LOCAL EXTERIOR SQUARE GAMMA FACTORS FOR NON-SPLIT COVERS AND LEVEL ZERO REPRESENTATIONS

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ABSTRACT. A purely local approach has been developed by Krishnamurthy and Kutzko to compute Langlands-Shahidi local coefficient for SL(2) via Bushnell-Kutzko’s types and covers. We extend their method to the non-split case. In the course of computation, we also investigate the structure of Gelfand-Graev space, which generalizes the results of Chan-Savin and Mishra-Pattanayak for SL(2). Turning to the case of the Siegel Levi subgroup inside Sp(2n), we establish that Langland-Shahidi local exterior square $\gamma$-factors for level zero supercuspidal representations of GL(n) agree with those through Jacquet-Shalika integrals in characteristic 0, and Bump-Friedberg and Jacquet-Shalika integrals in characteristic $p > 0$. Furthermore, all our analysis support Bump and Friedberg conjecture.

1. INTRODUCTION

In 1988, Jacquet and Shalika [17] first established an integral representation for the exterior square $L$-function $L(s, \Pi, \Lambda^2)$ associated to a (unitary) cuspidal automorphic representation $\Pi$ on $GL_n(\mathbb{A}_k)$, where $\mathbb{A}_k$ is the adèles of a number field $k$. Just a year later, Bump and Friedberg [2] independently constructed a different Rankin-Selberg integral representing the exterior square $L$-function $L(s_1, \Pi)L(s_2, \Pi, \Lambda^2)$ with two variables $s_1$ and $s_2$ for a (unitary) cuspidal automorphic representation $\Pi$ on $GL_n(\mathbb{A}_k)$, where $k$ is a global field this time. After factoring $\Pi$ into its local component $\Pi \cong (\otimes_{v|\infty} \Pi_v) \otimes (\otimes'_{v<\infty} \Pi_v)$ with $\otimes'$ a restricted tensor product, the local calculation for Jacquet-Shalika and Bump-Friedberg integrals attached to $\Pi_v$ has been only carried out at the non-archimedean ($v < \infty$) unramified places. Somewhat surprisingly, as it turned out, it was the Bump-Friedberg integrals for $\Pi_v$ at archimedean ($v|\infty$) places that came to fruition first. Unlike Jacquet-Shalika integrals, the computation of Archimedean Bump-Friedberg integrals for spherical representations was substantially performed by Stade [43] around the same time of early 1990’ s. The definitions for $L$-factors $L(s, \Pi_v, \Lambda^2)$ and $L(s, t, \Pi_v, BF)$, $\gamma$-factors $\gamma(s, \Pi_v, \Lambda^2, \psi)$ and $\gamma(s, t, \Pi_v, BF, \psi)$ and their functional equations were much less well understood, when $v < \infty$ is a ramified place [2, 17].

Recently there has been renewed interest to complete the local theory of the exterior square $L$-function via these integral representations over non-archimedean ($v < \infty$) ramified places. In particular, in a flurry of work of Matringe [7, 30–33] and his joint paper with Cogdell [7], Matringe used his result on the connection between linear periods and Shalika periods [30] to formulate the precise definition of local exterior $L$-function and its local functional equation at ramified places. Furthermore, he subsequently gave local exterior square $\gamma$-factors and $\varepsilon$-factors which had long been expected.

We now illustrate the result more accurately. We let $F$ be a non-archimedean local field, and then fix a non-trivial additive character $\psi$. We denote by $q$ the cardinality of the residue field of $F$. Let $\pi$ be an irreducible generic representation of $GL_n(F)$ and $\pi^\vee$ its contragredient representation of $\pi$ with associated Whittaker models $W(\pi, \psi)$ and $W(\pi, \psi^{-1})$. For each Whittaker function $W \in W(\pi, \psi)$, and each Schwartz-Bruhat function $\Phi$ belonging to a space of locally constant and compactly supported functions, we associate Jacquet-Shalika integrals $J(s, W, \Phi)$, which generate

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a $\mathbb{C}[q^{\pm s}]$-fractional ideal $J(\pi)$. The ideal $J(\pi)$ is spanned by normalized generators of the form $P(q^{-s})^{-1}$, where the polynomial $P(X) \in \mathbb{C}[X]$ satisfies $P(0) = 1$. The local exterior square $L$-function $L(s, \pi, \wedge^2)$ is defined to be such a unique greatest common divisor $P(q^{-s})^{-1}$ [7,18,20,30]. Further, two Jacquet-Shalika integrals $J(s, W, \Phi)$ and $J(1-s, \pi_n W, \Phi)$ are related by a functional equation, where $W$ is a Whittaker function for a dual representation $\pi_n$, and $\Phi$ is a Fourier transform of $\Phi$. Taking the one dimensionality of Shalika periods for granted, a local exterior square gamma factor $\gamma(s, \pi, \wedge^2, \psi)$ is a rational function in $\mathbb{C}(q^{-s})$ given as a proportionality factor [7,30]. The $L$-factor $L(s, \pi, \wedge^2)$ and the $\gamma$-factor $\gamma(s, \pi, \wedge^2, \psi)$ are associated in a standard way:

\begin{equation}
\gamma(s, \pi, \wedge^2, \psi) = \frac{\varepsilon(s, \pi, \wedge^2, \psi)L(1-s, \bar{\pi}, \wedge^2)}{L(s, \pi, \wedge^2)},
\end{equation}

where $\varepsilon(s, \pi, \wedge^2, \psi)$ is a monomial factor in $\mathbb{C}[q^{\pm s}]$.

Matringe put local Bump-Friedberg factors $L(s, \pi, BF), \gamma(s, \pi, BF, \psi)$, and $\varepsilon(s, \pi, BF, \psi)$ on the equal footing with the Jacquet-Shalika setting by working out the case of one parameter $s$ for $s_1 = s + 1/2$ and $s_2 = 2s$ [32]. Afterwords, local Bump-Friedberg factors are extended to their corresponding ones $L(s, t, \pi, BF), \gamma(s, t, \pi, BF, \psi)$, and $\varepsilon(s, t, \pi, BF, \psi)$ with two complex variables $s$ and $t$ for $s_1 = s + t + 1/2$ and $s_2 = 2s$ [33]. In order to interpolate two variables, one needs to introduce an auxiliary character $\delta_t$, and then fixes an arbitrary value $t$, but varies $s$. On the other hand, long before Rankin-Selberg convolutions for exterior square $L$-function were actively investigated, Shahidi had already developed an Eisenstein series on a symplectic group whose global functional equation implies that of the exterior square $L$-function [38,39,42]. This framework is nowadays known as the Langlands-Shahidi method of analyzing global $L$-functions through the Fourier coefficients of Eisenstein series. Locally, the Langlands-Shahidi method relies on the theory of the Langlands-Shahidi local coefficient $C_\psi(s, \pi, w_0)$, which is another constant of proportionality arising from the uniqueness of Whittaker models for parabolically induced representations. A list of Langlands-Shahidi local exterior square factors $L_{LS}(s, \pi, \wedge^2), \gamma_{LS}(s, \pi, \wedge^2, \psi)$, and $\varepsilon_{LS}(s, \pi, \wedge^2, \psi)$ can be thereupon defined through the local coefficient $C_\psi(s, \pi, w_0)$ satisfying

\begin{equation}
C_\psi(s, \pi, w_0) = \gamma(s, \pi, \psi)\gamma_{LS}(2s, \pi, \wedge^2, \psi)
\end{equation}

\begin{equation}
\gamma_{LS}(s, \pi, \wedge^2, \psi) = \frac{\varepsilon_{LS}(s, \pi, \wedge^2, \psi)L_{LS}(1-s, \bar{\pi}, \wedge^2)}{L_{LS}(s, \pi, \wedge^2)},
\end{equation}

where $\gamma(s, \pi, \psi)$ denotes a Godement-Jacquet $\gamma$-factor built in [15]. Lomelí thereafter generalized the Langlands-Shahidi method to a global function field for the case of a Siegel Levi subgroup of a split classical group or a quasi-split unitary group [26-28]. All aforementioned three approaches are fundamentally different, because the first two methodologies construct local factors by studying various integrals of automorphic forms, whereas the last methodology construct local factors as normalizing factors of local intertwining operators. There are many situations of considerable interest where both Rankin-Selberg and Langlands-Shahidi constructions are available, and it is a non-trivial problem to verify that the resulting local factors ought to be identical, perhaps up to normalization of certain Haar measures. Along the line of this philosophy, Bump and Friedberg has predicted the following pattern of local $\gamma$-factors especially from a spherical situation that $\gamma(s, t, \pi, BF, \psi)$ is a product of the exterior square $\gamma$-factor and the standard $\gamma$-factor.

**Conjecture.** ([2, Conjecture 4]) Let $F$ be a non-archimedean local field and $\pi$ an irreducible generic representation of $GL_n(F)$. Then we have

\[\gamma(s, t, \pi, BF, \psi) = \gamma(s + t + 1/2, \pi, \psi)\gamma_{LS}(2s, \pi, \wedge^2, \psi) = \gamma(s + t + 1/2, \pi, \psi)\gamma(2s, \pi, \wedge^2, \psi).\]

The main aim of this article is to shed some lights on the above identities. The analogous statement for local $L$-functions has been settled by combining the work of Matringe [32,33] with
The author [18,20]. With (1.1) and (1.2), a reformulation for local \( \varepsilon \)-factor can be viewed as a simple consequence of Conjecture. A keen audience may notice the discrepancy of the Tate \( \gamma \)-factor appearing in [2, Conjecture 4]. The issue boils down to the choice between Schwartz-Bruhat functions \( \Phi \) and good sections in the sense of Piatetski-Shapiro and Rallis attached to local zeta integrals. This subject has been widely taken up in [19]. At least, Conjecture can be confirmed for any characters \( \eta \) of \( GL_1(F) \cong F^\times \), and we advise the reader to consult Section 3 for undefined terminologies in Theorem A below.

**Theorem A** (Theorems 3.1 and 3.6). Let \( G = SL_2(F) \) over a non-archimedean local field \( F \). Suppose that \( \psi \) is a fixed additive character of level 0. Let \( \eta \) be a non-trivial (ramified) character of \( \sigma^\times \) satisfying \( \eta^2 = 1 \) with level \( n_\eta \). Let \( \tilde{\eta} \) be any extension of \( \eta \). Then we have

\[
C_\psi(s, \tilde{\eta}, w_0) = \tilde{\eta}(-w^{-n_\eta}) \tau(\eta, \psi, \omega^{-n_\eta}) q^{-n_\eta(s-1)} = \gamma(s, \tilde{\eta}, \psi).
\]

Moreover, we put \( \epsilon = \tilde{\eta}(-1) \). As \( H(G, \lambda_\eta) \)-module isomorphisms, we get

\[
(c\text{-ind}_{G_\eta}^G \psi)^{\lambda_\eta} \cong H(G, \lambda_\eta) \otimes_{H(K, \lambda_\eta)} C_\epsilon \cong H(G, \lambda_\eta) \otimes_{H(K, \lambda_\eta)} C_{-1}.
\]

Our strategy to tackle the problem of computing local coefficient on \( F^\times \) inside \( SL_2(F) \) is inspired by the pioneering work of Casselman [5], where local coefficients in the context of unramified principal series representations are explicitly computed. The computation there relies on finding the effect of the intertwining operator on the subspace of vectors fixed by the Iwahori subgroup. The role played by the trivial representation of the Iwahori subgroup in Casselman’s trick can be regarded as an extreme incident of the theory of types and covers. Later, types and covers are adapted by Krishnamurthy and Kutzko [23] for a split cover (\( \eta^2 \neq 1 \)) of \( SL_2(F) \), and by the author’s joint work with Krishnamurthy [21] for a cover of a homogeneous pair (\( \pi_1, \pi_2 \)) of the Levi subgroup \( GL_n(F) \times GL_n(F) \) inside \( GL_{2n}(F) \). In the course of the computation for \( \pi_1 \times \pi_2 \) of \( GL_n(F) \times GL_n(F) \) [21, §5.3], one of the most crucial step in the proof is to express the principal series block of the Gelfand-Graev representation, \( c\text{-ind}_G^G \psi \), as a cyclic Hecke algebra module. This types of results was originally treated by Chan-Savin for unramified principal series blocks of split reductive groups [6], and was further refined by Mishra-Pattanaya for principal series blocks for connected reductive groups over \( F \) whose residue characteristic is large enough [34]. Benefiting from [21], it is our belief that this certain Hecke algebra module structure of the isotypic component of the space of compactly supported Whittaker functions, \( c\text{-ind}_G^G \psi \), is expected to be responsible for computing local coefficients in more general settings, notably, Siegel Levi subgroups inside \( Sp_{2n}(F) \). Section 2 is devoted to reviewing the primary concepts of types, covers, and intertwining operators for \( SL_2(F) \). In Section 3, we complete the remaining case of non-split cover (\( \eta^2 = 1 \)) for ramified characters \( \tilde{\eta} \) (see [23, §3.4.(2)]), and then extend the structure theory of the Gelfand-Graev representation for \( SL_2(F) \) without any assumption on non-archimedean local fields \( F \).

For higher rank symplectic groups in \( 2n \)-variables, Conjecture holds for all level zero supercuspidal representations of \( GL_n(F) \). The reader consults the end of [47, §5] for pertinent discussion.

**Theorem B** (Theorems 4.3, 4.8, and 4.10). Let \( F \) be a non-archimedean local field and \( \pi \) a level zero supercuspidal representation of \( GL_n(F) \). Then we have

\[
\gamma(s, \pi, \lambda^2, \psi) = \gamma_{LS}(s, \pi, \lambda^2, \psi).
\]

In particular, if the characteristic of the field \( F \) is positive, we obtain

\[
\gamma(s, t, \pi, BF, \psi) = \gamma(s + t + 1/2, \pi, \psi) \gamma_{LS}(2s, \pi, \lambda^2, \psi) = \gamma(s + t + 1/2, \pi, \psi) \gamma(2s, \pi, \lambda^2, \psi).
\]

In contrast to purely local means in Theorem A, a key feature of our tactic is to utilize a globalization of level zero supercuspidal representations over function fields equipped with a close field theory. In a series of versions of globalizing (non)supercuspidal representation over number fields, one typically loses control of the local component of a (unitary) cuspidal automorphic representation \( \Pi \) exactly at one place of \( k \), an archimedean place. While the Langlands-Shahidi theory over
archimedean fields has been fairly well navigated since the seminal work of Shahidi [41], there has been little progress on the desired archimedean input for Bump-Friedberg and Jacquet-Shalika integrals. However the globalization of level zero supercuspidal representations in positive characteristic gives rather good control at all places. Although one may sacrifice a few local places, the necessary equalities of exterior square $\gamma$-factors for irreducible constituents of spherical representations at those bad places do not appear to be insurmountable.

Having a solid matching of $\gamma$-factors for level zero supercuspidal representations over characteristic $p > 0$ in hand, we then incorporate $\gamma$-factors arising from Langlands-Shahidi methods and integral representations with Deligne-Kazhdan theory over close local fields. Deligne [9, Proposition 3.7.1] proved that Artin local factors remain the same for corresponding representations over close local fields via Deligne isomorphisms. An analogous result has been studied by Ganapathy and Lomelí [10, 11] for Langlands-Shahidi local factors on analytic sides, though that time they only consider Kazhdan isomorphisms over sufficiently closed local fields. So far, there seems to have been no previous discourse in the literature about a very basic case of “1”-close local fields. To be precise, we will show that local exterior square $\gamma$-factors for level zero supercuspidal representations via Jacquet-Shalika integrals are compatible with the Kazhdan correspondence over 1-close local fields. In doing so, we are ultimately allowed to transport the identity of $\gamma$-factors over positive characteristic to characteristic zero. This work will be done in Section 4. Overall, this procedure might be quite contrary to the general perception from local $L$-factors [18, 32] that the number fields are easier to deal with than function fields, and having some archimedean places is a blessing instead of a curse, because of the breakthrough work over archimedean fields done by Shahidi [41].

With regard to Bump-Friedberg $\gamma$-factor in Theorem B, the positivity assumption on the characteristic of the field $F$ might be possibly redundant. It likely involves a better understanding of Bump-Friedberg integrals for level zero supercuspidal representations (refer to Remark 4.11 for related discussion). Nonetheless, our method in Jacquet-Shalika $\gamma$-factors opens the door to proving similar identities in characteristic zero, and consider this manuscript as ground work in this direction. In spirit of [13, Remark 2.6], it is also interesting to see that one might implement the equality of $\gamma$-factors for whole supercuspidal representations, once so called “multiplicativity” of $\gamma$-factors for just principal series representations is established. Generalizing the recursive formula on radial parts [14, Lemmata 6.1 and 7.4] will enable us to make substantial progress on multiplicativity. The author currently pursues these topics with E. Zelingher [46, 47]. Although all this analysis seems to be doable, if somewhat involved, we think that our present formulation keeps our exposition a reasonable length and sufficient for further applications.

2. Intertwining Operators for SL(2)

2.1. Types and Covers. Let $F$ be a non-archimedean local field with its residual finite field $\mathbb{F}_q$ of order $q$. The base field $F$ is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, called a $p$-adic field in characteristic 0, or a local function field in characteristic $p > 0$. Let $\mathfrak{o}_F$ be its ring of integers, $\mathfrak{p}_F$ its maximal ideal, $q_F$ the cardinality of its residue field. We fix a generator $\varpi_F$ of $\mathfrak{p}_F$ and normalize the absolute value $| \cdot |$ of $F$ so that $|\varpi_F| = q_F^{-1}$. When there is no possibility of confusion, we sometimes drop the subscripts, while working over a fixed $F$. We let $G = \text{SL}_2(F)$. Let $B$ be the subgroup of $F$-points of the Borel subgroup of upper triangular matrices. Then $B = TU$, where

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \bigg| a \in F^\times \right\}; \quad U = \left\{ u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \bigg| x \in F \right\}.$$ 

In particular, we identify $T$ with $F^\times$ when no confusion can arise. Let $\overline{B} = TTU$ denote the opposite Borel subgroup of lower triangular matrices, where

$$\overline{U} = \left\{ \overline{u}(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \bigg| x \in F \right\}.$$
Let $K = G(\sigma)$ be a maximal compact subgroup and let $W = N_G(T)/T = \{1_2, w_0\}$, where

$$w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

The character $\chi_s$ is given by the formula

$$\chi_s \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = |a|^s.$$ 

We may view it as a character of $B$ by extending $\chi_s$ trivially to $U$. Let $\delta$ denote the modulus character of $B$: explicitly,

$$\delta \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = |a|^2.$$ 

For any topological group $H$, we write $\hat{H}$ to denote the group of continuous homomorphism from $H$ to $\mathbb{C}^\times$. In particular, we are interested in the character group $\hat{\sigma}^\times$. Let $0T := T \cap K$. In particular, $0T \simeq \sigma^\times$. Then $T/0T$ is a free abelian group of rank 1. Let $P(T)$ denote the group of continuous homomorphisms of $T$ into $\mathbb{C}^\times$ which are trivial on $0T$—called the group of unramified characters of $T$. Moreover $X(T)$ is equipped with the structure of a complex variety whose ring of regular functions is $\mathbb{C}[T/0T] \cong \mathbb{C}[t, t^{-1}]$.

By a *cuspidal pair* in $G$, we mean a pair $(M, \sigma)$ in $G$, where $M$ is either $T$ or $G$, and $\sigma$ is a supercuspidal representation of $M$. Two such pairs $(M_i, \sigma_i)$, $i = 1, 2$, are said to be *inertially equivalent* if there exist $g \in G$ and an unramified character $\chi$ of $G$ such that $M_2 = M^g_2 = g^{-1}M_1g$, and $\sigma_2$ is equivalent to the representation $\sigma_1^g \otimes \chi : x \mapsto \sigma_1(xgx^{-1})\chi(x)$ of $M_2$. We denote by $[(M, \sigma)]_G$ the $G$-inertial equivalence class of a cuspidal pair $(M, \sigma)$ in $G$. Let $\mathfrak{B}(G)$ denote the set of inertial equivalence classes of cuspidal pairs in $G$.

It is a fundamental result of Bernstein (cf. [24, §1.4]) that $\mathcal{R}(G)$ of smooth complex representations of $G$ decomposes into a product of full subcategories

$$\mathcal{R}(G) \cong \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{R}^{\mathfrak{s}}(G).$$

We divide the equivalence classes into what is known as *supercuspidal blocks* and *principal series blocks*. For each irreducible supercuspidal representation $\sigma$ of $G$, we write $\mathfrak{s}(\sigma)$ for the equivalence class of $\sigma$ in $\mathcal{R}(G)$. We let $\mathcal{R}^{\mathfrak{s}(\sigma)}(G)$ be the full subcategory of $\mathcal{R}(G)$ whose objects are isomorphic to sums of copies of $\sigma$.

Let $i_B^G$ denote the functor of normalized parabolic induction and for any character $\chi$ of $T$, and let $F_B(\chi)$ denote the space of $i_B^G(\chi)$. Let $\eta$ be an extension of a character $\eta$ to $F^\times$, where $\eta = \bar{\eta}|_{\sigma^\times}$. Let $\eta_i$, $i = 1, 2$, be characters of $T$. Then $\eta_2$ is $G$-inertially equivalent to $\eta_1$ if and only if $\eta_2 = \eta_1^{-1}$. We denote this equivalent class by $\mathfrak{s}_\eta$. We let $\mathcal{R}^{\mathfrak{s}_\eta}(G)$ be the full subcategory of $\mathcal{R}(G)$ whose irreducible objects are exactly those that occur as a subquotient of some $i_B^G(\bar{\eta} \otimes \chi_s)$, where $Q$ is either $B$ or $\overline{B}$.

With $\eta_i$, $i = 1, 2$, as above, $\bar{\eta}_1$ is $T$-inertially equivalent to $\eta_2$ if and only if there exist $s \in \mathbb{C}$ such that $\eta_2 = \bar{\eta}_1\chi_s$. Let $t_\eta$ be the corresponding $T$-inertially equivalent class. Let $\mathcal{R}^{t_\eta}(T)$ be the full subcategory of $\mathcal{R}(T)$ whose object $(\pi, V)$ has the property that $\pi(x)v = \eta(x)v$ for all $x \in 0T$ and $v \in V$. Then the category $\mathcal{R}(T)$ similarly decomposes as a product of its subcategories $\mathcal{R}^{t_\eta}(T)$:

$$\mathcal{R}(T) \cong \prod_{\eta \in \hat{\sigma}^\times} \mathcal{R}^{t_\eta}(T).$$

Let $J$ be a compact open subgroup of $G$, let $(\lambda, W)$ be a smooth irreducible representation of $J$, and write $(\bar{\lambda}, W)$ for the contragredient representation. Then $\mathcal{H}(G, \lambda)$ is the space of compactly supported function $f : G \to \text{End}_\mathbb{C}(W)$ that satisfy

$$f(hxk) = \bar{\lambda}(h)f(x)\lambda(k), \quad x \in G, \ h, k \in J.$$
It is a unital (associative) algebra with respect to the standard convolution operation

\[ f \ast g(y) = \int_G f(x)g(x^{-1}y) \, dx, \quad f, g \in \mathcal{H}(G, \lambda), \quad y \in G, \]

where we normalized the Haar measure on \( G \) such that \( \text{vol}(J) = 1 \). An element \( x \in G \) is called intertwine \( \lambda \) if \( \lambda(xkx^{-1}) = \lambda(k) \), \( k \in J \cap x^{-1}Jx \). It is equivalent to saying that the double coset \( JxJ \) supports a non-zero function in \( \mathcal{H}(G, \lambda) \) [24, §2.2]. A pair \((J, \lambda)\) is said to be a type for \( s \in \mathcal{B}(G) \), or simply a s-type if for every irreducible object \((\pi, V) \in \mathcal{R}(G)\), we have \( (\pi, V) \in \mathcal{R}^s(G) \) if and only if \( \pi \) contains \( \lambda \), that is to say, the space of \( \lambda \)-covariants \( V_\lambda := \text{Hom}_J(W, V) \) is non-trivial. We similarly define the Hecke algebra \( \mathcal{H}(G, \lambda) \). There is a canonical anti-isomorphism \( f \mapsto \breve{f} \) from \( \mathcal{H}(G, \lambda) \to \mathcal{H}(G, \lambda) \) given by \( \breve{f}(g) = (f(g^{-1}))^\vee \). For \( a \in \text{End}_\mathbb{C}(W) \), \( a^\vee \) denote the transpose of \( a \) with respect to the canonical pairing between \( W \) and \( \hat{W} \). There is a natural left \( \mathcal{H}(G, \lambda) \to \text{Mod} \) structure (also denoted as \( \pi \)) given by

\[ (f \ast \phi)(w) := (\pi(f)\phi)(w) = \int_G \pi(g)f(g^\vee w) \, dg \]

for \( \phi \in V_\lambda \), \( w \in W \), and \( f \in \mathcal{H}(G, \lambda) \). For s-type \((J, \lambda)\), the map \( V \mapsto V_\lambda \) induces an equivalence of categories \( \mathcal{R}^s(G) \cong \mathcal{H}(G, \lambda) \to \text{Mod} \).

We fix a non-trivial (ramified) character \( \eta \) of \( o^\times \) such that \( \eta^2 = 1 \). We let \( n_\eta \) be the smallest number \( n \) so that \( 1 + p^n \subset \ker \eta \). The compact open subgroup is given by

\[ J_\eta = \left\{ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in G \mid c_{11}, c_{22} \in o^\times, c_{12}, c_{21} \in p^{n_\eta} \right\}, \]

and \( \lambda_\eta \) is a function on \( J_\eta \) given by

\[ \lambda_\eta \left( \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \right) = \eta(c_{11}). \]

It follows from [24] that the pair \((J_\eta, \lambda_\eta)\) is a \( G \)-cover for \((0^T, \eta)\). We recall certain crucial properties of a \( G \)-cover \((J_\eta, \lambda_\eta)\):

(a) \( J_\eta = (J_\eta \cap \overline{U})0^T(J_\eta \cap U) \).

(b) The representation \( \lambda_\eta \) is trivial on \( J_\eta \cap \overline{U} \) and \( J_\eta \cap U \), while \( \lambda_\eta \mid_{0^T} = \eta \). The pair \((J_\eta, \lambda_\eta)\) is a type for \( s_\eta \).

(c) There is a support preserving injective algebra map \( t_Q : \mathcal{H}(T, \eta) \to \mathcal{H}(G, \lambda_\eta) \) that realizes the parabolic induction functor \( t_Q^G \) at the level of Hecke algebras. It means that the following diagram commutes:

\[ \begin{array}{ccc} \mathcal{R}^{s_\eta}(T) & \cong & \mathcal{H}(T, \eta) \to \text{Mod} \\
\mathcal{H}(T, \eta) \overset{(t_Q)_*}{\longrightarrow} & & \mathcal{H}(G, \lambda_\eta) \to \text{Mod} \\
\mathcal{H}(G, \lambda_\eta) \overset{(t_Q)_*}{\longrightarrow} & & \mathcal{H}(G, \lambda_\eta) \to \text{Mod} \\
\end{array} \]

where \((t_Q)_*\) is an adjoint of the restriction functor \( t_Q^G : \mathcal{H}(G, \lambda_\eta) \to \text{Mod} \to \mathcal{H}(T, \eta) \to \text{Mod} \).

### 2.2. Intertwining Operators

We write \( \mathcal{F}(s, \tilde{\eta}) \) to denote the induced space \( \mathcal{F}_B(\tilde{\eta} \otimes \chi_s) \). As \( \lambda_\eta \) is one-dimensional, it is straightforward from [4, (2.13)] that there is a natural isomorphism \( V_{\lambda_\eta} \cong V_{\lambda_\eta} \otimes \mathbb{C} \lambda_\eta \cong V^{\lambda_\eta} \) given by \( \phi \otimes w \mapsto \phi(w) \). For \( w \in W \), the functions \( f_{12, \tilde{\eta}} \) and \( f_{W, \tilde{\eta}} \) in \( \mathcal{F}(s, \tilde{\eta})^{\lambda_\eta} \) are given by

\[ f_{12, \tilde{\eta}}(w)(g) = \begin{cases} \tilde{\eta}\chi_s \delta^{1/2}(b)\lambda_\eta(j) & \text{if } g = bj \in BJ_\eta \\ 0 & \text{otherwise}, \end{cases} \]
and
\[ f_{w_0}^T(w)(g) = \begin{cases} \tilde{\eta}_\chi b^{3/2}(b) \lambda(h) & \text{if } g = b_0 w_0 j \\
0 & \text{otherwise.} \end{cases} \]

For \( w \in W \), let \( T_w \in \mathcal{H}(G, \lambda_\eta) \) denote the function supported on the double coset \( J_\eta w J_\eta \) and given by the formula \( T_w(hwK) = \lambda(h)\lambda(k) \), \( h, k \in J_\eta \). Let \( \mathcal{H}(K, \lambda_\eta) \) be the subalgebra in \( \mathcal{H}(G, \lambda_\eta) \) spanned by the function \( T_w \) for \( w \in W \). Since any element \( w \) in \( W \) certainly intertwines the representation \( \lambda_\eta \), [24, Lemma 2.3] (cf. [4, (11.6)]) implies that \( J_\eta W J_\eta = J_\eta K J_\eta \) (cf. [34, (5.1)]). We summarize the properties of \( \mathcal{H}(K, \lambda_\eta) \) from [4, §11.5 and §11.6].

**Lemma 2.1.** The set \( \{ T_{1_2}, T_{w_0} \} \) is a \( C \)-basis of \( \mathcal{H}(K, \lambda_\eta) \). In particular, \( \dim_C(\mathcal{H}(K, \lambda_\eta)) = 2 \).

We observe that
\[ \{ g \in G \mid \text{There is } f \in \mathcal{F}(s, \tilde{\eta})^{\lambda_\eta} \text{ with } f(g) \neq 0 \} \]
is a subset of \( BGJ_\eta = BKJ_\eta \). Hence, any \( f \in \mathcal{F}(s, \tilde{\eta})^{\lambda_\eta} \) is determined by its restriction to \( K \). In this regard, we define
\[(2.1) \quad \iota: \text{Hom}_{J_\eta}(W, \mathcal{F}(s, \tilde{\eta})) \rightarrow \mathcal{H}(K, \lambda_\eta) \]
by \( \iota(\phi)(k)(w) = \phi(w)(k) \) for \( \phi \in \text{Hom}_{J_\eta}(\mathbb{C}_{\lambda_\eta}, \mathcal{F}(s, \tilde{\eta})), k \in J_\eta K J_\eta, \) and \( w \in \mathbb{C}_{\lambda_\eta} \).

**Lemma 2.2.** (cf. [22, Lemma 3.2.6]) The map \( \iota \) is well defined and is a \( \mathcal{H}(K, \lambda_\eta) \)-module homomorphism.

**Proof.** Let \( j, j' \in J_\eta \) and \( w \in W \). Thanks to (a), we decompose \( j = j_U j_T j_U \) with \( j_U \in J_\eta \cap U \), \( j_T \in J_\eta \cap T \), and \( j_T \in J_\eta \cap T \). We see that elements \( w^{-1} j_T w j' \) intertwine the character \( \lambda_\eta \) and the right translation \( R \). With these in hand, we have
\[ \iota(\phi)(j w j')(w) = \lambda_\eta(j_T)\phi(w)(j_T w j')(w) = \lambda_\eta(j_T)R(w^{-1} j_T w j')\phi(w)(w) \]
\[ = \lambda_\eta(j_T)\phi(w)(w^{-1} j_T w j')w)(w). \]
Appealing to (b), \( \lambda_\eta(w^{-1} j_T w) \) is trivial, and it becomes
\[ \iota(\phi)(w)(j w j') = \lambda_\eta(j_T)\phi(\lambda_\eta(j')w)(w) = \lambda_\eta(j_T)\iota(\phi)(w)\lambda_\eta(j')(w) = \lambda_\eta(j)\iota(\phi)(w)\lambda_\eta(j')(w) \]
having used the fact that \( \lambda_\eta(j_T) = \lambda_\eta(j) \). We confirm that \( \iota \) is well defined, that is to say, \( \iota(\phi) \) belongs to \( \mathcal{H}(K, \lambda_\eta) \).

It remains to show that \( \iota \) is a \( \mathcal{H}(K, \lambda_\eta) \)-module homomorphism. To this end, for \( f \in \mathcal{H}(K, \lambda_\eta) \),
\[ \iota(\pi(f)\phi)(k)(w) = \int_G \pi(g)f(g)^{\lambda_\eta}(w)(k)dg = \int_G \phi(f(g)^{\lambda_\eta}(w)(kg)dg \]
\[ = \int_G \phi(f(g)^{-1}(w)(kg)dg = \int_G \iota(\phi)(kg)f(g^{-1})(w) = \iota(\phi)\ast f)(k)(w) \]
from which the desired result follows. \( \Box \)

The following Proposition can be thought of as the \( \text{SL}_2(F) \)-analogue of [22, Lemma 3.2.9] (cf. [21, Proposition 5.9]).

**Proposition 2.3.** As \( \mathcal{H}(K, \lambda_\eta) \)-modules, \( \mathcal{H}(K, \lambda_\eta) \) is isomorphic to \( \mathcal{F}(s, \tilde{\eta})^{\lambda_\eta} \). Consequently, the \( \lambda_\eta \)-isotypic subspace \( \mathcal{F}(s, \tilde{\eta})^{\lambda_\eta} \) is two dimensional with a \( C \)-basis \( \{ f_{1_2, \tilde{\eta}}, f_{w_0, \tilde{\eta}} \} \).

**Proof.** To verify that (2.1) is an isomorphism, it is sufficient to check that it takes a basis of \( \mathcal{H}(K, \lambda_\eta) \) to a basis of \( \mathcal{F}(s, \tilde{\eta})^{\lambda_\eta} \). This is immediate from Lemma 2.1, since \( \iota(f_{w, \tilde{\eta}}) = T_w \) for \( w \in W \). \( \Box \)
Since $U, \mathcal{U} \cong F$, we may identify the measure on $U$ and $\mathcal{U}$ with the additive measure $dx$ on $F$, and we take $d^\times x$ to be $dx/|x|$ on $F^\times$. The Haar measure $dx$ is normalized so that $J_\eta \cap U \cong \mathfrak{o}$ has volume one. We define a standard $G$-intertwining operator $A(s, \chi) : \mathcal{F}(s, \chi) \rightarrow \mathcal{F}(-s, \chi^{-1})$ by

$$A(s, \chi)(f)(g) = \int_U f(w_0 u g) \, du,$$

for all $f \in \mathcal{F}(s, \chi)$. The integral converges absolutely for $\text{Re}(s) \gg 0$ and defines a rational function on a non-empty Zariski open dense subset of the complex torus $X(T)$. The character $\chi$ is said to be regular if $\chi \neq \chi^{-1}$. Since $\chi$ is regular, every $G$-morphism from $\mathcal{F}(s, \chi)$ to $\mathcal{F}(-s, \chi^{-1})$ is a scalar multiple of $A(s, \chi)$. Our calculation is inspired from the idea in [21].

**Proposition 2.4.** Let $\eta$ be a non-trivial (ramified) character of $\mathfrak{o}^\times$ such that $\eta^2 = 1$. Let $\tilde{\eta}$ be any extension of $\eta$. We assume that $\chi_s$ is regular so that $\mathcal{F}(s, \tilde{\eta})$ and $\mathcal{F}(-s, \tilde{\eta}^{-1})$ are all irreducible. Then we have

$$A(s, \tilde{\eta})(f_{12, \tilde{\eta}}) = \tilde{\eta}(-1) \text{vol}(U \cap J_\eta)f_{w_0, \tilde{\eta}^{-1}} = \tilde{\eta}(-1)q^{-n_s}f_{w_0, \tilde{\eta}^{-1}}$$

and

$$A(s, \tilde{\eta})(f_{w_0, \tilde{\eta}}) = \tilde{\eta}(-1) \text{vol}(U \cap J_\eta)f_{12, \tilde{\eta}^{-1}} = \tilde{\eta}(-1)f_{12, \tilde{\eta}^{-1}}.$$

**Proof.** Since $\mathfrak{R}^\mathfrak{o}(G)$ is equivalent to $\mathcal{H}(G, \lambda_\eta)$-module, we have

$$\text{Hom}_G(\mathcal{F}(s, \tilde{\eta}), \mathcal{F}(-s, \tilde{\eta}^{-1})) \cong \text{Hom}_{\mathcal{H}(G, \lambda_\eta)}(\mathcal{F}(s, \tilde{\eta})\lambda_\eta, \mathcal{F}(-s, \tilde{\eta}^{-1})\lambda_\eta).$$

Therefore, $A(s, \tilde{\eta})$ induces an intertwining map $\mathcal{F}(s, \tilde{\eta})\lambda_\eta \rightarrow \mathcal{F}(-s, \tilde{\eta}^{-1})\lambda_\eta$, which by abuse of notation we will again denote by $A(s, \tilde{\eta})$. It follows from Proposition 2.3 that $A(s, \tilde{\eta})(f_{12, \tilde{\eta}}) = a_1 f_{12, \tilde{\eta}^{-1}} + a_w f_{w_0, \tilde{\eta}^{-1}}$. We may evaluate both sides of this equation at $w_0$ to obtain

$$a_w = \int_U f_{12, \tilde{\eta}}(w_0 u w_0) \, du = \tilde{\eta}(-1) \int_U f_{12, \tilde{\eta}}(u) \, du = \tilde{\eta}(-1) \text{vol}(U \cap J_\eta).$$

For $x \neq 0$, we have the Iwasawa decomposition:

$$(2.2) \quad w_0 u(x) = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}. $$

In particular, for $x^{-1} \in \mathfrak{p}^{n_q}$, we have $f_{12, \tilde{\eta}}(w_0 u(x)) = \tilde{\eta}\chi_s \delta^{1/2}(x^{-1})$. We may rewrite $x \in F^\times$ as $x = \varpi^k y$ with $k \in \mathbb{Z}$ and $y \in \mathfrak{o}^\times$, and then use the relation of additive Haar measure $d(ax) = |a| dx$ for $a \in F^\times$ to determine $a_1$:

$$a_1 = \int_U f_{12, \tilde{\eta}}(w_0 u) \, du = \int_{\{x: x \neq 0, x^{-1} \in \mathfrak{p}^{n_q}\}} \tilde{\eta}(x^{-1})|x|^{-s-1} \, dx$$

$$= \int_{|x| \geq q^{n_q}} \tilde{\eta}(x^{-1})|x|^{-s-1} \, dx = \sum_{n = -\infty}^{n_q} \eta(\varpi^{-n})|\varpi^{-n}|^{-s} \int_{\mathfrak{o}} \tilde{\eta}(x) \, d^\times x.$$

But $\tilde{\eta}$ being ramified implies that the series equality of above integrals is zero. In other words, $a_1 = 0$. \hfill \Box

The **Plancherel constant** is a scalar valued function $\mu(s, \tilde{\eta}) \in \mathbb{C}$ attached to $\eta$ is by the defining relation

$$A(-s, \tilde{\eta}^{-1}) \circ A(s, \tilde{\eta}) = \mu(s, \tilde{\eta})^{-1}$$

on a Zariski open dense subset of $\mathbb{C}$. It is a rational function in $q^{-s}$ and clearly depends on the measure defining intertwining operators.

**Theorem 2.5** (The Plancherel Constant I). Let $\eta$ be a non-trivial (ramified) unitary character of $\mathfrak{o}^\times$ satisfying $\eta^2 = 1$. Let $\tilde{\eta}$ be any extension of $\eta$. Then we have

$$\mu(s, \tilde{\eta}) = \text{vol}(U \cap J_\eta)^{-1} \text{vol}(U \cap J_\eta)^{-1} = q^{n_q}.$$
2.4 \[\text{The Plancherel constant in Theorem 2.5 coincides with that in split cases [23, \S 3.3] on the common Zariski open dense subset of } \mathbb{C} \text{ (refer to [25, Theorem 4.5] for the precise description), and this recovers the work of Kutzko and Morris [25, Theorem 4.5.(2)-(ii)].}\]

3. THE WHITTAKER MODEL AND LOCAL COEFFICIENT FOR SL(2)

3.1. The local coefficient. Let \( \psi \) be a non-trivial additive character of \( F \) trivial on \( \mathfrak{o} \) but not on \( \mathfrak{p}^{-1} \), and \( \chi \) a character of \( F^\times \). It is a theorem of Rodier [36] that the dimension of the space of \( \psi \)-Whittaker functionals \( \Omega_s \) on \( \mathcal{F}(s, \chi) \) is one. We may define a basis vector for of \( \psi \)-Whittaker functionals \( \Omega_s \) on \( \mathcal{F}(s, \chi) \) by the formula

\[
\Omega_s(f) = \int_U f(w_0u)\psi^{-1}(u) \, du.
\]

This integral may not converge for all \( f \) but can be extended to the whole space as a principal value integral. We also have the following convenient reinterpretation for \( \Omega_s \) on \( \mathcal{F}(s, \chi) \) by the formula

\[
\Omega_s(f) = \int_{U_*} f(w_0u)\psi^{-1}(u) \, du
\]

for all \( s \) and for all \( f \in \mathcal{F}(s, \chi)^{K_\sigma} \). We similarly define \( \Omega'_s \) on \( \mathcal{F}(s, \chi^{-1}) \) via

\[
\Omega'_s(f) = \int_U f(w_0u)\psi^{-1}(u) \, du
\]

as a principal value integral in the above sense. Appealing to the aforementioned result of Rodier [36], there exists a non-zero constant \( C_\psi(s, \chi) \) called the Langlands-Shahidi local coefficient [38–40, 42] satisfying

\[
C_\psi(s, \chi)(\Omega'_s \circ A(s, \chi)) = \Omega_s.
\]

For \( c \in F \), the Gauss sum attached to \( \chi \) is defined by

\[
\tau(\chi, \psi, c) = \int_{\mathfrak{o}^\times} \chi(x^{-1})\overline{\psi(cx)} \, dx.
\]

Theorem 3.1 (The Local Coefficient). Suppose that \( \psi \) is a fixed additive character of level 0. Let \( \eta \) be a non-trivial (ramified) character of \( \mathfrak{o}^\times \) satisfying \( \eta^2 = 1 \) with level \( n_\eta \). Let \( \tilde{\eta} \) be any extension of \( \eta \). Then we have

\[
C_\psi(s, \tilde{\eta}) = \tilde{\eta}(-\overline{w^{n_\eta}}) \tau(\eta, \psi, \overline{w^{-n_\eta}})q^{-n_\eta(s-1)}.
\]

Proof. Since \( C_\psi(s, \tilde{\eta}) \) is a rational function in \( \mathbb{C}(q^{-s}) \), it suffices to prove the assertion on a Zariski open dense subset of \( X(T) \). In particular, we impose the assumption that \( \chi_s \) is regular so that \( \mathcal{F}(s, \tilde{\eta}) \) and \( \mathcal{F}(-s, \tilde{\eta}^{-1}) \) are all irreducible. Essentially, we outline the proof in [23, Proposition 3.1]. Evaluating one side is straightforward, since the function \( f_{w_0, \tilde{\eta}} \) is supported on \( Bw_0J_\eta \). Namely,

\[
(\Omega'_s \circ A(s, \tilde{\eta})(f_{12, \tilde{\eta}}) = \tilde{\eta}(-1)\operatorname{vol}(\overline{U} \cap J_\eta)\int_{\mathfrak{o}^\times} f_{w_0, \tilde{\eta}^{-1}}(w_0u)\psi^{-1}(u) \, du = \tilde{\eta}(-1)\operatorname{vol}(\overline{U} \cap J_\eta)\int_{J_\eta \cap U} \psi^{-1}(u) \, du.
\]

We see that \( \psi \) is trivial on \( J_\eta \cap U \), and consequently,

\[
(\Omega'_s \circ A(s, \tilde{\eta})(f_{12, \tilde{\eta}}) = \tilde{\eta}(-1)\operatorname{vol}(\overline{U} \cap J_\eta)\operatorname{vol}(\overline{U} \cap J_\eta) = \tilde{\eta}(-1)q^{-n_\eta}.
\]
This brings us to the central issue of computing the other side $\Omega_s(f_{12,\bar{\eta}})$. In contrast to $f_{\omega,\bar{\eta}}$, this is not immediate, because $f_{12,\bar{\eta}}$ is supported near the identity element. To this end, we deduce from (2.2) that $f_{12,\bar{\eta}}(w_0 u(x))$ is $\tilde{\eta}_1 x 2 \cdot 1/2(x^{-1})$, if $x^{-1} \in p^{m_n}$, and 0, otherwise. Then $\Omega_s(f_{12,\bar{\eta}})$ equals to

$$
\int_{\{p^{-m} \{0\} \cap \{x \mid x^{-1} \in p^{m_n}\}} \tilde{\eta}(x^{-1})|x|^{-s-1}\psi^{-1}(x) dx
$$

for some large positive integer $m \gg 0$. For convenience, let $\mathcal{D}$ denote $\{p^{-m} \{0\} \cap \{x \mid x^{-1} \in p^{m_n}\}$.

For any integer $r$, let $\mathcal{D}_r$ denote the shell, $\{x \in \mathbb{Z}^d \cap x = p^{-m} \{0\}, x^{-1} \in p^{m_n}\}$. Then our domain of the integration can be decomposed as shells $\mathcal{D} = \bigcup_{-m \leq r \leq -m_n} \mathcal{D}_r$. The crux of the proof of [33, Proposition 3.1] is that the last shell $\mathcal{D}_{-m_n}$ is expressed as $\mathbb{Z}^{-n_n} \mathcal{O}$ which only contributes $\Omega_s(f_{12,\bar{\eta}})$. Assembling all of this information, we obtain

$$
(3.2) \quad \Omega_s(f_{12,\bar{\eta}}) = \int_{\mathbb{Z}^{-n_n} \mathcal{O}} \tilde{\eta}(x^{-1})|x|^{-s-1}\psi^{-1}(x) d^x x = \tilde{\eta}(\mathbb{Z}^{-n_n}) q^{-n_n s} \int_{\mathbb{Z}^d} \eta(x^{-1})\psi^{-1}(\mathbb{Z}^{-n_n} x) d^x x
$$

$$
= \tilde{\eta}(\mathbb{Z}^{-n_n}) q^{-n_n s} (\eta, \psi, \mathbb{Z}^{-n_n}).
$$

We can draw the conclusion from (3.1) aligned with (3.2). \qed

This result should be compared with the work of Shahidi [38, Lemma 4.4]. In addition, the local coefficient coincides with the corresponding Hecke-Tate local $\gamma$-factor [44], which can be viewed as Bump and Friedberg exterior square local factors for $GL_1(F) \cong \mathbb{F}_r$ (cf. [32, Proof of Theorem 5.4]). Indeed, Theorem 3.1 confirms Conjecture 3.4 in [2]. The local coefficient has a great influence on the development of modern number theory, and is closed related to the theory of local factors à la the Langlands-Shahidi method [26–29, 38–40, 42].

Let $P(X) \in \mathbb{C}[X]$ be the unique polynomial satisfying $P(0) = 1$ such that $P(q^{-s})$ is the numerator of $C_{\psi}(s, \chi)$. Whenever $\chi$ is unitary, the local $L$-factor is defined by

$$
L(s, \chi) := \frac{1}{P(q^{-s})}.
$$

The local $\epsilon$-factor is defined to satisfy the relation:

$$
C_{\psi}(s, \chi) = \epsilon(s, \chi, \psi)L(1-s, \chi^{-1}) L(s, \chi).
$$

**Corollary 3.2** (Local Factors). Keeping notations and the hypothesis of Theorem 3.1, we have

$$
\epsilon(s, \tilde{\eta}, \psi) = C_{\psi}(s, \tilde{\eta}) \quad \text{and} \quad L(s, \tilde{\eta}) = 1.
$$

Corollaries 3.3 and 3.4 match with the corresponding formulae [40, §Introduction] and [39, Proposition 3.1.1].

**Corollary 3.3** (The Functional Equation). Keeping notations and the hypothesis of Theorem 3.1, we have

$$
C_{\psi}(s, \tilde{\eta}) C_{\psi}(1-s, \tilde{\eta}^{-1}) = \tilde{\eta}(-1).
$$

**Proof.** The functional equation for the local constant $\epsilon(s, \tilde{\eta}, \psi)$ (cf. [3, Corollary 23.4.2]) gives rise to

$$
C_{\psi}(s, \tilde{\eta}) C_{\psi}(1-s, \tilde{\eta}^{-1}) = \epsilon(1/2, \tilde{\eta}, \psi) \epsilon(1/2, \tilde{\eta}^{-1}, \psi) = \tilde{\eta}(-1).
$$

The local coefficient is related to the Plancherel constant which is more or less saying that “the square root of the local coefficient equals the associated Plancherel constant” as given in Corollary 3.4.

**Corollary 3.4** (The Plancherel Constant II). We retain notations and the hypothesis of Theorem 3.1. We further assume that $\tilde{\eta}$ is unitary. Then we have

$$
\mu(s, \tilde{\eta}) = C_{\psi}(s, \tilde{\eta}) C_{\psi}(1-s, \tilde{\eta}^{-1}) = |C_{\psi}(s, \tilde{\eta})|^2.
$$
Proof. We observe from [3, (23.6.3)] that
\[
\tau(\eta, \psi, \varpi^{-n})\tau(\eta^{-1}, \psi^{-1}, \varpi^{-n}) = |\tau(\eta, \psi, \varpi^{-n})|^2 = q^{-n \eta}.
\]

We conclude this section with reviewing the Langlands-Shahidi method to the exterior square $\gamma$-factor in a more general context, which we use later in §4. Let $G = \text{Sp}_{2n}(F)$ be a symplectic group over $F$ in $2n$ variables. The group $M \cong \text{GL}_n(F)$ can be embedded as a Levi component of a maximal Siegel parabolic subgroup $P = MN$ with an unipotent radical $N$. Let $B = TU$ denote the $F$-points of the standard Borel subgroup of upper triangular matrices with with maximal torus $T$ and unipotent radical $U$. By abuse of notation, a character $\psi : U \to \mathbb{C}$ can be defined through $\psi$ in §3.1 in a canonical way. We write $\iota^G_M(\cdot)$ to denote the normalized parabolic induction functor. For $s \in \mathbb{C}$ and $g \in \text{GL}_n(F)$, let $\chi_s$ denote the unramified character $\chi_s(g) = |\det(g)|^s$. For an irreducible generic representation $\pi$ of $\text{GL}_n(F) \cong M$, we let $\iota(s, \pi) := \iota^G_M(\pi \otimes \chi_s)$. and we denote $V(s, \pi)$ denote the space of the unitarily induced representation $\iota(s, \pi)$. We hope the standard use of $\pi$ for the representation of $\text{GL}_n(F)$ (§4) and our use of $\pi$ for the natural left $\mathcal{H}(G, \lambda)$-Mod structure on $V_\lambda$ (§2.1, §2.2, §3.2) will not cause confusion. The distinction should be clear from context. Let
\[
w_0 = \begin{pmatrix} 1_n & -1_n \\ -1_n & 1_n \end{pmatrix}.
\]

We let $w_0(\pi)$ denote the representation of $M$ given by $\pi(w_0 m w_0^{-1})$, $m \in M$. The standard intertwining operator $A(s, \pi, w_0)$ between $V(s, \pi)$ and $V(-s, w_0(\pi))$ is then defined by
\[
A(s, \pi, w_0)f(g) = \int_N f(w_0 ng) \ dn,
\]
where $f \in V(s, \pi)$. It converges for $\text{Re}(s)$ large enough and extends to a rational operator on $q^{-s}$. If $\Omega^M$ is a Whittaker functional for $\pi$, then $\iota(s, \pi)$ is $\psi$-generic for the Whittaker functional $\Omega$ given by
\[
\Omega(s, \pi)(f) = \int_N \Omega^M(f(w_0 n)) \psi(n) \ dn,
\]
where $f \in V(s, \pi)$. The integral converges as a principal value integral over compact open subgroup of $N$ and it is a non-zero Laurent polynomial function on $q^{-s}$ (cf. [26, Theorem 1.4]). Appealing to Rodier’s multiplicity-one theorem [36], the Langlands-Shahidi local coefficient is given by
\[
C_\psi(s, \pi, w_0)(\Omega(-s, w_0(\pi)) \circ A(s, \pi, w_0)) = \Omega(s, \pi).
\]

Let $\gamma(s, \pi, \psi)$ denote a Godement-Jacquet $\gamma$-factor [15]. The Langlands-Shahidi exterior square $\gamma$-factor $\gamma_{LS}(s, \pi, \lambda^2, \psi)$ is a rational function in $\mathbb{C}(q^{-s})$, which is defined by (cf. [27, §2.2])
\[
C_\psi(s, \pi, w_0) = \gamma(s, \pi, \psi) \gamma_{LS}(2s, \pi, \lambda^2, \psi).
\]

3.2. The Gelfand-Graev representation. We turn our attention to the case $G = \text{SL}_2(F) \cong \text{Sp}_2(F)$ as in §2.1. The Gelfand-Graev representation $\text{c-ind}_U^G \psi$ [6, §4.1] is provided by the space of smooth functions $f : G \to \mathbb{C}$ which are compactly supported modulo $U$. They also satisfy
\[
f(u g) = \psi(u) f(g) \quad \text{for all } u \in U \text{ and } g \in G.
\]

Let $\text{Ind}_U^G \psi$ denote the full space of smooth functions $f : G \to \mathbb{C}$ which satisfy (3.3). As a potential application, the contragredient ($\text{c-ind}_U^G \overline{\psi})^\vee \cong \text{Ind}_U^G \overline{\psi}$ of $\text{c-ind}_U^G \psi$ appears in the target space of the Whittaker map $\omega_s : \mathcal{F}(s, \overline{\eta}) \to \text{Ind}_U^G \psi$ corresponding to $\Omega_s$ via Frobenius reciprocity. We take this occasion to explore the structure of the Gelfand-Graev space in $\text{SL}_2(F)$-cases, which extends the result of [6] and [34]. For $w \in W$, we define the functions $\varphi_{\psi_{12}, \overline{\eta}}$ and $\varphi_{\psi_{w_0}, \eta} : G \to \mathbb{C}$ in the $\lambda_{\eta}$-co-invariant Gelfand-Graev space $\text{c-ind}_U^G \psi)^{\lambda_{\eta}}$ given by
\[
\varphi_{\psi_{12}, \overline{\eta}}(w)(g) = \begin{cases} 
\psi(u) \lambda_{\eta}(j) & \text{if } g = uj \in U \cap J^\eta \\
0 & \text{otherwise},
\end{cases}
\]

\[
\varphi_{\psi_{w_0}, \eta}(w)(g) = \begin{cases} 
\psi(u) \lambda_{\eta}(j) & \text{if } g = uj \in U \cap J^\eta \\
0 & \text{otherwise},
\end{cases}
\]
and
\[ \varphi_{w_0, \eta}(w)(g) = \begin{cases} \psi(u)\lambda_\eta(j) & \text{if } g = uw_0j \in Uw_0J_\eta \\ 0 & \text{otherwise.} \end{cases} \]

According to [4, Lemma 10.3] (cf. [6, Proposition 4.2]), the map from \( \text{c-ind}_T^G \psi \) to \( \text{c-ind}_T^T \mathbb{C} \cong C_c(T) \) defined by
\[ \Psi : f \mapsto f_{\Psi}(t) = \delta^{1/2}(t) \int_T f(\pi t) \, d\pi, \quad t \in T \]
descends to an isomorphism \( (\text{c-ind}_T^G \psi)_T \) and \( C_c(T) \). Let \( \text{ch}_{(T \cap J_\eta)}^\eta \in (C_c(T))^\eta \) be a test function supported on \( T \cap J_\eta \) such that \( \text{ch}_{(T \cap J_\eta)}^\eta(t) = \text{Vol}(U \cap J_\eta)\eta(t) \) for all \( t \in T \cap J_\eta \). Owing to [34, Theorem 1] (cf. [6, The discussion preceding Lemma 4.3]), \( \Psi \) in turn endows \( \mathcal{H}(T, \eta) \)-module isomorphisms from \( \text{c-ind}_T^G \psi \) to \( (C_c(T))^{\eta_0} \cong \mathcal{H}(T, \eta) \) of which \( \text{ch}_{(T \cap J_\eta)}^\eta \) is a generator.

**Proposition 3.5.** Suppose that \( \psi \) is a fixed additive character of level 0. Let \( \eta \) be a non-trivial (ramified) character of \( \mathbb{Q}^* \) satisfying \( \eta^2 = 1 \) with level \( n_\eta \). We set \( \epsilon = \eta(-1) \). Then we have

1. \( T_w \ast (\varphi_{12, \psi, \eta} + \epsilon \varphi_{w_0, \eta}) = \epsilon(\psi)(\varphi_{12, \psi, \eta} + \epsilon \varphi_{w_0, \eta}) \) for \( w \in W \).
2. \( \Psi(\varphi_{12, \psi, \eta}) = \text{ch}_{(T \cap J_\eta)}^\eta \) and \( \Psi(\varphi_{w_0, \eta}) = 0 \).

Consequently, \( \text{c-ind}_T^G \psi \) is a free \( \mathcal{H}(T, \eta) \)-module generated by \( \varphi_{12, \psi, \eta} + \epsilon \varphi_{w_0, \eta} \).

**Proof.** The function \( T_{12} \ast \varphi_{12, \psi, \eta} \) is supported on \( UJ_\eta J_\eta \subset UJ_\eta \). Let us evaluate at \( 12 \):
\[
(\pi(T_{12})\varphi_{12, \psi, \eta})(w)(12) = \int_G \varphi_{12, \eta}(T_{12}(x^{-1})w(x) \, dx \\
= \int_{J_\eta} \varphi_{12, \eta}(T_{12}(j^{-1})w)(j) \, dj = \int_{J_\eta} \lambda(j)^{-1} \lambda(j) \varphi_{12, \eta}(w)(12) \, dj = \text{vol}(J_\eta)\varphi_{12, \eta}(w)(12) = 1.
\]

We conclude that \( T_{12} \ast \varphi_{12, \psi, \eta} = \varphi_{12, \psi, \eta} \). In a similar manner, the function \( T_{12} \ast \varphi_{w_0, \psi, \eta} \) is supported on \( UJ_\eta(J_\eta w_0 J_\eta) \subset Uw_0 J_\eta \). Plugging in \( w_0 \) yields
\[
(\pi(T_{12})\varphi_{w_0, \psi, \eta})(w)(w_0) = \int_G \varphi_{w_0, \eta}(T_{12}(x^{-1})w)(w_0 x) \, dx \\
= \int_{J_\eta} \varphi_{w_0, \eta}(T_{12}(j^{-1})w)(w_0)(j) \, dj = \int_{J_\eta} \lambda(j)^{-1} \varphi_{w_0, \eta}(w)(w_0j) \, dj = \text{vol}(J_\eta)\varphi_{w_0, \eta}(w)(w_0) = 1,
\]
which implies that \( T_{12} \ast \varphi_{w_0, \psi, \eta} = \varphi_{w_0, \psi, \eta} \).

We turn to the effect of inter-twiners \( T_{w_0} \) on the generator \( \varphi_{12, \psi, \eta} + \epsilon \varphi_{w_0, \psi, \eta} \). Taking into account the support, it is sufficient to specialize the values at \( 12 \) and \( w_0 \). We first treat off-diagonal terms. We make the change of variables \( x \mapsto w_0^{-1}x \) and then \( x \mapsto x^{-1} \) to deduce
\[
(\pi(T_{w_0})\varphi_{12, \psi, \eta})(w)(w_0) = \int_G \varphi_{12, \eta}(T_{w_0}(x^{-1})w)(w_0 x) \, dx = \int_G \varphi_{12, \eta}(T_{w_0}(x^{-1}w_0)x) \, dx \\
= \int_{J_\eta} \varphi_{12, \eta}(T_{w_0}(jw_0)(j^{-1}) \, dj = \text{vol}(J_\eta)\varphi_{12, \eta}(w)(12) = 1.
\]

We now use the relation \( w_0^2 = -1 \). Then the support of \( \varphi_{w_0, \eta} \) forces that
\[
(\pi(T_{w_0})\varphi_{w_0, \psi, \eta})(w)(12) \\
= \int_G \varphi_{w_0, \eta}(T_{w_0}(x^{-1})) \, dx = \int_{J_\eta} \int_{J_\eta} \varphi_{w_0, \eta}(T_{w_0}(j^{-1}w_0^{-1}j^{-1})w)(jw_0j') \, dj' \, dj.
\]
\[
\begin{align*}
\eta(-1)J_{\eta,j_1,j_2} = J_{\eta,j_1,j_2} &\left( J_{\eta,j_1,j_2}^{-1}w_{j_1,j_2}^{-1}w_{j_1,j_2} \right) J_{\eta,j_1,j_2} = \epsilon \langle U \cap J_{\eta}, \delta \rangle \varphi_{\omega,\eta}(\omega_0) = \epsilon.
\end{align*}
\]

In this circumstance, we can see that the sign change occurs. On the other hand, the diagonal term \( (T_{\omega_0} \times \varphi^{(1)}_{1,j}) \) is 0, as supported \( J_{\eta,j_1,j_2} \) and \( U \cap J_{\eta} \). For another diagonal term \( (\pi(T_{\omega_0})\varphi^{(1)}_{1,j}) \), we use the fact that the integrand below is supported on \( J_{\eta,j_1,j_2} \) and is right invariant under \( J_{\eta} \). Thus \( J_{\eta}/(J_{\eta} \cap w_0 J_{\eta} w_0^{-1}) \rightarrow J_{\eta} J_{\eta} J_{\eta} \) (cf. [24, Lemma 3.2]) to see that

\[
(\pi(T_{\omega_0})\varphi^{(1)}_{1,j})(\omega_0) = \eta(-1)[J_{\eta,j_1,j_2} \cap w_0 J_{\eta} w_0^{-1}] \int_{J_{\eta,j_1,j_2}} \varphi^{(1)}_{1,j} w_0 (\omega_0) du.
\]

The integrand is 0, unless \( x \notin \sigma^{+} \). Then for \( x \in \sigma^{+} \), we apply the Bruhat decomposition (cf. [24, P. 606]):

\[
\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} w_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

whence \( \varphi^{(1)}_{1,j}(w_0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \eta(-x^{-1}) \psi(-x) \). Therefore it can be expressed in terms of Gauss sums attached to the ramified character \( \eta \) and then eventually this term vanishes

\[
(\pi(T_{\omega_0})\varphi^{(1)}_{1,j})(\omega_0) = \eta(-1)[J_{\eta,j_1,j_2} \cap w_0 J_{\eta} w_0^{-1}] \int_{\sigma^{+}} \eta(-x^{-1}) \psi(x) \varphi^{(1)}_{1,j}(\omega_0) dx = 0,
\]

as was to be shown. Altogether, we obtain

\[
T_{\omega_0} \varphi^{(1)}_{1,j} = \varphi^{(1)}_{1,j} = \epsilon^2 \varphi^{(1)}_{1,j} \quad \text{and} \quad T_{\omega_0} \varphi^{(1)}_{1,j} = \epsilon \varphi^{(1)}_{1,j}.
\]

The second assertion is immediate from Claim in the proof of [6, Lemma 4.1].

Let \( \epsilon \)-sgn denote the one-dimensional representation of \( \mathcal{H}(K, \lambda_n) \) on \( \mathbb{C} \) wherein \( T_w \) acts as \( \epsilon^{\ell(w)} \), \( w \in \mathcal{W} \). Here \( \ell \) denotes the length function on \( \mathcal{W} \). When \( \epsilon = -1 \), we simply write \( (\text{sgn}, \mathbb{C}_{-1}) \) for \( (\epsilon \text{-sgn}, \mathbb{C}_{-1}) \).

**Theorem 3.6.** Suppose that \( \psi \) is a fixed additive character of level 0. Let \( \eta \) be a non-trivial (ramified) character of \( \sigma^{+} \) satisfying \( \eta^2 = 1 \) with level \( n_\eta \). We set \( \epsilon = \eta(-1) \). As \( \mathcal{H}(G, \lambda_\eta) \)-module isomorphisms, we have

\[
(c \text{-ind}_{U}^{G} \psi)^{\lambda_\eta} \cong \mathcal{H}(G, \lambda_\eta) \otimes \mathcal{H}(K, \lambda_n) \mathbb{C} \cong \mathcal{H}(G, \lambda_\eta) \otimes \mathcal{H}(K, \lambda_n) \mathbb{C}_{-1}.
\]

**Proof.** In virtue of Proposition 3.5-(1), we construct an element in \( \operatorname{Hom}_{\mathcal{H}(K, \lambda_n)}(\mathbb{C}, c \text{-ind}_{U}^{G} \psi)^{\lambda_\eta} \) given by \( 1 \mapsto \varphi^{\psi}_{1,\eta} + \epsilon \varphi^{\psi}_{0,\eta} \). We have the following Frobenius reciprocity

\[
\operatorname{Hom}_{\mathcal{H}(K, \lambda_n)}(\mathbb{C}, c \text{-ind}_{U}^{G} \psi)^{\lambda_\eta} \cong \operatorname{Hom}_{\mathcal{H}(G, \lambda_\eta)}(\mathcal{H}(G, \lambda_\eta) \otimes \mathcal{H}(K, \lambda_n) \mathbb{C}, c \text{-ind}_{U}^{G} \psi)^{\lambda_\eta},
\]

where the element \( 1 \mapsto \varphi^{\psi}_{1,\eta} + \epsilon \varphi^{\psi}_{0,\eta} \) corresponds to \( T_{1} \otimes 1 \mapsto \varphi^{\psi}_{1,\eta} + \epsilon \varphi^{\psi}_{0,\eta} \). The conclusion follows from Proposition 3.5-(2), since both \( \mathcal{H}(G, \lambda_\eta) \otimes \mathcal{H}(K, \lambda_n) \mathbb{C} \) and \( \mathcal{H}(G, \lambda_\eta) \otimes \mathcal{H}(K, \lambda_n) \mathbb{C}_{-1} \) are free \( \mathcal{H}(G, \lambda_\eta) \)-modules generated by \( T_{1} \otimes 1 \). \( \square \)

When the characteristic of the field is 0 and that of the residual field is \( p > 3 \), the above theorem has been settled by Mishra and Pattanayak [34, Theorem 3]. To the best of our knowledge, Theorem 3.6 is new for positive characteristic, or characteristic 0 with \( p = 2, 3 \).
Remark 3.7. If $\gamma^2 \neq 1$, then $t_Q$ in (c) becomes an isomorphism [24, Corollary 3.1] in which case the cover $(J_\eta, \lambda_\eta)$ is said to be a split cover. Then Proposition 3.5-(1) seems to be superfluous, because the map $t_Q$ already produce a series of $H(T, \eta) \cong H(G, \lambda_\eta)$-module isomorphism

$$\text{(c-ind}_{U}\psi)^{\lambda_\eta} \cong H(G, \lambda_\eta) \otimes_{H(K, \lambda_\eta)} C_{\tau} \cong H(T, \eta) \otimes_{H(K, \lambda_\eta)} C_{\tau} \cong H(T, \eta) \cong (C^\infty_c(T))^\eta.$$ 

4. The Local Exterior Square Gamma Factors for Level Zero Representations

4.1. Jacquet-Shalika integrals. Let $\tau$ be an irreducible cuspidal representation of $GL_n(F)$. A level zero supercuspidal representation $\pi$ of $GL_n(F)$ is given by

$$\pi \cong \text{c-Ind}_{F^{\times\times}GL_n(F)}^{GL_n(F)} \tilde{\lambda},$$

where $\tilde{\lambda}$ is a representation of $F^{\times\times}GL_n(o)$ such that $\lambda := \tilde{\lambda}|_{GL_n(o)}$ is an inflation of $\tau$ via the canonical projection $GL_n(o) \mod p \to GL_n(F)$. Let $\omega_\pi$ denote the central character of $Z_n(F) \cong F^{\times\times}$ the center of $GL_n(F)$ such that $\omega_\pi|_{GL_n(o) \cap F^{\times\times}} = \tilde{\lambda}|_{GL_n(o) \cap F^{\times\times}}$. In comparison with §3, we switch a level zero nontrivial additive character $\psi$ to a level one character that is trivial on $p$ but not on $o$.

We briefly overview the theory of $\gamma$-factor [18], using the formulation of Jacquet and Shalika [17]. Let $\pi$ be a level zero supercuspidal representation of $GL_n(F)$ with associated Whittaker models $W(\pi, \psi)$. We let $M_n(F)$ be $n \times n$ matrices, $N_n(F)$ the subspace of upper triangular matrices of $M_n(F)$. We denote by $U_n(F)$ the maximal unipotent subgroup of upper triangular unipotent matrices. As before, we let $K_n = GL_n(o)$ be the standard maximal compact subgroup of $GL_n(F)$. Let $\sigma_n$ be the permutation matrix given by

$$\sigma_{2m} = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & 2m \\ 1 & 3 & \cdots & 2m-1 & 2 & 4 & \cdots & 2m \end{pmatrix}$$

when $n = 2m$ is even, and by

$$\sigma_{2m+1} = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & 2m & 2m+1 \\ 1 & 3 & \cdots & 2m-1 & 2 & 4 & \cdots & 2m & 2m+1 \end{pmatrix}$$

when $n = 2m + 1$ is odd. Let $S(F^m)$ be the space of locally constant and compactly supported functions $\Phi : F^m \to \mathbb{C}$. Let $\{e_i | 1 \leq i \leq m\}$ be the standard row basis of $F^m$. For each $W \in W(\pi, \psi)$ and $\Phi \in S(F^m)$, we define Jacquet-Shalika integrals $J(s, W, \Phi)$ by

$$\int_{U_n(F) \backslash GL_n(F)} \int_{N_n(F) \backslash M_n(F)} \int_{F^m} W \left( \sigma_{2m+1} \begin{pmatrix} 1 & X \cr m & 1_m \end{pmatrix} \begin{pmatrix} g & \cr \cr \cr g & 1 \end{pmatrix} \begin{pmatrix} 1 & 1_m \cr \cr z & 1 \end{pmatrix} \right) \psi^{-1}(\text{Tr} X) \Phi(z)|\det(g)|^{s-1} \, dz \, dX \, dg$$

in the odd case $n = 2m + 1$ and

$$\int_{U_n(F) \backslash GL_n(F)} \int_{N_n(F) \backslash M_n(F)} \int_{F^m} W \left( \sigma_{2m} \begin{pmatrix} 1 & X \cr m & 1_m \end{pmatrix} \begin{pmatrix} g & \cr \cr \cr g & 1 \end{pmatrix} \psi^{-1}(\text{Tr} X) \Phi(e_m g)|\det(g)|^s \, dX \, dg$$

in the even case $n = 2m$. These integrals converge absolutely for $\text{Re}(s) \gg 0$, and it defines a rational function in $\mathbb{C}(q^{-s})$. We define the Fourier transform on $S(F^m)$ by

$$\hat{\Phi}(y) = q^{-m/2} \int_{F^m} \Phi(x) \psi(x^t y) \, dx$$

so that the measure on $F^m$ is the self-dual Haar measure. Then the Fourier inversion takes the form $\hat{\hat{\Phi}}(x) = \Phi(-x)$. Let

$$w_t := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$$
denote the long Weyl element in $\text{GL}_n(F)$. Let $\tilde{\pi}$ denote the contragredient representation of $\pi$. If $W \in \mathcal{W}(\pi, \psi)$, then $W'(g) := W(W_{\psi}^{-1}g^{-1})$ belongs to $\mathcal{W}(\tilde{\pi}, \psi^{-1})$. The local exterior square $\gamma$-factor $\gamma(s, \pi, \wedge^2, \psi)$ is defined as a proportionality. The $\gamma$-factor is a rational function in $\mathbb{C}(q^{-s})$ satisfying

$$J(1 - s, \pi, \tau_n)W, \Phi) = \gamma(s, \pi, \wedge^2, \psi)J(s, W, \Phi),$$

where $\tau_n$ is the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ if $n = 2m$, and the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $n = 2m + 1$.

**Lemma 4.1.** Let $F$ be a local function field. Let $\pi$ be an irreducible subquotient of the unramified principal series representation $\text{Ind}_{B_n(F)}^{\text{GL}_n(F)}(\mu_1 \otimes \cdots \otimes \mu_n)$, where each $\mu_i$ is an unramified character of $F^\times$, that is to say, it is invariant under the maximal compact subgroup $o^\times$ of $F^\times$. Then we have

$$\gamma(s, \pi, \wedge^2, \psi) = \prod_{1 \leq j < k \leq n} \gamma(s, \mu_j \times \mu_k, \psi).$$

**Proof.** Let us set $\Xi = \text{Ind}_{B_n(F)}^{\text{GL}_n(F)}(\mu_1 \otimes \cdots \otimes \mu_n)$. Let $V_\pi$ and $V_\Xi$ denote their underlying space of $\pi$ and $\Xi$, respectively. By the uniqueness of Whittaker functionals, a non-zero Whittaker functional on $V_\pi$ induces a non-zero Whittaker functional on $V_\Xi$. As this representation has a unique Whittaker functional, this must be it and we may conclude that $\gamma(s, \pi, \wedge, \psi) = \gamma(s, \Xi, \wedge, \psi)$. Now it is worthwhile noting that a spherical representation of Whittaker types $\Xi$ (not necessarily irreducible) has a $K_n$-fixed vector. Furthermore, the subspace of $K_n$-fixed vectors must be one-dimensional. This $K_n$-fixed vector, which is unique up to scalar multiplication, is called the spherical vector of $\pi$. For such a spherical representation $\pi$, we normalized the spherical Whittaker function $W^\circ$ in the Whittaker model $\mathcal{W}(\Xi, \psi)$ so that $W^\circ(1_n) = 1$. Upon taking $W^\circ \in \mathcal{W}(\Xi, \psi)$ and $\Phi^\circ \in \mathcal{S}(F^m)$ of a characteristic function on $\sigma^m$ and using [17, §2], we have the identity

$$\prod_{i=1}^n L(1 - s, \mu_i^{-1}, \wedge^2) \prod_{1 \leq j < k \leq n} L(1 - s, \mu_j^{-1} \times \mu_k^{-1}) = J(1 - s, \tau_n)W^\circ, \Phi^\circ$$

$$= \gamma(s, \Xi, \wedge^2, \psi)J(s, W^\circ, \Phi^\circ) = \gamma(s, \Xi, \wedge^2, \psi) \prod_{i=1}^n L(s, \mu_i, \wedge^2) \prod_{1 \leq j < k \leq n} L(s, \mu_j \times \mu_k)$$

from which the result we seek for follows. \(\square\)

We let $k$ denote a global function field with field of constant $F_q$ and ring of ad` eles $A_k$. We summarize the following standard globalization due to Lomeli [29, Lemma 3.1 and Remark 3.2] (cf. [1, Lemma 9.28]), which is stemmed from the work of Henniart and Vignéras [42, Proposition 5.1].

**Theorem 4.2.** Let $\pi$ be a level zero unitary supercuspidal representation of $\text{GL}_n(F)$ over a local function field $F$. There is a global field $k$ with a set of three places $S = \{v_0, v_1, v_{\infty}\}$ such that $k_{v_0} \cong F$. There exists an irreducible cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ of $\text{GL}_n(A_k)$ satisfying the following properties:

(i) $\Pi_{v_0} \cong \pi$;
(ii) $\Pi_v$ is an irreducible unramified principal series representation at every $v \notin S$;
(iii) $\Pi_{v_1}$ and $\Pi_{v_{\infty}}$ are irreducible quotients of unramified principal series representations.
(iv) If $\pi$ is generic, then $\Pi$ is globally generic.

The aforementioned globalization is required to prove a purely local statement, namely, that local exterior square $\gamma$-factors $\gamma(s, \pi, \wedge^2, \psi)$ via Rankin-Selberg methods due to Jacquet and Shalika [17].
agrees with those $\gamma_{LS}(s, \pi, \wedge^2, \psi)$ via Langlands-Shahidi methods [12] in positive characteristic at hand. We eventually generalize the equality to characteristic zero in Theorem 4.8.

**Theorem 4.3.** Let $\pi$ be a level zero supercuspidal representation of $GL_n(F)$ over a local function field $F$. Then we have

$$\gamma(s, \pi, \wedge^2, \psi) = \gamma_{LS}(s, \pi, \wedge^2, \psi).$$

**Proof.** Twisting by an unramified character does not affect the conclusion, so there is no harm to assume that $\pi$ is unitary (cf. [28, §6.6-(vi)]). We select a set $S = \{v_0, v_1, v_\infty\}$ of three places as in Theorem 4.2. We take a non-trivial additive character $\Psi$ of $\mathbb{A}_k/k$, and assume, as we may, that $\Psi_{v_0} = \psi$. Applying Theorem 4.2 to the irreducible level zero, we obtain an irreducible unitary cuspidal automorphic representation $\Pi$. The global functional equation via the Langlands-Shahidi method can be read from [12, §4-(vi)] as

$$L^S(s, \Pi, \wedge^2) = \gamma_{LS}(s, \Pi_{v_0}, \wedge^2, \Psi_{v_0}) \prod_{v \in S - \{v_0\}} \gamma_{LS}(s, \Pi_v, \wedge^2, \Psi_v)L^S(1 - s, \Pi, \wedge^2).$$

(4.1)

Since for $v \notin S$ we know that $\Pi_v$ and $\psi_v$ are unramified so that $\varepsilon(s, \Pi_v, \wedge^2, \psi_v) \equiv 1$, [20, Theorem 3.3] is rephrased as

$$L^S(s, \Pi, \wedge^2) = \gamma(s, \Pi_{v_0}, \wedge^2, \Psi_{v_0}) \prod_{v \in S - \{v_0\}} \gamma(s, \Pi_v, \wedge^2, \Psi_v)L^S(1 - s, \Pi, \wedge^2).$$

(4.2)

Applying Lemma 4.1 gives us $\gamma_{LS}(s, \Pi_v, \wedge^2, \Psi_v) = \gamma(s, \Pi_v, \wedge^2, \Psi_v)$ for $v \in S - \{v_0\}$. In doing so, the result that we seek for is immediate, once we divide (4.1) by (4.2). 

\[\square\]

**4.2. Deligne-Kazhdan close field theory.** We turn our attention to Deligne-Kazhdan close local field theory. Two non-archimedean local fields $F$ and $F'$ are $m$-close if $\mathfrak{o}_F/\mathfrak{p}_F^m \cong \mathfrak{o}_{F'}/\mathfrak{p}_{F'}^m$. For example, the fields $\mathbb{F}_p((t))$ and $\mathbb{Q}_p(p^{1/m})$ are $m$-close. We follow the elaboration about Deligne’s theory in [10, §2.1] and [11, §6.3]. Deligne consider the triplet

$$\text{Tri}(F) := (\mathfrak{o}_F/\mathfrak{p}_F, \mathfrak{p}_F/\mathfrak{p}_F^2, \xi)$$

where $\xi$ is the trivial map of $\mathfrak{p}_F/\mathfrak{p}_F^2$ onto $\mathfrak{p}_F/\mathfrak{p}_F$. If $F$ and $F'$ is 1-close, the isomorphism of triplet $\text{Tri}(F) = \text{Tri}(F')$ gives rise to an isomorphism:

$$\text{Gal}(\overline{F}/F)/I_{F}^1 \xrightarrow{\text{Del}} \text{Gal}(\overline{F'}/F')/I_{F'}^1,$$

where $\overline{F}$ is a separable algebraic closure of $F$, $I_{F}^1 := I_F$ is the inertia subgroup, and $I_{F'}^1 := P_F$ is the wild inertia subgroup. This produces a bijection:

$$\gamma_{LS}(s, \wedge^2, \psi) = \gamma(s, \wedge^2, \psi).$$

Moreover, the above holds when $\text{Gal}(\overline{F}/F)$, the absolute Galois group, is replaced by $W_F$, the Weil group of $F$. Elements $\phi$ and $\phi'$ are so-called *tamely ramified*. The Weil representation $\phi$ is tamely ramified if the kernel of $\phi$ contains the wild inertia subgroup $I_F^1$ (cf. [47, §2]). The triplet $(F, \phi, \psi)$ is said to be Del-associate to $(F', \phi', \psi')$ if

(a) $F$ and $F'$ are 1-close;
(b) $\phi' = \text{Del}(\phi)$;
(c) an additive character $\psi'$ of $F'$ satisfies $\text{cond}(\psi') = 1$ and $\psi'|_{\mathfrak{o}_F/\mathfrak{p}_F} = \psi'|_{\mathfrak{o}_F/\mathfrak{p}_{F'}}$.

Let $\gamma(s, \wedge \circ \phi, \psi)$ denote the Artin exterior square $\gamma$-factor. We then have a following proposition [11, Proposition 6.2].

**Proposition 4.4.** For $(F, \phi, \psi)$ that is Del-associate to $(F', \phi', \psi')$, we have

$$\gamma(s, \wedge \circ \phi, \psi) = \gamma(s, \wedge \circ \phi', \psi').$$
The analogous isomorphism of Deligne on the analytic side over close local fields has been studied by Kazhdan [10, §2.3]. We provide a variant of the Kazhdan isomorphism [11, §6.2], which can be directly verified from the construction as in [46, Theorem 3.5]:

\[
\begin{align*}
\left\{ \text{level zero supercuspidal representations } (\pi, V) \text{ of } \text{GL}_n(F) \right\}_{Kaz} \rightarrow \left\{ \text{level zero supercuspidal representations } (\pi', V') \text{ of } \text{GL}_n(F') \right\},
\end{align*}
\]

where \( \pi \cong \text{c-Ind}_{F \times \text{GL}_n(\mathfrak{o}_F)}^{\text{GL}_n(F)} \tilde{\lambda} \) and \( \pi' \cong \text{c-Ind}_{F' \times \text{GL}_n(\mathfrak{o}_{F'})}^{\text{GL}_n(F')} \tilde{\lambda}' \) under the isomorphism “Kaz” satisfy

\[
\begin{align*}
& (i) \quad \omega_{\pi}(\varpi_F) = \omega_{\pi'}(\varpi_{F'}); \\
& (ii) \quad \lambda := \tilde{\lambda}|_{\text{GL}_n(\mathfrak{o}_F)} \text{ and } \lambda' := \tilde{\lambda}'|_{\text{GL}_n(\mathfrak{o}_{F'})} \text{ are an inflation of a common irreducible cuspidal representation } \tau \text{ via the canonical projections:}
\end{align*}
\]

\[
(GL_n(\mathfrak{o}_F), \lambda) \underset{\text{mod } \mathfrak{p}_F}{\sim} (GL_n(\mathbb{F}_q), \tau) \underset{\text{mod } \mathfrak{p}_{F'}}{\sim} (GL_n(\mathfrak{o}_{F'}), \lambda').
\]

We say that the triplet \((F, \pi, \psi)\) is Kaz-associated to \((F', \pi', \psi')\) if

(a) \( F \) and \( F' \) are 1-close; \\
(b) \( \pi' = \text{Kaz}(\pi) \); \\
(c) an additive character \( \psi' \) of \( F' \) satisfies \( \text{cond}(\psi') = 1 \) and \( \psi'|_{\mathfrak{o}_{F'/\mathfrak{p}_{F'}}} = \psi|_{\mathfrak{o}_{F'/\mathfrak{p}_{F'}}} \).

Let \( \tau_1 \) be an irreducible cuspidal representation of \( \text{GL}_n(\mathbb{F}_q) \) and \( \tau_2 \) an irreducible cuspidal representation of \( \text{GL}_t(\mathbb{F}_q) \) with \( n > t \). Then there exists a complex number \( \Gamma(\tau_1 \times \tau_2, \psi) \in \mathbb{C} \) such that

\[
\Gamma(\tau_1 \times \tau_2, \psi) \sum_{g \in U_t(\mathbb{F}_q) \backslash \text{GL}_t(\mathbb{F}_q)} W\left( \begin{pmatrix} g & 0 \\ 0 & 1_{n-t} \end{pmatrix} \right) W'(g) = \sum_{g \in U_t(\mathbb{F}_q) \backslash \text{GL}_t(\mathbb{F}_q)} W\left( \begin{pmatrix} 0 & 1_{n-t} \\ g & 0 \end{pmatrix} \right) W'(g),
\]

for all \( W \in \mathcal{W}(\tau_1, \psi) \) and \( W' \in \mathcal{W}(\tau_2, \psi^{-1}) \) [35, Theorem 2.3]. Let \( \pi_1 \) be a level zero supercuspidal representation of \( \text{GL}_n(F) \) associated to \( \tau_1 \) and \( \pi_2 \) a level zero supercuspidal representation of \( \text{GL}_t(F) \) associated to \( \tau_2 \), with \( n > t \). Let \( \gamma(s, \pi_1 \times \pi_2, \psi) \) denote the Rankin-Selberg \( \gamma \)-factor defined by Jacquet, Piatetski-Shapiro, and Shalika [35, Theorem 2.1].

**Lemma 4.5.** For \((F, \pi_1, \pi_2, \psi)\) that is Kaz-associated to \((F', \pi'_1, \pi'_2, \psi')\), we have

\[
\gamma(s, \pi_1 \times \pi_2, \psi) = \gamma(s, \pi'_1 \times \pi'_2, \psi').
\]

**Proof.** With aid of [35, Theorem 4.2], we can relate gamma factors for a pair of level zero supercuspidal representations with those for the corresponding cuspidal representations over finite fields:

\[
\omega_{\tau_2}^{n-1}(-1) \gamma(s, \pi_1 \times \pi_2, \psi) = \text{Vol}(\mathfrak{p}_F)^{-t(n-t-1)} \Gamma(\tau_1 \times \tau_2, \psi).
\]

Upon recalling from §2.2 that we normalize the Haar measure on \( F \) so that the volume of \( \mathfrak{o}_F \) is 1, we find that

\[
\omega_{\tau_2}^{n-1}(-1) \gamma(s, \pi_1 \times \pi_2, \psi) = \mathfrak{p}_F^{-t(n-t-1)} \Gamma(\tau_1 \times \tau_2, \psi),
\]

in which case it is equal to \( \omega_{\tau_2}^{n-1}(-1) \gamma(s, \pi'_1 \times \pi'_2, \psi') \). \( \square \)

Following the literature in [47], we let \( \Pi_0(\text{GL}_n(F)) \) be the set of equivalence classes of level zero supercuspidal representations of \( \text{GL}_n(F) \). We set \( \Phi'(\text{GL}_n(F)) \) to be the isomorphism classes of tamely ramified Weil representation of \( W_F \) of degree \( n \). We apologize for the double usage of “\( t \)”, but we hope that the reader can separate the meaning from the context. Since the local Langlands reciprocity map preserves the conductor and the depth of the representation [10, Theorem 7.3], the correspondence induces a natural bijective map \( \Pi_0(\text{GL}_n(F)) \rightarrow \Phi'(\text{GL}_n(F)) \) [37, Appendix A] (cf. [10, §7]). The assignment “LLC” is now reconciled with the Deligne-Kazhdan theory (4.3) and (4.4).
Proposition 4.6. We assume that non-archimedean local fields $F$ and $F'$ are 1-close. Then the following diagram commutes:

\[
\begin{array}{ccc}
\Pi_0(\text{GL}_n(F)) & \xrightarrow{\text{LLC}} & \Phi^t(\text{GL}_n(F)) \\
\text{Kaz} \cong & & \cong \text{Del} \\
\Pi_0(\text{GL}_n(F')) & \xrightarrow{\text{LLC}} & \Phi^t(\text{GL}_n(F')) \\
\end{array}
\]

Proof. We will prove this theorem by induction on $n$. When $n = 1$, the Deligne-Kazhdan philosophy is compatible with local class field theory [10, Property (i) of §2.1]. Now we assume that the Proposition holds for $1 \leq t \leq n - 1$. Let $\pi_1 \in \Pi_0(\text{GL}_n(F))$ and $\tau \in \Pi_0(\text{GL}_t(F))$. Let $\phi_{\pi_1}$ and $\phi_\tau$ denote the local Langlands parameter attached to $\pi_1$ and $\tau$, respectively. We put $\pi'_1 = \text{Kaz}(\pi_1)$ and $\tau' = \text{Kaz}(\tau)$. Writing $\pi_2 = \text{LLC}^{-1} \circ \text{Del}^{-1}(\phi_{\pi'_1})$, the corresponding local Langlands parameter $\phi_{\pi_2}$ is Del-associated to $\phi_{\pi'_1}$. In view of [10, (d) of Theorem 7.1] along with [10, Property (i) of §2.1] again, $\pi_1$ and $\pi_2$ share the same central character $\omega_{\pi_1} = \omega_{\pi_2}$. By induction hypothesis, we have

$$\tau = \text{LLC}^{-1} \circ \text{Del}^{-1}(\phi_{\tau'}).$$

This leads us to a chain of identities:

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi'_1 \times \tau', \psi') = \gamma(s, \phi_{\pi'_1} \otimes \phi_{\tau'}, \psi')$$

$$= \gamma(s, \phi_{\pi_2} \otimes \text{Del}^{-1}(\phi_{\tau'}), \psi') = \gamma(s, \pi_2 \times \text{LLC}^{-1} \circ \text{Del}^{-1}(\phi_{\tau'}), \psi') = \gamma(s, \pi_2 \times \tau, \psi)$$

for all $\tau \in \Pi_0(\text{GL}_t(F))$ and $1 \leq t \leq n - 1$. Here, the second and fourth equalities are a part of local Langlands correspondence [10, (b) of Theorem 7.1], the third equality follows from [10, Property (iii) of §2.1] due to Deligne, and the first equality is clear from Lemma 4.5. Then by the local converse theorem for level zero supercuspidal representations [45, Theorem 5.3], we conclude that $\pi_1 \cong \pi_2$ from which the desired commutative diagram follows. \qed

We define the Fourier transform on $S(F_q^n)$ by

$$\hat{\Phi}(y) = q^{-m/2} \sum_{x \in F_q^n} \Phi(x)\psi(x^t y).$$

The Fourier inversion formula for this normalization is given by $\hat{\Phi}(x) = \Phi(-x)$. Let $\tau$ be an irreducible cuspidal representation of $\text{GL}_n(F_q)$. For all $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in S(F_q^n)$, there exists a complex number $\Gamma(\tau, \lambda^2, \psi) \in \mathbb{C}$ such that

\[
\Gamma(\tau, \lambda^2, \psi) \sum_{g \in U_m(F_q) \backslash \text{GL}_m(F_q)} \sum_{X \in \mathcal{N}_m(F_q) \backslash \mathcal{M}_m(F_q)} \sum_{z \in F_q^n}
W \left( \begin{array}{cc} 1_m & X \\ 1_m & 1 \end{array} \right) \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_m & 1_m \\ z & 1 \end{pmatrix} \psi^{-1}(\text{Tr} X)\Phi(z)
= \sum_{g \in U_m(F_q) \backslash \text{GL}_m(F_q)} \sum_{X \in \mathcal{N}_m(F_q) \backslash \mathcal{M}_m(F_q)} \sum_{z \in F_q^n}
W \left( \begin{array}{c} 1 \\ 1_m \end{array} \right) \sigma_{2m+1} \left( \begin{array}{cc} 1_m & X \\ 1_m & 1 \end{array} \right) \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_m & 1_m \\ -^t Z & 1 \end{pmatrix} \psi^{-1}(\text{Tr} X)\Phi(z)
\]

in the odd case $n = 2m + 1$, and
\[
\Gamma(\tau, \wedge^2, \psi) \sum_{g \in U_m(F_q) \backslash GL_m(F_q)} \sum_{X \in N_m(F_q) \backslash M_m(F_q)} W\left(\sigma_{2m}\begin{pmatrix} 1_m & X \\ 1_m & \end{pmatrix}\begin{pmatrix} g & \\ g & \end{pmatrix}\right) \psi^{-1}(\text{Tr} X) \Phi(e_m g) = \sum_{g \in U_m(F_q) \backslash GL_m(F_q)} \sum_{X \in N_m(F_q) \backslash M_m(F_q)} W\left(\sigma_{2m}\begin{pmatrix} 1_m & X \\ 1_m & \end{pmatrix}\begin{pmatrix} g & \\ g & \end{pmatrix}\right) \psi^{-1}(\text{Tr} X) \hat{\Phi}(e_1 t g^{-1})
\]
in the even case \( n = 2m \) \([46, \text{Theorem 2.8}].\)

We define a Shalika subgroup of \( S_{2m}(F) \) of \( GL_{2m}(F) \) by

\[ S_{2m}(F) = \left\{ \begin{pmatrix} 1_m & X \\ 1_m & \end{pmatrix}\begin{pmatrix} g & \\ g & \end{pmatrix} | X \in M_m(F), g \in GL_m(F) \right\}. \]

Let \( \Theta \) be a Shalika character of \( S_{2m}(F) \) given by

\[ \Theta\left(\begin{pmatrix} 1_m & X \\ 1_m & \end{pmatrix}\begin{pmatrix} g & \\ g & \end{pmatrix}\right) = \psi(\text{Tr} X). \]

In contrast to \( \S 2.2 \), we normalize the multiplicative Haar measure \( d^\times x \) on \( F^\times \) so that the volume of \( \sigma_F^\times \) is 1 for the sake of convenience (cf. \([46, \text{§3.1.1}]\)).

**Proposition 4.7.** For \((F, \pi, \psi)\) that is Kaz-associated to \((F', \pi', \psi')\), we have

\[ \gamma(s, \pi, \wedge, \psi) = \gamma(s, \pi', \wedge, \psi') \quad \text{and} \quad \gamma_{\text{LS}}(s, \pi, \wedge, \psi) = \gamma_{\text{LS}}(s, \pi', \wedge, \psi'). \]

**Proof.** We consider the first equality. Proposition 3.23 in \([46]\) yields the equivalent condition that \( \text{Hom}_{S_{2m}(F)}(\pi \otimes |\det(\cdot)|^{s/2}, \Theta) \neq 0 \) for some \( s \in \mathbb{C} \) if and only if \( \text{Hom}_{S_{2m}(F')}(\pi' \otimes |\det(\cdot)|^{s'/2}, \Theta') \neq 0 \) for some \( s' \in \mathbb{C} \). If this is the case, we use \([46, \text{Theorem 3.17}]\) in conjunction with the fact that \( \omega_n(\varpi F') = \omega_n(\varpi F) \) to prove

\[ \gamma(s, \pi, \wedge, \psi) = \frac{q_F^{m(s-1/2)}}{\omega_n(\varpi F)} \frac{L(m(1-s), \omega_n^{-1})}{\omega_n^2(\varpi)} = \frac{q_F^{m(s-1/2)}}{\omega_n(\varpi F')} \frac{L(m(1-s), \omega_n^{-1})}{\omega_n^2(\varpi)} = \gamma(s, \pi', \wedge, \psi'). \]

Otherwise, owing to \([46, \text{Theorem 3.16}]\), we are guided to

\[ \gamma(s, \pi, \wedge, \psi) = \Gamma(\tau, \wedge^2, \psi) = \gamma(s, \wedge \circ \phi, \psi). \]

Next, we deal with the second equality. Let \( \pi(\phi) \) be a level zero supercuspidal representation of \( GL_n(F) \) obtained from a tamely ramified Weil representation \( \phi \) of \( W_F \) of degree \( n \) via the local Langlands correspondence (LLC). The identity

\[ \gamma_{\text{LS}}(s, \pi, \wedge, \psi) = \gamma(s, \wedge \circ \phi, \psi) \]

relating analytic \( \gamma \)-factors \( \gamma_{\text{LS}}(s, \pi, \wedge, \psi) \) with corresponding Artin factors \( \gamma(s, \wedge \circ \phi, \psi) \) has been established for non-archimedean local fields \( F \) of characteristic 0 in \([8]\) and positive characteristic in \([12]\). For \((F, \phi, \psi)\) that is Del-associated to \((F', \phi', \psi')\), a similar notation \( \pi'(\phi') \) applies to \( \phi' \). We see from Proposition 4.6 that \( \pi(\phi) \) is Kaz-associated to \( \pi'(\phi') \), at which point Proposition 4.4 together with (4.7) completes the proof.

It is time to bring Deligne-Kazhdan close field theory and Theorem 4.3 back together for good use.

**Theorem 4.8.** Let \( \pi \) be a level zero representation of \( GL_n(F) \) over a \( p \)-adic field \( F \). Then we have

\[ \gamma(s, \pi, \wedge^2, \psi) = \gamma_{\text{LS}}(s, \pi, \wedge^2, \psi). \]

**Proof.** Given a local field \( F' \) of characteristic \( p \) and an integer \( m \geq 1 \), there exists a local field \( F \) of characteristic 0 such that \( F' = m \)-close to \( F \) \([11, \text{p.1123}]\). The converse also holds for \( m = 1 \). Specifically, for a field \( F \) of characteristic 0, its residue field \( \sigma_F/p_F \) is isomorphic to \( F_q \) with \( q = p^k \) for some prime \( p \) and integer \( k \geq 1 \). Then we take \( F' \) to be \( F_q((t)) \) of characteristic \( p \). Combining Theorem 4.3 with Proposition 4.7, we can ultimately transfer the identity over a local function field \( F' \) in Theorem 4.3 to a \( p \)-adic field \( F' \). \( \square \)
4.3. Bump-Friedberg integrals. We introduce another construction of $\gamma$-factor, using the formulation of Bump and Friedberg [2]. We define the embedding $J: GL_m \times GL_m \rightarrow GL_n$ by

$$J(g, g')_{k,l} = \begin{cases} g_{i,j} & \text{if } k = 2i - 1, l = 2j - 1, \\ g'_{i,j} & \text{if } k = 2i, l = 2j, \\ 0 & \text{otherwise,} \end{cases}$$

for $n = 2m$ even and $J: GL_{m+1} \times GL_m \rightarrow GL_n$ by

$$J(g, g')_{k,l} = \begin{cases} g_{i,j} & \text{if } k = 2i - 1, l = 2j - 1, \\ g'_{i,j} & \text{if } k = 2i, l = 2j, \\ 0 & \text{otherwise,} \end{cases}$$

for $n = 2m + 1$ odd. For the purpose of keeping consistent terminology with [33], we interchange the role of $g$ and $g'$ in [2]. We set $m' := m$ if $n = 2m$ and $m' := m + 1$ if $n = 2m + 1$. For each Whittaker function $W \in W(\pi, \psi)$ and Schwartz-Bruhat function $\Phi \in S(F^n)$, we define Bump-Friedberg integrals $Z(s_1, s_2, W, \Phi)$ by

$$\int_{U_m(F) \backslash GL_m(F)} \int_{U_m(F) \backslash GL_m(F)} W(J(g, g')) \Phi(e_m g') |\det(g)|^{s_1 - 1/2} |\det(g')|^{1/2 + s_2 - s_1} dg dg'$$

in the even case $n = 2m$ and

$$\int_{U_m(F) \backslash GL_m(F)} \int_{U_{m+1}(F) \backslash GL_{m+1}(F)} W(J(g, g')) \Phi(e_m g') |\det(g)|^{s_1} |\det(g')|^{s_2 - s_1} dg dg'$$

in the odd case $n = 2m + 1$. In accord with the standard language, we retain notations from Matringe [33]. For $n = 2m$ even, we denote by $M_{2m}(F)$ the standard Levi of $GL_{2m}(F)$ associated with the partition $(m, m)$ of $2m$. Let $w_{2m} = \sigma_{2m}$ and then we set $H_{2m}(F) = w_{2m}M_{2m}(F)w_{2m}^{-1}$. Let $w_{2m+1} = w_{2m+2}GL_{2m+1}(F)$ so that

$$w_{2m+1} = \begin{pmatrix} 1 & 2 & \cdots & m + 1 & m + 2 & m + 3 & \cdots & 2m & 2m + 1 \\ 1 & 3 & \cdots & 2m + 1 & 2 & 4 & \cdots & 2m - 2 & 2m \end{pmatrix}.$$

In the odd case, $w_{2m+1} \neq \sigma_{2m+1}$ and we let $M_{2m+1}(F)$ denote the standard Levi associated to the partition $(m + 1, m)$ of $2m + 1$. We set $H_{2m+1}(F) = w_{2m+1}M_{2m+1}(F)w_{2m+1}^{-1}$. We note that $H_n(F)$ is compatible with the intersection in the sense that $H_n(F) \cap GL_{n-1}(F) = H_{n-1}(F)$ and we can easily see that $J(g, g') = w_n \text{diag}(g, g')w_n^{-1}$ for $\text{diag}(g, g') \in M_n(F)$. If $r$ a real number, we denote by $\delta_r$ the character,

$$\delta_r : J(g, g') \mapsto \frac{\det(g)}{|\det(g')|^r}.$$

We denote by $\chi_n$ characters of $H_n$:

$$\chi_n \left( \begin{pmatrix} w_n(g & g') \end{pmatrix} w_n^{-1} \right) = \begin{cases} 1_{H_n(F)} & \text{for } n = 2m; \\ \frac{\det(g)}{\det(g')} & \text{for } n = 2m + 1. \end{cases}$$

We turn toward the case for $s_1 = s + t + 1/2$ and $s_2 = 2s$. We combine the previous Bump-Friedberg zeta integrals with one single integral in the fashion:

$$Z(s, t, W, \Phi) = \int_{J(U_n(F) \cap H_n(F)) \backslash H_n(F)} W(h) \Phi(e_n h) \chi_n^{1/2}(h) \delta_t(h)|\det(h)|^s dh.$$

This integral converges absolutely for $\text{Re}(s)$ and $\text{Re}(t)$ sufficiently large, and it admits a meromorphic extension to $\mathbb{C} \times \mathbb{C}$ as an element of $\mathbb{C}(q^{-s}, q^{-t})$ [33, Proposition 3.1]. According to
[33, Corollary 3.2], there exists a rational function \( \gamma(s, t, \pi, BF, \psi) \) in \( \mathbb{C}(q^{-s}, q^{-t}) \) such that for every \( W \) in \( W(\pi, \psi) \), and every \( \Phi \) in \( \mathcal{S}(F^{m'}) \), we have the following functional equation

\[
Z(1/2 - s, -1/2 - t, \hat{W}, \hat{\Phi}) = \gamma(s, t, \pi, BF, \psi)Z(s, t, W, \Phi).
\]

Unlike the Jacquet and Shalika local factor \( \gamma(s, \pi, \Lambda^2, \psi) \), the Bump and Friedberg local factor \( \gamma(s, t, \pi, BF, \psi) \) possesses two parameters, \( s \) and \( t \). For this reason, \( \gamma(s, t, \pi, BF, \psi) \) is not really defined as a proportionality, but rather the functional equation for the \( \varepsilon \)-factor, \( \varepsilon(s, t, \pi, BF, \psi) \), need to be established beforehand.

**Lemma 4.9.** Let \( F \) be a local function field. Let \( \pi \) be an irreducible subquotient of the unramified principal series representation \( \text{Ind}_{E_n(F)}^{\text{GL}_n(F)}(\mu_1 \otimes \cdots \otimes \mu_n) \), where each \( \mu_i \) is an unramified character of \( F^\times \), that is to say, it is invariant under the maximal compact subgroup \( \mathfrak{o}^\times \) of \( F^\times \). Then we have

\[
\gamma(s, t, \pi, BF, \psi) = \gamma(s + t + 1/2, \pi, \psi)\gamma(2s, \pi, \Lambda^2, \psi)
= \prod_{1 \leq i \leq n} \gamma(s + t + 1/2, \mu_i, \psi) \prod_{1 \leq j < k \leq n} \gamma(2s, \mu_j \times \mu_k, \psi).
\]

**Proof.** The proof of Lemma 4.1 literally goes through word by word except that we use the unramified computation of Bump and Friedberg [2, Theorem 3] in place of [17, §2] (cf. Readers may refer to [16, Proposition 1.2] for an alternative proof of unramified computations).

We are now in a position to formulate the main factorization formula conjectured by Bump and Friedberg [2, Conjecture 4].

**Theorem 4.10.** Let \( \pi \) be a level zero supercuspidal representation of \( \text{GL}_n(F) \) over a local function field \( F \). Then we have

\[
\gamma(s, t, \pi, BF, \psi) = \gamma(s + t + 1/2, \pi, \psi)\gamma_{LS}(2s, \pi, \Lambda^2, \psi) = \gamma(s + t + 1/2, \pi, \psi)\gamma(2s, \pi, \Lambda^2, \psi).
\]

**Proof.** With the help of [28, §6.6-(vii)], twists by unramified characters do not affect on the first equality. For this reason, we may assume that \( \pi \) is unitary without loss of generality. We find a finite set \( S = \{v_0, v_1, v_\infty\} \) as in Theorem 4.2. We choose a non-trivial additive character \( \Psi \) of \( \mathbb{A}_k/k \), and assume, as we may, that \( \Psi_{v_0} = \psi \). Applying Theorem 4.2 to the irreducible level zero, we obtain an irreducible unitary cuspidal automorphic representation \( \Pi \). The global functional equation for exterior square \( L \)-functions via the Langlands-Shahidi method can be read from [12, §4-(vi)] as (4.1), while that for standard \( L \)-functions due to Godement and Jacquet is extracted from [15] as

\[
L^S(s, \Pi) = \gamma(s, \Pi_{v_0}, \Psi_{v_0}) \prod_{v \in S - \{v_0\}} \gamma(s, \Pi_v, \Psi_v)L^S(1 - s, \Pi).
\]

In the meantime, taking into account local functional equations (4.8), the global functional equation for Bump-Friedberg \( L \)-functions via the Rankin-Selberg method in [2, Theorem 6] (cf. [16, (1.1)]) takes the following explicit form:

\[
L^S(s + t + 1/2, \Pi, \psi)L^S(2s, \Pi, \Lambda^2)
= \gamma(s, \Pi_{v_0}, BF, \Psi_{v_0}) \prod_{v \in S - \{v_0\}} \gamma(s, t, \Pi_v, BF, \Psi_v)L^S(1/2 - s - t, \Pi)L^S(1 - 2s, \Pi, \Lambda^2).
\]

Invoking Lemma 4.9, each places \( v \) in \( S - \{v_0\} \) can be controlled in such a way that

\[
\gamma(s, t, \Pi_v, BF, \Psi_v) = \gamma(s + t + 1/2, \Pi_v, \Psi_v)\gamma_{LS}(2s, \Pi_v, \Lambda^2, \Psi_v).
\]

After suitably adjusting the parameter \( s \), the case for positive characteristics is at least done by dividing (4.10) by the product of (4.1) and (4.9).
Remark 4.11. While we do not prove this, our approach using Deligne-Kazhdan close field theory can very probably go through to show that the Bump and Friedberg exterior square $\gamma$-factor $\gamma(s,t,\pi, BF, \psi)$ is factored into the Godement and Jacquet standard $\gamma$-factor $\gamma(s + t + 1/2, \pi, \psi)$, and the Jacquet and Shalika exterior square $\gamma$-factor $\gamma(2s, \pi, \wedge^2, \psi)$ over non-archimedean local fields $F$ of characteristic zero just as in the proof of Theorem 4.8. We are unable to deduce the decomposition, that is to say, Theorem 4.10 in characteristic zero, because at least to the best of our knowledge, the computation of Bump and Friedberg integrals for level zero supercuspidal representations analogue to Jacquet and Shalika cases (4.5) and (4.6) [46, Theorems 3.16 and 3.17] is not known at this moment. Jointly with E. Zelingher [46,47], this project in connection with the product formula in terms of Gauss sums is currently underway.

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