Scalarization approach for approximation of weakly efficient solutions of set-valued optimization problems

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Abstract: In this paper, by using the scalarization method and normal subdifferential for set-valued maps, we consider an extension of Minty variational-like inequalities and obtain some relations between their solutions and set-valued optimization problems. An existence result for generalized Minty variational-like inequalities and set-valued optimization problems is also given. Moreover, the concept of approximate efficient solutions due to Kutateladze is investigated and by the Tammer–Weidner nonlinear functional, we characterize them for cone constrained set-valued optimization problems.

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1. Introduction

Variational inequalities are identified either in the form presented by the Minty (1967) or in the form by Stampacchia (1960). Giannessi (1980) was the first author who obtained the equivalence between solutions of a Minty variational inequality and efficient solution of differentiable, convex optimization problem. Afterward, some authors focused their works to nonsmooth functions (see, e.g. Al-Homidan & Ansari, 2010; Alshahrani, Ansari, & Al-Homidan, 2014; Chen & Huang, 2012; Yang & Yang, 2006). Al-Homidan and Ansari (2010) obtained these results for invex functions with Clarke’s generalized directional derivative. By using the scalarization method, Santos, Rojas-Medar,
Ruiz-Garzón, and Rufián-Lizana (2008) studied scalarized variational-like inequalities presented in terms of Clarke’s generalized directional derivative and showed that the set of their solutions is equal to weak efficient solutions set. In Alshahrani et al. (2014) further extended results in Santos et al. (2008) to establish some existence results for solutions of nonsmooth variational-like inequalities under a dense pseudomonotonicity assumption. Moreover, Oveisiha and Zafarani (2014) considered generalized Stampacchia variational-like inequalities and set-valued optimization problems and also obtained some characterizations of the solution sets of pseudoinvex extremum problems. Very recently, Ruiz-Garzón, Osuna-Gómez, Rufián-Lizana, and Hernández-Jiménez (2015) introduced a new concept of generalized invexity for continuous-time programming problems and show the equivalence of efficient and weak efficient solutions for the multiobjective continuous-time programming problem, and solutions of Stampacchia and Minty variational-like type inequality problems.

The notion of approximate solutions has been defined in several ways (see, e.g. Helbig, 1992; Kutateladze, 1979; Tanaka, 1994; White, 1986). The first concept was introduced by Kutateladze (1979) and has been used to construct approximate Kuhn–Tucker type conditions, approximate duality theorems and so forth, see, for instance (Bednarczuk & Przybyla, 2007; Bolintineanu, 2001; Chen, Huang, & Yang, 2005; Göpfert, Riahi, Tammer, & Zălinescu, 2003; Gutiérrez, Jiménez, & Novo, 2008; Vályi, 1987). Rong and Wu (2000) considered the notion of $\varepsilon$-weak efficient solution and studied Lagrangian multiplier and duality properties for set-valued optimization problems with cone subconvexlike mappings based on the separation theorems of convex sets. In Gutiérrez, Jiménez, and Novo (2011) introduced a new concept of $\varepsilon$-efficient point based on set-valued mapping. They obtained some existence results and properties on the behavior of these approximate efficient and weak efficient solutions. Very recently, Huerga, Gutiérrez, Jiménez, and Novo (2015), studied the concept of $\varepsilon$-efficiency, defined by Kutateladze, and proved that the limit of them, when the precision $\varepsilon$ tends to zero is the set of weak efficient solutions for single vector-valued optimization problems and also obtained Kuhn–Tucker optimality conditions for $\varepsilon$-efficient solutions of nondifferentiable convex Pareto multiobjective problems with inequality constraints.

In this paper, we consider generalized variational-like inequalities in terms of normal subdifferential for set-valued maps. By introducing a scalarized Minty variational-like inequalities (MVLI), we show that any solution of a scalarized set-valued optimization problems (SOP) is also a solution of Minty variational-like inequalities under standard assumptions and that the inverse implications hold under the additional generalized $K$-convexity assumption, where $K$ is an ordering cone of the considered image space. Also, by using the Tammer–Weidner nonlinear scalarization functional a characterization of $\varepsilon$-efficient solutions of cone constrained set-valued optimization problems is given. The paper is organized as follows: in Section 2, some basic definitions and preliminary results are presented. Section 3 is devoted to study several relationships between scalarized Minty variational-like inequalities and set-valued optimization problems. Also, we obtain an existence result for (MVLI) and set-valued optimization problems. In Section 4, we follow an approach presented in Gutiérrez et al. (2011), Huerga et al. (2015). We state a kind of penalization scheme for approximate solutions of a cone constrained set-valued optimization problems. Finally, in Section 5, some conclusions are presented, which summarize this work.

2. Preliminaries
Let $X$ be a Banach space and $X^*$ be its topological dual space. The norm in $X$ and $X^*$ will be denoted by $\| \cdot \|$. We denote by $(\cdot, \cdot), (x, y)$ and $\{x, y\}$ the dual pair between $X$ and $X^*$, the line segment for $x, y \in X$, and the interior of $[x, y]$, respectively. Now, we recall some concepts of subdifferentials and coderivatives that we need in next sections.

**Definition 2.1** Mordukhovich (2006) Let $X$ be a Banach space, $\Omega$ be a nonempty subset of $X$, $x \in \Omega$ and $\varepsilon \geq 0$. The set of $\varepsilon$-normals to $\Omega$ at $x$ is
\[ \hat{N}(x; \Omega) := \{ x^* \in X^* \mid \limsup_{\omega \to x} \frac{(x^*, u - x)}{\|u - x\|} \leq \epsilon \}. \]

If \( \epsilon = 0 \), the above set is denoted by \( \tilde{N}(x; \Omega) \) and called regular normal cone to \( \Omega \) at \( x \). Let \( x \in \Omega \), the basic normal cone to \( \Omega \) at \( x \) is

\[ \mathcal{N}(x; \Omega) := \text{Limsup}_{\omega \to x} \tilde{N}(x; \Omega). \]

**Definition 2.2** Mordukhovich (2006) Let \( X \) be a Banach space and \( \varphi: X \to \mathbb{R} \) be finite at \( \bar{x} \in X \). The basic (limiting, Mordukhovich) subdifferential due to Mordukhovich (2006) of \( \varphi \) at \( x \) is defined by

\[
\partial^{\text{lim}} \varphi(x) := \{ x^* \in X^* \mid (x^*, -1) \in \mathcal{N}(\bar{x}, \varphi(x)); \ \text{epi} \varphi \}. 
\]

Given a set-valued mapping \( F: X \rightrightarrows Y \) between Banach spaces with the range space \( Y \) partially ordered by a nonempty, closed and convex cone \( K \). Denoting the ordering relation on \( Y \) by \( \preceq \), we have \( y_1 \preceq y_2 \) if and only if \( y_2 - y_1 \in K \).

Now, we present some definitions and results about coderivatives and subdifferentials of set-valued mappings.

**Definition 2.3** Mordukhovich (2006) Let \( F: X \rightrightarrows Y \) be a set-valued mapping between Banach spaces and \( (\bar{x}, \bar{y}) \in \text{gr} F \). Then the Fréchet coderivative of \( F \) at \( (\bar{x}, \bar{y}) \) is the set-valued mapping \( \tilde{D}^* F(\bar{x}, \bar{y}): Y^* \rightrightarrows X^* \) given by

\[
\tilde{D}^* F(\bar{x}, \bar{y})(y^*) := \{ x^* \in X^* \mid (x^* - y^*, -1) \in \hat{N}(\bar{x}, \tilde{\varphi}(\bar{y}); \text{gr} F) \},
\]

and furthermore, the normal coderivative of \( F \) at \( (x, y) \) is the set-valued mapping \( D^*_K F(\bar{x}, \bar{y}): Y^* \rightrightarrows X^* \) given by

\[
D^*_K F(\bar{x}, \bar{y})(y^*) := \{ x^* \in X^* \mid (x^* - y^*, -1) \in \mathcal{N}(\bar{x}, \bar{y}; \text{gr} F) \}.
\]

**Definition 2.4** Bao and Mordukhovich (2007) Let \( F: X \rightrightarrows Y \) be a set-valued mapping. Then, the epi-graphical multifunction \( \mathcal{E}_F(X \rightrightarrows Y) \) is defined by

\[
\mathcal{E}_F(x) := \{ y \in Y \mid y \in F(x) \cup K \}.
\]

The Fréchet and normal subdifferentials of \( F \) at the point \((\bar{x}, \bar{y}) \in \text{epi} F \) in the direction \( y^* \in Y^* \) are, respectively, defined by

\[
\hat{\partial} F(\bar{x}, \bar{y})(y^*) := \tilde{D}^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*) \quad \text{and} \quad \partial F(\bar{x}, \bar{y})(y^*) := D^*_K \mathcal{E}_F(\bar{x}, \bar{y})(y^*).
\]

**Definition 2.5** Mordukhovich (2006) Let \( F: X \subset X \rightrightarrows Y \) with \( \text{dom} F \neq \emptyset \) and \( B_y \) be the closed unit ball of \( Y \).

(i) \( F \) is said to be Lipschitz around \( \bar{x} \in \text{dom} F \) iff there are a neighborhood \( U \) of \( \bar{x} \) and \( \epsilon' \geq 0 \) such that

\[
F(x) \subset F(\bar{x}) + \epsilon' \|x - \bar{x}\| B_y, \quad \text{for all } x, u \in X \cap U.
\]

(ii) \( F \) is said to be epi-Lipschitz around \( \bar{x} \in \text{dom} F \) iff \( \mathcal{E}_F \) is Lipschitz around this point.

Let \( K \) be a closed, convex and pointed cone in \( Y \) and denote the positive polar cone of \( K \) by
\[ K^\ast = \{y^\ast \in Y^\ast \mid (y^\ast, k) \geq 0, \forall k \in K\}. \]

The next object is the marginal function associated with a set-valued mapping. Given \( F: X \rightrightarrows Y \) and \( y^\ast \in Y^\ast \). We associate to \( F \) and \( y^\ast \) a marginal function \( f_{y^\ast}: X \to \mathbb{R} \cup \{\pm \infty\} \)

\[ f_{y^\ast}(x) = \inf \{y^\ast(y) \mid y \in F(x)\}, \]

and the minimum set \( M_{y^\ast}(x) = \{y \in F(x) \mid f_{y^\ast}(x) = y^\ast(y)\} \).

Throughout this paper, we suppose that \( \text{gr} F \) is closed, and for all \( x \in \text{dom} F \) and \( y^\ast \in K^\ast \), \( M_{y^\ast}(x) \) is nonempty.

**Lemma 2.6** Oveisaha and Zafarani (2013) Suppose that \( F: \Omega \subseteq X \rightrightarrows Y \) is a set-valued map and \( \tilde{x} \in \text{dom} F \). If \( F \) is epi-Lipschitz around \( \tilde{x} \) and \( y^\ast \in K^\ast \), then the scalar-function \( f_{y^\ast} \) is locally Lipschitz at \( \tilde{x} \).

The next theorem gives some relations between normal subdifferential and normal coderivative of \( F \) and limiting subdifferential of its marginal functions. (see also Theorem 3.4 and Corollary 3.5 in Oveisaha & Zafarani, 2013)

**Theorem 2.7** Oveisaha and Zafarani (2013) Let \( X, Y \) be Asplund spaces, \( F: X \rightrightarrows Y \) and \( y^\ast \in K^\ast \). Suppose that \( \tilde{x} \in \text{dom} F \) and \( y^\ast \in M_{y^\ast}(\tilde{x}) \).

1. If \( F \) is Lipschitz around \( \tilde{x} \), then \( \partial f_{y^\ast}(\tilde{x}) \subseteq D_{\text{ps}} f_{y^\ast}(\tilde{x}, y^\ast) \).

2. If \( F \) is epi-Lipschitz around \( \tilde{x} \), then \( \partial f_{y^\ast}(\tilde{x}) \subseteq \partial F(\tilde{x}, y^\ast) \).

**Definition 2.8** Weir and Mond (1988) Let \( \eta: X \times X \to \mathbb{R} \) be an invex set with respect to \( \eta \) iff, for any \( x, y \in \Omega \) and \( \lambda \in [0, 1] \), \( y + \lambda \eta(x, y) \in \Omega \).

**Definition 2.9** (see Oveisaha & Zafarani, 2013) Let \( \Omega \subseteq X \) be an invex set with respect to \( \eta \) and \( F: \Omega \subseteq X \rightrightarrows Y \) be a set-valued mapping. Then:

1. \( F \) is said to be \( K \)-preinvex with respect to \( \eta \) on \( \Omega \) if for any \( x_1, x_2 \in \Omega \) and \( \lambda \in [0, 1] \), one has

   \[ \lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \lambda \eta(x_1, x_2) + K. \]

2. \( F \) is said to be \( K \)-invex with respect to \( \eta \) on \( \Omega \) if for any \( y^\ast \in K^\ast \), \( x_i \in \Omega, y_i \in M_{y^\ast}(x_i), (i = 1, 2) \) and \( \xi \in \partial F(x_i, y_i) \), one has

   \[ \langle \xi, \eta(x_2, x_1) \rangle \leq y^\ast(y_2) - y^\ast(y_1); \]

3. \( F \) is said to be weakly \( K \)-invex with respect to \( \eta \) on \( \Omega \) if for any \( y^\ast \in K^\ast \), \( x_i \in \Omega, y_i \in M_{y^\ast}(x_i), (i = 1, 2) \) and \( \xi \in \partial F(x_i, y_i) \), there exists \( \xi \in \partial F(x_i, y_i) \) such that

   \[ \langle \xi, \eta(x_2, x_1) \rangle \leq y^\ast(y_2) - y^\ast(y_1); \]

4. The set-valued map \( \partial F: X \times Y \rightrightarrows X^\ast \) is said to be invariant \( K \)-monotone on \( \Omega \) with respect to \( \eta \) if for any \( y^\ast \in K^\ast \), \( x_i \in \Omega, y_i \in M_{y^\ast}(x_i) \), and \( \xi \in \partial F(x_i, y_i) \), one has

   \[ \langle \xi_1, \eta(x_2, x_1) \rangle + \langle \xi_2, \eta(x_1, x_2) \rangle \leq 0. \]

**Remark 1** If \( F = f: X \to \mathbb{R} \) is a real-valued function and \( K = [0, +\infty] \), then the above definition reduces to preinvexity, invexity, weak invexity, and invariant monotonicity, respectively, for real-valued functions, that has been investigated in Jabarootian and Zafarani (2006), Soleimani-damanekh (2010).
Condition A. Jabarootian and Zafarani (2009) A mapping $F: \Omega \rightrightarrows Y$ from an inconvex set $\Omega$ with respect to $\eta$ to an ordered Banach space is said to enjoy Condition A if

$$F(x_1) \subset F(x_2 + \eta(x_1, x_2)) + K,$$

for all $x_1, x_2 \in \Omega$.

Condition C. Mohan and Neogy (1995) Let $\eta: X \times X \rightarrow X$. Then, for any $x, y \in X, \lambda \in [0, 1]$

$$\eta(y, y + \lambda \eta(x, y)) = -\lambda \eta(x, y), \quad \eta(x, y + \lambda \eta(x, y)) = (1 - \lambda) \eta(x, y).$$

Remark 2 By some computation, we can see that if Condition C holds, then for any $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$

$$\eta(x_1 + \lambda_1 \eta(x_2, x_1), x_1 + \lambda_2 \eta(x_2, x_1)) = (\lambda_1 - \lambda_2) \eta(x_2, x_1).$$

Let $\Omega$ be a convex subset of a vector space $X$. Then a mapping $F: \Omega \rightrightarrows \Omega$ is called a KKM mapping iff for each nonempty finite subset $A$ of $\Omega$, $\text{conv}(A) \subseteq F(A)$, where $\text{conv}(A)$ denotes the convex hull of $A$, and $F(A) = \bigcup \{F(x) | x \in A\}$.

**Lemma 2.10** (see e.g. Fakhar & Zafarani, 2005) Let $\Omega$ be a nonempty and convex subset of a Hausdorff topological vector space $X$. Suppose that $\Gamma, \tilde{\Gamma}: \Omega \rightrightarrows \Omega$ are two set-valued mappings such that the following conditions are satisfied:

1. (A1) $\hat{\Gamma}(x) \subseteq \tilde{\Gamma}(x), \quad \forall x \in \Omega$,
2. (A2) $\hat{\Gamma}$ is a KKM map,
3. (A3) $\Gamma$ is closed-valued,
4. (A4) there is a nonempty compact convex set $\mathcal{B} \subseteq \Omega$, such that $\text{cl}_{\text{int}}(\bigcap \text{ext}(\Gamma(x)))$ is compact.

Then, $\bigcap \text{ext}(\hat{\Gamma}(x)) \neq \emptyset$.

### 3. (MVLI) and set-valued optimization problems

This section is devoted to get some relations between scalarized Minty variational-like inequalities and scalarized set-valued optimization problems.

Suppose that $F: \Omega \rightrightarrows Y$ is a set-valued map between Banach spaces. We consider the following set-valued optimization problem

$$\min F(x) \quad \text{s.t. } x \in \Omega \subset X. \quad (1)$$

**Definition 3.1**

(i) Chen, Huang, and Yang (2005) A point $x$ is said to be a weakly efficient solution of problem (1) iff there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(\Omega) - \bar{y}) \cap \text{int} K = \emptyset.$$

(ii) $\bar{x}$ is said to be a scalarized solution of problem (1) ($\bar{x}$ is a solution of (SOP)) iff, for any $y^* \in K^* \setminus \{0\}$, there exists $\bar{y} \in F(\bar{x})$ such that

$$y^*(\bar{y}) \leq y^*(y) \quad \text{for all } y \in F(\Omega).$$

Suppose that $\eta: X \times X \rightarrow X$. Now, we consider the following scalarized Minty variational-like inequality (MVLI): Find a vector $\bar{x} \in \Omega$ such that, for any $x \in \Omega$ and $y^* \in K^* \setminus \{0\}$, there exist $y \in M_y(x)$ and $x^* \in dF(x,y)(y^*)$ such that

$$\langle x^*, \eta(x, x) \rangle \leq 0.$$
When \( F: X \rightarrow Y \) is a vector-valued function, Santos et al. (2008) and Alshahrani et al. (2014) by using a similar way, considered nonsmooth Stampacchia variational-like inequality in which the limiting nonconvex subdifferentials was replaced by the convex Clarke’s generalized directional derivative.

**Example 3.2** Suppose that \( X = \Omega = \mathbb{R}^2 \), \( Y = \mathbb{R} \), \( K = [0, +\infty] \), \( u = (0,0) \) and \( F: \Omega \rightarrow Y \) given by \( F(x_1,x_2) = \left[ \sqrt{x_1^2 + x_2^2}, +\infty \right] \). Then, the normal subdifferential of \( F \) is

\[
\begin{align*}
\partial F(x_1,x_2) & = \left( \frac{\alpha_1}{\sqrt{x_1^2 + x_2^2}}, \frac{\alpha_2}{\sqrt{x_1^2 + x_2^2}} \right) & \text{if} (x_1,x_2) \neq (0,0), \\
\partial F(0,0) & = \{(x_1,x_2) : x_1^2 + x_2^2 \leq r \} & \text{if} (x_1,x_2) = (0,0),
\end{align*}
\]

which \( r \) is a positive real number. Let \( \eta: \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined as \( \eta(x,x') = \lambda(x-x') \) such that \( 0 < \lambda \leq 1 \). Then, by some computation, we deduce that \( \bar{x} \) is a solution of (MVLI).

**Lemma 3.3** Oveisigha and Zafarani (2014) Every solution of (SOP) is a weakly efficient solution of problem (1).

**Theorem 3.4** Let \( X \) be an Asplund space, \( \Omega \subseteq X \) be an invex set and \( F: \Omega \subseteq X \rightarrow Y \). If \( F \) is weakly \( K \)-invex and \( \bar{x} \in \Omega \) is a solution of (SOP), then it is a solution of (MVLI).

**Proof** Assume that \( \bar{x} \) is a solution of (SOP). Suppose to the contrary that \( \bar{x} \) is not a solution of (MVLI). Hence, there exist \( x \in \Omega \) and \( y^* \in K^* \setminus \{0\} \) such that

\[
\langle x^*,\eta(\bar{x},x) \rangle > 0,
\]

for all \( y \in M_{\eta^*}(x) \) and \( x^* \in \partial F(x,y)(y^*) \). Now, weak \( K \)-invexity of \( F \) implies that, there exists \( x^* \in \partial F(x,y)(y^*) \) such that

\[
\langle x^*,\eta(\bar{x},x) \rangle \leq y^*(\bar{y}) - y^*(y).
\]

Relations (3.1) and (3.2) contradicts that \( \bar{x} \) is a solution of (SOP).

**Theorem 3.5** Let \( X \) be an Asplund space, \( \Omega \subseteq X \) be an invex set and \( F: \Omega \subseteq X \rightarrow Y \) be epi-Lipschitz. Suppose that \( F \) satisfies Condition A and is \( K \)-invex, \( \eta \) satisfies Condition C and \( \bar{x} \) is a solution of (MVLI), then it is a solution of (SOP) and hence, a weakly efficient solution of problem (1).

**Proof** Assume that \( x \in \Omega \) is a solution of (MVLI). Suppose to the contrary that \( x \) is not a solution of (SOP). Hence, there exists \( y^* \in K^* \setminus \{0\} \) such that for all \( y \in F(x) \)

\[
y^*(y) < y^*(\bar{y})
\]

for a \( y \in F(x) \) that \( x \in \Omega \). Set \( x(t) = x + t_0(x,x) \). Choose \( t_0 \in [0,1] \) arbitrary. Since \( F \) is epi-Lipschitz, by Lemma 2.6, \( f_{x^*} \) is Lipschitz on \( \Omega \). Now, by using mean value inequality for limiting subdifferential (Corollary 3.5.1 in Mordukhovich (2006)), there exist \( t_1, t_2, \in [0,1] \) and \( \zeta \in \partial_{x^*} f_{x^*}(\bar{x} + t_0\eta(x,\bar{x})) \) such that

\[
t_0\langle \zeta,\eta(x,\bar{x}) \rangle \leq f_{x^*}(\bar{x} + t_0\eta(x,\bar{x})) - f_{x^*}(\bar{x}).
\]

Since \( F \) is \( K \)-invex and epi-Lipschitz, Theorem 2.7 implies that \( f_{x^*} \) is invex. Now, by a similar proof as that of Lemma 3.2 in Jabarootian and Zafarani (2006), invexity of \( f_{x^*} \) implies its preinvexity. Hence, we obtain

\[
f_{x^*}(\bar{x} + t_0\eta(x,\bar{x})) - f_{x^*}(\bar{x}) \leq t_0(f_{x^*}(x) - f_{x^*}(\bar{x})).
\]
Now, by using relations (3.3), (3.4), and (3.5), we have
\[
\langle \xi, \eta(x, \tilde{x}) \rangle < 0. \tag{3.6}
\]

Choose \( t_2 \in ]0, 1[ \) such that \( t_2 < t_1 \). By using Remark 2, we get
\[
\eta(x(t_2), x(t_1)) = (t_2 - t_1)\eta(x, \tilde{x}), \quad \eta(x(t_1), x(t_2)) = (t_1 - t_2)\eta(x, \tilde{x}). \tag{3.7}
\]

From relations (3.6) and (3.7), we obtain
\[
\langle \xi, \eta(x(t_2), x(t_1)) \rangle > 0. \tag{3.8}
\]

Since \( F \) is \( K \)-invex, one can easily show that \( \partial F \) is invariant \( K \)-monotone (see Lemma 3.7 in Oveisiha and Zafarani (2013)). Hence, we get
\[
\langle \xi, \eta(x(t_2), x(t_1)) \rangle + \langle \xi, \eta(x(t_1), x(t_2)) \rangle \leq 0, \tag{3.9}
\]

for any \( y_1 \in M^+ \Phi_1(x(t_1)) \) and \( \xi_1 \in \partial F(x(t_2), y_1) \). Now, by relations (3.8) and (3.9), we deduce that
\[
\langle \xi_1, \eta(x(t_1), x(t_2)) \rangle < 0 \tag{3.10}
\]

for any \( y_1 \in M^+ \Phi(x(t_1)) \) and \( \xi_1 \in \partial F(x(t_2), y_1) \). Because \( \eta(x, x(t_2)) = -t_2\eta(x, x) \), by using relations (3.7) and (3.10), we can obtain that
\[
\langle \xi, \eta(x, x(t_2)) \rangle > 0,
\]

for any \( y_1 \in M^+ \Phi(x(t_1)) \) and \( \xi_1 \in \partial F(x(t_2), y_1) \). This contradicts the fact \( x \) is a solution of (MVLI). Therefore, \( x \) is a solution of (SOP) and from Lemma 3.3 we deduce that \( x \) is a weakly efficient solution of problem (1).

\[ \square \]

**Remark 1** Theorems 3.4 and 3.5 generalize Theorem 3.1 in Al-Homidan and Ansari (2010) and Theorem 3.1 in Chen and Huang (2012) to set-valued maps.

**Example 3.6** Let \( X = Y = \mathbb{R} \), \( \Omega = ]-1, 1[ \), \( K = [0, +\infty[ \) and \( F: \Omega \rightarrow X \) such that \( F(x) = [x^2 + x, 3] \) for \( x \geq 0 \) and \( F(x) = [-x, 3] \) for \( x < 0 \). Let \( \eta: X \times X \rightarrow \mathbb{R} \) be defined as
\[
\eta(x, y) = \begin{cases} 
x - y & \text{if } x \geq 0, y \geq 0 \text{ or } x \leq 0, y \leq 0, \\
-y & \text{otherwise.}
\end{cases}
\]

Then, the normal subdifferential of \( F \) is
\[
\partial F(x, x^2 + x)(r) = r(2x + 1) \quad \text{if } x > 0,
\]
\[
\partial F(0, 0)(r) = [-r, r] \quad \text{if } x = 0,
\]
\[
\partial F(x, -x)(r) = -r \quad \text{if } x < 0,
\]

which \( r \) is a positive real number. Then \( \eta \) satisfies Condition C, \( F \) satisfies Condition A and is \( K \)-invex with respect to \( \eta \). Hence, by some computation we can see that all assumptions of Theorem 3.5 are fulfilled and \( x = 0 \) is a solution of (MVLI), therefore it is a solution of (SOP) and weakly efficient solution of problem (1).

Here, we obtain an existence theorem for the solution of (MVLI) and therefore a weak efficient solution of problem (1).
LEMMA 3.7 Let $F:X \rightrightarrows Y$ and $0 \neq y^* \in K^\ast$. Suppose that $\bar{x} \in \text{dom}F$ and $\bar{y}_1, \bar{y}_2 \in M_{y^*}(\bar{x})$. Then, one has

$$\partial F(\bar{x}, \bar{y}_1)(y^*) = \partial F(\bar{x}, \bar{y}_2)(y^*).$$

Proof Since the proof is direct, it is omitted.

For normal subdifferential, we need the following condition to get an existence result for (MVLI).

**Condition D.** Let $F:X \rightrightarrows Y$ and $0 \neq y^* \in K^\ast$. Then, for any $\bar{x} \in \text{dom}F$ and $\bar{y}_1, \bar{y}_2 \in M_{y^*}(\bar{x})$, we have

$$\partial F(\bar{x}, \bar{y}_1)(y^*) = \partial F(\bar{x}, \bar{y}_2)(y^*).$$

THEOREM 3.8 Let $F:X \rightrightarrows Y$ be K-invex and satisfy Condition D. Assume that the following conditions are satisfied:

1. $\eta$ is affine and continuous in the first argument and skew.
2. There are a nonempty compact set $M \subset X$ and a nonempty compact convex set $B \subset X$ such that for each $x' \in X \setminus M$, there exists $x \in B$ and $y^* \in K^\ast \setminus \{0\}$ such that for any $y \in M_{y^*}(x)$ and $x' \in \partial F(x, y)(y^*)$, we have $(x', \eta(x', y^*)) > 0$.

Then, (MVL) has a solution. Also, the set of (MVL) solutions is compact.

Proof Define two set-valued mappings $\Gamma, \bar{\Gamma}: X \rightrightarrows X$ by

$$\Gamma(x) = \{ x' \in X : \forall y^* \in K^\ast \setminus \{0\}, \exists \gamma \in M_{y^*}(x) \text{ and } x^* \in \partial F(x, y)(y^*) ; (x^*, \eta(x', x^*)) \leq 0 \},$$

$$\bar{\Gamma}(x) = \{ x' \in X : \forall y^* \in K^\ast \setminus \{0\}, \exists \gamma \in M_{y^*}(x') \text{ and } x^* \in \partial F(x', y^*)(y^*) ; (x^*, \eta(x', x^*)) \geq 0 \},$$

for each $x \in X$. $\Gamma(x)$ and $\bar{\Gamma}(x)$ are nonempty because they contain $x$. The proof is divided in the following steps.

(i) $\bar{\Gamma}$ is a KKM mapping on $X$. Suppose that $\bar{\Gamma}$ is not a KKM mapping. Then, there exist $(x_1, x_2, \ldots, x_m)$ and $\lambda_i \geq 0, i = 1, \ldots, m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $x_0 = \sum_{i=1}^m \lambda_i x_i \notin \bigcup_{i=1}^m \bar{\Gamma}(x_i)$. Hence, it follows that $x_0 \notin \bar{\Gamma}(x)$ for all $i = 1, \ldots, m$, i.e.

$$\exists y^* \in K^\ast \setminus \{0\}; \forall y_0 \in M_{y^*}(x_0), x^* \in \partial F(x_0, y_0)(y^*) ; (x^*, \eta(x_0, x^*)) < 0;$$

for each $i = 1, \ldots, m$. Therefore, for any $y_0 \in M_{y^*}(x_0)$ and $x^* \in \partial F(x_0, y_0)(y^*)$, one has

$$0 = \langle x^*, \eta(x_0, x^*) \rangle = \sum_{i=1}^m \lambda_i \langle x_i^*, \eta(x_i, x_0) \rangle < 0,$$

which yields a contradiction. Hence, $\bar{\Gamma}$ is a KKM mapping.

(ii) Because K-invexity of $F$ implies invariant $K$-monotonicity of $\partial F$ (Lemma 3.7 in Oveisgha and Zafarani (2013)), hence we obtain $\bar{\Gamma}(x) \subseteq \bar{\Gamma}(x)$ and therefore, $\bar{\Gamma}$ is also a KKM mapping.

(iii) $\Gamma$ is closed valued: Let $(x_n)$ be a sequence in $\Gamma(x)$ which converges to $x_0$. Therefore, for any $y^* \in K^\ast \setminus \{0\}$, there exist $y \in M_{y^*}(x)$ and $x_n^* \in \partial F(x, y)(y^*)$ such that

$$\langle x_n^*, \eta(x_n, x) \rangle \leq 0.$$

Since $F$ satisfies Condition D, we can fix $y \in M_{y^*}(x)$ such that $x_n^* \in \partial F(x, y)(y^*)$ for any $n \geq 1$. Now, by using epi-Lipschitzian property of $F$, $\partial F(x, y)(y^*)$ is $w^\ast$-compact and therefore $\{x_n^*\}$ has a convergent
subsequence \(\{x^*_m\}\), that its limit \(x^*_t\) should be in \(\partial F(x, y)(y^*)\). Since \(\eta\) is continuous in the first argument, \(\{\eta(x^*_m, x)\}\) is a convergent sequence. Hence, we obtain

\[\langle x^*_t, \eta(x^*_t, x) \rangle \leq 0.\]

Thus, \(x^*_t \in \Gamma(x)\), this means that \(\Gamma\) is closed valued.

(iv) From condition 2, there exists a nonempty compact convex set \(B\), such that \(\text{cl}(\bigcap_{x \in X} \Gamma(x))\) is compact.

(v) Thus, all of the conditions of Lemma 2.10 are fulfilled by mapping \(\Gamma\). Therefore,

\[\bigcap_{x \in X} \Gamma(x) \neq \emptyset.\]

Hence, there exists \(\bar{x}\) such that for any \(x \in X\) and \(y^* \in K^* \setminus \{0\}\) there exist \(y \in M_p(x)\) and \(x^* \in \partial F(x, y)(y^*)\) such that

\[\langle x^*, \eta(\bar{x}, x) \rangle \leq 0.\]

Thus, (MVLI) has a solution. From (iii), \(\Gamma\) is closed valued and therefore, the set of solutions of (MVLI), i.e. \(\bigcap_{x \in X} \Gamma(x)\) is closed. Now, from condition 2, the set of solutions must be contained in the compact set \(M\), hence it is compact. \(\square\)

### 4. Approximation of weakly efficient solutions

In this section, we present the concept of approximate efficiency for set-valued maps that is a generalization of the same notion due to Kutateladze (1979).

**Definition 4.1** Let \(v \in Y \setminus \{0\}\) and \(\varepsilon \geq 0\). It is said that \((x_0, y_0) \in \text{gr}F\) is an \(\varepsilon v\)-efficient solution of problem (1), denoted by \((x_0, y_0) \in \text{WE}(F, \Omega, \varepsilon v, K)\), if we have

\[\text{WE}(F, \Omega, \varepsilon v, K) = \{ (x_0, y_0) : \exists \text{ sequences } (x_i, y_i) \subset \text{gr}F, (\varepsilon_i) \subset \mathbb{R}_+ \setminus \{0\}, \varepsilon_i \to \varepsilon \text{ such that } (x_i, y_i) \in \text{WE}(F, \Omega, \varepsilon_i v, K), (x_i, y_i) \to (x_0, y_0) \}.\]

In the next theorem, we present some properties of approximate efficient solutions.

**Theorem 4.2** Assume that \(q \in \text{int} K\). The following properties hold:

1. \(\text{WE}(F, \Omega, \varepsilon_1 q, K) \subset \text{WE}(F, \Omega, \varepsilon_2 q, K)\) for all \(\varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1 < \varepsilon_2\).
2. \(\text{WE}(F, \Omega, \varepsilon q, K) = \bigcap_{\varepsilon > 0} \text{WE}(F, \Omega, \varepsilon q, K)\).
3. Let \((x_0, y_0) \in \text{gr}F\), \(\varepsilon \geq 0\) and sequences \((x_i, y_i) \subset \text{gr}F\) and \((\varepsilon_i) \subset \mathbb{R}_+\) such that \((x_i, y_i) \in \text{WE}(F, \Omega, \varepsilon_i q, K), \varepsilon_i \to \varepsilon\) and \(y_i \to y_0\). Then \((x_0, y_0) \in \text{WE}(F, \Omega, \varepsilon q, K)\).
4. Suppose that \(\text{gr}F\) is closed and \(\varepsilon \geq 0\). Then, \(\text{WE}(F, \Omega, \varepsilon q, K)\) is closed and

\[\text{lim sup}_{\varepsilon \to 0^+} \text{WE}(F, \Omega, \varepsilon q, K) = \text{WE}(F, \Omega, 0 q, K)\].

**Proof** Parts (1) and (2) follow from the definition of \(\text{WE}(F, \Omega, \varepsilon q, K)\).
(3) Suppose that \((x_0, y_0) \not\in \text{WE}(F, \Omega, \bar{\varepsilon}q, K)\). Since
\[
\bar{\varepsilon}q + \text{int}K = \bigcup_{i \geq 2} (\varepsilon q + \text{int}K),
\]
there exist \(x \in \Omega\), \(y \in F(x)\) and \(\varepsilon_0 > \bar{\varepsilon}\) such that
\[
y - y_0 \in -\varepsilon_0 q - \text{int}K.
\]
Because \(y_i \to y_0\) and \(-\varepsilon_0 q - \text{int}K\) is open, there exists \(i_1\) such that
\[
y - y_i \in -\varepsilon_0 q - \text{int}K, \quad \forall i \geq i_1.
\]
Since \(\varepsilon_i \to \bar{\varepsilon}\) and \(\varepsilon_0 > \bar{\varepsilon}\), there exists \(i_2\) such that \(\varepsilon_i < \varepsilon_0\) for all \(i \geq i_2\). Let \(i_0\) be such that \(i_0 \geq \max\{i_1, i_2\}\). Therefore
\[
y - y_i \in -\varepsilon_0 q - \text{int}K \subseteq -\varepsilon_0 q - \text{int}K,
\]
which is a contradiction, since \((x_i, y_i) \in \text{WE}(F, \Omega, \varepsilon_0 q, K)\). Thus, \((x_0, y_0) \in \text{WE}(F, \Omega, \bar{\varepsilon}q, K)\).

(4) Set
\[
\Gamma(x) = \{(x_0, y_0) \in \text{gr}F: (x_0, y_0) \in \text{WE}(F, \Omega, \bar{\varepsilon}q, K)\}, \quad \forall x \in \Omega.
\]
The set \(\Gamma(x)\) is closed, for every \(x \in \Omega\). Let \((x_i, y_i) \subset \Gamma(x)\) be a sequence such that \((x_i, y_i) \to (x, y)\). Therefore
\[
(F(x) - y_i + \bar{\varepsilon}q) \cap -\text{int}K = \emptyset, \quad \forall i.
\]
Hence \((F(x) - y_i + \bar{\varepsilon}q) \subseteq -\text{int}K^c\). Since \((-\text{int}K)^c\) and \(\text{gr}F\) are closed and \(y_i \to y\), we obtain that \(y \in F(x)\) and
\[
(F(x) - y + \bar{\varepsilon}q) \cap -\text{int}K = \emptyset,
\]
i.e. \((x, y) \in \Gamma(x)\).
Also, it is clear that
\[
\text{WE}(F, \Omega, \bar{\varepsilon}q, K) = \left(\bigcap_{x \in \Omega} \Gamma(x)\right),
\]
and therefore \(\text{WE}(F, \Omega, \bar{\varepsilon}q, K)\) is closed. Now, we show that if \(\varepsilon_0 > \varepsilon\), then
\[
\limsup_{\varepsilon \to \varepsilon_0} \text{WE}(F, \Omega, \varepsilon q, K) \subset \text{WE}(F, \Omega, \varepsilon_0 q, K).
\]
Suppose that \((x_i, y_i) \to (x, y)\), with \((x_i, y_i) \in \text{WE}(F, \Omega, \varepsilon q, K)\) and \(\varepsilon_i \to \varepsilon\). Then, there exists \(i_0\) such that \((x_i, y_i) \in \text{WE}(F, \Omega, \varepsilon q, K)\) for any \(i \geq i_0\). Since \(\text{WE}(F, \Omega, \varepsilon q, K)\) is closed, it follows that \((x, y) \in \text{WE}(F, \Omega, \varepsilon q, K)\).

Finally, by using part (2), we obtain that
\[
\text{WE}(F, \Omega, \bar{\varepsilon}q, K) \subset \limsup_{\varepsilon \to \varepsilon_0} \text{WE}(F, \Omega, \varepsilon q, K)
\]
\[
\subset \left(\bigcap_{\varepsilon \to \varepsilon_0} \text{WE}(F, \Omega, \varepsilon q, K)\right) = \text{WE}(F, \Omega, \bar{\varepsilon}q, K),
\]
and the proof is complete.
Corollary 4.3 Assume that $\text{gr} F$ is closed and consider $q \in \text{int} K$. Then

$$\limsup_{\epsilon \to 0} \text{WE}(F, \Omega, \epsilon q, K) = \bigcap_{\epsilon < 0} \text{WE}(F, \Omega, \epsilon q, K) = \text{WE}(F, \Omega, K).$$

Now, consider the following scalarization mapping $\varphi: Y \to \mathbb{R}$ and $\delta \geq 0$. Set

$$\delta - \text{argmin}(\varphi_{\epsilon} F) := \{(x_0, y_0) \in \text{gr} F : \varphi(y_0) - \delta \leq \varphi(y), \forall y \in F(\Omega)\}. \quad (4.1)$$

Notice that (4.1) is the set of optimal solutions with error $\delta$ of the set-valued optimization problem $\varphi_{\epsilon} F$.

The first part of the next proposition gives a sufficient condition for approximate efficient solutions of problem (1) by scalarization and the second part a necessary condition. For each $y_0 \in F(\Omega)$, let $F_{\epsilon, \lambda} X \Rightarrow Y$ be a set-valued map that $F_{\epsilon, \lambda}(x) = F(x) - \{y_0\}$, for all $x \in X$.

Proposition 4.4 Consider $v \in Y \backslash (-\text{int}K)$ and $\epsilon \geq 0$.

1. Suppose that $\varphi(0) \geq 0$, $-\epsilon v - \text{int} K \subset \{y \in Y : \varphi(y) < 0\}$.

   If $(x_0, 0) \in \varphi(0) - \text{argmin}(\varphi_{\epsilon} F_{\epsilon, \lambda})$, then $(x_0, y_0) \in \text{WE}(F, \Omega, \epsilon v, K)$.

2. Suppose that $\{y \in Y : \varphi(y) < 0\} \subset -\epsilon v - \text{int} K$. \hspace{1cm} (4.2)

   If $(x_0, y_0) \in \text{WE}(F, \Omega, \epsilon v, K)$, then $(x_0, 0) \in \varphi(0) - \text{argmin}(\varphi_{\epsilon} F_{\epsilon, \lambda})$.

Proof Because the proof follows directly from definitions, it is omitted. \hfill $\Box$

Remark 1

1. Observe that statement (4.2) implies that $\varphi(0) \geq 0$.

2. If we want to use the both parts of Proposition 4.4, we need a mapping $\varphi: Y \to \mathbb{R}$ such that

$$\{y \in Y : \varphi(y) < 0\} = -\epsilon v - \text{int} K,$$ \hspace{1cm} (4.3)

where $v \notin \text{int} K$.

Now, by using the Tammer–Weidner nonlinear separation functional (Gerth & Weidner, 1990) satisfy this property and we denote it by $\varphi_e: Y \to \mathbb{R}$ defined by $e \in \text{int} K$ and

$$\varphi_e(y) = \inf\{t \in \mathbb{R} : y \in te - \text{cl} K\}, \hspace{1cm} \forall y \in Y. \quad (4.4)$$

The main properties of the scalar function $\varphi_e$ are given in the next lemma.

Lemma 4.5 Göpfert et al. (2003) Let $K \subset Y$ be a closed convex cone with nonempty interior and $e \in \text{int} K$. Consider the functional $\varphi_e$ defined by (4.4). This map is continuous, convex, Lipschitz and for every $t \in \mathbb{R}$

$$\{y : \varphi_e(y) \leq t\} = te - K, \hspace{0.5cm} \{y : \varphi_e(y) < t\} = te - \text{int} K.$$ \hspace{1cm} (4.1)

Moreover, for every $u \in Y$

$$\partial \varphi_e(u) = \{v^* \in K^* : \langle v^*, e \rangle = 1, v^*(u) = \varphi_e(u)\},$$

where $v^* \in K^*$ is the Fenchel conjugate of $v$. \hfill $\Box$


where $\partial$ means the convex subdifferential.

**Theorem 4.6** Consider $v \in Y \setminus (-\text{int}K)$, $\varepsilon \geq 0$ and the mapping $\varphi: Y \rightarrow \mathbb{R}$, $\varphi(y) = \varphi_{\varepsilon}(y + \varepsilon v)$ for all $y \in Y$. Then

$$(x_0, y_0) \in \text{WE}(F, \Omega, \varepsilon v, K) \iff (x_0, 0) \in \varepsilon \varphi_{\varepsilon}(v) \text{-argmin}(\varphi_{\varepsilon}F_{y_0}).$$

**Proof** By using Lemma (4.5), we have

$$\{y \in Y : \varphi(y) < 0\} = -\varepsilon v - \text{int}K,$$

and by Proposition 4.4, we deduce that

$$(x_0, y_0) \in \text{WE}(F, \Omega, \varepsilon v, K) \iff (x_0, 0) \in \varphi(0) - \text{argmin}(\varphi_{\varepsilon}F_{y_0}).$$

Also, notice that $\varphi(0) = \varphi_{\varepsilon}(\varepsilon v) = \varepsilon \varphi_{\varepsilon}(v)$, and the proof is complete. □

Let $Z$ be a real locally convex Hausdorff topological linear space, $D \subset Z$ be a proper convex cone with nonempty topological interior and consider a mapping $G: X \rightrightarrows Z$ and the set

$$\Omega_d = \{x \in X : G(x) \cap -D \neq \emptyset\}.$$

Now, we will characterize the $\varepsilon q$-efficient solutions of the set-valued optimization problem

$$\text{Min}_q \{F(x) : G(x) \cap -D \neq \emptyset\},$$

where $q \in \text{int}K$, $F$ is $K$-preinvex and $G$ is $D$-preinvex and the Slater constraint qualification is satisfied, i.e. there exists $x \in \Omega_d$ such that $G(x) \cap -D \neq \emptyset$.

**Lemma 4.7** Let $\varepsilon > 0$.

1. We have

$$(x_0, y_0) \in \text{WE}(F, \Omega_d, \varepsilon v, K) \Rightarrow (x_0, (y_0, z_0)) \in \text{WE}((F, G), X, \varepsilon(v, (1/\varepsilon)z_0), K \times D) \cap (\Omega_d \times Y \times Z),$$

that $z_0 \in G(x_0) \cap -D$.

2. Consider $q \in \text{int}K$ and suppose that $F$, $G$ are $K$-preinvex and $D$-preinvex, respectively, with respect to the same $\eta$ and the Slater constraint qualification is satisfied. Then

$$(x_0, y_0) \in \text{WE}(F, \Omega_d, \varepsilon q, K) \Rightarrow (x_0, (y_0, z_0)) \in \text{WE}((F, G), X, \varepsilon(q, (1/\varepsilon)z_0), K \times D) \cap (\Omega_d \times Y \times Z),$$

that $z_0 \in G(x_0) \cap -D$.

**Proof**

1. Suppose that $(x_0, y_0) \in \text{WE}(F, \Omega_d, \varepsilon v, K)$, therefore

$$(F(\Omega_d) - y_0 + \varepsilon v) \cap (-\text{int}K) = \emptyset. \quad (4.5)$$

Set $A = ((F(X), G(X)) - (y_0, z_0) + \varepsilon(v, (1/\varepsilon)z_0)) \cap (-\text{int}K \times (-\text{int}D))$. Then

$$A = (F(X) - y_0 + \varepsilon v, G(X)) \cap (-\text{int}K \times (-\text{int}D)).$$

If $A \neq \emptyset$, then there exists $x_1 \in X$ such that
\( G(x_1) \cap (- \text{int} D) \neq \emptyset \) and \( F(x_1) - y_o + \varepsilon v \cap (- \text{int} K) \neq \emptyset \),

which is a contradiction with (4.5). Hence \( A = \emptyset \) and therefore

\[(x_o, (y_o, z_o)) \in \mathcal{WE}(F, G, X, e(t, (1/\varepsilon)z_o), K \times D) \cap (\Omega_o \times Y \times Z).
\]

(2) The necessary condition is given by part 1, hence we prove the sufficient condition. Let

\[(x_o, y_o) \in \mathcal{WE}(F, G, X, e(q, (1/\varepsilon)z_o), K \times D) \cap (\Omega_o \times Y \times Z)
\]

and suppose that \((x_o, y_o) \notin \mathcal{WE}(F, \Omega_o, e, K)\). By using Theorem 4.2, there exists \( \varepsilon_0 > \varepsilon \) such that \((x_o, y_o) \notin \mathcal{WE}(F, \Omega_o, e, q, K)\) and therefore there exist \( x \in X, z \in G(x) \cap -D \) and \( y \in F(x) \) such that

\[y - y_o \in -\varepsilon_0 q - \text{int} K.
\]

Suppose that \( \bar{x} \in X \) and \( \bar{z} \in G(\bar{x}) \) be such that \( \bar{z} \in - \text{int} D \) and set \( x_t = x + t\eta(x, x) \) that \( t \in (0, 1) \). Since \( G \) is \( D \)-preinvex there exists \( z_t \in G(x_t) \) such that

\[z_t \in (1 - t)z + t \bar{z} - D \subset -\text{int} D.
\]

Also, from \( K \)-preinvexity of \( F \), for any \( \bar{y} \in F(\bar{x}) \), there exists \( y_t \in F(x_t) \) such that

\[y_t - y_o \in (1 - t)y + t\bar{y} - y_o - K
\]

\[= t(y - y) + y - y_o - K
\]

\[\subset t(y - y) - \varepsilon_0 q - \text{int} K
\]

for \( t \) small enough. Hence

\[(y_t, z_t) \in (y_o, z_o) - \varepsilon(q, (1/\varepsilon)z_o) - \text{int}(K \times D),
\]

which is a contradiction. Therefore, \((x_o, y_o) \in \mathcal{WE}(F, \Omega_o, e, q, K)\).

Now, suppose that \( e_1 \in \text{int} K \) and \( e_2 \in \text{int} D \). We consider the mapping \( \varphi_{e_1, e_2}: Y \times Z \to \mathbb{R} \) defined as follows:

\[\varphi_{e_1, e_2}(y, z) = \inf\{t \in \mathbb{R} : y \in te_1 - \text{cl} K, z \in te_2 - \text{cl} D\}, \quad \forall y \in Y, z \in Z.
\]

**Theorem 4.9** Let \( q \in \text{int} K, \varepsilon > 0, x_o \in \Omega_o \) and suppose that \( F \) is \( K \)-preinvex and \( G \) is \( D \)-preinvex with respect to the same \( \eta \) and the Slater constraint qualification is satisfied. Consider the mapping \( \varphi: Y \times Z \to \mathbb{R} \) defined as \( \varphi(y, z) = \varphi_{e_1, e_2}(y, z) = e(q, (1/\varepsilon)z_o) \), that \( z_o \in G(x_o) \cap -D \), for all \( y \in Y, z \in Z \). Then

\[(x_o, y_o) \in \mathcal{WE}(F, \Omega_o, e, q, K) \iff (x_o, 0, 0) \in e \varphi_{e_1, e_2}(q, (1/\varepsilon)z_o) - \text{argmin}(\varphi_{e_1, e_2}(F, G))_{y=0}\).

**Proof** By using Theorem 4.6 and Lemma 4.7, we can deduce the proof.

5. Conclusions

In this work we have studied relations between (MVLI) and set-valued optimization problems. For this objective, a new notion of (MVLI) based on normal subdifferential for set-valued maps has been presented, which extends a lot of (MVLI) in the literature. Also, the behavior of the approximate efficient solutions set is studied. We showed that the upper limit of \( \varepsilon \)-efficient points of a set-valued
optimization problem, when $\varepsilon$ tends to zero is equal to weak efficient points. Finally, by the Tammer–Weidner nonlinear scalarization functional, we obtained a kind of penalization for $\varepsilon$-efficient solutions of a cone constrained set-valued optimization problem.

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