Quantum network coding for quantum repeaters

Takahiko Satoh, François Le Gall, and Hiroshi Imai

Department of Computer Science, Graduate School of Information Science and Technology, The University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo, Japan

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This paper considers quantum network coding, which is a recent technique that enables quantum information to be sent on complex networks at higher rates than by using straightforward routing strategies. Kobayashi et al. have recently showed the potential of this technique by demonstrating how any classical network coding protocol gives rise to a quantum network coding protocol. They nevertheless primarily focused on an abstract model, in which quantum resource such as quantum registers can be freely introduced at each node. In this work, we present a protocol for quantum network coding under weaker (and more practical) assumptions: our new protocol works even for quantum networks where adjacent nodes initially share one EPR-pair but cannot add any quantum registers or send any quantum information. A typically example of networks satisfying this assumption is quantum repeater networks, which are promising candidates for the implementation of large scale quantum networks. Our results thus show, for the first time, that quantum network coding techniques can increase the transmission rate in such quantum networks as well.

I. INTRODUCTION

Quantum communications hold potentialities which are qualitatively different from classical communications. For example, quantum key distribution (QKD) provides shared, secret bits (useful for classical cryptography) whose secrecy does not depend on the presumed difficulty of factoring large numbers or other number-theoretic problems, as the commonly-used Diffie-Hellman key exchange protocol does.

Urban scale and complex topology QKD networks have already been constructed experimentally [1, 2]. However, it is still difficult to realize long distance quantum communication. Quantum repeaters [3, 4] are a potential approach for dealing with this problem. Quantum repeaters have three important functions: the first is the basic physical creation of entanglement over long distances, the second is management of imperfections in the created quantum states (e.g., purification [5–7] or recent works using error correction different from purification [8–10]), and the third is extending entanglement from the endpoints of a single channel to distant nodes in a topologically complex network (e.g., entanglement swapping [11, 12]).

In quantum repeater networks, EPR-pairs are consumed as a source of quantum communication and require a high cost for sharing and conservation. The communication capacity of a quantum repeater network is limited by the maximum number of qubits the quantum repeater can store and operate on at one time. Hence, in the future, large and complex quantum repeater networks will be confronted with the bottleneck problem caused by shortage of quantum resources.

Meanwhile, large scale classical networks such as the Internet have continued to increase their communication volume, and also have the bandwidth bottleneck problem. To address this problem, classical network coding [14] is drawing attention. One of the most useful applications of this method is throughput enhancement for certain traffic patterns: network coding is able to achieve higher throughput than independent forwarding of every data packet, by active encoding of the packets at intermediate nodes. We show an example of multiple-unicast transmission over the directed butterfly network by using this technique in Fig. 1.

Recently, researchers expanded network coding to include quantum information [15–18] and showed that network coding using quantum information is available with-
out infringement of the non-cloning theorem (which forbids duplication of an unknown quantum state). After that, [13][21] proved that, for any graph, quantum perfect network coding is feasible, if free classical channels are available, whenever classical network coding is possible. But these later works do not consider concrete implementation issues and, especially, assume the availability of additional quantum resources such as quantum registers at each node of the network. Implementation is nevertheless a fundamental problem. Indeed, in order to be able to use network coding on real quantum networks, the amount of quantum resources required by the protocol need to be minimized.

In this work, we study quantum network coding for practical quantum networks where adjacent nodes initially share one EPR-pair but cannot add any quantum registers or send any quantum information. Typical examples are the quantum repeater networks discussed above. Since this setting forbids the introduction of quantum registers, the methods from [13][21] cannot be applied directly. Our results nevertheless demonstrate that quantum network coding can be realized in this model as well and are, to the best of our knowledge, the first application of network coding to increase the transmission rate in quantum repeater networks. This may become an effective countermeasure against communication congestion in quantum repeater networks.

Our results are obtained by constructing a version of the protocol in [21] that does not require the introduction of any quantum register. This is non-trivial and requires new ideas. The key idea is to convert, using only local operations and classical communication, the EPR-pairs between adjacent nodes into appropriate entangled states of higher dimension shared between distant nodes. To do this, we introduce two new techniques inspired by quantum teleportation [22] and one-way quantum computation [23], which we call “Connection” and “Removal”, that enable us to manipulate such entangled states and systematize the methods of encoding.

II. PRELIMINARIES

A. Notations

We suppose that the reader is familiar with the basics of quantum information theory and refer to [24] for a good reference. In classical information science, the fundamental unit of information is described as a binary digit (bit). In the case of quantum information, a quantum bit (qubit) is the equivalent of a bit. A qubit is expressed as a superposition of two orthonormal quantum states \( |0 \rangle \) and \( |1 \rangle \) with amplitudes \( \alpha \) and \( \beta \) as follows:

\[
|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,
\]

where \( \alpha \) and \( \beta \) are complex numbers such that \( |\alpha|^2 + |\beta|^2 = 1 \). A general quantum state of \( n \) qubits can be written as

\[
|\psi\rangle = \sum_{x \in \{0, 1\}^n} \alpha_x |x\rangle,
\]

where \( \alpha_x \) are complex numbers such that \( \sum_{x \in \{0, 1\}^n} |\alpha_x|^2 = 1 \).

In this paper, we will use the Pauli operators \( \sigma_X \) and \( \sigma_Z \) and the Hadamard operator, which are the following single qubit transformations:

\[
\sigma_X := |1\rangle\langle 0| + |0\rangle\langle 1|,
\]

\[
\sigma_Z := |0\rangle\langle 0| - |1\rangle\langle 1|,
\]

\[
H := \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1| - |1\rangle\langle 0|).
\]

We denote by \(|+\rangle, |−\rangle\) the Hadamard basis:

\[
|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),
\]

\[
|−\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).
\]

We will also use the Control-NOT gate, which is the following two qubits transformation.

\[
\text{CNOT}^{(A, B)} := |0\rangle_A |0\rangle_B |0\rangle_A |B\rangle_A |1\rangle_B |B\rangle_A + |1\rangle_A |B\rangle_B |1\rangle_A |0\rangle_B |1\rangle_A |B\rangle_B.
\]

We denote by \( |\Psi^+\rangle \) and \( |\Phi^+\rangle \) the following two qubits state (EPR-pairs):

\[
|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),
\]

and by \( |GHZ\rangle \) the following three qubits state:

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).
\]

B. Quantum repeater network

We define a quantum repeater network as a network consisting of a number of quantum repeaters, undirected classical channels and EPR-pairs \( |\Psi^+\rangle \) (each pair of adjacent quantum repeaters shares one EPR-pair). We show an example of network with three quantum repeaters in Fig. 2.

On the network of Fig. 2, quantum communications are possible between adjacent repeaters \( (s-r \) and \( r-t) \) by teleportation using shared EPR-pairs. Furthermore, quantum communication between the non-adjacent repeaters \( s \) and \( t \) is possible by applying entanglement swapping (the relay repeater \( r \) converts the two EPR-pairs \( |\Psi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} \) to one EPR-pair \( |\Psi^+\rangle_{AD} \)). In this way, each repeater performs quantum communication by EPR-pairs and LOCC on this network.

The present paper will show that, for specific networks, network coding can achieve a better throughput than this simple entanglement swapping strategy.
III. QUANTUM REPEATER NETWORK CODING

We present the setting for our protocol in Fig. 3. The goal of this work is to simultaneously send quantum information between two pairs of repeaters \((s_1, t_1)\) and \((s_2, t_2)\) located diagonally on a butterfly quantum repeaters network. For this purpose, we construct a protocol for quantum repeater network coding that generates EPR-pairs \(|\Psi^+\rangle_{AB}\) and \(|\Psi^+\rangle_{CD}\) between \(s-r\) and \(r-t\), respectively.

We cannot generate these EPR-pairs simultaneously because these methods require the introduction of intermediate quantum registers, which is not possible in our model of quantum repeater networks. In this work, we construct a protocol for sharing EPR-pairs without additional quantum registers.

In subsection A and B we first show two new techniques. In subsection C we give an overview of our protocol. In subsection D we present a preliminary protocol. In subsection E we give the final version of our protocol.

**TABLE I. Con\(C_{R\rightarrow T}\)**

| Step 1. | \(u\) applies CNOT\((C,R)\). |
|---------|-------------------------------|
| Step 2. | \(u\) measures \(R\) in the \(|0\rangle, |1\rangle\) basis. Let \(a \in \{0,1\}\) be the outcome. |
| Step 3. | \(u\) sends \(a\) to \(v\) by a classical channel. |
| Step 4. | If \(a = 1\) then \(v\) applies \(\sigma_X\) to \(T\) |

**A. Technique 1: Connection**

Our first technique is called Connection. Connection is a non-unitary operation between two repeaters \((u\) and \(v\), respectively). Repeater \(u\) has Control and Resource qubits \((C\) and \(R\), respectively). Repeater \(v\) has a Target qubit \((T)\). We show the procedure for Connection as Table I. This technique corresponds to sending one bit in the original classical network coding scheme and is utilizing the basis manipulation method of quantum teleportation [22]. The following lemma shows the action of Connection.

**Lemma 1.** Let \(|\Psi_{init}\rangle\) be a state of the form

\[|\Psi_{init}\rangle = (\alpha|\psi_0\rangle|0\rangle_C + \beta|\psi_1\rangle|1\rangle_C) \otimes |\Psi^+\rangle_{RT} \otimes |\Phi\rangle,\]

where \(\alpha^2 + \beta^2 = 1\) and \(|\psi_0\rangle\), \(|\psi_1\rangle\) and \(|\Phi\rangle\) are arbitrary quantum states. Then the state after applying Con\(C_{R\rightarrow T}\) to \(|\Psi_{init}\rangle\) is

\[|\Psi_{final}\rangle = (\alpha|\psi_0\rangle|00\rangle_{CT} + \beta|\psi_1\rangle|11\rangle_{CT}) \otimes |\Phi\rangle,\]

where register \(R\) can be disregarded.

**Proof.** At step 1, we apply CNOT\((C,R)\). The state becomes

\[|\Psi_1\rangle = \alpha|\psi_0\rangle|0\rangle_C \otimes |\Psi^+\rangle_{RT} \otimes |\Phi\rangle + \beta|\psi_1\rangle|1\rangle_C \otimes |\Psi^+\rangle_{RT} \otimes |\Phi\rangle.\]

At step 2, we measure \(R\). When \(a = 0\) the state becomes

\[|\Psi_2\rangle = (\alpha|\psi_0\rangle|00\rangle_{CT} + \beta|\psi_1\rangle|11\rangle_{CT}) \otimes |\Phi\rangle,\]

and when \(a = 1\) the state becomes

\[|\Psi_2\rangle = (\alpha|\psi_0\rangle|01\rangle_{CT} + \beta|\psi_1\rangle|10\rangle_{CT}) \otimes |\Phi\rangle,\]

where register \(R\) has been disregarded since it is not entangled anymore. At step 4, if \(a = 1\) then we apply \(\sigma_X\) to \(T\). The state becomes

\[|\Psi_3\rangle = (\alpha|\psi_0\rangle|00\rangle_{CT} + \beta|\psi_1\rangle|11\rangle_{CT}) \otimes |\Phi\rangle = |\Psi_{final}\rangle.\]

For example, Lemma 1 shows that applying Con\(A\) to two EPR-pairs

\[|\Psi_{init}\rangle = |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}\]
From Lemma 1, the state becomes

\[ \text{Lemma 1, the state becomes} \]

**Proof.**

We now show two variants of the above Connection operation. The first variant is “multiple resource and target qubits” Connection and called Connection:Fanout (or **Fanout**). We show the procedure for Connection:Fanout as table III.

Note that applying **Fanout** is equivalent to applying **Con** in the state.

We can derive the following lemma.

**Lemma 2.** Let \( |\Psi_{\text{init}}\rangle \) be a state of the form

\[ |\Psi_{\text{init}}\rangle = (\alpha |\psi_0\rangle |0\rangle_A + \beta |\psi_1\rangle |1\rangle_A) |\Psi^+\rangle_{BC} |\Psi^+\rangle_{DE} \otimes |\Phi\rangle, \]

where \( \alpha^2 + \beta^2 = 1 \) and \( |\psi_0\rangle, |\psi_1\rangle \) and \( |\Phi\rangle \) are arbitrary quantum states. Then the state after applying **Fanout** to \( |\Psi_{\text{init}}\rangle \) is

\[ |\Psi_{\text{final}}\rangle = (\alpha |\psi_0\rangle |000\rangle_{AC} + \beta |\psi_1\rangle |111\rangle_{AC}) \otimes |\Phi\rangle, \]

where registers B and D are disregarded.

**Proof.** At step 1, we apply **Con** to \( |\Psi_{\text{init}}\rangle \). From Lemma, the state becomes

\[ |\Psi_1\rangle = (\alpha |\psi_0\rangle |00\rangle_{AC} + \beta |\psi_1\rangle |11\rangle_{AC}) |\Psi^+\rangle_{DE} \otimes |\Phi\rangle, \]

where register B has been disregarded since it is not entangled anymore. At step 2, we apply **Con** to \( |\Psi_1\rangle \). From Lemma, the state becomes

\[ |\Psi_2\rangle = (\alpha |\psi_0\rangle |00\rangle_{AC} + \beta |\psi_1\rangle |11\rangle_{AC}) \otimes |\Phi\rangle = |\Psi_{\text{final}}\rangle, \]

Here, \( |\phi_0\rangle = \alpha |0\rangle + \beta |1\rangle \).

**FIG. 4.** The quantum circuit for getting GHZ-states from two EPR-pairs by Connection.

**TABLE II.** **Fanout**

| \( R_1 \rightarrow T_1 \) | \( R_2 \rightarrow T_2 \) |
|-----------------|-----------------|
| \( C \) is a 1-qubit register owned by u. | \( T_1 \) is a 1-qubit register owned by v. |
| \( T_2 \) is a 1-qubit register owned by w. |

**Step 1.** \( u \) and \( v \) apply **Con**.

**Step 2.** \( u \) and \( v \) apply **Con**.

We show the corresponding quantum circuit in Fig. 4.

**FIG. 5.** The circuit for Connection:Fanout.

**TABLE III.** **Add**

| \( C_1 \) and \( C_2 \) are 1-qubit registers owned by u. |
| \( T \) is a 1-qubit register owned by v. |

**Step 1.** \( u \) applies **CNOT**.

**Step 2.** \( u \) and \( v \) apply **Con**.

where register D has been disregarded since it is not entangled anymore.

For example, Lemma shows that applying **Fanout** to

\[ |\Psi_{\text{init}}\rangle = (\alpha |0\rangle_A + \beta |1\rangle_A) |\Psi^+\rangle_{BC} |\Psi^+\rangle_{DE} \]

\[ |\Psi_{\text{final}}\rangle = \alpha |000\rangle_{AC} + \beta |111\rangle_{AC}. \]

We show the corresponding circuit in Fig. 5.

The next variant is “multiple control qubits” Connection and called Connection:Add (or **Add**). We show the procedure for Connection:Add as Table III and prove the following lemma.

**Lemma 3.** Let \( |\Psi_{\text{init}}\rangle \) be a state of the form

\[ |\Psi_{\text{init}}\rangle = (\alpha |\psi_0\rangle |0\rangle_A + \beta |\psi_1\rangle |1\rangle_A) \otimes (\gamma |\phi_0\rangle |0\rangle_B + \delta |\phi_1\rangle |1\rangle_B) |\Psi^+\rangle_{CD} \otimes |\Phi\rangle, \]

where \( \alpha^2 + \beta^2 = \gamma^2 + \delta^2 = 1 \) and \( |\psi_0\rangle, |\psi_1\rangle, |\phi_0\rangle, |\phi_1\rangle \) and \( |\Phi\rangle \) are arbitrary quantum states. Then the state after applying **Add** to \( |\Psi_{\text{init}}\rangle \) is

\[ |\Psi_{\text{final}}\rangle = (\alpha \gamma |\psi_0\rangle |\phi_0\rangle |0\rangle_{AB} + \beta \delta |\psi_1\rangle |\phi_1\rangle |1\rangle_{AB}) |0\rangle_D \]

\[ + (\alpha \delta |\psi_0\rangle |\phi_1\rangle |0\rangle_{AB} + \beta \gamma |\psi_1\rangle |\phi_0\rangle |1\rangle_{AB}) |1\rangle_D |\Phi\rangle, \]

where register C can be disregarded.
Proof. At step 1, we apply \( CNOT^{(A,C)} \). The state becomes

\[
|\Psi_1\rangle = (\alpha|\psi_0\rangle|0\rangle_A (\gamma|\phi_0\rangle|0\rangle_B + \delta|\phi_1\rangle|1\rangle_B) \otimes |\Psi_+\rangle_{CD} \\
+ \beta|\psi_1\rangle|1\rangle_A (\gamma|\phi_0\rangle|0\rangle_B + \delta|\phi_1\rangle|1\rangle_B) \otimes |\Psi_+\rangle_{CD} |\Phi\rangle.
\]

From Lemma 4, the final state becomes:

\[
|\Psi_4\rangle = ((\alpha\gamma|\psi_0\rangle|\phi_0\rangle|00\rangle_{AB} + \beta\delta|\psi_1\rangle|\phi_1\rangle|11\rangle_{AB})|0\rangle_D \\
+ (\alpha\delta|\psi_0\rangle|\phi_1\rangle|01\rangle_{AB} + \beta\gamma|\psi_1\rangle|\phi_0\rangle|10\rangle_{AB})|1\rangle_D)|\Phi\rangle = |\Psi_{\text{final}}\rangle.
\]

where register \( C \) has been disregarded since it is not entangled anymore. \( \square \)

For example, Lemma 4 shows that applying \( \text{Add}^{A,B}_{C\rightarrow D} \) to

\[
|\Psi_{\text{init}}\rangle = (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes (\gamma|0\rangle_B + \delta|1\rangle_B) |\Psi_+\rangle_{CD}
\]

gives the following quantum state:

\[
|\Psi_{\text{final}}\rangle = (\alpha\gamma|00\rangle_{AB} + \beta\delta|11\rangle_{AB})|0\rangle_D \\
+ (\alpha\delta|01\rangle_{AB} + \beta\gamma|10\rangle_{AB})|1\rangle_D.
\]

We show the corresponding circuit in Fig. 6.

**B. Technique 2: Removal**

Our second technique is called Removal. Removal is a non-unitary operation between two repeaters \((u \text{ and } v)\), respectively, which deletes a resource qubit \((R)\) of a quantum state using measurement in the Hadamard basis and \(\sigma_2\). The procedure for Removal is shown as Table 4. This technique is inspired by the qubit removal method using pauli measurements in one-way quantum computing \((\text{OQC})\) (e.g., qubit removal from the graph states by using a \(Z\) basis) measurement. The following lemma shows the action of Removal.

**Lemma 4.** Let \(|\Psi_{\text{init}}\rangle\) be a state of the form

\[
|\Psi_{\text{init}}\rangle = (\alpha|00\rangle_{AB}|\psi_{00}\rangle + \beta|11\rangle_{AB}|\psi_{11}\rangle) \otimes |\Phi\rangle,
\]

where \(|\alpha|^2 + |\beta|^2 = 1\), and \(|\psi_{00}\rangle, |\psi_{11}\rangle, |\Phi\rangle\) are arbitrary quantum states. Then by applying \(\text{Rem}_{A\rightarrow B}\) on \(|\Psi_{\text{init}}\rangle\), we obtain the state

\[
|\Psi_{\text{final}}\rangle = (\alpha|0\rangle_B|\psi_{00}\rangle + \beta|1\rangle_B|\psi_{11}\rangle) \otimes |\Phi\rangle.
\]

where register \( C \) can be disregarded.

**Proof.** At step 1, we apply the Hadamard gate to \( A \). The state becomes

\[
|\Psi_1\rangle = (\alpha|\psi_0\rangle|0\rangle_A (\gamma|\phi_0\rangle|0\rangle_B + \delta|\phi_1\rangle|1\rangle_B) \otimes |\Psi_+\rangle_{CD} \\
+ \beta|\psi_1\rangle|1\rangle_A (\gamma|\phi_0\rangle|0\rangle_B + \delta|\phi_1\rangle|1\rangle_B) \otimes |\Psi_+\rangle_{CD} |\Phi\rangle.
\]

After this step, when \( a = 0 \) the state is

\[
|\Psi_2\rangle = (\alpha|0\rangle_B|\psi_{00}\rangle + \beta|1\rangle_B|\psi_{11}\rangle) \otimes |\Phi\rangle,
\]

and when \( a = 1 \) the state is

\[
|\Psi_2\rangle = (\alpha|0\rangle_B|\psi_{00}\rangle - \beta|1\rangle_B|\psi_{11}\rangle) \otimes |\Phi\rangle,
\]

where register \( A \) can be disregarded since it is not entangled anymore. At step 4, if \( a = 1 \) then we apply \(\sigma_2\) to \( T \). The state becomes

\[
|\Psi_4\rangle = (\alpha|0\rangle_B|\psi_{00}\rangle + \beta|1\rangle_B|\psi_{11}\rangle) \otimes |\Phi\rangle = |\Psi_{\text{final}}\rangle.
\]

**Lemma 4** shows that Removal is able to “delete” the target qubit used in a Connection operation (compare with Lemma 4). In fact, by applying \(\text{Rem}_{A\rightarrow B}\) on the GHZ-state

\[
|\Psi_{\text{init}}\rangle = |\text{GHZ}\rangle_{ABC},
\]

we obtain the EPR-pair

\[
|\Psi_{\text{final}}\rangle = |\Psi^+\rangle_{BC}.
\]

The corresponding circuit is shown in Fig. 7.

We now present a variant of \(\text{Rem}\) that will delete the target qubit used in Connection:Add operation. We call this operation Removal:Add (\(\text{RemAdd}\)) and show the procedure as Table 4. We can derive the following lemma.
At step 2, we measure $R$ in \{0, 1\} basis. Let $a$ be the outcome.

**Proof.** At step 1, we apply the Hadamard gate to $C$. The state becomes

\[
|\Psi_1\rangle = \left( \sum_{i,j=0}^{1} a_{ij} |i\rangle_2 |j\rangle_3 |\psi_{ij}\rangle \right) \otimes |\Phi\rangle ,
\]

where $\sum_{i,j}|a_{ij}|^2 = 1$, and $|\psi_{ij}\rangle$, $|\Phi\rangle$ are arbitrary quantum states. Then by applying RemAdd$_{C \rightarrow A,B}$, we obtain the state

\[
|\Psi_{\text{final}}\rangle = \left( \sum_{i,j=0}^{1} a_{ij} |i\rangle_2 |j\rangle_3 |\psi_{ij}\rangle \right) \otimes |\Phi\rangle ,
\]

where register $C$ can be disregarded.

For example, Lemma $5$ shows that applying RemAdd$_{C \rightarrow A,B}$ to

\[
|\Psi_{\text{init}}\rangle = (\alpha\gamma|00\rangle_{AB} + \beta\delta|11\rangle_{AB}) |0\rangle_C + (\alpha\delta|01\rangle_{AB} + \beta\gamma|10\rangle_{AB}) |1\rangle_C
\]
gives the following quantum state:

\[
|\Psi_{\text{final}}\rangle = (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes (\gamma|0\rangle_B + \delta|1\rangle_B) .
\]

The corresponding circuit is shown in Fig. 8.

C. Overview of our encoding protocol

We now give an overview of our protocol for the butterfly quantum repeater network of Fig. 3. We will give a complete description of our protocol in subsections D and E.

The first half of our protocol simulates the classical strategy of Fig. 1. For this purpose, each repeater applies Add or Fanout operations. We show the correspondence between classical and quantum operations in Fig. 9. Applying Add$_{B \rightarrow D}$ to

\[
|\Psi_{\text{init}}\rangle = |\Psi^+ \rangle_{AB} \otimes |\Psi^+ \rangle_{CD} \otimes |\Psi^+ \rangle_{EF}
\]
gives the following quantum state:

\[
|\Psi_{\text{final}}\rangle = \frac{1}{2} (|0000\rangle_{ABCD} + |1111\rangle_{ABCD}) \otimes |0\rangle_F
+ \frac{1}{2} (|1100\rangle_{ABCD} + |0100\rangle_{ABCD}) \otimes |1\rangle_F .
\]

Thus Add$_{B \rightarrow D}$ corresponds to the classical parity operation (computing the parity of $B$ and $D$ into register $F$). Applying Fanout$_{B \rightarrow D,E \rightarrow F}$ to

\[
|\Psi_{\text{init}}\rangle = |\Psi^+ \rangle_{AB} \otimes |\Psi^+ \rangle_{CD} \otimes |\Psi^+ \rangle_{EF}
\]
gives the following quantum state:

\[
|\Psi_{\text{final}}\rangle = \frac{1}{\sqrt{2}} (|0000\rangle_{ABDF} + |1111\rangle_{ABDF}) .
\]
Table VI, and show below the evolution of the quantum performed by the senders. We present our procedure as will be used as control qubits for the E register coding.

FIG. 9. The correspondence between classical and quantum operation (copying B register in subsection E. that our protocol can work without additional registers col for this setting in subsection D. We will then show and describe quantum repeater network coding proto-

and describe quantum repeater network coding protocol for this setting in subsection D. We will then show that our protocol can work without additional registers in subsection E.

D. Encoding with additional registers

In this subsection we assume that s1 has an additional register A’ with state |+〉_A’ and s2 has an additional register E’ with state |+〉_E’. The registers A’ and E’ will be used as control qubits for the Fanout operations performed by the senders. We present our procedure as Table VI and show below the evolution of the quantum state of the system.

The input state is

\[ |\Psi_0 \rangle = |+\rangle_{A'}|\Psi^+\rangle_{AB}|\Psi^+\rangle_{CD}|+\rangle_{E'}|\Psi^+\rangle_{EF}|\Psi^+\rangle_{GH} \otimes |\Psi^+\rangle_{IJ}|\Psi^+\rangle_{KL}|\Psi^+\rangle_{MN}. \] (1)

At step 1, s1 and t2 apply Fanout \(_{E\rightarrow B,C\rightarrow D,F,G\rightarrow H}^B_\). s2 and r1 apply Fanout \(_{E\rightarrow F,G\rightarrow H}^B_\). The second half of our protocol will delete redundant registers by Rem and RemAdd operations.

A difficulty is that this correspondence of Fig. 9 cannot be used for the encoding performed at the sender nodes because the sender nodes have no control qubit (i.e., the sender nodes have no incoming edges). To deal with this, we will introduce additional registers as control qubits and describe quantum repeater network coding protocol for this setting in subsection D. We will then show that our protocol can work without additional registers in subsection E.

| Step | Action |
|------|--------|
| 1    | s1 and r1 apply Fanout \(_{E\rightarrow B,C\rightarrow D,F,G\rightarrow H}^B_\). s2 and r1 apply Fanout \(_{E\rightarrow F,G\rightarrow H}^B_\). |
| 2    | r1 and r2 apply Add \(_{E\rightarrow F}^B_\). |
| 3    | r2, t1 and t2 apply Fanout \(_{L\rightarrow M,N\rightarrow H}^B_\). |
| 4    | t1 applies CNOT \(_{N,F}\), t2 applies CNOT \(_{L,B}\). |
| 5    | t2 and t2 apply Rem \(_{L\rightarrow J}^E_\), t1 and t2 apply Rem \(_{N\rightarrow J}^E_\). |
| 6    | r2 and r1 apply RemAdd \(_{D,H\rightarrow J}^B_\). |
| 7    | r1 and s1 apply Rem \(_{A\rightarrow A'}^E_\), r1 and s2 apply Rem \(_{H\rightarrow E'}^B_\). |

At step 1, s1 and r2 apply Connection:Fanout. (s2 and t1 do the same.) From Lemma 2, the state becomes

\[ |\Psi_1 \rangle = (GHZ)_{A'BD}(GHZ)_{E'FH}|\Psi^+\rangle_{IJ}|\Psi^+\rangle_{KL}|\Psi^+\rangle_{MN}. \]

At step 2, r1 and r2 apply Connection:Add. From Lemma 3, the state becomes

\[ |\Psi_2 \rangle = \frac{1}{2}((000000)_{A'BDE'F'H} + |111111)_{A'BDE'F'H}) |0\rangle_J \otimes |\Psi^+\rangle_{KL}|\Psi^+\rangle_{MN} \]

\[ + \frac{1}{2}((001111)_{A'BDE'F'H} + |111000)_{A'BDE'F'H}) |1\rangle_J \otimes |\Psi^+\rangle_{KL}|\Psi^+\rangle_{MN}. \]

At step 3, r2, t1 and t2 apply Connection:Fanout. From Lemma 2 the state becomes

\[ |\Psi_3 \rangle = \frac{1}{2}((000000)_{A'BDE'F'H} + |111111)_{A'BDE'F'H}) \otimes |000\rangle_{JLN} \]

\[ + \frac{1}{2}((001111)_{A'BDE'F'H} + |111000)_{A'BDE'F'H}) \otimes |111\rangle_{JLN}. \]

At step 4, t1 and t2 apply C-NOT. The state becomes

\[ |\Psi_4 \rangle = \frac{1}{2}((000000)_{A'BDE'F'H} + |111111)_{A'BDE'F'H}) \otimes |000\rangle_{JLN} \]

\[ + \frac{1}{2}((010101)_{A'BDE'F'H} + |101010)_{A'BDE'F'H}) \otimes |111\rangle_{JLN}. \]

At step 5, t1 and t2 delete redundant registers N and F using Removal. From Lemma 3 the state becomes

\[ |\Psi_5 \rangle = \frac{1}{2}((000000)_{A'BDE'F'H} + |111111)_{A'BDE'F'H}) |0\rangle_J \]

\[ + \frac{1}{2}((010101)_{A'BDE'F'H} + |101010)_{A'BDE'F'H}) |1\rangle_J. \]

At step 6, r2 deletes the redundant register J using Removal:Add. From Lemma 4 the state becomes

\[ |\Psi_6 \rangle = \frac{1}{2}((000000)_{A'BDE'F'H} + |111111)_{A'BDE'F'H}) \]

\[ + \frac{1}{2}((010101)_{A'BDE'F'H} + |101010)_{A'BDE'F'H}). \]
the network of Fig. [3] (i.e., without additional registers). When there are no additional registers, the input state is

\[ |\Psi_0\rangle = |\Psi^+\rangle_{AB} |\Psi^+\rangle_{CD} |\Psi^+\rangle_{EF} |\Psi^+\rangle_{GH} |\Psi^+\rangle_{IJ} |\Psi^+\rangle_{KL} \otimes |\Psi^+\rangle_{MN}. \]

Suppose that \( s_1 \) and \( t_2 \) apply \( \text{Con}_D^{A} \), and \( s_2 \) and \( t_1 \) apply \( \text{Con}_E^{B} \). The state becomes

\[ |\Psi_1\rangle = \left|GHZ\right>_{ABD} |GHZ\rangle_{EFH} |\Psi^+\rangle_{IJ} |\Psi^+\rangle_{KL} |\Psi^+\rangle_{MN}. \]

Compare with state \([1]\), the two states are the same if we take \( A = A' \) and \( E = E' \). Then, if we apply steps 2-7 as in the previous section, we obtain the state

\[ |\Psi_T\rangle = |\Psi^+\rangle_{AF} \otimes |\Psi^+\rangle_{BE}. \]

Thus, the only modification we have to make is to replace step 1 in the procedure of the previous subsection. The whole procedure for network coding over the network of Fig. [3] is described as Table [VII]. We show the corresponding circuit in Fig. [10].

IV. CONCLUSION

Our protocol shows that quantum network coding techniques are operational using only LOCC and shared EPR-pairs between adjacent repeaters (i.e., without additional quantum registers). This method has been described for the butterfly network but can be actually extended to other linear network coding schemes on other graphs. We expect that this protocol will be a fundamental tool to apply techniques from network coding to real quantum repeater networks.

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| TABLE VII. Encoding without additional registers |
|-----------------------------------------------|
| **Step 1.** | \( s_1 \) and \( t_1 \) apply \( \text{Con}_D^{A} \). |
| **Step 2.** | \( s_2 \) and \( t_1 \) apply \( \text{Con}_E^{B} \). |
| **Step 3.** | \( r_2 \), \( t_1 \), and \( t_2 \) apply \( \text{Fanout}_K^{L,M,N} \). |
| **Step 4.** | \( t_1 \) applies \( \text{CNOT}^{N,F} \), \( t_2 \) applies \( \text{CNOT}^{(L,B)} \). |
| **Step 5.** | \( t_2 \) and \( t_2 \) apply \( \text{Rem}_s^{J} \). |
| **Step 6.** | \( t_1 \) and \( t_2 \) apply \( \text{Add}_s^{J} \). |
| **Step 7.** | \( r_1 \) and \( s_1 \) apply \( \text{Rem}_s^{A} \). |
| | \( r_1 \) and \( s_2 \) apply \( \text{Rem}_s^{H} \). |

At step 7, \( r_1 \) deletes redundant registers \( D \) and \( H \) using Removal the same way as in step 5. The state becomes

\[ |\Psi_7\rangle = \frac{1}{2} \left( |0000\rangle_{ABEF} + |1111\rangle_{ABEF} \right) + \frac{1}{2} \left( |0110\rangle_{ABEF} + |1001\rangle_{ABEF} \right) = |\Psi^+\rangle_A \otimes |\Psi^+\rangle_{BE}. \]

We obtain separated EPR-pairs. The first one is owned by \( (s_1, t_1) \), and the second by \( (s_2, t_2) \).

In the next subsection, we describe an encoding protocol without additional registers based on the above protocol.

E. Encoding without additional registers

We now show how to use the result of the previous subsection to construct a network coding scheme over
FIG. 10. Overall view of the encoding circuit. Parenthetic numbers refer to the state after each step of the encoding procedure of Table VII.

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