Fields of a tightly focused radially polarized laser beam: the truncated series versus the complex-source-point spherical wave representation

Yousef I Salamin
Physics Department, American University of Sharjah, PO Box 26666, Sharjah, United Arab Emirates
E-mail: ysalamin@aus.edu

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Abstract. In this paper, the fields of a radially polarized fundamental Gaussian beam are derived with a complex-source-point spherical wave approach and then compared and contrasted with those derived recently from a Lax series to the order of $\epsilon^{15}$, where $\epsilon$ is the diffraction angle. It is shown that the domain of validity of the derived fields is restricted by a discontinuity built into the vector potential from which they are obtained. Furthermore, in disagreement with recent experimental observations, the axial field intensity decreases with tighter focusing (decreasing waist radius at focus).

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1. Introduction

It has recently been demonstrated experimentally that a radially polarized laser beam can be focused to a smaller waist radius than a linearly polarized one [1]–[3]. It has also transpired from these studies that the strength of the axial field component increases with focusing. The need to focus a laser beam to a sub-wavelength waist radius is essential for a number of applications such as particle trapping and acceleration [4]. Interest in radially polarized beams has witnessed some recent resurgence [5]–[10] stimulated by calculations in the field of electron acceleration [11, 12] and acceleration of atomic nuclei [13] for application in radiotherapy [14].

The aim of this paper is to examine the recent theoretical representations, suggested to describe a radially polarized laser beam, for scope and accuracy. While it is well known that the paraxial description is adequate for describing a beam focused to a waist radius \( w_0 \) larger than the wavelength \( \lambda \), focusing to \( w_0 < \lambda \) necessitates a description that should admit higher-order corrections. The representation of a Gaussian beam (GB) based on the complex-source-point spherical wave (CSPSW) description, introduced many years ago by Couture and Belanger [15], will be used here to represent the radially polarized beam and will be compared with the recently developed representation [7] based on the Lax series (LS) [16]–[18].

This paper is organized as follows. In section 2, the field components of the radially polarized beam will be derived in closed analytic form from the vector potential derived many years ago by Couture and Belanger [15]. Field intensities calculated from the expressions derived in this way will then be compared in section 3 and contrasted with their counterparts of fields derived from the generalized LS. A discussion of our results and concluding remarks will be given in section 4.

2. The accurate fields

A GB of wavelength \( \lambda \), frequency \( \omega \) (and wavenumber \( k = 2\pi/\lambda \)) and a stationary focus at the origin of a cylindrical coordinate system \((r, \theta, z)\), in which \( z \) is taken along the propagation axis of the beam, may be described in terms of three parameters, namely, the waist radius at focus \( w_0 \), the Rayleigh length \( z_r = kw_0^2/2 \) and the diffraction angle \( \varepsilon = w_0/z_r \). The field components may be derived from a vector potential \( A \) that satisfies the Helmholtz equation \( \nabla^2 A + k^2 A = 0 \), which, in turn, is derived from the Maxwell equations in vacuum, together with a Lorentz condition. An axial vector potential (polarized along the beam axis) leads to a satisfactory description of the radially polarized beam [19, 20]. Recall, on the other hand, that a transverse vector potential (in the \( E \)-field polarization direction) gives the fields of a linearly polarized GB [21]–[26].

2.1. The LS representation

Lowest-order corrections to the paraxial field components of a linearly polarized fundamental GB were found many years ago [16]–[18] as a series in powers of the diffraction angle \( \varepsilon \), with the proviso \( \varepsilon < 1 \). The same procedure has recently been used to obtain the field components of a radially polarized beam [5, 7]. With corrections to order \( \varepsilon^{15} \) incorporated, the field components take essentially the following forms:

\[
E_r = E_0 e^{-(r^2 - ikz)} \sum_{\ell=0}^{n} \varepsilon^{2\ell+1} E_{2\ell+1},
\]

\[
E_\theta = 0,
\]

\[
E_z = E_0 e^{-(r^2 - ikz)} \sum_{\ell=0}^{n} \varepsilon^{2\ell+1} E_{2\ell+1},
\]

\[
E_\phi = 0.
\]

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\[ E_z = -iE_0 e^{-f_0^2 - ikz} \sum_{\ell=0}^{n-1} e^{2\ell+2} E_{2\ell+2}, \]  
(2)

\[ B_{\theta} = \frac{E_0}{c} e^{-f_0^2 - ikz} \sum_{\ell=0}^{n} e^{2\ell+1} B_{2\ell+1}. \]  
(3)

In equations (1)–(3), the time dependence \( \exp(i\omega t) \) has been dropped altogether, \( f = i/(i + (z/z_r)) \), \( \rho = r/w_0 \), \( c \) is the speed of light in vacuum and \( E_0 \) is a constant amplitude. Explicit expressions for \( E_{2\ell+1}, E_{2\ell+2} \) and \( B_{2\ell+1} \), in other words all the corrections to order \( \varepsilon^7 \) \((n = 7)\), have recently been reported [7] and a procedure to produce the corrections to any arbitrary order has more recently been given [27]. An interesting result reported in [27] is the compact closed-form expression giving the total power output of the beam as a series in even powers of \( \varepsilon \).

It should be borne in mind that the LS is strictly valid as long as the condition \( \varepsilon < 1 \) is satisfied. This condition is equivalent to focusing the beam to a waist radius at focus \( w_0/\lambda > 1/\pi \). Thus, in all applications in which the LS fields have been used [5, 6, 14], care was taken to avoid situations in which the validity of the series was in doubt, based on this condition.

2.2. The CSPSW approach

The LS fields were arrived at by turning the Helmholtz equation satisfied by the axially polarized vector potential into a set of coupled differential equations. The expressions were required to reproduce the well-known paraxial fields in the appropriate limits. Furthermore, the propagation properties of a spherical wave emanating from the focal point played an important role in the derivation. Alternatively, the propagation characteristics of a CSPSW were shown many years ago [15] to produce a solution in closed analytic form to the Helmholtz equation for the vector potential. Such a solution may be utilized to describe the radially polarized fundamental GB, provided that the vector potential is axially polarized. Using the work of Couture and Belanger [15], the vector potential may be written as

\[ A = \hat{z} A_0 \frac{\exp(-ikR_c)}{R_c}; \quad R_c = \sqrt{r^2 + (z + iz_r)^2}, \]  
(4)

where \( A_0 \) is a constant amplitude and \( \hat{z} \) is a unit vector in the direction of propagation. The field components follow from the vector potential via the equations

\[ E = -\frac{i\omega}{k^2} \nabla (\nabla \cdot A) - i\omega A, \quad B = \nabla \times A. \]  
(5)

After some algebra, one obtains the following expressions from equations (4) and (5):

\[ E_r = -\frac{i\omega}{k^2} \frac{\partial^2 A}{\partial r \partial z} = -\frac{i\omega A_0}{k^2} \frac{(z + iz_r)r \exp(-ikR_c)}{R_c^3} \left[ \frac{k^2}{R_c^3} + \frac{3}{R_c^5} \right], \]  
(6)

\[ E_z = -\frac{i\omega}{k^2} \frac{\partial^2 A}{\partial z^2} - i\omega A \]

\[ = -\frac{i\omega A_0}{k^2} \frac{\exp(-ikR_c)}{R_c} + \frac{i\omega A_0}{k^2} \exp(-ikR_c) \left[ \frac{ik}{R_c^2} + \frac{1}{R_c^4} \right] \]

\[ -(z + iz_r)^2 \left[ \frac{k^2}{R_c^3} + \frac{3}{R_c^5} \right], \]  
(7)

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\[ B_0 = -\frac{\partial A}{\partial r} = A_0 r \exp(-ikR_c) \left[ \frac{ik}{R_c^2} + \frac{1}{R_c^3} \right]. \] (8)

These expressions are quite compact and appear to be deceptively simple. Unfortunately, they have their shortcomings. One major drawback stems from the fact that \( R_c \) vanishes at (what, for now, appear to be complex) points for which \( r = \pm i(z + i\zeta) \). This state of affairs causes the field components to diverge. The issue at hand will be discussed further below.

At this stage, we point out that equations (6)–(8) differ slightly from their counterparts in \[9\] due to the fact that \[9\] starts with a vector potential in which the exponential factor reads \( \exp(i k R_c) \) while we use the factor \( \exp(-i k R_c) \), originally developed by Couture and Belanger \[15\]. As a result, a few terms in equations (6)–(8) differ in sign, as well, from their counterparts in \[9\].

One may recover the expected paraxial approximation fields as follows. To begin with, for points near the beam axis, the following approximation is made:

\[ \exp(-ikR_c) \simeq \exp(kz_r) \exp(-ikz) \exp(-f\rho^2). \] (9)

Furthermore,

\[ \frac{1}{R_n^c} = \left( \frac{f}{iz_r} \right)^n \left[ 1 - (\varepsilon f \rho)^2 \right]^{-n/2}, \quad n = 1, 2, \ldots, 5. \] (10)

When only the leading term(s) in a Taylor series expansion of the square-bracketed factor in equation (10) are retained, the paraxial approximation fields take on the following forms:

\[ E_r \rightarrow \varepsilon E_0 \exp(-ikz)\rho f^2 \exp(-f\rho^2), \] (11)

\[ E_z \rightarrow -i\varepsilon^2 E_0 \exp(-ikz)(f - \rho^2 f^2) f \exp(-f\rho^2), \] (12)

\[ B_\theta \rightarrow \varepsilon \frac{E_0}{c} \exp(-ikz)\rho f^2 \exp(-f\rho^2). \] (13)

In order to recover the well-known paraxial fields precisely, the replacement \( \omega A_0 \rightarrow i\varepsilon E_0 z_r \exp(-kz_r) \) has been made. This substitution ought to be also made in equations (6)–(8) whenever they are employed in numerical calculations, as will be done shortly.

3. Calculations

In this section, the normalized electric field intensities calculated using the LS, to order \( \varepsilon^{15} \), and the CSPSW, equations (6)–(8), will be compared and contrasted. Normalization is effected by dividing \( E_r \) and \( E_z \) by the corresponding maximum paraxial field strengths \( E_{r0} = \varepsilon E_0/\sqrt{2\varepsilon} \) and \( \varepsilon^2 E_0, \) respectively, where \( \varepsilon = 2.7183 \) \[11\]. Only those field strengths in the transverse plane through the focus (the \( z = 0 \) plane) will be considered. In figure 1, normalized intensities are shown for the cases of focusing to \( \varepsilon = 0.1 \) and 0.4, considered in \[9\]. As shown in figures 1(a) and (b), the two representations give almost identical results. This should come as no surprise, because \( \varepsilon = 0.1 \) corresponds to a waist radius at focus \( w_0/\lambda = 10/\pi \sim 3.183 \), for which the paraxial fields (identical in both representations) suffice. However, the two representations begin to show discrepancies as the focusing becomes tighter. Figures 1(c) and (d) show the normalized...
Figure 1. Normalized field intensities in the plane $z = 0$ versus the radial coordinate $\rho = r/w_0$ for two values of the diffraction angle $\epsilon$. Discrepancies between the LS and CSPSW representations begin to show up clearly with tighter focusing.

Intensities for $\epsilon = 0.4$, which corresponds to a waist radius at focus $w_0/\lambda = 2.5/\pi \sim 0.796$, i.e. fourfold tighter than the cases in figures 1(a) and (b). Obviously, the two representations describe the intensities quite differently at this level of focusing.

Furthermore, note that figures 1(c) and (d) reveal that both CSPSW field intensities fall down with tighter focusing. This is to be contrasted with the trend followed by the corresponding LS intensities: the decrease in the radial field intensity is accompanied by an increase in the axial field intensity. To further elucidate this point, the normalized intensities are shown for each representation separately, but for four increasing values of the diffraction angle. The trends just described seem to be general. The decrease in the CSPSW axial field intensity with tighter focusing conflicts with recent experimental results [1]–[3]. This casts some doubt on the validity of the CSPSW representation.

Another source of doubt about the scope of the CSPSW representation is related to the issue, alluded to above, of divergence of the fields as $r \to \pm i(z + izr)$. For the cases considered thus far, in figures 1 and 2, divergence occurs as $r \to \pm iz$, or as $\rho \to \rho_c = 1/\epsilon$, thus making $\rho_c$ a discontinuity or cutoff value on the transverse coordinate, close to and beyond which the CSPSW representation is not valid. That is why in figures 1 and 2 the domain of $\rho$ is restricted to values between 0 and 2. Note that, for $0.1 \leq \epsilon \leq 0.4$, the cutoff is at values in the range $2.5 \leq \rho_c \leq 10$. These limitations become quite severe when it comes to calculating experimentally measurable quantities such as the total output power of the laser system.
Figure 2. Normalized field intensities in the plane \( z = 0 \) versus the radial coordinate \( \rho = r / w_0 \) for four values of the diffraction angle \( \varepsilon \). Evolutions of the field intensities of the LS and CSPSW representations with increasing diffraction angle are shown separately in this figure. Note that, with tighter focusing, the intensity of the axial LS field increases while that of the corresponding CSPSW field decreases.

The total power may be computed through integrating the (time-averaged) axial component of the Poynting vector over the entire \( z = 0 \) plane. In this situation, \( \rho_c \) serves as an upper limit on the integration. Contribution to the power integral from points in the plane outside the circle of radius \( r_c = w_0 \rho_c \) cannot, therefore, be calculated. One may, however, argue that this contribution should be negligible in view of the fact that the real fields diminish to zero below the cutoff, as may be inferred from figures 1 and 2.

The LS representation, on the other hand, does not suffer from this discontinuity/cutoff problem. The power, for example, may be calculated by integrating over the entire \( z = 0 \) plane and one may actually use the fields to calculate an analytic closed form expression for the power as a function of the diffraction angle \([5, 7, 8, 11]\).

Further support for the LS representation over the CSPSW representation may be gathered from the knowledge, supported by experiment \([1]–[3]\), that tighter focusing actually leads to more structure in the transverse intensity distributions. This feature is demonstrated clearly in figure 3 in which the intensities are shown for \( \varepsilon = 0.8 \), which is within the domain of validity of the LS representation. This diffraction angle, however, corresponds to a cutoff \( \rho_c = 1.25 \). The CSPSW fields present no information on the transverse normalized intensities beyond \( \rho_c \).
Figure 3. Normalized field intensities in the plane $z = 0$ versus the radial coordinate $\rho = r/w_0$ for a large diffraction angle $\varepsilon = 0.8$, which corresponds to a waist radius at focus $w_0 = 1.25/\pi \sim 0.398\lambda$. The CSPSW representation becomes entirely of no use as $\rho \rightarrow \rho_c = 1.25$, whereas the LS series continues to describe the fields reasonably accurately everywhere.

4. Discussion and conclusions

Fields of a tightly focused radially polarized fundamental GB have been derived based on a CPSPW approach. The normalized field intensity distributions in the transverse plane through the stationary beam focus have been compared and contrasted with those obtained from an LS representation. The following sums up our findings. Firstly, the validity of the CSPSW representation is limited to points within a certain transverse distance from the beam axis. For the $z = 0$ plane, the domain of validity extends merely to include points within a circle of radius $r_c = w_0/\varepsilon$. Secondly, the normalized axial field intensity decreases with tighter focusing, in sharp disagreement with well-established experimental observations [1]–[3]. Thirdly, due to the lack of information from the CSPSW field expressions about, for example, points on the $z = 0$ plane outside a circle of radius $r_c = w_0/\varepsilon$, those expressions cannot account for the observed structure in the normalized field intensities upon tighter focusing.

In conclusion, while the CSPSW approach leads to simple expressions for the fields in closed analytic form, these expressions are quite restricted in their domain of validity. The LS, on the other hand, is valid so long as focusing is such that the diffraction angle $\varepsilon < 1$, which corresponds to the condition on the waist radius given by $w_0 > \lambda/(\pi \varepsilon)$. Finally, note that real
analytic expressions can easily be extracted from the corresponding complex LS fields, whereas
the same thing is difficult to ascertain for CSPSW fields.

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