ON THE SET OF SUBARCS
IN SOME NON-POSTCRITICALLY FINITE DENDRITES

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Abrosimov, N.V., Chanchieva, M.V., Tetenov, A.V., On the set of subarcs in some non-postcritically finite dendrites.

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This work was supported by the Ministry of Education and Science of Russia (state assignment No. 1.13557.2019/13.1).

Received November 29, 2018, published August 3, 2019.

Abstract. We construct a family $F$ of non-PCF dendrites $K$ in a plane, such that for any dendrite $K \in F$ all its subarcs have the same Hausdorff dimension $s$, while the set of $s$-dimensional Hausdorff measures of subarcs connecting the given point and a self-similar Cantor subset in $K$ is a Cantor discontinuum.

Keywords: self-similar dendrite, ramification point, Hausdorff dimension, postcritically finite set.

1. Introduction

Post-critically finite (PCF) self-similar sets occupy significant position in the theory of self-similar sets. They have a very clear structure which allows to build productive models of analysis and differential equations for such sets [3, 7]. Such sets can also have very attractive geometric features: as it was proved by C. Bandt in [1], the set of dimensions of minimal subarcs of a PCF set is finite. In particular, this holds for any postcritically finite self-similar dendrite $K$, and the set of cut points of such dendrite may be represented as a countable union of images of arcs $\gamma_k, k = 1, \ldots, n$ which are the components of attractor of some graph-directed IFS [6, 8, 2].

In this connection, much less is known about non-postcritically finite self-similar dendrites. Nevertheless, it turns out that in non-PCF dendrites which satisfy one-point intersection property the subarcs may have similar properties [5].
In this paper, we show that in the case of non-postcritically finite dendrites such properties can also occur. We construct a sufficiently wide family of non-postcritically finite systems of contraction similarities \( S = \{S_0, S_1, S_2, S_3\} \), whose attractors \( K \) are dendrites, lying in a triangle \( \Delta \subset \mathbb{R}^2 \) with the vertices \((0,0), (1,0), (1/2, \sqrt{3}/2)\) and for which the following properties hold:

1. All subarcs \( \gamma_{xy} \subset K, x \neq y \) and the set of cut points of the dendrite \( K \) have the same Hausdorff dimension \( s \) (see Theorem 5, Corollary 6).
2. The set of \( s \)-dimensional measures \( \mu_O \) of paths connecting the point \( O = \text{Fix}(S_0) \) with the points \( x \in K \cap [0,1] \) lying on the base of the triangle \( \Delta \) either is a one point set or it is a self-similar Cantor discontinuum (see Theorem 7).

2. Preliminaries

Let \( S = \{ S_1, S_2, \ldots, S_m \} \) be a system of (injective) contraction maps on the complete metric space \( (X,d) \). A nonempty compact set \( K \subset X \) is called the attractor of the system \( S \), if \( K = \bigcup_{i=1}^{m} S_i(K) \). We also call the subset \( K \subset X \) self-similar with respect to \( S \). Throughout the whole paper, the maps \( S_i \in S \) are supposed to be similarities and the set \( X \) to be \( \mathbb{R}^2 \).

Let \( I = \{1, 2, \ldots, m\} \) be the set of indices, \( I^* = \bigcup_{n=1}^{\infty} I^n \) be the set of all finite \( I \)-tuples, or multiindices \( j = j_1 j_2 \ldots j_n \). We write \( S_j = S_{j_1} S_{j_2} \ldots S_{j_n} \) and for the set \( A \subset X \) we denote \( S_j(A) \) by \( A_j \). We also denote by \( G_S = \{ S_j : j \in I^* \} \) the semigroup, generated by \( S \).

Let \( I^\infty = \{ \tilde{\alpha} = \alpha_1 \alpha_2 \ldots : \alpha_i \in I \} \) be the index space, and \( \pi : I^\infty \to K \) is the index map, which sends \( \tilde{\alpha} \) to the point \( \bigcap_{n=1}^{\infty} K_{\alpha_1 \ldots \alpha_n} \).

By \( ij \) we mean the concatenation of strings \( i \) and \( j \), the same \( ji \) is a concatenation \( j_1 \ldots j_n \alpha_1 \alpha_2 \ldots \).

A non-empty compact \( K \subset \mathbb{R}^2 \) is called the attractor of the system \( S \), if \( K = \bigcup_{i=1}^{m} S_i(K) \). The system \( S \) is called post-critically finite (PCF), if the set \( \{ x \in K : \exists_{i_1, \ldots, i_n, j, I} : S_{i_1 \ldots i_n}(x) \in K_j \cap K_i \} \) is finite.

Let \( \mathcal{C} \) be the union of all intersections \( S_i(K) \cap S_j(K) \), \( i, j \in I, i \neq j \). The post-critical set \( \mathcal{P} \) of the system \( S \) is the set of all \( \alpha \in I^\infty \) such that for some \( j \in I^* \), \( S_j(\alpha) \in \mathcal{C} \). In other words, \( \mathcal{P} = \{ \sigma^k(\alpha) : \alpha \in \mathcal{C}, k \in \mathbb{N} \} \), where the map \( \sigma^k : I^\infty \to I^\infty \) is defined by \( \sigma^k(\alpha_1 \alpha_2 \ldots) = \alpha_{k+1} \alpha_{k+2} \ldots \) A system \( S \) is called post-critically finite (PCF) [3] if its post-critical set is finite.

Let the set \( K \) be connected. Then \( K \) is arcwise connected and for any pair of points \( x, y \in K \) we can consider a set \( P_{xy} \) of paths form \( x \) to \( y \). Let \( \beta_{xy} = \inf \{ \dim_H(\gamma) : \gamma \in P_{xy} \} \). We will refer a path \( \gamma \in P_{xy} \) as minimal one, if \( \dim_H(\gamma) = \beta_{xy} \). It was proved by C. Bandt [1], that the set of dimensions of all minimal paths in postcritically finite self-similar sets is finite, which gives the finiteness of the set of dimensions of all arcs in PCF dendrites.

3. Construction

Let \( \Delta \) be the triangle on the plane \( \mathbb{R}^2 \) with the vertices
\[
A_1 = (0,0), \quad A_2 = (1,0), \quad A_3 = (1/2, \sqrt{3}/2).
\]
Denote the set of vertices \( \{A_1, A_2, A_3\} \) of \( \Delta \) by \( V_\Delta \).

Let \( p_1, p_2, p_3 \) be such positive numbers that \( p_1 + p_2 + p_3 < 1 \). Define contraction similarities with fixed points at the vertices of \( \Delta \) as follows.

\[
S_1 = p_1 z, \quad S_2 = p_2 z - p_2 + 1, \quad S_3 = p_3 z - p_3 e^{i \frac{\pi}{2}} + e^{i \frac{\pi}{2}}.
\]

Denote \( \Delta_k = S_k(\Delta) \). Let \( K' \) be the Cantor set generated by maps \( S_1, S_2, S_3 \).

Each point \( x \in K' \) is defined by unique sequence of indices \( \tilde{a} = a_1 a_2 \ldots \) where \( a_k \in \{1, 2, 3\} \).

We now consider an equilateral triangle \( \Delta_0 \) such that its vertices \( B_1, B_2, B_3 \) satisfy the conditions

\[
(1) \quad B_1 \in K' \cap S_{12}([A_2 A_3]), \quad B_2 \in K' \cap S_{23}([A_1 A_3]), \quad B_3 \in K' \cap S_{31}([A_1 A_2]).
\]

Let \( S_0 \) be a similarity that maps the points \( A_1, A_2, A_3 \) to points \( B_1, B_2, B_3 \) respectively, and let \( O = \text{fix}(S_0) \) be its fixed point.

Denote by \( \tilde{a} = a_1 a_2 \ldots, \tilde{b} = b_1 b_2 \ldots, \tilde{c} = c_1 c_2 \ldots \) the addresses of points \( B_1, B_2, B_3 \) in the set \( K' \). According to (1), \( a_1 = 1, a_2 = 2; b_1 = 2, b_2 = 3 \) and \( c_1 = 3, c_2 = 1 \).

There are sufficiently many triples of points \( B_1, B_2, B_3 \) satisfying the conditions (1). For example, if \( p_1 = p_2 = p_3 \) then for any address \( \tilde{a} = a_1 a_2 \ldots \) of point \( B_1 \), the points \( B_2, B_3 \) such addresses that \( b_k = \sigma(a_k), c_k = \sigma(b_k) \) where \( \sigma = (1, 2, 3) \) is a cyclic permutation of indices \( 1, 2, 3 \), the points \( B_1, B_2, B_3 \) form an equilateral triangle. The following Lemma shows that it is possible to choose the parameters \( p_1, p_2, p_3, B_1, B_2, B_3 \) in which the coefficients \( p_1, p_2, p_3 \) do not match.

**Lemma 1.** For any \( \tilde{c}' \in \{1, 2\}^\infty \) different from 1111\ldots, there exists a set of parameters \( p_1, p_2, p_3, B_1, B_2, B_3 \) such that \( \tilde{c} = 31\tilde{c}' \) and \( p_1 > p_2 \).

**Proof.** Denote by \( Z_{pq} \) the self-similar zipper on the interval \([0,1]\) with the vertices \( \{0, p_1, 1 - p_2, 1\} \) and the signature \((0,0,0)\). Let the map \( \varphi_{p_1, p_2} \) be a homeomorphism of the interval \([0,1]\) onto itself performing an isomorphism of the zipper \( Z_{p_1, p_2} \) onto the zipper \( Z_0 \) with the vertices \( \{0, \frac{1}{3}, \frac{2}{3}, 1\} \) and the same signature \((0,0,0)\).

The map \( \varphi_{p_1, p_2}(t) \) bijectively and continuously maps the Cantor set \( K_{p_1, p_2} \) generated by the maps \( S_1(z) = p_1 z \) and \( S_2(z) = p_2 z - p_2 + 1 \) to the standard middle-third Cantor set \( K_{1/3} \), and the point with the address \( a_1 a_2 \ldots \) in \( K_{pq} \) corresponds to a point with the same address in \( K_{1/3} \). Denote by \( \psi_{p_1, p_2}(t) \) the map inverse to \( \varphi_{p_1, p_2} \).

Consider some pair of coefficients \( p_1, p_2 \) and construct an equilateral triangle with vertices \( S_1(\varphi_{p_2, p_3}(0)) \) (i.e. with the address 1222\ldots) and \( S_2(\varphi_{p_1, p_1}(0)) \) (i.e. 1111\ldots), and let \( \tilde{a} = a_1 a_2 \ldots \) be such positive numbers that \( a_1 = a_2 = 1, a_3 = a_4 = 2 \) and \( a_k \in \{1, 2, 3\} \) for \( k > 4 \). Let \( \tilde{b} \) and \( \tilde{c} \) be such addresses that \( \tilde{b} = b_1 b_2 \ldots, \tilde{c} = c_1 c_2 \ldots \) respectively and \( \tilde{c} = 111 \ldots \) in \( K' \). Then there exists a similarity \( S_0 \) such that \( \tilde{a} = \varphi_{p_1, p_2}(S_0) \). Consider the set \( S(X) \) of points that are images of \( X \) under the map \( \varphi_{p_1, p_2} \). The proof follows from Lemma 1.
with the address 2333 . . . ). The coordinates of these points are \( p_1 \) and \( 1 + p_2 e^{2i\pi/3} \), respectively.

From the condition of equality of the sides of the triangle \( \Delta_0 \), we find the coordinates of the third vertex \( C(p_1,p_2) = e^{i\pi/3} - p_2 + p_1 e^{-i\pi/3} \), hence \( p_1 = p_3 \). Notice that if \( p_1 = p_2 \), the third vertex coincides with the point \( S_3(\varphi_{p_1,p_2}(0)) \), and as \( p_2 \) tends to 0, the third vertex tends to \( S_3(\varphi_{p_1,p_2}(1)) \).

Obviously, the coordinates of the point \( C(p_1,p_2) \) continuously depend on \( p_1 \) and \( p_2 \), as does its projection \( S_3^{-1}(C(p_1p_2)) \) on the interval \([0,1]\).

The function \( \varphi_{p_1,p_2}(t) : \left[0, 1\right] \rightarrow \left[0, 1\right] \) is continuous in \( p_2 \) and monotone in \( t \) for any \( p_1, p_2 \). Note that \( \varphi_{p_1,p_2}(0) = 0 \) and \( \varphi_{p_1,p_2}(1) = 1 \).

Since the function \( \psi_{p_1,p_2}(1 - \frac{p_2}{p_1}) \) is continuous in \( p_2 \) and vanishes at \( p_1 = p_2 \), and at \( p_2 \rightarrow 0 \) it tends to 1, then for any \( a \in (0, 1) \) there is \( p_2 \) such that \( \psi_{p_1,p_2}(1 - \frac{p_2}{p_1}) = a \).

Therefore, for any point \( t > 0 \) from \( C_2 \) with the address \( c_2c_3 \ldots \in \{1, 2\}^\infty \) there is such \( p_2 < p_1 \) that the triangle with the vertices \( S_1(\varphi_{p_1,p_2}(0)), S_2(\varphi_{p_1,1}(0)) \) and \( S_3(\varphi_{p_1,p_2}(t)) \) is equilateral. \( \square \)

**Theorem 1.** Let \( S = \{S_1, S_2, S_3, S_0\} \) be a system with parameters \( p_1, p_2, p_3, B_1, B_2, B_3 \), where each of the points \( B_k \) lies in the corresponding set \( S_k(\varphi_{p_1,1}(a_k)) \), where \( a_k \in C_2 \), and addresses of points \( B_1, B_2, B_3 \) satisfy condition (1). Then the attractor \( K \) of the system \( S \) is a dendrite.

**Proof.** Let for every \( j \in I^* \), \( \Delta_j = S_j(\Delta) \). Let \( T(A) = \bigcup_{k=0}^4 S_k(A) \) be a Hutchinson operator of the system \( S \). We set \( \tilde{\Delta}^k = T^k(\Delta) = \bigcup_{j \in I^k} \Delta_j \). Each system \( \tilde{\Delta}^k \) has the following properties.

1. For each \( k \), the set \( \tilde{\Delta}^k \) is connected, locally connected, and simply connected.
2. The diameter of each \( \Delta_i = S_i(\Delta) \leq (\text{Lip}S_0)^k \).
3. If \( |i_1| = |i_2| = k \), and \( i_1 \neq i_2 \) then \( \Delta_{i_1} \cap \Delta_{i_2} \) is either empty or one-point set, which is a vertex of one of the triangles \( \Delta_{i_1 \setminus 1} \Delta_{i_2} \).
4. The sets \( \tilde{\Delta}^k \) form a decreasing sequence \( \tilde{\Delta}^1 \supset \tilde{\Delta}^2 \supset \ldots \).

The attractor \( K \) of the system \( S \) coincides with the intersection of the decreasing sequence of the sets \( \tilde{\Delta}^k \), and therefore, according to [8, Lemma 2, Theorem 5], is a dendrite. \( \square \)

**Lemma 2.** If at least one of the addresses \( \bar{a}, \bar{b}, \bar{c} \) is not periodic, then the system \( S \) is not postcritically finite. \( \square \)

4. **Dimensions and measures of subarcs**

Let \( O \) be a fixed point of the map \( S_0 \). Denote by \( \gamma_k \) the subarcs \( \gamma_{O,A_k} \subset K \) with endpoints \( O \) and \( A_k \), \( k = 1, 2, 3 \). It is easy to verify that these arcs have equal dimensions.

**Lemma 3.** \( \dim_H(\gamma_1) = \dim_H(\gamma_2) = \dim_H(\gamma_3) \).
Proof. Indeed, according to the conditions (BB) we have
\[ \gamma_1 \supset S_1 S_0(\gamma_2), \]
\[ \gamma_2 \supset S_2 S_0(\gamma_3), \]
\[ \gamma_3 \supset S_3 S_0(\gamma_1). \]

Therefore, \( \dim_H(\gamma_1) \geq \dim_H(\gamma_2) \geq \dim_H(\gamma_3) \geq \dim_H(\gamma_1) \), that gives the result of the Lemma. \( \square \)

Denote by \( s \) the \( \dim_H(\gamma_1) \). Note that \( s \geq 1 \).

**Theorem 2.** The dimension of the set \( CP(K) \) of cut points in \( K \) is \( s \).

Proof. Let \( j = j_1 \ldots j_k \) be the multiindex of length \( k \). If the index \( j_k \) is 0 then intersection of the triangle \( \Delta_j \) with the set \( \bigcup_{i \in I^k \backslash \{i\}} \Delta_i \) consists of no more than three points, each of which is a vertex of the triangle \( \Delta_j \). If \( j_k \neq 0 \) then for each \( i \in I^k \backslash \{j\} \) for which \( \Delta_j \cap \Delta_i \neq \emptyset \), the index \( i_k = 0 \), and the intersection is a one-point set consisting of a single vertex of the triangle \( \Delta_i \). Therefore, for every \( j = j_1 \ldots j_k \), the set \( K_j \cap K \backslash K_j \) consists of no more than three points, each of which lies in \( G_8(V_\Delta) \).

Let us now take some point \( x \in CP(K) \) not lying in \( G_8(V_\Delta) \). This point has the single address \( j_1 j_2 j_3 \ldots \in I^\infty \). Let \( U_1, U_2 \) be any two connected components of \( K \backslash \{x\} \), and \( 0 < d < \min(\text{diam} U_1, \text{diam} U_2) \). Suppose there is such \( k \), that for \( j = j_1 \ldots j_k, j_k = 0 \), and \( \text{diam} \Delta_j < d \). Then each of the intersections \( U_i \cap K_j \cap K \backslash K_j \) is one of the vertices \( A'_i \) of the triangle \( \Delta_j \), and the arc \( \gamma_{A'_i} \) is a subarc \( S_j(\hat{\gamma}) \), containing the point \( x \).

If there is no such \( k \) then for some \( N, k > N \) implies \( j_k \neq 0 \) which means that \( x \in G_8(K') \).

Thus \( CP(K) \subset G_8(\hat{\gamma} \cup K') \). Since \( \dim \hat{\gamma} = s > 1 \), and \( \dim K' \leq 1 \) then \( \dim_H CP(K) = s \). \( \square \)

**Corollary 1.** For any subarc \( \gamma_{xy} \in K \), \( \dim_H(\gamma_{xy}) = s \).

Proof. Since the set \( G_8(K') \) is totally disconnected then the set of all points, whose addresses contain infinite number of zeros is dense in \( \gamma_{xy} \). As it was shown in the previous proof, for any \( z \in \gamma_{xy} \), different from \( x \) and \( y \), there is such multiindex \( j \), that \( \gamma_{xy} \cap K_j \subset S_j(\hat{\gamma}) \). Therefore, \( \dim_H(\gamma_{xy}) \geq s \). Since \( \gamma_{xy} \subset CP(K) \) then \( \dim_H(\gamma_{xy}) = s \). \( \square \)

Let \( M \) be the set of arcs \( \gamma_{Ox} \) with endpoints in \( O \) and \( x \in K \cap [0, p_1] \), \( N \) be the set of arcs \( \gamma_{Ox} \) with endpoints in \( O \) and \( x \in K \cap [1 - p_2, 1] \). Let \( M = \{ \ell_{Ox} : x \in K \cap [0, p_1] \} \) be the set \( s \)-dimensional Hausdorff measures of the arcs \( \gamma_{Ox} \in M \) and \( N = \{ \ell_{Ox} : x \in K \cap [1 - p_2, 1] \} \) be the set \( s \)-dimensional Hausdorff measures of the arcs \( \gamma_{Ox} \in N \).

**Theorem 3.** The sets \( M, N \) are the components of the attractor of some graph-directed system of contraction similarities in \( \mathbb{R} \), and
\[ \dim_H M = \dim_H N \leq \dim_H (K' \cap [0, 1]) / s < 1 / s. \]

Proof. First suppose, that \( B_1 \neq p_1 \). Then there is such \( n \) that \( B_1 \in S_1 S_2^n(K) \) and \( B_1 \notin S_1 S_2^{n+1}(K) \). Let \( \lambda_k \), where \( k = 0, \ldots, n \) be the arc in \( K \) whose endpoints are \( S_1 S_2^{n}(O) \) and \( O \). Let \( \lambda' \) be the arc with endpoints \( O \) and \( S_2(O) \). Since each
arc, connecting \( O \) and \( S_1S_2^0(x) \) where \( x \in K \cap [0, p_1] \), is a sum of subarcs \( \lambda_k \) and \( S_1S_2^0(\gamma_{Ox}) \) then allowing some liberty of notation we can represent the set of arcs \( \{ \gamma_{Ox} : x \in S_1S_2^0(K \cap [0, p_1]) \} \) as \( \lambda_k + S_1S_2^0(M) \). The same way we get the representation of the form \( \lambda' + S_2(M) \) for the set of arcs \( \{ \gamma_{Ox} : x \in S_2(K \cap [0, p_1]) \} \), \( \lambda_n + S_1S_2^0(N) \) for the set of arcs \( \{ \gamma_{Ox} : x \in S_1S_2^0(K \cap [1 - p_2, 1]) \} \), and \( \lambda' + S_2(N) \) for the set of arcs \( \{ \gamma_{Ox} : x \in S_2(K \cap [1 - p_2, 1]) \} \).

Therefore we can write the following system of representations for the sets \( M \) and \( N \).

\[
\begin{align*}
M &= (\lambda_0 + S_1(M)) \cup (\lambda_1 + S_1S_2(M)) \cup \ldots \\
N &= (\lambda' + S_2(M)) \cup (\lambda' + S_2(N))
\end{align*}
\]

We denote \( \ell_k = H^s(\lambda_k) \) and \( \ell' = H^s(\lambda') \).

Therefore, defining the contraction linear maps

\[
\sigma'(x) = \ell' + p_1^s x, \quad \sigma_0(x) = \ell_0 + p_1^s x, \quad \ldots, \quad \sigma_n(x) = \ell_n + p_1^s p_2^n x
\]

in \( \mathbb{R} \), we come to the following graph-directed system of similarities \( S_{MN} \) for the sets \( M \subset \mathbb{R} \) and \( N \subset \mathbb{R} \).

\[
\begin{align*}
M &= \sigma_0(M) \cup \sigma_1(M) \cup \ldots \cup \sigma_n(M) \\
N &= \sigma'(M) \cup \sigma'(N)
\end{align*}
\]

Let us evaluate the similarity dimension \( d \) of the system \( S_{MN} \). According to [4], \( d \) is the unique value of the parameter \( t \), such that for certain \( \mu, \nu > 0 \) the following equations hold.

\[
\begin{align*}
\mu &= p_1^s (1 + p_2^s + \ldots + p_2^{s \ell}) \mu + p_1^s p_2^{s \ell} \nu \\
\nu &= p_2^s \mu + p_2^s \nu
\end{align*}
\]

Expressing \( \nu = \frac{p_2^s \mu}{1 - p_2^s} \) in the second equation and substituting to the first one, we get \( (1 - p_2^s) \mu = p_1^s \mu \). Thus, \( d \) is the unique solution of the equation \( p_1^s \mu + p_2^s \mu = 1 \).

Therefore, \( d \leq \dim_H(K' \cap [0, 1]) / s < \dim_H(K') / s < 1 / s \).

In the case when \( B_1 = p_1 \), denote by \( M' \) the set of arcs \( \gamma_{A_2x} \) where \( x \in K' \cap [0, 1] \).

Notice that \( M = \gamma_{B_1O} + S_1(M') \), and each arc in \( M' \) either belongs to \( S_2(M') \) or
is the sum of some subarc in $S_1(M')$ and the arc $\gamma_{A_1B_1} \subset K$. Putting $\sigma_1(x) = p_2^1x$, $\sigma_2(x) = \ell_{A_1B_1} + p_2^1x$, $\sigma_3(x) = \ell_{OS_2(B_1)} + p_2^1p_2^2x$ and $\sigma_4(x) = \ell_{OO_2} + p_2^2x$, we get the following graph-directed system of similarities for $M'$ and $N$.

\[
\begin{align*}
M' &= \sigma_1(M') \cup \sigma_2(M') \\
N &= \sigma_3(M') \cup \sigma_4(N)
\end{align*}
\]

Thus, the set $M'$ is the attractor of the system $\{\sigma_1, \sigma_2\}$ whose similarity dimension $t$ satisfies the equation $p_1^t + p_2^t = 1$. \hfill \Box

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