In this article we review how categorical equivalences are realized by renormalization group flow in physical realizations of stacks, derived categories, and derived schemes. We begin by reviewing the physical realization of sigma models on stacks, as (universality classes of) gauged sigma models, and look in particular at properties of sigma models on gerbes (equivalently, sigma models with restrictions on nonperturbative sectors), and ‘decomposition,’ in which two-dimensional sigma models on gerbes decompose into disjoint unions of ordinary theories. We also discuss stack structures on examples of moduli spaces of SCFTs, focusing on elliptic curves, and implications of subtleties there for string dualities in other dimensions. In the second part of this article, we review the physical realization of derived categories in terms of renormalization group flow (time evolution) of combinations of D-branes, antibranes, and tachyons. In the third part of this article, we review how Landau–Ginzburg models provide a physical realization of derived schemes, and also outline an example of a derived structure on a moduli spaces of SCFTs.

1 Introduction

Over the last twenty years, we have gained a much better appreciation of how many abstract mathematical concepts play a role in various aspects of modern physics. In particular, there seems to be a general story that in physical realizations of categorical structures, notion of homotopy are often realized by the renormalization group. We illustrate the relationship in Table 1. We will review and explore this yoga of physical realizations of homotopy in the rest of this paper.

We begin in Section 2 by describing sigma models on Deligne–Mumford stacks [1–3]. We describe how renormalization group flow realizes equivalences between presentations, and then discuss novel physical properties of sigma models on special stacks (gerbes), most importantly, the ‘decomposition’ conjecture relating sigma models on gerbes to sigma models on disjoint unions of spaces. We also discuss four-dimensional sigma models, and the concrete example of moduli spaces of SCFTs for elliptic curves, explicitly identifying the Bagger–Witten line bundle, and some implications for string dualities.

In Section 3 we describe the physical realization of derived categories [4, 5], in terms of systems of branes, antibranes, and tachyons, and identify localization on quasi-isomorphisms with renormalization group flow. As the physical realization of derived categories has been described in many places, we confine ourselves to a brief overview.

In Section 4, we outline some physical realizations of derived schemes implicit in the mathematics literature [6–9]. Specifically, we discuss how some properties of two-dimensional Landau–Ginzburg models are encapsulated by derived schemes, as derived critical loci and derived zero loci, and how renormalization group flow again realizes equivalences. We also discuss derived structures on moduli spaces of SCFTs, and in particular, walk through how massless spectrum computations in Landau–Ginzburg orbifolds realize cotangent complex structures.
fields on compact spaces. This application is not what this paper concerns.

Both stacks and, for the purposes of this paper, gerbes can be thought of as generalized spaces – pseudo-geometric constructions that are locally, though not necessarily globally, spaces. This section outlines work done to understand to what extent strings can propagate on these generalized geometries, the definition and properties of quantum field theories describing strings on a stack or a gerbe. One of the original motivations was to understand whether compactifications on stacks or gerbes describe new superconformal field theories (SCFTs), new string compactifications.

2.1 Brief introduction to (Deligne–Mumford) stacks

There are several ways to define analogues of geometries that do not involve point/set topology. For one example, in noncommutative geometry, one defines a space via the ring of functions on that space. Briefly, we define a stack via all the maps from other spaces to the stack.

This is nicely set up for sigma models, in which the path integral sums over maps into the target. For a stack, in principle the stack defines the maps into itself. Examples of stacks include ordinary spaces, orbifolds, and gerbes.

Let us briefly consider an example to demonstrate a few key points. Consider the global quotient stack \( [X/G] \), where \( G \) is finite. A map \( Y \to [X/G] \) is a pair

\[
\text{principal } G - \text{bundle } E \to Y, \\
G - \text{equivariant map } \text{Tot}(E) \to X.
\]

In physics, a sigma model on \( [X/G] \) coincides with a global orbifold by \( G \): the bundle \( E \to Y \) defines the twisted section on worldsheet \( Y \), and the map \( \text{Tot}(E) \to X \) defines a map from that twisted sector to the covering space.

See for example [10–19] for introductions to stacks.

2.2 Sigma models (in two dimensions)

Now, how can we define the (two-dimensional) quantum field theory of a nonlinear sigma model with target an arbitrary (Deligne–Mumford) stack?

Stacks can be locally presented as spaces, so it may be tempting to imagine ‘glueing’ nonlinear sigma models on various open patches. Unfortunately, it is not known how to perform such a glueing for a full quantum field theory. (Something like this was proposed in [20] for the perturbative part of a quantum field theory, but gluing nonperturbative physics is currently unknown.)

Another tempting option is to utilize the fact that stacks can be presented as groupoids. It is tempting to then just try to implement the groupoid relations in physics. Unfortunately, it is not clear how to proceed in this fashion either. To implement the relations defined by a group, not a groupoid, one gauges the group action, which requires various ghost and gauge-fixing techniques ala Faddeev–Popov and Batalin–Vilkovisky. To im-

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**Table 1 Some illustrations of how categorical equivalences are realized in physics.**

| Math                                      | Physics                   |
|-------------------------------------------|---------------------------|
| **Stacks** (Deligne–Mumford):            |                           |
| presentation of a stack                   | gauged sigma model [1–3]  |
| equivalences of presentations             | RG                        |
| **Derived categories** (of coherent sheaves): |                        |
| complex of sheaves                        | branes/antibranes/tachyons [1, 5] |
| quasi-isomorphism                         | RG                        |
| **Derived schemes** (-1)-shifted symplectic): |                      |
| presentation of a derived scheme          | Landau–Ginzburg model [6–9] |
| equivalences                             | RG                        |
plement groupoid relations would seem to require a significant generalization of gauging, Faddeev–Popov, and Batalin–Vilkovisky techniques, a generalization which does not seem to be known, at least to this author, at this time.

Given the constraints above, the reader might well wonder at this point why one should believe that a quantum field theory for a sigma model on a stack should exist. One answer is that since before this work began, the Gromov–Witten community had been working on (and now possesses) a notion of Gromov–Witten invariants of stacks [21, 22]. Now, a notion of Gromov–Witten theory is neither necessary nor sufficient for the existence of a full quantum field theory, but it is usefully suggestive.

With all that in mind, the following proposal was made in [1–3]. Under mild conditions (see e.g. [23–25]), smooth (Deligne–Mumford) stacks can be presented as a (stacky) global quotient \([X/G]\), for \(X\) a space and \(G\) a group. \(G\) need not be finite, and need not act effectively. To such a presentation, we associate a \(G\)-gauged sigma model on \(X\). (Since \(G\) need not be finite, this class includes both orbifolds as well as more general gauge theories; since \(G\) does not necessarily act effectively, it also probes classes of gauge theories not considered prior to [1–3].)

Different presentations of the same stack can yield very different quantum field theories. As a simple example, consider the following two supersymmetric theories:

i) the orbifold \([\mathbb{C}^2/\mathbb{Z}_2]\), a \(\mathbb{Z}_2\) gauge theory with two free chiral superfields,

ii) a \(U(1)\)-gauged supersymmetric sigma model with target

\[
X = \mathbb{C}^2 \times \mathbb{C}^\times / \mathbb{Z}_2,
\]

where the generator of \(\mathbb{Z}_2\) acts as

\[
(x, y, t) \in \mathbb{C}^2 \times \mathbb{C}^\times \mapsto (-x, -y, -t),
\]

and the \(U(1)\) acts only on the \(\mathbb{C}^\times\) factor.

The first of these examples defines a conformal field theory; the second, because of D-terms in the gauge action, is not conformal. These are therefore two different quantum field theories, but we expect that the second theory flows in the IR to the first.

Thus, we cannot associate gauged sigma models themselves to stacks, but must do something a shade more subtle. To be precise, we associate stacks to universality classes of renormalization group flow of such gauged sigma models. Put another way, physical realizations of different (physically-realizable) presentations of stacks are related by renormalization group flow, realizing the first row of Table 1.

In the remainder of this section, we will describe some interesting examples and applications of these ideas.

### 2.3 Sigma models on gerbes

Returning to stacks, let us consider the important special case of gerbes. Mathematically, a global quotient stack \([X/G]\) will be a gerbe when a nontrivial subgroup of \(G\) acts trivially on \(X\) (i.e. the group action is technically ineffective). A sigma model on a gerbe can be described in several equivalent ways in two-dimensional quantum field theories:

i) a gauged sigma model in which a subgroup of the gauge group \(G\) acts trivially on the target \(X\),

ii) a gauged sigma model with a restriction on nonperturbative sectors,

iii) a gauged sigma model ’coupled to a topological field theory.’

For the moment, we will focus on the first description, as a gauge theory in which a subgroup of the gauge group acts trivially, and later will return to the second description in terms of restricted nonperturbative sectors.

In thinking about that first description, we quickly run into a puzzle: stacks may remember trivial group actions, but why should a quantum field theory? Why in physics is a gauge theory with a trivially-acting subgroup of the gauge group, any different from a gauge theory in which one only gauges the effectively-acting coset?

For example, in a \(U(1)\) gauge theory, if one decides that all fields have charges that are multiples of two rather than one, what physical difference can that make? It sounds solely like a choice of convention.

In fact, there can be a physical difference, arising solely in how the nonperturbative sector is defined [1–3].

In two-dimensional theories, there are essentially three different approaches to see this distinction.

First, on a compact worldsheet, to specify the matter fields uniquely, one must specify the vector bundle to which the matter fields couple. In essence, if the gauge field is associated to a line bundle \(L\), then the unambiguous way to say that a matter field has charge \(Q\) is to say that it is a section of the bundle \(L^\otimes Q\). Comparing theories with fields of charge one versus fields of charge two, the two theories have fields coupling to different bundles, hence have different zero modes, different anomalies, and different physics.
Second, on a noncompact worldsheet, we can distinguish these cases using the periodicity of the theta angle. In two dimensions, the theta angle acts as an electric field, and its periodicity is determined by the matter content of the theory. To be precise, if we build a capacitor, then as the theta angle is increased, the field density increases, and eventually the capacitor will pair-produce matter fields once the field density is high enough. One can pair produce arbitrarily massive fields, even fields with masses above the cutoff scale. We can distinguish the following consequences, among others:

- Massless fields of charge $k > 1$ from massless fields of charge $1$.
- Nonminimal charges have to distinguish the case of massless fields of charge $k > 1$ from massless states of charge $1$, and the theta angle periodicity will detect their presence, even though their mass is above the cutoff scale.

Third, in either case, one can add defects. Here, for example, one can add Wilson lines for fields of charge $±1$ to distinguish the case of massless fields of charge $k > 1$ from massless fields of charge $1$.

Let us consider a concrete example, namely an analogue of the two-dimensional supersymmetric $\mathbb{CP}^{N-1}$ model. This is a supersymmetric gauge theory with one gauged $U(1)$ and $N$ chiral superfields of charge $1$ (The chiral superfields behave like homogeneous coordinates on the projective space.) The gerby analogue of this theory [1–3] is a $U(1)$ supersymmetric gauge theory with $N$ chiral superfields of charge $k$, distinguished from the charge $1$ case as above. These nonminimal charges have the following consequences, among others:

| Ordinary $\mathbb{CP}^{N-1}$ | Gerby $\mathbb{CP}^{N-1}$ |
|-------------------------------|-----------------------------|
| Anomalous global $U(1)$:      | $U(1)\to\mathbb{Z}_{2N}$    |
| $U(1)_A\to\mathbb{Z}_{2N}$    | $U(1)_A\to\mathbb{Z}_{2N}$ |
| A model correlation functions:| $\langle x^{N(d+1)} \rangle = q^d$ | $\langle x^{N(kd+1)} \rangle = q^d$ |
| Quantum cohomology rings:     | $\mathbb{C}[x]/(x^N-q)$     | $\mathbb{C}[x]/(x^{kN}-q)$ |

Concretely, these two models have different physics.

In passing, the A model correlation functions of the gerby theory correlate with a different (equivalent) description of these gerby theories: as theories with restrictions on nonperturbative sectors, one of the alternative descriptions we mentioned at the start.

Now, restricting nonperturbative sectors violates cluster decomposition, one of the fundamental axioms of quantum field theory. Similarly, if one computes chiral rings and spectra in such theories, one finds multiple dimension-zero operators, again signaling a violation of cluster decomposition.

### 2.4 Decomposition

The resolution of this cluster decomposition issue lies in ‘decomposition’ [26]. Briefly,

$$\text{strings on gerbes} = \text{strings on disjoint unions of spaces}.$$ Strings on disjoint unions of spaces also violate cluster decomposition, but in a manner that is straightforward to understand and control.

This decomposition can take different forms under different circumstances, but some form of it is commonplace. We give two families of examples below:

1. For nonlinear sigma models on spaces and orbifolds [26]. Consider the global quotient $[X/G]$ where $G$ is an extension

$$1 \to K \to G \to H \to 1,$$ and $K$ acts trivially on $X$. For simplicity, also assume that the gerbe is ‘banded.’ Then, in these circumstances, for $Y = [X/H]$ (the effectively-acting quotient, decomposition predicts

$$\text{QFT}([X/G]) = \text{QFT} \left( \prod_{\hat{G}} (Y, B) \right),$$

where $\hat{G}$ is the set of irreducible representations of $G$, and the $B$ field $B$ is determined by the image of the characteristic class of the gerbe under the map

$$H^2(Y, Z(G)) \stackrel{Z(G) \to U(1)}{\longrightarrow} H^2(Y, U(1)).$$

2. For nonabelian two-dimensional gauge theories [27]. Briefly, if the matter is invariant under a subgroup of the center of the gauge group, then the QFT will decompose with factors differing by discrete theta angles. For example, schematically,

$$\text{pure }SU(2) = \text{pure }SO(3)_+ + \text{pure }SO(3)_.$$

It should be noted that decomposition will be altered if the original theory has a nontrivial $B$ field background or nonzero theta angle. It should also be noted that decomposition is only claimed for two-dimensional theories, not theories in higher dimensions.

Decomposition has a straightforward understanding in terms of path integrals. Consider for example a nonlinear sigma model on a gerbe over a space or orbifold $Y$.\[4\]
Briefly, summing over nonlinear sigma models on $Y$ with different $B$ fields projects out many nonperturbative sectors, realizing the gerbe theory (as a theory with a restriction on nonperturbative sectors). Schematically, the path integral of the theory with a restriction on nonperturbative sectors takes the form [27]

$$
\int [D\phi] \exp(-S) \sum_B \exp \left( \int \phi^* B \right)
$$

$$
= \sum_B \int [D\phi] \exp \left( -S + \int \phi^* B \right),
$$

which is the path integral for a nonlinear sigma model on a disjoint union of spaces with variable $B$ fields.

Let us now consider a more concrete example. We will recover decomposition for the case of a global orbifold, describing a $\mathbb{Z}_2$ gerbe over $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$. Specifically, consider $[X/D_4]$, where

$$1 \rightarrow \mathbb{Z}_2 \rightarrow D_4 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow 1,$$

(8)

where $\mathbb{Z}_2$ (equal to the center of $D_4$) acts trivially on $X$.

In this example, decomposition predicts

$$\text{CFT}([X/D_4]) = \text{CFT}([X/\mathbb{Z}_2 \times \mathbb{Z}_2] \bigcup [X/\mathbb{Z}_2 \times \mathbb{Z}_2])$$

(9)

where one of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds has discrete torsion and the other does not.

Next, we shall see how to recover decomposition at the level of string one-loop partition functions. Label the group elements as follows:

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\},$$

(10)

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = D_4/\mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab} = \overline{ba}\},$$

(11)

where $z \in D_4$ generates the center ($\mathbb{Z}_2$), and we use bars to denote cosets, e.g., $\overline{z} = \{z, az\}$. The string one-loop partition function has the form

$$Z([X/D_4]) = \frac{1}{|D_4|} \sum_{g,h} Z_{g,h},$$

(12)

where each $Z_{g,h}$ is a sum over maps with boundary conditions determined by $g$, $h$. Note that since $Z_{g,h}$ is determined only by boundary conditions, each $Z_{g,h}$ for the $D_4$ orbifold is the same as a $Z_{g,h}$ of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$, except for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sectors defined by the pairs $(\overline{a}, \overline{b})$, $(\overline{a}, \overline{ab})$, and $(\overline{b}, \overline{ab})$, which do not have a lift to $D_4$. (The preimages of the group elements in $D_4$ do not commute.)

As a result of the counting above, we see

$$Z([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \cdot (Z([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors})),
$$

(13)

$$= 2 (Z([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors})).
$$

(14)

Discrete torsion acts as a sign on the omitted twisted sectors above of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, hence

$$Z([X/D_4]) = Z([X/\mathbb{Z}_2 \times \mathbb{Z}_2] \bigcup [X/\mathbb{Z}_2 \times \mathbb{Z}_2])$$

(15)

with discrete torsion in one component, consistent with the prediction of decomposition.

We list here two sets of applications of decomposition.

i) Gromov–Witten theory. In particular, one prediction of decomposition is that Gromov–Witten invariants of gerbes match Gromov–Witten invariants of disjoint unions of spaces. This has since been proven in work of H.-H. Tseng, Y. Jiang, and collaborators in e.g. [28–33].

ii) Phases of gauged linear sigma models (GLSMs). Phases of certain GLSMs, which were previously obscure, now have a solid understanding utilizing decomposition. The prototype [34] is the Landau–Ginzburg point of the GLSM for a complete intersection of $n$ quadrics in a projective space $\mathbb{P}^{2n-1}$. Briefly, utilizing decomposition, this Landau–Ginzburg point can be interpreted as a (possibly noncommutative resolution of a) branched double cover of $\mathbb{P}^{n-1}$, branched over a degree $2n$ locus. The double cover structure is a local application of decomposition for a $\mathbb{Z}_2$ gerbe. For other applications see e.g. [35–39].

2.5 Four-dimensional sigma models on stacks

So far we have focused on two-dimensional nonlinear sigma models, that is, QFTs on two-dimensional spaces with targets of targets of possibly other dimension.

In principle, we can also consider four-dimensional low-energy effective nonlinear sigma models, that is, quantum field theories on four-dimensional spaces with targets of possibly other dimension. These arise in e.g. four-dimensional supergravity theories, describing the space of scalar field vevs, and analogous considerations apply there. (See [40] for a more detailed discussion.)

A four-dimensional nonlinear sigma model on a gerbe can be distinguished from an ordinary nonlinear sigma model in much the same way as two-dimensional cases:

i) Compact four-dimensional spaces: to specify matter fields, one must specify bundles, and different bundles give rise to different anomalies and zero modes, and hence different physics, just as in two dimensions.

ii) Noncompact four-dimensional spaces: In four dimensions, the theta angle no longer acts like an
electric field as it does in two dimensions. However, instead of using theta angles, we can use charged Reissner-Nordstrom black holes. As before, to distinguish nonminimal charges from minimal charges, we add massive minimally-charged fields. These fields can be emitted in Hawking radiation from black holes. To be specific, consider a four-dimensional $U(1)$ gauge theory with massless fields whose charges are multiples of $k$. If there are massive minimally-charged fields, then a charged black hole can Hawking radiate down to charge 1, whereas if there are no massive minimally-charged fields, then it can only Hawking radiate to charge $k$.

iii) Defects: we can for example add Wilson lines of charges that are only well-defined in certain theories, thereby distinguishing between different four-dimensional theories with the same low-energy matter content.

Next, we will consider some concrete examples.

In a perturbative string compactification, the low-energy effective nonlinear sigma model whose target is four-dimensional supergravity contains a low-energy four-dimensional theories with the same low-energy matter. In physics, the holomorphic top-form corresponds to the spectral flow operator of the $N=2$ algebra, so in order to construct a moduli space of SCFTs for sigma models on elliptic curves, over which one has a family of spectral flow operators, one must work with the $SL(2,\mathbb{Z})$ quotient, which is a $\mathbb{Z}_2$ gerbe over the $PSL(2,\mathbb{Z})$ quotient.

Mathematically, it turns out that the moduli space $\mathcal{M}_{1,1}$ of elliptic curves is identified with the $SL(2,\mathbb{Z})$ quotient:

$$\mathcal{M}_{1,1} = [\mathcal{H}/SL(2,\mathbb{Z})].$$

(20)

In addition, since there is a naturally-defined line bundle of spectral flow operators (holomorphic top-forms), this is also at least part of what we need for a good moduli space of SCFTs.

However, it turns out that to get a good moduli space of SCFTs (of sigma models on elliptic curves), we need more. The issue is that the chiral Ramond vacuum is not well-defined under either the $PSL(2,\mathbb{Z})$ or $SL(2,\mathbb{Z})$ quotients.

Specifically, under $SL(2,\mathbb{Z})$,

$$(\tau, z, |0\rangle) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \pm |0\rangle \sqrt{\frac{c\tau + d}{c\tau + d}}\right).$$

(21)

In principle, this is a consequence of the fact that the Fock vacuum in a sigma model with target $X$ couples to the pullback of $K_X^{1/2}$. (This is implicit in the NSR formalism [42], and follows ultimately from the fact that on a Kähler manifold $X$, the spinor bundle can be described as [43][Equation (D.16)]

$$\wedge^* TX \otimes \sqrt{K_X},$$

(22)

with the wedge interpreted as the complex exterior power, not the real exterior power. In the worldsheet realization, the $\wedge^* TX$ is formed by multiplying a Ramond vacuum by various worldsheet fermions $\psi$, and the Fock vacuum itself corresponds to $\sqrt{K_X}$.)
We can understand the role of the central element of $SL(2, \mathbb{Z})$ more concretely as follows. First, in a sigma model on $T^2$, there is a single complex fermion $\psi$, and in a chiral Ramond sector, strictly speaking there are two chiral Ramond vacua $|\pm\rangle$, which we define as follows:

$$\psi|+\rangle = 0, \quad \psi|-\rangle = |+\rangle, \quad \overline{\psi}|-\rangle = 0.$$  \hfill (23)

Ultimately because these vacua have fractional charges, under the transformation $\psi \rightarrow -\psi$, these vacua transform as

$$|\pm\rangle \rightarrow \pm \exp(i\pi/2)|\pm\rangle.$$  \hfill (24)

This is consistent with the transformation (21) above, for the central element of $SL(2, \mathbb{Z})$.

In any event, this means there is a sign ambiguity in the action of $SL(2, \mathbb{Z})$ on the Fock vacuum; the action of $SL(2, \mathbb{Z})$ is not well-defined. To get a well-defined action, we must replace $SL(2, \mathbb{Z})$ by a $\mathbb{Z}_2$ extension. In fact, it can be shown that [44] chiral Ramond vacua couple to a naturally-defined line bundle over the quotient $[h/Mp(2, \mathbb{Z})]/PSL(2, \mathbb{Z})$, where $Mp(2, \mathbb{Z})$ is the metaplectic group, the unique nontrivial $\mathbb{Z}_2$ extension of $SL(2, \mathbb{Z})$.

We conjecture [44] that the moduli space of (complex structures on) SCFTs for sigma models on elliptic curves is $[h/Mp(2, \mathbb{Z})]/PSL(2, \mathbb{Z})$.

In fact, we have also argued [45] that the metaplectic group arises elsewhere in string dualities:

1) $T$-duality of $T^2$: for essentially the same reasons as above, one should replace the $SL(2, \mathbb{Z})$ factors in the $T$-duality group $SO(2, 2; \mathbb{Z})$ by $Mp(2, \mathbb{Z})$.

2) Ten-dimensional IIB S-duality: because of analogous sign ambiguities in S-duality actions on ten-dimensional fermions, one should also replace $SL(2, \mathbb{Z})$ by $Mp(2, \mathbb{Z})$. (This has also been observed by D. Morrison.)

3) M theory on $T^2$: one can argue that the action of the mapping class group on fermions is $Mp(2, \mathbb{Z})$.

4) U-duality in nine dimensions: $Mp(2, \mathbb{Z})$ instead of $SL(2, \mathbb{Z})$.

These dualities are interrelated: the U-duality group in nine dimensions can be understood either from M theory on $T^2$ or from ten-dimensional IIB S-duality, and so it is a nontrivial consistency check that these different duality groups match.

2.7 Open questions

A natural question to ask is, which moduli stack is sensed by defects in ten-dimensional IIB? The stack $[h/PSL(2, \mathbb{Z})]$, $[h/SL(2, \mathbb{Z})]$, or $[h/Mp(2, \mathbb{Z})]$? In F-theory compactifications, there is a defect which senses the center of $SL(2, \mathbb{Z})$, corresponding to a Kodaira fiber $I_0$, or a D7-brane on an $O7$-plane. One can then ask whether there is a D7-brane configuration that senses the $\mathbb{Z}_2$ specific to $Mp(2, \mathbb{Z})$.

Another natural question for future work is whether there is any three-dimensional analogue of decomposition. One does not expect the three-dimensional theory to decompose as a disjoint union of quantum field theories, but there might be some sort of decomposition for certain classes of defects within the theory, for example. A first pass at answering this question is implicit in [46].

Finally, let us conclude with a conjecture motivated by recent work. The Bagger–Witten and Hodge line bundles over moduli stacks of Calabi–Yau’s are known in only a few examples [44, 47], but in those examples, they are nontrivial but generate a finite subgroup of the Picard group. We conjecture that this is true more generally:

Conjecture: over any Calabi–Yau moduli space, the Bagger–Witten and Hodge line bundles are holomorphically nontrivial but admit flat connections.

This is a refinement of ideas expressed in [48], which gave a physical argument that the Bagger–Witten line bundle should be flat (but did not require nontriviality). This would also be an analogue of, and related to, the weak gravity conjecture [49]) for existence of UV completions of four-dimensional supergravity theories.

3 Derived categories

Of the various topics discussed in this overview, the physical realization of derived categories, as combinations of branes, antibranes, and tachyons, is relatively well-known, so we will be comparatively brief. (See e.g. [5] for a more detailed review of the physical realization, and [50–53] for more information on the mathematics of derived categories.)
The physical realization of derived categories, first described in [4], was originally motivated by two separate developments.

i) Kontsevich’s homological mirror symmetry [54], relating the derived category of coherent sheaves of one Calabi–Yau to a derived Fukaya category of the mirror Calabi–Yau. When this was originally proposed, mirror symmetry was only understood as a relationship between closed string theories, and the physical meaning, if any, of derived categories of coherent sheaves and derived Fukaya categories, was unknown. Part of the motivation of [4] was to find a physical underpinning for homological mirror symmetry.

ii) Sen’s work on antibranes [55–58]. Sen introduced the idea of antibranes and pertinent facts about brane–antibrane annihilation, which were interpreted by Witten mathematically in terms of K-theory [59]. However, this work only kept track of smooth information, and so another motivation was to find a holomorphic analogue, which in physics could keep track of e.g. information about connections (morally) on smooth bundles.

Derived categories provided a holomorphic analogue of K-theory, and an interpretation as some sort of version of tachyon condensation answered the riddle about the physical meaning of Kontsevich’s proposal.

Before talking about derived categories of sheaves, let us first quickly review the dictionary between ordinary coherent sheaves and D-branes. Briefly, we know such a dictionary for various special cases. The most common case is as follows. Let \( i : S \hookrightarrow X \) be a submanifold of some Calabi–Yau \( X \), with holomorphic vector bundle \( \mathcal{E} \to S \). Then, the dictionary [60] equates the sheaf \( i_* \mathcal{E} \) to a D-brane on \( S \) with gauge bundle \( \mathcal{E} \otimes K_S^{-1/2} \). The factor of \( \sqrt{K_S} \) is ultimately a reflection of the Freed–Witten anomaly [61], as discussed in [60], and is important in order to match open string B model boundary chiral rings with Ext groups between sheaves. Another set of known special cases relates structures sheaves of nonreduced subschemes to D-branes with nilpotent Higgs vevs [62, 63]. Simple statements are not known for other cases (except via projective resolutions, as we will discuss next.)

The dictionary is summarized in Table 3.

| Sheaf / D-brane dictionary. |
|--------------------------------|
| \( i_* \mathcal{E} \)              | D-brane on \( S \) with bundle \( \mathcal{E} \otimes K_S^{-1/2} \) [60] |
| nonreduced scheme              | Nilpotent Higgs vev, T-brane [62, 63] |

Maps in the complex. Now, there are subtleties, including both the fact that we do not know a simple dictionary between all possible sheaves and D-branes, only certain sheaves, and that whether or not one has a tachyon in the brane/antibrane spectrum depends upon the difference in dimensions between the brane and antibrane.

In broad brushstrokes, we deal with these issues as follows. We replace any complex of sheaves on a Calabi–Yau by a projective resolution consisting of locally-free sheaves. For locally-free sheaves, we know the corresponding D-branes (which are defined by the sheaves themselves), and physically there exist tachyons between all branes and antibranes here since they all have the same dimension.

Let us now illustrate these ideas in greater detail. Boundary actions for brane, antibrane, tachyon systems were constructed in [64][Section 5.1.2], [65][Section 4], [66][Section 2], and [67], and take the form

\[
\int_{\partial \Sigma} d\bar{u}[\bar{\pi} d\eta + i \psi^i (\partial_i P) \eta + i \psi^j (\partial_j \Pi) \bar{\eta}] - i |P|^2 - i |Q|^2, \tag{26}
\]

where \( \psi^i = \psi_+^i + \psi_-^i \) is the restriction of the bulk worldsheet fermions to the boundary, and \( \eta, \bar{\eta} \) are fermions living only on the boundary. There are two vector bundles (associated with the branes and antibranes), which we will label \( \mathcal{E}_0, \mathcal{E}_+ \). The boundary fermion \( \eta \) couples to \( \mathcal{E}_0^* \otimes \mathcal{E}_1 \), and \( \bar{\eta} \) couples to \( \mathcal{E}_0 \otimes \mathcal{E}_1^* \). The field \( P \) is a section of \( \mathcal{E}_0^* \otimes \mathcal{E}_1 \), and \( Q \) is a section of \( \mathcal{E}_0 \otimes \mathcal{E}_1^* \). Under a supersymmetry transformation, for which the boundary fermions transform as

\[
\delta \eta = -i \bar{P} \alpha - i Q \bar{\alpha}, \tag{27}
\]

\[
\delta \bar{\eta} = -i P \alpha - i \bar{Q} \bar{\alpha}, \tag{28}
\]

the supersymmetry variation of the boundary action takes the form

\[
\int_{\partial \Sigma} \left[ -\alpha \psi^j (\partial_j (P \bar{Q}) - \bar{\alpha} \psi^i (\partial_i (P \bar{Q})) \right]. \tag{29}
\]

If the bulk worldsheet theory had a superpotential, as in a Landau–Ginzburg model, then one could solve the
Warner problem \cite{68, 69} by requiring $PQ = W1d$, up to a constant shift, which leads to matrix factorizations. If the bulk theory is just a nonlinear sigma model without superpotential, then instead we require $PQ = 0$ (up to a constant).

Now, the bulk worldsheet sigma model also has a pair of $U(1)_R$ symmetries, acting on the left- and -right-moving worldsheet fermions, which in principle on the boundary restrict to a common $U(1)_R$. To recover the grading implicit in a derived category, we require that the bundles $E_{0,1}$ be $U(1)_R$ equivariant, and that the maps $P_i$, $Q$ each have $U(1)_R$ charge $+1$. The fermions $y_i$ then have $U(1)_R$ charge $-1$. (In the case of a matrix factorization, this is consistent with the convention that the worldsheet superpotential have $U(1)_R$ charge two. See e.g. \cite{70, 71} for further information on the $U(1)_R$ action for matrix factorizations.)

Now, in a (bulk) nonlinear sigma model without superpotential, the $U(1)_R$ acts only on the fermions, not the bosons, so we are requiring that the bundles $E_{0,1}$ be equivariant with respect to a group that acts trivially on the space over which they are defined. In such a case, the action is $\mathbb{Z}$-graded. In general, $E_0$ and $E_1$ may decompose, so we write

$$E_0 = \oplus_i A_i, \quad E_1 = \oplus_j B_j.$$  

Each summand $A_i$, $B_j$ can have a different integral weight under $U(1)_R$, so without loss of generality, we will identify the $U(1)_R$ charge with the integer index $i$–meaning, for example, we will take $A_i$ to have $U(1)_R$ weight $i$ and $B_j$ to have $U(1)_R$ weight $j$. Then, since $P$ is a section of $E_0 \otimes E_1$, it defines a set of maps

$$P_i : A_i \longrightarrow B_{i+1},$$

and similarly $Q$ defines a set of maps

$$Q_i : B_i \longrightarrow A_{i+1}.$$  

Since $PQ = 0$, we see that these maps form a complex

$$\cdots \longrightarrow A_i \xrightarrow{P_i} B_{i+1} \xrightarrow{Q_{i+1}} A_{i+2} \xrightarrow{P_{i+2}} B_{i+3} \xrightarrow{Q_{i+3}} \cdots.$$  

(In principle, we have sufficient data for two complexes of this form. However, for simplicity, we assume here that, for example, $A_{odd}$ and $B_{even}$ all vanish, which also allows us to cleanly distinguish branes from anti-branes (one is encoded by the $A_i$, the other by the $B_i$). In this fashion, we see that $U(1)_R$-equivariant boundary data defines a complex of bundles.

Now, there are essentially two classes of isomorphisms between the brane-antibrane systems above. The first is defined by homotopies of complexes. Maps between complexes can again be physically realized in terms of tachyons, and it can be shown (see e.g. \cite{72}) that chain-homotopic maps are BRST-equivalent in that realization.

The second class of isomorphisms we shall discuss is that of quasi-isomorphisms. Quasi-isomorphisms between complexes are realized in physics by renormalization group flow (realizing an entry in Table 1). This is straightforward to outline in an example. Consider a brane described by the structure sheaf $\mathcal{O}$, and an anti-brane on some ideal sheaf $\mathcal{O}(-D)$, for some divisor $D$, together with a tachyon map corresponding to the inclusion $\mathcal{O}(-D) \rightarrow \mathcal{O}$. Physically, we expect that such a brane-anti-brane collection should mostly annihilate, physically evolving over time into a single brane supported along the divisor $D$. Such time-evolution in spacetime corresponds to renormalization group flow on the worldsheet, and mathematically we are identifying

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow 0$$

with

$$0 \longrightarrow \mathcal{O}_D \longrightarrow 0.$$  

These two complexes are quasi-isomorphic to one another, and so we see in this example that quasi-isomorphism is the mathematical implementation of renormalization group flow.

So far we have just discussed the physical realization of derived categories. Perhaps the most well-known application is to stability questions. That is beyond the scope of this review; we instead refer interested readers to e.g. \cite{73, 74}.

4 Derived schemes

In Section 3, we saw one realization of complexes – in terms of branes, anti-branes, and tachyons. In principle, there are other places where complexes enter physics. One example is via Yukawa couplings in two-dimensional Landau–Ginzburg theories. Another is in massless spectrum computations in string compactifications. Typically, we describe massless spectra and BRST cohomology as the cohomology of some complex. That complex has meaning inside the quantum field theory – each element of the complex represents some set of states or operators, which may or may not be massless or BRST closed, but which can be explicitly represented within the QFT. Mathematically, the complexes above can be interpreted in terms of derived geometry. In this section we will outline these two examples, their relation to derived geometry, and their roles in two- and four-dimensional
theories, in close analogy with Section 2 on the role of stacks in physics.

Now, to be clear, we do not claim these are the only places where complexes or derived geometry enter physics, and indeed, one suspects there are many places in physics where derived geometry can play a role. Other physical realizations of derived geometry in different contexts have also appeared in e.g. [75–77].

4.1 Brief introduction to derived geometry

In a nutshell, in derived schemes, instead of relating spaces to ordinary algebras of functions, one associates spaces to dg-algebras of functions. As a practical matter, this means that a given derived space can be presented as a variety of different spaces, of potentially different dimension, all with different analogues of structure sheaves, analogous to the manner in which a given stack can have a variety of different presentations.

One essential aspect of derived geometry is the 'cotangent complex,' often denoted $\mathbb{L}$, a complex generalizing the cotangent bundle of smooth manifolds. Later we will see this arise as the BRST complex in various situations. If the derived space is smooth, then the cotangent complex will have cohomology only in degree 0, and that cohomology sheaf will be precisely the ordinary cotangent complex. If the cotangent complex has nonzero cohomology sheaves in other degrees, then the space is not smooth, at least in the ordinary sense. Very readable introductions to the cotangent complex and derived geometry can be found in [78–80].

Let us briefly describe an example of a prototypical form, following [78][Example 5.5]. Consider a complete intersection of hypersurfaces in $\mathbb{C}^n$, defined by the ring

$$ A = \mathbb{C}[x_1, \cdots, x_n]/(f_1, f_2, \cdots, f_k) $$

A dg algebra resolving the ring above is

$$ \mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_k] $$

with $x$'s of degree 0, $y$'s of degree $-1$, and differential

$$ s(x_i) = 0, \quad s(y_j) = \delta_{ij}. $$

We therefore identify the affine space $\mathbb{C}^{n+k}$ as an equivalent derived scheme for the complete intersection, with corresponding cotangent complex given by

$$ 0 \to \oplus_i \text{Ad} y_j \to \oplus_i \text{Ad} x_i \to 0, $$

where the $dx_i$ are in degree 0 and the $dy_j$ in degree $-1$. The cokernel of the differential, the cohomology at the second step, is easily checked to be the differential one-forms on the complete intersection, defined by $dx_i$ subject to the equivalence

$$ \sum_j \frac{\partial f_j}{\partial x_i} \, dx_i \sim 0 $$

(reflecting the fact that we are restricted to the complete intersection of hypersurfaces $(f_j = 0)$). If the complete intersection is singular, then at the singularity, the rank of the cotangent bundle defined by the constraints above is wrong, and precisely in such a case, the differential $s$ has a nonzero kernel. Thus, in this example, the cohomology at degree 0 (identified with the second term) is the cotangent sheaf, and the complete intersection is singular if and only if there is nonzero cohomology at degree $-1$, matching the description of the cotangent complex above. (For a compact space (such as a complete intersection in a projective space), the same story applies patch-by-patch.)

Readers familiar with gauged linear sigma models [81] will find the structure above familiar – a GLSM$^2$ describes a complete intersection in $\mathbb{C}^n$ by a theory on $\mathbb{C}^{n+k}$ with a superpotential, in which one adds a new field (here, corresponding to the $y_j$) for each hypersurface in the complete intersection. Furthermore, the cotangent complex is realized implicitly in Yukawa couplings in the GLSM. The superpotential for a complete intersection in $\mathbb{P}^{n-1}$, for example, is of the form

$$ W = \sum_j p_j f_j(x), $$

which has Yukawa couplings such as

$$ \sum_{i,j} \frac{\partial f_j}{\partial x_i} \psi^i \psi^j. $$

Following standard tricks, we can identify $\psi^i$ with $dx^i$, and then the Yukawa coupling is a mass term that, in describing the tangent bundle, realizes equation (40). In this fashion we see these elementary aspects of derived geometry appearing explicitly in standard GLSM constructions. We will describe physical analogues of other constructions in the next section.

---

$^2$ Strictly speaking, since we are describing affine spaces rather than projective spaces, nothing is being gauged physically, so the term GLSM is perhaps not perfectly appropriate. On the other hand, we can also consider derived structures for complete intersections in projective spaces in almost an identical fashion, for which the language of GLSMs is absolutely appropriate.
Next, we will describe two more concrete examples of derived spaces. The first is the derived critical locus, which is defined as follows. (See e.g. [82] for additional information.) Let \( X \) be a variety, and \( W \) a holomorphic function on \( X \), whose critical locus is \( Z \). Then, the cotangent complex is given by

\[
0 \longrightarrow TX|_Z \xrightarrow{\partial^2 W} \Omega^1_X|_Z \longrightarrow 0,
\]

where \( \Omega^1_X|_Z \) is in degree 0 and \( TX|_Z \) is in degree \(-1\).

Suppose that \( X \) and \( Z \) are smooth, then the complex above is quasi-isomorphic to the one-element complex giving the cotangent bundle of \( Z \). Let us outline this explicitly, at least for the special case that \( W \) consists of (fat) points. In this case, we dualize the short exact sequence

\[
0 \longrightarrow T Z \longrightarrow TX|_Z \xrightarrow{\partial^2 W} \Omega^1_X|_Z \longrightarrow 0,
\]

and rewrite it as a quasi-isomorphism between complexes:

\[
0 \longrightarrow TX|_Z \longrightarrow \Omega^1_X|_Z \longrightarrow 0
\]

Thus, we see that in this special case, the cotangent complex is quasi-isomorphic to \( \Omega^1_X|_Z \).

The second example of a derived manifold we shall encounter is the derived zero locus. Given a variety \( X \), a vector bundle \( E \to X \), and a regular section \( s \in \Gamma(E) \), the zero scheme \( Z \subset X \) of \( s \) is a local complete intersection whose cotangent complex is given by

\[
0 \longrightarrow E^1|_Z \xrightarrow{ds} \Omega^1_X|_Z \longrightarrow 0,
\]

where \( \Omega^1_X|_Z \) is in degree 0, and \( E^1|_Z \) is in degree \(-1\), and of course the codimension of \( Z \) in \( X \) equals the rank of \( E \).

Suppose that \( X \) is smooth, and the zero locus \( Z \) is also smooth, of codimension equal to the rank of \( E \). Then in this case, the cotangent complex is quasi-isomorphic to the one-element complex giving the cotangent bundle of \( Z \). To see this, we dualize the short exact sequence

\[
0 \longrightarrow T Z \longrightarrow TX|_Z \xrightarrow{ds} E|_Z \longrightarrow 0
\]

and rewrite as a quasi-isomorphism between complexes:

\[
0 \longrightarrow E^1|_Z \longrightarrow \Omega^1_X|_Z \longrightarrow 0
\]

Thus, we see that in this special case, the cotangent complex is quasi-isomorphic to \( \Omega^1_X|_Z \). (If the codimension of \( Z \) is different from the rank of \( E \), then there is cohomology in degree \(-1\), as we shall see in an example in the next section.)

### 4.2 Two-dimensional Landau–Ginzburg models

In this section we review how a sigma model with a superpotential can be interpreted in terms of derived manifolds.

Consider a two-dimensional (2,2) supersymmetric Landau–Ginzburg model, a nonlinear sigma model on \( X \) with superpotential \( W : X \to \mathbb{C} \). Let us consider the tangent bundle arising in the IR limit. The Landau–Ginzburg theory has the Yukawa coupling term

\[
\psi^i \bar{\psi}^j D_i \partial_j W,
\]

so applying standard methods of (0,2) theories, if we let \( Z \equiv \{ dW = 0 \} \) denote the critical locus, then at least semiclassically, since the Yukawa coupling gives a mass to elements of \( TX|_Z \) that are not annihilated by

\[
D \partial W|_Z = \partial^2 W|_Z,
\]

the tangent bundle of the IR limit should be described as the kernel of the map

\[
TX|_Z \xrightarrow{\partial^2 W} \Omega^1_X|_Z.
\]

Mathematically, the complex above is the tangent complex of the derived critical locus of \( W \). The cotangent complex is the dual complex, namely

\[
TX|_Z \xrightarrow{\partial^2 W} \Omega^1_X|_Z.
\]

Let us consider a special case. Suppose \( Z \) is the total space of a vector bundle \( V \to M \), and \( W = pf \) where \( f \in \Gamma(V) \) and \( p/s \) are fiber coordinates on \( V^* \), so that in the IR this theory should flow to a nonlinear sigma model on \( Z' = \{ f = 0 \} \subset M \) (for essentially the same reasons as in analysis of large-radius limits of GLSMs). Let us also

3 In more detail, the Yukawa coupling is defined by

\[
D_i \partial_j W = \partial_i \partial_j W + \Gamma^k_{ij} \partial_k W,
\]

but when we restrict to the critical locus \( Z \), the second term drops out, yielding just the ordinary Hessian.
assume that $Z'$ is smooth, so that $Z'$ coincides with the critical locus $Z$. In this case, the restriction of the Hessian $D\partial W$ to the critical locus is of the form

$$\begin{bmatrix} 0 & \partial_1 f \\ \partial_j f & 0 \end{bmatrix}. \quad (54)$$

Thus, in this case, the mass matrix defined by the Yukawa coupling is describing the cotangent complex (53).

There is also an analogue of these considerations for (0,2) supersymmetric theories. Consider a (0,2) Landau–Ginzburg model, defined by a space $X$, holomorphic vector bundle $\mathcal{E} \to X$ satisfying the anomaly cancellation condition

$$\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX), \quad (55)$$

and $J \in \Gamma(\mathcal{E}^\ast)$ defining a (0,2) superpotential\footnote{In general, a (0,2) theory has potentials defined by both $J \in \Gamma(\mathcal{E}^\ast)$ as well as $E \in \Gamma(\mathcal{E})$, satisfying $E \cdot J = 0$, but in this case for simplicity we restrict to the special case $E \equiv 0$.}. The Yukawa couplings

$$\psi^i_\alpha \lambda^a D_\alpha J_a \quad (56)$$

(where $\psi^i_\alpha$ s are right-moving fermions and $\lambda^a$ s are left-moving fermions) define a mass matrix that implies the left-movers couple to the kernel of

$$\mathcal{E}|_Z \searrow \Omega^1_{Z|X} \quad (57)$$

where $Z \equiv \{ J = 0 \}$, and the right-movers couple to the cokernel. (Fermions not in either the kernel or cokernel get a mass, and so are integrated out along RG flow.)

It will be helpful to consider a concrete example arising in two-dimensional (0,2) theories. Suppose we wish to describe an IR theory on a space $Z$ with bundle $\mathcal{E}$ given as the kernel

$$0 \to \mathcal{E} \to M \xrightarrow{F} L \to 0, \quad (58)$$

where $M$, $L$ are holomorphic vector bundles on $B$. This theory arises as the IR endpoint of a (0,2) Landau–Ginzburg model on a space

$$X \equiv \text{Tot}(\pi : L^* \to B), \quad (59)$$

with (0,2) superpotential

$$W = p \Lambda^a \pi^* F_a, \quad (60)$$

where the $\lambda^a$ are Fermi superfields coupling to the bundle $M$, and $p$ is a fiber coordinate on $L^*$. The $F_a$ are simply indexed components of the map $F : M \to L$. The (0,2) superpotential is defined by $J_a = p \pi^* F_a$. In principle, the theory should flow in the IR to the zero locus of $J$ (just as in a (2,2) theory, a Landau–Ginzburg model flows to the critical locus of the superpotential). In this case, since $F$ is surjective, the zero locus is $B = \{ p = 0 \}$. Note that $J \in \Gamma(M^*)$, and so we define $E = M^*$. The cotangent complex of the derived zero locus of $J$ is given by

$$E^*|_B \xrightarrow{df} \Omega^1_X|_B \quad (61)$$

and since the codimension of $B$ is different from the rank of $E$ in general, there is nonzero cohomology in both degree 0 and −1. Specifically, the cohomology at degree 0 is $\Omega^1_B$, and the cohomology at degree −1 is $\mathcal{E}$. Thus, we see that the cotangent complex of the derived zero locus is encoded in the physics of two-dimensional (0,2) Landau–Ginzburg models.

4.3 Derived structures on moduli spaces of SCFTs

So far we have discussed two-dimensional Landau–Ginzburg models as giving a physical realization of derived schemes. Next, we turn our attention to moduli spaces of SCFTs, much as we did in Section 2.6 in our examination of stacks, as would arise in four-dimensional $N = 1$ supergravity theories obtained from string compactification. We will outline how derived structures on such moduli spaces (specifically, the tangent and cotangent complexes) seem to be encoded in worldsheet physics, and outline how cohomology at nonzero degree corresponds to singular points and enhanced gauge symmetries. (In other words, if the cohomology of the cotangent complex at degree 0 corresponds to scalars in the target-space theory – infinitesimal moduli of the compactification – then cohomology at degree −1 corresponds to vectors arising at enhanced symmetry points.)

In four dimensions, it is important to note that a derived structure on the moduli space cannot be described as a holomorphic derived critical locus of the spacetime superpotential, simply because the superpotential is a section of a line bundle [83], the line bundle of holomorphic top-forms (often called the Hodge line bundle, and identified with the tensor square of the Bagger–Witten line bundle). As a result, the critical locus is defined by not only the superpotential, but also a choice of connection on the Bagger–Witten line bundle (defined physically by the Kähler potential). Although the superpotential itself is holomorphic, the connection is not, and so holomorphic derived geometry cannot be relevant here.

To see, for example, the cotangent complex, it is difficult to work directly with states in a nonlinear sigma
model on a Calabi–Yau, simply because in the case that the complex has cohomology at degree $-1$, the Calabi–Yau is singular, and so the nonlinear sigma model becomes ill-behaved. It may still be possible to work directly with such singular theories, but we will take a different approach. Instead of working with IR nonlinear sigma models, we will work with UV theories, which will sidestep this issue.

In general terms, we will identify BRST complexes of states with the cotangent complex. From a more global perspective, we are proposing a shift in emphasis from describing states and operators in terms of BRST cohomology, so we expect that BRST complexes of states, without taking cohomology at degree $-1/2$ has dimension 305. As discussed in [84], the states at degree $-1/2$ are represented by five quartic functions $P_i(\phi_{-1/10})$ subject to the relation

$$P_i \sim P_i + A_i \frac{\partial W}{\partial \phi_j} + \phi^k B_k \frac{\partial^2 W}{\partial \phi^i \partial \phi^j},$$

where $W$ is the superpotential, and $A, B$ are arbitrary constant matrices.

The cohomology at degree $-3/2$ has dimension 5. Of those five elements of cohomology at degree $-3/2$, one state is present for generic complex structures.

We can understand these states somewhat more systematically as follows. Since we are on the $(2,2)$ locus, albeit at a Landau–Ginzburg orbifold point, morally we expect these singlets to be related to complex structure moduli $H^1(T)$, Kähler moduli $H^1(T^*)$, and bundle moduli $H^1(\text{End} T)$. (Rather, these are the moduli in the corresponding large-radius $(2,2)$ supersymmetric nonlinear sigma model, so barring massing up of pairs, one expects a similar, though not necessarily identical, counting of moduli at Landau–Ginzburg.)

In this language, following [84], we can understand the complex structure deformations as states of the form

$$P_i(\phi_{-1/10})\psi_{-3/5}^j,$$

where $P_i = \partial_i S$ for some quintic polynomial $S$, subject to the equivalence

$$S \sim S + \phi^i A_i \partial_j W + \phi^k B_k \frac{\partial^2 W}{\partial \phi^i \partial \phi^j}.$$  

Similarly, the analogue of bundle deformations is encoded [84] in the space of five quartic polynomials $P_i$ such that $\phi^i P_i = 0$, subject to the equivalence relation

$$P_i \sim P_i + A_i \phi_j \partial W \sim -\frac{1}{5} \partial_i \left( \phi^k A_k \partial_j W \right).$$

---

5 Strictly speaking, the authors of [84] are using a spectral sequence to compute BRST cohomology, so $\overline{Q_{+L}}$ is only part of the BRST operator, but the rest has already been taken into account, so without loss of generality we may as well consider this to be the BRST operator.

6 Off the $(2,2)$ locus, only a subset of the complex and bundle moduli will be present in the CFT in general, because not all complex structure deformations are compatible with a given holomorphic bundle. See e.g. [85, 86] for recent discussions.
For special values of the superpotential $W$, this equivalence relation is less powerful, so that the Landau–Ginzburg theory has extra $E_6$ singlets. For example, the Fermat quintic has five extra states

$$\left(\frac{1}{4} \phi_{i/10}^{10} \bar{\psi}_{-9/10, i} - \bar{\psi}_{-2/5, i} \psi_{-3/5}^i\right)[0].$$

(71)

In this case, there are extra gauge bosons arising at $q_+ = -3/2$ and extra scalars at $q_+ = -1/2$. In the target-space supergravity theory, in accord with standard lore that at special/singular points in the moduli space, the target-space theory has enhanced gauge symmetries.

4.4 Derived stacks

We have outlined in this section how at least certain derived schemes appear to have a physical realization in terms of Landau–Ginzburg models, and we have also seen that certain (Deligne–Mumford) stacks have a physical realization in terms of gauged sigma models. With this in mind, the physical realization of certain derived stacks should be as gauged Landau–Ginzburg models or gauged sigma models with superpotential, of which the most common examples are gauged linear sigma models [81].

5 Conclusions

In this paper we have outlined three examples of physical realizations of mathematical structures in which renormalization group flow realizes categorical or homotopy equivalences: stacks, derived categories, and derived schemes.

The fact that renormalization group flows seems to often realize categorical or homotopy equivalences would appear to suggest that there may be some way to define a model structure on a category of renormalization group flows, such that weak equivalences relate theories which RG flow to the same endpoint. We have not so far succeeded in finding completely sensible definitions, but in general terms, one can outline an idea. Let $\mathcal{C}$ be a category of renormalization-group flows of ‘one’ theory, meaning that the objects are quantum field theories at various scales and the morphisms correspond to RG flows toward lower energies. One expects that such a category should have an initial object (a UV fixed point, which should be a cofibrant object in the sense of model categories) and a terminal object (an IR fixed point, which should be a fibrant object in the sense of model categories). One also expects it will have pushouts and pullbacks. For example, if a field theory $A$ RG flows to both theories $B$ and $C$ at different scales, then they should flow to a common theory at a lower scale, so in a nutshell one expects pushouts to exist, and for similar reasons, one also expects pullbacks to exist. Now, if we want to associate weak equivalences with maps between theories related by RG flow, we have a minor issue: the morphisms only run from higher to lower energies. We can solve this by localizing on the morphisms. In other words, if $S$ denotes the set of morphisms, we could consider the category $S^{-1} \mathcal{C}$, and then associate weak equivalences with all morphisms. It is less clear how one should define fibrations and cofibrations in this category, however, in a way that is both nontrivial and satisfies axiom MC5 of [87].

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