OPTIMALITY OF PIECEWISE THERMAL CONDUCTIVITY IN A SNOW-ICE THERMODYNAMIC SYSTEM

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Abstract. This article is intended to provide the optimality of piecewise thermal conductivity in a snow-ice thermodynamic system. Based on the temperature distribution characteristics of snow and sea ice, we construct a piecewise smooth thermodynamic system coupled by snow and sea ice. Taking the piecewise thermal conductivities of snow and sea ice as control variables and the temperature deviations obtained from the system and the observations as the performance criterion, an identification model with state constraints is given. The dependency relationship between state and control variables is proven, and the existence of the optimal control is discussed. The work can provide a theoretical foundation for simulating temperature distributions of snow and sea ice.

1. Introduction. Sea ice is both an important actor and an indicator of climate change [20]. It changes the physical properties of the surface of the polar ocean, and modifies the energy and mass transfer between the ocean and the atmosphere. To improve the accuracy of sea ice models and general circulation models, accurate representations of both mechanical and thermodynamic sea ice properties and physics are required. Thermal conductivity is an important parameter. It controls the thermodynamic growth rate, equilibrium thickness, and conductive heat flux through the ice. Moreover, the conductivity representation uses almost universally in both small- and large-scale sea ice modeling remains essentially unvalidated by measurements [19].

Many researchers have been devoted to studying thermal conductivities of sea ice and snow. Maykut and Untersteiner [15] established a classical one-dimension sea ice thermodynamical model, in which the thermal conductivity expression was given and they applied it to study multiyear Arctic ice. Most of this work has done in [18, 12, 22, 16, 21]. Pringle et al. [19] analyzed the thermal conductivity in landfast Antarctic and Arctic sea ice, and put forward a better model with wider application. Many researchers studied the thermophysical properties of snow through the snow-sea ice system, because the snow cover on sea ice plays a key role in the sea-ice system and has significant impact on sea-ice heat/mass balance. Fichefet et al. [7] found that Antarctic sea ice shows up a strong sensitivity to

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snow thermal conductivity. Calonne et al. [1] carried out numerical simulations of the snow thermal conductivity under considering the conduction through ice and interstitial air. Lei et al. [11] used the mass balance measurements and temperature profiles to calculate thermal conductivity of snow. Lecomte et al. [10] formulated the snow thermal conductivity in large-scale sea ice models.

The above researchers determined thermal conductivity according to other physical parameters. Thermal conductivity cannot be measured by hand directly, which could not help people to thoroughly understand the physical evolution of thermal conductivity. Identification method can solve this kind of problem. Identification refers to the determination from observed data of unknown parameters in the system model such that the predicted response of the model is close, in some well-defined sense, to the process observations. Using this method, some researchers [17, 2, 23, 4] considered the identification of thermal conductivities of heat equations. Refs. [13] and [14] provided the mathematical theory to calculate the thermal conductivities. Semion [3] considered the heat conduction with a piecewise constant conductivity and proved the identifiability for the conductivities by using eigenvalues and eigenfunctions method. In [6], a distributed parameter coupled system concerning Arctic sea ice temperature field was established, and they proved the existence and continuity of the weak solution for the coupled system. In this paper, we consider a snow-ice thermodynamic system with matching conditions, in which thermal conductivities of snow and sea ice are identified. We prove the existence and uniqueness of the weak solution of the system and give the basic structure of the solution. After that, taking the piecewise thermal conductivities of snow and sea ice as control variables and the temperature deviations as the performance criterion, an identification model is given. The dependency relationship between state and control variables is proven, and the existence of the optimal control is discussed.

The objective of this study is to present the optimality of piecewise thermal conductivity in a snow-ice thermodynamic system. Section 2 presents the piecewise smooth thermodynamic system of snow and sea ice, and gets the existence and uniqueness of weak solution. Furthermore, the basic structure of the solution will be given.

2. Piecewise Snow-Ice Thermodynamic System and Its Properties. In this section, we will describe the snow-ice thermodynamic system in detail and prove the existence and uniqueness of weak solution. Furthermore, the basic structure of the solution will be given.

2.1. Snow-Ice Thermodynamic System. The system is consisted of two layers: snow and sea ice (FIGURE. 1). Because the gradient variation in the vertical direction is far greater than the one in the horizontal direction, we only consider the heat flux in the vertical direction. Let some point on the snow surface represent the origin, \(z\) representing the depth of the system to be the vertical coordinate taken as positive downward. Let \(t\) denote time, \(t_f (> 0)\) the final time, \(T(z, t)\) the temperature of the snow-ice thermodynamic system at depth \(z\) and time \(t\). For the subsequent analysis, we may use the notation \(T\) and \(T(z, t; k)\) to denote \(T(z, t)\). Let \(\Omega = (0, l) \subset \mathbb{R}, I = (0, t_f).\) By the Fourier’s law of heat conduction, the snow-ice thermodynamic system is described as the following partial differential equations denoted by PWTS:
FIGURE 1 The configuration of the snow-ice thermodynamic system.

\[ C(z) \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} (k(z) \frac{\partial T}{\partial z}) + f(z,t), \text{ in } (\Omega \setminus \{l_1\}) \times I, \]

\[ T(l_1+, t) = T(l_1-, t), \text{ in } I, \]

\[ k(l_1+)T(l_1+, t) = k(l_1-)T(l_1-, t), \text{ in } I, \]

\[ T(z, 0) = T_0(z), \text{ in } \Omega, \]

\[ T(0, t) = T_U(t), \text{ in } \bar{T}, \]

\[ T(l, t) = T_L(t), \text{ in } \bar{T}, \]

where \( C(z) \) is volumetric heat capacity, \( k(z) \) is thermal conductivity and \( f(z,t) \) is heat source term.

According to the physical properties of snow and sea ice, we give the following assumptions:

(A1) \( C(z) \) and \( k(z) \) are positive and continuous on interval \([0, l]\). \( C'(z) \) and \( k'(z) \) are continuous on open subintervals \((0, l_1)\) and \((l_1, l)\), both can be continuously extend to the closed intervals \([0, l]\). For definiteness, we assume that \( C'(z) \) and \( k'(z) \) are continuous from the right, i.e \( C'(z) = C'(z+) \) and \( k'(z) = k'(z+) \).

(A2) \( T_L(t), T_U(t) \in C^1[0, t_f], T_0(z) \in L^2[0, l], f(z, t) = h(z)r(t), h(z) \in L^2[0, l], r(t) \in C^1[0, t_f] \).

Remark 1. From Assumption (A1), we can get that there exist \( 0 < k_L \leq k_U \) and \( 0 < c_L \leq c_U \), such that \( k_L \leq k(z) \leq k_U \) and \( c_L \leq C(z) \leq c_U \). Define thermal conductivities \( k(z) \) are in the set \( A_{ad} = \{k(z) \in L^\infty[0, l] : k_L \leq k(z) \leq k_U \} \).

From the above, we can get PWTS is a piecewise smooth system. Next we will discuss the properties of the piecewise smooth system.

2.2. Properties of Snow-Ice Thermodynamic System. Let \( H = L^2(\Omega), H_C = \{v \in L^2(\Omega) : \int_\Omega C(z)v^2 \, dz < \infty\}, V = H_0^1[0, l], V_k = \{v \in H_0^1[0, l] : \int_\Omega k(z)|v'(z)|^2 \, dz < \infty\} \). The inner product and the norm on \( H \) are denoted as \( \langle \cdot, \cdot \rangle \) and \( \|\cdot\| \). The inner product and the norm on \( H_C \) are defined by

\[ (u, v)_C = \int_\Omega C(z)uv \, dz, \forall u, v \in H_C, \]
The inner product and induced norm on $V_k$ are defined by:

$$(u, v)_{V_k} = \int_{\Omega} k(z)u'v'dz, \ \forall u, v \in V_k,$$

$$||u||_{V_k} = \int_{\Omega} k(z)(u')^2dz, \ \forall u \in V_k.$$ 

The existence and uniqueness of the weak solution established as well.

**Theorem 2.1.** Suppose that assumptions (A1) and (A2) hold, then there exists a unique weak solution $T(z, t) \in C((0, t_f]; C[0, l])$ of PWTS.

**Proof.** Let

$$\Phi(z, t) = \frac{T_L(t) - T_U(t)}{\int_{\Omega} \frac{ds}{k(s)}} \int_{0}^{z} \frac{ds}{k(s)} + T_U(t),$$

and $u(z, t) = T(z, t) - \Phi(z, t)$ is defined by the following system (denote as AS):

$$u_t - C^{-1}(z)(k(z)u_z)_z = C^{-1}(z)f(z, t) - \Phi_t, \ \text{in} \ \Omega \times I,$$

$$u(0, t) = 0, \ \text{in} \ \mathcal{I},$$

$$u(l, t) = 0, \ \text{in} \ \mathcal{I},$$

$$u(z, 0) = T_0(z) - \Phi(z, 0), \ \text{in} \ \overline{\Omega},$$

where $u_z$ and $u_t$ express derivatives on depth $z$ and time $t$ of $u(z, t)$, $\Phi_t$ express the derivative on time $t$. For any $v \in H^1_0(\Omega)$, we multiply equation (1) by $v$ and integrate them over $\Omega$, we have

$$\int_{\Omega} u_1vdz - \int_{\Omega} C^{-1}(z)(k(z)u_z)_zvdz = \int_{\Omega} (C^{-1}(z)f(z, t) - \Phi_t)vdz.$$

For the second term on the left,

$$\int_{\Omega} C^{-1}(z)(k(z)u_z)_zvdz = - \int_{\Omega} C^{-1}(z)k(z)u_zv_zdz.$$

Let

$$A(u, v; t) = \int_{\Omega} C^{-1}(z)k(z)u_zv_zdz,$$

$$< u_t, v > = \int_{\Omega} u_tvdz,$$

$$(C^{-1}(z)f(z, t) - \Phi_t, v) = \int_{\Omega} (C^{-1}(z)f(z, t) - \Phi_t)vdz,$$

where $< \cdot, \cdot >$ denote the dual paring between space $H^1_0(0, l)$ and its dual $H^{-1}(0, l)$. Then the AS system can be transformed to

$$\begin{cases}
< u_t, v > + A(u, v; t) = (C^{-1}(z)f(z, t) - \Phi_t, v), \\
\ u(0) = T_0(z) - \Phi(z, 0), \ z \in \overline{\Omega},
\end{cases}$$

The existence and uniqueness of weak solution $u(z, t)$ of AS is established in [5], where $u(z, t) \in C((0, t_f]; L^2[0, l]) \cap L^2([0, t_f]; H^1_0[0, l])$. It is obviously that the existence and uniqueness of the weak solution $T(z, t) \in C((0, t_f]; C[0, l])$ of PWTS is established as well. □
The associated Strum-Liouville problem of PWTS (SLP for short) is
\[
\begin{cases}
(k(z)\omega'(z))' = -\lambda C(z)\omega(z), \\
\omega(0) = 0, \\
\omega'(0) = A.
\end{cases}
\]
and
\[
\begin{cases}
(k(z)\omega'(z))' = -\lambda C(z)\omega(z), \\
\omega(l_1) = B_1, \\
\omega'(l_1) = B_2.
\end{cases}
\]
both have a unique solution. Suppose that \(\psi_1(z)\) and \(\psi_2(z)\) correspond to the same eigenvalue \(\lambda_n\), so \(\psi_1(0) = \psi_2(0) = 0\). Find the numbers \(\rho_1\) and \(\rho_2\) \((\rho_1^2 + \rho_2^2 \neq 0)\) so that
\[\rho_1 \psi_1'(0) + \rho_2 \psi_2'(0) = 0.\]
Let \(\psi(z) = \rho_1 \psi_1(z) + \rho_2 \psi_2(z)\), then \(\psi(z)\) is a nontrivial solution of SLP corresponding to eigenvalue \(\lambda_n\). Thus \(\psi(0) = \rho_1 \psi_1(0) + \rho_2 \psi_2(0) = 0\) and \(\rho'(0) = \rho_1 \psi_1'(0) + \rho_2 \psi_2'(0) = 0.\) Since problem
\[
\begin{cases}
(k(z)\psi'(z))' = -\lambda C(z)\psi(z), \\
\psi(0) = 0, \\
\psi'(0) = 0.
\end{cases}
\]
has only solution \( \psi(z) \equiv 0 \), \( \psi_1(z) \) and \( \psi_2(z) \) are linear dependence. Thus \( \psi_2(z) = \rho \psi_1(z) \) on \((0, l_1)\), \( \psi_2(l_1-) = \rho \psi_1(l_1-) \), \( \psi_2'(l_1-) = \rho \psi_1'(l_1-) \). According to the linear matching conditions one gets that \( \psi_2(l_1+) = \rho \psi_1(l_1+), \psi_2'(l_1+) = \rho \psi_1'(l_1+) \). The uniqueness of the solution implies that \( \psi_2(z) = \rho \psi_1(z) \) on \((l_1, l)\). Thus \( \psi_2(z) = \rho \psi_1(z) \) on \((0, l)\). So the eigenvalue \( \lambda_n \) is simple.

According to the uniqueness of the solution and the matching conditions one gets that for \((k(z)\omega'(z))' = -\lambda C(z) \omega(z), \omega(0) = 0, \omega'(0) = 0\), its solution is \( \omega \equiv 0 \) on \((0, l)\). Thus there nonexistent eigenfunction \( \omega_n(z) \) satisfied \( \omega_n'(0) = 0 \). Assuming that the eigenfunction \( \omega_n(z) \) is normalized in \( H_C \), it leaves us with the choice of its sign for \( \omega_n'(0) \). Let \( \omega_n'(0) > 0 \) makes the eigenfunction unique.

(iii) In [9], the eigenvalue of (2) satisfy

\[
\lambda_n = \min_{V_n} \max \left\{ \frac{\int_{\Omega} k(z) (\omega'(z))^2 \, dz}{\int_{\Omega} C(z) (\omega(z))^2 \, dz} : \omega \in V_n \right\},
\]

where \( V_n \) varies over all subspace of \( H^1_0(0, l) \) of finite dimension \( n \). Thus, for \( k(z) \leq b(z), z \in [0, l] \), we have \( \lambda_n(k(z)) \leq \lambda_n(b(z)) \). Let \( k(z) = C(z) = 1 \), the eigenvalue of (2) is \( n^2 \pi^2 / l^2 \). According to Remark 1, we have \( k_L/c_U \leq k(z)/C(z) \leq k_U/c_L \), therefore the inequality is obtained.

We have gotten the existence and uniqueness of weak solution of PWTS, next we will give the basic structure of the solution. If there is no risk of confusion, we will use the notation \( k \) to be \( k(z) \).

**Theorem 2.3.** Suppose that assumptions (A1) and (A2) are valid, let \( \{\lambda_n, \omega_n\}_{n=1}^{\infty} \) be eigenvalues and eigenfunctions of SLP,

\[
\mu_n = (u(0), \omega_n)C, \phi_n(t) = (\Phi(\cdot, t), \omega_n)C, \quad f_n(t) = (f(\cdot, t), \omega_n).
\]

The solution of PWTS is given by

\[
T(z, t; k) = \Phi(z, t) + \sum_{n=1}^{\infty} b_n(t; k) \omega_n(z),
\]

where

\[
b_n(t; k) = e^{-\lambda_n t} \mu_n + \int_0^t e^{-\lambda_n (t-\tau)} (f_n(\tau) - \phi_n'(\tau; k)) \, d\tau.
\]

And for \( 0 < t_0 \leq t \leq t_f \), the series in (4) is uniform convergence in \( C[0, l] \).

**Proof.** Let \( \{\omega_n\}_{n=1}^{\infty} \) be the orthonormal basis of eigenfunctions in \( H_C \) corresponding to the \( k(z) \in A_{ad} \). For the equation

\[
C(z)u_t - (k(z)u_z)_z = f(z, t) - C(z)\Phi_t,
\]

we multiply the two sides of above equation by \( \omega_n(z) \) and integrate them over \( \Omega \),

\[
\int_{\Omega} C(z) u_t \omega_n(z) \, dz - \int_{\Omega} (k(z)u_z)_z \omega_n(z) \, dz = \int_{\Omega} f(z, t) \omega_n(z) \, dz - \int_{\Omega} C(z) \Phi_t \omega_n(z) \, dz.
\]

For the second term on the left,

\[
\int_{\Omega} (k(z)u_z)_z \omega_n(z) \, dz = - \int_{\Omega} \lambda_n C(z) u(z) \omega_n(z) \, dz.
\]

Combining the above two equations, we have

\[
\int_{\Omega} C(z) u_t \omega_n(z) \, dz + \int_{\Omega} \lambda_n C(z) u(z) \omega_n(z) \, dz = \int_{\Omega} f(z, t) \omega_n(z) \, dz - \int_{\Omega} C(z) \Phi_t \omega_n(z) \, dz.
\]
Let \( b_n(t) = (u(\cdot, t), \omega_n) = \int_\Omega C(z) \omega_n(z) dz \), thus
\[
\begin{align*}
\begin{cases}
\dot{b}_n(t) + \lambda_n b_n(t) = f_n(t) - \phi'_n(t), \\
  b_n(0) = \mu_n.
\end{cases}
\end{align*}
\]
Therefore \( b_n(t) \) has the representation stated in (5).

Then \( u = \sum_{n=1}^\infty b_n(t) \omega_n(z) \). Because \( V \) is continuously imbedded in \( C[0, 1] \) and \( 0 < k_L \leq k(z) \leq k_U \), the original norm in \( V \) is equivalent to the norm in \( V_k \). Thus the uniform convergence of the series in \( V_k \) can be proved. For the eigenvalue \( \lambda_n \) and corresponding eigenfunction \( \omega_n \) one has \( (k(z)\omega'_n(z))' = -\lambda_n \omega_n(z) \), we multiply the two sides by \( \omega_j(z) \) and integrate them over \( \Omega \), thus
\[
\int_\Omega k(z) \omega'_n(z) \omega_j(z) dz = \int_\Omega \lambda_n C(z) \omega_n \omega_j \text{ for all } n \text{ and } j.
\]
And \( \| \frac{\omega_n}{\sqrt{\lambda_n n}} \|_{V_k} = \int_\Omega k(z) \frac{\omega_n^2}{\lambda_n} dz = \int_\Omega C(z) (\omega_n)^2 dz = 1 \), thus \( \{ \frac{\omega_n}{\sqrt{\lambda_n n}} \}_{n=1}^\infty \) is an orthonormal basis in \( V_k \). Therefore the series converges in \( V_k \) if and only if \( \sum_{n=1}^\infty \lambda_n |b_n(t)|^2 = \|u(\cdot, t; k)||_{V_k} < \infty \) for any \( t > 0 \).

\[
\begin{align*}
\sum_{n=1}^\infty \lambda_n |b_n(t)|^2 \\
&\leq \sum_{n=1}^\infty \lambda_n e^{-2\lambda_n t} \mu_n^2 + 2 \sum_{n=1}^\infty \lambda_n |e^{-\lambda_n t} \mu_n \int_0^t e^{-\lambda_n (t-\tau)} (f_n(\tau) - \phi'_n(\tau; k)) d\tau| \\
&\quad + \sum_{n=1}^\infty \lambda_n \int_0^t e^{-\lambda_n (t-\tau)} (f_n(\tau) - \phi'_n(\tau; k)) d\tau| \\
&\leq \sum_{n=1}^\infty \lambda_n e^{-2\lambda_n t} \mu_n^2 \\
&\quad + 2 \left( \sum_{n=1}^\infty \lambda_n e^{-2\lambda_n t} \mu_n^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty \lambda_n \int_0^t e^{-2\lambda_n (t-\tau)} (f_n(\tau) - \phi'_n(\tau; k)) d\tau| \right)^{\frac{1}{2}} \\
&\quad + \sum_{n=1}^\infty \lambda_n \int_0^t e^{-2\lambda_n (t-\tau)} (f_n(\tau) - \phi'_n(\tau; k)) d\tau|.
\end{align*}
\]

According to the fact the function \( s \to \sqrt{se^{-\theta s}} \) is bounded on \([0, \infty)\) for any \( \theta > 0 \), we can get that \( \sum_{n=1}^\infty \lambda_n e^{-2\lambda_n t} \mu_n^2 \) and \( \sum_{n=1}^\infty \lambda_n \int_0^t e^{-2\lambda_n (t-\tau)} (f_n(\tau) - \phi'_n(\tau; k)) d\tau| \) are bounded. Thus \( \sum_{n=1}^\infty \omega_n(z)|b_n(t)|^2 < \infty \), the uniform convergence is proved.

In next section, the continuous mapping \( k \to T(z, t; k) \) will be established and an identification model will be given.

3. **Identification Model.** In this section, we consider the identification problem of PWTS. The performance criterion is given by
\[
J(k) = \| \frac{T(z, t; k) - T_{\text{mea}}(z, t)}{T_{\text{max}}} \|_{L^2((0, t) \times (0, l_j))},
\]
where \( T(z, t; k) \) is obtained from PWTS, \( T_{\text{mea}}(z, t) \) is a fitted function of the observed temperature data of snow and sea ice, and \( T_{\text{max}} \) is the maximum temperature
among the field data. To make the temperature \( T(z, t; k) \) be close to the observations, an identification model is established and denoted as IM:

\[
\begin{align*}
\min \ & J(k), \\
\text{s.t} \ & T(z, t; k) \in C([0, t_f]; C[0, l]), \\
\ & k \in A_{ad}.
\end{align*}
\]

Next we will establish the continuity of mapping \( k \rightarrow T(z, t; k) \).

3.1. **Continuity of the solution mapping on thermal conductivity.** In this section, we will establish the continuity of the mapping \( k \rightarrow T(z, t; k) \). For this purpose, Theorem 3.1 and 3.2 will present the continuity of the mappings \( k \rightarrow \lambda_n(k) \) and \( k \rightarrow \omega_n(k) \).

**Theorem 3.1.** Suppose that (A1) and (A2) hold, \( \{\lambda_n(k)\}_{n=1}^{\infty} \) be the eigenvalues of SLP, then the mapping \( k \rightarrow \lambda_n(k) \) is continuous for \( n = 1, 2, \ldots \).

**Proof.** Let \( k, \tilde{k} \in A_{ad}, \{\lambda_n, \omega_n\}_{n=1}^{\infty} \) and \( \{\tilde{\lambda}_n, \tilde{\omega}_n\}_{n=1}^{\infty} \) be the eigenvalues and eigenfunctions corresponding to \( k \) and \( \tilde{k} \) respectively. From the Lemma 2.2, the eigenfunctions form a complete orthonormal set in \( H_C \). Since for any \( \omega \in H_C^1(0, l) \),

\[
\int_\Omega k(z)\omega_j(z)dz = \int_\Omega \lambda_j(z)\omega_j(z)dz,
\]

thus for any \( \omega \in W_n \) can be expressed by

\[
\omega(z) = \sum_{i=1}^{n} a_i \omega_i(z), \quad a_i \in \mathbb{R}.
\]

Because the eigenvalues of SLP satisfy the min-max principle

\[
\lambda_n = \min_{W_n} \max_{\omega \in W_n} \left\{ \frac{\int_\Omega k(z)\omega(z)dz}{\int_\Omega C(z)\omega(z)^2dz} : \omega \in W_n \right\}.
\]

Therefore

\[
\tilde{\lambda}_n \leq \max_{\omega \in W_n} \frac{\int_\Omega \tilde{k}(z)\omega(z)dz}{\int_\Omega C(z)\omega(z)^2dz} \leq \max_{\omega \in W_n} \frac{\int_\Omega \tilde{k}(z)\omega(z)dz}{\int_\Omega C(z)\omega(z)^2dz} + \max_{\omega \in W_n} \frac{\int_\Omega (\tilde{k}(z) - k(z))\omega(z)dz}{\int_\Omega C(z)\omega(z)^2dz}.
\]

According to Lemma 2.2 and \( \int_\Omega k(z)\omega_j^i(z)dz = 0 \) \( (i \neq j) \), the first term on the right can be transformed to

\[
\max_{\omega \in W_n} \frac{\int_\Omega k(z)\omega(z)dz}{\int_\Omega C(z)\omega(z)^2dz} = \max \left\{ \frac{\sum_{i=1}^{n} a_i^2 \lambda_i}{\sum_{i=1}^{n} a_i^2} : a_i \in \mathbb{R} \right\} = \lambda_n,
\]

Thus

\[
\tilde{\lambda}_n \leq \lambda_n + \|\tilde{k}(z) - k(z)\|_{L^1} \max_{a_i} \frac{\sum_{i=1}^{n} a_i w_i'(z)^2}{\sum_{i=1}^{n} a_i} \leq \lambda_n + \|\tilde{k}(z) - k(z)\|_{L^1} \sum_{i=1}^{n} |w_i'(z)|^2.
\]

According to Lemma 2.2, \( \omega_n(z) \) and \( k(z)\omega_n'(z) \) are all continuous on \([0, l]\), there exists \( p \in (0, l) \) such that \( \omega_n'(p) = 0 \). Let \( z \in (0, l) \) and \( z \neq l_1 \), integrating
According to the Arzela-Ascoli Theorem, its closure is compact in $L^2[0, l]$. By the inequality (3) and the above inequality, the set $\{\omega_n(z; k)\}$ is precompact in $L^2[0, l]$. Because $m \to \infty$, we get $k(z)\omega'_n(z) = \lambda_n \int_z^p C(s)\omega_n(s)ds$. According to the Cauchy's inequality

$$k_L|\omega'_n(z)| \leq |k(z)\omega'_n(z)| = |\lambda_n \int_z^p C(s)\omega_n(s)ds| \leq \lambda_n \int_{\Omega} |C(z)\omega_n(z)|dz \leq \lambda_n \sqrt{c_L l}.$$ 

By the inequality (3) and the above inequality

$$\sum_{i=1}^n |\omega'(z)|^2 \leq \sum_{i=1}^n \frac{\lambda^2_c U l}{k^2_L} \leq \frac{k^2_L \pi^4 n^5}{k^2_L c^2 U l} = F(n).$$

Substituting (8) to (7), we can get that the mapping $k \to \lambda_n(k)$ is continuous. \qed

**Theorem 3.2.** Suppose that (A1) and (A2) hold, $\{\omega_n(z; k)\}_{n=1}^\infty$ be the normalized eigenfunctions of system SL$(P)$ satisfying $\omega'_n(0+; k) > 0$, then the mapping $k \to \omega_n(k)$ is continuous for $n = 1, 2, \ldots$.

**Proof.** Fix a positive number $n$.

$$|\omega_n(z_2; k) - \omega_n(z_1; k)| = |\int_{z_1}^{z_2} \omega_n'(s; k)ds| \leq \int_{z_1}^{z_2} |\omega'_n(s; k)|ds \leq \lambda_n \sqrt{c_U l} |z_2 - z_1|.$$ 

Thus the set $\{\omega_n(z; k) : k \in A_{ad}\}$ is equicontinuous and equibounded in $C[0, l]$. According to the Arzela-Ascoli Theorem, its closure is compact in $C[0, l]$.

$$|k(z_2) \omega'_n(z_2; k) - k(z_1) \omega'_n(z_1; k)| = |\int_{z_1}^{z_2} [k(s)\omega'_n(s; k)]'ds| \leq \lambda_n \int_{z_1}^{z_2} |C(s)\omega_n(s; k)|ds \leq \lambda_n \sqrt{c_U l} |z_2 - z_1|.$$ 

The set $\{k\omega'_n(z; k) : k \in A_{ad}\}$ is equibounded and equicontinuous in $C[0, l]$. Therefore its closure is compact in $C[0, l]$.

Let sequence $k_m \to k$ as $m \to \infty$. The sequences of eigenvalues and eigenfunctions $\omega_n(k_m)$ and $k_m\omega_n(k_m)$ are located within a compact set in $C[0, l]$. Thus the mapping $k \to \omega_n(k)$ is continuous if $\omega_n(k_m)$ and $k_m\omega_n(k_m)$ both have a unique cluster point $\omega_n(k)$ and $k\omega_n(k)$ respectively. Assumption $\omega_n(k_m) \to \eta$, $k_m\omega_n(k_m) \to \xi$, $\eta, \xi \in C[0, l]$.

From $\|k_m - k\|_{L^1} \to 0$, we have $\|1/k_m - 1/k\|_{L^1} \to 0$. Because $0 < k_L \leq k(z) \leq k_U$, we have the convergence $k_m \to k$ and $1/k_m \to 1/k$ is also in $L^2[0, l]$. From above analysis, $\omega'_n(k_m)$ also converges in $L^2[0, l]$ and $\omega'_n(k_m) \to \xi/k$ as $m \to \infty$. So $\omega'_n(k_m) \to \eta$ in $H^1_0[0, l]$ and $\eta k' = \xi$. From Lemma 2.2, $\int_{\Omega} k_m(s)(\omega'_n(s; k_m))^2ds = \lambda_n(k_m)$, we have

$$\int_{\Omega} k(s)(\omega'_n(s; k_m))^2ds = \lambda_n(k_m) + \int_{\Omega} (k(s) - k_m(s))(\omega'_n(s; k_m))^2ds. \quad (9)$$ 

Because $k_m\omega'_n(k_m)$ is convergent in $L^2[0, l]$, the set $\{k\omega'_n(k)\}$ is precompact in $L^2[0, l]$. The mapping $k \to 1/k$ is continuous, so the set $\{\omega'_n(k) : k \in A_{ad}\}$ is precompact in $L^2[0, l]$. $\omega'_n(k)$ is bounded, so $\{\omega'_n(k) : k \in A_{ad}\}$ is precompact in $L^1[0, l]$. According to Lemma 3.1 in [8], for the second term on the right of (9), $\frac{1}{2}\pi^4 n^5 \int_{\Omega} (k(s) - k_m(s))(\omega'_n(s; k_m))^2ds \to 0$ as $\|k(s) - k_m(s)\|_{L^1} \to 0$. Thus we take $m \to \infty$ in (9), we get $\int_{\Omega} k(s)\eta'(s)^2ds = \lambda_n(k)$. Since $\|\omega_n(k_m)\|_{C^1} = 1$ for any
n, we have \( \| \eta \|_{C^1} = 1 \). Suppose the continuous dependence \( k \to \omega_n(k) \) is already shown for \( j = 1, 2, \ldots, n - 1 \). Then
\[
\int_{\Omega} C(z) \omega_j(k) \eta \, dz = \lim_{m \to \infty} \int_{\Omega} C(z) \omega_j(k \omega_m) \eta \, dz = 0.
\]
Thus \( \eta \) is an eigenfunction for \( \lambda_n(k) \). From Lemma 2.2, each eigenvalue is simple, therefore \( \eta = \omega_n(k) \) or \( \eta = -\omega_n(k) \). From the above discuss that \( k_m \omega'_n(k_m) \to \xi = kn' \) in \( C[0,l] \). \( k_n(0) \omega'_n(0; k_m) \to \xi(0) = k(0) n'(0+) \leq 0 \) since \( \omega'_n(0; k_m) > 0 \). Only \( \eta = \omega_n(k) \), the above is possible, thus the theorem is proved.

3.2. Optimality of thermal conductivity. Now we will establish the optimality of the thermal conductivity. Next Theorem describes the existence of thermal conductivity.

**Theorem 3.3.** Suppose that assumptions (A1) and (A2) hold, then there exists a solution \( k^* \in A_{ad} \) satisfying IM.

**Proof.** First we will show that the mapping \( k \to T(z,t;k) \) is continuous.

Let \( u^N(z,t;k) = \sum_{n=1}^N b_n(z;k) \omega_n(z) \), the \( b_n(z;k) \) is given by (5). According to Theorems 3.1 and 3.2 and the representation of \( b_n(z;k) \), we have the mapping \( k \to u^N(k) \) is continuous. By Theorem 2.3, we can get \( u^N(k) \to u \) as \( N \to \infty \).

Because the solution \( T(z,t;k) \) of PWTS is given by \( T(z,t;k) = u(z,t;k) + \Phi(z,t;k) \), therefore the mapping \( k \to T(z,t;k) \) is continuous.

Let \( X(T;k) = T(z,t;k) - T_{mea}(z,t) \), then \( J(k) = \| X(T;k) \|_{L^2(\Omega \times I)} \geq 0 \). Obviously, \( X(T;k) \) is continuous on \( A_{ad} \), thus \( J(k) \) is continuous on \( A_{ad} \). Since \( A_{ad} \) is a nonempty bounded closed set, there exists a solution \( k^* \in A_{ad} \), such that \( J(k^*) \leq J(k) \) for all \( k \in A_{ad} \), and this proof is completed.

4. Conclusions. In this paper, we aim to provide the optimality of piecewise thermal conductivity in a snow-ice thermodynamic system. Based on the characteristics of the snow-sea ice coupling temperature distribution, we have constructed a snow-ice thermodynamic system and presented the existence and uniqueness of the weak solution of the system. Then we have taken the thermal conductivities as the control variable and the temperature deviation between the system and the observations of snow and sea ice as the performance criterion, an identification model has been established. Finally, we have gotten the optimality of thermal conductivity.

However, the volumetric heat capacities and the thermal conductivities of snow and sea ice not only are relevant to depth but also temperature and salinity. In addition, our mathematical framework do not include the impact of the oceanic heat flux. These limitations will be subject to the future work. The optimization algorithm and numerical results will be presented in a forthcoming paper.

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