Estimating the small–\(x\) exponent of the structure function \(g_{1}^{\text{NS}}\) from the Bjorken Sum Rule

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Abstract

We present a new estimate of the exponent governing the small–\(x\) behavior of the nonsinglet structure function \(g_{1}^{p-n}\) derived under the assumption that the Bjorken Sum Rule is valid. We use the world wide average of \(\alpha_s\) and the NNNLO QCD corrections to the Bjorken Sum Rule. The structure function \(g_{1}^{\text{NS}}\) is found to be clearly divergent for small \(x\).

Key words: small–\(x\) behavior, nonsinglet structure function, Bjorken Sum Rule, strong coupling

Introduction

During the last decades many deep inelastic scattering experiments have been performed to elucidate the internal spin structure of the nucleon and to verify certain Sum Rules. One of them is the Bjorken Sum Rule [1] (BSR) which has been verified with an uncertainty of 8% [2] and which relates the structure function \(g_{1}^{p-n}\) to the axial vector decay constant. Perturbative corrections to the BSR have been calculated up to \(O(\alpha_s^3)\) [3] and \(O(\alpha_s^4)\)–corrections have been estimated [4].

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\[ \Gamma_1^{p-n}(Q^2) = \frac{g_A}{6g_V} \left[ 1 - \frac{\alpha_s(Q^2)}{\pi} - 3.5583 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 \right. \\
-20.2153 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^3 - \mathcal{O}(130) \left( \frac{\alpha_s(Q^2)}{\pi} \right)^4 \]  

(1)

with the numerical values for \( n_f = 3 \).

Above perturbative series can be used to determine \( \alpha_s \) from an experimental value for \( \Gamma_1^{p-n} \). In [2] it was found that such an analysis suffers from the poorly restrained small-\( x \) behavior of the structure function \( g_1 \). Only some years ago it was quite common to expect \( g_1 \) to be in agreement with Regge–theory [5] which predicts \( g_1 \sim x^\lambda \) with \( \lambda = 0 \ldots 0.5 \). Experimental data seems to contradict this expectation at least for the neutron structure function. A divergent behavior was also found by QCD analyzes of parton distributions, see e.g. [6,7]. There have been attempts to explain the obviously divergent behavior of \( g_1 \) by considering perturbative effects of higher orders that may dominate over the leading or even next-to-leading order QCD at small \( x \). A resummation of double logarithmic terms originating from ladder diagrams [8] led to \( \lambda \approx -0.5 \). It is not clear yet how large the influence of single logarithmic terms is and how one could perform a consistent resummation of those.

In the first part of this letter we want to repeat the analysis from [2] and show how the small-\( x \) behavior prevents a conclusive extraction of \( \alpha_s \) from the BSR. We will furthermore show that even other uncertainties such as higher twist corrections or experimental errors prevent this method of determining \( \alpha_s \) from being competitive.

In the second part we will make use of the fact that \( \alpha_s \) is known quite accurately from other experiments and we can therefore estimate the small-\( x \) exponent of \( g_1^{NS} \) by assuming the BSR to be valid.

**Extracting \( \alpha_s \)**

The perturbative expansion (1) allows for a determination of \( \alpha_s \) from an experimental value for \( \Gamma_1 \). This has been done in [9], where only experimental results from EMC, E142 and E143 have been used, and in [2]. One has to critically analyze the way those experimental values were obtained. The problem is, that the structure function \( g_1 \) cannot be measured at one fixed \( Q^2 \) over the whole \( x \)-range. Therefore one has to evolve the data points to one common scale and extrapolate into the unmeasured \( x \)-region. The first problem is solved using the DGLAP–equations [10] that describe the \( Q^2 \)-dependence of the parton distributions:
Table 1
World average for higher twist terms. The last two rows show values extracted from experimental data for comparison, they have not been included in the average. Some models are able to produce $Q^2$-dependent twist matrix elements. In such cases values for 5 GeV$^2$ are stated.

| Method        | Year | $f_{A,n}^{p-n}$ | $d_{A,n}^{p-n}$ | $d_{A,2}^{p-n}$ | ref. |
|---------------|------|-----------------|-----------------|-----------------|------|
| QCD sumrule   | 1990 | $-0.068 \pm 0.034$ | 0.025 $\pm 0.012$ |               | [15] |
|               | 1994 | $-0.120 \pm 0.006$ | 0.057 $\pm 0.003$ |               | [16] |
|               | 1995 | $-0.024 \pm 0.004$ | 0.024 $\pm 0.010$ |               | [17] |
| MIT bag       | 1994 | $+0.035$         | 0.021           | 0.059          | [18] |
| Modified bag  | 1993 |                | 0.0052          |               | [19] |
| Diquark       | 1995 | $+0.006$         |                |               | [20] |
| Soliton       | 1998 | $-0.033$         | 0.0003          |               | [21] |
| Renormalon    | 1996 | $\pm 0.043$      |                |               | [22] |
| Lattice       | 2001 |                | 0.0046 $\pm 0.0090$ | 0.034 $\pm 0.008$ | [23] |
| E142, E143,   | 1999 |                | 0.003 $\pm 0.010$ |               | [24] |
| E154, E155    |      |                |                |               |      |
| E143          | 1996 |                | 0.0024 $\pm 0.0200$ | 0.0311 $\pm 0.0040$ | [25] |

\[
d\Delta q_{NS}(x, Q^2) / d\ln Q^2 = \frac{\alpha_s(Q^2)}{2\pi} P_{qq}^{NS} \otimes \Delta q_{NS}, \tag{2}
\]

\[
d\left(\begin{array}{c}
\Delta \Sigma \\
\Delta G
\end{array}\right) / d\ln Q^2 = \frac{\alpha_s(Q^2)}{2\pi} \left(\begin{array}{cc}
P_{qq} & P_{qq} \\
P_{qq} & P_{qq}
\end{array}\right) \otimes \left(\begin{array}{c}
\Delta \Sigma \\
\Delta G
\end{array}\right), \tag{3}
\]

with $P_{ij}$ being the splitting functions that have been calculated in next-to-leading order (NLO) [11]. The symbol $\otimes$ denotes convolution with respect to $x$. With

\[
g_1^{p(n)} = \frac{\langle e^2 \rangle}{2} \left[ \frac{1}{4} C_{NS} \otimes (\pm 3\Delta q_3 + \Delta q_8) + C_S \otimes \Delta \Sigma + 2n_f C_G \otimes \Delta G \right], \tag{4}
\]

the structure functions $g_1^p$ and $g_1^n$ can be related to the polarized nonsinglet and singlet parton distributions and gluon distributions $\Delta q_{3,8}$, $\Delta \Sigma$ and $\Delta G$, respectively. The corresponding Wilson coefficients have been calculated in NNLO [12]. The average quark charge is defined by $\langle e^2 \rangle = (n_f)^{-1} \sum_{i=1}^{n_f} e_i^2$.

With these ingredients one can construct an algorithm that performs the Altarelli–Parisi evolution of the parton distributions and calculates a best
fit for $g_1(x, Q^2)$ which can then be integrated to yield the first moment

$$\Gamma_{p(n)}(Q^2) = \int_0^1 dx \, g_1^{p(n)}(x, Q^2).$$

(5)

We repeated an analysis performed by the ABFR–group [6] with the N LO code that had been kindly provided by Stefano Forte. It parametrizes the input distributions as

$$\Delta q_i = \eta_i x^{\alpha_i}(1 - x)^{\beta_i}(1 + \gamma_i x^{\delta_i}),$$

(6)

with $i = \Sigma, G, 3$ and 8. The distributions are normalized in order for $\eta_3$ and $\eta_8$ to resemble their first moments which are known from other experiments and can be fixed to $\eta_8 = 0.579 \pm 0.025$ [13] and $\eta_3 = g_A/g_V = 1.2670 \pm 0.0035$ [14]. Of course, $\eta_3$ should not be fixed if one wants to test the BSR. But since we assume its validity and want to extract $\alpha_s$ we may use the known value for $g_A/g_V$ and shall not fix $\alpha_s$ during the evolution procedure.

We use the ABFR–procedure which fixes 5 more parameters in 4 different fit types, so we are left with 7 fixed and 14 free parameters (including $\alpha_s$). See [6] for details about the fits. This procedure results in a world average for data from SMC, E142, E143, E154, E155 and HERMES (where only data taken at $Q^2 > 1$ GeV$^2$ have been used in the fit) of

$$\Gamma_{p-n} = 0.1847 \pm 0.0035,$$

(7)

where systematic and statistical uncertainties from experiment have been added quadratically.

This is the result of a purely perturbative NLO analysis. We may include higher twist corrections which are given by twist matrix elements of the form

$$\int_0^1 dx \, g_1^{p-n} = \frac{g_A}{6g_V} + \frac{m^2}{9Q^2} \left(a_{A,2} + 4d_A + 4f_A\right),$$

(8)

where perturbative corrections have not been noted explicitly. The matrix elements $a_{A,2}$ and $d_A$ are so–called target mass corrections and arise from operators of twist 2 and 3, respectively. The higher twist corrections originate from an operator of twist 4 and are given by $f_A$. Even higher twist corrections are suppressed with $1/Q^4$ and have been neglected in our analysis. There are many different models how to calculate higher twist corrections. An overview is given in Table 1. The errors produced by different models are due to some approximation used within the respective models and do not give a statement about the quality of the model. Therefore we did not consider them.
Table 2
Fit results for $g_1$ at lowest $x$ data points and the theoretical value for $\Gamma_1$ at the corresponding $Q^2$.

| experiment | $x$ | $Q^2$/GeV$^2$ | $g_1^{(p-n)}$ (fit) | $\Gamma_1^{\text{theory}}$ |
|------------|-----|---------------|---------------------|---------------------------|
| SMC        | 0.005 | 1.3 | 1.1968 ± 0.8516 | 0.1580 ±0.0043 (match) ±0.0162 (twist) |
| SMC        | 0.008 | 2.1 | 1.1330 ± 0.6560 | 0.1734 ±0.0003 (match) ±0.0101 (twist) |
| SMC        | 0.014 | 3.5 | 1.0161 ± 0.3985 | 0.1806 ±0.0005 (match) ±0.0060 (twist) |
| E155       | 0.015 | 1.2 | 0.6798 ± 0.1909 | 0.1548 ±0.0048 (match) ±0.0176 (twist) |

but calculated the average value with one standard deviation as characteristic uncertainty. That results in a correction to the BSR at 5 GeV$^2$ of

$$\Delta \Gamma_1 = \frac{m^2}{9Q^2} (a_{A,2} + 4d_A + 4f_A) = 0.0004 \pm 0.0042.$$  \hspace{1cm} (9)

The absolute size of the correction is indeed small but the uncertainty due to different models becomes so large that higher twist corrections cannot be neglected and result in a non-negligible uncertainty on $\alpha_s$.

Furthermore, it has become clear that at small–$x$ effects which are beyond NLO QCD become important. A resummation of double–logarithmic terms $\alpha_s^n (\ln^2 1/x)^n$ [8] yields a small–$x$ behavior for the nonsinglet structure function with a power–like divergence $g_1^{NS} \sim x^\lambda$ with $\lambda \approx -0.5$. This result contradicts Regge theory [5]. Running the evolution program again with the nonsinglet exponent fixed to the two extreme cases $\pm 0.5$ results in an uncertainty for $\Gamma_1^{p-n}$ of

$$\Delta \Gamma_1^{p-n} = \pm \frac{0.0034}{0.0282}.$$  \hspace{1cm} (10)

This induces a large error on $\alpha_s$ and basically spoils the whole analysis. Our result for $\alpha_s$ is summarized as follows.
\[ \alpha_s(M_Z) = 0.1160 \pm 0.0180 \pm 0.0043 \quad (\text{small } x) \pm 0.0034 \pm 0.0041 \quad (\text{twist}) \]
\[ \pm 0.0008 \quad (\text{match}) \pm 0.0006 \quad (g_A/g_V) \]

The evolution error has been estimated by repeating the fit in different fit types and renormalization schemes and varying factorization and renormalization scales. The matching error results from varying the matching thresholds for the evolution of \( \alpha_s \) to the \( Z \)-pole from single to double quark masses, where we used the matching procedure from [26]. The evolution was performed with the three-loop \( \beta \)-function.

This result impressively states the need for more accurate experimental data in the small-\( x \) region and/or better theoretical predictions. Even if the small-\( x \) uncertainty could be significantly reduced, the experimental and higher twist uncertainty prevent this method of “measuring” \( \alpha_s \) from being competitive to other methods. However, our result is in agreement with the world wide average \( \alpha_s(M_Z) = 0.1181(20) \) [14].

Also note [27] which investigates the \( Q^2 \)-dependence of certain error sources when extracting \( \alpha_s \) from the BSR.

**Estimating the small-\( x \) exponent**

Above result enables us to turn the question around. If a determination of \( \alpha_s \) suffers from the unknown small-\( x \) behavior one can use the precise knowledge of \( \alpha_s \) to predict this behavior. We may ask, what behavior does \( g_1 \) have to exhibit in order to fulfill the BSR?

To solve this question we start from the world wide average of \( \alpha_s \) and compute the theoretical BSR via (1) at \( Q^2 \) given in Table 2. There, also the data points with the smallest \( x \) are shown, with \( g_1^{p-n} \) being obtained from our best fit and the experimental errors assigned to it. Note, that \( \Gamma_1^{\text{theory}} \) is defined with already included higher twist corrections that were calculated as in (9).

We now assume the difference between \( \Gamma_1^{\text{theory}} \) and the integral over the measured region (where we assume the extrapolation to large \( x \) to be in agreement with the NLO QCD fit)

\[ \Gamma_1(x_0, \ldots, 1, Q^2) = \int_{x_0}^1 dx g_1^{p-n}(Q^2, x) \quad (11) \]
to be given by the power–like behavior of $g_1 \sim x^\lambda$ alone. Consequently, we obtain the small–$x$ exponent by solving

$$
\Gamma_1^{\text{theory}}(Q^2) - \Gamma_1(x_0 \ldots 1, Q^2) = \text{const} \int_{x_0}^x dx \ x^\lambda
$$

(12)

for $\lambda$, where the constant is determined from a fit through the lowest–$x$ data point. The most accurate result with this method is obtained if we choose the data point $x_0 = 0.014$, $Q^2 = 3.5 \text{ GeV}^2$ from SMC because this point combines the need for a low $x$–value with the desire for a large $Q^2$ to minimize the influence of higher twist corrections or experimental uncertainties. We find

$$
\lambda = -0.40 \pm 0.24 \ (\text{exp}) \pm 0.19 \ (\text{twist}) \pm 0.06 \ (\alpha_s).
$$

Other uncertainties, e.g. matching, errors on $g_A/g_V$ or the quark masses, turned out to be comparably negligible, i.e. less than 3%.

At other data points $\lambda$ cannot be determined with sufficient precision, mostly due to higher twist corrections which become more important at such low $Q^2$. The errors associated with the other data points are too large to identify any $Q^2$–dependence of the exponent $\lambda$.

An important advantage of this method is the use of the NLO code only above $x_0$, where $Q^2$ is relatively large.

To check on this result we also determine the nonsinglet exponent that best fulfills the BSR with the help of the evolution code. We fix $\eta_3 = 1.2670(35)$ and $\alpha_s = 0.1181(20)$ in the evolution program where we simultaneously fixed the parameters governing the large–$x$ behavior — being $\beta_i, \gamma_i, \delta_i$ — to the best fit output values. This produces a nonsinglet exponent

$$
\alpha_3 = -0.563 \pm 0.016,
$$

(13)

where the given uncertainty results from experiments only. This method neglects higher twist corrections. We may estimate the influence of the evolution code by varying the large–$x$ parameters or the constants $\alpha_s$ and $\eta_3$ in the input values. This leads to only minor changes and we estimate the total uncertainty to be

$$
\alpha_3 = -0.56 \pm 0.04.
$$

(14)

The exponent $\alpha_3$ agrees in LO with the nonsinglet exponent $\lambda$ and confirms our first result. As stated before, the data does not allow for a $Q^2$–dependent
determination of the exponent, that’s why it remains pointless to check for
the agreement of $\lambda$ and $\alpha_3$ in higher order QCD.

A similar (only code based) method has been used in [7] with a different NLO
evolution code. They found $g_1$ to be even more divergent with an error of the
same size.

**Conclusion**

We showed that the Bjorken Sum Rule is not suited to derive a value for $\alpha_s$
due to the uncertainties associated with the small–$x$ behavior. We found

$$\alpha_s(M_Z) = 0.1160 \pm 0.0193,$$  \hspace{1cm} (15)

with the major uncertainties added in quadrature. This shows the need for
more experimental data in the small–$x$ region. Unfortunately, the new HERMES–
results were obtained at too low $Q^2$ to considerably improve this situation.
Even if the small–$x$ uncertainty could be significantly reduced, higher twist
corrections are a non–negligible obstacle, they are not well enough constrained
due to the variety of existing models.

We provide a new QCD based estimate for the exponent governing the small–$x$
behavior of $g_1^{p-n}$ by assuming the validity of the BSR. We find

$$\lambda = -0.40 \pm 0.29,$$  \hspace{1cm} (16)

with errors from experiment, higher twist corrections and $\alpha_s$ combined. This
result was derived under the assumption that $g_1$ is dominated by a power–
like behavior for $x < 0.014$ and is confirmed by the evolution code which
neglects higher twist corrections and agrees with $\lambda$ in LO only. Our result
is in agreement with the behavior predicted by the resummation of double–
logs and contradicts Regge–theory. The assumption of the validity of the BSR
shrinks the width of possible exponents considerably.

The critical point of our analysis is that we depend on the experimental value
of $g_1$ at one data point with low $x$ and preferably high $Q^2$. Since there is no
neutron data at our preferred data point available we calculated $g_1^{p-n}$ from the
fit. Repeating the analysis with different fit results does not change $\lambda$ beyond
the quoted error. One could also extract $g_1^n$ from deuteron measurements but
we considered this method to be too unprecise.

One might also question how justified the use of a NLO code in combination
with the NNNL0 corrections to the BSR is. It has been argued in [6] that
this procedure is justified because of the relatively small $Q^2$–span of experiments which leads to an only minor higher order uncertainty in the evolution procedure. Furthermore, we did rely on the NLO code only above $x_0$ when determining $\lambda$, in this regime one does not expect higher order QCD terms to be of importance due to the higher $Q^2$ that comes naturally with larger $x$. Besides, logarithmic terms in $x$ do not dominate in this region.

Of course, using the BSR in NNLO only when determining $\alpha_s$ may change the absolute values in (15) but not the dominance of the small–$x$ uncertainty, which is the major pillar in the second part of our analysis. However, we do not believe the influence of higher order uncertainties to be of major importance for the determination of $\lambda$ because of the reliability of the NLO code above $x_0$. Repeating the calculation in consistency with the NLO code, i.e. with the NNLO BSR and the two–loop $\beta$–function changes $\lambda$ to -0.52, which lies within the given error.

Our analysis would profit from an improvement of estimates for higher twist corrections or a NNLO evolution procedure that could at least restrain the large–$x$ behavior better. One might even consider to perform a two–parametric fit that includes higher twist corrections at the parton distribution level and fits higher twist corrections and the nonsinglet exponent simultaneously, although it is not a priori clear if current experimental data are already suitable for such an analysis.

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