AUTOMORPHISMS OF CATEGORIES OF FREE MODULES
AND FREE LIE ALGEBRAS.

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Abstract. Let Θ^0 be a category of finitely generated free algebras in the variety of algebras Θ. Solutions to problems in algebraic geometry over Θ are often determined by the structure of the group of automorphisms Aut Θ^0 of category Θ^0. Here we consider two varieties Θ: noetherian modules and Lie algebras. We show that every automorphism in Aut Θ^0, where Θ is the variety of modules over noetherian rings, is semi-inner. A similar result for the variety of Lie algebras over a infinite field has been recently obtained in [9]. Here we present a different approach allowing us to shorten the proof in the Lie case.

1. Introduction

1.1. Automorphisms of category. Let Φ be a small category. Denote by End Φ the semigroup of endofunctors of the category Φ and by Aut Φ the subgroup of invertible elements of End Φ.

Recall that functor isomorphism s : ϕ_1 → ϕ_2, ϕ_1, ϕ_2 ∈ End Φ, is a collection of isomorphisms s_A : ϕ_1(A) → ϕ_2(A) defined for all A ∈ Ob Φ and such that for every ν : A → B, ν ∈ Mor Φ, holds

s_B · ϕ_1(ν) = ϕ_2(ν) · s_A,

i.e., the following diagram is commutative

ϕ_1(A) −→ s_A ϕ_2(A)
|         v         |
ϕ_1(ν)  |         v         | ϕ_2(ν)
|         v         |
ϕ_1(B) −→ s_B ϕ_2(B)

A relation ∼ of the isomorphism on functors of a category Φ is a congruence on the semigroup End Φ. Denote End^s Φ = End Φ / ∼. We have a natural homomorphism δ : End Φ → End^s Φ which induces a homomorphism Aut Φ → Aut^s Φ, where Aut^s Φ is the group of invertible elements of the semigroup End^s Φ.

Definition 1.1. An automorphism ϕ ∈ Aut^s Φ of the category Φ is isomorphic to the identical one is called inner.

That means an automorphism ϕ : Φ → Φ is inner if there exists a function s which chooses for each object A ∈ Ob Φ an isomorphism s_A : A → ϕ(A) such that for every ν : A → B holds

A −→ s_A ϕ(A)
|       v       |
ϕ(ν)  |       v       |
|       v       |
B −→ s_B ϕ(B)
Here $\varphi(\nu) = s_B \cdot \nu \cdot s_B^{-1}$. It is clear that the kernel $\text{Ker} \nu$ of the homomorphism $\delta$ consists all inner automorphisms of the category $\Phi$.

Denote by $\text{Int} \Phi$ the group of inner automorphisms of the category $\Phi$.

**Definition 1.2.** Group $\text{Out} \Phi = \text{Aut} \Phi / \text{Int} \Phi$ is called the group of outer automorphisms of the category $\Phi$.

**Definition 1.3.** An automorphism $\varphi$ of the category $\Phi$ is called a special automorphism of this category if for each object $A \in \Phi$ we have $A \simeq \varphi(A)$.

**Definition 1.4.** The category is called special if each of its automorphisms is special.

**Definition 1.5.** The category is called perfect if each of its automorphisms is inner.

It is easy to prove

**Proposition 1.6.** If $\varphi$ is a special automorphism of the category $\Phi$ then it can be represented in the following form: $\varphi = \varphi_0 \cdot \varphi_1$, where $\varphi_0$ is an inner automorphism of the category $\Phi$ and $\varphi_1$ is an automorphism which changes no object of $\Phi$.

Below we consider the category $\Theta^0$ of free algebras $W = W(X)$ generated by finite sets $X$, where each $X$ belongs to the universe $X^0$. In this case the category $\Theta^0$ is a small category.

It has been proven [10] that the homomorphism $\delta : \text{Aut} \Theta^0 \rightarrow \text{Aut}^0 \Theta^0$ is a surjective.

Varieties of three following types are important for our consideration below.

**Definition 1.7.** A variety $\Theta$ is notherian if each object of the category $\Theta^0$ is notherian with respect to its congruences.

**Definition 1.8.** A variety $\Theta$ is hopfian if each object of the category $\Theta^0$ is hopfian, i.e. every surjective endomorphism $\mu : W \rightarrow W$ is an automorphism.

**Definition 1.9.** A variety $\Theta$ is regular if the condition $W(X) \simeq W(Y)$ implies $|X| \simeq |Y|$.

The following implications are true: $\text{NP} \Rightarrow \text{IS} \Rightarrow \text{KJ}$

We recall several results related to the category $\Theta^0$.

**Theorem 1.10.** [10] Let $W_0 = W(x_0)$ be a cyclic algebra generated by $x_0$, $\Theta$ a hopfian variety and $\varphi$ an automorphism of $\Theta^0$. Then from $\varphi(W_0) \simeq W_0$ follows that $\varphi$ is a special automorphism.

Let $\Theta$ be a variety generated by a free algebra $W^0 = W(x_1, ..., x_k)$ and $\nu_0 : W^0 \rightarrow W_0$ be a homomorphism determined by the equalities $\nu_0(x_i) = x_0, i = 1, ..., k$.

**Theorem 1.11.** [10] Let $\Theta$ be a hopfian variety and $\varphi$ an automorphism of the category $\Theta^0$ not changing its objects and acting on $\text{End} W^0$ as an identical automorphism. In addition let $\varphi(\nu_0) = \nu_0$. Then $\varphi$ is an inner automorphism.

Note that any automorphism $\varphi$ of the category $\Theta^0$ which does not changes objects of $\Theta^0$ induces an automorphism $\varphi_W$ of the semigroup $\text{End} W$ for every $W = W(X)$, $|X| < \infty$. Furthermore, $\varphi$ induces a substitution on the set $\text{Hom}(W_0, W)$. Denote this substitution by $\sigma_X$. 
Theorem 1.12. Let \( \nu : W(X) \rightarrow W(Y) \) be a homomorphism. Then \( \varphi(\nu) = \sigma_Y \cdot \nu \cdot \sigma_X^{-1} \).

1.2. Geometries over algebras. Recall the main definitions from the universal algebraic geometry \([11]\). Let \( \Theta \) be a variety of algebras and \( W = W(X) \in \Theta \) be a free algebra from \( \Theta \) over a finite set of generators \( X, |X| = n \). The set \( \text{Hom}(W, G) \), \( G \in \Theta \) can be treated as an affine space whose points are homomorphisms. Every solution of an equation \( w = w' \) in \( W \) is a homomorphism \( \mu : W \rightarrow G \) such that \( w^\mu = w'^\mu \). The set \( A \subseteq \text{Hom}(W, G) \) is called an algebraic set over the algebra \( G \) if there exists a system of equations \( T \) (or a binary relation \( T \) on \( W \)) such that \( T' = A \), where \( T' = \{ \mu \mid T \subseteq \text{Ker} \mu \} \). It is possible to define also the relation \( A' \) on \( W \): \( A' = T = \bigcap_{\mu \in A} \text{Ker} \mu \) and if \( A \) is an algebraic set then \( A'' = A \).

Denote by \( K_\Theta(G) \) the category of pairs \((A, X)\), where \( A \) is an algebraic set over \( G \), i.e., \( A \subseteq \text{Hom}(W(X), G) \). Morphisms in this category are mappings \((\overline{s}, s) : (A, X) \rightarrow (B, Y)\),

where \( s : W(Y) \rightarrow W(X) \) is a mapping in category \( \Theta^\circ \) of free algebras from \( \Theta \) and \( \overline{s} : A \rightarrow B \) is such a mapping from \( A \) to \( B \) induced by \( s \) that the following diagram is commutative:

\[
\begin{array}{ccc}
W(Y) & \xrightarrow{s} & W(Y) \\
\mu_Y \downarrow & & \mu_X \\
W(Y)/B' & \xrightarrow{\overline{s}} & W(X)/A'
\end{array}
\]

Here \( \mu_X \) and \( \mu_Y \) are the natural homomorphisms. The category \( K_\Theta(G) \) is a geometric invariant of \( G \). Algebras \( G_1 \) and \( G_2 \) are categorically equivalent if the categories \( K_\Theta(G_1) \) and \( K_\Theta(G_2) \) are isomorphic. If algebras \( G_1 \) and \( G_2 \) are categorically equivalent then these algebras define the same geometry. Algebras \( G_1 \) and \( G_2 \) are geometrically equivalent if

\[
T''_{G_1} = T''_{G_2}
\]

holds for all finite sets \( X \) and for all relations \( T \) in \( W = W(X), W \in \Theta \).

In some cases the categorical equivalence of algebras \( G_1 \) and \( G_2 \) coincides with the geometric equivalence of these algebras. Whether these two notions are or are not equivalent in \( \Theta \), depends on the structure of the group \( \text{Aut} \Theta^\circ \) \([11]\). The group \( \text{Aut} \Theta^\circ \) is known in the following cases: the variety of all groups, the variety of all \( F \)-groups, where \( F \) is a free group of constants, the variety of all semigroups, the variety of commutative associative algebras with unit element over an infinite field, the variety of all Lie algebras over an infinite field \([2, 9, 10, 11]\).

Here we give a description of the group \( \text{Aut} \Theta^\circ \) for the variety of all \( K \)-modules, where \( K \) is a noetherian ring, and present a different approach allowing to simplify the proof \([9]\) of the corresponding theorem for the variety of Lie algebras over an infinite field.

2. Modules

2.1. Preliminary remarks. Let \( K \) be an arbitrary ring with unit. Denote by \( C = \text{Mod} - K \) the category of all left \( K \)-modules. Every free module is a direct sum of cyclic modules over \( K \). The ring \( K \) is noetherian if it is noetherian as a
$K$-module. Thus the variety $\mathcal{C}$ is noetherian iff the ring $K$ is noetherian. In this case the variety $\mathcal{C}$ is also hopfian and regular.

We wish to study the semigroup of endomorphisms of a cyclic module over $K$. Let $F = Kx_0$ be a free cyclic $K$-module generated by $x_0 \in X$ and $\nu_\alpha : Kx_0 \to Kx_0$ be an endomorphism of this module such that $\nu_\alpha(x_0) = \alpha x_0$, $\alpha \in K$. Since $(\nu_\alpha \cdot \nu_\beta)(x_0) = \nu_{\beta \cdot \alpha}(x_0)$, the semigroup $\text{End} Kx_0$ is antiisomorphic to the multiplicative semigroup $K^\times$ of the ring $K$. We will show first that there exists isomorphism $\text{Aut} \ (\text{End} Kx_0) \simeq \text{Aut} K^\times$ between groups of automorphisms of the semigroups $\text{End} Kx_0$ and $K^\times$ respectively.

Let $\sigma$ be an automorphism of the semigroup $K^\times$, i.e., $(\beta \cdot \alpha)\sigma = \beta^\sigma \cdot \alpha^\sigma$, $\alpha, \beta \in K^\times$. Define a mapping $\sigma^*$ on $\text{End} Kx_0$ in the following way: $\nu_\alpha^* = \nu_{\alpha^*}$. We may write

$$(\nu_\alpha \cdot \nu_\beta)\sigma^* = \nu_{(\beta \cdot \alpha)}^* = \nu_{\beta \cdot \alpha}^* = \nu_\alpha^* \cdot \nu_\beta^*,$$

i.e., $\sigma^* \in \text{Aut} (\text{End} Kx_0)$.

Next we assume that $\sigma^* \in \text{Aut} (\text{End} Kx_0)$, i.e., $(\nu_\alpha \cdot \nu_\beta)\sigma^* = \nu_{(\beta \cdot \alpha)}^* \cdot \nu_\beta^*$, for all $\nu_\alpha, \nu_\beta \in \text{End} Kx_0$. Then there exists a mapping $\sigma : K^\times \to K^\times$ such that $\nu_\alpha^* = \nu_{\alpha^*}$. We will show that $\sigma \in \text{Aut} K^\times$.

We have

$$\nu_\alpha^* \cdot \nu_\beta^* = \nu_{\alpha^*} \cdot \nu_{\beta^*} = \nu_{\beta^*} \cdot \alpha^*.$$

On the other hand

$$(\nu_\alpha \cdot \nu_\beta)\sigma^* = \nu_{(\beta \cdot \alpha)}^* = \nu_{(\beta \cdot \alpha)^*}.$$

Thus $(\beta \cdot \alpha)^\sigma = \beta^\sigma \cdot \alpha^\sigma$ and $\sigma \in \text{Aut} K^\times$. It is clear that $(\sigma_1 \cdot \sigma_2)^* = \sigma_1^* \cdot \sigma_2^*$.

**Proposition 2.1.** Let $\mathcal{C}^0 = (\text{Mod} - K)^0$ be a category of finite generated free modules over noetherian ring $K$. If $\phi$ is an automorphism of the category $\mathcal{C}^0$ then $\phi_{Kx_0} = \sigma$ determines an automorphism of the ring $K$.

**Proof.** Let $A = Kx_1 \oplus Kx_2$, $x_1, x_2 \in X$ be a 2-generated free $K$-module and $s^0$ be its automorphism of the following form:

$$s^0(x_1) = x_1 + \alpha \cdot x_2, \quad \alpha \in K$$

$$s^0(x_2) = x_2$$

Denote by the same symbol the matrix of the automorphism $s^0$:

$$s^0 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

Consider the following commutative diagram

$$\begin{array}{ccc}
Kx_0 & \xrightarrow{\varepsilon_1} & Kx_1 \oplus Kx_2 \\
\downarrow{\delta_1} & & \downarrow{\pi_1} \\
& Kx_1 & \\
\end{array}$$

where $\delta_1 = \pi_1 \cdot \varepsilon_1$, $\varepsilon_1(x_0) = x_1$ and $\pi_1$ is the natural projection $Kx_1 \oplus Kx_2$ on $Kx_1$, i.e., $\pi_1(x_1) = x_1$, $\pi_1(x_2) = 0$. Applying the automorphism $\phi$ to the last diagram, we obtain

$$\begin{array}{ccc}
Kx_0 & \xrightarrow{\phi(\varepsilon_1)} & Kx_1 \oplus Kx_2 \\
\downarrow{\phi(\delta_1)} & & \downarrow{\phi(\pi_1)} \\
& Kx_1 & \\
\end{array}$$

Let $\phi(\varepsilon_1)(x_0) = y_1$ and $\phi(\pi_1)(y_1) = \phi(\delta_1)(x_0) = y_1^0 \in Kx_1$. 


Now, we consider another commutative diagram:

\[
\begin{array}{c}
Kx_0 \xrightarrow{\varepsilon_2} Kx_1 \oplus Kx_2 \\
\downarrow \delta_2 \\
Kx_2
\end{array}
\]

where \( \delta_2 = \pi_2 \cdot \varepsilon_2 \), \( \varepsilon_2(x_0) = x_2 \) and \( \pi_2 \) is the natural projection \( Kx_1 \oplus Kx_2 \) on \( Kx_2 \), i.e., \( \pi_2(x_2) = x_2 \), \( \pi_2(x_1) = 0 \). Applying \( \phi \) to the last diagram, we obtain

\[
\phi(\varepsilon_2)(x_0) = y_2, \quad \phi(\pi_2)(y_2) = \phi(\delta_2)(x_0) = y_2^0 \in Kx_2.
\]

Since the ring \( K \) is noetherian, we receive

\[
Kx_1 \oplus Kx_2 = Ky_1 \oplus Ky_2
\]

Now, consider the diagram

\[
\begin{array}{c}
Kx_0 \xrightarrow{\varepsilon_1} Kx_1 \oplus Kx_2 \\
\downarrow \delta \\
Kx_2
\end{array}
\]

where \( \pi_2 \cdot \varepsilon_1 = \delta \). It is clear that \( \pi_1 \cdot \varepsilon_2 = 0 \) and so

\[
\phi(\varepsilon_1) \cdot \phi(\varepsilon_2) = 0 \quad \text{and} \quad \phi(\varepsilon_2) = 0.
\]

Thus

\[
\phi(\varepsilon_1) \cdot \phi(\pi_2)(y_2) = 0 \quad \text{and} \quad \phi(\varepsilon_2) = 0.
\]

We obtained the following relations:

\[
\phi(\pi_1)(y_2) = 0, \quad \phi(\pi_1)(y_1) = y_1^0 \in Kx_1, \\
\phi(\pi_2)(y_2) = 0, \quad \phi(\pi_2)(y_2) = y_2^0 \in Kx_2.
\]

By definition of \( s^\alpha \)

\[
s^\alpha(x_1) = s^\alpha \varepsilon_1(x_0) = x_1 + \alpha \cdot x_1, \\
\pi_1 s^\alpha \varepsilon_1(x_0) = x_1 = \delta_1(x_0).
\]

Thus \( \pi_1 s^\alpha \varepsilon_1 = \delta_1 \) and so \( \phi(\pi_1) \phi(s^\alpha) \phi(\varepsilon_1) = \phi(\delta_1) \). Applying the last equality to \( x_0 \), we get

\[
\phi(\pi_1) \phi(s^\alpha)(y_1) = \phi(\delta_1)(x_0) = y_1^0 = \phi(\pi_1)(y_1).
\]

Further we have

\[
\pi_2 s^\alpha \varepsilon_1(x_0) = \alpha \cdot x_2 = \alpha \cdot \delta_2(x_0) = \delta_2(\alpha \cdot x_0) = \delta_2 \nu_\alpha(x_0).
\]

Therefore \( \pi_2 s^\alpha \varepsilon_1 = \delta_2 \nu_\alpha \), hence

\[
\phi(\pi_2) \phi(s^\alpha)(\varepsilon_1) = \phi(\delta_2) \phi(\nu_\alpha) = \phi(\delta_2) \nu_\alpha^\sigma.
\]

Here \( \sigma \) is an automorphism of the semigroup \( K^\times \) corresponding to \( \phi \). Now, applying equalities (2.3) to the element \( x_0 \), we obtain

\[
\phi(\pi_2) \phi(s^\alpha)(y_1) = \phi(\delta_2) \nu_\alpha^\sigma(x_0) = \\
= \phi(\delta_2)(\alpha \sigma x_0) = \alpha \sigma \phi(\delta_2)(x_0) = \alpha \sigma y_2^0.
\]

We come to

\[
\phi(\pi_2) \phi(s^\alpha)(y_1) = \alpha \sigma y_2^0 = \alpha \sigma \phi(\pi_2)(y_2).
\]

Further we get

\[
s^\alpha(x_2) = x_2, \quad s^\alpha \varepsilon_2(x_0) = \varepsilon_2(x_0) \\
\]

\[
s^\alpha \varepsilon_2 = \varepsilon_2, \quad \phi(s^\alpha) \phi(\varepsilon_2) = \phi(\varepsilon_2).
\]
Applying the last equality to \( x_0 \), we obtain
\[
\varphi(s^\sigma)(y_2) = y_2. 
\] (2.5)

Now we need to represent the element \( \varphi(s^\sigma)(y_1) \) in the basis \( y_1, y_2 \). Let \( \varphi(s^\sigma)(y_1) = \lambda_1 \cdot y_1 + \lambda_2 \cdot y_2 \). By the equalities (2.1), (2.2), (2.4), (2.5) we calculate \( \lambda_1 \)
\[
\varphi(\pi_1)\varphi(s^\sigma)(y_1) = y_0^1 = \lambda_1 \varphi(\pi_1)(y_1) + \lambda_2 \varphi(\pi_2)(y_2) = \lambda_1 y_1^0, \text{ i.e. } \lambda_1 = 1,
\]
and \( \lambda_2 \)
\[
\varphi(\pi_2)\varphi(s^\sigma)(y_1) = \varphi(\pi_2)(y_1 + \lambda_2 y_2) = \lambda_2 \varphi(\pi_2)(y_2) = \lambda_2 y_2^0 = \alpha^\sigma y_2^0, \text{ i.e. } \lambda_2 = \alpha^\sigma.
\]

Finally, we obtain
\[
\varphi(s^\sigma)(y_1) = y_1 + \alpha^\sigma y_2
\]
\[
\varphi(s^\sigma)(y_2) = y_2.
\]

We showed that the matrix of \( \varphi(s^\sigma) \) relative to the basis \( y_1, y_2 \) has the same triangular form as \( s^\sigma \):
\[
s^\sigma = \begin{pmatrix} 1 & \alpha^\sigma \\ 0 & 1 \end{pmatrix}.
\]

We have \( s^\alpha s^\beta = s^{\alpha+\beta} \). Since
\[
\varphi(s^\alpha)\varphi(s^\beta) = \varphi(s^{\alpha+\beta}),
\]
we must have
\[
s^\alpha s^\beta = s^{\alpha+\beta} = s^{(\alpha+\beta)^\sigma}.
\]

Thus \( (\alpha + \beta)^\sigma = \alpha^\sigma + \beta^\sigma \) and \( \varphi_{Kx_0} = \sigma \) is an automorphism of the ring \( K \).

\[\square\]

2.2. Semi-inner automorphism of the category of free modules over a ring. Let us define the notion of a semi-inner automorphism of the category \( C^0 = (\text{Mod} - K)^0 \). Consider the category of modules with semimorphisms (semi-linear maps).

**Definition 2.2.** A semimorphism of the category \( \text{Mod} - K \) is a pair \((\sigma, s)\), where \( \sigma \in \text{Aut } K \) and \( s : A \rightarrow B \) an addition-compatible map such that \( s(\alpha \cdot a) = \alpha^\sigma s(a) \), \( a \in A \), \( \alpha \in K \).

**Definition 2.3.** An automorphism \( \varphi \) of the category \( (\text{Mod} - K)^0 \) is called semi-inner if there exists \( \sigma \in \text{Aut } K \) and a collection \((\sigma, s)\) of semi-isomorphisms such that \( (\sigma, s) : 1_{C^0} \rightarrow \varphi \) is semi-isomorphism of functors, i.e., for every object \( KX \in \text{Ob } C^0 \) there exists semi-isomorphism \( (\sigma, s_X) : KX \rightarrow \varphi(KX) \) and for each \( \nu : KX \rightarrow KY \) the following diagram is commutative:

\[
\begin{array}{ccc}
KX & \xrightarrow{s_X} & \varphi(KX) \\
\downarrow{\nu} & & \downarrow{\varphi(\nu)} \\
KY & \xrightarrow{s_Y} & \varphi(KY)
\end{array}
\]
Let \( \sigma \) be an automorphism of the ring \( K \) and \( KX = Kx_1 \oplus \ldots \oplus Kx_n \) be a free module over a ring \( K \). Define a mapping \( \sigma_X : KX \to KX \) such that if \( u = \sum_{i=1}^{n} \lambda_i x_i \) then \( \sigma_X(u) = \sum_{i=1}^{n} \lambda_i^\sigma x_i \). It is evident that
\[
\sigma_X(u + v) = \sigma_X(u) + \sigma_X(v) \quad \text{and} \quad \sigma_X(\lambda u) = \lambda^\sigma \sigma_X(u),
\]
for each \( u, v \in KX \) and \( \lambda \in K \).

Now, consider a mapping
\[
\varphi(\nu) = \sigma_Y^\nu \sigma_X^{-1} : KX \to KY.
\]
We show that this mapping is a homomorphism of modules. Clearly, it preserves the additive structure. Taking \( \lambda u \) for each \( \sigma \), i.e.,
\[
\nu \sigma \in \{ \subset K, \nu \}
\]
for all \( \nu \in KX, \lambda \in K, u \in KX \), we obtain
\[
\sigma_Y^\nu \sigma_X^{-1}(\lambda u) = \sigma_Y^\nu(\lambda^\sigma^{-1} \sigma_X^{-1}(u)) = 
\]
\[
= \sigma_Y^\lambda \sigma^{-1} \nu \sigma_X^{-1}(u) = \lambda \sigma_Y^\nu \sigma_X^{-1}(u),
\]
\[
\text{i.e., } \sigma_Y^\nu \sigma_X^{-1}(\lambda u) = \lambda \sigma_Y^\nu \sigma_X^{-1}(u).
\]
Therefore \( \sigma_Y^\nu \sigma_X^{-1} \) is a module homomorphism.

Denote by \( \hat{\sigma} \) an automorphism of the category of free \( K \)-modules of the following form:

1. \( \hat{\sigma}(KX) = KX, \ KX \in Ob \ C^0 \), i.e., \( \hat{\sigma} \) does not change objects of the category of the free \( K \)-modules.

2. \( \hat{\sigma}(\nu) = \sigma_Y^\nu \sigma_X^{-1} : KX \to KY \) for all \( \nu : KX \to KY; \ KX, KY \in Ob \ C^0. \)

It is clear that \( \hat{\sigma} \) is a semi-inner automorphism of the category \( C^0 \).

Note that the set \( S \) of the semi-inner automorphisms of the category \( C^0 \) is a subgroup in \( Aut \ C^0 \). The subgroup \( S \) contains the invariant subgroup of the inner automorphisms of the category \( C^0 \).

Now, we shall describe the automorphisms of the category \( (\text{Mod} - K)^0 \) over a noetherian ring \( K \).

**Theorem 2.4.** Let \( K \) be a noetherian ring. Every automorphism of the category \( C^0 = (\text{Mod} - K)^0 \) is semi-inner.

**Proof.** Suppose that \( \varphi \in Aut \ C^0 \). We shall show that \( \varphi(Kx_0) \simeq Kx_0 \). Recall that \( Aut(End Kx_0) \simeq Aut K^\times \). It is clear that \( Aut(End KX) \not\simeq Aut K^\times \) if \( |X| \neq 1 \). Therefore \( \varphi(KX) \not\simeq Kx_0 \) if \( |X| \neq 1 \) and \( \varphi(Kx_0) \simeq Kx_0 \).

Since \( K \) is a noetherian ring, the variety \( C \) is hopfian. By Theorem \( 1.10 \) every automorphism of the category \( C^0 \) is special. By Proposition \( 2.1 \) the automorphism \( \varphi \) can be represented as \( \varphi = \varphi_1 \cdot \varphi_2 \), where \( \varphi_1 \) is an inner automorphism of \( C^0 \) and \( \varphi_2 \) is an automorphism which does not change the objects of this category. Now, without loss of generality, we can assume that \( \varphi \) itself does not change the objects of the category \( C^0 \).

Take \( \varphi_{Kx_0} = \sigma \). Denote by the same letter the automorphism of the ring \( K \) corresponding to \( \sigma \) (see Proposition \( 2.4 \)). Let \( \hat{\sigma} \) be a semi-inner automorphism of the category \( C^0 \) corresponding to \( \sigma \). We shall show that \( \hat{\sigma} \) and \( \sigma \) act in the same way on \( End Kx_0 \).

Let \( \nu_0 : Kx_0 \to Kx_0 \) be an endomorphism of the cyclic module \( Kx_0 \). Consider the endomorphism \( \sigma x_0 \nu_0 \sigma^{-1} \). Here for the convenience we denote by \( x_0 \) the subset \( \{ x_0 \} \). We have
\[
\sigma x_0 \nu_0 \sigma^{-1}(x_0) = \sigma x_0 \nu_0(x_0) = \sigma x_0 (\alpha x_0) = \alpha^\sigma x_0 = \nu_0^\sigma(x_0).
\]
Therefore
\[ \sigma_{x_0} \nu_\alpha \sigma_{x_0}^{-1} = \nu_\alpha = \nu_\alpha \]

Thus \( \hat{\sigma}_{Kx_0} = \sigma \) and \( \varphi_1 = \hat{\sigma}^{-1} \varphi \) is an automorphism of \( C^0 \) acting identically on the semigroup \( \text{End}(Kx_0) \).

Recall that the variety \( C \) generated by a cyclic module \( Kx_0 \). Consider a homomorphism \( \nu_0 : Kx_0 \rightarrow Kx_0 \) such that \( \nu_0(x_0) = x_0 \). It is clear that \( \varphi_1(\nu_0) = \nu_0 \).

Hence all conditions of Theorem 1.11 hold, and the automorphism \( \varphi_1 = \hat{\sigma}^{-1} \varphi \) is an inner automorphism of the category \( C^0 \). Thus \( \varphi = \hat{\sigma} \varphi_1 \) is semi-inner. The proof is complete. \( \square \)

**Corollary 2.5.** If a ring \( K \) has non-trivial automorphisms, the variety \( \text{Mod} - K \) is perfect, i.e., all automorphisms of \( (\text{Mod} - K)^0 \) are inner.

**Corollary 2.6.** The variety of abelian groups is perfect.

Let now \( K \) be an arbitrary ring. Recall that the group \( \text{Out} K \) of outer automorphisms of the ring \( K \) is a factor group \( \text{Aut} K / \text{Int} K \), were \( \text{Aut} K \) and \( \text{Int} K \) are the groups of all automorphisms and the group of inner automorphisms respectively of the ring \( K \).

**Theorem 2.7.** If automorphisms of the variety \( (\text{Mod} - K)^0 \) are semi-inner then the group of outer automorphisms of the category \( C = (\text{Mod} - K)^0 \) is isomorphic to the group of outer automorphisms of the ring \( K \).

**Proof.** We should prove the isomorphism
\[ \text{Aut} C / \text{Int} C \simeq \text{Out} K \]

Denote by \( Sp C \) the group of the all special automorphisms of the category \( C \) and by \( St C \) the group of the all automorphisms which do not change objects of \( C \). We have
\[ Sp C = St C \cdot \text{Int} C \]
\[ Sp C / \text{Int} C \simeq St C / \text{Int} C \cap St C. \]

As a consequence of these formulas we get
\[ Sp C = \text{Aut} C \]
\[ \text{Aut} C / \text{Int} C \simeq St C / \text{Int} C \cap St C. \]

Now, it is sufficient establish an isomorphism
\[ St C / \text{Int} C \cap St C \simeq \text{Out} K. \]

Let us construct a group homomorphism
\[ \pi : St C \rightarrow \text{Aut} K. \]

Any automorphism \( \varphi \in St C \) induces an automorphism \( \varphi_{Kx_0} \) of the semigroup \( \text{End} Kx_0 \). Since \( \text{Aut}(\text{End} Kx_0) \simeq \text{Aut} K^\times \), there exists \( \sigma \in \text{Aut} K^\times \) corresponding to \( \varphi_{Kx_0} \). Put \( \pi(\varphi) = \sigma \).

Since \( \pi(\hat{\sigma}) = \sigma \) for all \( \sigma \in \text{Aut} K \), homomorphism \( \pi \) is surjective. Now consider the natural homomorphism \( \psi \)
\[ \psi : \text{Aut} K \rightarrow \text{Aut} K / \text{Int} K. \]

Denote by \( \overline{\pi} = \psi \cdot \pi \). We have to describe the kernel \( \text{Ker} \overline{\pi} \). By definition \( \varphi \in \text{Ker} \overline{\pi} \) iff the automorphism \( \sigma \) of the ring \( K \) corresponding to \( \varphi \) is an inner automorphism of
this ring. Equivalently, \( \varphi_{Kx_0} \) is an inner automorphism of the semigroup \( \text{End} Kx_0 \).

By Theorem 1.12 we have that \( \varphi \in \text{Int} C \cap \text{St} C \). Finally, we get

\[
\text{Ker } \varphi = \text{Int} C \cap \text{St} C.
\]

This completes the proof of the theorem.

**Corollary 2.8.** If \( K \) is a noetherian ring then the group of the outer automorphisms of the category \((\text{Mod} - K)^0\), is isomorphic to the group of the outer automorphisms of the ring \( K \).

\[\Box\]

3. Lie algebras

3.1. Automorphisms of semigroup of endomorphisms of 2-generated Lie algebras.

We consider the variety \( \Theta = \text{Lie} - K \) of all Lie algebras over field \( K \).

**Definition 3.1.** Let \( W = W(X) \) be a free Lie algebra over a field \( K \) and \( \delta \in \text{Aut} K \). Denote a mapping \( \delta \) of the algebra \( W \) to \( W \) in the following way:

1. \( \delta \) for \( u, w \) \in \( W \), \( \delta \) is an inner automorphism of the semigroup \( \text{End}Kx_0 \).
2. \( \delta \) for \( \lambda \cdot w \) \in \( W \), \( \lambda \in K \).

Obviously, that if \( u_1, ..., u_n \) is a bases of \( W \) then from \( w = \sum_{i=1}^{n} \lambda_i \cdot u_i \) follows

\[
\delta \cdot w = \sum_{i=1}^{n} \lambda_i \cdot u_i.
\]

**Definition 3.2.** Let \( F_1 \) and \( F_2 \) be Lie algebras over a field \( K \), \( \delta \) be an automorphism of \( K \) and \( \varphi : F_1 \rightarrow F_2 \) be a ring homomorphism of these algebras. A pair \( (\delta, \varphi) \) is called semi-homomorphism from \( F_1 \) to \( F_2 \) if

\[
\varphi(\lambda \cdot u) = \lambda^\delta \cdot \varphi(u), \quad \text{for } \alpha \in K, \ u \in F_1.
\]

**Proposition 3.3.** A pair \( (\delta, \delta W) \) is an semi-automorphism of \( W \).

**Proof.** It is sufficient to check that

\[
(3.1) \quad \delta \cdot w = [\delta \cdot u, \delta \cdot w]
\]

for \( u, w \in W \). Let \( V =< v_1, v_2, ..., v_n, ... > \) be the Hall-Shirshov base \([1], [3]\).

Represent \( u \) and \( w \) in the basis \( V \). It is well-known that a structural constants for this basis are integers \([3]\). The formula \([3.1]\) is a consequence of this fact. \[\Box\]

We will call \( \delta \) the field semi-automorphism of \( F \) defined by a field automorphism \( \delta \).

**Definition 3.4.** An automorphism \( T \) of \( \text{End} W \) is called semi-inner if it can be presented in the form: \( T = \rho \cdot \delta \), where \( \delta \) is a field semi-automorphism of \( \text{End} W \) and \( \rho \) is an inner automorphism of \( \text{End} W \).

**Theorem 3.5.** Every automorphism of semigroup \( \text{End} W \), where \( W = W(x, y) \), is semi-inner.
We prove the Theorem 3.5 in several steps.

I. It is known [4] that \( \text{Aut} W \cong GL_2(K) \). Therefore

\[
\text{Aut} \text{Aut} W \cong \text{Aut} GL_2(K)
\]

However the description of \( \text{Aut} GL_2(K) \) is known [5]. Namely, if \( S \in \text{Aut} GL_2(K) \), then there are two possibilities:

1. \( S(\beta) = \chi(\beta) \cdot \tau \cdot f(\beta) \cdot \tau^{-1} \), where \( \beta, \tau \in \text{Aut} GL_2, \chi : \text{Aut} W \rightarrow K^* \) is a multiplicative homomorphism, \( f \) is an automorphism of the field \( K \) and if \( \beta = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(K) \), then \( f(\beta) = \begin{pmatrix} f(a_{11}) & f(a_{12}) \\ f(a_{21}) & f(a_{22}) \end{pmatrix} \).

2. \( S(\beta) = \chi(\beta) \cdot f(\bar{\beta}) \cdot \tau^{-1} \), where \( \chi, \tau, \beta \) and \( f \) are as above and \( \bar{\beta} \) is the transformation contragradient to \( \beta \).

Note that if \( \beta \in GL_2(K) \) then \( \bar{\beta} = \delta^{-1} \cdot \beta \cdot \delta \) for some \( \delta \in GL_2(K) \). This follows from the conjugacy of Jordan forms of the matrix \( \beta \) and its contragradient matrix \( \bar{\beta} \) in \( GL_2(K) \). Therefore, up to conjugacy, it is sufficient to consider only the case 1. By virtue of the isomorphism (3.2) we conclude that if \( T \in \text{Aut} \text{Aut} W \) and \( \xi \in \text{Aut} W \), then there exists \( \tau \in GL_2(K) \) and \( \gamma \in \text{Aut} K \) such that

\[
T(\xi) = \chi(\xi) \cdot \tau^{-1} \cdot \gamma_W^{-1} \cdot \xi \cdot \gamma_W \cdot \tau.
\]

Putting

\[
T_1(\xi) = \gamma_W \cdot \tau \cdot T(\xi) \cdot \tau^{-1} \cdot \gamma_W^{-1},
\]

we obtain

\[
T_1(\xi) = \chi(\xi) \cdot \xi, \quad \xi \in \text{Aut} W.
\]

**Definition 3.6.** Let \( T \in \text{Aut} \text{End} W \) and the restriction \( T|_{\text{End} W} = T_1 \). Automorphism \( T \) is called a characteristic automorphism of \( \text{End} W \) if \( T_1 = \chi_1 \cdot I \), where \( I \) is the identical automorphism on \( \text{Aut} W \) and \( \chi_1 \) is a multiplicative homomorphism \( \text{Aut} W \) on \( K^* \).

Let \( T \) be an automorphism of the semigroup of endomorphisms of \( \text{End} W \). Then, obviously, \( T \in \text{Aut} (\text{Aut} F) \) and, replacing \( T \) by \( T_1 \), as above we can regard \( T \), up to conjugacy, as a characteristic automorphism on \( \text{Aut} W \).

II. Now consider a L-endomorphism of \( W \).

**Definition 3.7.** An endomorphism \( \sigma \in \text{End} W \) is called L-endomorphism of \( W \) if

\[
\begin{align*}
\sigma(x) &= a_{11}x + a_{12}y \\
\sigma(y) &= a_{21}x + a_{22}y,
\end{align*}
\]

where \( a_{ij} \in K \).

Denote by \( D \) the semigroup of L-endomorphisms of \( W \). There exists the natural isomorphism \( \varphi : D \rightarrow \text{Mp}_2(K) \), where \( \text{Mp}_2(K) \) is the semigroup of \( 2 \times 2 \) matrix over the field \( K \).

**Proposition 3.8.** Let \( T \in \text{Aut} (\text{End} W) \) be a characteristic automorphism of \( \text{End} W \), where \( W \) is a free 2-generated Lie algebra over a infinite field \( K \), and \( \sigma \in D \). Then there exists \( \gamma \in \text{Aut} K \) and \( \tau \in \text{Aut} W \) such that

\[
(3.3) \quad T(\sigma) = \tau \cdot \gamma_W \cdot \sigma \cdot \gamma_W^{-1} \cdot \tau^{-1}
\]
Proof. 1. We check first that our assertion holds for the idempotent \( \sigma \in \text{End} W \): 
\[
\sigma(x) = x \quad \text{and} \quad \sigma(y) = 0.
\]
Let 
\[
T \sigma(x) = a_{11} x + a_{12} y + \sum_{n=3}^{\infty} a_{1n} u_n(x, y)
\]
(3.4) 
\[
T \sigma(y) = a_{21} x + a_{22} y + \sum_{n=3}^{\infty} a_{2n} u_n(x, y)
\]
be the representations of \( T \sigma(x) \) and \( T \sigma(y) \) relative to same basis of \( W \), \( a_{ij} \in K \), and \( \deg u_n \geq 2 \) for \( n \geq 3 \). Denote by \( \text{cont}_x(u) \) (or \( \text{cont}_y(u) \)) a number of appearances of \( x \) (or \( y \)) in the word \( u \). Now consider the automorphism \( \varphi_\lambda \) of \( W_2 \): 
\[
\varphi_\lambda(x) = x, \quad \varphi_\lambda(y) = \lambda y, \quad \lambda \neq 0, 1.
\]
Then \( \varphi_\lambda \cdot \sigma = \sigma \). Therefore, \( T \varphi_\lambda \cdot T \sigma = T \sigma \), i.e. 
\[
k \cdot \varphi_\lambda \cdot T \sigma = T \sigma,
\]
where \( k = \chi(\varphi_\lambda) \neq 0 \). Substituting (3.4) into (3.3) and comparing the corresponding coordinates we obtain 
\[
k a_{1n} \lambda' = a_{2n}, \quad \text{where} \quad r = \text{cont}_y(u_n), \quad n \geq 2.
\]
Since the field \( K \) is infinite, \( a_{1n} = a_{2n} = 0 \) for \( n \geq 2 \). Hence, \( T \sigma \) is \( L \)-endomorphism of \( W \).

2. Now we consider the general case. Let \( \sigma = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) be an \( L \)-endomorphism of \( W \). If \( \det \sigma \neq 0 \) then \( \sigma \) is automorphism of \( W \) and, therefore, \( T \sigma \) is \( L \)-endomorphism. If \( \det \sigma = 0 \), then there exists \( d \in K \) such that 
\[
\sigma(x) = a_{11} x + a_{12} y
\]
\[
\sigma(y) = d a_{11} x + d a_{12} y.
\]
The eigenvalues of matrix \( \sigma = \begin{pmatrix} a_{11} & a_{12} \\ d a_{11} & d a_{12} \end{pmatrix} \) are \( \lambda_1 = a + d b \), \( \lambda_2 = 0 \). Therefore the possible Jordan forms of the matrix \( \sigma \) are 
\[
\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \lambda_1 \\ 0 & 0 \end{pmatrix}.
\]
Consider only the first case. Let \( \delta(x) = \lambda_1 x, \quad \delta(y) = 0, \quad \lambda_1 \neq 0 \). If \( \varphi_\lambda_1(x) = \lambda_1 x, \quad \varphi_\lambda_1(y) = y \) and \( \sigma_1(x) = x, \quad \sigma_1(y) = 0 \) then \( \delta = \varphi_\lambda_1 \cdot \sigma_1 \). Therefore, 
\[
T \delta = T \varphi_\lambda_1 \cdot T \sigma_1 = \chi(\varphi_\lambda_1) \cdot \varphi_\lambda_1 \cdot T \sigma_1.
\]
Since \( \sigma_1 \) is idempotent of \( W \), \( T \sigma_1 \) is an \( L \)-endomorphism of \( W \) (see point 1), \( T \delta \) is also an \( L \)-endomorphism, so \( T \sigma \) is \( L \)-endomorphism as well. We had proved that \( T D = D \) for all \( T \in \text{Aut} \text{End} W \). Now you need in the following result

**Proposition 3.9.** Let \( M_n(K) \) be a full matrix semigroup over a field \( K \) and \( f : M_n(K) \to M_n(K) \) is semigroup automorphisms of \( M_n(K) \). Then for some \( \gamma \in \text{Aut} K \) and for some \( \Phi \in GL_n(K) \) holds 
\[
f(A) = \Phi \cdot \gamma(A) \cdot \Phi^{-1}, \quad A \in M_n(K),
\]
where if \( A = (a_{ij}) \) then \( \gamma(A) = (\gamma(a_{ij})) \).

Now the formula (3.3) follows from Proposition 3.9. The proof of the theorem is complete. 

**Remark 3.10.** From Proposition 3.8 follows that any characteristic automorphism \( T \) of \( \text{End} W \) is the identical automorphism on the semigroup \( D \) of \( L \)-endomorphisms of \( W \) up to conjugacy.
Proposition 3.11. If $T$ is a characteristic automorphism of $End\mathcal{W}$ and $\sigma \in End\mathcal{W}$ then $T\sigma = \tau^{-1}_\lambda \sigma \tau_\lambda$, where $\tau_\lambda(x) = \lambda x$, $\tau_\lambda(y) = \lambda y$, $\lambda \neq 0 \in K$.

Proof. From now on we assume that any characteristic automorphism of $End\mathcal{W}$ is the identical automorphism on the semigroup $L$-endomorphisms of $W$ (see. Remark 3.10).

1. Denote by $d(p)$ an endomorphism of $W$ such that $d(p)(x) = p$, $d(p)(y) = y$, $p \in W$. We would like to show that

$$
T d(p) = \tau^{-1}_\lambda d(p) \tau_\lambda .
$$

(i) We assume first that $p = [x, y]$ i.e. $d(p)(x) = [x, y]$, $d(p)(y) = y$. Let

$$
T d(p)(x) = \alpha [x, y] + A(x, y),
$$

where $A(x, y) \in W$. Let $\varphi_\lambda(x) = \lambda \cdot x$, $\varphi_\lambda(y) = y$. Then $d([x, y]) \cdot \varphi_\lambda = \varphi_\lambda \cdot d([x, y])$.

Therefore

$$
\lambda \cdot A(x, y) = A(\lambda \cdot x, y).
$$

Let $A(x, y) = \sum m_i$ be a linear combination of Lie monomials $m_i$ from $W$. From the formula 3.7 follows that $cont_\lambda(m_i) = 1$ for all $i$. Now consider two endomorphisms $\varphi_1$, $\varphi_2$ of $W$: $\varphi_1(x) = \frac{1}{k} x$, $\varphi_1(y) = ky$ and $\varphi_2(x) = x$, $\varphi_2(y) = ky$ for $k \neq 0$. Then $\varphi_1 \cdot d([x, y]) = d([x, y]) \cdot \varphi_2$. Therefore, $(\varphi_1 \cdot T d([x, y]))(x) = (T d([x, y]) \cdot \varphi_2)(x)$.

Finally we have $\varphi_1 A(x, y) = A(x, y)$. Since $cont_\lambda(m_i) = 1$ for all $i$ we obtain $A(x, y) = \beta [x, y]$, i.e., $T d(p)(x) = \lambda [x, y]$, $(\lambda = \alpha + \beta)$.

Now we consider the endomorphism $\varphi$ of $W$: $\varphi(x) = \varphi(y) = y$. Since $d(p) \cdot \varphi = \varphi$, $(T d(p)) \varphi = \varphi$. Therefore $T d(p)(y) = y$. We have that the equality 3.6 is fulfilled in this case.

(ii) Let $p$ be a right-normalized monomial of length $n$ from $W$. We prove the validity of (3.6) using induction on $n$. We have the basis of induction in the part (i). Two cases are possible: $p = [v, y]$ or $p = [v, x]$, where $deg(v) < n$. We investigate only the first case.

Consider the endomorphisms $d(v)$ and $d([x, y])$. It is obvious that $d(v) \cdot d([x, y]) = d(p)$. Therefore,

$$
T d(v) \cdot T d([x, y]) = T d(p).
$$

According to (i), we have $T d([x, y]) = \tau^{-1}_\lambda d([x, y]) \tau_\lambda$ and by induction $T d(v) = \tau^{-1}_\lambda d(v) \tau_\lambda$. Substituting the last equalities in (3.6) we obtain the desired.

(iii) Now consider an endomorphism $\varphi$: $\varphi(x) = v_1$, $\varphi(y) = v_2$, where $v_1$, $v_2$ are right-normalized monomials. We have

$$
T d(\varphi(x)) = T (\varphi d(x)) = (T \varphi)d(x) = d(T \varphi(x))
$$

However from (ii) follows that

$$
T d(\varphi(x)) = \tau^{-1}_\lambda d(\varphi(x)) \tau_\lambda = d(\tau^{-1}_\lambda \varphi \tau_\lambda(x)).
$$

Comparing (3.9) and (3.10) we obtain $T \varphi(x) = \tau^{-1}_\lambda \varphi \tau_\lambda(x)$. In a similar way we may obtain $T \varphi(y) = \tau^{-1}_\lambda \varphi(y) \tau_\lambda$, i.e., $T \varphi = \tau^{-1}_\lambda \varphi \tau_\lambda$.

(iv) Let $p = v_1 + v_2$, where $v_1$, $v_2$ are right-normalized monomials. Consider the endomorphisms $\delta$ and $\varphi$: $\delta(x) = x + y$, $\delta(y) = y$ and $\varphi(x) = v_1$, $\varphi(y) = v_2$. Then $d(p) = \varphi \cdot \delta$. From (iii) follows that $T \varphi = \tau^{-1}_\lambda \varphi \tau_\lambda$ and $T \delta = \delta$. This proves our assertion for two summands. Easy induction on number of summands of $p$ leads to the formula 3.6 in the general case.
2. Let $\sigma \in \text{End} W$ such that $\sigma(x) = p_1$, $\sigma(y) = p_2$, where $p_1$, $p_2 \in W$. The same arguments as in part (iii) prove the Proposition 3.11.

Proof of Theorem 3.5 Let $T$ be an automorphism of $\text{End} W$ and $T_1$ be the restriction of $T$ on $\text{Aut} W$, i.e. $T_1 = T \mid_{\text{Aut} W}$. We have for all $\beta \in \text{Aut} W$: $T_1(\beta) = \tau^{-1} \cdot \delta_w^{-1} \cdot \beta \cdot \delta_w \cdot \tau$, $\beta \in \text{End} W$, where $\delta_w$ is a field semi-automorphism of $\text{End} W$, $\tau \in \text{Aut} W$.

Let $T_2$ be mapping of $\text{End} W$ the following form: $T_2(\beta) = \tau^{-1} \cdot \delta_w^{-1} \cdot \beta \cdot \delta_w \cdot \tau$, $\beta \in \text{End} W$, for the same $\delta_w$, $\tau$ as above and for all $\beta \in \text{End} W$. It is easy to check that $T_2$ is an automorphism of $\text{End} W$.

It is clear that $T_2 \mid_{\text{Aut} W} = T \mid_{\text{Aut} W}$. Therefore, $T \cdot T_2^{-1}(\beta) = \beta$, $\beta \in \text{Aut} W$. By Proposition 3.11 we have $T \cdot T_2^{-1} = \tau_\lambda$. Hence, $T = \tau_\lambda \cdot T_2 = \tau_\lambda \cdot \tau \cdot \delta_w \cdot \delta_w = \delta \cdot \delta_w$, where $\phi = \tau_\lambda \cdot \tau$ is the inner automorphism of $\text{End} W$ and $\delta_w$ is as above. The proof of Theorem 3.5 is complete.

3.2. Semi-inner automorphisms of the category of free Lie algebras. Consider the category $\Theta = \text{Lie} - K$ of free Lie algebras over a field $K$ with the semimorphisms defined in 3.1. We can to define semi-inner automorphisms of the category $\Theta^0$ on the same way as in the case of the category of free modules.

Definition 3.12. An automorphism $\varphi$ of the category $\Theta^0$ is called semi-inner if there exists $\sigma \in \text{Aut} K$ and a collection $(\sigma, s) = \{(\sigma, s_x) : s_x : F \rightarrow F, F \in \text{Ob} \Theta\}$ of semi-isomorphisms such that $(\sigma, s) : 1_{\Theta^0} \rightarrow \varphi$ is semi-isomorphism of functors, i.e., for every object $F \in \text{Ob} \Theta$ there exists semi-isomorphism $(\sigma, s_x) : F \rightarrow F$ and for each $\nu : F_1 \rightarrow F_2$ the following diagram is commutative:

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\varphi} & \varphi(F_1) \\
\downarrow{\nu} & & \downarrow{\varphi(\nu)} \\
F_2 & \xrightarrow{\varphi} & \varphi(F_2)
\end{array}
$$

Denote by $\hat{\sigma}$ an automorphism of the category of the free $K$-modules of the following form:

1. $\hat{\sigma}(KX) = KX$, $KX \in \text{Ob} \Theta^0$, i.e., $\hat{\sigma}$ does not change objects of the category of the free $K$-modules.

2. $\hat{\sigma}(\nu) = \sigma \cdot \nu \cdot \delta^{-1}_w : KX \rightarrow KY$ for all $\nu : KX \rightarrow KY$. $KX, KY \in \text{Ob} \Theta^0$

It is clear that $\hat{\sigma}$ is a semi-inner automorphism of the category $\Theta^0$.

Now, we shall describe the group of automorphisms of the category of free Lie algebras over an infinite field $K$.

Theorem 3.13. Let $K$ be a infinite field. Then every automorphism of the category $\Theta^0 = (\text{Lie} - K)^0$ is semi-inner.

Proof. A free Lie algebra over field is hopfian. It is clear that if $\varphi \in \text{Aut} \Theta^0$ then for any one-dimensional Lie algebra $Kx_0$ we have $\varphi(Kx_0) \simeq Kx_0$. By Theorem 1.10 every automorphism of the category $\Theta^0$ is special. By Proposition 1.6 an automorphism $\varphi$ can be represented in the form $\varphi = \varphi_1 : \varphi_2$, where $\varphi_1$ is an inner automorphism of $\Theta^0$ and $\varphi_2$ is an automorphism which does not change the objects of this category. Now, without loss of generality, we can assume that $\varphi$ itself does not change the objects of the category $\Theta^0$. From this follows that $\varphi$ induces an automorphism $\varphi_W$ of $\text{End} W$ for all $W \in \Theta$. Take 2-generated Lie
algebra $W^0 = W(x, y)$. By Theorem 3.3 the automorphism $\varphi_{W^0}$ is semi-inner. This automorphism is defined by $\delta \in \text{Aut} K$ and $\tau_W \in \text{Aut} W^0$.

Making use of the pair $(\delta, \tau_W)$ we build an inner automorphism of the category $\Theta^0$. Let $W \in \Theta$. For any $W \in \Theta$ set $(\delta, \tau)W = (\delta, \tau_W)$ if $W = W^0$ and $(\delta, \tau_W)$ otherwise. The collection $(\delta, \tau)W$ of automorphisms of this category defines an inner automorphism $\phi$ of $\Theta^0$. Denote by $\varphi_1 = \phi^{-1}\varphi$. Then $\varphi_1$ acts identical on $\text{End} W^0$.

Let $\nu_0 : W^0 \to Kx_0 = W_0$ be a homomorphism such that $\nu_0(x) = \nu_0(y) = x_0$ and $\rho : W^0 \to W^0$ be a homomorphism satisfying $\rho(x) = y, \rho(y) = x$. From $\nu_0 \rho = \nu_0$ follows $\varphi(\nu_0) \rho = \varphi(\nu_0)$. Therefore $\varphi_1(\nu_0(x)) = \varphi_1(\nu_0(y)) = \alpha x$ for some $\alpha \neq 0$ from $K$.

Consider the collection of automorphisms $f_W : W \to W, W = W(X) \in \Theta$ defined in the following way: if $W = W^0$, then $f_W^0(x) = \alpha x_0$ and if $W \neq W^0$, then $f_W(x) = x$ for all $x \in X$. This collection of automorphisms defines an inner automorphism $\hat{f}$ of the category $\Theta^0$ which does not change objects of $\Theta^0$ and $\hat{f}(\nu) = f_{Kx_{0}}^{\nu}f_{Kx_{0}}^{-1}$ for every morphism $\nu \in \text{Mor} \Theta^0$. It is clear that $\hat{f}(\nu_0) = \varphi_1(\nu_0)$.

By Theorem 3.3 the automorphism $\psi = \hat{f}^{-1}\varphi_1$ is an inner automorphism of the category $\Theta^0$ and thus, $\varphi = \phi \hat{f}^{-1}\psi$ is semi-inner. The proof is complete. \[\square\]

References

[1] Yu. Bahturin, Identical relations in Lie algebras, VNU Science Press, Utrecht, 1987.
[2] A. Berzins, B. Plotkin, E. Plotkin, Algebraic geometry in varieties of algebras with the given algebra of constants, Journal of Math. Sciences, 102:3, (2000), 4039-4070.
[3] L. Bokut, G. Kukin, Algorithmic and combinatorial algebra, Kluver, (1994).
[4] P. Cohn, Subalgebras of free associative algebras, Proc. London Math. Soc, 14, (1985), 618-632.
[5] J. Dieudonne, On the automorphisms of the classical groups, Memoirs Amer. Math. Soc., 2, (1951), 1-95.
[6] E. Formanek, A question of B. Plotkin about the semigroup of endomorphisms of a free group, Proc. American Math. Soc., 130, (2001), 935-937.
[7] M. Jodiet, T. Lam, Multiplicative maps of matrix semigroups, Arch. Math., 20, (1969), 10-16.
[8] G. Mashevitzky, B. Schein, Automorphisms of the endomorphism semigroup of a free monoid or a free semigroup, to appear.
[9] G. Mashevitzky, B. Plotkin, E. Plotkin, Automorphisms of the category of free Lie algebras, to appear.
[10] G. Mashevitzky, B. Plotkin, E. Plotkin, Automorphisms of the category of free algebras of varieties, Electron. Res. Announs. Amer. Math. Soc., 8, (2002), 1-10.
[11] B. Plotkin, Seven lectures in universal algebraic geometry, Preprint, Hebrew University, Jerusalem, (2000).
[12] A. Shirshov, Subalgebras of free Lie algebras, Uspekhi Mat. Nauk, 8, (1953), 173-176.
[13] M. Hall, A basis for free Lie ring and higher commutators in free groups, Proc. London Math. Soc., 1, (1950), 575-581.
[14] G. Zhitomirskii, Autoequivalences of categories of free algebras of varieties, to appear.

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