A characterization of spaces with discrete topological fundamental group

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Abstract

The fundamental group of a locally path connected metric space inherits the discrete topology in a natural way if and only if $X$ is semilocally simply connected. We also provide a counterexample to a similar theorem in the literature.

1 Introduction

In general $\pi_1(X, p)$, the based fundamental group of a space $X$, admits a canonical topology and becomes the topological fundamental group. To date significant progress has been made in the investigation of the topological fundamental group of certain non semilocally simply connected spaces such as the Hawaiian earring and the harmonic archipelago. For example, the Hawaiian earring group is not a Baire space [4] and also fails to embed topologically in the inverse limit of free groups [5] despite injectivity of the canonical homomorphism [5]. The harmonic archipelago is a compact metric space whose fundamental group is uncountable but has only two open sets [2]. In particular neither of these groups has the discrete topology.

When does $\pi_1(X, p)$ have the discrete topology? The fundamental groups of locally contractible spaces such as $n$—manifolds have the discrete topology. Indeed, a published theorem of another author (Theorem 5.1 [1]) indicates that $\pi_1(X, p)$ has the discrete topology precisely when $X$ is semilocally simply connected.
Unfortunately Theorem 5.1 is false. This paper contains a counterexample, an alternate version of the theorem, and some remarks and examples that lend further perspective on the results in question.

2 Definitions

Suppose $X$ is a metrizable space and $p \in X$. Let $C_p(X) = \{f : [0,1] \to X \text{ such that } f \text{ is continuous and } f(0) = f(1) = p\}$. Endow $C_p(X)$ with the topology of uniform convergence. The **topological fundamental group** $\pi_1(X, p)$ is the set of path components of $C_p(X)$ topologized with the quotient topology under the canonical surjection $q : C_p(X) \to \pi_1(X, p)$ satisfying $q(f) = q(g)$ if and only if $f$ and $g$ belong to the same path component of $C_p(X)$.

Thus a set $U \subset \pi_1(X, p)$ is open in $\pi_1(X, p)$ if and only if $q^{-1}(U)$ is open in $\pi_1(X, p)$.

A space $X$ is **semilocally simply connected** at $p$ if $p \in X$ and there exists an open set $U$ such that $p \in U$ and $j_U : U \to X$ induces the trivial homomorphism $j_U^* : \pi_1(U, p) \to \pi_1(X, p)$. Roughly speaking, this means small loops in $X$ are inessential, but might bound large disks.

A topological space $X$ is **discrete** if every subset is both open and closed.

The above definitions are consistent with those found in Munkres [6].

3 A counterexample

Looking beyond the notational ambiguity in case $X$ is not path connected, it is claimed falsely in Theorem 5.1 [1] that $\pi_1(X)$ is discrete if and only if the topological space $X$ is semilocally simply connected.

Here is a counterexample to Theorem 5.1.

**Example 1** Let $X$ denote the following subset of the plane:

$$X = \{(0,1),\frac{1}{2},\frac{1}{3},...\} \times [0,1] \cup ([0,1] \times \{0\}) \cup ([0,1] \times \{1\}).$$

Let $p = (0,0)$. Consider the inessential map $f$ which starts at $(0,0)$, goes straight up to $(0,1)$ and then comes back down again. Notice we may construct maps $f_n \in C_p(X)$ such that $f_n$ is inessential but $f_n \to f$ uniformly. (Let $f_n$ trace a rectangle with sides $\{0\} \times [0,1]$ and $\{\frac{1}{n}\} \times [0,1]$). This shows
that the path component of the constant map is not open in \( C_p(X) \). Consequently \( \pi_1(X, p) \) does not have the discrete topology. On the other hand \( X \) is semilocally simply connected since small loops in \( X \) are null homotopic in \( X \).

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Theorem 5.1 [\( \square \)] can be repaired by making the added assumption that \( X \) is locally path connected.

Theorem 2 Suppose \( X \) is a locally path connected metrizable space and \( p \in X \). Then \( \pi_1(X, p) \) is discrete if and only if \( X \) is semilocally simply connected at \( p \).

Proof. Let \( F : C_p(X) \to \pi_1(X, p) \) denote the canonical quotient map. Let \( P \in C_p(X) \) denote the constant map. Let \([P]\) denote the path component of \( P \) in \( C_p(X) \). Let \( d \) be any metric on \( X \).

Suppose \( \pi_1(X, p) \) is discrete. Then the trivial element of \( \pi_1(X, p) \) is an open subset. Hence, since \( F \) is continuous, \([P]\) is open in \( C_p(X) \). Suppose, in order to obtain a contradiction, that \( \pi_1(X, p) \) is not semilocally simply connected at \( p \). Then there exists a sequence of maps \( \alpha_n \to P \) such that \( \alpha_n \in C_p(X) \) and \( \alpha_n \) is essential in \( X \) for each \( n \). On the other hand, since \([P]\) is open in \( C_p(X) \) we can be sure that eventually \( \alpha_n \) is inessential in \( X \).

Conversely, suppose \( X \) is semilocally simply connected at \( p \). To prove \( \pi_1(X, p) \) is discrete we must show that each one point subset is open. Since \( \pi_1(X, p) \) is a topological group (Proposition 3.1 [\( \square \)]) \( \pi_1(X, p) \) is homogeneous. Thus it suffices to prove that the trivial element of \( \pi_1(X, p) \) is an open subset. Since \( F \) is a quotient map it suffices to prove \([P]\) is open in \( C_p(X) \). Suppose \( f \in [p] \) and suppose \( f_n \in C_p(X) \) and suppose \( f_n \to f \) uniformly. We must prove that for large \( n \) \( f \) and \( f_n \) are path homotopic.

Observation 1. Since \( X \) is locally path connected and since \( \text{im}(f) \) is compact, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( x \in \text{im}(f) \) and \( d(x, y) < \delta \) then there exists a path \( \alpha_{xy} \) from \( x \) to \( y \) such that \( \text{diam}(\alpha_{xy}) < \varepsilon \).

Observation 2. Since \( X \) is semilocally simply connected and since \( \text{im}(f) \) is compact, there exists \( \varepsilon > 0 \) such that if \( x \in \text{im}(f) \) and if \( \alpha_{xx} \) is any path from \( x \) to \( x \) such that \( \text{diam}(\alpha_{xx}) < \varepsilon \) then \( \alpha_{xx} \) is null homotopic.
Observation 3. The set \( f \cup \{ f_1, f_2, \ldots \} \) is an equicontinuous collection of maps.

Combining observations 1, 2, and 3, for sufficiently large \( n \) we can manufacture a homotopy from \( f_n \) to \( f \) described roughly as follows. Observation 1 says we can connect \( f(\frac{i}{2n}) \) and \( f_n(\frac{i}{2n}) \) by a small path and we can connect \( f(\frac{i+1}{2n}) \) and \( f_n(\frac{i+1}{2n}) \) by a small path. Observation 3 says the restriction of \( f \) and \( f_n \) to \( [\frac{i}{2n}, \frac{i+1}{2n}] \) is small. Observation 2 says the boundary of the rectangle with corners \( f(\frac{i}{2n}), f_n(\frac{i}{2n}), f(\frac{i+1}{2n}) \) and \( f_n(\frac{i+1}{2n}) \) determines an inessential loop. Thus we can fill in the homotopy across \( [\frac{i}{2n}, \frac{i+1}{2n}] \times [0, 1] \).

Remark 3 Local path connectivity is only used in the 2nd part of the proof of Theorem 2. Thus all path connected spaces with discrete topological fundamental group must be semilocally simply connected.

Remark 4 There exist spaces which are not locally path connected but which have discrete fundamental group. For example the cone over \( X \) in example 4 (i.e. let \( Y \) be quotient of the space \( X \times [0, 1] \) with \( X \times \{ 1 \} \) considered as a point. Note \( Y \) is simply connected.)

Remark 5 Theorem 5.1 of [1] cannot be repaired by replacing the notion of ‘discrete’ with ‘totally disconnected’. The space \( X = S^1 \times S^1 \) has totally disconnected fundamental group but \( X \) is not semilocally simply connected.

References

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