SMOOTH STATIONARY STOCHASTIC PROCESSES WITH
POLYNOMIAL CONDITIONAL MOMENTS

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Abstract. We are studying stationary random processes with conditional polynomial moments that allow a continuous path modification. Processes with continuous path modification are important because they are relatively easy to simulate. One does not have to care about the distribution of their jumps which is always difficult to find. Among them are the Ornstein-Uhlenbeck process, the Gamma process, the process with Arcsin margins and the Theta function transition densities and others. We give a simple criterion for the stationary process to have a continuous path modification expressed in terms of skewness and excess kurtosis of the marginal distribution.

1. Introduction

The paper is organized as follows. First, we fix notation, then we recall some facts from the theory stochastic processes and the theory of probability. In particular, we recall the notion of the stationary stochastic process with polynomials conditional moments, the main object of the research presented in this paper. Then we formulate some necessary conditions, expressed in terms of the first four moments of the marginal distribution that allow continuous path modification of the process with this given marginal. Finally, we present examples of such stationary processes that allow continuous path modification. These include Gaussian, Gamma, Laplace and Arcsine processes. The name of the process refers naturally, to the name of the marginal distribution.

2. Notation and the basic facts

Let us start with the following remarks concerning notation. We will be considering only probability measures, that is, nonnegative measures that integrate over their supports to 1. Moreover, the integrals with respect to such measure μ, will be exchangeably denoted either traditionally as \( \int f(x)d\mu(x) \) or as \( E_f \). Here we assume that the random variable \( X \) has the so-called distribution \( \mu \) that is, more precisely i.e. \( P(X \leq x) = \mu((-\infty, x]) = \int_{-\infty}^{x} d\mu(x) \). In the above-mentioned formulae, \( f : \text{supp} X \rightarrow \mathbb{R} \) denoted a \( \mu \)-measurable function. Since we will be considering Markov stochastic processes the majority of measures considered will be at most 2-dimensional. Moreover, we will be often using the so-called tower
property in integration with respect to such 2-dimensional measure. Namely, if \( X = (Y_1, Y_2) \) then

\[
Ef(Y_1, Y_2) = \int_{\text{supp}(X)} f(y_1, y_2) d\mu(y_1, y_2)
= \int_{\text{supp}(Y_2)} \int_{\text{supp}(Y_1)} f(y_1, \cdot) d\mu_{Y_1|Y_2}(y_1|y_2) d\lambda_{Y_2}(y_2) = E(Ef(Y_1, Y_2)|Y_2)).
\]

Here \( \lambda_{Y_2}(y_2) \) denotes a marginal measure of the random variable \( Y_2 \) and \( d\mu_{Y_1|Y_2}(y_1|y_2) \) denotes the so-called conditional measure of \( Y_1 \) given \( Y_2 = y_2 \). The existence of such a measure is guaranteed by the theory of measure at almost every point of \( \text{supp} Y_2 \) mod \( d\lambda_{Y_2} \). Moreover, \( Ef(Y_1, Y_2)|Y_2) \) denotes the so-called conditional expectation of a random variable \( f(Y_1, Y_2) \) given \( Y_2 \).

There were four incentives to write this paper.

The first one is the so-called continuity Kolmogorov Theorem that reads the following:

**Theorem 1.** Let \((S, d)\) be a metric space and let \( X : [0, \infty) \times \Omega \rightarrow S \) be a stochastic process. Suppose, that for all \( T > 0 \) there exist 3 positive constants \( \alpha, \beta, K \) such that \( \forall 0 \leq s, t \leq T : \)

\[
Ed^\gamma (X_s, X_t) \leq K |s - t|^{1+\beta}.
\]

Then, there exists a modification \( \hat{X} \) of \( X \) that has continuous paths, \( \forall t \geq 0 : P(X_t = \hat{X}_t) = 1 \). Moreover, every path of \( \hat{X} \) is \( \delta \)-Hölder, for \( \delta \in (0, 1) \).

Note, that by a continuous mapping of time \( s : [0, \infty) \rightarrow \mathbb{R} \), like, for example, \( s(x) = \log x, x \geq 0 \), that doesn’t affect the continuity of the paths of the stochastic process, we can extend the formulation of the above-mentioned theorem to the stochastic process defined for all real \( t \).

Another incentive for writing this paper is the following auxiliary result:

**Lemma 1.** Let \((X, Y)\) have bivariate Gaussian distribution \( N(0, 0; 1, 1, \rho) \), that is \( EX = EY = 0, EX^2 = EV^2 = 1, EXY = \rho \).

Then

\[
E(X - Y)^2 = \frac{(2k)^k}{k!}(1 - \rho)^k.
\]

**Proof.** We have

\[
E(X - Y)^2k = E(X - \rho Y - (1 - \rho)Y)^2k = \\
\sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1 - \rho)^{2k-j} E(X - \rho Y)^j Y^{2k-j} = \\
\sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1 - \rho)^{2k-j} E(E(X - \rho Y)^j|Y) Y^{2k-j}.
\]

Now, we know that random variables \((X - \rho Y)\) and \( Y \) are independent, hence

\[
(E(X - \rho Y)^j|Y) = \begin{cases} 
0 & \text{if } j \text{ is odd} \\
1 & \text{if } j = 0 \\
(2j-1)!!(1 - \rho^2)^{j/2} & \text{if } j \text{ is even}
\end{cases}
\]
and \( EY^j = \begin{cases} 
0 & \text{if } j \text{ is odd} \\
1 & \text{if } j = 0 \\
(2j - 1)!! & \text{if } j \text{ is even} 
\end{cases} \). Thus we have

\[
E(X - Y)^{2k} = \sum_{j=0}^{k} \binom{2k}{2j} (1 - \rho)^{2k - 2j} (1 - \rho^2)^j C_j C_{k-j},
\]

where we denoted \( C_n = \begin{cases} 
1 & \text{if } n = 0 \\
(2n - 1)!! & \text{if } n > 0 
\end{cases} \). Setting \((-1)!! = (0)!! = 1\), we get:

\[
\binom{2k}{2j} C_j C_{k-j} = \frac{(2k)!}{(2j)!(2k - 2j)!} \frac{(2j - 1)!!(2k - 2j - 1)!!}{(k-j)!}.
\]

Summarizing we get:

\[
E(X - Y)^{2k} = \frac{(2k)!}{k!2^k} (1 - \rho)^k \sum_{j=0}^{k} \binom{k}{j} (1 + \rho)^j (1 - \rho)^{k-j} = \frac{(2k)!}{k!} (1 - \rho)^k.
\]

Now, keeping in mind, that the Ornstein-Uhlenbeck process is the stationary Gaussian process with \( 2\)-dimensional density \( N(0, 0; 1, 1, \rho) \) with \( \rho = \exp(-\alpha t) \), for some positive \( \alpha \), we see that

\[
E(X_\tau - X_{\tau+t})^{2k} = \frac{(2k)!}{k!} (1 - \exp(-\alpha t))^k \equiv F_k(|t|^k).
\]

Thus, the Ornstein-Uhlenbeck process allows modification with the continuous path of the \( \gamma \)-Hölder class with \( \gamma < \inf_{k} \frac{k}{2k} = 1/2 \).

The third incentive for writing this paper were the following two results. The first one is the so-called formula for the expansion of the \( 2\)-dimensional distribution \( dG(x, y) \) of, say \( (X_\tau, X_{\tau+t}) \), where \( X_\tau \) and \( X_{\tau+t} \) belong to some normalized Ornstein-Uhlenbeck process. Namely, we have the following Lancaster-type expansion

(2.1) \[
dG(x, y) = \frac{1}{2\pi} \exp\left(-\frac{x^2 + y^2}{2}\right) \sum_{j=0}^{\infty} H_j(x) H_j(y) \exp(-j\alpha t) / j! dx dy,
\]

where \( H_j(x)/\sqrt{j!} \) are orthonormal with respect to the measure with the density \( \sqrt{2\pi} \exp(-x^2/2) \). Now, we can easily deduce that this example has a nice feature that for every \( k \), conditions for having a continuous path modification are expressed in terms of moments of the marginal distribution. One knows that \( H_j \) are the so-called monic Hermite polynomials of the probabilistic type.

The fourth result that spurred to write this paper is slightly more complicated and requires a small introduction. But is basically simple and concerns, so to say, a generalization of expansion (2.1).

As stated above, in this paper we will be examining the path continuity of the processes defined on the whole real line. But out of all defined so stochastic processes, we will confine our considerations to stationary stochastic processes additionally having the property of possessing polynomial conditional moments. The class of Markov stochastic processes having polynomial conditional moments has
been described and analyzed in the series of papers [6], [7] and [5]. Recently yet another property of such class of stochastic processes has been added. Namely, in [8] it has been shown that under some regularity conditions, the two-dimensional distributions of a Markov stochastic process with the property that all its conditional moments are polynomials of the conditional random variable, must be of the Lancaster-type, that will be explained and defined below. The example of a Lancaster-type distribution is given by (2.1), above.

Since there are several points in fixing notation and exposing necessary assumptions that will enable necessary regularity, let’s present them first.

Let \( X = (X_t)_{t \in \mathbb{R}} \) be a real stochastic process defined on some probability space \((\Omega, \mathcal{F}, P)\). By the stationary Markov processes we mean those Markov processes \( X = (X_t)_{t \in \mathbb{R}} \) that have marginal distributions, that do not depend on the time parameter and the property that the conditional distributions of say \( X_t \) given \( X_s \) does depend only on \( t - s \).

We will assume that \( \forall n \in \mathbb{N}, t \in \mathbb{R} : E|X_t|^n < \infty \). More precisely, we assume that the distributions of \( X_t \) will be identifiable by their moments. This assumption is a slightly stronger assumption than the existence of all moments. For example, it is known that if \( \exists \beta > 0 : \int \exp(\beta |x|) d\mu(x) < \infty \), then measure \( \mu \) is identifiable by its moments. Here \( \mu \) denotes the distribution of \( X_0 \). In fact, there exist other conditions assuring this. For details see e.g. [11].

In [7] one considers the general case of the cardinality of \( \text{supp} \mu \). But in order to avoid unnecessary complications, we will confine ourselves to the infinite numbers of points of the set \( \text{supp} \mu \).

To fix further notation, let us denote \( \mathcal{F}^{\leq s} = \sigma(X_r : r \in (-\infty, s] \cap \mathbb{R}) \), \( \mathcal{F}^{\geq s} = \sigma(X_r : r \in [s, \infty) \cap \mathbb{R}) \) and \( \mathcal{F}_{s,u} = \sigma(X_r : r \notin (s, u), r \in \mathbb{R}) \).

Let us also denote by \( \mu(.) \) and by \( \eta(., y, \tau) \) respectively marginal stationary distribution and transition distribution of our Markov process. That is \( P(X_t \in A) = \int_A \mu(dx) \) and \( P(X_{t+\tau} \in A | X_t = y) = \int_A \eta(dx | y, \tau) \). Stationarity of \( X \) means thus that \( \forall T \ni \tau \neq 0, B \in \mathcal{B} \) (\( \mathcal{B} \) denotes here Borel \( \sigma \)-field)

\[
\mu(B) = \int \eta(B | y, \tau) \mu(dy).
\]

By \( L_2(\mu) \) let us denote the space spanned by the real functions that are square-integrable (more precisely by the set of equivalence classes) with respect to \( \mu \) i.e.

\[
L_2(\mu) = \{ f : \mathbb{R} \rightarrow \mathbb{R}, \int |f|^2 d\mu < \infty \}.
\]

Our assumption on the existence of all moments of \( X_0 \) in terms of \( L_2(\mu) \) implies that there exists a set of orthogonal polynomials that constitute the orthogonal base of this space. Let us denote these polynomials by \( \{ h_n \}_{n \geq -1} \). Additionally, let us assume that polynomials \( h_n \) are orthonormal and \( h_{-1}(x) = 0, h_0(x) = 1 \). Thus we will assume that for all \( i, j \geq 0 \):

\[
\int h_i(x)h_j(x) d\mu(x) = \delta_{ij},
\]

where, as usually, \( \delta_{ij} \) denotes Kronecker’s delta.

Thus, the class of Markov processes that we consider, is a class of stochastic processes that satisfy some mild technical assumptions that were described and interpreted in [7] and moreover satisfying the following conditions: \( \forall t \in \mathcal{T}, n \in \mathbb{N} : E(X_t^n) = m_n \) and \( \forall n \geq 1, s < t : \)
where $Q_n(x,t-s)$ is a polynomial of order not exceeding $n$ in $x$.

It has been shown in [7] that under the above-mentioned regularity assumptions and also under the following assumption that $\eta \ll \mu$

\begin{equation}
\int (d\eta d\mu)^2 d\mu < \infty,
\end{equation}

where, as above, $\mu(dx)$ and $\eta(dx|y,t)$ denote respectively marginal and transitional measures of $X$.

The following expansion holds

\begin{equation}
\frac{d\eta}{d\mu}(x|y,t) = \sum_{n \geq 0} \exp(-\alpha_n t) h_n(x)h_n(y).
\end{equation}

In this formula, we will set $\alpha_0 = 0$ and there appear certain positive constants $\{\alpha_i\}_{i \geq 1}$ whose existence is guaranteed by the mentioned above technical assumptions. The constants $\{\alpha_i\}_{i \geq 1}$ allow to define orthogonal martingales defined by the formula:

\begin{equation}
M_n(X_t, t) = \exp(\alpha_n t) h_n(X_t), \quad n \geq 1.
\end{equation}

The class of such stationary stochastic process will be briefly called SMPR. More precisely, since from (2.5) it follows that such processes are completely characterized by the distribution $\mu$ and a sequence $\{\alpha_n\}$ of positive numbers. We will write to denote such a process $X = \{X_t\}_{t \in \mathbb{R}} = \text{SMPR}(\{\alpha_n\}, \mu)$.

Thus, under the above-mentioned assumption, the two-dimensional distribution of say $(X_\tau, X_{\tau+t})$ is given by the formula

\begin{equation}
\rho(dx, dy) = d\mu(x)d\mu(y) \sum_{n \geq 0} \exp(-\alpha_n t) h_n(x)h_n(y).
\end{equation}

**Remark 1.** Notice, that from the fact that $M_n(X_t, t)$ is a martingale, it follows that

\begin{equation}
E(M_n(X_{t+\tau}, t + \tau)|\mathcal{F}_{\leq \tau}) = M_n(X_\tau, \tau),
\end{equation}

**hence, following (2.6), we see that**

\begin{equation}
Eh_n(X_{\tau+t})|\mathcal{F}_{\leq \tau}) = \exp(-\alpha_n t) h_n(X_\tau),
\end{equation}

a.s. mod $d\mu$.

Now, let us recall that in [2] the following numbers $\{c_{j,k}\}_{j \geq 0, 0 \leq n \leq j}$ were introduced and analyzed. Their interpretation is the following:

\begin{equation}
x^j = \sum_{n=0}^{j} c_{j,n} h_n(x).
\end{equation}

for all $j \geq 0$. Let us set $c_{j,n} = 0$ for $n > j$. Let us also denote by $L_k$ the following lower-triangular matrix $[c_{j,n}]_{j=0,...,k,n=0,...,k}$. It has been remarked in [2] (Propositions 1 and 2) that

\[ L_k L_k^T = M_k, \]
where $\mathbf{M}_k$ is the moment matrix, i.e., $\mathbf{M}_k = [m_{i+j}]_{i=0,\ldots,k,j=0,\ldots,k}$, where $m_j = \int x^j d\mu(x)$ that is equal to $j$–th moment of the distribution $\mu$. Hence, the coefficients $c_{j,n}$ can be computed directly from the moments’ matrix. Besides, we know by \cite{2},

$$\sum_{n=0}^{\min(j,k)} c_{j,n} c_{k,n} = m_{i+k},$$

for all $i, k \geq 0$. We have the following observation:

**Remark 2.**

\begin{equation}
E(X^j_{t+\tau} | \mathcal{F}_{\leq \tau}) = E(\sum_{n=0}^{j} c_{j,n} h_n(X_{t+\tau})| \mathcal{F}_{\leq \tau}) = \sum_{n=0}^{j} c_{j,n} \exp(-\alpha_n t) h_n(X_{\tau})
\end{equation}

$$= X^j_t - \sum_{n=1}^{j} c_{j,n} (1 - \exp(-\alpha_n t)) h_n(X_{\tau}),$$

since we have $\alpha_0 = 0$.

In the sequel, we will use the following almost trivial lemma. It has been presented in assertion 6 of Proposition 1 in \cite{10}. Since it is important in the present context, we will present its generalized version once more with its simple proof.

**Lemma 2.** Suppose that a sequence \{\(b_n\)\}_{n \geq 0} is a positive moment sequence sequence.

i) Suppose that $b_0 = 1$, and $b_{4m+2} = b_{2m+1}^2$ for some $m \geq 0$. Then $\forall n \geq 0 : b_n = b_{2m+1}^{n/(2m+1)}$.

ii) Suppose that $b_0 = 1$ and $b_{4m} = b_{2m}^2$ for some $m \geq 1$. Let us set $p = (b_1 + b_{2m}^{1/(2m)})/(2b_{2m}^{1/(2m)})$. Then for all $n \geq 0 : b_{2n} = b_{2n}^{n/m}$ and $b_{2n+1} = b_{2m}^{(2n+1)/(2m)} p - b_{2m}^{(2n+1)/(2m)} (1 - p)$.

**Proof.** Let $X$ denote a random variable whose moments are $b_n$, that is $EX^n = b_n$. In the case i) we have $E(X^{4m+2}) = (E(X^{2m+1})^2$. That is $E(X^{2m+1} - EX^{2m+1})^2 = \text{var}(X^{2m+1}) = 0$. But this equality means that the distribution of $X$ is a one-point distribution, i.e. $P(X = b_{2m+1}^{1/(2m+1)}) = 1$. In the case ii) we deduce that $E(X^{2m} - EX^{2m})^2 = 0$. That is that the distribution of $X^2$ is a one point distribution. One can find that $X^2 = b_{2m}^{1/(2m)}$ and that $P(X = b_{2m}^{1/(2m)}) = p = 1 - P(X = b_{2m}^{1/(2m)})$, for some $p \in [0, 1]$. But it means that $b_1 = EX = b_{2m}^{1/(2m)} p - b_{2m}^{1/(2m)} (1 - p)$. Parameter $p$ can be found to be equal to $(b_1 + b_{2m}^{1/(2m)})/(2b_{2m}^{1/(2m)})$. Then and $b_{2n+1}$ is a $2n + 1$–th moment of such variable that is $b_{2m}^{(2n+1)/(2m)} p - b_{2m}^{(2n+1)/(2m)} (1 - p)$. □

**Remark 3.** 1) It has been remarked in \cite{8} that the sequence \{\(\exp(\alpha_n t)\)\}_{n \geq 0} must be such that \(1 \sum_{n \geq 0} \exp(-2\alpha_n t) < \infty\) for all $t \in \mathbb{R}$. 2) If $\sup \mu$ is unbounded, then \{\(\exp(-\alpha_n t)\)\}_{n \geq 0} must be a moment sequence. Moreover, since all \{\(\alpha_n\)\}_{n \geq 1} are positive then we deduce that the support of the measure with respect to which \{\(\exp(-\alpha_n t)\)\}_{n \geq 0} is a moment sequence must have support contained in $[0, 1]$. Consequently, not only the matrix \{\(\exp(-\alpha_{i+j} t)\)\}_{0 \leq i, j \leq n} but also the matrix \{\(\exp(-\alpha_{i+j+1} t)\)\}_{0 \leq i, j \leq n} must be nonnegative definite. In particular, we deduce that

\[2\alpha_{2n+1} \geq \alpha_{2n} + \alpha_{2n+2} \text{ and } 2\alpha_{2n} \geq \alpha_{2n-1} + \alpha_{2n+1}.\]
But this means that for all $m \geq 0$ we have
\[ 2\alpha_m \geq \alpha_{m-1} + \alpha_{m+1}, \]
which means that the sequence $\{\alpha_n\}_{n \geq 0}$ must be a concave sequence. We must also have for all $i, n \geq 0$, $i + n \geq 1$
\[ \alpha_{2n} + \alpha_{2i} \leq 2\alpha_{n+i} \text{ and } \alpha_{2i+1} + \alpha_{2i+2n+1} \leq 2\alpha_{2i+2n}. \]
The last inequality follows the fact that for $\{\exp(-\alpha_n t)\}_{n \geq 0}$ being a moment sequence is equivalent to the fact that for all $m$ the matrices $[\exp(-\alpha_{i+j} t)]_{0 \leq i, j \leq m}$ must be nonnegative defined. This, in particular, all central $2 \times 2$ minors must be nonnegative. Now one of such minors is the following ones
\[ \begin{bmatrix} \exp(-t\alpha_{2i}) & \exp(-\alpha_{2i+m} t) \\ \exp(-\alpha_{2i+m} t) & \exp(-\alpha_{2i+2n} t) \end{bmatrix} \text{ and } \begin{bmatrix} \exp(-t\alpha_{2i+1}) & \exp(-\alpha_{2i+2n} t) \\ \exp(-\alpha_{2i+2n} t) & \exp(-\alpha_{2i+2n+1} t) \end{bmatrix}. \]
In the first of these matrices we set $n = i + m$.

**Remark 4.** Notice also that if the support of the measure $\mu$ is unbounded then, if only we are able to show that $\alpha_2 = 2\alpha_1$, then, as it follows from the assertion 6 of Proposition 1, we must have $\alpha_n = n\alpha_1$ for all $n \geq 1$ and consequently, $\text{SMPR}(\{\alpha_n\}, \mu)$ is additionally a harness (for details see [7]).

### 3. Necessary conditions

Having said this, we can state that the paper is dedicated to defining conditions under which a given $\text{SMPR}(\{\alpha_n\}, \mu)$ allows continuous path medication.

Thus we can calculate quantities $E|X_t - X_{t+T}|^n$ that are needed in order to apply Kolmogorov’s continuity theorem and examine the dependence of these quantities on $t$.

Of course, in order to simplify calculations, we consider only $\alpha = 2k$ for some natural $k$.

It turns out that for the further analysis, we will need the following moments defined by the formula
\[ (3.1) \quad \int x^j h_n(x) d\mu(x) = EX^j h_n(X). \]
Notice that we have
\[ EX^j h_n(X) = c_{j,n} \]
where $c_{j,n}$ are defined above, by (2.8). Let us observe that from the definition of the orthogonal polynomials it follows that $\forall 0 \leq j < n$ we have $EX^j h_n(X) = 0$. Thus in the sequel we will be using parameters $c_{j,n}$ set as 0 for $j < n$.

**Remark 5.** Notice that regardless of the boundedness of the support of marginal measure $\mu$ we observe that the following sequence
\[ (3.2) \quad \left\{ m_{2j} - \sum_{n=1}^{j} (1 - \exp(-\alpha_n t)) c_{j,n}^2 \right\}_{j \geq 0} \]
must be a moment sequence for every $t > 0$. This simple observation follows the fact that
\[ E(X_t X_{t+T})^j = \int \int (xy)^j \rho(dx, dy), \]
where $\rho(dx, dy)$ is given by (2.7). Then we use the definition of coefficients $c_{j,n}$ and the fact that $\sum_{n=0}^{j} c_{j,n}^2 = m_{2j}$. 

Now we can formulate sufficient conditions for the process $X = \text{SMPR}(\{\alpha_n\}, \mu)$ to allow continuous path modification. Namely, we have the following lemma.

**Lemma 3.** Let $X = \{X_t\}_{t \in \mathbb{R}} = \text{SMPR}(\{\alpha_n\}, \mu)$ with polynomial $\{h_n(x)\}$ orthonormal with respect to $\mu$. Let us define numbers $\{H_{j,n}\}_{j \geq 0, 0 \leq n \leq j}$ by (3.3), if only for some $k \geq 1$ and $r > 1$ the following conditions hold:

$$\sum_{n=1}^{2k} \alpha_n^k \sum_{j=n}^{2k} (-1)^j \binom{2k}{j} c_{j,n} c_{2k-j,n} = 0,$$

for $s = 1, \ldots, r$, with the assumption that $c_{j,n} = 0$, for $n > j$, then the process $X$ allows continuous path modification and the so-modified process has paths of the $m$-th Holder class where $m < \frac{1}{2k}$.

**Proof.** Let us start with the following calculation:

$$E(X_{\tau+t} - X_{\tau})^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} E(X_{\tau}^{m-j} E(X_{\tau+t}^{j} | F_{\tau}^r) =$$

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} E(X_{\tau}^{m-j} (X_{\tau}^{j} - \sum_{n=1}^{j} (1 - \exp(-\alpha_n t)) c_{j,n} h_n(X_{\tau}) =$$

$$- \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{n=1}^{j} (1 - \exp(-\alpha_n t)) c_{m-j,n} c_{j,n}$$

$$= - \sum_{n=1}^{m} (1 - \exp(-\alpha_n t)) \sum_{j=n}^{m} (-1)^j \binom{m}{j} c_{j,n} c_{m-j,n}$$

$$= \sum_{k=1}^{r} (-1)^{k-1} k! \sum_{n=1}^{m} \alpha_n^k \sum_{j=n}^{m} (-1)^j \binom{m}{j} c_{j,n} c_{m-j,n} + O(t^{r+1}).$$

Notice that, the crucial from the point of view of this paper, numbers $H_{j,n}$ can be expressed in terms of moments if the marginal distribution $\mu$. Since the first two particular cases of $k$ are the most important let us find numbers $H_{j,n}$ for $n = 0, 1, 2$ and $j = 0, 1, 2, 3, 4$.

The conditions presented in (3.3) are somewhat difficult to satisfy in the general case and have rather a theoretical character. The most important case is the case $k = 2$. Notice that then we have only two constants $\alpha_1$ and $\alpha_2$ involved. They are thus the most important from the point of view of the continuity of the paths of the analyzed process. Hence let us analyze this case in more detail.

**Proposition 1.** Let $X \sim \mu$. Let us denote $E X = c$, $m_j = E (X - c)^j$, and $h_j(x)$ the polynomial of degree $j$ that is orthonormal with respect to the measure $\mu$. Let the numbers $H_{j,n}$ be defined by (3.3). Then:

i) $H_{j,0} = EX^j = \sum_{k=0}^{j} \binom{j}{k} c^k m_{j-k}$, with an obvious fact that $m_1 = 0$.

ii) $H_{3,1} = \sqrt{m_2}$, $H_{2,1} = \left( m_3 + 2cm_2 \right) / \sqrt{m_2}$, $H_{3,1} = \left( m_4 + 3cm_3 + 3c^2 m_2 \right) / \sqrt{m_2}$, consequently we have $E(X_{\tau} - X_{\tau-t})^2 = 2m_2(1 - \exp(-\alpha_1 t))$.

iii) $H_{2,2} = \sqrt{m_2 m_4 - m_3^2 - m_2^2} / \sqrt{m_2}$, consequently we have

$$E(X_{\tau} - X_{\tau+t})^4 = 2(4m_4 + 3m_2^2) - 2\exp(-\alpha_1 t)(4m_2 m_4 - 3m_2^3) / m_2$$

$$+ 6\exp(-\alpha_2 t) \left( m_2 m_4 - m_3^2 - m_2^2 \right) / m_2.$$
Proof: i) Since \( h_0(x) = 1 \), we must have \( H_{j,0} = EX^j h_0(X) = E(X - c + c)^j = \sum_{k=0}^j \binom{j}{k} c^{j-k} \) with \( H_{1,0} = c \). Moreover, we have following (3.4)

\[
E(X_\tau - X_{\tau+t})^2 = 2H_{2,0} - 2H_{1,0}^2 + \exp(-\alpha_1 t) m_2 = 2m_2 - 2m_2 \exp(-\alpha_1 t).
\]

ii) We start with the well-known fact that polynomials that are orthogonal with respect to the measure \( \mu \) are given by the formula

\[
\eta_n(x) = a_n \det \begin{bmatrix} 1 & EX & \ldots & EX^n \\ EX & EX^2 & \ldots & EX^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x & \ldots & x^n \end{bmatrix},
\]

for some constants \( a_n \). Hence, orthonormal polynomials \( h_n(x) \) are given as \( b_n \eta_n(x) \) for suitably chosen constants \( b_n \). Thus we have \( h_1(x) = b_n \det \begin{bmatrix} 1 & c \\ 1 & x \end{bmatrix} = b_1(x - c) \).

Further, we have to have \( Eh_2^n(X) = 1 \), so \( b_1 = 1/\sqrt{m_2} \). Hence, we have \( H_{j,1} = (EX^j(X - c))/\sqrt{m_2} = (EX^{j+1} - cEX^j)/\sqrt{m_2} \). In particular, we have: \( H_{1,1} = \sqrt{m_2} \), \( H_{2,1} = (m_3 + 2cm_2)/\sqrt{m_2} \) and \( H_{3,1} = (m_4 + 3cm_3 + 3c^2m_2)/\sqrt{m_2} \).

iii) Using (3.8), we get

\[
h_2(x) = b_2 \det \begin{bmatrix} 1 & c & m_2 + c^2 \\ c & m_2 + c^2 & m_3 + 3cm_2 + c^3 \\ 1 & x & x^2 \end{bmatrix} = b_2(m_2(x^2 - m_2 - c^2) - (m_3 + 2cm_2)(x - c)).
\]

Since we are interested in orthonormal \( h_2 \) we have to calculate \( Eh_2(X)^2 \). Simple, but the lengthy calculation gives

\[
Eh_2(X)^2 = b_2^2 m_2(m_2m_4 - m_3^2 - m_2^2).
\]

Hence \( b_2 = 1/\left(\sqrt{m_2(m_2m_4 - m_3^2 - m_2^2)}\right) \). Now, we have \( H_{2,2} = b_2 EX^2 h_2(X) = b_2(m_2m_4 - m_3^2 - m_2^2) = \sqrt{(m_2m_4 - m_3^2 - m_2^2)/\sqrt{m_2}} \).

**Corollary 1.** Under the assumptions about Lemma 8 and upon applying Kolmogorov Theorem 7 we get

\[
E(X_\tau - X_{\tau+t})^4 = t(\alpha_1(8m_4m_2 - 6m_3^2)/m_2 - 6\alpha_2(m_2m_4 - m_3^2 - m_2^2)) + O(t^2).
\]

Consequently, the process allows continuous path modification if only

\[
\frac{4m_4m_2 - 3m_3^2}{3(m_4m_2 - m_3^2 - m_2^2)} > 0
\]

and

\[
\alpha_2 = \alpha_1 \frac{4m_4m_2 - 3m_3^2}{3(m_4m_2 - m_3^2 - m_2^2)}.
\]

Now let’s introduce, popular in mathematical statistics, parameters \( \kappa \)-kurtosis (excess kurtosis more precisely) defined as \( \kappa = m_4/m_2^2 - 3 \) and Fisher’s skewness parameter \( s \) defined as \( s = m_3/m_2^{3/2} \). (3.9) takes now, the following form:

\[
\alpha_2 = \alpha_1 \frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2}.
\]
Since, as mentioned earlier, if only the support of the marginal measure \( \mu \) is unbounded the sequence \( \{ \exp(-\alpha_n t) \} \) must be a moment sequence.

**Corollary 2.** If the support of the measure \( \mu \) is unbounded and \( 3s^2 = 2\kappa \) then the only \( \text{SMPR}(\{\alpha_n\}, \mu) \) process having continuous path modifications is the one with \( \alpha_n = n\alpha_1 \). From (\ref{3.10}) it follows that such process is also a harness.

**Proof.** Notice that, following (\ref{3.10}). when \( 3s^2 = 2\kappa \) we must have \( \alpha_2 = 2\alpha_1 \). Now, by Lemma 2 we have \( \alpha_n = n\alpha_1 \). \qed

Now let us show some examples and consider particular cases.

4. Examples

**Example 1 (1.).** **Gaussian case.** Let us consider 
\[ dx = \exp(-x^2/2)dx/\sqrt{2\pi}. \]

Consequently, polynomials \( h_n(x) \) are the so-called probabilistic Hermite polynomials satisfying the following three-term recurrence
\[ h_{n+1}(x) = xh_n(x) - nh_n(-1)(x), \]
with \( h_{-1}(x) = 0 \) and \( h_0(x) = 1 \). It is a common knowledge, that for \( \forall n \geq 0 \) :
\[ x^j = j! \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{1}{2^m m!(j-2m)} h_{j-2m}(x). \]

Consequently, the numbers \( H_{j,n} \) are given by
\[
H_{j,n} = \begin{cases} 
0 & \text{if } n > j \text{ or } j - n \text{ odd} \\
\frac{1}{x^{(j-n)/2}(j-n)/2!} & \text{if } j - n \text{ is even}
\end{cases}
\]

There is \( \sqrt{n!} \) in the denominator since polynomials \( h_n \) are not orthonormal. They become orthonormal after dividing \( n \)-th polynomial by \( \sqrt{n!} \). We have \( \kappa = 0 \) and \( s = 0 \) so \( \alpha_2 = \alpha_1 \). From the Remark it follows that we must have \( \alpha_n = n\alpha_1 \) and consequently we must be dealing with the Ornstein-Uhlenbeck (OU) process. We can thus conclude that the Ornstein-Uhlenbeck process is the only SMPR process with Gaussian marginals that allow continuous path modifications.

**Example 2 (2.).** **Gamma distribution.** Let us consider the Gamma distribution with rate parameter zero and shape parameter \( \beta > 0 \), i.e., the distribution with the following density:
\[ f_\beta(x, \beta) = x^{\beta-1} \exp(-x)/\Gamma(\beta), \]
for \( x > 0 \) and 0 otherwise. In order to simplify notation let us introduce the so-called raising factorial, which is the following function:
\[ (x)^{(n)} = x(x+1)\ldots(x+n-1), \]
defined for all complex \( x \). Notice, that we have for all \( x \neq 0 \):
\[ (x)^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}. \]

It is well known that if \( X \sim f_\beta(x, \beta) \), then \( \text{EX}^j = (\beta)^{(j)} \). Thus we have \( m_2 = \beta \), \( m_3 = 2\beta \), \( m_4 = 3\beta(2+\beta) \). Hence the skewness parameter \( s \) is equal to \( 2/\sqrt{\beta} \) and the kurtosis \( \kappa = 3(2+\beta)/\beta - 3 = 6/\beta \). Consequently, the parameter \( \alpha_2/\alpha_1 \) is equal to
\[
\frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2} = \frac{12 + 24/\beta - 12/\beta}{6 + 18/\beta - 12/\beta} = 2.
\]
so-called generalized Laguerre polynomials $L_n^{\alpha}$ with $n$ and following [7](Thm. 2) such process is also a harness (a class of processes introduced by Hammersley in [3]).

Moreover, we know also that the polynomials $h_n(x|\beta)$ are proportional to the $n$th generalized Laguerre polynomials $L_n^{\alpha}$, defined by the following recurrence relation:

$$L_n^{(\beta)} x_{n+1} = \frac{2n + \beta - x}{n + 1} L_n^{(\beta)} x_n - n + \beta - 1 L_{n-1}^{(\beta)},$$

with $L_{-1}^{(\beta)} = 0$ and $L_0^{(\beta)} = 1$. We also know that

$$\frac{1}{\Gamma(\beta)} \int_0^\infty L_n^{(\beta)}(x) L_m^{(\beta)}(x)x^{\beta-1} \exp(-x)dx = \frac{\Gamma(n + \beta)}{n!} \delta_{nm}.$$

Thus

$$h_n(x|\beta) = \sqrt{\frac{n!}{\Gamma(n + \beta)}} L_n^{(\beta)}(x).$$

Moreover, we also know the exact form of the two-dimensional density of such SMPR($\{\alpha_n\}, f_\alpha$) process. It has complicated form and is commonly known under the name Hardy-Hille formula. Namely we have

$$f_\beta(x,y) = \sum_{j \geq 0} h_j(x|\beta)h_j(y|\beta) \exp(-n\alpha_1 t)$$

$$= \frac{1}{(1 - \exp(-\alpha_1 t))(xy \exp(-\alpha_1 t))^{(\beta-1)/2}} \exp(-(x+y)$$

$$\times \frac{\exp(-\alpha_1 t)}{1 - \exp(-\alpha_1 t)} I_{\beta-1}(2 \exp(-\alpha_1 t/\sqrt{y}) / 1 - \exp(-\alpha_1 t)),$$

where $I_\alpha$ denotes modified Bessel function of the first kind. Recently, it has been shown in [9] that, as in the Gaussian case, we have

$$E(X_t - X_{t+\tau})^k = \frac{(2k)!}{k!} (\beta)^k (1 - \exp(-\alpha_1 t))^k.$$

Hence, the gamma process with transition distribution given above is as smooth as the Ornstein-Uhlenbeck process.

**Example 3 (3.). Laplace distribution.** Laplace distribution is defined by its density given by the formula $f_L(x) = \exp(-|x|)/2$, for $x \in \mathbb{R}$. It is also known that it is a symmetric distribution, hence its all odd moments are equal to 0 while its even moments, say of degree $2n$ are equal to $(2n)!$. Consequently, we have $m_2 = 2$, $m_4 = 0$ and $m_4 = 24$, hence $k = 3$, and $s = 0$. Hence, we have

$$\frac{\alpha_2}{\alpha_1} = \frac{12 + 4 \times 3}{6 + 3 \times 3} = \frac{8}{5} < 2.$$

Let us denote by $\{h_n\}_{n \geq 0}$ the family of polynomials orthonormal with respect to the measure with the density $f_L$. Thus, for every moment sequence $\{1, \exp(-\alpha_1 t), \exp(-\alpha_2 t), \ldots\}$, such that the bivariate function:

$$(4.2) \quad 1 + \sum_{j \geq 1} \exp(-\alpha_j t)h_j(x)h_j(y),$$

with $\alpha_2 = 8\alpha_1/5$ is nonnegative for all $t$, the SMPR($\{\alpha_n\}, f_L$) allows continuous path modification with $E(X_{t+t} - X_t)^4 = O(t^2)$. 



Interestingly, if we calculate (with the help of Mathematica) the coefficients \( \{c_{i,n}\}_{i=0,...,n=0,...} \), which can be easily done using moments of the Laplace distribution, and then solving system of equations (3.3) we get
\[
\alpha_2/\alpha_1 = (35 - \sqrt{105})/28, \quad \alpha_3/\alpha_1 = (15 - \sqrt{105})/12.
\]
Consequently, every SMPR(\( \{\alpha_n\}, f_L \)) with these parameters \( \alpha_2, \alpha_3 \) (for which naturally function \( f_L \) is nonnegative), would allow continuous path modification and \( E(X_{t+\tau} - X_t)^6 = O(\tau^3) \).

We can continue this procedure. There are however some numerical problems since there is no known a nice general form of polynomials that are orthonormal with respect to the Laplace distribution.

Example 4 (4). Arcsine distribution. Arcsine distribution is an example of the so-called beta distribution. So first let us start with the general case of the Beta distribution. Under this name function two distributions related to one another. Namely, primarily the distribution with the density
\[
f_B(x|\gamma, \beta) = 2^{1-\gamma-\beta}(1 + x)^{\gamma-1}(1 - x)^{\beta-1}/B(\gamma, \beta),
\]
for \( x \in (0, 1) \) and \( \gamma, \beta > 0 \), where \( B(\gamma, \beta) = \Gamma(\gamma)\Gamma(\beta)/\Gamma(\gamma + \beta) \) is the so-called beta function. Often under the name Beta distribution functions also the distribution with the following density
\[
f_B(x|\gamma, \beta) = 2^{1-\gamma-\beta}(1 + x)^{\gamma-1}(1 - x)^{\beta-1}/B(\gamma, \beta),
\]
for \( x \in (-1, 1) \) and \( \gamma, \beta > 0 \). As one can easily notice if random variable \( X \sim f_B \) then \( Y = 2X - 1 \) has density \( f_B \). It is also known that the so-called Jacobi polynomials are orthogonal with respect to the measure with the density \( f_B \). In particular, we know that
\[
f_B(x|1/2, 1/2) = \frac{1}{\pi \sqrt{1 - x^2}},
\]
that is, we are dealing with the so-called arcsine distribution while \( f_B(x|3/2, 3/2) = 2\sqrt{1 - x^2} \), that is we are dealing with the so-called Wigner or semicircle distribution. It is known (see e.g. [4]) that the skewness of the beta distribution is equal to:
\[
s = \frac{2(\beta - \gamma)\sqrt{\gamma + \beta + 1}}{(\gamma + \beta + 2)\sqrt{\gamma\beta}},
\]
while the excess kurtosis is given by the formula:
\[
\kappa = \frac{\gamma\beta(\gamma + \beta + 2)\gamma\beta}{{6((\gamma - \beta)^2(\gamma + \beta + 1) - (\gamma + \beta + 2)\gamma\beta)}}.
\]

Now
\[
\frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2} = \frac{2(\gamma + \beta + 1)}{\gamma + \beta} = 2 + \frac{2}{\gamma + \beta}.
\]

Let us concentrate on the case \( \gamma = 1/2, \beta = 1/2 \). Let us recall that in this case, the sequence of polynomials orthogonal with respect to the measure with the density \( f_B(1/2, 1/2) \) are the so-called Chebyshev polynomials of the first kind \( \{T_n(x)\}_{n \geq -1} \). Following [1] let us recall the most important properties of these polynomials. They are defined by the following three-term recurrence
\[
2xT_n(x) = T_{n+1}(x) + T_{n-1}(x),
\]
with \( T_0(x) = 1 \) and \( T_1(x) = x \). What is however important for our purposes, is the following property of polynomials \( \{T_n\}_{n \geq 0} \). Namely, we have

\[
T_n(\cos \varphi) = \cos(n \varphi).
\]

for \( n \geq 0 \) and \( \varphi \in \mathbb{R} \). Moreover, it is known that

\[
\frac{1}{\pi} \int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 1 & \text{if } n = m = 0 \\ \delta_{mn}/2 & \text{if } n + n > 0 \end{cases}
\]

Thus, polynomials are now given by \( h_n(x) = \sqrt{2} T_n(x) \) for \( n \geq 1 \).

As one can see in order to have a continuous path modification of the SMPR (\( \{\alpha_n\}, f_B(1/2, 1/2) \)) one has to have \( \alpha_2 = 4\alpha_1 \). We will define the sequence of numbers \( \{\alpha_n\}_{n \geq 1} \) that satisfies the above-mentioned conditions and above all will lead to a nonnegative transition density. To do it, let us recall the definition of the Jacobi Theta function. Namely, we have for \( |q| < 1 \) and complex \( \alpha \):

\[
\theta(q; \alpha) = \sum_{j=-\infty}^{\infty} q^j \exp(2i\pi j \alpha).
\]

One can easily notice, that

\[
\theta(q; \alpha) = 1 + 2 \sum_{j=1}^{\infty} q^j \cos(j \pi \alpha).
\]

Consequently, we have in terms of the new variables \( \varphi \) and \( \phi : 2T_n(\cos \varphi)T_n(\cos \phi) = 2 \cos(n \varphi) \cos(n \phi) = \cos(n(\varphi - \phi)) + \cos(n(\varphi + \phi)) \) and

\[
1 + 2 \sum_{n \geq 1} \rho^{n^2} T_n(\cos \varphi) T_n(\cos \phi)
\]

\[
= 1 + \sum_{n \geq 1} \rho^{n^2} \cos(n(\varphi - \phi)) + \sum_{n \geq 1} \rho^{n^2} \cos(n(\varphi + \phi))
\]

\[
= (\theta(\rho, ((\varphi - \phi)/\pi)) + \theta(\rho, ((\varphi + \phi)/\pi))/2.
\]

Thus

\[
(4.4) \quad 1 + 2 \sum_{n \geq 1} \rho^{n^2} T_n(x) T_n(y)
\]

\[
= \frac{1}{2} \left( \theta(\rho, (\arccos(x) - \arccos(y))/\pi) + \theta(\rho, (\arccos(x) + \arccos(y))/\pi) \right).
\]

Now, we see that the plot of this function has a saddle shape on the square \([-1, 1] \times [-1, 1] \) and has minimum there at the points \((-1, 1) \) or \((1, -1)\) equal to \(\sum_{j \geq 0} (-1)^j \rho^{j^2} = 1 - \rho + \rho^2 - \rho^4 + \ldots > 0 \) and maximum at points \((1, 1) \) or \((-1, -1)\) equal to \(\sum_{j \geq 0} \rho^{j^2} < \infty \) for \(|\rho| < 1 \). Setting \( \rho = \exp(-\alpha_1 t) \) in \((4.4)\) and multiplying by, say \(\pi / \sqrt{1-y^2} \), we get a transitional density of \(X_{t+\tau} \) given \(X_\tau = y \).

**Remark 6.** Let us note that we have \( E(X_{t+\tau} - X_\tau)^4 = O(t^2) \) as it follows directly from \((4.3)\). However, we know parameters \( c_{i,n} \) defined above since we have:

\[
x^j = 2^{1-j} \sum_{n \equiv j \mod 2} \binom{j}{(j-n)/2} h_n(x) \overset{df}{=} \sum_{n=0}^{j} c_{i,n} h_n(x),
\]
where the ′ above the sum means that when $n = 0$ then the appropriate coefficient is divided by 2. Thus consequently, we can, using Mathematica, check that for $k = 3, 4, 5$ that also $E(X_{t+\tau} - X_\tau)^{2k} = O(t^k)$. Thus one can utter the following conjecture:

**Conjecture 1.** Let us consider $X = \text{SMPR} \left( \left\{ n^2\alpha_1 \right\}, f_B(x|1/2, 1/2) \right)$. Then for all $k \geq 1$ we have $E(X_{t+\tau} - X_\tau)^{2k} = O(t^k)$.

Thus, this process is as 'smooth' as the Ornstein-Uhlenbeck process. It is not however a harness.

**Remark 7.** Notice also, that the sequence $\left\{ \exp(-n^2\alpha_1) \right\}_{n \geq 0}$ is not a moments' sequence since, for example, the matrix $\begin{bmatrix} 1 & \exp(-a_1t) \\ \exp(-a_1t) & \exp(-4a_1t) \end{bmatrix}$ is not nonnegative definite. Recall, that this feature of $\text{SMPR} \left( \left\{ \alpha_n \right\}, \mu \right)$, as pointed out in [8], has been allowed because the support of the measure $\mu$ is bounded in this case.

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