Some examples in the integral and Brown-Peterson cohomology of \( p \)-groups.

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Introduction.

For a finite group \( G \), we define the Chern ring, \( \text{Ch}(G) \), to be the subring of \( H^{\text{even}}(G; \mathbb{Z}) \) generated by Chern classes of representations of \( G \). We say that \( G \) has \( p \)-rank \( n \) if \( n \) is maximal such that \( G \) contains a subgroup isomorphic to \((\mathbb{C}_p)^n\). In [3] Atiyah showed that for any finite group \( G \), \( K^0(BG) \) is the completion of the representation ring of \( G \) with respect to a certain topology. The filtration of \( K^0(BG) \) given by the \( E_\infty \) page of the Atiyah-Hirzebruch spectral sequence (AHSS) gives rise to a filtration of the representation ring of \( G \). Atiyah conjectured that this filtration coincided with another filtration defined algebraically, and remarked that this conjecture is equivalent to the conjecture that \( \text{Ch}(G) \) maps onto the \( E_\infty \) page of the AHSS. (It is clear that \( \text{Ch}(G) \) consists of universal cycles because the AHSS for \( BU(n) \) collapses.) Weiss discovered that the alternating group \( A_4 \) gives a counterexample to this conjecture [16], and Thomas has exhibited many counterexamples, all of which have order divisible by more than one prime [14].

Thomas showed that the split metacyclic \( p \)-groups and various other \( p \)-groups of \( p \)-rank two have the property that the Chern subring is the whole of the even degree integral cohomology, and conjectured that this property would hold for all \( p \)-groups of \( p \)-rank two [12], [13], [15]. The group \( A_4 \) shows that the conjecture cannot be extended to groups of non-prime power order. AlZubaidy claimed to have verified this conjecture, but some of his proofs are flawed [1], [2]. Recently Huebschmann and Tezuka-Yagita have shown that
Ch(G) = H_{even}(G; \mathbb{Z}) for any metacyclic $p$-group $G$ [6], [11]. For $p \geq 5$ Blackburn’s classification [4] implies that the only $p$-groups of $p$-rank two not covered by the above theorems are the groups $G(n, \epsilon)$ defined below. We show that $\text{Ch}(G(n, \epsilon))$ is strictly contained in $H_{even}(G(n, \epsilon); \mathbb{Z})$ for each such group. These groups also afford $p$-group counterexamples to the conjecture of Atiyah described above.

Similar calculations may be made in the Brown-Peterson cohomology rings of these groups. These enable us to give a negative answer to a question of Landweber [7], who asked if Chern classes generate the Brown-Peterson cohomology of every $p$-group.

The examples.

The groups which we shall consider may be presented as

$$G(n, \epsilon) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [B, C] = 1 \quad [A, C^{-1}] = B \quad [B, A] = C^{\epsilon p^{n-3}} \rangle,$$

where $p$ is a prime not equal to $2$ or $3$, $n \geq 4$, and for fixed $p$ and $n$ there are two isomorphism classes of such groups, depending whether $\epsilon$ is either $1$ or a quadratic non-residue modulo $p$. The group $G(n, \epsilon)$ has order $p^n$. In the sequel we shall refer to $G$ instead of $G(n, \epsilon)$ unless the values of $n$ and $\epsilon$ are important. The subgroup $M$ generated by $B$ and $C$ is maximal (and hence normal) and is isomorphic to $C_p \oplus C_{p^{n-2}}$. We define one dimensional representations $\theta$ and $\phi$ of the group $M$ by

$$\theta : B^j C^k \mapsto \exp(2\pi ij/p)$$

$$\phi : B^j C^k \mapsto \exp(2\pi ik/p^{n-2}).$$

The action of the quotient group $G/M$ on the representation ring of $M$ is that conjugation by $A$ sends $\theta$ to $\theta \otimes \phi^{\otimes p^{n-3}}$ and sends $\phi$ to $\phi \otimes \theta^{\otimes \epsilon}$. Later we shall define elements of $H^2(M; \mathbb{Z})$ and $BP^2(BM)$ as Chern classes, and the action of $G/M$ on these elements will be determined by its action on the representations $\theta$ and $\phi$.

The group $G$ has only $1$- and $p$-dimensional irreducible representations because it has an abelian subgroup ($M$ in fact) of index $p$. A one dimensional representation of $G$ must
restrict trivially to \( \langle B \rangle \), and a \( p \)-dimensional representation of \( G \) restricts to \( \langle B \rangle \) as either \( p \) copies of the same representation of \( \langle B \rangle \), or as the sum of one copy of each of the one-dimensional representations of \( \langle B \rangle \). The examples \( \text{Ind}_G^M(\theta) \) and \( \text{Ind}_G^M(\phi) \) show that both these alternatives do occur.

Now define generators \( \beta, \gamma \) for \( H^2(M; \mathbb{Z}) \) by

\[
\beta = c_1(\theta) \quad \gamma = c_1(\phi),
\]

so that \( H^{\text{even}}(M; \mathbb{Z}) \cong \mathbb{Z}[\beta, \gamma]/(p\beta, p^{n-2}\gamma) \),

and let \( \beta' \) be the restriction to \( \langle B \rangle \) of \( \beta \), so that

\[
H^*(\langle B \rangle; \mathbb{Z}) \cong \mathbb{Z}[\beta']/(p\beta').
\]

**Lemma 1.** With notation as above, the image of \( \text{Ch}(G) \) under restriction to \( \langle B \rangle \) is the subring of \( H^*(\langle B \rangle; \mathbb{Z}) \) generated by \( \beta'^{p-1} \) and \( \beta'^p \). For all \( m \geq 0 \),

\[
\beta'^{m+p-1} = -\text{Res}_{\langle B \rangle}^G \text{Cor}^G_M(\gamma^{p-1}\beta^m).
\]

**Proof.** If \( \rho \) is a 1-dimensional representation of \( G \) then its Chern class restricts trivially to \( \langle B \rangle \). If \( \rho \) is a \( p \)-dimensional representation of \( G \), then either \( \rho \) restricts to \( \langle B \rangle \) as \( p \)-copies of the same representation, in which case

\[
\text{Res}(c.(\rho)) = (1 + i\beta')^p = 1 + i\beta'^p,
\]

or as one copy of each representation, in which case

\[
\text{Res}(c.(\rho)) = \prod_{i=0}^{p-1} (1 + i\beta') = 1 - \beta'^{p-1}.
\]

By applying the double coset formula we see that

\[
\text{Res}_{\langle B \rangle}^G \text{Cor}^G_M(\gamma^{p-1}\beta^m) = \text{Res}_{\langle B \rangle}^M \left( \sum_{i=0}^{p-1} c^*_A(\gamma^{p-1}\beta^m) \right)
= \text{Res}_{\langle B \rangle}^M \left( \sum_{i=0}^{p-1} (\gamma + i\epsilon\beta)^{p-1}(\beta + ip^{n-3}\gamma)^m \right)
= -\beta'^{m+p-1}.
\]

\[\blacksquare\]
Remarks. The image of $\text{Res}^G_{\langle B \rangle}$ is precisely the subring of $H^*(\langle B \rangle; \mathbb{Z})$ generated by $\beta^{p-1}, \beta^p, \beta^{p+1}, \ldots, \beta^{2p-3}$. One way to show this is by considering the subgroup $N$ of $G$ generated by $A$ and $B$. This subgroup is normal in $G$, and is the non-abelian group of order $p^3$ and exponent $p$. Using Lewis’ calculation of $H^*(N; \mathbb{Z})$ [9], it may be shown that the image of $H^*(N; \mathbb{Z})^{G/N}$ under restriction to $\langle B \rangle$ does not contain $\beta^i$ for $i < p - 1$.

Corollary 2. $\text{Ch}(G)$ is strictly contained in $H^\text{even}(G; \mathbb{Z})$. Moreover, $\text{Ch}(G)$ does not map onto $H^*(G; \mathbb{Z})$ modulo its nilradical.

Proof. We know that $\beta^{p+1}$ is in $\text{Res}^G_{\langle B \rangle}(H^\text{even}(G; \mathbb{Z}))$, but not in $\text{Res}^G_{\langle B \rangle}(\text{Ch}(G))$. ■

Corollary 3. In the AHSS for $G$, write $B_\infty(G)$ for the universal boundaries, and $Z_\infty(G)$ for the universal cycles. Then $\text{Ch}(G) + B_\infty(G)$ is strictly contained in $Z_\infty(G)$.

Proof. The AHSS for $\langle B \rangle$ collapses, so $\text{Res}^G_{\langle B \rangle}(B_\infty(G))$ is trivial. Corestrictions of Chern classes are universal cycles, so $\beta^{p+1} \in \text{Res}^G_{\langle B \rangle}(Z_\infty(G))$, but $\beta^{p+1} \notin \text{Res}^G_{\langle B \rangle}(\text{Ch}(G) + B_\infty(G))$. ■

For any generalised cohomology theory $\mathcal{H}$ and any group $K$, we may define $\text{Ch}_{\mathcal{H}}(K)$ to be the subring of $\mathcal{H}^*(BK)$ generated by $\rho^*(\mathcal{H}^*(BU))$ for all representations $\rho$ of $K$ in a unitary group $U$. We now give a result analogous to Corollary 2 for Brown-Peterson cohomology.

Lemma 4. $\text{Ch}_{BP}(G)$ is strictly contained in $BP^*(BG)$.

Proof. As in the integral cohomology case, define elements $\beta$ and $\gamma$ in $BP^2(M)$ by $\beta = c_1(\theta)$, $\gamma = c_1(\phi)$, and also define $\beta' = \text{Res}^M_{\langle B \rangle}(\beta)$, so that

$$BP^*(BM) \cong BP_*[[\beta, \gamma]]/(\langle p \rangle \beta, [p^{n-2}] \gamma), \quad BP^*(B\langle B \rangle) \cong BP_*[[\beta']]/(\langle p \rangle \beta'),$$

where $[r]x$ stands for the $BP$ formal group sum of $r$ copies of $x$. Let ‘$\equiv$’ stand for congruence modulo the ideal of $BP^*(B\langle B \rangle)$ generated by $p, v_1, v_2, \ldots$. As in Lemma 1, if $\rho$ is a
$p$-dimensional representation of $G$, then either

$$\text{Res}_{(G/B)}^G (c.(p)) = (1 + [i] \beta')^p \equiv 1 + i \beta'^p$$

or

$$\text{Res}_{(G/B)}^G (c.(p)) = \prod_{i=0}^{p-1} (1 + [i] \beta') \equiv 1 - \beta'^{p-1}.$$ 

Also, we have that

$$\text{Res}_{(G/B)}^G \text{Cor}^G_M (\gamma^{p-1} \beta^2) = \text{Res}_{(G/B)}^M \left( \sum_{i=0}^{p-1} (\gamma + BP [i \epsilon] \beta)^{p-1} \left( \beta + BP [i p^{\alpha-3}] \gamma \right)^2 \right) \equiv -\beta'^{p+1}.$$ 

Our original proofs of these results involved calculation with $BP^*(BN)$, which has been determined by Tezuka-Yagita [10], and with the integral cohomology of the non-abelian maximal subgroups of $G$, determined by Leary [8]. Using these methods we obtain more information concerning $BP^*(BG)$ and $H^*(G; \mathbb{Z})$, which we intend to publish later.

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