Spectral characterization some new classes of multicone graphs and algebraic properties of F(2.Ok) graphs

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Abstract

If all the eigenvalues of the adjacency matrix of a graph Γ are integers, then we say that Γ is an integral graph. It seems hard to prove a graph to be determined by its spectrum. In this paper, we investigate some new classes of multicone graphs determinable by their spectra, such as: K_n △ O_k, K_n ▽ mO_k, K_n △ □_m, and K_n ▽ m□_m, where O_k denoted the Odd graph O_{2k+1} and □_m denoted the folded 2d + 1 cube. Also, the notation of the folded graph of 2.O_k will be defined and denoted by F(2.O_k). Moreover, we study some algebraic properties of the graph F(2.O_k), in fact we show that F(2.O_k) is an integral graph.

Keywords: adjacency spectrum, Laplacian spectrum, multicone graph, Odd graph, folded n cube.

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1. Introduction

The adjacency matrix A(Γ) of a (simple, undirected) graph Γ is the n × n symmetric matrix whose rows and columns are indexed by the vertices of Γ, and where A(Γ)_{xy} = 1 if and only if x is adjacent to y (that is xy ∈ E), and A(Γ)_{xy} = 0 otherwise. If all the eigenvalues of the adjacency matrix of a graph Γ are integers, then we say that Γ is an integral graph. The notion of integral graphs was first introduced by F. Harary and A. J. Schwenk in 1974 [14]. In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area [1, 2]. The square matrix A(Γ) is called diagonalizable if it is similar to a diagonal matrix, that is, if there exists an invertible matrix P such that P^{-1}A(Γ)P is a diagonal matrix. The eigenvalues of a graph Γ are the eigenvalues of the adjacency matrix of Γ. The characteristic polynomial of Γ with respect to the adjacency matrix A(Γ) is the polynomial P(Γ) = P(Γ, λ) = det(A_n − A(Γ)), where I_n denotes the n × n identity matrix. Because A(Γ) is real and symmetric, its eigenvalues are real numbers. Moreover, we can show that if v and w are eigenvectors corresponding to the distinct eigenvalues λ and μ of A(Γ), respectively, then v and w are orthogonal and we can conclude A(Γ) is diagonalizable. The join Γ_1 ⊔ Γ_2 is the graph obtained from Γ_1 ⊔ Γ_2 by joining every vertex of Γ_1 with every vertex of Γ_2. A multicone graph is defined to be the join of a complete graph and a regular graph [23]. For a graph Γ let L(Γ) = D(Γ) − A(Γ) be the Laplacian matrix of Γ, where D(Γ) is the diagonal matrix of vertex degrees with \{d_1, d_2, ..., d_n\} as diagonal entries. The polynomial P(L(Γ), μ) = det(μI_n − L(Γ)), is called the characteristic polynomial of the graph Γ with respect to the Laplacian matrix. The spectrum of Γ is the list of the eigenvalues of the adjacency matrix of Γ together with their multiplicities, and it is denoted by Spec(Γ), see [10, 13]. Laplacian spectra and their applications are involved indiverse theoretical problems on complex networks [12, 26]. Many results have been devoted to studying Laplacian spectra for complex networks [18, 25]. Calculating the Laplacian spectra of networks has many applications in lots of aspects, such as the topological structures and dynamical processes [7, 19].

Two graphs with the same spectrum are called cospectral. However, it is not hard to see that the spectrum of a graph does not determine its isomorphism class, see [13]. The authors in [11] proposed the question: which graphs are determined by their spectrum? It seems hard to prove a graph to be determined by its spectrum. Up to now, only
few graphs are proved to be determined by their spectra, such as: the path $P_n$, the complete graph $K_n$ and the cycle $C_n$. Let us abbreviate ‘determined by the spectra’ to DS, see [17, 21, 22]. Here, of course, ‘spectra’ (and DS) depends on the type of matrix. If the matrix is not specified, we mean the ordinary adjacency matrix. In particular a spectral characterization of multicone graph is studied in [23] and some classes of multicone graphs have been studied by various authors. In this paper, we investigate some new classes of multicone graphs determinable by their spectra.

Suppose $k$ is an integer not less than 2 and $[n]$ is a set of odd cardinality $2k - 1$, that is, $[n] = \{1, 2, ..., 2k - 1\}$. We denote by $O_k$ the Odd graph as follows: the vertex set $V$ of $O_k$, is the set of subsets $v$ of $[n]$ with cardinality $|v| = k - 1$, and two vertices are adjacent when the subsets are disjoint $[3, 5]$. The graph $2.O_k (k \geq 2)$, where $2.O_k$ is the bipartite double graph of the Odd graph $O_k$, has $2^{[n]} \cap \{v \mid v \neq \emptyset\}$ vertices indexed by the set $(v, i)$, where $v$ is a $(k - 1)$-subset of $[n]$ and $i \in \{0, 1\}$. Two vertices $(v, i)$ and $(w, j)$ are adjacent if and only if $v \cap w = \emptyset$ and $i \neq j$ [3]. In this paper, the notation of the folded graph of $2.O_k$ will be defined and denoted by $F(2.O_k)$, as the graph whose vertex set is identical to the vertex set of $2.O_k$, and with edge set $E_1 = E_0 \cup \{(v, i), (v, i')\} | (v, i), (v, i') \in V(2.O_k), i, i' \in \{0, 1\}$, where $i'$ is the complement of $i (1' = 0, 0' = 1)$, and $E_1$ is the edge set of $2.O_k$. It is clear that this graph is a regular bipartite graph of degree $k + 1$. The bipartite double graph of the Odd graph $O_k$ is known as the middle cube $M_{Q_k}$ (Dalfó, Fiol, Mitjana [9]). The graph $M_{Q_k}$ has been studied by various authors [3, 5, 9, 24]. One of our goals in this paper is to obtain all eigenvalues of the graph $F(2.O_k)$, by using theory of equitable partition of graphs. First, we show that multicone graphs $K_v \cup O_k$ and $K_v \cup mO_k$ are determined by their adjacency spectra as well as their Laplacian spectra, where $mO_k$ denoted union $m$ copies of $O_k$ and $K_v$ is the complete graph on $v$ vertices. Also, we study some of the properties of the bipartite double graph of the Odd graph $O_k$. Moreover, we show that $F(2.O_k)$ is a vertex transitive graph, and we determine the automorphism group of $F(2.O_k)$, in fact we prove that $\text{Aut}(F(2.O_k)) \cong \mathbb{Z}_2 \times \text{Sym}(2k - 1)$, where $\mathbb{Z}_2$ is the cyclic group of order 2. In particular, we completely determine all the eigenvalues of $F(2.O_k)$ in the light of the theory of equitable partition of graphs, indeed, we prove that $F(2.O_k)$ is an integral graph. In the sequel, we show that multicone graphs $K_v \cup \square$, and $K_v \cup \overline{\square}$, are determined by their adjacency spectra as well as their Laplacian spectra, where $\overline{\square}$ denoted union $m$ copies of $\square$.

2. Definitions And Preliminaries

**Definition 2.1.** [13] Let $\Gamma$ and $\Lambda$ be two graphs. A mapping $f$ from $V(\Gamma)$ to $V(\Lambda)$ is a homomorphism if $f(x)$ and $f(y)$ are adjacent in $\Lambda$ whenever $x$ and $y$ are adjacent in $\Gamma$. If $f$ is a homomorphism from $\Gamma$ to $\Lambda$, then the preimages $f^{-1}(w)$ of each vertex $w$ in $\Lambda$ are called the fibres of $f$. A homomorphism $f$ from $\Gamma$ to $\Lambda$ is a local isomorphism if for each vertex $w$ in $\Lambda$, the induced mapping from the set of neighbours of a vertex in $f^{-1}(w)$ to the neighbours of $w$ is bijective. We call $f$ a covering map if it is a surjective local isomorphism, in which case we say that $\Gamma$ covers $\Lambda$.

**Definition 2.2.** [4, 6] For any vertex $v$ of a connected graph $\Gamma$, we define the $r$-distance graph as

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid d(u, v) = r\},$$

where $r$ is a non-negative integer not exceeding $d$, the diameter of $\Gamma$. It is clear that $\Gamma_0(v) = \{v\}$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(v), ..., \Gamma_d(v)$, for each $v$ in $V(\Gamma)$. The graph $\Gamma$ is called distance regular with diameter $d$ and intersection array $[c_0, c_1, ..., c_{d-1}; a_1, ..., a_d]$ if it is regular of valency $k$ and, for any two vertices $u$ and $v$ in $\Gamma$ at distance $r$, we have $[\Gamma_{r+1}(v) \cap \Gamma_1(u)] = b_r$, and $[\Gamma_{r-1}(v) \cap \Gamma_1(u)] = c_r$ ($0 \leq r \leq d$). The intersection numbers $a_r, b_r$ and $c_r$ satisfy

$$a_r = k - b_r - c_r \quad (0 \leq r \leq d),$$

where $a_r$ is the number of neighbours of $u$ in $\Gamma_r(v)$ for $d(u, v) = r$.

**Theorem 2.1.** [6] Let $\Gamma$ be a distance regular graph with valency $k$, diameter $d$, adjacency matrix $A$, and intersection array

$$[b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d].$$

Then, the tridiagonal $(d + 1) \times (d + 1)$ matrix
Theorem 2.7. \[ \Lambda \]

Let \( \Lambda \) be the spectral characterization of multicone graph of two graphs. Then it is well known that \( \Lambda \) is either a regular graph or a bidegree graph, in which, each vertex is of degree either \( 1 \) or \( \infty \).

Theorem 2.6. \[ \Gamma \]

Theorem 2.5. \[ \Omega \]

Theorem 2.4. \[ \Pi \]

Theorem 2.3. \[ \Omega \]

Theorem 2.2. \[ \Omega \]

Theorem 2.1. \[ \Omega \]

Definition 2.3. \[ \Theta \]

Definition 2.2. \[ \Theta \]

Definition 2.1. \[ \Theta \]

Determines all the eigenvalues of \( \Gamma \).

Definition 2.3. \[ \Theta \]

A partition \( \pi = \{ C_1, C_2, \ldots, C_\ell \} \) of the \( n \) vertices of a graph \( \Gamma \) is equitable, if the number of neighbors in \( C_j \) of a vertex \( u \) in \( C_j \) is a constant \( b_j \) independent of \( u \). Let \( \pi \) be an equitable partition of the graph \( \Gamma \) as follows, the directed graph with vertex set \( \pi \) with \( b_j \) arcs from \( C_j \) to \( C_i \) is called the quotient of \( \Gamma \) over \( \pi \) and is denoted by \( \Gamma/\pi \); also we denote the adjacency matrix of the directed graph \( \Gamma/\pi \) by the matrix \( (B_\pi)_{ij} = (b_{ij}) \). It is clear that if \( \pi \) is an equitable partition, then every vertex in \( C_i \) has the same valency. Note that the partition matrix \( (B_\pi)_{ij} = (b_{ij}) \) is well-defined if and only if the partition \( \pi \) is equitable.

Definition 2.4. \[ \Theta \]

Let \( \Gamma \) be a graph. If \( H \leq \text{Aut}(\Gamma) \) is a group of automorphisms of \( \Gamma \), then \( H \) partition the vertex set of \( \Gamma \) into orbits. The partition of \( \Gamma \) consisting of the set of orbits which are constructed by \( H \), is called an orbit partition of \( \Gamma \).

Theorem 2.2. \[ \Theta \]

Let \( \Gamma \) be a graph with equitable partition \( \pi \). Let \((B_\pi)_{ij} = (b_{ij})\) be the adjacency matrix of the directed graph \( \Gamma/\pi \) and \( \Lambda \) be the adjacency matrix of \( \Gamma \). Then, each eigenvalue of the matrix \((B_\pi)_{ij} = (b_{ij})\) is an eigenvalue of the matrix \( \Lambda \).

Theorem 2.3. \[ \Theta \]

Let \( \Gamma \) be a vertex transitive graph and \( \pi \) the orbit partition of some subgroup \( H \) of \( \text{Aut}(\Gamma) \). If \( \pi \) has a singleton cell \( \{a\} \), then every eigenvalue of \( \Gamma \) is an eigenvalue of \( \Gamma/\pi \).

Theorem 2.4. \[ \Theta \]

Let \( \Gamma \) be an \( r_i \)-regular graph of \( n_i \) \((i = 1, 2)\) vertices. Then
\[
P(\Gamma_1 \vee \Gamma_2, \lambda) = \frac{P(\Gamma_1, \lambda)P(\Gamma_2, \lambda)}{(\lambda - r_1)(\lambda - r_2)}(\lambda^2 - r_1r_2 - n_1n_2).
\]

Theorem 2.5. \[ \Theta \]

Let \( \Gamma \) be a graph with \( n \) vertices, \( m \) edges and \( \delta = \delta(\Gamma) \) be the minimum degree of vertices of \( \Gamma \). Then
\[
\varrho(\Gamma) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}},
\]
where the spectral radius \( \varrho(\Gamma) \) of \( \Gamma \) is the largest eigenvalue of its adjacency matrix. Equality holds, if and only if \( \Gamma \) is either a regular graph or a bidegree graph, in which, each vertex is of degree either \( \delta \) or \( n - 1 \).

Theorem 2.6. \[ \Theta \]

Let \( \Gamma_1 \) and \( \Gamma_2 \) be two graphs with the Laplacian spectrum \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) and \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \), respectively. Then, the Laplacian spectrum of \( \Gamma_1 \vee \Gamma_2 \), is \( n + m, m + \lambda_1, m + \lambda_2, \ldots, m + \lambda_{n-1}, n + \mu_1, n + \mu_2, \ldots, n + \mu_{m-1}, 0 \).

Theorem 2.7. \[ \Theta \]

Let \( \Gamma \) be a graph on \( n \) vertices. Then, \( n \) is a Laplacian eigenvalue of \( \Gamma \) if and only if \( \Gamma \) is the join of two graphs.

3. Main results

Spectral Characterization of Multicone Graph \( K_v \vee O_k \)

It seems hard to prove a graph to be determined by its spectrum. In this paper, we investigate new classes of multicone graphs determinable by their spectra. Suppose \( \Gamma \) is a connected, \( k \)-regular graph such that \( \text{Spec}(\Gamma) \equiv \text{Spec}(\Lambda) \) where \( \Lambda \) is a distance regular graph of diameter \( d \) with parameters \( a_1 = a_2 = \ldots = a_{d-1} = 0 \) and \( a_d > 0 \). It is well known that \( \Gamma \) must be distance regular, with the same parameters as \( \Lambda \). Furthermore, \( \Gamma \) is isomorphic to \( \Lambda \) if \( \Lambda \) is one of the Odd graph \( O_{d+1} \), the folded \( 2d + 1 \)-cube and see [16]. In this section we show that multicone graph \( K_v \vee O_k \) is determined by their adjacency spectra as well as their Laplacian spectra.
Proposition 3.1. Let $\Gamma$ be a graph cospectral with the multicone graph $K\circ O_k$ with respect to its adjacency matrix spectrum. Then
\[
\Spec(\Gamma) = \{\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, -1^{(r-1)}, (P + \sqrt{P^2 + 4Q}), (P - \sqrt{P^2 + 4Q})\},
\]
where $P = k + v - 1$, $Q = \frac{(2k-1)!}{(k-1)!v - k(v - 1)}$, $\lambda_i = (-1)^{j(i - 1)}$ and $m(\lambda_i) = \frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ for $1 \leq i \leq k - 1$.

Lemma 3.1. Let $\Gamma$ be a graph cospectral with multicone graph $K\circ O_k$. Then $\Gamma$ is a bipartite graph.

Theorem 3.1. Consider the multicone graph $K\circ O_k$. Then $K\circ O_k$ is DS with respect to its adjacency matrix spectrum.

Proof. We proceed by induction on the number of vertices in $K\circ$. Let $K\circ$ have one vertex and $\Gamma$ be a graph cospectral with multicone graph $K\circ O_k$ with respect to its adjacency matrix spectrum. By Lemma 3.1, it is easy to see that $\Gamma$ has one vertex of degree $\frac{(2k-1)!}{(k-1)!v - k(v - 1)}$, say $j$. Hence, if $\Spec(\Gamma - j) = \Spec(O_k)$, then $\Gamma - j \cong O_k$, because $O_k$ is a distance regular DS graph see [11].

Note that if $|n| = \{1, 2, \ldots, k - 1\}$ then the diameter of $O_k$ is $k - 1$ and hence $O_{k+1} = O_k$. So $\Gamma \cong K\circ O_k$. We assert inductively that this claim holds for $K\circ$, that is, if $\Spec(\Gamma_1) = \Spec(K\circ O_k)$, then $\Gamma_1 \cong K\circ O_k$, where $K\circ$ is a graph cospectral with multicone graph $K\circ O_k$ with respect to its adjacency matrix spectrum. We show that the claim is true for $K\circ$, that is, if $\Spec(\Gamma) = \Spec(K\circ O_k)$, then $\Gamma \cong K\circ O_k$, where $\Gamma$ is a graph cospectral with multicone graph $K\circ O_k$ with respect to its adjacency matrix spectrum. It is obvious that $\Gamma$ has one vertex and $\frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ edges more than $\Gamma_1$. On the other hand, by Lemma 3.1, we know that $\Gamma_1$ has $v$ vertices of degree $\frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ vertices of degree $k + v$. In particular, $\Gamma$ has $v + 1$ vertices of degree $\frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ vertices of degree $k + v - 1$. So, we must have $\Gamma \cong K\circ \circ \Gamma_1$. Now, by the induction hypothesis, we conclude that $\Gamma \cong K\circ O_k$, and complete the proof.

Theorem 3.2. Consider the multicone graph $K\circ O_k$. Then $K\circ O_k$ is DS with respect to its Laplacian spectrum.

Proof. We know that the Laplacian matrix spectrum of $O_k$ is $\mu_i = k - \lambda_i$, where $m(\mu_i) = \frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ for $0 \leq i \leq k - 1$. Also, the Laplacian matrix spectrum of $K\circ O_k$ is
\[
\Spec(L(\Gamma)) = \Spec(L(K\circ O_k)),
\]
where $m(\mu_i + v) = \frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ for $1 \leq i \leq k - 1$. We proceed by induction on the number of vertices in $K\circ$. Let $K\circ$ have one vertex and $\Gamma$ be a graph cospectral with the multicone graph $K\circ O_k$ with respect to its adjacency Laplacian spectrum, that is, $\Spec(L(\Gamma)) = \Spec(L(K\circ O_k))$. By Theorem 2.7, we can show that $\Gamma \cong K\circ O_k$. We assume inductively that this claim holds for $K\circ$, that is, if $\Spec(L(\Gamma_1)) = \Spec(L(K\circ O_k))$, then $\Gamma_1 \cong K\circ O_k$, where $\Gamma_1$ is a graph cospectral with the multicone graph $K\circ O_k$ with respect to its Laplacian spectrum. We show that the claim is true for $K\circ$, that is, if
\[
\Spec(L(\Gamma)) = \Spec(L(K\circ O_k))
\]
where $m(\mu_i + v + 1) = \frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ for $1 \leq i \leq k - 1$.

Then $\Gamma \cong K\circ O_k$, where $\Gamma$ is a graph cospectral with the multicone graph $K\circ O_k$ with respect to its Laplacian spectrum. By Theorem 2.7, we know that $\Gamma_1$ and $\Gamma$ are join of two graphs, because $\frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ and $\frac{(2k-1)!}{(k-1)!v - k(v - 1)} + v$ are eigenvalues of $\Gamma_1$ and $\Gamma$, respectively. In addition, $\Gamma$ has one vertex of degree $v + \frac{(2k-1)!}{(k-1)!v - k(v - 1)}$ more than $\Gamma_1$, say $j$. Hence $\Spec(L(\Gamma - j)) \cong \Spec(L(K\circ O_k))$. Now, by the induction hypothesis, we conclude that $\Gamma - j \cong K\circ O_k$ and complete the proof.

Spectral Characterization of Multicone Graph $K\circ mO_k$

In this section, it is shown that multicone graph $K\circ mO_k$ is determined by its adjacency spectra as well as their Laplacian spectra.
Proposition 3.2. Let \( \Gamma \) be a graph cospectral with the multicone graph \( K_v \vee mO_k \) with respect to its adjacency matrix spectrum. Then

\[
\text{Spec}(\Gamma) = \{d_0, d_1, d_2, \ldots, d_{k-1}, -1^{(v-1)}\}, \quad \left(\frac{P + \sqrt{P^2 + 4Q}}{2}, \frac{P - \sqrt{P^2 + 4Q}}{2}\right),
\]

where \( P = k + v - 1, Q = \binom{2k-1}{k-1}mv - k(v-1), A_i = (-1)^i(k-i) \) for \( 0 \leq i \leq k-1 \), and \( m(\lambda_0) = m-1, m(\lambda_i) = m(\binom{2k-1}{i} - \binom{2k-1}{i-1}) \) for \( 1 \leq i \leq k-1 \).

Lemma 3.2. Let \( \Gamma \) be a graph cospectral with multicone graph \( K_v \vee mO_k \). Then \( \Gamma \) is a bidirected graph.

Theorem 3.3. Consider the multicone graph \( K_v \vee O_k \). Then \( K_v \vee mO_k \) is DS with respect to its adjacency matrix spectrum.

Proof. We proceed by induction on the number of vertices in \( K_v \). Let \( K_v \) have one vertex and \( \Gamma \) be a graph cospectral with multicone graph \( K_v \vee mO_k \) with respect to its adjacency matrix spectrum by Lemma 3.2. It is easy to see that \( \Gamma \) has one vertex of degree \( \binom{2k-1}{k-1}m \), say \( j \). Hence, if \( \text{Spec}(\Gamma - j) = \text{Spec}(mO_k) \), then \( \Gamma - j \cong mO_k \), because \( mO_k \) is a distance regular DS graph and the union of regular DS graphs with the same degree is always DS, see [11]. So \( \Gamma \cong K_1 \vee mO_k \). We assume inductively that this claim holds for \( K_v \), that is, if \( \text{Spec}(\Gamma_1) = \text{Spec}(K_v \vee mO_k) \), then \( \Gamma_1 \cong K_v \vee mO_k \), where \( \Gamma_1 \) is a graph cospectral with multicone graph \( K_v \vee mO_k \) with respect to its adjacency matrix spectrum. We show that the claim is true for \( K_{v+1} \), that is, if \( \text{Spec}(\Gamma) = \text{Spec}(K_{v+1} \vee mO_k) \), then \( \Gamma \cong K_{v+1} \vee mO_k \), where \( \Gamma \) is a graph cospectral with multicone graph \( K_{v+1} \vee mO_k \) with respect to its adjacency matrix spectrum. It is obvious that \( \Gamma \) has one vertex and \( \binom{2k-1}{k-1}m + v \) edges more than \( \Gamma_1 \). On the other hand, by Lemma 3.2, we know that \( \Gamma_1 \) has \( v \) vertices of degree \( \binom{2k-1}{k-1}m + v - 1 \) and \( \binom{2k-1}{k-1}m \) vertices of degree \( k + v \). In particular, \( \Gamma \) has \( v + 1 \) vertices of degree \( \binom{2k-1}{k-1}m + v \) and \( \binom{2k-1}{k-1}m \) vertices of degree \( k + v + 1 \). So, we must have \( \Gamma \cong K_1 \vee \Gamma_1 \). Now, by the induction hypothesis, we conclude that \( \Gamma \cong K_{v+1} \vee mO_k \), and complete the proof. \( \square \)

Theorem 3.4. Consider the multicone graph \( K_v \vee mO_k \). Then \( K_v \vee mO_k \) is DS with respect to its Laplacian spectrum.

Proof. We know that the Laplacian matrix spectrum of \( mO_k \) is \( \mu_i = k - \lambda_i \), where \( m(\mu_i) = m(\binom{2k-1}{i} - \binom{2k-1}{i-1}) \) for \( 0 \leq i \leq k-1 \). Also, the Laplacian matrix spectrum of \( K_v \) is \( \{v^{(v-1)}, 0\} \). So, by Theorem 2.6, the Laplacian matrix spectrum of \( K_v \vee mO_k \) is

\[
\{(\binom{2k-1}{k-1}m + v)^{(v)}, (\mu_1 + v), (\mu_2 + v), (\mu_3 + v), \ldots, (\mu_{k-1} + v), v^{(m-1)}, 0\},
\]

where \( m(\mu_i + v) = m(\mu_i) \) for \( 1 \leq i \leq k-1 \). We proceed by induction on the number of vertices in \( K_v \). Let \( K_v \) have one vertex and \( \Gamma \) be a graph cospectral with the multicone graph \( K_v \vee mO_k \) with respect to its Laplacian spectrum, that is, \( \text{Spec}(L(\Gamma_1)) = \text{Spec}(L(K_v \vee mO_k)) \). By Theorem 2.7, we can show that \( \Gamma \cong K_1 \vee mO_k \). We assume inductively that this claim holds for \( K_v \), that is, if \( \text{Spec}(L(\Gamma_1)) = \text{Spec}(L(K_v \vee mO_k)) \), then \( \Gamma_1 \cong K_v \vee mO_k \), where \( \Gamma_1 \) is a graph cospectral with the multicone graph \( K_v \vee mO_k \) with respect to its Laplacian spectrum. We show that the claim is true for \( K_{v+1} \), that is, if

\[
\text{Spec}(L(\Gamma)) = \text{Spec}(L(K_{v+1} \vee mO_k))
\]

\[
= \{(\binom{2k-1}{k-1}m + v + 1)^{(v+1)}, (\mu_1 + v + 1), (\mu_2 + v + 1), (\mu_3 + v + 1), \ldots, (\mu_{k-1} + v + 1), (v + 1)^{(m-1)}, 0\},
\]

where \( m(\mu_i + v + 1) = m(\mu_i) \) for \( 1 \leq i \leq k-1 \). Then \( \Gamma \cong K_{v+1} \vee mO_k \), where \( \Gamma \) is a graph cospectral with the multicone graph \( K_{v+1} \vee mO_k \) with respect to its Laplacian spectrum. By Theorem 2.7, we know that \( \Gamma_1 \) and \( \Gamma \) are join of two graphs, because \( \binom{2k-1}{k-1}m + v \) and \( \binom{2k-1}{k-1}m + v + 1 \) are eigenvalues of \( \Gamma_1 \) and \( \Gamma \), respectively. In addition, \( \Gamma \) has one vertex of degree \( v + \binom{2k-1}{k-1}m \) more than \( \Gamma_1 \), say \( j \). Hence, \( \text{Spec}(L(\Gamma - j)) \cong \text{Spec}(L(K_v \vee mO_k)) \). Now, by the induction hypothesis, we conclude that \( \Gamma - j \cong K_v \vee mO_k \). Thus, it can be concluded \( \Gamma \cong K_{v+1} \vee mO_k \). \( \square \)
Vertex Transitivity and Automorphism of $F(2,O_k)$

Let $\Gamma$ be a graph with automorphism group $\text{Aut}(\Gamma)$. We know that $\Gamma$ is vertex transitive if for any $x, y \in V(\Gamma)$, there is some $\varphi \in \text{Aut}(\Gamma)$, the automorphism group of $\Gamma$, such that $\varphi(x) = y$. In the sequel, we show that $F(2,O_k)$ is a vertex transitive graph and we determine the automorphism group of the graph $\Lambda = F(2,O_k)$.

**Proposition 3.3.** The graph $F(2,O_k)$ is vertex transitive.

*Proof.* Let $\Lambda = F(2,O_k)$ and $[n] = \{1, 2, ..., 2k-1\}$. It is easy to prove that the graph $\Lambda$ is a regular bipartite graph of degree $k + 1$. In fact, if $V_1 = \{(v, i) \mid v \in [n], |v| = k - i; i \in \{0, 1\}\}$ and $V_2 = \{(v, i) \mid v \in [n], |v| = k - i; i \in \{0, 1\}\}$, then $\text{Aut}(\Lambda) = V_1 \cup V_2$, and each edge of $\Lambda$ has one end in $V_1$ and the other end in $V_2$, and $|V_1| = |V_2| = \binom{2k-1}{k-i}$. Suppose $(u, i), (v, i) \in V(\Lambda)$, by the following steps we show that $\Lambda$ is a vertex transitive graph.

(i) If both vertices $(u, i)$ and $(v, i)$ lie in $V_1$ and $|u \cap v| = t$, where $0 \leq t \leq k - 2$, then we may assume $u = \{x_1, ..., x_t, u_1, ..., u_{(k-1)-t}\}$ and $v = \{x_1, ..., x_t, v_1, ..., v_{(k-1)-t}\}$, where $x_i, u_i, v_i \in [n]$. Let $\sigma$ be a permutation of $\text{Sym}([n])$, such that $\sigma(x_i) = x_i$, $\sigma(u_i) = v_i$ and $\sigma(w_i) = w_i$, where $w_i \in [n] - (u \cup v)$. So, $\sigma$ induces an automorphism $\sigma: \text{Aut}(\Lambda) \rightarrow \text{Aut}(\Lambda)$ such that $\sigma((u, i), (v, i)) = (v, i)$. Moreover, let $L = \left\{f_r \mid r \in \text{Sym}([n])\right\}$, where $f_r: \text{Aut}(\Lambda) \rightarrow \text{Aut}(\Lambda)$, $f_r((u, i), (v, i)) = (\sigma(u), \sigma(v))$ for every $(u, i) \in V(\Lambda)$. It is easy to prove that $L$ is an automorphism of $\text{Aut}(\Lambda)$ such that $L(u, i) = (\sigma(u), \sigma(i))$. Thus, $\sigma(u, i) = (v, i)$. Therefore, $\sigma(u, i) = (v, i).

(ii) We define the mapping $\theta: V(\Lambda) \rightarrow V(\Lambda)$ by $\theta(v, i) = (v, i)$ for every $(v, i) \in V(\Lambda)$. It is easy to prove that $\theta$ is an automorphism of $\Lambda$. So, if both vertices $(u, i)$ and $(v, i)$ lie in $V_2$, then $\theta(u, i) = (v, i) \in V_1$. Therefore, there is an automorphism $\theta: \text{Aut}(\Lambda) \rightarrow \text{Aut}(\Lambda)$ such that $\theta(u, i) = (v, i)$. Thus, $\theta^{-1}(\theta(u, i)) = (v, i).

(iii) Now, let $(u, i) \in V_1$ and $(v, i) \in V_2$, so $\theta(v, i) \in V_1$, and there is an automorphism $\theta \in \text{Aut}(\Lambda)$ such that $\theta(u, i) = (v, i)$. Therefore, $(\theta^{-1}(\theta)(u, i)) = (v, i)$. \hfill $\square$

**Theorem 3.5.** The automorphism group of the graph $\Lambda = F(2,O_k)$ is the automorphism group of $\Gamma = 2.O_k$.

*Proof.* Let $H = \text{Aut}(\Gamma)$, and $[n] = \{1, 2, ..., 2k-1\}$. We know that $H \cong \mathbb{Z}_2 \times \text{Sym}([n])$ (see [6], p. 260). Moreover, let $G = \text{Aut}(\Lambda)$. It is not hard to see that $G \leq H$, because if $f \in G$ and $e = [u, v]$ is an edge in $\Gamma$, then $e$ is an edge in $\Lambda$. Therefore, $f(e) = f([u, v])$ is an edge in $\Lambda$, hence, $G \leq H$. Thus, $|G| \leq |H| = 2(2k-1)!$. Now, if we find a subgroup $M$ of $G$ of order $2(2k-1)!$, then we can conclude that $G = M = H$. Let $K$ be the group that is generated by $\theta$, where $\theta: V(\Lambda) \rightarrow V(\Lambda)$, by $\theta((v, i)) = (v, i)$ for every $(v, i) \in V(\Lambda)$, is an automorphism of order $2$ in the graph $\Lambda$. Moreover, let $L = \left\{f_r \mid r \in \text{Sym}([n])\right\}$, where $f_r: V(\Lambda) \rightarrow V(\Lambda)$, for $f_r((u, i), (v, i)) = (\sigma(u), \sigma(v))$ for every $(u, i) \in V(\Lambda)$. It is not hard to see that $K$ and $L$ are normal subgroups of $G$ and $K \cap L = 1$. Therefore, $M = K \times L = G \cong \mathbb{Z}_2 \times \text{Sym}([n]) = \mathbb{Z}_2 \times \text{Sym}(2k-1)$. \hfill $\square$

**Integrality of $F(2,O_k)$**

In this section, by the using theory of equitable partition of graphs we show that $F(2,O_k)$ is an integral graph.

**Proposition 3.4.** The bipartite double graph of the Odd graph $O_k$ is a cover for the Odd graph $O_k$.

*Proof.* The bipartite double graph of the Odd graph $O_k$ has the property that, for each vertex $(v, i)$, there is a unique vertex in $2.O_k$ at distance $2k - 1$ from $(v, i)$, say $(v, i)$. Thus $V(2.O_k)$ can partitioned into $\binom{2k}{k}$ pairs, and these pairs are the fibres of a covering map from $2.O_k$ onto $O_k$. \hfill $\square$

**Proposition 3.5.** All the eigenvalues of $2.O_k$ are the integers $\lambda_i = \pm(k-i)$ with multiplicity $m(\lambda_i) = \binom{2k-1}{i} - \binom{2k-1}{i-1}$, for $0 \leq i \leq k-1$.

*Proof.* We know that all the eigenvalues of $O_k$ are the integers $\lambda_i = (-1)^{i}(k-i)$ with multiplicity $m(\lambda_i) = \binom{2k-1}{i} - \binom{2k-1}{i-1}$ for $0 \leq i \leq k-1$ (see [5], p. 74), and it is well known that the eigenvalues of the base graph are also eigenvalues of the cover. Therefore, all the eigenvalues of $2.O_k$ are the integers $\lambda_i = \pm(k-i)$ with multiplicity $m(\lambda_i) = \binom{2k-1}{i} - \binom{2k-1}{i-1}$ for $0 \leq i \leq k-1$, because the bipartite double graph of the Odd graph $O_k$ is a bipartite cover graph over $O_k$. \hfill $\square$
Lemma 3.3. Let \( k \) be a positive integer with \( k \geq 2 \). Then, all the eigenvalues of the tridiagonal \((2k) \times (2k)\) matrix

\[
B = \begin{bmatrix}
0 & k & 0 & 0 & \cdots \\
1 & 0 & k-1 & 0 & \cdots \\
0 & 1 & 0 & k-1 & \\
& \ddots & \ddots & \ddots & \ddots \\
& & & k-1 & 0 & 1 \\
& & & 0 & k-1 & 1 \\
& & & 0 & 0 & k
\end{bmatrix}
\]

are the integers \( \lambda_i = \pm(k - i) \) for \( 0 \leq i \leq k-1 \).

Proof. It is well known that the bipartite double graph of the Odd graph \( O_k \) is a distance regular graph, such that the valency of each vertex is \( k \), with diameter \( 2k - 1 \), and whose intersection array is

\[
\iota(\Gamma) = \begin{bmatrix}
* & 1 & 1 & 2 & 2 & \cdots & k-1 & k-1 & k \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix}.
\]

Then, by Theorem 2.1, the tridiagonal \((2k) \times (2k)\) matrix \( B \) determines all the eigenvalues of \( 2.0_k \). On the other hand, by Proposition 3.5, all the eigenvalues of \( 2.0_k \) are the integers \( \lambda_i = \pm(k - i) \). Therefore, all the eigenvalues of the tridiagonal \((2k) \times (2k)\) matrix \( B \) are the integers \( \lambda_i = \pm(k - i) \) for \( 0 \leq i \leq k-1 \).

\[\square\]

Theorem 3.6. All the eigenvalues of \( F(2,O_k) \) are the integers \( \theta_i = \pm(k - i + (-1)^r) \), for \( 0 \leq i \leq k-1 \).

Proof. Let \( \Gamma \) be the bipartite double graph of the Odd graph \( O_k \), that is, \( \Gamma = 2.0_k \). For any vertex \( v \) of the graph \( \Gamma \), we define

\[
\Gamma_r(v) = \{ u \in V(\Gamma) \mid d(u,v) = r \},
\]

where \( r \) is a non-negative integer not exceeding \( 2k - 1 \), the diameter of \( \Gamma \). It is clear that \( \Gamma_0(v) = \{ v \} \), and \( V(\Gamma) \) is equitable partitioned into the disjoint subsets \( \Gamma_0(v), \ldots, \Gamma_{2k-1}(v) \) for each \( v \) in \( V(\Gamma) \), since \( \Gamma \) is a distance regular graph. Let \( \pi = \{ \Gamma_0(v), \ldots, \Gamma_{2k-1}(v) \} \) be a partition of the graph \( \Gamma \). It is easy to show that the partition matrix \( (B_\pi)_{\Gamma} \) of the graph \( \Gamma \) is equal to the tridiagonal \((2k) \times (2k)\) matrix

\[
(B_\pi)_{\Gamma} = \begin{bmatrix}
0 & k & 0 & 0 & \cdots \\
1 & 0 & k-1 & 0 & \cdots \\
0 & 1 & 0 & k-1 & \\
& \ddots & \ddots & \ddots & \ddots \\
& & & k-1 & 0 & 1 \\
& & & 0 & k-1 & 1 \\
& & & 0 & 0 & k
\end{bmatrix}.
\]

So, by Lemma 3.3, all the eigenvalues of \( (B_\pi)_{\Gamma} \) are the integers \( \lambda_i = \pm(k - i) \). On the other hand, let \( \Lambda \) be the graph \( F(2,O_k) \), that is, \( \Lambda = F(2.O_k) \). We can show the orbit partition \( \pi = \{ \Gamma_0(v), \ldots, \Gamma_{2k-1}(v) \} \) for each \( v \) in \( V(\Gamma) \) is equitable partitioned for the graph \( \Lambda \), because \( \Lambda \) is a vertex transitive graph. Then, the partition matrix \( (B_\pi)_{\Lambda} \) of graph \( \Lambda \) is equal to the \((2k) \times (2k)\) matrix,

\[
(B_\pi)_{\Lambda} = \begin{bmatrix}
0 & k & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
1 & 0 & k-1 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & k-1 & \cdots & 0 & 1 & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 1 & 0 & k-1 & 0 & 1 \\
& & & 0 & 1 & 0 & 0 & k-1 & 0 \\
& & & 0 & 0 & 0 & \cdots & 0 & k
\end{bmatrix}.
\]
So, by Proposition 3.3, we know that $\Lambda$ is a vertex transitive graph and, hence, by Theorem 2.3 every eigenvalue of $\Lambda$ is an eigenvalue of $(B_{2})_{\frac{\Lambda}{2}}$. Also, we have $(B_{2})_{\frac{\Lambda}{2}} = (B_{2})_{\frac{\Lambda}{2}} + C$, where $C$ is equal to the $(2k) \times (2k)$ matrix,

\[
C = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{bmatrix}
\]

It is easy to prove that $(B_{2})_{\frac{\Lambda}{2}}$ and $C$ are diagonal matrices, and $(B_{2})_{\frac{\Lambda}{2}} C = C(B_{2})_{\frac{\Lambda}{2}}$. So, $(B_{2})_{\frac{\Lambda}{2}}$ and $C$ are simultaneously triangularizable. Therefore, there is a base $\{w_{1}, w_{2}, \ldots, w_{2k}\}$ for $\mathbb{R}^{2k}$, such that all of them are numbers of the special vectors of $(B_{2})_{\frac{\Lambda}{2}}$ and $C$. Moreover, we know that

$$\theta_{i} = (B_{2})_{\frac{\Lambda}{2}} w_{i} = (B_{2})_{\frac{\Lambda}{2}} w_{i} + C w_{i} = \lambda_{i} \pm 1 = \pm(k - i + (-1)^{j}).$$

Thus, $F(2.O\lambda_{i})$ is an integral graph.

**Spectral Characterization of Multicone Graph $K_{v} \triangle \Box_{n}$**

An interesting family of Cayley graphs for the elementary abelian group $(\mathbb{Z}_{2})^{n}$ is provided by the $n$ cubes $Q_{n}$. The vertex set of $Q_{n}$ is the set of all $2^{n}$ binary $n$-tuples, with two being adjacent if they differ in precisely one coordinate. The $n$-cubes, $Q_{n}$, are a well known and frequently studied family of graphs. They are in fact the Hamming graphs $H(n; 2)$, so their vertices can be regarded as binary $n$-tuples, adjacent if their Hamming distance is 1. Let $n = 2d + 1$, the folded $2d + 1$-cube denoted by $\Box_{n}$, is the graph defined on the partitions of an $2d + 1$-set into two subsets, and two partitions being adjacent when their common refinement contains a set of size one. Its intersection array is given by

\[
\begin{bmatrix}
\star & 1 & 2 & 3 & 4 & \ldots & d - 2 & d - 1 & d \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
2d + 1 & 2d & 2d - 1 & 2d - 2 & 2d - 3 & \ldots & d + 3 & d + 2 & \star
\end{bmatrix}
\]

and its eigenvalues and multiplicities are $\lambda_{i} = n - 4i$ with $m(\lambda_{i}) = \binom{n}{i}$ for $0 \leq i \leq d$, (see [6], p. 264). In this section we show that multicone graphs $K_{v} \triangle \Box_{n}$ and $K_{v} \triangle m\Box_{n}$ are determined by their adjacency spectra as well as their Laplacian spectra.

**Proposition 3.6.** Let $\Gamma$ be a graph cospectral with the multicone graph $K_{v} \triangle \Box_{n}$ with respect to its adjacency matrix spectrum. Then

$$\text{Spec}(\Gamma) = \{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{d}, -1^{(n-1)}(P + \frac{\sqrt{P^{2} + 4Q}}{2}, P - \frac{\sqrt{P^{2} + 4Q}}{2})\},$$

where $P = v - 1 + n$, $Q = 2^{n+1}v - n(v - 1)$, $\lambda_{i} = n - 4i$ and $m(\lambda_{i}) = \binom{n}{i}$ for $1 \leq i \leq d$.

**Lemma 3.4.** Let $\Gamma$ be a graph cospectral with multicone graph $K_{v} \triangle \Box_{n}$. Then $\Gamma$ is a bipartite graph.

**Theorem 3.7.** Consider the multicone graph $K_{v} \triangle \Box_{n}$. Then $K_{v} \triangle \Box_{n}$ is DS with respect to its adjacency matrix spectrum.

**Proof.** We proceed by induction on the number of vertices in $K_{v}$. Let $K_{v}$ have one vertex and $\Gamma$ be a graph cospectral with multicone graph $K_{1} \triangle \Box_{n}$ with respect to its adjacency matrix spectrum. By Lemma 3.4, it is easy to see that $\Gamma$ has one vertex of degree $2^{n-1}$, say $j$. Hence, if $\text{Spec}(\Gamma - j) = \text{Spec}(\Box_{n})$, then $\Gamma - j \cong \Box_{n}$, because $\Box_{n}$ is a distance regular DS graph see [11]. So $\Gamma \cong K_{1} \triangle \Box_{n}$. We assume inductively that this claim holds for $K_{v}$, that is, if $\text{Spec}(\Gamma_{1}) = \text{Spec}(K_{v} \triangle \Box_{n})$, then $\Gamma_{1} \cong K_{v} \triangle \Box_{n}$, where $\Gamma_{1}$ is a graph cospectral with multicone graph $K_{v} \triangle \Box_{n}$ with
respect to its adjacency matrix spectrum. We show that the claim is true for $K_{v+1}$, that is, if $Spec(K) = Spec(K_{v+1} \lor \Box_n)$, then $\Gamma \cong K_{v+1} \lor \Box_n$, where $\Gamma$ is a graph cospectral with multicone graph $K_{v+1} \lor \Box_n$ with respect to its adjacency matrix spectrum. It is obvious that $\Gamma$ has one vertex and $2^{v-1} + v$ edges more than $\Gamma_1$. On the other hand, by Lemma 3.4, we know that $\Gamma_1$ has $v$ vertices of degree $2^{v-1} + v - 1$ and $2^{v-1}$ vertices of degree $n + v$. In particular, $\Gamma$ has $v + 1$ vertices of degree $2^{v-1} + v$ and $2^{v-1}$ vertices of degree $n + v + 1$. So, we must have $\Gamma \cong K_1 \lor \Gamma_1$. Now, by the induction hypothesis, we conclude that $\Gamma \cong K_{v+1} \lor \Box_n$, and complete the proof.

$$\square$$

**Theorem 3.8.** Consider the multicone graph $K_v \lor \Box_n$. Then $K_v \lor \Box_n$ is DS with respect to its Laplacian spectrum.

**Proof.** We know that the Laplacian matrix spectrum of $\Box_n$ is $\mu_i = n - \lambda_i$, where $m(\mu_i) = \binom{n}{i}$ for $0 \leq i \leq d$. Also, the Laplacian matrix spectrum of $K_v$ is $\{v^{i-1}, 0\}$. So, by Theorem 2.6, the Laplacian matrix spectrum of $K_v \lor \Box_n$ is

$$\{2^{v-1} + v, (\mu_1 + v), (\mu_2 + v), (\mu_3 + v), ..., (\mu_d + v), 0\},$$

where $m(\mu_i + v) = \binom{n}{i}$ for $1 \leq i \leq d$. We proceed by induction on the number of vertices in $K_v$. Let $K_v$ have one vertex and $\Gamma$ be a graph cospectral with the multicone graph $K_1 \lor \Box_n$ with respect to its adjacency matrix spectrum, that is, $Spec(\Gamma) = Spec(K_1 \lor \Box_n)$. By Theorem 2.7, we can show that $\Gamma \cong K_1 \lor \Box_n$. We assume inductively that this claim holds for $K_v$, that is, if $Spec(\Gamma_1) = Spec(K_v \lor \Box_n)$, then $\Gamma_1 \cong K_v \lor \Box_n$, where $\Gamma_1$ is a graph cospectral with the multicone graph $K_v \lor \Box_n$ with respect to its Laplacian spectrum. We show that the claim is true for $K_{v+1}$, that is, if $Spec(K_v) = Spec(K_{v+1} \lor \Box_n)$, then $\Gamma \cong K_{v+1} \lor \Box_n$. Thus, it can be concluded $\Gamma \cong K_{v+1} \lor \Box_n$.

$$\square$$

**Spectral Characterization of Multicone Graph $K_v \lor m\Box_n$**

**Proposition 3.7.** Let $\Gamma$ be a graph cospectral with the multicone graph $K_v \lor m\Box_n$ with respect to its adjacency matrix spectrum. Then

$$Spec(\Gamma) = \{\lambda_0, \lambda_1, \lambda_2, ..., \lambda_d, -1^{(v-1)}, (P + \sqrt{P^2 + 4Q})/2, (P - \sqrt{P^2 + 4Q})/2\},$$

where $P = v - 1 + n$, $Q = m2^{v-1}v - n(v - 1)$, $\lambda_i = n - 4i$, $m(\lambda_0) = m - 1$, $m(\lambda_i) = \binom{n}{i}m$ for $1 \leq i \leq d$.

**Lemma 3.5.** Let $\Gamma$ be a graph cospectral with multicone graph $K_v \lor m\Box_n$. Then $\Gamma$ is a bidegreed graph.

**Theorem 3.9.** Consider the multicone graph $K_v \lor m\Box_n$. Then $K_v \lor m\Box_n$ is DS with respect to its adjacency matrix spectrum.

**Proof.** We proceed by induction on the number of vertices in $K_v$. Let $K_v$ have one vertex and $\Gamma$ be a graph cospectral with multicone graph $K_v \lor m\Box_n$, with respect to its adjacency matrix spectrum. By Lemma 3.5, it is easy to see that $\Gamma$ has one vertex of degree $m2^{v-1}$, say $j$. Hence, if $Spec(\Gamma - j) = Spec(m\Box_n)$, then $\Gamma - j \cong m\Box_n$, because $\Box_n$ is a distance regular DS graph and the union of regular DS graphs with the same degree is always DS, see [11]. So $\Gamma \cong K_v \lor m\Box_n$. We assume inductively that this claim holds for $K_v$, that is, if $Spec(\Gamma_1) = Spec(K_v \lor m\Box_n)$, then $\Gamma_1 \cong K_v \lor m\Box_n$, where $\Gamma_1$ is a graph cospectral with multicone graph $K_v \lor m\Box_n$ with respect to its adjacency matrix spectrum. We show that the claim is true for $K_{v+1}$, that is, if $Spec(\Gamma) = Spec(K_{v+1} \lor m\Box_n)$, then $\Gamma \cong K_{v+1} \lor m\Box_n$, where $\Gamma$ is a graph cospectral with multicone graph $K_{v+1} \lor m\Box_n$, with respect to its adjacency matrix spectrum. It is obvious that $\Gamma$ has one vertex and $m2^{v-1} + v$ edges more than $\Gamma_1$. On the other hand, by Lemma 3.5, we know that $\Gamma_1$ has $v$ vertices of degree $m2^{v-1} + v - 1$ and $m2^{v-1}$ vertices of degree $n + v$. In particular, $\Gamma$ has $v + 1$ vertices of degree $m2^{v-1} + v$ and $m2^{v-1}$ vertices of degree $n + v + 1$. So, we must have $\Gamma \cong K_1 \lor \Gamma_1$. Now, by the induction hypothesis, we conclude that $\Gamma \cong K_{v+1} \lor m\Box_n$, and complete the proof.

$$\square$$
Theorem 3.10. Consider the multicone graph $K_v \triangledown m \Box n$. Then $K_v \triangledown m \Box n$ is DS with respect to its Laplacian spectrum.

Proof. We know that the Laplacian matrix spectrum of $m \Box n$ is $\mu_i = n - \lambda_i$, where $m(\mu_i) = m_1^{(i)}$ for $0 \leq i \leq d$. Also, the Laplacian matrix spectrum of $K_v$ is $\{v^{i-1}, 0\}$. So, by Theorem 2.6, the Laplacian matrix spectrum of $K_v \triangledown m \Box n$ is

$$[(m2^{n-1} + v^{(i)}), (\mu_1 + v), (\mu_2 + v), (\mu_3 + v), \ldots, (\mu_d + v), v^{(m-1)}, 0],$$

where $m(\mu_i + v) = m(\mu_i)$ for $1 \leq i \leq d$. We proceed by induction on the number of vertices in $K_v$. Let $K_v$ have one vertex and $\Gamma$ be a graph cospectral with the multicone graph $K_1 \triangledown m \Box n$ with respect to its adjacency Laplacian spectrum, that is, $Spec(L(\Gamma)) = Spec(L(K_1 \triangledown m \Box n))$. By Theorem 2.7, we can show that $\Gamma \cong K_1 \triangledown m \Box n$. We assume inductively that this claim holds for $K_v$, that is, if $Spec(L(\Gamma_1)) = Spec(L(K_v \triangledown m \Box n))$, then $\Gamma_1 \cong K_v \triangledown m \Box n$, where $\Gamma_1$ is a graph cospectral with the multicone graph $K_v \triangledown m \Box n$ with respect to its Laplacian spectrum. We show that the claim is true for $K_{v+1}$, that is, if

$$Spec(L(\Gamma)) = Spec(L(K_{v+1} \triangledown m \Box n))$$

$$= [(m2^{n-1} + v + 1)^{(v+1)}, (\mu_1 + v + 1), (\mu_2 + v + 1), (\mu_3 + v + 1), \ldots, (\mu_d + v + 1), (v + 1)^{(m-1)}, 0],$$

where $m(\mu_i + v + 1) = m(\mu_i)$ for $1 \leq i \leq d$. Then $\Gamma \cong K_{v+1} \triangledown m \Box n$, where $\Gamma$ is a graph cospectral with the multicone graph $K_{v+1} \triangledown m \Box n$, with respect to its Laplacian spectrum. By Theorem 2.7, we know that $\Gamma_1$ and $\Gamma$ are join of two graphs, because $m2^{n-1} + v + 2m\Box n + v + 1$ are eigenvalues of $\Gamma_1$ and $\Gamma$, respectively. In addition, $\Gamma$ has one vertex of degree $v + m2^{n-1}$ more than $\Gamma_1$, say $j$, hence $Spec(L(\Gamma - j)) \cong Spec(L(K_v \triangledown m \Box n))$. Now, by the induction hypothesis, we conclude that $\Gamma - j \cong K_v \triangledown m \Box n$. Thus, it can be concluded $\Gamma \cong K_{v+1} \triangledown m \Box n$. 

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