Spherical functions on $U(n, n)/(U(n) \times U(n))$ and hermitian Siegel series

Yumiko Hironaka
Department of Mathematics,
Faculty of Education and Integrated Sciences, Waseda University
Nishi-Waseda, Tokyo, 169-8050, JAPAN

Abstract: We study spherical functions on the space isogeneous to $U(n, n)/(U(n) \times U(n))$ over a $p$-adic field, especially those functional equations with respect to the action of the Weyl group, the location of possible poles and zeros, and explicit formulas for some special points. Then, as an application, we give a functional equation of $p$-adic local hermitian Siegel series.

§0 Introduction

Let $k'$ be an unramified quadratic extension over a $k$ be a non-archimedian local field of characteristic 0. We fix a prime element $\pi$ of $k$, and the additive value $v_{\pi}(-)$ and the normalized absolute value $| |$ on $k^\times$, where $|\pi|^{-1} = q$ is the cardinality of the residue class field of $k$. We consider hermitian matrices with respect to the involution $*$ on $k'$ which is identity on $k$, and set

$$\mathcal{H}_m = \{ A \in M_m(k') \mid A^* = A \}, \quad \mathcal{H}_m^{ad} = \mathcal{H}_m \cap GL_m(k'),$$

(0.1)

where, for a matrix $A = (a_{ij}) \in M_{mn}(k')$, we denote by $A^*$ the matrix $(a_{ji}) \in M_{nm}(k')$.

For $T \in \mathcal{H}_n^{ad}$, we define the space

$$\mathfrak{X}_T = \{ x \in M_{2n,n}(k') \mid x^* H_n x = T \}, \quad X_T = \mathfrak{X}_T/U(T),$$

where $U(T) = \{ g \in GL_n(k') \mid g^* T g = T \}$. Then we see that $X_T$ is isomorphic to $U(H_n)/(U(T) \times U(T))$ over $k$ (cf. Lemma 1.1). We set $G = U(H_n)$ and $K = G(O_{k'})$, and introduce the spherical function $\omega_T(\overline{\pi}, s)$ on $X_T$ by

$$\omega_T(\overline{\pi}, s) = \int_{K} |f_T(kx)|^{s+\varepsilon} \, dk, \quad (\overline{\pi} \in X_T, \ s \in \mathbb{C}^n).$$

(0.2)
Here $dk$ is the normalized Haar measure on $K$,
\[
\varepsilon = (-1, \ldots, -1, -\frac{1}{2}) + (\frac{\pi \sqrt{-1}}{\log q}, \ldots, \frac{\pi \sqrt{-1}}{\log q}) \in \mathbb{C}^n,
\]
\[
|f_T(x)|^s = \prod_{i=1}^{n} |d_i(x_2 T^{-1} x_2^*)|^{s_i} \quad (s \in \mathbb{C}^n),
\]
where $x_2$ is the lower half $n$ by $n$ block of $x \in \mathfrak{X}_T$ and $d_i(y)$ is the determinant of the upper left $i$ by $i$ block of $y$. The right hand side of (0.2) is absolutely convergent if $\text{Re}(s_i) \geq 1$ ($1 \leq i \leq n - 1$) and $\text{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and becomes a common eigen function with respect to the action of Hecke algebra $\mathcal{H}(G, K)$; thus we have a spherical function on $X_T$ (as well as $\mathfrak{X}_T$).

We introduce a new variable $z$ which is related to $s$ by
\[
s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n - 1), \quad s_n = -z_n
\]
and write $\omega_T(\overline{x}; z) = \omega_T(\overline{x}; s)$. We denote by $W$ the Weyl group of $G$ with respect to the maximal $k$-split torus in $G$. The group $W$ is isomorphic to $S_n \times (C_2)^n$, $S_n$ acts on $z_i$ by permutation of indices and $W$ is generated by $S_n$ and $\tau : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, -z_n)$. We denote by $\Sigma^+$ the set of positive roots of $G$ with respect to the Borel group, and regard it a subset of $\mathbb{Z}^n$ and set $\langle \alpha, z \rangle = \sum_{i=1}^{n} \alpha_i z_i$ for $\alpha \in \Sigma^+$ (for details, see §2.3).

We will prove the following in §1 and §2 (Theorem 1.3, Theorem 2.5, Theorem 2.8).

**Theorem 1** (i) For any $T \in \mathcal{H}_n^{nd}$, the function
\[
\prod_{1 \leq i < j \leq n} \frac{(1 + q^{z_i - z_j})}{(1 - q^{z_i - z_j - 1})} \times \omega_T(\overline{x}; z)
\]
is holomorphic for all $z$ in $\mathbb{C}^n$ and $S_n$-invariant, and the function
\[
|2|^{-z_1 - z_2 - \cdots - z_n} \prod_{1 \leq i < j \leq n} \frac{(1 + q^{z_i - z_j})(1 + q^{z_i + z_j})}{(1 - q^{z_i - z_j - 1})(1 - q^{z_i + z_j - 1})} \times \omega_T(\overline{x}; z)
\]
is also holomorphic for all $z$ in $\mathbb{C}^n$ and $W$-invariant. In particular the latter is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$.

(ii) For any $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the following functional equation holds
\[
\omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \quad (0.4)
\]
where
\[
\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma^+} f_\sigma(\langle \alpha, z \rangle), \quad f_\sigma(t) = \begin{cases} \frac{1 - q^{t-1}}{q^t - q^{-1}} & \text{if } \alpha \text{ is short} \\ |2|^t & \text{if } \alpha \text{ is long} \end{cases}
\]

Further, we will give an explicit expression of $\omega_T(x_T; s)$ based on functional equations and data of the group $G$ in §3 (Theorem 3.2).
As an application, we consider the hermitian Siegel series in §4. For each $T \in \mathcal{H}_n$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\text{tr}(TR))dR,$$

where $\psi$ is an additive character on $k$ of conductor $\mathcal{O}_k$, $\text{tr}()$ is the trace of matrix and $\nu_\pi(R)$ is the "denominator" of $R$, which is certain non-negative powers of $q$ (cf. (4.1)) As for Siegel series (for symmetric matrices), F. Satô and the author have given a new integral expression and related it to a spherical function on the symmetric space $O(2n)/(O(n) \times O(n))$ (cf. [HS]). In the present paper we develop the similar argument for hermitian Siegel series. Since we know well about the functional equations of spherical functions $\omega_T(x; s)$ with respect to $W$ as above, we can bring out the functional equation of $b_\pi(T; s)$ as an application; thus we will give an integral expression of $b_\pi(T; s)$ and a new proof of the functional equation from the view point of spherical functions in the follows(Theorem 4.2, Theorem 4.4).

**Theorem 2**

(i) If $\text{Re}(s) > 2n$, one has

$$b_\pi(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \cdot \int_{X_T(O_{k'})} |N_{k'/k}(\text{det} x_2)|^{\frac{s}{2}} \cdot |\Theta_T|(x),$$

where $X_T(O_{k'}) = X_T \cap M_{2n,n}(O_{k'})$, $\zeta_n(k'; \cdot)$ is the zeta function of the matrix algebra $M_n(k')$, and $|\Theta_T|(x)$ is a certain normalized measure on $X_T$.

(ii) For any $T \in \mathcal{H}_n^{nd}$, one has

$$\prod_{i=0}^{n-1} \frac{(1 - (-1)^i q^{-s+i})}{b_\pi(T; 2n - s)} \prod_{i=0}^{n-1} \frac{(1 - (-1)^i q^{-(2n-s)+i})}{\chi_\pi(\text{det}(T/2))^{s-n}}.$$
§1

Let \( k'/k \) be an unramified quadratic extension of \( p \)-adic fields with involution \( * \), and for each \( A = (a_{ij}) \in M_{mn}(k') \), we denote by \( A^* \) the matrix \( (a_{ji}^*) \in M_{nm}(k') \). We set
\[
\mathcal{H}_m = \{ A \in M_m(k') \mid A^* = A \}, \quad \mathcal{H}_{m,n}^d = \mathcal{H}_m \cap GL_m(k').
\]

For \( A \in \mathcal{H}_m \) and \( X \in M_{mn}(k') \), we write
\[
A[X] = X^* AX = X^* \cdot A \in \mathcal{H}_n,
\]
and define the unitary group
\[
U(A) = \{ g \in GL_m(k') \mid A[g] = A \}.
\]

In particular we write
\[
G = U(n, n) = U(H_n) \quad \text{with} \quad H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.
\]

For \( T \in \mathcal{H}_{n}^d \), we set
\[
\mathfrak{X}_T = \{ x \in M_{2n,n}(k') \mid H_n[x] = T \}, \quad X_T = \mathfrak{X}_T / U(T), \quad (1.2)
\]
\[
x_T = \begin{pmatrix} \frac{1}{2} T \\ 1_n \end{pmatrix} \in \mathfrak{X}_T.
\]

The group \( G \) acts on \( \mathfrak{X}_T \), as well as on \( X_T \), through left multiplication, which is transitive by Witt’s theorem for hermitian matrices (cf. [Sch], Ch.7, §9).

**Lemma 1.1** The stabilizer subgroup of \( G \) at \( x_T U(T) \in X_T \) is given as
\[
\left\{ \tilde{T}^{-1} \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \tilde{T} \mid h_1, h_2 \in U(T) \right\},
\]
where
\[
\tilde{T} = \begin{pmatrix} 1_n & \frac{1}{2} T \\ 1_n & -\frac{1}{2} T \end{pmatrix} \in GL_{2n}(k').
\]

In particular, the space \( X_T \) is isomorphic to \( U(H_n) / (U(T) \times U(T)) \).

**Proof.** First we note
\[
\tilde{T} x_T = \begin{pmatrix} T \\ 0 \end{pmatrix} \in M_{2n,n},
\]
\[
H_n[\tilde{T}^{-1}] = H_n \begin{bmatrix} \frac{1}{2} 1_n & \frac{1}{2} 1_n \\ \tilde{T}^{-1} & -\tilde{T}^{-1} \end{bmatrix} = \begin{pmatrix} T^{-1} & 0 \\ 0 & -T^{-1} \end{pmatrix}.
\]

Then, we have for any \( h \in U(T) \)
\[
x_T h = \tilde{T}^{-1} \begin{pmatrix} T h \\ 0 \end{pmatrix} = \tilde{T}^{-1} \begin{pmatrix} h^{*-1} & 0 \\ 0 & 1 \end{pmatrix} \tilde{T} x_T.
\]
For $g \in G$ such that $gx_T = x_T$, since $\tilde{T}g\tilde{T}^{-1}$ belongs to $U(H_n[\tilde{T}^{-1}])$ and stabilizes $\tilde{T}x_T$, we see
\[
\tilde{T}g\tilde{T}^{-1} = \begin{pmatrix} 1_n & 0 \\ 0 & d \end{pmatrix}, \text{ for some } d \in U(-T^{-1}).
\]
Hence any element in the stabilizers at $x_T U(T)$ has a form
\[
\tilde{T}^{-1} \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \tilde{T}, \quad h_1, h_2 \in U(T),
\]
and the assertion follows from this.

We fix the Borel subgroup $B$ of $G$ as
\[
B = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \bigg| \begin{array}{c} b \text{ is upper triangular of size } n, \\ a + a^* = 0 \end{array} \right\}. \tag{1.3}
\]
For each element $x \in \mathfrak{X}_T$, we denote by $x_2$ the lower half $n$ by $n$ block of $x$. We consider the following function on $\mathfrak{X}_T$
\[
f_{T,i}(x) = d_i(x_2 T^{-1} x_2^*) \quad 1 \leq i \leq n, \tag{1.4}
\]
where $d_i(y)$ is the determinant of the upper left $i$ by $i$ block of a matrix $y$. Then, $f_{T,i}$ is a relative $B$-invariant associated with $k$-rational character $\psi_i$ of $B$ as follows
\[
f_{T,i}(bx) = \psi_i(b) f_{T,i}(x), \quad \psi_i(b) = \prod_{j=1}^{i} N(b_j)^{-1}, \tag{1.5}
\]
where $b_j$ is the $j$-th diagonal component of $b \in B$ and $N = N_{k'/k}$. Since $f_{T,i}(xh) = f_{T,i}(x)$ for any $h \in U(T)$, we understand $f_{T,i}(x), 1 \leq i \leq n$, as $B$-relative invariants on $X_T$.

**Remark 1.2** Though we can realize above objects as the sets of $k$-rational points of algebraic sets defined over $k$ and develop the arguments, we write down to earth way for simplicity of notations. We only note here that $\{x \in X_T \mid f_{T,i}(x) \neq 0, 1 \leq i \leq n\}$ is a Zariski open $B$-orbit over the algebraic closure of $k$.

For the absolute value $||$ on $k^\times$, we set $|0| = 0$ for convenience. For simplicity of notations, we write an element $\mathfrak{X} = x U(T)$ in $X_T$ by its representative $x$ in $\mathfrak{X}_T$ in the following. We denote by $\delta$ the modulus character on $B$ (i.e., $d_i(bb') = \delta(b')^{-1} d_i(b)$ for the left invariant measure $d_i(b)$ on $B$), then
\[
\delta^\frac{1}{2}(b) = \prod_{i=1}^{n-1} |\psi_i(b)|^{-1} \times |\psi_n(b)|^{-\frac{1}{2}}.
\]
Now we introduce the spherical function $\omega(x; s)$ on $X_T = \mathfrak{X}_T / U(T)$: for $s \in \mathbb{C}^n$ set
\[
\omega_T(x; s) = \omega_T^{(n)}(x; s) = \int_{K} |f_T(kx)|^{s+\epsilon} \, dk, \tag{1.6}
\]
where $dk$ is the normalized Haar measure on $K = G \cap GL_{2n}(O_K)$,

$$
\varepsilon = (-1, \ldots, -1, -\frac{1}{2}) + (\frac{\pi \sqrt{-1}}{\log q}, \ldots, \frac{\pi \sqrt{-1}}{\log q}) \in \mathbb{C}^n,
$$

$$f_T(x) = \prod_{i=1}^{n} f_{T,i}(x), \quad |f_T(x)|^s = \prod_{i=1}^{n} |f_{T,i}(x)|^{s_i}.
$$

The right hand side of (1.6) is absolutely convergent if $\text{Re}(s_i) \geq 1$ $(1 \leq i \leq n-1)$ and $\text{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and becomes a common eigenfunction with respect to the action of the Hecke algebra $H(G, K)$ (cf. [H2], §1).

It is easy to see

$$
\omega_{T[h]}(x; s) = \omega_T(xh^{-1}; s), \quad h \in GL_n(k'), \ x \in \mathcal{X}_{T[h]}, \quad (1.7)
$$

since $\mathcal{X}_{T[h]} = (\mathcal{X}_T)h$. Hence, in order to study functional properties of $\omega_T(x; s)$ (e.g., Theorem 1 in the introduction), it suffices to consider only for diagonal $T$’s.

We introduce a new variable $z$ which is related to $s$ by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n \quad (1.8)
$$

and write $\omega_T(x; z) = \omega_T(x; s)$. The Weyl group $W$ of $G$ relative to the maximal $k$-split torus in $B$ acts on rational characters of $B$ as usual (i.e., $\sigma(\psi)(b) = \psi(n^{-1}_\sigma b)$ by taking a representative $n_\sigma$ of $\sigma$), so $W$ acts on $z$ and on $s$ as well. We will determine the functional equations of $\omega_T(x; s)$ with respect to this Weyl group action. The group $W$ is isomorphic to $S_n \ltimes \mathbb{C}_n^2$, $S_n$ acts on $z$ by permutation of indices and $W$ is generated by $S_n$ and $\tau: (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, -z_n)$.

By using a result on spherical functions on the space of hermitian forms, we obtain the following theorem.

**Theorem 1.3** For any $T \in H_n^{ad}$, the function

$$
\prod_{1 \leq i < j \leq n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i}} \times \omega_T(x; z)
$$

is holomorphic for any $z$ in $\mathbb{C}^n$ and $S_n$-invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}$.

**Proof.** By the embedding

$$K_0 = GL_n(O_K') \hookrightarrow K, \quad h \mapsto \tilde{h} = \left( \begin{array}{cc} h^{s-1} & 0 \\ 0 & h \end{array} \right),
$$

we obtain

$$\omega_T(x; z) = \omega_T(x; s) = \int_{K_0} \int_K |f_T(kx)|^{s+\varepsilon} \, dk$$

$$= \int_{K_0} \int_K |f_T(\tilde{hk}x)|^{s+\varepsilon} \, dk = \int_K \int_{K_0} |f_T(\tilde{hk}x)|^{s+\varepsilon} \, dh \, dk$$

$$= \int_K \zeta^{(\varepsilon)}(D(kx); s) \, dk.$$
Here \( D(kx) = (kx)_2 \cdot T^{-1} \), which may be assumed in \( \mathcal{H}_{nd}^n \), and \( \zeta^{(n)}(y; s) \) is a spherical function on \( \mathcal{H}_{nd}^n \) defined by

\[
\zeta^{(n)}(y; s) = \int_{K_0} \prod_{i=1}^{n} |d_i(h \cdot y)|^{s_i+\varepsilon_i} \, dh,
\]

and we keep the relation of variables \( s \) and \( z \) as before. Then the assertion of Theorem 1.3 follows from the next proposition.

**Proposition 1.4** (cf. [H1] or [H3]) For any \( y \in \mathcal{H}_{nd}^n \), the function

\[
\prod_{1 \leq i < j \leq n} q^{z_j} + q^{z_i} \times \zeta^{(n)}(y; s) \text{ is holomorphic for } s \in \mathbb{C}^n \text{ and invariant under the action of } S_n.
\]

In [H3] §4.2, we considered the spherical function

\[
\omega^{(H)}(y; s) = \int_{K_0} \prod_{i=1}^{n} \chi_\pi(d_i(h \cdot y)) |d_i(h \cdot y)|^{s_i+\varepsilon_i} \, dh,
\]

where \( \chi_\pi \) is the character of \( k^\times \) defined by \( \chi_\pi(a) = (-1)^{v_\pi(a)} \) and \( \varepsilon' = (-1, \ldots, -1, \frac{n-1}{2}) \) comes from the modulus character of the Borel subgroup of \( GL_n(k') \) consisting of lower triangular matrices. The function \( \zeta^{(n)}(x; s) \) satisfies the same functional properties as \( \omega^{(H)}(y; s) \), since \( \omega^{(H)}(y; s) = |\det y|^{\frac{n}{2}} \zeta^{(n)}(y; s) \).

**Remark 1.5** For the transposition \( \tau_i = (i \ i+1) \in W, \ 1 \leq i \leq n-1 \), the following functional equation holds by Theorem 1.3

\[
\omega_T(x; z) = \frac{1 - q^{z_i-z_{i+1}} - 1}{q^{z_i-z_{i+1}} - q^{-1}} \times \omega_T(x; \tau_i(z)), \ 1 \leq i \leq n-1.
\]

On the other hand, one can obtain (1.10) directly in the similar way to the case of \( \tau \) in § 3, then Theorem 1.3 follows from (1.10), through the similar line to the proof of Proposition 1.4 In fact, Proposition 1.4 was proved by using functional equations of type (1.10).

**§2**

We calculate the functional equation for \( \tau \in W \), and give the functional equations with respect to the whole \( W \). We fix a unit \( \epsilon \in \mathcal{O}_k^\times \) for which \( k' = k(\sqrt{\epsilon}) \). If \( k \) is not dyadic, it is well-known that \( \mathcal{O}_k' = \mathcal{O}_k + \mathcal{O}_k \sqrt{\epsilon} \). If \( k \) is dyadic, we may take \( \epsilon \) as \( \epsilon \in 1 + 4 \mathcal{O}_k^\times \), and then \( k\sqrt{\epsilon} \cap \mathcal{O}_k' = \mathcal{O}_k \sqrt{\epsilon} \) (cf. [Om], 63.3 and 63.4).

**2.1.** First we calculate the spherical function for \( n = 1 \). Since \( N(\mathcal{O}_k') = \mathcal{O}_k^\times \), we may assume that

\[ T = \pi^\lambda. \]
We collect the data:

$$G = U(1, 1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{*^{-1}} \end{pmatrix}, \begin{pmatrix} 1 & w\sqrt{\epsilon} \\ v\sqrt{\epsilon} & 1 + v\epsilon \end{pmatrix} \right\} \mid a \in k^\times, v, w \in k \right\},$$

$$K = K_1 = K_{1,1} \cup K_{1,2},$$

where

$$K_{1,1} = \left\{ \begin{pmatrix} \alpha & \alpha v/\sqrt{\epsilon} \\ \alpha^{*^{-1}} u\sqrt{\epsilon} & \alpha^{*^{-1}}(1 + uv) \end{pmatrix} \right\} \mid \alpha \in \mathcal{O}^\times_k, u, v \in \mathcal{O}_K,$$

$$K_{1,2} = \left\{ \begin{pmatrix} \alpha \pi u\sqrt{\epsilon} & \alpha(1 + \pi uv) \\ \alpha^{*^{-1}} v/\sqrt{\epsilon} \end{pmatrix} \right\} \mid \alpha \in \mathcal{O}^\times_k, u, v \in \mathcal{O}_K.$$

$$\mathfrak{x}_T = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in M_{21}(k') \mid x_1^2 + x_1 x_2 = \pi^\lambda \right\},$$

$$X_T = \mathfrak{x}_T/\mathcal{O}_k,$$

where \(\mathcal{O}_k = \{ \varepsilon \in \mathcal{O}_k \mid N(\varepsilon) = 1 \},\)

\(f_1(x) = \pi^{-\lambda} N(x_2) \) for \(x \in \mathfrak{x}_T,\)

and

$$\omega_T^{(1)}(x; s) = \int_{K_1} \chi_\pi(f_1(hx)) |f_1(hx)|^{s-\frac{1}{2}} dh,$$

where \(dh\) is the Haar measure on \(K_1.\)

**Proposition 2.1** (i) A set of complete representatives of \(K_1 \backslash \mathfrak{x}_T\) for \(T = \pi^\lambda\) can be taken as follows:

$$\left\{ x_e = \pi^e \left( \begin{pmatrix} 1 \\ \frac{1}{2}\pi \lambda-2e \end{pmatrix} \right) \mid e \in \mathbb{Z}, 2e \leq \lambda - v_\pi(2) \right\}.$$

(ii) For \(x_e \in \mathfrak{x}_T\) with \(T = \pi^\lambda\) as above, one has

$$\omega_T^{(1)}(x_e; s) = \frac{(-1)^\lambda q^{-\frac{s}{2}} |2|^{-s}}{1 + q^{-1}} \times \frac{1}{q^s - q^{-s}}, (q^{(\lambda-2e-e_0+1)s}(1 - q^{-2s-1}) - q^{-(\lambda-2e-e_0+1)s}(1 - q^{2s-1})),$$

where \(e_0 = v_\pi(2).\)

(iii) For any \(T \in \mathcal{H}_1^{ad}, \omega_T^{(1)}(x; s)\) is holomorphic for all \(s \in \mathbb{C}\) and satisfies the functional equation

$$\omega_T^{(1)}(x; s) = |2|^{-2s} \omega_T^{(1)}(x; -s).$$

**Proof.** It is easy to see that the above set contains a representative of each coset in \(K_1 \backslash \mathfrak{x}_T\), and the explicit formula in (ii) indicates us there is no redundancy within it.

For \(h \in K_{1,1}\) written as in the above list, we have

$$f_1(hx_e) = \pi^{2e-\lambda} N(\alpha^{*^{-1}}(u\sqrt{\epsilon} + (1 + uv)\frac{1}{2}\pi^{\lambda-2e})).$$
Since $\text{vol}(K_{1,1}) = \frac{1}{i+q^r}$, we obtain
\[
\int_{K_{1,1}} \chi_\pi(f_1(hx_\tau)) |f_1(hx_\tau)|^{s-\frac{1}{2}} dh = \frac{(-1)^\lambda q^{(\lambda-2e)(s-\frac{1}{2})}}{1 + q^{-1}} \cdot \sum_{r \geq 0} q^{-r} (1 - q^{-1}) q^{-2\min\{r, \lambda-2e-e_0\}(s-\frac{1}{2})}
\]
\[
= \frac{(-1)^\lambda q^{(\lambda-2e)(s-\frac{1}{2})}}{1 + q^{-1}} \cdot \left( \frac{(1-q^{-1})(1-q^{-(\lambda-2e-e_0)s})}{1-q^{-2s}} + q^{-2(\lambda-2e-e_0)s} \right).
\]

For $h \in K_{1,2}$, since
\[
f_1(hx_\tau) = \pi^{2e-\lambda} N(\alpha^{s-1})(1 + \frac{1}{2} \pi^{\lambda-2e} v / \sqrt{\varepsilon}) \in \pi^{2e-\lambda} O_k^\times,
\]
we obtain
\[
\int_{K_{1,2}} \chi_\pi(f_1(hx_\tau)) |f_1(hx_\tau)|^{s-\frac{1}{2}} dh = \frac{q^{-1}}{1 + q^{-1}} \cdot (-1)^\lambda q^{(\lambda-2e)(s-\frac{1}{2})}.
\]

Thus we obtain
\[
\omega_T^{(1)}(x_\tau; s) = \frac{(-1)^\lambda q^{(\lambda-2e)(s-\frac{1}{2})}}{1 + q^{-1}} \cdot \frac{1}{1-q^{-2s}} \cdot \left( 1 - q^{-2s-1} + q^{-2(\lambda-2e-e_0)s-1} - q^{-(\lambda-2e-e_0+1)s} \right)
\]
\[
= \frac{(-1)^\lambda q^{\lambda-\frac{s}{2}} |2|^{-s}}{1 + q^{-1}} \cdot \frac{1}{q^s - q^{-s}} \cdot \left( q^{(\lambda-2e-e_0+1)s} (1 - q^{-2s-1}) - q^{-(\lambda-2e-e_0+1)s} (1 - q^{2s-1}) \right),
\]

which proves (ii), and the assertion (iii) follows from (ii) and (1.7).

2.2. Assume that $n \geq 2$ and set
\[
w_\tau = \left( \begin{array}{c|c} 1_{n-1} & u \\ \hline 0 & 1_{n-1} \\ \hline v & 0 \end{array} \right) \in G, \quad u, v \in O_k^\times, \quad uv^* = 1
\]
\[(e.g., u = \sqrt{\varepsilon}, v = -\frac{1}{\sqrt{\varepsilon}}; u = v = 1),
\]
then $w_\tau$ gives the element $\tau \in W$ and $\tau(z) = (z_1, \ldots, z_{n-1}, -z_n)$. We will prove the following.

**Theorem 2.2** For any $T \in H_n^{ad}$, the spherical function satisfies the following functional equation:
\[
\omega_T(x; z) = |2|^{2z_n} \omega_T(x; \tau(z)).
\]
The parabolic subgroup $P = P_\tau$ attached to $\tau$ is given as follows (cf. [Bo], 21.11):

$$P = B \cup Bw_\tau B$$

$$= \left\{ \left( \begin{array}{cc} q & b \\ a & c \end{array} \right) \left( \begin{array}{cc} 1_{n-1} & \alpha \\ \alpha^* & 1 \end{array} \right) \left( \begin{array}{cc} 1_{n} & B \\ -\beta^* & 1_{n} \end{array} \right) \in G \right\}$$

$q$ is upper triangular in $GL_{n-1}(k')$, 
$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in U(1, 1), \alpha, \beta \in M_{n-1,1}(k'),$$

$$B \in M_{n-1}(k'), B + B^* = 0$$

where each empty place in the above expression means zero-entry.

Hereafter we fix a diagonal $T \in H_{n}^d$, write $f_i(x) = f_{T,i}(x)$ by abbreviating the suffix $T$, and set

$$X_T^{op} = \left\{ x \in X_T \mid \prod_{i=1}^{n} f_i(x) \neq 0 \right\}.$$

We consider the following action of $\tilde{P} = P \times GL_1$ on $\tilde{X}_T = X_T \times V$ with $V = M_{21}(k')$:

$$(p, r) \cdot (x, v) = (px, \rho(p)vr^{-1})$$

where $\rho(p) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ for the decomposition as in (2.1). For $(x, v) \in \tilde{X}_T$, set

$$g(x, v) = \det \left[ \left( \frac{1_{n-1}}{I_{n}} + T \right) \left( \begin{array}{c} x_2 \\ -y \end{array} \right) \right],$$

where $x_2$ is the lower half $n$ by $n$ block of $x$ (the same before) and $y$ is the $n$-th row of $x$.

**Lemma 2.3**

(i) $g(x, v)$ is a relative $\tilde{P}$-invariant on $\tilde{X}_T$ associated with character

$$\tilde{\psi}(p, r) = N(p_1 \cdots p_{n-1})^{-1}N(r)^{-1} = \psi_{n-1}(p)N(r)^{-1}, \quad (p, r) \in \tilde{P} = P \times GL_1,$$

where $p_i$ is the $i$-th diagonal component of $p$ and $\psi_{n-1}$ is well-defined on $P$, and satisfies

$$g(x, v_0) = f_n(x), \quad v_0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

(ii) $g(x, v)$ is expressed as

$$g(x, v) = D(x)[v],$$

with some hermitian matrix

$$D(x) = \left( \begin{array}{cc} a & b \sqrt{c} \\ b - c\sqrt{\epsilon} & d \end{array} \right) \quad (a, b, c, d \in k),$$

such that $\det D(x) = 0$ and $b = -\frac{1}{2} f_{n-1}(x)$.  

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Proof. (i) It is easy to see that \( g((1, r) \cdot (x, v)) = N(r)^{-1}g(x, v) \). In order to consider the action of \( P \), we write an element \( \tilde{p} \in P \) as

\[
\tilde{p} = \begin{pmatrix}
p & * & * & * \\
0 & a & \lambda & b \\
p & 0 & c & d \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad p \in GL_{n-1}, \ \lambda, \mu \in M_{1,n-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1, 1),
\]

and for \( x \in X_T \), we denote by \( x' \) the matrix consisting of the upper \((n - 1)\) rows of \( x_2 \), by \( z \) (resp. \( y \)) the \( n \)-th row of \( x_2 \) (resp. \( x \)). Then we obtain

\[
g((\tilde{p}, 1) \cdot (x, v)) = \det \begin{pmatrix}
\frac{1}{n-1} & \mu & d \cdot v'(x') \\
\cdot v' & (d - \lambda) & \mu \\
\cdot v' & -b & -b \\
\end{pmatrix} \cdot T^{-1}
\]

\[
= \det \begin{pmatrix}
\frac{1}{n-1} & \mu & d \cdot v'(x') \\
\cdot v' & (d - \lambda) & \mu \\
\cdot v' & -b & -b \\
\end{pmatrix} \cdot T^{-1}
\]

\[
= \det \begin{pmatrix}
p^{-1} & \mu & x' \\
\cdot & \mu & z \\
\cdot & -b & -b \\
\end{pmatrix} \cdot T^{-1}
\]

\[
= N(\det p)^{-1}g(x, v),
\]

where \( \varepsilon = ad - bc \in O^1_{k^r} \).

(ii) Since \( g(x, v) \) is a linear form with respect to both \( v_1, v_2 \) and \( v_1^*, v_2^* \), and \( g(x, v)^* = g(x, v) \), we have an expression (2.3) with some \( D(x) \in H_2 \). Writing \( T = Diag(t_1, \ldots, t_n) \), we have

\[
g(x_T, v) = (t_1 \cdots t_n)^{-1}(v_1 - v_2 \frac{1}{2}t_n)(v_1^* - v_2^* \frac{1}{2}t_n),
\]

\[
= \begin{pmatrix}
(t_1 \cdots t_n)^{-1} & -\frac{1}{2}(t_1 \cdots t_n)^{-1} \\
-\frac{1}{2}(t_1 \cdots t_n^{-1})^{-1} & \frac{1}{4}(t_1 \cdots t_n^{-1})^{-1} t_n
\end{pmatrix}[v], \quad (2.5)
\]

in particular \( \det D(x_T) = 0 \). Since \( g(x, v) \) is a relative \( P \)-invariant, we have

\[
\psi_{n-1}(p)D(x) = D(px)[\rho(p)], \quad p \in P, \quad (2.6)
\]

in particular we see (cf. Remark 1.2)

\[
\det D(x) = 0, \quad \text{for any} \ x \in X_T^{op}.
\]

Since the expression (2.3) is determined by \( x \), let us denote \( b(x) \) in stead of \( b \) in the expression of \( D(x) \). Any element \( b \) of \( B \) has a form (cf. (1.5))

\[
b = p\tilde{u}, \quad \tilde{p} = \begin{pmatrix}
* & 0 \\
0 & *
\end{pmatrix}, \quad \tilde{u} = \begin{pmatrix}
1_n & * \\
0 & 1_n
\end{pmatrix},
\]
and we write the \((2n, n)\)-entry of \(\tilde{u} \) by \(u \sqrt{\epsilon} (\in k \sqrt{\epsilon}) \) and the \((n, n)\)-entry of \(p \) by \(q (\in k' \times) \).

Then by (2.6), we have

\[
D(\tilde{u}x) = D(x)[\begin{pmatrix} 1 & -u \sqrt{\epsilon} \\ 0 & 1 \end{pmatrix}], \quad b(\tilde{u}x) = b(x),
\]

\[
D(px) = \psi_{n-1}(p) D(x)[\begin{pmatrix} q^{-1} & 0 \\ 0 & q^* \end{pmatrix}], \quad b(px) = \psi_{n-1}(p) b(x).
\]

Thus \(b(x) \) has the same relative \(B\)-invariance with \(f_{n-1}(x) \), and we see \(b(x) = -\frac{1}{2} f_{n-1}(x) \) since \(b(x_T) = -\frac{1}{2} f_{n-1}(x_T) \neq 0 \) by (2.5).

For \(A \in H_2 \) and \(s \in \mathbb{C} \), we consider

\[
\zeta_{K_1}(A; s) = \int_{K_1} |d_1(h \cdot A)|^{s-\frac{1}{2}} \, dh,
\]

where \(dh \) is the normalized Haar measure on \(K_1 = U(1, 1) \cap GL_2(O_{k'}) \).

**Proposition 2.4** Assume \(x \in X_T^{op} \) and \(D(x) \) is given by (2.4). Set \(m = \min\{v_\pi(a), v_\pi(d)\} \) and \(\lambda = v_\pi(b) - m \). Then \(\lambda \geq 0 \) and

\[
\zeta_{K_1}(D(x); s) = \frac{q^m}{1 + q^{-1}} \left| f_{n-1}(x) \right|^s \frac{q^{(\lambda+1)s} (1 - q^{-2s-1}) - q^{-\lambda s} (1 - q^{2s-1})}{q^s - q^{-s}}.
\]

In particular, one has

\[
\zeta_{K_1}(D(x), s) = |2|^{-2s} |f_{n-1}(x)|^{2s} \zeta_{K_1}(D(x), -s).
\]

**Proof.** Let \(x \in X_T^{op} \) and write \(D(x) = \left( \begin{array}{cc} a & b + c \sqrt{\epsilon} \\ b - c \sqrt{\epsilon} & d \end{array} \right) \). Since \(b^2 = ad + c^2 \epsilon \) and \(\epsilon \notin O_k^{\times 2} \), we have \(v_\pi(c) \geq m = \min\{v_\pi(a), v_\pi(d)\} \), and \(v_\pi(b) \geq m \). Since \(N(O_{k'}) = O_k^{\times} \), we see that \(D(x) \) is \(K_1\)-equivalent to a matrix

\[
A = \left( \begin{array}{cc} \pi^m & b \\ b & f \end{array} \right), \quad b \text{ is the same as in } D(x), \quad v_\pi(b) = m + \lambda \geq m, \quad b^2 = \pi^m f,
\]

and \(\zeta_{K_1}(D(x); s) = \zeta_{K_1}(A; s) \).

We recall the data for \(K_1 = K_{1,1} \cup K_{1,2} \) in §2.1, and set

\[
\zeta_{K_{1,i}}(A; s) = \int_{K_{1,i}} |d_1(h \cdot A)|^{s-\frac{1}{2}} \, dh, \quad i = 1, 2.
\]

Since \(f \in \pi^m(O_k)^2 \), we have \(|d_1(h \cdot A)| = q^{-m} \) for any \(h \in K_{1,1} \) and

\[
\zeta_{K_{1,1}}(A; s) = \frac{1}{1 + q^{-1}} \cdot q^{-m(s-\frac{1}{2})}.
\]

Assume \(h \in K_{1,2} \) has the form as in §2.1. Then \(d_1(h \cdot A) = N(\alpha) (\pi^{m+2} u^2 \epsilon + (1 + \pi u)^2 f) \), and

\[
\zeta_{K_{1,2}}(A; s) = \frac{q^{-m(s-\frac{1}{2})-1}}{1 + q^{-1}} \quad \text{if } \lambda = 0.
\]
If $\lambda \geq 1$, then

$$
\zeta_{K_1,2}(A; s) = \frac{q^{-1}}{1 + q^{-1}} \left( \sum_{\ell=0}^{\lambda-1} q^{-\ell} (1 - q^{-1}) q^{-(m+2\ell)(s-\frac{1}{2})} + q^{-\lambda} q^{-(m+2\lambda)(s-\frac{1}{2})} \right)
$$

$$
= \frac{q^{-m(s-\frac{1}{2})}}{(1 + q^{-1})(1 - q^{-2s})} \left( q^{-2s} - q^{-2s-1} + q^{-2\lambda s-1} - q^{-2(\lambda+1)s} \right), \quad (2.9)
$$

which is also valid for $\lambda = 0$.

By (2.8) and (2.9), we obtain

$$
\zeta_{K_1}(A; s) = \frac{q^{-\lambda(s+\frac{m}{2})}}{1 + q^{-1}} \times q^{(\lambda+1)s} (1 - q^{-2s-1}) - q^{-\lambda(s+\frac{m}{2})} (1 - q^{-2s-1}).
$$

Now by Lemma 2.3, we obtain

$$
\zeta_{K_1}(D(x); s) = \frac{q^{\frac{m}{2}}}{1 + q^{-1}} \left| \frac{f_{n-1}(x)}{2} \right|^s \times q^{(\lambda+1)s} (1 - q^{-2s-1}) - q^{-\lambda(s+\frac{m}{2})} (1 - q^{-2s-1}).
$$

The identity (2.7) follows from the above explicit formula.

Now we will prove Theorem 2.2. We consider the embedding

$$
K_1 \longrightarrow K = K_n, \quad h = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \longmapsto \tilde{h} = \left( \begin{array}{cc} 1_{n-1} & a \\ c & 1_{n-1} \end{array} \right).
$$

Then we have

$$
\omega_T(x; s) = \int_{K_1} dh \int_K |f(kx)|^{s+\varepsilon} dk
$$

$$
= \int_{K_1} dh \int_K |f(\tilde{h}kx)|^{s+\varepsilon} dk
$$

$$
= \int_K \chi_\pi(\prod_{i<n} f_i(kx)) \prod_{i<n} |f_i(kx)|^{s_i-1} \left( \int_{K_1} \chi_\pi(f_n(\tilde{h}kx)) |f_n(\tilde{h}kx)|^{s_n-\frac{1}{2}} dh \right) dk.
$$

By definition of $f_n(x)$ and $g(x, v)$ and Lemma 2.3, we have for $h = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_1$

$$
f_n(\tilde{h}x) = \det \left[ \begin{array}{cc} x' \\ cy + dz \end{array} \right] \cdot T^{-1} = g(x, \left( \begin{array}{c} d \\ -c \end{array} \right))
$$

$$
= (d^s - c^s)D(x) \left( \begin{array}{c} d \\ -c \end{array} \right) = d_1(h^{s-1} \cdot D(x)).
$$

Since $\{h^{s-1} \mid h \in K_1\} = K_1$, we have

$$
\omega_T(x; s) = \int_K \chi_\pi(\prod_{i<n} f_i(kx)) \prod_{i<n} |f_i(kx)|^{s_i-1} \zeta_{K_1}(D(kx); s_n + \frac{\pi \sqrt{-1}}{\log q}) dk,
$$
and by Proposition 2.4 we obtain
\[ \omega_T(x; s) = |2|^{-2s_n} \int_K \chi_K \prod_{i<n} f_i(kx) \prod_{i>n-2} |f_i(kx)|^{s_{i-1}} |f_{n-1}(kx)|^{s_{n-1}+2s_n-1} \times \zeta_{K_1}(D(kx); -s_n + \frac{\pi \sqrt{-1}}{\log q}) \, dk \]
\[ = |2|^{-2s_n} \omega_T(x; s_1, \ldots, s_{n-2}, s_{n-1} + 2s_n, -s_n). \]

In variable \( z \), we have
\[ \omega_T(x; z_1, \ldots, z_{n-1}, z_n) = |2|^{2s_n} \omega_T(x; z_1, \ldots, z_{n-1}, -z_n), \]
which completes the proof.

2.3. In order to describe functional equations of \( \omega_T(x; z) \) with respect to \( W \), we prepare some notations. We denote by \( \Sigma \) the set of roots of \( G \) with respect to the \( k \)-split torus of \( G \) contained in \( B \) and by \( \Sigma^+ \) the set of positive roots with respect to \( B \). Let \( e_i \in \mathbb{Z}^n \) be the element whose \( j \)-th component is given by the Kronecker delta \( \delta_{ij} \). Then we may understand
\[ \Sigma^+ = \{ e_i - e_j, e_i + e_j \mid 1 \leq i < j \leq n \} \cup \{ 2e_i \mid 1 \leq i \leq n \}, \]
and the set
\[ \Sigma_0 = \{ e_i - e_{i+1} \mid 1 \leq i \leq n - 1 \} \cup \{ 2e_n \} \]
forms the set of simple roots. We denote by \( \Delta \) the subset of \( W \) consisting of the reflections associated with elements in \( \Sigma_0 \). Then \( \Delta = \{ \tau_i \mid 1 \leq i \leq n \} \cup \{ \tau \} \) generates \( W \). We write \( \alpha < 0 \) if \( \alpha \in \Sigma \) is negative. We consider the pairing on \( \Sigma \times \mathbb{C}^n \) defined by
\[ \langle \alpha, z \rangle = \sum_{i=1}^{n} \alpha_i z_i, \quad (\alpha \in \Sigma, z \in \mathbb{C}^n), \]
which is \( W \)-invariant, i.e.,
\[ \langle \alpha, z \rangle = \langle \sigma(\alpha), \sigma(z) \rangle, \quad (\alpha \in \Sigma, \sigma \in W, z \in \mathbb{C}^n) \quad (2.10) \]

**Theorem 2.5** For \( T \in \mathcal{H}^\text{nd}_n \) and \( \sigma \in W \), the spherical function \( \omega_T(x; z) \) satisfies the following functional equation
\[ \omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \quad (2.11) \]
where
\[ \Gamma_\sigma(z) = \prod_{\alpha \in \Sigma^+, \langle \alpha, z \rangle < 0} f_\alpha(\langle \alpha, z \rangle), \]
\[ f_\alpha(t) = \begin{cases} |2|^t & \text{if } \alpha = 2e_i \text{ for some } i \\ 1 - q^{t-1} & \text{otherwise}, \end{cases} \]
in particular, the Gamma factor \( \Gamma_\sigma(z) \) does not depend on \( T \) nor \( x \).
Proof. We determine \( \Gamma_\sigma(z) \) by \((2.11)\), which is a rational function of \( q^{z_1}, \ldots, q^{z_n} \) and satisfies the cocycle relation

\[
\Gamma_{\sigma_2 \sigma_1}(z) = \Gamma_{\sigma_2}(\sigma_1(z)) \cdot \Gamma_{\sigma_1}(z), \quad (\sigma_1, \sigma_2 \in W).
\]

(2.12)

For an element \( \sigma \in \Delta \) associated with some \( \alpha_0 \in \Sigma_0 \), we have known, by Remark 1.5 and Theorem 2.3

\[
\Gamma_\sigma(z) = f_{\alpha_0}(\langle \alpha_0, z \rangle) \left( \prod_{\substack{\alpha \in \Sigma^+ \\ \sigma(\alpha) < 0}} f_\alpha(\langle \alpha, z \rangle) \right).
\]

In general, assume that \( \sigma \in W \) has the shortest expression

\[
\sigma = \sigma_\ell \cdots \sigma_1,
\]

where \( \sigma_i \in \Delta \) is the reflection of \( \alpha_i \in \Sigma_0 \). Then by cocycle relations \((2.12)\) together with \((2.10)\) and the result for elements in \( \Delta \), we have

\[
\Gamma_\sigma(z) = \Gamma_{\sigma_\ell}(\sigma_{\ell-1} \cdots \sigma_1(z)) \cdots \Gamma_{\sigma_2}(\sigma_1(z)) \cdot \Gamma_{\sigma_1}(z)
= f_{\alpha_\ell}(\langle \alpha_\ell, \sigma_{\ell-1} \cdots \sigma_1(z) \rangle) \cdots f_{\alpha_2}(\langle \alpha_2, \sigma_1(z) \rangle) \cdot f_{\alpha_1}(\langle \alpha_1, z \rangle)
= f_{\alpha_\ell}(\langle \sigma_1 \cdots \sigma_{\ell-1}(\alpha_\ell), z \rangle) \cdots f_{\alpha_2}(\langle \sigma_1(\alpha_2), z \rangle) f_{\alpha_1}(\langle \alpha_1, z \rangle).
\]

Since

\[
\{ \alpha \in \Sigma^+ \mid \sigma(\alpha) < 0 \} = \{ \sigma_1 \cdots \sigma_{k-1}(\alpha_k) \mid 1 \leq k \leq \ell \},
\]

we obtain, by definition of \( f_\alpha \),

\[
\Gamma_\sigma(z) = \prod_{\substack{\alpha \in \Sigma^+ \\ \sigma(\alpha) < 0}} f_\alpha(\langle \alpha, z \rangle).
\]

We note the explicit Gamma-factor for a particular element for the later use in §4.

**Corollary 2.6** Set \( \rho \in W \) by

\[
\rho(z_1, \ldots, z_n) = (-z_n, -z_{n-1}, \ldots, -z_1).
\]

(2.13)

Then

\[
\Gamma_\rho(z) = 2^{(z_1 + \cdots + z_n)} \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i+z_j-1}}{q^{z_i+z_j} - q^{-1}}.
\]

(2.14)

**Proof.** Since

\[
\{ \alpha \in \Sigma^+ \mid \rho(\alpha) < 0 \} = \{ e_i + e_j \mid 1 \leq i \leq j \leq n \},
\]

the assertion follows from Theorem 2.5.
Remark 2.7 The above $\rho$ gives the functional equation of the hermitian Siegel series (cf. §4), and it is interesting that such $\rho$ corresponds to the unique automorphism of the extended Dynkin diagram of the root system of type $(C_n)$, which was pointed out by Y. Komori.

2.4. By Theorem 1.3 and Theorem 2.2 we obtain the following theorem.

Theorem 2.8 Set

$$F(z) = \prod_{\alpha \in \Sigma^+} g_\alpha(z),$$

where, for $\alpha \in \Sigma$,

$$g_\alpha(z) = \begin{cases} |2|^{\frac{\langle \alpha, z \rangle}{2}} & \text{if } \alpha = \pm 2e_i \text{ for some } i \\ \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle}} & \text{otherwise} \end{cases}.$$  

Then, for any $T \in H_{n}^{ad}$, the function $F(z)\omega_T(x; z)$ is holomorphic for all $z$ in $\mathbb{C}^n$ and $W$-invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$.

Proof. For $\sigma \in \Delta$ associated with some $\alpha \in \Sigma_0$, $F(z)\omega_T(x; \sigma)$ is $\sigma$-invariant, since

$$g_\alpha(\sigma z) = g_{\sigma \alpha}(z) = g_{-\alpha}(z), \quad \Gamma_\sigma(z) = \frac{g_{-\alpha}(z)}{g_\alpha(z)}.$$  

Thus $F(z)\omega_T(x; \sigma)$ is $W$-invariant, since $\Delta$ generates $W$. Set

$$F_1(z) = \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j}},$$

$$F_2(z) = |2|^{-z_1 - \cdots - z_n} \prod_{1 \leq i < j \leq n} \frac{1 + q^{z_i + z_j}}{1 - q^{z_i + z_j}}.$$  

Then $F(z) = F_1(z)F_2(z)$ and $F_1(z)\omega_T(x; z)$ is holomorphic in $z \in \mathbb{C}^n$ and $S_n$-invariant by Theorem 1.3. Hence $F(z)\omega_T(x; z)$ is holomorphic in $z \in \mathbb{C}^n$, since it is $W$-invariant and holomorphic for certain region e.g., $\{z \in \mathbb{C}^n \mid \text{Re}(z_i) \leq 0\}$. 

§3

3.1. In this section we give an explicit formula of $\omega_T(x; s)$ at $x_T$ by using the general formula of Proposition 1.9 in [H2] (or Theorem 2.6 in [H4]). In order to apply it, we have to check several conditions ((A1) – (A4) in [H4]-§1), and it is obvious our $(B, X_T)$ satisfies them except (A3), which is the same as (C) below.
**Proposition 3.1** The following condition \((C)\) is satisfied.

\((C)\) : For \(y \in X_T\) not contained in \(X_T^{op}\), i.e., \(f_T(y) = 0\), there exists a character \(\psi \in \langle \psi_i \mid 1 \leq i \leq n \rangle\) whose restriction to the identity component of the stabilizer of \(B\) at \(y\) is not trivial.

We denote by \(U\) the Iwahori subgroup of \(K\) compatible with \(B\), and state our main result in this section.

**Theorem 3.2** Let \(T = \text{Diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n})\) with \(\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq \nu_\pi(2)\). Then

\[
\omega_T(x_T; z) = \frac{(-1)^{\sum_i \lambda_i(n-i+1) + \sum_i \lambda_i(n-i+\frac{1}{2})}}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) \Gamma_{\sigma}(z) q^{<\lambda, \sigma(z)>},
\]

where \(<\lambda, z> = \sum_{i=1}^{n} \lambda_i z_i\), \(\Gamma_{\sigma}(z)\) is defined in Theorem 2.5 and

\[
Q = \prod_{1 \leq i < j \leq n} \frac{(1 - q^{2z_i - 2z_j})(1 - q^{2z_i + 2z_j})}{(1 - q^{2z_i - 2z_j})(1 - q^{2z_i + 2z_j})} \cdot \prod_{i=1}^{n} \frac{1 - q^{2z_i - 1}}{1 - q^{2z_i}},
\]

\[
\gamma(z) = \prod_{\alpha \in \Sigma^+, \text{short}} \frac{1 - q^{2(\alpha,z)-2}}{1 - q^{2(\alpha,z)}} \cdot \prod_{\alpha \in \Sigma^+, \text{long}} \frac{1 - q^{(\alpha,z)-1}}{1 - q^{(\alpha,z)}}.
\]

We admit Proposition 3.1 for the moment and prove Theorem 3.2

The \(B\)-orbits in \(X_T^{op}\) are parametrized by \(U = (\mathbb{Z}/2\mathbb{Z})^{n-1}\): for \(u \in U\) set

\[X_{T,u} = \{ x \in X_T \mid \psi_i(f_{T,x}(x)) \equiv u_1 + \cdots + u_i \pmod{2}, \ 1 \leq i \leq n - 1 \},\]

then \(X_T^{op}\) is the disjoint union of these \(X_{T,u}\)'s. We set

\[
\omega_{T,u}(x; s) = \int_K |f_T(kx)|^{s+\varepsilon}_u dk,
\]

where

\[
|f_T(y)|^{s+\varepsilon}_u = \begin{cases} |f_T(y)|^{s+\varepsilon} & \text{if } y \in X_{T,u}, \\ 0 & \text{otherwise}. \end{cases}
\]

For a character \(\chi = (\chi_1, \ldots, \chi_{n-1})\) of \(U\), we set

\[
L_T(x; \chi; z) = \int_K \chi(f_T(kx)) |f_T(kx)|^{s+\varepsilon} dk = \sum_{u \in U} \chi(u) \omega_{T,u}(x; z),
\]

where \(\chi(u) = \prod_{i=1}^{n-1} \chi_i(u_1 + \cdots + u_i)\). Adjusting \(z\) according to \(\chi\), by adding \(\frac{\pi \sqrt{-1}}{\log q}\) to \(z_i\) if necessary, we may write

\[
L_T(x; \chi; z) = \omega_T(x; z_{\chi}).
\]
Then, by the functional equations of $\omega_T(x; z)$ (Theorem 2.5), we have
\[
L_T(x; \chi; z) = \Gamma_\sigma(z_\chi)L_T(x; \sigma(\chi); \sigma(z)), \quad \sigma \in W
\tag{3.2}
\]
by taking suitable character $\sigma(\chi)$ of $U$. If $\chi$ is the trivial character $1$, then (3.2) coincides with the original functional equation of $\omega_T(x; z)$. We obtain
\[
(\omega_{T,u}(x_T; z)) = (A^{-1} \cdot G(\sigma, z) \cdot \sigma A) (\omega_{T,u}(x_T; \sigma(z)))u,
\]
where
\[
A = (\chi(u))_{\chi,u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi,u} \in GL_{2n}(\Z),
\]
and $G(\sigma, z)$ is the diagonal matrix of size $2^n$ whose $(\chi, \chi)$-component is $\Gamma_\sigma(z_\chi)$. For $T$ given as in Theorem 3.2, we obtain
\[
\int_U |f_{T,u}(x_T)|^{s+\varepsilon} \, du = |f_T(x_T)|^{s+\varepsilon}
= (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+1/2)} q^{<\lambda,z>},
\]
where $du$ is the normalized Haar measure on $U$. Setting
\[
\delta_u(x_T, z) = \begin{cases} 
(-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+1/2)} q^{<\lambda,z>} & \text{if } xTU(T) \in X_{T,u} \\
0 & \text{otherwise},
\end{cases}
\]
we have, by Proposition 1.9 in [H2] (or its generalization Theorem 2.6 in [H4]),
\[
(\omega_{T,u}(x_T; z)) = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) \left( A^{-1} \cdot G(\sigma, z) \cdot \sigma A \right) (\delta_u(x_T, \sigma(z)))u,
\]
where
\[
Q = \sum_{\sigma \in W} [U\sigma U : U]^{-1} = \prod_{i=1}^{2n} \frac{(1 - (-1)^i q^{-i})}{(1 - q^{-2})^n}.
\]
Hence we obtain
\[
\omega_T(x_T; z) = \sum_{u \in U} 1(u) \omega_u(x_T; z)
= \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+1/2)}}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) \Gamma_\sigma(z) q^{<\lambda,\sigma(z)>}.
\]

3.2. Now we will prove Proposition 3.1.

We consider the action of $G \times U(T)$ on $\mathfrak{X}_T$ by $(g, h) \circ x = gxh^{-1}$. Then, the stabilizer $B_y$ of $B$ at $yUT(T) \in X_T$ coincides with the image $B_{(y)}$ of the projection to $B$ of the stabilizer $(B \times U(T))_y$ at $y \in \mathfrak{X}_T$ to $B$. Hence the condition (C) is equivalent to the following:
(C′) : For \( y \in \mathfrak{X}_T \) such that \( f_T(y) = 0 \) there exists \( \psi \in \langle \psi_i \mid 1 \leq i \leq n \rangle \) whose restriction to the identity component of \( B_{(y)} \) is not trivial.

It is sufficient to prove the condition \((C) \) (or \((C')\)) over the algebraic closure \( \overline{k} \), since, for a connected linear algebraic group \( \mathbb{H}, \mathbb{H}(k) \) is dense in \( \mathbb{H}(\overline{k}) \). In the rest of this section, we consider algebraic sets over \( \overline{k} \), extend the involution \( * \) on \( k' \) to \( \overline{k} \), denote it by \( ꞉ \), and write \( \sigma = (\sigma_{ij}) \) for any matrix \( x = (x_{ij}) \). Since \( \mathfrak{X}_T \) is isomorphic to \( \mathfrak{X}_{T[h]} \) by \( x \mapsto xh \) and \( B_{(h)} = B(h) \) for \( h \in GL_n \), we may assume that \( T = 1_n \). Then, our situation is the following:

\[
\mathfrak{X} = \mathfrak{X}_{1_n} = \{ x \in M_{2n,n} \mid H_n[x] = 1_n \},
\]

\[
(U(H_n) \times U(1_n)) \times \mathfrak{X} \to \mathfrak{X}, \quad ((g, h), x) \mapsto (g, h) \circ x = gxh^{-1},
\]

and \( B \) is the Borel subgroup of \( U(H_n) \) (as in (1.5)). We consider the set

\[
\tilde{\mathfrak{X}} = \{ (x, y) \in M_{2n,n} \oplus M_{2n,n} \mid t_y H_n x = 1_n \}
\]

\[
= \left\{ \left( \begin{array}{cc} x_1 & \vspace{1em} \\
                        & x_2 \\
\end{array} \right), \left( \begin{array}{c} y_1 \\
                        y_2 \end{array} \right) \in M_{2n,n} \oplus M_{2n,n} \mid t_y x_2 + t_y^1 x_2 = 1_n \right\},
\]

together with \((GL_{2n} \times GL_n)\)-action defined by

\[
(g, h) \ast (x, y) = (gxh^{-1}, g^y h), \quad \dot{g} = H_n t_y^1 H_n, \quad (3.3)
\]

and take the Borel subgroup \( P \) of \( GL_{2n} \) by

\[
P = \left\{ \left( \begin{array}{cc} p_1 & r \\
                        0 & p_2 \end{array} \right) \in GL_{2n} \mid p_1, p_2 \in B_n, \ r \in M_n \right\},
\]

where \( B_n \) is the Borel subgroup of \( GL_n \) consisting of upper triangular matrices.

Then, the embedding \( i : \mathfrak{X} \hookrightarrow \tilde{\mathfrak{X}}, \ x \mapsto (x, \sigma) \) is compatible with the actions, i.e., we have the commutative diagram

\[
\begin{array}{ccc}
(U(H_n) \times U(1_n)) & \times & \mathfrak{X} \to \mathfrak{X} \\
\downarrow{id} & & \downarrow{i} \circ \downarrow{i} \\
(GL_{2n} \times GL_n) & \times & \tilde{\mathfrak{X}} \to \tilde{\mathfrak{X}}.
\end{array}
\]

For \( (x, y) \in \tilde{\mathfrak{X}} \) and \( p \in P \), set

\[
\tilde{f}_i(x, y) = d_i(x_2 \ t_y y_2), \quad \tilde{\psi}_i = \prod_{1 \leq j \leq i} p_{j-1}^{-1} p_{n+j}, \quad (1 \leq i \leq n),
\]

where \( p_j \) is the \( j \)-th diagonal component of \( p \). Then \( \tilde{f}_i(x, y) \)'s are basic relative \((P \times GL_n)\)-invariants on \( \tilde{\mathfrak{X}} \) associated with characters \( \tilde{\psi}_i, \ f_i(x, x) = f_i(x) \) for \( x \in \mathfrak{X} \), and \( \tilde{\psi}_i|_B = \psi_i \).

We set

\[
S = \bigg\{ (x, y) \in \tilde{\mathfrak{X}} \mid \prod_{i=1}^n \tilde{f}_i(x, y) = 0, \quad (P \times GL_n) \ast (x, y) \cap \mathfrak{X} \neq \emptyset \bigg\}.
\]
For \( \alpha = (x, y) \in \tilde{X} \), we denote by \( H_\alpha \) the stabilizer of \( P \times GL_n \) at \( \alpha \), and by \( P_\alpha \) its image of the projection to \( P \). In order to prove the condition \((C)\), it is sufficient to show the following:

\((\tilde{C})\) : For each \( \alpha \in S \), there exists some \( \psi \in \langle \tilde{\psi}_i \mid 1 \leq i \leq n \rangle \) whose restriction to the identity component of \( P_\alpha \) is not trivial.

It is sufficient to show \((\tilde{C})\) for representatives under the action of \( P \times GL_n \). In the following we consider the case \( n \geq 2 \), since \( X_T = \tilde{X}_T^{op} \) for \( n = 1 \) and there is nothing to prove.

**Lemma 3.3** The condition \((\tilde{C})\) is satisfied for \( (x, y) \in S \) for which \( \det x_2 \neq 0 \) or \( \det y_2 \neq 0 \).

**Proof.** Let \( \alpha = (x, y) \in S \) and \( \det x_2 \neq 0 \). Then by the action of \( P \times GL_n \), we may assume that \( x_2 = y_1 = 1_n, x_1 = 0 \). Since \( (P \times GL_n) \times (x, y) \cap \tilde{X} \neq \emptyset \), \( y_2 \) is written as \( y_2 = \psi_0 h \) by some \( p_0 \in B_n \) and \( h \in GL_n \) satisfying \( \tilde{h} = \psi h \) (i.e., \( h \) is hermitian). Since \( \tilde{f}_i(\alpha) = d_i(y_2) \), we have \( \prod_i d_i(h) = 0 \). Thus, in the same \( P \times GL_n \)-orbit containing \( \alpha \), we may take

\[
\beta = \left( \begin{array}{cc} 0 & 1_n \\ 1_n & h \end{array} \right)
\]

with hermitian matrix \( h \) satisfying

\[
h = 1_r \perp (0) \perp 1_{n-1}, \quad \text{or} \quad h = 1_r \perp h_2,
\]

where \( 0 \leq r \leq n-1 \), and for \( h_2 \), there is some \( i \), \( (1 < i \leq n-r) \) such that each component in the first row and column or in the \( i \)-th row and column in 0 except at \((1, i)\) or \((i, 1)\) which are 1. Then \( H_\beta \) contains the following elements, according to the above type of \( h \),

\[
\left( \begin{array}{c} \delta_{r+1}(a) \\ 1_n \end{array} \right), 1_n \) or \( \left( \begin{array}{c} \delta_{r+1}(a) \\ \delta_{r+i}(a) \end{array} \right), \delta_{r+i}(a) \),
\]

where \( \delta_j(a) \) is the diagonal matrix in \( GL_n \) whose diagonal components are 1 except the \( j \)-th which is \( a \in GL_1 \). Hence we see \( \tilde{\psi}_{r+1} \neq 1 \) on the identity component of \( P_\beta \).

The case \( \alpha = (x, y) \in S \) with \( \det y_2 \neq 0 \) is reduced to the case \( \det x_2 \neq 0 \), since \( \beta = (y, x) \in S \) and \( H_\beta = \{ (\tilde{p}, r^{-1}) \mid (p, r) \in H_\alpha \} \) and \( \tilde{\psi}_i(\tilde{p}) = \tilde{\psi}_i(p)^{-1} \).

Now we have to consider for \( (x, y) \in S \) having \( \det x_2 = \det y_2 = 0 \). We set

\[
S_0 = \{ (x, y) \in S \mid \det x_2 = \det y_2 = 0 \}.
\]

**Lemma 3.4** Under the action of \( P \times GL_n \), every element in \( S_0 \) is equivalent to an element \( (x, y) \) of the following type. There exist integers

\[
1 \leq e_1 < e_2 < \cdots < e_k \leq n \quad (1 \leq k < n),
\]

\[
1 \leq r_1 < r_2 < \cdots r_\ell \leq n \quad (\ell = n-k),
\]

for which

\[
x_1 : 1 \text{ at } (r_j, k+j) \text{-component for } 1 \leq j \leq \ell \text{ and } 0 \text{ at any other component},
\]

\[
x_2 : 1 \text{ at } (e_j, j) \text{-component for } 1 \leq j \leq k \text{ and } 0 \text{ at any other component},
\]
\( y_1 : \) the \( e_j \)-th row is the same as in \( x_2 \) for \( 1 \leq j \leq k \), and \((i,j)\)-component is 0 if \( i < e_j \) or \( j > k \).

\( y_2 : \) the \( r_j \)-th row is the same as in \( x_1 \) for \( 1 \leq j \leq \ell \), and \((i,k+j)\)-component is 0 if \( i > r_j \).

**Proof.** First we normalize \( x_2 \) as above and make the first \( k \) columns of \( x_1 \) into 0, where \( k \) must be \( 1 \leq k < n \), since \( \det x_2 = 0 \). Then each of the rest \( k - n \) columns of \( x_1 \) has nonzero component, since \( t_x H_n y = 1_n \), and we can normalize \( x \) as the required form.

Then, fixing this \( x \), we can make \( y \) as above.

**Lemma 3.5** The condition \((\tilde{C})\) is satisfied for elements in \( \mathcal{S}_0 \).

**Proof.** We may assume that \( \alpha \in \mathcal{S}_0 \) has the form given in Lemma 3.4. If \( e_1 > 1 \) or \( r_1 = 1, \tilde{\psi}_1 \neq 1 \) on the identity component of \( P_\alpha \), since

\[
\left( \begin{array}{c} \frac{1}{n} \\ 1/a(n) \\
\end{array} \right), 1_n) \in H_\alpha \text{ if } e_1 > 1, \quad \left( \begin{array}{c} \delta_i(a) \\ 1_n \\
\end{array} \right), \delta_{k+1}(a) \in H_\alpha \text{ if } r_1 = 1.
\]

In the following we assume that \( e_1 = 1 \) and \( r_1 > 1 \), and we continue the standardization of the representative. We denote by \( a^{ij} \) the \((i,j)\)-component for any matrix \( a \).

We take the following procedure.

(i) If \( y_2^{ij} = 0 \), \( 1 \leq j \leq k \), we leave \((x,y)\).

(ii) If \( y_2^{ij} \neq 0 \) for some \( j \) with \( 1 \leq j \leq k \), take the smallest possible \( j \) (say \( j_1 \)), and move the \( j_1 \)-th column into the first one and the \( j \)-th column into the \((j + 1)\)-th one for \( 1 \leq j < j_1 \); which is done by right multiplication of some \( h = t_{h^{-1}} \in GL_n \) satisfying \( t_{h} = h^{-1} \). Changing \( x \) by this \( h \), we reset \((x,y)\) by the new one.

To make \( y_2^{i1} \) into 0 for \( i \geq 2 \), we take a matrix \( p \in P \) of type

\[
p = \left( \begin{array}{cc} 1_n & 0 \\
0 & 1_{p^1} \\
\end{array} \right), \quad p_1 = \left( \begin{array}{cc} 1 & 0 \\
\eta & 1_{n-1} \\
\end{array} \right) \in GL_n, \eta \in M_{n-1,1}.
\]

This \( p \) acts on \( x \) as left multiplication by \( \tilde{p} = H_n t_p^{-1} H_n \) (cf. (3.3)), and stabilizes \( x \), since every \( p_{ij}^{i1} = 0 \) for \( i = r_j \). We reset \((x,y)\) into the new one.

(iii) We look at the second row of \( y_2 \). If \( y_2^{ij} = 0 \), \( 1 \leq j \leq k \), we leave \((x,y)\). Otherwise, we move the first nonzero column into the first one after (i), or the second one after (ii), and make its components below the second into 0. We continue the similar procedure until the \( r_\ell \)-th row and reset \((x,y)\) by the new one. Then \( y_2 \) satisfies

- \( y_2^{ij} = 0 \) for \( i \leq r_\ell \) and \( 1 \leq j \leq k \), or
- for some integer(s) \( d_1 < \cdots < d_s < r_\ell \) with \( 1 \leq s \leq k \),

\[
y_2^{d_{ij}} \neq 0, \quad y_2^{d_{ij}^j} = 0 \quad \text{for } 1 \leq j \leq s, \quad j' < j,
\]

\[
y_2^{ij} = 0 \quad \text{for } 1 \leq i \leq r_\ell, \quad i \neq d_1, \ldots, d_s, \quad 1 \leq j \leq k,
\]

the each \( e_j \)-th row of \( y_2 \) remains as before, and

\[
y_2^{ij} = 0 \quad \text{for } i > r_\ell, \quad k + 1 \leq j \leq n.
\]

(iv) Since \( x \) has changed only by a permutation of first \( k \) columns, we return \( x \) into the original one by multiplying some \( h_1 = t_{h_1^{-1}} \in GL_n \) from the right, and reset \( y \) by \( yh_1 \).

We write

\[
x = \left( \begin{array}{c} 0 \\
J_2 \\
\end{array} \right), \quad y = \left( \begin{array}{c} z_1 \\
z_2 \\
z_3 \\
\end{array} \right), \quad (J_1, z_3 \in M_{n\ell}, \quad J_2, z_1, z_2 \in M_{nk}).
\]
Then, by a suitable chosen
\[ h = \begin{pmatrix} 1_k & C \\ 0 & 1_\ell \end{pmatrix} \in GL_n, \quad \text{with } C = (c_{ij}) \in M_{k\ell}, \]
we make \( z'_3 = z_2C + z_3 \) satisfy

\[ \begin{align*}
\text{its } i\text{-th row is 0, if that of } z_2 \text{ is not 0, for } 1 \leq i \leq n, \\
\text{its } r_j\text{-th row remains as before, 0 except the } j\text{-th entry which is 1, for } 1 \leq j \leq \ell.
\end{align*} \tag{3.4} \]

To make the latter \( \ell \) rows of \( y_1h \) into 0, we take \( D = (d_{ij}) \in M_n \) such that its \( r_j\)-th column is the same as the \( j\)-th column of \( z_1C \) for \( 1 \leq j \leq \ell \), and any other column is 0; hence \( d_{e_i,j} = c_{ij} \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq \ell \). Setting
\[ p = \begin{pmatrix} 1_n & -D \\ 0 & 1_n \end{pmatrix}, \]
we have
\[ pยห = \begin{pmatrix} z'_1 & 0 \\ z_2 & z'_3 \end{pmatrix}, \quad z'_1 = z_1 - Dz_2, \tag{3.5} \]
and
\[ \dot{p}x^th^{-1} = \begin{pmatrix} 1_n & tD \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 0 & J_1 \\ J_2 & 0 \end{pmatrix} \begin{pmatrix} 1_k & 0 \\ -tC & 1_\ell \end{pmatrix} = \begin{pmatrix} -J_1tC + tDJ_2 & J_1 \\ J_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & J_1 \\ J_2 & 0 \end{pmatrix} = x. \]

We set \( \beta = (x, pยห) \), where \( x \) remains the same as the original and \( z'_3 \) in \( pยห \) satisfies (3.4). Hence \( H_\beta \) contains \((A, A_2)\) satisfying
\[ A = \begin{pmatrix} A_1 & 0 \\ 0 & 1_n \end{pmatrix}, \quad A_1 = Diag(a_1, \ldots, a_n), \quad a_i = \begin{cases} a & \text{if the } i\text{-th row of } z_2 \text{ is 0} \\ 1 & \text{otherwise} \end{cases}, \]
\[ A_2 = Diag(1, \ldots, 1, a, \ldots, a), \]
and \( \bar{\psi}_i \neq 1 \) on the identity component of \( P_\beta \) at least for \( i = r_j, 1 \leq j \leq \ell \).

Thus we have shown the condition \((\bar{C})\) is satisfied for every \((x, y) \in \mathcal{S}\), which completes the proof of Proposition 3.1.
§4

We recall the hermitian Siegel series, and give its integral representation and a proof of the functional equation as an application of spherical functions.

Let $\psi$ be an additive character of $k$ of conductor $\mathcal{O}_k$. For $T \in \mathcal{H}_n(k')$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\text{tr}(TR))dR,$$

where $\text{tr}(\ )$ is the trace of matrix and $\nu_\pi(R)$ is defined as follows: if the elementary divisors of $R$ with negative $\pi$-powers are $\pi^{-e_1}, \ldots, \pi^{-e_r}$, then $\nu_\pi(R) = q^{e_1+\cdots+e_r}$, and $\nu_\pi(R) = 1$ otherwise (cf. [SH]-§13).

In the following we assume that $T$ is nondegenerate, since the properties of $b_\pi(T; s)$ can be reduced to the nondegenerate case. We give an integral expression of $b_\pi(T; s)$ by following the argument for Siegel series in [HS]-§2.

We recall the set $\mathfrak{X}_T$ for $T \in \mathcal{H}_n^\text{nd}(k')$

$$\mathfrak{X}_T = \mathfrak{X}_T(k') = \{ x \in M_{2n,n}(k') \mid H_n[x] = T \},$$

which is the fibre space $g^{-1}(T)$ for the polynomial map $g : M_{2n,n}(k') \longrightarrow \mathcal{H}_n(k')$, $g(x) = H_n[x]$ defined over $k$. We may take the measure $|\Theta_T|$ on $\mathfrak{X}_T$ induced by a $k$-rational differential form $\omega$ on $M_{2n,n}(k')$ satisfying $\omega \wedge g^*(dT) = dx$ where $dT$ is the canonical gauge form on $\mathcal{H}_n(k')$, $dx$ is the canonical gauge form on $M_{2n,n}(k')$. Then the following identity holds (cf. [Ym], [HS]-§2):

$$\int_{\mathfrak{X}_T(k')} \phi(x) |\Theta_T| (x) = \lim_{e \to -\infty} \int_{\mathcal{H}_n(\pi^{-e})} \psi(-\text{tr}(Ty)) \int_{M_{2n,n}(k')} \phi(x) \psi(\text{tr}(H_n[x]y))dxdy,$$

where $\phi \in \mathcal{S}(M_{2n,n}(k'))$, a locally constant compactly supported function on $M_{2n,n}(k')$ and $\mathcal{H}_n(\pi^{-e}) = \mathcal{H}_n(k') \cap M_n(\pi^{-e}\mathcal{O}_{k'})$.

The following lemma can be proved in the similar line to the case of symmetric matrices (cf. [HS]-§2).

**Lemma 4.1** If $\text{Re}(s) > n$, one has

$$\int_{\mathfrak{X}_T(O_{k'})} |N_{k'/k}(\det x_2)|^{s-n} |\Theta_T| (x) = \lim_{e \to -\infty} \int_{\mathcal{H}_n(\pi^{-e}O_{k'})} \psi(-\text{tr}(Ty))dy \int_{M_{2n,n}(O_{k'})} |N_{k'/k}(\det x_2)|^{s-n} \psi(\text{tr}(H_n[x]y))dx.$$

Let $\zeta(k'; s)$ be the zeta function of the matrix algebra $M_n(k')$:

$$\zeta(k'; s) = \int_{M_n(O_{k'})} |\det x|_{k'}^{s-n} dx = \int_{M_n(O_{k'})} |N_{k'/k}(\det x)|^{s-n} dx,$$
whose explicit formula is well-known:

\[ \zeta_n(k'; s) = \prod_{i=1}^{n} \frac{1 - q^{-2i}}{1 - q^{-2(s-i+1)}}. \]

Then we have the following integral expression of hermitian Siegel series.

**Theorem 4.2** If \( \text{Re}(s) > 2n \), we have

\[ b_k(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \times \int_{\mathcal{O}(\mathcal{O})} |N_{k'/k}(\text{det } x_2)|^{\frac{s}{2} - n} |\Theta_T|(x). \]

**Proof.** We define the Fourier transform of \( \phi \in S(M_n(k')) \) by

\[ \hat{\phi}(z) = \int_{M_n(k')} \phi(y) \psi(T_{k'}/k(\text{tr}(yz^*)) dy, \]

where \( T_{k'}/k \) is the trace of the extension \( k'/k \). Since

\[ \text{tr}(H_n[x]y) = \text{tr}(x_1^*x_2y) + \text{tr}(x_2^*x_1y) = \text{tr}((x_1^*)^*x_2) + \text{tr}(x_2^*x_1y) = T_{k'}/k(\text{tr}(x_2(x_1y))^*), \]

the second integral in the right hand side of (4.2) becomes

\[
\int_{M_n(\mathcal{O})} |N_{k'/k}(\text{det } x_2)|^{s-n} \text{ch}_{M_n(\mathcal{O})}(x_2y)dx_2 \\
= \int_{M_n(\mathcal{O})} |N_{k'/k}(\text{det } x_2)|^{s-n} \text{ch}_{M_n(\mathcal{O})}(x_2y)dx_2 \\
= \int_{M_n(\mathcal{O})} |N_{k'/k}(\text{det } x_2)|^{s-n} dx_2 \\
= \int_{M_n(\mathcal{O})} |\text{det } x_2|^{s-n} dx_2,
\]

where \( D_y = \text{Diag}(\pi^{e_1}, \ldots, \pi^{e_r}, 1, \ldots, 1) \) if the elementary divisors of \( y \) with negative \( \pi \)-powers are \( \pi^{-e_1}, \ldots, \pi^{-e_r} \). Hence the second integral in the right hand side of (4.2) is equal to

\[ |\text{det } D_y|^{s} \int_{M_n(\mathcal{O})} |\text{det } x_2|^{s-n} dx_2 = \nu_\pi(y)^{-2s} \times \zeta_n(k'; s). \]

Now by Lemma 4.1, we obtain

\[
\int_{\mathcal{O}(\mathcal{O})} |N_{k'/k}(\text{det } x_2)|^{s-n} |\Theta_T|(x) \\
= \zeta_n(k'; s) \times \lim_{e \to \infty} \int_{\mathcal{H}_n(\pi^{-e}\mathcal{O})} \nu_\pi(y)^{-2s} \cdot \psi(-\text{tr}(T_y)) dy \\
= \zeta_n(k'; s) \times b_\pi(T; 2s),
\]

which gives the required identity.
We introduce the spherical function on $X_T$ with respect to the Siegel parabolic subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \bigg| a, b, d \in M_n(k') \right\}$ by

$$\tilde{\omega}_T(x; s) = \int_K |N_{k'/(\det(kx))_2}|^{s-n} dk.$$  

Then we have

$$\tilde{\omega}_T(x; s) = |\det T|^{s-n} \omega_T(x; 1 - \frac{\pi \sqrt{-1}}{\log q}, \ldots, 1 - \frac{\pi \sqrt{-1}}{\log q}, s - n + \frac{1}{2} - \frac{\pi \sqrt{-1}}{\log q}), \quad (4.3)$$

which is holomorphic for $s \in \mathbb{C}$ by Theorem 1.3.

**Proposition 4.3** Denote the $k$-orbit decomposition of $X_T(O_{k'})$ as

$$X_T(O_{k'}) = \bigsqcup_{i=1}^r Kx_i.$$  

Then one has

$$b_\pi(T; s) = c_i \tilde{\omega}_T(x_i; \frac{s}{2}) + \text{c.c.}, \quad c_i = \int_{Kx_i} |\Theta_T|\, (y).$$

**Proof.** Since $X_T(O_{k'})$ is compact, it is a finite union of $K$-orbits as above. By Theorem 4.2, we have

$$b_\pi(T; s) = \sum_{i=1}^r \int_{Kx_i} |N_{k'/(\det(ky))_2}|^{\frac{s}{2}-n} |\Theta_T|\, (y)$$

$$= \sum_{i=1}^r \int_{Kx_i} \int_K |N_{k'/(\det(ky))_2}|^{\frac{s}{2}-n} dk |\Theta_T|\, (y)$$

$$= \sum_{i=1}^r c_i \cdot \tilde{\omega}_T(x_i; \frac{s}{2}).$$

Now we give the functional equation of hermitian Siegel series by using the results of functional equations of spherical functions on $X_T$.

**Theorem 4.4**

$$\frac{b_\pi(T; s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-s+i})} = \chi_\pi(\det T)^{n-1} |\det(T/2)|^{s-n} \times \frac{b_\pi(T; 2n-s)}{\prod_{i=0}^{n-1} (1 - (-1)^i q^{-(2n-s)+i})}.$$  

**Proof.** Let us recall $\rho \in W$ given in Corollary 2.6. For $s \in \mathbb{C}$, set

$$s^* = (1, \ldots, 1, \frac{s}{2} - n + \frac{1}{2}) - (\frac{\pi \sqrt{-1}}{\log q}, \ldots, \frac{\pi \sqrt{-1}}{\log q}) \in \mathbb{C}^n.$$
Then $s^*$ corresponds to $z^* \in \mathbb{C}^n$ in $z$-variable with $z^*_i = -\frac{s}{2} + i - \frac{1}{2} - (n - i + 1)\frac{\sqrt{-1}}{\log q}$, and $\rho(z^*)$ corresponds to

$$(1, \ldots, 1, -\frac{s}{2} + \frac{1}{2}) - (\frac{\sqrt{-1}}{\log q}, \ldots, \frac{\sqrt{-1}}{\log q}, n\frac{\sqrt{-1}}{\log q})$$

in $s$-variable. We set $F_n(s) = \Gamma_{\rho}(z^*)$, Then by (4.3) and Corollary 2.6, we have the following functional equation

$$\tilde{\omega}_T(x; \frac{s}{2}) = \chi_{\pi}(\det T)^{n-1}|\det T|^{-n} F_n(s) \times \tilde{\omega}_T(x; n - \frac{s}{2}).$$

Hence we obtain by Proposition 4.3,

$$b_\pi(T; s) = \chi_{\pi}(\det T)^{n-1}|\det T|^{-n} \frac{\zeta_n(k'; n - \frac{s}{2})}{\zeta_n(k'; \frac{s}{2})} F_n(s) \times b_\pi(T; 2n - s). \quad (4.4)$$

Now, by definition,

$$F_n(s) = |2|^{-ns+n^2} \prod_{i<j} \frac{1 - (-1)^{i+j}q^{-s+i+j-2}}{(1-q^{-s+i+j-1})^{-1}}.$$

Since we have

$$\frac{\zeta_n(k'; n - \frac{s}{2})}{\zeta_n(k'; \frac{s}{2})} = \prod_{i=1}^{n} \frac{1 - q^{-s+2(i-1)}}{1 - q^{s-2(n-i+1)}} = (-1)^n q^{-ns+n(n+1)} \frac{1 - q^{-s}}{1 - q^{-s+2n}},$$

we obtain

$$F_n(s) \times \frac{\zeta_n(k'; n - \frac{s}{2})}{\zeta_n(k'; \frac{s}{2})} = |2|^{-ns+n^2} (-1)^{\frac{n(n+1)}{2}} q^{-ns+n(3n+1)} \prod_{i=1}^{n} \frac{1 - (-1)^{i-1}q^{-s+i-1}}{1 - (-1)^{i+n}s^{-n+i}}$$

$$= |2|^{-ns+n^2} (-1)^{\frac{n(n+1)}{2}} q^{-ns+n(3n+1)} \prod_{i=1}^{n}\frac{1 - (-1)^{i-1}q^{-s+i-1}}{1 - (-1)^{i+n}q^{-2(n-s)+n-1}}$$

$$= |2|^{-ns+n^2} \prod_{i=0}^{n-1} \frac{1 - (-1)^{i}q^{-s+i}}{1 - (-1)^{i}q^{-(2n-s)+i}}.$$

Substituting this value into (4.4), we have

$$b_\pi(T; s) = \chi_{\pi}(\det T)^{n-1}|\det(T/2)|^{-n} \prod_{i=0}^{n-1} \frac{1 - (-1)^{i}q^{-s+i}}{1 - (-1)^{i}q^{-(2n-s)+i}} \times b_\pi(T; 2n - s),$$

which completes the proof.
References

[Bo] A. Borel: Linear Algebraic Groups, Second enlarged edition, Graduate Texts in Mathematics 126, Springer, 1991.

[H1] Y. Hironaka: Spherical functions of hermitian and symmetric forms III, Tôhoku Math. J. 40 (1988), 651–671.

[H2] Y. Hironaka: Spherical functions and local densities on hermitian forms, J. Math. Soc. Japan 51 (1999), 553 – 581.

[H3] Y. Hironaka: Functional equations of spherical functions on $p$-adic homogeneous spaces, Abh. Math. Sem. Univ. Hamburg 75 (2005), 285 – 311.

[H4] Y. Hironaka: Spherical functions on $p$-adic homogeneous spaces, arXiv:0904.0102 (2009).

[HS] Y. Hironaka and F. Sato: The Siegel series and spherical functions on $O(2n)/(O(n) \times O(n))$, "Automorphic forms and zeta functions – Proceedings of the conference in memory of Tsuneo Arakawa –", World Scientific, 2006, p. 150 – 169.

[Ik] T. Ikeda: On the lifting of hermitian modular forms, Comp. Math. 114 (2008), 1107-1154.

[Kr] M. L. Karel: Functional equations of Whittaker functions on $p$-adic groups, Amer. J. Math. 101 (1979), 1303 –1325.

[Om] O. T. O’Meara: Introduction to quadratic forms, Grund. math. Wiss. 117, Springer-Verlag, 1973.

[KS] S. S. Kudla and W. J. Sweet: Degenerate principal series representations for $U(n,n)$, Israel J. Math. 98 (1997), 253 –306.

[Sch] W. Scharlau: Quadratic and hermitian forms, Grund. math. Wiss. 270, Springer-Verlag, 1985.

[Sh] G. Shimura: Euler products and Eisenstein series, CBMS 93 (AMS), 1997.

[Ym] T. Yamazaki: Integrals defining singular series, Memoirs Fac. Sci. Kyushu Univ. 37 (1983), 113 – 128.