Quantum corrections to the entropy of charged rotating black holes

M. Akbar\textsuperscript{a} and K. Saifullah\textsuperscript{b}

\textsuperscript{a}Centre for Advanced Mathematics and Physics
National University of Sciences and Technology, Rawalpindi, Pakistan
\textsuperscript{b}Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Electronic address: makbar@camp.nust.edu.pk, saifullah@qau.edu.pk

ABSTRACT: Hawking radiation from a black hole can be viewed as quantum tunneling of particles through the event horizon. Using this approach we provide a general framework for studying corrections to the entropy of black holes beyond semiclassical approximations. Applying the properties of exact differentials for three variables to the first law thermodynamics, we study charged rotating black holes and explicitly work out the corrections to entropy and horizon area for the Kerr-Newman and charged rotating BTZ black holes. It is shown that the results for other geometries like the Schwarzschild, Reissner-Nordström and anti-de Sitter Schwarzschild spacetimes follow easily.
1. Introduction

Hawking’s ground-breaking work [1] on black hole evaporation and information loss is based on the idea that a pair of particles is created just inside the event horizon and from this pair the positive energy particle tunnels out of the hole and appears as Hawking radiation. The negative energy particle tunnels inwards and results in decrease of the mass of the black hole. The energy of the particle changes sign as it crosses the horizon. These particles follow trajectories which cannot be explained classically. This process of particle evaporation from black holes is thus a phenomenon of quantum tunneling of particles through the event horizon [2]. This contributes to the change in mass, angular momentum and charge of the black hole, which change its thermodynamics as well. The particle travels in time so that the action becomes complex and the dynamics of the outgoing particle is governed by the imaginary part of this action. This action has been calculated using the radial null geodesics [2] or the so-called Hamilton-Jacobi method [3] for various spacetimes. Using the method of the radial null geodesics Hawking radiation as a tunneling process has been analyzed for the Kerr and Kerr-Newman black holes [4]. They have done the analysis using transformed forms of these metrics and they do not consider the question of quantum corrections at all. The Hamilton-Jacobi method has been used to calculate quantum corrections to the Hawking temperature and the Bekenstein-Hawking area law for the Schwarzschild, anti-de Sitter Schwarzschild and Kerr black holes [3]. These corrections have been worked out for the uncharged BTZ black hole as well [5].

In this paper we extend this analysis to black holes that are not spherically symmetric and are charged and rotating. For this purpose we set up a criterion for exactness of differential of black hole entropy, from the first law of thermodynamics for three parameters (mass, angular momentum and charge), so that entropy can be written in a convenient form. Using this we provide a sufficiently general framework for calculating quantum corrections beyond the semiclassical limit of entropy. As a result of quantum effects the famous Bekenstein-Hawking area law is also modified. We apply this to the Kerr-Newman and the charged rotating BTZ black holes, in particular. We find that the leading correction term is logarithmic, which is consistent with the results found using quantum geometry techniques and field theoretic methods. The other terms involve ascending powers of inverse of the area. Further, we show that the earlier investigations for finding (quantum) corrections are recovered as special cases of the present study.

The paper is organized as follows. In the next section we briefly provide motiva-
2. Quantum corrections for entropy of black holes

In semiclassical treatment the radial trajectories of a massless particle in a spacetime described by the metric, $g_{\mu\nu}$, have the wave function

$$\phi(r, t) = e^{-\frac{i}{\hbar} S(r, t)},$$

which satisfies the Klein-Gordon equation

$$-\frac{\hbar^2}{\sqrt{-g}} \partial_{\mu} \left[ g^{\mu\nu} \sqrt{-g} \partial_{\nu} \right] \phi = 0. \tag{2.2}$$

When a particle with positive energy crosses the horizon and tunnels out, it escapes to infinity and appears as Hawking radiation. However, when a particle with negative energy tunnels inwards it is absorbed by the black hole and as a result the mass of the black hole decreases. The essence of the quantum tunneling argument for Hawking radiation is the calculation of the imaginary part of the action, which for a black hole of radius $r$ and momentum $p_r$ is defined by [2]

$$ImS = Im \int_{r_{in}}^{r_{out}} p_r dr$$

$$= Im \int_{r_{in}}^{r_{out}} \int_{p_r}^{0} dp_r' dr. \tag{2.3}$$

If we use Hamilton’s equation

$$\dot{r} = \frac{dH}{dp_r} \bigg|_r,$$ \hspace{1cm} \tag{2.4}

in the above action, we get

$$ImS = Im \int_{r_{in}}^{r_{out}} \int_{0}^{H} \frac{dH'}{r} dr. \tag{2.5}$$
For the Schwarzschild spacetime, for example, $r_{in} = 2M$ and $r_{out} = 2(M - \omega)$, where $\omega$ denotes the energy. We expand the action $S(r, t)$ in powers of $\hbar$

$$S(r, t) = S_0(r, t) + \hbar S_1(r, t) + \hbar^2 S_2(r, t) + \ldots,$$  

(2.6)

such that $S_0$ gives the semiclassical value and higher order terms correspond to quantum corrections. An analysis similar to the one done earlier [3] shows that the correction terms $S_i$ are of the order of $\hbar^i$ and are proportional to the semiclassical contribution $S_0$. The Klein-Gordon equation and the dimensional analysis suggest that the constants of proportionality for charged rotating black holes have the dimensions of $(r_+^2 + a^2)^{-i}$, where $r_+$ represents the horizon. In order to make the corrections dimensionless, we write the series expansion as

$$S = S_0 \left[ 1 + \sum_i \frac{\alpha_i \hbar^i}{(r_+^2 + a^2)^i} \right].$$

(2.7)

This expansion will be used later to work out quantum corrections in entropy and horizon area in different geometries.

3. Exact differentials in three variables and the first law of thermodynamics

We will be using the first law of thermodynamics in three parameters, for which we need the analysis of differentials in three variables. The three dimensional differential of a function $f$,

$$df(x, y, z) = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz$$

(3.1)

is exact if the following conditions hold

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}, \quad \frac{\partial A}{\partial z} = \frac{\partial C}{\partial x}, \quad \frac{\partial B}{\partial z} = \frac{\partial C}{\partial y}. \quad (3.2)$$

Here we have

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial y} = B, \quad \frac{\partial f}{\partial z} = C.$$

(3.3)

For $df$ to be exact (3.1) will have the solution of the form
\[ f(x, y, z) = \int A \, dx + \int B \, dy + \int C \, dz \]
\[ - \int \left( \frac{\partial}{\partial y} \left( \int A \, dx \right) \right) \, dy - \int \left( \frac{\partial}{\partial z} \left( \int A \, dx \right) \right) \, dz - \int \left( \frac{\partial}{\partial z} \left( \int B \, dy \right) \right) \, dz \]
\[ + \int \frac{\partial}{\partial z} \left( \int \left( \frac{\partial}{\partial y} \left( \int A \, dx \right) \right) \, dy \right) \, dz. \tag{3.4} \]

We apply this criteria to the first law of thermodynamics for charged rotating black holes, which for three parameters \( M, J, Q \), the mass, angular momentum and charge of the black hole, respectively, can be written in the form

\[ dM = TdS + \Omega dJ + \Phi dQ, \tag{3.5} \]

where, \( T \) is the temperature, \( S \) entropy, \( \Omega \) angular velocity and \( \Phi \) electrostatic potential of the black hole. Or, we can rewrite as

\[ dS(M, J, Q) = \frac{1}{T} dM - \frac{\Omega}{T} dJ - \frac{\Phi}{T} dQ. \tag{3.6} \]

Written in this way the \( A, B, C \) of conditions (3.2) will be replaced by \( 1/T, \ -\Omega/T, \ -\Phi/T \), in which case \( M, J, Q \) will play the role of \( x, y, z \), respectively. In other words, we note that in order for \( dS \) to be an exact differential (3.6) the following conditions must be satisfied

\[ \frac{\partial}{\partial J} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Omega}{T} \right), \tag{3.7} \]
\[ \frac{\partial}{\partial Q} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Phi}{T} \right), \tag{3.8} \]
\[ \frac{\partial}{\partial Q} \left( -\frac{\Omega}{T} \right) = \frac{\partial}{\partial J} \left( -\frac{\Phi}{T} \right). \tag{3.9} \]

Now we can use (3.4) to write entropy for the black hole in integral form as

\[ S(M, J, Q) = \int \frac{1}{T} \, dM - \int \frac{\Omega}{T} \, dJ - \int \frac{\Phi}{T} \, dQ \]
\[ - \int \left( \frac{\partial}{\partial J} \left( \int \frac{1}{T} \, dM \right) \right) \, dJ \]

\[ - 5 - \]
\[ - \int \left( \frac{\partial}{\partial Q} \left( \int \frac{1}{T} dM \right) \right) dQ + \int \left( \frac{\partial}{\partial Q} \left( \int \frac{\Omega}{T} dJ \right) \right) dQ + \int \frac{\partial}{\partial Q} \left( \int \left( \frac{\partial}{\partial J} \left( \int \frac{1}{T} dM \right) \right) dJ \right) dQ. \tag{3.10} \]

We will see in the next sections that this provides a convenient way to calculate quantum corrections for entropy of charged rotating black holes.

4. Entropy corrections for the Kerr-Newman black hole

We want to calculate quantum corrections to entropy beyond the semiclassical limit for charged rotating black holes. The corresponding Bekenstein-Hawking area law will also undergo corrections and will be modified. For this purpose, we will use the criterion described above for exactness of differential of entropy. We first consider the Kerr-Newman black hole, which in the Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) is given by

\[ ds^2 = -\frac{\Delta^2}{\rho^2} (dt - asin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta^2} dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2} (adt - (r^2 + a^2)d\phi)^2, \]

where

\[ \Delta(r)^2 = (r^2 + a^2) - 2Mr + Q^2, \]
\[ \rho^2(r, \theta) = r^2 + a^2 cos^2\theta, \]
\[ a = \frac{J}{M}. \]

To obtain the horizons for this metric, we put \(g^{rr} = 0\), and get

\[ r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{4.1} \]

The Hawking temperature is defined as

\[ T = \left( \frac{\hbar}{4\pi} \right) \left( \frac{r_+ - r_-}{r_+^2 + a^2} \right), \tag{4.2} \]

which, in our case takes the form

\[ T = \left( \frac{\hbar}{2\pi} \right) \frac{\sqrt{M^4 - J^2 - Q^2M^2}}{M \left( 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2M^2} \right)}. \tag{4.3} \]
The angular velocity, \( \Omega = a / (r_+^2 + a^2) \), for the above metric takes the form

\[
\Omega = \frac{J}{M \left( 2M^2 - Q^2 + 2 \sqrt{M^4 - J^2 - Q^2 M^2} \right)},
\]

and the electrostatic potential, \( \Phi = r_+ Q / (r_+^2 + a^2) \), becomes

\[
\Phi = \frac{Q \left( M^2 + \sqrt{M^4 - J^2 - Q^2 M^2} \right)}{M \left( 2M^2 - Q^2 + 2 \sqrt{M^4 - J^2 - Q^2 M^2} \right)}.
\]

(4.5)

It can easily be verified that the above quantities for the Kerr-Newman black hole satisfy conditions (3.7)-(3.9). Thus \( dS \) is an exact differential and we can use the integral form (3.10) to work out the semiclassical entropy for this black hole. In order to do this note that the expressions for \( T \), \( \Omega \) and \( \Phi \) as given above satisfy

\[
\frac{\partial}{\partial J} \int \frac{dM}{T} = -\frac{\Omega}{T},
\]

(4.6)

\[
\frac{\partial}{\partial Q} \int \frac{dM}{T} = -\frac{\Phi}{T},
\]

(4.7)

\[
\frac{\partial}{\partial Q} \int \left( -\frac{\Omega dJ}{T} \right) = -\frac{\Phi}{T}.
\]

(4.8)

If we use these in (3.10), it reduces to

\[
S(M, J, Q) = \int \frac{dM}{T} = \frac{2\pi}{\hbar} \int \frac{M \left( 2M^2 - Q^2 + 2 \sqrt{M^4 - J^2 - Q^2 M^2} \right)}{\sqrt{M^4 - J^2 - Q^2 M^2}} dM
\]

\[
= \frac{\pi}{\hbar} \left( 2M^2 - Q^2 + 2 \sqrt{M^4 - J^2 - Q^2 M^2} \right),
\]

(4.9)

which is the standard black hole entropy given in the literature [6, 7]. Using (4.1) this can also be written as

\[
S = \frac{\pi}{\hbar} (r_+^2 + a^2).
\]

(4.10)

Our aim now is to calculate quantum corrections to this formula. In order to do this we need to use the corrected form of the Hawking temperature [3] which in our case is given by
\[ T_c = T \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right)^{-1}, \]  
(4.11)

where \( \alpha_i \) correspond to higher order loop corrections to the surface gravity of black holes \( \mathcal{K} = 2\pi T \), and the modified form of the surface gravity \[8\] due to quantum effects becomes

\[ \mathcal{K} = \mathcal{K}_0 \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right)^{-1}. \]  
(4.12)

Now in the first law of thermodynamics (3.6) temperature, \( T \), will be replaced by the corrected temperature, \( T_c \). If we include the correction terms the conditions (3.7)-(3.9) take the following form

\[
\frac{\partial}{\partial J} \left( \frac{1}{T} \right) \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) = \frac{\partial}{\partial M} \left( \frac{-\Omega}{T} \right) \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right), \]  
(4.13)

\[
\frac{\partial}{\partial Q} \left( \frac{1}{T} \right) \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) = \frac{\partial}{\partial M} \left( \frac{-\Phi}{T} \right) \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right), \]  
(4.14)

\[
\frac{\partial}{\partial Q} \left( \frac{-\Omega}{T} \right) \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) = \frac{\partial}{\partial J} \left( \frac{-\Phi}{T} \right) \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right). \]  
(4.15)

These expressions replace the earlier conditions for exactness of the entropy. It is not difficult to verify these relations, for all orders, by using from (4.3)-(4.5). In general, quantum corrections to the temperature are accompanied by corrections to the angular velocity and the electrostatic potential, in which case they always make \( dS \) exact. It is interesting to note that in our case we do not require corrections for \( \Omega \) and \( \Phi \). This is due to the particular functional form of (4.11). Thus the corrected form of entropy also satisfies the exactness criteria for differentials in three variables and, therefore, can be written in the following integral form.

\[
S(M, J, Q) = \int \frac{1}{T} \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) dM - \int \frac{\Omega}{T} \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) dJ
- \int \frac{\Phi}{T} \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) dQ
- \int \left( \frac{\partial}{\partial J} \left( \int \frac{1}{T} \left( 1 + \sum \frac{\alpha_i h^i}{(r_+^2 + a^2)^i} \right) dM \right) \right) dJ
\]
\[- \int \left( \frac{\partial}{\partial Q} \left( \int \frac{1}{T} \left( 1 + \sum \frac{\alpha_i h_i}{(r^2_+ + a^2)^i} \right) dM \right) \right) dQ \]
\[+ \int \left( \frac{\partial}{\partial Q} \left( \int \frac{\Omega}{T} \left( 1 + \sum \frac{\alpha_i h_i}{(r^2_+ + a^2)^i} \right) dJ \right) \right) dQ \]
\[+ \int \frac{\partial}{\partial Q} \left( \int \left( \frac{\partial}{\partial J} \left( \int \frac{1}{T} \left( 1 + \sum \frac{\alpha_i h_i}{(r^2_+ + a^2)^i} \right) dM \right) \right) \right) dJ \right) dQ. \quad (4.16)\]

This is the corrected and modified form of (3.10). These complicated integrals can be simplified by employing the exactness criterion described above. Using argument similar to the one adopted to write the integral in (4.9) above, we find that (4.16) reduces to

\[S(M, J, Q) = \int \frac{1}{T} \left( 1 + \sum \frac{\alpha_i h_i}{(r^2_+ + a^2)^i} \right) dM, \quad (4.17)\]

which can be written in the expanded form as

\[S(M, J, Q) = \int \frac{1}{T} dM + \int \frac{\alpha_1 h}{T(r^2_+ + a^2)} dM + \int \frac{\alpha_2 h^2}{T(r^2_+ + a^2)^2} dM \]
\[+ \int \frac{\alpha_3 h^3}{T(r^2_+ + a^2)^3} dM + \cdots \]
\[= I_1 + I_2 + I_3 + I_4 + \cdots, \quad (4.18)\]

where the first integral \(I_1\) has been evaluated in (4.9). We work out the other integrals one by one after substituting values from (4.1) and (4.3). Thus

\[I_2 = 2\pi\alpha_1 \int \frac{MdM}{\sqrt{M^4 - J^2 - Q^2 M^2}}. \quad (4.19)\]

This can be integrated by substituting \(M^2 - Q^2 / 2 = w\), say, so that after some steps it yields

\[I_2 = \pi\alpha_1 \ln \left| 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right|, \quad (4.20)\]

which is nothing but

\[I_2 = \pi\alpha_1 \ln (r^2_+ + a^2). \quad (4.21)\]
Let us evaluate the k-th integral \( I_k \) where \( k = 3, 4, \ldots \).

\[
I_k = \int \frac{\alpha_{k-1} h^{k-1} dM}{T(r_+^2 + a^2)^{k-1}}
\]

\[
= \frac{2\pi}{\hbar} \int \frac{\alpha_{k-1} h^{k-1} M dM}{\sqrt{M^4 - J^2 - Q^2 M^2} \left[ 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right]^{k-2}}, k > 2, \quad \text{(4.22)}
\]

or, in terms of \( r_+ \), this can be written as

\[
I_k = \frac{\pi \alpha_{k-1} h^{k-2}}{(2 - k)(r_+^2 + a^2)^{k-2}}, k > 2, \quad \text{(4.23)}
\]

Thus the entropy including the correction terms becomes

\[
S = \frac{\pi}{\hbar} (r_+^2 + a^2) + \pi \alpha_1 ln(r_+^2 + a^2) + \cdots + \sum_{k>2} \frac{\pi \alpha_{k-1} h^{k-2}}{(2 - k)(r_+^2 + a^2)^{k-2}} + \cdots . \quad \text{(4.24)}
\]

Note that this is a rather general formula for entropy corrections. If we put charge, \( Q = 0 \), we recover the corrections for the case of the Kerr black black hole [3]. Further, if the angular momentum is also put equal to zero we get results for the Schwarzschild black hole (\( a = Q = 0 \)). In this case the proportionality constants in the above series involve inverse powers of square of the mass, \( M \). On the other hand if only the angular momentum vanishes (i.e. \( a = 0 \)), we get corresponding corrections for Reissner-Nordström black hole, in which case the power series involve the charge \( Q \) also, in addition to \( M \), which are the only two parameters for this object.

If we use the Bekenstein-Hawking area law relating entropy and horizon area, \( A \), by

\[
S = \frac{A}{4\hbar}, \quad \text{(4.25)}
\]

where, the area in our case is

\[
A = 4\pi (r_+^2 + a^2), \quad \text{(4.26)}
\]

from (4.24) we obtain
\[ S = \frac{A}{4\hbar} + \pi \alpha_1 \ln A - \frac{4\pi^2 \alpha_2 \hbar}{A} - \frac{8\pi^3 \alpha_3 \hbar^2}{A^2} - \ldots, \] (4.27)

which gives quantum corrections for the area law. Note that the first term is the usual semiclassical area, the second term is the logarithmic correction found earlier [9] by field theoretic arguments. The issue of the value of \( \alpha_i \)'s is highly disputatious. There are different interpretations found in the literature. The prefactor of the logarithmic term, \( \alpha_1 \), for example, is given to be \(-\frac{3}{2}\) in Ref. 10, and \(-\frac{1}{2}\) in Ref. 11. Similarly, some authors [12] take it to be a positive integer, while others find it even to be zero [13].

5. Entropy corrections for the charged rotating BTZ black hole

Next we consider the (1+2) Bañados-Teitelboim-Zanelli (BTZ) black hole [14], when it is charged and rotating. The metric can be written as [15]

\[
ds^2 = -(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r) dt^2 + (-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r)^{-1} dr^2 + r^2 (d\phi - \frac{J}{2r^2} dt)^2, \] (5.1)

where \( M \) is the mass, \( J \) the angular momentum, \( Q \) the charge of the black hole, and \( 1/l^2 = -\Lambda \) is the negative cosmological constant.

If \( r_+ \) and \( r_- \) are the event horizon and the inner horizon, respectively, then they must satisfy

\[- M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r = 0. \] (5.2)

We will not find roots of this expression to obtain \( r_+ \) as was done easily in the case of uncharged BTZ black hole [5], where the entropy depended only upon two parameters \( M \) and \( J \). Here we will circumvent the difficulty of finding the roots by using the results obtained above on exactness of the entropy function for three parameters to work out the integrals involved in the corrections. Secondly, instead of substituting the value of \( r_+ \) directly into different expressions we will be using chain rules of differentiation. Let us write
\[ f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2}Q^2 \ln r. \]  
(5.3)

The angular velocity of a rotating black hole is [16]
\[ \Omega = \left[-\frac{g_{\phi t}}{g_{\phi \phi}} - \sqrt{\left(\frac{g_{\phi t}}{g_{\phi \phi}}\right)^2 - \frac{g_{tt}}{g_{\phi \phi}}} \right]_{r=r_+}. \]  
(5.4)

For the BTZ black hole this becomes
\[ \Omega = -\frac{g_{\phi t}}{g_{\phi \phi}} \bigg|_{r=r_+} = \frac{J}{2r^2} \bigg|_{r=r_+}. \]  
(5.5)

The event horizon is related with temperature \( T \) by [6, 17]
\[ T = \frac{\hbar f'(r)}{4\pi} \bigg|_{r=r_+}, \]  
(5.6)

where \( f'(r) \) denotes the derivative of \( f \) with respect to \( r \). The electric potential is given by [15]
\[ \Phi = -\frac{\partial M}{\partial Q} \bigg|_{r=r_+} = -\pi Q \ln r_+. \]  
(5.7)

With these thermodynamic quantities the BTZ black hole satisfy the first law of thermodynamics of the form (3.5). Therefore, the semiclassical entropy becomes
\[ S = \int \frac{1}{T} dM = \frac{8\pi l^2}{\hbar} \int \frac{r_+^3 dM}{4r_+^4 - J^2l^2 - \pi Q^2l^2r_+^2}. \]  
(5.8)

In order to evaluate this integral, we note that from (5.2) we can write
\[ dM = \left(\frac{2r_+}{l^2} - \frac{J^2}{2r_+^2} - \frac{\pi Q^2}{2r_+} \right) dr_+. \]  
(5.9)

Using this in the integral in (5.8) we readily get
\[ S = \frac{4\pi r_+}{\hbar}, \]  
(5.10)

which is the standard formula for entropy in this geometry.
Now we calculate quantum corrections to the entropy. We first note that for the case of three dimensional BTZ black hole the correction terms are proportional to \((r_+)^{-i}\), so that the corrected action can be written as

\[
S = S_0 \left[ 1 + \sum \frac{\alpha_i \hbar^i}{(r_+)^i} \right].
\]  

(5.11)

We use the differential of the entropy \(dS\) from (3.6) and using the exactness criteria obtain an integral expression of the form (4.16). This expression is simplified as before, so that we obtain the following form which includes quantum corrections to the entropy as well.

\[
S(M, J, Q) = \int \frac{1}{T} \left( 1 + \sum \frac{\alpha_i \hbar^i}{(r_+)^i} \right) dM = I_1 + I_2 + I_3 + I_4 + \cdots.
\]  

(5.12)

where the uncorrected semiclassical term \(I_1\) is given by (5.10). For the correction terms we proceed as follows. The integral \(I_2\) is given by

\[
I_2 = \alpha_1 \hbar \int \frac{1}{r_+ T} dM.
\]  

(5.13)

Using (5.6) and (5.9) this simplifies to

\[
I_2 = 4 \pi \alpha_1 \int \frac{1}{r_+} dr_+ = 4 \pi \alpha_1 \ln r_+.
\]  

(5.14)

Similarly,

\[
I_3 = \alpha_2 \hbar^2 \int \frac{1}{r_+^2 T} dM = - \frac{4 \alpha_2 \hbar \pi}{r_+},
\]  

(5.15)

and so on. Thus the series becomes

\[
S = \frac{4 \pi r_+}{\hbar} + 4 \pi \alpha_1 \ln r_+ - \frac{4 \alpha_2 \hbar \pi}{r_+} - \cdots.
\]  

(5.16)

Putting \(8G_3 = 1\), where \(G_3\) is the three dimensional Newton’s gravitational constant, the area formula is

\[
A = 2 \pi r_+.
\]  

(5.17)

If we include \(G_3\) this becomes
\[ A = 16\pi G^3 r_+, \quad (5.18) \]

and the above series takes the form

\[ S = \frac{A}{4\hbar G^3} + 4\pi\alpha_1 \ln A - \frac{64\alpha_2 \hbar \pi^2 G^3}{A} - \ldots. \quad (5.19) \]

Note that if we put \( Q = 0 \), the results for the uncharged BTZ black holes [5] are recovered.

6. Conclusion

One way to describe Hawking radiation is by the process of quantum tunneling thereby particles cross event horizons and traverse “forbidden” trajectories. The positive energy particles tunnel out of the event horizon, whereas, the negative energy particles tunnel in and result in black hole evaporation. Thus a system which is stable classically becomes unstable quantum mechanically, and this changes the thermodynamics of the system as well. We have used this analysis to study quantum corrections in entropy for charged and rotating black holes. For this purpose we use the criterion for exactness of differential of entropy. With the help of this procedure the formidable integrals involving quantum corrections become manageable and we obtain power series for entropy and Bekenstein-Hawking horizon area of black holes for a quite general framework. We first applied this to the Kerr-Newman black holes. The first term obtained in the power series is the semiclassical value, while the leading correction term is logarithmic as has been found using other methods. The other terms involve ascending powers of \((r_+^2 + a^2)^{-1}\). This reduces to \((r_+^2)^{-1}\) for non-rotating objects like Reissner-Nordström black hole. If the charge and angular momentum both become zero, we obtain results for the Schwarzschild black hole, in which case the power series involve mass only as it is the only macroscopic parameter for this object. This is true for the entropy corrections of the anti-de Sitter Schwarzschild spacetimes also. An important feature of our approach is that we have obtained the corrections in entropy without requiring the corrections in the angular momentum and the electrostatic potential.

Another significant application of the procedure is for the three dimensional BTZ black hole, when it is charged and rotating. Here without having to find the horizon we have been able to calculate quantum corrections beyond semiclassical approximation by using the exactness criterion for the entropy differential. In this
case also the modified Bekenstein-Hawking area law has been derived. The leading order correction term is again found to be logarithmic in this case.

Acknowledgments

The authors are grateful to the organizers of the First NCP Scientific Spring, National Centre for Physics, Islamabad, April 6-9, 2009, where this work was presented.

References

[1] S.W. Hawking, *Black hole explosions*, Nature **248** (1989) 30;
S.W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. **43** (1975) 199;
*Erratum ibid. 46* (1976) 206.

[2] M.K. Parikh and F. Wilczek, *Hawking radiation as tunneling*, Phys. Rev. Lett. **85** (2000) 5042 [hep-th/9907001];
M.K. Parikh, *A secret tunnel through the horizon*, Gen. Rel. Grav., **36** (2004) 2419 [hep-th/0405160].

[3] R. Banerjee and B.R. Majhi *Quantum tunneling beyond semiclassical approximation*, JHEP **06** (2008) 095 [arXiv:0805.2220].

[4] Q.Q. Jiang, S.Q. Wu and X. Cai, *Hawking radiation as tunneling from the Kerr and Kerr-Newman black holes*, Phys. Rev. D **73** (2006) 064003.

[5] S.K. Modak, *Corrected entropy of BTZ black hole in tunneling approach*, arXiv:0807.0959.

[6] D. Kothawala, S. Sarkar and T. Padmanabhan, *Einstein's equations as a thermodynamic identity: The case of stationary axisymmetric horizons and evolving spherically symmetric horizons*, Phys. Lett. B **652** (2007) 338.

[7] R. G. Cai, *Critical behavior in black hole thermodynamics*, J. Korean Phys. Soc. **33** (1998) S477 [gr-qc/9901026].

[8] J.W. York, Jr., *Black hole in thermal equilibrium with a scalar field*, Phys. Rev. D **31** (1985) 775;
C.O. Lousto and N.G. Sanchez, *Back reaction in black hole space-times*, Phys. Lett. B **212** (1988) 411.
[9] D.V. Fursaev, *Temperature and entropy of a quantum black hole and conformal anomaly*, Phys. Rev. **D 51** (1995) 5352;
R.B. Mann and S.N. Solodukhin, *Universality of quantum entropy for extreme black holes*, Nucl. Phys. **B 523** (1998) 293.

[10] R.K. Kaul and P. Majumdar, *Logarithmic correction to the Bekenstein-Hawking entropy*, Phys. Rev. Lett. **84** (2000) 5255.

[11] A. Ghosh and P. Mitra, *Log correction to the black hole area law*, Phys. Rev. **D 71** (2005) 027502;
K.A. Meissner, *Black-hole entropy in loop quantum gravity*, Class. Quant. Grav. **21** (2004) 5245.

[12] S. Hod, *High-order corrections to the entropy and area of quantum black holes*, Class. Quant. Grav. **21** (2004) L97.

[13] A.J.M. Medved, *A comment on black hole entropy or does nature abhor a logarithm?*, Class. Quant. Grav. **22** (2005) 133.

[14] M. Bañados, C. Teitelboim and J. Zanelli, *Black hole in three-dimensional spacetime*, Phys. Rev. Lett. **69** (1992) 1849.

[15] M. Akbar and A.A. Siddiqui, *Charged rotating BTZ black hole and thermodynamic behavior of field equations at its horizon*, Phys. Lett. **B 656** (2007) 217.

[16] S.M. Carroll, *An introduction to general relativity: spacetime and geometry*, Addison-Wesley 2004.

[17] M. Akbar, *Thermodynamic interpretation of field equations at horizon of BTZ black hole*, Chin. Phys. Lett. **24** (2007) 1158.