Glassy transition and aging in a model without disorder.

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Abstract

We study the off-equilibrium relaxational dynamics of the Amit-Roginsky $\phi^3$ field theory, for which the mode coupling approximation is exact. We show that complex phenomena such as aging and ergodicity breaking are present at low temperature, similarly to what is found in long range spin glasses. This is a generalization of mode coupling theory of the structural glass transition to off-equilibrium situations.

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Recently there has been important progress in the understanding of the off-equilibrium dynamics of disordered systems. Although aging phenomena were discovered experimentally in spin glasses in 1983 [1] and subsequently studied by several groups in spin glasses [2] and many other different disordered systems, only quite recently has a dynamical mean field theory (DMFT) for non-equilibrium phenomena, based on microscopic spin glass models, been put forward [3, 4, 5]. Among the most interesting features which emerge from the theory are the violations at low temperature of time translation invariance (TTI) and of the fluctuation-dissipation theorem (FDT), even for very large times, implying breaking of ergodicity. This breaking of ergodicity can not be described simply as a separation of the phase space into different ergodic components within which the system thermalizes as in usual broken-symmetry phases. Rather, the distribution function continues to spread over larger regions of configuration space as time goes on. This spreading
gets slower and slower but never stops; thus the system never comes to equilibrium, and its properties depend on the time elapsed since the quench to low temperature.

Far from being confined to the realm of disordered system aging phenomena have been observed in a number of different systems without disorder, in experiments [6] and numerical simulations [7].

At present there is a great deal of theoretical research on spin-glass like phenomena in non-random systems [8, 9, 10]. In [8, 9] a mechanism has been proposed in which an effective disorder can be induced in a priori uniform systems (although with “complicated” interactions among the variables). In this paper we focus on the dynamical mean field theory for glasses, usually called mode coupling theory (MCT), showing how it is possible to extend it in order to include non equilibrium phenomena.

It has been noted [11] that the mean field dynamical equations of many disordered mean field models have, in the high temperature phase, a structure which is surprisingly similar to that of the equation of the mode coupling theory often used to describe the structural glass transition [12, 13]. The mode coupling equations have been derived from molecular or hydrodynamical models based on an approximation in which vertex correction are neglected in the perturbation expansion of the self-energy [14]. Systematic use is made of the hypothesis that the system under study is at thermal equilibrium. On the other hand there is experimental evidence that real glasses are out of equilibrium [13]. Even in MCT the ideal glass transition is described as one into a non-ergodic state. Thus current mode coupling theory can at best give a good description of the physics at high enough temperature.

This letter is a step towards the extension of the MCT below the glassy temperature, including the possibility of studying nonequilibrium phenomena. Technically, we construct a theory in which we do not assume TTI or the FDT ab initio in the derivation of the equations. As an example we have chosen to study a simple model without quenched disorder, where the mode coupling approximation is exact. This is the Amit-Roginsky $\phi^3$ field theory [15], originally proposed to study the critical properties of the 3-state Potts model. We will find dynamical equations identical to that of the spherical $p$-spin glass model [16, 11, 3] for $p = 3$. At high temperature, supposing TTI and FDT one gets usual MCT equations, similar to those first proposed by Leutheusser to study the glassy transition. At low temperature there is a phase where ergodicity is broken in a way now known from mean field spin glass dynamics.

Amit and Roginsky (AR) consider an $O(3)$ symmetric $\phi^3$ field theory where the basic fields transform under rotation according to representations of high angular momentum quantum number $l$. Recently the same method has been exploited in two interesting papers [17] to find soluble limits of nonlinear stochastic equations.
The method is applicable any time the nonlinearity is represented by a bilinear term. In this letter we discuss for simplicity the dynamics of the original AR model. In a forthcoming paper [18] we will discuss the case of the fluctuating compressible hydrodynamics of Das and Mazenko [14], which, although formally much more complicated than the present model, gives rise to equations of with the same structure.

The AR model is defined by the $O(3)$ invariant Lagrangian

\[
\mathcal{L}[\psi, \psi^*] = \sum_{m=-l}^{l} \psi_m^* (x) [\mu^2 - \nabla^2] \psi_m (x) + \frac{g}{3!} \sqrt{N} \sum_{m_1, m_2, m_3} \left( \begin{array}{lll} l & l & l \\ m_1 & m_2 & m_3 \end{array} \right) [\psi_{m_1}(x) \psi_{m_2}(x) \psi_{m_3}(x) + \text{c.c.}] \tag{1}
\]

with $N = 2l + 1$, and $\psi_m$ transforms according to the irreducible $N$-dimensional representation of $O(3)$. In the following we will denote by $\mathcal{L}_I[\psi, \psi^*]$ the cubic part of the lagrangian. The coefficients $\left( \begin{array}{lll} l & l & l \\ m_1 & m_2 & m_3 \end{array} \right)$ are the Wigner 3$j$-symbols.

To have a non vanishing interaction term $l$ must be even. Amit and Roginsky have shown that in the limit $l \to \infty$ vertex corrections in the perturbation expansion can be neglected. Thus, defining the equilibrium correlation function $G(x,y) = \sum_{m} \langle \psi_m(x) \psi_m^*(y) \rangle_{\text{Gibbs}}$, the perturbative series can be resummed, and the Dyson equation of the model in real space is simply:

\[
G^{-1}(x,y) = G_0^{-1}(x,y) - \frac{g^2}{2} [G(x,y)]^2, \tag{2}
\]

where $G_0^{-1}(x,y) = \delta(x-y)(\mu^2 - \nabla^2)$. Similarly one finds that the free energy as a functional of $G$ can be written

\[
F = -\text{Tr} \text{Log}(G) + \int \text{d}x \text{d}y G_0^{-1}(x,y) G(y,x) - \frac{g^2}{6} \int \text{d}x \text{d}y G(x,y)^3 \tag{3}
\]

and (2) is its associated variational equation. It is assumed in deriving (3) that $\langle \psi_m(x) \rangle = 0$.

In order to avoid problems connected with the unnormalizability of the Gibbs measure for a $\phi^3$ theory, we will consider a “spherical” variation of the AR model, imposing the constraint $\sum_m \psi_m(x) \psi_m^*(x) = N \bar{q}$. In this situation, $\mu^2$ can be viewed as a Lagrange multiplier which enforces the constraint. Eq. (2) for $x = y$ should in this case be read as an equation for $\mu^2$.

We study the relaxational dynamics of this model, restricting our attention to the zero-dimensional case (we drop the $x$ dependence and the gradient term in (1)).
The extension of our results to finite dimension is immediate. We consider the Langevin equation

$$\frac{d}{dt} \psi_m(t) = -\mu^2(t) \psi_m(t) - \frac{\partial L}{\partial \psi_m^*} + \eta_m(t)$$

(4)

where $\eta_m$ is a white noise with correlation function $\langle \eta_m(t) \eta_n(s) \rangle = 2T \delta_{mn} \delta(t-s)$, and $T$ is the temperature. We will see a posteriori that it is possible to choose $\mu$ as a function of $t$ to impose the spherical constraint at all times. In order to generate the perturbation expansion of eq. (4), we consider the Martin-Siggia-Rose generating functional

$$Z[h] = \int \mathcal{D}\psi \mathcal{D}\psi^* \left\langle \prod_{m,t} \delta \left( \frac{d}{dt} \psi_m(t) + \frac{\partial L}{\partial \psi_m^*} - \eta_m(t) \right) \right\rangle \exp \left( \int dt \psi_m^*(t) \psi_m(t) \right) \mathcal{J}[\psi]$$

(5)

$\mathcal{J}[\psi]$ is the functional determinant of $d/dt + \partial^2 L_t/\partial \psi_m \partial \psi_m^*$ which, with the Ito convention, is equal to 1, and we denoted by $L_t$ the time dependent “Lagrangian” $L_t = \mu^2(t) \sum_m \psi_m^* \psi_m + L_I$. We now follow a standard procedure [19] to prove that the structure of the diagrammatic expansion for dynamics is the same as that for statics. We introduce an auxiliary field $\hat{\psi}_m(t)$, to get $Z = \int \mathcal{D}\psi \mathcal{D}\psi^* \mathcal{D}\hat{\psi} \mathcal{D}\hat{\psi}^* \exp \left( -\int dt \hat{\psi}_m^* \left[ \dot{\psi}_m - i \hat{\psi}_m + \left( \partial L/\partial \psi_m^* \right) \right] \right)$. For simplicity we have set $h_m = 0$. This will not affect our subsequent manipulations. We then introduce an even grassman variable $\theta$, and the “superfield” $\phi_m(t) = \psi_m(t) + \theta \hat{\psi}_m(t)$. In terms of these, $Z$ reads,

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^* \exp \left( -\int dt \int d\theta \{ \phi_m^*(t, \theta) \left[ (1 - \theta^2 \frac{\partial}{\partial \theta}) \frac{\partial}{\partial t} - T \theta \frac{\partial}{\partial \theta} + \mu^2(t) \} \phi_m(t, \theta) + L_I[\phi, \phi^*] \} \right)$$

(6)

The time integration here should be intended to start from an initial time $t = 0$, where the system is in a random initial condition. This corresponds to the experimental situation of a sudden quench of the system from very high temperature at the initial time.

The interaction term in (3) is represented by the same function $L_I$ as in statics, but its arguments are now the superfields $\phi$ and $\phi^*$ rather than just of $\psi$ and $\psi^*$. Thus the structure of the dynamical perturbation expansion in terms of the superfield is formally identical to that of statics in terms of the field. We define at this point the correlator

$$G(t, \theta; t', \theta') = \langle \phi_m(t, \theta) \phi_m^*(t', \theta') \rangle$$

$$= \langle \psi_m(t) \psi_m^*(t') \rangle + \theta \langle i \hat{\psi}_m(t) \psi_m^*(t') \rangle + \theta' \langle i \hat{\psi}_m(t) i \hat{\psi}_m^*(t') \rangle + \theta \theta' \langle i \hat{\psi}_m(t) i \hat{\psi}_m^*(t') \rangle$$

(7)
We see that $G$ codes for the correlation function $C(t,t') = \langle \psi_m(t)\psi^*_m(t') \rangle$ and the response function $r(t,t') = \langle i\psi_m(t)i\psi^*_m(t') \rangle = \langle \delta\psi_m(t)/\delta\eta_m(t') \rangle$, while the conservation of probability implies $\langle i\psi_m(t)i\psi^*_m(t') \rangle = 0$. Causality and the Ito convention imply $r(t,t') = 0$ for $t' > t$ and $r(t^+, t) = 1$. The zeroth order correlator in dynamics is defined by $G^{-1}_0 = (1 - \theta \partial / \partial \theta) \partial / \partial t - T \theta \partial / \partial \theta + \mu^2(t)$. In terms of $G_0$ and $G$ defined in this way, the Dyson equation of the theory has the same form as (2), replacing $x$ and $y$ respectively by $(t, \theta)$ and $(t', \theta')$. The appearance of $C$ and $r$ only in the combination given by $G$ is a consequence of the gradient character of the Langevin equation.

Inverting the equation (2), multiplying it by $G$, and disentangling the superfield notation, we get coupled equations for the correlation and the response function; for $t > t'$ we have:

$$\frac{\partial C(t,t')}{\partial t} = -\mu^2(t)C(t,t') + \frac{1}{2} g^2 \left\{ \int_0^t ds C(t,s)r(t,s)C(t',s) + \int_0^{t'} ds C^2(t,s)r(t',s) \right\}$$

$$\frac{\partial r(t,t')}{\partial t} = -\mu^2(t)r(t,t') + \frac{1}{2} g^2 \int_0^t ds C^2(t,s)r(t,s)r(t',s)$$

(8)

while for arbitrary $\mu^2(t)$, $C(t,t)$ satisfies:

$$\frac{1}{2} \frac{dC(t,t)}{dt} = -\mu^2(t)C(t,t) + \frac{1}{2} g^2 \int_0^t ds 3C^2(t,s)r(t,s) + T.$$  

(9)

As mentioned above, we can choose $\mu(t)$ to satisfy $\sum_m \psi^*_m \psi_m = N\tilde{q}$ at all times. We recognize in (8,9) well studied equations in mean field spin glass theory: they are identical to those found in the dynamics of the so called spherical $p$-spin model [11] for $p = 3$. This is defined by the Hamiltonian

$$H = -\sum_{i<j<k} J_{ijk} S_i S_j S_k$$

(10)

where the “spins” $S$’s are real variables satisfying the constraint $\sum_{i=1}^N S_i^2 = N\tilde{q}$, and the couplings $J_{ijk}$ are independent Gaussian variables, symmetric under interchange of any pair of indices, with zero mean and variance $\langle J^2_{ijk} \rangle = 3g^2/N^2$. The equivalence for large $N$ of the $O(3)$ symmetric vertex with a random one in dynamics had already been noted in [17]. Note that eq. (8,9) are causal first order integro-differential equations which admit a unique solution for finite times.

In a seminal paper [3] Cugliandolo and Kurchan have found an asymptotic solution to the equation of the $p$-spin model, showing the existence of a low temperature phase in which TTI and FDT are violated even in the infinite time limit. The explicit numerical integration of equations with the same structure obtained
for a related model \[4\] support the asymptotic analysis of \[3\], which we review briefly here in the present context. At high temperature, the asymptotic dynamics is completely characterized by the two functions \(C_{as}(\tau) = \lim_{t \to \infty} C(t + \tau, t)\) and \(r_{as}(\tau) = \lim_{t \to \infty} r(t + \tau, t)\) related by FDT, and such that \(\lim_{\tau \to \infty} C_{as}(\tau) = 0\).

The long time limit of \(\mu\) is given by
\[
\mu^2 = \left(\frac{g^2}{2T}\right) \tilde{q}^2 + \frac{T}{\tilde{q}}.
\]
This is just the zero-dimensional limit of eq. (2) for the statics. Accordingly the “energy” of the system,
\[
E = \langle L_I(t) \rangle = -\left(\frac{g^2}{3T}\right) \int_0^t ds C^2(t, s)r(t, s)
\]
reduces to a single one e.g. for the response function, almost identical to the one originally proposed by Lautheusser \[12\].

Below the critical temperature \(T_c = g(\tilde{q}/2)^{3/2}\), such a regime still exists for \(t, t' \to \infty\) with \(\tau/t\) and \(\tau/t' \to 0\), but with \(\lim_{\tau \to \infty} C_{as}(\tau) = q \neq 0\). In addition, there is an aging regime, in which TTI and FDT are violated. In this regime the limit \(t, t' \to \infty\) is taken fixing the ratio
\[
h(t')/h(t) = e^{-\sigma} \quad (\sigma \geq 0).
\]
Here one has
\[
C(t, t') = \dot{C}(\sigma),
\]
\[
Tr(t, t') = u \frac{\partial C(t, t')}{\partial t'} = -u \left(\frac{d\ln(h(t'))}{dt'}\right) \frac{\partial \dot{C}(\sigma)}{\partial \sigma}.
\]
Continuity between the homogeneous and aging regimes, implies \(\dot{C}(0) = q\). The value of \(q\) is specified at low temperature by the largest root of the equation
\[
q = \frac{T^2}{g^2 (\tilde{q} - q)^2}
\]
while \(\mu^2\) and \(u\) are given by
\[
\mu^2 = \frac{1}{2T} g^2 (\tilde{q}^2 - q^2) + \frac{T}{(\tilde{q} - q)}; \quad u = \frac{\tilde{q} - q}{q}.
\]
It is worth noticing that the former equations can be found in the replica formalism for the \(p\)-spin glass by imposing the condition of marginal stability. The energy takes contribution both from the asymptotic and the aging regions of times, and tends to the limit
\[
E_{Dyn} = -g^2 (\tilde{q}^3 - (1 - u)q^3)/3T,
\]
different from its static counterpart. Figure 1 shows the results of numerical integration of eqs. (8) and (9) at a temperature where aging occurs.

We have seen that the dynamics of the spherical AR model quite unexpectedly displays glassy behaviour, with a sharp phase transition from ergodic to non-ergodic

\[1\] \(h(t)\) is an arbitrary increasing function which the present theory is not able to predict. On a numerical basis it was claimed in \[3\] that \(h(t) = t\), leading to the suggestive result \(C(t, t) = C(t'/t)\).
behaviour. For a correct picture of the model at low temperature, a set of coupled equations for the correlation and response functions have to be solved. It has been often discussed in the literature whether glasses undergo a real phase transition, or whether a better picture is one of a progressive freezing \cite{14,13}. MC equations which fit better with this scenario have been proposed \cite{14,13}. Even though in this case one can not expect aging behaviour for infinite times no matter how the limit is taken, the relaxation to equilibrium can be extremely slow. On an experimental time scale aging phenomena (or eventually ”interrupted aging”) can be present, and the use of the off-equilibrium equations can be more appropriate than that of the asymptotic one.

A natural generalization of the MC eq. (2), and consequently of (8,9) is obtained by replacing the non linear term by some generic ”self-energy” function \( \Sigma(G(x,y)) \). This can be introduced on a purely phenomenological ground, as it has been done by Götze in the equilibrium case \cite{20}, or derived from some microscopic theory. Examples of equations of this form have been analyzed extensively in \cite{4,5} in the context of mean field disordered systems. Depending on the form of \( \Sigma \), the long time aging regime can assume different forms. In the model discussed here, we have found the simplest among the scenarios proposed in \cite{4,5}. Further work is necessary to determine the most appropriate form of \( \Sigma \) to interpret real glass experiments. Our results however, open the door for the systematic study, starting from a mean-field level, of aging effects in real glasses.

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Figure 1. A sketch of the correlation and the response functions for $g = \tilde{q} = 1$, $T = 0.2 < T_c$. (a). Normalized correlation function $C(t_w + \tau, t_w)/\tilde{q}$ and integrated response function $m(t_w + \tau, t_w) = (T/\tilde{q}) \int_{t_w}^{t_w} r(t_w + \tau, s) ds$ as functions of $\tau$, for different waiting times $t_w$. If TTI and the FDT held, these functions would be equal up to a constant shift and independent of $t_w$. (b). $m$ plotted against $C$ for different waiting times. For large values of $C$ (short times), the FDT holds, and the curves approach a straight line of unit slope which intersects the $C$ axis at $q$. The deviation from this line at smaller $C$ (long times) indicates off-equilibrium behaviour. In the region $C < \tilde{q}$ the behaviour predicted by the asymptotic solution would be a straight line with slope $u$, starting from the origin. The times displayed in the figure are very short compared to the ones needed to see the asymptotic behaviour.