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Pythagorean fuzzy Muirhead mean operators in multiple attribute decision making for evaluating of emerging technology commercialization

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ABSTRACT
In today’s world, with the advancement of technology, several emerging technologies are coming. Faced with massive emerging technologies which are the component of the technology pool, how to identify the commercial potential of emerging technologies in theory and practice is an important problem. The scientific approach to the selection of these emerging technologies is one of the main objectives of the research. In this paper, we extend Muirhead mean (MM) operator and dual Muirhead mean (DMM) operator to process the Pythagorean fuzzy numbers (PFNs) and then to solve the multiple attribute decision making (MADM) problems. Firstly, we develop some Pythagorean fuzzy Muirhead mean operators by extending MM and DMM operators to Pythagorean fuzzy information. Then, we prove some properties and discuss some special cases with respect to the parameter vector. Moreover, we present some new methods to deal with MADM problems with the PFNs based on the proposed MM and DMM operators. Finally, we verify the validity and reliability of our methods by using an application example for potential evaluation of emerging technology commercialization, and analyze the advantages of our methods by comparing with other existing methods.

1. Introduction
Atanassov (1986, 2000) introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set (Zadeh, 1965). Each element in the IFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and a non-membership degree. The sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to 1. More recently, a Pythagorean fuzzy set (PFS) (Yager, 2013, 2014) has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of
squares is less than or equal to 1; the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot; for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu (2014) provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN). Meanwhile, they also developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Peng and Yang (2015) proposed the division and subtraction operations for PFNs, and also developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multicriteria group decision making problem with PFNs. Afterwards, Beliakov and James (2014) focused on how the notion of ‘averaging’ should be treated in the case of PFNs and how to ensure that the averaging aggregation functions produce outputs consistent with the case of ordinary fuzzy numbers. Reformat and Yager (2014) applied the PFNs in handling the collaborative-based recommender system. Gou, Xu, and Ren (2016) investigated the Properties of Continuous PFN. Ren, Xu, and Gou (2016) proposed the Pythagorean fuzzy TODIM approach to multi-criteria decision making. Garg (2016a) proposed the new generalized Pythagorean fuzzy information aggregation by using Einstein Operations. Zeng, Chen, and Li (2016) developed a hybrid method for Pythagorean fuzzy multiple-criteria decision making. Garg (2016b) studied a novel accuracy function under interval-valued PFNs for solving multicriteria decision making problems. Wei (2017a) utilized arithmetic and geometric operations (Wang, Wei, & Lu, 2018a; Wei, Gao, & Wei, 2018a; Wei, Lu, Tang, & Wei, 2018b; Wu, Wang, Wei, & Wei, 2018) to develop some Pythagorean fuzzy interaction aggregation operators: Pythagorean fuzzy interaction weighted average (PFIWA) operator, Pythagorean fuzzy interaction weighted geometric (PFIWG) operator, Pythagorean fuzzy interaction ordered weighted average (PFIOWA) operator, Pythagorean fuzzy interaction ordered weighted geometric (PFIOWG) operator, Pythagorean fuzzy interaction hybrid average (PFIHA) operator and Pythagorean fuzzy interaction hybrid geometric (PFIHG) operator. Wei and Lu (2018a) extended Maclaurin symmetric mean (Maclaurin, 1729) to Pythagorean fuzzy environment to propose the Pythagorean fuzzy Maclaurin symmetric mean (PFMSM) operator and Pythagorean fuzzy weighted Maclaurin symmetric mean (PFWMSM) operator. Wei and Lu (2018b) utilized power aggregation operators (Yager, 2001) to develop some Pythagorean fuzzy power aggregation operators: Pythagorean fuzzy power average (PPPA) operator, Pythagorean fuzzy power geometric (PFPG) operator, Pythagorean fuzzy power weighted average (PFPWA) operator, Pythagorean fuzzy power weighted geometric (PFPG) operator, Pythagorean fuzzy power ordered weighted average (PFPOWA) operator, Pythagorean fuzzy power ordered weighted geometric (PFPOWG) operator, Pythagorean fuzzy power hybrid average (PFPHA) operator and Pythagorean fuzzy power hybrid geometric (PFPHG) operator. Lu, Wei, Alsaadi, Hayat, and Alsaeedi (2017) proposed some hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making. Wei and Lu (2017a) defined some dual hesitant Pythagorean fuzzy
Hamacher aggregation operators in multiple attribute decision making. Wei and Lu (2017b) developed some Pythagorean hesitant fuzzy Hamacher aggregation operators in multiple attribute decision making. Wu and Wei (2017) gave some Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Wei, Lu, Alsaaedi, Hayat, and Alsaedi (2017a) proposed some Pythagorean 2-tuple linguistic aggregation operators: Pythagorean 2-tuple linguistic weighted average (P2TLWA) operator, Pythagorean 2-tuple linguistic weighted geometric (P2TLMGA) operator, Pythagorean 2-tuple linguistic ordered weighted average (P2TLOWA) operator, Pythagorean 2-tuple linguistic ordered weighted geometric (P2TLOWGA) operator, Pythagorean 2-tuple linguistic hybrid average (P2TLHMA) operator and Pythagorean 2-tuple linguistic hybrid geometric (P2TLHMG) operator. Lu and Wei (2017) proposed some Pythagorean uncertain linguistic aggregation operators for multiple attribute decision making.

In some real decision making, there exist the interrelationships among the attributes in MADM problems. Bonferroni mean (BM) operators (Bonferroni, 1950; Deng, Wei, Gao, & Wang, 2018; Liu, Chen, & Liu, 2017; Wang, Wei, & Wei, 2018b; Wei, 2017b; Zhu, Xu, & Xia, 2012) and the Heronian mean (HM) (Beliakov, Pradera, & Calvo, 2007; Chu & Liu, 2015; Liu, Liu, & Zhang, 2014; Liu, Zhu, Liu, & Hao, 2013; Yu, 2013; Yu, Zhou, Chen, & Wang, 2015) operators provided a tool to consider the interrelationships of aggregated arguments; however, they can only consider the interrelationships between two attributes and cannot process the interrelationships among three or more than three attributes. Muirhead mean (MM) (Muirhead, 1902) is a well-known aggregation operator which can consider interrelationships among any number of arguments assigned by a variable vector, and some existing operators, such as arithmetic and geometric operators (not considering the interrelationships), both BM operator and Maclaurin symmetric mean (Maclaurin, 1729) are the special cases of MM operator. Therefore, the MM can offer a flexible and robust mechanism to process the information fusion problem and make it more adequate to solve MADM problems. However, the original MM can only deal with the numeric arguments, in order to make the MM operator to process the linguistic information, Qin and Liu (2016) extended the MM operator to process the 2-tuple linguistic information, and proposed some 2-tuple linguistic MM operators and applied the proposed operators to solve the MADM problems.

Because PFNs can easily describe the fuzzy information, and the MM operator and dual MM (DMM) operator can capture interrelationships among any number of arguments assigned by a variable vector, it is necessary to extend the MM and DMM operator to deal with the PFNs. The purpose of this paper is to propose some Pythagorean fuzzy MM operators by extending MM and DMM operators to Pythagorean fuzzy information, then to study some properties of these operators, and applied them to solve the MADM problems in which the attributes take the form of PFNs.

In order to achieve this purpose, the rest of this paper is set out as follows. Section 2 reviews some basic concepts and theory of PFSs. In Section 3, we propose the some Pythagorean fuzzy MM operators, and study some properties of these operators. In Section 4, we develop two MADM methods for PFNs based on the PFWMM operator.
and PFWM operator. In Section 5, an illustrative example for potential evaluation of emerging technology commercialization is given to verify the validity of the proposed methods and to show their advantages. In Section 6, we give some conclusions of this study.

2. Preliminaries

In this section, we review some fundamental concept of Pythagorean fuzzy set and MM, which will be used in the next section.

2.1. Pythagorean fuzzy set

The basic concepts of PFSs (Yager, 2013, 2014) are briefly reviewed in this section. Afterwards, novel score and accuracy functions for PFNs are proposed. Furthermore, a new comparison method for PFNs is developed.

Definition 1 (Yager, 2013, 2014). Let $X$ be a fix set. A PFS is an object having the form

$$ P = \{ (x, (\mu_p(x), \nu_p(x))) | x \in X \} $$

(1)

where the function $\mu_p : X \rightarrow [0, 1]$ defines the degree of membership and the function $\nu_p : X \rightarrow [0, 1]$ defines the degree of non-membership of the element $x \in X$ to $P$, respectively, and, for every $x \in X$, it holds that

$$ (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1. $$

(2)

Definition 2 (Wei, 2017a). Let $\tilde{a} = (\mu, \nu)$ be a Pythagorean fuzzy number, a score function $S$ of a Pythagorean fuzzy number can be represented as follows:

$$ S(\tilde{a}) = \frac{1}{2} (1 + \mu^2 - \nu^2), S(\tilde{a}) \in [0, 1]. $$

(3)

Definition 3 (Ren et al., 2016). Let $\tilde{a} = (\mu, \nu)$ be a Pythagorean fuzzy number, an accuracy function $H$ of a Pythagorean fuzzy number can be represented as follows:

$$ H(\tilde{a}) = \mu^2 + \nu^2, H(\tilde{a}) \in [0, 1]. $$

(4)

to evaluate the degree of accuracy of the Pythagorean fuzzy number $\tilde{a} = (\mu, \nu)$, where $H(\tilde{a}) \in [0, 1]$. The larger the value of $H(\tilde{a})$ is, the more the degree of accuracy of the Pythagorean fuzzy number $\tilde{a}$.

Based on the score function $S$ and the accuracy function $H$, in the following, we shall give an order relation between two Pythagorean fuzzy numbers, which is defined as follows:
Definition 4 (Wei, 2017a). Let \( \tilde{a}_1 = (\mu_1, \nu_1) \) and \( \tilde{a}_2 = (\mu_2, \nu_2) \) be two Pythagorean fuzzy numbers, \( s(\tilde{a}_1) = \frac{1}{2}(1 + (\mu_1)^2 - (\nu_1)^2) \) and \( s(\tilde{a}_2) = \frac{1}{2}(1 + (\mu_2)^2 - (\nu_2)^2) \) be the scores of \( \tilde{a} \) and \( \tilde{b} \), respectively, and let \( H(\tilde{a}_1) = (\mu_1)^2 + (\nu_1)^2 \) and \( H(\tilde{a}_2) = (\mu_2)^2 + (\nu_2)^2 \) be the accuracy degrees of \( \tilde{a} \) and \( \tilde{b} \), respectively, then if \( S(\tilde{a}) < S(\tilde{b}) \), then \( \tilde{a} \) is smaller than \( \tilde{b} \), denoted by \( \tilde{a} < \tilde{b} \); if \( S(\tilde{a}) = S(\tilde{b}) \), then

1. if \( H(\tilde{a}) = H(\tilde{b}) \), then \( \tilde{a} \) and \( \tilde{b} \) represent the same information, denoted by \( \tilde{a} = \tilde{b} \);
2. if \( H(\tilde{a}) < H(\tilde{b}) \), \( \tilde{a} \) is smaller than \( \tilde{b} \), denoted by \( \tilde{a} < \tilde{b} \).

Definition 5 (Reformat & Yager, 2014). Let \( \tilde{a}_1 = (\mu_1, \nu_1) \), \( \tilde{a}_2 = (\mu_2, \nu_2) \), and \( \tilde{a} = (\mu, \nu) \) be three Pythagorean fuzzy numbers, and some basic operations on them are defined as follows:

1. \( \tilde{a}_1 \oplus \tilde{a}_2 = (\sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2(\mu_2)^2}, \nu_1 \nu_2) \);
2. \( \tilde{a}_1 \odot \tilde{a}_2 = (\mu_1 \mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2(\nu_2)^2}) \);
3. \( \lambda \tilde{a} = (\sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda), \lambda > 0 \);
4. \( (\tilde{a})^\lambda = (\mu^\lambda, \sqrt{1 - (1 - \nu^2)^\lambda}), \lambda > 0 \);
5. \( \tilde{a}^\ast = (\nu, \mu) \).

2.2. Muirhead mean (MM)

The MM was first introduced by Muirhead (1902), the advantage of the MM operator is that it can capture the overall interrelationships among the multiple input arguments and it is a generalization of some existing aggregation operators. It was defined as follows:

Definition 6 (Muirhead, 1902). Let \( a_j (j = 1, 2, \ldots, n) \) be a set of crisp numbers and \( [\lambda] = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R \), then the Muirhead mean (MM) operator is defined as

\[
\text{MM}^{\lambda}(a_1, a_2, \ldots, a_n) = \left( \frac{1}{n!} \sum_{\varnothing \in S_n} \prod_{j=1}^{n} a_j^{\lambda_j} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}
\]

(5)

Where \( \varnothing(j) (j = 1, 2, \ldots, n) \) is any permutation of \((1, 2, \ldots, n)\) and \( S_n \) is the set of all permutation of \((1, 2, \ldots, n)\).

By assigning some special vectors to \( \lambda \), we can obtain some special cases of the MM operator:

1. If \( \lambda = (1, 0, 0, \ldots, 0) \) the MM is reduced to

\[
\text{MM}^{(1,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} a_j
\]

(6)
Which is the arithmetic averaging operator.

2. If \( \lambda = (1, 1, 0, 0, \ldots, 0) \) the MM is reduced to

\[
\text{MM}^{(1,1,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \frac{1}{n(n+1)} \sum_{i,j = 1}^{n} a_i a_j
\]

Which is the BM operator (Bonferroni, 1950).

3. If

\[
\text{MM}^{(1,1,\ldots,1-k,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \left( \frac{\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k}}{C_k^n} \right)^{1/k}
\]

the MM is reduced to

\[
\text{MM}^{(1,1,\ldots,1-k,0,0,\ldots,0)}(a_1, a_2, \ldots, a_n) = \left( \frac{\sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} a_{i_1} \cdots a_{i_k}}{C_k^n} \right)^{1/k}
\]

which is the Maclaurin symmetric mean (MSM) operator (Maclaurin, 1729).

4. If \( P = (1/n, 1/n, \ldots, 1/n) \) the MM is reduced to

\[
\text{MM}^{(1/n,1/n,\ldots,1/n)}(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} a_j^{1/n}
\]

which is the arithmetic averaging operator.

3. **Pythagorean fuzzy Muirhead mean operators**

In this section, we shall develop some Pythagorean fuzzy Muirhead mean operators based on the operations of PFNs.

3.1. **PFMM operator**

The MM operator has usually been applied to a situation where the aggregation assessments exhibit interaction relationship. Next, we extend MM operator to PFS. From definition 5 and 6, we can obtain:

**Definition 7.** Let \( p_j = (\mu_j, v_j) (j = 1, 2, \ldots, n) \) be a set of PFN and \( \hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n) \in \mathbb{R} \) be a vector of parameters, then the Pythagorean Fuzzy Muirhead mean (PFMM) operator is defined as

\[
\text{PFMM}^\hat{\lambda}(p_1, p_2, \ldots, p_n) = \left( \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{n} p_{\sigma(j)}^{j} \right)^{1/\sum_{j=1}^{n} j^\hat{\lambda}_j}
\]
Theorem 1. Let $p_{\vartheta}(j = 1, 2, \ldots, n)$ be a collection of PFNs, then their aggregated value by using the PFMM operator is also a PFN, and

$$PFMM^2(p_1, p_2, \ldots, p_n) = \left( \frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} P_{\vartheta(j)}^{\lambda_j} \right) \frac{1}{n!} \sum_{j=1}^{n} \lambda_j$$

Therefore,

$$\sqrt{1 - \left( \prod_{\vartheta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j} \right) \right) \frac{1}{n!} \sum_{j=1}^{n} \lambda_j}$$

Proof:

$$p_{\vartheta(j)}^{\lambda_j} = \left( \mu_{\vartheta(j)}^{\lambda_j}, \sqrt{1 - \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j}} \right)$$

$$\prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} = \left( \prod_{j=1}^{n} \mu_{\vartheta(j)}^{\lambda_j}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j}} \right)$$

Thereafter,

$$\sum_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} = \left( \prod_{\vartheta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j} \right) \prod_{\vartheta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j} \right) \right) \frac{1}{n!} \sum_{j=1}^{n} \lambda_j$$

$$\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\vartheta(j)}^{\lambda_j} = \left( \prod_{\vartheta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j} \right) \prod_{\vartheta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - v_{\vartheta(j)}^2 \right)^{\lambda_j} \right) \right) \frac{1}{n!} \sum_{j=1}^{n} \lambda_j$$

Therefore,
\[
\left( \frac{1}{n!} \sum_{\varnothing \in S_n} \prod_{j=1}^{n} P_{\varnothing(j)}^{\lambda_j} \right) \sum_{j=1}^{n} \lambda_j = \left( \left( \sqrt{1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right)^{\frac{1}{n}}} \right) \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{n}},
\]

\[
\sqrt{1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right)^{\frac{1}{n}}} \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{n}} = 1.
\]

And then, we can know:

\[
0 \leq \left( \sqrt{1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right)^{\frac{1}{n}}} \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{n}} \leq 1
\]

\[
0 \leq \sqrt{1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right)^{\frac{1}{n}}} \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{n}} \leq 1
\]

We can obtain \( \mu_{\varnothing(j)}^{2} + \nu_{\varnothing(j)}^{2} \leq 1 \) from the definition of PFS, so

\[
\left( \left( \sqrt{1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right)^{\frac{1}{n}}} \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{n}} \right)^{2}
\]

\[
\left( \sqrt{1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right)^{\frac{1}{n}}} \sum_{j=1}^{n} \lambda_j \right)^{\frac{1}{n}} \leq \left( 1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j
\]

\[
+ \left( 1 - \left( \prod_{\varnothing \in S_n} \left( 1 - \frac{2}{n} \mu_{\varnothing(j)}^{2} \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j = 1
\]

We complete the proof.
Example 1. Let \(x_1 = (0.6, 0.7), x_2 = (0.4, 0.3), x_3 = (0.8, 0.1)\) be three PFNs, and \([\lambda] = (0.2, 0.5, 0.3)\), then we have

\[
\text{PFMM}^{(0.2,0.5,0.3)}(x_1, x_2, x_3) = \left(1 - \left(1 - 0.6^{0.4} \times 0.4^{0.4} \times 0.8^{0.6} \times (1 - 0.4^{0.4} \times 0.6^{0.4} \times 0.8^{0.6}) \right)ight)^{1/3},
\]

\[
= (0.5818, 0.4673)
\]

In the following, we give some properties of PFMM operator.

Property 1. (Idempotency) let \(p_j = (\mu_{p_j}, \nu_{p_j}) = p = (\mu_p, \nu_p)\) \((j = 1, 2, 3, ..., n)\), then

\[
\text{PFMM}^\lambda(p_1, p_2, ..., p_n) = p
\]

Proof:

\[
\text{PFMM}^\lambda(p_1, p_2, ..., p_n) = \left(\frac{1}{n!} \sum_{\theta \in S_n} \prod_{j=1}^{n} p_j^{\lambda_j} \right)^{1/n} = \left(\frac{1}{n!} \cdot n! \cdot p \right)^{1/n} = p
\]

Property 2. (Monotonicity) let \(p_j = (\mu_{p_j}, \nu_{p_j})\) and \(q_j = (\mu_{q_j}, \nu_{q_j})\) \((j = 1, 2, 3, ..., n)\) be two sets of PFNs, If \((\mu_{p_j})^2 \leq (\mu_{q_j})^2\) and \((\nu_{p_j})^2 \geq (\nu_{q_j})^2\) then

\[
\text{PFMM}^\lambda(p_1, p_2, ..., p_n) \leq \text{PFMM}^\lambda(q_1, q_2, ..., q_n)
\]

Proof:

\[
\prod_{j=1}^{n} \mu_{\lambda(p_j)}^{2\lambda_j} \leq \prod_{j=1}^{n} \mu_{\lambda(q_j)}^{2\lambda_j}
\]
\[
\left( \sum_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \mu_{\vartheta}(p_j) \right) \right)^{\frac{1}{n}} \geq \left( \sum_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \mu_{\vartheta}(q_j) \right) \right)^{\frac{1}{n}}
\]

(24)

Therefore,

\[
\left( \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} \mu_{\vartheta}(p_j) \right) \right)^{\frac{1}{n}} \leq \left( \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} \mu_{\vartheta}(q_j) \right) \right)^{\frac{1}{n}}
\]

(25)

Similarly, we also can obtain

\[
\left( \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} \mu_{\vartheta}(p_j) \right) \right)^{\frac{1}{n}} \geq \left( \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} \mu_{\vartheta}(q_j) \right) \right)^{\frac{1}{n}}
\]

(26)

then, the proof is completed. Then

If \( \mu_{p_j}^2 < \mu_{q_j}^2 \) and \( v_{p_j}^2 > v_{q_j}^2 \), then

\[
\text{PFMM}^2(P_1, P_2, \ldots, P_n) < \text{PFMM}^2(q_1, q_2, \ldots, q_n);
\]

If \( \mu_{p_j}^2 < \mu_{q_j}^2 \) and \( v_{p_j}^2 = v_{q_j}^2 \), then

\[
\text{PFMM}^2(P_1, P_2, \ldots, P_n) = \text{PFMM}^2(q_1, q_2, \ldots, q_n);
\]

If \( \mu_{p_j}^2 = \mu_{q_j}^2 \) and \( v_{p_j}^2 > v_{q_j}^2 \), then

\[
\text{PFMM}^2(P_1, P_2, \ldots, P_n) < \text{PFMM}^2(q_1, q_2, \ldots, q_n);
\]

If \( \mu_{p_j}^2 = \mu_{q_j}^2 \) and \( v_{p_j}^2 = v_{q_j}^2 \), then

\[
\text{PFMM}^2(P_1, P_2, \ldots, P_n) = \text{PFMM}^2(q_1, q_2, \ldots, q_n);
\]

Property 3. (Boundedness) Let \( p_j = (\mu_j, v_j) (j = 1, 2, \ldots, n) \) be a set of PFNs. If \( p^+ = (\max_j(\mu_j), \min_j(v_j)) \) and \( p^- = (\min_j(\mu_j), \max_j(v_j)) \), According the process of property of Monotonicity and Idempotency, it is easy to got that
\[ p^- \leq \text{P2TLMM}^\lambda (p_1, p_2, \ldots, p_n) \leq p^+ \] 

### 3.2. PFWMM operator

In Section 3.1, it can be seen that the PFMM operator doesn’t consider the importance of the aggregated arguments. However, in many real practical situations, especially in multiple attribute decision making, the weights of attributes plays an important role in the process of aggregation. To overcome the limitation of PFMM, we shall propose the Pythagorean fuzzy weighted MM (PFWMM) operator as follows.

**Definition 8.** Let \( p_j = (\mu_j, v_j) (j = 1, 2, \ldots, n) \) be a set of PFNs with weights vector being \( W = (w_1, w_2, \ldots, w_n)^T \), \( w_j \in [0, 1] \), \( \sum_{j=1}^n w_j = 1 \) and \( [\lambda] = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R \), then the Pythagorean fuzzy weighted Muirhead mean (PFWMM) operator is defined as

\[
\text{PFWMM}^\lambda_w (p_1, p_2, \ldots, p_n) = \left( \frac{1}{n!} \sum_{\varnothing \in S_n} \prod_{j=1}^n (nw_{0(j)} p_{0(j)})^{\lambda_j} \right)^{\sum_{j=1}^n \lambda_j} 
\]

(28)

Based on the operations of the PFN described, we can drive the **Theorem 2.**

**Theorem 2.** Let \( p_j = (\mu_j, v_j) (j = 1, 2, \ldots, n) \) be a collection of PFNs, then their aggregated value by using the PFWMM operator is also a PFN, and

\[
\text{PFWMM}^\lambda_w (p_1, p_2, \ldots, p_n) = \left( \frac{1}{n!} \sum_{\varnothing \in S_n} \prod_{j=1}^n (nw_{0(j)} p_{0(j)})^{\lambda_j} \right)^{\sum_{j=1}^n \lambda_j} 
\]

(29)

**proof:**

\[
\text{nw}_{0(j)} p_{0(j)} = \left( \sqrt{1 - (1 - \mu_{0(j)}^2)^{nw_{0(j)}}}, v_{0(j)}^{nw_{0(j)}} \right) 
\]

(30)
\[
(nw_{\vartheta(j)} p_{\vartheta(j)})^{\lambda_j} = \left( \sqrt{1 - \left( 1 - \mu^2_{\vartheta(j)} \right)^{nw_{\vartheta(j)}}} \right)^{\lambda_j}, \sqrt{1 - \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}}
\]

(31)

Thereafter,
\[
\prod_{j=1}^{n} (nw_{\vartheta(j)} p_{\vartheta(j)})^{\lambda_j} = \left( \prod_{j=1}^{n} \left( \sqrt{1 - \left( 1 - \mu^2_{\vartheta(j)} \right)^{nw_{\vartheta(j)}}} \right)^{\lambda_j}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right)
\]

(32)

\[
\sum_{\varrho \in S_n} \prod_{j=1}^{n} (nw_{\vartheta(j)} p_{\vartheta(j)})^{\lambda_j} = \left( \prod_{\varrho \in S_n} \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right), \prod_{\varrho \in S_n} \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right) \right)
\]

(33)

Thus,
\[
\frac{1}{n!} \sum_{\varrho \in S_n} \prod_{j=1}^{n} (nw_{\vartheta(j)} p_{\vartheta(j)})^{\lambda_j} = \left( \prod_{\varrho \in S_n} \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right), \prod_{\varrho \in S_n} \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right) \right)^{\frac{1}{n}}
\]

(34)

Therefore,
\[
\left( \frac{1}{n!} \sum_{\varrho \in S_n} \prod_{j=1}^{n} (nw_{\vartheta(j)} p_{\vartheta(j)})^{\lambda_j} \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j
\]

\[
= \left( \prod_{\varrho \in S_n} \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right), \prod_{\varrho \in S_n} \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \nu_{\vartheta(j)}^{2nw_{\vartheta(j)}} \right)^{\lambda_j}} \right) \right)^{\frac{1}{n}} \frac{1}{n} \sum_{j=1}^{n} \lambda_j
\]

(35)

and we can get followed easily,
Example 2. Let \( x_1 = (0.6, 0.7), x_2 = (0.4, 0.3), x_3 = (0.8, 0.1) \) be three PFNs, and \( \lambda = (0.2, 0.5, 0.3) \), \( W = (0.3, 0.4, 0.3) \) then we have

\[
0 \leq \left( \sqrt{1 - \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\theta(j)}^2 \right)^{\text{nw}_{\theta(j)}} \lambda_j \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j \leq 1 \quad (36)
\]

\[
0 \leq \sqrt{1 - \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\theta(j)}^2 \right)^{2\text{nw}_{\theta(j)}} \lambda_j \right) \right) \}^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j \leq 1 \quad (37)
\]

Therefore,

\[
\left( \sqrt{1 - \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\theta(j)}^2 \right)^{\text{nw}_{\theta(j)}} \lambda_j \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j \right)^2 \\leq \left( \sqrt{1 - \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\theta(j)}^2 \right)^{\text{nw}_{\theta(j)}} \lambda_j \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j \right)^2 \quad (38)
\]

\[
\left( \sqrt{1 - \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\theta(j)}^2 \right)^{2\text{nw}_{\theta(j)}} \lambda_j \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j \right)^2 \leq \left( \sqrt{1 - \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\theta(j)}^2 \right)^{\text{nw}_{\theta(j)}} \lambda_j \right) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j \right)^2 \quad (39)
\]

We complete the proof.
PFWMM$^{(0.2,0.5,0.3)}_{(0.3,0.4,0.3)}(x_1, x_2, x_3)$

\[
= \left( 1 - \left( \begin{array}{c}
(1 - 0.6013^{0.2} \times 0.1888^{0.5} \times 0.3308^{0.3}) \\
(1 - 0.1888^{0.2} \times 0.6013^{0.5} \times 0.3308^{0.3}) \\
(1 - 0.3308^{0.2} \times 0.1888^{0.5} \times 0.6013^{0.3}) \\
(1 - 0.3308^{0.2} \times 0.1888^{0.5} \times 0.6013^{0.3}) \\
\end{array} \right)^{1/3} \right)
\]

\[
= (0.5821, 0.4766)
\]

The PFWMM operator has the property of boundedness and monotonicity, but it does not satisfy the property of idempotency. In the following, we omitted the process of proof, because it is similar with the PFMM monotonicity property.

Property 4. (Monotonicity) let $p_j = (\mu_{p_j}, v_{p_j})$ and $q_j = (\mu_{q_j}, v_{q_j})$ ($j = 1, 2, 3, ..., n$) be two sets of PFNs, If $\mu_{p_j} \leq \mu_{q_j}$ and $v_{p_j} \geq v_{q_j}$ then

PFWMM$^+_w(p_1, p_2, ..., p_n) \leq$ PFWMM$^+_w(q_1, q_2, ..., q_n)$  \hspace{1cm} (40)

Property 5. (Boundedness) Let $p_j = (\mu_{p_j}, v_{p_j}) (j = 1, 2, ..., n)$ be a set of PFNs with weights vector being $W = (w_1, w_2, ..., w_n)^T$, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. If $p^+ = (\text{max}_j(\mu_j), \text{min}_j(v_j))$ and $p^- = (\text{min}_j(\mu_j), \text{max}_j(v_j))$, because of property 4, then

\[
P_2TLWMM^+_w(p^-, p^-, ..., p^-) \leq P_2TLWMM^+_w(p_1, p_2, ..., p_n) \leq P_2TLWMM^+_w(p^+, p^+, ..., p^+)
\]  \hspace{1cm} (41)

### 3.3. PFDMM operator

Qin and Liu (2016) proposed the dual Muirhead mean (DMM) based on MM operator.
Definition 9. Let \( a_j (j = 1, 2, ..., n) \) be a set of non-negative real numbers, and \( P = (p_1, p_2, ..., p_n) \in \mathbb{R}^n \) be a vector of parameters. If

\[
\text{DMM}^P (a_1, a_2, ..., a_n) = \frac{1}{\sum_{j=1}^{n} p_j \sigma(j)} \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} p_j a_{\sigma(j)} \right)^{\frac{1}{n}} \tag{42}
\]

Then we called \( \text{DMM}^P \) the dual Muirhead mean (DMM) operator, where \( \sigma(j) (j = 1, 2, ..., n) \) is any a permutation of \( \{1, 2, ..., n\} \) and \( S_n \) is the set of all permutation of \( \{1, 2, ..., n\} \).

In the following, we proposed the Pythagorean fuzzy dual MM (PFDMM) operator for PFNs.

Definition 10. Let \( p_j = (\mu_j, v_j) (j = 1, 2, ..., n) \) be a collection of PFNs and there exists parameter vector \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n \), then

\[
\text{PFDMM}^\lambda (p_1, p_2, ..., p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j} \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j p_{\sigma(j)} \right)^{\frac{1}{n}} \tag{43}
\]

Based on the operations of the PFN described, we can drive the Theorem 3.

Theorem 3. Let \( p_j = (\mu_j, v_j) (j = 1, 2, ..., n) \) be a collection of PFNs, then their aggregated value by using the PFDMM operator is also a PFN, and

\[
\text{PFDMM}^\lambda (p_1, p_2, ..., p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j} \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j p_{\sigma(j)} \right)^{\frac{1}{n}}
\]

\[
= \left( \sqrt{\prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \mu_j^2 \sigma(j) \right) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j} \right)^{\frac{1}{n}}
\]

\[
= \left( \sqrt{\prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \frac{\lambda_j}{v_j} \sigma(j) \right)^{\frac{1}{n}} \sum_{j=1}^{n} \lambda_j} \right)^{\frac{1}{n}} \tag{44}
\]

Proof:

\[
\lambda_j p_{\sigma(j)} = \left( \sqrt{1 - \left( 1 - \mu_j^2 \sigma(j) \right)^{\lambda_j} v_j} \sigma(j) \right) \tag{45}
\]
Therefore,

\[
\prod \sum_{\sigma \in S_n} \lambda_j \sigma(j) = \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} (1 - \mu_{\sigma(j)}^2) \right)^{\lambda_j} \right)^\frac{1}{\pi} \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} v_{\sigma(j)}^{2\lambda_j} \right) \right)^\frac{1}{\pi} \sum_{j=1}^{\infty} \lambda_j
\]  

(47)

Then, we can get

\[
\frac{1}{\sum_{j=1}^{n} \lambda_j} \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j \sigma(j) \right)^\frac{1}{\pi} = \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} (1 - \mu_{\sigma(j)}^2) \right)^{\lambda_j} \right)^\frac{1}{\pi} \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} v_{\sigma(j)}^{2\lambda_j} \right) \right)^\frac{1}{\pi} \sum_{j=1}^{\infty} \lambda_j
\]  

(48)

From the aggregation result above, we prove the result of PFDMM aggregation is also a PFN in the following, then

\[
0 \leq \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} (1 - \mu_{\sigma(j)}^2) \right)^{\lambda_j} \right)^\frac{1}{\pi} \sum_{j=1}^{\infty} \lambda_j \leq 1
\]  

(50)

\[
0 \leq \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} v_{\sigma(j)}^{2\lambda_j} \right)^{\lambda_j} \right)^\frac{1}{\pi} \sum_{j=1}^{\infty} \lambda_j \leq 1
\]  

(51)
And, we can prove

\[
\left(1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}^2 \lambda_j \right)\right)\right)\right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right) \right)^2 \\
+ \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}^2 \lambda_j \right)\right)\right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right) \right)^2 \\
\leq 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}^2 \lambda_j \right)\right)\right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right) \right)^2 \\
+ \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}^2 \lambda_j \right)\right)\right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right) \right)^2 = 1
\]

(52)

So, we proved that the aggregation result of PFDMM is also a PFN.

**Example 3.** Let \(x_1 = (0.6, 0.7), x_2 = (0.4, 0.3), x_3 = (0.8, 0.1)\) be three PFNs, and \([\lambda] = (0.2, 0.5, 0.3)\), then we have

\[
\text{PFDMM}^{(0.2,0.5,0.3)}(x_1, x_2, x_3) = \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}^2 \lambda_j \right)\right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right) \right) \right)^2 \\
+ \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}^2 \lambda_j \right)\right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right) \right) \right)^2 = (0.6447, 0.2891)
\]

Property 6. (Idempotency) if all \(p_j (j = 1, 2, ..., n)\) are equal, i.e., \(p_j = p = (\mu, \nu)\), then

\[
\text{PFDMM}^{\nu}(p_1, p_2, ..., p_n) = p
\]

(53)

Property 7. (Monotonicity) let \(p_j = (\mu_{p_j}, \nu_{p_j})\) and \(q_j = (\mu_{q_j}, \nu_{q_j})\) \((j = 1, 2, 3, ..., n)\) be two sets of PFNs, If \(\mu_{p_j} \leq \mu_{q_j}\) and \(\nu_{p_j} \geq \nu_{q_j}\) then
Property 8. (Boundedness). Let \( p_j = (\mu_j, \nu_j) \) \( (j = 1, 2, ..., n) \) be a set of PFNs. If \( p^+ = (\max_j(\mu_j), \min_j(\nu_j)) \) and \( p^- = (\min_j(\mu_j), \max_j(\nu_j)) \), because of property 7 and property 8, then PFDM\( \mu_k(p^-, p^-, ..., p^-) = p^- \leq \text{PFDM} \mu_k(p^+, p^+, ..., p^+) = p^+ \).

\[
p^- \leq \text{PFDM} \mu_k(p_1, p_2, ..., p_n) \leq p^+ \tag{55}
\]

### 3.4. PFWDMM operator

In Section 3.3, it can be seen that the PFDMM operator doesn’t consider the importance of the aggregated arguments. However, in many real practical situations, especially in multiple attribute decision making, the weights of attributes play an important role in the process of aggregation. To overcome the limitation of PFDMM operator, we shall propose the Pythagorean fuzzy weighted DMM (PFWDMM) operator as follows.

**Definition 11.** Let \( p_j = (\mu_j, \nu_j) \) \( (j = 1, 2, ..., n) \) be a collection of PFNs with weights vector being \( W = (w_1, w_2, ..., w_n)^T \), \( w_j \in [0, 1], \sum_{j=1}^n w_j = 1 \) and there exists parameter vector \([\lambda] = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n\), then

\[
\text{PFWDMM}^{[\lambda]}_w(p_1, p_2, ..., p_n) = \frac{1}{\sum_{j=1}^n \lambda_j} \left( \prod_{\sigma \in \mathbb{S}_n} \sum_{j=1}^n \lambda_j \mu_{\sigma(j)}^{w_{\sigma(j)}} \right)^{\frac{1}{\lambda}} \tag{56}
\]

Based on the operations of the PFN described, we can derive the **Theorem 4.**

**Theorem 4.** Let \( p_j = (\mu_j, \nu_j) \) \( (j = 1, 2, ..., n) \) be a collection of PFNs, then their aggregated value by using the PFWDMM operator is also a PFN, and

\[
\text{PFMM}^{[\lambda]}(p_1, p_2, ..., p_n) = \left( \frac{1}{n} \sum_{\sigma \in \mathbb{S}_n} \prod_{j=1}^n \mu_{\sigma(j)}^{\lambda_j} \right)^{\frac{1}{\sum_{j=1}^n \lambda_j}}

= \left( \left( \prod_{\sigma \in \mathbb{S}_n} \left( 1 - \prod_{j=1}^n \left( 1 - \mu_{\sigma(j)}^{\lambda_j} \right) \right) \right)^{\frac{1}{\sum_{j=1}^n \lambda_j}} \right)^{\frac{1}{\sum_{j=1}^n \lambda_j}}

\right)
\]

\[
= \left( \left( \prod_{\sigma \in \mathbb{S}_n} \left( 1 - \prod_{j=1}^n \left( 1 - \nu_{\sigma(j)}^{\lambda_j} \right) \right) \right)^{\frac{1}{\sum_{j=1}^n \lambda_j}} \right)^{\frac{1}{\sum_{j=1}^n \lambda_j}} \right)^{\frac{1}{\sum_{j=1}^n \lambda_j}} \tag{57}
\]
Proof:

\[ \hat{\lambda}_j P_{\sigma(j)}^{nw_{\sigma(j)}} = \left( \sqrt{1 - \left( 1 - \mu_{\sigma(j)}^{2nw_{\sigma(j)}} \right)} \right) \hat{\lambda}_j \left( \sqrt{1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{nw_{\sigma(j)}}} \right) \tag{58} \]

\[ \sum_{j=1}^{n} \hat{\lambda}_j P_{\sigma(j)}^{nw_{\sigma(j)}} = \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \mu_{\sigma(j)}^{2nw_{\sigma(j)}} \right)} \right) \prod_{j=1}^{n} \left( 1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{nw_{\sigma(j)}} \right) \hat{\lambda}_j \tag{59} \]

Thereafter

\[ \prod_{\sigma \in S_n} \prod_{j=1}^{n} \hat{\lambda}_j P_{\sigma(j)}^{nw_{\sigma(j)}} \]

\[ = \left( \prod_{\sigma \in S_n} \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \mu_{\sigma(j)}^{2nw_{\sigma(j)}} \right)} \right) \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{nw_{\sigma(j)}} \right) \hat{\lambda}_j \right)^{\frac{1}{\pi}} \tag{60} \]

\[ \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} \hat{\lambda}_j P_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{\frac{1}{\pi}} = \left( \prod_{\sigma \in S_n} \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \mu_{\sigma(j)}^{2nw_{\sigma(j)}} \right)} \right)^{\frac{1}{\pi}} \]

\[ \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{nw_{\sigma(j)}} \right) \hat{\lambda}_j \right) \right)^{\frac{1}{\pi}} \tag{61} \]

Therefore,

\[ \frac{1}{\sum_{j=1}^{n} \hat{\lambda}_j} \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} \hat{\lambda}_j P_{\sigma(j)}^{nw_{\sigma(j)}} \right)^{\frac{1}{\pi}} = \left( \sqrt{1 - \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \mu_{\sigma(j)}^{2nw_{\sigma(j)}} \right) \right) \right)^{\frac{1}{\pi}}} \right) \sum_{j=1}^{n} \hat{\lambda}_j \]

\[ \left( \sqrt{1 - \left( 1 - \prod_{\sigma \in S_n} \left( 1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{nw_{\sigma(j)}} \right) \hat{\lambda}_j \right) \right)^{\frac{1}{\pi}}} \sum_{j=1}^{n} \hat{\lambda}_j \tag{62} \]

Then, we can get

\[ 0 \leq \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \mu_{\sigma(j)}^{2nw_{\sigma(j)}} \right) \right) \right)^{\frac{1}{\pi}} \sum_{j=1}^{n} \hat{\lambda}_j \leq 1 \tag{63} \]
\[
0 \leq \left( \sqrt{1 - \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\sigma(j)}^{2} \right)^{\lambda_{j}} \right) \right) \right)} \right)^{\frac{1}{\pi}} \sum_{j=1}^{\frac{1}{\pi}} \lambda_{j} \leq 1 \tag{64}
\]

Because \(\mu^2 + \nu^2 \leq 1\), therefore

\[
\left( \sqrt{1 - \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{\lambda_{j}} \right) \right) \right)} \right)^{\frac{1}{\pi}} \sum_{j=1}^{\frac{1}{\pi}} \lambda_{j} \leq \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \nu_{\sigma(j)}^{2} \right)^{\lambda_{j}} \right) \right) \right)^{\frac{1}{\pi}} \sum_{j=1}^{\frac{1}{\pi}} \lambda_{j} + \left( 1 - \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - \mu_{\sigma(j)}^{2} \right)^{\lambda_{j}} \right) \right) \right)^{\frac{1}{\pi}} \sum_{j=1}^{\frac{1}{\pi}} \lambda_{j} = 1 \tag{65}
\]

So, the aggregation result of PFWDMM is also PFN.

**Example 4.** Let \(x_1 = (0.6, 0.7), x_2 = (0.4, 0.3), x_3 = (0.8, 0.1)\) be three PFNs, and \([\lambda] = (0.2, 0.5, 0.3), W = (0.3, 0.4, 0.3)\) then we have

\[
\text{PFWDMM}_{\{0.2,0.5,0.3\}}^{\{0.3,0.4,0.3\}}(x_1, x_2, x_3)
\]

\[
= \left( \sqrt{1 - \left( (1 - 0.8181^{0.2} \times 0.3330^{0.5} \times 0.6314^{0.3}) \times (1 - 0.8181^{0.2} \times 0.3330^{0.5} \times 0.6314^{0.3}) \right)^{\frac{1}{\pi}} \sum_{j=1}^{\frac{1}{\pi}} \lambda_{j} \right) \quad ,
\]

\[
= \left( \sqrt{1 - \left( (1 - 0.0090^{0.2} \times 0.1070^{0.5} \times 0.4545^{0.3}) \times (1 - 0.0090^{0.2} \times 0.1070^{0.5} \times 0.4545^{0.3}) \right)^{\frac{1}{\pi}} \sum_{j=1}^{\frac{1}{\pi}} \lambda_{j} \right) \quad ,
\]

\[
= (0.6567, 0.2890)
\]

PFWDMM is also satisfying property boundedness and monotonicity, but it is not satisfying the property of idempotency.
Property 10. (Monotonicity) let \( p_i = (\mu_{p_i}, \nu_{p_i}) \) and \( q_j = (\mu_{q_j}, \nu_{q_j}) \) \( (j = 1, 2, 3, ..., n) \) be two sets of PFNs with weights vector being \( W = (w_1, w_2, ..., w_n)^T \), \( w_j \in [0, 1], \sum_{j=1}^n w_j = 1 \), If \( \mu_{p_i} \leq \mu_{q_j} \) and \( \nu_{p_i} \geq \nu_{q_j} \) then

\[
\text{PFWDMM}^2_w(p_1, p_2, ..., p_n) \leq \text{PFWDMM}^2_w(q_1, q_2, ..., q_n)
\]  

(66)

Property 11. (Boundedness) Let \( p_i = (\mu_{p_i}, \nu_{p_i}) \) \( (j = 1, 2, ..., n) \) be a set of PFNs with weights vector being \( W = (w_1, w_2, ..., w_n)^T \), \( w_j \in [0, 1], \sum_{j=1}^n w_j = 1 \). If \( p^+ = (\max_j(\mu_j), \min_i(\nu_j)) \) and \( p^- = (\min_j(\mu_j), \max_i(\nu_j)) \), because of property 10, then

\[
\text{PFWDMM}_w(p^-, p^-, ..., p^-) \\
\leq \text{PFWDMM}_w(p_1, p_2, ..., p_n) \\
\leq \text{PFWDMM}_w(p^+, p^+, ..., p^+) 
\]  

(67)

5. Models for MADM with PFNs

Based the PFWM and PFWDMM operators, in this section, we shall propose the model for MADM with PFNs. Let \( A = \{A_1, A_2, ..., A_m\} \) be a discrete set of alternatives, and \( G = \{G_1, G_2, ..., G_n\} \) be the set of attributes, \( \omega = (\omega_1, \omega_2, ..., \omega_n) \) is the weighting vector of the attribute \( G_j \) \( (j = 1, 2, ..., n) \), where \( \omega_j \in [0, 1] \), \( \sum_{j=1}^n \omega_j = 1 \). Suppose that \( P = (p_{ij})_{m \times n} = (\mu_{ij}, \nu_{ij})_{m \times n} \) is the Pythagorean fuzzy decision matrix, where \( \mu_{ij} \) indicates the degree that the alternative \( A_i \) satisfies the attribute \( G_j \) given by the decision maker, \( \nu_{ij} \) indicates the degree that the alternative \( A_i \) doesn’t satisfy the attribute \( G_j \) given by the decision maker, \( \mu_{ij} \in [0, 1], \nu_{ij} \in [0, 1], (\mu_{ij})^2 + (\nu_{ij})^2 \leq 1, i = 1, 2, ..., m, j = 1, 2, ..., n \).

In the following, we apply the PFWM (PFWDMM) operator to the MADM problems with PFNs.

Step 1. We utilize the PFNs given in matrix \( \tilde{R} \), and the PFWM operator

\[
p_i = \text{PFWM}^2_w(p_{i1}, p_{i2}, ..., p_{in}) = \left( \frac{1}{n} \sum_{\theta \in S_n} \prod_{j=1}^n (nw_{\theta(j)}p_{\theta(j)})^{\lambda_j} \right)^{\frac{1}{n} \sum_{j=1}^n \lambda_j},
\]

\[
= \left( \frac{1}{n} \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^n \left( 1 - \left( (1 - \mu_{\theta(j)})^2 \right)^{\lambda_j} \right) \right) \right)^{\frac{1}{n} \sum_{j=1}^n \lambda_j},
\]

\[
= \left( 1 - \left( \prod_{\theta \in S_n} \left( 1 - \prod_{j=1}^n \left( 1 - \nu_{\theta(j)}^2 \right)^{\lambda_j} \right) \right) \right)^{\frac{1}{n} \sum_{j=1}^n \lambda_j}, i = 1, 2, ..., m.
\]  

(68)

Or
\[ p_i = \text{PFWDMM}_W(p_{i1}, p_{i2}, \ldots, p_{in}) \]
\[ = \frac{1}{\sum_{j=1}^{n} \lambda_j} \left( \prod_{\sigma \in S_n} \sum_{j=1}^{n} \lambda_j p_{\sigma(ij)} \right)^{\frac{1}{\mu}} \]
\[ = \left( \sqrt{1 - \left( \prod_{\sigma \in S_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \mu \frac{2n w_{\sigma(ij)}}{\lambda_j} \right) \right) \right) \sum_{j=1}^{n} \lambda_j} \right), \quad i = 1, 2, \ldots, m. \quad (69) \]

to derive the \( p_i (i = 1, 2, \ldots, m) \) of the alternative \( A_i \).

**Step 2.** Calculate the scores \( S(p_i) (i = 1, 2, \ldots, m) \) of the overall PFNs \( p_i (i = 1, 2, \ldots, m) \) to rank all the alternatives \( A_i (i = 1, 2, \ldots, m) \) and then to select the best one(s). If there is no difference between two scores \( S(p_i) \) and \( S(p_j) \), then we need to calculate the accuracy degrees \( H(p_i) \) and \( H(p_j) \) of the overall PFNs \( p_i \) and \( p_j \), respectively, and then rank the alternatives \( A_i \) and \( A_j \) in accordance with the accuracy degrees \( H(p_i) \) and \( H(p_j) \).

**Step 3.** Rank all the alternatives \( A_i (i = 1, 2, \ldots, m) \) and select the best one(s) in accordance with \( S(p_i) (i = 1, 2, \ldots, m) \).

**Step 4.** End.

### 6. Numerical example and comparative analysis

#### 6.1. Numerical example

For the time being, emerging technologies have mushroomed up gradually with the rapid development of science and technology, and emerging technologies have brought obvious impacts on states, industries and societies. There are no countries which can oversee the development of emerging technologies, it can be said that the competition between countries and states is the one of emerging technologies, especially the quality, quantity and speed of commercialization and industrialization of emerging technologies. It is not only a difficult task for countries and enterprises to identify, further evaluate and select emerging technologies, finally commercialize and industrialize emerging technologies; but also a weak issue for researchers to study theoretically. Thus, in this section we shall present a numerical example to show potential evaluation of emerging technology commercialization with Pythagorean fuzzy information in order to illustrate the method proposed in this paper. There is a panel with five possible emerging technology enterprises \( O_i (i = 1, 2, 3, 4, 5) \) to select. The experts selects four attributes to evaluate the five possible emerging technology enterprises: \( \odot G_1 \) is the technical advancement; \( \circ G_2 \) is the potential market and market risk; \( \boxtimes G_3 \) is the industrialization infrastructure, human resources and financial
conditions; \( G_4 \) is the employment creation and the development of science and technology. The five possible emerging technology enterprises \( O_i (i = 1, 2, 3, 4, 5) \) are to be evaluated using the PFNs according to four attributes (whose weighting vector \( W = (0.2, 0.1, 0.3, 0.4), \lambda = (0.2, 0.2, 0.3, 0.3) \), as shown in Table 1.

In the following, in order to show potential evaluation of emerging technology commercialization of five possible emerging technology enterprises, we utilize the PFMM, PFWMM, PFDMM and PFWDMM operators to solve MADM problem with PFNs, which concludes the following calculating steps:

**Step 1.** According to Table 1, aggregate all PFNs \( p_{ij} (j = 1, 2, ..., n) \) by using the PFMM, PFWMM, PFDMM and PFWDMM operators to derive the overall PFNs \( p_i (i = 1, 2, 3, 4) \) of the emerging technology enterprises \( O_i \). The aggregating results are shown in Table 2.

**Step 2.** According to the aggregating results shown in Table 2 and the score functions of the emerging technology enterprises are shown in Table 3.

According the result of emerging technology enterprises order, we can know that the best choice is emerging technology enterprise \( O_4 \), we get same result by different aggregation, that proved the effectiveness of the result.

### 6.2. Influence of the parameter on the final result

The aggregation method of extend PFS with MM has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result,
the other is that it can capture the interrelationship between PFNs. These aggregation operators have a parameter vector, which make extended operator more flexible, so the different vector leads to different aggregation results, different scores and ranking results. In order to illustrate the influence of the parameter vector \( R \) on the ranking result, we discuss the influence with several parameter vectors, the result you can find in Tables 4 and 5.

We can see that the different parameters lead to a different result and different ranking order. More attributes we consider, the bigger the scores, the bigger the attribute value and the more lower the scores. Therefore, the parameter vector can be considered as decision maker’s risk preference.

6.3. Comparative analysis

The prominent characteristic of the PFMM, PFWMM, PFDMM and PFWDMM operators is that they can consider the interrelationship among the PFNs. We investigate some comparative analyses to demonstrate the advantages of the proposed operators. Table 6 presents further details.

Table 4. Ranking results by utilizing different parameter vector \( R \) in the PFWMM operator.

| \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) | \(O_1\) | \(O_2\) | \(O_3\) | \(O_4\) | \(O_5\) | Order |
|---|---|---|---|---|---|---|
|(1,0,0,0) | 0.4126 | 0.5869 | 0.6269 | 0.6000 | 0.5147 | \(O_3 \succ O_2 \succ O_4 \succ O_5 \succ O_1\) |
|(1,1,0,0) | 0.3793 | 0.5401 | 0.5521 | 0.5708 | 0.4696 | \(O_4 \succ O_2 \succ O_3 \succ O_5 \succ O_1\) |
|(2,0,0,0) | 0.4389 | 0.5962 | 0.6440 | 0.6166 | 0.5412 | \(O_1 \succ O_2 \succ O_3 \succ O_4 \succ O_5\) |
|(3,0,0,0) | 0.4640 | 0.6052 | 0.6574 | 0.6309 | 0.5611 | \(O_1 \succ O_2 \succ O_3 \succ O_4 \succ O_5\) |
|(1,1,1,0) | 0.3638 | 0.5273 | 0.4993 | 0.5548 | 0.4374 | \(O_4 \succ O_2 \succ O_3 \succ O_5 \succ O_1\) |
|(1,1,1,1) | 0.3524 | 0.5197 | 0.4576 | 0.5438 | 0.4026 | \(O_4 \succ O_2 \succ O_3 \succ O_5 \succ O_1\) |

Table 5. Ranking results by utilizing different parameter vector \( R \) in the PFWDMM operator.

| \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) | \(O_1\) | \(O_2\) | \(O_3\) | \(O_4\) | \(O_5\) | Order |
|---|---|---|---|---|---|---|
|(1,0,0,0) | 0.3750 | 0.5261 | 0.5875 | 0.5353 | 0.4572 | \(O_3 \succ O_4 \succ O_2 \succ O_5 \succ O_1\) |
|(1,1,0,0) | 0.4317 | 0.6145 | 0.6137 | 0.6081 | 0.5234 | \(O_1 \succ O_2 \succ O_3 \succ O_4 \succ O_5\) |
|(2,0,0,0) | 0.3535 | 0.4827 | 0.5734 | 0.4913 | 0.4233 | \(O_1 \succ O_2 \succ O_3 \succ O_4 \succ O_5\) |
|(3,0,0,0) | 0.3368 | 0.4486 | 0.5609 | 0.4603 | 0.3939 | \(O_1 \succ O_2 \succ O_3 \succ O_4 \succ O_5\) |
|(1,1,1,0) | 0.4647 | 0.6502 | 0.6285 | 0.6582 | 0.5401 | \(O_2 \succ O_3 \succ O_4 \succ O_5 \succ O_1\) |
|(1,1,1,1) | 0.4991 | 0.6730 | 0.6379 | 0.6832 | 0.5515 | \(O_4 \succ O_2 \succ O_3 \succ O_5 \succ O_1\) |
PFDMM, PFWMM, PFWDMM operators have a parameter vector, thereby enabling the aggregation process to be substantially flexible.

At the same time, we compare our proposed method with other existing methods including the Pythagorean fuzzy weighted averaging (PFWA) operator (Yager, 2014), Pythagorean fuzzy weighted geometric (PFWG) operator (Yager, 2014), symmetric Pythagorean fuzzy weighted averaging (SPFWA) operator (Ma & Xu, 2016) and symmetric Pythagorean fuzzy weighted geometric (SPFWG) operator (Ma & Xu, 2016).

From the Table 7, we can get the same optimal emerging technology enterprises and four methods’ ranking results are slightly different from the proposed approaches. However, the existing aggregation operators, such as PFWA operator, PFWG operator, SPFWA operator and SPFWG operator, do not consider the information about the relationship between arguments being aggregated, and thus cannot eliminate the influence of unfair arguments on decision results. Our proposed operators, such as PFMM operator, PFDMM operator, PFWMM operator and PFWDMM operator consider the information about the relationship among arguments being aggregated.

### Table 6. The Comparison of the different aggregation operators under PFNs.

| Aggregation operators | Whether the operator can capture the interrelationship between any two or three PFNs | Whether the operator can capture the interrelationship among all PFNs | Whether a parameter vector exists to manipulate the ranking results |
|-----------------------|----------------------------------------------------------------------------------------|-----------------------------------------------------------------|--------------------------------------------------------------------------------|
| PFWA and PFWG (Yager, 2014)  | No                                                                                   | No                                                              | No                                                                                  |
| SPFWA and SPFWG (Ma & Xu, 2016) | No                                                                                   | No                                                              | No                                                                                  |
| PFOWAWAD (Zeng et al., 2016) | No                                                                                   | No                                                              | No                                                                                  |
| PFEWA, PFEOWA (Garg, 2016a) | No                                                                                   | No                                                              | No                                                                                  |
| PFHWA and PFHWG (Wu & Wei, 2017) | No                                                                                   | No                                                              | No                                                                                  |
| PFHWA, PFHGWG (Wu & Wei, 2017) | No                                                                                   | No                                                              | No                                                                                  |
| DPHFWA, DPHFWG (Wei & Lu, 2017a) | No                                                                                   | No                                                              | No                                                                                  |
| DPHFWA, DPHFWG (Wei & Lu, 2017a) | No                                                                                   | No                                                              | No                                                                                  |
| HPFWWA, HPFWHG (Lu et al., 2017) | No                                                                                   | No                                                              | No                                                                                  |
| HPHFA, HPFHFA (Lu et al., 2017)  | No                                                                                   | No                                                              | No                                                                                  |
| PFCIA and PFCIG (Peng & Yang, 2016) | Yes                                                                                   | No                                                              | No                                                                                  |
| PFPWA, PFPWG (Wei & Lu, 2018b) | Yes                                                                                   | No                                                              | No                                                                                  |
| PFIWA, PFIWG (Wei, 2017a) | Yes                                                                                   | No                                                              | No                                                                                  |
| PFIWA, PFIWG (Wei, 2017a) | Yes                                                                                   | No                                                              | No                                                                                  |
| PFIWA, PFIWG (Wei, 2017a) | Yes                                                                                   | No                                                              | No                                                                                  |
| PFPMSM, PFPWMSM (Wei & Lu, 2018a) | Yes                                                                                   | Yes                                                             | No                                                                                  |
| GPFWBGM, GPFWBGM (Zhang, Wang, Zhu, Xia, & Yu, 2017) | Yes                                                                                   | Yes                                                             | Yes                                                                                  |
| PFMM, PFDMM  | Yes                                                                                   | Yes                                                             | Yes                                                                                  |
| PFWM, PFWDMM | Yes                                                                                   | Yes                                                             | Yes                                                                                  |

### Table 7. Ordering of the emerging technology enterprises.

| Ordering |
|----------|
| PFWA operator (Yager, 2014) | \( A_3 > A_4 > A_2 > A_5 > A_1 \) |
| PFWG operator (Yager, 2014) | \( A_3 > A_4 > A_2 > A_5 > A_1 \) |
| SPFWA operator (Ma & Xu, 2016) | \( A_3 > A_4 > A_2 > A_5 > A_1 \) |
| SPFWG operator (Ma & Xu, 2016) | \( A_3 > A_4 > A_2 > A_5 > A_1 \) |
7. Conclusion

Aggregation operators have become a hot issue and an important tool in the decision making fields in recent years. However, they still have some limitations in practical applications. For example, some aggregation operators suppose the attributes are independent of each other. However, the MM operator and dual MM operator have a prominent characteristic that it can consider the interaction relationships among any number of attributes by a parameter vector \( \lambda \). Motivated by the studies about MM operator and dual MM operator, in this paper, we proposed some new MM and DMM aggregation operators to deal with MADM problems under a Pythagorean fuzzy environment, included the Pythagorean fuzzy MM (PFMM) operator, Pythagorean fuzzy weighted MM (PFWMM) operator, Pythagorean fuzzy dual MM (PFWDMM) operator and the Pythagorean fuzzy weighted dual MM (PFWDMM) operator. Then, the desirable properties were proved. Moreover, these proposed operators are utilized to solve the MADM problems with PFNs. Finally, we used an illustrative example for potential evaluation of emerging technology commercialization to show the feasibility and validity of the proposed operators by comparing with the other existing methods. In the future, we shall extend the proposed operators to dynamic and complex decision making (Chen, 2015; De & Sana, 2014; Gao, 2018; Gao, Wei, & Huang, 2018; Huang & Wei, 2018; Tang & Wei, 2018; Wang, Wei, & Lu, 2018c; Wei, 2017c, 2018a, 2018b; Wei, Gao, Wang, & Huang, 2018c; Yue & Jia, 2013), risk analysis (Wei, Liu, Lai, & Hu, 2017b; Wei, Yu, Liu, & Cao, 2018d) and many other fields under uncertain environment (Chen, 2017; Mardani et al., 2015, 2018; Peng & Selvachandran, 2017; Rostamzadeh, Esmaeili, Nia, Saparauskas, & Ghorabaee, 2017; Wang, Wei, & Gao, 2018d; Wei, 2018c; Wei & Wei, 2018; Zeng, Mu, & Balezentis, 2018).

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