A prediction theory for a coloured noise-driven stochastic differential system

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The standard approaches to analyse the Itô stochastic differential system are the Fokker–Planck equation and multi-dimensional Itô differential rule. In contrast to the Itô stochastic differential system, this paper develops a mathematical theory of a coloured noise process-driven stochastic differential system. More precisely, the coloured noise process-driven stochastic differential equation $\dot{x}_t = f(x_t) + g(x_t) \xi_t$ is the subject of investigations. The statistical properties of the input noise process $\xi_t$ are stationary and finite, non-zero, relatively smaller correlation time. The notion of “stochastic equivalence” coupled with stochastic differential rule plays the key role to develop the theory of this paper. The theory of the paper will be of use to analysing and control of dynamical systems embedded in the coloured noise environment.

Keywords: the Markov process; prediction algorithm; stochastic equation; the Ornstein–Uhlenbeck process

1. Introduction

The stochastic differential equation (SDE) formalism is exploited to analyse stochastic problems arising from satellite mechanics, mathematical control theory, wireless communications and mathematical finance. The notion of random initial conditions and noise perturbations lead to the concept of the SDEs. Rigorous mathematical treatments of the SDEs from the mathematicians’ viewpoint can be found in authoritative books and seminal papers, e.g. multi-dimensional diffusion processes authored by DW Strook and SRS Varadhan (Karatzas & Shreve, 1991) as well as results published in probability journals. From the mathematicians’ viewpoint, the white noise-driven SDE is recast as the Itô SDE. The Itô SDE has displayed striking surprises, supreme beauty for analysing stochastic differential systems. Mumford (2000) recommends stochasticity considerations in dynamical systems, since stochasticity is an intrinsic property of nature. White noise has found applications in dynamical systems to model stochastic perturbations (Wax, 1954). However, it is argued that the white noise process is an approximation of the real noise process associated with dynamical systems (Kloeden & Platen, 1991, p. 262). Furthermore, it is argued that the real noise process has finite, non-zero correlation time. In the theory of random noise, the statistical properties of the coloured noise process confirm the real noise process statistics. Most notably, ignoring the random perturbation, inaccurate choice of stochastic integrals as well as inaccurate choice of noise models for dynamical equations will lead to poor decisions about the state estimates, stochastic stability and control; see Kushner (1967). For these reasons, it is worthwhile to develop algorithmic procedures for the coloured noise-driven SDE as well. The stochastic stability properties of a specific and non-trivial case in the real noise environment can be found in Liu and Liew (2005). The problem of analysing the coloured noise-driven stochastic differential system involves the augmented state vector approach. That accounts for the Ornstein–Uhlenbeck (OU) process SDE, i.e.

$$d\xi_t = -\frac{1}{\tau_{cor}} \xi_t \, dt + \sqrt{2D} \, dB_t$$

as well. The terms $t_{cor}$ and $(\sqrt{2D}/\tau_{cor})$ are the OU process correlation time and the process noise coefficient of the Brownian motion process, respectively. Subsequently, combining with the non-Markovian stochastic differential system leads to the modified Itô SDE. The extended phase space approach brings the Markovian property in the augmented state vector. On the other hand, the extended phase space increases the dimensionality of the state vector leading to a greater complexity. The problem of analysing the vector SDE becomes quite difficult and several complexities arise. For this reason, it becomes imperative to explore alternative approaches that allow the Markovian state vector as well as preserve the dimensionality of the state vector. There is no formal and systematic method available in “stochastic dynamics and control” literature yet

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on “prediction algorithms for coloured noise-driven SDE” that exploits the theory of Markov processes as well as preserves the dimension of the phase space. This paper does that.

In this paper, we wish to develop the prediction theory for a coloured noise-driven stochastic differential system, where the coloured noise process has the finite, non-zero, relatively smaller correlation time. In the topics of random noise, the “coloured noise process with smaller correlation time” is attractive for the problem of formalizing dynamical systems embedded in real noise process for three reasons. This allows one to exploit the notion of “stochastic equivalence” leading to a quite simplified analysis. Secondly, it preserves “the dimensionality of the state vector.” The statistical property of the real noise process is preserved by accounting the correlation term in “equivalent dynamical equations.”. Thirdly, the state vector satisfying the stochastically equivalent dynamical equation will be a Markov process. In Patel and Sharma (2012), the extended phase space was the cornerstone formalism to analyse a specific coloured noise-perturbed dynamical system. In contrast to the extended phase space approach, the mathematical theory of this paper hinges on two results of stochastic processes: (i) replace an actual process with a Markov process, a consequence of the notion of stochastic equivalence and (ii) introduce the concept of stochastic differential rules in the Itô setting.

2. A prediction theory for the coloured noise SDE

The prediction algorithm for the Itô stochastic differential system can be accomplished using the Kolmogorov forward and backward equations (Karatzas & Shreve, 1991) that are celebrated results in stochastic processes. The prediction algorithm is useful for two cases: (i) observation rates are less and (ii) the initial conditions are known and observations are not available. The evolution $d\langle \varphi(x_t) \rangle$ of the coloured noise-driven SDE, the prediction algorithm, can be obtained using the stochastic equation, the definition of conditional expectation and integration by part formula. Thus, $d\langle \varphi(x_t) \rangle = \int \varphi(x_t) d\rho(x_t) dx$. The term “$\varphi(x_t)$” has an interpretation as a scalar function and the stochastic equation $d\rho = \mathcal{L}(\rho) dt$ is the term “$\mathcal{L}(\rho)$” as a functional operator. The operator $\mathcal{L}(\cdot) = \sum_{n \geq 1} (1/n!)[k_n(x_t)\partial^n \cdot/\partial x^n]$ and the notation $\langle \cdot \rangle$ denotes the inner product for finite as well as infinite dimensional vectors, e.g. Hilbert space. The notation $\langle \cdot \rangle$ must be carefully differentiated. The former denotes the action of the expectation operator adopted in the theory of stochastic processes. The latter denotes the inner product between two vectors. The above expression suggests an infinite series structure of the evolution $d\langle \varphi(x_t) \rangle$ of conditional moment. The stochastic equation involves system non-linearity-process noise coefficient coupling terms. In this paper, the authors try to develop a prediction algorithm for a coloured noise-driven SDE, where the coloured noise process is a stationary process with zero mean, smaller correlation time. The restriction, the smaller correlation time on the input coloured noise process, allows to exploit the notion of “stochastic equivalence” that results in attractive structures of conditional mean and variance evolutions. The coloured noise process has finite, non-zero correlation time as well as the autocorrelation of the stationary coloured noise process depends upon the time interval and the correlation time. The power spectral density of the coloured noise process depends on the frequency term and correlation time. In this regard, Hänggi and Jung (1995, p. 247, 2007), Demir, Liu, and S.-Vincentelli (1996) and Fox (1986) will be useful, see Papoulis (1991) as well. The OU process is a coloured noise process that satisfies the Itô SDE, $d\xi_t = -(1/\tau_{cor})\xi_t dt + (\sqrt{2D}/\tau_{cor}) dB_t$.

That denotes a relationship between the Brownian motion $B_t$ and the coloured noise $\xi_t$. The term Brownian motion process $B = \{B_t, \mathcal{F}_t, 0 \leq t < \infty\}$ and the term $\mathcal{F}_t$ has interpretation as a sigma algebra up to time $t$ (Karatzas & Shreve, 1991). The terms $\tau_{cor}$ and $\sqrt{2D}/\tau_{cor}$ denote the OU process correlation time and the input noise strength coefficient, respectively. Furthermore, the auto correlation $R_{\xi \xi}(\tau)$ of the OU process, a stationary process, is $D/\tau_{cor} e^{-|\tau|/\tau_{cor}}$. This expression suggests that the correlation time of the stationary coloured noise can be explained using the concept of the autocorrelation of the coloured noise process. The prediction algorithm is cornerstone formalism for the noise analysis of dynamical systems under two cases: (i) no observations and (ii) value-less observations. The conditional moment evolution characterizes the prediction algorithm, where the conditional mean and variance evolutions are special cases.

**Theorem 1** Consider the stochastic differential system described by

$$\dot{x}_t = f(x_t) + g(x_t)\xi_t,$$

(1)
where the input process $\xi_t$ is a stationary process with zero mean, relatively smaller correlation time. The right-hand side $g(x_t)\xi_t$ of Equation (1) has a multiplicative noise character.

Suppose $f: R \rightarrow R$ is a continuous, bounded function as well as the function $f$ has bounded continuous double derivative. Let $g: R \rightarrow R$ be a bounded continuous function as well as the function $g$ has a bounded continuous double derivative. Suppose the scalar function $\varphi(x_t)$ is twice continuously differentiable with respect to the state $x_t$, the evolution $d(\varphi(x_t))$ of conditional moment can be stated as

$$
d(\varphi(x_t)) = \left( \frac{f(x_t) - c_2}{2} g^2(x_t) g'(x_t) \left( \frac{f(x_t)}{g(x_t)} \right)^n \right) + \frac{1}{2} c_1 g^2(x_t) x_t^2 \left( \frac{d^2 \varphi(x_t)}{dx_t^2} \right) dt. \tag{2}
$$

More precisely, $\varphi: R \rightarrow R$, i.e. $\varphi(x_t) \in R$, $R$ is the solution space of Equation (1),

$$
\langle \varphi(x_t) \rangle = E(\varphi(x_t)|x_{t_0}, t_0),
$$

$$
c_1 = 2 \int_{-\infty}^{0} R_{\xi x}(\tau) d\tau, \quad c_2 = \int_{-\infty}^{0} |\tau| R_{\xi x}(\tau) d\tau,
$$

$$
R_{\xi x}(\tau) = E(\xi_t|x_{t_0}, t_0). \tag{3}
$$

The terms $c_1$ and $c_2$ are associated with Equation (2) for the OU process, the input coloured noise process becomes $c_1 = 2D$, $c_2 = D_{\text{cor}}$. Note that the right-hand side term of Equation (2) is a consequence of the action of conditional expectation operator $\mathbb{E}$.

Proof The proof of the theorem exploits the stochastic equation, which is the evolution of conditional probability density for given initial states for the non-Markov process assuming the structure of an infinite series (Stratonovich, 1963). It is interesting to note that the stochastic equation reduces to the Fokker–Planck equation for the coloured noise process $\xi_t$ with a smaller correlation time. For the smaller correlation time, the higher order term $k_n(x)$ of the stochastic equation vanishes, where $n > 2$. The Fokker–Planck equation describes the evolution of conditional probability density for a Markov process. Note that the notion of the generalized Fokker–Planck equations is useful for non-Markovian stochastic systems (Sluisarenko, 2011).

Importantly, Equation (1), the coloured noise-driven SDE, would be “stochastically” equivalent to a Markovian SDE (Patel & Sharma, 2012, p. 1250020-17; Stratonovich, 1963)

$$
\dot{x}_t = \left( k_1(x_t) - \frac{k_2(x_t)}{4} \right) + \sqrt{k_2(x_t)} \, w_0(t), \tag{3}
$$

where the term $w_0(t)$ has been interpreted as the white noise process. Note that the prime sign denotes the spatial derivative throughout the paper. Thanks to Equation (4.180) of Stratonovich (1963), the coefficients $k_1(x)$ and $k_2(x)$ for the stochastic differential system $\dot{x}_t = f(x_t) + g(x_t) \xi_t$ can be recast as

$$
k_1(x) = f + \frac{c_1}{2} gg' + c_2 g^2 g' \left( \frac{f}{g} \right),
$$

$$
k_2(x) = c_1 g^2 + 2c_2 g^3 \left( \frac{f}{g} \right). \tag{4}
$$

A good source about the notion of stochastic equivalence can be found in an older, but an authoritative book authored by Stratonovich (1963). A recent paper discusses succinctly about the notion of stochastic equivalence as well, see Section (4) of the paper (Patel & Sharma, 2012). Equation (4) is a special case of Equation (4.180) of Stratonovich (1963) with the input noise process, $\xi_t$, having smaller correlation time. From Equations (3)–(4), we are led to

$$
\dot{x}_t = f(x_t) - c_2 \frac{g^2(x_t)}{g(x_t)} g'(x_t) \left( \frac{f(x_t)}{g(x_t)} \right) + \frac{1}{2} c_1 \frac{g^2(x_t)}{g(x_t)} \frac{\sqrt{c_1 g(x_t)}}{\sqrt{1 + 2c_2 c_1 g(x_t)} \left( \frac{f(x_t)}{g(x_t)} \right)} w_0(t). \tag{5}
$$

A careful observation reveals that the above white noise-driven SDE accounts for two additional correction terms, i.e. (i)

$$
- \frac{c_2}{2} g^2(x_t) g'(x_t) \left( \frac{f(x_t)}{g(x_t)} \right),
$$

$$
- \frac{c_2}{2} g^2(x_t) \left( \frac{f(x_t)}{g(x_t)} \right)''
$$

(ii)

$$
\sqrt{c_1 g(x_t)} \sqrt{1 + \frac{2c_2 c_1 g(x_t)}{c_1 g(x_t)}} \left( \frac{f(x_t)}{g(x_t)} \right) w_0(t).
$$

On the other hand, two additional terms are not accounted in the coloured noise-driven SDE of the paper. Thus, the qualitative properties of the input coloured noise process are accounted by considering these two additional terms in the white noise-driven SDE. The white noise $w_0(t)$ is an informal non-existent time derivative of the Brownian motion process $B_t$, i.e. $dB_t/\, dt = w_0(t)$ (Kuo, 2009). Equation (5) is an informal SDE. For convenience, the above equation can be restated as

$$
dx_t = \left( f(x_t) - \frac{c_2}{2} g^2(x_t) g'(x_t) \left( \frac{f(x_t)}{g(x_t)} \right) \right) \, dt + \left( \sqrt{c_1 g(x_t)} \sqrt{1 + \frac{2c_2 c_1 g(x_t)}{c_1 g(x_t)}} \left( \frac{f(x_t)}{g(x_t)} \right) w_0(t) \right) \, dt.
$$
From the mathematicians’ stochastic interpretation viewpoint, Kiyoshi Itô considered the term dB_t = w_0(t) dt. As a consequence of the Itô theory, the above equation reads

$$\begin{align*}
\text{d}x_t &= \left( f(x_t) - c_2 g^2(x_t) g'(x_t) \frac{f(x_t)}{g(x_t)} \right) \, \text{d}t \\
&\quad + c_2 g^2(x_t) \left( \frac{f(x_t)}{g(x_t)} \right)'' \, \text{d}t \\
&\quad + \left( \sqrt{c_2} g(x_t) \sqrt{1 + \frac{2c_2}{c_1} g(x_t)} \left( \frac{f(x_t)}{g(x_t)} \right) \right) \, dB_t, \\
&= a(x_t) \, \text{d}t + b(x_t) \, dB_t.
\end{align*}$$

(6)

Note that the terms c_1 and c_2 of Equation (6) are defined in the notational descriptions of Equation (2) of the paper. For notational brevity, we adopt the notations a(x_t) and b(x_t) for Equation (6). The notations a(x_t) and b(x_t) are the coefficients of the terms \text{d}t and dB_t of Equation (6), respectively. Note that Equation (6) describes a formal stochastic interpretation of the Itô setting. This is one of the reasons that white noise-driven SDE is further recast as the Itô SDE (Kloeden & Platen, 1991; Kuo, 2009). The Itô calculus has demonstrated its surprising power in applied mathematics (Kunita, 2010). The first term of the right-hand side of Equation (6) describes the system’s non-linearity, a(x_t), and the second term denotes the process noise coefficient, b(x_t). An application of Equations (1) and (6), which are stochastically equivalent, is explained briefly in Section 3 of the paper. Thanks to Theorem 3.6 of Karatzas and Shreve (1991, p. 153) and Lemma 4.2 of Jazwinski (1970, p. 112), the stochastic evolution of the scalar function \phi(x_t) can be stated as

$$\begin{align*}
\text{d}\phi(x_t) &= \left( \sum_i a_i(x_t) \frac{\partial \phi(x_t)}{\partial x_i} + \frac{1}{2} \sum_{i<j} (bb^T)_{ij}(x_t) \frac{\partial^2 \phi(x_t)}{\partial x_i \partial x_j} \right) \, \text{d}t \\
&\quad + \sum_{i<j} (bb^T)_{ij}(x_t) \frac{\partial^2 \phi(x_t)}{\partial x_i \partial x_j} \, \text{d}t \\
&\quad + \sum_{1\leq i\leq m, 1\leq y\leq r} b_{iy}(x_t) \frac{\partial \phi(x_t)}{\partial x_i} \, dB_y, \\
&= \left( \sqrt{c_1} g(x_t) \sqrt{1 + \frac{2c_2}{c_1} g(x_t)} \left( \frac{f(x_t)}{g(x_t)} \right) \right) \, dB_t.
\end{align*}$$

(7)

The stochastic evolution \text{d}\phi(x_t) for Equation (1), which is stochastically equivalent to Equation (6), becomes

$$\begin{align*}
\text{d}\phi(x_t) &= \left( \left( f(x_t) - \frac{c_2}{2} g^2(x_t) g'(x_t) \frac{f(x_t)}{g(x_t)} \right) \right) \, \text{d}t \\
&\quad - c_2 g^2(x_t) \left( \frac{f(x_t)}{g(x_t)} \right)'' \, \text{d}t + \frac{c_1}{2} g^2(x_t) \\
&\quad \times \left( 1 + \frac{2c_2}{c_1} g(x_t) \left( \frac{f(x_t)}{g(x_t)} \right) \right) \, dB_t.
\end{align*}$$

(8)

Equation (8) is a consequence of two equations, Equation (6) and the scalar version of Equation (7). Note that the summation sign associated with Equation (7) will vanish for the scalar random process. The contribution to the term \text{d}\phi(x_t) stems from the terms a(x_t) and b(x_t) of the Itô SDE. As a result of this, the expectation and differential operators can be interchanged (Jazwinski, 1970, p. 137), i.e.

$$\begin{align*}
\text{d}(\phi(x_t)) &= E(\text{d}(\phi(x_t)|x_0, t_0)), \\
&= E \left( \sqrt{c_1} g(x_t) \sqrt{1 + \frac{2c_2}{c_1} g(x_t)} \left( \frac{f(x_t)}{g(x_t)} \right) \frac{\text{d}\phi(x_t)}{\text{d}x_t} \right) \, \text{d}B_t.
\end{align*}$$

(9)

After plugging Equation (8) in the right-hand side of Equation (9), utilizing the definition of conditional expectation, we arrive at Equation (2). Note that the action of the conditional expectation operator E on the last term of the right-hand side of Equation (8) vanishes, since the terms \sqrt{c_1} g(x_t) \sqrt{1 + \frac{2c_2}{c_1} g(x_t)} \left( \frac{f(x_t)}{g(x_t)} \right) \frac{\text{d}\phi(x_t)}{\text{d}x_t} and \, dB_t are independent random variables, i.e.

$$\begin{align*}
E \left( \sqrt{c_1} g(x_t) \sqrt{1 + \frac{2c_2}{c_1} g(x_t)} \left( \frac{f(x_t)}{g(x_t)} \right) \frac{\text{d}\phi(x_t)}{\text{d}x_t} \, dB_t \bigg| x_0, t_0 \right) &= 0.
\end{align*}$$

(10)

From Equations (8)–(10), we get Equation (2) of the paper, i.e.

$$\begin{align*}
\text{d}(\phi(x_t)) &= E(\text{d}(\phi(x_t)|x_0, t_0)) \\
&= \left( \left( f(x_t) - \frac{c_2}{2} g^2(x_t) g'(x_t) \frac{f(x_t)}{g(x_t)} \right) \right) \, \text{d}t \\
&\quad - c_2 g^2(x_t) \left( \frac{f(x_t)}{g(x_t)} \right)'' \, \text{d}t + \frac{c_1}{2} g^2(x_t) \\
&\quad \times \left( 1 + \frac{2c_2}{c_1} g(x_t) \left( \frac{f(x_t)}{g(x_t)} \right) \right) \, dB_t.
\end{align*}$$
This completes the proof of the conditional moment evolution, the prediction result.

**Remark 1** Equation (2) is the main result of the paper. In the integral setting, the conditional moment evolution equation can be recast as

\[
\langle \varphi(x_t) \rangle = \left[ \left( f(x_t) - \frac{c_2}{2} g^2(x_t) g'(x_t) \frac{f(x_t)}{g(x_t)} \right) + \frac{c_2}{2} g^3(x_t) \frac{f(x_t)}{g(x_t)} \right] \frac{d\varphi(x_t)}{dx_t} ds + \frac{1}{2} \int_0^t \left[ c_1 g^2(x_s) \left( 1 + \frac{2c_2}{c_1} g(x_s) \frac{f(x_s)}{g(x_s)} \right) \right] ds + \langle \varphi(x_0) \rangle.
\]

Note that the term within the integral sign associated with the above equation is an immediate consequence of the action of the conditional expectation operator.

**Remark 2** The special cases of the conditional moment evolution equation, stated in the theorem of the paper, are coupled conditional mean and conditional variance evolutions for the coloured noise-driven SDE. The coupled equations assume the following structures:

\[
d(x_t) = \left( f(x_t) - \frac{c_2}{2} g^2(x_t) g'(x_t) \frac{f(x_t)}{g(x_t)} \right) dr,
\]

\[
dP_t = \left( \frac{c_2}{2} g^2(x_s) g'(x_t) \frac{f(x_s)}{g(x_s)} \right) dt.
\]

Equation (11) can be treated as a system of two Equations (11a)–(12). A greater detail about Equation (11), its approximate counterpart as well as its usefulness are explained and listed briefly in the appendix.

**Remark 3** Here, we state some comments on mathematical methods available in the literature (Hänggi & Jung, 1995, 2007) for coloured noise analysis of dynamical systems.

The major ingredients of this paper are stochastic equation, the concept of stochastic equivalence with the smaller correlation time restriction. One can arrive at the stochastic equation by using the notion of conditional probability density, conditional expectation and conditional characteristic function.

Alternative methods to accomplish coloured noise analysis in dynamical systems are smaller correlation time approximation, decoupling approximation and unified coloured noise approximation. In these three methods, we exploit the function calculus approach that involves the functional derivative to construct the conditional probability density evolution equation for the coloured noise process. For the small correlation time approximation, higher order diffusion correction terms of the conditional probability density are ignored. As a result of this, we get a Fokker–Planck-type equation.

Secondly, the decoupling approximation method of coloured noise analysis adopts the concept of independent random variables for simplifying diffusion correction terms of the conditional probability density evolution. Under long-time limit considerations, we get a Fokker–Planck-type equation for the coloured noise process. Note that the structure of the Fokker–Planck equation resulting from the smaller correlation time approximation is different from that of the Fokker–Planck equation arising from the decoupling approximation. Importantly, both hold under the smaller correlation time restriction.

The third method is the unified coloured noise approximation. Note that the Fokker–Planck-type equation under this assumption accounts for the “drift correction term” of the system dynamics in lieu of the “diffusion correction term.” The unified coloured noise approximation holds for smaller as well as moderate to large correlation times. A good demonstration of coloured noise methods can be found in Gudyma, Maksymov, and Enachescu (2010).

3. An illustrative example

In order to demonstrate the usefulness of the concept of stochastic equivalence, we analyse the nonlinear sampling mixer, an appealing electronic circuit under the coloured noise influence.

When the gate voltage is high, the output node equation for a sampling mixer, an appealing example of Complementary Metal Oxide Semiconductor (CMOS) digital circuits, can be stated as

\[
\frac{dV_d}{dt} = -\frac{K}{C}(V_g - V_i + \alpha \xi_i)(V_d - V_s) + \frac{K}{C}V_s V_d - \frac{K}{C}V_i^2 + \frac{K}{2C}V_i^2 + \frac{K}{C}V_d V_s.
\]

The deterministic version of the above can be found in Yu and Leung (1999). After accounting for the noise contribution to the gate terminal, we replace the deterministic gate terminal voltage \( V_g \) with the term \( V_g + \sigma \xi_i \), where \( \sigma \xi_i \) is a...
noise process. As a result of this, we get
\[
\frac{dV_d}{dt} = \frac{K}{C} (V_g - V_t) V_s - \frac{K}{C} (V_g - V_t) V_d - \frac{K}{C} V_d \xi_t \\
+ \frac{K}{2C} V_d \xi_t - \frac{K}{2C} V_s^2 + \frac{K}{2C} V_d^2.
\]

Thus, the above equation can be recast in a more convenient notation by choosing the state variable notation \( x_t \) for the sampling mixer state voltage \( V_d \). Thus,
\[
\dot{x}_t = f(x_t, t) + g(x_t, t) \xi_t,
\]
where
\[
f(x_t, t) = \mu_t + \alpha_t x_t + \lambda_t x_t^2, \quad g(x_t, t) = (\beta_t + \gamma_t x_t) \xi_t,
\]
\[
\mu_t = \frac{g_t}{C} u_t - \frac{K}{2C} u_t^2, \quad \alpha_t = \frac{g_t}{C}, \quad \lambda_t = \frac{K}{2C}, \quad \beta_t = \frac{K}{C} u_t,
\]
\[
\gamma_t = -(K\sigma/C), \quad K \text{ and } V_t \text{ are the CMOS constant and the gate threshold voltage, respectively. Note that the sampling mixer is a nonlinear time-varying system. As a result of this, the input arguments of the right-hand side of Equation (12) involve the time variable } t \text{ as well. In systems and control, time-varying systems have variable coefficients and constant coefficients are associated with time-invariant systems. From Equations (1) and (6) of the paper, we get}
\]
\[
\begin{align*}
\frac{dx_t}{dt} &= \left( \frac{g_t}{C} u_t - \frac{K}{2C} u_t^2 - \frac{g_t}{C} x_t + \frac{K}{2C} x_t^2 \right) \\
&\quad - \frac{c K^2 \sigma^2 \gamma_t^2}{4C} x_t^2 - \frac{c K^3 \sigma^2 \gamma_t^2}{4C^3} + \frac{c K^3 \sigma^2 \gamma_t x_t^2}{2C^3} \right) dt \\
&\quad + \sqrt{\xi_t} \left( \frac{K}{C} u_t + \frac{K}{C} u_t \right) \\
&\times \sqrt{1 + \frac{2c_2}{c_1} (K/2C) x_t^2 + (K/2C) u_t^2 - (K/C) u_t x_t} d\xi_t.
\end{align*}
\]

Note that Equations (12) and (13), two settings, of the paper would be stochastically equivalent. The stochastic equivalence suggests that qualitative characteristics of the nonlinear time-varying sampling mixer are preserved in both settings. Here, we demonstrate numerical experimentations for the SDEs of Equations (12)–(13).

The initial conditions and system parameters for the sampling mixer SDE are the following:
\[
K = 860 \text{nA/V}^2, \quad C = 2.2 \text{ pF}, \quad \mu = 1, \\
v_t = 1.05 \text{ V}, \quad \alpha = 0.1, \quad \beta = 0.01, \quad \sigma = 0.02, \\
frequency = 3 \text{ MHz}, \quad x_t(0) = 0.01 \text{ V}, \quad \xi_t(0) = 0.5, \quad \tau_{cor} = 0.1, \quad D = 0.016.
\]

Note that the parameters \( \tau_{cor}, D \) are the OU process parameters.

The numerical simulations of this paper are graphically illustrated in Figures 1–2. In the first figure, we consider that the correlation time \( \tau_{cor} \) of the input coloured noise is smaller than that of the second figure. Note that the solid line trajectory (-) of Figure 1 demonstrates the numerical experimentation of the SDE stated in Equation (12). On the other hand, the dotted line trajectory (–) of Figure 1 denotes the numerical experimentation of the SDE stated in Equation (13) of the paper. The terms \( c_1 = 2D_1 \), \( c_2 = D\tau_{cor} \) corresponding to the correlation time \( \tau_{cor} = 0.1 \) become \( c_1 = 0.032 \) and \( c_2 = 0.0016 \). Figure 1 suggests that the qualitative characteristics of both SDEs confirm each other. At initial time instants, both trajectories of the SDEs are quite apart and at later instants, the trajectories coincide. The correlation time of the input coloured noise process \( \xi_t \) is taken to be smaller, i.e. \( \tau_{cor} = 0.1 \). Thus, the numerical experimentation suggests that both SDEs are stochastically equivalent for the input coloured noise process with a smaller correlation time. Furthermore, we simulate the sampling mixer SDEs, Equations (1) and (2), by choosing the second set of initial conditions and system parameters, i.e.
\[
K = 860 \text{nA/V}^2, \quad C = 2.2 \text{ pF}, \quad \mu = 1, \\
v_t = 1.05 \text{ V}, \quad \alpha = 0.1, \quad \beta = 0.01, \quad \sigma = 0.02, \\
frequency = 3 \text{ MHz}, \quad x_t(0) = 0.01 \text{ V}, \quad \xi_t(0) = 0.5, \quad \tau_{cor} = 1, \quad D = 0.16.
\]

The second set of system parameters is different from that of the first in the sense that the second set utilizes the larger correlation time of the input coloured noise process, e. g. \( \tau_{cor} = 1 \). The terms \( s c_2 = D\tau_{cor} \) corresponding to the correlation time \( \tau_{cor} = 1 \) become \( c_1 = 0.032 \) and \( c_2 = 0.016 \).

Figure 2 of the paper is a consequence of the second set of initial conditions and system parameters. The graphical notations of Figure 2, the solid line trajectory (-) and the
dotted line trajectory (—) confirm the notations of Figure 1. Figure 2 reveals that the differences between the state trajectories resulting from the SDEs stated in Equations (1) and (2) are larger. On the other hand, Figure 1 suggests the differences between the trajectories are smaller.

Thus, the numerical experimentation suggests that both SDEs are stochastically equivalent for the input coloured noise process with “smaller correlation time.” That validates the concept of stochastic equivalence by considering a specific, non-trivial case of nonlinear dynamic circuits.

4. Conclusion

The main contribution of this paper is to develop a prediction algorithm for the coloured noise-driven SDE. The paper is the first of its kind in sense that the method of this paper exploits a “formal” stochastic interpretation, which preserves the dimension of the solution vector of the coloured noise SDE as well, in lieu of “informal” stochastic interpretations available in engineering and physics literature. The main result of the paper is new and classical prediction results available in the literature are special cases of the result of this paper, see the appendix. The prediction algorithm of this paper will be useful for three cases: (i) nonlinear stochastic control problems, where the statistical properties of acting random perturbations confirm the “coloured noise statistics with smaller correlation time,” (ii) observations are available at discrete-time instants and (iii) observations are valueless. Despite the restricted nature of the results obtained, the research communities in systems and control will find the results of the paper revealing, i.e. exploring many-sided aspects of “dynamical systems embedded in coloured noise environment arising from finance and technology.”

More importantly, this paper will encourage mechanists, dynamists and control theorists to account stochastic considerations in their research, since stochasticity will shape future research in dynamical systems. Another recommendation of the paper is that the unified theory of interconnected systems, that encompass dynamical systems, can be developed by “bringing mechanists, dynamists and stochasticians together in lieu of the gap between them.”

The problem of developing prediction algorithms for fractional noise process-driven dynamical systems deserves investigations as well. The theory of dynamical systems driven by the fractional noise process is constructed using the “fractional stochastic calculus” (Guerra & Nualart, 2008). Importantly, this paper is aimed not to develop prediction algorithm for the fractional noise process-driven dynamical system, but only to develop a prediction algorithm for the coloured noise-driven dynamical system.

In this paper, we have attempted to list as well as cite important, relevant papers and celebrated books on the related topic. However, we abandon our claim for the completeness of references.

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### Appendix

Equation (2) describes the conditional moment evolution for a coloured noise-driven SDE. In systems and control literature, the conditional mean and variance equations are utilized for the noise analysis (Demir et al., 1996). For the "scalar" exact mean and variance equations, $d\langle x_t \rangle$ and $dP_t$, consider $\varphi(x_t)$ as $x_t$ and $x_t^2$, respectively, where $\vec{x}_t = x_t - \langle x_t \rangle$. Thus, we get Equation (11) of the paper. Alternatively, Equation (11) can be recast as

\[
\langle x_t \rangle = \langle x_0 \rangle + \int_0^t \left( f(x_s) - \frac{c_2}{2} g^2(x_s)g''(x_s) \left( \frac{f(x_s)}{g(x_s)} \right) \right) ds, \tag{A.1}
\]

\[
P_t = P_0 + \int_0^t \left[ 2x_s \left( f(x_s) - \frac{c_2}{2} g^2(x_s)g''(x_s) \left( \frac{f(x_s)}{g(x_s)} \right) \right) \right] ds,
\]

\[
-2\langle x_t \rangle \int_0^t \left( f(x_s) - \frac{c_2}{2} g^2(x_s)g''(x_s) \left( \frac{f(x_s)}{g(x_s)} \right) \right) ds
\]

\[
+ \int_0^t c_1 g^2(x_s) \left( 1 + 2\frac{c_2}{c_1} g(x_s) \left( \frac{f(x_s)}{g(x_s)} \right) \right) ds \tag{A.2}
\]

Note that the coefficients associated with the term “ds” of Equations (A.1)–(A.2) denote the conditional expectation of the terms within brackets. For notational convenience and to make the exposition simple, the scalar SDE is the subject of investigations. The results of this paper can be extended for the vector case by introducing the component-wise analysis (Karatzas & Shreve, 1991, p. 284). The exact moment evolutions involve the higher order moments as well as they are not convenient for numerical simulations. This paper introduces an approximation to the right-hand side terms of Equations (11a)–(11b) as well as the "nearly" Gaussian assumption. Approximate moment evolution equations for the stochastic differential system of Equation (1) can be obtained by introducing the second-order partials of the right-hand side terms of Equation (11), which are evaluated at the term $\langle x_t \rangle$. The approach to arrive at the approximate evolution equations is straightforward, involves the notion of successive differentiations only. Here, we simply state the evolution equations. A detail about deriving the approximate evolution equations from the exact evolution equations can be found in Pugachev and Semitsyn (1987, p. 452). A few steps of calculations as well as a re-arrangement of resulting terms lead to

\[
d\langle x_t \rangle = \left( f(\langle x_t \rangle) - \frac{c_2}{2} g^2(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right) \right) ds
\]

\[
- \frac{c_2}{2} g^3(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
+ \frac{1}{2} P_t \left( f''(\langle x_t \rangle) - \frac{c_2}{2} g^3(\langle x_t \rangle) \frac{d^4}{dx_t^4} \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right) \right)
\]

\[
+ 7g^2(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
+ 5g^2(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
+ 10g(\langle x_t \rangle)g'(\langle x_t \rangle)^2 \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
+ g^2(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
+ 2g(\langle x_t \rangle)g'(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
\times g'(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
\times g'(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
\times g'(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
\times g'(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]

\[
\times g'(\langle x_t \rangle)g''(\langle x_t \rangle) \left( \frac{f(\langle x_t \rangle)}{g(\langle x_t \rangle)} \right)'' ds
\]
\[ + P_{1} \left( c_{2}g^{3}(x_{t}) \left( \frac{f(x_{t})}{g(x_{t})} \right) \right)'' + 6c_{2}g^{2}(x_{t})g'(x_{t}) \left( \frac{f(x_{t})}{g(x_{t})} \right)'' \\
+ 3c_{2}g^{2}(x_{t})g''(x_{t}) \left( \frac{f(x_{t})}{g(x_{t})} \right)' + 6c_{2}g(x_{t})g'(x_{t}) \left( \frac{f(x_{t})}{g(x_{t})} \right)' + c_{1}g(x_{t})g''(x_{t}) \\
+ c_{1}(g'(x_{t}))^{2} \right) \, dt. \] (A.4)

Notably, the numerical coefficients of Equations (A.3)–(A.4), approximate evolutions, are attributed to the successive differentiation with respect to the term \( \langle x_{t} \rangle \). Here, we state two special cases of evolution Equations (11a)–(11b): (i) for the terms \( c_{1} = 1 \) and \( c_{2} = 0 \), Equations (11a)–(11b) reduce to the classical exact evolution equations, see Equation (4.159) of Jazwinski (1970, p. 137) and (ii) the terms \( c_{1} = 2D \) and \( c_{2} = D_{\text{cov}} \) lead to the exact evolution equations of the OU process-driven SDE. Remarkably, the results of this paper, Equations (2), (6), (11), are not available in the literature. These three equations have additional correction terms that account for the qualitative characteristics of the coloured noise processes. Generally, the stochastic analysis of the coloured noise processes-driven dynamical systems involves additional correction terms in contrast to the stochastic analysis of white noise process-driven stochastic differential systems.

The results of the paper will be useful for “coloured noise analysis” of nonlinear dynamical systems, e.g. dynamic circuits (Yu & Leung, 1999), switched systems (Scherpen, Jeltsema, & Klaassens, 2003), satellite dynamics under the influence of fluctuating atmospheric density and helicopter rotor dynamics under turbulence (Kloeden & Platen, 1991, pp. 261–262). A nice compilation of celebrated books and seminal papers on the related topic, i.e. coloured noise-driven dynamical systems, is available in Demir et al. (1996) and Hänggi and Jung (1995). For a greater detail about stochastic stability and control of physical models arising from diverse fields, a famous book authored by Kushner (1967) will be also useful. Equations (A.3)–(A.4) are approximate coloured noise estimation equations for the exact SDE. Notably, Equations (A.3)–(A.4) have the ability to account for linear terms and square non-linearity contributions “completely.” Thus, Equations (A.3)–(A.4) will be exact estimation equations for coloured noise-driven SDEs arising from practical problems involving linear as well as square non-linearity terms.