Abstract: The purpose of this paper is twofold. First we consider a ruin theory approach along with risk measures in order to determine the solvency capital of long-term guarantees such as life insurances or pension products. Secondly, for such products, we challenge the definition of the Solvency Capital Requirement (SCR) under the Solvency II (SII) regulatory framework based on a yearly viewpoint. Several methods for the calculation of the solvency capital are presented. We start our study with risk measures as considered in the SII framework and then proceed with the ruin theory approach. Instead of considering the continuous time setting of the ruin theory, we consider the discrete time—the yearly basis—of the accounting viewpoint. We finally give an illustration with a fixed guaranteed rate product along with the equity, interest rate and longevity risks. The latter risk brings us to consider zero-coupon longevity bonds in which we invest the capital. We show that long-term guarantees might be overloaded under the SII regulation.

Keywords: solvency capital, risk measures, ruin theory, longevity, life insurance, pension funds, Solvency II

MSC: 91B30

1 Introduction

Modern solvency frameworks, such as the Solvency II (SII) legislation for European insurers or the Swiss Solvency Test (SST) for Switzerland, require a fixed confidence level in order to determine the amount of solvency capital. This amount of capital has to be put aside by the insurer in order to be solvent with a certain probability given by this confidence level. For instance, the SII regulatory framework considers a probability of 99.5% over a one-year period.

These new regulations are designed such that they recognize that an insurer faces different kinds of risks and take them into account for the determination of the solvency capital. We say that such a regulation is risk-sensitive. However, concerning life insurers, but also pension funds, an important characteristic that has to be considered is the time horizon of the liabilities, which is the topic of this paper. We focus more particularly on long-term life insurances or pension products.

Furthermore, regarding Solvency II, concerns have been expressed during the Quantitative Impact Study 2 (QIS2), conducted by the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS), now called the European Insurance and Occupational Pensions Authority (EIOPA), and reported in...
the Impact Assessment Report (IAR) accompanying the proposal of the SII directive.¹ These concerns relate
to the shock that has to be applied on the market value of equity investments matching long-term liabilities.
It is considered to be inappropriate and overly conservative, compared to the expected holding period of the
equity position. This method could give incentives for insurance undertakings to move out of equities into
bonds. This reallocation is currently observed and the financial crisis of 2007 intensified this movement.

The purpose of this paper is twofold. We first link the classic ruin theory with the determination of the
solvency capital by means of risk measures. This has been considered for non-life insurances in [17] and in a
continuous setting in [5] for life insurances. The idea is to start from the time continuous setting of the ruin
theory, which continuously checks the solvency of the company. However, from an accounting viewpoint, we
are interested in the solvency of the company on a yearly basis. This approach boils down to the computation
of the well-known VaR measure of an infimum of net values, which is a unique random variable at a particular
date. Secondly, we challenge the definition of the Solvency Capital Requirement under the SII regulatory
framework for long-term guarantees and we show that long-term guarantees might be overloaded under the
SII regulation.

Firstly, we focus on the equity risk and model the financial market with a Black-Scholes-Merton model
([2] and [12]). This allows us to highlight the impact of the equity risk for long-term products. We also consider
long-term liabilities with one single payment (outflow) at maturity, such as some life insurances or pension
products. As an illustration, we consider a fixed guaranteed rate over the whole lifetime of the product on
one single payment at the inception of the contract.

We then consider a more complete model in which we include the interest rate and longevity risks. The
latter allows us to consider a new approach for the determination of the capital taking into account zero-
coupon longevity bonds. Instead of investing the solvency capital in zero-coupon bonds, we invest it in zero-
coupon longevity bonds. This allows us to protect ourselves against the longevity risk and to reduce the level
of the solvency capital.

In the following, we set a complete atomless probability space $(\Omega, \mathcal{F}, P)$. Equalities and inequalities be-
tween random variables (r.v.’s) are understood in the almost sure (a.s.) sense. We define $L^1 = L^1(\Omega, \mathcal{F}, P)$
as the space of all real-valued $\mathcal{F}$-measurable r.v.’s $X$ such that $\mathbb{E}[|X|] < +\infty$, where two r.v.’s are identified if
they coincide a.s. We consider an equivalence class $X \in L^1$ as a r.v. We also call $X$ a risk, meaning that if $X$ is
positive, we face a gain (or profit) while if it is negative or zero, we face a loss.

The paper is organized as follow. In Section 2, we start our study with the SII regulatory framework with
the one-year period, and follow with the maturity viewpoint. We also present our approach from the ruin
theory. The impact of the equity risk is analyzed in Section 3. In Section 4, we set the model with the interest
rate and longevity risks and then determine the solvency capital according to zero-coupon bonds and zero-
coupon longevity bonds.

2 Solvency and time horizon

Throughout this paper, we consider long-term guarantees such as life insurances or pension products. More
precisely, we consider a product with one single cash flow at maturity. The life insurer or pension fund offers a
fixed guaranteed rate on the premium or contribution, and the payment of the liability might be contingent to
the survival of the policyholder or affiliate, depending on the hypotheses made concerning the mortality risk.
This product could correspond to a Defined Contribution (DC) pension scheme with a maturity guarantee. We
then consider indifferently the viewpoint of a life insurer or a pension fund. The premium or contribution is
invested in a portfolio exposed to market risks, such as the equity and interest rate risks.

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¹ Commission staff working document: Accompanying document to the Proposal for a Directive of the European Parliament and of
the Council concerning life assurance on the taking-up and pursuit of the business of Insurance and Reinsurance—Solvency II—Impact
assessment report, European Commission, SEC/2007/0871.
According to the regulatory authority of the company, it has to compute a capital, called the solvency capital. This capital is the amount the company has to put aside in order to be solvent according to the measure considered by the regulator. This capital is generally invested following a conservative strategy—like (government) zero-coupon bonds—and comes in addition to the initial value of the portfolio of assets, it is then also exposed to market risks.

We present the one-year period approach of the Solvency II regulatory framework in Section 2.1 and the maturity approach in Section 2.2. We finally consider the ruin theory approach in Section 2.3.

2.1 Solvency II regulatory framework

The Solvency II (SII) regulatory framework is designed to be risk-sensitive and recognizes that the insurer faces different kinds of risks, like the market and longevity risks.² This regulation introduced a risk-based approach to compute the solvency capital requirement (SCR) the insurer should hold. The solvency capital requirement is the minimum amount of extra cash the insurer has to put aside in order to be allowed to proceed its plans. This amount comes in addition to the initial value of the liabilities. The method used to compute this capital is based on a risk measure—the VaR—on the own funds of the company.³

Let $A_0 \in \mathbb{R}$ be the initial market value of the assets of the insurer ($t = 0$) and $A_t \in L^1$ be the market value of its assets at time $t > 0$. We also define $L_0 \in \mathbb{R}$ as the market value of the liabilities of the insurer at time $t = 0$ and $L_t \in L^1$ as the market value of its liabilities at time $t > 0$. We now define the net value of the insurer.

**Definition 2.1.** The net value of the insurer at time $t \geq 0$ is defined as the difference between the market values $A_t$ and $L_t$, i.e.

$$N_t = A_t - L_t.$$ 

In this paper, we identify the net value with the basic own funds of the insurer as defined in the SII regulation. The SCR under SII is defined as the VaR of the evolution of the net value of the insurer with a confidence level of 99.5% over a one-year period.

**Definition 2.2.** The SCR under SII is given by

$$\text{SCR}^{\text{SII}} = \text{VaR}^{0.995} (N_1 P(0, 1) - N_0),$$

where $P(0, 1) > 0$ is the price of a risk-free zero-coupon bond at time $t = 0$ with maturity $t = 1$, and the value at risk (VaR) of $X \in L^1$ at level $p \in (0, 1)$ is given by the lower quantile of $-X$,

$$\text{VaR}^p(X) = q_{-X}(p),$$

where the lower quantile of $-X$ at level $p$ is defined as

$$q_{-X}(p) = \inf \{ x \in \mathbb{R} : \mathbb{P}[-X \leq x] \geq p \}.$$

**Remark 2.3.** For $X \in L^1$ and $p \in (0, 1)$, we have that (see [9])

$$q_{-X}(p) = -q_X(1 - p),$$

where the upper quantile of $X$ at level $1 - p$ is

$$q_X(1 - p) = \inf \{ x \in \mathbb{R} : \mathbb{P}[X \leq x] > 1 - p \}.$$

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² Directive 2009/138/EC of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II), OJ L 335, 17 December 2009, pp. 1-155.
³ Ibid. Art. 101.
Remark 2.4. The SCR under SII is the amount of initial net value the insurer should hold in order to be solvent in one year with a probability of 99.5%. Indeed, equations (1) and (2) give us that

\[
\text{SCR}_{\text{SII}} = q_{N_0 - N_1 \cdot P(0,1)}(0.995) \\
= q_{N_0 - N_1 \cdot P(0,1)}(0.995) \\
= \inf \{x \in \mathbb{R} : P(N_0 - N_1 \cdot P(0,1) \leq x) \geq 0.995\}.
\]

We then compute that

\[
P\left[N_0 - N_1 \cdot P(0,1) \leq \text{SCR}_{\text{SII}}\right] = P\left[N_1 \geq \frac{N_0 - \text{SCR}_{\text{SII}}}{P(0,1)}\right] \\
\geq 0.995.
\]

Since \(P(0,1) > 0\), if

\[N_0 - \text{SCR}_{\text{SII}} \geq 0,
\]

we have that

\[P\left[N_1 \geq 0\right] \geq P\left[N_1 \geq \frac{N_0 - \text{SCR}_{\text{SII}}}{P(0,1)}\right] \geq 0.995,
\]

and the insurer does not need to bring additional capital. In this situation, the insurer might be allowed to reduce its assets while staying solvent with a probability of 99.5%. We emphasize that the insurer can only withdraw the quantity \(N_0 - \text{SCR}_{\text{SII}}\) of \(A_0\) if the corresponding amount in \(A_0\) was invested in the same financial instrument (the zero-coupon bond \(P(0,1)\)). Otherwise, a change in the investment strategy of \(A_0\) might lead to a different SCR and we then face another optimization problem. However, if

\[N_0 - \text{SCR}_{\text{SII}} < 0,
\]

then

\[P\left[\tilde{N}_1 \geq 0\right] \geq 0.995,
\]

with

\[
\tilde{N}_1 = N_1 - \frac{N_0 - \text{SCR}_{\text{SII}}}{P(0,1)}
\]

which means that at time \(t = 0\), the insurer has to invest the quantity

\[\text{SCR}_{\text{SII}} - N_0 > 0
\]

in a zero-coupon bond \(P(0,1)\).

We then define the solvency capital under SII as the capital the shareholders have to bring and to invest in \(P(0,1)\).

Definition 2.5. The solvency capital under SII is given by

\[
\text{SC}_{\text{SII}} = \left(\text{SCR}_{\text{SII}} - N_0\right)_+.
\]

Since \(N_0\) is deterministic, we also compute by the property of cash invariance

\[
\text{SCR}_{\text{SII}} - N_0 = \text{VaR}^{0.995}(N_1 \cdot P(0,1) - N_0) - N_0 \\
= \text{VaR}^{0.995}(N_1 \cdot P(0,1)) \\
= \text{VaR}^{0.995}((A_1 - L_1) \cdot P(0,1)) \tag{3}
\]

The amount given by the sum of the market value of the liabilities \(L_0\) and the SCR gives us the initial total amount of assets the insurer should hold.
2.2 Risk measures at maturity

In this paper we focus on long-term liabilities with a single payment (outflow) at maturity, such as some life insurances or pension products. We then consider here the viewpoint of either a life insurer or a pension fund.

The SII approach presented in Section 2.1 consider a one-year period for the determination of the SCR. Due to the long-term characteristics of our products, we consider here a long-term point of view for the calculation of the solvency capital. We follow the maturity approach presented in [5, 7] and also considered in the SII legislation with the duration-based equity risk sub-module for assets and liabilities with long-term horizons and quite strong characteristics. In that approach, we consider again a risk measure in order to compute the solvency capital. However, the confidence level we consider is seen as a survival probability (see also the Probabilistic criterion of [10]).

We fix an integer $T > 0$ which denotes the maturity (in year) of a long-term guarantee, such as a pure endowment policy or a Defined Contribution scheme. In this setting, $T$ being the maturity of the liability, $L_T$ is the cash-flow that the insurer has to pay at time $T$ to the policyholder.

**Definition 2.6.** The SCR at maturity is given by

$$SCR_{\text{mat}} = \text{Var}^{0.995^T} (N_T P(0,T) - N_0),$$

where $P(0,T) > 0$ is the price of a risk-free zero-coupon bond at time $t = 0$ with maturity $t = T$.

Following a same reasoning as in Remark 2.4, the SCR at maturity is then the amount of initial net value the insurer should hold in order to be solvent at maturity according to an adjusted confidence level of $0.995^T$,

$$P[N_T \geq 0] \geq 0.995^T.$$

In particular, we now consider the difference between the net value at the maturity date and the initial net value, we do not consider any intermediate value of the net value.

We then define the solvency capital at maturity as the capital the shareholders have to bring and to invest in $P(0,T)$.

**Definition 2.7.** The solvency capital at maturity is defined by

$$SC_{\text{mat}} = (SCR_{\text{mat}} - N_0)_{+}.$$

Since $N_0$ is deterministic, we compute, in the same way as equation (3),

$$SC_{\text{mat}} = (\text{Var}^{0.995^T} ((A_T - L_T)P(0,T))_{+}.$$  

The solvency capital computed according to Definition 2.7 is the amount invested in a zero-coupon bond $P(0,T)$ such that the insurer will be solvent at maturity with a probability of $0.995^T$. This amount has to be compared with (3) which guarantees a solvency of $99.5\%$ in one year, even if the maturity $T$ is much greater than 1. In particular, if $T = 1$, both definitions coincide. This level of $0.995^T$ is considered as a survival probability, assuming a constant and independent yearly probability of default. However, we see in Definition 2.6 that we only consider the initial and final net values. This leads us to the following section in which we tackle the problem through a ruin theory approach, as introduced in [5] and [17].

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4 Ibid. Art. 304.
5 CEIOPS’ Advice for Level 2 Implementing Measures on Solvency II: Article 111 and 304 Equity Risk Sub-Module, Committee of European Insurance and Occupational Pensions Supervisors, CEIOPS-DOC-65/10, 29 January 2010.
6 The underlying assumptions in the standard formula for the Solvency Capital Requirement calculation, European Insurance and Occupational Pensions Authority, EIOPA–14–322, 25 July 2014, pp. 19–20.
2.3 Ruin theory approach

In this section, we consider the determination of the solvency capital of an insurer or a pension fund, through a ruin theory approach. In Section 2.3.1 we consider first the probability of default at the maturity date, while in Section 2.3.2 we define the solvency capital according to a ruin measure.

2.3.1 Probability of default at maturity

Under the classical ruin theory, we are interested in the probability of defaulting at any time,
\[ P[\exists t > 0 : N_t < 0] = 1 - P[\forall t > 0, N_t \geq 0] . \]

However, we focus in this paper on long-term liabilities with a single payment (outflow) at maturity. We then fix an integer \( T > 0 \) which denotes the maturity (in year) of our long-term guarantee, and we are interested in the probability of defaulting over the lifetime of the product,
\[ P[\exists 0 < t \leq T : N_t < 0] = 1 - P[\forall 0 < t \leq T, N_t \geq 0] . \]

As there is only one single outflow at maturity, we can also consider the solvency of the life insurer or the pension fund when the payment has to be made. This leads us to the following definition.

**Definition 2.8.** The probability of default at maturity \( T \) is given by
\[ \Psi = P[N_T < 0] = P[A_T < L_T] . \] (6)

This measure of the insolvency of the insurer can not be easily translated into a monetary value. For instance, it does not consider the magnitude of a loss when it occurs. Nevertheless, as it will be illustrated in Sections 3.4 and 4.4, it will give us some hints that the solvency capital might be lowered for long-term horizons. In the following section, we consider a yearly accounting viewpoint, meaning that we study the probability of defaulting not only at maturity, but on a yearly basis,
\[ P[\exists t \in \{1, \ldots, T\} : N_t < 0] = 1 - P[\forall t \in \{1, \ldots, T\}, N_t \geq 0] \]
\[ = P[\inf_{t \in \{1, \ldots, T\}} N_t < 0] . \] (7)

2.3.2 Solvency capital from a ruin theory approach

We fix a random variable \( D > 0 \) with integer values, i.e. \( D(\omega) \in \mathbb{N} \) for all \( \omega \in \Omega \). We see this random variable \( D \) as the year of default of the company. For instance, let \( T \in \mathbb{N} \) and \( \omega \in \Omega \). If \( D(\omega) = T \), then the company defaulted during year \( T \) under scenario \( \omega \). In particular, we have that,
\[ P[D > T] = P[N_1 \geq 0, \ldots, N_T \geq 0] \]
\[ = P[\inf_{t \in \{1, \ldots, T\}} N_t \geq 0] , \] (7)

since we only consider the solvency of the company on a yearly basis following the accounting viewpoint.

We now assume that the event of defaulting during year \( t \) is independent of the event of defaulting during year \( t - 1 \), for any \( t \in \mathbb{N} \). We also assume that the yearly default probability is constant and given by \( 1 - a \), with \( a \in (0, 1) \). Then the r.v. \( D \) is a sequence of Bernoulli trials and follows a geometric distribution with parameter \( 1 - a \). We have that
\[ P[D = T] = a^{T-1}(1 - a) , \] (8)
which gives the probability of defaulting during year $T$, and

$$ P[D = T | D \geq T] = 1 - \alpha , $$

(9)

which is the probability of defaulting during year $T$ knowing that the company has survived until the beginning of year $T$. Furthermore, the probability of surviving until year $T$ is

$$ P[D > T] = \alpha^T . $$

(10)

The confidence level $0.995^T$ used in Definition 2.6 of the SCR at maturity is based on this assumption of temporal independence of events, which is equivalent to assume a geometric distribution for $D$. In particular, the SCR at maturity is such that we should have

$$ P[N_T \geq 0] \geq 0.995^T . $$

(11)

However,

$$ P[D > T] \neq P[N_T \geq 0] , $$

and from equations (7) and (10), we should not only consider $N_T$, but we should also consider the intermediate values of the net value, i.e.

$$ P[D > T] = P\left[ \inf_{t \in \{0, \ldots, T\}} N_t \geq 0 \right] \geq 0.995^T . $$

(12)

The SCR at maturity should satisfy this inequality instead of satisfying the inequality (11), in order to be coherent with the assumption of temporal independence of events. We then define the SCR according to this ruin theory approach, by means of the inequality (12).

**Definition 2.9.** The SCR at ruin is given by

$$ \text{SCR}^{\text{ruin}} = \text{VaR}^{0.995^T} \left( \inf_{t \in \{0, \ldots, T\}} \frac{P(0, T)}{P(t, T)} - N_0 \right) , $$

(13)

where $P(t, T) > 0$ is the price of a risk-free zero-coupon bond at time $t \in \{0, \ldots, T\}$ with maturity $t = T$.

From the definition of the VaR, we compute

$$ P\left[ N_0 - \inf_{t \in \{0, \ldots, T\}} N_t \frac{P(0, T)}{P(t, T)} \leq \text{SCR}^{\text{ruin}} \right] = P\left[ \inf_{t \in \{0, \ldots, T\}} \left( N_t + \frac{\text{SCR}^{\text{ruin}} - N_0}{P(0, T)} P(t, T) \right) \geq 0 \right] \geq 0.995^T , $$

and again, we can define the solvency capital the insurer or the pension fund has to invest in a zero-coupon bond $P(0, T)$ in addition to the initial value of its assets.

**Definition 2.10.** The solvency capital at ruin is defined by

$$ \text{SC}^{\text{ruin}} = \left( \text{SCR}^{\text{ruin}} - N_0 \right) . $$

Since $N_0$ is deterministic, we compute, in the same way as equations (3) and (5),

$$ \text{SC}^{\text{mat}} = \left( \text{VaR}^{0.995^T} \left( \inf_{t \in \{0, \ldots, T\}} N_t \frac{P(0, T)}{P(t, T)} \right) \right) . $$

(14)

**Remark 2.11.** A similar starting point has been considered in [17] for non-life insurances. Since non-life products are short-term, the problem in this paper lies in the long development process of the claims. In our paper, we study long-term products with one single cash flow at the maturity of the contract.
We emphasize that considering the confidence level of $0.995^T$, we have shown that the SCR at maturity should satisfy the inequality (12). However, if we want to keep looking at $N_T$ instead of the infimum over the whole time interval, the level of confidence should be adapted in two ways.

We could consider (8) since the company will default at maturity date $T$ if $N_T < 0$. We then have under the strong assumption of independence that the SCR at maturity with the first adapted confidence level should satisfy

$$
P[N_T < 0] \leq 0.995^{T-1}(1 - 0.995) = 0.995^{T-1} \times 0.005 ,$$

or equivalently,

$$
P[N_T \geq 0] \geq 1 - 0.995^{T-1} \times 0.005 > 0.995 \geq 0.995^T .$$

Alternatively we could consider (9). We then look at the solvency at time $T$, assuming that the insurer was solvent at intermediate date $t \in \{1, \ldots, T-1\}$. Then under the assumption of independence, the SCR at maturity with the second adapted confidence level should satisfy

$$
P \left[ N_T \geq 0 \left| \inf_{t \in \{1, \ldots, T-1\}} N_t \geq 0 \right. \right] \geq 0.995 \geq 0.995^T .$$

### 3 Equity risk

We focus in this paper on long-term guarantees and we want to understand the impact of the equity risk on the solvency of these products through the different definitions presented in the previous section. We then only consider this risk in this section. This risk is modeled through a geometric Brownian motion. We call the model presented here the equity model. We first set the financial market in Section 3.1, the liabilities in Section 3.2 and the assets in Section 3.3. We then consider the probability of default at maturity in Section 3.4 and the determination of the solvency capital in Section 3.5.

#### 3.1 Financial market

The financial market is composed of a bank account $B$ and a stock $S$. We follow the same assumptions as in the model of Black, Scholes and Merton (see [2] and [12]). There are no transaction costs, taxes or problems with indivisibilities of assets and trading takes place continuously in time. The short-term interest rate is known and is constant through time. The dynamic of the bank account is then given by

$$
\frac{dB(t)}{B(t)} = r(t) \ dt ,
$$

for $t \geq 0$, with $B(0) = 1$ and $r \in \mathbb{R}$ being the constant (risk-free) short-term interest rate. We have that

$$
B(t) = e^{rt} ,
$$

for each $t \geq 0$. As the term-structure is flat and known with certainty, the price at time $t \geq 0$ of a zero-coupon bond paying one unit of currency at maturity $s \geq t$, $t \leq s$, is given by

$$
P(t, s) = \frac{B(t)}{B(s)} = e^{-r(s-t)} .
$$

We assume the stock pays no dividends or other distributions and is modeled by a geometric Brownian motion (GBM) with the following stochastic differential equation (SDE),

$$
\frac{dS_t}{S_t} = \mu_S S_t \ dt + \sigma_S S_t \ dW^S_t ,
$$

for $t \geq 0$, with $S_0 = 1$, $\mu_S, \sigma_S \in \mathbb{R}$, $\sigma_S > 0$ and $W^S$ a standard Brownian motion on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We also consider its completed natural filtration $\mathbb{F}^S = \left( \mathcal{F}^S_t \right)_{t \geq 0}$. We then have that

$$
S_t = \exp \left( \left( \mu_S - \frac{\sigma_S^2}{2} \right) t + \sigma_S W^S_t \right) ,
$$
Table 1: Parameters of the GBM obtained by the MLE method with the corresponding standard errors between brackets.

| $\mu_S$  | $\sigma_S$ | $\mu_S$  | $\sigma_S$ |
|----------|------------|----------|------------|
| 0.06561  | 0.03857    | 0.21045  | 0.00172    |

for each $t \geq 0$. We finally assume that short selling of all assets is allowed without any penalties.

We assume that the risk-free interest rate is equal to 1.5%,$r = 0.015$.

The GBM model has been calibrated on daily log-returns of the Euro Stoxx 50 (STOXX50E) index from 1 January 1987 to 23 December 2015 by means of the Maximum Likelihood Estimation (MLE) method (see Table 1).

3.2 Liabilities

We consider a life insurer or pension fund which offers a fixed guaranteed rate $r_G \in \mathbb{R}$ over a time horizon $T \in \mathbb{N}, T > 0$. As we focus here on the equity risk, we assume that the company is fully hedged against the mortality and underwriting risks. Then the liability at maturity is given by

$$L_T = l_0 \pi_0 e^{r_G T},$$

where $\pi_0 > 0$ is the initial unique premium (contribution) paid by each policyholder or for each affiliate and $l_0 \in \mathbb{N}$ is the number of subscribers—there is no need to make a distinction according to the age of each subscriber due to the absence of mortality risk.

**Remark 3.1.** From a pension viewpoint, this setting corresponds to a Defined Contribution (DC) pension plan. However we could have considered a defined benefit (DB) plan as well.

The market value of the liability at time $t \in \{0, \ldots, T\}$ is then given by

$$L_t = l_0 \pi_0 e^{r_G t} P(t, T).$$

We assume that $l_0 = 1000$, $\pi_0 = 1000$ and that the guaranteed rate is equal to the risk-free rate,

$$r_G = r,$$

such that

$$L_0 = l_0 \pi_0 = 1000 \times 1000.$$

3.3 Assets

The $l_0$ initial unique contributions (premiums) $\pi_0 > 0$ are invested in a portfolio $A$ over the period $[0, T]$ made up of the bank account $B$ and the stock $S$. This portfolio follows a constant allocation strategy and is self-financing, i.e. its SDE is given by, for $t \in [0, T],$

$$dA_t = x_S A_t \left( \frac{dS_t}{S_t} + (1 - x_S) A_t \frac{dB(t)}{B(t)} \right)$$

$$= \left( x_S \mu_S + (1 - x_S) r \right) A_t \, dt + x_S \sigma_S A_t \, dW_t^S,$$

where $x_S \in [0, 1]$ is the deterministic proportion of the portfolio invested in the stock. At any time $t \in [0, T]$, a proportion $x_S$ of the portfolio $A_t$ is invested in the stock and a proportion $1 - x_S$ is invested in the bank account. We then have

$$A_t = l_0 \pi_0 \exp \left[ \left( x_S \mu_S + (1 - x_S) r - \frac{x_S^2 \sigma_S^2}{2} \right) t + x_S \sigma_S W_t^S \right],$$
Figure 1: Probabilities of default for the equity model along with the maturity of the product.

for all \( t \in [0, T] \). The natural logarithm of the assets follows a normal distribution
\[
\ln A_t \sim N \left( \ln(l_0 \pi_0) + m_t^A, \left( s_t^A \right)^2 \right),
\]
where
\[
m_t^A = \left( x_S \mu_S + \left( 1 - x_S \right)r - \frac{x_S^2 \sigma_S^2}{2} \right) t,
\]
and
\[
\left( s_t^A \right)^2 = x_S^2 \sigma_S^2 t.
\]
We assume that 15% of the portfolio is invested in the stock,
\[
x_S = 0.15.
\]

3.4 Probability of default

We compute that
\[
\Psi = P [A_T < L_T] = \Phi \left( r_G T - m_T^A \right),
\]
where \( \Phi \) denotes the cumulative distribution function of a standard normal r.v., i.e. for \( y \in \mathbb{R} \),
\[
\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} \, dx.
\]

Figure 1 illustrates the evolution of this probability along with the maturity of the product. We clearly see the decreasing trend of the probability of default with the maturity, which shows us a kind of maturity effect. Of course this probability can not be used directly for solvency measurement and has to be translated into a monetary value. But its trend indicates that a smaller amount of capital might be required for long-term horizons, compared to short maturities.

3.5 Solvency capital

We now compute the solvency capital given by Definitions 2.5, 2.7 and 2.10. This amount has to be brought by shareholders and invested in risk-free reference instruments.
Proposition 3.2. We have that

\[
\text{SC}^{\text{SII}} = (L_0 - P(0, 1)A_0 \exp \left( m_1^A + s_1^A \Phi^{-1}(0.005) \right))_+,
\]

and

\[
\text{SC}^{\text{mat}} = (L_0 - P(0, T)A_0 \exp \left( m_1^A + s_1^A \Phi^{-1} \left( 1 - 0.995^T \right) \right))_+.
\]

Proof. From Definition 2.5, equation (3), the cash invariance and positive homogeneity of the VaR, we compute that

\[
\text{VaR}^{0.995}(A_1 - L_1)P(0, 1) = L_1 P(0, 1) + P(0, 1) \text{VaR}^{0.995}(A_1)
\]

\[
= L_0 - P(0, 1)q_{A_1}(1 - 0.995)
\]

\[
= L_0 - P(0, 1)l_0^p \pi_0 \exp \left( m_1^A + s_1^A \Phi^{-1}(0.005) \right),
\]

since

\[
\ln A_1 \sim N \left( \ln(l_0^p \pi_0) + m_1^A, \left( s_1^A \right)^2 \right).
\]

The same reasoning applies for the solvency capital at maturity.

We clearly see that under our model, the solvency capital under SII does not depend on the maturity of the product, since \( L_0 = l_0^p \pi_0 \). It is then constant and is equal to about 7% of \( A_0 \) (dotted line in Figure 2).

![Figure 2: Solvency capital with the equity model in proportion of the initial value of the portfolio under SII (dotted line), at maturity (dashed line) and at ruin (solid line).](image)

Unlike SII, the solvency capital at maturity varies with the maturity of the product. We see in Figure 2 that it is higher than SII for short maturities (for instance, about 10% of \( A_0 \) for a maturity of 6 years), while for long maturities it goes down and becomes zero for maturities longer than 26 years. It is then more conservative than SII for short time horizons although it is less expensive for long-term products. This is due to the advantage of an equity investment over long horizons, which is not taken into account in the SII computation. This observation is in line with the decreasing trend of the probability of default depicted in Figure 1.

In order to compute the solvency capital at ruin, we used Monte Carlo simulations. The solid line in Figure 2 shows that the capital increases for short maturities while it decreases for long maturities. We consider maturities ranging from 1 to 150 years. In life insurance or considering pension products, maturities longer than 50 are quite rare and it does not make sense to look at maturities longer than 100 years. The purpose is to highlight the effect of the maturity and the equity risk on the level of the capital. The confidence level of a maturity of 100 years equals about 61% and 47% for 150 years. The higher level of the capital is obtained for a maturity of 13 years and is equal to about 12%. This capital is much more expensive than SII or the solvency
capital at maturity, which is intuitive as we look yearly at the solvency of the insurer over the whole life of the product, even with a confidence level decreasing with the maturity.

As the SII capital does not include the maturity of the product and the capital at ruin is quite expensive, it appears that the capital at maturity could be the more reasonable. However, the latter erroneously stems from a quite strong assumption of independence.

4 Market and longevity risks

We now consider the interest rate and longevity risks in addition to the equity risk. The equity risk is modeled as in the previous section, while the interest rate and longevity risks are driven by Ornstein-Uhlenbeck processes. The model composed of these three risks is called the complete model. The particular case when the interest rates are assumed to be constant is called the equity-longevity model.

Following the same reasoning as in Section 3, we first set the financial market in Section 4.1, the liabilities in Section 4.2 and the assets in Section 4.3. We then consider the probability of default at maturity in Section 4.4. Finally, we determine the solvency capital according to an investment in zero-coupon bonds in Section 4.5 and in zero-coupon longevity bonds in Section 4.6.

4.1 Financial market

The financial market is now composed of a bank account $B$, zero-coupon bonds $P$ and a stock $S$. Again, there are no transaction costs, taxes or problems with indivisibilities of assets and trading takes place continuously in time. The stock $S$ is modeled as in Section 3.1. The short-term interest rate $r$ is modeled with an Ornstein-Uhlenbeck process and follows this SDE

$$dr_t = a_r(b_r - r_t) dt + \sigma_r dW^r_t,$$

for $t \geq 0$, with $r_0 \in \mathbb{R}$, $a_r, b_r, \sigma_r \in \mathbb{R}$, $a_r, \sigma_r > 0$ and $W^r$ a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is correlated with $W^S$ with correlation $\rho \in [-1, 1]$. We also define its completed natural filtration $\mathcal{F}^r = (\mathcal{F}^r_t)_{t \geq 0}$. The dynamic of the bank account is then given by

$$dB_t = r_t B_t dt,$$

for $t \geq 0$, with $B_0 = 1$.

We fix $T > 0$. Following the Girsanov theorem (see [13]), we define the risk-neutral probability measure $\mathbb{Q}^r$ on $\mathcal{F}^r_T$ with the Radon-Nikodym density

$$\frac{d\mathbb{Q}^r}{d\mathbb{P}} = \exp \left( \lambda_r W^r_T - \frac{1}{2} \lambda_r^2 T \right),$$

such that the process $W^{Q^r}$ given by

$$W^{Q^r}_t = W^r_t - \lambda_r t,$$

for $t \in [0, T]$, is a standard Brownian motion on $(\Omega, \mathcal{F}^r_T, \mathbb{Q}^r)$, where $\lambda_r \in \mathbb{R}$ denotes the constant market price of interest rate risk. The price at time $t \in [0, T]$ of a zero-coupon bond paying one unit of currency at maturity $s \in [0, T]$, $t \leq s$, is given by (see [16]),

$$P(t, s) = \mathbb{E}^{Q^r} \left[ B_s \mid \mathcal{F}^r_t \right] = A_r(t, s) \exp \left[ -B_r(t, s)r_t \right],$$

with $\mathbb{E}^{Q^r}$ the expectation under the probability measure $\mathbb{Q}^r$,

$$A_r(t, s) = \exp \left[ \left( B_r + \frac{\lambda_r \sigma_r}{a_r} - \frac{\sigma_r^2}{2a_r^2} \right) \left( B_r(t, s) - (s - t) \right) - \frac{\sigma_r^2}{4a_r} B_r(t, s)^2 \right],$$
the force of mortality at the inception date to keep notations simple. We again consider its completed natural filtration independent of \(W\) for non mean reverting as suggested in [3]. The SDE is then given by

\[
\frac{dX_t}{X_t} = (\mu_t + \sigma_t B_t(t, s)) \, dt - \sigma_t B_t(t, s) \, dW_t^F.
\]

The Ornstein-Uhlenbeck process has been calibrated on daily Euro OverNight Index Average (EONIA) rates from 4 January 1999 to 31 December 2015 by means of the MLE method. The market price of interest rate risk has been estimated on AAA-rated euro area central government bonds on 31 December 2015 by minimizing the residual sum of squares (see Table 2). We also estimated a correlation of about \(-3\%\) between the stock and interest rates,

\[\rho = -0.03.\]

In particular, for the equity-longevity model, we assume that \(r_0 = b_r = 0.015\) and \(\sigma_r = 0.\) We then get back the financial market of Section 3.1.

### 4.2 Liabilities

As in Section 3.2, we consider a life insurer or pension fund which offers a fixed guaranteed rate \(r_G \in \mathbb{R}\) over a time horizon \(T \in \mathbb{N}, T > 0.\) We now add a mortality credit. We assume that the company is fully hedged against the diversifiable part of the mortality risk arising from possible deviations around the expected mortality rates, called the process risk, insurance risk or non-systematic risk (see [3] or [14]). This may be achieved if the company is big enough and through the realization of a proper pooling effect (application of the law of large numbers). However, unlike Section 3.2, the company is not hedged against the non-diversifiable part of the mortality risk arising from a modification of the average lifetime of the population considered, called the longevity risk or systematic risk. We also assume that it is fully hedged against the underwriting risk.

#### 4.2.1 Mortality process and survivor index

We fix an integer \(x > 0\) which denotes the initial age of the policyholders or affiliates that subscribe the product. In order to model the longevity risk, we consider a second Ornstein-Uhlenbeck process for the dynamic of the force of mortality (see for instance [1], [15], [11] or [6]). We assume that the force of mortality process is non mean reverting as suggested in [3]. The SDE is then given by

\[
d\mu^x_t = a_x \mu^x_t \, dt + \sigma_x \, dW^x_t,
\]

for \(t \geq 0, \mu^0_0 \in \mathbb{R}, a_x \sigma_x > 0\) and \(W^x\) a standard Brownian motion on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) which is independent of \(W^F\) and \(W^r.\) This assumption of independence could be dropped but it is convenient in order to keep notations simple. We again consider its completed natural filtration \(\mathbb{F}^x = (\mathcal{F}^x_t)_{t \geq 0}\). The quantity \(\mu^x_0\) is the force of mortality at the inception date \(t = 0\) for a policyholder aged \(x\) and \(\mu^x_t\) is the force of mortality at date \(t\) for a policyholder aged \(x + t\) at date \(t.\)
Remark 4.1. We observe that if $\sigma_x = 0$, then
\[ d\mu_t^x = a_x \mu_t^x \, dt, \]
and
\[ \mu_t^x = \mu_0^x e^{a_t}, \]
for $t \geq 0$. In this way we get back the well-known Gompertz model. The Ornstein-Uhlenbeck process we consider here for the modeling of the mortality is then the simplest model which generalizes the Gompertz model with an additive noise. It will also allow us to have understandable and tractable formulae while the main drawback is that we can obtain negative rates with a positive probability due to the normal distribution of the force of mortality. Nevertheless, from a numerical viewpoint, this model could be easily replaced with a Cox-Ingersoll-Ross model, for instance. This will be part of future researches.

We define the survivor index at time $t \geq 0$ for an initial age $x$ by
\[ I_t^x = \exp \left( - \int_0^t \mu_s^x \, ds \right). \]
The natural logarithm of the survivor index follows a normal distribution
\[ \ln I_t^x \sim N \left( m_t^x, (s_t^x)^2 \right), \]
where
\[ m_t^x = \frac{\mu_0^x}{a_x} \left( 1 - e^{a_t} \right), \]
and
\[ (s_t^x)^2 = \frac{\sigma_x^2}{a_x^2} \left( t + \frac{e^{2a_t} - 1}{2a_x} + 2 \frac{1 - e^{a_t}}{a_x} \right). \]
The number of survivors at time $t \geq 0$ is then given by
\[ l_t^x = l_0^x I_t^x, \]
where $l_0^x \in \mathbb{N}$ is the number of policyholders of age $x$ subscribing the policy.

4.2.2 Survival probabilities and zero-coupon longevity bonds

Following [3], the true or real-world survival probability of a policyholder initially aged $x$ and being alive at time $t$ (age $x + t$) of surviving until time $s \geq t$ (age $x + s$) is then given by
\[ p_x(t, s) = \mathbb{E} \left[ I_s^x \bigg| F_t^x \right] = A_x(t, s) \exp \left[ -B_x(t, s) \mu_t^x \right], \quad (15) \]
with
\[ A_x(t, s) = \exp \left[ -\frac{\sigma_x^2}{2a_x^2} \left( B_x(t, s) - (s - t) \right) + \frac{\sigma_x^2}{4a_x^2} B_x(t, s)^2 \right], \]
and
\[ B_x(t, s) = -\frac{1}{a_x} \left( 1 - e^{a_x(s-t)} \right). \]

As in Section 4.1 for interest rates, we define the risk-neutral probability measure $Q^x$ on $\mathcal{F}_T^x$ with the Radon-Nikodym density
\[ \frac{dQ^x}{dt} = \exp \left( \lambda_x W_t^x - \frac{1}{2} \lambda_x^2 T \right). \]
such that the process $W^Q, x$ given by

$$W^Q_t = W^F_t - \lambda x t,$$

for $t \in [0, T]$, is a standard Brownian motion on $(\Omega, {\mathcal F}_T^x, Q^x)$, where $\lambda x \in \mathbb{R}$ denotes here the constant market price of longevity risk.

Let $\mathbb{F}^{r, x} = (\mathcal{F}^{r, x}_t)_{t \geq 0}$ be the filtration generated from both $W^r$ and $W^x$. Since the interest rate and longevity risks are independent, we can de/\_fine the risk-neutral measure $Q$ on $\mathbb{F}^{r, x}_t$ with the density

$$\frac{dQ}{dF} = \frac{dQ^r}{dF} \frac{dQ^x}{dF},$$

such that the processes $W^{Q, r}$ and $W^{Q, x}$ are standard Brownian motions on $(\Omega, \mathcal{F}_T^{r, x}, Q)$.

The price of a zero-coupon longevity bond $L_x(t, s)$ at time $t \in [0, T]$ for initial age $x$, paying the amount $I^x_t$ at time $s \in [0, T]$, $t \leq s$ is given by

$$L_x(t, s) = E^Q \left[ I^x_t B^t_t \mid \mathcal{F}^{r, x}_t \right],$$

and

$$L_x(t, s) = E^Q \left[ I^x_t \frac{B^t_t}{\bar{B}^t_t} \mid \mathcal{F}^{r, x}_t \right] E^Q \left[ \frac{B^t_t}{\bar{B}^t_t} \mid \mathcal{F}^{r, x}_t \right] = I^x_t S_x(t, s) P(t, s),$$

(16)

due again to the independence between the interest rates and the mortality intensity process, where

$$S_x(t, s) = \tilde{A}_x(t, s) \exp \left[ -\bar{B}_x(t, s) \mu^x_t \right],$$

(17)

with

$$\tilde{A}_x(t, s) = \exp \left[ -\frac{\lambda x \sigma_x}{\alpha x} - \frac{\sigma_x^2}{2 \alpha^2 x} \right. \left. \left( B_x(t, s) - (s - t) \right) + \frac{\sigma_x^2}{4 \alpha x} B_x(t, s)^2 \right],$$

and

$$\bar{B}_x(t, s) = B_x(t, s).$$

**Remark 4.2.** We observe that

$$L_x(0, T) = S_x(0, T) P(0, T),$$

is close to the classical pricing formula

$$A_{x, \pi}^{-1} = T E_x = \tau p_x \nu^\tau,$$

of a pure endowment insurance product in actuarial science, with $\tau p_x$ the survival probability until time $T$ of a policyholder initially aged $x$ given through a deterministic mortality model such as the Gompertz law (see Remark 4.1), and where

$$\nu = \frac{1}{1 + i},$$

with $i \in \mathbb{R}$ the technical interest rate (we refer to [8]). In particular, if the interest rates and the force of mortality are not independent, this equivalence does not hold anymore since equation (16) would not be a simple product between an interest rates factor and a mortality factor, while under our models it would still give a closed-form expression.

### 4.2.3 Liabilities

The liability at maturity is given by

$$L_T = \pi_0 \frac{e^{\nu T}}{p_x(0, T)} I^x_T = \pi_0 \frac{e^{\nu T}}{p_x(0, T)} \frac{I^x_T}{p_x(0, T)},$$
The natural logarithm of the liability then follows a normal distribution

\[ \ln L_T \sim \mathcal{N} \left( \ln \left( l_0^T \pi_0 \right) + r_G T - \ln p_s(0, T) + m^x_T, (s^x_T)^2 \right) . \]

The market value of the global liability at time \( t \in \{0, \ldots, T\} \) is then given by

\[
L_t = \mathbb{E}^Q \left[ L_T \mathbb{B}_t \mathbb{B}_T \mathbb{J}^{x,t}_t \right] = \pi_0 \frac{e^{r_G t}}{p_s(0, T)} \mathbb{E}^{Q^x} \left[ I^x_T \mathbb{J}^{x,t}_t \right] P(t, T)
\]

\[
= l_0^T \pi_0 \frac{e^{r_G t}}{p_s(0, T)} l^x_T S_x(t, T) P(t, T)
\]

\[
= l_0^T \pi_0 \frac{e^{r_G t}}{p_s(0, T)} L_x(t, T) .
\]

### 4.2.4 Calibration choice

Parameters of the Ornstein-Uhlenbeck process and the market price of longevity risk have been chosen such that the initial term structure of mortality \( T \mapsto S_x(0, T) \) is similar to the corresponding curve under the legal Belgian MR-5 table (see Table 3). We observe that if the market price of longevity risk is equal to zero, then the probability measures \( \mathbb{P} \) and \( Q^x \) are equal. Furthermore, if again \( \lambda_x = 0 \), we also have that

\[ p_s(t, s) = S_x(t, s) , \]

since the insurer guarantees the expected value of the survivor index (15). However, we have chosen \( \lambda_x \) to be equal to \(-1.5\%\), such that

\[ p_s(t, s) < S_x(t, s) , \]

as illustrated in Figure 3. This choice correspond to a safety loading in the computation of the market value of liabilities. We are more conservative under the risk-neutral probability measure \( Q \) than under the real-world measure \( \mathbb{P} \), for life insurance operations.

We also assume that \( x = 20, \pi_0 = 1000, l_0^T = 1000 \) and that the guaranteed rate is equal to the risk-free rate,

\[ r_G = -\frac{1}{T} \ln P(0, T) , \]

such that the guaranteed rate is now linked to the maturity of the product. We then have that

\[ L_0 = l_0^T \pi_0 S_x(0, T) = 1000 \times 1000 \times S_x(0, T) / p_s(0, T) > 1000 \times 1000 . \]
Remark 4.3. We emphasize that if we set \( \alpha_t = 0, r_0 = b_t = 0.015 \) and \( \sigma_t = 0 \), then we get back the equity model described in Section 3. However we chose to begin with the equity model in order to first highlight the impact of the equity risk on the different measures introduced at the beginning of this paper.

4.3 Assets

The \( t_0^0 \) initial contributions (premiums) \( \pi_0 > 0 \) are invested in a portfolio \( A \) over the period \([0, T]\) made up of the bank account \( B_t \), zero-coupon bonds \( P \) and the stock \( S \). This portfolio follows a constant allocation strategy, is self-financing and follows a rolling bond strategy, i.e. its SDE is given by, for \( t \in [0, T] \),

\[
\begin{align*}
\text{d}A_t &= x_S A_t \frac{\text{d}S_t}{S_t} + x_p A_t \frac{\text{d}P(t, t + K)}{P(t, t + K)} + (1 - x_S - x_p) A_t \frac{\text{d}B(t)}{B(t)} \\
&= ((1 - x_S) r_t + x_S \mu_S + x_p \lambda_t \sigma_t B_t(t, t + K)) A_t \text{d}t + x_S \sigma_S A_t \text{d}W_t^S - x_p \sigma_t B_t(t, t + K) A_t \text{d}W_t^S,
\end{align*}
\]

where \( x_S \in [0, 1] \) is the deterministic proportion of the portfolio invested in the stock, \( x_p \in [0, 1 - x_S] \) is the deterministic proportion of the portfolio invested in the zero-coupon bonds and \( K > 0 \) is the time to maturity of the rolling bond strategy. At any time \( t \in [0, T] \), a proportion \( x_S \) of the portfolio \( A_t \) is invested in the stock, a proportion \( x_p \) is invested in the zero-coupon bonds and a proportion \( 1 - x_S - x_p \) is invested in the bank account. We have that

\[
B_t(t, t + K) = B_t(K) = \frac{1}{d_t} \left(1 - e^{-\sigma_t K}\right),
\]

and then

\[
\begin{align*}
A_t &= t_0^0 \pi_0 \exp \left[ \left(1 - x_S \right) \int_0^t \text{d}s + \left( x_S \mu_S + x_p \lambda_t \sigma_t B_t(K) - \frac{1}{2} \left( x_S^2 \sigma_S^2 + x_p^2 \sigma_t^2 B_t^2(K) - 2 p x_S x_p \sigma_S \sigma_t B_t(K) \right) \right) t \\
&\quad + x_S \sigma_S W_t^S - x_p \sigma_t B_t(K) W_t^S \right],
\end{align*}
\]

for all \( t \in [0, T] \). The natural logarithm of the assets follows a normal distribution

\[
\ln A_t \sim \mathcal{N}(\ln (t_0^0 \pi_0) + m_t^A, \left( s_t^A \right)^2),
\]

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7 Arrêté royal du 14 novembre 2003 relatif à l’activité d’assurance sur la vie, Moniteur belge, 14 November 2003, p. 55201.
where
\[
m^A_t = \left( (1 - x_S) b_r + x_S \lambda S + x_p \lambda \sigma_r B_r(K) - \frac{1}{2} \left( x_S^2 \sigma_S^2 + x_p^2 \sigma_r^2 B^2_r(K) - 2 \rho x_S x_p \sigma_S \sigma_r B_r(K) \right) \right) t \\
+ (1 - x_S) \frac{r_0 - b_r}{a_r} \left( 1 - e^{-a_r t} \right),
\]
and
\[
\left( s^A_t \right)^2 = x_S^2 \sigma_S^2 (1 - \rho^2) t + \left( (1 - x_S) \frac{\sigma_r}{a_r} + x_S \sigma_S \rho - x_p \sigma_r B_r(K) \right)^2 t \\
+ (1 - x_S)^2 \sigma_r^2 \frac{1 - e^{-2a_r t}}{2a_r^2} - 2 \left( (1 - x_S) \frac{\sigma_r}{a_r} + x_S \sigma_S \rho - x_p \sigma_r B_r(K) \right) \left( 1 - x_S \right) \sigma_r \frac{1 - e^{-a_r t}}{a_r^2} .
\]
We set \( K = 8 \) and we assume that 15% of the portfolio is invested in the stock, 80% in zero-coupon bonds and the remaining 5% in cash,
\[
x_S = 0.15 ,
\]
and
\[
x_p = 0.8 .
\]
Due to the safety loading with the market price of longevity risk included in the computation of the market value of the liabilities and since the insurer guarantees the expected value of the survivor index (15), the initial net value of the company is negative,
\[
N_0 = A_0 - L_0 = l_0^2 \pi_0 \left( 1 - \frac{S_x(0, T)}{p_x(0, T)} \right) < 0 .
\]
We then assume that shareholders are going to bring enough capital after the computation of the SCR, such that the company is solvent at the inception time. The insurer could have chosen other values for \( p_x(0, T) \), by including its own loading. This would decrease the level of the solvency capital. The purpose of our choice for \( p_x(0, T) \) allows us to highlight the effect of each measure. We emphasize that there is no outflow at time \( t = 0 \).

### 4.4 Probability of default

As in Section 3.4, we compute that
\[
\Psi = P [ A_T < L_T ] = \Phi \left( \frac{r_T - \ln p_x(0, T) + m^x_T - m^A_T}{\sqrt{(s^x_T)^2 + (s^A_T)^2}} \right) ,
\]
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Figure 5: Solvency capital with the complete model in proportion of the initial value of the portfolio: solvency capital under SII (black dotted line), at maturity (black dashed line) and at ruin (black solid line). The corresponding curves without the longevity risk ($\sigma_x = 0$) are given in grey.

Figure 4 is a generalization of Figure 1. It illustrates the evolution of this probability along with the maturity of the product for the equity-longevity model (dotted line) and the complete model (dashed line). As in Section 3.4, we clearly see a decreasing trend of the probability of default with the maturity.

However, we observe that the longevity risk has a slight impact on this probability. The curve under the equity-longevity model (dotted line) is slightly higher than under the equity model (solid line) for long maturities. We then see that, as expected, the longevity risk tends to moderately neutralize the benefit of an investment in the stock—the equity premium. We conclude from this plot that a greater level of capital under the complete model should be required, compared with the equity model. This is in line with our intuition as we have added two main risks that an insurer or pension fund faces.

### 4.5 Solvency capital

The approach considered here is the same as the one presented in Section 3.5. We compute the solvency by considering the assets $A_t$ and liabilities $L_t$, $t \in \{0, \ldots, T\}$, and by investing the capital in zero-coupon bonds. However, unlike Section 3.5, it is not possible anymore to have closed-form expressions for the solvency capital given by Definitions 2.5, 2.7 and 2.10. It is due to the fact that the net value at time $t \in \{1, \ldots, T\}$,

$$N_t = A_t - L_t,$$

is now a difference between two log-normal variables and that the price of $P(t, T)$ is also log-normally distributed. Nevertheless, it is easy to compute these capital through a numerical approach, such as Monte Carlo simulations.

Figure 5 illustrates these capital for maturities between 1 and 45 years. We also give these quantities when the longevity risk is removed—only the equity and interest rate risks are maintained, i.e. we set $\sigma_x = 0$.

We clearly observe the same trends we observed in Figure 2. The impact of the longevity risk appears clearer for long maturities. The level of capital required increases significantly with the maturity. This increase is less obvious for the solvency capital at maturity (dashed lines) as it becomes zero around a maturity of 30 years. These observations are in line with the probability of default obtained in Section 4.4.

### 4.6 Solvency capital with zero-coupon longevity bonds

We now change our reference instrument. Instead of investing the solvency capital in a risk-free zero-coupon bond $P(0, 1)$ for the SII approach or $P(0, T)$ for the maturity and ruin approaches, we invest it in a zero-coupon longevity bond (ZCLB). We make the implicit assumption that such a market exists and is liquid.
In particular, we write $\text{SCR}_{\text{ZCLB}}^\text{mat}$ the SCR at maturity considering a ZCLB with final value 
\[ \frac{L_x(0, T)}{I_T^x} \]
and the corresponding SC is defined by
\[ \text{SC}_{\text{ZCLB}}^\text{mat} = \left( \text{SCR}_{\text{ZCLB}}^\text{mat} - N_0 \right)_+. \]

**Proposition 4.4.** We have that
\[ \text{SC}_{\text{ZCLB}}^\text{mat} = (L_0 - L_x(0, T)A_0 e^{cr})_+, \]
with
\[ c_T = m^x_T - m_T^A + \sqrt{(s^A_T)^2 + (s^x_T)^2} \Phi^{-1}\left(1 - 0.995^T\right). \]

**Proof.** Since $N_0$ is deterministic and from the cash invariance of the VaR, we compute that
\[ \text{SCR}_{\text{ZCLB}}^\text{mat} - N_0 = \text{VaR}^{0.995^T} \left( (A_T - L_T) \frac{L_x(0, T)}{I_T^x} \right) = -q^+_{(A_T - L_T)L_x(0, T)/I_T^x}(1 - 0.995^T). \]

We also have that
\[ \frac{L_T}{I_T^x} = \frac{\ell_0^x \pi_0}{P_x(0, T)} e^{c_T T} > 0 \]
is a constant and
\[ \ln \frac{A_T}{I_T^x} \sim N \left( \ln \left( \ell_0^x \pi_0 \right) + m^A_T - m_T^x, \left( s_A^x \right)^2 + \left( s_T^x \right)^2 \right), \]
since the market and longevity risks are independent, then
\[ -q^+_{(A_T - L_T)L_x(0, T)/I_T^x}(1 - 0.995^T) = \frac{\ell_0^x \pi_0}{P_x(0, T)} e^{c_T T} \left( L_x(0, T) - L_x(0, T) \ell_0^x \pi_0 e^{c_T} \right) = L_0 - L_x(0, T)A_0 e^{cr}, \]
and the proof is complete. \( \square \)
Figure 7: Solvency capital invested in zero-coupon longevity bonds (black curves) compared with those invested in zero-coupon bonds (grey curves) with the complete model in proportion of the initial value of the portfolio, and with $x_S = 0$ and $x_P = 95\%$: solvency capital under SII (dotted line), at maturity (dashed line) and at ruin (solid line).

Figure 6 illustrates these quantities (black lines) compared with an investment in zero-coupon bonds (grey lines) from the previous section.

We observe that the impact of this approach is negligible for the capital computed under SII and at maturity. For SII, this is due to the one-year horizon approach, as the impact of the longevity risk over such a period is very low (we only consider zero-coupon longevity bonds over one year). For the maturity approach, this is again due to the advantage of an equity investment over long horizons with a measurement of the solvency at maturity only. Figure 7 illustrates this observation by removing the investment in the stock ($x_S = 0$ and $x_P = 95\%$). We observe that the solvency capital at maturity is lower with zero-coupon longevity bonds, while the impact for the SII approach is negligible.

However, for the capital at ruin, we see in both figures that considering an investment of the capital in zero-coupon longevity bonds slightly decreases the level of the capital required for long maturities. This smaller level for the solvency capital with this approach was expected as we protect ourselves against the longevity risk by investing in zero-coupon longevity bonds.

5 Conclusion

In this paper, we followed a ruin theory approach in order to determine the solvency capital of a life insurer or pension fund who offers a fixed guaranteed rate over a certain time horizon. We compared it to the Solvency II legislation and studied the impact of an equity risk. In a second step, we added the interest rate and longevity risks. We were then able to compute the solvency capital by considering either zero-coupon bonds or zero-coupon longevity bonds in order to protect ourselves against the longevity risk.

It appears that the maturity of the long-term guarantees has a significant impact on the capital computed through the maturity and ruin theory approaches, while it is not the case for the SII capital. In future researches, the case of long-term guarantees with multiple cash-flows will be considered. The credit risk will also be included. This risk will allow us to consider corporate bonds in order to have a more representative portfolio.

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