On a regularization approach to the inverse transmission eigenvalue problem

S A Buterin¹,*, A E Choque-Rivero² and M A Kuznetsova¹

¹ Department of Mathematics, Saratov State University, Saratov, Russia
² Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C3A, Cd. Universitaria. C. P. 58040 Morelia, Mich., Mexico

E-mail: buterinsa@info.sgu.ru, abdon@ifm.umich.mx and kuznetsovama@info.sgu.ru

Received 30 April 2020, revised 28 July 2020
Accepted for publication 13 August 2020
Published 24 September 2020

Abstract

We consider the irregular (in the Birkhoff and even the Stone sense) transmission eigenvalue problem of the form

\[-y'' + q(x)y = \rho^2 y, \quad y(0) = y(1)\cos \rho a - y'(1)\rho^{-1} \sin \rho a = 0.\]

The main focus is on the ‘most’ irregular case \(a = 1\), which is important for applications. The uniqueness questions of recovering the potential \(q(x)\) from transmission eigenvalues were studied comprehensively. Here we investigate the solvability and stability of this inverse problem. For this purpose, we suggest the so-called regularization approach, under which there should first be chosen some regular subclass of eigenvalue problems under consideration, which actually determines the course of the study and even the precise statement of the inverse problem. For definiteness, by assuming \(q(x)\) to be a complex-valued function in \(W^2_1[0,1]\) possessing the zero mean value and \(q(1) \neq 0\), we study properties of transmission eigenvalues and prove the local solvability and stability of recovering \(q(x)\) from the spectrum along with the value \(q(1)\). In the appendices, we provide some illustrative examples of regular and irregular transmission eigenvalue problems, and we also obtain necessary and sufficient conditions in terms of the characteristic function for the solvability of the inverse problem of recovering an arbitrary real-valued square-integrable potential \(q(x)\) from the spectrum for any fixed \(a \in \mathbb{R}\).

Keywords: inverse spectral problem, transmission eigenvalue problem, Birkhoff and Stone regularity, local solution, stability, Nevanlinna function, global solution

*Author to whom any correspondence should be addressed.
1. Introduction

Consider the boundary value problem $R(a, q)$ of the form
\[
\ell y := -y'' + q(x)y = \lambda y, \quad 0 < x < 1, \tag{1}
\]
\[
y(0) = 0, \quad V(y) := y(1)\cos \rho a - y'(1)\frac{\sin \rho a}{\rho} = 0, \tag{2}
\]
where $\rho^2 = \lambda$ is the spectral parameter, $q(x) \in L^2(0, 1)$ and $a \in \mathbb{R}$. For $a > 0$, the problem $R(a, q)$ belongs to the so-called transmission eigenvalue problems. Recently, it has attracted much attention in connection with the inverse acoustic scattering problem (see [1–17] and references therein). A special place among these works is occupied by studying the inverse transmission eigenvalue problem, when the potential $q(x)$ is to be found either on the entire interval $(0, 1)$ or on its subinterval from eigenvalues of the problem $R(a, q)$ or their subset.

The most complete results in the inverse spectral theory are known for the Sturm–Liouville operator $\ell$ with regular boundary conditions both in self-adjoint and in non-self-adjoint cases (see, e.g., [18–29]). In particular, Borg [18] proved that the real-valued potential $q(x) \in L^2(0, 1)$ is uniquely determined by specifying the spectra $\{\lambda_{k,j}\}, j = 0, 1$, of two boundary value problems $L_j(q)$, $j = 0, 1$, for equation (1) with one common boundary condition, for example:
\[
y(0) = y'(1) = 0,
\]
respectively. For complex-valued potentials, i.e. in the non-self-adjoint case, this uniqueness result was generalized by Karaseva [19]. It is known that the following asymptotics holds:
\[
\lambda_{k,j} = \pi^2 \left( k - \frac{J_j^2}{2} \right) + \omega + x_{k,j}, \quad \{x_{k,j}\} \in l^2, \ k \geq 1, \ j = 0, 1. \tag{3}
\]
Moreover,
\[
\omega = \int_0^1 q(x)dx. \tag{4}
\]
Borg [18] also established the local solvability and stability of the corresponding inverse problem. Specifically, the following theorem holds (see also [25]).

**Theorem 1.** For any model real-valued potential $q(x) \in L^2(0, 1)$, there exists $\delta > 0$ such that if arbitrary real sequences $\{\tilde{\lambda}_{k,j}\}_{k \geq 1}$, $j = 0, 1$, satisfy the condition
\[
\Omega := \sqrt{\sum_{k=1}^{\infty} \left( |\lambda_{k,0} - \tilde{\lambda}_{k,0}|^2 + |\lambda_{k,1} - \tilde{\lambda}_{k,1}|^2 \right)} \leq \delta,
\]
then there exists a unique function $\tilde{q}(x) \in L^2(0, 1)$ such that $\{\tilde{\lambda}_{k,j}\}_{k \geq 1}$ are the spectra of the problems $L_j(\tilde{q})$, $j = 0, 1$, respectively. Moreover,
\[
\|q - \tilde{q}\|_2 \leq C_{q,\delta} \Omega,
\]
where $C_{q,\delta}$ is independent of $\tilde{q}(x)$ and $\|\cdot\|_\nu := \|\cdot\|_{L^\nu(0,1)}$. 

The original proof of theorem 1 is also applicable for complex-valued potentials but under the requirement of simplicity of the spectra. In [29], theorem 1 was generalized for arbitrary multiple spectra: it remains completely true after replacing all entries of ‘real’ with ‘complex’. In the self-adjoint case, i.e. when the function \(q(x)\) is real-valued, however, one can prove global solvability of this inverse problem. Namely, the following theorem holds (see [20]).

**Theorem 2.** For two arbitrary sequences \(\{\lambda_{k,0}\}_{k \geq 1}\) and \(\{\lambda_{k,1}\}_{k \geq 1}\) to be the spectra of the boundary value problems \(L_0(q)\) and \(L_1(q)\), respectively, with a real-valued potential \(q(x) \in L_2(0,1)\), it is necessary and sufficient to be real, to have asymptotics (3) and to interface:

\[
\lambda_{k,1} < \lambda_{k,0} < \lambda_{k+1,1}, \quad k \geq 1.
\]

Unlike the classical Sturm–Liouville problem (when \(a = 0\)), the boundary conditions (2) for \(a > 0\) can generally be classified as irregular in the Birkhoff (and even the Stone) sense (see, e.g., [30, 31]) since Green’s function of the problem \(R(a, q)\) may exponentially grow (see appendix A). This results in more complicated behavior of the spectrum \(\{\lambda_i\}\) of \(R(a, q)\). The eigenvalues \(\lambda_i\) with an account of multiplicity coincide with the zeros of the characteristic function \(\Delta(\lambda) := V(S(x, \lambda))\), where \(y = S(x, \lambda)\) is the solution of equation (1) under the initial conditions \(S(0, \lambda) = 0\) and \(S'(0, \lambda) = 1\). Since \(\Delta(\lambda)\) is an entire function of the order not exceeding \(1/2\), according to Hadamard’s factorization theorem, we have

\[
\Delta(\lambda) = \gamma \Theta(\lambda), \quad \Theta(\lambda) = \lambda^{s} \prod_{\lambda_i \neq 0} \left(1 - \frac{\lambda}{\lambda_i}\right),
\]

where \(s \geq 0\) is the algebraic multiplicity of the zero eigenvalue. Using the transformation operator (see formula (10) below) for the solution \(S(x, \lambda)\), one can also get the representation

\[
\Delta(\lambda) = \frac{\sin \rho(1 - a)}{\rho} - \omega \frac{\cos \rho(1 - a)}{2\rho^2} + \int_{a-1}^{a+1} w(t) \frac{\cos \rho t - 1}{\rho^2} dt, \quad w(t) \in L_2(a - 1, a + 1).
\]

Unlike the regular case when \(a \leq 0\), the term \(\rho^{-1} \sin \rho(1 - a)\) in (7) for \(a > 0\) is no longer the global main part of the asymptotics for \(\Delta(\lambda)\). Moreover, it even disappears when \(a = 1\).

Under the real-valuedness of \(q(x)\), however, properties of \(R(a, q)\) tend to the self-adjoint case. Namely, McLaughlin and Polyakov [1] proved that for real-valued potentials and \(a \neq 1\) the problem \(R(a, q)\) has infinitely many real eigenvalues \(\{\mu_n\}_{n \geq 0}\) of the form

\[
\mu_n = \frac{\pi^2 n^2}{(1 - a)^2} + \frac{\omega}{1 - a} + \kappa_n, \quad \{\kappa_n\} \in l_2,
\]

which can always be supplemented by other, possibly nonreal, eigenvalues \(\mu_1, \ldots, \mu_{a-1}\) with account of multiplicity up to the sequence \(\{\mu_n\}_{n \geq 1}\). Moreover, as was illustrated in [3], the problem \(R(a, q)\) may additionally have an infinite number of nonreal eigenvalues. However, if \(a \neq 1\), then the spectrum of \(R(a, 0)\), obviously, coincides with \(\{(1 - a)^{-2}\pi^2 n^2\}_{n \geq 1}\). In view of this, the sequence \(\{\mu_n\}_{n \geq 1}\) was referred to in [9] as an almost real subspectrum of \(R(a, q)\). In [1], it was proved that specification of \(\{\mu_n\}_{n \geq 1}\) determines the potential \(q(x)\) uniquely on the subinterval \((0, |a - 1|/2)\) if \(q(x)\) is known on \([|a - 1|/2, 1]\) \textit{a priori}. In particular, if \(a \geq 3\), then \(q(x)\) is determined on the entire interval \((0, 1)\). For \(a = 0\) the corresponding fact was known as the Hochstadt–Lieberman theorem [22]. The minimality of the input data \(\{\mu_n\}_{n \geq 1}\) for this uniqueness result was established in [12]. Moreover, in [12] the local solvability and stability
of the corresponding inverse problem were proved, having become the first result dealing with the solvability and stability of the inverse transmission eigenvalue problem.

Aktosun and coauthors [3] studied the uniqueness of recovering $q(x)$ from the full spectrum $\{\lambda_k\}$ of the problem $R(a, q)$ for $a \geq 1$. They reduced the inverse problem to the classical inverse Sturm–Liouville problem [18] and proved that $q(x)$ is uniquely determined by $\{\lambda_k\}$ if $a > 1$, as well as by $\{\lambda_k\}$ along with the constant $\gamma$ in (6) if $a = 1$. In [8], it was shown that in the case $a = 1$ for each nonzero real-valued potential $q(x)$ one can construct infinitely many different real-valued potentials $\tilde{q}(x) \in L_2(0, 1)$ such that the corresponding problems $R(1, \tilde{q})$ have one and the same spectrum coinciding with the spectrum of $R(1, q)$, which means the necessity of specifying $\gamma$. In [9], however, it was shown that for $a > 1$ the uniqueness theorem in [3] can be improved. Namely, if $a > 1$, then for the unique determination of the potential it is sufficient to specify only $\{\lambda_k\} \setminus \{\mu_k\}_{k \geq 1}$, i.e. the full spectrum with the exception of the entire almost real subspectrum. Moreover, even though the authors of [3] assumed the real-valuedness of the potential $q(x)$, their uniqueness results remain true also for complex-valued potentials. In appendix B, we show, in particular, that it holds for $a \leq -1$ as well.

The case $a = 1$ is exceptional because in general it allows saying almost nothing about the spectrum. For example, the spectrum of $R(1, 0)$ coincides with the entire plane $C$, while the spectrum of the problem $R(1, x - 1/2)$ is $\{\pi^2k^2/4 + kx_k\}_{k \geq 2}$, where $\{x_k\}_{k \geq 2}$ is a square-summable sequence. Unlike $R(1, 0)$, the problem $R(1, x - 1/2)$ obeys some regularization conditions on the potential, which are stated in the hypothesis of the following theorem.

**Theorem 3.** Let $q(x) \in W_2^1[0, 1]$ and $q(1) \neq 0$, while $\omega = 0$. Then the spectrum $\{\lambda_k\}_{k \geq 2}$ of the problem $R(1, q)$ has the form

$$
\lambda_k = \frac{(\pi k)^2}{4} + kx_k, \quad \{x_k\} \in l_2, \quad k \geq 2.
$$

Note that under the hypothesis of theorem 3 the problem $R(1, q)$ is Stone regular, i.e. its Green’s function grows polynomially as $\lambda \to \infty$ (see example A3 in appendix A).

In the present paper, we demonstrate the so-called regularization approach that consists of choosing and studying an appropriate regular subclass of generally speaking irregular eigenvalue problems. The definition of such a subclass can be given in terms of some restrictions on the potential $q(x)$. Note that in [13] this idea was used for $a \neq 1$. Here, however, we apply it to studying the solvability and stability of the inverse problem. For definiteness, we confine ourselves to the class $R$ of problems $R(1, q)$ that is determined by the hypothesis of theorem 3 and consider the following inverse problem.

**Inverse problem 1.** Given the spectrum $\{\lambda_k\}_{k \geq 2}$ of a problem $R(1, q) \in R$ along with the value $\eta := q(1)/4$, find the function $q(x)$.

For our purpose, one can use the complex generalization of theorem 1 (see [29]). Therefore, we are able to work in the class of complex-valued potentials. It is more convenient, however, to reduce inverse problem 1 to the problem of recovering $q(x)$ from the so-called Cauchy data (see section 2). As will be seen below, specification of the value $\gamma$ is equivalent to the specification of $\eta$. Our main result is the following theorem, which gives the local solvability and stability of inverse problem 1.

**Theorem 4.** Let $\{\lambda_k\}_{k \geq 2}$ be the spectrum of a certain model problem $R(1, q)$ with a fixed complex-valued potential $q(x) \in W_2^1[0, 1]$ obeying $\eta \neq 0$ and $\omega = 0$. Then there exists $\delta > 0$
such that for any sequence \( \{ \tilde{\lambda}_k \}_{k \geq 2} \) and for an arbitrary number \( \tilde{\eta} \) satisfying
\[
\Lambda := |\eta - \tilde{\eta}| + \sqrt{\frac{\sum_{k=2}^{\infty} |\lambda_k - \tilde{\lambda}_k|^2}{k^2}} \leq \delta
\]  
(9)

there exists a unique problem \( R(1, \tilde{q}) \in \mathcal{R} \), whose spectrum coincides with the sequence \( \{ \tilde{\lambda}_k \}_{k \geq 2} \) and \( \tilde{q}(1) = 4\tilde{q} \). Moreover, the estimate
\[
\| q - \tilde{q} \|_{W^1_2[0,1]} \leq C_{q,\delta} \Lambda
\]
is fulfilled, where \( C_{q,\delta} \) is independent of \( \tilde{q}(x) \), and \( \| f \|_{W^1_2[a,b]} = \| f \|_{L^2_2[a,b]} + \| f' \|_{L^2_2[a,b]} \).

Theorem 4, in particular, illustrates the minimality of the input data in inverse problem 1. Moreover, it is the first local solvability and stability result in the inverse transmission eigenvalue problem for complex-valued potentials and for \( a = 1 \). The proof of theorem 4 is constructive.

We note that the suggested regularization approach is vital for finding conditions for the solvability of an inverse problem in terms of the spectrum. Sometimes, however, one can alternatively formulate conditions for the solvability in terms of the characteristic function. For example, in the recent work [16] this was suggested for the problem of recovering an arbitrary real-valued potential \( q(x) \in L^2_2(0, 1) \) from the spectrum of \( R(1, q) \). But, unfortunately, the corresponding theorem (theorem 4.1) contains a mistake. In appendix B, by using the results of [8] we correct the mentioned mistake in [16] and extend this result to all other real values of the parameter \( a \).

The paper is organized as follows. In the next section, we provide some auxiliary results and obtain an algorithm for solving inverse problem 1. The proof of theorem 4 is given in section 3. In appendix A, we provide several examples of regular and irregular problems \( R(a, q) \). In appendix B, we obtain necessary and sufficient conditions in terms of the characteristic function for the solvability of the inverse problem of recovering an arbitrary real-valued potential \( q(x) \in L^2_2(0, 1) \) from the spectrum of the problem \( R(a, q) \) for any real \( a \).

Throughout the paper, one and the same symbol \( C_{q,\delta} \) denotes different positive constants in estimates, which depend only on \( q(x) \) and \( \delta \).

2. Constructive solution of the inverse problem

We start with the following well known representation (see, e.g., [21]):
\[
S(x, \lambda) = \sin \frac{\rho x}{\rho} + \int_0^x K(x, t) \sin \frac{\rho t}{\rho} \, dt, \quad 0 \leq x \leq 1,
\]  
(10)

where \( K(x, t) \) is a continuous function and \( K(x, 0) = 0 \). More precisely, by virtue of formulae (1.2.9) and (1.2.17) in [21], after the odd continuation
\[
K(x, t) \equiv -K(x, -t), \quad -1 \leq -x \leq t < 0,
\]  
(11)

and then the continuation by zero outside the triangle \( 0 \leq |t| \leq x \leq 1 \), the kernel \( K(x, t) \) will satisfy the integral equation
\[
K(x, t) = \frac{1}{2} \int_{-x}^{x+t} q(\tau) \, d\tau + \frac{1}{2} \int_0^t q(\tau) \, d\tau \int_t^{t+(x-\tau)} K(\tau, \xi) \, d\xi, \quad 0 \leq |t| \leq x \leq 1.
\]  
(12)
Note that the domain of integration in the double integral in (12) includes subdomains in which $|\xi| > \tau$ and, hence, $K(\tau, \xi)$ may possess first-order discontinuities. In order to remove them, it is sufficient to rewrite equation (12) in the following equivalent form

$$K(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} q(\tau) d\tau + \frac{1}{2} \int_{-\infty}^{\xi} q(\tau) d\tau \int_{t-(x-\tau)}^{\tau} K(\tau, \xi) d\xi - \frac{1}{2} \int_{\xi}^{\infty} q(\tau) d\tau \int_{t+x-\tau}^{\infty} K(\tau, \xi) d\xi,$$

on the right-hand side of which the inequality $|\xi| \leq \tau$ automatically holds.

Alternatively, by substituting (10) directly into equation (1) and integrating by parts, one can show that after the continuation (11) the kernel $K(x, t)$ becomes a solution of the following Goursat problem (see [23]):

$$K_{xx}(x, t) - K_{tt}(x, t) = q(x)K(x, t), \quad 0 < |t| < x \leq 1,$$

$$K(x, \pm x) = \pm \frac{1}{2} \int_{0}^{x} q(t) \, dt. \quad (15)$$

For potentials $q(x) \notin W_1^1[0, 1]$, we emphasize that the second partial derivatives in (14) do not exist in the usual sense. Therefore, finding and studying the kernel $K(x, t)$ are more convenient directly via the integral equation (13), which is equivalent to the Goursat problem (14), (15) and can be derived independently (see [21]).

Let $j \in \{0, 1\}$. Eigenvalues of the problem $L_j(q)$ coincide with zeros of its characteristic function $\Delta_j(\lambda) := S_j^0(1, \lambda)$. Integrating by parts and differentiating in (10), we obtain

$$\Delta_0(\lambda) = \sin \frac{\rho}{\rho} - \omega \frac{\cos \rho}{2\rho^2} + \int_{0}^{1} w_0(t) \frac{\cos \rho t}{\rho^2} \, dt,$$

$$\Delta_1(\lambda) = \cos \frac{\rho}{\rho} + \omega \frac{\sin \rho}{2\rho} + \int_{0}^{1} w_1(t) \frac{\sin \rho t}{\rho} \, dt,$$

where $w_0(t) = K_0(1, t)$ and $w_1(t) = K_1(1, t)|_{x=1}$. Thus, the kernel $K(x, t)$ is also a solution of the Cauchy problem for equation (14) along with the initial conditions

$$K(1, t) = \int_{0}^{\tau} w_0(\tau) d\tau, \quad K_j(x, t)|_{x=1} = w_j(t), \quad t \in [-1, 1], \quad (17)$$

where $w_j(t) = (-1)^j w_j(-t)$ for $t \in [-1, 0]$ and $j = 0, 1$. Within this context, the ordered pair of functions $\{w_0, w_1\}$ is sometimes referred to as Cauchy data related to the potential $q(x)$. Thus, after assuming $\omega$ to be fixed, Borg’s statement of the inverse problem, which consists of recovering $q(x)$ from the spectra $\{\lambda_{x,0}\}$ and $\{\lambda_{x,1}\}$, is equivalent to the following inverse problem from the Cauchy data.

**Inverse problem 2.** Given the functions $w_0(x)$ and $w_1(x)$, find the function $q(x)$ such that the solution $K(x, t)$ of the Goursat problem (14), (15) satisfies the conditions (17).

By substituting (16) into $\Delta(\lambda) = V(S(x, \lambda)) = \Delta_0(\lambda) \cos \rho a - \rho^{-1} \Delta_1(\lambda) \sin \rho a$, we arrive at representation (7), in which we have

$$w(t) = \frac{1}{2} \begin{cases} w_0(a-t) - w_1(a-t), & t \in [a-1, a], \\ w_0(t-a) + w_1(t-a), & t \in (a, a+1]. \end{cases}$$
For $a = 1$, it takes the form

$$\Delta(\lambda) = -\frac{\omega}{2\rho^2} + \int_0^2 w(t) \frac{\cos \rho t}{\rho^2} dt,$$

(19)

where

$$w(t) = \frac{1}{2} \begin{cases} w_0(1 - t) - w_1(1 - t), & t \in [0, 1], \\ w_0(t - 1) + w_1(t - 1), & t \in (1, 2]. \\ \end{cases}$$

(20)

In what follows, we assume $R(1, q) \in \mathbb{R}$, i.e. the function $q(x)$ obeys the hypothesis of theorem 3. Then, by using (13), one can show that $w(x) \in W^2_2[0, 2]$. Indeed, since $q(x) \in W^2_2[0, 1]$, we have $w_0(x), \ w_1(x) \in W^2_2[0, 1]$. Thus, it remains to note that $w(1 - 0) = w(1 + 0)$ since $w_1(0) = 0$. Let us calculate $w(2)$. By differentiating (13), we arrive at

$$K_1(x, t) = \frac{1}{4} \left( q \left( \frac{x + t}{2} \right) - q \left( \frac{x - t}{2} \right) \right) + \frac{1}{2} \int_{-t}^{t} q(\tau, t + \tau - x) d\tau + \frac{1}{2} \int_{-t}^{t} q(\tau, t + x - \tau) d\tau,$$

(21)

$$K_2(x, t) := K_1(x, t) - \frac{1}{4} \left( q \left( \frac{x + t}{2} \right) + q \left( \frac{x - t}{2} \right) \right) - \frac{1}{2} \int_{-t}^{t} q(\tau, t + \tau - x) d\tau + \frac{1}{2} \int_{-t}^{t} q(\tau, t + x - \tau) d\tau.$$

Hence, in particular,

$$w_0(1) = \frac{q(1) + q(0)}{4} - \frac{1}{2} \int_0^1 q(\tau) K(\tau, \tau) d\tau, \quad w_1(1) = \frac{q(1) - q(0)}{4} + \frac{1}{2} \int_0^1 q(\tau) K(\tau, \tau) d\tau.$$

By virtue of (20), we finally get

$$w(2) = \frac{w_0(1) + w_1(1)}{2} = \frac{q(1)}{4} = \eta.$$

(22)

Thus, we have

$$w(x) = \eta + \int_x^2 v(t) dt, \quad v(t) = -w'(t).$$

(23)

By integrating by parts in (19) and by taking into account that $\omega = 0$ for $R(1, q) \in \mathbb{R}$, we get

$$\Delta(\lambda) = \eta \sin \frac{2\rho}{\rho^3} + \int_0^2 v(t) \frac{\sin \rho t}{\rho^3} dt, \quad \eta \neq 0, \quad v(t) \in L_2(0, 2).$$

(24)

By the standard approach involving Rouche’s theorem (see, e.g., [25]), one can show that any entire function $\Delta(\lambda)$ of the form (24) has infinitely many zeros $\lambda_k$, $k \geq 2$, of the form (8), which gives the assertion of theorem 3. Moreover, using Hadamard’s factorization theorem, by the
standard approach (see, e.g., [25]) one can prove that the function $\Delta(\lambda)$ is uniquely determined by its zeros along with the constant $\eta$. Moreover, the following formula holds:

$$\Delta(\lambda) = -\frac{8\eta}{\pi} \sum_{k=2}^{\infty} \frac{4(\lambda_k - \lambda)}{(\pi k)^2}. \quad (25)$$

Conversely, the following lemma can be obtained as a corollary from lemma 3.3 in [32].

**Lemma 1.** For any complex sequence $\{\lambda_k\}_{k \geq 2}$ of the form (8), the function $\Delta(\lambda)$ determined by formula (25) with some $\eta > 0$ has the form (24) with some function $v(t) \in L_2(0, 2)$.

Stability of recovering the function $v(t)$ in (24) from zeros $\{\lambda_k\}_{k \geq 2}$ of the functions $\Delta(\lambda)$ along with the value $\eta$ can be obtained as a corollary from proposition 1 in [33]. Moreover, by using lemma 1 in [34] one can obtain even uniform stability, i.e. the following lemma holds.

**Lemma 2.** For any $r > 0$, there exists $C_r > 0$ such that

$$\|v - \tilde{v}\|_{L_2(0, 2)} \leq C_r \Lambda$$

as soon as $|\eta| \leq r$ (or alternatively, $|\tilde{\eta}| \leq r$) and

$$\sum_{k=2}^{\infty} \frac{|4\lambda_k - (\pi k)^2|^2}{k^2} \leq r, \quad \sum_{k=2}^{\infty} \frac{|4\lambda_k - (\pi k)^2|^2}{k^2} \leq r.$$

Here $\Lambda$ is determined in (9), while the function $\tilde{v}(x)$ is determined by the relation

$$\tilde{\Delta}(\lambda) := -\frac{8\tilde{\eta}}{\pi} \sum_{k=2}^{\infty} \frac{4(\tilde{\lambda}_k - \lambda)}{(\pi k)^2} = \tilde{\eta} \sin\frac{2\rho}{\rho^3} + \int_0^2 \tilde{v}(x) \sin\frac{\rho x}{\rho^3} \, dx. \quad (26)$$

Now we are in the position to provide an algorithm for solving inverse problem 1. Fix a model problem $\mathcal{R}(1, q) \in \mathfrak{R}$ with the spectrum $\{\lambda_k\}_{k \geq 2}$. Let an arbitrary nonzero complex number $\tilde{\eta}$ and a complex sequence $\{\tilde{\lambda}_k\}_{k \geq 2}$ be given that obey inequality (9) with a sufficiently small $\delta > 0$. Thus, the corresponding potential $\tilde{q}(x)$ can be found by the following algorithm.

**Algorithm 1.**

(a) Construct the function $\tilde{v}(x)$ by the formula

$$\tilde{v}(x) = \frac{\pi^3}{8} \sum_{k=1}^{\infty} k^4 \tilde{\Delta} \left(\frac{\pi^2 k^2}{4}\right) \sin\frac{\pi k x}{2}, \quad (27)$$

where the function $\tilde{\Delta}(\lambda)$ is determined by the first equality in (26).

(b) Calculate the functions $\tilde{w}_0(x)$ and $\tilde{w}_1(x)$ by the formulae

$$\tilde{w}_j(x) = \tilde{w}(1 + x) + (-1)^j \tilde{w}(1 - x), \quad x \in [0, 1], \quad j = 0, 1, \quad (28)$$

where the function $\tilde{w}(x)$ is determined by the formula

$$\tilde{w}(x) = \tilde{\eta} + \int_x^2 \tilde{v}(t) \, dt. \quad (29)$$

(c) For $j = 0, 1$, find zeros $\{\tilde{\lambda}_{j,k}\}_{k \geq 1}$ of the function $\tilde{\Delta}_j(\lambda)$, where

$$\tilde{\Delta}_0(\lambda) = \frac{\sin \rho}{\rho} + \int_0^1 \tilde{w}_0(x) \frac{\cos \rho x}{\rho^2} \, dx, \quad \tilde{\Delta}_1(\lambda) = \cos \frac{\rho x}{\rho} + \int_0^1 \tilde{w}_1(x) \frac{\sin \rho x}{\rho} \, dx. \quad (30)$$
(d) Put \(\tilde{q}(x) = q(x) + r(x)\), where \(r(x)\) is a solution of the Borg equation (38) in [29].

By using lemma 2 along with a \(W_1\text{-analogue of lemma 4.6 in [35]}, as well as theorem 1 for complex-valued potentials (see theorem 3 in [29]), one can show that for sufficiently small \(\delta > 0\) the Borg equation in step (d) of algorithm 1 is uniquely solvable. At the same time, the following example shows, in particular, that the choice of sufficiently small \(\delta > 0\) is important.

**Example 1.** Let \(\tilde{\lambda}_k = \pi^2 k^2/4\), \(k \geq 2\), then formulae (26) and (27) give

\[
\tilde{\Delta}(\lambda) = -\frac{8\tilde{\eta}}{\pi^2} \prod_{k=2}^{\infty} \left(1 - \frac{4k^2}{(\pi^2 k^2)}\right) = \tilde{\eta} \sin \frac{2\rho}{\rho^3} + \int_0^2 \tilde{\nu}(x) \sin \frac{\rho x}{\rho^3} \, dx, \quad \tilde{\nu}(x) = -\frac{\pi \tilde{\eta}}{2} \sin \frac{\pi x}{2}.
\]

Then, by using formulae (28) and (29), we calculate

\[
\tilde{w}(x) = -\tilde{\eta} \cos \frac{\pi x}{2}, \quad \tilde{\nu}(x) = -\tilde{\eta} \left(\cos \frac{\pi}{2}(1 + x) + (-1)^j \cos \frac{\pi}{2}(1 - x)\right) = 2\tilde{\eta} \sin \frac{\pi x}{2},
\]

where \(j = 0, 1\), which along with (30) give \(\tilde{\Delta}(\lambda) = \rho^{-1} \sin \rho\) and

\[
\tilde{\Delta}(\lambda) = \cos \rho + \tilde{\eta} \int_0^1 \left(\cos \left(\frac{\pi}{2} - \rho\right) x - \cos \left(\frac{\pi}{2} + \rho\right) x\right) \, dx
\]

\[
= \cos \rho + \tilde{\eta} \left(\frac{\pi}{2} - \rho\right)^{-1} \sin \left(\frac{\pi}{2} - \rho\right) - \left(\frac{\pi}{2} + \rho\right)^{-1} \sin \left(\frac{\pi}{2} + \rho\right)
\]

\[
= \cos \rho + \tilde{\eta} \left(\frac{\pi}{4} - \lambda\right)^{-1} \left(\sin \left(\frac{\pi}{2} - \rho\right) + \sin \left(\frac{\pi}{2} + \rho\right)\right) = (\pi^2 - 4\lambda + 8\tilde{\eta}) \frac{\cos \rho}{\pi^2 - 4\lambda}.
\]

Thus, the third step of algorithm 1 gives

\[
\tilde{\lambda}_{k,0} = \pi^2 k^2, \quad k \geq 1, \quad \tilde{\lambda}_{1,1} = \frac{\pi^2}{4} + 2\tilde{\eta}, \quad \tilde{\lambda}_{k,1} = \pi^2 \left(k - \frac{1}{2}\right)^2, \quad k \geq 2.
\]

According to theorem 2, there exists a real-valued potential \(\tilde{q}(x)\) such that the constructed sequences \(\{\tilde{\lambda}_{k,0}\}_{k \geq 1}\) and \(\{\tilde{\lambda}_{k,1}\}_{k \geq 1}\) are the spectra of the problems \(\mathcal{L}_0(\tilde{q})\) and \(\mathcal{L}_1(\tilde{q})\), respectively, if and only if \(\tilde{\eta} < 3\pi^2/8\). Thus, one can see that the solvability of inverse problem 1 with the input data, consisting of the sequence \(\{\pi^2 k^2/4\}_{k \geq 2}\) along with the number \(\tilde{\eta}\), in the class of real-valued potentials is equivalent to \(\tilde{\eta} \in (-\infty, 0) \cup (0, 3\pi^2/8)\).

For example, taking \(\eta = \pi^2/4\) and \(\lambda_k = \pi^2 k^2/4 = \tilde{\lambda}_k, k \geq 2\), as the model input data, one can see that \(\delta\) in (9) should be less than \(\pi^2/8\). Otherwise, it would admit the value \(\Lambda = \pi^2/8\), allowing \(\delta\) to be equal to \(3\pi^2/8\), i.e. \(\tilde{\lambda}_{1,1} = \pi^2 = \tilde{\lambda}_{1,0}\), which leads to the nonexistence of \(\tilde{q}(x)\) since the problems \(\mathcal{L}_0(\tilde{q})\) and \(\mathcal{L}_1(\tilde{q})\) cannot possess common eigenvalues.

Finally, let us obtain the potential \(\tilde{q}(x)\) such that for \(j = 0, 1\) the spectrum \(\{\tilde{\lambda}_{k,j}\}_{k \geq 1}\) of the boundary value problem \(\mathcal{L}_j(\tilde{q})\) has the form (31) or, equivalently, the problem

\[
-\gamma'' + \tilde{Q}(x) \gamma = \lambda \gamma, \quad \gamma(0) = \gamma(2) = 0, \quad \tilde{Q}(x) = \begin{cases} \tilde{q}(x), & x \in (0, 1), \\ \tilde{q}(2-x), & x \in (1, 2), \end{cases}
\]

has spectrum \(\{\tilde{\lambda}_k\}_{k \geq 1}\), where \(\tilde{\lambda}_{2k-j} = \tilde{\lambda}_{k,j}\) for \(k \geq 1\) and \(j = 0, 1\). According to (31), this implies \(\tilde{\lambda}_1 = \alpha^2\), where \(\alpha = \sqrt{\pi^2/4 + 2\tilde{\eta}}\), and \(\tilde{\lambda}_k = \pi^2 k^2/4\) for \(k \geq 2\). The corresponding
potential \( \tilde{Q}(x) \) is unique and given by the formula (see [24], p 115):
\[
\tilde{Q}(x) = -\frac{d^2}{dx^2} \ln \left( \frac{\sin \alpha(2 - x) - \sin \alpha x}{\alpha} \right), \quad \langle y, z \rangle := y'z - yz.
\]

Thus, the potential \( \tilde{q}(x) = \tilde{Q}(x), \) \( 0 < x < 1, \) where \( \tilde{Q}(x) \) is determined by formula (32) and \( \alpha = \sqrt{\pi^2/4 + 2\tilde{\eta}}, \) is the solution of inverse problem 1 with the input data \( \lambda_k = \pi^2k^2/4, k \geq 2, \) and \( \tilde{\eta} = \alpha^2/2 - \pi^2/8. \) By differentiation in (32), one can obtain the representation
\[
\tilde{q}(x) = \frac{\pi^2 - 4\alpha^2}{2} \left( \frac{2\alpha \sin^2 \alpha(1 - x) - 4\alpha^2 \sin^2 \frac{\alpha x}{2}}{\alpha(1 - x) \cos \frac{\alpha x}{2} + 2\alpha \sin \frac{\alpha x}{2} \cos \alpha(1 - x)} \right)^2,
\]
which, for example, takes the following form as soon as \( \alpha = 0: \)
\[
\tilde{q}(x) = \frac{\pi^2}{2} \left( \frac{\pi(1 - x)^3 - 4 \sin^2 \frac{\pi x}{2}}{(\pi(1 - x) \cos \frac{\pi x}{2} + 2 \sin \frac{\pi x}{2})^2} \right)^2.
\]

For proving theorem 4, it is convenient, however, to replace steps (c) and (d) in algorithm 1 with direct recovering the potential \( \tilde{q}(x) \) from the Cauchy data \( \{\tilde{w}_0, \tilde{w}_1\}. \) Recently, Bondarenko [36] proved the following theorem, which gave the local solvability and stability of inverse problem 2 for complex-valued potentials (see theorem 5.1 in appendix of [36]).

**Theorem 5.** For each complex-valued potential \( q(x) \in L_2(0, 1), \) there exists \( \varepsilon > 0 \) such that for any functions \( \tilde{w}(x) \in L_2(0, 1), \) \( j = 0, 1, \) satisfying the estimate
\[
\Xi := \max_{j=0,1} ||w_j - \tilde{w}_j||_2 \leq \varepsilon,
\]
there exists a unique function \( \tilde{q}(x) \in L_2(0, 1) \) such that \( \int_0^1 q(x) \, dx = \int_0^1 \tilde{q}(x) \, dx \) and \( \{\tilde{w}_0, \tilde{w}_1\} \) are the Cauchy data for \( \tilde{q}(x). \) Moreover, the following estimate holds:
\[
\|q - \tilde{q}\|_2 \leq C_{q, \varepsilon} \Xi.
\]

Here the pair \( \{w_0, w_1\} \) is the Cauchy data related to the potential \( q(x). \)

In the next section, leaning on this result we give the proof of theorem 4. One of the main technical difficulties is connected with our dealing with \( W^1_2 \)-potentials. It will be seen that for our purpose, however, there is no need to derive any \( W^1_2 \)-analogue of theorem 5.

### 3. Proof of theorem 4

Fix a problem \( R(1, q) \in \mathfrak{R} \) with the spectrum \( \{\lambda_k\}_{k \geq 2}. \) Let us be given a certain nonzero complex number \( \tilde{\eta} \) and some complex sequence \( \{\tilde{\lambda}_k\}_{k \geq 2} \) for which the value \( \Lambda \) determined in (9) is finite. It is then easy to see that \( \tilde{\lambda}_k = (\pi k)^2/4 + k\tilde{\eta}, \) where \( \{\tilde{\lambda}_k\} \in l_2. \) By virtue of lemma 1, there exists a unique function \( \tilde{\varphi}(x) \) for which representation (26) is fulfilled. Construct the functions \( \tilde{w}_0(x) \) and \( \tilde{w}_1(x) \) by formula (28), where the function \( \tilde{w}(x) \) is determined by formula (29). By using (20) combined with (28) and (23) along with (29), it is easy to estimate \( ||w_j - \tilde{w}_j||_{W^1_2[0,1]} \leq \sqrt{2} ||w - \tilde{w}||_{W^1_2[0,2]} \) for \( j = 0, 1 \) and \( ||w - \tilde{w}||_{W^1_2[0,2]} \leq \sqrt{2} ||\eta - \tilde{\eta}|| + (\sqrt{2} + 1)||v - \tilde{v}||_{L^2[0,2]}, \) respectively. By combining these estimates, we get
\[
||w_j - \tilde{w}_j||_{W^1_2[0,1]} \leq 2 ||\eta - \tilde{\eta}|| + (2 + \sqrt{2}) ||v - \tilde{v}||_{L^2[0,2]}, \quad j = 0, 1,
\]
which along with lemma 2 implies the estimate
\[
\|w_j - \tilde{w}_j\|_{W^1_{2[0,1]}} \leq C_{q,\delta} \Lambda, \quad j = 0, 1,
\]  
(35)
as soon as inequality (9) is fulfilled. Thus, by virtue of theorem 5, for sufficiently small \( \delta > 0 \), inequality (9) implies the existence of a unique potential \( \tilde{q}(x) \in L_2(0, 1) \) with the Cauchy data \( \{ \tilde{w}_0, \tilde{w}_1 \} \). Moreover, by virtue of (33)–(35), we have the estimate
\[
\|q - \tilde{q}\|_2 \leq C_{q,\delta} \Lambda.
\]  
(36)

Furthermore, since \( \tilde{w}_0(x), \tilde{w}_1(x) \in W^1_2[0, 1] \), we have \( \tilde{q}(x) \in W^1_2[0, 1] \). Indeed, this can be easily obtained as a consequence from the corollary to theorem 1.5.1 in [21]. It is easy to see that the corresponding problem \( R(1, \tilde{q}) \) belongs to the class \( \mathcal{R} \) and has the spectrum \( \{ \tilde{\lambda}_k \}_{k \geq 2} \). Moreover, by virtue of (22), we have \( \tilde{q}(1) = 4\tilde{q}_1 \).

Thus, for finishing the proof of theorem 4 it remains to establish the estimate
\[
\|q' - \tilde{q}'\|_2 \leq C_{q,\delta} \Lambda.
\]  
(37)

We agree that if some symbol \( \alpha \) denotes an object related to the potential \( q(x) \), then this symbol with tilde \( \tilde{\alpha} \) denotes the analogous object corresponding to \( \tilde{q}(x) \), and \( \tilde{\alpha} := \alpha - \alpha \). The subsequent arguments partially repeat those in Borg’s method (see [29]).

Since \( \ell S(x, \lambda) = \lambda S(x, \lambda) \) and \( \ell S(x, \lambda) = \lambda S(x, \lambda) \), we get
\[
\int_0^1 \tilde{q}(x) S(x, \lambda) S(x, \lambda) \, dx = \tilde{S}(1, \lambda) S(1, \lambda) - \tilde{S}(1, \lambda) S(1, \lambda) = \tilde{\Delta}(\lambda) S(1, \lambda) - \Delta(\lambda) S(1, \lambda).
\]  
(38)

Put
\[
\varphi(x, \lambda) := 1 - 2\lambda S(x, \lambda) \tilde{S}(x, \lambda) = \cos \ 2\rho x + \int_0^x Q(x, t) \cos \ 2\rho t \, dt,
\]  
(39)
where \( Q(x, t) \) is a continuous function. Moreover, by substituting (10) into (39) and using (11), one can calculate
\[
Q(x, t) = 2 \left( K(x, 2t - x) + \tilde{K}(x, 2t - x) + \int_{2t-x}^x K(x, \tau) \tilde{K}(x, 2t - \tau) \, d\tau \right), \quad 0 \leq t \leq x \leq 1.
\]  
(40)

By substituting (39) into (38) and taking into account the zero mean value of \( \tilde{q}(x) \), we get
\[
\int_0^1 \tilde{q}(x) \varphi(x, \lambda) \, dx = 2\lambda \left( \Delta(\lambda) \tilde{\Delta}(\lambda) - \tilde{\Delta}(\lambda) \Delta(\lambda) \right).
\]  
(41)

It is easy to show that the function \( Q(x, t) \) is continuous and possesses square-integrable partial derivatives \( Q_x(x, t) \) and \( Q_{xx}(x, t) \) on the triangle \( 0 < t < x < \pi \) if \( q(x), \tilde{q}(x) \in L_2(0, 1) \). Moreover, under our standing condition \( q(x), \tilde{q}(x) \in W^1_2[0, 1] \), the kernel \( Q(x, t) \) acquires an additional degree of smoothness. By integrating in (41) by parts and multiplying with \( 2\rho \), we arrive at
\[
\int_0^1 \tilde{q}(x) \varphi(x, \lambda) \, dx = \omega_1 \left( \frac{2\rho}{\pi} \right) - \omega_2 \left( \frac{2\rho}{\pi} \right),
\]  
(42)
where \(\omega_1(2\rho/\pi) = \hat{q}(1)\phi(1, \lambda)\) and \(\omega_2(2\rho/\pi) = 4\rho^3(\Delta_0(\lambda)\hat{\Delta}_1(\lambda) - \hat{\Delta}_0(\lambda)\Delta_1(\lambda))\), while

\[
\phi(x, \lambda) = 2\rho \int_0^x \varphi(t, \lambda) dt = \sin 2\rho x + \int_0^x U(x, t) \sin 2\rho dt, \tag{43}
\]

\[
U(x, t) = -\frac{d}{dt} \int_t^x Q(\tau, \lambda) d\tau = Q(t, t) - \int_t^x Q(\tau, \lambda) d\tau. \tag{44}
\]

Thus, the functional sequence \(\{\phi(x, (\pi n)^2/4)\}_{n \in \mathbb{N}}\) is a Riesz basis in \(L_2(0, 1)\). Hence, formula (42) implies the estimate

\[
\|\hat{q}''\|_2 \leq A \left(\|\omega_1(n)\|_{L^2_\lambda} + \|\omega_2(n)\|_{L^2_\lambda}\right), \quad (45)
\]

where, according to (43), we have

\[
A = \sqrt{2}\|I + U^\ast\|^{-1} = \sqrt{2}\|I + U\|^{-1}, \quad Uf = \int_0^x U(x, t) f(t) dt,
\]

while \(I\) is the identity operator and \(\|\cdot\| : = \|\cdot\|_{L^2_\lambda} - L^2_\lambda(0, 1)\) (see, e.g., section 1.8.5 in [25]). Furthermore, by virtue of lemma 1 in [37], we have the estimate

\[
\|I + U\|^{-1} \leq 1 + \|I + U\|^{-1} - I \leq 1 + F(\|U(\cdot, \cdot)\|_{L^2_\lambda(0, 1)}), \quad F(x) = x + \sum_{k=0}^{\infty} \frac{x^{k+1}}{\sqrt{k!}}.
\]

On the other hand, by solving the integral equation (13) with the method of successive approximations (see, e.g., theorem 1.2.2 in [21]), one can get the estimate

\[
|K(x, t)| \leq \|q\|_1 \exp(\|q\|_1), \quad 0 \leq t \leq x \leq 1,
\]

which along with (21) yields

\[
\|K_2(\cdot, \cdot)\|_{L^2_\lambda(0, 1)^2} \leq \|q\|_2 + \|q\|_1^2 \exp(\|q\|_1).
\]

Thus, by using (9), (11), (36), (40) and (44), we get

\[
\|U(\cdot, \cdot)\|_{L^2_\lambda(0, 1)^2} \leq C_{q, \delta}, \quad \|U(1, \cdot)\|_2 \leq C_{q, \delta}.
\]

Hence, in (45) we have

\[
A \leq C_{q, \delta}, \quad (46)
\]

and it remains to prove the estimates

\[
\|\omega_j(n)\|_{L^2_\lambda} \leq C_{q, \delta} \Lambda, \quad j = 1, 2. \tag{47}
\]

For \(j = 1\), we get

\[
\|\omega_1(n)\|_{L^2_\lambda} \leq |\hat{q}(1)| \cdot \|\{\phi(1, (\pi n)^2/4)\}_{n \in \mathbb{N}}\|_{L^2_\lambda} = 4\sqrt{2}\|\phi(1, \cdot)\|_2 \|\hat{q}\|_{L^2_\lambda} \leq C_{q, \delta} \Lambda.
\]

For \(j = 2\), we have

\[
\frac{1}{4\rho^3} 2\rho \omega_2(2\rho/\pi) = \Delta_0(\lambda)\hat{\Delta}_1(\lambda) - \hat{\Delta}_0(\lambda)\Delta_1(\lambda) = \hat{\Delta}_0(\lambda)\Delta_1(\lambda) - \Delta_0(\lambda)\hat{\Delta}_1(\lambda),
\]
where, by using (16) with \( \omega = 0 \) and (30), we obtain

\[
\Delta_0(\lambda) = \frac{\sin \rho}{\rho} + \int_{0}^{1} w_0(t) \cos \frac{\rho t}{\rho^2} \, dt, \quad \Delta_1(\lambda) = \cos \rho + \int_{0}^{1} w_1(t) \sin \frac{\rho t}{\rho} \, dt,
\]

\[
\tilde{\Delta}_0(\lambda) = \int_{0}^{1} \tilde{w}_0(t) \cos \frac{\rho t}{\rho^2} \, dt = \tilde{w}_0(1) \frac{\sin \rho}{\rho} - \int_{0}^{1} \tilde{w}_0'(t) \sin \frac{\rho t}{\rho} \, dt,
\]

\[
\tilde{\Delta}_1(\lambda) = \int_{0}^{1} \tilde{w}_1(t) \sin \frac{\rho t}{\rho} \, dt = -\tilde{w}_1(1) \frac{\cos \rho}{\rho^2} + \int_{0}^{1} \tilde{w}_1'(t) \cos \frac{\rho t}{\rho^2} \, dt.
\]

Therefore, we have

\[
\omega_2(n) = -4 \int_{0}^{1} \tilde{w}_0'(t) \sin \frac{n \pi t}{2} \, dt \left( (-1)^n + \frac{2}{\pi n} \int_{0}^{1} w_1(t) \sin \frac{n \pi t}{2} \, dt \right)
\]

\[
- 4 \int_{0}^{1} w_0(t) \cos \frac{n \pi t}{2} \int_{0}^{1} \tilde{w}_1(t) \sin \frac{n \pi t}{2} \, dt
\]

for even \( n \), and

\[
\omega_2(n) = 4 \int_{0}^{1} \tilde{w}_0(t) \cos \frac{n \pi t}{2} \int_{0}^{1} w_1(t) \sin \frac{n \pi t}{2} \, dt
\]

\[
- 4 \left( (-1)^n + \frac{2}{\pi n} \int_{0}^{1} w_0(t) \cos \frac{n \pi t}{2} \, dt \right) \int_{0}^{1} \tilde{w}_1'(t) \cos \frac{n \pi t}{2} \, dt
\]

for odd \( n \). Hence, we get the estimates

\[
\| \{ \omega_2(2n) \}_{n \in \mathbb{N}} \|_{L_2} \leq C_q(\| \tilde{w}_0' \|_2 + \| \tilde{w}_1' \|_2), \quad \| \{ \omega_2(2n - 1) \}_{n \in \mathbb{N}} \|_{L_2} \leq C_q(\| \tilde{w}_0 \|_2 + \| \tilde{w}_0' \|_2).
\]

Thus, by virtue of (35), we arrive at the estimate (47) also for \( j = 2 \). According to (45)--(47), we have (37), which finishes the proof. \( \square \)

**Funding**

The first and the third authors were supported by Russian Foundation for Basic Research (Project No. 20-31-70005). The second author is supported by CONACYT Project A1-S-31524 and CIC-UMSNH, Mexico.

**Appendix A**

Here we provide several illustrative examples of both regular and irregular problems \( R(a, q) \). Denote by \( G(x, t, \lambda) \) Green’s function of \( R(a, q) \), which is determined by the formula

\[
y(x) = \int_{0}^{1} G(x, t, \lambda) f(t) \, dt,
\]

where \( y(x) \) is the solution of the boundary value problem

\[
\ell y = \lambda y + f(x), \quad 0 < x < 1, \quad y(0) = V(y) = 0, \quad f(x) \in L_2(0, 1).
\]
In accordance with the classical direct spectral theory of ordinary differential operators (see, e.g., [30, 31]), we refer to the problem $R(a, q)$ as Birkhoff regular if it possesses Green’s function and there exist expanding contours $\{ \lambda : |\lambda| = r_k \}$, where $r_k \to \infty$ as $k \to \infty$, on which the estimate

$$G(x, t, \lambda) = O(\lambda^\theta), \quad \lambda \to \infty,$$

(48)

is fulfilled for $\theta = -1/2$. If Green’s function exists and estimate (48) holds for at least some finite $\theta$, then the problem $R(a, q)$ is referred to as Stone regular. By substitution, it is easy to check that the function $G(x, t, \lambda)$ has the form

$$G(x, t, \lambda) = \frac{1}{\Delta(\lambda)} \begin{cases} \psi(x, \lambda)S(t, \lambda), & t \leq x, \\ S(x, \lambda)\psi(t, \lambda), & t \geq x, \end{cases}$$

(49)

where $\psi(x, \lambda)$ is a solution of equation (1) under the initial conditions

$$\psi(1, \lambda) = \frac{\sin \rho a}{\rho}, \quad \psi'(1, \lambda) = \cos \rho a.$$

The following asymptotics holds (see [12]):

$$\psi(x, \lambda) = \frac{\sin \rho(a + x - 1)}{\rho} + O\left(\frac{1}{\rho^2} \exp(|\text{Im}\rho|(|a| + 1 - x))\right), \quad \lambda \to \infty,$$

(50)

uniformly with respect to $x \in [0, 1]$, which along with the classical asymptotics

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + O\left(\frac{1}{\rho^2} \exp(|\text{Im}\rho|x)\right), \quad \lambda \to \infty,$$

and (49) gives the asymptotic formula

$$G(x, t, \lambda) = \frac{1}{2\lambda \Delta(\lambda)} \begin{cases} \cos \rho(a + x - 1 + t) - \cos \rho(a + x - 1 - t) \\ + O\left(\frac{1}{\rho^2} \exp(|\text{Im}\rho|(a| + 1 - x + t))\right), & t \leq x, \\ \cos \rho(a + x - 1 + t) - \cos \rho(a + t - 1 - x) \\ + O\left(\frac{1}{\rho^2} \exp(|\text{Im}\rho|(|a| - x + t - 1))\right), & t \geq x. \end{cases}$$

(51)

Consider the set

$$D_\varepsilon(\sigma) := \left\{ \lambda = \rho^2 : \rho - \frac{\pi k}{\sigma} \geq \varepsilon, k \in \mathbb{Z} \right\}, \quad \varepsilon > 0, \sigma \neq 0.$$

(52)

Example A1. Let $a \leq 0$. Then the problem $R(a, q)$ is Birkhoff regular for any $q(x)$. Indeed, according to (7), we have the estimate

$$|\Delta(\lambda)| \geq \frac{C_\varepsilon}{|\rho|} \exp(|\text{Im}\rho|(1 - a)), \quad \lambda \in D_\varepsilon(1 - a), \quad |\lambda| \geq r_\varepsilon,$$

for sufficiently large $r_\varepsilon$, which along with (51) gives the estimate (48) for $\theta = -1/2$. 
Example A2. For any $a > 0$, the problem $R(a, 0)$ is not regular even in the Stone sense. Indeed, for the zero potential, Green’s function has the form

$$G(x, t, \lambda) = \frac{1}{\rho \sin \rho(a - 1)} \begin{cases} \sin \rho(a + x - 1) \sin \rho t, & t \leq x, \\ \sin \rho(a + t - 1) \sin \rho x, & t \geq x, \end{cases}$$

as soon as $a \neq 1$, while it does not exist for $a = 1$ because in this case $\Delta(\lambda) \equiv 0$. Hence, the problem $R(1, 0)$ is automatically irregular. For $a \neq 1$, the latter representation implies the following estimates:

$$|G(1, 1, \lambda)| \geq C_r \frac{\rho}{|\rho|^3} \exp(2|\Im \rho|), \quad \lambda \in D_r(1) \cap D_r(a), \quad a \in (0, 1) \cup [3, \infty),$$

$$\left|G \left(\frac{a - 1}{2}, \frac{a - 1}{2}, \lambda\right)\right| \geq C_r \exp(|\Im \rho|(a - 1)), \quad \lambda \in D_r \left(\frac{a - 1}{2}\right) \cap D_r \left(\frac{3a - 3}{2}\right), \quad a \in (1, 3),$$

which imply the impossibility of estimate (48) for any finite $\theta$.

Example A3. Any problem $R(1, q) \in R$ is Stone regular. Indeed, since (51) takes the form

$$G(x, t, \lambda) = \frac{1}{2\lambda \Delta(\lambda)} \left(\cos \rho(x + t) - \cos \rho(x - t) + O \left(\frac{1}{\rho} \exp(|\Im \rho|(2 - |x - t|))\right)\right)$$

and formula (24) implies the estimate

$$|\Delta(\lambda)| \geq C_r \frac{|\rho|^3}{|\rho|^3} \exp(2|\Im \rho|), \quad \lambda \in D_r(2), \quad |\lambda| \geq r_*,$$

we arrive at (48) with $\theta = 1/2$.

Appendix B

In what follows, we let $q(x)$ be an arbitrary real-valued function in $L_2(0, 1)$, and we consider the following inverse problem.

Inverse problem B1. Given the spectrum $\{\lambda_k\}$ of the problem $R(a, q)$, find $q(x)$.

We show that the solution of inverse problem B1 is unique if and only if $a \in (-\infty, -1] \cup (1, \infty)$, and we obtain necessary and sufficient conditions of its solvability for all real $a$. The case $a = 1$ is exceptional and treated separately in the following theorem.

Theorem B1. For any sequence $\{\lambda_k\}$ of complex numbers to be the spectrum of the boundary value problem $R(1, q)$ with a real-valued square-integrable potential $q(x)$, it is necessary and sufficient that the infinite product in (6) is convergent for each complex $\lambda$ and that the corresponding function $\Theta(\lambda)$ has the form

$$\Theta(\lambda) = \int_0^2 u(x) \sin \frac{\rho x}{\rho} dx, \quad u(x) \in W^1_{2,R}[0, 1], \quad u(2) = 0,$$

where $W^1_{2,R}[0, 1]$ is the real version of the space $W^1[0, 1]$.  

15
Proof. For the necessity part, it is sufficient to note that, according to the first equality in (6) along with (19), we have

\[ \Theta(\lambda) = \frac{\alpha}{\rho^2} + \int_0^2 g(t) \frac{\cos \rho t}{\rho^2} dt, \]

where

\[ \alpha = -\frac{\omega}{2\gamma}, \quad g(t) = \frac{u(t)}{\gamma}, \quad \int_0^2 g(t) dt = -\alpha. \]

Thus, integrating by parts in (54), we get (53) with

\[ u(x) = -\int_2^x g(t) dt. \]

Let us prove the sufficiency. Integrating by parts in (53), we obtain (54) with \( g(x) = u'(x) \) and \( \alpha = u(0) \). Calculate the functions \( g_0(x) \) and \( g_1(x) \) by the formula

\[ g_j(x) = g(1 + x) + (-1)^j g(1 - x), \quad x \in (0, 1), \quad j = 0, 1. \]

For \( j = 0, 1 \) and \( \gamma \in \mathbb{R} \setminus \{0\} \), denote by \( \{\lambda_{k,j}\}_{k \geq 1} \) the sequence of all zeros (with an account of multiplicity) of the function \( \Delta_j(\lambda) \) determined by the corresponding formula in (16) with \( \omega = -2\gamma \alpha \) and \( w_j(t) = \gamma g_j(t) \). Hence, the asymptotics (3) holds, and, according to the proof of theorem 1 in [8], for sufficiently small positive \( |\gamma| \), the zeros are real and interlacing as in (5). Thus, by virtue of theorem 2, there exists a real-valued potential \( q_j(x) \in L_2(0, 1) \) such that \( \{\lambda_{k,j}\}_{k \geq 1} \) is the spectrum of the problem \( R_j(q_j) \) for \( j = 0, 1 \). Moreover, as in section 2, one can show that the characteristic function of the problem \( R(1, q_j) \) coincides with \( \gamma \Theta(\lambda) \). \( \square \)

We note that, as it was first established in [3], the constructed potential \( q_j(x) \) is uniquely determined by fixing the value \( \gamma \). Otherwise, there are infinitely many potentials corresponding to one and the same spectrum \( \{\lambda_k\} \) (see [8]), although not every \( \gamma \) may lead to some potential \( q_j(x) \) (see [15]). The specification of the spectrum of \( R(a, q) \), however, does determine the potential uniquely as soon as \( a > 1 \) or \( a < -1 \). Moreover, inverse problem B1 is overdetermined for \( a \in (-\infty, -1) \cup (1, \infty) \). Nevertheless, one can obtain necessary and sufficient conditions for its solvability. In spite of the overdetermination, the consistency is achieved by the assumption that the given sequence \( \{\lambda_k\} \) coincides with the sequence of zeros of an entire function of a special form. More precisely, the following theorem holds.

Theorem B2. For any sequence \( \{\lambda_k\} \) of complex numbers to be the spectrum of the boundary value problem \( R(a, q) \) with \( a \in (-\infty, -1) \cup (1, \infty) \) and a real-valued square-integrable potential \( g(x) \), it is necessary and sufficient that the following two conditions are fulfilled:

(a) The infinite product in (6) is convergent for each complex \( \lambda \) and the corresponding function \( \Delta(\lambda) \) with

\[ \gamma = 2^{1-a} \frac{1-a}{\pi} \lim_{n \to \infty} \frac{(-1)^{n+1}}{2n-1} \left( \Theta \left( \frac{\pi^2}{(1-a)^2} \left( n - \frac{1}{2} \right) \right) \right)^{-1} \]

has the form (7) with some real-valued function \( w(t) \in L_2(a - 1, a + 1) \) and \( \omega \in \mathbb{R} \); and

(b) The sequence \( \{\lambda_k\} \) coincides with the sequence of all zeros of the function

\[ g_1(x) = \frac{\sin \Theta \left( \frac{\pi^2}{(1-a)^2} \left( n - \frac{1}{2} \right) \right)}{\gamma \Theta(\lambda) \Theta \left( \frac{\pi^2}{(1-a)^2} \right)}. \]
(b) The zeros of the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ constructed by (16) with $w_0(t)$ and $w_1(t)$ determined by (18), i.e.

$$w_j(t) = w(a + t) + (-1)^j w(a - t), \quad t \in (0, 1), \quad j = 0, 1, \tag{56}$$

are real and interlacing.

**Proof.** By necessity, both the representations (6) and (7) are already established. Thus, for (a), it is sufficient to prove (55). According to (7), for any fixed positive $\varepsilon$ we have

$$\frac{\rho \Delta(\lambda)}{\sin \rho(1 - a)} \to 1, \quad \lambda \to \infty, \quad \lambda \in D_\varepsilon(1 - a), \tag{57}$$

where the set $D_\varepsilon(\sigma)$ is determined in (52). In particular, we have $\{\eta_n\}_{n \in \mathbb{N}} \subset D_\varepsilon(1 - a)$ as soon as $\varepsilon \in (0, \pi(a - 1)/2)$, where

$$\eta_n = \theta_n^2, \quad \theta_n = \frac{\pi}{1 - a} \left(n - \frac{1}{2}\right). \tag{58}$$

Thus, (57) implies

$$(-1)^n \theta_n \Delta(\eta_n) \to 1, \quad n \to \infty,$$

which along with (58) and the first identity in (6) gives (55). For (b), it remains to note that the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are the characteristic functions of the problems $\mathcal{L}_0(q)$ and $\mathcal{L}_1(q)$, respectively. According to theorem 2, their zeros are real, simple and interlace.

Let us prove the sufficiency of conditions (a) and (b). For $j = 0, 1$, we let $\{\lambda_{k,j}\}_{k \geq 1}$ be zeros of the function $\Delta_j(\lambda)$ determined in (b). They then have asymptotics (3), and, hence, the interlacement implies (5). According to theorem 2, there exists a real-valued potential $q(x) \in L_2(0, 1)$ such that the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are the characteristic functions of the problems $\mathcal{L}_0(q)$ and $\mathcal{L}_1(q)$, respectively. Consider the problem $\mathcal{R}(a, q)$, and let $\Delta(\lambda)$ be its characteristic function, which possesses the representation

$$\Delta(\lambda) \equiv \frac{\sin \rho(1 - a)}{\rho} - \tilde{\omega} \frac{\cos \rho(1 - a)}{2 \rho^2} + \int_{a-1}^{a+1} \tilde{w}(t) \frac{\cos \rho t}{\rho^2} dt,$$

where

$$\tilde{\omega} = \int_0^1 q(x) \, dx, \quad \tilde{w}(t) = \frac{1}{2} \begin{cases} w_0(a - t) - w_1(a - t), & t \in (a - 1, a), \\ w_0(t - a) + w_1(t - a), & t \in (a, a + 1). \end{cases}$$

By comparing these formulae with (4) and (18), respectively, according to representation (7), we arrive at $\Delta(\lambda) \equiv \Delta(\lambda)$, and hence the sequence $\{\lambda_k\}$ is the spectrum of $\mathcal{R}(a, q)$. \hfill $\square$

Finally, we consider the case $a \in (-1, 1)$, when inverse problem B1 is not uniquely solvable. Indeed, according to representation (7), specification of $\Delta(\lambda)$ determines only the even part $w_+(t)$ of the function $w(t)$ on the interval $(-b, b)$, where

$$b = \min\{1 - a, 1 + a\}, \quad w_+(t) = \frac{w(t) + w(-t)}{2}.$$
Hence, (7) takes the form

$$\Delta(\lambda) = \frac{\sin \rho(1-a)}{\rho} - \omega \frac{\cos \rho(1-a)}{2\rho^2} + 2 \int_0^b w_{\pm}(t) \frac{\cos \rho t}{\rho^2} dt + \int_{\alpha_1}^{\alpha_2} w(t) \frac{\cos \rho t}{\rho^2} dt, \quad (59)$$

where $\alpha_1 = \text{sgn}(a)(1-a)$ and $\alpha_2 = \text{sgn}(a)(1+a)$. Therefore, in order to recover $w(t)$ completely and, thus, to fix a unique solution of inverse problem B1, one should additionally specify the odd part $w_-(t)$ of $w(t)$ on $(-b, b)$:

$$w_-(t) = \frac{w(t) - w(-t)}{2}.$$

Analogously to theorem B2, one can prove the following theorem, which gives necessary and sufficient conditions for the solvability (not unique) of inverse problem B1 when $a \in (-1, 1)$.

**Theorem B3.** For any sequence $\{\lambda_k\}$ of complex numbers to be the spectrum of the problem $R(a, q)$ with $a \in (-1, 1)$ and a real-valued square-integrable potential $q(x)$, it is necessary and sufficient that the following two conditions are fulfilled:

(a’) The infinite product in (6) is convergent for each complex $\lambda$ and the corresponding function $\Delta(\lambda)$ with the constant $\gamma$ determined by formula (55) has the form (59) with some real-valued functions $w_+(t) \in L_2(0, b)$ and $w(t) \in L_2(a_1, a_2)$; and

(b’) There exists a real-valued function $w_-(t) \in L_2(0, b)$ such that zeros of the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$, constructed by (16) with $w_0(t)$ and $w_1(t)$ determined by (56), are real and interlacing. Here $w(t)$ is determined on $(a_1, a_2)$ by the relation (59), while on $(-b, b)$ it is determined by the formula

$$w(t) = \begin{cases} w_+(-t) - w_-(t), & t \in (-b, 0), \\ w_+(t) + w_-(t), & t \in (0, b). \end{cases}$$

**Remark B1.** For $a = 0$, it is well known that inverse problem B1 is solvable if and only if the numbers $\lambda_k$ are real, simple and obey the asymptotics

$$\lambda_k = \pi^2 k^2 + \omega + \Delta_k, \quad \{\Delta_k\} \in l_2, \quad k \geq 1.$$

Meanwhile, under such assumptions, conditions (a) and (b’) can be checked independently. Indeed, the fulfillment of (a) can be proved by using lemma 3.3 in [32], while (b’) follows from theorem 2.

**Remark B2.** Conditions (b) and (b’) in theorems B2 and B3 can be formulated in terms of a Nevanlinna function. By definition, a complex function belongs to the Nevanlinna class if it is analytic on the open upper half-plane and has a nonnegative imaginary part there. Consider the meromorphic function

$$M(\lambda) := \frac{\Delta_0(\lambda)}{\Delta_1(\lambda)},$$

where the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are determined by formulae (16) with some real number $\omega$ and real-valued square-integrable functions $w_0(x)$ and $w_1(x)$. Thus, each of conditions (b) and (b’) is equivalent to the inclusion of the function $M(\lambda)$ in the Nevanlinna class. Indeed, for the latter it is necessary and sufficient that zeros and poles of $M(\lambda)$ interlace, which can be proved analogously to theorem 1 on page 308 in [38]. For convenience of the reader, we provide the crucial arguments of the proof. First of all, we note that the following representations hold:
\[ \Delta_j(\lambda) = \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{\pi^2(k - j/2)^2}, \quad j = 0, 1, \]

(see, e.g., [25]). Hence, we arrive at the formula

\[ M(\lambda) = \prod_{k=1}^{\infty} \frac{\lambda_k,0 - \lambda}{\lambda_k,1 - \lambda} \left( 1 - \frac{1}{2k} \right)^2. \]

Assume that (5) is fulfilled. Then we have

\[ \arg M(\lambda) = \sum_{k=1}^{\infty} d_k \in (0, \pi), \quad d_k := \arg(\lambda - \lambda_k,0) - \arg(\lambda - \lambda_k,1) > 0, \quad \text{Im} \lambda > 0, \]

which proves the sufficiency. Let \( M(\lambda) \) now be a Nevanlinna function, i.e. \( \arg M(\lambda) \in [0, \pi] \) for \( \text{Im} \lambda > 0 \) and, symmetrically, \( \arg M(\lambda) \in \{ \pi \} \cup (-\pi, 0] \) for \( \text{Im} \lambda < 0 \) since \( M(\lambda) = \overline{M(\lambda)} \). Therefore, all its zeros and poles should be real. Otherwise, a circuit around any single zero or pole lying in the open upper or lower half-plane would increment \( \arg M(\lambda) \) by no less than \( 2\pi \), which is impossible. By the same means, we establish that all zeros and poles of \( M(\lambda) \) are simple, and on any interval their numbers differ by no more than one, i.e. zeros and poles interlace.

Finally, we note that \( M(\lambda) \) can be referred as the Weyl function of the operator generated by the differential expression \( \ell \) and the boundary conditions \( y(0) = y'(1) = 0 \) (see, e.g., [25]).

**ORCID iDs**

S A Buterin 🎓 https://orcid.org/0000-0002-5771-7083
A E Choque-Rivero 🎓 https://orcid.org/0000-0003-0226-9612
M A Kuznetsova 🎓 https://orcid.org/0000-0003-1083-0799

**References**

[1] McLaughlin J R and Polyakov P L 1994 On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues *J. Differ. Equ.* **107** 351–82
[2] Cakoni F, Colton D and Monk P 2007 On the use of transmission eigenvalues to estimate the index of refraction from far field data *Inverse Problems* **23** 507–22
[3] Aktosun T, Gintides D and Papanicolaou V G 2011 The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation *Inverse Problems* **27** 115004
[4] Cakoni F and Haddar H 2012 *Transmission Eigenvalues in Inverse Scattering Theory, Inside Out II* vol 60 ed G Uhlmann (Cambridge: Cambridge University Press) pp 527–78
[5] Colton D and Leung Y-J 2013 Complex eigenvalues and the inverse spectral problem for transmission eigenvalues *Inverse Problems* **29** 104008
[6] Wei G and Xu H-K 2013 Inverse spectral analysis for the transmission eigenvalue problem *Inverse Problems* **29** 115012
[7] Colton D, Leung Y-J and Meng S 2015 Distribution of complex transmission eigenvalues for spherically stratified media *Inverse Problems* **31** 035006
[8] Buterin S A, Yang C-F and Yurko V A 2015 On an open question in the inverse transmission eigenvalue problem *Inverse Problems* **31** 045003
[9] Buterin S A and Yang C-F 2017 On an inverse transmission problem from complex eigenvalues *Results Math.* **71** 859–66
[10] Gintides D and Pallikarakis N 2017 The inverse transmission eigenvalue problem for a discontinuous refractive index Inverse Problems 33 055006.

[11] Pallikarakis N 2017 The inverse spectral problem for the reconstruction of the refractive index from the interior transmission problem PhD Thesis NTUA, Athens.

[12] Bondarenko N and Buterin S 2017 On a local solvability and stability of the inverse transmission eigenvalue problem Inverse Problems 33 115010.

[13] Xu X-C, Yang C-F, Buterin S and Yurko V 2019 Estimates of complex eigenvalues and an inverse spectral problem for the transmission eigenvalue problem Electron. J. Qual. Theory Differ. Equ. 38 1–15.

[14] Wang Y P and Shieh C T 2019 The inverse interior transmission eigenvalue problem with mixed spectral data Appl. Math. Comput. 343 285–98.

[15] Yang C-F and Buterin S A 2020 Isospectral sets for transmission eigenvalue problem J. Inverse Ill-Posed Problems 28 63–9.

[16] Wei Z and Wei G 2020 Unique reconstruction of the potential for the interior transmission eigenvalue problem for spherically stratified media Inverse Problems 36 035017.

[17] Xue X-C and Yang C-F 2020 On the inverse spectral stability for the transmission eigenvalue problem with finite data Inverse Problems 36 8.

[18] Borg G 1946 Eine Umkehrung der Sturm-Liouvillechen Eigenwertaufgabe: Bestimmung der Differentialgleichung durch die Eigenwerte Acta Math. 78 1–96.

[19] Karaseva T M 1953 On the inverse Sturm–Liouville problem for a non-Hermitian operator Mat. Sbornik 32 477–84.

[20] Marchenko V A and Ostrovskii I V 1975 A characterization of the spectrum of the Hill operator Mat. Sb. 97 540–606.

Marchenko V A and Ostrovskii I V 1975 Math. USSR-Sb 26 493–554 (Engl. transl.)

[21] Marchenko V A 1977 Sturm–Liouville Operators and Their Applications (Kiev: Naukova Dumka).

Marchenko VA 1986 Sturm–Liouville Operators and Their Application (Basel: Birkhäuser) (Engl. transl.)

[22] Hochstadt H and Lieberman B 1978 An inverse Sturm-Liouville problem with mixed given data SIAM J. Appl. Math. 34 676–80.

[23] Levitan B M 1984 Inverse Sturm–Liouville Problems (Moscow: Nauka).

Levitan B M 1987 Inverse Sturm–Liouville Problems (Utrecht: VNU Science Press) (Engl. transl.)

[24] Pöschel J and Trubowitz E 1987 Inverse Spectral Theory (New York: Academic).

[25] Freiling G and Yurko V A 2001 Inverse Sturm–Liouville Problems and Their Applications (New York: Nova Science Publishers).

[26] Makin A S 2006 An inverse problem for the Sturm-Liouville operator with regular boundary conditions Dokl. Math. 73 372–5.

[27] Buterin S A 2007 On inverse spectral problem for non-self adjoint Sturm-Liouville operator on a finite interval J. Math. Anal. Appl. 335 739–49.

[28] Buterin S A, Shieh C-T and Yurko V A 2013 Inverse spectral problems for non-selfadjoint second-order differential operators with Dirichlet boundary conditions Bound. Value Probl. 2013 180.

[29] Buterin S and Kuznetsova M 2019 On Borg’s method for non-selfadjoint Sturm-Liouville operators Anal. Math. Phys. 9 2133–50.

[30] Naimark M A 1967, 1968 Linear Differential Operators, Parts I, II (New York: Ungar).

[31] Freiling G 2012 Irregular boundary value problems revisited Results Math. 62 265–94.

[32] Buterin S A 2007 On an inverse spectral problem for a convolution integro-differential operator Results Math. 50 173–81.

[33] Bondarenko N and Buterin S 2020 Numerical solution and stability of the inverse spectral problem for a convolution integro-differential operator Commun. Nonlinear Sci. 89 105298.

[34] Buterin S A 2020 Uniform stability of the inverse spectral problem for a convolution integro-differential operator Appl. Math. Comp. 390 125592.

[35] Yang C-F, Bondarenko N P and Xu X-C 2020 An inverse problem for the Sturm-Liouville pencil with arbitrary entire functions in the boundary condition Inverse Problem Imaging 14 153–69.

[36] Bondarenko N P 2020 Inverse Sturm–Liouville problem with analytical functions in the boundary condition Open Math. 18 512–28.

[37] Buterin S and Malyugina M 2018 On global solvability and uniform stability of one nonlinear integral equation Results Math. 73 117.

[38] Levin B J 1964 Distribution of Zeros of Entire Functions (Translation of Mathematical Monographs) (Providence, RI: American Mathematical Society)