MAXIMAL REPRESENTATION DIMENSION OF FINITE $p$-GROUPS

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Abstract. The representation dimension $\text{rdim}(G)$ of a finite group $G$ is the smallest positive integer $m$ for which there exists an embedding of $G$ in $\text{GL}_m(\mathbb{C})$. In this paper we find the largest value of $\text{rdim}(G)$, as $G$ ranges over all groups of order $p^n$, for a fixed prime $p$ and a fixed exponent $n \geq 1$.

1. Introduction

The representation dimension of a finite group $G$, denoted by $\text{rdim}(G)$, is the minimal dimension of a faithful complex linear representation of $G$. In this paper we determine the maximal representation dimension of a group of order $p^n$. We are motivated by a recent result of N. Karpenko and A. Merkurjev [KM07, Theorem 4.1], which states that if $G$ is a finite $p$-group then the essential dimension of $G$ is equal to $\text{rdim}(G)$. For a detailed discussion of the notion of essential dimension for finite groups (which will not be used in this paper), see [BR97] or [JLY02, §8]. We also note that a related invariant, the minimal dimension of a faithful complex projective representation of $G$, has been extensively studied for finite simple groups $G$; for an overview, see [TZ00, §3].

Let $G$ be a $p$-group of order $p^n$ and $r$ be the rank of the centre $Z(G)$. A representation of $G$ is faithful if and only if its restriction to $Z(G)$ is faithful. Using this fact it is easy to see that a faithful representation $\rho$ of $G$ of minimal dimension decomposes as a direct sum

$$\rho = \rho_1 \oplus \cdots \oplus \rho_r$$

of exactly $r$ irreducibles; cf. [MR09, Theorem 1.2]. Since the dimension of any irreducible representation of $G$ is $\leq \sqrt{|G:Z(G)|}$ (see, e.g., [W03, Corollary 3.11]) and $|Z(G)| \geq p^r$, we conclude that

$$\text{rdim}(G) \leq rp^{(n-r)/2}.$$
Let 
\[ f_p(n) := \max_{r \in \mathbb{N}} (r p^{\lfloor (n-r)/2 \rfloor}). \]

It is easy to check that \( f_p(n) \) is given by the following table:

| \( n \) | \( p \) | \( f_p(n) \) |
|-------|-------|-------------|
| even  | arbitrary | \( 2p^{(n-2)/2} \) |
| odd   | odd   | \( p^{(n-1)/2} \) |
| odd, \( \geq 3 \) | 2     | \( 3p^{(n-3)/2} \) |
| 1     | 2     | 1           |

We are now ready to state the main result of this paper.

**Theorem 1.** Let \( p \) be a prime and \( n \) be a positive integer. For almost all pairs \((p, n)\), the maximal value of \( \text{rdim}(G) \), as \( G \) ranges over all groups of order \( p^n \), equals \( f_p(n) \). The exceptional cases are
\[(p, n) = (2, 5), (2, 7) \text{ and } (p, 4), \text{ where } p \text{ is odd.}\]

In these cases the maximal representation dimension is 5, 10, and \( p + 1 \), respectively.

The proof will show that the maximal value of \( \text{rdim}(G) \), as \( G \) ranges over all groups of order \( p^n \), is always attained for a group \( G \) of nilpotency class \( \leq 2 \). Moreover, if \((p, n)\) is non-exceptional, \( n \geq 3 \) and \((p, n) \neq (2, 3), (2, 4)\), the maximum is attained on a special class of \( p \)-groups of nilpotency class 2. We call these groups **generalized Heisenberg groups** since their representation theory looks very similar to the usual Heisenberg group (the group of unipotent upper triangular \( 3 \times 3 \) matrices); see Section 2.4.

The rest of this paper is structured as follows. In §2 we introduce generalized Heisenberg groups and study their irreducible representations. In §3, we prove Theorem 1.

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## 2. Generalized Heisenberg Groups

### 2.1. Spaces of alternating forms

Let \( V \) be a finite dimensional vector space over an arbitrary field \( F \). Let \( \mathcal{A}(V) \) denote the space of bilinear alternating forms on \( V \); that is, linear maps \( b : V \otimes V \to F \) satisfying \( b(v, v) = 0 \).

Let \( K \) be a subspace of \( \mathcal{A}(V) \). Then \( K \) defines a map \( \omega_K : V \times V \to K^* \) as follows. Let \( j : \mathcal{A}(V)^* \to K^* \) denote the dual of the natural injection \( K \hookrightarrow \mathcal{A}(V) \). Then \( \omega_K \) is defined to be the composition
\[
(3) \quad V \times V \xrightarrow{\Lambda^2(V)} \mathcal{A}(V)^* \xrightarrow{j} K^*.
\]

\( \omega_K \)
where the first map is the natural projection and the second one is the canonical identification of the two spaces.

2.2. Symplectic subspaces.

**Definition 2.** A subspace \( K \subseteq \mathcal{A}(V) \) is **symplectic** if every nonzero element of \( K \) is non-degenerate, as a bilinear form on \( V \).

**Remark 3.** Equivalently, \( K \subset \mathcal{A}(V) \) is symplectic if and only if for every nonzero linear map \( K^* \rightarrow F \) the composition \( V \times V \xrightarrow{\omega_K} K^* \rightarrow F \) is non-degenerate.

Clearly nontrivial symplectic subspaces of \( \mathcal{A}(V) \) can exist only if \( \dim(V) \) is even.

**Lemma 4.** Suppose \( V \) is an \( F \)-vector space of dimension \( 2m \). If \( F \) admits a field extension of degree \( m \) then there exists an \( m \)-dimensional symplectic subspace \( K \subset \mathcal{A} \).

**Proof.** Choosing a basis of \( V \), we can identify \( \mathcal{A}(V) \) with the space of alternating \( 2m \times 2m \)-matrices. Let \( f: \text{M}_m(F) \rightarrow \mathcal{A}(V) \) be the linear map
\[
A \mapsto \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}.
\]
If \( W \) is a linear subspace of \( \text{M}_m(F) = \text{End}_F(F^m) \) such that \( W \setminus \{0\} \subset \text{GL}_m(F) \) then \( K = f(W) \) is a symplectic subspace.

It thus remains to construct an \( m \)-dimensional linear subspace \( W \) of \( \text{M}_m(F) \) such that \( W \setminus \{0\} \subset \text{GL}_m(F) \). Let \( E \) be a degree \( m \) field extension of \( F \). Then \( E \) acts on itself by left multiplication. This gives an \( F \)-vector space embedding of \( \Psi: E \hookrightarrow \text{End}_F(E) \) such that \( \Psi(e) \) is invertible for all \( e \neq 0 \). \( \square \)

2.3. Groups associated to spaces of alternating forms. Let \( V \) be a finite-dimensional vector space over a field \( F \). Let \( K \) be a subspace of \( \mathcal{A}(V) \) and let \( \omega_K \) denote the induced map \( V \times V \rightarrow K^* \), see (3). Choose a bilinear map \( \beta: V \times V \rightarrow K^* \) such that
\[
\omega_K(v, w) = \beta(v, w) - \beta(w, v).
\]
To see that this can always be done, note that if \( \{e_i\} \) is a basis of \( V \), we can define \( \beta \) by
\[
\beta(e_i, e_j) = \begin{cases} 
\omega_K(e_i, e_j), & \text{if } i > j \\
0, & \text{otherwise.}
\end{cases}
\]
We also remark that \( \beta \) is uniquely determined by \( \omega_K \), up to adding a symmetric bilinear form \( V \times V \rightarrow K^* \).

**Definition 5.** Let \( H = H(V, K, \beta) \) denote the group whose underlying set is \( V \times K^* \) and whose multiplication is given by
\[
(v, t) \cdot (v', t') = (v + v', t + t' + \beta(v, v')).
\]
If $K$ is a symplectic subspace, we will refer to $H$ as a generalized Heisenberg group.

**Example 6.** Suppose $\omega$ is a nondegenerate alternating bilinear form on $V = F \oplus F$, where $F$ is a field of characteristic not equal to 2. Let $K$ be the span of $\omega$ in $A(V)$. Then $H(V,K,\frac{1}{2}\omega)$ is isomorphic to the group of unipotent upper triangular $3 \times 3$ matrices over $F$. This group is known as the Heisenberg group.

**Remark 7.** It is easy to see that (5) is indeed a group law with the inverse given by $(v,t)^{-1} = (-v,-t+\beta(v,v))$ and the commutator given by

$$(6) \quad [(v_1,t_1),(v_2,t_2)] = (0,\omega_K(v_1,v_2)).$$

As $\omega_K$ is surjective, we see that $[H,H] = K^*$. Moreover, (6) also shows that $K^* \subset Z(H)$, and that equality holds unless the intersection $\cap_{k \in K} \ker(k)$ is nontrivial. In particular, $Z(H) = K^*$ if $K$ contains a symplectic form.

**Remark 8.** A non-abelian finite $p$-group $S$ is called special if $Z(S) = [S,S]$ and $S/[S,S]$ is elementary abelian; see [1156, §2.3]. Suppose $K$ is a subspace of $A(V)$ such that $\cap_{k \in K} \ker(k)$ is trivial. Then over the finite field $F_p$, the groups $H(V,K,\beta)$ are examples of non-abelian special $p$-groups. We are grateful to the referee for pointing this out.

**Remark 9.** If $\beta$ and $\beta'$ both satisfy (4) then $H(V,K,\beta)$ may not be isomorphic to $H(V,K,\beta')$. For example, let $V$ be a 2-dimensional vector space over $F = \mathbb{F}_2$, $K$ be the one-dimensional (symplectic) subspace generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\beta$, $\beta'$ be bilinear forms on $V$ defined by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, respectively. Then $\beta$ and $\beta'$ both satisfy (4), but $H(V,K,\beta)$ is isomorphic to the quaternion group while $H(V,K,\beta')$ is isomorphic to the dihedral group of order 8.

On the other hand, it is easy to see that $H(V,K,\beta)$ and $H(V,K,\beta')$ are always isoclinic. (Two groups $S$ and $T$ are isoclinic if there are isomorphisms $f : S/Z(S) \to T/Z(T)$ and $g : [S,S] \to [T,T]$ such that if $a, b \in S$ and $a', b' \in T$ with $f(aZ(S)) = a'Z(T)$ and $f(bZ(S)) = b'Z(T)$, then we have $g([a,b]) = [a',b']$, see [1140].)

2.4. **Representations.** Let $p$ be an arbitrary prime and let $F = \mathbb{F}_p$ be the finite field of $p$ elements. Fix, once and for all, a homomorphism $\tau : (\mathbb{F}_p,+) \to \mathbb{C}^\times$. Let $W$ be a vector space over $F$. Using $\tau$, we identify the algebraic dual $W^* = \text{Hom}(W,F)$ with the Pontryagin dual $\text{Hom}(W,\mathbb{C}^\times)$. It is clear that a bilinear alternating map $W \times W \to \mathbb{F}_p$ is non-degenerate if and only if the composition $W \times W \to \mathbb{F}_p \xrightarrow{\tau} \mathbb{C}^\times$ is non-degenerate.

Now let $V$ be a vector space over $F$, $K$ a subspace of $A(V)$, and $\omega = \omega_K$ the associated map. Choose $\beta$ satisfying (4) and let $G = H(V,K,\beta) = V \times K^*$. Recall that $K^*$ is in the center of $G$ (Remark 7); in particular, it acts via a character on every irreducible representation of $G$. 
Lemma 10. Let \( \rho \) be an irreducible representation of \( G \) such that \( K^* \) acts by \( \psi \). Assume \( \psi \circ \omega : V \times V \to \mathbb{C}^\times \) is non-degenerate.

(a) If \( g \in G, \ g \notin K^* \), then \( \text{Tr}(\rho(g)) = 0 \).
(b) \( \dim(\rho) = \sqrt{|V|} \).
(c) \( \rho \) is uniquely determined (up to isomorphism) by \( \psi \).

Proof. (a) Let \( g \in G \setminus K^* \). Since \( \psi \circ \omega \) is non-degenerate there exists \( h \in G \) such that \( \psi \circ \omega (gK^*, hK^*) \neq 1 \). Observe that \( \rho([g, h]) = \psi([g, h]) \text{Id} \), and that \( \rho(h^{-1}gh) = \rho(g) \rho([g, h]) \). Taking the trace of both sides, we have \( \text{Tr}(\rho(g)) = \psi([g, h]) \text{Tr}(\rho(g)) \). Since \( \psi([g, h]) \neq 1 \) we must have \( \text{Tr}(\rho(g)) = 0 \).

(b) Since \( \rho \) is irreducible, and the trace of \( \rho \) vanishes outside of \( K^* \), we have:

\[
1 = \frac{1}{|G|} \sum_{g \in G} \text{Tr}(\rho(g)) \overline{\text{Tr}(\rho(g))} = \frac{1}{|G|} \sum_{g \in K^*} \text{Tr}(\rho(g)) \overline{\text{Tr}(\rho(g))} = \frac{1}{|G|} \dim(\rho)^2 \sum_{g \in K^*} \text{Tr}(\psi(g)) \overline{\text{Tr}(\psi(g))} = \dim(\rho)^2 \frac{|K^*|}{|G|}.
\]

Thus \( \dim \rho = \sqrt{\frac{|G|}{|K^*|}} = \sqrt{|V|} \).
(c) We have completely described the character of \( \rho \), and it follows that \( \rho \) is uniquely determined by \( \psi \). Indeed,

\[
\text{Tr}(\rho(g)) = \begin{cases} \sqrt{|V|} \cdot \psi(g), & \text{if } g \in K^* \text{ and } \\ 0, & \text{otherwise.} \end{cases}
\]

In view of Remark \ref{rem:...}, the following proposition is a direct consequence of the above lemma.

Proposition 11. The irreducible representations of a generalized Heisenberg group \( H = H(V, K, \beta) \) are exhausted by the following list:

(i) \( |V| \) one-dimensional representations, one for every character of \( V \).
(ii) \(|K| - 1 \) representations of dimension \( \sqrt{|V|} \), one for every nontrivial character \( \psi : K^* \to \mathbb{C}^\times \).

The next corollary is also immediate upon observing the centre of a generalized Heisenberg groups \( H = H(G, K, \beta) \) equals \( K^* \); see Remark \ref{rem:...}

Corollary 12. The representation dimension of a generalized Heisenberg group \( H = H(V, K, \beta) \) equals \( \dim(K) \sqrt{|V|} \).
If $G$ is a finite Heisenberg group in the usual sense (as in Example 6) then for each nontrivial character $\chi$ of $Z(G)$ there is a unique irreducible representation $\psi$ of $G$ whose central character is $\chi$; cf. [GH07, §1.1]. This is a finite group variant of the celebrated Stone-von Neumann Theorem. For a detailed discussion of the history and the various forms of the Stone-von Neumann theorem we refer the reader to [R04]. We conclude this section with another immediate corollary of Proposition 11 which tells us that over the field $\mathbb{F}_p$ every generalized Heisenberg group has the Stone-von Neumann property. This corollary will not be needed in the sequel.

**Corollary 13.** Two irreducible representations of a generalized Heisenberg group with the same nontrivial central character are isomorphic.

Corollary 13 is the reason we chose to use the term “generalized Heisenberg group” in reference to the groups $H(V,K,\beta)$, where $K$ is a symplectic subspace. Special $p$-groups (Remark 8) which are not generalized Heisenberg groups may not have the Stone-von Neumann property; see Remark 18.

### 3. Proof of Theorem 1

The case where $n \leq 2$ is trivial; clearly $\text{rdim}(G) = \text{rank}(G)$ if $G$ is abelian. We will thus assume that $n \geq 3$.

In the non-exceptional cases of the theorem, in view of the inequality (2), it suffices to construct a group $G$ of order $p^n$ with $\text{rdim}(G) = f_p(n)$. Here $f_p(n)$ is the function defined just before the statement of Theorem 1.

If $(p,n) = (2,3)$ or $(2,4)$, we take $G$ to be the elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^3$ and $(\mathbb{Z}/2\mathbb{Z})^4$, yielding the desired representation dimension of 3 and 4, respectively. For all other non-exceptional pairs $(p,n)$, we take $G$ to be a generalized Heisenberg group as described in the table below. Here $H(V,K)$ stands for $H(V,K,\beta)$, for some $\beta$ as in (4). In each instance, the existence of a symplectic subspace $K$ of suitable dimension is guaranteed by Lemma 4 and the value of $\text{rdim}(H(V,K))$ is given by Corollary 12.

| $n$  | $p$    | $\dim(V)$ | $\dim(K)$ | $\text{rdim}(H(V,K))$ |
|------|--------|-----------|-----------|------------------------|
| even, $\geq 6$ | arbitrary | $n - 2$ | 2 | $2p^{(n-2)/2}$ |
| odd, $\geq 3$ | odd | $n - 1$ | 1 | $p^{(n-1)/2}$ |
| odd, $\geq 9$ | 2 | $n - 3$ | 3 | $3p^{(n-3)/2}$ |

This settles the generic case of Theorem 1. We now turn our attention to the exceptional cases. We will need the following upper bound on $\text{rdim}(G)$, strengthening (2).

Let $\Omega_1(Z(G))$ be the subgroup of elements $g \in Z(G)$ such that $g^p = 1$.

**Lemma 14.** Let $G$ be a $p$-group and $r = \text{rank}(Z(G)) = \text{rank}(\Omega_1(Z(G)))$.

(a) Let $\rho_1$ be an irreducible representation of $G$ such that $\text{Ker}(\rho_1)$ does not contain $\Omega_1(Z(G))$. Then there are irreducible representations $\rho_2, \ldots, \rho_r$
of $G$ such that $\rho_1 \oplus \cdots \oplus \rho_r$ is faithful. In particular,
\[ \mathrm{rdim}(G) \leq \dim(\rho_1) + (r-1)\sqrt{|G:Z(G)|}. \]
(b) If $\Omega_1(Z(G))$ is not contained in $[G,G]$, then
\[ \mathrm{rdim}(G) \leq 1 + (r-1)\sqrt{|G:Z(G)|}. \]

The lemma can be deduced from [KM07, Remark 4.7] or [MR09, Theorem 1.2]; for the sake of completeness we give a self-contained proof.

Proof. (a) Let $\chi_1$ be the restriction to $\Omega_1(Z(G))$ of the central character of $\rho_1$. By our assumption $\chi_1$ is nontrivial. Complete $\chi_1$ to a basis $\chi_1, \chi_2, \ldots, \chi_r$ of the $r$-dimensional $\mathbb{F}_p$-vector space $\Omega_1(Z(G))^*$ and choose an irreducible representation $\rho_i$ such that $\Omega_1(Z(G))$ acts by $\chi_i$. (The representation $\rho_i$ can be taken to be any irreducible component of the induced representation $\text{Ind}_{G_i}^{G(Z)(G)}(\chi_i)$.) The restriction of $\rho : = \rho_1 \oplus \cdots \oplus \rho_r$ to $\Omega_1(Z(G))$ is faithful. Hence, $\rho$ is a faithful representation of $G$. As we mentioned in the introduction $\dim(\rho_i) \leq \sqrt{|G:Z(G)|}$ for every $i \geq 2$, and part (a) follows.

(b) By our assumption there exists a one-dimensional representation $\rho_1$ of $G$ whose restriction to $\Omega_1(Z(G))$ is nontrivial. Now apply part (a). \hfill \Box

We are now ready to prove Theorem 1 in the three exceptional cases.

3.1. Exceptional case 1: $p$ is odd and $n = 4$.

Lemma 15. Let $p$ be an odd prime and $G$ be a group of order $p^4$.

(a) Then $\mathrm{rdim}(G) \leq p + 1$.

(b) Suppose $Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$ and $G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Then $\mathrm{rdim}(G) = p + 1$.

Proof. (a) We argue by contradiction. Assume there exists a group of order $p^4$ such that $\mathrm{rdim}(G) \geq p + 2$. If $|Z(G)| \geq p^3$ or $G/Z(G)$ is cyclic then $G$ is abelian and $\mathrm{rdim}(G) = \text{rank}(G) \leq 4 \leq p + 1$, a contradiction. If $Z(G)$ is cyclic then $\mathrm{rdim}(G) \leq p$ by (2), again a contradiction.

Thus $Z(G) \cong G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$. This reduces part (a) to part (b).

(b) Here $\Omega_1(Z(G)) = Z(G)$ has rank 2. Hence, a faithful representation $\rho$ of $G$ of minimal dimension is the sum of two irreducibles $\rho_1 \oplus \rho_2$, as in (1), each of dimension 1 or $p$.

Clearly $\dim(\rho_1) = \dim(\rho_2) = 1$ is not possible, since in this case $G$ would be abelian, contradicting $[G : Z(G)] = p^2$. It thus remains to show that $\mathrm{rdim}(G) \leq p + 1$. Since $G/Z(G)$ is abelian, $[G,G] \subset Z(G)$. Hence, by Lemma 14(b) we only need to establish that $[G,G] \not\subseteq Z(G)$.

To show that $[G,G] \not\subseteq Z(G)$, note that the commutator map
\[ \Psi : G/Z(G) \times G/Z(G) \to [G,G] \]
\[ (gZ(G), g'Z(G)) \to [g,g'] \]
can be thought of as an alternating bilinear map from $\mathbb{F}_p^2$ to itself. Viewed in this way, $\Psi$ can be written as $\Psi(v, v') = (w_1(v, v'), w_2(v, v'))$ for alternating
maps $w_1$ and $w_2$ from $(\mathbb{F}_p)^2$ to $\mathbb{F}_p$. Since the space of alternating maps is a one-dimensional vector space over $\mathbb{F}_p$, $w_1$ and $w_2$ are scalar multiples of each other. Hence, the image of $\Psi$ is a cyclic group of order $p$, and $[G,G] \subseteq Z(G)$, as claimed.

To finish the proof of Theorem 1 in this case, note that $G_0$ is a non-abelian group of order $p^3$, satisfies the conditions of Lemma 15(b). Thus the maximal representation dimension of a group of order $p^4$ is $p + 1$, for any odd prime $p$.

3.2. Exceptional case 2: $p = 2$ and $n = 5$.

**Lemma 16.** Let $G$ be a group of order 32. Then $\text{rdim}(G) \leq 5$.

**Proof.** We argue by contradiction. Assume there exists a group of order 32 and representation dimension $\geq 6$. Let $r = \text{rank}(Z(G))$. Then $1 \leq r \leq 5$ and (2) shows that $\text{rdim}(G) \leq 5$ for every $r \neq 3$.

Thus we may assume $r = 3$. If $|Z(G)| \geq 16$ or $G/Z(G)$ is cyclic then $G$ is abelian, and $\text{rdim}(G) = \text{rank}(G) \leq 5$. We conclude that $Z(G) \simeq (\mathbb{Z}/2\mathbb{Z})^3$ and $G/Z(G) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Applying the same argument as in the proof of Lemma 15(b), we see that $[G,G] \subseteq Z(G)$, and hence $\text{rdim}(G) \leq 5$ by Lemma 14(b), a contradiction.

To finish the proof of Theorem 1 in this case, note that the elementary abelian group of order $2^5$ has representation dimension 5. Thus the maximal representation dimension of a group of order $2^5$ is 5.

3.3. Exceptional case 3: $p = 2$ and $n = 7$.

**Lemma 17.** If $|G| = 128$ then $\text{rdim}(G) \leq 10$.

**Proof.** Again, we argue by contradiction. Assume there exists a group $G$ of order 128 and representation dimension $\geq 11$. Let $r$ be the rank of $Z(G)$. By (2), $r = 3$; otherwise we would have $\text{rdim}(G) \leq 10$.

As we explained in the introduction, this implies that a faithful representation $\rho$ of $G$ of minimal dimension is the direct sum of three irreducibles $\rho_1$, $\rho_2$ and $\rho_3$, each of dimension $\leq \sqrt{2^7/|Z(G)|}$. If $|Z(G)| > 8$, then $\dim(\rho_1) \leq 2$ and $\text{rdim}(G) = \dim(\rho_1) + \dim(\rho_2) + \dim(\rho_3) \leq 6$, a contradiction.

Therefore, $Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^3$ and $\dim(\rho_1) = \dim(\rho_2) = \dim(\rho_3) = 4$. By Lemma 14(a) this implies that the kernel of every irreducible representation of $G$ of dimension 1 or 2 must contain $Z(G)$. In other words, any such representation factors through the group $G/Z(G)$ of order 16. Consequently, if $m_i$ is the number of irreducible representations of $G$ of dimension $i$ then $m_1 + 4m_2 = 16$. We can now appeal to [JNO90, Tables I and II], to show that no group of order $2^7$ has these properties. From Table I we can determine which groups $G$ (up to isoclinism, cf. Remark 4) have $|Z(G)| = 8$ and using Table II we can determine $m_1$ and $m_2$ for these groups. There is no group $G$ with $|Z(G)| = 8$ and $m_1 + 4m_2 = 16$. □
We will now construct an example of a group $G$ of order $2^7$ with $\operatorname{rdim}(G) = 10$. Let $V = (\mathbb{F}_2)^4$ and let $K$ be the 3-dimensional subspace of $A(V)$ generated by the following three elements:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{bmatrix}, \\
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 
\end{bmatrix}, \\
\begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 
\end{bmatrix}.
$$

Let $G := H(V, K, \beta) = V \times K^*$ for some $\beta$ as in (14). Note that $K$ contains only one non-zero degenerate element (the sum of the three generators).

In other words, there is only one non-trivial character $\chi$ of $K^*$ such that $\chi \circ \omega : V \times V \to \mathbb{C}^\times$ is degenerate. By Remark (7)

(7) $[G, G] = Z(G) = K^*$.

Let $\rho$ be a faithful representation of $G$ of minimal dimension. As we explained in the Introduction, $\rho$ is the sum of rank $(Z(G)) = 3$ irreducibles. Denote them by $\rho_1$, $\rho_2$, and $\rho_3$, and their central characters by $\chi_1$, $\chi_2$ and $\chi_3$, respectively. Since $\rho$ is faithful, $\chi_1$, $\chi_2$ and $\chi_3$ form an $\mathbb{F}_2$-basis of $\Omega_1(Z(G))^* \cong (\mathbb{Z}/2\mathbb{Z})^3$. By Lemma (10), for each nontrivial character $\chi$ of $K^*$ except one, there is a unique irreducible representation $\psi$ of $G$ such that $\chi$ is the central character to $\psi$, and $\dim \psi = 4$. Thus at least 2 of the irreducible components of $\rho$, say, $\rho_1$ and $\rho_2$ must have dimension 4. By Lemma (17), $\dim(\rho) \leq 10$, i.e., $\dim(\rho_3) \leq 2$. But every one-dimensional representation of $G$ has trivial central character. We conclude that $\dim(\rho_3) = 2$ and consequently $\operatorname{rdim}(G) = \dim(\rho) = 4 + 4 + 2 = 10$.

Thus the maximal representation dimension of a group of order $2^7$ is 10.

**Remark 18.** The group $G$ constructed above has 16 one-dimensional representations with trivial central character, 4 two-dimensional representations with non-trivial degenerate central character, and 6 four-dimensional representations with pair-wise distinct non-degenerate central characters. In view of (7), $H$ is a non-abelian special 2-group which does not enjoy the Stone-Von Neumann property (Corollary 13).

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