LONG-TIME BEHAVIOR OF AN SIR MODEL WITH PERTURBED DISEASE TRANSMISSION COEFFICIENT

NGUYEN HUU DU*
Faculty of Mathematics, Mechanics, and Informatics
University of Science-VNU
334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

NGUYEN THANH DIEU
Department of Mathematics
Vinh University
182 Le Duan, Vinh, Nghe An, Vietnam

(Communicated by Peter Hinow)

ABSTRACT. In this paper, we consider a stochastic SIR model with the perturbed disease transmission coefficient. We determine the threshold \( \lambda \) that is used to classify the extinction and permanence of the disease. Precisely, \( \lambda < 0 \) implies that the disease-free \((\frac{\alpha}{\beta}, 0, 0)\) is globally asymptotic stable, i.e., the disease will disappear and the entire population will become susceptible individuals. If \( \lambda > 0 \) the epidemic takes place. In this case, we derive that the Markov process \((S(t), I(t))\) has a unique invariant probability measure. We also characterize the support of a unique invariant probability measure and prove that the transition probability converges to this invariant measure in total variation norm. Our result is considered as a significant improvement over the results in \([6, 7, 11, 18]\).

1. Introduction. There have been many kinds of infectious diseases, which have permanent immunity upon recovered, like morbilli, HBV, rubella, whooping cough, smallpox, etc. Such epidemic spreads are often described by SIR (Susceptible-Infective-Recovered) epidemic models. Under the assumption that population develop in a certain environment, the densities \( S(t), I(t), R(t) \) of individuals in susceptible, infective, recovered classes, respectively at time \( t \), satisfy the following differential equations:

\[
\begin{align*}
    dS(t) &= (\alpha - \beta S(t)I(t) - \mu S(t))dt \\
    dI(t) &= (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt \\
    dR(t) &= (\gamma I(t) - \mu R(t))dt,
\end{align*}
\]

where \( \alpha \) is the per capita birth rate of the population, \( \mu \) is the per capita disease-free death rate and \( \rho \) is the excess per capita death rate of infective class, \( \beta \) is the
effective per capita contact rate, $\gamma$ is the per capita recovery rate of the infective individuals.

However, it is well-known that this assumption is in general not true since the evolution of a population is always subject to unpredictable factors. The problem to learn how randomness affects to the long term behavior of population in stochastic models is interesting. If the disease transmission coefficient $\beta$ in the equation (1) is subject to the environmental white noise, then it becomes $\beta \rightarrow \beta + \text{white noise}$. Thus, under the language of stochastic calculus, the epidemic model (1) becomes

$$\begin{align*}
    dS(t) &= (\alpha - \beta S(t)I(t) - \mu S(t))dt - \sigma S(t)I(t)dB(t) \\
    dI(t) &= (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt + \sigma S(t)I(t)dB(t) \\
    dR(t) &= (\gamma I(t) - \mu R(t))dt,
\end{align*}$$

(2)

where $B(t)$ is a Brownian motion.

This model has been investigated in [6, 7, 11, 18] and extended in [1, 8, 16, 20, 19, 21]. When studying epidemic models, it is naturally important to know whether the population will result in a disease free state or the disease will remain permanently.

For the deterministic model (1), the asymptotic behavior has been classified completely by the value $\lambda_d = \frac{\alpha \mu}{\rho} - (\mu + \rho + \gamma)$. Precisely, if $\lambda_d \leq 0$ then the population will result in the disease-free equilibrium $(\frac{\alpha}{\rho}, 0, 0)$ while the population approach an endemic equilibrium in case $\lambda_d > 0$.

For the stochastic case, in [7, 11, 18, 21], the authors attempted to answer the afore-mentioned question for the model (2). In particular, in [11], by using Lyapunov-type functions, they provided some sufficient conditions for the exponential extinction of the disease as well as sufficient conditions for the existence of a stationary distribution to the system (2) and described the support of the invariant density. Unfortunately, their conditions are restrictive and not close to any necessary one. With these results it is unable to classify completely stochastic SIR models similar to the deterministic case. Furthermore, there is a gap in [11]. Specially, when Hörmander’s condition was verified and when a control system was investigated, they used drift and diffusion coefficients of Itô stochastic differential equation instead of those of Stratonovich one.

Our main goal in this paper is to provide a sufficient and almost necessary condition for permanence (as well as ergodicity) and extinction of the disease in the stochastic SIR model (2) in using a value $\lambda$ to be similar to $\lambda_d$ in the deterministic model. Note that such kind of results are obtained for a stochastic SIS model in [3]. However, the model studied there can be reduced to one-dimensional equation that is much easier to investigate.

The rest of the paper is arranged as follows. Section 2 derives a threshold $\lambda$ that is used to classify the model. Clearly, it is shown that if $\lambda < 0$, the disease is eradicated as a disease-free equilibrium $(\frac{\alpha}{\rho}, 0, 0)$, which is exponentially asymptotically stable. This situation is the eradication of the disease among the population. Meanwhile, in case $\lambda > 0$, the solution converges to a stationary distribution in total variation, i.e., the disease is permanent. The ergodicity of the solution process is also examined. Finally, Section 3 is reserved for some discussion and comparison to existing results in the literature. Some numerical examples and figures are also provided to illustrate our results.
2. Sufficient and almost necessary conditions for permanence. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and \(B(t)\) be an one-dimensional Brownian motion. Because the dynamics of recover class has no effect on the disease transmission dynamics, we need only considering the following system

\[
\begin{cases}
    dS(t) = (\alpha - \beta S(t)I(t) - \mu S(t))dt - \sigma S(t)I(t)dB(t), \\
    dI(t) = (\beta S(t)I(t) - (\mu + \rho + \gamma) I(t))dt + \sigma S(t)I(t)dB(t).
\end{cases}
\] (3)

It is shown in [7] that \(\mathbb{R}^2_+ = \{(x, y) : x \geq 0, y \geq 0\}\) is invariant under the equation (3), i.e., if \(S(t) \geq 0, I(t) \geq 0\) for all \(t \geq 0\), the existence and uniqueness of positive solution are proved.

We first rewrite the equation (3) in Stratonovich’s form:

\[
\begin{cases}
    dS(t) = [\alpha - \beta S(t)I(t) - \mu S(t) - \frac{\sigma}{2} S(t)I(t)(I(t) - S(t))]dt - \sigma S(t)I(t) \circ dB(t) \\
    dI(t) = [\beta S(t)I(t) - (\mu + \rho + \gamma) I(t) + \frac{\sigma}{2} S(t)I(t)(I(t) - S(t))]dt \\
         + \sigma S(t)I(t) \circ dB(t).
\end{cases}
\] (4)

Denote by \((S^{s,i}(t), I^{s,i}(t))\) the solution with initial value \((s, i)\) to (4). In case there is no confusion, we simply write \((S(t), I(t))\) for \((S^{s,i}(t), I^{s,i}(t))\). Let \(P(t, s, i, \cdot)\) be its transition probability and let

\[A(s,i) = \begin{pmatrix} -\sigma si \\ \sigma si \end{pmatrix}, \quad B(s,i) = \begin{pmatrix} \alpha \beta si - \mu s - \frac{\sigma}{2} si(i - s) \\ \beta si - (\mu + \rho + \gamma)i + \frac{\sigma}{2} si(i - s) \end{pmatrix} .\]

By direct calculation we see that

\[ [A, B](s, i) = \begin{pmatrix} -\sigma \alpha i - \sigma^3 s^2 i^2 - \sigma (\mu + \rho + \gamma)i \\ \alpha \sigma i + \sigma \mu si + \sigma^3 s^2 i^2 \end{pmatrix} , \]

where \([\cdot, \cdot]\) is the Lie bracket of the vector fields. Therefore,

\[ \det(A, [A, B])(s, i) = \sigma^2 s^2 i^2(\gamma + \rho) \neq 0 \quad \forall (s, i) \in \mathbb{R}^2_+ .\]

Thus, the Lie algebra \(\mathcal{L}(s, i)\) generated by vector fields \(A(s, i)\) and \(B(s, i)\) and the ideal \(\mathcal{L}_0(s, i)\) in \(\mathcal{L}(s, i)\) generated by \(B(s, i)\) are non degenerate. In particular, the Hörmander condition (see [14, 15]) holds for the diffusion equation (4). As a result, the transition probability \(P(t, s, i, \cdot)\) of \((S^{s,i}(t), I^{s,i}(t))\) has density \(p(t, s, i, u, v)\) which is smooth in \((s, i, u, v) \in \mathbb{R}_+^4 \).

In order to study the control set and the support of diffusion process (4), we analyze the following control system

\[
\begin{cases}
    \dot{u}_\phi(t) = \alpha - \beta u_\phi(t)v_\phi(t) - \mu u_\phi(t) - \frac{\sigma^2}{2} u_\phi(t)v_\phi(t)(v_\phi(t) - u_\phi(t)) \\
    \dot{v}_\phi(t) = \beta u_\phi(t)v_\phi(t) - (\mu + \rho + \gamma)v_\phi(t) + \frac{\sigma^2}{2} u_\phi(t)v_\phi(t)(v_\phi(t) - u_\phi(t)) \\
                  - \sigma u_\phi(t)v_\phi(t)\phi(t) \\
    \dot{\phi}(t) = \beta u_\phi(t)v_\phi(t) - (\mu + \rho + \gamma)v_\phi(t) + \frac{\sigma^2}{2} u_\phi(t)v_\phi(t)(v_\phi(t) - u_\phi(t)) + \sigma u_\phi(t)v_\phi(t)\phi(t) ,
\end{cases}
\] (5)

where \(\phi\) is taken from the set of piecewise continuous real valued functions defined on \(\mathbb{R}_+\). Let \((u_\phi(t, u, v), v_\phi(t, u, v))\) be the solution to the equation (5) with control \(\phi\) and initial value \((u, v)\). Denote by \(\mathcal{O}_T^+(u, v)\) the reachable set from \((u, v) \in \mathbb{R}^2_+\), that is the set of \((u', v') \in \mathbb{R}^2\) such that there exist a \(t \geq 0\) and a control \(\phi(\cdot)\) satisfying \(u_\phi(t, u, v) = u', v_\phi(t, u, v) = v'\).

We now recall some concepts introduced in [10]. Let \(X\) be a subset of \(\mathbb{R}^2\) having the property: \(w_2 \in \overline{\mathcal{O}_T^+(w_1)}\) for any \(w_1, w_2 \in X\). Then, there is a unique maximal...
set $Y \supset X$ such that this property still holds for $Y$. Such $Y$ is called a control set. A control set $W$ is said to be invariant if $\mathcal{O}_t^W(w) \subset W$ for all $w \in W$.

Putting $z_\phi(t) = u_\phi(t) + v_\phi(t)$ yields the following equivalent control system

$$
\left\{ \begin{array}{l}
\dot{z}(t) = h(z_\phi(t), v_\phi(t)), \\
\dot{v}_\phi(t) = g(z_\phi(t), v_\phi(t)) + \sigma v_\phi(t)(z_\phi(t) - v_\phi(t))\phi(t),
\end{array} \right. \quad (6)
$$

where

$$
g(z, v) = \beta(z - v)v - (\mu + \rho + \gamma)v + \frac{\sigma^2}{2}(z - v)(2v - z),
$$

and

$$
h(z, v) = \alpha - \mu z - (\rho + \gamma)v.
$$

Let $(z_\phi(t, z_0, v_0), v_\phi(t, z_0, v_0))$ be the solution to the equation (6) with control $\phi$ and initial value $(z_0, v_0)$.

It is easy to see that for all $0 < v_0 < z_0$ we have $v_\phi(t, z_0, v_0) < z_\phi(t, z_0, v_0)$ for all control $\phi$ and $t > 0$. We have the following claims.

**Claim 1.** For all $v_0, v_1, z_0 \in (0, \frac{\alpha}{\rho + \gamma})$, $v_0, v_1 < z_0$, and $0 < \varepsilon < z_0 - v_1$, there exist a control $\phi$ and $T > 0$ such that $v_\phi(T, z_0, v_0) = v_1$, $|z_\phi(T, z_0, v_0) - z_0| < \varepsilon$.

Indeed, suppose that $v_0 < v_1 < z_0$ and let $\rho_1 = \sup\{|g(z, v)|, |h(z, v)| : v_0 \leq v \leq v_1, |z - z_0| \leq \varepsilon\}$, $\rho_2 = \min\{|v(z - v) : v_0 \leq v \leq v_1, |z - z_0| \leq \varepsilon, v < z\}$. Let’s choose $\phi(t) = \rho_3(t)$ with $2(\rho_1 + \rho_2 + 1)\varepsilon \geq v_1 - v_0$. It is easy to check that with this control, there is a $0 < T < \frac{\varepsilon}{\rho_1}$ such that $v_\phi(T, z_0, v_0) = v_1$, $|z_\phi(T, z_0, v_0) - z_0| < \varepsilon$.

**Claim 2.** For any $0 < z_0 < z_1 < \frac{\alpha}{\rho + \gamma}$, we have $h(z, v) = \alpha - \mu z - (\rho + \gamma)v > 0 \forall z \in [z_0, z_1]$, if $v_0$ is sufficiently small. As a result, there is a control $\phi$ and a $T > 0$ such that $z_\phi(T, z_0, v_0) = z_1$ and $v_\phi(t, z_0, v_0) = v_0 \forall 0 \leq t < T$.

**Claim 3.** Let $\frac{\alpha}{\rho + \gamma} < z_1 < z_0 < \frac{\alpha}{\rho}$ arbitrary. Since $h(z, v) = \alpha - \mu z - (\rho + \gamma)v < 0 \forall z \in [z_1, z_0]$ if $v$ is very close to $z$, there are a control $\phi$ and $T > 0$ satisfying $z_\phi(T, z_0, v_0) = z_1$.

**Claim 4.** For any $z_0 > \frac{\alpha}{\rho}$ and $0 < v_0 < z_0$, there are a control $\phi$, and a $T > 0$ such that $z_\phi(T, z_0, v_0) < \frac{\alpha}{\rho}$ and $v_\phi(t, z_0, v_0) = v_0 \forall 0 \leq t < T$.

Indeed, if $\tilde{z}(t)$ is the solution of the following equation

\[
\begin{aligned}
\frac{d\tilde{z}(t)}{dt} &= \alpha - \mu \tilde{z}(t) - (\rho + \gamma)v_0 \\
\tilde{z}(0) &= z_0, \quad t \geq 0,
\end{aligned}
\]

then $\lim_{t \to \infty} \tilde{z}(t) = \frac{\alpha - (\rho + \gamma)v_0}{\mu + \rho + \gamma}$. Therefore, there exists $T > 0$ such that $\tilde{z}(T) < \frac{\alpha}{\rho}$. By choosing $\phi(t) = \frac{\alpha - g(\tilde{z}(t), v_0)}{\sigma v_0(\tilde{z}(t) - v_0)}$, we have $z_\phi(t, z_0, v_0) = \tilde{z}(t)$ and $v_\phi(t, z_0, v_0) = v_0$. The claim satisfied.

**Claim 5.** It is seen that $h(z, v) > 0 \forall z \leq \frac{\alpha}{\rho + \rho + \gamma}, 0 \leq v < z$ and $h(z, v) < 0 \forall z \geq \frac{\alpha}{\rho}, 0 < v < z$. Therefore, we cannot find any control $\phi$ and $T > 0$ satisfying $z_\phi(T, z_0, v_0) = z_1$, where $z_1 = \min\left\{z_0, \frac{\alpha}{\rho + \rho + \gamma}\right\}$ or $z_1 = \max\left\{z_0, \frac{\alpha}{\rho}\right\}$.

From these five claims, we conclude that the control system (6) has a unique invariant control set $\mathcal{C} = \{(z, v) : z \in [\frac{\alpha}{\rho + \rho + \gamma}, \frac{\alpha}{\rho}], 0 < v < z\}$. This means the control system (5) has only one invariant control set, namely $\mathcal{S} = \{(u, v) \in \mathbb{R}^2_+ : u + v \in \left[\frac{\alpha}{\rho + \rho + \gamma}, \frac{\alpha}{\rho}\right]\}$. 

We are now in position to provide a condition for the existence of a unique invariant probability measure for the process \((S(t), I(t))\) and investigate some properties of the invariant probability measure. Firstly, define the threshold
\[
\lambda := \frac{\alpha \beta}{\mu} - \left( \mu + \rho + \gamma + \frac{\sigma^2 \alpha^2}{2\mu^2} \right).
\] (7)

By adding side by side in system (3), we have
\[
\alpha - (\mu + \rho + \gamma)(S(t) + I(t)) \leq \frac{d}{dt}(S(t) + I(t))
= \alpha - \mu(S(t) + I(t)) - (\rho + \gamma)I(t) \leq \alpha - \mu(S(t) + I(t)). \tag{8}
\]

Using the comparison theorem yields
\[
\frac{\alpha}{\mu + \rho + \gamma} \leq \liminf_{t \to \infty} (S(t) + I(t)) \leq \limsup_{t \to \infty} (S(t) + I(t)) \leq \frac{\alpha}{\mu} \text{ a.s.} \tag{9}
\]

**Theorem 2.1.** Let \((S(t), I(t))\) be the solution to equation (3). If \(\lambda > 0\), the process \((S(t), I(t))\) has an unique invariant probability measure whose support is \(S\).

**Proof.** We derive from (3) that
\[
S(t) = S(0) + \int_0^t (\alpha - \beta S(\tau) I(\tau) - \mu S(\tau)) d\tau - \int_0^t \sigma S(\tau) I(\tau) dB(\tau).
\]
From (9) and the strong law of large numbers for local martingales we have
\[
0 = \lim_{t \to \infty} \frac{S(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t (\alpha - \beta S(\tau) I(\tau) - \mu S(\tau)) d\tau. \tag{10}
\]
Thus,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left( -\alpha + \mu S(\tau) + \beta S(\tau) I(\tau) \right) d\tau = 0. \tag{11}
\]
Otherwise, by Itô’s formula, we obtain
\[
\ln I(t) = \ln I(0) + \int_0^t \left( \beta S(\tau) - (\mu + \rho + \gamma) - \frac{\sigma^2 S^2(\tau)}{2} \right) d\tau + \int_0^t \sigma S(\tau) dB(\tau). \tag{12}
\]
By (9) and (12), we conclude that
\[
0 \geq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( \beta S(\tau) - (\mu + \rho + \gamma) - \frac{\sigma^2 S^2(\tau)}{2} \right) d\tau. \tag{13}
\]
Multiplying (10) with \(-\frac{\beta}{\mu}\) and adding it into (13) we obtain
\[
0 \geq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( \beta S(\tau) - (\mu + \rho + \gamma) - \frac{\sigma^2 S^2(\tau)}{2} \right) d\tau
+ \lim_{t \to \infty} \frac{1}{t} \int_0^t \left( \frac{\alpha \beta}{\mu} - \frac{\beta^2}{\mu} S(\tau) I(\tau) - \beta S(\tau) \right) d\tau
= \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( \frac{\alpha \beta}{\mu} - (\mu + \rho + \gamma) - \frac{\sigma^2 S^2(\tau)}{2} - \frac{\beta^2}{\mu} S(\tau) I(\tau) \right) d\tau
= \lambda + \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( \frac{\sigma^2}{2} \left( \frac{\alpha^2}{\mu^2} - S^2(\tau) \right) - \frac{\beta^2}{\mu} S(\tau) I(\tau) \right) d\tau.
\]
Therefore,
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t [S(\tau)I(\tau)] d\tau \geq \frac{\lambda \mu}{\beta^2} := \mathbf{m} > 0.
\]  
(14)

Let \( h \) be a positive number,
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t S(\tau)I(\tau) d\tau \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t S(\tau)I(\tau) \mathbf{1}_{\{S(\tau) \geq \bar{h}, I(\tau) \geq h\}} d\tau
\]
\[
+ \limsup_{t \to \infty} \frac{1}{t} \int_0^t S(\tau)I(\tau) \left( \mathbf{1}_{\{S(\tau) \leq \bar{h}\}} + \mathbf{1}_{\{I(\tau) \leq h\}} \right) d\tau
\]
\[
\leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t S(\tau)I(\tau) \mathbf{1}_{\{S(\tau) \geq \bar{h}, I(\tau) \geq h\}} d\tau + h \limsup_{t \to \infty} \frac{1}{t} \int_0^t (S(\tau) + I(\tau)) d\tau
\]
\[
\leq \liminf_{t \to \infty} \frac{\alpha^2}{\mu^2 t} \int_0^t \mathbf{1}_{\{S(\tau) \geq \bar{h}, I(\tau) \geq h\}} d\tau + \frac{\alpha h}{\mu}.
\]  
(15)

Thus,
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{S(\tau) \geq \bar{h}, I(\tau) \geq h\}} d\tau \geq \frac{\mu}{\alpha} \left( \frac{\mu \bar{m}}{\alpha} - h \right) := \overline{\lambda} > 0
\]  
(16)

for \( h \) sufficiently small. Also, using (9) yields \( \lim_{t \to \infty} E \mathbf{1}_{\{S(t) + I(t) > \frac{\alpha}{\mu}\}} = 0 \), which implies that
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t E \mathbf{1}_{\{S(\tau) + I(\tau) > \frac{\alpha}{\mu}\}} d\tau \leq \frac{\overline{\lambda}}{2}.
\]  
(17)

Let \( D = \{(s, i) : s, i \geq \bar{h}, s + i \leq \frac{\alpha}{\mu}\} \). From the inequality \( 1_D(s, i) \geq 1_{\{s \geq \bar{h}, i \geq h\}} - 1_{\{s + i > \frac{\alpha}{\mu}\}} \), it follows that
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t E \mathbf{1}_{\{(S(\tau), I(\tau)) \in D\}} d\tau
\]
\[
\geq \liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{S(\tau) \geq \bar{h}, I(\tau) \geq h\}} d\tau - \limsup_{t \to \infty} \frac{1}{t} \int_0^t E \mathbf{1}_{\{S(\tau) + I(\tau) > \frac{\alpha}{\mu}\}} d\tau \geq \frac{\overline{\lambda}}{2} > 0.
\]  
(18)

We show that \( (S(t), I(t)) \) is a strong Feller Markov process. Indeed, from the right-hand side of inequalities (8), we have
\[
0 < S(t, s, i) + I(t, s, i) \leq (s + i) \exp\{-\mu t\} + \frac{\alpha}{\mu} (1 - \exp\{-\mu t\}).
\]

As a consequence, for any \( m > 0 \) the solution \( (S(t), I(t)) \), starting in the domain \( A_m = \{(s, i) : s + i < \frac{\alpha}{\mu} + m\} \) remains in \( A_m \) for all \( t \). Therefore, the solution \( (S(t), I(t)) \) of equation (4) with initial conditions in \( A_m \) forms a Markov process with states space \( A_m \). By virtue of boundedness of \( A_m \), the smoothness of the coefficients and the Hörmander condition for the diffusion equation (4), it follows that \( (S(t), I(t)) \) is a strong Feller Markov process (see [4, 10, 9] for details).

Since \( \mathbb{R}^{2, \circ}_+ \) is invariant under equation (3), \( (S(t), I(t)) \) is a Markov process with states space \( \mathbb{R}^{2, \circ}_+ \). Thus, by virtue of the compactness of \( D \) in \( \mathbb{R}^{2, \circ}_+ \), the inequality (18) and the support theorem [5, Theorem 8.1, p. 518] there is a unique invariant probability measure, namely \( \varphi^* \) on \( \mathbb{R}^{2, \circ}_+ \) (see [17] or [13]). Further, \( \varphi^* \) has the support \( S \).
\( \square \)
By the well-known results in [4] and [10] and the Hörmander condition, it follows that for any $\varphi^*$-integrable function $f$, and $(s, i) \in S$ we have

$$
P\left\{ \lim_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \int_0^t f(S^{s,i}(\tau), I^{s,i}(\tau))d\tau \right] = \int_{\mathbb{R}^2} f(u, v)\varphi^*(du, dv) \right\} = 1 \tag{19}$$

and

$$
\|P_t(s, i, \cdot) - \varphi^*(\cdot)\| \to 0 \text{ as } t \to \infty, \tag{20}
$$

where $\| \cdot \|$ is the total variation norm.

We now show that for any initial value $(s, i) \in \mathbb{R}^{2,0}_+$, $(S^{s,i}(t), I^{s,i}(t))$ eventually enters $S$. Indeed, suppose in the contrary that $\mathbb{P}(\Omega_1) > 0$ where $\Omega_1 = \{ S(t) + I(t) \geq \frac{\alpha}{\mu} \forall t > 0 \}$. Then, in $\Omega_1$ we have

$$
\frac{d}{dt}(S(t) + I(t)) = [\alpha - \mu(S(t) + I(t)) - (\rho + \gamma)I(t)] \leq -(\rho + \gamma)I(t). \tag{21}
$$

On the other hand from (14) it yields $\lim_{t \to \infty} \int_0^t I(\tau)d\tau = \infty$ a.s., which implies that $\lim_{t \to \infty}(S(t) + I(t)) = -\infty$ for all $\omega \in \Omega_1$. This is a contradiction.

Similarly, we can show that if $\Omega_2 = \{ S(t) + I(t) \leq \frac{\alpha}{\rho+\gamma} \forall t > 0 \}$, then $\mathbb{P}(\Omega_2) = 0$. Thus, for any initial value $(s, i) \in \mathbb{R}^{2,0}_+$, $(S^{s,i}(t), I^{s,i}(t))$ eventually enters $S$. By virtue of invariant property of $S$, it follows that $S$ is an absorptive set.

Summing up we obtain the following result.

**Theorem 2.2.** If $\lambda > 0$, then $S$ is an absorptive set and (19), (20) hold for all $(s, i) \in \mathbb{R}^{2,0}_+$.

We now give conditions for the disease-free equilibrium to be globally asymptotically stable.

**Theorem 2.3.** If $\lambda < 0$, then $(S^{s_0,i_0}(t), I^{s_0,i_0}(t)) \to (\frac{\alpha}{\mu}, 0)$ a.s. as $t \to \infty$ for all $(s_0, i_0) \in \mathbb{R}^{2,0}_+$, i.e., the disease will be extinct. Moreover, for any $(s_0, i_0) \in \mathbb{R}^{2,0}_+$,

$$
\lim_{t \to \infty} \frac{\ln I^{s_0,i_0}(t)}{t} = \lambda < 0 \text{ with probability 1.} \tag{22}
$$

**Proof.** We proceed in the following steps.

(i) By using a Lyapunov function, we can show that for any $\varepsilon > 0$, there exists a $\delta' > 0$ such that

$$
P\left\{ \lim_{t \to \infty} (S^{s,i}(t), I^{s,i}(t)) = \left( \frac{\alpha}{\mu}, 0 \right) \right\} \geq 1 - \varepsilon \forall (s, i) \in U_{\delta'} := \left\{ \left( \frac{\alpha}{\mu} - \delta', \frac{\alpha}{\mu} + \delta' \right) \times [0, \delta'] \right\}. \tag{23}
$$

(ii) There is $0 < \delta < \delta'$ such that for any $(s_0, i_0) \in \mathbb{R}^{2,0}_+$, the process $(S^{s_0,i_0}(t), I^{s_0,i_0}(t))$ is recurrent relative to $\hat{S}_\delta = \{ (s, i) \in \mathbb{R}^2_+ : s + i \leq \frac{\alpha}{\mu} + \delta, s \geq \delta \}.

(iii) There exists a $T > 0$ such that for any $(u, v) \in \hat{S}_\delta$, there exists a control $\phi$ such that $(u_\phi(t, u, v), v_\phi(t, u, v)) \in U_{\delta}$ for some $t \in [0, T]$.

(iv) Using Markov property of the solution and the support theorem we show that $U_{\delta}$ is absorptive and then we obtain the desired conclusion.

Indeed, for item (i), by virtue of the inequality $\lambda = \frac{\alpha^2}{\mu} - (\mu + \rho + \gamma + \frac{\sigma^2}{\mu}) < 0$ and the continuity of the function $F(x, y) = \beta(\frac{\alpha}{\mu} + x) - (\mu + \rho + \gamma + (1 - p)\frac{\sigma^2}{2})(\frac{\alpha}{\mu} + \delta_1)^2 < 0$ in $(x, y)$, we can choose $0 < p < 1$ and $0 < \delta_1 < \frac{\alpha}{\mu}$ satisfying

$$
-c_3(\delta_1) := \beta(\frac{\alpha}{\mu} + \delta_1) - (\mu + \rho + \gamma + (1 - p)\frac{\sigma^2}{2})(\frac{\alpha}{\mu} - \delta_1)^2 < 0.
$$
Consider the Lyapunov function $V(s, i) = (\alpha - \mu s)^2 + ip_i$. By direct calculation we have

$$LV(s, i) = 2\mu(\mu s - \alpha)|\alpha - \mu s - \beta s| + p_i^2|\beta s - (\mu + \rho + \gamma)| + \frac{p(p - 1)\sigma^2 s^2 p_i^2}{2} + \mu^2 \sigma^2 s^2 i^2$$

$$\leq -\mu(\alpha - \mu s)^2 + p_i^2\left[\beta s - (\mu + \rho + \gamma(1 - p)\frac{\sigma^2 s^2}{2}\right]i^2 + \mu(\sigma^2 + \beta^2)s^2 i^2 \forall (s, i) \in \mathbb{U}_{\delta}.$$  

Since $i^2 = o(p_i^2)$ as $i \to 0$ and $-c_3(\delta_1) \downarrow$ as $\delta_1 \downarrow$, we can choose $0 < \delta_2 < \delta_1$ such that

$$p_i^2\left[\beta s - (\mu + \rho + \gamma(1 - p)\frac{\sigma^2 s^2}{2}\right]i^2 + \mu(\sigma^2 + \beta^2)s^2 i^2 \leq -\frac{p(c_3(\delta_1))}{2},$$

for any $(s, i) \in \mathbb{U}_{\delta_2}$. Therefore,

$$LV(s, i) \leq -\mu(\alpha - \mu s)^2 - \frac{c_3(\delta_1)p_i^2}{2} \leq -\theta V(s, i) \forall (s, i) \in \mathbb{U}_{\delta},$$

where $\theta = \min\{\mu, \frac{c_3(\delta_1)p_i^2}{2}\}$. By [12, Theorem 2.3. p. 112], for any $\varepsilon > 0$, there is $0 < \delta' < \delta_2$ such that

$$\mathbb{P}\left\{\lim_{t \to \infty} (S^{s,i}(t), I^{s,i}(t)) = \left(\frac{\alpha}{\mu}, 0\right)\right\} \geq 1 - \varepsilon \forall (s, i) \in U_{\delta'}. \quad (24)$$

The proof for part (ii) is motivated by that of [2, Proposition 3.3]. Indeed, from (9) and (11), we obtain

$$\limsup_{t \to \infty} \int_0^t S(\tau) d\tau \geq \frac{\alpha}{\frac{\alpha^2}{p} + \mu} > 0. \quad (25)$$

Therefore, there is $\delta < \delta'$ such that $\limsup_{t \to \infty} S(t) > 2\delta$. Further, from (9), there exists $t_0 > 0$ such that

$$S(t) + I(t) \leq \frac{\alpha}{\mu} + \delta \forall t \geq t_0. \quad (26)$$

Combining (25) and (26) follows that $(S^{s_i, i_0}(t), I^{s_i, i_0}(t))$ is recurrent relative to $\hat{S}_{\delta} := \{(s, i): s \geq \delta, 0 \leq s + i \leq \frac{\alpha}{p} + \delta\}$. To show (iii) we have Claim 6. For any $z_0 \in \left(0, \frac{\alpha}{p} + \delta\right)$, there is a $T > 0$ satisfying $z_\phi(T, z_0, 0) \in (\frac{\alpha}{p} - \delta, \frac{\alpha}{p} + \delta)$ and clearly $v_{\phi}(T, z_0, 0) = 0$.

Indeed from (6) we see that, for any control $\phi(\cdot)$ and $z \in \mathbb{R}_+$, $v_{\phi}(t, z, 0) = 0 \forall t$. Moreover $h(z, 0) = \alpha - \mu z > 0, \forall z \in (0, \frac{\alpha}{p})$ then for any $z_0 \in (0, \frac{\alpha}{p})$, there is a $T > 0$ satisfying $z_\phi(T, z_0, 0) \in (\frac{\alpha}{p} - \delta, \frac{\alpha}{p})$. If $z_0 \in [\frac{\alpha}{p}, \frac{\alpha}{p} + \delta)$ then by the continuity of the solution of (6), we can choose $T$ small enough such that $z_\phi(T, z_0, 0) \in [\frac{\alpha}{p}, \frac{\alpha}{p} + \delta)$.

By using Claims 1-6, it follows that for each $(s, i) \in \hat{S}_{\delta}$, we can choose a control $\phi(\cdot)$ and $T_{s,i} > 0$ such that

$$(u_{\phi}(T_{s,i}, s, i), v_{\phi}(T_{s,i}, s, i)) \in \mathbb{U}_i.$$  

In view of the support theorem [5, Theorem 8.1, p. 518], for all $(s, i) \in \hat{S}_{\delta}$ there is a $T_{s,i} > 0$ such that

$$\mathbb{P}\{\{(S^{s,i}(T_{s,i}), I^{s,i}(T_{s,i})) \in \mathbb{U}_i\} > 2p^{s,i} > 0.$$  

Since the process $(S(t), I(t))$ is strong Feller, there is a neighborhood $V_{s,i}$ of $(s, i)$ such that

$$\mathbb{P}\{(S^{s', i'}(T_{s,i}), I^{s', i'}(T_{s,i})) \in \mathbb{U}_i\} > p^{s,i}, \forall (s', i') \in V_{s,i}.$$
By virtue of compactness of $\hat{S}_\delta$ and Heine-Borel theorem, we can find a finite number of $V_{s,j}, \ j = 1, \ldots, n$ such that $\hat{S}_\delta \subset \bigcup_{j=1}^{n} V_{s,j}$. Let

$$T^* = \max\{T_{s,j}, j = 1, \ldots, n\}, \ p^* = \min\{p_{s,j}, j = 1, \ldots, n\}. $$

For $(s, i) \in \mathbb{R}^{2,\circ}_+$, set

$$\tau^{s,i}_\delta = \inf\{t > 0 : (S^{s,i}(t), I^{s,i}(t)) \in \mathcal{U}_\delta\}.$$  

Then, it is easy to see that

$$\mathbb{P}\{\tau^{s,i}_\delta < T^*\} \geq p^* > 0, \ \forall (s; i) \in \hat{S}_\delta.$$  

Define a sequence of stopping times

$$\eta_0 = 0, \eta_k = \inf\{t > \eta_{k-1} + T^* : (S^{s_0,i_0}(t), I^{s_0,i_0}(t)) \in \hat{S}_\delta\}, k \in \mathbb{N}. $$

Since $(S^{s_0,i_0}(t), I^{s_0,i_0}(t))$ is recurrent relative to $\hat{S}_\delta$, $\eta_k$ is finite for every $k$. Consider the events

$$A_k = \{(S^{s_0,i_0}(t), I^{s_0,i_0}(t)) \notin \mathcal{U}_\delta, \ \forall t \in [\eta_k, \eta_k + T^*]\}. $$

To obtain last item (iv), we deduce from the strong Markov property of $(S(t), I(t))$ and (27) that

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) \leq (1 - p^*)^n \to 0 \text{ as } n \to \infty.$$  

As a result,

$$\mathbb{P}\{\tau^{s_0,i_0}_\delta < \infty\} = 1.$$  

In light of the strong Markov property of $(S(t), I(t))$, (23) and (28) yield

$$\mathbb{P}\{\lim_{t \to \infty} (S^{s_0,i_0}(t), I^{s_0,i_0}(t)) = (\frac{\alpha}{\mu}, 0)\} \geq 1 - \varepsilon \ \forall (s_0, i_0) \in \mathbb{R}^{2,\circ}_+.$$  

Since $\varepsilon$ can be taken arbitrarily, we obtain

$$\mathbb{P}\{\lim_{t \to \infty} (S^{s_0,i_0}(t), I^{s_0,i_0}(t)) = (\frac{\alpha}{\mu}, 0)\} = 1 \ \forall (s_0, i_0) \in \mathbb{R}^{2,\circ}_+.$$  

Finally, it follows from Itô’s formula that

$$\frac{\ln I^{s_0,i_0}(t)}{t} = \frac{\ln i_0}{t} + \frac{1}{t} \int_{0}^{t} \left[\beta S(\tau) - \mu - \rho - \gamma - \frac{\sigma^2 S^2(\tau)}{2}\right] d\tau + \frac{\sigma}{t} \int_{0}^{t} S(\tau) dB(\tau).$$  

Hence (22) follows straightforward from (30) and (31). \hfill \square

3. Discussion and numerical examples. We have classified whenever the disease in a stochastic SIR model is extinct or permanent by the sign of a threshold value $\lambda$. Only the critical case $\lambda = 0$ is not studied in this paper.

To illustrate the significance of our results, let us compare our results with those in [11], where authors considered the conditions for which the semi-group $\{P(t)\}$ is either asymptotically stable or sweeping to compact sets. Specially, they claimed that

**Theorem 3.1 ([11, Theorem 2.1]).** Let $(S(t), I(t))$ be a solution of system (3). Then for every $t > 0$ the distribution of $(S(t), I(t))$ has a density $u(t, x, y)$ which satisfies Fokker-Planck equation. If $R_0 - 1 > \frac{\sigma^2}{2\mu^2(\mu + \rho + \gamma)}$ and $\sigma^2 < \frac{2\mu(\mu + \rho)}{\alpha(\mu S^2 + (\mu + \rho + \gamma) \int I^2)}$, then
then there exists a unique density \( u_*(x, y) \) which is a stationary solution of Foraker-Planck equation and
\[
\lim_{t \to \infty} \int \int_{\mathbb{R}^2} \left| u(t, x, y) - u_*(x, y) \right| dx dy = 0,
\]
where
\[
R_0 = \frac{\alpha \beta}{\mu (\mu + \rho + \gamma)}; \quad S^* = \frac{\mu + \rho + \gamma}{\beta}; \quad I^* = \frac{\alpha}{\mu + \rho + \gamma} - \frac{\mu}{\beta}; \quad \alpha = \frac{2 \mu + \rho + \gamma}{\beta}.
\]
In addition, we have
\[
\text{supp} u_* = \left\{ (x, y) \in \mathbb{R}^2 : \frac{\alpha}{\mu + \rho + \gamma} < x + y < \frac{\alpha}{\mu} \right\}.
\]

**Theorem 3.2** ([11, Theorem 2.2]). Assume that

(a) \( R_0 - 1 < \frac{\sigma^2 \alpha^2}{(\mu + \rho + \gamma)} \) and \( \sigma^2 > \frac{\beta}{\alpha} \), or

(b) \( \sigma^2 > \max\left\{ \frac{2 \mu + \rho + \gamma}{\beta}, \frac{\beta \mu}{\alpha} \right\} \).

Then
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq (\mu + \rho + \gamma)(R_0 - 1 - \frac{\sigma^2 \alpha^2}{2 \mu^2 (\mu + \rho + \gamma)}) < 0 \ a.s. \text{ if (a) holds;}
\]
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq \frac{\beta^2}{2 \sigma^2} - (\mu + \rho + \gamma) < 0 \ a.s. \text{ if (b) holds;}
\]
In addition, if \( \lim_{t \to \infty} I(t) = 0 \ a.s. \), then \( \lim_{t \to \infty} S(t) = \frac{\alpha}{\mu} \ a.s. \).

It is easy to see that the condition \( R_0 - 1 > \frac{\sigma^2 \alpha^2}{2 \mu^2 (\mu + \rho + \gamma)} \) in Theorem 3.1 is equivalent to condition \( \lambda > 0 \). Also, the condition \( R_0 - 1 < \frac{\sigma^2 \alpha^2}{2 \mu^2 (\mu + \rho + \gamma)} \) in Theorem 3.2 implies that \( \lambda < 0 \). Theorems 2.1, 2.2 in this paper states that with only the condition \( \lambda > 0 \), we can conclude that the semi-group \( \{ P(t) \} \) is asymptotically stable. Thus, if \( \lambda < 0 \) holds, then Theorem 2.3 in this paper confirms that \( \{ P(t) \} \) is sweeping to compact sets. In these theorems, we do not require any further assumption.

Let us finish this paper by providing some numerical examples.

**Example 1.** Consider (3) with parameters \( \alpha = 40, \beta = 50, \mu = 20, \rho = 10, \gamma = 2, \sigma = 5.8 \). For these parameters, we get \( \sigma^2 > \frac{2 \mu + \rho + \gamma}{\alpha (\mu + \rho + \gamma) (\mu + \rho + \gamma)} \). This implies that the conditions in Theorem 3.1 are not satisfied. Otherwise, direct calculating shows that \( \lambda = 0.72 > 0 \). As a result of Theorem 2.1, the system (3) has a unique invariant probability measure \( \pi^* \) whose support is \( S = \{(s, i) : 1.25 \leq s + i \leq 2\} \). Moreover, the strong law of large numbers and the convergence in total variation norm of the transition probability hold.

A sample path of solution to (3) is illustrated by Figures 1, while the phase portrait in Figure 2 demonstrates that the support of \( \pi^* \) is a domain surrounded by the lines \( s+i = 1.25; i = 0; s = 0 \) and \( s+i = 2 \). The empirical density of \( \pi^* \) is shown in this picture as well.

**Example 2.** Consider (3) with parameters \( \alpha = 4, \beta = 0.5, \mu = 1, \rho = 0.2, \gamma = 0.3, \sigma = 0.4 \). For these parameters, we get \( \frac{2 \mu}{\rho} < \sigma^2 < \frac{\beta}{4(\mu + \rho + \gamma)} \). It means the conditions in Theorem 3.2 are not satisfied. We obtain \( \lambda = -0.78 < 0 \). As a result of Theorem 2.3 that \( \lim_{t \to \infty} (S(t), I(t)) = (4, 0) \) a.s. A sample path of solution to (3) is illustrated by Figures 3.
Figure 1. Trajectories of $S(t), I(t)$ in Example 1.

Figure 2. Phase portrait of (3); the boundary $s + i = 1, 25$ and $s + i = 2$ of the support of $\pi^*$ and the empirical density of $\pi^*$ in Example 1.

Figure 3. Trajectories of $S(t), I(t)$ in Example 2.

Acknowledgments. We gratefully thank the reviewers and the editor for constructive comments and detailed suggestions, which help to improvement of the presentation of the paper.
REFERENCES

[1] N. T. Dieu, D. H. Nguyen, N. H. Du and G. Yin, Classification of Asymptotic Behavior in a Stochastic SIR Model, SIAM J. Appl. Dyn. Syst., 15 (2016), 1062–1084.

[2] N. H. Du, D. H. Nguyen and G. Yin, Conditions for permanence and ergodicity of certain stochastic predator-prey models, J. Appl. Probab., 53 (2016), 187–202. Available from: http://projecteuclid.org/euclid.jap/1457470568.

[3] A. Gray, D. Greenhalgh, L. Hu, X. Mao and J. Pan, A stochastic differential equation SIS epidemic model, SIAM J. Appl. Math., 71 (2011), 876–902.

[4] K. Ichihara and H. Kunita, A classification of the second order degenerate elliptic operators and its probabilistic characterization, Z. Wahrsch. Verw. Gebiete, 30 (1974), 235–254; Corrections in 39 (1977), 81–84.

[5] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, second edition, North-Holland Publishing Co., Amsterdam, 1989.

[6] C. Ji and D. Jiang, Threshold behaviour of a stochastic SIR model, Appl. Math. Model., 38 (2014), 5067–5079.

[7] C. Y. Ji, D. Q. Jiang and N. Z. Shi, The behavior of an SIR epidemic model with stochastic perturbation, Stochastic Anal. Appl., 30 (2012), 755–773.

[8] C. Y. Ji, D. Q. Jiang and N. Z. Shi, Multigroup SIR epidemic model with stochastic perturbation, J. IFAC, 48 (2012), 121–131.

[9] R. Z. Khas’minskii, Stochastic Stability of Differential Equations, Springer-Verlag Berlin Heidelberg, 2012.

[10] W. Kliemann, Recurrence and invariant measures for degenerate diffusions, Ann. Probab., 15 (1987), 690–707.

[11] Y. G. Lin and D. Q. Jiang, Long-time behaviour of a perturbed SIR model by white noise, Discrete Contin. Dyn. Syst. Ser. B, 18 (2013), 1873–1887.

[12] X. Mao, Stochastic Differential Equations and Their Applications, Horwood Publishing Chichester, 1997.

[13] S. P. Meyn and R. L. Tweedie, Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes, Adv. Appl. Prob., 25 (1993), 518–548.

[14] J. Norris, Simplified Malliavin calculus, In: Séminaire de probabilités XX, Lecture Notes in Mathematics, Springer, New York, 1204 (1986), 101–130.

[15] D. Nualart, The Malliavin Calculus and Related Topics, Springer-Verlag, Berlin Heidelberg, 2006.

[16] H. Schurz and K. Tosun, Stochastic asymptotic stability of SIR model with variable diffusion rates, J. Dynam. Differential Equations, 27 (2015), 69–82.

[17] L. Stettner, On the existence and uniqueness of invariant measure for continuous time Markov processes, LICS Report, No. 86-16, April 1986, Brown University, Providence. Available from: https://www.amazon.co.uk/existence-uniqueness-invariant-continuous-processes/dp/ B000722C66

[18] E. Tornatore, S. M. Buccellato and P. Vetro, Stability of a stochastic SIR system, Phys. A, 354 (2005), 111–126.

[19] Q. Yang and X. Mao, Stochastic dynamics of SIRS epidemic models with random perturbation, Math. Biosci. Eqs., 11 (2014), 1003–1025.

[20] X. Zhong and F. Deng, Extinction and persistent of a stochastic multi-group SIR epidemic model, Journal of Control Science and Engineering, 1 (2013), 13–22.

[21] Y. Zhou, W. Zhang and S. Yuan, Survival and stationary distribution of a SIR epidemic model with stochastic perturbations, Appl. Math. Comput., 244 (2014), 118–131.

Received December 2015; revised June 2016.

E-mail address: dunh@vnu.edu.vn
E-mail address: dieunguyen2008@gmail.com