FPRAS for computing a lower bound for weighted matching polynomial of graphs

Shmuel Friedland
Department of Mathematics, Statistics, and Computer Science,
University of Illinois at Chicago
Chicago, Illinois 60607-7045, USA

April 12, 2007

Abstract

We give a fully polynomial randomized approximation scheme to compute a lower bound for the matching polynomial of any weighted graph at a positive argument. For the matching polynomial of complete bipartite graphs with bounded weights these lower bounds are asymptotically optimal.

2000 Mathematics Subject Classification: 05A15, 05C70, 15A52, 68Q10.
Keywords and phrases: Perfect matchings, k-matchings, permanents, hafnians, weighted matching polynomial of graph, fully polynomial randomized approximation scheme.

1 Introduction

Let \( G = (V, E) \) be an undirected graph, (with no self-loops), on the set of vertices \( V \) and the set of edges \( E \). A set of edges \( M \subseteq E \) is called a matching if no two distinct edges \( e_1, e_2 \in M \) have a common vertex. \( M \) is called a \( k \)-matching if \( \#M = k \). For \( k \in \mathbb{N} \) let \( M_k(G) \) be the set of \( k \)-matchings in \( G \). (\( M_k(G) = \emptyset \) for \( k > \lfloor \frac{\#V}{2} \rfloor \).) If \( \#V = 2n \) is even then an \( n \)-matching is called a perfect matching.

Let \( \omega : E \to (0, \infty) \) be a weight function, which associate with edge \( e \in E \) a positive weight \( \omega(e) \). We call \( G_\omega = (V, E, \omega) \) a weighted graph. Denote by \( \iota \) the weight \( \iota : E \to \{1\} \). Then \( G \) can be identified with \( G_\omega \).

Let \( M \in M_k(G) \). Then the weight of the matching is defined as \( \omega(M) := \prod_{e \in M} \omega(e) \). The total weighted \( k \)-matching of \( G_\omega \) is defined:

\[
\phi(k, G_\omega) := \sum_{M \in M_k(G)} \omega(M), \ k \in \mathbb{N}
\]
where $\phi(k, G_\omega) = 0$ if $\mathcal{M}_k(G) = \emptyset$ for any $k \in \mathbb{N}$. Furthermore we let $\phi(0, G_\omega) := 1$. Note that $\phi(k, G_i) = \# \mathcal{M}_k(G)$, i.e. the number of $k$-matchings in $G$ for any $k \in \mathbb{N}$. The weighted matching polynomial of $G_\omega$ is defined by:

$$\Phi(t, G_\omega) := \sum_{k=0}^{n} \phi(k, G_\omega)t^{n-k}, \quad n = \lfloor \frac{\# V}{2} \rfloor.$$  

This polynomial is fundamental in the monomer-dimer model in statistical physics [3, 12], and for $\omega = 1$ in combinatorics. Note that if $\# V$ is even then $\Phi(0, G_\omega)$ is the total weighted perfect matching of $G$. (Some authors consider the polynomial $t^{\lfloor \frac{\# V}{2} \rfloor} \Phi(t^{-1}, G_\omega)$ instead of $\Phi(t, G_\omega)$.) It is known that nonzero roots of a weighted matching polynomial of $G$ are real and negative [12]. Observe that $\Phi(1, G_i)$ the total number monomer-dimer coverings of $G$.

Let $G$ be a bipartite graph, i.e., $V = V_1 \cup V_2$ and $E \subset V_1 \times V_2$. In the special case of a bipartite graph where $n = \# V_1 = \# V_2$, it is well known that $\phi(n, G)$ is given as perm $B(G)$, the permanent of the incidence matrix $B(G)$ of the bipartite graph $G$. It was shown by Valiant that the computation of the permanent of a $(0,1)$ matrix is $\#P$-complete [17]. Hence, it is believed that the computation of the number of perfect matching in a general bipartite graph satisfying $\# V_1 = \# V_2$ cannot be polynomial.

In a recent paper Jerrum, Sinclair and Vigoda gave a fully-polynomial randomized approximation scheme (ffras) to compute the permanent of a non-negative matrix [13]. (See also Barvinok [1] for computing the permanents within a simply exponential factor, and Friedland, Rider and Zeitouni [9] for concentration of permanent estimators for certain large positive matrices.) [13] yields the existence a fpras to compute the total weighted perfect matching in a general bipartite graph satisfying $\# V_1 = \# V_2$. In a recent paper of Levy and the author it was shown that there exists fpras to compute the total weighted $k$-matchings for any bipartite graph $G$ and any integer $k \in [1, \frac{\# V}{2}]$. In particular, the generating matching polynomial of any bipartite graph $G$ has a fpras. This observation can be used to find a fast computable approximation to the pressure function, as discussed in [8], for certain families of infinite graphs appearing in many models of statistical mechanics, like the integer lattice $\mathbb{Z}^d$.

The MCMC, (Monte Carlo Markov Chain), algorithm for computing the total weighted perfect matching in a general bipartite graph satisfying $\# V_1 = \# V_2$, outlined in [13], can be applied to estimate the total weighted perfect matchings in a weighted non-bipartite graph with even number of vertices. However the proof in [13], that shows this algorithm is frpas for bipartite graphs, fails for non-bipartite graphs. Similarly, the proof of concentration results given in [9] do not seem to work for non-bipartite graphs. The technique introduced by Barvinok in [1] to estimate the number of weighted perfect matching in bipartite graphs, does extend to the estimate of total weighted perfect matchings in a general non-bipartite graph with even number of vertices, when one uses real or complex Gaussian distribution. (See the discussion in §5.)
In this paper we give a fpras for computing a lower bound $\tilde{\Phi}(t, G_\omega)$ for the weighted generated function $\Phi(t, G_\omega)$ for a fixed $t > 0$. We show that this lower bound has a multiplicative error at most $\exp(N \min(\frac{a^2}{2t}, C_1))$, see (1.7), where $a^2$ is the maximal weight of edges of $G$

$$C_1 = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \log(x^2)e^{-\frac{x^2}{2}}dx = 1.270362845\ldots \quad (1.1)$$

These estimates are similar in nature to heuristic computations of Baxter [2], where he showed that his computation for the dimers on $\mathbb{Z}_2$ lattice are very precise away from only dimer configurations, i.e. perfect matchings. (The results of heuristic computations of Baxter were recently confirmed in [8].)

We show that that for the matching polynomial of complete bipartite graphs with weights in $[b^2, a^2]$, $0 < b \leq a$, this lower bound is asymptotically optimal.

We now describe briefly our technical results. With each weighted graph $G_\omega$ associate a skew symmetric matrix $A = [a_{ij}]_{i,j=1}^N \in \mathbb{R}^{N \times N}$, $A^\top = -A$, where $N := \#V$, as follows. Identify $E$ with $\langle N \rangle := \{1, \ldots, N\}$, and each edge $e \in E$ with the corresponding unordered pair $(i, j), i \neq j \in \langle N \rangle$. Then $a_{ij} \neq 0$ if and only $(i, j) \in E$. Furthermore for $1 \leq i < j \leq N$ let $x_{ij}$ be a set of $(N^2)$ independent random variables with

$$E x_{ij} = 0, \quad E x_{ij}^2 = 1, \quad 1 \leq i \leq j \leq N. \quad (1.2)$$

Let $x := (x_{11}, \ldots, x_{1N}, x_{22}, \ldots, x_{NN})$. We view $x$ as a random vector variable with values $\xi = (\xi_{11}, \ldots, \xi_{NN}) \in \mathbb{R}^{(N^2)}$. Let $Y_A$ be the following skew-symmetric random matrix

$$Y_A := [a_{ij}x_{\min(i,j)\max(i,j)}]_{i,j=1}^N \in \mathbb{R}^{N \times N}. \quad (1.3)$$

A variation of the Godsil-Gutman estimator [10] states

$$E \det(\sqrt{t}I_N + Y_A)) = \Phi(t, G_\omega) \text{ if } N = \#V \text{ is even,} \quad (1.4)$$
$$E \det(\sqrt{t}I_N + Y_A)) = \sqrt{t}\tilde{\Phi}(t, G_\omega) \text{ if } N = \#V \text{ is odd.} \quad (1.5)$$

for any $t \geq 0$. Here $I_N$ stands for $N \times N$ identity matrix.

We show the concentration of $\log \det(\sqrt{t}I_N + Y_A))$ around

$$\log \tilde{\Phi}(t, G_\omega) := E \log \det(\sqrt{t}I_N + Y_A) \quad (1.6)$$

using [11]. These concentration results show that $\tilde{\Phi}(t, G_\omega)$ has a fpras. Jensen inequalities yield that $\tilde{\Phi}(t, G_\omega) \leq \Phi(t, G_\omega)$. Together with an upper estimate we have the following bounds:

$$\frac{1}{N} \log \tilde{\Phi}(t, G_\omega) \leq \frac{1}{N} \log \Phi(t, G_\omega) \leq \frac{1}{N} \log \tilde{\Phi}(t, G_\omega) + \min(\frac{a^2}{2t}, C_1) \quad (1.7)$$

where $a = \max |a_{ij}|$. The above inequality hold also for $t = 0$. (For $N$ even and $t = 0$ this result is due to Barvinok [11, §7].) It is our hope that by refining the techniques we are using one can show that $\Phi(t, G_\omega)$ has fpras for any $t > 0$. 


2 Preliminary results

Lemma 2.1 Let $G = (V, E)$ be an undirected graph on $N$ vertices. Let $\omega : V \to (0, \infty)$ be a given weight function. Let $A = -A^\top \in \mathbb{R}^{N \times N}$ be the corresponding real skew symmetric matrix defined in §1. Assume that $x_{ij}, i = 1, \ldots, j, j = 1, \ldots, N$ are $\binom{N+1}{2}$ independent random variables, normalized by the conditions (1.2). Let $Y_A \in \mathbb{R}^{N \times N}$ be the skew symmetric real matrix defined by (1.3). Then (1.4-1.5) hold.

Proof. Let $\sqrt{t} = s$. Observe first that $\det(sI_N + Y_A)$ is a sum of $N!$ monomials, where each monomial is of degree at most 2 in the variables $x_{ij}$ for $i < j$ and of degree $m$ invariable $s$. The total degree of each monomial is $N$. The expected value of such a monomial is zero if at least the degree of one of the variables $x_{ij}$ is one. So it is left to consider the expected value of all monomials, where the degree if each $x_{ij}$ is 0 or 2, which are called nontrivial monomials.

Assume first that $N$ is even. Observe that if a monomial contains $s$ of odd power than it must be linear at least in one $x_{ij}$. Hence its expected value is zero. Thus $E\det(sI_N + Y_A)$ is a polynomial in $s^2$. Consider a nontrivial monomial such that the power of $s$ is $N - 2m$. Note that this monomial is of the form $\tau s^{N-2m} \prod_{(i,j) \in M} \omega((i,j))x_{ij}^2$, for some $m$ matching $M \in \mathcal{M}_m$. Here $(-1)^m \tau$ is the sign of the corresponding permutation $\sigma : \langle N \rangle \to \langle N \rangle$. Since $\sigma(i) = j, \sigma(j) = i$ for any edge $(i,j) \in M$, and $\sigma(i) = i$ for all vertices $i$ which are not covered by $M$ we deduce that $\tau = 1$. Hence the expected value of this monomial is $s^{N-2m} \prod_{e \in M} \omega(e)$. This proves (1.4). The identity (1.5) is shown similarly. \hfill \Box

Recall the following well known result:

Lemma 2.2 Let $A = -A^\top \in \mathbb{R}^{N \times N}$ be a skew symmetric matrix. Then $B := iA$, where $i := \sqrt{-1}$, is a hermitian matrix. Arrange the eigenvalues of $B$ in a decreasing order: $\lambda_1(B) \geq \ldots \geq \lambda_N(B)$. Then

$$\lambda_{N-i+1}(B) = -\lambda_i(B) \text{ for } i = 1, \ldots, N. \quad (2.1)$$

In particular

$$\det(\sqrt{t}I_N + A) = \prod_{i=1}^{N} \sqrt{t + \lambda_i(B)^2}. \quad (2.2)$$

Proof. Clearly, $B$ is hermitian. Hence all the eigenvalues of $B$ are real. Arrange these eigenvalues in a decreasing order. So $-i\lambda_j(B), j = 1, \ldots, N$ are the eigenvalues of $A$. Since $A$ is real valued, the nonzero eigenvalues of $A$ must be in conjugate pairs. Hence equality (2.1) holds. Observe next that if $\lambda_k(A) = -i\lambda_k(B) \neq 0$ then

$$(\sqrt{t} + \lambda_k(A))(\sqrt{t} + \lambda_{N-k+1}(A)) = \sqrt{t + \lambda_k(B)^2} \sqrt{t + \lambda_{N-k+1}(B)^2}.$$
As the eigenvalues of $\sqrt{t}I_N + A$ are $\sqrt{t} + \lambda_k(A), k = 1, \ldots, N$ we deduce (2.2).

3 Concentration for Gaussian entries

In this section we assume that each $x_{ij}$ is a normalized real Gaussian variable, i.e satisfying (1.2). Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called Lipschitz function, or Lipschitzian, if there exists $L \in [0, \infty)$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x \neq y \in \mathbb{R}$. The smallest possible $L$ for a Lipschitz function is denoted by $|f|_L$. Let $A_N \subset \mathbb{R}^{n \times n}, iA_N \subset \mathbb{C}^{n \times n}$ denote the set of $N \times N$ real skew symmetric matrices, and the set of $N \times N$ hermitian matrices of the form $iA, A \in A_N$. With each $A \in A_N$ we associate a weighted graph $G_{\omega} = (V, E, \omega)$, where $V = \langle N \rangle$, $(i, j) \in V \iff a_{ij} \neq 0, \omega((i, j)) = |a_{ij}|^2$. Denote by $a := \max |a_{ij}|$. To avoid the trivialities we assume that $a > 0$. Note that $a^2$ is the maximal weight of the edges in $G_{\omega}$. Let $Y_A$ be the random skew symmetric matrix given by (1.3) and denote by $X_A$ the random hermitian matrix $X_A := \frac{1}{\sqrt{N}}Y_A$.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. As in [11] consider the following $F : iA_N \rightarrow \mathbb{R}$ given by the trace formula:

$$F(B) = \text{tr}_N f(B) := \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i(B)), \quad B \in iA_N.$$  

Denote by $E \text{ tr}_N(f(X_A))$ the expected value of the function $\text{tr}_N(f(X_A))$. The concentration result [11, Thm 1.1(b)] states:

$$\Pr(|\text{tr}_N(f(X_A)) - E \text{ tr}_N(f(X_A))| \geq r) \leq 2e^{-\frac{r^2}{8a^2}}.$$  

(Recall that the normalized Gaussian distribution satisfies the log Sobolev inequality with $c = 1$.) We remark that since the entries of $X_A$ are either zero or pure imaginary one can replace the factor 8 in the inequality (3.1) by the factor 2. See for example the results in [13, 8.5].

**Lemma 3.1** Let $0 \neq A = [a_{ij}] \in A_N, a = \max |a_{ij}|, t \in (0, \infty), x_{ij}, 1 \leq i \leq j \leq N$ be independent Gaussian satisfying (1.2). Let $Y_A \in A_N$ be the random skew symmetric matrix given by (1.3). Then

$$\Pr(|\log \det(\sqrt{t}I_N + Y_A) - E \log \det(\sqrt{t}I_N + Y_A)| \geq Nr) \leq 2e^{-\frac{tN^2r^2}{8a^2}}.$$  

**Proof.** Let $f_t(x) := \frac{1}{t} \log(\frac{t}{N} + x^2)$. $f_t$ is differentiable and

$$|(f_t)_L| = \max_{x \in \mathbb{R}} |f_t''(x)| = \frac{\sqrt{N}}{2\sqrt{t}}.$$
Apply (3.1) to $f_t$. Observe that the right-hand side of (3.1) is equal to the right-hand side of (3.2). Use (2.2) to deduce that

$$N \text{tr}_N(f_t(X_A)) = \sum_{i=1}^{N} \log \sqrt{\frac{t}{N} + \lambda_i(X_A)^2} = \sum_{i=1}^{N} \log \sqrt{\frac{t}{N} + |\lambda_i(Y_A)|^2}$$

$$= -\frac{1}{2} N \log N + \log \prod_{i=1}^{N} \sqrt{t + |\lambda_i(Y_A)|^2} = -\frac{1}{2} N \log N + \log \det(\sqrt{t}I_N + Y_A).$$

Hence the left-hand sides of (3.1) and (3.2) are equivalent.

The following lemma is well known, e.g. [9, p’1566], and we bring its proof for completeness.

**Lemma 3.2** Let $U$ be a real random variable with a finite expected value $E U$. Then $e^{eU} \leq E e^U$. Assume that the following condition hold

$$\Pr(U - E U \geq r) \leq 2e^{-Kr^2} \text{ for each } r \in (0, \infty) \text{ and some } K > 0. \quad (3.3)$$

Then

$$e^{E U} \leq E e^U \leq e^{E U}(1 + \frac{2e^{-r^2}}{\sqrt{K \pi}}). \quad (3.4)$$

**Proof.** Since $e^u$ is convex, the inequality $e^{E U} \leq E e^U$ follows from Jensen inequality. Let $\mu := E U$ and $F(u) := \Pr(U \leq u)$ be the cumulative distribution function of $U$. We claim that

$$E e^U \leq e^\mu + \int_{\mu < u} e^u (1 - F(u)) du. \quad (3.5)$$

Clearly

$$E e^U = \int_{-\infty}^{\infty} e^u dF(u) = \int_{u \leq \mu} e^u dF(u) + \int_{\mu < u} e^u dF(u). \quad (3.6)$$

Since $e^u \leq e^\mu$ for $u \leq \mu$ we deduce that

$$\int_{u \leq \mu} e^u dF(u) \leq e^\mu F(\mu).$$

We now estimate the second integral in the right-hand side of (3.6). Recall that $F(u)$ is an nondecreasing function continuous from the right satisfying $F(+\infty) = 1$. Hence $e^u(F(u) - 1) \leq 0$ for all $u \in \mathbb{R}$. For any $R > \mu$ use integration by parts to deduce

$$\int_{\mu < u \leq R} e^u dF(u) = e^\mu (F(u) - 1)|_{\mu}^{R} + \int_{\mu < u \leq R} e^u (1 - F(u)) du \leq$$

$$e^\mu (1 - F(\mu)) + \int_{\mu < u} e^u (1 - F(u)) du.$$
So
\[ \int_{\mu < u} e^u dF(u) \leq e^{\mu}(1 - F(\mu)) + \int_{\mu < u} e^u (1 - F(u)) du, \]
and (3.5) holds.
Assume now that (3.3) holds. Thus
\[ 1 - F(u) = \Pr(U > u) \leq 2e^{-K(u - \mu)^2} \text{ for any } u > \mu. \]
Hence
\[ \int_{\mu < u} e^u (1 - F(u)) du \leq 2 \int_{\mu < u} e^{u-K(u-\mu)^2} du \leq 2e^\mu \frac{1}{\sqrt{K\pi}}. \]
Combine the above inequality with (3.5) to deduce the right-hand side of (3.4).

Corollary 3.3 Let the assumptions of Lemma 3.1 hold. Then
\[ \frac{1}{N} \log \tilde{\Phi}(t, G_\omega) \leq \frac{1}{N} \log \Phi(t, G_\omega) \leq \frac{1}{N} \log \tilde{\Phi}(t, G_\omega) + \frac{1}{N} \log\left(1 + \frac{\sqrt{8Nae^{2N}}}{\sqrt{\pi t}}\right). \]

4 FPRAS for computing \( \log \tilde{\Phi}(t, G_\omega) \)

Let \( B \in \mathbb{R}^{N \times N} \). For \( k \in \mathbb{N} \) denote by \( \oplus_k B \in \mathbb{R}^{kN \times kN} \) the block diagonal matrix \( \text{diag}(B, \ldots, B). \) (\( \oplus_k B \) is a direct sum of \( k \) copies of \( B. \)) Note that if \( B \in A_N \) then \( \oplus_k B \in A_{kN}. \) Clearly,
\[ \det(sI_{kN} + \oplus_k B) = (\det(sI_N + B))^k \text{ for any } B \in \mathbb{R}^{N \times N} \text{ and } s \in \mathbb{R}. \quad (4.1) \]
Let \( A \in A_N, \) and \( Y_A \) be the random matrix defined by (1.3). By \( Y_A(\xi) \) we mean the skew symmetric matrix \( [a_{ij}\xi_{\min(i,j)}\max(i,j)]_{i,j=1}^{N} \), which is a sampling of \( Y_A. \) Let \( x_{ij}, 1 \leq i \leq j \leq kN \) be \( \binom{kN+1}{2} \) normal Gaussian independent random variables. Consider the random matrix \( Y_{\oplus_k A}. \) Then a sampling
\[ Y_{\oplus_k A}(\xi), \xi \in \mathbb{R}^{\binom{kN+1}{2}} = \text{diag}(Y_A(\xi_1), \ldots, Y_A(\xi_k)), \xi_i \in \mathbb{R}^{\binom{N+1}{2}}, i = 1, \ldots, k \]
is equivalent to \( k \) sampling of \( Y_A. \)

Theorem 4.1 Let \( 0 \neq A = [a_{ij}] \in A_N, a = \max |a_{ij}|, t \in (0, \infty), x_{ij}, 1 \leq i \leq j \leq N \) be independent Gaussian satisfying (1.2). Let \( Y_A \in A_N \) be the
random skew symmetric matrix given by (1.3). Let \( Y_A(\xi_1), \ldots, Y_A(\xi_k) \) be \( k \) samplings of \( Y_A \). Then

\[
\Pr\left( \left| \frac{1}{k} \sum_{i=1}^{k} \log \det(\sqrt{t}I_N + Y_A(\xi_i)) - \log \Phi(t, G_\omega) \right| \geq N r \right) \leq 2e^{-tkN^2/t^2}. \tag{4.2}
\]

In particular the inequality

\[
\frac{1}{N} \log \Phi(t, G_\omega) \leq \frac{1}{N} \log \Phi(t, G_\omega) \leq \frac{1}{N} \log \Phi(t, G_\omega) + \frac{a^2}{2t} \tag{4.3}
\]

holds.

Hence an approximation of \( \Phi(t, G_\omega) \) by \( \left( \prod_{i=1}^{k} \det(\sqrt{t}I_N + Y_A(\xi_i)) \right)^\frac{1}{k} \) is a fully-polynomial randomized approximation scheme.

**Proof.** Use (4.1) to obtain

\[
\log \det(\sqrt{t}I_N + Y_{\oplus A}(\xi)) = \sum_{i=1}^{k} \log \det(\sqrt{t}I_N + Y_A(\xi_i))
\]

Hence

\[
E \log \det(\sqrt{t}I_N + Y_{\oplus A}) = kE \log \det((\sqrt{t}I_N + Y_A)^k = k \log \Phi(t, G_\omega) \tag{4.4}
\]

Apply (3.2) to \( Y_{\oplus A} \) to deduce (4.2). Observe next that

\[
E \det(\sqrt{t}I_N + Y_{\oplus A}) = E \det((\sqrt{t}I_N + Y_A)^k = \Phi(t, G_\omega)^k. \tag{4.5}
\]

Use Lemma 3.2 for the random variable \( \log \det(\sqrt{t}I_N + Y_{\oplus A}) \) to deduce

\[
\frac{1}{N} \log \Phi(t, G_\omega) \leq \frac{1}{N} \log \Phi(t, G_\omega) \leq \frac{1}{N} \log \Phi(t, G_\omega) + \\
+ \frac{1}{kN} \log(1 + \frac{\sqrt{8kN}ae^{\frac{a^2kN}{2t}}}{\sqrt{\pi t}}).
\]

Let \( k \to \infty \) to deduce (4.3).

We now show that (4.2) gives fpras for computing \( \Phi(t, G_\omega) \) in sense of [14].

Let \( \epsilon, \delta \in (0, 1) \). Choose

\[
r = \frac{\epsilon}{2N}, \quad k = \left\lceil \frac{8a^2N \log \frac{4}{\delta}}{\epsilon t^2} \right\rceil.
\]

Then

\[
\Pr(1 - \epsilon < \frac{\prod_{i=1}^{k} \det(\sqrt{t}I_N + Y_A(\xi_i))^\frac{1}{k}}{\Phi(t, G_\omega)} < 1 + \epsilon) > 1 - \frac{\delta}{2}.
\]
Observe next that
\[ \Pr(|x_{ij}| > \sqrt{2 \log \frac{N^2 k}{\delta}}) < \frac{\delta}{N^2 k}. \]
Hence with probability \(1 - \frac{\delta}{4}\) at least, the absolute of each off-diagonal of \(Y_A(\xi_i), i = 1, \ldots, k\) is bounded by \(a\sqrt{2 \log \frac{N^2 k}{\delta}}\). In this case all the entries of \(\sqrt{I_N + Y_A(\xi_i)}\) are polynomial in \(a, \sqrt{t}, N, \frac{1}{\epsilon}, \log \frac{1}{\delta}\). The length of the storage of each entry is logarithmic in the above quantities.

Finally observe that we need \(O(N^3)\) to compute \(\det(\sqrt{I_N + Y_A(\xi_i)})\). Hence the total number of computations for our estimate is of order
\[ t^{-1}a^2N^4\epsilon^{-2}\log^{-1}. \]

\[ \square \]

The quantity \(\frac{1}{N} \log \Phi(t, G_\omega)\) can be viewed as the \textbf{exponential growth} of \(\log \Phi(t, G_\omega)\) in terms of the number of vertices \(N\) of \(G\). Note that since the total number of matching of a graph \(G\) is given by \(\Phi(1, G_i)\), Theorem 4.1 combined with (1.7) yields that the exponential growth of the computable lower bound \(\tilde{\Phi}(1, G_i)\) differs by \(\frac{1}{2}\) at most from the exponential growth of \(\Phi(1, G_i)\). Note that for complete graphs on \(2n\), the exponential growth of the number of perfect matching matchings is of order \(\log 2n - 1\). For \(k\)-regular bipartite graphs on \(2n\) vertices the results of [4, 7] imply the inequality that for \(n\) big enough the exponential growth of the total number of matchings is at least \(\log k - 1\). Thus for graphs \(G\) on \(2n\) vertices containing, bipartite \(k\)-regular graphs on \(2n\) vertices, with \(k \geq 5\) and \(n\) big enough, \(\tilde{\Phi}(1, G_i)\) has a positive exponential growth.

5 \textbf{Another estimate of} \(\log \Phi(t, G_\omega) - \log \tilde{\Phi}(t, G_\omega)\)

\textbf{Lemma 5.1} Let \(X\) be a real Gaussian random variable. Then
\[ \log E X^2 - E \log X^2 \leq C_1, \] (5.1)
where \(C_1\) is given by (1.7). Equality holds if and only if \(E X = 0\).

\textbf{Proof.} Clearly, it is enough to prove the lemma in the case \(X = Y + a\), where \(Y\) is a normalized by (1.2) and \(a \geq 0\). In that case the left-hand side of (5.1) is equal to
\[ g(a) := \log(1 + a^2) - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \log((x + a)^2) e^{-x^2/2} dx. \]
We used the software Maple to show that \(f(a)\) is a decreasing function on \([0, \infty)\). So \(f(0) = C_1\) and \(\lim_{a \to \infty} f(a) = 0\). This proves the inequality (5.1).
Equality holds if and only if $X = bY$ for some $b \neq 0$. \hfill \Box

Denote by $S_n \subset \mathbb{R}^{n \times n}$ the space of $n \times n$ real symmetric matrices. A polynomial $P : \mathbb{R}^n \to \mathbb{R}$ is of degree 2 if

$$P(x) = x^\top Qx + 2a^\top x + b,$$

$x = (x_1, \ldots, x_n)^\top, a = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n, Q \in S_n, b \in \mathbb{R}.$

(We allow here the case $Q = 0$.) The quadratic form $P_h : \mathbb{R}^{n+1} \to \mathbb{R}$ induced by $P$ is given by

$$P_h(y) = y^\top Q_h y, Q_h = \begin{bmatrix} Q & a \\ a^\top & b \end{bmatrix} \in S_{n+1}, y = (y_1, \ldots, y_{n+1})^\top.$$

Clearly, $P(x) = P_h((x^\top, 1)^\top)$. $P$ is called a nonnegative polynomial if $P(x) \geq 0$ for all $x \in \mathbb{R}^n$. It is well known and a straightforward fact that $P$ is nonnegative if and only if $Q_h$ is a nonnegative definite matrix.

The following lemma is a generalization of [1, Thm 4.2, (1)].

**Lemma 5.2** Let $P : \mathbb{R}^n \to \mathbb{R}$ be a nonzero nonnegative quadratic polynomial. Let $X_1, \ldots, X_n$ be $n$-Gaussian random variables, and denote $X := (X_1, \ldots, X_n)^\top$. Then

$$E \log P(X) \leq \log E P(X) \leq E \log P(X) + C_1,$$

where $C_1$ is given by (1.1).

**Proof.** We may assume without a loss of generality that $E P = 1$. In view of the concavity of log we need to show the right-hand side of (5.2). Since $Q_h$ is nonnegative definite it follows that

$$P(x) = \sum_{i=1}^m \lambda_i (a_i^\top x + b_i)^2, a_i \in \mathbb{R}^n, b_i \in \mathbb{R}, \lambda_i > 0, i = 1, \ldots, m,$$

$$E (a_i^\top X + b_i)^2 = 1, i = 1, \ldots, m, \sum_{i=1}^m \lambda_i = 1.$$

Note that one can have at most one $a_i = 0$, and in that case then $b_i^2 = 1$. The concavity of log yields

$$\log P(X) \geq \sum_{i=1}^m \lambda_i \log (a_i^\top X + b_i)^2.$$

(We assume that $\log 0 = -\infty$.) Note that if $a_i \neq 0$ then $a_i X + b_i$ is Gaussian. Lemma 5.1 yields $E \log P(X) \geq -C_1$. \hfill \Box
Theorem 5.3 Let the assumptions of Theorem 4.1 hold. Then (1.7) holds.

Proof. In view of (4.3) it is left to show
\[
\log \Phi(t, G_\omega) \leq \log \tilde{\Phi}(t, G_\omega) + (N - 1)C_1. \tag{5.3}
\]
Let \( A = [a_{ij}]_{i,j=1}^n \in A_N \). Recall that \( \det A = (\text{pfaf } A)^2 \), where \( \text{pfaf } A \) is the pfaffian. (So \( \text{pfaf } A = 0 \) if \( n \) is odd.) Let \( a_i = (a_{1i}, \ldots, a_{(i-1)i})^\top \in \mathbb{R}^{i-1}, i = 2, \ldots, n \). We view \( \text{pfaf } A \) as multilinear polynomial \( \text{Pf}(a_2, \ldots, a_n) \) of total degree \( \frac{n}{2} \), which is linear in each vector variable \( a_i \). (Any polynomial of noninteger total degree is zero polynomial by definition.)

Denote by \( Q_{k,n} \) the set of subsets of \( \{n\} \) of cardinality \( k \in [1, n] \). Each \( \alpha \in Q_{k,n} \) is viewed as \( \alpha = \{i_1, \ldots, i_k\}, 1 \leq i_1 < \ldots < i_k \leq m \). For any matrix \( B = [b_{ij}] \in \mathbb{R}^{n \times n} \) and \( \alpha \in Q_{k,n} \) we define \( B[\alpha] \in \mathbb{R}^{k \times k} \) as the principal submatrix \( [b_{\alpha \alpha}]_{k \times k} \). Then for \( A = [a_{ij}] \in A_n \) denote
\[
\text{Pf}_\alpha(a_2, \ldots, a_n) := \text{pfaf } A[\alpha].
\]
Then \( \text{Pf}_\alpha(a_2, \ldots, a_n) \) is a multilinear polynomial of total degree \( \frac{k}{2} \), which is linear in each \( a_i \). Hence
\[
\det(sI_N + A) = s^N + \sum_{k=1}^n s^{N-k} \sum_{\alpha \in Q_{k,n}} \text{Pf}_\alpha(a_2, \ldots, a_N)^2, \text{ for any } A \in A_N. \tag{5.4}
\]
View \( a_i \in \mathbb{R}^{i-1} \) as a variable while all other \( a_2, \ldots, a_N \) are fixed. Then for \( s \geq 0 \) the above polynomial is quadratic and nonnegative. Group the \( \binom{N}{2} \) independent normalized random Gaussian variables \( X_{ij}, 1 \leq i < j \leq N \) into \( N - 1 \) random vectors \( X_i := (X_{1i}, \ldots, X_{(i-1)i})^\top, i = 2, \ldots, N \). Consider now \( Y_A \). Let
\[
P(X_2, \ldots, X_N) := \det(\sqrt{t}I_N + Y_A) \quad t \geq 0.
\]
Then \( P(X_2, \ldots, X_N) \) is a nonnegative quadratic polynomial in each \( X_j, j = 2, \ldots, N \). Denote by \( E_i \) the expectation with respect to the variables \( X_{1i}, \ldots, X_{(i-1)i} \). (5.4) yields that
\[
P_i(X_2, \ldots, X_i) := E_{i+1} \ldots E_N P(X_2, \ldots, X_N)
\]
is a nonnegative quadratic polynomial in each \( X_j, j = 2, \ldots, i \). Lemma 5.2 yields
\[
\log E_i P_i(X_2, \ldots, X_i) \leq E_i \log P_i(X_2, \ldots, X_i) + C_1, \quad i = 2, \ldots, N.
\]
Hence
\[
\log \Phi(t, G_\omega) = \log E_2 P_2(X_2) \leq E_2 \log P_2(X_2) + C_1 \leq E_2 E_3 \log P_3(X_2, X_3) + 2C_1 \leq \ldots \leq E_2 E_3 \ldots E_N \log P(X_2, X_3, \ldots, X_N) + (N - 1)C_1 = \log \tilde{\Phi}(t, G_\omega) + (N - 1)C_1.
\]
\( \square \)
6 Bipartite graphs

Assume that \( G = (V, E) \) is a bipartite graph. So \( V = V_1 \cup V_2, E \subset E_1 \times E_2 \) and \( N = m + n \). Assume for convenience of notation that \( m : \# V_1 \leq n := \# V_2 \). Thus \( E \subset \langle m \rangle \times \langle n \rangle \), so each \( e \in E \) is identified uniquely with \( (i, j) \in \langle m \rangle \times \langle n \rangle \).

Let \( C = [c_{ij}] \in \mathbb{R}^{m \times n} \) be the weight matrix associated with the weights \( \omega : E \rightarrow (0, \infty) \). So \( c_{ij} = 0 \) if \( (i, j) \notin E \) and \( c_{ij} = \sqrt{\omega(i, j)} \) if \( (i, j) \in E \). Let \( x_{ij}, i = 1, \ldots, m, j = 1, \ldots, n \) be \( mn \) independent normalized real Gaussian variables. Let \( U_C := [c_{ij} x_{ij}] \in \mathbb{R}^{m \times n} \) be a random matrix. Then the skew symmetric matrix \( A \) associated with \( G_\omega \) is given by and the corresponding random matrices \( Y_A, X_A \) are given as

\[
A = \begin{bmatrix} 0 & C \\ -C^T & 0 \end{bmatrix}, \ Y_A = \begin{bmatrix} 0 & U_C \\ -U_C^T & 0 \end{bmatrix}, \ X_A = \frac{1}{\sqrt{m + n}} Y_A. \tag{6.1}
\]

Denote by

\[
\sigma_1(U_C) \geq \ldots \geq \sigma_m(U_C) \geq 0 \tag{6.2}
\]

be the first \( m \) singular values of \( U_C \). Then the eigenvalues of \( Y_A \) consists of \( n - m \) zero eigenvalues and the following \( 2m \) eigenvalues:

\[
\pm \sigma_1(U_C), \ldots, \pm \sigma_m(U_C). \tag{6.3}
\]

Hence

\[
\det(\sqrt{t} I_{m+n} + Y_A) = t^{\frac{n-m}{2}} \prod_{i=1}^{m} (t + \sigma_i(U_C)^2). \tag{6.4}
\]

In [9] the authors considered the random matrix \( V_C := U_C U_C^T \in \mathbb{R}^{m \times m} \). Note that the eigenvalues of \( V_C \) are

\[
\sigma_1^2(U_C) \geq \ldots \geq \sigma_m^2(U_C). \tag{6.5}
\]

Furthermore, one has the equality \( \mathbb{E} \det V_C = \phi(m, G_\omega) \). Let \( K_{m,n} \) be the complete bipartite graph on \( V_1 = \langle m \rangle, V_2 = \langle n \rangle \) vertices. Assume that \( 1 \leq m \leq n \). Let \( 0 < b \leq a \) be a fixed. Denote by \( \Omega_{m,n,[b^2,a^2]} \) the sets of all weights \( \omega : \langle m \rangle \times \langle n \rangle \rightarrow [b^2, a^2] \). Recall that each \( \omega \in \Omega_{m,n,[b^2,a^2]} \) induces the positive matrix \( C(\omega) = [c_{ij}(\omega)] \in \mathbb{R}^{m \times n} \), where \( c_{ij}(\omega) \in [b, a] \). It was shown in [9] that

\[
\frac{1}{n} \log \det V_C(\omega) \quad \text{concentrates at} \quad \frac{1}{n} \log \phi(m, K_{m,n,\omega}) \quad \text{with probability 1 as} \quad n \rightarrow \infty.
\]

More precisely

\[
\limsup_{n \rightarrow \infty} \sup_{m \leq n, \omega \in \Omega_{m,n,[b^2,a^2]}} \Pr\left( \left| \frac{1}{n} \log \det V_C(\omega) - \log \phi(m, K_{m,n,\omega}) \right| > \delta \right) = 0 \tag{6.6}
\]

for any \( \delta > 0 \).

**Theorem 6.1** Let \( 0 < b \leq a \) be given. For \( \omega \in \Omega_{m,n,[b^2,a^2]} \) let \( C(\omega) \) be a positive \( m \times n \) matrix defined above and \( A(\omega) = A_{m+n} \) be given by [6, 7], \( (C = C(\omega)) \). Assume that \( x_{ij}, 1 \leq i \leq j \leq (m + n) \) are independent Gaussian
satisfying (1.2). Let \( Y_A \in \mathbb{A}_N \) be the random skew symmetric matrix given by (1.3). Then for any \( t > 0 \)

\[
\limsup_{n \to \infty} \sup_{m \leq n, \omega \in \Omega_{m,n,[k^2,a^2]}} \Pr\left( \frac{1}{m+n} \left| \log \det(\sqrt{t}I_N + Y_A) - \log \Phi(t, K_{m,n,\omega}) \right| > \delta \right) = 0
\]  

(6.7)

Equivalently

\[
\limsup_{n \to \infty} \sup_{m \leq n, \omega \in \Omega_{m,n,[k^2,a^2]}} \frac{1}{m+n} \left( \log \Phi(t, K_{m,n,\omega}) - \log \tilde{\Phi}(t, K_{m,n,\omega}) \right) = 0.
\]  

(6.8)

**Proof.** Our proof follows the arguments in [9], and we point out the modifications that one has to make. Let \( N = m + n \). Since \( 1 \leq m \leq n \) we have that

\[
\frac{1}{2n} \leq \frac{1}{N} < \frac{1}{n}.
\]  

(4.2) with \( k = 1 \) implies:

\[
\limsup_{n \to \infty} \sup_{m \leq n, \omega \in \Omega_{m,n,[k^2,a^2]}} \Pr\left( \frac{1}{m+n} \left| \log \det(\sqrt{t}I_N + Y_A) - \log \Phi(t, K_{m,n,\omega}) \right| > \delta \right) = 0
\]  

(6.9)

Thus it is enough to show equality (6.8).

Denote by \( X_A \) the random hermitian matrix \( X_A := \frac{1}{\sqrt{N}} Y_A \). For \( \epsilon > 0 \) define

\[
\det_{\epsilon}(\sqrt{t}I_N + Y_N) := \prod_{i=1}^{N} \sqrt{t + \max(|\lambda_i(Y_N)|, \sqrt{N}\epsilon)^2},
\]

\[
\det_{\epsilon}(\frac{\sqrt{t}}{\sqrt{N}} I_N - 1X_N) := \prod_{i=1}^{N} \sqrt{\frac{t}{N} + \max(|\lambda_i(X_N)|, \epsilon)^2}.
\]

Clearly,

\[
\det_{\epsilon}(\sqrt{t}I_N + Y_N) = N^{\frac{N}{2}} \det_{\epsilon}(\frac{\sqrt{t}}{\sqrt{N}} I_N - 1X_N).
\]  

(6.10)

Let \( f_{N,t,\epsilon}(x) := \frac{1}{2} \log(\frac{t}{N} + \max(|x|, \epsilon)^2) \). Then

\[
|f_{N,t,\epsilon}|_{\mathcal{L}} \leq \frac{1}{\epsilon} \text{ for } N \geq \frac{t}{\epsilon^2}.
\]

In what follows we assume that \( N \geq \frac{t}{\epsilon^2} \). Observe next that

\[
\frac{1}{N} \log \det_{\epsilon}(\frac{\sqrt{t}}{\sqrt{N}} I_N - 1X_N) = \text{tr}_N f_{N,t,\epsilon}(X_A).
\]

Combine the concentration inequality (3.1) with (6.10) to obtain

\[
\Pr\left( \frac{1}{N}(\log \det_{\epsilon}(\sqrt{t}I_N + Y_N) - E \log \det_{\epsilon}(\sqrt{t}I_N + Y_N)) \geq r \right) \leq 2e^{-N^2 r^2 / 8a^2}
\]  

(6.11)
Let
\[ \epsilon_N = \frac{1}{(\log N)^2}. \] (6.12)

Note that for a fixed \( t \) one has \( N \geq \frac{1}{\epsilon_N^2} \) for \( N >> 1 \). Hence
\[
\limsup_{N \to \infty} \Pr\left( \frac{1}{N} \log \det_{\epsilon_N}\left( \sqrt{t}I_N + Y_N \right) - E \log \det_{\epsilon_N}\left( \sqrt{t}I_N + Y_N \right) \geq \delta \right) = 0
\]
for any \( \delta > 0 \). As in [9, Prf. of Lemma 2.1] use (6.11) and Lemma 3.2 to deduce that
\[
\lim_{N \to \infty} \frac{1}{N} \left( \log E \det_{\epsilon_N}\left( \sqrt{t}I_N + Y_N \right) - E \log \det_{\epsilon_N}\left( \sqrt{t}I_N + Y_N \right) \right) = 0,
\]
which is equivalent to
\[
\lim_{N \to \infty} \frac{1}{N} \left( \log E \det_{\epsilon_N}\left( \sqrt{t}I_N - \mathbf{1}X_N \right) - E \log \det_{\epsilon_N}\left( \sqrt{t}I_N - \mathbf{1}X_N \right) \right) = 0. \quad (6.13)
\]
It is left to show that under the assumption of the theorem
\[
\lim_{N \to \infty} \frac{1}{N} \left( \log E \det_{\epsilon_N}\left( \sqrt{t}I_N + Y_N \right) - \log E \det\left( \sqrt{t}I_N + Y_N \right) \right) = 0. \quad (6.14)
\]
Clearly, the above claim is equivalent to
\[
\lim_{N \to \infty} \frac{1}{N} \left( \log E \det_{\epsilon_N}\left( \sqrt{t}I_N - \mathbf{1}X_N \right) - \log E \det\left( \sqrt{t}I_N - \mathbf{1}X_N \right) \right) = 0. \quad (6.15)
\]
To prove the above equality we use the results of [9]. First observe that \( X_N \) has at least \( n - m \) eigenvalues which are equal to zero, while the other \( 2m \) eigenvalues are \( \pm \lambda_1(X_N), \ldots, \pm \lambda_m(X_N) \). Furthermore \( \lambda_1(X_N)^2, \ldots, \lambda_m^2(X_N) \) are the \( m \) eigenvalues of \( \frac{1}{N} \mathbf{1}U_\mathbf{C}U_\mathbf{C}^\top \), denoted in [9] as \( Z(\tilde{A}_{n,m}) \). Clearly
\[
\det_{\epsilon_N}\left( \sqrt{t}I_N - \mathbf{1}X_N \right) = \left( \frac{\sqrt{t}}{\sqrt{N}} \right)^{n-m} \prod_{i=1}^{m} \left( \frac{t}{N} + \max(\lambda_i(X_N)^2, \epsilon^2) \right) \geq \left( \frac{\sqrt{t}}{\sqrt{N}} \right)^{n-m} \prod_{i=1}^{m} \left( \frac{t}{N} + \lambda_i(X_N)^2 \right). \quad (6.16)
\]
Hence for \( \epsilon \leq 1 \)
\[
0 \leq \frac{1}{N} \left( \log \det_{\epsilon_N}\left( \sqrt{t}I_N - \mathbf{1}X_N \right) - \log \det\left( \sqrt{t}I_N - \mathbf{1}X_N \right) \right) = \\
\frac{1}{N} \sum_{\lambda_i(X_N)^2 \leq \epsilon^2} \log \frac{\frac{t}{N} + \epsilon^2}{\frac{t}{N} + \lambda_i(X_N)^2} \leq \\
\frac{1}{N} \sum_{\lambda_i(X_N)^2 \leq \epsilon^2} \log \frac{\epsilon^2}{\lambda_i(X_N)^2} \leq \\
\frac{1}{N} \sum_{\lambda_i(X_N)^2 \leq \epsilon^2} \log \frac{1}{\lambda_i(X_N)^2}.
\]
is equivalent to
\[
\limsup_{n \to \infty} \sup_{m \leq n, \omega \in \Omega_{m,n}, \beta \omega \leq a^2} E \frac{1}{m+n} \sum_{\lambda_i(X_{m+n})^2 \leq t_{m+n}} \log \frac{1}{\lambda_i(X_{m+n})^2} = 0.
\]
Hence
\[
\lim_{N \to \infty} \frac{1}{N} (E \log \det_{\epsilon_N} (\sqrt{\frac{t}{N}} I_N - 1X_N) - E \log \det (\sqrt{\frac{t}{N}} I_N - 1X_N)) = 0. \quad (6.17)
\]
Combine (6.16) with Jensen’s inequality to deduce
\[
E \log \det (\sqrt{\frac{t}{N}} I_N - 1X_N) \leq \log E \det (\sqrt{\frac{t}{N}} I_N - 1X_N) \leq \log E \det (\sqrt{\frac{t}{N}} I_N - 1X_N)
\]
Hence
\[
\limsup_{N \to \infty} \frac{1}{N} (E \log \det_{\epsilon_N} (\sqrt{\frac{t}{N}} I_N - 1X_N) - E \log \det (\sqrt{\frac{t}{N}} I_N - 1X_N)) \geq
\]
\[
\limsup_{N \to \infty} \frac{1}{N} (E \log \det_{\epsilon_N} (\sqrt{\frac{t}{N}} I_N - 1X_N) - \log E \det (\sqrt{\frac{t}{N}} I_N - 1X_N)) \geq 0.
\]
Use (6.13) and (6.17) to deduce (6.15).

References

[1] A. Barvinok, Polynomial time algorithms to approximate permanents and mixed discriminants within a simply exponential factor, *Random Structures Algorithms* 14 (1999), 29-61.

[2] R. J. Baxter, Dimers on a rectangular lattice, *J. Math. Phys.* 9 (1968), 650–654.

[3] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York, 1982.

[4] S. Friedland, A proof of a generalized van der Waerden conjecture on permanents, *Linear Multilin. Algebra* 11 (1982), 107-120.

[5] S. Friedland and L. Gurvits, Generalized Friedland-Tverberg inequality: applications and extensions, submitted.

[6] S. Friedland and D. Levy, A polynomial-time approximation algorithm for the number of k-matchings in bipartite graphs, *Mathematical papers in honour of Eduardo Marques de Sá, Textos de Matemática* #39, Coimbra University, Portugal, 2006, 61-67.
[7] S. Friedland and U.N. Peled, Theory of Computation of Multidimensional Entropy with an Application to the Monomer-Dimer Problem, (with U. Peled), *Advances of Applied Math.* 34(2005), 486-522.

[8] S. Friedland and U.N. Peled, The pressure associated with multidimensional SOFT, *in preparation.*

[9] S. Friedland, B. Rider and O. Zeitouni, Concentration of permanent estimators for certain large matrices, *Annals of Applied Probability,* 14(2004), 1559-1576.

[10] C.D. Godsil and I. Gutman, On the matching polynomial of a graph, *Algebraic Methods in Graph Theory I-II,* North Holland, 1981, 67-83.

[11] A. Guionnet and O. Zeitouni, Concentration of the spectral measure for large matrices, *Electronic J. Prob.* (2000), 119-136.

[12] O.J. Heilman and E.H. Lieb, Theory of monomer-dimer systems, *Comm. Math. Phys.* 25 (1972), 190–232; Errata 27 (1972), 166.

[13] M. Jerrum, A. Sinclair and E. Vigoda, A polynomial-time approximation algorithm for the permanent of a matrix with non-negative entries, *J. ACM* 51 (2004), 671-697.

[14] R. Karp and M. Luby, Monte Carlo algorithms for the planar multi-terminal network reliability problem, *J. Complexity* 1 (1985), 45–64.

[15] M. Ledoux, *The concentration of measure phenomenon,* Mathematical Surveys and Monographs, 89, American Mathematical Society, Providence, RI, 2001.

[16] M. Talagrand, A new look at independence, *Ann. Probab.* 24 (1996), 1–34.

[17] L.G. Valiant, The complexity of computing the permanent, *Theoretical Computer Science* 8 (1979), 189-201.