Minimum uncertainty states for free particles with quantized mass-energy

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Quantum particles with dynamical internal energy are promising for first direct tests of quantum theory and general relativity and, in particular, for quantum effects of time dilation. However, all studies of the free propagation of these particles found they are rendered unsuitable for experiments by spatial delocalization of their internal energy components. Here we derive a new class of states for particles with dynamical internal energy which completely overcome the delocalization problem, follow semi-classical trajectories, and can be prepared in harmonic traps. Our results will further be relevant to future tests with microparticles in a quantum regime—where the delocalisation effects due to thermal internal energies could become detrimental.

Continuous progress in capabilities of quantum technologies allows for quantum experiments with increasingly complex systems and over increasingly large distance scales—quantum interference has been observed with molecules comprising 2000 atoms [1], coherence of spatial superpositions has been verified over tens of centimetres [2]. This progress paves the path towards testing joint quantum and general relativistic phenomena thus far inaccessible to direct experimental verification. In particular, tests of time dilation effects on quantum coherence of composite particles (e.g. atoms or molecules) [3–7] are becoming feasible with Very Long Baseline Interferometry [8].

The theory of relativistic composite particles has been extensively explored, motivated by the particles’ experimental utility for the above mentioned tests of time dilation, and their appeal as a model of relativistic quantum clocks. This exploration has ventured into foundational and applied topics such as: The relevance of time dilation for the quantum-to-classical transition [6, 9, 10], the notion of passive gravitational mass for these systems [11, 12], and practical and fundamental limitations to time-keeping of relativistic quantum clocks [13–18]. It was also shown that testing the equivalence principle for composite quantum systems requires more, and different, experiments compared to the classical case [19–22]. Finally, the first experiments were performed testing the equivalence principle [23], or testing analogue implementations of time dilation in matter-wave interference experiments [24].

For the upcoming experiments with composite quantum particles it is crucial to understand how these particles propagate. Surprisingly, all studies which looked at this question have found that the position degree of freedom (DOF) of a low-energy composite particle delocalizes into separate internal energy components travelling at different speeds [21, 25, 26]. If that was an unavoidable behaviour of these particles their suitability as a theoretical model of a quantum clock, as well as their usefulness for experiments, would be drastically limited.

Here we show that the previously reported delocalization of composite particles’ wave-packets can be fully avoided. The problem at hand is: what are the states of composite particles whose trajectories remain as localised as quantum mechanics allows? Since the notion of a trajectory is defined in configuration space, we first formulate an uncertainty inequality for position and velocity of a composite particle and then derive a new class of states that minimize it. We show that these new states indeed follow semi-classical trajectories, discuss how to prepare them, and compare our findings to previous studies.

In prior works reporting the delocalization, the states of the particles were taken to be Gaussian wave-packets. These states minimise the Heisenberg Uncertainty Principle (HUP), and in the non-relativistic limit indeed represent semi-classical states of free particles. Yet, by considering states minimising the HUP, one is looking at paths in phase-space rather than in configuration space, while for relativistic composite particles, even at low energies, these notions are not the same. The regime where time dilation affects the internal dynamics of a particle is fully equivalent to the regime where mass-energy equivalence applies, and internal energy contributes to particle’s mass [27–32]. In such a case the relation between momentum and velocity is no longer trivial—failure to incorporate it can lead to inconsistent results [33, 34]. In the context of this work the relativistic mass-energy equivalence means that the internal mass-energy has its own spectrum and uncertainty, and therefore uncertainty in momentum (in phase space) and uncertainty in velocity (in configuration space) are also non-trivially related.

**Low-energy composite particles**—A composite particle can be described in a tensor product Hilbert space $\mathcal{H} = \mathcal{H}_{int} \otimes \mathcal{H}_{ext}$, where $\mathcal{H}_{int}$ is the Hilbert space describing the states of the internal DOFs and $\mathcal{H}_{ext}$ those of the external ones (i.e. the centre of mass states). In the low-energy regime the Hamiltonian of a composite particle in the homogeneous gravitational field $g$ reads [12, 19, 21, 35] (see also Appendix A)

$$\hat{H} = \hat{M}c^2 + \frac{\hat{p}^2}{2\hat{M}} + \hat{M}gx,$$

(1)

where $\hat{M} = m_0\hat{1} + H_{int}/c^2$, with $m_0$ the ground state of the mass-energy (its ‘rest mass’ parameter), $H_{int}$ describing the energy levels of the internal states, and $c$ the speed of light.
The velocity operator, \( \dot{x} \), is defined
\[
\dot{x} = -i\hbar [\hat{x}, \hat{H}] \equiv \hat{M}^{-1}\hat{p}.
\]

**Position and velocity uncertainty and its minimizing states**—The Heisenberg uncertainty principle (HUP) places an upper bound on the precision to which we can know both the position and momentum of a particle. States which have this minimum uncertainty follow semi-classical trajectories in phase space, and are the well-known coherent and squeezed states [36].

We now derive the states which minimise the uncertainty in position and velocity, which will have a semi-classical trajectory in configuration space.

The minimum uncertainty states (MUSs) for two arbitrary quantum observables \( \hat{X} \) and \( \hat{Y} \) are the generalized intelligent states which minimise the Schrödinger-Robertson uncertainty inequality [37]. This inequality is a ‘stronger’ formulation of the more familiar Heisenberg-Robertson uncertainty inequality carrying additional covariant terms [38],
\[
(\Delta X)^2(\Delta Y)^2 - (\Delta XY)^2 \geq \frac{1}{4} \left| \left\langle \left[ \hat{X}, \hat{Y} \right] \right\rangle \right|^2. \tag{2}
\]

Its minimum uncertainty (intelligent) states are the solutions to the eigenvalue equation [39]
\[
(u\hat{A} + v\hat{A}^\dagger) |\Psi\rangle = z |\Psi\rangle, \tag{3}
\]
where \( z, u, c \in \mathbb{C} \) and \( |u|^2 - |v|^2 = 1 \), \( \hat{A} = \hat{X} + i\hat{Y} \) and \( \hat{A}^\dagger = \hat{X} - i\hat{Y}. \)

As we are interested in MUSs for position and velocity, we define the operators in Eq. (3) as
\[
\hat{A} = \left( \hat{x} + i\frac{\hat{p}}{\Omega} \right) ; \quad \hat{A}^\dagger = \left( \hat{x} - i\frac{\hat{p}}{\Omega} \right) \tag{4}
\]
with \( \Omega \) introduced to match the units, and set to \( \Omega = 1 \) for the remainder of the paper.

The commutator on the right hand side of Eq. (2) is
\[
\left[ \hat{x}, \hat{x} \right] = \frac{\hbar}{\Omega} \text{ and, } \left[ \hat{A}, \hat{A}^\dagger \right] = -2i\left[ \hat{x}, \hat{p} \right] = \frac{2\hbar}{M},
\]
It is then convenient to define operators \( \hat{a}_M := \sqrt{\frac{M}{2\hbar}} \hat{A} \)
and \( \hat{a}^\dagger_M := \sqrt{\frac{M}{2\hbar}} \hat{A}^\dagger \) such that,
\[
\left[ \hat{a}_M, \hat{a}^\dagger_M \right] = \frac{\hbar}{2M} \left[ \hat{A}, \hat{A}^\dagger \right] = \hat{1}. \tag{5}
\]
This leads to the eigenvalue equation for the position and velocity case:
\[
(u\hat{a}_M + v\hat{a}^\dagger_M) |\Psi\rangle = z_M |\Psi\rangle, \tag{6}
\]
where \( z_M := z\sqrt{\frac{M}{2\hbar}}. \)

As \( \hat{M} = \sum_m m \hat{\Pi}_m \), where \( \{\hat{\Pi}_m\}_m \) is a set of orthonormal projectors, we recast
\[
\hat{a}_M = \sum_m \left( \hat{a}_m \otimes \hat{\Pi}_m \right), \tag{7}
\]
where \( \hat{a}_m = \hat{A}\sqrt{\frac{M}{2\hbar}} \) and, similarly, \( z_m = z\sqrt{\frac{M}{2\hbar}}. \)

Additionally, we can represent \( |\Psi\rangle = \sum_m c_m |\psi_m\rangle |m\rangle \), so Eq. (6) takes a more telling form:
\[
\sum_m (u\hat{a}_m + v\hat{a}^\dagger_m) c_m |\psi_m\rangle |m\rangle = \sum_m z_m c_m |\psi_m\rangle |m\rangle, \tag{8}
\]
where the full MUS is made up of superposed states each with its own associated eigenvalue equation:
\[
(u\hat{a}_m + v\hat{a}^\dagger_m) |\psi_m\rangle |m\rangle = z_m |\psi_m\rangle |m\rangle. \tag{9}
\]
(Recall that \( |m\rangle \) are eigenstates of the mass-energy of the particle.)

Since operators \( \hat{a}_m \) satisfy the canonical commutation relations, Eq. (5), each \( |\psi_m\rangle \) is a squeezed Gaussian state with a squeezing parameter \( \alpha_m = z_m \) [36, 40] (see also Appendix B). Since each state has a different but fixed value of the mass-energy \( m \), the joint state \( |\Psi\rangle \) exhibits a small amount of entanglement between the internal and the centre-of-mass DOFs.

In the position representation the eigenstates of Eq. (8) take the form of normalised wave functions:
\[
|\psi_m(x)\rangle = \frac{1}{\sqrt{N_m}} e^{\frac{1}{2\sigma^2} \left[ -\frac{\hbar}{2\sigma^2} (x - \bar{x})^2 + i\frac{\hbar^2}{2\sigma^2 \Omega^2} (x - \bar{x}) \right]}, \tag{10}
\]
with \( \mathfrak{R}[\cdot] \) denotes the imaginary part of a complex number, \( \alpha := (u + v), \beta := (u - v) \), and the normalisation factor \( N_m \) can be found in Appendix B.

Let us compare these new states to a regular Gaussian, such as would minimise the HUP,
\[
|\psi_G(x)\rangle = \frac{1}{(\pi \sigma^2)^{\frac{1}{4}}} e^{\left[ -\frac{x^2}{2\sigma^2} + \frac{\hbar^2}{4\pi^2 \sigma^4} \right]}, \tag{11}
\]
which has the peak momentum \( p \), and the position variance, \( \sigma \). For our MUS state, each mass-energy component, Eq. (10), is also a Gaussian wave packet, however the whole of the exponential term is dependent on the mass. This means that each \( |\psi_m\rangle \) has a mass-dependent peak momentum and variance.

**Particle propagation**—To obtain the propagated states, we use the path integral formalism. The general form of a propagator is an integral over all possible trajectories for a given time interval [41], and for a composite particle at low-energies it takes the form
\[
K^{m}(x_f, t_f; x_i, t_i) = N_m e^{-i \Delta x^2 / (2\alpha^2 \Delta t^2) + i \Delta t (x_f + x_i) + \frac{\hbar^2}{2\sigma^2 \Omega^2} (x_f - x_i)^2}, \tag{12}
\]
and where \( \Delta x := x_f - x_i, \Delta t := t_f - t_i \), and \( N_m = \sqrt{\frac{M}{2\hbar \Delta t}} \) (see Appendix C for derivation).
The propagator is applied by convolving it with an initial wave function \( \psi(x_i, t_i) \) to yield the final state

\[
\Psi(x_f, t_f) = \int dx_i \ K(x_f, t_f; x_i, t_i) \Psi(x_i, t_i).
\]  

We first propagate a superposition of mass-energies for the centre of mass in a generic Gaussian state Eq. (11),

\[
\psi_G(x) \left( \sum_{i=1}^{N} \alpha_i |m_i\rangle \right),
\]

where \( \sum_i |\alpha_i|^2 = 1 \). The individual propagated mass states for \( i = 1, 2, 3 \) and \( \alpha_i \equiv 1/\sqrt{3} \) are shown in Figure 1 top-panel; the analytical form of the wave function is given in Appendix C. As expected from Eq. (11) the centres of the mass component wave functions shift in time as \( x_i = pt/m_i \), i.e. each travel with a different velocity \( p/m_i \) as they all have the same initial momentum \( p \) but different mass-energy. This is the delocalization effect found in prior studies [21, 25, 26]. Furthermore, the squared position variance of the Gaussian state Eq. (11),

\[
\Delta x^2_G(t) = \sum_{i=1}^{N} |\alpha_i|^2 \Delta x^2(t; m_i, 0)
\]

where \( \Delta x^2(t; m, 0) \equiv \Delta x^2(m, 0) \) and \( \alpha_i \equiv 1/\sqrt{3} \); the analytical form of the state is again given in Appendix C. The position variance of each mass-energy component of our MUS evolves as \( \sigma^2(m_i, t) = \sigma^2(m_i, 0) (1 + e^{-4t^2}) \), where \( \sigma^2(m, 0) \propto 1/m_i \) (cf. Eq. 10) and \( \cosh[\beta] \equiv u \).

Thus the position variance of the entire state reads

\[
\Delta x^2_f(t) = \sum_{i=1}^{N} \alpha_i \Delta x^2_G(t; m_i, 0)
\]

with equality holding for the case \( \alpha_j = 1 \) for some \( j \in 1, ..., N \), and \( \alpha_{i \neq j} = 0 \). This shows that our MUSs are indeed more localised than the regular Gaussian states, even when the mass-dependent propagation velocity does not play a role (initial momentum \( p = 0 \)).

What is the extent of the delocalization between the propagating mass-energy components in the Gaussian state that is avoided by our minimum uncertainty state? Denoting the ground state mass-energy \( m_g \), and its velocity \( v_g = p_0/m_g \), and some higher mass-energy \( m_e = m_g + \Delta E \), with velocity \( v_e = p_0/m_e \), the difference in the velocities up to order \( 1/c^2 \) is \( v_g - v_e \approx v_g \Delta E/m_g^2 \). Using a Strontium atom as an example, due to its stable excited state with \( \Delta E = 10^{15} \text{ Hz} \) and a lifetime of \( \approx 100 \text{ s} \) [42], we will have \( m_g \approx 10^{-25} \text{ kg} \), and the laboratory source will determine the initial centre of mass velocity of the atoms. If \( v_g \) is the most probable velocity corresponding to \( T = 800 \text{ K} \) [43], we find \( v_g - v_e \approx 10^{-3} \text{ m/s} \). This means that in a Gaussian state after around \( 10^{-3} \text{ s} \), the peak separation of the internal mass-energy states will become comparable with the atom’s de Broglie wavelength, which is here around \( 10^{-12} \text{ m} \), thus suppressing longitudinal coherence [44].

Analogous estimations can be made for a molecule. The variance in the molecule’s velocity arising due to a thermal distribution of its internal mass-energies in a high temperature \( T \) limit and up to order \( 1/c^2 \) is \( \Delta v \approx \sqrt{3N - 6} \cdot v_g \frac{k_B T}{m_N} \) where \( N \) is the number of atoms and \( k_B \) is the Boltzmann constant. Taking as an example data from ref. [45]: \( N = 810, m = 1.7 \cdot 10^{-23} \text{ kg}, T = 600 \text{ K}, \) de Broglie wavelength of the molecule \( \lambda_{DB} = 5 \cdot 10^{-13} \text{ m} \) and its size \( 10^4 \lambda_{DB} \), we find that the delocalization of the molecule \( \Delta v \cdot t \) would be of the order \( \lambda_{DB} \) after \( t = 0.02 \text{ s} \) and would be as large as the size of the molecule after \( t = 3.3 \text{ minutes} \).
**Wigner functions in phase and configuration space**—Our position-and-velocity MUSs do not only stay more localised in position, but also have a more defined space-time trajectory—i.e., path in configuration space—than the Gaussian states. We illustrate this using the Wigner quasi-probability distributions in phase space and in configuration space.

The general form of the phase-space Wigner function for a mass-energy superposition state (see Appendix D) is a sum of weighted Wigner functions for each mass-energy component: \( W(x, p) = \sum_j |\alpha_j|^2 W^{(j)}(x, p) \).

In Appendix D we derive a configuration space Wigner function, which takes the form: \( \tilde{W}(x, v) = \sum_j |\alpha_j|^2 W^{(j)}(x, m_j v) \). A similar function was used in ref. [21] in the context of the Weak Equivalence Principle for quantum particles.

Figure 2 shows results for time-evolved states from Eqs. (14) and (16). In configuration space, our MUS exhibits no separation of the mass states in either position or velocity, while the generic Gaussian state spreads out in both parameters. This demonstrates that our MUSs indeed follow a semi-classical space-time trajectory, in contrast to generic Gaussian states whose trajectory delocalizes. In phase space, a generic Gaussian shows a spread in position, as observed in Figure 1, whereas our MUS remains localized in position and exhibits correlations between the individual mass-energy components and peak wavepacket momenta.

**Discussion**—Previous studies [25, 26] looked at how particles in superpositions of internal mass-energy states interfere in double-slit type experiments. Since the initial centre-of-mass state were taken to be Gaussian, due to the difference in propagation velocities, it was found that in the presence of gravity, the interference pattern is suppressed. That is, due to the different travel times of the internal mass-energy states towards the screen, each interfere at different point on the screen when the particle is in free fall, thus producing a mixture of interference fringes.

Our result shows that delocalization can be avoided by taking instead of a Gaussian wave packet, our position-and-velocity MUS. The interference pattern obtained in our case thus significantly differs from the case of refs [25, 26]—as the mass-components arrive at the screen at the same point and time—but it also differs from the fully non-relativistic case where the internal states have the same mass and only differ in their rest energy; see Appendix E. The difference can be directly interpreted from the propagator: The internal state evolves for a different proper time depending on the path taken towards the screen. This difference, encoded in the internal state, affects the spatial interference as expected from the complementarity between interference and which-path information. In this sense, our minimum uncertainty states facilitate a double-slit version of time-dilation induced decoherence found in ref [6].

One could think that difficulties would be encountered in the preparation of the MUSs derived in this work. However, in typical scenarios where one would think a Gaussian state would be prepared, our MUS would in fact arise when the dynamical nature of the mass-energy is taken into account. Specifically, the ground state of a harmonic potential for a massive particle is a Gaussian with squared width \( \sigma^2 \propto 1/m \), e.g. [46]. Thus a particle in a superposition of internal mass-energies, cooled down to the motional ground state of a harmonic trap that has a fixed frequency would be prepared exactly in our MUS state Eq. (10) [18], with initial velocity given by the velocity of the trap in the laboratory reference frame.

**Conclusion**—By including the mass-energy equivalence in the dynamics of low energy particles we have found a new class of quantum states minimising position-velocity uncertainty. This allowed us to show that in contrast to previous studies, particles in internal energy superpositions can propagate with no delocalization of the component mass-energy states.

Here derived MUS can be beneficial for experiments testing interference of complex molecules [1, 47], nano-
and microparticles [48–50], and particles in internal energy superpositions (quantum ‘clocks’) [3, 6, 8]—in which the delocalization effect which the MUS overcame would become detrimental. Our findings are also relevant to theoretical studies of the equivalence principle in quantum mechanics [19, 21, 51], and may further find applications in the study of neutrino propagation.

**Appendix A: Low-energy Hamiltonian of a composite particle**

Recall first that the square of the relativistic four momentum \( p^\mu, \mu = 0, \ldots, 3 \) is a relativistic invariant. It describes the energy of a particle in its rest frame [52] \( H_{\text{rest}}c^2 = -\sum p^\mu g_{\mu\nu}p^\nu \), where \( g_{\mu\nu} \) is a spacetime metric with signature \((-+, +, +, +)\), and \( c \) is the speed of light. In an arbitrary reference frame, the energy is \( H \equiv cp_0 \). Assuming a static symmetric metric we obtain

\[
H = \sqrt{-g_{00}(c^2 p_0^2 + H_{\text{rest}}c^4)}, \tag{A1}
\]

where \( p_0^2 \equiv \sum_{i,j=1,2,3} p_i^j g_{ij}p^j \). For a derivation of this dispersion relation from quantum field theory (as energy in a one-particle subspace) see [19, 21, 35, 53], for a derivation in a small-size limit of a bound system of \( N \) relativistic particles see [54].

At low energies the relativistic Hamiltonian in Eq. (A1) reduces to \( H_{\text{rest}} + p^2/2H_{\text{rest}}c^2 + H_{\text{rest}}\phi(x)/c^2 \), with \( \phi(x) \) denoting the gravitational potential. For a structureless particle \( H_{\text{rest}} \equiv mc^2 \), where \( m \) is the rest mass parameter. For a particle with internal DOFs, the rest energy comprises not only the masses of all the constituents but also the internal energies, as dictated by the relativistic mass-energy equivalence. For an atom or a molecule these include electronic and vibrational energies. We can thus write \( H_{\text{rest}} = M_0c^2 + H_{\text{int}} \), where \( M_0 \) is the mass-energy of the system when the internal DOFs are in a ground state of rest energy; \( M_0 \) thus defines the usual mass parameter familiar from the non-relativistic physics. The remaining \( H_{\text{int}} \) describes the dynamical part of the rest energy and can be identified as the internal Hamiltonian driving time evolution of the internal DOFs. For an atom, \( H_{\text{int}} \) can describe the electronic level structure, and for a molecule – the vibrational energy levels.

The low-energy limit \( H \approx H_{\text{rest}} + p^2/2H_{\text{rest}}c^2 + H_{\text{rest}}\phi(x)/c^2 \) applies when the centre of mass energy is small enough to warrant the non-relativistic approximation but when the internal energy contributions to the kinetic and potential terms are non-negligible – when mass-energy equivalence between internal energy and mass of the system cannot be neglected. For this reason we denote the rest energy as \( H_{\text{rest}} \equiv M_0c^2 \) and can write

\[
H \approx Mc^2 + p^2/2M + M\phi(x), \tag{A2}
\]

which is the Hamiltonian in Eq. (1) in the main text. For derivation up to \( O(1/c^2) \) in terms of an atom in a post-Newtonian metric see also [12].

**Appendix B: Minimum Uncertainty States**

1. **The Eigenvalue Equation**

The solutions to Eq. (8) in the main text are the MUS for the position and velocity uncertainty inequality, comprising superpositions of mass-energy states and their corresponding centre of mass wave-functions.

Each mass component within the sum has its own associated eigenvalue equation:

\[
(u\hat{a}_m + v\hat{a}_m^\dagger) |\psi_m\rangle |m\rangle = z_m |\psi_m\rangle |m\rangle. \tag{B1}
\]

Compare this with the eigenvalue equation for a canonical MUS, a squeezed state, where \(|\mu|^2 - |\nu|^2 = 1\)

\[
(\mu\hat{a} + \nu\hat{a}^\dagger) |\alpha, \mu, \nu\rangle = \alpha |\alpha, \mu, \nu\rangle. \tag{B2}
\]

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In the notation of Stoler [36] and Hollenhorst [55], |α, μ, ν⟩ = ˆS(ξ)|ν⟩, where ˆS(ξ) = e^\frac{i}{\hbar}(ξ^a\tilde{a}^* - ξ^a\tilde{a}^a), is the squeezing operator, ξ ∈ C, and μ = cosh(|ξ|), ν = -e^{i\theta} sinh(|ξ|), and where |ν⟩ is a coherent state with amplitude α.

If we recast the states |ψ_m⟩ as ˆS_m(ξ)|α_m⟩, where ˆS_m(ξ) = e^\frac{i}{\hbar}(ξ^a\tilde{a}^a - ξ^a\tilde{a}^a), the left hand side of Eq. (B1) becomes:

(\hat{u}\hat{m} + \hat{v}\tilde{m}^\dagger)|ψ_m⟩ |m⟩ = (\hat{u}\hat{m} + \hat{v}\tilde{m}^\dagger) ˆS_m(ξ)|α_m⟩ |m⟩

(B3)

The squeeze operators are unitary, ˆS^\dagger ˆS = 1 = ˆS ˆS^\dagger, with

\[ S_m\hat{a}_m S_m^\dagger = \hat{a}_m \cosh(\xi) + \hat{a}_m^\dagger e^{-i\theta} \sinh(\xi) \]
\[ S_m\hat{a}_m^\dagger S_m^\dagger = \hat{a}_m \cosh(\xi) + \hat{a}_m e^{i\theta} \sinh(\xi) \]

(B4)

Using the Eq. (B4) relations and substituting in \( u = \cosh(|ξ|) \) and \( v = -e^{i\theta} \sinh(|ξ|) \), Eq. (B3) becomes

\[ (\hat{u}\hat{m} + \hat{v}\tilde{m}^\dagger) ˆS_m(ξ)|α_m⟩ |m⟩ = α_m |ψ_m⟩ |m⟩, \]

(B5)

which gives us the right hand side of Eq. (B1), where \( α_m \equiv z_m \). The position-and-velocity MUS are squeezed states, with squeezing that depends on the mass-energy eigenvalue \( m \).

2. The Wave Functions

The position representation of the wave functions |ψ_m⟩ is derived in the usual way, by writing Eq. (B1) as a differential equation, with \( \hat{x} = \frac{\hat{p}}{M} = \frac{\hbar}{iM} \frac{\partial}{\partial x} \), such that,

\[ [(u + v)\hat{x} + i(u - v)\hat{z} - \mathcal{Z}] |ψ_m⟩ |m⟩ = \left[ αx + \frac{\hbar}{m} \beta \frac{d}{dx} - z \right] ψ_m(x) = 0 \]

dropping the |m⟩, and defining \( α := (u + v) \) and \( β := (u - v) \).

The resulting wave function is

\[ ψ_m(x) = ψ_m(0) e^{-\frac{m}{\hbar}[\frac{\alpha x^2}{2} - \frac{\beta}{2}x^2]}, \]

and after normalisation,

\[ ψ_m(x) = \frac{ψ_m(0)}{|ψ_m(0)|} \left( \frac{m}{\pi \hbar} \sqrt{\frac{α}{β}} \right)^{\frac{1}{4}} e^{\frac{m}{\hbar} \left[ -\frac{αx^2}{2\sin^2(\alpha/2)} + R(\frac{α}{β}) \right]} e^{\frac{m}{\hbar} \left[ -\frac{β}{2}(x^2 - 2idR(\frac{α}{β})) \right]} \]

(B6)

Appendix C: The Propagator

The propagator for our system is derived via the following expression, with Eq. (1) in the main text as the Hamiltonian, and taking \( \Delta t = (t_f - t_1) \):

\[ \langle x_f, t_f, m'|x_i, t_i, m \rangle = \langle x_f, m' | e^{\frac{-iM\Delta t}{\hbar}} | x_i, m \rangle = \langle x_f, m' | e^{\frac{-i\Delta t}{\hbar}(\mathcal{M}c^2 + \frac{\Delta m^2}{2m} + Mg^2}) | x_i, m \rangle, \]

and thus is diagonal in the mass-energy components \( \langle x_f, t_f, m'|x_i, t_i, m \rangle \equiv K^m(x_f, t_f; x_i, t_i)δ_{m,m'}, \) where \( |x_i, t_i, m \rangle \equiv |x_i, t_i⟩ |m⟩ \).

Via the BCH/Zassenhaus formula, and noting that \( \mathcal{M} \) commutes with both \( \hat{x} \) and \( \hat{p} \),

\[ \langle x_f, m' | e^{\frac{-i\Delta t}{\hbar}(\mathcal{M}c^2 + \frac{\Delta m^2}{2m} + Mg^2}) | x_i, m \rangle = \langle x_f, m' | e^{\frac{-i\Delta t}{\hbar}mc^2} e^{\frac{-i\Delta t}{\hbar}g^2} e^{\frac{-i\Delta t}{\hbar}mg^2} e^{\frac{i\Delta t}{\hbar}mg^2} e^{\frac{i\Delta t}{\hbar}mc^2} | x_i, m \rangle δ_{m,m'} \]

\[ = \frac{1}{2\pi i} \int dp e^{\frac{-i\Delta t}{\hbar} \left[ \frac{p^2}{2m} - p(x_f - x_i) + mg^2 + \frac{\Delta m^2}{2m} + m^2c^2 \right]} δ_{m,m'}. \]

Solving the integral gives us \( K^m \): our propagator for a particle with dynamic internal mass-energy,

\[ K^m(x_f, t_f; x_i, t_i) = \left( \frac{m}{2\pi i\hbar(t_f - t_i)} \right)^\frac{1}{2} e^{\frac{-im^2(t_f - t_i)}{\hbar}} \left[ 1 - \frac{(x_f - x_i)^2}{2c^2(t_f - t_i)^2} + \frac{g^2}{2es^2(t_f - t_i)^2} + \frac{\Delta m^2}{24mc^2(t_f - t_i)^2} \right]. \]
1. Propagating a Free Particle with Mass-Energy States

States are propagated through convolution with the propagator in Eq. (C1).

Each mass-energy component in a generic Gaussian wavepacket (Equation (11) in the main text) after the propagator has been applied reads

\[
\psi_G(x_f, t) = \frac{1}{\sqrt{\pi} \sqrt{\sigma}} \frac{1}{\sqrt{1 + \frac{it}{m\sigma^2}}} \exp \left[ -\frac{(px_t - x)^2}{2(\sigma^2 + \frac{t^2}{m^2\sigma^2})} + \frac{i}{\hbar} \left( 2mc^2t + \frac{p(x_t + x)}{2} + \frac{c^2x_t^2}{2m^2} \right) \right] \tag{C2}
\]

For x-v MUS (Equation (10) in the main text), each mass-energy component after propagation reads:

\[
\psi_{MUS}(x_f, t) = \frac{1}{\sqrt{\frac{\pi}{m} \sqrt{1 + it}}} \frac{1}{\sqrt{1 + \frac{it}{m\sigma^2}}} \exp \left[ -\frac{m(x_t - tv_0)^2}{2(1 + t^2)} - \frac{im}{2\hbar} \left( 2c^2t + \frac{-2v_0x_t + \sqrt{3}x_t^2}{1 + t^2} \right) \right] \tag{C3}
\]

From the above states for each individual mass-energy, the total superposition states are constructed as in Equations (14) and (16) in the main text.

Appendix D: Wigner Representation

Wigner quasi-probability distributions allow us to compare the minimum uncertainty states with the generic Gaussian states in both phase space and in configuration (position and velocity) space.

1. Phase-space Wigner functions

For a state \(|\Psi\rangle\) of the composite particle the Wigner function is defined as

\[
W(x, p) = \int \frac{d\xi}{2\pi} e^{ip\xi} \text{Tr}_m \left\{ (x + \frac{1}{2}\xi) |\Psi\rangle \langle x - \frac{1}{2}\xi| \right\}. \tag{D1}
\]

Expressing the state as \(|\Psi\rangle = \sum_i \alpha_i |\psi_i\rangle |m_i\rangle\), the partial trace over the mass-energy gives

\[
\text{Tr}_m \{|\Psi\rangle \langle \Psi|\} = \sum_j |\alpha_j|^2 \langle x + \frac{1}{2}\xi |\psi_j\rangle \langle \psi_j |x - \frac{1}{2}\xi \rangle,
\]

leaving an overall function comprised of a convex combination of Wigner functions for each mass-energy component,

\[
W(x, p) = \sum_j |\alpha_j|^2 \int \frac{d\xi}{2\pi} e^{ip\xi} \psi_j(x + \frac{1}{2}\xi) \psi_j^\dagger(x - \frac{1}{2}\xi)
= \sum_j |\alpha_j|^2 W^{(j)}(x, p).
\]

The Wigner representation of the propagated Gaussian function, where \(\psi_j\) is given in Eq. (C2), reads

\[
W_G(x, p) = \sum_j |\alpha_j|^2 \frac{1}{\pi\hbar} e^{ - \frac{(\frac{\hbar p_j}{m} + x)^2}{2\sigma^2} - \frac{\hbar^2}{m^2} (x - p_j - v_0)^2}. \tag{D2}
\]

Similarly, the Wigner function for our propagated minimum uncertainty state Eq. (C3) is

\[
W_{Int}(x, p) = \sum_j |\alpha_j|^2 \frac{1}{\pi\hbar} e^{ - \frac{m}{\hbar^2} \left( \frac{\hbar p_j}{m} + x \right)^2 + \left( \frac{\hbar p_j}{m} - v_0 \right)^2} \tag{D3}
\]
2. Wigner functions in position and velocity space

For a configuration (position and velocity) space the Wigner function we change variables in Eq. (D1) to \( \xi' = m \xi \):

\[
\widetilde{W}(x, v) = \int \frac{d\xi'}{2\pi m} e^{iv\xi'} \text{Tr}_m \left\{ \langle x + \frac{\xi'}{2m} | \Psi \rangle \langle \Psi | x - \frac{\xi'}{2m} \rangle \right\}.
\]

For one mass, the Wigner function the above gives

\[
\int \frac{d\xi'}{2\pi m} e^{iv\xi'} \psi_j(x + \frac{\xi'}{2m}) \psi_j^*(x - \frac{\xi'}{2m}) \equiv W^{(j)}(x, m_j v),
\]

which is simply the Wigner function where momentum is non-trivially dependent on the individual mass energies, such that \( v = \frac{p_j}{m_j} \), as expected.

Consequently, the full Wigner function is again a sum of Wigner functions each corresponding to a different mass-energy state,

\[
\widetilde{W}(x, v) = \sum_j |\alpha_j|^2 W^{(j)}(x, m_j v). \tag{D4}
\]

The Wigner function for our MUS in configuration space is thus

\[
\widetilde{W}_{\text{Int}}(x, v) = \sum_j |\alpha_j|^2 \frac{1}{\pi \hbar} e^{-\frac{m_j}{\hbar} [(x-vt)^2 + (v-v_0j)^2]},
\]

where we note its similarity to the phase space Wigner function for our MUS, Equation (D3).

The X–V Wigner function of the generic Gaussian is, similarly,

\[
\widetilde{W}_{\text{G}}(x, v) = \sum_j |\alpha_j|^2 \frac{1}{\pi \hbar} e^{-\frac{m_j}{\hbar} [\frac{(x-vt)^2}{\sigma^2} + \frac{(v-v_0j)^2}{\hbar^2}]}, \tag{D5}
\]

where \( v_{0j} := p_0/m_j \).

Appendix E: Double-slit interference

To place this work in the context of previous studies, we look at how the minimum uncertainty states for position and velocity interfere in a double-slit experiment.

FIG. 3. Interference between spatial superpositions of mass superpositions in our MUS. Masses related as \( m_2 = 10 m_1 \), in equal superposition |\( \Psi \rangle = \sum_{i=1,2} \frac{1}{\sqrt{2}} (|\psi_m^L\rangle + |\psi_m^R\rangle) |m_i\rangle \). Left plot at \( t = 0 \), right at \( t = 20 \).

The initial state is a superposition of mass states which are themselves each in a superposition of left and right displacement, |\( \Psi \rangle = \sum_m c_m (|\psi_m^L\rangle + |\psi_m^R\rangle) |m\rangle \). This is analogous to a double slit experiment where the slits are the same width as the wave packets and the resulting probability function is the interference pattern seen at the screen. Figure 3 shows how these states interfere.
Figure 4 plots analogous interferences comparing our intelligent state to a generic Gaussian. In the latter we consider common velocity of propagation for the different mass-energy states, to take out the dominant effect of different arrival times, already studied in refs [25, 26]. We note that the only effect of gravity on all the studied wave-packets is thus to shift the entire interference pattern by a classical free-fall distance $-g t^2/2$ where $g$ is gravitational acceleration and $t$ the propagation time. The plots can thus be equivalently interpreted as centred at $z_0 = 0$ in a gravity-free case and at $z_0 = -g t^2/2$ in the case where the interfering particle is subject to a homogeneous gravitational field along the screen at which the interference is observed (perpendicular to the initial velocity of the wavepackets). The beating visible in the interference pattern obtained for our MUS state comes from time dilation between different paths that interfere at the screen that is encoded in the evolution of the internal mass-energy superposition.

![FIG. 4. Interference between spatial superpositions of mass superposition states, comparing generic Gaussian case to our MUSs. Masses related as $m_2 = 10 m_1$, in equal superposition $|\Psi\rangle = \sum_{i=1,2} \sqrt{2} |\psi_{m_i}\rangle |m_i\rangle$, at $t = 10$ (left) and $t = 20$ (right).](image)

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