Arbitrary-order Hilbert spectral analysis for time series possessing scaling statistics: a comparison study with detrended fluctuation analysis and wavelet leaders

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In this paper we present an extended version of Hilbert-Huang transform, namely arbitrary-order Hilbert spectral analysis, to characterize the scale-invariant properties of a time series directly in an amplitude-frequency space. We first show numerically that due to a nonlinear distortion, traditional methods require high-order harmonic components to represent nonlinear processes, except for the Hilbert-based method. This will lead to an artificial energy flux from the low-frequency (large scale) to the high-frequency (small scale) part. Thus the power law, if it exists, is contaminated. We then compare the Hilbert method with structure functions (SF), detrended fluctuation analysis (DFA), and wavelet leader (WL) by analyzing fractional Brownian motion and synthesized multifractal time series. For the former simulation, we find that all methods provide comparable results. For the latter simulation, we perform simulations with an intermittent parameter $\mu = 0.15$. We find that the SF underestimates scaling exponent when $q > 3$. The Hilbert method provides a slight underestimation when $q > 5$. However, both DFA and WL overestimate the scaling exponents when $q > 5$. It seems that Hilbert and DFA methods provide better singularity spectra than SF and WL. We finally apply all methods to a passive scalar (temperature) data obtained from a jet experiment with a Taylor’$s$ microscale Reynolds number $Re_{\lambda} \approx 250$. Due to the presence of strong ramp-cliff structures, the SF fails to detect the power law behavior. For the traditional method, the ramp-cliff structure causes a serious artificial energy flux from the low-frequency (large scale) to the high-frequency (small scale) part. Thus DFA and WL underestimate the scaling exponents. However, the Hilbert method provides scaling exponents $\zeta_q(q)$ quite close to the one for longitudinal velocity, indicating a less intermittent passive scalar field than what was believed before.

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I. INTRODUCTION

Multifractal properties have been found in many fields, such as turbulence [1, 5], rainfaill [2, 6], financial time series [3, 4], physiology [7], etc. Conventionally, multifractal properties of such time series are characterized by the scaling exponents $\zeta_q(q)$, which are extracted by structure function (SF) analysis: $\Delta V_{\ell}^{q} \sim \ell^{\zeta_q(q)}$, where $\Delta V_{\ell} = \left| V(x+\ell) - V(x) \right|$ are the increment for scale separation $\ell$, and $\zeta_q(q)$ is a nonlinear function $[1, 2, 3]$. The function $\zeta_q(q)$ is linear for monoscaling processes and nonlinear for multifractal processes. We may also mention the detrended fluctuation analysis (DFA) $[14, 16]$ or the multifractal detrended fluctuation analysis $[17]$, which are sometimes also employed for scaling time series analysis. The DFA method is similar to SFs since it involves increments and characterizes the scale invariance in the physical domain.

Other widely used methods are wavelet-based methods, e.g. wavelet transform modulus maxima (WTMM), wavelet leader (WL), or gradient modulus wavelet projection (GMWP), to extract the scaling exponents from a scaling time series $[18, 30]$. However, as we will show in this paper, the wavelets share the same drawback with Fourier transform, which requires high-order harmonic components to represent nonlinear processes.

Some of us have proposed recently a new methodology, namely arbitrary-order Hilbert spectral analysis (HSA), to characterize the scale invariant properties directly in amplitude-frequency space $[31, 35]$. It is an extended version of the Hilbert-Huang transform (HHT), which provides a joint probability density function (pdf) in an amplitude-frequency space $[31, 35]$. We have applied part of this new methodology to several different time series to show its efficiency and validity: turbulence experimental database $[31]$, synthesized fractional Brownian mo-
A. Arbitrary-order Hilbert spectral analysis

The most innovative part of the Hilbert-Huang transform is the so-called empirical mode decomposition (EMD). In the real world most of the signals are multi-components, which means that different scales can co-exist simultaneously [32, 33]. This may be considered as fast oscillations superposed to slower ones at a local level [34]. Meanwhile, for decomposition methods, a characteristic scale (CS) is always defined implicitly or explicitly before the decomposition. For example, the CS of the classical Fourier analysis is a period of sine wave. The CS of wavelet transform is the shape of the mother wavelet [35]. In the present method, the CS is defined as the distance between two successive maxima (respectively minima) points. Then the so-called intrinsic mode functions (IMF) are proposed to represent each mono-component signal. An IMF satisfies the following two conditions: (i) the difference between the number of local extrema and the number of zero-crossings must be zero or one; (ii) the running mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero [36].

A subpart of the EMD algorithm, called “sifting process,” is then designed to decompose a given signal into several IMF modes [37, 38]. The first step of the sifting process is to identify all the local maxima (respectively, minima) points for a given time series \( x(t) \). Once all the local extrema points are identified, the upper envelope \( e_{\text{max}}(t) \) and the lower envelope \( e_{\text{min}}(t) \) are constructed, respectively, for the local maxima and minima points by using a cubic spline algorithm. The mean between these two envelopes is defined as

\[
m_1(t) = \frac{e_{\text{max}}(t) + e_{\text{min}}(t)}{2}
\]

Thus the first component is estimated by

\[
h_1(t) = x(t) - m_1(t)
\]

Ideally, \( h_1(t) \) should be an IMF as expected. However, \( h_1(t) \) may not satisfy the above-mentioned conditions to be an IMF. The function \( h_1(t) \) is then taken as a new time series and this sifting process is repeated \( j \) times, until \( h_{1j}(t) \) is an IMF

\[
h_{1j}(t) = h_{1(j-1)}(t) - m_{1j}(t)
\]

The first IMF component \( C_1(t) \) is then written as

\[
C_1(t) = h_{1j}(t)
\]

and the residual \( r_1(t) \) as

\[
r_1(t) = x(t) - C_1(t)
\]

from the data \( x(t) \). The sifting procedure is then repeated on the residual until \( r_n(t) \) becomes a monotonic function or at most has one local extreme point, which means that no more IMF can be extracted from \( r_n(t) \). There are finally \( n - 1 \) IMF modes with one residual \( r_n(t) \). The original signal \( x(t) \) is rewritten at the end of the process as

\[
x(t) = \sum_{i=1}^{n-1} C_i(t) + r_n(t)
\]

To guarantee that the IMF modes retain enough physical sense, a certain stopping criterion has to be introduced to stop the sifting process properly. Different types of stopping criteria have been introduced by several authors [39, 40, 41, 42]. The first stopping criterion is a Cauchy-type convergence criterion. We introduce the standard deviation (SD), defined for two successive sifting processes as

\[
\text{SD} = \frac{\sum_{t=0}^{T} [h_{i(j-1)}(t) - h_{ij}(t)]^2}{\sum_{t=0}^{T} h_{ij}^2(t)}
\]

If a calculated SD is smaller than a given value, then the sifting stops, and gives an IMF. A typical value is
0.2 ~ 0.3, proposed based on Huang et al.’s experiences [38, 39]. Another widely used criterion is based on three thresholds $\alpha$, $\theta_1$, and $\theta_2$, which are designed to guarantee globally small fluctuations meanwhile taking into account locally large excursions [40]. The mode amplitude and evaluation function are given as

$$a(t) = \frac{c_{\text{max}}(t) - c_{\text{min}}(t)}{2} \quad (8a)$$

and

$$\sigma(t) = |m(t)/a(t)| \quad (8b)$$

So that the sifting is iterated until $\sigma(t) < \theta_1$ for some prescribed fraction $1 - \alpha$ of the total duration, while $\sigma(t) < \theta_2$ for the remaining fraction. The typical values proposed by Rilling et al. [40] are $\alpha \approx 0.05$, $\theta_1 \approx 0.05$ and $\theta_2 \approx 10 \theta_1$, respectively based on their experience. In our practice, if one of these criteria is satisfied, then the sifting process will stop. We also set a maximal iteration number (e.g., 300) to avoid over-decomposing the time series.

The above-described EMD algorithm performs the decomposition on a very local level in the physical domain without a priori basis. This means that the present decomposition is a posteriori: The basis is induced by the data itself [38, 39, 41]. It is thus a scale-based decomposition. Since its introduction, this method has attracted large interests in various research fields: waves [34, 41, 45], biological applications [46, 48], financial studies [49], meteorology and climate studies [50–54], mechanical engineering [55, 56], acoustics [57], aquatic environment [58], and turbulence [59], to quote a few. More detail about the EMD algorithm can be found in several methodological papers [38, 41, 43, 59].

2. Hilbert spectral analysis

After having extracted the IMF modes, one can apply the associated Hilbert spectral analysis to each component $C_i$ in order to extract the energy time-frequency information from the data [38, 39, 60]. The Hilbert transform of a function $C(t)$ is written as

$$\tilde{C}(t) = \frac{1}{\pi} P \int \frac{C(t')}{t - t'} dt' \quad (9)$$

where $P$ means the Cauchy principle value [37, 38, 60]. For each mode function $C_i(t)$, one can then construct the analytical signal [37, 61], $C_i(t)$, as

$$C_{\tilde{i}}(t) = C_i(t) + j\tilde{C}_i(t) = A_i(t)e^{j\theta_i(t)} \quad (10)$$

where

$$\left\{ \begin{array}{l} A_i(t) = |C_i(t)|^2 + \tilde{C}_i^2(t) \quad (11a) \\ \theta_i(t) = \arctan \left( \frac{\tilde{C}_i(t)}{C_i(t)} \right) \end{array} \right.$$
other words, similarly with the SF analysis, it has no resolution on the right part of the singularity spectrum. The main drawback of the Hilbert-based method is its absence of solid theoretical ground, since the EMD part is almost empirical [13]. It has been found experimentally that the method, especially for the HSA, is statistically stable with different stopping criteria [12]. Recently, Flandrin et al. have obtained new theoretical results on the EMD method [11, 26, 27, 76]. However, more theoretical work is still needed to fully mathematically understand this method.

### B. Structure function analysis

The conventional way to extract scaling exponents is the classical SF analysis, which has been proposed in the field of turbulence and is now quite classical for intermittency studies [13]. The $q$th order SF is written as

$$S_q(\ell) = \langle |x_\ell(t)|^q \rangle \sim \ell^{q/\tau}$$  (18)

where $x_\ell(t)$ is a signal with a trend $\ell$ and $q$ is the time separation. The scaling exponent $\tau(q)$ characterizes the fluctuation statistic at all scales; it is linear for monofractal processes such as fractional Brownian motion, and non-linear and concave (as a second Laplace characteristic function) for multifractal processes [66]. This approach has been widely used in turbulent research [11, 26, 13] and also other research fields [9, 77, 68]. However, the increment operation acts as a filter and thus SF characterizes the scale-invariant properties in an indirect way; see detailed discussion in Refs. [33, 36].

As we have shown elsewhere, the increment operation in SF acts as a filter and is a global operation. It thus measures the scale invariant property in an indirect way. It is also found that it is strongly influenced by energetic large scale structures [33, 36]. Therefore the SF analysis is not suitable for those data which possess energetic large scale structures. We will show an example of passive scalar turbulence data with strong ramp-cliff structures in Sec. V. More discussion can be found in Refs. [33, 36].

### C. Multifractal detrended fluctuation analysis

DFA was first introduced by Peng et al. [14] to study the scaling properties of DNA sequence, in which only the second-order moment $q = 2$ was considered. Later this was generalized into a multifractal version by considering the arbitrary order $q$, namely multifractal detrended fluctuation analysis (MFDFA) [17, 69]. It then became a more common technique for scaling data analysis [14, 17, 69, 75]. For a given discrete time series $x(i)$, $i = 1 \cdots N$, we first estimate its cumulative function

$$Y(j) = \sum_{i=1}^{j} (x(i) - \bar{x}), \quad j = 1, \cdots N$$  (19)

where $\bar{x}$ is the mean value of $x$. We then divide it into $M_n$ segments of length $n$ ($n < N$) starting from both the beginning and the end of the time series. Each segment $v$ has its own local trend that can be approximated by fitting a $p$th-order polynomial $P_v^p$ which is removed from the data. The variances for all the segments $v$ and for all segment lengths $n$ are then calculated by

$$F^2(v, n) = \frac{1}{n} \sum_{j=1}^{n} \{Y[(v - 1)n + j] - P_v^p(j)\}^2$$  (20)

The $q$th-order fluctuation function is then defined as

$$F_q(n) = \left( \frac{1}{2M_n} \sum_{v=1}^{2M_n} [F^2(v, n)]^{q/2} \right)^{1/q}$$  (21)

For discussion convenience, we redefine the $q$th-order fluctuation function as

$$F_q(n) = F^q(n)$$  (22)

In case of scale invariance, we have power law scaling within a significant range of $n$

$$F_q(n) \sim n^{h(q)}$$  (23)

in which $h(q)$ is the corresponding scaling exponent function.

### D. Discrete wavelet transform and wavelet leaders

Wavelets have been widely used in data analysis and turbulence research [18, 23, 25, 27, 76, 78]. Several wavelet-based methods have been proposed by several researchers to extract the scaling exponents from a scaling time series, for example, wavelet coefficients (WC), WTMM [18, 19, 76], WL [25, 28, 78], etc. We consider here WC and WL.

The discrete wavelet transform (DWT) is defined as

$$\psi(k, j) = \int_{\mathbb{R}} x(t) \varphi(2^{-j} t - k) \, dt$$  (24)

where $\varphi$ is the chosen wavelet, $\psi(k, j)$ is the wavelet coefficient, $k$ is the position index, $j$ is the scale index, and $\ell = 2^j$ is the corresponding scale [77, 79]. The first way to detect the scale-invariant properties is to consider the wavelet coefficients

$$Z_q(j) = \langle |\psi(k, j)|^q \rangle \sim 2^{j\tau(q)}$$  (25)

where $\tau(q)$ are the corresponding scaling exponents.

Every discrete wavelet coefficient $\psi(k, j)$ can be associated with the dyadic interval $\varrho(k, j)$

$$\varrho(k, j) = [2^j k, 2^j (k + 1)]$$  (26)
Thus the wavelet coefficients can be represented as
\[
\psi(q) = \psi(k, j).
\]
Wavelet leaders are defined as
\[
l(k, j) = \sup_{q' \in 3q(k, j), j' \leq j} |\psi(q')|
\]
where \(3q(k, j) = q(k-1, j) \cup q(k, j) \cup q(k+1, j)\) \(26\,28\,78\). Thus power law behavior is expected
\[
Z_q(j) = \langle l(k, j)^q \rangle \sim 2^{\tau(q)}
\]
in which \(\tau(q)\) is the corresponding scaling exponent. Its efficiency has been shown for various types of data set \(26\,28\,78\).

Let us recall some previous comparison studies between WTMM, MFDSA and WL. Oświecimka et al. \(69\) performed a comparison study between WTMM and MFDSA by analyzing synthesized data. They stated that the MFDSA provides a better estimation of singularity spectrum than WTMM. Jaffard et al. \(25\) stated that WL provides a better singularity spectrum than WTMM. Ser- rano and Figliola \(27\) performed a comparison study between MFDSA and WL. They found that WL performs better than MFDSA. However, for a short time series, MFDSA is proposed to extract multifractal spectrum. A detailed comparison can be found in Ref. \(26\,69\,27\), respectively, for WTMM and WL, MFDSA and WTMM, and WL and MFDSA.

However, we argue here that DWT violates two facts of the time-frequency representation of a time series. First, the scale of a time series from complex system, for example, turbulent flows, is continuous in a statistical sense, but not discrete on several scales \(31\,33\). The other one is that for a certain scale, it may not exist all the time \(33\,38\,61\); see also the discussion in the next section. Thus to represent a signal by using a DWT is not consistent with the physical aspect.

### III. NONLINEAR EFFECTS

![FIG. 1. (Color online) (a) A fifth order numerical solution (thick solid line) for Duffing equation, (b) An enlarged portion. For comparison, a sine wave with the same mean frequency is also shown as a thin solid line. The departure from a pure sine wave profile is the result of nonlinear interactions, which are nonlinear distortion.](image)

We first consider nonlinear effects by using the classical Duffing equation, which reads
\[
\frac{d^2x}{dt^2} + x(1 + \epsilon x^2) = b \cos(\Omega t)
\]
in which \(\epsilon\) is a nonlinear parameter. It can be considered as a pendulum with forcing function \(b \cos(\Omega t)\), in which its pendulum length varies with the angle. Figure \(3\) shows a fifth-order Runge-Kutta numerical solu-
A. Fractional Brownian motion

We have shown in previous works that the arbitrary-order HSA can be applied to the fractional Brownian motion \( H \neq 1/2 \). Here we briefly recall these results. FBm is a Gaussian self-similar process with a normal distribution increment, which is characterized by \( H \), the Hurst number \( 0 < H < 1 \). Note that the singularity spectra for the above mentioned methods are

\[
\alpha = \zeta'(q), \quad f(\alpha) = \min\{\alpha q - \zeta(q) + 1\} \quad (30a)
\]

for SFs, and

\[
\alpha = \xi'(q), \quad f(\alpha) = \min\{\alpha q - \xi(q) + 2\} \quad (30b)
\]

for the Hilbert-based method, and

\[
\alpha = h'(q) - 1, \quad f(\alpha) = \min\{(\alpha + 1)q - h(q) + 1\} \quad (30c)
\]

for DFA, and

\[
\alpha = \tau'(q) - 1, \quad f(\alpha) = \min\{(\alpha + 1)q - \tau(q) + 1\} \quad (30d)
\]

for WC and WL, respectively. Ideally, we should have \( \alpha = H \) and \( f(\alpha) = 1 \).

We performed 500 realizations each of length \( 2^{14} \) data points by applying a Fourier-based Wood-Chan algorithm \([84]\) with \( H = 1/3 \), which corresponds to the Hurst number of turbulent velocity. We apply the above mentioned methods to each realization of the data series. The final spectra and statistical errors are then estimated from these 500 realizations. Figure 4 show results for (a) SF: (left) \( S_\alpha(\ell) \) with \( q = 0 \) (\( \circ \)), \( q = 2 \) (\( \square \)), \( q = 4 \) (\( \triangle \)) and \( q = 6 \) (\( \diamond \)), (middle) the corresponding scaling exponents \( \zeta(q) \) on the range \( 0 \leq q \leq 8 \), (right) the corresponding singularity spectrum \( f(\alpha) \), (b) HSA, (c) DFA, and (d) wavelet, respectively. The symbols are the same as the SF symbols. Graphically, all methods provide comparable estimation of \( f(\alpha) \). However, we note that the Hilbert-based method slightly overestimates \( \zeta(q) \) when \( q > 6 \).

IV. VALIDATION AND CALIBRATION

In this section, we will validate the Hilbert-based method by performing a comparison study of simulated FBm with Hurst number \( H = 1/3 \) and synthesized multifractal random walk with an intermittent parameter \( \mu = 0.15 \). For comparison convenience, spectral curves (or the \( q \)-th order statistical moment) provided by SFs, MFDFA and wavelet are converted from the physical domain into the spectral domain by taking \( f = 1/\ell \), \( f = 1/n \), and \( f = 1/\pi \), respectively. The corresponding scaling exponents are estimated on the range \( 0.001 < f < 0.1 \) (we set here the sampling frequency as 1). Wavelet transform is performed by using the db3 wavelet. Due to the limitation of the SF analysis and the HSA, we only consider here the non-negative \( q \)-th order moment, \( q \geq 0 \), the left part of the singularity spectrum.
for this simple monofractal process. It seems that they provide a better estimation than Hilbert and DFA methods. This result is not in full agreement with Oświęcimka et al. [69], who stated that the MF DFA provides a better estimation of $H$ than WTMM.

The above results show that all methods provide comparable prediction of singularity spectra for fBm with $H = 1/3$. However, it seems that SF and wavelet based methods provide a better estimation.
FIG. 7. (Color online) Multifractal random walk with $\mu = 0.15$. (a) Structure function, (b) Hilbert spectral analysis, (c) multifractal detrended fluctuation analysis and (d) wavelet coefficients and wavelet leaders. The symbols are the same as in Fig. 4. The statistical error bars are estimated from the total 100 realizations.

B. Multifractal simulation

We show now that the new method applies to multifractal time series. First, let us consider a multiplicative discrete cascade process to simulate a multifractal measure $\epsilon(x)$. Figure 3 illustrates the cascade process algorithm. The larger scale corresponds to a unique cell of size $L = \ell_0 \lambda_1^n$, where $\ell_0$ is a fixed scale and $\lambda_1 > 1$ is a dimensional scale ratio. For discrete models, this ratio is often taken as $\lambda_1 = 2$. The models being discrete, the next scale involved corresponds to $\lambda_1$ cells, each of size $L/\lambda_1 = \ell_0 \lambda_1^{n-1}$. This is iterated and at step $p$ ($1 \leq p \leq n$) there are $\lambda_1^p$ cells, each of size $L/\lambda_1^p = \ell_0 \lambda_1^{n-p}$. There are $n$ cascade steps, and at step $n$ there are $\lambda_1^n$ cells, each of size $\ell_0$, which is the smallest scale of the cascade. To reach this scale, all intermediate scales have been involved. Finally, at each point the multifractal measure writes as the product of $n$ cascade random variables

$$\epsilon(x) = \prod_{p=1}^{n} W_{p,x}$$

where $W_{p,x}$ is the random variable corresponding to position $x$ and level $p$ in the cascade [35]. Following multifractal random walk ideas [36, 37], we generate a non-stationary multifractal time series as

$$u(x) = \int_0^x \epsilon(x')^{1/2} dB(x')$$

where $B(x)$ is Brownian motion. Taking lognormal statistic for $\epsilon$, the scaling exponent $\zeta(q)$ such as $\langle |\Delta u_x(t)|^q \rangle \sim \tau(q)$ can be shown to be written as

$$\zeta(q) = \frac{q}{2} - \frac{\mu}{2} \left( \frac{q^2 - \mu}{2} \right)$$

where $\mu$ is the intermittency parameter (0 $\leq \mu \leq 1$) characterizing the lognormal multifractal cascade.

Synthetic multifractal time series are generated following Eq. (32). For each realization, we choose $n = 17$ levels, corresponding to data sets with data length 131,072 points each. A sample for one realization is shown in Fig. 6 (a) for the multifractal measure and (b) for the nonstationary multifractal time series with $\mu = 0.15$. We perform 100 realizations with intermittent parameter $\mu = 0.15$. Except for the structure functions, we apply all methods to each realization by dividing one realization into eight subsets with 214 data points each. The spectra for each realization are averaged over these eight subsets. The final spectra and error bars are respectively ensemble average and standard deviation estimated from these 100 realizations.
V. PASSIVE SCALAR TURBULENCE WITH RAMP-CLIFF STRUCTURES

We now apply the above-mentioned methods to a real time data set, a temperature time series as a turbulent passive scalar. The data are obtained from a jet experiment performed at Joseph Fourier University Grenoble, France. The bulk Reynolds number is about $Re \approx 60000$. The corresponding Taylor’s microscale Reynolds number is about $Re_\lambda \approx 250$. The initial temperature of the two flows are $T_j = 27.8^\circ C$ and $T = 14.8^\circ C$. The measurement location is in the mixing layer and close to the nozzle of the jet. The sampling frequency is 50 kHz. The total data length is 10 s, corresponding to 500,000 data points. Figure 8 shows a 0.1 s portion temperature data, illustrating strong ramp-cliff structures. For comparison, a pure sine wave is also shown. Obviously, the so-called ramp-cliff structure is a large-scale structure with a very sharp interface. The original time series is divided into 122 non-overlapping segments with $2^{12}$ data points each. The finally spectra and statistical errors (the standard deviation) are then estimated from these 122 realizations. Figure 9 shows the energy spectra (or the second-order statistical moment) provided by several methods. The inset shows the compensated spectra by multiplying the result by $f^{\alpha/2}$ for Hilbert and Fourier, $f^{\beta/2}$ for DFA and wavelet, and $f^{\gamma/2}$ for SFs, respectively. For clarity, the curves have been vertically shifted. Both Fourier and Hilbert methods predict a clear power law on the range $80 < \omega < 2000$ Hz. Due to the presence of ramp-cliff structures, the SF analysis fails to capture the power law behavior and DFA and wavelet predict a short inertial range on the range $100 < f < 1000$ Hz. The corresponding scaling exponents are $\beta_h = 1.56$ for Fourier, $\xi(2) = 1.70$ for Hilbert, $\tau_f(2) = 2.46$ for WL with db3 wavelet, and $h_b(2) = 2.47$ for the first-order DFA, respectively. The corresponding scaling exponents are $\alpha_f = 2.70$ for Fourier, $\xi_f(2) = 2.56$ for Fourier, $\xi(2) = 2.56$ for Hilbert, and $\tau_f(2) = 2.46$ for WL with db3 wavelet. The energy spectra (arbitrary unit) are shown in Figure 9. The inset shows the compensated spectra by multiplying the result by $f^{\alpha/2}$ for Hilbert and Fourier, $f^{\beta/2}$ for DFA and wavelet, and $f^{\gamma/2}$ for SFs, respectively. For clarity, the curves have been vertically shifted. Both Fourier and Hilbert methods predict a clear power law on the range $80 < \omega < 2000$ Hz. Due to the presence of ramp-cliff structures, the SF analysis fails to capture the power law behavior and DFA and wavelet predict a short inertial range on the range $100 < f < 1000$ Hz. The corresponding scaling exponents are $\beta_h = 1.56$ for Fourier, $\xi(2) = 1.70$ for Hilbert, $\tau_f(2) = 2.46$ for WL with db3 wavelet, and $h_b(2) = 2.47$ for the first-order DFA, respectively. Therefore, the Fourier-based methodologies, it is inevitable that one requires high-order harmonic components to represent their difference, in which the underlying idea is a linear asymptotic approximation. This linear asymptotic approximation process thus leads to an artificial energy flux from low frequencies (large scales) to higher frequencies (small scales). It means that the Fourier-based spectrum may be contaminated by this artificial energy flux. As another direct consequence, the artificial redistribution of the energy will lead to an unreal correlation if we consider cross-correlation between two scales.

The original time series is divided into 122 non-overlapping segments with $2^{12}$ data points each. The finally spectra and statistical errors (the standard deviation) are then estimated from these 122 realizations. Figure 9 shows the energy spectra (or the second-order statistical moment) provided by HSA (solid line), Fourier transform (dashed line), WL ( ), the first-order DFA ( ) and SF ( ), respectively. The inset shows the corresponding compensated spectra by multiplying a Kolmogorov-Obukhov-Corrsin nonintermittent scaling exponent 5/3 for Hilbert spectrum, Fourier power spectrum, 8/3 for WL and DFA, and 2/3 for SF, respectively. Except for the SF, all methods display a clear power law on the range $80 < f < 2000$ Hz or $100 < f < 1000$ Hz, a more than one decade inertial range. The corresponding scaling exponents are $\xi(2) \approx 1.70$.
For Hilbert, $\beta_1 \simeq 1.56$ for Fourier, $\tau_0(2) \simeq 2.46$ for WL and $h_0(2) \simeq 2.47$ for DFA, respectively obtained by using a least square fitting algorithm. We note that only the Hilbert based scaling exponent $\xi_0(2)$ is close to the corresponding nonintermittent scaling exponent $\xi_0(2) = 5/3$. It is also comparable with the scaling exponent of longitudinal velocity in fully developed turbulence [2]. Due to the presence of strong ramp-cliff structures, the SF fails to detect the correct scaling behavior. The influence of large energetic structures on SF has been studied in detail by Huang et al. and Huang. It is interesting to note that DFA and WL provide almost the same scaling exponent, which indicates that the ramp-cliff structure may have the same influence on them. We believe that there exists an artificial energy flux as we discussed above in both Fourier and DFA and WL spectra. Thus they may underestimate the scaling exponents.

Figure 10 shows the analysis results of (a) Hilbert spectral analysis, (b) MFDFA, and (c) wavelet transform, respectively. The symbols are the same as in Fig. 4.

As we already mentioned previously, the wavelet and DFA spectra are strongly influenced by nonlinear large scale structures (e.g. ramp-cliff structures in passive scalar turbulence). Their scaling exponents are thus contaminated by high-order harmonics. In other words, the
VI. CONCLUSION

In summary, we introduced in this paper a new method, namely arbitrary-order Hilbert spectral analysis, to characterize scale-invariant properties directly in the amplitude-frequency space. It is an extended version of Hilbert-Huang transform. The main advantage of the Hilbert-based methodology is its fully adaptive and very local ability both in spectral and physical domains. Thus, it is not necessary to require high-order harmonics to represent nonlinear and nonstationary processes, which is usually required by conventional Fourier-based methods, such as Fourier transform, wavelet transform, etc. We illustrated the nonlinear effect by using the Duffing equation. It is found that not only Fourier-based methods, but also SF analysis and DFA are influenced by nonlinear processes. It is also found that the HSA can constrain the high-order harmonics by using the intrawave-frequency-modulation mechanism for the nonlinear distortion.

Then we performed a comparison study of the Hilbert-based methodology with SF analysis, MFDFA, and WL, by analyzing fBm simulations with Hurst number $H = 1/3$ and a synthesized multifractal lognormal random walk with intermittent parameter $\mu = 0.15$, respectively. For the former simulation, we considered the scaling exponents and singularity spectrum on the range $0 < q < 8$. It was found that all methods provide comparable scaling exponents and singularity spectra. For the latter synthesized multifractal random walk data, HSA and MFDFA provide a better estimation of singularity spectra than SF and WL. However, none of these methods recover the whole spectrum. We finally applied all methods to the passive scalar (temperature) data set with strong ramp-cliff structure, which is an important signature of passive scalar turbulence. We found that except for HSA, all the methods require high-order harmonics to represent the ramp-cliff structures. Therefore, the singularity spectra provided by DFA and WL are contaminated by this large nonlinear structure. In fact, it already has been reported by several authors that for passive scalar turbulence the second-order SF and Fourier power spectrum are not consistent with each other. It seems that the idea of the Reynolds number of the present passive scalar data set is about $Re \approx 250$. Thus the strong ramp-cliff structure may be recognized as an effect of the finite Reynolds number. We will add this issue elsewhere.

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