Multidimensional SU(2) wormhole between two null surfaces

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Abstract

The multidimensional gravity on principal bundle with structural SU(2) gauge group is considered. The Lorentzian wormhole solution disposed between two null surfaces is founded in this case. As the nondiagonal components are similar to a gauge fields (in Kaluza - Klein’s sense) then in some sense this solution is dual to the black hole in 4D Einstein - Yang - Mills gravity: 4D black hole has the stationary area outside of event horizon but multidimensional wormhole inside null surface (on which \( ds^2 = 0 \))

I. INTRODUCTION

In gravity various physical phenomena can be connected with wormholes (WH). In quantum gravity they form spacetime foam. According to J. Wheeler, the classical WH with electrical field can describe the “charge without charge”. The force line flows into one side of WH (“minus” charge) and outflow on other side (“plus” charge). The WH have been studied in many papers. The Euclidean WH interacting with classical fields are considered in [1]- [5]. In some works the multidimensional WH are examined [6]- [7]. In this article I shall receive the WH in multidimensional gravity in which the SU(2) gauge group is supplementary coordinates. In such multidimensional gravity the total space is the fiber bundle with fiber = gauge group

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and base = 4D Einstein’s spacetime. Since any group is symmetric space, its metric is the conformally Euclidean metric and it can be written in the following view:

$$ds^2_{fiber} = h(x^\mu)\sigma^a\sigma_a,$$  (1)

where conformal factor $h(x^\mu)$ depends only on spacetime coordinates $x^\mu; \mu = 0, 1, 2, 3; \sigma_a = \gamma_{ab}\sigma^b; \gamma_{ab}$ is Euclidean metric; $a = 4, 5, \ldots N$ indexes of supplementary coordinates; $\sigma^a$ are one-form satisfies Maurer - Cartan structure equations:

$$d\sigma^a = f^a_{bc}\sigma^b \wedge \sigma^c,$$  (2)

where $f^a_{bc}$ is a structural constant.

**II. U(1) WORMHOLE**

We remind the solution for 5D Kaluza - Klein’s theory derived in [7]. The metric is:

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\psi(r)}(d\chi - 
\omega(r)dt)^2 - dr^2 - a^2(r)(d\theta^2 + \sin^2 \theta d\varphi^2),$$  (3)

where $\chi$ is the 5th supplementary coordinate; $r, \theta, \varphi$ are 3D polar coordinates; $t$ is the time.

The solution 5D Einstein’s equations is:

$$a^2 = r_0^2 + r^2,$$  (4)

$$e^{-2\psi} = e^{2\nu} = \frac{2r_0 r^2 + r_0^2}{q r_0^2 - r^2},$$  (5)

$$\omega = \frac{4r_0^2 r}{q r_0^2 - r^2}.$$  (6)

This solution is the WH between two null surfaces.

**III. SU(2) WORMHOLE**

We can introduce the Euler angles on SU(2) group. Then one-forms $\sigma^a$ can be written as a follows:
\[ \sigma^1 = \frac{1}{2}(\sin \alpha \beta - \sin \beta \cos \alpha \gamma), \]  
\[ \sigma^2 = -\frac{1}{2}(\cos \alpha \beta + \sin \beta \sin \alpha \gamma), \]  
\[ \sigma^3 = \frac{1}{2}(d\alpha + \cos \beta d\gamma), \]

where \( 0 \leq \beta \leq \pi, 0 \leq \gamma \leq 2\pi, 0 \leq \alpha \leq 4\pi. \) Thus, 7D metric on the total space can be written in the following view:

\[ ds^2 = ds^2_{\text{fibre}} + 2G_{A\mu}dx^A dx^\mu, \]

where \( A = 0, 1, \ldots, N \) is multidimensional index. There are the following independent degrees of freedom in multidimensional gravity on the bundle: conformal factor \( h(x^\mu) \) and components of the multidimensional metric \( G_{A\mu}. \) Variation of the action with respect to \( G_{A\mu} \) leads to corresponding Einstein’s equations:

\[ R_{A\mu} - \frac{1}{2}G_{A\mu}R = 0, \]

where \( R_{A\mu} \) is Ricci tensor and \( R \) is Ricci scalar. Variation of the actions with respect to conformal factor \( h(x^\mu) \) is represented in the following view:

\[ \delta S = \delta G^{ab} \frac{\delta S}{\delta G^{ab}} \propto \delta h^{ab} \frac{\delta S}{\delta h^{ab}}, \]

where \( G_{ab} = h^{\gamma}_{ab} \) is metric on fibre of the bundle. Hence, the corresponding equation has the following form:

\[ R^a_a = R^4_4 + R^5_5 + R^6_6 = 0. \]

Thus, the equations system described the multidimensional gravity on principal bundle with fibre = nonabelian gauge group are \( \square \), \( \square \) equations. If the gauge group is Abelian \( U(1) \) group, we have the standard 5D Kaluza - Klein’s theory.

We remind the following result \cite{8} - \cite{9}. Let \( G \) group be the fibre of principal bundle. Then there is the one-to-one correspondence between \( G \)-invariant metrics on the total space \( \mathcal{X} \) and the triples \( (g_{\mu\nu}, A^a_{\mu}, h^{\gamma}_{ab}). \) Where \( g_{\mu\nu} \) is Einstein’s pseudo - Riemannian metric; \( A^a_{\mu} \) is gauge field of the \( G \) group; \( h^{\gamma}_{ab} \) is symmetric metric on the fibre.
We see a solution of the form:

\[
d s^2 = e^{2ν(r)}dt^2 - r_0^2e^{2ψ(r)} \sum_{a=1}^{3} \left( σ^a - A^a_μ(r)dx^μ \right)^2 - \\
dr^2 - a^2(r) \left(dθ^2 + \sin^2 θdϕ^2\right).
\] (14)

We choose the “potentials” \(A^a_μ\) in following monopole-like form:

\[
A^a_θ = \frac{1}{2}(f(r) + 1)\{\sin φ; - \cos φ; 0\}, \\
A^a_ϕ = \frac{1}{2}(f(r) + 1)\sin θ\{\cos φ \cos θ; \sin φ \cos θ; - \sin θ\}, \\
A^a_t = v(r)\{\sin θ \cos φ; \sin θ \sin φ; \cos θ\},
\] (15, 16, 17)

Let us introduce tetrads \(e^A_A\):

\[
ds^2 = η_{\bar{A}\bar{B}}Σ^AΣ^B, \\
Σ^A = e^A_Adx^A,
\] (18, 19)

where \(\bar{A}, \bar{B} = 0, 1, \ldots, 6\) are tetrads indexes; \(η_{\bar{A}\bar{B}}\) is 7D Minkowski metric. The input equations are written below in the following form:

\[
R_{\bar{A}μ} = 0, \\
R^a_μ = 0.
\] (20, 21)

7D gravity equations become:

\[
ν'' + ν'v' + 3νψ' + 2aν' + \frac{r_0^2}{a^2}v^2e^{2(ψ - ν)}ν'^2 - \\ \frac{r_0^2}{a^2}v^2f^2e^{2(ψ - ν)} = 0, \\
ν'' + 3ψ'' + 3ψν'' + 2a'' + \frac{r_0^2}{a^2}v^2e^{2(ψ - ν)}ν'^2 + \\ \frac{r_0^2}{4a^2}f^2e^{2ψ} = 0, \\
\frac{a''}{a} + \frac{a'}{a}(ν' + 3ψ') + \frac{a'^2}{a^2} - \frac{1}{a^2} + \\ \frac{r_0^2}{8a^2}e^{2ψ}f'^2 - \frac{r_0^2}{2a^2}νe^{2(ψ - ν)} +
\] (22, 23)
\[
\frac{r_0^2}{8a^4} \left( f^2 - 1 \right)^2 = 0, \quad (24)
\]
\[
\psi'' + 3\psi'f' + 2\frac{a'}{a}\psi' + \psi'\nu' + \frac{r_0^2}{6} e^{2(\psi-\nu)}v'^2 - \frac{2}{r_0^2} e^{-2\psi} - \frac{r_0^2 f'^2}{12a^2} e^{2\psi} + \frac{r_0^2 v^2 f'^2 - 8}{3a^2} e^{2(\psi-\nu)} - \frac{r_0^2}{24a^4} \left( f^2 - 1 \right)^2 = 0, \quad (25)
\]
\[
f'' + f'(\nu' + 5\psi') + 2v^2 f e^{-2\nu} = \frac{f}{2a^2} (f^2 - 1), \quad (26)
\]
\[
v'' - v'(\nu' - 5\psi' - 2\frac{a'}{a}) = 2\frac{v}{a} f, \quad (27)
\]

Here the Eq’s (26) and (27) are “Yang - Mills” equations for nondiagonal components of the multidimensional metric. For simplicity we consider \( f = 0 \) case. This means that we have “color electrical” field \( A_i^a \) only (\( i=1,2,3 \)). In this case it is easy to integrate Eq.(27):

\[
v' = \frac{q}{r_0 a^2} e^{\nu - 5\psi}, \quad (28)
\]

where \( q \) is the constant of the integration (“color electrical” charge). Let us examine the most interesting case when the linear dimensions of fibers \( r_0 \) are vastly smaller than the space dimension \( a_0 \) and “charge” \( q \) is sufficiently large:

\[
\left( \frac{q}{a_0} \right)^{1/2} \gg \left( \frac{a_0}{r_0} \right)^2 \gg 1, \quad (29)
\]

where \( a_0 = a(r = 0) \) is the throat of the WH.

On this approximation we deduce the equations system:

\[
\nu'' + \nu'^2 + 3\nu'\psi' + 2\frac{a'}{a}\nu' - \frac{q^2}{2a^4} e^{-8\psi} = 0, \quad (30)
\]
\[
\nu'' + \nu'^2 + 3\psi'^2 + 2\frac{a'}{a} - \frac{q^2}{2a^4} e^{-8\psi} = 0, \quad (31)
\]
\[
\psi'' + 3\psi'^2 + 2\frac{a'}{a} + \psi'\nu' + \frac{q^2}{6a^4} e^{-8\psi} = 0, \quad (32)
\]
\[
\frac{a''}{a} + \frac{a'}{a}(\nu' + 3\psi') + \frac{a'^2}{a^2} - \frac{1}{a^2} = 0. \tag{33}
\]

This system has the following solution:

\[
\nu = -3\psi, \tag{34}
\]

\[
a^2 = a_0^2 + r^2, \tag{35}
\]

\[
e^{-\frac{4}{3}\nu} = \frac{q}{2a_0} \cos \left( \sqrt{\frac{8}{3}} \arctan \frac{r}{a_0} \right), \tag{36}
\]

\[
v = \sqrt{6} \frac{a_0}{r_0 q} \tan \left( \sqrt{\frac{8}{3}} \arctan \frac{r}{a_0} \right). \tag{37}
\]

Let us define value \( r \) in which metric has null surfaces. From condition:

\[
G_{tt}(rg) = e^{2\nu(rg)} - r_0^2 e^{2\psi(r)} \sum_{a=1}^{3} (A^a_t(rg))^2 = 0 \tag{38}
\]

it follows that:

\[
\frac{r_g}{a_0} = \tan \left( \sqrt{\frac{3}{8}} \arcsin \sqrt{\frac{2}{3}} \right) \approx 0.662. \tag{39}
\]

It is easy to verify that \( \exp(2\nu) = 0 \) (\( \exp(2\psi = \infty) \) by \( r/a_0 = \tan(\pi \sqrt{3/32}) \approx 1.434 \). This value lies beyond the null surfaces. This means that the small terms in (24), (25) will stay also small even near the null surfaces.

**IV. DISCUSSION**

One can concede that the solution of this sort has certain meaning in nature: it can be a source of the gauge fields. As it is pointed out above, there is one-to-one correspondence between the multidimensional metric on bundle and 4D gravity + gauge field + scalar field. It is possible that the mechanism of supplementary coordinates compactification exists in nature. In Ref. [10] the possible mechanism of this phenomenon is discussed from the algorithmical point of view. In this case the composite \( WH \) realizing J.Wheeler’s idea on “charge without charge” and “mass without mass” can exists in nature. Such \( WH \) can be constructe in the
following way: The multidimensional region is found in center of the $WH$ and two $4D$ regions are disposed on each side of center. It can be supposed that the compactification happens on some chosen surface. The event horizon can be such surface. Such composite $WH$ in $5D$ Kaluza - Klein’s theory was examined in [7]. Thus, the exterior observer cannot detects the presence of this multidimensional region under event horizon, he can registers only electrical charge, angular momentum and mass of the black hole. It should be noted that in this model the compactification of the supplementary coordinates is a quantum phenomenon (this is a jump and not a classical step - by - step splitting off the supplementary coordinates).
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