ISOPERIMETRIC INEQUALITY ON CR-MANIFOLDS WITH NONNEGATIVE $Q'$-CURVATURE

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Abstract. In this paper, we study contact forms on the three-dimensional Heisenberg manifold with its standard CR structure. We discover that the $Q'$-curvature, introduced by Branson, Fontana and Morpurgo on the CR three-sphere and then generalized to any pseudo-Einstein CR three manifold by Case and Yang, controls the isoperimetric inequality on such a CR-manifold. To show this, we first prove that the nonnegative Webster curvature at infinity deduces that the metric is normal, which is analogous to the behavior on a Riemannian four-manifold.

1. Introduction

On a four dimensional manifold, the Paneitz operator $P_g$ and Branson’s $Q$-curvature have many properties analogous to those of the Laplacian operator $\Delta_g$ and the Gaussian curvature $K_g$ on surfaces. The Paneitz operator is defined as

$$P_g = \Delta^2 + \delta \left( \frac{2}{3} R_g - 2Ric \right) d,$$

where $\delta$ is the divergence, $d$ is the differential, $R$ is the scalar curvature of $g$, and $Ric$ is the Ricci curvature tensor. The $Q$-curvature is defined as

$$Q_g = \frac{1}{12} \left\{ -\Delta R + \frac{1}{4} R^2 - 3 |E|^2 \right\},$$

where $E$ is the traceless part of $Ric$, and $| \cdot |$ is taken with respect to the metric $g$. The two most important properties for the pair $(P_g, Q_g)$ are that under the conformal change $g_w = e^{2w} g_0$,

1. $P_g$ transforms by $P_{g_w}(\cdot) = e^{-4w} P_{g_0}(\cdot)$;
2. $Q_g$ satisfies the fourth order equation

$$P_{g_0} w + 2Q_{g_0} = 2Q_{g_w} e^{4w}.$$

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As proved by Beckner \cite{Bec93} and Chang-Yang \cite{CY13}, the pair \((P_g, Q_g)\) also appears in the Moser-Trudinger inequality for higher order operators.

On CR manifolds, it is a fundamental problem to study the existence and properties of CR invariant pairs analogous to \((P_g, Q_g)\). Graham and Lee \cite{GL88} has studied a fourth-order CR covariant operator with leading term \(\Delta b^2 + T^2\) and Hirachi \cite{Hir93} has identified the \(Q\)-curvature which is related to \(P\) through a change of contact form. However, although the integral of the \(Q\)-curvature on a compact three-dimensional CR manifold is a CR invariant, it is always equal to zero. And in many interesting cases when the CR three manifold is the boundary of a strictly pseudoconvex domains, by the work of \cite{FH03}, the \(Q\)-curvature vanishes everywhere. As a consequence, it is desirable to search for some other invariant operators and curvature invariants on a CR manifold that are more sensitive to the CR geometry. The work of Branson, Fontana and Morpurgo \cite{BFM13} aims to find such a pair \((P', Q')\) on the CR sphere. Later, the definition of \(Q'\)-curvature is generalized to all pseudo-Einstein CR manifolds by the work of Case-Yang \cite{CY13} and that of Hirachi \cite{Hir14}. The construction uses the strategy of analytic continuation in dimension by Branson \cite{Bra95}, restricted to the subspace of the CR pluriharmonic functions:

\begin{equation}
(1.1) \quad P'_4 := \lim_{n \to 1} \frac{2}{n - 1} P_{4,n}|_\mathcal{P}.
\end{equation}

Here \(P_{4,n}\) is the fourth-order CR covariant operator that exists for every contact form \(\theta\) by the work of Gover and Graham \cite{GG05}. By \cite{GL88}, the space of CR pluriharmonic functions \(\mathcal{P}\) is always contained in the kernel of \(P_{4,1}\). On the Heisenberg spaces with its standard contact structure, the expression of \(P'\) simplifies to be

\begin{equation}
(1.2) \quad P'u = 2\Delta b^2 u.
\end{equation}

In this paper, we want explore the geometric meaning of this newly introduced conformal invariant \(Q'\)-curvature.

In Riemannian geometry, a classical isoperimetric inequality on a complete simply connected surface \(M^2\), called Fiala-Huber’s \cite{Fia41, Hub57} isoperimetric inequality, states that

\begin{equation}
(1.3) \quad \text{Vol}(\Omega) \leq \frac{1}{2(2\pi - \int_{M^2} K_g^+ dv_g)} \text{Area}(\partial \Omega)^2,
\end{equation}

where \(K_g^+\) is the positive part of the Gaussian curvature \(K_g\). Also \(\int_{M^2} K_g^+ dv_g < 2\pi\) is the sharp bound for the isoperimetric inequality to hold.
In [Wan15], the first author generalizes the Fiala-Huber’s isoperimetric inequality to all even dimensions, replacing the role of the Gaussian curvature in dimension two by that of the $Q$-curvature in higher dimensions:

Let $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let $Q^+$ and $Q^-$ denote the positive and negative part of $Q_g$ respectively, and let $dv_g$ denote the volume form of $M$. Suppose $g = e^{2u}|dx|^2$ is a normal metric, i.e.

$$u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x - y|} Q_g(y) dv_g(y) + C,$$

where $c_n = 2^{n-2}(\frac{n-2}{2})!\pi^{\frac{n}{2}}$, and $C$ is some constant. If

$$\beta^+ := \int_{M^n} Q^+ dv_g < c_n,$$

and

$$\beta^- := \int_{M^n} Q^- dv_g < \infty,$$

then $(M^n, g)$ satisfies the isoperimetric inequality with isoperimetric constant depending only on $n, \beta^+$ and $\beta^-$. Namely, for any bounded domain $\Omega \subset M^n$ with smooth boundary,

$$|\Omega|^g \leq C(n, \beta^+, \beta^-)|\partial\Omega|^\frac{n}{n-1}.$$

It is well known that if the scalar curvature is nonnegative at infinity, then one can show that the metric is a normal metric. For interested readers, the proof of such a fact when $n = 4$ was given in [CQY00]. For higher even dimensions, one can prove by a similar manner.

In the main result of this paper, we prove that the $Q'$-curvature and $P'$ operator are the relevant CR scalar invariant and CR covariant operator to study the isoperimetric inequalities in the CR setting. The Webster curvature at infinity imposes important geometric rigidity on the CR manifold. We also notice that the class of pluriharmonic functions $P$ is the relevant subspace of functions for the conformal factor $u$. We derive the following isoperimetric inequality on any CR three manifold with $Q'$ curvature assumptions.

**Theorem 1.1.** Let $(\mathbb{H}^1, e^{u}\theta)$ be a complete CR manifold, where $\theta$ denotes the standard contact form on the Heisenberg group $\mathbb{H}^1$ and $u$ is a pluriharmonic function on $\mathbb{H}^1$. Suppose additionally the $Q'$ curvature is nonnegative, the Webster scalar curvature is nonnegative at infinity
\[ \int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta < c_1'. \]

Then the isoperimetric inequality is valid, i.e. for any bounded domain \( \Omega \),
\[ \text{Vol}(\Omega) \leq C \text{Area}(\partial \Omega)^{4/3}. \]
Here \( C \) depends only on the integral of the \( Q' \)-curvature, and \( c_1' \) is the constant in the fundamental solution of \( P' \) operator. (See Section 2.)

**Remark 1.2.** It is worth noting that the homogeneous dimension \( N \) of \( M^3 \) is 4. Therefore the power on the right hand side of the isoperimetric inequality is equal to \( \frac{N}{N-1} = 4/3 \).

**Remark 1.3.** We also remark that \( c_1' \) is the critical constant for the validity of the isoperimetric inequality. In fact, there is a CR contact form \( e^{4u} \theta \) with \( \int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta = c_1' \), that does not satisfy the isoperimetric inequality. We give this example in Example 4.6.

In fact, we have proved a stronger result.

**Theorem 1.4.** Suppose the \( Q' \)-curvature of \( (\mathbb{H}^1, e^{4u}) \) is nonnegative. Suppose additionally the metric is normal and \( u \) is a pluriharmonic function on \( \mathbb{H}^1 \). If
\[ \int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta < c_1', \]
then \( e^{4u} \) is an \( A_1 \) weight.

We will introduce the meaning of \( A_1 \) weight in Section 4.

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2. **Fundamental solution of \( P' \) operator**

In this section, we compute the fundamental solution of the Paneitz operator \( P' \) on the Heisenberg group \( \mathbb{H}^1 \). Let \( p, q \) be two points on \( \mathbb{H}^1 \). \( \rho \) denotes the distance function on \( \mathbb{H}^1 \). We show that \( P'(\log \rho(q^{-1}p)) \) is equal to the real part of Szegö kernel. Therefore, \( P' \) restricted to the space of pluriharmonic functions has the fundamental solution \( \log \rho(q^{-1}p) \).
Let us first consider the case for \( p = (z, t) \in \mathbb{H}^1 \), and \( q = (0, 0) \in \mathbb{H}^1 \). Note that

\[
\Delta_b \log \rho(q^{-1}p) = \Delta_b \log(|z|^4 + t^2) \tag{2.1}
\]

\[
= \frac{1}{4} (\partial_x + 2y \partial_t)(\partial_x + 2y \partial_t) \log(|z|^4 + t^2)
+ \frac{1}{4} (\partial_y - 2x \partial_t)(\partial_y - 2x \partial_t) \log(|z|^4 + t^2).
\]

\[
(\partial_x + 2y \partial_t)(\partial_x + 2y \partial_t) \log(|z|^4 + t^2)
= \frac{1}{4} (\partial_x + 2y \partial_t) \left[ \frac{1}{(|z|^4 + t^2)^2} (4x|z|^2 + 4yt) \right]
= \frac{-1}{(|z|^4 + t^2)^2} (4x|z|^2 + 4yt)^2 + \frac{1}{|z|^4 + t^2} (4|z|^2 + 8x^2 + 8y^2)
= \frac{1}{(|z|^4 + t^2)^2} [-16(x^2|z|^4 + 4xyt|z|^2 + y^2t^2) + 12|z|^2(|z|^4 + t^2)].
\]

Similarly, one can see

\[
(\partial_y - 2x \partial_t)(\partial_y - 2x \partial_t) \log(|z|^4 + t^2)
= \frac{1}{(|z|^4 + t^2)^2} [-16(y^2|z|^4 - 2xyt|z|^2 + x^2t^2) + 12|z|^2(|z|^4 + t^2)].
\]

Thus, we obtain

\[
\Delta_b \log(|z|^4 + t^2)^{\frac{1}{2}} = \frac{1}{4(|z|^4 + t^2)^2} [-16(|z|^6 + |z|^2t^2) + 24|z|^2(|z|^4 + t^2)]
= \frac{2|z|^2}{|z|^4 + t^2}.
\]

We now need to compute \( \Delta_b |z|^2 |z|^4 + t^2 \).
\[ (\partial_x + 2y\partial_t)(\partial_x + 2y\partial_t) \frac{|z|^2}{|z|^4 + t^2} \]
\[ = (\partial_x + 2y\partial_t)[\frac{2x}{|z|^4 + t^2} - \frac{2|z|^2}{(|z|^4 + t^2)^2} (4x|z|^2 + 4yt)] \]
\[ = \frac{-2x}{(|z|^4 + t^2)^2} (4x|z|^2 + 4yt) + \frac{2}{|z|^4 + t^2} + \frac{2|z|^2}{(|z|^4 + t^2)^3} (4x|z|^2 + 4yt)^2 \]
\[ + \frac{-2x}{(|z|^4 + t^2)^2} (4|z|^2 + 8x^2 + 8y^2) + \frac{-2x}{(|z|^4 + t^2)^2} (4x|z|^2 + 4yt) \]
\[ = \frac{2}{|z|^4 + t^2} + \frac{1}{(|z|^4 + t^2)^2} [-8x^2|z|^2 - 16xyt - 12|z|^4 - 8x^2|z|^2] \]
\[ + \frac{32|z|^2}{(|z|^4 + t^2)^3} (x|z|^2 + yt)^2. \]

Similarly,

\[ (\partial_y - 2x\partial_t)(\partial_y - 2x\partial_t) \frac{|z|^2}{|z|^4 + t^2} \]
\[ = (\partial_y - 2x\partial_t)[\frac{2y}{|z|^4 + t^2} - \frac{2|z|^2}{(|z|^4 + t^2)^2} (4y|z|^2 - 4xt)] \]
\[ = \frac{-2y}{(|z|^4 + t^2)^2} (4y|z|^2 - 4xt) + \frac{2}{|z|^4 + t^2} + \frac{2|z|^2}{(|z|^4 + t^2)^3} (4y|z|^2 - 4xt)^2 \]
\[ + \frac{-2y}{(|z|^4 + t^2)^2} (4|z|^2 + 8x^2 + 8y^2) + \frac{-2y}{(|z|^4 + t^2)^2} (4y|z|^2 - 4xt) \]
\[ = \frac{2}{|z|^4 + t^2} + \frac{1}{(|z|^4 + t^2)^2} [-8y^2|z|^2 + 16xyt - 12|z|^4 - 8y^2|z|^2] \]
\[ + \frac{32|z|^2}{(|z|^4 + t^2)^3} (y|z|^2 - xt)^2. \]
Therefore, by (2.5) and (2.6) we have
\[
\Delta_{b} \frac{|z|^2}{|z|^4 + t^2} = \frac{4}{|z|^4 + t^2} + \frac{1}{(|z|^4 + t^2)^2}(-8|z|^4 - 24|z|^4 - 8|z|^4) \\
+ \frac{32|z|^2}{(|z|^4 + t^2)^3}(|z|^6 + |z|^2t^2) \\
= \frac{4}{|z|^4 + t^2} - \frac{8|z|^4}{(|z|^4 + t^2)^2} \\
= 4 \frac{t^2 - |z|^4}{(|z|^4 + t^2)^2}.
\]

(2.7)

So we’ve show that
\[
P'(\log(|z|^4 + t^2)^\frac{1}{4}) = 2\Delta_{b} \frac{|z|^2}{|z|^4 + t^2} \\
= 8 \frac{t^2 - |z|^4}{(|z|^4 + t^2)^2}.
\]

(2.8)

Note that this is equal to the real part of Szegö kernel \(Re(S_{\mathbb{H}^1}(p, q))\), up to a multiplicative constant. So we’ve proved that \(\log(|z|^4 + t^2)^\frac{1}{4}\) is propositional to the fundamental solution of the operator \(P'\) on the space of pluriharmonic functions at point \(p = (z, t)\) and \(q = (0, 0)\). Since the norm \(\rho\) and \(P'\) are both left invariant, this computation is also valid for arbitrary value of \(q\). Thus we’ve proved that \(\log(\rho(q^{-1} p))\) is propositional to the fundamental solution of \(P'\). We denote \(G_{\mathbb{H}^1}(u, v) = c_1' \cdot \log \rho(q^{-1} p)\).

3. **Nonnegative Webster scalar curvature at \(\infty\)**

In this section, we describe the property of CR-manifolds with nonnegative Webster scalar curvature at infinity. We will see this geometric condition has a strong analytic implication. We denote the volume form \(\theta \wedge d\theta\) of \(\mathbb{H}^1\) by \(dv\).

**Proposition 3.1.** Let \(\theta\) be the standard contact form of the Heisenberg group \(\mathbb{H}^1\), and \(\hat{\theta} = e^{u}\theta\) be the conformal change of it. Suppose \(u \in \mathcal{P}\) is a pluriharmonic function on \(\mathbb{H}^1\), \(\Delta_{b}^2 u \in L^1(\mathbb{H}^1)\) and \(\hat{\theta}\) has nonnegative Webster scalar curvature near \(\infty\), i.e. \(-\Delta_{b} u \geq |\nabla_{b} u|^2\). Then \(\hat{\theta}\) is a normal, i.e.

\[
u(p) = \int_{\mathbb{H}^1} G_{\mathbb{H}^1}(p, q) P' u(q) dv(q) + C,
\]

where \(C\) is a constant.
It is proved by [BFM13] that the Green’s function for \( P_{S^3} \) is given by
\[
(3.2) \quad G_{S^3}(\zeta, \eta) = \log |1 - \zeta \cdot \bar{\eta}|.
\]
It satisfies the equation
\[
(3.3) \quad P_{S^3}' G_{S^3}(u, v) = S_{S^3}(u, v) - \frac{1}{\text{vol}(S^3)},
\]
where \( S_{S^3}(u, v) \) is the real part of the Szegö kernel. We proved in section 2 that the fundamental solution for \( P_{H^1} \) is given by \( \log \rho(v^{-1}u) \).

We recall that the homogeneous norm on \( H^1 \) is given by \( \rho(z, t) = (|z|^4 + t^2)^{1/4} \).

**Definition 3.2.** Given \( u \in P \) such that \( P'u \in L^1(H^1) \). Define
\[
v(p) := \int_{H^1} G_{H^1}(p, q) P'u(q) dv(q).
\]
This is well-defined when \( P'u \in L^1(H^1) \). We want to prove that \( w := u - v \) is a linear function in \( t \).

**Lemma 3.3.** Under the same assumption as Proposition 3.1, we have \( \Delta_b w = \text{constant} \).

**Proof.** First, we observe that
\[
P'w = P'u - P'v = 0.
\]
We can then apply the mean value property to the function \( \Delta_b w \) which satisfies the equation \( \Delta_b(\Delta_b w) = 0 \). Let \( K_r(x, y) \) denotes the Poisson kernel. We apply the Poisson integral formula to \( \Delta_b w \) and derive
\[
\Delta_b w(p) = \int_{\partial B(p, r)} \Delta_b w(q) K_r(p, q) dv(q),
\]
for arbitrary sphere \( B(p, r) \) of radius \( r \). Here the radius is with respect to the distance given by \( \rho(\cdot) \) on \( H^1 \). Note that \( \Delta_b u \leq -|\nabla_b u|^2 \leq 0 \), and \( \Delta_b v \) tends to zero for large spheres \( \partial B(p, r) \). Thus by taking \( r \to \infty \),
\[
\Delta_b w \leq 0,
\]
at \( \infty \). Thus \( \Delta_b w \) is bounded from above by \( 3.4 \) and the fact that the Poisson kernel is nonnegative.

Now \( \Delta_b w \) is bounded from above and \( (\Delta_b(\Delta_b w)) = 0 \). Thus, analogous to the harmonic function on the Euclidean spaces, by the Liouville’s theorem for \( \Delta_b \) operator, we have
\[
(3.5) \quad \Delta_b w = c_1.
\]
\[\square\]
Next we observe that besides $\Delta b w = c_1$, $Tw$ is also a constant, because $\Delta b^2 w + T^2 w = 0$. We denote the constant of $Tw$ by $c_2$. This allows us to show that

**Lemma 3.4.** $w_x(x, y, t)$ is independent of $t$ variable, i.e. $w_x(x, y, t) = w_x(x, y, 0)$.

**Proof.** We recall that

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t, \quad T = \partial_t.$$ 

Since $X$ and $T$ commute, we have

$$0 = XT w = TX w = T(w_x + 2yw_t)$$

$$= Tw_x.$$ 

Thus $w_x$ is independent of $t$ variable. In other words, for any $(x, y, t)$,

$$w_x(x, y, t) = w_x(x, y, 0).$$

□

Similarly since $Y$ and $T$ commute, $w_y$ is independent of $t$ variable.

**Lemma 3.5.** $w_{xx} + w_{yy}$ is independent of $t$ variable, i.e. $w_{xx}(x, y, t) + w_{yy}(x, y, t) = w_{xx}(x, y, 0) + w_{yy}(x, y, 0)$.

**Proof.** This can be seen from the following computation

$$0 = T\Delta b w$$

$$= T[(XX + YY)] w$$

$$= T[(\partial_x + 2yT)(\partial_x + 2yT) + (\partial_y - 2xT)(\partial_y - 2xT)] w$$

$$= T[w_{xx} + 2yT\partial_x w + \partial_x(2yT w) + 2yT(2yT w)$$

$$+ w_{yy} - \partial_y(2xT w) - 2xT(\partial_y w) + 2xT(2xT w)].$$

By the fact that $Tw$ is a constant, and that $T$ commutes with both $\partial_x$ and $\partial_y$, we obtain the above is equal to

$$T(w_{xx} + w_{yy}).$$

Thus the lemma holds. □

**Lemma 3.6.**

$$\Delta b \partial_x w = 0 \quad \Delta b \partial_y w = 0.$$
Proof. If $\Delta_b$ and $\partial_x$ commute, then since $\Delta_b w = c_1$, we prove the lemma. In general, $\Delta_b$ and $\partial_x$ might not commute. However, we will use the fact that $T w$ is a constant to achieve the goal.

(3.8)
$$\Delta_b \partial_x w = [(\partial_x + 2yT)(\partial_x + 2yT)\partial_x w + (\partial_y - 2xT)(\partial_y - 2xT)\partial_x w]$$
$$= w_{xxx} + \partial_x(2yT \partial_x w) + 2yT \partial_x(\partial_x w) + 2yT(2yT \partial_x w)$$
$$+ w_{xxy} - 2xT \partial_y \partial_x w - \partial_y(2xT \partial_x w) + 2xT(2xT \partial_x w)$$
$$= w_{xxx} + w_{xxy}.$$

The last equality uses the fact that $T$ commutes with both $\partial_x$ and $\partial_y$; and the fact that $T w$ is a constant. Thus cross terms
$$\partial_x(2yT \partial_x w); \quad 2yT \partial_x(\partial_x w); \quad 2xT \partial_y \partial_x w; \quad \partial_y(2xT \partial_x w); \quad 2xT(2xT \partial_x w)$$
vanishes. \hfill $\square$

Lemma 3.7. $|w_x|$ and $|w_y|$ are at most of linear growth.

Proof. \hfill (3.9) \hfill $|\nabla_b w|^2 = w_x^2 + w_y^2 + 4c_2(x^2 + y^2) - 4c_2(xw - yw).$

The right hand side is greater than
$$\left(1 - \alpha\right)(w_x^2 + w_y^2) + 4c_2\left(-\frac{1}{\alpha} + 1\right)(x^2 + y^2),$$
for any $\alpha > 0$. Let us fix $\alpha = 1/2$. Note that $|\nabla_b w|^2 \leq 2|\nabla_b u|^2 + 2|\nabla_b v|^2$ and
$$|\nabla_b u|^2 \leq -\Delta_b u$$
near $\infty$. Also, $|\nabla_b v|$ tends to 0 near $\infty$. Thus $|\nabla_b w|^2 \leq -2c_1 + 1$ near $\infty$, where $c_1 \leq 0$ is the constant value of function $\Delta_b w$. Thus $|\nabla_b w|$ has an upper bound. It follows that $|\partial_x w|$ and $|\partial_y w|$ are at most of linear growth. \hfill $\square$

This together with Lemma 3.6 implies that $\partial_x w$ is a linear function. Similarly, $\partial_y w$ is also a linear function. Suppose both $\partial_x w$ and $\partial_y w$ are not constant, then $w$ is a quadratic function. Since $c_1 \leq 0$, we see that $e^{u\theta}$ gives rise to an incomplete metric. This is a contradiction. Thus both $\partial_x w$ and $\partial_y w$ are constant. So $w$ is linear in both $x$ and $y$. Again, $e^{u\theta}$ is incomplete unless $w$ is a constant in both $x$ and $y$. In other words, $w$ only depends on $t$. On the other hand, we also have $T w = c_2$. So $w$ is a linear function of $t$. We now use the assumption that the Webster scalar curvature $R$ is nonnegative to show that $w$ must be a constant.
To do this, we first note that by a simple computation,
\[-\Delta_b(e^{c^2t}) = -4c_2^2(x^2 + y^2)e^{c^2t} < 0.\]

Also
\[Re^{2u} = -\Delta_b(e^u) = \Delta_b(e^{c^2t + u}) = -\Delta_b(e^{c^2t})e^u - 2X(e^{c^2t})X(e^u) - 2Y(e^{c^2t})Y(e^u) - \Delta_b(e^u)e^{c^2t}\]
\[= -4c_2^2(x^2 + y^2)e^{c^2t}e^u - 4c_2ye^{c^2t}X(e^u) + 4c_2xe^{c^2t}Y(e^u) - (\Delta_b u + |\nabla_b u|^2)e^u e^{c^2t}.\] (3.10)

**Lemma 3.8.**
\[
\frac{1}{|\partial B_r|} \int_{\partial B_r} |\nabla_b v(x)|d\sigma(x) = O\left(\frac{1}{r}\right) \quad \text{as} \quad r \to \infty.
\] (3.11)

**Proof.** By a direct computation, we have
\[X(\log(|z|^4 + t^2)^{1/4}) = \frac{1}{\rho^4}(|z|^2x + ty) \leq \frac{|z|}{\rho^2} \leq \frac{1}{\rho},\]
\[Y(\log(|z|^4 + t^2)^{1/4}) = \frac{1}{\rho^4}(|z|^2y - tx),\]
and
\[|\nabla_b(\log(|z|^4 + t^2)^{1/4})| = \frac{|z|}{\rho^2} \leq \frac{1}{\rho}.\]

Therefore
\[
\frac{1}{|\partial B_r|} \int_{\partial B_r} |\nabla_b v(x)|d\sigma(x) \leq \int_{\mathbb{R}^1} \frac{1}{|\partial B_r|} \int_{\partial B_r} \frac{1}{\rho(y^{-1}x)}|Q'(y)|e^{4u(y)}dv(y)dv(x). \]
(3.12)

Now we need to show
\[
\frac{1}{|\partial B_r|} \int_{\partial B_r} \frac{1}{\rho(y^{-1}x)}d\sigma(x) \leq O\left(\frac{1}{r}\right).
\]
where \(C\) is independent of \(y\).

This is true because we can dilate and take the integration over the unit sphere.
\[
\frac{1}{|\partial B_r|} \int_{\partial B_r} \frac{1}{\rho(y^{-1}x)}d\sigma(x) = \frac{1}{r} \cdot \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1}{\rho((r^{-1}y)^{-1}x)}d\sigma(x).
\]
If \(|r^{-1}y| \geq 1 + \delta\) or \(\leq 1 - \delta\), then it is easy to see that
\[
\frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1}{\rho((r^{-1}y)-1x)}d\sigma(x) \leq C
\]
for a constant \(C\) independent of \(x\).

If \(1 - \delta \leq |r^{-1}y| \leq 1 + \delta\), then we need to use spherical coordinates to prove
\[
(3.13) \quad \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1}{\rho((r^{-1}y)-1x)}d\sigma(x) \leq C.
\]
It is obvious that we only need to deal with the limiting case when \(r^{-1}y\) is on the unit sphere \(\partial B_1\). Let \(r^{-1}y = (y_1, y_2, s)\) and \(x = (x_1, x_2, t)\). Let \((r', \theta')\) be the polar coordinates centered at \((y_1, y_2)\) in the \(xy\)-plane (by our notation \(x = (x_1, x_2, t)\), it is the \(x_1x_2\)-plane).

(3.14) \quad \rho((y_1, y_2, s), (x_1, x_2, t)) \geq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = r'.

The area form of the unit sphere is given by
\[
d\sigma = \sqrt{(u_{x_1} - x_2)^2 + (u_{x_2} + x_1)^2}dx_1dx_2,
\]
where \(u(x_1, x_2) = t = \pm \sqrt{1 - (x_1^2 + x_2^2)^2}\). One can directly compute that
\[
d\sigma = \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}}rdrd\theta.
\]
Here \((r, \theta)\) are polar coordinates of \((x_1, x_2)\) centered at \((0, 0)\). It is obvious that \(rdrd\theta = r'dr'd\theta'\). Therefore,
\[
(3.15) \quad \int_{\partial B_1} \frac{1}{\rho((r^{-1}y)-1x)}d\sigma(x) \leq 2 \int_{x_1^2 + x_2^2 \leq 1} \frac{1}{r'} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}}r'dr'd\theta'.
\]

Case 1: \(\sqrt{y_1^2 + y_2^2} < 1\).
We can denote \(\sqrt{y_1^2 + y_2^2} = 1 - \eta\), where \(\eta > 0\). Then the integral (3.15) is bounded by
\[
(3.16) \quad C + 2 \int_{1 - \frac{\eta}{2} \leq r \leq 1} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}}dr'd\theta'.
\]
Here \(r\) is a function of \((r', \theta')\) by the change of variable formula. The last inequality in (3.16) is because \(r = 1\) is the only singularity of such an integration.
Now, since \( \sqrt{\eta_1^2 + \eta_2^2} = 1 - \eta \) and \( 1 - \frac{\eta}{2} \leq r \leq 1 \), we have \( r' \geq \frac{\eta}{2} \). Thus \( dr'd\theta' = \frac{r}{r'} drd\theta \leq \frac{2}{\eta} rdrd\theta \). Therefore

\[
\int_{1 - \frac{\eta}{2} \leq r \leq 1} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} dr'd\theta' \leq \frac{2}{\eta} \int_{1 - \frac{\eta}{2} \leq r \leq 1} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} rdrd\theta.
\]

The last integral is bounded, because

\[
(3.18) \quad 2 \int_{r \leq 1} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} rdrd\theta = 2 \int_{r \leq 1} d\sigma = |\partial B_1| < \infty.
\]

Case 2: \( \sqrt{\eta_1^2 + \eta_2^2} = 1 \).

Without loss of generality, we can assume that \((\eta_1, \eta_2) = (1, 0)\). We adopt the notation that \( \theta' \) is the angle between the ray and the positive \( x_2 \)-axis. Since the unit sphere on the \( x_1 x_2 \)-plane is completely on the left hand side of \((1, 0)\), we have \( \theta' \in [0, \pi] \).

Now

\[
(3.19) \quad \int_{x_1^2 + x_2^2 \leq 1} \frac{1}{r^2} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} r'dr'd\theta' \leq \int_0^\pi \int_{r' > \epsilon/2} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} dr'd\theta' + \int_0^\pi \int_{r' \leq \epsilon/2} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} dr'd\theta'.
\]

Note that

\[
\int_0^\pi \int_{r' > \epsilon/2} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} dr'd\theta' \leq C
\]

because when \( r' > \epsilon/2 \), we can apply the argument in Case 1 again, using \( dr'd\theta' = \frac{r}{r'} drd\theta \). For \( r' \leq \epsilon/2 \), by a direct computation, for very small \( \epsilon \), \( 1 - r \approx r'\theta' \).

\[
(3.20) \quad \int_0^\pi \int_{r' \leq \epsilon/2} \sqrt{\frac{r^2(1 + 3r^4)}{(1 - r^2)(1 + r^2)}} dr'd\theta' \leq \int_0^\pi \int_{r' \leq \epsilon/2} \sqrt{\frac{1}{r'\theta'}} dr'd\theta'.
\]
Since we have
\[ r^2(1 + 3r^4) \left[ (1 + r)(1 + r^2) \right]^{1/2} < C, \]
\[ \int_{r' < \epsilon/2} \frac{1}{\sqrt{r'}} dr' < \infty, \]
and
\[ \int_0^\pi \frac{1}{\sqrt{\theta'}} d\theta' < \infty, \]
the integration in the second line of (3.20) is finite. This completes the proof of (3.13).

\[ \square \]

By a similar proof, one can show the average estimate of $|\Delta_b v|$ and $|v|$ as well.

**Lemma 3.9.**

(3.21) \[ \frac{1}{|\partial B_r|} \int_{\partial B_r} |\Delta_b v|(x) d\sigma(x) = O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty. \]

**Lemma 3.10.**

(3.22) \[ \frac{1}{|\partial B_r|} \int_{\partial B_r} |v|(x) d\sigma(x) = O(1) \quad \text{as} \quad r \to \infty. \]

So there exists a sequence of points \( \{p_i\} \), \( |p_i| \to \infty \), such that
(3.23) \[ |v|(p_i) \leq C, \]
(3.24) \[ |\nabla_b v|(p_i) + |\Delta_b v|(p_i) \leq \epsilon. \]

Moreover, we can choose \( p_i \), such that they lie in the half space \( c_2 t \geq 0 \), and away from the \( t \)-axis. In other words, we can require that \( c_2 t(p_i) \geq 0 \), and that \( (x(p_i), y(p_i)) \) does not tend to \( (0, 0) \). Here we adopt the notation that \( p_i = (x(p_i), y(p_i), t(p_i)) \).

When \( |x| + |y| \geq L \) for some \( L > 0 \), we have
(3.25) \[ |4c_2 ye^{c_2 t} X(e^v)| \leq |y|e^{c_2 t}e^v|\nabla_b v| \leq \epsilon |y|e^{c_2 t}e^v \leq \epsilon (x^2 + y^2)e^{c_2 t} e^v; \]

(3.26) \[ |4c_2 xe^{c_2 t} Y(e^v)| \leq |x|e^{c_2 t}e^v|\nabla_b v| \leq \epsilon |x|e^{c_2 t}e^v \leq \epsilon (x^2 + y^2)e^{c_2 t} e^v; \]

and
\[ |\Delta_b(e^v)e^{c_2 t}| = |(\Delta_b v + |\nabla_b v|^2)e^v e^{c_2 t}| \leq ee^v e^{c_2 t}. \]

Thus
(3.27) \[ |2X(e^{c_2 t})X(e^v) + 2Y(e^{c_2 t})Y(e^v) + \Delta_b(e^v)e^{c_2 t}| \leq 3\epsilon (x^2 + y^2)e^{c_2 t} e^v. \]
We want to show \( c_2 = 0 \). We prove this by contradiction. Suppose \( c_2 \neq 0 \). Then, by applying (3.27) in (3.10), we obtain that

\[
\text{Re} e^{2u}(p_i) = -4c_2^2(x^2 + y^2)e^{c_2t}e^v - 4c_2ye^{c_2t}X(e^v) + 4c_2xe^{c_2t}Y(e^v)
- (\Delta_{bV} + |\nabla_{bV}|^2)e^ve^{c_2t}e^v(p_i)
\leq -3c_2^2(x(p_i)^2 + y(p_i)^2)e^{c_2t(p_i)}e^v(p_i),
\]

when \( \epsilon \) is small enough.

By our choice of \( \{p_i\}, |v(p_i)| \leq C \) and \( c_2t(p_i) \geq 0 \) for all \( i \). Thus \( e^v \geq \eta > 0 \), and \( e^{c_2t(p_i)} \geq 1 \). Since \( c_2 \neq 0 \), \( -3c_2^2(x(p_i)^2 + y(p_i)^2)e^{c_2t(p_i)}e^v(p_i) < 0 \), as \( i \to \infty \). In fact, this quantity goes to \(-\infty\) unless \((x(p_i), y(p_i))\) tends to \((0, 0)\). Because if \((x(p_i)^2 + y(p_i)^2)\) is bounded, then \( c_2t(p_i) \to +\infty \). This contradicts the assumption on the nonnegativity of Webster scalar curvature \( R \). Therefore \( c_2 = 0 \).

This completes the proof of Proposition 3.1.

4. Main Results

To begin this section, we recall some preliminary Poincaré inequalities for Heisenberg groups \( \mathbb{H}^n \) of arbitrary dimension. Let us denote the homogenous dimension by \( N \). For \( \mathbb{H}^n \), \( N = 2n + 2 \).

**Proposition 4.1.** For any ball \( B \) in Heisenberg group,

\[
\int_B \int_B |g(x) - g(y)|dv(x)dv(y) \leq C|B|^{\frac{N+1}{N}} \int_{2B} |\nabla_{bV}|dv(x).
\]

Here \( 2B \) denotes the concentric ball of \( B \) with double radius, and \( |\cdot| \) denotes the volume with respect to the Harr measure on \( \mathbb{H}^n \).

In fact, the above inequality is a direct consequence of the following 1-Poincaré inequality.

**Proposition 4.2.** [Jer86] For any ball \( B \) in Heisenberg group,

\[
\int_B |g(x) - g_B|dv(x) \leq C|B|^\frac{1}{N} \int_{2B} |\nabla_{bV}|dv(x).
\]

Here \( 2B \) denotes the concentric ball of \( B \) with double radius, \( g_B \) denotes the average of \( g(x) \) on \( B \), and \( |\cdot| \) denotes the volume with respect to the Harr measure on \( \mathbb{H}^n \).
This implies Proposition 4.1 because
\[
\int_B \int_B |g(x) - g(y)| dv(x) dv(y) \leq \int_B \int_B |g(x) - g_B| + |g(y) - g_B| dv(x) dv(y) \leq C |B|^{\frac{N+1}{N}} \int_{2B} |\nabla g| dv(x).
\]

David Jerison [Jer86] proved the stronger version of 2-Poincaré inequality:
\[
\int_B |g(x) - g_B|^2 dv(x) \leq C |B|^{\frac{2}{N}} \int_B |\nabla g|^2 dv(x).
\]
The same method also implies a stronger version of 1-Poincaré inequality. (See [HK00].)
\[
\int_B |g(x) - g_B| dv(x) \leq C |B|^{\frac{1}{N}} \int_B |\nabla g| dv(x).
\]

For the purpose of this paper, we only need the weaker statement Proposition 4.1 in which the integration is over $2B$ on the right hand side of the inequality.

Given a bounded domain with smooth boundary, as a special case of the above proposition, one can take $g$ to be (a smooth approximation of) the characteristic function $\chi_{\Omega}$, and derive
\[
|B \cap \Omega| \cdot |B \cap \Omega^c| \leq C |\partial \Omega \cap 2B| \cdot |B|^{\frac{N+1}{N}}.
\]
This immediately gives rise to the following corollary.

**Corollary 4.3.** For all balls $B \subset \mathbb{H}^n$, such that,
\[
|B \cap \Omega| \geq \frac{1}{2} |B| \quad \text{and} \quad |B \cap \Omega^c| \geq \frac{1}{2} |B|,
\]
we have, by (4.6),
\[
|B|^{\frac{N+1}{N}} \leq C |\partial \Omega \cap 2B|.
\]

**Theorem 4.4.** Suppose $\omega(x) \geq 0$ is an $A_1$ weight on $\mathbb{H}^n$. Namely, there exists a constant $C_0$ (independent of $B$), so that for any ball $B \subset \mathbb{H}^n$,
\[
\frac{1}{|B|} \int_B \omega(p) dv(p) \leq C_0 \inf_{z \in B} \omega(z).
\]
Then the weighted isoperimetric inequality holds for \( \omega(x) \): for any domain \( \Omega \subset \mathbb{H}^n \) with smooth boundary,

\[ \int_{\Omega} \omega(x)dv(x) \leq C_1 \left( \int_{\partial \Omega} \omega(x) \frac{N-1}{N} d\sigma(x) \right)^{\frac{N}{N-1}}, \]

where \( C_1 \) only depends on the \( A_1 \) bound \( C_0 \) of \( \omega(x) \) and the homogeneous dimension \( N = 2n + 2 \).

We now give the proof of this theorem by Proposition 4.1.

**Proof.** Consider a covering \( \bigcup_{\alpha \in \Lambda} B_\alpha \) of the domain \( \Omega \) such that each \( B_\alpha \) satisfies the properties:

\[ \left| \frac{1}{2} B_\alpha \cap \Omega \right| \geq \frac{1}{2} \left| \frac{1}{2} B_\alpha \right|, \quad \left| \frac{1}{2} B_\alpha \cap \Omega^c \right| \geq \frac{1}{2} \left| \frac{1}{2} B_\alpha \right|. \]

In other words, \( \left| \frac{1}{2} B_\alpha \cap \Omega \right| \) and \( \left| \frac{1}{2} B_\alpha \cap \Omega^c \right| \) are both comparable to \( \left| \frac{1}{2} B_\alpha \right| \). By Vitali covering theorem, there exists a countable subset \( \bigcup_{i=1}^{\infty} B_i \) such that \( \Omega \subset \bigcup_{i=1}^{\infty} B_i \), and \( \left\{ \frac{1}{2} B_i \right\} \) are mutually disjoint. Therefore,

\[ \omega(\Omega) = \int_{\Omega} \omega(x)dv(x) \leq \sum_{i=1}^{\infty} \int_{B_i \cap \Omega} \omega(x)dv(x) \]

\[ \leq \sum_{i=1}^{\infty} \int_{B_i} \omega(x)dv(x) \]

\[ \leq \sum_{i=1}^{\infty} C_0 |B_i| \omega(p_i) \]

\[ \leq C_2(n) \sum_{i=1}^{\infty} \left| \frac{1}{4} B_i \right| \omega(p_i). \]

Here \( \omega(p_i) = \inf_{x \in B_i} \omega(x) \).

By using Corollary 4.3 to \( B = \frac{1}{4} B_i \),

\[ \omega(\Omega) \leq C_3 \sum_{i=1}^{\infty} |\partial \Omega \cap \frac{1}{2} B_i| \omega(p_i) \]

\[ \leq C_3 \sum_{i=1}^{\infty} \left( \int_{\partial \Omega \cap \frac{1}{2} B_i} \omega(x) \frac{N-1}{N} d\sigma(x) \right)^{\frac{N}{N-1}} \]

\[ \leq C_3 \left( \sum_{i=1}^{\infty} \int_{\partial \Omega \cap \frac{1}{4} B_i} \omega(x) \frac{N-1}{N} d\sigma(x) \right)^{\frac{N}{N-1}} \]

\[ \leq C_3 \left( \int_{\partial \Omega} \omega(x) \frac{N-1}{N} d\sigma(x) \right)^{\frac{N}{N-1}}. \]
Lemma 4.5. $\frac{1}{\rho(u)}$ is an $A_1$ weight for $0 < \alpha < N = 2n + 2$ on the Heisenberg group $\mathbb{H}^n$.

One can directly check this fact by estimating the maximal function of $\frac{1}{\rho(u)}$.

In the following, we will give a proof of Theorem 1.4. Theorem 1.1 is then a consequence of Theorem 1.4 because if $e^{4u}$ is an $A_1$ weight, by Theorem 4.4, on such a conformal Heisenberg group, the isoperimetric inequality is valid. Moreover, the isoperimetric constant depends only on the integral of the $Q'$-curvature.

Proof of Theorem 1.4. The PDE that the conformal factor $u$ satisfies is

$$P'u = Q'e^{4u}.$$  

Since $u$ is a pluriharmonic function, one has $\Delta_2^2 u = T^2 u$. Recall that the fundamental solution of Paneitz operator $P' = 2\Delta_2^2$ is given by $c'_1 \log \frac{1}{\rho(y^{-1}x)}$. By section 3 as the Webster scalar curvature at $\infty$ is nonnegative, we have the metric is normal. Namely, $u$ has an integral representation

$$u(x) = \frac{1}{c'_1} \int_{\mathbb{H}^1} \log \frac{\rho(y)}{\rho(y^{-1}x)} Q'(y)e^{4u(y)} dv(y) + C.$$  

We now want to prove $e^{4u}$ is an $A_1$ weight. In other words, for any ball $B \subset \mathbb{H}^1$,

$$M(e^{4u})(x) \leq C(\alpha)e^{4u(x)},$$  

for a.e. $x \in \mathbb{H}^1$, where

$$M(f)(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dv(y).$$

Define $\alpha := \int_{\mathbb{H}^1} Q'e^{4u} dv(x)$. By assumption, $\alpha < c'_1$. Note that we can assume $\alpha \neq 0$. As if $\alpha = 0$, then $u$ is a constant. So the conclusion follows directly.

$$\frac{M(e^{4u})(x)}{e^{4u(x)}} = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} \exp \left( \frac{4}{c'_1} \int_{\mathbb{H}^1} \log \frac{\rho(p)}{\rho(p^{-1}y)} Q'(p)e^{4u(p)} dv(p) \right) dv(y)$$

$$= \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} \exp \left( \frac{4\alpha}{c'_1} \int_{\mathbb{H}^1} \log \frac{\rho(p^{-1}x)}{\rho(p^{-1}y)} \cdot \frac{Q'(p)e^{4u(p)}}{\alpha} dv(p) \right) dv(y).$$
This is bounded by

\[
\sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \int_{\mathbb{H}^1} \left( \frac{\rho(p^{-1}x)}{p(p^{-1}y)} \right)^{\frac{4\alpha}{c'_1}} Q'(p) e^{4u(p)} \frac{\rho(p^{-1}y)}{\alpha} d\nu(p) d\nu(y)
\]

\[
= \sup_{r>0} \int_{\mathbb{H}^1} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left( \frac{\rho(p^{-1}x)}{p(p^{-1}y)} \right)^{\frac{4\alpha}{c'_1}} d\nu(y) Q'(p) e^{4u(p)} \frac{\rho(p^{-1}y)}{\alpha} d\nu(p).
\]

We know that by Lemma 4.5, \(\frac{1}{\rho(x)^{\frac{4\alpha}{c'_1}}}\) is an \(A_1\) weight. And so is \(\frac{1}{\rho(p^{-1}x)^{\frac{4\alpha}{c'_1}}}\) for each fixed \(p\). This means

\[
\frac{1}{|B(x,r)|} \int_{B(x,r)} \frac{1}{\rho(p^{-1}y)^{\frac{4\alpha}{c'_1}}} d\nu(y) \leq C(\alpha),
\]

for each fixed \(p\). Observe that \(C\) is independent of \(p\), one can substitute this inequality to the estimate (4.15) and obtain that (4.15) is bounded by

\[
\int_{\mathbb{H}^1} C(\alpha) Q'(p) e^{4u(p)} \frac{\rho(p^{-1}y)}{\alpha} d\nu(p) = C(\alpha).
\]

This shows that \(e^{4u}\) is an \(A_1\) weight. Once we have the \(A_1\) property of \(e^{4u}\), we can apply Theorem 4.4 to it. It completes the proof of Theorem 1.4. \(\square\)

Finally, we give the example, that shows \(c'_1\) is the critical constant for the validity of the isoperimetric inequality.

**Example 4.6.** Let \(e^{u}\theta\) be a contact form on \(\mathbb{H}^1\). And suppose \(u\) is given by the following integral formula.

\[
u(x) = \frac{1}{c_1} \int_{\mathbb{H}^1} \log \frac{\rho(y)}{\rho(y^{-1}x)} c'_1 \delta_0 d\nu(y),
\]

where \(\delta_0\) denotes Dirac delta function. It is obvious that the volume form \(e^{4u\theta} = \frac{1}{\rho(x)^{\frac{4\alpha}{c'_1}}}\) on \(\mathbb{H}^1\) is not an \(A_1\) weight. Moreover, such a CR manifold does not satisfy the isoperimetric inequality. This is because \(e^{u\theta} = \frac{1}{\rho}\theta\) is the standard contact form on the cylinder \(\mathbb{R} \times S^2 \cong \mathbb{H}^1 \setminus \{(0,0,0)\}\). In particular, one can choose a sequence of rotationally symmetric annular domains \(A(r_0, r)\) on \(\mathbb{H}^1\), \(r \to \infty\). The area of \(\partial A(r_0, r)\) with respect to \(e^{u\theta}\) is bounded in \(r\). But the volume of \(A(r_0, r)\) with respect to \(e^{u\theta}\) tends to \(\infty\) as \(r \to \infty\). This gives a counterexample to the isoperimetric inequality. In this construction, \(u\) is singular at the origin. But we can use the approximation argument
to deal with the issue. By choosing $\phi_\epsilon(y)$ to be a sequence of compactly supported smooth functions approximating $c'_1 \delta_0$, and defining

\begin{equation}
(4.18) 
    u_\epsilon(x) = \frac{1}{c'_1} \int_{\mathbb{H}^1} \log \frac{\rho(y)}{\rho(y^{-1}x)} \phi_\epsilon(y) dv(y),
\end{equation}

we construct a sequence of $u_\epsilon$ that approximates $u(x) = \log \frac{1}{\rho(x)}$ locally uniformly away from the origin. Since $\phi_\epsilon(y)$ are compactly supported, when the annular domains $A(r_0, r)$, $r \to \infty$ are chosen such that $r_0$ is big enough (but fixed), the CR manifold $(\mathbb{H}^1, e^{u_\epsilon} \theta)$ does not satisfy the isoperimetric inequality.

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