Laughlin States Change Under Large Geometry Deformations and Imaginary Time Hamiltonian Dynamics

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Abstract: We study the dependence of the Laughlin states on the geometry of the sphere and the plane, for one-parameter Mabuchi geodesic families of curved metrics with Hamiltonian $S^1$-symmetry. For geodesics associated with convex functions of the symmetry generator, as the geodesic time goes to infinity, the geometry of the sphere becomes that of a thin cigar collapsing to a line and the Laughlin states become concentrated on a discrete set of $S^1$-orbits, corresponding to Bohr–Sommerfeld orbits of geometric quantization.

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1. Introduction

The importance of the surface geometry dependence of the Fractional Quantum Hall effect has been emphasized and studied by several authors [1–8]. In the present paper, we consider a deformation of the background metric of an oriented surface with an Hamiltonian $S^1$-symmetry and with a uniform magnetic field with respect to the area 2-form. This endows the configuration space of the system with the structure of an effective phase space whose Kähler quantization determines the lowest Landau level (LLL). Natural families of deformations of the physical metric are provided by paths which are geodesic relative to the Mabuchi metric on the space of Kähler structures [9, 10]. The latter are generated by imaginary time Hamiltonian flows [11].

We use methods from geometric quantization to study the evolution, along the deformation, of the single-particle and many-particle states, in particular for the case of Laughlin states, along the geodesic family. This is done via a generalized coherent state transform (GCST) [12], which lifts the Mabuchi geodesics to the bundle of quantum states. In the flat case with quadratic Hamiltonians, these transforms are projectively unitary and coincide with the parallel transport with respect to the Knizhnik–Zamolodchikov–Hitchin (KZH) connection [13,14]. In the case of cotangent bundles of Lie groups of compact type, these transforms are also, for appropriate choices of Hamiltonians, unitary transforms which correspond to the classical Segal–Bargmann transform and to the Hall coherent state transform [15,16].

The Mabuchi geodesics on the space of Kähler metrics on $\Sigma$ which we consider correspond to Hamiltonian motion in imaginary time generated by $H = x^2/2$, where $x$ is the function generating the Hamiltonian $S^1$-symmetry. The geodesics exist for infinite (geodesic=imaginary-Hamiltonian) time $s$. As $s \to +\infty$, the lengths of the $S^1$-orbits converge to zero and the scalar curvature converges, in the case of the plane and the sphere, to a $\delta$-function supported on the $S^1$-fixed points. In both cases, as the geometry becomes more and more deformed, the holomorphic fractional Laughlin wave functions converge to distributional wave functions supported on the “Bohr–Sommerfeld leaves”, corresponding to integer values of $x$.

This asymptotic behaviour will have important consequences for the density profiles for extreme deformations of the geometry. In particular, this geometric quantization inspired evolution of single-particle and many-particle LLL states leads to qualitatively different asymptotic features in the density profiles in the case of fractional filling factor. The study of quantum Hall states in different geometries has had a long history, in particular in [17,18], Haldane considered the case of the sphere with the round metric and the torus with a flat metric, respectively; in [19] the cylinder with standard metric was also considered. Recently, the change of the quantum Hall states with respect to small deformations of the geometry has received a lot of attention because it is deemed to be relevant for a full diagnosis of topological order [20]. In particular, in [21] the effect of deforming the geometry of the cylinder was considered. Klevtsov [1,2,7,22] has given a general prescription for constructing Laughlin states in general Riemann surfaces which takes into account holomorphic data on the surface. When changing the geometry, this prescription corresponds to taking into account analytic continuation from one complex structure to another. In the language of geometric quantization, this is implemented by evolution in imaginary time under the prequantum operator. The latter is only part of the full GCST, and, although it correctly provides transport along a path of geometries, it fails to correct the norms of the 1-particle states as given by the evolution with respect to the quantum operator. This will lead to different asymptotic density profiles in the two approaches—which would be interesting to probe experimentally.
In this work, we consider the deformation of LLL states for surfaces with a toric structure, that is with a Kähler structure invariant under an $S^1$–action. More concretely, we consider the case of the sphere and the plane with $S^1$–invariant Riemannian metric, and study the associated particle densities for both the integer quantum Hall effect and the filling fraction $\nu = 1/3$ generalized Laughlin states, along Hamiltonian evolution in imaginary time $\tau = -is$, $s > 0$. As $s$ becomes large, the densities also concentrate on the Bohr–Sommerfeld leaves, and have scalar coefficients that depend on both the combinatorial properties of the Laughlin states and on the Kähler metric geometry of the surface.

The manuscript is organized follows. In Sect. 2, we review aspects of the Kähler geometry of toric manifolds, Hamiltonian flows in complex time and geometric quantization. In Sect. 3, we study the evolution of one-particle states along families of deformed geometries with $S^1$-symmetry. The corresponding Laughlin states are studied in Sect. 4 and in Sect. 5 we present the density profiles for the deformed geometries. We end with some conclusions in Sect. 6.

2. Preliminaries

In this section, we review basic material from the symplectic and complex geometry of toric Kähler surfaces and of their geometric quantization. We also review the application of Hamiltonian flows in imaginary time to the deformation of toric Kähler structures. This will provide the main ingredients to describe the one-particle states for the quantum Hall effect in deformed geometries and later on also the associated many-particle Laughlin states.

2.1. Toric Kähler surfaces. Recall that a toric Kähler surface is a connected 2-dimensional Kähler manifold $(M, \omega, J, \gamma)$, with a compatible triple consisting of a symplectic structure $\omega$, complex structure $J$ and Riemannian metric $\gamma$, such that the three structures are invariant under an Hamiltonian $S^1$-action on $M$.

The image of the resulting moment map $\mu: M \to P$ is always a Delzant polytope (see, for instance, [23]). In the two-dimensional case, that we are considering, the only three possibilities for $P$ are the following:

(i) Closed interval $P = [a, b] \subset \mathbb{R}$, $(a < b)$, if $M = \mathbb{CP}^1 \cong S^2$.
(ii) Half–line: $P = \mathbb{R}_{\geq a} = [a, \infty)$, if $M = \mathbb{C}$.
(iii) $P = \mathbb{R}$, if $M = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

There are two natural systems of coordinates, that we recall below, and that allow one to encode the toric Kähler structure of the surface by means of a convex function on the polytope. First, one has the toric holomorphic coordinate. This is given by $w = e^z$, $z = y + i\theta$, on the dense subset $M^0 = \mu^{-1}(\text{int}(P)) \cong \mathbb{C}^*$ in which the $S^1$-action takes the form [23,24]

$$e^{it} \cdot (e^{y+i\theta}) = e^{y+it\theta}, \quad e^{it} \in S^1, \quad y \in \text{int}(P), \quad 0 \leq \theta < 2\pi. \quad (2.1)$$

Furthermore, since $\omega$ is $S^1$-invariant, one can choose the Kähler potential $\kappa$ over $M^0$ to also be $S^1$-invariant, i.e. $\kappa = \kappa(y)$, implying that

$$\omega|_{M^0} = i\partial\overline{\partial}\kappa = i \frac{\partial^2 \kappa}{\partial z \partial \overline{z}} dz \wedge d\overline{z} = \frac{i}{4} \kappa'' dz \wedge d\overline{z} = \frac{1}{2} \kappa'' dy \wedge d\theta \quad (2.2)$$
and
\[ \gamma|_{M^0} = \omega(\cdot, J \cdot)|_{M^0} = \kappa'' dy^2, \quad (2.3) \]
or, in matrix form,
\[ \gamma = \begin{bmatrix} \kappa'' & 0 \\ 0 & \kappa'' \end{bmatrix}. \quad (2.4) \]
All possible Kähler toric structures with fixed complex structure are captured by the definition of this strictly convex function \( \kappa \).

We now start by considering the above coordinates \((y, \theta)\): \(M^0 \cong \mathbb{R} \times S^1\). Since \( \kappa \) is strictly convex, one can define, via Legendre transformation,
\[ x = \frac{\partial \kappa}{\partial y}. \quad (2.5) \]
Obviously, we have
\[ e^{it} \cdot (x, \theta) = (x, \theta + t), \quad e^{it} \in S^1 \quad (2.6) \]
and, in fact, it can be shown that, in coordinates \((x, \theta)\), known as action-angle coordinates, \(\omega\) takes the simple form
\[ \omega|_{M^0} = dx \wedge d\theta. \quad (2.7) \]
From this, it becomes clear that \(x\) is actually a moment map for this action and thus we get a diffeomorphism
\[ M^0 \cong \text{int}(P) \times \mathbb{T}^n. \quad (2.8) \]
Furthermore, we can define \(g : \text{int}(P) \to \mathbb{R}\) as the Legendre transform of \(\kappa\), which is known as the symplectic potential. Then
\[ y = \frac{\partial g}{\partial x} \Rightarrow \frac{\partial}{\partial x} = g'' \frac{\partial}{\partial y}, \quad (2.9) \]
and, applying this change of basis to \(J\), we get
\[ J = \begin{bmatrix} (g'')^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} g'' & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & (g'')^{-1} \\ -g'' & 0 \end{bmatrix}. \quad (2.10) \]
Consequently, the Kähler metric becomes, in matrix form,
\[ \gamma = \begin{bmatrix} g'' & 0 \\ 0 & (g'')^{-1} \end{bmatrix}. \quad (2.11) \]
Furthermore, from the general theory \([23,24]\), it is known that for every \(P\) of the form
\[ P = \{x \in \mathbb{R}^n : \ell_i(x) = \langle v_i, x \rangle + a_i \geq 0, \quad i = 1, \ldots, d\}, \]
where the \(v_i\)’s are the exterior pointing normals to the \(d\) facets of \(P\), there is a “canonical” symplectic potential \(g_P\) with fixed singular behaviour on its boundaries and given by
\[ g_P(x) = \frac{1}{2} \sum_{i=1}^{d} \ell_i(x) \log \ell_i(x), \quad (2.12) \]
describing a toric Kähler metric on a Kähler manifold of dimension $2n$.\footnote{Note that the number, $d$, of facets of $P$ can be different for different underlying 2n-dimensional toric manifolds. See, for instance, the cases for $n = 1$ described above.} For the case of toric Kähler surfaces considered in this work, we have $n = 1$ and we will use this “canonical” symplectic potential as defining the unperturbed geometry for the three different possible choices of $P$.

From \cite{24}, it follows that one can deform the geometry by adding to a symplectic potential any function $H$, smooth on the whole of $P$

$$g \mapsto g + H,$$

such that the sum remains convex. These deformations preserve the symplectic form since $\omega$ does not depend on $g$ (cf. Eq. (2.7)). These are the type of deformations we will find in the following discussion by considering Hamiltonian flows in imaginary time.

\subsection*{2.2. Hamiltonian flows in imaginary time.} First, let us recall a few facts from the theory of flows in complex time. Let $(M, \omega, J, \gamma)$ be a compact Kähler manifold, where $\omega$ is the symplectic structure, $J$ is the complex structure and $\gamma$ is the Riemannian metric and such that all the three structures are real analytic. Let $H$ be a real analytic Hamiltonian function on $M$ and denote by $X_H$ the associated Hamiltonian vector field, i.e., $\iota_{X_H} \omega = dH$.

It follows from the theory of ODEs that the flow of $X_H$, $\phi_{X_H}^t$, is also real analytic. Additionally, we have that, within regions of convergence \cite{25},

$$f \in C^\omega(M) \implies (\phi_{X_H}^t)^* f = e^{tX_H} f = \sum_{k=0}^{\infty} \frac{X_H^k(f)}{k!} t^k,$$

which, for some $T > 0$ and $\tau \in \mathbb{C}, |\tau| < T$, can we analytically continue to get

$$e^{\tau X_H} f = \sum_{k=0}^{\infty} \frac{X_H^k(f)}{k!} \tau^k \in C^\omega(M) \otimes \mathbb{C}. \quad (2.13)$$

We can, in particular, apply (2.13) to local $J$-holomorphic coordinates on $M$, such that for small enough $\tau$, one obtains new local functions

$$z_j^\tau = e^{\tau X_H} z_j. \quad (2.14)$$

It is shown in \cite{11} that this local construction in fact globalizes to produce a well-defined new global complex structure $J_\tau$, such that $z_j^\tau$ are local $J_\tau$-holomorphic coordinates, and that it defines a diffeomorphism $\phi_{X_H}^\tau : M \to M$, such that $(\phi_{X_H}^\tau)^* J_\tau = J$. Moreover, one obtains a new global Kähler structure $(M, \omega, J_\tau, \gamma_\tau)$ where the original symplectic form is unchanged. Therefore, complex time Hamiltonian flows give us a systematic way of deforming Kähler structures.

We now apply the concepts discussed above specifically to the case of 2-dimensional Kähler toric manifolds. For that, let us consider a Kähler toric surface $(M, \omega, J, P)$ as described in Sect. 2.1, with toric holomorphic coordinate $w = e^z$, with $z = y + i\theta$ and action-angle coordinates $(x, \theta)$, and a Hamiltonian $H = H(x)$ on $M$. Then

$$X_H = -H'(x) \frac{\partial}{\partial \theta}. \quad (2.15)$$
and applying (2.14) to the toric holomorphic coordinate, one obtains

\[
ws = \left( e^{\tau X_H \cdot w} \right) \bigg|_{\tau = -is} = \left( e^{y + i(\theta - \tau H'(x))} \right) \bigg|_{\tau = -is} = e^{y + sH'(x) + i\theta}
\]

(2.16)

which also motivates the definitions \( y_s = y + sH'(x) \) and \( z_s = y_s + i\theta \).

Since we are keeping the symplectic structure fixed, the same action-angle coordinates are applicable and thus

\[
y_s = \frac{\partial g_s}{\partial x} \iff y + sH'(x) = \frac{\partial g_s}{\partial x} \iff g_s = g + s(H(x) + c)
\]

(2.17)

where \( c \in \mathbb{R} \) is an integration constant, which we take to be zero since it does not alter the system. Consequently, the deformed metric has the form [26,27]

\[
\gamma_s(x) = g_s''(x)dx^2 + \frac{1}{g_s''(x)}d\theta^2
\]

(2.18)

with scalar curvature given by Abreu’s formula [27],

\[
Sc(x) = -\left( \frac{1}{g_s''(x)} \right)''
\]

(2.19)

As for the Kähler potential, \(^2\) since it is the Legendre dual of \( g \), it is given by \( \kappa_s = xy_s - g_s \).

2.3. Geometric quantization. Given a symplectic manifold \((M, \omega)\), describing the phase space of some classical system, the mathematical problem of finding its quantization can be addressed in a geometric framework known as geometric quantization. Here, one assumes that there exists a complex line bundle \( L \rightarrow M \), equipped with a compatible connection \( \nabla \) and Hermitian structure such that the curvature of \( \nabla \) is \(-i\hbar \omega\). \( L \) is known as the pre-quantum line bundle. Note that this places an integrality condition \( \left[ \frac{\omega}{2\pi \hbar} \right] \in H^2(M, \mathbb{Z}) \).

Geometric quantization produces a Hilbert space of quantum states, \( \mathcal{H}_P \), depending on the choice of a polarization: an integrable Lagrangian distribution \( P \subset TM \otimes \mathbb{C} \). The Hilbert space is then

\[
\mathcal{H}_P = \{ s \in \Gamma(M, L) : \nabla_X s = 0, \text{ for any } X \in \Gamma(M, P) \text{ and } ||s||^2_{L^2} < \infty \},
\]

where the completion is with respect to the \( L^2 \)-norm. When \((M, \omega)\) is Kähler, a natural choice is to take \( P \) to be the holomorphic tangent bundle of \( M, P = T^{(1,0)}M \), so that one obtains \( H_P = H^0(M, L) \).

Given \( f \in C^\infty(M) \), its prequantum operator is a complex-linear map \( Q_{\text{pre}}(f) : \Gamma(M, L) \rightarrow \Gamma(M, L) \) defined by

\[
Q_{\text{pre}}(f) = i\hbar \nabla_X f + f.
\]

(2.20)

Note that, for general \( f \), \( Q_{\text{pre}}(f) \) will not preserve the space of quantum states \( \mathcal{H}_P \). This will happen if \( [X_f, P] \subset P \).

\(^2\) \( g_s \) and \( \kappa_s \) denote the potentials that induce the new structures on the manifold, and do not correspond to the changes of \( g \) and \( \kappa \) with respect to imaginary time Hamiltonian flows.
Above, we have described how Hamiltonian flows in imaginary time $\tau$ can be used to deform the complex structure of a Kähler manifold. As the complex structure changes along the deformation, we will obtain a family of Kähler polarizations $P_\tau$ and the corresponding Hilbert spaces of quantum states, $\mathcal{H}_{P_\tau}$, will vary in the space of smooth sections of $L, \Gamma(M, L)$. To relate these Hilbert spaces to each other, we need, given a complex time Hamiltonian flow $\phi^X_{\tau}$ as described in Sect. 2.2, a way to lift this deformation to the system quantized via geometric quantization. Since the initial polarization $P$ changes to a polarization $P_\tau$, we will naturally get a map $U_\tau: \mathcal{H}_{P} \rightarrow \mathcal{H}_{P_\tau}$.

In this work we considered evolutions in imaginary time $\tau = -is$ and the appropriate choice turns out to be a GCST of the form (see [12])

$$U_s = \left(e^{\frac{i}{\hbar}Q_{\text{pre}}(H)} e^{-\frac{i}{\hbar}Q(H)}\right)|_{\tau = -is},$$

(2.21)

where $Q(H)$ is an appropriate quantum operator for $H$ that we will be describe explicitly, for the case of toric deformations of the initial toric Kähler structure, below. Note that the operator

$$U_{\text{pre}} = e^{\frac{i}{\hbar}Q_{\text{pre}}(H)}$$

(2.22)

is a generalization of time evolution in quantum mechanics and we know it does not preserve the Hilbert space of polarized sections, but rather maps $J$-polarized states to $J_\tau$-polarized states (cf. Eq.(3.27)) in a natural way. The Laughlin states defined in [7] for some geometry deformation generated from an initial geometry by imaginary time Hamiltonian flow, can be obtained by the action of this operator on the Laughlin states for the initial Kähler geometry.

However, this evolution by prequantization of the Hamiltonian is highly non-unitary. The inclusion of $e^{-\frac{i}{\hbar}Q(H)}$, which we know to preserve the Hilbert space, “reverts” the effect of the prequantum evolution operator on quantum states without preventing the geometry deformation of our system (Eq. (3.28)) and restores unitarity asymptotically. We will analyse closely the non-unitarity of (2.22), as well as the consequences of taking the above GCST instead, in Proposition 1 and also Section in 5 where particle density profiles will be presented.

3. The Lowest Landau Level for Deformed Geometries

We consider charged particles living on a Kähler surface $(M, \omega, J, \gamma)$ and subject to a uniform external magnetic field. By a uniform magnetic field we mean that the Faraday 2-form $F$ is proportional to the area form of the surface, i.e., $F = B\omega$. The curvature of $L$ is $\Omega = -i\frac{qF}{\hbar} = -i\frac{q\omega}{\hbar_{\text{eff}}}$, where $q$ is the charge of the carriers and $B$ is the magnetic field, and where we introduced an effective Planck’s constant, denoted $\hbar_{\text{eff}}$, associated to the configuration space which, effectively, behaves like a phase space in the physics of the lowest Landau level. Thus, $\hbar_{\text{eff}} = \hbar/(qB) := \ell_B$ where $\ell_B$ is the so-called magnetic length. Below, in the case of the sphere, the dimension of the one-particle Hilbert space is $N = h^0(L) = c_1(L) + 1$ where $c_1(L) = qBA/\hbar$, in which $A$ denotes the area of the surface. Thus, $N = 1 + qBA/\hbar$. The single-particle Hilbert space of the quantum theory is described by the square integrable sections of the electromagnetic line bundle $L \rightarrow M$, over which the electromagnetic gauge field describes a unitary connection.
whose curvature 2-form is given by $-\frac{i}{\hbar_{\text{eff}}} \omega$. The single-particle Hamiltonian is described by the Bochner Laplacian, whose groundstate subspace, known in the physics literature as the lowest Landau level (LLL), is described by $H^0(M, L)$ [22]. It is now clear the relation to geometric quantization: the lowest Landau level is nothing but the Hilbert space of quantum states for the Kähler quantization of $M$.

In this section, we describe the LLLs on the sphere and the plane with (toric) geometries deformed by Hamiltonian flow in imaginary time. We begin with the round sphere $S^2$ in Sect. 3.1, for which the process is described in detail. We determine explicit variations of the structure, along with holomorphic states and Hermitian product.

In Sect. 3.2, we make the analogy with the case of the plane, since we will also compute evolution of density profiles for this system in Sect. 5. For more details on deformations of the plane see [28].

In Sect. 3.3, we explicitly apply GCST to the states described previously, namely in the limit $s \to \infty$. We also compare this evolution with the one obtained by using just the prequantum evolution operator of Eq. (2.22), as this is the evolution operator that produces the Laughlin states used by Klevtsov (see [7]).

3.1. One particle states on the deformed sphere. We start with a first analysis of the sphere $M = S^2 \cong \mathbb{C}P^1$.

As mentioned in Sect. 2.1, the corresponding polytope in this case is $P = [a, b] \subset \mathbb{R}$, where $a < b$. Right away, we can use (2.7) to compute the symplectic area

$$\int_M \omega = 2\pi (b - a).$$

One immediately concludes that, for the system to be quantizable, we must have

$$\frac{b - a}{\hbar_{\text{eff}}} \in \mathbb{Z}. \tag{3.2}$$

For simplicity of computations, we will now assume Planck units so that $\hbar_{\text{eff}} = 1$ and hence $P = [a, a + N]$, for some $N \in \mathbb{N}$. The symplectic potential (2.12) then reads

$$g_P(x) = \frac{1}{2} ((x - a) \log (x - a) + (N + a - x) \log (N + a - x)) \tag{3.3}$$

Furthermore, as we have seen in Sect. 2, the toric holomorphic coordinates are obtained from the action angle ones, by Legendre transform, hence we get

$$x \mapsto y = \frac{\partial g_P}{\partial x} = \frac{1}{2} \log \left(\frac{x - a}{N + a - x}\right) \implies w = e^{y + i\theta} = \sqrt{\frac{x - a}{N + a - x}} e^{i\theta} \tag{3.4}$$

Now, if we consider an imaginary time Hamiltonian flow induced by $H(x) = \frac{1}{2} x^2$, then the results of Eqs. (2.16), (2.17) and (2.18) from Sect. 2 yield

$$y_s = y + sx \tag{3.5}$$

$$g_s = g_P + s \frac{x^2}{2} \tag{3.6}$$

$$w_s = \sqrt{\frac{x - a}{N + a - x}} e^{sx + i\theta} \tag{3.7}$$
\[ \gamma_s = \left( \frac{1}{2} \frac{N}{(x-a)(N+a-x)} + s \right) dx^2 + \left( \frac{1}{2} \frac{N}{(x-a)(N+a-x)} + s \right)^{-1} d\theta^2 \]

(3.8)

And the scalar curvature (2.19) becomes

\[ Sc(x) = -\left( \frac{1}{\frac{1}{2} \frac{N}{(x-a)(N+a-x)} + s} \right)^{\prime\prime}, \]

(3.9)

which is constant at \( s = 0 \) and concentrates around the poles as \( s \to \infty \) (see [29] for more details). Writing the induced metric as

\[ \gamma_s(x) = g_s'' \, dx^2 + \frac{1}{g_s''} \, d\theta^2 = \left( \frac{1}{2} \frac{u}{(N-u)} + s \right) \, du^2 + \left( \frac{1}{2} \frac{N}{(N-u)} + s \right)^{-1} \, d\theta^2, \]

(3.10)

where we made the substitution \( u = x - a \), we see that, if \( s = 0 \), it is the Fubini-Study metric and, in general, it does not depend on the choice of \( a \). Furthermore, if \( x \) is a moment map, then so is \( x - a \). This translates the polytope by \( a \), but (3.10) shows that (assuming we are using the canonical symplectic potential) the Kähler structure remains entirely equivalent. We can also see that the choice of \( N \) amounts to a simple rescaling of the sphere and the imaginary time unit. It is then common practice to let these (so far irrelevant) parameters denote a specific choice of line bundle \( L \) and basis of holomorphic sections in a very natural way. We now present this construction (more details and the generalization for \( n \)-dimensional Delzant polytopes can be found in [30]).

First, consider a formal sum

\[ D^L := \lambda^L_1 D_1 + \lambda^L_2 D_2 \]

(3.11)

where \( D_1 = x^{-1}(a) \) and \( D_2 = x^{-1}(a+N) \). By taking \( \lambda^L_1, \lambda^L_2 \in \mathbb{Z} \), we have what is called a toric divisor in algebraic geometry. It is a notation that is used represent sections of some line bundle \( L \) on \( M \) that can be given in local holomorphic coordinates around each \( D_j \) by \( z^j_s \). This proves to be very convenient, as it can be shown that there exists only one such line bundle and only one section \( \sigma_{D^L} \) of that line bundle, up to multiplication by a constant, with divisor \( D^L \). We will now construct \( L \) explicitly.

The toric holomorphic coordinates \( w_s \) given in (3.4) cover \( S^2 \setminus \{D_1, D_2\} \) and can be trivially extended to \( U_1 := S^2 \setminus \{D_2\} \). These are simply stereographical projection coordinates and thus the atlas is completed by considering a similar map \( \tilde{w}_s : U_2 \to \mathbb{C} \), with \( U_2 := S^2 \setminus \{D_1\} \) and \( w = \tilde{w}^{-1} \) in the intersection \( U_1 \cap U_2 \). If the section \( \sigma_{D^L} \) is given by \( w^L_{s_1} \) in \( U_1 \) and \( \tilde{w}^L_{s_2} \) in \( U_2 \), then

\[ \tilde{w}^L_{s_2} = f_{1,2} w^L_{s_1} \iff f_{1,2} = w^{-L_{s_{1}}^{L_{1}} L_{s_{2}}^{L_{2}}} \]

(3.12)

in \( U_1 \cap U_2 \), where \( f_{1,2} \) is the corresponding transition function for \( L \), which uniquely defines the line bundle up to equivalence. Evidently, the space of holomorphic sections of this line bundle is given by \( H^0(M, L) := \text{span}_\mathbb{C} \{ w^m_s \sigma_{D^L} : \text{div}(w^m_s \sigma_{D^L}) \geq 0 \} \), with

\[ \text{div}(w^m_s \sigma_{D^L}) \geq 0 \iff m + L_1^s \geq 0 \text{ and } -m + L_2^s \geq 0. \]

(3.13)
where \( \text{div} \) denotes the divisor of a given section. Thus we see that there is a bijection between this \( S^1 \)-equivariant basis of \( H^0(\mathbb{C}P^1, L) \) and the integral points (we want \( m \in \mathbb{Z} \) for the sections to be single-valued) of the Delzant polytope \( P_L = [-\lambda_1^L, \lambda_2^L] \).

When considering half form however, it becomes appropriate to have \( \lambda_j^L \in \mathbb{Z} + \frac{1}{2} \). This is because, since \( w_s = e^{\zeta s} \), we have that \( \text{div}(\sqrt{d\zeta_s}) = \text{div}\left(\sqrt{\frac{dw_s}{w_s}}\right) = -\frac{1}{2}(D_1^L + D_2^L) \), meaning that \( \sigma_{D^L} \otimes \sqrt{d\zeta_s} \) is single-valued only when the \( \lambda_j^L \)'s are half-integers. And although the rest of the argument is still valid, (3.13) now becomes

\[
\text{div}\left(w_s^m \sigma_{D^L} \otimes \sqrt{d\zeta_s}\right) \geq 0 \iff m + \lambda_1^L - \frac{1}{2} \geq 0 \text{ and } -m + \lambda_2^L - \frac{1}{2} \geq 0. \tag{3.14}
\]

and so once again we obtain a bijection with integral points of the Delzant polytope \( P_L = [-\lambda_1^L, \lambda_2^L] \).

Hence, in either case, the divisor (3.11) defines a line bundle \( L \) on \( M \) and a basis of \( H^0(\mathbb{C}P^1, L) \) indexed by the integer points of \( P_L \). Naturally, we let \( a = \lambda_1^L \) and \( N = \lambda_2^L + \lambda_1^L \), so that, as promised, the choice of polytope determines the line bundle we are considering. Note that, in this way, the choice of area of our surface (i.e. choice of \( N \)) determines how many one-particle states our system has.

In our case, we will be considering half-form correction and taking \( a = -\frac{1}{2} \) for simplicity. Then \( \lambda_2^L = N - \frac{1}{2} \) and, denoting \( \sigma_{D^L} \) by \( \sigma_{P,s} \), we write the elements of the basis as

\[
\sigma_s^m = w_s^m \sigma_{P,s} \otimes \sqrt{d\zeta_s}, \quad m \in P_L \cap \mathbb{Z} = \{0, \ldots, N - 1\}. \tag{3.15}
\]

Next we introduce an Hermitian structure on \( L \) without half-form correction. These results are still applicable when including the half-form correction, by considering the product with the norm of the half-form correction (cf. Eq. (3.20)) (see [29]). For that, let \( \nu: \Gamma(L) \times \Gamma(L) \to C^\infty(M, \mathbb{C}) \) denote the pointwise Hermitian product with \( \| \cdot \| \) the corresponding norm. Note that, since \( L \) is a line bundle, it is enough to give the value of \( \| \sigma_{P,s} \| \), since the Hermitian product then follows by skew-linearity. If \( \tilde{s}_0 \) is a local unitary trivializing section, then the connection is given locally by

\[
\nabla_X (f \tilde{s}_0) = (X(f) - i \theta(X)f) \tilde{s}_0. \tag{3.16}
\]

Since \( \omega = i \partial \bar{\partial} \kappa_s \), we can take

\[
\theta = i (\bar{\partial} - \partial) \kappa_s. \tag{3.17}
\]

Therefore, the holomorphic sections we determined before to be given by \( H^0(M, L) = \text{span}_\mathbb{C} \{w_s^m \sigma_{P,s} : m = 0, \ldots, N - 1\} \) are, locally, solutions of

\[
\nabla_{\bar{\partial}} (f s_0) = 0, \quad \text{or} \quad \frac{\partial f}{\partial \bar{\zeta}} + \frac{\partial h_s}{\partial \bar{\zeta}} f = 0.
\]

Defining \( F = e^{h_s} f \), the above reduces to

\[
\frac{\partial F}{\partial \bar{\zeta}} = 0. \tag{3.18}
\]

Hence

\[
\sigma_{P,s} = e^{-h_s} s_0 \implies \nabla_{\bar{\partial}} w_s^m \sigma_{P,s} = 0. \tag{3.19}
\]
and so we can define the hermitian structure by \( \| \sigma_{s,P} \| = e^{-\kappa_s} \). With half-form correction, this becomes

\[
\| \sigma_{s,P} \otimes \sqrt{dz} \| = e^{-\kappa_s} \| dz \|
\]

Summarizing, the \( S^1 \)-equivariant basis of the space of holomorphic sections is given by

\[
\begin{cases}
\sigma^m_s = \tilde{\sigma}^m_s \otimes \sqrt{dz}, & m \in P \cap \mathbb{Z} = PL \cap \mathbb{Z} = \{0, \ldots, N-1\} \\
\tilde{\sigma}^m_s = w_s^m \sigma_{s,P}, \\
w_s = e^{^i\theta_s} = e^{y_s+i\theta}, \\
y_s = g'_s,
\end{cases}
\]

where

\[
\begin{cases}
g_s(x) = g_P(x) + \frac{s}{2} x^2 \\
\| \sigma_{s,P} \otimes \sqrt{dz} \| = e^{-2\kappa_s} \| dz \|
\end{cases}
\]

with \( g_P \) defined in (3.3).

### 3.2. One-particle states on the deformed plane.

The results of the previous subsection are converted to the case of the plane if we replace in the formulas of the previous section, the polytope \( P = \left[-\frac{1}{2}, N - \frac{1}{2}\right] \) by \( P = [-\frac{1}{2}, \infty) \). We see that the divisor \( D_2 \) does not exist and so we get a bijection between basis for the space of holomorphic sections and \( P \cap \mathbb{Z} = \mathbb{Z}_{\geq 0} \). Therefore, as expected, since the symplectic area is infinite, the space of holomorphic sections is infinite dimensional. As usual in the quantum Hall effect we will describe approximately finite discs by taking \( N \)-dimensional subspaces of the space of holomorphic sections, namely those corresponding to integers \( m \) smaller than \( N \). This also has the advantage of making the treatment done in the next chapter for the sphere entirely analogous to the one that would be done for rotationally invariant non-flat deformations of the plane.

From (2.12), the canonical symplectic potential reads,

\[
g_P(x) = \frac{1}{2} \left( x + \frac{1}{2} \right) \log \left( x + \frac{1}{2} \right),
\]

which corresponds to the standard, \( S^1 \)-equivariant, flat metric on the plane. To simplify the relation between the symplectic coordinates \((x, \theta)\) and the holomorphic ones \(w\) let us change the canonical potential in (3.23) by a linear term that leaves the metric unchanged,

\[
g_P(x) \rightsquigarrow \tilde{g}_P(x) = \frac{1}{2} \left( x + \frac{1}{2} \right) \log 2 \left( x + \frac{1}{2} \right) - \frac{x}{2}.
\]

We consider deformations of the geometry induced by the same convex function of the \( S^1 \)-Hamiltonian, \( H(x) = \frac{1}{2} x^2 \). The expressions (3.21) and (3.22) remain valid with \( g_P \) replaced by \( \tilde{g}_P \) in (3.23) so that, in particular,

\[
w = e^{\tilde{g}'_P(x)+i\theta} = \sqrt{2 \left( x + \frac{1}{2} \right)} e^{i\theta}
\]

with imaginary time evolution given by

\[
w_s = e^{isX_H} = \sqrt{2 \left( x + \frac{1}{2} \right)} e^{sx} e^{i\theta} = e^{sx} w,
\]
For the metric and its scalar curvature we obtain from (2.18) and (2.19)

\[ g_s(x) = \tilde{g}_P(x) + \frac{s}{2} x^2. \]

and

\[ \gamma_s(x) = \frac{2s(x + \frac{1}{2}) + 1}{2(x + \frac{1}{2})} dx^2 + \frac{2(x + \frac{1}{2})}{2s(x + \frac{1}{2}) + 1} d\theta^2 \]

(3.25)

and

\[ Sc(x) = \frac{8s}{(2s(x + \frac{1}{2}) + 1)^{\frac{3}{2}}} \]

(3.26)

3.3. GCST in the limit \( s \to \infty \). We now analyse the evolutions described in the previous sections in the limit \( s \to \infty \). Geometric quantization in this scenario has a particular combinatorial flavour, and extreme deformations of the geometry leading to degenerations of the Kähler structure lead to the convergence of sections, in a distributional sense, to distributional sections localized in integer points of the polytope [29,31], as is shown in the next proposition. It also states how both the GCST operator \( U_s \) (cf. Eq. (2.21)) and the prequantum evolution operator \( U^\text{pre}_s \) (cf. Eq. (2.22)) act on one particle states, justifying the claims made in Sect. 1. In fact, we see that norms converge as \( s \to \infty \) only for the GCST operator.

Proposition 1. (See [29,32]) Consider the GCST operator \( U_s \) defined in (2.21), with \( H = \frac{1}{2} x^2 \) in action-angle coordinates. Then

\[ U^\text{pre}_s (\sigma^m_0) = \sigma^m_s \]

(3.27)

where \( U^\text{pre}_s \) is given by (2.22) \( \sigma^m_s \) denotes the one particle states defined in (3.21). Furthermore,

\[ U_s (\sigma^m_0) = e^{-s m^2 x} \sigma^m_s \]

(3.28)

and

\[ \lim_{s \to \infty} U_s (\sigma^m_0) = \sqrt{\frac{2\pi}{s}} e^{g^s \delta_m} s_0 \otimes \sqrt{dx}, \]

(3.29)

where \( s_0 \) is a unitary section (cf. Eq. (3.19)), \( g = g_P \) for the sphere and \( g = \tilde{g}_P \) for the plane, and \( \delta_m \) denotes a Dirac delta distribution supported on \( x = m \).

Proof. We have

\[ X_H = -x \frac{\partial}{\partial \theta}. \]

(3.30)

Substituting in (2.20), we get, with respect to the trivializing section defined by (3.19),

\[ Q^\text{pre}(H) = -ix \frac{\partial}{\partial \theta} + i \left( \frac{\partial}{\partial \tilde{\theta}} - \frac{\partial}{\partial \theta} \right) \kappa \left( x \frac{\partial}{\partial \theta} \right) + H \]

\[ = -ix \frac{\partial}{\partial \theta} + x d\theta \left( -x \frac{\partial}{\partial \theta} \right) + \frac{x^2}{2} \]

\[ = -ix \frac{\partial}{\partial \theta} - \frac{1}{2} x^2 \]
Furthermore, as (3.30) does not preserve the complex polarization, we define
\[
Q(H) = \frac{1}{2} \left( Q_{\text{pre}}(x) \right)^2 = \frac{1}{2} \left( -i \frac{\partial}{\partial \theta} \right)^2 = -\frac{1}{2} \frac{\partial^2}{\partial \theta^2}
\]
and so the GCST, given in (2.21), becomes
\[
U_s = \exp \left( -isx \frac{\partial}{\partial \theta} - s \frac{x^2}{2} \right) \exp \left( s \frac{\partial^2}{2 \partial \theta^2} \right).
\]
(3.31)

Evidently, all of the operators in the exponents commute between themselves and thus
\[
U_s^{\text{pre}}(\sigma_0^m) = e^{-\frac{x^2}{2} e^{-isXH} e^{m(y+i\theta)} e^{-xy+g\tilde{s}0} \otimes \sqrt{dz} = e^{m(y+i\theta)} e^{-xy+g\tilde{s}0} \otimes \sqrt{dz} = \sigma_s^m
\]

Furthermore, only the factor \(e^{mi\theta}\) is not \(S^1\)-equivariant, and so Eq. (3.28) follows immediately.

Finally, using \(\|\sqrt{dz}\| = \sqrt{g_s^m}\) (see [29]) and \(\lim_{s \to \infty} \frac{g_s^m}{s} = 1\) (which is true for both the plane and the sphere), then
\[
\lim_{s \to \infty} \|U_s(\sigma_0^m)\| = \lim_{s \to \infty} \sqrt{se^{-\frac{s^2}{2}(x-m)^2} e^{-y(x-m)+g}} = \sqrt{2 \pi} e^{g(m)} \delta_m
\]
(3.32)

Thus, Eq. (3.29) follows from \(\lim_{s \to \infty} \frac{\sqrt{s}}{s} = x\).

4. Evolution of Laughlin States on the Sphere

We will now study many particle lowest energy states of the quantum Hall effect and their geometry dependence. We only consider the case of the sphere in the first 2 sections, as the case of the plane was treated in detail by Gabriel Matos in [28] and we already described the analogies with the sphere in Sect. 3.1.

We first study, in Sect. 4.1, the integer quantum Hall effect, in which the ground states are fully filled. Some of the mathematical notions used to represent many-particle states are introduced as well as how they evolve by imaginary time Hamiltonian flow, in particular in the limit \(s \to \infty\).

In Sect. 4.2, we describe the FQHE, in which we consider states filling only \(1/m\) of ground states, with \(m\) odd, since, for these cases, there is a very good approximation for the lowest energy states known as Laughlin states. And using a result of Dunne ([33]), that gives a Slater decomposition of Laughlin states, we are able to describe their evolution under imaginary time Hamiltonian flow for an arbitrary number of particles.

These results are then used to study and discuss density profiles in Sect. 5. One very important aspect of this discussion is the comparison with evolutions using only the
prequantum evolution operator (2.22), which corresponds to the Laughlin states in [7]. In fact, the results of the present section showcase a convergence of Laughlin states to specific Slater determinants under the evolution of the prequantum operator, which is entirely a consequence of the non-unitarity of this operator.

4.1. LLL states of the IQHE. The fully filled LLL state corresponds to the unique state in $\Lambda^N H_0$, where $H_0 = H^0(CP^1, L_0)$, with $L_0$ the holomorphic line bundle described in Sect. 3.1, corresponding to the $s = 0$ initial geometry. It is, intrinsically, an $N$-particle fermionic state.

Let $(CP^1)^N$ denote the $N$-fold Cartesian product of $CP^1$ and $\pi_i : (CP^1)^N \rightarrow CP^1$ the $i$th canonical projection. We have a line bundle $L^{\otimes N} := \bigotimes_{j=1}^{N} \pi_j^* L_0 \rightarrow (CP^1)^N$. Given $N$ sections of $L \rightarrow CP^1$, $s_1, \ldots, s_N$, we have a natural section $s_1 \otimes s_2 \otimes \cdots \otimes s_N := \pi_1^* s_1 \otimes \cdots \otimes \pi_N^* s_N \in \Gamma(L^{\otimes N})$, given by

$$(w_1, \ldots, w_N) \mapsto s_1(w_1) \otimes s_2(w_2) \otimes \cdots \otimes s_N(w_N).$$

Any other section can be written as linear combination of such sections and we learn that, as a vector space, we have $\Gamma(L^{\otimes N}) \cong \Gamma(L)^{\otimes N}$. Actually, we can identify the section $\pi_1^* s_1 \otimes \cdots \otimes \pi_N^* s_N$ as the tensor product $s_1 \otimes \cdots \otimes s_N \in \Gamma(L)^{\otimes N}$. Since we are looking at fermionic particles, we wish to consider only those sections which are completely antisymmetric under permutations of the particles (i.e. permutations of variables $w_j$). Therefore the state belongs to $\Lambda^N H \subset \Gamma(L)^{\otimes N}$ and corresponds to the wedge product

$$\Psi_{IQHE} := \sigma_0^0 \wedge \cdots \wedge \sigma_0^{N-1},$$

where $\{\sigma_0^0\}_{j=0}^{N-1}$ is a basis for $H_0$. As a section of $L^{\otimes N} \rightarrow (CP^1)^N$ it is written as

$$(w_1, \ldots, w_N) \mapsto \sum_{\tau \in S_N} \text{sgn}(\tau) \sigma_0^{\tau(1)}(w_1) \otimes \sigma_0^{\tau(2)}(w_2) \otimes \cdots \otimes \sigma_0^{\tau(N)}(w_N).$$

Given a local trivialization of $L \rightarrow CP^1$ over an open set $U$ provided by a local section $s_0$, we have an induced local trivialization of $L^{\otimes N} \rightarrow (CP^1)^N$ over $U \times \cdots \times U$ given by $s_0^{\otimes N} := \pi_1^* s_0 \otimes \cdots \otimes \pi_N^* s_0$. If over $U$ we have $\sigma_0^j = f^j s_0$, for some local functions $f^j$, $i = 1, \ldots, N$, then obviously,

$$\sum_{\tau \in S_N} \text{sgn}(\tau) \sigma_0^{\tau(0)}(w_1) \otimes \sigma_0^{\tau(2)}(w_2) \otimes \cdots \otimes \sigma_0^{\tau(N-1)}(w_N)$$

$$= s_0^{\otimes N}(w_1, \ldots, w_N) \cdot \left( \sum_{\tau \in S_N} \text{sgn}(\tau) f^{\tau(0)}(w_1) f^{\tau(1)}(w_2) \cdots f^{\tau(N-1)}(w_N) \right)$$

$$= s_0^{\otimes N}(w_1, \ldots, w_N) \cdot \det \begin{bmatrix} f^0(w_1) & \cdots & f^0(w_N) \\ \vdots & \ddots & \vdots \\ f^{N-1}(w_1) & \cdots & f^{N-1}(w_N) \end{bmatrix},$$

hence it locally looks like the Slater determinant of the associated local representatives. We will denote by $w_{s,j} = \pi_j^* w_s$ the imaginary time $\tau = -is$ evolved local holomorphic
coordinate associated to the \(j\)-th particle, \(j = 1, \ldots, N\). We will adopt a similar notation for the other coordinates \(y_s, j, x_j\) and \(\theta_j\) defined for the single particle case. Analogously to what is done in (3.19), we consider the local trivialization \(U \otimes \sqrt{dz_0}\) (where \(\sqrt{dz_0}\) denotes the product \(\sqrt{dz_1} \cdots \sqrt{dz_{N-1}}\) at \(s = 0\)) of \(L\) over \(U\) so that its local representative is

\[
\prod_{i=1}^{N} e^{-h_0(y_{0,i})} \det \begin{bmatrix} 1 & \cdots & 1 \\ w_{0,1} & \cdots & w_{0,N} \\ \vdots & \ddots & \vdots \\ w_{N-1} & \cdots & w_{N-1} \end{bmatrix} = e^{-\sum_{i=1}^{N} h_0(y_{0,i})} \prod_{1 \leq i < j \leq N} (w_{0,j} - w_{0,i}).
\]

The GCST operator \(U_s\) acts on \(\Gamma(L^\otimes) \cong \Gamma(L)^\otimes N\) in a natural way, namely, given \(s_1, \ldots, s_N\), we have, in terms of sections of \(L^\otimes\)

\[
U_s(s_1 \boxtimes \cdots \boxtimes s_N) = U_s\left(\pi_s^* s_1 \otimes \cdots \otimes \pi_s^* s_N\right) := \pi_s^*(U_s s_1) \otimes \cdots \otimes \pi_s^*(U_s s_N) = (U_s s_1) \boxtimes \cdots \boxtimes (U_s s_N),
\]

or in terms of \(\Gamma(L)^\otimes N\),

\[
U_s(s_1 \otimes \cdots \otimes s_N) = (U_s s_1) \otimes \cdots \otimes (U_s s_N).
\]

Consequently, we also get

\[
U_s(\sigma_0^0 \wedge \cdots \wedge \sigma_0^{N-1}) = (U_s \sigma_0^0) \wedge \cdots \wedge (U_s \sigma_0^{N-1}). \tag{4.1}
\]

If \(\sigma_s^m\) denotes the states described in Sect. 3.1 (cf. Eq. (3.21)), then it follows from Proposition 1, that

\[
U_s(\Psi_{\text{IQHE}}) = e^{-\frac{i}{\hbar} \sum_{m=0}^{N} m^2 \sigma_s^0 \wedge \cdots \wedge \sigma_s^{N-1}}, \tag{4.2}
\]

and

\[
\lim_{s \to \infty} U_s(\Psi_{\text{IQHE}}) = (2\pi)^N e^{\sum_{m=1}^{N-1} g(m)} \sum_{\tau \in S_N} \sgn(\tau) (\pi_s^* \delta_{\tau(0)} \otimes \sqrt{dx_1}) \wedge \cdots \wedge (\pi_s^* \delta_{\tau(N-1)} \otimes \sqrt{dx_N}).
\]

4.2. FQHE Laughlin states. We will consider now a Laughlin state with \(N_e\) electrons and filling fraction \(\nu := N_e/N = 1/m\) with \(m\) odd. We have that \(N_e = N/m\), and we will assume that \(m\) divides \(N\), where \(N = \text{deg } L\), as before. At the \(s = 0\) geometry, i.e. the round sphere geometry (cf. Eq. (3.10)), the Laughlin state is given in a trivializing open set \(U^{N_e}\) by

\[
\Psi_{\text{Laughlin}} := e^{-\sum_{i=1}^{N_e} h_0(y_{0,i})} \prod_{1 \leq i < j \leq N_e} (w_{0,j} - w_{0,i})^m \left(1_{i}^{U(1)} \otimes \sqrt{dz_{0,i}}\right),
\]

Since \(m\) is odd it is clearly skew-symmetric. The pre-factor is just the norm of the holomorphic section \(\sigma_{P,0}^N\) while the remainder is holomorphic. Therefore we conclude that this state belongs to \(\Lambda^{N_e} H^0(\mathbb{C}P^1, L_0)\).
We will now consider the GCST operator in the \( N_e \)-particle sector. From Proposition 1, it is clear that at imaginary time \( \tau = -is \), the operator assumes the simple representative

\[
U_s = \exp \left( -s \sum_{i=1}^{N_e} \frac{x_i^2}{2} - is \sum_{i=1}^{N_e} x_i \frac{\partial}{\partial \theta_i} \right) \exp \left( \frac{s}{2} \sum_{i=1}^{N_e} \frac{\partial^2}{\partial \theta_i^2} \right),
\]

where, evidently, the exponentials commute with each other. Hence, a computation entirely analogous to that of Proposition 1 gives

\[
U_s \left( \Psi_{\text{Laughlin}} \right) = \exp \left( -s \sum_{i=1}^{N_e} h_z(y_{s,i}) \right) \exp \left( \frac{s}{2} \sum_{i=1}^{N_e} \frac{\partial^2}{\partial \theta_i^2} \right) \prod_{1 \leq i < j \leq N_e} (w_{x,i} - w_{x,j})^m \left( \bigotimes_{i=1}^{N_e} (I_1^{U(1)} \otimes \sqrt{dz_{x,i}}) \right).
\]

Given \( \lambda = (\lambda_1, \ldots, \lambda_{N_e}) \) with \( 0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{N_e} \leq N \), where \( N = \deg L \), we have an associated Slater determinant state

\[
\Psi_s^\lambda := \sigma_s^{\lambda_1} \wedge \cdots \wedge \sigma_s^{\lambda_{N_e}},
\]

where \( \{ \sigma_s^m \}_{m=0}^N \) is again the monomial basis of \( H^0(\mathbb{C}P^1, L_s) \) given in (3.21). The states \( \{ \Psi_s^\lambda \}_{0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{N_e} \leq N} \) form a basis of \( \Lambda^{N_e} H^0(\mathbb{C}P^1, L_s) \) and moreover,

\[
U_s \left( \Psi_0^\lambda \right) = e^{-\frac{s}{2} \sum_{i=1}^{N_e} \lambda_i^2} \sum_{\lambda} a_{\lambda}^{\nu} \Psi_s^\lambda.
\]

Since the Laughlin states with odd filling fraction belong to \( \Lambda^{N_e} H^0(\mathbb{C}P^1, L_s) \), it follows that they can be expanded in the basis of anti-symmetric states given by \( \{ \psi_0^\lambda \} \), as shown by Dunne in Section 3 of [33]. (This is a consequence of the fact that Schur polynomials, which can be written as ratios of Slater determinants, form a basis of the space of symmetric polynomials.) Therefore, there exist coefficients \( \{ a_{\lambda}^{\nu} \} \) such that we will be able to write

\[
\Psi_s^{\nu = 1/3}_{\text{Laughlin}} = \sum_{\lambda} a_{\lambda}^{\nu} \Psi_s^\lambda,
\]

so that

\[
U_s \left( \Psi_s^{\nu = 1/3}_{\text{Laughlin}} \right) = \sum_{\lambda} e^{-\frac{s}{2} \sum_{i=1}^{N_e} \lambda_i^2} a_{\lambda}^{\nu} \Psi_s^\lambda.
\]

The paper [33] also gives a method to determine these coefficients in a combinatorial manner. These become harder to compute the bigger \( N_e \) is, and so we only compute exact density profiles for \( N_e = 2 \) and \( N_e = 3 \).

For \( N_e = 2 \), one obtains

\[
\Psi_s^{\nu = 1/3}_{\text{Laughlin}} = \Psi_0^{(0,3)} - 3 \Psi_0^{(1,2)}
\]
and thus the evolved state is (cf. Eq. (4.4))

\[ U_s(\Psi_{\text{Laughlin}}^{1/3}) = e^{-s/2} \Psi_s^{(0,3)} - 3e^{-s/2} \Psi_s^{(1,2)}. \]

For \( N_e = 3 \),

\[ \Psi_{\text{Laughlin}}^{1/3} = \psi_0^{(0,3,6)} - 3\psi_0^{(1,2,6)} - 3\psi_0^{(0,4,5)} + 6\psi_0^{(1,3,5)} - 15\psi_0^{(2,3,4)}, \]

so that

\[ U_s(\Psi_{\text{Laughlin}}^{1/3}) = e^{-s/2}45 \Psi_s^{(0,3,6)} - 3e^{-s/2}41 \Psi_s^{(1,2,6)} - 3e^{-s/2}41 \Psi_s^{(0,4,5)} + 6e^{-s/2}35 \Psi_s^{(1,3,5)} - 15e^{-s/2}29 \Psi_s^{(2,3,4)}. \]

5. Evolution of Density Profiles Through Imaginary Time Flows on the Sphere and on the Plane

We wish to evaluate the density profiles given by

\[ \rho_s(x) := \frac{\langle U_s(\Psi_{\text{Laughlin}}^{1/3}), \sum_{i=1}^{N_e} \delta(x, x_i) U_s(\Psi_{\text{Laughlin}}^{1/3}) \rangle}{\| U_s(\Psi_{\text{Laughlin}}^{1/3}) \|^2_{L^2}}. \]

with

\[ U_s(\Psi_{\text{Laughlin}}^{1/3}) = \sum_{\lambda} a_{\lambda} e^{-s/2} \sum_{i=1}^{N_e} \lambda_i^2 \psi_{\lambda i}, \]

We also have orthogonality of the Slater determinant states, i.e.,

\[ \langle \psi_{\lambda}, \psi_{\lambda'} \rangle = \delta_{\lambda, \lambda'} f_{\lambda} \]

with

\[ f_{\lambda} = \prod_{i=1}^{N_e} \| \sigma_{\lambda i} \|^2_{L^2}, \]

due to rotational symmetry (orthogonality of \( e^{im\theta} \) and \( e^{in\theta}, m \neq n \)). Therefore, we find that

\[ \rho_s(x) = \frac{1}{\| U_s(\Psi_{\text{Laughlin}}^{1/3}) \|^2_{L^2}} \sum_{\lambda, \lambda'} \bar{a}_{\lambda} a_{\lambda'} e^{-s/2} \sum_{i=1}^{N_e} (\lambda_i^2 + \lambda_i'^2) \langle \psi_{\lambda}, \sum_{j=1}^{N_e} \delta(x, x_j) \Psi_{\lambda'} \rangle, \]

and since \( \{ \psi_{\lambda} \} \) form an orthogonal basis (cf. Eq. (5.2)), we have

\[ \rho_s(x) = \frac{1}{\| U_s(\Psi_{\text{Laughlin}}^{1/3}) \|^2_{L^2}} \sum_{\lambda} |a_{\lambda}^\dagger|^2 e^{-s/2} \sum_{i=1}^{N_e} \lambda_i^2 \langle \psi_{\lambda}, \sum_{j=1}^{N_e} \delta(x, x_j) \Psi_{\lambda} \rangle. \]
where

\[ \| U_s (\Psi^{1/3}_{\text{Laughlin}}) \|^2_{L^2} = \sum_\lambda |a^w_\lambda|^2 e^{-s} \sum_{i=1}^{N_e} \lambda_i^2 \| \Psi^L_s \|^2_{L^2} = \sum_\lambda |a^w_\lambda|^2 e^{-s} \sum_{i=1}^{N_e} \lambda_i^2 \prod_{i=1}^{N_e} \| \sigma^L_s \|^2_{L^2}. \]

Finally, we note that

\[ \left( \Psi^L_s , \sum_{j=1}^{N_e} \delta(x, x_j) \Psi^L_s \right) \]

\[ = \sum_{j=1}^{N_e} \int (\mathbb{C}P^1)^{N_e} \delta(x - x_j) \prod_{i=1}^{N_e} v(\sigma^L_s (x_i), \sigma^L_s (x_j)) \, dx_i \]

\[ = \sum_{j=1}^{N_e} v(\sigma^L_s (x), \sigma^L_s (x_j)) \int (\mathbb{C}P^1)^{N_e-1} \prod_{i=1, i \neq j}^{N_e} v(\sigma^L_s (x_i), \sigma^L_s (x_j)) \, dx_i \]

\[ = \sum_{j=1}^{N_e} v(\sigma^L_s (x), \sigma^L_s (x_j)) \prod_{i=1}^{N_e} \| \sigma^L_s \|^2_{L^2} \| \sigma^L_s \|^2_{L^2} \]

so (5.3) becomes

\[ \rho_s (x) = \frac{\sum_\lambda |a^w_\lambda|^2 e^{-s} \sum_{i=1}^{N_e} \lambda_i^2 \prod_{i=1}^{N_e} \| \sigma^L_s \|^2_{L^2} \sum_{j=1}^{N_e} v(\sigma^L_s (x), \sigma^L_s (x_j))}{\sum_\lambda |a^w_\lambda|^2 e^{-s} \sum_{i=1}^{N_e} \lambda_i^2 \prod_{i=1}^{N_e} \| \sigma^L_s \|^2_{L^2}}. \]

(5.4)

We can still make the result a bit more explicit by using

\[ v(\sigma^L_s (x), \sigma^L_s (x_j)) = e^{-2h_s} |d z_s| |w_s|^{2\lambda_j} \]

and thus obtaining

\[ \rho_s (x) = e^{-2h_s} |d z_s| \frac{\sum_\lambda |a^w_\lambda|^2 e^{-s} \sum_{i=1}^{N_e} \lambda_i^2 \prod_{i=1}^{N_e} \| \sigma^L_s \|^2_{L^2} \sum_{j=1}^{N_e} |w_j|^{2\lambda_j}}{\sum_\lambda |a^w_\lambda|^2 e^{-s} \sum_{i=1}^{N_e} \lambda_i^2 \prod_{i=1}^{N_e} \| \sigma^L_s \|^2_{L^2}}. \]

(5.5)

5.1. Large s asymptotics. We will now consider the large s limit of \( \rho_s (x) \). From (3.28) and (3.29), we get, as \( s \to \infty \)

\[ v(\sigma^L_s , \sigma^L_s) \sim \sqrt{\pi} e^{s |\lambda_i|^2} e^{2g p (\lambda_i)} \delta_{\lambda_i}, \]

(5.6)

and

\[ \| \sigma^L_s \|^2_{L^2} \sim \sqrt{\pi} e^{s |\lambda_i|^2} e^{2g p (\lambda_i)}, \]

(5.7)
Using both to take the limit $s \to \infty$ in (5.4), we obtain

$$
\lim_{x \to \infty} \rho_s(x) = \frac{\sum_{\lambda} |a_{\nu,\lambda}^v|^2 e^{2g_P(\lambda_s)} \sum_{j=1}^{N_v} \delta_{\lambda,j}}{\sum_{\lambda} |a_{\nu,\lambda}^v|^2 e^{2g_P(\lambda_s)}}.
$$

(5.8)

In particular, for the IQHE state, we have $N_v = N$ with $\lambda = (0, 1, \ldots, N)$ being the only possible state, so we immediately obtain

$$
\lim_{x \to \infty} \rho_s(x) = \sum_{j=1}^{N} \delta_{\lambda,j}
$$

(5.9)

corresponding to a uniform distribution of Bohr–Sommerfeld leaves on $\mathbb{C}P^1$, in particular those corresponding to the integer points $P \cap \mathbb{Z} = [-1/2, N - 1/2] \cap \mathbb{Z} = \{0, \ldots, N - 1\} = [0, N - 1] \cap \mathbb{Z}$.

5.2. Examples with $\nu = 1/3$ filling. We will now analyze (5.8) in the specific case where $\nu = 1/3$.

First, we look at examples with few particles, for which the combinatoric coefficients $a_{\nu,\lambda}^v$ are easily determined (see Ref. [33]). We can then plot the density profiles of these states for different values of $s$, as is done in Figs. 1 and 2, for systems of 2 and 3 particles, respectively.

The first thing we observe is that the ratios between heights of different peaks indeed seem to approach well-defined limits, very close to

$$
R_{m,n} = \frac{\sum_{\lambda: m \in \lambda} |a_{\nu,\lambda}^v|^2 e^{2\sum_{i=1}^{N_v} g_P(\lambda_i)}}{\sum_{\tilde{\lambda}: n \in \tilde{\lambda}} |a_{\nu,\tilde{\lambda}}^v|^2 e^{2\sum_{i=1}^{N_v} g_P(\tilde{\lambda}_i)}}.
$$

(5.10)
between peaks at $x = m$ and $x = n$ corresponding to the density profile (5.8), which yields

\[ N_{\epsilon} = 2 \implies R_{0,1} \approx 1.08 \]
\[ N_{\epsilon} = 3 \implies \begin{cases} R_{0,1} & \approx 1.03 \\ R_{1,2} & \approx 1.01 \end{cases} \]

for the sphere and

\[ N_{\epsilon} = 2 \implies R_{0,1} \approx 0.35 \]
\[ N_{\epsilon} = 3 \implies \begin{cases} R_{0,1} & \approx 0.81 \\ R_{1,2} & \approx 0.50 \end{cases} \]
for the plane.

In Figs. 5 and 6 we show imaginary time Hamiltonian flow evolutions using only the prequantum evolution operator (2.22), for 2 electrons on the plane and 3 on the sphere. We see that, as expected from the results found in Proposition 1, the Laughlin states converge to a single slater determinant, namely the one with the largest $|\lambda|^2$. This is purely a consequence of non-unitarity of the evolution operator used.
Fig. 6. Density profiles of states of 3 particles on the sphere evolved only with prequantum operator for $s = 0$, $s = 5$, $s = 10$, $s = 50$ and $s = 100$

Fig. 7. Ratio of $\frac{S(\lambda^m)}{|a^M_{\lambda} |^2 S(\lambda^M)}$ as a function of $N_\theta$, on the sphere
We now consider the large $N_e$ limit. We see from (5.8) that how much a certain state $\lambda$ contributes to the height of its peaks depends on the coefficient $a_{\lambda}^{\nu}$ and the factor

$$S(\lambda) = \exp\left(2 \sum_{i=1}^{N_e} g_P(\lambda_i)\right).$$

For large $N_e$, the dominant $|a_{\lambda}^{\nu}|^2$ comes from the maximally “bunched” state $\lambda^M = (0, 1, \ldots, N_e - 1) + (N_e - 1, N_e - 1, \ldots, N_e - 1)$, whose coefficient satisfies $|a_{\lambda^M}^{1/3}| = (2N_e - 1)!$ (see Ref. [33]). In contrast, the minimum $|a_{\lambda}^{\nu}|^2$ comes from the most uniform state $\lambda^m = 3(0, 1, \ldots, N_e - 1)$, for which $|a_{\lambda^m}^{1/3}| = 1$. By plotting the ratio of the factors $|a_{\lambda}^{1/3}|^2 S(\lambda)$ for these two states as a function of $N_e$ (Fig. 7), we see that the change in the $S$ factor greatly outweighs the change in the combinatorial factor $|a_{\lambda}^{1/3}|^2$.

Since the function $g_P$ has a minimum in the center of the polytope and increases away from it, the state $\lambda^M$ is the one for which $S$ is the smallest. However, there are states with higher concentration in the poles that have much larger $S$ than the value of $S(\lambda^m)$. As this is the leading factor in determining the heights of peaks, we expect these states to have much more relevance and thus the density to be higher near the poles than in the center.

A similar analysis can be made for many particle states on the plane, since the combinatorial factors are the same and do not change significantly for different Slater
determinants in comparison with the function
\[ \tilde{S}(\lambda) = \exp \left( 2 \sum_{i=1}^{N_e} \tilde{g}_P(\lambda_i) \right), \]
as can be seen in the plot in Fig. 8.

6. Conclusion

With this work, we achieved our goal of using the GCST to obtain a detailed description of the evolution of Laughlin states on the sphere under deformations of the geometry.

Starting with one particle states, the results of Proposition 1, not only describe how the GCST acts on the quantum Hilbert space of the sphere, but also allow us compare this evolution with the one obtained via the prequantum evolution operator defined by (2.22). We see that, although the latter gives a very intuitive evolution, yielding states considered in the literature to be Laughlin states (see [7]), it is highly non-unitary, which is an issue that we observe to be precisely fixed by considering the GCST instead.

The results of chapter 4 then highlight the consequences of the aforementioned proposition to many particle states, in particular Laughlin states. The density profile obtained in expression (5.4) provides a very useful description of the evolution of Laughlin states, and cases with few particles are computed in detail (see Figs. 1, 2, 3 and 4) for the sphere and the plane, with results for the latter matching those obtained by Gabriel Matos in [28].

We pay special attention to extreme deformations (\( s \to \infty \)), where we see the density converge to integer points on the polytope with peak ratios given by well-defined limits (cf. Eq. (5.10)), dependent purely on physical properties of the system. This heavily contrasts with the evolution obtained by only considering the prequantum operator, where non-unitarity causes convergence of the Laughlin states to specific Slater determinants (see Figs. 5 and 6).

Finally, we see that the density profiles we obtained also provide information related to systems with a large number of particles as \( s \to \infty \), namely higher concentrations near the poles for the sphere and near the center for the plane.

Further work would be useful to obtain a systematic and reliable way to compute the coefficients \( a_\lambda \) present in (5.4). This would allow for an exact computation of the evolution of states for an arbitrary number of particles as \( s \to \infty \). The study of analogous deformations to those considered here, namely on the torus for which a Laughlin state was also given by Haldane (see [17]), or induced by different Hamiltonians would also be extremely relevant. Additionally, it would be interesting to use the GCST, (2.21), to study model quantum Hamiltonians for the FQH states obtained above. More concretely, one could act by conjugation with \( U_s \) on the underformed quantum Hamiltonians for which the undeformed states are ground states producing new Hamiltonians with the correct ground states. Finally, it would also be interesting to apply the method of the present work to other trial wave-functions, such as the Moore-Read Pfaffian states, which also have a holomorphic description and to which, hence, the method can be applied.

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Conflicts of interest

We hereby declare that there are no conflicts of interest associated with our paper entitled “Laughlin states change under large geometry deformations and imaginary time Hamiltonian dynamics”.

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