Photo-acoustic tomography in a rotating measurement setting

Guillaume Bal\textsuperscript{1} and Amir Moradifam\textsuperscript{2,3}

\textsuperscript{1}Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, USA
\textsuperscript{2}Department of Mathematics, University of California, Riverside, CA, USA

E-mail: gb2030@columbia.edu and moradifam@math.ucr.edu

Received 17 December 2015, revised 20 July 2016
Accepted for publication 15 August 2016
Published 8 September 2016

Abstract
Photo-acoustic tomography (PAT) aims to leverage the photo-acoustic coupling between optical absorption of light sources and ultrasound (US) emission to obtain high contrast reconstructions of optical parameters with the high resolution of sonic waves. Quantitative PAT often involves a two-step procedure: first the map of sonic emission is reconstructed from US boundary measurements; and second optical properties of biological tissues are evaluated. We consider here a practical measurement setting in which such a separation does not apply. We assume that the optical source and an array of ultrasonic transducers are mounted on a rotating frame (in two or three dimensions) so that the light source rotates at the same time as the US measurements are acquired. As a consequence, we no longer have the option to reconstruct a map of sonic emission corresponding to a given optical illumination. We propose here a framework where the two steps are combined into one and an absorption map is directly reconstructed from the available US measurements.

Keywords: photo-acoustic tomography (PAT), microlocal analysis, stability estimate

1. Introduction
Photo-acoustic tomography (PAT) is a novel medical imaging modality that aims to image the optical properties of biological tissues with high resolution. It combines the high contrast of optical (mostly absorption) parameters with the high resolution of ultrasound (US). As radiation propagates into tissues, a small fraction is transformed into sonic waves by the
photo-acoustic effect. These sonic waves propagate through the domain and are measured by an array of transducers at the boundary of the domain of interest \( \Omega \subset \mathbb{R}^n \), where \( n \) is spatial dimension.

Mathematically, sound propagation is modeled by the following scalar wave equation:

\[
\begin{align*}
\left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) v &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^n, \\
v_{t=0} &= H(x) \quad \text{in } \mathbb{R}^n, \\
\partial_t v|_{t=0} &= 0 \quad \text{in } \mathbb{R}^n.
\end{align*}
\]

Here the function \( c(x) \) is assumed to be a smooth function and \( H(x) \) is the amount of acoustic signal generated by the absorbed radiation of a short pulse of light propagating throughout the domain. Its expression is given by

\[
H(x) = \lambda(x) \sigma(x) u(x),
\]

where for each \( x \in \Omega \), \( u(x) \) is the density of radiation reaching point \( x \), \( \sigma(x) \) is the absorption coefficient, and \( \lambda(x) \) is the Grüneisen coefficient, which characterizes the amount of US generated by each absorbed photon.

A reasonable model for the propagation of radiation is given by the following second-order elliptic equation

\[
\begin{align*}
- \nabla \cdot (\gamma(x) \nabla u) + \sigma u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n (n \geq 2) \), \( \gamma(x) \) is a (scalar) diffusion coefficient and \( f \) describes how light enters the domain \( \Omega \).

Photoacoustic tomography aims to reconstruct \( (\gamma(x), \sigma(x)) \) as well as possibly \( \lambda(x) \) from the measurements \( v(t, x)|_{\partial \Omega} \) of the acoustic pressure leaving the domain \( \Omega \) at each point \( x \in \partial \Omega \) and each positive time \( t > 0 \); as well as for each available probing illumination \( f \).

Such reconstructions are typically done in two steps. In the first step, the sonic source \( H \) is reconstructed from the measurements \( v(t, x)|_{\partial \Omega} \). This is a well-posed problem, at least in the presence of complete data and when sound speed \( c(x) \) is known; see, e.g., [9, 11, 12, 13, 14, 15, 18, 19] and their references for experimental and theoretical works on the first, qualitative, step of PAT.

Once \( H(x) \) has been reconstructed for one or more illuminations \( f \) on \( \partial \Omega \), the second, quantitative, step of PAT allows us to obtain explicit reconstructions of \( \gamma, \sigma \), and \( \lambda \) in some cases. We refer to [2, 3, 5–7, 21] as well as their references for several works on the problem. Note that high resolution reconstructions can typically not be achieved by purely optical measurements, which are modeled by a problem that has similar stability properties to the standard electrical impedance tomography problem; see [1].

In this paper, we consider an experimental setting [4, 20] in which such a separation into first and second steps is not feasible. The reason is that the source of radiation \( f \) and the small array of transducers performing the acoustic measurements are mounted on a rotating frame. In other words, as the pressure \( v(t, x) \) is measured by a rotating array of transducers on \( \partial \Omega \), it corresponds to a radiation source \( f \) that also rotates. We thus no longer acquire a pressure \( v(t, x) \) that corresponds to a single illumination \( f \) and cannot even define a meaningful initial condition \( H(x) \). The objective of this paper is to present a mathematical framework for this experimental setting, whose main advantage is that it allows for a clear spatial separation between the light source and the array of detectors.

As we mentioned above, the qualitative and quantitative steps of PAT need to be merged into one reconstruction. In order to simplify the presentation, we assume that the Grüneisen
coefficient is known and set to 1. We also assume that the diffusion coefficient $\gamma$ is known and is normalized to 1. Under these assumptions, we present a theory for the reconstruction of the absorption coefficient $\sigma$ from knowledge of pressure measurements at the domain’s boundary in the aforementioned rotating setting.

The rest of the paper is structured as follows. The measurement setting and our main results are presented in section 2. The proof of our main result is split into a proof of a linearized version in section 3 and a proof of the full nonlinear inverse problem in section 4.

2. Measurement setting and main results

We consider the stable reconstruction of the absorption coefficient $\sigma(x)$ from multiple partial measurements under the assumption $\gamma = \lambda = 1$ inside a ball in $\mathbb{R}^n$ ($n \geq 2$). More precisely, let $B_\rho$ be the ball of radius $\rho$ in $\mathbb{R}^n$ and assume $\Omega \subseteq B_\rho$. Let $\sigma \in W_0^{1,\infty}(\Omega)$, and define

$$\Theta := \{ R_i \in SO(n), \ i = 1, 2, \ldots, m \},$$

a finite number of rotations around the origin in $\mathbb{R}^n$. Now fix $0 \leq g \in H^{1/2}(\partial B_\rho) \cap L^\infty(\partial B_\rho)$ and for $R_i \in \Theta$, define $u_i(x) = g_{R_i}(x)$ to be the unique solution of

$$\begin{align*}
- \Delta u_i + \sigma u_i &= 0 \quad \text{in } B_\rho, \\
u &= g_{R_i} \quad \text{on } \partial B_\rho,
\end{align*}$$

where $g(x) = g(R_i x), \ R_i \in \Theta$. We shall assume that $g$ is not identically zero and non-negative. In practical applications, we may envision $g$ to have a small support on $\partial B_\rho$, and certainly to be supported away from the location of the US transducers we now consider.

Let $v_i$ be the solution of the wave equation

$$\begin{align*}
\left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) v_i &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^n, \\
v_{i|t=0} &= \sigma u_i, \\
\frac{\partial v_i}{\partial t}|_{t=0} &= 0,
\end{align*}$$

where $c \geq c_0 > 0$ is the sound speed, and $c - 1$ is assumed to be supported in $\tilde{B}_\rho$. In general the time-dependent wave solution at the boundary of the ball $B_\rho$ is given by

$$\Lambda_i^\rho \sigma := \Lambda(\sigma u_i) := v_i|_{(0, \infty) \times \partial B_\rho}, \quad 1 \leq i \leq m.$$ 

In practical setting that we consider here, we have access to $v_i$ on $[0, T] \times \Gamma$ for some $\Gamma \subseteq \partial B_\rho$ and not on the whole domain $(0, \infty) \times \partial B_\rho$. Typically the support of the US transducers $\Gamma$ is relatively small and away from the support of the optical source to avoid measurement interferences.

To model this restriction, fix $\Gamma \subseteq \partial B_\rho$ and define $\Gamma_i = R_i(\Gamma)$ for $R_i \in \Theta$. Here, we are interested to know if the absorption coefficient $\sigma(x)$ can be stably determined from the finite number of rotating partial measurements

$$\chi_i \Lambda_i^\rho \sigma = \chi_i \Lambda_i u_i \sigma = \chi_i v_i|_{(0, \infty) \times \partial B_\rho}, \quad R_i \in \Theta,$$

where $\chi_i \in C_0^\infty([0, \infty) \times B_\rho)$ are cut-off functions with supp $(\chi_i) \subset [0, \infty) \times \Gamma_i$. This is a reasonably faithful model for the experimental setups described in [4, 20], to which we refer the reader for additional details.
We now introduce additional hypotheses and notation in order to state our main result; theorem 2.1 below. One of our main theoretical tools is a description of acoustic wave propagation in the domain $B_\rho$ mostly following the presentation in [15]. As in [15], and for a given $h$, we define $v$ to be the unique solution of

$$
\begin{align*}
\begin{cases}
(\partial_t^2 - c^2 \Delta) v &= 0 \text{ in } (0, T) \times \mathbb{R}^n \\
v(0,T) \times \partial B_\rho &= h, \\
v|_{t=T} &= \varphi, \\
\partial_t v|_{t=T} &= 0,
\end{cases}
\end{align*}
$$

(8)

where $\varphi$ is the harmonic extension of $h(T, \cdot)$ in $B_\rho$. We also define $Ah = v(0, \cdot)$ in $\tilde{B}_\rho$, the wave solution at time $t = 0$, and set

$$
\mathcal{G} := \{(t, y) : y \in \bigcup_{R \in \Theta} \Gamma_i, 0 < t < s(y)\},
$$

(10)

where $s(y)$ is a continuous function on $\bigcup_{R \in \Theta} \Gamma_i$ indicating how long measurements need to last at every measurement point. It is known from, e.g., the work in [15] that longer times are necessary for stability purposes than for injectivity purposes.

As in [15] and to guarantee injectivity of the measurement operator, we will assume that there exists $\mu_j$ such that

$$
\text{dist}(x, \cdot) \leq W \text{ with } \text{dist}(x, \cdot) \leq \mu_j,
$$

where $\text{dist}(x, \cdot)$ denotes the distance with respect to the metric $c^{-2} g$ ($g$ is the Euclidean metric in $\mathbb{R}^n$). This assumption is partially technical: it imposes that the measurements for one of the rotation step $j$ are taken for a sufficiently long duration that the measurement operator (mapping $\sigma$ to the available measurements) is injective.

To guarantee stability of the reconstruction of $\sigma$, we need the following stronger assumption on $s(y)$:

$$
\forall (x, \xi) \in WF(\sigma) \cap (\Omega \times \mathbb{R}^n), (\tau_\eta(x, \xi), \gamma_{\xi, \eta}(\tau_\eta(x, \xi))) \in \mathcal{G} \text{ for } \eta = + \text{ or } \eta = -, \quad (12)
$$

where $\gamma_{\xi, \eta}(\tau_\eta(x, \xi))$ are the integral curves of the corresponding Hamilton vector field (see chapter 6 in [8]) and

$$
\tau_\eta(x, \xi) = \max \{t \geq 0 : \gamma_{\xi, \eta}(\pm t) \in \tilde{B}_\rho\}.
$$

We refer to [15] for additional details on this assumption, which here simply means that any singularity of $\sigma$ at position $x \in B_\rho$ and in direction $\xi$ propagates to a singularity in the measurement set $\mathcal{G}$.

Note that since $\Omega$ is compact there exists an open set $\mathcal{G}' \subset \mathcal{G}$ such that (12) still holds. Define

$$
\Gamma_j' := \Gamma_j \cap \pi_\epsilon(\mathcal{G}'),
$$

where $\pi_\epsilon$ is the projection map. Now fix the cut-off functions $\chi_i \in C_0^\infty(\mathbb{R}^n \times (0, \infty) \times B_\rho)$ such that

$$
\text{supp}(\chi_i) \subset \Gamma_j \times (0, \infty) \cap \mathcal{G} \text{ and } \chi_i = 1 \text{ on } \Gamma_j' \times (0, \infty) \cap \mathcal{G}.
$$

The following is the main result of this paper.
Theorem 2.1. Let $B_{r}$ be the ball of radius $r$ in $\mathbb{R}^{n}$, $\Omega \subseteq B_{r}$, and $\sigma \in H^{1}_0(\Omega)$. Let $m \in \mathbb{N}$ and assume that (11) and (12) hold. If

$$\Lambda^{a}_{i}(\sigma) = \Lambda^{a}_{i}(\bar{\sigma}) \text{ on } \Gamma_{i} \times (0, \infty) \cap \mathcal{G}_{i}, \quad 1 \leq i \leq m,$$

then $\sigma = \bar{\sigma}$ in $\Omega$. Moreover, there exist a constant $\eta > 0$ such that for all $\bar{\sigma} \in W^{1,\infty}(\Omega)$ with

$$C_{\Omega}||\bar{\sigma}||_{W^{1,\infty}(\Omega)} < \eta,$$

where $C_{\Omega}$ is the best constant in the classical Poincaré inequality on $H^{1}_0(\Omega)$. Then the following stability estimate holds

$$||\sigma - \bar{\sigma}||_{H^{1}(\Omega)} \leq C \sum_{i=1}^{m} ||\chi_{i} \Lambda^{a}_{i}(\sigma) - \chi_{i} \Lambda^{a}_{i}(\bar{\sigma})||_{H^{1}(0,T) \times \partial B_{r}},$$

where $C > 0$ is independent of $\bar{\sigma}$ and $\sigma$.

Remark 2.2. Notice that $C_{\Omega}$ is small for a small region $\Omega (\Omega \subset B_{r}$ for some small $r$), and therefore the condition (13) is satisfies if the support of $\sigma$ is small in $B_{r}$. This is consistent with the experiments in [4] where the method is applied on small animals.

3. Stability of the linearized problem

In this section we study the linearized problem associated with (4)–(7). Fix $\sigma \in C^\infty_0(\Omega)$ and let $\bar{\sigma} \in C^\infty_0(\Omega)$. Assume $u_{i}, \bar{u}_{i}$ be the corresponding solutions of (4). Then

$$\left\{ \begin{array}{ll}
-\Delta u_{i} + \sigma \delta u_{i} = -u_{i} \delta \sigma & \text{in } B_{r}, \\
\delta u_{i} = 0 & \text{on } \partial B_{r},
\end{array} \right.$$  

where $\delta u_{i} = u_{i} - \bar{u}_{i}$, and $\delta \sigma = \sigma - \bar{\sigma}$. Thus

$$\delta u_{i}(x) = (2\pi)^{-n} \int \int e^{i(x-y)\xi}q(x, \xi)u_{i}(y) \delta \sigma(y)dyd\xi,$$

where $q(x, \xi) = \frac{i}{1 + \xi^{2}} \text{ mod } S^{-3}(B_{r} \times \mathbb{R}^{n})$, i.e.

$$q(x, \xi) + \frac{1}{1 + \xi^{2}} \in S^{-3}(B_{r} \times \mathbb{R}^{n}).$$

Recall that a symbol $P(x, \xi) \in S^{m}$ if

$$|D^{\alpha}_{x}D^{\beta}_{\xi} P(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi^{2}|)^{m-|\alpha|},$$

for all $\alpha, \beta \in \mathbb{N}^{n}$. See chapter 1 in [8] for more details about symbols and oscillatory integrals. Now let $\delta \bar{p}_{i}(t, x)$ be the solution of the wave equation

$$\left\{ \begin{array}{ll}
(\partial^{2}_{t} - c^{2}\Delta)\delta \bar{p}_{i}(t, x) = 0 & \text{in } (0, T) \times \mathbb{R}^{n} \\
\delta \bar{p}_{i}|_{t=0} = \sigma u_{i} - \bar{\delta} \bar{u}_{i} & \\
\partial_{t}\delta \bar{p}_{i}(t, x)|_{t=0} = 0.
\end{array} \right.$$  

Notice that

$$\sigma u_{i} - \bar{\delta} \bar{u}_{i} = \Delta (\delta u_{i}).$$
Hence modulo smooth terms
\[ \delta p_j(t, x) := (2\pi)^{-n} \sum_{\tau = \pm} \int \int e^{i\varphi_{\tau}(t, x, \xi)} a_{\tau}(t, x, \xi) [\sigma u_i(y) - \partial u_i(y)] d\xi dy \]
\[ = (2\pi)^{-n} \sum_{\tau = \pm} \int \int e^{i\varphi_{\tau}(t, x, \xi)} a_{\tau}(t, x, \xi) \Delta(\partial u_i(y)) d\xi dy, \]
where the phase function \( \varphi_{\pm} \) are homogeneous of order 1 and solve the eikonal equations
\[ \mp \partial_\tau \varphi_{\pm} = |D_x \varphi_{\pm}|, \quad \varphi_{\pm}|_{h=0} = x \cdot \xi, \]
\( a_{\pm} \) are amplitudes of order zero satisfying the corresponding transport equations (see equation (V.1.50) in [16]). Now define \( F_{\pm} \) to be the Fourier integral operators
\[ F_{\pm}(w) := (2\pi)^{-n} \int \int e^{i\varphi_{\pm}(t, x, \xi)} a_{\pm}(t, x, \xi) w(y) d\xi dy, \]
and \( P_{\pm} \) to be the pseudodifferential operator
\[ P_{\pm}(w) := (2\pi)^{-n} \int \int e^{i(x-y)\xi} a(x, y, \xi) w(y) dy d\xi, \]
where \( a(x, y, \xi) = u_i(y) \mod S^{-1}(B_p \times B_p \times \mathbb{R}^n) \). Then
\[ \delta p_j(t, x) = F_{\pm} P_{\pm}(\delta \sigma) + F P(\delta \sigma). \]
Eliminate the dependence on \( y \) in the symbol \( a(x, y, \xi) \) to get
\[ P_{\pm}(w) := (2\pi)^{-n} \int \int e^{i(x-y)\xi} b(x, \xi) w(y) dy d\xi, \]
where \( b(x, \xi) = u_i(x) \mod S^{-1}(B_p \times \mathbb{R}^n) \). The composition of the Fourier integral operator \( F_{\pm} \) with the pseudodifferential operator \( P_{\pm} \) is a Fourier integral operator with the same phase \( \varphi_{\pm} \) and amplitude \( b_{\pm}(t, x, \xi) = u_i(y) a_{\pm}(t, x, \xi) \mod S^{-1}(B_p \times \mathbb{R}^n) \) (see [16]). Therefore
\[ \delta p_j(t, x) = (2\pi)^{-n} \sum_{\tau = \pm} \int \int e^{i\varphi_{\pm}(t, x, \xi)} b_{\tau}(t, x, \xi) u_i(y) \delta \sigma d\xi dy, \quad (17) \]
where \( b_{\tau}(t, x, \xi) = a_{\tau}(t, x, \xi) \mod S^{-1}(B_p \times \mathbb{R}^n) \).

The measurements are modeled by the operator
\[ A_{\pm}(\sigma) := \sum_{i=1}^{m} \chi_i A_{\pm}^{i}(\sigma) = \sum_{i=1}^{m} \chi_i A(u_i(\sigma)). \]
Hence
\[ A(\delta \sigma) = A(\sigma) - A(\delta \sigma) = \sum_{i=1}^{m} \chi_i A(u_i(\sigma) - \delta u_i) \]
\[ = \sum_{i=1}^{m} \chi_i A(\delta u_i) + \sum_{i=1}^{m} \chi_i A(u_i(\delta \sigma)). \]
The above pseudodifferential calculus in (17) indicates that $H_{\delta \sigma} := \sum_{i=1}^{m} \chi_i \Lambda(u_i \delta \sigma)$ is the higher order term in $\Lambda(\sigma - \tilde{\sigma})$, and $L_{\delta \sigma} := \sum_{i=1}^{m} \chi_i \Lambda(\delta \tilde{u}_i)$ may be controlled by $H_{\delta \sigma}$ for small $\delta \sigma$ (see section 4). Hence we first study invertibility of the operator

$$p(\delta \sigma) := \sum_{i=1}^{m} (2\pi)^{-n} \chi_i \sum_{\tau = \pm} \int e^{i\tau (t \cdot \xi)} b_{\tau}(t, x, \xi) u_i(y) \delta \sigma dyd\xi = \sum_{i=1}^{m} \chi_i \Lambda(u_i \delta \sigma)$$

and find an approximate inverse. Let $A$ be the back-propagation operator defined in (9) and set

$$\kappa(\delta \sigma) := A \left( \sum_{i=1}^{m} \chi_i \Lambda(u_i \delta \sigma) \right).$$

**Proposition 3.1.** The operator $\kappa$ is a zero order pseudo-differential operator in a neighborhood of $\Omega$ with principal symbol

$$\frac{1}{2} \sum_{i=1}^{n} \left[ \chi_i \left( \gamma_{i,\xi}(\tau_i(x, \xi)) \right) + \chi_i \left( \gamma_{i,\xi}(\tau_i(x, \xi)) \right) \right] u_i(x).$$

Consequently if (11) and (12) hold, then $\kappa$ is an elliptic Fredholm operator on $H_0^1(\Omega)$, and there exists $C > 0$ such that

$$\|\delta \sigma\|_{H^1(\Omega)} \leq C \|\sum_{i=1}^{m} \chi_i \Lambda(u_i \delta \sigma)\|_{L^2(\Omega)}.$$  

**Proof.** First note that

$$\kappa(\delta \sigma) = \sum_{i=1}^{m} \chi_i \delta p_i(t, x).$$

It follows from theorem 3 in [15] that the principle symbol of $\kappa$ is given by (20). It follows from maximum principle that $u_i \geq \beta > 0$ in $\bar{\Omega}$ for all $1 \leq i \leq m$. Since $\tilde{\mathcal{G}}$ satisfies (12),

$$\frac{1}{2} \sum_{i=1}^{n} \left[ \chi_i \left( \gamma_{i,\xi}(\tau_i(x, \xi)) \right) + \chi_i \left( \gamma_{i,\xi}(\tau_i(x, \xi)) \right) \right] u_i(x) \geq \frac{\beta}{2} > 0.$$ 

Thus the operator $\kappa$ is elliptic, and therefore it follows from the mapping properties of the back-propagation operator $A$ (see [10, 15]) that

$$\|\delta \sigma\|_{H^1(\Omega)} \leq C \left( \left\| \sum_{i=1}^{m} \chi_i \Lambda(u_i \delta \sigma) \right\|_{L^2(\Omega)} + \|\delta \sigma\|_{L^2(\Omega)} \right).$$

Since (11) holds for some $1 \leq j \leq m$, by theorem 2 in [15], the measurement $\chi_j \Lambda(\sigma u_j)$ uniquely determines $\sigma u_j$ in $\Omega$. On the other hand, $u_j$ is the unique solution of

$$-\Delta u_j = -\sigma u_j \quad u_j |_{\partial B_0} = g_j \geq 0.$$ 

By strong maximum principle we have $u_j > 0$ in $\Omega$. Hence if (11) holds, then $\sigma = \frac{u_j}{u_j}$ is uniquely determined from a single measurement $\chi_j \Lambda(\sigma u_j)$. Consequently the operator $\sum_{i=1}^{m} \chi_i \Lambda(u_i \delta \sigma)$ is also injective. Therefore it follows from proposition V.3.1 in [17] that the estimate (21) holds for some constant $C > 0$, possibly different from the above constant. □
4. Stability of the nonlinear problem

In this section we present the proof of theorem 2.1. Let $B_\rho$ be the ball of radius $\rho$ in $\mathbb{R}^n$ and assume $\sigma$ and $\tilde{\sigma}$ are essentially bounded in $B_\rho$ and $\Omega := \text{supp}(\sigma - \tilde{\sigma}) \subseteq B_\rho$. For fixed $0 \leq g \in H^{1/2}(\partial B)$ and for $R_i \in \Theta$ define $u_t(x)$ to be the unique solution of
\[
\begin{cases}
-\Delta u_t + \sigma u_t = 0 & \text{in } B_\rho \\
u = g_t & \text{on } \partial B_\rho,
\end{cases}
\]
where $g_t(x) = g(R_t x), R_t \in \Theta$. Similarly let $\tilde{u}_t$ be the unique solution of
\[
\begin{cases}
-\Delta \tilde{u}_t + \tilde{\sigma} \tilde{u}_t = 0 & \text{in } B_\rho \\
\tilde{u}_t = g_t & \text{on } \partial B_\rho.
\end{cases}
\]
Then $\delta u_t = u_t - \tilde{u}_t$ satisfies
\[
\begin{cases}
-\Delta \delta u_t + \tilde{\sigma} \delta u_t = -u_t \delta \sigma & \text{in } B_\rho \\
\delta u_t = 0 & \text{on } \partial B_\rho.
\end{cases}
\] (22)

Proof of theorem 2.1. Since the mapping $\Lambda : H^1(\Omega) \rightarrow H^1([0, T] \times \partial B_\rho)$ is bounded (see remark 5 in [15]), it is enough to prove the theorem for $\sigma, \tilde{\sigma} \in C^\infty_0(\Omega)$. The general result will follow from a standard density argument.

Multiply (22) by $\delta u_t$ and integrate by parts and use Hölder’s inequality to get
\[
\|\nabla \delta u_t\|^2_{L^2(B_\rho)} + \int_{B_\rho} \tilde{\sigma} (\delta u_t)^2 \, dx \leq \|u_t \delta \sigma\|_{L^2(B_\rho)} \|\delta u_t\|_{L^2(B_\rho)}.
\]
By Poincaré inequality, there exists $C_\Omega$ such that
\[
\|\delta u_t\|_{L^2(B_\rho)} \leq C_\rho \|\nabla \delta u_t\|_{L^2(B_\rho)},
\]
where $C_\rho$ is dependent of $\delta u_t$. Thus we have
\[
\|\nabla \delta u_t\|_{L^2(B_\rho)} \leq C_\rho \|u_t \delta \sigma\|_{L^2(B_\rho)}.
\] (23)

Therefore
\[
\|\tilde{\sigma} \delta u_t\|_{L^2(\Omega)} \leq \|\tilde{\sigma}\|_{W^{1,\infty}(\Omega)} \|\nabla \delta u_t\|_{L^2(B_\rho)} \\
\leq C_\rho \|\tilde{\sigma}\|_{W^{1,\infty}(\Omega)} \|u_t \delta \sigma\|_{L^2(B_\rho)} \\
\leq C_\rho C_{\Omega} \|\tilde{\sigma}\|_{W^{1,\infty}(\Omega)} \|\nabla (u_t \delta \sigma)\|_{L^2(B_\rho)},
\]
where $C_\Omega$ is the best constant in the classical Poincaré inequality on $H^1_0(\Omega)$. Thus we have
\[
\|\tilde{\sigma} \delta u_t\|_{L^2(\Omega)} \leq C_\rho C_{\Omega} \|\tilde{\sigma}\|_{W^{1,\infty}(\Omega)} \|u_t \delta \sigma\|_{L^2(\Omega)}.
\] (24)

On the other hand, since the mapping $\Lambda : H^1(\Omega) \rightarrow H^1([0, T] \times \partial B_\rho)$ is bounded
\[
\left\| \sum_{i=1}^m \chi_i \Lambda(u_t \delta \sigma) \right\|_{L^p([0, T] \times \partial B_\rho)} \leq C \|\delta \sigma\|_{H^1_0},
\]
for some $\tilde{C} > 0$. Hence it follows from proposition 3.1 and (24) that

$$\left\| \sum_{i=1}^{m} \chi_i \Lambda (\sigma u_i - \tilde{\sigma} u_i) \right\|_{H^k([0,T] \times \partial \Omega)} \geq \left\| \sum_{i=1}^{m} \chi_i \Lambda (\tilde{\sigma} \delta u_i) \right\|_{H^k([0,T] \times \partial \Omega)} - \left\| \sum_{i=1}^{m} \chi_i \Lambda (\sigma \delta u_i) \right\|_{H^k([0,T] \times \partial \Omega)} \geq C \| \delta \sigma \|_{L^2(\Omega)} - \tilde{C} \| \sum_{i=1}^{m} \delta \sigma u_i \|_{L^2(\Omega)}$$

$$\geq C \| \delta \sigma \|_{L^2(\Omega)} - \tilde{C} \sum_{i=1}^{m} \| u_i \|_{H^1(\Omega)} \sum_{i=1}^{m} \| \delta \sigma \|_{L^2(\Omega)} \geq (C - mM_{\delta} \tilde{C} C_{\Omega} \| \sigma \|_{L^2(\Omega)}) \| \delta \sigma \|_{L^2(\Omega)},$$

where $M_{\delta}$ is the maximum of $g$ on $\partial \Omega$. Therefore there exists $\eta > 0$ such that if

$$C_{\Omega} \| \sigma \|_{H^{1,\infty}(\Omega)} < \eta$$

then

$$\| \sigma - \tilde{\sigma} \|_{L^2(\Omega)} \leq C^* \left\| \sum_{i=1}^{m} \chi_i \Lambda (\sigma u_i - \tilde{\sigma} u_i) \right\|_{H^k([0,T] \times \partial \Omega)},$$

for some $C^* > 0$ independent of $\tilde{\sigma}$ and $\sigma$. $\square$

**Acknowledgments**

GB’s work was supported in part by the NSF grant DMS-1408867. AM’s work is supported by a start-up grant from University of California, Riverside. The authors would like to thank the anonymous referees for careful reading of the manuscript and helpful comments.

**References**

[1] Arridge S R and Schotland J C 2010 Optical tomography: forward and inverse problems Inverse Problems 25 123010
[2] Bal G and Ren K 2011 Multiple-source quantitative photoacoustic tomography Inverse Problems 27 075003
[3] Bal G and Uhlmann G 2013 Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions Commun. Pure Appl. Math. 66 1692–52
[4] Brecht H P, Su R, Fronheiser M, Ermilov C A., S A and Oraevsky A A 2009 Whole-body three-dimensional optoacoustic tomography system for small animals J. Biomed. Opt. 14 064007
[5] Cox B T, Arridge S R and Beard P C 2009 Estimating chromophore distributions from multiwavelength photoacoustic images J. Opt. Soc. Am. A 26 443–55
[6] Cox B T, Laufer J G and Beard P C 2009 The challenges for quantitative photoacoustic imaging Proc. SPIE 7177 717713
[7] Cox B T, Laufer J G and Beard P C 2010 Quantitative photoacoustic image reconstruction using fluence dependent chromophores Biomed. Opt. Express 1 201–8
[8] Grigs A and Sjstrand J 1994 Microlocal Analysis For Differential Operators: An Introduction (Cambridge: Cambridge University Press)
[9] Kuchment P and Kunyansky L 2008 Mathematics of thermoacoustic tomography *Eur. J. Appl. Math.* 19 191–224
[10] Lasiecka I, Lions J-L and Triggiani R 1986 Nonhomogeneous boundary value problems for second order hyperbolic operators *J. Math. Pures Appl.* 65 149–92
[11] Oksanen L and Uhlmann G 2014 Photoacoustic and thermoacoustic tomography with an uncertain wave speed *Math. Res. Lett.* 21 1199–1214
[12] Qian J, Stefanov P, Uhlmann G and Zhao H-K 2011 An efficient Neumann-series based algorithm for the thermoacoustic and photoacoustic tomography with variable sound speed *SIAM J. Imaging Sci.* 4 850–83
[13] Patch S and Scherzer O 2007 Photo- and thermo- acoustic imaging *Inverse Problems* 23 S1–10
[14] Stefanov P and Uhlmann G 2013 Recovery of a source or a speed with one measurement and applications *Trans. AMS* 365 5737–58
[15] Stefanov P and Uhlmann G 2009 Thermoacoustic tomography with variable sound speed *Inverse Problems* 25 075011
[16] Treves F 1980 *Introduction to Pseudodifferential and Fourier Integral Operators*. Vol. 2. *Fourier Integral Operators (The University Series in Mathematics)* (New York: Plenum)
[17] Taylor M E 1981 *Pseudodifferential Operators (Princeton Mathematical Series* vol 34) (Princeton, NJ: Princeton University Press)
[18] Wang L V 2004 Ultrasound-mediated biophotonic imaging: a review of acousto-optical tomography and photo-acoustic tomography *J. Dis. Markers* 19 123–38
[19] Xu M and Wang L V 2006 Photoacoustic imaging in biomedicine *Rev. Sci. Instrum.* 77 041101
[20] Yang L, Nadvoreskiy V, Wang K, Emilov S, Oraevsky A and Anastasio M 2015 Effect of rotating partial illumination on image reconstruction for optoacoustic breast tomography *Proc. SPIE* 9323 93233L
[21] Zemp R J 2010 Quantitative photoacoustic tomography with multiple optical sources *Appl. Opt.* 49 3566–72