Some further curiosities
from the world of integrable lattice systems
and their discretizations

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Abstract. Unexpected relations are found between the Toda lattice, the relativistic Toda
lattice and the Bruschi–Ragnisco lattice, as well as between their integrable discretizations.
1 Introduction

We continue in this paper playing some curious game named "reparametrizations" with the well-known integrable lattice equations of the classical mechanics. This game was started in the papers [1], [2], where several "new" integrable lattices were introduced together with their integrable discretizations.

In the present paper we shall demonstrate that the equations of motion of one of simplest flow of the relativistic Toda hierarchy may be brought into the form:

\[ \ddot{x}_k = \dot{x}_k (x_{k+1} - 2x_k + x_{k-1}) + \left( \frac{\dot{x}_{k+1} \dot{x}_k}{x_{k+1} - x_k} - \frac{\dot{x}_k \dot{x}_{k-1}}{x_k - x_{k-1}} \right). \]  (1.1)

This form of equations of motion is very remarkable, since the right-hand side may be seen as a sum of the right-hand sides of another two well-known integrable lattices. The first one of them is the usual Toda lattice, the equations of motion for which can be put into the form:

\[ \ddot{x}_k = \dot{x}_k (x_{k+1} - 2x_k + x_{k-1}). \]  (1.2)

The second one is exactly the so called Bruschi–Ragnisco lattice [3], [4]:

\[ \ddot{x}_k = \frac{\dot{x}_{k+1} \dot{x}_k}{x_{k+1} - x_k} - \frac{\dot{x}_k \dot{x}_{k-1}}{x_k - x_{k-1}}. \]  (1.3)

Even more bizarre becomes the situation when looking at the discrete time counterparts of these systems. Namely, we shall demonstrate that the discretization of the relativistic Toda flow derived in [5] in the parametrization corresponding to (1.1) takes the form

\[ \frac{\bar{x}_k - x_k}{x_k - \bar{x}_k} = \frac{(x_{k+1} - x_k)}{(x_{k+1} - x_k)} \frac{(x_k - \bar{x}_{k-1})}{(x_k - \bar{x}_{k-1})} \frac{1 + h(x_{k+1} - x_k)}{1 + h(x_k - \bar{x}_{k-1})}. \]  (1.4)

The right-hand side of this equation may be seen as a product of the right-hand sides of the following two equations:

\[ \frac{\bar{x}_k - x_k}{x_k - \bar{x}_k} = \frac{1 + h(x_{k+1} - x_k)}{1 + h(x_k - \bar{x}_{k-1})}, \]  (1.5)

and

\[ \frac{\bar{x}_k - x_k}{x_k - \bar{x}_k} = \frac{(x_{k+1} - x_k)}{(x_{k+1} - x_k)} \frac{(x_k - \bar{x}_{k-1})}{(x_k - \bar{x}_{k-1})}. \]  (1.6)
Here (1.3) turns out to be exactly the same discretization of the Toda lattice as derived in [8], [8] and discussed further in [8], but expressed in the parameterization leading to (1.2). And (1.6) is an integrable discretization of the Bruschi–Ragnisco lattice which, to the author’s knowledge, has not yet appeared in the literature, and can be derived on the basis of a general recipe worked out in [7], [5] and the subsequent publications of the author.

All the systems above (continuous and discrete time ones) may be considered either on an infinite lattice ($k \in \mathbb{Z}$), or on a finite one ($1 \leq k \leq N$). In the last case one of the two types of boundary conditions may be imposed: open–end ($x_0 = \infty, x_{N+1} = -\infty$) or periodic ($x_0 \equiv x_N, x_{N+1} \equiv x_1$). We shall be concerned only with the finite lattices here, consideration of the infinite ones being to a large extent similar.

2 Simplest flow of the Toda hierarchy

We consider in this section the simplest flow of the Toda hierarchy. All the results here are not new but collected in the form convenient for our present purposes. For the relevant references see [8].

The simplest flow of the Toda hierarchy is:

$$\dot{a}_k = a_k(b_{k+1} - b_k), \quad \dot{b}_k = a_k - a_{k-1}. \quad (2.1)$$

It may be considered either under open–end boundary conditions ($a_0 = a_N = 0$), or under periodic ones (all the subscripts are taken (mod $N$), so that $a_0 \equiv a_N, b_{N+1} \equiv b_1$).

The flow (2.1) is Hamiltonian with respect to the following Poisson bracket:

$$\{a_k, b_k\} = -\{a_k, b_{k+1}\} = a_k \quad (2.2)$$

(only the non–vanishing brackets are written down), with the Hamiltonian function

$$H_T = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k. \quad (2.3)$$

An integrable discretization of the flow (2.1) is given by the difference equations [7]

$$\tilde{a}_k = a_k \frac{\beta_{k+1}}{\beta_k}, \quad \tilde{b}_k = b_k + h \left( \frac{a_k}{\beta_k} - \frac{a_{k-1}}{\beta_{k-1}} \right), \quad (2.4)$$

where $\beta_k = \beta_k(a, b)$ is defined as a unique set of functions satisfying the recurrent relation

$$\beta_k = 1 + h b_k - \frac{h^2 a_{k-1}}{\beta_{k-1}} \quad (2.5)$$
together with an asymptotic relation

\[ \beta_k = 1 + h b_k + O(h^2). \]  

(2.6)

In the open–end case, due to \( a_0 = 0 \), we obtain from (2.5) the following finite continued fractions expressions for \( \beta_k \):

\[ \beta_1 = 1 + h b_1; \quad \beta_2 = 1 + h b_2 - \frac{h^2 a_1}{1 + h b_1}; \quad \ldots; \]

\[ \beta_N = 1 + h b_N - \frac{h^2 a_{N-1}}{1 + h b_{N-1} - \frac{h^2 a_{N-2}}{1 + h b_{N-2} - \ldots \frac{h^2 a_1}{1 + h b_1}}}. \]

In the periodic case (2.3), (2.6) uniquely define \( \beta_k \)'s as \( N \)-periodic infinite continued fractions. It can be proved that for \( h \) small enough these continued fractions converge and their values satisfy (2.6).

It can be proved [7] that the map (2.4) is Poisson with respect to the bracket (2.2).

Let us recall also the Lax representations of the flow (2.1) and of the map (2.4). They are given in terms of the \( N \times N \) Lax matrix \( T \) depending on the phase space coordinates \( a_k, b_k \) and (in the periodic case) on the additional parameter \( \lambda \):

\[ T(a, b, \lambda) = \sum_{k=1}^{N} b_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k} + \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1}. \]  

(2.7)

Here \( E_{jk} \) stands for the matrix whose only nonzero entry on the intersection of the \( j \)th row and the \( k \)th column is equal to 1. In the periodic case we have \( E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1} \); in the open–end case we set \( \lambda = 1 \), and \( E_{N+1,N} = E_{N,N+1} = 0 \).

The flow (2.1) is equivalent to the matrix differential equation

\[ \dot{T} = [T, B], \]  

(2.8)

where

\[ B(a, b, \lambda) = \sum_{k=1}^{N} b_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \]  

(2.9)

and the map (2.4) is equivalent to the matrix difference equation

\[ \tilde{T} = B^{-1}TB, \]  

(2.10)
where

$$B(a, b, \lambda) = \sum_{k=1}^{N} \beta_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}. \quad (2.11)$$

The spectral invariants of the matrix $T(a, b, \lambda)$ serve as integrals of motion for the flow (2.1), as well as for the map (2.4). In particular,

$$H_T = \frac{1}{2} \text{tr}(T^2).$$

An involutivity of the spectral invariants of the matrix $T$ follows from the $r$–matrix theory. Namely, it turns out that the matrices $T(a, b, \lambda)$ form a Poisson submanifold for a certain linear $r$–matrix bracket on the algebra of $N \times N$ matrices (in the open–end case) or on the loop algebra over the previous one (in the periodic case) (cf. [8]), and (2.2) gives exactly the coordinate representation of this $r$–matrix bracket restricted to the set of the matrices $T(a, b, \lambda)$.

3 Continuous and discrete time Toda flows in a new parametrization

The usual parametrization of the variables $(a_k, b_k)$ by the canonically conjugated ones $(x_k, p_k)$ is:

$$a_k = \exp(x_{k+1} - x_k), \quad b_k = p_k.$$  

This corresponds to the bracket (2.2) and results in the most usual form of the Toda lattice and in its discretization studied in [7].

Here we chose a different parametrization, which, however, corresponds to the same bracket (2.2) and is in a sense dual to the parametrization just mentioned:

$$a_k = \exp(p_k), \quad b_k = x_k - x_{k-1}. \quad (3.1)$$

The corresponding Hamiltonian function $H_T$ takes the form

$$H_T(x, p) = \sum_{k=1}^{N} \exp(p_k) + \frac{1}{2} \sum_{k=1}^{N} (x_k - x_{k-1})^2, \quad (3.2)$$

which generates the canonical equations of motion

$$\dot{x}_k = \frac{\partial H_T}{\partial p_k} = \exp(p_k), \quad \dot{p}_k = -\frac{\partial H_T}{\partial x_k} = x_{k+1} - 2x_k + x_{k-1}.$$
These equations of motion imply the Newtonian ones (1.2).

We turn now to a less straightforward case of discrete equations of motion.

**Theorem 1.** In the parametrization (3.1) the equations of motion (2.4) may be presented in the form of the following two equations:

\[ h \exp(p_k) = (x_k - x_k - 1)(1 + h(x_k - x_{k-1})) \],

\[ h \exp(p_k) = (x_k - x_k + 1)(1 + h(x_{k+1} - x_k)) \],

which imply also the Newtonian equations of motion (1.5).

**Proof.** The second equation of motion in (2.4) together with the second equation in (3.1) implies that

\[ \bar{x}_k - x_k - \frac{ha_k}{\beta_k} = \bar{x}_{k-1} - x_{k-1} - \frac{ha_{k-1}}{\beta_{k-1}}. \]

Hence the expression on the left–hand side does not depend on \( k \), and setting it equal to 0, we get:

\[ \frac{ha_k}{\beta_k} = \bar{x}_k - x_k. \] (3.5)

Substituting this into the recurrent relation (2.5), we get:

\[ \beta_k = 1 + h(x_k - \bar{x}_{k-1}). \] (3.6)

The last two formulas imply the expression (3.3) for \( ha_k = h \exp(p_k) \). Finally, the first equation of motion in (2.4) implies the expression (3.4), if one takes into account (3.6). \( \blacksquare \)

## 4 Simplest flow of the Bruschi–Ragnisco hierarchy

This section is devoted to reminding the known facts about the simplest flow of the Bruschi–Ragnisco hierarchy [3], [4].

It is given by the equations of motion:

\[ \dot{b}_k = b_{k+1}c_k - b_k c_{k-1}, \quad \dot{c}_k = c_k(c_k - c_{k-1}). \] (4.1)

It may be considered either under open–end boundary conditions (\( c_0 = b_{N+1} = 0 \)), or under periodic ones (all the subscripts are taken (mod \( N \)), so that \( c_0 \equiv c_N, b_{N+1} \equiv b_1 \)).

The flow (4.1) is Hamiltonian with respect to the following Poisson bracket:

\[ \{c_k, b_k\} = -\{c_k, b_{k+1}\} = c_k, \] (4.2)
with the Hamiltonian function

\[ H_{BR} = \sum_{k=1}^{N} b_{k+1} c_k. \]  \hfill (4.3)

The Lax representation of the flow (4.1) is given in terms of the \( N \times N \) rank 1 Lax matrix \( T(b, c) \) with the entries given by:

\[ T_{kj} = \begin{cases} 
  b_j \prod_{i=k}^{j-1} c_i, & k \leq j \\
  b_j \left( \prod_{i=j}^{k-1} c_i \right)^{-1}, & k > j
\end{cases} \]  \hfill (4.4)

The flow (4.1) is equivalent to the following matrix differential equation:

\[ \dot{T} = [T, M], \]  \hfill (4.5)

where \( M \) is a constant matrix, so that the Lax equation is in fact linear. For example, in the open–end case

\[ M = \sum_{k=1}^{N-1} E_{k+1,k}. \]  \hfill (4.6)

Hence the solution of the flow (4.1) can be read off the following explicit formula:

\[ T(t) = \exp(-tM)T(0)\exp(tM). \]  \hfill (4.7)

The integrals of motion for the flow (4.1) are given not by the spectral invariants of the matrix \( T(b, c) \) (they are all dependend because this matrix has rank 1), but by the following set of linear functionals on the Lax matrix: \( \text{tr}(M^m T) \). In particular,

\[ H_{BR} = \text{tr}(MT). \]

The involutivity of such linear functionals on the matrix \( T \) with respect to the bracket (4.2) follows from its interpretation as a coordinate representation of a standard Lie–Poisson bracket on the algebra \( \mathfrak{gl}(N) \), restricted to a Poisson submanifold of rank 1 matrices. Taking such functions as Hamiltonians, i.e. replacing in the previous formulas \( M \) by \( M^m, 1 \leq m \leq N - 1 \), we get the whole Bruschi–Ragnisco hierarchy.

According to the philosphy of [7], [5], the discrete time Bruschi–Ragnisco lattice belongs to the same hierarchy, i.e. it shares the Lax matrix with the continuous time one, and its explicit solution should be given by

\[ T(nh) = (I + hM)^{-n}T(0)(I + hM)^n. \]  \hfill (4.8)

Hence the corresponding discrete Lax equation should have the form

\[ \tilde{T} = (I + hM)^{-1}T(I + hM). \]  \hfill (4.9)
A direct calculation shows that the last equation in the coordinates \((b_k, c_k)\) has the form:

\[
\tilde{b}_k (1 + h\tilde{c}_{k-1}) = b_k + h b_{k+1} c_k, \quad \tilde{c}_k = c_k \frac{1 + h\tilde{c}_k}{1 + h\tilde{c}_{k-1}}.
\] (4.10)

By construction, this map is Poisson with respect to the bracket (4.2).

5 Continuous and discrete time Bruschi–Ragnisco lattices in the Newtonian form

Observe now that the bracket (4.2) for the Bruschi–Ragnisco lattice formally is identical with that for the Toda lattice (2.2).

This suggest the parametrization

\[
c_k = \exp(p_k), \quad b_k = x_k - x_{k-1}.
\] (5.1)

The corresponding Hamiltonian function \(H_{BR}\) takes the form

\[
H_{BR}(x, p) = \sum_{k=1}^{N} \exp(p_k) (x_{k+1} - x_k),
\] (5.2)

which generates the canonical equations of motion

\[
\dot{x}_k = \frac{\partial H_{BR}}{\partial p_k} = \exp(p_k) (x_{k+1} - x_k),
\]
\[
\dot{p}_k = -\frac{\partial H_{BR}}{\partial x_k} = \exp(p_k) - \exp(p_{k-1}).
\]

These equations of motion imply the Newtonian ones (1.3).

Proof. We start with the chain of identities following from the equations of motion (4.10), independently of any parametrization:

\[
b_{k+1} - \frac{b_{k+1}}{1 + h\tilde{c}_k} = \frac{h b_{k+1} c_k}{1 + h\tilde{c}_k} = \frac{h b_{k+1} c_k}{1 + h\tilde{c}_{k-1}} = \frac{b_k - b_k}{1 + h\tilde{c}_{k-1}}.
\]
Now we substitute \( b_{k+1} = x_{k+1} - x_k \) into the leftmost part of this chain, \( \bar{b}_k = \bar{x}_k - \bar{x}_{k-1} \) into the rightmost part, and arrive at the conclusion that

\[
\frac{b_{k+1}}{1 + h\bar{c}_k} - (x_{k+1} - \bar{x}_k) = \frac{b_k}{1 + h\bar{c}_{k-1}} - (x_k - \bar{x}_{k-1}).
\]

So, the expression on the left hand side of this equality does not depend on \( k \). Setting it equal to 0, we obtain:

\[
1 + h\bar{c}_k = \frac{b_{k+1}}{x_{k+1} - \bar{x}_k} = \frac{x_{k+1} - x_k}{x_{k+1} - \bar{x}_k},
\]

which is, clearly, equivalent to (5.4), and, together with the second equation of motion in (4.10), implies also (5.3).

### 6 Simplest flow of the relativistic Toda hierarchy

In this section we recall some known facts about the relativistic Toda lattice. The sources are: [5] and the references therein.

The simplest flow of the relativistic Toda hierarchy is:

\[
\dot{d}_k = d_k(c_k - c_{k-1}), \quad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}). \tag{6.1}
\]

It may be considered either under open–end boundary conditions \( (d_{N+1} = c_0 = c_N = 0) \), or under periodic ones (all the subscripts are taken \( \text{mod} \ N \)), so that \( d_{N+1} \equiv d_1, \ c_0 \equiv c_N, \ c_{N+1} \equiv c_1 \).

It is Hamiltonian with respect to the Poisson bracket

\[
\{c_k, d_{k+1}\} = -c_k, \quad \{c_k, d_k\} = c_k, \quad \{d_k, d_{k+1}\} = c_k, \tag{6.2}
\]

with a Hamiltonian function

\[
H_{RT} = \frac{1}{2} \sum_{k=1}^{N} (d_k + c_{k-1})^2 + \sum_{k=1}^{N} (d_k + c_{k-1})c_k. \tag{6.3}
\]

An integrable discretization of the flow (6.1) derived in [5] is given by the difference equations

\[
\bar{d}_k = d_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k}, \quad \bar{c}_k = c_k \frac{a_{k+1} + hc_{k+1}}{a_k + hc_k}, \tag{6.4}
\]

where \( a_k = a_k(c, d) \) is defined as a unique set of functions satisfying the recurrent relation

\[
a_{k+1} = 1 + hd_{k+1} + \frac{hc_k}{a_k} \tag{6.5}
\]
together with an asymptotic relation

\[ a_k = 1 + h(d_k + c_{k-1}) + O(h^2). \]  (6.6)

In the open–end case, due to \( c_0 = 0 \), one obtains from (6.5) the following finite continued fractions expressions for \( a_k \):

\[
\begin{align*}
a_1 &= 1 + hd_1; \\
a_2 &= 1 + hd_2 + \frac{hc_1}{1 + hd_1}; \quad \ldots; \\
a_N &= 1 + hd_N + \frac{hc_{N-1}}{1 + hd_{N-1} + \frac{hc_{N-2}}{1 + \ldots + \frac{hc_1}{1 + hd_1}}}.
\end{align*}
\]

In the periodic case (6.5), (6.6) uniquely define \( a_k \)'s as \( N \)-periodic infinite continued fractions. It can be proved that for \( h \) small enough these continued fractions converge and their values satisfy (6.6).

The map (6.4) is Poisson with respect to the bracket (6.2).

Both the continuous time flow (6.1) and the discrete time one (6.4) admit Lax representations with the same Lax matrix, given by

\[
\begin{align*}
T&(c, d, \lambda) = L(c, d, \lambda)U^{-1}(c, d, \lambda), \\
\text{where } L \text{ and } U \text{ are two } N \times N \text{ matrices depending on the phase space coordinates } c_k, d_k \\
&\text{and (in the periodic case) on the additional parameter } \lambda:
\end{align*}
\]

\[
\begin{align*}
L(c, d, \lambda) &= \sum_{k=1}^{N} d_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \\
U(c, d, \lambda) &= \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}.
\end{align*}
\]  (6.8) (6.9)

The flow (6.1) is equivalent to the matrix differential equation

\[ \dot{T} = [T, A], \]  (6.10)

where

\[
\begin{align*}
A&(c, d, \lambda) = \sum_{k=1}^{N} (d_k + c_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}.
\end{align*}
\]  (6.11)
The map (6.4) is equivalent to the matrix difference equation
\[ \bar{T} = A^{-1}TA, \]  
where
\[ A(c, d, \lambda) = \sum_{k=1}^{N} a_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \]  
and the quantities \( a_k \) are defined by the recurrent relations (6.5).

The spectral invariants of the matrices \( T(c, d, \lambda) \) serve as integrals of motion for the flow (6.1), as well as for the map (6.4). In particular,
\[ H_{RT} = \frac{1}{2} \text{tr}(T^2). \]

As in the case of the Toda lattice, the involutivity of the spectral invariants of the matrix \( T \) with respect to the bracket (6.2) follows from the \( r \)-matrix theory. Namely, it turns out that the matrices \( T(c, d, \lambda) \) form another Poisson submanifold for the same bracket on the matrix algebra as in the Toda case, and (6.2) serves as a coordinate representation of this bracket restricted to this manifold (cf. [9]).

### 7 Continuous and discrete time relativistic Toda lattices in a new parametrization

It is easy to see that the following parametrization of the variables \( (c_k, d_k) \) by the canonically conjugated variables \( (x_k, p_k) \) leads to the linear bracket (6.2):
\[ d_k = x_k - x_{k-1} - \exp(p_{k-1}), \quad c_k = \exp(p_k). \]  

Let us find out how do the equations of motion look in this parametrization. We start with (6.1).

Obviously, the function \( H_{RT} \) takes the form
\[ H_{RT} = \sum_{k=1}^{N} \exp(p_k)(x_k - x_{k-1}) + \frac{1}{2} \sum_{k=1}^{N} (x_k - x_{k-1})^2. \]  

Correspondingly, the flow (6.1) takes the form of canonical equations of motion:
\[
\begin{align*}
\dot{x}_k &= \frac{\partial H_{RT}}{\partial p_k} = \exp(p_k)(x_k - x_{k-1}), \\
\dot{p}_k &= -\frac{\partial H_{RT}}{\partial x_k} = x_{k+1} - 2x_k + x_{k-1} + \exp(p_{k+1}) - \exp(p_k).
\end{align*}
\]
As an immediate consequence of these equations one gets the Newtonian equations of motion (1.1).

We turn now to a less straightforward case of discrete equations of motion.

**Theorem 3.** In the parametrization (7.1) the equations of motion (6.4) may be presented in the form of the following two equations:

\[ h \exp(p_k) = \frac{(\tilde{x}_k - x_k)}{(x_k - \tilde{x}_{k-1})} \left( 1 + h(x_k - \tilde{x}_{k-1}) \right), \]  

(7.3)

\[ h \exp(\tilde{p}_k) = \frac{(\tilde{x}_k - x_k)}{(x_{k+1} - \tilde{x}_k)} \frac{(\tilde{x}_{k+1} - \tilde{x}_k)}{(\tilde{x}_k - \tilde{x}_{k-1})} \left( 1 + h(x_{k+1} - \tilde{x}_k) \right). \]  

(7.4)

which imply also the Newtonian equations of motion (1.4).

**Proof.** We start with derivation of several useful formulas which do not depend on the parametrization of the variables \((c_k, d_k)\).

The equations of motion (6.4) together with the recurrent relation (6.5) imply:

\[ \tilde{d}_k + \tilde{c}_{k-1} = \left( d_k + c_{k-1} \frac{a_k}{a_{k-1}} \right) \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k}. \]  

(7.5)

But, applying once more (6.5), we derive:

\[ hd_k + hc_{k-1} \frac{a_k}{a_{k-1}} = hd_k + a_k(a_k - 1 - hd_k) = (a_k - 1)(a_k - hd_k), \]

so that (7.5) may be rewritten as

\[ h(\tilde{d}_k + \tilde{c}_{k-1}) = (a_k - 1)(a_{k+1} - hd_{k+1}). \]  

(7.6)

It is easy to see that, according to (7.5),

\[ (a_k - 1)(a_{k+1} - hd_{k+1}) = h(d_{k+1} + c_k) - (a_{k+1} - a_k), \]

so that (7.6) implies:

\[ a_{k+1} - a_k = h(d_{k+1} + c_k) - h(\tilde{d}_k + \tilde{c}_{k-1}). \]  

(7.7)

Now we are in a position to make use of the concrete parametrization (7.1). Namely, due to \(d_{k+1} + c_k = x_{k+1} - x_k\) the last formula implies that

\[ a_k - h(x_k - \tilde{x}_{k-1}) = a_{k+1} - h(x_{k+1} - \tilde{x}_k), \]

i.e. the expression on the left–hand side does not depend on \(k\). Setting this equal to 1, we arrive at:

\[ a_k = 1 + h(x_k - \tilde{x}_{k-1}). \]  

(7.8)
Now the formula (6.5) upon use of (7.1), (7.8) may be presented as
\[ 1 + h(x_{k+1} - \bar{x}_k) = 1 + h(x_{k+1} - x_k - \exp(p_k)) + \frac{h \exp(p_k)}{1 + h(x_k - \bar{x}_{k-1})}, \]
which can be easily resolved for \( \exp(p_k) \) and gives (7.3). Next, (7.8) together with the expression (7.3) for \( c_k = \exp(p_k) \) implies
\[ a_k + hc_k = \frac{(\bar{x}_k - \bar{x}_{k-1})}{(x_k - \bar{x}_{k-1})} \left( 1 + h(x_k - \bar{x}_{k-1}) \right). \]

Substituting this into the second equation of motion in (5.4), we finally obtain (7.4).

8 Conclusion

The relations between three well known lattice systems, established in the present paper, may be symbolically presented as
\[ \text{RTL} = \text{TL} + \text{BRL}, \quad \text{dRTL} = \text{dTL} \times \text{dBRL}. \]

They are, in my opinion, rather beautiful and unexpected. However, they probably are nothing more than a pure curiosity, which demonstrates once more: the field of integrable systems of classical mechanics, even in its best studied parts, is far from being exhausted. Many important and less important but funny findings await us in this fascinating world.

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