Generalized Calabi-Yau structures
and mirror symmetry

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ABSTRACT

We use the differential geometrical framework of generalized (almost) Calabi-Yau structures to reconsider the concept of mirror symmetry. It is shown that not only the metric and B-field but also the algebraic structures are uniquely mapped. As an example we use the six-torus as a trivial generalized Calabi-Yau 6-fold and an appropriate B-field.
1 Introduction

For a long time it has been believed that compactifications of string theory on compact six-dimensional spaces lead to a satisfactory model describing several important features of the real 4-dimensional world. Applying supersymmetry arguments to vacuum structure without fluxes, lead to internal manifolds having special holonomy. When one considers in addition to the metric, the $B$-field and the dilaton, the backgrounds can be characterized by the theory of $G$-structures.

The problem of having several, distinct but physically relevant superstring theories formulated in $\mathbb{M}^{1,9}$ was solved by duality maps. Moreover, using e.g. SCFT, it turned out that (in the flux free case) in particular the low-energy effective theories for IIA and IIB can be (mirror)dual, although the compactified background spaces are even topologically inequivalent. By using moduli space investigations of D-branes it was conjectured in [1] that the low-energy effective theories are mirror symmetric to each other if the Calabi Yau spaces are $T^3$ fibered and the action of T-duality on to these fibres is the mirror map. This means that firstly only a very restricted subspace of the huge moduli space of Calabi-Yau spaces is relevant (being compatible with the duality). Also considering the remaining (background) fields, no satisfactory mathematical formalism to combine the concept of $G$-structures and duality maps in a natural way has been found yet.

We consider this problem by using the concept of generalized complex spaces introduced by Hitchin in [2] and further developed by Gualtieri [3]. On the one hand, this concept provides us with a natural map, called the $B$-field transformation, where $B \in \Lambda^2 T^*$. On the other hand, we introduce a well-defined map $\mathcal{M}$ relating generalized Calabi-Yau spaces to each other which we conjecture therefore as a mirror map.

After reviewing the necessary definitions in section two, we apply the theory of generalized Calabi-Yau manifolds (for convenience only) to the simple example of $T^6$ in section three. Here we investigate the case without a $B$-field first. From a theory inherited $B$-field transformation, we introduce a $B$-field in a systematic way. Moreover,

\footnote{It is convenient to consider this $B$-field in the following as the physical background $B$-field}
we give an explicit description of the mirror symmetry map and derive the mirrored
generalized structures where we are able to rederive the Buscher rules as well as the
mirrored (classical) complex and symplectic structures. In section four we discuss
obvious generalizations.

Recent developments using generalized complex structures and related topics can
be found e.g. in [4, 5, 6, 7, 2, 8, 9, 10, 11, 12].

Note added. While preparing this paper we became aware of the independent
work done by Oren Ben-Bassat [9] which has some overlap with our results.

2 Preliminaries

The theory of generalized complex (and Calabi-Yau) structures was introduced by
Hitchin [2]. Recently, Gualtieri [3] introduced in his thesis the notation of generalized
Kähler structures where he discussed also integrability conditions and torsion. In
what follows we will stick (almost) to the definitions given there and find it usefull to
remember here the relevant concepts.

2.1 Basic definitions

Let $T$ be a 6-dim real vector space and $T^*$ its dual. Because of treating manifolds
we also use (in abuse of notation) the same symbols for the (co-)tangent bundle. By
introducing local coordinates and using the canonical basis for $T$ and $T^*$ we also have
the natural pairings:

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu, \quad \partial_\mu(dx^\nu) = \delta_\mu^\nu$$  \hspace{1cm} (1)

Using this fact we define the non-degenerate, symmetric bilinear form of signature $(6, 6)$
on the vector space $T \oplus T^*$ by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X))$$  \hspace{1cm} (2)

The group preserving this bilinear form and the orientation is the non-compact special
orthogonal group $SO(6, 6)$. The Lie algebra $\mathfrak{so}(6, 6)$ can be decomposed in a direct
sum of three terms. An element \( g \in \mathfrak{so}(6,6) \) is given by

\[
g = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}
\] (3)

Exponentiation gives three distinguished transformations: the diagonal embedded \( GL(T) \) action and the two shear transformations generated by \( B \in \Lambda^2 T^* \) and \( \beta \in \Lambda^2 T \). The \( B \)-transformation is given by

\[
\exp(B) = \begin{pmatrix} 1 \\ B \\ 1 \end{pmatrix}
\] (4)

which means that: \( \exp(B)(X + \xi) = X + \xi + X \Box B \). Correspondingly for the \( \beta \)-transformation we obtain

\[
\exp(\beta) = \begin{pmatrix} 1 & \beta \\ 1 \end{pmatrix}
\] (5)

where: \( \exp(\beta)(X + \xi) = X + \xi + \xi \Box \beta \).

### 2.2 Spinors and associated bilinear form

Let us act with \( X + \xi \in T \oplus T^* \) on \( \varphi \in \Lambda T^* \) by

\[
(X + \xi) \cdot \varphi = X \Box \varphi + \xi \wedge \varphi
\] (6)

and note that \( (X + \xi)^2 \cdot \varphi = (X + \xi)\varphi \), where we used (2). This defines the spin representation on the exterior algebra \( \Lambda T^* \). Taking the argument of dimension and signature into account the spin representation splits into two chiral irreducible parts \( S = S^+ \oplus S^- \). Furthermore, we consider elements of \( S^+(S^-) \) as even(odd) forms. In what follows we prefer the notation of “even” (“odd”) forms instead of using \( S^\pm \). Let us define on the spinor bundles \( S^\pm \) an invariant bilinear form

\[
(\cdot, \cdot) : S \otimes S \to \det T^*.
\] (7)

Because of dealing here with manifolds of real dimension \( n = 6 \) the bilinear form is skew-symmetric:

\[
\langle \varphi, \psi \rangle = (\sigma \varphi \wedge \psi)_{top}
\] (8)

where we used the anti-automorphism

\[
\sigma(\varphi_{2m}) = (-1)^m \varphi_{2m}, \quad \sigma(\varphi_{2m+1}) = (-1)^m \varphi_{2m+1}.
\] (9)
2.3 Purity

In this subsection we introduce (less familiar for physicists) the powerful tool of pure spinors. Let $L_\varphi \subset T \oplus T^*$ be defined by using the clifford multiplication such that

$$L_\varphi = \{ X + \xi \in T \oplus T^* | (X + \xi) \cdot \varphi = 0 \} \quad (10)$$

It can be easily checked that $L_\varphi$ is isotropic. Spinors with maximally isotropic associated annihilator $L_\varphi$ (or null space) are called pure, i.e. $\varphi$ is pure when $\dim(L_\varphi) = 6$.

The power of pure spinors come into play by using the fact that: Every maximal isotropic subspace of $T \oplus T^*$ is generated by a unique pure spinor line. Using the bilinear form on the spinor bundle we can distinguish two maximal isotropics $L_\varphi, L_\psi$:

$$L_\varphi \cap L_\psi = 0 \quad \Leftrightarrow \quad 0 \neq \langle \varphi, \psi \rangle = (\sigma \varphi \wedge \psi)_{top} \quad (11)$$

where $\varphi, \psi$ are pure spinors. Note that every maximal isotropic subspace $L$ of type $k$ has the form

$$L(E, \varepsilon) = \{ X + \xi \in E \oplus T^* : \xi|_E = \varepsilon(X) \} \quad (12)$$

where $E \subset T, \varepsilon \in \Lambda^2 T^*$ and $k$ is the codimension of its projection onto $T$. In what follows we only consider the special case when $\varepsilon = B$ and only introduce the extremal isotropics of lowest and highest type. It is possible to extend our previous facts by complexification to $(T \oplus T^*) \otimes \mathbb{C}$. This provides us also with the action of complex conjugation. We are now prepared to define one of the important working tools by which we construct later on generalized Calabi-Yau structures: The complex maximal isotropic subspace $L \subset (T \oplus T^*) \otimes \mathbb{C}$ ($E$ denotes its projection on $T \otimes \mathbb{C}$ with $\dim_{\mathbb{C}}(E) = 3 - k$) is defined by the complex spinor line $U_L \subset \Lambda(T^* \otimes \mathbb{C})$ which is generated by

$$\varphi_L = c \cdot \exp(B + i \omega) \theta_1 \wedge \ldots \wedge \theta_k, \quad (13)$$

where $c \in \mathbb{C}, (B + i \omega) \in \Lambda^2 (T^* \otimes \mathbb{C})$ and $\theta_i$ are linearly independent complex one-forms.

Note: If $\theta_1 \wedge \ldots \wedge \theta_k$ is pure then one can easily check that clifford multiplication by $\exp(B + i \omega)$ keeps the property of purity. But obviously we may single out different isotropics. The reader might now wonder if the real form $\omega \in \Lambda^2 T^*$ can be the symplectic form. The answer in our case is ‘yes’. Later on we define two spinors, one is of type $k = 0$ (symplectic type) and one is of type $k = 3$ (complex type). We do not need in this note non-extremal types.
2.4 Integrability

In the ordinary sense (at least in differential geometry) we call a structure integrable if smooth vector fields are closed under the Lie bracket. The structure of the Lie bracket, however, is invariant only under diffeomorphisms. The situation changes if we ask for integrability of smooth sections of \((T \oplus T^*) \otimes \mathbb{C}\). The answer of this question is the Courant bracket. Our main concern are smooth sections of maximal isotropic sub-bundles which are closed under this bracket. It is also shown that the Courant bracket is additionally invariant under \(B\)-transformations iff \(dB = 0\). Clifford multiplication of \(X + \xi \in T \oplus T^*\) on a spinor is a map taking \(\Lambda^{ev/od} \to \Lambda^{od/ev}\) (see (6)). Also the exterior derivative is such a map. There is the following correspondence between isotropic \(L\) being involutive (closed under Courant bracket) and smooth sections in the spin bundle:

\[ L_\rho \text{ is involutive } \iff \exists (X + \xi) \in C^\infty(T \oplus T^*) \otimes \mathbb{C} : d\rho = (X + \xi) \cdot \rho \quad (14) \]

for any local trivialization \(\rho\). We can also extend this definition by twisting the Courant bracket with a gerbe. Integrability forces the substitution of the ordinary differential operator \(d\) by the twisted differential operator \(d^H\):

\[ d \cdot \to d^H \cdot = d \cdot + H \wedge \cdot \quad (15) \]

where \(H \in \Lambda^3T^*\) is real and closed.

2.5 Generalized Calabi-Yau structures

Consider the natural indefinite metric \([2]\) on \(T \oplus T^*\). In what follows we are interested not only in involutive but additionally also in positive (negative) definite subspaces/subbundles, called \(C_{\pm}\). This forces the structure group to reduce globally to the maximal compact subgroup \(O(6) \times O(6)\). We get therefore the splitting \(T \oplus T^* = C_+ \oplus C_-\). This serves us to define the positive definite metric \(G\) on \(T \oplus T^*\). The metric \(G\) has the properties of being symmetric \((G^* = G)\) and squares to one \((G^2 = 1)\). (Note that \(G\) is an automorphism).

A generalized complex structure is an endomorphism \(\mathcal{J}\) on \(T \oplus T^*\) which commutes (is compatible) with \(G\). So \(C_\pm\) is stable under the action of \(\mathcal{J}\). It satisfies \(\mathcal{J}^2 = -1\)
and its dual $J^*$ is symplectic ($J^* = -J$). Using the properties of $G$ and $J$ we can define another generalized complex structure. Additionally, requiring that the two commuting generalized complex structures are integrable we have a generalized Kähler structure and $G$ is given by

$$G = -J_1 J_2.$$  

This reduces the structure group to $U(3) \times U(3)$.

A generalized Calabi Yau structure is a generalized Kähler structure with the following additional constraint for the generating spinor lines,

$$\left(\sigma \varphi_1 \wedge \bar{\varphi}_1\right)_{\text{top}} = c \cdot \left(\sigma \varphi_2 \wedge \bar{\varphi}_2\right)_{\text{top}} \quad \text{on each point}$$

where $c \in \mathbb{R}$. The property

$$\left(\sigma \varphi_{1/2} \wedge \bar{\varphi}_{1/2}\right)_{\text{top}} \neq 0$$

trivializes the determinant bundle of the maximal isotropic and thus reduces the structure group to $SU(3) \times SU(3)$.

## 3 Application

Let us use the powerful machinery introduced in the previous section to reproduce commonly known facts. Getting started, we choose the six-torus $T^6$ and the complex structure $J$, symplectic structure $\omega$ and metric $g$. We consider this manifold as a trivial fibration of $T^3 \hookrightarrow T^6$ over the base space $B = T^3$. Later on we investigate manifolds having also non-trivial $T^3$ fibrations (also a non-vanishing $B$-field). The basic idea is to embed the given structures into generalized ones and consider their behavior under a special map, the mirror symmetry map $\mathcal{M}$.

### 3.1 Warm up: $T^6$ without $B$-field

We start with $T^6$ and vanishing $B$-field. For later convenience we denote the usual structures by $(2 \times 2)$-blocks, strictly speaking, these denote the coordinate matrix of
the considered tensor respecting the base-fibre split of coordinates. In local coordinates we have
\[
g = \begin{pmatrix}
\delta_{ij} & 0 \\
0 & \delta_{\alpha\beta}
\end{pmatrix}, \quad g^{-1} = \begin{pmatrix}
\delta^{ij} & 0 \\
0 & \delta^{\alpha\beta}
\end{pmatrix}
\]
(19)
where \((y_i, x_\alpha) ((i, \alpha) \in \{1, 2, 3\})\) denotes the coordinates on the base \(\mathcal{B}\) and fibre \(\mathcal{F}\), respectively.

The complex and symplectic structures we use are therefore given by
\[
J = \begin{pmatrix}
\delta_{i\alpha} & -\delta_{i\alpha} \\
-\delta_{i\alpha} & \delta_{i\alpha}
\end{pmatrix}, \quad J^T = \begin{pmatrix}
\delta_{i\alpha} & \delta_{i\alpha} \\
-\delta_{i\alpha} & -\delta_{i\alpha}
\end{pmatrix}
\]
(20)
and
\[
\omega = \begin{pmatrix}
\delta_{\alpha i} & 0 \\
0 & \delta_{\alpha i}
\end{pmatrix}, \quad \omega^{-1} = \begin{pmatrix}
\delta^{\alpha i} & 0 \\
0 & \delta^{\alpha i}
\end{pmatrix}
\]
(21)
It is not difficult to check that the identities
\[
J\omega^{-1} = g^{-1} \quad \text{and} \quad J^T\omega = g
\]
(22)
hold. Let us embed these structures into the generalized structures \(\mathcal{J}_J, \mathcal{J}_\omega\) by:
\[
\mathcal{J}_J = \begin{pmatrix}
J & 0 \\
0 & -J^T
\end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix}
\omega & -\omega^{-1} \\
-\omega^{-1} & \omega
\end{pmatrix}
\]
(23)
which makes \(\mathcal{J}_J, \mathcal{J}_\omega\) into generalized complex structures. Using the above identities we can prove easily the identity:
\[
G = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix}
g & g^{-1}
\end{pmatrix}
\]
(24)
The triple \((\mathcal{J}_J, \mathcal{J}_\omega, G)\) provides us with a simple example of a generalized Kähler structure. The generating spinor lines are given by
\[
\varphi_1 = e^{i\omega} \Leftrightarrow \mathcal{J}_\omega, \quad \varphi_2 = \Omega^{(3,0)} \Leftrightarrow \mathcal{J}_J.
\]
(25)
Naturally, by introducing a trivialization and squaring the generating spinor lines according to \((17)\) we get global non-vanishing sections of the canonical bundle and furthermore
\[
\omega^3 = \frac{i3!}{2^3} \Omega \wedge \bar{\Omega}
\]
(26)
where $c = 1$. The sections are closed and thus reduces the structure group to $SU(3) \times SU(3)$, but obviously the simple example of $T^6$ has structure group the identity, thus, a trivial generalized Calabi-Yau space.

This fixes our setup in the generalized sense and we are now prepared to define a map $\mathcal{M}$ which assigns to the generalized Calabi-Yau structure an other generalized Calabi-Yau structure. By specializing only on spaces which are $T^3$ fibrations over the base $\mathcal{B}$ and acting only on the fibre we will call this map $\mathcal{M}$ a mirror symmetry map. This map is an isomorphism of the bundle $T \oplus T^*$ and also maps the triple $(\mathcal{J}_J, \mathcal{J}_\omega, G)$ in a well defined way. This means that on the mirror side this triple is completely fixed.

Let the mirror map $\mathcal{M} : T \oplus T^* \rightarrow T \oplus T^*$ be given by

$$\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

where we distinguish the vielbeins (as above) in $T$ and $T^*$, more precisely,

$$\mathcal{M} : T_B \oplus T_F \oplus T_B^* \oplus T_F^* \rightarrow T_B \oplus T_F^* \oplus T_B^* \oplus T_F$$

This is simply a map acting on the base as the identity and on the fibre as a “flip”. Note: The identity maps in $\mathcal{M}$ are tensors of adequate type to make $\mathcal{M}$ a well defined isomorphism. The property $\mathcal{M} = \mathcal{M}^{-1}$ makes the mirror map an involution, $\mathcal{M}^2 = 1$.

The action of the mirror map on a generalized structure $(\mathcal{J}_J, \mathcal{J}_\omega, G)$ is defined by,

$$(\hat{\mathcal{J}}_J, \hat{\mathcal{J}}_\omega, \hat{G}) = \mathcal{M} (\mathcal{J}_J, \mathcal{J}_\omega, G) \mathcal{M}^{-1}.$$  

Let us act now by $\mathcal{M}$ on the generalized structures defining the $T^6$. We obtain the mirror metric $\hat{G}$ by (because of $\mathcal{M} = \mathcal{M}^{-1}$ we abuse notation and write only $\mathcal{M}$),

$$\hat{G} = \mathcal{M} G \mathcal{M} = \begin{pmatrix} \delta^{ij} & \delta_{ij} \\ \delta_{ij} & \delta^{\alpha\beta} \end{pmatrix}.$$
The mirror metric $\hat{g}$ on the tangent bundle is therefore given by

$$\hat{g} = \begin{pmatrix} \delta_{ij} & -\delta^\alpha_i \\ -\delta_i^a & \delta_i^a \end{pmatrix},$$

where we do have now the inverse metric in the trivial fibre which is expected by using the Buscher rules. By a similar mirror transformation of $(\mathcal{J}, \mathcal{J}_\omega)$ we get

$$(\hat{\mathcal{J}}, \hat{\mathcal{J}}_\omega) = (\mathcal{M} \mathcal{J}, \mathcal{M} \mathcal{J}_\omega, \mathcal{M} \mathcal{J} \mathcal{M}),$$

where we obtain

$$\hat{\mathcal{J}} = \begin{pmatrix} \delta^{\alpha i} \\ -\delta^a_i & \delta^a_i \end{pmatrix}, \quad \hat{\mathcal{J}}_\omega = \begin{pmatrix} \delta^i_\alpha & -\delta^i_\alpha \\ -\delta^a_i & \delta^a_i \end{pmatrix}. \quad (33)$$

Considering these structures as an embedding of usual complex and symplectic structures (acting in the tangent bundle) we see immediately that $\hat{\mathcal{J}}$ is of pure symplectic type ($k = 0$) while $\hat{\mathcal{J}}_\omega$ is of pure complex type ($k = 3$). This makes the mirrored structures of generalized type and agrees with the literature (see e.g. [12][9]) that on the mirror space the (generalized) algebraic structures are interchanged:

$$\varphi_1 = e^{i\omega} \leftrightarrow \hat{\varphi}_1 = \Omega \quad \varphi_2 = \Omega \leftrightarrow \hat{\varphi}_2 = e^{i\omega} \quad g_B + g_F \leftrightarrow g_B + g_F^{-1}$$

trivial GCY $\leftrightarrow$ trivial GCY. \quad (34)

We verify (for $T^6$) therefore the work of [12] (see also [9]). But there the authors must introduce the Buscher rules by hand, in contrast to the mappings above, where these rules are already included.

### 3.2 $B$-field transform of $T^6$

Next in our line is the introduction of a globally defined and flat $B$-field. We are considering only closed $B$-fields because such a transformation of a pure spinor will not affect the integrability. We only single out different involutive $C_\pm$. 

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We do not consider general $B$-fields within this article. Let $B = B_{i\alpha}dy^i \wedge dx^\alpha$ be the $B$-transformation of our interest:

$$
e^B = \begin{pmatrix} 1 & 1 & B_{\alpha i} \\ B_{i\alpha} & 1 \end{pmatrix}, \quad e^{-B} = \begin{pmatrix} 1 & 1 \\ -B_{\alpha i} & 1 \end{pmatrix} \quad (35)$$

The $B$-transformed metric $G$ of the $T^6$ can be computed by

$$G^B = e^B G e^{-B} \quad (36)$$

but we suppress here the explicit form. By a following mirror transformation of this metric we obtain

$$\hat{G}^B = \mathcal{M}G^B \mathcal{M}, \quad (37)$$

which is completely off-diagonal. It is not hard to proof that it is again a generalized metric and more important of pure Riemannian type. Or in other words $\hat{G}^B$ is a “pure” Riemannian metric (not $B$-transformed) and thus can be constructed by embedding of only a Riemannian metric $\hat{g}$,

$$\hat{g} = \begin{pmatrix} g_B - B g_F^{-1} B & B g_F^{-1} \\ -g_F^{-1} B & g_F^{-1} \end{pmatrix} \quad (38)$$

where $g_B = \delta_{ij}$ is the metric in the base $B$ and $g_F = \delta_{\alpha\beta}$ denotes the metric in the fibre $F$. These transformation rules do verify the Buscher rules exactly and means that on the mirror side there is no $\hat{B}$-field and the “old” one is completely absorbed in the metric $\hat{G}$.

**Example 1** Let $B = (B_{i\alpha}dy^i) \wedge dx^\alpha$ (in local coordinates) only depends on coordinates $y$ on the base. The mirror metric $\hat{g}$ has the shape of the following form

$$\hat{g} = (g_B)_{ij}dy^i dy^j + (g_F^{-1})_{\alpha\beta}(dx^\alpha + A^\alpha)(dx^\beta + A^\beta) \quad (39)$$

where $A$ is a local, flat connection-one-form. Thus, the $B$-field transforms into the metric and re-appears in a non-trivial fibration.
Moreover, because of $\mathcal{M}$ being an isomorphism and an involution we can reverse the procedure. The initial data, a flat non-trivial fibration and vanishing $B$-field, mirrors in a trivial fibred $T^6$ with non-trivial $B$-field. Naturally, the combination of both “effects” is possible and independent of each other.

The $B$-field transformation of the generalized complex structures $(\mathcal{J}_J, \mathcal{J}_\omega)$ can be calculated by

$$\left(\mathcal{J}_J^B, \mathcal{J}_\omega^B\right) = \left(e^B \mathcal{J}_J e^{-B}, e^B \mathcal{J}_\omega e^{-B}\right) \tag{40}$$

which are no longer diagonal/off-diagonal, respectively. An action of the mirror map $\mathcal{M}$ on these is given by

$$\left(\hat{\mathcal{J}}_J, \hat{\mathcal{J}}_\omega\right) = \left(\mathcal{M} \mathcal{J}_J^B \mathcal{M}, \mathcal{M} \mathcal{J}_\omega^B \mathcal{M}\right). \tag{41}$$

It can also be shown that the mirror structures are generalized complex structures and become again completely off-diagonal/diagonal, or equivalently, structures of pure symplectic and complex type, respectively (see also [9]). The embedded structures are given in components by

$$\hat{\mathcal{J}} = \begin{pmatrix} \delta B & -\delta \\ B\delta^{-1}B + \delta & -B\delta^{-1} \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} B\delta + \delta B & -\delta \\ \delta & 0 \end{pmatrix} \tag{42}$$

We used a condensed notation where $\delta$ denotes the appropriate tensors (see (20), (21)) and we also suppressed here the identity maps coming from $\mathcal{M}$. Note: The object $\hat{\mathcal{J}}$ is embedded in $\hat{\mathcal{J}}_\omega$ (which makes it a complex structure (type $k = 3$)) and should therefore not mixed up with the subscript $\omega$ which denotes the symplectic form on the $T^6$ by which we started with.

## 4 Generalizations

The concepts that were introduced in the previous section (see also [9]) hold in more generality, and we only applied it to a simple example to make the mappings of the generalized structures clear. So one can immediately use the framework for more complicated generalized (almost) Calabi-Yau structures, being torsion-full and non-integrable in general. For example if the base is no longer $T^3$ and is a more general
3-dimensional manifold. Naturally, some base allow for non-trivial fibrations of $T^3$ (or even more general (S)LAG fibrations). The $B$-field can be treated in more general sense as well by dropping the restrictions and moreover the flatness assumption.

The previous discussion means we have to discuss integrability (and torsion) in more detail. Focusing on the above considered cases, obviously, these are integrable and torsion-less and so is its mirror. The case of $NS$-fluxes is already worked out in the amazing paper of [12] where the authors also used the concept of generalized complex structures but they had to introduce the Buscher rules by hand. There it was shown how torsion-full 6-manifolds are interchanged, and the authors gave a precise description of each $SU(3)$ torsion component. So it must be possible to recalculate these results by the concepts given in this article and by dropping the above assumed restrictions (made only for convenience).

Relating the introduced notation with that in physics makes it immediately clear that the above $B$-field is defined globally. But the physical meaning of a $B$-field is actually different. There a $B$-field is only defined patchwise and serves as a connection of a 1-gerbe $H$ which takes values in $H^3(M^6, \mathbb{Z})$ (see also [7]). In the language of branes, this corresponds to topologically inequivalent embeddings of branes carrying $NS$-flux. Furthermore, if also D-branes, SUSY-compatible submanifolds carrying vector-bundles charged under K-theory, are present, the $NS$-flux serves as a twist. This means that it is more close to physics to bring the physical $B$-field via the twisted differential operator (15) into play. In the case of $T^6$ it becomes immediately clear why it is not possible to have $dB = H \neq 0$. Using once more generalized structures, D-branes should (actually) be treated like generalized submanifolds (see e.g. [5, 6, 8, 9] for definitions and recent developments).

**Acknowledgments**

The author would thank F. Gmeiner, D. Lüst and S. Stieberger for very valuable discussions as well as J. Babington and I. Runkel for proofreading the draft version. Special thanks go to F. Witt for very usefull correspondence, mathematical support and proofreading.
The work of C.J. is supported by a Graduiertenkolleg grant of the DFG (The Standard Model of Particle Physics - structure, precision tests and extensions).

References

[1] A. Strominger, S.-T. Yau, and E. Zaslow, “Mirror symmetry is T-duality,” *Nucl. Phys. B479* (1996) 243–259, hep-th/9606040.

[2] N. Hitchin, “Generalized Calabi-Yau manifolds,” *Q. J. Math. 54 no.3* (2003) 281–308, math.dg/0209099.

[3] M. Gualtieri, *Generalized complex geometry*. PhD thesis, Oxford University, 2004. math.dg/0401221.

[4] U. Lindstrom, R. Minasian, A. Tomasiello, and M. Zabzine, “Generalized complex manifolds and supersymmetry,” hep-th/0405085.

[5] A. Kapustin, “Topological strings on noncommutative manifolds,” hep-th/0310057.

[6] M. Zabzine, “Geometry of D-branes for general $N = (2, 2)$ sigma models,” hep-th/0405240.

[7] N. Hitchin, “Lectures on special Lagrangian submanifolds,” in *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, A. Stud. Adv. Math., ed., vol. 23, pp. 151–182. Amer. Math. Soc., Providence, RI, 2001. math.dg/9907034.

[8] O. Ben-Bassat and M. Boyarchenko, “Submanifolds of generalized complex manifolds,” math.dg/0309013.

[9] O. Ben-Bassat, “Mirror symmetry and generalized complex manifolds,” math.ag/0405303.

[10] S. F. Hassan, “$O(d, d : R)$ deformations of complex structures and extended world sheet supersymmetry,” *Nucl. Phys. B454* (1995) 86–102, hep-th/9408060.
[11] S. F. Hassan, “$SO(d,d)$ transformations of Ramond-Ramond fields and space-time spinors,” *Nucl. Phys.* B**583** (2000) 431–453, [hep-th/9912236](https://arxiv.org/abs/hep-th/9912236).

[12] S. Fidanza, R. Minasian, and A. Tomasiello, “Mirror symmetric $SU(3)$-structure manifolds with NS fluxes,” [hep-th/0311122](https://arxiv.org/abs/hep-th/0311122).