Algorithmic Chaos

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Abstract

Many physical theories like chaos theory are fundamentally concerned with the conceptual tension between determinism and randomness. Kolmogorov complexity can express randomness in determinism and gives an approach to formulate chaotic behavior.

1 Introduction

Ideally, physical theories are abstract representations—mathematical axiomatic theories for the underlying physical reality. This reality cannot be directly experienced, and is therefore unknown and even in principle unknowable. Instead, scientists postulate an informal description which is intuitively acceptable, and subsequently formulate one or more mathematical theories to describe the phenomena.

Deterministic Chaos: Many phenomena in physics (like the weather) satisfy well accepted deterministic equations. From initial data we can extrapolate and compute the next states of the system. Traditionally it was thought that increased precision of the initial data (measurement) and increased computing power would result in increasingly accurate extrapolation (prediction). Unfortunately it turns out that for many systems this is not the case. In fact, it turns out that any long range prediction with any confidence better than what we would get by flipping a fair coin is practically impossible: this phenomenon is known as chaos (see [3] for an introduction). There are two, more or less related, causes for this:

Instability In certain deterministic systems, an arbitrary small error in initial conditions can exponentially increase during the subsequent evolution of the system, until it encompasses the full range of values achievable by the system. This phenomenon of instability of a computation is in fact well known in numerical analysis: computational procedures inverting ill-conditioned matrices (with determinant about zero) will introduce exponentially increasing errors.

Unpredictability Assume we deal with a system described by deterministic equations which can be finitely represented (like a recursive function). Even if fixed-length initial segments of the infinite binary representation of the real parameters describing past states of the system are perfectly known, and the computational procedure used is perfectly error free, for many such systems it will still be impossible to effectively predict (compute) any significantly long extrapolation of system states with any confidence higher than using a random coin flip. This is the core of chaotic phenomena: randomness in determinism.

Probability: Classical probability theory deals with randomness in the sense of random variables. The concept of random individual data cannot be expressed. Yet our intuition about the latter is very strong: An adversary claims to have a true random coin and invites us to bet on the outcome. The coin produces a hundred heads in a row. We say that the coin cannot have been fair. The adversary, however, appeals to probability theory which says that each sequence of outcomes of a hundred coin flips is equally likely, 1/2^{100}, and one sequence had to come up. Probability theory gives us no basis to challenge an outcome after it has happened. We could only exclude unfairness in advance by putting a penalty side-bet on an outcome of 100 heads. But what about 1010...? What about an initial segment of the binary expansion of π?

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Regular sequence
\[ \Pr(00000000000000000000000000000000) = \frac{1}{2^{26}} \]

Regular sequence
\[ \Pr(01000110110000010100111001) = \frac{1}{2^{26}} \]

Random sequence
\[ \Pr(10010011011000111011010000) = \frac{1}{2^{26}} \]

The first sequence is regular, but what is the distinction of the second sequence and the third? The third sequence was generated by flipping a quarter. The second sequence is very regular: 0, 1, 00, 01, ...

In fact, classical probability theory cannot express the notion of randomness of an individual sequence. It can only express expectation of properties of the total set of sequences under some distribution.

This is analogous to the situation in physics above. How can ‘an individual object be random?’ is as much a probability theory paradox as ‘how can an individual sequence of states of a deterministic system be random?’ is a paradox of deterministic physical systems.

In probability theory the problem has found a satisfactory resolution by combining notions of computability and information theory to express the complexity of a finite object. This complexity is the length of the shortest binary program from which the object can be effectively reconstructed. It may be called the algorithmic information content of the object. This quantity turns out to be an attribute of the object alone, and recursively invariant. It is the Kolmogorov complexity of the object. It turns out that this notion can be brought to bear on the physical riddles too.

2 Kolmogorov Complexity

To make this paper self-contained we briefly review notions and properties required. For details and further properties see the textbook [12]. We identify the natural numbers \( \mathcal{N} \) and the finite binary sequences as
\[ (0, \epsilon), (1, 0), (2, 1), (3, 00), (4, 01), \ldots, \]
where \( \epsilon \) is the empty sequence. The length \( l(x) \) of a natural number \( x \) is the number of bits in the corresponding binary sequence. For instance, \( l(\epsilon) = 0 \). If \( A \) is a set, then \( |A| \) denotes the cardinality of \( A \). Let \( \langle \cdot \rangle : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \) denote a standard computable bijective ‘pairing’ function. Throughout this paper, we will assume that \( \langle x, y \rangle = 1^{l(\langle x \rangle)} 0 x y \).

Define \( (x, y, z) \) by \( \langle x, \langle y, z \rangle \rangle \).

We need some notions from the theory of algorithms, see [15]. Let \( \phi_1, \phi_2, \ldots \) be a standard enumeration of the partial recursive functions. The (Kolmogorov) complexity of \( x \in \mathcal{N} \), given \( y \), is defined as
\[ C(x|y) = \min \{ l(\langle n, z \rangle) : \phi_n(\langle y, z \rangle) = x \}. \]

This means that \( C(x|y) \) is the minimal number of bits in a description from which \( x \) can be effectively reconstructed, given \( y \). The unconditional complexity is defined as \( C(x) = C(x|\epsilon) \).

An alternative definition is as follows. Let
\[ C_\psi(x|y) = \min \{ l(z) : \psi(\langle y, z \rangle) = x \} \]
be the conditional complexity of \( x \) given \( y \) with reference to decoding function \( \psi \). Then \( C(x|y) = C_\psi(x|y) \) for a universal partial recursive function \( \psi \) that satisfies \( \psi(\langle y, n, z \rangle) = \phi_n(\langle y, z \rangle) \).
We will also make use of the prefix complexity \( K(x) \), which denotes the shortest self-delimiting description. To this end, we consider so called prefix Turing machines, which have only 0’s and 1’s on their input tape, and thus cannot detect the end of the input. Instead we define an input as that part of the input tape which the machine has read when it halts. When \( x \neq y \) are two such input, we clearly have that \( x \) cannot be a prefix of \( y \), and hence the set of inputs forms what is called a prefix code. We define \( K(x) \) similarly as above, with reference to a universal prefix machine that first reads \( 1^0 \) from the input tape and then simulates prefix machine \( n \) on the rest of the input.

We need the following properties. Throughout ‘\( \log \)’ denotes the binary logarithm. We often use \( O(f(n)) = -O(f(n)) \), so that \( O(f(n)) \) may denote a negative quantity. For each \( x, y \in \mathcal{N} \) we have

\[
C(x|y) \leq l(x) + O(1).
\]

(2)

For each \( y \in \mathcal{N} \) there is an \( x \in \mathcal{N} \) of length \( n \) such that \( C(x|y) \geq n \). In particular, we can set \( y = \epsilon \). Such \( x \)'s may be called random, since they are without regularities that can be used to compress the description. Intuitively, the shortest effective description of \( x \) is \( x \) itself. In general, for each \( n \) and \( y \), there are at least \( 2^n - 2^{n-\epsilon} + 1 \) distinct \( x \)'s of length \( n \) with

\[
C(x|y) \geq n - \epsilon.
\]

(3)

In some cases we want to encode \( x \) in self-delimiting form \( x' \), in order to be able to decompose \( x'y \) into \( x \) and \( y \). Good upper bounds on the prefix complexity of \( x \) are obtained by iterating the simple rule that a self-delimiting (s.d.) description of the length of \( x \) followed by \( x \) itself is a s.d. description of \( x \). For example, \( x' = 1^{l(x)}0x \) and \( x'' = 1^{l(l(x))}0l(x)x \) are both s.d. descriptions for \( x \), and this shows that \( K(x) \leq 2l(x) + O(1) \) and \( K(x) \leq l(x) + 2l(l(x)) + O(1) \).

Similarly, we can encode \( x \) in a self-delimiting form of its shortest program \( p(x) \) \((l(p(x)) = C(x))\) in \( 2C(x) + 1 \) bits. Iterating this scheme, we can encode \( x \) as a selfdelimiting program of \( C(x) + 2 \log C(x) + 1 \) bits, which shows that \( K(x) \leq C(x) + 2 \log C(x) + 1 \), and so on.

The string \( sqi \) has length at most \( n - \delta(n) - O(1) \) and can be padded

### 2.1 Random Sequences

We would like to call an infinite sequence \( \omega \in \{0,1\}^\infty \) random if \( C(\omega_{1:n}) \geq n - O(1) \) for all \( n \). It turns out that such sequences do not exist. This occasioned the celebrated theory of randomness of P. Martin-Löf, [14]. Later it turned out, [1], that we can yet precisely define the Martin-Löf random sequences, but using prefix Kolmogorov complexity. We need

**Theorem 1** An infinite binary sequence \( \omega \) is random in the sense of Martin-Löf iff there is an \( n_0 \) such that \( K(\omega_{1:n}) \geq n \) for all \( n > n_0 \).

That \( \omega \) is random in Martin-Löf’s sense means that it will pass all effective tests for randomness: both the tests which are known now and the ones which are as yet unknown [14].

Similar properties hold for high-complexity finite strings, although in a less absolute sense.

For every finite set \( S \subseteq \{0,1\}^* \) containing \( x \) we have \( K(x|S) \leq \log |S| + O(1) \). Indeed, consider the selfdelimiting code of \( x \) consisting of its \( \lfloor \log |S| \rfloor \) bit long index of \( x \) in the lexicographical ordering of \( S \). This code is called data-to-model code. The lack of typicality of \( x \) with respect to \( S \) is the amount by which \( K(x|S) \) falls short of the length of the data-to-model code. The randomness deficiency of \( x \) in \( S \) is defined by

\[
\delta(x|S) = \log |S| - K(x|S),
\]

(4)

for \( x \in S \), and \( \infty \) otherwise. If \( \delta(x|S) \) is small, then \( x \) may be considered as a typical member of \( S \). There are no simple special properties that single it out from the majority of elements in \( S \). This is not just terminology: If \( \delta(x|S) \) is small, then \( x \) satisfies all properties of low Kolmogorov complexity that hold with high probability for the elements of \( S \). For example: Consider strings \( x \) of length \( n \) and let \( S = \{0,1\}^n \) be a set of such strings. Then \( \delta(x|S) = n - K(x|ni) \pm O(1) \).

(i) If \( P \) is a property satisfied by all \( x \) with \( \delta(x|S) \leq \delta(n) \), then \( P \) holds with probability at least \( 1 - 1/2^{\delta(n)} \) for the elements of \( S \).

(ii) Let \( P \) be any property that holds with probability at least \( 1 - 1/2^{\delta(n)} \) for the elements of \( S \). Then, every such \( P \) holds simultaneously for every \( x \in S \) with \( \delta(x|S) \leq \delta(n) - K(P|n) - O(1) \).
3 Algorithmic Chaos Theory

For convenience assume that time is discrete: \( N \). In a deterministic system \( X \) the state of the system at time \( t \) is \( X_t \). The orbit of the system is the sequence of subsequent states \( X_0, X_1, X_2, \ldots \). For convenience we assume the states are elements of \( \{0, 1\} \). The definitions below are easily generalized. For each system, be it deterministic or random, we associate a measure \( \mu \) with the space \( \{0, 1\}^\infty \) of orbits. That is, \( \mu(x) \) is the probability that an orbit starts with \( x \in \{0, 1\}^\infty \).

Given an initial segment \( X_{0:t} \) of the orbit we want to compute \( X_{t+1} \). Even if it would not be possible to compute \( X_{t+1} \), we would like to compute a prediction of it which does better than a random coin flip.

**Definition 1** Let the set of orbits be \( S = \{0, 1\}^\infty \) with the Lebesgue measure \( \lambda \). Let \( \phi \) be a partial recursive function and let \( \omega \in S \). Define

\[
\zeta_i = \begin{cases} 
1 & \text{if } \phi(\omega_1:i−1) = \omega_i \\
0 & \text{otherwise}
\end{cases}
\]

A deterministic system is chaotic if, for every computable function \( \phi \), we have

\[
\lim_{t \to \infty} \frac{1}{t−1} \sum_{i=0}^{t−1} \zeta_i = \frac{1}{2},
\]

with probability 1.

A well-known example of a chaotic system is the doubling map, [5]. Consider the deterministic system \( X \) with initial state \( X_0 = 0.\omega \) a real number in the interval \([0, 1]\) where \( \omega \in S \) is the binary representation.

\[
X_{n+1} = 2X_n \pmod{1} \tag{5}
\]

where \( \pmod{1} \) means drop the integer part. Thus, all iterates of equation 5 lie in the unit interval \([0, 1]\). In physics, this is called the ‘energy surface’ of the orbit. We can partition this energy surface into two cells, a left cell \([0, \frac{1}{2})\) and a right cell \(\left[\frac{1}{2}, 1\right)\). Thus \( X_n \) lies in the left cell if and only if the \( n \)th digit of \( \omega \) is 0.

One way to derive the doubling map is as follows: In chaos theory, [3], people have for years being studying the discrete logistic equation

\[
Y_{n+1} = \alpha Y_n (1 − Y_n)
\]

which maps the unit interval upon itself when \( 0 \leq \alpha \leq 4 \). When \( \alpha = 4 \), setting \( Y_n = \sin^2(\pi X_n) \), we obtain:

\[
X_{n+1} = 2X_n \pmod{1}.
\]

**Lemma 1** There are chaotic systems (like \( X \) and \( Y \) above).

**Proof.** We prove that \( X \) is a chaotic system. Since \( Y \) reduces to \( X \) by specialization, this shows that \( Y \) is chaotic as well. Assume \( \omega \) is random. Then by Theorem 1,

\[
C(\omega_{1:n}) > n − 2 \log n + O(1). \tag{6}
\]

Let \( \phi \) be any partial recursive function. Construct \( \zeta \) from \( \phi \) and \( \omega \) as in Definition 2.

Assume by way of contradiction that there is an \( \epsilon > 0 \) such that

\[
\left|\frac{1}{n} \lim_{n \to \infty} \sum_{i=1}^{n} \zeta_i - \frac{1}{2}\right| \geq \epsilon.
\]

Then, there is a \( \delta > 0 \) such that

\[
\lim_{n \to \infty} \frac{C(\zeta_{1:n})}{n} \leq (1 − \delta). \tag{7}
\]

We prove this as follows. The number of binary sequences of length \( n \) where the numbers of 0’s and 1’s differ by at least an \( \epsilon n \) is

\[
N = 2 \cdot 2^n \sum_{m=(\frac{1}{2}+\epsilon)n}^{n} b(n, m, \frac{1}{2})
\]
where \( b(n, m, p) \) is the probability of \( m \) successes out of \( n \) trials in a \((p, 1-p)\) Bernoulli process: the Binomial distribution. A general estimate of the tail probability of the binomial distribution, with \( m \) the number of successful outcomes in \( n \) experiments with probability of success \( 0 < p < 1 \) and \( q = 1 - p \), is given by Chernoff’s bounds, [4, 2],

\[
\Pr(|m - np| \geq \epsilon n) \leq 2e^{-\epsilon^2n^2/3n}.
\]

Therefore, we can describe any element \( \zeta_{1:n} \) concerned by giving \( n \) and \( \epsilon n \) in \( 2\log n + 4\log\log n \) bits self-delimiting descriptions, and pointing out the string concerned in a constrained ensemble of at most \( N \) elements in \( \log N \) bits. Therefore,

\[
C(\zeta_{1:n}) \leq n - \epsilon^2 n \log e + 2 \log n + 4 \log\log n + O(1). \tag{8}
\]

That is, we can choose

\[
\delta = \epsilon^2 \log e + \frac{2 \log n + 4 \log\log n + O(1)}{n}.
\]

Next, given \( \zeta \) and \( \phi \) we can reconstruct \( \omega \) as follows:

\[
\text{for } i := 1, 2, \ldots \text{ do:}
\]

\[
\text{if } \phi(\omega_{1:i-1}) = a \text{ and } \zeta_i = 0 \text{ then } \omega_i := \neg a
\]

\[
\text{else } \omega_i := a.
\]

Therefore,

\[
C(\omega_{1:n}) \leq C(\zeta_{1:n}) + K(\phi) + O(1). \tag{9}
\]

Now Equations 6, 7, 9 give the desired contradiction. By Theorem 1, the set of \( \omega \)'s satisfying Equation 6 has uniform measure one, which proves the lemma.

In [5] the argument is as follows. Assuming that the initial state is randomly drawn from \([0, 1)\) according to the uniform measure \( \lambda \), we can use complexity arguments to show that the doubling map’s observable orbit cannot be predicted better than a coin toss. Namely, with \( \lambda \)-probability 1 the drawn initial state will be a Martin-Löf random infinite sequence. Such sequences by definition cannot be effectively predicted better than a random coin toss, see [14].

But in this case we do not need to go to such trouble. The observed orbit essentially consists of the consecutive bits of the initial state.uniform measure is isomorphic to flipping a fair coin to generate it. This raises the challenging problem of a meaningful application of Kolmogorov complexity to chaos problems.

From a practical viewpoint it may be argued that we really are not interested in infinite sequences: in practice the input will always be finite precision. Now an infinite sequence which is random may still have an arbitrary long finite initial segment which is completely regular. Therefore, we analyse the theory for finite precision inputs in the following section.

### 3.1 Chaos with Finite Precision Input

In the case of infinite precision real inputs, the distinction between chaotic and non-chaotic systems can be precisely drawn. In the case of finite precision inputs the distinction is necessarily a matter of degree. This occasions the following definition.

**Definition 2** Let \( S, \lambda, \phi, \omega \) and \( \zeta \) be as in Definition 1. A deterministic system with input precision \( n \) is \((\epsilon, \delta)\)-chaotic if, for every computable function \( \phi \), we have

\[
|\sum_{i=1}^{n} \zeta_i - \frac{1}{2}| \geq \epsilon,
\]

with probability at least \( 1 - \delta \).

So systems are chaotic in the sense of Definition 1, like the doubling map above, iff they are \((0, 0)\)-chaotic with precision \( \infty \). The system is probably approximately unpredictable: a pai-chaotic system.
Theorem 2 Systems $X$ and $Y$ above are $(\sqrt{\delta(n) + O(1)}) \ln 2/n, 1/2^{\delta(n)}$-chaotic for every function $\delta$ such that $0 < \delta(n) < n$.

Proof. We prove that $X$ is $(\epsilon, \delta)$-chaotic. Since $Y$ reduces to $X$, this implies that $Y$ is $(\epsilon, \delta)$-chaotic as well.

Assume that $x$ is a binary string of length $n$ with

$$C(x) \geq n - \delta(n). \tag{10}$$

Let $\phi$ be a polynomial time computable function, and define $z$ by:

$$z_i = \begin{cases} 
1 & \text{if } \phi(x_{1:i-1}) = x_i \\
0 & \text{otherwise}
\end{cases}$$

Then, $x$ can be reconstructed from $z$ and $\phi$ as before, and therefore:

$$C(x) \leq C(z) + K(\phi) + O(1).$$

By Equation 10 this means

$$C(z) \geq n - \delta(n) - K(\phi) + O(1). \tag{11}$$

We analyse the number of zeros and ones in $z$. Using Chernoff’s bounds, Equation 8, with $p = q = \frac{1}{2}$, the number of $z$’s which have an excess of $\epsilon n$ of ones over zeros is:

$$N \leq 2^{n+1} e^{-\epsilon(n)^2/n}$$

with

$$|\#\text{ones}(x) - \frac{n}{2}| < \epsilon n.$$  

Then, we can give an effective description of $z$ by giving a description of $\phi$, $\delta$ and $z$’s index in the set of size $N$ in this many bits

$$n - \epsilon^2 n \log e + K(\phi) + K(\delta) + 2 \log K(\phi) K(\delta) + O(1). \tag{12}$$

From Equations 11, 12 we find

$$\epsilon \leq \frac{\sqrt{\delta(n) + 2K(\phi) + K(\delta) + 2 \log K(\phi) K(\delta) + O(1)}}{n \log e} \tag{13}$$

Making the simplifying assumption that $K(\phi), K(\delta) = O(1)$ this yields

$$|\#\text{ones}(z) - \frac{n}{2}| < \sqrt{(\delta(n) + O(1)) n \ln 2} \tag{14}$$

The number of binary strings $x$ of length $n$ with $C(x) < n - \delta(n)$ is at most $2^{n-\delta(n)} - 1$ (there are not more programs of length less than $n - \delta(n)$). Therefore, the uniform probability of a real number starting with an $n$-length initial segment $x$ such that $C(x) \geq n - \delta(n)$ is given by:

$$\lambda \{ \omega : C(\omega_{1:n} \geq n - \delta(n)) \} > 1 - \frac{1}{2^{\delta(n)}}. \tag{15}$$

Therefore, system $X$ is $(\epsilon, \delta)$-chaotic with $\epsilon = \sqrt{(\delta(n) + O(1)) \ln 2/n}$ and $\delta = 1/2^{\delta(n)}$. □
References

[1] G.J. Chaitin, A theory of program size formally identical to information theory, *J. Assoc. Comp. Mach.*, 22(1975), 329-340.

[2] Corman, C. Leiserson, R. Rivest, *Introduction to Algorithms*, 1990.

[3] R.L. Devaney, An Introduction to Chaos Dynamical Systems, Addison-Wesley, 2nd Edition, 1989.

[4] P. Erdős and J. Spencer, *Probabilistic Methods in Combinatorics*, Academic Press, New York, 1974.

[5] J. Ford, How random is a random coin toss? *Physics Today*, 36(1983), April, 40-47.

[6] P. Gács, On the symmetry of algorithmic information, *Soviet Math. Dokl.*, 15(1974), 1477-1480.

[7] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989.

[8] D.E. Knuth, *Seminumerical Algorithms*, Addison-Wesley, 1981.

[9] A.N. Kolmogorov, Three approaches to the definition of the concept ‘quantity of information’, *Problems in Information Transmission*, 1:1(1965), 1-7.

[10] A.N. Kolmogorov, Combinatorial foundation of information theory and the calculus of probabilities, *Russian Math. Surveys*, 38:4(1983), 29-40.

[11] L.A. Levin, Laws of Information conservation (non-growth) and aspects of the foundation of probability theory, *Problems in Information Transmission*, 10(1974), 206-210.

[12] M. Li and P.M.B. Vitányi, *An Introduction to Kolmogorov Complexity and its Applications*, Second Edition, Springer-Verlag, New York, 1997.

[13] M. Li and P.M.B. Vitányi, Kolmogorov complexity arguments in Combinatorics, *J. Comb. Th.*, Series A, 66:2(1994), 226-236. Correction, *Ibid.*, 69(1995), 183.

[14] P. Martin-Löf, On the definition of random sequences, *Information and Control*, (1966).

[15] H.J. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[16] C.E. Shannon, A mathematical theory of communication, *Bell System Tech. J.*, 27(1948), 379-423, 623-656.