{\tau_1}^{\tau_2} \left\{ \frac{\partial \tilde{L}}{\partial \chi^\mu} \delta \chi^\mu + \frac{\partial \tilde{L}}{\partial \dot{\chi}^\mu} \left[ \frac{d (\delta \chi^\mu)}{d\tau} - \frac{1}{c^2} \dot{\chi}^\nu \frac{d (\delta \chi^\nu)}{d\tau} \right] \right\} d\tau \\
= \int_{\tau_1}^{\tau_2} \frac{\partial \tilde{L}}{\partial \chi^\mu} \delta \chi^\mu d\tau + \int_{\tau_1}^{\tau_2} \frac{\partial \tilde{L}}{\partial \dot{\chi}^\mu} - \frac{1}{c^2} \frac{\partial \tilde{L}}{\partial \chi^\nu} \dot{\chi}^\nu \frac{d (\delta \chi^\mu)}{d\tau} d\tau \\
= \int_{\tau_1}^{\tau_2} \left\{ \frac{\partial \tilde{L}}{\partial \chi^\mu} \delta \chi^\mu - \frac{d}{d\tau} \left[ \frac{\partial \tilde{L}}{\partial \dot{\chi}^\mu} - \frac{1}{c^2} \dot{\chi}^\nu \frac{\partial \tilde{L}}{\partial \chi^\nu} \right] \right\} \delta \chi^\mu d\tau, \\
\] (53)

where used if made of the following identity,

\[
\delta \dot{\chi}^\mu = \frac{d (\delta \chi^\mu)}{d\tau} - \frac{1}{c^2} \dot{\chi}^\nu \frac{d (\delta \chi^\nu)}{d\tau} = \frac{d (\delta \chi^\mu)}{d\tau} - \frac{1}{c^2} \dot{\chi}^\nu \frac{d (\delta \chi^\nu)}{d\tau}. \\
\] (54)

Substituting Eqs. (52) and (53) into Eq. (47), we obtain

\[
0 = \delta A = \int_{\tau_1}^{\tau_2} \left\{ \frac{\partial \tilde{L}}{\partial \chi^\mu} - \frac{d}{d\tau} \left[ \frac{\partial \tilde{L}}{\partial \dot{\chi}^\mu} - \frac{1}{c^2} \dot{\chi}^\nu \frac{\partial \tilde{L}}{\partial \chi^\nu} \right] \right\} \delta \chi^\mu d\tau, \\
\] which implies

\[
\frac{\partial \tilde{L}}{\partial \chi^\mu} - \frac{d}{d\tau} \left[ \frac{\partial \tilde{L}}{\partial \dot{\chi}^\mu} - \frac{1}{c^2} \dot{\chi}^\nu \frac{\partial \tilde{L}}{\partial \chi^\nu} \right] \dot{\chi}^\mu = 0, \\
\] (55)
due to the arbitrariness of \( \delta \chi^\mu \). Equation (55) will be called geometric EL equation. We note that it has been derived by Infeld using the method of Lagrange multiplier [9].

**IV. GEOMETRIC LAGRANGIAN DENSITY**

For particle-field systems, we need to find the density of the geometric Lagrangian. We start from Eq. (22), the non-geometric form of the action of our system. It can be written as [5]

\[
A = - \sum_{s,p} \int m_s c^2 d\tau - \sum_{s,p} \int \frac{q_s}{c} A_\mu d\chi^\mu - \frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} d\Omega, \\
\] (56)
where $\chi_{sp}^\mu (\mu = 0, 1, 2, 3)$ are components of the world-line of $p$th particle of $s$-species, $\tau$ is the proper time, the 4-potential $A_\mu$ and the field-strength tensor $F^{\mu\nu}$ are the functions on the Minkowski space. The boundary of the domain is taken to be at the infinity.

Equation (56) can be easily translated into another form,

$$\mathcal{A} = \sum_{s,p} \int \left( -m_s c^2 - \frac{q_s}{c} A_\mu \dot{\chi}^\mu(\tau) \right) d\tau - \frac{1}{4\pi c} \int \partial^{[\mu} A^{\nu]} \partial_\mu A_\nu d\Omega, \tag{57}$$

by using the relations $\dot{\chi}^\mu = d\chi^\mu/d\tau$ and $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where the symbol $[\mu \nu]$ in Eq. (57) means the anti-symmetrization of the indexes of $\mu$ and $\nu$. The action is manifestly covariant or geometric. However, to obtain the Lagrangian density of the particle-field systems, we should change it into another form by multiplying the first term in Eq. (57) with the equation

$$\int \delta(\chi_{sp} - \chi) d\Omega = 1, \tag{58}$$

where $\chi$ is an arbitrary world point in Minkowski space, and $\delta(\chi_{sp} - \chi)$ is the Dirac delta function. Then equation (57) becomes

$$\mathcal{A} = \int \left[ \sum_{s,p} \int \left( -m_s c^2 - \frac{q_s}{c} A_\mu \dot{\chi}^\mu(\tau) \right) \delta(\chi_{sp} - \chi) d\tau \right] d\Omega - \frac{1}{4\pi c} \int \partial^{[\mu} A^{\nu]} \partial_\mu A_\nu d\Omega. \tag{59}$$

The geometric Lagrangian density is easy to read off from Eq. (59) as

$$\mathcal{L} = \sum_{s,p} \int \left( -m_s c^2 - \frac{q_s}{c} A_\mu \dot{\chi}_{sp}^\mu(\tau) \right) \delta(\chi_{sp} - \chi) d\tau - \frac{1}{4\pi c} \partial^{[\mu} A^{\nu]} \partial_\mu A_\nu. \tag{60}$$

The geometric Lagrangian above can be simplified as

$$\mathcal{L} = \mathcal{L}_P + \mathcal{L}_F = \int \hat{\mathcal{L}}_P d\tau + \mathcal{L}_F \tag{61}$$

by defining

$$\hat{\mathcal{L}}_P = \sum_{s,p} \left( -m_s c^2 - \frac{q_s}{c} A_\mu \dot{\chi}_{sp}^\mu(\tau) \right) \delta(\chi_{sp} - \chi), \tag{62}$$

$$\mathcal{L}_P = \int \hat{\mathcal{L}}_P (\chi_{sp}, \dot{\chi}_{sp}, \chi, A) d\tau \tag{63}$$

and

$$\mathcal{L}_F = -\frac{1}{4\pi c} \partial^{[\mu} A^{\nu]} \partial_\mu A_\nu. \tag{64}$$
Here, \( \hat{L}_P \) is a function of \( \chi, \chi_{sp}, \dot{\chi}_{sp} \) and \( A \). On the other hand, \( L_P \) can be regarded as a functional of the world-lines of particles and a function of the space-time and the field \( A \). Note that \( L_F \) in Eq. (64) is different from \( L_F \) in Eq. (9) by a constant \( c \). This is caused by the fact that in the geometric form of the action given by Eq. (59), the volume form \( d\Omega \) has the dimension of [length]\(^4\).

V. GEOMETRIC WEAK EULER-LAGRANGIAN EQUATION AND ENERGY-MOMENTUM CONSERVATION

Now we determine how the action given by Eq. (59) and geometric Lagrangian density vary in response to the variation of \( \delta\chi_{sp} \) and \( \delta A \). From Eqs. (59)-(64), the variation of action of the particle-field system can be written as

\[
\delta A = \int \frac{\delta}{\delta A} \left[ \int \hat{L}_P (\chi_{sp}, \dot{\chi}_{sp}, \chi, A) \, d\tau \right] \, d\Omega + \int \delta L_F d\Omega, \tag{65}
\]

where the notation \( \delta \left[ \int \hat{L}_P (\chi_{sp}, \dot{\chi}_{sp}, \chi, A) \, d\tau \right]_\alpha \) (\( \alpha = A, \chi_{sp}, \dot{\chi}_{sp} \)) means keeping \( \alpha \) fixed.

For the first term on the right-hand side of Eq. (65), it can be treated by the same procedure for the derivation of Eq. (55),

\[
\int \frac{\delta}{\delta A} \left[ \int \hat{L}_P (\chi_{sp}, \dot{\chi}_{sp}, \chi, A) \, d\tau \right] \, d\Omega = \int \int \left\{ \frac{\partial \hat{L}_P}{\partial \dot{\chi}_{sp}} \frac{D}{D\tau} \left[ \frac{\partial \hat{L}_P}{\partial \chi_{sp}} - \frac{1}{c^2} \left( \chi_{sp} \frac{\partial \hat{L}_P}{\partial \chi_{sp}} - \hat{L}_P \right) \dot{\chi}_{sp} \right] \right\} \delta \chi_{sp} \, d\tau \, d\Omega \tag{66}
\]

The second and third terms on the right-hand side of Eq. (65) are actually

\[
\int \left\{ \frac{\partial L}{\partial A^\mu} - \frac{D}{D\chi^\nu} \left[ \frac{\partial L}{\partial (\partial_\nu A^\mu)} \right] \right\} \delta A^\mu \, d\Omega. \tag{67}
\]

If we define

\[
E_A^\mu = \frac{\partial L}{\partial A^\mu} - \frac{D}{D\chi^\nu} \left[ \frac{\partial L}{\partial (\partial_\nu A^\mu)} \right], \tag{68}
\]

\[
E_{\chi_{sp}^\mu} = \frac{\partial \hat{L}_P}{\partial \chi_{sp}^\mu} - \frac{D}{D\tau} \left[ \frac{\partial \hat{L}_P}{\partial \chi_{sp}^\mu} - \frac{1}{c^2} \left( \chi_{sp} \frac{\partial \hat{L}_P}{\partial \chi_{sp}^\mu} - \hat{L}_P \right) \dot{\chi}_{sp}^\mu \right], \tag{69}
\]

\[
E_{A^\mu} \chi_{sp}^\nu = \frac{\partial \hat{L}_P}{\partial A^\mu} \chi_{sp}^\nu - \frac{D}{D\tau} \left[ \frac{\partial \hat{L}_P}{\partial A^\mu} \chi_{sp}^\nu - \frac{1}{c^2} \left( A^\mu \frac{\partial \hat{L}_P}{\partial A^\nu} - \hat{L}_P \right) \dot{\chi}_{sp}^\nu \right], \tag{70}
\]

\[
E_{\chi_{sp}^\mu} A^\nu = \frac{\partial \hat{L}_P}{\partial \chi_{sp}^\mu} A^\nu - \frac{D}{D\tau} \left[ \frac{\partial \hat{L}_P}{\partial \chi_{sp}^\mu} A^\nu - \frac{1}{c^2} \left( \chi_{sp} \frac{\partial \hat{L}_P}{\partial \chi_{sp}^\mu} - \hat{L}_P \right) \dot{\chi}_{sp}^\nu \right]. \tag{71}
\]
and substitute Eqs. (66)-(69) into Eq. (65), then

\[
\delta A = \int \left[ \int E_{\chi^p} d\Omega \right] \delta \chi^p d\tau + \int E_A^\mu \delta A^\mu d\Omega. \tag{70}
\]

Thus,

\[
E_A^\mu = \frac{\partial L}{\partial A^\mu} - \frac{D}{D\tau} \left[ \frac{\partial L}{\partial (\partial_\nu A^\mu)} \right] = 0, \tag{71}
\]

\[
\int E_{\chi^p} d\Omega \equiv \int \left\{ \frac{\partial \hat{L}_P}{\partial \chi^\nu_{\chi^p}} - \frac{D}{D\tau} \left[ \frac{\partial \hat{L}_P}{\partial \dot{\chi}^\nu_{\chi^p}} - \frac{1}{c^2} \left( \chi^\nu_{\chi^p} \frac{\partial \hat{L}_P}{\partial \chi^\nu_{\chi^p}} - \hat{L}_P \right) \dot{\chi}^p \right] \right\} d\Omega = 0, \tag{72}
\]

by the arbitrariness of \(\delta \chi^\mu_{\chi^p}\) and \(\delta A^\mu\).

Equation (71) is the EL equation of the 4-potential of the electromagnetic field. Substituting Eq. (60) into Eq. (71) gives the Maxwell equation,

\[
\partial^\nu (\partial_\nu A^\mu - \partial_\mu A^\nu) = \frac{4\pi}{c} \sum_{s,p} q_s \int \dot{\chi}^p \delta (\chi^p - \chi) \, ds, \tag{73}
\]

where \(s\) is the length of the world-line, i.e.,

\[
ds = cd\tau. \tag{74}
\]

The 4-current is

\[
J^\mu = \sum_{s,p} q_s \int \dot{\chi}^p \delta (\chi^p - \chi) \, ds. \tag{75}
\]

Equation (72), will be called geometric sub-manifold EL equation because it is defined only on the world-line after the integrating over the space-time variable \(\chi^p\). If \(\chi^p\) were a function of the entire space-time domain, then \(E_{\chi^p}\) would vanish everywhere, as in the case for 4-potential \(A\). Generally, we expect that \(E_{\chi^p} \neq 0\).

We now derive an expression for \(E_{\chi^p}\) by substituting Eq. (62) into Eq. (69). For the first term in \(E_{\chi^p}\),

\[
\frac{\partial \hat{L}_P}{\partial \chi^\nu_{\chi^p}} = \left( -m_s c^2 - \frac{q_s}{c} A^\nu_{\chi^p} \right) \frac{\partial \delta_2}{\partial \chi^p} \tag{76}
\]

where \(\delta_2 \equiv \delta (\chi^p - \chi)\). The second term of \(E_{\chi^p}\) is given by

\[
- \frac{D}{D\tau} \left( \frac{\partial \hat{L}_P}{\partial \dot{\chi}^\nu_{\chi^p}} \right) = \frac{q_s}{c} A^\mu_{\chi^p} \frac{\partial \delta_2}{\partial \chi^\nu} \tag{77}
\]

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The third and fourth terms came from the mass-shell condition, and can be rewritten as
\[-\frac{D}{D\tau} \left( -\frac{1}{c^2} \dot{\chi}_s^\nu \frac{\partial \mathcal{L}_P}{\partial \dot{\chi}_s^\nu} \right) \]
\[= -\frac{q_s}{c^2} A_\nu \ddot{\chi}_s^\nu \dot{\chi}_s^\mu \delta_2 - \frac{q_s}{c^2} A_\nu \ddot{\chi}_s^\nu \dot{\chi}_s^\mu \delta_2 - \frac{q_s}{c^2} A_\nu \ddot{\chi}_s^\nu \dot{\chi}_s^\mu \frac{D \delta_2}{D\tau} \]
\[= -\frac{q_s}{c^2} A_\nu \ddot{\chi}_s^\nu \dot{\chi}_s^\mu \delta_2 - \frac{q_s}{c^2} A_\nu \ddot{\chi}_s^\nu \dot{\chi}_s^\mu \delta_2 + \frac{D}{D\chi^\alpha} \left( \frac{q_s}{c^2} A_\nu \ddot{\chi}_s^\nu \dot{\chi}_s^\mu \dot{\chi}_s^\sigma \delta_2 \right) \]
\[-\frac{q_s}{c^2} \chi_s^\nu \frac{\partial A_\nu}{\partial \chi_s^\sigma} \dot{\chi}_s^\sigma \dot{\chi}_s^\mu \delta_2, \] (78)
and
\[-\frac{D}{D\tau} \left( \frac{1}{c^2} \ddot{\mathcal{L}}_P \dot{\chi}_s^\mu \right) \]
\[= -\frac{D}{D\chi^\nu} \left[ \left( m_s + \frac{q_s}{c^2} A_\sigma \dot{\chi}_s^\sigma \right) \dot{\chi}_s^\mu \ddot{\chi}_s^\nu \delta_2 \right] + \frac{D}{D\chi^\alpha} \left( \frac{q_s}{c^2} \ddot{\chi}_s^\nu \frac{\partial A_\sigma}{\partial \chi_s^\nu} \dot{\chi}_s^\sigma \dot{\chi}_s^\mu \delta_2 \right) \]
\[+ m_s \ddot{\chi}_s^\mu \delta_2 + \frac{q_s}{c^2} A_\sigma \dot{\chi}_s^\sigma \dot{\chi}_s^\mu \delta_2 + \frac{q_s}{c^2} A_\sigma \ddot{\chi}_s^\sigma \dot{\chi}_s^\mu \delta_2. \] (79)

Therefore,
\[
E_{\chi_s^\mu} = \frac{D}{D\chi^\nu} \left\{ \left[ \left( m_s c^2 + \frac{q_s}{c^2} A_\sigma \dot{\chi}_s^\sigma \right) \eta_\mu^\nu - \left( \frac{q_s}{c^2} A_\mu + m_s \dot{\chi}_s^\mu \right) \dot{\chi}_s^\nu \right] \ddot{\chi}_s^\nu \right\} \]
\[+ \left[ m_s \ddot{\chi}_s^\mu \delta_2 - \frac{q_s}{c^2} \ddot{\chi}_s^\nu \left( \frac{\partial A_\mu}{\partial \chi_s^\mu} - \frac{\partial A_\mu}{\partial \chi_s^\nu} \right) \right] \ddot{\chi}_s^\nu \delta_2. \] (80)

Substituting Eq. (80) into the geometric sub-manifold EL equation (72), we immediately obtain the equation of motion for particles,
\[m_s \ddot{\chi}_s^\mu = \frac{q_s}{c} \left( \frac{\partial A_\nu}{\partial \chi_s^\mu} - \frac{\partial A_\mu}{\partial \chi_s^\nu} \right) \dot{\chi}_s^\nu. \] (81)
The right term in Eq. (81) is the 4-Lorenzen force on particles. Then, \(E_{\chi_s^\mu}\) reduces to
\[
E_{\chi_s^\mu} = \frac{\partial \dot{\mathcal{L}}_P}{\partial \dot{\chi}_s^\mu} - \frac{D}{D\tau} \left[ \frac{\partial \dot{\mathcal{L}}_P}{\partial \dot{\chi}_s^\nu} - \frac{1}{c^2} \left( \chi_s^\nu \frac{\partial \dot{\mathcal{L}}_P}{\partial \dot{\chi}_s^\nu} - \dot{\mathcal{L}}_P \right) \dot{\chi}_s^\mu \right] \]
\[= \frac{D}{D\chi^\nu} \left\{ \left[ \left( m_s c^2 + \frac{q_s}{c^2} A_\sigma \dot{\chi}_s^\sigma \right) \eta_\mu^\nu - \left( \frac{q_s}{c^2} A_\mu + m_s \dot{\chi}_s^\mu \right) \dot{\chi}_s^\nu \right] \ddot{\chi}_s^\nu \right\}. \] (82)
As expected, \(E_{\chi_s^\mu} \neq 0\). We will refer to Eq. (82) as the geometric weak EL equation [8], which play a significant role in subsequent analysis of the local conservation laws. The qualifier “weak” is used to indicate the fact that only the space-time integral of \(E_{\chi_s^\mu}\) is zero. Unlike the non-relativistic case (see Eq. (31) ), the geometric weak EL equation here is written as a differential equation about \(\dot{\mathcal{L}}_P\), instead of the Lagrangian density.
Another crucial step to systematically obtain conservation laws is finding the symmetries. A symmetry group of action $\mathcal{A}$ is defined by a continuous transformation

$$(\chi, [\chi_{sp}], A, \partial A) \mapsto (\bar{\chi}, [\bar{\chi}_{sp}], \bar{A}, \partial \bar{A})$$

(83)

such that

$$\int \mathcal{L} (\chi, [\chi_{sp}], A, \partial A) \, d\Omega = \int \tilde{\mathcal{L}} (\bar{\chi}, [\bar{\chi}_{sp}], \bar{A}, \partial \bar{A}) \, d\Omega,$$

(84)

where the indexes for physical quantities has been omitted to simplify the notations. The symbol $[\beta]$ ($\beta = \chi_{sp}, \dot{\chi}_{sp}, \bar{\chi}_{sp}, \dot{\bar{\chi}}_{sp}$) indicates that the geometric Lagrangian density $\mathcal{L}$ is a functional of $\beta$. For the following transformation defined by

$$(\bar{\chi}, [\bar{\chi}_{sp}], \bar{A}, \partial \bar{A}) = (\chi + \epsilon X, [\chi_{sp} + \epsilon X], [\dot{\chi}_{sp}], A, \partial A), \quad \epsilon \in \mathbb{R}$$

(85)

where $X$ is a given constant 4-vector field, the condition (84) is satisfied because

$$\mathcal{L} = \int \hat{\mathcal{L}}_P (\chi_{sp}, \dot{\chi}_{sp}, \chi, A) \, d\tau + \mathcal{L}_F = \int \hat{\mathcal{L}}_P (\chi + \epsilon X, \chi_{sp} + \epsilon X, \dot{\chi}_{sp}, A) \, d\tau + \mathcal{L}_F,$$

(86)

with $\tilde{A}(\bar{\chi}) = A(\chi) = A(\bar{\chi} - \epsilon X)$ and $\tilde{\partial}A(\bar{\chi}) = \partial A(\chi) = \partial A(\bar{\chi} - \epsilon X)$. Equation (86) should be transformed into a partial differential equation before we apply it in deriving conservation laws. The symmetry condition is

$$\left[ \frac{d\mathcal{L}}{d\epsilon} \right]_{\epsilon=0} \equiv 0,$$

(87)

which is equivalent to

$$0 = \int d\tau \left[ \frac{d(\chi^\mu + \epsilon X^\mu)}{d\epsilon} \frac{\partial \hat{\mathcal{L}}_P}{\partial (\chi^\mu + \epsilon X^\mu)} + \sum_{s,p} \frac{d(\chi_{sp}^\mu + \epsilon X_{sp}^\mu)}{d\epsilon} \frac{\partial \hat{\mathcal{L}}_P}{\partial (\chi_{sp}^\mu + \epsilon X_{sp}^\mu)} \right]_{\epsilon=0}$$

$$= X^\mu \int \left( \frac{\partial \hat{\mathcal{L}}_P}{\partial \chi^\mu} + \sum_{s,p} \frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu} \right) \, d\tau.$$

(88)

Therefore,

$$\int \left[ \frac{\partial \hat{\mathcal{L}}_P}{\partial \chi^\mu} + \sum_{s,p} \frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu} \right] \, d\tau = 0,$$

(89)

or

$$\frac{\partial \mathcal{L}}{\partial \chi^\mu} + \sum_{s,p} \int \frac{\partial \hat{\mathcal{L}}_P}{\partial \chi_{sp}^\mu} \, d\tau = 0$$

(90)

Due to the integral, Eq. (90) indicates that the corresponding vector field of this symmetry for the particle-field systems is infinite dimensional.
Next, we establish the connection between the symmetry of the particle-field system, i.e., Eq. (90), and geometric conservation laws. The first term in Eq. (90) is

$$\frac{\partial L}{\partial \chi^\mu} - \frac{D L}{D \chi^\mu} \partial_A \frac{\partial L}{\partial A} - \frac{D (\partial_A A)}{D \chi^\mu} = \frac{D}{D \chi^\nu} \left[ L \eta^\nu_{\mu} - \partial_A^\nu \eta_A^\mu \right], \quad (91)$$

where use is made of the EL equation, i.e., Eq. (71). For second term in Eq. (90), we use the geometric weak EL equation (82) to obtain

$$\int_{p_1}^{p_2} \frac{\partial \hat{L}_P}{\partial \chi^\mu} d\tau = \frac{D}{D \chi^\nu} \left\{ \int \left[ \left( m_s c^2 + \frac{q_s}{c} A^\sigma \hat{x}^\sigma \right) \eta^\nu_{\mu} - \left( \frac{q_s}{c} A + m_s \hat{x}^\sigma \right) \hat{x}^\nu_{\mu} \right] d\tau \right\}$$

$$+ \left[ \frac{\partial \hat{L}_P}{\partial \chi^\nu} - \frac{1}{c^2} \left( \chi^\nu_{\mu} \eta_{\sigma} - \hat{L}_P \right) \hat{x}^\nu_{\mu} \right]_{p_1}^{p_2}. \quad (92)$$

The boundaries, i.e., $p_1$ and $p_2$ of the integral of Eq. (92) should be extended to infinity, which will make the last term vanishes because of the existence of the Dirac delta function in $\hat{L}_P$.

Bringing Eqs. (91), (92) and (60) into Eq. (90), we obtain

$$\frac{D}{D \chi^\nu} \left[ \sum_{s,p} m_s \int \hat{x}^\nu_{\mu} \hat{x}^\nu_{\mu} \delta_2 dS - \frac{1}{2\pi} \partial_A A^\sigma \hat{x}^\sigma - \frac{1}{4\pi} A^\mu \partial_A^\nu \hat{F}^\nu_{\mu} + \frac{1}{16\pi} F_{\rho\sigma} F^\rho_{\sigma} \eta^\nu_{\mu} \right] = 0, \quad (93)$$

which is equivalent to

$$\frac{D}{D \chi^\nu} \left\{ \sum_{s,p} m_s \int \hat{x}^\nu_{\mu} \hat{x}^\nu_{\mu} \delta_2 d\tau + \frac{1}{4\pi} \left( -F^\sigma_{\mu} F^\nu_{\sigma} + \frac{1}{4} \eta^\nu_{\mu} F_{\rho\sigma} F^\rho_{\sigma} \right) - \frac{1}{4\pi} \left[ \eta_{\sigma} (A^\mu F^\nu_{\mu}) \right] \right\} = 0, \quad (94)$$

with the identity

$$- \frac{1}{2\pi} \partial_A A^\sigma \hat{x}^\sigma - \frac{1}{4\pi} A^\mu \partial_A^\nu \hat{F}^\nu_{\mu} = - \frac{1}{4\pi} F_{\mu\sigma} F^\nu_{\nu} - \frac{1}{4\pi} \partial_A (A^\mu F^\nu_{\nu}). \quad (95)$$

The last term in Eq. (94) is zero because

$$- \frac{1}{4\pi} \frac{D}{D \chi^\nu} \partial_A (A^\mu F^\nu_{\nu}) = - \frac{1}{4\pi} \frac{D}{D \chi^\nu} \left( \frac{D}{D \chi^\sigma} (A^\mu F^\nu_{\nu}) \right) \equiv 0, \quad (96)$$

where $(\nu \mu)$ is the total symmetrization of the indexes of $\nu$ and $\mu$. Finally, we arrive at the geometric, or manifestly covariant, energy-momentum conservation laws

$$\frac{\partial}{\partial \chi^\nu} \left\{ \sum_{s,p} m_s \int \hat{x}^\nu_{\mu} \hat{x}^\nu_{\mu} \delta_2 ds + \frac{1}{4\pi} \left( -F^\sigma_{\mu} F^\nu_{\sigma} + \frac{1}{4} \eta^\nu_{\mu} F_{\rho\sigma} F^\rho_{\sigma} \right) \right\} = 0. \quad (97)$$
where we regard the field as defined on $\chi$, and $\partial/\partial\chi^\nu \equiv D/D\chi^\nu$. In terms of energy-momentum tensor, Eq. (97) is
\begin{equation}
\partial_\nu T^{\mu\nu} = 0, \tag{98}
\end{equation}
where
\begin{equation}
T^{\mu\nu} = T_P^{\mu\nu} + T_E^{\mu\nu}, \tag{99}
\end{equation}
\begin{equation}
T_P^{\mu\nu} = \sum_{s,p} m_s \int \dot{\chi}^\mu_{sp} \dot{\chi}^\nu_{sp}{\delta_2 ds}, \tag{100}
\end{equation}
\begin{equation}
T_E^{\mu\nu} = \frac{1}{4\pi} \left( -F^{\mu\sigma} F^\nu_{\sigma} + \frac{1}{4} \eta^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \right). \tag{101}
\end{equation}
Here, $T_P^{\mu\nu}$, $T_E^{\mu\nu}$ and $T^{\mu\nu}$ are the energy-momentum tensors of the particles, electromagnetic field and the particle-field systems written in the geometric form, respectively, and it’s easy to check that all of these tensors are symmetric. The energy-momentum tensor $T_E^{\mu\nu}$ for electromagnetic field described by Eq. (101) is well-known. However, to the best of our knowledge, the geometric energy-momentum tensor for particles given in Eq. (100) has not been derived previously. Instead, the energy-momentum tensor for particle is typically written in the existing literature \[5\] as
\begin{equation}
T_P^{\mu\nu} = \sum_{s,p} m_s \delta (X_{sp}(t) - \mathbf{x}) \dot{\chi}^\mu_{sp}(t) \dot{\chi}^\nu_{sp}(t) \frac{d\tau(t)}{dt}, \tag{102}
\end{equation}
where a Lorentzian coordinate system $\{t, \mathbf{x}\}$ is chosen. Obviously, Eq. (102) is not a geometric form as Eq. (100). We can prove that Eq. (100) recovers Eq. (102) when a coordinate system is chosen. For this purpose, we first show that
\begin{equation}
\int \delta (\chi_{sp} - \chi) g (\chi_{sp}) d\delta s = \gamma_{sp}^{-1}(t) \delta (X_{sp}(t) - \mathbf{x}) g (\chi_{sp}(t)), \tag{103}
\end{equation}
where $\gamma_{sp}^{-1}(t) \equiv \sqrt{1 - \dot{\mathbf{x}}_{sp}^2(t)/c^2}$, and $g(\chi_{sp})$ is an arbitrary field. The left-hand side of Eq. (103) is
\begin{align}
&\int \delta (\chi_{sp} - \chi) g (\chi_{sp}) d\delta s \\
&\equiv \int \delta [\chi_{sp} (s_{sp}) - \chi] g [\chi_{sp} (s_{sp})] d\delta s_{sp} \tag{104} \\
&= \int \delta [\chi_{sp} (s_{sp} (t_{sp})) - \chi] g [\chi_{sp} (s_{sp} (t_{sp}))] \frac{ds_{sp}}{dt_{sp}} dt_{sp} \\
&= \int \delta [c (t_{sp} - t)] \delta (X_{sp} (t_{sp}) - \mathbf{x}) g [\chi_{sp} (t_{sp})] \frac{cd\tau_{sp} (t_{sp})}{dt_{sp}} dt_{sp}. \tag{105}
\end{align}
where $t_{sp}$ is the time parameter for each world-line, and $x_{sp}(t_{sp})$ is the space position for $sp$-particle at time $t_{sp}$. Because

$$\frac{d\tau_{sp}(t_{sp})}{dt_{sp}} = \gamma_{sp}^{-1}(t_{sp}),$$

(106)

and

$$\delta [c (t_{sp} - t)] = \frac{1}{c} \delta (t_{sp} - t),$$

(107)

we have

$$\int \delta (x_{sp} - \chi) g (x_{sp}) \, ds = \int \gamma_{sp}^{-1}(t_{sp}) \delta [X_{sp}(t_{sp}) - x] g [x_{sp}(t_{sp})] \delta (t_{sp} - t) \, dt_{sp}$$

$$= \gamma_{sp}^{-1}(t) \delta [X_{sp}(t) - x] g [x_{sp}(t)],$$

(108)

which is Eq. (103). If we take $g (x_{sp}) = m_{s}\dot{\chi}_{sp}^{\mu} \dot{\chi}_{sp}^{\nu}$, then the geometric energy-momentum tensor for particles is

$$T_{\mu\nu}^{sp} = \sum_{s,p} m_{s} \int \delta (x_{sp} - \chi) \dot{\chi}_{sp}^{\mu} \dot{\chi}_{sp}^{\nu} \, ds = m_{s}\gamma_{sp}^{-1}(t) \delta [X_{sp}(t) - x] \dot{\chi}_{sp}^{\mu}(t) \dot{\chi}_{sp}^{\nu}(t)$$

$$= \sum_{s,p} m_{s} \delta [X_{sp}(t) - x] \dot{\chi}_{sp}^{\mu}(t) \dot{\chi}_{sp}^{\nu}(t) \frac{d\tau_{sp}(t)}{dt}.$$  

(109)

This confirms that the geometric energy-momentum tensor for particles recovers the non-geometric form in a chosen coordinate system.

**VI. CONCLUSIONS**

In this paper, we have developed a manifestly covariant, or geometric, field theory for the relativistic classical particle-field systems often encountered in plasma physics, accelerator physics, and astrophysics. The connection between space-time symmetry and energy-momentum conservation laws is demonstrated geometrically. In our theoretical formalism, space and time are treated with equal footing, i.e., space-time is treated as one identity without choosing a coordinate system. This is different from existing field theories where it is necessary to split space and time coordinates at certain stage, and thus the manifestly covariant property is lost.

There are several unique features in the geometric field theory developed. The first is the mass-shell condition, which induces two new terms in the geometric EL equation.
for particles. The geometric Lagrangian density of particle-field systems is a functional of particles’ world-lines (see equation (60)), which makes the symmetry vector field of the systems lies on the infinitive dimensional space (see Eq. (92)). Another feature of the theory is that particles and fields reside on different manifolds. The domain of the particle field can be the proper time or any other parameterization for the world-lines, and the electromagnetic field, on the other hand, are defined on space-time. In order to establish geometrically the connection between symmetries and energy-momentum conservations, a geometric weak EL equation (81) for particles is derived. Combining the EL equation (71) for field and the geometric weak EL equation (81) for particles, symmetries and conservation laws could be established geometrically. Using the theory, we derived for the first time a geometric expression for the energy-momentum tensor for particles in Eq. (100), which recovers the non-geometric form in the existing literature [5] for a chosen coordinate system.

In the present study, we make use of proper time. We note that different particles have different proper time, which is not synchronized in the laboratory frame. This brings difficulties if one would like to use proper time for particle-in-cell (PIC) simulations. However, proper time can be beneficial in certain situations. For example, proper time has been utilized to construct explicit symplectic integrators for relativistic dynamics of charged particles [19, 24]. As a matter of fact, the geometric field theory for classical particle-field systems in the present study can only be established with the help of proper time. As for the specific application of proper time in PIC simulations, more investigation is needed.

The very facts that proper time can be used to construct explicit symplectic integrators and that it is essential in establishing the geometric field theory for classical particle-field systems suggest that proper time could play a role in developing advanced PIC algorithms [25, 27]. For example, we can investigate the possibility of using different proper time-steps in the symplectic integrators for different particles such that they are synchronized in the laboratory frame. This topic will be explored in future study.

ACKNOWLEDGMENTS

This research is supported by National Magnetic Confinement Fusion Energy Research Project (2015GB111003, 2014GB124005), National Natural Science Foundation of China (NSFC-11575185, 11575186, 11305171), JSPS-NRF-NSFC A3 Foresight Program (NSFC-
11261140328), Key Research Program of Frontier Sciences CAS (QYZDB-SSW-SYS004), the Geo-Algorithmic Plasma Simulator (GAPS) Project, and National Magnetic Confinement Fusion Energy Research Project (2013GB111002B).

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