THE INDEPENDENCE NUMBER OF NON-UNIFORM UNCROWDED HYPERGRAPHS AND AN ANTI-RAMSEY TYPE RESULT

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ABSTRACT. We prove the following: Fix an integer \( k \geq 2 \), and let \( T \) be a real number with \( T \geq 1.5 \). Let \( H = (V, E_1 \cup E_2 \cup \cdots \cup E_k) \) be a non-uniform hypergraph with the vertex set \( V \) and the set \( E_i \) of edges of size \( i = 2, \ldots, k \). Suppose that \( H \) has no 2-cycles (regardless of sizes of edges), and neither contains 3-cycles nor 4-cycles consisting of 2-element edges. If the average degrees \( t_i^{-1} := \frac{|E_i|}{|V|} \) satisfy that \( t_i^{-1} \leq (\ln T)^{\frac{i-1}{i}} \) for \( i = 2, \ldots, k \), then there exists a constant \( C_k > 0 \), depending only on \( k \), such that \( \alpha(H) \geq \frac{C_k (\ln T)^{\frac{k-1}{k}}}{k} \), where \( \alpha(H) \) denotes the independence number of \( H \). This extends results of Ajtai, Komlós, Pintz, Spencer and Szemerédi [J. Comb. Theory Ser. A 32, 1982, 321–335] and Duke, Rödl and the second author [Random Struct. Algorithms 6, 1995, 209–212] for uniform hypergraphs.

As an application, we consider an anti-Ramsey type problem on non-uniform hypergraphs. Let \( H = (V; 2, \ldots, \ell) \) be the hypergraph on the \( n \)-vertex set \( V \) in which, for \( s = 2, \ldots, \ell \), each \( s \)-subset of \( V \) is a hyperedge of \( H \). Let \( \Delta \) be an edge-coloring of \( H \) satisfying the following: (a) two hyperedges sharing a vertex have different colors; (b) two hyperedges with distinct size have different colors; (c) a color used for a hyperedge of size \( s \) appears at most \( u_s \) times. For such a coloring \( \Delta \), let \( f_\Delta(n; u_2, \ldots, u_\ell) \) be the maximum size of a subset \( U \) of \( V \) such that each hyperedge of \( H[U] \) has a distinct color, and let \( f(n; u_2, \ldots, u_\ell) := \min_\Delta f_\Delta(n; u_2, \ldots, u_\ell) \). We determine \( f(n; u_2, \ldots, u_\ell) \) up to a multiplicative logarithm factor, which is a non-uniform version of a result for edge-colorings of graphs by Babai [Graphs Comb. 1, 1985, 23–28], and for uniform hypergraphs by Alon, Rödl and the second author [Coll. Math. Soc. János Bolyai, 60. Sets, Graphs and Numbers, 1991, 9–22] and by Rödl, Wyosocka and the second author [J. Comb. Theory Ser. A 74, 1996, 209–248].

1. INTRODUCTION

Let \( H = (V, E) \) be a hypergraph with its vertex set \( V \) and its edge set \( E \). Let \( E_i \subset E \) be the set of all \( i \)-element edges in \( H \). For a vertex \( v \in V \), let \( d_i(v) \) denote the number of \( i \)-element edges \( E \in E_i \) containing \( v \). A hypergraph \( H = (V, E) \) is called \( k \)-uniform if \( E = E_k \). A subset \( V' \subset V \) is called independent if for no edge \( E \in E \) it is \( E \subset V' \). The independence number \( \alpha(H) \) of \( H \) is the maximum size of an independent set of \( H \). For a subset \( V^* \subset V \) of the vertex set, let \( H[V^*] \) be the subhypergraph of \( H \) induced on \( V^* \).

The independence number has been well-studied for uniform hypergraphs, however, for non-uniform hypergraphs it was not that studied correspondingly. The goal of this paper is to extend known results on the independence number of a uniform hypergraph to a non-uniform hypergraph. Turán’s theorems by Turán [8] and Spencer [9] imply the following theorem about the independence number of a \( k \)-uniform hypergraph \( H \).

Theorem 1 (Turán [8] and Spencer [9]). Let \( k \geq 2 \) be an integer. Let \( H = (V, E_k) \) be a \( k \)-uniform hypergraph with \( N \) vertices and average degree \( t^{k-1} := k|E_k|/N \), where \( t \geq 1 \). Then,
\[
\alpha(H) \geq \frac{k - 1}{k} \frac{N}{t}.
\]
Later, a better lower bound on the independence number of a $k$-uniform hypergraph was obtained if the hypergraph does not contain cycles of small lengths. We introduce the definition of cycles of a given length.

**Definition 2.** Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A $j$-cycle in $\mathcal{H}$ is a family of pairwise distinct edges $E_1, \ldots, E_j \in \mathcal{E}$ such that the following hold:

- $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \ldots, j - 1$ and $E_j \cap E_1 \neq \emptyset$.
- There are pairwise distinct vertices $v_1, \ldots, v_j$ such that $v_i \in E_i \cap E_{i+1}$ for $i = 1, \ldots, j - 1$ and $v_j \in E_j \cap E_1$.

**Definition 3.** A hypergraph $\mathcal{H}$ is called uncrowded if it does not contain any 2-, 3- or 4-cycles. A hypergraph $\mathcal{H}$ is called linear if it does not contain any 2-cycles.

Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] obtained a lower bound on the independence number of a $k$-uniform uncrowded hypergraph as follows. Later, Bertram-Kretzberg and Lefmann [4] and Fundia [6] provided a deterministic polynomial time algorithm.

**Theorem 4** (Ajtai, Komlós, Pintz, Spencer and Szemerédi [1]). Let $k \geq 2$ be a fixed integer. Let $t$ and $N$ satisfy $t > t_0(k)$ and $N > N_0(k, t)$. Let $\mathcal{H} = (V, \mathcal{E}_k)$ be an uncrowded $k$-uniform hypergraph on $N$ vertices with average degree $t^{k-1} := k|\mathcal{E}_k|/N$. Then, there exists a constant $C_k > 0$ such that

$$\alpha(\mathcal{H}) \geq C_k \frac{N}{t} \left( \ln t \right)^{\frac{k}{k-1}}. \quad (2)$$

We remark that in [1] the constant $C_k$ is bounded from below by $C_k \geq \frac{0.08}{k^{1/2}}$.

**Theorem 5** (Duke, Lefmann, and Rödl [5]). Let $k \geq 3$ be a fixed integer. Let $t$ and $N$ satisfy $t > t_0(k)$ and $N > N_0(k, t)$. Let $\mathcal{H} = (V, \mathcal{E}_k)$ be a linear $k$-uniform hypergraph on $N$ vertices with average degree $t^{k-1} := k|\mathcal{E}_k|/N$. Then, there exists a constant $C'_k > 0$ such that

$$\alpha(\mathcal{H}) \geq C'_k \frac{N}{t} \left( \ln t \right)^{\frac{1}{k-1}}. \quad (3)$$

In this paper we extend Theorems 4 and 5 to non-uniform hypergraphs as follows.

**Theorem 6.** Let $k \geq 2$ be a fixed integer. Let $T$ be a real number with $T \geq 1.5$ and $N$ be a positive integer. Let $\mathcal{H} = (V, \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_k)$ be a linear hypergraph on $N$ vertices such that there are no 3-cycles and 4-cycles consisting of 2-element edges. Let the average degrees $t_i^{i-1} := i|\mathcal{E}_i|/N$ satisfy that for $i = 2, \ldots, k$

$$t_i^{i-1} \leq T^{i-1} \left( \ln T \right)^{\frac{k}{k+1}}. \quad (4)$$

Then, there exists a constant $C_k > 0$ such that

$$\alpha(\mathcal{H}) \geq C_k \frac{N}{T} \left( \ln T \right)^{\frac{1}{k-1}}. \quad (5)$$

**Remark 7.** For the range of $0 < T < 1.5$ in Theorem 6, a simple greedy algorithm gives that there exists a constant $C'_k > 0$, depending only on $k$, such that $\alpha(\mathcal{H}) \geq C'_k N$.

As an application of the main theorem (Theorem 6), we consider an anti-Ramsey type problem of a non-uniform hypergraph. Let $\mathcal{H} = \mathcal{H}(n; 2, \ldots, \ell)$ be the hypergraph on the vertex set $V$, $|V| = n$, in which, for $s = 2, \ldots, \ell$, each $s$-subset of $V$ is a hyperedge of $\mathcal{H}$. Suppose that $\Delta$ is an edge-coloring of $\mathcal{H}$ satisfying the following conditions:

(a) Two hyperedges sharing a vertex have different colors. In other words, each color class is a matching.

(b) Two hyperedges with distinct size have different colors.
(c) The coloring is \((u_2, \ldots, u_\ell)\)-bounded, that is, a color used for a hyperedge of size \(s\) appears at most \(u_s\) times.

For such a coloring \(\Delta\), let \(f_\Delta(n; u_2, \ldots, u_\ell)\) be the maximum size of a vertex set \(U\) in \(V\) such that the subhypergraph \(\mathcal{H}[U]\) of \(\mathcal{H}\) induced on \(U\) is totally multicolored, which means that each hyperedge of \(\mathcal{H}[U]\) has a distinct color. Let
\[
f(n; u_2, \ldots, u_\ell) := \min_{\Delta} f_\Delta(n; u_2, \ldots, u_\ell),
\]
where we minimize it over all edge-colorings \(\Delta\) of \(\mathcal{H}\) satisfying (a)–(c).

We will show the following which is a non-uniform version of a result for edge-colorings of graphs in [7].

**Theorem 8.**

(a) Suppose that
\[
\max_{2 \leq i \leq \ell} \left[ \left( \frac{n^{i-1} u_i}{u_s} \right)^{\frac{1}{2s-1}} \right] = \left( \frac{n^{s-1} u_s}{u_s} \right)^{\frac{1}{2s-1}}.
\]
Then, there exist positive constants \(c_1\) and \(c_2\), depending only on \(\ell\), such that for every sufficiently large \(n\),
\[
c_1 \left( \frac{n^s}{u_s} \right)^{\frac{1}{2s-1}} \leq f(n; u_2, \ldots, u_\ell) \leq c_2 \left( \frac{n^s \ln n}{u_s} \right)^{\frac{1}{2s-1}}.
\]
(b) Suppose that
\[
\max_{2 \leq i \leq \ell} \left[ \left( \frac{n^{i-1} u_i}{\ln n} \right)^{\frac{1}{2s-1}} \right] = \left( \frac{n^{s-1} u_s}{\ln n} \right)^{\frac{1}{2s-1}}
\]
and
\[
u_s \geq n^{1/2+\varepsilon}\quad \text{for some absolute constant } \varepsilon > 0.
\]
Then, there exist positive constants \(c_3 = c_3(\ell, \varepsilon)\) and \(c_4 = c_4(\ell)\) such that for every sufficiently large \(n\),
\[
c_3 \left( \frac{n^s \ln n}{u_s} \right)^{\frac{1}{2s-1}} \leq f(n; u_2, \ldots, u_\ell) \leq c_4 \left( \frac{n^s \ln n}{u_s} \right)^{\frac{1}{2s-1}}.
\]

**Remark 9.** Statements (a) and (b) in Theorem 8 have different assumptions. We remark that under assumption (8), our argument for the lower bound in (7) gives
\[
c_3 \left( \frac{n^s}{u_s} \right)^{\frac{1}{2s-1}} \left( \ln n \right)^{\frac{1}{2s-1}} \leq f(n; u_2, \ldots, u_\ell),
\]
which is less than the lower bound in (10), while under assumption (9), our argument for the lower bound in (10) gives
\[
c_3 \left( \frac{n^s}{u_s} \right)^{\frac{1}{2s-1}} \max \left[ 1, \left( \frac{u_s^2 n}{\ln n} \right)^{\frac{1}{2s-1}} \right] \leq f(n; u_2, \ldots, u_\ell),
\]
which is bigger than the lower bound in (7) if \(u_s \gg \sqrt{n}\).

If we assume \(u_2 = \ldots = u_\ell = n\), Theorem 8 (b) immediately implies the following corollary:

**Corollary 10.** There exist positive constants \(c_3\) and \(c_4\), depending only on \(\ell\), such that for every sufficiently large \(n\),
\[
c_3 \left( n \ln n \right)^{\frac{1}{\ell-1}} \leq f(n; u_2, \ldots, n) \leq c_4 \left( n \ln n \right)^{\frac{1}{\ell-1}}.
\]
For the special case of $\ell = 2$, i.e., edge-colorings of graphs, the upper bound in [11] was shown in 1985 by Babai [3], and the lower bound was proved in 1991 by Alon, Rödl and the second author [2].

The organization of this paper is as follows. Section 2 contains an extension of Theorem 1 to non-uniform hypergraphs. In Section 3, we show an extension of Theorem 4 to non-uniform hypergraphs. Section 4 contains our proof of an extension of Theorem 5 to non-uniform hypergraphs. In Section 5, we show Theorem 6 in full. Section 6 contains the proof of Theorem 8.

### 2. Arbitrary non-uniform hypergraphs

The next theorem by Spencer [9] provides a lower bound on the independence number $\alpha(H)$ of a non-uniform hypergraph $H$.

**Theorem 11** (Spencer [9]). Let $H = (V, E_2 \cup \cdots \cup E_k)$ be a hypergraph with $N$ vertices and average degree $t_i^{-1} = |E_i|/N$ for the $i$-element edges, where $i = 2, \ldots, k$. Let $T := \max \{t_i \mid 2 \leq i \leq k\} \geq 1/2$. Then,

$$\alpha(H) \geq \frac{1}{4} \frac{N}{T}.$$

**Proof.** For convenience we provide the sketch of a proof. Choose each vertex uniformly at random and independently with probability $p := 1/(2T) \leq 1$. Let $V^*$ be the random set of chosen vertices and let $E_i^*$, $i = 2, \ldots, k$, be the sets of $i$-element edges in the induced random subhypergraph $H[V^*]$. Then we have in expectation

$$\mathbb{E} \left[ |V^*| - \sum_{i=2}^{k} |E_i^*| \right] = \mathbb{E}[|V^*|] - \sum_{i=2}^{k} \mathbb{E}[|E_i^*|] = pN - \sum_{i=2}^{k} \frac{p^i N t_i^{i-1}}{i} \geq pN - \frac{1}{2T} \sum_{i=2}^{k} \frac{1}{i2^i} \frac{N}{T} \geq \frac{N}{4T}.$$

Thus there exists a subset $V^* \subseteq V$ such that

$$|V^*| - \sum_{i=2}^{k} |E_i^*| \geq \frac{N}{4T}.$$

By deleting one vertex from each edge $E \in E_i^*$, $i = 2, \ldots, k$, we destroy all edges in $H[V^*]$, and hence we obtain an independent set $V^*$ of $H$ with $|V^*| \geq N/(4T)$.

### 3. Uncrowded non-uniform hypergraphs

In this section we show the following lemma which gives a lower bound on the independence number of an uncrowded non-uniform hypergraph.

**Lemma 12.** Let $k \geq 2$ be a fixed integer. Let $T$ and $N$ satisfy $T > T_0(k)$ and $N > N_0(k, T)$. Let $H = (V, E_2 \cup \cdots \cup E_k)$ be an uncrowded hypergraph on $N$ vertices with average degrees $t_i^{-1} := |E_i|/N$ such that for $i = 2, \ldots, k$

$$t_i^{i-1} \leq c_i T^{i-1} (\ln T)^{(k-1)/k-1},$$

where $c_i$ are constants satisfying $0 < c_i < \frac{1}{10c_0} \left(\frac{k-1}{i-1}\right)^{3(4k-1)/k-1}$. Then, there exists a constant $C_k > 0$ such that

$$\alpha(H) \geq C_k \frac{N}{T} (\ln T)^{(k-1)/k-1}. \quad (12)$$

In order to prove Lemma 12, we will use the following lemma which was used to show Theorem 4 in Ajtai, Komlós, Pintz, Spencer and Szemerédi [11]. Since the lemma in [11] has been written densely (with several typos), we provide a slightly modified statement, based on Lemma 6 in Fundia [6].
Lemma 13 (Ajtai, Komlós, Pintz, Spencer and Szemerédi [1]). Let $k \geq 2$ be a fixed integer. Let $T$, $N$, and $s$ be such that

$$\tag{13} T \geq T_0(k), \quad N \geq N_0(k, T), \quad 0 \leq s \leq 0.01 \ln T.$$

Let $n$ and $t$ be integers satisfying

$$\frac{1}{2} N \leq \frac{N}{e^s} \quad \text{and} \quad \frac{T}{e^s} \leq t \leq 2 \frac{T}{e^s}. \tag{14}$$

Let $H = (V, E_2 \cup \cdots \cup E_k)$ be an uncrowded hypergraph with $n$ vertices satisfying the following: for each vertex $v \in V$ and $2 \leq i \leq k$, the numbers $d_i(v)$ of $i$-element edges $E \in E_i$ containing vertex $v$ fulfill

$$d_i(v) \leq \binom{k-1}{i-1} s^k t^{i-1}. \tag{15}$$

Then, there exists an independent set $I \subseteq V$ and a vertex set $V^* \subset V$ with $I \cap V^* = \emptyset$ satisfying the following:

(i) $\alpha(H) \geq |I| + \alpha(H^*)$ where $H^* = H[V^*]$

(ii) $|I| \geq 0.09 \frac{n}{e^s} w_s n \frac{1}{t}, \quad \text{where} \quad w_s := (s+1)\frac{1}{t} - s \frac{1}{t}$

(iii) $\frac{n}{e^s} (1 - \varepsilon) \leq |V^*| \leq \frac{n}{e^s} (1 + \varepsilon)$ where $\varepsilon = 1/(10^6 \ln T)$

(iv) for every vertex $v \in V^*$ and $i = 2, \ldots, k$ it is

$$d_i^*(v) \leq \binom{k-1}{i-1} (s+1)^{k-i} \left( t \frac{1+\varepsilon}{e^s} \right)^{i-1},$$

where $d_i^*(v)$ is the number of $i$-element edges in $H^*$ containing $v$.

In order to apply Lemma 13, we are going to obtain an induced subhypergraph of $H$ in Lemma 12 that satisfies the assumptions of Lemma 13.

Lemma 14. Let $k \geq 2$ be a fixed integer. Let $T$, $N$ and $s$ be positive integers satisfying

$$\tag{13} T \geq T_0(k), \quad N \geq N_0(k, T), \quad s = 0.001 \ln T.$$

Let $H = (V, E_2 \cup \cdots \cup E_k)$ be a hypergraph with $N$ vertices and average degrees $i_{i-1} := i |E_i| / N$ satisfying that

$$t_{i-1}^i \leq c_i T^{i-1} (\ln T)^{\frac{k-i}{i-1}},$$

where the constants $c_i > 0$ are such that $c_i < \frac{1}{10^6} \frac{(k-1)}{(i-1)} 10^{-3 \frac{k-1}{i-1}}$.

Then, $H^*$ contains a subhypergraph $H^* = (V^*, E_2^* \cup \cdots \cup E_k^*)$ induced on $V^*$ with $|V^*| = n$ such that the following hold.

(a) $\frac{3 N}{4 e^s} \leq n \leq \frac{N}{e^s}$, and

(b) for each vertex $v \in V^*$,

$$d_i^*(v) \leq \binom{k-1}{i-1} s^k t^i \left( \frac{T}{e^s} \right)^{i-1}, \tag{15}$$

where $d_i^*(v)$ is the number of $i$-element edges in $H^*$ containing $v$.

Proof. Let $V'$ be a random set obtained by choosing each vertex in $H$ independently with probability $p := 1/e^s$. For its expected size we have $E[|V'|] = Np = N/e^s$, thus by Chernoff’s inequality it is

$$\Pr(|V'| \leq E[|V'|] - u) \leq e^{-u^2/N}.$$
for every real $u \geq 0$. Then, with $s = 10^{-3} \ln T$ and a sufficiently large $N \geq N_0(k, T)$ we have

$$\Pr \left( |V'| \leq \mathbb{E}[|V'|] - \frac{N}{8\varepsilon^s} \right) \leq e^{-\frac{N^2/64\varepsilon^2}{k}} = e^{-N/(64\varepsilon^2)} < \frac{1}{k}. \quad (16)$$

Let $\mathcal{H}' = (V', \mathcal{E}_2' \cup \cdots \cup \mathcal{E}_k')$ be the subhypergraph of $\mathcal{H}$ induced on $V'$. For $i = 2, \ldots, k$, we have

$$\mathbb{E}[|\mathcal{E}_i'|] = p_i^j t_i^{-1} N/i \leq e^{-s_i c_i T_i^{-1} \ln T} \frac{N}{k/i}. \quad (17)$$

By Markov's inequality we infer

$$\Pr \left( |\mathcal{E}_i'| > k \mathbb{E}[|\mathcal{E}_i'|] \right) \leq \frac{1}{k}. \quad (18)$$

From now on, we are going to obtain an induced subhypergraph $\mathcal{H}'$ of $\mathcal{H}'$ such that the maximum degree is bounded. From (18), we infer

$$\mathbb{E}[d_i'(v)] \leq k c_i e^{-s_i T_i^{-1} \ln T} \frac{N}{k/i}. \quad (19)$$

Let $Y_i$ be the number of vertices $v \in V'$ of high degree, i.e., such that $d_i'(v) > 7k \cdot k c_i e^{-s_i T_i^{-1} \ln T} \frac{N}{k/i}$. Thus, the total number of vertices of high degree is bounded as

$$\sum_{i=2}^{k} Y_i \leq \sum_{i=2}^{k} \frac{|V'|}{7k} \leq \frac{|V'|}{7}. \quad (20)$$

By deleting these vertices of high degree from $V'$, we obtain an induced subhypergraph $\mathcal{H}' = (V^*, \mathcal{E}_2^* \cup \cdots \cup \mathcal{E}_k^*)$ of $\mathcal{H}'$ with $|V^*| \geq (6/7)|V'|$ such that with (18) we infer

$$|V^*| \geq \frac{6}{7} |V'| \geq \frac{3 N}{4 \varepsilon^s} \quad (21)$$

$$d_i^*(v) \leq 7k^2 c_i e^{-s_i T_i^{-1} \ln T} \frac{N}{k/i} \quad (19)$$

$$d_i^*(v) \leq 8k^2 c_i \left( \frac{T_i}{\varepsilon^s} \right)^{i-1} \frac{N}{k/i}, \quad (18)$$

where $d_i^*(v)$ is the number of $i$-element edges in $\mathcal{H}^*$ containing $v$. Recalling $s := 10^{-3} \ln T$ and the upper bound on $c_i$, we have

$$d_i^*(v) \leq \left( \frac{k - 1}{i - 1} \right)^{k-i} \left( \frac{T_i}{\varepsilon^s} \right)^{i-1},$$

which proves condition (b) in (19). We can obtain the condition $|V^*| \leq N/e^s$ by possibly deleting some more vertices from $V^*$. This together with (21) implies condition (a). \hfill \square

Now we are ready to prove Lemma 12.

**Proof of Lemma 12**: We apply Lemma 14 with $s = 10^{-3} \ln T$ to $\mathcal{H}$ on $N$ vertices. Then we obtain a subhypergraph $\mathcal{H}' = (V^*, \mathcal{E}_2^* \cup \cdots \cup \mathcal{E}_k^*)$ induced on $V^*$ with $|V^*| = n$ such that the following hold:
(a) \(3 \frac{N}{4 e^s} \leq n \leq \frac{N}{e^s}\), and
(b) for each vertex \(v \in V^s\), it is
\[
d^i_s(v) \leq \left( \frac{k-1}{i-1} \right) s^{r-s} \left( \frac{T}{e^s} \right)^{i-1},
\]
where \(d^i_s(v)\) is the number of \(i\)-element edges in \(H^s\) containing \(v\).

Set \(n_s = n\) and \(t_s = T/e^s\) and \(H_s = H^s = (V_s, \mathcal{E}_{2,s} \cup \cdots \cup \mathcal{E}_{k,s})\). We apply Lemma 13 with \(\varepsilon = 1/(10^6 \ln T)\) to \(H_s\), iteratively. Let \(n_r\) and \(t_r\) be such that
\[
\frac{3 N}{4 e^s} (1 - \varepsilon)^{r-s} \leq n_r \leq \frac{N}{e^r}
\]
and
\[
t_r = \frac{T}{e^r} (1 + \varepsilon)^{r-s}.
\]
For each \(r = s, \ldots, 10^{-2} \ln T\), we obtain an independent set \(I_r \subseteq V_r\) and a vertex set \(V_{r+1} \subseteq V_r\) with \(I_r \cap V_{r+1} = \emptyset\) satisfying the following:

(i) \(\alpha(H_r) \geq |I_r| + \alpha(H_{r+1})\) where \(H_{r+1} = H_r[V_{r+1}]\)
(ii) \(|I_r| \geq \frac{0.99}{e} w_r t_r\) where \(w_r := (r+1)\frac{1}{e^{r-1}} - r\)
(iii) \(\frac{n_r}{e^s} (1 - \varepsilon) \leq |V_{r+1}| \leq \frac{n_r}{e^r}\) recalling \(\varepsilon = 1/(10^6 \ln T)\)
(iv) for every vertex \(v \in V_{r+1}\) and \(2 \leq i \leq k\),
\[
d^i_{r+1}(v) \leq \left( \frac{k-1}{i-1} \right) s^{r-s} \left( \frac{T}{e^r} \right)^{i-1},
\]
where \(d^i_{r+1}(v)\) is the number of \(i\)-element edges in \(H_{r+1}\) containing \(v\).

We first check how many times we can iteratively apply Lemma 13 to \(H_r\). Observe that we can apply Lemma 13 as far as inequalities (13) and (14) are satisfied.

- The inequality (13) is \(s + r \leq 0.01(\ln T)\).
- From (22), the inequalities about \(n\) in (14) are satisfied if \(\frac{1}{2} \leq \frac{3}{4} (1-\varepsilon)^{r-s}\). Using \(1-p \geq e^{-2p}\) for \(0 \leq p \leq 0.5\), one can check that it suffices to have \(r \leq 10^5 \ln T\).
- From (23), the inequalities about \(t\) in (14) are satisfied if \((1 + \varepsilon)^{r-s} \leq 2\). One can check that it suffices to have \(r \leq 10^5 \ln T\).

Therefore, we can apply Lemma 13 for \(r + s \leq 0.01 \ln T\).

Now we estimate the size of an independent set in \(H\) obtained by the above procedure. Notice that by using \((1 + \varepsilon)^n \geq 1 + \varepsilon n\) and \(1 + \varepsilon \leq e^\varepsilon\) and \(r \leq 10^{-2} \ln T\) and \(\varepsilon = 10^{-6} / \ln T\) we have
\[
\frac{n_r}{t_r} \geq \frac{(3/4)N(1-\varepsilon)^{r-s}/e^r}{T(1+\varepsilon)^{r-s}/e^r} \geq \frac{3 N}{4 T} \frac{(1-\varepsilon)^r}{(1+\varepsilon)^r} \geq \frac{3 N (1-\varepsilon)}{4 T (1+\varepsilon)} \geq \frac{3 N 1 - \varepsilon r}{4 T e^{\varepsilon r}} \geq \frac{3 N 1 - 10^{-8}}{4 T e^{10^{-8}}} \geq 0.74 \frac{N}{T}.
\]
and regard it as a vertex in $E$, we obtain an independent set $I = I_s \cup \cdots \cup I_{0.01 \ln T}$ in $\mathcal{H}$ with

$$
\alpha(\mathcal{H}) \geq |I| = \sum_{r=s}^{0.01 \ln T} |I_r| \geq \frac{0.99}{e} 0.74 N \frac{0.01 \ln T}{T} \sum_{r=s} w_r
$$

$$
\geq \frac{0.73}{e} N \frac{0.01 \ln T}{T} \sum_{r=s} \left((r+1)^{\frac{1}{r-1}} - r^{\frac{1}{r-1}}\right)
$$

$$
\geq \frac{0.73}{e} N \frac{\ln T}{T} \frac{1}{r-1} \left(0.01 \frac{1}{r-1} - 0.001 \frac{1}{r-1}\right),
$$

which gives the lower bound $|H|$ in Lemma [12] \hfill \square

4. A Weak Version of Theorem [6]

Now we show a (seemingly) weaker version of Theorem [6] in which $T$ and $N$ are large and the assumptions of the upper bounds on $t_i^{i-1}$ are a bit different.

**Lemma 15.** Let $k \geq 2$ be a fixed integer. Let $T$ and $N$ satisfy $T > T_0(k)$ and $N > N_0(k,T)$. Let $\mathcal{H} = (V, E_2 \cup \cdots \cup E_k)$ be a linear hypergraph on $N$ vertices such that there are no 3-cycles and 4-cycles consisting of 2-element edges. Let the average degrees $t_i^{i-1} := i|E_i|/N$ satisfy for $i = 2, \ldots, k$

$$
t_i^{i-1} \leq c_i T^{i-1} (\ln T)^{\frac{k-i}{k-1}},
$$

where $c_i$ are constants satisfying $0 < c_i < \frac{1}{2^{2k^2} (i-1) 10^{-3} k^{k-i-1}}$. Then, there exists a constant $C_k > 0$ such that

$$
\alpha(\mathcal{H}) \geq C_k \frac{N}{T} (\ln T)^{\frac{k}{k-1}}.
$$

**Proof.** Let $d_i(v)$ denote the number of $i$-element edges in $\mathcal{H}$ containing vertex $v \in V$. We delete all vertices $v \in V$ with $d_i(v) > k c_i T^{i-1} (\ln T)^{\frac{k-i}{k-1}}$ for some $i = 2, \ldots, k$. Let $V^*$ be the set of remaining vertices, hence we have $|V^*| \geq N/k$. Then, the subhypergraph $\mathcal{H}^* = (V^*, E_2^* \cup \cdots \cup E_k^*)$ of $\mathcal{H}$ induced on $V^*$ satisfies that for each vertex $v \in V^*$

$$
d_i^*(v) \leq k c_i T^{i-1} (\ln T)^{\frac{k-i}{k-1}},
$$

where $d_i^*(v)$ is the number of $i$-element edges in $\mathcal{H}^*$ containing vertex $v$.

Now we estimate the numbers of 3-cycles and of 4-cycles not containing a 3-cycle in $\mathcal{H}$. First we consider the number of 3-cycles in $\mathcal{H}$. Let $C^*(g,h,i)$ denote the number of 3-cycles $\{E_1, E_2, E_3\}$ in $\mathcal{H}$ such that $|E_1| = g$, $|E_2| = h$, and $|E_3| = i$. To estimate $C^*(g,h,i)$, we fix a vertex $v \in V^*$ and regard it as a vertex in $E_1 \cap E_2$. There are at most $k c_g T^{g-1} (\ln T)^{\frac{k-g}{k-1}}$ edges in $E_2^*$ containing vertex $v$. Let $E_1$ be one of these edges. Similarly, there are at most $k c_h T^{h-1} (\ln T)^{\frac{k-h}{k-1}}$ edges in $E_2^*$ containing vertex $v$. Let $E_2$ be one of these edges. Moreover, we fix one vertex $w \in E_1$ with $w \neq v$ and another vertex $x \in E_2$ with $x \neq v$ in at most $(k-1)^2$ ways, and consider $v, x \in E_3$. Since $\mathcal{H}^*$ is linear, there is at most one $i$-element edge $E_3 \in E_i^*$ containing both vertices $w$ and $x$. Hence, for each $2 \leq g \leq h \leq i \leq k$,

$$
C^*(g,h,i) \leq N \cdot k c_g T^{g-1} (\ln T)^{\frac{k-g}{k-1}} \cdot k c_h T^{h-1} (\ln T)^{\frac{k-h}{k-1}} \cdot (k-1)^2
$$

$$
< N k^4 c_g c_h T^{g+h-2} (\ln T)^2.
$$

Next, we consider the number of 4-cycles in $\mathcal{H}$. Let $C^*(g,h,i,j)$ be the number of 4-cycles $\{E_1, E_2, E_3, E_4\}$ in $\mathcal{H}$ such that $|E_1| = g$, $|E_2| = h$, $|E_3| = i$, and $|E_4| = j$ and any three of
\( \{E_1, E_2, E_3, E_4\} \) do not form a 3-cycle. With an argument similar to the argument to obtain (27), one can show that for each \( 2 \leq g \leq h \leq i \leq j \leq k, \)

\[
C^*(g, h, i, j) < 3nk^6c_g c_h c_i T^{g + h + i - 3}(\ln T)^3.
\]

(28)

Note that the factor 3 in (28) arises due to the possible arrangements of the edges of possibly different sizes.

Now we choose each vertex in \( V^* \) independently with probability \( p = T^{-1+\varepsilon} \) for some constant \( \varepsilon > 0 \). Let \( V^{**} \subseteq V^* \) be the set of chosen vertices, and let \( \mathcal{H}^{**} = (V^{**}, \mathcal{E}_2^{**} \cup \cdots \cup \mathcal{E}_k^{**}) \) be the subhypergraph of \( \mathcal{H} \) induced on \( V^{**} \). Note that

\[
\mathbb{E}[|V^{**}|] = p|V^*| \geq \frac{N}{k} T^{-1+\varepsilon}.
\]

(29)

Let \( C^{**}(g, i, j) \) and \( C^{**}(g, h, i, j) \) be the numbers of 3-cycles and 4-cycles (not containing a 3-cycle) in \( \mathcal{H}^{**} \), respectively. Since a 3-cycle covers exactly \( (g + h + i - 3) \) vertices in a linear hypergraph, inequality (27) yields that for \( i \geq 3, \)

\[
\mathbb{E}[C^{**}(g, h, i)] < p^{g + h + i - 3} Nk^4 c_g c_h c_i T^{g + h - 2}(\ln T)^2 \leq k^4 c_g c_h NT^{-(i-1) + \varepsilon(g+2+i-3)} (\ln T)^2
\]

\[
< k^4 c_g c_h NT^{-2+3\varepsilon}(\ln T)^2.
\]

Moreover, since there are no 3-cycles with 3 edges, each of size 2, in \( \mathcal{H} \), we have \( \mathbb{E}[C^{**}(2, 2, 2)] = 0. \) Hence, we infer

\[
\sum_{2 \leq g \leq h \leq i \leq k} \mathbb{E}[C^{**}(g, h, i)] < k^7 \max_{2 \leq g \leq k} \{c_g^2\} \cdot NT^{-2+3\varepsilon}(\ln T)^2.
\]

(30)

Similarly, inequality (28) implies that for \( j \geq 3, \)

\[
\mathbb{E}[C^{**}(g, h, i, j)] < p^{g + h + i - 3} Nk^6 c_g c_h c_i T^{g + h + i - 3}(\ln T)^3
\]

\[
= 3k^6 c_g c_h c_i NT^{-(i-1) + \varepsilon(g+2+i-3)} (\ln T)^3
\]

\[
\leq 3k^6 \max_{2 \leq g \leq k} \{c_g^3\} \cdot NT^{-2+4\varepsilon}(\ln T)^3.
\]

Also since there are no 4-cycles with 4 edges, each of size 2, in \( \mathcal{H} \), we have \( \mathbb{E}[C^{**}(2, 2, 2, 2)] = 0. \) Thus, we infer

\[
\sum_{2 \leq g \leq h \leq i \leq j \leq k} \mathbb{E}[C^{**}(g, h, i, j)] < 3k^{10} \max_{2 \leq g \leq k} \{c_g^3\} \cdot NT^{-2+4\varepsilon}(\ln T)^3.
\]

(31)

By (26), we have for \( i = 2, \ldots, k \) that

\[
\mathbb{E}[|\mathcal{E}_i^{**}|] = p^i|\mathcal{E}_i^*| \leq p^i k c_i T^{i-1}(\ln T)^{k-i} |V^*|^{k-i} = k c_i T^{-1+\varepsilon} (\ln T)^{k-i} |V^*|^{k-i}.
\]

(32)

Chernoff’s and Markov’s inequalities with (29)–(32) imply that there exists a subhypergraph \( \mathcal{H}^{**} = (V^{**}, \mathcal{E}_2^{**} \cup \cdots \cup \mathcal{E}_k^{**}) \) induced on \( V^{**} \) such that

\[
|V^{**}| \geq \frac{1}{2k} NT^{-1+\varepsilon}
\]

\[
|\mathcal{E}_i^{**}| \leq (k + 2) k c_i T^{-1+\varepsilon} (\ln T)^{k-i} |V^*|^{k-i}
\]

\[
\sum_{2 \leq g \leq h \leq i \leq k} C^{**}(g, h, i) \leq (k + 2) k^7 \max_{2 \leq g \leq k} \{c_g^2\} \cdot NT^{-2+3\varepsilon}(\ln T)^2
\]

\[
\sum_{2 \leq g \leq h \leq i \leq j \leq k} C^{**}(g, h, i, j) \leq (k + 2) 3k^{10} \max_{2 \leq g \leq k} \{c_g^3\} \cdot NT^{-2+4\varepsilon}(\ln T)^3.
\]
For $0 < \varepsilon < 1/(4k-1)$ and $T > T_0(k)$, the number of 3-cycles and the number of 4-cycles (not containing a 3-cycle) are much less than $|V^*|$.

Let $\mathcal{H}^{**} = (V^{**}, E_2^{**} \cup \cdots \cup E_k^{**})$ be the subhypergraph obtained from $\mathcal{H}^{**}$ by removing one vertex from each 3-cycle and 4-cycle (not containing a 3-cycle). With $\varepsilon := 1/(4k)$, we have

$$|V^{**}| \geq \frac{1}{4k} NT^{-1+\varepsilon}. \quad (34)$$

We infer that the average degrees $(i_i^{**})^{i-1}$ of $\mathcal{H}^{**}$ satisfy $i = 2, \ldots, k$:

$$(t_i^{**})^{i-1} = \frac{i|iE_i^{**}|}{|V^{**}|} \leq \frac{i|E_i^{**}|}{|V^{**}|} \leq (k+2)kc_iT^{-1+\varepsilon} (\ln T)^{\frac{k-1}{k-1}} |V^{**}|$$

$$\leq (k+2)kc_iT^{-1+\varepsilon} (\ln T)^{\frac{k-i}{k-i}}$$

$$\leq 4k^2(k+2)c_i T^{\varepsilon(i-1)} (\ln T)^{\frac{k-i}{k-i}}$$

$$= \frac{4k^2(k+2)}{\varepsilon} c_i(T^\varepsilon)^{i-1} (\ln(T^\varepsilon))^\frac{k-i}{k-i}$$

$$\leq 32k^4 c_i(T^\varepsilon)^{i-1} (\ln(T^\varepsilon))^\frac{k-i}{k-i}.$$  

Since the assumption $32k^4 c_i < \frac{1}{\ln^2(k-1)} 10^{-3^k}$ of Lemma 12 is satisfied, this implies that there exists a constant $C_k > 0$ such that

$$\alpha(\mathcal{H}^{**}) \geq C_k \frac{|V^{**}|}{T^\varepsilon} (\ln(T^\varepsilon))^\frac{1}{k-i} \geq C_k \frac{NT^{-1+\varepsilon}/(4k)}{T^\varepsilon} (\ln(T^\varepsilon))^\frac{1}{k-i}$$

$$\geq C_k \frac{N}{T} (\ln T)^{\frac{1}{k-i}},$$

which completes our proof of Lemma 15.  \hfill \Box

5. PROOF OF THEOREM 6

We are going to show Theorem 6 from Lemma 15 that is a weaker version of Theorem 6. We need to modify two assumptions of Lemma 15: the first assumption is about the upper bounds on $t_i^{i-1}$, and the second assumption is about the ranges of $T$ and $N$.

We first change the assumptions on the upper bounds on $t_i^{i-1}$, and show the following.

**Lemma 16.** Let $k \geq 2$ be a fixed integer. Let $T$ and $N$ satisfy $T > T_0(k)$ and $N > N_0(k, T)$. Let $\mathcal{H} = (V, E_2 \cup \cdots \cup E_k)$ be a linear hypergraph on $N$ vertices such that there are no 3-cycles and 4-cycles consisting of 2-element edges. Let the average degrees $t_i^{i-1} := i|iE_i||N$ satisfy that for $i = 2, \ldots, k$

$$t_i^{i-1} \leq T^{i-1}(\ln T)^{\frac{k-i}{k-i}}. \quad (35)$$

Then, there exists a constant $C_k > 0$ such that

$$\alpha(\mathcal{H}) \geq C_k \frac{N}{T} (\ln T)^{\frac{1}{k-i}}.$$

**Proof.** We are going to use Lemma 15. To this end, we need to change the assumption (35) to the shape of the assumption (24) in Lemma 15. We have

$$t_i^{i-1} \leq T^{i-1}(\ln T)^{\frac{k-i}{k-i}} = c_i \cdot \frac{2}{c_i} T^{i-1} \cdot \frac{1}{2} (\ln T)^{\frac{k-i}{k-i}}$$

$$\leq c_i \left( \frac{T}{c_i} \right)^{i-1} \left( \frac{1}{2} \ln T \right)^{\frac{k-i}{k-i}} \leq c_i \left( \frac{T}{c_i} \right)^{i-1} \left( \ln \frac{T}{c_i} \right)^{\frac{k-i}{k-i}}.$$
where \( c_n^* := (c_n/2)^{1/2} \) and the last inequality holds because \( T \) is sufficiently large depending on \( k \).

Now we apply Lemma 15 and we infer
\[
\alpha(\mathcal{H}) \geq C_k \frac{N}{T/c_n^*} \left( \ln \frac{T}{c_n^*} \right)^{1/4} \geq C_k c_n^* \frac{N}{T} \cdot \frac{1}{2} (\ln T)^{1/4} \\
\geq C_k \frac{N}{T} (\ln T)^{1/4},
\]
where \( C_k' := C_k c_n^*/2 \) and the second inequality holds since \( T \) is sufficiently large depending on \( k \).

This completes the proof of Lemma 16. \( \square \)

In order to show Theorem 6, it only remains to enlarge the ranges of \( T \) and \( N \) in Lemma 16 as \( T \geq 1.5 \) and every \( N \).

First, we enlarge the range of \( T \) from \( T > T_0(k) \) to \( T \geq 1.5 \). If \( 1.5 \leq T \leq T_0(k) \), then \( T \ln T \geq 1/2 \), and hence, Theorem 11 with \( t_i^{k-1} \leq (T \ln T)^{i-1} \) implies that

\[
\alpha(\mathcal{H}) \geq \frac{1}{4} \frac{N}{T \ln T} \geq \frac{1}{4} \frac{N}{T \ln T_0} = \left( \frac{1}{4(\ln T_0)^{1/4}} \right) \frac{N}{T} (\ln T_0)^{1/4} \\
\geq \left( \frac{1}{4(\ln T_0)^{1/4}} \right) \frac{N}{T} (\ln T)^{1/4} = C_k \frac{N}{T} (\ln T)^{1/4}.
\]

Next, we enlarge the range of \( N \) from \( N > N_0(k, T) \) to be every \( N \). Let \( \mathcal{H} = (V, \mathcal{E}) \) be a hypergraph satisfying the assumption in Lemma 16 except for \( T > T_0(k) \) and \( N > N_0(k, T) \), and suppose that \( N \leq N_0(k, T) \) and \( T \geq 1.5 \).

Let \( L > N_0(k, T)/N \), and consider the hypergraph \( \mathcal{H}' \) obtained by \( L \) vertex-disjoint copies of \( \mathcal{H} \). Observe that \( \mathcal{H}' \) has \( LN \) vertices and its average degree is the same as the average degree of \( \mathcal{H} \). Hence \( \mathcal{H}' \) satisfies the assumption of Lemma 16 thus

\[
\alpha(\mathcal{H}') \geq C_k \frac{LN}{T} (\ln T)^{1/4}.
\]

Since \( \alpha(\mathcal{H}) = \alpha(\mathcal{H}')/L \), we infer

\[
\alpha(\mathcal{H}) \geq C_k \frac{N}{T} (\ln T)^{1/4},
\]
which completes the proof of Theorem 6. \( \square \)

6. Proof of Theorem 8

It will be convenient here to work with the \( O \)-notation. For functions \( f, g: \mathbb{N} \rightarrow \mathbb{N} \) and a fixed integer \( \ell > 0 \), let \( f = O_\ell(g) \) mean that there exists a constant \( c > 0 \), depending only on \( \ell \), such that \( f(n) \leq cg(n) \) for every sufficiently large \( n \in \mathbb{N} \).

**Proof of the lower bound of Theorem 8** We will use in our arguments some ideas from [2] (compare also [7]). Recall that \( \mathcal{H} = \mathcal{H}(n; 2, \ldots, \ell) \) is the hypergraph on the vertex set \( V \), \( |V| = n \), in which, for \( s = 2, \ldots, \ell \), each \( s \)-subset of \( V \) is a hyperedge of \( \mathcal{H} \), and let a \( (u_2, \ldots, u_\ell) \)-bounded edge-coloring \( \Delta \) of \( \mathcal{H} \) be given. We define another hypergraph \( \mathcal{G} = (V, E) \) as follows: If \( e_1 \) and \( e_2 \) are hyperedges in \( \mathcal{H} \) with the same color in \( c \), then \( e_1 \cup e_2 \in E(\mathcal{G}) \). Let \( E_{2i} \) be the set of hyperedges of \( \mathcal{G} \) of size \( 2i \), hence \( E = \bigcup_{i=0}^{\ell-2} E_{2i} \). Observe that if \( I \subset V \) is an independent set of \( \mathcal{G} \), then the subhypergraph \( \mathcal{H}[I] \) of \( \mathcal{H} \) induced on \( I \) is totally multicolored. Hence, a lower bound on \( f_\Delta(n; u_2, \ldots, u_\ell) \) can be obtained by finding an independent set in \( \mathcal{G} \). Therefore, it suffices to show the following:


(i) Under assumption (6), we have
\[ \alpha(G) \geq c_1 \left( \frac{n^s}{u_s} \right)^{\frac{1}{2s-1}}. \]

(ii) Under assumptions (8) and (9), we have
\[ \alpha(G) \geq c_3 \left( \frac{n^s \ln n}{u_s} \right)^{\frac{1}{2s-1}}. \]

We first prove (i). Let assumption (6) hold. Set
\[ p = \left( \frac{1}{n^{s-1}u_s} \right)^{\frac{1}{2s-1}}. \]

Let \( R \) be a random subset of \( V \) obtained by choosing each vertex independently with probability \( p \). Note that with high probability \(|R| = np(1 + o(1))\).

For \( i = 2, \ldots, \ell \), let \( E_{2i}^R \) be the set of all \( 2i \)-element hyperedges in the subhypergraph \( G[R] \) induced on \( R \). To estimate the expected numbers \( \mathbb{E}(|E_{2i}^R|) \) of \( 2i \)-element hyperedges, choose an \( i \)-element hyperedge \( e \) in \( \binom{n}{i} \) ways. Less than \( u_i \) other hyperedges have the same color as \( e \), thus
\[ |E_{2i}^R| \leq \frac{n^i u_i}{2(i!)} \]

Consequently, Markov’s inequality gives for \( i = 2, \ldots, \ell \) that
\[ \Pr \left[ |E_{2i}^R| > \frac{n^i u_i p^{2i}}{2(i!)} \right] \leq \frac{1}{\ell}. \]

Therefore, there exists a vertex set \( R \subset V \) such that
- \(|R| \geq np/2\),
- \(|E_{2i}^R| \leq \ell \cdot \frac{n^i u_i p^{2i}}{2(i!)} \) for \( i = 2, \ldots, \ell \).

The average degree \( t_{2i} \) for the \( 2i \)-element hyperedges in the subhypergraph \( G[R] \) satisfies
\[ t_{2i} \leq c_\ell \left( n^{i-1} u_i \right)^{1/(2i-1)} \]

and hence, for \( i = 2, \ldots, \ell \), and some constant \( c_\ell > 0 \), it is
\[ t_{2i} \leq c_\ell \left( n^{s-1} u_s \right)^{1/(2s-1)} p. \]

By Theorem 11 we infer that
\[ \alpha(G) \geq \frac{1}{4 c_\ell \left( n^{s-1} u_s \right)^{1/(2s-1)}} \frac{np/2}{p} = c_1' \left( \frac{n^s}{u_s} \right)^{1/(2s-1)}, \]

for some constant \( c_1' > 0 \), depending only on \( \ell \), which completes the proof of (i).

Next we show (ii). Suppose that (8) and (9) hold. Set
\[ p = \left( \frac{1}{n^{s-1}u_s} \right)^{\frac{1}{2s-1}} \omega, \quad \text{where } \omega := \left( \frac{u_s^2}{n} \right)^{\frac{1}{2(2s-1)(2s+1)}}. \quad (36) \]

As above, let \( R \) be a random subset of \( V \) obtained by choosing each vertex independently with probability \( p \). We again have that
- \(|R| = np(1 + o(1))\) with high probability,
• For \( i = 2, \ldots, \ell \),
\[
\Pr \left[ |E^{2i}_{2i}| > 3\ell \cdot \frac{n^i u_i p^{2i}}{2(i!)} \right] \leq \frac{1}{3\ell}.
\]

Next, we consider 2-cycles in \( \mathcal{G} \). For integers \( 2 \leq i, j, k \leq \ell \), let \( C(2i, 2j, k) \) be the family of all 2-cycles which consist of two distinct hyperedges in \( \mathcal{G} \) with one of size \( 2i \) and the other of size \( 2j \) sharing exactly \( k \) vertices. Note that the condition on \( k \) is either \( 2 \leq k \leq 2i - 1 \) for \( i = j \) or \( 2 \leq k \leq 2i \) for \( i < j \). We estimate \( |C(2i, 2j, k)| \) as follows. Fix the first hyperedge \( e \in E_{2i} \) as an arbitrary one in at most \( n^i u_i \) ways. Let \( e_1 \cup e_2 \in E_{2j} \) \((e_1, e_2 \in E_j(\mathcal{H}))\) be the second hyperedge in \( \mathcal{G} \) and let \( k_1 = |e_1 \cap e| \) and \( k_2 = |e_2 \cap e| \). Without loss of generality, we assume \( k_1 \geq k_2 \).

• Suppose \( k_1 = k \) and \( k_2 = 0 \). The number of choices of \( e_1 \) is at most \( O_\ell(n^{j-k}) \). Since the color of \( e_1 \) is determined, the number of choices of \( e_2 \) is at most \( u_j \leq n \) because of the assumption that each color class is a matching. Hence, the number of choices of \((e_1, e_2)\) is at most \( O_\ell(n^{j-k+1}) \).

• Otherwise, we have \( k_2 \neq 0 \). The number of choices of \( e_1 \) is at most \( O_\ell(n^{j-k_1}) \). Then, the number of choices of \( e_2 \) is at most \( O_\ell(1) \). Hence, the number of choices of \((e_1, e_2)\) is at most \( O_\ell(n^{j-k_1}) \). By minimizing \( k_1 \), we have the upper bound \( O_\ell(n^{j-[k/2]}) \).

Consequently, we have that for all \( k \geq 2 \),
\[
|C(2i, 2j, k)| = O_\ell \left( |E_{2i}| \cdot n^{j-[k/2]} \right) = O_\ell \left( u_i n^{i+j-[k/2]} \right).
\]

Let \( C^R(2i, 2j, k) \) be the random set of all 2-cycles in \( C(2i, 2j, k) \) that are contained in \( R \). Let
\[
C^R(2i, 2i) = \bigcup_{k=2}^{2i-1} C^R(2i, 2i, k) \quad \text{for} \quad 2 \leq i \leq \ell, \quad \text{and}
\]
\[
C^R(2i, 2j) = \bigcup_{k=2}^{2i} C^R(2i, 2j, k) \quad \text{for} \quad 2 \leq i < j \leq \ell.
\]

Since \( \mathbb{E} \left[ |C^R(2i, 2j, k)| \right] = |C(2i, 2j, k)| p^{2i+2j-k} \), we infer that for \( 2 \leq i \leq \ell \),
\[
\mathbb{E} \left[ |C^R(2i, 2i)| \right] = \sum_{k=2}^{2i-1} \mathbb{E} \left[ |C^R(2i, 2i, k)| \right] \overset{\text{(np\gg 1)}}{=} O_\ell \left( u_i n^{2i-1} p^{4i-2} + u_i n^{i+1} p^{2i+2} \right)
\]
\[
\overset{\text{(np\ll 1)}}{=} O_\ell \left( u_i n^{i+1} p^{2i+2} \right),
\]

and that for \( 2 \leq i < j \leq \ell \),
\[
\mathbb{E} \left[ |C^R(2i, 2j)| \right] = \sum_{k=2}^{2i} \mathbb{E} \left[ |C^R(2i, 2j, k)| \right] \overset{\text{(np\gg 1)}}{=} O_\ell \left( u_i n^{j+i-1} p^{2j+2i-2} + u_i n^j p^{2j} \right)
\]
\[
\overset{\text{(np\ll 1)}}{=} O_\ell \left( u_i n^j p^{2j} \right).
\]

Using Markov’s inequality, it simultaneously holds with probability bigger than \( 2/3 \) that
\[
|C^R(2i, 2i)| = O_\ell \left( u_i n^{i+1} p^{2i+2} \right) \quad \text{for} \quad 2 \leq i \leq \ell, \quad \text{and}
\]
\[
|C^R(2i, 2j)| = O_\ell \left( u_i n^j p^{2j} \right) \quad \text{for} \quad 2 \leq i < j \leq \ell.
\]

Therefore, there exists a subset \( R \subset V \) such that for some constant \( c > 0 \), depending only on \( \ell \), we have that
\[
|R| \geq np/2
\]
\[
|E^{2i}_{2i}| \leq cu_i n^i p^{2i} \quad \text{for} \quad 2 \leq i \leq \ell
\]
indeed, we have that
\[ |C^R(2i, 2i)| \leq cu_i n^{i+1} p^{2i+2} \quad \text{for } 2 \leq i \leq \ell, \quad \text{and} \quad |C^R(2i, 2j)| \leq cu_i n^j p^{2j} \quad \text{for } 2 \leq i < j \leq \ell. \]

We claim that \( \sum_{i=2}^{\ell} |C^R(2i, 2i)| \ll np \) and \( \sum_{2 \leq i < j \leq \ell} |C^R(2i, 2j)| \ll np \). For the first inequality, it suffices to check that for \( 2 \leq i \leq \ell \),
\[ u_i n^{i+1} p^{2i+2} \ll np, \quad \text{that is,} \quad u_i p(n^2)^i \ll 1, \]

indeed, we have that
\[ u_i p \leq \left( \frac{u_s^2}{n} \right)^{\frac{1}{2(2i-1)}} \omega(\ln n)^{\frac{2(s-1)}{2^{s-1}}} \quad \text{and} \quad np^2 = \left( \frac{n}{u_s^2} \right)^{\frac{1}{2(2i-1)}} \omega^2. \]

Hence,
\[ u_i p(n^2)^i \leq \left( \frac{n}{u_s^2} \right)^{\frac{1}{2(2i-1)}} \omega^{2i+1}(\ln n)^{\frac{2(s-1)}{2^{s-1}}} \ll 1, \]

where the last inequality follows from \( u_s \geq n^{1+\varepsilon} \) and \( \omega = \left( \frac{u_s^2}{n} \right)^{\frac{1}{2(2i-1)(2i+1)}} \). Next, for the second inequality, similarly it follows that \( u_i n^jp^{2j} \ll np \) for \( 2 \leq i < j \leq \ell \).

After deleting a vertex from each member of \( C^R(2i, 2i) \) or \( C^R(2i, 2j) \), there exists a vertex set \( U \subset R \) such that
- \( |U| \geq \frac{np}{4} \)
- \( |E^S| \leq cu_i n^i p^{2i} \) for \( 2 \leq i \leq \ell \).
- There is no 2-cycle in the subhypergraph \( G[U] \) induced on \( U \).

Set
\[ T := c^* p(n^s-1) u_s^{\frac{1}{2s-1}} (\ln n)^{\frac{2s-2}{2s-1}}, \]

where \( c^* > 0 \) is a suitable large constant to be fixed later. We now check that the average degree \( t_{2i-1}^{2i-1} \) of the subhypergraph \( G[U] \) with hyperedges of size \( 2i \) satisfies (38), that is,
\[ t_{2i-1}^{2i-1} \leq T^{2i-1} \cdot (\ln T)^{\frac{2s-2}{2s-1}}. \]

First, observe that
\[ t_{2i-1}^{2i-1} \leq 8\ell cu_i n^{i-1} p^{2i-1} = 8\ell c(u_i p)(np^2)^{i-1} \leq 8\ell c\omega^{2i-1}(\ln n)^{\frac{2(s-1)}{2^{s-1}}}. \]

On the other hand, since \( u_s \geq n^{1/2+\varepsilon} \), we have \( \ln T \geq c' \ln n \) for some constant \( c' = c'(\ell, \varepsilon) > 0 \), and hence,
\[ T^{2i-1} \cdot (\ln T)^{\frac{2s-2}{2s-1}} \geq (c^*)^{2i-1}(c')^{\frac{2s-2}{2s-1}} \omega^{2i-1}(\ln n)^{\frac{2(s-1)}{2^{s-1}}}. \]

With a suitable large choice of \( c^* = c^*(\ell, \varepsilon) > 0 \), depending on \( c \) and \( c' \), we infer (39).

Theorem 8 implies that there exist positive constant \( c_\ell \), depending only on \( \ell \), such that
\[ \alpha(G) \geq c_\ell \frac{np}{T} (\ln T)^{\frac{1}{2s-1}} = c_3 \left( \frac{n^s u_s}{\ln n} \right)^{\frac{1}{2s-1}}, \]

where \( c_3 > 0 \) is a constant, depending only on \( \ell \) and \( \varepsilon \), which completes the proof of (ii).

**Proof of the upper bound of Theorem 8.** We will show the following using some ideas from [2] and [3]:

For each integer \( k \) with \( 2 \leq k \leq \ell \), there exists a positive constant \( C = C(k) \) such that for every sufficiently large \( n \),
\[ f(n; u_2, \ldots, u_\ell) \leq C \left( \frac{n^k}{u_k \ln n} \right)^{\frac{1}{2s-1}}. \]

This with \( k = s \) implies the upper bound in Theorem 8.
For a proof of (40), let $H_k$ be the subhypergraph of $H$ on the vertex set $V$ only with all $k$-element hyperedges. We define a random edge-coloring of $H_k$ as follows. Set

$$m = \frac{c_0 x^k}{u_k},$$

where $c_0$ is a constant with $0 \leq c_0 \leq 1/(8e^2(k!))$. (We ignore divisibility constraints in our arguments, as there is enough room in the calculations.) Let $M_1, \ldots, M_m$ be random matchings chosen uniformly and independently from the set of all matchings of size $u_k$ on $V$, and let $U_0 = \emptyset$ and $U_i = \bigcup_{j \leq i} M_j$ for $i = 1, \ldots, m$. We color all hyperedges in $M_i \setminus U_{i-1}$ by color $i$, and color the remaining ones with distinct new colors.

In order to prove (40), it suffices to show that, for $x = C \left( \frac{n^k}{u_k} \ln n \right)^{\frac{1}{2k-1}}$, where $C > 0$ is a suitable constant to be fixed later, we have

$$\Pr \left[ \exists X \subset V \text{ such that } |X| = x \text{ and } X \text{ is totally multicolored} \right] = o(1).$$

For its proof, let $X \subseteq V$ be an arbitrarily fixed subset of $V$ of size $x$. We will show that $X$ is totally multicolored with very small probability. For $i = 1, \ldots, m$, let $Y_i$ be the number of pairs $\{S, T\}$ of hyperedges in $M_i \setminus U_{i-1}$ contained in $X$. Observe that $X$ is totally multicolored if and only if simultaneously $Y_i = 0$ for $i = 1, \ldots, m$. We can show that $Y_i = 0$, $i = 0, \ldots, m$, hold with very small probability under the condition that the intersection size $|U_m \cap [X]^k|$ is small, where $[X]^k$ denotes the family of all $k$-element hyperedges in $X$. To this end, let $A$ be the event that $|U_m \cap [X]^k| \leq c_1 x^k$ where $c_1 = 1/(4(k!))$. We have the following:

$$\Pr [X \text{ is totally multicolored}] = \Pr [Y_1 = 0, \ldots, Y_m = 0] \leq \Pr [A^c] + \Pr [Y_1 = 0, \ldots, Y_m = 0, A] \leq \Pr [A^c] + \Pr [Y_1 = 0, \ldots, Y_m = 0 | A]. \tag{41}$$

We will use the following two claims:

**Claim 17.** For every sufficiently large $n$,

$$\Pr [A^c] \leq \exp \left( -c_1 x^k \right). \tag{42}$$

**Claim 18.** For every sufficiently large $n$,

$$\Pr [Y_1 = 0, \ldots, Y_m = 0 | A] \leq \exp \left( -\frac{c_0 c_1^2 u_k x^{2k}}{4 n^k} \right). \tag{43}$$

The union bound and Claims 17 and 18 together with (41) yield that

$$\Pr \left[ \exists X \subset V \text{ such that } |X| = x \text{ and } X \text{ totally multicolored} \right] \leq \binom{n}{x} \left( \exp \left( -\frac{c_0 c_1^2 u_k x^{2k}}{4 n^k} \right) + \exp \left( -c_1 x^k \right) \right) \leq \exp (x \ln n) \cdot \left( \exp \left( -\frac{c_0 c_1^2 u_k x^{2k}}{4 n^k} \right) + \exp \left( -c_1 x^k \right) \right). \tag{44}$$

By choosing the constant $C > (4/(c_0 c_1^2))^{\frac{1}{2k-1}}$, the term (44) goes to 0 as $n$ tends to $\infty$, which completes our proof of (40). It remains to prove Claims 17 and 18.
First, we prove Claim 17. Since \( |U_m \cap |X|^k| \leq \sum_{i=1}^m |M_i \cap |X|^k| \) and the events \( |M_i \cap |X|^k| \geq t_i \), \( i = 1, \ldots, m \), are independent, we infer that

\[
\Pr \left[ |U_m \cap |X|^k| > t \right] \leq \Pr \left[ \sum_{i=1}^m |M_i \cap |X|^k| \geq t \right] \leq \sum_{i=1}^m \Pr \left[ |M_i \cap |X|^k| \geq t_i \right].
\]

Now we estimate \( \Pr \left[ |M_i \cap |X|^k| \geq t_i \right] \) for integers \( 1 \leq i \leq m \) and \( t_i \geq 0 \). There are \( \binom{n}{k} \) choices for selecting \( t_i \) hyperedges in \( M_i \). Then the \( t_i \) hyperedges are contained in \( X \) with probability \( \left( \frac{x}{k!} \right) / \binom{n}{k} \), hence

\[
\Pr \left[ |M_i \cap |X|^k| \geq t_i \right] \leq \left( \frac{u_k x^k}{n^k} \right)^{t_i} \leq \left( \frac{u_k x^k}{n^k} \right)^{t_i} \tag{45}
\]

Thus, we infer that

\[
\Pr \left[ |U_m \cap |X|^k| > t \right] \leq \sum_{i=1}^m \prod_{j=1}^{t_i} \left( \frac{u_k x^k}{n^k} \right)^{t_i} \leq \left( \frac{u_k x^k}{n^k} \right)^{t_i} \leq \left( \frac{e(t + m) u_k x^k}{tn^k} \right)^{t_i}.
\]

Take \( t = c_1 x^k \) and note that \( t = o(m) \). Consequently,

\[
\Pr \left[ |U_m \cap |X|^k| > c_1 x^k \right] \leq \left( \frac{2em u_k x^k}{c_1 x^k n^k} \right)^{c_1 x^k} = \left( \frac{2em}{c_1} \right)^{c_1 x^k} \leq e^{-c_1 x^k},
\]

where the last inequality follows from \( 0 \leq c_0 \leq 1/(8e^2 (k!)) \) and \( c_1 = 1/(4 (k!)) \). This completes our proof of Claim 17.

Next, we prove Claim 18. First, we have that

\[
\Pr \left[ Y_1 = 0, \ldots, Y_m = 0 \mid A \right] = \prod_{i=1}^m \Pr \left[ Y_i = 0 \mid A, Y_j = 0 \text{ for } 1 \leq j \leq i - 1 \right]. \tag{46}
\]

For simplification, let \( B_i \) denote the event that \( A \) happens and \( Y_j = 0 \) for \( 1 \leq j \leq i \). Next we upper bound \( \Pr[Y_i = 0 \mid B_{i-1}] \), or equivalently, we lower bound \( \Pr[Y_i \geq 1 \mid B_{i-1}] \).

To this end, it is useful to consider \( \mathbb{E}[Y_i \mid B_{i-1}] \). The condition that \( A \) holds implies \( |U_{i-1} \cap |X|^k| \leq c_1 x^k \), that is,

\[
|X|^k \setminus U_{i-1} \geq \left( \frac{x}{k} \right) - c_1 x^k \geq \left( \frac{1}{2(k!)} - c_1 \right) x^k = c_1 x^k.
\]

For each hyperedge \( S \in |X|^k \) there are at most \( k \binom{x-1}{k-1} \) hyperedges in \( X \) which are not disjoint from \( S \). Hence, the number of pairs \( \{S, T\} \in |X|^k \setminus U_{i-1} \) of hyperedges with \( S \cap T = \emptyset \) is, for every sufficiently large \( n \), at least

\[
\frac{1}{2} c_1 x^k \left( c_1 x^k - k \left( \frac{x - 1}{k - 1} \right) \right) > \frac{c_1^2 x^{2k}}{3}.
\]

For disjoint hyperedges \( S \) and \( T \),

\[
\Pr[S, T \in M_i] = \frac{u_k (u_k - 1)}{\binom{n}{k} \binom{n-k}{k}} \geq \frac{u_k^2}{n^{2k}},
\]
and therefore
\[ \mathbb{E}[Y_i | B_{i-1}] > \frac{c^2}{3} \left( \frac{u_k x^k}{n^k} \right)^2. \] (47)

Now we estimate \( \Pr[Y_i \geq 1 | B_{i-1}] \) by using \( \mathbb{E}[Y_i | B_{i-1}] \). We have that
\[
\Pr[Y_i \geq 1 | B_{i-1}] = \mathbb{E}[Y_i | B_{i-1}] - \sum_{j \geq 2} (j - 1) \Pr[Y_i = j | B_{i-1}]
= \mathbb{E}[Y_i | B_{i-1}] - \sum_{j \geq 2} \Pr[Y_i \geq j | B_{i-1}]. \] (48)

Observe that for \( j \) pairwise distinct two-element sets, the underlying set has cardinality at least \( \lceil \sqrt{2j+1} \rceil \). Hence,
\[
\Pr[Y_i \geq j | B_{i-1}] \leq \Pr[M_i \cap [X]^k] \geq \lceil \sqrt{2j+1} \rceil \leq \left( \frac{u_k x^k}{n^k} \right)^{\sqrt{2j+1}}. \] (49)

Consequently, it follows from (47)–(49) and \( x^k = o \left( \frac{n^k}{u_k} \right) \) that
\[
\Pr[Y_i \geq 1 | B_{i-1}] \geq \frac{c^2}{4} \left( \frac{u_k x^k}{n^k} \right)^2,
\]
that is,
\[
\Pr[Y_i = 0 | B_{i-1}] \leq 1 - \frac{c^2}{4} \left( \frac{u_k x^k}{n^k} \right)^2.
\]

Therefore, (46) gives that
\[
\Pr \left[ Y_1 = 0, \ldots, Y_m = 0 \mid A \right] \leq \left( 1 - \frac{c^2}{4} \frac{u_k x^{2k}}{n^{2k}} \right)^m \leq \exp \left( -\frac{c^2}{4} \frac{u_k x^{2k} m}{n^{2k}} \right) \leq \exp \left( -\frac{c_0 c^2}{4} \frac{u_k x^{2k}}{n^k} \right),
\]
which completes our proof of Claim 18. \( \square \)

REFERENCES

[1] M. Ajtai, J. Komlós, J. Pintz, J. Spencer and E. Szemerédi, *Extremal Uncrowded Hypergraphs*, Journal of Combinatorial Theory Ser. A 32, 1982, 321–335.

[2] N. Alon, H. Lefmann and V. Rödl, *On an Anti-Ramsey Type Result*, Coll. Math. Soc. János Bolyai, 60. Sets, Graphs and Numbers, 1991, 9–22.

[3] L. Babai, *An Anti-Ramsey Theorem*, Graphs and Combinatorics 1, 1985, 23–28.

[4] C. Bertram-Kretzberg and H. Lefmann, *The Algorithmic Aspects of Uncrowded Hypergraphs*, SIAM Journal on Computing 29, 1999, 201–230.

[5] R. A. Duke, H. Lefmann and V. Rödl, *On Uncrowded Hypergraphs*, Random Structures & Algorithms 6, 1995, 209–212.

[6] A. Fundia, *Derandomizing Chebychev’s Inequality to Find Independent Sets in Uncrowded Hypergraphs*, Random Structures & Algorithms 8, 1996, 131–147.

[7] H. Lefmann, V. Rödl and B. Wysocka, *Multicolored Subsets in Colored Hypergraphs*, Journal of Combinatorial Theory Series A 74, 1996, 209–248.

[8] P. Turán, *On an Extremal Problem in Graph Theory*, Mat. Fiz. Lapok 48, 1941, 436–452.

[9] J. Spencer, *Turán’s Theorem for k-Graphs*, Discrete Mathematics 2, 1972, 183–186.

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