Nonlocal Gravitational Models and Exact Solutions

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Abstract—A nonlocal gravity model with a function \( f(\Box^{-1}R) \), where \( \Box \) is the d’Alembert operator, is considered. The algorithm, allowing to reconstruct \( f(\Box^{-1}R) \), corresponding to the given Hubble parameter and the state parameter of the matter, is proposed. Using this algorithm, we find the functions \( f(\Box^{-1}R) \), corresponding to de Sitter solutions.

DOI: 10.1134/S1063779612050371

1. NONLOCAL GRAVITATIONAL MODELS

In this paper we consider nonlocal gravity models, which are describing by the action

\[
S = \int \! d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R(1 + f(\Box^{-1}R)) - 2\Lambda \right) + \mathcal{L}_{\text{matter}} \right],
\]

where \( \kappa^2 = 8\pi M_p^2 \), the Planck mass being \( M_p = 1.2 \times 10^{19} \text{ GeV} \). We use the signature \((- + + +)\), \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( \Lambda \) is the cosmological constant, \( f \) is a differentiable function, and \( \mathcal{L}_{\text{matter}} \) is the matter Lagrangian. Note that the modified gravity action (1) does not include a new dimensional parameter. This nonlocal model has a local scalar-tensor formulation. Introducing two scalar fields, \( \eta \) and \( \xi \), we can rewrite action (1) in the following local form:

\[
S = \int \! d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R(1 + f(\eta)) - \xi \Box \eta - 2\Lambda \right) + \mathcal{L}_{\text{matter}} \right].
\]

By varying the action (2) over \( \xi \), we get \( \Box \eta = R \). Substituting \( \eta = \Box^{-1}R \) into action (2), one reobtains action (1). Varying action (2) with respect to the metric tensor \( g_{\mu\nu} \), one gets

\[
\frac{1}{2} g_{\mu\nu} \left[ R(1 + f(\eta)) - \partial_\mu \xi \partial_\nu \eta - 2\Lambda \right] - R_{\mu\nu} (1 + f(\eta)) \xi \partial_\nu \eta + \frac{1}{2} \left( \partial_\mu \xi \partial_\nu \eta + \partial_\mu \eta \partial_\nu \xi \right) \]

\[- (g_{\mu\nu} \Box - \nabla_\mu \partial_\nu) (f(\eta) - \xi) + \kappa^2 T_{\text{matter} \mu\nu} = 0,
\]

where \( \nabla_\mu \) is the covariant derivative and \( T_{\text{matter} \mu\nu} \) the energy-momentum tensor of matter.

Variation of action (2) with respect to \( \eta \) yields \( \Box \xi + f'(\eta) R = 0 \), where the prime denotes derivative with respect to \( \eta \). If the scalar fields \( \eta \) and \( \xi \) depend on time only, then in the spatially flat Friedmann–Lemaître–Robertson–Walker metric with the interval\[ds^2 = -dt^2 + a^2(t) (dx_1^2 + dx_2^2 + dx_3^2),\]
eq 0, \quad \text{where the prime denotes derivative with respect to time},

Eqs. (3) are equivalent to the following ones:

\[-3H^2 (1 + \Psi) + \frac{1}{2} \dot{\xi} \dot{\eta} - 3H \Psi + \Lambda + \kappa^2 \rho_m = 0,\]

\[(2H + 3H^2)(1 + \Psi) + \frac{1}{2} \dot{\xi} \dot{\eta} + \ddot{\Psi} \]

\[+ 2H \dot{\Psi} - \Lambda + \kappa^2 \rho_m = 0,\]

where \( \Psi(t) = f(\eta(t)) - \xi(t) \), \( H = \dot{a} / a \) is the Hubble parameter, differentiation with respect to time \( t \) is denoted by a dot. For a perfect matter fluid, we have \( T_{\text{matter} \mu\nu} = \rho_m(t) g_{\mu\nu} \). The equation of state (EoS) is

\[\dot{\rho}_m = -3H (\rho_m + \rho_m).\]

The equations of motion for the scalar fields \( \eta \) and \( \xi \) are as follows

\[\ddot{\eta} + 3H \dot{\eta} = -6(\dot{H} + 2H^2),\]

\[\ddot{\xi} + 3H \dot{\xi} = 6\left( \dot{H} + 2H^2 \right) f'(\eta).\]

Note that the considered system of equations does not include the function \( \eta \), but only \( f(\eta) \), \( f'(\eta) \) and time derivatives of \( \eta \). Also, one can add a constant to \( f(\eta) \) and the same constant to \( \xi \), without changing of equations. So, \( f(\eta) \) can be determined up to a constant. For the model with action (2), contained a perfect fluid with a constant state parameter \( w_m \), a reconstruction procedure has been made [1] in terms of functions of the scale factor \( a \).

1The article is published in the original.
For the model, describing by the initial nonlocal action (1), a technique for choosing the distortion function so as to fit an arbitrary expansion history has been derived in [2].

Our goal is to demonstrate how one can reconstruct \(f(\eta)\) and get a model with the exact solution for the given Hubble parameter \(H(t)\) and the state parameter \(w_m(t) = P_m(t)/\rho_m(t)\). We show that to do this it is enough to solve only linear equations.

The algorithm is as follows:
- Assume the explicit form of \(H(t)\) and \(w_m(t)\).
- Solve (7) and get \(\rho_m(t)\).
- Solve (8) and get \(\eta(t)\).
- Subtracting Eq. (5) from Eq. (6), get a linear differential equation
  \[
  \Psi + 5H \Psi + (2 \dot{H} + 6H^2)(1 + \Psi) - 2 \Lambda + \kappa^2 (w_m - 1)\rho_m = 0,\]
where \(\Phi(\eta) = -\Phi(-4H_0 t) = \Psi + 3H_0 \Psi\). We get the following solution
  \[
  f(\eta) = \frac{1}{16H_0^2} \int \bigg[ \Phi(\tilde{\zeta}) e^{-3\tilde{\zeta}/2} \tilde{\zeta}^2 + 16C_3 H_0^2 \bigg] e^{3\tilde{\zeta}/2} d\tilde{\zeta} + C_4,\]
where \(C_3\) and \(C_4\) are arbitrary constants. We can fix \(C_4\) without loss of generality.

Following [3], we consider the matter with the state parameter \(w_m = P_m/\rho_m\) to be a constant, not equal to \(-1\). Thus, Eq. (7) has the following general solution
  \[
  \rho_m = \rho_0 e^{-3/(1 + w_m)H_0 t},\]
where \(\rho_0\) is an arbitrary constant. Equation (10) has the following general solution:
- At \(w_m \neq 0\) and \(w_m \neq -1/3\),
  \[
  \Psi_1(t) = C_1 e^{-\Lambda H_0 t} + C_2 e^{-2\Lambda H_0 t} - 1 + \frac{\Lambda}{3H_0^2} \frac{\kappa^2 \rho_0 (w_m - 1)}{3H_0^2 w_m (1 + 3w_m)} e^{-3H_0 (w_m + 1) H_0 t},\]
- At \(w_m = -1/3\),
  \[
  \Psi_2(t) = C_1 e^{-\Lambda H_0 t} + C_2 e^{-2\Lambda H_0 t} - 1 + \frac{\Lambda}{3H_0^2} + 4\kappa^2 \rho_0 e^{-2\Lambda H_0 t},\]
- At \(w_m = 0\),
  \[
  \Psi_3(t) = C_1 e^{-\Lambda H_0 t} + C_2 e^{-2\Lambda H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2 \rho_0}{H_0^2} e^{-3\Lambda H_0 t},\]
where \(C_1\) and \(C_2\) are arbitrary constants.

Substituting the explicit form of \(\Psi(t)\), we get
  \[
  f(\eta) = \frac{C_2}{4} e^{\eta/2} + C_3 e^{3\eta/2} + C_4 + \frac{\kappa^2 \rho_0 e^{3w_m + 1\eta}}{3(1 + 3w_m)H_0^2 e^{3\eta/2}};\]
at \(w_m \neq -1/3\),
  \[
  f(\eta) = \frac{C_2}{4} e^{\eta/2} + C_3 e^{3\eta/2} + C_4 + \frac{\kappa^2 \rho_0}{4H_0^2} \left(1 - \frac{1}{3} \eta\right) e^{3\eta/2};\]
at \(w_m = -1/3\),
where \(C_3\) and \(C_4\) are arbitrary constants. Note that \(C_3\) is an arbitrary constant as well.

One can see that the key ingredient of all functions \(f(\eta)\) is an exponent function. For the models with \(f(\eta)\) equal to an exponential function or a sum of exponential functions, particular de Sitter solutions have been found in [3, 4]. De Sitter solutions in the case of the exponential function \(f\) have been generalized and those stability have been analysed in [5].
3. CONCLUSIONS

Exact solutions play an important role in modern cosmological models, in particular, in nonlocal cosmological models [3–9]. The main result of this paper is the algorithm, using which one can reconstruct $f(\square^{-1} R)$, corresponding to the given Hubble parameter and the state parameter of the matter. We have found that the function $f$ corresponding to de Sitter solution is an exponential function or a sum of exponential functions. In the case of the exponential function $f$, expanding universe solutions $a \sim t^2$ have been found in [4, 8]. We plan to analyse possible forms of the corresponding function $f$ in future investigations.

The author is grateful to the organizers of the Dubna International Workshop “Supersymmetries and Quantum Symmetries” (SQS’2011) for hospitality and financial support. The author wishes to express his thanks to Emilio Elizalde, S.D. Odintsov, E.O. Pozdeeva, Richard P. Woodard, and Ying-li Zhang for useful and stimulating discussions. The work is supported in part by the RFBR grant 11-01-00894, a grants of the Russian Ministry of Education and Science NSh-4142.2010.2 and NSh-3920.2012.2, and by contract CPAN10-PD12 (ICE, Barcelona, Spain).

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