New Mass Relation 
for Meson 25-plet

L. Burakovsky*, T. Goldman†
Theoretical Division, MS B285
Los Alamos National Laboratory
Los Alamos, NM 87545, USA

and

L.P. Horwitz‡

School of Physics and Astronomy
Tel-Aviv University
Ramat-Aviv, 69978 Israel

Abstract

By assuming the existence of (quasi)-linear Regge trajectories for 25-plet mesons in the low energy region, we derive a new, 14th power, meson mass relation. This relation may be reduced to a quadratic Gell-Mann–Okubo type formula by fitting the values of the Regge slopes of these (quasi)-linear trajectories. Such a formula holds with an accuracy of ~ 2% for vector mesons, and

*E-mail: BURAKOV@PION.LANL.GOV
†E-mail: GOLDMAN@T5.LANL.GOV
‡E-mail: HORWITZ@TAUNIVM.TAU.AC.IL. Also at Department of Physics, Bar-Ilan University, Ramat-Gan, Israel
also suggests that the quantum number \( I(J^P) = \frac{1}{2}(2^+) \) should be assigned to the both \( B_J(5732) \) and \( B_{sJ}(5850) \) mesons discovered recently. We also discuss reasons for the failure of group theory to produce correct mass relations for heavy quarkonia.

**Key words:** flavor symmetry, quark model, charmed mesons, Gell-Mann–Okubo, Regge phenomenology

**PACS:** 11.30.Hv, 11.55.Jy, 12.39.-x, 12.40.Nn, 12.40.Yx, 14.40.Lb

The generalization of the standard \( SU(3) \) Gell-Mann–Okubo mass formula \[1\] to higher symmetry groups, e.g., \( SU(4) \) and \( SU(5) \), became a natural subject of investigation after the discovery of the fourth and fifth quark flavors in the mid-70's \[2\]. Attempts have been made in the literature to derive such a formula, either quadratic or linear in mass, by a) using group theoretical methods \[3, 4, 5, 6, 7\], b) generalizing the perturbative treatment of \( U(3) \times U(3) \) chiral symmetry breaking and the corresponding Gell-Mann-Oakes-Renner relation \[8\] to \( U(4) \times U(4) \) \[9, 10\], c) assuming the asymptotic realization of \( SU(4) \) symmetry in the algebra \( [A_\alpha, A_\beta] = i f_{\alpha\beta\gamma} V_\gamma \) (where \( V_\alpha, A_\beta \) are vector and axial-vector charges, respectively) \[11\], d) extending the Weinberg spectral function sum rules \[13\] to accommodate the higher symmetry breaking effects \[12\], and e) applying alternative methods, such as the linear mass spectrum for meson multiplets\[14, 15\]. In the following\[2\], \( \eta, \eta_s, \eta_c, \eta_b, K, D, D_s, B, B_s, B_c \) stand for the masses of the \( n\bar{n} \) (\( n \equiv u \) or \( d \)), \( s\bar{s}, c\bar{c}, s\bar{n}, c\bar{n}, c\bar{s}, b\bar{n}, b\bar{s}, b\bar{c} \) mesons, respectively\[2\]. The linear mass relations

\[
\begin{align*}
D &= \frac{\eta + \eta_c}{2}, & D_s &= \frac{\eta_s + \eta_c}{2}, \\
B &= \frac{\eta + \eta_b}{2}, & B_s &= \frac{\eta_s + \eta_b}{2}, & B_c &= \frac{\eta_c + \eta_b}{2}
\end{align*}
\]

found in \[2, 14, 15\], although perhaps justified for vector mesons, since a vector meson mass is given approximately by a sum of the corresponding constituent quark masses,

\[ m(ij) \simeq m(i) + m(j) \]

\[1\]Here, we speak of linear spectrum over the multiplet quantum numbers, taking proper account of degeneracy, not (directly) make use of linear Regge trajectories.

\[2\]Here \( \eta \) stands for the masses of both isovector and isoscalar \( n\bar{n} \) states which coincide on a naive quark model level.

\[3\]Since these designations apply to all spin states, vector mesons will be confusingly labelled as \( \eta \)'s. We ask the reader to bear with us in this in the interest of minimizing notation.
for vector mesons, the relations (1),(2) hold with an accuracy of up to $\sim 4\%$), are expected to fail for other meson multiplets, as confirmed by direct comparison with experiment. Similarly, the quadratic mass relations
\[ B_s^2 - B^2 = D_s^2 - D^2 = K^2 - \eta^2 \] (3)
obtained in ref. [9] by generalizing the $SU(3)$ Gell-Mann-Oakes-Renner relation [8] to include the $D$ and $D_s$ mesons (here $\pi$ stands for the mass of the $\pi$, etc.),
\[ \frac{\pi^2}{2n} = \frac{K^2}{n + s} = \frac{D^2}{n + c} = \frac{D_s^2}{s + c}, \] (4)
(and therefore $D_s^2 - D^2 = K^2 - \pi^2 \propto (s - n)$, also found in refs. [4, 11]), does not agree with experiment. For pseudoscalar mesons, for example, one has (in GeV$^2$) 0.388 for $(D_s^2 - D^2)$ vs. 0.226 for $(K^2 - \pi^2)$. For vector mesons, the corresponding quantities are 0.424 vs. 0.199, with about 100% discrepancy. The reason that the relations (3) do not hold is apparently due to the impossibility of perturbative treatment of $U(4) \times U(4)$ and $U(5) \times U(5)$ symmetry breaking, as a generalization of that of $U(3) \times U(3)$, due to very large bare masses of the $c$- and $b$-quarks as compared to those of the $u$, $d$- and $s$-quarks. In ref. [15] by the application of the linear spectrum to $SU(4)$ meson 16-plet, the following relation was obtained,
\[ 12\bar{D}^2 = 7\eta_0^2 + 5\eta_c^2, \] (5)
where $\bar{D}$ is the average mass of the $D$ and $D_s$ states (which are mass degenerate when $SU(4)$ flavor symmetry is broken to $SU(3)$), and $\eta_0$ is the mass average of the corresponding $SU(3)$ nonet. As shown in ref. [15], this relation holds with an accuracy of up to $\sim 5\%$ for all well established meson 16-plets. The accuracy of the generalization of Eq. (5) to a meson 25-plet is, however, much worse than that of (5), perhaps due to the contribution of the higher power corrections to the linear spectrum in the mass range corresponding to the $b$-flavored mesons [15].

It is well known that the hadrons composed of light ($u$, $d$, $s$) quarks populate linear Regge trajectories; i.e., the square of the mass of a state with orbital momentum $\ell$ is proportional to $\ell : M^2(\ell) = \ell/\alpha' + \text{const}$, where the slope $\alpha'$ depends weakly on the flavor content of the states lying on the corresponding trajectory,
\[ \alpha'_{n\bar{n}} \approx 0.88 \text{ GeV}^{-2}, \quad \alpha'_{s\bar{n}} \approx 0.84 \text{ GeV}^{-2}, \quad \alpha'_{s\bar{s}} \approx 0.80 \text{ GeV}^{-2}. \] (6)
In contrast, the data on the properties of Regge trajectories of hadrons containing heavy quarks are almost nonexistent at the present time, although it is established [10] that the slope of the trajectories decreases with increasing quark mass (as seen in Eq. (6)) in the mass region of the lowest excitations. This is due to an increasing (with mass) contribution of the color Coulomb interaction, leading to a curvature of the trajectory near the ground state. However, as the analyses show [10, 17, 18], in
the asymptotic regime of the highest excitations, the trajectories of both light and heavy quarkonia are linear and have the same slope $\alpha' \simeq 0.9 \text{ GeV}^{-2}$, in agreement with natural expectations from the string model.

Knowledge of Regge trajectories in the scattering region, i.e., at $t < 0$, and of the intercepts $a(0)$ and slopes $\alpha'$ is also useful for many non-spectral purposes, for example, in the recombination [19] and fragmentation [20] models. Therefore, as pointed out in ref. [16], the slopes and intercepts of the Regge trajectories are the fundamental constants of hadron dynamics, perhaps generally more important than the mass of any particular state. Thus, not only the derivation of a mass relation but also the determination of the parameters $a(0)$ and $\alpha'$ of heavy quarkonia is of great importance, since they afford opportunities for better understanding of the dynamics of the strong interactions in the processes of production of charmed and beauty hadrons at high energies.

Here we apply Regge phenomenology for the derivation of a mass relation for the $SU(5)$ meson 25-plet, by assuming the (quasi)-linear form of Regge trajectories for heavy quarkonia with slopes which are generally different from (less than) the standard one, $\alpha' \simeq 0.9 \text{ GeV}^{-2}$. We show that for this relation to avoid depending on the values of the slopes, it must be of 14th power in meson masses. The relation may be reduced to a quadratic Gell-Mann–Okubo type formula by fitting the values of the slopes. We note that the corresponding formula for the $SU(4)$ meson 16-plet, which is of sixth power in meson masses, was derived in our previous paper [21].

Let us assume the (quasi)-linear form of Regge trajectories for hadrons with identical $J^{PC}$ quantum numbers (i.e., belonging to a common multiplet). Then for the states with orbital momentum $\ell$ one has

$$\ell = \alpha'_{ii} m_{ii}^2 + a_{ii}(0),$$

$$\ell = \alpha'_{ji} m_{ji}^2 + a_{ji}(0),$$

$$\ell = \alpha'_{jj} m_{jj}^2 + a_{jj}(0).$$

Using now the relation among the intercepts [22, 23],

$$a_{ii}(0) + a_{jj}(0) = 2a_{ji}(0),$$

one obtains from the above relations

$$\alpha'_{ii} m_{ii}^2 + \alpha'_{jj} m_{jj}^2 = 2\alpha'_{ji} m_{ji}^2.$$  

In order to eliminate the Regge slopes from this formula, we need a relation among the slopes. Two such relations have been proposed in the literature,

$$\alpha'_{ii} \cdot \alpha'_{jj} = \left(\alpha'_{ji}\right)^2,$$

which follows from the factorization of residues of the $t$-channel poles [24, 25], and

$$\frac{1}{\alpha'_{ii}} + \frac{1}{\alpha'_{jj}} = \frac{2}{\alpha'_{ji}}.$$
based on topological expansion and the $q\bar{q}$-string picture of hadrons \cite{23}.

For light quarkonia (and small differences in the $\alpha'$ values), there is no essential difference between these two relations; viz., for $\alpha'_{ji} = \alpha'_{ij}/(1 + x)$, $x \ll 1$, Eq. (10) gives $\alpha'_{jj} = \alpha'_{ii}/(1 + 2x)$, whereas Eq. (9) gives $\alpha'_{jj} = \alpha'_{ii}/(1 + x^2) \approx \alpha'/(1 + 2x)$, i.e., differing only in the interpretation of the value of $x$, to order $x^2$. However, for heavy quarkonia (and expected large differences from the $\alpha'$ values for the light quarkonia) these relations are incompatible; e.g., for $\alpha'_{ji} = \alpha'_{ij}/2$, Eq. (9) will give $\alpha'_{jj} = \alpha'_{ii}/4$, whereas Eq. (10) produces $\alpha'_{jj} = \alpha'_{ii}/3$. One has therefore to choose one of these relations in order to proceed further. Here we use Eq. (10), since it is much more consistent with (8) than is Eq. (9), which we tested by using measured quarkonia masses in Eq. (8). We shall justifiy this choice in more detail in a separate publication \cite{20}.

Since we are interested in $SU(4)$ and $SU(5)$ breaking, and since it simplifies the discussion, we take average slope in the light quark sector:

$$\alpha'_{nn} \cong \alpha'_{sh} \cong \alpha'_{ss} \cong \alpha' \cong 0.85 \text{ GeV}^{-2}. \quad (11)$$

Note that this should lead us to expect accuracy to be limited to $\sim 5\%$ in what follows.

It then follows from the relations based on (8),

$$\alpha'_{ii} \eta_i^2 + \alpha'_{jj} \eta_j^2 = 2\alpha'_{ij} D^2; \quad (12)$$
$$\alpha'_{jj} \eta_j^2 + \alpha'_{ji} \eta_i^2 = 2\alpha'_{ji} D^j_i; \quad (13)$$

that

$$\alpha'_{ij} \cong \frac{\alpha'_{ji}}{\alpha'_{ij}} \cong \alpha' \frac{(\eta_i^2 - \eta_j^2)}{2(D_i^2 - D_j^2)}; \quad (14)$$

$$\alpha'_{ji} = \alpha' \left[ \frac{(\eta_i^2 - \eta_j^2)}{(D_i^2 - D_j^2)} \frac{D_i^2}{\eta_i^2} - 2 \right]. \quad (15)$$

Using these values of the slopes in Eq. (10) with $i = n, j = c$, we obtain

$$\left(\eta_s^2 D^2 - \eta_n^2 D_n^2\right) \left(\eta_s^2 - \eta_n^2\right) + \eta_c^2 \left(D_s^2 - D_s^2\right) \left(\eta_s^2 - \eta_n^2\right) = 4 \left(\eta_s^2 D^2 - \eta_n^2 D_n^2\right) \left(D_s^2 - D_s^2\right), \quad (16)$$

which is the new mass relation for the $SU(4)$ meson 16-plet obtained in our previous paper \cite{21}. As shown in \cite{21}, this relation holds with an accuracy of up to $\sim 5\%$ for the four well-established meson multiplets. The use of $B, B_s, B_c$ in place of $D, D_s, D_c$, respectively, in Eq. (12),(13) leads to

$$\alpha'_{nn} \cong \alpha'_{sb} = \frac{\alpha'_{n s} (\eta_s^2 - \eta_n^2)}{2(B_s^2 - B_n^2)}; \quad (17)$$

$$\alpha'_{bs} = \alpha' \left[ \frac{(\eta_s^2 - \eta_n^2)}{(B_s^2 - B_n^2)} \frac{B_s^2}{\eta_s^2} - \eta_n^2 \right]. \quad (18)$$
Using now these values of the slopes in Eq. (10) with \( i = n, \ j = b \), we obtain

\[
\left( \eta_s^2 B^2 - \eta_s^2 B_s^2 \right) \left( \eta_s^2 - \eta_s^2 \right) + \eta_b^2 \left( B_s^2 - B^2 \right) \left( \eta_s^2 - \eta_b^2 \right) = 4 \left( \eta_s^2 B^2 - \eta_s^2 B_s^2 \right) \left( B_s^2 - B^2 \right),
\]

which is a new mass relation for the \( SU(4) \) meson 16-plet built of the \( u-, \ d-, \ s-, \) and \( b \)-quarks \[21\]. At present, this relation may only be tested for vector mesons alone since the masses of all of the beauty states involved are established experimentally only for vector mesons \[27\]. As shown in \[21\], in this case the accuracy is \( \sim 3.5\% \).

In order to extend the results to \( SU(5) \) and to incorporate the remaining \( b\bar{c} \)-state of the 25-plet, we use a relation based on (8),

\[
\alpha_{cc}^\prime \eta_c^2 + \alpha_{bb}^\prime \eta_b^2 = 2 \alpha_{bc}^\prime B_c^2,
\]

which leads, through (15),(18), to

\[
\alpha_{bc}^\prime = \alpha^\prime \left[ \frac{\eta_s^2 - \eta_s^2}{B_s^2 - B^2} \frac{B^2}{2B_c^2} + \frac{\eta_b^2 - \eta_b^2}{D_s^2 - D^2} \frac{D^2}{2B_c^2} - \frac{\eta_b^2}{B_c^2} \right].
\]

Using now this expression for \( \alpha_{bc}^\prime \) together with those for \( \alpha_{cc}^\prime \) and \( \alpha_{bb}^\prime \), Eqs. (15),(18), in relation, as follows from (10),

\[
\frac{1}{\alpha_{bb}^\prime} + \frac{1}{\alpha_{cc}^\prime} = \frac{2}{\alpha_{bc}^\prime},
\]

we finally arrive at

\[
\left[ (\eta_s^2 - \eta_s^2) \left( \eta_s^2 B_s^2 - B_s^2 \right) + \eta_b^2 D^2 (B_s^2 - B^2) \right] - \eta^2 (\eta_s^2 + \eta_b^2) (D_s^2 - D^2) (B_s^2 - B^2)
\]

\[
\times \left[ (\eta_s^2 - \eta_s^2) \left( B_s^2 (D_s^2 - D^2) + D^2 (B_s^2 - B^2) \right) - 2\eta^2 (D_s^2 - D^2) (B_s^2 - B^2) \right]
\]

\[
= 4B^2 (D_s^2 - D^2) (B_s^2 - B^2) \left[ D^2 (\eta_s^2 - \eta_s^2) - \eta^2 (D_s^2 - D^2) \right] \left[ B^2 (\eta_s^2 - \eta_s^2) - \eta^2 (B_s^2 - B^2) \right],
\]

which is a new, 14th power in meson masses, relation for the \( SU(5) \) 25-plet. As applied to vector mesons, this relation yields for the mass of the \( B_c^\prime \):

\[
m(B_c) = 6.285 \text{ GeV},
\]

in agreement with a rough estimate, as follows from (2),

\[
m(B_c^\prime) \simeq \frac{m(J/\psi) + m(\Upsilon(1S))}{2} \simeq 6.280 \text{ GeV}.
\]

We emphasize that the formulas (16),(19),(22) do not depend on the values of the Regge slopes, but only on a relation between them, Eq. (10), which justifies their use in both the low energy region where the slopes are different and the high energy
region where all the slopes coincide. In the latter case, for example, as follows from (12),(13), \( \eta_s^2 - \eta^2 = 2(D_s^2 - D^2) \), and Eq. (16) reduces to

\[
\eta^2 + \eta_s^2 = 2D^2,
\]

consistent with Eq. (12) in this limit. One may also find from Eqs. (12),(13) and similar relations for the beauty mesons with equal slopes, and the standard \( SU(3) \) Gell-Mann–Okubo relation,

\[
\eta^2 + \eta_s^2 = 2K^2,
\]

that the relations (3) also hold in this limit.

Let us now discuss the question of the generalization of the standard \( SU(3) \) Gell-Mann–Okubo mass formula which is quadratic in mass to the case of heavier quarkonia. We shall continue to assume the validity of Eq. (11) and introduce \( x > 0 \) through the relation

\[
\alpha'_{\bar{c}n} = \alpha'_{\bar{c}s} = \frac{\alpha'}{1 + x}.
\]

It then follows from (10) that

\[
\alpha'_{\bar{c}\bar{c}} = \frac{\alpha'}{1 + 2x},
\]

and one obtains from (12),(13),

\[
(1 + x)\left( \eta^2 + \eta_s^2 \right) + \frac{2(1 + x)\eta_s^2}{1 + 2x} = 2\left( D^2 + D_s^2 \right).
\]

Results of the calculations of the Regge slopes of heavy quarkonia in refs. \[25\]: \( \alpha'_{\bar{c}b}/\alpha' \simeq \alpha'_{\bar{c}s}/\alpha' \simeq 0.73, \alpha'_{\bar{c}\bar{c}}/\alpha' \simeq 0.58 \), and \[23, 28\]: \( \alpha'_{\bar{c}\bar{c}} \simeq 0.5 \text{ GeV}^{-2} \), support the value

\[
x \simeq 0.355.
\]

With this \( x \), it follows from (29) and the standard \( SU(3) \) Gell-Mann–Okubo formula (19) that

\[
8.13K^2 + 4.75\eta_s^2 = 6\left( D^2 + D_s^2 \right).
\]

As shown in ref. \[21\], this formula holds at a 1% level for the four well-established meson multiplets. Also, the formula (31) is in qualitative agreement with the relation (5) obtained by two of the present authors in ref. \[15\] by the application of the linear spectrum to a meson 16-plet.

The entire analysis may, of course, be repeated for \( B, B_s, \eta_b \) in place of \( D, D_s, \eta_c \). In this case, introducing \( y > 0 \) through the relation

\[
\alpha'_{\bar{b}h} = \alpha'_{\bar{b}s} = \frac{\alpha'}{1 + y},
\]
one finds
\[ \alpha_{bb}' = \frac{\alpha_b'}{1 + 2y}, \tag{33} \]
and
\[ (1 + y) \left( \eta^2 + \eta_s^2 \right) + \frac{2(1 + y)}{1 + 2y} \eta_b^2 = 2 \left( B^2 + B_s^2 \right). \tag{34} \]

It follows from Eqs. (15),(18) for the measured vector meson masses, and \( \alpha_{cc}' \approx 0.5 \) GeV\(^{-2} \) that \( \alpha_{bb}' \approx 0.182 \) GeV\(^{-2} \), consistent with the value \( \alpha_{bb}' = 1/5.82 \approx 0.172 \) GeV\(^{-2} \) found in ref. [29]. This value of \( \alpha_{bb}' \) supports the value \( y \approx 1.85 \). \tag{35} \]

With this \( y \), one further obtains from (34) a relation similar to (31),
\[ 17.1 K^2 + 3.64 \eta_b^2 = 6 \left( B^2 + B_s^2 \right). \tag{36} \]

Thus, the new sixth and 14th order mass relations may be accurately reduced to quadratic ones by use of specific values for the Regge slopes.

For vector mesons, the values of the l.h.s. and r.h.s. of the formula (36) are, respectively, (in GeV\(^2\)) 339.4 vs. 346.1; the accuracy is therefore \( \sim 2\% \).

This formula (as well as Eqs. (19) and (22)) may be applied, for example, to the question of the correct \( q\bar{q} \) assignments for the meson multiplets. There are two states in the most recent Review of Particle Physics [27] discovered recently whose quantum numbers are uncertain, the \( B^*_J(5732) \) and \( B^*_{sJ}(5850) \) having masses \( 5698 \pm 12 \) MeV and \( 5853 \pm 15 \) MeV, respectively. Since the dominant decay modes of the \( B^*_J(5732) \) are \( B^*\pi \) and \( B\pi \) [27], a natural suspicion would be that this state is a tensor meson. Moreover, the relative proximity of the masses of these states would suggest that the both belong to a common multiplet, i.e., to the tensor meson 25-plet. This suspicion may be tested in two ways:

1) one may use the relation, as follows from the assumption on the (quasi)-linear \( B^* \) and \( B^*_s \) Regge trajectories (with equal slopes) on which the discussed mesons lie,
\[ m^2(B^*_J) - m^2(B^*) \approx m^2(B^*_2) - m^2(B^*_s), \tag{37} \]
which in this case gives (in GeV\(^2\)) \( 4.11 \pm 0.14 \) vs. \( 4.92 \pm 0.18 \), or

2) one may use the formulas (19) and (36). For the values (as those for the masses of the \( B^*_J \) and \( B^*_{sJ} \), respectively) \( B = 5.7 \) GeV and \( B_s = 5.85 \) GeV, the values of the l.h.s. and r.h.s., respectively, of Eq. (19) are (in GeV\(^2\)) 129.1 vs. 124.7, with an accuracy of \( \sim 3.5\% \). For the same values of \( B \) and \( B_s \), the formula (36) yields (in GeV\(^2\)) 418.7 vs. 400.3, with a similar accuracy of \( \sim 4.5\% \), which does not exceed a limit of \( \sim 5\% \) expected from the assumption on equality of the Regge slopes in the light quark sector.
Finally, we discuss the failure of group theory to produce correct mass relations for heavy quarkonia. Consider, along with Iwao [3], sub-SU(3) symmetry which incorporates the u-, d- and c-quarks, i.e., assume that a world consists of the u-, d- and c-quarks, and the c-quark plays the same role as s-quark in the real world (the difference in the masses of the s- and c-quarks is not reflected in a group theoretical approach). Then charm (C) will play the role of strangeness (S), the “supercharge” Z may be introduced in place of hypercharge, viz., (B stands for baryon number)

\[ Z = B + C, \]

the modified Gell-Mann–Nishijima formula will become

\[ Q = I_3 + \frac{Z}{2}, \]

and the Gell-Mann–Okubo mass formula will take on the form

\[ m^\ell = a + bZ + c \left[ \frac{Z^2}{4} - I(I + 1) \right], \quad (38) \]

where \( \ell = 1 \) or 2, and \( a, b, c \) are independent of \( I \) and \( Z \) but, in general, depend on \( (p, q) \), where \( (p, q) \) is any irreducible representation of \( SU(3) \). The whole difference of the above formula from the standard Gell-Mann–Okubo one [1] is the presence of \( Z \) in place of \( Y \) in the standard formula. Then, in the same way Eq. (26) follows from the standard Gell-Mann–Okubo formula, the following relation may be derived from (38) [7]:

\[ \eta^\ell + \eta^\ell_c = 2D^\ell, \]

which for \( \ell = 2 \) disagrees with experiment, and for \( \ell = 1 \), although perhaps justified for vector mesons, fails for other multiplets, as discussed in the beginning of the paper. Thus, the failure of group theory to produce correct mass relations for heavy quarkonia is solely due to the large bare masses of the c- and b-quarks as compared to those of u-, d- and s-quarks. On the other hand, group theory is rather successful in producing a correct mass formula in the \( (u,d,s) \)-sector, since the difference in the masses of the non-strange and strange quarks (and symmetry breaking in sub-SU(3) sector of full \( SU(4) \) or \( SU(5) \)) is negligibly small as compared to that of the strange and charm quarks. The difference in the quark masses seems to be the only reason for the different slopes of the corresponding trajectories. We also remark that we do not see an alternative way to Regge phenomenology to obtain mass relations for heavy quarkonia, at least at present.

In conclusion, we note that the derived Regge slopes in the charm sector are [21]

\[ \alpha'_{c\bar{c}} \simeq \alpha'_{c\bar{s}} \simeq 0.63 \text{ GeV}^{-2}, \quad \alpha'_{c\bar{c}} \simeq 0.50 \text{ GeV}^{-2}, \quad (39) \]

and in the beauty sector

\[ \alpha'_{b\bar{n}} \simeq \alpha'_{b\bar{s}} \simeq 0.30 \text{ GeV}^{-2}, \quad \alpha'_{b\bar{c}} \simeq 0.27 \text{ GeV}^{-2}, \quad \alpha'_{b\bar{b}} \simeq 0.18 \text{ GeV}^{-2}. \quad (40) \]
References

[1] S. Okubo, Prog. Theor. Phys. 27 (1962) 949, 28 (1962) 24
M. Gell-Mann and Y. Ne’eman, The Eightfold Way, (Benjamin, NY, 1964)

[2] J.J. Aubert et al., Phys. Rev. Lett. 33 (1974) 1404
J.-E. Augustin et al., Phys. Rev. Lett. 33 (1974) 1406
C. Bacci et al., Phys. Rev. Lett. 33 (1974) 1408
G.S. Abrams et al., Phys. Rev. Lett. 33 (1974) 1453
S.W. Herb et al., Phys. Rev. Lett. 39 (1977) 252

[3] A.J. Macfarlane, J. Phys. G 1 (1975) 601

[4] S. Borchardt, V.S. Mathur and S. Okubo, Phys. Rev. Lett. 34 (1975) 38
V.S. Mathur, S. Okubo and S. Borchardt, Phys. Rev. D 11 (1976) 2572
H. Hayashi et al., Ann. Phys. 101 (1976) 394

[5] D.B. Lichtenberg, Nuovo Cim. Lett. 13 (1975) 346
A. Kazi, G. Kramer and D.H. Schiller, Acta Phys. Austr. 45 (1976) 65
D.H. Boal and R. Torgerson, Phys. Rev. D 15 (1977) 327
D.H. Boal, Phys. Rev. D 18 (1978) 3446
S.A. Rashid, Indian J. Pure Appl. Phys. 19 (1981) 172

[6] S. Iwao, Nuovo Cim. Lett. 20 (1977) 347, 21 (1978) 239, 245, 22 (1978) 192

[7] G. Dattoli, R. Mignani and D. Prosperi, Nuovo Cim. Lett. 24 (1979) 193

[8] M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. 175 (1968) 2195

[9] M.K. Gaillard, B.W. Lee and J.L. Rosner, Rev. Mod. Phys. 47 (1975) 277

[10] C. Montonen, M. Roos and N. Törnqvist, Nuovo Cim. Lett. 12 (1975) 627

[11] H.L. Hallock and S. Oneda, Phys. Rev. D 18 (1978) 841, 19 (1979) 347, 20
(1979) 2932
M. Majewski and W. Tybor, Acta Phys. Pol. B 9 (1978) 177

[12] B. Bagchi, V.P. Gautam and A. Nandi, Nuovo Cim. Lett. 24 (1979) 175
V.P. Gautam, B. Bagchi and A. Nandy, J. Phys. G 5 (1979) 885

[13] S. Weinberg, Phys. Rev. Lett. 18 (1967) 507

[14] L. Burakovsky and L.P. Horwitz, Nucl. Phys. A 609 (1996) 585, 614 (1997) 373;
Found. Phys. Lett. 9 (1996) 561

[15] L. Burakovsky and L.P. Horwitz, Gell-Mann–Okubo mass formula for $SU(4)$
meson hexadecuplet, Found. Phys. Lett, in press
[16] J.-L. Basdevant and S. Boukraa, Z. Phys. C 28 (1985) 413; Ann. Phys. (Paris) 10 (1985) 475

[17] J.S. Kang and H. Schnitzer, Phys. Rev. D 12 (1975) 841

[18] C. Quigg and J.L. Rosner, Phys. Rep. 56 (1979) 167

[19] K.P. Das and R.C. Hwa, Phys. Lett. B 68 (1977) 459
   T. Tashiro et al., Z. Phys. C 35 (1987) 21
   A.B. Batunin, B.B. Kiselyev and A.K. Likhoded, Yad. Phys. 49 (1989) 554

[20] A. Capella and J. van Tran Thanh, Z. Phys. C 10 (1981) 249; Phys. Lett. B 114 (1982) 450
   A.B. Kaidalov, Phys. Lett. B 116 (1982) 459
   A.B. Kaidalov and K.A. Ter-Martirosyan, Sov. J. Nucl. Phys. 39 (1984) 979
   A.B. Kaidalov and O.I. Piskounova, Z. Phys. C 30 (1986) 145
   G.I. Lykasov and M.N. Sergeenko, Z. Phys. C 52 (1991) 635

[21] L. Burakovsky, T. Goldman and L.P. Horwitz, New mass relations for heavy quarkonia, Preprint LA-UR-97-1494, to be published

[22] K. Kawarabayashi, S. Kitakado and H. Yabuki, Phys. Lett. B 28 (1969) 432
   R.C. Brower, J. Ellis, M.G. Schmidt and J.H. Weis, Nucl. Phys. B 128 (1977) 175
   V.V. Dixit and L.A. Balazs, Phys. Rev. D 20 (1979) 816

[23] A.B. Kaidalov, Z. Phys. C 12 (1982) 63

[24] J. Pasupathy, Phys. Rev. Lett. 37 (1976) 1336
   K. Igi, Phys. Lett. B 66 (1977) 276; Phys. Rev. D 16 (1977) 196

[25] M. Kuroda and B.-L. Young, Phys. Rev. D 16 (1977) 204

[26] L. Burakovsky, T. Goldman and L.P. Horwitz, in preparation

[27] Particle Data Group, Phys. Rev. D 54 (1996) 1

[28] N.-P. Chang and C.A. Nelson, Phys. Rev. Lett. 35 (1975) 1492; Phys. Rev. D 13 (1976) 1345

[29] N.-P. Chang and C.A. Nelson, Phys. Rev. D 19 (1979) 3336