Separability of Solvable Subgroups in Linear Groups

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Abstract

Let \( \Gamma \) be a finitely generated linear group over a field of characteristic 0. Suppose that every solvable subgroup of \( \Gamma \) is polycyclic. Then any solvable subgroup of \( \Gamma \) is separable. This conclusion is false without the hypothesis that every solvable subgroup of \( \Gamma \) is polycyclic.

1 Introduction

A subgroup \( S \) of a finitely generated group \( \Gamma \) is called separable (in \( \Gamma \)) if \( S \) is the intersection of all the subgroups of finite index containing \( S \). This can be restated: for every \( x \not\in S \), there is a finite quotient \( \pi : \Gamma \to G \) of \( \Gamma \) with \( \pi(x) \not\in \pi(S) \); that is, \( x \) and \( S \) are separated in some finite quotient \( G \). When \( S \) is the trivial subgroup, separability of \( S \) in \( \Gamma \) is the same as \( \Gamma \) being residually finite.

It is a basic problem of infinite group theory to determine which subgroups \( S \) of a given \( \Gamma \) are separable; see, e.g., \[\text{ALR, Bd, G, H, HW, MR}\] for some recent progress on this problem. Apart from applications in geometric topology (see, e.g. \[\text{AH, Mc}\] and the references therein, and Corollary 1.2 below), separability also arises when studying the profinite topology on \( \Gamma \): a subgroup \( S \) is closed in this topology precisely when it is separable. In this paper we will consider finitely generated linear groups \( \Gamma \) over a field \( K \) of characteristic 0, that is subgroups \( \Gamma < \text{GL}(n, K) \).

Malcev proved that every finitely generated linear group \( \Gamma \) is residually finite. However, even for \( \Gamma = \text{GL}(n, \mathbb{Z}), n \geq 4 \), there are products \( S = F_2 \times F_2 \) of free groups in \( \Gamma \) which are not separable in \( \Gamma \). To see this, note that for a finitely presented \( \Gamma \) with separable subgroup \( S \), the generalized word problem is solvable for \( S \) in \( \Gamma \), i.e. there is an algorithm to decide

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wether or not any given element \( x \in \Gamma \) lies in \( S \). The group \( F_2 \times F_2 \) contains finitely generated subgroups for which the generalized word problem is not solvable (see, e.g., \([LS]\)) . As the property of subgroup separability (of all subgroups) of \( \Gamma \) is inherited by its subgroups, it follows that the property of subgroup separability for arbitrary subgroups is not valid for \( \Gamma \).

Thus we must restrict the class of subgroups we consider in order to obtain positive separability results for linear groups. In this paper we will consider the separability of solvable subgroups \( S \) in linear \( \Gamma \). For solvable groups themselves, Malcev proved (see \([R]\), Theorem 5.4.16) that every subgroup of a finitely generated polycyclic group \( \Gamma \) is separable in \( \Gamma \). In contrast, as shown in \([A]\), if \( \Gamma \) is any finitely generated solvable group which is not polycyclic, then \( \Gamma \) has a subgroup \( A \) and an element \( x \) so that \( xAx^{-1} \) is properly contained in \( A \). This easily implies that non-polycyclic solvable groups contain subgroups which are not separable. In particular,

*For a finitely generated solvable subgroup \( S \) to be separable in a linear group \( \Gamma \), it is necessary that \( S \) be polycyclic.*

It is not true in general that any polycyclic subgroup of a linear group is separable; the Baumslag-Solitar group \( \langle a, x \mid xax^{-1} = a^2 \rangle \) is linear (over \( \mathbb{R} \)) but contains the cyclic subgroup \( \langle a^2 \rangle \) which cannot be separated from \( a \).

Our main result gives a sufficient condition for separability of polycyclic subgroups.

**Theorem 1.1.** Let \( \Gamma \) be a finitely generated linear group over a field of characteristic 0. Suppose that every solvable subgroup of \( \Gamma \) is polycyclic. Then any solvable subgroup of \( \Gamma \) is separable.

Theorem \([A]\) had been known in some special cases: for \( \Gamma \) a discrete subgroup of \( \text{SL}(2, \mathbb{C}) \) (Allman-Hamilton \([AH]\) ), for \( \Gamma \) an arithmetic lattice in a rank one simple Lie group (McReynolds \([Mc]\) ), and for subgroups of the Borel subgroup of the \( O \)-points of a connected algebraic group (McReynolds \([Mc]\) ).

A *solvmanifold* is a manifold homeomorphic to \( S/\Lambda \) where \( S \) is a connected, simply-connected, linear solvable Lie group with cocompact discrete subgroup \( \Lambda \). Auslander \([Au]\) proved that the fundamental group of any solvmanifold must contain a finite index, normal subgroup which is polycyclic. Applying standard topological arguments (see, e.g. \([H]\) ), Theorem \([A]\) has the following topological consequence:

**Corollary 1.2.** Let \( M \) be any closed manifold with linear fundamental group in which all finitely generated solvable subgroups are polycyclic. Then any
\[ \pi_1 \text{-injective immersion } f : S/\Lambda \to M \text{ of a solvmanifold into } M \text{ lifts to an embedding in a finite cover of } M. \]

In particular, Corollary [1.2] gives that any isometrically immersed torus in a finite volume, locally symmetric manifold \( N \) lifts to an embedded torus in some finite cover of \( N \); this follows since every solvable subgroup of such an \( N \) is polycyclic.

## 2 A special case

While the proof of Theorem [1.1] involves a number of technical points, one of the main ideas in this proof is rather simple to illustrate in the following special case. For a ring \( R \), let \( T_n(R) \) denote the group of upper triangular matrices over \( R \).

**Theorem 2.1.** Suppose \( O \) is the ring of integers in a number field. Then any solvable subgroup \( S \subset T_n(O) \) of \( \Gamma = GL_n(O) \) is subgroup separable.

**Proof.** A solvable subgroup of \( GL_n(O) \) is polycyclic [KM] and polycyclic groups are subgroup separable by Malcev (see [R], Theorem 5.4.16), i.e. all subgroups of a polycyclic group are separable. If \( x \notin T_n(O) \), then some entry of \( x \) below the diagonal is non-zero and therefore not in some finite index ideal \( I \); reducing modulo \( I \) via \( \phi : O \to O/I \), then in the finite group \( GL_n(O/I) \) the element \( \phi(x) \notin T_n(O/I) \). Thus \( \phi(x) \notin \phi(S) \) for any subgroup \( S \subset T_n(O) \). Since the ideal is of finite index, we are done in this case.

Suppose now \( x \in T_n(O) \). Then the group \( T = gp(x,S) \) generated by \( x \) and \( S \) is a subgroup of \( T_n(O) \) and therefore is a polycyclic subgroup of \( \Gamma \) properly containing \( S \). Now every subgroup of \( T \) is separable, so we can separate \( x \) from \( S \) in a finite quotient \( F \) of \( T \) using a homomorphism having kernel \( K \). By Theorem 5 of [F], the profinite topology and the congruence topology on subgroups of \( T_n(O) \) are the same so there is a congruence kernel, by a non-zero ideal \( I \), which is contained in \( K \). Thus using \( \phi : GL_n(O) \to GL_n(O/I) \) yields \( \phi(x) \notin \phi(S) \). \( \diamond \)

## 3 Specializations, profinite and congruence topologies

If \( \Gamma \) is a finitely generated subgroup of \( GL_n(\mathbb{C}) \), then \( \Gamma \) is in fact contained in \( GL_n(A) \) for some finitely generated subring \( A \) of \( \mathbb{C} \), namely the subring...
generated by the entries of the matrices representing the (finitely many) generators of $\Gamma$.

We consider two topologies on $\Gamma$. The *profinite topology* on $\Gamma$ is given with the basis of all (cosets of) subgroups of finite index. The *(cofinite) congruence topology* is given by the basis of congruence kernels for ideals of finite index; that is, for any ideal $I$ of finite index in $A$, the subgroup of finite index

$$GL_n(A, I) = \ker(GL_n(A) \to GL_n(A/I))$$

gives by restriction $\Gamma$ a basis of open sets. It is clear that the profinite topology is finer than the congruence topology.

Let $T_n(A)$ denote the upper triangular matrices of $GL_n(A)$. It is well known that all non-zero ideals in $O$, the ring of integers in a number field, are of finite index. Formanek showed in [F] that subgroups of the triangular group $T_n(O)$ have the *congruence subgroup property*, that is, the profinite and the congruence topologies are the same. We shall show (Theorem 3.4) that this result of Formanek can be extended to any finitely generated subring of some number field $K$.

### 3.1 Specializations

To understand finitely generated subrings $A$ of $\mathbb{C}$, we will need the following consequence of the Noether Normalization Theorem: for any such $A$, there is an integer $m$ and there are algebraically independent elements $x_1, \ldots, x_N \in A$ so that $A[\frac{1}{m}]$ is an integral extension of $\mathbb{Z}[\frac{1}{m}][x_1, x_2, \ldots, x_N]$. When $A$ is contained in a number field, $A[\frac{1}{m}]$ is an integral extension of $\mathbb{Z}[\frac{1}{m}]$, and so $A$ is contained in $O[\frac{1}{m}]$, for $O$ the ring of integers in a number field.

The main result, Theorem B, of Gruenwald and Segal [GS], implies that for (any finite extension of) a polycyclic subgroup of a linear group over a field of characteristic zero, there is a *specialization* into an algebraic number field which is faithful. In our case, we are given a finitely generated linear group $\Gamma$ containing the polycyclic subgroup $S$, and we realize $\Gamma$ as a subgroup of $GL_n(A)$, where $A$ is a finitely generated ring. The specialization provided by Theorem B [GS] is simply a homomorphism $\psi : A \to K$, where $K$ is an algebraic number field, which induces a homomorphism $\hat{\psi} : GL_n(A) \to GL_n(K)$ which by the statement of the theorem is injective on $S$. Composing with inclusion gives a homomorphism

$$\rho : \Gamma \to GL_n(K)$$

which is injective on $S$. We can regard the image of $\rho$ as taking values in
matrices with entries in $O[\frac{1}{m}]$, where $O$ is the ring of integers in the number field $K$.

3.2 Units

The following is a consequence of a more general theorem of Chevalley [C]. We denote the units of $A$ by $A^*$.

**Theorem 3.1.** Suppose $O$ is the ring of integers in a number field. Let $A \subset O[\frac{1}{m}]$ and let $r > 0$ an integer. Then there is a positive integer $\alpha = \alpha(r)$, relatively prime to $m$, so that any element $x \in A^*$ with $x \equiv 1 \mod \alpha$ is an $r^{th}$ power of an element of $A^*$.

3.3 Congruence Topology

For the remainder of this section we assume that $B = O[\frac{1}{m}]$, where $O$ is the ring of integers in a number field.

Let $D_n(B)$ denote the $n \times n$ diagonal matrices over $B$; let $T_n(B)$ denote the group of upper-triangular matrices over $B$. The natural homomorphism $\Delta : T_n(B) \to D_n(B)$ has kernel $UT_n(B)$, the uni-triangular subgroup of $T_n(B)$ of matrices with 1’s along the diagonal.

The proof of the next lemma follows Formanek’s argument in the proof of Lemma 4 of [F], with suitable changes for extending this to the case of a finitely generated ring. The main ingredient in the proof is Theorem 3.1 and the use of an isolator subgroup. We refer the interested reader to that article for the necessary details.

**Lemma 3.2 ([F], Lemma 4).** Let $N$ be any subgroup of $D_n(B)$. Then for any $r > 0$ there is an integer $\mu(r)$ so that if $x \in N$ and $x \equiv I \mod \mu(r)$ then $x$ is an $r^{th}$ power of an element of $N$.

3.4 Nilpotents

Suppose that $x \in UT_n(B)$ and that $x \equiv I \mod r^2s$. Then we can solve $x = y^r$ where $y \in T_n(B)$ and $y \equiv 1 \mod rs$. Write $x = I + M$ and $y = I + N$. Then

$$I + M = (I + N)^r = I + rN + \ldots$$

Since $M \equiv 0 \mod r^2s$, we can solve for the upper diagonals of $N$ inductively and $N \equiv 0 \mod rs$. 

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This result is extended by Formanek in \cite{F}; we shall need the following modified version of Lemma 2 of \cite{F}. The proof follows Formanek’s argument and again uses the isolator subgroup.

**Lemma 3.3.** Let $H$ be a subgroup of $UT_n(B)$. Then for any $r > 0$ there is an integer $\nu(r)$, divisible by $r$, so that if $x \in H$ and $x \equiv I \mod \nu(r)$, then $x$ is an $r$th power of an element of $H$.

### 3.5 Quotients

Let $B = \mathcal{O}_{\frac{1}{m}}$ as above, and let $\pi(m)$ be the set of prime divisors of $m$. A number is called a $\pi(m)'$ number if it has some prime factor not lying in $\pi(m)$. The quotient ring $B/rB$ is a nonzero finite ring when $r$ is a $\pi(m)'$ number. In this case we will let $f(r)$ be the exponent of the finite group $T_n(B/rB)$.

**Theorem 3.4.** Consider any subgroup $G$ of $T_n(B)$. Suppose $H$ is a normal, finite index subgroup of $G$. If the exponent of $G/H$ is a $\pi(m)'$ number, then $H$ is a congruence subgroup.

**Proof.** Suppose $H$ is a normal finite index subgroup $H$ with quotient $G/H$ of exponent $r$ in $G$, for which $r$ is a $\pi(m)'$ number. Then so is $\nu(r)$. Also $rf(\nu(r)) > 0$, so $\mu(rf(\nu(r)))$ exists by Theorem 3.1. Let $M = \nu(r)\mu(rf(\nu(r)))$.

Given a subgroup $G$ of $T_n(B)$ and a normal subgroup $H$ with finite quotient having exponent $r$, we construct a congruence subgroup $L$ contained in $H$. Specifically we use the congruence subgroup $L$ of level $M$ to detect the subgroup generated by $r$th powers $G^r$ in $G$; i.e. $L \subset G^r \subset H$. Consider $g \in G \subseteq T_n(B)$, with $g \equiv I \mod M$. We claim that $g \in G^r$, which shows that level $M$ works.

To prove the claim, suppose $g \equiv I \mod M$. Then

$$g \equiv I \mod \mu(rf(\nu(r)))$$

and so $\Delta(g)$ is an $rf(\nu(r))$ power by Theorem 3.1. Hence $\Delta(g) = \Delta(zrf(\nu(r)))$, and $g^{-1}z^{-r}f(\nu(r)) \in UT_n(B)$, where we can choose $z \in G$ by Lemma 3.2 with $N = \Delta(G)$.

Let $k = z^{f(\nu(r))}$. By definition of the exponent, $k \equiv I \mod \nu(r)$. Thus also

$$x = g^{-1}k^r \equiv I \mod \nu(r).$$

Furthermore, $k \in G$, so $k^r \in G^r$. Now, $x \equiv I \mod \nu(r)$ so by Lemma 3.3 we have $x = h^r$, $h \in G$ and finally $g = k^r h^{-r} \in G^r$ as desired. ∘
4 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. We first show how to reduce the problem to a simpler case.

Let $S$ be a solvable subgroup of $\Gamma$. As remarked above, $\Gamma$ is contained in $\text{GL}_n(A)$ for a finitely generated subring $A$ of a finitely generated field $K$. The Lie-Kolchin-Malcev Theorem is an extension of the Lie-Kolchin Theorem (see Theorem 21.1.5 of [KM]) to the case of finitely generated fields. It states that every solvable subgroup $S < \text{GL}_n(A)$ contains a finite index subgroup $S_1$ which can be realized as a group of upper triangular matrices in $\text{GL}_n(B)$, where $B$ is a finitely generated ring containing $A$. As $\Gamma$ is residually finite, it is easy to see that it is enough to prove that $S_1$ is separable in $\Gamma$.

Since any finitely generated field of characteristic zero can be regarded as a subfield of $\mathbb{C}$ we may assume that the finitely generated linear group $\Gamma$ is a subgroup of $\text{GL}_n(\mathbb{C})$. Thus, as shown previously, we may assume also that there is an integer $m > 0$ so that $B[\frac{1}{m}]$ is an integral extension of $\mathbb{Z}[\frac{1}{m}]$. So, without loss of generality, we assume that $\Gamma < \text{GL}_n(B)$ and (by hypothesis) that $S$ is an infinite polycyclic subgroup which is realized as a subgroup of the group $T_n(B)$ of upper-triangular matrices in $\text{GL}_n(B)$.

The structure of the proof of Theorem 1.1 is similar to the special case of a number field. Since $S \subset T_n(B)$ is a subgroup of $\Gamma$ we can separate any $x \in \Gamma$ where $x \notin T_n(B)$ using a congruence quotient as in the first paragraph of the proof of Theorem 2.1. Thus we may consider the other case where $x \in \Gamma - S$ and $x \in T_n(B)$.

Consider now the subgroup $T = < x, S >$ of $T_n(B)$. From our hypothesis that all finitely generated solvable subgroups are polycyclic it follows immediately that $T$ is also polycyclic. We now specialize using the results of [GS], to obtain $\rho : \Gamma \to \text{GL}_n(O[\frac{1}{m}])$ so that the map is injective on $T$. Now every subgroup of $T$ is separable, so we can separate $x$ from $S$ in a finite quotient $F$ of $T$ using a homomorphism having kernel $H$. By Theorem 3.3, the profinite topology and the congruence topology on subgroups of $T_n(O[\frac{1}{m}])$ are the same for subgroups whose index is a $\pi(m)'$ number. However, it may not be the case that the index of $H$ has that property. Therefore we shall show below that there is another subgroup $H_1$ whose index is a $\pi(m)'$ number which can be used to separate $x$ and $S$.

Let $T_k$, $k \geq 0$, denote the $k^{th}$ term of the derived series for the group $T$. Since $T$ is polycyclic the layers $T_k/T_k'$ in the derived series consist of finitely generated abelian groups. Let $r$ be the smallest integer for which $T_r/T_r'$ is
infinite; such an $r$ exists since $T$ is infinite. Consider the $p^{th}$ power map
\[ \pi : T_r/T_r^l \to T_r/T_r^l \]
where $\pi(x) := x^p$. The image of $\pi$ has finite index (since layers of the solvable series are finitely generated) and its index is a proper power of $p$ for infinitely many primes $p$. Furthermore, $W_p = T_r^pT_r^l$ is invariant under every automorphism of $T_r$, so it is a normal subgroup of $W = T_r$ and has index a power of $p$.

Considering the finite set $W/W_p = \{1\}$ in the residually finite group $T/W_p$, we can find a finite index subgroup $N < T$, so that
\[ N/W_p \cap W/W_p = \{1\} \]
That is, we can choose $N < T$ of finite index so that $N \cap W = W_p$. Then $NW/N \cong W/W_p$, so that $N$ has index divisible by $p$. Since $H_1 = N \cap H < H$ is of finite index, then in $T/H_1$ the images of $x$ and $S$ will be separated (since they are already separated in $T/H$) and the group $T/H_1$ has finite order divisible by $p$.

Finally we now use Theorem 3.4 to find a congruence kernel, by a non-zero ideal $I$ of finite index, which is contained in $H_1$. The inverse image of $I$ in $B$ of $I$ (under the homomorphism $\rho : B \to \mathcal{O}([1/m])$) is an ideal of finite index in $B$. Thus using the homomorphism $\bar{\rho} : GL_n(B) \to GL_n(B/I)$ yields $\bar{\rho}(x) \notin \bar{\rho}(S)$. ♦

One of the ideas used in the proof followed from discussions and suggestions from J. S. Wilson regarding his proof of subgroup separability for polycyclic groups \[ \mathbb{R} \]; that method gives us the freedom in the choice of subgroup of finite index to achieve the separation. It is a pleasure to thank him.

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