Generalization of channel estimation weights for multiple differential detection based on per-survivor processing

Nanase Yumoto¹, Toshiki Mori¹, Takuma Yamagishi¹, Kazuki Shimomura¹, and Hiroshi Kubo²a)

¹ Graduate School of Science and Engineering, Ritsumeikan University, 1–1–1 Noji-Higashi, Kusatsu-shi, Shiga 525–8577, Japan
² Faculty of Science and Engineering, Ritsumeikan University, 1–1–1 Noji-Higashi, Kusatsu-shi, Shiga 525–8577, Japan
a) kubohiro@fc.ritsumei.ac.jp

Abstract: This paper proposes novel channel estimation weights for multiple differential detection based on per-survivor processing (PSP-MDD). Channel estimation weights for PSP-MDD can control a trade-off between performance in the required signal-to-noise power ratio (SNR) and tracking capability. There have been proposed the two channel estimation weights: weights of simple first-order channel prediction and weights of higher-order channel prediction based on the binomial theorem. In order to control this trade-off widely and precisely, this paper proposes channel estimation weights generalizing the two conventional weights. Finally, computer simulation results confirm that the proposed weights can control the trade-off widely and precisely.

Keywords: differential encoding, multiple differential detection, per-survivor processing, channel prediction, fast time-varying channels

Classification: Transmission Systems and Transmission Equipment for Communications

References

[1] A. C. Singer, J. K. Nelson, and S. S. Kozat, “Signal processing for underwater acoustic communications,” IEEE Commun. Mag., vol. 47, pp. 90–96, Feb. 2009. DOI:10.1109/MCOM.2009.4752683
[2] S. Sheng, “Mobile television receivers: A free-to-air overview,” IEEE Commun. Mag., vol. 47, pp. 142–149, Oct. 2009. DOI:10.1109/MCOM.2009.5277468
[3] H. Kubo, A. Okazaki, K. Tanada, B. Penther, and K. Murakami, “A multibit differential detection based on channel prediction for fast time-varying fading,” IEICE Trans. Commun., vol. E88-B, no. 8, pp. 3393–3400, Aug. 2005. DOI:10.1093/ietcom/e88-b.8.3393
1 Introduction

Fast time-varying fading channels are serious issues for underwater acoustic communications (UWAC) and wide-area point-to-multi point (P-MP) communications [1, 2]. Multiple differential detection based on per-survivor processing (PSP-MDD) employing channel prediction is one of good approaches for fast-time varying channels [3]. PSP-MDD can control a trade-off between performance in the required signal-to-noise power ratio (SNR) and tracking capability according to its channel estimation weights. In the absence of channel state information (CSI), channel estimation weights are derived according to parameters of a maximum order of channel prediction $L$ and an observation symbol number of the received signals $N$. The following two channel estimation weights have been proposed in [3]:

1) channel estimation weights of simple first-order channel prediction for an arbitrary $N$;
2) channel estimation weights of higher order channel prediction employing the binominal theorem for an arbitrary $L$ and a fixed $N$ ($N = L + 1$).

Thus, the channel estimation weights have the following constraints:
- an arbitrary $N$ but $L = 1$;
- an arbitrary $L$ but $N = L + 1$.

Thus, these channel estimation weights cannot be derived for an arbitrary $L$ and an arbitrary $N$.

This paper proposes generalized channel estimation weights for an arbitrary $L$ and an arbitrary $N$ expanding the simple first-order channel prediction. Computer simulation results confirm that the proposed weights can widely and precisely control the trade-off between performance in the required SNR and tracking capability.

2 Communication system model

Fig. 1 shows a single-input multiple-output (SIMO) communication system model with $N_R$ receive antennas. In Fig. 1, the SIMO channel has the channel impulse responses (CIRs) at symbol time $k$ received at the $q$th ($q = 1, 2, \cdots, N_R$) receive antenna in the absence of intersymbol interference (ISI), $h_k[q]$. The transmitted information signal $b_k$ ($b_k \in \{0, 1, \cdots, 2^m - 1\}$, i.e., $m$ bps/Hz) is converted to the following modulated information signal $u_k$:

$$u_k = \exp\left(j\frac{2\pi}{M} b_k\right). \quad (1)$$

where the modulated information signal $u_k$ is assumed $M$-ary phase shift keying (PSK) signals ($M = 2^m$). The transmitted modulation signal $x_k$ is:

$$x_k = u_k x_{k-1}, \quad (2)$$

where $x_0 = 1$. The transmitted modulation signal $x_k$ is corrupted by additive white Gaussian noise (AWGN) at symbol time $k$ received at the $q$th receive antenna, $w_k[q]$, resulting in the following received signal at symbol time $k$ received at the $q$th receive antenna, $r_k[q]$: 

$$r_k[q] = h_k[q]x_k + w_k[q]. \quad (3)$$
A demodulator estimates the information signal, i.e., decisions $\hat{b}_k$, according to the received signals $r_k[q]$. 

3 Conventional channel estimation weights for PSP-MDD

3.1 Branch metric of PSP-MDD

PSP-MDD estimates information sequence based on the Viterbi algorithm (VA) [3]. This paper denotes a candidate of the transmitted information sequence as \{\hat{b}_k\}. In this paper, $\hat{a}_k$ denotes a candidate of signal $a_k$ corresponding to \{\hat{b}_k\}. The path metric $H$ and the branch metric $\Gamma_k$ of the VA are defined as follows:

\[
H = \sum_k \Gamma_k, \tag{4}
\]

\[
\Gamma_k = - \sum_{q=1}^{N_t} |r_k[q] - \hat{r}_k[q]|^2. \tag{5}
\]

PSP-MDD calculates the path metric $H$ for each \{\hat{b}_k\} and selects \{\hat{b}_k\} with the maximum $H$ as the estimated information sequence \{\hat{b}_k\}. An estimate of the received signal $\hat{r}_k[q]$ in Eq. (5) denotes a candidate of the received signal $r_k[q]$. In the absence of ISI, $\hat{r}_k[q]$ can be denoted as follows:

\[
\hat{r}_k[q] = \hat{h}_k[q] \bar{x}_k, \tag{6}
\]

where $\hat{h}_k[q]$ is a candidate of the estimated CIR and $\bar{x}_k$ is a candidate of the transmitted modulation signal. $\hat{h}_k[q]$ is written as the following equation:

\[
\hat{h}_k[q] = \sum_{n=1}^{N} v_n \tilde{G}_{k-n}[q], \tag{7}
\]

\[
\tilde{G}_{k}[q] = r_k \bar{x}^*_k, \tag{8}
\]

where $a^*$ denotes the complex conjugate of $a$, $\tilde{G}_{k-n}[q] \ (n = 1, 2, \cdots, N)$ is a reverse-modulated value, $N$ is an observation symbol number of the received signals, and $v_n$ is channel estimation weights for the reverse-modulated value $\tilde{G}_{k-n}[q]$. PSP-MDD predicts the CIR at time $k$ by the channel estimation weights $v_n$ and the reverse-modulated value $\tilde{G}_{k-n}[q]$ at time $(k - n)$ for the CIR estimation. For channel prediction, the reverse-modulated value from time $(k - N)$ to time $(k - 1)$ are approximated to $L$-th order function.
3.2 Channel estimation weights of simple first-order channel prediction

Ref. [3] has proposed the following channel estimation weights \( v_n \):

\[
v_n = \begin{cases} 
  N/(N-1) & n = 1 \\
  -1/(N-1) & n = N \\
  0 & \text{otherwise}.
\end{cases}
\]  

(9)

Let us denote that the conventional channel estimation weights of Eq. (9) are channel estimation weights of simple first-order channel prediction (SFP). For SFP, the order of channel prediction \( L \) is 1. SFP can improve performance in the required SNR for a larger \( N \). However, the larger \( N \) causes degradation of tracking capability on fast time-varying channels.

3.3 Channel estimation weights of higher order channel prediction employing the binominal theorem

Ref. [3] has also proposed the following channel estimation weights \( v_n \):

\[
v_n = (-1)^{n-1} \binom{N}{n} \quad (n = 1, 2, \cdots, N).
\]  

(10)

Let us denote that the conventional channel estimation weights of Eq. (10) are channel estimation weights of higher order channel prediction employing the binominal theorem (HBT). For HBT, the order of channel prediction \( L \) is equal to \((N-1)\). HBT can improve tracking capability for a larger \( L \). However, the larger \( L \) causes degradation of performance in the required SNR.

4 Generalized channel estimation weights for PSP-MDD

This section proposes generalized channel estimation weights for PSP-MDD. The generalized channel estimation weights are derived for an arbitrary \( L \) and an arbitrary \( N \). In this section, \([q]\) is omitted for its simplicity.

The conventional SFP estimates a value of first-order channel variation by calculating a difference of the reverse-modulated values at time \((k-N)\) and that at time \((k-1)\). Expanding the concept of SFP, this section proposes channel estimation weights which estimate a value of higher-order channel variation. Similar to SFP in Eq. (9), an first-order average channel variation \( \Delta G_k \) for unit time interval is calculated as the follows:

\[
\Delta G_k = \frac{1}{N-L} (G_{k-1} - G_{k-N+L-1}).
\]  

(11)

An estimated value of \(l\)-th order channel variation \( \Delta G^{(l)}_k \) can be recursively derived as follows:

\[
\Delta G^{(l)}_k = \Delta G^{(l-1)}_k - \Delta G^{(l-1)}_{k-1} \quad (l = 2, 3, \cdots, L),
\]  

(12)

where an estimated value of first order channel variation for unit time interval, \( \Delta G^{(1)}_k \), is:

\[
\Delta G^{(1)}_k = \Delta G_k.
\]  

(13)
The estimated CIR at time \( k \), \( \hat{h}_k \), is calculated by summation of \( G_{k-1} \) and overall \( \Delta G^{(l)}_k \)’s as follows:

\[
\hat{h}_k = G_{k-1} + \sum_{l=1}^{L} \Delta G^{(l)}_k. \tag{14}
\]

Although it is difficult to provide exact formulation of \( v_n \) for Eq. (14), these \( v_n \) can be obtained by the following relationship:

\[
\sum_{n=1}^{N} v_n G_{k-n} = G_{k-1} + \sum_{l=1}^{L} \Delta G^{(l)}_k. \tag{15}
\]

Let us denote that the proposed channel estimation weights \( v_n \) for Eq. (14) are generalized channel estimation weights. For \( L = 1 \), the channel estimation weights \( v_n \) of Eq. (9) is equivalent to that for Eq. (14). For \( L = N - 1 \), the channel estimation weights \( v_n \) of Eq. (10) is equivalent to that for Eq. (14). Therefore, the proposed weights can generalize the conventional SFP and HBT.

5 **Computer simulation results**

This section evaluates the BER performances of PSP-MDD employing the proposed generalized channel estimation weights. This paper assumes that a modulation scheme is BPSK, \( N_K \) is 1, and channels suffers from independent Rayleigh fading in the absence of ISI, where the maximum Doppler frequency normalized by symbol rate, \( f_D T \), of 0% corresponds to quasi-static fading channels. Fig. 2 shows BER performances of PSP-MDD with the proposed weights as a function of average \( E_b/N_0 \) on quasi-static fading channels, where BER of differential detection (DD) is also plotted for reference. On the other hand, Fig. 3 shows BER performances in the absence of noises as a function of \( f_D T \), where BER of DD is also plotted for reference.

From Figs. 2 and 3, we can obtain the following results:

- A larger \( L \) improves tracking capability, and a smaller \( L \) improves performance in the required SNR;

![Fig. 2. BER performance as a function of average \( E_b/N_0 \) on quasi-static fading channels (\( f_D T = 0\% \)).](image-url)
A smaller $N$ improves tracking capability, and a larger $N$ improves performance in the required SNR. Thus, the proposed weights can control the trade-off between performance in the required SNR and tracking capability more widely and more precisely than the conventional SFP and HBT.

6 Conclusion

This paper has proposed the channel estimation weights for PSP-MDD which generalize the two conventional channel estimation weights. The proposed weights are derived for an arbitrary order of channel prediction $L$ and an arbitrary observation symbol number of the received signals $N$. Computer simulation results have confirmed the proposed weights can control the trade-off between performance in the required SNR and tracking capability widely and precisely.

Acknowledgments

This work was partly supported by JSPS Grants-in-Aid for Scientific Research (KAKENHI) Grant Number 16K06374.
Dynamic channel properties based on diffuse scattering from vehicles for high frequency bands in NLOS urban environments

Minoru Inomata\textsuperscript{a)}, Tetsuro Imai, Koshiro Kitao, and Yukihiko Okumura

\textit{NTT DOCOMO, INC.},
3–6 Hikarino-oka, Yokosuka-shi, Kanagawa 239–8536, Japan
\texttt{a) minoru.inomata.gn@nttdocomo.com}

Abstract: We describe dynamic channel properties based on diffuse scattering from vehicles for high frequency bands in a non-line-of-sight (NLOS) urban environment. We investigate the dominant paths based on the measured power delay angular profile in experiments using a 20-GHz band channel sounder. Based on the results, we clarify that arrival waves from buildings and diffuse scattering from vehicles represent dominant paths in NLOS urban environments. Also, we find that the fading of scattering paths from vehicles is approximately 26 dB in those paths between birth and death.

Keywords: 5G mobile communication systems, high frequency bands, NLOS urban environments, dynamic channel property

Classification: Antennas and Propagation

References

[1] NTT DOCOMO, INC., “DOCOMO 5G white paper, 5G radio access: Requirements, concept and technologies,” July 2014.
[2] M. Inomata, T. Imai, K. Kitao, Y. Okumura, M. Sasaki, and Y. Takatori, “Radio propagation prediction for high frequency bands using hybrid method of ray-tracing and ER model with point cloud of urban environments,” 12th European Conference on Antennas and Propagation (EuCAP), London, UK, April 2018. DOI:10.1587/transcom.2017EBP3436
[3] 3GPP TR 38.901 V14.0.0, “Study on channel model for frequencies from 0.5 to 100 GHz (Release 14),” March 2017.
[4] White paper on “5G channel model for bands up to 100 GHz,” Global Communications Conference, 3rd Workshop on Mobile Communications in Higher Frequency Bands, Washington DC, U.S.A., May 2016 (http://www.5gworkshops.com/5GCM.html).
[5] ITU-R M.2412-0, “Guidelines for evaluation of radio interface technologies for IMT-2020,” Oct. 2017.
1 Introduction

The amount of traffic in wireless communication systems has been rapidly increasing in recent years [1]. One avenue of research being pursued to address this problem is applying frequency bands above 6 GHz to the next generation mobile communications systems (5G) [1]. These frequency bands include the millimeter wave bands, which use a wider frequency bandwidth and provide attractive higher data rates. Presently, it is assumed that the main service areas for 5G will be in line-of-sight (LOS) urban environments. However, in terms of deployment cost, it is also desirable to be able to cover non-line-of-sight (NLOS) environments. In addition, a massive multiple-input multiple-output (MIMO) technique using a large number of antenna elements is being investigated to compensate for the large propagation loss in high frequency bands, and the application of beamforming is being considered to obtain a higher antenna gain. Therefore, in order to evaluate these techniques, it is necessary to clarify dynamic channel properties based on the effect of moving objects around the mobile station in a NLOS urban environment.

Since the wavelength is shorter in high frequency bands, diffuse scattering due to various scattering objects such as buildings, roadside trees, road signs, and vehicles must be considered. There have been several studies on diffuse scattering [2, 3, 4, 5]. Reference [2] described the effect of diffuse scattering on channel properties in a LOS urban environment. Also, the shielding effect of vehicles and humans on channel properties is described in [3, 4, 5], and these studies report on a prediction method based on those effects. However, it is not clear how diffuse scattering from moving objects affects the dynamic channel properties in NLOS urban environments. Therefore, we investigate the dynamic channel properties based on diffuse scattering from moving objects in high frequency bands in a NLOS urban environment and clarify the dominant paths based on the measured power delay angular profile using a 20-GHz band channel sounder.

2 Measurement campaign

Fig. 1(a) is a photograph of the measurement area in Japan, around Tokyo station. Fig. 1(b) shows the measurement point. To analyze diffuse scattering in high frequency bands, we set the measurement frequencies to the 19.85 GHz band. The transmitter (Tx) antenna is an omni-directional antenna and the receiver (Rx) antenna is a planar array antenna that has 256 elements. An OFDM signal is transmitted from the Tx antenna, which is established on the roof of a vehicle positioned along the roadside. The Rx antenna is affixed to a pole mounted on the car. The Tx antenna height is 2.5 m and the Rx antenna height is 5 m. The Tx antenna and the Rx antenna is placed as shown in Fig. 1(b). The Tx signal power is 30 dBm and the bandwidth is 50 MHz. For the measurements, in order to obtain data from 360 degrees in the horizontal plane, the planar array antenna records measurements in 3 directions: 0, 90 and −90 degrees. For data processing, the angular delay profile is obtained using the IFFT and beamforming processing in three different Rx directions. The three angular delay profiles are concatenated based on the azimuth angle. The profile of the maximum power value of the 180-
degree elevation angle for each propagation delay is extracted. Then, the power delay profile is obtained by extracting the maximum power value of the azimuth angle for each propagation delay. The angular profile is obtained by extracting the maximum power value of the propagation delay for each azimuth angle. In order to obtain the angular profile, we use beamforming processing on the received signal. The recording duration per direction is 30 s and 15 snapshots of the power angular delay profiles are obtained.

3 Investigation of dynamic channel properties

In order to evaluate the dynamic channel properties based on diffuse scattering from the moving objects, we compared the 15 snapshots of the power angular delay profiles and analyzed the time fading characteristics. Fig. 2(a) shows the measurement power delay profile. In the figure, all 15 snapshots of the power delay profiles are represented. From Fig. 2(a), we confirm (1) 1-bounce reflection and then the diffraction at the building in the same lane as the Rx antenna, (2) the diffraction and then 1-bounce reflection at the building in the opposite lane, and (3) the 2-bounce reflections. The 2-bounce reflections represent a propagation path that is first reflected from the building opposite the Rx antenna, reflected from the building on the same side as the Rx antenna, and then arrives at the Rx antenna. We find that the time fading of paths (1), (2), and (3) is relatively stable; however, the fading of the paths at the propagation distance of 47 m is relatively greater. The fading of
paths (1), (2) and (3) is approximately 3 dB. On the other hand, the fading of the scattering paths between (1) and (2) is approximately 7 dB. Since these scattering paths are between the path (1) and (2), and the scattering paths fade in accordance with the time variation, we assume that these scattering paths occur from vehicles on the road. In order to analyze the arrival angle of the scattering paths from vehicles, we analyze the angular profiles. Fig. 2(b) shows the angular profiles we obtained. The angular profiles at the propagation distance of 47 m are extracted and 15 snapshots of the angular profiles are shown. From Fig. 2(b), we confirm that the scattering paths from vehicles arrive from the direction of −50 degrees. This indicates that based on the propagation distance of the scattering paths from the vehicles and the arrival angle, the scattering paths from the vehicles are first reflected from the buildings around the Tx antenna, scattered from the vehicle, reflected from the building on the same side as the Rx antenna, and arrive at the Rx antenna. In order to obtain the time fading characteristics, Figs. 2(c) and 2(d) show the power angular delay profiles at 2 different snapshots, 3 and 4, that we obtained over a 2 s measurement duration, respectively. At the propagation distance of 47 m and the arrival angle of −50 degrees, we find that the scattering paths from the vehicles are born and die between each snapshot. Fig. 2(e) shows the time fading of the received power at the propagation distance of 47 m and the arrival angle of −50 degrees. As a result, we find that the fading of the received power is approximately 26 dB when those paths are born and die. Fig. 3 shows the dominant paths in a NLOS urban environment. Based on these results, we confirm that (1) 1-bounce reflection and then the diffraction at the building in the same lane as the Rx antenna, (2) the diffraction and then 1-bounce reflection at the building in the opposite lane, (3) the 2-bounce reflections, and the scattering path from the vehicles on the road are dominant for high frequency bands in a NLOS urban environment.
(b) Angular profiles at propagation distance of 47 m

(c) Power angular delay profile at snapshot 3

(d) Power angular delay profile at snapshot 4
4 Conclusions

In this paper, we clarified the dynamic channel properties based on diffuse scattering from vehicles for high frequency bands in a NLOS urban environment. We investigated the dominant paths based on the measured power delay angular profile using a 20-GHz band channel sounder. We confirm based on these results that the arrival waves from the building on the same or opposite side of the Rx antenna and the scattering path from the vehicles on the road are dominant in a NLOS urban environment. We also found that the fading of the scattering paths is approximately 26 dB in those paths when they are born and die. In order to design a service area, constructing a prediction method considering scattering from vehicles on the road will be the subjects for our future work.

Acknowledgments

This paper includes a part of the results from “The research and development project for realization of the fifth-generation mobile communications system” commissioned by The Ministry of Internal Affairs and Communications, Japan.
Revisiting the sensitivity analysis of Google’s PageRank

Hirotada Honda
Faculty of Information Networking for Innovation and Design, Toyo University, 1–7–11 Akabanedai, Kita-ku, Tokyo 115–0053, Japan
a) honda.hirotada@iniad.org

Abstract: PageRank is widely known as a prominent method used for indexing web pages and has significantly contributed to the development of Google’s search engine. Significant contributions have been made to the analyses of PageRank, in particular, its sensitivity analyses. However, the arguments presented thus far are within the range of the matrix perturbation theory or the analyses of Markov chains. This paper explores the application of Kato’s perturbation theory to study the sensitivity analyses of PageRank, which enables the consideration of a larger number of general situations than those in established arguments.

Keywords: PageRank, perturbation, linear operator, spectrum

Classification: Fundamental Theories for Communications

References
[1] S. Brin and L. Page, “The anatomy of a large-scale hypertextual web search engine,” Seventh International World-Wide Web Conference (WWW 1998), Brisbane, Australia, 1998.
[2] A. N. Langville and C. D. Meyer, Google’s PageRank and Beyond: The Science of Search Engine Rankings, Princeton University Press, Princeton, 2006.
[3] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag Berlin Heiderberg, New York, 1980. DOI:10.1007/978-3-642-66282-9
[4] A. Bátkai, M. K. Fijavž, and A. Rhandi, Positive Operator Semigroups: From Finite to Infinite Dimensions, Birkhäuser Basel, 2017. DOI:10.1007/978-3-319-42813-0
[5] T. Kato, “Estimation of iterated matrices, with application to the von Neumann condition,” Numer. Math., vol. 2, pp. 22–29, 1960. DOI:10.1007/BF01386205
[6] R. Lempel and S. Moran, “Rank-stability and rank-similarity of link-based web ranking algorithms in authority-connected graphs,” Second Workshop on Algorithms and Models for the Web-Graph (WAW 2003), Budapest, Hungary, May 2003.
[7] A. N. Langville and C. D. Meyer, “Deeper inside PageRank,” Internet Math., vol. 1, pp. 335–380, 2004. DOI:10.1080/15427951.2004.10129091
[8] T. H. Haveliwala and S. D. Kamvar, “The second eigenvalue of the Google Matrix,” Technical Report 2003-20, Stanford University, 2003.
1 Introduction

The PageRank algorithm was originally proposed by authors Brin and Page [1]. Although the sensitivity analyses of the PageRank vector have been conducted thoroughly [2], we consider it worthy of further discussion, since more general situations are expected to appear in practical situations. Our contribution includes the following: (i) we consider more general situations than past arguments, (ii) we limit ourselves to the case of sufficiently small perturbation, but precisely discuss the dependency of the PageRank vector on the perturbation, and (iii) we consider the dependency of the power method on the perturbation. For instance, they obtained the estimate

\[ \|\pi(\chi) - \pi\|_1 \leq \frac{a}{1 - a} \sum_{i \in I_i} \pi_i \]

[2]. However, this estimate does not provide us with sufficient information regarding how the PageRank vector differs under the perturbation, nor how the perturbed vector is asymptotically denoted.

2 Terms and notations

2.1 Notations of norms

Hereafter, \( i = \sqrt{-1} \) is the imaginary unit. For a vector \( u = (u_1, u_2, \ldots, u_N)^T \) in general, we define the \( p \)-norm of the form as

\[ \|u\|_p \equiv \left( \sum_{j=1}^{N} |x_j|^p \right)^{\frac{1}{p}} \quad (p \in [1, \infty]), \]

and \( \|u\|_\infty \equiv \max_{1 \leq j \leq N} |u_j| \). For a squared matrix \( M = [m_{ij}] \in M_{N \times N} \) (hereafter, a set of \( N \) times \( N \) matrices is denoted as \( M_{N \times N} \)), we define

\[ \|M\|_\infty = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |m_{ij}|, \quad \|M\|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |m_{ij}|. \]

We also define the operator norm, denoted as \( \|M\| \equiv \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|} \) where the norm of the right-hand side is the usual norms of vector spaces: \( \|x\|_p \). Additionally, we denote the vector norm as \( \|\cdot\|_2 \) by \( \|\cdot\| \), and apply the corresponding operator norms. For a matrix \( M \), we use notations \( \sigma(M) \), \( \text{spr}(M) \) and \( \text{tr}(M) \) to denote the set of the eigenvalues of \( M \), the spectral radius, and the trace, respectively (See [3] for the definition of spectral radius). Hereafter, \( c \)'s with some indices stand for positive constants.

**Remark 2.1.** It is often useful to use \( \|\pi\|_1 = 1 \) for a PageRank vector \( \pi \), but Tikhonov’s theorem [4] states that 2-norm is equivalent to 1-norm.

2.2 Notations of Google matrices

The Google matrix \( G \) is defined by \( G = aW + \frac{(1-a)}{N} ev^T \), where \( e \) is a column vector whose elements are all 1, and \( v \) is a non-negative vector satisfying \( \|v\|_1 = 1 \). The constant \( a \in (0, 1) \), is called the damping factor, which represents the probability that an internet surfer navigates from one web page \( (i) \) to another \( (j) \).
The row stochastic matrix \( W \) is defined by \( W = H + \frac{a}{N} e^T \), where \( a = (a_j) \), which satisfies \( a_j = 1 \) if web page \( j \) is a dangling node and \( a_j = 0 \) otherwise [2]. The hyperlink matrix, \( H = [h_{ij}] \), is defined by \( h_{ij} = 1/|P_i| \) if there is a link from web page \( i \) to web page \( j \) and \( h_{ij} = 0 \) otherwise; it is a type of weighted adjacency matrix. The following lemma explains the foundation of the theory of the Google matrix.

**Lemma 2.1.** The Google matrix, \( G \), has a simple principal eigenvalue \( \lambda_1 = 1 \). In addition, the eigenvalues of \( G \) satisfy \( \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \ldots \). Additionally, there exists a non-negative left-eigenvector \( \pi \) that corresponds to the principal eigenvalue \( \lambda_1 = 1 \), which is known as the PageRank vector.

### 3 Formulation

#### 3.1 Formulation and settings

In this paper, we consider a situation in which a small perturbation is imposed on the Google matrix \( G \in M_{N \times N} \). The Google matrix under this perturbation is denoted as \( G(\chi) \), where \( \chi \in \mathbb{C} \). Likewise, \( W \) under the perturbation is denoted as \( W(\chi) \). Note that this perturbation is imposed so that \( G(\chi) \) and \( W(\chi) \) preserve their original characteristics. (both matrices are row stochastic, \( G(\chi) \) is irreducible and primitive.)

#### 3.2 Perturbation theory in finite dimensional space

We denote the resolvent of \( G(\chi) \) by \( R(\zeta, \chi) \equiv (G(\chi) - \zeta I)^{-1} \), where \( \zeta \in \mathbb{C} \), and \( I \) is the \( N \)-dimensional squared unit matrix. \( R(\zeta, \chi) \) is defined for all \( \zeta \) that are not equal to any eigenvalue of \( G(\chi) \). We know that \( R(\zeta, \chi) \) is holomorphic with respect to \( \zeta \) and \( \chi \) for such \( \zeta \) (see Theorem II.1.5 in [3]), which can be expanded as follows:

\[
R(\zeta, \chi) = R(\zeta)[I + W(\chi)R(\zeta)]^{-1} = R(\zeta) + \sum_{l=1}^{\infty} \chi^{l} R^{(l)}(\zeta). \tag{1}
\]

By integrating the infinite series (1) over a closed loop \( \Gamma_h \) that encloses a single eigenvalue \( \lambda_h \) of \( G \), we render the perturbation of the projection operator \( P_h(\chi) \):

\[
P_h(\chi) = -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta, \chi) \, d\zeta = P_h(\chi) + \sum_{l=1}^{\infty} \chi^l P_h^{(l)}(\chi).
\]

When \( \chi = 0 \), it is equal to the eigenprojection \( P_h(0) = P_h = -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta) \, d\zeta \), where \( R(\zeta) = (G - \zeta I)^{-1} \). We also use

\[
S_h = -\frac{1}{2\pi i} \int_{\Gamma_h} (\zeta - \lambda_h)^{-1} R(\zeta) \, d\zeta,
\]

which satisfies \( P_h S_h = S_h P_h = 0 \). Additionally, we have \( GP_h = \lambda_h P_h + D_h \), where \( D_h \) is referenced as the *eigennilpotent* operator of \( G \). For later use, for \( q \in \mathbb{N} \) we define:

\[
S_h^{(0)} = -P_h, \quad S_h^{(q)} = S_h^{(q)} (q > 0), \quad S_h^{(q)} = D_h^{(q)} (q < 0).
\]

For simplicity, we omit the subindex \( h \) hereafter.
4 Sensitivity of PageRank vector

First, we consider the case of the analytic perturbation:

\[
G(\chi) = a(\chi)W(\chi) + \frac{(1 - a(\chi))}{N} ev(\chi)^T
= G + \chi G^{(1)} + \chi^2 G^{(2)} + \ldots \equiv G + \tilde{G}(\chi).
\tag{2}
\]

**Theorem 4.1.** For \( \chi \in \mathbb{C} \) with sufficiently small \( |\chi| \), the PageRank vector \( \pi(\chi) \) under the perturbation of the form (2) is represented as

\[
\pi(\chi)^T = \pi^T - \pi^T \tilde{G}(\chi)S[I + \tilde{G}(\chi)S]^{-1},
\tag{3}
\]

where \( \tilde{G}(\chi) = G(\chi) - G \). We also have the estimate:

\[
\Vert \pi(\chi) - \pi \Vert \leq \frac{\Vert S \Vert \Phi_1(\chi)}{1 - \Phi_2(\chi)},
\]

where \( \Phi_1(\chi) \) and \( \Phi_2(\chi) \) are superior series of \( \tilde{G}(\chi)S \) and \( \tilde{G}(\chi)\pi \), respectively.

**Proof.** In virtue of Lemma 2.1, the principal eigenvalue of the Google matrix is \( \lambda_1 = 1 \) and simple. Thus, we have \( \pi(\chi)^T = ((\pi P(\chi))^T, \pi)^{-1} \pi^T P(\chi) \), where \((\cdot, \cdot)\) denotes the inner product of the vectors. Starting from \( \pi(\chi)^T (G(\chi) - I) = 0 \) and by using \((G - zI)S = I - P\), we have

\[
(\pi(\chi) - \pi)^T (G - I)S + \pi(\chi)^T \tilde{G}(\chi)S = 0.
\tag{4}
\]

By noting \( \pi(\chi) = (\pi(\chi) - \pi) + \pi \) in (4), we arrive at

\[
(\pi(\chi) - \pi)^T = -\pi^T \tilde{G}(\chi)S[I + \tilde{G}(\chi)S]^{-1}.
\]

This proves the first part. The latter part is verified by applying the method of the superior series.

We next consider a special case, \( G(\chi) = G + \chi G^{(1)} \).

**Theorem 4.2.** For sufficiently small \( |\chi| > 0 \), the PageRank vector under the perturbation above is represented as

\[
\pi(\chi)^T = \pi^T - \sum_{j=1}^{\infty} \pi^T (\chi a G^{(1)}) S^j.
\tag{5}
\]

We also have the estimate:

\[
\Vert \pi(\chi) - \pi \Vert \leq \frac{\chi a c_1 \Vert G^{(1)} \Vert}{(1 - a) - \chi a c_2 \Vert G^{(1)} \Vert}.
\]

**Proof.** Let us substitute \( \tilde{G} = \chi G^{(1)} \) into (3). By expanding the resultant equality with respect to \( \chi \) and estimating its norm, we have

\[
\Vert \pi(\chi) - \pi \Vert \leq \frac{\chi a \Vert G^{(1)} S \Vert \Vert \pi \Vert}{1 - \chi a \Vert G^{(1)} S \Vert}.
\]

From the definition of \( S \), we then have

\[
\Vert S \Vert \leq \frac{1}{2\pi} \int_{\Gamma_1} \frac{\Vert R(\zeta) \Vert}{|\zeta - 1|} |d\zeta|,
\tag{6}
\]

where \( \Gamma_1 \) is a closed loop that encloses \( \lambda_1 = 1 \) with a sufficiently small radius \( \delta > 0 \). (Since \( |\lambda_2| \leq \alpha \) [2], it suffices to take \( \delta = \frac{1-\alpha}{2\pi} \).) Thus, \( \zeta = 1 + \delta e^{i\theta} \) on \( \Gamma_1 \).
Note that the eigenvalues of $R(\zeta)$ are $(\frac{1}{\delta-1})_k$ [3]. For $\lambda_1 = 1$, the corresponding eigenvalue of $R(\zeta)$ is $\frac{1}{1-\zeta} = -\delta^{-1}e^{-i\theta}$ and simple, which is the only eigenvalue of $R(\zeta)$ located on its spectral radius. Therefore, we can apply the following inequality [5]:

$$\|R(\zeta)\| \leq c_{43}\text{spr}(R(\zeta)).$$

(note that $c_{43}$ is independent of $\zeta$.) Noting that [3] $\text{spr}(R(\zeta)) = \min_{k} \frac{1}{|\xi_{-k}|} = \frac{2}{1-\alpha}$, and estimating the right-hand side of (6) from above, we estimate

$$\|S\| \leq \frac{2c_{43}}{1-\alpha}.$$ Consequently, we acquire the desired result. \hfill \Box

**Remark 4.1.** If we assume that $H$ is normal, in addition to the assumptions in Theorem 4.2, then we have a sharper estimate: $H$ is normal, then, we have a shaper estimate:

$$\|S\| = \min_{\lambda \in \sigma(G) \neq 1} \frac{1}{|\lambda_1 - \lambda|} = \frac{1}{1-\alpha}.$$ If $H$ is normal, then so are $W$ and $G$. Therefore, we obtain the equality above. In addition, the constant $c_{43}$ is larger than 1 [5].

### 5 Sensitivity of power method convergence rate

Next, we estimate the sensitivity of the convergence rate of the power method. This is a concept similar to running-time stability [6]. The power method is a well-known approach used to find the PageRank vector $\pi$ numerically through iterative calculations. Here, we consider the analytic perturbation of the form of (2) again. It is understood that the convergence rate of the power method for $G$ is $|\frac{\lambda_1}{\lambda_2}|^t$ [1]. As we have mentioned, $\lambda_1 = 1$. If $W$ is reducible, $|\lambda_2| = \alpha$, otherwise, $|\lambda_2| < \alpha$ [7, 8]. Therefore, if the perturbation is imposed by preserving the reducibility of $W(\chi)$ for all $\chi$ under consideration, we state that the convergence rate under the perturbation is $\alpha(\chi)$. Otherwise, we need more careful discussions.

In the case that $\lambda_2$ forms a $\lambda$-group of cycle $p$ and $\chi = 0$ is an exceptional point, we have the following Puiseux series in general [3]:

$$\lambda_{2\alpha}(\chi) = \lambda_2 + a_1\alpha^h\chi^{1/p} + a_2\alpha^{2h}\chi^{2/p} + \ldots \quad (h = 0, 1, 2, \ldots, p - 1) \quad (7)$$

for $p \geq 2$, where $\omega = e^{\frac{2\pi i}{p}}$.

**Remark 5.1.** For the definitions of the exceptional point, $\lambda$-group and the cycle of the $\lambda$-group, see [3].

**Theorem 5.1.** Let $\lambda_2$ be a subdominant eigenvalue of $W$. Let us impose the perturbation of the form (2) on $G$, and let $\chi = 0$ be an exceptional point. We also assume that its perturbed value, $\lambda_2(\chi)$, forms a $\lambda$-group of cycle $p$, each of which takes the form of the Puiseux series (7). Then, the convergence rate of the power method under the smallness of the perturbation lies in the interval

$$\left(\Re(\lambda_2) - \sum_{l=1}^{\infty} |a_l| |\chi|^l, \Re(\lambda_2) + \sum_{l=1}^{\infty} |a_l| |\chi|^l\right).$$

where $a_l$ ($l = 1, 2, \ldots$) are provided in (7).
Theorem 5.1 is a direct consequence of the facts stated above.

**Corollary 5.1.** Let us assume the same assumptions as in Theorem 5.1, and let $\lambda_2 \in \sigma(W)$ be simple. Then, the convergence rate of the power method under the smallness of the perturbation lies in the interval

$$\left( |\lambda_2| - \sum_{l=1}^{\infty} |\lambda^{(l)}_l| |x|^l, |\lambda_2| + \sum_{l=1}^{\infty} |\lambda^{(l)}_l| |x|^l \right),$$

where $\lambda_l (l = 1, 2, \ldots)$ take the following form:

$$\lambda^{(l)} = \sum_{r=1}^{l} \frac{(-1)^r}{r} \sum_{r_1=1}^{r} \sum_{r_{1,2} \leq r_{1,2}} \sum_{k_{1,2}} \text{tr} \ G^{(r_1)} S^{(k_{1,2})} \ldots G^{(r_{1,2})} S^{(k_{1,2})}. \text{tr}$$

In this case, we have simpler representations for lower $l$’s; for instance,

$$\hat{\lambda}^{(1)} = \frac{1}{N} \text{tr} \ G^{(1)} P, \quad \hat{\lambda}^{(2)} = \frac{1}{N} \text{tr} \ [G^{(1)} P - G^{(1)} S G^{(1)} P].$$

**Corollary 5.2.** Let us assume the same assumptions as in Theorem 5.1, and let $\lambda_2 \in \sigma(W)$ be simple. The small perturbation $\chi$ is imposed on $G$ as the linear form: $G(\chi) = G + \chi G^{(1)}$. We then have the same form of the convergence rate of the power method as in Corollary 5.1, with a simpler form of $\lambda^{(l)}$:

$$\hat{\lambda}^{(l)} = \frac{1}{N} \text{tr} \ G^{(1)} P^{(l-1)}, \quad l = 1, 2, 3, \ldots$$

We limit ourselves to mentioning that if $\lambda_2$ is simple, the asymptotic representation of the weighted mean of the $\lambda$-group matches with that of $\lambda_2(\chi)$. In the case that the multiplicity of $\lambda$ is $m$ in general, the weighted mean of the $\lambda$-group is denoted as $\hat{\lambda}(\chi) = \frac{1}{m} \text{tr}(G(\chi) P(\chi))$. This yields the Corollaries 5.1 and 5.2.

**6 Conclusion**

In this paper, we discussed the PageRank vector under the small perturbation in the presence of the general form of the perturbation on the Google matrix. We also discussed the effect of the perturbation on the convergence rate of the power method.