Sharing quantum nonlocality and genuine nonlocality with independent observables

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Recently the authors in [Phys. Rev. Lett. 125, 090401 (2020)] considered the following scenario: Alice and Bob each have half of a pair of entangled qubit state. Bob measures his half and then passes his part to a second Bob who measures again and so on. The goal is to maximize the number of Bobs that can have an expected violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality with the single Alice. By taking the maximally entangled pure two-qubit state\textsuperscript{1,\dagger}\textsuperscript{3,\#}

\begin{equation}
|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\end{equation}

as an example, it has been constructively proved that arbitrarily many independent Bobs can share the nonlocality with the single Alice. Here we demonstrate that arbitrarily many independent observers can share the nonlocality of a single arbitrary dimensional bipartite entangled but not necessary two-qubit entangled state. Further, taking the generalized GHZ states as an example, we show that at most two Charlies can share the genuine nonlocality of a single generalized GHZ state with an Alice and a Bob.

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INTRODUCTION

Quantum nonlocality, such as that revealed by violating the Bell inequalities of quantum entangled states\textsuperscript{1,\dagger}, is one of the most startling predictions of quantum mechanics. Recently, as confirmed in loophole-free experiments\textsuperscript{2}, nonlocality has been proven to be useful in many quantum tasks such as device-independent cryptography\textsuperscript{3} and randomness certification\textsuperscript{4,\textsuperscript{\#}}. A successful and secure quantum network relies on quantum correlations distributed and shared among many sites\textsuperscript{5}. Different kinds of multipartite quantum correlations have been considered as valuable resources for various applications in quantum communication tasks. A key property is that such quantum correlations cannot be freely shared among the multipartite systems, see e.g.\textsuperscript{6} and references therein.

Recently, in Ref\textsuperscript{10–14}, the authors studied the fundamental limits on nonlocality, asking whether a single pair of entangled qubits could generate a long sequence of nonlocal correlations. This sequential scenario (see FIG. 1) was introduced first in\textsuperscript{10}. With the same sharpness of the two measurements applied by each Bob, in Ref.\textsuperscript{11} the authors shown that at most two Bobs can achieve an expected CHSH\textsuperscript{13} violation with a single Alice, in line with the numerical evidence from\textsuperscript{10}. Equal sharpness two-outcome measurements were also adopted in\textsuperscript{12} to show that at most two Bobs can share the Bell nonlocality of a maximally entangled state with a single Alice by using local realist inequalities with three and four dichotomic measurements per observer\textsuperscript{13}. More recently, in\textsuperscript{14} the authors studied such scenario and shown that if the Bobs’ apply different measurements, then arbitrarily many independent Bobs can share the nonlocality of the maximally entangled pure two-qubit state\textsuperscript{1,\dagger}\textsuperscript{3,\#}

\begin{equation}
|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\end{equation}

with the single Alice.

High-dimensional quantum entanglement and nonlocality provide a playground for fundamental research and also lead to technological advances, with stronger locality violations that can be exploited to tolerate larger amounts of noise in quantum communication protocols. Various physical and technical approaches on how to manipulate multilevel quantum states in different degrees of freedom have been presented, inspiring new synergies.
that create new technologies such as teleporting the complete quantum information stored in a single “photon” [10]. Therefore, in this article we study such nonlocal correlation sharing scenario for arbitrary high dimensional bipartite entangled pure states. We show that arbitrarily many independent observers can share the nonlocality of any single arbitrary bipartite entangled states. Furthermore, we investigate the genuine nonlocality sharing among Alice, Bob and Charlie, see FIG. 2. By using the Svetlichny inequality [17], we show that at least two Charlies can share the genuine nonlocality of a single generalized GHZ state with Alice and Bob.

NONLOCAL SHARING OF BIPARTITE HIGH-DIMENSIONAL PURE STATES

We first consider the measurement scenario such that Alice attempts to share the nonlocal correlations of an entangled pure state with n independent Bobs (FIG. 1). Any bipartite pure state $|\psi\rangle \in H_A \otimes H_B$ with $\dim(H_A) = s$ and $\dim(H_B) = t$ ($s \leq t$) has Schmidt decomposition form, $|\psi\rangle = \sum_{i=1}^{s} c_i |i_A\rangle |i_B\rangle$, where $c_i \in [0, 1]$, $\sum_i c_i^2 = 1$, $\{i_A\}_1^s$ and $\{i_B\}_1^t$ are the orthonormal bases of $H_A$ and $H_B$, respectively. $|\psi\rangle$ is entangled if and only if at least two $c_i$s are nonzero. Without loss of generality, below we assume that $c_i$ are arranged in descending order.

To begin with, Alice and Bob(1) share an arbitrary entangled bipartite pure state $\rho_{AB}^{(1)} = |\psi\rangle \langle \psi| = \sum_{i,j=1}^{s,t} c_i c_j |i\rangle \langle i| |j\rangle \langle j|$. Bob(1) proceeds by choosing a uniformly random input, performing the corresponding measurement and recording the outcome. Denote the binary input and output of Alice (Bob(1)) by $X$ ($Y^{(1)}$) and $A$ ($B^{(1)}$), respectively. Suppose Bob(1) performs the measurement according to $Y^{(1)} = y$ with the outcome $B^{(1)} = b$. Averaged over the inputs and outputs of Bob(1), the unnormalized state shared between Alice and

Bob(2) is given by

$$\rho_{AB}^{(2)} = \frac{1}{2} \sum_{b,y} (I_s \otimes \sqrt{B^{(1)}_{by}}) \rho_{AB}^{(1)} (I_s \otimes \sqrt{B^{(1)}_{by}})$$

where $B^{(1)}_{by}$ is the positive operator-valued measure (POVM) effect corresponding to outcome $b$ of Bob(1)’s measurement for input $y$. $I_s$ is the $s \times s$ identity matrix. Repeating this process, one gets the state $\rho_{AB}^{(k)}$ shared between Alice and Bob(k).

To detect the nonlocality we employ the CHSH inequality [15], $I_{CHSH} = \langle B \rangle \leq 2$, where $\langle B \rangle = Tr(\rho B)$, $B = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$, $A_i$ and $B_i$, $i = 0, 1$, are Hermitian operators with eigenvalues $\in [-1, 1]$. If for some binary observables $A_i$ and $B_i$, $i = 0, 1$, $I_{CHSH}^{(k)} = Tr(\rho_{AB}^{(k)}) > 2$, then the state $\rho_{AB}^{(k)}$ is nonlocally correlated.

For the case that $s$ and $t$ are even, we employ the POVMs with measurement operators $\{E, I - E\}$, where $E$ has the form $E = \frac{1}{2}(I_m + \gamma (I_\theta \otimes I_\gamma \otimes I_\phi \otimes I_\delta))$, $\vec{r} \in R^3$ with $||\vec{r}|| = 1$, $\vec{r} \cdot \vec{d} = r_1 \sigma_3 + r_2 \sigma_2 + r_3 \sigma_3$, $\sigma_i, i = 1, 2, 3$, are the standard Pauli matrices, $\gamma \in [0, 1]$ is the sharpness of the measurement, $I_m$ stands for the $m \times m$ identity matrix, $m = s, t$. We set the Alice’s POVMs to be

$$A_{0|0} = \frac{1}{2}(I_s + (I_\theta \otimes (\cos \theta \sigma_3 + \sin \theta \sigma_1)))$$

$$A_{0|1} = \frac{1}{2}(I_s + (I_\theta \otimes (\cos \theta \sigma_3 - \sin \theta \sigma_1)))$$

for some $\theta \in (0, \frac{\pi}{2}]$. For each $k = 1, 2, \cdots, n$, Bob(k)’s POVMs are defined as

$$B_{0|0}^{(k)} = \frac{1}{2}(I_t + (I_\theta \otimes \gamma_k \sin \theta \sigma_1))$$

$$B_{0|1}^{(k)} = \frac{1}{2}(I_t + (I_\theta \otimes \gamma_k \cos \theta \sigma_3))$$

where $m = s, t$, $|p|$ represents the integer less or equal to $p$. The Alice’s POVMs are defined as

$$A_{0|0} = \frac{1}{2}(I_s + (I_\theta \otimes \gamma_k \cos \theta \sigma_3 + \sin \theta \sigma_1))$$

$$A_{0|1} = \frac{1}{2}(I_s + (I_\theta \otimes \gamma_k \cos \theta \sigma_3 - \sin \theta \sigma_1))$$

FIG. 2: A quantum state $\rho_{ABC}^{(1)}$ is initially shared by Alice, Bob and Charlie(1). Charlie(1) performs a measurement on her part and passes it to Charlie(2). The post-measurement state is $\rho_{ABC}^{(2)}$. Charlie(2) measures $\rho_{ABC}^{(2)}$ on her part and passes it to Charlie(3) and so on.
for some $\theta \in (0, \frac{\pi}{4}]$. The Bob$^{(k)}$'s POVMs set to be
\[ B_{0|0}^{(k)} = \frac{1}{2} \left[ I_{t} + \left( \frac{I_{\hat{x}}}{2} \otimes \sigma_{3} \right) \right], \tag{7} \]
\[ B_{0|1}^{(k)} = \frac{1}{2} \left[ I_{t} + \gamma_{k} \left( \frac{I_{\hat{x}}}{2} \otimes \sigma_{1} \right) \right], \tag{8} \]
where $k = 1, 2, \cdots, n$.

The observables are then given by $A_{x} = A_{0|x} - A_{1|x}$ and $B_{y}^{(k)} = B_{0|y}^{(k)} - B_{1|y}$, $x, y = 0, 1$, for both even and odd $k$, respectively. We have the following conclusion for the expected CHSH value for Alice and Bob$^{(k)}$, see proof in Appendix.

Theorem 1. For any initial entangled bipartite pure quantum state $|\psi\rangle \in H_{A} \otimes H_{B}$ with Schmidt decomposition $|\psi\rangle = \sum_{i=1}^{n} c_{i} |i_{A}\rangle |i_{B}\rangle$, the expected CHSH value of $\rho_{AB}^{(k)}$ is given by
\[ I_{CHSH}^{(k)} \geq 2^{2-k} \left[ \gamma_{k} L \sin \theta + \cos \theta \prod_{j=1}^{k-1} \left( 1 + \sqrt{1 - \gamma_{j}^{2}} \right) \right], \tag{9} \]
where $L = 2(c_{1} c_{2} + c_{3} c_{4} + \cdots + c_{2l-1} c_{2l})$.

Next we show that there exist suitable parameters $\gamma_{k}$ and $\theta$ such that $I_{CHSH}^{(k)} > 2$ for arbitrary $k$. From Theorem 1 we require that
\[ \gamma_{k} > \frac{2^{k-1} - \cos \theta \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_{j}^{2}})}{L \sin \theta}. \tag{10} \]
Set $\gamma_{1}(\theta) = (1 + \epsilon) \frac{1 - \cos(\theta)}{L \sin(\theta)}$ for $\epsilon > 0$. One has
\[ \gamma_{k}(\theta) = (1 + \epsilon) \frac{2^{k-1} - \cos \theta \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_{j}^{2}})}{L \sin \theta} \tag{11} \]
for $\gamma_{k-1}(\theta) \in (0, 1), k \in \{2, \cdots, n\}$.

Theorem 2. For each $n \in \mathbb{N}$, there exists a sequence $\{\gamma_{k}\}_{k=1}^{n}$ and $\theta_{n} \in (0, \frac{\pi}{4}]$ such that $I_{CHSH}^{(k)} > 2$ for all $k = 1, 2, \cdots, n$, with $\theta \in (0, \theta_{n})$ and $\gamma_{k}(\theta) < 1$ for all $k \leq n$.

The proof is given in Appendix. Theorem 2 shows that arbitrarily many independent observers can share the nonlocality of a single arbitrary dimensional bipartite entangled state, as long as at least two Schmidt coefficients $c_{i}$ of the state are nonzero. Moreover, the state is not necessary to be maximally entangled.

Genuine Nonlocal Sharing of Three-Qubit States

Multipartite nonlocal correlations have not only foundational implications [18] but also novel applications in quantum communication and quantum computation [19–22], as well as in phase transitions and criticality in many-body systems [21]. Essentially different from the bipartite case, one has so called genuine multipartite nonlocality for multipartite systems. In the following, we consider the sharing ability of the genuine nonlocality of the three-qubit generalized GHZ state, $|\psi_{\alpha}\rangle = \cos \alpha |000\rangle + \sin \alpha |111\rangle$.

Let $A_{i} = a_{i} \cdot \hat{d}, B_{i} = b_{i} \cdot \hat{d}$ and $C_{i} = c_{i} \cdot \hat{d}$, $i = 0, 1$, be the measurement observables on the first, second, and third qubit, respectively, with $a_{i}, b_{i}$, and $c_{i}$ the real unit vectors. The Svetlichny operator is defined by [17],
\[ S = A_{0}(B_{0} + B_{1})C_{0} + A_{0}(B_{0} - B_{1})C_{1} + A_{1}(B_{0} - B_{1})C_{0} - A_{1}(B_{0} + B_{1})C_{1}. \tag{12} \]
If a state $|\psi\rangle$ admits bi-local hidden variable model, then the expectation value of the Svetlichny operator satisfies the Svetlichny’s inequality,
\[ \mathbb{E}(\psi) = \langle \psi | S | \psi \rangle \leq 4. \tag{13} \]
If (13) is violated, $|\psi\rangle$ must be a genuine three-qubit nonlocally correlated state. The maximal violation of the Svetlichny inequality (13) for the state $|\psi_{\alpha}\rangle$ has been studied in [23]. It has been shown that when $\sin^{2} 2\alpha > \frac{1}{2}$, the state $|\psi_{\alpha}\rangle$ is genuine three-qubit nonlocal.

We consider the following measurement scenario, see FIG. 2: Alice and Bob want to share the genuine three-qubit nonlocality of single three-qubit state $|\psi_{\alpha}\rangle$ with possible $n$ independent Charlies. Denote the binary input and output of Alice (Bob) by $X$ ($Y$) and $A$ ($B$), respectively. For each $k \in N$ we denote the binary input and output of Charlie$^{(k)}$ by $Z^{(k)}$ and $C^{(k)}$, respectively. At the beginning, the three-qubit state $|\psi\rangle$ is shared among Alice, Bob and Charlie$^{(1)}$. Charlie$^{(1)}$ proceeds by choosing a uniformly random input, performing the corresponding measurement and recording the outcome. The postmeasurement qubit is then sent to Charlie$^{(2)}$. Suppose Charlie$^{(1)}$ performs the measurement according to $Z^{(1)} = z$ and receives the outcome $C^{(1)} = c$. The postmeasurement state can be described by the Lüders rule. Averaged over the inputs and outputs of Charlie$^{(1)}$'s, the postmeasurement unnormalized state $\rho_{ABC}^{(2)}$ shared among Alice, Bob and Charlie$^{(2)}$ is given by
\[ \rho_{ABC}^{(2)} = \frac{1}{2} \sum_{c,z} (I_{2} \otimes I_{2} \otimes \sqrt{C_{c,z}^{(1)}}) \rho_{ABC}^{(1)} (I_{2} \otimes I_{2} \otimes \sqrt{C_{c,z}^{(1)}}), \]
where $C_{c,z}^{(1)}$ is the POVM effect corresponding to the outcome $c$ of Charlie$^{(1)}$'s measurement for input $z$. Repeating this process, one can compute the state $\rho_{ABC}^{(k)}$ shared among Alice, Bob and Charlie$^{(k)}$. The expected value of Svetlichny operator associated with the state $\rho_{ABC}^{(k)}$ and the binary random observables $A_{i}$, $B_{i}$ and $C_{i}^{(k)}$, $i = 0, 1$, is given by $S^{(k)} \equiv \text{Tr}(S \rho_{ABC}^{(k)})$. 

To see the maximal $k$ such that $S^{(k)} > 4$, we consider again two-outcome POVMs $\{E, I - E\}$. Let Alice’s POVMs be given by

$$A_{0|0} = \frac{1}{2}(I + \sigma_1), \quad A_{0|1} = \frac{1}{2}(I + \sigma_2)$$

and Bob’s POVMs by

$$B_{0|0} = \frac{1}{2}(I + \cos \theta \sigma_1 - \sin \theta \sigma_2), \quad B_{0|1} = \frac{1}{2}(I + \cos \theta \sigma_1 + \sin \theta \sigma_2)$$

for some $\theta \in (0, \frac{\pi}{4})$. The Charlie's POVMs are defined by

$$C^{(k)}_{0|0} = \frac{1}{2}(I + \sigma_1), \quad C^{(k)}_{0|1} = \frac{1}{2}(I + \gamma_k \sigma_2)$$

for $k = 1, 2, \cdots, n$.

Set $A_x = A_{0|x} - A_{1|x}$, $B_y = B_{0|y} - B_{1|y}$ and $C^{(k)}_z = C^{(k)}_{0|z} - C^{(k)}_{1|z}$, $x, y, z = 0, 1$. We have the following conclusion, see detailed proof in Appendix.

**Theorem 3.** For the initially shared generalized GHZ state $|\psi_{\alpha}\rangle$, the expected value of the Svetlichny operator with respect to the state $\rho_{ABC}^{(k)}$ is given by

$$S^{(k)} = 2^{2-k} \sin 2\alpha (\cos \theta + \sin \theta) (\gamma_k + \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_j^2})).$$

For $k = 1$, we have $S^{(1)} = 2 \sin 2\alpha (\cos \theta + \sin \theta) (\gamma_1 + 1)$. If $\gamma_k = 1$ and $\theta = \frac{\pi}{4}$, then $S^{(1)} > 4$ as long as $\sin^2(2\alpha) > 0.5$. By detailed analysis, we have, see proof in Appendix,

**Theorem 4.** There are at most two Charlies sharing the genuine nonlocality with Alice and Bob, $S^{(k)} > 4$, $k = 1, 2$, when $\sin^2 2\alpha \in \left(\frac{3}{8}, 1\right]$ for the initially shared generalized GHZ state $|\psi_{\alpha}\rangle$.

In [24] the authors considered this problem from the formalism of weak or unsharp measurements for the GHZ stat, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Our conclusion is for the generalized GHZ states: when $\sin^2 2\alpha \in \left(\frac{3}{8}, 1\right]$, at most two Charlies can share the three-qubit genuine nonlocality of a single generalized GHZ state with an Alice and a Bob, which coincides with the result derived in [24] as a special case ($\alpha = \frac{\pi}{4}$).

**CONCLUSIONS AND DISCUSSIONS**

Quantum nonlocality is a fundamental feature in quantum mechanics. We have demonstrated that it is possible for arbitrarily many independent Bobs to violate the CHSH inequality with a single Alice by using any bipartite high-dimensional pure states that are either maximally or non-maximally entangled. As the approach used for qubit case can not be used for high dimensional case since relations like $\text{Tr}(\rho (\sigma_x \otimes \sigma_y)) = (\hat{a}, T(\rho)\hat{b})$ is only correct for two-qubit states, where $T(\rho)$ is a matrix with entries given by $T_{ij}(\rho) = \text{Tr}[\rho (\sigma_i \otimes \sigma_j)]$, we have presented a new approach in derivations. Our innovation also lies in choosing the POVM measurement operators and in calculating the expected CHSH values. Our approach can be also extended to the case of mixed states.

We have also investigated the shareability of genuine tripartite nonlocality. For the generalized GHZ state, it has been shown that from our measurement schemes two Charlies can share the genuine nonlocality with Alice and Bob. Our results may also highlight researches on sharing general multipartite quantum nonlocalities and other quantum correlations such as quantum steerability [25, 26], entanglement [27, 28] and coherence [29].

Finally, in this article we have constructed the higher-dimensional dichotomic POVM measurement operators in terms of the Pauli operators. As the Pauli operators are easily implemented in experiments, the POVM operators we constructed may have potential advantages in some specific experimental implementations [30, 31]. Our approach may also highlight the related applications in randomness generation [32], quantum teleportation [33] and random access codes [34]. It would be also interesting to give some insight into larger multipartite scenarios and explore the relationship between our methodology and, for instance, nonlocality depth sharing [35] and monogamy relations [36].

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[1] J. S. Bell, Physics, 1, 195 (1964).
[2] B. Hensen, et al, Nature, 526, 682 (2015);
M. Giustina, et al, Phys. Rev. Lett 115, 250401, (2015);
L. K. Shalm, et al, Phys. Rev. Lett. 115, 250402, (2015).
[3] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. 98, 230501 (2007).
[4] S. Pironio, et al, Nature 464, 1021 (2010).
[5] R. Colbeck, arXiv:0911.3814 [quant-ph] (2019).
[6] L. K. Shalm, et al, arXiv:1912.11158 [quant-ph] (2019).
[7] W.Z. Liu, et al, arXiv:1912.11159 [quant-ph] (2019).
[8] H.J. Kimble, Nature (London) 453, 1023 (2008).
[9] Z.X. Jin, S.M. Fei, Phys. Rev. A 99, 032343 (2019).
APPENDIX

Proof of Theorem 1

We first consider the simplest case that both \( t \) and \( s \) are even. From (1), (2), (3) and (4) we have

\[
A_0 = A_{0|0} - A_{1|0} = I_\varpi \otimes (\cos \theta \sigma_3 + \sin \theta \sigma_1),
\]

\[
A_1 = A_{0|1} - A_{1|1} = I_\varpi \otimes (\cos \theta \sigma_3 - \sin \theta \sigma_1)
\]

and

\[
B^{(k)}_0 = B_{0|0} - B_{1|0} = I_\varpi \otimes \sigma_3,
\]

\[
B^{(k)}_1 = B_{0|1} - B_{1|1} = I_\varpi \otimes \gamma k \sigma_1.
\]

we have

\[
I_{CHSH}^{(1)} = Tr[\rho^{(1)}_{AB}(A_0 + A_1) \otimes B^{(k)}_0] + Tr[\rho^{(1)}_{AB}(A_0 - A_1) \otimes B^{(k)}_1]
\]

\[
= 2 \cos \theta \sin \theta \left( I_\varpi \otimes \sigma_3 \right) \left( I_\varpi \otimes \sigma_3 \right)
\]

\[
+ 2 \gamma k \sin \theta \left( I_\varpi \otimes \sigma_1 \right) \left( I_\varpi \otimes \sigma_1 \right).
\]

For the case \( k = 1 \), we get

\[
I_{CHSH}^{(1)} = 2 \cos \theta \sin \theta \left( I_\varpi \otimes \sigma_3 \right) \left( I_\varpi \otimes \sigma_3 \right)
\]

\[
+ 2 \gamma \sin \theta \left( I_\varpi \otimes \sigma_1 \right) \left( I_\varpi \otimes \sigma_1 \right)
\]

\[
= 2 \cos \theta \sin \theta L \sin \theta,
\]

where we have used \( \rho^{(1)}_{AB} = \Sigma_{i,j} c_i c_j |ii\rangle\langle jj| \) and \( L = 2(2c_1c_2 + c_3c_4 + \cdots) \).

To obtain the relation between \( \rho^{(1)}_{AB} \) and \( \rho^{(k)}_{AB} \) we use the following equation,

\[
\frac{1}{2}(I_\varpi \pm \gamma k I_\varpi \otimes \sigma) = \frac{1}{2\sqrt{2}} (I_\varpi \pm \sqrt{1 + \gamma k + \sqrt{1 - \gamma k}}) I_\varpi \pm \sqrt{1 + \gamma k - \sqrt{1 - \gamma k}} (I_\varpi \otimes \sigma)
\]

for \( \sigma = \sigma_1, \sigma_3 \). By using the Lüders update rule we have

\[
\rho^{(k)}_{AB} = \frac{1}{2} \Sigma_{b,y} (I_\varpi \otimes \sqrt{B^{(k)}_{b|y}}) \rho^{(k-1)}_{AB} (I_\varpi \otimes \sqrt{B^{(k-1)}_{b|y}})
\]

\[
= \frac{1}{4} (I_\varpi \otimes (I_\varpi \otimes \sigma_3)) \rho^{(k-1)}_{AB} (I_\varpi \otimes (I_\varpi \otimes \sigma_3))
\]

\[
+ \frac{1 - \sqrt{1 - \gamma^2}}{4} (I_\varpi \otimes (I_\varpi \otimes \sigma_1)) \rho^{(k-1)}_{AB} (I_\varpi \otimes (I_\varpi \otimes \sigma_1))
\]

\[
+ \frac{2 + \sqrt{1 - \gamma^2}}{4} \rho^{(k-1)}_{AB}.
\]
Substituting (A3) into (A1) and taking into account the relations $\sigma_3 \sigma_3 \sigma_3 = \sigma_3$ and $\sigma_1 \sigma_3 \sigma_1 = -\sigma_3$, we get

$$\text{Tr}[\rho_{AB}^{(k)}(I_\mathbb{Z} \otimes \sigma_3) \otimes (I_\mathbb{Z} \otimes \sigma_3)] = \frac{1}{4} \left( 1 - \sqrt{1 - \gamma_k^2} + 2 + \sqrt{1 - \gamma_k^2} \right)$$

$$\cdot \text{Tr}[\rho_{AB}^{(k-1)}(I_\mathbb{Z} \otimes \sigma_3) \otimes (I_\mathbb{Z} \otimes \sigma_3)]$$

$$= \frac{1}{2} \left( 1 + \sqrt{1 - \gamma_k^2} \right)$$

$$\cdot \text{Tr}[\rho_{AB}^{(k-1)}(I_\mathbb{Z} \otimes \sigma_3) \otimes (I_\mathbb{Z} \otimes \sigma_3)].$$

Similarly we can obtain

$$\text{Tr}[\rho_{AB}^{(k)}(I_\mathbb{Z} \otimes \sigma_1) \otimes (I_\mathbb{Z} \otimes \sigma_1)] = \frac{1}{2} \text{Tr}[\rho_{AB}^{(k-1)}(I_\mathbb{Z} \otimes \sigma_1) \otimes (I_\mathbb{Z} \otimes \sigma_1)].$$

By recursion, we get

$$\text{Tr}[\rho_{AB}^{(k)}(I_\mathbb{Z} \otimes \sigma_3) \otimes (I_\mathbb{Z} \otimes \sigma_3)] = 2^{1-k} \prod_{j=1}^{k-1} \left[ (1 + \sqrt{1 - \gamma_j^2}) \text{Tr}[\rho_{AB}^{(j)}(I_\mathbb{Z} \otimes \sigma_3) \otimes (I_\mathbb{Z} \otimes \sigma_3)] \right]$$

and

$$\text{Tr}[\rho_{AB}^{(k)}(I_\mathbb{Z} \otimes \sigma_1) \otimes (I_\mathbb{Z} \otimes \sigma_1)] = 2^{1-k} \text{Tr}[\rho_{AB}^{(1)}(I_\mathbb{Z} \otimes \sigma_1) \otimes (I_\mathbb{Z} \otimes \sigma_1)].$$

Substituting (A1) and (A3) into (A1) and using (A2), we obtain (9).

Now we prove the most complex case that both $s$ and $t$ are odd. From (6), (7), (8) and (9) we have

$$A_0 = A_{0|0} - A_{1|0} = \left( I_\mathbb{Z} \otimes (\cos \theta \sigma_3 + \sin \theta \sigma_1) \otimes (0 \ 1) \right)$$

$$A_1 = A_{0|1} - A_{1|1} = \left( I_\mathbb{Z} \otimes (\cos \theta \sigma_3 - \sin \theta \sigma_1) \otimes (0 \ 1) \right)$$

$$B_0^{(k)} = B_{0|0} - B_{1|0} = \left( I_\mathbb{Z} \otimes \sigma_3 \otimes (0 \ 1) \right)$$

and

$$B_1^{(k)} = B_{0|1} - B_{1|1} = \gamma_k \left( I_\mathbb{Z} \otimes \sigma_1 \otimes (0 \ 1) \right).$$

Correspondingly, we have

$$I_{CHSH}^{(k)} = \text{Tr}[\rho_{AB}^{(k)}((A_0 + A_1) \otimes B_0^{(k)})]$$

$$+ \text{Tr}[\rho_{AB}^{(k)}((A_0 - A_1) \otimes B_1^{(k)})]$$

$$= 2 \text{Tr} \left[ \rho_{AB}^{(k)} \left( I_\mathbb{Z} \otimes (\cos \theta \sigma_3) \otimes (0 \ 1) \right) \otimes \left( I_\mathbb{Z} \otimes \sigma_3 \otimes (0 \ 1) \right) \right]$$

$$+ 2 \gamma_k \sin \theta \text{Tr} \left[ \rho_{AB}^{(k)} \left( I_\mathbb{Z} \otimes \sigma_1 \otimes (0 \ 1) \right) \otimes \left( I_\mathbb{Z} \otimes \sigma_1 \otimes (0 \ 1) \right) \right].$$

For the case $k = 1$, one has

$$I_{CHSH}^{(1)} = 2 \text{Tr} \left[ \rho_{AB}^{(1)} \left( I_\mathbb{Z} \otimes (\cos \theta \sigma_3) \otimes (0 \ 1) \right) \otimes \left( I_\mathbb{Z} \otimes \sigma_3 \otimes (0 \ 1) \right) \right]$$

$$+ 2 \gamma_1 \sin \theta \text{Tr} \left[ \rho_{AB}^{(1)} \left( I_\mathbb{Z} \otimes \sigma_1 \otimes (0 \ 1) \right) \otimes \left( I_\mathbb{Z} \otimes \sigma_1 \otimes (0 \ 1) \right) \right]$$

$$= \begin{cases} 
2 \cos \theta (1 - c_2^2) + 2c_3^2 + 2\gamma_1 L \sin \theta, & \text{if } s = t \text{ and } c_s \neq 0, \\
2 \cos \theta + 2\gamma_1 L \sin \theta, & \text{otherwise.}
\end{cases}$$

where we have used $\rho_{AB}^{(1)} = \Sigma_{i,j=1}^2 c_i c_j |ij\rangle \langle jj|$ and $L = 2(c_1 c_2 + c_3 c_4 + \cdots + c_2 |12\rangle - c_1 |21\rangle)$.

Using the following identity

$$\begin{aligned}
\frac{1}{2} (I_\mathbb{Z} \otimes \sigma_3 \otimes (0 \ 1)) &= \frac{(\sqrt{1 + \gamma_k} + \sqrt{1 - \gamma_k})I_\mathbb{Z}}{2\sqrt{2}} \\
&\pm \frac{(\sqrt{1 + \gamma_k} - \sqrt{1 - \gamma_k})I_\mathbb{Z}}{2\sqrt{2}}
\end{aligned}$$

and repeating the similar process for even $t$ and $s$, we get the relations similar to Eq. (A1) and Eq. (A5). Moreover, we get

$$I_{CHSH}^{(k)} \geq 2^{2-k} (\gamma_k L \sin \theta + \cos \theta \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_j^2})).$$

It is straightforward to prove that the above inequality holds also for the cases of even (odd) $t$ and odd (even) $s$.

**Proof of Theorem 2**

The Theorem can be proved by using two lemmas in [14]. Since the sequence $\gamma_k(\theta)$, defined by [11], reduces to the ones given in [14] when $L = \lambda_1$, we have that $\gamma_1(\theta) = (1 + e^{1 - \cos \theta}) \frac{1}{2 \sin \theta}$ is positive and an increasing function of $\theta$ for $\theta \in (0, \frac{\pi}{4})$ and $\varepsilon > 0$. Moreover, the subsequence $\{c_k(\theta)\}^2 N$ consisting of all finite terms
is a strictly increasing sequence. In addition to the sequence being monotonically increasing, each term in the sequence also has a vanishing limit as $\theta$ approaches 0. For any $n \in \mathbb{N}$ there exists some $\theta_n \in (0, \frac{\pi}{4}]$ such that for all $k \leq n$ and $\theta \in (0, \theta_n)$, $\gamma_k(\theta) < 1$. Moreover, we have $\lim_{\theta \to 0^+} \gamma_n(\theta) = 0$ for all $n \in \mathbb{N}$.

Therefore, there exists some $\theta_n \in (0, \frac{\pi}{4}]$ such that $\gamma_k(\theta) < 1$ and $0 < \gamma_1(\theta_n) \leq \gamma_2(\theta_n) \leq \cdots \leq \gamma_n(\theta_n)$, with each $\gamma_k(\theta_n)$ satisfying the condition (10) and thus giving rise to Bell violations.

**Proof of Theorem 3**

First of all, we have

$$A_0 = A_{0|0} - A_{1|0} = \sigma_1,$$

$$A_1 = A_{0|1} - A_{1|1} = \sigma_2,$$

$$B_0 = B_{0|0} - B_{1|0} = \cos \theta \sigma_1 - \sin \theta \sigma_2,$$

$$B_1 = B_{0|1} - B_{1|1} = \cos \theta \sigma_1 + \sin \theta \sigma_2$$

and

$$C_0^{(k)} = C_{0|0} - C_{1|0} = \sigma_1,$$

$$C_1^{(k)} = C_{0|1} - C_{1|1} = \gamma_k \sigma_2.$$  

Then

$$S^{(k)} = Tr[S^{(k)} \rho_{ABC}^{(k)}]$$

$$= 2 \cos \theta Tr[\rho_{ABC}^{(k)} \sigma_1 \otimes \sigma_1 \otimes \sigma_1]$$

$$- 2 \sin \theta Tr[\rho_{ABC}^{(k)} \sigma_1 \otimes \sigma_2 \otimes \sigma_2]$$

$$- 2 \sin \theta \gamma_k Tr[\rho_{ABC}^{(k)} \sigma_2 \otimes \sigma_2 \otimes \sigma_1]$$

$$- 2 \cos \theta \gamma_k Tr[\rho_{ABC}^{(k)} \sigma_2 \otimes \sigma_1 \otimes \sigma_2].$$  \hspace{1cm} (A6)

For the case $k = 1$, substituting $\rho_{ABC}^1 = |\psi_y \rangle \langle \psi_y|$ into (A6) we get

$$S^{(1)} = 2 \cos \theta \sin 2 \alpha + 2 \sin \theta \sin 2 \alpha$$

$$+ 2 \sin \theta \gamma_1 \sin 2 \alpha + 2 \cos \theta \gamma_1 \sin 2 \alpha$$

$$= 2 \sin 2 \alpha (\cos \theta + \sin \theta) (1 + \gamma_1).$$

By using the following identity

$$\sqrt{1 + \gamma_k + 1 - \sqrt{1 - \gamma_k}} I \otimes I \pm (\sqrt{1 + \gamma_k} - \sqrt{1 - \gamma_k}) \sigma$$

$$= \sqrt{2} \gamma_k \sigma$$

for $\sigma = \sigma_1, \sigma_2$ and repeating the same processes in the proof of Theorem 1, we obtain

$$Tr[\rho_{ABC}^{(k)} \sigma_1 \otimes \sigma_1 \otimes \sigma_1]$$

$$= \left( \frac{1}{4} - \frac{1 - \sqrt{1 - \gamma_k^2 - 1}}{4} + 2 \frac{\sqrt{1 - \gamma_k^2 - 1}}{4} \right) \cdot Tr[\rho_{ABC}^{(k-1)} \sigma_1 \otimes \sigma_1 \otimes \sigma_1]$$

$$= \left( \frac{1}{2} - \frac{\gamma_k - 1}{2} \right) Tr[\rho_{ABC}^{(k-1)} \sigma_1 \otimes \sigma_1 \otimes \sigma_1].$$

Then

$$Tr[\rho_{ABC}^{(k)} \sigma_1 \otimes \sigma_1 \otimes \sigma_1]$$

$$= 2^{1-k} Tr[\rho_{ABC}^{(1)} \sigma_1 \otimes \sigma_1 \otimes \sigma_1] \Pi_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_j^2}).$$

At last we have

$$S^{(k)} = 2^{2-k} \cos \theta + \sin \theta (\gamma_k + \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_j^2})).$$

**Proof of Theorem 4**

From (11) Alice, Bob and Charlie share a genuine non-locally correlated state if

$$\gamma_k > \frac{\gamma_k}{\sin 2 \alpha (\cos \theta + \sin \theta)} - \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_j^2}).$$

Let $\varepsilon > 0$ and $\gamma_1(\theta) = (1 + \varepsilon) (\frac{2}{\sin 2 \alpha (\cos \theta + \sin \theta)} - 1).$ For $k \in \{2, \cdots , n\}$ recursively set

$$\gamma_k(\theta) = \begin{cases} (1 + \varepsilon)(\frac{2}{\sin 2 \alpha (\cos \theta + \sin \theta)} - P_k), & \text{if } \gamma_{k-1}(\theta) \in (0, 1), \\ \infty, & \text{otherwise}, \end{cases}$$

where $P_k = \prod_{j=1}^{k-1} (1 + \sqrt{1 - \gamma_j^2}).$

Suppose there is a finite integer number $m \geq 2$, such that $0 < \gamma_j(\theta) < 1$ for all $j = 1, 2, \cdots , m$. Then $1 < 1 + \sqrt{1 - \gamma_j^2} < 2$. The bound $1 + \sqrt{1 - \gamma_j^2} < 2$ implies that $\gamma_j(\theta)/\gamma_{j-1}(\theta) > 2$.

Take $\sin^2 2 \alpha = \frac{8}{9}$. We have

$$\gamma_1(\theta) > \gamma_1 = \frac{3 \sqrt{2}}{2} \frac{2}{\cos \theta + \sin \theta} - 1.$$