A NOTE ON NODAL DETERMINANTAL HYPERSURFACES

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ABSTRACT. We prove that a general determinantal hypersurface of dimension 3 is nodal. Moreover, in terms of Chern classes associated with bundle morphisms, we derive a formula for the intersection homology Euler characteristic of a general determinantal hypersurface.

1. INTRODUCTION

A large and important class of varieties is the class of determinantal varieties. These are a particular case of degeneracy loci $D_i(\sigma)$ associated with a morphism $\sigma : \mathcal{E} \to \mathcal{F}$ of vector bundles on $M$, which is locally defined by the ideal generated by all $(i+1)$-minors of a matrix for $\sigma$.

On the other hand, it is well known that a hypersurface $x_0g_\delta(x_0, \ldots, x_4) + x_1f_\delta(x_0, \ldots, x_4) = 0$ in $\mathbb{P}^4$ is a nodal determinantal hypersurface, that is, all its singular points are ordinary double points (ODPs). Here, $g_\delta$ and $f_\delta$ are general polynomials of degree $\delta$. In this note, we generalize this to a general determinantal hypersurface of dimension 3.

To state our result, we need an appropriate notion of the generality of morphisms. Suppose that $M$ is a smooth proper variety and $\mathcal{E}, \mathcal{F}$ have the same rank $n + 1$. A morphism $\sigma : \mathcal{E} \to \mathcal{F}$ is said to be $n$-general if for all $0 \leq i \leq n$, the subset $D_i(\sigma) \setminus D_{i-1}(\sigma)$ is smooth of codimension $(n + 1 - i)^2$ in the smooth proper variety $M$ (see Definition 3.1).

Theorem 1.1 (Theorem 4.4). Suppose that the smooth proper variety $M$ has dimension 4. If a morphism $\sigma$ is $n$-general, then the determinantal hypersurface $D_n(\sigma)$ is nodal, provided that it is irreducible.

This gives us a way to construct 3-dimensional Calabi–Yau (or Fano) hypersurfaces with ODPs (see Example 4.7 and 4.8) admitting a natural small resolution (see (3.1) and Proposition 3.6).

To prove Theorem 1.1 we will give a formula for the intersection homology Euler characteristic of $D_n(\sigma)$, for a $n$-general morphism $\sigma$. The global topology of the determinantal hypersurface imposes strong restrictions on the singularities $D_n(\sigma)$ can have.

We set $\mathcal{L} := \det \mathcal{E}^\vee \otimes \det \mathcal{F}$. Note that the maximal degeneracy locus $D_n(\sigma)$ is defined by $\det(\sigma)$ associated to a global section of $\mathcal{L}$. It is known
that
\[ \chi(M|L) := \int_M c_1(L)(1 + c_1(L))^{-1}c(TM) \]
is the topological Euler characteristic of the smooth hypersurface defined by a global section of \(L\) (cf. [10, Example 4.2.6]).

Let \(d := \dim M \geq 4\). We let \(\chi_{IH}(\sigma)\) denote the intersection homology Euler characteristic. As an analogy of the generalized Milnor number of singular hypersurfaces in [20], we write
\[ \mu_{IH}(D_n(\sigma), M) := (-1)^d(\chi_{IH}(D_n(\sigma)) - \chi(M|L)) \]
In terms of Schur polynomials \(s_\lambda(\mathcal{F} - \mathcal{E})\) (see Notation 2.4) we prove:

**Theorem 1.2** (Theorem 4.1). If \(\sigma : E \to \mathcal{F}\) is \(n\)-general, then
\[ \mu_{IH}(D_n(\sigma), M) = \sum_{\lambda \supseteq (2,2)} (-1)^{d+|\lambda|} f^\lambda \int_M s_\lambda(\mathcal{F} - \mathcal{E}) c_{d-|\lambda|}(TM), \]
where \(f^\lambda\) denotes the number of standard Young tableaux of shape \(\lambda\).

Here, we write \(\lambda \supseteq \mu\) if the Young diagram of \(\mu\) is contained in that of \(\lambda\) and the integer \(|\lambda|\) for the total number of boxes of the diagram \(\lambda\). We adopt the convention that \(c_i(TM) = 0\) for \(i < 0\).

In the following cases, \(\mu_{IH}(D_n(\sigma), M)\) is expressed in terms of the singular locus of \(D_n(\sigma)\), being \(D_{n-1}(\sigma)\) (see Remark 3.3), for a \(n\)-general \(\sigma\).

**Corollary 1.3** (Corollary 4.2). Assume that the smooth proper variety \(M\) satisfies one of the following assumptions (1) \(\dim M = 4\), or (2) \(\dim M = 5\) and
\[ (1.1) \quad c_1(TM) = c_1(\mathcal{F}) - c_1(\mathcal{E}). \]
If \(\sigma\) is \(n\)-general, then
\[ \mu_{IH}(D_n(\sigma), M) = (\dim M - 2) \int_M c(TM) \cap [D_{n-1}(\sigma)]. \]

The equality (1.1) is called the Calabi–Yau condition.

The above results are motivated by considering the special case where the bundle \(\mathcal{F}\) is trivial and \(\mathcal{E}\) is a direct sum of line bundles. We proved the case [1] in this situation [22, Proposition 3.4]. It supply a proof of the famous result that all 3-dimensional Calabi–Yau complete intersections in product of projective spaces are connected through conifold transitions (see [12] or [22, Theorem 5.6]). When the rank of \(\mathcal{F}\) is two, the case [2] was proved in [2] for \(M\) being a complete intersection in a product of projective spaces.

The paper is organized as follows. In Section 2 we review some basic materials on Young diagrams, Schur polynomials and intersection cohomology groups. Section 3 contains a brief exposition of the required \(n\)-generality of degeneracy loci. We conclude with Section 4 where we prove main results and derive interesting formulas (see Proposition 4.5) for maximal degeneracy loci under the Calabi–Yau condition.
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2. Preliminaries

Throughout this paper all varieties are irreducible and defined over complex field $\mathbb{C}$.

2.1. Young Diagrams. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ be a partition of size $|\lambda| = \sum \lambda_i$, identified with its Young diagram. It consists of $|\lambda|$ boxes arranged in $k$ left adjusted rows of length $\lambda_1, \cdots, \lambda_k$. We abbreviate $(i, \cdots, i)$ ($k$-times) to $(i)^k$ and $(\lambda_1, \lambda_2, \cdots, i + \mu_1, \cdots, i + \mu_k)$ to $\lambda, (i)^k + \mu$. If $\lambda$ and $\mu$ are two Young diagrams, we write $\mu \subseteq \lambda$ if the diagram of $\mu$ is contained in that of $\lambda$, that is, $\mu_i \leq \lambda_i$ for all $i$.

A standard Young tableau with shape $\lambda$ is a numbering of a Young diagram $\lambda$ with $1, 2, \cdots, |\lambda|$ in which the numbers appear in ascending order within each row or column from left to right and top to bottom. Let $f^\lambda$ denote the number of standard Young tableaux of shape $\lambda$. The numbers $f^\lambda$ satisfy the following inductive formula
\begin{equation}
    f^\mu = \sum_{\lambda \subseteq \mu, |\lambda| = |\mu|-1} f^\lambda,
\end{equation}

since giving a standard tableau with $n$ boxes is the same as giving one with $n-1$ boxes and saying where to put the $n$th box.

For a Young diagram, each box determines a hook, which consists of that box and all boxes in its column below the box or in its row to right of the box. The hook length of a box is the number of boxes in its hook. Let $h(\lambda)$ be the product of the hook lengths of all boxes in $\lambda$. We have the following closed formula for the number $f^\lambda$, the hook length formula [9 p. 53].

**Proposition 2.1 ([9]).** If $\lambda$ is a Young diagram with $n$ boxes, then the number $f^\lambda$ of standard Young tableaux with shape $\lambda$ is $n!$ divided by the product of the hook lengths of the boxes. In particular, $f^\lambda = \binom{n-1}{k-1}$ for each hook $\lambda = (k, 1, \cdots, 1)$.

Let us illustrate the above definitions and notations with an example.

**Example 2.2.** $\lambda = (3, 2)$ is drawn as $\begin{array}{c|c|c} 1 & 2 & 3 \\ \hline 4 & 5 \\ \end{array}$. The number $f^\lambda$ of standard Young tableaux is 5, which has the inductive formula $f^{(3,2)} = f^{(3,1)} + f^{(2,2)}$.

\begin{center}
\begin{tabular}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 4 \\ \hline 1 & 3 & 4 \\ \hline 1 & 2 & 5 \\ \hline 1 & 3 & 5 \\ \hline 4 & 5 \\ \hline 3 & 5 \\ \hline 2 & 5 \\ \hline 3 & 4 \\ \hline 2 & 4 \\ \hline
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The product of the hook lengths of all boxes in $\lambda$ is $h(\lambda) = 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 24$.

2.2. **Schur Polynomials.** Let $\{c_i\}_{i \in \mathbb{N}}$ and $\{s_i\}_{i \in \mathbb{N}}$ be commuting variables and $c_0 = s_0 = 1$, related by the identity $(\sum_{t=0}^{\infty} c_t^i) \cdot (\sum_{t=0}^{\infty} (-1)^i s_t^i) = 1$.

**Example 2.3.** For vector bundles $F$, Schur Polynomials.

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Example 2.6. Let $V$ be a proper variety of dimension $m$ with isolated singularities and let $\Sigma = \operatorname{Sing}(V)$. For the lower middle perversity, we have the following description of intersection cohomology groups in terms of ordinary ones (cf. [16, Proposition 4.4.1]):

$$IH^i(V) = \begin{cases} 
H^i(V) & i > m, \\
\operatorname{Im}(H^i(V) \to H^i(V \setminus \Sigma)) & i = m, \\
H^i(V \setminus \Sigma) & i < m.
\end{cases}$$

For an isolated singularity $a \in V$, let us denote by $\mu_0(V, a)$ the $(m-1)$th Betti number of the link of the germ of the singularity $(V, a)$. The following proposition is a straightforward generalization of [7, (6.4.23)], where it is stated for hypersurfaces.

Proposition 2.7 ([7]). Let $m \geq 3$ and $V$ be an $m$-dimensional proper variety with only isolated complete intersection singularities. Then

$$\chi_{IH}(V) = \chi(V) + (-1)^{m+1} \sum_{a \in \operatorname{Sing} V} \mu_0(V, a),$$

where $\chi_{IH}(-)$ is the intersection homology Euler characteristic.

For the convenience of the reader, we include a proof.

Proof. Let $\Sigma = \operatorname{Sing}(V)$, and let $h^i(V), h^i_{\Sigma}(V)$ and $Ih^i(V)$ be the dimension of $H^i(V), H^i_{\Sigma}(V)$ and $IH^i(V)$ respectively. According to Example 2.6 and the following long exact sequence

$$\cdots \to H^i_{\Sigma}(V) \to H^i(V) \to H^i(V \setminus \Sigma) \to \cdots \to H^m(V) \to IH^m(V) \to 0,$$

it follows that

$$\chi_{IH}(V) - \chi(V) = \sum_{i=0}^{m} (-1)^i (h^i(V \setminus \Sigma) - h^i(V)) + (-1)^m (IH^m(V) - h^m(V))$$

$$= \sum_{i=0}^{m} (-1)^{i+1} h^i_{\Sigma}(V) = \sum_{i=0}^{m} (-1)^{i+1} \left( \sum_{a \in \Sigma} h^i_{\{a\}}(V) \right).$$

Since $V$ has only isolated singularities, the $(i-1)$th reduced cohomology group of the link of the germ $(V, a)$ is isomorphic to $H^i_{\{a\}}(V)$ (by replacing $V$ with a contractible open neighborhood of $a \in \Sigma$). Then the proposition follows from the fact that the link of $(V, a)$, being a local complete intersection singularity, is $(m-2)$-connected (see [13, Corollary 1.3]).

\[\Box\]

Remark 2.8. For an isolated rational singularity $(V, a)$, the number $\mu_0(V, a)$ measures how far $(V, a)$ is from being $\mathbb{Q}$-factorial. Indeed, we have the defect

$$\sigma(V, a) := \operatorname{rank}(\operatorname{Weil}(V, a)/\operatorname{Cart}(V, a))$$

of the germ $(V, a)$, where $\operatorname{Weil}(V, a)$ (resp. $\operatorname{Cart}(V, a)$) is the Abelian group of Weil (resp. Cartier) divisors of $(V, a)$. It is a finite number [15, Lemma 1.12]. If $\sigma(V, a) = 0$, the germ $(V, a)$ is called $\mathbb{Q}$-factorial. On the other hand, the defect $\sigma(V, a)$ equals $h^m_{\{a\}}(V)$ (cf. the proof of [18, Proposition 3.10]) and thus equals...
the \((m - 1)\)th Betti number \(\mu_0(V, a)\) of the link of \((V, a)\) as seen in the proof of Proposition 2.7.

We recall the definition of small morphisms in the sense of Goresky and MacPherson [11].

**Definition 2.9.** Let \(\pi : W \to V\) be a proper morphism of varieties. For every integer \(k \geq 1\) we define

\[
V^k := \{ x \in V \mid \dim \pi^{-1}(x) = k \}.
\]

We say that \(\pi\) is small if \(\dim W - \dim V^k > 2k\) for each \(k\).

**Remark 2.10.** Suppose that \(V\) is a 3-dimensional projective variety with isolated rational singularities that admits a small resolution \(\pi : W \to V\). Then the exceptional set of the small resolution \(\pi\) consists of irreducible (rational) curves, and the defect \(\sigma(V, a)\), as well as \(\mu_0(V, a)\), equals the number of the irreducible components of \(\pi^{-1}(a)\) (see [15, Lemma 3.4]).

The following proposition connects the topology of \(W\) and \(V\).

**Proposition 2.11 ([11, §6.2]).** If a proper morphism \(\pi : W \to V\) is a small resolution, then \(\text{IH}^i(V) = \text{IH}^i(W) = H^i(W)\) for all \(i\).

In particular, the topological Euler characteristic \(\chi(W)\) of the smooth variety \(W\) equals the intersection homology Euler characteristic \(\chi_{\text{IH}}(V)\) of \(V\).

### 2.4. Chern Classes

We collect some basic material from Fulton’s book [10] that we will need in Section 3.

**Lemma 2.12.** Let \(\mathcal{E}\) and \(\mathcal{F}\) be vector bundles with the same rank \(r\) and \(\mathcal{L}\) a line bundle. Set \(\xi = c_1(\mathcal{L})\). Then for each integer \(k \geq 1\)

\[
(2.3) \quad c_k(\mathcal{F} \otimes \mathcal{L} - \mathcal{E} \otimes \mathcal{L}) = \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} c_i(\mathcal{F} - \mathcal{E}) \xi^{k-i},
\]

\[
(2.4) \quad c_r(\mathcal{E} \otimes \mathcal{L}) = \sum_{i \geq 0} c_i(\mathcal{E}) \xi^{-i}.
\]

**Proof.** By [10, Example 3.2.2], the Chern polynomial of \(\mathcal{F} \otimes \mathcal{L}\) is

\[
c_i(\mathcal{F} \otimes \mathcal{L}) = (1 + \xi t)^r c_{\tau(t)}(\mathcal{F})
\]

where \(\tau(t) = t/(1 + \xi t)\), and similarly for \(\mathcal{E} \otimes \mathcal{L}\). Then, by substituting one into \(t\), we get

\[
(2.5) \quad c(\mathcal{F} \otimes \mathcal{L} - \mathcal{E} \otimes \mathcal{L}) = c_{\tau(1)}(\mathcal{F}) / c_{\tau(1)}(\mathcal{E}) = \sum_{i \geq 0} c_i(\mathcal{F} - \mathcal{E}) (1 + \xi)^{-i}.
\]

The equation (2.3) follows from the binomial expansion of \((1 + \xi)^{-i}\) and (2.5). From the splitting principle, it is easy to get the equation (2.4) (see [10, Remark 3.2.3 (b)]).
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Let \( \mathcal{V} \) be a vector bundle of rank \( r \) on a smooth proper variety \( M \). We say that a subvariety \( V \) of \( M \) is a \textit{complete intersection} with respect to the vector bundle \( \mathcal{V} \) if it is the zero scheme of a regular section of \( \mathcal{V} \), that is, \( V \) has codimension \( r \) in \( M \) (see [10] B.3.4]). Then the fundamental class of \( V \) is

\[
[V] = c_r(\mathcal{V}) \cap [M]
\]

(see [10] Example 3.2.16]) and the restriction \( \mathcal{V}|_V \) is the normal bundle of \( V \) in \( M \). The \textit{Fulton class} of \( V \) is defined by

\[
c_F(V) := c(T_M|_V - \mathcal{V}|_V) \cap [V],
\]

where \( T_M \) is the tangent bundle of \( M \) (cf. [10] Example 4.2.6]). We will denote by \( \chi(M|_V) \) the degree of 0-dimensional component of the Fulton class \( c_F(V) \) and

\[
\mu_{IH}(V, M) := (-1)^{\dim M}(\chi_{IH}(V) - \chi(M|_V)).
\]

We will write \( \mu_{IH}(V) \) instead of \( \mu_{IH}(V, M) \) when no confusion can arise.

**Remark 2.13.** If we suppose that \( V \) is smooth, at least a rational homology manifold, then \( \mu_{IH}(V) \) is equal (up to a sign) to the degree of the 0-dimensional components of the (motivic) Milnor class of \( V \) (see [23] and references therein).

The following proposition is probably well known. For the readers convenience, we provide a proof of this proposition.

**Proposition 2.14.** Let \( m \geq 3 \) and \( V \) be a \( m \)-dimensional complete intersection with respect to \( \mathcal{V} \). If \( V \) has only isolated singularities, then

\[
\mu_{IH}(V) = (-1)^{\text{rank } \mathcal{V} - 1} \sum_{a \in \text{Sing}(V)} (\mu_0(V, a) + \mu(V, a)),
\]

where \( \mu(V, a) \) is the Milnor number and \( \mu_0(V, a) \) is the \((m - 1)\)th Betti number of the link of the singularity \((V, a)\).

**Proof.** Note that \( m = \dim M - \text{rank } \mathcal{V} \). By [21] Theorem 2.4], we have

\[
\chi(V) = \chi(M|_V) + (-1)^{m+1} \sum_{a \in \text{Sing}(V)} \mu(V, a).
\]

The formula (2.7) follows from Proposition 2.7 and the definition of \( \mu_{IH}(V) \). \( \square \)

**Remark 2.15.** For a hypersurface \( V \) with (possibly nonisolated) singularities, there is a formula for \( \mu_{IH}(V) \) in terms of a stratification of the singular locus (cf. [17] Corollary 4.4).

3. DETERMINANTAL CONTRACTIONS

Let \( \sigma : \mathcal{E} \to \mathcal{F} \) be a morphism of vector bundles of ranks \( e \) and \( f \) on a smooth proper variety \( M \). Note that there is a natural bijection between morphisms \( \mathcal{E} \to \mathcal{F} \) and global sections of \( \mathcal{E}^\vee \otimes \mathcal{F} \).
For $i \leq \min(e, f)$, we define the $i$th degeneracy locus of $\sigma$ by
$$D_i(\sigma) = \{ x \in M \mid \text{rank}(\sigma(x)) \leq i \}$$
with the convention $D_{-1}(\sigma) = \emptyset$. Its ideal is locally generated by $(i + 1)$-minors of a matrix for $\sigma$. Notice that the $0$th degeneracy locus of $\sigma$ is the zero scheme $Z(\sigma)$. The codimension of $D_i(\sigma)$ in $M$ is less than or equal to $(e - i)(f - i)$ (see Remark 3.2 below or [10] Theorem 14.4 (b)), which is called its expected codimension.

**Definition 3.1 ([19])**. For a given integer $r \geq 0$, we say that $\sigma$ is $r$-general if for every $i = 0, 1, \cdots, r$, the subset $D_i(\sigma) \setminus D_{i-1}(\sigma)$ is smooth of (expected) codimension $(e - i)(f - i)$ in the smooth variety $M$.

Before proceeding further, let us remark on the generality of morphisms.

**Remark 3.2.** Parusiński and Pragacz [19] defined the notion of $r$-general morphisms for an arbitrary variety $M$ with a fixed Whitney stratification $S$. Indeed, consider the vector bundle $\text{Hom}(\mathcal{E}, \mathcal{F})$ and the universal degeneracy loci $D_i$ whose fiber over $x \in M$ consists of all homomorphisms $\mathcal{E}(x) \to \mathcal{F}(x)$ of rank less than or equal to $i$. The codimension of $D_i$ in $\text{Hom}(\mathcal{E}, \mathcal{F})$ is $(e - i)(f - i)$ and each $D_i \setminus D_{i-1}$ is smooth. If $\sigma : \mathcal{E} \to \mathcal{F}$ is viewed as
$$s_{\sigma} : M \to \text{Hom}(\mathcal{E}, \mathcal{F})$$
a section of the vector bundle $\text{Hom}(\mathcal{E}, \mathcal{F})$, then the degeneracy locus $D_i(\sigma)$ is the pullback $s_{\sigma}^{-1}(D_i)$, which is isomorphic to the intersection of $s_{\sigma}(M)$ with $D_i$, and thus it has codimension at most $(e - i)(f - i)$ in $M$. Then $\sigma$ is $r$-general in the sense of [19] if the section $s_{\sigma}$ induced by $\sigma$ intersects, on each stratum of $S$, the subset $D_i \setminus D_{i-1}$ transversely for $i = 0, 1, \cdots, r$. It coincides with Definition 3.1 if $M$ is smooth [19] Lemma 2.9.

**Remark 3.3.** According to Definition 3.1, it follows that for $i = 0, \cdots, r$, the singular locus of $D_i(\sigma)$ is contained in $D_{i-1}(\sigma)$ if $\sigma$ is $r$-general. Moreover, the inclusion is an equality. Indeed, since the statement is local, we pick a connection on the vector bundle $\text{Hom}(\mathcal{E}, \mathcal{F})$. Then
$$\nabla(\wedge^{i+1}s_{\sigma}) = (i + 1)(\nabla s_{\sigma}) \wedge (\wedge^i s_{\sigma}),$$
by the Leibniz rule. Therefore the derivative of $\wedge^{i+1}s_{\sigma}$ vanishes along $D_i(\sigma)$, that is, $\text{Sing}(D_i(\sigma)) \supseteq D_{i-1}(\sigma)$.

If the $(r - 1)$th degeneracy locus has codimension exactly $\dim M$, then the number of points of it is the degree of $[D_{n-1}(\sigma)]$, because $s_{\sigma}(M)$ and $D_{r-1}$ intersect transversely.

From now on we assume that $\mathcal{E}$ and $\mathcal{F}$ have the same rank $n + 1$.

**Definition 3.4.** For a morphism $\sigma : \mathcal{E} \to \mathcal{F}$, we call the maximal degeneracy locus $D_n(\sigma)$ a determinantal hypersurface if it has the expected codimension (which equals one).

Note the $D_n(\sigma)$ is the zero scheme of the section $\det \sigma$ of the line bundle $\det \mathcal{E}^\vee \otimes \det \mathcal{F}$.
Example 3.5. The prototypical setting is where $M = \mathbb{P}^d$ and the vector bundles are sums of line bundles. A morphism of rank $n + 1$ vector bundles

$$\sigma : \bigoplus \mathcal{O}_{\mathbb{P}^d}(-a_i) \to \bigoplus \mathcal{O}_{\mathbb{P}^d}(b_j)$$

is represented as a $(n + 1) \times (n + 1)$ matrix $[\sigma_{ij}(z)]$ of homogeneous polynomials such that $a_i + b_j$ is the degree of $\sigma_{ij}(z)$ for $1 \leq i, j \leq n + 1$. Then $n$th degeneracy locus $D_n(\sigma)$ is defined by the determinant of the matrix $[\sigma_{ij}(z)]$.

To get small resolutions of maximal degeneracy loci, we consider the following geometric construction (cf. [10, Example 14.4.10]). Let $\mathbb{P}(\mathcal{F})$ be the projective bundle with the structure morphism $p$. Fix a morphism $\sigma$ and let $Z_n(\sigma)$ denote the zero scheme in $\mathbb{P}(\mathcal{F})$ of the global section of $p^*\mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ induced by the composition of $p^*\sigma$ with $p^*\mathcal{F} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

We can interpret $Z_n(\sigma)$ as the projectivization of the cokernel sheaf $\mathcal{E}$ of $\sigma$. Indeed, thinking of $\mathbb{P}(\mathcal{F})$ as the bundle of 1-dimensional quotients of $\mathcal{F}$

$$\mathbb{P}(\mathcal{F}) = \{(x, \lambda) \mid \lambda : \mathcal{F}(x) \to \mathbb{C}\},$$

the zero scheme $Z_n(\sigma)$ induced by $p^*\mathcal{E} \to p^*\mathcal{F} \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is exactly

$$\{(x, \lambda) \mid \lambda \circ \sigma(x) : \mathcal{E}(x) \to \mathbb{C} \text{ is zero}\},$$

which is nothing but the subscheme $\mathbb{P}(\mathcal{E})$ of $\mathbb{P}(\mathcal{F})$. So $Z_n(\sigma)$ maps onto $D_n(\sigma)$ by $p$, because the linear map $\sigma : \mathcal{E}(x) \to \mathcal{F}(x)$ is not surjective if and only if there is a nonzero functional $\lambda : \mathcal{F}(x) \to \mathbb{C}$ with $\lambda \circ \sigma(x)$ being the zero map.

Let $d = \dim M \geq 4$. Denoting by $\pi$ the restriction of $p$ to $Z_n(\sigma)$, we have the following commutative diagram:

\[
\begin{array}{ccc}
Z_n(\sigma) & \xrightarrow{\iota} & \mathbb{P}(\mathcal{F}) \\
\pi \downarrow & & \downarrow p \\
D_n(\sigma) & \xrightarrow{\iota} & M.
\end{array}
\]

The morphism $\pi$ is called the determinantal contraction of $Z_n(\sigma)$.

The following proposition is a reformulation of [19, Theorem 2.12] for maximal degeneracy loci.

Proposition 3.6 ([19]). With the above notation, if $\sigma$ is $n$-general then $\pi$ is a small resolution and

\[
\chi_{\text{IH}}(D_n(\sigma)) = \chi(Z_n(\sigma)).
\]

Moreover, the pushforward of the total Chern class of $Z_n(\sigma)$ is

\[
p_* \mathcal{O}(Z_n(\sigma)) = \sum_{0 \leq i+j \leq d-1} (-1)^{i+j} \binom{i+j}{i} s_{1+i_1}(\mathcal{F} - \mathcal{E}) \cap c(M).
\]
Proof. Since $Z_n(\sigma)$ is the projectivization of $C = \text{Coker}(\sigma)$, the fiber of $\pi$ over $x \in D_n(\sigma)$ is just the projective space $\mathbb{P}(\mathcal{C}(x))$. On the other hand, the projective space $\mathbb{P}(\mathcal{C}(x))$ has dimension $\geq k$ if and only if $x \in D_{n-k}(\sigma)$. Therefore, if $x \in D_n(\sigma) \setminus D_{n-1}(\sigma)$ then the fiber $\pi^{-1}(x)$ is a point. In other word, $\pi$ is birational.

By the assumption that $\sigma$ is $n$-general, the dimension of $D_k(\sigma) \setminus D_{k-1}(\sigma)$ is $d - (n + 1 - k)^2$. It is straightforward to show that the morphism $\pi$ is small (see Definition 2.9). Thanks to [19, Lemma 2.9], it follows that $Z_n(\sigma)$ is smooth (for details see [19, pp. 810-811]). Hence the morphism $\pi$ is a small resolution and (3.2) follows from Proposition 2.11. The formula (3.3) of the total Chern class $\pi^*c(T_{Z_n(\sigma)}) \cap [Z_n(\sigma)]$ is a special case of [19, Proposition 2.5].

Remark 3.7. It is known that the pushforward of the Chern class of a small (or crepant) resolution is independant of the resolution. The class obtained is called the stringy Chern class and the integration of such class is called the stringy Euler number [1, 8]. In the situation (3.1), the stringy Chern class of $D_n(\sigma)$ and $\pi_*c(T_{Z_n(\sigma)}) \cap [Z_n(\sigma)]$ coincide [8, Proposition 4.5].

4. MAIN RESULT

Keeping the notation introduced in Section 3 we can now prove the formula for the intersection homology Euler characteristic of general determinantal hypersurfaces.

**Theorem 4.1.** If the morphism $\sigma : \mathcal{E} \rightarrow \mathcal{F}$ is $n$-general, then

$$\mu_{IH}(D_n(\sigma), M) = \sum_{\lambda \geq (2,2)} (-1)^{d+|\lambda|} f^\lambda \int_M s_\lambda(\mathcal{F} - \mathcal{E}) c_{d-|\lambda|}(T_M),$$

where $f^\lambda$ denotes the number of standard Young tableaux of shape $\lambda$.

**Proof.** Let $\mathcal{L} = \det \mathcal{E}^\vee \otimes \det \mathcal{F}$. We abbreviate $D_n(\sigma)$ and $Z_n(\sigma)$ to $D$ and $Z$ respectively. Recall that the Fulton class of $D$ is

$$c^F(D) = c(T_M|D - \mathcal{L}|D) \cap [D].$$

According to Proposition 3.6 and the commutative diagram (3.1), it follows that

$$\mu_{IH}(D) = (-1)^d \int_M (p_*j_*c(Z) - \iota_*c^F(D)).$$

Our goal is to find the difference of classes $p_*j_*c(Z) - \iota_*c^F(D)$. First, we observe that $\iota_*[D] = c_1(\mathcal{L}) \cap [M]$ in $A_{d-1}(M)$,

$$c_1(\mathcal{L}) = c_1(\mathcal{F}) - c_1(\mathcal{E}) = s_{(1)}(\mathcal{F} - \mathcal{E})$$
and the inverse of total Chern class of \( \mathcal{L} \) is \((1 + c_1(\mathcal{L}))^{-1}\). By the projection formula,

\[
\iota_* c^\ell(D) = \iota_* (\iota^* c(T_M - \mathcal{L}) \cap [D]) \\
= c(T_M - \mathcal{L}) \cap (c_1(\mathcal{L}) \cap [M]) \\
= (1 + c_1(\mathcal{L}))^{-1} c_1(\mathcal{L}) \cap (c(T_M) \cap [M]) \\
= \sum_{\ell=0}^{\infty} (-1)^\ell s_{(1)}^{\ell} (\mathcal{F} - \mathcal{E})^{\ell+1} \cap c(M).
\]

Then, by Proposition 3.6 and the hook length formula (see Proposition 2.1),

\[
p_* f_* c(Z) = \sum_{0 \leq i+j \leq d-1} (-1)^{i+j} f^{1+i,1,j} s_{1+i,1,j}(\mathcal{F} - \mathcal{E}) \cap c(M).
\]

Denote \( s_{\lambda}(\mathcal{F} - \mathcal{E}) \) briefly by \( s_{\lambda} \). According to Lemma 2.5, it follows that for \( \ell \geq 0 \)

\[
\sum_{i+j=\ell} (-1)^{i+j} f^{1+i,1,j} s_{1+i,1,j} - (-1)^\ell s_{(1)}^{\ell+1} \\
= (-1)^{\ell+1} \left( - \sum_{i+j=\ell} f^{1+i,1,j} s_{1+i,1,j} + \sum_{|\lambda|=\ell+1} f^{\lambda} s_{\lambda} \right) \\
= (-1)^{\ell+1} \sum_{\lambda \geq (2,2), |\lambda|=\ell+1} f^{\lambda} s_{\lambda}.
\]

The desired formula follows now by (4.1) and taking 0-dimensional components of the difference of classes. This completes the proof.

\[\square\]

The following is an immediate consequence of Theorem 4.1

**Corollary 4.2.** Assume that the smooth proper variety \( M \) satisfies one of the following assumptions (1) \( \dim M = 4 \), or (2) \( \dim M = 5 \) and

\[
c_1(T_M) = c_1(\mathcal{F} - \mathcal{E}).
\]

If \( \sigma \) is \( n \)-general, then

\[
\mu_{IH}(D_n(\sigma), M) = (\dim M - 2) \int_M c(T_M) \cap [D_{n-1}(\sigma)].
\]

**Proof.** For simplicity, we let \( s_{\lambda} \) stand for \( s_{\lambda}(\mathcal{F} - \mathcal{E}) \). For the case \( \dim M = 5 \), we first observe that the first Chern class of \( T_M \) is \( s_{(1)} \) by (4.2) (see Example 2.3). From Proposition 2.1 \( f^{(2,2)} = 2 \) and \( f^{(3,2)} = f^{(2,2,1)} = 5 \). Theorem 4.1 now yields

\[
\mu_{IH}(D_n(\sigma)) = f^{(3,2)} \int_M s_{(3,2)} + f^{(2,2,1)} \int_M s_{(2,2,1)} - f^{(2,2)} \int_M s_{(2,2)} c_1(T_M) \\
= 5 \int_M s_{(2,2)} s_{(1)} - 2 \int_M s_{(2,2)} c_1(T_M) = 3 \int_M s_{(2,2)} c_1(T_M).
\]
Here the second equality follows from Pieri’s formula (2.2). Then the corollary follows from the Giambelli-Thom-Porteous formula [10, Theorem 14.4],

\[(4.3) \quad [D_{n-1}(\sigma)] = s_{(2,2)}(\mathcal{F} - \mathcal{E}) \cap [M].\]

Similarly, for the case dim \(M = 4\), twice the degree of \([D_{n-1}(\sigma)]\) is equal to \(\mu_{1H}(D_n(\sigma))\).

**Remark 4.3.** Corollary 4.2 also can be computed by standard tools in [10], e.g., (4.5) and (4.8) below (cf. the proof of Proposition 4.5).

We are now ready to prove the main result. Recall that if \(\sigma : \mathcal{E} \to \mathcal{F}\) is \(n\)-general, then \(\pi : Z_n(\sigma) \to D_n(\sigma)\) is a small resolution by Proposition 3.6.

**Theorem 4.4.** Suppose that the smooth proper variety \(M\) has dimension 4. Given a \(n\)-general morphism \(\sigma\), we assume that \(Z_n(\sigma)\) is connected. Then the determinantal hypersurface \(D_n(\sigma)\) is nodal.

**Proof.** We abbreviate \(D_i(\sigma)\) to \(D_i\) for all \(i\). By assumption, the determinantal hypersurface \(D_n\) is irreducible. Since \(\sigma\) is \(n\)-general and the codimension of \(D_{n-1}\) in \(M\) is 4, the number of singular locus \(D_{n-1}\) of \(D_n\) equals the degree of the zero cycle \([D_{n-1}]\) (see Remark 4.3). The theorem will be proved by showing that the Milnor number \(\mu(D_n,x)\) is 1 for \(x \in D_{n-1}\).

By Proposition 2.14 and Corollary 4.2

\[(4.4) \quad \sum_{x \in D_{n-1}} (\mu_0(D_n,x) + \mu(D_n,x)) = \mu_{1H}(D_n) = 2 \cdot [D_{n-1}].\]

According to that integers \(\mu(D_n,x)\) and \(\mu_0(D_n,x)\) are greater than zero for all \(x \in D_{n-1}\) (see Remark 2.10 and (4.4), it follows that these are equal to one. Hence the 3-dimensional hypersurface \(D_n\) has only ODPs.

We are going to give a formula of intersections products on the smooth variety \(Z_n(\sigma)\). Recall that \(j : Z_n(\sigma) \hookrightarrow \mathbb{P}(\mathcal{F})\) is the inclusion, \(p : \mathbb{P}(\mathcal{F}) \to M\) the bundle map and \(d = \dim M\). To simplify notations, we write \(Z\) (resp. \(D\)) instead of \(Z_n(\sigma)\) (resp. \(D_n(\sigma)\)).

**Proposition 4.5.** Fix a \(n\)-general morphism \(\sigma\). Let \(\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))\) and \(L = j^*\xi\). Given a line bundle \(\mathcal{O}\) on \(M\), let \(H_M = c_1(\mathcal{O})\) and \(H_Z = j^*p^*H_M\).

1. For \(0 \leq k \leq d - 1\),

\[\int_Z H_Z^k L^{d-1-k} = \int_M H_M^k c_{d-k}(\mathcal{E}^\vee - \mathcal{F}^\vee).\]

2. Under the assumption \(d = 4\) and \(c_1(\mathcal{F} - \mathcal{E}) = c_1(T_M)\),

\[\int_Z c_2(T_Z).H_Z = \int_M c_2(T_M).c_1(\mathcal{E}^\vee - \mathcal{F}^\vee).H_M,\]

\[\int_Z c_2(T_Z).L = \int_M c_2(T_M).c_2(\mathcal{E}^\vee - \mathcal{F}^\vee) - |\text{Sing}(D)|.\]
Proof. For simplicity of notation, we write $\mathcal{O}_F(1)$ instead of $\mathcal{O}_{\mathbb{P}(F)}(1)$. Recall that $Z$ is the zero scheme of the global section of $p^*\mathcal{E}^\vee \otimes \mathcal{O}_F(1)$, which is induced by $\sigma : \mathcal{E} \to \mathcal{F}$. Then we have, by (2.4) and (2.6),

$$
J^*Z = c_{n+1}(p^*E^\vee \otimes \mathcal{O}_F(1)) \cap [\mathbb{P}(F)] \\
= \sum_{i \geq 0} c_i(p^*E^\vee, Z^{n+1-i}) \cap [\mathbb{P}(F)] \\
= \sum_{i \geq 0} Z^{n+1-i} \cap p^*(c_i(E^\vee) \cap [M]).
$$

(4.5)

Hence, by the projection formula, (4.5) and the definition of $c_{d-k}(E^\vee - F^\vee)$,

$$
p_*J^*(L^{d-1-k} \cap [Z]) = p_*(\sum_{i \geq 0} Z^{n+d-k-i} \cap p^*(c_i(E^\vee) \cap [M]))
\begin{align*}
&= \sum_{i \geq 0} s_{d-k-i}(F^\vee, F^\vee) \cdot \mathcal{E}_i(E^\vee) \cap [M] \\
&= c_{d-k}(E^\vee - F^\vee) \cap [M],
\end{align*}
$$

(4.6)

which establishes the formula (1) of $\int_Z H^k_L(L^{d-1-k} \cap [Z])$ for $0 \leq k \leq d-1$.

Now assume that $d = 4$ and $c_1(F - E) = c_1(T_M)$. We claim that

$$
c_2(T_Z) = j^*p^*(c_2(T_M) - c_2(E^\vee - F^\vee)) - j^*p^*c_1(E^\vee - F^\vee).L.
$$

(4.7)

Assume the claim is proved. Then the zero cycle $p_*J^*(c_2(T_Z).L \cap [Z])$ is

$$
\{ [c_2(T_M) - c_2(E^\vee - F^\vee)] \cdot c_2(E^\vee - F^\vee) - c_1(E^\vee - F^\vee) \cdot c_3(E^\vee - F^\vee) \} \cap [M].
$$

(4.8)

by (4.6). Since $c_1(F - E) = -c_1(E^\vee - F^\vee)$, we get

$$
\int_Z c_2(T_Z).L = \int_M (c_2(T_M) \cdot c_2(E^\vee - F^\vee) - s_{(2,2)}(E^\vee - F^\vee)).
$$

It is easily seen that $s_{(2,2)}(E^\vee - F^\vee) = s_{(2,2)}(F - E)$ (cf. [10] Lemma 14.5.1). Then the formula of $\int_Z c_2(T_Z).L$ follows from the singular locus of $D$ being the $(n-1)$th degeneracy locus of $\sigma$ and Giambelli-Thom-Porteous formula (4.3). The formula of $\int_Z c_2(T_Z).H_Z$ is proved similarly.

To show the formula of $c_2(T_Z)$, we first recall that by [10] Example 3.2.11,

$$
c(T_{\mathbb{P}(F)}) = c(p^*T_M) \cdot c(p^*F^\vee \otimes \mathcal{O}_{\mathcal{F}}(1)).
$$

From the normal exact sequence

$$
0 \to T_Z \to j^*T_{\mathbb{P}(F)} \to j^*(p^*E^\vee \otimes \mathcal{O}_F(1)) \to 0,
$$

where the right hand term is the normal bundle of $Z$ in $\mathbb{P}(F)$, we have

$$
c(T_Z) = j^*c(T_{\mathbb{P}(F)}) / c(p^*E^\vee \otimes \mathcal{O}_F(1))
\begin{align*}
&= j^*(c(p^*F^\vee \otimes \mathcal{O}_F(1) - p^*E^\vee \otimes \mathcal{O}_F(1)) \cdot c(p^*T_M)).
\end{align*}
$$

Applying the formula (2.3) to (4.8), we get

$$
c_2(T_Z) = j^*p^*(c_2(T_M) + c_1(F^\vee - E^\vee).c_1(T_M) + c_2(F^\vee - E^\vee))
\begin{align*}
&- j^*p^*c_1(F^\vee - E^\vee).L.
\end{align*}
$$

(4.9)
According to the observation that \( c(\mathcal{F} - \mathcal{E}) \cdot c(\mathcal{E} - \mathcal{F}) = 1 \) and \( c_1(T_M) = c_1(\mathcal{E} - \mathcal{F}) \) by our assumption, it follows that
\[
c_1(\mathcal{F} - \mathcal{E}) \cdot c_1(T_M) + c_2(\mathcal{F} - \mathcal{E}) = -c_2(\mathcal{E} - \mathcal{F}),
\]
and (4.7) is proved.

\[\square\]

**Remark 4.6.** If we replace (4.7) with (4.9), then the same proof works when we drop the assumption \( c_1(T_M) = c_1(\mathcal{E} - \mathcal{F}) \). For simplicity, we state Proposition 4.5 (2) under the Calabi–Yau condition.

Let us illustrate the above results with examples.

**Example 4.7.** We use the same notation as in Proposition 4.5 where \( M = \mathbb{P}^4 \) and \( \mathcal{Q} = \mathcal{O}(1) \), and construct quintic hypersurfaces with 46 ODPs.

Let \( \mathcal{F} = \mathcal{O} \oplus 4 \) and \( \mathcal{E} = \mathcal{O}(-1) \oplus 3 \oplus \mathcal{O}(-2) \). Given a 3-general morphism \( \sigma : \mathcal{E} \to \mathcal{F} \), let \( Z = Z_3(\sigma) \) and \( D = D_3(\sigma) \). Observe that \( Z \) is a complete intersection of a \((2,1)\) and three \((1,1)\) ample divisors in \( \mathbb{P}(\mathcal{F}) = \mathbb{P}^4 \times \mathbb{P}^3 \), and thus it is connected by Lefschetz hyperplane theorem. By Theorem 4.4 and Proposition 4.5, we obtain Table 1 (cf. [6, Lemma 4.1]).

| L^3 | L^2.H | L.H^2 | H^3 | L.c_2(T_Z) | H.c_2(T_Z) | # of ODPs |
|-----|-------|-------|-----|------------|------------|----------|
| 2   | 7     | 9     | 5   | 44         | 50         | 46       |

This construction includes the example in [6], which is the blow-up of \( D \) along a smooth surface \( B \subseteq \mathbb{P}^4 \) (cf. [6, (4.4)]). More precisely, let \( B \) be the blow-up of \( \mathbb{P}^2 \) at 10 general points, which is called a Bordiga surface. We can describe \( B \) in terms of a degeneracy locus in \( \mathbb{P}^4 \). Indeed, the surface \( B \) is the locus where the \( 4 \times 3 \) matrix \( \tau = [\tau_{ij}] \) of generic linear forms on \( \mathbb{P}^4 \) has rank \( \leq 1 \). In other words, we have the resolution
\[
0 \to \mathcal{O}(-1) \oplus 3 \xrightarrow{\tau} \mathcal{O} \xrightarrow{\pi} \mathcal{I}_B \to 0
\]
and \( B = D_2(\tau) \), where \( \mathcal{I}_B \) is the ideal sheaf of \( B \) and \( \tau \) is 2-general.

Choose four generic homogeneous polynomials of degree 2 on \( \mathbb{P}^4 \) which induce a morphism \( \sigma_1 : \mathcal{O}(-2) \to \mathcal{F} \). Then it gives rise to a 3-general morphism \( \sigma = \tau \oplus \sigma_1 \) from \( \mathcal{E} \) to \( \mathcal{F} \). Observe that \( D = D_3(\sigma) \) contains the Bordiga surface \( B \) and there is a natural morphism \( \varphi : Z \to \Bl_B D \) factoring the determinantal contraction \( \pi : Z \to D \). Since \( \varphi \) is a birational morphism between projective varieties with the same Picard number, the blow-up of \( D \) along \( B \) is isomorphic to \( Z \).

A series of examples of 3-dimensional Calabi–Yau varieties will be discussed in detail in a forthcoming paper.

**Example 4.8.** We also can construct Fano determinantal hypersurfaces in a similar way as Example 4.7. For instance, we consider the case that \( M = \mathbb{P}^4 \) and vector bundles \( \mathcal{E}, \mathcal{F} \) of rank \( n+1 \) are direct sums of line bundles with \( c_1(\mathcal{F} - \mathcal{E}) = \)
Example 4.9. We use the same notation with above examples. Let \( E = \mathcal{O}_{\mathbb{P}^4}(0, 0) \) and \( F = \mathcal{O}_{\mathbb{P}^4}(2, 2) \). We pick a 1-general morphism \( \sigma : E \to F \) such that \( D_1(\sigma) \) is a nodal quartic hypersurface in \( \mathbb{P}^4 \) containing a smooth del Pezzo surface \( F_{22} \) of degree 4 and \( |\text{Sing}(D_1(\sigma))| = 16 \). Also, \( Z_1(\sigma) \to D_1(\sigma) \) is a small resolution of \( D_1(\sigma) \).

Let us apply the formula of \( \mu_{1H}(D_1(\sigma)) \) to compute \( \chi(Z_1(\sigma)) \). Note that the topological Euler characteristics of a smooth quartic hypersurface is \(-56\). Since \( D_1(\sigma) \) has exactly 16 ODPs and Corollary 4.7 we get \( \mu_{1H}(D_1(\sigma)) = 32 \). By Proposition 3.6 \( \chi(Z_1(\sigma)) = \mu_{1H}(D_1(\sigma)) - 56 = -24 \).

Furthermore, the sufficiently general hypersurface \( D_1(\sigma) \) is nonrational [3, Theorem 11]. Indeed, I. Cheltsov (see [3, §3]) observed that the restriction of \( Z_1(\sigma) \) to the second projection \( \mathbb{P}(F) \to \mathbb{P}^1 \) is a standard del Pezzo fibration of degree 4. Then \( D_1(\sigma) \) is nonrational if \( \chi(Z_1(\sigma)) \neq 0, -4 \) or \(-8\).

Remark 4.10. We mention the rationality problem for other (sufficiently general) quartic threefolds \( D \) in Table 2. If \( D \) contains \( \Pi \) or \( Q_2^2 \), then it is nonrational (see [3, Theorem 6, Example 10], [4, Corollary 1.11]). On the other hand, \( D \) is rational if it contains \( F \) or \( B \) (see [4, Entry 30 in Table 1], [3, Example 7]).

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