Nondigital Implementation of the Arithmetic of Real Numbers by Means of Quantum Computer Media

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Abstract—In the framework of a model for quantum computer media, a nondigital implementation of the arithmetic of the real numbers is described. For this model, an elementary storage "cell" is an ensemble of qubits (quantum bits). It is found that to store an arbitrary real number it is sufficient to use four of these ensembles and the arithmetical operations can be implemented by fixed quantum circuits.

Key words: quantum media, quantum computer, arithmetic, qubit, q-ensemble.

INTRODUCTION

In this note an implementation of the arithmetic of real numbers is described in the framework of a model for quantum computer media (QCM). This model is an extension of the well-known standard model for quantum computers. For such quantum computer media an elementary storage “cell” is an ensemble of qubits (i.e., quantum bits). It is found that to store an arbitrary real number it is sufficient to use four of these ensembles and the arithmetical operations can be carried out with a fixed number of elementary steps. Here any number is represented in nondigital form. The representation of such a number in digital form (e.g., in the form of a binary or decimal fraction) is a separate problem of statistical estimation. Another approach to quantum computations over continuous variable is presented, e.g., in [1, 2].

1. STANDARD MODEL OF QUANTUM COMPUTATIONS

The idea of quantum computing was first put forward by P. Benioff [3, 4], Yu. I. Manin [5], R. P. Feynman [6, 7], and A. Peres [8]. In Feynman’s paper [7] this idea was discussed in detail. D. Deutsch [9] stated a general formal definition of the so-called quantum Turing machine. In [10] he presented another (equivalent but more convenient) model, which is now regarded as standard.

We shall recall some basic concepts and the corresponding notation for a version of the standard model presented in [11–13] (see these papers for details, as well as, e.g., [14]).

Let $X$ be a finite set. Denote by $\mathcal{B}(X)$ the set of all Boolean functions defined on $X$ and taking values 0 or 1. Let $\mathcal{H}(X)$ be the complex Hilbert space with $\mathcal{B}(X)$ as an orthonormal basis; so if $X$ contains $n$ elements, then the dimension of the space $\mathcal{H}(X)$ is $2^n$. Let $\mathcal{L}(X)$ be the algebra of all linear operators in $\mathcal{H}(X)$, $\mathcal{U}(X)$ the group of all unitary operators in $\mathcal{H}(X)$, and $\mathcal{D}(X)$ the set of all density operators, i.e., positive selfadjoint operators in $\mathcal{H}(X)$ whose trace is equal to 1.
If $\Delta$ is a set of bits forming the storage of a classical computer, then the states of this storage can be described by elements of $B(\Delta)$. But if $\Delta$ is a set of qubits forming a quantum storage of a quantum computer, then mixed states of this storage can be described by elements of $D(\Delta)$. Of course, pure states are characterized by elements of $H(\Delta)$ (up to a nonzero number coefficient).

A linearly ordered subset $X$ of $\Delta$ is called register. Pure states of $X$ correspond to elements of $H(X)$; general mixed states of $X$ correspond to elements of $D(X)$, i.e., density operators in $H(X)$. In particular, each qubit is a register. In this case $X$ consists of a single element and $H(X)$ is two-dimensional. If $X$ consists of $n$ qubits, then $H(X)$ is a tensor product of $n$ two-dimensional Hilbert spaces corresponding to each qubit.

Each qubit has two basic states denoted by $|0\rangle$ and $|1\rangle$ (Dirac’s notation is used). States of a quantum storage (or its register) are called classical states if they are tensor products of these basic states corresponding to each qubit. It is assumed that the quantum storage can be prepared (initiated) in an arbitrary classical state.

Any unitary operator $U \in U(X)$ defines a transformation $S \mapsto S^U$ on the set $D(X)$ of all states of the register $X$ by the formula $S^U = USU^{-1}$. We shall say that a unitary operator $U \in U(\Delta)$ is concentrated on a register $X \subset \Delta$, if $U$ can be represented in the form $U = U_X \otimes id_Z$, where $U_X \in U(X)$, $Z = \Delta \setminus X$, and $id_Z$ is the identity operator in $H(Z)$. Respectively, we shall also consider any unitary operator $U_X \in U(X)$ as an operator of the form $U_X \otimes id_Z$ belonging to $U(\Delta)$, where $Z = \Delta \setminus X$. We shall say that the register $X$ is a support of $U \in U(\Delta)$ and denote it by $\text{supp}(U)$, if $X$ is the minimal register on which $U$ is concentrated.

A quantum computer performs unitary transformations in $H(\Delta)$. It is assumed that in one step an elementary unitary transformation can be made and there is a fixed collection (basis) of such unitary operators which are called logic gates, or simply gates. It is also assumed that every gate has a short support (usually consisting of one or two qubits). Combinations of these gates define quantum circuits.

Every bijection $\sigma$ of the set of all classical states $B(X)$ onto itself leads to a permutation of the elements of the corresponding orthonormal basis in $H(X)$ and generates a unitary operator $\sigma \in U(X)$. Operators of this type are called classical operators (transformations).

**Example 1.** For a register $\{x\}$ consisting of a single qubit $x$ the permutation $|0\rangle \mapsto |1\rangle$, $|1\rangle \mapsto |0\rangle$ defines the so-called negation operator (or NOT operator) denoted by $\neg_x$.

**Example 2.** Another important example is the so-called controlled NOT (or CNOT) operator, see, e.g., [7]. For a two-bit register $X = \{x, y\}$ this operator is induced by the bijection $\tau : |x, y\rangle \mapsto |x, x + y\rangle$, where “$+$" denotes addition modulo 2. For classical states this bijection $\tau$ allows to copy the content of one bit into another, provided the second bit is empty. Of course, a similar operator can be defined for a pair of arbitrary registers of the same size by applying $\tau$ to each pair of bits.

According to the above, the operators described in these examples can be treated as unitary operators belonging to $U(\Delta)$.

Thus, for the standard model, (mixed) states of a finite storage $\Delta$ are defined by density operators belonging to $D(\Delta)$, whereas elementary operations (gates) are defined by a fixed collection of unitary operators concentrated on short registers. In the framework of this model, the execution of an algorithm starts from a preparation of the storage $\Delta$ in a classical state. Then a sequence of unitary quantum gates is applied. Finally, a measurement operation (which is a specific type of interaction between the quantum computer and an external physical device) is performed. The result of this measurement operation is a classical state of the register. For the corresponding details, see, e.g., [11–13].
2. QUANTUM COMPUTER MEDIA MODEL

There are rather many different paradigms and models for quantum computer systems, see e.g. [2, 11–19]. We shall say that a computer medium including a system of parallel quantum computers (processors) and classical components is a quantum computer medium (briefly QCM). We shall consider a version of this model convenient for our aims. This QCM has a storage \( \Delta \) which is a set of large ensembles called \( q \)-ensembles. Roughly speaking, any \( q \)-ensemble can be regarded as a flow of independent qubits, whereas operations with \( q \)-ensembles can be treated as actions independently affecting each qubit in the same way under the same conditions (of course, the number of qubits in a \( q \)-ensemble is finite but large enough). There is a similar situation, say, in so-called bulk quantum computation, where one can manipulate a large number of indistinguishable quantum computers by parallel unitary operations; see, e.g., [15] for details and implementations using nuclear magnetic resonance.

Denote by \( \Delta \) the set of all \( q \)-ensembles forming the storage of our QCM and denote by \( \hat{\Delta} \) the set of all qubits belonging to this storage. Any \( q \)-ensemble \( x \in \Delta \) forms a subset \( \hat{\Delta}_x \) in \( \hat{\Delta} \). We say that its state \( \hat{S}_x \in D(\hat{\Delta}_x) \) is admissible, if \( \hat{S}_x \) is a tensor power of a state \( S \in D(\{a\}) \), where \( a \in \hat{\Delta}_x \), so that there is a one-to-one correspondence between the set of all admissible \( q \)-ensemble states and the set of all states for each qubit belonging to this \( q \)-ensemble. Every mixed state \( \hat{S} \in D(\hat{\Delta}) \) can be restricted to any \( q \)-ensemble (by the partial trace formula, see, e.g., [11, 12] and below). If all such restrictions are admissible, then we say that the state \( \hat{S} \) is admissible. Denote by \( \mathbf{D}^*(\Delta) \) the set of all admissible states of our QCM. We shall say that a unitary operator \( U \in U(\hat{\Delta}) \) is admissible, if \( \mathbf{D}^*(\Delta) \) is invariant under the action of this operator.

It is clear that the corresponding standard quantum computer model can be embedded into the QCM model, so that \( \mathbf{D}(\Delta) \) and \( U(\Delta) \) correspond to the sets of admissible states and admissible unitary operators respectively. It is assumed that the set of all classical states in the standard model can be identified with the corresponding set of states in the QCM model. So every algorithm implemented in the framework of the standard model can be transferred to the QCM model.

However, for the QCM case, it is possible to construct a cloning (copying) operation which transfers any \( q \)-ensemble in an admissible state to a pair of \( q \)-ensembles such that each of their qubits has the same state. This copying operation may be treated as the division of the initial \( q \)-ensemble into large parts. Note that in the framework of the standard model, perfect cloning is impossible: an unknown quantum state can not be cloned (unless this state is already known, i.e., there exists a classical information which specifies it). However, it is possible to make approximate copies. For details see, e.g., [13, 20–23].

3. MODELING THE ARITHMETIC OF REAL NUMBERS

Thus, in the framework of the QCM model, it is possible to implement the standard model with mixed states and a cloning operation. Below we need to obtain a collection of qubits prepared in identical classical states and to manipulate these copies by parallel unitary operations. So, for the sake of simplicity, we shall consider the standard model extended by this cloning operation. This operation is not a quantum unitary operation. However, we shall include it in quantum circuits (a similar trick was used in [13] with respect to measurement operations). For this quantum computer an implementation of the real number arithmetic is presented below. Moreover, for the arithmetical operations, the execution time does not depend on the complexity (in the usual sense) of operands. In particular, for the function \( n \mapsto a^n \), where \( n \in \mathbb{N} \) and \( a \) is an arbitrary real number, it is possible to get a polynomial algorithm of its calculation (with respect to the size of the number \( n \), i.e., \( \log n \)) using the well-known standard trick: \( a \mapsto a^2 \mapsto a^4 = (a^2) \cdot (a^2) \) etc.

Using the notation introduced above in Sec. 1, denote by \( \mathcal{B}(n) \) the set \( \mathcal{B}(\{1, \cdots , n\}) \) and
by $\mathcal{H}(n)$ the Hilbert space $\mathcal{H}\{\{1, \cdots, n\}\}$. Denote by $|0\rangle$ and $|1\rangle$ elements of $\mathcal{B}(1)$ and by $|\alpha_1, \cdots, \alpha_n\rangle$ elements of $\mathcal{B}(n)$, where $|\alpha_i\rangle \in \mathcal{B}(1)$. Using Dirac’s bra/ket notation, denote by $|x\rangle$ elements of the Hilbert space $\mathcal{H}(X)$ (ket-vectors), and by $\langle x|y\rangle$ the scalar product of the vectors $|x\rangle, |y\rangle \in \mathcal{H}(X)$. Any bra-vector $\langle x|$ corresponds to the linear functional $y \mapsto \langle x|y\rangle$ on $\mathcal{H}(X)$, whereas the notation $|a\rangle\langle b|$ corresponds to the linear operator $|x\rangle \mapsto (b, x)|a\rangle$.

We have assumed that each register is linearly ordered; therefore $\mathcal{B}(X)$ and $\mathcal{H}(X)$ can be naturally identified with $\mathcal{B}(n)$ and $\mathcal{H}(n)$, where $n$ is the length of the register $X$ (i.e., number of elements of $X$).

Let $S$ be a (mixed) state of the storage $\Delta$ and let $X \subset \Delta$ be an arbitrary register. The restriction of the state $S$ to the register $X$ is defined by the partial trace formula

$$S \mapsto S(X) = \text{Tr}_Z(S) \in \mathcal{D}(X), \quad \text{where} \quad Z = \Delta \setminus X,$$

(see, e.g., [11, 12]) and we shall say that $S(X)$ is the state of $X$.

Any state of a one-point register $\{x\}$, i.e., of a qubit, is defined by the corresponding density matrix $S = (S_{ij})$ with respect to the basis $\mathcal{B}\{\{x\}\} = \{\{0\}, \{1\}\}$. Here the matrix element $S_{00}$ is equal to the probability that the measured value of the qubit is $|0\rangle$. Similarly, the probability that the measured value of the qubit is $|1\rangle$ coincides with $S_{11}$. Any classical state of $X$, i.e., any element $f \in \mathcal{B}(X)$, corresponds to the operator $\otimes_{x \in X} |f(x)\rangle\langle f(x)|$. In particular, we set

$$0_X = \otimes_{x \in X} |0\rangle\langle 0|, \quad 1_X = \otimes_{x \in X} |1\rangle\langle 1|.$$  

These operators correspond to the Boolean functions on $X$ (i.e., classical states) which are identically equal to 0 or 1.

Consider a partition $X = \bigcup_{i=1}^n X_i$ of an arbitrary register $X$ into disjoint registers $X_i$. It is clear that in this case the space $\mathcal{H}(X)$ can be decomposed in the form of the tensor product

$$\mathcal{H}(X) = \bigotimes_{i=1}^n \mathcal{H}(X_i)$$

of the spaces $\mathcal{H}(X_i)$. Therefore, for any collection of operators $A_i \in \mathcal{L}(X_i)$, their tensor product

$$A = \bigotimes_{i=1}^n A_i \in \mathcal{L}(X)$$

exists. We shall say that this operator $A$ is decomposable with respect to the partition $X = \bigcup_{i=1}^n X_i$. Note that the tensor product of density operators is a density operator, and in the same way the tensor product of unitary operators is a unitary operator. We say that a state (i.e., an element of $\mathcal{D}(X) \subset \mathcal{L}(X)$) is (simply) decomposable, if it is decomposable with respect to the partition of $X$ into its points, i.e., into one-qubit registers. Note that every classical state is decomposable.

Let $A = A_1 \cup A_2$ be a partition of a register $A$ into disjoint registers $A_1$ and $A_2$, $X_1 \subset A_1$ and $X_2 \subset A_2$. It is easy to check that if a state $S \in \mathcal{D}(A)$ is decomposable with respect to the partition $A = A_1 \cup A_2$, then the restriction of this state to the register $X = X_1 \cup X_2$ is decomposable with respect to the partition $X = X_1 \cup X_2$. We say that a state $S \in \mathcal{D}(\Delta)$ is decomposable with respect to a register $X \subset \Delta$, if its restriction $S(X)$ to $X$ is decomposable.

We say that a register $F$ is free with respect to a state $S$ and that this state $S$ is free with respect to the register $F$ if

$$S = S' \otimes 0_F, \quad \text{where} \quad S' \in \mathcal{D}(\Delta \setminus F).$$
Suppose that $F$ is free with respect to $S$, then the register $F \setminus \text{supp}(U)$ is free with respect to the state $S^{U} = USU^{-1}$.

Now we can discuss our implementation of the arithmetic of real numbers. Let $S$ be a state of a register $X$; then denote by $S(x)$ the restriction of $S$ to a qubit $x \in X$. Suppose

$$X = \{x_1, x_2, x_3, x_4\}$$

is a register consisting of four qubits, $S$ is a decomposable state of $X$, $S(x)_{ij}$ is the corresponding density matrix.

We shall say that any decomposable state $S$ of $X = \{x_1, x_2, x_3, x_4\}$ represents the following real number:

$$r(S) = \frac{S(x_1)_{11} - S(x_2)_{11}}{S(x_3)_{11} - S(x_4)_{11}}. \tag{1}$$

Of course, different states of $X$ may represent the same real number. In particular, every real number can be represented by a pure state. We say that real numbers represented in the form (1) are numbers of real4 type.

Any arithmetical operation $\oplus$ (e.g. multiplication or addition) is implemented by a circuit $U$. Suppose that $F$ is a free storage and numbers are located in the disjoint registers $A$ and $B$. Assume that $S$ is an initial state which is free with respect to $F$ and decomposable with respect to $A \cup B$. The corresponding circuit $U$ transfers $S$ to a state $\tilde{S}$ such that the restriction $\tilde{S}(A)$ of $S$ to the register $A$ is decomposable and

$$r(\tilde{S}(A)) = r(S(A)) \oplus r(S(B)).$$

The circuit $U$ is a fixed finite combination of unitary operators belonging to a fixed collection of gates. It is natural to say that this collection is a set of instructions for the corresponding arithmetical processor.

Along with the numbers of real4 type, we shall also consider numbers of real1 type and real2 type. Any state $S(x)$ of a one-qubit register $\{x\}$ represents the following real number of real1 type:

$$r(S(x)) = (S(x))_{11}, \tag{2}$$

so $0 \leq r(S(x)) \leq 1$.

Let $B = \{b^+, b^-\}$ be a two-qubit register, $S = S(B)$ its decomposable state; then we say that $S(B)$ represents the following real number of real2 type:

$$r(S(B)) = (S(b^+))_{11} - (S(b^-))_{11} = r(S(b^+)) - r(S(b^-)). \tag{3}$$

Of course, $-1 \leq r(S(B)) \leq 1$ and every such number can be represented by a pure state. It is clear that every number $r$ of real4 type can be treated as a pair of numbers $(r', r'')$ of real2 type, where $r = r'/r''$.

Let $S$ be a state of a one-qubit register; then we say that $S$ is diagonal if the corresponding matrix $S_{ij}$ $(i, j = 1, 2)$ is diagonal. Suppose $S$ is a state of an arbitrary register; then we say that the state $S$ is diagonal if it is decomposable and its restriction to every element (qubit) of the register is diagonal. We shall say that states of one-qubit registers are equivalent, if their density matrices have the same diagonal elements, i.e., represent the same number of real1 type. We say that states of registers of the same length are equivalent if the restrictions of these states to the corresponding components (qubits) are equivalent.

Let us describe an operation which transfers states of qubits to equivalent diagonal states using a free storage. If $S$ is a state of a two-qubit register $X = \{a, b\}$ and $\{b\}$ belongs to the free storage, then this operation transfers $S$ to a diagonal state $S'$ such that $S'(a), S'(b)$, and $S(a)$ are equivalent. To this end the CNOT operation (described in the example 2 above) can be used. It is elementary to check that the following proposition is true.
Proposition 1. Let $S = S(a) \otimes 0 \{b\}$ be a state of a register $X = \{a, b\}$, $S^\tau = USU^{-1}$, where $U = \hat{\tau}$ is the classical CNOT operator described in Example 2. Then

$$S^\tau(a) = S^\tau(b) = (\delta_{ij}(S(a)))_{ij},$$

where $i, j = 1, 2$ and $\delta_{ij}$ is the Kronecker symbol.

4. IMPLEMENTATION OF ARITHMETICAL OPERATIONS

4.1. We shall describe a set of instructions (i.e., unitary operators) ensuring the implementation of arithmetical operations. Finally, we shall describe two elementary operations for all numbers $x, y$ of \texttt{real}1 type. Set

$$\sigma_1 : x, y \mapsto \frac{x + y}{2},$$

$$\sigma_2 : x, y \mapsto 1 - (x + y) + 2xy.$$ (4) (5)

The corresponding unitary operators act on two-qubit registers $X = \{a, b\}$, i.e., on the space $\mathcal{H}(X) = \mathcal{H}(2)$.

A direct verification shows that if an input state $S$ is decomposable with respect to $X = \{a, b\}$, i.e., $S(X) = S(a) \otimes S(b)$, then the operation $\sigma_2$ can be implemented by a classical operator; this operator is generated by the following permutation of elements of the standard orthonormal basis in $\mathcal{H}(2)$:

$$|1, 1\rangle \mapsto |1, 1\rangle, \quad |1, 0\rangle \mapsto |0, 0\rangle, \quad |0, 0\rangle \mapsto |0, 1\rangle, \quad |0, 1\rangle \mapsto |1, 0\rangle.$$

Similarly, let $S$ be an input state decomposable with respect to $X = \{a, b\}$ and diagonal for each its qubit. In this case the operation $\sigma_1$ is implemented by the unitary operator $U$ which acts on the standard basis in $\mathcal{H}(2)$ by the following way:

$$|1, 1\rangle \mapsto |1, 1\rangle, \quad |0, 0\rangle \mapsto |0, 0\rangle,$n

$$|1, 0\rangle \mapsto \lambda(|1, 0\rangle + |0, 1\rangle), \quad |0, 1\rangle \mapsto \lambda(-|1, 0\rangle + |0, 1\rangle),$$

where $\lambda = 1/\sqrt{2}$. The operation $\sigma_1$ is implemented by the operator $U$ only in the case of diagonal state $S$. However, the CNOT operator $\hat{\tau}$ (see Example 2 and Proposition 1 above) with a qubit of a free storage transfers the qubit in the state $S$ to a diagonal state $S'$ such that $r(S') = r(S)$. Therefore, combining $U$ with $\hat{\tau}$, we obtain an implementation of the operation $\sigma_1$.

Combining the operations $\sigma_1$ and $\sigma_2$ with cloning operations, it is possible to compute the following function

$$\mu_1(x, y) = \sigma_1(\sigma_1(\sigma_2(x, y), 0), \sigma(x, y))$$

$$= ((1 - (x + y) + 2xy + 0)/2 + (x + y)/2)/2 = xy/2 + 1/4.$$

Thus, we have proved the following

Proposition 2. For all numbers of the \texttt{real}1 type the operations of arithmetic mean $x, y \mapsto (x + y)/2$ and displaced multiplication $x, y \mapsto xy/2 + 1/4$ can be implemented by fixed quantum circuits\footnote{This means that every circuit is a fixed combination of gates (instructions).}.
4.2. Let us show now that for all numbers of \texttt{real2} type a similar proposition is valid.

**Proposition 3.** For all numbers of \texttt{real2} type the operations of arithmetic mean \(x, y \mapsto (x + y)/2\) and quasimultiplication \(x, y \mapsto xy/4\) can be implemented by fixed quantum circuits.

Recall that any number \(z\) of \texttt{real2} type can be represented as the difference \(z^+ - z^-\) of numbers of \texttt{real1} type.

By abuse of language, we denote by \(\sigma\) the operation of arithmetic mean and by \(\mu_2\) the operation of quasimultiplication from the proposition 3.

An implementation of the operations stated in Proposition 3 can be given by the following formulas:

\[
\begin{align*}
\sigma^+(x, y) &= \sigma(x^+, y^+) = (x^+ + y^+)/2, \\
\sigma^-(x, y) &= \sigma(x^-, y^-) = (x^- + y^-)/2; \\
\mu^+(x, y) &= \mu_1(x^+, y^+), \mu_1(x^-, y^-) = \sigma(x^+ y^+ / 2 + 1/4, x^- y^- / 2 + 1/4) \\
&= (x^+ y^+ + x^- y^-)/4 + 1/4, \\
\mu^-(x, y) &= \mu_1(x^+, y^-), \mu_1(x^-, y^+) = \sigma(x^+ y^- / 2 + 1/4, x^- y^+ / 2 + 1/4) \\
&= (x^+ y^- + x^- y^+)/4 + 1/4.
\end{align*}
\]

Indeed,

\[
\begin{align*}
\sigma(x, y) &= \sigma^+(x, y) - \sigma^-(x, y) = (x + y)/2, \\
\mu_2(x, y) &= \mu^+(x, y) - \mu^-(x, y) = (x^+ y^+ + x^- y^- - x^+ y^- - x^- y^+)/4 = xy/4,
\end{align*}
\]

as was to be proved. Of course, this calculation needs copying operations.

4.3. From Proposition 2 and 3 we can easily deduce the following

**Theorem.** For all numbers of \texttt{real4} type, the operations of addition, multiplication, subtraction, and division can be implemented by fixed quantum circuits.

Recall that any number \(z\) of \texttt{real4} type can be represented in the form \(z' / z''\), where \(z'\) and \(z''\) are numbers of \texttt{real2} type. The operation of multiplication is given by the following formulas:

\[
(xy)' = \mu_2(x', y'), \quad (xy)'' = \mu_2(x'', y'').
\]

The arithmetic mean \((x + y)/2\) is given by the formulas

\[
((x + y)/2)' = \sigma(\mu_2(x', y''), \mu_2(x'', y')), \quad ((x + y)/2)'' = \mu_2(x'', y '').
\]

Finally, the sum \(x + y\) can be obtained by the multiplication of the numbers \((x + y)/2\) and 2. Note that the number 2 can be easily implemented as a number of \texttt{real4} type.

From the basic formula (1) it is clear that for any real number \(r\) and its representation in the form (1) we can easily construct the corresponding representations for the numbers \(-r\) and \(r^{-1}\); so the operations of subtraction and division can also be implemented by fixed quantum circuits.

**Remark.** Note that our exact definitions and constructions of the operations are not stable with respect to small perturbations. However, all the elementary operations are continuous and in practice we shall deal with approximate values and operations, errors, etc. (as usual for calculations with real numbers). So we need to examine the corresponding methods for fault-tolerant calculations. This will be the subject of our subsequent publications.
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