SUSY SHAPE-IN Variant Hamiltonians for the Generalized Dirac-Coulomb Problem

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Abstract

A spin \( \frac{1}{2} \) relativistic particle described by a general potential in terms of the sum of the Coulomb potential with a Lorentz scalar potential is investigated via supersymmetry in quantum mechanics.

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Supersymmetry in Quantum Mechanics (SUSY QM) [1] is of intrinsic mathematical interest in its own as it connects otherwise apparently unrelated second-order differential equations.

The (1+3) and (1+1) dimensional Dirac equations with both scalar-like and vector-like potentials are well known in the literature for a long time [2]. The connection between position-dependent-effective-mass and shape invariant condition under parameter translation has been discussed in non-relativistic quantum mechanics [3, 4]. Recently, some relativistic shape invariant potentials have been investigated [5].

Exact solutions for the bound states in this mixed potential can be obtained by the method of separation of variables [6, 7, 8] and also by the use of the dynamical algebra $SO(2, 1)$ [9]. In a recent paper the solution of the scattering problem for this potential has been obtained by an analytic method and also by an algebraic method [10], the problem of a relativistic Dirac electron with a $1/r$ scalar potential, as well as a Dirac magnetic monopole and an Aharonov-Bohm potential has also been investigated [11], and the bound eigenfunctions and spectra of a Dirac hydrogen atom have been found via $su(1, 1)$ Lie algebra [12].

Recently exact solutions have been found for fermions in the presence of a classical background which is a mixing of the time-dependent of a gauge potential and a scalar potential [13]. Also, exactly solvable Eckart scalar and vector potentials in the Dirac equation have been investigated via SUSY QM [14], the $S$-wave Dirac equation has been solved exactly for a single particle with spin and pseudospin symmetry moving in a central Woods-Saxon potential [15].

The special case of the non-relativistic [16] and relativistic Coulomb problems have been treated recently via SUSY QM [17, 18, 19]. In this work, the relativistic Coulomb potential with a Lorentz scalar potential is investigated via shape invariance conditions of the SUSY QM.

The time independent Dirac equation may be written in the form $H \Psi = E \psi$, where the Hamiltonian is given by

$$H = \rho_1 \otimes \vec{\sigma} \cdot \vec{p} + \left( M - \frac{A_2}{r} \right) \rho_3 \otimes 1_{2x2} - \frac{A_1}{r} \otimes 1_{4x4},$$

and we have used a direct product notation in which $\rho_i$ and $\sigma_i, (i = 1, 2, 3)$ are the Pauli spin matrices obeying $[\rho_i, \sigma_j]_- = 0$, with $\hbar = c = 1$. 

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We consider
\[ \Psi = \begin{pmatrix} iG_{\ell j} \phi_{jm}^\ell \\ F_{\ell j} r \sigma \cdot \vec{n} \phi_{jm}^\ell \end{pmatrix}, \]
where \( \phi_{jm}^\ell = \phi_{jm}^{(\pm)} \) for \( j = \ell \pm \frac{1}{2} \). Next, using the relation \([1 + \vec{L} \cdot \sigma, \vec{n} \cdot \vec{l}]_+ = 0\) we obtain \( K \Psi = -k \Psi \) and the following radial equations
\[ \begin{align*}
\frac{dG_{\ell j}}{dr} + \frac{k}{r} G_{\ell j} - \left( E + M - \frac{A_2}{r} + \frac{A_1}{r} \right) F_{\ell j} &= 0, \\
\frac{dF_{\ell j}}{dr} - \frac{k}{r} F_{\ell j} + \left( E - M - \frac{A_2}{r} + \frac{A_1}{r} \right) G_{\ell j} &= 0. \end{align*} \]
(2)
Note that the interaction in these two equations can be diagonalized so that we obtain
\[ A^+ \hat{G} \propto \hat{F}, \quad A^- \hat{F} \propto \hat{G} \]
(3)
where
\[ A^\pm = \pm \frac{d}{dr} + \frac{\lambda}{r} - \frac{EA_1 + MA_2}{\lambda}. \]
(4)
These relations are similar to the relations between the two components of the eigenfunctions of a "supersymmetric" Hamiltonian which satisfies the following Lie graded algebra
\[ \mathcal{H} = [Q, Q^\dagger]_+ = QQ^\dagger + Q^\dagger Q, \quad [\mathcal{H}, Q^\dagger]_- = 0 = [\mathcal{H}, Q]_- \]
(5)
with the following representation
\[ Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H_+ = A^+ A^- & 0 \\ 0 & H_- = A^- A^+ \end{pmatrix}, \quad \Phi_{\text{SUSY}} = \begin{pmatrix} F \\ G \end{pmatrix}. \]
(6)
Note that the supercharges are nilpotent operators, viz., \((Q^\dagger)^2 = 0 = Q^2\).

Thus, using the shape invariant Hamiltonians \( H_\pm \) we obtain the energy eigenvalues associated to the component \( \hat{F}^n \) given by
\[ E_n = \sqrt{\frac{M^2}{1 + \left( \frac{\gamma_n^2}{(k^2 - \gamma_n^2 + n)^2} \right)}}, \quad n = 0, 1, 2, \cdots, \quad \gamma_n(E) = A_1 + \frac{MA_2}{E_n}. \]
(7)
In conclusion, we obtain the complete set of the energy eigenvalues of the Dirac equation for a potential which is the sum of the Coulomb potential with a Lorentz scalar potential inversely proportional to $r$ via shape invariance property as applied in [17]. One of us (RLR) will make elsewhere a detailed analysis for this problem as applied to the relativistic Coulomb potential via SUSY shape-invariant potentials [17].

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