Leafwise quasigeodesic foliations in dimension three and the funnel property

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Abstract. We construct one-dimensional foliations which are subfoliations of two-dimensional foliations in 3-manifolds. The subfoliation is by quasigeodesics in each two-dimensional leaf, but it is not funnel: not all quasigeodesics share a common ideal point in most leaves.

Key words: quasigeodesics, subfoliations, Anosov flows
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1. Introduction

The goal of this article is to analyze whether certain geometric conditions imply that a one-dimensional foliation in a 3-manifold is the foliation by flow lines of a topological Anosov flow. We do this analysis for one-dimensional foliations whose leaves lie inside leaves of two-dimensional foliations and whose leaves are quasigeodesics in these two-dimensional foliations. In other words, the goal of this article is to analyze whether some strictly geometric behavior implies strong dynamical systems behavior in this setting. This has important connections with partial hyperbolicity in dimension three.

A foliation $\mathcal{G}$ subfoliates a foliation $\mathcal{F}$ if each leaf of $\mathcal{F}$ has a foliation made up of leaves of $\mathcal{G}$. We call $\mathcal{G}$ the subfoliation and $\mathcal{F}$ the super foliation. This situation is very common, for example, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two foliations which are transverse to each other everywhere, then their intersection forms a subfoliation of each of them. This article aims to study geometric properties of leaves of subfoliations inside the leaves of the super foliation.

One very common and extremely important example is the following: let $\Phi$ be an Anosov flow and let $\mathcal{F}^{w_s}, \mathcal{F}^{w_u}$ be the weak stable and weak unstable foliations of $\Phi$ respectively [Ano63, KH95]. Then $\mathcal{F}^{w_s}, \mathcal{F}^{w_u}$ are transverse to each other—the intersection is the foliation by flow lines of $\Phi$ which is a subfoliation of each of them. This example has connections with geometry or large-scale geometry: the leaves of $\mathcal{F}^{w_s}, \mathcal{F}^{w_u}$ are Gromov hyperbolic. In rough terms, this means that they are negatively curved. The subfoliation by flow lines in, say, $\mathcal{F}^{w_s}$ satisfies an additional strong geometric property:
in each leaf of $\mathcal{F}^{w_s}$, the flow lines are quasigeodesics. This means that when lifted to the universal cover of the leaves, the flow lines are uniformly efficient up to a bounded multiplicative distortion in measuring length in the weak stable leaves. In other words, the flow lines are quasi-isometrically embedded in these weak stable leaves. The quasigeodesic property has many important consequences, for example, the flow lines are within a bounded distance from length minimizing geodesics when lifted to the universal cover of their respective weak stable leaves [Gro87, Thu82, Thu97]. Hence the flow lines have well-defined distinct ideal points in the Gromov boundary of the weak stable leaves in both directions. These properties and others are very strong and useful in many contexts. Obviously, this also works for the flow subfoliation of the weak unstable foliation.

A (one-dimensional) subfoliation made of quasigeodesics in the leaves of a superfoliation by Gromov hyperbolic leaves is called a **leafwise quasigeodesic foliation**. The Anosov case has an additional geometric property: in (say) a weak stable leaf, all flow lines are forward asymptotic, which is a defining property of the weak stable foliation. In particular, all flow lines in a given weak stable leaf have the same forward ideal point in the ideal boundary of the weak stable leaf (when lifted to the universal cover).

When all leaves of a leafwise quasigeodesic subfoliation in a leaf of the superfoliation have a common ideal point, we call that leaf a **funnel leaf**. If all leaves of the superfoliation are funnel leaves, then the leafwise quasigeodesic foliation is said to have the **funnel property**.

The motivation for this article is the following question: is the funnel property an additional property or is it a consequence of the leafwise quasigeodesic property? The importance of this is the following: in dimension three, we have a much stronger connection between some of these properties as follows. Suppose that $\mathcal{G}$ is a leafwise quasigeodesic foliation (which is a one-dimensional subfoliation of a two-dimensional foliation) which has the funnel property. The ambient manifold is three-dimensional. Suppose that the foliation $\mathcal{G}$ is orientable or, in other words, it is the foliation of a non-singular flow. Then one can prove that the flow in question is expansive. (We refer to [BFP20] for definitions of the terms used here and for detailed proofs.) This implies that the flow is a topological Anosov flow [IM90, Theorem 15], [Pat93, Lemma 7]. If the flow is transitive (the union of periodic orbits is dense), then the topological Anosov flow is in addition orbitally equivalent to a (smooth) Anosov flow [Sha20]. This means that if the leafwise quasigeodesic property implies the funnel property, then this purely geometric condition implies a very strong dynamical systems property: the foliation is the flow foliation of an Anosov flow, up to topological equivalence.

In this article, we prove that the funnel property is not a consequence of leafwise quasigeodesic behavior.

**Theorem 1.1.** There are examples of leafwise quasigeodesic foliations in dimension three which do not have the funnel property. In these examples, the two-dimensional foliations are $C^0$ with $C^1$ leaves and the subfoliation is by $C^1$ curves in the two-dimensional leaves.

We now briefly explain one class of examples: start with the Franks–Williams example of a non-transitive Anosov flow $\Phi$. This is obtained as follows: start with a suspension
Anosov flow and do a DA (derived from Anosov) blow up of a periodic orbit transforming it into (say) a repelling orbit $\alpha$. Remove a tubular neighborhood of $\alpha$ so that the resulting semiflow is incoming in the complement of the removed tubular neighborhood of $\alpha$. Glue this manifold with boundary with a copy of it which has a reversed flow. One fundamental result is that the ensuing flow $\Phi$ in the final manifold $\mathcal{M}$ is Anosov [BBY17, FW80]. This holds for certain isotopy classes of gluings and certain gluing maps satisfying transversality conditions. These were the first examples of non-transitive Anosov flows in dimension three. Our examples use this flow. There is a smooth torus $T$ in $\mathcal{M}$ transverse to the flow. There is a single two-dimensional attractor and a single two-dimensional repeller of the flow $\Phi$ in $M$. Start with a one-dimensional foliation $Z$ in $T$ which is transverse to the intersections of both the weak stable and the weak unstable foliations of $\Phi$ with $T$. Saturate $Z$ by the flow producing a collection of two-dimensional sets embedded in $\mathcal{M}$. The flow saturation of $T$ is an open subset $V$ of $M$, and the collection of the two-dimensional subsets described is a two-dimensional foliation in $V$. In addition, $V$ is exactly the complement of the union of the attractor and the repeller of $\Phi$. Complete the foliation in $V$ to a foliation $\mathcal{F}$ in $\mathcal{M}$ which is the weak unstable foliation of $\Phi$ in the attractor of $\Phi$ and the weak stable foliation in the repeller of $\Phi$. The proof that this is in fact a foliation of $\mathcal{M}$ depends on a careful choice of the one-dimensional foliation $Z$ in $T$. There is a subtle point here in that if one chooses an arbitrary foliation $Z$ in $T$, then when lifting to $\tilde{\mathcal{M}}$, the lifted sets may not be properly embedded in $\tilde{\mathcal{M}}$ and so $\mathcal{F}$ would not be a foliation in $\mathcal{M}$. This is carefully analyzed in §3 and there we prove that for appropriate choices of $Z$, the object $\mathcal{F}$ we construct is a foliation. The super foliation is this two-dimensional foliation $\mathcal{F}$. The subfoliation $\mathcal{G}$ of $\mathcal{F}$ is the foliation by flow lines of $\Phi$. Each leaf of $\mathcal{F}$ is saturated by flow lines. We prove that $\mathcal{G}$ is a leafwise quasigeodesic subfoliation of $\mathcal{F}$, but $\mathcal{G}$ does not have the funnel property. There is an Anosov flow $\Phi$ in this example; however, notice that the super foliation $\mathcal{F}$ is neither the weak stable nor the weak unstable foliation of $\Phi$, but rather a different foliation. In fact in the same way, one can construct an infinite number of inequivalent examples with the same starting flow $\Phi$. The foliations are pairwise distinguished because of how they intersect the torus $T$ in foliations which are not equivalent.

In this article, we consider more general examples. We prove that one can construct examples starting with any non-transitive Anosov flow $\Phi$ in dimension three so that all the basic sets have dimension two. As in the case of the Franks–Williams example, we construct super foliations which have Gromov hyperbolic leaves and whose leaves are saturated by flow lines of $\Phi$. We show that the subfoliation $\mathcal{G}$ by flow lines of $\Phi$ is by quasigeodesics in each leaf of the super foliation $\mathcal{F}$. This is the hardest step to prove. This involves a very careful analysis of the geometry in these examples. The proof that $\mathcal{G}$ is not funnel is simpler than proving it is leafwise quasigeodesic as a subfoliation of $\mathcal{F}$.

In the course of the proof of Theorem 1.1, we prove another independent result which can be used in other contexts. In Definition 6.1, we define the notion of continuity properties for a pair of foliations $(\mathcal{F}, \mathcal{G})$ on $\mathcal{M}$ where $\mathcal{F}$ is a two-dimensional foliation sub-foliated by a one-dimensional foliation $\mathcal{G}$. We can show that the continuity property implies leafwise quasigeodesity.
**THEOREM 1.2.** Suppose $\mathcal{F}$ is a two-dimensional foliation on a 3-manifold $\mathcal{M}$ and $\mathcal{F}$ is subfoliated by a one-dimensional foliation $\mathcal{G}$. If the pair $(\mathcal{F}, \mathcal{G})$ satisfies the continuity properties as defined in Definition 6.1, then $\mathcal{G}$ is leafwise quasigeodesic on $\mathcal{F}$.

We finish this introduction mentioning another reason why we analyzed this question: this comes from partially hyperbolic dynamics. Let $f$ be a partially hyperbolic diffeomorphism in a closed 3-manifold $\mathcal{M}$ (we refer to [BFP20] for definitions and properties of partially hyperbolic diffeomorphisms). Under very general orientability conditions, there is a pair of transverse two-dimensional branching foliations (center stable and center unstable foliations) associated with the partially hyperbolic diffeomorphism which intersect in an one-dimensional branching foliation, called the center foliation [BI08]. The center foliation subfoliates both the center stable and center unstable foliations. In some situations [BFP20], it is shown that the center foliation is a leafwise quasigeodesic subfoliation of both the center stable and center unstable foliations. However, in [BFP20], it is proved that in the partially hyperbolic setting the leafwise quasigeodesic property implies that the center foliation has the funnel property (as a subfoliation of both super foliations). The proof of this also uses dynamical system properties, namely partial hyperbolicity. An open question from the article [BFP20] was to whether the funnel property could be derived strictly from the leafwise quasigeodesic property in (say) the center stable foliation. In this article, we prove that this is not the case by constructing counterexamples for general foliations.

### 2. Preliminaries

A $C^1$-flow $\Phi_t : \mathcal{M} \to \mathcal{M}$ on a Riemannian manifold $\mathcal{M}$ is *Anosov* if the tangent bundle $T\mathcal{M}$ splits into three $D\Phi_t$-invariant sub-bundles $T\mathcal{M} = E^s \oplus E^0 \oplus E^u$ and there exists two constants $C, \lambda > 0$ such that:

- $E^0$ is generated by the non-zero vector field defined by the flow $\Phi_t$;
- for any $v \in E^s$ and $t > 0$,
  \[ \| D\Phi_t(v) \| \leq C e^{-\lambda t} \| v \|; \]
- for any $w \in E^u$ and $t > 0$,
  \[ \| D\Phi_t(w) \| \geq C e^{\lambda t} \| w \|. \]

The definition is independent of the choice of the Riemannian metric $\| . \|$ as the underlying manifold $\mathcal{M}$ is compact. For a point $x \in \mathcal{M}$, the set $\gamma_x = \{ \Phi_t(x) | t \in \mathbb{R} \}$ is called the *flow line* of $x$. The collection of all flow lines of a flow defines a one-dimensional foliation on $\mathcal{M}$. For an Anosov flow, there are several flow invariant foliations associated with the flow and these foliations play a key role in the study of Anosov flows.

**Property 2.1.** [Ano63] For an Anosov flow $\Phi_t$ on $\mathcal{M}$, the distributions $E^u$, $E^s$, $E^0 \oplus E^u$, and $E^0 \oplus E^s$ are uniquely integrable. The associated foliations are denoted by $\mathcal{F}^u$, $\mathcal{F}^s$, $\mathcal{F}^{wu}$, and $\mathcal{F}^{ws}$ respectively and they are called the strong unstable, strong stable, weak unstable, and weak stable foliation on $\mathcal{M}$.
For the remainder of this article, we will assume that \( \mathcal{M} \) is a closed three-dimensional Riemannian manifold.

We also assume that \( \mathcal{M} \) is equipped with an Anosov flow \( \Phi_t \), and \( \tilde{\Phi}_t \) is the lift of the flow \( \Phi_t \) in \( \tilde{\mathcal{M}} \), the universal cover of \( \mathcal{M} \). The strong unstable, strong stable, weak unstable, and weak stable foliations of \( \tilde{\Phi} \) are the lifts of the foliations \( F^u, F^s, F^{wu}, \) and \( F^{ws} \) in the universal cover \( \tilde{\mathcal{M}} \), and these foliations in \( \tilde{\mathcal{M}} \) are denoted by \( \tilde{F}^u, \tilde{F}^s, \tilde{F}^{wu}, \) and \( \tilde{F}^{ws} \) respectively.

A map \( f : (X_1, d_1) \to (X_2, d_2) \) is called a \((K, s)\)-quasi-isometric embedding if there exists \( K > 1 \) and \( s > 0 \) such that for all \( x, y \in X_1 \),

\[
\frac{1}{K} d_1(x, y) - s \leq d_2(f(x), f(y)) \leq K d_1(x, y) + s.
\]

A \((K, s)\)-quasigeodesic in \( X \) is the image of a \((K, s)\)-quasi-isometric embedding \( \gamma : [a, b] \to X \), where \([a, b]\) is a closed interval on \( \mathbb{R} \) with the Euclidean metric. The interval could be infinite (that is, \( a = -\infty \), \( b = \infty \), or both), in which case the notation would be of a half open or open interval. If we have a map \( \mathbb{R} \to X \) with rectifiable image, we consider the arclength metric in the domain \( \mathbb{R} \).

**Lemma 2.2.** Flow lines on the leaves in \( \tilde{F}^{ws} \) and \( \tilde{F}^{wu} \) are quasigeodesics with respect to the induced path metric from \( \tilde{\mathcal{M}} \) in their respective leaves.

**Proof.** Reparameterize the flow to have unit speed. The new flow is still Anosov with the same flow lines and the same weak stable and weak unstable foliations; however, the strong stable and strong unstable leaves may change [Ano63, AS67].

Any leaf \( L \) of \( \tilde{F}^{wu} \) is subfoliated by \( \tilde{F}^u \) and by the flow lines, these two foliations are transversal to each other. We can define a metric \( ds' \) on \( L \) by \( ds' = dw + dy \), where \( dw \) measures length along flow lines and \( dy \) measures length along unstable curves. Suppose \( ds \) is the Riemannian metric induced on \( L^{wu} \) from \( \tilde{\mathcal{M}} \). The two path metrics induced in \( L \) from \( ds' \) and \( ds \) are uniformly quasi-isometric to each other [Fen94]. Moreover, each flow line in the leaf \( L \) is a length-minimizing curve in the \( ds' \) metric, and hence flow lines are uniform quasigeodesics with respect to the metric induced by \( ds \). Similarly, it can be shown that flow lines on leaves in \( \tilde{F}^{ws} \) are quasigeodesic with respect to the induced metric on their respective leaves. \( \square \)

**Definition 2.3.** Suppose \( \mathcal{F} \) is a two-dimensional foliation on \( \mathcal{M} \) with Gromov hyperbolic leaves when lifted to the universal cover. Suppose that \( \mathcal{G} \) is a one-dimensional foliation on \( \mathcal{M} \) which subfoliates \( \mathcal{F} \). In this situation, we say that leaves of \( \mathcal{G} \) are leafwise quasigeodesic in \( \mathcal{F} \) if every leaf of \( \mathcal{G} \) is a quasigeodesic in the respective leaf of \( \mathcal{F} \) containing it when lifted to the universal cover of the leaf. In that case, we say that \( \mathcal{G} \) is a leafwise quasigeodesic subfoliation of \( \mathcal{F} \).

In Lemma 2.2, the flow lines of \( \Phi_t \) are shown to be leafwise quasigeodesics in the leaves of \( \tilde{F}^{ws} \) and \( \tilde{F}^{wu} \).

The leaves in \( \tilde{F}^{ws} \) and \( \tilde{F}^{wu} \) are Gromov hyperbolic with respect to the Riemannian metric on the leaves induced from the metric on \( \tilde{\mathcal{M}} \) [Fen94]. Suppose that \( L \) is a leaf either in \( \tilde{F}^{ws} \) or in \( \tilde{F}^{wu} \). As the leaves are Gromov hyperbolic, we can define the ideal boundary
of L which is homeomorphic to the circle and we denote it as $S^1(L)$. The compactification $L \cup S^1(L)$ is homeomorphic to a closed disk. As the flow lines are quasigeodesics in L, they define two distinct ideal points on $S^1(L)$: if $\gamma$ is a flow line in L, then the forward ray of $\gamma$ defines an unique ideal point on $S^1(L)$ as $\gamma$ is a quasigeodesic, which is called the forward or positive ideal point of $\gamma$. Similarly, we define the backward or negative ideal point as the limit of the ray in the backward direction. The following statement describes the equivalence between the forward and backward flow rays in the leaves of $\tilde{F}_{ws}$ and $\tilde{F}_{wu}$, and the points on their ideal boundaries.

**Property 2.4.** [Fen94] For a leaf $L$ either in $\tilde{F}_{ws}$ or $\tilde{F}_{wu}$ all the points on $S^1(L)$ correspond to forward or backward flow rays on L. If $L \in \tilde{F}_{ws}$, then all the flow lines on L have a common forward ideal point and all the other ideal points are backward ideal points on $S^1(L)$ of the flow lines. No two different flow lines define a common negative or backward ideal point.

If $L \in \tilde{F}_{wu}$, then all the flow lines have a common backward ideal point and all the forward flow lines define all the other ideal points on $S^1(L)$. No two different flow lines define the same positive or forward ideal point.

The property for forward ideal points in $\tilde{F}_{ws}$ is immediate as these flow lines are forward asymptotic, a direct consequence of the definitions. The property for backward ideal points in leaves of $\tilde{F}_{ws}$ is not as immediate and is proved in [Fen94].

**Definition 2.5.** Suppose that $G$ is a leafwise quasigeodesic subfoliation of $F$. If a leaf $L$ of $\tilde{F}$ has all leaves of $G$ in it sharing a common ideal point, then the projected leaf $\pi(L)$ of $F$ in $M$ is called a funnel leaf. In this case, the common ideal point shared by all the flow lines in $L$ is called the funnel point of $L$.

**Corollary 2.6.** By Property 2.4, for an Anosov flow $\Phi_t$ on a 3-manifold $M$, with the flow foliation a leafwise quasigeodesic subfoliation of both $F_{ws}$ and $F_{wu}$, the following happens: all the leaves in weak stable foliation $F_{ws}$ and weak unstable foliation $F_{wu}$ are funnel leaves, as shown in Figure 1.

2.1. Basic sets of Anosov flows on 3-manifolds. The Anosov flow $\Phi$ is called transitive if there exists a flow line $\gamma$ dense in $M$, otherwise the flow is non-transitive. The first example of a non-transitive Anosov flow was constructed by John Franks and Bob Williams in their 1980’s article [FW80].

A point $x \in M$ is called non-wandering if for any open neighborhood $U$ of $x$ and any $t_0 > 0$, there exists $t > t_0$ such that $\Phi_t(U) \cap U \neq \emptyset$, the set of all non-wandering points is denoted by $\Omega(\Phi)$. For a non-transitive Anosov flow $\Phi_t$, the non-wandering set $\Omega(\Phi)$ is not equal to the whole manifold $M$ and according to spectral decomposition theorem [Sma67], $\Omega(\Phi)$ is decomposed into finitely many closed, disjoint, $\Phi_t$-invariant, and transitive basic sets $\{\Lambda_i, i = 1, \ldots, n\}$, so $\Omega(\Phi) = \bigsqcup_{i=1}^n \Lambda_i$.

Suppose $\Lambda$ is a basic set of a non-transitive Anosov flow $\Phi_t$ on a 3-manifold. Then $\Lambda$ can be characterized into four different types [Bru93, Sma67]:

- $\dim(\Lambda) = 2$, and the basic set $\Lambda$ is an attractor, i.e. there exists an open set $U$ containing $\Lambda$ such that $\bigcap_{t>0} \Phi_t(U) = \Lambda$;
\textbf{Property 2.7.} If $\Lambda$ is an attractor, then $\Lambda$ is saturated by weak unstable leaves. If $\Lambda$ is a repeller, then $\Lambda$ is saturated by weak stable leaves.

From now on we assume the following.

\textbf{Assumption 2.8.} We assume throughout that the Anosov flow $\Phi$ on $\mathcal{M}$ is non-transitive and its non-wandering set $\Omega$ consists of two-dimensional basic sets only.

In other words, we assume that $\Phi$ has no one-dimensional basic set. As $\mathcal{M}$ is compact, there exists at least one attracting basic set and one repelling basic set. Suppose $\mathcal{A}$ denotes the union of all attracting basic sets and $\mathcal{R}$ denotes the union of all repelling basic sets. We will denote the collection of all lifts of $\mathcal{A}$ in $\tilde{\mathcal{M}}$ by $\tilde{\mathcal{A}}$. Here, $\tilde{\mathcal{A}}$ is the attracting set for $\tilde{\Phi}_t$ defined on $\tilde{\mathcal{M}}$. The union of all lifts of $\mathcal{R}$ is denoted by $\tilde{\mathcal{R}}$ similarly.

\textbf{Property 2.9.} \textbf{[KH95]} Suppose $\gamma$ is a flow line not contained in $\mathcal{A}$ or $\mathcal{R}$. Then there exists a flow line in $\mathcal{A}$, say $\alpha$, such that the forward rays of $\gamma$ and $\alpha$ are asymptotic in $\mathcal{M}$. Similarly, there exists a flow line $\beta$ in $\mathcal{R}$ such that the backward rays of $\gamma$ and $\beta$ are asymptotic in $\mathcal{M}$.

\textbf{Proof.} This is classical [KH95], we explain briefly. Given the orbit $\gamma$, it gets closer and closer to the attractor $\mathcal{A}$ in future time. Fix $x$ in $\gamma$. Every point in the attractor has a local product structure, see, for example, Proposition 6.4.21 of [KH95]. Hence for $t$ sufficiently

\begin{itemize}
  \item $\dim(\Lambda) = 2$, and the basic set $\Lambda$ is a repeller, i.e $\Lambda$ is an attractor for the reversed flow $\Psi_t = \Phi_{-t}$;
  \item $\dim(\Lambda) = 1$, and $\Lambda$ is a saddle with local cross section a Cantor set;
  \item $\dim(\Lambda) = 1$, and $\lambda$ is a hyperbolic periodic orbit.
\end{itemize}

\textbf{Property 2.7.} If $\Lambda$ is an attractor, then $\Lambda$ is saturated by weak unstable leaves. If $\Lambda$ is a repeller, then $\Lambda$ is saturated by weak stable leaves.

From now on we assume the following.

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\textbf{Proof.} This is classical [KH95], we explain briefly. Given the orbit $\gamma$, it gets closer and closer to the attractor $\mathcal{A}$ in future time. Fix $x$ in $\gamma$. Every point in the attractor has a local product structure, see, for example, Proposition 6.4.21 of [KH95]. Hence for $t$ sufficiently
big, $\Phi_t(x)$ is $\epsilon$ near the attractor where $\epsilon$ is smaller than the size of product boxes of the hyperbolic set $A$. Hence, $\Phi_t(x)$ is $\epsilon$ near some point $z$ in $A$ and there is $w$ in $A$ near $z$ so that $\Phi_t(x)$ is in the stable manifold of $w$ because of the local product structure in sets of size $\epsilon$. This proves the result.

The attractor is saturated by leaves of $F^{wu}$ and the repeller saturated by leaves of $F^{ws}$. In the property above, one can choose the flow line $\alpha$ in the attractor $A$ to be contained in the boundary of the attractor. This means the following: let $x \in \alpha$ and $L$ the $F^{ws}$ leaf through $x$. Let $D$ be a small disk in $L$ with $x$ in the interior. The local flow line of $x$ cuts $D$ into two components $D_1$, $D_2$ (which are also disks). The condition is that one of $D_1$ or $D_2$ does not intersect the attractor $A$. Suppose it is $D_1$. The ‘$D_1$ side’ of $\alpha$ in $L$ is the side so that $\gamma$ is getting increasingly closer to $\alpha$.

3. The foliation $F$

Throughout the article, we will fix a non-transitive Anosov flow $\Phi$ as in the previous section, that is, $\Phi$ has only two-dimensional basic sets.

To prove our results, we will consider a two-dimensional foliation $F$ in $M$ such that:

- on the attractor $A$, $F|_{\tilde{A}} = F^{wu}|_{\tilde{A}}$;
- on the repeller $R$, $F|_{\tilde{R}} = F^{ws}|_{\tilde{R}}$;
- on $\tilde{M} \setminus \{\tilde{A} \cup \tilde{R}\}$, $F$ is transversal to both $F^{ws}$ and $F^{wu}$;
- every leaf $L \in F$ is subfoliated by the flow lines of $\Phi$, i.e. every leaf $L$ is $\Phi_{\tilde{R}}$-invariant.

We will denote the lift of $F$ in the universal cover $\tilde{M}$ by $\tilde{F}$. Leaves of $\tilde{F}$ in $\tilde{A}$ and $\tilde{R}$ look like the leaves in Figure 1. Leaves in $\tilde{M} \setminus (\tilde{A} \cup \tilde{R})$ are described in Figure 2. It is not immediate from the definition why the leaves not contained in $\tilde{A}$ and $\tilde{R}$ are as described in Figure 2, but we will prove this later in this article.

**THEOREM 3.1.** There exists foliations $F$ with the properties described above.

**Start of the proof of Theorem 3.1.** We start with an Anosov flow as described above. For simplicity, assume that $M$ is orientable as well. There is a finite collection of disjoint tori $\{T_i\}$ transverse to the flow $\Phi$ which separate the basic sets [Bru93, Sma67]. We choose $T_i$ to be smooth. The collection of tori is supposed to be minimal respective to the property that if an orbit is not in $R$ or $A$, then it intersects one of the $\{T_i\}$. Let $\gamma$ be such an orbit intersecting a specific $T_i$, let $x$ be a point in the intersection. Then the forward orbit of $x$ is asymptotic to a component $A$ of the attractor $A$—this uses the important fact that there are no one-dimensional components of the non-wandering set of $\Phi$ by assumption. The set of such $x$ so that the forward ray of $x$ is asymptotic to $A$ is open in $T_i$. This holds for any component $A$ of the attractor $A$. Since the union over such components of $A$ is all of $T_i$ and $T_i$ is connected, it follows that all orbits in $T_i$ are forward asymptotic to a single component $A$ of $\tilde{A}$.

In a similar way, one proves that if $T_1$, $T_2$ are tori contained in the complement of the union of the attractor and repeller, and $T_1$, $T_2$ intersect a common orbit of $\Phi$, then $T_1$, $T_2$ intersect exactly the same set of orbits of $\Phi$. In other words, if $B$ is a component of $\tilde{M} \setminus (\tilde{A} \cup \tilde{R})$, then there is a torus $T$ contained in $B$, transverse to $\Phi$ so that $B$ is the flow saturation of $T$. Hence, we can choose a minimal collection $\{T_i\}$ of tori transverse to $\Phi$ and
3.1. Construction of $\mathcal{F}$. Now we construct $\mathcal{F}$. The foliations $\mathcal{F}^{wu}$, $\mathcal{F}^{wu}$ are $C^0$ with $C^1$ leaves [KH95], and so are the intersections with each $T_i$. On each $T_i$, choose a one-dimensional $C^1$ foliation $F_i$ transverse to both $\mathcal{F}^{wu} \cap T_i$, and $\mathcal{F}^{wu} \cap T_i$.

Saturate $F_i$ by the flow to produce a two-dimensional foliation in the flow saturation of $T_i$. Note that the leaves in the flow saturation are either a plane or an infinite annulus. Let $\mathcal{F}$ be this foliation in the complement of the attractor union the repeller. Figure 2 describes a possible leaf in the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $\tilde{\mathcal{M}}$, where $R_2$, the blue line, represents its intersection with some lift $\tilde{T}_i$ of some torus $T_i$.

interacting all orbits in the complement of $A \cup R$, and any such orbit intersects a unique $T_i$ and only once.
3.2. Properties of $\mathcal{F}$. At this point, $\mathcal{F}$ is just a collection of two-dimensional subsets of $\mathcal{M}$. We will prove that $\mathcal{F}$ is a foliation of $\mathcal{M}$. Clearly, $\mathcal{F}$ is a foliation in the complement of the union of the attractor and the repeller, because this is an open set and because of the definition of $\mathcal{F}$: each component $C$ of $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ is equal to $\Phi_{\mathbb{R}}(T_i)$ for some $T_i$ and this is homeomorphic to $T_i \times \mathbb{R}$ with the product topology (the topology in $T_i$ is induced from $\mathcal{M}$). The foliation $\mathcal{F}$ in $C$ is equivalent to the foliation $F_i \times \mathbb{R}$ in $T_i \times \mathbb{R}$.

The interaction between $\mathcal{F}$ in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and $\mathcal{F}$ in $\mathcal{A} \cup \mathcal{R}$ is more complex. There is a subtle point here, which we now explain. Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\tilde{\mathcal{M}}$. If $\mathcal{F}$ is a foliation, then it will follow that $\tilde{\mathcal{F}}$ is a foliation of $\tilde{\mathcal{M}}$ by properly embedded planes. By construction, the ‘leaves’ of $\mathcal{F}$ intersecting the attractor are contained in the attractor and similarly for the repeller. Therefore, the leaves of $\mathcal{F}$ in the complement of $\mathcal{A} \cup \mathcal{R}$ are entirely contained in the complement of $\mathcal{A} \cup \mathcal{R}$ as well. In particular, if $L$ is a lift of a leaf of $\mathcal{F}$ in the complement of the attractor and repeller, then it should be properly embedded when lifted to $\tilde{\mathcal{M}}$. As it turns out, this property is not true if one starts with an arbitrary foliation $F_i$ in $T_i$. Let us review the construction: we start with a foliation $F_i$ in $T_i$ and saturate it by the flow to produce a foliation in an open set in $\mathcal{M}$. Then consider a lift $L$ of a leaf of this foliation to the universal cover. Is $L$ always properly embedded in $\tilde{\mathcal{M}}$? In general, this is not true. For example, start with the Franks–Williams non-transitive flow [FW80], consider a smooth torus $T$ which separates the attractor and repeller, and start with say the intersection of the stable foliation of $\Phi$ with $T$, which we call $F$. Then for some of the leaves of $F$, it follows that if $L$ is a lift of the flow saturation to $\tilde{\mathcal{M}}$, then $L$ is not properly embedded in $\mathcal{M}$. For example, [FW80, Fig. 3, p. 164] depicts the foliations induced by the weak stable and unstable foliations in $T$ for the Franks–Williams flow. Each has two Reeb components. Take $\alpha$ to be a leaf of the stable foliation which is not in the interior of a Reeb component, that is, a horizontal line in the figure, and also that $\alpha$ is a closed curve. Lift it to $\tilde{\alpha}$ in $\tilde{\mathcal{M}}$. If $C$ is the flow saturation of $\tilde{\alpha}$, then $C$ is not properly embedded in $\tilde{\mathcal{M}}$: there is an orbit $\gamma$ of $\tilde{\Phi}$ which is not in $C$ but is contained in the closure of $C$. This orbit $\gamma$ is the lift of a periodic orbit contained in the attractor of the Franks–Williams flow. The same would happen if we took $F$ to be the intersection of the unstable foliation with $T$, $\alpha$ a closed leaf of $F$, and considering the repeller of $\Phi$ instead of the attractor.

The reason why our construction of $\mathcal{F}$ as above produces a foliation is because we start with a foliation $F_i$ in $T_i$ which is transverse to both the stable and unstable foliations in $T_i$. We first prove the following result.

**Lemma 3.2.** Let $\ell$ be a leaf of $F_i$ and let $E$ be the flow saturation of $\ell$. Then, with the induced path metric from $\mathcal{M}$, it follows that $E$ is complete.

**Proof.** Let $E$ be the flow saturation of $\ell$. Since $\ell$ is smooth and the flow is $C^1$ it follows that $E$ is $C^1$. For any $x, y$ in $\ell$ if

$$\Phi_t(x) = \Phi_{s}(y),$$

then $x = y$ and $t = s$, since the component of $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ containing $\ell$ is homeomorphic to $T \times \mathbb{R}$ and $\ell$ is injectively immersed in $T$. Hence, $E$ is parameterized as $\ell \times \mathbb{R}$, that is, every point $p$ in $E$ can be represented as $(x, t)$ where $x \in \ell$ and $t \in \mathbb{R}$.
The Riemannian metric in $M$ induces a Riemannian metric in $E$ and a path metric in $E$. What we prove is the following.

**CLAIM 1.** There is $a_0 > 0$ so that any point $p = (x, t)$ in $E$ is the center of a metric disk of radius $a_0$ in $E$.

*Proof.* This is obvious for any point $p$ in $\ell$ or, in other words, if $t = 0$.

We now prove the claim for $t > 0$ using the unstable foliation. The analogous proof shows the result for $t < 0$ using the stable foliation. The foliation $F_i$ is transverse to both the stable and unstable foliations induced in $T_i$, hence uniformly transverse to these foliations (which means the angles between $F_i$ and $\mathcal{F}^{wu} \cap T_i$ or $\mathcal{F}^{cs} \cap T_i$ are uniformly bounded away from 0 on $T_i$). Given any smoothly embedded curve $\alpha$ in $M$, let $l_u(\alpha)$ be its unstable length: we integrate only the component of the tangent vector in the direction of the unstable bundle. For example, if $\alpha$ is contained in a weak stable leaf, then $l_u(\alpha)$ is zero, while if $\alpha$ is contained in a strong unstable leaf, then $l_u(\alpha)$ is the same as its length under the Riemannian metric of $M$. In particular, if $\alpha$ is a curve not contained in a strong stable or unstable leaf, then the original length $l(\alpha)$ is always strictly greater than the unstable length $l_u(\alpha)$.

By the definition of an Anosov flow, there exist constants $C > 0$, $\lambda > 1$ such that if we flow forward a segment with $t$ amount of time, the new unstable length is at least $C \lambda t$ times the original unstable length. Hence if we let $a_1 = C$, then for any smooth segment, any flow forward of that segment has unstable length which is at least $a_1$ times the original unstable length.

Since $F_i$ is uniformly transverse to $\mathcal{F}^{us} \cap T_i$ by our construction, it follows that any point $x$ in $T_i$ is the midpoint of a segment $\beta$ in its leaf of $F_i$ of unstable length 2. For any $t \geq 0$, the unstable length of $\Phi_t(\beta)$ is at least $2a_1$. This constant $a_1$ is defined globally. In addition, if $v$ is a non-zero vector tangent to $\beta$, then $v$ makes a definite positive angle with the flow direction. Since flowing forward increases the size of unstable vectors more than the size of tangent vectors (where $t > t_0 > 0$ for some $t_0$ big enough), it follows that there is a global constant $\theta > 0$ so that $D\Phi_t v$ also makes an angle $> \theta$ with the tangent to the flow. Consider the infinitesimal arclengths $dt$, $ds$, $du$ along the flow, stable, and unstable bundles. The (non-Riemannian) metric

$$|dt| + |ds| + |du|$$

is quasicomparable (this means Lipschitz equivalent) with the Riemannian metric in $M$: there is $a_2 > 0$ so that the Riemannian length is at least $a_2$ times the length in this metric. Consider the following set:

$$A = \Phi_{[t-1,t+1]}(\beta)$$

for $t \geq 0$. The segment $\beta$ of $F_i$ is contained in the leaf $E$ of $F$. From any point in the boundary of $A$ to $\Phi_t(x)$ along $E$, one has to have at least $a_1$ unstable length and flow length of at least 1. It follows that there is a global constant $a_0$ (depending only on $a_1$) so that $A$ contains a disk in the Riemannian metric of radius $a_0$ and centered at $\Phi_t(x)$.

For $t < 0$, we use the stable foliation and flow backward instead of forward. This finishes the proof of the claim. □
The claim shows that $E$ is complete and finishes the proof of the lemma.

**Continuation of the proof of Theorem 3.1** We consider the collection $\mathcal{F}$ as constructed in the beginning of this section. This object $\mathcal{F}$ is a foliation restricted to $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and this is an open set.

The only remaining thing to prove is that if a sequence $x_n$ in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ converges to $x$ in $\mathcal{A} \cup \mathcal{R}$, then the leaves of $\mathcal{F}$ through $x_n$ converge to the leaf of $\mathcal{F}$ through $x$. Without loss of generality, we may assume that $x$ is in an attractor.

Let $p_n \in T_i$ so that $x_n$ are in $\Phi_{\mathbb{R}}(p_n)$. There are $t_n \in \mathbb{R}$ with $x_n = \Phi_{t_n}(p_n)$. Since $x$ is in the attractor, then $t_n$ converges to positive $\infty$. The leaf of $\mathcal{F}$ through $p_n$ is the $\Phi$-flow saturation of the leaf of $F_i$ through $p_n$. The tangent to this two-dimensional set through $p_n$ is generated by the Anosov vector field generating $\Phi$ and a tangent vector $v$ to $F_i$ at $p_n$. The leaf of $\mathcal{F}$ is $\Phi$-flow invariant. Flowing forward, the flow vector remains invariant. The vector $v$ is transverse to the weak stable foliation and hence it flows increasingly more (does not matter how fast) to the weak unstable direction. So flowing forward, these leaves become increasingly more tangent to the $E^0 \oplus E^u$ bundle and limit to leaves of $\mathcal{F}^{wu}$. Since flowing forward limits to the attractor, this shows that the leaves of $\mathcal{F}$ through $x_n$ converge to the leaf of $\mathcal{F}$ through $x$.

In addition, the previous lemma shows that the leaves of $\mathcal{F}$ through $x_n$ are complete in their path metrics. This shows that $\mathcal{F}$ defines a foliation. We stress that Lemma 3.2 is needed to ensure that $\mathcal{F}$ is a foliation. Otherwise, even if the tangent directions of $\mathcal{F}$ in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and $\mathcal{F}$ in $\mathcal{A} \cup \mathcal{R}$ match continuously, one would have that leaves of the first set ‘arrive’ at leaves of the second set in finite distance. In other words, the union of a leaf in $\mathcal{M} - (\mathcal{A} \cup \mathcal{R})$ and a leaf in $(\mathcal{A} \cup \mathcal{R})$ would form a branched surface. This would produce a branching foliation, instead of a foliation.

This finishes the proof of Theorem 3.1.

We remark that the construction of $\mathcal{F}$ highlights why our methods do not work when there are one-dimensional basic sets. For simplicity, suppose that there is a basic set which is a periodic orbit $\gamma$. There is a torus $T$ so that negative saturation limits on $\gamma$. If we start with $F$ in $T$ transverse to both $\mathcal{F}^{ws} \cap T$ and $\mathcal{F}^{wu} \cap T$, then flowing backward will make it limit to the weak stable leaf of $\gamma$. So the weak stable leaf of $\gamma$ is in the collection $\mathcal{F}$ so constructed. However, there is also a torus $T'$ so that the forward flow saturation limits on $\gamma$. The similar argument shows that the weak unstable foliation of $\gamma$ also has to lie in the collection $\mathcal{F}$. Hence the collection $\mathcal{F}$ has sets which intersect transversely and cannot be a foliation.

**Remark 3.3.** By construction, the foliation $\mathcal{F}$ does not have compact leaves: any leaf in $\mathcal{R} \cup \mathcal{A}$ is not compact as they are weak stable leaves of an Anosov flow. Each leaf in $\mathcal{M} - (\mathcal{R} \cup \mathcal{A})$ limits on $\mathcal{R}$ and hence cannot be compact. Since $\mathcal{F}$ does not have compact leaves, it follows from Novikov’s theorem [Cal01] that leaves of $\widetilde{\mathcal{F}}$ are properly embedded planes in $\widetilde{\mathcal{M}}$.

To prove Theorem 1.1, we will prove the following properties for such a foliation $\mathcal{F}$:

1. the flow lines are leafwise quasigeodesics in leaves of $\mathcal{F}$;
2. every leaf of $\mathcal{F}$ not contained in $\mathcal{A}$ or $\mathcal{R}$ is a non-funnel leaf, as in Figure 2.
4. Gromov hyperbolicity of the leaves of $F$

We will consider a foliation $F$ as constructed in the previous section.

In this section, we will show that there exists a metric $g$ such that every leaf of the foliation $F$ is Gromov hyperbolic. By Candel’s uniformization theorem, this condition is equivalent to the fact that every holonomy invariant non-trivial measure $\mu$ on $M$ has Euler characteristic $\chi_{\mu}(M, F) < 0$, which includes the case when there exists no invariant measure. For more details about the Euler characteristic, see [Can93] or [CC00]. In our context, we will prove that there is no holonomy invariant transverse measure to $F$. Candel’s theorem requires that the foliation has $C^\infty$ leaves. To obtain that, we use Calegari’s result [Cal01b] which implies that $F$ is isotopic to a foliation with $C^\infty$ leaves. This does not change the property that $F$ has or does not have holonomy invariant transverse measures. Once this is obtained, Candel proved that there is a metric in $M$ inducing a smooth Riemannian metric in the leaves so that curvature in each leaf of $F$ is constant equal to $-1$. A precise statement can be found in [Can93], [CC00], or [Cal01]. We call such a metric a Candel metric. This Candel metric is not smooth in the transverse direction.

Here is the precise statement on the equivalence of Gromov hyperbolicity of leaves of a foliation and negative Euler characteristic of a positive invariant measure.

**Proposition 4.1.** [Can93] Let $(M, F)$ be a compact oriented surface lamination with a Riemannian metric $g$. Then $\chi(M, \mu) < 0$ for every positive invariant transverse measure $\mu$ if and only if there is a metric in $M$ which induces a metric in each leaf of $F$ which makes it into a hyperbolic surface. In particular, this holds true if $M$ has no invariant measure.

To prove that all the leaves of $F$ are Gromov hyperbolic, we will show that there does not exist any invariant measure. We will argue by contradiction, we assume that there exists an invariant measure $\mu$, and we will attain a contradiction.

The support of $\mu$ on $M$, denoted by $\text{supp}(\mu)$, is defined as the collection of all points $x \in M$ such that if $\tau$ is a one-dimensional manifold transverse to $F$ which contains $x$ in its interior, then $\mu(\tau) > 0$. The support of a holonomy invariant transverse measure is a closed set and it is saturated by $F$, which means $\text{supp}(\mu)$ is a union of leaves of $F$. The orientation hypothesis is not essential as it can be achieved by a double cover. The double cover does not change the conformal type of any leaf.

**Lemma 4.2.** The support of $\mu$ on $M$ contains at least one leaf from the attractor $A$ or the repeller $R$.

**Proof.** Consider a point $x \in \text{supp}(\mu)$ and suppose $L_x$ is the leaf in $F$ which contains $x$, then $L_x \subset \text{supp}(\mu)$ as $\text{supp}(\mu)$ is $F$-saturated. If $x \in A$, then $L_x \subset A$ and the claim is true. Similarly if $x$ is in $R$, then its leaf is contained in $\text{supp}(\mu)$. Finally suppose that $x \notin (A \cup R)$. Then consider the sequence $\{\Phi_n(x)\}$ as $n \to \infty$. Let $z$ be an accumulation point of $\{\Phi_n(x)\}$. As $\text{supp}(\mu)$ is closed, $z$ is in $\text{supp}(\mu)$ and hence $L_z \subset \text{supp}(\mu)$. Since $z$ is an accumulation point of $\Phi_n(x)$, it implies that $z$ is a non-wandering point, and hence $z \in A \cup R$ and $L_z \subset (A \cup R) \cap \text{supp}(\mu)$. In fact, since $n \to \infty$, it follows that $z$ is in the attractor, so $L_z \subset A$. \qed
Suppose $L$ is a leaf in $\text{supp}(\mu)$ which is contained in $A$ (assume in $A$ without loss of generality). By [Pla75, Theorem 6.3], we know that if $\mu$ is a holonomy invariant transverse measure on a compact manifold foliated by a codimension-one foliation $F$, then any leaf contained in $\text{supp}(\mu)$ has polynomial growth. Then the leaf $L_\varepsilon$ in the attractor $A$, as obtained in the previous paragraph, has polynomial growth. Recall that the leaves of $F$ are either planes or annuli. At the same time, $L_\varepsilon$ is contained in the attractor and each leaf in the attractor belongs to the weak unstable foliation of the Anosov flow $\Phi$. However, weak stable and weak unstable leaves of Anosov flows have exponential growth, a contradiction.

As each leaf $L \in \tilde{\mathcal{F}}$ is Gromov hyperbolic with respect to the path metric from the induced Riemannian metric from $\tilde{\mathcal{M}}$, we can define the circle at infinity or the ideal boundary $S^1(L)$ of each leaf $L$.

Next we will describe the topology we will use on the spaces $S^1(\tilde{\mathcal{M}}) = \bigcup_{L \in \tilde{\mathcal{F}}} S^1(L)$ and $\tilde{\mathcal{M}} \cup S^1(\tilde{\mathcal{M}}) = \bigcup_{L \in \tilde{\mathcal{F}}} (L \cup S^1(L))$.

For this, we will assume first that $\mathcal{M}$ has a Candel metric.

Suppose $\tau$ is an open segment homeomorphic to $(0,1)$ and transversal to $\tilde{\mathcal{F}}$. We define the the following sets $P_\tau = \bigcup_{y \in \tau} S^1(L_y)$ and $Q_\tau = \bigcup_{y \in \tau} (L_y \cup S^1(L_y))$.

If $T^1(\tau)$ denotes the unit tangent bundle of $\tilde{\mathcal{F}}$ restricted to $\tau$, then $T^1(\tau)$ is naturally homeomorphic to the standard cylinder. The natural identification between $T^1(\tau)$ and $P_\tau$ induces the topology on $P_\tau$ homeomorphic to the standard annulus. In [Fen02], it is proved that this topology is independent of the particular transversal $\tau$ that is chosen intersecting the same sets of leaves of $\tilde{\mathcal{F}}$. This is because the metrics induced in $S^1(L)$ from the visual metric in any point are Hölder equivalent.

Similarly, $Q_\tau$ has a natural topology homeomorphic to the standard solid cylinder.

The collection of all $P_\tau$ sets over a $\pi_1(\mathcal{M})$-invariant discrete collection of transversals defines a topology on $S^1(\tilde{\mathcal{M}})$. Similarly, the collection of $Q_\tau$ sets over the same collection of transversals defines a topology on $\tilde{\mathcal{M}} \cup S^1(\tilde{\mathcal{M}})$. Deck transformations act by homeomorphisms on both sets. For more details, see [Fen02], [Cal00], or [Cal01].

After the fact, it is easy to see that the topologies described are independent of the specific metric in $\mathcal{M}$ chosen and also work for any Riemannian metric in $\mathcal{M}$.

5. Properties of flow lines

This section describes the behavior of forward rays of flow lines, in particular their asymptotic behavior toward the the boundary at infinity $\bigcup \{S^1(L) | L \in \tilde{\mathcal{F}}\}$. In particular, we will prove that the rays are quasigeodesics in their respective leaves of $\tilde{\mathcal{F}}$. Notice that this is definitely much weaker than saying that full flow lines are quasigeodesics in their respective leaves. We will also show that in some leaves, the forward ideal points are
pairwise distinct and the negative ideal points are also pairwise distinct. In particular, even if the flow foliation is a leafwise quasigeodesic subfoliation of \( \mathcal{F} \), it will not have the funnel property.

We now introduce a family of sets in \( \tilde{\mathcal{M}} \) which will be extremely useful for us.

5.1. The sets \( \mathcal{U} \). Consider an arbitrary point \( x \in \tilde{\mathcal{A}} \subset \tilde{\mathcal{M}} \) and the forward ray from \( x \) which is denoted by

\[
\gamma^+_x = \tilde{\Phi}_{[0,\infty)}(x)
\]

starting at \( x \), and let \( L_x \subset \tilde{\mathcal{A}} \) be the leaf containing \( \gamma^+_x \). Recall that in the attractor \( \mathcal{A} \), the foliation \( \mathcal{F} \) is equal to \( \mathcal{F}^{wu} \), and hence transverse to \( \mathcal{F}^{ws} \). Therefore, the foliations \( \tilde{\mathcal{F}} \) and \( \tilde{\mathcal{F}}^{ws} \) are transversal to each other near \( \tilde{\mathcal{A}} \).

Let \( U \) be a compact rectangle transverse to the flow and with \( x \) in the interior of \( U \). We assume that \( U \) is contained in the foliation boxes of all foliations \( \tilde{\mathcal{F}} \), \( \tilde{\mathcal{F}}^s \), and \( \tilde{\mathcal{F}}^{ws} \) such that \( U \) is made up of a union of stable segments, every one of which intersects the local strong unstable segment through \( x \). Consider the set

\[
\mathcal{U} = \tilde{\Phi}_{[0,\infty)}(U).
\]

The set \( \mathcal{U} \) is a neighborhood of the forward ray \( \tilde{\Phi}_{[0,\infty)}(x) \). We can assume that \( \mathcal{U} \) is homeomorphic to \([-1, 1] \times [-1, 1] \times [0, \infty)\) with \( x = (0, 0, 0) \) and we can define coordinates on \( \mathcal{U} \) such that the following hold.

- \( U \) is identified with \([-1, 1] \times [-1, 1] \times \{0\} \) and points on \( U \) are represented as \( (r, s, 0) \) for \( r, s \in [-1, 1] \). In particular, \( x = (0, 0, 0) \).
- For a point \( y = (r, s, 0) \), \( \tilde{\Phi}_t(y) \) has coordinates \( (r, s, t) \), that is, the ray \( \{(r, s, t) | t \in [0, \infty)\} \) represents the ray \( \Phi_{[0,\infty)}(y) \).
- For a point \( y' = (r', s', t') \in \mathcal{U} \), \( P_{y'} \) denotes the horizontal infinite strip

\[
P_{y'} = \{(r, s, t) | r \in [-1, 1], t \in [0, \infty)\}.
\]

The infinite strip \( P_{y'} \) is contained in the leaf \( L_{y'} \in \tilde{\mathcal{F}} \) which contains \( y' \).
- For a point \( y' = (r, s', t) \in \mathcal{U} \), \( Q_{y'} \) denotes the vertical infinite strip

\[
Q_{y'} = \{(r, s, t) | s \in [-1, 1], t \in [0, \infty)\}.
\]

The infinite strip \( Q_{y'} \) is contained in the leaf \( E_{y'} \in \tilde{\mathcal{F}}^{ws} \) which contains \( y' \).

As \( x = (0, 0, 0) \in \tilde{\mathcal{A}} \), the leaf of \( \tilde{\mathcal{F}} \) through \( x \) is actually the weak unstable leaf of \( \tilde{\Phi} \) through \( x \), and hence \( P_x \) is contained in the \( \tilde{\mathcal{F}}^{wu} \) leaf through \( x \).

The sets \( \mathcal{U} \) will be used throughout this section. Any such particular set \( \mathcal{U} \) is completely determined by the rectangle \( U \).

We can define a projection map \( \Pi : \mathcal{U} \to P_x \) by the formula \( \Pi(y) = S_y \cap P_x \), where \( S_y \) is the one-dimensional leaf of the strong stable foliation \( \tilde{\mathcal{F}}^s \) containing \( y \). This is possible because one can do that in the original rectangle \( U \) as it is a union of strong stable segments, and then \( \mathcal{U} \) is the flow forward saturation of \( U \) and the maps \( \tilde{\Phi}_t \) preserve the strong stable foliation in \( \tilde{\mathcal{M}} \). These projection maps are well defined and continuous because of the foliation structures on \( \mathcal{U} \).
Observation 5.1. Here we list out all of the important observations from the above construction of $\mathcal{U}$ which we need in the rest of the article.

1. For any $y \in \mathcal{U}$, the rays $\tilde{\Phi}_{(0,\infty)}(y)$ and $\tilde{\Phi}_{[0,\infty)}(\Pi(y))$ are asymptotic as they lie on the same weak stable leaf.

2. We can assume that lengths of all the line segments $\{(r, s, 0) | s \in [-1, 1]\}$ are less than a fixed $\epsilon > 0$ in $\tilde{\mathcal{M}}$. Without loss of generality, we assume $\epsilon$ is small enough that for any point $p \in \mathcal{M}$, the $\epsilon$-neighborhood of $p$ is contained in a covering neighborhood of all the foliations $\mathcal{F}, \mathcal{F}^u, \mathcal{F}^s, \mathcal{F}^{wu}, \mathcal{F}^{ws}$.

3. We have considered a Candel metric on the leaves of $\tilde{\mathcal{M}}$ and the Candel metric varies continuously on the leaves transversally.

The line segment $\lambda = \{(0, s, 0) | s \in [-1, 1]\}$ is transversal to $\tilde{\mathcal{F}}$. Consider the open sets $\mathcal{V} = \bigcup_{x' \in \lambda} S^1(L_{x'})$ and $\mathcal{V} = \bigcup_{x' \in \lambda} (L_{x'} \cup S^1(L_{x'}))$.

Definition 5.2. Let $\gamma$ be a flow line of $\tilde{\Phi}$ contained in a leaf $L$ of $\tilde{\mathcal{F}}$. Given $z$ in $\gamma$, if the forward ray of $\gamma$ converges to a single point of $S^1(L)$, we let this be $\eta^+(z)$. Similarly define $\eta^-(z)$. In addition, given a point $a$ in $\tilde{\mathcal{M}}$, let $\gamma_a$ be the flow line of $\tilde{\Phi}$ containing $a$.

At this point, we only know that flow lines on $\tilde{\mathcal{A}} \cup \tilde{\mathcal{R}}$ are leafwise quasigeodesics (by Lemma 2.2). Hence for $x \in \tilde{\mathcal{A}} \cup \tilde{\mathcal{R}}$, both $\eta^+(x)$ and $\eta^-(x)$ are well-defined points in $S^1(L)$. The next lemma shows that every flow ray on $\tilde{\mathcal{M}}$ is leafwise quasigeodesic (in the respective leaf of $\tilde{\mathcal{F}}$), and hence $\eta^+(x)$ and $\eta^-(x)$ are well defined for any arbitrary $x \in \tilde{\mathcal{M}}$. The proof of this lemma is quite involved.

Lemma 5.3. For any $w \in \tilde{\mathcal{M}}$, the forward and the backward rays of the flow line $\gamma_w = \tilde{\Phi}_{\mathcal{R}}(w)$ are quasigeodesics on the leaf $L_w$ in $\tilde{\mathcal{F}}$ which contains the flow line.

Proof. In §4, we proved that $\mathcal{F}$ does not admit any holonomy invariant transverse measure, so by Candel’s theorem, $\mathcal{M}$ admits a Candel metric. To prove the current lemma, we assume a Candel metric in $\mathcal{M}$ so that leaves of $\mathcal{F}$ are hyperbolic surfaces. This metric is not Riemannian, but the result is independent of the metric.

By Lemma 2.2, every forward or backward ray in a leaf in $\mathcal{F}^{wu}$ or $\mathcal{F}^{ws}$ is quasigeodesic in its respective leaf. In particular, every flow line is a quasigeodesic in the respective leaf of $\mathcal{F}$ if contained in the attractor or repeller.

So we may assume that the ray is in a leaf not in the attractor or repeller. Property 2.9 shows that every forward (respectively backward) ray is asymptotic with a ray in the attractor (respectively repeller). We will prove the result for a forward ray and the backward case is similar. Suppose $\gamma$ denotes a forward ray not contained in the attractor. By taking a subray, we may assume that the ray $\gamma$ is in the weak stable leaf of a point $x$ in the attractor and the initial point $w$ of the ray $\gamma$ is very near $x$ and contained in the strong stable segment of $x$. Hence we may assume that the initial point is contained in a local cross section $U$ to $\tilde{\Phi}$ which is a rectangle centered at $x$, as described in the construction of the set $\mathcal{U}$ in the beginning of this section. Let $L_x$ be the leaf of $\tilde{\mathcal{F}}$ containing $x$, and similarly define $L_w$.

Recall that $L_x$ is also the weak unstable leaf of $\tilde{\Phi}$ containing $x$. 

Therefore, it is sufficient to show that every forward ray in the set $\mathcal{U}$ described above is quasigeodesic in its respective leaf of $\tilde{\mathcal{F}}$. In the leaf $L_x$ through $x$, we consider a curve $c$ as follows.

Let $I$ be the compact unstable segment $U \cap L_x$ which has endpoints $z, y$. Let $r_1 = \tilde{\Phi}_{[0,\infty)}(z)$ and $r_2 = \tilde{\Phi}_{[0,\infty)}(y)$ be the forward rays of $\Phi$ through $z$ and $y$. Then $c := r_1 \cup I \cup r_2$ is the bi-infinite curve on $L_x$, as shown in Figure 3.

The two rays $r_1$ and $r_2$ are quasigeodesics in $L_x$ by Lemma 2.2 as $L_x$ is a weak unstable leaf of $\Phi$. Moreover, they converge to distinct ideal points in $S^1(L_x)$. Let $v$ be the interval in $S^1(L_x)$ bounded by these ideal points and containing the ideal point of $\tilde{\Phi}_{[0,\infty)}(x)$.

The curve $c$ bounds a region $A_x$ in $L_x$ (as in Figure 3) which is exactly $\tilde{\Phi}_{[0,\infty)}(I)$. The region $A_x$ is contained in $\mathcal{U}$, in fact,

$$A_x = \tilde{\Phi}_{[0,\infty)}(I) = \mathcal{U} \cap L_x.$$

This region contains a half plane in $L_x$, which we denote as $P_x$, as shown in Figure 3.

Recall that we are considering $w$, a point in $U \cap \tilde{\mathcal{F}}^s(x)$, where $\tilde{\mathcal{F}}^s(x)$ is the strong stable leaf of $x$, in other words, $\Pi(w) = x$ in $\mathcal{U}$ according to the definition of the map $\Pi$ above. Let $J$ be the intersection of $L_w \cap \mathcal{U}$, where $L_w$ is the leaf of $\tilde{\mathcal{F}}$ through $w$. Then $B_w := \tilde{\Phi}_{[0,\infty)}(J)$ is contained in $L_w$ and contained in $\mathcal{U}$. In addition, $\Pi(B_w) = A_x$. 

---

**Figure 3.** The region $A_x$ in $L_x$ and the half-space $P_x$. The region $A_x$ is the region bounded by the curve $c = r_1 \cup I \cup r_2$. 

Since every point in $J$ is in the strong stable leaf of a point in $I$, it follows that every flow ray in $B_w$ is asymptotic to a flow ray in $A_x$ by Observation 5.1(2). In fact, as points leave compact sets in $B_w$, they become closer and closer to $A_x$.

The flow ray $r_x = \tilde{\Phi}_{(0,\infty)}(x)$ is quasigeodesic on $L_x$ by Lemma 2.2 as $L_x$ is a weak unstable leaf of $\tilde{\Phi}$. We want to show that $r_w = \tilde{\Phi}_{(0,\infty)}(w)$ is also quasigeodesic with respect to the induced path metric on $L_w$. The key idea of the proof is as follows: the induced metrics on the leaves $\tilde{F}$ vary continuously and the region $A_x \subset L_x$ is very close to $B_w \subset L_w$. As $r_x$ is quasigeodesic in its leaf and asymptotic to the ray $r_w = \tilde{\Phi}_{(0,\infty)}(w)$, it follows that the other ray is also a quasigeodesic in its $\tilde{F}$ leaf.

Next we provide more specific details. In the leaf $L_x$, choose two points $x_1, x_2$ in $I$ with $x$ in between them so that the geodesic $\beta_x$ in $L_x$ with ideal points $\eta^+(x_1), \eta^+(x_2)$ is contained in the interior of $A_x$. This is possible since the flow lines in $L_x$ are uniform quasigeodesics and they spread out in the forward direction. We stress that, in general, it is not possible to choose $x_1, x_2$ as the endpoints of $\nu$ as the flow lines are only quasigeodesics and not geodesics in $L_x$. Recall that $\nu$ is the interval of $S^1(L_x)$ defined previously. Let $P_x$ be the half plane of $L_x$ bounded by $\beta_x$ and containing a forward ray from $x$. We also may assume that every point in $P_x$ is $\epsilon_1$ close to $L_w$ with $\epsilon_1$ very close to zero. Then $\beta_x$ is $\epsilon_1$ close to a curve $\beta'$ in $L_w$ which has geodesic curvature in $L_w$ very close to zero. To obtain this property of $\beta'$ with small geodesic curvature in $L_w$ was one of the reasons to choose a Candel metric with hyperbolic leaves varying continuously, and hence uniformly continuously, since $M$ is compact. Notice that using this continuity only gives us a curve $\beta'$ with small geodesic curvature, but not necessarily one which has zero geodesic curvature. However, since the induced path metric in $L_w$ is hyperbolic, it now follows that this curve $\beta'$ with very small geodesic curvature is very close in $L_w$ to an actual geodesic in $L_w$. This geodesic is denoted by $\beta_w$. Let $P_w$ be the union of the size-1 neighborhood of $\beta_w$ in $L_w$ and the half plane of $L_w$ which is very close to $P_x$, as shown in Figure 4.

Note that $\Pi^{-1}(r_x) = r_w$. We choose $\epsilon_1$ small enough so that $\Pi^{-1}$ is defined in $P_x$ and $\Pi^{-1}(L_x) \subset P_w$. This is the reason to include a neighborhood of $\beta_w$ in $L_w$.

We will now show that the map $\Pi^{-1} : P_x \rightarrow P_w$ is a quasi-isometry. Using the fact that $P_w$ is very close to $L_x$, continuity of leafwise Riemannian metric on $M$, and the compactness of $M$, we obtain the following: given $\epsilon > 0$, we can consider $\epsilon_1 > 0$ small enough, such that

$$\text{if } d_{L_x}(a, b) \leq 1, \quad \text{for } a, b \in P_x, \quad \text{then } d_{L_w}(\Pi^{-1}(a), \Pi^{-1}(b)) \leq 1 + \epsilon.$$
Next consider \(a_0\) and \(b_0\) on \(P_x\) and let \(\rho\) be the geodesic connecting them on \(P_x\). Partition \(\rho\) in \(n\) subintervals \(a_0, a_1, a_2, \ldots, a_{n+1} = b_0\) such that \(d_{L_x}(a_i, a_{i+1}) = 1\) for all \(0 \leq i < n - 1\) and \(0 < d_{L_x}(a_n, b_0) \leq 1\). As \(d(a_i, a_{i+1}) \leq 1\) for all \(i\),

\[
d_{L_w}(\Pi^{-1}(a_0), \Pi^{-1}(b_0)) \leq \sum_{i=0}^{n} d_{L_w}(\Pi^{-1}(a_i), \Pi^{-1}(a_{i+1})) \leq (n + 1)(1 + \epsilon) = n(1 + \epsilon) + (1 + \epsilon).
\]

By construction, \(n < d_{L_x}(a_0, b_0)\). Hence, we conclude

\[
d_{L_w}(\Pi^{-1}(a_0), \Pi^{-1}(b_0)) \leq d_{L_x}(a_0, b_0)C_0 + C_0,
\]

where \(C_0 = (1 + \epsilon)\) is a fixed constant. Similarly, if \(a_0, b_0\) are in \(\Pi^{-1}(P_x)\), we get that \(d_{L_x}(\Pi(a_0), \Pi(b_0)) < C_1d_{L_w}(a_0, b_0) + C_2\) for some globally fixed constants \(C_1, C_2\). Since \(\Pi^{-1}(P_x)\) is \(2\)-dense in \(P_w\), it follows that \(\Pi^{-1}\) is a quasi-isometry from \(P_x\) to \(P_w\).

Finally, as we know that \(r_x\) is a quasigeodesic on \(P_x\), then its image via the quasi-isometry \(\Pi^{-1}\), \(r_w = \Pi^{-1}(r_x)\) is a quasigeodesic on \(P_w \subset L_w\) with respect to the metric \(d_{L_w}\). Since \(P_w\) is a quasi-isometrically embedded in \(L_w\), it now follows that \(r_w\) is a quasigeodesic in \(L_w\).

If we reverse the flow, every backward ray becomes a forward ray, and hence leafwise quasigeodesic.

This finally finishes the proof of Lemma 5.3. \(\square\)

By compactness and continuity, there is global \(K_0, s_0 > 0\) so that given any flow line \(\gamma\), there is a forward ray \(\gamma^+\) and a backward ray \(\gamma^-\) of \(\gamma\) which are \((K_0, s_0)\) quasigeodesics in leaf \(L_{\gamma}\), the leaf of \(\hat{\mathcal{F}}\) containing \(\gamma\). Note that all the leaves \(L \in \hat{\mathcal{F}}\) are hyperbolic and we can define their boundary at infinity \(S^1(L)\). As the flow rays \(\gamma^+\) and \(\gamma^-\) are quasigeodesics on \(F_{\gamma}\), they define unique points on the ideal boundary \(S^1(L_{\gamma})\). Hence for all \(a \in \gamma \subset \hat{\mathcal{M}}\), the forward subray \(\gamma_a^+\) limits on a single point in \(S^1(L_{\gamma})\) and \(\eta^+(a)\) is well defined as in Definition 5.2. Similarly, \(\eta^-(a)\) is also well defined by the backward subray \(\gamma_a^-\).

In the next proposition, we consider the sets \(P_y\) contained in \(\mathcal{U}\).

**Proposition 5.4.** Suppose \(a, b \in P_y \subset L_y\) but \(\gamma_a \neq \gamma_b\), then \(\eta^+(a) \neq \eta^+(b)\) in \(S^1(L_y)\).

**Proof.** By the previous Lemma 5.3, we already know that all rays are quasigeodesics in their respective leaves. We do the proof by contradiction and assume that \(\eta^+(a) = \eta^+(b)\) on \(S^1(L_y)\). Since the rays \(\tilde{\Phi}_{[0,\infty)}(a), \tilde{\Phi}_{[0,\infty)}(b)\) are quasigeodesics in \(L_y\) and by assumption they have the same ideal point in \(S^1(L_y)\), the following happens: there is \(d_0 > 0\) and points \(p_i, q_i\) in \(\tilde{\Phi}_{[0,\infty)}(a), \tilde{\Phi}_{[0,\infty)}(b)\) respectively, escaping in the rays so that \(d_{L_y}(p_i, q_i) < d_0\). Consider the points \(\Pi(a)\) and \(\Pi(b)\) on \(P_x\). Since

\[
\tilde{\Phi}_{[0,\infty)}(a), \tilde{\Phi}_{[0,\infty)}(\Pi(a))
\]

are asymptotic in the weak stable leaf of \(\tilde{\Phi}\) in \(\hat{\mathcal{M}}\), there are \(p_i'\) in \(\tilde{\Phi}_{[0,\infty)}(\Pi(a))\) with \(d(p_i, p_i') \to 0\). Here \(d\) is the ambient distance in \(\hat{\mathcal{M}}\). Similarly, there are \(q_i'\) in \(\tilde{\Phi}_{[0,\infty)}(\Pi(b))\) with \(d(q_i, q_i') \to 0\). By the local product structure of the foliation \(\mathcal{F}\), it follows that \(d_{L_x}(p_i', q_i') < d_0 + 1\) for \(i\) sufficiently big.
We explain this in more detail. We choose a finite cover of $\mathcal{M}$ by foliated boxes of $\mathcal{F}$, each of which contains a ball of radius $2/m$, where $m$ is a fixed integer. Any disk in a leaf of $\mathcal{F}$ which has diameter less than $1/m$ which is product foliated and any path in the disk is approximated by a path in another leaf with length very close to the length of the original path. Let $n$ be the smallest positive integer bigger than $d_0$. Using compactness of $\mathcal{M}$, it follows that any connected union of at most $nm$ such disks in a leaf has a transversal neighborhood of fixed size which is product foliated and has the property above on lengths of paths. Therefore, the paths from $p_i$ to $q_i$ in $L_y$ can be approximated by paths from $p_i'$ to $q_i'$ in $L_x$ with length very close, resulting in $d_{L_x}(p_i', q_i') < d_0 + 1$ for $i$ sufficiently big.

Therefore, the rays $\Phi_{[0,\infty)}(\Pi(a)), \Phi_{[0,\infty)}(\Pi(b))$ converge to the same ideal point in $S^1(L_x)$. However, $L_x$ is also a weak unstable leaf of $\tilde{F}$ and as the flow lines $\Phi_{\mathbb{R}}(\Pi(a))$ and $\Phi_{\mathbb{R}}(\Pi(b))$ are distinct flow lines in $L_x$, by the description of ideal points of flow lines in weak unstable leaves as in Property 2.4, the forward limit points are distinct, that is,

$$\eta^+(\Pi(a)) \neq \eta^+(\Pi(b)) \text{ in } S^1(L_x).$$

This is a contradiction and shows that $\eta^+(a) \neq \eta^+(b)$ in $S^1(L_y)$. \qed

**Lemma 5.5.** In each leaf $L$ of $\tilde{F}$, the leaf space of the flow foliation is Hausdorff and homeomorphic to the real line $\mathbb{R}$.\hfill\vspace{0.2cm}

**Proof.** For the leaves of $\tilde{F}$ in lifts $\tilde{A}$ and $\tilde{R}$ of the attractor and repeller, the result is obvious, since the foliation by flow lines satisfies this property in weak stable and weak unstable leaves of Anosov flows [Fen94]. Any other leaf $L$ of $\tilde{F}$ is the lift of a leaf of $\mathcal{F}$ which intersects a torus $T$ from the collection of tori $\{T_i\}$ which separates $A$ and $R$. Hence $L$ intersects a lift $\tilde{T}$ of $T$ in a curve $\beta$. The flow saturation of $\beta$ is exactly $L$, since every flow line in $M$ is either in the attractor or repeller; or intersects a torus in $\{T_i\}$. The curve $\beta$ is transverse to the weak stable and weak unstable foliations, and hence intersects a flow line exactly once. Hence $\beta$ parameterizes the flowlines in $L$. This proves the result. \hfill\vspace{0.2cm}

For each $L$ of $\tilde{F}$, the map $\eta^+$ induces a map from the flow foliation leaf space in $L$ (which is $\cong \mathbb{R}$) to $S^1(L)$. Since flow lines are disjoint, this map is weakly monotone.

**Corollary 5.6.** For all $y \in \mathcal{U}$, $\eta^+(y) \neq \eta^-(y)$ in $S^1(L_y)$.

**Proof.** If $\eta^+(y) = \eta^-(y)$, then the flow line $\gamma_y$ bounds a disk $D$ on $L_y \cup S^1(L_y)$ such that the closure of $D$ in $L \cup S^1(L)$ intersects $S^1(L)$ only in $\eta^+(y) = \eta^-(y)$. For any $z$ in the interior of $D$, the flow line $\gamma_z$ is contained in $D$, and hence $\eta^+(z) = \eta^-(z) = \eta^+(y) = \eta^-(y)$. This contradicts Proposition 5.4, because if $z, y \in \mathcal{U}$, and $\gamma_z \neq \gamma_y$, then $\eta^+(z) \neq \eta^-(y)$. \hfill\vspace{0.2cm}

We now extend the map $\eta^+$ to a map from $\tilde{\mathcal{M}}$ to $S^1(\tilde{\mathcal{M}})$. For each $x$ in $\tilde{\mathcal{M}}$, $\eta^+(x)$ is in $S^1(L_x) \subset S^1(\tilde{\mathcal{M}})$.

**Proposition 5.7.** $\eta^+$ and $\eta^-$ are continuous on $\tilde{\mathcal{M}}$.\

Proof. In this proof, we again use a Candel metric in $M$.

Suppose $x_i \to x_0$ in $\tilde{M}$. We will show that $\eta^+(x_i) \to \eta^+(x_0)$ in $S^1(\tilde{M})$. There are two different cases depending on whether $x_0 \in \tilde{R}$ or $x_0 \notin \tilde{R}$.

We first prove the result for $x_0 \notin \tilde{R}$. As $x_0 \notin \tilde{R}$, the forward ray starting at $x_0$ is asymptotic to a forward flow ray in $\tilde{A}$. Therefore, it is enough to assume that $\{x_i\}$ and $x_0$ belong to a neighborhood $U$ as constructed above, since this is true for every ray asymptotic to $\tilde{A}$.

For $z$ in $\tilde{M}$, let $L_z$ be the leaf of $\tilde{F}$ containing $z$.

For $i \in \mathbb{N} \cup \{0\}$, let $\gamma_i^+$ denote the forward flow ray starting from $x_i$ and let $\zeta_i$ denote the geodesic ray on $L_{x_i}$ starting at $x_i$ and with ideal point $\eta^+(x_i)$ in $S^1(L_{x_i})$. Each $\zeta_i$ defines the ideal point $\eta^+(x_i)$ on $S^1(L_{x_i})$, therefore it is enough to show that any convergent subsequence of $(\zeta_i)$ converges to $\zeta_0$ in the compact open topology. Since all $x_i$ are contained in a compact subset of $\tilde{M}$, existence of convergent subsequences of $\{\zeta_i\}$ is assured.

Suppose that a subsequence $(\zeta_i(k))$ converges to $\zeta'$. We have to prove that $\zeta' = \zeta_0$. We assume that the neighborhood $\mathcal{U}$ constructed above has a point $x \in \tilde{A}$, as in the construction of $\mathcal{U}$. Then all flow rays in $L_x \cap \mathcal{U}$ are $(K, s)$-quasigeodesics in $L_x$ for some fixed $K, s$. Since all flow rays in $\mathcal{U}$ are forward asymptotic to flow rays in $L_x$, there are $K', s'$ so that all flow rays in $\mathcal{U}$ are $(K', s')$-quasigeodesics in their respective $\tilde{F}$ leaves. It follows that there exists a constant $d' > 0$ such that

$$\gamma_i^+(k) \subset N_{d'}(\zeta_i(k)) \quad \text{and} \quad \gamma_0^+ \subset N_{d'}(\zeta_0),$$

where $N_{d'}$ denotes the neighborhood of radius $d$ in the respective leaf of $\tilde{F}$. For any $d_1 > 0$, the segment of length $d_1$ on $\gamma_i^+(k)$ starting at $x_i(k)$ is within $d'$-distance from $\zeta_i(k)$. Therefore in the limit, the segment of $\gamma_0^+$ of length $d_1$ starting from $x_0$ is contained in $N_{d'}(\zeta')$ in the respective leaf. This is true for all $d_1$, so $\zeta'$ is at Hausdorff distance $d'$ from $\gamma_0^+$ on $L_{x_0}$. However, $\gamma_0^+$ is also at a bounded distance from $\zeta_0$ on $L_{x_0}$; therefore, $\zeta'$ and $\zeta_0$ are at a finite Hausdorff distance from each other on $L_{x_0}$. Hence $\zeta' = \zeta_0$, because they have the same starting point. As this is true for all convergent subsequences of $(\zeta_i)$, we get our result for $x_0$ not in $\tilde{R}$.

Before dealing with the remaining case, let us note the following.

Observation 5.8. By the construction of $\mathcal{U}$ starting with $x$ in $\tilde{A}$ and continuity of $\eta^+$ near $\tilde{A}$, we observe that the set $\mathcal{U} \cup \{\eta^+(z) | z \in \mathcal{U}\}$ is homeomorphic to $[0, 1] \times [0, 1] \times [0, 1]$ inside $\mathcal{W} = \bigcup_{y \in \lambda} (L_y \cup S^1(L_y))$, which is homeomorphic to a compact solid cylinder $[0, 1] \times \{\text{theclosedunitdisc}\}$.

The set $\mathcal{U} \cup \{\eta^+(z) | z \in \mathcal{U}\}$ above is saturated by forward flow lines and all the ideal points contained in this neighborhood are defined by forward flow rays. Hence, we conclude the following.

1. If $L \in \tilde{F} \setminus \tilde{A}$ and $p \in L$, then there exists a neighborhood $N_p$ of $\eta^+(p)$ in $\bigcup_{L \in \tilde{F}} S^1(L)$ such that $N_p \subset \eta^+(\tilde{M})$ and $N_p \cap \eta^-(\tilde{M}) = \emptyset$.

2. Similarly, for $q \in L_q \in \tilde{R}$, there exists a neighborhood $N_q$ of $\eta^-(q)$ in $\bigcup_{L \in \tilde{F}} S^1(L)$ such that $N_q \subset \eta^-(\tilde{M})$ and $N_q \cap \eta^+(\tilde{M}) = \emptyset$. Moreover, the backward ray $\gamma_q^-$ starting from $q$ is contained in an infinite cubical neighborhood in $\tilde{M}$ saturated by backward flow rays.
To continue the proof of Proposition 5.7, we next assume that \( x_0 \in \tilde{\mathcal{R}} \). Suppose that a subsequence \((\eta^+(x_{i(k)}))\) converges to \( q \) where \( q \) is not \( \eta^+(x_0) \). As \( x_0 \) is in \( \tilde{\mathcal{R}} \), then \( L_{x_0} \) is a leaf of the weak stable foliation \( \mathcal{F}^w \). Hence by Property 2.4 on \( L_{x_0} \), all the forward flow rays converge to a single ideal point in \( S^1(L_{x_0}) \) and all the other ideal points in \( S^1(L_{x_0}) \) are ideal points of backward flow rays. As \( q \neq \eta^+(x_0) \), \( q \) is defined by a backward ray, that is, \( q = \eta^-(z) \) for some \( z \in L_{x_0} \). By Observation 5.8(2) starting with \( z \) in \( \tilde{\mathcal{R}} \) (notice that \( z \) is in the repeller, not the attractor), there exits a neighborhood \( \mathcal{V} \) saturated by backward flow rays around \( z \) in \( \bigcup \{L_y \cup S^1(L_y) \mid y \in \kappa' \} \) for some transversal \( \kappa' \). By Observation 5.8(2), all limit points are backward ideal points in \( \mathcal{V} \) and no limit point is a forward ideal point. This contradicts the fact that the forward rays \( y_{i(k)}^+ \) have ideal points in these intervals of ideal points for \( k \) big enough by construction. This contradiction shows that a subsequence \((\eta^+(x_{i(k)}))\) converging to \( q \neq \eta^+(x_0) \) is not possible, and hence \( q = \eta^+(x_0) \).

Hence \( \eta^+ \) is continuous on \( \tilde{\mathcal{M}} \). If we consider the flow \( \Psi_t = \Phi_{-t} \), then backward ideal points of \( \Phi_t \) are forward ideal points of \( \Psi_{-t} \) and the continuity of \( \eta^- \) follows. This completes the proof of Proposition 5.7.

\[ \square \]

6. Flow lines are leafwise quasigeodesic

In this section, we prove a general result of quasigeodesic behavior of some subfoliations. This result will imply that in the examples we constructed associated with non-transitive Anosov flows, the flow lines are uniform quasigeodesics in their respective two-dimensional leaves. We first consider some general continuity properties.

**Definition 6.1.** (Continuity properties) Let \( \mathcal{G} \) be a one-dimensional oriented subfoliation of a two-dimensional foliation \( \mathcal{F} \) with Gromov hyperbolic leaves on a 3-manifold \( \mathcal{M} \). Suppose that leaves of \( \mathcal{G} \) are \( C^1 \) curves in leaves of \( \mathcal{F} \). Suppose that the following three properties are satisfied.

1. For each \( x \) in \( \tilde{\mathcal{M}} \), let \( \ell \) be the leaf of \( \tilde{\mathcal{G}} \) containing it, and \( L \) the leaf of \( \tilde{\mathcal{F}} \) containing \( x \). Then in the forward direction (given by the orientation of \( \tilde{\mathcal{G}} \)), the leaf \( \ell \) has a unique limiting point in \( S^1(L) \) and this is denoted by \( \eta^+(x) \). Similarly, in the negative direction, there is a unique limiting point in \( S^1(L) \) denoted by \( \eta^-(x) \).
2. For each \( x \) in \( \tilde{\mathcal{M}} \), the points \( \eta^+(x) \), \( \eta^-(x) \) are distinct points in \( S^1(L) \) \( L \) the leaf of \( \tilde{\mathcal{F}} \) containing \( x \).
3. The functions \( \eta^+, \eta^- : \tilde{\mathcal{M}} \to \bigcup_{L \in \tilde{\mathcal{F}}} S^1(L) \) are continuous.

Then we say that \((\mathcal{F}, \mathcal{G})\) has the continuity properties.

From the foliation \( \mathcal{G} \), we can produce a flow with flow lines which are the leaves of \( \mathcal{G} \): for example, just flow forward along leaves of \( \mathcal{G} \) with speed 1 in the positive direction. Any reparameterization of the flow produces a time parameter which is quasi-isometric with this one, so the result on the quasigeodesic behavior of flow lines depends only on \( \mathcal{G} \) and not the particular parameterization, or description of \( \mathcal{G} \) as the flow foliation of a flow.

Notice that the two-dimensional foliation \( \mathcal{F} \) constructed in §3 with the one-dimensional subfoliation \( \mathcal{G} \) by the flow lines of \( \Phi_t \) satisfies the continuity properties as follows. In the previous section, we proved the pair \((\mathcal{F}, \mathcal{G})\) satisfies the properties (1), (2), and (3).
of Definition 6.1: property (1) was proved in Lemma 5.3, property (2) was proved in Corollary 5.6, and property (3) was proved in Proposition 5.7.

The next result is a general result that will imply that the foliations we constructed in §3 are leafwise quasigeodesic foliations.

**THEOREM 6.2.** Suppose that \( \tilde{\mathcal{G}} \) is a one-dimensional subfoliation of a two-dimensional foliation \( \tilde{\mathcal{F}} \) satisfying the continuity properties of Definition 6.1. Then \( \tilde{\mathcal{G}} \) is a leafwise quasigeodesic foliation.

The proof will be attained by the following three results. In the next lemma, we combine all the results of the previous section to obtain a key property that will be used to show that all the flow lines are quasigeodesic on their respective leaves of \( \tilde{\mathcal{F}} \).

We stress that the quasigeodesic behavior is proved using only the continuity properties, irrespective of how these continuity properties are obtained. Therefore, Theorem 6.2 is applicable not only to the examples constructed in §3, but theoretically to many other situations as well.

To prove Theorem 6.2, again we use a Candel metric. Given \( x \) in \( \tilde{\mathcal{M}} \), let \( \gamma_x \) be the leaf of \( \tilde{\mathcal{G}} \) through it. In addition, let \( L_x \) be the leaf of \( \tilde{\mathcal{F}} \) containing \( x \). Using property (i) of Definition 6.1, we let \( \eta^+(x), \eta^-(x) \) be the unique limiting points of the two rays of \( \gamma_x \) in \( S^1(L_x) \). Notice that they are distinct points in \( S^1(L_x) \) by property (ii) of Definition 6.1. Since \( L_x \) has a hyperbolic metric, there is a unique geodesic in \( L_x \), denoted by \( g_x \), whose ideal points in \( S^1(L_x) \) are \( \eta^+(x), \eta^-(x) \).

**LEMMA 6.3.** There exists \( \delta > 0 \) such that for all \( x \in \tilde{\mathcal{M}} \), we have that
\[
\gamma_x \subset N_\delta(g_x),
\]
where \( g_x \) is the geodesic on \( L_x \) connecting \( \eta^+(x) \) and \( \eta^-(x) \) and \( N_\delta(g_x) \) is the \( \delta \)-neighborhood of \( g_x \) on \( L_x \).

**Proof.** Suppose that there does not exist any such \( \delta \). Then there exists a sequence \( (x_i) \) in \( \tilde{\mathcal{M}} \) with \( x_i \) in leaves \( L_{x_i} \) of \( \tilde{\mathcal{F}} \) such that \( d_{L_{x_i}}(x_i, g_{x_i}) > i \). Up to deck transformations, there exists a convergent subsequence of \( (x_i) \) which we assume is the original sequence, and we assume \( x_i \to x \). By property (iii) of Definition 6.1, we know that
\[
\eta^+(x_i) \to \eta^+(x) \quad \text{and} \quad \eta^-(x_i) \to \eta^-(x).
\]
Since \( x_i \) converges to \( x \), we assume that all \( x_i \) are in leaves of \( \tilde{\mathcal{F}} \) which intersect a fixed transversal \( \lambda \) to \( \tilde{\mathcal{F}} \).

Since \( \eta^+(x_i) \) converges to \( \eta^+(x) \), \( \eta^-(x_i) \) converges to \( \eta^-(x) \), and \( \eta^+(x) \neq \eta^-(x) \), it follows that \( \{g_{x_i}\} \) converges to \( g_x \). This uses that the topology defined on \( \bigcup_{y \in \lambda} (S^1(L_y)) \) is given by the trivialization of the unit tangent bundle to \( \tilde{\mathcal{F}} \) along \( \lambda \). By convergence we mean convergence in the compact open topology. However, this contradicts that \( d_{L_{x_i}}(x_i, g_{x_i}) \) converges to infinity, since \( d_{L_x}(x, g_x) \) is finite and the sequence \( d_{L_{x_i}}(x_i, g_{x_i}) \) converges to it. This finishes the proof.

We now prove a weak quasigeodesic property of the leaves of \( \tilde{\mathcal{G}} \) in the leaves of \( \tilde{\mathcal{F}} \) containing them.
PROPOSITION 6.4. For any \( b > 0 \), there exists a \( c_b > 0 \) such that if \( γ \) is the segment in a leaf of \( \tilde{G} \) connecting \( x \) and \( y \) with \( \text{length}(γ) > c_b \), then \( d_{L_x}(x, y) > b \), where \( L_x \) is the leaf of \( \tilde{F} \) which contains \( x \).

Proof. Fix \( b > 0 \). We do the proof by contradiction. Suppose the statement is not true for some \( b > 0 \). Then for all \( i ∈ \mathbb{N} \), there exists two points \( x_i \) and \( y_i \) in leaves \( L_i \) of \( \tilde{F} \), with \( x_i, y_i \) in the same flow line defining a flow line segment \( γ_i \) satisfying \( \text{length}(γ_i) > i \) but \( d_{L_i}(x_i, y_i) < b \). Up to deck transformations and a subsequence, we assume that \( (x_i) \) is convergent and \( x_i \to x_0 \). Since \( d_{L_i}(x_i, y_i) < b \), we can similarly assume that \( (y_i) \) is convergent and let \( y_i \to y_0 \).

CLAIM 2. \( x_0 \) and \( y_0 \) are on the same leaf \( L_0 \) of \( \tilde{F} \).

Proof. If we consider a compact ball \( B_{x_0} \) on \( L_0 \) containing \( x_0 \) and a product neighborhood \( N(B_{x_0}) \) with respect to \( \tilde{F} \), then for all large \( i \), \( L_i \) intersects \( N(B_{x_0}) \) and \( x_i \in L_i \cap N(B_{x_0}) \). If we consider \( B_{x_0} \) sufficiently large, the assumption \( d_{L_i}(x_i, y_i) < b \) for all \( i \) forces that \( y_i \) has to be contained in \( N(B_{x_0}) \). Hence by the product structure on \( N(B_{x_0}) \), \( y_0 \) also has to lie on \( L_0 \) as \( y_i \to y_0 \).

CLAIM 3. \( x_0 \) and \( y_0 \) cannot be on the same flow line in \( L_0 \).

Proof. If not, there exists a flow line segment \( γ \) connecting \( x_0 \) and \( y_0 \) and consider a compact neighborhood \( N \) around \( γ \) which has a product structure with respect to the flow lines. As \( x_i \to x_0 \) and \( y_i \to y_0 \), the flow segments \( γ_i \) are contained in \( N \) for all large \( i \). By continuity of length of flow lines, \( \text{length}(γ_i) \to \text{length}(γ) \). However, that is not possible as \( \text{length}(γ_i) \to \infty \) and \( γ \) is compact, a contradiction.

CLAIM 4. \( x_0 \) and \( y_0 \) cannot be connected by a curve on \( L_0 \) everywhere transversal to the flow lines in \( L_0 \).

Proof. Suppose that there exists a line segment \( σ \) on \( L_0 \) everywhere transversal to the flow lines on \( L_0 \) and connecting \( x_0 \) and \( y_0 \). By the local product structure of \( \tilde{F} \) near \( σ ∈ L_0 \), there should be a segment \( σ_i \) in \( L_i \) connecting \( x_i \) and \( y_i \), and everywhere transversal to flow lines on \( L_i \). Up to taking a sub-segment of \( γ_i \) if necessary and then a sub-segment of \( σ_i \), we may assume that \( γ_i \) does not intersect the interior of \( σ_i \). It follows that the union of \( σ_i \) and \( γ_i \) bounds a disk \( D_i \) on \( L_i \) as their end points are the same. All the flow lines which enter \( D_i \) transversally intersecting \( σ_i \) have to exit \( D_i \) transversally intersecting \( σ_i \). The Poincaré–Hopf theorem says that there exists at least one flow line tangent to \( σ_i \), a contradiction.

By Lemma 5.5, the leaf space of the flow foliation in \( L_0 \) is homeomorphic to the reals. Hence any two distinct flow lines in \( L_0 \) are connected by a transversal.

This contradiction proves Proposition 6.4.

Now we are ready to prove our final claim.
PROPOSITION 6.5. The leaves of $\tilde{G}$ are uniformly quasigeodesics in their respective leaves of $\tilde{F}$.

Proof. We prove the theorem by contradiction. Recall that we are using a Candel metric in leaves of $F$. We assume that the leaves of $\tilde{G}$ are not uniform quasigeodesic on their leaves. From this assumption, we will construct a sequence of pairs $\{(x_i, y_i)\}$ such that $x_i$ and $y_i$ are connected by a flow segment $\gamma_i$, where $\text{length}(\gamma_i) \to \infty$ but $d_{L_i}(x_i, y_i)$ is bounded. Here $L_i$ is the $\tilde{F}$ leaf containing both $x_i, y_i$. However, this will contradict Proposition 6.4.

A very similar result was proved in [FM01], we reconstruct the same arguments in our specific case.

By our assumption, the leaves of $\tilde{G}$ are not uniform quasigeodesics. We get that for any $K > 0$, there exists a segment of a leaf of $\tilde{G}$ with endpoints $x, y$ denoted by $\gamma[x, y]$, contained in a leaf of $\tilde{F}$ denoted by $L_x$, and such that

$$\text{length}(\gamma[x, y])/d_{L_x}(x, y) > 2K \text{ and length}(\gamma[x, y]) > K.$$ 

For each $K$, one can find such $x, y, L_x$, which obviously depend on $K$, but we omit the explicit dependence on $K$ for notational simplicity. Consider the geodesic $g_x = g_y$ on $L_x$ with ideal points

$\eta^+(x) = \eta^+(y) \quad \text{and} \quad \eta^-(x) = \eta^-(y) \quad \text{or} \quad S^1(L_x).$

By Lemma 6.3, there exists $\delta > 0$ such that $\gamma \subset N_\delta(g_x)$, where the neighborhood is in $L_x$. This $\delta$ is global, it works for any segment in a leaf of $\tilde{G}$ in its respective leaf of $\tilde{F}$. Let $\rho : L_x \to g_x$ be the 'closest point map projection', which means $\rho(p)$ is the orthogonal projection in $L_x$ to $g_x$, a bi-infinite length-minimizing geodesic on $L_x$. The map is well defined as the leaves $L_x$ are of constant curvature $-1$ and so isometric to the hyperbolic plane: the 'closest point map' on to a length-minimizing geodesic is well defined in the hyperbolic plane. It follows that:

$$d_{L_x}(\rho(x), \rho(y)) \leq d_{L_x}(x, y) \leq d_{L_x}(\rho(x), \rho(y)) + 2\delta. \quad (6.1)$$

Let us assume that $d_{L_x}(x, y) > 1 + 2\delta$. Hence, $d_{L_x}(\rho(x), \rho(y)) > 1$ by equation (6.1) and

$$\frac{\text{length}(\gamma[x, y])}{d_{L_x}(\rho(x), \rho(y))} \geq \frac{\text{length}(\gamma[x, y])}{d_{L_x}(x, y)} \geq 2K > K + \frac{K}{d_{L_x}(\rho(x), \rho(y))}.$$ 

Therefore,

$$\frac{\text{length}(\gamma[x, y])}{K} > d_{L_x}(\rho(x), \rho(y)) + 1 > [d_{L_x}(\rho(x), \rho(y))],$$

where $[a]$ denotes the integer $n$ such that $n - 1 < a \leq n$.

Suppose $n_0 = [d_{L_x}(\rho(x), \rho(y))]$, then $\text{length}(\gamma[x, y]) > n_0K$. Also,

$$n_0 - 1 < [d_{L_x}(\rho(x), \rho(y))] \leq n_0,$$

and hence we can construct a sequence $\{\rho(x) = z_0, z_1, \ldots, z_n = \rho(y)\}$ of points in $g_x$, such that $d_{L_x}(z_{i-1}, z_i) = 1$ for all $i < n_0$ and $d_{L_x}(z_{n_0-1}, z_{n_0}) \leq 1$. Next we consider the sequence $x = x_0, x_1, \ldots, x_{n_0}$, where $x_i$ is the last point on $\gamma[x, y]$ such that $\rho(x_i) = z_i$. 


If \( \gamma_i \) denotes the flow segment joining \( x_{i-1} \) and \( x_i \), we have \( \gamma_{[x,y]} = \gamma_1 \ast \gamma_2 \ast \cdots \ast \gamma_{n_0} \). Hence,

\[
\sum_{n=1}^{n_0} \text{length}(\gamma_i) = \text{length}(\gamma_{[x,y]}) > n_0K.
\]

By the pigeonhole principle, there exists \( x_{i-1} \) and \( x_i \) such that \( \text{length}(\gamma_{[x_{i-1},x_i]}) > K \). However, from \((*)\), we get that for all \( i \),

\[
d_{L_\chi}(x_{i-1}, x_i) \leq d_{L_\chi}(\rho(x_{i-1}, x_i)) + 2\delta = d_{L_\chi}(z_{i-1}, z_i)) + 2\delta < 1 + 2\delta.
\]

As the choice of \( K > 0 \) was arbitrary, this proves that the ‘weak quasigeodesic property’ in Lemma 6.4 is not true for \( b = 1 + 2\delta \), a contradiction. We conclude that leaves of \( \tilde{\mathcal{G}} \) are uniformly quasigeodesic on their respective leaves of \( \tilde{\mathcal{F}} \).

This finishes the proof of Proposition 6.5.

7. Conclusion

We now apply the results of this section to the two-dimensional foliation \( \mathcal{F} \) with a subfoliation \( \mathcal{G} \) as constructed in §3. Section 4 shows that every leaf in \( \mathcal{F} \) is Gromov hyperbolic when lifted to the universal cover. Proposition 6.5 proves that the flow foliation (that is, the foliation \( \mathcal{G} \)) is a leafwise quasigeodesic subfoliation of \( \mathcal{F} \). Moreover, Proposition 5.4 proves that all leaves of \( \mathcal{F} \) which are not contained in \( \mathcal{A} \) or \( \mathcal{R} \) are non-funnel. This is because of the following: if \( \gamma_a, \gamma_b \) are distinct flow lines in some leaf \( L \) of \( \tilde{\mathcal{F}} \) which are not in the lift of the attractor or the repeller, then Proposition 5.4 shows that \( \eta^+(a) \neq \eta^+(b) \) in \( S^1(L) \). Applying the same result to negative flow rays, one obtains that \( \eta^-(a) \neq \eta^-(b) \) in \( S^1(L) \). Hence \( L \) cannot be a funnel leaf. However, all leaves in \( \mathcal{A} \) or \( \mathcal{R} \) are funnel by Corollary 2.6. This completes the proof of the Theorem 1.1.

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REFERENCES

[Ano63] D. V. Anosov. Ergodic properties of geodesic flows on closed Riemannian manifolds of negative curvature. Dokl. Akad. Nauk SSSR 151 (1963), 1250–1252.

[AS67] D. V. Anosov and J. G. Sinaï. Certain smooth ergodic systems. Uspekhi Mat. Nauk 22 (5(137)) (1967), 107–172.

[BBY17] F. Béguin, C. Bonatti and B. Yu. Building Anosov flow on 3-manifolds. Geom. Topol. 21 (2017), 1837–1930.

[BFP20] T. Barthelmé, S. Fenley and R. Potrie. Collapsed Anosov flows and self orbit equivalences. Preprint, 2022, arXiv:2008.0654. Comment. Math. Helv., to appear.

[Bru93] M. Brunella. Separating the basic sets of a nontransitive Anosov flow. Bull. Lond. Math. Soc. 25(5) (1993), 487–490.

[Cal00] D. Calegari. The geometry of R-covered foliations. Geom. Topol. 4 (2000), 457–515.

[Cal01] D. Calegari. Foliations and the Geometry of 3-Manifolds (Oxford Mathematical Monographs). Oxford University Press, Oxford, 2007.

[Cal01b] D. Calegari. Leafwise smoothing laminations. Algebr. Geom. Topol. 1 (2001), 579–585.
A. Candel. Uniformization of surface laminations. *Ann. Sci. Éc. Norm. Supér. (4)* 26(4) (1993), 489–516.

| Ref. | Author(s) | Title | Journal | Volume | Issue | Pages |
|------|-----------|-------|---------|--------|-------|-------|
| [Can93] | A. Candel | Uniformization of surface laminations. | *Ann. Sci. Éc. Norm. Supér. (4)* | 26(4) | | 489–516 |
| [CC00] | A. Candel and L. Conlon | Foliations. I (Graduate Studies Mathematics, 23). | American Mathematical Society, Providence, RI, 2000. |
| [Fen94] | S. R. Fenley | Anosov flows in 3-manifolds. | *Ann. of Math. (2)* | 139(1) | | 79–115 |
| [Fen02] | S. R. Fenley | Foliations, topology and geometry of 3-manifolds: R-covered foliations and transverse pseudo-Anosov flows. | *Comment. Math. Helv.* | 77(3) | | 415–490 |
| [FM01] | S. Fenley and L. Mosher | Quasigeodesic flows in hyperbolic 3-manifolds. | *Topology* | 40(3) | | 503–537 |
| [FW80] | J. Franks and B. Williams | Anomalous Anosov flows. | *Global Theory of Dynamical Systems (Proceedings of an International Conference Held at Northwestern University, Evanston, Illinois, June 18-22, 1979)* (Lecture Notes in Mathematics, 819). Springer, Berlin, 1980, pp. 158–174. |
| [Gro87] | M. Gromov | Hyperbolic groups. | *Essays in Group Theory (Mathematical Sciences Research Institute Publications, 8)*. Springer, New York, 1987, pp. 75–263. |
| [IM90] | T. Inaba and S. Matsumoto | Nonsingular expansive flows on 3-manifolds and foliations with circle prong singularities. | *Jpn. J. Math. (N.S.)* | 16(2) | | 329–340 |
| [KH95] | A. Katok and B. Hasselblatt | Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and Its Applications, 54). Cambridge University Press, Cambridge, 1995. With a supplementary chapter by A. Katok and L. Mendoza. |
| [Pat93] | M. Paternain | Expansive flows and the fundamental group. | *Bol. Soc. Brasil. Mat. (N.S.)* | 24(2) | | 179–199 |
| [Pla75] | J. F. Plante | Foliations with measure preserving holonomy. | *Ann. of Math. (2)* | 102(2) | | 327–361 |
| [Sha20] | M. Shannon | Dehn surgeries and smooth structures on 3-dimensional transitive Anosov flows. | *PhD Thesis*, 2020, [https://tel.archives-ouvertes.fr/tel-02951219/document](https://tel.archives-ouvertes.fr/tel-02951219/document). |
| [Sma67] | S. Smale | Differentiable dynamical systems. | *Bull. Amer. Math. Soc. (N.S.)* | 73 (1967), | | 747–817 |
| [Thu82] | W. P. Thurston | Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. | *Bull. Amer. Math. Soc. (N.S.)* | 6(3) | | 357–381 |
| [Thu97] | W. P. Thurston | Three Dimensional Geometry and Topology (Princeton Mathematical Series, 35). Vol. 1. Princeton University Press, Princeton, NJ, 1997. |