On the OPE of 1/2 BPS Short Operators in $N = 4$ SCFT$_4$

B. Eden$^1$ and E. Sokatchev$^2$

Laboratoire d’Annecy-le-Vieux de Physique Théorique$^3$ LAPTH, Chemin de Bellevue, B.P. 110, F-74941 Annecy-le-Vieux, France

Abstract

The content of the OPE of two 1/2 BPS operators in $N = 4$ SCFT$_4$ is given by their superspace three-point functions with a third, a priori long operator. For certain 1/2 BPS short superfields these three-point functions are uniquely determined by superconformal invariance. We focus on the cases where the leading ($\theta = 0$) components lie in the tensor products $[0, m, 0] \otimes [0, n, 0]$ and $[m, 0, 0] \otimes [0, 0, n]$ of SU(4).

We show that the shortness conditions at the first two points imply selection rules for the supermultiplet at the third point. Our main result is the identification of all possible protected operators in such OPE’s. Among them we find not only BPS short multiplets, but also series of special long multiplets which satisfy current-like conservation conditions in superspace.

$^1$email: burkhard@lapp.in2p3.fr

$^2$email: sokatche@lapp.in2p3.fr

$^3$UMR 5108 associée à l’Université de Savoie
1 OPE, three-point functions and 1/2 BPS short superfields

The OPE of two conformal primary operators \( \phi(1) \) and \( \psi(2) \) can be written symbolically in the following form \[1, 2, 3, 4\]:

\[
\phi(1)\psi(2) = \sum_{(\sigma)} \int_{3, 3'} \langle \phi(1)\psi(2)O(3) \rangle \langle O(\sigma)(3)O(\sigma)(3') \rangle^{-1} O(\sigma)(3').
\] (1)

Thus, the OPE has the meaning of the tensor product of two UIR’s of the four-dimensional conformal group \( SO(1, 5) \) (Euclidean case) or \( SO(2, 4) \) (Minkowski case) decomposed into an infinite (discrete or continuous) sum of UIR’s denoted by \( O(\sigma) \). In this decomposition the rôle of the Clebsch-Gordon coefficients (the kernel of the integral decomposition \([1]\)) is played by the three-point function of the two operators with each operator appearing in the OPE, multiplied (in the sense of convolution) by the inverse two-point function of the latter.

It should be pointed out that there is a subtle but very important difference between the Euclidean and Minkowski cases \([3, 2]\). In the Euclidean case the inverse two-point function is simply obtained by changing the dimension of the operator (“shadow” operator), but the sum \( \sum_{(\sigma)} \) should be understood as an integral over the imaginary conformal dimension. In the Minkowski case the spectrum of the dimension becomes discrete, but the inverse two-point function is a non-trivial object which makes the kernel in \([1]\) transform non-locally. Nevertheless, in both cases the content of the OPE can be determined by examining the three-point functions that the two operators can possibly form with a third, a priori arbitrary operator.

We shall assume that the above applies to the \( N = 4 \) superconformal algebra \( PU(2, 2/4) \)\(^3\) although to our knowledge no rigorous mathematical analysis exists in the literature. So, we propose to reduce the problem of studying the OPE of two superconformal primary operators to constructing all possible three-point functions of these operators with a third operator. In order to make conformal supersymmetry manifest, it is natural to carry out the analysis in superspace.\(^4\) At first sight, this task may seem difficult for the following reason. A superspace three-point function depends on three sets of Grassmann coordinates \( \theta^i, \bar{\theta}^{\dot{i}} \), \( i = 1, \ldots, 4 \), i.e., on \( 3 \times 4 \times 4 = 48 \) odd variables. On the other hand, the superconformal algebra contains two odd generators (\( Q \) of Poincaré and \( S \) of conformal supersymmetry) acting on the \( \theta \)'s as \( 2 \times 4 \times 4 = 32 \) shifts. Thus, one can form \( 48 - 32 = 16 \) invariant combinations of the Grassmann coordinates. This implies that when expanding a super-three-point function in the \( \theta \)'s, at each step new arbitrary nilpotent terms may appear, up to the level \( \theta^{16} \), thus allowing for an enormous arbitrariness.

The situation radically changes if one puts certain types of 1/2 BPS short operators at points 1 and 2, still keeping the third operator arbitrary. By definition, an \( N \)-extended 1/2 BPS short superfield depends only on half of the Grassmann variables, namely, on \( p \) left-handed \( \theta^a \)'s and on \( N - p \) right-handed \( \bar{\theta}^{\dot{a}} \)'s. Using the terminology of \([3]\) we can call them \( (N|N-p,p) \) superfields.

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\(^1\)The more familiar form of the OPE \( \phi \psi = \sum C(x, \partial_x)O \) is obtained by Taylor-expanding the three-point function in the limit \( 1 \to 2 \).  

\(^2\)The R charge of the \( N = 4 \) superconformal algebra is a central charge. In \( N = 4 \) SYM this charge vanishes, therefore one usually considers the superalgebra \( PSU(2, 2/4) \) without central charge. However, we prefer to work with \( PU(2, 2/4) \) because we want to study the OPE of more general superconformal operators, some of which may carry central charge. We are grateful to S. Ferrara for turning our attention to this point.

\(^3\)OPE’s in SCFT have mostly been investigated in ordinary space in terms of the components of the supermultiplets \([3]\). In this way it is not easy to fully exploit the power of conformal supersymmetry. An early attempt on a superspace OPE was made in \([3]\).
Let us put a \((4|4-p_1, p_1)\) and a \((4|4-p_2, p_2)\) superfield at points 1 and 2, respectively. Repeating the above counting, we find \((p_1 + p_2 + 4) \times 4\) left-handed and \((4 - p_1 + 4 - p_2 + 4) \times 4\) right-handed odd coordinates in the three-point function. If we choose \(p_1\) and \(p_2\) such that \(p_1 + p_2 = 4\), then the above numbers exactly match the numbers of left- and right-handed supersymmetries. So, in this case there are no nilpotent invariants and \(\text{the three-point superfunction is uniquely determined by its leading component (obtained by setting all } \theta = 0)\). Our aim here will be to construct such three-point functions and study the restrictions on the operators at point 3.

The 1/2 BPS short \(N = 4\) superfields fall into two essentially different classes. One of them, familiar from \(N = 1\) supersymmetry, are the so-called chiral superfields\(^4\), i.e. \((4|4, 0)\) or \((4|0, 4)\) superfields. It should be stressed that the conformal dimension of a chiral superfield is only limited by the unitarity bound, so such objects do not have a protected dimension in SCFT. According to our counting above, the three-point functions of a chiral and an antichiral superfield with a third, general superfield are uniquely determined by conformal supersymmetry. Three-point functions involving only (anti)chiral superfields have been studied in \[9, 10\], but no detailed explicit results are available for the case relevant to the OPE where the third superfield is general. A preliminary examination of this case does not indicate any interesting selection rules on the superconformal UIR’s in such OPE’s.

Extended \((N > 1)\) supersymmetry provides a new class of 1/2 BPS short superfields called Grassmann (G-)analytic (they were introduced for the first time in the \(N = 2\) case in \[11\]). Examples of such G-analytic objects are the \(N = 2\) matter multiplet (hypermultiplet) \[12\] and the \(N = 3\) \[13\] and \(N = 4\) super-Yang-Mills multiplet \[14, 8, 28\]. The essential difference from the chiral case is that G-analytic superfields depend on at least one \(\theta\) and one \(\bar{\theta}\). Moreover, they carry no spin and their conformal dimension is expressed in terms of the Dynkin labels of the R symmetry representation of their first component, i.e., they have a \textit{protected dimension} in SCFT \[15\]. Such objects have an important rôle in the AdS/CFT correspondence because they are the SCFT duals of the KK states in the AdS bulk \[16, 17\].

The G-analytic superfields are of the type \((4|4 - p, p)\) with \(p \neq 0, 4\). With the additional requirement that the three-point function of two such operators should be uniquely determined by conformal supersymmetry, our choice for the two 1/2 BPS operators whose OPE we want to study is restricted to: (i) a pair of self-conjugate superfields \((4|2, 2)\); (ii) one \((4|1, 3)\) and one \((4|3, 1)\) superfield. The OPE content of such operators is the subject of the present paper. Here we develop the idea put forward in the recent publication \[18\] and apply it to the case of \(N = 4\) supersymmetry. Our main result is the identification of all possible protected operators in this OPE. Among them we find not only BPS short multiplets, but also series of special long multiplets which satisfy current-like conservation conditions in superspace. One example of the latter is the multiplet starting with a scalar of dimension 4 in the \(20\) of \(SU(4)\) whose “exceptional” non-renormalization properties have been discovered in \[19\] and later discussed in \[20, 21, 22\].

The paper is organized as follows. In Section 2 we recall the key points of the harmonic superspace \[12, 23\] treatment of BPS short (or G-analytic) superfields. We also give some basic information about the UIR’s of the superconformal algebra \(PU(2, 2/4)\) and identify the types of 1/2 BPS short objects whose OPE we wish to discuss. Section 3 is devoted to the construction of the three-point functions of these 1/2 BPS operators with an arbitrary third operator.\(^5\) Here

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\(^4\)The term “chiral primary operator” is often improperly used in the literature to denote all kinds of BPS short objects.

\(^5\)Superconformal two- and three-point functions in superspace have been discussed in many papers. These
we do not present the complete form of these functions. Instead, we use conformal supersymmetry to reconstruct the dependence on the Grassmann coordinates at one short point only. This is sufficient to examine the consequences of the shortness conditions. In this way we derive selection rules for the superconformal UIR’s at the third point. The results are summarized in Section 4. There we also comment on the application of our results to the four-point function of \( N = 4 \) SYM stress-tensor multiplets and to the so-called “extremal” \(^{32}\) and “next-to-extremal” \(^{33}\) correlators.

2 Grassmann-analytic superfields

2.1 On-shell \( N = 4 \) SYM and Grassmann analyticity

The simplest example of an \( N = 4 \) G-analytic superfield is the SYM field strength on shell. In ordinary superspace it is described by a real scalar superfield \( W^{ij} (x, \theta, \bar{\theta}) = -W^{ji} = 1/2 \epsilon^{ijkl} \tilde{W}_{kl} \) in the 6 of \( SU(4) \) (Dynkin labels \([0,1,0]\)) satisfying the following on-shell constraints (massless field equations):

\[
D_\alpha^k W^{ij} = 0, \quad \bar{D}_{\dot{\alpha}} (k W ^{ij}) = 0
\]  

(2)

where (\()\) means symmetrization, {\} denotes the traceless part and \( D_\alpha^k, \bar{D}_{\dot{\alpha}} k \) are the superspace covariant derivatives obeying the algebra

\[
\{ D_\alpha^i, D_\beta^j \} = \{ \bar{D}_{\dot{\alpha}} \bar{\beta}, \bar{D}_{\dot{\beta}} \bar{\beta} \} = 0, \quad \{ D_\alpha^i, \bar{D}_{\dot{\beta}} \bar{\beta} \} = -2i \delta_i^j (\sigma^k)_{\alpha \bar{\beta}} \partial_\mu .
\]  

(3)

These constraints can be equivalently rewritten in the form of G-analyticity conditions without losing the manifest \( SU(4) \) invariance with the help of the so-called harmonic variables (first introduced in \(^{12}\) in the case \( N = 2 \) and then generalized to \( N > 2 \) in \(^{13},^{14},^{3}\)). The harmonics \( u^I_i \) (and their conjugates \( u^I_\bar{i} = (u^I_i)^* \)) form an \( SU(4) \) matrix where \( i \) is an index in the fundamental representation of \( SU(4) \) and \( I = 1, \ldots, 4 \) are the projections of the second index onto the subgroup \([U(1)]^3\). Further, we define two independent \( SU(4) \) groups, a left one acting on the index \( i \) and a right one acting on the projected index \( I \) of the harmonics:

\[
(u^I_i)_I' = \Lambda^I_i u^I_\bar{i} \Sigma^I_J , \quad \Lambda \in SU(4)_L , \quad \Sigma \in SU(4)_R .
\]  

(4)

In particular, the charge operators (the three generators of the Cartan subgroup \([U(1)]^3 \subset SU(4)_R \)) act on the harmonics as follows:

\[
m_K u^I_i = (\delta^I_K - \delta^I_\bar{I}) u^I_\bar{i} , \quad m_K u^I_\bar{i} = - (\delta_K I - \delta_I \bar{I}) u^I_i ,
\]  

(5)

so that \( m_4 \equiv 0 \). The harmonics satisfy the following \( SU(4) \) defining conditions:

\[
u \in SU(4) : \quad u^I_i u^I_\bar{j} = \delta^I_j , \quad u^I_i u^I_\bar{i} = \delta^I_i , \quad \epsilon^{I_1 \cdots I_4} u^I_{i_1} \cdots u^I_{i_4} = 1 .
\]  

\(^{3}\) include the case of \( N = 1 \) chiral superfields \(^{1},^{10}\) as well as a method valid for all types of \( N = 1 \), \(^{22}\), \(^{23}\) and \( N > 1 \) superfields \(^{20}\). This method has recently been adapted to G-analytic superfields in \(^{27}\). A construction of analytic \( N = 2 \) and \( N = 4 \) two- and three-point functions is presented in \(^{28},^{29}\) (see also \(^{30}\)). Ref. \(^{31}\) discusses the \( N = 2 \) case in the context of three protected (current or stress-tensor) operators. However, no explicit results are available for the case of interest to us, namely, when two of the operators are short but the third one is generic.
Now, let us use the harmonic variables to covariantly project all the $SU(4)$ indices in the constraints $\mathfrak{g}_2^{[3]}$. In this way all objects become $SU(4)_L$ invariant but $[U(1)_R]^{3}$ covariant. For example, the projection

$$W^{12} = W^{\alpha i}(x, \theta, \bar{\theta})u_1^{\alpha}u_2^{i}$$

satisfies the constraints

$$D_\alpha^1 W^{12} = D_\alpha^2 W^{12} = D_{\bar{\alpha}}^3 W^{12} = D_{\bar{\alpha}}^4 W^{12} = 0$$

where

$$D_\alpha^I = D_\alpha^i u_i^I, \quad \bar{D}_{\bar{\alpha}}^i = \bar{D}_{\bar{\alpha}}^i u_i^I.$$

From (3), (6) and (9) it follows that the projected derivatives appearing in (8) anticommute. This in turn implies the existence of a G-analytic basis in superspace,

$$x_A^\mu = x_A^{\mu} - 2i(\theta_1 \sigma^{\mu} \bar{\theta}_1 + \theta_2 \sigma^{\mu} \bar{\theta}_2 - \theta_3 \sigma^{\mu} \bar{\theta}_3 - \theta_4 \sigma^{\mu} \bar{\theta}_4), \quad \theta_I^\alpha = \theta_I^i u_i^I, \quad \bar{\theta}_{\bar{\alpha}}^i = \bar{\theta}_{\bar{\alpha}}^i u_i^I,$$

where these derivatives become just $D_\alpha^1, D_\alpha^2, \bar{D}_{\bar{\alpha}}^3, \bar{D}_{\bar{\alpha}}^4$. Consequently, in this basis the superfield $W^{12}$ is an unconstrained Grassmann-analytic function of two $\theta$'s and two $\bar{\theta}$'s, as well as of the harmonic variables:

$$W^{12} = W^{12}(x_A, \theta_3, \theta_4, \bar{\theta}_1, \bar{\theta}_2, u).$$

So, this is an example of a 1/2 BPS short superfield of the type $(4|2,2)$.

### 2.2 $SU(4)$ irreducibility and harmonic analyticity

It is important to realize that the G-analytic superfield (11) is an $SU(4)_L$ invariant object only because it is a function of the harmonic variables. In order to recover the original harmonic-independent but constrained superfield $W^{ij}(x, \theta, \bar{\theta})$ (3), we need to impose harmonic differential constraints equivalent to the definition of a highest weight state (HWS) of $SU(4)$. To this end we introduce derivatives acting on the harmonics as follows:

$$\partial_I^J u_i^K = \delta_I^K u_i^J - \frac{1}{4} \delta_I^K u_i^K, \quad \bar{\partial}_I^J u_i^I = -\delta_I^K u_i^J + \frac{1}{4} \delta_I^K u_i^K.$$

It is easy to see that they generate the group $SU(4)_R$ acting on the $[U(1)_R]^{3}$ projected indices of the harmonics.

A harmonic function defined on $SU(4)$ can be restricted to the coset $SU(4)/[U(1)]^{3}$ by assuming that it transforms homogeneously under $[U(1)_R]^{3} \subset SU(4)_R$. In terms of the harmonic derivatives this condition is translated into the requirement that the harmonic functions $f(u)$ are eigenfunctions of the diagonal derivatives $\partial_I^I - \partial_A^A$ which count the $U(1)_R$ charges (3). Then the independent harmonic derivatives on the coset are the six complex derivatives $\partial_I^J$, $I < J$ corresponding to the raising operators in the Cartan decomposition of the algebra of $SU(4)_R$.

In general, the harmonic functions have an infinite “harmonic” expansion on the coset $SU(4)/[U(1)]^{3}$, thus giving rise to an infinite set of $SU(4)$ irreps. They can be restricted to
a single irrep by requiring that they are annihilated by the raising operators of $SU(4)_R$, i.e. that they correspond to a HWS,

$$\partial^I_J f(u) = 0, \quad I < J. \quad (13)$$

For example, the harmonic function $f^1(u)$ subject to the constraint (13) is reduced to the 4: $f^1 u^1_1$; the function $f^{12}(u)$ to the 6: $f^{ij} u^1_2 u^2_2$ where $f^{ij} = -f^{ji}$; the function $f^{123}(u) \equiv f_4^{(4)}(u)$ to the 4: $f^{ijk} u^1_4 u^2_2 u^3_3 = f_4^{(4)}$ where $f^{ijk} = \epsilon^{ijk} f_i$, etc. The generalization is straightforward: the harmonic function

$$f_{m_1 \ldots m_2 \ldots m_3 \ldots} (u), \quad m_1 \geq m_2 \geq m_3 \geq 0 \quad (14)$$

subject to the irreducibility condition (13) corresponds to the HWS of the $SU(4)$ irrep carrying the Young tableau labels $m_1, m_2, m_3$ (recall (5)) or the Dynkin labels $a_1 = m_1 - m_2$, $a_2 = m_2 - m_3$, $a_3 = m_3$.

The $SU(4)$ irreducibility conditions (13) have an alternative interpretation obtained by introducing complex coordinates $z_m$, $m = 1, \ldots , 6$ on the harmonic coset $SU(4)/[U(1)]^3$. Then the six derivatives in (13) correspond, roughly speaking, to $\partial/\partial z_m$. So, conditions (13) are (covariant) Cauchy-Riemann analyticity conditions on the compact complex manifold $SU(4)/[U(1)]^3$.

The only regular solutions allowed by such constraints are homogeneous harmonic polynomials, i.e., $SU(4)$ irreps. In this sense one may call (13) harmonic (H-)analyticity conditions.

It is important to realize that there also exist singular harmonic functions which do not correspond to $SU(4)$ irreps. To give an example, consider two sets of harmonics, $u^1_i$ and $v^1_i$, and form the $SU(4)_L$ invariant combination, e.g., $u^1_1 v^1_2$. Obviously, it is H-analytic, i.e., $(\partial^I_J) u^1_{i_1} v^1_{i_2} = 0$, $I < J$. However, this is not true for the function $(u^1_i v^1_i)^{-1}$ which becomes singular when $u = v$ (recall (5)). In fact, the raising operators acting on this singular function produce harmonic delta functions, in accordance with the formula $\partial/\partial z_m z_m^{-1} = \pi \delta(z_m)$. In other words, the singular harmonic functions cannot be expanded on the coset $SU(4)/[U(1)]^3$ into a harmonic series of $SU(4)$ irreps. The issue of harmonic singularities [28, 34] will be the central point in our study of the three-point functions in Section 3.

Let us now come back to the $N = 4$ SYM field-strength in the form of the G-analytic superfield (11). The analog of (13) in this case are the H-analyticity conditions

$$D^I_J W^{12} = 0, \quad I < J \quad (15)$$

where $D^1_J = \partial^1_J$, $D^2_J = \partial^2_J$ and

$$D^I_J = \partial^I_J + 2i \theta_J \sigma^\mu \partial^I \partial_\mu - \theta_J \partial^I + \partial^J \partial_I, \quad I = 1, 2, \quad J = 3, 4 \quad (16)$$

are the supercovariant versions of the harmonic derivatives in the G-analytic basis (10). The presence of space-time derivatives in (16) implies massless field equations on the component fields, which are the six physical scalars, the four spinors and the vector of the $N = 4$ SYM on-shell multiplet.

We remark that the conditions of G-analyticity and H-analyticity are compatible because the spinor derivatives from (8) commute with the harmonic ones from (13). One says that they form a Cauchy-Riemann (CR) structure [35].
2.3 Other types of G- and H-analytic objects

The $N = 4$ SYM field-strength is the simplest example of a G- and H-analytic $N = 4$ superfield. Before considering its generalizations, we remark that the projection (7) is not the only possible way to convert $W^{ij}$ into a G-analytic superfield. An equivalent projection is

$$W^{13} = W^{ij}(x, \theta, \bar{\theta})u_i^1 u_j^3$$

and the resulting G-analytic superfield depends on a different half of the Grassmann variables,

$$W^{13}(x, \theta, \bar{\theta}, u_1 u_3).$$

According to our conventions, this projection is not a HWS of an $SU(4)$ irrep, so one of the raising operators converts it into the HWS $W^{12}$:

$$D_2^3 W^{13}(\theta, \bar{\theta}) = W^{12}(\theta, \bar{\theta})$$

whereas all the remaining raising operators still annihilate it.

Besides the self-conjugate $N = 4$ SYM multiplet, there exist two other massless conjugate $N = 4$ multiplets which have no spin but carry non-trivial $SU(4)$ quantum numbers [36]. In ordinary superspace they are described by a superfield $W_i(x, \theta, \bar{\theta})$ in the fundamental of $SU(4)$ and by its conjugate $\bar{W}_i \equiv -\frac{1}{6} \epsilon_{ijkl} W^{ijkl}$ satisfying on-shell constraints similar to (2). In harmonic superspace this corresponds to the G-analytic superfields

$$W^1(x_A, \theta_2, \theta_4, \bar{\theta}_1, \bar{\theta}_3, u) \quad \text{or} \quad W^{123}(x_A, \theta_4, \theta_1, \bar{\theta}_3, \bar{\theta}_4, u) \equiv \bar{W}_4$$

satisfying the H-analyticity conditions $D_I^J W^1 = 0, D_I^J \bar{W}_4 = 0, I < J$ (cf. (15)). Note that the G-analytic basis (10) needs to be adapted to the new types of G-analyticity. We can say that $W^1$ is 1/2 BPS short of the type $(4|1,3)$ and its conjugate $\bar{W}_4$ of the type $(4|3,1)$.

Further BPS short objects can be constructed by multiplying the above ones. For instance, an interesting class of 1/2 BPS superfields of the type $(4|2,2)$ is obtained as gauge-invariant composite operators built out of the SYM field-strength [28]:

$$O_{1/2}^{[m,0]} = \text{Tr} [W^{12}(\theta, \bar{\theta})]^m.$$  \hspace{1cm} (21)

Since the elementary superfield $W^{12}$ is subject to the irreducibility constraints (13), the same applies to the composite as well. We conclude that its leading component is in the $SU(4)$ irrep with Dynkin labels $[0,m,0]$. Further, the leading component of $W^{12}$ is a massless scalar of canonical dimension $\ell = 1$, therefore the composite object (21) has dimension $\ell = m$. If $m > 1$ the composites (21) are not massless. The case $m = 2$ corresponds to the $N = 4$ SYM stress-tensor multiplet. Note that the operators of this series are self-conjugate, just like the SYM field-strength itself.

Similarly, using the alternative massless multiplets (20), we obtain two conjugate series:

$$O_{1/2}^{[m,0]} = [W^1(\theta, \bar{\theta})]^m$$

and

$$C_{1/2}^{[0,m]} = [\bar{W}_4(\theta, \bar{\theta})]^m.$$  \hspace{1cm} (22)

Note that we have dropped the YM trace in (22), treating the multiplication of $W$’s as a formal way of obtaining new superconformal UIR’s [24, 28, 29]. As before, these objects have dimension $\ell = m$. 

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Next, mixing different types of G-analytic objects we can obtain BPS objects with a lower degree of shortening. For instance, the following 1/4 BPS superfields:

\[ O^{[q,p,q]}_{1/4} = [W^{12}(\theta_{3,4}, \bar{\theta}^{1,2})]^{p+q} [W^{13}(\theta_{2,4}, \bar{\theta}^{1,3})]^q, \quad \ell = p + 2q \]  

and

\[ O^{[p,0,q]}_{1/4} = [W^1(\theta_{2,3,4}, \bar{\theta}^1)]^p [W_4(\theta_{4}, \bar{\theta}^{1,2,3})]^q, \quad \ell = p + q \]

are G-analytic of the type (4|1, 1). They appear in the OPE of 1/2 BPS objects, as we show in the next section.

### 2.4 Series of UIR’s of \( PU(2, 2/4) \)

The UIR’s of \( PU(2, 2/4) \) are labeled [15] by the conformal dimension \( \ell \), the two Lorentz \( (SO(4) \) or \( SO(1, 3) \)) quantum numbers (“spins”) \( j_1, j_2 \) and the \( SU(4) \) Dynkin labels \( a_1, a_2, a_3 \):

\[ D(\ell; j_1, j_2; a_1, a_2, a_3). \]

In addition, in \( PU(2, 2/4) \) there is a central charge which does not intervene in our discussion, so we do not list it among the UIR labels. Above we saw examples of G-analytic BPS operators of the type \( (4|p, q) \), \( pq \neq 0 \). According to the classification of [15] (in the notation of [37, 38, 39]), they belong to the series C of UIR’s. The most general representative of this series can be viewed as a product of the three kinds of massless superfields \( W^1, W^{12} \) and \( W^{123} \):

\[ [W^1]^{a_1} [W^{12}]^{a_2} [W^{123}]^{a_3}. \]

This object has no spin, \( j_1 = j_2 = 0 \), but carries a non-trivial \( SU(4) \) irrep with Dynkin labels \([a_1, a_2, a_3]\). Moreover, its conformal dimension is fixed:

\[ \ell = a_1 + a_2 + a_3. \]

In a quantum theory this means that such operators cannot acquire an anomalous dimension or, as one says, they are protected by conformal supersymmetry.

The other class of BPS short multiplets form the series B of UIR’s of \( PU(2, 2/4) \). They are of the type \( (4|0, p) \) (or \( (4|p, 0) \)). The familiar chiral (or antichiral) superfields correspond to \( p = 4 \). This time they can carry both spin \( (j_1, 0) \) (or \( (0, j_2) \)) and \( SU(4) \) quantum numbers, and their dimension is limited only by the unitarity bound, e.g., \( \ell \geq 1 + j_1 + a_1 + a_2 + a_3 \). Consequently, operators from this series are in general not protected.

Finally, the series A involves multiplets without any BPS shortening. They carry arbitrary spin and \( SU(4) \) quantum numbers. In this case the unitarity bound is (upon elimination of the central charge)

\[ \ell \geq 2 + j_1 + j_2 + a_1 + a_2 + a_3. \]

There is no reason for these operators to be protected unless the unitarity bound is saturated. To explain this, recall the well-known example of a conformal vector field \( J_\mu(x) \). Its dimension

\footnote{Note that the situation changes in the case of the superalgebra \( PSU(2, 2/4) \). There the R (or central) charge vanishes and the dimension of the series B UIR’s gets fixed, i.e., they become protected.}
should satisfy the unitarity bound $\ell \geq 3$. In general, this field is irreducible as a representation of the conformal group, but it becomes reducible (although indecomposable) when the unitarity bound is saturated, $\ell = 3$. In this case the divergence of the vector $\partial^\mu J_\mu(x)$ transforms covariantly and can thus be set to zero. The resulting transverse (conserved) vector is an exceptional UIR with fixed (i.e., “protected”) dimension $7$. In CFT this corresponds to a conserved current made out of elementary massless physical fields, e.g., $J_\mu = \phi \partial_\mu \phi - \phi \partial_\mu \bar{\phi}$. Furthermore, the three-point function $\langle \phi_d(x) \phi_d(y) J_\mu^d(z) \rangle$ of two scalars and a vector of dimension $d$ and $\ell$, respectively, is completely determined by conformal invariance. At the unitarity bound $\ell = 3$ it is automatically conserved, $\langle \phi_d(x) \phi_d(y) J_\mu^d(z) \rangle \nabla_x \mu = 0$.

Exactly the same phenomenon takes place in series A. When the unitarity bound is saturated,

$$\ell = 2 + j_1 + j_2 + a_1 + a_2 + a_3,$$

(29)

the operator $J_{[a_1, a_2, a_3]}^{[a_3 \ldots a_2, a_1]} \ldots \bar{a_2} \bar{a_1}$ realizes a reducible representation of $PU(2,2/4)$. As in the case of the current, we can impose the “conservation” conditions:

$$D^1 \alpha J_{[a_1, a_2, a_3]}^{[a_3 \ldots a_2, a_1]} \ldots \bar{a_2} \bar{a_1} = \bar{D}^1 \bar{\alpha} J_{[a_1, a_2, a_3]}^{[a_3 \ldots a_2, a_1]} \ldots \bar{a_2} \bar{a_1} = 0$$

(30)

if $s \neq 0$ or

$$D^1 \alpha D^1 \bar{\alpha} J_{[a_1, a_2, a_3]}^{[a_3 \ldots a_2, a_1]} \ldots \bar{a_2} \bar{a_1} = \bar{D}_4 \bar{\alpha} \bar{D}_4 \bar{\alpha} J_{[a_1, a_2, a_3]}^{[a_3 \ldots a_2, a_1]} \ldots \bar{a_2} \bar{a_1} = 0$$

(31)

if $s = 0$, or similar ones involving more projections of the spinor derivatives for some restricted sets of Dynkin labels $[39]$. In this sense we may call objects of the type (30), (31) “current-like”. It should be stressed that in general none of the components of such an object is a conserved tensor. This only happens when $J$ is an $SU(4)$ singlet. In this case the complete spinor derivatives appear in (30), (31) and $J$ can be viewed as a bilinear composite made out of massless chiral superfields, i.e., as a conserved generalized supercurrent.

The differential constraints (30), (31) “protect” the dimension of such objects, just like the conservation condition $\partial^\mu J_\mu = 0$ protects the dimension of the current. In Section 3 we show that they can appear in the OPE of two 1/2 BPS operators.

3 Three-point functions involving two 1/2 BPS operators

The main aim of the present paper is to study the OPE of two 1/2 BPS operators of the type $O_{1/2}^{[0,m,0]} O_{1/2}^{[0,n,0]}$ or $O_{1/2}^{[m,0,0]} O_{1/2}^{[0,0,n]}$. To this end we have to construct all possible superconformal three-point functions

$$\langle O_{1/2}^{[0,m,0]}(1) O_{1/2}^{[0,n,0]}(2) O^{\mathcal{D}}(3) \rangle$$

(32)

and

$$\langle O_{1/2}^{[m,0,0]}(1) O_{1/2}^{[0,0,n]}(2) O^{\mathcal{D}}(3) \rangle$$

(33)

where $O^{\mathcal{D}}$ is an arbitrary operator carrying the $PU(2,2/4)$ UIR $\mathcal{D}$ $[23]$. Our task is to find out what the allowed UIR’s are. As explained in Section 1, conformal supersymmetry

\footnote{We call such objects “protected”, although the mechanism keeping the current conserved needs to be explained.}
completely determines such three-point functions. The restrictions on the third UIR, apart from
the standard conformal ones, stem from the requirement of H-analyticity (SU(4) irreducibility)
at points 1 and 2 combined with G-analyticity. As we shall see later on, the expansion of (32)
and (33) in, e.g., the \( \theta \)'s at point 1 may contain harmonic singularities. If this happened, the
basic assumption that we are dealing with SU(4) UIR's would not be true. To put it differently,
harmonic singularities violate H-analyticity, i.e., SU(4) irreducibility. Demanding that such
singularities be absent implies selection rules for the UIR's at point 3.

Although the case (32) is the physically interesting one (the operators \( O^{[m,0,0]}_{1/2} \) can be viewed as
composites made out of the \( N = 4 \) SYM field-strength, see (21)), the second case (33) is simpler
and we present it in more detail to explain our method. The generalization to the case (32) is
straightforward.

3.1 Two-point functions in the case \( O^{[m,0,0]}_{1/2} \otimes O^{[0,0,n]}_{1/2} \)

Before discussing the three-point function (33) itself, it is instructive to examine the two-point
function

\[
\langle W^1(x, \theta_{2,3,4}, \theta^1, 1) \bar{W}_4(y, \zeta_4, \bar{\zeta}^{1,2,3}, 2) \rangle
\]

where the two sets of harmonic variables are denoted by \( 1^I \) and \( 2^I \), respectively. Its leading
component (obtained by setting \( \theta = \zeta = 0 \)) is easily found:

\[
\langle W^1 \bar{W}_4 \rangle_{\theta=\zeta=0} = \frac{(1^1 2_4)}{(x-y)^2} \quad (35)
\]

where

\[
(1^1 2_4) \equiv 1^1 2^4_4 \quad (36)
\]
is the \( SU(4)_L \) invariant contraction of the harmonics at points 1 and 2. From (22) it is clear
that it satisfies the \( SU(4) \) irreducibility (H-analyticity) condition (13) at point 1,

\[
(\partial^I_j)_{1}(1^1 2_4) = 0, \quad I < J \quad (37)
\]

and similarly at point 2. The space-time factor in (35) is determined by the other quantum
numbers of \( W \), namely, spin 0 and dimension 1.

The next question is how to restore the odd coordinate dependence starting from (35). The
counting argument of Section 1 shows that the two-point function (34) is overdetermined since
it only depends on 16 odd variables, compared to the 32 supersymmetries. Thus, it is sufficient
to use only \( Q \) supersymmetry to fully restore the odd coordinates. Furthermore, our aim is
to study the harmonic singularities at point 1 (or, equivalently, at point 2), so it is enough to
restore the \( \theta \) dependence. To see this, imagine a frame in which \( \zeta \) has been set to zero by a finite
\( Q \)-supersymmetry transformation. Such a transformation depends on the harmonic \( 2^I \)
only, so it cannot introduce singularities at point 1. In this frame the residual \( Q \) supersymmetry is given
by the conditions (recall (11))

\[
\delta_Q \zeta 4 = 0 \quad \Rightarrow \quad \epsilon' = (2^1 2_4^1 + 2^2 2_4^2 + 2^3 2_4^3) \epsilon_j,
\]

\[
\delta_Q \bar{\zeta}^{1,2,3} = 0 \quad \Rightarrow \quad \bar{\epsilon}' = 2^4 \bar{\epsilon}^j. \quad (38)
\]

*The full \( \theta \) dependence can be obtained by adapting the standard method of [24, 25] and will be given elsewhere.
It is rather complicated and is irrelevant to the issue of harmonic singularities at points 1 or 2.
Now, in the analytic basis at point 1 (the analog of (10)) \( x^\mu \) transforms as follows:
\[
\delta_Q x^\mu = -2i[\theta_2 \sigma^\mu \vec{e}^1 1^2 + \theta_3 \sigma^\mu \vec{e}^3 1^3 + \theta_4 \sigma^\mu \vec{e}^1 1^4 - i \epsilon_i \sigma^\mu \theta_i].
\] (39)
Replacing the parameters in (33) by the residual ones from (38) we can find \( \delta_Q x^\mu \). Then it is easy to verify that the following combination
\[
x^\mu - y^\mu + \frac{2i}{(1^{12}_4)} \xi_{2/4} \sigma^\mu \bar{\theta}^1,
\] \( \xi_{2/4} \equiv [(1^2 2_4) \theta_2 + (1^3 2_4) \theta_3 + (1^4 2_4) \theta_4] \) (40)
is invariant under the residual \( Q \) supersymmetry (note that \( \delta_Q y^\mu = 0 \)). This allows us to write the two-point function (34) in the form of a coordinate shift of the leading component (35):
\[
\langle W^1(\theta) W_4(0) \rangle = \exp \left\{ \frac{2i}{(1^{12}_4)} \xi_{2/4} \sigma^\mu \bar{\theta}^1 \frac{\partial}{\partial x^\mu} \right\} \frac{(1^{12}_4)}{(x-y)^2} \] (41)
Now it becomes clear that the coordinate shift can be the source of harmonic singularities of the type \((1^{12}_4)^{-1}\) or \((1^{12}_4)^{-2}\), since \((1^{12}_4) = 0\) when points 1 and 2 coincide. However, this does not happen in (41) because the leading component (35) already contains one factor \((1^{12}_4)\) and because \( \square_x (x-y)^{-2} = 0 \) if \( x^\mu \neq y^\mu \). So, we can say that the two-point function (34) is free from harmonic singularities at point 1 (and similarly at point 2), at least up to space-time contact terms. This can also be translated into the statement that the full superfunction (34) satisfies the H-analyticity condition (37) with covariantized harmonic derivatives (cf. (16)).

### 3.2 Three-point functions in the case \( \mathcal{O}_{1/2}^{[m,0,0]} \mathcal{O}_{1/2}^{[0,0,n]} \)

The discussion of the two-point function above can easily be adapted to the three-point function (33). Firstly, the \( SU(4) \) irrep carried by \( \mathcal{O}(3) \) should be in the decomposition of the tensor product of the two irreps at points 1 and 2:
\[
[m, 0, 0] \otimes [0, 0, n] = \bigoplus_{k=0}^{\min(m,n)} [m-k, 0, n-k].
\] (42)
From (24) we see that irreps of the type \([p, 0, q]\) can be realized as products of \( W^1 \)'s and \( W^4 \)'s. This suggests to build up the \( SU(4) \) structure of the function (33) in the form of a product of two-point functions of \( W \)'s.

Secondly, in addition to the \( SU(4) \) quantum numbers, the operator \( \mathcal{O}(3) \) also carries spin and dimension. Since the leading components at points 1 and 2 are scalars, the Lorentz irrep at point 3 must be of the type \((s/2, s/2)\), i.e., a symmetric traceless tensor of rank \( s \). The corresponding conformal tensor structure is built out of the conformally covariant vector
\[
Y^\mu = \frac{(x-z)^\mu}{(x-z)^2} - \frac{(y-z)^\mu}{(y-z)^2}
\] (43)
where \( z^\mu \) is the space-times coordinate at point 3. Thus, to the \( PU(2,2/4) \) UIR \( \mathcal{D} \) at point 3 corresponds the following leading term:
\[
\langle \mathcal{O}_{1/2}^{[m,0,0]} \mathcal{O}_{1/2}^{[0,0,n]} \mathcal{O}(s/2,s/2; m-k,0,n-k) \rangle_0 =
\] (44)
\[
\left[ \frac{(1^1 2^1)}{(x - y)^2} \right]^k \left[ \frac{(1^3 3^1)}{(x - z)^2} \right]^{m-k} \left[ \frac{(1^1 2^4)}{(z - y)^2} \right]^{n-k} (Y^2)^{\frac{1}{2}(\ell-s-m-n+2k)} Y^{(\mu_1 \ldots \mu_s)}
\]

where \(\{\mu_1 \ldots \mu_s\}\) denotes traceless symmetrization.

Thirdly, to study the harmonic singularities at point 1 we need only restore the \(\theta\) dependence at this point. This can be done by analogy with the two-point function above. We start by fixing a frame in which the odd coordinates are set to zero at points 2 and 3. This time, in order to reach the frame we have to use both \(Q\) and \(S\) supersymmetry. It is important that the harmonics \(1^1_1\) do not participate in the frame fixing, thus there is no danger of creating artificial harmonic singularities at point 1. The residual \(Q + S\) supersymmetry involves as many parameters as the remaining number of \(\theta\)'s at point 1. Next, the vector \(Y^\mu\) is invariant under conformal boosts at points 1 and 2 and covariant at point 3 (it is translation invariant as well). So, we need to find a superextension of \(Y^\mu\) which is invariant under \(Q + S\) supersymmetry at points 1 and 2. Remarkably, the combination \((41)\) that was \(Q\) invariant in the two-point case turns out to be \(Q + S\) invariant in this new frame. Thus, performing the shift \((41)\) on the vectors \(Y\) in \((44)\) we obtain the desired superextension. In addition, the two-point factor \((1^1 2^1)/(x - y)^2\) in \((44)\) undergoes the same shift, whereas the shift of the factor \((1^1 3^1)/(x - z)^2\) is obtained from \((41)\) by replacing the harmonics \(2^1_1\) by \(3^1_1\) (the factor \((3^1 2^4)/(z - y)^2\) requires no supersymmetrization in this frame).

In this way we arrive at the following form of the three-point function where the full \(\theta\) dependence at point 1 is restored:

\[
\langle O_{1/2}^{[m,0,0]}(\theta) O_{1/2}^{[0,0,n]}(0) O^{(\ell; s/2,s/2; m-k,0,n-k)}(0) \rangle
= \left[ \frac{(1^3 2^1)}{(z - y)^2} \right]^{n-k} \times \exp \left\{ \frac{2i}{(1^3 2^1)} \xi_3/4 \sigma^\mu \bar{\theta}^1 \partial_x \mu \right\} \left[ \frac{(1^1 2^4)}{(x - z)^2} \right]^{m-k} \times \exp \left\{ \frac{2i}{(1^1 2^4)} \xi_2/4 \sigma^\nu \bar{\theta}^1 \partial_x \nu \right\} \left[ \frac{(1^1 2^4)}{(x - y)^2} \right]^{k} (Y^2)^{\frac{1}{2}(\ell-s-m-n+2k)} Y^{(\mu_1 \ldots \mu_s)} \right\}.
\]

Let us now turn to the crucial question of harmonic singularities. In \((45)\) they may originate from the two shifts. Recalling \((41)\), we see that no singularity arises when \((1^1 3^1) = 0\) (up to space-time contact terms), just like in the two-point function. However, the presence of a singularity when \((1^1 2^4) = 0\) depends on the value of \(k\) and there are three distinct cases.

(i) If \(k = 0\) the singularity occurs in the \(\bar{\theta} \theta\) term. In order to remove it we must require:

\[
\partial^\nu \left\{ (Y^2)^{\frac{1}{2}(\ell-s-m-n)} Y^{(\mu_1 \ldots \mu_s)} \right\} = 0 \quad \text{(46)}
\]

which implies

\[
s = 0, \quad \ell = a_1 + a_2 + a_3 \quad \text{(47)}
\]

where \([a_1, a_2, a_3]\) is the \(SU(4)\) irrep at point 3 (in the case at hand \(a_1 = m, a_2 = 0, a_3 = n\)). In this case we can immediately write down the complete three-point function \((54)\) in the form of a product of two-point functions of the type \((54)\):

\[
\langle O_{1/2}^{[m,0,0]}(1) O_{1/2}^{[0,0,n]}(2) O^{(m+n; 0; m,0,n)}(3) \rangle = (W^1(1)\bar{W}_4(3))^m (W^1(3)\bar{W}_4(2))^n. \quad \text{(48)}
\]

Consequently, \(O^{(m+n; 0; m,0,n)}\) is a 1/4 BPS short protected operator.
(ii) If $k = 1$ the singularity occurs in the $(\theta \bar{\theta})^2$ term. In order to remove it we must require:

$$\Box_x \left\{ (x - y)^{-2} \left( Y^2 \right)^{\frac{1}{2} \left( \ell - s - m - n + 2 \right)} Y^{\mu_1 \ldots \mu_s} \right\} = 0. \tag{49}$$

This equation admits two solutions: $\ell = s + m + n$, i.e.,

$$\ell = 2 + s + a_1 + a_2 + a_3. \tag{50}$$

Note that this corresponds to the unitarity bound \(\ref{eq:unitarity_bound}\). From Section 2 we know that this is a necessary condition for $O_D(3)$ to be a current-like object. Remarkably, it turns out to be sufficient as well, as can be seen by restoring the $\theta$ dependence only at point 3 and then checking that the constraints \(\ref{eq:solution_a}\) or \(\ref{eq:solution_b}\) are automatically satisfied (the details of this calculation will be given elsewhere). Thus, we conclude that $O^{(s+m+n; s/2, s/2; m-1,0,n-1)}$ - being forced to saturate the unitarity bound - is a current-like protected operator.

The second solution is

$$\ell = -s + a_1 + a_2 + a_3. \tag{51}$$

This solution clearly violates the unitarity bound for series A \(\ref{eq:unitarity_bound}\). It can only be compatible with the series C if we set $s = 0$:

$$(ii.b) \quad s = 0, \quad \ell = a_1 + a_2 + a_3. \tag{51}$$

Once again, the three-point function is reduced to a product of two-point functions:

$$\langle O^{[m,0,0]}_{1/2}(1) O^{[0,0,n]}_{1/2}(2) O^{(m+n-2; 0,0; m-1,0,n-1)}(3) \rangle = \langle W^1(1) W^4(2) \rangle \langle W^1(1) W^4(3) \rangle^{m-1} \langle W^1(3) W^4(2) \rangle^{n-1}. \tag{52}$$

From this form it is clear that $O^{(m+n-2; 0,0; m-1,0,n-1)}$ is a 1/4 BPS short protected operator of the type \(\ref{eq:protected_operator}\). Exceptionally, if $\min(m,n) = 1$, the operator becomes 1/2 BPS short.

(iii) If $k \geq 2$ the harmonic factor $(1^{12}_4)^k$ is sufficient to suppress all the singularities originating from the coordinate shift. Then the operator $O(3)$ can either belong to series C (if $s = 0$ and $\ell = m + n - 2k$) or to series A. In the latter case its dimension is only limited by the unitarity bound \(\ref{eq:unitarity_bound}\), so in general it is an unprotected operator. If the unitarity bound happens to be saturated, $\ell = 2 + s + m + n - 2k$, the operator $O(3)$ becomes current-like.

The results of this subsection are summarized in Table 1.

### 3.3 The case $O^{[0,m,0]}_{1/2} O^{[0,n,0]}_{1/2}$

The three-point function \(\ref{eq:three_point_function}\) can be treated in very much the same way. Leaving the details for a future publication, here we only mention a few key points.

The analog of the two-point function \(\ref{eq:two_point_function}\) now is (leading component only)

$$\langle W^{12}(x, 0, 1) W^{12}(y, 0, 2) \rangle = \frac{(1^{12}_2 1^{12}_2)}{(x - y)^2} \tag{53}$$

where

$$\left(1^{12}_2 1^{12}_2\right) = \epsilon^{ijkl}1^{1}_{i}1_{j}^{2}\epsilon^{12}_{k}2^{1}_{l}. \tag{54}$$
The $\theta$ dependence at point 1 can be restored by making the shift (cf. (40))
\[
x^\mu - y^\mu - \frac{2i}{(1^{12}2^{12})}[\xi^{2/12}\sigma^\mu \bar{\theta}^1 - \xi^{1/12}\sigma^\mu \bar{\theta}^2], \quad \xi^{1/12} \equiv [(1^{13}2^{12})\theta_3 + (1^{14}2^{12})\theta_4].
\] (55)

The corresponding exponential (cf. (41)) now contains terms up to the level $(\theta \bar{\theta})^4$, but careful examination shows that in it one does not encounter singularities worse than $(1^{12}2^{12})^{-2}$.

The $SU(4)$ irrep carried by $\mathcal{O}(3)$ should be in the decomposition of the tensor product (we assume that $m \geq n$):
\[
[0, m, 0] \otimes [0, n, 0] = \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{n-k} [j, m + n - 2j - 2k, j].
\] (56)

From (23) we see that irreps of the type $[q, p, q]$ can be obtained as products of the two $G$-analytic realizations of the $N = 4$ SYM field strength (11) and (18). So, we also need the mixed two-point function
\[
\langle W^{12}(x, 0, u_1)W^{13}(y, 0, u_2) \rangle = \frac{(1^{12}2^{13})}{(x - y)^2}.
\] (57)

Then, generalizing (44), we can write down the leading term:
\[
\langle \mathcal{O}^{[0, m, 0]}_{1/2}(1)\mathcal{O}^{[0, n, 0]}_{1/2}(2)\mathcal{O}(t; s/2, s/2; j, m+n-2j-2k, j)(3) \rangle_0 = \\
\left[ (1^{12}2^{12}) \right]^k \left[ (1^{12}3^{12}) \right]^{m-j-k} \left[ (2^{12}3^{12}) \right]^{n-j-k} \\
\times \left\{ \left[ (1^{12}2^{12}) \right] \left[ (2^{12}3^{12}) \right] - \left[ (1^{12}3^{12}) \right] \left[ (2^{12}3^{12}) \right] \right\}^j \\
\times (Y^2)^{t(s-m-n+2k)} Y^{\mu_1 \ldots \mu_s}.
\] (58)

Finally, we restore the $\theta$ dependence at point 1 by making shifts of the type (55) and study the resulting harmonic singularities in the variable $(1^{12}2^{12})$. They depend on the value of $k$ and we find exactly the same conditions (47), (50) and (51) as in the preceding case (see the summary in Table 2).

---

9Note that if $m = n$ and the two operators at points 1 and 2 are considered identical, some of the terms in (56) drop out due to crossing symmetry.
Table 1: $\mathcal{O}_{1/2}^{[m,0,0]} \mathcal{O}_{1/2}^{[0,0,n]} \rightarrow \sum_{k=0}^{\min(m,n)} \mathcal{O}(\ell,s/2,s/2;m-k,0,n-k)$

| k  | Spin | Dimension | Protection | Type |
|----|------|-----------|------------|------|
| $k = 0$ | $s = 0$ | $\ell = m + n$ | protected | 1/4 BPS |
| $k = 1$ | $s = 0$ | $\ell = m + n - 2$ | protected | 1/4 BPS |
|        | $s \geq 0$ | $\ell = s + m + n$ | protected | 1/2 BPS if $\min(m,n) = 1$ current-like |
| $k \geq 2$ | $s = 0$ | $\ell = m + n - 2k$ | protected | 1/4 BPS |
|        | $s \geq 0$ | $\ell = 2 + s + m + n - 2k$ | “protected” | 1/2 BPS if $\min(m,n) = k$ current-like |
|        |        | $\ell > 2 + s + m + n - 2k$ | unprotected | long |

Table 2: $\mathcal{O}_{1/2}^{[0,m,0]} \mathcal{O}_{1/2}^{[0,n,0]} \rightarrow \sum_{k=0}^{n} \sum_{j=0}^{n-k} \mathcal{O}(\ell,s/2,s/2;j,m+n-2j-2k,j)$, $m \geq n$

| k  | Spin | Dimension | Protection | Type |
|----|------|-----------|------------|------|
| $k = 0$ | $s = 0$ | $\ell = m + n$ | protected | 1/4 BPS |
|        |        |           |            | 1/2 BPS if $j = 0$ |
| $k = 1$ | $s = 0$ | $\ell = m + n - 2$ | protected | 1/4 BPS |
|        | $s \geq 0$ | $\ell = s + m + n$ | protected | 1/2 BPS if $j = 0$ current-like |
| $k \geq 2$ | $s = 0$ | $\ell = m + n - 2k$ | protected | 1/4 BPS |
|        | $s \geq 0$ | $\ell = 2 + s + m + n - 2k$ | “protected” | 1/2 BPS if $j = 0$ current-like |
|        |        | $\ell > 2 + s + m + n - 2k$ | unprotected | long |

4 Conclusions

In this paper we have established the OPE content of two types of 1/2 BPS short operators. The results are summarized in Tables 1 and 2. The most interesting case is that of $\mathcal{N} = 4$ SYM (Table 2), where the central charge vanishes and we are actually dealing with the superalgebra $PSU(2,2/4)$.

Let us give two examples of operators widely discussed in the literature. They appear in the OPE of two $\mathcal{N} = 4$ SYM stress-tensor multiplets ($m = n = 2$).
The case $s = 0$, $\ell = 4$, $k = 1$ and $SU(4)$ irrep $[0,2,0]$ corresponds to the so-called $O_{20}'$ whose non-renormalization was first conjectured in [10]. As we can see from Table 2, it is a protected current-like operator because the OPE fixes its dimension.

The case $s = 0$, $\ell \geq 2$, $k = 2$ and $SU(4)$ irrep $[0,0,0]$ corresponds to the so-called Konishi multiplet [40]. This time the OPE does not fix the dimension, therefore the operator is either current-like if $\ell = 2$ (which only takes place in the free theory) or long if $\ell > 2$ (in the presence of interactions this operator picks an anomalous dimension).

We stress that in the OPE of two $N = 4$ SYM stress-tensor multiplets the only unprotected operators are $SU(4)$ singlets (since $m = n = k = 2$ implies $j = 0$, i.e., Dynkin labels $[0,0,0]$).

As an application of this result, consider the correlator of four stress-tensor multiplets

$$G^{(N=4)} = \langle O^{[0,2,0]}(1) O^{[0,2,0]}(2) O^{[0,2,0]}(3) O^{[0,2,0]}(4) \rangle.$$ (59)

Performing a double OPE, we see that the six $SU(4)$ representations $[j,4-2k-2j,j]$, with $k = 0,1,2$ and $0 \leq j \leq 2-k$ can be exchanged. According to our classification, the only “unprotected channel” is the singlet $j = 0, k = 2$ which is just the operator $O_{20}'$ mentioned above.

As another application of our results, we sketch the explanation of the non-renormalization of the so-called “extremal” and “next-to-extremal” $n$-point correlators [32, 33].

Consider an “extremal” 4-point function of operators $O^{[0,m,0]} = [W^{12}]^m$, where the exponents (viz charges) obey $m_1 = m_2 + m_3 + m_4$.

In the OPE between the operators at points 1 and 2 we find a range of representations with Dynkin labels $[j,m_1 + m_2 - 2j - 2k,j]$. Since $m_1 > m_2$ we must have $k \leq m_2$. The number of indices of the corresponding Young tableau is $2(m_1 + m_2 - 2k)$. The representation with the smallest number of boxes in its tableau therefore has

$$2(m_1 + m_2 - 2m_2) = 2(m_3 + m_4)$$ (60)

indices. This equals the total number of indices of the remaining operators at points 3 and 4. The only operators in their OPE contributing to the correlator carry therefore $SU(4)$ Dynkin labels $[j,m_i + m_j - 2j,j]$, i.e. the case $k = 0$ in Table 2, because $k > 0$ means a dualization by the $\epsilon$ symbol. Only 1/4 BPS or 1/2 BPS operators are exchanged, and they always have protected dimension.

For the “next-to-extremal” case $m_1 = m_2 + m_3 + m_4 - 2$ the same argument leads to one $\epsilon$-dualization in the product of the operators at points 3,4. Hence, the exchanged operators carry Dynkin labels $[j,m_i + m_j - 2j - 2, j]$ corresponding to the case $k = 1$ in Table 2. Again, all these operators are protected.

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