Variation of the relative yield of charged and neutral $B$ mesons across the $\Upsilon(4S)$ resonance

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Abstract

It is shown that the ratio of the production rates of the pairs $B^+B^-$ and $B^0\bar{B}^0$ should experience a substantial and rapid variation with energy within the width of the $\Upsilon(4S)$ resonance, crossing the value of one near the center of the resonance. This behavior is due to an interference of the rapidly changing with energy Breit-Wigner phase with the phase introduced in the wave function of charged mesons by their Coulomb interaction.
The ratio of the production rates of charged and neutral $B$ mesons in $e^+e^-$ annihilation at the $\Upsilon(4S)$ resonance,

$$R_{c/n} = 1 + \delta R_{c/n} = \frac{\sigma(e^+e^- \to B^+B^-)}{\sigma(e^+e^- \to B^0\bar{B}^0)},$$

is an important parameter in detailed studies of the properties of $B$ mesons. Recent dedicated measurements\cite{1, 2, 3} of $R_{c/n}$ at the maximum of the resonance report values ranging from $1.04 \pm 0.07 \pm 0.04$\cite{1} to $1.10 \pm 0.06 \pm 0.05$\cite{2}, which leave enough room for further studies of the quantity of interest $\delta R_{c/n}$.

Theoretically the difference $\delta R_{c/n}$ of the discussed ratio from one arises as dominantly an effect of the Coulomb interaction, clearly different for charged and neutral $B$ mesons, since the mass difference $m_{B^0} - m_{B^+} = 0.33 \pm 0.28 \text{MeV}$\cite{4} is quite small, and its effect can be accounted separately. In the most simple approach\cite{5}, where the $B$ mesons are treated as point particles, and the existence of the resonant interaction is ignored, the estimate of $\delta R_{c/n}$

$$\delta R_{c/n} = \frac{\pi\alpha}{2v} + O\left(\frac{\alpha^2}{v^2}\right),$$

At the excitation energy of the $\Upsilon(4S)$ resonance, $E_0 = M_{\Upsilon(4S)} - 2m_B \approx 20 \text{MeV}$, one has $v \approx 0.06$, and the simple estimate\cite{2} would yield $\delta R_{c/n} \approx 0.19$. It was subsequently argued\cite{6} that the estimate of $\delta R_{c/n}$ can in fact be substantially reduced from eq.(2) if one accounts for a finite size of the $B$ mesons (through their electromagnetic form factor) and also for the finite size of the $\Upsilon(4S)$. The latter effect was further discussed in a specific model of heavy quarkonium\cite{7}. Recently the problem of calculation of the ratio $R_{c/n}$ was revisited\cite{8} in the context of a chiral-type model for strong interaction of $B$ mesons at short distances, including the $B^*B\pi$ vertex and the coupled channels with pairs of pseudoscalar and/or vector mesons, although still considering all the mesons as point-like with respect to the Coulomb interaction.

It should be noted that in all previous theoretical studies of the ratio $R_{c/n}$ the presence of the resonance in the wave function of the $B$ meson pair was essentially ignored. In other words, the coupling of the resonance to the $B$ mesons was either treated perturbatively (although with a form factor\cite{6, 7}), or the considered model of the strong interaction did not contain a resonance at all\cite{8}. For this reason the results predicted a smooth behavior of $\delta R_{c/n}$ with energy in the region of the $\Upsilon(4S)$ resonance. It is however well known (see
e.g. in the two last chapters of the textbook [9]) that the presence of a resonance produces a large effect on the wave function of the scattering states, which is rapidly changing across the resonance with energy. The scale for the variation is set by the resonance width $\Gamma$. In particular, the relative phase $2\delta$ between the outgoing and incoming spherical waves (twice the scattering phase $\delta$) changes by $2\pi$ at the scale $\Gamma$ when the excitation energy $E$ passes the central value $E_0$. It is the purpose of the present paper to properly take into account the resonant behavior of the wave function of the scattering states along the lines of the standard non-relativistic scattering theory [9]. It will be shown that an interplay between the rapidly changing relative phase of incoming/outgoing wave and of the effects of the Coulomb interaction gives rise to a rather non-trivial behavior of $\delta R_{c/n}$ with the energy changing across the resonance. Namely $\delta R_{c/n}$ has to change sign at energy within a fraction of the width $\Gamma$ from the ‘nominal’ resonance center energy $E_0$. It will also be argued that the effect of a rapid variation of $\delta R_{c/n}$ should be model independent, while the details, such as the overall magnitude of $\delta R_{c/n}$ and the precise position of its zero(s), do depend on yet unknown details of the strong and electromagnetic interactions of the $B$ mesons at short distances and of the structure of the $\Upsilon(4S)$ resonance. Thus a detailed experimental study of the behavior of $\delta R_{c/n}$ in the resonance region could in principle provide a certain insight into those finer properties of the hadron dynamics.

The standard physical picture for considering the scattering in the resonance region (c.f. Ref. [9]) is that the strong interaction, responsible for the existence of the resonance has a short range $a$, and the essential effects of the interaction at distances $r < a$ can be parameterized in terms of phenomenologically measurable parameters of the resonance, the most important being its energy $E_0$ above the threshold and the width $\Gamma$. At distances larger than $a$ the motion is described by a known potential $V(r)$: either $V(r) = 0$, or a Coulomb potential (with a possible modification due to form factor at short distances), where the wave function of the scattering state can be found explicitly from the Schrödinger equation. The boundary (matching) conditions at $r \approx a$ for the ‘outer’ wave function are related to the measurable parameters of the resonance. In the discussed process the $B\bar{B}$ pairs are produced in the $P$ wave. Also beyond the region of strong interaction, i.e. at $r > a$, there is no strong interaction mixing between the “neutral”, $B^0\bar{B}^0$, and the “charged”, $B^+B^-$, channels. Thus the ‘outer’ wave function at $r > a$ is described by the spherical wave with $L = 1$, whose radial part can be written as $R(r) = \chi(r)/r$, with a separate function $\chi(r)$ for each of the channels: $\chi_n(r)$ and $\chi_c(r)$, each satisfying at energy $E = p^2/m$ the corresponding
one-dimensional Schrödinger equation
\[ \chi''_n + \left( p^2 - \frac{2}{r^2} \right) \chi_n = 0 , \quad \chi''_c + \left( p^2 + m \frac{\alpha}{r} - \frac{2}{r^2} \right) \chi_c = 0 , \quad (3) \]
where the prime denotes derivative over \( r \), and \( m = m_B \approx 5280 \text{ MeV} \) is the mass of either of the \( B \) mesons (a possible small mass difference between the charged and neutral \( B \) mesons is completely ignored throughout the present discussion).

The coupling between the “neutral” and the “charged” channels takes place in the region of strong interaction at short distances. At those distances the light quark parts of the mesons strongly overlap and become a part of (presumably) quite complicated dynamics of light quarks and gluons. Thus in this region it would be inappropriate to continue description in terms of individual \( B \) mesons. The boundary condition for the ‘outer’ dynamics at distances \( r > a \) is however dictated by the isotopic invariance of the strong interaction. Namely, one can assume with a rather high degree of accuracy that when the \( B \) mesons emerge from the region of strong dynamics as individual particles their wave function is an isotopic singlet. In other words, the isospin condition for the functions \( \chi_n \) and \( \chi_c \) is that they evolve from one and the same function at a certain short distance \( r = a \), i.e. that
\[ \chi_c(a) = \chi_n(a) \quad \text{and} \quad \chi'_c(a) = \chi'_n(a) , \quad (4) \]
which boundary conditions can be viewed as our formal definition of the short distance parameter \( a \). Although there can be a small ‘intrinsic’ isospin violation also in the region of strong interaction, its effect in \( \delta R^{c/n} \), as discussed in Ref.\([8]\), is noticeably smaller than that of the Coulomb interaction, and can be studied as a further adjustment, using the approximation of exact isospin symmetry at short distances in eq.\((4)\) as a starting point.

As is known from the standard Breit-Wigner description of a resonance scattering\([9]\) at energy \( E \) near the position \( E_0 \) of the resonance, the relevant ‘outer’ solution of the Schrödinger equations \([4]\) for stationary wave functions has the form
\[ \chi_n(r) = (\Delta - i \gamma) b_n f_n(r) + (\Delta + i \gamma) b^*_n f^*_n(r) , \]
\[ \chi_c(r) = (\Delta - i \gamma) b_c f_c(r) + (\Delta + i \gamma) b^*_c f^*_c(r) , \quad (5) \]
where \( \Delta = E - E_0, \gamma = \Gamma/2 \) and the complex coefficients \( b_{n(c)} \) are generally functions of the energy, which however have no zeros at \( \Delta = i \gamma \). Finally, each of the functions \( f_n \) and \( f_c \) is the solution of the corresponding equation in \([3]\), which contains only the outgoing wave,
i.e. at \( r \to \infty \) they contain only the factor \( \exp(ipr) \) (while their complex conjugates \( f_{n(c)}^* \) contain only the incoming wave factor \( \exp(-ipr) \)).

The function \( f_n(r) \) specified by this condition is well known for the free motion with \( L = 1 \),

\[
f_n(r) = \left( 1 + \frac{i}{pr} \right) e^{ipr},
\]

and with this condition for its phase, the phase of the coefficient \( b_n \) coincides with the non-resonant scattering phase \( \delta_1 \) at \( L = 1 \),

\[
\exp(2i\delta_1) = \frac{b_n}{\bar{b}_n}.
\]

The corresponding function \( f_c(r) \) for the motion in the Coulomb potential is also well known (see e.g. in Ref.\[9\]), however for our present purpose it would be more convenient to make use of the perturbation theory in the Coulomb interaction, rather than to do an expansion of the explicit expression. In specifying the phase convention for the function \( f_c \) a minor technical point arises due to the well known fact that its phase at \( r \to \infty \) contains a slowly varying logarithmic Coulomb phase: \( f_c(r) \sim \exp[ipr + (\alpha/2v) \ln 2pr + \text{const}] \). This however can be readily resolved by assuming that the Coulomb interaction is cut off at a large distance \( r = R \). Then at still larger \( r \) the Coulomb phase does not change and can be considered as constant. Clearly the physical results, including the discussed here effect in \( \delta R^{c/n} \) do not depend on this infrared cutoff. Imposing such regularization, the function \( f_c(r) \) in eq.(5) can be chosen to exactly coincide (both in phase and in normalization) with \( f_n(r) \) at asymptotically large distances: \( f_c(r) = \exp(ipr) \). Thus any difference in phase and magnitude that arises from the Coulomb interaction in the “charged” channel is encoded in the coefficient \( b_c \).

In order to find from the wave function of a stationary state (eq.(5)) the relative rate of production of the pairs of charged and neutral \( B \) mesons in \( e^+e^- \) annihilation, i.e. by a source localized well inside the region of strong interaction, it is necessary to note that the rate in each channel is proportional to the inverse of the norm squared of the coefficient in front of the incoming wave. This can be understood by considering the reverse process: annihilation of a meson pair into \( e^+e^- \), in complete analogy with an explanation of the “\( |\psi(0)|^2 \) rule” for production of bound states. In the reverse process the incoming wave has a fixed flux, i.e. a fixed norm of the coefficient in front of \( \exp(-ipr) \) at large \( r \). Matching this wave to the incoming part of the wave function in eq.(5) implies that the corresponding function \( \chi(r) \) has to be divided by the coefficient of its incoming wave part. Under the normalization
conventions adopted here for the functions $f(r)$ this results in the annihilation rate being proportional to the factor $|\left(\Delta + i\gamma\right) b^*|^2$, with $b$ equal to $b_n$ or $b_c$, depending on the chosen initial state for the incoming wave. Clearly, this factor in the rate contains both the standard Breit-Wigner resonance curve and the normalization factor $|b|^2$. Thus the discussed here ratio of the yields in the two channels is given by\(^1\)

$$R^{c/n} = \frac{|b_n|^2}{|b_c|^2}. \quad (8)$$

The relation between the coefficients $b_n$ and $b_c$ is found from the matching conditions \(\square\). After substituting the wave functions from eq.(5) into the conditions \(\square\), the ratio of the coefficients (in fact the inverse of that entering eq.(8)) is found as

$$\frac{b_c}{b_n} = \frac{f_c^{**} \left(f_n + f_n^{*} e^{-2i\delta_{BW} - 2i\delta_1}\right) - f_c^{*} \left(f_n^{*} + f_n^{**} e^{-2i\delta_{BW} - 2i\delta_1}\right)}{f_c^{*n} f_c - f_c^{*} f_c^{*n}}, \quad (9)$$

where all the functions and their derivatives are taken at the matching point $r = a$, $\delta_{BW}$ is the standard Breit-Wigner resonance phase: $\exp(2i\delta_{BW}) = (\Delta - i\gamma)/(\Delta + i\gamma)$, and the non-resonant phase $\delta_1$ is defined by eq.(7). Finally, in the last transition a use is made of the fact that the denominator in the intermediate expression is the Wronskian, which is constant in $r$ and can thus be found from the asymptotic form of the function $f_c$ at large $r$.

The equation (9) contains no approximation with regards to the Coulomb interaction, and can be used down to arbitrarily small values of $v$, i.e. for arbitrary values of the Coulomb parameter $\alpha/v$, provided that the exact Coulomb function is used for $f_c(r)$. However for the practical purpose of discussing the Coulomb effects at the $\Upsilon(4S)$ resonance it is sufficient to consider only the effect of first order in $\alpha$. In this order one can write $f_c(r) = f_n(r) + \phi(r)$ with $\phi$ being formally a small perturbation of order $\alpha$ of the wave function. Using this expression in eq.(9) and also assuming an expansion of the ratio $b_c/b_n$ to the first order in $\alpha$, one readily finds

$$\delta R^{c/n} = \frac{|b_n|^2}{|b_c|^2} - 1 = \frac{1}{p} \Im \left\{ e^{2i\delta_{BW} + 2i\delta_1} \left[ \phi(a) f_n^{*}(a) - \phi^{*}(a) f_n(a) \right] \right\}. \quad (10)$$

\(^1\)In a somewhat more widely familiar non-resonant situation when point-like particles are produced by a point source, the functions $\chi_n(r)$ and $\chi_c(r)$ are the regular at $r = 0$ solutions of the equations (8). In this case one has (for a $P$ wave): $|b_n/b_c|^2 = |\psi_c(0)/\psi_n(0)|^2$, where $\psi_c(r)$ and $\psi_n(r)$ are the wave functions for corresponding stationary states, having the same relative normalization at infinity.
The combination of the functions in this expression can be found directly from the equations \(3\) by also a rather standard method. Indeed, the function \(f_n(r)\) satisfies the first of those equations, while \(f_c(r)\) satisfies the second equation. Multiplying the first equation by \(f_c(r)\) and the second equation by \(f_n(r)\), subtracting the results, and expanding in the difference \(\phi\) between \(f_c\) and \(f_n\), one arrives at the relation

\[
\frac{d}{dr} \left[ \phi(r) f_n'(r) - \phi'(r) f_n(r) \right] = m \frac{\alpha}{r} f_n^2(r). \tag{11}
\]

Integrating this relation over \(r\) from \(a\) to infinity, using the fact that under our conventions \(\phi(r) \to 0\) at \(r \to \infty\), and also using the explicit expression \(6\) for \(f_n(r)\), one finds in terms of eq.\(10\) the formula

\[
\delta R^{c/n} = -\frac{\alpha}{v} \text{Im} \left[ e^{2i\delta_{BW} + 2i\delta_1} \int_a^\infty e^{2ipr} \left( 1 + \frac{i}{pr} \right)^2 \frac{dr}{r} \right] \tag{12}
\]

where in the latter expression the coefficients \(A\) and \(B\) are given (with the opposite sign) by respectively the imaginary and the real part of the integral with complex exponent:

\[
A = -\int_{pa}^\infty \left[ \left( 1 - \frac{1}{u^2} \right) \sin 2u + \frac{2 \cos 2u}{u} \right] \frac{du}{u}, \quad B = \int_{pa}^\infty \left[ \frac{2 \sin 2u}{u} - \left( 1 - \frac{1}{u^2} \right) \cos 2u \right] \frac{du}{u}. \tag{13}
\]

At small values of the product \(pa\) the coefficients \(A\) and \(B\) have the expansion:

\[
A = \frac{\pi}{2} - \frac{2pa}{3} + O(p^3a^3), \quad B = \frac{1}{2p^2a^2} - \ln 2pa - \gamma_E + 1 + O(p^4a^4), \tag{14}
\]

where \(\gamma_E = 0.577\ldots\) is the Euler constant.

One can see from eq.\(12\) that in the limit, where the effects of the strong interaction in the wave function of the scattering state are ignored, corresponding to \(\gamma \to 0\) and \(\delta_1 \to 0\), the simple estimate \(2\) of \(\delta R^{c/n}\) is recovered, assuming production of point-like particles by a point source (i.e. also \(a \to 0\)). However this limit of a vanishing resonance width is totally inadequate for resolving the behavior of \(\delta R^{c/n}\) inside the resonance curve, i.e. at the energy scale of \(\gamma\). In particular, the second term in the final expression in eq.\(12\), proportional to \(\gamma\Delta/(\Delta^2 + \gamma^2)\) vanishes at energies far away from the resonance, i.e. at \(|\Delta| \ll \gamma\), as well as at the center of the resonance, i.e. at \(\Delta = 0\). On the other hand, the first term contains
the factor $(\Delta^2 - \gamma^2)/(\Delta^2 + \gamma^2)$, which changes from $+1$ away from the resonance to $-1$ at $\Delta = 0$. Thus $\delta R^{c/n}$ necessarily has to change sign within the width of the resonance.

Given that the coefficient $B$ is singular at small momenta, it is instructive to analyze the behavior of $\delta R^{c/n}$ described by the equations (12-14) in the limit of small $p$ (but still considering $\alpha/v$ as small for the applicability of the perturbative treatment of the Coulomb interaction). At this point one has to recall that the width parameter $\gamma$ for a $P$ wave resonance has to vanish at small $p$ as $p^3$, and that the non-resonant scattering phase $\delta_1$ also vanishes as $p^3$. Thus the effect of the $p^{-2}$ singularity in the coefficient $B$ results in a constant at small velocity term in the correction: $\text{const} \cdot \alpha$, which is small in comparison with the dominant part of the correction behaving as $\alpha/v$.

It should be emphasized, prior to discussing the behavior at the realistic $\Upsilon(4S)$ resonance, that the expressions (12) - (14) are formally applicable only in the limit of small $pa$. Indeed, in this limit the details of the transition between an isotopically symmetric strong dynamics at short distances and the Coulomb behavior at the relevant distances of order $p^{-1}$ are not essential, and the parameter $a$ enters only the leading singularity of the coefficient $B$ resulting in a subleading at low energy term in $\delta R^{c/n}$, which can be studied phenomenologically\(^2\). When the parameter $pa$ cannot be considered as small, the details of the actual behavior of the wave functions at the transition distances would generally depend on the shape of the transition, and a more elaborate matching at short distances, than the conditions (4) at a fixed distance, might be required.

The position of the actual $\Upsilon(4S)$ resonance corresponds to the momentum $p_0 \approx 330\, MeV$. Thus, most likely, the relevant values of the parameter $pa$ are of order one, i.e. the parameter is neither small nor large in the region of interest. Under these circumstances one can approach this region of phenomenological interest from the side of small values of $pa$ aiming at an at least qualitative description of what kind of behavior should be expected for $\delta R^{c/n}$ at the resonance. In addition, the width of the $\Upsilon(4S)$, $\Gamma \approx 14\, MeV$, is not small as compared to its excitation energy. Combined with the not too small value of $p_0$ this can generally lead to that effects of higher terms of expansion in $p^2$ of the quantities $\gamma(p)$, $\delta_1(p)$, and $\Delta(p)$ may become essential. (An attempt at taking into account higher than $p^2$ terms in the width parameter was done in connection with the experimental measurement of $\Gamma$ in

\(^2\)It can be noticed that a non-removable dependence on a short-distance parameter is a general property of scattering in the states with $L \neq 0$, and in fact it also appears, although as a logarithmic dependence, in the $S$ wave scattering in the presence of a Coulomb interaction\(^4\).
Ref.[10]. However the effect of those higher terms turned out to be quite small within the experimental accuracy.)

Figure 1: The dependence of the ratio $R_{c/n}$ on the excitation energy $E = \sqrt{s} - 2m_B$ in the region of the $\Upsilon(4S)$ resonance (the center position is assumed to be at $E_0 = 20$ MeV) for some values of $a$ and $\delta_1(E_0)$: $a^{-1} = 200$ MeV, $\delta_1(E_0) = 0$ (solid), $a^{-1} = 400$ MeV, $\delta_1(E_0) = 0$ (dashed), $a^{-1} = 300$ MeV, $\delta_1(E_0) = 30^0$ (dashdot), and $a^{-1} = 300$ MeV, $\delta_1(E_0) = -30^0$ (dotted).

With all the stated reservations about uncertainties involved in a quantitative description of the behavior of $\delta R_{c/n}$ in the region of the $\Upsilon(4S)$ resonance, a qualitative illustration of the expected variation of $R_{c/n}$ in the resonance region is provided by the plots in Fig.1. The curves in the plots are calculated with various values of $a$ and $\delta_1$ under the following assumptions: only the leading terms in the expansion of $A$ and $B$ at small $pa$, explicitly shown in eq.(14), are retained, the width parameter is parameterized as $\gamma = (\Gamma/2) (p/p_0)^3$, and the non-resonant phase as $\delta_1 = \delta_1(E_0) (p/p_0)^3$. (It should be mentioned that the range of $\delta_1(E_0)$ from $-30^0$ to $+30^0$ most likely is unrealistically broad, and is used here for an illustration of the effect of the phase under extreme assumptions.) One can clearly see
from the shown curves that being undoubtedly different in details, they exhibit quite similar qualitative behavior\(^3\). Thus, as expected on general grounds from eq. (12), the very fact of a substantial and rapid variation of $\delta R^{c/n}$ within the resonance width stays robust under assumptions about the presently unknown parameters.

The strong variation of $\delta R^{c/n}$ at the scale of few MeV near the center of the $\Upsilon(4S)$ resonance can be important in a comparison of the data obtained in different experiments, especially at different electron-positron colliders, given the differences in the beam energy spread and possible slight differences in the values (and stability) of the central energy of the beams, at which the data are collected. The thus far available results\(^1\)\(^2\)\(^3\), are consistent with each other mainly due to the substantial statistical and systematic errors. This agreement however can change if the measurements of the discussed relative yield are pursued further with better accuracy, possibly permitting a quantitative study of the variation of this yield. As is argued here, the effect of the variation is not small, and the detailed behavior is sensitive to properties of the heavy and light quark and hadron dynamics, which would be difficult, if possible at all, to study by other means.

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\(^3\)The second zero of $\delta R^{c/n}$ at $E > E_0$ is not reached in some of the curves in Fig.1 within the shown range of the energy. The behavior at large energies however becomes very sensitive to higher in $p^2$ terms in $\gamma$, $\Delta$, and $\delta_1$, and the location of that zero cannot be presently estimated with any certainty.
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