Lévy targeting and the principle of detailed balance

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We investigate confined Lévy flights under premises of the principle of detailed balance. The master equation admits a transformation to Lévy - Schrödinger semigroup dynamics (akin to a mapping of the Fokker-Planck equation into the generalized diffusion equation). We solve a stochastic targeting problem for arbitrary stability index 0 < μ < 2 of Lévy drivers: given an invariant probability density function (pdf), specify the jump - type dynamics for which this pdf is a long-time asymptotic target. Our (“µ-targeting”) method is exemplified by Cauchy family and Gaussian target pdfs. We solve the reverse engineering problem for so-called Lévy oscillators: given a quadratic semigroup potential, find an asymptotic pdf for the associated master equation for arbitrary μ.

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I. INTRODUCTION

Many complex physical systems (like-wise non-physical, e.g. economic [1]) can be satisfactorily described in terms of the dynamics of a certain fictitious particle under the action of random forces (noise), originating from its environment. Whenever we can identify a Gaussian noise as an emergent property of the environment-particle coupling, the interrelated notions of (thermal) equilibrium, Boltzmann asymptotic probability density functions (pdfs) and detailed balance generally follow. That is the case in the standard Brownian motion picture, based upon kinetic theory derivations, in the presence of (conservative) external forces.

However, in many stochastic systems the experimental data show that the description based on the introduction of the Gaussian noise is insufficient, since the involved fluctuations turn out to generate have heavy-tailed distributions of Lévy - stable type. Those distributions are widespread, in a broad range of systems of varied levels of complexity: physical, chemical, biological [2, 3], geophysical, economic [1]. That is why a deeper understanding of properties of general complex systems with non-Gaussian noises is extremely desirable.

For example, contrary to the case of systems with Gaussian fluctuations, in the context of Lévy flights the notion of “equilibrium”, although natural under confining conditions, has no obvious thermal connotation, see however [1]. It is clear that any conceivable “thermal equilibrium” concept for non-Gaussian jump-type processes needs to be addressed with care and should account for a number of precautions. In particular, an issue of physically motivated thermalization mechanisms for (confined) Lévy flights has received only a residual attention in the literature, [2,3] and [4,5]. The main obstacle here may be that the source of Lévy noise is interpreted as extrinsic to the physical system under consideration, with no reliable kinetic theory background, i.e. with no identifiable microscopic channels of an energy exchange with the environment.

Lévy flights are pure jump (jump-type) processes. Therefore, it seems useful to recall that various model realizations of standard jump processes (jump size is bounded from below and above) can be thermalized, by means of a locally defined scenario of an energy exchange with the thermostat, [6,7, see also 8]. It amounts to a suitable re-definition of transition rates for the jump process which enforces the principle of detailed balance to be respected by a random motion. We shall elaborate upon extension of this idea to Lévy - stable processes, with a focus on the existence of asymptotic (large time limit) invariant pdfs, of the manifest Boltzmann form.

Our approach is close to that used to analyze the Lévy motion in systems with topological complexity like polymers (see, e.g. [9]), but remains distinctively different from a standard theory of confined Lévy flights which is based on the Langevin modeling. There is no known (additive or multiplicative) Langevin representation for Lévy processes respecting the canonical form of detailed balance.

The considered class of confined Lévy flights is well suited for the description of jump-type processes that are equilibrated (eventually, to a thermal equilibrium state) by a mild spatial disorder of the physical environment in which jumps take place. The inhomogeneity of the environment is quantified by turning over from the master equation to the affiliated semigroup dynamics. It is a suitable functional form of the semigroup potential (which we consider to be a continuous function) that allows for a unique asymptotic invariant state. That ensures the existence of an asymptotic invariant pdf for the master equation in question.

The structure of the paper is as follows. First we discuss an issue of detailed balance for standard jump processes and next define its immediate generalization to Lévy flights (µ-family of Lévy-stable laws with 0 < µ ≤ 2), Sections II and III. A mapping of the resultant master equation to a fractional version of the generalized diffusion equation follows in Section IV. For clarity of presentation, we make a Brownian detour in Section V to indicate how the semigroup framework is related to the standard Fokker-Planck dynamics of diffusion-type processes. In Section VI we describe the Lévy µ - targeting under an assumption that target pdfs are selected.
from so-called Cauchy $\alpha$-family of pdfs. For a computationally advantageous example of $\alpha = 2$ and arbitrary $\mu \in (0, 2)$ we provide analytic formulas for the associated semigroup potentials (they define the semigroup dynamics which makes the considered pdfs to be genuine asymptotic targets of the jump-type process). In Section VII the Lévy targeting is considered for Gaussian target pdfs. Section VIII presents a complete solution of the reverse engineering problem for the $\mu$-family of Lévy oscillators, corresponding to quadratic semigroup potential. The obtained analytic formulas for asymptotic pdfs are depicted in Figs. 4 and 5. Not to overburden the paper with formal arguments, a general solution of the reverse engineering problem for arbitrary semigroup potential has been moved to another publication.

II. JUMP PROCESSES AND DETAILED BALANCE

Let $K$ be a finite state space, with $x, y \in K$. We consider Markovian stochastic dynamics for a finite random system, with transition rates $k(x|y) \equiv k(y \to x)$. Given an initial probability distribution $\rho_0(x)$, its time evolution for times $t \geq 0$ is governed by the master equation:

$$\frac{d}{dt} \rho_t(x) = \sum_{y \in K} [k(x|y)\rho_t(y) - k(y|x)\rho_t(x)].$$

(1)

Given a stationary solution $\rho_{eq}(x)$ of the master equation, $\rho_{eq}(x) = 0$. If we have

$$k(x|y)\rho_{eq}(y) = k(y|x)\rho_{eq}(x)$$

(2)

one says that the condition of detailed balance is fulfilled.

Let $\rho_{eq}(x) \propto \exp\left[-U(x)\right]$, where $U$ is a suitable function on $K$. (The inverse temperature $\beta$ can be safely absorbed in the definition of $U$. As well, for clarity of discussion, we can set $\beta = 1$). Accordingly:

$$k(x|y) = k(y|x)\exp[U(y) - U(x)].$$

(3)

We note that $k_0(x|y) = k_0(y|x)$, in a finite state space, yields a uniform distribution $\rho_{eq}(x) = \text{const}$ for all $x \in K$. Let us consider a simple multiplicative modification of a symmetric transition intensity $k_0(x|y)$:

$$k_0(x|y) \Rightarrow k_U(x|y) = k_0(x|y)\exp\left[\frac{U(y) - U(x)}{2}\right].$$

(4)

By inspection (simply replace $k(x|y)$ by $k_U(x|y)$ in Eqs. (1)-(3)) one verifies the validity of the detailed balance condition, with $\rho_{eq}(x) \propto \exp[-U(x)]$ as the corresponding stationary distribution.

We assume that an equilibrium density $\rho_{eq}(x) > 0$ is unique and presume the detailed balance condition [2], [4] to be respected. Then, the relative entropy (negative of the Kullback-Leibler entropy) becomes

$$S(\rho_t|\rho_{eq}) = \sum_{x \in K} \rho_t(x) \ln \frac{\rho_t(x)}{\rho_{eq}(x)} = F(\rho_t) - F(\rho_{eq}) \geq 0.$$  

(5)

Here an obvious analogue of the familiar Helmholtz free energy $F(\rho_t) = \sum_{x \in K} U(x)\rho_t(x) - S(\rho_t)$ has been introduced, with $S(\rho_t) = -\sum_{x \in K} \rho_t(x) \ln \rho_t(x)$ being the Shannon entropy of the probability distribution $\rho_t(x)$. We have $F(\rho_t) \geq F(\rho_{eq}) = -\ln \sum_{x \in K} \exp[-U(x)]$. The relative entropy is monotonous in time and converges to zero, which is accompanied by a decrease of the free energy $F(\rho_t)$ to its minimal value $F(\rho_{eq})$.

It is useful to mention an interesting inverse stationary problem of Refs. [10, 11]. Namely, for an arbitrary positive probability distribution $\rho_{eq}(x) > 0$ on $K$ there exists a function $U(x)$ such that $\rho_{eq}(x)$ is invariant under the jump dynamics with the transition rate $k_U(x,y)$ of the form (11). In the original formulation of Ref. [11], the reference transition rate $k_0(x,y)$ needs not to be symmetric.

III. DETAILED BALANCE FOR LÉVY FLIGHTS

The above reasoning gives an immediate justification to the strategy adopted before in the context of Lévy-stable processes, albeit with no explicit reference to the detailed balance principle, in a number of papers [13–15]. We also note Refs. [4, 8, 16], where ”stochastic targeting” and related ”inverse engineering” (terms, originally coined in Ref. [17]) have been exploited to this end.

To proceed further, we recall that a characteristic function of a random variable $X$ completely determines a probability distribution of that variable. If this distribution admits a pdf $\rho(x)$, we can write $<\exp[ipX]> = \int_\mathbb{R} \rho(x) \exp[ipx] dx$. A classification of infinitely divisible probability laws is provided by the Lévy-Khintchine formula for the exponent $-F(p)$ of $<\exp[ipX]> = \exp[-F(p)]$.

We restrict subsequent considerations to a subclass of stable probability distributions with $F(p) = |p|\mu$, with $0 < \mu \leq 2$. The induced jump-type dynamics $<\exp[ipX_t]> = \exp[-F(p)]$ is conventionally interpreted in terms of Lévy flights and quantified by means of a pseudo-differential (fractional) analog of the heat equation for corresponding pdf

$$\partial_t \rho = -|\Delta|^{\mu/2} \rho = \int [w_\mu(x|y)\rho(y) - w_\mu(y|x)\rho(x)] dy,$$

(6)

which has been rewritten as a master equation for a random system on real axis, with a pure jump dynamics. The jump rate $w_\mu(x|y) \propto 1/|x-y|^{1+\mu}$ is a symmetric function, $w_\mu(x|y) = w_\mu(y|x)$ akin to $k_0(x|y)$ of the previous subsection. We recall that the action of a fractional operator $|\Delta|^{\mu/2}$ on a function from its domain is defined by means of the Cauchy principal value of an involved integral:

$$-(|\Delta|^{\mu/2} f)(x) = \frac{\Gamma(\mu+1)}{\pi} \int \frac{f(z) - f(x)\,|z-x|^{1+\mu}}{|z-x|} dz.$$  

(7)
Mimicking the previous step [1], we open a possibility of a locally controlled energy exchange with an environment, by modifying the jump rate $w_{\mu}(x|y)$ of the free (neither external forces nor potentials) fractional dynamics to the non-symmetric form $w_{\mu}^{U}(x|y) \neq w_{\mu}^{U}(y|x)$:

$$
\partial_{t} \rho = -|\Delta|^{\mu/2}/2 = \int [w_{\mu}^{U}(x|y)\rho(y)-w_{\mu}^{U}(y|x)\rho(x)]dy = -[\exp(-U/2)]|\Delta|^{\mu/2}[\exp(2U/\rho)+\rho \exp(2U/2)|\Delta|^{\mu/2}\exp(-U/2)].
$$

The above transport equation cannot be transformed to any known form of the fractional Fokker-Planck dynamics, based on the standard (Lévy-stable) Langevin modeling, (c.f. [10, 22] for literature sample). These two dynamical patterns of behavior are inequivalent, [4, 16].

For a suitable (to secure normalization) choice of $U(x)$, $\rho_{eq}(x) \propto \exp[-U(x)]$ is a stationary solution of Eq. (6).

The detailed balance principle of the form (2), (3) holds with pure time-dependent Schrödinger equation [8, 14, 15, 18]. The difference terms of a Lévy-stable semigroup or a fractional (Lévy-) Schrödinger-type equation, [8, 14, 18], is a legitimate generator of a dynamical semigroup asymptotic of a locally controlled energy exchange with an environment, (c.f. [19]-[23] for literature sample). These two dynamical patterns of behavior are inequivalent, [4, 16].

Looking for stationary solutions of the affiliated semigroup equation $\partial_{t} \Psi = \hat{H}_{\mu} \Psi$, we realize that if a square root of a positive invariant pdf $\rho_{s}(x)$ is asymptotically to come out via the semigroup dynamics $\Psi \rightarrow \rho_{s}^{1/2}$, then the resulting fractional Sturm-Liouville equation $\hat{H}_{\mu} \rho_{s}^{1/2} = 0$ imposes a compatibility condition upon the functional form of $V(x)$, that needs to be respected. Namely, the potential function and invariant pdf $\rho_{s}^{1/2}$ should be related as

$$
\mathcal{V} = -|\Delta|^{\mu/2}/\rho_{s}^{1/2}.
$$

The resulting semigroup dynamics provides a solution for the Lévy stable targeting problem, with a predefined invariant pdf.

Inversely, if we predefine a concrete potential function $\mathcal{V}(x)$, then the functional form of an asymptotic invariant pdf $\rho_{s}(x)$ (actually $\rho_{s}^{1/2}(x)$) comes out from the above compatibility condition. We call the problem of derivation of $\rho_{s}$ from a predefined semigroup potential $\mathcal{V}(x)$ as reverse engineering problem, see Ref. [17] where this idea had been put forward.

For $\mathcal{V} = \mathcal{V}(x)$ bounded from below, the integral kernel $k(y, s, x, t) = \{\exp[-(t-s)\hat{H}]\}(y, x), s < t$, of the dynamical semigroup $\exp(-t\hat{H})$ is positive. The semigroup dynamics reads: $\Psi(x, t) = \int \Psi(y, s)k(y, s, x, t)dy$ so that for all $0 \leq s < t$ we can reproduce the dynamical pattern of behavior, actually set by Eq. (8), but now in terms of Markovian pdfs $p(x, s, y, t)$:

$$
\rho(x, t) = \rho^{1/2}(x)\Psi(x, t) = \int p(y, s, x, t)\rho(y, s)dy,
$$

where

$$
p(y, s, x, t) = k(y, s, x, t)\rho^{1/2}(y).
$$

An asymptotic behavior of $\Psi(x, t) \rightarrow \rho_{s}^{1/2}(x)$ implies $\rho(x, t) \rightarrow \rho_{s}(x)$.
A remark is in place here. The spectral theory of fractional operators of the form \( \mathcal{L}_\mu \) has received a broad coverage in the mathematical literature \([24, 28]\) and mathematical physics literature \([29, 30]\). An explicit functional form of asymptotic invariant pdfs of confined Lévy flights \( \rho_\alpha(x) = \rho_{1/2}(x) \) in the semigroup notations is seldom accessible, with a notable exception of those for Cauchy flights \([16, 21]\). Therefore it is wise to rely on accumulated data that are available, about the near-equilibrium behavior and the decay of pdfs as \( |x| \to \infty \), under very general circumstances. Various rigorous estimates pertaining to the decay at infinites of the eigenfunctions, quantify the number of moments of the associated pdfs for different classes of potential functions \( \mathcal{V}(x) \). As well, fractional versions of Feynman-Kac formula determining an integral kernel of the semigroup operator, and hence the transition probability which generates (by virtue of Eq. \([11]\)) the pdf \( \rho(x, t) \) dynamics consistent with Eq. \([8]\), have an ample coverage therein.

V. BROWNIAN DETOUR

The aim of this section is to describe the relation between above Lévy - Schrödinger semigroup framework and standard Fokker-Planck dynamics of diffusion-type processes. To make this description clear, here we put explicit relations, translating things from the language of partial differential equations (like Fokker-Planck one) and dealing explicitly with pdfs into the operator language, inherent in (both normal and fractional) quantum mechanics and ultimately in Lévy - Schrödinger semigroup.

In the theory of standard Brownian motion, the Langevin equation or the like (stochastic differential equation with the Wiener noise input) allows to infer a corresponding Fokker-Planck equation. This in turn can be transformed into a Hermitian (strictly-speaking, self-adjoint) spectral problem, \([22]\). Contrary to the Lévy - Schrödinger semigroup framework involving above Lévy - Schrödinger semigroup framework and standard Fokker-Planck dynamics of diffusion-type processes. To make this description clear, here we put explicit relations, translating things from the language of partial differential equations (like Fokker-Planck one) and dealing explicitly with pdfs into the operator language, inherent in (both normal and fractional) quantum mechanics and ultimately in Lévy - Schrödinger semigroup.

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If we set \( \mathcal{V}(x) = 0 \) identically, a purely continuous spectral problem arises. Then, one arrives at the familiar heat kernel

\[
\rho_\alpha(x) = \rho_{1/2}(x) \]

which is a well-known transition probability density of the Wiener process (actually, upon setting \( t \to (t-s) \)).

When confining potentials are present, either entire spectrum or its part turns out to be discrete, the corresponding eigenfunctions being real-valued. A standard example is the harmonic oscillator i.e. the Ornstein-Uhlenbeck process in its original stochastic version. Consider

\[
\hat{H} = (1/2)(-\Delta + x^2 - 1).
\]

The integral kernel of \( \exp(-t\hat{H}) \) is given by the classic Mehler formula \([31]\):

\[
k(y, x, t) = k(x, y, t) = \exp(-t\hat{H})(y, x) = \frac{1}{\pi \sqrt{1 - e^{-2t}}} \exp\left[-\frac{x^2 - y^2}{2} - \frac{(xe^{-t} - y)^2}{1 - e^{-2t}}\right].
\]

The normalization condition

\[
\int k(y, x, t) \exp[(y^2 - x^2)/2] dy = 1
\]

directly employs (and defines upon setting \( t \to (t-s) \)) the transition probability density of the Ornstein-Uhlenbeck process,

\[
p(y, x, t) = k(y, x, t) \frac{\rho_{1/2}(x)}{\rho_{1/2}(y)}
\]

with \( \rho_\alpha(x) = \pi^{-1/2} \exp(-x^2) \) being its (Gaussian) invariant pdf.

VI. CAUCHY FAMILY OF PDFS AND LÉVY \( \mu \)-TARGETING

Here we describe in some detail the Lévy stable (with stability index \( \mu \)) targeting strategy with the predetermined one-parameter family of Cauchy target pdfs:

\[
\rho_\alpha(x) = \rho_\alpha(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi} \Gamma(\alpha - 1/2)} \frac{1}{(1 + x^2)\alpha}, \; \alpha > 1/2.
\]

We consider functions \([12]\) as asymptotic invariant pdfs for the stochastic jump-type process of Eq. \([8]\). We wish to demonstrate that any \( \mu \)-stable driver can be employed to this end.

Instead of addressing directly Eq. \([8]\), we use the semigroup dynamics \( \exp(-t\hat{H}_\mu) \) generated by the fractional operator \([9]\), i.e. the integro-differential equation

\[
\frac{\partial \Psi}{\partial t} = -|\Delta|^{\mu/2} \Psi - \mathcal{V}_\mu \Psi,
\]

where \( \mathcal{V}_\mu(x) = \rho(x, t) / \rho_{1/2}^{1/2}(x) \) and \( \mathcal{V}_\mu(x) = -(|\Delta|^{\mu/2} \rho_{1/2}^{1/2}) / \rho_{1/2}^{1/2}, \; 0 < \mu < 2 \).

We note that the Cauchy family \([12]\) has been chosen for computational convenience only. In principle, there
is no restriction on the choice of any other target pdf $\rho_\ast(x)$. The qualitative outcome will be the same as that provided in terms of family (12). Hereafter we call such general procedure "$\mu$-targeting".

Let us add, as a side comment, that the Cauchy family of pdfs has played an important role in the previously mentioned search for "thermodynamic equilibria", that may possibly be associated with confined Lévy flights, [4, 7]. It is known [3–8], that an exponent $\alpha$ in principle can be directly related to the thermal equilibrium label $\alpha \propto 1/k_BT$. An analogous observation has been reported in Refs. [4, 5], after transforming pdfs (12) into an "exponential form", which resembles Boltzmann one $\rho_\ast \propto \exp(-U)$, with $U(x) = \alpha \ln(1 + x^2)$.

To pass over to the semigroup description we need to infer $V_{\mu}(x)$, given $\rho_\ast$. This can be done analytically by means of the Fourier transform, specifically because Fourier images of functions (12) for arbitrary $\alpha \times \pi/\alpha$ exist in a closed analytical form of MacDonald functions $K_{\frac{\alpha}{2}}$. The Fourier image $g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{ikx} dx$ of a function $g(x)$, when adopted to $g(x) = |\Delta|^\mu/2 f(x)$ reads $|k|^\mu f(k)$. Fourier images of the square roots of pdfs (12) read

$$\rho_\alpha^{1/2}(k) = \sqrt{\frac{2\Gamma(\frac{1+\alpha}{2})}{\pi \Gamma(\alpha-1/2) \Gamma(\alpha/2)}} |k|^{\alpha-1} K_{\frac{\alpha}{2}}(|k|).$$

An explicit expression for the $\alpha$-family of "$\mu$-potentials"

$$V_{\mu,\alpha}(x) = -(|\Delta|^\mu/\rho_\alpha^{1/2})/\rho_\alpha^{1/2} \text{(readily follows)}$$

$$V_{\mu,\alpha}(x) = -\frac{2^\mu}{\sqrt{\pi}} (1 + x^2)^{\alpha/2} \frac{\Gamma(\frac{1+\mu}{2}) \Gamma(\frac{\alpha+\mu}{2})}{\Gamma(\frac{\alpha}{2})} \times \frac{1 + x^2}{\alpha + \mu, 1/2 - x^2},$$

where $\frac{1}{2}F_1(a; b; c; x)$ is a hypergeometric function [32].

The expression (15) gives the general form of the semigroup potentials $V_{\mu,\alpha}(x)$ for arbitrary $\alpha$ and $\mu$. To have a better feeling about the properties of the function (15), we should express this expression for some specific values of parameter $\alpha$. Further discussion is limited to the case of $\alpha = 2$, i.e.

$$\rho_2^{1/2}(x) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + x^2} \rightarrow \rho_2^{1/2}(k) = e^{-|k|}. \quad (16)$$

We note that the Fourier image $\rho_2^{1/2}(k)$ directly comes from the general expression (15), if we use $K_{1/2}(x) = (\pi/2x)^{1/2} e^{-x}$. Then for all $0 < \mu < 2$ we have

$$V_{\mu,2}(x) = -(1 + x^2)^{\frac{\mu}{2}} \Gamma(1 + \mu) \cos((1 + \mu) \arctan x). \quad (17)$$

For $\mu = 1$ from (17) we recover our elder result, originally obtained in the context of Cauchy flights, (16):

$$V_{1,2}(x) = \frac{x^2 - 1}{1 + x^2}. \quad (18)$$

The plots of the $\mu$-dependence of (14) are reported in Fig.1.

The stability index $\mu$ is constrained to stay within an interval $0 < \mu \leq 2$. The boundary value $\mu = 2$ takes us beyond the jump-type "territory" to continuous (Wiener noise) stochastic processes. It is interesting to observe that on the level of "$\mu$-potentials", the transition from $\mu < 2$ to $\mu = 2$ is actually smooth.

Analytically, recalling the fractional derivative transcription $(-\Delta)^{\mu/2} \equiv -\partial^{\mu}/\partial|x|^\mu$ and then setting "blindly" $\mu = 2$ in (10), we arrive at the semigroup potential for the operator $H = -\Delta + V_{2,2}$:

$$V_{2,2}(x) = V_{FP}(x) = \frac{\frac{d^2}{dx^2} V_{2,2}(x)}{\rho_2^{1/2}(x)} = 2(3x^2 - 1) \frac{2(3x^2 - 1)}{(1 + x^2)^2}. \quad (19)$$

The notation $V_{FP}(x)$ refers to the fact that this potential appears in the semigroup (self-adjoint) version, (c.f. Ref. [22]) of the standard Fokker-Planck equation for a diffusion-type process. The same result (19) can be obtained from Eq. (17) at $\mu = 2$.

The expression (17) permits us to expand the potential $V_{\mu,2}(x)$ near $\mu = 2$ to obtain

![FIG. 1: Dependence $V_{\mu,2}(x)$ for $\rho_2$ terminating pdf. Figures near curves correspond to $\mu$ values. The potentials for $\mu = 1$ and $2$ are given by Eqs. (15) and (19) respectively.](image-url)
\[ V_{\mu \to 2,2}(x) \approx \frac{2(3x^2 - 1)}{(1 + x^2)^2} - \frac{\mu - 2}{(1 + x^2)^2} \left[ 2x(x^2 - 3) \arctan x + (3x^2 - 1)(2\gamma - 3 + \ln(1 + x^2)) \right], \quad (20) \]

where \( \gamma \approx 0.577216 \) is Euler constant. This (along with numerical curves from Fig.1) demonstrates the continuous transition from \( \mu < 2 \) to \( \mu = 2 \) in \( V_{\mu,2}(x) \).

VII. GAUSSIAN \( \mu \)-TARGETING FOR LÉVY FLIGHTS

In the previous publications [4, 8, 16] we have investigated various patterns of jump-type and diffusive behavior that would produce a priori selected, basically heavtailed pdfs in the large time asymptotics. While an association of jump type-processes with pdfs possessing a finite number of moments is rather natural, an observation of Ref. [8] that diffusion-type processes may as well admit such asymptotic pdfs, may be classified as "unnatural".

Here we proceed in the very same "unnatural" vein, asking for a Lévy-stable jump-type dynamics, whose asymptotic pdf would have a definite Gaussian form. Let us select the Gaussian target pdf

\[ \rho_s = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}. \quad (21) \]

whose square root \( \rho_s^{1/2}(x) \equiv f(x) = (2\pi\sigma^2)^{-1/4} \exp(-x^2/(4\sigma^2)) \) has Fourier image \( (\rho^s)^{1/2}(k) \equiv f(k) = (2\sigma^2/\pi)^{1/4} \exp(-k^2/2\sigma^2) \). That gives

\[ V_{\mu G}(x) = -\frac{\sigma^{-\mu}}{\sqrt{\pi}} e^{\frac{x^2}{4\sigma^2}} \Gamma \left( 1 + \frac{\mu}{2} \right) \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1+k)}{(4\sigma^2)^{1+k} F_1} \left[ \frac{1}{2}, \frac{1}{2} \frac{x^2}{4\sigma^2} \right], \quad (22) \]

where \( F_1(a,b,x) \) is a hypergeometric function [32]. This \( \mu \)-family of semigroup potentials sets solution to the Lévy stable targeting problem, if the desired target has the Gaussian form.

Minor comments are necessary for a qualitative assessment of the above analytic result. The potential \( V_{\mu G}(x) \) depends on two parameters: order of fractional derivative \( \mu \) and variance \( \sigma \). It can be seen from Eqs. (21) and (22) that the variance \( \sigma \) simply alters the width of the potential curve and does not influence its shape. The same is true for the factor \( \sigma^{-\mu} \) in front of Eq. (22). That is why in Fig. 2 we report the shape of the potential (22) in normalized variables \( z = x/(2\sigma) \) and \( y_\mu = \sigma^\mu V_{\mu G}(x) \). These universal curves are the same for any \( \sigma \) and depend on the single parameter \( \mu \). Note, that in these variables the \( \mu = 2 \) parabola assumes the form \( y_2 = z^2 - 1/2 \).

It is also seen from Fig. 2 that at small \( \mu \) the potential \( y_\mu \) is around \(-1 \) (we recollect that at \( \mu = 0 \) \( V_{\mu G}(x) \equiv -1 \)), while at larger \( x \) it has very steep growth like \( \exp(z^2) \). These steep tails flatten as \( \mu \) grows and around \( \mu = 1.5 \) the exponential growth of the potential is replaced by power-law \( z^\mu \) so that at \( \mu = 2 \) we have the correct asymptotics \( z^2 \).

VIII. REVERSE ENGINEERING: ASYMPTOTIC \( \mu \)-TARGETS FOR LÉVY OSCILLATORS

Now we pass to a detailed discussion of a particular class of solvable examples of the reverse engineering problem which well illustrates the following general strategy (its full description is moved to another publication): given a priori a concrete semigroup with Lévy driver, infer an asymptotic pdf for the associated master equation (8).

Our main idea is to adopt an approach we have developed before, [4] (see also [28, 34]) to the Lévy oscillator with \( V(x) = x^2/2 \) and arbitrary stability index \( \mu \).

We begin with the equation for a terminal pdf \( \rho_s \), inferred from the \( \mu \)-Lévy semigroup with a predefined harmonic potential

\[ V_\mu(x)\rho_{s}^{1/2} = \frac{x^2}{2} \rho_{s}^{1/2} = -|\Delta|^{\mu/2} \rho_{s}^{1/2}, \quad 0 < \mu \leq 2. \quad (23) \]

We take Fourier images of both sides of Eq. (23) to obtain

\[ u_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx = -\frac{1}{2} \frac{\partial^2 f(k)}{\partial k^2}. \quad (24) \]
The right-hand side of Eq. (26) has the form $-|k|^\mu f(k)$ so that

$$\frac{\partial^2 f(k)}{\partial k^2} = \frac{d^2 f(k)}{dk^2} = 2|k|^\mu f(k).$$

(25)

The idea to solve the Eq. (26) for arbitrary $0 < \mu \leq 2$ is borrowed from Ref. [34], where the solution for $\mu = 1$ had been obtained in terms of Airy functions. The method of Ref. [34] is based on the consideration of 1D Schrödinger problem with a potential being even function of the coordinate, which implies that the corresponding eigenfunctions should be either even or odd (see e.g. [35, 36]). In particular, the ground state wave function should be even as it does not have nodes [35]. It can be shown that solution $f(k)$, defining the Fourier image of desired terminal pdf, corresponds to the ground state wave function of the above Schrödinger problem. Generalizing the method of Ref. [34] for arbitrary $\mu$, we can show that to obtain this function for even potential like $|k|^\mu$ we should consider instead of (26) the equation

$$\left\{ \begin{array}{ll}
\frac{d^2 f(k)}{dk^2} = 2k^\mu f(k), & k > 0 \\
\frac{d^2 f(k)}{dk^2} = -2(-k)^\mu f(k), & k < 0.
\end{array} \right.$$

(26)

Now the scenario of obtaining the desired $f(k)$ is as follows. After finding the exponentially decaying solution of Eq. (26) for $k > 0$ and oscillatory one at $k < 0$, we should require the continuity of the function $f(k)$ and its derivative at $k = 0$. This is because the Eq. (26) is of the second order. After that we should find the position $k_m$ of the first maximum of oscillating part and shift the solution to the right by $k_m$ so that the first maximum of oscillating part is at $k = 0$. Then, ”chopping” the rest of oscillating part and reflecting the obtained piece about the vertical axis to obtain the even ”bell-shaped” function. The resultant solution in the $k$ space should be Fourier-inverted and squared to yield the desired terminal pdf in the $x$-space.

To fulfill this scenario, we observe the following form of solutions of Eq. (26) for $k > 0$ and $k < 0$ [32]. Namely, for $k \geq 0$

$$f(k) = \sqrt{k} \left[ C_{11} J_{\frac{\mu}{2}} (\frac{\sqrt{\mu}}{q} k^q) + C_{12} K_{\frac{\mu}{2}} (\frac{\sqrt{2}}{q} k^q) \right],$$

(27)

while for $k < 0$

$$f(k) = \sqrt{|k|} \left[ C_{21} J_{\frac{\mu}{2}} (\frac{\sqrt{\mu}}{q} |k|^q) + C_{22} N_{\frac{\mu}{2}} (\frac{\sqrt{2}}{q} |k|^q) \right],$$

(28)

where $q = (\mu + 2)/2$. Here $J_\nu(x)$ and $N_\nu(x)$ are Bessel functions and $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions, see Ref. [32]. At $x \to \infty$ $I_\nu(x)$ is exponentially growing function [32] while $K_\nu(x)$ is exponentially decaying [32]. On the other hand, as $x \to -\infty$ the functions $J_\nu(x)$ and $N_\nu(x)$ have ”needed” oscillatory asymptotics [32]. This means that to have a localized pdf, we should leave the term with $K_\frac{\mu}{2}$ in (27) only. Then $f(k)$ assumes the following form
Now we join (glue) the obtained solutions at $k = 0$ to secure a continuity of a function and its first derivative.

The gluing procedure yields

$$f(k) = C \sqrt{|k|} \begin{cases} \frac{\sqrt{2}}{q} \left[ \cot \frac{\pi \nu}{2} J_{\nu}(u) - N_{\nu}(u) \right], & k < 0, \\ C_{12} \frac{\sqrt{K}}{q} \left( \frac{\sqrt{2} q}{2} k \right), & k \geq 0 \end{cases}$$

where $C \equiv C_{12}$,

$$\nu = \frac{1}{2q} \equiv \frac{1}{\mu + 2}, \quad u = \frac{\sqrt{2}}{q} k = \frac{2 \sqrt{2}}{\mu + 2} |k|^{1 + \frac{\nu}{2}}.$$  

We note here that for the Cauchy driver, i.e. $\mu = 1$ we obtain from (30) the result

$$f(k) = C \frac{\sqrt{K}}{q} \left( \frac{2 \sqrt{2}}{3} k \frac{1}{2} \right) = C \frac{\sqrt{2}}{q} \left( \frac{2 \sqrt{2}}{3} \right) \Lambda\left( 2 \frac{1}{2} k \right),$$

known from our earlier publication [4].

The ”raw” solutions (30) are plotted on the main panel of Fig. 3 for different values of $\mu$. It is seen from the inset that for $\mu \to 0$ (thick black line corresponding to $\mu=0.01$) the potential has the shape of almost rectangular barrier, corresponding to decaying solution (localized particle inside the barrier) at $k > 0$ and oscillating one (free particle) at $k < 0$ [35, 36]. We note here that for potentials depicted on the inset to Fig. 3 the above kind of solution exist only if its eigenenergy lies between the limiting values of a barrier at $|x| \to \infty$ [35, 36]. In this case the zeroth eigenenergy, which is the case for Eqs. (24) and (26) perfectly suits the problem under consideration not only for $\mu \to 0$, where the barrier is almost rectangular, but also at higher $\mu$. This explains the fact that as the shape of barrier deviates from rectangular one at $\mu$ increase, the oscillations at $k < 0$ start to decay, the strongest one being at $\mu = 2$. Also, with the growth of $\mu$, the period of the oscillations lowers, the minimum being achieved at $\mu = 2$ also.

Now we find the position $k_m$ of the first maximum of oscillating part. Equating to zero the first derivative of an oscillating part we arrive at

$$N_{\nu-1}(u) - \cot \frac{\pi \nu}{2} J_{\nu-1}(u) = 0,$$

where $\nu$ and $u$ are defined by (31). The roots of Eq. (33) can easily be obtained numerically for different $\mu$.

The normalization of the obtained function can be achieved through the condition $C^2 \int_{-\infty}^{\infty} f^2(k) dk = 1$ or

$$2C^2 \left[ \int_{0}^{k_m} f^2_1(k) dk + \int_{-k_m}^{\infty} f^2_2(k) dk \right] = 1,$$  

where $f_1$ and $f_2$ denote oscillatory and decaying parts of Eq. (30) respectively. Normalized solutions in the $k$-space for different $\mu$’s are reported in Fig. 4. It is seen that for small $k$ and on the tails, the distribution functions for higher $\mu$’s run below those for smaller $\mu$’s, while in the intermediate $k$ range the situation is opposite.

The final step of the procedure is to invert the $k$-space solutions to the $x$-space and square them to obtain the desired terminal pdf. For general $\mu$ this procedure can be accomplished only numerically.

Fig. 5 displays both the inverted functions $f(k)$, corresponding to square roots of the inferred terminal pdfs (panel (a)) and those pdfs themselves (panel (b)). The opposite (if compared to this in the $k$-space) tendency is seen in the $x$ space, where the curve corresponding to lowest $\mu$ lies below all other curves in the small $x$ region and has slowest decay. As $\mu$ grows, the central part of the curve rises and tails become steeper.

Panel (c) of Fig. 5 reports a comparison between the shapes of functions $f(k)$ and $f(x)$. The situation here is the same as that for the Airy function, as discussed in [18]. Namely, the function in $k$-space decays quicker then in $x$-space and its value at the center is larger then that in $x$ - space. We plot here the exemplary case of $\mu = 0.5$, the situation for other $\mu$ is qualitatively the same.

**IX. OUTLOOK**

The next natural step in our $\mu$-targeting procedure is to obtain (numerically) the dynamics of a function $p(x,t)$ for Lévy oscillators with different values of $\mu$. This can be done both for the semigroup process [13] and for the Langevin-driven one (e.g. fractional Fokker-Planck dynamics). Those patterns of temporal behavior are inequivalent, although both processes may terminate at common pdfs with a predefined decay at infinites. The latter pdfs may have heavy tails, but generically admit an arbitrary (finite, eventually infinite) number of moments.

A more general problem would be that of the existence of terminal pdfs, after passing from the master equation to the (fractional) Hamiltonian dynamics [9] with an arbitrary potential $\mathcal{V}$, in one, two or three spatial dimensions.

We note that in the case of $\mu = 2$, when fractional Hamiltonian [9] reduces to ordinary quantum-mechanical Hamiltonian operator. In the standard quantum mechanical setting (see, e.g., Refs. [35, 36]) the above question is equivalent to an issue of the existence
of bound states in a given potential. The quantum mechanical language appears because we can convert the parabolic equation of the Fokker-Planck type to the generalized Schrödinger equation.

The wave function of a bound state should be localized to ensure a normalization of its squared expression, i.e. the corresponding stationary pdf of the Fokker-Planck equation. It is known (see, e.g., Ref. [35]) that in 1D case the bound state exist in the potential well $U(x)$ of not only finite but an infinitesimal depth. The only restriction is that the integral $\int_{-\infty}^{\infty} U(x) dx$ should exist.

The latter condition is equivalent to the requirement that $U(x)$ should have the same asymptotics at infinities and potential zero point $U(\pm \infty) = 0$. In the 2D case, when the potential $U = U(x, y)$, the situation is similar to that in 1D one, while in 3D ($U = U(x, y, z)$) the situation is to some extent opposite - if the potential well is not sufficiently deep (see Ref. [35] for details), the particle cannot be "captured", so that bound state does not exist. Confining potentials in 3D, where bound states exist, form the so-called Kato class of potentials.

The presence of fractional derivatives with $0 < \mu \leq 2$ alters the picture both in 1D (2D) and in 3D. In 1D they definitely "spoil" the bound states. It is not only that the pdfs (if in existence) may have heavier tails if compared to the conventional ($\mu = 2$) case. The pdfs in question may not exist at all, if a normalizability of the bound state is lost. In 3D and in equations with fractional derivatives there may typically be no normalizable bound states (and thus terminal pdfs), except for a carefully selected (Kato)-subclass of conceivable potentials.

Some peculiarities pertaining to the (non)-existence of invariant pdfs in the case of Lévy drivers (Langevin-driven fractional dynamics) were discussed for 1D case.
We have encountered the same problem in connection with the Cauchy family of pdfs \[4, 8\], see also Ref. \[37\] for a discussion of so-called infinite covariant densities.

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