VALUATIONS OF SEMIRINGS

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Abstract. We develop a notion of valuations on a semiring. In particular, we classify valuations on the semifield $\mathbb{Q}_{\text{max}}$ and also valuations on the (suitably defined) ‘function field’ $\mathbb{Q}_{\text{max}}(T)$ which are trivial on $\mathbb{Q}_{\text{max}}$. As a byproduct, we reinterpret the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ over $\mathbb{F}_1$ as an abstract curve associated to $\mathbb{Q}_{\text{max}}(T)$.

1. Introduction

The existence of mythical geometry over $\mathbb{F}_1$, the field with one element, was first hinted at by J.Tits as a special case of his geometric construction of (semi) simple algebraic groups in 1956 (cf. [19]). Much later in 2004, C.Soulé gave the first mathematical definition of an algebraic variety over $\mathbb{F}_1$ and a Hasse-Weil type zeta function associated to it (cf. [17]). His work was partially motivated by Y.Manin’s work in [15] which suggests that one might obtain some clues towards the Riemann hypothesis by realizing $\text{Spec} \mathbb{Z}$ as ‘a curve over $\mathbb{F}_1$’ and translating the geometric proof of Weil’s conjectures to ‘$\text{Spec} \mathbb{Z} \times_{\mathbb{F}_1} \text{Spec} \mathbb{Z}$’.

Soulé’s idea was to enlarge the category of commutative rings to more general algebraic structures so that $\mathbb{Z}$ is no longer an initial object. In the following years, the foundation of algebraic geometry over more general algebraic structures than commutative rings has been a main theme of $\mathbb{F}_1$-geometry. In particular, algebraic geometry over commutative (pointed) monoids has been intensively investigated (cf. [1], [2], [5], [6], [17], [20]). The most recent studies on the notion of $\mathbb{F}_1$-geometry are to consider geometries over algebraic structures which maintain the additive structure rather than forgetting it completely. In that perspective, algebraic geometry over commutative semirings has been investigated in [4], [7], and [13]. Finally, in [14], O.Lorscheid unified commutative monoids and commutative semirings by means of his newly introduced structures; blueprints. For the overall view of various approaches, we refer the reader to [16].

The goal of this paper is to initiate the study of valuations for semirings with a goal to develop a concept of an abstract curve by means of valuations. We shall provide three different perspectives and classify all valuations of $\mathbb{Q}_{\text{max}}$ and valuations of $\mathbb{Q}_{\text{max}}(T)$ which are trivial on $\mathbb{Q}_{\text{max}}$ up to equivalence. By appealing to this result, we obtain the following description of a projective line over $\mathbb{F}_1$.

Theorem. (cf. Example 4.2, Remark 4.3) The abstract curve associated to a (suitably defined) function field $\mathbb{Q}_{\text{max}}(T)$ over $\mathbb{Q}_{\text{max}}$ is the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ over $\mathbb{F}_1$.

Acknowledgment. This is a part of the author’s Ph.D. thesis [12].

2. Valuation theory over semirings

Throughout the paper, we assume all semirings are commutative unless stated otherwise. As in the classical case, one might expect that a theory of valuations over semirings provides...
some geometric information. To shape a theory of valuations over semirings, one first needs to find proper definitions. We provide three possible approaches and compute toy examples for each.

The first definition directly extends the definition of classical valuation (cf. Definition 2.2). The second definition comes from the observation that for a valuation \( \nu \), we have \( \nu(a + b) \in \{ \nu(a), \nu(b) \} \) if \( \nu(a) \neq \nu(b) \) (cf. Definition 2.3). In the last approach (cf. Definition 2.9), we shall make use of rather unknown algebraic structures, hyperfields instead of the field \( \mathbb{R} \) of real numbers. Roughly speaking, a hyperfield is an algebraic object which generalizes a field by stating that the addition is multi-valued (cf. [11] or Appendix [11] for basic notions of hyperfields). This is related with the probabilistic intuition: when \( \nu(a) = \nu(b) \), the value \( \nu(a + b) \) is not solely determined by \( \nu(a) \) and \( \nu(b) \), but \( \nu(a + b) \in [-\infty, \nu(a)] \).

Recall that an idempotent semiring is a semiring \( M \) on a commutative ring, we have 
\[
x \odot y = \begin{cases} 
\max\{x, y\} & \text{if } x \neq y \\
[-\infty, x] & \text{if } x = y
\end{cases}
\]
The multiplication \( \odot \) is given by the usual addition of real numbers with a \( \odot (-\infty) = -\infty \) for \( \forall a \in \mathbb{R} \cup \{-\infty\} \).
Remark 2.7. The addition of $\mathbb{R}_{+,\text{val}}$ is designed to capture the information we lose when $\nu(x) = \nu(y)$ since, in this case, $\nu(x + y)$ can be any number less than or equal to $\nu(x)$.

Proposition 2.8. $\mathbb{R}_{+,\text{val}}$ is a hyperfield.

Proof. A more generalized result has been proven in [22]. □

Next, we define a valuation of an idempotent semiring with values in $\mathbb{R}_{+,\text{val}}$.

Definition 2.9. Let $M$ be an idempotent semiring and $H = \mathbb{R}_{+,\text{val}}$. A valuation of $M$ with values in $H$ is a function $\nu : M \rightarrow H$ which satisfies the following conditions:

$$\nu(x + y) \in \nu(x) \oplus \nu(y), \quad \nu(xy) = \nu(x) \odot \nu(y), \quad \nu(x) = -\infty \iff x = 0_M. \quad (1)$$

Remark 2.10. Note that in Definition 2.2, 2.4, and 2.9, we say that two valuations $\nu_1$, $\nu_2$ are equivalent if there exists a real number $\rho > 0$ such that $\nu_1(x) = \rho \nu_2(x) \forall x \in M$, where $\rho \nu_2(x)$ is the usual multiplication of real numbers.

3. Examples

3.1 The first example, $\mathbb{Q}_{\text{max}}$.

Proposition 3.1. Let $M = \mathbb{Q}_{\text{max}}$. Then,

1. With Definition 2.2, the set of valuations on $M$ is equal to $\mathbb{R}$. There are exactly three valuations on $M$ up to equivalence.
2. With Definition 2.4, the set of strict valuations on $M$ is equal to $\mathbb{R}_{\geq 0}$. There are exactly two strict valuations on $M$ up to equivalence.
3. With Definition 2.9, the set of valuations on $M$ with values in $\mathbb{R}_{+,\text{val}}$ is equal to $\mathbb{R}_{\geq 0}$. There are exactly two valuations on $M$ up to equivalence.

Proof. To avoid any possible confusion, let us denote by $\oplus$, $\odot$ the addition and the multiplication of $M$ respectively.

1. In this case, as we previously remarked, the third condition is redundant since $M$ is totally ordered. We claim that any valuation $\nu$ on $M$ only depends on the value $\nu(1)$. In fact, since $\mathbb{Z}$ is (multiplicatively) generated by 1 in $\mathbb{Q}_{\text{max}}$, it follows from the second condition that the value $\nu(1)$ determines $\nu(m) \forall m \in \mathbb{Z}$. Moreover, for $\frac{1}{m}$, we have $\nu(1) = \nu(\frac{1}{m} \odot \cdots \odot \frac{1}{m}) = m \nu(\frac{1}{m})$ and hence $\nu(\frac{1}{m}) = \frac{1}{m} \nu(1)$. This implies that for $\frac{m}{n} \in \mathbb{Q}$, we have $\nu\left(\frac{m}{n}\right) = \frac{m}{n} \nu(1)$. Conversely, let $\nu : M \rightarrow \mathbb{R} \cup \{\infty\}$ be a function such that $\nu\left(\frac{m}{n}\right) := \frac{m}{n} \nu(1)$ for some $\nu(1) \neq \infty$. Then, clearly $\nu$ is a valuation on $M$. It follows that the set of valuations on $M$ is equal to $\mathbb{R}$.

Next, suppose that $\nu_1$, $\nu_2$ are valuations on $M$ such that $\nu(1) > 0, \nu(2) > 0$, then they are equivalent. In fact, if we take $\rho := \frac{\nu_1(1)}{\nu_2(1)}$, then for $x \in \mathbb{Q}_{\text{max}} \setminus \{-\infty\}$, we have $\nu_1(x) = x \nu_1(1) = x \rho \nu_2(1) = \rho \nu_2(x)$. Similarly, valuations $\nu_1$ and $\nu_2$ on $M$ with $\nu_1(1) < 0$ are equivalent. Finally, $\nu(1) = 0$ gives a trivial valuation. Therefore, we have exactly three valuations up to equivalence.

2. In this case, we claim that a strict valuation $\nu$ is an order-preserving map. Indeed, we have $x \leq y \iff x \oplus y = y$. Suppose that $x \leq y$. Then we have $\nu(y) = \nu(x \oplus y) = \nu(x) + \nu(y) \iff \nu(x) \leq \nu(y)$. On the other hand, as in the above case, a strict valuation $\nu$ only depends on $\nu(1)$. Since $\nu$ is an order-preserving map and $\nu(0) = 0$, it follows that $\nu(1) \geq 0$. Therefore, the set of valuations on $M$ is equal to $\mathbb{R}_{\geq 0}$. Moreover, if $\nu(1) = 0$, then we have a trivial valuation and strict valuations $\nu$ on $M$ such that $\nu(1) > 0$ are equivalent as in the above case. Thus, in this case, there are exactly two strict valuations on $M$ up to equivalence.
(3) In this case, a valuation $\nu$ on $M$ is determined by $\nu(1)$ and $\nu(1) \geq 0$. In fact, suppose that $x \leq y$. Then we have

$$\nu(x \oplus y) = \nu(y) \in \nu(x) + \nu(y).$$

Assume that $\nu(y) < \nu(x)$. Then we have $\nu(x) + \nu(y) = \nu(x)$ and it follows from (2) that $\nu(x) = \nu(y)$ which is a contradiction. This shows that $\nu$ is an order-preserving map, where an order on $\mathbb{R}_{+,val}$ is the usual order of $\mathbb{R}$. Furthermore, we have $\nu(0) = 0$ since $\nu(0 \odot 0) = \nu(0) = \nu(0) + \nu(0)$ (since $\cdot$ is the usual addition of real numbers).

It follows that $\nu(1) \geq 0 (= 1_{\mathbb{R}_{+,val}})$. Finally, similar to the first case, we have $\nu(\frac{a}{b}) = \frac{1}{b}\nu(1)$. Conversely, it is clear that all maps which satisfy such properties are valuations on $M$. Hence, the set of valuations on $M$ is equal to $\mathbb{R}_{\geq 0}$. Furthermore, two valuations $\nu_1, \nu_2$ on $M$ with $\nu_1(1), \nu_2(1) > 0$ are equivalent as in the first case. Hence, there are exactly two valuations on $M$ up to equivalence.

$\square$

3.2 The second example, $\mathbb{Q}_{\text{max}}(T)$.

We begin by investigating $\mathbb{Q}_{\text{max}}[T]$, the idempotent semiring of polynomials with coefficient in $\mathbb{Q}_{\text{max}}$. In the sequel, we use the notations $+$ and $\cdot$ for the usual operations of $\mathbb{Q}$. We use the notations $\oplus, \odot$ for the operations of $\mathbb{Q}_{\text{max}}[T]$ and $+_{t}, \cdot_{t}$ for $\mathbb{Q}_{\text{max}}$.

For $f(T) = \sum_{i=0}^{n} a_{i}T^{i}, g(T) = \sum_{i=0}^{m} b_{i}T^{i} \in \mathbb{Q}_{\text{max}}[T]$, suppose that $n \leq m$. The addition and the multiplication of $\mathbb{Q}_{\text{max}}[T]$ are given as follows:

$$\begin{align*}
(f \oplus g)(T) &= \sum_{i=0}^{n} \max\{a_{i}, b_{i}\}T^{i} \oplus \sum_{i=n+1}^{m} b_{i}T^{i}, \\
(f \odot g)(T) &= \sum_{i=0}^{n+m} \left( \sum_{r+l=i} a_{r} \cdot b_{l} \right)T^{i} = \sum_{i=0}^{n+m} \left( \max\{a_{r} + b_{l}\} \right)T^{i}.
\end{align*}$$

(3) In this case, a valuation $\nu$ on $M$ is determined by $\nu(1)$ and $\nu(1) \geq 0$. In fact, suppose that $x \leq y$. Then we have

$$\nu(x \oplus y) = \nu(y) \in \nu(x) + \nu(y).$$

Assume that $\nu(y) < \nu(x)$. Then we have $\nu(x) + \nu(y) = \nu(x)$ and it follows from (2) that $\nu(x) = \nu(y)$ which is a contradiction. This shows that $\nu$ is an order-preserving map, where an order on $\mathbb{R}_{+,val}$ is the usual order of $\mathbb{R}$. Furthermore, we have $\nu(0) = 0$ since $\nu(0 \odot 0) = \nu(0) = \nu(0) + \nu(0)$ (since $\cdot$ is the usual addition of real numbers).

It follows that $\nu(1) \geq 0 (= 1_{\mathbb{R}_{+,val}})$. Finally, similar to the first case, we have $\nu(\frac{a}{b}) = \frac{1}{b}\nu(1)$. Conversely, it is clear that all maps which satisfy such properties are valuations on $M$. Hence, the set of valuations on $M$ is equal to $\mathbb{R}_{\geq 0}$. Furthermore, two valuations $\nu_1, \nu_2$ on $M$ with $\nu_1(1), \nu_2(1) > 0$ are equivalent as in the first case. Hence, there are exactly two valuations on $M$ up to equivalence.

$\square$

3.2 The second example, $\mathbb{Q}_{\text{max}}(T)$.

We begin by investigating $\mathbb{Q}_{\text{max}}[T]$, the idempotent semiring of polynomials with coefficient in $\mathbb{Q}_{\text{max}}$. In the sequel, we use the notations $+$ and $\cdot$ for the usual operations of $\mathbb{Q}$. We use the notations $\oplus, \odot$ for the operations of $\mathbb{Q}_{\text{max}}[T]$ and $+_{t}, \cdot_{t}$ for $\mathbb{Q}_{\text{max}}$.

For $f(T) = \sum_{i=0}^{n} a_{i}T^{i}, g(T) = \sum_{i=0}^{m} b_{i}T^{i} \in \mathbb{Q}_{\text{max}}[T]$, suppose that $n \leq m$. The addition and the multiplication of $\mathbb{Q}_{\text{max}}[T]$ are given as follows:

$$\begin{align*}
(f \oplus g)(T) &= \sum_{i=0}^{n} \max\{a_{i}, b_{i}\}T^{i} \oplus \sum_{i=n+1}^{m} b_{i}T^{i}, \\
(f \odot g)(T) &= \sum_{i=0}^{n+m} \left( \sum_{r+l=i} a_{r} \cdot b_{l} \right)T^{i} = \sum_{i=0}^{n+m} \left( \max\{a_{r} + b_{l}\} \right)T^{i}.
\end{align*}$$

(3) In this case, a valuation $\nu$ on $M$ is determined by $\nu(1)$ and $\nu(1) \geq 0$. In fact, suppose that $x \leq y$. Then we have

$$\nu(x \oplus y) = \nu(y) \in \nu(x) + \nu(y).$$

Assume that $\nu(y) < \nu(x)$. Then we have $\nu(x) + \nu(y) = \nu(x)$ and it follows from (2) that $\nu(x) = \nu(y)$ which is a contradiction. This shows that $\nu$ is an order-preserving map, where an order on $\mathbb{R}_{+,val}$ is the usual order of $\mathbb{R}$. Furthermore, we have $\nu(0) = 0$ since $\nu(0 \odot 0) = \nu(0) = \nu(0) + \nu(0)$ (since $\cdot$ is the usual addition of real numbers).

It follows that $\nu(1) \geq 0 (= 1_{\mathbb{R}_{+,val}})$. Finally, similar to the first case, we have $\nu(\frac{a}{b}) = \frac{1}{b}\nu(1)$. Conversely, it is clear that all maps which satisfy such properties are valuations on $M$. Hence, the set of valuations on $M$ is equal to $\mathbb{R}_{\geq 0}$. Furthermore, two valuations $\nu_1, \nu_2$ on $M$ with $\nu_1(1), \nu_2(1) > 0$ are equivalent as in the first case. Hence, there are exactly two valuations on $M$ up to equivalence.

$\square$
\[ f(T) = \sum_{i=0}^{n} a_i T^i, \quad g(T) = \sum_{i=0}^{m} b_i T^i. \] It is enough to show that
\[ (f \oplus g)(x) = f(x) + g(x), \quad (f \odot g)(x) = f(x) \cdot g(x) \quad \forall x \in \mathbb{Q}_{\text{max}}. \]

We may assume that \( n \leq m \). Then we have
\[ (f \oplus g)(T) = \sum_{i=0}^{n} \max\{a_i, b_i\} T^i \oplus \sum_{i=n+1}^{m} b_i T^i. \]

For \( x \in \mathbb{Q}_{\text{max}} \), we have, by letting \( a_i = -\infty \) for \( i = n+1, \ldots, m \),
\[ (f \oplus g)(x) = \max_{i=0, \ldots, n} \{ \max\{a_i, b_i\} + i x \}. \]

However, \( f(x) = \max_{i=0, \ldots, n} \{ a_i + i x \} \) and \( g(x) = \max_{i=0, \ldots, m} \{ b_i + i x \} \), thus
\[ f(x) + g(x) = \max\{f(x), g(x)\} = \max_{i=0, \ldots, n} \{ a_i + i x \} \quad \max_{i=0, \ldots, m} \{ b_i + i x \} \]
\[ = \max_{i=0, \ldots, m} \{ \max\{a_i, b_i\} + i x \} = (f \oplus g)(x). \]

This proves the first part. Next, we have
\[ (f \odot g)(T) = \sum_{i=0}^{n+m} ( \sum_{r+l=i} a_r b_l ) T^i = \sum_{i=0}^{n+m} ( \max\{a_r, b_l\} ) T^i. \]

It follows that for \( x \in \mathbb{Q}_{\text{max}} \), we have
\[ (f \odot g)(x) = \max_{0 \leq i \leq n+m} \{ a_i + b_i + i x \} = \max_{0 \leq i \leq n+m} \{ a_r + b_l + r x + b_l + l x \}. \]

On the other hand, we have
\[ f(x) \cdot g(x) = \max_{0 \leq i \leq n} \{ a_i + i x \} + \max_{0 \leq j \leq m} \{ b_j + j x \}. \]

Thus, if \( f(x) = a_{i_0} + i_0 x \) and \( g(x) = b_{j_0} + j_0 x \) for some \( i_0 \) and \( j_0 \), then we have
\[ f(x) \cdot g(x) = (a_{i_0} + i_0 x) + (b_{j_0} + j_0 x) = (a_{i_0} + b_{j_0}) + (i_0 + j_0) x. \]

It follows that \( f(x) \cdot g(x) \leq (f \odot g)(x) \). But, if \( (f \odot g)(x) = (a_{r_0} + r_0 x) + (b_{l_0} + l_0 x) \), then
\[ (f \odot g)(x) \leq f(x) \cdot g(x). \]

Hence, \( (f \odot g)(x) = f(x) \cdot g(x) \). \( \square \)

From Proposition 3.3, the set \( \overline{\mathbb{Q}_{\text{max}}}[T] := \mathbb{Q}_{\text{max}}[T]/\sim \) is an idempotent semiring. In fact, \( \mathbb{Q}_{\text{max}}[T] \) is a semiring since \( \sim \) is a congruence relation. Furthermore, for \( f(T) \in \mathbb{Q}_{\text{max}}[T] \), we have \( f(x) + f(x) = f(x) \forall x \in \mathbb{Q}_{\text{max}} \). This implies that \( f(T) \oplus f(T) \sim f(T) \) and hence \( \overline{\mathbb{Q}_{\text{max}}}[T] \) is an idempotent semiring. It is known that, for \( \mathbb{Q}_{\text{max}}[T] \), the fundamental theorem of tropical algebra holds, i.e. a polynomial \( \overline{P}(T) \in \overline{\mathbb{Q}_{\text{max}}}[T] \) can be uniquely factored into linear polynomials in \( \overline{\mathbb{Q}_{\text{max}}}[T] \) (cf. [18] or [21]). In particular, this implies that the notion of a degree of \( \overline{f}(T) \in \overline{\mathbb{Q}_{\text{max}}}[T] \) is well defined. Furthermore, \( \overline{\mathbb{Q}_{\text{max}}}[T] \) does not have any multiplicative zero-divisor. Indeed, suppose that \( \overline{f}(T) \cdot \overline{g}(T) = \overline{(f g)(T)} \sim (\infty, -\infty) \). Then, for \( x \in \mathbb{Q}_{\text{max}} \), we have \( f(x) \cdot g(x) = f(x) + g(x) = -\infty \). In other words, for \( x \in \mathbb{Q}_{\text{max}} \), we have \( f(x) = -\infty \) or \( g(x) = -\infty \). However, this only happens when \( f(T) = -\infty \) or \( g(T) = -\infty \). Thus, \( \overline{\mathbb{Q}_{\text{max}}}[T] \) does not have a multiplicative zero-divisor. In fact, in Corollary 3.7 we shall prove that \( \overline{\mathbb{Q}_{\text{max}}}[T] \) satisfies the stronger condition: \( \overline{\mathbb{Q}_{\text{max}}}[T] \) is multiplicatively cancellative. We will define \( \mathbb{Q}_{\text{max}}(T) := \text{Frac}(\overline{\mathbb{Q}_{\text{max}}}[T]) \). We first prove several lemmas to classify valuations on \( \mathbb{Q}_{\text{max}}(T) \).

**Lemma 3.4.** Let \( M := \overline{\mathbb{Q}_{\text{max}}}[T] \). For \( \overline{f}(T) \in M \), let \( r_f \) be the maximum natural number such that \( T^{r_f} \) can divide \( \overline{f}(T) \). Then, for \( \overline{f}(T), \overline{g}(T) \in M \), we have
\[ r_{f \oplus g} = \min\{r_f, r_g\}, \quad r_{f \odot g} = r_f + r_g. \]
Proof. Let \( f(T), g(T) \in Q_{\text{max}}[T] \). We first claim that if \( f(T) \) has a constant term different from \(-\infty\) and \( g(T) \) does not have a constant term, then \( f(T) \) and \( g(T) \) are not functionally equivalent. Indeed, if \( f(T) = \sum a_i T^i \) and \( g(T) = \sum b_i T^i \), then \( f(-\infty) = a_0 \neq -\infty = g(-\infty) \). One can further observe that if \( f(T) \sim T \), then \( f(T) = T \). In fact, from the fundamental theorem of tropical algebra, we know that the degree of \( f(T) \) should be one. Hence, \( f(T) = a \odot T \oplus b \) for some \( a, b \in Q_{\text{max}} \). Then \( b = -\infty \) since, otherwise, \( f(-\infty) = b \neq -\infty \) and therefore \( f(T) \neq T \). Furthermore, \( a = 0 \) since, otherwise, we have \( f(-a) = 0 \). However, this is different from the evaluation of \( T \) at \(-a\).

Next, we claim that \( f(T) \in M \) has the factor \( T \) if and only if any representative of \( f(T) \) does not have a constant term. To see this, suppose that \( f(T) \) has the factor \( T \). Then, \( f(T) \sim T \odot g(T) \) for some \( g(T) \in Q_{\text{max}}[T] \). Since \( T \odot g(T) \) does not have a constant term, from the first claim, \( f(T) \) also does not have a constant term. Conversely, suppose that any representative of \( f(T) \) does not have a constant term. We can write \( f(T) = T \odot g(T) \) for some \( g(T) \in Q_{\text{max}} \). Hence, \( f(T) \) has a factor \( T \). From the above argument and the fundamental theorem of tropical algebra, it is now clear that \( r_f \) is well defined. Moreover, for \( \overline{f(T), g(T)} \in M \), we can write \( \overline{f(T)} = T^l \odot h(T), \overline{g(T)} = T^m \odot p(T) \) for some \( h(T), p(T) \) such that \( h(T) \) and \( p(T) \) do not have \( T \) as a factor. From our previous claim, this is equivalent to that \( h(T) \) and \( p(T) \) do have a constant term different from \(-\infty\). Assume that \( l \leq m \), then we have \( \overline{f(T)} \oplus \overline{g(T)} = T^l \odot (h(T) \oplus T^{m-l} p(T)) \).

Since \( h(T) \) has a constant term different from \(-\infty\), it follows that \( h(T) \oplus T^{m-l} p(T) \) has a constant term different from \(-\infty\) and therefore \( h(T) \oplus T^{m-l} p(T) \) does not have a factor \( T \). This shows that \( r_f \odot g = \min \{r_f, r_g\} \). The second assertion \( r_f \odot g = r_f + r_g \) is clear from the fundamental theorem of tropical algebra. \( \square \)

Remark 3.5. Lemma 3.4 is different from the classical case. Essentially, this is due to the absence of additive inverses. In the classical case, if \( f(T) = T^l h(T), g(T) = T^m p(T) \in Q[T] \) with \( l < m \), then \( f(T) + g(T) = T^l (h(T) + T^{m-l} p(T)) \). Hence, we have \( r_f + r_g = \min \{r_f, r_g\} \). The problem is when \( l = m \). For example, if \( f(T) = T(T + 1), g(T) = T(T - 1) \in Q[T] \), then \( r_f = r_g = 1 \). However, \( f(T) + g(T) = 2T^2 \) and hence \( r_f + r_g = 2 > \min \{r_f, r_g\} = 1 \) from the additive cancellation which is impossible in the case of idempotent semirings.

Lemma 3.6. Let \( M := Q_{\text{max}}[T] \). Then, for \( f(T) \in M \), \( \deg f(T) \) is well defined. Furthermore, for \( f(T), g(T) \in M \), we have \( \deg (f(T) \oplus g(T)) = \max \{\deg f(T), \deg g(T)\} \) and \( \deg (f(T) \odot g(T)) = \deg f(T) \odot \deg g(T) \).

Proof. This is straightforward from the fundamental theorem of tropical algebra and the fact that no additive cancellation happens when we add two tropical polynomials. \( \square \)

Corollary 3.7. Let \( M := Q_{\text{max}}[T] \). Then \( M \) is multiplicatively cancellative.

Proof. For \( f(T) \odot h(T) = g(T) \odot h(T) \) with \( h(T) \neq -\infty \), we have to show that \( f(T) = g(T) \). We continue using the notation as in Lemma 3.4. We know that \( f(T) \odot h(T) = g(T) \odot h(T) \) is equivalent to the following condition:

\[
\forall x \in Q_{\text{max}}, \quad f(x) + h(x) = g(x) + h(x) \tag{5}
\]

where \( + \) is the usual addition. Thus, if \( h(x) \neq -\infty \), we have \( f(x) = g(x) \). Since \( h(x) = -\infty \) happens only when \( x = -\infty \), it follows that \( f(x) = g(x) \) as long as \( x \neq -\infty \). Hence, all we have to show is that \( f(-\infty) = g(-\infty) \). From Lemma 3.4, we have \( r_f + r_h = r_g + r_h \) and therefore \( r_f = r_g \). Fix a representative \( f(T) = \sum a_i T^i \) of \( f(T) \). We then have \( f(-\infty) = a_0 \).
if \( r_f = 0 \) and \( f(-\infty) = -\infty \) if \( r_f \neq 0 \). Thus, we may assume that \( r_f = r_g = 0 \). Fix a representative \( g(T) = \sum b_i T^i \) of \( g(T) \). From Lemma 3.2 of [21], there exists a real number \( N \) such that if \( x > N \), then \( f(x) = a_0 \) and \( g(x) = b_0 \). Since we know that \( f(T) \) and \( g(T) \) agree on all elements of \( \Q_{\max} \) but \( -\infty \), we conclude that \( f(x) = a_0 = b_0 = g(x) \) for \( x > N \). Therefore, we have \( f(-\infty) = a_0 = b_0 = g(-\infty) \) and hence \( f(T) = g(T) \). \( \square \)

Let \( M := \Q_{\max}[T], S := M \setminus \{-\infty\} \), and \( \Q_{\max}(T) := S^{-1}M \). It follows from Corollary 3.7 that the localization map \( S^{-1} : M \to S^{-1}M \) is injective and \( \Q_{\max}(T) \) is an idempotent semifield.

**Proposition 3.8.** Let \( M \) be a multiplicatively cancellative idempotent semiring. Let \( S := M \setminus \{0_M\} \) and \( N := S^{-1}M \). Let \( \nu \) be a valuation on \( N \) in the sense of any of Definitions 2.2, 2.4, and 2.9. Then, a valuation \( \nu \) on \( N \) only depends on the image \( i(M) \) of the canonical injection \( i : M \to S^{-1}M = N, m \mapsto \frac{m}{1} \).

**Proof.** Since \( i \) is an injection, one can identify an element \( m \in M \) with \( \frac{m}{1} \in S^{-1}M = N \) under the canonical injection \( i \). We have \( 1_N = \frac{a}{a} \forall a \in S = M^\times \). Then, with Definitions 2.2, 2.4, and 2.9 we have \( \nu(1_N) = \nu(a) + \nu(\frac{1}{a}) = 0 \), where + is the usual addition of real numbers. It follows that \( \nu(\frac{1}{a}) = -\nu(a) \) and hence \( \nu(\frac{1}{a}) = \nu(a) - \nu(b) \). \( \square \)

**Remark 3.9.** In the theory of commutative rings, to be multiplicatively cancellative and to have no (multiplicative) zero divisors are equivalent conditions. This is in contrast to the theory of semirings, where the first condition implies the second condition but not conversely in general. However, even when \( M \) is only a semiring without (multiplicative) zero divisors, one can derive the statement as in Proposition 2.8 in the following sense: Let \( M \) be a semiring without (multiplicative) zero divisors and \( \Val(M) \) be the set of valuations on \( M \) (with respect to Definition 2.2 or 2.9). Then, there exists a set bijection between \( \Val(M) \) and \( \Val(S^{-1}M) \). Indeed, for \( \nu \in \Val(M) \), one can define a valuation \( \nu(\frac{1}{a}) = \nu(a)\nu(b)^{-1} \). Conversely, for \( \nu \in \Val(S^{-1}M) \), we define \( \nu = \nu \circ i \in \Val(M) \), where \( i : M \to S^{-1}M \). One can easily check that these two are well defined and inverses to each other.

**Proposition 3.10.** Let \( M := \Q_{\max}[T], S := M \setminus \{-\infty\} \), and \( \Q_{\max}(T) := S^{-1}M \). Then, with Definition 2.4, the set of strict valuations on \( \Q_{\max}(T) \) which are trivial on \( \Q_{\max} \) is equal to \( \R \). There are exactly three strict valuations on \( \Q_{\max}(T) \) which are trivial on \( \Q_{\max} \) up to equivalence.

**Proof.** From Proposition 3.8 and Corollary 3.7 a strict valuation \( \nu \) on \( \Q_{\max}(T) \) only depends on values of \( \nu \) on \( M \). Then, from the fundamental theorem of tropical algebra, we have
\[
\tilde{f}(T) = l_1(T) \odot l_2(T) \odot \ldots \odot l_n(T),
\]
where \( l_i(T) = a_i T \oplus b_i \) for some \( a_i \in \Q, b_i \in \Q_{\max} \). It follows that
\[
\nu(\tilde{f}(T)) = \nu(l_1(T)) + \nu(l_2(T)) + \ldots + \nu(l_n(T)).
\]
Let us first assume that \( \nu(T) < 0 \). If \( b_i \neq -\infty \), since \( \nu \) is trivial on \( \Q_{\max} \), we have
\[
\nu(a_i T \oplus b_i) = \max\{\nu(a_i) + \nu(T), \nu(b_i)\} = \max\{\nu(T), 0\} = 0.
\]
Thus, we have
\[
\nu(\tilde{f}(T)) = r_f(\nu(T)),
\]
where \( r_f \) is as in Lemma 3.4. Conversely, any map \( \nu : \Q_{\max}(T) \to \R_{\max} \) satisfying the following conditions:
\[
\nu(q) = 0 \quad \forall q \in \Q, \quad \nu(-\infty) = -\infty, \quad \nu(T) < 0, \quad \nu(\tilde{f}(T)) = r_f(\nu(T))
\]

is indeed a strict valuation. In fact, from Lemma 3.4 we know that \( r_{f \oplus g} = \min\{r_f, r_g\} \).
Since \( \nu(T) < 0 \) and \( r_f, r_g \in \mathbb{N} \), this implies that
\[
\nu(f(T) \oplus g(T)) = \nu(f(T) \oplus g(T)) = r_{f \oplus g} \nu(T) = \min\{r_f, r_g\} \nu(T)
\]
\[
= \max\{r_f \nu(T), r_g \nu(T)\} = \max\{\nu(f(T)), \nu(g(T))\}.
\]
Moreover, \( \nu(f(T) \odot g(T)) = \nu(f(T) \odot g(T)) = r_{f \odot g} \nu(T) = (r_f + r_g) \nu(T) = r_f \nu(T) + r_g \nu(T) = \nu(f(T)) + \nu(g(T)) \). Furthermore, all such valuations on \( \mathbb{Q}_{\max}(T) \) are equivalent. Indeed, let \( \nu_1, \nu_2 \) be strict valuations on \( \mathbb{Q}_{\max}(T) \) such that \( \nu_1(T) = \alpha < 0 \) and \( \nu_2(T) = \beta < 0 \). Since \( \alpha, \beta \) are negative numbers, \( \rho := \frac{\beta}{\alpha} \) is a positive number and \( \nu_2(f(T)) = r_f f(T) = (r_f \rho) \alpha = \rho \nu_1(f(T)) \).
Secondly, suppose that \( \nu(T) = 0 \). Then, we have
\[
\nu(a_i T \oplus b_i) = 0.
\]
In other words, \( \nu \) is a trivial valuation since \( 0 = 1_{\mathbb{R}_{\max}} \). Clearly, this is not equivalent to the first case.
The final case is when \( \nu(T) > 0 \). Then we have
\[
\nu(a_i T \oplus b_i) = \max\{\nu(a_i) + \nu(T), \nu(b_i)\} = \max\{\nu(T), 0\} = \nu(T).
\]
It follows that
\[
\nu(f(T)) = \deg(f(T)) \nu(T).
\]
Conversely, any map \( \nu : \mathbb{Q}_{\max}(T) \rightarrow \mathbb{R}_{\max} \) satisfying the following conditions:
\[
\nu(q) = 0 \quad \forall q \in \mathbb{Q}, \quad \nu(-\infty) = -\infty, \quad \nu(T) > 0, \quad \nu(f(T)) = \deg(f(T)) \nu(T)
\]
is indeed a strict valuation from Lemma 3.6. Furthermore, suppose that \( \nu_1, \nu_2 \) are strict valuations on \( \mathbb{Q}_{\max}(T) \) such that \( \nu_1(f(T)) = \alpha > 0, \nu_2(f(T)) = \beta > 0 \). Then, with \( \rho = \frac{\beta}{\alpha} \), \( \nu_1, \nu_2 \) are equivalent. Furthermore, this case is not equivalent to any of the above. In summary, the set of strict valuations on \( \mathbb{Q}_{\max}(T) \) which are trivial on \( \mathbb{Q}_{\max} \) is equal to \( \mathbb{R} \) (by sending \( \nu \) to \( \nu(T) \)). There are exactly three strict valuations depending on a sign of a value of \( T \).

**Proposition 3.11.** Let \( M := \overline{\mathbb{Q}_{\max}(T)} \), \( S := M \setminus \{-\infty\} \), and \( \mathbb{Q}_{\max}(T) := S^{-1} M \). Then, with Definition 2.3, the set of valuations on \( \mathbb{Q}_{\max}(T) \) with values in \( \mathbb{R}_{+, \text{val}} \) which are trivial on \( \mathbb{Q}_{\max} \) is equal to \( \mathbb{R} \). There are exactly three valuations on \( \mathbb{Q}_{\max}(T) \) which are trivial on \( \mathbb{Q}_{\max} \) up to equivalence.

**Proof.** To avoid notational confusion, we denote by \( \oplus, \odot \) the addition and the multiplication of idempotent semirings and by \( \vee, \cdot \) the addition and the multiplication of \( \mathbb{R}_{+, \text{val}} \). From Proposition 3.8, a valuation \( \nu \) on \( \mathbb{Q}_{\max}(T) \) only depends on values of \( \nu \) on \( M \). Let \( \nu \) be a valuation on \( \mathbb{Q}_{\max}(T) \) which is trivial on \( \mathbb{Q}_{\max} \). For \( f(T) \in M \), from the fundamental theorem of tropical algebra, we have \( f(T) = l_1(T) \odot l_2(T) \odot \ldots \odot l_n(T) \), where \( l_i(T) = a_i T \oplus b_i \) for some \( a_i \in \mathbb{Q}, b_i \in \mathbb{Q}_{\max} \). Hence, \( \nu \) is entirely determined by values on linear polynomials. Similar to Proposition 3.10, we divide the cases up to a sign of \( \nu(T) \). The first case is when \( \nu(T) < 0 \). Since \( \nu \) is trivial on \( \mathbb{Q}_{\max} \), if \( b \neq -\infty \), we have
\[
\nu(a \overline{T} \oplus b) \in \nu(a \overline{T}) \lor \nu(b) = (\nu(a) \cdot \nu(T)) \lor \nu(b) = \max\{\nu(T), 0\} = 0.
\]
With the same notation as in Lemma 3.4, we have
\[
\nu(f(T)) = r_f \nu(T) \tag{7}
\]
Conversely, any map \( \nu : \mathbb{Q}_{\max}(T) \rightarrow \mathbb{R}_{+, \text{val}} \) given by (7) is a valuation when \( \nu(T) < 0 \). Indeed, from the fundamental theorem of tropical algebra, we have \( \nu(f(T) \odot g(T)) = (r_f \oplus r_g) \nu(T) = r_f \nu(T) + r_g \nu(T) = \nu(f(T)) \cdot \nu(g(T)) \). Moreover, from Lemma 3.4 we have \( \nu(f(T) \oplus g(T)) = \max\{\nu(f(T)), \nu(g(T))\} \).
\[ g(T) = r(f, g)\nu(T) = \min\{r_f, r_g\} \nu(T) = \max\{r_f \nu(T), r_g \nu(T)\} = \max\{\nu(f(T)), \nu(g(T))\} \in \nu(f(T)) \vee \nu(g(T)). \]

Similar to Proposition 3.10, all these cases are equivalent.

The second case is when \( \nu(T) = 0 \). Then we have \( \nu(a_i T + b_i) = 0 \) and this case gives us a trivial valuation since \( 0 = 1_{\mathbb{R}^+_{val}} \). Clearly this is not equivalent to the first case.

The final case is when \( \nu(T) > 0 \). Then, as in Proposition 3.10, we have \( \nu(f(T)) = \deg(f(T)) (\nu(T)) \). Conversely, any map \( \nu : \mathbb{Q}_\text{max}(T) \rightarrow \mathbb{R}^+_{val} \) given in this way is indeed a valuation by Lemma 3.6. These are all equivalent from the exact same argument in Proposition 3.10. \( \square \)

4. The projective line \( \mathbb{P}^1_{\mathbb{F}_1} \) as an abstract curve

Our motivation in developing a valuation theory of semirings is to make an analogue of abstract (nonsingular) curves in characteristic one. Recall that an abstract curve associated to a function field \( K \) of dimension 1 over an algebraically closed field \( k \) is the set \( C_K \) of all discrete valuation rings of \( K/k \) for a discrete valuation over \( \mathbb{F}_1 \).

Propositions 3.10 and 3.11 are the direct analogue of the set \( C_K \) with \( \nu(e) = \nu(T) \) and \( k = \mathbb{Q}_\text{max} \) is the set \( \text{Val}(\mathbb{Q}_\text{max}(T)) := \{\nu_+, \nu_-\} \), where \( \nu_+ \) is the class of valuations \( \nu \) such that \( \nu(T) > 0 \) and \( \nu_- \) is the class of valuations \( \nu \) such that \( \nu(T) < 0 \). Furthermore, since their image is the integers as a set, they can be considered as discrete valuations. In the spirit of the above theorem, one may expect that the set of valuations \( \text{Val}(\mathbb{Q}_\text{max}(T)) \) gives some geometric information about the projective line \( \mathbb{P}^1_{\mathbb{Q}_\text{max}} \) over \( \mathbb{Q}_\text{max} \). However, one can observe that \( X := \text{Spec}(\mathbb{Q}_\text{max}[T]) \) contains many points. For example, in \[8\], the authors proved that there is one-to-one correspondence between principle prime ideals of \( \mathbb{Q}_\text{max}[T] \) and points of \( \mathbb{Q}_\text{max} \).

Hence, the points of the projective line \( \mathbb{P}^1_{\mathbb{Q}_\text{max}} \) over \( \mathbb{Q}_\text{max} \) are a lot more than the points of \( \text{Val}(\mathbb{Q}_\text{max}(T)) \). This implies that \( \text{Val}(\mathbb{Q}_\text{max}(T)) \) can not be understood as an abstract curve analogue of \( \mathbb{P}^1_{\mathbb{Q}_\text{max}} \). However, one may notice that \( \text{Val}(\mathbb{Q}_\text{max}(T)) \) can be interpreted as the projective line \( \mathbb{P}^1 T \) over \( \F_1 \) rather than over \( \mathbb{Q}_\text{max} \). Let us first recall the construction of the projective line \( \mathbb{P}^1_{\F_1} \) over \( \F_1 \).

Example 4.2. (An example from \[8\]) Let \( C_\infty := \{\ldots, t^{-1}, 1, t, \ldots\} \) be an infinite cyclic group generated by \( t \) and let \( C_{\infty, +} := \{1, t, t^2, \ldots\}, C_{\infty, -} := \{1, t^{-1}, t^{-2}, \ldots\} \) be sub-monoids of \( C_\infty \).

Let \( U_+ := \text{Spec}(C_{\infty, +}), U_- := \text{Spec}(C_{\infty, -}) \), and \( U := \text{Spec}(C_\infty) \) (see \[8\] for the notion of monoid spectra). One defines the projective line \( \mathbb{P}^1_{\F_1} \) over \( \F_1 \) by gluing \( U_+ \) and \( U_- \) along \( U \).

The space \( U_+ \) has two points, a generic point \( c_0 \) and a closed point \( c_+ \) containing \( t \). Similarly, the space \( U_- \) has two points, a generic point \( c_0 \) and a closed point \( c_- \) containing \( t^{-1} \). Hence, the projective line \( \mathbb{P}^1_{\F_1} \) over \( \F_1 \) consists of three points \( \{c_+, c_0, c_-\} \).

Remark 4.3. We can observe that the number of closed points of \( \mathbb{P}^1_{\F_1} \) is exactly same as the number of points of \( \text{Val}(\mathbb{Q}_\text{max}(T)) = \{\nu_+, \nu_-\} \). Furthermore, \( \nu_+ \) corresponds to \( c_+ \) (which is the prime ideal containing \( t \)) and \( \nu_- \) corresponds to \( c_- \) (which is the prime ideal containing \( t^{-1} \)). In fact, one may consider that \( \nu_0 \), which is an equivalence class of a trivial valuation, corresponds to \( c_0 \) which is the prime ideal consists of \( \{1 = t^0\} \). This correspondence can
be justified since Theorem 4.1 only concerns closed points of a projective nonsingular curve. Hence, \( \text{Val}(\mathbb{Q}_{\text{max}}(T)) \) can be considered as the projective line \( \mathbb{P}^1_{\mathbb{F}_1} \) understood as an abstract curve.

**Appendix A. Basic definitions of semirings**

The following are the basic definitions of semirings in this paper.

**Definition A.1.** By a monoid we mean a non-empty set equipped with a binary operation \( \cdot \) which satisfies the following conditions: (1) \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \) \( \forall x, y, z \in M \) and (2) \( \exists ! 1_M \in M \) such that \( a \cdot 1_M = 1_M \cdot a \) \( \forall a \in M \). If \( x \cdot y = y \cdot x \) \( \forall x, y \in M \), we call \( M \) commutative. When \( M \) only satisfies the first condition then \( M \) is called a semigroup.

**Definition A.2.** A semiring \( (M, +, \cdot) \) is a non-empty set \( M \) endowed with an addition \( + \) and a multiplication \( \cdot \) such that

1. \( (M, +) \) is a commutative monoid with the neutral element 0.
2. \( (M, \cdot) \) is a monoid with the identity 1.
3. The operations \( + \) and \( \cdot \) are compatible, i.e. \( x \cdot (y + z) = x \cdot y + x \cdot z \) and \( (x + y) \cdot z = x \cdot z + y \cdot z \) \( \forall x, y, z \in M \).
4. 0 is an absorbing element, i.e. \( m \cdot 0 = 0 \cdot m = 0 \) \( \forall m \in M \).
5. \( 0 \neq 1 \).

When \( (M, \cdot) \) is commutative, then we call \( M \) a commutative semiring. If any non-zero element of \( M \) is multiplicatively invertible, then a semiring \( M \) is called a semifield.

**Definition A.3.** (cf. [8]) Let \( M, N \) be semirings. A map \( \varphi : M \rightarrow N \) is a homomorphism of semirings if: \( \forall a, b \in M \),

\[
\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(0) = 0, \quad \varphi(1) = 1.
\]

By an idempotent semiring, we mean a semiring \( M \) such that \( x + x = x \) \( \forall x \in M \).

**Example A.4.** Let \( \mathbb{B} := \{0, 1\} \). An addition is defined by: \( 1 + 1 = 1, 1 + 0 = 0 + 1 = 1, \) and \( 0 + 0 = 0 \). A multiplication is given by \( 1 \cdot 1 = 1, 1 \cdot 0 = 0, \) and \( 0 \cdot 0 = 0 \). \( \mathbb{B} \) is called a boolean semi-field and it is the initial object in the category of idempotent semirings.

**Example A.5.** The tropical semifield \( \mathbb{R}_{\text{max}} \) is \( \mathbb{R} \cup \{-\infty\} \) as a set. An addition \( \oplus \) is given by \( a \oplus b = \max\{a, b\} \) with the usual order of the real numbers and the smallest element \(-\infty \). A multiplication \( \odot \) is defined as the usual addition of \( \mathbb{R} \) as follows: \( a \odot b := a + b \), where \( + \) is the usual addition of real numbers and \( (-\infty) \odot a = a \odot (-\infty) = (-\infty) \forall a \in \mathbb{R}_{\text{max}} \). By \( \mathbb{Q}_{\text{max}}, \mathbb{Z}_{\text{max}} \) we mean the sub-semifields of \( \mathbb{R}_{\text{max}} \) with the underlying sets \( \mathbb{Q} \cup \{-\infty\}, \mathbb{Z} \cup \{-\infty\} \) respectively.

**Appendix B. Basic definitions of hyperrings**

In this section, we give a very brief introduction to the theory of hyperrings. In a word, a hyperring is an algebraic object which generalizes a ring by assuming that an addition is multi-valued.

A hyper-operation on a non-empty set \( H \) is a map \( + : H \times H \rightarrow \mathcal{P}(H)^* \), where \( \mathcal{P}(H)^* \) is the set of non-empty subsets of \( H \). In particular, \( \forall A, B \subseteq H \), we denote \( A + B := \bigcup_{a \in A, b \in B} (a + b) \).

**Definition B.1.** (cf. [3]) A canonical hypergroup \((H, +)\) is a non-empty pointed set with a hyper-operation \( + \) which satisfies the following properties:

1. \( x + y = y + x \quad \forall x, y \in H. \) (commutativity)
2. \( (x + y) + z = x + (y + z) \quad \forall x, y, z \in H. \) (associativity)
3. \( \exists ! 0 \in H \) such that \( 0 + x = x = x + 0 \quad \forall x \in H. \) (neutral element)
(4) \( \forall x \in H \ \exists! y(\vdash -x) \in H \) such that \( 0 \in x + y \). (unique inverse)
(5) \( x \in y + z \rightarrow z \in x - y \). (reversibility)

The definition of a hypergroup is more general than that of a canonical hypergroup. For example, reversibility and commutativity are not assumed. However, in this paper, by a hypergroup we always mean a canonical hypergroup.

**Definition B.2.** (cf. [3]) A hyperring \((R, +, \cdot)\) is a non-empty set \(R\) with a hyperaddition + and a usual multiplication \(\cdot\) such that \((R, +)\) is a canonical hypergroup and \((R, \cdot)\) is a monoid such that two operations are distributive. When \((R \setminus \{0\}, \cdot)\) is a group, we call \((R, +, \cdot)\) a hyperfield.

**Definition B.3.** (cf. [3]) For hyperrings \((R_1, +_1, \cdot_1)\), \((R_2, +_2, \cdot_2)\) a map \(f : R_1 \rightarrow R_2\) is called a homomorphism of hyperrings if

1. \(f(a +_1 b) \subseteq f(a) +_2 f(b) \quad \forall a, b \in R_1.\)
2. \(f(a \cdot_1 b) = f(a) \cdot_2 f(b) \quad \forall a, b \in R_1.\)
3. We call \(f\) strict if \(f(a +_1 b) = f(a) +_2 f(b) \quad \forall a, b \in R_1.\)

Let \(R\) be a hyperring. For \(x, y \in R\), if \(x + y\) consists of a single element \(z\), we let \(x + y = \{z\}\).

**Example B.4.** (cf. [3])

1. Let \(K := \{0, 1\}\). A multiplication is commutative and given by
   \[
   1 \cdot 1 = 1, \quad 0 \cdot 1 = 1 \cdot 0 = 0,
   \]
   and a commutative hyperaddition is given by
   \[
   0 + 1 = \{1\}, \quad 0 + 0 = \{0\}, \quad 1 + 1 = \{0, 1\}.
   \]
   Then \((K, +, \cdot)\) is a hyperfield called the Krasner’s hyperfield.
2. Let \(S = \{-1, 0, 1\}\). A multiplication is commutative and given by
   \[
   1 \cdot 1 = (-1) \cdot (-1) = 1, \quad (-1) \cdot 1 = (-1), \quad a \cdot 0 = 0 \quad \forall a \in S.
   \]
   A hyperaddition + is commutative and given by
   \[
   0 + 0 = 0, \quad 1 + 0 = 1 + 1 = 1, \quad (-1) + 0 = (-1) + (-1) = (-1), \quad 1 + (-1) = \{-1, 0, 1\}.
   \]
   We call \(S\) the hyperfield of signs.

One can easily obtain hyperrings from classical commutative algebras as follows.

**Theorem B.5.** (cf. [3], Proposition 2.6) Let \(A\) be a commutative ring and \(G \subseteq A^\times\) be a subgroup of the multiplicative group \(A^\times\). Then, the set \(A/G\) is a hyperring with the following operations:

1. \(xG \cdot yG := xyG \quad \forall x, y \in A.\) (multiplication)
2. \(xG + yG := \{zG | z = xa + yb \text{ for some } a, b \in G\} \quad \forall x, y \in A.\) (hyperaddition)

A hyperring which arises in this way is called a quotient hyperring.

Note that, for a field \(k\) with \(|k| \geq 3\), we can identify the Krasner’s hyperfield \(K\) with the quotient hyperring \(k/k^\times\).

In [3], Connes and Consani constructed a crucial link between a quotient hyperring and an incidence geometry. More precisely, they showed that for a hyperfield extension \(\mathbb{H}\) of the Krasner’s hyperfield \(K\), one can canonically associate a (incidence) projective geometry \(P\). Moreover, if \(P\) is Desarguesian of dimension \(\geq 2\), then there exists a unique pair \((F, K)\) of a field \(F\) and a subfield \(K \subseteq F\) such that \(\mathbb{H} \simeq F/K^\times\). This immediately implies that the classification of all finite hyperfield extensions of \(K\) is equivalent to the existence of Non-Desarguesian finite projective planes with a simply transitive abelian group \(G\) of collineations.
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