Relativistic diffusion of quarks in random gluon fields

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Abstract
We consider Wong equations for a particle with a continuous mass spectrum in a random Yang-Mills field approximating the quantum field at finite temperature. We show that particle time evolution can be approximated by a relativistic diffusion. Kubo’s generator of the relativistic diffusion is defined as an expectation value of the square of the Liouville operator. We discuss hydrodynamics, energy and entropy of the gas of diffusing particles.

1 Introduction
In non-relativistic physics the dynamics of a test (tracer) particle in a medium of some other particles is often approximated by diffusion [1]. We encounter such a problem in high-energy physics as well. The diffusion approximation has been applied to a description of the quark-gluon plasma and to heavy ion collisions [2][3][4][5]. If QCD is considered as the fundamental theory of strong interactions then the diffusion should result from the QCD transport theory [6][7][8]. The form of the diffusion operator can in principle be determined on the basis of the Boltzmann kinetic equation after a calculation of the scattering amplitude [3][9]. A semi-classical approach to the transport theory based on the Wong equation (derived from quantum dynamics in [10][11][12]) leads to an analog of the Vlasov equation [6][7][11][13]. Quantized gauge fields in the Vlasov equation have been discussed in [14][15][16]. These authors have shown that quark’s time evolution in a quantum gauge field leads to a diffusion of momentum and color. However, the generator of the diffusion has not been determined in an unambiguous way. It is well-known [17][18] that the Vlasov equation describing a tracer particle moving in plasma in a random electromagnetic field produced by a chaotic motion of other particles can be approximated by a diffusion equation. The generator of the diffusion has been calculated in [17][18]. A detailed discussion
of the diffusion approximation to a random non-relativistic dynamics can be found in [19][20] and [21].

The mathematical theory of a relativistic diffusion is not so well developed as the non-relativistic one (see the reviews in [22][23]). A relativistic diffusion in the configuration space does not exist [24][25]. If we assume that the diffusing particle has a fixed mass then an analog of the Kramers diffusion is uniquely defined by its generator being the Laplace-Beltrami operator on the mass-shell as shown by Schay [26] and Dudley [27]. There are some other suggestions for a definition of the relativistic diffusion without the assumption that the diffusion takes place on the mass-shell [22][23][28].

In this paper we start from the basic principles of quantum gauge theory at finite temperature. We assume the gauge and Lorentz covariance in the most general form. It is known that the thermodynamic equilibrium is defined in the rest frame. Then, if the finite temperature theory is Lorentz covariant then it can be transformed to an arbitrary Lorentz frame described by a unit time-like four-vector $w^\mu$. We assume that the two-point function of gauge invariant field strength $[29][30] F_{\mu\nu}(x)$ depends on two four-vectors the position $x$ and the frame $w$. It comes out that the diffusion generator depends only on the form of the two-point function at coinciding points. The diffusion tensor can be related to the energy-momentum tensor of gluons in the thermal state. We do not assume that the particles (quarks) are on the mass-shell as we did in the case of (zero-temperature) QED in [31]. Only free stable particles satisfy the mass-shell condition. Relaxing this condition allows an explicit use of the Lorentz covariance (four independent components of the momentum; the transport theory for particles with a continuous mass spectrum has been discussed also in ref.[32]). The plan of the paper is the following. In sec.2 we formulate the Wong equation in the random field. In sec.3 we calculate the expectation value of the random evolution (the square of the Liouville operator) which determines the diffusion generator. In sec.4 we take the diffusion model resulting from the quantum gauge theory at finite temperature as the starting point of the statistical mechanics of diffusing particles. We define the energy, free energy and entropy satisfying the second law of thermodynamics. We initiate a hydrodynamic description of the gas of diffusing particles. The aim is to develop a macroscopic hydrodynamics on the basis of quantum gauge theory to supplement the phenomenological approach of Israel and Stuart [33] usually applied in heavy ion collisions [34].

## 2 Wong equation in a random Yang-Mills field

The dynamics of a particle in a Yang-Mills field $A^\mu_\mu$ is described by the equations [10]

$$\frac{dx^\mu}{d\tau} = p^\mu, \tag{1}$$
\[
\frac{dp_\mu}{d\tau} = Q_a F^a_{\mu\nu} p^\nu, \quad (2)
\]
\[
\frac{dQ_a}{d\tau} = -f^{abc} p^\mu A^b_{\mu} Q_c, \quad (3)
\]
where $\mu = 0, 1, 2, 3$. It follows from eq.(2) that
\[
\frac{d}{d\tau} (\eta_{\mu\nu} p_\mu p_\nu) = 0. \quad (4)
\]
where $\eta_{\mu\nu} = (1, -1, -1, -1)$. From eqs.(1) and (4) it follows that $\tau$ is the proper time
\[
d\tau^2 = dx^\mu dx_\mu p^{-2}. \quad (5)
\]
If we consider an observable as a function $\phi$ on the product of the phase space and the internal (color) space $(x, p) \times SU(n)$ then while the coordinates, momenta and charges $(x(\tau), p(\tau), Q(\tau))$ evolve in the proper time according to eqs.(1)-(3) a function of observables $\phi_\tau \equiv \phi(x(\tau), p(\tau), Q(\tau))$ evolves as
\[
\partial_\tau \phi = p^\mu \frac{\partial \phi}{\partial x^\mu} + Q_a F^a_{\mu\nu} p^\nu \frac{\partial \phi}{\partial p_\mu} - f^{abc} p^\mu A^b_{\mu} Q_c \frac{\partial \phi}{\partial Q_a}. \quad (6)
\]
In eq.(6) we assumed that $p_0$ is an independent variable. This means a continuous mass spectrum as in a description of unstable particles (resonances) in quantum field theory [32]. The Wigner function approach to the kinetic equation [9][35] also leads to the continuous mass spectrum well-known from the Källen-Lehman representation of expectation values of quantum fields. We define an expectation value of the observable $\phi$ in a state (probability distribution) $\Omega$ as
\[
\Omega(\phi) = (\Omega, \phi) = \int dx d\sigma(p) \Omega(\phi). \quad (7)
\]
where the momentum distribution $\sigma$ will be expressed in the form (we assume $m^2 \geq 0$)
\[
d\sigma(p) = dp dm^2 \rho(m^2) \delta(p^2 - m^2 c^2) \quad (8)
\]
In order to describe a particle with a fixed mass $m_0$ it is sufficient to choose $\rho(m^2) = \delta(m^2 - m_0^2)$. The evolution of $\Omega$ is defined as an adjoint of the evolution of $\phi$
\[
(\Omega_\tau, \phi) = (\Omega, \phi_\tau). \quad (9)
\]
Then
\[
\partial_\tau \Omega_\tau = -p^\mu \frac{\partial \Omega}{\partial x^\mu} + Q_a F^a_{\mu\nu} p^\nu \frac{\partial \Omega}{\partial p_\mu} + f^{abc} Q_a p^\mu A^b_{\mu} \frac{\partial \Omega}{\partial Q_c}. \quad (10)
\]
The requirement of the $\tau$-independence of the probability distribution ($\partial_\tau \Omega_\tau = 0$) gives the kinetic equation in the laboratory time $t$. This is the same equation as the one which can be derived by an elimination of $\tau$ in favor of $t$ in the
evolution equations (1)-(3). It will be useful to work with a matrix notation. Let $T_a$ be a representation of the algebra of the group $SU(n)$

$$[T_a, T_b] = f^{abc} T_c$$

normalized by $Tr(T_a T_b) = -\delta_{ab}$. We define $Q = Q_a T_a$ and $A_\mu = T_a A^a_\mu$. We consider the path-ordered exponential along a curve $\gamma$

$$U_{\gamma y} = T\left(\exp(- \int A_\mu d\gamma^\mu)\right).$$

(11)

with $y$ as the endpoint of the curve. Then,

$$\frac{\partial}{\partial y^\mu} U_{\gamma y} = -A_\mu(y) U_{\gamma y}.$$

(12)

The field strength $F$ is constructed from $A_\mu$. This assumption is expressed as the Bianchi identities

$$\epsilon^{\mu\nu\sigma\rho} D_\mu F_{\sigma\rho} = 0,$$

where $D_\mu F_{\sigma\rho} = \partial_\mu F_{\sigma\rho} + [A_\mu, F_{\sigma\rho}]$. We consider random gluon fields as an approximation to quantum fields. In general, correlation functions of gauge fields being gauge dependent are not well-defined. We are going to define a two-point correlation function of gauge invariant variables

$$\hat{F}_{\mu\nu}(\gamma, y) = U_{\gamma y}^{-1} F_{\mu\nu}(y) U_{\gamma y}.$$  

(13)

$\hat{F}$ satisfies the Bianchi identities with an ordinary derivative (instead of the covariant one;this has been shown first in [29], see also [30])

$$\epsilon^{\mu\nu\sigma\rho} \partial_\mu \hat{F}_{\sigma\rho} = 0$$

(14)

The Yang-Mills equations are [30]

$$\partial^\mu \hat{F}_{\mu\nu}(\gamma, y) = U_{\gamma y}^{-1} J_\nu(y) U_{\gamma y}.$$  

(15)

In terms of the path ordered exponential the solution of eq.(3) is

$$Q(\tau) = U_p(\tau)Q(0)U_p^{-1}(\tau),$$

(16)

where $U_p(\tau) = U_{\gamma y}$ is calculated along the straight line in the direction $p$, i.e., $\gamma(s) = x - ps$ with $y = x - p\tau$.

We define the corresponding gauge invariant field strength transported along the straight line

$$\hat{F}_{\mu\nu}(p, x - ps) = U_p(s)^{-1} F_{\mu\nu}(x - ps) U_p(s).$$

(17)

We treat the Wong equations (1)-(3) in a random Yang-Mills field as an approximate description of the dynamics of quarks in quantum gauge theory at
finite temperature. Then, the expectation value of an observable $O$ is defined as

$$\langle O \rangle_\beta = Tr \left( \exp(-\beta w^\mu P^\mu) O \right)$$

(18)

where $P^\mu$ is the generator of space-time translations and the four-vector $w^\mu$ ($w^\mu w_\mu = 1$) describes the reference frame [36]. In QED (and perturbative QCD) the expectation value (18) leads to the distribution of photons (gluons) according to the covariant Bose-Einstein law

$$n(p; w) = (\exp(\beta w^\mu p_\mu) - 1)^{-1}$$

(19)

The two-point correlation functions of gauge invariant variables $\hat{F}_{\mu\nu}(p, x - ps)$ are invariant under transformations of the Poincare group (acting upon $x, p$ and $w$) and under transformations of the (internal) SU(n) symmetry. We assume that the non-commutativity of quantum gauge fields at finite temperature can be neglected in the calculation of the particle time evolution. Then, the symmetry properties with respect to an exchange of indices together with the Bianchi identities (14) lead to the representation (see a more detailed discussion of the Abelian case in [31])

$$\langle \hat{F}_{\mu\nu}(p, x - ps) \rangle = \delta^{\mu\nu} G_{\rho\sigma;x\rho}(x - x'; s, s', w),$$

(20)

where

$$G_{\mu\nu;\sigma\rho}(y; s, s', w) = \int dk \hat{G}(0, 0, k, p, w)(\eta_{\mu\sigma} k_\rho k_\rho - \eta_{\mu\rho} k_\nu k_\nu + \eta_{\nu\sigma} k_\mu k_\mu - \eta_{\nu\rho} k_\mu k_\rho) \exp(iky)$$

(21)

$\hat{G}$ is a Lorentz invariant function of $k, p$ and $w$. Let us note that

$$G_{\mu\nu;\sigma\rho}(0; 0, 0, w) = \eta_{\mu\sigma} T_{\nu\rho} - \eta_{\mu\rho} T_{\nu\sigma} + \eta_{\nu\sigma} T_{\mu\rho} - \eta_{\nu\rho} T_{\mu\sigma}$$

(22)

and

$$T_{\mu\nu} = \int dk \hat{G}(0, 0, k, 0, w) k_\mu k_\nu = \rho w_\mu w_\nu - \pi_E (\eta_{\mu\sigma} w_\rho - \eta_{\mu\rho} w_\sigma)$$

(23)

Eq.(23) gives a decomposition of the tensor $T_{\mu\nu}$ in terms of $\eta_{\mu\nu}$ and $w_\mu w_\nu$. If $\hat{G}$ is non-negative then $\rho$ and $\pi_E$ must be non-negative. $\hat{G}$ at $s = s' = 0$ does not depend on $p$ because $p$ can enter $F$ only through the path starting in $0$ and ending in $sp$. $T_{\mu\nu}$ is related to the expectation value (22) of the Yang-Mills energy-momentum tensor. In order to see this let us calculate

$$-Tr(F_{\mu\rho} F_{\nu\sigma} \eta^{\rho\sigma} - \frac{1}{4} \eta_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}) = 2T_{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} T_{\rho\rho}$$

(24)

The expectation value of $F_{\mu\nu}$ can be different from zero if there are some source terms in the Lagrangian. So, adding the term $j_\mu A^\mu$ to the Lagrangian of QED gives

$$\langle F_{\mu\nu}(x) \rangle_\beta = \int dy j^\mu(y)(\partial^\rho G_{\mu\nu}(x - y) - \partial^\rho G_{\rho\nu}(x - y))$$

(25)
where \( G_{\mu\nu} \) is the two-point function of the potential \( A_\mu \). A gauge and Poincare invariant choice \( j^\mu = w^\mu \) in eq.(25) leads to

\[
\langle \partial_\sigma F_{\mu\nu}(x) \rangle^\beta = (2\pi)^{-3} \int dk \delta(k^2) \delta(k)n(k,w)(k_\sigma k_\nu w_\mu - k_\sigma k_\mu w_\nu) = \lambda (\eta_\mu \sigma w_\nu - \eta_\nu \sigma w_\mu)
\]

(26)

with a certain constant \( \lambda \) (rhs of eq.(26) satisfies the Bianchi identities (14)).

In non-abelian gauge theories we may insert in the formula for the expectation value the Wilson loop (11) with the curve \( \gamma^\mu(s) = sw^\mu \) (this is the Polyakov loop applied for a description of confinement in finite temperature gauge theories)

\[
\text{Tr} \left( \exp \left( \int_0^\beta A_\mu w^\mu dt \right) \right)
\]

(27)

In a perturbative QCD we again derive eq.(26) with a certain constant \( \lambda \).

In finite temperature QED \( \rho = w^\mu T_{\mu\nu} w^\nu \) in eq.(23) has the meaning of the photon energy density when \( \tilde{G} \simeq \delta(k^2)n(k,w) \). \( \pi_E \) has the meaning of the pressure of photons. Such an interpretation is expected to hold true for gluons.

### 3 Time evolution in a random gluon field

In this section we study functions of particle trajectories. We consider kinetic evolution equations (6) which are of the form

\[
\partial_t \phi = (X + Z + Y)\phi \equiv (K + Y)\phi.
\]

(28)

where \( K = X + Z \). Let

\[
Y(s) = \exp(-sK)Y \exp(sK).
\]

(29)

Then, the solution of eq.(28) can be expressed as

\[
\phi_t = \exp(tK)\phi^I_t,
\]

(30)

and

\[
\partial_s \phi^I_s = Y(s)\phi^I_s.
\]

(31)

In eq.(28) we have according to eq.(7)

\[
X = p^\mu \partial^\nu_{\mu}
\]

(32)

\[
Y = -\text{Tr}(QF_{\mu\nu}(x))p^\rho \frac{\partial}{\partial p^\rho}
\]

(33)

\[
Z = -f^{abc}Q_a p^\mu A_\mu^b \frac{\partial}{\partial Q_c^I}
\]

(34)

6
Let us calculate \( \exp(\tau K) \) (with \( K = X + Z \)). We have

\[
\exp(\tau K)\psi = \exp(\tau X)\psi^I + \partial_\tau \psi^I = Z(\tau)\psi^I
\]

with

\[
Z(\tau) = \exp(-\tau X)Z \exp(\tau X) = -f^{abc}p^\mu A^b_\mu(x - p\tau)Q_a \frac{\partial}{\partial Q_c}.
\]

From eq.(28) and (30) we obtain

\[
Y(s) = -Tr\left(Q(s)F_{\mu\nu}(x - sp)p^\nu\left(\frac{\partial}{\partial p^\mu} + s\frac{\partial}{\partial x^\mu}\right)\right)
\]

We are going to approximate the evolution (28) by a diffusion. In general, this will be a diffusion on the product of the phase space and the group \( SU(n) \). As follows from eqs.(1)-(3) such an evolution is covariant but not invariant with respect to the gauge transformations. We show that the diffusion in the momentum space can be defined in a gauge invariant way.

We apply the approach of Kubo [19]-[20] which approximates the random Liouville operator on the rhs of eq.(6) by the expectation values of \( Y \) and its square

\[
\langle \phi^I_\tau \rangle \approx \exp\left(\int_0^\tau ds \langle Y(s) \rangle + \frac{1}{2}(\int_0^\tau ds \tilde{Y}(s))^2\right)
\]

where

\[
\tilde{Y} = Y - \langle Y \rangle
\]

If the mean value of \( Y(s) \) is different from zero then in the expectation value \( \langle \phi_t \rangle \) we could obtain the term of the first order in \( \tau \)

\[
K_1 = \langle Tr(Q\tilde{F}_{\mu\nu})p^\mu \partial^\nu \rangle
\]

(from now on we denote the momentum derivatives by \( \partial^\nu \) and space-time derivatives by \( \partial^\mu \)). However, such a term must vanish owing to the Poincare covariance because it is impossible to express the antisymmetric tensor \( Tr(Q\tilde{F}_{\mu\nu}) \) by \( \eta^{\mu\nu} \) and \( w^\mu \). Then, we have the second order term \( K_2 \tau^2 \) coming from the expansion of \( \tilde{F}_{\mu\nu}(x - sp) \) in \( s \)

\[
K_2 = \langle Tr(Q\tilde{F}_{\mu\nu})\rangle p^\mu p^\sigma \partial^\nu = \lambda(\eta_{\mu\sigma}w_{\nu} - \eta_{\nu\sigma}w_{\mu})p^\mu p^\sigma \partial^\nu \equiv \lambda P_{\mu\sigma} w^\sigma \partial^\mu
\]

where

\[
P^{\mu\nu} = \eta^{\mu\nu} - p^{-2}p^\mu p^\nu
\]
The second order term resulting from the square of $Y$ reads
\[
\lim_{\tau \to 0} \frac{1}{2} \langle (\int_0^\tau Y(s)ds)^2 \rangle = \lim_{\tau \to 0} \frac{1}{2} \int_0^\tau ds \int_0^\tau ds' \langle Y(s)Y(s') + Y(s')Y(s) \rangle
\]
\[
= (\eta_{\mu\nu} T_{\nu\rho} - \eta_{\mu\rho} T_{\nu\nu} + \eta_{\nu\rho} T_{\mu\mu} - \eta_{\nu\nu} T_{\mu\rho}) p^\nu \partial^\mu p^\rho \partial^\sigma \equiv -A_w
\]

(42)

Elementary calculations using eq.(22) give
\[
A_w = -2\pi E q^2 \partial_\mu P_{\mu\nu} p^\nu \partial_\rho + q^2 (p + \pi E) ((wp)^2 \partial_\mu \partial^\mu - 2wpw^\mu \partial_\mu \partial_\rho + p^2 w^\mu - w^2 \partial_\mu \partial_\rho - 2wpw^\nu \partial_\nu)
\]

(43)

where $q^2 = Q_a(\tau)Q_a(\tau) = Q_a(0)Q_a(0)$ is the square of the charge (which is a constant). The term $s\partial_\mu$ of eq.(37) does not contribute to $A_w$. This is so because it is of higher (third) order in $\tau$. The limit in eq.(42) depends only on $DG(0)$ corresponding to $s = s' = 0$ because taking into account $s$ in $G$ would give higher orders in $\tau$. As a consequence the $T_{\mu\nu}$ coefficients do not depend on $p$ because the dependence on $p$ is always multiplied by $s$. According to Kubo [19]-[20] the $\tau^2$ behaviour at short time (small in comparison to the correlation time) of the random evolution goes over to the linear behaviour at large time with the same generator $A_w + K_2$. In order to justify a replacement of the Liouville evolution (6) by the diffusion (42) at an arbitrarily large time we have to apply the Markov approximation. In such a case the dynamics is viewed as a composition of independent short time evolutions (see the discussion in Kubo [19]-[20]). The rigorous proof [21] requires a proper scaling of fields and the time in order to determine precisely the conditions required for the limiting diffusive behavior.

In the lowest order in the expectation value of the expansion of the solution (28) (besides the term (42)) we shall also have
\[
\int_0^\tau ds \int_0^s ds' \langle Z(s)Z(s') \rangle
\]

(44)

and
\[
\int_0^\tau ds \int_0^s ds' \langle Z(s)Y(s') \rangle.
\]

(45)

The term (44) would describe a charge diffusion in a Markovian approximation for the $\langle AA \rangle$ correlations. We would have to determine a form of the $\langle AA \rangle$ correlation function of the gauge field which would not be gauge invariant. The whole (non-perturbative) evolution $\exp(\tau K)$ is gauge dependent although different gauges just rotate the charges (as can be seen from eq.(16)). In this sense fixing the gauge and approximating the charge evolution by the Brownian motion on $SU(n)$ (a well-studied problem in the diffusion theory [37]) could still make sense (as discussed in [14]). The term (45) describes a mixing of the momentum diffusion with the color diffusion. In this paper we do not go into details of the color diffusion because of its complicated gauge dependence.
4 Relativistic diffusion at finite temperature

The calculations of sec.3 lead to the following diffusion equation in the proper
time describing an evolution of a particle observable $\phi$ (a function of particle’s
position and momentum)

$$
\partial_\tau \phi \equiv G^w \phi = p^\mu \partial^\mu \phi - a \partial_\mu P^{\mu\nu} p^\nu \partial_\nu \phi - \lambda p^2 P^{\mu\nu} w_\nu \partial_\mu \phi \\
+ b \left((pw)^2 \partial^\mu \partial_\mu - (wp)(w^\mu p^\rho + w^\rho p^\mu) \partial_\rho \partial_\mu + p^2 w^\mu w^\rho \partial_\mu \partial_\rho\right) \\
- w^2 p^\rho \partial^\mu \partial_\mu - 2 wpw^\nu \partial^\mu \partial^\nu \phi \nonumber \\
\equiv p^\mu \partial^\mu \phi + A_w \phi 
$$

(46)

where $a = 2q^2 \pi E$, $b = q^2 (\rho + \pi E)$ and $\lambda$ is determined by the expectation value
(41).

The probability distribution evolves according to the adjoint equation (9). The
independence of the proper-time parametrization ($\partial_\tau \Omega = 0$) is equivalent
to the replacement of the proper time by the physical time $t$ in the kinetic
equation (10). This requirement gives the transport equation

$$
G^+ \Omega = 0 
$$

(47)

where the adjoint is either in $L^2(dx \sigma)$ or $L^2(dx dp)$ (it does not depend on $\sigma$).
Explicitly,

$$
p^\mu \partial^\mu \Omega = \partial_\mu \left(D^\mu + \lambda p^2 P^{\mu\nu} w_\nu \right) \Omega \equiv \partial_\mu \left(-a P^{\mu\nu} p^\nu + \lambda p^2 P^{\mu\nu} w_\nu + b (pw)^2 \partial^\mu \\
- (wp)(w^\mu p^\rho + w^\rho p^\mu) \partial_\rho + p^2 w^\mu w^\rho \partial_\rho \right) \Omega 
$$

(48)

In eq.(48) we have

$$
D^\mu = \alpha^{\mu\nu} \partial_\nu 
$$

(49)

where from positive definiteness of $-A_w$ it follows that the momentum dependent
coefficients $\alpha^{\mu\nu}$ satisfy the positivity condition $a_\mu a_\nu \alpha^{\mu\nu} \geq 0$. We find the
equilibrium solution from the requirement

$$
(D^\mu + \lambda p^2 P^{\mu\nu} w_\nu) \Omega_E = 0 
$$

(50)

There is a solution (which is the Jüttner equilibrium [38]), if

$$
\lambda = \beta (b - a) 
$$

(51)

Then,

$$
\Omega_E = \exp(-\beta wp - \gamma p^2), 
$$

(52)

where $\gamma$ is an arbitrary constant.

At zero temperature (realized by setting $b = 0$ and $\lambda = 0$ in eq.(46)) the
diffusion with a continuous mass distribution (8) can be decomposed into the
diffusions on the mass shell using the formula

$$
- \int d\sigma (p) f \partial_\mu p^\mu P^{\mu\nu} \partial_\nu g = \int dp dm^2 \rho(m^2) \delta(p^2 - m^2 c^2) f \triangle m g 
$$

(53)
where
\[ \triangle m_H^j = (\delta_{jl} m^2 c^2 + p^j p^l) \frac{\partial^2}{\partial p^l \partial p^j} + 3p^j \frac{\partial}{\partial p^j} \] (54)

\( j, k = 1, 2, 3 \), and \( \triangle m_H^j \) is the generator of the relativistic diffusion defined by Schay[26] and Dudley [27] (see [39] for recent mathematical results). The diffusion corresponding to \( b = 0 \) and \( \lambda \neq 0 \) has been derived in [40] (and in a covariant form in [41]) from the requirement of the detailed balance leading to the Jüttner equilibrium distribution.

On the basis of the diffusion theory we can develop a statistical description of a gas of diffusing particles. We define the current
\[ N^\mu = \int d\sigma(p) p^\mu \Omega \] (55)
The current can be expressed by the mean momentum \( v \) and the velocity \( u \) (\( u^2 = 1 \))
\[ N^\mu = \rho(\Omega) v^\mu = \rho(\Omega) n u^\mu \] (56)
where \( \rho(\Omega) = \int d\sigma(p) \Omega \), \( v^\mu = n u^\mu \) and \( n^2 = v^\mu v_\mu \). Then, using
\[ p_\mu (D^\mu + \lambda p^2 P^\nu w_\nu) = 0 \] (57)
we show that
\[ \partial_\mu N^\mu = 0 \] (58)

It follows that
\[ N = \int d\Omega N_0 \] (59)
is a constant (in such a case the probability distribution \( \Omega \) can be normalized).
Then, let us define the energy-momentum tensor
\[ \Theta^{\mu\nu} = \int d\sigma(p) p^\mu p^\nu \Omega \] (60)
From eq.(48) we obtain
\[ \partial_\mu \Theta^{\mu\nu} = -(b - 3a) N^\nu - \lambda \Theta^\nu_\mu w^\nu + \lambda \Theta^\mu w_\mu - 2bw^\nu w_\mu N^\mu \] (61)
The statistical description can proceed with some thermodynamic notions (the definitions follow the ones for mass-shell diffusion of ref.[42], see also [43]).
We define the relative entropy current
\[ S_0^{\mu} = N^{-1} \int d\sigma(p) p^\mu \Omega \ln(N^{-1}\Omega N E \Omega^{-1}) \] (62)
It can be shown that \( \int d\Omega S_0^{\mu} \geq 0 \) and
\[ \partial_\mu S_0^{\mu} = -N^{-1} \int d\sigma(p) \Omega \alpha^{\mu\nu} \partial_\nu \ln R \partial_\nu \ln R \leq 0 \] (63)
where \( R = \Omega\Omega^{-1}_E \) and \( \alpha^{\mu\nu} \) (defined in eq.(49)) is positive definite. It follows that \( \int d\Omega S^0_K \) is a non-negative function decreasing monotonically to zero (it can be interpreted as the entropy of the system plus its heat bath). The entropy of the particle system is

\[
S = N^{-1} \int d\Omega \int d\sigma(p)p_0 \Omega \ln \Omega
\]  \hspace{1cm} (64)
We define the free energy

\[
\mathcal{F} = \beta^{-1}N \int d\Omega S^0_K - N\beta^{-1} \ln(NN_E^{-1})
\]  \hspace{1cm} (65)
and the energy

\[
\mathcal{W} = \int d\Omega T_{00}
\]  \hspace{1cm} (66)
It follows from eqs.(64)-(66) that the thermodynamic relation

\[
\beta^{-1}S = \mathcal{W} - \mathcal{F}
\]  \hspace{1cm} (67)
comes out as an identity. The time evolution of each term in eq.(67) is determined by eqs.(61) and (63).

The energy-momentum tensor can be expressed in the form

\[
\Theta^{\mu\nu} = \epsilon u^\mu u^\nu + \tau^{\mu\nu}
\]  \hspace{1cm} (68)
where \( \epsilon = \rho(\Omega)n^2 \) and

\[
\tau^{\mu\nu} = \int d\Omega \sigma(p) (p^\mu - v^\mu)(p^\nu - v^\nu)\Omega
\]  \hspace{1cm} (69)
Let

\[
h_{\nu\alpha} = \eta_{\nu\alpha} - u_\nu u_\alpha
\]  \hspace{1cm} (70)
The divergence of the energy-momentum (68) can be compared with the one calculated in eq.(61). As a result we obtain hydrodynamics equations for the velocity (relativistic Navier-Stokes, see [34])

\[
\epsilon u^\mu \partial^\nu u_\alpha + h_{\nu\alpha} \partial^\nu \tau^{\mu\nu} = \lambda h_{\nu\alpha} (\Theta^{\mu\nu} w_\mu - \Theta^{\nu\nu} w^\mu) - 2bh_{\mu\alpha} w^\nu w^\mu N^\mu
\]  \hspace{1cm} (71)
If \( \lambda = b = 0 \) and \( \tau^{\mu\nu} = -\pi_E(\Omega)h^{\mu\nu} \) (ideal fluids) then in the non-relativistic limit eq.(71) coincides with the Euler equation. The terms on the rhs of eq.(71) describe a friction which brings the gas of particles to the equilibrium described by the state (52)( this is the state of an ideal fluid with the velocity \( u^\mu = w^\mu \)). In order to describe the fluid motion close to the equilibrium we can apply an expansion of the solution of eq.(48) in \( p^\mu \partial^\nu \). In such a case we could determine the viscosity by means of the parameters entering the diffusion equation (48) which can be calculated in quantum gauge theory at finite temperature.
5 Discussion and summary

There is a basic difficulty in a formulation of a transport theory of quark-gluon plasma which starting from quantum gauge theory at finite temperature could reach a form useful for calculations. First of all the non-perturbative QCD is formulated in the Euclidean framework on the lattice. In such a formulation it is impossible to treat quarks dynamically in the real time. Some approximations are unavoidable. Our approximation relies on the Wong equation which can be derived from formal continuum quantum gauge theory \cite{11}\cite{12}. In this paper we have shown that the diffusion results from a model of a particle moving in a gauge field in the same way as a non-relativistic diffusion results from an evolution in a random electric and electromagnetic fields \cite{17}\cite{18}. The Wong equation has the gauge coupling constant as the only parameter. Then, we derive the diffusion equation which depends on three parameters $\rho, \pi_E$ and $\lambda$ which could be calculated from the correlation functions in the Euclidean framework on the lattice (the diffusion constant has been expressed by correlation functions of Wilson loops in finite temperature QCD in an earlier paper \cite{16}). When the diffusion equation is applied then it could be directly compared with experimental data (as some diffusion equations are \cite{5}) or we could use it for some other approximations. We have in particular in mind the hydrodynamic equations which could be derived from the diffusion equation. The parameters entering the fluid equations could be expressed by the ones of the diffusion equation.

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