Research Article

An Application of Variational Minimization: Quasi-Harmonic Coons Patches

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Received 3 March 2022; Accepted 30 March 2022; Published 17 May 2022

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For a minimal surface, the mean curvature of the surface vanishes for all possible parameterizations which results in a second-order nonlinear partial differential equation (pde), whose solution in general is the desired surface as the unknown function of surface parameters. The solution of this partial differential equation is known only for very few cases. Instead of solving the corresponding partial differential equation, we exploit an ansatz method (Ahmad et al. (2013), Ahmad et al. (2014) Ahmad et al. (2015)), used for Coons patches spanned by finite number of boundary curves for a quasi-minimal surface as the extremal of rms of mean curvature. The ansatz method targets a slightly perturbed surface (the rational blending interpolants-based Coons patch, Hermite cubic polynomials interpolants-based Coons patch, and Ferguson surface (vanishing twist vectors) in our case) that comprises initially a nonminimal surface plus the product of a real parameter with a variational function of surface parameters (vanishes at the boundary curves along the unit normal to the slightly perturbed surface). The variational function can be deliberately chosen as the product of linear functions and the mean curvature of the initial nonminimal surface such that it is zero at the boundary curves and then replace this mean curvature by the mean curvature of the resulting surface to find a surface of reduced area, and the process can be repeated for further improvement. In this article, we extend the ansatz method (1) for the rational blending functions interpolants-based Coons patch with a parameter in their form for its different values and (2) for the blending functions comprising Hermite cubic polynomial interpolants-based Coons patch (bicubically blended Coons patch (BBCP) and the Ferguson surface). The ansatz method can be extended for the variational extremal of the surfaces for fuzzy optimal control problems (Filev et al. (1992), Emamizadeh (2005), Farhadinia (2011), Mustafa et al. (2021)).

1. Introduction

Mathematics, computer science, operations research, economics, and its affiliated disciplines depend heavily on the decision sciences, the science of decision-making problems. These decision-making problems rely upon the techniques from the differential and integral calculus and the optimization theory. Optimization theory is related to many diverse areas of minimization and optimization that encompasses the fields like calculus of variations, control theory, convex optimization theory, decision theory, linear programming, network analysis, and many other disciplines of mathematics. In its simplest form, an optimization problem is to find a function called an objective function that maximizes or minimizes for a given constraint, usually in the form of an integral called a functional of functions for a variety of domains and the objective functions. The goal of optimization theory is (1) to obtain a set of conditions of an objective function in a domain D in which the solution of an optimization problem is guaranteed, (2) the characterization of the set of optimal values that indicates the behavior of the function within the given domain values. In optimization theory, we maximize or minimize the values of a given function. The function itself, as mentioned above, is called an objective function. These extreme values are called optimal values. Among the diverse applications and active research area of optimization theory in mathematics, one of the important uses is found in variational improvement of an objective
function [1, 2]. The basic idea behind the variational improvement in a surface to obtain the corresponding quasi-minimal surface is carried out by guessing first an appropriate trial function that includes a set of adjustable parameters (called the variational parameters). These variational parameters are found subject to the conditions given for a variational problem that minimize the objective function. These conditions may be algebraic, differential, or in the form of integrals. The examples already known for such problems are (1) finding the curve of the shortest length connected by two points whose solution may be a straight line in case of no constraints, however, in case of constraints imposed, there is a possibility of obtaining many solutions, generally known as geodesics, and (2) finding the path of stationary optical length connecting two points in accordance with Fermat’s principle of least time (1657) that a light ray takes the path that minimizes the travel time of the light ray among all the possible paths connecting the two points and (3) the Hamilton’s principle of least action that set the stage for Lagrangian and Hamiltonian formalism that has many applications in classical mechanics, theory of relativity, and quantum mechanics. In the integral principle, the quantity to be varied is of the dimension of action which is energy multiplied by time (i.e., energy x time), the reason for the name the principle of least action. Hamilton’s principle is an example of such a principle which requires that the time integral over the Lagrangian given by $\int_{t_i}^{t_f} L(t, q_i(t), \dot{q}_i(t)) dt$ shall have extremum, usually expressed as $\delta \int_{t_i}^{t_f} L(t, q_i(t), \dot{q}_i(t)) dt = 0$, which results in the famous path equation of the system called the Lagrange’s equation of the motion, where $L$, the Lagrangian of the dynamical system, denotes the difference of kinetic and the potential energy of the system and the $q_i(t)$ denote the generalized coordinates, i.e., the minimum number of coordinates required for configuration space. (4) The so-called action principle can be applied to the fields such as gravity or electromagnetic field to obtain equations of motions for these fields. The famous Einstein field equations (EFEs) $\mathbf{R}_{\mu\nu} = \frac{1}{2} \mathbf{g}_{\mu\nu} \mathbf{\nabla}^2 \mathbf{g} + (\delta \mathbf{g}_{\mu\nu} + (8\pi G/c^4) \mathbf{T}_{\mu\nu}$ can be derived by requiring $\delta S = 0$, where $\delta S$ denotes the variation of Einstein-Hilbert action $S = \int [(1/2)\mathbf{(R - 2\Lambda)} + \mathbf{\nabla}_\mu \mathbf{\nabla}_\nu - \mathbf{\nabla}_\mu \mathbf{\nabla}^\mu \mathbf{\nabla}_\nu] \sqrt{-\mathbf{g}} d^4 x$; $\mathbf{\nabla}_\mu$, the matter fields; $\Lambda$, the cosmological constant; $\mathbf{g} = \det (\mathbf{g}_{\mu\nu})$, the determinant of the metric of spacetime; $\mathbf{R}$, the Ricci scalar; $\kappa = 8\pi G c^{-4}$; $G$, the gravitational constant; and $c$, the speed of light in vacuum; for details see the section (21.2) of the ref. [3]. Another related application of the variational methods is the Plateau problem [4, 5]. The famous Plateau problem consists of finding a surface of minimal area among all the surfaces spanned by prescribed border. The Belgian physicist Joseph Plateau demonstrated the minimal surfaces in 1849 by immersing wire frames of different shapes in some foamy water and then to take out these wire frames, so that the wire frames act as the boundary of the minimal surfaces having prescribed borders [6]. The minimal surfaces find broad spectrum applications in surface design, materials science, civil engineering, ship manufacture, and so on.

The theory of minimal surfaces [4, 5] began with Lagrange (1762) who is known for his equation, now called Euler-Lagrange (EL) equation in an attempt to find the surface $z = z(u, v)$ of minimal area spanned by a given curve, by an appeal to variational approach. In 1776, Meusnier found that the catenoid $x(u, v) = (c \cos u \ cosh (\nu v), c \sin u \ cosh (\nu v), v)$ and the helicoid $x(u, v) = (\rho \cos (\alpha t), \rho \sin (\alpha t), \theta)$ satisfy the Euler-Lagrange equation. This Euler-Lagrange equation for these surfaces appeared as the twice of the mean curvature. He concluded that the minimal surfaces are surfaces for which the mean curvature is zero everywhere on the surface. Even in its simple form, the vanishing condition for mean curvature for the surface given in the form $z = z(u, v)$ reduces to the Euler-Lagrange equation $(1 + z_w^2)z_u - 2z_u z_w z_v + (1 + z_v^2)z_{ww} = 0$, a second-order partial differential equation. The general solution of the Euler-Lagrange equation is the desired minimal surface $z = z(u, v)$ in this case. However, the general solution of the Euler-Lagrange even in this case cannot be found for all $z = z(u, v)$, though we can solve the equation for some special cases and the possible numerical solutions can be analyzed for the nonlinear partial differential equations for an equivalent optimization problem [7].

If a surface is minimal, then its minimal isothermal patch [8] satisfies the Laplace equation $\nabla^2 \lambda = 0$, and the components of minimal isothermal patch are harmonic functions. The minimal surfaces can be spanned by a variety of curves from closed contours to a boundary composed of finite number of boundary curves. The contribution of mathematicians in the field of minimal surfaces spanned by versatile boundaries is significant. For the review or the comprehensive survey of the material related to minimal surfaces obtained from the extremal of certain quantity, nonminimum type minimal surfaces, regularity conditions of the minimal surfaces, and higher dimensional Plateau problem, one can see the references [5, 9–20].

The variational calculus is an optimization discipline that deals with the pivotal extermination of integrals. The majority of variational problems required optimality with relation to their surroundings, or given conditions, which is not always easy to achieve in a normal approach. In creating new systems based on one’s choice, the traditional optimum control problems play a vital role. As a methodical manner of dealing with issues in the area of calculus of variations, Euler and Lagrange discovered the Euler-Lagrange equation, which is considered a needed condition for the variational problem. Following that, several renowned academics like Hamilton, Jacobi, Legendre, and Weierstrass expanded the study in various directions. An application of variational extremal is in fuzzy calculus of variations, an extension of classical variational calculus. An instance is the work of Fielev and Angelov [21]; they formulated the fuzzy optimal control problem and found the solution on the basis of fuzzy mathematical programming. Emamizadeh [22] used an integrable fuzzy set to solve a variational problem by replacing the classical environment with a fuzzy set of cuts and then considered a special case in which the variational problem can be transformed into a one-dimensional setting. Farhadinia [23] derived the fuzzy
Euler-Lagrange conditions for fuzzy variational problems (constrained and unconstrained variational problems). Mustafa et al. [24] studied the fuzzy variational problem for the granular Euler-Lagrange equation and derived necessary conditions for Pontryagin-type fuzzy optimal control problems. Sahiner et al. [25] utilize FLA to optimize the energy of a molecule along with the DFT calculations to obtain the results for untested data. Waelder [26] interpolate surface over a grid and get an approximate analytical surface. For the related concepts for basic construction rules for consistent fuzzy surfaces and derivatives, the refs. [27–29] are worth mentioning.

As expressed above, a surface is said to be minimal if its mean curvature is zero everywhere on the surface. The way around is to find a minimal surface spanned by finite number of curves as has been done by [30–33]. In order to find such a minimal surface by utilizing a variational technique for a surface spanned by a finite number of boundary curve, we need to consider slightly deformed surfaces with differential geometry-related quantities as much closer as desired as has been done in ref. [34] for a Coons patch without imposing any condition on the twist vectors (mixed partial derivatives) at the corners. The Coons patches were introduced by Steven Anson Coons in 1967. Coons patch [35, 36] is spanned through four different boundary curves. Coons method generates a patch which is a parametrization of the interpolation of its four boundary curves joined at the four corner points. Qiu and Zhu [37] propose a class of rational quadratic/linear trigonometric Hermite functions based on two shape parameters and the interpolation operators which they call FCSS, SCSS, FCSV, and SCSV to construct $C^1$ Coons surface over triangular domain. For an improvement in the shape of the Coons patch, Wang et al. [38] study the second-order trigonometric blending functions with adjustable parameters for a biquadratic trigonometric polynomial Coons surface. Zou et al. [39] deal with the problem of lack of shape adjustability of the classical Coons patch by defining rational hybrid basis functions for the R-Coons patch.

A relevant application is found in surface modeling by utilizing already known surface patches. One such instance is the Coons patch, which can be used to model a variety of surfaces in computer graphics with the significant applications in CAGD, CAD, and CAM. Coons patch with bilinear blending interpolants (for which the information is based on the four corner points join the four interpolating curves as the data points) is less flexible in its nature; however, a better smooth surface is obtained for bicubically blended interpolants like cubic Hermite polynomials. In a bicubically blended Coons patch, the information is based on the interpolating curves and the tangent vectors along the twist vectors which can be computed for the given corner points. The surface patch [40] constructed by using higher order polynomials for boundary curves and the blending functions is a surface that is more flexible and smooth. The zero twist vectors in a bicubically blended Coons patch lead to the surface called the Ferguson surface. The bilinearly blended Coons patches, when they are joined together, might not necessarily have same tangent planes along the interpolating curves. In order to overcome this problem, we consider the Coons patches and analyze the resulting surfaces with rational interpolants as blending functions with a shape parameter that can adjust the shape of the surface without affecting the function values and the partial derivatives of the boundary curves. Instead of linear blending functions, we exploit an ansatz indicated in the ref. [34]. The ansatz consists of a slightly deformed surface that is the sum of an initial surface (nonminimal surface) and a multiple of a variational parameter, a function that has zero variation at the given boundary curves and the mean curvature of the initial surface along the slightly perturbed direction of unit normal to the initial surface. The ansatz scheme for an initial surface gives us variationally improved and more smooth surface, and the process can be continued for further optimized surface which is an approximation of minimal surface, which may be called a quasi-minimal surface. The techniques available in mathematical literature can be used to find the twist vectors for a given bicubically blended surface. In particular, we consider the BBCP and compute twist vectors at the four corner points by the already known techniques and then find the Coons patch as the extremal of a desired functional.

As mentioned above, for the implementation of ansatz, the target surface is a nonminimal surface taken as an initial surface to initiate a process to reduce the area of a surface spanned by a finite number of boundary curves, which is four in case of a BBCP. For this purpose, we construct blending surfaces that depend on rational blending functions along with a shape parameter in it. The ansatz approach is used to find a surface of reduced area and we apply this scheme to a variety of surfaces and compare the results for the Coons patch, bilinearly blended Coons patch, BBCP, Ferguson surface, and hump-like surface. For the rational blending functions-based Coons patch taken as an initial surface, the reduced area of the surface after implementation of the ansatz comes out to be less than that of the initial surface, shown in Table 1 with reduced areas of other nonminimal surfaces. The related works in the literature of quasi-minimal surfaces are the surfaces obtained as the extremal of some energy functionals [41–46], such as the surfaces originating from the vanishing condition of gradient of the Dirichlet functional, extended Dirichlet functional, quasi-harmonic functional, extended quasi-harmonic, and Willmore energy integral [47] for some desired characteristic of the surface. We apply the ansatz method to quasi-harmonic functional for the surfaces hump-like surface, bilinear interpolation surface, Coons patch, bicubically blended Coons patch, and Ferguson surface for rational blending functions as there is a strong relationship between the minimal surfaces and the harmonicity, and the problem of finding a minimal surface is related to the Plateau problem.

For the bicubically blended Coons patch [40] along with twist vectors as mentioned above, there are known techniques to find first the twist vectors such as by Forrest’s method, bilinear surface formula or the Adini’s method [48] and then to construct the BBCP using one of these techniques. However, instead of finding these twist vectors, we can introduce the bicubically blended Coons patch for unknown twist vectors at the four known corner points.
The corresponding quasi-minimal surface then can be found as the extremal of quasi-harmonic functional by solving the vanishing condition of the gradient of the quasi-harmonic functional \( w.r.t. \) the twist vectors. The vanishing condition of gradient of quasi-harmonic functional results in four linear constraints on the twist vectors or the mixed partial derivatives (MPDs), which can be solved simultaneously for optimal values of twist vectors, giving thus a quasi-harmonic Coons patch. If the mixed partial derivatives (MPDs) are taken zero in a surface patch then that surface is known as Ferguson surface. In Ferguson surface, the shape of the surface is flattened but a surface having twist vectors at four corners of the patch becomes rounded.

The paper is organized in the following manner: the schematic procedure of the ansatz method is given in Section 2 for the surfaces as the extremal of quasi-harmonic functional. The technique is applied to the nonminimal surface of known minimal area in Section 3 and for the surfaces of unknown minimal area are carried out in Section 4 and Section 5. The final remarks of the results are discussed in Section 6.

2. Variational Minimization of Quasi-Harmonic Functional

The ansatz method targets a slightly perturbed surface that comprises initially a nonminimal surface \( \mathbf{x}_0(u, v) \equiv \mathbf{x}(u, v) \) and the product of a variational parameter \( t \) with a function \( m_n(u, v) \) which is zero at the boundary curves and the unit normal

\[
\mathbf{N}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|},
\]

where \( \mathbf{N}(u, v) \) is represented by

\[
\mathbf{x}_{n+1}(u, v, t) = \mathbf{x}_n(u, v) + t m_n(u, v) \mathbf{N},
\]

(2)

where \( t \) is our variational parameter multiplying the variational function \( m_n(u, v) \) and \( \mathbf{N} \) is the unit normal to the surface; however, for convenience, we replace this vector with one of the unit vectors (say \( \mathbf{k} \)) along one of the coordinate axes (\( z \)-axis) that makes a small angle with the normal to the initial surface. The familiar expression for the mean curvature of the unperturbed surface \( \mathbf{x}(u, v) \) is represented by

\[
H = (2\alpha)^{-1} \mu,
\]

(3)

for its numerator

\[
\mu = G e - 2Ff + Eg,
\]

(4)

\[
\alpha = EG - F^2.
\]

The subscript \( n \) in \( \mathbf{x}_{n+1}(u, v) \) takes care of the iterative scheme where \( n = 0 \) corresponds to the first iteration, \( n = 1 \) corresponds to the second iteration, and so on. The variational function \( m_n(u, v) \) is chosen in such a way that it is zero at the boundary curves. We can take \( m_n(u, v) \), for example, as the product of linear functions \( u, v, (1-u), (1-v) \) (or quadratic functions as well, but we preferably choose linear functions) with numerator \( \mu_n \)

\[
\mu_n = G_n e_n - 2F_n f_n + E_n g_n,
\]

(5)

of the mean curvature \( H_n \) of the iterated surface \( \mathbf{x}_n(u, v) \) and \( \mathbf{E} = E(u, v) = \mathbf{x}_u \cdot \mathbf{x}_v, F = F(u, v) = \mathbf{x}_u \cdot \mathbf{x}_u, G = G(u, v) = \mathbf{x}_u \cdot \mathbf{x}_u, \mathbf{N}, f = f(u, v) = e_n \cdot \mathbf{N}, \mathbf{N}, g = g(u, v) = e_n \cdot \mathbf{N}, \mathbf{N}, \mathbf{N} \) are the fundamental coefficients of the surface \( \mathbf{x}(u, v) \), whereas \( E_n = E_n(u, v, t), F_n = F_n(u, v, t), G_n = G_n(u, v, t), e_n = e_n(u, v, t), f_n = f_n(u, v, t), g_n = g_n(u, v, t) \) are the fundamental coefficients of the surface \( \mathbf{x}_n(u, v) \) for \( n \neq 0 \); however, \( E = E_0(u, v), F = F_0(u, v), G = G_0(u, v), e = e_0(u, v), f = f_0(u, v), g = g_0(u, v) \) for the initial surface \( \mathbf{x} \equiv \mathbf{x}_0(u, v) \) when \( n = 0 \). Equation (3) is the outcome of averaging of the principal curvatures \( \kappa_1 \) and \( \kappa_2 \) (the maximal and minimal curvatures) at a certain point of the
surface, usually expressed as $H = 1/2(\kappa_1 + \kappa_2)$. The variational function $m_n(u, v)$ of equation (2) is chosen in such a way that it vanishes at the boundary curves; thus, for instance, $m_0(u, v) = uv(1 - u)(1 - v)\mu_0$ for $n = 0$ can be written as $m_0(u, v) = uv(1 - u)(1 - v)\mu_0$, and the variational function $m_1(u, v)$, for $n = 1$, can be written as $m_1(u, v) = uv(1 - u)(1 - v)\mu_1$ and so on; in the general case, it takes the form

$$m_n(u, v) = uv(1 - u)(1 - v)\mu_n.$$  

As mentioned above, $m_n(u, v)$ have zero variation at the boundary curves $u = 0$, $u = 1$, $v = 0$, and $v = 1$. To initiate the first iteration, we calculate the initial mean curvature $\mu_0$ from the equation (5) for $n = 0$ and substitute it in the first iteration for $m_0(u, v) = uv(1 - u)(1 - v)\mu_0$ along with $x_0(u, v)$ (equation (2) for $n = 0$) and then replace this mean curvature $\mu_0$ by the mean curvature $\mu_n$ of the resulting surface $x_1(u, v)$ (equation (2) for $n = 1$). Instead of using the mean curvature $H$, we can use its numerator $G_n$ given by the equation (5), denoted by $\mu_n$ in this equation, as we know that the denominator $\beta = E\gamma - F^2$ of the mean curvature $H$ is always positive for all values of the surface parameters and it serves as a scaling factor. In principle, we can variationally minimize the area functional [51],

$$A(x(u, v)) = \int_0^1 \int_0^1 \sqrt{EG - F^2}dudv, \quad (7)$$

which generates a surface for which the mean curvature is zero. This should give us an iterative surface for which the value of the variational parameter $t$ is minimum as for $n \neq 0$, $E_n(u, v, t)$, $F_n(u, v, t)$, and $G_n(u, v, t)$ depend on the variational parameter $t$ along with the surface parameter $u, v$. However, note that the integrand in the area functional is nonlinear; otherwise, we could expect to achieve an iterative surface of less area than that of the surface under consideration for known fundamental coefficients $E \equiv E(u, v), F \equiv F(u, v, t), G \equiv G(u, v)$ of the surface. We can choose some other functional instead of area functional which is quasi-harmonic in our case, as there is a well-established relationship between the harmonicity and minimal surface. This is related to the Plateau problem of finding the minimal surface spanned by its prescribed boundary among all such surfaces. We start by writing the quasi-harmonic functional

$$Q(x(u, v)) = \int_0^1 (x_{uu} + x_{vv})^2dudv, \quad (8)$$

for the target nonminimal surface $x(u, v)$, where $x_{uu}$ and $x_{vv}$ are the second-order partial derivatives of the surface $x(u, v)$ w.r.t. the surface parameters $u, v$. The integrand $(x_{uu} + x_{vv})^2$ of the functional (8) is a function of the surface parameters $u, v$. The quasi-harmonic functional (8) for setting up an iterative scheme can be written as

$$Q(t) = \int_R [(x_n(u, v, t))_{uu} + (x_n(u, v, t))_{vv}]^2dudv, \quad n \neq 0, 0 \leq u, v \leq 1, \quad (9)$$

in which the integrand $(x_n(u, v, t))_{uu} + (x_n(u, v, t))_{vv} = p_n(u, v, t)$, a function of the surface parameters $u, v$ and the variational parameter $t$, and it can be arranged in terms of powers of the variational parameter $t^k$ with the coefficients $q_k^n(u, v)$ for $k = 0, 1, 2$, written as

$$(x_n(u, v, t))_{uu} + (x_n(u, v, t))_{vv} = t^k \sum_{k=0}^{2} q_k^n(u, v). \quad (10)$$

Plugging equation (10) in equation (9), integration $w.r.t.$ the surface parameters $u, v$ for $0 \leq u, v \leq 1$, gives us the polynomial function in variational parameter $t$

$$Q(t) = \int_0^1 \int_0^1 \sum_{k=0}^{2} q_k^n(u, v)dudv, \quad (11)$$

which can be minimized for $t$ to find $t_{(\min)}$, thus variationally minimizing the surface $x_0(u, v, t_{(\min)})$. This allows us to find the completely new variationally minimized iterative surfaces $x_1(u, v), x_2(u, v), x_3(u, v), \ldots$ specified by the resulting values of $t_{(\min)}$. For a given $t_{(\min)}$, the surfaces $x_1(u, v), x_2(u, v), x_3(u, v), \ldots$ are expected to have area less than that of the initial surface $x_0(u, v)$. The area functional (7) of the surface $x(u, v)$ can now be used to find the reduced areas of the variationally minimized iterative surface $x_n(u, v)$ for $E_n \equiv E_n(u, v, t_{(\min)}), F_n \equiv F_n(u, v, t_{(\min)}), G_n \equiv G_n(u, v, t_{(\min)})$, denoting these areas by writing it in the form

$$S_n = \int_0^1 \int_0^1 \sqrt{E_n G_n - F_n^2}dudv, \quad (12)$$

such that $S_0$ represents the initial area of the surface $x_0(u, v)$ and $S_1, S_2, S_3, \ldots$; denoting the reduced areas of the surfaces $x_1(u, v), x_2(u, v), x_3(u, v), \ldots$, we can find the geometrically appreciable quantity, the dimensionless area, and the relative change in area. For an already known surface of minimal area $S_0$ (flat surface is minimal in our case), the relative decrease $\alpha_{jk}$ (for $j < k$ and $j, k = 0, 1, 2, \ldots$) in area expressed as a percentage is

$$\alpha_{jk} = (\Delta S_{\max})^{-1} \Delta S_{jk} \times 100 = (S_0 - S_m)^{-1}(S_j - S_k) \times 100,$$

in which $\Delta S_{\max} = S_0 - S_m$ is the maximum possible change in the area used as a reference value and $S_{jk} = S_j - S_k$ represents the change in area for the $j^\text{th}$ and $k^\text{th}$ iterations. However, for a nonminimal surface, we can use $S_0$ as a reference value, and in this case, relative decrease $\beta_{jk}$ (for $j < k$ and $j, k = 0, 1, 2, \ldots$) in area expressed as a percentage is $\beta_{jk} = S_0^2(S_j - S_k) \times 100$. 
3. A Surface of Known Minimal Area—Variational Minimization

In this section, we apply the ansatz method to determine the variational parameter $t$ from the quasi-harmonic functional for a nonminimal surface of known minimal area, namely, the hump-like surface, to ensure the effectiveness of the ansatz method. The target minimal surface in this case is flat and is a square of known minimal area. Then, we consider the nonminimal surfaces of unknown minimal area (in the forthcoming section 4), the rational Coons patch surfaces called R-Coons patch surfaces, and the Coons patch surfaces with rational blending functions. The R-Coons patch surface for the special case for $a = 0$ reduces to the already known classical Coons patch. The two cases $a = 0$ and $a = 1$ in the rational blending functions have been included for the illustration of the technique, discussed shortly after this section.

Let us consider a hump-like surface, a nonminimal surface, bounded by four coplanar lines $0 \leq u, v \leq 1$; the minimal surface in this case is a flat surface and is a square spanned by the same boundary straight lines. The minimal area in this case is 1-square units. We choose, the nonminimal surface, the hump-like surface spanned by the four bounding straight lines $0 \leq u, v \leq 1$ (shown in Figure 1) as

$$x(u, v) = (u, v, 16uv(1 - u)(1 - v)). \quad (13)$$

Starting with initial surface $x_0(u, v) = x(u, v)$, the fundamental coefficients of the initial surface (13) are

$$E_0(u, v) = 1 + 256u^2 - 1024uv + 1024u^2v^2 - 512v^3 + 2048uv^3 - 2048u^3v^2 + 256v^4 - 1024uv^4 + 1024u^2v^4;$$
$$F_0(u, v) = 256uv(1 - 3u + 2a - 3v + 9uv - 6u^2v + 2v^2 + 6uv - 4uv^2);$$
$$G_0(u, v) = 1 + 256u^2 - 512u^2 + 256u - 1024uv + 2048uv^3 - 1024u^2v + 1024u^2v^2 - 2048u^3v^2 + 2048u^4v^2;$$

$$e_0(u, v) = -32v + 32u, f_0(u, v) = 16(1 - 2u - 2v + 4uv),$$

$$g_0(u, v) = -32u + 32v; \quad (14)$$
$$\alpha_0(u, v) = 1 + 256u^2(1 - 2u + u^2 - 4v + 8uv - 4u^2v + 4v^2 - 8uv^2 + 4au^2v) + 256v^2(1 - 4u + 4u^2 - 2v^2 + 8uv - 4v^2 - 4u^2v + 4uv^2);$$
$$\beta_0(u, v) = 256(-1 + 4u - 4u^2 + 4v - 12uv + 12u^2v - 4v^2 + 12uv^2);$$

$$\gamma_0 = 1024u^4 + 2048uv^3 + 2048u^3v^2 - 2048u^2v^2 + 1024u^4 + 1024uv^4 + 2048uv^3 + 2048u^3v^2 + 2048u^2v^3 + 2048u^2v^2.$$ 

The numerator $m_0(u, v)$ of the mean curvature $H_0$, the mean curvature $H_0 = \mu_0/(2C_0)$, the variational functional $m_0(u, v)$, $\alpha_0(u, v)$, $\beta_0(u, v)$, $\gamma_0(u, v)$, and the Gaussian curvature $k_0 = \beta_0/\alpha_0$ of the initial hump-like surface $x_0(u, v)$ are shown in Figures 2 and 3.

Plugging the value of $\mu_0(u, v)$ from equation (15) and that of $m_0(u, v)$ from equation (16) in equation (2) for $n = 0$ to obtain the first-order variational surface $x_1(u, v, t)$ in which we replaced $N(u, v)$ of equation (1) with the unit vector $k = (0, 0, 1)$ which makes a small angle with the normal $N(u, v)$ to the initial surface $x_0(u, v)$, we are left with

$$x_1(u, v, t) = (u, v, 16(1 - u)(1 - v)) + t(1 - u)u(1 - v)(-2(16 - 32u - 32v + 64uv) \cdot (1 - u)u(1 - v) - 16(1 - u)(1 - v)v) + (1 - v)v - 16u(1 - v)v + (-32v + 32u^2)(1 + (16(1 - u)u(1 - v) - 16(1 - u)(1 - v)v)^2) + (1 - v)v - 16u(1 - v)(1 - v)v^2). \quad (17)$$
We can find the fundamental coefficients $E_1(u, v, t), F_1(u, v, t), G_1(u, v, t), e_1(u, v, t), f_1(u, v, t)$ and $g_1(u, v, t)$ of the surface $x_1(u, v, t)$ given above in equation (17) and substituting these values in the quasi-harmonic functional equation (9), and after performing the integration w.r.t. the surface parameters $u, v$ for $0 \leq u, v \leq 1$, we find the polynomial function in $t$ as stated in equation (11), $Q(t) = 8525.16t^2 - 339.75t + 125.15$. The minimum value of this polynomial function comes out to be $t_{\text{min}} = 0.01993$ (as shown in Figure 4). For this minimum value of $t_{\text{min}}$, we can find the first iterative surface $x_1(u, v)$, and hence, $E_1(u, v, t_{\text{min}}), F_1(u, v, t_{\text{min}}), G_1(u, v, t_{\text{min}}), e_1(u, v, t_{\text{min}}), f_1(u, v, t_{\text{min}}), g_1(u, v, t_{\text{min}})$ can be computed, and they are given in the Appendix. The area equation (12) (for $n = 1$) of the first iterative surface $x_1(u, v)$ comes out to be $S_1 = \ldots$
2.40628 of $x_1(u, v)$; this area is less than $S_0 = 2.4945$ of the initial surface $x_0(u, v)$. The change in area of the initial surface and that of first-order variational surface is $\Delta S_{01} = S_0 - S_1 = 2.4945 - 2.4063 = 0.0882$, and the maximum possible change in area is $\Delta S_{\text{max}} = S_0 - S_m = 1.4945$. Thus, the dimensionless decrease in area is $\Delta S_{01}/\Delta S_{\text{max}} = 0.0882/1.4945 = 0.059$, and hence, the percentage decrease in area for the first-order variation is $\alpha_{01} = 5.9$, which is a significant reduction in the area for the first-order variation of the hump-like surface for the quasi-harmonic functional as the objective function. In the similar, we can continue the iterative process for the second-order variation in the surface to obtain the surface $x_2(u, v)$ (as shown in Figure 5); in this case, the minimum value of the variational parameter $t_{\text{min}} = 0.0149$ is obtained from the polynomial $Q(t) = 4526.3t^2 - 134.663t + 121.77$ (see Figure 6), and for this value of $t_{\text{min}}$, the area of the second-order variational surface $x_2(u, v)$ is 2.3424 a further decrease that of first-order variation of the surface $x_1(u, v)$. The percentage decrease for the second-order variational surface is thus, $\alpha_{12} = 4.2756$, which is the percentage decrease when compared the first and second perturbed surfaces; however, it is interesting to note that the percentage decrease of the second perturbed surface w.r.t. the initial surface, $\alpha_{02} = 10.18$, which is obtained by finding the percentage ratio of the difference of areas of the second perturbed surface and the initial surface to the reference value of the area which is defined as the maximum change in area (difference of areas of the initial surface and the surface of least area).

4. Variational Improvement to Birationally Blended Coons Patch Surfaces

A Coons patch generates a surface patch fitted between four arbitrary boundary curves. It is based on the information obtained from its boundary curves and the auxiliary functions. These functions are called the basis functions, and a continuous function can be expressed as the linear combination of these basis functions; as in a vector space, we are used to express a vector as a linear combination of the basis vectors. In interpolation works, the basis functions are usually termed as blending functions. The effect of the blending functions is to process the four distinct boundary curves to get a well-defined surface. We would like to deal with the rational blending functions of the form

$$f_1(u) = 1 - \frac{u + au^2}{1 + u}, f_2(u) = \frac{u + au^2}{1 + u}, \quad (18)$$

$$g_1(v) = 1 - \frac{v + av^2}{1 + v}, g_2(v) = \frac{v + av^2}{1 + v}. $$
a is a scalar. The rational functions as defined in equation (18) have poles at 
\[ u = -1 \text{ and } v = -1, \]
but it does not effect the mapping of the functions in our case, as 
\[ u, v \in \left[0, 1\right]. \]
For instance, for \( a = 0 \), the rational blending functions (18) reduce to the blending functions of the form
\[
\begin{align*}
  f_1(u) &= 1 - \frac{u}{1 + u}, \\
  f_2(u) &= \frac{u}{1 + u}, \\
  g_1(v) &= 1 - \frac{v}{1 + v}, \\
  g_2(v) &= \frac{v}{1 + v}.
\end{align*}
\]
and for \( a = 1 \), equation (18) reduces to the well-known linear blending functions
\[
\begin{align*}
  f_1(u) &= 1 - u, \\
  f_2(u) &= 1, \\
  g_1(v) &= 1 - v, \\
  g_2(v) &= 1.
\end{align*}
\]
These blending functions \( f_1(u), f_2(u), g_1(v), \) and \( g_2(v) \)
satisfy the following conditions:

\[ f_1(0) = 1, f_2(0) = 1, g_1(0) = 1, g_2(0) = 1, f_1(1) = 0, f_2(1) = 1, g_1(1) = 0, g_2(1) = 1, \]

which can be summarized as

\[ f_j(j) = g_j(j) = \delta_{i,j}, \quad \text{(22)} \]

An amalgam of basis functions (blending functions) along with the prescribed boundary curves (of known characteristics) gives an interpolating surface. A blending surface is a surface spanned by the boundary curves carrying information about the surface along with the given blending and indicates a blend of the boundary curves and the blending functions. The shape of such a blending surface thus depends on the prescribed boundary curves and the blending functions. We would like to examine a Coons patch (a blending surface), composed essentially from the information passed by its boundary curves and the auxiliary functions, the blending functions which blend the four separate boundary curves to give a well-defined surface. For this context, let us write the four boundary curves denoted by \( a_1(u), a_2(u), b_1(v), \) and \( b_2(v) \) for the surface parameters \( u \), \( v \in [0, 1] \). The boundary curves \( a_1(u), a_2(u), b_1(v), \) and \( b_2(v) \) are taken in the manner that \( a_1(u) = x(u, 1), a_2(u) = x(u, 0), b_1(v) = x(0, v), b_2(v) = x(1, v) \). Now for the given blending functions (arbitrary functions) \( f_1(u), f_2(u), g_1(v), \) and \( g_2(v) \) along with boundary curves prescribed by \( a_1(u) = x(u, 0), a_2(u) = x(u, 1), b_1(v) = x(0, v), \) and \( b_2(v) = x(1, v) \), the Coons patch can be defined as follows:

\[ x(u, v) = [f_1(u)f_2(u)] \begin{bmatrix} x(0, v) \\ x(1, v) \end{bmatrix} + [x(u, 0)x(u, 1)] \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix} - [f_1(u)f_2(u)] \begin{bmatrix} x(0, 0) \\ x(1, 0) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix}, \quad \text{(23)} \]

where \( x(0, 0), x(0, 1), x(1, 0), \) and \( x(1, 1) \) are corner points of the four boundary curves. For instance, we can consider the four boundary curves \( a_1(u) = x(u, 0) \), \( a_2(u) = x(u, 1) \), \( b_1(v) = x(0, v) \), and \( b_2(v) = x(1, v) \) satisfying the condition that these curves have the same corner points \( x(0, 0), x(0, 1), x(1, 0), \) and \( x(1, 1) \) for all values of \( u, v \in [0, 1] \); they are defined by

\[ x(u, 0) = (u, 0, u - u^2), \]
\[ x(u, 1) = (u, 1, u), \]
\[ x(0, v) = (0, v^2, v - v^2), \]
\[ x(1, v) = (1, v, v). \]

For the prescribed boundary curves \( x(u, 0), x(u, 1), x(0, v), \) and \( x(1, v) \), as defined above in equations (24) and the rational blending functions \( f_1(u), f_2(u), g_1(v), g_2(v) \) of the equation (18), the expression for the birationally blended Coons patch can be worked out from Coons patch equation (23) and it can be expressed as

\[ x(u, v) = (1 + u)^{-1}(1 + v)^{-1}(u(1 + 2v + av^2) \]
\[ + u^2(1 + v + 2av + 2a^2v^2) - v(1 + av), v(1 + v) \]
\[ \cdot (u + v + au(1 - v)), v(1 + u - v + auv)(1 + v) \]
\[ + u(1 + v + u(auv^2 - 1))(1 + u) - v(1 + au)(1 + v). \]

Equation (25) is the birationally blended Coons patch, the Coons patch (equation (23)) for the rational blending functions (18), for the scalar parameter \( a \) in it. The rational Coons patch (equation (25)) can be reduced to the known bilinearly blended Coons patch for \( a = 1 \) (the linear blending functions (20)). We, now, find the iterative surfaces of lesser area for the R-Coons patch equation (25) for the case when the scalar parameter \( a = 0 \) and \( a = 1 \).

4.1. Birationally Blended Coons Patch Surfaces for the Scalar Parameter \( a = 0 \). As the first instance of an R-Coons patch for variational minimization, we take \( a = 0 \) in the equation for the Coons patch (25), which assumes the following form as the initial surface

\[ x_0(u, v) = \left( \frac{u + u^2 - v - 2uv + u^2v}{(1 + u)(1 + v)} \right), \quad v(u + v) + v - u^2 - v^2 + u(1 + uv) + v(1 + v), \]

(26)

and it is shown in Figure 7.

The fundamental coefficients of initial surface \( x_0(u, v) \) (equation (26)) in this case can be computed, and they are

\[ E_0(u, v) = 1 + 4u^6 - 4u^5(-3 + v) + 2u^4(-5 + v)(-1 + v) \]
\[ - 4u^3(-1 + v) + 4u(1 + v)(1 + 3v + v^2) \]
\[ + 2u^2(3 + v(8 + v(4 - 2v(v)))) + v(6) \]
\[ + v(10 + v^2(-1 + 2v(v)))/((1 + u)^4(1 + v)^2), \]

\[ F_0(u, v) = -1 - 2u^6 + u^5(-5 + v) - u^4(5 + v + 2v^2) \]
\[ - 2u^3(2 + v - 3v - 3v^2) - u(-1 + v) \]
\[ \cdot (-1 + 3v(v(1 + v))) - v(3 + v(v + v^2)(-3 + 4v)) \]
\[ - 2u^2(1 + v(3 - 3 + 4v(v))/(1 + u)^3(1 + v)^3), \]

\[ G_0(u, v) = 2 + 2u^5 + u^6 - 4u^5(-1 + v)(1 + v)^2 + 2uv(2 + v) \]
\[ \cdot (2 + v(v + v)) + u^4(3 + 2v(2 + v)) + u^2(5) \]
\[ + 2v(4 + v(3 + v(2 + v))) + v^2(-2 + 2v(12) \]
\[ + v(33 + 4v(7 + 2v))))/(1 + u)^2(1 + v)^4, \]

\[ c_0(u, v) = -2u^2(1 + v)^2 + u^2(1 + v)(3 + v(4 + 3v)) + (-1 + v) \]
\[ v(1 + v(v + v(5 + 3v))) + u^2(3 + v(v + v(24 + 7v))) + uv(7 + v(17 + v(5 + 2v)))/(1 + u)^4(1 + v)^3, \]
\[ f_0(u, v) = -1 + u(-2 + u(1 + u)(3 + u(3 + 2u))) + 4u(1 + u(7 + u(8 + u(4 + u))))v + (11 + u(2 + u) \cdot (14 + u(4 + u)(5 + 2u))v^2 + 2(15 + u(24 + u(22 + u(8 + 3u)))v^3 + (32 + 15u(2 + u))v^4 + 4(3 + u(2 + u))v^4/(1 + u)^4 + (1 + v)^4. \]

\[ g_0(u, v) = -2(1 + u(4 + u(3 + u(1 + u)) (4 + u))) + 6v + u(21 + u(19 + u(26 + u(22 + u(7 + u)))v) + (16 + u(39 + u(35 + 3u(8 + u(3 + u))))v^2) + 2(7 + u(13 + u(9 + u))v^3 + (1 + 2u) \cdot (3 + u(2 + u))v^4/(1 + u)^4 + (1 + v)^4. \]

(27)

Let \( \mu_0 \) be the mean curvature number \( \mu \) (equations (5) and (3)) of the initial surface \( x_0(u, v) \) (eq. (26)), and it is

\[ \mu_0 = 2(1 + u)^{-8}(1 + v)^{-7}[u^{12}(v + 1)^{-3} + u^{11}(v^2 + 11v + 29 + 19) + u^9 (6v - 9u^3 - 57v^2 + 43v^3 + 209v + 120) + u^8(8v^6 + 72v^5 + 133v^4 + 98v^3 + 230v^2 + 313v + 122) + u^7(26v^9 + 167v^8 + 366v^7 + 539v^6 + 571v^5 + 338v^4 + 81) + u^6(10v^8 + 70v^7 + 355v^6 + 122v^5 + 2497v^4 + 3278v^3 + 2715v^2 + 1170v + 219) + u^5(6v^9 + 100v^8 + 561v^7 + 1713v^6 + 5373v^5 + 5333v^4 + 5497v^3 + 3489v^2 + 1159v + 177) + u^4(34v^9 + 439v^8 + 1948v^7 + 4552v^6 + 6957v^5 + 7709v^4 + 5962v^3 + 8287v^2 + 725v + 99) + u^3(4v^{11} + 24v^{10} + 163v^9 + 1057v^8 + 3550v^7 + 6774v^6 + 8518v^5 + 7656v^4 + 4719v^3 + 1829v^2 + 424v + 62) + u^2(18v^{11} + 60v^{10} + 201v^9 + 1050v^8 + 3144v^7 + 5381v^6 + 6096v^5 + 4889v^4 + 2732v^3 + 1034v^2 + 256v + 35) + u(-48v^{10} - 122v^9 + 149v^8 + 1013v^7 + 1941v^6 + 2277v^5 + 1842v^4 + 1038v^3 + 402v^2 + 104v + 12) - 18v^{11} - 96v^{10} - 206v^9 - 213v^8 - 42v^7 - 220v^6 + 385v^5 + 328v^4 + 164v^3 + 54v^2 + 14v + 2]. \]

(28)

It can be seen that the mean curvature (equation (28)) is zero only for the lines \( u = -0.941707, -0.131211 \) and \( u = -0.941707, -0.131211 \) and \( v = -0.0178658, -1.25424 \) but not otherwise. Substituting the value of \( \mu_0 \) from equation (28) in equation (6) for \( n = 0 \) and then by using equation (2) for \( n = 0 \), the new perturbed surface with the variational parameter \( t \) is \( x_t(u, v, t) \). The minimum value of \( t \) may be calculated by variationally minimizing the quasi-harmonic functional. Finding the fundamental magnitudes \( E_1(u, v, t), F_1(u, v, t), G_1(u, v, t), E_1(u, v, t), f_1(u, v, t), \) and \( g_1(u, v, t) \) for the surface \( x_t(u, v, t) \) and after performing the requisite integration as stated above in equation (11), we find the polynomial function in \( t \), \( Q(t) = 37.5718t^2 - 24.4657t + 8.43423 \), which can be solved for minimum value of the variational parameter \( t \); we find that \( t_{min} = 0.3256 \). The area of the first variationally improved surface is \( S_1 = 1.6451 \) which is less than that of the area of the initial surface, \( S_0 = 1.7239 \), and thus, the decrease in area which is the difference of areas of initial surface and the first variationally improved surface is \( \Delta S_{01} = S_0 - S_1 = 0.0788 \). The percentage decrease in area is thus \( \beta_{01} = 4.5687 \).

4.2. Bilinear Interpolation—A Special Case of Bilinearly Blended Coons Patches (for the Variational Parameter \( a = 1 \)). As a special case of bilinearly blended Coons patch \( x(u, v) \), when all the three terms in equation (23) are equal, we obtain a surface called the bilinear interpolation for the given four corner points \( x(0, 0), x(0, 1), x(1, 0), x(1, 1) \) of the four arbitrary curves defined for all values of \( u, v \in [0, 1] \) as

\[ x(u, v) = [f_1(u)f_2(u)] \begin{bmatrix} x(0, 0) \\ x(0, 1) \\ x(1, 0) \\ x(1, 1) \end{bmatrix} + g_1(v) \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix}. \]

(29)

The bilinear interpolation (equation (29)) is the nonminimal surface taken as the initial surface \( x_0(u, v) = x(u, v) \) with the corners \( x(0, 0), x(0, 1), x(1, 0), x(1, 1) \). Ahmad and Masud [49] consider two varieties of corners for variational minimization of the bilinear interpolation as an extremal of \( rms \) of the mean curvature functional for interpretation of string-arrangement surfaces. However, for quasi-harmonic bilinear interpolants, the two varieties of bilinear interpolants can be variationally improved by variationally minimizing the quasi-harmonic functional. One of the bilinear interpolants, for the real scalars \( r \) and \( d \) and the corner points

\[ r_1 = (0, 0, 0), r_2 = (r, d, 0), r_3 = (0, d, d), r_4 = (r, 0, d), \]

(30)

for

\[ x(0, 0) = r_1, x(1, 1) = r_2, x(1, 0) = r_3, x(0, 1) = r_4, \]

(31)

the bilinear interpolation patch (equation (29)) is labeled as \( r_{ud} \). For the rational blending functions \( f_1(u), f_2(u), g_1(v), g_2(v) \) of equation (18) for already known boundary curves \( x(u, 0), x(u, 1), x(0, v), \) and \( x(1, v) \) of equation (24), the bilinear interpolation \( x(u, v) \) (equation (29)) can be computed and written in the form

\[ x(u, v) = \left( ru(au + 1) \bigg/ u + 1 \right) \bigg/ \left( dv(au + 1) \bigg/ v + 1 \right) - d \left( 2a^2u^2v^2 + au^2v^2 - au^2 + auv^2 - av^2 + u - v \right) \bigg/ \left( u + 1 \right) \bigg/ \left( v + 1 \right) \bigg/ \left( \left( u + 1 \right) \bigg/ \left( v + 1 \right) \right). \]

(33)
along with the fundamental coefficients

\[
E = E(u, v) = \frac{(au^2 + 2au + 1)^2(4a^2d^2v^4 + 4ad^2v^3 - 4ad^2v^2 + d^2v^2 - 2d^2v + d^2 + r^2v^2 + 2r^2v + r^2)}{(u+1)^4(v+1)^2},
\]

\[
F = F(u, v) = \frac{d^2(au^2 + 2au + 1)(2au^2 + u - 1)(av^2 + 2av + 1)(2av^2 + v - 1)}{(u+1)^4(v+1)^4},
\]

\[
G = G(u, v) = \frac{(au^2 + 2au + 1)^2(4a^2d^2v^4 + 4ad^2v^3 - 4ad^2v^2 + d^2v^2 - 2d^2v + d^2 + r^2v^2 + 2r^2v + r^2)}{(u+1)^4(v+1)^2},
\]

\[
e = c(u, v) = 0,
\]

\[
f = f(u, v) = -\frac{2a^2r(au^2 + 2au + 1)^2(au^2 + 2av + 1)^2}{(u+1)^4(v+1)^4},
\]

\[
g = g(u, v) = 0.
\]

The mean curvature of the \textit{ruled}_1 (equation (33)) is

\[
H_{\text{ruled}_1} = \frac{2d^2r(au^2 + 2au + 1)(2au^2 + u - 1)(av^2 + 2av + 1)(2av^2 + v - 1)}{2r^2(v+1)^4(2a^2u^2 + 2au^2 + (1 - 2a)u^2 + 1) + d^2(u+1)^4(2av^2 + v - 1)^2},
\]
and the Gaussian curvature of the \textit{ruled}_1 (equation (33)) is

\[ K_{\text{ruled}_1} = -\frac{4r^2d^2(au^2 + 2au + 1)^2 (av^2 + 2av + 1)^2}{(u + 1)^2(v + 1)^2 (2r^2(v + 1)^2(2a^2u^4 + 2au^3 - 2au^2 + u^2 + 1) + d^2(u + 1)^2(2av^2 + v - 1)^2)^2} . \]  

(36)

For the corner points (32) and the rational blending functions (equation (18)), the bilinear interpolation \( x(u, v) \) denoted by \textit{ruled}_2 surface can be determined from equation (29) as follows:

\[ x(u, v) = \left( \frac{rv(av + 1)}{v + 1}, \frac{du(au + 1)}{u + 1}, \frac{-d(2a^2u^2v^2 + auv - av^2 - au^2 - av^2 - u - v)}{(u + 1)(v + 1)} \right). \]

(37)

and the mean curvature of the above \textit{ruled}_2 is

\[ H_{\text{ruled}_2} = \frac{-2d^2r(au^2 + 2au + 1)(2au^2 + u - 1)(av^2 + 2av + 1)(2av^2 + v - 1)}{(1 + u)(1 + v)2r^2(u + 1)^2(2a^2v^4 + 2av^3 + (1 - 2a)v^2 + 1) + d^2(v + 1)^2(2au^2 + u - 1)^2} . \]

(38)

and the Gaussian curvature

\[ K_{\text{ruled}_2} = \frac{-4d^2r^2(au^2 + 2au + 1)^2 (av^2 + 2av + 1)^2}{(u + 1)^2(v + 1)^2 (2r^2(u + 1)^2(2a^2v^4 + 2av^3 - 2av^2 + v^2 + 1) + d^2(v + 1)^2(2au^2 + u - 1)^2)^2} . \]

(39)

The zero mean curvature condition for the bilinear interpolation for the two cases is satisfied only for the vertical lines \( u = -1 \pm \sqrt{1 + 8a/4a} \), \(-a \pm \sqrt{a(a - 1)/a}\) and horizontal lines \( v = -1 \pm \sqrt{1 + 8a/4a} \), \(-a \pm \sqrt{a(a - 1)/a}\) in the \( uv \)-plane. The mean curvature is not identically equal to zero for all values of \( u, v \). It is not a minimal surface. To apply the technique to see the variational improvement in the area of the bilinear interpolation, we accept in particular, the variational parameter, \( a = 0 \), and \( r = 1 \) and \( d = 1 \); in this case, the bilinear interpolation surface \textit{ruled}_1 (equation (33)) (shown in Figure 8) can be taken as the initial surface

\[ x(u, v) = \left( \frac{u}{u + 1}, \frac{v}{v + 1}, \frac{u + v}{(u + 1)(v + 1)} \right) , \]

(40)

It can be readily seen that the fundamental coefficients of the above initial surface (40) are

\[ E_0(u, v) = \frac{2(1 + v^2)}{(1 + u)^2(1 + v)^2}, \quad F_0(u, v) = \frac{(1 - u)(1 - v)}{(1 + u)^7(1 + v)^7}, \quad G_0(u, v) = \frac{2(1 + u^2)}{(1 + u)^2(1 + v)^2}, \]

\[ e_0(u, v) = 0, \quad f_0(u, v) = \frac{-2}{(1 + u)^3(1 + v)^3}, \quad g_0(u, v) = 0, \]

(41)

with its mean curvature (36) given by

\[ H_0(u, v) = \frac{2(u - 1)(v - 1)}{(u + 1)(v + 1)(3u^2v^2 + 2u^3v + 3u^2 + 2uv^2 - 4uv + 2u + 3v^2 + 2v + 3)} . \]

(42)
Mean curvature of our initial surface is zero only for \( u = 1 \) and \( v = 1 \) in the \( uv \)-plane. However, it is not a minimal surface, as it is not zero for all values of \( u \) and \( v \). Same is the case with the other ruled surface. The mean curvature of the two ruled surfaces are shown in Figures 9 and 10.

Numerator of the mean curvature of the initial surface \( x(u, v) \) is denoted by \( \mu_0 \)
\[
\mu_0 = \frac{4(u - 1)(v - 1)}{(u + 1)^2(v + 1)^2},
\]
however, for variationally improved surface, it is suffice to take the numerator of \( \mu_0 \) of equation (43) and it is when substituted in equation (6) for \( n = 0 \) gives us the variational function \( m_0(u, v) \) as
\[
m_0(u, v) = u(1 - u)^2v(1 - v)^2,
\]
and then the first variational surface from equation (2) for \( n = 0 \) can be written as
\[
x_1(u, v, t) = \left( \frac{u}{u + 1}, \frac{v}{v + 1}, t \frac{(1 - u)^2u(u + 1)v(v + 1)(1 - v)^2 + (u + v)}{(u + 1)(v + 1)} \right).
\]

After finding the fundamental magnitudes \( E_1(u, v, t), F_1(u, v, t), G_1(u, v, t), e_1(u, v, t), f_1(u, v, t) \) and \( g_1(u, v, t) \) of the surface \( x_1(u, v, t) \), the minimum value of the variational parameter \( t \) can be computed by variationally minimizing the quasi-harmonic functional, as illustrated above and it comes out to be \( t_{\text{min}} = 2.01874 \). The area of the surface \( x_1(u, v) \) is 0.32008, and the maximum decrease in area in this case is 0.000108988. It shows that it is already a quasi-minimal surfaces.

5. Bicubically Blended Coons Patch

In a bicubically blended Coons patch, we have four corner vertices, eight tangent vectors, and four twist vectors. The shape of the surface can be controlled by using these tangent vectors and magnitude and direction of the twist vectors of the surface. A bicubically blended Coons patch can be written in the form
\[
x(u, v) = (H_0(u) \quad G_0(u) \quad G_1(u) \quad H_1(u))
\[
\begin{pmatrix}
x(0, 0) & x(0, 1) & x_v(0, 0) & x_v(0, 1) \\
x(1, 0) & x(1, 1) & x_v(1, 0) & x_v(1, 1) \\
x_u(0, 0) & x_u(0, 1) & x_w(0, 0) & x_w(0, 1) \\
x_u(1, 0) & x_u(1, 1) & x_w(1, 0) & x_w(1, 1)
\end{pmatrix}
\begin{pmatrix}
H_0(v) \\
G_0(v) \\
G_1(v) \\
H_1(v)
\end{pmatrix},
\]
and for the vanishing twist vectors $\mathbf{x}_{uv}(0, 0)$, $\mathbf{x}_{uv}(0, 1)$, $\mathbf{x}_{uv}(1, 0)$, and $\mathbf{x}_{uv}(1, 1)$, it is termed as the Ferguson patch, that is

$$
\mathbf{x}(u, v) = \begin{pmatrix}
\mathbf{x}(0, 0) & \mathbf{x}(0, 1) & \mathbf{x}_u(0, 0) & \mathbf{x}_u(0, 1) \\
\mathbf{x}(1, 0) & \mathbf{x}(1, 1) & \mathbf{x}_u(1, 0) & \mathbf{x}_u(1, 1) \\
\mathbf{x}_u(0, 0) & \mathbf{x}_u(0, 1) & 0 & 0 \\
\mathbf{x}_u(1, 0) & \mathbf{x}_u(1, 1) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
H_0(v) \\
G_0(v) \\
G_1(v) \\
H_1(v)
\end{pmatrix},
$$

(47)
where $H_0(u), G_0(u), G_1(u),$ and $H_1(u)$ are the cubic Hermite polynomials used as blending functions for a Coons; they are
\[ H_0(u) = 1 - 3u^2 + 2u^3, \quad G_0(u) = u - 2u^2 + u^3, \]
\[ G_1(u) = -u^2 + u^3, \quad H_1(u) = 3u^2 - 2u^3. \] (48)

For illustration of the technique for a bicubically blended Coons patch spanned by the boundary curves given by the Hermite segments (48), connected by the position vectors $x(0,0) = (1,0,0), x(1,0) = (2,0,0), x(0,1) = (1,1,0),$ and $x(1,1) = (2,1,0), the tangent vectors of these four corner points $x(0,0), x(1,0), x(0,1), and x(1,1)$ are taken along $u$-direction and $v$-direction as follows:
\[
\begin{align*}
x_u(0,0) &= (1,0,-2), \quad x_u(0,1) = (1,1,0),
\quad x_v(0,0) = (0,0,1), \quad x_v(0,1) = (2,1,0), \\
x_u(0,1) &= (2,0,-1), \quad x_u(1,1) = (2,1,0),
\quad x_v(0,1) = (1,1,0), \quad x_v(1,1) = (0,1,1). \end{align*} \] (49)

It is well known that the twist vectors can be determined by various techniques that includes the Forrest’s method, the method using bilinear surface formula, the method using the twist vectors of spline surface, and the Adini’s method [48]. The Adini’s method is known for better approximation as compared to the other methods. For our initial bicubically blended Coons patch as the nonminimal surface, we exploit the Adini’s method to find the four twist vectors
\[
\begin{align*}
x_{uw}(0,0) &= -x_u(0,0) + x_v(1,0) - x_u(0,1) + x_u(1,1) - T, \\
x_{uw}(0,1) &= -x_u(0,1) + x_v(1,1) - x_u(0,0) + x_u(1,0) - T, \\
x_{uw}(1,0) &= -x_u(0,0) + x_v(1,0) - x_u(1,0) + x_u(1,1) - T, \\
x_{uw}(1,1) &= -x_u(0,1) + x_v(1,1) - x_u(1,0) + x_u(1,1) - T, \end{align*} \] (50)

where $T$ is given by the following expression
\[ T = x(0,0) - x(1,0) - x(0,1) + x(1,1). \] (51)

and these twist vectors (50) by virtue of equations (49) and (51) for our initial surface are
\[
\begin{align*}
x_{uw}(0,0) &= (2,0,1), \quad x_{uw}(0,1) = (-1,0,2), \\
x_{uw}(1,0) &= (1,-1,0), \quad x_{uw}(1,1) = (-2,-1,1). \end{align*} \] (52)

The bicubically blended Coons patch (46) (Figure 11) taken as the initial surface, for equations (48), (49), and (52) when plugged in equation (46), can be written in the form
\[
\begin{align*}
x(u,v) &= (1 + u + v + 2uv - 2u^2v + u^3v - 4v^2 + 3v^3) \\
&\quad - uv^3 - 3u^2 + 3u^3v + v^2 + u^3v - 2u + 4u^2, \\
&\quad - 2u^2 + uv - 2u^2v + u^2v - uv^2 + uv^3. \end{align*} \] (53)

The fundamental coefficients of the initial surface (53), the bicubically blended Coons patch are
\[
\begin{align*}
E_{0h}(u,v) &= 5 + (-1 + v)v^2(-9 + 2v^2) + 9u^4(13 + v(-10 + 3v(-10 + 3v)))) - 12u^3(17 + v(-14 + 5v)) + 2u^2(62 + v(-53 + 33v - 9v^2)) + 8u(-4 + v(3 + v(-5 + 3v))) \end{align*} \]
and
\[
\begin{align*}
F_{0h}(u,v) &= 1 + u(-6 + u(19 + u(-29 + 5(7 - 3u)u))) - 6v + u(7 + u(-34 + u(42 + u(-25 + 9u))))v \\
&\quad + (-7 + 5u(4 - 2 - 5u)u)v^2 + (17 + u(-38 + 27u))v^3 + (8 - 5u)v^4 + 3(-3 + 2u)v^5. \end{align*} \]
\[G_0(u, v) = 2 + u(4 + u(3 + u(-12 + u(15 + u(-10 + 3u)))))
- 16v - 4u(8 + u(-7 + u(2 + u)))v + 2(41)
+ u(15 + u(-19 + 9u))v^2 - 12(12 + (-4 + u)u)v^3
+ 9(9 + 2(-3 + u)u)v^4\]

\[e_0(u, v) = -2(-10 + 6u^5(-3 + v) + 3u^3(19 + v(-5 + (3 - 4v)v))
+ u^3(-89 + 2v(13 + v(-22 + v))))
+ v(51 + v(-96 + v(87 + v(-73 + 44 - 9v)v))))
+ u^2(41 + v(124 + v(220 + 3(-18 + v)v))
+ u(9 + (-1 + v)v(145 + v(-159 + v(116
+ v(-106 + 27v))))))\]

\[f_0(u, v) = 5 - 6u^6 - v(2 + (-5 + v)(-1 + v)v) + 2u^5(7
+ 18v(-1 + 2v)) - 2u^4(4 + v(-66 + v(97 + 2v))
- 2u^3(8 + v(86 + v(-100 + (-6 + v)v))
- 2u(13 + v(14 + v(-81 + 2v(72 + v(-53 + 9v))))
+ u^2(37 + v(108 + v(-289 + v(414
+ v(-317 + 54v))))\]))\]

\[g_0(u, v) = -2(9u^6(-1 + 4v) + (-4 + 9v)(-2 + v - v^2 + v^3)
+ 2u^5(18 + v(-65 + 3v)) + u(-31 + v(93
+ v(-45 + 2v))) + u^4(-71 + v(224
+ v(-54 + 7v)) + u^3(-1 + v)(-14 + v(60
+ v(-10 + 9v))) + u^3(48 - v(109
+ v(13 + 99(-2 + v)v)))\).}

The numerator \(\mu_0\) of the mean curvature \(H_0\) of the initial surface \(x(u, v)\)

\[\mu_0(u, v) = 2(-25 + 36u^{11}(-1 + v) - 120v + 1719v^2 - 7140v^3
+ 17230v^4 - 25948v^5 + 26693v^6 - 21258v^7
+ 13006v^8 - 4825v^9 + 711v^{10} - 3u^{10}(-374
+ 1911v - 1620v^2 + 528v^3) + u^2(-5725 + 30830v
- 27666v^2 + 9708v^3 - 168v^4) + u^8(15243
- 79407v + 78308v^2 - 34157v^3 + 34157v^4
+ 4912v^4 - 180v^5) + u^7(-21815 + 109965v
- 117834v^2 + 66147v^3 - 19017v^4 + 1430v^5
- 324v^6) + u^6(14586 - 77225v + 94303v^2
- 75637v^3 + 30146v^4 - 2401v^5 + 1492v^6
- 288v^7) + u^5(711 + 11067v - 25985v^2
+ 37077v^3 - 19590v^4 + 10968v^5 - 13112v^6
+ 3672v^7 - 240v^8) + u^4(-8448 + 24952v
- 23624v^2 + 18319v^3 - 20453v^4 + 1282v^5 + 8335v^6
+ 22v^7 - 176v^8 - 56v^9) + u(489 - 1626v - 210v^2
+ 13042v^2 - 45768v^3 + 78919v^4 - 84594v^5
+ 71613v^2 - 48755v^3 + 19667v^4 - 2997v^{10})\]

when plugged from equation (55) in equation (6) for \(n = 0\) and then by using equation (2) for \(n = 0\); the first variationally improved surface is \(x_1(u, v, t, m)\), in which \(t_m\), the minimum value of the variational parameter \(t\), can be determined by variationally minimizing the quasi-harmonic functional. We repeat the process of computing the fundamental coefficients \(E_1(u, v, t), F_1(u, v, t), G_1(u, v, t), e_1(u, v, t), f_1(u, v, t),\) and \(g_1(u, v, t)\) for the given \(t_m\), for the surface \(x_1(u, v, t)\). The area of the surface \(x_1(u, v, t_m)\) can be determined by performing the integration of the area integral. The area of first variational surface \(x_1(u, v)\) in this case is 1.64739 (for \(n = 1\) in (7)). In this case, the maximum decrease in area is 0.0542277, which is of significant decrease.

6. Conclusion

An ansatz scheme for the reduction of area of a nonminimal surface \(x(u, v)\) which we call a variationally improved surface \(x_1(u, v)\) is presented for differential geometric quantities. For illustration of the technique, the scheme is tested for various surfaces that includes the Coons patch, bilinearly blended Coons patch, bicubically blended patch, Ferguson surface and hump-like surface for rational blending functions as the extremal of quasi-harmonic functional, extendable for various applications of these surfaces including the fuzzy optimal problems.

Appendix

A. Fundamental Coefficients and the Variationally Improved Surface for the Hump-Like Surface (Section 3)

\[E_1(u, v, t, m) = 6.13942 \times 10^7 u^{10} v^{12} - 3.06791 \times 10^6 u^9 v^{12}
+ 6.59988 \times 10^5 u^8 v^{12} - 7.98125 \times 10^4 u^7 v^{12}
+ 5.94757 \times 10^5 u^6 v^{12} - 2.80111 \times 10^4 u^5 v^{12}
+ 8.15392 \times 10^4 u^4 v^{12} - 1.343 \times 10^3 u^3 v^{12}
+ 959285 u^2 v^{12} - 3.68365 \times 10^8 u^{10} v^{11}
+ 1.84183 \times 10^9 u^9 v^{11} - 3.95993 \times 10^8 u^8 v^{11}
+ 4.78875 \times 10^7 u^7 v^{11} - 3.56854 \times 10^6 u^6 v^{11}
+ 1.68067 \times 10^5 u^5 v^{11} - 4.89235 \times 10^9 u^4 v^{11}
+ 8.05799 \times 10^4 u^3 v^{11} - 5.75571 \times 10^2 u^2 v^{11}
+ 9.66959 \times 10^3 u^{10} v^{10} - 4.8348 \times 10^9 u^9 v^{10}\]
\begin{equation}
F_1(u, v, t_{\text{min}}) = 6.13942 \times 10^{7} \nu^{11} u^{11} - 3.37668 \times 10^{8} v^{10} u^{11} + 2.87785 \times 10^{5} \nu^{11} u^{11} - 3.37668 \times 10^{8} v^{10} u^{11} + 1.85178 \times 10^{6} \nu^{11} u^{11} - 4.4319 \times 10^{8} v^{10} u^{11} + 6.01472 \times 10^{9} v^{8} u^{10} - 5.09668 \times 10^{9} v^{8} u^{10} + 2.76993 \times 10^{9} v^{8} u^{10} - 9.44416 \times 10^{8} v^{8} u^{10} + 1.84662 \times 10^{9} v^{4} u^{10} - 1.58282 \times 10^{7} v^{10} u^{10} + 8.05799 \times 10^{9} v^{11} u^{8} - 4.4319 \times 10^{6} \nu^{11} u^{8} + 1.05748 \times 10^{9} v^{8} u^{10} - 1.43475 \times 10^{10} \nu^{8} u^{10} + 1.21519 \times 10^{9} v^{7} u^{9} - 6.59961 \times 10^{9} v^{7} u^{9} + 2.24794 \times 10^{9} v^{7} u^{9} - 4.38936 \times 10^{9} v^{7} u^{9} + 3.75395 \times 10^{7} \nu^{9} u^{9} + 4371.74 v^{2} u^{9} - 1.09358 \times 10^{9} v^{11} u^{7} + 6.01472 \times 10^{9} v^{7} u^{8} - 1.44475 \times 10^{10} \nu^{7} u^{8} - 1.64853 \times 5.90244 \times 10^{8} v^{7} u^{8} - 5.02163 \times 10^{9} v^{7} u^{8} - 1.9672.87 v^{8} u^{8} + 9.26669 \times 10^{8} v^{11} u^{7} - 5.09668 \times 10^{9} v^{10} u^{7} + 10^{7} v^{8} u^{7} + 8.92375 \times 10^{9} v^{8} u^{7} - 5.09668 \times 10^{9} v^{8} u^{7} + 10^{7} v^{8} u^{7} + 4.09432 \times 10^{7} v^{3} u^{7} + 114573 v^{2} u^{7} + 1.62639 v^{2} u^{7} - 5.03625 \times 10^{9} v^{11} u^{6} + 2.76993 \times 10^{9} v^{10} u^{6} - 6.59961 \times 10^{9} v^{9} u^{6} + 8.92375 \times 10^{9} v^{8} u^{6} - 7.5127 \times 10^{9} v^{7} u^{6} + 4.03979 \times 10^{9} v^{6} u^{6} - 1.35305 \times 10^{9} v^{5} u^{6} + 2.55572 \times 10^{9} v^{5} u^{6} - 1.97559 \times 10^{9} v^{5} u^{6} - 309197. v^{5} u^{6} - 5.69237 v^{5} u^{6} + 1.71712 \times 10^{9} v^{11} u^{5} - 9.44416 \times 10^{9} v^{10} u^{5} + 2.42794 \times 10^{9} v^{9} u^{5} - 3.03261 \times 10^{9} v^{8} u^{5} + 2.54124 \times 10^{9} v^{7} u^{5} - 1.35305 \times 10^{9} v^{6} u^{5} + 4.42764 \times 10^{9} v^{5} u^{5} - 7.82931 \times 10^{9} v^{4} u^{5} + 4.32805 \times 10^{9} v^{4} u^{5} + 392016. v^{5} u^{5} + 68.5332 v^{5} u^{5} - 3.3575 \times 10^{9} v^{11} u^{4} + 1.84662 \times 10^{9} v^{10} u^{4} - 4.38936 \times 10^{9} v^{9} u^{4} + 5.90244 \times 10^{9} v^{8} u^{4} - 4.90688 \times 10^{9} v^{7} u^{4} + 2.55572 \times 10^{9} v^{6} u^{4} - 7.82931 \times 10^{9} v^{5} u^{4} + 1.07274 \times 10^{9} v^{4} u^{4} + 539350. v^{4} u^{4} - 252950. v^{4} u^{4} - 157.102 v u^{4} + 2.8778 \times 10^{9} v^{11} u^{3} - 1.58282 \times 10^{7} v^{10} u^{3} + 3.75395 \times 10^{5} v^{9} u^{3} - 5.02163 \times 10^{5} v^{8} u^{3} + 4.09432 \times 10^{5} v^{7} u^{3} - 1.97559 \times 10^{5} v^{6} u^{3} + 4.32805 \times 10^{5} v^{5} u^{3} + 539350. v^{4} u^{3} - 507797. v^{3} u^{3} + 79615.7 v^{2} u^{3} + 635.242 v u^{3}
\end{equation}
\[ G_1(u, v, t_{\text{min}}) = 6.13942 \times 10^7 v^{10} u^{12} - 3.06971 \times 10^8 v^8 u^{12} + 6.59988 \times 10^7 v^8 u^{12} - 7.98125 \times 10^6 v^7 u^{12} + 5.94757 \times 10^6 v^6 u^{12} - 2.80111 \times 10^6 v^6 u^{12} + 8.15392 \times 10^5 v^5 u^{12} - 1.343 \times 10^5 v^5 u^{12} + 959285 \times 10^4 v^4 u^{12} - 3.684 \times 10^4 v^4 u^{12} + 1.84183 \times 10^3 v^3 u^{12} - 3.95993 \times 10^3 v^3 u^{12} + 4.78875 \times 10^2 v^2 u^{12} - 3.56854 \times 10^2 v^2 u^{12} + 1.68067 \times 10^1 v u^{12} - 4.89235 \times 10^0 v u^{12} + 8.05799 \times 10^0 v^2 u^{12} - 5.75571 \times 10^0 v^2 u^{12} + 9.66959 \times 10^{-1} v^3 u^{12} - 4.8348 \times 10^{-1} v^3 u^{12} + 1.03935 \times 10^{-1} v^4 u^{12} - 1.25654 \times 10^{-1} v^4 u^{12} + 9.35908 \times 10^{-2} v^5 u^{12} - 4.40466 \times 10^{-2} v^5 u^{12} + 1.2809 \times 10^{-2} v^6 u^{12} - 2.10683 \times 10^{-2} v^6 u^{12} + 1.50176 \times 10^{-2} v^7 u^{12} + 1.249.07 vv^10 - 1.45811 \times 10^{-3} v^8 u^{12} + 7.29056 \times 10^{-3} v^8 u^{12} - 1.56683 \times 10^{-3} v^9 u^{12} + 1.89299 \times 10^{-3} v^9 u^{12} - 1.40838 \times 10^{-4} v^{10} u^{12} + 6.61717 \times 10^{-4} v^{10} u^{12} - 1.91982 \times 10^{-4} v^{10} u^{12} + 3.14765 \times 10^{-4} v^{10} u^{12} - 2.23271 \times 10^{-5} v^{11} u^{12} - 6245.34 vv^10 + 3.69002 \times 10^{-5} v^{11} u^{12} + 1.49292 \times 10^{-5} v^{11} u^{12} - 1.80168 \times 10^{-5} v^{11} u^{12} + 1.33784 \times 10^{-5} v^{11} u^{12} - 6.26658 \times 10^{-5} v^{12} u^{12} + 1.80953 \times 10^{-5} v^{12} u^{12} + 2.94337 \times 10^{-5} v^{12} u^{12} + 2.04901 \times 10^{-5} v^{12} u^{12} + 4424.72 uu^0 + 0.40698 uu^0 - 8.63356 \times 10^{-1} v^9 u^7 - 4.31678 \times 10^{-1} v^9 u^7 - 9.26621 \times 10^{-1} v^8 u^7 + 1.11642 \times 10^{-1} v^8 u^7 - 8.26491 \times 10^{-1} v^8 u^7 + 3.85066 \times 10^{-1} v^7 u^7 - 1.10081 \times 10^{-1} v^7 u^7 + 1.751 \times 10^{-1} v^6 u^7 - 1.13108 \times 10^{-1} v^6 u^7 - 139517.7u^6 - 1.62639 u^6 + 3.43424 \times 10^{-1} v^5 u^6 - 1.71712 \times 10^{-1} v^5 u^6 + 3.68228 \times 10^{-1} v^5 u^6 - 4.4264 \times 10^{-1} v^5 u^6 + 3.26169 \times 10^{-1} v^5 u^6 - 1.50456 \times 10^{-1} v^5 u^6 + 4.2014 \times 10^{-1} v^5 u^6 - 6.25591 \times 10^{-1} v^5 u^6 + 2.90906 \times 10^0 v^6 u^6 + 207148 v^6
\]

\[ e_1(u, v, t_{\text{min}}) = 7.6518uv - 33.2753v - 7.6518uv_0
\]

\[ f_1(u, v, t_{\text{min}}) = 117532.5v^4 - 47012.7u^5 v - 105779.3u^4 v^3 + 41136.1u^2 v - 5876.59u v + 117532.5u^4 v^5 - 293829.u^4 v^4 + 264446.u^4 v^3 - 102840.u^4 v^2 + 14691.5u^3 v - 105779.3u^3 v^3 + 264446.u^3 v^2 + 237675.u^3 v^3 + 92066.5u^3 v^2 + 13054.u^3 v + 2.5506u^3 + 41136.1u^2 v^5 - 105779.3u^2 v^4 + 92066.5u^2 v^3 + 35259.5u^2 v^2 + 4889.5u^2 v + 3.826 + 5876.59uv_0^5 + 14691.5uv^4 - 13054.uv_0^3 + 4889.5uv_0^2 - 583.853uv
\]

\[ v^2 - 33.2753v + 16
\]

(A.2)

(A.3)

(A.4)

(A.5)
\[ g_1(u, v, t_{\text{min}}) = 78354.5u^4v^3 - 39177.2u^6v^4 - 52889.3u^8v^5 + 13712.6u^7v^4 - 979.431u^6v + 117532.3u^5v^4 - 235063.1u^5v^5 + 1587 \times u^6v^2 - 41136.1u^7v + 2938.29u^6v - 132223.5u^4v^4 + 264446.7u^4v^5 - 178256.3u^3v^6 + 46033.3u^4v^4 - 3263.49u^5v + 68560.2u^4v^4 - 137120.3u^3v^5 + 92066.5u^3v^7 - 23506.3u^2v + 1629.83u^3v - 14691.5u^4v^3 + 29382.9u^3v^5 - 19581.6u^3v^6 + 4889.5u^4v \] 
\[ - 291.926u^3 - 7.6518uv + 7.6518uv - 33.2753u \]

(A.6)

\[ x_1(u, v) = (u, v, 16uv - 16.6377u^2v + 1.2753u^3v - 0.6377u^4v - 16.6377u^2v - 145.963u^3v + 814.917u^4v) \]
\[ - 1631.75u^4v^2 + 1469.15u^5v^3 - 489.715u^6v^2 + 1.2753u^3v^4 + 814.917u^4v^3 - 3917.72u^5v^4 + 7672.21u^4v^5 - 6856.02u^5v^6 + 2285.34u^6v^5 \]
\[ - 0.6377uv^4 - 1631.75u^4v^3 + 7672.21u^4v^5 \]
\[ - 14854.7u^5v^4 + 13222.3u^5v^4 - 4407.44 \times u^6v^4 + 1469.15u^5v^5 - 6856.02u^5v^6 + 13222.3u^4v^5 \]
\[ - 11753.2u^5v^5 + 3917.72u^6v^6 - 489.715u^7v^6 + 2285.3 \times u^3v^6 - 4407.44u^4v^6 + 3917.72u^5v^6 \]
\[ - 1305.91u^6v^6) \]

(A.7)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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