QED
and ortho-para- positronium
mass difference

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Abstract

Bound state problem in the relativistic QED is investigated by the
functional integral methods. The ortho- para- positron mass differ-
ence is calculated. Contribution of the ”nonphysical” time variable
turned out to be important and leads to the nonanalytic dependence
of the bound state mass of the order $\alpha^2$. It is shown that the relativis-
tic and non-relativistic QED gives different results for this mass shift.
In addition so-called abnormal states as ”time excitations” arise. Se-
quential application of relativistic QED to bound state problem is in
contradiction with real ortho- and para- positronium bound states.

The conclusion: the relativistic QED is not suited to describe real
bound states correctly.

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1 Introduction.

We believe that the relativistic quantum electrodynamics (QED) is a uniquely correct universal theory giving an exhaustive description of all interactions between electrons and photons including possible bound states like positronium. Only our inability to calculate something out of perturbation method does not permit us to obtain all the desired details. Earlier, some scientists considered that QED should have its own applicability region. A short review of the history and the development of quantum field theory is done in [1]. Supporting these doubts we will show in this paper that the sequential use of the standard QED does not give a correct description of the positronium spectra, namely, the ortho-para-positronium mass difference.

First of all let us realize what is the status of bound states in non-relativistic quantum mechanics (QM) and relativistic quantum field theory (QFT). In what follows, we restrict ourselves to the discussion of positronium in QED. The difference between QM and QFT is shown in Table I. Let us give some comments.

The total Hamiltonian $H = H_0 + g H_I$ can be constructed in QM and QFT. However, in QM $H$ is a well defined operator, so that the non-relativistic Schrödinger equation is mathematically correct and time development of a quantum system can be described. Solutions of the Schrödinger equation contain both free and bound states. One can remark that in QM a bound state (positronium) is created by real particles (electron-positron), i.e. constituent particles are on mass shell but are no virtual particles.

In QFT the Fock space $F$ is defined by the noninteracting free Hamiltonian $H_0$ and contains the free particles only. However, $H_I$ is not defined on $F$. As a result, the bound state as an eigenvalue problem of the relativistic Schrödinger equation on the Fock space cannot be formulated mathematically in a correct way (see [2]). Besides, the time development of quantum field system cannot be obtained. The only way to overcome these problems is to construct the $S$-matrix which contains all elastic and inelastic scattering amplitudes of free particles from the time $t \to -\infty$ to the time $t \to \infty$. It is important that the $S$-matrix is a unitary operator on the Fock space. It means that the bound states like positronium, which is a unstable particle, cannot belong to any Fock space in principle. In addition, our computing abilities are restricted to the perturbation theory.

Nevertheless, we believe that the $S$-matrix amplitudes should contain some correct information on possible bound states. The simplest way to re-
alize this idea is to postulate that a bound state is a simple pole of an elastic scattering amplitude of constituent particles with appropriate quantum numbers. It means, that the amplitudes out of mass shell and out of perturbation approach should be calculated. Standard methods to go out of perturbation calculations are reduced to sum appropriate classes of Feynman diagrams and this summation can be formulated in a form of integral equations. The best known approaches are the Bethe-Salpeter and Schwinger-Dyson equations. There is numerous literature devoted these equations (see, for example, [3, 4, 5, 6, 7, 9, 8]). The important difference comparable with the nonrelativistic case is that bound states in these equations are created by particles which are out of mass shell so that the role of time becomes important.

One remark on these equations. We know that the perturbation series are asymptotic series so that the problem is how to sum them? The exact amplitudes should have some singularity at the point $\alpha = 0$ in QED (see [10]). What is a precise character of this singularity is not known up to now. Standard perturbation expansions are connected with Feynman diagrams. Usual methods are reduced to summation of an appropriate class of Feynman diagrams. Result of a summation of any definite class of Feynman diagrams is a kind of geometrical progression, i.e. it is an analytic function at the point $\alpha = 0$. However, it should be stressed that the generally accepted point of view - non-perturbed behavior is a sum of a definite class of Feynman diagrams - is not true.

One of probably successful proposals to calculate the relativistic corrections to bound state problem is the so-called non-relativistic QED (NRQED) (see [11]). The basic idea is that the QM is correct, only non-relativistic momenta are responsible for bound state properties. In other words, the Hamiltonian should not depend on time and the problem is to find somehow relativistically small corrections to the non-relativistic Coulomb potential. The basic idea is that for small coupling constants the Born approximation is a good approximation which is directly defined by the Fourier transform of the potential. The aim is to extract from the relativistic $S$-matrix some relativistic corrections to non-relativistic Hamiltonian. The hypothesis is that the scattering amplitudes in the non-relativistic Schrödinger theory and the relativistic $S$-matrix theory should coincide in the low energy limit. The procedure is to write down the non-relativistic Lagrangian with a set of all possible terms, and coefficients in front of them are calculated by identification with appropriate amplitudes of relativistic $S$-matrix. This prescription allows one to remove effectively time out of the relativistic equations, in
other words, to place all intermediate particles on their mass shell. It seems
NRQED is supported by experimental data.

Another quantum field idea is that a bound state is defined by an asymptotic behavior of the vacuum mean value of the corresponding relativistic currents (see, for example, [12]) with desired quantum numbers:

\[
\langle 0 | \mathcal{J}(x) \mathcal{J}(0) | 0 \rangle = \sum_n e^{-E_n|x|} \langle 0 | \mathcal{J}(0) | n \rangle | \langle n | \mathcal{J}(0) | 0 \rangle |^2 \sim e^{-M_{\text{min}} |x|} \langle 0 | \mathcal{J}(0) | \text{min} \rangle | \langle \text{min} | \mathcal{J}(0) | 0 \rangle |^2 \quad \text{for } |x| \to \infty .
\]

This formula gives a possibility to calculate the mass of the lowest bound state $|\text{min}\rangle$ if $M_{\text{min}} < 2m$. Essentially, the space of states $\{ |n\rangle \}$ is supposed to contain possible bound states although we saw that the Fock space cannot contain unstable bound states. These vacuum mean values (1) can be represented in closed forms by functional methods. The functional methods permit one to get formally the exact representations for Green functions which are not connected directly with Feynman diagrams, so that it is possible to go out of standard perturbation expansions using asymptotic methods. Development of functional methods permits one to get the exact character of non-analyticity at the point $\alpha = 0$ and to clarify the role of "time" in bound state formation. Exactly this approach will be used in this paper.

The practically unique experimental object to investigate the bound state problems is the positronium which is the result of pure QED interaction. On the one hand, the positronium is not a stable state. It cannot belong to the asymptotic Fock space. Nevertheless, it exists. The binding energy of positronium itself is not measured with great accuracy but the mass difference of two possible states, ortho-positronium ($1^3S_1$) and para-positronium ($1^1S_0$), is known with very large accuracy

\[
\Delta \epsilon = \epsilon_{\text{ortho}} - \epsilon_{\text{para}} = 203.38910 \, \text{GHz} = 8.4115 \cdot 10^{-4} \, \text{eV}
\]

\[
= 0.580487 \, \alpha^4 m_e = \frac{7}{12} \alpha^4 m_e \cdot 0.99512...
\]

The main contribution can be explained by the non-relativistic Breit potential approach (see, for example, [13,14,15,16]) taking into account scattering and annihilation channels

\[
\Delta \epsilon = \epsilon_{\text{ortho}} - \epsilon_{\text{para}} = \frac{7}{12} \alpha^4 m_e, \quad \frac{7}{12} = \left( \frac{1}{3} \right)_{\text{scatt}} + \left( \frac{1}{4} \right)_{\text{annih}}.
\]
If we apply the relativistic current formula (1) to the positronium problem, we can write
\[
\langle 0 | J(x) J(0) | 0 \rangle = \sum_{\text{particles}} e^{-iE_n|x|} | \langle 0 | J(0) | n \rangle |^2 + \sum_{\text{photons}} e^{-iE_n|x|} | \langle 0 | J(0) | n \rangle |^2
\]
where
\[
\sum_{\text{particles}} e^{-iE_n|x|} | \langle 0 | J(0) | n \rangle |^2 \sim e^{-M_{\text{lowest}}|x|}
\]
and annihilation channel looks like
\[
\sum_{\text{photons}} e^{-iE_n|x|} | \langle 0 | J(0) | n \rangle |^2 \sim \frac{1}{|x|^2}
\]
It means that the annihilation channel does not take part in the bound state formation in contradiction with the non-relativistic potential approach.

Another point: we want to understand what is the role of TIME in formation of bound states.

In this paper we apply functional methods to calculate the asymptotic behavior of vacuum mean value (1) of relativistic currents for positronium and clarify the role of time in the formation of bound states.

## 2 Lagrangian and bound states

All our calculations will be performed in the Euclidean space. The Lagrangian of the electron field \( \psi \) and the electromagnetic photon field \( A_\mu \) looks like
\[
L = -\frac{1}{4} F_{\mu\nu}^2(x) + (\bar{\psi}(x)[i(\hat{p} + e\hat{A}(x)) - m] \psi(x)),
\]
\[
F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).
\]
The electron propagator has the standard form
\[
S(x - x') = \langle T[\bar{\psi}(x)\psi(x')] \rangle = \int \frac{dp}{(2\pi)^4} \frac{e^{i\hat{p}(x-x')}}{m - i\hat{p}}
\]
The propagator of the photon vector field is
\[
D_{\mu\nu}(x - y) = \langle A_\mu(x) A_\nu(y) \rangle = \delta_{\mu\nu} D(x - y) + \frac{\partial^2}{\partial x_\mu \partial x_\nu} D_d(x - y),
\]
\[
D(x) = \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{k^2} = \frac{1}{(2\pi)^2 x^2}, \quad D_d(x) = \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{k^2} \frac{d(k^2)}{k^2}.
\]
2.1 Two-point Green function

The object of our interest is the gauge invariant two-point Green function

\[ G_\Gamma(x-y) = \iint \frac{D\bar{\psi}DA}{C} e^{-\frac{i}{2}(A_\mu D^{\mu\nu}_A A_\nu) + (\bar{\psi}[i(\not{p} + e\not{A}) - m] \psi)} \cdot (\bar{\psi}(x) \Gamma \psi(x))(\bar{\psi}(y) \Gamma \psi(y)) \] (6)

Here \( \Gamma \) is a Dirac matrix which defines the local vertex with quantum numbers of the state \( J_\Gamma = (\bar{\psi} \Gamma \psi) \). We have for para-positronium \( \Gamma = i\gamma_5 \) and for ortho-positronium \( \Gamma = \gamma_\mu \).

After integration over the electron fields \( \psi \) and \( \bar{\psi} \) we get

\[ G_\Gamma(x-y) = B_\Gamma(x-y) + H_\Gamma(x-y), \] (7)

where

\[ B_\Gamma(x-y) = \iint \frac{DA}{C} e^{-\frac{i}{2}(A_\mu D^{\mu\nu}_A A_\nu) + T[A]} \cdot \text{Tr}[\Gamma S(x,y|A) \Gamma S(y,x|A)], \] (8)

and

\[ H_\Gamma(x-y) = \iint \frac{DA}{C} e^{-\frac{i}{2}(A_\mu D^{\mu\nu}_A A_\nu) + T[A]} \cdot \text{Tr}[\Gamma S(x,x|A)] \cdot \text{Tr}[\Gamma S(y,y|A)]. \]

Here \( S(x,y|A) \) is the electron propagator in the external field \( A_\mu \):

\[ S(x,y|A) = \frac{1}{i(\not{p} + e\not{A}(x)) - m} \delta(x-y) \] (9)

The functional

\[ T[A] = \text{Tr} \ln \frac{i(\not{p} + e\not{A}) - m}{i\not{p} - m} = \text{Tr} \ln \left[ 1 + ie\not{A} \frac{1}{i\not{p} - m} \right] + O(e^4 A^4) \]

\[ = \frac{e^2}{2} \text{Tr} \left[ \frac{1}{i\not{p} - m} \not{A} \frac{1}{i\not{p} - m} \right] + O(e^4 A^4) \]

\[ = \frac{e^2}{2} \iint dx dy A_\mu(x) \Pi_{\mu\nu}(x-y) A_\nu(y) + O(e^4 A^4), \]

\[ S_0(x-y) = \frac{1}{i\not{p} - m} \delta(x-y) = \int \frac{dp}{(2\pi)^4} \frac{e^{-ip(x-y)}}{i\not{p} - m}, \]

\[ \Pi_{\mu\nu}(x-y) = \text{Tr} \left[ \gamma_\mu S_0(x-y) \gamma_\nu S_0(y-x) \right], \]
Figure 1: Terms $B$ and $H$

describes radiation corrections to the photon propagator and to the photon-photon interaction. In this paper, we neglect this term because it does not contain spin-spin interaction and, therefore, does not contribute to ortho-para- positronium mass difference in the lowest corrections.

The loop $B_\Gamma$ contains all possible $(\overline{\psi}\Gamma\psi)$-bound states. If the mass of the lowest state $M_\Gamma < 2m$, then the asymptotic behavior of this loop for large $|x - y|$ looks like

$$B_\Gamma(x - y) \sim e^{-M_\Gamma|x-y|}$$

where $M_\Gamma$ is the mass of the lowest state in the current $(\overline{\psi}\Gamma\psi)$, i.e. the mass of a possible bound state. This mass can be calculated by the formula

$$M_\Gamma = -\lim_{|x|\to\infty} \frac{1}{|x|} \ln B_\Gamma(x) = 2m - \epsilon_\Gamma.$$
The loop $H$ describes so the called annihilation channel and contains long-range contributions of photons:

$$H(r)(x-y) \sim \frac{1}{|x-y|^2}$$

This term does not contain any bound state.

Graphic representations of the loops $B$ and $H$ are shown on Fig.1.

3 The electron propagator

The propagator of the electron fermion field satisfies the equation

$$[i(\hat{p} + e\hat{A}(x)) - m]S(x,y|A) = \delta(x-y), \quad (12)$$

For the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu f(x)$$

it is transformed as

$$S(x,y|A + \partial f) = e^{ie(x)}S(x,y|A)e^{-ie(y)},$$

so that the loop (8) is gauge invariant.

The solution of the equation (12) can be represented by the functional integral (see, for example, [17]):

$$S(x,y|A) = \frac{1}{i(\hat{p} + e\hat{A}(x)) - m} \delta(x-y)$$

$$= \left[ i(\hat{p}_x + e\hat{A}(x)) + m \right] \cdot \frac{1}{(p + eA(x))^2 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}(x) + m^2} \delta(x-y),$$

$$= \left[ i(\hat{p}_x + e\hat{A}(x)) + m \right] \int_0^\infty ds \frac{1}{8\pi^2 s^2} e^{-\frac{1}{2}} \left[ m^2 + s^2 + (x-y)^2 \right]$$

$$\cdot \int \frac{D\eta}{C} e^{-\frac{1}{2} \int_0^t dt \dot{z}(t)^2 + ie \int_0^t dt \dot{z}_\mu(t)A_\mu(z(t))} T \left\{ \begin{array}{c} e^{i \int_0^t dt \sigma_{\mu\nu}(z(t))F_{\mu\nu}(z(t))} \\
\end{array} \right\}, \quad (13)$$

$$z(t) = x - \frac{t}{s} + y\left(1 - \frac{t}{s}\right) + \eta(t).$$
The boundary conditions are \( \eta(0) = \eta(\alpha) = 0 \) and the normalization is

\[
\int \frac{D\eta}{C} \exp \left\{ -\int_0^s dt \dot{\eta}^2(t) \right\} = 1.
\]

The symbol \( T_t \) means the time-ordering of the matrix \( \sigma_{\mu\nu}(\tau) \) to the time variable \( t \).

The representation (13) is quite close to functional representation of the propagator for a scalar charged particle (see [9]). The main functional structure is the same.

The representation (13) is obviously gauge covariant because

\[
\delta \int_0^s dt \dot{z}_\mu(t) A_\mu(z(t)) = \int_0^s dt \dot{z}_\mu(t) \frac{\partial}{\partial z_\mu} f(z(t))
\]

\[
= \int_0^s dt \frac{d}{dt} f(z(t)) = f(z(s)) - f(z(0)) = f(x) - f(y).
\]

As it was said above, our aim is to calculate the functional integral (8) in the limit \( |x| \rightarrow \infty \) (we put \( y = 0 \)). We want to calculate the main contributions to the binding energy assuming the coupling constant \( \alpha \) to be small. In this case, for large \( x \rightarrow \infty \) and small \( \alpha \) the saddle-point in the integral over \( s \) is realized for \( s = \frac{X}{m} \). Putting

\[
x = (x, x_4) \Rightarrow (0, x_4), \quad \sqrt{x^2} \Rightarrow x_4 = X > 0, \quad t = \frac{X}{m} \tau,
\]

one can get for \( X \rightarrow \infty \)

\[
S(x, 0|A) = \frac{\text{const}}{X^{\frac{1}{2}}} (1 + \gamma_0) e^{-mX} \cdot S(x)
\]

\[
S(0, x|A) = \frac{\text{const}}{X^{\frac{1}{2}}} (1 - \gamma_0) e^{-mX} \cdot S(x)
\]

\[
S(x) = \int \frac{D\eta}{C} e^{-\int_0^X dr \frac{m\dot{z}_\nu(\tau)^2}{2} + ie \int_0^X d\tau \dot{z}_\mu(\tau) A_\mu(z(\tau))} \cdot \frac{e^{-\int_0^s dt \dot{\eta}^2(t)}}{C} R[z],
\]

\[
R[z] = T_\tau \left\{ e^{-\int_0^X d\tau \sigma_{\mu\nu}(\tau) F_{\mu\nu}(z(\tau))} \right\}
\]
we shall use this representation in what follows.

3.1 Mass of the bound state

The next step is to substitute electron propagators $S(x, 0|A)$ and $S(0, x|A)$ in the form (14) into the representation (8) for the Green function $B_\Gamma(x)$ and then to integrate over the photon field $A$. We have for large $X \to \infty$

$$B_\Gamma(X) \sim e^{-2mX} \int D\eta_1 D\eta_2 \frac{-m}{\tau} \int^X_0 d\tau [\dot{\eta}_1^2(\tau)+\dot{\eta}_2^2(\tau)] \mathcal{F}_\Gamma[X, \eta_1, \eta_2],$$

with

$$\mathcal{F}_\Gamma[X, \eta_1, \eta_2] = \frac{1}{4} \text{Tr} \left[ \Gamma (1 + \gamma_0) R[z^{(1)}] \Gamma (1 - \gamma_0) R[z^{(2)}] \right].$$

The mass $M_\Gamma$ of the bound state with quantum number $\Gamma$ is defined by the formula (11).

The integral (17) over the photon field $A$ can be calculated explicitly. The result of the calculations is shown in Fig.2. We will not write down this simple long formula. Our aim is to find the ortho-para mass difference in the lowest approximation of the functional method. Therefore, we omit all terms connected with contributions to the electron propagator in the loop and take into account the dominant terms responsible for positronium formation and desired ortho-para mass difference (see Fig.3). We get in the lowest approximation over spin-spin interaction

$$\mathcal{F}_\Gamma[X; \eta_1, \eta_2] = \int \frac{DA}{C} e^{-\frac{1}{2}(A_\mu D^{-1}_{\mu\nu} A_\nu)} \cdot e^{\frac{ie}{0} \int^X_0 d\tau z^{(1)}_\mu(\tau) A_\mu(z^{(1)}(\tau)) + \frac{ie}{0} \int^X_0 d\tau z^{(2)}_\mu(\tau) A_\mu(z^{(2)}(\tau))}.$$
\[ \frac{1}{4} \text{Tr} \left\{ \Gamma(1 + \gamma_0) \Gamma(1 - \gamma_0) \right. \\
+ \frac{e^2}{16m^2} \int_0^X d\tau_1 d\tau_2 \cdot \Gamma(\gamma_0 + 1) \sigma_{\mu\nu} \Gamma(-\gamma_0 + 1) \sigma_{\rho\sigma} \cdot F^{\mu\nu}(z^{(1)}(\tau_1)) F^{\rho\sigma}(z^{(2)}(\tau_2)) \left. \right\} \\
= e^{W[X;\eta_1,\eta_2]} \cdot \left\{ \Sigma^{(0)}_\Gamma + K[s;\eta_1,\eta_2] + O(e^4) \right\}. \]

Here
\[ \Sigma^{(0)}_\Gamma = \frac{1}{4} \text{Tr} \Gamma(1 + \gamma_0) \Gamma(1 - \gamma_0). \]

The main functional responsible for the bound state formation looks like
\[ W[X;\eta_1,\eta_2] = e^2 \int_0^X d\tau_1 d\tau_2 \cdot z^{(1)}_{\mu}(\tau_1) z^{(2)}_{\nu}(\tau_2) D_{\mu\nu}(z^{(1)}(\tau_1) - z^{(2)}(\tau_2)) \]
\[ = e^2 \int_0^X d\tau_1 d\tau_2 \cdot z^{(1)}_{\mu}(\tau_1) z^{(2)}_{\nu}(\tau_2) D(z^{(1)}(\tau_1) - z^{(2)}(\tau_2)) \] (18)
Figure 3: Diagrams which are responsible for the bound state

\[ e^2 \int_0^X dt_1 dt_2 \int_0^X d\tau_1 d\tau_2 \frac{dk}{(2\pi)^4} \frac{e^{ik(z(1)-(\tau_1)-z(2)-(\tau_2))}}{k^2}. \]

The functional responsible for the ortho-para mass difference is

\[ K_\Gamma [X; \eta_1, \eta_2] = e^2 \int_0^X dt_1 dt_2 \cdot \frac{1}{4} \text{Tr} \Gamma(1 + \gamma_0)\sigma_{\mu\rho} \Gamma(1 - \gamma_0)\sigma_{\nu\rho} \]

\[ \frac{\partial^2}{\partial z_\mu(1) \partial z_\nu(2)} D(z(1)(\tau_1) - z(2)(\tau_2)) \]

\[ = e^2 \int_0^X dt_1 dt_2 \int \frac{dk}{(2\pi)^4} \cdot \frac{e^{ik(z(1)-(\tau_1)-z(2)-(\tau_2))}}{k^2} \Sigma_\Gamma (k), \]

where

\[ \Sigma_\Gamma (k) = \frac{k_\mu k_\rho}{k^2} \cdot \frac{1}{4} \text{Tr} \Gamma(1 + \gamma_0)\sigma_{\mu\nu} \Gamma(1 - \gamma_0)\sigma_{\rho\nu} = \Sigma_\Gamma^{(0)} \cdot \Delta_\Gamma (k). \]

The term \( W \) is responsible for positronium bound states. The term \( K_\Gamma \) describes the spin-spin interaction and it is responsible for the ortho-para mass difference. All neglected terms give the next to \( \alpha = e^2/4\pi \) perturbation contributions.

Let us introduce the notation

\[ d\sigma_{\eta_1 \eta_2}^{W} = \frac{D\eta_1 D\eta_2}{C} e^{-\frac{X}{\hbar} \int_0^t dt [\eta_1^2(t) + \eta_2^2(t)] + W[X;\eta_1, \eta_2]}\]
with

\[ J_0(X) = \int d\sigma_{\eta_1\eta_2}^W = J e^{\epsilon_0 X} \]  \hfill (22)

then one can write

\[
B_\Gamma(X) = e^{-2mX} \int \int d\sigma_{\eta_1\eta_2}^W \cdot \left\{ \Sigma^{(0)}_\Gamma + K_\Gamma[X; \eta_1, \eta_2] + O(e^4) \right\} \\
= J \Sigma^{(0)}_\Gamma e^{-(2m-\epsilon_0-\epsilon_\Gamma)X} \]  \hfill (23)

The binding energy \( \epsilon_0 \) does not depend on the spin of electron-positron system and it is defined mainly by the Coulomb interaction. The binding energy \( \epsilon_\Gamma \) depends on the spin of constituents and defines the ortho-para mass difference. It looks like

\[
\epsilon_\Gamma = \lim_{X \to \infty} \frac{e^2}{4m^2} \frac{1}{X} \int_0^X d\tau_1 d\tau_2 \int \frac{dk}{(2\pi)^4} \cdot \left\langle e^{ik(z^{(1)}(\tau_1)-z^{(2)}(\tau_2))} \right\rangle_{\eta_1\eta_2} \Delta_\Gamma(k), \]  \hfill (24)

where the average is

\[
\left\langle e^{ik(z^{(1)}(\tau_1)-z^{(2)}(\tau_2))} \right\rangle_{\eta_1\eta_2} = \frac{1}{J_0(X)} \int d\sigma_{\eta_1\eta_2}^W e^{ik(z^{(1)}(\tau_1)-z^{(2)}(\tau_2))} \]  \hfill (25)

The functional \( W \) in (21) contains terms defining the Coulomb and "time" interactions. The measure \( d\sigma_{\eta_1\eta_2}^W \) contains the space \( \eta \) and "time" \( \eta_4 \) functional variables. Our aim is to evaluate the contribution of "time" interaction to the bound state formation. Thus our direct problem is to calculate the integral (25).

### 4 Para- and ortho-positronium mass difference

In the quantum field theory there is the problem how to define the bound state. The point is that the relativistic currents of the type \( \bar{\psi} \Gamma \psi \) are classified by the relativistic group in the space \( \mathbb{R}^4 \) while the physical bound states are classified by the non-relativistic group in the space \( \mathbb{R}^3 \). As a result the physical states are described by an appropriate mixture of different components of different relativistic currents. In addition, the angles of mixture
are not known a priori. In Table 2 the non-relativistic quantum numbers of different components of relativistic currents are listed.

One can see that each current $\langle \bar{\psi} \Gamma \psi \rangle$ with quantum number $J^P$ is determined by a mixture of two relativistic currents

\begin{align*}
S(0^+) & : \quad \Gamma_S = I \cos \theta_S + \gamma_0 \sin \theta_S, \\
A(1^+) & : \quad \Gamma_A = \gamma_5 \gamma \cos \theta_A + i[\gamma \times \gamma] \sin \theta_A, \\
V(1^-) & : \quad \Gamma_V = \gamma \cos \theta_V + i\gamma_0 \gamma \sin \theta_V, \\
P(0^-) & : \quad \Gamma_P = i\gamma_5 \cos \theta_P + \gamma_5 \gamma_0 \sin \theta_P.
\end{align*}

Generally speaking, in order to define the angles $\theta_S, \theta_A, \theta_V, \theta_P$, some additional argumentation should be used. However, we shall see that the desired masses do not depend on these angles at least in the lowest approximation.

Let us come to formula (20). The results for $\Sigma^{(0)}$ and $\Sigma^k$ with currents (26) are listed in Table 3. One can see that the states $S(0^+)$ and $A(1^+)$ do not exist at all. The masses of bound states $P(0^-)$ and $V(1^-)$ do not depend on the mixing angles $\theta_P$ and $\theta_V$.

According to (21), the desired mass difference is defined by the formula

\begin{align*}
\delta M &= \epsilon_V - \epsilon_P \\
&= \frac{e^2}{4} \lim_{X \to \infty} \frac{1}{X} \int_0^X d\tau_1 d\tau_2 \int \frac{dk}{(2\pi)^4} \cdot \left< e^{ik_4(\tau_1 - \tau_2)} \right>_{\eta_4} (\Delta_V(k) - \Delta_P(k)) \\
&= \frac{8}{3} \cdot \frac{e^2}{4} \lim_{X \to \infty} \frac{1}{X} \int_0^X d\tau_1 d\tau_2 \int \int \frac{dk_4 dk_2}{(2\pi)^4} \cdot \frac{k^2}{k^2 + k_4^2} \left< e^{ik_4(\eta_1(\tau_1) - \eta_2(\tau_2))} \right>_{\eta_4} \left< e^{ik_4(\eta(\tau_2))} \right>_{\eta_4} \left< e^{ik_4(\eta(\tau_2))} \right>_{\eta_4} \left< e^{ik_4(\eta(\tau_2))} \right>_{\eta_4} \left< e^{ik_4(\eta(\tau_2))} \right>_{\eta_4}
\end{align*}

\( \text{(27)} \)

5 The lowest contribution

The lowest main contribution to positronium bound state is defined by the integrals (22) and (25). It is convenient to extract the $\alpha^2$ dependence in order to extract the non-relativistic Coulomb potential (see [9]). For this aim in the representation (21) let us introduce the new variables:

\begin{align*}
Y &= \alpha^2 mX, \quad \tau = \frac{v}{\alpha^2 m}, \quad k_4 = \alpha^2 mq, \quad k = \alpha m q. \\
\eta(t) &= \frac{1}{am} \xi(v), \quad \eta(t) = \frac{1}{am} \xi(v).\end{align*}

14
The parameters $X$ and $Y$ are infinitely large quantities.

We get

$$J_0(Y) = \int \int \frac{D\xi_1 D\xi_2 D\xi_1 D\xi_2}{C} e^{\frac{1}{2} \int_0^Y \left[ \frac{\dot{\xi}_1^2(v) + \dot{\xi}_2^2(v) + \dot{\xi}_1^2(\tau) + \dot{\xi}_2^2(\tau)}{2} \right] + W[\xi_1, \xi_2, \xi_1, \xi_2; \alpha]}$$

(28)

with

$$W[\xi_1, \xi_2, \xi_1, \xi_2; \alpha] = \int_0^Y dv \int_0^Y dv \left[ \frac{\dot{\xi}_1^2(v_1) + \dot{\xi}_2^2(v_2) + \dot{\xi}_1^2(\tau) + \dot{\xi}_2^2(\tau)}{2} \right]$$

$$\cdot \int \int \frac{dq dq e^{iq(v_1 - v_2) + \alpha(\xi_1(v_1) - \xi_2(v_2)) + i\alpha(\dot{\xi}_1(v_1) - \dot{\xi}_2(v_2))}}{4\pi^2}$$

(29)

It is important to stress that the functional $W[\xi_1, \xi_2, \xi_1, \xi_2; \alpha]$ is not analytic at the point $\alpha = 0$, so that the relativistic corrections cannot be obtained by a regular method. Calculation of the integral (28) is not a simple problem. Therefore, we restrict ourselves to calculation of the next relativistic correction to $\alpha$, so that we neglect terms with $\alpha$ in square brackets in (29) and introduce the variables

$$\xi_1(\tau) = R(\tau) + \frac{1}{2}\rho(\tau), \quad \xi_2(\tau) = R(\tau) - \frac{1}{2}\rho(\tau),$$

$$\xi_1(\tau) = R(\tau) + \frac{1}{2}\rho(\tau), \quad \xi_2(\tau) = R(\tau) - \frac{1}{2}\rho(\tau),$$

The variables $R$, $\rho$ and $R$, $\rho$ describe the center of mass and relative coordinates in configuration and "time" spaces, respectively. We shall see that the "time" variables give the important contribution to the desired mass correction. We get

$$J_0(Y) = \int \int \frac{D\rho D\rho_1 D\rho_2 D\rho_2}{C} e^{\frac{1}{2} \int_0^Y \left[ \dot{\rho}^2(v) + R^2(\rho) + \frac{1}{2}\dot{\rho}_1^2(v) + R^2(\rho) \right] + W[R, \rho, R, \rho; \alpha]}$$

(30)

where

$$W[R, \rho, R, \rho; \alpha] = \int_0^Y dv_1 dv_2$$

(31)
\[ J_0(Y) = \int \frac{d\rho_1 d\rho_2}{C} e^{-\frac{Y}{2} \left[ \frac{1}{4} \rho^2(v_1) + \frac{1}{4} \rho^2(v_2) \right]} W[\rho, \rho, \alpha]. \] (32)

Here

\[ W[\rho, \rho; \alpha] = \int d\rho_1 d\rho_2 \frac{q^2 + \alpha^2 q^2}{4\pi^3} e^{-\frac{Y}{2} \left[ \frac{1}{4} \rho^2(v_1) + \frac{1}{4} \rho^2(v_2) \right]} e^{-\frac{i\alpha q}{2} (\rho(v_1) + \rho(v_2))} e^{-\frac{i\alpha q}{2} (\rho(v_1) + \rho(v_2))}. \]

The functional \( W[\rho, \rho; \alpha] \) is not analytic at the point \( \alpha = 0 \). Nevertheless, one can extract the lowest terms to \( \alpha \).

\[
W[\rho, \rho; \alpha] \Rightarrow W_0[\rho, \rho; \alpha]
\]

\[
= \int d\rho_1 d\rho_2 \int \frac{d\rho_1 d\rho_2}{2\pi^2 q^2} e^{i\rho(v_1 - v_2) - \frac{1}{2} q^2 |v_1 - v_2|} e^{-\frac{i\alpha q}{2} (\rho(v_1) + \rho(v_2))} e^{-\frac{i\alpha q}{2} (\rho(v_1) + \rho(v_2))}
\]

\[
= \int \frac{d\rho_1 d\rho_2}{2\pi^2 q^2} \int d\rho_1 d\rho_2 \delta(v_1 - v_2 - \frac{\alpha}{2} (\rho(v_1) + \rho(v_2))) e^{-\frac{i\alpha q}{2} |v_1 - v_2|} e^{-\frac{i\alpha q}{2} (\rho(v_1) + \rho(v_2))}
\]

\[
= \int \frac{d\rho_1 d\rho_2}{2\pi^2 q^2} \int d\rho_1 d\rho_2 \delta(v_1 - v_2 - \frac{\alpha}{2} (\rho(v_1) + \rho(v_2))) e^{-\frac{i\alpha q}{2} |v_1 - v_2|} e^{-\frac{i\alpha q}{2} (\rho(v_1) + \rho(v_2))}
\]

Taking into account the correlations (see Appendix III) one can get

\[
\delta(\rho(v)) = \langle \delta(\rho(v)) \rangle \rho + \left[ \delta(\rho(v)) - \langle \delta(\rho(v)) \rangle \right] \rho
\]

\[
= \frac{1}{8\pi} + \left[ \delta(\rho(v)) - \frac{1}{8\pi} \right]
\]
because
\[ \langle \delta(\rho(v)) \rangle_{\rho} = \Psi_0^2(0) = \frac{1}{8\pi} \]

Finally, we have
\[ W_0[\rho, \rho; \alpha] = \int_0^\gamma \frac{dv}{|\rho(v)|} - \frac{\alpha}{8} \int_0^\gamma dv |\rho(v)| + O(\alpha^{1+\delta}) \quad (33) \]

The measure in (32) is reduced to
\[ d\sigma^W = \frac{D\rho D\rho}{C} e^{-\int_0^\gamma dv \left[ \frac{1}{4} \dot{\rho}^2(v) + \frac{1}{8} \rho^2(v) \right] + W[\rho, \rho; \alpha]} \quad (34) \]
\[ \rightarrow d\sigma = d\sigma_{\rho} d\sigma_{\rho}, \]

Here
\[ d\sigma_{\rho} = \frac{D\rho}{C\rho} e^{-\int_0^\gamma dv \left[ \frac{1}{4} \dot{\rho}^2(v) - \frac{1}{8} \rho^2(v) \right]}, \]
\[ d\sigma_{\rho} = \frac{D\rho}{C\rho} e^{-\int_0^\gamma dv \left[ \frac{1}{4} \dot{\rho}^2(v) + \frac{1}{8} |\rho(v)| \right]}. \]

This measure consists of two components - the standard non-relativistic Coulomb potential term and the ”time” term which is not taken into account in any usual calculations. This term corresponds to one-dimension linear potential with coupling constant $\alpha$ and leads to non-analytical behavior of energy on $\alpha$.

One can calculate (see Appendix III)
\[ J_{\text{Coulomb}} = \int d\sigma_{\rho} = \int \frac{D\rho}{C\rho} e^{-\int_0^\gamma dv \left[ \frac{1}{4} \dot{\rho}^2(v) - \frac{1}{8} \rho^2(v) \right]} \]
\[ = \sum_{n\ell} (2\ell + 1) e^{-X \frac{\alpha^2}{m^2}} |\Psi_{n\ell}(0)|^2, \]
\[ J_{\text{time}} = \int d\sigma_{\rho} = \int \frac{D\rho}{C\rho} e^{-\int_0^\gamma dv \left[ \frac{1}{4} \dot{\rho}^2(v) + \frac{1}{8} |\rho(v)| \right]} = \sum_{\kappa} e^{-X \alpha^2 + \frac{1}{8} \epsilon_{\kappa} m |\Phi_{\kappa}(0)|^2} \quad (35) \]
Thus the general spectrum of the Coulomb and "time" potentials is

\[ E_{n\kappa} = \left[ -\frac{\alpha^2}{n^2} + \alpha^2 + \frac{3}{4} \epsilon_\kappa \right] m \]  

(36)

As a result we have the "time excitations", or abnormal states, connected with the fourth component of 4-dimensional space. These states appear in solutions of the Bethe-Salpeter equation. Up to now it is not known exactly these states does or does not exist in reality. It is the second reason why relativistic QED does not describe correctly the real bound states.

6 Mass difference

The desired mass difference is defined by the formula (27) which can be represented as

\[ \delta M = \frac{1}{3} \alpha^4 m \cdot \Delta(\alpha), \]

\[ \Delta(\alpha) = \lim_{Y \to \infty} \frac{1}{Y} \int_0^Y dv_1 dv_2 \int \frac{dq dq}{2\pi^3} \cdot \frac{q^2 e^{iq(v_1-v_2)-\frac{1}{4}[(\alpha^2 q^2+q^2)|v_1-v_2|}}{q^2 + \alpha^2 q^2} \]

\[ \langle e^{-i\frac{\alpha}{2}}(\rho(v_1)+\rho(v_2)) \rangle_{\rho} \langle e^{-i\alpha\frac{3}{2} (\rho(v_1)+\rho(v_2))} \rangle_{\rho} \]

The averaging over the fields \( \rho \) and \( \rho \) gives (see Appendix III)

\[ \langle e^{i\frac{\alpha}{2} (\rho(\tau_1)+\rho(\tau_2))} \rangle_{\rho} \Rightarrow \sum_{n\ell} e^{-|\tau_1-\tau_2|(E_n-E_0)} (-1)^\ell (2\ell + 1) C_{n\ell}^2 \left( \frac{k}{2} \right). \]

\[ \langle e^{-i\frac{\alpha}{2} (\rho(\tau_1)+\rho(\tau_2))} \rangle_{\rho} \Rightarrow \sum_{n\ell} e^{-|\tau_1-\tau_2|(E_n-E_0)} (-1)^\kappa A_{\kappa} \left( \alpha^3 q \right)^2 \]

Finally we have

\[ \Delta(\alpha) = \frac{1}{\pi^3} \int_0^\infty d\tau \int dk dq \cdot \frac{k^2}{k^2 + \alpha^2 q^2} \cdot e^{i\tau r - \frac{1}{2}(\alpha^2 q^2+k^2)|\tau|} \]

\[ \cdot \sum_{n\ell} e^{-\frac{i\alpha}{4} (1-\frac{1}{4})} (-1)^\ell (2\ell + 1) C_{n\ell}^2 \left( \frac{k}{2} \right) \sum_{\kappa} (-1)^\kappa e^{-\frac{\alpha^3 q^2}{4} (\epsilon_\kappa-\epsilon_0)} A_{\kappa} \left( \alpha^3 q \right)^2 \]

\[ = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{\kappa=0}^{\ell+\kappa} (-1)^{\ell+\kappa} \Delta_{n\ell\kappa}(\alpha) \]
After integration over \( \tau \) and angles one can get

\[
\Delta_{n\ell\kappa}(\alpha) = 32 \frac{k^4}{\pi^2} \int_0^\infty \frac{dkdq}{k^2 + \alpha^2 q^2} \cdot \frac{k^2 + \alpha^2 q^2 + 1 - \frac{1}{n^2} + \alpha^2 (\epsilon_\kappa - \epsilon_0)}{(k^2 + \alpha^2 q^2 + 1 - \frac{1}{n^2} + \alpha^2 (\epsilon_\kappa - \epsilon_0))^2 + 16q^2} \cdot (2\ell + 1) C_{nl}^2 \left( \frac{k}{2} \right) |A_\kappa(\alpha^2 q)|^2
\]  

(38)

The numerical results are shown in Table 4. For the function \( \Delta(\alpha) \) we get

\[
\Delta_0(0) = \sum_{n=1}^{3} \sum_{\ell=0}^{n-1} (-1)^\ell \Delta_{n\ell0}(0) = 1.0,
\]

\[
\Delta_0(\alpha) = \sum_{n=1}^{3} \sum_{\ell=0}^{n-1} (-1)^\ell \Delta_{n\ell0}(\alpha) = 0.9641,
\]

\[
\Delta(\alpha) = \sum_{n=1}^{3} \sum_{\ell=0}^{n-1} \sum_{\kappa=0}^{2} (-1)^{\ell+\kappa} \Delta_{n\ell\kappa}(\alpha) = 0.952754.
\]

Obviously, this result is in contradiction with the existing experimental number \( (\Delta = 0.99512...) \).

7 Breit potential approach

One of the attempts to describe the bound state problem is the Breit potential approach (see, for example, \[13\]). Let us consider the elastic electron-positron scattering

\[ e_p + \bar{e}_k \rightarrow e_{p'} + \bar{e}_{k'} \]

The scattering amplitude in the lowest order of relativistic \( S \)-matrix theory is described by the Feynman diagrams shown on Fig.2 and looks like

\[
M = -e^2[\bar{u}(p')\gamma_\mu u(p)]D_{\mu\nu}(p - p')[\bar{u}(-k)\gamma_\nu u(-k')]
\]  

(39)

\[
+ e^2[\bar{u}(-k)\gamma_\mu u(p)]D_{\mu\nu}(p + k)[\bar{u}(p')\gamma_\nu u(-k')]
\]

where the first term (a) is connected with scattering and the second one (b) with annihilation channels. The spinors \( \bar{u}(p') \) and \( u(k') \) are the solutions
of the Dirac equation. This amplitude in the non-relativistic limit should coincide with the non-relativistic Born approximation which defines the effective electron-positron potential. This potential should be introduced into the non-relativistic Schrödinger equation.

The Dirac spinors $\bar{u}(p')$ and $u(k')$ define the relativistic corrections to the non-relativistic potential. It should emphasize that electrons and positrons are on the mass shell, i.e. they are real physical particles. Thus the time is removed from the interaction Hamiltonian.

The part of the Breit potential which is responsible for the ortho- and para- mass difference looks like

$$U(r) = U_{sc}(r) + U_{an}(r) = -\frac{\alpha}{r} + \frac{7}{12} \cdot \frac{2\alpha \pi}{m^2} (\sigma_- - \sigma_+) \delta(r). \quad (40)$$

where

$$U_{sc}(r) = -\frac{\alpha}{r} + \frac{1}{3} \cdot \frac{2\alpha \pi}{m^2} (\sigma_- - \sigma_+) \delta(r),$$

$$U_{an}(r) = \frac{1}{4} \cdot \frac{2\alpha \pi}{m^2} (\sigma_- - \sigma_+) \delta(r).$$

We want to stress that the coefficient $\frac{7}{12}$ is the sum of the contributions from scattering and annihilation channels

$$\frac{7}{12} = \left(\frac{1}{3}\right)_{sc} + \left(\frac{1}{4}\right)_{an}$$
Taking into account
\[ \Psi^2(0) = \frac{\alpha^3 m^3}{8\pi}, \]
\[ \langle (\sigma - \sigma^+) \rangle_{\text{ortho}} - \langle (\sigma - \sigma^+) \rangle_{\text{para}} = 4, \]
one can get for the ortho- and para- mass difference
\[ \Delta = \epsilon_{\text{ortho}} - \epsilon_{\text{para}} = \frac{7}{12} \cdot \frac{2\alpha\pi}{m^2} \cdot \frac{\alpha^3 m^3}{8\pi} \cdot 4 = \frac{7}{12} \alpha^4 m \]
This result is in good agreement with the experimental data and, therefore, supports the point of view that the annihilation channel plays the essential role in the formation of the positronium.

8 Conclusion
In conclusion, one can say that the functional approach is the best mathematical representation to preserve the gauge invariance. The developed technique of calculations permits one to get accurate results in QED where the coupling constant \( \alpha \) is small. The lowest approximation of this functional representation is the pure non-relativistic Feynman path integral representation of the non-relativistic Schrödinger equation with the Coulomb potential. One can see that any regular series for next corrections to \( \alpha \) do not exist and these corrections cannot be reduced to some terms to the non-relativistic potential in the Schrödinger picture. In other words, the “nonphysical” time coordinate is important and leads to corrections which is not analytic of the order \( \alpha^4 \).

There exists a contradiction in the current algebra formula. On one hand, it is supposed that the space of states \{\ket{n}\} can contain possible bound states. However on the other hand, in reality it is the Fock space of free electrons and photons which does not contain any unstable bound states. Nevertheless calculations of the functional representation for an appropriate Green function in the limit \( t \to \infty \) indicate that a bound with \( M_{\text{bound}} < 2m \) does exist really. Besides, the current algebra in QFT excludes influence of the annihilation channel for the bound state formation.

Our calculations show that the role of time is very important and give essential contribution into bound state mass. The next radiation corrections, connected with time excitations, to electromagnetic mass difference to positronium are of the order \( \alpha^2 \), i.e. they are to large.
In addition, the "time excitations", or abnormal states arise in QFT calculations but they are not exist in reality.

The experimental value of ortho- para-positronium mass difference is described in the framework of the Breit potential picture with attraction of the annihilation channel. Thus, explanation of experimental value para- ortho-positronium mass difference requires to take into account annihilation channel for effective potential.

One can conclude that in the relativistic QED time corrections are important, but the bound state problem requires the non-relativistic potential description where the time variable does not play any essential role.

The conclusion: the relativistic QED is not suited to describe real bound states correctly.

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9 Appendix I

We use the following representation for $\gamma$ matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),$$

$$\sigma_{0j} = -i\sigma_j\gamma_5, \quad \sigma_{ij} = -\epsilon_{ijk}\sigma_k$$

We should calculate the traces

$$\Sigma^{(0)}_{\Gamma} = \frac{1}{4}\text{Tr} \Gamma(1 + \gamma_0)\Gamma(1 - \gamma_0)$$

and

$$\Sigma_{\Gamma}(k) = \frac{k_\mu k_\nu}{k^2} \cdot \frac{1}{4}\text{Tr} \Gamma(1 + \gamma_0)\sigma_{\mu\rho}\Gamma(1 - \gamma_0)\sigma_{\nu\rho} = \Sigma^{(0)}_{\Gamma} \cdot \Delta_{\Gamma}(k).$$

For the para-positronium $(P)$ with $\Gamma_P = i\gamma_5$ we get

$$\Sigma^{(0)}_{\Gamma_P} = \frac{1}{4}\text{Tr} i\gamma_5(1 + \gamma_0)i\gamma_5(1 - \gamma_0) = -2$$
and
\[ \Sigma_P(k) = \frac{k_\mu k_\nu}{k^2} \cdot \frac{1}{4} \text{Tr} \, i \gamma_5 (1 + \gamma_0) \sigma_{\mu\rho} i \gamma_5 (1 - \gamma_0) \sigma_{\nu\rho} = \frac{4k^2}{k^2 + k_4^2} \]

For the ortho-positronium (V) with \( \Gamma_V = \gamma_j \) we have
\[ \Sigma_P^{(0)} = \frac{1}{4} \text{Tr} \, \gamma_i (1 + \gamma_0) \gamma_j (1 - \gamma_0) = -2\delta_{ij} \]
and
\[ \Sigma_V(k) = \frac{k_\mu k_\nu}{k^2} \cdot \frac{1}{4} \text{Tr} \, \gamma_i (1 + \gamma_0) \sigma_{\mu\rho} \gamma_j (1 - \gamma_0) \sigma_{\nu\rho} = -\delta_{ij} \frac{4k^2}{3(k^2 + k_4^2)} \]
The results are collected in Table 2.

\[ \Delta_V(k) - \Delta_P(k) = \frac{8}{3} \cdot \frac{k^2}{k^2 + k_4^2} \]

10 Appendix II

Let us consider the contribution of the longitudinal part of the photon propagator to the integral (17). We have equality
\[ \dot{z}_\mu(\tau) \partial_\mu f(z(\tau)) = \frac{\partial}{\partial \tau} f(z(\tau)) \]
Then the term with the gauge dependent part \( \partial_\mu \partial_\nu D_d \) looks like
\[ W_{ij} = \frac{e^2}{2} \int_0^{s_i} d\tau_1 \int_0^{s_j} d\tau_2 \dot{z}_\mu(\tau_1) \dot{z}_\mu(\tau_2) \partial_\mu \partial_\nu D_d(z^{(i)}(\tau_1) - z^{(j)}(\tau_2)) \]
\[ = \frac{e^2}{2} \int_0^{s_i} d\tau_1 \int_0^{s_j} d\tau_2 \dot{z}_\mu(\tau_1) \dot{z}_\mu(\tau_2) \frac{\partial^2}{\partial \tau_1 \partial \tau_2} D_d(z^{(i)}(\tau_1) - z^{(j)}(\tau_2)) \]
\[ = \frac{e^2}{2} \left[ D_d(z^{(i)}(s_i) - z^{(j)}(s_j)) - D_d(z^{(i)}(0) - z^{(j)}(s_j)) \right. \]
\[ \left. - D_d(z^{(i)}(s_i) - z^{(j)}(0)) + D_d(z^{(i)}(0) - z^{(j)}(0)) \right] \]
\[ = e^2 \left[ D_d(0) - D_d(x - y) \right] = e^2 \int \frac{dk d(k^2)}{(2\pi)^4} \frac{1 - e^{ik(x-y)}}{k^2} \]
This term does not contribute to the bound state mass and should be omitted.
11 Appendix III

We consider the integral

\[ I = \int d\sigma \ e^{\Phi} = \int \frac{D\rho}{C} e^{-\int_0^t d\tau \left[ \frac{1}{2} \dot{\rho}^2(\tau) - U(|\rho(\tau)|) \right]} + \Phi[\rho] \]

(41)

where

\[ \Phi[\rho] = \frac{1}{2} \int_0^t d\tau_1 d\tau_2 \ A(|\tau_1 - \tau_2|) \cdot e^{i \frac{k}{\tau} (\rho(\tau_1) + \rho(\tau_2))} \]

The assumption is that the functional \( \Phi \) is small \( ||\Phi[\rho]|| \ll 1 \). We have in this case

\[ I = \int d\sigma \ e^{\Phi} = N \exp \left\{ \frac{1}{N} \int d\sigma \ \Phi + O(\Phi^2) \right\} \]

\[ \Rightarrow N = \int d\sigma. \]

and the problem is to calculate the integral

\[ W = \langle \Phi[\rho] \rangle = \frac{1}{N} \int d\sigma \ \Phi \]

(42)

\[ = \frac{1}{N} \int_{\rho(0)=0, \ \rho(t)=0} \frac{D\rho}{C} e^{-\int_0^t d\tau \left[ \frac{1}{2} \dot{\rho}^2(\tau) - U(|\rho(\tau)|) \right]} \Phi[\rho]. \]

Let the Hamiltonian

\[ H = -\left( \frac{\partial}{\partial x} \right)^2 + U(|x|) \]

have the spectrum \( \{E_n\} \) with the wave functions \( \Psi_n(x) \)

\[ H\Psi_n(x) = E_n \Psi_n(x). \]

The time Green function can be represented in two forms

\[ G_{t-t'}(x, x') = e^{-H(t-t')} \delta(x - x') = \int_{\rho(t')} = x', \rho(t) = x} \frac{D\rho}{C} e^{-\int_{t'}^t d\tau \left[ \frac{1}{2} \dot{\rho}^2(\tau) - U(|\rho(\tau)|) \right]} \]

\[ = \sum_n \Psi_n(x) e^{-(t-t')E_n} \Psi_n^+(x'). \]

(43)
The Green function satisfies the correlation for \( t > t'' > t' \)

\[
G_{t''-t'}(x,x') = \int dy \ G_{t-t''}(x,y)G_{t'-t'}(y,x')
\]

We have for the function \( W \)

\[
W(t) = \int_0^t d\tau_1 \int_\tau_1^\tau_2 A(|\tau_1 - \tau_2|)H(\tau_1, \tau_2)
\]

with

\[
H(\tau_1, \tau_2) = \frac{1}{N} \int d\rho_1 \int d\rho_2 \ G_{t-t_1}(0,\rho_1) e^{i\frac{\xi}{2}\rho_1} G_{t_1-t_2}(\rho_1,\rho_2) e^{i\frac{\xi}{2}\rho_2} G_{t_2}(\rho_2,0)
\]

\[
= \sum_{n_1n_2n_3} \frac{1}{N} \int d\rho_1 d\rho_2 \ e^{-E_{n_1}(X-t_1)} \Psi_{n_1}(0) \Psi_{n_2}^*(\rho_1) e^{i\frac{\xi}{2}\rho_1} \Psi_{n_2}(\rho_2) e^{i\frac{\xi}{2}\rho_2} e^{-E_{n_3}t_2} \Psi_{n_3}(\rho_2) \Psi_{n_3}^*(0)
\]

\[
= \frac{1}{N} \sum_{n_1n_2n_3} e^{-E_{n_1}(X-t_1)-E_{n_3}(t_1-t_2)-E_{n_3}t_2} \Psi_{n_1}(0)
\]

\[
\cdot \left( \int d\rho_1 \Psi_{n_1}^*(\rho_1) e^{i\frac{\xi}{2}\rho_1} \Psi_{n_2}(\rho_1) \right) \left( \int d\rho_2 \Psi_{n_2}^*(\rho_2) e^{i\frac{\xi}{2}\rho_2} \Psi_{n_3}(\rho_2) \right) \Psi_{n_3}^*(0)
\]

The function \( W(t) \) for \( t \to \infty \) behaves as \( W(t) \sim tW_0 \). It means that in the above stated sum in this limit only terms with \( n_1 = n_3 = 0 \) survive:

\[
H(\tau_1, \tau_2) \Rightarrow \sum_n e^{-(E_n-E_0)(\tau_1-\tau_2)}
\]

\[
\cdot \left( \int d\rho_1 \Psi_n^*(\rho_1) e^{i\frac{\xi}{2}\rho_1} \Psi_n(\rho_1) \right) \left( \int d\rho_2 \Psi_n^*(\rho_2) e^{i\frac{\xi}{2}\rho_2} \Psi_0(\rho_2) \right)
\]

\[
= \sum_n e^{-(E_n-E_0)(\tau_1-\tau_2)} C_{0n} \left( \frac{k}{2} \right) C_{0n}^* \left( -\frac{k}{2} \right)
\]

where

\[
C_{0n} \left( \frac{k}{2} \right) = \int dx \ \Psi_0(x) e^{i\frac{\xi}{2}x} \Psi_n(x)
\]

Finally, we have for \( t \to \infty \)

\[
W(t) = \int_0^t d\tau_1 \int_\tau_1^\tau_2 A(|\tau_1 - \tau_2|) \sum_n e^{-(\tau_1-\tau_2)(E_n-E_0)} C_{0n}^2 \left( \frac{k}{2} \right) = t \ W_0
\]

\[
W_0 = \int_0^\infty d\tau A(\tau) \sum_n e^{-\tau(E_n-E_0)} C_{0n} \left( \frac{k}{2} \right) C_{0n}^* \left( -\frac{k}{2} \right)
\]
Thus,

$$I = \int d\sigma \, e^{\Phi} = \Psi_0^2(0) \cdot e^{-t(E_0 - W_0 + O(\Phi^2))}$$

### 11.1 Spherically symmetric potentials

If the potential is spherically symmetric $U = U(|\rho|)$ then the spectrum $\{E_{n\ell}\}$ and the eigenfunctions are

$$\Psi_{n\ell m}(\rho) = R_{n\ell}(\rho)Y_{\ell m}(n), \quad \sum_m Y_{\ell m}^*(n)Y_{\ell m}(n) = \frac{2\ell + 1}{4\pi}$$

with ortho-normal conditions

$$\int d\rho \, \Psi_{n\ell m}^*(\rho)\Psi_{n'\ell m'}(\rho) = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'},$$

$$\sum_{n\ell m} \Psi_{n\ell m}(\rho)\Psi_{n\ell m}(\rho') = \delta(\rho - \rho') = \frac{1}{\rho^2}\delta(\rho - \rho')\delta(n - n')$$

For radial functions we get $(\rho = |\rho|)$

$$\int_0^{\infty} d\rho \rho^2 R_{n\ell}(\rho)R_{n'\ell}(\rho) = \delta_{nn'},$$

$$\sum_{n=0}^{\infty} R_{n\ell}(\rho)R_{n\ell}(\rho') = \frac{1}{\rho^2}\delta(\rho - \rho')$$

The form-factors looks like

$$C_{\ell n m}\left(\frac{k}{2}\right) = C_{000, n\ell m}\left(\frac{k}{2}\right) = \int d\rho \, \Psi_{000}(\rho)e^{i\frac{k}{2}\rho}\Psi_{n\ell m}(\rho)$$

$$= \int_0^{\infty} d\rho \rho^2 R_{00}(\rho)R_{n\ell}(\rho)\sqrt{\frac{4\pi}{\rho}} \cdot e^{i\frac{k}{2}\rho}Y_{\ell m}(n)$$

$$= \sqrt{4\pi}i^{\ell}\chi_{\ell m}(n)C_{\ell n}(\frac{k}{2}),$$

with

$$C_{\ell n}\left(\frac{k}{2}\right) = \int_0^{\infty} d\rho \rho^2 R_{00}(\rho)R_{n\ell}(\rho)j_\ell\left(\frac{k}{2}\rho\right)$$

$$j_\ell\left(\frac{k}{2}\rho\right) = \sqrt{\frac{\pi}{k\rho}}J_{\ell + \frac{1}{2}}\left(\frac{k}{2}\rho\right)$$
Finally we get

\[
W_0 = \int_0^\infty d\tau A(\tau) \sum_{n\ell} \int_0^1 e^{-\tau(E_{n\ell} - E_0)} (-1)^\ell (2\ell + 1) C_{n\ell}^2 \left( \frac{k}{2} \right)
\]

### 11.2 The Coulomb potential

In the representation (41) the Hamiltonian is

\[
H \Psi_n(\rho) = E_n \Psi_n(\rho), \quad H = -\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho},
\]

The Coulomb wave functions

\[
\Psi_{n\ell m}(\rho) = R_{n\ell}(\rho) Y_{\ell m}(\mathbf{n})
\]

are solutions of the non-relativistic Schrödinger equation \((E_n < 0)\)

\[
\left[ \frac{d^2}{d\rho^2} + \frac{2}{\rho} \frac{d}{d\rho} - \frac{\ell(\ell + 1)}{\rho^2} + \frac{1}{\rho} - \frac{|E_{n\ell}|}{\rho^2} \right] R_{n\ell}(\rho) = 0
\]

or according to [18] one can put \(\rho = \frac{r}{2\sqrt{|E_{n\ell}|}} = nr\) so that we have

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell(\ell + 1)}{r^2} + \frac{n}{r} - \frac{1}{4} \right] R_{n\ell}(r) = 0,
\]

where the spectrum is

\[
n = \frac{1}{\sqrt{4E_{n\ell}}} \quad \text{or} \quad E_{n\ell} = E_n = -\frac{1}{4n^2}
\]

with \(n = 1, 2, 3, ...\) and \(\ell = 1, ..., n - 1.\)

Solutions are

\[
R_{n\ell}(\rho) = \frac{1}{n^2(2\ell + 1)!} \sqrt{\frac{(n + \ell)!}{2(n - \ell - 1)!}} \left( \frac{\rho}{n} \right)^\ell e^{-\frac{\rho^2}{2n}} F \left( -n + \ell + 1, 2\ell + 2, \frac{\rho}{n} \right)
\]

\[
C_{n\ell}(k) = \int_0^\infty d\rho \rho^2 R_{00}(\rho) R_{n\ell}(\rho) j_\ell \left( \frac{k\rho}{2} \right)
\]
Several particular functions are

\[
C_{10}(k) = \frac{16}{(4 + k^2)^2}, \\
C_{20}(k) = \frac{256\sqrt{2} k^2}{(9 + 4k^2)^3}, \quad C_{21}(k) = \frac{128\sqrt{6} k}{(9 + k^2)^3} \\
C_{30}(k) = \frac{432\sqrt{3} k^2(16 + 27k^2)}{(16 + 9k^2)^4}, \quad C_{31}(k) = \frac{288\sqrt{6} k(16 + 27k^2)}{(16 + 9k^2)^4}, \quad C_{32}(k) = \sqrt{5} \cdot \frac{6912 k^2}{(16 + 9k^2)^4}
\]

11.3 ”Time” potential

We have the integral

\[
J_X(\alpha) = \frac{e^{\frac{X}{4}}}{8\pi} I_X(\alpha),
\]

\[
I_X(\alpha) = \int \frac{D\rho}{C} e^{-\frac{1}{4} \int_0^X d\tau \dot{\rho}^2(\tau) + U[\rho, \alpha]},
\]

\[
U[\rho, \alpha] = W[\rho; \alpha] - W[\rho; 0].
\]

In the paper [] it is shown that for small \(\alpha\)

\[
U[\rho, \alpha] = W[\rho; \alpha] - W[\rho; 0] = -\frac{\alpha}{8} \int_0^X d\tau |\rho(\tau)|.
\]

The last integral corresponds to the one-dimensional non-relativistic quantum system with the Lagrangian

\[
L = \frac{\dot{\rho}^2}{4} - \frac{\alpha}{8} |\rho|
\]

for which the Hamiltonian reads

\[
H = \rho^2 + \frac{\alpha}{8} |\rho|
\]

The Schrödinger equation looks as

\[
\left[ -\frac{d^2}{d\rho^2} + \frac{\alpha}{8} |\rho| \right] \Psi(\rho) = \mathcal{E} \Psi(\rho)
\]
Let us introduce
\[ \rho = \frac{2v}{\alpha^\frac{3}{2}}, \quad \mathcal{E} = \frac{\alpha^\frac{3}{4}}{4} \epsilon \]
then \( \epsilon \) is the eigenvalue of the equation
\[ \left[ -\frac{d^2}{dv^2} + v \right] Y(v, \epsilon) = \epsilon Y(v, \epsilon), \quad v \in [0, \infty). \]

The non-normalized solution of the equation looks like
\[ Y(v, \epsilon) = \begin{cases} 
\pi \sqrt{\frac{\epsilon - v}{3}} \left[ J_{\frac{1}{2}} \left( \frac{2}{3}(\epsilon - v)^{\frac{3}{2}} \right) + J_{-\frac{1}{2}} \left( \frac{2}{3}(\epsilon - v)^{\frac{3}{2}} \right) \right], & v < \epsilon \\
\sqrt{v - \epsilon} K_{\frac{1}{2}} \left( \frac{2}{3}(v - \epsilon)^{\frac{3}{2}} \right), & v > \epsilon 
\end{cases} \]
The spectrum is defined by the equations
\[ \frac{d}{dv} Y(v, \epsilon_{2n}) \bigg|_{v=0} = 0 \quad \text{for even states} \quad n \to 2n \quad (47) \]
\[ Y(0, \epsilon_{2n+1}) = 0 \quad \text{for odd states} \quad n \to 2n + 1. \]
The even eigenfunctions \( \Phi_{2n}(v) = Y(v, \epsilon_{2n}) \) should have \( n \) zeros and the odd eigenfunctions \( \Phi_{2n+1}(v) = Y(v, \epsilon_{2n+1}) \) should have one zero for \( v = 0 \) and \( n \) zeros for \( 0 < v < \infty \).

The wave functions are
\[ \Phi_n(\rho) = \sqrt{\frac{\alpha^{\frac{1}{2}}}{4N_n}} Y\left( \frac{\alpha^{\frac{1}{2}}}{2} \rho, \epsilon_n \right), \quad \rho \in (-\infty, \infty) \]
The form-factors are defined like
\[ A_n \left( \frac{q}{2} \right) = \int_{-\infty}^{\infty} d\rho \ e^{i\frac{q}{2} \rho} \Phi_0(\rho) \Phi_n(\rho) \]
\[ = \frac{\alpha^{\frac{3}{4}}}{4\sqrt{N_0 N_n}} \int_{-\infty}^{\infty} d\rho \ e^{i\frac{q}{2} \rho} Y\left( \frac{\alpha^{\frac{1}{2}}}{2} \rho, \epsilon_0 \right) Y\left( \frac{\alpha^{\frac{1}{2}}}{2} \rho, \epsilon_n \right) \]
\[ = \left\{ \begin{array}{l}
\frac{1}{\sqrt{N_0 N_{2n}}} \int_0^\infty dv \ \cos \left( \alpha^{\frac{3}{2}}qv \right) Y(v, \epsilon_0) Y(v, \epsilon_{2n}) \\
\frac{i}{\sqrt{N_0 N_{2n+1}}} \int_0^\infty dv \ \sin \left( \alpha^{\frac{3}{2}}qv \right) Y(v, \epsilon_0) Y(v, \epsilon_{2n+1})
\end{array} \right. \]
\[ = A_n \left( \alpha^{\frac{3}{2}} q \right), \]
with the symmetry condition
\[ A_n(-\alpha^2 q) = (-1)^n A_n(\alpha^2 q), \]
For \( \alpha \ll 1 \) form-factors behave like
\[ A_0(\alpha^2 q) = 1 - \alpha^2 a_0, \quad a_0 = \int_0^\infty dv \frac{v^2}{2} \frac{Y^2(v, \epsilon_0)}{N_0} = 0.374939, \]
\[ A_1(\alpha^2 q) = i\alpha^2 q a_1, \quad a_1 = \int_0^\infty dv \int_0^\infty Y(v, \epsilon_0) Y(v, \epsilon_1) \sqrt{N_0 N_1} = 0.862863, \]
\[ A_2(\alpha^2 q) = \alpha^4 q^2 a_2, \quad a_2 = \int_0^\infty dv \int_0^\infty Y(v, \epsilon_0) Y(v, \epsilon_2) \sqrt{N_0 N_2} = 0.569709, \]
\[ A_3(\alpha^2 q) = i\alpha^2 q a_3, \quad a_3 = \int_0^\infty dv \int_0^\infty Y(v, \epsilon_0) Y(v, \epsilon_3) \sqrt{N_0 N_3} = -0.0685378, \]
The next integral for \( t \rightarrow \infty \) behaves like
\[
W(t) = \int \frac{D\rho}{NC} e^{-\int_0^t d\tau [\frac{1}{2}p^2(\tau) - U(\rho(\tau))]} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H(|\tau_1 - \tau_2|) e^{i\frac{\alpha q^2}{2}(\rho(\tau_1) + \rho(\tau_2))} \]
\[
= \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H(|\tau_1 - \tau_2|) \]
\[
\cdot \frac{1}{N} \int d\rho_1 \int d\rho_2 \; G_{t-\tau_1} (0, \rho_1) e^{i\frac{\alpha q^2}{2}\rho_1} G_{t-\tau_2} (\rho_1, \rho_2) e^{i\frac{\alpha q^2}{2}\rho_2} G_{\tau_2} (\rho_2, 0) \]
\[
\Rightarrow \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H(|\tau_1 - \tau_2|) \sum_\kappa e^{-(\tau_1 - \tau_2)(E_\kappa - E_0)} A_\kappa(\alpha^2 q) A_\kappa^* (-\alpha^2 q) = t W_0 \]
where
\[
W_0 = \int_0^\infty d\tau \; H(\tau) \sum_\kappa e^{-\tau(E_\kappa - E_0)} A_\kappa(\alpha^2 q) A_\kappa^* (-\alpha^2 q) \]
Thus, for \( t \rightarrow \infty \) one can write
\[
\langle e^{i\frac{\alpha q^2}{2}(\rho(\tau_1) + \rho(\tau_2))} \rangle = \sum_\kappa (-1)^\kappa e^{-\frac{1}{2}(|\tau_1 - \tau_2|\alpha^2 (\epsilon_\kappa - \epsilon_0))} |A_\kappa(\alpha^2 q)|^2 \quad (48) \]
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Table 1. QM and QFT.

| Quantum Mechanics | Quantum Field Theory |
|-------------------|---------------------|
| \[ H = H_0 + g H_I \] | \[ H_0 \Psi_E^{(0)} = E \Psi_E^{(0)} \] |
| \( (H_0 + g H_I) \Psi_E = E \Psi_E \) | \( Fock \ space = \{ free \ particles \} \) |
| \( \{ \Psi_E \} = \{ free \ particles \} \oplus \{ bound \ states \} \) | \( H_I \) is not operator on Fock space |
| \( \Psi(t) = e^{-iH(t-t_0)} \Psi(t_0) \) Development in time | \( S = \lim_{t_0 \to -\infty} e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} \oplus \) renormalization |
| intermediate particles ON mass shell | \( \implies S \) is operator on Fock space Relation between asymptotically free states |

Relativistic corrections

| interaction is transmitted instantly | retarded interaction |
|-------------------------------------|----------------------|
| Small corrections to \( H \) Elimination TIME out of Hamiltonian Effective theories | \( A_{in \to out}(p_1, p_2; k_1, k_2) \sim \frac{1}{M^2 - (p_1 + p_2)^2} \) |
| Breit potential Nonrelativistic QED | \( \langle 0 | J(x) J(0) | 0 \rangle \sim e^{-M|x|}, \ |x| \to \infty \) Bethe-Salpeter equation |
| classification of states \( \mathbb{R}^4 \) | classification of states \( \mathbb{R}^4 = \mathbb{R}^3 \otimes \mathbb{R}_t \) "Time" excitations \( \Rightarrow \) abnormal states |
Table 2. Quantum numbers of relativistic currents

| $J$ | $(\bar{\psi}O_J\psi)$ | $S$ | $L$ | $J$ | $P = (-1)^{1+L}$ | $J^P$ |
|-----|-----------------|-----|-----|-----|----------------|-------|
| $S$ | $(\bar{\psi}\psi) \Rightarrow (\sigma k)$ | 1 | 1 | 0 | +1 | 0$^+$ |
| $V$ | $(\bar{\psi}\gamma_0\psi) \Rightarrow (\sigma k)$ | 1 | 1 | 0 | +1 | 0$^+$ |
|     | $(\bar{\psi}\gamma_0\psi) \Rightarrow \sigma$ | 1 | 0 | 1 | −1 | 1$^-$ |
| $T$ | $(\bar{\psi}\gamma_0\gamma^0\psi) \Rightarrow \sigma$ | 1 | 0 | 1 | −1 | 1$^-$ |
|     | $(\bar{\psi}\sigma_{ij}\psi) \Rightarrow [\sigma \times k]$ | 1 | 1 | 1 | +1 | 1$^+$ |
| $A$ | $(\bar{\psi}\gamma_5\gamma_0\gamma^0\psi) \Rightarrow 1$ | 0 | 0 | 0 | −1 | 0$^-$ |
|     | $(\bar{\psi}\gamma_0\gamma^5\psi) \Rightarrow [\sigma \times k]$ | 1 | 1 | 1 | +1 | 1$^+$ |
| $P$ | $(\bar{\psi}\gamma_5\gamma^5\psi) \Rightarrow 1$ | 0 | 0 | 0 | −1 | 0$^-$ |
Table 3. Functions $\Sigma_{\Gamma}^{(0)}$ and $\Sigma_{\Gamma}(k)$

| $J^P$  | $\Gamma$                              | $\Sigma_{\Gamma}^{(0)}$ | $\Delta_{\Gamma}(k)$ |
|--------|---------------------------------------|--------------------------|-----------------------|
| $P(0^-)$ | $i\gamma_5 \cos \theta_P + \gamma_5 \gamma_0 \sin \theta_P$ | $-2$                     | $-2 \cdot \frac{k^2}{k^2 + k_f^2}$ |
| $V(1^-)$ | $\gamma_j \cos \theta_V + i \gamma_0 \gamma_j \sin \theta_V$ | $-2\delta_{ij}$          | $\frac{2}{3} \cdot \frac{k^2}{k^2 + k_f^2}$ |
| $S(0^+)$ | $I \cos \theta + \gamma_0 \sin \theta$ | 0                        | 0                     |
| $A((1^+))$ | $\gamma_5 \gamma \cos \theta + i[\gamma \times \gamma] \sin \theta$ | 0                        | 0                     |

Table 4. The function $\Delta_{n\ell 0}(\alpha)$

| $(n\ell)$ | (10) | (20) | (21) | (30) | (31) | (32) |
|-----------|------|------|------|------|------|------|
| $\Delta_{n\ell 0}(0)$ | 1.   | 0.0987 | 0.0987 | 0.0283 | 0.0307 | 0.00244 |
| $\Delta_{n\ell 0}(\alpha)$ | 0.9999 | 0.09868 | 0.09864 | 0.02829 | 0.03071 | 0.002438 |
| $\Delta_{n\ell 1}(\alpha)$ | 0.96707 | 0.09387 | 0.09617 | 0.2683 | 0.002986 | 0.00238 |
| $\Delta_{n\ell 2}(\alpha)$ | 0.01421 | 0.002089 | 0.001173 | 0.0006319 | 0.0004049 | 0.00002632 |

Table 5. Norm $N_n$

| $n$ | $\epsilon_n$ | $N_n = \int_0^\infty dv \Phi_n^2(v)$ |
|-----|--------------|-------------------------------------|
| 0   | 1.0188       | 8.655                               |
| 1   | 2.3381       | 14.558                              |
| 2   | 3.2482       | 16.886                              |
| 3   | 4.0879       | 19.097                              |
| 4   | 4.8201       | 20.652                              |