Meromorphic solutions of certain types of complex functional equations

Zhi-Bo Huang$^a$, Shuang-Ting Lan$^b$, Ran-Ran Zhang$^c$

$^a$ School of Mathematical Sciences, South China Normal University, Guangzhou 510631 China
$^b$ School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665 China
$^c$ Department of Mathematics, Guangdong University of Education, Guangzhou 510303 China

*Corresponding authors, e-mail: huangzhibo@scnu.edu.cn, zhangranran@gdei.edu.cn

Received 3 Feb 2021
Accepted 26 Apr 2021

ABSTRACT: In this paper, we investigate the properties of meromorphic solutions on complex functional equations of Malmquist type of the form

$$
\sum_{|J|} a_J(z) \left( \prod_{j \in J} w(q^j z) \right) = \frac{a_0(z) + a_1(z)(w \circ p) + \cdots + a_n(z)(w \circ p)^r}{b_0(z) + b_1(z)(w \circ p) + \cdots + b_n(z)(w \circ p)^r},
$$

where $|J|$ is a collection of all non-empty subsets of $\{1, 2, \ldots, n\}$, $q \in \mathbb{C}$, $|q| > 1$, and all coefficients are small functions relative to $w(z)$ such that $a_j(z)b_j(z) \neq 0$, $p(z) = d_n z^n + \cdots + d_1 z + d_0$ is a polynomial with constant coefficients $d_k (\neq 0), \ldots, d_1, d_0$ and of degree $k$. Furthermore, we prove that the meromorphic solutions having Borel exceptional zeros and poles appear in special situations. Some other $q$-difference versions of complex difference equations of Malmquist type are also presented.

KEYWORDS: difference equation, functional equation, meromorphic solution, Nevanlinna theory, growth

MSC2010: 30D35 34M05

INTRODUCTION

Recently, Ablowitz et al [1] applied Nevanlinna theory to investigate the properties on complex difference equations reminiscent of the classical Malmquist theorem in complex differential equations. A typical example of their results tells us that if a complex difference equation

$$
w(z + 1) + w(z - 1) = R(z, w)
$$

with $R(z, w)$ rational in both arguments admits a transcendental meromorphic solution of finite order, then $\deg_{w} R(z, w) \leq 2$. Heittokangas et al [2] improved and extended the above results, see Propositions 8 and 9, and showed that solutions having Borel exceptional zeros and poles seem to appear in special situations only. Zhang and Huang [3] focused on Theorem 13 in [2] to present exact form of difference equations by proving some results on deficiencies of the meromorphic solutions. Laine et al [4] generalized the key lemma that $w(z)$ has to be infinite order, provided that $\deg_{w} R(z, w) \leq 2$ and that a certain growth condition for the counting function of distinct poles of $w(z)$ holds (see [5]) to higher order difference equations of more general type (see [4]), and presented related complex functional equations. The properties on the meromorphic solutions of complex functional difference equations composited with polynomials are also investigated in [6].

Bergweiler et al [7] considered nonlinear $q$-difference equation

$$
\sum_{j=0}^{n} a_j(z) w(q^j z) = Q(z),
$$

where $0 < |q| < 1$, $a_j(z)(j = 0, 1, \ldots, n)$ and $Q(z)$ are rational functions with $a_0(z) \neq 0$, $a_n(z) \equiv 1$. They gave sufficient conditions for the existence of meromorphic solutions of (2), and also pointed out that all meromorphic solutions of (2) satisfy $T(r, w) = O((\log r)^2)$. This implies that all meromorphic solutions of (2) are of zero order of growth.

If $|q| > 1$, Gundersen et al [8] showed that the order of growth of generalized Schröder $q$-
difference equation

\[ w(qz) = R(z, w(z)) \] (3)

is equal to \( \log \deg_{w} (R) / \log |q| \), while Zheng and Chen [9] showed that the lower order \( \mu(w) \) of solutions of (4) below is not less than \( \log d_{0} / (n \log |q|) \) if \( |q| > 1 \). Now, we recall their results.

**Theorem 1 ([8])** Suppose that \( w(z) \) is a transcendental meromorphic solution of an equation of the form

\[ w(qz) = R(z, w(z)) \]

where \( q \in \mathbb{C}, |q| > 1, R(z, w(z)) \) is irreducible in \( w(z) \) with meromorphic coefficients \( a_{u}(z)(u = 0, 1, \ldots, s) \) and \( b_{v}(z)(v = 0, 1, \ldots, t) \) such that \( a_{u}(z)b_{v}(z) \neq 0 \). Then

\[ \sigma(w) = \frac{\log \deg_{w}(R)}{\log |q|}. \]

**Theorem 2 ([9])** Suppose that \( w(z) \) is a transcendental meromorphic solution of equation

\[ \sum_{j=1}^{n} a_{j}(z)w(q^{j}z) = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \] (4)

where \( q \in \mathbb{C}, |q| > 1, the coefficients a_{j}(z) are rational functions and P(z, w(z)), Q(z, w(z)) \) are relatively prime polynomials in \( w(z) \). Suppose that \( w(z) \) has infinitely many poles, then for sufficiently large \( r \),

\[ n(r, w) \geq K d_{0} \log r / n \log |q| \]

holds for some constant \( K \).

A meromorphic function means meromorphic in the whole complex plane \( \mathbb{C} \). For a meromorphic function \( w(z) \), let \( \sigma(w) \) be the order of growth and \( \mu(w) \) be the lower order of \( w(z) \). Further, let \( \lambda(w) \) (respectively, \( \lambda(1/w) \)) be the exponent of convergence of the zeros (respectively, poles) of \( w(z) \). We also assume that the reader is familiar with the standard symbols and fundamental results such as \( m(r, w), N(r, w), N(r, w) \) and \( T(r, w) \), etc., of Nevanlinna theory, see e.g. [10, 11]. We now recall that a meromorphic function \( a(z) \) is said to be a small function relative to \( w(z) \) if \( T(r, a) = S(r, w) \), where \( S(r, w) \) is used to denote any quantity satisfying \( S(r, w) = o(T(r, y)) \) as \( r \to \infty \), possibly outside of a set of finite logarithmic measure, furthermore, possibly outside of a set of logarithmic density 0, i.e. outside of a set \( E \) such that \( \lim_{r \to \infty} \int_{|z|=r} \frac{dt}{t} / \log r = 0 \). Moreover, suppose that \( R(z, w(z)) \) is rational in \( w(z) \) with small functions relative to \( w(z) \) as its coefficients. We use the notation \( d = \deg_{w}(R(z, w(z))) \) for the degree of \( R(z, w(z)) \) with respect to \( w(z) \). In the follows, we always assume that \( R(z, w(z)) \) is irreducible in \( w(z) \).

The present paper mainly deal with functional equations of the general form

\[ \sum_{\{J\}} a_{j}(z) \left( \prod_{j \in J} w(q^{j}z) \right) = R(z, w \circ p) = \frac{P(z, w \circ p)}{Q(z, w \circ p)} \] (5)

where \( \{J\} \) is a collection of all non-empty subsets of \( \{1, 2, \ldots, n\} \), \( q \in \mathbb{C}, |q| > 1, P(z, w) \) and \( Q(z, w) \) are relatively prime polynomials in \( w(z) \), and all coefficients in (5) are small functions relative to \( w(z) \), \( P(z) = d_{0} z^{k} + \cdots + d_{1} z + d_{0} \) is a polynomial with constant coefficients \( d_{i} \neq 0, \ldots, d_{1}, d_{0} \) and of degree \( k \). We permit different expressions on both sides of equation (5).

**MEROMORPHIC SOLUTIONS WITH CERTAIN GROWTH CONDITION FOR COUNTING FUNCTION OF DISTINCT POLES**

Halburd and Korhonen [5] showed that the existence of sufficiently many meromorphic solutions of finite order is enough to single out a discrete form of the second Painlevé equation from a more general class (1). A key lemma in their reasoning is to show that \( w(z) \) has to be of infinite order, provided that \( \deg_{w}(R, w) \leq 2 \) and that a certain growth condition for the counting function of distinct poles \( w(z) \) holds. Laine et al [4] extended it into a more general type. Zheng and Chen [9] proved a \( q \)-difference counterpart of the above results. In this section, we proceed to extend Theorem 4 in [9] into a more general type again.

**Theorem 3** Suppose that \( w(z) \) is a transcendental meromorphic solution of (5), \( \{J\} \) is a collection of all non-empty subsets of \( \{1, 2, \ldots, n\} \), \( q \in \mathbb{C}, |q| > 1, P(z, w(z)) \) and \( Q(z, w(z)) \) are relatively prime polynomials in \( w(z) \), and all coefficients in (5) are small functions relative to \( w(z) \). Moreover, we assume that \( t = \deg_{w}(Q) > 0, b_{i}(z) = 1, and \)

\[ n = \max\{s, t\} := \max\{\deg_{r}(P), \deg_{r}(Q)\}. \]

If there exists \( \alpha \in [0, n) \) such that, for all sufficiently
large $r$,
\[
\sum_{j=1}^{n} N(r, w(q^j z)) \leq a N(r, w(z)), \tag{6}
\]
then the order of growth $\sigma(w) > 0$ and
\[
Q(z, w(z)) = (w(z) - s(z))',
\]
where $s(z)$ is a small function relative to $w(z)$.

At this point, we first need to recall the following lemmas. Weisseborn obtained the following result.

**Lemma 1 ([12])** Let $f(z)$ be a meromorphic function and $\phi$ be given by
\[
\phi = w^n + a_{n-1}w^{n-1} + \cdots + a_0,
\]
\[
T(r, a_j) = S(r, w), \quad j = 0, 1, \ldots, n-1.
\]
Then either
\[
\phi \equiv \left( w + \frac{a_{n-1}}{n} \right)^n,
\]
or
\[
T(r, w) \leq N(r, \frac{1}{\phi^n}) + N(r, w) + S(r, w).
\]

**Lemma 2 ([4])** Let $f(z)$ be a nonconstant meromorphic function and let $P(z, w), Q(z, w)$ be two polynomials in $w(z)$ with meromorphic coefficients small relative to $w(z)$. If $P(z, w)$ and $Q(z, w)$ have no common factors of positive degree in $w(z)$ over the field of small functions relative to $w(z)$, then
\[
N(r, \frac{1}{Q(z, w)}) \leq N(r, \frac{P(z, w)}{Q(z, w)}) + S(r, w).
\]

**Lemma 3 ([13])** Given distinct meromorphic functions $w_1, \ldots, w_n$, let $\{J\}$ denote the collection of all non-empty subsets of $\{1, 2, \ldots, n\}$, and suppose that $\alpha_j \in \mathbb{C}$ for each $J \in \{J\}$. Then
\[
T \left( r, \sum_{J} \alpha_j \left( \prod_{j \in J} w_j \right) \right) \leq \sum_{k=1}^{n} T(r, w_k) + O(1)
\]

**Lemma 4 ([14])** If $T : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a piecewise continuous increasing function such that
\[
\lim_{r \to \infty} \frac{\log T(r)}{\log r} = 0,
\]
then the set
\[
E := \{ r : T(C_1 r) \geq C_2 T(r) \}
\]
has logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

**Remark 1** By using similar method of Theorem 1.1 and Theorem 1.3 in [15] and $q$-difference version of lemma on logarithmic derivatives [16, 17], we deduce from Lemma 4 that, for $|q| > 1$,
\[
T(r, w(qz)) = T(r, w(z)) + S(r, w)
\]
and
\[
\overline{N}(r, w(qz)) = \overline{N}(r, w(z)) + S(r, w)
\]
on a set of logarithmic density 1.

**Proof of Theorem 3:** Assume that the second alternative of the assertion is incorrect. Then, we deduce from Lemmas 1–3, (5) and (6) that
\[
T(r, w) \leq \overline{N}(r, \frac{1}{Q(z, w)}) + \overline{N}(r, w(z)) + S(r, w)
\]
\[
\leq \overline{N}(r, \frac{P(z, w)}{Q(z, w)}) + \overline{N}(r, w(z)) + S(r, w)
\]
\[
= \overline{N}(r, \sum_{j \in J} \alpha_j \left( \prod_{j \in J} w(q^j z) \right)) + \overline{N}(r, w(z)) + S(r, w)
\]
\[
\leq \sum_{j=1}^{n} (\alpha N(r, w(z)) + \overline{N}(r, w(z)) + S(r, w)
\]
\[
\leq \alpha \overline{N}(r, w(z)) + \overline{N}(r, w(z)) + S(r, w).
\]
Therefore,
\[
T(r, w) - \overline{N}(r, w(z)) \leq \alpha \overline{N}(r, w(z)) + S(r, w).
\]

Now, assume in contrary to the assertion that $\sigma(w) = 0$, we get from Remark 1 that for all $j = 1, 2, \ldots, n$, $S(r, w(q^j z)) = S(r, w)$ and
\[
T(r, w(q^j z)) - \overline{N}(r, w(q^j z)) \leq \alpha \overline{N}(r, w(q^j z)) + S(r, w)
\]
\[
= \alpha \overline{N}(r, w(z)) + S(r, w) \tag{7}
\]
on a set of logarithmic density 1.

We also conclude from Remark 1, Lemma 3, (6) and (7) that
\[
n T(r, w) = T \left( r, \sum_{j \in J} \alpha_j \left( \prod_{j \in J} w(q^j z) \right) \right) + S(r, w)
\]
\[
\leq \sum_{j=1}^{n} T(r, w(q^j z)) + S(r, w)
\]
\[
= \sum_{j=1}^{n} \left[ T(r, w(q^j z)) - \overline{N}(r, w(q^j z)) \right]
\]
\[
+ \sum_{j=1}^{n} \overline{N}(r, w(q^j z)) + S(r, w)
\]
\[
\leq \sum_{j=1}^{n} (\alpha \overline{N}(r, w(z)) + \alpha \overline{N}(r, w(z)) + S(r, w)
\]
\[
= (n+1) \alpha \overline{N}(r, w(z)) + S(r, w) \tag{8}
\]
on a set of logarithmic density 1. Thus,
\[
T(r, w) - \overline{N}(r, w(z)) \leq \frac{n + 1}{n} \alpha \overline{N}(r, w(z)) - \overline{N}(r, w(z)) + S(r, w)
\]  
(9)
on a set of logarithmic density 1.
Moreover, we obtain from (6), (8), (9) and Remark 1 that
\[
nT(r, w) \leq \sum_{j=1}^{n} \left[ T(r, w(q^{j}z)) - \overline{N}(r, w(q^{j}z)) \right] + \sum_{j=1}^{n} \overline{N}(r, w(q^{j}z)) + S(r, w)
\]
\[
\leq \sum_{j=1}^{n} \left[ \frac{n + 1}{n} \alpha \overline{N}(r, w(q^{j}z)) - \overline{N}(r, w(q^{j}z)) \right] + \alpha \overline{N}(r, w(z)) + S(r, w)
\]
\[
= (n+2)\alpha \overline{N}(r, w(z)) - n\overline{N}(r, w(z)) + S(r, w)
\]on a set of logarithmic density 1. Thus,
\[
T(r, w) - \overline{N}(r, w(z)) \leq \frac{n + 2}{n} \alpha \overline{N}(r, w(z)) - 2\overline{N}(r, w(z)) + S(r, w)
\]on a set of logarithmic density 1. By repeating this process for \(m\) times, we deduce that
\[
T(r, w) - \overline{N}(r, w(z)) \leq \frac{n + m}{n} \alpha \overline{N}(r, w(z)) - m\overline{N}(r, w(z)) + S(r, w)
\]  
(10)
on a set of logarithmic density 1. Since \(\alpha \in [0, n]\), we immediately see from (10) that, for sufficiently large \(m\),
\[
\overline{N}(r, w(z)) \leq \frac{n + m}{n(m - 1)} \alpha \overline{N}(r, w(z)) < \overline{N}(r, w(z)),
\]on a set of logarithmic density 1, a contradiction.

On the other hand, if the second alternative of the assertion is valid, then we must have \(\sigma(w) > 0\). Otherwise, by Remark 1 and the \(q\)-version of Mohon'ko lemma, we again obtain a contradiction. \(\square\)

**MEROMORPHIC SOLUTIONS WITH FINITELY MANY POLES**

Theorem 3 shows that either the order of growth \(\sigma(w) > 0\), and
\[
Q(z, w(z)) \equiv (w(z) - s(z))^j,
\]provided that \(\deg Q(z, w) > 0\) and that a certain growth condition for the counting function of distinct poles \(w(z)\) holds. However, if \(w(z)\) just has finitely many zeros, we further obtain the following theorem.

**Theorem 4** Suppose that \(w(z)\) is a transcendental meromorphic solution of (5), where \(|J|\) is a collection of all non-empty subsets of \(\{1, 2, \ldots, n\}\), \(q \in \mathbb{C}\), \(|q| > 1\), \(P(z, w(z))\) and \(Q(z, w(z))\) are relatively prime polynomials in \(w(z)\), and all coefficients in (5) are rational functions. Moreover, we assume that \(t = \deg P(Q) > 0\) and \(b_z(z) \equiv 1\). If \(w(z)\) has finitely many poles only, then

(1) \(Q(z, w(z)) = (w(z) - s(z))^j\), where \(s(z)\) is a rational function;

(2) \(w(z)\) must be of the form
\[
w(z) = r(z)e^{\tau(z)} + s(z),
\]where \(s(z)\) is a rational function, \(r(z)\) is a small function relative to \(w(z)\), \(g(z)\) is a transcendental entire function satisfying a \(q\)-difference equation of the form
\[
k_0g(z) + k_1g(qz) + \cdots + k_ng(q^nz) = \tau,
\]where \(\tau \in \mathbb{C}\), and \(k_j, j \in \{0, 1, \ldots, n\}\) are integers and not identically zeros.

We firstly recall the following lemmas.

**Lemma 5** Suppose that \(w_1(z), w_2(z), \ldots, w_n(z)\) are meromorphic functions and that \(g_1(z), g_2(z), \ldots, g_n(z)\) are entire functions satisfying the following conditions.

(i) \(\sum_{j=1}^{n} w_j(z)e^{\xi_j(z)} \equiv 0\);

(ii) \(g_j(z) - g_k(z)\) are not constants for \(1 \leq j < k \leq n\);

(iii) for \(1 \leq j \leq n\), \(1 \leq h < k \leq n\),
\[
T(r, w_j) = o(T(r, e^{\xi_h(z)})\} \quad (r \to \infty, r \notin E),
\]where \(E \subset (1, \infty)\) is of finite linear measure or finite logarithmic measure.

Then \(w_j(z) \equiv 0\) \((j = 1, 2, \ldots, n)\).

**Proof of Theorem 4:** (1) Since \(P(z, w(z))\) and \(Q(z, w(z))\) are relatively prime polynomials in \(w(z)\) with coefficients are rational functions, it follows from Lemma 2 that \(P(z, w(z))\) and \(Q(z, w(z))\) have finitely many common zeros only. Thus, from (5),
Lemma 3 and the assumption that $w(z)$ has finitely many poles, we conclude that

$$ N \left( r, \frac{1}{Q(z,w(z))} \right) \leq N \left( r, \frac{P(z,w(z))}{Q(z,w(z))} \right) + O(\log r) $$

$$ = N \left( r, \sum_{j \in J} \alpha_j(z) \left( \prod_{j \in J} w(q^i z) \right) \right) + O(\log r) $$

$$ \leq \sum_{j=1}^n N(r, w(q^i z)) + O(\log r) = O(\log r). \quad (12) $$

Thus, we deduce from Lemma 1 that

$$ Q(z,w(z)) = (w(z) - s(z))^\gamma, $$

where $s(z)$ is a rational function.

(2) Since $w(z)$ is transcendental with finitely many poles only, so is $Q(z,w(z)) = (w(z) - s(z))^\gamma$. We also note that $Q(z,w(z)) = (w(z) - s(z))^\gamma$ has finitely many zeros only from (12). Thus, there exists a rational function $h(z)$ and a nonconstant entire function $k(z)$ such that

$$ w(z) - s(z) = \beta h(z)^{1/t} e^{k(z)/t}, $$

where $\beta$ is the $t$-th root of unity. Denoting $g(z) = k(z)/t$, and noting that $r(z) := \beta h(z)^{1/t}$ is small function relative to $w(z)$, we get the desired form (11).

Now, substituting (11) into (5), and noting that $Q(z,f(z)) = (f(z) - s(z))^\gamma$, we conclude an equation of form

$$ h(z) a_M(z) \left( \prod_{j \in M} r(q^i z) \right) \exp \left( t g(z) + \sum_{j \in M} g(q^i z) \right) $$

$$ + \sum_{j \in |K|} H_j(z) \exp \left( t g(z) + \sum_{j \in J} g(q^i z) \right) $$

$$ = \sum_{j=0}^s p_j^*(z) \exp(j g(z)), \quad (13) $$

where the cardinality of the set $M$ is maximal among the sets in the collection $\{J\}$ such that $\alpha_M(z) \neq 0$, $|K|$ is a collection of non-empty subsets of $\{1, 2, \ldots, n\}$ such that $M \neq \{K\}$, $H_j(z)$ is rational function for every $J$, $p_j^*(z)(j = 0, 1, \ldots, s)$ are rational functions with $p_j^*(z) \neq 0$. Therefore, we deduce from Lemma 5 that there must exist at least two exponents in (13) that cancel each other to a constant $\tau \in \mathbb{C}$ such that

$$ \sum_{j=1}^n g(q^i z) = \sum_{j \in K} g(q^i z) + \tau, \quad \text{or} $$

$$ \sum_{j=1}^n g(q^i z) = (j_0 - \tau) g(z) + \tau. $$

These mean that there are at most $n + 1$ integers $k_0, k_1, \ldots, k_n$, which are not identically zeros such that

$$ k_0 g(z) + k_1 g(qz) + \cdots + k_n g(q^n z) = \tau. $$

In the follows, we prove that $g(z)$ is a transcendental entire function. Assume that $g(z) = c_k z^{k} + c_{k-1} z^{k-1} + \cdots + c_1 z + c_0$ is a nonconstant polynomial with degree $k$. Then for every $j \in \{1, 2, \ldots, n\}$, we may write

$$ g(q^i z) = a_k q^{ik} g(z) + g_j(z), \quad (14) $$

where $g_j(z)(j = 1, 2, \ldots, n)$ are polynomials in $z$ with degree no greater than $k - 1$. Substituting (11) and (14) into (5) again, we conclude that

$$ h(z) e^{\eta(z)} \sum_{j \in K} \alpha_j(z) \prod_{j \in J} \left( r(q^i z) e^{g_j(z)} e^{a_k q^{ik} g(z)} + g(q^i z) \right) $$

$$ = \sum_{j=0}^p a_j(z) \left( r(z) e^{g(z)} + s(z) \right)^\gamma. $$

Since polynomials $P(z,w(z))$ and $Q(z,w(z))$ are relatively prime, there is no common factor of positive degree in $w(z)$ for $P(z,w(z))$ and $Q(z,w(z))$. But, we deduce from Lemma 5 that $\sum_{j=0}^p g_j(z)s(z)^\gamma \equiv 0$, a contradiction.

**MEROMORPHIC SOLUTIONS WITH FEW POLES AND ZEROS**

Gundersen et al [8] proved the reduction theorems for functional equation of the form

$$ w(qz) = R(z,w(z)) $$

$$ = a_0(z) + a_1(z)w(z) + \cdots + a_r(z)w(z)^r, $$

which admits meromorphic solutions with relatively few distinct zeros and poles only, see Theorem 5.2 in [8]. As an application of Tumura-Clunie theorem, Rieppo extended the above result and proved the reduction theorems for certain functional equation that admit meromorphic solutions with relatively few distinct poles only [19]. The reasoning relies on the combination of Nevanlinna theory and algebraic field theory.

We now proceed to consider the reduction theorems for functional equations of form

$$ \prod_{i=0}^n w(q^i z)^{\lambda_i} = R(z,w \circ p) $$

$$ = a_0(z) + a_1(z)(w \circ p) + \cdots + a_r(z)(w \circ p)^r, $$

$$ b_0(z) + b_1(z)(w \circ p) + \cdots + b_r(z)(w \circ p)^r, \quad (15) $$

www.scienceasia.org
where \( p(z) = d_k z^k + \cdots + d_1 z + d_0 \) is a polynomial with constant coefficients \( d_k(\neq 0), \ldots, d_1, d_0 \) and of degree \( k \), \( l \) is a finite set of multi-indices \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \), and all coefficients of (15) are small meromorphic functions relative to \( w(z) \) such that \( a(z) b(z) \neq 0 \). The following results tell us that solutions having Borel exceptional zeros and poles appear in special situations only.

**Theorem 5** Let \( q \in \mathbb{C}, \ |q| > 1 \), and \( w(z) \) be a transcendental meromorphic solution of (15). If

\[
\max \left\{ \frac{\lambda(w)}{\lambda(w)}, \frac{1}{w} \right\} < \sigma(w),
\]

then (15) is either of the form

\[
\prod_{i=0}^{n} w(q^i z)^{\lambda_i} = \frac{a_0(z)}{b_0(z)}(w \circ \rho)^{p+1} \quad \text{or}
\]

\[
\prod_{i=0}^{n} w(q^i z)^{\lambda_i} = \frac{a_0(z)}{b_0(z)} \left( \frac{1}{w \circ \rho} \right)^{p+1},
\]

where \( \alpha \) is some nonzero constant.

We now give **Example 1** and **Example 2** to show Theorem 5 is sharp. **Example 3** shows that condition (16) is necessary and cannot be replaced by

\[
\min \left\{ \frac{\lambda(w)}{\lambda(w)}, \frac{1}{w} \right\} < \sigma(w).
\]

**Example 1** Let \( w(z) = e^z \). Then \( \lambda(w) = \lambda(1/w) = 0 < 1 = \sigma(w) \) and \( w(z) \) solves equation

\[
w(z)^4 w(2z)^2 = w(4z)^2 \quad \text{or} \quad w(z)^4 w(-2z)^2 = \frac{1}{w(2z)^2},
\]

which is the form of (17).

**Example 2** Let \( w(z) = e^z \). Then \( \lambda(w) = \lambda(1/w) = 0 < 1 = \sigma(w) \) and \( w(z) \) solves equation

\[
w(z)w(-2z)w(4z) = w(z)^3 \quad \text{or} \quad w(z)w(-3z) = \frac{1}{w(z)^2},
\]

which is the form of (17).

**Example 3** \( w(z) = \cos z \) solves the equation

\[
w(2z)w(4z) = 2w(2z)^3 - w(2z).
\]

Clearly, \( \lambda(1/w) = 0 < 1 = \lambda(w) = \sigma(w) \).

However, if we consider the inverse problem of Theorem 5, we can use similar techniques of Theorem 5.3 in [8] and Theorem 2 in [20] to get the following result.

**Theorem 6** Let \( q \in \mathbb{C}, \ |q| > 1 \), \( c(z) \) be nontrivial meromorphic functions, and \( w(z) \) be a transcendental meromorphic solution of equation

\[
\prod_{i=0}^{n} w(q^i z)^{\lambda_i} = c(z)w(z)^m, \quad m \in \mathbb{Z} \setminus \{0\}.
\]

If \( \sigma(c) < \sigma(w) \), then

\[
\max \left\{ \frac{\lambda(w)}{\lambda(w)}, \frac{1}{w} \right\} < \sigma(w).
\]

We now proceed to prepare some Lemmas.

**Lemma 6** Let \( w(z) \) be a transcendental meromorphic function, \( p(z) = d_k z^k + \cdots + d_1 z + d_0 \) be a polynomial of degree \( k \). Given \( 0 < \delta < |d_k| \), denote \( \nu := |d_k| + \delta \quad \text{and} \quad \mu := |d_k| - \delta \). Then, given \( \epsilon > 0 \) and \( a \in \mathbb{C} \cup \{\infty\} \), we have for all \( r \geq r_0 > 0 \),

\[
k \left( \mu r^k, a, w \right) \leq n(r, a, w \circ \rho) \leq kn \left( \nu r^k, a, w \right),
\]

\[
kN \left( \mu r^k, a, w \right) + O(\log r) \leq N(r, a, w \circ \rho)
\]

\[
\leq kn \left( \nu r^k, a, w \right) + O(\log r),
\]

\[
(1 - \epsilon) T \left( \mu r^k, w \right) \leq T(w \circ \rho) \leq (1 + \epsilon) T \left( \nu r^k, w \right).
\]

**Lemma 7** Suppose that \( w(z) \) is a transcendental meromorphic solution of equation

\[
\sum_{j=0}^{n} a_j(z)w(q^i z) = Q(z),
\]

where \( q \in \mathbb{C}, \ |q| \neq 0, 1 \), and all coefficients \( a_0, \ldots, a_n \) are meromorphic and of finite order \( \leq \rho \). Then \( \sigma(w) \leq \rho \).

**Proof of Theorem 5:** Let \( \tau \) be the multiplicity of pole of \( w(z) \) at the origin, and let \( q(z) \) be a canonical product of \( w(z) \) formed by the nonzero poles of \( w(z) \). Since \( \max \{\lambda(w), \lambda(1/w)\} < \sigma(w) \), then \( h(z) = z^a q(z) \) is an entire function such that

\[
\sigma(h) = \lambda \left( \frac{1}{w} \right) < \sigma(w)
\]

and \( g(z) = h(z)w(z) \) is a transcendental entire function with

\[
T(r, g) = T(r, w) + S(r, w), \quad \sigma(g) = \sigma(w), \quad \lambda(g) = \lambda(w).
\]

We now conclude from the last assertion of Lemma 6, (19) and (20) that

\[
\sigma(h \circ \rho) = k \sigma(h) = k \lambda \left( \frac{1}{w} \right) < k \sigma(g) = \sigma(g \circ \rho).
\]

www.scienceasia.org
Therefore,
\[ T(r, h \circ p) = S(r, g \circ p). \]  
(21)
Substituting \( w(z) = g(z)/h(z) \) into (15), we conclude that
\[
\frac{h \circ p}{\prod_{i=0}^{n} h(q^i z)^{\lambda_i}} = \frac{a_0(z)(h \circ p)^{i} + \cdots + a_i(z)(g \circ p)^{i}}{b_0(z)(h \circ p)^{i} + \cdots + b_i(z)(g \circ p)^{i}}.
\]  
(22)

Obviously, it follows from (19), (20) and (21) that
\[
\begin{cases}
T(r, \prod_{i=0}^{n} h(q^i z)^{\lambda_i}) = S(r, g \circ p), \\
T(r, h \circ p)^{-1} = S(r, g \circ p), \\
T(r, a_0(z)(h \circ p)^{-1} u = 0, 1, \ldots, s, \\
T(r, b_i(z)(h \circ p)^{-1} = S(r, g \circ p), \quad v = 0, 1, \ldots, t.
\end{cases}
\]  
(23)

Denoting \( A(z) = (h \circ p)^{-1} / \prod_{i=0}^{n} h(q^i z)^{\lambda_i} \), we get from (23) that
\[ T(r, A) = S(r, g \circ p). \]  
(24)

Since zeros and poles are Borel exceptional values of \( w(z) \) by (16), we may apply a result due to Whittaker, see Satz 13.4 in [23], to deduce that \( w(z) \) is of regular growth. Thus, we use (23) again to get
\[
T \left( r, \frac{w'}{w} \right) = \frac{N(r, F)}{F(r)} + \frac{1}{N(r, F)} + S(r, w) = S(r, g \circ p). \]  
(25)

Similarly, if we set \( B(z) = A(z)(\prod_{i=0}^{n} g(q^i z)^{\lambda_i}) \), we also deduce from the lemma of the logarithmic derivative, (16), (20) and (24) that
\[
T \left( r, \frac{B'}{A} \right) = T \left( r, \frac{A'}{A} + \sum_{i=0}^{n} \lambda_i q^i \frac{g'(q^i z)}{g(q^i z)} \right) = S(r, g \circ p). \]  
(26)

Denote \( F(z) = g \circ p \),
\[
P(z, F) = \frac{a_0(z)}{a_i(z)} (h \circ p)^{y} + \frac{a_i(z)}{a_i(z)} (h \circ p)^{-1} F(z) + \cdots + F(z)^{y},
\]
and
\[
Q(z, F) = \frac{b_0(z)}{b_i(z)} (h \circ p)^{y} + \frac{b_i(z)}{b_i(z)} (h \circ p)^{-1} F(z) + \cdots + F(z)^{y}.
\]
Therefore, we deduce from (20) and (21) that the coefficients of \( P(z, F) \) and \( Q(z, F) \) are small functions relative to \( g \circ p \). Thus, (22) can be written in the form
\[
\frac{b_i(z)}{a_i(z)} B(z) = \frac{P(z, F)}{Q(z, F)} = u(z, F). \]  
(27)

By denoting
\[
\psi(z) = \frac{F'(z)}{F(z)} \quad \text{and} \quad U(z) = \frac{u'(z, F)}{u(z, F)},
\]
we get \( T(r, U) = S(r, w) \) from (26) and (27). We also conclude from the lemma of logarithmic derivative, Lemma 6, (16), (20) and (21) that
\[
T(r, U) = T \left( r, \frac{F'}{F} \right) = m(r, \frac{F'}{F}) + N(r, \frac{F'}{F}) + N(r, \frac{1}{F}) + S(r, F)
\]
and
\[
\leq N(r, F) + N(r, \frac{1}{F}) + S(r, g \circ p)
\]
\[
\leq N \left( r, \frac{1}{g \circ p} \right) + S(r, g \circ p)
\]
\[
\leq N \left( \frac{w \lambda_z}{g} \right) + S(r, g \circ p) = S(r, g \circ p),
\]
where \( \nu \) is defined as Lemma 6. Since
\[
P'Q - PQ' = u' = Uu = UPQ,
\]
we conclude that
\[
P'Q - PQ' = UPQ. \]  
(28)

Now, writing \( F' = \psi F \) in (28), regarding then (28) as an algebraic equation in \( F \) with coefficients of growth \( S(r, F) \) (in fact \( S(r, w) \)), and comparing the leading coefficients, we deduce that
\[
(s - t)\psi = U.
\]

By integrating both sides of the above equality, we conclude that
\[
u(z, F) = g \circ p = \alpha F(z)^{y-t}, \]  
(29)
for some \( \alpha \in \mathbb{C} \setminus \{0\} \). Therefore, by combing the representations of \( F, B, A, g \) with (29), we conclude that
\[
\prod_{i=0}^{n} w(q^i z)^{\lambda_i} = \alpha \frac{a_i(z)}{b_i(z)} (w \circ p)^{y-t}. \]  
(30)
If \( st \neq 0 \), we deduce from (15) and (30) that
\[
\frac{a_i(z)}{b_i(z)} (w \circ p)^{y-t} = R(z, w \circ p)
\]
and
\[
\frac{a_0(z) + a_1(z)(w \circ p)^{y-t} + \cdots + a_i(z)(w \circ p)^{y-t}}{b_0(z) + b_1(z)(w \circ p)^{y-t} + \cdots + b_i(z)(w \circ p)^{y-t}}.
\]
From this, we get that \( R(z, w \circ p) \) is not irreducible in \( w \circ p \), a contradiction. Thus, \( s = 0 \) or \( t = 0 \). Therefore, we deduce from (30) that

\[
\prod_{i=0}^{n} w(q^i)\lambda_i = \frac{a_0(z)}{b_0(z)}(w \circ p)^s \quad \text{or} \quad \prod_{i=0}^{n} w(q^i)\lambda_i = \frac{a_0(z)}{b_i(z)}(w \circ p)^t.
\]

Applying Valiron-Mohon’ko theorem [24] to (31), we obtain \( s = n + 1 \) or \( t = n + 1 \). Thus, the desired forms (17) are obtained.

**Proof of Theorem 6:** Denote \( y(z) = w'(z)/w(z) \). We conclude from (18) that

\[
\sum_{i=1}^{n} \lambda_i q^i y(q^i z) + (\lambda_0 - m) y(z) = \frac{c'(z)}{c(z)}.
\]

Thus, we deduce from Lemma 7 that

\[
\sigma(y) \leq \sigma\left(\frac{c'(z)}{c(z)}\right) < \sigma(w).
\]

Therefore,

\[
\max\left\{ \lambda(w), \frac{1}{w} \right\} = \lambda\left(\frac{w'}{w}\right) \leq \sigma(y) < \sigma(w).
\]

**GROWTH OF MEROMORPHIC SOLUTIONS**

At this point, we briefly introduce some notations used below. Let \( I \) be a finite set of multi-indexes \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \). A \( q \)-difference monomial of a meromorphic function \( w(z) \) is defined as

\[
\prod_{i=0}^{n} w(q^i z)\lambda_i,
\]

and a \( q \)-difference polynomial \( H_q(z, w(z)) \) of a meromorphic function \( w(z) \), a finite sum of \( q \)-difference monomials, is defined as

\[
H_q(z, w(z)) = \sum_{\lambda \in I} a_\lambda(z) \prod_{i=0}^{n} w(q^i z)^{\lambda_i}, \tag{32}
\]

where the coefficients \( a_\lambda(z) \) are small functions relative to \( w(z) \). The degree of the \( q \)-difference polynomial (32) is defined by

\[
\deg_w(H_q) = \max_{\lambda \in I} \left\{ \sum_{i=0}^{n} \lambda_i \right\}.
\]

For instance, the degree of the \( q \)-difference polynomial \( w^2(z)w(qz)w\left(q^2z\right) + w(z)w\left(q^3z\right) \) is four.

In follows, we consider the growth of meromorphic solutions of some functional difference equations.

**Theorem 7** Let \( q \in \mathbb{C} \), \( |q| > 1 \), and \( w(z) \) be a transcendental meromorphic solution of equation

\[
\sum_{\lambda \in I} a_\lambda(z) \prod_{i=0}^{n} w(q^i z)^{\lambda} = R(z, w(z))
\]

\[
= \frac{a_0(z) + a_1(z)w(z) + \cdots + a_s(z)w(z)^s}{b_0(z) + b_1(z)w(z) + \cdots + b_t(z)w(z)^t}, \tag{33}
\]

where \( I \) is a finite set of multi-indexes \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \), and all coefficients in (33) are small meromorphic functions relative to \( w(z) \) such that \( a_i(z)b_i(z) \neq 0 \). If \( d = \max\{s, t\} > (n + 1)\deg_{w}(H_q) \), then \( \sigma(w) > 0 \).

**Example 4** \( w(z) = e^z/z \) solves the functional equation

\[
f(z)^2f(-2z)f(4z) + f(-8z)^2 = \frac{1 - 8z^{18}f(z)^{20}}{64z^{18}f(z)^{16}}
\]

of type (33). Here, \( q = -2 \), \( d = 20 > 16 = (3+1)4 = (n+1)\deg_{h_i} \), and \( \sigma(f) = 1 > 0 \).

In fact, the following Example 5 shows that the assertion of Theorem 7 may occur if \( d = \deg_{i}(H_q) \). But we can not find a proper method to prove it.

**Example 5** \( w(z) = \cos z \) solves functional equation

\[
w(2z)w\left(2^2z\right) = 16w(z)^6 - 24w(z)^4 + 10w(z)^2 - 1
\]

of type (33). Here, \( q = 2 \), \( d = 6 = (2 + 1)2 = (n + 1)\deg_{f}(H_q) \), and \( \sigma(f) = 1 > 0 \).

**Theorem 8** Let \( q \in \mathbb{C} \), \( |q| > 1 \), and \( w(z) \) be a transcendental meromorphic solution of equation

\[
\sum_{\lambda \in I} a_\lambda(z) \prod_{i=0}^{n} w(q^i z)^{\lambda} = R(z, w(z))
\]

\[
= \frac{a_0(z) + a_1(z)(w \circ p) + \cdots + a_s(z)(w \circ p)^s}{b_0(z) + b_1(z)(w \circ p) + \cdots + b_t(z)(w \circ p)^t}, \tag{34}
\]

where \( p(z) = d_kz^k + \cdots + d_1z + d_0 \) is a polynomial with constant coefficients \( d_k \neq 0, \ldots, d_1, d_0 \) and of degree \( k \geq 2 \), \( I \) is a finite set of multi-indexes \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \), and all coefficients in (34) are small meromorphic functions relative to \( f(z) \) such that \( a_i(z)b_i(z) \neq 0 \). Moreover, we assume that \( kd =
Lemma 8 Let \( q \in \mathbb{C}, |q| > 1 \), and \( w(z) \) be a meromorphic function. Then the characteristic function of \( q \)-difference polynomial (32) satisfies
\[
T\left(r, \sum_{\lambda \in \ell} \alpha_{\lambda}(z) \prod_{i=0}^{n} w(q^i z)^{\lambda} \right) \\
\leq (n+1) \text{deg}_w(H_q) T(|q|^n r, w) + S(r, w).
\]

We now prepare some lemmas. By denoting \( w_i = w(q^i z) \) \((i = 0, 1, \ldots, n)\), it is easy to prove the following result from Lemma 3.

Lemma 9 ([24, 25]) Let \( g(r) \) and \( h(r) \) be monotone nondecreasing functions on \([0, \infty)\) such that \( g(r) \leq h(r) \) for all \( r \notin E \cup [0, 1] \), where \( E \subset (1, \infty) \) is a set of finite logarithmic measure. Let \( \alpha > 1 \) be a given constant. Then there exists an \( r_0 = r_0(\alpha) > 0 \) such that \( g(r) \leq h(ar) \) for all \( r \geq r_0 \).

Lemma 10 ([26]) Let \( \psi(r) \) be a function of \( r \geq r_0 \), positive and bounded in every finite interval.

(i) Suppose that \( \psi(\mu^m) \leq A \psi(r) + B(r \geq r_0) \), where \( \mu > 0 \), \( m \geq 1 \), \( A \geq 1 \), \( B \) are constants. Then \( \psi(r) = O((\log r)^\alpha) \) with \( \alpha = \log A/\log m \), unless \( A = 1 \) and \( B > 0 \); and if \( A = 1 \) and \( B > 0 \), then for any \( \varepsilon > 0 \), \( \psi(r) = O((\log r)^\varepsilon) \).

(ii) Suppose that (with the notation of (i)) \( \psi(\mu^m r^K) \geq A \psi(r)(r \geq r_0) \). Then for all sufficiently large values of \( r \), \( \psi(r) \geq K(\log r)^\alpha \) with \( \alpha = \log A/\log m \) for some positive constant \( K \).

Proof of Theorem 7: Assume in contrary to the assertion that \( w(z) \) is meromorphic with \( \sigma(w) = 0 \). For any \( \eta(0 < \varepsilon < (d - (n + 1) \text{deg}_w(H_q))/(d + (n + 1) \text{deg}_w(H_q))) \), we may apply Valiron-Mohon’ko lemma, Lemma 6, (32) and (34) to conclude that
\[
d(1 - \varepsilon) T(r, w) \leq d T(r, w) + S(r, w) \\
= T\left( r, \sum_{\lambda \in \ell} \alpha_{\lambda}(z) \prod_{i=0}^{n} w(q^i z)^{\lambda} \right) \\
\leq (n+1) \text{deg}_w(H_q) T(|q|^n r, w) + S(r, w) \\
\leq (n+1) \text{deg}_w(H_q)(1 + \varepsilon) T(|q|^n r, w),
\]
on a set of logarithmic density 1. So, we get
\[
T(r, w) \leq \frac{(n+1) \text{deg}_w(H_q)(1 + \varepsilon)}{d(1 - \varepsilon)} T(|q|^n r, w) \\
=: \gamma T(|q|^n r, w)
\]
on a set of logarithmic density 1, where
\[
\gamma := \frac{(n+1) \text{deg}_w(H_q)}{d(1 - \varepsilon)} < 1,
\]
since \( \varepsilon(0 < \varepsilon < (d - (n + 1) \text{deg}_w(H_q))/(d + (n + 1) \text{deg}_w(H_q))) \) and the assumption that \( d > (n + 1) \text{deg}_w(H_q) \). Thus, we deduce from Lemma 4 that \( \sigma(w) > 0 \), a contradiction.

Proof of Theorem 8: For any \( \varepsilon(0 < \varepsilon < 1) \), we may apply Valiron-Mohon’ko lemma, Lemma 6, Lemma 8, (32) and (34) to conclude that
\[
d(1 - \varepsilon) T(\mu^k, w) \leq d T(r, w \circ p) + S(r, w \circ p) \\
= T\left( r, \sum_{\lambda \in \ell} \alpha_{\lambda}(z) \prod_{i=0}^{n} w(q^i z)^{\lambda} \right) + S(r, w) \\
\leq (n+1) \text{deg}_w(H_q) T(|q|^n r, w) + S(r, w) \\
\leq (n+1) \text{deg}_w(H_q)(1 + \varepsilon) T(|q|^n r, w)
\]
holds for all sufficiently large \( r \), possibly outside of an exceptional set of finite logarithmic measure, where \( \mu \) is defined as Lemma 6. Now, we may apply Lemma 9 to deal with the exceptional set, and conclude that, for every \( \eta > 1 \), there exists an \( r_0 > 0 \) such that
\[
d(1 - \varepsilon) T(\mu^k, w) \leq (n+1) \text{deg}_w(H_q)(1 + \varepsilon) T(|q|^n r, w) \tag{35}
\]
holds for all \( r > r_0 \). Denoting \( \tau = \eta|q|^n r \). Then (35) can be written in the form
\[
T\left( \mu^k, \frac{\tau}{|q|^n r} \right) \leq \frac{(n+1) \text{deg}_w(H_q)(1 + \varepsilon)}{d(1 - \varepsilon)} T(\tau, w).
\]
Since \( d k \leq (n+1) \text{deg}_w(H_q) \), we get \( (n+1) \text{deg}_w(H_q)(1 + \varepsilon)/d(1 - \varepsilon) > 1 \) for all \( 0 < \varepsilon < 1 \). Thus, we now apply Lemma 10 (i) to conclude that
\[
T(r, w) = O((\log r)^{a + \varepsilon}),
\]
and
\[ \alpha = \log \frac{(n+1) \deg_w(H_q)(1+\epsilon)}{d(1-\epsilon)} \log k = \frac{\log(n+1) + \log \deg_w(H_q) - \log d}{\log k} + o(1). \]

Acknowledgements: The research were supported by National Natural Science Foundation of China (11871260, 11801110, 11801093) and Characteristic Innovation Project (Natural Science) of Guangdong Province (2019KTSCX119).

REFERENCES

1. Ablowitz M, Halburd RG, Herbst B (2000) On the extension of Painlevé property to difference equations. Nonlinearity 13, 889–905.
2. Heittokangas J, Korhonen R, Laine I, Rieppo J, Tohge K (2001) Complex difference equations of Malmquist type. Comput Methods Funct Theory 1, 27–39.
3. Zhang RR, Huang ZB (2014) Meromorphic solutions of a complex difference equations of Malmquist type. Bull Korean Math Soc 51, 1735–1748.
4. Laine I, Rieppo J, Silvennoinen H (2005) Remarks on complex difference equations. Comput Methods Funct Theory 5, 77–88.
5. Halburd R, Korhonen R (2006) Existence of finite-order meromorphic solutions as a detector of integrability in difference equations. Physica D 218, 191–203.
6. Huang ZB, Zhang RR (2014) Some properties on complex functional difference equations. Abstr Appl Anal 2014, ID 283895.
7. Bergweiler W, Ishizaki K, Yanagihara N (1998) Meromorphic solutions of some functional equations. Methods Appl Anal 5, 248–258.
8. Gundersen GG, Heittokangas J, Laine I, Rieppo J, Yang DG (2002) Meromorphic solutions of generalized Schröder equations. Aequationes Math 63, 110–135.
9. Zheng XM, Chen ZX (2010) Some properties of meromorphic solutions of q-difference equations. J Math Anal Appl 361, 472–480.
10. Hayman WK (1964) Meromorphic Functions, Clarendon Press, Oxford.
11. Yang L (1982) Value Distribution Theory and New Research, Science Press, Beijing. [in Chinese]
12. Weissenborn G (1986) On the theorem of Tumura and Clunie. Bull London Math Soc 18, 371–373.
13. Gromak V, Laine I, Shimomura S (2002) Painlevé Differential Equations in the Complex Plane, Walter de Gruyter, Berlin.
14. Barnett DC, Halburd RG, Morgan W, Korhonen RJ (2007) Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations. Proc Roy Soc Edinburgh 137, 457–474.
15. Zhang JL, Korhonen R (2010) On the Nevanlinna characteristic of f(qz) and its application. J Math Anal Appl 369, 537–544.
16. Korhonen R (2009) An extension of Picard’s theorem for meromorphic functions of small hyper-order. J Math Anal Appl 357, 244–253.
17. Tu H, Gu Y (2020) On the meromorphic solutions of generalized q-difference equations. ScienceAsia 46, 626–633.
18. Yang CC, Yi HX (2003) Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers Group, Dordrecht.
19. Rieppo J (2007) On a class of complex functional equations. Ann Acad Sci Fenn 32, 151–170.
20. Zheng XM, Chen ZX (2012) On properties of q-difference equations. Acta Math Sci 32, 724–734.
21. Goldstein R (1971) Some results on factorisation of meromorphic functions. J London Math Soc 2–4, 357–364.
22. Heittokangas J, Laine I, Rieppo J, Yang DG (2000) Meromorphic solutions of some linear functional equations. Aequationes Math 60, 148–166.
23. Jank G, Volkmann L (1985) Einführung in die Theorie der Ganzen und Meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser Verlag, Basel-Boston.
24. Laine I (1993) Nevanlinna Theory and Complex Differential Equations, de Gruyter, Berlin.
25. Gundersen GG (1988) Finite order solutions of second order linear differential equations. Trans Amer Math Soc 305, 415–429.
26. Goldstein R (1978) On meromorphic solutions of certain functional equations. Aequationes Math 18, 112–157.