ON THE RESTRICTION MAP FOR JACOBI FORMS

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Abstract. In this article we give a description of the kernel of the restriction map for Jacobi forms of index 2 and obtain the injectivity of $D_0 \oplus D_2$ on the space of Jacobi forms of weight 2 and index 2. We also obtain certain generalization of these results on certain subspace of Jacobi forms of square-free index $m$.

1. Introduction

Let $D_0$ be the restriction map from the space of Jacobi forms of weight $k$, index $m$ on the congruence subgroup $\Gamma_0(N)$ to that of elliptic modular forms of the same weight on $\Gamma_0(N)$ given by $\phi(\tau,z) \mapsto \phi(\tau,0)$. More generally, one obtains modular forms of weight $k + \nu$ from Jacobi forms of weight $k$ by using certain differential operators $D_\nu$. Then it is known that the direct sum $\oplus_{\nu=0}^m D_\nu$ is injective for $k$ even, but in general the restriction map $D_0$ is not injective (see [1, 2, 3] for details). However, J. Kramer [4] and T. Arakawa and S. Böcherer [2] observed that when $k = 2$ the situation may be different. In fact, when $m = 1$, Arakawa and Böcherer [1] provided two explicit descriptions of $\text{Ker}(D_0)$: one in terms of modular forms of weight $k - 1$ and the other in terms of cusp forms of weight $k + 2$ (by applying the differential operator $D_2$ on $\text{Ker}(D_0)$). In a subsequent paper [2], they proved that $D_0$ is injective in the case $k = 2$, $m = 1$ and gave some applications. In a private communication to the authors, Professor Böcherer informed that one of his students gave a precise description of the image of $D_0 \oplus D_2$ in terms of vanishing orders at the cusps ($k$ arbitrary, $m = 1$). Based on this, he conjectured that in the case $k = 2$, one can remove one of the $D_2\nu$ from the direct sum $\oplus_{\nu=0}^m D_\nu$ without affecting the injectivity.

In this paper, we generalize the results of [1] to higher index. In §3, we consider the case $m = 2$ and show that $\text{Ker}(D_0)$ is isomorphic to the space of vector-valued modular forms of weight $k - 1$ and $D_2(\text{Ker}(D_0))$ is isomorphic to a certain subspace of cusps forms of weight $k + 2$ and these two spaces are related with each other by a simple isomorphism (Theorem 3.3). In §3.3, we obtain the injectivity of $D_0 \oplus D_2$ on $J_{2,2}(\Gamma_0(2N))$, where $N = 2$ or an odd square-free positive integer (Theorem 3.4). This confirms the conjecture made by Professor Böcherer partially in the index 2 case (i.e., we can omit the operator $D_4$). In §4, we consider a subspace of $J_{k,m}(\Gamma_0(mN), \chi)$, where $m$ is a square-free positive integer and $N$ is any positive integer and obtain results similar to [1] and prove the injectivity of $D_0$ on this subspace when $k = 2$, and $mN$ is square-free (Theorem 4.2 and Corollary 4.4). In §5, we make several remarks concerning the subspace studied in §4.

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2. Preliminaries

Let \( k, m \) and \( N \) be positive integers and \( \chi \) be a Dirichlet character modulo \( N \). We denote the standard generators of the full modular group \( SL_2(\mathbb{Z}) \) by \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and let \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) be the identity matrix. One has the relation \( S^2 = (ST)^3 = -I_2 \). Let \( \mathbb{H} \) denote the complex upper half-plane. Define the action of the full modular group \( SL_2(\mathbb{Z}) \times \mathbb{Z}^2 \) on \( \mathbb{H} \times \mathbb{C} \) by

\[
[\gamma, (\lambda, \mu)](\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right),
\]

where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) \( \in SL_2(\mathbb{Z}) \), \( \lambda, \mu \in \mathbb{Z} \) and \( (\tau, z) \in \mathbb{H} \times \mathbb{C} \). We simply write \( \gamma(\tau, z) \) for \( [\gamma, (0, 0)](\tau, z) \). We denote the space of Jacobi forms of weight \( k \), index \( m \) and Dirichlet character \( \chi \) for the Jacobi group \( \Gamma_0(N) \times \mathbb{Z}^2 \) by \( \text{J}_{k,m}(\Gamma_0(N), \chi) \). It is well known that any such Jacobi form \( \phi(\tau, z) \) can be (uniquely) written as

\[
\phi(\tau, z) = \sum_{r=0}^{2m-1} h_{m,r}(\tau) \theta_{m,r}^J(\tau, z),
\]

with

\[
\theta_{m,r}^J(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i m((n+\frac{r}{2m})^2 + 2(n+\frac{r}{2m})z)} e^{2\pi i(n-\frac{r^2}{4m})\tau},
\]

\[
h_{m,r}(\tau) = \sum_{n \in \mathbb{Z}} c_{\phi}(n, r) e^{2\pi i(n-\frac{r^2}{4m})\tau},
\]

where \( c_{\phi}(n, r) \) denotes the \((n, r)\)-th Fourier coefficient of the Jacobi form \( \phi \).

The (column) vector \( \Theta^J(\tau, z) = (\theta_{m,r}^J(\tau, z))_{0 \leq r < 2m} \) satisfies the transformation

\[
\Theta^J(\gamma(\tau, z)) = e^{2\pi i m \tau} (c\tau + d)^\frac{1}{2} U_m(\gamma) \Theta^J(\tau, z)
\]

for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) \( \in SL_2(\mathbb{Z}) \). Here \( U_m : SL_2(\mathbb{Z}) \rightarrow U(2m, \mathbb{C}) \) is a (projective) representation of \( SL_2(\mathbb{Z}) \). Note that \( U_m \) is an example of a Weil representation (associated with the discriminant form \((D, Q)\) with \(D = \mathbb{Z}/2m\mathbb{Z}\) and \(Q : D \rightarrow \mathbb{Q}/\mathbb{Z}\) given by \(Q(x) = x^2/4m\)). In the cases \( m = 1 \) and \( 2 \), it is given by

\[
U_1(T) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad U_1(S) = \frac{e^{-\pi i/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

\[
U_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{i} \end{pmatrix}, \quad U_2(S) = \frac{e^{-\pi i/4}}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.
\]
The (column) vector \( \mathbf{h} = (h_{m,r})_{0 \leq r < 2m} \) satisfies the following transformation property.

For any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \), we have

\[
\mathbf{h}(\gamma \tau) = \chi(d)(c\tau + d)^{k-\frac{1}{2}}U_m(\gamma)\mathbf{h}(\tau),
\]

where \( \gamma \tau = \frac{ar + b}{cr + d} \).

For any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m) \), let \( \gamma_m \) denote the \( SL_2(\mathbb{Z}) \) matrix \( \begin{pmatrix} a & bm \\ c & d \end{pmatrix} \). Let the matrices \((u_{ij})_{0 \leq i,j < 2m}\) and \((u_{ij}^m)_{0 \leq i,j < 1}\) represent \( U_m(\gamma) \) and \( U_1(\gamma) \) respectively. Since \( \theta_m^J(\tau, z) = \theta_{1,0}^J(m\tau, mz) \) and \( \theta_m^J(\gamma, z) = \theta_{1,1}^J(m\tau, mz) \), for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m) \), we have

\[
(\theta_m^J, \theta_m^J)^J(\gamma(\tau, z)) = (\theta_{1,0}^J, \theta_{1,1}^J)^J(m\gamma(\tau, z))) = (\theta_{1,0}^J, \theta_{1,1}^J)^J(m\tau, mz))
\]

\[
eq e^{2\pi i m c t d}(c\tau + d)^{\frac{1}{2}}U_1(\gamma)(\theta_m^J, \theta_m^J)^J(\tau, z).
\]

Now, comparing the transformation properties for the action of \( \gamma \in \Gamma_0(m) \) as given in (2) and (3) for \( \theta_m^J \) and \( \theta_m^J \), we have two linear equations in \( (\theta_m^J, \theta_m^J)^J(m\tau, mz) \) as follows:

\[
(u_{00} - u_{00}^m)\theta_m^J, 0 + (u_{0m} - u_{01}^m)\theta_m^J, m + \sum_{j \neq 0, m} u_{0j}\theta_m^J, j = 0.
\]

\[
(u_{m0} - u_{10}^m)\theta_m^J, 0 + (u_{mm} - u_{11}^m)\theta_m^J, m + \sum_{j \neq 0, m} u_{mj}\theta_m^J, j = 0.
\]

Since the set \( \{\theta_m^J\}_{0 \leq r < 2m} \) is linearly independent over the field of complex numbers \( \mathbb{C} \) (see for example [10, Lemma 3.1]), we have

\[
u_{00} = u_{00}^m, u_{0m} = u_{01}^m, u_{0j} = 0 \text{ for all } j \neq 0, m;
\]

\[
u_{m0} = u_{10}^m, u_{mm} = u_{11}^m, u_{mj} = 0 \text{ for all } j \neq 0, m.
\]

Using (3) and the above observation for \( U_m \), we also have

\[
(h_{m,0}, h_{m,m})^J(\gamma \tau) = \chi(d)(c\tau + d)^{k-\frac{1}{2}}U_1(\gamma)(h_{m,0}, h_{m,m})^J(\tau),
\]

for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN) \).

For any two matrices \( \gamma, \gamma' \in \Gamma_0(m) \), \( (\gamma \gamma')_m = \gamma_m \gamma'_m \). Using this property we define a character \( \omega_m \) on \( \Gamma_0(m) \) by

\[
\omega_m(\gamma) = \det(U_1(\gamma)) \quad (\gamma \in \Gamma_0(m)).
\]

Note that the character \( \omega_1 \) is the same as the character \( \omega \) as defined in [11 p. 311].

Set \( \theta_{m,r}(\tau) := \theta_m^J(\tau, 0) \). Putting \( z = 0 \) in (2), we get the following transformation property for the column vector \( \Theta = (\theta_{m,r})_{0 \leq r < 2m} \).

\[
\Theta(\gamma \tau) = (c\tau + d)^{1/2}U_m(\gamma)\Theta(\tau),
\]
for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Let $\eta(\tau) = e^{2\pi i \tau/24} \prod_{n \geq 1} (1 - e^{2\pi in\tau})$ denote the Dedekind eta function. Combining the equations (76.1), (78.4) and (78.6) of [6], one has the following identity:

$$\eta^3(2\tau) = \frac{1}{2} \theta_{1,0}(\tau)\theta_{1,1}(\tau) \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi in^2\tau}. \tag{10}$$

The above identity implies that $\theta_{1,0}$ and $\theta_{1,1}$ have no zeros in the upper half-plane.

Let $\xi(\tau) = (\theta_{1,1}\theta_{1,0}' - \theta_{1,0}\theta_{1,1}')(\tau)$ be the cusp form of weight 3 for $SL_2(\mathbb{Z})$ with character $\omega$ (see [1, Proposition 2]). For any $m \geq 1$, define

$$\xi_m^*(\tau) := (\theta_{m,m}\theta_{m,0}' - \theta_{m,0}\theta_{m,m}')(\tau). \tag{11}$$

Since $\theta_{m,0}(\tau) = \theta_{1,0}(m\tau)$ and $\theta_{m,m}(\tau) = \theta_{1,1}(m\tau)$, we get that $\xi_m^*(\tau) = m\xi(\tau)$. Therefore, $\xi_m^*$ is a cusp form of weight 3 for the group $\Gamma_0(m)$ with character $\omega_m$. Note that $\xi_1^* = \xi$. It is proved in [1, Proposition 2] and [5] that $\xi(\tau) = -\pi im\eta^6(\tau)$. Therefore,

$$\xi_m^*(\tau) = -\pi im\eta^6(m\tau) \tag{12}$$

and hence $\xi_m^*$ has no zeros in $\mathbb{H}$.

Let $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) denote the vector space of modular forms (resp. cusp forms) of weight $k$ for the group $\Gamma_0(N)$ with character $\chi$. Let $D_0 : J_{k,m}(\Gamma_0(N), \chi) \rightarrow M_k(\Gamma_0(N), \chi)$ be the restriction map given by $\phi(\tau, z) \mapsto \phi(\tau, 0)$ and $D_2$ be the differential operator

$$D_2 = \left( \frac{1}{2\pi i} \frac{\partial^2}{\partial z^2} - \frac{4}{\partial \tau} \right) \bigg|_{z=0}, \tag{13}$$

which acts on holomorphic functions on $\mathbb{H} \times \mathbb{C}$. Note that $D_2$ maps the space Jacobi form $J_{k,m}(\Gamma_0(N), \chi)$ into the space of cusp forms $S_{k+2}(\Gamma_0(N), \chi)$. We denote the kernel of the restriction map $D_0 : J_{k,m}(\Gamma_0(N), \chi) \rightarrow M_k(\Gamma_0(N), \chi)$ by $J_{k,m}(\Gamma_0(N), \chi)^0$. Without loss of generality we assume that the space $M_k(\Gamma_0(N), \chi)$ is non-empty, otherwise $\ker(D_0)$ is the entire space $J_{k,m}(\Gamma_0(N), \chi)$. In particular, this implies that $\chi(-1) = (-1)^k$. Using the transformation property of any Jacobi form for the matrix $-I_2$, we get the symmetry relation $h_{m,r}(\tau) = h_{m,2m-r}(\tau)$ for all $r \in \mathbb{Z}/2m\mathbb{Z}$.

3. THE SPACE OF JACOBI FORMS OF INDEX 2

Throughout this section, we assume that $N = 2$, or an odd square-free positive integer. In this section, we study the kernel of the restriction map $D_0$ for the space of index 2 Jacobi forms on $\Gamma_0(2N)$ and deduce the injectivity of $D_0 \oplus D_2$ in the weight 2 case.

It is elementary to verify that the set

$$X = \{ V = (v_{ij})_{0 \leq i,j \leq 3} : v_{ij} = 0 \iff i + j \equiv 0 \pmod{2}, v_{11} = v_{33}, v_{13} = v_{31} \}$$

is a subgroup of $GL_4(\mathbb{C})$ and hence it is easy to verify that the function

$$r(V) = v_{11} + v_{13} \tag{14}$$
Define two new functions as

\[ \theta \]

From (19), we have

\[ X \] is a character on the subgroup \( H \)

and implies that

\[ U_2(I_2) = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \]

\[ U_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{i} \end{pmatrix} \]

Therefore, it follows that \( U_2(\gamma) \in X \) for every \( \gamma \in \Gamma_0(2) \). Thus, we have a (projective) representation of \( \Gamma_0(2) \) defined by

\[ \rho_2(\gamma) = r(U_2(\gamma))^{-1}U_1(\gamma_2), \]

where \( \gamma_2 = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), with \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \) and \( r \) is the character defined by (14). Since \( U_2(\gamma) \in X \) for all \( \gamma \in \Gamma_0(2) \), equation (9) gives us the following transformation properties.

\[ \theta_{2,1}(\gamma \tau) = (c\tau + d)^{1/2}(u_{11}\theta_{2,1} + u_{13}\theta_{2,3})(\tau), \]

\[ \theta_{2,3}(\gamma \tau) = (c\tau + d)^{1/2}(u_{13}\theta_{2,1} + u_{11}\theta_{2,3})(\tau), \]

where \( U_2(\gamma) = (u_{ij})_{0 \leq i, j \leq 3} \) as assumed before. One also has the following relations.

\[ \theta_{2,1}(\tau) = \sum_{n \in \mathbb{Z}} e^{4\pi i (n + \frac{1}{2})^2 \tau} = \sum_{n \in \mathbb{Z}} e^{4\pi i (n - \frac{1}{2})^2 \tau} = \sum_{n \in \mathbb{Z}} e^{4\pi i (n + \frac{1}{2})^2 \tau} = \theta_{2,3}(\tau), \]

\[ \theta_{1,1}(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i (n + \frac{1}{2})^2 \tau} = \sum_{n \in \mathbb{Z}} e^{2\pi i (2n + \frac{1}{2})^2 \tau} + \sum_{n \in \mathbb{Z}} e^{2\pi i ((2n + 1) + \frac{1}{2})^2 \tau} = \theta_{2,1}(2\tau) + \theta_{2,3}(2\tau) \]

\[ = 2\theta_{2,1}(2\tau). \]

From (19), we have \( \theta_{2,1}(\tau) = \frac{1}{2}\theta_{1,1}(\tau/2) \) and since \( \theta_{1,1} \) has no zeros in \( \mathbb{H} \), \( \theta_{2,1} \) also has no zeros in the upper half-plane \( \mathbb{H} \).

3.1. Connection to the space of vector valued modular forms. We start now from a Jacobi form \( \phi \in J_{k,2}(\Gamma_0(2N),\chi^0) \), the kernel of the restriction map \( D_0 \). This implies that

\[ 0 = \phi(\tau, 0) = h_{2,0}(\tau)\theta_{2,0}(\tau) + 2h_{2,1}(\tau)\theta_{2,1}(\tau) + h_{2,2}(\tau)\theta_{2,2}(\tau). \]

Define two new functions as

\[ \varphi_0(\tau) := \frac{h_{2,0}(\tau)}{\theta_{2,1}(\tau)}, \quad \varphi_2(\tau) := \frac{h_{2,2}(\tau)}{\theta_{2,1}(\tau)}. \]
As observed before, $\theta_{2,1}$ has no zeros in the upper half-plane, and hence, $\varphi_0$ and $\varphi_2$ are holomorphic in the upper half-plane.

**Proposition 3.1.** Let $\varphi_0$ and $\varphi_2$ be as in (20). Then, $(\varphi_0, \varphi_2)^t$ is a vector valued modular form on $\Gamma_0(2N)$ of weight $(k-1)$ with character $\chi$ and representation $\rho_2$. We denote the space of all such vector valued modular forms by $M_{k-1}(\Gamma_0(2N), \chi; \rho_2)$.

**Proof.** We have to check the transformation property and holomorphy condition at the cusps. First we check the transformation property. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$ with $\gamma_2 = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$ as defined in §2. Using the definition of $\varphi_0, \varphi_2$ and using the transformation (7) with $m = 2$, we have

$$\theta_{2,1}(\gamma \tau)(\varphi_0, \varphi_2)^t(\gamma \tau) = \chi(d)(c \tau + d)^{k-\frac{1}{2}}U_1(\gamma_2)\theta_{2,1}(\tau)(\varphi_0, \varphi_2)^t(\tau).$$

(21)

Using (16) with (18), we get $\theta_{2,1}(\gamma \tau) = (u_{11} + u_{13})(c \tau + d)^{\frac{k}{2}}\theta_{2,1}(\tau)$. Since, $(u_{11} + u_{13}) = r(U_2(\gamma))$, by using the definition of the representation $\rho_2$ given by (15), the above formula (21) gives the following transformation property:

$$(\varphi_0, \varphi_2)^t(\gamma \tau) = \chi(d)(c \tau + d)^{k-1}\rho_2(\gamma)(\varphi_0, \varphi_2)^t(\tau).$$

(22)

It remains to investigate the behaviour of $\varphi_0$ and $\varphi_2$ at each cusp of $\Gamma_0(2N)$. A complete set of cusps of $\Gamma_0(2N)$ is given by the numbers $\frac{n}{c}$ where $c$ runs over positive divisors of $2N$ and for a given $c$, $a$ runs through integers with $1 \leq a \leq 2N$, gcd$(a, 2N) = 1$ that are inequivalent modulo gcd$(c, \frac{2N}{c})$. For any $N$, the cusps corresponding to the divisors 1 and $2N$, (i.e., the cusps 1 and $\frac{1}{c}$) are equivalent to 0 and $\infty$ respectively. So we can assume that all the cusps of $\Gamma_0(2N)$ are given by $\infty$, 0 and $\frac{1}{c}$ with $c | 2N$ and $c \neq 1, 2N$. Now choose any cusp $s$ from the above list. We divide our cusp condition verification into the following two cases:

**Case 1:** Suppose that $s = \infty$. Let us denote $e^{2\pi i \tau}$ by $q$. If $\phi \in \text{Ker}(D_0)$, then $\phi(0, \tau) = \sum c_\phi(n, r)q^n = 0$ implies that $c_\phi(0, 0) = 0$. Therefore, $h_{2,0}(\tau) = \sum c_\phi(n, 0)q^n = q\sum_{n \geq 1} c_\phi(n, 0)q^{n-1}$. Also we have $h_{2,2}(\tau) = \sum c_\phi(n, 2)q^{n-\frac{1}{2}} = q^{\frac{1}{2}}\sum_{n \geq 1} c_\phi(n, 2)q^{n-1}$ and $\theta_{2,1}(\tau) = q^{2(n+\frac{1}{2})^2} = q^{\frac{1}{2}}\sum_{n \in \mathbb{Z}} q^{2n^2+n}$. Now the holomorphy at the cusp $\infty$ of the functions $\varphi_0 = \frac{h_{2,0}}{\theta_{2,1}}$ and $\varphi_2 = \frac{h_{2,2}}{\theta_{2,1}}$ follow from the $q$ expansions of $h_{2,0}$, $h_{2,2}$ and $\theta_{2,1}$.

**Case 2:** Suppose that $s \neq \infty$ and choose a matrix $g \in SL_2(\mathbb{Z})$ such that $g(\infty) = s$. Explicitly, if $s = 0$ or $s = \frac{1}{c}$ we choose $g$ as $S$ or $ST^{-c}S$. Then we have

$$(c \tau + d)^{-k+\frac{1}{2}}(-2h_{2,1}(g \tau)) = (c \tau + d)^{-k+\frac{1}{2}}(\varphi_0(g \tau), \varphi_2(g \tau))(\theta_{2,0}(g \tau), \theta_{2,2}(g \tau))^t$$

$$= (c \tau + d)^{-k+1}(\varphi_0(g \tau), \varphi_2(g \tau))(c \tau + d)^{-\frac{1}{2}}(\theta_{2,0}(g \tau), \theta_{2,2}(g \tau))^t.$$
If \( g = ST^{-c}S \) then the (0, 0)-th and (2, 0)-th entries of \( U_2(g) = U_2(S)U_2(T^{-c})U_2(S) \) are equal (upto some constants) \( 1 + (-1)^c + 2i^{c/2} \) and \( 1 + (-1)^c - 2i^{c/2} \) respectively. Thus, for the above choices of \( g \), the matrix \( U_2(g) \) will have nonzero entries at the (0, 0)-th place and the (2, 0)-th place since either \( g \) is \( S \) or \( 2|c \) or \( c \) is odd. This fact can also be obtained by using general formulas for the Weil representations (see for example, [7, 9]). Using the transformation property given by \([9]\) together with the above observations, we see that \((c\tau + d)^{-1/2}(\theta_{2,0}(g\tau), \theta_{2,2}(g\tau))^t\) is a column vector such that each component have \( \theta_{2,0}(\tau) \). Since \( \theta_{2,0} \to 1 \) and all other theta components tend to 0 as \( \text{Im}(\tau) \to \infty \), each component of the above column vector tends to a non-zero limit as \( \text{Im}(\tau) \) tends to \( \infty \). Together with the holomorphicity of \( h_{2,1} \), this shows that \((c\tau + d)^{-k+1}(\varphi_0(g\tau), \varphi_2(g\tau))\) tends to a finite limit as \( \text{Im}(\tau) \) goes to \( \infty \). The required cusp conditions now follow. \( \square \)

Conversely, let \((\varphi_0, \varphi_2)^t\) be a vector valued modular form in \( M_{k-1}(\Gamma_0(2N), \chi; \rho_2) \). We now define \( \phi(\tau, z) \) by

\[
\phi(\tau, z) = \varphi_0(\tau)\theta_{2,1}(\tau)\theta_{2,0}^J(\tau, z) - \frac{1}{2}(\varphi_0\theta_{2,0} + \varphi_2\theta_{2,2})(\tau)(\theta_{2,1} + \theta_{2,3})(\tau, z) + \varphi_2(\tau)\theta_{2,1}(\tau)\theta_{2,2}^J(\tau, z).
\]

(23)

Using the transformation properties for \((\varphi_0, \varphi_2)\) and the theta functions with respect to \( \Gamma_0(2N) \), we see that \( \phi \in J_{k,2}(\Gamma_0(2N), \chi) \). Clearly, by definition, \( \phi \in \ker(D_0) \). Thus, we have obtained the following theorem.

**Theorem 3.2.** There is a linear isomorphism

\[
\Lambda_2 : J_{k,2}(\Gamma_0(2N), \chi)^0 \longrightarrow M_{k-1}(\Gamma_0(2N), \chi; \rho_2),
\]

given by \( \phi \mapsto (\varphi_0, \varphi_2)^t \), where \( \varphi_0 \) and \( \varphi_2 \) are defined by \([20]\). The inverse of \( \Lambda_2 \) is given by \((23)\).

### 3.2. Connection to the space of cusp forms.

Let \( D_2 \) be the differential operator as defined in \([13]\) and \( \phi \in \ker(D_0) \) be given by \((23)\). Then proceeding as in \([1]\) Section 3] and using the differential equations

\[
\frac{\partial^2}{\partial z^2}\theta_{m,r}^j = 4m(2\pi i)\frac{\partial}{\partial \tau}\theta_{m,r}^j \quad \text{for} \quad r \in \{0, 1, \ldots, 2m - 1\},
\]

(24)

we obtain

\[
D_2(\phi) = 8k \left( \varphi_0(\theta_{2,1}\theta_{2,0}' - \theta_{2,0}\theta_{2,1}') + \varphi_2(\theta_{2,1}\theta_{2,2}' - \theta_{2,2}\theta_{2,1}') \right).
\]

We define \( \xi_0 := \theta_{2,1}\theta_{2,0}' - \theta_{2,0}\theta_{2,1}' \) and \( \xi_2 := \theta_{2,1}\theta_{2,2}' - \theta_{2,2}\theta_{2,1}' \). Then

\[
D_2(\phi) = 8k(\varphi_0, \varphi_2)(\xi_0, \xi_2)^t.
\]

(25)

Proceeding as in \([1]\) Proposition 2], for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2) \), we have

\[
(\xi_0, \xi_2)^t(\gamma \tau) = (c\tau + d)^3(\rho_2(\gamma)^{-1})^t(\xi_0, \xi_2)^t(\tau).
\]

(26)
Analysing the behaviour of \(\xi_0\) and \(\xi_2\) at the cusps of \(\Gamma_0(2)\), we find that the vector \((\xi_0, \xi_2)^t\) is a vector valued cusp form for \(\Gamma_0(2)\) of weight 3 and representation \((\rho_2^{-1})^t\). Define the space

\[
S_{k+2}(\Gamma_0(2N), \chi)^0 := \{ f \in S_{k+2}(\Gamma_0(2N), \chi) : f = \varphi_0 \xi_0 + \varphi_2 \xi_2 \text{ with } (\varphi_0, \varphi_2)^t \in M_{k-1}(\Gamma_0(2N), \chi; \rho_2) \}.
\]

We shall now prove that \(\psi\) is a modular function of weight 3 and representation \((\rho_2)\) of weight 2 case. Let \(\phi \in \text{Ker}(D_0 \oplus D_2)\). Then \(\phi \in \text{Ker}(D_0)\) and so by (25) we get \(D_2(\phi) = 8k(\varphi_0 \xi_0 + \varphi_2 \xi_2)\). Now using the fact that \(\phi \in \text{Ker}(D_2)\), we obtain

\[
0 = D_2(\phi) = 8k(\varphi_0 \xi_0 + \varphi_2 \xi_2),
\]

which gives \(\varphi_0 \xi_0 + \varphi_2 \xi_2 = 0\). Define

\[
\psi(\tau) = \frac{\varphi_0}{\xi_2}(\tau) = \frac{-\varphi_2}{\xi_0}(\tau).
\]

(27)

Using the definitions of \(\xi_0\) and \(\xi_2\), we have

\[
(\theta_{2,2} \xi_0 - \theta_{2,0} \xi_2)(\tau) = \theta_{2,1}(\tau)(\theta_{2,2} \xi_0 - \theta_{2,0} \xi_2)(\tau) = 2 \theta_{2,1} \xi_2^*(\tau),
\]

where \(\xi_2^*\) is defined by (11) and is a cusp form of weight 3 on \(\Gamma_0(2)\) with character \(\omega_2\). If for any \(\tau \in \mathbb{H}\), \(\xi_0(\tau) = \xi_2(\tau) = 0\), then the above equation implies (using (12)) that \(\theta_{2,1}(\tau) \eta^6(2\tau) = 0\), which is not true. Therefore the function \(\psi\) (defined by (27)) is holomorphic in the upper half-plane. We shall now prove that \(\psi\) is a modular function of weight \(k - 4\) with respect to the group \(\Gamma_0(2N)\). Let \(\gamma \in \Gamma_0(2N)\). Since \(\psi \cdot (\xi_2, -\xi_0)^t = (\varphi_0, \varphi_2)^t\), using the transformation (22) we get the following.

\[
\psi(\gamma \tau)(\xi_2, -\xi_0)^t(\gamma \tau) = \chi(d)(c\tau + d)^{k-1} \rho_2(\gamma) \psi(\tau)(\xi_2, -\xi_0)^t(\tau).
\]

Using (26) and following the argument similar to that given in [1] p. 312, we get the following transformation property (and using the definitions of the representation \(\rho_2\),
characters $r$ and $\omega_2$ given respectively by \([15], [14]\) and \([8]\):

$$\psi(\gamma \tau) = \chi(d) \frac{\bar{\omega}_2(\gamma)}{(u_{11} + u_{13})^2} (ct + d)^{k-4} \psi(\tau).$$  \hfill (28)

Now, let $k = 2$ and $\chi = 1$. Consider the function $\psi \xi(\xi_2^2)^3$, which is a weight 10 cusp form on $\Gamma_0(2N)$ and is divisible by $n^{18}(2\tau)$, which follows from \([12]\). Therefore, for $N$ odd square-free, by using Corollary 2.3 of \([2]\), we get $\psi \xi(\xi_2^2)^3 = 0$, which implies that $\psi = 0$. When $N = 2$, we consider the function $\psi \xi^2$ which is a weight 4 cusp form on $\Gamma_0(4)$. Since $S_4(\Gamma_0(4)) = \{0\}$, in this case also we get $\psi = 0$. In other words, $\varphi_0 = \varphi_2 = 0$ and hence \([23]\) implies that $\phi = 0$. Thus, we have the following theorem.

**Theorem 3.4.** For $N = 2$ or an odd square-free positive integer, the differential map $D_0 \oplus D_2$ is injective on $J_{2,2}(\Gamma_0(2N))$.

### 4. A certain subspace of the space of Jacobi forms of square-free index

Throughout this section we assume that $m$ is a square-free positive integer and $N$ is any positive integer. Consider the following subspace of Jacobi forms of index $m$ on $\Gamma_0(mN)$:

$$J_{k,m}(\Gamma_0(mN),\chi) := \{ \phi \in J_{k,m}(\Gamma_0(mN),\chi) : h_{m,r} = 0 \text{ for all } r \neq 0, m \}.$$  \hfill (29)

When $m = 1$, we have $J_{k,1}^*(\Gamma_0(N),\chi) = J_{k,1}(\Gamma_0(N),\chi)$. In the case $m = 2$, we relate the subspace $J_{k,2}^*(\Gamma_0(2N),\chi)$ with the space $J_{k,2}(\Gamma_0(2N),\chi)^0$ in §5. Denote the intersection of the space $J_{k,m}(\Gamma_0(mN),\chi)^0$ with the subspace $J_{k,m}^*(\Gamma_0(mN),\chi)$ by $J_{k,m}^*(\Gamma_0(mN),\chi)^0$. In this section we study the space $J_{k,m}^*(\Gamma_0(mN),\chi)^0$ and relate this to the space $J_{k,1}(\Gamma_0(N),\chi)^0$, which was studied by Arakawa and Böcherer \([1,2]\).

#### 4.1. Connection to the space of modular forms.

Suppose $\phi \in J_{k,m}^*(\Gamma_0(mN),\chi)^0$. Then $0 = \phi(\tau, 0) = h_{m,0}(\tau)\theta_{m,0}(\tau) + h_{m,m}(\tau)\theta_{m,m}(\tau)$. We define a new function by

$$\varphi := \frac{h_{m,0}}{\theta_{m,0}} = -\frac{h_{m,m}}{\theta_{m,m}}.$$  \hfill (30)

Since $\theta_{m,0}(\tau) = \theta_{1,0}(m\tau)$ and $\theta_{m,m}(\tau) = \theta_{1,1}(m\tau)$, it follows that $\theta_{m,0}$ and $\theta_{m,m}$ have no zeros in the upper half-plane. Therefore $\varphi$ defines a holomorphic function in the upper half-plane. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN)$, let $\gamma_m = \begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}$ (as in §2). Then using the transformation \((7)\) and \((29)\), we get the following.

$$\varphi(\gamma \tau)(\theta_{m,m}, -\theta_{m,0})^t(\gamma \tau) = \chi(d)(ct + d)^{k-2}U_1(\gamma_m)\varphi(\tau)(\theta_{m,m}, -\theta_{m,0})^t(\tau)$$  \hfill (31)

We now proceed as in the proof of \([1, Proposition 1]\). Using \((14)\) with $z = 0$ in \((30)\) gives the following transformation property. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN)$,

$$\varphi(\gamma \tau) = \chi(d)\bar{\omega}_m(\gamma)(ct + d)^{k-1}\varphi(\tau).$$  \hfill (31)
We now study the behaviour of \( \varphi \) at the cusps of \( \Gamma_0(mN) \). We can assume that all the cusps of \( \Gamma_0(mN) \) are of the form \( \frac{a}{c} \) with \( \gcd(a, c) = 1 \) and \( c \) varies over positive divisors of \( mN \). For such a cusp \( s = \frac{a}{c} \), choose a matrix \( g \in SL_2(\mathbb{Z}) \) such that \( g(\infty) = s \).

Explicitly, we choose \( g = \begin{pmatrix} a & b \\ c & md' \end{pmatrix} \), where \( \alpha = \gcd(m, c) \) and \( b, d' \) are integers such that \( a\frac{m}{\alpha}d' - bc = 1 \). Note that \( \gcd(a\frac{m}{\alpha}, c) = 1 \), since \( m \) is square-free. Let \( d = \frac{m}{\alpha}d' \). We have

\[
(c\tau + d)^{-k + \frac{1}{2}}(h_{m,0}(g\tau), h_{m,m}(g\tau)) = (c\tau + d)^{-k + \frac{1}{2}}\varphi(g\tau)(\theta_{m,m}(g\tau), -\theta_{m,0}(g\tau)) = (c\tau + d)^{-k + \frac{1}{2}}\varphi(g\tau)(\theta_{m,0}(\frac{m}{\alpha}g\tau), \theta_{0,\alpha}(\frac{m}{\alpha}g\tau)) S.
\]

Since \( \begin{pmatrix} m/\alpha & 0 \\ 0 & 1 \end{pmatrix} g\tau = g'\tau' \), where \( g' = \begin{pmatrix} a\frac{m}{\alpha} & b \alpha \\ c/\alpha & d' \end{pmatrix} \in \Gamma_0(\alpha) \) and \( \tau' = \frac{\alpha}{m}\tau \). Using (1), we get the following equation.

\[
(c\tau + d)^{-k + \frac{1}{2}}(h_{m,0}(g\tau), h_{m,m}(g\tau)) = (c\tau + d)^{-k + 1}\varphi(g\tau)(\theta_{m,0}(\frac{m}{\alpha}\tau), \theta_{0,\alpha}(\frac{m}{\alpha}\tau)) \theta_{1}(g'_{\alpha})^T S,
\]

where \( g'_\alpha = \begin{pmatrix} am/\alpha & b\alpha \\ c/\alpha & d' \end{pmatrix} \). Now, by a similar argument as in the proof of [1, Proposition 1], we see that \( \varphi \) is holomorphic at all the cusps of \( \Gamma_0(mN) \). This shows that the function \( \varphi \) is a modular form of weight \( k - 1 \) for \( \Gamma_0(mN) \) with character \( \chi_{m,m} \). We denote the space of all such modular forms by \( M_{k-1}(\Gamma_0(mN), \chi_{m,m}) \).

Conversely, starting with a modular form \( \varphi \in M_{k-1}(\Gamma_0(mN), \chi_{m,m}) \), we obtain a Jacobi form

\[
\phi(\tau, z) = \varphi(\tau)(\theta_{m,m}(\tau)\theta_{0,\alpha}'(\frac{m}{\alpha}\tau, z) - \theta_{m,0}(\tau)\theta_{m,m}'(\tau, z)), \tag{32}
\]

which belongs to \( J_{k,m}^*(\Gamma_0(mN), \chi)^0 \). We summarize the result of this subsection in the following theorem.

**Theorem 4.1.** There is a linear isomorphism

\[
\Lambda_m^* : J_{k,m}^*(\Gamma_0(mN), \chi)^0 \rightarrow M_{k-1}(\Gamma_0(mN), \chi_{m,m})
\]

given by \( \phi \mapsto \varphi \), where \( \varphi \) is defined by (29). The inverse of \( \Lambda_m^* \) is given by (32).

4.2. **Connection to the space of cusp forms.** Let \( \phi \in J_{k,2}^*(\Gamma_0(mN), \chi)^0 \) be of the form given by (32). By applying \( D_2 \) and using the differential equations given by (24), we have

\[
D_2(\phi)(\tau) = 4mk\varphi(\tau)(\theta_{m,m}\theta_{0,\alpha}'(m\tau, 0) - \theta_{m,0}\theta_{m,m}'(m\tau, 0)) = 4m^2k\varphi(\tau)\xi_m^*(\tau), \tag{33}
\]

where \( \xi_m^*(\tau) \) is the cusp form defined by (11). Now define the space

\[
S_{k+2}^*(\Gamma_0(mN), \chi)^0 := \{ f \in S_{k+2}(\Gamma_0(mN), \chi) : f/\xi_m^* \in M_{k-1}(\Gamma_0(mN), \chi_{m,m}) \}.
\]

We summarize the results of §4.1 and §4.2 in the following theorem.
Theorem 4.2. The map $D_2 : J_{k,m}(\Gamma_0(mN), \chi) \rightarrow S_{k+2}(\Gamma_0(mN), \chi)$ induces an isomorphism between $J_{k,m}^*(\Gamma_0(mN), \chi)^0$ and $S_{k+2}^*(\Gamma_0(mN), \chi)^0$. Combining this with Theorem 4.1, we get the following commutative diagram of isomorphisms:

\[
\begin{array}{c}
\Lambda_m^* \\
M_{k-1}(\Gamma_0(mN), \chi_m) & \xrightarrow{D} & S_{k+2}^*(\Gamma_0(mN), \chi)^0 \\
J_{k,m}^*(\Gamma_0(mN), \chi)^0 & \xrightarrow{\phi} & J_{k,m}^*(\Gamma_0(mN), \chi)^0
\end{array}
\]

where the isomorphism in the bottom is given by $\varphi \mapsto 4m^2k\xi_m^*\varphi$.

4.3. Connection to the space $J_{k,1}(\Gamma_0(N), \chi)^0$. Following [3] Theorem 3.4] for the congruence subgroup $\Gamma_0(N)$, we see that the operator $D_0 \oplus D_2$ is injective on $J_{k,1}(\Gamma_0(N))$ for all positive even integers $k$. Also by [2] Theorem 4.3], $D_0$ is injective on $J_{2,1}(\Gamma_0(N))$ for square-free $N$. Now we deduce similar kind of results for the space $J_{k,m}^*(\Gamma_0(mN))$ in the following two corollaries.

Corollary 4.3. The differential map $D_0 \oplus D_2 : J_{k,m}^*(\Gamma_0(mN)) \rightarrow M_k(\Gamma_0(mN)) \oplus S_{k+2}(\Gamma_0(mN))$ is injective.

Proof. It is observed in §2 that $\xi_m^*$ has no zeros in the upper half-plane (see [12]). The corollary follows by using this fact along with (33). \[\square\]

Corollary 4.4. The restriction map $D_0 : J_{2,m}^*(\Gamma_0(mN)) \rightarrow M_2(\Gamma_0(mN))$ is injective, when $mN$ is square-free, i.e., the kernel space $J_{2,m}^*(\Gamma_0(mN))^0 = \{0\}$.

Proof. By Theorem 4.2 we see that the spaces $J_{2,m}^*(\Gamma_0(mN))^0$ and $S_4^*(\Gamma_0(mN))^0$ are isomorphic, where $S_4^*(\Gamma_0(mN))^0$ is the subspace of $S_4(\Gamma_0(mN))$ whose functions are divisible by $\xi_m^*$; in other words, divisible by $\eta^0(m\tau)$. By applying the operator $\tau \mapsto -\frac{1}{m\tau}$, the subspace $S_4^*(\Gamma_0(mN))^0$ is equal to the subspace whose functions are divisible by $\eta^0(\tau)$. Since $mN$ is square-free, it follows from [2] Corollary 2.3, (2.4)] that $S_4^*(\Gamma_0(mN))^0 = \{0\}$. \[\square\]

Let $J_{k,m}(\Gamma_0(mN), \nu)$ denote the space of Jacobi forms of index $m$ on $\Gamma_0(mN)$ with character $\nu$.

Corollary 4.5. The two kernel spaces $J_{k,1}(\Gamma_0(mN), \chi)^0$ and $J_{k,m}^*(\Gamma_0(mN), \chi_m^*\omega_m\overline{\nu})^0$ are isomorphic.

Proof. Using [11] Theorem 1] and Theorem 4.1 both the spaces $J_{k,1}(\Gamma_0(mN), \chi)^0$ and $J_{k,m}^*(\Gamma_0(mN), \chi_m^*\omega_m\overline{\nu})^0$ are isomorphic to the same space $M_{k-1}(\Gamma_0(mN), \chi\overline{\nu})$. This proves the corollary. \[\square\]

Corollary 4.6. Let $N$ be a square-free positive integer and coprime to $m$. Then $J_{2,m}^*(\Gamma_0(mN), \omega_m\overline{\nu})^0 = \{0\}$.

Proof. Since $mN$ is square-free, the corollary follows by using Corollary 4.5 together with [2] Theorem 4.3]. \[\square\]
5. Concluding Remarks

Remark 5.1. Note that Theorem 4.2 reduces to [1, Theorem 2] in the case of index 1. Moreover, Corollary 4.5 shows that there may exist non-trivial examples of isomorphic subspaces in the spaces of Jacobi forms of different index.

Remark 5.2. The space $J^*_{k,m}(\Gamma_0(mN), \chi)$ as defined in the section 4 can be quite large for some values of $m$. For example, if $k$ is even and $m$ is square-free, it is easy to verify that $\dim J^*_{k,m}(\Gamma_0(mN)) \geq \dim J_{k,m}(SL_2(\mathbb{Z}))$. Take any Jacobi form $\phi = \sum_{j=0}^{2m-1} h_{m,j} \theta^{1}_{m,j}$ of even weight $k$ and square-free index $m$ for the full Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Using [3] and [7], the function defined by $\psi_{0,m} := h_{m,0} \theta^{J}_{m,0} + h_{m,m} \theta^{J}_{m,m}$ is a Jacobi form of weight $k$ and index $m$ for the Jacobi group $\Gamma_0(m) \ltimes \mathbb{Z}^2$. Define a mapping $J_{k,m}(SL_2(\mathbb{Z})) \to J^*_{k,m}(\Gamma_0(m))$ by $\phi(\tau, z) \mapsto \psi_{0,m}(\tau, z)$. By [3] Theorem 1 we know that this map is injective. Hence, for any positive integer $N$ we have the inequalities $\dim J^*_{k,m}(\Gamma_0(mN)) \geq \dim J^*_{k,m}(\Gamma_0(m)) \geq \dim J_{k,m}(SL_2(\mathbb{Z}))$.

Remark 5.3. Suppose $N$ is either 2 or an odd square-free positive integer. By using [7], it is easy to see that the space $J^*_{k,2}(\Gamma_0(2N), \chi)$ is isomorphic to $M_{k-\frac{1}{2}}(\Gamma_0(2N), \chi; U^*_2)$, the space of vector valued modular forms of weight $k - \frac{1}{2}$ and (projective) representation $U^*_2$ defined on $\Gamma_0(2)$ by $U^*_2(\gamma) = U^*_1(\gamma_2)$. It is also not hard to verify that the map $(\varphi_0, \varphi_2)^t \mapsto (\varphi_0^{U^*_2}, \varphi_2^{U^*_2})^t$ is an isomorphism from the space $M_{k-\frac{1}{2}}(\Gamma_0(2N), \chi; U^*_2)$ onto the space $M_{k-1}(\Gamma_0(2N), \chi; \rho_2)$. Combining this observation with Theorem 3.2 we see that the spaces $J^*_{k,2}(\Gamma_0(2N), \chi)^0$ and $J^*_{k,2}(\Gamma_0(2N), \chi)$ are isomorphic.

Remark 5.4. Suppose that $N$ is either 2 or an odd square-free positive integer. If we consider the space $J^*_{k,2}(\Gamma_0(2N), \chi)$ as a subspace of the full kernel space $J_{k,2}(\Gamma_0(2N), \chi)^0$, then under the isomorphism diagram of Theorem 3.3 any $\phi \in J^*_{k,2}(\Gamma_0(2N), \chi)^0$ will correspond to a vector valued modular form $(\varphi_0, \varphi_2)^t$ which satisfies the property $\varphi_0 \theta^{2}_{2,0} + \varphi_2 \theta^{2}_{2,2} = 0$. Moreover, the image of the space $J^*_{k,2}(\Gamma_0(2N), \chi)^0$ under $\Lambda_2$, that is, the space $\Lambda_2(J^*_{k,2}(\Gamma_0(2N), \chi)^0)$ is isomorphic to the space $M_{k-1}(\Gamma_0(2N), \chi^2)$ and the isomorphism is given by $(\varphi_0, \varphi_2)^t \mapsto \varphi_0^{\theta^{2}_{2,0}} \varphi_2^{\theta^{2}_{2,2}}$.

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