Ward-Takahashi identity and dynamical mass generation in Abelian gauge theories

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We derive Ward-Takahashi identities including composite fields in Abelian gauge theories and the matching condition between the elementary field description and the composite field description. With these we develop an approach to dynamical symmetry breaking in Abelian gauge theories including the study of the dynamically generated masses of the gauge boson, the fermions and the composite Higgs field. The Cornwall-Norton, Jackiw-Johnson and Schwinger models are taken as examples of the application. The obtained gauge boson masses are in agreement with the existing results. In this approach, we are able to further obtain new results for the mass of the composite Higgs boson and the Goldstone boson decay constant $F_{\pi}$. 

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I. INTRODUCTION

Dynamical symmetry breaking plays an important role in particle physics. In strong interaction, chiral symmetry is realized in this pattern and many low-energy phenomena are determined by the dynamically broken chiral symmetry. On the other hand, dynamical symmetry breaking provides an alternative mechanism of mass generation in the electroweak theory. In this approach, all interactions are governed by the gauge principle so that the number of parameters can be reduced, and the problems of triviality and unnaturalness due to elementary Higgs fields do not occur. Because of these attractive properties, many dynamical electroweak symmetry breaking models have been proposed.

The reason that a gauge field can obtain a mass dynamically was first given by Schwinger. He made a crucial observation that if the polarization tensor of vector mesons has a pole at $q^2 = 0$, the gauge symmetry will be spontaneously broken and the vector mesons acquire a mass. This is called Schwinger mechanism or the dynamical Higgs mechanism. He also gave an example in two-dimensional quantum electrodynamics. In the Schwinger model, the symmetry breaking leading to massive vector field is due to anomalies, rather than bound state Goldstone bosons. Inspired by the pioneering work of Numbu concerning the gauge invariance of the BCS theory of super-conductivity, Jackiw and Johnson and Cornwall and Norton extended the work of Numbu and Jona-Lasinio to dynamical gauge symmetry breaking. They constructed two simple Abelian gauge theoretical models and showed that there are nontrivial self-consistent solutions with the vector mesons acquiring masses dynamically.

Since dynamical symmetry breaking is a nonperturbative phenomenon, it is difficult to study it quantitatively and most of the investigations on dynamical symmetry breaking are based on the analogy of the low-energy phenomenology and current algebra. In order to study dynamical symmetry breaking from the first principles of the fundamental theory, it is interesting to explore new approaches.

Recently, we derived Ward-Takahashi identities including composite fields and applied them to the study of chiral symmetry breaking and the properties of fermion bound states. In some model investigations, this method can be...

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applied to obtain the mass spectra of both fermion and bound state bosons, and the results are in agreement with those from other methods. It is also convenient to use it to study the phase structure, mass spectrum and PCAC in the presence of explicit breaking terms [3]. In this paper, we generalize this method to the case of dynamical breaking of gauge symmetry and study the mechanism of gauge boson mass generations and the properties of the composite bosons.

As a test of this approach, we take the simple and well understood Cornwall-Norton, Jackiw-Johnson and Schwinger models as the application. We shall calculate the dynamically generated masses of the gauge bosons in these models to test our method and calculate new results related to the composite Higgs boson to show the ability of this new approach.

In Sec. II, we study Cornwall-Norton model. As there are bound states in dynamical symmetry breaking, we introduce composite external sources in the generating functional. From gauge symmetry, we derive the Ward-Takahashi identity with composite fields in this model, and with which we study the dynamical symmetry breaking and the mass spectra of the fermions, the vector mesons and bound state boson. It is shown that when the gauge symmetry is dynamically broken, one of the vector mesons acquires a mass. This is in agreement with the picture of the Schwinger mechanism. In the platform approximation, vector meson mass is identical to those given by Cornwall and Norton. For a dynamically broken gauge theory, we can either describe it by extracting the composite field degree of freedom in the effective action or describe it only by elementary fields in which the bound states reside implicitly in the effective action. These two descriptions should be equivalent. From this requirement we derive a matching condition between two descriptions. Some physical quantities like $F_\pi$ can be expressed in terms of basic quantities by means of the matching condition.

In Sec. III, we apply the approach to the Jackiw-Johnson model. Considering that in two space-time dimensions anomalies can provide a pole in the vacuum polarization tensor of the vector boson, we apply this approach to the Schwinger model in Sec. IV, and find that the vector boson does acquire a mass from the anomaly, which is identical to other approaches. Some comments and conclusions are presented in Sec. V.

II. THE CORNWALL-NORTON MODEL

A. Ward-Takahashi Identity

The Cornwall-Norton model contains two fermions with equal bare masses and two massless Abelian gauge fields. The gauge fields couple to the fermions via different currents. There are two $U(1)$ ($O(2)$) symmetries in this model. The first one is related to the rotation within the two fermion fields. This $O(2)$ symmetry is dynamically broken which makes the associated gauge field massive. The other one is an unbroken $U_V(1)$ symmetry and the associated gauge field remains massless. The Cornwall-Norton model is described by the following Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial - m_0)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g \bar{\psi} \gamma_\mu \psi A_\mu + g' \bar{\psi} \gamma_\mu \tau_2 \psi B_\mu,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$

$\psi$ represents the two fermion fields $\psi_1$ and $\psi_2$, $\tau_2$ is the Pauli matrix and $A_\mu, B_\mu$ are Abelian gauge fields.

The Lagrangian density is locally invariant under the following transformations

$$\delta \psi(x) = i(\alpha(x) + \tau_2 \beta(x))\psi(x),$$
$$\delta A_\mu(x) = \frac{1}{g} \partial_\mu \alpha(x),$$
$$\delta B_\mu(x) = \frac{1}{g'} \partial_\mu \beta(x),$$

where $\alpha(x), \beta(x)$ are the infinitesimal parameters of $U_V(1)$ and $O(2)$ respectively. The $U_V(1)$ symmetry is related to the fermion number conservation, and it remains unbroken.

Considering that the $O(2)$ symmetry is dynamically broken, there can be fermion pair condensates and bound states in the broken phase. In order to describe the fermion pair condensates, we introduce the external source $K_a$ coupling to the composite operator $\bar{\psi} \tau_a \psi$ in the generating functional [7].
From the above equation, one finds where we have introduced a local field $\sigma$. We denote a dimensionful coefficient making $\text{dim.}\sigma = 1$. Then we have mass spectra of fermion and gauge field.

Under the $O(2)$ transformation, we can obtain the following generating equation of the Ward-Takahashi identities

$$
\Gamma[\phi] = W[J] - \int d^4x[\bar{\psi}_c(x)\eta(x) + \bar{\eta}(x)\psi_c(x) + J_\mu(x)B^\mu_c(x) + I_\mu(x)A^\mu_c(x) + (G_a(x) + \bar{\psi}_c(x)\tau_a\psi_c(x))K_a(x)].
$$

Making the Legendre transformation, we get

$$
\frac{\delta W[J]}{\delta K_a(x)} = G_a(x) + \bar{\psi}_c(x)\tau_a\psi_c(x).
$$

Then we have

$$
\frac{\delta \Gamma[\phi]}{\delta \psi_c(x)} = \bar{\eta}(x) + \bar{\psi}_c(x)\tau_aK_a(x),
$$

$$
\frac{\delta \Gamma[\phi]}{\delta \bar{\psi}_c(x)} = -\eta(x) - \tau_a\psi_c(x)K_a(x),
$$

$$
\frac{\delta \Gamma[\phi]}{\delta B^\mu_c(x)} = -J_\mu(x),
$$

$$
\frac{\delta \Gamma[\phi]}{\delta A^\mu_c(x)} = -I_\mu(x),
$$

$$
\frac{\delta \Gamma[\phi]}{\delta G_a(x)} = -K_a(x).
$$

Under the $O(2)$ transformation, we can obtain the following generating equation of the Ward-Takahashi identities

$$
\frac{\delta \Gamma[\phi]}{\delta \psi_c(x)} i\tau_2\bar{\psi}_c(x) + \bar{\psi}_c(x)i\tau_2\frac{\delta \Gamma[\phi]}{\delta \bar{\psi}_c(x)} + \frac{1}{g'}\delta^{\mu
\nu}\frac{\delta \Gamma[\phi]}{\delta B^\nu_c(x)} - 2\epsilon_{2ab}\sigma_a^c(x)\frac{\delta \Gamma[\phi]}{\delta \sigma_b^c(x)} = 0,
$$

where we have introduced a local field $\sigma^a(x) \equiv aG^a(x)$ to describe the bound state degree of freedom $G^a(x)$ with a dimensionful coefficient making $\text{dim.}\sigma^a = 1$. From (8) we can get the Ward-Takahashi identity for the two-point one-particle-irreducible (1PI) vertex. As the two-point 1PI vertex is related to the mass spectrum, we can get the mass spectra of fermion and gauge field.

B. Mass Spectrum

Taking derivatives of (8) with respect to $\psi_c(y)$ and $\bar{\psi}_c(z)$, one gets
\[ \tilde{\psi}_c(x)i\tau_2 \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \psi_c(x)} - \delta(x - z)i\tau_2 \frac{\delta^2 \Gamma[\phi]}{\delta \psi_c(y) \delta \psi_c(x)} - \delta(x - y)i\tau_2 \frac{\delta^2 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(x)} + \delta(x - z)i\tau_2 \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \psi_c(x)} + \delta(x - y)i\tau_2 \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \psi_c(x)} + 2\epsilon_{2ab} \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \sigma^a_c(x) \delta \sigma^b_c(x)} \sigma^a_c(x) \sigma^b_c(x) = 0. \] (10)

In the absence of the external sources
\[ \psi_c(x) = \bar{\psi}_c(x) = 0, \] (11)
eq \[ \text{eq. (10) reduces to} \]
\[ \delta(x - z)i\tau_2 \frac{\delta^2 \Gamma[\phi]}{\delta \psi_c(y) \delta \psi_c(x)} + \delta(x - y)i\tau_2 \frac{\delta^2 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(x)} + 2\epsilon_{2ab} \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \sigma^a_c(x) \delta \sigma^b_c(x)} \sigma^a_c(x) = 0. \] (12)

Making Fourier transformation, we can express (12) as
\[ \Gamma^{(2)}_{\psi,\bar{\psi}}(p)i\tau_2 - i\tau_2 \Gamma^{(2)}_{\psi,\bar{\psi}}(p + k) = 2\epsilon_{2ab} \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_a}(p + k, -p; -k) \sigma^b_c - \frac{i}{g} k_{\mu} \Gamma^{(3)}_{\psi,\bar{\psi},B_\mu}(p + k, -p; -k). \] (13)

Since we have extracted the composite field \( \sigma^a(x) \) as an independent degree of freedom, there is no pole corresponding to this degree of freedom in the proper vertices. If \( k_{\mu} \rightarrow 0 \), (13) becomes
\[ -i[\tau_2, \Gamma^{(2)}_{\psi,\bar{\psi}}(p)] = 2\epsilon_{2ab} \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_a}(p, -p; 0) \sigma^b_c. \] (14)

In the limit \( p \rightarrow 0 \), (14) is
\[ -i[\tau_2, Z^{-1}_\psi m_f] = 2\epsilon_{2ab} \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_a}(0, 0; 0) \sigma^b_c. \] (15)

Since the only nonvanishing condensate is \( \langle \bar{\psi} \tau_3 \psi \rangle \), the fermion mass can be expressed as
\[ m_f = m^0_f + \tau_3 \delta m_f. \] (16)

Thus from (15) and (16) we get
\[ \delta m_f = -\tau_3 Z_\psi \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_a}(0, 0; 0) \sigma^3_c. \] (17)

In order to get the mass spectrum of the bound states, we take derivatives of (14) with respect to \( \sigma_s(y), \sigma_s(z) \)
\[ \frac{1}{g} \frac{\partial}{\partial \sigma_s(y) \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(y) \delta B^\mu(x)}} + \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(y) \delta \psi_c(x)} i\tau_2 \psi_3(x) + \bar{\psi}_3(x) i\tau_2 \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(y) \delta \psi_c(x)} + \delta(x - y) \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(y) \delta \sigma_s(x)} = 0, \] (18)
\[ \frac{1}{g} \frac{\partial}{\partial \sigma_s(z) \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(z) \delta B^\mu(x)}} + \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(z) \delta \psi_c(x)} i\tau_2 \psi_3(x) + \bar{\psi}_3(x) i\tau_2 \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(z) \delta \psi_c(x)} + \delta(x - z) \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_s(z) \delta \sigma_s(x)} = 0. \] (19)

From the above equations, we see that the mass spectrum of the bound states is
\[ m^2_{\sigma_1} = 0, \] (20a)
\[ m^2_{\sigma_3} = -Z_s^{-1} \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_1}(0, 0; 0) \sigma_3. \] (20b)
where \( \sigma_1 \) corresponds to the massless Goldstone boson, and the Feynman diagram of \( \Gamma^{(3)}_{\sigma_1, \sigma_1; \sigma_3} (0; 0) \) is shown in Fig. 1. In (20b), we have used the relation

\[
Z_{\sigma_1} = Z_{\sigma_3},
\]

which is determined by the O(2) symmetry. As \( \sigma_1 \) is the massless Goldstone boson, we can get the wavefunction renormalization constant \( Z_{\sigma_1} \) from the self-energy

\[
Z_{\sigma_1} = \frac{d}{dp^2} \Gamma^{(2)}_{\sigma_3} (p) \bigg|_{p^2 = 0}.
\]

FIG. 1. The vertex \( \Gamma^{(3)}_{\sigma_1, \sigma_1; \sigma_3} (0; 0) \).

Similar to the above discussion, taking derivative of (9) with respect to \( B^\nu_{\nu} \), we have

\[
\frac{i}{g'} \frac{\partial}{\partial B^\mu (y)} \Gamma^{(2)}_{\sigma_3} (p) = 2 \epsilon_{2ab} \sigma^a (x) \delta B^\nu_{\nu} (y) \delta \sigma^b_c (x).
\]

Making Fourier transformation and turning off the external sources, we have

\[
\frac{i}{g'} \frac{\partial}{\partial B^\mu (y)} \Gamma^{(2)}_{\sigma_3} (p) = -2 \epsilon_{2ab} \sigma^a (x) \delta B^\nu_{\nu} (y) \delta \sigma^b_c (x) = 0.
\]

Note that if there is no fermion pair condensate, \( \langle \bar{\psi} \tau_3 \psi \rangle = 0 \) and the symmetry remains, i.e.

\[
P^\mu \Gamma^{(2)}_{\sigma_1; \sigma_1} (p) = 0.
\]

This shows that the gauge field \( B^\mu \) is transverse. If there is nonvanishing fermion pair condensate, \( \langle \bar{\psi} \tau_3 \psi \rangle \neq 0 \), then

\[
P^\mu \Gamma^{(2)}_{\sigma_1; \sigma_1} (p) \neq 0,
\]

which shows that the O(2) gauge symmetry is dynamically broken.

Multiplying \( p^\nu \sigma^\nu = p_{\mu} / p^2 \) to (24) and taking \( p^\mu \to 0 \), we have

\[
\frac{p^\mu p^\nu}{p^2} \Gamma^{(2)}_{\sigma_1; \sigma_1} (p) = i 2 \frac{p^\nu}{p^2} \Gamma^{(2)}_{\sigma_1} (p) \sigma^3_c, \tag{27}
\]

Using the relation

\[
\lim_{p \to 0} \Gamma^{(2)}_{\sigma_1; \sigma_1} (p) = -Z_B \delta_{\mu \nu} m_B^2, \tag{28}
\]

we get the mass of gauge boson \( B^\mu \)

\[
m_B^2 = - \lim_{p \to 0} Z_B^{-1} \frac{p^\mu p^\nu}{p^2} \Gamma^{(2)}_{\sigma_1; \sigma_1} (p) \sigma^3_c. \tag{29}
\]

Eq. (29) looks different from the standard formula. In the next subsection, we shall see that once the matching relation between the two descriptions is concerned, \( m_B^2 \) reduces to the standard formula.
C. Matching Condition

In the above subsections, we have extracted the composite field $\sigma_a$ and treat it as an independent degree of freedom in the effective action. We regard this as the \textit{composite field description}. On the other hand, we can also describe the same system by including only the classical fields of the elementary fields in the effective action, and the bound state property resides implicitly in the effective action. We regard this as the \textit{elementary field description}. The two descriptions should be equivalent in describing the same system. The requirement of equivalence between the two descriptions leads to matching conditions for various 1PI vertices which relate the corresponding quantities in the two descriptions. These conditions are crucially important in the present approach.

In order to derive the matching conditions, we consider now the elementary field description in which the effect of the external source $K_a$ is implicitly contained in the effective action. The generating functional in this description is

$$\bar{\Gamma}[\tilde{\phi}] = W[J] - \int d^4x [\bar{\psi}_c(x) \eta(x) + \bar{\eta}(x) \psi_c(x) + I_\mu(x) A^\mu_c(x) + J_\mu B^\mu_c(x)],$$

(30)

where $\tilde{\phi} \equiv (\bar{\psi}_c, \psi_c, A^\mu_c, B^\mu_c, K_a)$. In the limit of $K_a = 0$, the effective action $\bar{\Gamma}[\tilde{\phi}]$ is just the usual effective action of the fundamental theory.

Comparing (30) with (7), we have

$$\bar{\Gamma}[\tilde{\phi}] = \int d^4x \left[ \left( \frac{1}{a} \sigma_a(x) + \bar{\psi}_c(x) \tau_a \psi_c(x) \right) K_a(x) \right].$$

(31)

This is the basic relation between the effective actions in the two different descriptions. Taking derivatives with respect to the elementary fields, we can obtain the matching conditions for various 1PI vertices.

We first examine the two-point 1PI vertex in the two descriptions. For a given $K_a$, taking derivatives of (32) with respect to $\bar{\psi}_c(x)$ and $\psi_c(y)$, we get

$$\frac{\delta^2 \bar{\Gamma}[\tilde{\phi}] - \delta^2 \bar{\Gamma}[\tilde{\phi}]}{\delta \bar{\psi}_c(y) \delta \psi_c(x)} \bigg|_{K_a} = \int d^4x' \left[ \left( \frac{\delta^2 \sigma_a(x')}{\delta \bar{\psi}_c(y) \delta \psi_c(x)} \right) a + \tau_a \delta(x' - y) \delta(x' - x) \right] K_a(x).$$

(32)

In $\Gamma[\phi]$, $\bar{\psi}_c, \psi_c, A^\mu_c$, and $\sigma_a$ are taken to be independent variables. When fixing $K_a$, $\sigma_a$ is a function of $\bar{\psi}_c, \psi_c$ and $A^\mu_c$. Then

$$\frac{\delta^2 \Gamma[\phi]}{\delta \bar{\psi}_c(y) \delta \psi_c(x)} \bigg|_{K_a} - \frac{\delta^2 \bar{\Gamma}[\tilde{\phi}]}{\delta \bar{\psi}_c(y) \delta \psi_c(x)} = \int d^4x_1 \left[ \frac{\delta \sigma_a(x_1)}{\delta \bar{\psi}_c(y)} \right] \frac{\delta^2 \Gamma[\phi]}{\delta \sigma_a(x_1) \delta \psi_c(x)} + \int d^4x_1 d^4x_2 \frac{\delta \sigma_a(x_1)}{\delta \bar{\psi}_c(y)} \frac{\delta^2 \bar{\Gamma}[\tilde{\phi}]}{\delta \sigma_a(x_1) \delta \psi_c(x)} + \int d^4x_1 \left[ \frac{\delta \sigma_a(x_1)}{\delta \psi_c(x)} \right] \frac{\delta^2 \sigma_a(x_1)}{\delta \psi_c(x)}.$$}

(33)

Note that in the absence of the external sources, the derivative of $\Gamma[\phi]$ with respect to one fermion field $\bar{\psi}_c(x)$ or $\psi_c(x)$ vanishes. Thus when turning off the external sources we have

$$\frac{\delta^2 \Gamma[\phi]}{\delta \bar{\psi}_c(y) \delta \psi_c(x)} \bigg|_{K_a} = 0,$$

(34)

and

$$\frac{\delta^2 \bar{\Gamma}[\tilde{\phi}]}{\delta \bar{\psi}_c(y) \delta \psi_c(x)} = \frac{\delta^2 \bar{\Gamma}[\tilde{\phi}]}{\delta \bar{\psi}_c(y) \delta \psi_c(x)},$$

(35)

which indicates that the fermion two-point 1PI vertex in the composite field description is the same as that in the elementary field description. This provides a consistent relation between the two descriptions.

Next we look at the three-point 1PI vertex. Similar to the above discussion, one can get the Ward-Takahashi identity in the elementary field description \[1\]. According to (13), we have

$$\frac{k_\mu}{2g'} \int_{\psi_c, B_c} \Gamma^{(3)}_{\psi_c, B_c}(p + k, -p; -k) = \frac{k_\mu}{2g'} \int_{\bar{\psi}_c, B_c} \Gamma^{(3)}_{\bar{\psi}_c, B_c}(p + k, -p; -k) + e_{2ab} \int_{\psi_c, \bar{\psi}_c} \Gamma^{(3)}_{\psi_c, \bar{\psi}_c}(p + k, -p; -k) \sigma_b.$$}

(36)
Differentiating \( (33) \) and \( (35) \) with respect to \( B^\mu_c(z) \) and in the limit \( I, J \to 0 \), we have

\[
\frac{\delta^3(\Gamma[\phi] - \Gamma[\tilde{\phi}])}{\delta B^\mu_c(z)\delta\psi_c(y)\delta\psi_c(x)} \bigg|_{K_a=0} = 0, \tag{37}
\]

\[
\frac{\delta^3\Gamma[\phi]}{\delta B^\mu_c(z)\delta\psi_c(y)\delta\psi_c(x)} - \frac{\delta^3\Gamma[\tilde{\phi}]}{\delta B^\mu_c(z)\delta\psi_c(y)\delta\psi_c(x)} = \int d^4x_1 d^4x_2 \frac{\delta^2\Gamma[\phi]}{\delta B^\mu_c(z)\delta\sigma_a(x_1)} D_{ab}(x_1, x_2) \frac{\delta^3\Gamma[\phi]}{\delta\psi_c(y)\delta\psi_c(x)\delta\sigma_b(x_2)}. \tag{38}
\]

Therefore

\[
\tilde{\Gamma}^{(3)}_{\psi, \bar{\psi}; B_\mu}(p + k, -p; -k) = \Gamma^{(3)}_{\psi, \bar{\psi}; B_\mu}(p + k, -p; -k) + \Gamma^{(3)}_{\psi, \bar{\psi}; \sigma_a}(p + k, -p; -k)iD_{ab}(k)i\Gamma^{(2)}_{B, \sigma_b}(k). \tag{39}
\]

Note that in the elementary field description, when \( \Gamma^{(2)}_{\psi, \bar{\psi}; B_\mu}(p) \) has a symmetry breaking solution, \( \Gamma^{(3)}_{\psi, \bar{\psi}; B_\mu}(p + k, -p; -k) \) has a pole as \( k_\mu \to 0 \). Since there is no pole of elementary fields in the 1PI vertex, the pole is attributed to a bound state \( \bar{\Gamma}^{(3)}_{\psi, \bar{\psi}; B_\mu}(p + k, -p; -k) \). There are two kinds of contribution in the vertex \( \bar{\Gamma}^{(3)}_{\psi, \bar{\psi}; B_\mu}(p + k, -p; -k) \): the first comes from the exchange of the bound state (corresponding to \( \sigma_a \)); the other is the regular term coming from the quantum fluctuations of the elementary fields which is the same as that in the composite field description as is shown in Fig.(2). From the right-hand-side of \( (33) \), we see that the pole comes from the massless Goldstone boson \( \sigma_1 \). In the elementary field description, this pole term implicitly resides in the 1PI vertex \( -2\tilde{\Gamma}^{(3)}_{\psi, \bar{\psi}; \sigma_1}(p + k, -p; -k)\sigma_3 \) as shown in \( (33) \).

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{Fig2a.png} \\
\text{(k)}
\end{array}
+ \sum_{a, b}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{Fig2b.png} \\
\text{(r)}
\end{array}

\text{FIG. 2. The two-point 1PI vertex in the two descriptions.}
\end{align*}
\]

Since the two-point vertex \( \Gamma^{(2)}_{B, \sigma_1}(k) \) reflects the mixing between \( B_\mu \) and \( \partial_\mu \sigma_1 \), it must be of the following Lorentz structure

\[
\Gamma^{(2)}_{B, \sigma_1}(k) = k_\mu \Gamma^{(2)}_{B, \sigma_1}(k). \tag{40}
\]

Then from the equivalence between the pole terms in the two descriptions mentioned above, we obtain

\[
2\sigma_3 = -\frac{1}{g^2} \Gamma^{(2)}_{B, \sigma_1}(0). \tag{41}
\]

Eq. (41) gives the relation between the composite field \( \sigma_3 \) and the basic quantity \( \Gamma^{(2)}_{B, \sigma_1} \) which is of crucial importance in the present approach. From (41) we see that if \( \Gamma^{(2)}_{B, \sigma_1}(0) \neq 0 \), \( \sigma_3 \) is nonvanishing and the gauge symmetry is dynamical broken; otherwise \( \sigma_3 = 0 \) and the gauge symmetry remains unbroken. Note that this provides a method to calculate the fermion condensate parameter \( \sigma_3 \).

Inserting the expression of \( \sigma_3 \) into eq.\((41)\), we get

\[
m_B^2 = \lim_{p \to 0} Z^\text{B}_E \Gamma^{(2)}_{B, \sigma_1}(p) \frac{1}{p^2} \Gamma^{(2)}_{B, \sigma_1}(p), \tag{42}
\]

in which we have used \( (40) \), and here \( \Gamma^{(2)}_{B, \sigma_1}(p) \) is the regular term. The gauge boson mass generation can be expressed as Fig.(3).
This is the dynamical Higgs mechanism [3].

On the other hand, we can take the composite field description to express the gauge field mass as

\[ m_B^2 = \frac{1}{4} g'^2 F_{\pi}^2, \]  

where

\[ F_{\pi} = 4 Z_B^{-1/2} \sigma_3. \]  

This is the familiar gauge boson mass formula in the theory where the generated mass results from a non-zero vacuum expectation value of a scalar field.

**D. Platform Approximation**

In order to compare our approach with that of Cornwall and Norton, we follow them and make the platform approximation [5]

\[ \Sigma(p) = \tau_3 \delta m \left( \frac{p^2}{m^2} \right)^{-\epsilon}, \]  

\[ Z_{\psi} = 1, \]  

\[ \epsilon = \frac{3}{(4\pi)^2} (g^2 - g'^2), \]

where \( \delta m = (m_1 - m_2)/2 \). From [13], we know that

\[ \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_1} (p + q, -p; -q) = \frac{\tau_1}{2 \sigma_3} \delta m \left[ \left( \frac{p^2}{m^2} \right)^{-\epsilon} + \left( \frac{(p + q)^2}{m^2} \right)^{-\epsilon} \right], \]

\[ \Gamma^{(3)}_{\psi,\bar{\psi},\sigma_3} (p + q, -p; -q) = \frac{\tau_3}{2 \sigma_3} \delta m \left[ \left( \frac{p^2}{m^2} \right)^{-\epsilon} + \left( \frac{(p + q)^2}{m^2} \right)^{-\epsilon} \right]. \]

As are shown in Figs. 1 and 4, the 1PI vertices \( \Gamma^{(2)} B_{\mu,\sigma_1} (-q), \Gamma^{(3)}_{\sigma_2,\sigma_1,\sigma_3} (0, 0; 0) \) and \( \Gamma^{(2)}_{\sigma_1} (q) \) are

\[ \lim_{q \to 0} i \Gamma^{(2)}_{B_{\mu,\sigma_1}} (-q) = \lim_{q \to 0} -\text{tr} \int \frac{d^4 p}{(2\pi)^4} i g \tau_2 \gamma_{\mu} \frac{1}{\gamma \cdot (p + q) - i \left( m + \tau_3 \delta m \left( \frac{(p + q)^2}{m^2} \right)^{-\epsilon} \right)} \]
\[
\times i\Gamma^{(3)}_{\psi,\bar{\psi},\sigma_1}(p + q, -p; -q) \frac{1}{\gamma_p - i\left(m + \tau_3 \delta m \left(\frac{p^2}{m^2}\right)^{-\epsilon}\right)}
\]
\[
= \lim_{q \to 0} iq\mu' \frac{1 - \epsilon}{\sin(2\pi\epsilon \sigma_3/2\pi)} \left(\frac{\delta^4 m}{\sin(2\pi\epsilon \sigma_3^3)}\right) \quad (47a)
\]
\[
\Gamma^{(3)}_{\sigma_1,\sigma_3}(0, 0; 0) = \frac{1 - \epsilon}{2\pi \sin(2\pi\epsilon \sigma_3)} \quad (47b)
\]
\[
i\Gamma^{(2)}_{\sigma_1}(q) = -\text{tr} \int \frac{d^4p}{(2\pi)^4} i\Gamma^{(3)}_{\psi,\bar{\psi},\sigma_1}(p, -p - q, q) \frac{1}{\gamma_p + i\left(m + \tau_3 \delta m \left(\frac{(p+q)^2}{m^2}\right)^{-\epsilon}\right)}
\]
\[
\times i\Gamma^{(3)}_{\psi,\bar{\psi},\sigma_1}(p + q, -p; -q) \frac{1}{\gamma_p - i\left(m + \tau_3 \delta m \left(\frac{p^2}{m^2}\right)^{-\epsilon}\right)} \quad (47c)
\]

In the platform approximation, \( \epsilon \ll 1 \), \( Z_B = 1 \), so the mass spectrum and the Goldstone boson decay constant \( F_\pi \) [cf. eq. (44)] are

\[
m_B^2 = \frac{2}{3} \left(\frac{g^2}{g^2 - g'^2}\right) (m_1 - m_2)^2, \quad (48a)
\]
\[
m_{\sigma_1}^2 = 0, \quad (48b)
\]
\[
m_{\sigma_3}^2 = \frac{(m_1 - m_2)^2}{4}, \quad (48c)
\]
\[
\sigma_3^2 = \frac{(m_1 - m_2)^2}{32\pi^2\epsilon}, \quad (48d)
\]
\[
Z_{\sigma_1} = \frac{1}{32\pi^2\epsilon} \left(\frac{m_1 - m_2}{\sigma_3}\right)^2, \quad (48e)
\]
\[
F_\pi = 4\sigma_3 = \frac{m_1 - m_2}{\sqrt{2\pi}}. \quad (48f)
\]

We see that the obtained gauge boson mass \( m_B^2 \) is exactly the same as that in the paper by Cornwall and Norton \( [5] \), the massless scalar field \( \sigma_1 \) is the Goldstone boson and the wavefunction renormalization constant \( Z_{\sigma_1} = 1 \). From Fig. (4), one can see that the scalar field \( \sigma_1 \) is eaten by the gauge boson and becomes the longitudinal component of the massive gauge field \( B_\mu \). The scalar \( \sigma_3 \) is a composite Higgs boson with the mass proportional to \( m_1 - m_2 \). The Goldstone boson decay constant \( F_\pi \) is also proportional to \( m_1 - m_2 \) or the Higgs boson mass. The results (48c)-(48f) have not been given by Cornwall and Norton. They are new results from the present approach.

### III. THE JACKIW-JOHNSON MODEL

#### A. Ward-Takahashi Identity

Jackiw and Johnson proposed a model with chiral symmetry which is dynamically broken. The model contains a massless fermion field and a neutral vector-meson field. The Lagrangian density is

\[
\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g J_{5\mu} A_\mu, \quad (49)
\]
\[
J_{5\mu} = i\bar{\psi} \gamma_\mu \gamma_5 \psi, \quad (50)
\]
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (51)
\]

As there is an axial-vector coupling, the axial-vector anomaly occurs in the theory. The anomaly can be eliminated by introducing additional fermions. For the time being, we ignore the effect of anomaly. We shall discuss the anomaly and mass generation in the Schwinger model in the next section.

The Lagrangian is invariant under a \( U_A(1) \) chiral transformation

\[ \delta\psi(x) = i\alpha(x)\gamma_5\psi(x), \quad \delta A_\mu(x) = -i\frac{\gamma_\mu}{g}\alpha(x), \]

We introduce external sources coupling to the composite operators in the generating functional

\[ Z[J] = e^{iW[J]} \]

\[ = \int D\bar{\psi} D\psi D\alpha D\beta \exp \left( i \int d^4x [L + J_\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta + \bar{\psi}\gamma_5\psi K + \bar{\psi}\gamma_5\psi K_5] \right). \]  

From (54), we have

\[ \Gamma[\phi] = W[J] - \int d^4x [\bar{\psi}_c(x)\eta(x) + \bar{\eta}(x)\psi_c(x) + J_\mu(x) A_\mu^\nu(x)] + (G(x) + \bar{\psi}_c(x)\psi_c(x))K(x) + (G_5(x) + \bar{\psi}_c(x)\psi_c(x))K_5(x) \]

and rewrite (58) as

\[ \frac{\delta\Gamma[\phi]}{\delta\psi_c(x)} = \frac{\delta\Gamma[\phi]}{\delta\bar{\psi}_c(x)} = \frac{\delta\Gamma[\phi]}{\delta\bar{\psi}_c(x)} = \frac{\delta\Gamma[\phi]}{\delta\bar{\psi}_c(x)} = 0. \]

This is the Ward-Takahashi identity in the Jackiw-Johnson model.

Now we introduce the scalar and pseudoscalar composite fields \( \sigma(x) \) and \( \pi(x) \), i.e.

1Note that we have ignored the anomaly.
\[ \sigma(x) = aG(x), \]
\[ \pi(x) = aG_\pi(x). \]  

(61a)
(61b)

With the composite fields, the Ward-Takahashi identity can be re-expressed as

\[ \frac{\delta \Gamma[\phi]}{\delta \psi_c(x)} 2 \gamma_5 \psi_c(x) - \tilde{\psi}_c(x) 2 \gamma_5 \frac{\delta \Gamma[\phi]}{\delta \psi_c(x)} - \frac{i}{2g} \frac{\delta \Gamma[\phi]}{\delta A^\mu_c(x)} + \sigma_c(x) \frac{\delta \Gamma[\phi]}{\delta \sigma_c(x)} - \pi_c(x) \frac{\delta \Gamma[\phi]}{\delta \pi_c(x)} = 0. \]

(62)

We shall get the mass spectra of fermion, vector meson and scalar mesons from this Ward-Takahashi identity.

**B. Mass Spectrum**

Differentiating (62) with respect to \( \psi_c(y) \) and \( \tilde{\psi}_c(z) \), we have

\[ \delta (x - z) \frac{i}{2} \gamma_5 \frac{\delta^2 \Gamma[\phi]}{\delta \psi_c(y) \delta \psi_c(x)} - \delta (x - y) \frac{\delta^2 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(x)} \frac{i}{2} \gamma_5 - \tilde{\psi}_c(x) \frac{i}{2} \gamma_5 \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \psi_c(x)} + \]
\[ + \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \sigma_c(x)} \pi_c(x) + \frac{\delta^3 \Gamma[\phi]}{\delta \psi_c(z) \delta \psi_c(y) \delta \pi_c(x)} \sigma_c(x) = 0. \]

(63)

We choose the symmetry breaking direction to be

\[ \langle \tilde{\psi}(x) \psi(x) \rangle \neq 0, \]
\[ \langle \psi(x) i \gamma_5 \psi(x) \rangle = 0. \]

(64a)
(64b)

Making Fourier transformation we have, for vanishing external sources

\[ \frac{i}{2} \gamma_5 \Gamma^{(2)}_{\psi, \tilde{\psi}}(p + k) + \Gamma^{(2)}_{\psi, \psi}(p) \frac{i}{2} \gamma_5 = \frac{1}{2g} k_\mu \Gamma^{(3)}_{\psi, \tilde{\psi}, A_\mu}(p + k, -p; k) + \Gamma^{(3)}_{\psi, \psi, \pi}(p + k, -p; k) \sigma_c. \]

(65)

As \( k_\mu \to 0 \), (65) becomes

\[ \frac{i}{2} \{ \gamma_5, \Gamma^{(2)}_{\psi, \tilde{\psi}}(p) \} = \Gamma^{(3)}_{\psi, \tilde{\psi}, \pi}(p, -p; 0) \sigma_c. \]

(66)

The fermion mass is now

\[ m_f = Z_\psi^{-1} \gamma_5 \Gamma^{(3)}_{\psi, \tilde{\psi}; \pi}(0, 0; 0; 0) \sigma_c, \]

(67)

where \( Z_\psi \) is the wavefunction renormalization constant. This is the analogy of the Goldberg-Treiman relation [11].

Similar to Sec. II when we take derivatives of (62) with respect to \( A^\mu_c \), we have

\[ \sigma_c \Gamma^{(2)}_{A_\mu, \pi}(p) - \frac{i}{2g} p_\mu \Gamma^{(2)}_{A_\mu, \pi}(p) = 0. \]

(68)

From the above equation we see that if chiral gauge symmetry is unbroken, the gauge field only has transverse components. Using the relation (28), we obtain the gauge boson mass

\[ m_A^2 = \lim_{q \to 0} Z_A^{-1} 2g \frac{q_\mu \Gamma^{(2)}_{A_\mu, \pi}(p)}{q_\mu} \sigma_c, \]

(69)

where \( Z_A \) is the wavefunction renormalization constant of gauge field.

In order to determine \( \sigma_c \), we can use the matching condition

\[ \frac{k_\mu}{2g} \Gamma^{(3)}_{\psi, \tilde{\psi}, A_\mu}(p + k, -p; k) = \frac{k_\mu}{2g} \Gamma^{(3)}_{\psi, \psi, A_\mu}(p + k, -p; -k) + \Gamma^{(3)}_{\psi, \psi, \pi}(p + k, -p; -k) \sigma_c. \]

(70)
\[ \Gamma^{(3)}_{\bar{\psi}, \psi; A_{\mu}} (p + k, -p; -k) = \Gamma^{(3)}_{\bar{\psi}, \psi; A_{\mu}} (p + k, -p; -k) + \Gamma^{(3)}_{\bar{\psi}, \psi; \pi} (p + k, -p; -k) \frac{-i}{k^2} \Gamma^{(2)}_{A_{\mu; \pi}} (k), \] (71)

we have

\[ \sigma_c = \frac{1}{2g} \Gamma^{(2)}_{A_{\pi}} (0), \] (72)

where we have denoted \( q_\mu \Gamma^{(2)}_{A_{\mu; \pi}} (q) = \Gamma^{(2)}_{A_{\mu; \pi}} (q) \). Therefore, we can express the vector meson mass as

\[ m_A^2 = \lim_{p \to 0} Z^{-1}_A \Gamma^{(2)}_{A_{\mu; \pi}} (p) \frac{1}{p^2} \Gamma^{(2)}_{A_{\mu; \pi}} (p), \] (73)

which is of the standard form. Introducing the Goldstone boson decay constant \( F_\pi \), the gauge boson mass can be expressed as

\[ m_A^2 = \frac{1}{4} g^2 F^2_\pi, \] (74)

\[ F_\pi = 4Z^{-1/2} \sigma_c. \] (75)

From the Ward-Takahashi identity, we can also obtain the scalar and pseudoscalar meson masses

\[ m_\pi^2 = 0, \] (76a)

\[ m_\sigma^2 = -Z^{-1}_\pi \Gamma^{(3)}_{\bar{\sigma}, \pi; \pi} (0, 0; 0) \sigma_c, \] (76b)

in which the wavefunction renormalization constant \( Z_\pi \) is

\[ Z_\pi = \frac{d}{dp^2} \Gamma^{(2)}_\pi (p) \bigg|_{p^2 = 0}. \] (77)

From eqs. (65), (66) and (76b), we can express the 1PI vertex \( \Gamma^{(3)}_{\bar{\psi}, \psi; \pi} \) in terms of \( \Gamma^{(2)}_{\bar{\psi}, \psi} \) and obtain all the mass spectrum, the fermion condensate and the Goldstone boson decay constant \( F_\pi \).

In order to compare our results with those in the paper by Jackiw and Johnson, we follow them and take the approximation for \( \Sigma (p) \) [6]

\[ \Sigma (p) = m \left( \frac{p^2}{m^2} \right)^{-\epsilon (g^2)} , \] (78)

where \( \epsilon (g^2) \) is a positive, coupling constant-dependent quantity. Then

\[ \Gamma^{(2)}_{\bar{\psi}, \psi} (p) = \gamma \cdot p - i \Sigma (p), \] (79a)

\[ Z_\psi = 1, \] (79b)

\[ Z_A = 1. \] (79c)

Using (70), we get

\[ \Gamma^{(3)}_{\bar{\psi}, \psi; \pi} (p, -p; 0) = i \gamma_5 \frac{\Sigma (p)}{\sigma_c}, \] (80a)

\[ i \Gamma^{(2)}_{A_{\mu; \pi}} (p) = -\text{tr} \int \frac{d^4 q}{(2 \pi)^4} \left( \frac{1}{\gamma \cdot q - i m \frac{q^2}{m^2}} \right) \Gamma^{(3)}_{\bar{\psi}, \psi; \pi} (p + q, -q; -p) \times \gamma \cdot (p + q - i m \frac{q^2}{m^2}) \frac{-ig \gamma_5}{\sigma_c}, \] (80b)

\[ i \Gamma^{(3)}_{\sigma, \pi; \pi} (0, 0; 0) = -2 \text{tr} \int \frac{d^4 q}{(2 \pi)^4} \left( \frac{1}{\gamma \cdot p - i \Sigma (p)} \right) \frac{\gamma_5 \Sigma (p)}{\sigma_c} \frac{1}{\gamma \cdot p - i \Sigma (p)} \times \gamma_5 \frac{\Sigma (p)}{\sigma_c} \frac{1}{\gamma \cdot p - i \Sigma (p)} \frac{1}{\sigma_c} \Sigma (p), \] (80c)
\[ i\Gamma^{(2)}_\pi(q) = -\text{tr} \int \frac{d^4p}{(2\pi)^4} \left( i\Gamma^{(3)}_{\psi,\bar{\psi};\pi}(p,-p-q;\bar{q}) \frac{1}{\gamma.(p+q) - i \left( m + m^2 \left( \frac{(p+q)^2}{m^2} \right)^{\tau} \right)} \right) \]
\times \left( i\Gamma^{(3)}_{\psi,\bar{\psi};\pi}(p+q,-p;\bar{q};q) \frac{1}{\gamma.p - i \left( m + m^2 \left( \frac{p^2}{m^2} \right)^{\tau} \right)} \right). \]

(80d)

Then, we have the results of the mass spectrum and the Goldstone boson decay constant

\[ m_A^2 = \frac{g^2}{4\pi^2} \frac{m_\pi^2}{\epsilon(g)}, \]
\[ m_\pi^2 = 0, \]
\[ m_\sigma^2 = 2m^2, \]
\[ F_\pi = \frac{m}{\pi \epsilon(g)^{1/2}}, \]
\[ Z_\pi = \frac{1}{(4\pi^2\epsilon) \left( \frac{m}{\epsilon} \right)^2} \]

(81a)-(81e)

where \( m \) is the dynamical fermion mass and \( m_\sigma \) is the composite Higgs boson mass. The gauge boson mass is exactly the same as that in the paper by Jackiw and Johnson. The composite Higgs boson mass and \( \pi \) decay constant \( F_\pi \) are our new results. From the above equations, it is easy to see that \( Z_\pi = 1 \).

### IV. THE SCHWINGER MODEL

In the last section, we have ignored the chiral anomaly, i.e. ignoring the variation of the measure in the path integral under the chiral transformation. If a theory is not anomaly free, the integral measure is not invariant under the chiral transformation and the Ward-Takahashi identity will be modified. In 1+1 dimensions, it has been shown that the chiral anomaly affects the mass spectrum [13]. For example, in the Schwinger model, the vector boson acquires a mass from the anomaly. Since the Schwinger model is exactly solvable, it has been extensively studied in different approaches, such as functional method, operator method and bosonization techniques. In this section, we apply the Ward-Takahashi identity to study the generation of the vector boson mass.

The Schwinger model is described by the Lagrangian density

\[ \mathcal{L} = \bar{\psi}i\gamma.D\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

(82)

where

\[ D_\mu = \partial_\mu - ieA_\mu \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

(83a)-(83b)

The generating functional is

\[ Z[J] = e^{iW[J]} = \int D\bar{\psi}D\psi DA_\mu \exp \left( i \int d^2x \left[ \mathcal{L} + J_\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta \right] \right). \]

(84)

We define the classical fields

\[ \frac{\delta W[J]}{\delta J_\mu(x)} = A_\mu^c(x), \]
\[ \frac{\delta W[J]}{\delta \bar{\eta}(x)} = \psi_c(x), \]
\[ \frac{\delta W[J]}{\delta \eta(x)} = -\psi_c(x). \]

(85a)-(85c)
Taking derivative with respect to $A$

In the absence of the external sources, (92) becomes

\[ \Gamma[\phi] = W[J] - \int d^2x [\bar{\psi}_c(x)\eta(x) + \bar{\eta}(x)\psi_c(x) + J_\mu(x)A^\mu_c(x)]. \] (86)

Under the chiral gauge transformation

\[ \delta\psi(x) = i\gamma_5\beta(x)\psi(x), \] (87a)
\[ \delta\bar{\psi}(x) = i\bar{\psi}(x)\gamma_5\beta(x), \] (87b)
\[ \delta A_\mu(x) = \frac{i}{e}\epsilon_{\mu\nu}\partial_\nu\beta(x), \] (87c)

the Lagrangian is invariant and the generating functional is

\[ Z[J] = \int D\bar{\psi}'D\psi'DA'_\mu \exp\left( i \int d^2x [\mathcal{L} + J_\mu A'_\mu + \bar{\eta}\psi' + \psi'\eta] \right) \]
\[ = \int D\bar{\psi}D\psi DA_\mu \exp\left( i \int d^2x \left[ \frac{ie}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}\beta(x) + J_\mu\delta A_\mu + \bar{\eta}\delta\psi + \delta\bar{\psi}\eta \right] \right)e^{is_{eff}}, \] (88)

where we have considered the variation of the integral measure

\[ D\bar{\psi}'D\psi'DA'_\mu = D\bar{\psi}D\psi DA_\mu \exp\left( - \int d^2x \frac{e}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}\beta(x) \right). \] (89)

From (88), we can obtain

\[ i\frac{e}{\pi}\epsilon_{\mu\nu}\partial_\mu A_{\nu\lambda}(x) - \frac{i}{e}\epsilon_{\mu\nu}\partial_\nu J_\mu(x) + \bar{\eta}(x)i\gamma_5\psi_c(x) + \bar{\psi}_c(x)i\gamma_5\eta(x) = 0. \] (90)

This is the Slavnov-Taylor identity including the effect of the anomaly. Expressing the external sources in terms of the derivatives of the effective action, we get the Ward-Takahashi identity

\[ i\frac{e}{\pi}\epsilon_{\mu\nu}\partial_\mu A_{\nu\lambda}(x) + \frac{i}{e}\epsilon_{\mu\nu}\partial_\nu J_\mu(x) + \frac{\delta\Gamma[\phi]}{\delta\psi_c(x)}i\gamma_5\psi_c(x) - \bar{\psi}_c(x)i\gamma_5 = 0. \] (91)

Taking derivative with respect to $A_{\nu\lambda}(y)$, we have

\[ \frac{\delta^2\Gamma[\phi]}{\delta A_{\nu\lambda}(x)\delta\psi_c(x)}i\gamma_5\psi_c(x) + \bar{\psi}_c(x)i\gamma_5 = \frac{\delta^2\Gamma[\phi]}{\delta A_{\nu\lambda}(x)\delta A_{\mu}(x)} \]
\[ + i\frac{e}{\pi}\epsilon_{\mu\nu}\partial_\mu\delta(x-y)\delta_{\nu\lambda} + \frac{i}{e}\epsilon_{\mu\nu}\partial_\nu \frac{\delta^2\Gamma[\phi]}{\delta A_{\nu\lambda}(y)\delta A_{\mu}(x)} = 0. \] (92)

In the absence of the external sources, (92) becomes

\[ i\frac{e^2}{\pi}k_\mu k_\nu + ik_\mu k_\nu \Gamma_{\mu\nu}^{(2)}(k) = 0. \] (93)

Thus the vector boson mass is

\[ m_A^2 = -\lim_{k\to0} \frac{k^\mu k^\nu}{k^2} \Gamma_{\mu\nu}^{(2)}(k) \]
\[ = \frac{e^2}{\pi}. \] (94)

This coincides with the results in other approaches. In this approach, we see clearly that, in the Schwinger model, the vector meson acquires a mass from the anomaly related to the measure of the path integral as it should be.
V. CONCLUSIONS

In this paper, we have developed a formal approach to dynamical breaking of Abelian gauge symmetry based on the Ward-Takahashi identity including composite fields and the matching condition for the equivalence between the elementary field description and the composite field description. The approach is easily applied to the Cornwall-Norton model, the Jackiw-Johnson model and the Schwinger model. In this approach, we can obtain not only the dynamically generated gauge boson and fermion masses as in other approaches, but also the masses of the composite scalars and the Goldstone boson decay constant $F_\pi$ which have not been given in the corresponding papers.

Of course, the explicit results of the mass spectra and the Goldstone boson decay constant depends on the evaluation of the 1PI vertices $\Gamma^{(3)}_{\psi,\bar{\psi},\pi}$ and $\Gamma^{(3)}_{\psi,\bar{\psi},\sigma_1}$, which requires certain approaches to the dynamics. In this paper, for the sake of comparison with the existing results, we simply take the simple approximations following the corresponding papers. Developing further improved approximations will be of interest in future investigations.

The approach developed in this paper can in principle be generalized to the study of dynamical symmetry breaking in non-Abelian gauge theories.

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\[ m_t \sim \sqrt{\frac{h^2}{4\pi}} \]

\[ \frac{h^2}{4\pi} = 0.022 \]

\[ \frac{h^2}{4\pi} = 1 \]