TRACE FORMULAE FOR CURVATURE OF JET BUNDLES OVER PLANAR DOMAIN

DINESH KUMAR KESHWAR

Abstract. For a domain $\Omega$ in $\mathbb{C}$ and an operator $T$ in $\mathcal{B}_n(\Omega)$, Cowen and Douglas construct a Hermitian holomorphic vector bundle $E_T$ over $\Omega$ corresponding to $T$. The Hermitian holomorphic vector bundle $E_T$ is obtained as a pull-back of the tautological bundle $S(n, H)$ defined over $\mathcal{G}(n, H)$ by a nondegenerate holomorphic map $z \mapsto \ker(T - z)$, $z \in \Omega$. To find the answer to the converse, Cowen and Douglas studied the jet bundle in their foundational paper. The computations in this paper for the curvature of the jet bundle are somewhat difficult to comprehend. They have given a set of invariants to determine if two rank $n$ Hermitian holomorphic vector bundle are equivalent. These invariants are complicated and not easy to compute. It is natural to expect that the equivalence of Hermitian holomorphic jet bundles should be easier to characterize. In fact, in the case of the Hermitian holomorphic jet bundle $J_k(L_f)$, we have shown that the curvature of the line bundle $L_f$ completely determines the class of $J_k(L_f)$. In case of rank $n$ Hermitian Holomorphic vector bundle $E_f$, We have calculated the curvature of jet bundle $J_k(E_f)$ and also have generalized the trace formula for jet bundle $J_k(E_f)$.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. The following important class of operators was introduced in [3].

Definition 1.1. For a connected open subset $\Omega$ of $\mathbb{C}$ and a positive integer $n$, let

$$\mathcal{B}_n(\Omega) = \left\{ T \in \mathcal{L}(\mathcal{H}) \mid \Omega \subset \sigma(T), \begin{array}{l} \text{ran}(T - w) = \mathcal{H} \text{ for } w \in \Omega, \\ \bigvee \limits_{w \in \Omega} \ker(T - w) = \mathcal{H}, \\ \dim \ker(T - w) = n \text{ for } w \in \Omega \end{array} \right\},$$

where $\sigma(T)$ denotes the spectrum of the operator $T$.

We recall (cf. [3]) that an operator $T$ in the class $\mathcal{B}_n(\Omega)$ defines a Hermitian holomorphic vector bundle $E_T$ in a natural manner. It is the sub-bundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by

$$E_T = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker(T - w)\}$$

with the natural projection map $\pi : E_T \to \Omega$, $\pi(w, x) = w$. It is shown in [3] Proposition 1.12] that the mapping $w \mapsto \ker(T - w)$ defines a rank $n$ Hermitian holomorphic vector bundle $E_T$ over $\Omega$ for $T \in \mathcal{B}_n(\Omega)$. In [3], it was also shown that the equivalence class of the Hermitian holomorphic vector bundle $E_T$ and the unitary equivalence class of the operator $T$ determine each other.

Theorem 1.2. The operators $T$ and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if the corresponding Hermitian holomorphic vector bundles $E_T$ and $E_{\tilde{T}}$ are equivalent.

Key words and phrases. Cowen-Douglas class, curvature, Hermitian holomorphic vector bundle, Jet Bundle.

The work of author was supported by IISc Research Associate Fellowship at the Indian Institute of Science.
In general, it is not easy to decide if two Hermitian holomorphic vector bundles are equivalent except when the rank of the bundle is 1. In this case, the curvature

$$\mathcal{K}(w) = -\frac{\partial^2 \log \| \gamma(w) \|^2}{\partial w \bar{w}}$$

of the line bundle $E$, defined with respect to a non-zero holomorphic section $\gamma$ of $E$, is a complete invariant. The definition of the curvature is independent of the choice of the section $\gamma$: If $\gamma_0$ is another holomorphic section of $E$, then $\gamma_0 = \phi \gamma$ for some holomorphic function $\phi$ on some open subset $\Omega_0$ of $\Omega$, consequently the harmonicity of $\log |\phi|$ completes the verification.

For a domain $\Omega$ in $\mathbb{C}$ and an operator $T$ in $B_n(\Omega)$, the Hermitian holomorphic vector bundle $E_T$ is obtained as a pull-back of the tautological bundle $S(n, \mathcal{H})$ defined over $\mathfrak{g}(n, \mathcal{H})$ by a nondegenerate holomorphic map $z \mapsto \ker(T - z)$, $z \in \Omega$ as in Definition 2.2. To find the answer to the converse, namely, when a given Hermitian holomorphic vector bundle is a pull-back of the tautological bundle by a nondegenerate holomorphic map, Cowen and Douglas studied the jet bundle in their foundational paper [3, pp. 235]. The computations in this paper for the curvature of the jet bundle are somewhat difficult to comprehend. They have given a set of invariants to determine if two rank $n$ Hermitian holomorphic vector bundle are equivalent. These invariants are complicated and not easy to compute. It is natural to expect that the equivalence of Hermitian holomorphic jet bundles should be easier to characterize. In fact, in the case of the Hermitian holomorphic jet bundle $J_k(\mathcal{L}_f)$, where the line bundle $\mathcal{L}_f$ is a pull-back of the tautological bundle on $\mathfrak{g}(1, \mathcal{H})$, we have shown that the curvature of the line bundle $\mathcal{L}_f$ completely determines the class of $J_k(\mathcal{L}_f)$. In general, however, our results are not as complete. Relating the complex geometric invariants inherent in the short exact sequence

$$0 \to E_I \to E \to E_{II} \to 0.$$  

is an important problem. In the paper [1], it is shown that the Chern classes of these bundles must satisfy

$$c(E) = c(E_I) c(E_{II}).$$

Donaldson [5] obtains similar relations involving what are known as secondary invariants. We obtain a refinement, in case $E_I = J_k(E_f)$ and $E = J_{k+1}(E_f)$, namely,

$$(\text{trace } \otimes \text{Id}_{n \times n})(\mathcal{K}_{J_k(E_f)}) - (\text{trace } \otimes \text{Id}_{n \times n})(\mathcal{K}_{J_{k-1}(E_f)}) = \mathcal{K}_{J_k(E_f)/J_{k-1}(E_f)}.$$

2. Definitions and Notations

Here we give the definition of a jet bundle closely following [3]. An equivalent description, in a slightly different language, may be found in [2].

Let $E$ be a Hermitian holomorphic bundle of rank $n$ over a bounded domain $\Omega \subset \mathbb{C}$. For each $k = 0, 1, \ldots$ we associate to $E$ a $(k+1)n$ dimensional holomorphic bundle $J_k(E)$, the holomorphic k-jet bundle of $E$, defined as follows:

If $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a holomorphic frame for $E$, on an open subset $U$ contained in $\Omega$, then $J_k(E)$ has an associated frame

$$J_k(\sigma) = \{\sigma_{10}, \ldots, \sigma_{n0}, \ldots, \sigma_{1k}, \ldots, \sigma_{nk}\}$$

defined on $U$. If $\tilde{\sigma}$ is another frame for $E$ defined on $\tilde{U}$, then on $U \cap \tilde{U}$, we have $\tilde{\sigma}_j = \sum a_{ij} \sigma_i$, where $A = (a_{ij})$ is a holomorphic, $n \times n$, nonsingular matrix. Symbolically

$$\tilde{\sigma} = \sigma A.$$

Let $J_k(A)$ be the $(k+1)n \times (k+1)n$, non-singular, holomorphic matrix
\[ J_k(A) = \begin{pmatrix}
A & A' & A'' & \cdots & \binom{k}{k} A^{(k)} \\
\vdots & A & 2A' & \cdots & \binom{k}{k-1} A^{(k-1)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & A
\end{pmatrix}. \]

Then, by definition, the frames \( J_k(\sigma) \) and \( J_k(\tilde{\sigma}) \) are related on \( U \cap \tilde{U} \) by

\[ J_k(\tilde{\sigma}) = J_k(\sigma) J_k(A). \]

A straightforward computation yields that if \( A \) and \( \tilde{A} \) are holomorphic \( n \times n \) matrices, then

\[ J_k(A\tilde{A}) = J_k(A) J_k(\tilde{A}) \]

so the bundle \( J_k(E) \) is well-defined.

The Hermitian metric \( h \) on \( E \) induces a Hermitian form \( J_k(h) \) on \( J_k(E) \) such that if \( h(\sigma) \) is the matrix of inner products \( \langle (\sigma_j, \sigma_i) \rangle \) for \( i, j = 1, \ldots, n \), then

\[ J_k(h)(J_k(\sigma)) = \begin{pmatrix}
h(\sigma) & \cdots \frac{\partial h(\sigma)}{\partial z^k} \\
\vdots & \ddots \vdots \\
\frac{\partial h(\sigma)}{\partial z^k} & \cdots \frac{\partial h(\sigma)}{\partial z^k} \frac{\partial h(\sigma)}{\partial z^{k+j}} \end{pmatrix} \]

is the matrix of \( J_k(h) \) relative to the frame \( J_k(\sigma) \). To see that \( J_k(h) \) is well-defined, we need

\[ J_k(h)(J_k(\tilde{\sigma})) = J_k(A)^* \{ J_k(h)(J_k(\sigma)) \} J_k(A) \]

which follows from the computation:

For \( 0 \leq l_1, l_2 \leq k \)

\[ \frac{\partial^{l_1 + l_2}}{\partial z^{l_1} \partial z^{l_2}} h(\tilde{\sigma}) = \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} \binom{l_1}{i} \binom{l_2}{j} \frac{\partial^{j+\sum_{i=1}^{l_1} \binom{l_1}{i}}}{\partial z^j \partial z^i} h(\sigma) \frac{\partial^{l_1-\sum_{i=1}^{l_1} \binom{l_1}{i}}}{\partial z^{l_1-\sum_{i=1}^{l_1} \binom{l_1}{i}}} A. \]

Using equation (2.1), we have

\[ J_k(h)(J_k(\tilde{\sigma})) = J_k(A)^* \{ J_k(h)(J_k(\sigma)) \} J_k(A). \]

In general, the form \( J_k(h)(z) \) on the jet bundle \( J_k(E) \) need not be positive definite for \( z \in \Omega \). Thus \( J_k(E) \) has no natural Hermitian metric, just a Hermitian form.

For \( \mathcal{H} \) a complex Hilbert space and \( n \) a positive integer, let \( \mathcal{G}r(n, \mathcal{H}) \) denote the Grassmann manifold, the set of all \( n \)-dimensional subspaces of \( \mathcal{H} \).

**Definition 2.1.** For \( \Omega \) an open connected subset of \( \mathbb{C} \), we say that a map \( f : \Omega \to \mathcal{G}r(n, \mathcal{H}) \) is holomorphic at \( \lambda_0 \in \Omega \) if there exists a neighborhood \( U \) of \( \lambda_0 \) and \( n \) holomorphic \( \mathcal{H} \)-valued functions \( \sigma_1, \ldots, \sigma_n \) on \( U \) such that \( f(\lambda) = \sqrt[n]{\{ \sigma_1(\lambda), \ldots, \sigma_n(\lambda) \}} \) for \( \lambda \) in \( U \). If this holds for each \( \lambda_0 \in \Omega \) then we say that \( f \) is holomorphic on \( \Omega \).

If \( f : \Omega \to \mathcal{G}r(n, \mathcal{H}) \) is a holomorphic map, then a natural \( n \)-dimensional Hermitian holomorphic vector bundle \( E_f \) is induced over \( \Omega \), namely,

\[ E_f = \{ (x, \lambda) \in \mathcal{H} \times \Omega : x \in f(\lambda) \}. \]

and

\[ \pi : E_f \to \Omega \text{ where } \pi(x, \lambda) = \lambda. \]
The metric for the jet bundle with respect to the metric holomorphic map

\[ j_k(f) : \Omega \to \mathcal{G}r((k+1)n, H) \]

Remark 2.4. If \( E \) be the set \( \{ \sigma_1, \ldots, \sigma_n^k(w) \} \) such that \( \sigma_1(w), \ldots, \sigma_n(w) \) are independent for each \( w \) in the open set \( U \). If this holds for all \( k = 0, 1, \ldots \), then we say that \( f \) is nondegenerate.

If \( f \) is \( k \)-nondegenerate, then \( f \) induces a holomorphic map

\[ j_k(f) : \Omega \to \mathcal{G}r((k+1)n, H) \]

such that \( j_k(f)(w) \) is the span of \( \sigma_1(w), \ldots, \sigma_n^k(w) \). If \( \sigma \) is a frame for \( E_f \) on \( \Omega \), let \( j_k(\sigma) = \{ \sigma_1, \ldots, \sigma_n, \sigma_1^{(k)}, \ldots, \sigma_n^{(k)} \} \) be the induced frame for \( E_{j_k(f)} \). Then \( j_k(E_f) \) and \( E_{j_k(f)} \) are naturally equivalent Hermitian holomorphic bundles by identifying \( \sigma_{ir}^{(r)} \) with \( \sigma_i^{(r)}(w) \), since \( \langle \sigma_{ir}, \sigma_{js} \rangle = \partial z^r \partial \bar{z}^s = \partial z^r \partial \bar{z}^s = \langle \sigma_i^{(r)}, \sigma_j^{(s)} \rangle \). In this case \( j_k(h) \) is a Hermitian metric for \( j_k(E_f) \), that is, \( j_k(h) \) is positive definite.

Definition 2.3. Let \( H \) be a Hilbert space and \( \Omega \) be a bounded domain in \( \mathbb{C}^m \). Let \( \mathfrak{S}_n(\Omega, H) \) be the set of all Hermitian holomorphic vector bundles of rank \( n \) over \( \Omega \) which arise as a pull-backs of the tautological bundle by nondegenerate holomorphic maps. That is, for any nondegenerate holomorphic map \( f : \Omega \to \mathcal{G}r(n, H) \) the vector bundle \( E_f = \{(x, \lambda) \in H \times \Omega : x \in f(\lambda) \} \) is in \( \mathfrak{S}_n(\Omega, H) \).

Remark 2.4. If \( E_f \) is in \( \mathfrak{S}_n(\Omega, H) \), then the preceding calculation shows that \( j_k(E_f) \) is in \( \mathfrak{S}_n(k+1)(\Omega, H) \).

3. Line Bundles

Let \( L_f \) be a Hermitian holomorphic line bundle over a bounded domain \( \Omega \subset \mathbb{C} \). Assume that \( L_f \in \mathfrak{S}_1(\Omega, H) \). Let \( j_k(L_f) \) be a jet bundle of rank \( k+1 \) obtained from \( L_f \). Let \( \sigma \) be a frame for \( L_f \) over an open subset \( \Omega_0 \) of \( \Omega \). A frame for \( j_k(L_f) \) over the open set \( \Omega_0 \) is easily seen to be the set \( \{ \sigma, \partial \sigma, \partial^2 \sigma, \ldots, \partial^k \sigma \} \). Let \( h \) be a metric for \( L_f \), which is of the form

\[ h(z) = \langle \sigma(z), \sigma(z) \rangle. \]

The metric for the jet bundle \( j_k(h) \) is then of the form

\[ j_k(h)(z) = \begin{pmatrix} h(z) & \cdots & \frac{\partial^k}{\partial z^k} h(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial z^k} h(z) & \cdots & \frac{\partial^{2k}}{\partial z^{2k}} h(z) \end{pmatrix}. \]

Let \( \mathcal{K}_{j_k(L_f)} \) be the curvature of the jet bundle \( j_k(L_f) \). An explicit formula for the curvature of a Hermitian holomorphic vector bundle \( E \) is given in [9, proposition 2.2, pp. 79]. The curvature \( \mathcal{K}_{j_k(L_f)} \) of the jet bundle therefore takes the form

\[ \mathcal{K}_{j_k(L_f)}(z) = \frac{1}{2} \{ (j_k(h)(z))^{-1} \partial j_k(h)(z) \}, \]

with respect to the metric \( j_k(h) \) obtained from frame \( \{ \sigma, \partial \sigma, \partial^2 \sigma, \ldots, \partial^k \sigma \} \). Set \( j^k(z) = (j_k(h)(z))^{-1} \partial j_k(h)(z) \) and note that

\[ (j_k(h)(z))^{-1} \partial (j_k(h)(z)) = \begin{pmatrix} 0 & 0 & \cdots & 0 & (j^k(z))_{1,k+1} \\ 0 & \cdots & 0 & (j^k(z))_{2,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & (j^k(z))_{k+1,k+1} \end{pmatrix} dz, \]
where \((J^k(z))_{i,k+1}\) is the \((i,k+1)\)th entry of the matrix \(J^k(z)\). The matrix product in the first equation is of the form \(A^{-1}B\), where the first \(k\) columns of \(B\) are the last \(k\) column of \(A\).

Therefore the curvature of the jet bundle \(\mathcal{J}_k(\mathcal{L}_f)\) is seen to be of the form

\[
\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) = \begin{pmatrix}
0 & \cdots & 0 & b_1(z) \\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & b_k(z) \\
0 & \cdots & 0 & \mathcal{K}_{\text{det}(\mathcal{J}_k(\mathcal{L}_f))}(z)
\end{pmatrix} \, d\zeta \wedge dz,
\]

where \(b_i(z) = \frac{\partial}{\partial z}[(J^k(z))_{i,k+1}], \ 1 \leq i \leq k.\)

**Theorem 3.1.** As before, let \(\mathcal{L}_f\) and \(\mathcal{L}_j\) be two Hermitian holomorphic line bundles over a bounded domain \(\Omega \subset \mathbb{C}\). Let \(\mathcal{J}_k(\mathcal{L}_f)\) and \(\mathcal{J}_k(\mathcal{L}_j)\) be the corresponding jet bundles of rank \(k+1\). If \(\mathcal{J}_k(\mathcal{L}_f)\) is locally equivalent to \(\mathcal{J}_k(\mathcal{L}_j)\), then \(\mathcal{J}_{k-1}(\mathcal{L}_f)\) is locally equivalent to \(\mathcal{J}_{k-1}(\mathcal{L}_j)\).

**Proof.** Since \(\mathcal{J}_k(\mathcal{L}_f)\) and \(\mathcal{J}_k(\mathcal{L}_j)\) are locally equivalent, for each \(z_0 \in \Omega\), there exists a neighborhood \(\Omega_0\) and a holomorphic bundle map \(\phi: \mathcal{J}_k(\mathcal{L}_f)|_{\Omega_0} \rightarrow \mathcal{J}_k(\mathcal{L}_j)|_{\Omega_0}\) such that \(\phi\) is an isomorphism. Let \(\mathcal{J}_k(\sigma) = \{\sigma, \frac{\partial \sigma}{\partial z}, \frac{\partial^2 \sigma}{\partial z^2}, \ldots, \frac{\partial^k \sigma}{\partial z^k}\}\) and \(\mathcal{J}_k(\tilde{\sigma}) = \{\tilde{\sigma}, \frac{\partial \tilde{\sigma}}{\partial z}, \frac{\partial^2 \tilde{\sigma}}{\partial z^2}, \ldots, \frac{\partial^k \tilde{\sigma}}{\partial z^k}\}\) be frames for \(\mathcal{J}_k(\mathcal{L}_f)\) and \(\mathcal{J}_k(\mathcal{L}_j)\) over the open subset \(\Omega_0\) of \(\Omega\) respectively.

Now

\[
\phi(\frac{\partial^i \sigma}{\partial z^i}(z)) = \sum_{i=0}^{k} \phi_{ij}(z) (\frac{\partial^i \tilde{\sigma}}{\partial z^i}(z)).
\]

So the matrix representing \(\phi\) with respect to the two frames \(\mathcal{J}_k(\sigma)\) and \(\mathcal{J}_k(\tilde{\sigma})\) is

\[
\phi(z) = \begin{pmatrix}
\phi_{0,0}(z) & \cdots & \phi_{0,k}(z) \\
\vdots & & \vdots \\
\phi_{k,0}(z) & \cdots & \phi_{k,k}(z)
\end{pmatrix}.
\]

Therefore we can write

\[
(\phi(\sigma(z)), \phi(\frac{\partial \sigma}{\partial z}(z)), \ldots, \phi(\frac{\partial^k \sigma}{\partial z^k}(z))) = (\tilde{\sigma}(z), \frac{\partial \tilde{\sigma}}{\partial z}(z), \ldots, \frac{\partial^k \tilde{\sigma}}{\partial z^k}(z)) \phi(z).
\]

But we know that

\[
\phi(z) \mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) = \mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) \phi(z).
\]

Now

\[
(\phi(z) \mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) )_{ij} = \begin{cases}
0 & \text{if } 0 \leq i, j \leq k - 1, \\
\sum_{l=0}^{k-1} b_{l+1}(z) \phi_{i,l}(z) + \mathcal{K}_{\text{det}(\mathcal{J}_k(\mathcal{L}_f))}(z) \phi_{i,k}(z) d\zeta \wedge dz & \text{if } 0 \leq i \leq k, j = k.
\end{cases}
\]

and

\[
\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) \phi(z) = \begin{pmatrix}
b_1(z) & \phi_{1,0}(z) & \cdots & \phi_{1,k}(z) \\
\vdots & & \vdots & \vdots \\
b_{k-1}(z) & \phi_{k-1,0}(z) & \cdots & \phi_{k-1,k}(z) \\
\mathcal{K}_{\text{det}(\mathcal{J}_k(\mathcal{L}_f))}(z) & \phi_{k,0}(z) & \cdots & \phi_{k,k}(z)
\end{pmatrix} d\zeta \wedge dz.
\]

Hence from equations \((3.1), (3.5)\) and \((3.6)\), it follows that

\[
\phi_{k,0}(z) = \phi_{k,1}(z) = \cdots = \phi_{k,k-1}(z) = 0.
\]
In this notation, we have the following equalities:

\[
A = \begin{pmatrix}
\phi_{0,0}(z) & \phi_{0,1}(z) & \cdots & \phi_{0,k}(z) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{k-1,0}(z) & \phi_{k-1,1}(z) & \cdots & \phi_{k-1,k}(z) \\
0 & 0 & \cdots & \phi_{k,k}(z)
\end{pmatrix}
\]

with respect to the frames \(j_k(\sigma)\) and \(j_k(\tilde{\sigma})\). Finally from equations (3.3) and (3.7), we see that

\[
\phi|_{j_{k-1}(L_f)|\Omega_0} : j_{k-1}(L_f)|\Omega_0 \rightarrow j_{k-1}(L_{\tilde{f}})|\Omega_0.
\]

Since \(\phi\) is a bundle isomorphism, it follows that

\[
\phi|_{j_{k-1}(L_f)|\Omega_0} : j_{k-1}(L_f)|\Omega_0 \rightarrow j_{k-1}(L_{\tilde{f}})|\Omega_0
\]

is also a bundle isomorphism.

\[\square\]

**Corollary 3.2.** Let \(L_f\) and \(L_{\tilde{f}}\) be Hermitian holomorphic line bundles. Let \(j_k(L_f)\) and \(j_k(L_{\tilde{f}})\) be the corresponding jet bundles of rank \(k + 1\). The two jet bundles \(j_k(L_f)\) and \(j_k(L_{\tilde{f}})\) are locally equivalent as Hermitian holomorphic vector bundles if and only if the two line bundles \(L_f\) and \(L_{\tilde{f}}\) are locally equivalent as Hermitian holomorphic vector bundles.

**Proof.** Suppose \(j_k(L_f)\) and \(j_k(L_{\tilde{f}})\) are locally equivalent. Then for each \(z_0 \in \Omega\) there exists a neighborhood \(\Omega_0\) and a holomorphic map \(\phi : j_k(L_f)|\Omega_0 \rightarrow j_k(L_{\tilde{f}})|\Omega_0\) such that \(\phi\) is an isomorphism.

Using Theorem 3.1 \(\phi|_{j_{k-1}(L_f)|\Omega_0} : j_{k-1}(L_f)|\Omega_0 \rightarrow j_{k-1}(L_{\tilde{f}})|\Omega_0\) is an isomorphism. Since \(\phi|_{j_{k-1}(L_f)|\Omega_0} : j_{k-1}(L_f)|\Omega_0 \rightarrow j_{k-1}(L_{\tilde{f}})|\Omega_0\) is an isomorphism, by the same argument which is given in the proof of the Theorem 3.1 it follows that

\[
\phi|_{j_{k-2}(L_f)|\Omega_0} : j_{k-2}(L_f)|\Omega_0 \rightarrow j_{k-2}(L_{\tilde{f}})|\Omega_0
\]

is an isomorphism. Repeating this argument, we see that \(\phi\) is an isomorphism from \(L_f|\Omega_0\) to \(L_{\tilde{f}}|\Omega_0\). \[\square\]

Let \(A\) be an \(n \times n\) matrix and \(A_{ij}\) be the \((n-1) \times (n-1)\) matrix which is obtained from \(A\) by removing the \(i\)th row and \(j\)th column of the matrix \(A\).

**Lemma 3.3.** Let \(A\) be an \(n \times n\) matrix and \(B\) be the \((n-2) \times (n-2)\) matrix which is obtained from \(A\) by removing the last two rows and last two columns of \(A\). Then

\[
\det(A_{\tilde{n},\tilde{n}}) \det(A_{n-1,n-1}) - \det(A_{\tilde{n},n-1}) \det(A_{n-1,\tilde{n}}) = \det(B) \det(A).
\]

**Proof.** Case(1): suppose \(B\) is invertible. Let

\[
A = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots \\
a_{n,1} & \cdots & a_{n,n}
\end{pmatrix}
\]

and

\[
x_1 = (a_{1,n-1}, a_{2,n-1}, \ldots, a_{n-2,n-1})^t, \quad x_2 = (a_{1,n}, a_{2,n}, \ldots, a_{n-2,n})^t,
\]

\[
y_1 = (a_{n-1,1}, a_{n-1,2}, \ldots, a_{n-1,n-1}), \quad y_2 = (a_{n,1}, a_{n,2}, \ldots, a_{n,n-2}).
\]

Thus the matrix \(A\) can be written in the form

\[
A = \begin{pmatrix}
B & x_1 & x_2 \\
y_1 & a_{n-1,n-1} & a_{n-1,n} \\
y_2 & a_{n,n-1} & a_{n,n}
\end{pmatrix}.
\]

In this notation, we have the following equalities:
Since determinant is a continuous function, taking
\[ m \]
clearly
that approximate
From equation (3.8), (3.9), (3.10), (3.11) and (3.12), it follows that
\[ (3.12) \]
Case(2):
\[ (3.13) \]
and
\[ \det(A) = \det(B) \left\{ \frac{\det(A_{\hat{n},\hat{n}}) \det(A_{\hat{n},\hat{n}-1})}{(\det B)^2} - \frac{\det(A_{\hat{n},\hat{n}-1}) \det(A_{\hat{n}-1,\hat{n}})}{(\det B)^2} \right\}, \]
that is,
\[ (3.13) \]
Case(2): Suppose \( B \) is not invertible. Then there exists a sequence of invertible matrices \( B_m \) that approximate \( B \), that is, \( \|B_m - B\| \to 0 \), as \( m \to \infty \). Let
\[ A_m = \begin{pmatrix} B_m & x_1 \\ y_1 & a_{n-1,n-1} \\ a_{n,n-1} & a_{n-1,n} \end{pmatrix} \]
clearly \( \|A_m - A\| \to 0 \) as \( m \to \infty \). From the proof of the previous case, we have
\[ \det\{(A_m)_{\hat{n},\hat{n}}\} \det\{(A_m)_{\hat{n}-1,\hat{n}-1}\} - \det\{(A_m)_{\hat{n},\hat{n}-1}\} \det\{(A_m)_{\hat{n}-1,\hat{n}}\} = \det(B_m) \det(A_m). \]
Since determinant is a continuous function, taking \( m \to \infty \), it follows that
\[ \det(A_{\hat{n},\hat{n}}) \det(A_{\hat{n}-1,\hat{n}-1}) - \det(A_{\hat{n},\hat{n}-1}) \det(A_{\hat{n}-1,\hat{n}}) = \det(B) \det(A). \]
Proposition 3.4. The curvature of the determinant bundle $\mathcal{J}_k(\mathcal{L}_f)$ is given by the following formula

$$\mathcal{K}_{\det \mathcal{J}_k(\mathcal{L}_f)}(z) = \frac{(\det \mathcal{J}_{k-1}h(z))(\det \mathcal{J}_{k+1}h(z))}{(\det \mathcal{J}_k h)^2(z)} \, d\bar{z} \wedge dz.$$  

Proof. The curvature of the determinant bundle $\det(\mathcal{J}_k(\mathcal{L}_f))$ is

$$\mathcal{K}_{\det \mathcal{J}_k(\mathcal{L}_f)}(z) = \frac{(\det \mathcal{J}_k h(z))(\frac{\partial}{\partial \bar{z}} \det \mathcal{J}_k h(z) - (\frac{\partial}{\partial z} \det \mathcal{J}_k h(z))(\frac{\partial}{\partial \bar{z}} \det \mathcal{J}_k h(z))}{(\det \mathcal{J}_k h)^2(z)} \, d\bar{z} \wedge dz.$$  

Here

$$\mathcal{J}_k h = \left((\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} h)^k\right)_{i,j=0} \text{ and } \mathcal{J}_{k+1} h = \left((\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} h)^{k+1}\right)_{i,j=0}.$$  

Now, we have

(3.14) $$\frac{\partial}{\partial z} (\det \mathcal{J}_k h) = \det((\mathcal{J}_{k+1} h)_{\bar{z}^{k+2}, \bar{z}^{k+1}}),$$  

(3.15) $$\frac{\partial}{\partial \bar{z}} (\det \mathcal{J}_k h) = \det((\mathcal{J}_{k+1} h)_{z^{k+1}, \bar{z}^{k+2}}),$$  

and

(3.16) $$\frac{\partial^2}{\partial z \partial \bar{z}} (\det \mathcal{J}_k h) = \det((\mathcal{J}_{k+1} h)_{z^{k+1}, \bar{z}^{k+1}}),$$  

Finally, note that

(3.17) $$\det \mathcal{J}_k h = \det((\mathcal{J}_{k+1} h)_{\bar{z}^{k+2}, \bar{z}^{k+2}}).$$  

By Lemma 3.3 we obtain

$$\det(\mathcal{J}_{k-1} h) \det(\mathcal{J}_{k+1} h) = \det((\mathcal{J}_{k+1} h)_{\bar{z}^{k+2}, \bar{z}^{k+2}}) \det((\mathcal{J}_{k+1} h)_{z^{k+1}, \bar{z}^{k+1}}) - \det((\mathcal{J}_{k+1} h)_{\bar{z}^{k+2}, \bar{z}^{k+1}}) \det((\mathcal{J}_{k+1} h)_{z^{k+1}, \bar{z}^{k+2}}).$$  

From equations (3.14), (3.15), (3.16), (3.17) and (3.18), it follows that

$$\det(\mathcal{J}_{k-1} h)(z)(\det \mathcal{J}_{k+1} h)(z)$$  

$$= (\det \mathcal{J}_k h)(z)(\frac{\partial^2}{\partial z \partial \bar{z}} \det \mathcal{J}_k h)(z)(\frac{\partial}{\partial z} \det \mathcal{J}_k h)(z)(\frac{\partial}{\partial \bar{z}} \det \mathcal{J}_k h)(z).$$  

Hence

$$\mathcal{K}_{\det \mathcal{J}_k(\mathcal{L}_f)}(z) = \frac{(\det \mathcal{J}_{k-1} h)(z)(\det \mathcal{J}_{k+1} h)(z)}{(\det \mathcal{J}_k h)^2(z)} \, d\bar{z} \wedge dz.$$  

□

Corollary 3.5. Let $\mathcal{L}_f$ and $\mathcal{L}_\bar{f}$ be Hermitian holomorphic line bundles over a domain $\Omega \subset \mathbb{C}$. The following statements are equivalent:

1. $\det \mathcal{J}_k(\mathcal{L}_f)$ is locally equivalent to $\det \mathcal{J}_k(\mathcal{L}_\bar{f})$ and $\det \mathcal{J}_{k+1}(\mathcal{L}_f)$ is locally equivalent to $\det \mathcal{J}_{k+1}(\mathcal{L}_\bar{f})$, for some $k \in \mathbb{N}$

2. $\mathcal{L}_f$ is locally equivalent to $\mathcal{L}_\bar{f}$.  

4. Rank $n$-Vector Bundles

We first recall some well known facts from linear algebra.

**Lemma 4.1.** Let $A, B, C$ and $D$ be matrices of size $n \times n, n \times m, m \times n$ and $m \times m$ respectively.

(i) \(\text{[3] pp. 138}\) If $A, D$ and $D - CA^{-1}B$ are invertible, then \[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} = \begin{pmatrix}
(A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1}
\end{pmatrix}.
\]

(ii) \(\text{[3] pp. 246}\) If $A$ is invertible then
\[
\det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det(A) \det(D - CA^{-1}B)
\]

(iii) \(\text{[3] pp. 247}\) If $D$ is invertible then
\[
\det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det(D) \det(A - BD^{-1}C).
\]

**Lemma 4.2.** \(\text{[3] pp. 240}\) If $V$ is a proper, non-zero subspace of an inner product space $W$ then it induces an inner product on the quotient $W/V$ by
\[
([w_1], [w_2]) = \|v_1 \wedge \ldots \wedge v_n\|^2 (v_1 \wedge \ldots \wedge v_n \wedge w_1, v_1 \wedge \ldots \wedge v_n \wedge w_2)
\]
where $[w_1], [w_2]$ denote the equivalence classes of $w_1$ and $w_2$ respectively in $W/V$ and $\{v_1, \ldots, v_n\}$ is a basis for $V$.

**Lemma 4.3.** Let $W$ be an inner product space and let $V$ be a subspace of $W$. Let $\{e_1, \ldots, e_r\}$ be a basis of $V$ and $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_n\}$ be a basis of $W$ extending the basis of $V$. Suppose $\sigma_i = e_1 \wedge \ldots \wedge e_r \wedge e_i, \ r + 1 \leq i \leq n$

and
\[
A = \left(\langle e_i, e_j \rangle\right)_{1 \leq i, j \leq r}, \quad B = \left(\langle e_i, e_j \rangle\right)_{r+1 \leq i \leq n, 1 \leq j \leq r}, \\
C = \left(\langle e_i, e_j \rangle\right)_{1 \leq i \leq r, r+1 \leq j \leq n}, \quad D = \left(\langle e_i, e_j \rangle\right)_{r+1 \leq i, j \leq n},
\]
\[
A_\sigma = \left(\langle \sigma_i, \sigma_j \rangle\right)_{r+1 \leq i, j \leq n}.
\]

Then
\[
\det \left(\langle e_i, e_j \rangle\right)_{1 \leq i, j \leq n} = \det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \frac{\det(A_\sigma)}{(\det(A))^{n-r-1}}.
\]

**Proof.** Suppose $x_i = (\langle e_1, e_i \rangle, \ldots, \langle e_r, e_i \rangle)$ and $y_i = x_i^{tr}, r + 1 \leq i \leq n$.
\[
\langle \sigma_i, \sigma_j \rangle = \det \begin{pmatrix}
A_{x_j} & y_i \\
x_j & \langle e_i, e_j \rangle
\end{pmatrix} = \det(A) \langle \langle e_i, e_j \rangle - x_j A^{-1} y_i \rangle.
\]

Next, note that
\[
\det \left(\langle e_i, e_j \rangle\right)_{1 \leq i, j \leq n} = \det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det(A) \det(D - CA^{-1}B) = \det(A) \det \left(\langle \langle e_i, e_j \rangle - x_j A^{-1} y_i \rangle\right)_{r+1 \leq i, j \leq n} = \det(A) \det \left(\langle \langle \sigma_i, \sigma_j \rangle/\det(A)\rangle\right)_{r+1 \leq i, j \leq n} = \frac{\det(A_\sigma)}{(\det(A))^{n-r-1}}.
\]
We define \( F \) a metric for \( E \) let \( E \) have \( h \) Then the curvature \( F \) of \( E \) is equivalent to \( F \) and \( E/F \) respectively.

**Proof.** Let \( \{s_1, \ldots, s_r\} \) be a frame for \( F \) over an open subset \( U \) of \( \Omega \) and let \( \{s_1, \ldots, s_r, s_{r+1}, \ldots, s_n\} \) be a frame of \( E \) obtained by extending the frame of \( F \). The quotient \( E/F \) admits a frame of the form \( \{[s_{r+1}], \ldots, [s_n]\} \), where \( [s_i], r+1 \leq i \leq n \), denotes the equivalence class of \( s_i \) in \( E/F \). Let \( h_E = (\langle [s_j], [s_i]\rangle)_{i,j=1}^n \), \( h_F = (\langle [s_j], [s_i]\rangle)_{i,j=1}^r \) and \( h_{E/F} = (\langle [s_j], [s_i]\rangle)_{i,j=r+1}^n \) be the metrics of \( E, F \) and \( E/F \) respectively. Then by the definition of the determinant bundle \( h_{det}E = det h_E, h_{det}F = det h_F \) and \( h_{det}E/F = det h_{E/F} \). By Lemma 4.2 and Lemma 4.3 we have

\[
h_{det}E/F = \frac{det h_{det}F}{det h_{det}E} = \frac{det (([s_j],[s_i]))_{i,j=r+1}^n}{det (([s_j],[s_i]))_{i,j=1}^n} = \frac{det \left( \frac{(s_1 \wedge \ldots \wedge s_r \wedge s_j, s_1 \wedge \ldots \wedge s_r \wedge s_i)}{|s_1 \wedge \ldots \wedge s_r|^2} \right)_{i,j=r+1}^n}{(det h_F)^{n-r}} = \frac{h_{det}E}{h_{det}F}.
\]

**Corollary 4.5.** Let \( 0 \to F \to E \to E/F \to 0 \) be an exact sequence of Hermitian holomorphic vector bundles. Then

\[
\mathcal{K}_{det(E/F)} = \mathcal{K}_{det(E)} - \mathcal{K}_{det(F)}
\]

which is equivalent to

\[
\text{trace}(\mathcal{K}_{E/F}) = \text{trace}(\mathcal{K}_E) - \text{trace}(\mathcal{K}_F).
\]

Let \( E_f \) be a Hermitian holomorphic vector bundle of rank \( n \) over an open subset \( \Omega \) in \( \mathbb{C} \) and let \( E_f \in \mathfrak{S}_n(\Omega, \mathcal{H}) \). Let \( \{\sigma_1, \ldots, \sigma_n\} \) be a frame for \( E_f \) over an open subset \( \Omega_0 \) of \( \Omega \). Let \( h \) be a metric for \( E_f \) which is defined as

\[
h(z) = ((\langle \sigma_j(z), \sigma_i(z)\rangle))_{i,j=1}^n
\]

We define \( F_i^k \) for each \( 1 \leq k < \infty \) and \( 1 \leq i \leq n \) by

\[
F_i^k = \sigma_1 \wedge \ldots \wedge \sigma_n \wedge \ldots \wedge \frac{\partial^{k-1} \sigma_n}{\partial z^{k-1}} \wedge \frac{\partial^k \sigma_i}{\partial z^k},
\]

where wedge products between \( \sigma_i \)'s and their derivatives are taken in the Hilbert space \( \wedge \mathcal{H} \). Let \( h_{k} \) be the matrix

\[
h_k(z) = ((\langle F_j^k(z), F_i^k(z)\rangle))_{i,j=1}^n
\]

**Proposition 4.6.** Let \( E_f \) be a Hermitian holomorphic vector bundle of rank \( n \) over \( \Omega \subset \mathbb{C} \). Then the curvature \( \mathcal{K}_{E_f} \) of \( E_f \) is given by

\[
\mathcal{K}_{E_f}(z) = (\det h(z))^{-1} h(z)^{-1} h_1(z) \quad d\bar{z} \wedge dz.
\]
Proof. Set $x_i = \left( \frac{\partial}{\partial z} (\sigma_1, \sigma_i), \ldots, \frac{\partial}{\partial z} (\sigma_n, \sigma_i) \right)$ and $y_i = x_i^*\tau$, $1 \leq i \leq n$. For $1 \leq i, j \leq n$,

$$\langle F^1_j(z), F^1_i(z) \rangle = \det \left( \frac{\partial^2}{\partial z \partial \bar{z}} (\sigma_j(z), \sigma_i(z)) \right) = \det(h(z)) \left( \frac{\partial^2}{\partial z \partial \bar{z}} (\sigma_j(z), \sigma_i(z)) - x_i h(z)^{-1} y_j \right).$$

Now we can derive the formula for the curvature of the vector bundle $E_f$:

$$\mathcal{K}_{E_f}(z) = h^{-1}(z) \left\{ \bar{\partial} \partial h(z) - \partial h(z) h^{-1}(z) \partial h(z) \right\} = h^{-1}(z) \left( \frac{\partial^2}{\partial z \partial \bar{z}} (\sigma_j(z), \sigma_i(z)) - x_i h(z)^{-1} y_j \right)_{i,j=1}^n d\bar{z} \wedge dz = (\det h(z))^{-1} h^{-1}(z) h_1(z) d\bar{z} \wedge dz \quad \square$$

**Corollary 4.7.** Let $E_f$ be a vector bundle of rank $n$ over a bounded domain $\Omega \subset \mathbb{C}$. Then the curvature of the bundle $E_f$ is of rank $r$ if and only if exactly $r$ elements are independent from the set $\{ F^1_1, \ldots, F^1_n \}$ of $n$ elements.

**Proof.** By Lemma 4.6 the rank of the curvature of the bundle $E$ is same as the rank of $h_1$. But rank of $h_1$ is $r$ if and only if $r$ elements are independent from the set $\{ F^1_1, \ldots, F^1_n \}$ of $n$ elements. □

A result from [3, page 238, Lemma 4.12], which appeared to be mysterious, now follows from the formula derived for the rank of the curvature. Thus we have the following corollary:

**Corollary 4.8.** Let $E_f$ be a vector bundle of rank $n$ over a bounded domain $\Omega \subset \mathbb{C}$. Then the rank of the curvature $\mathcal{K}_{\mathcal{J}_k(E_f)}$ of the jet bundle $\mathcal{J}_k(E_f)$, $1 \leq k < \infty$, is at most $n$.

### 4.1 Curvature Formula in General

Let $E_f \xrightarrow{\pi} \Omega$ be a Hermitian holomorphic vector bundle of rank $n$. Let $\{s_1, \ldots, s_n\}$ be a local frame of $E_f$ over an open subset $\Omega_0$ of $\Omega$. Let $h$ be a metric for $E_f$ which is defined as

$$h(z) = \left( (s_i(z), s_j(z)) \right)_{i,j=1}^n.$$

For $1 \leq p \leq n$ and $1 \leq j \leq m$ set

$$\tau^j_p = s_1 \wedge \cdots \wedge s_n \wedge \frac{\partial s_p}{\partial z_j}.$$

For $1 \leq i, j \leq m$

$$h_{ij}(z) = \left( (\tau^i_p(z), \tau^j_q(z)) \right)_{p,q=1}^n.$$

**Proposition 4.9.** Let $E_f \xrightarrow{\pi} \Omega$ be a Hermitian holomorphic vector bundle of rank $n$ over a domain $\Omega \subset \mathbb{C}^m$. Then curvature $\mathcal{K}_{E_f}$ of the vector bundle $E_f$ is given by

$$\mathcal{K}_{E_f}(z) = (\det h(z))^{-1} h^{-1}(z) \sum_{i,j=1}^m h_{ij}(z) d\bar{z}_j \wedge dz_i.$$

**Proof.** Set $x_p^j = \left( \frac{\partial}{\partial z_j} (s_1, s_p), \ldots, \frac{\partial}{\partial z_j} (s_n, s_p) \right)$ and $y_p^i = x_p^{i*}$ for $1 \leq p \leq n$. 
For $1 \leq i, j \leq m$, 
\[
\frac{\partial^2 h}{\partial z_i \partial z_j}(z) - \frac{\partial h}{\partial z_j}(z)h^{-1}(z)\frac{\partial h}{\partial z_i}(z) = \left(\left(\frac{\partial^2}{\partial z_i \partial z_j}(s_q(z), s_p(z)) - x_j^i h(z)^{-1} y^j_q\right)_{p, q = 1}^n\right) = \left(\left(\frac{\partial h(z)}{\partial z_j}(z)\tau^j_q(z)\right)_{p, q = 1}^n\right).
\]
Hence the curvature of the vector bundle $E_f$ takes the form:
\[
\mathcal{K}_{E_f}(z) = h^{-1}(z)\sum_{i, j = 1}^m \left(\frac{\partial^2 h}{\partial z_i \partial z_j}(z) - \frac{\partial h}{\partial z_j}(z)h^{-1}(z)\frac{\partial h}{\partial z_i}(z)\right) dar{z}_j \wedge dz_i = (\det h(z))^{-1} h^{-1}(z) \sum_{i, j = 1}^m h_{ij}(z) d\bar{z}_j \wedge dz_i.
\]

4.2. Curvature of the Jet Bundle. Let $\mathcal{J}_k(E_f)$ be a jet bundle of rank $n(k+1)$ over $\Omega$, where $\Omega$ is a bounded domain in $\mathbb{C}$. If $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a frame for $E_f$ then a frame for $\mathcal{J}_k(E_f)$ is of the form
\[
\mathcal{J}_k(\sigma) = \{\sigma_1, \ldots, \sigma_n, \frac{\partial}{\partial z_1^i} \sigma_1, \ldots, \frac{\partial}{\partial z_n} \sigma_n, \ldots, \frac{\partial^k}{\partial z_1^i \partial z_n} \sigma_1, \ldots, \frac{\partial^k}{\partial z_n} \sigma_n\}.
\]
By Lemma 4.6 the curvature $\mathcal{K}_{\mathcal{J}_k(E_f)}$ of the bundle $\mathcal{J}_k(E_f)$ is given by
\[
\mathcal{K}_{\mathcal{J}_k(E_f)}(z) = (\det \mathcal{J}_k(h)(z))^{-1}(\mathcal{J}_k(h)(z))^{-1} \left(\begin{array}{cc}
0_{nk \times nk} & 0_{nk \times n} \\
0_{n \times nk} & h_{k+1}(z)
\end{array}\right) d\bar{z} \wedge dz
\]
Let $A = \mathcal{J}_{k-1}(h)$,
\[
C = \left(\frac{\partial^k}{\partial z_1^i \partial z_n}, \ldots, \frac{\partial^{k+1}}{\partial z_n} \right),
\]
\[
B = \bar{C}^\text{tr}, \quad D = \frac{\partial^2}{\partial z_1^i \partial z_n} h,
\]
\[
x_i = \left(\frac{\partial}{\partial z_1^i}(\sigma_1, \sigma_i), \ldots, \frac{\partial^k}{\partial z_1^i \partial z_n} \sigma_1, \ldots, \frac{\partial^{k+1}}{\partial z_n} \sigma_1, \ldots, \frac{\partial^k}{\partial z_n^i} \sigma_n, \frac{\partial^k}{\partial z_n^i} \sigma_i\right), \quad 1 \leq i \leq n,
\]
and finally $y_i = \bar{x}_i^\text{tr}, \quad 1 \leq i \leq n$.

Now
\[
D - CA^{-1}B = \frac{\partial^2}{\partial z_1^i \partial z_n} h - CA^{-1} B = \left(\left(\frac{\partial}{\partial z_1^i}(\sigma_j, \sigma_i) - x_i A^{-1} y_j\right)_{i, j = 1}^n\right) = \left(\left(\frac{\partial}{\partial z_1^i}(\sigma_j, \sigma_i) - x_i A^{-1} y_j\right)_{i, j = 1}^n\right)
\]
\[
= (\det \mathcal{J}_{k-1}(h))^{-1} h_k.
\]
Consequently,
\[
(\mathcal{J}_k h)^{-1} = \left(\begin{array}{cc}
A & B \\
C & D
\end{array}\right)^{-1}
\]
\[
= \left(\begin{array}{cc}
(A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1}
\end{array}\right)
\]
\[
= \left(\begin{array}{cc}
(A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-D^{-1}C(A - BD^{-1}C)^{-1} & \det(\mathcal{J}_{k-1}(h))^{-1}
\end{array}\right).
\]
The curvature of the jet bundle $\mathcal{J}_k(E_f)$ is

$$\mathcal{K}_{\mathcal{J}_k(E_f)}(z) = \begin{pmatrix} 0_{nk \times nk} & -(\det \mathcal{J}_k(h)(z))^{-1} A^{-1}(z) B(z) (D(z) - C(z) A^{-1}(z) B(z))^{-1} h_{k+1}(z) \\ 0_{n \times nk} & (\det \mathcal{J}_k(h)(z))^{-1} \det(\mathcal{J}_{k-1}h(z)) h^{-1}_k(z) h_{k+1}(z) \end{pmatrix}$$

Here

$$\det \mathcal{J}_k h(z) = (\det \mathcal{J}_{k-1} h(z))^{1-n} \det h_k(z)$$

and

$$(\det \mathcal{J}_k h(z))^{-1} \det \mathcal{J}_{k-1} h(z) = (\det h(z))^{n(1-n)^{k-1}} (\det h_1(z))^{n(1-n)^{k-2}} \cdots (\det h_{k-2}(z))^{n(1-n)} (\det h_{k-1}(z))^{n(1-n)} (\det h_k(z))^{-1}.$$ 

4.3. The Trace Formula. Let $\operatorname{trace} \otimes \operatorname{Id}_{n \times n} : \mathcal{M}_{mn}(\mathbb{C}) \cong \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C}) \to \mathbb{C} \otimes \mathcal{M}_n(\mathbb{C}) \cong \mathcal{M}_n(\mathbb{C})$ be the operator defined as follows

$$(\operatorname{trace} \otimes \operatorname{Id}_{n \times n})(\sum_{i,j=1}^m E_m(i,j) \otimes A_{i,j}) = \sum_{i=1}^m A_{i,i},$$

where $E_m(i,j)$ is the $m \times m$ matrix which is defined as follows

$$(E_m(i,j))_{k,l} = \begin{cases} 0 & \text{if } (k,l) \neq (i,j), \\ 1 & \text{if } (k,l) = (i,j). \end{cases}$$

(An arbitrary element $A$ in $\mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ is of the form $A = \sum_{i,j=1}^m E_m(i,j) \otimes A_{i,j}.$)

**Theorem 4.10.** Let $0 \to \mathcal{J}_{k-1}(E_f) \to \mathcal{J}_k(E_f) \to \mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f) \to 0$ be an exact sequence of jet bundles. Then we have

$$(\operatorname{trace} \otimes \operatorname{Id}_{n \times n})(\mathcal{K}_{\mathcal{J}_k(E_f)}) - (\operatorname{trace} \otimes \operatorname{Id}_{n \times n})(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)}) = \mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}(z).$$

**Proof.**

$$(\operatorname{trace} \otimes \operatorname{Id}_{n \times n})(\mathcal{K}_{\mathcal{J}_k(E_f)}) - (\operatorname{trace} \otimes \operatorname{Id}_{n \times n})(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)})$$

$$= (\det \mathcal{J}_k(h)(z))^{-1} \det(\mathcal{J}_{k-1}h(z)) h^{-1}_k(z) h_{k+1}(z)$$

$$- (\det \mathcal{J}_{k-1}(h)(z))^{-1} \det(\mathcal{J}_{k-2}h(z)) h^{-1}_{k-1}(z) h_k(z)$$

$$= \mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}(z).$$

The last equality follows from [1] page 244, Proposition 4.19].

**Acknowledgements:** Result of this paper contained in the thesis titled “Infinitely Divisible Metrics, Curvature Inequalities and Curvature Formulae” submitted at Indian Institute of Science, Bangalore. The author would like to thank Professor Gadaghar Misra and Dr. Cherian Varghese for their valuable suggestions and numerous stimulating discussions relating to topic of this paper.

**References**

[1] Raoul Bott and S. S. Chern, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math. 114 (1965), 71–112. MR 0185607 (32 #3070)

[2] Karen Chandler and Pit-Mann Wong, *Finsler geometry of holomorphic jet bundles*, A sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ., vol. 50, Cambridge Univ. Press, Cambridge, 2004, pp. 107–196. MR 2132659 (2006i:32022)

[3] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), no. 3–4, 187–261. MR MR501368 (80f:47012)
[4] ______, *Operators possessing an open set of eigenvalues*, Functions, series, operators, Vol. I, II (Budapest, 1980), Colloq. Math. Soc. János Bolyai, vol. 35, North-Holland, Amsterdam, 1983, pp. 323–341. MR MR751007 (85k:47033)

[5] S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3) 50 (1985), no. 1, 1–26. MR 765366 (86h:58038)

[6] F. R. Gantmacher, *The theory of matrices. Vol. 1*, AMS Chelsea Publishing, Providence, RI, 1998, Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation. MR 1657129 (99f:15001)

[7] S. Kumaresan, *A course in differential geometry and Lie groups*, Texts and Readings in Mathematics, vol. 22, Hindustan Book Agency, New Delhi, 2002. MR 1891361 (2003e:53001)

[8] A. Ramachandra Rao and P. Bhimasankaram, *Linear algebra*, second ed., Texts and Readings in Mathematics, vol. 19, Hindustan Book Agency, New Delhi, 2000. MR 1781860

[9] R. O. Wells, Jr., *Differential analysis on complex manifolds*, third ed., Graduate Texts in Mathematics, vol. 65, Springer, New York, 2008, With a new appendix by Oscar Garcia-Prada. MR MR2359489 (2008g:32001)

(Keshari) **DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX -77843, USA**

*E-mail address: kesharideepak@gmail.com*