Cosmological constraints from evaporations of primordial black holes.

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The formula for the initial mass spectrum of primordial black holes (PBHs), which can be used for a general case of the scale dependent spectral index, and for a wide class of models of the gravitational collapse, is derived. The derivation is based on the Press and Schechter formalism. The comparative analysis of different types of initial mass spectra used in concrete calculations is carried out. It is shown that densities of background radiations (ν, γ) from PBH evaporations depend rather strongly on a type of the gravitational collapse and on a taking into account the spread of horizon masses at which PBHs can form. Constraints on parameters of the primordial density perturbation amplitudes based on PBH evaporation processes and on atmospheric and solar neutrino data are obtained.

I. INTRODUCTION

Studies of cosmological and astrophysical effects of primordial black holes (PBHs) are important because they enable one to constrain the spectrum of density fluctuations in the early Universe. If the PBHs form directly from primordial density fluctuations then they provide a sensitive probe of the primordial power spectrum on small scales, \( \gtrsim 10^{-9}\text{pc} \). In particular, limits on PBHs production can be used to constrain models of inflation, in which the perturbation amplitudes are relatively large at small and medium scales.

In the simplest case, if we assume that the cosmological PBH formation is dominated, approximately, by primordial perturbations of one particular scale (i.e., there is some characteristic epoch of the PBH formation), we can obtain limits on the initial mass fraction of PBHs,

\[
\rho_i = \rho_{PBH,i}/\rho_{tot,i},
\]

where \( \rho_{PBH,i} \) and \( \rho_{tot,i} \) are the PBH and total energy densities, respectively, at the time \( t_i \) of the formation. This fraction can be expressed by the integral

\[
\beta_i = \int_{\delta_c}^1 P(\delta) d\delta,
\]

where \( P(\delta) \) is a probability distribution for density fluctuations entering horizon at \( t_i \), \( \delta \) is a density contrast, and \( \delta_c \) is a minimum value of \( \delta \) required for the collapse. If the probability distribution is assumed to be Gaussian, one has

\[
P(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\delta^2}{2\sigma^2}},
\]

where \( \sigma \) is the rms fluctuation amplitude on a given scale. It is just this value that is determined by the primordial power spectrum. The connection between \( \beta \) and \( \sigma \) is very simple in a case of the Gaussian distribution:

\[
\beta_i \approx \sigma e^{-\frac{\delta_c^2}{2\sigma^2}}.
\]

The limits on \( \beta_i \) arise from the entropy production constraints [1], from a distortion of the microwave background [2], from the cosmological nucleosynthesis constraints [3]. In these cases PBHs which give constraints have evaporated completely to the present time. On the contrary, the gravitational constraint (\( \Omega_{PBH,0} = \rho_{PBH,0}/\rho_c < 1 \)),

\[
\beta_i < 10^{-19} \left( \frac{M}{10^{15}\text{g}} \right)^{1/2}
\]

[4], is valid for PBH masses \( M \gtrsim 10^{15}\text{g} \) which survive today.

One should stress that all these limits are based just on the approximation that all PBHs form at the same scale and, correspondingly, the initial mass spectrum of PBHs is \( \delta \)-function-like or, at least, is “non-extended” one. Only in
this case one can approximately express the observational constraints through the initial mass fraction $\beta_i$. Evidently, a more accurate treatment should operate with the initial PBH mass spectrum directly.

The most strong limit on a PBH formation in the early Universe is due to the possible contribution of evaporating PBHs to the extragalactic $\gamma$-ray and neutrino backgrounds at energies $\sim 100$ MeV. The limit of such kind was obtained in the work of Page and Hawking, authors of which assumed that the differential initial mass spectrum of PBHs has power law form predicted in Carr’s work:

$$n_{BH}(M_{BH}) = (\alpha - 2) \left( \frac{M_{BH}}{M_*} \right)^{-\alpha} M_*^{-2} \rho_c \Omega_{PBH,0},$$

where $M_* \lesssim M_{BH}$.

Here, $\alpha = 2.5$ for PBHs formed in radiation-dominated era, $M_*$ is the mass of a black hole whose life-time is equal to the present age of the Universe.

This PBH mass spectrum is, clearly, the example of an ”extended” spectrum (it was derived in by considering the PBH formation as a process stretched in time). Naturally, the famous Page-Hawking constraint was formulated in terms of a PBH number density rather than in terms of $\beta_i$. According to, the upper limit on the present PBH number density is $\sim 10^4$ pc$^{-3}$ (or, that is the same, $\Omega_{PBH,0} \lesssim 10^{-8}$). This constraint was improved in later works, where the same initial PBH mass spectrum was used as input.

The derivation of Eq.(1.6) was based on the assumption of exact scale invariance (scale-independence of the perturbation amplitudes). The Carr’s work appeared before an advent of the inflation hypothesis, and at that time it seemed improbable that the possible case of a growth of the perturbation amplitudes with a decrease of the scale can be of any importance (from the point of view of observational evidences of the PBHs existence). Now we know that, due to inflation, minimum values of PBH mass in the initial mass spectrum can be rather large ($10^{13} - 10^{14}$ g or even more). It means that the PBHs can have rather extended mass spectrum even if the primordial fluctuations are not strictly scale invariant.

In general, initial PBH mass spectrum depends on cosmological and astrophysical aspects of the model used for its derivation (and, through the model, on such parameters as a time of the end of inflation, a reheating temperature, a spectral index of the density perturbations (or parameters of the inflationary potential) and, last but not least, parameters characterizing the process of the gravitational collapse leading to the PBH’s birth). Evidently, observational constraints on the PBH production (especially those derived from measurements of extragalactic diffuse backgrounds of $\gamma$-rays and neutrino) can be used as constraints on at least one of these parameters. Clearly, these constraints depend on the initial PBH mass spectrum, i.e., on the parameters of this spectrum which are considered as free in a course of the constraint’s derivation.

Usually, the parameters characterizing primordial density fluctuation amplitudes (e.g., the spectral index, if it is scale independent) are objects of the constraining. Parameters of the gravitational collapse and a method used for a summation over epochs of the PBH formation (if such a summation is performed) determine, by definition, the type of the initial PBH mass spectrum. All other parameters (characterizing, in particular, the end of inflation and the beginning of radiation dominated era) are ”external” and considered as free ones.

In the present work we obtain some cosmological constraints following from the possible contribution of PBH evaporations in extragalactic diffuse neutrino background. Assuming that the power spectrum of primordial fluctuations has a power-law form, $P(k) \sim k^n$, we present the observational limits as constraints on values of the spectral index $n$ (using the normalization on COBE data). In a more general case of the non-power $P(k)$-dependence one can directly constrain parameters of a concrete inflation model. We consider, as an example, a case of the running mass inflation model and obtain constraints on parameters of the corresponding inflationary potential.

The paper is organized as follows. In Sec. we derive, using the Press and Schechter formalism, the general formula for the initial PBH spectrum which is valid for any law of $P(k)$-dependence and for a wide class of models of the gravitational collapse. In Sec. we give the comparative analysis of different types of initial PBH mass spectra used in concrete calculations. In Sec. the neutrino background spectra from PBH evaporations are derived and the corresponding constraints on cosmological parameters are obtained. Discussions and conclusions are presented in Sec.
II. INITIAL PBH MASS SPECTRUM FORMULA

According to the Press - Schechter theory [1], the mass distribution function \( n(M, \delta_c) \), which is defined such that \( n(M, \delta_c)dM \) is the comoving number density of gravitationally bound objects in the mass range \( (M, M + dM) \), is determined by the equations (see, e.g., [10])

\[
n(M, \delta_c) = \frac{\rho_i}{M} \left| \frac{\partial F}{\partial M} (M, \delta_c) \right|, \tag{2.1}
\]

\[
F(M, \delta_c) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_R(M)} e^{-\frac{\delta_c^2}{\sigma_R^2(M)}}. \tag{2.2}
\]

Here, \( \sigma_R(M) \) is the \( \text{rms} \) mass fluctuation on a mass scale \( M \), i.e., a standard deviation of the density contrast of the density field smoothed on a size \( R \) having mean mass \( M \), \( F(M, \delta_c) \) is a fraction of the volume occupied by the regions that will eventually collapse into bound objects with masses larger than \( M \) \((\delta_c \text{ is a minimum value of the density contrast needed for developing a nonlinear growth of the density fluctuations in the overdense region)} \). The density \( \rho_i \) is, in our case, the background energy density at the initial moment of time, \( t = t_i \) (it is a moment of a beginning of the growth of the density fluctuations).

By the construction, Eq.(2.1) supposes a step-by-step formation of the whole mass spectrum of gravitationally bound objects (PBHs, in our case). Heavier PBHs form at later times when corresponding masses of overdense regions cross horizon. The nonzero probability that small PBHs are included in large ones at later times is approximately taken into account by the differentiation of \( F(M, \delta_c) \) with \( M \). The correct normalization of the spectrum is ensured by the extra factor 2 in Eq.(2.2).

Introducing the double differential distribution \( n(M, \delta) \) [11] defined by the expression

\[
n(M, \delta) = \int_0^\infty n(M, \delta_c)d\delta, \tag{2.3}
\]

one obtains from Eqs.(2.1, 2.2)

\[
n(M, \delta) = \sqrt{\frac{2}{\pi}} \frac{\rho_i}{M \sigma_R^2(M)} \left| \frac{\partial \sigma_R(M)}{\partial M} \left( \frac{\delta_c^2}{\sigma_R^2(M)} - 1 \right) \right| e^{-\frac{\delta_c^2}{\sigma_R^2(M)}}. \tag{2.4}
\]

The \( \text{rms} \) mass fluctuation \( \sigma_R(M) \) is given by the integral [12]

\[
\sigma_R^2(M) = \int \left( \frac{k}{aH} \right)^4 \delta_H^2(k)W^2(kR)T^2(k)\frac{dk}{k}, \tag{2.5}
\]

where \( W(kR) \) is the smoothing window function, \( \delta_H(k) \) is the (nonsmoothed) horizon crossing amplitude, \( T(k) \) is the transfer function taking into account the microphysical processing of density perturbations after entering horizon, \( a \) and \( H \) are a cosmic scale factor and an expansion rate.

There are two time-dependent factors in the right hand side of Eq.(2.5): the factor \( (aH)^4 \) in the denominator describing a growth of the perturbations with time and the transfer function. We need the amplitude \( \sigma_R(M) \) at the initial moment of time, \( t = t_i \). A role of the transfer function is essential at large \( k \), if an integrand in Eq.(2.5) is not cut off by a \( k \)-dependence of the window function and \( \delta_H(k) \). In the case when a top-hat window function is used and \( \delta_H^2 \sim k^{n-1} \) with \( n > 1 \), the contribution of large \( k \) modes in the integrand is rather large and a convergence of the integral in Eq.(2.5) is provided just by the transfer function. The form of this function at an epoch of the PBH formation is, of course, model dependent. One can assume, for simplicity, that at \( t = t_i \) the transfer function has a simplest form:

\[
T(k)|_{t=t_i} = \Theta(k_c - k), \tag{2.6}
\]

where \( \Theta(x) \) is the step function, and the border value \( k_c \) is of the order of \( k_{\text{end}} \), the horizon scale at the end of inflation [43]. From our point of view, however, one cannot exclude the possibility that \( k_c >> k_{\text{end}} \). If Eq.(2.6) is assumed, the equation (2.5) can be rewritten, for a power-law of the primordial power spectrum and a top-hat window function, as

\[
\sigma_R^2(M) = \frac{k_f^4}{(aH)^4} \cdot C^2 \left( \frac{k_c}{k_{\text{end}}}, n \right) \cdot \delta_H^2(k_f), \tag{2.7}
\]
\[
C^2 \left( \frac{k_\epsilon}{k_{end}}, n \right) = \int_0^{k_\epsilon/k_{end}} x^{n+2} W^2(x) \, dx = 9 \int_0^{k_\epsilon/k_{end}} x^{n+2} \left[ \frac{\sin x}{x^3} - \frac{\cos x}{x^2} \right]^2 \, dx.
\] (2.8)

If the spectral index \( n \) is slightly more than one, \( n \approx 1.2 \), and if \( k_\epsilon/k_{end} \approx 10 \) one has \( C \sim 5 \), i.e., the value of \( C \) is close to the corresponding value \( 4 \) for the present epoch. Having no reliable model for the transfer function at early postinflationary epoch, we assume everywhere below, where a power-law form of \( P(k) \) is used, that \( C \) is constant, and equal to 4.7, throughout all times. Correspondingly,

\[
\sigma_R^2(M) \approx \frac{k_{fl}^4}{(aH)^4} C^2 \sigma_H^2(k_{fl}) = \frac{k_{fl}^4}{(aH)^4} \sigma_H^2(M_h) .
\] (2.9)

In Eq.(2.9) \( k_{fl} \) is the comoving wave number, characterizing the perturbed region,

\[
k_{fl} = \frac{1}{R}.
\] (2.10)

and \( M_h \) is the fluctuation mass at the moment when the perturbed region crosses horizon in radiation dominated era. The value of \( M_h \) is expressed through \( k_{fl} \) (see below). The function \( \sigma_H(M_h) \) introduced by Eq.(2.9) is, evidently, the smoothed horizon crossing amplitude (a standard deviation of the density contrast at a moment when the fluctuation crosses horizon).

The general formula for \( \sigma_H^2(M_h) \), which is valid for an arbitrary \( k \)-dependence of \( P(k) \), is

\[
\sigma_H^2(M_h) = \int_0^{k_\epsilon/k_{end}} x^3 W^2(x) \delta_H^2(k_{fl} x) \, dx.
\] (2.11)

At the moment \( t_h \) when the fluctuation crosses horizon one has

\[
Ra(t_h) \approx t_h,
\] (2.12)

and, in radiation dominated era \((a \sim \sqrt{t})\),

\[
M_h \sim \frac{1}{k_{fl}}.
\] (2.13)

At the same time, an initial value of the fluctuation mass, \( M_i \), is proportional to \( R^3 \), so \( M_h \sim M^{2/3} \). The exact connection is

\[
M_h = M_i^{1/3} M^{2/3} ,
\] (2.14)

where \( M_i \) is the horizon mass at the initial moment of time, \( t = t_i \),

\[
M_i \sim t_i \sim \frac{1}{(a_i H_i)^2} .
\] (2.15)

The denominator \((aH)^4\) in Eq.(2.9) must be taken also at \( t = t_i \) and, using the proportionality

\[
M \sim R^3 a^3(t_i) \rho(t_i) \sim \frac{R^3}{\sqrt{t_i}} \sim k_{fl}^{-3} \frac{1}{\sqrt{t_i}} ;
\] (2.16)

one can easily see that

\[
\frac{k_{fl}^2}{(a_i H_i)^2} = \left( \frac{M}{M_i} \right)^{-2/3} .
\] (2.17)

Due to Eq.(2.17), one has the simple connection between \( \sigma_R \) and \( \sigma_H \):

\[
\sigma_R(M) = \left( \frac{M}{M_i} \right)^{-2/3} \sigma_H(M_h) .
\] (2.18)
For the following we parametrize the $k$-dependence of $\delta_H(k)$ by the formula

$$\delta_H(k) = \delta_H(k_0) \left( \frac{k}{k_0} \right)^{n(k) - 1}.$$  \hfill (2.19)

Introducing now the new variable $\delta '$,

$$\delta ' = \delta \left( \frac{M}{M_i} \right)^{2/3},$$  \hfill (2.20)

and a general form of the connection between PBH mass, fluctuation mass and density contrast,

$$M_{BH} = f(M, \delta'),$$  \hfill (2.21)

one gets the PBH mass spectrum $n_{BH}(M_{BH})$:

$$n_{BH}(M_{BH}) = \int n_{BH}(M_{BH}, \delta') d\delta',$$  \hfill (2.22)

$$n_{BH}(M_{BH}, \delta') = n(M, \delta) \frac{d\delta}{d\delta'} dM_{BH}.$$  \hfill (2.23)

Using Eqs. (2.4), (2.9), (2.14), (2.18-2.21), one obtains the final expression for the PBH mass spectrum

$$n_{BH}(M_{BH}) = \sqrt{\frac{2}{\pi \rho_i}} \int \frac{1}{M \sigma_H(M_h)} \left\{ \frac{2}{3M} - \left[ \frac{n'}{2} \ln \frac{k_{fl}}{k_0} + \frac{n(k_{fl}) - 1}{2 k_{fl}} \right] \frac{\partial k_{fl}}{\partial M} \right\} \times$$

$$\left| \frac{\delta'^2}{\sigma^2(H(M_h))} - 1 \right| e^{-\frac{\delta'^2}{2 \sigma^2(H(M_h))}} \frac{d\delta'}{df(M, \delta')/dM}.$$  \hfill (2.24)

Here the following notation is used:

$$n' = \frac{dn(k)}{dk} \bigg|_{k=k_{fl}}.$$  \hfill (2.25)

The limits of integration in Eq. (2.24) are determined from Eqs. (2.20) and (2.21) using the conditions

$$\delta_{min} = \delta_c, \quad M_{min} = M_{h_{min}} = M_i.$$  \hfill (2.26)

The connection between $k_{fl}$ and $M$ is determined by the expressions

$$k_{fl} = (a_{eq} H_{eq}) \left( \frac{M_{eq}}{M_h} \right)^{1/2},$$  \hfill (2.27)

$$a_{eq} H_{eq} = \sqrt{2} H_0 \Omega_m \Omega_r^{-1/2}, \quad M_{eq} = \frac{1}{8} \frac{M_{pl} t_{eq}}{t_{pl}}.$$  \hfill (2.27)

together with Eq. (2.14). In these formulae $a_{eq}$ and $H_{eq}$ are the cosmic scale factor and the expansion rate at the moment of matter-radiation equality, $t = t_{eq}$, $H_0$ is the Hubble constant ($H_0 = 100 h k m s^{-1} Mpc^{-1}$), $M_{eq}$ is the horizon mass at $t_{eq}$, $\Omega_m$ and $\Omega_r$ are matter and radiation densities in units of $\rho_c$. We consider in this paper, for simplicity, the case of flat models with zero cosmological constant.

III. DIFFERENT TYPES OF INITIAL PBH MASS SPECTRA

A concrete form of the initial mass spectrum of primordial black holes is determined, first of all, by peculiarities of the gravitational collapse near a threshold of the black hole formation. According to analytic calculations of seventies
a critical size of the density contrast needed for the PBH formation, \( \delta_c \), is about 1/3. Besides, it was argued that all PBHs have mass roughly equal to the horizon mass at a moment of the formation, independently of the perturbation size. The approximate connection between the PBH mass and the horizon mass in such models is very simple \[18,11\]:

\[
M_{BH} = \gamma^{1/2} M_h, \tag{3.1}
\]

where \( \gamma \) is the ratio of the pressure to energy density (\( \gamma = 1/3 \) in radiation dominated era).

However, it was shown recently that near a threshold of the black hole formation the gravitational collapse behaves as a critical phenomenon \[16\]. In this case the initial mass function will be quite different from the analogous function in the analytic calculations \[15\].

\[
M_{BH} = \kappa M_h (\delta' - \delta_c)^{\gamma_k}. \tag{3.2}
\]

In this formula \( M_h \) is the horizon mass at the time when the fluctuation enters horizon; \( \delta_c, \kappa, \gamma \) are parameters of a concrete model of the critical collapse \[14,17\]. The corresponding \( f(M, \delta') \) function defined in Eq.\( 2.21 \) is determined using the relation (2.14):

\[
f(M, \delta') = \kappa M_k^{1/3} M^{2/3} (\delta' - \delta_c)^{\gamma_k}. \tag{3.3}
\]

It is seen from Eq.(3.2) that the PBH mass may be arbitrarily small independently of the value of \( M_h \). Besides, a value of the critical overdensity, \( \delta_c \), in such models is typically \( \sim 0.7 \), i.e., about a factor of 2 larger than the value used in the standard collapse case.

The second important ingredient of a PBH initial mass spectrum calculation is a taking into account the spread of horizon masses at which PBHs are formed. This problem exists independently of a nature of the gravitational collapse.

As is stated in the Introduction, it is assumed very often that a majority of the PBH formation processes occurs at the shortest possible scale. In particular, the authors of ref. \[10\] determined the PBH initial mass function under the assumption that all PBHs form at the same horizon mass. The accuracy of such an approximation was studied at the shortest possible scale. In particular, the authors of ref. \[16\] determined the PBH initial mass function under

\[
n_{BH}(M_{BH}) = \frac{n + 3}{4} \sqrt{\frac{2}{\pi}} \rho \xi^{5/2} M_{BH}^{-5/2} \int_{\delta_c}^{\delta_{max}} \frac{1}{\sigma_H(M_h)} \left| \frac{\delta' \xi^{3/2}}{\sigma_H(M_h)} - 1 \right| d\delta', \tag{3.4}
\]

\[
\xi \equiv \kappa (\delta' - \delta_c)^{\gamma_k}.
\]

An upper limit of the integration in Eq.(3.4) is determined by the expression

\[
\delta_{max} = min \left( \left( \frac{M_{BH}}{\kappa M_k} \right)^{1/\gamma_k} + \delta_c, 1 \right). \tag{3.5}
\]

The formula (3.4) was derived using the assumption that the primordial power spectrum \( P(k) \) has a power-law form (i.e., the spectral index \( n \) is constant throughout all scales). Correspondingly, Eq.(3.4) can be obtained from the general expression (2.24) omitting there the \( n' \)-term. The amplitude \( \sigma_H(M_h) \) can be normalized on COBE data using the connection between \( \sigma_H \) and \( \delta_H \) given by Eq.(2.3),

\[
\sigma_H(M_h) = C \delta_H(k_f), \tag{3.6}
\]
\[ \delta_H(k) \approx 2 \cdot 10^{-5} \left( \frac{M_{eq}}{M_{ho}} \right)^{1/2} \left( \frac{M_h}{M_{eq}} \right)^{1/2}. \]  

(3.7)

Here \( M_{ho} \) is the present horizon mass.

All numerical calculations the results of which are presented in this paper are carried out with the following values of the parameters [16]:

\[ \kappa = 3, \quad \gamma_k = 0.36, \quad \delta_c = 0.7. \]  

(3.8)

The known formula [18] for the PBH mass spectrum in the standard collapse case can be obtained from Eq. (3.4) by using the substitutions [21]

\[ \gamma_k \rightarrow 0, \quad \kappa \rightarrow \gamma \frac{1}{2}, \quad \delta_c \rightarrow \gamma \]  

(3.9)

and the approximate relation

\[ \int_1^\gamma d\delta' \left( \frac{(\delta'^2)}{\sigma_H^2} - 1 \right) e^{-\frac{\delta'^2}{2\sigma_H^2}} \approx \gamma e^{-\frac{\gamma^2}{2\sigma_H^2}}. \]  

(3.10)

Substituting Eqs. (3.9) and (3.10) in Eq. (3.4) one obtains

\[ n_{BH}(M_{BH}) = \frac{n + 3}{4} \sqrt{\frac{2}{\pi}} \gamma^4/4 \rho_i M_{BH}^{-1/2} M_{BH}^{-5/2} \sigma_H^{-1}(M_h) \exp \left( -\frac{\gamma^2}{2\sigma_H^2(M_h)} \right). \]  

(3.11)

The Carr’s formula [Eq. (1.6)] follows from Eq. (3.11) if two additional assumptions are used: i) the amplitude \( \sigma_H(M_h) \) does not depend on \( M_h \) (exact scale invariance) and ii) there is no minimum value of PBH mass in the initial spectrum. In this case the spectrum is \( AM_{BH}^{-5/2} \), as it follows from Eq. (3.11). The factor \( A \) is found by normalization of the present PBH mass spectrum on the present value of \( \rho_{PBH,0} = \Omega_{PBH,0} \rho_c \), using the approximate formula for the instantaneous PBH spectrum at \( t = t_0 \) [11],

\[ n_{BH}(m, t_0) = \frac{m^2}{(M_{*}^2 + m^2)^{2/3}} A(M_{*}^2 + m^2)^{-5/2} \Theta(m - M_{*}), \]  

(3.12)

(\( \Theta(x) \) is the step function), and the expression

\[ \rho_{PBH,0} = \int_{M_{*}}^{\infty} n_{BH}(m, t_0) dm. \]  

(3.13)

We carried out in this work the calculations of the neutrino background spectra from PBH evaporations using the mass spectra given by Eqs. (3.4), (3.11) and, in parallel, the PBH mass spectrum from ref. [16],

\[ n_{BH}(M_{BH}) = \frac{\rho_i \left( M_{BH} / M_{*} \right)^{1/2}}{\sqrt{2\pi} \sigma_H(M_{*}) M_{BH} M_{BH}^{-1/2} \gamma_k} e^{-\left( \frac{M_{BH} / M_{*}}{\sigma_H(M_{*})} \right)^{1/2} \Theta(M_{BH} / M_{*})^2}. \]  

(3.14)

In this expression the amplitude \( \sigma_H \) is determined by the same formula (3.6), as above, but with the constant value of \( M_h, M_h = M_{*} \), in accordance with the assumption of authors of ref. [16] that all PBHs form at the smallest scale.

We have, for calculations of \( n_{BH}(M_{BH}) \), using Eqs. (3.4), (3.11) and (3.14), two free parameters: the spectral index \( n \), giving the perturbation amplitude through the normalization on COBE data on large scales, and \( t_i \), the moment of time just after reheating, from which the process of PBH formation started. The value of \( t_i \) is connected, in our approach, with a value of the reheating temperature,

\[ t_i = 0.301 g^* / T_{RH} \]  

(3.15)
Several examples of $z$-distributions (the integrands of the integral over $z$ in Eq.(4.3)) are shown on Fig.2. Again one can see the rather strong difference of two cases: the existence of a tail of heavy masses in BK spectrum leads to a relative enhancement of low $z$-contributions. The effect becomes more distinct with a rise of the reheating temperature.

IV. NEUTRINO BACKGROUND SPECTRA FROM PBHS EVAPORATIONS

Evolution of a PBH mass spectrum due to the evaporation leads to the approximate expression for this spectrum at any moment of time:

$$n_{BH}(m,t) = \frac{m^2}{(3\alpha t + m^3)^{2/3}}n_{BH}\left((3\alpha t + m^3)^{1/3}\right),$$

where $\alpha$ accounts for the degrees of freedom of evaporated particles and, strictly speaking, is a function of a running value of the PBH mass $m$. In all our numerical calculations we use the approximation

$$\alpha = \text{const} = \alpha(M_{BH}^{\text{max}}),$$

where $M_{BH}^{\text{max}}$ is the value of $M_{BH}$ in the initial mass spectrum corresponding to a maximum of this spectrum. Special study shows that errors connected with such approximation are rather small.

The expression for a spectrum of the background radiation is [11]

$$S(E) = \frac{c}{4\pi} \int dt \frac{a_t}{a_0} \left( \frac{a_t}{a_0} \right)^3 \int dm \frac{m^2}{(3\alpha t + m^3)^{2/3}}n_{BH}\left((3\alpha t + m^3)^{1/3}\right) \cdot f(E \cdot (1 + z), m)e^{-\tau(E,z)}$$

$$\equiv \int F(E, z)d\log_{10}(z + 1).$$

In this formula $a_0, a$ and $a_0$ are cosmic scale factors at $t_i, t$ and at present time, respectively, and $f(E, m)$ is a total instantaneous spectrum of the background radiation (neutrinos or photons) from the black hole evaporation. It includes the pure Hawking term and contributions from fragmentations of evaporated quarks and from decays of pions and muons (see [11] and earlier papers [22–24] for details). The exponential factor in Eq.(4.3) takes into account an absorption of the radiation during its propagation in space. The processes of the neutrino absorption are considered, in a given context, in ref. [11].

In the last line of Eq.(4.3) we changed the variable $t$ on $z$ using the flat model with $\Omega_\Lambda = 0$ for which

$$\frac{dt}{dz} = -\frac{1}{H_0(1 + z)}\left(\Omega_m(z + 1)^3 + \Omega_r(z + 1)^4\right)^{-1/2},$$

$$\Omega_r = (2.4 \cdot 10^4h^2)^{-1}, \quad h = 0.67.$$

On first five figures (Figs.2–6) some results of calculations for the case of a scale independent spectral index are presented. Several examples of $z$-distributions (the integrands of the integral over $z$ in Eq.(4.3)) are shown on Fig.2. Again one can see the rather strong difference of two cases: the existence of a tail of heavy masses in BK spectrum leads to a relative enhancement of low $z$-contributions. The effect becomes more distinct with a rise of the reheating temperature.
The sharp cut-off of all $z$-distributions near $z \sim 10^7$ is entirely due to the neutrino absorption \cite{11}. The shrinkage of $z$-distributions at large $T_{RH}$ in NJ-case is due to the absence of large masses in the spectrum (PBHs of small masses evaporated earlier, and their radiation today is more redshifted).

Some results of calculations of the neutrino background spectra are shown on Figs.3-5. The functional form of these spectra is qualitatively the same as in the standard collapse case \cite{11}: $E^{-3}$ law at large neutrino energies and a flat part at low energies with the crossover energy depending on $T_{RH}$. The only new feature is the sharp steepening of the spectra at high values of $T_{RH}$ in NJ case (Fig.4) connected with the corresponding shrinkage of the $z$-range.

Fig.5 shows the sensitivity of background neutrino intensities to a chosen value of the spectral index $n$, at a fixed $T_{RH}$. One can see from this figure that the neutrino background flux from PBH evaporation is comparable with the theoretical curve \cite{24} for the atmospheric neutrino flux at $\sim 100$ MeV if $n \sim 1.29$ (at $T_{RH} \sim 10^9$ GeV). Such large values of the spectral index are, probably, excluded by the recent large-scale experiments.

Next figure shows the constraints on the spectral index following from data of neutrino experiments, as a function of the reheating temperature. As in our previous work, we used for a description of these constraints the data of the Kamiokande atmospheric neutrino experiment \cite{26} and the experiment on a search of an antineutrino flux from the Sun \cite{27} (see \cite{11} for details).

For completeness, we show on the same figure the constraints following from extragalactic diffuse gamma ray data; part of them (NJ case) is in qualitative agreement with the results of ref. \cite{28}. It is seen that the constraints are much more weak in NJ case (at $T_{RH} \gtrsim 10^{10}$ GeV). It is again the result of an absence of the tail of large masses in NJ spectrum, leading to a shrinkage of the $z$-distributions and to a steepening of the background neutrino spectra.

One can see from Fig.4 that, in general, spectral index constraints following from the comparison of neutrino background predictions with existing data of neutrino experiments are rather weak if the purely power law of primordial density fluctuations is assumed for all scales (and, correspondingly, the normalization on COBE data is used). However, it does not mean that neutrino background intensities from PBH’s evaporation cannot be noticeable. If, for example, the spectrum of primordial density fluctuations is a combination of two simple power law spectra \cite{20}, i.e.,

$$
\sigma_H(M_h) \sim M_h^{\frac{1-n_l}{2}}, \quad M_h < M_{h<} < M_{eq};
\sigma_H(M_h) \sim M_h^{\frac{1-n_l}{2}}, \quad M_{eq} > M_h > M_{h<},
$$

then the small value of $n_l \sim 1$ ($n_l - 1 \approx 0$) which follows, in particular, from COBE data, does not contradict with the possibility that $n_x - 1$ is large and, correspondingly, $\sigma_H$ at small scales is also large. If $M_{h<}$ and $n_l$ are known from some model, one can easily obtain from the neutrino data the constraints on the small scale spectral index $n_x$, in the same manner as it is done above for the all scale spectral index $n$. Several cosmological models of such kind, which predict a large density fluctuation amplitude just on small scales (while the amplitude on large scales is constrained by the small CMBR anisotropy) appeared recently. The general problem of all these models is that the large density perturbations on small scales can lead to an overproduction of the primordial black holes. An origin of such kind, which leaves horizon outside. The only new feature is the sharp steepening of the background neutrino spectra.

At the end of this section we consider, with more details, the case of running mass inflation models, which also give a spectral index with rather strong scale dependence. The inflationary potential in these models is of the form \cite{8}:

$$
V = V_0 \left[1 - \frac{1}{2} \frac{\phi^2}{Q}\right],
$$

where $c$ and $Q$ are positive constants, $\phi$ is inflation field. This form corresponds to a loop correction coming from softly broken global supersymmetry. The constant $c$ is, essentially, the coupling strength of the field in the loop which is expected to be of the order of $0.01 - 0.1$ \cite{31}. The spectral index is given by the simple expression:

$$
\frac{1}{2} [n(k) - 1] = \sigma e^{-cN(k)} - c,
$$

where $N(k)$ is a number of $e$-folds of inflation that occur after the epoch when a scale $k$ leaves horizon outside. The new parameter $\sigma$ in Eq.(4.7) is expressed through $Q$, $c$ and $N(k)$. Using the connection $d\ln k = -dN$ one obtains

$$
\frac{1}{2} n' = \frac{c\sigma}{k} e^{-cN(k)}.
$$
For normalization of the perturbation amplitude to COBE data we use the parameter \( N_{\text{COBE}} = \ln(k_{\text{end}}/k_{\text{COBE}}) \) connected with \( N(k) \) by the expression

\[
N(k) = N_{\text{COBE}} \ln \frac{k}{k_{\text{COBE}}},
\]

(4.9)

where the COBE scale (the center of the range explored by COBE) is

\[
k_{\text{COBE}} = \frac{7}{3000} h \text{Mpc}^{-1}.
\]

(4.10)

We can fix the parameter \( N_{\text{COBE}} \) assuming the instant reheating after the end of slow-roll inflation, and expressing \( N_{\text{COBE}} \) through \( T_{\text{RH}} \) using Eq.(2.27),

\[
k_{\text{fl}}^{\text{max}} = (a_{\text{eq}}H_{\text{eq}}) \sqrt{\frac{M_{\text{eq}}}{M_i}},
\]

(4.11)

the connection between \( M_i \) and \( T_{\text{RH}} \) [Eqs.(3.15),(3.16)], and the relation

\[
k_{\text{fl}}^{\text{max}} = k_{\text{end}} = e^{N_{\text{COBE}}a_0H_0}.
\]

(4.12)

Instead of this, we put, for simplicity, \( N_{\text{COBE}} = 45 \) (for all possible values of \( T_{\text{RH}} \)), i.e., we assume that, in a general case, \( k_{\text{fl}}^{\text{max}} < k_{\text{end}} \) (radiation era begins not immediately after the end of slow-roll inflation). The slow-roll condition,

\[
\eta = \frac{1}{8\pi} M_{\text{pl}}^2 \frac{V''}{V_0} < 1,
\]

(4.13)

holds if values of the parameter \( c \) are not too large. If, e.g., \( n_{\text{COBE}} = 1 \), this condition can be rewritten as

\[
\eta \approx ce^{N_{\text{COBE}}} < 1.
\]

(4.14)

The horizon crossing amplitude \( \delta_H(k) \) is calculated using Eq.(2.19), where we put \( k_0 = k_{\text{COBE}} \). The expression (4.7) for \( n(k) \) has two parameters: \( \sigma \) and \( c \). If we suppose that the spectral index at \( k = k_{\text{COBE}} \) is known from experiment (e.g., according to ref. [33], it follows from COBE data that \( n_{\text{COBE}} = 1.02 \pm 0.24 \)), we exclude one parameter (e.g., \( \sigma \)). Now we can use the same steps as above to obtain the constraints on the parameter \( c \). The only difference is that now one must use the general expressions for the PBH mass spectrum, Eq.(2.24), and for \( \sigma_H^2(M_h) \), Eq.(2.11).

These constraints are shown on Fig.7 for three possible values of \( n_{\text{COBE}} \). Solid lines on the figure correspond to the critical collapse case, dotted line (calculated for the value \( n_{\text{COBE}} = 1.0 \) only) correspond to the standard picture of the collapse. The lines show the upper limits of the parameter \( c \). One can clearly see the difference between two cases: the sharp rise of the dotted curve at \( T_{\text{RH}} \sim 10^8 \text{GeV} \) is directly connected with the sharp cut-off of the initial mass spectrum (for the standard collapse case) at \( M_{\text{BH}} = M_i \). Indeed, due to the proportionality

\[
M_i \sim t_i \sim T_{\text{RH}}^{-2},
\]

(4.15)

at small enough reheating temperatures the minimum values of the PBH mass in the spectrum are too large and, therefore, the large part of PBHs cannot evaporate until the present time.

Finally, on Fig.8 we show the resulting constraint curves for the horizon crossing amplitude \( \delta_H(k) \), for fixed values of \( n_{\text{COBE}} \) and \( T_{\text{RH}} \) (for the critical collapse case). The curves correspond to the upper limits of the parameter \( c \) shown on the previous figure. The end points of the power spectrum on the figure is determined from Eq.(4.11). It is seen from Fig.8 that allowed values of \( \delta_H^2(k_{\text{fl}}^{\text{max}}) \) at the shortest scale (for a given \( T_{\text{RH}} \)) are rather large,

\[
\delta_H^2(k_{\text{fl}}^{\text{max}}) \sim 3 \cdot 10^{-4}.
\]

(4.16)

The corresponding \( \sigma_H \) value is \( \sim 0.08 \). At \( T_{\text{RH}} = 10^{10} \text{GeV} \) the shortest scale, \( k_{\text{fl}}^{\text{max}} \sim 10^{18}(h \text{ Mpc}^{-1}) \sim k_{\text{end}} \), corresponds to a rather small value of the horizon mass, \( M_h^{\text{min}} = M_i \sim 10^{11} \text{g} \). PBHs of such a small mass evaporated at \( t \sim 10^6 \text{s} \).
V. CONCLUSIONS

The main conclusion of the work is the following: constraints on parameters of cosmological models from evaporations of primordial black holes depend rather strongly on a form of their initial mass spectrum. Number densities of PBHs predicted by the models are extremely sensitive to values of primordial density perturbation amplitudes and, therefore, even very steeply falling mass spectra of PBHs can give useful constraints. As it follows from Fig.3, constraints on the scale independent spectral index $n$ at high values of a reheating temperature are drastically different in NJ and BK cases. We thoroughly traced the origin of this result: the absence of a long high mass tail in the initial NJ spectrum (Fig.3) leads to a relative shrinkage of the redshift distributions of evaporated neutrinos (Fig.4) and, as a consequence, to a steepening of the neutrino background spectra and to a decrease of the background intensities (Fig.5), resulting, eventually, in a weakening of the spectral index constraint. So, the constraint values at $T_{RH} > 10^{10}$ GeV on Fig.4 are entirely due to an effect of the summation over all epochs of the PBH formation in the approach of ref. [20,21]. Analogously, we see, on an example of the parameter $c$ of running mass inflation models, that the constraints at low values of $T_{RH}$ also depend on a type of the gravitational collapse: in the standard collapse case the constraint at $T_{RH} < 10^8$ is practically absent (dotted curve on Fig.3) due to a corresponding absence of a low mass tail in the initial PBH mass spectrum of this case (compare KL and BK spectra on Fig.3) the behavior of the corresponding PBH mass spectra in a case of a scale dependent $n$ is qualitatively similar.

The initial number density of PBHs is larger in a case of the standard picture of the gravitational collapse (as compared with the critical collapse case). It is due to relatively small value of the critical overdensity ($\delta^{e}_{c} \approx 1/3$). In the region near the maximum of PBH mass spectra the ratio of intensities is given by the approximate relation

$$\frac{I_{BH}^{KL}}{I_{BH}^{BK}} \sim e^{\frac{1}{2} - \frac{\gamma}{2}}. \quad (5.1)$$

Correspondingly, intensities of the neutrino and gamma backgrounds produced by PBH’s evaporations are smaller and spectral index constraints are systematically weaker in the critical collapse case (see, for the comparison with the standard collapse case, Fig.10 of ref. [11]).

It is worth noting that spectral index constraints followed from neutrino evaporations are stronger at high $T_{RH}$ values (as compared with the constraints from $\gamma$-quanta), especially in the NJ case (Fig.4). The reason is simple: at high $T_{RH}$ the relative contribution of large redshifts in $z$-distributions is very large (Fig.4) and, correspondingly, the absorption of $\nu$, $\gamma$ during propagation in space is important. Naturally, absorption effects are more sufficient for $\gamma$-quanta than for neutrinos (for $\gamma$-quanta $z_{\text{max}} \sim 700$, while for neutrinos $z_{\text{max}} \sim 10^7$ [11]), as we can see, in particular, from curves of Fig.4.

One should note that, in general, the constraints on a scale independent spectral index followed from PBH evaporations (Fig.4) are rather weak. Probably, so large deviations of the spectral index from 1 ($n \sim 1.28$) are excluded by the latest data (see, e.g., [31]). Therefore, constraints from PBH evaporations are more useful for those cosmological models in which the spectral index is scale dependent. In such models large density perturbation amplitude at small scales (and, correspondingly, large effects from evaporations) can coexist with the small value of $\delta_H(k)$ at COBE scale (Fig.3). We showed in this paper, using, as an example, the running mass inflation model, that in this case the PBH evaporation process can be used for a constraining of model parameters [in particular, parameters of an inflationary potential, (Fig.8)]. We stress, once more, that all constraints of such type depend on assumptions used in a derivation of the initial PBH mass spectrum.

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FIG. 1. Examples of PBH mass spectra for two values of $T_{RH}$. Solid lines: KL spectra [18], dotted lines: NJ spectra [16], dashed lines: BK spectra [11]; $n = 1.30$ in all cases.
FIG. 2. Red-shift distributions (integrand of the expression (4.3) for the background spectrum) for neutrino energy $E = 10^{-1}$ GeV (solid lines) and $E = 10^{-2}$ GeV (dashed lines), for PBH mass spectrum (left column) and BK mass spectrum (right column), for three values of $T_{RH}$; $n = 1.30$ in all cases. The dimension of $F(z)$ is $(s^{-1}cm^{-2}sr^{-1}GeV^{-1})$. 
FIG. 3. Neutrino background spectra for three values of $T_{RH}$, for BK mass spectrum; $n = 1.30$. 

$T_{RH}=10^{10}$ GeV

$T_{RH}=10^8$ GeV

$T_{RH}=10^9$ GeV

BK spectrum

$n=1.3$
FIG. 4. Neutrino background spectra for three values of $T_{RH}$, for NJ mass spectrum; $n = 1.30$. 
FIG. 5. Neutrino background spectra for three values of the parameter $n$, for BK mass spectrum. Dashed line represents the theoretical atmospheric neutrino spectrum for Kamiokande site (averaged over all directions) [25].
FIG. 6. Constraints on the spectral index $n$ as a function of the reheating temperature $T_{RH}$. Solid and dotted lines: BK and NJ mass spectra, atmospheric $\nu_e$ and solar $\tilde{\nu}_e$ experiments. Dashed and dot-dashed lines: BK and NJ mass spectra, extragalactic diffuse gamma-ray background data.
FIG. 7. Upper limits on values of the parameter $c$ of the running mass inflation model, for different reheating temperatures and for several possible values of $n_{COBE}$. Solid lines: critical collapse case, dotted line: standard collapse case.
FIG. 8. Constraints on the power spectrum of primordial density fluctuations in the running mass inflation model, for several values of $n_{C}OBE$ and for $T_{RH} = 10^{10}$ GeV. All curves are calculated for the critical collapse case.