“Cold convection” in porous layers salted from above

Salvatore Rionero

Abstract A multicomponent fluid mixture saturating a porous rotating horizontal layer, heated from below and salted partly from below and partly from above, in the Darcy–Boussinesq scheme, is investigated. Conditions guaranteeing the “cold convection” i.e. the instability of the thermal conduction solution irrespective of the temperature gradient, are furnished.

Keywords Instability · Porous layers · Thermal conduction

1 Introduction

The heat and mass transfer by convection in horizontal layers, either because of its great geophysical relevance (engineering geology, subsurface fluid motions,...) or the frequent applications (industrial processes, crystal growth, thermal engineering,...), in the past as nowadays, has attracted the attention of many scientists (cfr. [1–23] and the references therein). The essential feature of the phenomenon is as follows: a horizontal layer \( L \) of fluid at rest state is heated from below in such a way that an adverse temperature gradient \( \beta \) is maintained. Because of thermal expansion the fluid at bottom expands as it becomes hotter. When \( \beta \) reaches a critical value \( \beta_c \), the buoyancy overcomes gravity, the fluid raises and a pattern of cellular motion can be seen. This is the onset of the thermal convection [6]. The phenomenon is called Bénard convection because of his experiments (1900). Although the exact parameter for determining the onset of the thermal convection, is not \( \beta_c \) but the non-dimensional Rayleigh number depending on \( \beta \) and other parameters (cf (2.5) 9), the onset of instability of the thermal conduction solution is normally associated to the growing of \( \beta \). On the other hand, when \( L \) is heated from below and salted from above and below then the thermal conduction solution is stabilized by the chemicals (“salts”) salting \( L \) from below and destabilized by the chemicals salting \( L \) from above. Therefore one expects that the growth of the gradient of a chemical species salting \( L \) from above (by alone or together with the growth of the gradients of other species salting \( L \) from above) can produce the instability of the thermal conduction solution.

The present paper is devoted to this type of instability of the thermal conduction solution, but under a further relevant request. Precisely, we look for conditions guaranteeing the instability of the thermal conduction solution irrespective of the temperature gradient i.e. \( \forall \beta > 0 \). As far as we know this instability (analogous but different from the Marangoni effect)—named by us...
“cold convection”—is neither mentioned nor investigated in the literature.\footnote{Since a phenomenon is observable only if it is stable, it follows that the conditions guaranteeing the “cold convection” do not allow the observability of the thermal conduction solution.}

We consider an $m$-component ($m = 1, 2, \ldots$) fluid mixture saturating a porous horizontal rotating layer $L$-heated from below and salted from below by the salts $S_1, S_2, \ldots, S_r$ ($0 \leq r < m$) and from above by the salts $S_{r+1}, \ldots, S_m$, in the Darcy–Boussinesq scheme. Denoting by $R$ the thermal Rayleigh number, $R_s$ the salt $S_s$ Rayleigh number, $T$ the Taylor–Darcy number (cfr. Sect. 2) and by

$$R_C = \pi \left( 1 + \sqrt{1 + T^2} \right), \quad (1.1)$$

the critical thermal Rayleigh number in the absence of salts in $L$ ($m = 0$), our aim is to obtain—among other results—the following general one:

**Theorem 1.1** Either

$$R_s^2 \geq \frac{1}{q_s} \sum_{s=1}^{r} R_s^2 + R_C^2, \quad \forall s \in \{r+1, \ldots, m\}, \quad (1.2)$$

with

$$q_s > 0, \quad \sum_{s=r+1}^{m} \frac{1}{q_s} = 1, \quad (1.3)$$

or (in particular)

$$\exists \bar{s} \in \{r+1, \ldots, m\} : R_{\bar{s}}^2 \geq \sum_{s=1}^{r} R_s^2 + R_C^2, \quad (1.4)$$

guarantees the onset of the “cold convection”.

Section 2 is devoted to the introduction of the Darcy–Boussinesq equations governing the perturbations to the thermal conduction solution. In Sect. 3 the Routh–Hurwitz instability conditions are applied. In the subsequent Sect. 4, conditions guaranteeing the onset of the “cold convection” irrespective of the temperature gradient, are furnished. Sect. 5 is devoted to the discussion. The paper ends with an Appendix in which some useful estimates are obtained.

### 2 Preliminaries

Let $Oxyz$ be an orthogonal frame of reference with fundamental unit vectors $i, j, k$ ($k$ pointing vertically upwards).

We assume that $m$ different chemical species (“salts”) $S_\alpha$ ($\alpha = 1, 2, \ldots, m$), have dissolved in the fluid and have concentrations $C_\alpha$ ($\alpha = 1, 2, \ldots, m$), respectively, and that the equation of state is

$$\rho = \rho_0 \left[ 1 - \alpha_s (T - T_0) + \sum_{x=1}^{m} A_x (C_x - \hat{C}_x) \right],$$

where $\rho_0, T_0, \hat{C}_x$ ($\alpha = 1, 2, \ldots, m$), are reference values of the density, temperature and salt concentrations, while the constants $\alpha_s, A_x$ denote the thermal and solute $S_x$ expansion coefficients respectively. Combining Darcy’s Law with (thermal) energy and mass balance together with the Boussinesq approximation, we obtain the fundamental equations governing the isochoric motions, when $L$ rotates about the vertical axis with constant velocity $\omega = \omega k$

$$\begin{cases}
\nabla P = -\frac{\mu}{K} \mathbf{v} - g \rho_0 [1 - \alpha_s (T - T_0)] \\
+ \sum_{x=1}^{m} A_x (C_x - \hat{C}_x)] - 2 \rho_0 \omega k \times \mathbf{v},
\n\nabla \cdot \mathbf{v} = 0,
\nT_t + \mathbf{v} \cdot \nabla T = k \Delta T,
\nC_\alpha t + \mathbf{v} \cdot \nabla C_\alpha = k_\alpha \Delta C_\alpha,
\end{cases} \quad (2.1)$$

$p_1$ is the pressure field, $\mu$ is the dynamic viscosity, $K$ is the porosity, $\mathbf{v}$ is the velocity, $P = p_1 - \frac{1}{2} \rho_0 [\omega \times \mathbf{x}]^2$, $g$ is the gravity, $k$ is the thermal diffusivity, $K_\alpha$ is the diffusivity of the solute $S_\alpha$.

To (2.1) we append the boundary conditions

$$\begin{cases}
T(0) = T_1, T(d) = T_2.
\end{cases} \quad (2.2)$$

with $T_1, T_2 (< T_1), C_\alpha, C_\alpha^d$ ($\alpha = 1, 2, \ldots, m$), positive constants and $d$ = layer depth. The boundary value problem (2.1), (2.2) admits the conduction solution ($\tilde{v}, \tilde{p}, \tilde{T}, \tilde{C}_x$) given by

$$\begin{cases}
\tilde{v} = 0, \quad \tilde{T} = T_1 - \beta \tilde{z}, \quad \beta = T_1 - T_2 \frac{d}{T_2 - T_1},
\end{cases}$$

$$\tilde{C}_\alpha = C_\alpha - \tilde{z} (\delta C_\alpha) \frac{d}{d}, \quad \tilde{C}_\alpha = C_\alpha - \hat{C}_\alpha = \delta C_\alpha,$$

$$\tilde{P} = p_0 + \rho_0 g \tilde{z} \left[ \frac{\alpha_s \beta}{2} + \sum_{x=1}^{m} A_x (\delta C_x) ^2 \right] + \rho_0 g \tilde{z} \left[ 1 - \alpha (T_1 - T_0) + \sum_{x=1}^{m} A_x (C_\alpha - \hat{C}_x ) \right], \quad (2.3)$$

$\tilde{z}$ is a function of $\tilde{T}$ such that $\tilde{T} = \tilde{T}(\tilde{z})$. The heat balance in the fluid can be written as

$$\begin{cases}
\int_{\tilde{T}}^{T_1} \tilde{q} \tilde{v} \cdot \nabla T d\tilde{z} + \int_{\tilde{T}}^{T_1} \tilde{q} \tilde{v} \cdot \nabla \tilde{T} d\tilde{z} = 0
\end{cases} \quad (2.4)$$

where $\tilde{q}$ is the normal component of the heat flux.
where \( p_0 \) is a constant. Setting
\[
v = \tilde{v} + u, \quad p = \tilde{P} + \Pi, \quad T = \tilde{T} + \theta, \quad C_x = \tilde{C}_x + \Phi_x,
\]
and introducing the scalings
\[
\begin{align*}
\begin{cases}
t = t' \frac{d^2}{K}, & \mathbf{u} = \mathbf{u}' \frac{K}{d}, \\ \Pi = \Pi' \frac{\mu K}{d^3}, & \mathbf{x} = \mathbf{x}' d, \quad \theta = \theta' T' \frac{d}{K}, \end{cases}
\end{align*}
\]
\[
\begin{align*}
\Phi_x = (\Phi_x)' \Phi_x', & \quad T' = \left( \frac{\mu K}{\chi' s_{p_0} K d} \right)^{\frac{1}{2}}, \\
\Phi_x' = \left( \frac{\mu K}{\chi' s_{p_0} K d} \right)^{\frac{1}{2}} R = \left( \frac{\mu K}{\chi' s_{p_0} K d} \right)^{\frac{1}{2}} \\
R_x = \frac{\mu K}{\chi' s_{p_0} K d} \delta C_x), & \quad T' = \frac{\rho_0 k}{\mu} \delta T = T_1 - T_2, \quad H = \text{sgn}(\delta T), \\
H_x = \text{sgn}(\delta C_x), & \quad P_x = \frac{k}{\mu},
\end{align*}
\]
(2.5)
since in the case at stake the layer is heated from below, salted from below by \( S_x \) (with \( \alpha = 1, 2, \ldots, r \)) and from above by \( S_x \) with \( (x = r + 1, \ldots, m) \), it follows that \( H = H_x = 1 \), (for \( \alpha = 1, \ldots, r \)) and \( H_x = -1 \) (for \( \alpha = r + 1, \ldots, m \)) and the equations governing the dimensionless perturbations \( \{u', \Pi', \theta', (\Phi_x')'\} \), omitting the stars, are
\[
\begin{align*}
\nabla \Pi &= -u + \left( R\theta - \sum_{x=1}^{m} R_x \Phi_x \right) \mathbf{k} + T u \times \mathbf{k}, \\
\nabla \cdot u &= 0, \\
\theta_x + u \cdot \nabla \theta &= R u \cdot \mathbf{k} + \Delta \theta, \\
R_x \left( \frac{\partial \Phi_x}{\partial t} + u \cdot \nabla \Phi_x \right) &= R_x u \cdot \mathbf{k} \\
+ \Delta \Phi_x, & \quad \alpha = 1, 2, \ldots, r, \\
R_x \left( \frac{\partial \Phi_x}{\partial t} + u \cdot \nabla \Phi_x \right) &= -R_x u \cdot \mathbf{k} \\
+ \Delta \Phi_x, & \quad \alpha = r + 1, \ldots, m,
\end{align*}
\]
(2.6)
under the boundary conditions
\[
u \cdot \mathbf{k} = \theta = \Phi_x = 0 \text{ on } z = 0, 1.
\]
(2.7)
In (2.5), (2.6) \( R \) and \( R_x \) are the thermal and salt \( S_x \) Rayleigh numbers respectively while \( P_x \) and \( T \) are the salt \( S_x \) Prandtl number and the Taylor–Darcy number.

We assume, as usually done in stability problems in layers, that the horizontal laminar flows are avoided and that

(i) the perturbations \((u, v, w, \theta, \Phi_1, \ldots, \Phi_m)\) are periodic in the \( x \) and \( y \) directions, respectively of periods \( 2\pi a_x, 2\pi a_y \);

(ii) \( \Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1] \) is the periodicity cell;

(iii) \( \mathbf{u}, \Phi_x, (x = 1, \ldots, m) \), \( \theta \) belong to \( W^{2,2}(\Omega) \) and are such that all their first derivatives and second spatial derivatives can be expanded in a Fourier series uniformly convergent in \( \Omega \), and denote by \( L^2(\Omega) \) the set of functions \( \Phi \) such that

\[
\Phi : (x, t) \in \Omega \times \mathbb{R}^+ \rightarrow \Phi(x, t) \in \mathbb{R}, \quad \Phi \in W^{2,2}(\Omega), \forall t \in \mathbb{R}^+, \quad \Phi \text{ is periodic in the } x \text{ and } y \text{ directions of period } \frac{2\pi}{a_x}, \frac{2\pi}{a_y} \text{ respectively and } \Phi|_{z=0} = \Phi|_{z=1} = 0;
\]

(2) \( \Phi \), together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly convergent in \( \Omega \), \( \forall t \in \mathbb{R}^+ \).

Let \( u = (u, v, w) \) and \( \Phi \in (\Phi_1, \ldots, \Phi_m) \). Since the sequence \( \{\sin(npx)\} \) is a complete orthogonal system for \( L^2(0,1) \) (under (2.7)), it follows that, \( \forall \Phi \in L^2(\Omega) \), there exists a sequence \( \{\Phi_n(x, y, t)\}_{n \in \mathbb{N}} \) such that
\[
\Phi = \sum_{n=1}^{\infty} \Phi_n = \sum_{n=1}^{\infty} \Phi_n(x, y, t) \sin n\pi z, \quad \forall \Phi \in (\Phi_1, \ldots, \Phi_m).
\]
On the other hand setting
\[
\begin{align*}
a_n^2 &= a_x^2 + a_y^2, \\
\xi_n &= a_x^2 + n^2 \pi^2, \\
\eta_n &= \frac{a_n^2}{\xi_n + n^2 \pi^2 T^2},
\end{align*}
\]
(2.9)

it follows that Appendix (Proof of (2.10))
\[
w_n = \eta_n \left( R\theta_n - \sum_{x=1}^{m} R_x \Phi_{nx} \right), \quad n \in \mathbb{N},
\]
(2.10)
and in view of (2.8)–(2.10) and
\[
\Delta \Phi_n = -\xi_n \Phi_n, \quad \Phi_n \in (w_n, \theta_n, \Phi_{nx}),
\]
(2.11)

it follows that (2.6), (2.7) are equivalent to
\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix} = \sum_{n=1}^{\infty} \left\{ \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix} - \begin{pmatrix} u \cdot \nabla \Phi_0 \\ u \cdot \nabla \Phi_1 \\ \vdots \\ u \cdot \nabla \Phi_m \end{pmatrix} \right\}
\end{align*}
\]
(2.12)
\[ \Phi_{zn} = 0, \quad \forall n \in \mathbb{N}, \text{ on } z = 0, 1; \]
\[ [\Phi_{zn}]_{r=0} = \Phi_2^{(0)}, \quad \alpha \in \{0, \ldots, m\}, \quad (2.13) \]

with
\[ \mathcal{L}_{rm}^{(n)} = \begin{pmatrix} a_{0,0} & \cdots & a_{0,m} \\ a_{1,0} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{m,0} & \cdots & a_{m,m} \end{pmatrix} \quad (2.14) \]

\[ a_{0,0} = R^2 \eta_1 - \xi_n, \quad a_{0,n} = -RR_2 \eta_n, \]
\[ a_{1,1} = \frac{R_2}{P_1} \eta_1, \quad a_{1,1} = \frac{R_2}{P_1} \eta_n, \quad a_{1,x} = -\frac{RR_1}{P_1}, \quad a_{1,x} = -\frac{RR_1}{P_1}, \]
\[ a_{m,0} = \frac{RR_m}{P_m} \eta_1, \quad a_{m,0} = \frac{RR_m}{P_m} \eta_n, \quad a_{m,m} = \frac{R^2 \eta_m - \xi_n}{P_m}, \]
\[ a_{r,s} = \frac{R^2 \eta_r - \xi_n}{P_r}, \quad a_{r,s} = \frac{R^2 \eta_r - \xi_n}{P_r}, \quad (r \neq s) \quad (3.15) \]

3 Routh–Hurwitz instability conditions

The equation governing the eigenvalues \( \lambda_{zn,0} \) of (2.14) and hence the stability–instability of the thermal conduction solution, can be written
\[ \prod_{\alpha=0}^{m} (\lambda - \lambda_{zn,0}) = 0, \quad (3.1) \]

i.e.
\[ \lambda^{m+1} - I_{1n} \lambda^m + I_{2n} \lambda^{m-1} + \cdots + (-1)^{m+1} I_{(m+1)n} = 0, \quad (3.2) \]

with \( I_{sn}, s \in \{1, 2, \ldots, m+1\} \) characteristic values (invariants) of (2.14) given by
\[ I_{1n} = \sum_{\alpha=0}^{m} \lambda_{zn,0}, \quad I_{2n} = \sum_{\alpha \neq \beta} \lambda_{zn,0} \lambda_{zn,0}, \ldots, I_{(m+1)n} = \prod_{\alpha=0}^{m} \lambda_{zn,0}. \quad (3.3) \]

Setting
\[ \bar{a}_{0n} = 1, \quad \bar{a}_{zn} = (-1)^z I_{zn}, \quad \alpha \in \{1, \ldots, m+1\} \quad (3.4) \]

and introducing the matrix
\[ \begin{pmatrix} \bar{a}_{1n} & \bar{a}_{3n} & \bar{a}_{5n} & \cdots & 0 \\ \bar{a}_{0n} & \bar{a}_{2n} & \bar{a}_{4n} & \cdots & 0 \\ 0 & \bar{a}_{1n} & \bar{a}_{3n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{a}_{(m+1)n} \end{pmatrix} \quad (3.5) \]

and the quantities
\[ \Delta_{1n} = \bar{a}_{1n}, \quad \Delta_{2n} = \begin{vmatrix} \bar{a}_{1n} & \bar{a}_{13} \\ \bar{a}_{0n} & \bar{a}_{2n} \end{vmatrix}, \ldots, \Delta_{mn} = \bar{a}_{mn} \Delta_{(3-1)n}, \quad (3.6) \]

the following Routh–Hurwitz theorem holds [24, 25].

**Theorem 3.1** The conditions
\[ \Delta_{1n} > 0, \quad \Delta_{2n} > 0, \ldots, \Delta_{(m+1)n} > 0, \quad \forall n \in \mathbb{N}, \quad (3.7) \]

are necessary and sufficient in order for all the roots of (3.1) to have negative real parts.

Theorem 31 implies the following ones.

**Theorem 3.2** The condition \((n \in \mathbb{N})\)
\[ \exists s \in \{1, 2, \ldots, m\} : (-1)^{s+1} I_{sn} > 0, \quad (3.8) \]

(i.e. one of the coefficients of (3.2) is negative for at least one \( n \in \mathbb{N} \)) it is sufficient for guaranteeing the instability of the thermal conduction solution.

**Remark 3.1** The characteristic values \( I_{zn} \) are given in terms of the entries of (2.14). Precisely \( I_{zn} \) is obtained by adding the principal minors of orders \( s \) of (2.14).

By virtue of the previous remark, the following theorem holds.

**Theorem 3.3** In terms of the entries of (2.14) it follows that
\[ (i) \quad \text{for } m = 1, r = 0 \ (\text{double diffusive-convection in a rotating porous horizontal layer heated from below and salted from above by } S_1) \]
\[ I_{1n} = \begin{pmatrix} R^2 + R_1^2 \beta \frac{\xi_n}{\eta_n} \end{pmatrix} \eta_n, \]
\[ I_{2n} = \frac{1}{P_1} \left( -R^2 + R_1^2 + \frac{\xi_n}{\eta_n} \right) \eta_n, \quad (3.9) \]
(ii) for \( m = 2, r = 1 \) (ternary diffusive-convection in rotating porous horizontal layers heated from below and salted from above by \( S_1 \) and from above by \( S_2 \)):

\[
\begin{align*}
I_{1n} &= \left\{ R^2 - \left[ \frac{R_1^2}{P_1^2} + \frac{R_2^2}{P_2^2} + \frac{\xi_n}{\eta_n} \left( 1 + \frac{1}{P_1^2} + \frac{1}{P_2^2} \right) \right] \right\} \eta_n, \\
I_{2n} &= \frac{P_1 + P_2}{P_1 P_2} \left[ 1 + \frac{P_1 + P_2 \xi_n}{\eta_n} \right] \frac{R_2}{P_1^2} - \frac{R_1}{P_2^2} \right\} \eta_n, \\
I_{3n} &= \frac{1}{P_1 P_2} \left[ R^2 - \left( \frac{R_1^2 + R_2^2 + \xi_n}{\eta_n} \right) \right] \eta_n, \\
\end{align*}
\]

(3.10)

(iii) for \( m = 2, r = 0 \) (ternary diffusive-convection in rotating porous horizontal layers heated from below and salted from above by \( S_1 \) and from above by \( S_2 \)):

\[
\begin{align*}
I_{1n} &= \left\{ R^2 - \left[ \frac{R_1^2}{P_1^2} + \frac{R_2^2}{P_2^2} + \frac{\xi_n}{\eta_n} \left( 1 + \frac{1}{P_1^2} + \frac{1}{P_2^2} \right) \right] \right\} \eta_n, \\
I_{2n} &= \frac{P_1 + P_2}{P_1 P_2} \left[ 1 + \frac{P_1 + P_2 \xi_n}{\eta_n} \right] \frac{R_2}{P_1^2} - \frac{R_1}{P_2^2} \right\} \eta_n, \\
I_{3n} &= \frac{1}{P_1 P_2} \left[ R^2 + \frac{R_1^2 + R_2^2 - \xi_n}{\eta_n} \right] \eta_n, \\
\end{align*}
\]

(3.11)

(iv) in the general case \((m \in \mathbb{N}, r < m)\) \( I_{1n} \) and \( I_{(m+1)n} \) are given by

\[
\begin{align*}
I_{1n} &= \left( R^2 - \sum_{x=1}^{r} \frac{R_x^2}{P_x^2} + \sum_{x=r+1}^{m} \frac{R_x^2}{P_x^2} \right) \eta_n \\
&\quad - \left( 1 + \sum_{x=1}^{m} \frac{1}{P_x^2} \right) \frac{\xi_n}{\eta_n}, \\
I_{(m+1)n} &= \frac{1}{P_1 P_2 P_3} \left[ R^2 - \left( \sum_{x=1}^{r} \frac{R_x^2}{P_x^2} - \sum_{x=r+1}^{m} \frac{R_x^2 + \xi_n}{\eta_n} \right) \right] \eta_n, \\
&\quad \cdot \eta_n^{\frac{m}{2n}}; \text{ for } m \text{ even}, \\
I_{(m+1)n} &= \frac{1}{P_1 P_2 P_3} \left[ R^2 + \sum_{x=1}^{r} \frac{R_x^2}{P_x^2} - \sum_{x=r+1}^{m} \frac{R_x^2 + \xi_n}{\eta_n} \right] \eta_n, \\
&\quad \cdot \eta_n^{\frac{m}{2n}}; \text{ for } m \text{ odd}. \\
\end{align*}
\]

(3.12)

4 Salts critical Rayleigh numbers for the onset of “cold convection”

Let \( S_2 \) be a chemical specie salting \( L \) from above. We will call critical Rayleigh number of \( S_2 \) for the onset of the “cold convection” the lowest value \( R_2^{C1} \) of \( R_2 \) such that

\[
R_2 \geq R_2^{C1} \Rightarrow \text{the onset of the ‘‘cold convection’’}. \hspace{1cm} (4.1)
\]

**Remark 4.1** For the sake of simplicity we confine ourselves to determine \( R_2^{C} \) only by the applications of theorem 3.3.

**Theorem 4.1** Let \( L \) be heated from below and salted from above by \( S_1 \). Then the critical Rayleigh number of \( S_1 \) for the onset of the “cold convection” is given by

\[
R_{1c} = R_{c}. \hspace{1cm} (4.2)
\]

**Proof** In fact the thermal conduction solution is stable if and only if

\[
I_{1n} < 0, \quad I_{2n} > 0, \quad \forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \hspace{1cm} (4.3)
\]

hence it is unstable if and only if, at least for one couple \((a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \) one of the following relations holds

\[
R_1^2 \geq \left( 1 + \frac{1}{P_1} \right) \frac{\xi_n}{\eta_n}, \quad R_1^2 \geq \frac{\xi_n}{\eta_n}. \hspace{1cm} (4.4)
\]

Since (cfr. Appendix, Proof of (4.5))

\[
\min_{(a^2, n) \in \mathbb{R}^+ \times \mathbb{N}} \frac{\xi_n}{\eta_n} = \pi^2 (1 + \sqrt{1 - T^2}) - T, \hspace{1cm} (4.5)
\]

(4.4) become respectively

\[
R_1^2 \geq (1 + P_1)R_{c}^2, \quad R_1^2 \geq R_{c}^2, \hspace{1cm} (4.6)
\]

and (4.2) immediately follows.

**Theorem 4.2** Let \( L \) be heated from below and salted from below by \( S_1 \) and from above by \( S_2 \). Then the critical Rayleigh number \( R_{2c} \) of \( S_2 \) for the onset of the “cold convection” is given by

\[
R_{2c} = \min \left\{ \frac{P_2}{P_1} R_1^2 + \left( 1 + P_2 + \frac{P_2}{P_1} \right) R_{c}^2, \left( 1 + \frac{P_2}{P_1} \right) R_{c}^2 + \frac{P_2}{P_1} R_1^2 \right\}. \hspace{1cm} (4.7)
\]

**Proof** In the case \((m = 2, r = 1)\)—in view of theorem 33 and (3.10)—it follows that the instability is guaranteed by one of the following conditions

\( \square \) Springer
\[
\begin{align*}
R_2^2 & \geq \frac{p_1}{p_1'} R_1^2 + \left(1 + \frac{p_2}{p_1'}\right) R_C^2, \\
R_2^2 & \geq \frac{1 + p_1'}{1 + p_1} R_1^2 + \left(1 + \frac{p_2}{1 + p_1'}\right) R_C^2, \\
R_2^2 & \geq R_1^2 + R_C^2. 
\end{align*}
\] (4.8)

Then (4.7) immediately follows.

**Remark 4.2** In view of

\[
\begin{align*}
\frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1} + \frac{p_2}{p_1'}\right) R_C^2 & \geq R_1^2 + R_C^2, \\
\Leftrightarrow (P_2 - P_1) R_1^2 + P_2(1 + P_1) R_C^2 & \geq 0, \\
\frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1} + \frac{p_2}{p_1'}\right) R_C^2 & \leq R_1^2 + R_C^2, \\
\Leftrightarrow (P_2 - P_1) R_1^2 + P_2(1 + P_1) R_C^2 & \leq 0, \\
\frac{1 + p_2}{1 + p_1} R_1^2 + \left(1 + \frac{p_2}{1 + p_1} + \frac{p_2}{p_1'}\right) R_C^2 & \geq R_1^2 + R_C^2, \\
\Leftrightarrow (P_2 - P_1) R_1^2 + P_2 R_C^2 & \geq 0, \\
\frac{1 + p_2}{1 + p_1} R_1^2 + \left(1 + \frac{p_2}{1 + p_1} + \frac{p_2}{p_1'}\right) R_C^2 & \leq R_1^2 + R_C^2, \\
\Leftrightarrow (P_2 - P_1) R_1^2 + P_2 R_C^2 & \leq 0. 
\end{align*}
\] (4.9)

it follows that

\[
(P_2 - P_1) R_1^2 + P_2 R_C^2 \geq 0 \Leftrightarrow (P_1 - P_2) R_1^2 \leq P_2 R_C^2, \\
\Rightarrow R_{2c}^2 = R_1^2 + R_C^2, 
\] (4.11)

\[
(P_2 - P_1) R_1^2 + P_2 R_C^2 < 0 \Leftrightarrow (P_1 - P_2) R_1^2 > P_2 R_C^2 \Rightarrow 
\] (4.12)

\[
R_{2c}^2 = \min \left\{ \frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1} + \frac{p_2}{p_1'}\right) R_C^2, \\
\frac{1 + p_2}{1 + p_1} R_1^2 + \left(1 + \frac{p_2}{1 + p_1} + \frac{p_2}{p_1'}\right) R_C^2 \right\}. 
\] (4.13)

On the other hand

\[
\begin{align*}
\frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1'}\right) R_C^2 & \geq \frac{1 + p_2}{1 + p_1} R_1^2 + \left(1 + \frac{p_1'}{1 + p_1}\right) R_C^2, \\
R_C^2 & \Leftrightarrow [(P_2 - P_1) R_1^2 + P_2 R_C^2] \\
& \Leftrightarrow P_2 P_1 (1 + P_1) R_C^2 \geq 0, \\
\frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1'}\right) R_C^2 & \leq \frac{1 + p_2}{1 + p_1} R_1^2 + \left(1 + \frac{p_1'}{1 + p_1}\right) R_C^2, \\
R_C^2 & \Leftrightarrow [(P_2 - P_1) R_1^2 + P_2 R_C^2] + P_2 P_1 (1 + P_1) R_C^2 \leq 0. 
\end{align*}
\] (4.14)

Therefore

\[
\begin{align*}
\{(P_2 - P_1) R_1^2 + P_2 R_C^2 < 0, \\
[(P_2 - P_1) R_1^2 + P_2 R_C^2] + P_2 P_1 (1 + P_1) R_C^2 \geq 0 \\
\Rightarrow R_{2c}^2 = \frac{1 + p_2}{1 + p_1} R_1^2 + \left(1 + \frac{p_1'}{1 + p_1}\right) R_C^2. 
\end{align*}
\] (4.15)

\[
\begin{align*}
\{(P_2 - P_1) R_1^2 + P_2 R_C^2 < 0, \\
[(P_2 - P_1) R_1^2 + P_2 R_C^2] + P_2 P_1 (1 + P_1) R_C^2 < 0 \\
\Rightarrow R_{2c}^2 = \frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1'}\right) R_C^2. 
\end{align*}
\] (4.16)

As concerns (4.11)–(4.16), we confine ourselves to remark that \( P_2 \geq P_1 \) implies (4.11) while (4.12) requires \( P_2 < P_1 \).

**Theorem 4.3** Let \( L \) be heated from below and salted from above by two salts \( S_a \) \((\alpha = 1.2)\). Then the Rayleigh critical numbers of \( S_1 \) and \( S_2 \) are given respectively either by

\[
R_{1c}^2 = \min \left\{ \left(1 + \frac{P_1}{P_2}\right) R_1^2 - \frac{P_1}{P_2} R_2^2, \\
\left(1 + \frac{1 + P_1}{P_1 + P_2}\right) R_2^2 - \frac{1 + P_1}{P_1 + P_2} R_1^2, \\
\right\}. 
\] (4.17)

\[
R_{2c}^2 = \min \left\{ \left(1 + \frac{P_2}{P_1}\right) R_2^2 - \frac{P_2}{P_1} R_1^2, \\
\left(1 + \frac{1 + P_2}{P_1 + P_2}\right) R_1^2 - \frac{1 + P_2}{P_1 + P_2} R_2^2, \\
\right\}, 
\] (4.18)

or by the smallest coupled value of \( R_1 \) and \( R_2 \) verifying one of the equations

\[
\begin{align*}
R_{1c}^2 = \frac{p_2}{p_1} R_1^2 + \left(1 + \frac{p_2}{p_1} + \frac{p_2}{p_1'}\right) R_C^2, \\
(1 + P_2) R_1^2 + (1 + P_1) R_2^2 = (1 + P_1 + P_2) R_C^2, \\
R_1^2 + R_2^2 = R_C^2. 
\end{align*}
\] (4.19)

**Proof** In view of theorem 33 and (3.11), the proof is easily reached following the procedure used in the proof of theorem 42.

**Remark 4.3** We remark that

\[
\begin{align*}
(i) \quad P_2 \geq P_1 \Rightarrow R_{1c}^{(2)} = R_C^2 - R_2^2, 
\end{align*}
\] (4.20)
(ii) \( P_1 \geq P_2 \Rightarrow R_{2c}^{(2)} = R_C^2 - R_1^2 \), \hspace{1cm} (4.21)

(iii) if \( \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 0 \), then

\[
R_1^2 = \frac{P_1}{p} \left( 1 + \frac{1}{P_1} + \frac{1}{P_2} \right) R_C^2,
R_2^2 = \frac{P_2}{q} \left( 1 + \frac{1}{P_1} + \frac{1}{P_2} \right) R_C^2,
\]

verify (4.20).

(iv) if (4.22) hold, then (4.19)\(_1\)–(4.19)\(_3\) are respectively verified by

\[
R_1^2 = \frac{(1 + P_1 + P_2) R_C^2}{p(1 + P_2)}, \quad R_2^2 = \frac{(1 + P_1 + P_2) R_C^2}{q(1 + P_1)},
\]

\[
R_1^2 = \frac{R_C^2}{p}, \quad R_2^2 = \frac{R_C^2}{q}.
\] \hspace{1cm} (4.24)

\[
R_1^2 = \frac{R_C^2}{q}.
\] \hspace{1cm} (4.25)

Remark 4.4 We remark that in view of Theorem 3.3, (3.12) and Remark 4.3, Theorem 1.1 is immediately obtained.

5 Discussion

(i) The paper is concerned with an \( m \)-component fluid mixture saturating a porous rotating horizontal layer \( L \), heated from below and salted partly from below and partly from above.

(ii) The instability of the thermal conduction solution—irrespective of the temperature gradient (analogous but different from the Marangoni instability)—named by us “cold convection”—is studied via the Routh–Hurwitz instability conditions.

(iii) It is shown that the “cold convection” is admissible by the Darcy–Boussinesq porous media and arises when the Rayleigh numbers of the chemicals salting \( L \) from above reach certain critical values. These values, in some prototype cases, are furnished.

(iv) The application of the Routh–Hurwitz instability conditions appear to be very appropriate for investigating the “cold convection”.

(v) The onset of the “cold convection” eliminates the thermal conduction observability.

(vi) In the applications, the “cold convection” could be useful in the devices in which is needed that \( \langle \beta \rangle_{m=0} \approx 0 \Rightarrow \langle \beta \rangle_{m} \approx 0, \forall t > 0 \).

(vii) The results obtained, as far as we know, appear to be new in the existing literature.

Acknowledgments This paper has been performed under the auspices of G.N.F.M. of I.N.D.A.M. and Leverhulm Trust, “Tipping points: mathematics, metaphors and meanings”.

Appendix

Proof of (2.10)

Setting

\[ u = (u, v, w), \quad \zeta = (\nabla \times u) \cdot k = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \] \hspace{1cm} (6.1)

the third component of the curl and of the double curl of (2.6)\(_1\) are respectively

\[ \zeta = T w_z, \] \hspace{1cm} (6.2)

\[ \Delta w + T \zeta = \Delta_1 (R \theta - R_1 \Phi_1 - R_2 \Phi_2), \] \hspace{1cm} (6.3)

where \( \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the planar laplacian. On the other hand (6.2) implies

\[ \zeta_z = T w_{zz}, \] \hspace{1cm} (6.4)

hence (6.3) becomes

\[ \Delta w + T^2 w_{zz} = \Delta_1 (R \theta - R_1 \Phi_1 - R_2 \Phi_2). \] \hspace{1cm} (6.5)

Passing to the Fourier components, one obtains

\[ \Delta = -\xi_n, \quad \Delta_1 = -a^2 \] \hspace{1cm} (6.6)

and hence

\[ \xi_n w_n + T^2 n^2 \pi^2 w_n = a^2 (R \theta_n - R_1 \Phi_{1n} - R_2 \Phi_{2n}), \] \hspace{1cm} (6.7)

i.e. (2.10).
Proof of (4.5)

$$\mathcal{A}^* = \min_{(a^2, n) \in \mathbb{R}^+ \times n} \frac{\xi_n}{\eta_n} = \left( \frac{\xi_n}{\eta_n} \right)_{(a^2 = \pi^2 \sqrt{1 + T^2}, n = 1)} = \pi^2 \left( 1 + \sqrt{1 + T^2} \right)^2. \quad (6.8)$$

Setting

$$\mathcal{A}(a^2, n, T) = \frac{\xi_n}{\eta_n} = \frac{1}{a^2} \left( \xi_n + n^2 \pi^2 T^2 \xi_n \right),$$

it follows that

$$\frac{\partial \mathcal{A}}{\partial n^2} = \frac{\pi^2}{a^2} \left[ 2 \xi_n + T^2 (\xi_n + n^2 \pi^2) \right] > 0,$$

hence

$$\mathcal{A}^* = \min \mathcal{A}_1,$$

with

$$\mathcal{A}_1 = \mathcal{A}(1, a^2, T) = \frac{(a^2 + \pi^2)^2}{a^2} + \pi^2 \frac{T^2 a^2 + \pi^2}{a^2} \left( a^2 + \pi^2 \right) + \pi^2 T^2 (a^2 + \pi^2).$$

Since

$$\mathcal{A}_1(0, T) = \lim_{a^2 \to 0} \mathcal{A}_1 = + \infty,$$

$$\mathcal{A}_1(\infty, T) = \lim_{a^2 \to \infty} \mathcal{A}_1 = + \infty,$$

by virtue of the continuity of $\mathcal{A}_1$ as function of $a^2$ in $\mathbb{R}^+$ and the Weierstrass compactness theorem on unbounded interval, it follows that $\mathcal{A}_1$ takes its minimum value in a point $a_\varepsilon \in [0, \infty[$. On the other hand

$$\left\{ \begin{array}{l}
\frac{d\mathcal{A}_1}{da^2} = 0, \iff a^2 = \pi^2 \left( 1 + T^2 \right), \\
\iff a^2 = \pi^2 \sqrt{1 + T^2}, \\
(\xi_n + n^2 \pi^2 T^2 \xi_n)_{(a^2 = a_\varepsilon^2, n = 1)} = \pi^2 \left( 1 + \sqrt{1 + T^2} \right)^2
\end{array} \right.$$}

and (4.5) immediately follows.

References

1. Chandrasekhar S (1981) Hydrodynamic and hydromagnetic stability, Dover, New York
2. Nield DA, Bejan A (2013) Conduction in porous media, 4th edn. Springer, New York
3. Straughan B (2008) Stability and wave motion in porous media, vol 165. Springer, New York
4. Lappa M (2012) Rotating thermal flows in natural and industrial processes. Wiley, Chichester
5. Straughan B (2004) The energy method, stability and nonlinear convection, 2nd edn. Springer, New York
6. Flavin J, Rionero S (1996) Qualitative estimates for partial differential equations. An introduction. CRC Press, Boca Raton
7. Lopez A, Romero L, Pearlstein A (1990) Effect of rigid boundaries on the onset of convection in a triply diffusive fluid layer. Phys Fluids A 2:897–902
8. Straughan B, Tracey J (1999) Multi-component convection-diffusion with internal heating or cooling. Acta Mecc Solida Sin 133:219–239
9. Pearlstein AJ, Harris RM, Terrones G (1989) The onset of convective instability in a triply diffusive fluid layer. J Fluid Mech 202:443–465
10. Noutly RA, Leaist DG (1987) Quaternary diffusion in aqueous KCl–H₂PO₄–H₃PO₄ mixtures. J Phys Chem 91:1655–1658
11. Tracey J (1996) Multi-component convection-diffusion in a porous medium. Continuum Mech Thermodyn 8:361–381
12. Lombardo S, Mulone G, Rionero S (2001) Global nonlinear exponential stability of the conduction-diffusion solution for Schmidt numbers greater than Prandtl numbers. J Math Anal Appl 262:1229
13. Mulone G, Rionero S (1998) Unconditional nonlinear exponential stability in the Bénard problem for a mixture: necessary and sufficient conditions. Atti Accad Naz Lincei Cl Sci Fis Mat Nat Rend Lincei 9:221
14. Rionero S (2012) Absence of subcritical instabilities and global nonlinear stability for porous ternary diffusive-convective fluid mixtures. Phys Fluids 24:104101
15. Qin Y, Kaloni P (1993) A nonlinear stability problem of convection in a porous vertical slab. Phys Fluids A 5:2067–2069
16. Kuznetsov AV, Nield DA (2011) The onset of double-diffusive convection in a nanofluid layer. Int J Heat Fluid Flow 32(4):751–776
17. Rionero S (2012) Symmetries and skew-symmetries against onset of convection in porous layers salted from above and below. Int J Non-linear Mech 47(4):61–67
18. Rionero S (2013) Multicomponent diffusive-convective fluid motions in porous layers: ultimately boundedness, absence of subcritical instabilities and global nonlinear stability for any number of salts. Phys Fluids 25:054104
19. Rionero S, (2012) Soret effects on the onset of convection in a rotating porous layer. Meccanica 48:201–210
20. Olali PB (2013) Double-diffusive convection induced by selective absorption of radiation in a fluid overlying a porous layer. Meccanica 48(1):201–210
21. Capone F, De Luca R (2012) Onset of convection for ternary fluid mixtures saturating horizontal porous layers with large pores. Rend Lincei Mat Appl 23:405–428
22. Capone F, De Luca R (2012) Ultimately boundedness and stability of triply diffusive mixtures in rotating porous layers under the action of Brinkman law. Int J Nonlinear Mech 47(7):799–805
23. Capone F, De Luca R (2013) On the stability–instability of vertical through flows in double diffusive mixtures saturating rotating porous layers with large pores. Ric Mat. doi:10.1007/s11587-013-0168-2
24. Merkin DR (1997) Introduction to the theory of stability. Texts in Applied Mathematics, vol 24. Springer-Verlag, New York
25. Gantmaker FR (2000) The theory of matrices, vol 1–2. AMS (Chelsea Publishing), New York