A NOTE ON COMPACTNESS AND SINGULAR POINTS OF COMPOSITION OPERATORS ON DOMAINS IN $\mathbb{C}^n$

TIMOTHY G. CLOS

ABSTRACT. Let $\Omega \subset \mathbb{C}^n$ for $n \geq 2$ be a bounded pseudoconvex domain with a $C^1$-smooth boundary. We study the boundedness of composition operators on the $p$-Bergman spaces of $\Omega$ for $p \in [1, \infty)$. We also study the compactness of composition operators with surjective, proper symbols on the Bergman spaces of smoothly bounded convex domains. We assume the symbol is sufficiently regular up to the closure of the domain and relate the operator theoretic properties of the composition operator to the behavior of the symbol on the boundary.

1. INTRODUCTION

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let $\mathcal{O}(\Omega)$ be the set of all holomorphic functions from $\Omega$ into $\mathbb{C}$. Let $dV$ be the Lebesgue volume measure on $\Omega$. For $p \in [1, \infty)$ we define

$$A^p(\Omega) := \{f \in \mathcal{O}(\Omega) : \int_{\Omega} |f|^p dV < \infty\}$$

to be the $p$-Bergman space. We denote the norm as

$$\|f\|_{p,\Omega} := \left(\int_{\Omega} |f|^p dV\right)^{\frac{1}{p}}.$$

Let $\phi : \Omega \to \Omega$ be holomorphic on $\Omega$ and continuous up to $\overline{\Omega}$ in each component function. Then we define the composition operator with symbol $\phi$ as

$$C_\phi(f) = f \circ \phi$$

for all $f \in A^p(\Omega)$.

2. SOME BACKGROUND

Compactness of composition operators was studied on the unit disk in $\mathbb{D}$ in the article [CEGY98]. Here, the authors of [CEGY98] study the angular derivative of the symbol near the boundary and obtain a compactness result. They then construct a counterexample to show that the converse of their theorem does not hold true. The authors of [ACT10] studied the closed range property of composition operators on the unit disk. In several variables, work on essential norm estimates and compactness of composition operator was studied in [vZ07] where they obtained bounds on the essential norm of the composition operator on strongly
pseudoconvex domains in $\mathbb{C}^n$. Our approach is to use the idea of the ‘generalized angular derivative’ (the Jacobian) of the symbol and its behavior on the boundary. We will study the boundedness and compactness of the composition operator on bounded pseudoconvex domains in $\mathbb{C}^n$ with symbols of $C^1$-regularity up to the closure of the domain. In particular, we will study the singular points of the symbol and relate it to compactness and boundedness of the associated composition operator.

3. THE MAIN RESULTS

Here are the main results.

**Theorem 1.** Let $\Omega \subset \mathbb{C}^n$ for $n \geq 2$ be a bounded pseudoconvex domain with a smooth boundary. Let $\phi := (\phi_1, \phi_2, ..., \phi_n) : \Omega \to \Omega$ be holomorphic in every component function and $C^1$-smooth in each component function on $\overline{\Omega}$. Furthermore, assume $\phi$ is a surjective proper map. If the Jacobian of $\phi$ is non-vanishing at every point in $\partial \Omega$, then the composition operator $C_\phi : A^p(\Omega) \to A^p(\Omega)$ is bounded for all $p \in [1, \infty)$.

We also study the compactness of $C_\phi$ on $A^2(\Omega)$. Namely, we have the following theorem.

**Theorem 2.** Let $\Omega \subset \mathbb{C}^n$ for $n \geq 2$ be a bounded convex domain with a smooth boundary. Let $\phi := (\phi_1, \phi_2, ..., \phi_n) : \Omega \to \Omega$ be holomorphic in every component functions and $C^1$-smooth in each component function on $\overline{\Omega}$. Furthermore, assume $\phi$ is a proper map. Then if $C_\phi$ is never compact on $A^2(\Omega)$.

By modifying the proof of Theorem 2 slightly, one can obtain the following result.

**Theorem 3.** Let $\Omega \subset \mathbb{C}^n$ for $n \geq 2$ be a bounded convex domain with a $C^2$-smooth boundary. Let $\phi := (\phi_1, \phi_2, ..., \phi_n) : \Omega \to \Omega$ be holomorphic in every component functions and $C^1$-smooth in each component function on $\overline{\Omega}$. Suppose $C_\phi : A^2(\Omega) \to A^2(\Omega)$ is compact. Then either $\phi(\overline{\Omega}) \subset \subset \Omega$ or $(\phi(\overline{\Omega})) \cap \partial \Omega$ consists of singular points.

We can use these ideas to establish upper and lower estimates for the essential norm of $C_\phi$ if $C_\phi$ is not compact.

**Theorem 4.** Let $\Omega \subset \mathbb{C}^k$ for $k \geq 2$ be a bounded pseudoconvex domain with a $C^2$-smooth boundary. Let $\phi : \Omega \to \Omega$ be holomorphic in each component function and $C^1$-smooth in each component function up to $\overline{\Omega}$. Suppose $\inf_{\Omega} |J(\phi)| > 0$ and $C_\phi$ is bounded but is not compact on $A^2(\Omega)$. Then there exists $\delta_0 > 0$ so that

$$0 < \frac{1}{M_{\delta_0}} \inf_{U_{\delta_0}} \{ |J(\phi)| \} \leq \| C_\phi \| \leq M_{\delta_0} \sup_{U_{\delta_0}} \{ |J(\phi)| \}$$
4. Preliminaries

We let \( J(\phi)(p) \) be the Jacobian matrix of \( \phi \) at point \( p \) and \( |J(\phi)(p)| := \det(J(\phi)(p)) \) be its complex determinant at point \( p \).

We note that
\[
\phi(z_1, z_2, ..., z_n) = (\phi_1(z_1, z_2, ..., z_n), \phi_2(z_1, z_2, ..., z_n), ..., \phi_n(z_1, z_2, ..., z_n))
\]
where
\[
\phi_j \in \mathcal{C}^\infty(\overline{\Omega})
\]
and are holomorphic on \( \Omega \) for every \( j = 1, 2, ..., n \).

By the smoothness of \( b\Omega \), we can extend \( \phi_j \) as a smooth function on \( \mathbb{C}^n \) for \( j \in \{1, 2, ..., m\} \), also called \( \phi_j \).

Since \( |J(\phi)(p)| \neq 0 \) for all \( p \in b\Omega \), we use the inverse function theorem applied to \( \phi \), to cover \( b\Omega \) with finitely many balls \( \{B(p_s, r_s)\}_{s=1}^{k} \) so that \( p_s \in b\Omega \) for all \( s = 1, ..., k \), \( r_s > 0 \) for all \( s \in \{1, 2, ..., k\} \), and the restriction \( \phi|_{B(p_s, r_s)} \) is invertible with inverse \( \psi_s \) for \( s \in \{1, 2, ..., k\} \). Then there exists \( \delta_0 > 0 \) so that
\[
U_{\delta_0} := \{(z_1, z_2, ..., z_n) \in \Omega : \text{dist}((z_1, z_2, ..., z_n), b\Omega) < \delta_0\} \subset \bigcup_{s=\{1,2,...,k\}} B(p_s, r_s) \cap \Omega
\]

**Lemma 1.** For \( U_{\delta_0} \) defined previously, the measure \( d\mu := \chi_{U_{\delta_0}} \, dV \) is reverse Carleson with respect to the Lebesgue volume measure \( dV \) on \( \Omega \). That is, for every \( p \in [1, \infty) \) and \( g \in A^p(\Omega) \), there exists \( M_{p,\delta_0} > 0 \) so that
\[
\|g\|_{p,\Omega} \leq M_{p,\delta_0} \|g\|_{p, U_{\delta_0}}.
\]

*Proof.* We consider the restriction operator \( R_{\delta_0} : A^p(\Omega) \to A^2(U_{\delta_0}) \). By the identity principle for holomorphic functions, \( R_{\delta_0} \) is injective. And by Hartog’s extension theorem, \( R_{\delta_0} \) is surjective. Therefore, \( R_{\delta_0} \) is invertible. It is clear that \( R_{\delta_0} \) is bounded. Therefore, by the Open Mapping theorem, \( R_{\delta_0} \) has a bounded inverse. Then there exists \( M > 0 \) so that
\[
\|f\|_{p,\Omega} = \|(R_{\delta_0})^{-1} R_{\delta_0} f\|_{p,\Omega} \leq M \|R_{\delta_0} f\|_{p,\Omega} = M \|f\|_{p, U_{\delta_0}}.
\]

This shows that \( d\mu \) is reverse Carleson.

\[ \Box \]

**Lemma 2.** Let \( \Omega_1, \Omega_2 \subset \mathbb{C}^n \) for \( n \geq 2 \) be bounded pseudoconvex domains. Furthermore, assume there exists a biholomorphism \( B : \Omega_1 \to \Omega_2 \) so that \( B \in \mathcal{C}^1(\overline{\Omega_1}) \). Suppose \( \phi := (\phi_1, \phi_2, ..., \phi_n) : \Omega_2 \to \Omega_2 \) is such that the composition operator \( C_\phi \) is compact on \( A^2(\Omega_2) \). Then, \( C_{\phi \circ B} \) is compact on \( A^2(\Omega_2) \).
Proof. Let \( h_j \in A^2(\Omega_2) \) so that \( h_j \to 0 \) weakly as \( j \to \infty \). Then we have,
\[
\|C_{\phi \circ B}(h_j)\|_{2, \Omega_1}^2 = \|h_j \circ \phi \circ B\|_{2, \Omega_1}^2 \leq \mathcal{M}\|C_{\phi}(h_j)\|_{2, \Omega_2}^2
\]
This shows that \( C_{\phi \circ B} \) is compact on \( A^2(\Omega) \).

5. PROOFS OF MAIN RESULTS

5.1. Proof of Theorem 1.

Proof. To show if \( f \in A^p(\phi(\Omega)) \) then \( f \circ \phi \in A^p(\Omega) \), it suffices to show \( f \circ \phi \in A^p(U_{\delta_0}) \) and apply Hartog’s extension theorem and identity principle. We have
\[
\int_{U_{\delta_0}} |f \circ \phi|^p dV 
\]
\[
\leq \sum_{s=1}^k \int_{B(p_s, r_s) \cap \Omega} |f \circ \phi|^p dV 
\]
\[
= \sum_{s=1}^k \int_{\phi(B(p_s, r_s) \cap \Omega)} |f|^p |J(\psi_s)|^2 dV 
\]
\[
\leq k \sup_{s=1,2,...,k} \sup_{\phi(B(p_s, r_s) \cap \Omega)} |J(\psi_s)|^2 \|f\|_{p, \phi(\Omega)}^p < \infty
\]

Then the proof of Theorem 1 follows from an application of Lemma 1 to the above string of inequalities.

5.2. Proof of Theorem 2.

Proof. Since the Jacobian of \( \phi \) is holomorphic, it suffices to show that the Jacobian is identically 0 on \( b\Omega \) and use the maximum principle for holomorphic functions to conclude that the determinant of the Jacobian matrix of \( \phi \) is 0 on \( \Omega \). Assume \( C_{\phi} \) is compact on \( A^2(\Omega) \) but for the sake of obtaining a contradiction, there exists \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in b\Omega \) so that \( |J(\phi)(\lambda)| \neq 0 \). Then by the inverse function theorem, there exists \( \varepsilon > 0 \) so that \( \phi \) is invertible on \( B(\lambda, \varepsilon) \) with inverse \( \psi_\varepsilon \). By rotating and translating the domain (all biholomorphisms) and using Lemma 2, we may assume \( \phi(\lambda) = (0, 0, ..., 0) \). Since \( \phi \) is surjective and proper, \( (0, 0, ..., 0) \in b\Omega \). Let \( \{g_j\}_{j \in \mathbb{N}} \subset A^2(\phi(\Omega)), g_j \to 0 \) weakly as \( j \to \infty \), and \( \|g_j\|_{2, \phi(\Omega)} = 1 \) for all \( j \in \mathbb{N} \). Furthermore, we can construct \( g_j \) so that the ‘mass’ of \( g_j \) accumulates around \( 0 := (0,0,...,0) \).
See [Ccc18] for the construction of $g_j$ on such a bounded convex domain. That is, for every $\delta > 0$ there exists $N_\delta > 0$ so that $\|g_j\|_{2,\Omega \cap B(0,\delta)} \geq N_\delta$ for all $j \in \mathbb{N}$. Then let $\delta > 0$ be so that $B(0,\delta) \subset \phi(B(\lambda, \varepsilon))$. Then we have

$$\begin{align*}
\|C_{\phi g_j}\|_{2,\Omega}^2 &\geq \|C_{\phi g_j}\|_{2,\Omega \cap B(\lambda, \varepsilon)}^2 \\
&\geq \int_{\Omega \cap B(\lambda, \varepsilon)} |g_j \circ \phi|^2 dV \\
&\geq \int_{\phi(\Omega \cap B(\lambda, \varepsilon))} |g_j|^2 |J(\psi_\varepsilon)|^2 dV \\
&\geq \frac{\inf_{\phi(\Omega \cap B(\lambda, \varepsilon))} |J(\psi_\varepsilon)|^2 \|g_j\|_{2,\Omega \cap B(0,\delta)}^2}{\inf_{\phi(\Omega \cap B(\lambda, \varepsilon))}} \\
&\geq MN_\delta > 0
\end{align*}$$

Thus $\|C_{\phi g_j}\|_{2,\Omega}$ does not converge to 0 as $j \to \infty$, a contradiction. Therefore, the Jacobian of $\phi$ is identically 0 on $b\Omega$. Since the Jacobian is a holomorphic function on $\Omega$ and is continuous up to the closure of $\Omega$, we have that $|J(\phi)| \equiv 0$ on $\Omega$. Thus an application of Sard’s theorem states that $\phi$ is singular everywhere. Thus $\phi$ is not surjective, which contradicts the assumption that $\phi$ is proper. \hfill \square

5.3. Proof of Theorem 3.

Proof. Assume $C_{\phi}$ is compact. Suppose $(\phi(\Omega)) \cap b\Omega \neq \emptyset$ and so let $(p_1, p_2, ..., p_n) \in (\phi(\Omega)) \cap b\Omega$. Without loss of generality and appealing to Lemma 2 we may assume $(p_1, p_2, ..., p_n) = (0, 0, ..., 0)$. Furthermore, since $\Omega$ is convex, we may assume $\Omega \subset \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : \Re(z_1) > 0\}$. Now assume $|J(\phi)(0, 0, ..., 0)| \neq 0$. Since $\Omega$ has a $C^2$-smooth boundary, by the inverse function theorem there exists $\varepsilon > 0$ so that $\phi$ is a diffeomorphism on $\Omega(\varepsilon)$.

Thus there exists $\varepsilon > 0$ so that $\phi$ is a diffeomorphism on $B((0, 0, ..., 0), \varepsilon) := \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : |z_1|^2 + ... + |z_n|^2 < \varepsilon^2\}$.

We define $g_j(z_1, z_2) = \frac{\alpha_j}{z_1^\beta_j}$ where $\beta_j = 1 - \frac{1}{j}$ and $\alpha_j$ chosen so that $\alpha_j \to 0$ as $j \to \infty$ and $\|g_j\|_2 = 1$ for all $j \in \mathbb{N}$. The convexity of $\Omega$ allows us to construct this $g_j$ so that $g_j \in A^2(\Omega)$ for all $j \in \mathbb{N}$ by taking appropriate branch cuts. Then one can show $g_j \to 0$ uniformly on compact subsets of $\Omega$ as $j \to \infty$ and so $g_j \to 0$ weakly as $j \to \infty$. Furthermore, one can show that there exists $\delta > 0$ so
that \( \|g_j\|_{L^2(B((0,0,\ldots,0),\varepsilon) \cap \Omega)} \geq \delta \) for all \( j \in \mathbb{N} \). Then there exists an \( M > 0 \) so that

\[
\int_{\phi^{-1}(B((0,0,\ldots,0),\varepsilon) \cap \Omega)} |g_j \circ \phi|^2 dV = \int_{B((0,0,\ldots,0),\varepsilon) \cap \Omega)} |g_j|^2 |J(\phi)|^2 dV \geq \frac{\inf_{B((0,0,\ldots,0),\varepsilon) \cap \Omega} \{|J(\phi)|^2\} \|g_j\|_{L^2(B((0,0,\ldots,0),\varepsilon) \cap \Omega)}^2}{M} \geq M \delta^2 > 0
\]

Thus \( \|C\phi g_j\|_{L^2(\phi^{-1}(B((0,0,\ldots,0),\varepsilon) \cap \Omega))} \) does not converge to 0. Since \( \phi^{-1}(B((0,0,\ldots,0),\varepsilon) \cap \Omega) \) is an open set, we have that \( \|C\phi g_j\|_{L^2(\Omega)} \) does not converge to 0, which contradicts the compactness of \( C\phi \).

\[\Box\]

5.4. Proof of Theorem 4

Proof. Let \( K : A^2(\Omega) \to A^2(\Omega) \) be an arbitrary compact operator. Then for every \( n \in \mathbb{N} \) there exists \( f_n \in A^2(\Omega) \) so that \( f_n \to 0 \) weakly as \( n \to \infty \), \( \|f_n\| = 1 \) for all \( n \in \mathbb{N} \) and

\[
\|C\phi\|_{e} \leq \|C\phi - K\| \leq \|C\phi f_n - K f_n\| + \frac{1}{n}.
\]

Then there exists \( \delta_0 > 0 \) so that \( \phi \) is a diffeomorphism on \( U_{\delta_0} \) and \( J(\phi) \neq 0 \) on \( \overline{U_{\delta_0}} \). Furthermore, we may assume each component function of \( \phi \) has a smooth extension to \( \overline{U_{\delta_0}} \). Then we have using Lemma 11,

\[
\|C\phi\|_{e} \leq \|C\phi - K\| \leq \|C\phi f_n - K f_n\| + \frac{1}{n} \leq M_{\delta_0} \|C\phi f_n\|_{L^2(U_{\delta_0})} + \|K f_n\| + \frac{1}{n} \leq M_{\delta_0} \sup_{U_{\delta_0}} \{ |J(\phi)| \} \|f_n\|_{L^2(\phi(U_{\delta_0}))} + \|K f_n\| + \frac{1}{n} \leq M_{\delta_0} \sup_{U_{\delta_0}} \{ |J(\phi)| \} + \|K f_n\| + \frac{1}{n}.
\]

Then letting \( n \to \infty \) we have our upper estimate.
We assume $C_\phi$ is not compact. Then there exists $\{f_n\}_{n \in \mathbb{N}} \subset A^2(\Omega)$ so that $\|f_n\| = 1$ for all $n \in \mathbb{N}$, $f_n \to 0$ weakly as $n \to \infty$, and $\|f_n\|_{L^2(U_{\delta_0})} \geq \alpha$ for some $\alpha > 0$ and for all $n \in \mathbb{N}$ (and perhaps passing to a subsequence if needed). Then we have, for sufficiently large $n \in \mathbb{N}$,

\[
\|C_\phi - K\| \geq \|C_\phi f_n - Kf_n\|_{L^2(\phi^{-1}(U_{\delta_0}))} \\
\geq \|C_\phi f_n\|_{L^2(\phi^{-1}(U_{\delta_0}))} - \|Kf_n\|_{L^2(\phi^{-1}(U_{\delta_0}))} \\
\geq \|f_n\|_{L^2(U_{\delta_0})} \inf_{U_{\delta_0}} \{|J(\phi)|\} - \|Kf_n\|_{L^2(\phi^{-1}(U_{\delta_0}))} \\
\geq \frac{1}{M_{\delta_0}} \|f_n\|_{L^2(\Omega)} \inf_{U_{\delta_0}} \{|J(\phi)|\} - \|Kf_n\|_{L^2(\phi^{-1}(U_{\delta_0}))} \\
= \frac{1}{M_{\delta_0}} \inf_{U_{\delta_0}} \{|J(\phi)|\} - \|Kf_n\|_{L^2(\phi^{-1}(U_{\delta_0}))} \\
\geq 0
\]

Letting $n \to \infty$, we have that

\[
\|C_\phi - K\| \geq \frac{1}{M_{\delta_0}} \inf_{U_{\delta_0}} \{|J(\phi)|\}
\]

for any compact operator $K : A^2(\Omega) \to A^2(\Omega)$. Thus taking the infimum over $K$ compact, we have our result.

\[\square\]

REFERENCES

[AGT10] John R. Akeroyd, Pratibha G. Ghatage, and Maria Tjani, Closed-range composition operators on $A^2$ and the Bloch space, Integral Equations Operator Theory 68 (2010), no. 4, 503–517. MR 2745476

[Ccc18] Timothy G. Clos, Mehmet Çelik, and Sönmez Şahutoğlu, Compactness of Hankel operators with symbols continuous on the closure of pseudoconvex domains, Integral Equations Operator Theory 90 (2018), no. 6, Art. 71, 14. MR 3877477

[CEGY98] Guangfu Cao, N. Elias, P. Ghatage, and Dahai Yu, Composition operators on a Banach space of analytic functions, Acta Math. Sinica (N.S.) 14 (1998), no. 2, 201–208. MR 1704802

[vZ07] Željko Ćučković and Ruhan Zhao, Essential norm estimates of weighted composition operators between Bergman spaces on strongly pseudoconvex domains, Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 3, 525–533. MR 2329700