On Relative Category and Morse Decompositions for Dynamical Systems

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Abstract. We employ the relative category to develop a relation of the Ważewski pair \((N, E)\) and the Morse decomposition of the maximal invariant set in \(\overline{N \setminus E}\). From this relation, we obtain a dynamical system version of critical point theorem with relative category.

Keywords: Local semiflows, relative category, Morse decomposition, Ważewski pairs, quotient flows.

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1 Introduction

In the field of dynamical systems, invariant sets are of great significance and particular interests, in that they can determine and describe much of the long-term dynamics of a system. Equilibria, (almost) periodic solutions, homoclinic (heteroclinic) orbits and attractors are typical examples of compact invariant sets. It is therefore of great importance to study the existence, number and the location of compact invariant sets for a given dynamical system.

Early in 1940s, Ważewski introduced the famous Ważewski’s Retract Theorem \((22,23)\) to give the existence of invariant sets. Roughly speaking, this theorem states that for a given flow and a closed subset \(N\) (a Ważewski set) of the phase space, the existence of a solution entirely contained in \(N\) can be deduced from the assumption that the exit set \(N^-\) of \(N\) is not a deformation retract of \(N\). The Ważewski’s Retract Theorem appeared to be a powerful way to develop topological...
methods to study invariant sets. The well-known Conley index theory ([4]) was originally inspired by this theorem, which was generalized to infinite-dimensional case by Rybakowski ([15]) and shape index theory by Robbin and Salamon ([14], see more generally, [16, 17, 20, 21]). Based on the (Conley) index pair, Sanjurjo studied the flow on locally compact metric spaces and applied Lusternik-Schnirelmann category in the shape theory ([1]) to the Morse decompositions of an isolated invariant set to detect the existence of connecting orbits between Morse sets ([18]). Li, Shi and Song in [11] used the Ważewski pair (a generalization of the Ważewski set) to develop a dynamical-system version of the linking theorem and mountain pass theorem.

In this present work, we consider the local semiflows on metric spaces and study the relation between the relative category of a Ważewski pair \((N, E)\) and the (Lusternik-Schnirelmann) category of the invariant sets in \(N \setminus E\). We shall generalize the corresponding consequences for flows on locally compact metric spaces stated in [18]. The definitions of Lusternik-Schnirelmann category and relative category we adopt here are referred to [3, 5, 6, 8, 24].

Now we give a more detailed description of our work. Let \(X\) be a complete metric space, and \(\Phi\) be a local semiflow on \(X\). Let the closed pair \((N, E)\) be a Ważewski pair, i.e., \(E\) is an exit set of \(N\) and moreover, since \(X\) can be infinite-dimensional, the set \(N \setminus E\) is imposed an appropriate compact condition — strong admissibility. The strong admissibility condition can be viewed as the asymptotical compactness of dynamical systems (see [19]), but locally. We use the notation of relative category given in [6], i.e., for a closed pair \((N, E)\) and \(A \subset N\) such that \(E \subset A \subset N\), \(\text{cat}_{N,E}(A)\) denotes the category of \(A\) in \(N\) relative to \(E\). Then the classical Lusternik-Schnirelmann category \(\text{cat}_N(A)\) is indeed equal to \(\text{cat}_{N,\emptyset}(A)\).

As to establish the relation of \(\text{cat}_{N,E}(N)\) and the invariant sets in \(N \setminus E\), we also suppose the Ważewski pair \((N, E)\) satisfies the transversality condition, which ensures that each point lying on the boundary of \(E\) in \(N\) will not stay on this boundary within an arbitrarily short time under the action of semiflow, namely, Definition [3, 5, 6, 8, 24]. Our main theorem is the following one.

**Theorem 1.1.** Let \(\Phi\) be a local semiflow on \(X\) and \((N, E)\) be a transversal Ważewski pair for \(\Phi\). Let \(\{M_1, \cdots, M_n\}\) be a Morse decomposition of the maximal invariant set in \(N \setminus E\). Suppose that \(N\) is an ANE. Then

\[
\text{cat}_{N,E}(N) \leq \sum_{i=1}^{n} \text{cat}_N(M_i).
\]

As a deduction of this theorem for gradient semiflows, we can give an lower bound estimate for the number of equilibria. This is also a generalization of critical point theorem with relative category (see [24], a sort of minimax theorems) in dynamical systems.

This paper is organized as follows. In Section 2, we present the preliminaries, including some basic topological concepts, dynamical systems on normal Hausdorff topological spaces, attractors and Morse decompositions. In Section 3, we introduce the Ważewski pair and the concept — transversality for a closed pair, and develop some results on the quotient flow. We state and prove the main theorem in the 4th section.
2 Preliminaries

This section is concerned with some preliminaries.

Let \( X \) be a topological space and \( A, B \subset X \) with \( A \subset B \). We denote by \( \overline{A} \) the closure of \( A \) in \( X \) and by \( \text{int}_B(A) \) the interior of \( A \) in \( B \), i.e., the maximal open subset of \( B \) contained in \( A \). A set \( U \) is called an (open, closed) neighborhood of \( A \) in \( B \), if \( U \) is (open, closed) in \( B \) and there is an open subset \( O \) of \( B \) such that \( A \subset O \subset U \).

The set \( A \) is said to be sequentially compact, if each sequence \( x_n \) in \( A \) has a subsequence converging to a point \( x \in A \). It is a basic knowledge that if \( X \) is a metric space, then sequential compactness coincides with compactness.

2.1 HEP and quotient spaces

Let \( X \) be a topological space. Given a closed subset \( A \) of \( X \), the pair \( (X, A) \) is said to have the homotopy extension property (HEP for short), if for every space \( Y \) and continuous mapping \( F : X \times \{0\} \cup A \times I \to Y \), there exists a continuous map \( \tilde{F} : X \times I \to Y \) such that \( \tilde{F} \) is an extension of \( F \).

Proposition 2.1 ([15]). The pair \( (X, A) \) has the HEP if and only if \( A \) is a strong deformation retract of one of its open neighbourhoods.

Let \( A \) and \( B \) be two closed subsets of \( X \). The quotient space \( B/A \) is defined as follows.

If \( A \neq \emptyset \), then the space \( B/A \) is obtained by collapsing \( A \) to a single point \( \llbracket A \rrbracket \) in \( B \cup A \). If \( A = \emptyset \), we choose a single isolated point \( * \notin B \) and define \( B/A \) to be the space \( B \cup \{*\} \) equipped with the sum topology. In the latter case we still use the notation \( \llbracket A \rrbracket \) to denote the base point \( * \).

2.2 Local semiflows and gradient semiflows

Let \( X \) be a topological space. The space \( X \) is always assumed to be a Hausdorff topological space and sometimes to be a metric space if necessary.

Definition 2.2. A local semiflow \( \Phi \) on \( X \) is a continuous map \( \Phi : \mathcal{D}(\Phi) \to X \), where \( \mathcal{D}(\Phi) \) is an open subset of \( \mathbb{R}^+ \times X \), and \( \Phi \) enjoys the following properties:

1. for each \( x \in X \), there exists \( 0 < T_x \leq \infty \) such that
\[
(t, x) \in \mathcal{D}(\Phi) \iff 0 \leq t < T_x;
\]
2. \( \Phi(0, x) = x \) for all \( x \in X \);
3. if \( (t + s, x) \in \mathcal{D}(\Phi) \), where \( t, s \in \mathbb{R}^+ \), then \( \Phi(t + s, x) = \Phi(t, \Phi(s, x)) \).

The number \( T_x \) in (1) is called the maximal existence time of \( \Phi(t, x) \). In the case when \( \mathcal{D}(\Phi) = \mathbb{R}^+ \times X \), we simply call \( \Phi \) a global semiflow.

Let \( \Phi \) be a given local semiflow on \( X \). For notational convenience, we will rewrite \( \Phi(t, x) \) as \( \Phi(t)x \). Given a subset \( N \) of \( X \), we say \( \Phi \) does not explode in \( N \), if \( T_x = \infty \), whenever \( \Phi(t)x \in N \) for all \( t \in [0, T_x) \).
A subset $N$ of $X$ is said to be *admissible*, if for arbitrary sequences $x_n \in N$ and $t_n \to +\infty$ with $\Phi([0, t_n])x_n \subset N$ for all $n$, the sequence of the end points $\Phi(t_n)x_n$ has a convergent subsequence. The subset $N$ is *strongly admissible* if $N$ is admissible and $\Phi$ does not explode in $N$.

Since the phase space $X$ may be infinite-dimensional, to overcome the difficulty due to the lack of compactness of $X$, we always assume that $\Phi$ is *asymptotically compact*, that is, each bounded subset $B$ of $X$ is admissible.

A solution (trajectory) on an interval $J \subset \mathbb{R}$ is a map $\gamma : J \to X$ satisfying

$$
\gamma(t) = \Phi(t - s)\gamma(s), \quad \text{for all } s, t \in J, \ s \leq t.
$$

A full solution $\gamma$ is a solution defined on the whole line $\mathbb{R}$. If $x \in X$ is such that $\Phi(t)x = x$ for all $t \geq 0$, we say $x$ is an *equilibrium*.

The $\omega$-limit set and $\alpha$-limit set of a solution $\gamma$ are defined as follows: if $\gamma$ is defined on an interval containing $[0, \infty)$, it is defined that

$$
\omega(\gamma) = \{y \in X : \text{there exists } t_n \to \infty \text{ such that } \gamma(t_n) \to y\};
$$

if $\gamma$ is defined on an interval containing $(-\infty, 0]$, it is defined that

$$
\alpha(\gamma) = \{y \in X : \text{there exists } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.
$$

For an $x \in X$ with $T_x = \infty$, we define $\omega(x) = \omega(\gamma)$ with $\gamma(t) = \Phi(t)x$ for every $t \geq 0$.

Given an invariant set $K \subset N \subset X$, we define the local stable and unstable manifold, $W^s_N(K)$ and $W^u_N(K)$ of $K$ in $N$ as follows:

$$
W^s_N(K) := \bigcup_{\omega(\gamma) \subset K} \{\gamma(t) : \gamma([0, \infty)) \subset N, \ t \in [0, \infty)\},
$$

and

$$
W^u_N(K) := \bigcup_{\alpha(\gamma) \subset K} \{\gamma(t) : \gamma((-\infty, 0]) \subset N, \ t \in (-\infty, 0]\},
$$

where $\gamma$ is a solution and $\omega(\cdot)$ and $\alpha(\cdot)$ are limit sets. If $N = X$ is the whole phase space, we simply write $W^s(K) = W^s_N(K)$ and $W^u(K) = W^u_N(K)$.

Let $M$ and $B$ be two subsets of $X$. We say that $M$ attracts $B$, if $T_x = \infty$ for all $x \in B$ and moreover, for each neighbourhood $V$ of $M$ there exists $T > 0$ such that

$$
\Phi(t)B \subset V, \quad t > T.
$$

A nonempty sequentially compact invariant set $A \subset X$ is said to be an *attractor* of $\Phi$, if it attracts a neighbourhood $U$ of $A$ and $A$ is the maximal sequentially compact invariant set in $U$. It can referred to [21, Remark 2.5] about the discussion and comparison of this definition and the previous ones of attractor.

Let $A$ be an attractor. Set

$$
\Omega(A) = \{x \in X : A \text{ attracts } x\}.
$$
\( \Omega(A) \) is called the region of attraction (or attraction basin) of \( A \). One can easily verify that \( \Omega(A) \) is open; moreover, \( A \) attracts each compact subset of \( \Omega(A) \), see [12]. In the case when \( \Omega(A) = X \), we simply call \( A \) the global attractor of \( \Phi \).

Let \( K \subset X \) be a closed subset and \( U \) be a subset of \( X \) with \( K \subset U \). A nonnegative function \( \zeta \in C(U) \) is called a \( K_0 \) function of \( K \) on \( U \), if

\[
\zeta(x) = 0 \iff x \in K.
\]

If moreover the level set

\[
\zeta^a = \{ x \in \Omega : \zeta(x) \leq a \}
\]

is closed in \( X \) for every \( a \geq 0 \), we say \( \zeta \) is a \( K_\infty^0 \) function of \( K \) on \( \Omega \).

If \( X \) is a metric space and \( A \) is a nonempty closed subset of \( X \), then the distance \( d(x, A) \) is a \( K_\infty^0 \) function of \( A \) on \( X \). If \( B \) is another nonempty closed subset of \( X \) with \( A \cap B = \emptyset \), then the function defined as

\[
d(x, A) \min\{1, d(x, B)\}, \quad x \in X \setminus B
\]

is a \( K_\infty^0 \) function of \( A \) on \( X \setminus B \). Thus we conclude a simple lemma.

**Lemma 2.3.** Let \( A \) be a closed subset and \( U \) be an open subset of a metric space \( X \) with \( A \subset U \). Then, there is a \( K_\infty^0 \) function \( \zeta \) of \( A \) on \( U \) such that \( \zeta(x) \geq d(x, A) \) for each \( x \in U \).

Let \( A \) be an attractor and \( \Omega : = \Omega(A) \) be the region of attraction of \( A \). A nonnegative continuous function \( \zeta : \Omega \rightarrow \mathbb{R}^+ \) is said to be a Lyapunov function of \( A \), if \( \zeta \) is a \( K_0 \) function of \( A \) on \( \Omega \), and

\[
\zeta(\Phi(t)x) < \zeta(x)
\]

for each \( x \in \Omega \setminus A \) and \( t > 0 \).

### 2.3 Morse decompositions of invariant sets

For the reader’s convenience, we recall briefly the definition of Morse decompositions of invariant sets.

Let \( K \) be a compact invariant set. Then the restriction \( \Phi|_K \) of \( \Phi \) on \( K \) is a semiflow on \( K \). A set \( A \subset K \) is called an attractor of \( \Phi \) in \( K \), if it is an attractor of \( \Phi|_K \).

**Definition 2.4.** Let \( K \) be a compact invariant set. An ordered collection \( \mathcal{M} = \{M_1, \ldots, M_n\} \) of subsets \( M_k \subset K \) is called a Morse decomposition of \( K \), if there exists an increasing sequence

\[
\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = K
\]

of attractors in \( K \) such that

\[
M_k = A_k \setminus W^s_K(A_{k-1}), \quad 1 \leq k \leq n.
\]

The attractor sequence of \( A_k \) \((k = 0, 1, \ldots, n)\) is often called the Morse filtration of \( K \), and each \( M_k \) is called a Morse set of \( K \).

**Proposition 2.5.** Let \( \mathcal{M} = \{M_1, \ldots, M_l\} \) be the Morse decomposition of an compact invariant set \( K \) with the Morse filtration \( \{A_0, \ldots, A_l\} \). Then the following statements hold:

1. \( M_k \) \((1 \leq k \leq l)\) are compact invariant sets and pairwise disjoint;
if $\gamma$ is a full solution in $K$, then one and only one of the following conditions holds:

1. $\gamma$ is contained in $M_k$.
2. There is $i > j$, such that $\alpha(\gamma) \subset M_i$ and $\omega(\gamma) \subset M_j$.
3. For $1 \leq k \leq l$, $A_k = \bigcup_{i=1}^{k} W^u_K(M_i)$, $W^u_K(M_i) = \bigcup_{\alpha(\gamma) \subset M_i} \{\gamma(t) : t \in (-\infty, \infty)\}$, where $\gamma$ is a full solution.

Remark 2.6. (1) It is well known that if $K$ is isolated, then so are the Morse sets $M_k$.

(2) If $K$ is an attractor in $X$, then the local unstable manifold $W^u_K(M_i) = W^u(M_i)$ for each Morse set $M_i$ of $K$.

3 Ważewski Pairs and Quotient Flows

In this section, we recall the Ważewski pair and some properties of quotient flows (see [20]), and develop some new related conclusions for this paper. We always assume the phase space $X$ to be a complete metric space with the metric $d(\cdot, \cdot)$.

Let $A, N$ be subsets of $X$. $A$ is said to be $N$-positively invariant, if $\Phi([0, t])x \subset N$ ensures $\Phi([0, t])x \subset A$, for every $x \in N \cap A$ and $t \geq 0$. When $N = X$, $N$-positively invariance is just the positively invariance. For an arbitrary subset $N \subset X$, define a function $t_N : X \rightarrow \mathbb{R}^+ \cup \{\infty\}$ as

$$t_N(x) = \inf\{t \geq 0 : \text{either } t \geq T_x \text{ or } \Phi(t)x \notin N\}, \quad \text{for all } x \in X.$$ 

Note that for each $x \in N$, $t_N(x)$ is the supremum of the time $t$ such that $\Phi([0, t])x \subset N$.

Let $N, E$ be subsets of $X$. We say $E$ is an exit set of $N$, if

1. $E$ is $N$-positively invariant;
2. For every $x \in N$ with $t_N(x) < T_x$, there exists $t \leq t_N(x)$ such that $\Phi(t)x \in E$.

Definition 3.1. A pair of closed subsets $(N, E)$ of $X$ is called a Ważewski pair, if

1. $E$ is an exit set of $N$; and
2. $N \setminus E$ is strongly admissible.

Given a subset $A$ of $X$, we always denote by $\mathcal{I}(A)$ the maximal invariant set in $A$. When $A$ is strongly admissible, $\mathcal{I}(A)$ is compact, see [15]. Let $(N, E)$ be a Ważewski pair with $H = N \setminus E$. We say $(N, E)$ is a Ważewski pair of $\mathcal{I}(H)$ if it is necessary to emphasize the compact invariant set $\mathcal{I}(H)$. 

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**Definition 3.2.** Let \((N, E)\) be a closed pair and \(H = N \setminus E\). We say that \((N, E)\) is transversal for \(\Phi\), if for every \(x \in H \cap E\) and \(t > 0\), there is \(s \in (0, t)\) such that \(\Phi(s)x \notin H \cap E\).

There are natural simple results about transversal Ważewski pairs as follows.

**Proposition 3.3.** Suppose that a Ważewski pair \((N, E)\) is transversal. Then

1. \(\mathcal{I}(H) \cap E = \emptyset\); and
2. \(t_H\) is continuous on \(N \setminus W^s_N(\mathcal{I}(H))\).

**Proof.** The first conclusion is a simple deduction. For the second one, given \(x \in N \setminus W^s_N(K)\), we easily know that \(t_H(x) < \infty\). Then the continuity is given by Ważewski’s theorem (see, [4]).

Given a Ważewski pair \((N, E)\), we now consider the pointed space \((N/E, [E])\). Define a quotient flow \(\Phi\) of \(\Phi\) on \(N/E\) as follows:

If \(\tilde{x} = [E]\), then

\[
\Phi(t)\tilde{x} = \tilde{x}
\]

for \(t \in \mathbb{R}^+\); and if \(\tilde{x} = [x]\) for some \(x \in \overline{N \setminus E}\), then

\[
\Phi(t)\tilde{x} = \begin{cases} 
[\Phi(t)x], & \text{for } t < \frac{1}{N \setminus E}(x); \\
[E], & \text{for } t \geq \frac{1}{N \setminus E}(x).
\end{cases}
\]

Since \(E\) is \(N\)‐positively invariant, it can be easily seen that \(\Phi(t)\) is a well defined semigroup on \(N/E\). Moreover, \(\Phi\) is a continuous on \(\mathbb{R}^+ \times N/E\) and \(N/E\) is strongly admissible for \(\Phi\).

**Lemma 3.4.** Let \((N, E)\) be a Ważewski pair, and \(K = \mathcal{I}(N \setminus E)\). Let \(\Phi\) be the quotient flow on \(N/E\). Define for \(\Phi\) on \(N/E\)

\[
\mathcal{A} : = W^u(\{K\}) \cup \{[E]\}.
\]

Then \(\mathcal{A}\) is the compact global attractor for \(\Phi\) in \(N/E\) satisfying

\[
\mathcal{A} = [W^u_N(K)] \cup \{[E]\}. \tag{3.1}
\]

Moreover, if \(K \cap E = \emptyset\), the attractor \(\mathcal{A}\) has a Morse decomposition \(\mathcal{M} : = \{[E], [K]\}\).

**Proof.** The first conclusion is similar to the results of Lemmas 4.6 and 4.7 in [20]. The proof is just a slight modification of the proofs of the two lemmas, by using the framework of topological dynamical systems (12). We thus omit it.

Now we only prove that for the case when \(K \cap E = \emptyset\), the attractor has the Morse decomposition \(\mathcal{M}\). In this case, it follows from [20] that \([E]\) is an attractor. We only need to show \(\{[E]\}, [K]\) is an attractor-repeller pair in \(\mathcal{A}\), i.e., \([K] = \mathcal{A} \setminus \Omega([E])\).

Indeed, it is obvious that \([K] \subset \mathcal{A} \setminus \Omega([E])\). Contrarily, for every \(\tilde{x} \in \mathcal{A} \setminus \Omega([E])\), there is a full solution \(\tilde{\gamma}\) not containing \([E]\) in \(\mathcal{A}\) such that \(\tilde{\gamma}(0) = \tilde{x}\) (otherwise \(\tilde{x} \in \Omega([E])\)). By the definition of quotient flow, there is a full solution \(\gamma\) of \(\Phi\) in \(N \setminus E\) such that \([\gamma(t)] = \tilde{\gamma}(t)\) for all \(t \in \mathbb{R}\). So \(\gamma\) is contained in \(K\), which means \(\tilde{\gamma}\) is contained in \([K]\). Hence \(\tilde{x} = \tilde{\gamma}(0) \in [K]\). The proof is complete.
Let \((N, E)\) be a Wazewski pair and \(\tilde{\Phi}\) be the quotient flow on \(N/E\). Then according to our previous paper \([21]\), we know that every attractor \(A\) of \(\tilde{\Phi}\) containing \([E]\) has a \(K_0^\infty\) Lyapunov function on \(\Omega(A)\). Actually, we have much more information for this Lyapunov function.

**Theorem 3.5.** Every attractor \(A\) of \(\tilde{\Phi}\) containing \([E]\) has a \(K_0^\infty\) Lyapunov function \(\zeta\) on \(\Omega(A)\) such that, for each \(a \geq 0\),
\[
\pi^{-1}(\zeta^a) \subset B(A, a),
\]
where \(A = \pi^{-1}(A)\), \(\pi : N \cup E \to N/E\) is the quotient map and \(B(A, a)\) is the set of all points in \(X\) with the distance from \(A\) less than \(a\).

**Proof.** The construction of \(\zeta\) can be referred to the proof of Theorem 2.6 in \([21]\). What we need to do is to check (3.2). For this, we are necessary to recall some necessary constructions of \(\zeta\).

If \(N \setminus E \cap E = \emptyset\) and \(A = \{[E]\}\), we have that \(\Omega(A) = \{[E]\}\). The function \(\psi([E]) = 0\) is just what we desire and satisfies (3.2).

Now we only consider the case when \(N \cap E \neq \emptyset\) or \(A \neq [E]\).

Let \(M = N \cup E\) and \(U = \pi^{-1}(\Omega(A))\). By Lemma 2.3, we have a \(K_0^\infty\) function \(\delta\) of \(A\) on \(U\) such that \(\delta(x) \geq d(x, A)\). Define a function \(\psi : \Omega(A) \to \mathbb{R}^+\) such that \(\psi([E]) = 0\) and
\[
\psi(x) = \delta(x) \text{ for } \bar{x} = \pi(x) \in \Omega(A) \text{ with } x \in U \setminus E.
\]

It is obvious that for each \(a \geq 0\),
\[
\pi^{-1}(\psi^a) \subset \delta^a \subset B(A, a).
\]

For every \(\bar{x} \in \Omega(A)\), define
\[
\xi(\bar{x}) = \sup_{t \geq 0} \psi(\tilde{\Phi}(t)\bar{x}) \quad \text{and} \quad \zeta(\bar{x}) = \xi(\bar{x}) + \int_0^\infty e^{-t} \xi(\tilde{\Phi}(t)\bar{x}) dt.
\]

Then \(\zeta\) is the \(K_0^\infty\) Lyapunov function required (see \([21\] Theorem 3.4)). For every \(\bar{x} \in \Omega(A)\),
\[
\psi(\bar{x}) \leq \xi(\bar{x}) \leq \zeta(\bar{x}).
\]

Combining (3.3) with (3.4), we can easily obtain (3.2). \(\square\)

### 4 Relative Category and Morse Decomposition

#### 4.1 Relative category

In this section we recall the concept of relative category (see \([24]\)). Let \(X\) be a topological space and \(I = [0, 1]\).

A closed subset \(A\) is *contractible* in \(X\), if there exists \(h \in C(I \times A, X)\), the set of all continuous maps from \(I \times A\) to \(X\), such that, for every \(u, v \in A\),
\[
h(0, u) = u, \quad h(1, u) = h(1, v).
\]
Definition 4.1. Let $A$, $B$, $Y$ be closed subsets of $X$. Then by definition, $A \prec_Y B$ in $X$ if $Y \subset A \cap B$ and there exists $h \in C(I \times A, X)$ such that

1. $h(0, x) = x$, $h(1, x) \in B$, for all $x \in A$, and
2. $h(s, Y) \subset Y$, for all $s \in I$.

Definition 4.2. Let $Y \subset A$ be closed subsets of $X$. The category of $A$ in $X$ relative to $Y$ is the least $n \in \mathbb{N}^+ \cup \{\infty\}$ such that there exists $n + 1$ closed subsets $A_0, A_1, \cdots, A_n$ of $X$ satisfying

1. $A = \bigcup_{j=0}^{n} A_j$,
2. $A_1, \cdots, A_n$ are contractible in $X$, and
3. $A_0 \prec_Y Y$ in $X$.

We denote the category of $A$ in $X$ relative to $Y$ by $\text{cat}_{X,Y}(A)$. The category of $A$ in $X$ is defined by $\text{cat}_{X}(A) := \text{cat}_{X,\emptyset}(A)$.

Let $A$, $B$, $Y$ be closed subsets of $X$ such that $Y \subset A$. The relative category has the following basic properties:

1. Normalisation: $\text{cat}_{X,Y}(Y) = 0$,
2. Subadditivity: $\text{cat}_{X,Y}(A \cup B) \leq \text{cat}_{X,Y}(A) + \text{cat}_{X}(B)$,
3. Homotopy: if $A \prec_Y B$ then $\text{cat}_{X,Y}(A) \leq \text{cat}_{X,Y}(B)$,
4. Monotonicity: if $A \subset B \subset X$, then $\text{cat}_{X,Y}(A) \leq \text{cat}_{B,Y}(A)$ and $\text{cat}_{X,Y}(A) \leq \text{cat}_{X,Y}(B)$.

Remark 4.3. It is well known that $\text{cat}_{S^{p-1}}(S^{p-1}) = 2$ and $\text{cat}_{S^p, \partial S^p}(B^p) = 1$, $p \geq 1$, where $S^p$ is the $p$-dimensional unit sphere and $B^p$ is the open $p$-dimensional unit ball.

Proposition 4.4 (\cite{6}). Let $Y$, $Y'$ be closed subspaces of $X$ and $X'$ respectively. If $(X, Y)$ and $(X', Y')$ have the same homotopy type then

$$\text{cat}_{X,Y}(X) = \text{cat}_{X',Y'}(X').$$

A metric space $X$ is an absolute neighborhood extensor, shortly an ANE, if, for every metric space $E$, every closed subset $F$ of $E$ and every map $f : F \to X$ there exists a continuous extension of $f$ defined on a neighborhood of $F$ in $E$. Important examples of ANE are closed convex subsets of normed spaces, Banach manifolds, manifolds with boundary and finite product of ANEs (\cite{6}).

Proposition 4.5 (\cite{6}). Let $Y$ be a closed subset of $X$ and suppose that both $X$, $Y$ are ANEs. Then for an arbitrary closed subset $A \subset X$, there exists a closed neighborhood $B$ of $A$ such that

$$\text{cat}_{X,Y}(B) = \text{cat}_{X,Y}(A).$$

The property given by Proposition 4.5 is called the continuity property.
4.2 Relative category and Morse decomposition

Let $X$ be a complete metric space with the metric $d(\cdot, \cdot)$. Let $\Phi$ be a semiflow on $X$, $(N, E)$ a transversal Ważewski pair for $\Phi$, $H = \overline{N \setminus E}$ and $K = \mathcal{I}(H)$. In the following discussion of this section, we always assume $E \subset N$.

The main theorem of this paper is the following one.

**Theorem 4.6.** Let $\Phi$ be a local semiflow on $X$ and $(N, E)$ be a transversal Ważewski pair of $K$ for $\Phi$. Let $\{M_1, \ldots, M_n\}$ be a Morse decomposition of $K$. Suppose that $N$ is an ANE. Then

$$\text{cat}_{N, E}(N) \leq \sum_{i=1}^{n} \text{cat}_{N}(M_i). \quad (4.1)$$

**Remark 4.7.** Given a compact isolated invariant set $K$, note that $(K, \emptyset)$ is also a transversal Ważewski pair. Let $\{M_1, \ldots, M_n\}$ be a Morse decomposition of $K$. Then by Theorem 4.6 we have

$$\text{cat}_{K}(K) \leq \sum_{i=1}^{n} \text{cat}_{K}(M_i) \leq \sum_{i=1}^{n} \text{cat}_{M_i}(M_i).$$

This is actually a generalization of Sanjurjo [18, Lemma 4].

Before demonstrating Theorem 4.6 we first provide some auxiliary definitions and constructions for semiflows and some necessary results.

**Lemma 4.8.** Let $A$ be an attractor in $K$. Then there is a closed neighborhood $F$ of $W^u_N(A) \cup E$ in $N$ such that $(N, F)$ and $(F, E)$ are transversal Ważewski pairs and $\mathcal{I}(F \setminus E) = A$.

**Proof.** If $A \cup E = \emptyset$, then one can easily verify that $K$ is an attractor in $N$ and hence $W^u_N(K) = K$. In such a case $F = \emptyset$ fulfills all the requirements of the lemma.

Assume that $A \cup E \neq \emptyset$. Now we consider the quotient flow $\tilde{\Phi}$ on $N/E$. Since $A$ is an attractor in $K$ for $\Phi$, $[A]$ must be an attractor in $[K]$ for $\tilde{\Phi}$. By Proposition 2.8 and Lemma 3.4, we know that $\{[E]\}$ and $[A]$ are the first two Morse sets of of the global attractor of $\tilde{\Phi}$. Then the set

$$\mathcal{A} = \tilde{W}^u([A]) \cup \{[E]\},$$

is an attractor including $[E]$ for $\tilde{\Phi}$. So $\mathcal{A}$ has a $\mathcal{K}_0^\infty$ Lyapunov function $\zeta$ on $\Omega(\mathcal{A})$.

For each $a > 0$, define $F^a = \pi^{-1}(\zeta^a)$, where $\pi : N \to N/E$ is the quotient map. Since $\zeta^a$ is a closed neighborhood of $\mathcal{A}$, $F^a$ is a closed neighborhood of $W^u_N(A) \cup E$ in $N$. Also since $\zeta$ is a $\mathcal{K}_0^\infty$ Lyapunov function $\zeta$ on $\Omega(\mathcal{A})$, then clearly $F^a$ is an exit set of $N$ and $(N, F^a)$ is transversal.

We now finish the proof. $\square$

Let $C$ be a closed neighborhood of $K$ in $H$ with $C \cap E = \emptyset$. Define a set

$$C^s_N = \{x \in N : \text{ there is } t \geq 0 \text{ such that } \Phi([0, t])x \subset N \text{ and } \Phi(t)x \in C\}. \quad (4.2)$$

Then we have the following consequences.

**Lemma 4.9.** (1) $C^s_N \cap E = \emptyset$ and $C^s_N$ is closed.
(2) \( W^s_N(K) \subset \text{int}_N(C^s_N) \).

**Proof.** We only consider the case when \( C \neq \emptyset \), since Lemma 4.9 clearly holds if \( C = \emptyset \).

(1) Since \( C \cap E = \emptyset \), it is clear that \( C^s_N \cap E = \emptyset \). Let \( x_n \in C^s_N \) be a sequence such that \( x_n \to x_0 \). It is sufficient to show \( x_0 \in C^s_N \) for the closedness. By definition (4.2), we have \( t_n \geq 0 \) such that \( \Phi(t_n)x_n \in C \). If \( t_n \) is bounded, we can assume \( t_n \to t_0 \) and then by the closedness of \( C \), \( \Phi(t_n)x_n \to \Phi(t_0)x_0 \in C \). This implies that \( x_0 \in C^s_N \). If \( t_n \) is unbounded, we can assume \( t_n \to \infty \) and then by the admissibility (see [15]), \( x_0 \in W^s_N(K) \). Note that \( W^s_N(K) \subset C^s_N \). Therefore \( C^s_N \) is closed.

(2) We only need to show that every point \( x \in W^s_N(K) \) allows an open neighborhood \( U \) in \( N \) such that \( U \subset C^s_N \). If this is not true, then there is a sequence \( N \ni x_n \to x \) such that for all \( t \geq 0 \) satisfying \( \Phi([0,t])x_n \subset N \), the end point \( \Phi(t)x_n \) is not contained in \( C \).

If \( t_H(x_n) \) is bounded, we can assume \( t_H(x_n) \to t_0 \) and then

\[ \Phi(t_H(x_n))x_n \to \Phi(t_0)x \in E, \]

which means \( t_H(x) \leq t_0 \). But it follows from \( x \in W^s_N(K) \) that \( t_N(x) = \infty \), a contradiction.

If \( t_H(x_n) \) is unbounded, we can assume \( t_H(x_n) \to \infty \). Then for every \( t > 0 \), we have \( t_H(x_n) > 0 \) when \( n \) is large enough. Hence we have \( \Phi(t)x_n \to \Phi(t)x \notin \text{int}_N(C) \). However, since \( x \in W^s_N(K) \), then \( w(x) \subset K \). This indicates that there is \( t > 0 \) such that \( \Phi(t)x \in \text{int}_N(C) \), which is also a contradiction and ends the proof.

**Lemma 4.10.** Let \( U \) be an open neighborhood of \( K \). Then there is a closed subset \( F \) of \( N \) such that \( (F,E) \) is a Ważewski pair of \( K \) and a closed neighborhood \( C \) of \( K \) in \( H \) with \( C \cap E = \emptyset \), such that \( C^s_F \subset U \).

**Proof.** If \( K = \emptyset \), let \( F = N \) and \( C = \emptyset \) and we are done. Hence we assume \( K \neq \emptyset \) in the following.

Note that \( K \) itself is an attractor in \( K \). We follow the proof of Lemma 4.8. By Theorem 3.5 setting \( A' = \pi^{-1}(A) \), we have \( F^a \subset B(A',a) \), where \( A \) is the global attractor of \( \Phi \) in \( N/E \). By (3.1), one has

\[ A = [W^u_F((K)) \cup \{E\} \quad \text{and} \quad A' = W^u_F(K) \cup E, \]

where \( H^a = F^a \setminus E \). By the properties of strong admissibility (see [15]), \( W^u_H(K) \) is compact and \( W^u_F(K) \) is closed. By Proposition 4.5 we can take an open neighborhood \( B \) of \( K \) in \( U \) such that

\[ 2\delta := d(B,E) > 0. \]

Suppose the conclusion does not hold true. Then we have two positive sequences \( \varepsilon_n \to 0 \) and \( a_n \to 0 \) with \( B(K,\varepsilon_n) \subset B \) and \( a_n < \delta \) such that there is \( x_n \in (\overline{B}_H(K,\varepsilon_n))^{s_F}_{\varepsilon_n} \setminus B \), where \( \overline{B}_H(K,\varepsilon) = \overline{B}(K,\varepsilon) \cap H \). By the definition (4.2), there is \( t_n \geq 0 \) such that

\[ \Phi([0,t_n])x_n \cap B = \emptyset \quad \text{and} \quad y_n := \Phi(t_n)x_n \in \partial B. \]
This indicates that
\[ y_n \in \partial B \cap (\overline{B}(K, \varepsilon_n))^s_{F^o_n} \]
\[ \subseteq B(A', a_n) \setminus B(E, \delta) \]
\[ \subseteq (B(W^u_H(K), a_n) \cup B(E, a_n)) \setminus B(E, \delta) \]
\[ \subseteq B(W^u_H(K), a_n) \]

Hence \( y_n \) has a convergent subsequence (still denoted by \( y_n \)) such that
\[ y_n \to y_0 \in W^u_H(K) \cap \partial B. \]  

(4.3)

Note still \( y_n \in (\overline{B}(K, \varepsilon_n))^s_{F^o_n} \setminus B(K, \varepsilon_n) \). By (4.2) again, there is \( n \geq 0 \) such that
\[ \Phi([0, n])y_n \subset F^o_n \quad \text{and} \quad z_n := \Phi(s_n)y_n \in \partial B(K, \varepsilon_n), \]

and so \( z_n \) can be assumed to converge to \( z_0 \in K \).

If \( n \) is bounded, we can assume \( s_n \to s_0 \) and then \( \Phi(s_n)y_n \to \Phi(s_0)y_0 = z_0 \in K \). This means \( \Phi([0, \infty))y_0 \subset H^s \) and \( \omega(y_0) \subset A \). So \( y_0 \in W^s_H(K) \); if \( n \) is unbounded, we can assume \( s_n \to \infty \), and by the admissibility, we also have \( y_0 \in W^s_H(K) \). Recalling (4.3), we have \( y_0 \in K \cap \partial B \), a contradiction!  

Lemma 4.11. Let \( (N, E) \) be a transversal Ważewski pair of a compact invariant set \( K \).

1. If \( K = \emptyset \), then \( E \) is an SDR of \( N \).

2. Let \( A \) be an attractor in \( K \). If there are two closed subsets \( F_1 \) and \( F_2 \) of \( N \) such that \( (F_i, E) \) is a Ważewski pair of \( A \) for \( i = 1, 2 \), then \( \text{cat}_{N, E}(F_1) = \text{cat}_{N, E}(F_2) \).

Proof. (1) Let \( H = N \setminus E \). Since \( K = \emptyset \), it is easy to see that \( t_H(x) < \infty \) for every \( x \in N \). By Proposition 3.3, \( t_H \) is continuous on \( N \). For \( s \in [0, 1] \) and \( x \in N \), let
\[ h(s, x) = \Phi(st_H(x))x. \]  

Then one can easily check that \( h \) is a strong deformation retraction of \( N \) onto \( E \).

(2) When \( (F_i, E) \) is a Ważewski pair of \( A \) for \( i = 1, 2 \), we see \( (F_1 \cap F_2, E) \) is also a Ważewski pair of \( A \). By Lemma 4.3, there is a subset \( F_3 \) of \( F_1 \cap F_2 \) such that \( (F_3, E) \) is a Ważewski pair of \( A \) and \( (F_i, F_3) \) is transversal for \( i = 1, 2 \). Moreover, \( (F_i, F_3) \) is a Ważewski pair of \( \emptyset \) for both \( i = 1, 2 \). By the conclusion above, we have \( F_i \prec E F_3 \) and hence by homotopy and monotonicity of relative category, we have
\[ \text{cat}_{N, E}(F_1) = \text{cat}_{N, E}(F_3) = \text{cat}_{N, E}(F_2). \]

Now the proof is finished.  

Lemma 4.12. Let \( M \) be a Morse set of \( K \) and \( F \), \( G \) be subsets of \( N \) with \( E \subset F \subset G \). Suppose that \( N \) is an ANE, \((N, F)\) and \((G, F)\) are transversal Ważewski pairs with \( I(G \setminus F) = M \). Then
\[ \text{cat}_{N, E}(G) \leq \text{cat}_{N, E}(F) + \text{cat}_N(M). \]

In particular, if \( (N, E) \) is transversal, we have \( \text{cat}_{N, E}(N) \leq \text{cat}_N(K) \).
Proof. By the continuity property of relative category, there is a closed neighborhood $B$ of $M$ in $N$ such that
\[ \text{cat}_N(M) = \text{cat}_N(B). \] (4.5)
By Lemma 4.10 there is a closed subset $G'$ of $G$ such that $(G', F)$ is a Ważewski pair of $M$ and a closed neighborhood $C$ of $M$ in $G$ such that $C^a_{G'} \subset B$, where $C^a_{G'}$ is defined as (4.2). Note $C^a_{G'}$ is closed. Hence
\[ \text{cat}_N(C^a_{G'}) \leq \text{cat}_N(B). \] (4.6)
Let $F' = G' \setminus \text{int}_{G'}(C^a_{G'})$, which is closed. We claim $(F', F)$ is a Ważewski pair of $\emptyset$. Then by the property of relative category, Lemma 4.11 (4.5), and (4.6),
\[ \text{cat}_{N,E}(G) = \text{cat}_{N,E}(G') \leq \text{cat}_{N,E}(F') + \text{cat}_N(C^a_{G'}) \leq \text{cat}_{N,E}(F) + \text{cat}_N(M), \]
which complete the proof.

Having only need to show the claim that $(F', F)$ is a Ważewski pair of $\emptyset$. By Lemma 4.9
\[ W^s_{G'}(M) \subset \text{int}_{G'}(C^a_{G'}) \]
Thereby,
\[ F' \subset G' \setminus W^s_{G'}(M). \]
Denote $H' = F' \setminus F$. If $\mathcal{I}(H') \neq \emptyset$, then $t_{H'}(x) = \infty$ for some $x \in H'$, since $F' \subset G'$ and $G'$ is $N$-positively invariant, we have
\[ \omega(x) \subset \mathcal{I}(G' \setminus F) = M. \]
We can infer that $x \in W^s_{G'}(M)$ and obtain a contradiction finally.

Now we are prepared to show Theorem 4.6.

Proof of Theorem 4.6 For $k = 0, 1, \cdots, n$, let
\[ \mathcal{A}_k = \bigcup_{i=1}^{k} W^u_{K}(M_i), \]
which is an attractor in $K$. We are to show the following assertion.

(A) for every Ważewski pair $(F, E)$ of $\mathcal{A}_k$,
\[ \text{cat}_{N,E}(F) \leq \sum_{i=1}^{k} \text{cat}_N(M_i), \quad k = 0, \cdots, n. \] (4.7)

When $k = n$, (A) implies the final consequence of this theorem. Now it suffices to show the assertion (A). We show it by induction.

(1) When $k = 0$, $\mathcal{A}_0 = \emptyset$. Then $(E, E)$ is a Ważewski pair of $\mathcal{A}_0$ and so $\text{cat}_{N,E}(E) = 0$. By Lemma 4.11 (A) surely holds true.

(2) We show that if (A) holds for $k = l$, $l = 0, \cdots, n - 1$, then (A) also holds for $k = l + 1$. 

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Let \((F, E)\) be an arbitrary Ważewski pair of \(\mathcal{A}_{l+1}\). Let \(F_i\) be given by Lemma 4.8 such that \((F_i, E)\) is a Ważewski pair of \(\mathcal{A}_i\) and \((N, F_i)\) is a transversal Ważewski pair. Since \((F, E)\) is a Ważewski pair of \(\mathcal{A}_{l+1}\), one can easily see that \((F, F_i)\) is a Ważewski pair of \(M_{l+1}\). By Lemma 4.12 and (4.7), we have

\[
\text{cat}_{N,E}(F) \leq \text{cat}_{N,E}(F_i) + \text{cat}_N(M_{l+1}) \leq \sum_{i=1}^{l+1} \text{cat}_N(M_i).
\]

The proof is complete.

Now suppose the local semiflow \(\Phi\) is gradient, i.e., there is a continuous function \(V : X \to \mathbb{R}\), usually called Lyapunov function, such that \(t \to V(\Phi(t)x)\) is non-increasing for each \(x \in X\), and if \(x\) is such that \(V(\Phi(t)x) = V(x)\) for all \(t \geq 0\), then \(x\) is an equilibrium of \(\Phi\). Each compact invariant set of \(\Phi\) contains at least one equilibrium. Consequently by using Theorem 4.6 we infer the following conclusion.

**Corollary 4.13.** Let \(\Phi\) be a gradient semiflow on \(X\) and \((N, E)\) be a transversal Ważewski pair for \(\Phi\). Suppose that \(N\) is an ANE. Then \(N \setminus E\) contains at least \(\text{cat}_{N,E}(N)\) equilibria of \(\Phi\).

**Proof.** Let \(\mathcal{E}(A)\) denote all the equilibria of \(\Phi\) in each subset \(A\) of \(X\) and \(#(A)\) denote the number of points in \(A\). Let \(K = \mathcal{I}(N \setminus E)\).

If \(#(\mathcal{E}(K)) = \infty\), we are done. Hence we only consider \(#(\mathcal{E}(K)) < \infty\). For the gradient system \(\Phi\), one has \(\mathcal{E}(N \setminus E) = \mathcal{E}(K)\). According to \([2\, Proposition 5.16]\), we impose an order on \(\mathcal{E}(K)\) such that \(\mathcal{E}(K) = \{e_1, \cdots, e_n\}\) is a Morse decomposition of \(K\). Thus the conclusion follows immediately from Theorem 4.6 since \(\text{cat}_N(\{e_k\}) = 1\) for \(k = 1, \cdots, n\).

**Remark 4.14.** By Corollary 4.13 a gradient semiflow on a compact manifold \(K\) possesses at least \(\text{cat}_K(K)\) equilibria, where we pick \((K, \emptyset)\) as the Ważewski pair.

**Remark 4.15.** Consider the solutions of the following equation

\[
-\Delta u(x) + f(x, u(x)) = 0 \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial \Omega.
\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain, \(f\) is a nonlinear function.

The corresponding evolution equation of (4.8)

\[
\frac{du}{dt} - \Delta u + f(x, u) = 0
\]

generates a \(C_0\) gradient semigroup \(\Phi\) on \(H^1_0(\Omega)\). Let

\[
\varphi(u) = \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 \, dx + \int_{\Omega} \int_0^u f(s) \, ds \, dx.
\]

Then \(\varphi\) is the variational functional of (4.8) and a Lyapunov function of \(\Phi\), which is thus gradient. Note that the critical points of \(\varphi\) is the solutions of (4.8) and the equilibria of \(\Phi\), and also that (PS) \((PS\) means Palais-Smale\) condition for \(\varphi\) is somehow equivalent to the strong admissibility condition for \(\Phi\). As a result, critical point theorem of \(\varphi\) with relative category (see \([24\, Theorem 5.19]\)) coincides with Corollary 4.13 by choosing appropriate Ważewski pairs.

In this sense, Corollary 4.13 can be seen as a generalization of critical point theorems with (Lusternik-Schnirelmann, relative) category in dynamical systems.
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