ON THE PATH ENERGY OF SOME GRAPHS

S. C. PATEKAR ∗ AND M. M. SHIKARE

Department of Mathematics, Savitribai Phule Pune University, Pune 411007, India

Abstract. Let G be a graph with vertex set V(G) = {v1, v2, ..., vn}. We define a matrix whose (i, j)th entry is the maximum number of vertex disjoint paths between the corresponding vertices if they are adjacent and is zero otherwise. We call this matrix as path matrix of G and its eigenvalues as path eigenvalues of G. In this paper, we investigate path eigenvalues and path energy of some graphs.

Keywords: symmetric matrix; eigenvalues; path eigenvalues of a graph; path energy of a graph.

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1. INTRODUCTION

The eigenvalues of a graph G are the eigenvalues of its adjacency matrix. The spectrum of a matrix is the list of its eigenvalues together with their multiplicities. The eigenvalues of graphs have several useful properties. For undefined terminology and notations, we refer to West [5] and Varga [4]. For an extensive survey on graph spectra we refer to Brouwer A. E. [3] and Beineke L. W. [6].

∗Corresponding author

E-mail address: shri82patekar@gmail.com

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We define a new matrix, called the path matrix ([1], [2]) of a graph in the following way. Let $G$ be a graph without loops and let $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of $G$. Define the matrix $P = (p_{ij})$ of size $n \times n$ such that

$$p_{ij} = \begin{cases} \text{maximum number of vertex disjoint paths from } v_i \text{ to } v_j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

We call $P$ as Path Matrix of $G$. The matrix $P$ is real and symmetric. Therefore, its eigenvalues are real. We call eigenvalues of $P$ as path eigenvalues of $G$.

Consider the graph as shown in the following figure

![Graph](image)

Then the path matrix of $G$ is

$$P = \begin{bmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix}.$$

The characteristic polynomial of the matrix $P$ is

$$C_P(x) = |P - xI| = (x - 12)(x + 3)^4.$$ The path eigenvalues of $G$ are $12, -3, -3, -3$ and $-3$. The eigenvalues of $G$ are $3.236, -2, -1.236, 0$ and $0$.

2. **Path Energy Obtained from Some Operations on Graphs**

   The ordinary energy ([7], [8]), $E(G)$, of a graph $G$ is defined to be the sum of the absolute values of the ordinary eigenvalues of $G$. Recently much work on ordinary graph energy appeared in the mathematical literature. In analogy, the path energy [2], $PE(G)$ is defined as the
sum of the absolute values of the path eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $G$, i.e.,

$$PE = PE(G) = \sum_{i=1}^{n} |\lambda_i|. \quad (2.1)$$

We know if $G$ is a $r$-regular, $r$-connected graph with $n$ vertices, then its path matrix has row sum $(n - 1)r$ and this row sum $(n - 1)r$ is one of the path eigenvalues of $G$ and the other path eigenvalues are $-1$ with multiplicity $n - 1$.

In the following Theorem, we investigate the path eigenvalues and path energy of a graph which is obtained by joining a vertex of $r_1$-regular, $r_1$-connected graph with a vertex of $r_2$-regular, $r_2$-connected graph by an edge.

**Theorem 2.1.** Let $G_1$ be $r_1$-regular, $r_1$-connected graph with $m$ vertices and $G_2$ be $r_2$-regular $r_2$-connected graph with $n$ vertices. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $\mu_1, \mu_2, \ldots, \mu_n$ are path eigenvalues of $G_1$ and $G_2$ respectively. Let $G$ be a graph obtained by joining a vertex of $G_1$ to a vertex of $G_2$ by an edge. Then the path eigenvalues of $G$ are $-r_1$ with multiplicity $m - 1$, $-r_2$ with multiplicity $n - 1$, $\frac{(m-1)r_1+(n-1)r_2+\sqrt{[(m-1)r_1+(n-1)r_2]^2+4mn-(m-1)r_1(n-1)r_2}}{2}$ with multiplicity 1 and $\frac{(m-1)r_1+(n-1)r_2-\sqrt{[(m-1)r_1+(n-1)r_2]^2+4mn-(m-1)r_1(n-1)r_2}}{2}$ with multiplicity 1. $PE(G) = 2[(m - 1)r_1 + (n - 1)r_2] = PE(G_1) + PE(G_2)$.

**Proof.** Let $P$, $Q$, and $R$ be the path matrices of $G$, $G_1$, and $G_2$ respectively. As $G_1$ is $r_1$-regular, $r_1$-connected with $m$ vertices, the path eigenvalues of $G_1$ are $\lambda_i = (m - 1)r_1$ with multiplicity 1, $-r_1$ with multiplicity $m - 1$ and as $G_2$ is $r_2$-regular, $r_2$-connected on $n$ vertices, the path eigenvalues of $G_2$ are $\mu_i = (n - 1)r_2$ with multiplicity 1, $-r_2$ with multiplicity $n - 1$. The path matrix $P$ can be written as

$$P = \begin{bmatrix} Q & J_{m \times n} \\ J_{n \times m} & R \end{bmatrix}$$

where $J_{m \times n}$ is $m$ by $n$ matrix with all entries 1. We know that $\mathbf{1}$ is an eigenvector of $Q$ corresponding to $(m - 1)r_1$, so we assume $\mathbf{1}X = 0$, where $X = [x_1, \ldots, x_m]'$ is an eigenvector of of $Q$ corresponding to $\lambda_i \neq (m - 1)r_1$. Now
implies that $\lambda_i = -r_1$ is a path eigenvalue of $P$ for $i = 2, \ldots, m$. Similarly $\mu_i = -r_2$ is a path eigenvalue of $P$ for $i = 2, \ldots, n$. Thus

$$tr(P) = 0 = tr(Q) + tr(R),$$

where $tr(Q) = 0, tr(R) = 0$ and $\lambda_1 + \mu_1 = (m - 1)r_1 + (n - 1)r_2$. Let $A$ be a square matrix, then the sum of all $2 \times 2$ principal minors of $A$ is equal to $s_2(A)$, where $s_2(A)$ is the second elementary symmetric function of the eigenvalues of $A$. Thus

$$s_2(P) = \lambda_1 \mu_1 + \lambda_1(-r_1)(m - 1) + \lambda_1(-r_2)(n - 1) + \mu_1(-r_1)(m - 1) + \mu_1(-r_2)(n - 1) + \sum_{2 \leq i < j} \lambda_i \lambda_j + \sum_{2 \leq i < j} \mu_i \mu_j + \sum_{2 \leq i < j} \lambda_i \mu_j.$$ We can write this as

$$s_2(P) = \lambda_1 \mu_1 + (\lambda_1 + \mu_1)[- (m - 1)r_1 - (n - 1)r_2] + \sum_{2 \leq i < j} \lambda_i \lambda_j + \sum_{2 \leq i < j} \mu_i \mu_j + [-(n - 1)r_2][-(m - 1)r_1] \quad (i)$$

Now for the path matrices $Q$ and $R$, we get

$$s_2(Q) = (m - 1)r_1(-r_1)(m - 1) + \sum_{2 \leq i < j} \lambda_i \lambda_j$$

$$= (m - 1)r_1( - (m - 1)r_1) + \sum_{2 \leq i < j} \lambda_i \lambda_j$$

and

$$s_2(R) = (n - 1)r_2(-r_2)(n - 1) + \sum_{2 \leq i < j} \mu_i \mu_j$$

$$= (n - 1)r_2(-n - 1)r_2) + \sum_{2 \leq i < j} \mu_i \mu_j.$$

Again every principal minor of size $2 \times 2$ of $P$ is either a $2 \times 2$ principal minor of $Q$ or $R$, or it has the form

$$\begin{bmatrix} q_{ii} & 1 \\ 1 & r_{jj} \end{bmatrix} = q_{ii} r_{jj} - 1, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \text{ where } Q = (q_{ij}) \text{ and } R = (r_{ij}).$$

Using this, we write

$$s_2(P) = s_2(Q) + s_2(R) + \sum_{i=1}^{m} \sum_{j=1}^{n} (q_{ii} r_{jj} - 1) = s_2(Q) + s_2(R) - mn. \quad (ii)$$

From (i) and (ii), we get

$$\lambda_1 \mu_1 + (\lambda_1 + \mu_1)[- (m - 1)r_1 - (n - 1)r_2] + [-(n - 1)r_2][-(m - 1)r_1] = (m - 1)r_1[-(m - 1)r_1] + (n - 1)r_2[-(n - 1)r_2] - mn.$$ This and $\lambda_1 + \mu_1 = (m - 1)r_1 + (n - 1)r_2$ gives $\lambda_1 \mu_1 = (m - 1)r_1(n - 1)r_2 - mn$. Solving this equations for $\lambda_1$ and $\mu_1$, we get
\[ \lambda_1 = \frac{(m-1)r_1 + (n-1)r_2 + \sqrt{(m-1)r_1 + (n-1)r_2^2 + 4mn - (m-1)(n-1)r}}{2} \] and
\[ \mu_1 = \frac{(m-1)r_1 + (n-1)r_2 - \sqrt{(m-1)r_1 + (n-1)r_2^2 + 4mn - (m-1)(n-1)r}}{2} \].

Hence \( PE(G) = 2[(m-1)r_1 + (n-1)r_2] = PE(G_1) + PE(G_2) \).

We know for a tree \( T \) with \( m \) vertices, its path matrix has row sum \( m-1 \) and this row sum \( m-1 \) is one of the path eigenvalue of \( T \) and the other path eigenvalues are \(-1\) with multiplicity \( m-1 \).

In the following Proposition, we investigate the path eigenvalues and path energy of a graph which is obtained by joining a vertex of a tree with a vertex of \( r \)-regular, \( r \)-connected graph by an edge.

**Proposition 2.2.** Let \( G_1 \) be a tree with \( m \) vertices and \( G_2 \) be \( r \)-regular, \( r \)-connected graph with \( n \) vertices. Let \( G \) be a graph obtained by joining a vertex of \( G_1 \) to a vertex of \( G_2 \) by an edge.

Then the path eigenvalues of \( G \) are \(-1\) with multiplicity \( m-1 \), \(-r\) with multiplicity \( n-1 \),
\[ \frac{(m-1)+(n-1)r + \sqrt{(m-1)+(n-1)r^2 + 4mn - (m-1)(n-1)r}}{2} \] with multiplicity \( 1 \) and
\[ \frac{(m-1)+(n-1)r - \sqrt{(m-1)+(n-1)r^2 + 4mn - (m-1)(n-1)r}}{2} \] with multiplicity \( 1 \).

\[ PE(G) = 2[(m-1) + (n-1)r] = PE(G_1) + PE(G_2). \]

**Proof.** Let \( P, Q \ (= J_m - I_m) \), and \( R \) be the path matrices of \( G, G_1 \), and \( G_2 \) respectively. Here \( \mu_1 = (n-1)r \). The path matrix \( P \) can be written as
\[
P = \begin{bmatrix}
Q & J_{m \times n} \\
J_{n \times m} & R
\end{bmatrix}
\]

where \( J_{m \times n} \) is \( m \) by \( n \) matrix with all entries 1. We know that \( 1 \) is an eigenvector of \( Q \) corresponding to \( \lambda_1 = (m-1) \), so we assume \( 1X = 0 \), where \( X = [x_1, \ldots, x_m]' \) is an eigenvector of \( Q \) corresponding to \( \lambda_i \neq (m-1) \). Now
Now for the path matrices $Q$ and $R$

$$
P = \begin{bmatrix} X & Q & J_{m \times n} \\ 0 & J_{n \times m} & R \end{bmatrix}
= \begin{bmatrix} X \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} X \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

implies that $\lambda_i = -1$ is a path eigenvalue of $P$ for $i = 2, \ldots, m$. Similarly $\mu_i = -r$ is a path eigenvalue of $P$ for $i = 2, \ldots, n$. Thus $\text{tr}(P) = 0 = \text{tr}(Q) + \text{tr}(R)$, where $0 = \text{tr}(Q) = (m - 1) + \lambda_2 + \ldots + \lambda_m$, $0 = \text{tr}(R) = (n - 1)r + \mu_2 + \ldots + \mu_n$ and $\lambda_1 + \lambda_2 = (m - 1) + (n - 1)r$. Let $A$ be a square matrix, then the sum of all $2 \times 2$ principal minors of $A$ is equal to $s_2(A)$, where $s_2(A)$ is the second elementary symmetric function of the eigenvalues of $A$. Thus

$$s_2(P) = \lambda_1 \mu_1 + \lambda_1(-1)(m - 1) + \lambda_1(-r)(n - 1) + \mu_1(-1)(m - 1) + \mu_1(-r)(n - 1) + \sum_{2 \leq i < j} \lambda_i \lambda_j + \sum_{2 \leq i < j} \mu_i \mu_j + \sum_{2 \leq i, j} \lambda_i \lambda_j.
$$

We can write this as

$$s_2(P) = \lambda_1 \mu_1 + (\lambda_1 + \mu_1)(-(m - 1) - (n - 1)r) + \sum_{2 \leq i < j} \lambda_i \lambda_j + \sum_{2 \leq i < j} \mu_i \mu_j + (-(n - 1)r)(-(m - 1)r) = \lambda_1 \lambda_2 + (\lambda_1 - \lambda_2)([m - 1] + [(n - 1)r]) + \sum_{2 \leq i < j} \lambda_i \lambda_j + \sum_{2 \leq i < j} \mu_i \mu_j + (n - 1)r(m - 1).
$$

(i)

Now for the path matrices $Q$ and $R$, we get

$$s_2(Q) = (m - 1)(-1)(m - 1) + \sum_{2 \leq i < j} \lambda_i \lambda_j
= (m - 1)(-(m - 1)) + \sum_{2 \leq i < j} \lambda_i \lambda_j
$$

and

$$s_2(R) = (n - 1)r(-r)(n - 1) + \sum_{2 \leq i < j} \mu_i \mu_j
= (n - 1)r(-(n - 1)r) + \sum_{2 \leq i < j} \mu_i \mu_j.
$$

Again every principal minor of size $2 \times 2$ of $P$ is either a $2 \times 2$ principal minor of $Q$ or $R$, or it has the form

$$
\begin{vmatrix} q_{ii} & 1 \\ 1 & r_{jj} \end{vmatrix}
= q_{ii}r_{jj} - 1, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,
$$

where $Q = (q_{ij})$ and $R = (r_{ij})$.

Using this, we write

$$s_2(P) = s_2(Q) + s_2(R) + \sum_{i=1}^m \sum_{j=1}^n (q_{ii}r_{jj} - 1) = s_2(Q) + s_2(R) - mn
$$

(ii)

From (i) and (ii), we get
\[ \lambda_1 \mu_1 + (\lambda_1 + \mu_1)(-(m-1)r_1 - (n-1)r_2) + (-(n-1)r_2)(-(m-1)r_1) = (m-1)r_1(-(m-1)r_1) + (n-1)r_2(-(n-1)r_2) - mn. \]

This and \( \lambda_1 + \mu_1 = (m-1) + (n-1)r \) gives \( \lambda_1 \mu_1 = (m-1)(n-1)r - mn \). Solving this equations for \( \lambda_1 \) and \( \mu_1 \), we get

\[ \lambda_1 = \frac{(m-1) + (n-1)r + \sqrt{[m-1 + (n-1)r] + 4mn - (m-1)(n-1)r}}{2} \]

and

\[ \mu_1 = \frac{(m-1) + (n-1)r - \sqrt{[m-1 + (n-1)r] + 4mn - (m-1)(n-1)r}}{2}. \]

Hence \( \text{PE}(G) = 2[(m-1) + (n-1)r] = \text{PE}(G_1) + \text{PE}(G_2) \). \( \square \)

We investigate the path eigenvalues and path energy of a graph which is obtained by taking \( k \) copies of \( r \)-regular, \( r \)-connected graph and joining a vertex of one graph with a vertex of other graph.

**Theorem 2.3.** Let \( G_1, G_2, \ldots, G_k \) be the \( k \) copies of some \( r \)-regular \( r \)-connected graph on \( n \) vertices and let \( G \) be a graph obtained by joining a vertex of \( G_i \) with a vertex of \( G_{i+1} \) \((1 \leq i \leq k-1)\) by an edge. Then the path eigenvalues of \( G \) are \( n(k-1) + r(n-1) \) with multiplicity 1, 

\(-r \) with multiplicity \( k(n-1) \) and \( n(r-1) - r \) with multiplicity \( k-1 \). \( \text{PE}(G) = \sum_{i=1}^{k} \text{PE}(G_i) \).

**Proof.** Let \( P \) be the path matrix of \( G \) and \( Q \) be the path matrix of \( G_i \), for \( i = 1, 2, \ldots, k \). Let \( J_n \) be the \( n \times n \) matrix with all entries 1. The path matrix \( P \) can be written as

\[
P = \begin{bmatrix}
Q & J_n & \cdots & J_n \\
J_n & Q & \cdots & J_n \\
\vdots & \vdots & \ddots & \vdots \\
J_n & J_n & \cdots & Q
\end{bmatrix}
\]

Adding 2\(^{nd}\), 3\(^{rd}\), ..., \( k^{th} \) columns to the first column, we get

\[
\begin{bmatrix}
Q + (k-1)J_n & J_n & \cdots & J_n \\
Q + (k-1)J_n & Q & \cdots & J_n \\
\vdots & \vdots & \ddots & \vdots \\
Q + (k-1)J_n & J_n & \cdots & Q
\end{bmatrix}
\]
Now subtracting the first row from 2nd, 3rd, ..., kth rows, we get
\[
\begin{bmatrix}
Q + (k - 1)J_n & J_n & \ldots & J_n \\
0 & Q - J_n & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & Q - J_n
\end{bmatrix}.
\]
This is a triangular block matrix. Hence the characteristic polynomial of \( P \) is
\[
C_P(x) = |Q + (k - 1)J_n - xI_n| |Q - J_n - xI_n| \ldots |Q - J_n - xI_n| \quad (\text{here } |Q - J_n - xI_n| \text{ appears } k - 1 \text{ times}).
\]
This implies that the path eigenvalues of \( G \) are the path eigenvalues of \( Q + (k - 1)J_n \) and the path eigenvalues of \( Q - J_n, \) \( k - 1 \) times. Now, the path eigenvalues of \( Q + (k - 1)J_n \) are \( n(k - 1) + r(n - 1) \) with multiplicity 1 and \(-r\) with multiplicity \( n - 1 \) whereas the path eigenvalues of \( Q - J_n \) are \( n(r - 1) - r \) with multiplicity 1 and \(-r\) with multiplicity \( n - 1. \) Hence the path eigenvalues of \( G \) are \( n(k - 1) + r(n - 1) \) with multiplicity 1, \(-r\) with multiplicity \( k(n - 1) \) and \( n(r - 1) - r \) with multiplicity \( k - 1. \) Hence \( PE(G) = n(k - 1) + r(n - 1) + rk(n - 1) + [n(r - 1) - r](k - 1) = 2kr(n - 1) = \sum_{i=1}^{k} PE(G_i). \)

3. Conclusion

In the present paper, path eigenvalues and path energy of graphs which are obtained by joining a vertices of some specific classes of graphs are obtained and studied.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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