Degenerate Hořava gravity

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Abstract

Hořava gravity breaks Lorentz symmetry by introducing a dynamical timelike scalar field (the khronon), which can be used as a preferred time coordinate (thus selecting a preferred space–time foliation). Adopting the khronon as the time coordinate, the theory is invariant only under time reparametrizations and spatial diffeomorphisms. In the infrared limit, this theory is sometimes referred to as khronometric theory. Here, we explicitly construct a generalization of khronometric theory, which avoids the propagation of Ostrogradski modes as a result of a suitable degeneracy condition (although stability of the latter under radiative corrections remains an open question). While this new theory does not have a general-relativistic limit and does not yield a Friedmann–Robertson–Walker-like cosmology on large scales, it still passes, for suitable choices of its coupling constants, local tests on Earth and in the Solar System, as well as gravitational-wave tests. We also comment on the possible usefulness of this theory as a toy model of quantum gravity, as it could be completed in the ultraviolet into a ‘degenerate Hořava gravity’ theory that could be perturbatively renormalizable without imposing any projectability condition.

Keywords: Hořava gravity, khronometric theory, modified gravity, degeneracy conditions

1. Introduction

Hořava gravity [1] is a gravitational theory that is power-counting renormalizable in the ultraviolet, at the expense of giving up Lorentz symmetry. The theory, written in terms of 3 + 1 Arnowitt–Deser–Misner (ADM) variables [2], is indeed invariant only under foliation-preserving diffeomorphisms (FDiffs), i.e. (monotonic) time reparametrizations and spatial diffeomorphisms, and not under full-fledged four-dimensional diffeomorphisms. The action

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of the theory involves up to six spatial derivatives of the ADM fields, but is only quadratic in the time derivatives (which only appear via the extrinsic curvature, i.e. via the time derivative of the spatial metric). It is this anisotropic scaling between space and time derivatives that ensures power-counting renormalizability.

Performing a Stuckelberg transformation, Hořava gravity can be recast as a Lorentz-violating scalar–tensor theory [3]. The scalar field, sometimes referred to as the khronon, is constrained by a Lagrange multiplier to be timelike (i.e. its gradient must be timelike), and it thus plays the role of a preferred time by selecting a preferred manifold slicing. Using this covariant formulation, as opposed to the original 3 + 1 one (i.e. the unitary gauge where the khronon is adopted as time coordinate), it becomes more clear why the 3 + 1 action is built without any time derivatives of the lapse. In fact, to avoid Ostrogradski instabilities [4] (or ‘ghosts’), one may naively require that the covariant action be quadratic in the unit-norm ‘æther’ vector field proportional to the khronon’s gradient. This vector field turns out to be \( u = -N \partial_t \) in the unitary gauge, with \( N \) the lapse, and one may naively try to obtain time derivatives of \( N \) by introducing the acceleration \( a^\nu = u^\mu \nabla_\mu u^\nu \). The latter, however, only includes spatial derivatives of \( N \) because of the unit norm condition (which implies \( a^\mu u_\mu = 0 \)).

The absence of time derivatives of the lapse poses a hurdle to proving perturbative renormalizability (beyond power counting). Calculations of the latter are technically involved and have so far only been performed in the unitary gauge [5, 6], where the lapse satisfies an elliptic equation as a result of the absence of its time derivatives in the action [7]. This leads to the ‘instantaneous’ propagator \( 1/(k, k') \). To overcome this problem, reference [5] proved perturbative renormalizability in Hořava gravity under the ‘projectability condition’, i.e. the assumption that the lapse is a function of time only. The resulting theory, while considered in the first paper by Hořava [1], is disjoint from general (i.e. non-projectable) Hořava gravity [7], and is strongly coupled on flat space [7–9].

It should be noticed, however, that including derivatives of the lapse (i.e. second time derivatives of the khronon scalar field) does not lead automatically to Ostrogradski ghosts, if the Lagrangian is degenerate. This fact is very well known in the context of Lorentz-symmetric scalar tensor theories, where it led first to beyond-Horndeski theories [10, 11] and then to degenerate higher-order scalar–tensor (DHOST) theories [12–14]. These theories have field equations that are higher than second order in time, but still propagate no ghosts.

In the following, we will apply this degeneracy program to the infrared (IR) limit of non-projectable (i.e. general) Hořava gravity. In that limit, the theory is sometimes referred to as khronometric theory, and while it still violates Lorentz symmetry (being only invariant under FDiffs in the unitary gauge), it is quadratic in both time and space derivatives (of the spatial metric and khronon) [3, 7]. We will show that khronometric theory can be modified to include second time derivatives of the khronon (i.e. time derivatives of the lapse), while still propagating no Ostrogradskii ghost. Although similar constructions were already obtained in [15–17], here we additionally show that the resulting theory is invariant under spatial diffeomorphisms and a special set of (monotonic) time reparametrizations (which will turn out to be given by ‘hyperbolic’ time compactifications). While invariance under this special group of transformations is sufficient to determine the form of the kinetic term for the lapse, it does not fix its coefficient unambiguously. Therefore, the radiative stability of the fine-tuning of the coupling constants needed to eliminate the ghost remains an open issue. We will comment on promising ways forward on this issue in the following.

As a result of this construction, the novel ‘degenerate Hořava gravity’ theory that we find does not have a limit to general relativity (GR). However, somewhat surprisingly, this does not prevent the theory from having the correct Newtonian limit, nor from reproducing (at least for specific values of the coupling constants) the dynamics of GR at the first
post-Newtonian (1PN) order and thus passing Solar System tests. However, the behavior on cosmological scales is wildly different from GR, at least at the background level (i.e. assuming isotropy and homogeneity). We will discuss this issue, its implications, and possible solutions below.

This paper is organized as follows. In section 2 we review Hořava gravity both in the unitary gauge and in the covariant (Stuckelberg) formalism. In section 3 we then consider theories invariant under a smaller gauge group, i.e. restricted foliation-preserving diffeomorphisms (RFDiffs), as an intermediate step toward section 4, where we introduce our degenerate generalization of Hořava gravity. We study its phenomenology in section 5, and we discuss our conclusions in section 6. We utilize units in which $c = 1$, except in appendix, where we discuss the PN expansion of degenerate Hořava gravity and we thus reintroduce $c$ as a book-keeping parameter.

2. Hořava gravity

In this section, we review the construction of Hořava gravity and the subgroup of the four-dimensional diffeomorphisms under which the theory is invariant. The distinction between space and time introduces a preferred frame, and thus a preferred time coordinate, which corresponds to endowing the space–time manifold with a preferred foliation by space-like surfaces. This means that the arbitrary reparameterization of time $t \rightarrow \tilde{t}(t,x)$ is not a symmetry of the theory anymore.

The basic ingredients to describe the space–time geometry are the spatial metric $\gamma_{ij}$, the shift $N^i$ and the lapse function $N$ entering the $3 + 1$ decomposition of the four-dimensional metric \[ ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \] (2.1)

The Hořava action is built from quantities invariant under the following unbroken symmetry, which is commonly referred to as FDiffs:

\[ x \rightarrow \tilde{x}(t,x), \quad t \rightarrow \tilde{t}(t), \] (2.2)

where $\tilde{t}(t)$ is a monotonic function of $t$. Notice that this is the largest possible unbroken gauge group that one can have, once a preferred foliation is introduced. Under this symmetry the fields in (2.1) transform as

\[ N \rightarrow \tilde{N} = N \frac{dt}{d\tilde{t}}, \quad N^i \rightarrow \tilde{N}^i = \left( N^i \frac{\partial \tilde{x}^j}{\partial x^i} - \frac{\partial \tilde{x}^j}{\partial t} \right) \frac{dt}{d\tilde{t}}, \quad \gamma_{ij} \rightarrow \tilde{\gamma}_{ij} = \gamma_{ij} \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j}. \] (2.3)

Up to dimension six, the action takes the form [1]

\[ S = \frac{1}{16\pi G} \int d^3x \sqrt{\gamma} \left[ (1 - \beta)K_{ij}K^{ij} - (1 + \lambda)K^2 + \alpha a^i a^i + (3)R - \mathcal{V} \right], \] (2.4)
where $K_{ij}$ is the extrinsic curvature of the surfaces of constant time

$$K_{ij} = \frac{1}{2N} \left( \dot{\gamma}_{ij} - D_i N_j - D_j N_i \right)$$

(with $D_i$ three-dimensional covariant derivatives compatible with the spatial metric $\gamma_{ij}$ and with an overdot denoting partial time derivatives), $K \equiv K_{ij} \gamma^{ij}$ is the trace of the extrinsic curvature, $(\gamma) R$ is the three-dimensional Ricci scalar and

$$a_i \equiv N^{-1} \partial_i N$$

is the acceleration vector. Besides the bare gravitational constant $G$, there are three free dimensionless constants: $\alpha$, $\beta$ and $\lambda$. The ‘potential’ $V$ depends on the three-dimensional Ricci tensor $(\gamma) R_{ij}$ and the acceleration $a_i$, with all possible operators of dimension four and six. This potential therefore involves only a finite number of these operators, which were fully classified e.g. in [20, 23]. While crucial for renormalizability, for our purposes the potential is completely irrelevant. Therefore, we omit to write explicitly its form here, and we focus on the IR limit of the theory (obtained by neglecting $V$) in the rest of this paper.

Also notice that the kinetic part of the action (i.e. the part where time derivatives appear) is fully contained in the first two terms of equation (2.4). In addition to the helicity-2 modes of the graviton, there is also a propagating scalar field, usually referred to as the ‘khronon’ [7]. This extra mode appears because the two first-class constraints (primary and secondary) associated with time diffeomorphisms in GR become here second-class constraints, because of the breaking of time diffeomorphisms. The theory therefore possesses six first-class constraints (associated with spatial diffeomorphisms) and two second-class constraints, leaving $[20 - (6 \times 2) - 2]/2 = 3$ dynamical degrees of freedom.

### 2.1. Stuckelberg formalism

This formalism allows one to single out explicitly the extra degree of freedom that appears because of the breaking of diffeomorphism invariance. It amounts to rewriting the action in a generally covariant form, at the expense of introducing a compensator field that transforms non-homogeneously under the broken part of the four-dimensional diffeomorphisms.

In more detail, one encodes the foliation structure of the space–time in a scalar field $\varphi$, such that the foliation surfaces are identified with those of constant $\varphi$. The action (2.4) then corresponds to the frame where the coordinate time coincides with $\varphi$ (i.e. $\varphi = t$). This choice of coordinates is often referred to as ‘unitary gauge’. The action in a generic frame is then obtained by performing the Stuckelberg transformation and reads

$$S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ R - \beta \nabla_\mu u^\nu \nabla_\nu u^\mu - \lambda \left( \nabla_\mu u^\mu \right)^2 + \alpha a_\mu a^\mu \right],$$

where

$$u_\mu = \frac{\partial_\mu \varphi}{\sqrt{-X}}, \quad X \equiv \partial_\mu \varphi \partial^\mu \varphi, \quad a^\mu \equiv u^\nu \nabla_\nu u^\mu,$$

and $R$ is the four-dimensional Ricci scalar. Notice that this is the action of Einstein–æther theory when the æther vector field is hypersurface orthogonal [24]. For later purposes, it is

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3 Note that our definition of $K_{ij}$ differs by an overall sign from the definition used in some textbooks (e.g. [18, 19]), although it agrees e.g. with [20–22].
convenient to write the action (2.7) explicitly in terms of the khronon field

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R + \beta \frac{\varphi_{\mu\nu} \varphi^{\mu\nu}}{X} + \lambda \frac{\Box \varphi}{X} - 2\lambda \frac{\Box \varphi \varphi_{\mu\nu} \varphi^{\mu\nu}}{X^2} \right. \]

\[ \left. + (\alpha - 2\beta) \frac{\varphi_{\mu} \varphi_{\nu} \varphi^{\mu\nu}}{X^2} + (\beta + \lambda - \alpha) \frac{\varphi_{\mu} \varphi_{\nu} \varphi_{\mu\nu} \varphi^{\mu\nu}}{X^3} \right], \quad (2.9) \]

where to avoid clutter we have introduced the notation \( \varphi_{\mu} \equiv \partial_{\mu} \varphi \) and \( \varphi_{\mu\nu} \equiv \nabla_{\nu} \partial_{\mu} \varphi \). Notice that the action above is invariant under reparameterizations of \( \varphi \),

\[ \varphi \rightarrow \tilde{\varphi} = f(\varphi), \quad (2.10) \]

where \( f \) is a (monotonic) arbitrary function. This reflects the invariance (in the unitary gauge) under the FDiffs (2.2).

Naively, the higher-order derivatives in the action (2.9) would suggest the presence of an Ostrogradski ghost in the theory. However, the counting of the degrees of freedom cannot be straightforwardly performed from the covariant action (2.9), because of the absence of a standard kinetic term for the khronon and the non-local \( 1/X \) dependence. However, one can easily perform the counting in the preferred frame, where \( \varphi \) cannot be constant and has a non-vanishing time profile \( \varphi = t \). In this frame, as can be seen in the unitary gauge action (2.4), the ghost mode is absent.

From the point of view of the covariant action (2.9), the absence of the Ostrogradski mode is guaranteed by the highly non-trivial tuning among the coefficients of the five operators in the action, which translates in the absence of \( \dot{N} \) terms in the unitary gauge action (2.4). Remarkably, this tuning is protected against radiative corrections by the reparameterization invariance (2.10). A detuning of the action coefficients would necessary break the symmetry (2.10), generate \( \dot{N} \) terms in the unitary gauge action, and generically reintroduce the ghost mode.

Finally, notice that the action (2.9) does not belong to any of the DHOST classes identified in [13, 25]. This is because when the full diffeomorphism invariance is broken, the degeneracy of the Hessian matrix of the velocities can be achieved in a less restrictive way. The full diffeomorphism invariant analog of action (2.9) is the Horndeski Lagrangian [26], which in the unitary gauge does not present \( \dot{N} \) terms.

3. RFDiff gravity

Noticeably, there exists also a smaller unbroken gauge group according to which we can construct our Lagrangian, namely the group of RFDiffs

\[ x \rightarrow \tilde{x}(t, x), \quad t \rightarrow \tilde{t} = t + \text{const}, \quad (3.1) \]

which differs from FDiffs because the invariance under general time reparametrizations is replaced by the invariance under time translations. In the Stuckelberg formulation, this symmetry of the khronon action reduces to the shift symmetry

\[ \varphi \rightarrow \tilde{\varphi} = \varphi + \text{const}, \quad (3.2) \]

which allows for a general dependence of the action on the derivatives of \( \varphi \).

Restricting the symmetry from FDiffs to RFDiffs therefore allows one to include in the action a kinetic term for the lapse \( N \). Moreover, all dimensionless couplings in the Lagrangian...
may now acquire an arbitrary dependence on $N$, and we can thus include in the potential $V$ a
generic function of $N$. The kinetic term for $N$ is fixed by the invariance under RFDiffs to be of the form

$$\left( \dot{N} - N^i \partial_i N \right)^2. \tag{3.3}$$

However, a general dependence of the action on this term inevitably leads to the propagation
of an additional ghost scalar degree of freedom [7]. In the Stuckelberg formulation, this is the
Ostrogradski mode associated with the higher derivatives of the khronon, which re-appears
because of the detuning of the coefficients of the action (2.9).

3.1. Degenerate RFDiff gravity and its downsides

It is well established that if the kinetic term (3.3) and the trace of the extrinsic curvature appear
in the Lagrangian in such a way as to enforce the existence of a primary constraint (which in turn generates a secondary constraint), then the ghost mode can be safely removed [12, 13, 27, 28]. The constraints structure—and so the number of degrees of freedom—becomes indeed the same as in Horava gravity, with the only difference that the second-class primary constraint given by the vanishing of the momentum conjugate to $N$ (i.e. $\partial L/\partial \dot{N} \approx 0$) is replaced by a linear combination of the momenta conjugate to $N$ and $\gamma$.

Therefore, it is possible to realize healthy theories within RFDiff gravity, provided that suitable degeneracy conditions are imposed. These models have been fully classified in [15–17], but they may not be very attractive for two reasons. First, the presence of arbitrary functions of $N$ in the Lagrangian results in an infinite number of coupling constants. Second, the degeneracy conditions are not protected by the RFDiff symmetry, so that radiative corrections will generically induce a detuning of the action and hence reintroduce the ghost. This is very different from Horava gravity, where the tuning of the action (2.9) is required by the FDiff symmetry and hence is protected by it.

4. Degenerate Horava gravity

In this section, we present a new class of gravity theories invariant under a symmetry intermediate between FDiff and RFDiff. In more detail, the transformation of time is restricted to take the form of a specific hyperbolic function $\tilde{t}(t)$, and the symmetry is realized up to a total derivative.

Our starting point is a generalization of the Horava action (2.4), which includes the time derivative of the lapse in the RFDiff invariant way (3.3). By introducing the definition

$$V \equiv -\frac{1}{N^2} \left( \dot{N} - N^i \partial_i N \right), \tag{4.1}$$

we write the action as

$$S = \frac{1}{16\pi G} \int d^3x \, dt \, N \sqrt{\gamma} \left[ \omega \, V^2 + 2 \, \sigma \, K \, V + (1 - \beta)K_{ij}K^{ij} \right.$$

$$\left. - (1 + \lambda)K^2 + \alpha \, a_i \, a^i \right) + \beta^{ij} R \right], \tag{4.2}$$

where $\omega$ and $\sigma$ are two additional dimensionless constants. Clearly, the first two terms in (4.2) break FDiff invariance, although they are RFDiff invariant. At this stage, two scalar degrees of freedom propagate, one of which is a ghost [7].
We can then impose the existence of a primary constraint by requiring that the determinant of the kinetic matrix of the two scalar modes in (4.2) vanishes [29–31]:

$$\det \begin{pmatrix} \omega & \sigma \\ \sigma & \lambda - \frac{\beta + 2}{3} \end{pmatrix} = 0.$$  \hfill (4.3)

In Hořava gravity, this condition is trivially realized since $\omega = \sigma = 0$, but a non-trivial solution is also possible and is given by

$$\omega = -\frac{3 \sigma^2}{3 \lambda + \beta + 2}.$$  \hfill (4.4)

To completely remove one degree of freedom, the primary constraint, enforced by the condition (4.4), must generate a secondary constraint. The conditions for this to happen were derived in complete generality for field theories in [31], and in the case at hand they are automatically satisfied because of the absence of the following couplings in the action:

$$V \partial_i N, \quad K \partial_i N, \quad (^{(3)}R \cdot V), \quad (^{(3)}R \cdot K).$$  \hfill (4.5)

Therefore, condition (4.4) is all that is needed to completely eliminate the ghost mode and remain with a single scalar field, the khronon.

Thus far, we have not made any progress with respect to degenerate RFDiff theories, of which the action (4.2)—with the condition (4.4) enforced—is a particular case. In fact, nothing prevents the couplings in the Lagrangian from being functions of $N$ and, more dangerously, quantum corrections from spoiling the condition (4.4). Ideally, our aim is to determine whether there exists a gauge group, smaller than the FDiffs one but larger than the RFDiffs one, that could protect the condition (4.4).

For this purpose, after imposing the condition (4.4), we transform the action (4.2) under the FDiffs (2.2) and obtain a new action, which now differs from the original one, because $V$ is not invariant. Indeed, using equation (2.3), we find that

$$V \rightarrow \tilde{V} = V - \frac{1}{N} \frac{d^2 t}{dt^2} \left( \frac{dt}{d\tilde{t}} \right)^2.$$  \hfill (4.6)

By requiring that the new terms generated by equation (4.6) in the action are a total derivative, one imposes that the equations of motion are invariant. In this way, we obtain a third-order differential equation for $\tilde{t}(t)$, which has a non-trivial solution only if the following condition is imposed on the coefficients of the Lagrangian

$$\sigma = -\lambda - \frac{\beta + 2}{3}.$$  \hfill (4.7)

In this case, there exists a unique family of solutions for $\tilde{t}(t)$ given by

$$\tilde{t}(t) = c_2 c_1 + t + c_3,$$  \hfill (4.8)

where $c_{1,2,3}$ are free integration constants. Being (4.8) a hyperbolic function, we refer to this family of time reparametrizations (together with spatial diffeomorphisms) as ‘hyperbolic-foliation-preserving Diffs’ (HFDiffs). Notice that equation (4.8) is a monotonic function on intervals not including its pole, which prevents violations of causality.
By inspection of the resulting Lagrangian, it is easy to realize that the conditions (4.4) and (4.7) force the kinetic term to be

\[(V + K)^2, \tag{4.9}\]

which is indeed invariant under HFDiffs up to a total derivative since

\[N\sqrt{\gamma}(V + K)^2 \rightarrow N\sqrt{\gamma}(V + K)^2 - 4 \left[ \partial_t \left( \sqrt{\gamma} \right) - \partial_i \left( \frac{\sqrt{\gamma}}{(c_1 + i)^N} \right) \right]. \tag{4.10}\]

However, being \(N\sqrt{\gamma}K^2\) FDiff invariant, if one starts from the generic Lagrangian (4.2), invariance under HFDiffs (up to a total derivative) only requires the kinetic term to be of the form \(V^2 + 2KV\), with an arbitrary coefficient in front. Therefore, this implies that radiative corrections may induce a running of the coefficients of the Lagrangian. This can potentially detune the degeneracy condition (4.4) and thus reintroduce the ghost.

A way out of this unappealing situation would be to enlarge the group of HFDiffs to a custodial symmetry sufficient to ensure stability of the degeneracy condition under quantum corrections. Such a gauge group must necessarily lie in between HFDiffs and full-fledged four-dimensional diffeomorphisms (since the latter would simply require the theory to be GR). Therefore, one might consider a specific class of time reparametrizations that are explicit functions of the spatial coordinate, e.g. \(\tilde{t}(t, x) = c_2/(c_1 + t) + c_3 + f(x)\), although we have been unable to identify a suitable function \(f(x)\) of the spatial coordinates.

To summarize, we have found a new unbroken gauge group that: (i) allows for a kinetic term for the lapse; (ii) avoids the presence of arbitrary functions of the lapse in the action; although (iii) it does not yet prevent the propagation of a ghost mode. These are the HFDiffs

\[x \rightarrow \tilde{x}(t, x), \quad t \rightarrow \tilde{t} = \frac{c_2}{c_1 + t} + c_3, \tag{4.11}\]

and the corresponding action is given by

\[S = \frac{1}{16\pi G} \int \frac{d^4x}{\sqrt{-g}} \left[ R + \beta \, \frac{\varphi^\mu \varphi^\nu}{X} + \lambda \frac{\Box \varphi^2}{X} + \frac{2(\beta + 2)}{3} \left( \Box \varphi \right)^2 \left( \varphi^\mu \right) \varphi^\nu \right]
+ (1 - \beta)K_{ij}K^{ij} - (1 + \lambda)K^2 + \alpha a_i a^i - \frac{3}{2} R \right]. \tag{4.12}\]

Comparing with the Ho\’rava action (2.4) we notice that, although there are two new operators, the number of coupling constants is the same. In the this new action, however, even when all couplings are set to zero, we do not recover the GR limit.

4.1. Stuckelberg formalism

It is now instructive to look at the new action (4.12) in the Stuckelberg formalism. As in section 2.1, we perform the Stuckelberg transformation and obtain

\[S = \frac{1}{16\pi G} \int \frac{d^4x}{\sqrt{-g}} \left[ R + \beta \, \frac{\varphi^\mu \varphi^\nu}{X} + \lambda \frac{\Box \varphi^2}{X} \right.
+ \frac{2(\beta + 2)}{3} \left( \Box \varphi \right)^2 \left( \varphi^\mu \right) \varphi^\nu
+ (\alpha - 2\beta) \frac{\varphi^\mu \varphi^\nu \varphi'^\mu \varphi'^\nu}{X^2} + \frac{2(2\beta - 3\alpha - 2)}{3} \left( \varphi^\mu \varphi^\nu \varphi'^\mu \varphi'^\nu \right)^2 \left. \right] . \tag{4.13}\]
Comparing with the khronon action for Hořava gravity, equation (2.9), we see that the coefficients of the third and fifth operators have changed. This new highly non-trivial tuning guarantees the absence of the Ostrogradski ghost at least at tree level. Moreover, the Lagrangian (4.13) is invariant under the transformation

\[ \varphi \to \tilde{\varphi} = \frac{c_2}{c_1 + \varphi} + c_3, \]  

up to the total derivative

\[ -4 \left( \lambda + \frac{\beta + 2}{3} \right) \nabla_\mu \left( \frac{1}{c_1 + \varphi} \partial^\mu \varphi \right). \]  

This reflects the invariance, up to a total derivative and in the unitary gauge, under the HFDiffs (4.11).

Finally, notice that also in this case, action (4.13) does not belong to any of the DHOST classes [13, 25]. Although the end result is the same as in the DHOST and beyond Horndeski constructions (i.e. higher order equations of motion without Ostrogradski ghost), there is no connection between those theories and ours. In fact, our theory is Lorentz-violating and the absence of ghosts is guaranteed precisely by the existence of a preferred foliation, where kinetic terms take their standard form and the non-local $1/X$ terms present in the Stuckelberg action disappear.

### 4.2. Conformal and disformal transformation

Looking at the form of the kinetic terms, equation (4.9), a natural question is whether the new theory is related to Hořava gravity by a conformal and/or disformal transformation. To check this, it is convenient to work in the Stuckelberg formalism, where these transformations read [32]

\[ \bar{g}_{\mu\nu} = \Omega(X)g_{\mu\nu} + \Gamma(X)\partial_\mu \varphi \partial_\nu \varphi, \]  

with $\Omega$ and $\Gamma$ free functions of $X$ only. (Notice that a $\varphi$ dependence would break even the shift symmetry and therefore it is not allowed.)

It is well known that Hořava gravity is invariant under the transformation given by [33]

\[ \Omega = 1, \quad \Gamma = \frac{\varsigma}{X}, \]  

where $\varsigma$ is a constant, provided the following rescaling of the coefficients

\[ \bar{\lambda} = \lambda + \varsigma (\lambda + 1), \quad \bar{\beta} = \beta + \varsigma (\beta - 1). \]  

A feature of the new theory is that it also enjoys this invariance for the very same choice of functions (equation (4.17)) and for the same rescaling of the coefficients (equation (4.18)). Moreover, any choice different from equation (4.17) would change the power of $X$ appearing in each of the operators of the Lagrangian, and therefore cannot connect the two theories.

As a consequence, a generic transformation of the form (4.16) cannot map Hořava gravity into its degenerate HFDiff generalization, and both theories are stable under the same transformation (4.17). Once again, the non-local form of $\Gamma$ in (4.17) is signaling that the two theories make sense only for $X \neq 0$, i.e. the khronon must always be timelike.
5. Phenomenology

To study the phenomenology of ‘degenerate’ HFDiff Hořava gravity, we first derive the field equations by varying the action (4.12) with respect to the spatial metric \( \gamma_{ij} \), the shift \( N_i \) and the lapse \( N \). We denote by \( D_t \) the covariant derivative defined with respect to \( \gamma_{ij} \), and \( D_t \equiv \partial_t - N_i D^i \). We also define the following quantities in terms of the variation of the matter action [21, 22]

\[
\mathcal{E} = -\frac{1}{\sqrt{\gamma}} \frac{\delta S_m}{\delta N} = N^2 T^{00},
\]

\[
\mathcal{J}^i = \frac{1}{\sqrt{\gamma}} \frac{\delta S_m}{\delta N^i} = N \left( T^{0i} + N^0 T^{00} \right),
\]

\[
T^{ij} = \frac{2}{N\sqrt{\gamma}} \frac{\delta S_m}{\delta \gamma_{ij}} = T^{ij} - N^i N^j T^{00},
\]

where \( T^{\mu\nu} \) is the four-dimensional matter energy–momentum tensor.

The variation with respect to \( N \) leads to

\[
\frac{(3)R}{1 - \beta} + \frac{\lambda + 1}{1 - \beta} K^2 - K_{ij}K^{ij} - \frac{\alpha(D_t N)(D^t N)}{(1 - \beta) N^2} - \frac{2\alpha D^t D_t N}{(1 - \beta) N} + \frac{2(2 + \beta + 3\lambda)}{3(1 - \beta)} \left[ K V + V^2 - K^2 - D_t \left( \frac{K + V}{N} \right) \right] = \frac{16\pi G \mathcal{E}}{(1 - \beta)c^4},
\]

which unlike in GR is not a constraint, but rather an evolution equation for \( N \) (cf the presence of both second-order space and time derivatives of \( N \)). Notice that in (non-degenerate) Hořava gravity\(^4\), this equation is instead an elliptic equation for \( N \), to be solved on each slice [21, 22]. In the covariant formalism, this equation becomes indeed (in both non-degenerate and degenerate Hořava gravity) the khronon evolution equation. Varying with respect to \( N_i \) one obtains the momentum constraint equation

\[
D_t \left[ \left( K^{ij} - \frac{\lambda + 1}{1 - \beta} \gamma^{ij} K \right) - \frac{(2 + \beta + 3\lambda)\gamma^{ij} V}{3(1 - \beta)} \right] = \frac{2\pi G \mathcal{J}^i}{(1 - \beta)c^4},
\]

while variation with respect to \( \gamma_{ij} \) yields the evolution equation

\[
\frac{1}{1 - \beta} \left[ \frac{(3)R}{1 - \beta} - \frac{1}{3} (3)R \right] + \frac{1}{N} D_t \left( K^{ij} - \frac{\lambda + 1}{1 - \beta} \gamma^{ij} - \frac{(2 + \beta + 3\lambda)}{3(1 - \beta)} V \gamma^{ij} \right) + \frac{2}{N} D_k \left( \gamma^{ij} \left( K^{jk} - \frac{\lambda + 1}{1 - \beta} K^{jk} \right) - \frac{(2 + \beta + 3\lambda)}{3(1 - \beta)} \gamma^{jk} \right) + 2K^{ik} K^{jk} - \frac{2\lambda + 1 + \beta}{1 - \beta} K K^{ij} = \frac{1}{2} \delta^{ij} \left( K_{ij} K^{ij} + \frac{\lambda + 1}{1 - \beta} K^2 \right)
\]

\(^4\)We often refer to the original Hořava gravity (equation (2.4)) as ‘non-degenerate’, in order to distinguish it from its degenerate extension (equation (4.12)). However, it should be by now well understood that even non-degenerate Hořava gravity is a degenerate theory that satisfies (although trivially) the degeneracy condition (4.3). This is even more evident from the tuning of the parameters in its khronometric formulation (2.9).
\[-\frac{1}{(1 - \beta)N} \left[ (D_i D_j N) - (D_i D_j N) \gamma^{ij} \right] + \frac{\alpha}{N^2 (1 - \beta)} \left[ (D_i N)(D_j N) - \frac{1}{2} (D_i N)(D_j N) \gamma^{ij} \right]
+ \frac{2(2 + \beta + 3\lambda)}{3(1 - \beta)} \left[ \frac{1}{4} V^2 \gamma^{ij} - K^{ij} V - \frac{K + V}{N^2} N^l (D_l N) \right] = \frac{8\pi G}{(1 - \beta)c^2} T^{ij}. \] (5.6)

5.1 Solar-System tests and gravitational-wave propagation

To check the experimental viability of the theory on Earth and in the Solar System, we perform a post-Newtonian expansion over flat space and compare to the parametrized PN metric (PPN) [34, 35]. This will allow us to extract the values of the PPN parameters in degenerate Horava gravity and compare them to their experimental bounds. The details of the calculation follow [22], which performs the same analysis for (non-degenerate) Horava gravity, and are presented in appendix. Here, we will simply summarize the main results.

We find that the only PPN parameters differing from GR are the preferred-frame parameters \(\alpha_1\) and \(\alpha_2\), which take the form
\[
\alpha_1 = \frac{4(\alpha - 2\beta)}{\beta - 1},
\]
\[
\alpha_2 = -\frac{(\alpha - 4\beta + 2)(3\alpha - 4\beta - 2)}{3(\alpha - 2)(\beta + \lambda)} + \frac{2(\alpha - 2)}{\beta - 1} + \frac{-27\alpha + 28\beta + 12\lambda + 38}{3(\alpha - 2)}. \] (5.8)

Experimental bounds on these parameters are \(|\alpha_1| \lesssim 10^{-4}\) and \(|\alpha_2| \lesssim 10^{-7}\) [35]. Comparing to their expressions in (non-degenerate) Horava gravity [22, 36], we see that while \(\alpha_1\) is unchanged, \(\alpha_2\) gets modified. In (non-degenerate) Horava gravity, \(\alpha_1\) and \(\alpha_2\) are both proportional to \(\alpha - 2\beta\), i.e. they are both small for \(\alpha \approx 2\beta\).

At this point, let us notice that constraints on the propagation speed of gravitational waves from GW170817 require \(|\beta| \lesssim 10^{-15}\) in (non-degenerate) Horava gravity [37, 38], as well as in the degenerate version of the theory that we are considering here. Indeed, the kinetic term of the tensor modes is given by \(K_{ij} K^{ij}\) and the spatial gradient is contained in \((^3 R)\), which gives (cf equations (2.4) and (4.12)) a gravitational-wave propagation speed \(c_{GW} = (1 - \beta)^{-1/2}\), which matches the speed of light only for \(\beta = 0\).

The Solar-System bound on \(\alpha_1\) then gives \(\alpha \approx \beta \approx 0\) in both non-degenerate [39, 40] and degenerate Horava gravity. For \(\alpha \approx \beta \approx 0\), equation (5.8) then yields
\[
\alpha_2 \approx -\frac{(1 + 2\lambda)(2 + 3\lambda)}{3\lambda}. \] (5.9)

The experimental constraint \(|\alpha_2| \lesssim 10^{-7}\) then selects \(\lambda \approx -1/2\) or \(\lambda \approx -2/3\). For the latter, the coefficient in front of \(V^2 + 2KV\) in the action disappears, i.e. one is left with the non-degenerate version of the theory. Therefore, there exists only one non-trivial set of parameters, namely \(\alpha \approx \beta \approx 0\) and \(\lambda \approx -1/2\), for which degenerate Horava gravity can satisfy Solar-System tests and the bound on the propagation speed of gravitational waves. For these values the action reads
\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \frac{(\Box \phi)^2}{X} \right. \\
+ \frac{4}{3} \frac{(\Box \phi \phi^{\mu} \phi_{\mu})^2}{X^2} - \left. \frac{2}{3} \frac{(\phi^{\mu} \phi_{\mu} \phi^2)^2}{X^3} \right] \]  
(5.10)

or, in the unitary gauge,

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-\gamma} \left[ -\frac{1}{6} (V^2 + 2K V) + K_{ij} K^{ij} - \frac{1}{2} K^2 + \frac{1}{2} R \right]. \]  
(5.11)

5.2. Cosmology

To test the behavior of the theory on cosmological scales, we assume a standard homogeneous and isotropic Robertson–Walker metric

\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \]  
(5.12)

where \( a(t) \) is the scale factor, and \( \gamma_{ij} = a^2(t) \delta_{ij} \). Notice that we have assumed flat spatial slices, but our conclusions are unchanged if we allow for curvature.

Replacing this ansatz in the field equations, the only non-trivial equations are provided by the khronon equation (5.4) and by the trace of the evolution equation (5.6), which give the system

\[ (2 + \beta + 3\lambda)(3H^2 + 2\dot{H}) = -16\pi G \rho, \]  
(5.13)

\[ (2 + \beta + 3\lambda)(3H^2 + 2\dot{H}) = -16\pi GP, \]  
(5.14)

with \( H = \dot{a}/a \) the Hubble rate, while \( \rho \) and \( P \) are the energy density and pressure of the cosmic matter.

As can be seen, this system is completely different from the Friedmann–Robertson–Walker equations of GR. This is no surprise since the theory does not reduce to GR for any values of the coupling constants. More worrisome is the fact that by taking the difference of the two equations, one obtains that the cosmic matter must necessarily have \( \rho = P \) (stiff fluid). In other words, the theory does not allow for the usual radiation and matter eras, nor for an early- or late-time accelerated expansion (even in the presence of a cosmological constant). As a curiosity, however, it is worth mentioning that if we set \( \rho = P \) in equations (5.13)–(5.14) and solve for \( H \), we find \( H(t) = 2/[3(t + C)] \), with \( C \) an integration constant. For \( C = 0 \) this reduces to the Hubble rate of the standard matter-dominated era, and is reminiscent of the appearance of dark matter as an integration constant in projectable Hořava gravity [41].

6. Conclusions

In this work, we have shown that it is possible to construct a novel khronometric theory with a dynamical lapse, which (via a degenerate Lagrangian) propagates only a graviton and a khronon. This theory is invariant under a special subgroup of the FDiff symmetry, equation (4.11), which we have referred to as HFDiffs. This new unbroken gauge group selects a specific kinetic term for the lapse (although it does not fix its overall coefficient), and it avoids an arbitrary dependence of the action on the lapse. HFDiffs are not sufficient by themselves to ensure stability of the degeneracy condition under radiative corrections, thus potentially letting the ghost re-appear beyond tree level. However, an enlarged gauge group lying between
HFDiffs and four-dimensional diffeomorphisms may protect the fine-tuning of the degeneracy condition. We have commented on this possibility above, and we will explore it further in future work.

Our construction has a two-fold interest for both phenomenology and theory. On the phenomenological side, it is a remarkable example of a theory which, despite being Lorentz breaking and not admitting a GR limit, does pass Earth-based, Solar-System and gravitational-wave tests, at least for a suitable choice of its coupling constants. Unfortunately, the theory fails to reproduce the standard Friedmann–Robertson–Walker cosmology and can provide an (effective) matter-dominated era only if the Universe contains stiff matter alone ($\rho = P$). However, while clearly the cosmology of the theory does not seem to work out of the box, a couple possibilities are worth mentioning. First, equations (5.13) and (5.14) assume a minimal coupling to matter. If matter is instead conformally coupled to gravity, it may be possible to obtain a matter-dominated era and a late-time accelerated expansion, although it would still be impossible to accommodate a radiation era and it might be tricky to pass Solar-System tests (at least in the absence of a screening mechanism protecting local scales from the conformal coupling). Second, and perhaps more importantly, since dark matter seems to arise naturally as an integration constant in our new theory, it may be worth trying to explain the observed late-time acceleration of the Universe in the context of non-standard cosmologies that violate the homogeneity/isotropy assumptions of the Robertson–Walker ansatz (see e.g. [42] for a review, and references therein).

On the theoretical side, if a custodial symmetry protecting the degeneracy condition is identified, this theory may provide a version of Hořava gravity that does not require the absence of time derivatives of the lapse to avoid ghosts, and hence may not present the same technical hurdles [5, 6] in proving perturbative renormalizability (beyond power counting) that one encounters in FDiff (i.e. non-degenerate) Hořava gravity (at least in its general non-projectable form).

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Data availability statement

No new data were created or analysed in this study.

Appendix. Post-Newtonian expansion

In order to calculate the PPN parameters, we follow [22] and consider a general perturbed flat metric in Cartesian coordinates ($x^0 = ct, x^i$).


g_{00} = -1 - \frac{2}{c^2} \phi - \frac{2}{c^4} \phi^{(2)} + \mathcal{O} \left( \frac{1}{c^6} \right),

g_{0i} = \frac{w_i}{c^2} + \frac{\partial_i \omega}{c^3} + \mathcal{O} \left( \frac{1}{c^5} \right),

g_{ij} = \left( 1 - \frac{2}{c^2} \psi \right) \delta_{ij} + \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \frac{\zeta}{c^2} + \mathcal{O} \left( \frac{1}{c^4} \right),

(A.1)

where under transformations of the spatial coordinates, \( \psi, \zeta, \omega, \phi, \phi^{(2)} \) transform as scalars, \( w_i, \zeta_j \) behave as transverse vectors (i.e. \( \partial_i w^i = \partial_i \zeta^i = 0 \)), and \( \zeta_{ij} \) is a transverse and traceless tensor (i.e. \( \partial_i \zeta_{ij} = \zeta_{ii} = 0 \)). Since we want to use this ansatz in the field equations in the unitary gauge (equations (5.4)–(5.6)), we are not allowed to perform a transformation of the time coordinate (which is fixed to coincide with the khronon), but we can perform a gauge transformation of the spatial coordinates to set \( \zeta = \zeta_i = 0 \) [22].

We supplement this ansatz with an expression for the energy–momentum tensor, which we assume to be given by the perfect fluid form



\[ T^{\mu\nu} = \left( \rho + \frac{P}{c^2} \right) u^\mu u^\nu + P g^{\mu\nu}, \]

(A.2)

where \( \rho \) is the matter energy density, \( P \) the pressure and \( u^\mu = dx^\mu/d\tau \) the four-velocity of the fluid elements (with \( \tau \) the proper time). In the following, we introduce a parameter \( \eta_{0,1} \) in the action

\[
S = \frac{1}{16\pi G} \int d^3x \, dt \, N \sqrt{\gamma} \left[ -\eta_{0,1} \left( \lambda + \frac{\beta + 2}{3} \right) (V^2 + 2 K V) + (1 - \beta) K_{ij} K^{ij} - (1 + \lambda) K^2 + \alpha a_i a^i + \mathcal{O} \left( \frac{1}{c^6} \right) \right],
\]

(A.3)

in order to distinguish between (non-degenerate) Hořava gravity (which corresponds to \( \eta_{0,1} = 0 \)) from its degenerate generalization (\( \eta_{0,1} = 1 \)).

Expanding the evolution equation (5.6) to lowest order in \( 1/c \), we find \( \zeta_{ij} = \mathcal{O}(1/c^2) \) from the off-diagonal part, while the trace gives

\[
\psi = \phi + \mathcal{O} \left( \frac{1}{c^2} \right).
\]

(A.4)

From this we can write

\[
\psi = \phi + \frac{\delta \psi}{c^2} + \mathcal{O} \left( \frac{1}{c^4} \right),
\]

(A.5)

which we can substitute in the other equations. Using this expression and expanding equation (5.4) to lowest order in \( 1/c \), we obtain the modified Poisson equation

\[
\nabla^2 \phi_N = 4\pi G_N \rho + \mathcal{O} \left( \frac{1}{c^2} \right),
\]

(A.6)

where we define the rescaled gravitational constant \( G_N = 2G/(2 - \alpha) \). Notice that \( G_N \) is the gravitational constant measurable by a local experiment, while \( G \) is merely the bare gravitational constant appearing in the action.
We can then expand the momentum constraint (5.5) to lowest order in $1/c$ to find the 1PN equation for the 'frame-dragging' potential $w_i$:

$$\nabla^2 w_i + 2 \left( \frac{\beta + \lambda}{\beta - 1} \right) \partial_i \nabla^2 \omega = \frac{16\pi G \rho v}{1 - \beta} \left( \frac{2(3 + \eta_{0,1})}{\beta - 1} \right) \partial_i \partial_t \phi. \quad (A.7)$$

This is the first place where one can see a modification with respect to (non-degenerate) Hořava gravity.

One can then expand both the trace of the evolution equation (5.6) and the krhronon equation (5.4) to next-to-leading order in $1/c$, obtaining respectively

$$2 \nabla^2 \delta \psi = -24\pi G p - 8\pi \rho v^2 - \left( 7 + \alpha \right) \partial_i \phi \partial_i \phi
- 8\phi \nabla^2 \phi + (2 + \beta + 3\lambda) \left( \partial_i \nabla^2 \omega + (3 + \eta_{0,1}) \partial_i^2 \phi \right), \quad (A.8)$$

and

$$\nabla \cdot \left[ \left( 1 - \frac{\alpha}{2} \right) \nabla \left( \phi + \frac{\phi_{(2)}}{c^2} \right) \right] = 4\pi G \rho + \frac{1}{c^2} \left( 8\pi G \rho v^2
+ 12\pi G p + (2 - \alpha) \nabla \phi \cdot \nabla \phi
- \frac{1}{6} (2 + \beta + 3\lambda) \left( (3 + \eta_{0,1}) \partial_i \nabla^2 \omega
+ (9 + 7\eta_{0,1}) \partial_i^2 \phi \right) \right). \quad (A.9)$$

Then, we use the same methods and notation described in appendix A of [22] and define the potentials

$$X(x, t) = G_N \int d^3x' \rho(x', t) |\vec{x} - \vec{x}'|, \quad (A.10)$$

$$V_i = G_N \int d^3x' \rho(x', t) v_i', \quad (A.11)$$

$$W_i = G_N \int d^3x' \frac{\rho(x', t) v_i'(x - x')}{|\vec{x} - \vec{x}'|}, \quad (A.12)$$

$$\Phi_1 = G_N \int d^3x' \rho(x', t) v^2', \quad (A.13)$$

$$\Phi_2 = \Phi(\vec{x}', t), \quad (A.14)$$

$$\Phi_4 = G_N \int d^3x' \frac{P(\vec{x}', t)}{|\vec{x} - \vec{x}'|}, \quad (A.15)$$

which obey the following relations

$$\nabla^2 X = -2\phi_N, \quad (A.16)$$

$$\nabla^2 V_i = -4\pi G_N \rho v_i, \quad (A.17)$$

$$\nabla^2 \Phi_1 = -4\pi G_N \rho v^2, \quad (A.18)$$

$$\nabla^2 \Phi_2 = 4\pi G_N \rho \phi_N, \quad (A.19)$$
We can then take the divergence of equation (A.7) and solve it for \( \omega \), obtaining

\[
\omega = \frac{3\alpha + 2\eta_{0,1} + (\beta + 3\lambda)(3 + \eta_{0,1})}{6(\beta + \lambda)} \partial_t X.
\]  

Replacing this solution again into equation (A.7), we obtain

\[
w_i = \frac{2 - \alpha}{\beta - 1} (V_i + W_i).
\]  

This allows us to evaluate \( g_{00} \) as

\[
g_{00} = \frac{w_i}{c^3} + \frac{\partial_t \omega}{c^3} + \mathcal{O} \left( \frac{1}{c^5} \right)
= -\frac{\eta_{0,1}(2 + 3\lambda - \beta) + \beta(\beta + 3\lambda)(3 + \eta_{0,1}) - 3\alpha(1 + \beta + 2\lambda) + 3\lambda + 9\beta W_i}{6(\beta - 1)(\beta + \lambda)}
+ \frac{\eta_{0,1}(2 + 3\lambda - \beta) - \beta(\beta + 3\lambda)(3 + \eta_{0,1}) + 3\alpha(1 - 3\beta - 2\lambda) + 21\lambda + 15\beta V_i}{6(\beta - 1)(\beta + \lambda)}
+ \mathcal{O} \left( \frac{1}{c^5} \right).
\]  

We can also solve equation (A.9) for \( \phi_{(2)} \)

\[
\phi_{(2)} = \phi_N^2 - 2 \Phi_1 - 2 \Phi_2 - 3 \Phi_4 + \frac{(3\alpha - 6\beta + \eta_{0,1}(\alpha - 6\beta - 4\lambda))(2 + \beta + 3\lambda)}{6(\alpha - 2)(\beta + \lambda)} \partial_t X.
\]  

While the solutions that we found completely describe the metric at 1PN order, in order to read off the PPN parameters one needs to transform the metric from the unitary gauge that we used for the calculation to the standard PN gauge \([22, 34, 35]\). We do that, following again \([22]\), by performing a gauge transformation \( t \rightarrow t + \delta t \), where we choose \( \delta t \propto \partial_t X \), with \( \eta_{0,1} \) appearing in the transformation. This finally yields

\[
g_{00} = -1 - \frac{2\phi_N}{c^2} - 2\frac{\phi_N^2}{c^4} + \frac{3\phi_1}{c^2} + \frac{4\phi_2}{c^4} + \frac{6\phi_4}{c^6} + \mathcal{O} \left( \frac{1}{c^8} \right),
\]  

\[
g_{0i} = -\frac{1}{2} (7 + \alpha_1 - \alpha_2) \frac{V_i}{c^3} - \frac{1}{2} (1 + \alpha_2) \frac{W_i}{c^5} + \mathcal{O} \left( \frac{1}{c^7} \right),
\]  

\[
g_{ij} = \left( 1 - 2 \frac{\phi_N}{c^2} \right) \delta_{ij} + \mathcal{O} \left( \frac{1}{c^4} \right),
\]  

from which we can read off the parameters \( \alpha_1 \) and \( \alpha_2 \).
\[ \alpha_1 = \frac{4(\alpha - 2\beta)}{\beta - 1}, \quad (A.31) \]
\[ \alpha_2 = \frac{2(1 - \alpha + 3\beta + 2\lambda)(2 + \beta + 3\lambda)}{3(\alpha - 2)(\beta + \lambda)} \]
\[ + \frac{(\alpha - 2\beta)(-\beta(3 + \beta + 3\lambda) - \lambda + \alpha(1 + \beta + 2\lambda))}{(\alpha - 2)(\beta - 1)(\beta + \lambda)}. \quad (A.32) \]

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