Anomaly non-renormalization in interacting Weyl semimetals

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For a class of interacting 3D lattice Weyl semimetals, we prove that the quadratic response of the quasi-particle flow between the Weyl points to an external electromagnetic field, which is the condensed-matter analogue of the chiral anomaly, is exactly independent of the model parameters, most notably of the interaction. Therefore, this response can be added to the limited list of universal transport coefficients in condensed matter. This universality phenomenon can be seen as a manifestation of the Adler-Bardeen non-renormalization property in a crystal, despite the breaking of relativistic symmetries due to the lattice. Our proof relies on constructive bounds on correlations, combined with lattice Ward Identities. Non-perturbative effects are rigorously excluded and irrelevant terms, which play a major role in the cancellation mechanism, are fully taken into account.

Dirac and Weyl semimetals are materials with an emergent description in terms of massless 3 + 1 dimensional Dirac fermions [1–3]. It has been proposed in [4], see also [5, 6], that the Adler-Bell-Jackiw (ABJ) [7, 8] axial anomaly has a counterpart in such materials: it arises also [5, 6], that the Adler-Bell-Jackiw (ABJ) [7, 8] axial Dirac fermions [1–3]. It has been proposed in [4], see e.g. [18], which says that radiative corrections cancel out at non-renormalization property [18], which says that radiative corrections cancel out at all orders in the fine structure constant. Such dramatic cancellation has deep consequences, such as the confirmation of the correct number of quark families, see e.g. [19, 20]. However, it is not known whether the AB non-renormalization property holds for Weyl semimetals, since its validity crucially relies on Lorentz and chiral gauge symmetries, both absent in lattice models. Moreover, the computation of transport coefficients for condensed matter systems with emergent Dirac descriptions, such as Weyl semimetals and graphene, is known to be subtle: predictions based on perturbation theory often give ambiguous results, see e.g. [21, 22]. Interaction [23–32] and disorder [33–50] effects in Weyl semimetals have been previously considered, but the issue of the anomaly non-renormalization has not been addressed yet.

In this Letter we consider a class of lattice models for Weyl semimetals with short-range interactions, and we show that the quadratic response to an external electromagnetic field of the internode quasiparticle flow is exactly universal, that is, it is independent of lattice or interaction effects. Therefore, this response can be added to the limited list of universal transport coefficients in condensed matter, like the Hall conductivity and graphene’s optical conductivity. Our result can be viewed as the analogue of the AB non-renormalization property for a non-relativistic, condensed matter system. In contrast with the universality of the anomaly, other physical quantities, such as the Fermi velocities and vertex functions, are renormalized non-trivially by the interaction.

Interacting lattice Weyl semimetals. We consider lattice models for Weyl semimetals with a minimal number of Weyl nodes, i.e. two nodes with opposite chiralities. For definiteness, we focus on the interacting version of the model in [51], describing fermions on a 3D lattice constituted of layers of face-centered square lattice with sub-lattices A and B, hopping between nearest and next-to-nearest neighbours, subject to a suitable magnetic flux pattern. In terms of the fermionic field $\psi^\pm = (a^\pm, b^\pm)$, whose components act on the A, B sub-lattices, the Hamiltonian of the system reads:

$$H = (\psi^+, h_0 \psi^-) + \lambda V + \nu N_3,$$

where $h_0$ is the hopping matrix [51], $V$ is a density-density, short-ranged, interaction, of strength $\lambda$, and $N_3 = N_A - N_B$ is the staggered fermion number, whose role is discussed soon. In momentum space:

$$\tilde{h}_0(k) = \begin{pmatrix} t_{\perp} \cos k_3 + \mu + \alpha(k) & t_1 \sin \frac{k_1}{\sqrt{2}} + i t_2 \sin \frac{k_2}{\sqrt{2}} - t_{\perp} \cos k_3 - \mu - \alpha(k) \\ t_1 \sin \frac{k_1}{\sqrt{2}} - i t_2 \sin \frac{k_2}{\sqrt{2}} + t_{\perp} \cos k_3 - \mu + \alpha(k) \\ t_1 \sin \frac{k_1}{\sqrt{2}} + i t_2 \sin \frac{k_2}{\sqrt{2}} - t_{\perp} \cos k_3 - \mu + \alpha(k) \\ t_1 \sin \frac{k_1}{\sqrt{2}} - i t_2 \sin \frac{k_2}{\sqrt{2}} + t_{\perp} \cos k_3 + \mu + \alpha(k) \end{pmatrix},$$

where $\alpha(k) = -t' \cos \frac{k_1}{\sqrt{2}} \cos \frac{k_2}{\sqrt{2}}$ and $t', t_1, t_2$ are in-layer hoppings, while $t_{\perp}$ is a transverse hopping parameter. The parameter $\mu$ is a staggered chemical potential, making the A, B sub-lattices inequivalent.

In the non-interacting case, if $|t' - \mu| < t_{\perp} < \mu + t'$, the dispersion relation has two bands touching at the Fermi points, or Weyl nodes, $p_F^\pm = \pm p_F$ with $p_F = \frac{1}{2}$, and  $t_1$, $t_2$, $t_{\perp}$ are in-layer hoppings. Around $p_F^\pm$ the dispersion relation is conical, with bare Fermi velocities $v_1^0 = t_1/\sqrt{2}$, $v_2^0 = t_2/\sqrt{2}$, $v_3^0 = t_{\perp} \sin p_F$. The distance between the Fermi points vanishes if $|t' - \mu| = t_{\perp}$, where the bands intersect quadratically. We let $\nu$ in (1) be a staggered potential counterterm, fixed in such a way that the interacting Fermi points are the same as the bare ones.

In the weakly interacting case [26, 27], the semimetallic phase survives, uniformly in the distance between the
Weyl nodes. In contrast to graphene [52, 53], there is no symmetry protecting the location of the interacting Fermi points: they are given by \( \pm p_F + b_\pm \lambda + \ldots \), with \( b_\pm \neq 0 \). The amplitudes of the Weyl cones are also renormalized: the interacting Fermi velocities are given by \( v_i = v_i^0 + a_i \lambda + O(\lambda^2) \), with \( a_i \neq 0 \) [26, 27].

**Main result.** Let us now couple the system to an external e.m. field \( A_\mu \), \( \mu = 0, 1, 2, 3 \). We denote by \( \langle \cdot \rangle_A \) the interacting Gibbs state of the system in the presence of the external field and \( \langle \cdot \rangle = \langle \cdot \rangle_0 \). The coupling is defined via the Peierls substitution, i.e. by replacing the hopping parameter \( t_b \) along the oriented bond \( b = (x, y) \) by \( t_b(A) = e^{i\frac{\pi}{2} \hbar A(t, y)} dt_b \). We let \( H_0(A) = (\psi^+, h_0(A) \psi^-) \) be the \( A \)-dependent hopping term. Next, following [4], we introduce the lattice analogue of the chiral density, \( \rho_0^A = Z^5 \frac{1}{2}(\psi^+_p \psi^-_{p+k} - \psi^+_{p+k} \psi^-_p) \), where \( Z^5 \) is an \( A \)-dimensional constant, to be fixed in such a way that the chiral density is proportional to \( \pm(\text{total density}) \), in the sense of correlations, at the Fermi points \( p_F^\pm \), that is,

\[
\langle \rho^A_p \rangle_A = i\hbar^2 \langle \rho^A_{p+k} \rangle_A \psi^+_p \psi^-_{p+k} + \frac{i}{2} \int dp_1 dp_2 \Gamma_{0,\nu,\sigma}(p_1, p_2) \delta(p + p_1 - p_2) \langle \psi^+_{p+k} \rangle_A \langle \psi^-_{p+k} \rangle_A + \langle \psi^+_{p+k} \rangle_A \langle \psi^-_{p+k} \rangle_A,
\]

up to higher orders in \( A \) (summation over repeated indices from 0 to 3 is understood). Note that \( \Gamma_{0,\nu,\sigma}(p_1, p_2) \) is not simply equal to \( -i\hbar^2 \langle \rho^A_{p+k} \rangle_A \psi^+_p \psi^-_{p+k} \rangle_A \) (here \( \Gamma_{p,\nu,\sigma} = (-i\hbar^2 \langle \rho^A_{p+k} \rangle_A \psi^+_p \psi^-_{p+k} \rangle_A \) is the lattice four-current, with \( \rho_{p,\nu} \) the lattice density and \( \Gamma_{p,\nu} = -\partial_{p,\nu} \langle \psi^+_p \psi^-_{p+k} \rangle_A \langle \psi^+_p \psi^-_{p+k} \rangle_A \) the lattice current): there are additional contributions, called Schwinger terms, arising from the nonlinear coupling to the external field, see (6) below. In the non-interacting case, if \( p_1 + p_2 = (p_0, 0) \), one has:

\[
p_0 \Gamma^5_{0,\nu,\sigma}(p_1, p_2) = \frac{\epsilon^2}{\hbar^2} \frac{1}{2\pi^2} \frac{1}{p_{1,\alpha} p_{2,\beta} \varepsilon^\alpha \varepsilon^{\beta \sigma}}
\]

with \( \varepsilon_{\alpha\beta\sigma} \) the Levi-Civita symbol, up to higher order terms in the momenta. Eq.(4) is independent of lattice details: the explicit prefactor coincides with the axial anomaly of QED4 [4].

After switching on the interaction, \( \Gamma^5_{0,\nu,\sigma} \) has a perturbative expansion in \( \lambda \): its leading contribution is the triangle graph, which is dressed by higher order terms, see Fig. 1. The Adler-Bardeen argument cannot be applied to this infinite series, since it heavily relies on the perfect linearity of the energy-momentum dispersion in QED, which is obviously violated by the lattice. Here, instead, the only possible mechanism for universality is the cancellation between the higher order terms arising from the expansion of \( \langle j^\mu_0 \rangle \) and those arising from the Schwinger terms, which is very hard, if not impossible, to check directly, on the basis of the Feynman diagram expansion. The next theorem, our main result, proves that this remarkable cancellation is actually valid.

**Theorem.** There exists \( \lambda_0 > 0 \), independent of the distance between the Fermi points, such that, if \( |\lambda| < \lambda_0 \), fixing \( \nu = \nu(\lambda) \) in such a way that the Fermi points do not move with the interaction, the ground state correlations and response functions are analytic in \( \lambda \). Fixing \( Z^5 = Z^5(\lambda) \) in such a way that (2) is verified, we have

\[
p_0 \Gamma^5_{0,\nu,\sigma}(p_1, p_2) = \frac{\epsilon^2}{\hbar^2} \frac{1}{2\pi^2} \frac{1}{p_{1,\alpha} p_{2,\beta} \varepsilon^\alpha \varepsilon^{\beta \sigma}},
\]

up to an error \( O(P^3 \log P) \), with \( P = \max(|p_1|, |p_2|) \).

In the special case of parallel constant external and magnetic fields, \( A_0 = A_1 \equiv 0 \), \( A_2(t, x) = B x_1 \), \( A_3(t, x) = -Et \), plugging (5) in (3), we get, at quadratic order,

\[
\partial_t \langle N^5(A)_A \rangle_A = \frac{\epsilon^2}{\hbar^2} \frac{1}{2\pi^2} \frac{1}{E B}
\]

where \( N^5 = \sum_k \rho^A_k \). Therefore, our result generalizes [4] to the interacting case and proves the perfect universality of the quadratic response of the lattice chiral density to an external field: all the lattice and interaction effects cancel exactly. It is the lattice analogue of the AB non-renormalization property of the axial anomaly, in a case where Lorentz symmetry is absent. Our result holds for \( |\lambda| \) small with respect to the hopping parameters \( t_1, t_2, t_3, t_4 \), but no smallness condition is required on \( |\lambda|/\epsilon_0^2 \). In particular, the result holds up to the collapse of the two Weyl nodes.

The mechanism ensuring universality for the lattice analogue of the chiral anomaly is related to the one for the optical conductivity of interacting graphene [54–57]. It is based on a combination of lattice Ward identities, together with non-perturbative regularity estimates on the correlation functions of the model.

**Proof.** For simplicity, we set \( e = h = c = 1 \). We define the gauge-invariant, chiral, four-current \( j^\mu_0(A) = (j^\mu_0, \tilde{j}^\mu_0(A), j^\mu_0, \tilde{j}^\mu_0(A)) \), where \( j^\mu_0(A) = -i\rho^A_0 \) and the spatial components are defined in such a way to: (i) be covariant under the lattice symmetries and (ii) reduce to the right expression for the relativistic chiral density in the low energy limit: \( \sum k \rho^A_k (0) \sim Z^5 \sum_{\omega = \pm} \int \frac{dk}{(2\pi)^2} \psi^+_{k,\omega} \psi^-_{k,\omega} \).
FIG. 1. Feynman graph expansion of $\langle \hat{J}_\alpha \hat{J}_{\beta}; \hat{J}_\sigma, \hat{P}_2 \rangle$.

and $\alpha_1^\dagger = \omega \sigma_1$, $\alpha_2^\dagger = \omega \sigma_2$, $\alpha_3^\dagger = -\sigma_3$, with $\sigma_\tau$ the Pauli matrices. For the precise definition of the lattice current, see the Supplementary Material. $Z_0^5$ are, in analogy with $Z^5$, normalization constants such that the chiral current is proportional to $\pm$ (vectorial current), in the same sense as (2). The generating function of correlations can be written as a Grassmann integral as

$$e^{W(A, A^5, \phi)} = \int P(d\psi) e^{V^{(0)}(\psi) + B(A, \psi) + (A_0^5, j_0^5(A)) + (\psi, \phi)},$$

where: $\psi, \phi$ are complex Grassmann fields; $P(d\psi)$ is the Gaussian integration with propagator $g(k) = (-i k_0 + \hat{h}_0(k))^{-1}$; $V^{(0)}(\psi) = -\lambda V(\psi - \nu N_2(\psi)$ is the interaction; $B(A, \psi) = -i (A_0, \mu) + (\psi^+, (\hat{h}_0 - h_0(\hat{A})) \psi)$, with $\rho_\mu = \psi^+_{\mu, \nu} \psi_{\nu, \mu}$, is the vectorial source term; $(A_0^5, j_0^5(A))$ is the chiral source term.

We let $\Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, p_2)$ (resp. $\Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, ..., p_n)$) be the derivatives of $W$ with respect to $A_0^5$ (resp. $A_\mu$, $A_{\mu, \nu, \sigma}$), computed at zero external fields. Our main object of interest is the quadratic chiral response, $\Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, p_2)$ that, letting $p = p_1 + p_2$, can be written as

$$\Gamma_{\mu, \nu, \sigma}^{(5, \phi; \nu, \sigma, \phi)}(p_1, p_2) = \langle \hat{j}_{\mu, \nu}^5 \hat{j}_{\nu, \sigma}^5; \hat{J}_{\sigma, \sigma}^5 \rangle + \langle \hat{j}_{\mu, \nu}^5 \hat{A}_{\nu, \nu}^5; \hat{J}_{\sigma, \sigma}^5 \rangle + \langle \hat{j}_{\mu, \nu}^5 \hat{A}_{\nu, \nu}^5; \hat{J}_{\sigma, \sigma}^5 \rangle,$$

where: $\Delta_{\nu, \nu}^5, \Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, p_2)$ is the derivative of $\hat{B}$ w.r.t. $A_{\nu, \nu}$.

The proof of universality combines two main ingredients: (a) invariance under local gauge transformation and Ward Identities; (b) regularity properties of the correlations $\Gamma_{\mu, \nu, \sigma}^{(5)}$. Let us start with (a). By gauge invariance:

$$W(A + \partial_\theta A^5, \phi e^{i\theta}) = W(A, A^5, \phi).$$

By differentiating (7) w.r.t. $\alpha$ many times as we like, we obtain a hierarchy of Ward Identities, among which

$$p_\mu \langle \hat{J}_{\mu, \nu}^5 \hat{\psi}_{\nu, \kappa}^+ + \hat{\psi}_{\nu, \kappa}^+ \hat{P}_2 \rangle = \langle \hat{\psi}_{\nu, \kappa}^+ \hat{\psi}_{\nu, \kappa}^+ \rangle + \langle \hat{\psi}_{\nu, \kappa}^+ \hat{\psi}_{\nu, \kappa}^+ \rangle,$$

and $p_\mu \Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, ..., p_n) = 0$, for $p = p_1 + ... + p_n$, $n \geq 1$, which implies that $p_\mu \langle \hat{J}_{\mu, \nu}^5(A) \rangle = 0$, i.e., the current is conserved. This argument does not apply to the chiral current, due to the absence of chiral gauge symmetry for our lattice model. However, note that gauge invariance implies

$$p_\mu \Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, ..., p_n) = 0, \quad \forall i = 1, ..., n.$$

Next, in order to compute the decay and regularity of correlations, we shall use an exact Renormalization Group (RG) analysis [26]. We rewrite $\psi^\dagger_k = \sum_{\omega=\pm} \sum_{h=-\infty}^\infty \psi^\dagger_{\omega, h} \xi^\dagger_{\omega, h} k$, where $\psi^\dagger_{\omega, h} \xi^\dagger_{\omega, h} k$ is supported on a shell of width $O(2^h)$ around the Fermi point $p^\dagger_2$. After having integrated out the fields on scales $0, 1, ..., h + 1$, we are left with an effective theory for the remaining, infrared, degrees of freedom: $e^{W(A, A^5, \phi)} = \int P_{\leq h}(dy) e^{V^{(4)}(A, A^5, \phi, \sqrt{y} \xi)}$, with $P_{\leq h}$ a dressed Gaussian interaction with propagator $g^{(5)}(k) = \chi_h(k^h)/(Z_h k_\mu^2 \alpha_{\mu, \omega})$, up to terms of higher order in $k$, where: $k^h = (k_0, v_1^h k_1, v_2^h k_2, v_3^h k_3)$, with $v^h$ the running Fermi velocities; $Z_h$ is the running wave function renormalization; $\chi_h$ is a smooth ultraviolet cutoff function; $\alpha_{0, \omega} = -\Im$, $\alpha_{1, \omega} = \sigma_1$, $\alpha_{2, \omega} = \sigma_2$, $\alpha_{3, \omega} = -\omega \sigma_3$. The effective potential $V^{(h)}$ has the following structure (for simplicity, we spell out explicitly only the $\phi$-independent terms): $V^{(h)} = RV^{(h)} + \sum_{\omega} \int d k$, with

$$2^h v_{\omega, k} \psi^\dagger_{\omega, k} \sigma_3 \psi^\dagger_{\omega, k} + Z^h \Delta_{\mu, \kappa}^\dagger \hat{J}_{\mu, \omega} \hat{J}_{\mu, \omega} + Z^h \Delta_{\mu, \kappa} \hat{J}_{\mu, \omega} \hat{J}_{\mu, \omega},$$

and $\Delta_{\mu, \kappa}^\dagger = \int d k, \psi^\dagger_{\omega, k} \sigma_3 \psi^\dagger_{\omega, k}$. $\alpha_{\mu, \omega} = \omega \alpha_{\mu, \omega}$. $RV^{(h)}$ is the sum of the irrelevant, non-local, terms, which we symbolically write $RV^{(h)} = \sum_{\omega} \int d x, d y W^{(h)}(x, y) \psi_{\kappa, \omega} \psi_{\kappa, \omega}$, with $W^{(h)}(x, y)$ depends only on the dressed local couplings $\mu_h, Z, \mu_h^2$, while $W^{(h)}(x, y)$ depends on the non-local quartic interaction, the following non-perturbative bound holds:

$$\int d x, d y W^{(h)}(x, y) e^{-\int d x, d y W^{(h)}(x, y) \leq C_{m,n} \sum_{h=0}^{n-m} |h|^{4 (2 - \theta_0)} \alpha_{\mu, \omega}^h.$$ (10)

Here $f^\dagger$ indicates the constraint that one of the points in $x$ or $y$ is not integrated over, $d(x, y)$ denotes the tree distance among the points in $(x, y)$, $\theta_0 = 0$ and $\theta_1 = 1/2$. The dimensional improvement $2^{\theta_0}$ is due to the irrelevance of the quartic interaction, and will play a crucial role in the following.

We choose the counterterm $\nu$ so that $v_\nu = O(\lambda 2^{8 h})$. All the marginal running constants converge to an infrared limit as $h \rightarrow -\infty$: $v^\dagger = v_\nu + O(\lambda 2^{8 h})$, $Z^\dagger_h = Z + O(\lambda 2^{8 h})$, $Z^\dagger_h = Z_\nu + O(\lambda 2^{8 h})$, and $Z^\dagger_h = Z^\dagger_0 + O(\lambda 2^{8 h})$. The fixed point values are all expressed by convergent series in $\lambda$, whose coefficients depend on all the microscopic details of the model. We fix the normalization constants $Z^\dagger_0$, at scale $0$ in such a way that at the infrared fixed point $Z_\nu = Z^\dagger_0$, which guarantees the validity of (2) and of its analogue for the current.

We now show how to use (10) in order to prove regularity properties of the correlations. Consider, e.g., $\Gamma_{\mu, \nu, \sigma}$, which is bounded by the sum over the scales of the right side of (10), with $n = 0, m = 3$ and $i = 0$: $|\Gamma_{\mu, \nu, \sigma}^{(5)}(p_1, p_2)| \leq \sum_{h=0}^{n-m} 2^h$, which is finite. Dimensionally, the derivative in $p_1, p_2$ of $\Gamma_{\mu, \nu, \sigma}$ is bounded.
by (const.) $\sum_{h} 2^{-h} 2^{h}$, which is infinite; this implies that $\Gamma_{\mu,\nu,\sigma}^5$ is continuous but non-differentiable in the momenta. The same argument shows that sum of all the contributions to $\Gamma_{\mu,\nu,\sigma}^5$ that involve at least one vertex $\lambda$ or $\nu_h$ is bounded by (const.) $\sum_{h} 2^{(1+\theta_1)h}$, while its derivative is bounded by (const.) $\sum_{h} 2^{2\theta_1 h}$, which are both finite; therefore, these terms are more regular, they are continuously differentiable in the momenta. A similar argument shows that the contribution to $\Gamma_{\mu,\nu,\sigma}^5$ from the Schwinger terms, that is, the second to fifth terms in the right side of (6), is continuously differentiable. In conclusion, $\Gamma_{\mu,\nu,\sigma}^5$ can be decomposed into a dominant, non-differentiable, part, which is simply the contribution of order zero in $\lambda, \nu_h$ to (5), and a remainder, which is continuously differentiable. We further decompose the dominant part into a “relativistic” contribution (linear propagators with ultraviolet cutoff) plus a differentiable remainder so that, in conclusion, see Fig. 2.

$$\Gamma_{\mu,\nu,\sigma}^5(p_1, p_2) = \Gamma_{\mu,\nu,\sigma}^{5,rel}(p_1, p_2) + H_{\mu,\nu,\sigma}^5(p_1, p_2), \quad (11)$$

where $H_{\mu,\nu,\sigma}^5$ is continuously differentiable, and $\Gamma_{\mu,\nu,\sigma}^{5,rel}$, which is just continuous with discontinuous derivative, is given by (no summation over $\mu, \nu, \sigma$ in the right side)

$$\Gamma_{\mu,\nu,\sigma}^{5,rel}(p_1, p_2) = \frac{Z_{\mu} Z_{\nu} Z_{\sigma}}{\nu_1 \nu_2 \nu_3} I_{\mu,\nu,\sigma}(p_1, p_2). \quad (12)$$

Here $I_{\mu,\nu,\sigma}(p_1, p_2)$ is the undressed relativistic chiral triangle graph with momentum cutoff, computed at $p_j = (p_{j0}, \nu_1 p_{j1}, \nu_2 p_{j2}, \nu_3 p_{j3})$.

We now combine the use of Ward Identities (WI) with the regularity properties of $\Gamma_{\mu,\nu,\sigma}^5$ and $H_{\mu,\nu,\sigma}^5$. The WI (8) implies that the vertex renormalizations are proportional to the velocities:

$$Z_0 = Z, \quad Z_1 = Z_2 = v_1 Z, \quad Z_3 = v_3 Z,$$

to which factor in the right side of (12) reduces to $\frac{Z_{\mu} Z_{\nu} Z_{\sigma}}{\nu_1 \nu_2 \nu_3}$, while $I_{\mu,\nu,\sigma}(p_1, p_2)$ can be computed explicitly, leading to $p_{\mu,\nu,\sigma}(p_1, p_2) = \frac{1}{2\pi^2} p_{\mu} p_{\nu} p_{\sigma}$, where $p_{\mu} = p_{\mu} + p_{\nu,\sigma}$, and $p_{\nu,\sigma} = p_{\mu} I_{\mu,\nu,\sigma}(p_1, p_2)$, up to higher order (h.o.) terms, cubic in the momenta. Plugging these into (12) we find:

$$p_{\mu,\nu,\sigma}^{5,rel}(p_1, p_2) = \frac{\nu_{\mu} \nu_{\sigma}}{6\pi^2 v_1 v_2 v_3} I_{\mu,\nu,\sigma}(p_1, p_2) + h.o. \quad (13)$$

and $p_{\mu,\nu,\sigma}^{5,rel}(p_1, p_2) = \frac{\nu_{\mu} \nu_{\sigma}}{6\pi^2 v_1 v_2 v_3} I_{\mu,\nu,\sigma}(p_1, p_2) + h.o.$.

(no summations over $\nu, \sigma$ and $\mu, \sigma$ in the right sides). Recall that $n = n_0 n_0$, and note that $\nu_{\mu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} = \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu}$; therefore, the dependence upon the velocities in the right sides of (13) simplifies.

This concludes the computation of the first term in (11). The second term is essentially impossible to evaluate directly, being an infinite series. However, its contribution is fixed by the WI (9). From $p_{\mu,\nu,\sigma} = 0$, expanding $H_{\mu,\nu,\sigma}^5$ up to second order in the external momenta, we find that

$$p_{\mu,\nu,\sigma}^5(p_1, p_2, \beta, \epsilon, \alpha, \beta) = \frac{1}{\epsilon^4} p_{\mu,\nu,\sigma}^5(p_1, p_2) + \frac{1}{\epsilon^4} p_{\mu,\nu,\sigma}^5(p_1, p_2) = \frac{1}{\epsilon^4} p_{\mu,\nu,\sigma}^5(p_1, p_2),$$

vanishes, up to h.o. terms in $p_1, p_2$. By equating the Taylor coefficients of the same order we find:

$$\frac{p_{\mu,\nu,\sigma}^5(p_1, p_2, \beta, \epsilon, \alpha, \beta)}{\epsilon^4} = \frac{1}{\epsilon^4} p_{\mu,\nu,\sigma}^5(p_1, p_2),$$

and note that $\epsilon = \epsilon_{\alpha, \beta, \epsilon}$, which is infinite; this implies $\epsilon = \epsilon_{\alpha, \beta, \epsilon}$. Therefore, $\epsilon = \epsilon_{\alpha, \beta, \epsilon}$ is an interesting open problem to extend it to unscreened Coulomb interactions.

Acknowledgements. This work has been supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (ERC CoG UniCoSMI, grant agreement n.724939 and ERC StG MaMBoQ, grant agreement.
n.802901). M.P. gratefully acknowledges financial support from the Swiss National Science Foundation, for the project “Mathematical Aspects of Many-Body Quantum Systems”.

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Supplementary material

I. A MODEL FOR LATTICE WEYL SEMIMETALS

A. The non-interacting model

We consider two sublattices: one is $\Lambda_1 = \{n_1(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, 0) + n_2(-\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, 0) + n_3(0, 0, 1), \ n_i \in \mathbb{Z}\}$, and the other is $\Lambda_2 = \Lambda_1 + \frac{1}{\sqrt{2}}e_1 \equiv \Lambda_1 + \delta_1$ (we let $e_1, e_2, e_3$ be the elements of the standard Euclidean basis and $\delta_1 = \frac{1}{\sqrt{2}}e_1, \ \delta_2 = \frac{1}{\sqrt{2}}e_2$). The non-interacting Hamiltonian of the system, in real space, is assumed to be

$$H_0 := H_1 + H_2 + H_3,$$

where

$$H_1 = \frac{1}{2} \sum_{x \in \Lambda_1} \left[ t_1 (i a_+^x b_{x+\delta_1} + i b_{x+\delta_1}^- a_x + c.c.) + t_2 (a_+^x b_{x+\delta_2} - b_{x+\delta_2}^- a_x + c.c.) \right], \quad (I.14)$$

$$H_2 = \frac{t_1}{2} \left[ \sum_{x \in \Lambda_1} a_+^x (a_{x+e_1}^- + a_{x-e_1}^-) - \sum_{x \in \Lambda_2} b_+^x (b_{x+e_3}^- + b_{x-e_3}^-) \right] - \frac{t'}{4} \left[ \sum_{x \in \Lambda_1} a_+^x (a_{x+\delta_1+\delta_2}^- + a_{x+\delta_1-\delta_2}^- + a_{x-\delta_1+\delta_2}^- + a_{x-\delta_1-\delta_2}^-) - \sum_{x \in \Lambda_2} b_+^x (b_{x+\delta_1+\delta_2}^- + b_{x+\delta_1-\delta_2}^- + b_{x-\delta_1+\delta_2}^- + b_{x-\delta_1-\delta_2}^-) \right] \quad (I.15)$$

and $H_3 = \mu N_3$, with $N_3$ the staggered density:

$$N_3 = \sum_{x \in \Lambda_1} a_+^x a_x^- - \sum_{x \in \Lambda_2} b_+^x b_x^- \quad (I.18)$$

For definiteness, we let $t_1 > 0$, $\mu + t' > 0$, and we assume that $\mu + t' > t_\perp$. We pass to Fourier space, via the conventions (note that $\psi_{x,j}^\pm$ is supported on $\Lambda_j, \ j = 1, 2$):

$$a_x^\pm \equiv \psi_{x,1}^\pm = \int \frac{dk}{(2\pi)^3} e^{\pm ikx} \psi_{k,1}^\pm, \quad b_x^\pm \equiv \psi_{x,2}^\pm = \int \frac{dk}{(2\pi)^3} e^{\pm ikx} \psi_{k,2}^\pm \quad (I.19)$$

where the integrals are over the first Brillouin zone $B = \{\xi_1 (\delta_1 + \delta_2) + \xi_2 (-\delta_1 + \delta_2) + \xi_3 e_3 : \ \xi_i \in [-\pi, \pi]\}$, so that the Hamiltonian reads:

$$H_0 = \int \frac{dk}{(2\pi)^3} \hat{c}^+_k \left( \begin{array}{cc} \alpha(k) & \beta(k) \\ \beta^*(k) & -\alpha(k) \end{array} \right) \hat{c}^-_k \equiv \int \frac{dk}{(2\pi)^3} \hat{\psi}^+_k h_0(k) \hat{\psi}^-_k, \quad (I.20)$$

where

$$\alpha(k) = t_\perp \cos k_3 + \mu - t' \cos \frac{k_1}{t_\perp} \cos \frac{k_2}{2}, \quad (I.21)$$

and

$$\beta(k) = t_1 \sin \frac{k_3}{t_\perp} - il_2 \sin \frac{k_1}{2} \sin \frac{k_2}{2}. \quad (I.22)$$

Note that, if $|\mu - t'| < t_\perp$, letting $p_F = \text{arccos} \frac{t_\perp - \mu}{t_\perp}$, $\alpha(k)$ can be rewritten as

$$\alpha(k) = t_\perp \cos k_3 - \cos p_F + t'(1 - \cos \frac{k_1}{t_\perp} \cos \frac{k_2}{2}),$$

so that

$$- \det h_0(k) = \left[ t_\perp (\cos k_3 - \cos p_F) + t'(1 - \cos \frac{k_1}{t_\perp} \cos \frac{k_2}{2}) \right]^2 + t_1^2 \sin^2 \frac{k_1}{2} + t_2^2 \sin^2 \frac{k_2}{2},$$

which implies $\det h_0(k) < 0$, and $=0$ if and only if $k = \pm(0,0,p_F) \equiv p_F^\pm$.

The paramagnetic current is obtained, as usual, via the Peierls' substitution: any hopping term of the form $t_{x,y}^x \psi_{x,i}^+ \psi_{y,j}^-$ is promoted to an $A$-dependent hopping $t_{x,y}^x \psi_{x,i}^+ \psi_{y,j}^- \exp \{ i \int_{x-y} A(\ell) \cdot d\ell \}$, where $A: \mathbb{R}^3 \to \mathbb{R}^3$, thus replacing the original Hamiltonian by an $A$-dependent one, denoted by

$$H_0(A) \equiv \langle \psi^+, h_0(A) \psi^\pm \rangle.$$
Then we let the paramagnetic current be $j_p := -\frac{\delta H_0(A)}{\delta A_p} \big|_{A=0}$, where we use the convention $A(x) = \int \frac{dp}{(2\pi)^3} A_p e^{-ipx}$. We write

$$j_p = \int \frac{dk}{(2\pi)^3} \hat{\psi}^+_{k+p} \gamma(k,p) \hat{\psi}^-_k$$

(1.23)

where $\gamma(k,p)$ is a matrix-valued vector, whose components read:

$$\gamma_1(k,p) = v_1(k,p) \sigma_1 + u_1(k,p) \sigma_3,$$

(1.24)

$$\gamma_2(k,p) = v_2(k,p) \sigma_2 + u_2(k,p) \sigma_3,$$

(1.25)

$$\gamma_3(k,p) = u_3(k,p) \sigma_3,$$

(1.26)

with $\sigma_i$ the Pauli matrices and, letting $\varphi(x) := \frac{\sin(x/2)}{x/2}$,

$$v_j(k,p) = t \varphi(p \cdot \delta_j) \cos (k \cdot \delta_j + p \cdot \delta_j/2),$$

(1.27)

$$u_j(k,p) = \frac{t'}{2} \sum_{\sigma=\pm} \sigma^{j-1} \varphi (p \cdot (\delta_1 + \sigma \delta_2)) \sin ((k + \frac{p_j}{2}) \cdot (\delta_1 + \sigma \delta_2)), $$

for $j = 1, 2$, and

$$u_3(k,p) = -t_\perp \varphi(p_3) \sin (k_3 + p_3/2).$$

(1.28)

B. The interacting model and the Grassmann representation

We consider an interacting version of the model, with Hamiltonian $H = H_0 + \lambda V_0 + \nu N_3$, where $N_3$ is the staggered density and $V_0$ is a density-density interaction:

$$V_0 = \frac{1}{2} \sum_{x,y \in \Lambda_1 \cup \Lambda_2} (n_x - 1/2) v(x-y)(n_y - 1/2),$$

with $n_x = a_x^+ a_x^-$ or $n_x = b_x^+ b_x^-$, depending on whether $x \in \Lambda_A$ or $x \in \Lambda_B$, and $v$ a potential, normalized in such a way that $\|v\|_1 = \sum_{x \in \Lambda_1 \cup \Lambda_2} |v(x)| = 1$. The generating function for correlations can be written as a Grassmann integral, in the form

$$e^{W(A,\phi)} = \int P(\psi) e^{-V(\psi)+B(A,\psi)+\langle \phi, \psi \rangle},$$

where $\psi^\pm$ is a two-component Grassmann field, whose components are labelled by $(x, i) = ((x_0, x), i)$ with $x_0 \in \mathbb{R}$, and $x \in \Lambda_1$ or $x \in \Lambda_2$, depending on whether $i = 1$ or $i = 2$; $P(\psi)$ is the Gaussian Grassmann integration with propagator $g(x) = \int \frac{d\kappa}{(2\pi)^3} e^{-i\kappa x} \hat{g}(\kappa)$,

$$\hat{g}(\kappa) = \begin{pmatrix} -ik_0 + \alpha(k) & \beta(k) \\ \beta^*(k) & -ik_0 - \alpha(k) \end{pmatrix}^{-1},$$

(1.29)

that is, $P(\psi^\pm) \propto D\psi e^{-S_0(\psi)}$, with $S_0(\psi) = (\psi^+, \partial_\tau \psi^-) + (\psi^+, \nu \psi^-)$; letting $\int d\kappa$ be a shorthand for $\int_\mathbb{R} d\kappa_0 \sum_{x \in \Lambda_1 \cup \Lambda_2}$, we have

$$V(\psi) = \frac{\lambda}{2} \int d\kappa_0 \int dy_n x_0 \delta(x_0 - y_0) v(x-y)n_y$$

(1.30)

$$+ \nu \int_\mathbb{R} d\kappa_0 \sum_{i=1,2} (-1)^{i-1} \sum_{x \in \Lambda_i} \psi^+_{x,i} \psi^-_{x,i},$$

(1.31)

with $n_x = \psi^+_{x,1} \psi^-_{x,1}$ or $n_x = \psi^+_{x,2} \psi^-_{x,2}$, depending on whether $x \in \Lambda_1$ or $x \in \Lambda_2$. Moreover,

$$B(A,\psi) = -i(A_0, n) + (\psi^+, (h_0 - h_0(A)) \psi^-),$$

where $A = (A_0, A)$, with $A_0 : \mathbb{R}^4 \to \mathbb{R}$ and $A : \mathbb{R}^4 \to \mathbb{R}^3$, and $(A_0, n) = \int d\kappa \int_{\Lambda_1} A_0(x)n_x$. Finally,

$$\langle \phi, \psi \rangle = \int d\kappa_0 \sum_{x \in \Lambda_1} (\psi^+_{x,1} \phi^-_{x,1} + \psi^+_{x,2} \phi^-_{x,2}).$$
Note that $W(A, \psi)$ is gauge invariant,
\begin{equation}
W(A + \partial \alpha, \phi e^{i\alpha}) = W(A, \phi),
\end{equation}
where the left side indicates the generating function computed at $A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu} \alpha(x)$, $\mu = 0, 1, 2, 3$, and $(\phi_{\pm}^\dagger)^{\mu} = e^{\pm i\alpha(x)} \phi_{x, \mu}^\dagger$, with $\alpha : \mathbb{R}^4 \rightarrow \mathbb{R}$.

C. Symmetries

The Grassmann action is invariant under the following transformations of the fields and of the parameters $t_1, t_2, t'_1, t'_2, \mu, \nu$:

1. $\psi_{x, \mu}^\pm \rightarrow \psi_{x, \mu}^\pm$, with $x = (x_0, -x_1, x_2, x_3)$, $A_{\mu}(x) \rightarrow (-1)^{\delta_{\mu, 1}} A_{\mu}(x)$, $t_1 \rightarrow -t_1$ and the other parameters left invariant.
2. $\psi_{x, \mu}^\pm \rightarrow \psi_{x, \mu}^\pm$, with $x = (x_0, x_1, -x_2, x_3)$, $A_{\mu}(x) \rightarrow (-1)^{\delta_{\mu, 2}} A_{\mu}(x)$, $t_2 \rightarrow -t_2$ and the other parameters left invariant.
3. $\psi_{x, \mu}^\pm \rightarrow \psi_{x, \mu}^\pm$, with $x = (x_0, x_1, x_2, -x_3)$, $A_{\mu}(x) \rightarrow (-1)^{\delta_{\mu, 3}} A_{\mu}(x)$, and all the parameters left invariant.
4. $\psi_{x, \mu}^\pm \rightarrow (\pm (1)^{\delta_{\mu, 1}})^{-1} \psi_{x, j}^\pm$, with $x = (-x_0, x_1, x_2, x_3)$, $A_{\mu}(x) \rightarrow (-1)^{\delta_{\mu, 0}} A_{\mu}(x)$, $(t', t_\perp, \mu, \nu) \rightarrow -(t', t_\perp, \mu, \nu)$ and the other parameters left invariant.
5. $\psi_{x, j}^\pm \rightarrow (\pm i)^{-1} \psi_{x, j}^\pm$, with $x = (x_0, x_1, x_2, x_3)$, $A_{\mu}(x) \leftrightarrow A_2(x)$, $A_{\mu}(x) \rightarrow A_{\mu}(x)$ for $\mu = 0, 3$, $t_1 \rightarrow t_2$, $t_2 \rightarrow -t_1$, and the other parameters left invariant.
6. $\psi_{x, 1}^\pm \rightarrow i \psi_{x, 1}^\pm$, $\psi_{x, 2}^\pm \rightarrow -i \psi_{x, 2}^\pm$, with $x = (-x_0, x_1 + 1, x_2, x_3)$, $A_{\mu}(x) \rightarrow (-1)^{\delta_{\mu, 3}} A_{\mu}(x)$, $t_2 \rightarrow -t_2$, and the other parameters left invariant.
7. $c \rightarrow c^*$ (where $c$ is a generic constant appearing in the Grassmann action), $A_{\mu}(x) \rightarrow -A_{\mu}(x)$, $t_1 \rightarrow -t_1$ and the other parameters left invariant.
8. $\psi_{x, \mu}^\pm \rightarrow \psi_{x, \mu}^\pm$, $A_{\mu}(x) \rightarrow -A_{\mu}(x)$, $(t_2, t_\perp, t'_1, \mu, \nu) \rightarrow -(t_2, t_\perp, t'_1, \mu, \nu)$ and the other parameter, $t_1$, left invariant.
9. $\psi_{x, j}^\pm \rightarrow (1)^{\delta_{\mu, 1}} \psi_{x, j}^\pm$, $A_{\mu}(x)$ left invariant, $(t_1, t_2) \rightarrow -(t_1, t_2)$ and the other parameters left invariant.

The actual symmetries of the model correspond to transformation of the fields that leave the parameters of the Hamiltonian, $t_1, t_2, t'_1, t'_2, \mu, \nu$, invariant. These can be determined by taking suitable combinations of the previous transformations:

Symmetry 1: Complex conjugation. This is obtained by combining the transformations 1 and 7 of the previous list. We let: $c \rightarrow c^*$, $\psi_{x, \mu}^\pm \rightarrow \psi_{x, \mu}^\pm$, with $x = (x_0, -x_1, x_2, x_3)$, and $A_{\mu}(x) \rightarrow (-1)^{\delta_{\mu, 1}} A_{\mu}(x)$. The transformation of the fields in Fourier space reads: $\hat{\psi}_{x, k}^\pm \rightarrow \hat{\psi}_{x, k}^\pm$ or, in the quasi-particle representation, $\hat{\psi}_{\omega, k}^\pm \rightarrow \hat{\psi}_{-\omega, -k}^\pm$; $\hat{A}_{\mu, p} \rightarrow (1)^{\delta_{\mu, 1}} \hat{A}_{\mu, -p}$.

Symmetry 2: Translations. This is obtained by combining the transformations 2 and 6 of the previous list. We let: $\psi_{x, 1}^\pm \rightarrow i \psi_{x, 1}^\pm e_{1, 2}$, $\psi_{x, 2}^\pm \rightarrow -i \psi_{x, 2}^\pm e_{1, 2}$, with $x = (-x_0, x_1, -x_2, x_3)$ and $e_1 = (0, 1, 0, 0)$, and $A_{\mu}(x) \rightarrow (1)^{\mu - 1} A_{\mu}(x)$. In Fourier space, the transformation reads: $\hat{\psi}_{\omega, k}^\pm \rightarrow e^{i k_1} \hat{\psi}_{\omega, k}^\pm \sigma_2$, $\hat{\psi}_{\omega, k}^\pm \rightarrow -e^{-i k_1} \sigma_2 \hat{\psi}_{-\omega, -k}^\pm$, $\hat{A}_{\mu, p} \rightarrow (1)^{\mu - 1} e^{-i p_1} \hat{A}_{\mu, p}$.

Symmetry 3: Inversion. This is obtained by combining the transformations 1,2 and 9 of the previous list. We let: $\psi_{x, j}^\pm \rightarrow (1)^{\delta_{\mu, 1}} \psi_{x, j}^\pm$, with $x = (x_0, -x_1, -x_2, x_3)$, and $A_{\mu}(x) \rightarrow (1)^{\delta_{\mu, 1} + \delta_{\mu, 2}} A_{\mu}(x)$. In Fourier space, the transformation reads: $\hat{\psi}_{x, k}^\pm \rightarrow \hat{\psi}_{x, k}^\pm \sigma_3$, $\hat{\psi}_{\omega, k}^\pm \rightarrow \sigma_3 \hat{\psi}_{-\omega, -k}^\pm$, $\hat{A}_{\mu, p} \rightarrow (1)^{\mu - 1} \hat{A}_{\mu, p}$.

Symmetry 4: Hole-particle. This is obtained by combining the transformations 2, 4 and 8 of the previous list. We let: $\psi_{x, j}^\pm \rightarrow (-1)^{\delta_{\mu, 1}} \psi_{x, j}^\pm$, with $x = (-x_0, x_1, -x_2, x_3)$, and $A_{\mu}(x) \rightarrow (1)^{\mu - 1} A_{\mu}(x)$. In Fourier space, the transformation reads: $\hat{\psi}_{x, k}^\pm \rightarrow \hat{\psi}_{x, k}^\pm \sigma_3$, $\hat{\psi}_{\omega, k}^\pm \rightarrow -\sigma_3 \hat{\psi}_{-\omega, -k}^\pm$, $\hat{A}_{\mu, p} \rightarrow (1)^{\mu - 1} \hat{A}_{\mu, p}$.

Symmetry 5: Reflection. This is just transformations 3 of the previous list. We let: $\psi_{x, \mu}^\pm \rightarrow \psi_{x, \mu}^\pm$, with $x = (x_0, x_1, x_2, -x_3)$, and $A_{\mu}(x) \rightarrow (1)^{\delta_{\mu, 3}} A_{\mu}(x)$. In Fourier space, the transformation reads: $\hat{\psi}_{x, k}^\pm \rightarrow \hat{\psi}_{x, k}^\pm \sigma_3$, $\hat{A}_{\mu, p} \rightarrow (1)^{\mu - 1} \hat{A}_{\mu, p}$.

Note that if $t_1 = t_2$ there is an additional symmetry (in-plane rotation), obtained by combining the transformations 2 and 5 of the previous list.
D. Consequences of the symmetries

Let us now discuss the implications of the previous symmetries on the local parts of the effective action.

1. The local $\psi^+\psi^-$ term.

Let us consider a quadratic term, symmetric under the previous symmetries, of the form
\[ \int \frac{dk}{(2\pi)^4} \hat{\psi}^+_\omega k R_{\omega} \hat{\psi}^-_{\omega k}, \]
with $M_\omega$ a complex $2 \times 2$ matrix. Imposing symmetry 5 we find that $M_\omega = M_{-\omega} \equiv M$. Imposing the validity of symmetries 1,2,3,4, we find:
\[ M = M^* = -\sigma_2 M \sigma_2 = \sigma_3 M \sigma_3 = \sigma_3 M^T \sigma_3, \quad (I.33) \]
where $M^*$ is the matrix whose elements are the complex conjugates of the corresponding elements of $M$ (not to be confused with the adjoint, which we denote by $M^T \equiv M^{*T}$). If we now expand $M$ in the ‘Pauli basis’, $M = a_0 \mathbb{I} + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3$, we see that (I.33) implies that $M = a_3 \sigma_3$, with $a_3 \in \mathbb{R}$.

2. The local $\psi^+ \partial \psi^-$ term.

Let us consider a quadratic term, symmetric under the previous symmetries, of the form
\[ \int \frac{dk}{(2\pi)^4} \hat{\psi}^+_\omega k R_{\omega} \hat{\psi}^-_{\omega k}, \]
where the summation over repeated indices from 0 to 3 is understood, and $m_{\mu,\omega}$ are complex $2 \times 2$ matrices. Imposing symmetry 5 we find that $m_{\mu,\omega} = (-1)^{\mu+\omega} m_{\mu,-\omega}$, so we let $m_{\mu,\omega} \equiv m_\mu$, for $\mu = 0, 1, 2$, and $m_{3,\omega} \equiv \omega m_3$. Imposing the validity of symmetries 1,2,3,4, we find:
\[ \begin{align*}
0 &= m_0 = -m_0 = \sigma_2 m_0 \sigma_2 = \sigma_3 m_0 \sigma_3 = \sigma_3 m_0^T \sigma_3, \\
1 &= m_1 = -\sigma_2 m_1 \sigma_2 = -\sigma_3 m_1 \sigma_3 = -\sigma_3 m_1^T \sigma_3, \\
2 &= m_2 = \sigma_2 m_2 \sigma_2 = \sigma_3 m_2 \sigma_3 = \sigma_3 m_2^T \sigma_3, \\
3 &= m_3 = -\sigma_2 m_3 \sigma_2 = \sigma_3 m_3 \sigma_3 = \sigma_3 m_3^T \sigma_3.
\end{align*} \]

If we now expand $m_\mu$ in the ‘Pauli basis’, $m_\mu = a_{0 \mu}^0 \mathbb{I} + a_{1 \mu}^1 \sigma_1 + a_{2 \mu}^2 \sigma_2 + a_{3 \mu}^3 \sigma_3$, we see that (I.34)–(I.37) imply that $m_0 = a_{0 \mu}^0$, $m_1 = a_{1 \mu}^1 \sigma_1$, $m_2 = a_{2 \mu}^2 \sigma_2$, $m_3 = a_{3 \mu}^3 \sigma_3$, with $a_{0 \mu}^0, a_{1 \mu}^1, a_{2 \mu}^2, a_{3 \mu}^3 \in \mathbb{R}$.

3. The local $A \psi^+ \psi^-$ term.

Let us consider a term quadratic in $\psi$ and linear in $A$, symmetric under the previous symmetries, of the form
\[ \int \frac{dp}{(2\pi)^4} \int \frac{dk}{(2\pi)^4} \hat{\psi}^+_{\omega k} \hat{\psi}^-_{\omega k}, \]
where the summation over repeated indices from 0 to 3 is understood, and $\Gamma_{\mu,\omega}$ are complex $2 \times 2$ matrices. Imposing symmetry 5 we find that $\Gamma_{\mu,\omega} = (-1)^{\mu+\omega} \Gamma_{\mu,-\omega}$, so we let $\Gamma_{\mu,\omega} \equiv \Gamma_\mu$, for $\mu = 0, 1, 2$, and $\Gamma_{3,\omega} \equiv \omega \Gamma_3$. Imposing the validity of symmetries 1,2,3,4, we find the same as (I.34)–(I.37), with $m_\mu$ replaced by $\Gamma_\mu$. Therefore, $\Gamma_0 = i c_0 \mathbb{I}$, $\Gamma_1 = c_1 \sigma_1$, $\Gamma_2 = c_2 \sigma_2$, $\Gamma_3 = c_3 \sigma_3$, with $c_0, c_1, c_2, c_3 \in \mathbb{R}$.

E. The chiral density and currents

Let us now add to the generating function a chiral source term:
\[ e^{W(A,A^\dagger,\phi)} = \int P(d\psi)e^{-V(\psi)+B(A,\psi)+\langle A^\dagger, j^A(A) \rangle + \langle \phi, \psi \rangle}, \]
where $A^\dagger : \mathbb{R}^4 \to \mathbb{R}^4$, $(A^\dagger, j^A(A)) = \sum_{\alpha=0}^3 (A_{\alpha}^{\dagger}, j_\alpha(A))$, and, letting $e_\mu$ be the elements of the standard Euclidean basis in $\mathbb{R}^4$ and $\delta_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$:
\[ (A_{\alpha}^{\dagger}, j_\alpha(A)) \equiv \frac{Z_0}{4} \int dx_0 \sum_{j=1}^2 \sum_{x \in A_j} \left[ A_0^{\dagger}(x) + A_0^{-1}(x + e_3) \right] \cdot \left( \psi_{x,j}^+ \psi_{x+e_3,j}^- e^{i \int_0^1 A_0(x+s e_3) ds} - \psi_{x,j}^+ \psi_{x+e_3,j}^- e^{-i \int_0^1 A_0(x+s e_3) ds} \right), \quad (I.38) \]
\[
\langle A_3^1, j_3^0(A) \rangle = \frac{Z_5^0}{4} \int dx_0 \sum_{x \in A_1} \sum_{\sigma, \sigma' = \pm} \sigma' \left[ A_3^0(x) + A_3^1(x + \sigma \delta_1 + \sigma' \varepsilon_3) \right] \cdot \\
\langle A_2^1, j_2^0(A) \rangle = \frac{Z_2^0}{4} \int dx_0 \sum_{x \in A_1} \sum_{\sigma, \sigma' = \pm} \sigma' \left[ A_2^0(x) + A_2^1(x + \sigma \delta_1 + \sigma' \varepsilon_3) \right] \cdot \\
\langle A_1^1, j_1^0(A) \rangle = \frac{Z_1^0}{4} \int dx_0 \sum_{x \in A_1} \sum_{\sigma, \sigma' = \pm} \sigma' \left[ A_1^0(x) + A_1^1(x + \sigma \delta_1 + \sigma' \varepsilon_3) \right] \cdot
\]
\[
(\psi_{\mathbf{x}, 1}^0 \psi_{\mathbf{x} + \sigma \delta_1 + \sigma' \varepsilon_3, 2}^0 - i \int f_{\mathbf{x}, 1}^{\sigma} A_1(x + \sigma \delta_1 + \sigma' \varepsilon_3) dx + \text{c.c.}),
\]
\[
W(A + \partial \phi, A^5, \phi \phi^\dagger) = W(A, \phi),
\]

in the same sense as (I.32). Moreover, the chiral source term is invariant under the symmetries 1 to 5 listed above, provided the chiral field is transformed as follows:

Symmetry 1. \( A_0^0(x) \to A_0^0(\bar{x}), A_1^0(x) \to -A_1^0(\bar{x}), A_2^0(x) \to A_2^0(\bar{x}), A_3^0(x) \to A_3^0(\bar{x}) \), with \( \bar{x} = (x_0, -x_1, x_2, x_3) \).

Symmetry 2. \( A_0^0(x) \to -A_0^0(\mathbf{x} + \varepsilon_1), A_1^0(x) \to A_1^0(\mathbf{x} + \varepsilon_1), A_2^0(x) \to -A_2^0(\mathbf{x} + \varepsilon_1), A_3^0(x) \to A_3^0(\mathbf{x} + \varepsilon_1) \), with \( \mathbf{x} = (-x_0, x_1, -x_2, x_3) \).

Symmetry 3. \( A_0^0(x) \to A_0^0(\mathbf{x}), A_1^0(x) \to -A_1^0(\mathbf{x}), A_2^0(x) \to -A_2^0(\mathbf{x}), A_3^0(x) \to A_3^0(\mathbf{x}) \), with \( \mathbf{x} = (x_0, -x_1, -x_2, x_3) \).

Symmetry 4. \( A_0^0(x) \to -A_0^0(\mathbf{x}), A_1^0(x) \to A_1^0(\mathbf{x}), A_2^0(x) \to -A_2^0(\mathbf{x}), A_3^0(x) \to A_3^0(\mathbf{x}) \), with \( \mathbf{x} = (-x_0, x_1, x_2, -x_3) \).

Symmetry 5. \( A_0^0(x) \to -A_0^0(\mathbf{x}), A_1^0(x) \to -A_1^0(\mathbf{x}), A_2^0(x) \to -A_2^0(\mathbf{x}), A_3^0(x) \to A_3^0(\mathbf{x}) \), with \( \mathbf{x} = (x_0, x_1, x_2, -x_3) \).

As a consequence of these symmetries, we can determine the structure of the local \( A^5 \bar{\psi} \psi \) terms: let us consider a term quadratic in \( \bar{\psi} \) and linear in \( A^5 \), symmetric under the previous symmetries, of the form

\[
\int \frac{dp}{(2\pi)^4} \int \frac{dk}{(2\pi)^4} \tilde{A}_\mu^5 p_\mu \bar{\psi}_{\mathbf{x}, \mathbf{k}} \psi_{\mathbf{x}, \mathbf{k}},
\]

where the summation over repeated indices from 0 to 3 is understood, and \( \Gamma^5_{\mu, \omega} \) are complex 2 \( \times \) 2 matrices.

Imposing symmetry 5 we find that \( \Gamma^5_{\mu, \omega} = (-1)^{\delta_{\mu, \omega} + 1} \Gamma^5_{\mu, -\omega} \), so we let \( \Gamma^5_{\mu, \omega} = \omega \Gamma^5_{\mu} \), with \( \mu = 0, 1, 2, 3 \). Imposing the validity of symmetries 1, 2, 3, 4, we find the same as (I.34)–(I.37), with \( m_\mu \) replaced by \( \Gamma^5_{\mu} \). Therefore, \( \Gamma^5_0 = i \sigma_0 \mathbf{1}, \Gamma^5_1 = c_1 \sigma_1, \Gamma^5_2 = c_2 \sigma_2, \Gamma^5_3 = c_3 \sigma_3 \), with \( c_1, c_1, c_2, c_3 \in \mathbb{R} \).

F. Relativistic representation

In the infrared limit, the dressed propagator and currents of the interacting model can be conveniently rewritten in terms of relativistic notation. This is particularly convenient for the comparison of \( \Gamma^\lambda_{\mu, \nu, \sigma} \) with the relativistic chiral triangle graph, see (11). We introduce the Euclidean gamma matrices \( \gamma_0, \ldots, \gamma_3 \),

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & i \sigma_j \\ -i \sigma_j & 0 \end{pmatrix}
\]

and the 4-spinors

\[
\Psi = (-i \psi_{a, -}^+, -i \psi_{b, -}^+, -i \psi_{a, +}^+, +i \psi_{b, +}^+), \quad \Psi = \begin{pmatrix} \psi_{a, +}^- \\ -\psi_{b, +}^- \\ \psi_{a, -}^+ \\ \psi_{b, -}^+ \end{pmatrix}.
\]
In this notation, the two chiral components of the infrared propagator \( g^{(\leq h)}(k) = \int P_{\leq h}(d\psi) \hat{\psi}^-_{\omega,k} \hat{\psi}^+_{\omega,k} \)
can be rewritten as
\[
\langle \Psi_k \bar{\Psi}_k \rangle^{(\leq h)} = \frac{1}{Z} \frac{\chi(h)(k)}{\gamma_{\mu} k_{\mu}} (1 + O(2^{h,h})) ,
\]
where \( Z \) is the infrared limit of the running wave function renormalization \( Z_\theta \) (which is reached at exponential speed \( 2^h \)) and, if \( v_1, v_2, v_3 \) are the infrared limits of the running Fermi velocities (which are also reached at exponential speed \( 2^h \)), then \( k = (k_0, v_1 k_1, v_2 k_2, v_3 k_3) \). In terms of these relativistic notations, the components of the vectorial and chiral current also take a particularly convenient form:
\[
\sum_{\omega=\pm} j_{\mu,\omega.p} = \int \frac{dk}{(2\pi)^4} \bar{\psi}_{k+p} \gamma_{\mu} \psi_k
\]
and
\[
\sum_{\omega=\pm} j_{5,\mu.p} = \int \frac{dk}{(2\pi)^4} \bar{\psi}_{k+p} \gamma_5 \gamma_{\mu} \psi_k ,
\]
where
\[
\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
is the fifth gamma matrix, which anticommutes with the other four.

II. THE TRIANGLE GRAPH FOR 4D EUCLIDEAN DIRAC FERMIONS WITH MOMENTUM CUTOFF

A. The triangle graph: anomaly of the axial Ward Identity

Consider 4D Euclidean Dirac fermions, with propagator \( \chi(k)/k \), where: \( \chi(k) \) is a smooth characteristic function, depending only on the absolute value of the argument, and such that \( \chi(k) = 1 \) for \( |k| \leq 1 \), and \( = 0 \) for \( |k| \geq 2 \); \( k = \gamma_{\mu} k_{\mu} \) (the summation over \( \mu \) from 0 to 3 is understood), with \( \gamma_{\mu} \) the Euclidean gamma matrices in (I.40), which satisfy the anti-commutation rules \( \{ \gamma_{\mu}, \gamma_{\nu} \} = 2 \delta_{\mu\nu} \). We also let \( \gamma_5 \) be defined as in (I.41). We introduce
\[
I_{\mu,\nu,\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \text{Tr} \left\{ \frac{\chi(k)}{k} \gamma_{\mu} \gamma_5 \frac{\chi(k+p_1)}{k+p_1} \gamma_{\nu} \frac{\chi(k+p_2)}{k+p_2} \gamma_{\sigma} \right\} + [(\nu, p_1) \leftrightarrow (\sigma, p_2)] ,
\]
which is the undressed chiral triangle graph entering eq.(12). If we contract \( I_{\mu,\nu,\sigma}(p_1, p_2) \) against \( p_\mu := p_{1,\mu} + p_{2,\mu} \), we get:
\[
p_\mu I_{\mu,\nu,\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \text{Tr} \left\{ \frac{\chi(k)}{k} \gamma_5 \frac{\chi(k+p)}{k+p} \gamma_{\nu} \frac{\chi(k+p_2)}{k+p_2} \gamma_{\sigma} \right\} + [(\nu, p_1) \leftrightarrow (\sigma, p_2)]
\equiv T_{\nu,\sigma}(p_1, p_2) + T_{\sigma,\nu}(p_2, p_1) .
\]
Using the anti-commutativity of the fifth gamma matrix,
\[
T_{\nu,\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \text{Tr} \left\{ \frac{\chi(k)}{k} \gamma_5 \frac{\chi(k+p)}{k+p} \gamma_{\nu} \gamma_5 \frac{\chi(k+p_2)}{k+p_2} \gamma_{\sigma} \right\} ,
\]
and now we can use the rewriting (Ward Identity with correction term):
\[
\frac{\chi(k)}{k} \gamma_5 \frac{\chi(k+p)}{k+p} = \left[ \frac{\chi(k)}{k} - \frac{\chi(k+p)}{k+p} \right] + \frac{\chi(k)}{k} C(k,p) \frac{\chi(k+p)}{k+p} ,
\]
where
\[
C(k,p) = k(\chi^{-1}(k) - 1) - (k+p)(\chi^{-1}(k+p) - 1)
\]
(note that this only makes sense after multiplication by $\chi(k)\chi(k+p)$). The contribution to $T_{\mu\nu}(p_1, p_2)$ due to the difference in square brackets in (II.3) is:

$$\int \frac{dk}{(2\pi)^4} \text{Tr}\left\{ \frac{\chi(k)}{k} \gamma_\nu \gamma_5 \frac{\chi(k+p_2)}{k+p_2} \gamma_\sigma \right\} - \int \frac{dk}{(2\pi)^4} \text{Tr}\left\{ \frac{\chi(k+p_1)}{k+p_1} \gamma_\nu \gamma_5 \frac{\chi(k+p_2)}{k+p_2} \gamma_\sigma \right\}. \quad (II.5)$$

Changing integration variable $k \rightarrow k - p_2$ in the second integral, and using the cyclicity of the trace we get:

$$\int \frac{dk}{(2\pi)^4} \text{Tr}\left\{ \frac{\chi(k)}{k} \gamma_\nu \gamma_5 \frac{\chi(k+p_2)}{k+p_2} \gamma_\sigma \right\} - \int \frac{dk}{(2\pi)^4} \text{Tr}\left\{ \frac{\chi(k)}{k} \gamma_\nu \gamma_5 \frac{\chi(k+p_1)}{k+p_1} \gamma_\sigma \right\} \equiv F_{\nu\sigma}(p_2) - F_{\nu\sigma}(p_1),$$

which gives zero contribution to (II.1). Therefore,

$$p_\mu I_{\mu,\nu,\sigma}(p_1, p_2) = T_{\nu\sigma}^C(p_1, p_2) + T_{\nu\nu}^C(p_2, p_1), \quad (II.7)$$

with

$$T_{\nu\sigma}^C(p_1, p_2) := \int \frac{dk}{(2\pi)^4} \text{Tr}\left\{ \frac{\chi(k)}{k} C(k, p) \frac{\chi(k+p)}{k+p} \gamma_\nu \gamma_5 \frac{\chi(k+p_2)}{k+p_2} \gamma_\sigma \right\}. \quad (II.8)$$

Using (II.4) we find

$$T_{\nu\sigma}^C(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \left\{ (1 - \chi(k))\chi(k+p_2)\chi(k) \left\{ \frac{1}{k+p} \gamma_\nu \gamma_5 \frac{1}{k+p_2} \gamma_\sigma \right\} - 
-(1 - \chi(k+p))\chi(k+p_2)\chi(k) \left\{ \frac{1}{k} \gamma_\nu \gamma_5 \frac{1}{k+p} \gamma_\sigma \right\} \right\} \equiv A_{\nu\sigma}(p_1, p_2) - B_{\nu\sigma}(p_1, p_2).$$

In the term in the first line we can shift the integration variable $k \rightarrow k - p_2$ and use the cyclicity of the trace, thus finding

$$T_{\nu\sigma}^C(p_1, p_2) := \int \frac{dk}{(2\pi)^4} \left\{ (1 - \chi(k - p_2))\chi(k + p_1)\chi(k) \left\{ \frac{1}{k} \gamma_\nu \gamma_5 \frac{1}{k+p_1} \gamma_\sigma \right\} - 
-(1 - \chi(k+p))\chi(k+p_2)\chi(k) \left\{ \frac{1}{k} \gamma_\nu \gamma_5 \frac{1}{k+p} \gamma_\sigma \right\} \right\}, \quad (II.9)$$

so that, after plugging back this expression into (II.7) and exchanging names to $(p_1, \nu) \leftrightarrow (p_2, \sigma)$ in one of the terms contributing to $T_{\nu\sigma}^C(p_1, p_2)$, we get

$$p_\mu I_{\mu,\nu,\sigma}(p_1, p_2) = \tilde{T}_{\nu\sigma}^C(p_1, p_2) + \tilde{T}_{\sigma\nu}^C(p_2, p_1),$$

with

$$\tilde{T}_{\nu\sigma}^C(p_1, p_2) := A_{\nu\sigma}(p_2, p_1) - B_{\nu\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} (k) \chi(k) \chi(k + p_2) \chi(k + p) \chi(k - p_1) \gamma_\nu \gamma_5 \frac{1}{k} \gamma_\sigma \right\} \equiv A_{\nu\sigma}(p_1, p_2) - B_{\nu\sigma}(p_1, p_2). \quad (II.10)$$

Note that $\tilde{T}_{\nu\sigma}^C(0, 0)$. Let us now expand $\tilde{T}_{\nu\sigma}^C(p_1, p_2)$ in Taylor series around $(p_1, p_2) = (0, 0)$ and let us focus on the terms of order 1 and 2 in the momenta, to be denoted by $[\tilde{T}_{\nu\sigma}^C(p_1, p_2)]^{(1)}$ and $[\tilde{T}_{\nu\sigma}^C(p_1, p_2)]^{(2)}$, respectively.

It is straightforward to check that $[\tilde{T}_{\nu\sigma}^C(p_1, p_2)]^{(1)} = 0$, by parity. Let us now consider $[\tilde{T}_{\nu\sigma}^C(p_1, p_2)]^{(2)}$, which consists of several terms:

$$[\tilde{T}_{\nu\sigma}^C(p_1, p_2)]^{(2)} = \tilde{A}_{\nu\sigma}(p_1, p_2) + \tilde{B}_{\nu\sigma}(p_1, p_2) + \tilde{C}_{\nu\sigma}(p_1, p_2) + \tilde{D}_{\nu\sigma}(p_1, p_2), \quad (II.11)$$

where:

$$\tilde{A}_{\nu\sigma}(p_1, p_2) = \frac{1}{2} (p_\mu p_\lambda - p_1, \mu p_1, \lambda) \int \frac{dk}{(2\pi)^4} \chi^2(k) \partial_\mu \partial_\lambda \chi(k) \gamma_\nu \gamma_5 \frac{1}{k} \gamma_\sigma,$$

$$\tilde{B}_{\nu\sigma}(p_1, p_2) = p_{2, \mu} (p_{1, \lambda} + p_{1, \lambda}) \int \frac{dk}{(2\pi)^4} \chi(k) \partial_\mu \chi(k) \partial_\lambda \chi(k) \gamma_\nu \gamma_5 \frac{1}{k} \gamma_\sigma,$$

$$\tilde{C}_{\nu\sigma}(p_1, p_2) = (p_\mu + p_1, \mu) \int \frac{dk}{(2\pi)^4} \chi^2(k) \partial_\mu \chi(k) \chi(k) \gamma_\nu \gamma_5 \frac{1}{k} \gamma_\sigma,$$

$$\tilde{D}_{\nu\sigma}(p_1, p_2) = -2 (p_\mu + p_1, \mu) p_{2, \lambda} \int \frac{dk}{(2\pi)^4} \chi^2(k) \partial_\mu \chi(k) \chi(k) \gamma_\nu \gamma_5 \frac{1}{k} \gamma_\sigma.$$
Now, $\tilde{A}_{\nu\sigma}(p_1, p_2) = \tilde{B}_{\nu\sigma}(p_1, p_2) = \tilde{D}_{\nu\sigma}(p_1, p_2) = 0$ by simple parity reasons: in fact, after computation of the trace,

$$\tilde{A}_{\nu\sigma}(p_1, p_2) = 2(p_\mu p_\lambda - p_{1\mu} p_{1\lambda})\varepsilon_{\alpha\nu\beta\sigma} \int \frac{dk}{(2\pi)^4} \frac{\chi'(k)\gamma_\mu k_\lambda + \chi(k)(\delta_{\mu\lambda} - \delta_{k\lambda})}{|k|^4} k_\alpha k_\beta,$$

$$\tilde{B}_{\nu\sigma}(p_1, p_2) = 4p_{2\mu}(p_\lambda + p_{1\lambda})\varepsilon_{\alpha\nu\beta\sigma} \int \frac{dk}{(2\pi)^4} \frac{\chi'(k)(\chi(k))^2(k_\mu k_\lambda)}{|k|^4},$$

$$\tilde{D}_{\nu\sigma}(p_1, p_2) = -8(p_\mu + p_{1\mu})p_2\varepsilon_{\alpha\nu\beta\sigma} \int \frac{dk}{(2\pi)^4} \frac{\chi(k)(\chi(k))^2 k_\mu k_\lambda k_\alpha k_\beta}{|k|^6},$$

which are all zero by the anti-symmetry in $\alpha \leftrightarrow \beta$. The only non-trivial term we are left with is

$$\tilde{C}_{\nu\sigma}(p_1, p_2) = 4(p_\mu + p_{1\mu})p_{2\beta}\varepsilon_{\alpha\nu\beta\sigma} \int \frac{dk}{(2\pi)^4} \frac{\chi(k)(\chi(k))^2 k_\mu k_\alpha}{|k|^4},$$

where, using that the angular integration in the 4D integral over $k$ gives $2\pi^2$, we can rewrite

$$\int \frac{dk}{(2\pi)^4} \frac{\chi(k)(\chi(k))^2 k_\mu k_\alpha}{|k|^4} = \frac{\delta_{\mu\alpha}}{4} \int \frac{dk}{(2\pi)^4} \frac{\chi^2(k)(\chi'(k))^2}{|k|^4} = \frac{\delta_{\mu\alpha}}{4} \frac{1}{(2\pi)^2} 2\pi^2 \left(-\frac{1}{3}\right) = -\frac{\delta_{\mu\alpha}}{96\pi^2},$$

where in the last equality we also used that $\int_0^\infty \chi^2(p)\chi'(p)dp = -1/3$. Plugging this back into (II.12) and using again the anti-symmetry in $\alpha \leftrightarrow \beta$, we find (recalling that $p = p_1 + p_2$)

$$\tilde{C}_{\nu\sigma}(p_1, p_2) = -\frac{1}{12\pi^2} p_{1\alpha} p_{2,\beta}\varepsilon_{\alpha\nu\beta\sigma},$$

(II.13)

so that, putting things together,

$$p_\mu I_{\mu,\nu,\sigma}(p_1, p_2) = -\frac{1}{6\pi^2} p_{1,\alpha} p_{2,\beta}\varepsilon_{\alpha\nu\beta\sigma},$$

(II.14)

up to higher order terms in $p_1, p_2$. This leads to the first equation in (13).

B. The triangle graph: anomaly of the vectorial Ward Identity

In this section we compute

$$p_{\mu,\nu,\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \frac{\chi(k)}{|k|} \gamma_{\mu,\gamma_5} \frac{\chi(k + p)}{|k + p|} \gamma_\sigma \frac{\chi(k + p_2)}{|k + p_2|}.$$  

In the first term we rename $k \to -k - p$ and use the cyclicity of the trace; in the second we rename $k \to -k - p_1$, use the cyclicity of the trace and the anti-commutation properties of $\gamma_5$; we get:

$$-p_{\mu,\nu,\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \frac{\chi(k + p)}{|k + p|} \gamma_{\mu,\gamma_5} \frac{\chi(k + p_1)}{|k + p_1|} \gamma_\sigma \frac{\chi(k + p_2)}{|k + p_2|}.$$  

(II.15)

Now, in both integrals we rewrite $\frac{\chi(k + p_1)}{|k + p_1|}$ by using (II.3):

$$\frac{\chi(k + p_1)}{|k + p_1|} = \frac{\chi(k + p_1)}{|k + p_1|} - \frac{\chi(k)}{|k|},$$

(II.16)

$$\frac{\chi(k)}{|k|}$$

(II.17)

It is easy to see that the contribution to $p_{\mu,\nu,\sigma}(p_1, p_2)$ coming from the term in square brackets in (II.17) vanishes. Therefore, we are left with

$$-p_{\mu,\nu,\sigma}(p_1, p_2) = \int \frac{dk}{(2\pi)^4} \frac{\chi(k + p)}{|k + p|} \gamma_{\mu,\gamma_5} \frac{\chi(k)}{|k|} C(k, p_1) \frac{\chi(k + p_1)}{|k + p_1|} \gamma_\sigma$$

(II.18)
that, using the explicit form of $C(k, p_1)$ becomes:

$$-p_{1,\nu}I_{\mu,\nu,\sigma}(p_1, p_2) =$$

$$= \int \frac{dk}{(2\pi)^4} \chi(k+p)(1-\chi(k))\chi(k+p)\mathrm{Tr}\left\{\frac{1}{k+p_1}\gamma_\mu\frac{1}{k+p_1}\gamma_\sigma\right\}$$

$$-\int \frac{dk}{(2\pi)^4} \chi(k+p)\chi(k)(1-\chi(k+p_1))\mathrm{Tr}\left\{\frac{1}{k+p_1}\gamma_\sigma\frac{1}{k+p_1}\gamma_\mu\right\}$$

$$+\int \frac{dk}{(2\pi)^4} \chi(k-p_2)(1-\chi(k))\chi(k+p_1)\mathrm{Tr}\left\{\frac{1}{k-p_2}\gamma_\sigma\frac{1}{k+p_1}\gamma_\mu\right\}$$

$$-\int \frac{dk}{(2\pi)^4} \chi(k-p_2)\chi(k)(1-\chi(k+p_1))\mathrm{Tr}\left\{\frac{1}{k-p_2}\gamma_\sigma\frac{1}{k+p_1}\gamma_\mu\right\}. \tag{II.19}$$

If we now rename $k \to k - p_1$ in the first line, and $k \to k + p_2$ in the third and fourth lines, we can rewrite this as

$$-p_{1,\nu}I_{\mu,\nu,\sigma}(p_1, p_2) =$$

$$= \int \frac{dk}{(2\pi)^4} \chi(k)\chi(k+p_2)\chi(k+p_1)\mathrm{Tr}\left\{\frac{1}{k+p_1}\gamma_\sigma\frac{1}{k+p_1}\gamma_\mu\right\}$$

$$+\int \frac{dk}{(2\pi)^4} \chi(k)\chi(k+p_1)\chi(k+p_1)\mathrm{Tr}\left\{\frac{1}{k+p_2}\gamma_\sigma\frac{1}{k+p_1}\gamma_\mu\right\}.$$  \tag{II.20}

The expression in the right side vanishes at $(p_1, p_2) = (0, 0)$. Also in this case, we expand it in Taylor series around the origin and focus on the terms of order 1 and 2 in the momenta. Also in this case, it is straightforward to check that the term of order 1, by parity. After having computed the trace, we find that the term of order 2 can be rewritten as

$$p_{1,\nu}I_{\mu,\nu,\sigma}(p_1, p_2)^{(2)} =$$

$$= -4[p_\nu + p_{1,\nu}p_{2,\beta} + (p_{1,\nu} - p_{2,\nu})p_\beta]\epsilon_{\alpha \beta \mu \sigma} \int \frac{dk}{(2\pi)^4} \chi^2(k)\chi'(k)\hat{k}_\nu k_\alpha. \tag{II.21}$$

Recalling that $p = p_1 + p_2$ and computing the integral over $k$ we finally get:

$$p_{1,\nu}I_{\mu,\nu,\sigma}(p_1, p_2) = \frac{1}{6\pi^2}p_{1,\alpha}p_{2,\beta}\epsilon_{\alpha \beta \mu \sigma}, \tag{II.22}$$

up to higher order terms in $p_1, p_2$. This leads to the second equation in (13).