Integrable discretization and deformation of the nonholonomic Chaplygin ball

A.V. Tsiganov
St.Petersburg State University, St.Petersburg, Russia
e-mail: andrey.tsiganov@gmail.com

Abstract
The rolling of a dynamically balanced ball on a horizontal rough table without slipping was described by Chaplygin using Abel quadratures. We discuss integrable discretizations and deformations of this nonholonomic system using the same Abel quadratures. As a by-product one gets new geodesic flow on the unit two-dimensional sphere whose additional integrals of motion are polynomials in the momenta of fourth order.

1 Introduction
The nonholonomic Chaplygin ball is that of a dynamically balanced three-dimensional ball that rolls on a horizontal table without slipping or sliding [8]. 'Dynamically balanced' means that the geometric center coincides with the center of mass but the mass distribution is not assumed to be homogeneous. Because of the roughness of the table this ball cannot slip, but it can turn about the vertical axis without violating the constraints. There is a large body of literature dedicated to the Chaplygin ball, including the study of its generalizations, see [3, 4, 6, 7, 9, 11, 14] and references within.

In [22] we discuss integrable discretizations and deformations of the nonholonomic Veselova system using standard divisor arithmetic on the hyperelliptic curve of genus two. Because nonholonomic Veselova system is equivalent to the nonholonomic Chaplygin ball with a pure mathematical point of view [15, 16], we can transfer obtained results on the Chaplygin case.

The main our aim is to describe integrable discretizations and deformations of the nonholonomic Chapligin ball in a such way that we can eliminate the corresponding nonholonomic constraint and turn back to the holonomic Euler top on any step of our construction. In addition, we discuss auto and hetero Bäcklund transformations for the free motion on the sphere (partial case of Euler top) associated with the arithmetic of divisors on an auxiliary plane curve. As a result, we obtain the new geodesic flow on the sphere with the quartic integral of motion.

The paper is organized as follows. The second Section is devoted to well-studied description of the Chaplygin ball motion in term of Abel’s quadratures. Section 3 deals with the particular case when the angular momentum vector is parallel to the plane. Using Chaplygin’s calculations, we will introduce Abel’s differential equations and an intersection divisor. Then we will briefly repeat main information about the arithmetic of divisors and will explicitly show one of the possible discretizations of the Chaplygin ball motion associated with this arithmetic. In Section 4 we will present new integrable nonholonomic and holonomic systems, which can be considered as integrable deformations of the Chaplygin ball and Euler top restricted to the sphere.

2 Chaplygin ball
Following [8] consider a dynamically non-symmetric ball rolling without slipping over a horizontal plane. Its mass, inertia tensor and radius will be denoted by $m, I = \text{diag}(I_1, I_2, I_3)$ and $r$ respectively. Assume also that the mass center and the geometric center of the sphere coincide with each other.
Let \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) is the vertical unit vector, \( \omega = (\omega_1, \omega_2, \omega_3) \) and \( v = (v_1, v_2, v_3) \) are respectively the angular velocity of the ball and the velocity of its center. The condition of non-slippering of the point of contact of the ball with the horizontal plane is

\[
v + r \omega \times \gamma = 0.
\]

(2.1)

Here and below \( \times \) denotes the vector product in \( \mathbb{R}^3 \) and all the vectors are expressed in the so-called body frame, which is firmly attached to the ball, and its axes coincide with the principal inertia axes of the ball.

Under the nonholonomic constraint (2.1) the equations of motion can be reduced to the following closed system of equations of motion

\[
\dot{\gamma} = \gamma \times \omega, \quad \dot{M} = M \times \omega,
\]

(2.2)

which have the same form as the Euler-Poisson equations in rigid body dynamics [8]. Here \( M = (M_1, M_2, M_3) \) is the angular momentum of the ball with respect to the contact point

\[
\omega = A_\gamma M,
\]

(2.3)

where in the Chaplygin case

\[
A_\gamma = A + dg^2 A \otimes \gamma A, \quad A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad a_k = (I_k + d)^{-1},
\]

(2.4)

and

\[
g = \frac{1}{\sqrt{1 - d(\gamma, A_\gamma)}} = \frac{1}{\sqrt{1 - d(a_1 \gamma_1^2 + a_2 \gamma_2^2 + a_3 \gamma_3^2)}}, \quad d = mr^2.
\]

(2.5)

According to [3] the system (2.2) determines conformally Hamiltonian vector field

\[
X = g \Pi dH, \quad H = \frac{1}{2} (M, \omega)
\]

with respect to the Poisson bivector

\[
\Pi = g^{-1} \begin{pmatrix} 0 & \Gamma \\ \Gamma & M \end{pmatrix} - dg(M, A_\gamma) \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix},
\]

(2.6)

where

\[
\Gamma = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}.
\]

Vector field \( X \) possesses four independent first integrals

\[
C_1 = (\gamma, \gamma) = 1, \quad C_2 = (\gamma, M), \quad H_1 = (\omega, M), \quad H_2 = (M, M).
\]

(2.7)

Integrals of motion \( C_{1,2} \) are the Casimir functions of \( \Pi \)

\[
\Pi dC_{1,2} = 0,
\]

which give rise to the trivial vector fields. The two remaining integrals of motion \( H_{1,2} \) are in the involution with respect to the corresponding Poisson bracket

\[
\{M_i, M_j\} = \varepsilon_{ijk} (g^{-1} M_k - dg(M, A x)x_k), \quad \{M_i, x_j\} = \varepsilon_{ijk} g^{-1} x_k, \quad \{x_i, x_j\} = 0.
\]

Here \( \varepsilon_{ijk} \) is a totally skew-symmetric tensor.

**Remark 1** At \( d = 0 \) one gets standard Euler-Poisson equations for the Euler-Poinsot top, which is a well-studied holonomic Hamiltonian dynamical system, and standard Lie-Poisson bracket on Lie algebra \( e^*(3) \).
2.1 Integration of the equations of motion

In [8] Chaplygin found linear trajectory isomorphism with between two dynamical systems (2.2) with \((\gamma, M) = 0\) and with \((\gamma, M) \neq 0\). Modern discussion of the Chaplygin’s transformation may be found in [9].

Namely, integral of motion \(H_1\) is equal to

\[
H_1 = (\omega, M) = (A', M, M) = (A''(\gamma, M) + g^2(\gamma, M)(\gamma, A''M)),
\]

where \(A'\) and \(A''\) are diagonal matrices

\[
A' = A + dg^2 \begin{pmatrix}
(a_1 - a_3)(a_1 - a_2)x_1^2 & 0 & 0 \\
0 & (a_2 - a_3)(a_2 - a_1)x_2^2 & 0 \\
0 & 0 & (a_3 - a_1)(a_3 - a_2)x_3^2
\end{pmatrix}
\]

and

\[
A'' = \begin{pmatrix}
a_1(a_1 + a_3) - a_2a_3 & 0 & 0 \\
0 & a_2(a_1 + a_3) - a_1a_3 & 0 \\
0 & 0 & a_2(a_1 + a_2) - a_1a_2
\end{pmatrix}.
\]

So, on the symplectic leaves defined by the following values of the Casimir functions

\[
C_1 = (\gamma, \gamma) = 1, \quad C_2 = (\gamma, M) = 0
\]

we can consider integrals of motion

\[
H_1 = (A', M, M), \quad H_2 = (M, M)
\]

with diagonal matrix \(A'\) instead integral \(H_1\) with non-diagonal matrix \(A''\) (2.4) when \((\gamma, M) \neq 0\).

Following [8] we can integrate equations of motion at \((\gamma, M) = 0\) using sphero-conical coordinates \(q_1\) and \(q_2\), which are the roots of the equation

\[
\frac{\gamma_1^2}{x - e_1} + \frac{\gamma_2^2}{x - e_2} = \frac{\gamma_3^2}{x - e_3} = 0, \quad e_i = \frac{d}{T_i}.
\]

These variables satisfy the following equations

\[
\begin{align*}
\dot{q}_1 &= \frac{q_2^2(1 + q_1)(e_1 - q_1)(e_2 - q_1)(e_3 - q_1)(h_2q_1 - dh_1(1 + q_1))}{d^2(q_1 - q_2)} \\
\dot{q}_2 &= \frac{q_1^2(1 + q_2)(e_1 - q_2)(e_2 - q_2)(e_3 - q_2)(h_2q_2 - dh_1(1 + q_2))}{d^2(q_2 - q_1)}
\end{align*}
\]

(2.9)

where \(h_{1,2}\) are values of the integrals of motion \(H_{1,2}\). These equations can be reduced to the Abel quadratures after suitable change of time [8]. Theta function solution for the Chaplygin system in this, as well as in the generic case, was obtained in [12].

We prefer to use other coordinates \(u_{1,2}\)

\[
u_1 = \frac{q_1}{d(q_1 + 1)}, \quad u_2 = \frac{q_2}{d(q_2 + 1)},
\]

which are the roots of the equation

\[
g^2 \left( \frac{\gamma_1^2(1 - da_1)}{\lambda - a_1} + \frac{\gamma_2^2(1 - da_2)}{\lambda - a_2} + \frac{\gamma_3^2(1 - da_3)}{\lambda - a_3} \right) = 0.
\]

In contrast with \(q_{1,2}\) these coordinates \(u_{1,2}\) remain well-defined when \(d = 0\) [14].

According [14] [16], let us consider Poisson map \((u_1, u_2, p_{u_1}, p_{u_2}) \rightarrow (\gamma_1, \gamma_2, \gamma_3, M_1, M_2, M_3)\):

\[
\gamma_i = \sqrt[\alpha_{ij}, \alpha_{jk}]{\frac{(1 - da_j)(1 - da_k)}{(1 - du_1)(1 - du_2)}}, \quad \frac{(u_1 - a_i)(u_2 - a_i)}{(a_j - a_i)(a_k - a_i)}, \quad i \neq j \neq k,
\]

(2.10)

\[
M_i = \frac{2\varepsilon_{ijk}\gamma_j\gamma_k(a_j - a_k)g}{u_1 - u_2} \left( (a_i - u_1)(1 - du_1)p_{u_1} - (a_i - u_2)(1 - du_2)p_{u_2} \right).
\]
on the genus two hyperelliptic curve defined by separation relations (2.13).

Summing up, we can study the motion of the Chaplygin ball when \((\gamma,M)\) is small in original Chaplygin paper \([8]\), see discussion in \([14]\).

\[
H_1 = \frac{4u_2(du_1-1)(u_1-a_2)(u_1-a_3)p_{u_1}^2}{u_1-u_2} + \frac{4u_1(du_2-1)(u_2-a_1)(u_2-a_3)p_{u_2}^2}{u_2-u_1},
\]

and separation relations

\[
4(du_k-1)(a_1-u_k)(a_2-u_k)(a_3-u_k)p_{u_k}^2 + u_kH_2 - H_1 = 0, \quad k = 1, 2.
\]

The corresponding equations of motion are equal to

\[
\dot{u}_1 = \frac{4gu_2(a_1-u_1)(a_2-u_1)(a_3-u_1)(du_1-1)p_{u_1}}{u_1-u_2},
\]

\[
\dot{u}_2 = \frac{4gu_1(a_1-u_2)(a_2-u_2)(a_3-u_2)(du_2-1)p_{u_2}}{u_2-u_1}.
\]

After change of time \(t \to \tau\)

\[
d\tau = gd\tau, \quad g = \sqrt{\frac{(1-du_1)(1-du_2)}{1-da_1)(1-da_2)(1-da_3)}}
\]

equations (2.14) yield Abel quadratures

\[
\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 4d\tau, \quad \frac{u_1du_1}{\sqrt{f(u_1)}} + \frac{u_2du_2}{\sqrt{f(u_2)}} = 0,
\]

on the genus two hyperelliptic curve

\[
X : \quad y^2 = f(x), \quad f(x) = 4(1-dx)(a_1-x)(a_2-x)(a_3-x)(xh_2-h_1)
\]

defined by separation relations (2.13).

Remark 2 On the any step here we can put \(d = 0\) and consider standard Euler top in contrast with equations (2.9) and Abel’s quadratures in original Chaplygin paper [8], see discussion in [14].

Summing up, we can study the motion of the Chaplygin ball when \((\gamma,M) = 0\) and then apply Chaplygin’s transformation in order to describe motion when \((\gamma,M) \neq 0\) \([5,8,9]\). It allows us to consider discretizations of the Chaplygin ball for \((\gamma,M) = 0\) and then apply Chaplygin’s transformation in order to describe its discretization for \((\gamma,M) \neq 0\).

3 Discretization of the Chaplygin ball motion at \((\gamma,M) = 0\)

Discretization of the Chaplygin ball motion, which preserves the same first integrals as the continuous model, except the energy, was obtained in [11] in the framework of the formalism of variational integrators (discrete Lagrangian systems). Our intermediate aim is to describe discretizations of the Chaplygin ball motion preserving all the integrals of motion in the framework of the standard arithmetic of divisors on a hyperelliptic curve.

Suppose that transformation of variables

\[
B : \quad (u_1, u_2, p_{u_1}, p_{u_2}) \to (\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})
\]
It allows us to relate transformations of variables $B$ where $X$ if we put

$$X$$

Let us give some brief background on arithmetics of divisors \[10, 13\].

### 3.1 Arithmetic of divisors

A divisor is a finite formal linear combination

$$D = \sum_i m_i Z_i, \quad m_i \in \mathbb{Z},$$

of prime divisors. **The group of divisors on $X$, which is the free group on the prime divisors, is denoted $\text{Div}X$.**

The group of divisors $\text{Div}X$ is an additive abelian group under the formal addition rule

$$\sum m_i Z_i + \sum n_i Z_i = \sum (m_i + n_i)Z_i.$$  

To define an equivalence relation on divisors we use the rational functions on $X$. Function $f$ is a quotient of two polynomials; they are each zero only on a finite closed subset of codimension one in $X$, which is therefore the union of finitely many prime divisors. The difference of these two subsets define a principal divisor $\text{div} f$ associated with function $f$. The subgroup of $\text{Div}X$ consisting of the principal divisors is denoted by $\text{Prin}X$.  

If we put

$$x_{1,2} = u_{1,2}, \quad y_{1,2} = 4(du_{1,2} - 1)(a_1 - u_{1,2})(a_2 - u_{1,2})(a_3 - u_{1,2}) p_{u_{1,2}},$$

$$x_{1,2}'' = \tilde{u}_{1,2}, \quad y_{1,2}'' = -4(\tilde{d}u_{1,2} - 1)(a_1 - \tilde{u}_{1,2})(a_2 - \tilde{u}_{1,2})(a_3 - \tilde{u}_{1,2}) \tilde{p}_{u_{1,2}},$$

into the difference of (2.15) and (3.18), one gets a system of Abel’s differential equations

$$\omega_1(x_1, y_1) + \omega_1(x_2, y_2) + \omega_1(x_1'', y_1'') + \omega_1(x_2'', y_2'') = 0,$$

$$\omega_2(x_1, y_1) + \omega_2(x_2, y_2) + \omega_2(x_1'', y_1'') + \omega_2(x_2'', y_2'') = 0,$$

where $\omega_1, 2$ are holomorphic differentials on hyperelliptic curve $X$ of genus two

$$\omega_1(x, y) = \frac{dx}{y}, \quad \omega_2(x, y) = \frac{xdx}{y}.$$  

According \cite{1} solutions of Abel’s equations \cite{1.20} are points of intersection of the curve $X$ with another curve $Y$ not containing $X$

$$Y: \quad y = \mathcal{P}(x), \quad \mathcal{P}(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0.$$  

Thus, we have an intersection divisor of $X$ and $Y$ defined by the following four points on the plane

$$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2) \quad \text{and} \quad P_1'' = (x_1'', y_1''), \quad P_2'' = (x_2'', y_2'').$$  

It allows us to relate transformations of variables $B$ \cite{1.17} with the standard arithmetic of divisors.

By viewing the new variables $\tilde{u}_{1,2}, \tilde{p}_{u_{1,2}}$ as the old one $u_{1,2}, p_{u_{1,2}}$ but computed at the next time-step, then transformation $B$ becomes a discretization of continuous Chaplygin ball motion.
Definition 2 Two divisors $D, D' \in \text{Div}_X$ are linearly equivalent

$$D \approx D'$$

if their difference $D - D'$ is principal divisor

$$D - D' = \text{div}(f) \equiv 0 \mod \text{Prin}_X.$$ 

The Picard group of $X$ is the quotient group

$$\text{Pic}_X = \frac{\text{Div}_X}{\text{Prin}_X} = \frac{\text{Divisors defined over } k}{\text{Divisors of functions defined over } k}.$$ 

For a general (not necessarily smooth) variety $X$, what we have defined is not the Picard group, but the Weil divisor class group. For an irreducible normal variety $X$, the Picard group is isomorphic to the group of Cartier divisors modulo linear equivalence. For a thorough treatment see [10, 13].

The Picard group is a group of divisors modulo principal divisors, and the group operation is formal addition modulo the equivalence relations. These group operations define so-called arithmetic of divisors in Picard group

$$D + D' = D'' \quad \text{and} \quad [\ell]D = D'',$$ (3.22)

where $D, D'$ and $D''$ are divisors, $+$ and $[\ell]$ denote addition and scalar multiplication by an integer, respectively.

Remark 3 There are some other equivalence relations on divisors, for instance homological equivalence, numerical equivalence, algebraic equivalence or rational equivalence, which are different from the linear equivalence $D \approx D'$ [10]. It allows us to define relations between divisors which are different from the standard arithmetic relations (3.22).

Let $X$ be a hyperelliptic curve of genus $g$ defined by equation

$$y^2 + h(x)y = f(x),$$ (3.23)

where $f(x)$ is a monic polynomial of degree $2g + 2$ with distinct roots, $h(x)$ is a polynomial with $\text{deg} h \leq g$. Prime divisors are rational point on $X$ denoted $P_i = (x_i, y_i)$, and $P_\infty$ is a point at infinity.

Definition 3 Divisor $D = \sum m_i P_i$, $m_i \in \mathbb{Z}$ is a formal sum of points on the curve, and degree of divisor $D$ is the sum $\sum m_i$ of the multiplicities of points in support of the divisor

$$\text{supp} \left( \sum m_i P_i \right) = \bigcup_{m_i \neq 0} P_i.$$ 

The degree is a group homomorphism $\text{deg} : \text{Div}_X \to \mathbb{Z}$. Its kernel is denoted by

$$\text{Div}^0 X = \{D \in \text{Div}_C : \text{deg}D = 0\}.$$ 

Quotient group of $\text{Div}_X$ by the group of principal divisors $\text{Prin}_X$ is called the divisor class group or Picard group. Restricting to degree zero, we also define $\text{Pic}^0 X = \text{Div}^0 X/\text{Prin}_X$. The groups $\text{Pic}_X$ and $\text{Pic}^0 X$ carry essentially the same information on $X$, since we always have

$$\text{Pic}_X/\text{Pic}^0 X \cong \text{Div}_X/\text{Div}^0 X \cong \mathbb{Z}.$$ 

The divisor class group, where the elements are equivalence classes of degree zero divisors on $X$, is isomorphic to the Jacobian of $X$. By abuse of notation, a divisor and its class in $\text{Pic}_X$ will usually be denoted by the same symbol.

In order to describe elements of Jacobian we can use semi-reduced divisors.

Definition 4 A semi-reduced divisor is divisor of the form

$$D = \sum m_i P_i - \left( \sum m_i \right) P_\infty = E - \left( \sum m_i \right) P_\infty,$$

where $m_i > 0$, $P_i \neq -P_j$ for $i \neq j$, no $P_i$ satisfying $P_i = -P_i$ appears more than once and $E$ is an effective divisor.
For each divisor $\tilde{D} \in \text{Div}^0 X$ there is a semi-reduced divisor $D$ so that $D \approx \tilde{D}$. However semi-reduced divisors are not unique in their equivalence class. In [20] we used this fact in order to get auto Bäcklнд transformations associated with an equivalence relations.

**Definition 5** A semi-reduced divisor $D$ is called reduced if $\sum m_i \leq g$, i.e. if the sum of multiplicities is no more that genus of curve $C$. The reduced degree or weight of reduced divisor $D$ is defined as $w(D) = \sum m_i$.

This is a consequence of Riemann-Roch theorem for hyperelliptic curves that for each divisor $\tilde{D} \in \text{Div}^0 X$ there is a unique reduced divisor $D$ so that $D \approx \tilde{D}$.

### 3.2 One example of discretization

Let us consider adding two full degree genus two divisors

$$D + D' = D''$$

with respective supports $\text{supp}(D) = \{P_1, P_2\} \cup \{P_\infty\}$ and $\text{supp}(D') = \{P'_1, P'_2\} \cup \{P_\infty\}$ such that no $P_i$ has the same $x$-coordinate as $P'_j$. By definition cubic polynomial

$$P(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0$$

interpolates four points $P_1, P_2, P'_1, P'_2$ and, therefore, it has the standard form

$$P(x) = \frac{(x - x_1)(x - x'_1)(x - x_2)}{(x - x'_1)(x_1 - x_2)} y_1 + \frac{(x - x'_2)(x - x_1)}{(x_2 - x'_2)(x_1 - x_2)} y_2 + \frac{(x - x_2)(x - x_1)}{(x_2 - x_1)(x_2 - x'_1)} y'_1 + \frac{(x - x'_1)(x - x_2)}{(x_2 - x'_1)(x_2 - x'_2)} y'_2,$$

due to Lagrange interpolation. Substituting $y = P(x)$ into the definition of genus two hyperelliptic curve $X$

$$y^2 = f(x), \quad f(x) = a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0,$$

we obtain the so-called Abel polynomial \[11\]

$$\psi(x) = P(x)^2 - f(x) = (b_3^2 - a_6)(x - x_1)(x - x'_1)(x - x_2)(x - x'_2)(x - x''_1)(x - x''_2).$$

Equating coefficients of $\psi$ gives abscissas of points $P''_1$ and $P''_2$:

$$x''_1 + x''_2 = -x_1 - x_2 - x'_1 - x'_2 + \frac{a_5 - 2 b_2 b_3}{b_3^2 - a_6}, \quad (3.25)$$

$$x''_1 x''_2 = \frac{2 b_1 b_3 + b_2^2 - a_4}{b_3^2 - a_6} -(x_1 + x_2 + x'_1 + x'_2)(x''_1 + x''_2) - x_1(x_2 + x'_1 + x'_2) - x_2(x'_1 + x'_2),$$

whereas the corresponding ordinates $y''_{1,2}$ are equal to

$$y''_{1,2} = -P(x''_{1,2}). \quad (3.26)$$

Using explicit formulae for the corresponding transformation $B$ \[3.17\] we can prove the following statement.

**Proposition 1** Equations \[3.19\] and \[3.25-3.26\] determine transformation $B : (u, p_u) \rightarrow (\tilde{u}, \tilde{p}_u)$ preserving the form of the canonical Poisson bracket \(2.10\), i.e.

$$\{\tilde{u}_i, \tilde{p}_{u_j}\} = \delta_{i,j}, \quad \{\tilde{u}_1, \tilde{u}_2\} = \{\tilde{p}_{u_1}, \tilde{p}_{u_2}\} = 0.$$

This canonical transformation also preserves the form of integrals of motion \[2.12\].
The proof is a straightforward calculation.

We can rewrite addition of divisors \((\gamma, M) \rightarrow (\tilde{g}, \tilde{M})\) using \((2.10)\) and any modern computer algebra system. We do not show these bulky expressions here because the main our aim is the construction of new integrable systems instead of explicit construction of possible discretizations of the Chaplygin ball motion.

**Remark 4** Construction of integrable discretizations associated with doubling of divisors and other arithmetic operations in Jacobian is discussed in \([19, 20, 21, 22]\).

### 4 Integrable deformation of the Chaplygin ball at \((\gamma, M) = 0\)

According \([17, 18, 19]\) we can apply hidden symmetries of the generic level set of integrals of motion to construct new canonical variables and new new integrable systems on the initial phase space. Indeed, let us consider transformations \(B\) \((3.25, 3.26)\) when \(P_1^* = P_\infty\) and \(P_2^* = (d^{-1}, 0)\) is the ramification point. In this case \(B\) has the following form in original variables

\[
\begin{align*}
\tilde{g}_1 &= \frac{1}{(a_1 d - 1)\tilde{g}^2} \left( \gamma_1^2 - \frac{(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)(M_2^2 + M_3^2)}{M_1^2 + M_2^2 + M_3^2} \right), \\
\tilde{g}_2 &= \frac{1}{(a_2 d - 1)\tilde{g}^2} \left( \gamma_2^2 - \frac{(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)(M_1^2 + M_3^2)}{M_1^2 + M_2^2 + M_3^2} \right), \\
\tilde{g}_3 &= \frac{1}{(a_3 d - 1)\tilde{g}^2} \left( \gamma_3^2 - \frac{(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)(M_1^2 + M_2^2)}{M_1^2 + M_2^2 + M_3^2} \right), \\
\tilde{M}_1 &= \frac{g}{\tilde{g}} \left( -\frac{\tilde{g}_1^2 + \tilde{g}_2^2}{\tilde{g}_1}\gamma_2\gamma_3 M_1 + \frac{\gamma_1\gamma_2\gamma_3 M_2}{\tilde{g}_1} + \frac{\gamma_1\gamma_2\tilde{g}_3 M_3}{\tilde{g}_2} \right), \\
\tilde{M}_2 &= \frac{g}{\tilde{g}} \left( \frac{\gamma_1\gamma_2\gamma_3 M_1}{\tilde{g}_1} - \frac{\tilde{g}_1^2 + \tilde{g}_3^2}{\tilde{g}_1}\gamma_1\gamma_3 M_2 + \frac{\gamma_1\gamma_2\gamma_3 M_3}{\tilde{g}_1} \right), \\
\tilde{M}_3 &= \frac{g}{\tilde{g}} \left( \frac{\gamma_1\gamma_2\gamma_3 M_1}{\tilde{g}_1} + \frac{\gamma_1\gamma_2\gamma_3 M_2}{\tilde{g}_1} - \frac{\tilde{g}_1^2 + \tilde{g}_2^2}{\tilde{g}_1}\gamma_1\gamma_2 M_3 \right),
\end{align*}
\]

(4.27)

where \(\tilde{g}\) is the conformal factor \((2.5)\) in \(\gamma\)-variables

\[
\tilde{g} = \frac{1}{\sqrt{1 - d(\gamma, A\gamma)}} = \frac{1}{\sqrt{1 - d(a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2)}},
\]

which can be also rewritten as a function on the original conformal factor and integrals of motion

\[
\tilde{g}^2 = \frac{1}{g^2} \left( \frac{dH_1 - H_2}{(a_1 d - 1)(a_2 d - 1)(a_3 d - 1)H_2} \right).
\]

This transformation of variables has the following properties:

**Proposition 2** If \((\gamma, M) = 0\) then transformation \((4.27)\) preserves the form of the Poisson bivector \(\Pi\) \((2.7)\), the form of the bracket \((2.8)\) and the form of integrals of motion, i.e.

\[
C_1 = (\gamma, \gamma) = (\tilde{\gamma}, \tilde{\gamma}) , \quad C_2 = (\gamma, M) = (\tilde{\gamma}, \tilde{M}) = 0,
\]

and

\[
H_1 = (A_\gamma M, M) = (A_\tilde{\gamma}\tilde{M}, \tilde{M}) , \quad H_2 = (M, M) = (\tilde{M}, \tilde{M}) .
\]

The proof is a straightforward calculation.

**Remark 5** At \(d = 0\) conformal factor is equal to unity \(g = 1\) and transformation \((4.27)\) preserves canonical Poisson brackets on cotangent bundle to sphere \(T^*S^2\)

\[
\{M_i, M_j\} = \varepsilon_{ijk} M_k , \quad \{M_i, x_j\} = \varepsilon_{ijk} x_k , \quad \{x_i, x_j\} = 0 .
\]

(4.28)
It also preserves the form of Casimir functions \((\gamma, \gamma) = (\hat{\gamma}, \hat{\gamma}), (\gamma, M) = (\hat{\gamma}, \hat{M}) = 0\) and integrals of motion
\[
H_1 = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 = a_1 \hat{M}_1^2 + a_2 \hat{M}_2^2 + a_3 \hat{M}_3^2, \quad H_2 = M_1^2 + M_2^2 + M_3^2 = \hat{M}_1^2 + \hat{M}_2^2 + \hat{M}_3^2.
\]

Using this hidden symmetry of the level set manifold we can construct a new integrable system on cotangent bundle \(T^*S^2\).

Let \((\bar{u}_1, \bar{u}_2, \bar{p}_{u_1}, \bar{p}_{u_2})\) are images of coordinates \((u_1, u_2, p_{u_1}, p_{u_2})\) after transformation (4.27). Suppose that two functions
\[
\lambda_k = 4(d\hat{u}_k - 1)(a_1 - \hat{u}_k)(a_2 - \hat{u}_k)(a_3 - \hat{u}_k)\hat{p}_{\hat{u}_k}^2
\]
are eigenvalues of the recursion operator \(N = \Pi'\Pi^{-1}\), where
\[
\Pi' = \begin{pmatrix}
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & \lambda_2 \\
-\lambda_1 & 0 & 0 & 0 \\
0 & -\lambda_2 & 0 & 0
\end{pmatrix}
\]
(4.29)
is the Poisson bivector in \((\bar{u}_1, \bar{u}_2, \bar{p}_{u_1}, \bar{p}_{u_2})\) variables. The Nijenhuis torsion of \(N\) vanishes as a consequence of the compatibility between \(\Pi'\) and \(\Pi\) (2.11).

So, we have a Poisson-Nijenhuis manifold (partial case of bi-Hamiltonian manifolds) endowed with a pair of compatible non-degenerate Poisson brackets. On such manifold functions we can determine functions \(J_m = 1/2mtrN\) satisfying the Lenard relations
\[
\Pi'dJ_i = \Pi dJ_{i+1},
\]
which are in involution with respect to both Poisson brackets.

In our case first such integral of motion is equal to
\[
\hat{H} = \lambda_1 + \lambda_2 = (\hat{\omega}, M) = (\hat{A}',M,M), \quad \hat{A}' = \frac{A - (\gamma, A\gamma)E}{2} + d\gamma g^2 B, \quad (4.30)
\]
where \(E\) is a unit matrix and
\[
B_{\gamma} = \begin{pmatrix}
(a_1 - a_3)(a_1 - a_2)x_1^2 & 0 & 0 \\
0 & (a_2 - a_3)(a_2 - a_1)x_2^2 & 0 \\
0 & 0 & (a_3 - a_1)(a_3 - a_2)x_3^2
\end{pmatrix}
\]

Second integral of motion is the following polynomial of fourth order in momenta
\[
\hat{K} = \lambda_1 \lambda_2 = 4g^2(a_1 x_1 M_1 + a_2 x_2 M_2 + a_3 x_3 M_3)^2 \left((1 - a_1 d - g^2 d^2(a_1 - a_3)(a_1 - a_2)x_1^2)M_1^2 + (1 - a_2 d - g^2 d^2(a_2 - a_1)(a_2 - a_3)x_2^2)M_2^2 + (1 - a_3 d - g^2 d^2(a_3 - a_1)(a_3 - a_2)x_3^2)M_3^2 \right) + \left((1 - a_1 d - g^2 d^2(a_1 - a_3)(a_1 - a_2)x_1^2)M_1^2 + (1 - a_2 d - g^2 d^2(a_2 - a_1)(a_2 - a_3)x_2^2)M_2^2 + (1 - a_3 d - g^2 d^2(a_3 - a_1)(a_3 - a_2)x_3^2)M_3^2 \right),
\]
or
\[
\hat{K} = 4g^2(A, M)^2 \left((E - dA - d^2 g^2 B, M, M) \right). \quad (4.31)
\]

It is easy to prove that functions \(\hat{H}\) and \(\hat{K}\) are in involution with respect to the Poisson bracket (2.8) and Poisson bracket associated with bivector \(\Pi'\) (4.29).

In this case new angular velocity of the ball
\[
\hat{\omega} = \omega + \Delta\omega = A_\gamma M + \frac{A + (\gamma, A\gamma)E}{2} M
\]
is the additive deformation of initial angular velocity \(\omega = A_\gamma M\), that allows us to say about integrable deformation of the Chaplygin ball at \((\gamma, M) = 0\). It will be interesting to study a physical meaning of the corresponding nonholonomic model.
4.1 Integrable Hamiltonian systems on the sphere

If \( d = 0 \), then the integrals of motion (4.30-4.31) are equal to

\[
\vec{H} = (\gamma, A\gamma)(M, AM) - (\gamma, A\gamma)(M, M), \quad \vec{K} = (\gamma, A\gamma)^2(M, M).
\]

These functions are in involution with respect to the canonical Poisson bracket (4.22) and, therefore, determine integrable Hamiltonian systems on cotangent bundle to unit sphere \( T^*\mathbb{S}^2 \). We can also add some potentials to \( \vec{H} \), for instance,

\[
\vec{H}' = \vec{H} + a(\gamma, A\gamma) + b(\gamma, A\gamma)^3
\]

without loss of integrability. Integrable geodesic flow associated with \( \vec{H} \) has been found in \cite{22} together with another geodesic flows with integrals

\[
\vec{H} = (\gamma, A\gamma)\vec{H} - (\gamma, A\gamma)(\gamma, AM)^2, \quad \vec{K} = (\gamma, AM)^2\left((\gamma, A\gamma)(M, AM) - (\gamma, AM)^2\right).
\]

One more integrable geodesic flow we can obtain directly applying arithmetic of divisors associated another equivalence relation, similar bi-Hamiltonian systems on the plane are discussed in \cite{20}. Indeed, let us consider the free motion on the sphere with integrals

\[
H_1 = M_1^2 + M_2^2 + M_3^2, \quad H_2 = -(a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2).
\]

In elliptic coordinates \( u_{1,2} \) and momenta \( p_{u_{1,2}} \),

\[
\gamma_i = \sqrt{\frac{(u_1 - a_i)(u_2 - a_i)}{(a_j - a_i)(a_k - a_i)}}, \quad i \neq j \neq k,
\]

\[
M_i = \frac{2\pi i j k}{u_1 - u_2} \frac{\gamma_j \gamma_k (a_j - a_k)}{(a_i - u_1)p_{u_1} - (a_i - u_2)p_{u_2}).
\]

these integrals read as

\[
H_1 = \frac{\phi_1 p_{u_1}^2}{u_1 - u_2} + \frac{\phi_2 p_{u_2}^2}{u_2 - u_1}, \quad H_2 = \frac{u_2 \phi_1 p_{u_1}^2}{u_1 - u_2} - \frac{u_1 \phi_2 p_{u_2}^2}{u_2 - u_1},
\]

where

\[
\phi_k = 4(a_1 - u_k)(a_2 - u_k)(a_3 - u_k), \quad k = 1, 2.
\]

The corresponding separation relations determine genus one hyperelliptic curve

\[
X : \quad y^2 = f(x), \quad f(x) = 4(a_1 - x)(a_2 - x)(a_3 - x)(H_1 x + H_2).
\]

Here \( x = u_{1,2} \) and \( y = \phi_{1,2} p_{u_{1,2}} \) and Abel’s quadratures on this curve have the form

\[
\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 0, \quad \frac{u_1 du_1}{\sqrt{f(u_1)}} + \frac{u_2 du_2}{\sqrt{f(u_2)}} = 2dt.
\]

In order to reduce non-trivial second quadrature to the standard holomorphic form

\[
\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 2d\tau
\]

we have to change time \( t \to \tau \) and replace the canonical Poisson bracket \{\ldots\} to the bracket \{\ldots\}_W

\[
\{u_i, p_{u_j}\}_W = u_i \delta_{ij}, \quad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0,
\]

see details in \cite{20}.

Supposing that variables \( \tilde{u}_{1,2} \) and \( \tilde{p}_{u_{1,2}} \) satisfy to the same separation relations and quadratures (4.33), one gets Abel’s differential equation

\[
\omega(x_1, y_1) + \omega(x_2, y_2) + \omega(x_1'' , y_1'') + \omega(x_2'' , y_2'') = 0
\]
where \( \omega = dx/y \) is a holomorphic differential on an elliptic curve and

\[
x_{1,2} = u_{1,2}, \quad y_{1,2} = \phi_{1,2}p_{u_{1,2}}, \quad x'_{1,2} = \tilde{u}_{1,2}, \quad y'_{1,2} = -\tilde{\phi}_{1,2}\tilde{p}_{u_{1,2}}.
\]

According to [1], solutions of this equation are the coordinates of points of intersection \( P_{1,2} = (x_{1,2}, y_{1,2}) \) and \( P'_{1,2}(x'_{1,2}, y'_{1,2}) \) of elliptic curve \( X \) with the plane curve (conic section)

\[
Y : \quad y = \mathcal{P}(x), \quad \mathcal{P}(x) = b_2x^2 + b_1x + b_0.
\]

The elimination of coefficients \( b_i \) leads to determinant

\[
\begin{vmatrix}
x_1^2 & x_1 & 1 & y_1 \\
x_2^2 & x_2 & 1 & y_2 \\
x_{1}'^2 & x_1' & 1 & y_1' \\
x_{2}'^2 & x_2' & 1 & y_2'
\end{vmatrix} = 0,
\]

as the integral relation corresponding to Abel’s differential equation (4.35).

We can determine coefficients \( b_i \) using a standard method [10] [13]. By definition four points \( P_1, P_2, P_1', P_2' \) form an intersection divisor of \( X \) and \( Y \)

\[
X \cdot Y = \sum_{P \in X \cap Y} \text{mult}_P(X, Y) P = P_1 + P_2 + P_1' + P_2'.
\]

Note that it is symmetric in \( X \) and \( Y \), so the result can be considered as an element of Picard groups \( \text{Pic}_X \) and \( \text{Pic}_Y \), simultaneously.

In the previous Section we consider the addition of divisors in Picard group of hyperelliptic curve \( X \)

\[
D + D' = D'', \quad D, D', D'' \in \text{Pic}^0 X,
\]

where \( \text{supp}(D) = \{ P_1, P_2 \} \cup \{ P_\infty \} \) and \( \text{supp}(D'') = \{ P_1', P_2' \} \cup \{ P_\infty \} \). Now we consider similar addition of divisors in the second Picard group of conic section \( Y \)

\[
D + D' = D'', \quad D, D', D'' \in \text{Pic}^0 Y.
\]

Here \( \text{supp}D' = \{ P_1' \} \cup \{ P_\infty \} \) and \( P_1' = (x_1', y_1') \) is a point on the plane out of \( X \). Parabola \( Y \) interpolates \( P_1, P_2 \) and \( P_1' \)

\[
y_1 = \mathcal{P}(x_1), \quad y_2 = \mathcal{P}(x_2), \quad y_1' = \mathcal{P}(x_1').
\]

According to [20], we take \( P_1' = (0, 0) \) and substitute \( y = \mathcal{P}(x) \), where

\[
\mathcal{P}(x) = x \left( \frac{(x-x_2)y_1}{(x-x_1)y_2} + \frac{(x-x_1)y_2}{(x-x_2)y_1} \right)
\]

due to Lagrange interpolation, into the equation \( y^2 - f(x) = 0 \) for \( X \). As a result we obtain Abel’s polynomial on \( x \)

\[
\psi(x) = \mathcal{P}^2 - f(x) = A_2(x-x_1)(x-x_2)(x-x_1')(x-x_2'),
\]

having four zeroes at the points of intersection. Equating coefficients of \( \psi \) gives abscissas \( x_1'' \) and \( x_2'' \), whereas ordinates are equal to

\[
y_k'' = -\mathcal{P}(x_k''), \quad k = 1, 2.
\]

In our case

\[
A_2 = \frac{(4u_2^2(u_2 - u_1) - \phi_1p_{u_1})\phi_1p_{u_1}}{u_1^2(u_2 - u_1)^2} + \frac{2\phi_1\phi_2p_{u_1}p_{u_2}}{u_1u_2(u_2 - u_1)^2} + \frac{(4u_2^2(u_2 - u_1) - \phi_2)\phi_2p_{u_2}^2}{u_2^2(u_2 - u_1)^2}
\]

and desired variables \( \tilde{u}_{1,2} = x_{1,2}' \) are the roots of the polynomial

\[
\tilde{U}(x) = \frac{\psi(x)}{(x-x_1)(x-x_2)} = A_2x^2 + A_1x + 4a_1a_2a_3 \left( \frac{\phi_1p_{u_1}^2}{u_1(u_2 - u_1)} + \frac{\phi_2p_{u_2}^2}{u_2(u_1 - u_2)} \right),
\]

(438)
where
\[ A_1 = \frac{(4u_1^2(a_1 + a_2 + a_3 - u_1) - \phi_1)p_{u_1}^2}{u_1^2(u_1 - u_2)} + \frac{(4u_2^2(a_1 + a_2 + a_3 - u_2) - \phi_2)p_{u_2}^2}{u_2^2(u_2 - u_1)}. \]

Using equations (4.37, 4.38) we obtain \( x_{1,2}'' \) and \( y_{1,2}'' \) as functions on \( x_{1,2} \) and \( y_{1,2} \). So, we can explicitly determine auto Bäcklund transformations for the motion on the sphere governed by Hamiltonians \( H_{1,2} \).

**Proposition 3** If variables \( \tilde{u}_{1,2} \) are the roots of polynomial (4.38) and
\[ \tilde{p}_{u_k} = -\frac{x}{4(a_1 - x)(a_2 - x)(a_3 - x)} \left( \frac{(x - u_2)\phi_1 p_{u_1}}{(u_1 - u_2)u_1} + \frac{(x - u_1)\phi_2 p_{u_2}}{(u_2 - u_1)u_2} \right) \bigg|_{x=\tilde{u}_k}, \]
then mapping \((u, p_u) \to (\tilde{u}, \tilde{p}_u)\) preserves the form of Hamiltonians \( H_{1,2} \) (4.32), Abel’s quadrature (4.33) and the form of the Poisson bracket \{., \} \( W \) (4.34), i.e.
\[ \{ \tilde{u}_1, \tilde{p}_{u_j} \}_W = \tilde{u}_1 \delta_{ij}, \quad \{ \tilde{u}_1, \tilde{u}_2 \}_W = \{ \tilde{p}_{u_1}, \tilde{p}_{u_2} \}_W = 0. \]

The proof is a straightforward calculation.

According [20], in order to get new canonical variables on \( T^*S^2 \) we can use additional Poisson map
\[ \rho : (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1 p_{u_1}, u_2 p_{u_2}), \]
which reduces canonical Poisson bracket \{., \} \( (2.11) \) to bracket \{., \} \( W \) (4.31). Indeed, using the composition of Poisson mappings \( \rho \) and \((u, p_u) \to (\tilde{u}, \tilde{p}_u)\) we determine variables \( \tilde{u}_{1,2} \), which are the roots of the polynomial
\[ \rho(\hat{U})(x) = \rho(A_2)x^2 + \rho(A_1)x + 4a_1a_2a_3 \left( \frac{u_1\phi_1 p_{u_1}^2}{u_2 - u_1} + \frac{u_2\phi_2 p_{u_2}^2}{u_1 - u_2} \right) = 0 \tag{4.39} \]
The corresponding momenta are equal to
\[ \tilde{p}_{u_k} = -\frac{1}{4(a_1 - \tilde{u}_k)(a_2 - \tilde{u}_k)(a_3 - \tilde{u}_k)} \left( \frac{\tilde{u}_k - u_2)\phi_1 p_{u_1}}{(u_1 - u_2)u_1} + \frac{\tilde{u}_k - u_1)\phi_2 p_{u_2}}{(u_2 - u_1)u_2} \right) \tag{4.40} \]
The straightforward calculation allows us to prove the following statement.

**Proposition 4** Canonical Poisson bracket
\[ \{ u_i, p_{u_j} \} = \delta_{ij}, \quad \{ u_1, u_2 \} = \{ p_{u_1}, p_{u_2} \} = 0 \]
has the same form
\[ \{ \tilde{u}_i, \tilde{p}_{u_j} \} = \delta_{ij}, \quad \{ \tilde{u}_1, \tilde{u}_2 \} = \{ \tilde{p}_{u_1}, \tilde{p}_{u_2} \} = 0 \]
in variables \( \tilde{u}_{1,2} \) and \( \tilde{p}_{u_{1,2}} \) (4.39, 4.40).

Let \((\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})\) are images of coordinates \((u_1, u_2, p_{u_1}, p_{u_2})\) after transformation (4.39, 4.40). Suppose that two functions
\[ \nu_k = 4\tilde{u}_k(a_1 - \tilde{u}_k)(a_2 - \tilde{u}_k)(a_3 - \tilde{u}_k)p_{u_k}^2 \]
are eigenvalues of the recursion operator \( N = \Pi'\Pi^{-1} \), where
\[ \Pi' = \begin{pmatrix} 0 & 0 & \nu_1 & 0 \\ 0 & 0 & 0 & \nu_2 \\ -\nu_1 & 0 & 0 & 0 \\ 0 & -\nu_2 & 0 & 0 \end{pmatrix} \]
is the Poisson bivector in \((\tilde{u}_1, \tilde{u}_2, \tilde{p}_{u_1}, \tilde{p}_{u_2})\) variables. The Nijenhuis torsion of \( N \) vanishes as a consequence of the compatibility between \( \Pi' \) and \( \Pi \) and, therefore, the following functions
\[ \hat{H} = \frac{\nu_1 + \nu_2}{2} = \sum_{i,j=1}^{3} \hat{g}_{ij} p_{u_i} p_{u_j}, \]
where

\[
\hat{g} = \begin{pmatrix}
\phi_1 u_1 (2u_1 + u_2 - u_1 u_2 (a_1^{-1} + a_2^{-1} + a_3^{-1})) & 0 & \frac{u_1 u_2 (u_1 + 2u_2 - u_1 u_2 (a_1^{-1} + a_2^{-1} + a_3^{-1}))}{u_2 - u_1} \\
 u_1 - u_2 & \phi_2 u_2 (u_1 + 2u_2 - u_1 u_2 (a_1^{-1} + a_2^{-1} + a_3^{-1})) & 0 \\
 0 & \frac{u_1 u_2 (u_1 + 2u_2 - u_1 u_2 (a_1^{-1} + a_2^{-1} + a_3^{-1}))}{u_2 - u_1}
\end{pmatrix}
\]

and

\[
\hat{K} = \frac{\phi_1 \phi_2}{a_1 a_2 a_3} \left( \frac{u_1^2 p_{1u} - u_2^2 p_{2u}}{u_1 - u_2} \right)^2 \frac{\phi_1 u_1 p_{1u}^2 - \phi_2 u_2 p_{2u}^2}{u_1 - u_2},
\]

are in the involution with respect to the canonical Poisson bracket on \(T^*S^2\).

In terms of original variables \(\gamma\) and \(M\) functions \(\hat{H}\) and \(\hat{K}\) have more complicated form.

**Proposition 5** If \((\gamma, \gamma) = 1\), \((\gamma, M) = 0\) and \(A\) is the nonsingular diagonal matrix then functions

\[
\hat{H} = \alpha (M, A M) - \left( (\gamma, A^\vee \gamma) - \alpha (\gamma, A \gamma) + \alpha \text{tr} A \right) \cdot (M, M),
\]

where

\[
\alpha = (\gamma, A^\vee \gamma) \text{tr} A^{-1} + 2(\gamma, A \gamma) - 2 \text{tr} A,
\]

and

\[
\hat{K} = \left( (\gamma, A^\vee \gamma) \cdot (\gamma, A M) - ((\gamma, A \gamma) - \text{tr} A) \cdot (\gamma, A^\vee M) \right)^2 \times \left( (M, A M) + ((\gamma, A \gamma) - \text{tr} A) \cdot (M, M) \right),
\]

are in involution with respect to the canonical Poisson bracket \((4.28)\). Here \(A^\vee = A^{-1} \cdot \det A\) is the cofactor matrix and \(\text{tr} A\) is a trace of matrix \(A\).

The proof is a straightforward calculation.

Construction of geodesic flows on Riemannian manifolds is a classical object \([2]\). A particular place among them is occupied by integrable geodesic flows. We obtain integrable geodesic flow on the sphere \(S^2\) with quartic in momenta integral of motion which is absent in the list of known integrable systems on the sphere.

The work was supported by the Russian Science Foundation (project 15-11-30007).

**References**

[1] Abel N. H., Mémoire sure une propriété générale d’une classe très étendue des fonctions transcendantes, Œuvres complétes, Tom I, Grondahl Son, Christiania (1881), pp.145-211.

[2] Bolsinov A.V., Jovanović B., Integrable geodesic flows on Riemannian manifolds: construction and obstructions, In book: Contemporary geometry and related topics, World Sci. Publ., River Edge, NJ, (2004), pp.57-103.

[3] Borisov A.V., Mamaev I.S., The Chaplygin problem of the rolling motion of a ball is Hamiltonian, *Math. Notes*, 2001, vol. 70, n. 5, pp. 720-723.

[4] Borisov A.V., Mamaev I.S., Conservation Laws, Hierarchy of Dynamics and Explicit Integration of Nonholonomic Systems, *Regular and Chaotic Dynamics*, 2008, vol. 13, no. 5, pp. 443-490.

[5] Borisov A. V., Kilin A. A., Mamaev I. S. Generalized Chaplygin’s Transformation and Explicit Integration of a System with a Spherical Support, *Regular and Chaotic Dynamics*, 2012, vol. 17, no. 2, pp. 170-190.

[6] Borisov A. V., Mamaev I. S., Bizyaev I. A., The Hierarchy of Dynamics of a Rigid Body Rolling without Slipping and Spinning on a Plane and a Sphere, Regular and Chaotic Dynamics, 2013, vol. 18, no. 3, pp. 277-328.
[7] Borisov A.V., Mamaev I.S., Tsiganov A.V., Non-holonomic dynamics and Poisson geometry, *Rus. Math. Surveys*, 2014, vol.69, n.3, pp.481-538.

[8] Chaplygin S.A., On a Ball’s Rolling on a Horizontal Plane, *Mathematical collection of the Moscow Mathematical Society*, 1902, vol.24, pp.139-168, (Russian) and Regul. Chaotic Dyn., 2002, vol.7, pp.131148, (English).

[9] Duistermaat, J. J., Chaplygin’s Sphere, arXiv:math/0409019v1, 2004.

[10] Eisenbud, D., Harris, J. *3264 and all that: A second course in algebraic geometry*, pp. 632. Cambridge University Press, 2016.

[11] Fedorov Y.N., A discretization of the nonholonomic Chaplygin sphere problem, *SIGMA*, 2007, vol.3, 044, 15 pages.

[12] Fedorov Y.N., A Complete Complex Solution of the Nonholonomic Chaplygin Sphere Problem, preprint, 2007-2009.

[13] Hartshorne R., Algebraic geometry, volume 52 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1977.

[14] Tsiganov A.V., Integrable Euler top and nonholonomic Chaplygin ball, *Journal of Geometric Mechanics*, vol.3, n.3, p.337-362, 2011.

[15] Tsiganov A.V., One family of conformally Hamiltonian systems, *Theor. Math. Phys.*, 2012, vol.173, pp.1481-1497.

[16] Tsiganov A.V., On the Poisson structures for the nonholonomic Chaplygin and Veselova problems, *Regular and Chaotic Dynamics*, 2012, vol.17, pp.439-450.

[17] Tsiganov A.V., Simultaneous separation for the Neumann and Chaplygin systems, *Regular and Chaotic Dynamics*, 2015, vol.20, n.1, pp.74-93.

[18] Tsiganov A.V., On the Chaplygin system on the sphere with velocity dependent potential, *J. Geom. Phys.*, 2015, vol.92, pp.94-99.

[19] Tsiganov A.V., On auto and hetero Bäcklund transformations for the Hénon-Heiles systems, *Phys. Letters A*, 2015, vol.379, pp.2903-2907.

[20] Tsiganov A.V., New bi-Hamiltonian systems on the plane, arXiv:1701.05716, 2017.

[21] Tsiganov A.V., Bäcklund transformations and divisor doubling, arXiv:1702.03642, 2017.

[22] Tsiganov A. V. Bäcklund transformations for the nonholonomic Veselova system, *Regular and Chaotic Dynamics*, 2017, vol. 22, no. 2, pp. 163-179.