INTERSECTION COHOMOLOGY AND SEVERI’S VARIETIES

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Abstract. Let $X^{2n} \subseteq \mathbb{P}^N$ be a smooth projective variety. Consider the intersection cohomology complex of the local system $R^{2n-1} \pi_\ast \mathbb{Q}$, where $\pi$ denotes the projection from the universal hyperplane family of $X^{2n}$ to $(\mathbb{P}^N)$. We investigate the cohomology of the intersection cohomology complex $IC(R^{2n-1} \pi_\ast \mathbb{Q})$ over the points of a Severi’s variety, parametrizing nodal hypersurfaces, whose nodes impose independent conditions on the very ample linear system giving the embedding in $\mathbb{P}^N$.

Keywords: Intersection cohomology, Decomposition Theorem, Normal functions, Hodge conjecture, Severi varieties

MSC2010: Primary 14B05; Secondary 14E15, 14F05, 14F43, 14F45, 14M15, 32S20, 32S60, 58K15.

Dedicated to Ciro Ciliberto on his seventieth birthday.

1. Introduction

In the last years a great deal of work has been devoted to focusing on the deep relationship among Hodge conjecture and singularities of normal functions (compare e.g. with [21], [20] and references therein). The theory of normal functions, which dates back to Poincaré, Lefschetz and Hodge, had a renewed interest in last years after a crucial remark of Green and Griffiths [21] that the Hodge conjecture is equivalent to the existence of appropriately defined singularities for the normal function defined by means of a primitive Hodge cycle.

One of the starting points of this remark is a fundamental result of Kleiman concerning the smoothability of algebraic cycles of intermediate dimension ([27], [19, Example 15.3.2]). In light of this, R. Thomas showed inductively that the Hodge conjecture reduces to the following statement concerning middle dimensional Hodge cycles: for all even dimensional smooth complex projective varieties $(X^{2n}, \mathcal{O}(1))$ and any class $A \in H^{n,n}(X, \mathbb{C}) \cap H^{2n}(X, \mathbb{Q})$, there is a nodal hypersurface $D \subset X$ in $|\mathcal{O}_X(N)|$ for some $N$, such that the Poincaré dual of $A$ is in the image of the pushforward map $H_{2n}(D, \mathbb{Q}) \to H_{2n}(X, \mathbb{Q})$ [34].

In their fundamental work [21], Green and Griffiths further clarified the question by relating it to the singularities of normal functions. Specifically, they showed that the above hypersurface $D$ would be a singular point of the normal function associated to the Hodge class $A$, thus reducing the Hodge conjecture to the existence of such singularities.

In the paper [21], one can find various definitions of singularities of normal functions. Some of them are formalized by de Cataldo and Migliorini by means of the Decomposition Theorem [6]. More precisely, let $X = X^{2n}$ be a $2n$-dimensional smooth complex projective
variety embedded in $\mathbb{P}_C^N \equiv \mathbb{P}$ via a complete linear system $| H |$. Denote by $X^\vee$ the dual variety of $X$ and consider the universal hyperplane family

$$
X \hookrightarrow X^\vee \xrightarrow{\pi} \mathbb{P}^\vee, \quad \dim X = 2n - 1 + N.
$$

The hyperplane sections are the fibers of $\pi$:

$$
X_H := \pi^{-1}(H), \quad \dim X_H = 2n - 1.
$$

We observe that the projection from the universal hyperplane family $X \xrightarrow{\pi} \mathbb{P}^\vee$ is a proper map with equidimensional fibers and it is smooth outside the dual variety $X^\vee$.

As explained in [6, Sec. 2], the Decomposition Theorem for $\pi$ provides a non-canonical decomposition

$$
R\pi_* \mathcal{Q}_X \cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} IC(L_{ij})[-i - (2n - 1 + N)], \quad \text{in } D^b_c(\mathbb{P}^\vee),
$$

where $L_{ij}$ denotes a local system on a suitable stratum of codimension $j$ in $\mathbb{P}^\vee$. By [6], the most important summand for our purposes is the one related to $L_{00} = (R^{2n-1}\pi_* \mathcal{Q}_X)|_U$ ($U := \mathbb{P}^\vee \setminus X^\vee$) [6, 2.5]. Specifically, if one fix an intermediate primitive Hodge class $\xi$, de Cataldo and Migliorini define the singular locus of the normal function of $\xi$ as the support of the component of $\xi$ belonging to $\mathcal{H}^{-N+1} IC(R^{2n-1}\pi_* \mathcal{Q}_X)$ in the decomposition above (which is well defined in view of the perverse filtration) [6, Definition 3.3, Remark 3.4]. The main result of [6] says such a singularity is able to detect the local triviality of $\xi$.

In view of the aforementioned work of Thomas, it is particularly interesting to take a closer look at the cohomology sheaf $\mathcal{H}^{-N+1} IC(R^{2n-1}\pi_* \mathcal{Q}_X)$, especially in correspondence of nodal hypersurfaces. This paper, which is a small step in this direction, is devoted to the study of $IC(R^{2n-1}\pi_* \mathcal{Q}_X)_D$ with $D$ nodal divisor whose nodes impose independent conditions on the linear system giving the embedding $X \subseteq \mathbb{P}$. Under this hypothesis, what we are going to do is to compute the cohomology of the complex $IC(R^{2n-1}\pi_* \mathcal{Q}_X)_D$ and, above all, to give it a geometric interpretation.

The hypothesis that the nodes impose independent conditions to the hypersurfaces of a very ample linear system has long been investigated in relation to the study of Severi’s varieties, parametrizing irreducible nodal curves on a smooth algebraic surface (compare e.g. with [36], [33], [3] and references therein). In particular, in the paper [33], Sernesi proves that such a condition implies that Severi’s variety is smooth of the expected dimension, by using deformation theory.

Our first result consists in a different proof of the same result, extended to smooth algebraic varieties of even dimension (compare with Theorem 3.2 and Remark 3.3). Our approach, which is independent of deformation theory and consists in a careful local study of the conormal map, allows us to prove that the dual variety, in a neighborhood of a nodal hypersurface whose nodes impose independent conditions, is a divisor with normal crossings (cf. Theorem 3.2).

Understanding the local structure of the dual manifold allows us in Section 3 to compute the cohomology of the complex $IC(R^{2n-1}\pi_* \mathcal{Q}_X)_D$, where $D$ denotes a nodal hypersurface whose nodes pose independent conditions, and above all to give it a geometric
interpretation. More precisely, we see that the cohomology is concentrated in degrees $-N$ and $-N+1$, that $\mathcal{H}^{-N}IC(C^{2n-1}\pi_{*}\mathbb{Q}_{X})_{D}$ is naturally isomorphic to $H^{2n-1}(D)$, and that $\mathcal{H}^{-N+1}IC(C^{2n-1}\pi_{*}\mathbb{Q}_{X})_{D}$ is related with the defect of the nodes (Remark 4.6). Furthermore, we prove that the perverse filtration of $H^{2n}(D)$ is as simple as possible because it consists of only two pieces that vary in local systems over any component of Severi’s variety (Corollary 4.5).

As a by-product, we see that under the hypothesis that a hyperplane cuts $X$ in a set of nodes imposing independent conditions, the pull-back of a primitive Hodge cycle coincides with its local Green-Griffiths invariant. In particular, we get a different proof of the fact that the local Green-Griffiths invariant detects the local triviality of a Hodge cycle, in our context [6, Proposition 3.8 (ii)].

Last but not least, in Section 4 we provide several examples of even-dimensional smooth projective varieties equipped with linear systems containing nodal hypersurfaces $D$, whose nodes impose independent conditions and such that $\mathcal{H}^{-N+1}IC(C^{2n-1}\pi_{*}\mathbb{Q}_{X})_{D}$ is non-trivial.

2. Notations and basic facts

Notations 2.1. From now on, unless otherwise stated, all cohomology and intersection cohomology groups are with $\mathbb{Q}$-coefficients.

(1) For a complex algebraic variety $X$, we denote by $H^{l}(X)$ and $IH^{l}(X)$ its cohomology and intersection cohomology groups. Let $D^{b}_{c}(X)$ be the constructible derived category of sheaves of $\mathbb{Q}$-vector spaces on $X$. For a complex of sheaves $F \in D^{b}_{c}(X)$, we denote by $H^{l}(F)$ the $l$-th cohomology sheaf of $F$ and by $H^{l}(F)$ the $l$-th hypercohomology group of $F$. Let $IC_{X}$ denotes the intersection cohomology complex of $X$. If $X$ is nonsingular, we have $IC_{X} \cong \mathbb{Q}_{X}[\dim_{\mathbb{C}} X]$, where $\mathbb{Q}_{X}$ is the constant sheaf $\mathbb{Q}$ on $X$.

(2) More generally, let $i : S \rightarrow X$ be a locally closed embedding of a smooth irreducible subspace of $X$ and let $L$ be a local system on $S$. Denote by $IC_{S}(L) := i_{!}L[\dim S] \in D^{b}_{c}(X)$ the intersection cohomology complex of $L$ [14, Sec. 5.2], [5, Sec. 2.7]. It is defined as the intermediary extension of $L$, that is the unique extension of $L$ in $D^{b}_{c}(X)$ with neither subobjects nor quotients supported on $S \setminus S$.

In this paper, $X$ denotes a $2n$-dimensional smooth complex projective variety embedded in $\mathbb{P}^{N}_{\mathbb{C}} \equiv \mathbb{P}$ via a complete linear system $|H|$. Denote by $X^{\vee}$ the dual variety of $X$ and consider the universal hyperplane family $\mathcal{X} \subset X \times \mathbb{P}^{\vee}$. We have natural projections:

$$X \leftrightarrow^{q} \mathcal{X} \to^{\pi} \mathbb{P}^{\vee}, \quad \dim \mathcal{X} = 2n - 1 + N.$$ 

The hyperplane sections are the fibers of $\pi$:

$$\mathcal{X}_{H} := \pi^{-1}(H) = X \cap H, \quad \dim \mathcal{X}_{H} = 2n - 1.$$ 

Let $\text{Con}(X) \subset \mathcal{X}$ be the conormal variety of $X$:

$$\text{Con}(X) := \{ (p, H) \in X \times \mathbb{P}^{\vee} : TX_{p} \subseteq H \}, \quad \dim \text{Con}(X) = N - 1,$$
where \( TX_p \) denotes the embedded tangent space to \( X \) at \( p \). We denote by \( \pi_1 : \text{Con}(X) \rightarrow \mathbb{P}^v \) the restriction of \( \pi : \mathcal{X} \rightarrow \mathbb{P}^v \) to \( \text{Con}(X) \). We have \( X^v = \pi_1(\text{Con}(X)) \). If \( \dim X^v < N-1 \), then, for a general \( H \in X^v \), the fiber \( \pi_1(H)^{-1} \) has positive dimension. This means that the general tangent hyperplane to \( X \) is tangent to a subvariety of positive dimension. Obviously, this cannot happen if we replace \( |H| \) with a sufficiently large multiple. So we can always assume that \( X^v \) is a hypersurface of \( \mathbb{P}^v \).

If \( H \in U := \mathbb{P}^v \setminus X^v \), then \( X_H \) is smooth hence \( \pi : \pi^{-1}(U) \rightarrow U \) is a smooth fibration. This implies that the sheaf \( R^{2n-1}\pi_*\mathcal{Q}_X \) restricts to a local system on \( U \).

As explained in [6, Sec. 2], the Decomposition Theorem for \( \pi \) provides a non-canonical decomposition

\[
R\pi_*\mathcal{Q}_X \cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}} IC(L_{ij})[-i - (2n - 1 + N)], \quad \text{in } D_c^b(\mathbb{P}^v),
\]

where \( L_{ij} \) denote a local system on a suitable stratum of codimension \( j \) in \( \mathbb{P}^v \). By [6], the most important summand for the purpose of detect the primitive Hodge classes of \( X \), is the one related to \( L_{00} = (R^{2n-1}\pi_*\mathcal{Q}_X)|_U \) [6, 2.5].

The main aim of this paper is to investigate the cohomology sheaves of the complex \( IC((R^{2n-1}\pi_*\mathcal{Q}_X)|_U)[-N] \in D_c^b(\mathbb{P}^v) \), near the points corresponding to nodal divisors.

One can find different approaches to the Decomposition Theorem [1], [4], [5], [31], [37], which is a very general result but also rather implicit. On the other hand, there are many special cases for which the Decomposition Theorem admits a simplified and explicit approach. One of these is the case of varieties with isolated singularities [30], [11], [12], [18], [13]. For instance, in the work [11], one can find a simplified approach to the Decomposition Theorem for varieties with isolated singularities, in connection with the existence of a natural Gysin morphism, as defined in [9, Definition 2.3]. One of the main ingredients of these arguments is a generalization of the Leray-Hirsch theorem in a categorical framework ([35, Theorem 7.33], [10, Lemma 2.5]).

3. Local study of the dual hypersurface

Let \( X \) be a \( 2n \)-dimensional smooth complex projective variety embedded in \( \mathbb{P}^N_\mathbb{C} \equiv \mathbb{P} \) via a complete linear system \( |H| \). Assume moreover that \( X^v \), the dual variety of \( X \), is a hypersurface in \( \mathbb{P}^v \) (compare with the previous section).

With notations as above, for any \( H \in X^v \), the corresponding hyperplane cuts \( X \) in a singular divisor \( X_H := \pi^{-1}(H) = X \cap H \).

**Definition 3.1.** Let \( X^v_r \subseteq X^v \) be the locally closed Zariski subset of \( X^v \) containing all the hyperplanes \( H \) s.t. \( \text{Sing} X_H \) consists in exactly \( r \) singular ordinary double points (\( r \) nodes for short). An irreducible component of \( X^v_r \) is said to have the the expected dimension if its dimension is equal to \( N - r \) (in what follows we assume \( r \leq N \)). If \( H \in X^v_r \), then we say that \( X^v_r \) has the expected dimension at \( H \), if it belongs to a component having the expected dimension and not in components having bigger dimension.
The following result is probably well known via deformation theory, we include a slightly different proof of it in the attempt of making the present paper reasonably self-contained.

**Theorem 3.2.** Assume that $\Delta := \text{Sing}(X_H)$ consists of $\delta$ nodes. If $\Delta$ imposes independent conditions to $| H |$, that is if $H^1(\mathcal{I}_{\Delta,X}(1)) = 0$, then for every small ball $B \subseteq \mathbb{P}^\nu$ containing $H$ we have:

1. $B \cap X^\nu$ is a divisor of $B$ with normal crossings;
2. For every $r \leq \delta$, $B \cap X^\nu_r$ is non empty smooth of pure dimension $N - r$.

In particular, $X^\nu_\delta$ is smooth and has the expected dimension at $H$.

**Proof.** By [25, Proposition 3.3], the projection $\pi_1 : \text{Con}(X) \to \mathbb{P}^\nu$ is unramified at $(x_i, H)$, where the $x_i$’s are the nodes of $X_H$ (observe that $(x_i, H) \in \text{Con}(X)$, $\forall 1 \leq i \leq \delta$). Hence, $\pi_1$ provides an embedding of a suitable analytic neighborhood $U_i$ of $(x_i, H)$ in $\mathbb{P}^\nu$:

$$(x_i, H) \in U_i \subset \text{Con}(X).$$

Hence, the image $U_i := \pi_1(U_i)$ provides a branch of $X^\nu$ passing through $H$, $\forall i$ such that $1 \leq i \leq \delta$.

By [25, Lemme 4.1.2], each $U_i$ intersects $X^\nu \setminus \text{Sing}(X^\nu)$ and we can find a sequence $(p^i_n, H^i_n) \in U_i$ such that

$$H^i_n \in X^\nu \setminus \text{Sing}(X^\nu) \quad \text{and} \quad (p^i_n, H^i_n) \to (x_i, H).$$

By [23] p. 209, the embedded tangent space of $U_i$ at $H^i_n$ is $p^i_n$ viewed as a hyperplane of $\mathbb{P}^\nu$. Hence, the embedded tangent space of the branch $U_i$ at $H$ (in the following denoted by $T_{H,U_i}$) is $x_i$ (viewed as a hyperplane of $\mathbb{P}^\nu$).

On the other hand, the hypothesis $H^1(\mathcal{I}_{\Delta,X}(1)) = 0$, applied to the short exact sequence

$$0 \to \mathcal{I}_{\Delta,X}(1) \to \mathcal{O}_X(1) \to \mathcal{O}_\Delta(1) \to 0,$$

implies that $h^0(\mathcal{I}_{\Delta,X}(1)) = N + 1 - \delta$ and the nodes span a linear subspace of dimension $\delta - 1$ in $\mathbb{P}$. By our previous argument, we have

$$\bigcap_{i=1}^{\delta} T_{H,U_i} = \langle x_1, \ldots, x_\delta \rangle^\nu \subset \mathbb{P}(H^0(\mathcal{I}_{\Delta,X}(1))) \quad \text{and} \quad \dim \bigcap_{i=1}^{\delta} T_{H,U_i} = N - \delta.$$

In other words, the branches of $X^\nu$ at $H$ are independent and $X^\nu$ is a divisor with normal crossings around $H$.

Consider now a subset $I \subseteq \{1, \ldots, \delta\}$ and set

$$X^\nu_I := \bigcap_{i \in I} U_i.$$

As the branches $U_i$ have independent tangent hyperplanes at $H$, then for every sufficiently small ball $B \subseteq \mathbb{P}^\nu$ containing $H$, $B \cap X^\nu_I$ is a smooth complete intersection. Furthermore, we have

$$(3) \quad \dim B \cap X^\nu_I = N - | I | \quad \text{and} \quad \dim B \cap U_j \cap X^\nu_I = N - 1 - | I |, \ \forall j \notin I.$$

We point out that one could also deduce [23] from the factoriality of the ring of holomorphic functions defined in $B$ [22, p. 10]. Indeed, by factoriality, each branch $U_i$ is defined in $B$ by
an analytic function $f_i$. Since $X^\vee$ is a divisor with normal crossings around $H$, the sequence $f_1, \ldots, f_\delta$ is a regular sequence of analytic functions in $B$. Thus, any subset of $f_1, \ldots, f_\delta$ is regular as well and (3) follows at once.

Finally, the following locally closed subset of $X^\vee$

$$B \cap X^\vee \setminus \left( \bigcup_{j \not\in I} B \cap U_j \cap X^\vee \right)$$

is a non-empty analytic subspace of $X^\vee_{\mid I}$, with the expected dimension. □

Remark 3.3. The final part of the proof of Theorem 3.2 shows that, under the hypothesis $H^1(I_{\Delta,X}(1)) = 0$, the nodes of $H$ can be independently smoothed. This fact is usually proved by means of deformation theory (compare with [36], [33], [3]). We preferred to use a different approach here, because it is more suited to our purposes.

4. Intersection cohomology complex on Severi’s varieties

Notations 4.1. (1) Assume now that the hypotheses of Theorem 3.2 are verified. In particular, there exists a tubular neighborhood $T$ of some connected component $C$ of $X^\vee_{\delta}$ such that $T \cap X^\vee$ is a divisor with normal crossings in $T$. Set $T^0 := T \setminus (T \cap X^\vee)$. The local system $(R^{2n-1} \pi_* \mathbb{Q})_{\mid T^0}$ has a canonical extension to a holomorphic vector bundle $V$ on $T$ (see [7] and [32]).

(2) Fix $H \in U$. Combining Hard Lefschetz Theorem with Lefschetz Hyperplane Theorem we have

$$H^{2n}(\mathbb{X}_{\mid H}) \simeq H^{2n-2}(\mathbb{X}_{\mid H}) \simeq H^{2n-2}(X).$$

Hence, $R^{2n} \pi_* \mathbb{Q}_X$ is a constant system on $U$. Put $h := h^{2n-2}(X)$.

Remark 4.2. By Thom’s first isotopy lemma [20, Theorem 5.2], the family $\mathbb{X} \mid_C$ is locally trivial thus both $(R^{2n-1} \pi_* \mathbb{Q}_X) \mid_C$ and $(R^{2n} \pi_* \mathbb{Q}_X) \mid_C$ are local systems over $C$. Nevertheless, in the following Theorem we give a direct proof of this result, that we believe of independent interest.

Theorem 4.3. Assume the hypotheses of Theorem 3.2 are verified for $H \in X^\vee_{\delta}$. Fix a connected component $C$ of $X^\vee_{\delta}$ containing $H$. With notations as above, we have an isomorphism of local systems on $C$

$$\mathcal{H}^0(IC((R^{2n-1} \pi_* \mathbb{Q}_X)\mid_U)[-N]) \mid_C \cong (R^{2n-1} \pi_* \mathbb{Q}_X) \mid_C,$$

and a short exact sequence of local systems on $C$

$$0 \to \mathcal{H}^1(IC((R^{2n-1} \pi_* \mathbb{Q}_X)\mid_U)[-N]) \mid_C \to (R^{2n} \pi_* \mathbb{Q}_X) \mid_C \to \mathbb{Q}^h \to 0.$$

Furthermore, we have

$$\mathcal{H}^i(IC((R^{2n-1} \pi_* \mathbb{Q}_X)\mid_U)[-N]) \mid_C = 0, \quad \forall i \geq 2.$$
In particular, for any nodal divisor \( H \in X^\vee_\delta \), we have
\[
\begin{align*}
\bullet & \quad h^0(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H = h^{2n-1}(\mathcal{X}_H), \\
\bullet & \quad h^1(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H = h^{2n}(\mathcal{X}_H) - h, \\
\bullet & \quad h^i(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H = 0, \quad \forall i \geq 2.
\end{align*}
\]

Proof. Assume that the hypotheses of Theorem 3.2 are verified. In particular, there exists a tubular neighborhood \( T \) of \( C \) such that \( T \cap X^\vee \) is a divisor with normal crossings in \( T \). Set \( T^0 := T \setminus (T \cap X^\vee) \). The local system \((R^{2n-1}\pi_*\mathbb{Q}_X)[T^0] \) has a canonical extension to a local system \( \mathcal{V} \) on \( T \) (see \[7\] and \[32\]).

Further, in a suitable neighborhood of \( H \in C \), the equation of \( X^\vee \) has the form \( t_1 \ldots t_\delta = 0 \) and the local system \((R^{2n-1}\pi_*\mathbb{Q}_X)[T^0] \) has monodromy operators \( T_1, \ldots, T_\delta \), with \( T_i \) given by moving around the hyperplane \( t_i = 0 \). If we denote by \( N_i \) the logarithm of the monodromy operator \( T_i \), by \[2\] and \[26\], p. 322] the cohomology

\[
\mathcal{H}^i(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H
\]

of the intersection cohomology complex at \( H \in C \) can be computed as the \( i \)-th cohomology of the complex of finite-dimensional vector spaces

\[
B^p := \bigoplus_{i_1 < i_2 < \cdots < i_p} N_{i_1}N_{i_2} \cdots N_{i_p} \mathcal{V}_H,
\]

with differential acting on the summands by the rule

\[
N_{i_1} \ldots \hat{N}_{i_j} \ldots N_{i_p+1} \mathcal{V}_H \rightarrow (-1)^{j-1}N_{i_j}N_{i_1} \ldots \hat{N}_{i_j} \ldots N_{i_p+1} \mathcal{V}_H.
\]

Fix \( H \in X^\vee_\delta \). Since \( \mathcal{X}_H \) is nodal, the logarithm of the monodromy operators \( N_i \) act according to the Picard-Lefschetz formula. Furthermore, as \( \mathcal{X}_H \) has \( \delta \) ordinary double points, the vanishing spheres are disjoint to each other and we have

\[
N_iN_j = 0, \quad \text{for any } i \neq j.
\]

So the complex above is concentrated in degrees 0 and 1. We have

\[
\mathcal{H}^i(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H = 0, \quad \forall H \in X^\vee_\delta, \quad \forall i \geq 2.
\]

Thus (6) is proved.

Furthermore, we have the following exact sequence

\[
(7) \quad 0 \rightarrow \mathcal{H}^0(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H \rightarrow \mathcal{V}_H \rightarrow \bigoplus_i N_i \mathcal{V}_H \rightarrow \mathcal{H}^1(\text{IC}((R^{2n-1}\pi_*\mathbb{Q}_X)[U])[-N])_H \rightarrow 0.
\]

On the other hand, consider a hyperplane \( H_i \in U \) very near to \( H \), such that \( \mathcal{X}_{H_i} \) is smooth, and denote by \( B_i \) a small ball around the \( i \)-th node of \( \mathcal{X}_H \). By excision, we have

\[
H^i(\mathcal{X}_H, \cup_i (\mathcal{X}_H \cap B_i)) \cong H^i(\mathcal{X}_{H_i}, \cup_i (\mathcal{X}_{H_i} \cap B_i)).
\]

From the exact sequence

\[
\cdots \rightarrow H^{i-1}(\cup_i (\mathcal{X}_H \cap B_i)) \rightarrow H^i(\mathcal{X}_H, \cup_i (\mathcal{X}_H \cap B_i)) \rightarrow H^i(\mathcal{X}_H) \rightarrow H^i(\cup_i (\mathcal{X}_H \cap B_i)) \rightarrow \cdots,
\]

we get

\[
H^i(\mathcal{X}_H, \cup_i (\mathcal{X}_H \cap B_i)) \rightarrow \mathbf{Z} \rightarrow 0.
\]
and recalling the conic nature of isolated singularities of a divisor \[29\], we get
\[H^l(\mathcal{X}_H) \cong H^l(\mathcal{X}_H, \cup_i (\mathcal{X}_H \cap B_i)) \cong H^l(\mathcal{X}_H, \cup_i (\mathcal{X}_H \cap B_i)),\]
if \(l \geq 2\).
Inserting in the relative cohomology sequence for the pair \((\mathcal{X}_H, \cup_i (\mathcal{X}_H \cap B_i))\), we find the exact sequence
\[
0 \to H^{2n-1}(\mathcal{X}_H) \to H^{2n-1}(\mathcal{X}_H) \to H^{2n-1}(\cup_i (\mathcal{X}_H \cap B_i)) \cong \mathbb{Q}^\delta \to H^{2n}(\mathcal{X}_H) \to H^{2n-2}(X) \to 0,
\]
where we have also taken into account the isomorphism (compare with Notations 4.1)
\[H^{2n-2}(X) \cong H^{2n}(\mathcal{X}_H).
\]
By the Picard-Lefschetz formula, the map \(H^{2n-1}(\mathcal{X}_H) \to \mathbb{Q}^\delta\) coincides, up to some irrelevant sign, with the map \(\pi_1^* \to \bigoplus_i N_i \pi_1^*\) of the sequence \(\mathcal{X}_\mathcal{C}\). Hence, comparing \(\mathcal{X}_\mathcal{C}\) and \(\mathcal{X}_\mathcal{C}\), we find
\[
\mathcal{H}^0(\mathcal{I}C((R^{2n-1}_\pi, \mathcal{Q}_X)_{|U})[-N])_H \cong H^{2n-1}(\mathcal{X}_H) \cong (R^{2n-1}_\pi, \mathcal{Q}_X)_H
\]
and
\[
0 \to \mathcal{H}^1(\mathcal{I}C((R^{2n-1}_\pi, \mathcal{Q}_X)_{|U})[-N])_H \to (R^{2n-1}_\pi, \mathcal{Q}_X)_H \to \mathcal{O}^h \to 0.
\]
Thus, \(\mathcal{X}_\mathcal{C}\) and \(\mathcal{X}_\mathcal{C}\) are proved for any \(H \in \mathcal{C}\) and we need only to prove a similar result at the level of local systems on \(\mathcal{C}\). First of all, the fact that \(T \cap X^\vee\) is a divisor with normal crossings implies that, if we fix a small neighborhood \(B\) of \(H\) in \(\mathbb{P}^\vee\), then the local fundamental group \(\pi_1((B \setminus (B \cap X^\vee)), H) \cong \mathbb{Z}^\delta\) is independent of \(H \in X^\vee\). So, the rank of \(\mathcal{V}_{|X^\vee} \to \bigoplus_i N_i \pi_1(\mathcal{V}_{|X^\vee})\) does not change as long as we let \(H\) vary in \(\mathcal{C}\), and its kernel is a vector bundle on \(\mathcal{C}\). Furthermore, from the description of the canonical extension given e.g. in \(\text{[32, sec. 2]}\), one infers that such a kernel is the \(\mathbb{Z}^\delta\)-invariant part of the local system \((R^{2n-1}_\pi, \mathcal{Q}_X)_{|T^0} \otimes \mathbb{C}\). Moreover, since the tubular neighborhood \(T\) is homeomorphic to a fiber bundle on \(X^\vee\), the long exact sequence of homotopy groups of \(T^0\)
\[
\cdots \to \mathbb{Z}^\delta \to \pi_1(T^0, H) \to \pi_1(\mathcal{C}, H) \to 0
\]
shows that the kernel of \(\mathcal{V}_{|\mathcal{C}} \to \bigoplus_i N_i \mathcal{V}_{|\mathcal{C}}\) descends to a local system on \(\mathcal{C}\), consisting in the \(\mathbb{Z}^\delta\)-invariant part of the local system \((R^{2n-1}_\pi, \mathcal{Q}_X)_{|T^0} \otimes \mathbb{C}\), i.e. with \((R^{2n-1}_\pi, \mathcal{Q}_X)_{|X^\vee} \otimes \mathbb{C}\) by taking also into account of \(\mathcal{X}_\mathcal{C}\). This concludes the proof of \(\mathcal{X}_\mathcal{C}\).

As for the proof of \(\mathcal{X}_\mathcal{C}\), we argue in a similar way. First of all we observe that the vector space \(\bigoplus_i N_i \mathcal{V}_{|\mathcal{C}}\) is contained in the \(\mathbb{Z}^\delta\)-invariant part of the local system \((R^{2n-1}_\pi, \mathcal{Q}_X)_{|T^0} \otimes \mathbb{C}\) as well. This follows just combining \(\text{[28, p. 42]}\) (recall that the dimension of \(\mathcal{X}_H\) is odd), with the fact that the vanishing spheres are disjoint to each other. By the same argument as above, \(\bigoplus_i N_i \mathcal{V}_{|\mathcal{C}}\) is the stalk at \(H\) of a local system \(\mathcal{C}\), on which \(\pi_1(\mathcal{C}, H)\) acts by “exchanging the branches”. By \(\mathcal{X}_\mathcal{C}\), \(\mathcal{H}^1(\mathcal{I}C((R^{2n-1}_\pi, \mathcal{Q}_X)_{|U})[-N])_H\) is a local system \(\mathcal{C}\) as well and \(\mathcal{X}_\mathcal{C}\) follows by comparison with \(\mathcal{X}_\mathcal{C}\).

\[\square\]

Remark 4.4. Theorem \(\text{[43]}\) implies that both \((R^{2n-1}_\pi, \mathcal{Q}_X)_{|\mathcal{C}}\) and \((R^{2n-1}_\pi, \mathcal{Q}_X)_{|\mathcal{C}}\) are local systems on any component \(\mathcal{C}\) of \(X^\vee\) having the expected dimension. We observed that this also follows from Thom’s first isotopy lemma.
Remark 4.6. Assume now that $X$ is a projective space. Then $H^{2n-2}(X)$ is generated by the hyperplane class and $H^{2n-2}(X \cap H) \neq H^{2n-2}(X)$ iff the hypersurface $X \cap H$ has “defect” \cite{14} §6.4. Specifically, $h^1(IC((R^{2n-1}\pi_*\mathbb{Q}_X)[U][-N])_H$ coincides with the defect of $X \cap H$, which is constant on some connected component of $\mathcal{X}_d$. We note in passing that a great deal of work has been devoted to the study of the defect of projective nodal complete intersections, in relation to the number and the position of the nodes. For instance, in the paper \cite{8} one can find a construction of factorial complete intersections of codimension 2, having (asymptotically) a maximal number of nodes (see also \cite{16} and \cite{17} for other results concerning smooth projective varieties of codimension 2).

5. Examples

In this section we provide several examples of even-dimensional smooth projective varieties equipped with linear systems containing nodal hypersurfaces $D$, whose nodes impose independent conditions.
5.1. Curves in a projective surface. Let $X$ denote a smooth projective surface embedded in some projective space via a very ample divisor $H$. Let $C \subset X$ be a smooth curve in $X$ and let $K_X$ denotes the canonical divisor of $X$. If $n \gg 0$, we can assume
\[
H^1(I_{C,X}(n)) = 0, \quad \text{and} \quad |H - C| \text{ very ample on } X.
\]
Under these conditions, the general curve $R \in |H - C|$ is smooth, $\Delta := R \cap C$ is reduced and $R \cup C \sim nH$ has only ordinary double points.

**Proposition 5.1.** With notations as above, assume conditions (9) verified. Assume additionally that $C \cdot K_X < 0$. Then the set of nodes $\Delta = R \cap C$ imposes independent conditions to the linear system $|nH|$, that is to say $H^1(I_{\Delta,C}(n)) = 0$.

**Proof.** From the cohomology exact sequence deduced from the following short exact sequence
\[
0 \to I_{C,X}(n) \to I_{\Delta,X}(n) \to I_{\Delta,C}(n) \to 0,
\]
and taking into account of $H^1(I_{C,X}(n)) = 0$ (remember (9)), we see that it suffices to prove $H^1(I_{\Delta,C}(n)) = 0$. On the other hand, by adjunction we have
\[
\omega_C \cong O_C(K_X + C) \cong O_C(K_X + nH - R) \cong I_{\Delta,C}(K_X + nH).
\]
In view of the hypothesis $C \cdot K_X < 0$, we conclude at once:
\[
H^1(I_{\Delta,C}(n)) \cong H^1(\omega_C(-K_X)) \cong H^0(O_C(K_X)) = 0.
\]
\[\square\]

**Remark 5.2.** We observe that the hypothesis of Proposition above is satisfied when either $K_X$ is negative or when $C$ is an exceptional curve.

5.2. Defective hypersurfaces of $\mathbb{P}^{2n}$. Let $X = \mathbb{P}^{2n}$, $L = \mathbb{P}^m$ a linear subspace of dimension $n \geq 2$, and
\[
f = a_0 x_0 + \cdots + a_{n-1} x_{n-1} = 0
\]
a general hypersurface of degree $k \geq 2$, containing $L$. The singular locus $\Delta$ of $f = 0$ is a set of $\delta = (k-1)^n$ nodes, complete intersection of type $(k-1, \ldots, k-1)$ in $L$. Let $\mathcal{I}_\Delta$ the ideal sheaf of $\Delta$ in $L$. Then, the Koszul complex of $\Delta$ in $L$ is:
\[
0 \to \wedge^n E^* \to \wedge^{n-1} E^* \to \cdots E^* \to \mathcal{I}_\Delta \to 0,
\]
with $E = O_{\mathbb{P}^n}(k-1)^n$. We observe that $\wedge^n E^* = O_{\mathbb{P}^n}(n(1-k))$.

The set $\Delta$ imposes independent conditions to the linear system $|O_X(k)|$ if and only if $h^1(\mathbb{P}^n, \mathcal{I}_\Delta(k)) = 0$. For this to happen it suffices
\[
h^n(\mathbb{P}^n, O_{\mathbb{P}^n}(n(1-k) + k)) = 0,
\]
namely that
\[
k < \frac{2n + 1}{n - 1}.
\]
This is true when $n = 2$ e $k \leq 4$, $n = 3$ and $k \leq 3$, $n \geq 4$ e $k = 2$. 
Since a hypersurface containing a $\mathbb{P}^n$ has defect, in all these cases the nodes impose independent conditions and $\mathcal{H}^1 IC(R^{2n-1} \pi_s \mathbb{Q})_D[-N]$ is non-trivial.

5.3. Defective hypersurfaces in a complete intersection of quadrics. Let $X = X^{2n} \subset \mathbb{P}^{2n+h}$ be a general complete intersection of $h$ quadrics containing a linear subspace $L = \mathbb{P}^n$:

$$X = \begin{cases} q_1 = a_0^1 x_0 + \cdots + a_1^1 x_{n+h-1} = 0 \\ \vdots \\ q_h = a_0^h x_0 + \cdots + a_h^h x_{n+h-1} = 0. \end{cases}$$

We cut $X$ with a further general quadric $Q$, containing $L$, with equation:

$$q = \alpha_0 x_0 + \cdots + \alpha_{n+h-1} x_{n+h-1} = 0.$$ 

The singular locus $\Delta$ of $X \cap Q$ is the degeneracy locus $D_h(\phi)$ of a general morphism

$$\phi : \mathcal{O}_{\mathbb{P}^n}^{h+1} \to \mathcal{O}_{\mathbb{P}^n}(1)^{h+n}.$$ 

Thus, $\Delta$ is a set of $\delta = \binom{n+h}{n}$ nodes. The Eagon-Northcott complex of $\phi$ is:

$$0 \to S^{n-1} \mathcal{O}_{\mathbb{P}^n}^{h+1} \to S^{n-2} \mathcal{O}_{\mathbb{P}^n}^{h+1} \otimes \mathcal{O}_{\mathbb{P}^n}(1)^{h+n} \to S^{n-3} \mathcal{O}_{\mathbb{P}^n}^{h+1} \otimes \wedge^2 \mathcal{O}_{\mathbb{P}^n}(1)^{h+n} \to \ldots \to \wedge^{n-1} \mathcal{O}_{\mathbb{P}^n}(1)^{h+n} \to \mathcal{I}_{\Delta}(h + n) \to 0,$$

where $\mathcal{I}_{\Delta}$ is the ideal sheaf of $\Delta$ in $L$.

The set $\Delta$ imposes independent conditions to the linear system $|\mathcal{O}_X(2)|$ if and only if $h^1(\mathbb{P}^n, \mathcal{I}_{\Delta}(2)) = 0$. By the Eagon-Northcott complex above, since $S^{n-1} \mathcal{O}_{\mathbb{P}^n}^{h+1}$ is a direct sum of $\mathcal{O}_{\mathbb{P}^n}$, we have that $h^1(\mathbb{P}^n, \mathcal{I}_{\Delta}(2)) = 0$ as soon as

$$h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-h - n + 2)) = 0,$$

namely if

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(h - 3)) = 0.$$

The last condition is satisfied if $1 \leq h \leq 2$. When $h = 1$ we have $n + 1$ nodes in $\mathbb{P}^n$, if $h = 2$ we have $\frac{1}{2}(n + 2)(n + 1)$ nodes in $\mathbb{P}^n$.

Also in this case, since a hypersurface containing a $\mathbb{P}^n$ has defect, the nodes impose independent conditions and $\mathcal{H}^1 IC(R^{2n-1} \pi_s \mathbb{Q})_D[-N]$ is non-trivial.

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