One-dimensional Josephson junction arrays: Lifting the Coulomb blockade by depinning

Nicolas Vogt,1, 2 Roland Schäfer,3 Hannes Rotzinger,4 Wanyin Cui,4, 3 Andreas Fiebig,4, 3 Alexander Shnirman,1, 2 and Alexey V. Ustinov4, 2

1 Institut für Theorie der Kondensierten Materie, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany
2 DFG Center for Functional Nanostructures (CFN), Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany
3 Institut für Festkörperphysik, Karlsruhe Institute of Technology, D-76021 Karlsruhe, Germany
4 Physikalisches Institut, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany

(Dated: July 15, 2014)

Keywords: Josephson-Junction-Arrays, Depinning

Experiments with one-dimensional Josephson junction arrays in the regime of dominating charging energy show that the Coulomb blockade is lifted at the threshold voltage, which is proportional to the array’s length and depends strongly on the Josephson energy. We explain this behavior by depinning of the charges in the array. We assume strong charge disorder and argue that physics around the depinning point is governed by a disordered sine-Gordon-like model. This allows us to employ the well-known theory of charge density wave depinning. Our model is in good agreement with the experimental data.

PACS numbers: 74.81.Fa, 74.50.+r, 73.23.Hk

One-dimensional Josephson junction arrays show a diverse range of electrical conductances. In the regime of dominating Josephson energy, they are highly conducting and attract continued experimental interest [1, 2]. In the regime of Josephson energy smaller or comparable to the charging energy, one-dimensional Josephson arrays show insulating (Coulomb blockade) behavior with activated transport [3]. Above a certain threshold value of the bias voltage, finite current appears even at zero temperature. Initially, this switching was interpreted in terms of propagation onset of charge solitons [4–7], i.e., the energy one has to pay in order to push one soliton into the array. However, further experiments showed that the threshold voltage is proportional to the array length and depends strongly on the value of the Josephson energy [1, 8]. In this paper, we interpret the experimentally found behavior as depinning in presence of strong charge disorder [9, 11].

We argue that the system is described by a model similar to a disordered sine-Gordon model. The only difference is the fact that, instead of the usual cosine potential, here we have another periodic function, i.e., the lowest Bloch band energy, which depends strongly on the Josephson energy. It is this dependence which gives rise to the dependence of the switching voltage on the Josephson energy. Previously, similar models were derived [1, 6–8] using an additional phenomenological inductance element which provided the necessary mass (kinetic energy) term. In Ref. [12], it was shown, however, that a mass term is generated in the adiabatic regime due to the Bloch inductance [13] and the phenomenological inductance in not needed. The adiabatic mechanism is sufficient to describe the physics prior and at the depinning point.

For this work, two series of experiments have been performed on two sets of Josephson junction chains. In each case, three arrays have been fabricated in parallel on the same silicon substrate covered by an insulating thermally grown SiO₂ layer. The individual cells of the array are implemented as SQUID loops, similarly to earlier experiments [1–4]. The two tunnel junctions in each SQUID loop are equivalent to a single junction with an effective Josephson energy $E_J(\Phi)$ tunable by the magnetic flux $\Phi$ penetrating the loop area $A$. That gives $E_J(\Phi) = E_J^0 |\cos(\pi\Phi/\Phi_0)|$, where $E_J^0$ is two times the Josephson energy of one of the bare Josephson junctions of the SQUID and $\Phi_0 = h/2e$ is the magnetic flux quantum. In an ideal situation, the area $A$ is identical for each loop along the chain. However, a slight misalignment in the fabrication process leads to an alternating pattern in the loop sizes with areas $A$ and $B$ [1–4].

The first set of samples contains nominally identical arrays (labeled A255, B255, and C255 in the following) comprising 255 SQUIDs. These arrays had very similar resistances. Nevertheless, slight variations in the junction parameters are reflected in the details of the $I$-$V$ characteristics [14]. In the second set, SQUID chains with different length have been realized. The arrays (labeled D39, E59, and F255) comprise 39, 59, and 255 SQUIDs, respectively.

The experiments have been performed in a $^3$He/$^4$He dilution refrigerator at 20mK temperature. A scanning electron microscope (SEM) picture of a section of one of the arrays is shown in the inset of Fig. [1]. All electrical connections to the samples are carefully filtered by a combination of lumped-element low pass $RC$-filters and...
metal powder filters covering a bandwidth of 10 kHz. I-V characteristics are measured by stepping the applied bias voltage and recording the resulting current with a homemade transimpedance amplifier. A typical I-V characteristic is shown in Fig. 1, where the blue curve is recorded while the bias voltage is ramped up and the red curve represents the behavior for decreasing bias. In all cases, the current vanishes below a certain threshold; for the horizontal branch, no current can be detected within the resolution of our current measurement which is of the order of 50 fA. At a value $V_{sw}(\Phi)$, the chain switches to a conducting state; the current after the switching is flux dependent and has a magnitude of at least several pA. Retrapping to the $I = 0$ branch happens at a much lower voltage $V_{rt} < V_{sw}$. In this paper, we focus on the magnitude and the flux dependence of the switching voltage $V_{sw}(\Phi)$. We expect $V_{sw}$ to be primarily a function of $E_J(\Phi)$. Thus, $V_{sw}(\Phi)$ is a periodic function in $\Phi$ with a period of $\Phi_0$. Experimentally, we observe the period (measured in units of the external magnetic field) to be of order $B_{ext} = 6.9 \text{mT}$, corresponding to an area of $\Phi_0/6.9 \text{mT} = 0.3 \mu\text{m}^2$. This agrees well with the total area per SQUID loop, $A_{\text{SQUID}} = 1.6 \mu\text{m} \cdot 200 \text{nm}$ defined by the sample layout.

The system is modeled theoretically as an array of Josephson junctions (represented as crosses in the inset of Fig. 1) with the effective Josephson energy $E_J$ (controlled by the magnetic field in experiment) and capacitance $C_J$. Each superconducting island (represented by open squares in the inset) has a capacitance to the ground $C_0$. We focus on the regime $C_0 \ll C_J$, which means a long screening length $\Lambda \equiv \sqrt{C_J/C_0}$. $C_0$ is dominated by two ground planes running alongside of the array and is estimated by considering the specific capacitance of a strip line where the central conductor has the width of the Josephson-junction chain and the ground planes form the outer conductors. We find $5 \text{aF} < C_0 < 20 \text{aF}$. The total self capacitance of the islands is dominated by the capacitance of the tunnel junctions, whereas in experiment $C_J$ is the total capacitance of the two Josephson junctions of the SQUID. The capacitance $C_J$ also determines the relevant energy scale of the array, namely the charging energy of the Josephson junction $E_C = 2e^2/C_J$. The area of the junctions is deduced from SEM micrographs. Assuming a specific capacitance of $45 \text{fF}/\mu\text{m}^2$ for typical Al/AlO$_x$/Al tunnel junctions[7] we estimate $C_J = 0.575 \text{fF} \pm 20 \%$ where the error margin reflects systematic as well as island-to-island fluctuations. A similar estimate is obtained by considering the high voltage behavior of I-V characteristics (data not shown) which show a linear behavior with a voltage offset $V_{off}$ characteristic for charging effects, $I(V > 100 \text{mV}) \sim G(V-V_{off})$. Approximating $E_{\text{cap}} \approx 2eV_{off}/N$[16] gives the estimate for the charging energy (and thus $C_J$) reported in Tab. I. In this estimate the high bias voltage regime with incoherent quasi-particle transport is used and the two physical

![FIG. 1. (color online) Hysteretic I-V characteristics of array F255 measured for increasing voltages (blue) and decreasing voltages (red). Upper left inset shows an SEM micrograph of the Josephson junction array. Lower right inset is a schematic representation of the array.](image)

In the theoretical model, we consider no disorder in $E_J$, $C$ or $C_0$ but include a strong disorder in the gate (frustration) charge $2e f_i$ on each superconducting island. The Hamiltonian then reads $H = H_c + H_J$, where

$$H_c = \frac{(2e)^2}{2} \sum_{i,j} \langle n_i - f_i \rangle C_{ij}^{-1} \langle n_j - f_i \rangle$$

and $H_J = -\sum_i E_J \cos (\theta_i - \theta_{i+1})$. The capacitance matrix is given by $C_{ij} = (2C_J + C_0^\prime) \delta_{i,j} - C_J (\delta_{i-1,j} + \delta_{i+1,j})$. In the regime $C_J \gg C_0$, i.e., $\Lambda \gg 1$, one obtains $C_{ij}^{-1} \approx C_J^{-1} (\Lambda/2) \exp[-|i-j|/\Lambda]$. Arrays with charge disorder in the limit $C_J \gg C_0$ have been considered long ago (see, e.g.,[17][18]). The onset of charge transport was calculated purely form the analysis of the stability of charge configurations. The crucial difference in our work is the strong renormalization of the disorder potential in the regime $E_J \sim E_C$.

In the regime $\Lambda \gg 1$, it is convenient to introduce the integrated charge variables $n_i = \sum_j n_j$ and $F_i = 2e \sum_{j=1}^i f_j$. Substituting $n_i = m_i - m_{i-1}$ and $2e f_i = F_i - F_{i-1}$ into (1) one easily gets convinced that the charging energy $H_c$ can be obtained by minimizing

$$H_c(Q) = \sum_{i=1}^N \frac{(2e m_i - F_i - Q_i)^2}{2C_J} + \frac{(Q_i - Q_{i+1})^2}{2C_0}$$

with respect to continuous charge variables $Q_i$. That is $H_c = \min Q[H_c(Q)]$. The quasi-charges $Q_i$ are well known in the theory of Coulomb blockade and appear naturally in the theories including a phenomenological inductance[9][11]. Their electrostatic meaning and the
derivation with inductance is explained in the Supplementary Material.

The introduction of $Q_i$ is equivalent to a Hubbard-Stratonovich transformation in the sense that the real time (Keldysh) partition function can be obtained as

$$Z = \int \prod_i Dm_i D\phi_i e^{i \int dt \left[ \sum_i m_i \phi_i - H_F - H_J \right]}$$

$$= N \int \prod_i DQ_i Dm_i D\phi_i e^{i \int dt \left[ \sum_i m_i \phi_i - H_F (Q) - H_J \right]} ,$$

(3)

where $N$ is a normalization factor. Here $\phi_i \equiv \theta_i - \theta_{i+1}$ is the phase drop on the Josephson junction conjugate to the charge variable $m_i$.

For each path of the quasi-charges $Q_i(t)$ the Hamiltonian separates into Hamiltonians of independent Josephson junctions biased each by charge $Q_i + F_i$, i.e.,

$$H_i = \frac{1}{2C_j} (2em_i - Q_i - F_i)^2 - E_j \cos \phi_i .$$

(4)

To obtain the effective quasi-charge theory one has to integrate out the discrete charge degrees of freedom $m_i, \phi_i$. In Ref. [12], this was done for the adiabatic paths $Q_i(t)$. These are paths that do not induce Landau-Zener transitions between the diabatic energy levels of [1], i.e., energy levels for fixed $Q_i$. In Ref. [12], the derivation in absence of disorder was provided. A similar derivation also holds in presence of disorder as for a single-site Hamiltonian $H_i$ disorder charge can be absorbed into the quasi-charge $Q_i^F \equiv Q_i + F_i$. Thus, the following effective Lagrangian emerges

$$\mathcal{L} = \sum_i \left[ \frac{L_B(Q_i^F)}{2} \dot{Q}_i^2 - \frac{(Q_i - Q_{i+1})^2}{2C_0} - U_Q [Q_i^F] \right] .$$

(5)

Here $L_B(Q)$ is the Bloch inductance [12, 13] whereas $U_Q [Q]$ is the Bloch band energy. Thus the mass term is generated and the phenomenological inductance is not needed. In this paper, we are interested in depinning and approach this transition from the non-dynamical pinned side. Thus, we conjecture that the description in terms of slow adiabatic paths $Q_i(t)$ is sufficient. Moreover, in Ref. [12] it has been shown that the periodic potential $U_Q [Q_i^F]$ is relevant in the RG sense in a wide range of parameters.

It has been noticed by Gurarie and Tsvelik [11] that the quasi-charge model of Eq. (5) is equivalent to the well studied case of charge density waves (CDW) pinned by a random potential. As we are interested only in the onset of transport (depinning) it is sufficient to focus on the potential energy part of (5)

$$H_e = \sum_i \left[ \frac{(Q_i - Q_{i+1})^2}{2C_0} + U_Q [Q_i + F_i] \right] - Q_{i=1} V .$$

(6)

Here the last term has been added to describe the voltage bias $V$ applied on the left edge of the array. To transform an edge voltage bias to a homogeneous electric field we perform the following transformations $\tilde{Q}_i \equiv Q_i - C_0 V(N + 1 - i)(N - i)/2N$ and $\tilde{F}_i \equiv F_i + C_0 V(N + 1 - i)(N - i)/2N$, where $N$ is the length of the array. This gives

$$H_e = \sum_i \left[ \frac{(\tilde{Q}_i - \tilde{Q}_{i+1})^2}{2C_0} + U_Q [\tilde{Q}_i + \tilde{F}_i] - E \tilde{Q}_i \right] ,$$

(7)

where $E \equiv V/N$ is the homogeneous depinning force (electric field). We assume a strong charge disorder which is equivalent to a homogeneous distribution of $F_i$ between $-e$ and $e$ and statistical independence of $F_i$ and $F_j$ for $i \neq j$. Indeed, the disorder $F_i$ is effectively limited to the interval $[-e, e]$ as any deviation thereof is compensated by adjusting the number of Cooper-pairs on the islands. With this assumption, the shift of the quasi-charge to include the voltages applied at the boundaries does not change the distribution function of the random charge $\tilde{F}_i$. This property of the maximally disordered model is also referred to as statistic tilt symmetry [19].

In the continuum limit justified by large $\Lambda$, we obtain the following well established continuum model for CDW-depinning [20] (tildes are omitted)

$$H_e = \int dx \left[ \frac{(\partial_x Q(x))^2}{2C_0} + U_Q [Q(x) + F(x)] - E Q(x) \right] ,$$

(8)

where the spatial coordinate $x$ is measured in units of the array lattice constant, so that capacitance of one island $C_0$ is also naturally the capacitance density per unit length. As discussed in detail in the literature [20, 25], the critical value of the depinning force is determined by the competition between the disorder pinning potential and the elastic energy. The two become comparable at the so called Larkin length $L_L$ (a.k.a. Fukuyama-Lee length or Imry-Ma length) [20, 23, 26] and depinning happens at $E \approx 2 e f_p$, where $f_p \approx (C_0 L_1^2)^{-1}$ [27].

In this formalism [20], the following correlation function of the pinning potentials plays an important role

$$R(Q_1, x_1, Q_2, x_2) \equiv \langle U_Q (Q_1, x_1) U_Q (Q_2, x_2) \rangle$$

$$= \int \mathcal{d} \{F\} p(\{F\}) U_Q [Q_1 + F(x_1)] U_Q [Q_2 + F(x_2)] ,$$

(9)

where $\mathcal{d} \{F\}$ and $p(\{F\})$ denote the integration over the disorder distribution. Due to the assumption of maximal disorder, the cumulative disorder $F(x)$ is delta-correlated in $x$ which in turn leads to a spatial delta correlation of the disordered pinning potential:

$$R(Q_1, x_1, Q_2, x_2) \approx R(Q_1 - Q_2) \delta(x_1 - x_2) .$$

(10)
Finally, one obtains the Larkin length \( L_L \) (in units of the array lattice constant)

\[
L_L = 3^{1/3} C_0^{-\frac{2}{3}} \left| \frac{1}{(2e)^2} \frac{\partial^2 R(Q)}{\partial Q^2} \right|_{Q=0}^{-\frac{1}{2}}. 
\]

The following representation is useful for analysis of the behavior of the switching voltage:

\[
\frac{\partial^2 R(Q)}{\partial Q^2} \bigg|_{Q=0} = \frac{E_C^m}{(2e)^2} \tilde{R} \left( \frac{E_J}{E_C} \right). 
\]

(12)

The dimensionless function \( \tilde{R} \) can be obtained numerically (see Supplementary Material). Thus, we arrive at the main theoretical result of this paper, i.e., the following estimate for the switching voltage

\[
V_{sw} = \frac{NE_C}{2e} \Lambda^{-\frac{2}{3}} 3^{-\frac{1}{3}} \left\{ \tilde{R} \left( \frac{E_J}{E_C} \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \right) \right\}^{\frac{2}{3}}. 
\]

(13)

We use Eq. (13) to fit the experimental data for arrays A255, B255, and C255 as shown in Fig. 2. We assume \( E_J^m \) to be known as it can be obtained relatively precisely from the measurements of the normal resistance of the junctions. Thus we use \( \Lambda \) and \( E_C \) as fitting parameters. For the other three arrays, i.e., D39, E59 and F255, we have to take into account the \( ABABAB \ldots \) pattern in the areas of the SQUIDs. As the disorder charges \( F_i \) on neighboring junctions are uncorrelated we can account for the \( ABABAB \ldots \) pattern by doubling the size of the unit cell and using a fitting function of the form

\[
V_{sw} = \frac{NE_C}{2e} \Lambda^{-\frac{2}{3}} 3^{-\frac{1}{3}} \left\{ \tilde{R} \left( \frac{E_J}{E_C} \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \right) \right\}^{\frac{2}{3}} + \frac{1}{2} \tilde{R} \left( \frac{E_J}{E_C} \cos \left( \frac{3\pi \Phi}{\Phi_0} \right) \right)^{\frac{2}{3}}. 
\]

(14)

In this case, we use the areas \( A \) and \( B \) as additional fitting parameters. As shown in Fig. 3 we are able to reproduce the double peak structure due to the areas pattern with \( A \neq B \). Our fitting results are shown in Tab. I.

**TABLE I.** The experimental estimates and fitted values for Josephson junction arrays A255 to F255.

| array | \( N \) | \( E_C^{\exp} \) | \( E_J^{m} \) | \( \Lambda \) | \( E_C \) |
|-------|-------|-------------|-------------|----------|--------|
| A255  | 255   | 380\( \mu \)eV | 125\( \mu \)eV | 3.16 ± 0.06 | 310 ± 4\( \mu \)eV |
| B255  | 255   | 372\( \mu \)eV | 113\( \mu \)eV | 6 ± 0.1   | 267 ± 3\( \mu \)eV |
| C255  | 255   | 368\( \mu \)eV | 125\( \mu \)eV | 2.018 ± 0.03 | 239 ± 2\( \mu \)eV |
| D39   | 39    | 88\( \mu \)eV  | 180\( \mu \)eV | 9.5 ± 0.3 | 103 ± 2\( \mu \)eV |
| E59   | 59    | 116\( \mu \)eV | 160\( \mu \)eV | 14.6 ± 0.4 | 121 ± 2\( \mu \)eV |
| F255  | 255   | 148\( \mu \)eV | 117\( \mu \)eV | 18.2 ± 1.2 | 168 ± 5\( \mu \)eV |

We obtain values of \( E_C \) and \( \Lambda \) that are comparable with the ones expected from geometrical estimates for the capacitances \( C_J \) and \( C_0 \). The latter predict \( \Lambda^{\exp} \approx 10 \) for all arrays and \( E_C^{\exp} \) as shown in Tab. I. Only for the arrays A255 and C255 we obtain \( \Lambda \)-values that are significantly smaller than those we expect. One possible
Polyakov, S.V. Syzranov, and T. Giamarchi for helpful investigations [29]. It will be interesting to match this systems to the continuum limit. Transport well above of very short systems or at the crossover from discrete possibly help us to study depinning physics in the limit think this could be particularly interesting as Joseph-
therefore be linked to a rich research area of physics. We density waves, vortices in type II superconductors etc. depinning effect, similar to that of depinning of charge lating state in Josephson junction arrays is a collective retically analyze the switching voltage and fit the exper-
employ the connection to the theory of charge density introducing artificial large inductances [1, 6, 7, 11]. We Ref. [12] we argue that this model can be applied without 

We notice the essential difference between the physics we describe here and a situation in which a single charge soliton is being de-pinned in a disordered array. The latter case was analyzed within the disordered sine-Gordon model [28]. It was shown that the depinning critical force grows with the soliton length $\Lambda$. In our case, however, the depinning transition is a collective phenomenon happening in the whole array. At the transition point the array contains, on average, one extra charge of $2e$ per Larkin length, $L_L \propto \Lambda^{4/3} \tilde{R}^{-1/3}$. Thus, the bigger is $\Lambda$, the fewer charges are pinned and the easier is the depinning, $f_p \propto \Lambda^{-\frac{2}{3}} \tilde{R}^{2/3}$.

In this paper we have compared the experimentally measured magnetic flux dependence of the switching voltage of an insulating (Coulomb blockaded) SQUID-array with our theoretical predictions based on a sine-Gordon-like model for a continuous quasi-charge field. Based on Ref. [12] we argue that this model can be applied without introducing artificial large inductances [1] [6] [7] [11]. We employ the connection to the theory of charge density wave depinning, first pointed out in Ref. [11], to theoretically analyze the switching voltage and fit the experimental data. We find that the breakdown of the insulating state in Josephson junction arrays is a collective depinning effect, similar to that of depinning of charge density waves, vortices in type II superconductors etc. The switching behavior of Josephson junction arrays can therefore be linked to a rich research area of physics. We think this could be particularly interesting as Josephson junction arrays are artificially fabricated and could possibly help us to study depinning physics in the limit of very short systems or at the crossover from discrete systems to the continuum limit. Transport well above the switching voltage remains the subject of continuing investigations [29]. It will be interesting to match this transport regime with the depinning physics analyzed in this paper.

We thank A.D. Mirlin, J.H. Cole, B. Kießig, D.G. Polyakov, S.V. Syzranov, and T. Giamarchi for helpful discussions of the subject.

[1] D. B. Haviland, K. Andersson, and P. Ågren, J. Low Temp. Phys. 118, 733 (2000)
[2] A. Ergül, D. Schaeffer, M. Lindblom, D. B. Haviland, J. Lidmar, and J. Johansson, Phys. Rev. B 88, 104501 (2013)
[3] A. Ergül, J. Lidmar, J. Johansson, Y. Azizoğlu, D. Schaeffer, and D. B. Haviland, New Journal of Physics 15, 095014 (2013)
[4] J. Zimmer, N. Vogt, A. Fiebig, S. V. Syzranov, A. Lukaschenko, R. Schäfer, H. Rotzinger, A. Shnirman, M. Marthaler, and A. V. Ustinov, Phys. Rev. B 88, 144506 (2013)
[5] E. Ben-Jacob, K. Mullen, and M. Amman, Physics Letters A 135, 390 (1989)
[6] Z. Hermon, E. Ben-Jacob, and G. Schön, Phys. Rev. B 54, 1234 (1996)
[7] D. B. Haviland and P. Delsing, Phys. Rev. B 54, R6857 (1996)
[8] P. Ågren, K. Andersson, and D. Haviland, J. Low Temp. Phys. 124, 291 (2001)
[9] I. L. Aleine and I. Ruzin, Physical Review Letters 72, 1056 (1994)
[10] M. Fogler, Physical Review Letters 88, 1 (2002)
[11] V. Gurarie and A. M. Tsvelik, J. Low Temp. Phys. 135, 245 (2004)
[12] J. Homfeld, I. Protopopov, S. Rachel, and A. Shnirman, Phys. Rev. B 83, 10 (2010) [arXiv:1008.5123]
[13] A. B. Zorin, Physical Review Letters 96, 1 (2006)
[14] R. Schäfer, W. Cui, K. Grube, H. Rotzinger, and A. V. Ustinov, “Dissipation mechanism above the current threshold in Josephson junction chains,” (2013), arXiv:1310.4295
[15] This does not exclude an exponentially suppressed current due to thermal activation near zero bias, observable at elevated temperatures above $\sim 200\text{mK}$, see for example Ref. [4].
[16] G.-L. Ingold and Y. V. Nazarov, in Single Charge Tunneling: Coulomb Blockade Phenomena in Nanostructures (NATO Science Series: B:); NATO Science Series: B, Vol. 294, edited by H. Grabert and M. H. Devoret (Plenum Press, New York, 1992) 1st ed., Chap. Chapter 2, pp. 21–107, [0508728 [cond-mat].
[17] A. Middleton and N. Wingreen, Physical Review Letters 71, 3198 (1993)
[18] J. Johansson and D. Haviland, Phys. Rev. B 63, 1 (2000).
[19] A. Fedorenko, P. Le Doussal, and K. Wiese, Phys. Rev. E 74, 061109 (2006).
[20] S. Brazovskii and T. Nattermann, Advances in Physics 53, 177 (2004).
[21] P. Chauve, T. Giamarchi, and P. Le Doussal, Phys. Rev. B 62, 6241 (2000).
[22] P. Chauve, P. Le Doussal, and K. Jörg Wiese, Physical Review Letters 86, 1785 (2001).
[23] A. I. Larkin and Y. N. Ovchinnikov, Sov Phys JETP 38, 854 (1973).
[24] A. I. Larkin and Y. N. Ovchinnikov, J. Low Temp. Phys. 34, 409 (1979).
[25] H. Fukuyama and P. Lee, Phys. Rev. B 17, 535 (1978).
[26] Y. Imry and S.-K. Ma, Physical Review Letters 35, 1399 (1975).
[27] We use here notations of Ref. [20] The estimate for the depinning force $f_p$ can be improved with the help of an RG treatment [20] [22], using the classical equation of motion obtained from Eq. [8] and adding a phenomenologi-cal linear friction. This improved estimate is called $f_\text{e}$ in Ref. [20].
[28] K. G. Fedorov, M. V. Fistul, and A. V. Ustinov, Phys. Rev. B 84 (2011), 10.1103/PhysRevB.84.014526
[29] J. H. Cole, J. Leppäkangas, and M. Marthaler, New Journal of Physics 16, 063019 (2014).
The introduction of inductances $L_0$ into the model (see Fig. 4) necessitates a description in terms of continuous as well as discrete charge variables. The discrete ones are the overall charges $2en_i$ of the islands. The continuous ones are the charges $q_i$ on the junction capacitances $C_J$ and charges $q^0_i$ on the capacitances to the ground $C_0$. Conservation of charge requires
\begin{equation}
2e n_i - f_i - q^0_i + q_{i-1} - q_i = 0 ,
\end{equation}
where $f_i$ are the random offset charges. Introducing the integrated charge variables $m_i = \sum_{j=1}^i n_j$, $Q_i = 2e \sum_{j=1}^i q^0_j$, and $F_i = 2e \sum_{j=1}^i f_j$ one can easily obtain the following hamiltonian
\begin{align}
H &= \sum_{i=1}^N \left[ \frac{1}{2C_J} (2em_i - F_i - Q_i)^2 - E_J \cos \phi_i \\
&\quad + \frac{1}{2C_0} (Q_i - Q_{i+1})^2 + \frac{1}{2L_0} \Phi_i^2 \right] ,
\end{align}
where $\Phi_i$ is the flux on the inductance $L_0$ of the $i$-th island whereas $\phi_i$ is the phase drop on the $i$-th Josephson junction. The pairs of canonically conjugated variables in (16) are $(Q_i, \Phi_i)$ and $(m_i, \phi_i)$. The physical meaning of $Q_i$ is clarified by the following relation
\begin{equation}
q_i = Q_i + F_i - 2em_i ,
\end{equation}
which can be obtained using (15). The charge on the junction capacitance $q_i$ is given by the total charge that has flown into the junction $Q_i + F_i$ minus the discrete charge $2em_i$ that has tunneled through the junction. As $F_i$ is a constant offset charge, we understand that $Q_i$ is the integral of current that has flown into the junction.

In Ref. [6] the inductance $L_0$ was assumed to be large, so that the dynamics of $(Q_i, \Phi_i)$ is adiabatic. In the current paper we assume $L_0 \to 0$ and claim that the emerging Bloch inductance, the large screening length $\Lambda$ and the pinning disorder render an adiabatic regime in the vicinity of the depinning point.

### SUPPLEMENTARY MATERIAL

#### Array with inductances

The dimensionless correlation function $\tilde{R}$ as a function of $E_J/E_C$ in the main plot and as a function of the magnetic flux $\phi$ in the inset.

The correlation function $R(Q, x = 0)$ can be obtained by numerically diagonalizing the single junction Hamiltonian
\begin{equation}
H(Q) = E_C \left( \left( n - \frac{Q}{2e} \right)^2 + \frac{E_J}{E_C} (|m+1\rangle \langle m| + h.c.) \right)
\end{equation}
for a dense set of $Q$-values in the interval $[-e, e]$. For diagonalisation we use 15 charge state $|m\rangle$ with lowest charging energy. Including more states does not change the ground state energy $E_Q(Q)$ within our level of numerical accuracy. Having numerically obtained $E_Q(Q)$, we integrate numerically over the disorder, differentiate with respect to $Q$ to obtain the value of the function $\tilde{R}$ for one fixed value of $E_J/E_C$. The result is shown in Fig. 5.