The Newton problem solution of the transformed complex curve parameters

Alexander Lozhkin¹, Alexander Korobeynikov and Ruslan Khaziyakhmetov
Software department, Kalashnikov Izhevsk State Technical University, Izhevsk

¹E-mail: lag.izh@gmail.com

Abstract. Many tasks from the natural and engineering sciences require precision solutions with complex curves. The main obstacle is the lack of the necessary mathematical apparatus. The analysis of symmetries on the Euclidean plane by Dieudonne and the figure by Weyl allowed us to formulate a new method for obtaining the parameters of linear transformation alternative to classical. It can be used for an ellipse, hyperbola, as well as complex flat curves. The method is analyzed for trajectories having symmetries. A theorem to obtain the parameters of the transformed curve in the general case is formulated. Theoretical calculations and the results of experimental studies using the method of geometric modeling are given. The method is very new, so it may not work for some curves. There is the possibility of obtaining it, since the research uses the simplest apparatus.

1. Introduction
Any engineering creations base a mathematical calculation. It was all the development civilization time [1]. The exact solution was difficult to get always. The ancient Greeks proposed the first methods of approximate calculations. Newton put forward a canonical form for finding the parameters of a complex curve on plane by the Cartesian theory. An analytical solution was not found and he proposed the first modern approximation methods [2].

The theory of approximation is developing in the present. New methods and approaches are proposed [3-7]. Their number is difficult to count. They are used in mechatronics, mechanics, instrument making, etc. The main feature of all methods is that they have miscalculations. It is obvious that the design will most satisfy the necessary conditions if the accuracy of the calculations is ideal and the product precession will depend on manufacturing tolerances only. Unfortunately, the search for an analytical solution in theory was halted almost a hundred years ago, but production conditions dictate heavier requirements.

The accuracy of manufacturing engineering products was established task to increase about fifty years ago. The flat differentiable curves were focused in the study. The fundamental base of the research is symmetries theory. The team of authors received the first results in the early 90s. They repeated the classical method in a different way.

Solution of the characteristic equation \( \mathbf{T}\vec{v} = \lambda \vec{v} \), where \( \mathbf{T} \) – transformation matrix, \( \lambda \) – scalars, \( \vec{v} \) – vector, is required to obtain practical results in many fields of knowledge, such as mechatronics, physics, cryptography, optimal management, etc. The classic method is to search for the transformation parameters for symmetric conic sections [8]. A new method for finding the parameters of linear transformations was obtained by an additional [9,10]. The main object of this study is the
Cartesian product in Euclidean plane $\mathbb{R} \times \mathbb{R}$ for reflections $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$. Characteristic equation can be solved by modern mathematical methods using projective transformations in spaces with lots of measurements [11]. Research is focus in the field of topology, so use them for engineering calculations is difficult. The author tried to build a simple non-projective method leading to formulas without radical dependencies. The approximate methods do not always satisfy design processes and, most importantly, destroy geometrical semantics. A new method should be based on the internal properties of the space is an author’s hypothesis.

2. Direct method of linear conversions

Let an arbitrary figure $\Phi$ be planar differentiable curve in the Euclidean plane $\mathbb{R} \times \mathbb{R}$ in the Cartesian coordinate linear system. The figure is defined by the parametric equation:

$$
\begin{align*}
x &= k_1 f_x(t) \\
y &= k_2 f_y(t)
\end{align*}
$$

where $x, y, t, k_1, k_2 \in \mathbb{R}$, $f_x \neq t$, $f_y \neq t$. The last two conditions allow us to avoid the transformation of a certain class of singular curves, for example, parabolas. Let is figure $\Phi$ was transformed by the matrix $T = \begin{pmatrix} a & h \\ g & b \end{pmatrix}$, where $a, b, h, g \in \mathbb{R}$. Let us find new parameters of system

$$
\begin{align*}
x &= k'_1 f'_x(t + \alpha) \\
y &= k'_2 f'_y(t + \alpha)
\end{align*}
$$

where $\alpha$ -- angle of permutation symmetry. Non orthogonal basis is form from the corners $\alpha$ and $\beta$.

The basis defined the direct method of linear transformations. Let us formulate the next system of parametric equations:

$$
\begin{align*}
&k'_1 f_x(t + \alpha) = ak_x f_x(t + \beta) + bk_y f_y(t + \beta) \\
&k'_2 f_y(t + \alpha) = gk_x f_x(t + \beta) + bk_y f_y(t + \beta)
\end{align*}
$$

(1)

where $\beta$ -- angle of permutation symmetry. Non orthogonal basis is form from the corners $\alpha$ and $\beta$.

Let us calculate the angle $\beta$ which is symmetrical own corner $\alpha$ for system

$$
\tan 2\beta = \frac{2(g b + h a)}{a^2 - h^2 - b^2 + g^2}, \quad \tan \alpha_1 = \frac{a \tan \beta - h}{b - g \tan \beta} \quad \text{and} \quad \tan \alpha_2 = \frac{b \tan \beta + g}{a + h \tan \beta}
$$

Angles are equal if the
calculation and transformation is correct if \( \beta \not\in (-\pi/2, \pi/2, \pi) \). The coefficients \( \lambda \in \{k'_x, k'_y\} \) from
\[
\mathbf{Tv} = \lambda \mathbf{v}
\]
are equal \( k'_x = k_x \frac{a \cos \beta + b \sin \beta}{\cos \alpha} \) and \( k'_y = k_y \frac{b \sin \beta + g \cos \beta}{\sin \alpha} \), where \( \sin \alpha \neq 0 \). The system (1) will be:
\[
\begin{align*}
\begin{cases}
x' &= k'_{x}(t + \alpha) & \\
y' &= k'_{y}(t + \alpha)
\end{cases}
\end{align*}
\]
are equivalent \( k'_x = k'_2 \) and \( k'_y = k'_2 \).

Investigations of scale and rotation transformations showed that the method does not work for an exact transformation, but they coincide with classical results in the neighborhood. A non-orthogonal basis does not exist for these transformations. Calculations of the compression along the axis lead to a mismatch of the own angles \( \alpha_1 \neq \alpha_2 \), but if the angle does not equal zero, then the result is similar to the classical method. One non-orthogonal basis exists only. It coincides with the orthogonal basis. The most significant discrepancy in the results was found for singular transformations. The classical method transforms each curve individually, but the new method offers eight groups of transformations independent of the shape of the curve [13, 16].

Four additional transformations \( \begin{pmatrix} m & n \\ m & -n \end{pmatrix}, \begin{pmatrix} m & n \\ -m & n \end{pmatrix}, \begin{pmatrix} k & m \\ -n & m \end{pmatrix}, \begin{pmatrix} -k & m \\ -n & m \end{pmatrix} \) were found for inconsistencies of the own angle, where \( \beta \in (-\pi/2, \pi/2) \). Two non-orthogonal bases exist as classical theory. Each of them coincides with an orthogonal basis. The exact parameters are given by the new method. Computer simulation has shown that it can be applied to Jordan curves for these matrix. This result allowed to continue studies for complex curves.

3. Linear transformations of complex curves
Let there be an arbitrary figure \( \Phi \) -- planar differentiable curve in the Euclidean plane \( \mathbb{R}\times\mathbb{R} \) in the Cartesian coordinate linear system defined by the parametric equations system:
\[
\begin{align*}
x &= k_x f_x(t) \\
y &= k_y f_y(t)
\end{align*}
\]
functions \( f_x \neq t \) and \( f_y \neq t \) at the same time. We carry out any transformation of the figure \( \Phi \) defined by the matrix \( \mathbf{P} = \begin{pmatrix} a & h \\ g & b \end{pmatrix} \), where \( a, b, h, g \in \mathbb{R} \), \( \det \mathbf{P} \neq 0 \). It is necessary to obtain the parameters of the transformed figure.

1. Let us split the transformation matrix into the product of two matrices:
\[
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } \begin{pmatrix} 1 & h/b \\ g/a & 1 \end{pmatrix}
\]

2. Let us find the transformation
\[
\begin{pmatrix} 1 \\ g/a \\ h/b \\ 1 \end{pmatrix} \begin{pmatrix} f_x(t) \\ f_y(t) \end{pmatrix}
\]
parameters as a centrally symmetric conic section: \( \alpha \) - own angle (angle of first ort); \( \beta \) - angle of permutation symmetry (second ort); \( k'_x \) and \( k'_y \) - scalars of the characteristic equation.

3. Let us consider a new curve description system:
\[
\begin{align*}
x &= ak_x f_x(t) \\
y &= bk_y f_y(t)
\end{align*}
\]

4. Let us perform an additional, but inverse, transformation [14] \( \mathbf{T} \) depending on the angle \( \beta \). If the angle \( \beta \) is negative, then the transformation is
\[
\mathbf{T} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \cos \alpha & \sin \alpha \end{pmatrix}
\]
if positive, then
\[
\mathbf{T} = \begin{pmatrix} -\cos \beta & \sin \beta \\ \cos \alpha & \sin \alpha \end{pmatrix}
\]
This is a transition from a non-orthogonal basis to an orthogonal basis.
5. Let us transform \[
\begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix}
\] for the parametric system of equations of the trajectory has the form
\[
\begin{cases}
x = k_x f_x(t) \\
y = k_y t
\end{cases}
\] and transformation \[
\begin{pmatrix}
1 & 0 \\
0 & -1 \\
\end{pmatrix}
\] if the system has an alternative description. Transformation can not be performed in the program, but the direction of the point movement can be changed to the opposite.

6. Let us rotate the curve by its own angle \( \alpha \) and multiply it by the coefficients \( k'_x \) and \( k'_y \) founded in item 2.

The analytical formula of the transformed complex curve is found.

4. **Computer modeling**

The method of geometric modeling was used to verify the found patterns. The experiments were carried out in AutoCAD 2007 environment in the AutoLisp language. The program algorithm is quite simple:

- The requested transformation parameters;
- We output the desired curve will transform every part of the line of it. Output is black;
- Forward transformation parameters;
- Output a new curve with parameters obtained in yellow.

![Figure 2. Lemniscates. Test 1.](image)

![Figure 3. Circle.](image)

![Figure 4. Limacon.](image)
Table 1. The experiments results.

| Curve                  | Matrix  | Angle $\beta^\circ$ | Angle $\alpha^\circ$ | Scalar $k_x, k_y$ |
|------------------------|---------|----------------------|-----------------------|-------------------|
| Lemniscates (figure 2) | $\begin{pmatrix} 1 & 1 \\ 0.5 & 1 \end{pmatrix}$ | 52.02                | 37.98                | 1.781, 0.281      |
| Lemniscates            | $\begin{pmatrix} 1 & -1 \\ -0.5 & 1 \end{pmatrix}$ | -52.02               | -37.98               | 1.781, 0.281      |
| Lemniscates            | $\begin{pmatrix} 1.5 & -1 \\ -0.5 & 1.2 \end{pmatrix}$ | -52.02               | -37.98               | 1.614, 0.447      |
| Circle (figure 3)      | $\begin{pmatrix} 1.5 & -1 \\ -0.5 & 1.2 \end{pmatrix}$ | -52.02               | -37.98               | 1.614, 0.447      |
| Limacon (figure 4)     | $\begin{pmatrix} 1.5 & -1 \\ -0.5 & 1.2 \end{pmatrix}$ | -52.02               | -37.98               | 1.614, 0.447      |
| Steiner curve          | $\begin{pmatrix} 1.5 & -1 \\ -0.5 & 1.2 \end{pmatrix}$ | -52.02               | -37.98               | 1.614, 0.447      |

Experiments were performed on the following curves [15]: tricuspoid or Steiner curve by system $\begin{cases} x = k_x (2 \cos t + \cos 2t) \\ y = k_x (2 \sin t - \sin 2t) \end{cases}$, limacon of Pascal by system $\begin{cases} x = k_x (1 + 2 \cos t) \cos t \\ y = k_x (1 + 2 \cos t) \sin t \end{cases}$, Jerome lemniscates $\begin{cases} x = k_x \cos t \\ y = k_x \sin t \cos t \end{cases}$, circle, where $t \in [0, 2\pi]$.

The test results are presented in table 1. The image of the transformed curve coincided with the image of the curve obtained theoretically. The ellipse did not change shape after transformations in steps 2-6.

5. Conclusions

Engineering calculations determine the person's material life. The accuracy of the calculation affects not only the comfort of existence and the possibility of life. The main non-projective method for obtaining accurate engineering calculations is the inverse matrix method. It has one major drawback: it is applicable only to conical sections. Newton proposed a classification of curves for solving this problem, but the calculation methods did not appear until now.

The parameters of the transformation of a complex curve depend exclusively on the type of transformation and properties of space. This is the approach of article. Initially, a new method of obtaining parameters was found for centrally symmetric conic sections only. Further theoretical and experimental studies made it possible to justify the applicability of this method to the Jordan curves. The main drawback of the types of transformations obtained earlier is that they did not change the orthogonal basis. The latest research made it possible to find an algorithm for calculating in a non-orthogonal basis. New methods were tested on robot trajectories in laboratory conditions [16, 17, 18]. The algorithm will be help in the discovery of meanings and practical solutions for many engineering sciences.

Many scientists investigated the problem described in the article, but no good results were obtained. The problem was closed because of the impossibility of finding a solution in the middle of the last century. An analytical solution was found for flat differentiable curves, but not everything has been done yet. The obtained results need theoretical substantiation. Additional studies need to be conducted to verify the correctness of work for both different linear transformations, and for the others...
Finding a system of parametric equations for spatial curves, similar to system (1), is the most important task. The system must contain four equations. This is the only statement available now.

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