A TITS ALTERNATIVE FOR RATIONAL FUNCTIONS

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Abstract. We prove an analog of the Tits alternative for rational functions. In particular, we show that if $S$ is a finitely generated semigroup of rational functions over $\mathbb{C}$, then either $S$ has polynomially bounded growth or $S$ contains a nonabelian free semigroup. We also show if $f$ and $g$ are polarizable maps over any field of any characteristic and $\text{Prep}(f) \neq \text{Prep}(g)$, then there is a positive integer $j$ such that $(f^j, g^j)$ is a free semigroup on two generators. In the special case of polynomials, we are able to prove slightly stronger results.

1. Introduction

The Tits alternative [Tit72] is a celebrated result in the theory of linear groups. It says that a finitely generated linear group contains either a solvable subgroup of finite index or a nonabelian free group. In general, a group $G$ is said to satisfy the Tits alternative if each of its finitely generated subgroups contains either a solvable subgroup of finite index or a nonabelian free group. Many classes of groups have now been shown to satisfy the Tits alternative [BFH00, KOZ09, Ivan84, Lam01, McC85].

When one instead considers the structure of linear groups as semigroups, an even stronger dichotomy is obtained. A result of Longobardi, Maj, and Rhemtulla [LMR95] (see also Milnor [Mil68] and Wolf [Wol68]) combined with the Tits alternative implies that a finitely generated linear group is either virtually nilpotent or contains a nonabelian free semigroup. Okniński and Salwa [OS95] later showed if $S$ is a finitely generated cancellative linear semigroup, then either $S$ contains a nonabelian free semigroup or the group generated by $S$ is virtually nilpotent. Results from the theory of growth of groups then give that the growth of a finitely generated cancellative linear semigroup is either exponential or is polynomially bounded. The cancellativity condition here is crucial as Okniński [Okn93] has also produced finitely generated non-cancellative linear semigroups of intermediate growth (see also [CO07]).

We note that a semigroup $S$ contains a nonabelian free semigroup if and only if it contains a free semigroup on two generators. As with groups, it is not difficult to see that a free semigroup on two generators must contain a free semigroup on $n$ generators for any positive integer $n$.

We prove the following variant of the Tits alternative for semigroups of rational functions over the complex numbers.

Theorem 1.1. Let $S$ be a finitely generated semigroup of rational functions in $\mathbb{C}(x)$. Then either $S$ has polynomially bounded growth or $S$ contains a nonabelian free semigroup.

We say that a rational function of degree greater than 1 is non-special if it is not conjugate to a monomial, a Chebychev polynomial, or a Lattès map. When $S$ contains a non-special rational function of degree greater than one, we obtain a stronger dichotomy.

Theorem 1.2. Let $S$ be a finitely generated semigroup of rational functions in $\mathbb{C}(x)$ such that some element of $S$ is a non-special rational function of degree greater than 1. Then either $S$ has linear growth or $S$ contains a nonabelian free semigroup.

Date: October 14, 2020.

2010 Mathematics Subject Classification. Primary: 20M05 Secondary: 14H37, 20D15.

Key words and phrases. Tits alternative, semigroups, preperiodic points, height functions, rational functions, free semigroups.

The first-named author thanks NSERC for its generous support.

The fourth-named author thanks the Mathematical Sciences Research Institute for its generous support.
We derive Theorem 1.1 from a result relating common preperiodic points of rational functions with free subsemigroups. The techniques used for this result work in the setting of morphisms of projective varieties that are polarized by the same ample line bundle. For a projective variety, a morphism \( f : V \to V \) is said to be polarized by the ample line bundle \( \mathcal{L} \) if there is a degree \( d > 1 \) such that \( f^* \mathcal{L} \cong \mathcal{O}^{d}. \) The notion of polarization is due to Zhang [Zha95]. Any morphism of degree greater than 1 on projective space \( \mathbb{P}^n \) is polarized by \( \mathcal{O}(1) \); it is also true that any morphism of degree greater than 1 on a variety \( V \) comes from restricting a morphism of projective space to \( V \) for some embedding of \( V \) into projective space (see [Pak20]). Polarized morphisms give rise to canonical height functions with good properties (see [CS93] and Section 5.2). In the theorem below and throughout this paper, we let \( \text{Prep}(f) \) denote the semigroup generated under composition by \( f \) when \( f \) is a function from a set to itself. We also let \( \text{Prep}(f) \) denote the set of preperiodic points of \( f \) and \( g \) whenever \( f \) and \( g \) are two functions from a set to itself. We also let \( \text{Prep}(f) \) denote the set of preperiodic points of \( f \) and \( g \), respectively.

**Theorem 1.3.** Let \( V \) be a projective variety, and let \( f, g : V \to V \) be polarized by the same line bundle \( \mathcal{L} \). If \( \text{Prep}(f) \neq \text{Prep}(g) \), then there is a positive integer \( j \) such that \( \langle f^j, g^j \rangle \) is a free semigroup on two generators.

In the case of polynomials, we can derive stronger results in some cases.

**Theorem 1.4.** Let \( S \) be a finitely generated semigroup of polynomials in \( K[x] \) for \( K \) a field such that \( char K \) does not divide the degrees of any element of \( S \). Then either \( S \) has polynomially bounded growth or \( S \) contains a nonabelian free semigroup.

**Theorem 1.5.** Let \( K \) be a field and let \( f, g \in K[x] \) have degree greater than 1. Suppose that \( char K \) does not divide \( \deg f \) or \( \deg g \). If \( \text{Prep}(f) \neq \text{Prep}(g) \), then \( \langle f, g \rangle \) is a free semigroup on two generators.

When \( f \in \mathbb{C}[x] \) has degree greater than 1, its Julia set is determined by \( \text{Prep}(f) \). Thus, Theorem 1.5 implies that if \( f, g \in \mathbb{C}[x] \) have degree greater than one and do not share the same Julia set, then \( \langle f, g \rangle \) is a free semigroup on two generators.

We are also able to prove the following for abelian varieties of any dimension in any characteristic.

**Theorem 1.6.** Let \( A \) be an abelian variety. Let \( S \) be a finitely generated semigroup of finite morphisms from \( A \) to itself. Then either \( S \) has polynomially bounded growth or \( S \) contains a nonabelian free semigroup.

Since every irreducible curve \( C \) with an infinite semigroup of nonconstant maps \( f : C \to C \) has genus 0 or 1, this means that any semigroup of morphisms from a curve to itself either has polynomial growth or contains a nonabelian free semigroup (see Corollary 6.2).

Ritt [Rit22] studied the semigroup of polynomials under composition and gave necessary and sufficient conditions for two polynomials to commute under composition and determined relations for the semigroup of polynomials under composition. It is very possible that in this case, some of the results here can be obtained using Ritt’s work, although there do appear to be some additional subtleties involved. We also point out that the Tits alternative has been considered for automorphism groups of algebraic varieties, with a complete result for projective varieties in characteristic 0 (see [Din12] for a survey) as well as some results in characteristic \( p \) (see [Hin20]). The Tits alternative has also been proved for the Cremona group \( \text{Bir}(\mathbb{P}^2) \) in all characteristics (see [Can11]).

Pakovich [Pak20] has proved if \( S \) is a semigroup of non-special polynomials over the complex numbers, then either \( S \) contains a nonabelian free semigroup or \( S \) is amenable. He also extends this to certain classes of rational functions. We note that one can use Følner conditions along with Theorem 1.2 to prove that if \( S \) is any finitely generated semigroup of non-special rational functions in characteristic 0, then either \( S \) contains a nonabelian free semigroup or \( S \) is left amenable. On the other hand, it is not clear how to adapt our techniques to semigroups that are not finitely generated, though it may be possible to do so in the non-special case. Hindes [Hin20] has proved that certain conditions on a semigroup of rational functions over \( \mathbb{Q} \) guarantee that the semigroup is free; the conditions are much more restrictive than those of Theorem 1.3.
but have the advantage of ensuring that certain semigroups are free (and don’t merely contain a nonabelian free semigroup).

An outline of the paper is as follows. We begin with some preliminaries on semigroups and growth in semigroups in Section 2. In Section 3 we prove Theorem 1.3. We begin the proof in Section 3.4 with a proof of Proposition 3.1, a variant on the ping-pong lemma (see [Tit72]), that can be applied to a wide variety of functions. Then we introduce canonical height functions, both Weil and Moriwaki, which allow us to use Proposition 3.1 to derive Theorem 1.3. We close the section with Examples 3.7 and 3.8 which show that a converse to Theorem 1.3 is not possible in higher dimensions in characteristic 0 or even in dimension 1 in characteristic p. In Section 4, we prove Theorems 1.2 and 1.1. We do so by proving Propositions 4.8 and 4.9 which may be thought of as converses to Theorem 1.3 in the special case of rational functions in characteristic 0; Theorems 1.2 and 1.1 follow immediately from combining Propositions 4.8 and 4.9 with Theorem 1.3. Proposition 1.8 treats the case of non-special rational functions; the proof uses a result due to Ye (see [Ye15] Theorem 1.5) along with an argument about the action of a semigroup of morphisms on a point with a finite orbit under that semigroup. The proof of Proposition 4.9 is then a case-by-case analysis of the different sorts of special rational functions. Section 5 contains the proofs of Theorems 1.4 and 1.5, which follow quickly from some lemmas of Jiang and Zieve [JZ20] about Böttcher coordinates and formal power series. We then prove Theorem 1.6 in Section 6 using standard results on abelian varieties along with a theorem from [OS95]. We conclude the paper in Section 7 with some questions.

Acknowledgments. We would like to thank Alex Carney, Dragos Ghioca, Wade Hindes, Liang-Chung Hsia, Patrick Ingram, Shouwu Zhang, and Michael Zieve for many helpful conversations. We would also like to thank the Simons Foundation and the University of Paderborn for hosting conferences at which some of the problems here were discussed.

2. Preliminaries

We give a brief overview of the basics of semigroups. Let $S$ be a finitely generated semigroup and let $S$ be a finite set of generators for $S$. Then we can form the growth function of $S$ with respect to the generating function $S$ as follows. We define $d_S(n) = |S^{\leq n}|$, where $S^{\leq n}$ is the set of elements of $S$ that can be expressed as a product of elements of $S$ of length at most $n$. The function $d_S(n)$ is weakly increasing as a function of $n$ and while this function depends upon our choice of generating set, we observe that if $T$ is another generating set for $S$ then there exists a positive integer $a$ such that $T \subseteq S^{\leq a}$ and $S \subseteq T^{\leq a}$ and so we have the inequalities

$$d_S(n) \leq d_T(an) \quad \text{and} \quad d_T(n) \leq d_S(an).$$

Thus if we declare that two weakly increasing functions $f, g : \mathbb{N} \to \mathbb{N}$ are asymptotically equivalent if there is a positive integer $C$ such that $f(n) \leq g(Cn)$ and $g(n) \leq f(Cn)$ then the growth function is independent of our choice of generating set up to this asymptotic equivalence. Given a finitely generated semigroup $S$ with finite generating set $S$, we say that $S$ has polynomially bounded growth if $d_S(n) = O(n^k)$ for some positive constant $k$; we say that $S$ has linear growth if there are positive constants $C_1, C_2$ such that $C_1n \leq d_S(n) \leq C_2n$ for all $n$; and we say that $S$ has exponential growth if there is a positive constant $C > 1$ such that $d_S(n) > C^n$ for all $n$ sufficiently large. It is not difficult to check that the properties of having linear growth, polynomially bounded growth, and exponential growth are all preserved under asymptotic equivalence and so we can speak unambiguously of $S$ having these properties without making reference to a generating set.

A semigroup $S$ is left cancellative if whenever $ax = ay$ with $a, x, y \in S$ we have $x = y$; right cancellativity is defined analogously. A cancellative semigroup is one that is both left and right cancellative. Note that if $S$ is a semigroup of surjective maps, then $S$ is right cancellative since $xa = ya$ implies that $x = y$ whenever $a$ is surjective. Hence, in particular, semigroups of nonconstant rational functions are right cancellative; on the other hand $X^2 \circ (-X) = X^2 \circ (X)$, so semigroups of rational functions are not always left cancellative.
We will introduce the theory of height functions in the next section. It seems more natural to us to present them in the context of the proof of Theorem 1.3 than to do so in advance.

3. Proof of Theorem 1.3

3.1. A variant of the ping-pong lemma. In our work here, the functions \( \tau, \tau_f, \) and \( \tau_g \) will be some sort of real-valued height functions (either Weil or Moriwaki). The arguments in this section work in a more general setting, and we state Proposition 3.1 accordingly.

Let \( \mathcal{U} \) be a set, let \( f, g : \mathcal{U} \to \mathcal{U} \) be surjective maps, and let \( \tau : \mathcal{U} \to \mathbb{R} \) be any function that is not bounded in absolute value. Suppose that there are positive real numbers \( d_1, d_2 > 1 \), and a real number \( C \) such that

\[
\begin{align*}
|\tau(f(z)) - d_1 \tau(z)| &< C, \\
|\tau(g(z)) - d_2 \tau(z)| &< C.
\end{align*}
\]

We say that \( d_1 \) is the degree of \( f \) and \( d_2 \) is the degree of \( g \) and write \( \deg f = d_1, \deg g = d_2. \)

Recall that one can use (3.1) to construct canonical functions as follows

\[
\tau_f(z) = \lim_{n \to \infty} \frac{\tau(f^n(z))}{d_1^n} = \tau(z) + \sum_{i=0}^{\infty} \frac{\tau(f^{i+1}(z)) - d_1 \tau(f^i(z))}{d_1^{i+1}},
\]

\[
\tau_g(z) = \lim_{n \to \infty} \frac{\tau(g^n(z))}{d_2^n} = \tau(z) + \sum_{i=0}^{\infty} \frac{\tau(g^{i+1}(z)) - d_2 \tau(g^i(z))}{d_2^{i+1}}.
\]

Note that

\[
\left| \sum_{i=0}^{\infty} \frac{\tau(f^{i+1}(z)) - d_1 \tau(f^i(z))}{d_1^{i+1}} \right| < C \sum_{i=1}^{\infty} \frac{1}{d_1^i}
\]

and

\[
\left| \sum_{i=0}^{\infty} \frac{\tau(g^{i+1}(z)) - d_2 \tau(g^i(z))}{d_2^{i+1}} \right| < C \sum_{i=1}^{\infty} \frac{1}{d_2^i}.
\]

The telescoping sum argument above is due to Tate and was used by Call and Silverman [CS93] in their construction of canonical heights. Kawaguchi [Kaw06, Kaw07] has further developed the theory of canonical heights in the context of semigroups.

Let \( d = \min(d_1, d_2) \), and let

\[
C' = \sum_{i=1}^{\infty} \frac{1}{d^i}.
\]

Using the same telescoping series argument, we see that for any \( n \), we have

\[
\left| \frac{\tau(f^n(z))}{d_1^n} - \tau_f(z) \right| < \frac{C'}{d_1^n},
\]

\[
\left| \frac{\tau(g^n(z))}{d_2^n} - \tau_g(z) \right| < \frac{C'}{d_2^n}.
\]

We will prove the following variant of Tits’ ping-pong lemma (see [Tit72]).

**Proposition 3.1.** Let \( \mathcal{U} \) be a set, let \( \tau : \mathcal{U} \to \mathbb{R} \) be a function that is unbounded in absolute value and let \( f, g : \mathcal{U} \to \mathcal{U} \) be surjective maps that satisfy (3.1). Let \( \tau_f \) and \( \tau_g \) be as defined in (3.2). Suppose that there is some \( z \in \mathcal{U} \) such that \( \tau_f(z) \neq \tau_g(z) \). Then there is a positive integer \( j \) such that

(i) we have \( af^j \neq bg^j \) for all \( a, b \in (f, g) \); and

(ii) the semigroup \( \langle f^j, g^j \rangle \) is a free semigroup on two generators.

We begin with one more definition. Let \( w = \varphi_m \cdots \varphi_1 \), where each \( \varphi_j \) equals \( f \) or \( g \). We define the degree of \( w \) as

\[
\deg w = \prod_{j=1}^{m} \deg \varphi_j.
\]
Since $|\tau(w(z)) - \deg w\tau(z)|$ is bounded for all $z$ (by (3.1)) and $\tau$ is unbounded, we see that the definition in (3.4) is independent of the word representing $w$.

**Lemma 3.2.** Let $w = \varphi_m \cdots \varphi_1$ where $\varphi_i$ is equal to $f$ or $g$ for each $i$. Let $s_i = \varphi_i \cdots \varphi_1$ (for $i \leq m$). With notation as above, we have

$$
(3.5) \quad \left| \frac{\tau(w(z))}{\deg w} - \frac{\tau(s_j(z))}{\deg s_j} \right| < C'
$$

for $j = 1, \ldots, m$.

**Proof.** Let $e_\ell = \deg \varphi_\ell$ for each $\ell = 1, \ldots, m$. Then $\deg s_j = \prod_{\ell=1}^j e_\ell$. Thus, as in (3.1), we have a telescoping series

$$
(3.6) \quad \frac{\tau(w(z))}{\deg w} - \frac{\tau(s_i(z))}{\deg s_i} = \sum_{j=i}^{m-1} \frac{\tau(s_{j+1}(z)) - e_{j+1} \tau(s_j(z))}{\prod_{\ell=1}^{j+1} e_\ell}.
$$

Now

$$
|\tau(s_{j+1}(z)) - e_{j+1} \tau(s_j(z))| = |\tau(\varphi_{j+1}(s_j(z)) - (\deg \varphi_{j+1}) \tau(s_j(z))| < C
$$

for all $j$ by (3.1). Thus

$$
\left| \sum_{j=i}^{m-1} \frac{\tau(s_{j+1}(z)) - e_{j+1} \tau(s_j(z))}{\prod_{\ell=1}^{j+1} e_\ell} \right| < \frac{1}{d^i} \sum_{i=1}^\infty C \leq C' d^i.
$$

This completes the proof, by (3.6).

Now, we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** We choose $\beta$ so that $\tau_f(\beta) \neq \tau_g(\beta)$. Let $\epsilon = |\tau_f(\beta) - \tau_g(\beta)|$. Choose $j$ so that $C'/d^j < \epsilon/4$, where $d = \min(d_1, d_2)$ as above. Let $af^j$ and $bg^j$ be words in $f$ and $g$ such that $\deg af^j = \deg bg^j$. Then, by (3.5) and (3.6), we have

$$
\left| \frac{\tau_f(\beta)}{\deg af^j} - \frac{\tau(g^j(\beta))}{\deg bg^j} \right| < \frac{\epsilon}{2},
$$

Thus,

$$
\frac{\tau_f(\beta)}{\deg af^j} \neq \frac{\tau(g^j(\beta))}{\deg bg^j}.
$$

Since $\deg af^j = \deg bg^j$, this means that

$$
(3.7) \quad af^j(\beta) \neq bg^j(\beta).
$$

Now, let $u = \varphi_m \cdots \varphi_1$ and $w = \theta_n \cdots \theta_1$, where each $\varphi_i$ and $\theta_k$ is equal to $f^j$ or $g^j$. Suppose that $u = w$. We will show by induction on $\max(m, n)$ that $m = n$ and $\theta_i = \varphi_i$ for $i = 1, \ldots, m$. If $m = n = 1$, then we must have $\theta_1 = \varphi_1$ since $f^j \neq g^j$, because $\tau_f \neq \tau_g$. For the inductive step, it will suffice to show that $\varphi_1 = \theta_1$ since we may then cancel (as $f^j$ and $g^j$ are surjective) to obtain $\varphi_m \cdots \varphi_2 = \theta_n \cdots \theta_2$ and apply the inductive hypothesis. We argue by contradiction. If $\varphi_1 \neq \theta_1$, then we may assume without loss of generality that $\varphi_1 = f^j$ and $\theta_1 = g^j$. But then since $\deg u = \deg w$ (because $u = w$), we must have

$$
\varphi_m \cdots \varphi_1(\beta) \neq \theta_n \cdots \theta_1(\beta),
$$

by (3.7), a contradiction, so $\theta_1 = \varphi_1$, and our proof is complete. \qed
3.2. Height functions. We will prove Theorem 1.3 by letting $\tau$ be a height function, either a Weil height $h$ or a Moriwaki height $h_\mathcal{L}$, and using Proposition 3.1 using the fact that the canonical heights attached to these will be zero at exactly the points that are preperiodic. There may be other sorts of functions where Proposition 3.1 may be used though. For example, if we let $\tau : \mathbb{C} \to \mathbb{R}$ be defined by $\tau(z) = \log(\max |z|, 0)$, and $f, g \in \mathbb{C}[x]$ are polynomials of degree greater than 1, we see that $\tau_f$ and $\tau_g$ vanish precisely on the filled Julia sets of $f$ and $g$ respectively. Since the Julia set is simply the boundary of the filled Julia set, Proposition 3.1 thus implies that if the Julia sets of $f$ and $g$ are not equal, then there is a $j$ such that $\langle f^j, g^j \rangle$ is a free semigroup on two generators.

For a more general exposition of the Weil height functions, see [Lan83] and [BG06]. The Moriwaki height functions we use were introduced in [Mor00,Mor01].

Let $V$ be a projective variety and let $f, g : V \to V$. Since $V$ is finitely presented, there is a finitely generated field $K$ such that $f$, $g$, and $V$ are all defined over $K$. If $K$ is not finite, then there is a set $M_K$ of nontrivial absolute values $| \cdot |_v$ on $K$ along with positive integers $e_v$ such that the product formula

$$\prod_{v \in M_K} |z|_v^{e_v} = 1$$

holds for all nonzero $z \in K$.

When $K$ is a number field, these are simply the usual archimedean and $p$-adic absolute values, suitably normalized. When $K$ is a function field over a field $k$, we choose the absolute values from prime divisors on a variety $V$ over $k$ whose function field is a finite extension of $\mathbb{Q}$ when we are in characteristic 0 and a finite extension of $\mathbb{F}_p$ when we are in characteristic $p$. The set of $x \in K$ such that $|x|_v = 1$ for all $v \in M_K$ is called the field of constants.

By extending the $| \cdot |_v$ to $\overline{K}$, we obtain a Weil height function on the projective space $\mathbb{P}^n$ by defining

$$h_{\mathbb{P}^n}(z_0 : \cdots : z_n) = \frac{1}{m} \sum_{v \in M_K} \sum_{i=1}^m \log \max(|z_0^i|_v, \cdots, |z_n^i|_v)$$

where $(z_0^i : \cdots : z_n^i)$, $i = 1, \ldots, m$ is the set of conjugates of $(z_0 : \cdots : z_n)$ in $\overline{K}$ over $K$ (note that while this does depend on our choice of coordinates, a change of coordinates will only change the definition by a bounded constant see [Lan83] or [BG06] for details).

When $\mathcal{L}$ is an ample line bundle on $V$, we can associate a height function to $h$ to $\mathcal{L}$ by letting $\iota : V \to \mathbb{P}^d$ be an embedding such that $\iota^* \mathcal{O}_{\mathbb{P}^d}(1) = \mathcal{L}^{\otimes e}$ (such an $\iota$ and $e$ exist when $\mathcal{L}$ is ample) and taking $h_{\mathcal{L}}(z) = \frac{1}{d} h_{\mathbb{P}^d}(\iota(z))$.

If $\mathcal{L}$ is an ample line bundle on $V$ with associated height function $h_{\mathcal{L}}$ and $\varphi^* \mathcal{L} \cong \mathcal{L}^{\otimes d}$, where $d > 1$, we have

$$(3.8) \quad |h_{\mathcal{L}}(\varphi(z)) - dh_{\mathcal{L}}(z)| < C$$

for all $V(\overline{K})$. We can attach a canonical height to $\varphi$ as in [CS03] by letting $h_{\varphi}(z) = \lim_{n \to \infty} h_{\mathcal{L}}(\varphi^n(z))$.

Note that $h_{\varphi}(\varphi(z)) = dh_{\mathcal{L}}(z)$ by construction, so if $z \in \text{Prep}(\varphi)$, then clearly $h_{\varphi}(z) = 0$.

When $K$ is a number field or a finitely generated function field of characteristic $p$ with a finite field of constants and $h$ is a height function associated to an ample line bundle $\mathcal{L}$, we have the following ([Nor50], [Bak09], Section 1.2)).

**Theorem 3.3. (Northcott)** Let $K$ be a number field or finitely generated function field in characteristic $p$. Let $h_{\mathcal{L}}$, $\varphi$, and $h_{\varphi}$ be as above; when $K$ is a function field, assume that its field of constants is finite. For any constants $A$ and $B$ there are at most finitely many $z \in V(\overline{K})$ such that $h_{\mathcal{L}}(z) \leq A$ and $[K(z) : K] \leq B$. Since $|h_{\varphi} - h_{\mathcal{L}}|$ is bounded, this means that $h_{\varphi}(z) = 0$ if and only if $z \in \text{Prep}(\varphi)$.

Northcott’s theorem does not hold over function fields of characteristic 0 for Weil heights. However, Moriwaki [Mor00,Mor01] has used metrics on line bundles and Arakelov intersection theory to associate a height function $h_{\mathcal{L}}$ to an ample line bundle $\mathcal{L}$ such that a form of Northcott’s
theorem does hold. As with Weil heights, if \( \varphi^* L \cong L^{\otimes d} \) for an ample line bundle \( L \), then by construction (see \cite{YZ13} Section 2.4) there is a constant \( C \) such that
\[
|h_L(\varphi(z)) - d h_L(z)| < C
\]
for all \( z \in V(K) \), where \( h_L \) is a Moriwaki height associated to \( L \). We may form a canonical height \( h_{\varphi} \) for \( h_L \) by taking the limit
\[
h_{\varphi}(z) = \lim_{n \to \infty} \frac{h_L(\varphi^n(z))}{d^n}.
\]
We have the following (see \cite{Mor00,Mor01}).

**Theorem 3.4.** (Moriwaki) For any constants \( A \) and \( B \) there are at most finitely many \( z \in V(K) \) such that \( h_L(z) \leq A \) and \( |K(z) : K| \leq B \). Since \( |h_{\varphi} - h_L| \) is bounded, this means that \( h_{\varphi}(z) = 0 \) if and only if \( z \in \text{Prep}(\varphi) \).

Now we are ready to prove Theorem 1.3. We use Weil heights to treat the case where the field \( K \) is a number field or function field of characteristic \( p \). For function fields of characteristic \( p \), we use Moriwaki heights. Note that we could treat the case of rational functions over function fields of characteristic 0 using Weil heights rather than Moriwaki heights, since Baker \cite{Bak09} has proved a dynamical form of Northcott’s theorem for Weil heights in the case of rational functions, assuming a non-isotriviality condition. This may be possible in higher dimensions, too, as there are more general dynamical Northcott-type results for non-isotrivial maps due to Chatzidakis and Hrushovski \cite{CH08a,CH08b} (see also \cite{GV19}), but non-isotriviality conditions there are a good deal more complicated.

**Proof of Theorem 1.3.** If \( K \) is a number field or a function field of characteristic \( p \) endowed with absolute values having a finite field of constants, we let \( \tau = h_L \) and apply Proposition 3.1 with \( \tau_f = h_f \) and \( \tau_g = h_g \). Note that we meet the conditions of the proposition since (3.8) holds. Thus, if \( h_f \neq h_g \), then there is a \( j \) such that \( \langle f^j, g^j \rangle \) is a free semigroup on two generators. By Theorem 3.3 if \( \text{Prep}(f) \neq \text{Prep}(g) \), then there is a \( z \in V(K) \) such that exactly one of \( h_f(z) \) and \( h_g(z) \) is zero, which means that \( h_f \) cannot equal \( h_g \).

If \( K \) is a function field, we argue the same way, letting \( \tau = h_L \) with \( \tau_f = h_f \) and \( \tau_g = h_g \). The conditions of Proposition 3.1 hold by (3.9) and the existence of a \( z \in V(K) \) that is preperiodic for exactly one of \( f \) and \( g \) implies the existence of a \( z \in V(K) \) such that \( h_f(z) \neq h_g(z) \) (by Theorem 3.4), which in turn implies that there is a \( j \) such that \( \langle f^j, g^j \rangle \) is a free semigroup on two generators by Proposition 3.1.

The following corollary answers a conjecture posed by Cabrera and Makienko \cite{CM19}.

**Corollary 3.5.** Let \( f, g \in \mathbb{C}(x) \) both have degree greater than 1. Let \( d \mu_f \) and \( d \mu_g \) be the measures of maximal entropy for \( f \) and \( g \), respectively. If \( d \mu_f \neq d \mu_g \) then there is a \( j \) such that \( \langle f^j, g^j \rangle \) is a free semigroup on two generators.

**Proof.** Theorem 1.5 of \cite{YZ13} (see also \cite{Car20}) states that if \( \text{Prep}(f) \cap \text{Prep}(g) \) is infinite for \( f, g \in \mathbb{C}(x) \), then \( \mu_f = \mu_g \). \( \square \)

**Remark 3.6.** In fact, Theorem 1.5 of \cite{YZ13} implies that if \( \text{Prep}(f) = \text{Prep}(g) \), then the canonical measures associated to \( f \) and \( g \) are equal to each other in much more generality. However, equality of these measures is a much weaker condition than equality of the set of preperiodic points. For example, equality of measures of maximal entropy at a non-archimedean place is much weaker than equality of the set of preperiodic points, since polarized morphisms having good reduction at a non-archimedean place \( v \) will have the same canonical measure at \( v \). Even over \( \mathbb{C} \), one can have \( \mu_f = \mu_g \) but \( \text{Prep}(f) \neq \text{Prep}(g) \) for special rational functions \( f \) and \( g \) (let \( f(X) = X^2 \) and \( g(X) = \omega X^2 \) where \( |\omega| = 1 \) but \( \omega \) is not a root of unity, for example). On the other hand, equality of measures of maximal entropy for non-special rational functions over \( \mathbb{C} \) has powerful consequences, due to work of Levin \cite{Lev90}; the results of Ye \cite{Ye15} that we use in the next section rely on Levin’s results.
3.3. Some counterexamples. Propositions 4.8 and 4.9 provide a converse to Theorem 1.3 for rational functions in characteristic 0. One might ask more generally, if it is true that when \( f, g : V \to V \) are morphisms polarized by the same line bundle for a variety \( V \) in any dimension over any field, the equality \( \text{Prep}(f) = \text{Prep}(g) \) must imply that \( \langle f, g \rangle \) cannot contain a nonabelian free semigroup. It turns out this is not true for polarized morphisms of varieties of dimension greater than one in characteristic 0, as Example 3.7 shows. In characteristic \( p \), it is not even true for polynomials, as Example 3.8 shows.

**Example 3.7.** Let \( A \) be an abelian variety defined over a number field such that \( \text{End}^0(A) \otimes \mathbb{R} \) is the Hamiltonian quaternion algebra \( \mathbb{H} \). (That such abelian varieties exist is well-known, see Theorem B.33 of [Lew99].) By Theorem 2 of [GMS99], \( 1 + 2i \) and \( 1 + 2j \) generate a free multiplicative subgroup of \( \mathbb{H} \) of order 2. We also have \( \phi^1 \phi = [1 - 2i][1 + 2i] = [5] \) and \( \psi^1 \psi = [1 - 2j][1 + 2j] = [5] \). Therefore \( \phi \) and \( \psi \) are both polarized by the theta divisor \( \Theta \) on \( A \) by Proposition 3.1 of [Paz13]. Since \( \phi \) and \( \psi \) both commute with \([m]\) for any \( m \), we must have \( \text{Prep}(\phi) = \text{Prep}(\psi) = A_{\text{tors}} \).

**Example 3.8.** Let \( K = \mathbb{F}_p \) and let \( d, e > 1 \) be integers such that \( p \nmid de \). Then if \( f(x) = x^d \) and \( g \in \mathbb{F}_p[x] \) is any polynomial of degree \( e \) that is not a monomial, the semigroup \( \langle f, g \rangle \) is a free semigroup on two generators by [JZ20, Lemma 3.1]. Note that \( \text{Prep}(f) = \text{Prep}(g) = \mathbb{F}_p \) since \( f \) and \( g \) are both defined over \( \mathbb{F}_p \).

4. Proofs of Theorems 1.1 and 1.2

We will now prove Theorems 1.1 and 1.2. We will do so by proving results on the growth of finitely generated semigroups \( S \) such that \( \text{Prep}(f) = \text{Prep}(g) \) for all \( f, g \in S \) with degrees greater than 1; these are Propositions 4.8 and 4.9. Combining with Theorem 1.3 then gives Theorems 1.1 and 1.2. We begin with a lemma about semigroups of maps that contain constant maps.

**Lemma 4.1.** Suppose \( S \) is a finite set of maps from a set \( U \) to itself and that \( f \) is a map that sends all of \( U \) to a single element of \( U \). Let \( S_1 = S \cup \{ f \} \). If \( S_1^{\leq n} \) is the set of distinct maps represented by words of length at most \( n \) in \( S \) and \( S_1^{\leq n} = \{ \} \) is the set of distinct maps represented by words of length at most \( n \) in \( S_1 \), then \( |S_1^{\leq n}| \leq 2|S_1^{\leq n}| \).

**Proof.** It will suffice to show that the number of words in \( S_1^{\leq n} \) containing \( f \) is bounded by \( |S_1^{\leq n}| \). Let \( w \in S_1^{\leq n} \) contain \( f \). We write \( w = w_1 f w_2 \) where \( w_1 \) does not contain \( f \) (note that \( w_1 \) may be empty). Let \( v \) be the element of \( U \) such that \( f(u) = v \) for all \( u \in U \). Then \( w(u) = w_1(v) \) for all \( u \in U \). Since \( w_1 \in S_1^{\leq n} \), there are at most \( |S_1^{\leq n}| \) such \( w_1(v) \), and our proof is done.

**Remark 4.2.** While Lemma 4.1 allows us to treat semigroups of rational functions containing constant maps, we cannot expect to obtain results in higher dimensions for semigroups containing morphisms that are neither constant nor finite, because of the examples in [Okn93].

Throughout this section, \( S \) will denote a finitely generated semigroup of rational functions in \( \mathbb{C}(x) \) and \( S^+ \) will denote the set of elements in \( S \) of degree greater than 1.

We begin with a simple lemma.

**Lemma 4.3.** Let \( f \in \mathbb{C}(x) \) be a non-special rational function of degree greater than 1. Then the set of \( \sigma \in \mathbb{C}(x) \) of degree 1 such that \( \mu_f = \mu_{\sigma f} \) is finite.

**Proof.** Let \( J \) denote the Julia set of \( f \) and \( \sigma f \) and let \( \mu \) denote their measure of maximal entropy. Then \( \sigma(J) = J \) and \( \mu(A) = \mu(\sigma(A)) \) for any set \( A \) in \( \mathbb{P}^1(\mathbb{C}) \). Thus, \( \sigma \) is a symmetry on \( J \) in the sense of [Lev90, Definition 1]. Since \( f \) is non-special the set of such symmetries is finite by [Lev90, Theorem 1].

We will use the following result due to Ye [Ye15]. Since his argument appears with slightly different notation as an implication in the proof of [Ye15, Theorem 1.5] (on page 393), rather than as a lemma or theorem, we provide a proof using the same argument here.
Lemma 4.4. Let \( f, g \in \mathbb{C}(x) \) be rational functions such that \( \mu_f = \mu_g \) and \( \deg f = \deg g > 1 \). Suppose that there is an \( \alpha \neq \infty \) in \( J_f = J_g \) such that \( f(\alpha) = g(\alpha) = \alpha \) and \( f'(\alpha) = g'(\alpha) \). Then \( f = g \).

Proof. We will show that \( R := f \circ g^{-1} \) is the identity map. Otherwise, since \( R \) has multiplier equaling 1 at \( \alpha \), it determines attracting and repelling flowers near \( \alpha \). Suppose that there is some point \( x \) near \( \alpha \) in the Julia set that is also in some attracting petal of the flowers determined by \( R \) (see [Mil99, Section 10]). Then there is some fundamental domain of \( R \) for this petal, which contains some neighborhood of this point \( x \). As the measure of maximal entropy \( \mu \) is supported in the Julia set, the fundamental domain won't have zero measure. Since \( R \) acts on this petal like a transformation (in appropriate coordinates) and \( R \) preserves the measure \( \mu \) (since \( \deg f = \deg g \)), the \( \mu \)-measure of this petal cannot be finite. □

Lemma 4.5. Let \( f, g \in \mathbb{C}(x) \) be non-special rational functions of degree greater than 1 such that \( \text{Prep}(f) = \text{Prep}(g) \) and \( \deg f = \deg g \). Let \( \mathcal{J} \) denote \( J_f = J_g \). Suppose there is a periodic cycle \( \{x_1, \ldots, x_r\} \) for \( f \) and \( g \) in \( \mathcal{J} \cap \mathbb{C} \) such that \( f(x_i) = g(x_i) = x_{i+1} \) for \( i = 1, \ldots, r-1 \) and \( f(x_r) = g(x_r) = x_1 \). Then we have the following:

1. \( f'(x_1)/g'(x_1) \) is a root of unity.
2. If \( f'(x_1) = g'(x_1) \), then \( f = g \).

Proof. Let \( \alpha_i = f'(x_i) \) and let \( \beta_i = g'(x_i) \) for \( i = 1, \ldots, r \). Note that none of the \( \alpha_i \) and \( \beta_i \) are zero since the \( x_i \) are in the Julia set for \( f \) and \( g \). Now let \( h_1 = fg^{-1} \) and let \( h_2 = g' \). Then \( h_1(x_2) = h_2(x_2) = x_2 \). We have

\[
(4.1) \quad h_1'(x_2) = \alpha_1 \prod_{i=2}^{n} \beta_i
\]

and

\[
(4.2) \quad h_2'(x_2) = \beta_1 \prod_{i=2}^{n} \beta_i
\]

by the chain rule.

Since \( \text{Prep}(h_1) = \text{Prep}(h_2) = \text{Prep}(f) = \text{Prep}(g) \) and \( \mathcal{J} \) contains a point that is periodic for both \( h_1 \) and \( h_2 \), it follows from [Ye15, Theorem 1.5] that there is an \( n \) such that \( h_1^n = h_2^n \). By (4.1) and (4.2), this means that \( (\alpha_1/\beta_1)^n = 1 \), so \( f'(x_1)/g'(x_1) \) is a root of unity, as desired. Furthermore, if \( f'(x_1) = g'(x_1) \), then (4.1) and (4.2) imply that \( h_1'(x_2) = h_2'(x_2) \), which means that \( h_1 = h_2 \), by Lemma 4.4. Now, if \( fg^{-1} = g' \), then \( f = g \) by right cancellation. □

Lemma 4.6. Let \( S \) be a finitely generated semigroup of rational functions in \( \mathbb{C}(x) \) such that the elements of \( S^+ \) are non-special. Suppose that \( S \) is not empty and that \( \text{Prep}(f) = \text{Prep}(g) \) for all \( f, g \in S^+ \). Then there is a constant \( N \) such that for all \( d \geq 1 \), the number of elements of \( S \) of degree \( d \) is less than or equal to \( N \).

Proof. There are finitely many elements in \( S \) of degree 1 by Lemma 4.3 since there is an \( f \in S \) of degree greater than 1 such that \( \text{Prep}(f) = \text{Prep}(f) \) for all \( \sigma \in S \) of degree 1.

Now, let \( \mathcal{J} \) be the Julia of the elements in \( S \) having degree greater than 1. Let \( \mathcal{P} \) be the set of preperiodic points of the elements \( S^+ \). Then \( f(\mathcal{P}) = \mathcal{P} \) for all \( f \in S^+ \). Choose a \( \gamma \in \mathbb{C} \cap \mathcal{P} \). Let \( K \) be a finitely generated field over which \( \gamma \) and every element of \( S \) is defined. Since \( \text{Prep}(f) \cap K \) is finite for each \( f \) of degree greater than 1, it follows that the orbit of \( \gamma \) under \( S^+ \) is finite. After change of coordinates, we may assume that \( O \) does not contain the point at infinity.

Let \( n \) be the number of roots of unity in \( K \). Let \( f \in S^+ \) have degree \( d \). We will show that there are at most \( n \) elements \( g \in S^+ \) such that \( \deg g = d \) and \( g|_O = f|_O \). Let \( \{x_1, \ldots, x_r\} \) be a periodic cycle for \( f \) in \( \mathcal{O} \) (note that there must be one since \( O \) is finite) such that \( f(x_i) = x_{i+1} \) for \( i = 1, \ldots, r-1 \) and \( f(x_r) = x_1 \). Then for any \( g \in S \) such that \( \deg g = d \) and \( g|_O = f|_O \), we may apply Lemma 4.5 to conclude that \( f'(x_1)/g'(x_1) \) is a root of unity in \( K \). Furthermore, given any \( g_1, g_2 \in S \) of degree \( d \) such that \( g_1|_O = g_2|_O = f|_O \), we have \( g_1 = g_2 \) whenever \( g_1'(x_1) = g_2'(x_1) \). Thus, the number of \( g \in S \) such that \( \deg g = d \) and \( g|_O = f|_O \) is bounded by the number of
roots of unity in $K$, which is $n$. Since $O$ is finite, there are $|O|^{|O|}$ maps from $O$ to itself, so there are at most $n|O|^{|O|}$ elements of $S$ of degree $d$ for any $d > 1$.

**Lemma 4.7.** Suppose that $S$ is a semigroup of rational functions in $\mathbb{C}(x)$ such that every element of $S^+$ is non-special and has the same set of preperiodic points. Then there is some $a \geq 2$ such that every element of $S$ has degree $a^n$ for some $n \geq 0$.

**Proof.** By [Lev90, Theorem 3], for any $f, g \in S^+$ there are $m$ and $n$ such that $f^m g^n = f^{2m}$ so $(\text{deg}(f))^m = (\text{deg}(g))^n$. Since the set $U$ of possible degrees of elements in $S^+$ is a finitely generated subsemigroup of the positive natural numbers under multiplication, it follows that $U$ is contained in a semigroup generated by a single element $a$.

Now we are ready to state and prove Proposition 4.8.

**Proposition 4.8.** Let $S$ be a finitely generated semigroup of rational functions in $\mathbb{C}(x)$ such that $S^+$ contains a non-special rational function of degree greater than 1. Suppose that $\mu_f = \mu_g$ for all $f, g \in S^+$. Then $S$ has linear growth.

**Proof.** It suffices to prove this when every element of $S$ is nonconstant, by Lemma 4.1. Now, since some element of $S^+$ is non-special, by hypothesis, we see that all elements of $S^+$ are non-special (see [Lev90], for example). Let $f_1, \ldots, f_s$ be generators for $S$. Then for each $i$, we have that there is some $a \geq 2$ such that $\text{deg}(f_i) = a^m$, by Lemma 4.7; let $M = \max(m_1, \ldots, m_s)$. Now consider the set $S^{\leq n}$ of elements in $S$ formed by taking a composition of length $\leq n$ of elements from $f_1, \ldots, f_s$. These elements all have degree in $\{a, a^2, \ldots, a^M\}$, so there is a natural number $N$ such that $|S^{\leq n}| \leq MNn$ for all $n$, by Lemma 4.6. Clearly, $|S^{\leq n}| \geq n + 1$ since the elements $id, f, f^2, \ldots, f^n$ are pairwise distinct for $f \in S^+$ and so we see that $S$ has linear growth as claimed.

We are now ready to prove state and prove Proposition 4.9.

**Proposition 4.9.** Let $S$ be a finitely generated semigroup of rational functions in $\mathbb{C}(x)$. Suppose that for any $f, g \in S^+$, we have $\text{Prep}(f) = \text{Prep}(g)$. Then $S$ has polynomially bounded growth.

**Proof.** Again, we may assume that every element of $S$ is nonconstant, by Lemma 4.1. Because of Proposition 4.8, we need only treat the cases where every element of $S$ is linear or $S^+$ contains a special rational function. We treat them case-by-case.

**Case I. Every element of $S$ is linear.** In this case, the result is contained in [OS95, Theorem 1.5].

**Case II. Some element of $S^+$ is conjugate to a Chebychev polynomial.**

Let $f \in S^+$ be conjugate to a Chebychev polynomial. Then there is a $\sigma \in \mathbb{C}(x)$ such that $\sigma^{-1} f \sigma = T_d \in \mathbb{Q}(x)$, where $T_d$ is the Chebychev polynomial of degree $d$. If there is some $g \in S^+$ such that $\sigma^{-1} g \sigma / T_d \notin \mathbb{Q}(x)$, then clearly $\text{Prep}(f) \neq \text{Prep}(g)$. Hence we may assume that $\sigma^{-1} S^+ \sigma \subset \mathbb{Q}(x)$. Furthermore we may assume that $h_{\sigma^{-1} \sigma} = h_{T_d}$ for all $g \in S^+$ by Theorem 3.3. Thus, we may apply [KS07, Theorem 1] to conclude that for all $g \in S^+$, we have some $d$ such that $\sigma^{-1} g \sigma = \pm T_d$. Let $S^{\leq n}$ be the words of length $n$ in some finite set $S$ of generators for $S$. Then the number of possible degrees for elements of $S^{\leq n}$ is bounded by $O(n^s)$ where $s = |S|$, so the number of elements of $S^{\leq n}$ of degree greater than 1 is bounded by $O(n^s)$ since there are at most two elements in $S^+$ having the same degree by the above. The number of elements of $S^{\leq n}$ of degree 1 is also bounded by a polynomial in $n$ by [OS95, Theorem 1.5]. Hence $S$ has polynomially bounded growth.

**Case III. Some element of $S^+$ is conjugate to $X^m$.**

There is some $f \in S^+$ and some $\sigma \in \mathbb{C}(x)$ such that $\sigma^{-1} f \sigma = X^m$ for some integer $m$ with $|m| > 2$. Then, as in the Chebychev case, either $\sigma g \sigma^{-1} \in \mathbb{Q}(x)$ for every $g \in S^+$. By [KS07, Theorem 2], we must also have $\sigma g \sigma^{-1} = \xi x^n$. Since all the elements of $S$ are defined over $K$, there are finitely many roots of unity $\xi$ that are possible. Thus there is a bound on the number of elements of $S^+$ of any given degree. Let $S^{\leq n}$ be the words in $S$ of length $n$ on some finite set of generators. As in the Chebychev case, the number of words of in $S^{\leq n}$ of degree
greater than 1 is bounded by $O(n^s)$ where $s$ is the number of generators and the number of elements of $S^{<n}$ of degree 1 is also bounded by a polynomial in $n$ by [OS95] Theorem 1.5.

**Case IV. Some element of $S^+$ is Lattès.**

Let $K$ be a finitely generated field such that every element of $S$ is defined over $K$. Recall that a Lattès map for an elliptic curve $E$ is a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than 1 such that there are nonconstant maps $\phi : E \to E$ and $\pi : E \to \mathbb{P}^1$ with the property that $f \pi = \pi \phi$. Suppose that some element $f$ of $S$ is a Lattès map for an elliptic curve $E$. Since Prep($g$) = Prep($f$) for every $g \in S^+$, it follows from [KS07] Theorem 27] that every element of $S^+$ is a Lattès map for $E$. Hence, for each $f \in S^+$, there is a $\pi_f : E \to \mathbb{P}^1$ and a $\phi_f : E \to E$ such that $f \pi_f = \pi_f \phi_f$.

By [Mil04] Theorem 3.1, the degree of $\pi_f$ is either 2,3,4, or 6 (note that the possibilities of 3, 4, and 6 can only arise when the endomorphism ring of $E$ is an order in an imaginary quadratic field containing roots of unity other than $\pm 1$). Furthermore, by Theorem [KS07] Theorem 30], if $\deg f = \deg g$ and $\deg \pi_f = \deg \pi_g$, then $g = \pi_g f \pi_g^{-1}$ for some $\pi_g \in K(x)$ of degree 1. After replacing $K$ by a suitable finite extension, we may assume that there are at least three points in Prep($f$) $\cap K$. Since $\sigma$ must send Prep($f$) $\cap K$ to itself, there are finitely many possible $\pi_g$.

So now we have that there is a finite set $\Pi$ of maps $\pi : E \to E$ such that for any $g \in S^+$ we have $g \pi_g = \pi_g \theta_g$ for some $\pi_g \in \Pi$ and some $\theta_g : E \to E$. Let $S^{\leq n}$ be the set of elements of $S$ represented by words of length $n$ in some set of generators for $S$. Since there are at most $|\Pi|$ possible choices for $\pi_g$ for $g \in S^+ \cap S^{\leq n}$, we will done if we can show that the number of $\theta_g$ for $g \in S^+ \cap S^{\leq n}$ is bounded by a polynomial in $n$ (as before, the number of elements of degree 1 in $S^{\leq n}$ is bounded by a polynomial in $n$ by [OS95] Theorem 1]). Let $s$ be the number of distinct degrees of elements in some generating set of $S$. The number of possible degrees of elements in $S^{\leq n}$ is then bounded by $O(n^s)$. Now let $M$ be the maximal degree of an element of our generating set and let $\mathcal{P} = \{p_1, \ldots, p_t\}$ be the set of primes dividing the degrees of the elements of our generating set. Then every element of $S^{\leq n} \cap S^+$ has degree bounded by $M^n$ and is divisible only by elements of $\mathcal{P}$. Now, each $\theta_g$ can be written as $m_g + t_g$ where $m_g \in \text{End}(E)$ and $t_g$ is translation by torsion point. Since there are only finitely many torsion points of $E$ in $K$, only finitely many translations arise, and we know that the number of different possible degrees of $m_g$ for $g \in S^+ \cap S^{\leq n}$ is bounded by a polynomial in $n$, we will be done if we can show that the number of $m_g$ having fixed degree $g \in S^+ \cap S^{\leq n}$ is bounded by a polynomial in $n$. If End$(E) = \mathbb{Z}$, this is clear (that number is 2). Otherwise End$(E)$ is an order in an imaginary field. In the ring of integers of an imaginary quadratic field the number of ideals having norm $N = p_1^{e_1} \cdots p_t^{e_t}$ is bounded by $(e_1 + 1) \cdots (e_t + 1)$ (this can be seen by noting that an ideal having norm $p^\ell$ has at most $\ell + 1$ possible factorizations – see [HW08] Chapter 17, for example). Since there at most 6 units in an order in an imaginary quadratic field, this means that the number of elements having norm $N$, where $N$ is a number whose only prime factors are in $\{p_1, \ldots, p_t\}$ is bounded by $O((\log_2 N)^6)$. For $N \leq M^n$, we see that $(\log_2 N)^6$ is bounded by a polynomial in $n$; thus, the number of $m_g : E \to E$ corresponding to a $g \in S^+ \cap S^{\leq n}$ having fixed degree is indeed bounded by a polynomial in $n$, and our proof is complete. \hfill $\Box$

**Remark 4.10.** The proof of Proposition [4.9] shows that for a finitely generated field $K$ and any rational function $f \in K(x)$ that is not a Lattès map, there is a constant $N(f, K)$ such that for any $d$, the set of rational functions $g$ of degree $d$ such that Prep($g$) = Prep($f$) has at most $N(f, K)$ elements. This is clearly not true for Lattès maps associated to elliptic curves with complex multiplication since the number of elements of an order in a quadratic number field having fixed norm can be arbitrarily large.

Now we can easily prove Theorems [1.2] and [1.4].

**Proof of Theorem 1.2.** By Theorem 1.3, the semigroup $S$ contains a free subsemigroup on two elements unless Prep($f$) = Prep($g$) for all $f, g \in S^+$. If Prep($f$) = Prep($g$) for all $f, g \in S^+$, then $S$ has linear growth by Proposition 1.8. \hfill $\Box$

**Proof of Theorem 1.4.** As above, the semigroup $S$ contains a free subsemigroup on two elements unless Prep($f$) = Prep($g$) for all $f, g \in S^+$, by Theorem 1.3. If Prep($f$) = Prep($g$) for all $f, g \in S^+$, then $S$ has polynomially bounded growth by Proposition 4.9. \hfill $\Box$
Corollary 4.11. Let \( f, g \in \mathbb{C}(X) \) be two rational functions, each having degree greater than 1. Then the following are equivalent:

(i) \( \text{Prep}(f) \cap \text{Prep}(g) \) is infinite;
(ii) \( \text{Prep}(f) = \text{Prep}(g) \);
(iii) for any \( \varphi_1, \varphi_2 \in \langle f, g \rangle \), we have \( \text{Prep}(\varphi_1) = \text{Prep}(\varphi_2) \);
(iv) \( \langle f, g \rangle \) has polynomial growth;
(v) \( \langle f, g \rangle \) does not contain a nonabelian free semigroup;
(vi) for any \( \ell > 0 \), the semigroup \( \langle f^\ell, g^\ell \rangle \) is not the free semigroup on two generators.

Proof. It is clear that (iii) implies (ii), that (iv) implies (v), and that (v) implies (vi). Theorem 1.2 of \cite{BD11} (see also \cite{MM13,YZ17,YZ13,Car20}) states that (i) and (ii) are equivalent. Proposition 4.9 shows that (iii) implies (iv). By Theorem 1.3, we have (vi) implies (iii). Hence, we will be done if we can show that (ii) implies (iii). Assume (ii) holds. Let \( \mathcal{U} = \text{Prep}(f) = \text{Prep}(g) \). Then \( f(\mathcal{U}) \subseteq \mathcal{U} \) and \( g(\mathcal{U}) \subseteq \mathcal{U} \). Let \( \varphi \in \langle f, g \rangle \). Then we have \( \varphi(\mathcal{U}) \subseteq \mathcal{U} \). Since \( \mathcal{U} \) contains at most finitely many points defined over any finitely generated field by Theorem 1.2 of \cite{BD11} it follows that for any \( z \in \mathcal{U} \), the orbit of \( z \) is finite under \( \varphi \), so \( \mathcal{U} \subseteq \text{Prep}(\varphi) \). Since \( \text{Prep}(f) \) and \( \text{Prep}(g) \) are infinite, it follows from Theorem 1.2 of \cite{BD11} that we have \( \text{Prep}(\varphi) = \text{Prep}(f) = \text{Prep}(g) \). Thus, \( \text{Prep}(\varphi_1) = \text{Prep}(\varphi_2) = \text{Prep}(f) = \text{Prep}(g) \) for any \( \varphi_1, \varphi_2 \in \langle f, g \rangle \).

5. Proofs of Theorems 1.4 and 1.5

We will prove a theorem that is slightly more general than Theorem 1.4. First a bit of notation. Let \( \varphi(\alpha) = \alpha \) for \( \varphi \) a nonconstant rational function in \( \mathbb{C}(x) \) and \( \alpha \in \mathbb{P}^1(\mathbb{C}) \). We let \( e_\varphi(\alpha) \geq 1 \) denote the degree of the term of lowest positive degree in the formal power series expansion of \( \varphi \) centered at \( \alpha \) (i.e. in \( \varphi \) written locally as an element of \( K[[X - \alpha]] \)).

Theorem 5.1. Let \( S \) be a finitely generated semigroup of rational functions in \( K(X) \). Suppose that one of the following statements holds:

1. Every nonconstant element of \( S \) has degree 1.
2. There is an \( \alpha \in \mathbb{P}^1(\mathbb{C}) \) fixed by every element of \( S \) such that \( e_f(\alpha) > 1 \) for some nonconstant \( f \in S \) and \( \text{char} \ K \nmid e_g(\alpha) \) for every nonconstant element of \( S \).

Then either \( S \) contains a nonabelian free semigroup or \( S \) has polynomially bounded growth.

We begin with some lemmas from Jiang and Zieve.

Lemma 5.2. (\cite{JJ20}, Lemma 2.1) Let \( f \in K[[X]] \) have a lowest degree term of degree \( m \geq 1 \) where \( \text{char} \ K \nmid m \). Then, there is a finite extension \( K' \) of \( K \) and an \( L \in K'[[[X]] \) with lowest degree term of degree 1 such that \( L^{-1} \circ f \circ L = X^m \).

Lemma 5.3. (\cite{JJ20}, Lemma 3.1) Let \( K \) be a field, let \( m, n > 1 \) be integers not divisible by \( \text{char} \ K \), and let \( g \in K[[X]] \) have lowest-degree term of degree \( n \). If \( g \) has at least two terms then \( \langle X^m, g \rangle \) is a free semigroup on two generators.

Lemma 5.4. (\cite{JJ20}, Lemma 3.2) Let \( K \) be a field, let \( m, n > 1 \) be integers, and let \( \alpha \in K^* \) have infinite order. Then \( \langle X^m, \alpha X^n \rangle \) is a free semigroup on two generators.

Note that Lemma 5.4 is not true when \( n = 1 \); for example if \( f(X) = X^2 \) and \( g(X) = 2X \), then \( g^2 \circ f = f \circ g \), but we still have the following.

Lemma 5.5. Let \( K \) be a field and let \( m \) be a positive integer that is not divisible by \( \text{char} \ K \). Then for any \( \alpha \in K^* \) of infinite order, the semigroup \( \langle X^m \circ \alpha X, \alpha X \circ X^m \rangle \) is a free semigroup on two generators.

Proof. We have \( X^m \circ \alpha X = \alpha^m X^m \) and \( \alpha X \circ X^m = \alpha X^m \), Let \( L = \alpha^{-1/(m-1)} X \). Then \( L^{-1} \circ \alpha X^m \circ L = X^m \) and \( L^{-1} \circ \alpha^m X^m \circ L = \alpha^{m-1} X^m \), and \( \langle X^m, \alpha^{m-1} X^m \rangle \) is a free semigroup on two generators by Lemma 5.4.

We are now ready to prove Theorem 5.1.
Proof. Again we may assume that every element of \( S \) is nonconstant, because of Lemma \ref{lem:nonconstant}. If all the elements of \( S \) have degree 1, then this is \cite[Theorem 1]{OS95}. We choose coordinates so that \( \alpha = 0 \) and write each element of \( S \) as a formal power series in \( K[[X]] \). After passing to a finite extension and conjugating by some \( L \in K[[X]] \), there is some \( f \in S \) and some \( L \in K[[X]] \) where the degree of the term of lowest degree of \( L \) is 1 such that \( L^{-1}fL = X^m \) where \( m > 1 \) and \( (\text{char} \ K) \nmid m \), by Lemma \ref{lem:degree}. If there is some \( g \in K[[X]] \) with \( e_g(\alpha) = n > 1 \) such that \( L^{-1}gL \) is not equal to \( \xi X^n \) for \( \xi \) a root of unity, then \( \langle f, g \rangle \) is a free semigroup on two generators by Lemmas \ref{lem:degree} and \ref{lem:free}. Similarly, if there is some \( g \in K[[X]] \) with \( e_g(\alpha) = 1 \) such that \( L^{-1}gL \) is not equal to \( \alpha X \) for some \( \alpha \in K^* \), then \( \langle f, g \rangle \) is a free semigroup on two generators by Lemma \ref{lem:degree} likewise, if there is some \( g \in K[[X]] \) with \( e_g(\alpha) = 1 \) such that \( L^{-1}gL \) is equal to \( \alpha X \) where \( \alpha \in K^* \) is not a root of unity, then \( \langle f, g \rangle \) is a free semigroup on two generators by Lemma \ref{lem:degree}.

Thus, we are reduced to showing that a semigroup of the form \( \langle \xi_r X^{e_1}, \ldots, \xi_r X^{e_t} \rangle \), where each \( \xi_i \) is a root of unity, has polynomially bounded growth. Let \( S \leq t \) be the set of words of length \( t \) or less in \( \langle \xi_r X^{e_1}, \ldots, \xi_r X^{e_t} \rangle \). Then the number of possible degrees of elements of \( S \) is bounded by \( O(t^v) \). Since the group generated by \( \{\xi_1, \ldots, \xi_t\} \) is finite, this means that the number of elements in \( S \) is also bounded by \( O(t^v) \) and we are done. \( \square \)

We are now ready to prove Theorem \ref{thm:tits} in a slightly more general form.

**Theorem 5.6.** Let \( f, g \in K(x) \) be nonconstant rational functions with a fixed point \( \alpha \) such that \( e_f(\alpha) \) and \( e_g(\alpha) \) are both greater than 1 and not divisible by \( \text{char} \ K \). If \( \text{Prep}(f) \neq \text{Prep}(g) \), then \( \langle f, g \rangle \) is a free semigroup on two generators.

*Proof.* By Lemma \ref{lem:degree} after taking a finite extension there is a \( L \in K[[X]] \) such that \( L^{-1}fL = X^m \) for \( m > 1 \) and not divisible by \( \text{char} \ K \). By Lemmas \ref{lem:degree} and \ref{lem:free} the semigroup \( \langle f, g \rangle \) must be free unless \( LgL^{-1} = \xi X^n \) for \( \xi \) a root of unity. As in the proof of Theorem \ref{thm:tits}, the semigroup \( \langle X^m, \xi X^n \rangle \) has polynomially bounded growth, so we see that \( \langle f, g \rangle \) is either free or has polynomially bounded growth.

If \( \text{Prep}(f) \neq \text{Prep}(g) \), then \( \langle f, g \rangle \) contains a free group on two elements by Theorem \ref{thm:tits} and thus does not have polynomially bounded growth, so \( \langle f, g \rangle \) must be free. \( \square \)

6. Proof of Theorem \ref{thm:tits}

*Proof of Theorem \ref{thm:tits}.* By \cite[Theorem 2]{Lit76}, any morphism \( f : A \rightarrow A \) can be written as \( \psi + t_a \) where \( \psi \) is a group endomorphism of \( A \) and \( t_a \) is the translation-by-\( a \) map for \( a \in A \). Thus, we can write \( S \) as \( \langle \psi_1 + ta_1, \ldots, \psi_m + ta_m \rangle \) where each \( \psi_i \) is a group endomorphism of \( A \). Let \( K \) be a finitely generated field such that \( A \) and all the \( \psi_i \) are defined over \( K \) and such that all the \( a_i \) are in \( A(K) \). After passing to a finite extension, we may assume that \( A(K) \) is Zariski dense in \( A \).

For any group abelian \( B \), we let \( T(B) \) denote the set of affine maps on \( B \) – that is, the set of maps from \( G \) to itself that are compositions of group homomorphisms and translations. Since \( A(K) \) is Zariski dense in \( A \), each element of \( S \) is uniquely determined by its action on \( A(K) \), so there is a natural embedding \( \iota : S \rightarrow T(A(K)) \). By \cite{Nor52}, the group \( A(K) \) is finitely generated; we let \( n \) denote its free rank. We may write \( A(K) = A(K)_{\text{tors}} \bigoplus G \), for some finitely generated free abelian group \( G \) of rank \( n \). Then the projection map \( \pi : A(K) \rightarrow G \) gives rise to a natural map \( \theta : T(A(K)) \rightarrow T(G) \) such that

\[
(6.1) \quad |\theta^{-1}(h)| \leq (n|A(K)_{\text{tors}}|)|A(K)_{\text{tors}}|
\]

for all \( h \in T(G) \). Let \( S' \) be the image of \( S \) under \( \theta \circ \iota \). Certainly, \( S \) will contain a nonabelian free semigroup whenever \( S' \) does; because of \cite{OS95}, \( S \) must also have polynomial growth whenever \( S' \) does.

Since each element of \( S \) is a finite morphism, each element of \( S' \) must be finite-to-one. Tensoring with \( \mathbb{Q} \), we thus see that \( S' \) is isomorphic to a cancellative semigroup of \( \text{Aff}_n(\mathbb{Q}) \), the group of invertible affine linear maps on an \( n \)-dimensional vector space over \( \mathbb{Q} \). Since \( \text{Aff}_n(\mathbb{Q}) \) embeds
Remark 6.1. The hypothesis that the morphisms \( f : A \to A \) are finite is necessary. There are examples of semigroups of \( 3 \times 3 \) matrices with integer coefficients having intermediate growth in [Okn93]. Thus, for any elliptic curve \( E \), the endomorphism ring \( \text{End}(E \times E \times E) \) contains multiplicative semigroups of intermediate growth.

Corollary 6.2. Let \( C \) be an irreducible curve over \( \mathbb{C} \) and let \( S \) be a finitely generated semigroup of morphisms from \( C \) to itself. Then either \( S \) has polynomially bounded growth or \( S \) contains a nonabelian free semigroup.

Proof. By Lemma 4.1 we may assume that every element of \( S \) is nonconstant. Any morphism \( f : C \to C \) extends to a morphism \( \tilde{f} : C' \to C' \) for \( C' \) the normalization of the projective closure of \( C \). Hence we may assume that \( C \) is projective and nonsingular. If \( C \) has genus greater than 1, then \( S \) must be finite (see [Har77, Ex. IV.5.2], for example), so we may assume that \( C \) is isomorphic either to \( \mathbb{P}^1 \) or to an elliptic curve. Applying Theorems 1.1 and 1.6 then gives the desired conclusion, since every nonconstant map on an elliptic curve is finite. \( \square \)

7. Further directions

We close with some general questions. In all of these questions, \( K \) will be a field of arbitrary characteristic.

Question 7.1. Are there rational functions \( f, g \in K(X) \) of degree greater than one such that \( \text{Prep}(f) \neq \text{Prep}(g) \) and \((f, g)\) is not a free semigroup on two generators?

One might also ask for something weaker, namely that there is a \( j \) depending only on \( K \) such that \((f^j, g^j)\) must be free whenever \( \text{Prep}(f) \neq \text{Prep}(g) \). This might be thought of as analogous to the uniform version of the Tits alternative proved by Breuillard and Gelander [BG08]. Recent work of DeMarco, Krieger, and Ye [DKY20] suggests it may be possible here to use quantitative equidistribution techniques to get good uniform bounds on \( \lim \inf |h_f - h_g| \) where \( h_f \) and \( h_g \) are the canonical heights associated to \( f \) and \( g \) as in Section 3.2, at least in the case of number fields (see also [FRL06] and [PST12]). This might allow for more precision in the conclusion of Proposition 5.1.

Question 7.2. Let \( V \) be a projective variety, let \( \mathcal{L} \) be an ample line bundle on \( V \), and let \( S \) be a finitely generated semigroup of morphisms \( f \) that are polarized by \( \mathcal{L} \). Is it true that \( S \) must either have polynomially bounded growth or contain a nonabelian free semigroup?

It might also be natural to ask for a version of the Tits alternative for semigroups of polarized maps that says something about the structure of the semigroups rather than the growth. For example, one might ask if it is true that any finitely generated semigroup of morphisms polarized by the same line bundle must contain either a nilpotent subsemigroup of finite index or a free subsemigroup on two generators (there is a notion of nilpotence for semigroups due to Malcev [Mal53]). Grigorchuk [Gri88] has shown that finitely generated cancellative semigroups have polynomially bounded growth if and only if they have a group of left quotients with a nilpotent subgroup of finite index; this extends well-known work of Gromov [Gro81] from the cancellative semigroup setting. It follows immediately that any cancellative finitely generated semigroup of rational functions contains either a nilpotent subsemigroup of finite index or a free semigroup on two generators. Thus, in the case of cancellative subgroups of rational functions over \( \mathbb{C} \), we do have a natural structural analog of the Tits alternative for linear groups.

Semigroups of polarized morphisms are not cancellative in general, however, as noted in Section 2. On the other hand, we can show that a finitely generated semigroup of polynomials in \( \mathbb{C}[z] \) contains either a nilpotent subsemigroup of finite index or a nonabelian free semigroup. We can also show that if a finitely generated semigroup \( S \) of polarized morphisms contains a nilpotent subsemigroup of finite index then all the elements of \( S \) have the same set of preperiodic points. Finally, the proof of Theorem 1.1 can be modified to show that if \( S \) is a finitely generated
semigroup of rational functions over $\mathbb{C}$ that does not contain a Lattès map, then either $\mathcal{S}$ contains a nonabelian free semigroup or there is an $N$ such that the degree map is at most $N$-to-1 on $\mathcal{S}^+$ (see Remark 4.10). It is not clear to us, though, what the right kinds of general structural questions are in the non-cancelling setting of polarized morphisms.

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