PARTIAL CONNECTION FOR \( p \)-TORSION LINE BUNDLES IN CHARACTERISTIC \( p > 0 \)

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To S. S. Chern, in memoriam

Abstract. The aim of this brief note is to give a construction for \( p \)-torsion line bundles in characteristic \( p > 0 \) which plays a similar rôle as the standard connection on an \( n \)-torsion line bundle in characteristic 0.

1. Introduction

In [3] (see also [4]) we gave an algebraic construction of characteristic classes of vector bundles with a flat connection \((E, \nabla)\) on a smooth algebraic variety \( X \) defined over a field \( k \) of characteristic 0. Their value at the generic point \( \text{Spec}(k(X)) \) was studied and redefined in [1], and then applied in [2] to establish a Riemann-Roch formula. One way to understand Chern classes of vector bundles (without connection) is via the Grothendieck splitting principle: if the receiving groups \( \bigoplus_n H^{2n}(X, n) \) of the classes form a cohomology theory which is a ring and is functorial in \( X \), then via the Whitney product formula it is enough to define the first Chern class. Indeed, on the flag bundle \( \pi : \text{Flag}(E) \to X \), \( \pi^*(E) \) acquires a complete flag \( E_i \subset E_{i+1} \subset \pi^*(E) \) with \( E_{i+1}/E_i \) a line bundle, and \( \pi^*: H^{2n}(X, n) \to H^{2n}(\text{Flag}(E), n) \) is injective, so it is enough to construct the classes on \( \text{Flag}(E) \). However, if \( \nabla \) is a connection on \( E \), \( \pi^*(\nabla) \) does not stabilize the flag \( E_i \). So the point of [3] is to show that there is a differential graded algebra \( A^* \) on \( \text{Flag}(E) \), together with a morphism of differential graded algebras \( \Omega^*_{\text{Flag}(E)} \to A^* \), so that \( R\pi_* A^* \cong \Omega^*_{X} \) and so that the operator defined by the composition \( \pi^*(E) \xrightarrow{\pi^*(\nabla)} \Omega^1_{\text{Flag}(E)} \otimes \mathcal{O}_{\text{Flag}(E)} \xrightarrow{\tau \otimes 1} A^1 \otimes \mathcal{O}_{\text{Flag}(E)} \xrightarrow{\pi^*(E)} \) stabilizes \( E_i \). We call the induced operator \( \nabla_i : E_i \to A^1 \otimes \mathcal{O}_{\text{Flag}(E)} E_i \) a (flat) \( \tau \)-connection. So it is a \( k \)-linear map which fulfills the \( \tau \)-Leibniz
for \( \lambda \) a local section of \( \mathcal{O}_{\text{Flag}(E)} \) and \( e \) a local section of \( E_i \). It is flat when \( 0 = \nabla_i \circ \nabla_i \in H^0(X, A^2 \otimes_{\mathcal{O}_X} \text{End}(E)) \), with the appropriate standard sign for the derivation of forms with values in \( E_i \). The last point is then to find the correct cohomology which does not get lost under \( \pi^* \). It is a generalization of the classically defined group \( H^1(X, \mathcal{O}_X \rightarrow \Omega^1_X \rightarrow \Omega^2_X \rightarrow \cdots) \) of isomorphism classes of rank one line bundles on \( X \) with a flat connection.

A typical example of such a connection is provided by a torsion line bundle: if \( L \) is a line bundle on \( X \) which is \( n \)-torsion, that is which is endowed with an isomorphism \( L^n \cong \mathcal{O}_X \), then the isomorphism yields an \( \mathcal{O}_X \)-étale algebra structure on \( A = \oplus_{i=0}^{n-1} L^i \), hence a finite étale covering \( Y = \text{Spec}_{\mathcal{O}_X} A \rightarrow X \), which is a principal bundle under the group scheme \( \mu_n \) of \( n \)-th roots of unity, thus is Galois cyclic as soon as \( \mu_n \subset k^\times \). Since the \( \mu_n \)-action commutes with the differential \( d_Y : \mathcal{O}_Y \rightarrow \Omega^1_Y = \sigma^* \Omega^1_X \), it defines a flat connection \( \nabla_L : L \rightarrow \Omega^1_X \otimes_{\mathcal{O}_X} L \). Concretely, if \( g_{\alpha,\beta} \in \mathcal{O}_X^\times \) are local algebraic transition functions for \( L \), with trivialization

\[
(1.3) \quad g_{\alpha,\beta}^n = u_{\beta} u_\alpha^{-1}, u_\alpha \in \mathcal{O}_X^\times, 
\]

then

\[
(1.4) \quad \left( g_{\alpha,\beta}, \frac{1}{n} \frac{du_\alpha}{u_\alpha} \right) \in \left( \mathcal{C}^1(\mathcal{O}_X^\times) \times \mathcal{C}^0((\Omega^1_X)_{\text{clsd}}) \right)_{\delta \log - \delta}, \quad \frac{dg}{g} = \delta \left( \frac{du}{u} \right)
\]

is a Cech cocyle for the class

\[
(1.5) \quad (L, \nabla_L) \in \mathbb{H}^1(X, \mathcal{O}_X^\times \xrightarrow{\delta \log} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots).
\]

Clearly (1.4) is meaningless if the characteristic \( p \) of \( k \) is positive and divides \( n \). The purpose of this short note is to give an Ersatz of this canonical construction in the spirit of the \( \tau \)-connections explained above when \( p \) divides \( n \).

2. A partial connection for \( p \)-torsion line bundles

Let \( X \) be a scheme of finite type over a perfect field \( k \) of characteristic \( p > 0 \). Let \( L \) be a \( n \)-torsion line bundle on \( X \), thus endowed with an
isomorphism
\begin{equation}
\theta : L^n \cong \mathcal{O}_X.
\end{equation}

Then \( \theta \) defines an \( \mathcal{O}_X \)-algebra structure on \( \mathcal{A} = \bigoplus_{i=0}^{n-1} L_i \) which is étale if and only if \((p, n) = 1\). It defines the principal \( \mu_n \)-covering
\begin{equation}
\sigma : Y = \text{Spec}_{\mathcal{O}_X} \mathcal{A} \to X
\end{equation}
which is étale if and only if \((p, n) = 1\), else decomposes into
\begin{equation}
\sigma : Y \twoheadrightarrow Z \xrightarrow{\sigma'} X
\end{equation}
with \( \sigma' \) étale and \( \iota \) purely inseparable. More precisely, if \( n = m \cdot p^r, (m, p) = 1, \) and \( M = L^{p^r} \), \( \theta \) defines an \( \mathcal{O}_X \)-étale algebra structure on \( \mathcal{B} = \bigoplus_{i=0}^{m-1} M_i \), which defines \( \sigma' : Z = \text{Spec}_{\mathcal{O}_X} \mathcal{B} \to X \) as an (étale) \( \mu_m \)-principal bundle. The isomorphism \( \theta \) also defines an isomorphism \( (L')^{p^r} \cong \mathcal{O}_Z \) as it defines the isomorphism \( (\sigma')^*(M) \cong \mathcal{O}_Z \), where \( L' = (\sigma')^*(L) \). So \( \mathcal{C} = \bigoplus_{i=0}^{p^r-1} (L')^i \) becomes a finite purely inseparable \( \mathcal{O}_Z \)-algebra defining the principal \( \mu_{p^r} \)-bundle \( \iota : Y = \text{Spec}_{\mathcal{O}_Z} \mathcal{C} \to Z \).

If \((n, p) = 1\), that is if \( r = 0 \), the formulae (1.3), (1.4) define \( (L, \nabla) \) as in (1.5). We assume from now on that \((n, p) = p\). Then, as is well known, as a consequence of (1.3) one sees that the form
\begin{equation}
\omega_L := \frac{du_\alpha}{u_\alpha} \in \Gamma(X, \Omega^1_X)_{\text{Cartier}} = 1
\end{equation}
is globally defined and Cartier invariant. Let \( e_\alpha \) be local generators of \( L \), with transition functions \( g_{\alpha, \beta} \) with \( e_\alpha = g_{\alpha, \beta} e_\beta \). The isomorphism \( \theta \) yields a trivialization
\begin{equation}
\sigma^* L \cong \mathcal{O}_Y
\end{equation}
thus local units \( v_\alpha \) on \( Y \) with
\begin{equation}
v_\alpha \in \mathcal{O}_Y^*, \quad g_{\alpha, \beta} = v_\beta v_\alpha^{-1}
\end{equation}
so that \( 1 = v_\alpha \sigma^*(e_\alpha) = v_\beta \sigma^*(e_\beta) \).

**Definition 2.1.** One defines the \( \mathcal{O}_X \)-coherent sheaf \( \Omega^1_L \) as the subsheaf of \( \sigma_* \Omega^1_Y \) spanned by \( \text{Im}(\Omega^1_Y) \) and \( \frac{dv_\alpha}{v_\alpha} \).

**Lemma 2.2.** \( \Omega^1_L \) is well defined and one has the exact sequence
\begin{equation}
0 \to \mathcal{O}_X \xrightarrow{\omega_L} \Omega^1_X \xrightarrow{\sigma^*} \Omega^1_L \xrightarrow{s} \mathcal{O}_X \to 0
\end{equation}
where \( s(\frac{dv_\alpha}{v_\alpha}) = 1 \).
Proof. The relation (2.6) implies

\[ \frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}} = \frac{dv_{\beta}}{v_{\beta}} - \frac{dv_{\alpha}}{v_{\alpha}} \]  

(2.8)

so \( \frac{dv_{\beta}}{v_{\beta}} \equiv \frac{dv_{\alpha}}{v_{\alpha}} \in \sigma_{*}\Omega^{1}_{Y}/\text{Im}(\Omega^{1}_{X}) \).

Hence the sheaf \( \Omega^{1}_{L} \) is well defined. If \( e'_{\alpha} \) is another basis, then one has \( e_{\alpha} = w_{\alpha}e'_{\alpha} \) for local units \( w_{\alpha} \in \mathcal{O}^{\times}_{X} \). The new \( v_{\alpha} \) are then multiplied by local units in \( \mathcal{O}^{\times}_{X} \), so the surjection \( s \) is well defined. It remains to see that \( \text{Ker}(\sigma^{*}) = \text{Im}(\omega_{L}) \). By definition, on the open of \( X \) on which \( L \) has basis \( e_{\alpha} \), one has

\[ Y = \text{Spec} \mathcal{O}_{X}[v_{\alpha}]/(v_{\alpha}^{n} - u_{\alpha}) \].

This implies \( \Omega^{1}_{Y} = \langle \text{Im}(\Omega^{1}_{X}), dv_{\alpha}\rangle_{\mathcal{O}_{Y}}/\langle du_{\alpha}\rangle_{\mathcal{O}_{Y}} \) on this open and finishes the proof.

\[ \square \]

Remarks 2.3.  
1) Assume for example that \( X \) is a smooth projective curve of genus \( g \), and \( n = p \). Recall that \( 0 \neq \omega_{L} \in \Gamma(X, \Omega^{1}_{X}) \). In particular, if \( g \geq 2 \), necessarily \( 0 \neq \Omega^{1}_{X}/\mathcal{O}_{X} \cdot \omega_{L} \) is supported in codimension 1. So \( \Omega^{1}_{L} \) contains a non-trivial torsion subsheaf.

2) The sheaf \( \Omega^{1}_{L} \) lies in \( \sigma_{*}\Omega^{1}_{Y} \) but is not equal to it. Indeed, on the smooth locus of \( X \) (assuming \( X \) is reduced) the torsion free quotient of \( \Omega^{1}_{L} \) has rank equal to the dimension of \( X \), while \( \sigma_{*}\Omega^{1}_{Y} \) has rank \( n \cdot \text{dim}(X) \) on the étale locus of \( \sigma \) (which is non-empty if \( L \) itself is not a \( p \)-power line bundle).

3) The class in \( \text{Ext}^{2}_{\mathcal{O}_{X}}(\mathcal{O}_{X}, \mathcal{O}_{X}) = H^{2}(X, \mathcal{O}_{X}) \) defined by (2.7) vanishes. Indeed, let us decompose (2.7) as an extension of \( \mathcal{O}_{X} \) by \( \Omega^{1}_{X}/\mathcal{O}_{X} \cdot \omega_{L} \), followed by an extension of \( \Omega^{1}_{X}/\mathcal{O}_{X} \cdot \omega_{L} \) by \( \mathcal{O}_{X} \cdot \omega_{L} \). The first extension class in \( H^{1}(X, \Omega^{1}_{X}/\mathcal{O}_{X} \cdot \omega_{L}) \) has cocycle \( \frac{dv_{\beta}}{v_{\beta}} - \frac{dv_{\alpha}}{v_{\alpha}} = \frac{du_{\alpha}}{g_{\alpha,\beta}} \) (see (2.8)), thus is the image of the Atiyah class of \( L \) in \( H^{1}(X, \Omega^{1}_{X}) \). Thus the second boundary to \( H^{2}(X, \mathcal{O}_{X}) \) dies.

Definition 2.4. We set \( \Omega^{0}_{L} := \mathcal{O}_{X} \) and for \( i \geq 1 \) we define the \( \mathcal{O}_{X} \)-coherent sheaf \( \Omega^{i}_{L} \) as the subsheaf of \( \sigma_{*}\Omega^{i}_{Y} \) spanned by \( \text{Im}(\Omega^{i}_{X}) \) and \( \frac{dv_{\alpha}}{v_{\alpha}} \wedge \text{Im}(\Omega^{i-1}_{X}) \).
Proposition 2.5. The sheaf $\Omega^i_L$ is well defined. One has an exact sequence
\begin{equation}
0 \to \omega_L \wedge \Omega^i_{X^{-1}} \to \Omega^i_X \overset{\sigma^*}{\to} \Omega^i_L \overset{s}{\to} \Omega^i_{X^{-1}} \to 0 \tag{2.10}
\end{equation}
Furthermore, the differential $\sigma_*(d_Y)$ on $\sigma_*\Omega^i_Y$ induces on $\bigoplus_{i \geq 0} \Omega^i_L$ the structure of a differential graded algebra $(\Omega^*_{\text{crist}}, d_L)$ so that $\sigma^*: (\Omega^*_{X}, d_X) \to (\Omega^*_{\text{crist}}, d_L)$ is a morphism of differential graded algebras.

Proof. One proves (2.10) as one does (2.7). One has to see that $\sigma^* (d_Y)$ stabilizes $\Omega^*_{\text{crist}}$. As $0 = d_X (\omega_L) \in \Omega^2_X$, $0 = d_Y (\frac{dv_{\alpha}}{v_{\alpha}}) \in \sigma_* \Omega^2_Y$, (2.10) extends to an exact sequence of complexes
\begin{equation}
0 \to (\omega_L \wedge \Omega^*_{X^{-1}}, -1 \wedge d_X) \to (\Omega^*_{X}, d_X) \overset{\sigma^*}{\to} (\Omega^*_{\text{crist}}, d_L) \overset{s}{\to} (\Omega^*_{X^{-1}}, -d_X) \to 0. \tag{2.11}
\end{equation}
This finishes the proof.

Remark 2.6. As $\frac{dg_{\alpha, \beta}}{g_{\alpha, \beta}} \in (\Omega^1_X)_\text{crist}$ the same proof as in Remark 2.3, 3) shows that the extension class $\text{Ext}^2(\Omega^*_{X^{-1}}, \omega_L \wedge \Omega^*_{X^{-1}})$ defined by (2.11) dies.

In order to tie up with the notations of the Introduction, we set
\begin{equation}
\tau = \sigma^*: \Omega^*_{X} \to \Omega^*_{\text{crist}}. \tag{2.12}
\end{equation}

Proposition 2.7. The formula $\nabla (e_\alpha) = -\frac{dv_{\alpha}}{v_{\alpha}} \otimes e_\alpha \in \Omega^1_L \otimes_{\mathcal{O}_X} L$ defines a flat $\tau$-connection $\nabla_L$ on $L$. So $(L, \nabla_L)$ is a class in $\text{H}^1(X, \mathcal{O}_X \xrightarrow{\tau_{\text{log}}} \Omega^1_L \xrightarrow{d_L} \Omega^2_L \xrightarrow{d_L} \cdots)$, the group of isomorphism classes of line bundles with a flat $\tau$-connection.

Proof. Formula (2.6) implies that this defines a $\tau$-connection. Flatness is obvious. A Cech cocycle for $(L, \nabla_L)$ is $(g_{\alpha, \beta}, \frac{dv_{\alpha}}{v_{\alpha}})$. \qed

Remarks 2.8. 1) The same formal definitions 2.1 and 2.4 of $\Omega^*_{\text{crist}}$ when $(n, p) = 1$ yield $(\Omega^*_{\text{crist}}, d_L) = (\Omega^*_{X}, d_X)$, and the flat $\tau$-connection becomes the flat connection defined in 1.3 and 1.5. So Proposition 2.7 is a direct generalization of it.

2) Let $X$ be proper reduced over a perfect field $k$, irreducible in the sense that $H^0(X, \mathcal{O}_X) = k$, and admitting a rational point $x \in X(k)$. A generalization of torsion line bundles to higher rank bundles is the notion of Nori finite bundles, that is bundles $E$ which are trivialized over principal bundle $\sigma: Y \to X$. 
under a finite flat group scheme $G$ (see [6] for the original definition and also [5] for a study of those bundles). So for the $n$-torsion line bundles considered in this section, $G \cong \mu_n$. If the characteristic of $k$ is 0, then again $\sigma$ is étale, the differential $d_Y : \mathcal{O}_Y \to \sigma^* \Omega_X^1 = \Omega^1_Y$ commutes with the action of $G$, inducing a connection $\nabla_E : E \to \Omega^1_X \otimes \mathcal{O}_X E$ and characteristic classes in our groups $\mathbb{H}^i(X, \mathcal{K}_i^m \xrightarrow{d_{\log}} \Omega^i_X \xrightarrow{d} \Omega^{i+1}_X \cdots)$ (see [3]). If the characteristic of $k$ is $p > 0$, then $\sigma$ is étale if and only if $G$ is smooth (which here means étale), in which case one can also construct those classes. If $G$ is not étale, thus contains a non-trivial local subschemes, then one should construct as in Proposition 2.5 a differential graded algebra $(\Omega^*_E, d_E)$ with a map $(\Omega^*_X, d_X) \xrightarrow{\tau} (\Omega^*_E, d_E)$, so that $E$ is endowed naturally with a flat $\tau$-connection $\nabla_E : E \to \Omega^1_E \otimes \mathcal{O}_X E$. The techniques developed in [3] should then yield classes in the groups $\mathbb{H}^i(X, \mathcal{K}_i^m \xrightarrow{\tau d_{\log}} \Omega^i_E \xrightarrow{d_E} \Omega^{i+1}_E \cdots)$.

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