External memory priority queues with decrease-key and applications to graph algorithms

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Abstract

We present priority queues in the external memory model with block size $B$ and main memory size $M$ that support on $N$ elements, operation UPDATE (a combination of operations INSERT and DECREASEKEY) in $O\left(\frac{1}{B} \log \frac{M}{B} \frac{N}{B}\right)$ amortized I/Os, operation EXTRACTMIN in $O\left(\frac{M^\varepsilon}{B} \log \frac{M}{B} \frac{N}{B}\right)$ amortized I/Os and operation DELETE in $O\left(\frac{M^\varepsilon}{B} \log^2 \frac{M}{B} \frac{N}{B}\right)$ amortized I/Os, for any real $\varepsilon \in (0, 1)$, using $O\left(\frac{N}{B} \log \frac{M}{B} \frac{N}{B}\right)$ blocks. Previous I/O-efficient priority queues either support these operations in $O\left(\frac{1}{B} \log 2 \frac{N}{B}\right)$ amortized I/Os [Kumar and Schwabe, SPDP ’96], or support only operations INSERT, DELETE and EXTRACTMIN in optimal $O\left(\frac{1}{B} \log \frac{M}{B} \frac{N}{B}\right)$ amortized I/Os, however without supporting DECREASEKEY [Fadel et al., TCS ’99].

We also present buffered repository trees that support on a multi-set of $N$ elements, operations INSERT and EXTRACT (on $K$ extracted elements) in $O\left(\frac{1}{B} \log \frac{M}{B} \frac{N}{B}\right)$ and $O\left(\log \frac{M}{B} \frac{N}{B} + K/B\right)$ amortized I/Os, respectively, using $O\left(\frac{N}{B}\right)$ blocks. This improves upon the previous I/O-bounds that achieve a base-2 logarithm instead [Buchsbaum et al., SODA ’00].

Our results imply improved $O\left(\frac{E}{B} \log \frac{M}{B} \frac{E}{B}\right)$ I/Os for single-source shortest paths, depth-first search and breadth-first search algorithms on massive directed graphs with an edge-to-node ratio $\frac{E}{V} = \Omega\left(M^\varepsilon\right)$, which is equal to the I/O-optimal bound for sorting $E$ values in external memory.

1 Introduction

Priority queues are fundamental data structures with numerous applications across computer science, most prominently in the design of efficient graph algorithms. They support the
following operations on a set of \( N \) stored elements of the type \((\text{key}, \text{priority})\), where “key” serves as an identifier and “priority” is a value from a total order:

- **INSERT** (element \( e \)): Insert element \( e \) to the priority queue.
- **DELETE** (key \( k \)): Remove all elements with key \( k \) from the priority queue.
- \( e = \text{EXTRACTMIN}() \): Remove and return the element \( e \) in the priority queue with the smallest priority.
- **DECREASEKEY** (element \((k, p)\)): Given that an element with key \( k \) and priority \( p' \) is stored in the priority queue, if priority \( p < p' \), replace the element’s priority \( p' \) with \( p \).

Operation **UPDATE** (element \((k, p)\)) is a combination of operations **INSERT** and **DECREASEKEY**, such that if the priority queue does not contain any element with key \( k \), **INSERT** \((k, p)\) is executed, otherwise **DECREASEKEY** \((k, p)\) is executed.

We study the problem of designing priority queues that support all these operations in external memory. In the **external memory model** (also known as the I/O model) [1] the amount of input data is assumed to be much larger than the main memory size \( M \). Thus, the data is stored in an external memory device (i.e. disk) that is divided into consecutive blocks of size \( B \) elements. Time complexity is measured in terms of I/O operations (or I/Os), namely block transfers from external to main memory and vice versa, while computation in main memory is considered to be free. Space complexity is measured in the number of blocks occupied by the input data in external memory. Algorithms and data structures in this model are considered **cache-aware**, since they are parameterized in terms of \( M \) and \( B \). In contrast, **cache-oblivious** algorithms and data structures [11] are oblivious to both these values, which allows them to be efficient along all levels of a memory hierarchy. I/O-optimally scanning and sorting \( x \) consecutive elements in an array are commonly denoted to take \( \text{Scan} \left( \frac{x}{B} \right) \) I/Os and \( \text{Sort} \left( \frac{x}{B} \right) \) I/Os, respectively [1, 11].

Priority queues are a basic component in several fundamental graph algorithms, including:

- The **single-source shortest paths** (SSSP) algorithm on directed graphs with positively weighted edges, which computes the minimum edge-weight paths from a given source node to all other nodes in the graph.
- The **depth-first search** (DFS) and **breadth-first search** (BFS) algorithms on directed unweighted graphs, which number all nodes of the graph according to a depth-first or a breadth-first exploration traversal starting from a given source node, respectively.

Another necessary component for these algorithms are I/O-efficient **buffered repository trees** (BRTs) [6, 2, 7]. They are used by external memory graph algorithms in order to confirm that a given node has already been visited by the algorithm. This avoids expensive random-access I/Os incurred by internal memory methods. In particular, BRTs support the following operations on a stored multi-set of \( N \) (key, value) elements, where “key” serves as an identifier and “value” is from a total order:
Table 1: Asymptotic amortized I/O-bounds of cache-aware and cache-oblivious priority queue operations (respectively, above and below the horizontal line) on \( N \) elements and real \( \epsilon \in (0, 1) \). *Expected I/Os.

- \textbf{INSERT}(element \( e \)): Insert element \( e \) to the BRT.
- \textbf{element} \( e_i = \text{EXTRACT}(\text{key} \ k) \): Remove and return all \( K \) elements \( e_i \) (for \( i \in [1, K] \)) in the BRT with key \( k \).

### 1.1 Previous work

Designing efficient external memory priority queues able to support operation \texttt{DECREASEKEY} (or at least operation \texttt{UPDATE}) has been a long-standing open problem [14, 10, 16, 9, 7, 13]. I/O-efficient adaptations of the standard heap data structure [10] or other sorting-based approaches [16], despite achieving optimal base-(\( M/B \)) logarithmic amortized I/O-complexity, fail to support operation \texttt{DECREASEKEY}. (Nevertheless, we use these priority queues as subroutines in our structure.) Adaptations of the tournament tree data structure support all operations, albeit in not so efficient base-2 logarithmic amortized I/Os [14, 7]. Indeed, in the recent work of Eenberg, Larsen and Yu [9] it is shown that for a sequence of \( N \) operations, any external-memory priority queue supporting \texttt{DECREASEKEY} must spend \( \max\{\text{INSERT, DELETE, EXTRACTMIN, DECREASEKEY}\} = \Omega\left(\frac{1}{B} \log_{\log B} N\right) \) amortized I/Os. Randomized priority queues with matching complexity were recently presented by Jiang and Larsen [13].

The BRTs introduced by Buchsbaum et al. [6, Lemma 2.1] and their cache-oblivious counterparts [2] support \texttt{INSERT} in \( O\left(\frac{1}{B} \log_2 N\right) \) amortized I/Os and \texttt{EXTRACT} on \( K \) extracted elements in \( O\left(\log_2 N + K/B\right) \) amortized I/Os on a multi-set of \( N \) stored elements.

### 1.2 Our contributions

We present I/O-efficient priority queues that support on \( N \) stored elements, operation \texttt{UPDATE} in optimal \( O\left(\frac{1}{B} \log_{\log B} N\right) \) amortized I/Os, operation \texttt{EXTRACTMIN} in \( O\left(\frac{M^\epsilon}{B} \log_{\log B} N\right) \) amortized I/Os and operation \texttt{DELETE} in \( O\left(\frac{M^\epsilon}{B} \log_2 \frac{N}{B}\right) \), for any real \( \epsilon \in (0, 1) \). Our priority queues are the first to support operation \texttt{UPDATE} (and hence \texttt{DECREASEKEY}) in a
cache-aware setting in optimal I/Os, while also I/O-optimally supporting operation Insert. These bounds improve upon previous priority queues supporting DecreaseKey [14], albeit at the expense of suboptimal I/O-efficiency for ExtractMin and Delete (respecting the lower bound of [9] for \( M = \Omega(B \log_2 N) \)). See Table 1 for a comparison with previous cache-aware and cache-oblivious I/O-efficient priority queues.

We also present I/O-efficient BRTs that support on a multi-set of \( N \) elements, operation Insert in \( O\left(\frac{1}{B} \log_2 \frac{N}{M} \frac{M}{B}\right) \) amortized I/Os and operation Extract in \( O\left(\log_2 \frac{M}{N} \frac{N}{B} \frac{K}{B}\right) \) amortized I/Os (on \( K \) extracted elements), improving upon previous cache-aware bounds [6]. Combined with our priority queues, we achieve \( O\left(\left(\frac{V}{B} M^\epsilon + \frac{E}{B}\right) \log_2 \frac{M}{N} \frac{N}{B} \frac{E}{B}\right) \) I/Os for external memory SSSP, DFS and BFS algorithms on graphs with \( V \) nodes and \( E \) directed edges. This improves upon previous \( O\left(\left(\frac{V}{B} + \frac{E}{B}\right) \log_2 E\right) \) I/Os for directed SSSP [14, 15] and \( O\left(\left(\frac{V}{B} M^\epsilon + \frac{E}{B}\right) \log_2 \frac{M}{N} \frac{N}{B} \frac{E}{B} + \frac{E}{B} \log_2 \frac{E}{B}\right) \) I/Os for directed DFS and BFS [6, 2]. These bounds are I/O-optimal for graphs with density \( \frac{E}{V} = \Omega\left(\frac{M^\epsilon - B \log_2 M}{\log_2 M - 1}\right) = \Omega(M^\epsilon), \ \epsilon \in (0, 1) \). For example, SSSP on a graph with \( E = \Omega(V^{1+\epsilon}) \) edges and \( V = \Omega(M) \) nodes takes optimal \( O\left(\frac{E}{B} \log_2 \frac{M}{N} \frac{E}{B}\right) \) I/Os, rather than \( O\left(\frac{E}{B} \log_2 E\right) \) I/Os [15].

1.2.1 Our approach

The main component of our priority queues is the \( x \)-treap, a recursive array inspired by similar cache-oblivious \( x \)-box [4] and cache-aware hashing data structures [12] that solve the dynamic dictionary problem in external memory (respectively, under predecessor and membership queries on a dynamic set of keys). To solve the priority queue problem, we adapt this recursive scheme to also handle priorities inspired by the cache-oblivious priority queues of Brodal et al. [5] that support Update, yet in suboptimal I/Os.

Specifically, every recursive level of our structure is divided into a front and a rear array, upon which appropriate invariants impose the functionality of a treap: lower priority elements ascend up the recursion occupying only front arrays, while higher priority elements descend down over rear arrays. The auxiliary operation Resolve ensures that at most one element per key occurs in every recursive level, by discarding high priority elements, whenever multiple elements with the same key appear in the same recursive level. Our algorithms are oblivious to the value of \( B \), but not of \( M \), since it is used to switch to the recursion’s base case. The base case structures introduce an \( O(M^\epsilon) \)-bottleneck to the I/O-efficiency of ExtractMin and Delete. Our BRTs are implemented by a cache-aware variant of the \( x \)-box [4].

2 \( x \)-Treap

Given real parameter \( \alpha \in (0, 1] \) and key range \( [k_{\min}, k_{\max}] \subseteq \mathbb{R} \), an \( x \)-treap \( D \) stores a set of at most \( 2(D.x)^{1+\alpha} \) elements \((*, k, p)\) associated with a key \( k \in [D.k_{\min}, D.k_{\max}] \) and a priority \( p \) from a totally ordered set. \( D \) represents a set \( D.rep \) of pairs (key, priority), such
that a particular key \( k \) contained in \( D \) is represented to have the smallest priority \( p \) of any element with key \( k \) stored in \( D \), unless an element with key \( k \) and a smaller priority has been removed from the structure. In particular, we call the key and priority represented, when the pair \((\text{key}, \text{priority}) \in D.rep\). A representative element contains a represented key and its represented priority. More formally, we define:

\[
D.rep := \bigcup_{\{k|(k,p) \in D\}} \left\{ (k, \min_p (k,p) \in D) \right\}
\]

The proposed representation scheme works under the assumption that a key that is not represented by the structure anymore, cannot become represented again. In other words, a key returned by operation ExtractMin cannot be Inserted to the structure again.

The following interface operations are supported:

- **Batched-Insert** \((D, e_1, e_2, \ldots, e_b)\): For constant \( c \in (0, \frac{1}{2}] \), insert \( b \leq c \cdot D.x \) elements \( e_1, e_2, \ldots, e_b \) to \( D \), given they are key-sorted with keys \( e_i,k \in [D.k_{\min}, D.k_{\max}), i \in [1, b] \). Batched-Insert adds the pairs \((e_i,k, e_i,p)\) to \( D.rep \) with key \( e_i,k \) that is not contained in \( D \) already. Batched-Insert decreases the priority of a represented key \( e_i,k \) to \( e_i,p \), if its represented priority is larger than \( e_i,p \) before the operation. More formally, let \( X_{\text{new}} \) contain the inserted pairs \((e_i,k, e_i,p)\) with \( e_i,k \notin D.rep \). Let \( X_{\text{old}} \) contain the pairs in \( D.rep \) with an inserted key, but with larger priority than the inserted one, and let \( X_{\text{dec}} \) contain these inserted pairs. After Batched-Insert, a new \( x\)-treap \( D' \) is created where \( D'.rep = D.rep \cup X_{\text{new}} \cup X_{\text{dec}} \setminus X_{\text{old}} \).

- **Batched-ExtractMin** \((D)\): For constant \( c \in (0, \frac{1}{4}] \), remove and return the at most \( c \cdot D.x \) elements \((k,p)\) with the smallest priorities in \( D \).

Batched-ExtractMin removes the pairs \( X_{\text{min}} \) from \( D.rep \) with the at most \( c \cdot D.x \) smallest priorities. Let \( X_{\text{key}} \) contain the pairs in \( D \) with keys in \( X_{\text{min}} \). After Batched-ExtractMin, a new \( x\)-treap \( D' \) is created where \( D'.rep = D.rep \setminus X_{\text{min}} \setminus X_{\text{key}} \).

**Theorem 1.** An \( x\)-treap \( D \) supports Batched-ExtractMin in \( O\left(\frac{D.x}{B} \log M D.x\right) \) amortized I/Os per element and Batched-Insert in \( O\left(\frac{1+\alpha}{B} \log M D.x\right) \) amortized I/Os per element, using \( O\left(\frac{(D.x)^{1+\alpha}}{B} \log M D.x\right) \) blocks, for any real \( \alpha \in (0, 1] \).

The structure is recursive. The base case is described separately in Subsection 2.3. The base case structure is used when \( D.x \leq c^{'} M^{\frac{1+\alpha}{1+\alpha}} \) (for an appropriately chosen constant \( c' > 0 \)). Thus assuming \( D.x > c^{'} M^{\frac{1+\alpha}{1+\alpha}} \), we define an \( x\)-treap to contain three buffers (which are arrays that store elements) and many \( \sqrt{x} \)-treaps (called subtreaps). Specifically, the top, middle and bottom buffers have sizes \( D.x, (D.x)^{1+\frac{\alpha}{2}} \) and \( (D.x)^{1+\alpha} \), respectively. Each buffer is divided in the middle into a front and a rear buffer. The subtreaps are divided into the upper and the lower level that contain at most \( \frac{1}{4}(D.x)^{\frac{3}{2}} \) and \( \frac{1}{4}(D.x)^{\frac{1+\alpha}{2}} \) subtreaps, respectively. Let
|b| denote the size of a buffer b. We define the capacity of an x-treap D to be the maximum number of elements it can contain, which is $D.x + \frac{5}{4} (D.x)^{1+\frac{\alpha}{2}} + \frac{5}{4} (D.x)^{1+\alpha} < 2 (D.x)^{1+\alpha}$.

We define a partial order (≤) using the terminology “above/below” among the buffers of an x-treap and all of the buffers in recursive subtreaps or base case structures. In this order we have top buffer ≤ upper level recursive subtreaps ≤ middle buffer ≤ lower level recursive subtreaps ≤ bottom buffer.

Along with all buffers of D, we also store several additional pieces of bookkeeping information: a counter with the total number of elements stored in D and an index indicating which subtreap is stored in which space in memory.

### 2.1 Invariants

An x-treap D maintains the following invariants with respect to every one of its top/middle/bottom buffers b. The invariants hold after the execution of each interface operation, but may be violated during the execution. They allow changes to D that do not change $D.rep$.

1. The front and rear buffers of b store elements left-justified.

2. The front buffer’s elements’ priorities are smaller than the rear buffer’s elements’ priorities.

3. The front buffer’s elements’ priorities are smaller than all elements’ priorities in buffers below b in D.
4. For a top or middle buffer $b$ with key range $[b.k_{\text{min}}, b.k_{\text{max}})$, the $r$ upper or lower subtreaps $D_i, i \in \{1, r\}$, respectively, have distinct key ranges $[D_1.k_{\text{min}}, D_1.k_{\text{max}}) < \ldots < [D_r.k_{\text{min}}, D_r.k_{\text{max}} = b.k_{\text{max}}$.

5. If the middle or bottom buffer $b$ is not empty, then at least one upper or lower subtreap is not empty, respectively.

### 2.2 Auxiliary operations

The operations Batched-Insert and Batched-ExtractMin make use of the following auxiliary operations (See Appendix A for their correctness):

- **Operation Resolve** $(D, b)$. We say that a buffer $b$ is resolved, when the front and rear buffers contain elements with pairs (key,priority) $(k, p)$, such that $k$ is a represented key, and when the front buffer contains those elements with smallest priorities in the buffer. To resolve $b$, operation Resolve assigns to the elements with represented keys, the key’s minimum priority stored in $b$. Also, it removes any elements with non-represented keys from $b$. Resolve restores Invariant 2 in $b$, when it is temporarily violated by other (interface or auxiliary) operations that call it.

- **Operation Initialize** $(D, e_1, e_2, \ldots, e_b)$ distributes to a new x-treap $D$, $\frac{1}{4} (D.x) \leq b \leq \frac{1}{2} (D.x)^{1+\alpha}$ elements $e_i, i \in [1, b]$ from a temporary array (divided in the middle into a front and a rear array, respecting Invariants 1 and 2). Reserve restores Invariant 2 in $b$, when it is temporarily violated by other (interface or auxiliary) operations that call it.

- **Operation Flush-Up** $(D)$ ensures that the front top buffer of $D$ contains at least $\frac{1}{4} D.x$ elements (unless all buffers of $D$ contains less elements altogether, in which case Flush-Up moves them all to the top front buffer of $D$). By Invariants 2 and 3 these are the elements in $D$ with smallest priority.

- **Operation Flush-Down** $(D)$ is called by Batched-Insert on an x-treap $D$ whose bottom buffer contains between $\frac{1}{2} (D.x)^{1+\alpha}$ and $(D.x)^{1+\alpha}$ elements. It moves to a new temporary array, at most $\frac{2}{3} (D.x)^{1+\alpha}$ elements from the bottom buffer of $D$. It ensures that the largest priority elements are removed from $D$.

- **Operation Split** $(D)$ is called by Batched-Insert on an x-treap $D$ that contains between $\frac{1}{2} (D.x)^{1+\alpha}$ and $(D.x)^{1+\alpha}$ elements. It moves to a new temporary (front and rear) array, the at most $\frac{1}{3} (D.x)^{1+\alpha}$ elements with largest keys in $D$.

#### 2.2.1 Resolving a buffer

Auxiliary operation Resolve on a buffer $b$ of an x-treap $D$ is called by the auxiliary operation Flush-Up and by the interface operation Batched-Insert. It makes use of two temporary auxiliary arrays of size $|b|$. Resolve$(D, b)$ implemented as following:
1. Determine the maximum priority $p_{\text{max}}$ in the front buffer (by a scan). Return, if the front buffer is empty.

2. 2-way merge the elements in the front and rear buffers into a temporary array (by simultaneous scans in increasing key-order). Empty the front and rear buffers.

3. Determine the representative elements in the temporary array (by a scan) and write them to a second temporary array (by another scan): specifically for each key, write only the element with the smallest priority to the secondary temporary array.

4. Scan the second temporary array, writing the elements with priority smaller or equal to $p_{\text{max}}$ to the front buffer, and with priority larger than $p_{\text{max}}$ to the rear buffer. Discard the temporary arrays.

5. Update the counter of $D$.

### 2.2.2 Initializing an $x$-treap

Auxiliary operation *INITIALIZE* is called by the auxiliary operation *FLUSH-UP* and by the interface operation *BATCHED-INSERT*. It allocates an empty $x$-treap $D$ and distributes the $b \in \left[\frac{1}{4} (D.x), \frac{1}{2} (D.x)^{1+\alpha}\right]$ elements $e_i$, $i \in [1, b]$ from a temporary key-sorted array (divided in the middle into a front and a rear array) to the buffers of $D$. *INITIALIZE*$(D, e_1, \ldots, e_b)$ is implemented as following:

1. Create a new $x$-treap $D$ and move all elements in the temporary rear array to the rear bottom buffer of $D$.

2. Find the $\left(\frac{1}{2} (D.x)^{1+\alpha}\right)$-th smallest priority in the temporary front array (by an order-statistics algorithm [3]) and move all elements in the array with larger priority to the front bottom buffer of $D$.

3. Find the $\left(\frac{1}{2} (D.x)\right)$-th smallest priority in the temporary front array and move all elements in the array with smaller priority to the front top buffer of $D$.

4. Find the $\left(\frac{1}{2} (D.x)\right)$-th smallest priority in the temporary front array and until the maximum number of upper level subtrees has been reached: *INITIALIZE* a new upper subtree with the $\frac{1}{2} (D.x)^{1+\alpha}$ key-next elements with smaller priority.

5. Find the $\left(\frac{1}{2} (D.x)^{1+\frac{\alpha}{2}}\right)$-th smallest priority in the temporary front array and move all elements in the array with smaller priority to the front middle buffer of $D$.

6. Find the $\left(\frac{1}{2} (D.x)^{1+\frac{\alpha}{2}}\right)$-th smallest priority in the temporary front array and until the maximum number of lower level subtrees has been reached: *INITIALIZE* a new lower subtree with the $\frac{1}{2} (D.x)^{1+\frac{\alpha}{2}}$ key-next elements with smaller priority.

7. Discard the temporary array and update the counters of $D$. 

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2.2.3 Flushing up an $x$-treap

Auxiliary operation Flush-Up on an $x$-treap $D$ is called only by the interface operation BATCHED-EXTRACTMIN. It is implemented by means of the recursive subroutine Flush-Up$(D, b)$ that also takes as argument a top or middle buffer $b$ of $D$ and moves to its front buffer, the elements with the (at most) $\frac{1}{4}|b|$ smallest represented priorities among the representative elements stored inside and below $b$ in $D$. The operation makes use of a temporary priority queue that supports only operations \textsc{Insert} and \textsc{ExtractMin} \cite{10, 16}. For a bottom buffer $b$, a non-recursive subroutine Flush-Up$(D, b)$ simply calls \textsc{Resolve} on $b$. Flush-Up$(D, b)$ is implemented as following:

1. \textsc{Resolve} $b$ and Flush-Up the top buffers of the subtreaps immediately below $b$.

2. If the front buffer of $b$ contains $k < \frac{1}{4}|b|$ elements: Allocate a temporary array of size $|b|$ and a temporary priority queue $Q$. For every subtreap immediately below $b$: Remove all elements from its front top buffer and \textsc{Insert} them to $Q$ (by simultaneous scans). While the temporary array contains no more than $\frac{1}{4}|b| - k$ elements, do:
   
   2.1. \textsc{ExtractMin} one element $e$ from $Q$ and write it in the temporary array.
   
   2.2. If $Q$ contains no more elements from the subtreap $D'$ that contained $e$: \textsc{Flush-Up} the top buffer of $D'$, remove all its elements and \textsc{Insert} them to $Q$.
   
   2.3. If $Q$ is empty and the temporary buffer contains $k' < \frac{1}{4}|b| - k$ elements: \textsc{Flush-Up} the buffer $b'$ immediately below $b$ in $D$, find the $(\frac{1}{4}|b| - k - k')$-th smallest priority in the front buffer of $b'$ (by an external memory order-statistics algorithm \cite{3}) and move its elements with smaller priority to the temporary array. Left-justify $b'$.

3. Sort the elements in the temporary array by key. 2-way merge into the front buffer of $b$, the elements in the front buffer of $b$ and the temporary array (by simultaneous scans in increasing key-order). Discard the temporary array.

4. If $Q$ is not empty: \textsc{ExtractMin} all elements from $Q$ into a new temporary array, sort the array by key, move the elements left-justified back to the front top buffers of the subtreaps they were taken from (by simultaneous scans in increasing key-order), update the subtreaps’ counters and discard the array.

5. Discard $Q$, update the counters of $D$ and remove all empty subtreaps immediately below $b$ (i.e. whose counter is 0).

6. If there are no subtreaps immediately below $b$ and the front buffer $b'$ immediately below $b$ is not empty: Find the $(\frac{1}{4}|b|^\frac{1+\alpha}{2})$-th smallest priority in the front buffer of $b'$ (by an external memory order-statistics algorithm \cite{3}), move the elements with smaller priority to a new temporary front array (by a scan), left-justify the front buffer of $b$ and \textsc{Initialize} a new subtreap with the elements in the array. Discard the temporary array.
2.2.4 Flushing down an $x$-treap

Auxiliary operation $\text{Flush-Down}$ on an $x$-treap $D$ is called only by the interface operation $\text{Batched-Insert}$ and returns a new temporary key-sorted array with at most $\frac{2}{3} (D.x)^{1+\alpha}$ elements. $\text{Flush-Down}(D)$ is implemented as following:

1. Move all elements from the bottom rear buffer of $D$ the the temporary rear array (by a scan).
2. If Step 1 did not move more than $\left(\frac{1}{6} (D.x)^{1+\alpha}\right)$ elements: Find the $\left(\frac{1}{3} (D.x)^{1+\alpha}\right)$-th smallest priority in the bottom front buffer of $D$ (by an external memory order-statistics algorithm [3]). Move all elements in the bottom front buffer with larger priority to the temporary front array and left-justify the bottom front buffer (by a scan).
3. 2-merge the two runs created by Steps 1 and 2 in the temporary array (by a scan).
4. Update the counter of $D$.

2.2.5 Splitting an $x$-treap

Auxiliary operation $\text{Split}$ is called only by the interface operation $\text{Batched-Insert}$. It moves to a new temporary key-sorted array (divided in the middle into a front and a rear array) at most $\frac{1}{4} |b_i|$ key-larger elements from all buffers $b_i$ in $D$. $\text{Split}(D)$ is implemented as following:

1. Find the $\left(\frac{1}{4} (D.x)^{1+\alpha}\right)$-th smallest key in the front top/middle/bottom buffer of $D$ (by a scan). Respectively, move all elements in the front buffers with larger key to three new front auxiliary arrays.
2. Repeat Step 1 with respect to rear buffers.
3. Update the counter of $D$.
4. $\text{Split}$ all upper and lower subtreaps.
5. Merge the elements in all front/rear auxiliary arrays to a new front/rear temporary array, respectively. Discard all auxiliary arrays.

2.3 Base case

The $x$-treap is a recursive structure. When the treap stores few enough elements so that it can be stored in internal memory, we use simple arrays to support the interface operations and operation $\text{Flush-Up}$.

Lemma 1. An $O \left( M^{\frac{1}{1+\alpha}} \right)$-treap fits in internal memory and supports operation $\text{Batched-Insert}$ in $O(1/B)$ amortized I/Os per element and operations $\text{Batched-ExtractMin}$ and $\text{Flush-Up}$ in $\text{Scan} \left( M^{\frac{\alpha}{1+\alpha}} \right)$ amortized I/Os per element.
Proof. For a universal positive constant \( c_0 \) and a constant parameter \( c' < c_0^{\frac{1}{\alpha} + 1} \), we allocate an array of size \( (c' \left( M^{\frac{1}{1+\alpha}} \right))^{\frac{1}{1+\alpha}} \leq c_0 M \) and divide it in the middle into a front and a rear buffer that store elements and maintain only Invariants [1] and [2]. To implement Batched-Insert on at most \( \frac{c'}{2} M^{\frac{1}{1+\alpha}} \) elements, we simply add them to the rear buffer and update the counter. This costs \( O \left( \frac{M^{\frac{1}{1+\alpha}}}{B} / \frac{1}{2} M^{\frac{1}{1+\alpha}} \right) = O \left( \frac{1}{B} \right) \) I/Os amortized per added element, since we only scan the part of the rear buffer where the elements are being added to. To implement Batched-ExtractMin on at most \( \frac{c'}{2} M^{\frac{1}{1+\alpha}} \) extracted elements, we Resolve the array (as implemented for Theorem [1]), remove and return all elements in the front buffer, and update the counter. By Lemma [4] (proven later in Subsection 2.5) this costs \( O \left( \frac{M}{B} / \frac{1}{2} M^{\frac{1}{1+\alpha}} \right) = O \left( \frac{M^{\frac{1}{1+\alpha}}}{B} \right) \) I/Os amortized per extracted element. Flush-Up is implemented like Batched-ExtractMin with the difference that the returned elements are not removed from the array.

2.4 Interface operations

2.4.1 Inserting elements to an \( x \)-treap

Interface operation Batched-Insert on an \( x \)-treap \( D \) is implemented by means of the recursive subroutine Batched-Insert\((D,e_1,\ldots,e_{c\cdot|b|},b)\) that also takes as argument a top or middle buffer \( b \) of \( D \) and inserts \( c\cdot|b| \) elements \( e_1,\ldots,e_{c\cdot|b|} \) (contained in a temporary array) inside and below \( b \) in \( D \), for constant \( c \in (0,\frac{1}{3}] \). For a bottom buffer \( b \), a non-recursive subroutine Batched-Insert\((D,b)\) simply executes Step 1 below and discards the temporary array. Batched-Insert\((D,e_1,\ldots,e_{c\cdot|b|},b)\) is implemented as following:

1. If \( D.x > c'M^{\frac{1}{1+\alpha}} + c|b| \):
   1.1. 2-way merge into the temporary array, the elements in the temporary array and in the rear buffer of \( b \) (by simultaneous scans in increasing key-order). Resolve \( b \) considering the temporary array as the rear buffer of \( b \).
   1.2. Implicitly partition the front buffer of \( b \) and the temporary array by the key ranges of the subtreaps immediately below \( b \). Consider the subtreaps in increasing key-order by reading the index of \( D \). For every key range (associated with subtreap \( D' \)) that contains at least \( \frac{1}{3} (D.x)^{\frac{1}{2}} \) elements in either the front buffer of \( b \) or the temporary array: While the key range in the front buffer of \( b \) and in the temporary array contains at most \( \frac{2}{3} (D.x)^{\frac{1}{2}} \) elements, do:
     1.2.1. Find the \( \left( \frac{2}{3} (D.x)^{\frac{1}{2}} \right) \)-th smallest priority within the key range in the front buffer of \( b \) and in the temporary array (by an external memory order-statistics algorithm [3]) and move the elements in the key range with larger priority to a new auxiliary array (by simultaneous scans in increasing key-order).
1.2.2. If the counter of $D'$ plus the auxiliary array’s size does not exceed the capacity of $D'$: **Batched-Insert** the elements in the auxiliary array to the top buffer of $D'$. Discard the auxiliary array.

1.2.3. Else, if there are fewer than the maximum allowed number of subtreaps in the level immediately below $b$: **Split** $D'$. Let $k$ be the smallest key in the array returned by **Split** (determined by a constant number of random accesses to the leftmost elements in the returned front/rear array). Move the elements in the auxiliary array with key smaller than $k$ to a new temporary array (by a scan). **Batched-Insert** these elements to $D'$ and discard this temporary array. 2-way merge the remaining elements in the auxiliary array into the returned rear array and discard the auxiliary array. **Initialize** a new subtreap with the elements in the returned array. Discard the returned array.

1.2.4. Else, **Flush-Down** all subtreaps immediately below $b$, which writes them to many returned arrays. 2-way merge into a new temporary array, all elements in $b$ and in all returned arrays (by simultaneous scans in increasing key-order). (When the scan on a subtreap’s temporary array is over, determine the subtreap with the key-next elements in the level by reading the index of $D$.) **Batched-Insert** the elements in the new temporary array to the buffer $b'$ immediately below $b$. Discard the new temporary array and all returned arrays.

1.3. Discard the temporary array and update the counter of $D$.

2. Else if $D.x \leq c'M^{1+\alpha} + c|b|$: **Batched-Insert** the elements to the base case structure.

### 2.4.2 Extracting minimum-priority elements from an $x$-treap

Interface operation **Batched-ExtractMin** on an $x$-treap $D$ is implemented as following:

1. If $D.x > c'M^{\frac{1}{1+\alpha}}$:
   - 1.1 If the front top buffer contains less than $\frac{1}{4}D.x$ elements: **Flush-Up** the top buffer.
   - 1.2 Remove and return all the elements $(e_i.k,e_i.p)$ from the front top buffer.
   - 1.3 Update the counter all of $D$.

2. Else if $D.x \leq c'M^{\frac{1}{1+\alpha}}$: **Batched-ExtractMin** the base case structure.

### 2.5 Analysis

See Appendices B, C, D and E for the proofs of Lemmata 2, 4, 5 and 6 respectively.

**Lemma 2.** An $x$-treap $D$ has $O\left(\log_M D.x\right)$ levels and occupies $O\left((D.x)^{1+\alpha} \log_M D.x\right)$ blocks of space.
Lemma 3. By the tall-cache assumption, scanning the buffers of an $x$-treap $D$ and randomly accessing $O \left( (D.x) \frac{1+\alpha}{2} \right)$ subtrees takes $\text{Scan}( (D.x)^{1+\alpha})$ I/Os, for any real $\alpha \in (0, 1]$.

Proof. We show that $O \left( \frac{(D.x)^{1+\alpha}}{B} + (D.x) \frac{1+\alpha}{2} \right) = O \left( \frac{(D.x)^{1+\alpha}}{B} \right)$. Indeed this holds when $(D.x) \frac{1+\alpha}{2} = O \left( \frac{(D.x)^{1+\alpha}}{B} \right)$. Otherwise $(D.x) \frac{1+\alpha}{2} = \Omega \left( \frac{(D.x)^{1+\alpha}}{B} \right) \Rightarrow (D.x)^{1+\alpha} = O (B^2)$ and by the tall-cache assumption that $M \geq B^2$, we get that $D.x = O \left( M^{\frac{1+\alpha}{1+\alpha}} \right) = O(M)$. Hence $D$ fits into main memory and thus randomly accessing its subtrees incurs no I/Os, meaning that the I/O-complexity is dominated by $O \left( \frac{(D.x)^{1+\alpha}}{B} \right)$. □

A buffer $b_i$ at level $i \leq h = O \left( \log_M D.x \right)$ with current number of elements in the front and rear buffers $b_f, b_r$, respectively, has potential $\Phi(b_i) = \Phi_f(b_i) + \Phi_r(b_i)$, such that (for constants $\varepsilon := \frac{\alpha}{1+\alpha}$ and $c_0 \geq 1$):

- $\Phi_f(b_i) = \begin{cases} 0, & \text{if } \frac{1}{2}|b_i| \leq b_f \leq \frac{1}{3}|b_i|, \\ \frac{c_0}{B} \left( \frac{|b_i|}{4} - b_f \right) \cdot (h - i), & \text{if } b_f < \frac{1}{4}|b_i|, \\ \frac{c_0}{B} \left( b_f - \frac{|b_i|}{3} \right) \cdot (h - i), & \text{if } b_f > \frac{1}{3}|b_i|, \end{cases}$

- $\Phi_r(b_i) = \begin{cases} 2c_0 \cdot \left( b_r - \frac{|b_i|}{2} \right) \cdot (h - i), & \text{if } b_r > 0. \end{cases}$

Lemma 4. Resolve on a buffer $b_i$ takes $\text{Scan}(|b_i|) + O(1)$ amortized I/Os.

Lemma 5. Batched-Insert on an $x$-treap $D$ takes $O \left( \frac{1+\alpha}{B} \log_M D.x \right)$ amortized I/Os per element, for any real $\alpha \in (0, 1]$.

Lemma 6. Batched-ExtractMin on an $x$-treap $D$ takes $O \left( M^{\frac{\alpha}{1+\alpha}} \log_M D.x \right)$ amortized I/Os per element, for any real $\alpha \in (0, 1]$.

3 Priority queues

Priority queues support operations Update and ExtractMin that are defined similarly to Batched-Insert and Batched-ExtractMin, respectively, but on a single element.

To support these operations, we compose a priority queue out of its batched counterpart in Theorem 1. The data structure on $N$ elements consists of $1 + \log_{1+\alpha} \log_2 N$ $x$-treaps of doubly increasing size with parameter $\alpha$ being set the same in all of them. Specifically, for $i \in \{0, \log_{1+\alpha} \log_2 N \}$, the $i$-th $x$-treap $D_i$ has $D_i.x = 2^{(1+\alpha)^i}$. We store all keys returned by ExtractMin in a hash table $X$. For $i \in \{0, \log_{1+\alpha} \log_2 N - 1 \}$, we define the top buffer of $D_i$ to be “below” the bottom buffer of $D_{i-1}$ and the bottom buffer of $D_i$ to be “above” the top buffer of $D_{i+1}$. We define
the set of represented pairs \((key, priority)\) \(rep = \bigcup_{i=0}^{\log_{1+\alpha} N} D_i.rep\\{(k,p)|k \in X\}\) and call represented the keys and priorities in \(rep\). We maintain the invariant that the maximum represented priority in \(D_i.rep\) is smaller than the smallest represented priority below it.

To implement Update on a pair \((key,priority)\) \(\in rep\), we Batched-Insert the corresponding element to \(D_0\). \(D_0\) handles single-element batches, since for \(i = 0 \Rightarrow x = \Theta(1)\). When \(D_i\) reaches capacity (i.e. contains \((D_i.x)^{1+\alpha}\) elements), we call Flush-Down on it, Batched-Insert the elements in the returned temporary array to \(D_{i+1}\) and discard the array. This process terminates at the first \(x\)-treap that can accommodate these elements without reaching capacity.

To implement ExtractMin, we call Batched-ExtractMin to the first \(x\)-treap \(D_i\) with a positive counter, add the extracted elements to the (empty) bottom front buffer of \(D_{i-1}\), until \(D_0\) returns at least one element. If the returned key does not belong to \(X\), we insert it. Else, we discard the element and repeat ExtractMin.

To implement Delete of a key, we add the key to \(X\).

See Appendix \(F\) for the proof of Theorem 2.

**Theorem 2.** There exist priority queues on \(N\) elements that support operation Update in \(O\left(\frac{1}{B} \log_{2^B} N\right)\) amortized I/Os per element, operation ExtractMin in \(O\left(\frac{M^{1+\alpha}}{B} \log_{2^B} N\right)\) amortized I/Os per element and operation Delete in \(O\left(M^{1+\alpha} \log_{2^B} N\right)\) amortized I/Os per element, using \(O\left(\frac{N}{B} \log_{2^B} N\right)\) blocks of space, for any real \(\alpha \in (0, 1]\).

4 Buffered repository trees

A buffered repository tree (BRT) \([6, 2, 7]\) stores a multi-set of at most \(N\) elements, each associated with a key in the range \([1 \ldots k_{max}]\). It supports the operations Insert and Extract that, respectively, insert a new element to the structure and remove and report all elements in the structure with a given key. To implement a BRT, we make use of the \(x\)-box \([4]\). Given positive real \(\alpha \leq 1\) and key range \([k_{min}, k_{max}] \subseteq \mathbb{R}\), an \(x\)-box \(D\) stores a set of at most \(\frac{1}{2} (D.x)^{1+\alpha}\) elements associated with a key \(k \in [D.k_{min}, D.k_{max}]\). An \(x\)-box supports the following operations:

- **Batched-Insert\((D, e_1, e_2, \ldots, e_b)\):** For constant \(c \in (0, \frac{1}{2}]\), insert \(b \leq c \cdot D.x\) elements \(e_1, e_2, \ldots, e_b\) to \(D\), given they are key-sorted with keys \(e_i, k \in [D.k_{min}, D.k_{max}], i \in [1, b]\).

- **Search\((D, \kappa)\):** Return pointers to all elements in \(D\) with key \(\kappa\), given they exist in \(D\) and \(\kappa \in [D.k_{min}, D.k_{max}]\).

To implement operation Extract\((D, \kappa)\) that extracts all elements with key \(\kappa\) from an \(x\)-box \(D\), we Search\((D, \kappa)\) and remove from \(D\) all returned pointed elements.

See Appendix \(G\) for the proof of Theorem 3.
Theorem 3. There exist buffered priority trees on a multi-set of $N$ elements and of $K$ extracted elements that support operations \texttt{Insert} and \texttt{Extract} in $O\left(\frac{1}{B} \log_{\frac{M}{B}} \frac{N}{B}\right)$ and $O\left(\frac{M}{B} \log_{\frac{M}{B}} \frac{N}{B} + \frac{K}{B}\right)$ I/Os amortized per element, using $O\left(\frac{N}{B}\right)$ blocks of space, for any real $\alpha \in (0, 1]$. 

5 Graph algorithms

5.1 Directed single-source shortest paths

Theorem 4. Single source shortest paths on a directed graph with $V$ nodes and $E$ edges can be computed in $O\left(\left(\frac{V}{M} \frac{1}{B} + \frac{E}{B}\right) \log_{\frac{M}{B}} \frac{E}{B}\right)$ I/Os, for any positive real $\alpha \leq 1$.

Proof. The algorithm of Vitter [15] (described in detail in [7, Lemma 4.1] for the cache-oblivious model) makes use of a priority queue that supports the \texttt{Update} operation and of a BRT on $O(E)$ elements. Specifically, it makes $V$ calls to \texttt{ExtractMin} and $E$ calls to \texttt{Update} on the priority queue and $V$ calls to \texttt{Extract} and $E$ calls to \texttt{Insert} on the BRT. Hence, we obtain the stated bounds, by using Theorems 2 and 3.

5.2 Directed depth-first and breadth-first search

Theorem 5. Depth-first search and breadth-first search numbers can be assigned to a directed graph with $V$ nodes and $E$ edges in $O\left(\left(\frac{V}{M} \frac{1}{B} + \frac{E}{B}\right) \log_{\frac{M}{B}} \frac{E}{B}\right)$ I/Os, for any positive real $\alpha \leq 1$.

Proof. The algorithm of Buchsbaum et al. [6] makes use of a priority queue and of a BRT on $O(E)$ elements. Specifically, it makes $2V$ calls to \texttt{ExtractMin} and $E$ calls to \texttt{Insert} on the priority queue and $2V$ calls to \texttt{Extract} and $E$ calls to \texttt{Insert} on the BRT [6, Theorem 3.1]. Hence, we obtain the stated bounds, by using Theorems 2 and 3. \hfill \square
References

[1] Alok Aggarwal and S. Vitter, Jeffrey. The input/output complexity of sorting and related problems. *Commun. ACM*, 31(9):1116–1127, September 1988. URL: http://doi.acm.org/10.1145/48529.48535

[2] Lars Arge, Michael A. Bender, Erik D. Demaine, Bryan Holland-Minkley, and J. Ian Munro. An optimal cache-oblivious priority queue and its application to graph algorithms. *SIAM Journal on Computing*, 36(6):1672–1695, 2007. URL: https://doi.org/10.1137/S0097539703428324, arXiv:https://doi.org/10.1137/S0097539703428324.

[3] Manuel Blum, Robert W. Floyd, Vaughan Pratt, Ronald L. Rivest, and Robert E. Tarjan. Time bounds for selection. *J. Comput. Syst. Sci.*, 7(4):448–461, August 1973. URL: http://dx.doi.org/10.1016/S0022-0000(73)80033-9

[4] Gerth Stølting Brodal, Erik D. Demaine, Jeremy T. Fineman, John Iacono, Stefan Langerman, and J. Ian Munro. Cache-oblivious dynamic dictionaries with update/query tradeoffs. In *Proceedings of the Twenty-first Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’10, pages 1448–1456, Philadelphia, PA, USA, 2010. Society for Industrial and Applied Mathematics. URL: http://dl.acm.org/citation.cfm?id=1873601.1873718

[5] Gerth Stølting Brodal, Rolf Fagerberg, Ulrich Meyer, and Norbert Zeh. Cache-oblivious data structures and algorithms for undirected breadth-first search and shortest paths. In Torben Hagerup and Jyrki Katajainen, editors, *Algorithm Theory - SWAT 2004*, pages 480–492, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

[6] Adam L. Buchsbaum, Michael Goldwasser, Suresh Venkatasubramanian, and Jeffery R. Westbrook. On external memory graph traversal. In *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’00, pages 859–860, Philadelphia, PA, USA, 2000. Society for Industrial and Applied Mathematics. URL: http://dl.acm.org/citation.cfm?id=338219.338650

[7] Rezaul A. Chowdhury and Vijaya Ramachandran. Cache-oblivious buffer heap and cache-efficient computation of shortest paths in graphs. *ACM Trans. Algorithms*, 14(1):1:1–1:33, January 2018. URL: http://doi.acm.org/10.1145/3147172, doi:10.1145/3147172

[8] Alex Conway, Martín Farach-Colton, and Philip Shilane. Optimal Hashing in External Memory. In Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella, editors, *45th International Colloquium on Automata, Languages, and Programming (ICALP 2018)*, volume 107 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 39:1–39:14, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-
[9] Kasper Eenberg, Kasper Green Larsen, and Huacheng Yu. Decreasekeys are expensive for external memory priority queues. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, pages 1081–1093, New York, NY, USA, 2017. ACM. URL: http://doi.acm.org/10.1145/3055399.3055437 doi:10.1145/3055399.3055437.

[10] R. Fadel, K. V. Jakobsen, J. Katajainen, and J. Teuhola. Heaps and heapsort on secondary storage. Theor. Comput. Sci., 220(2):345–362, June 1999. URL: http://dx.doi.org/10.1016/S0304-3975(99)00006-7 doi:10.1016/S0304-3975(99)00006-7.

[11] Matteo Frigo, Charles E. Leiserson, Harald Prokop, and Sridhar Ramachandran. Cache-oblivious algorithms. In Proceedings of the 40th Annual Symposium on Foundations of Computer Science, FOCS ’99, pages 285–, Washington, DC, USA, 1999. IEEE Computer Society. URL: http://dl.acm.org/citation.cfm?id=795665.796479.

[12] John Iacono and Mihai Paˇtraˇscu. Using hashing to solve the dictionary problem. In Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’12, pages 570–582, Philadelphia, PA, USA, 2012. Society for Industrial and Applied Mathematics. URL: http://dl.acm.org/citation.cfm?id=2095116.2095164.

[13] Shunhua Jiang and Kasper Green Larsen. A faster external memory priority queue with decreasekeys. In Timothy M. Chan, editor, Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, pages 1331–1343. SIAM, 2019. URL: https://doi.org/10.1137/1.9781611975482.81 doi:10.1137/1.9781611975482.81.

[14] Vijay Kumar and Eric J. Schwabe. Improved algorithms and data structures for solving graph problems in external memory. In Proceedings of the 8th IEEE Symposium on Parallel and Distributed Processing (SPDP ’96), SPDP ’96, pages 169–, Washington, DC, USA, 1996. IEEE Computer Society. URL: http://dl.acm.org/citation.cfm?id=829517.830723.

[15] Jeffrey Scott Vitter. External memory algorithms and data structures: Dealing with massive data. ACM Comput. Surv., 33(2):209–271, June 2001. URL: http://doi.acm.org/10.1145/384192.384193 doi:10.1145/384192.384193.

[16] Zhewei Wei and Ke Yi. Equivalence between priority queues and sorting in external memory. In Andreas S. Schulz and Dorothea Wagner, editors, Algorithms - ESA 2014, pages 830–841, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.
A \( x\)-Treap correctness

A.1 Auxiliary operations

After calling \texttt{Resolve} on a buffer \( b \), elements from \( b \) are allowed to be moved to other buffers, since Invariants 1 and 2 are maintained. This is because after Steps 2, 3 and 4, the front and rear buffers of \( b \) contain a representative element for every represented key in \( b \) separated by priority \( p_{\text{max}} \) (computed in Step 1). Step 5 accounts for the elements ignored in Step 3.

Operation \texttt{Initialize} moves the elements from the temporary array to a new \( x\)-treap in the following sequence: bottom rear buffer, top front buffer, upper subtrees’ front buffers, middle front buffer and lower subtrees’ front buffers, bottom front buffer. The recursive calls in Steps 4 and 6 ensure that the temporary array empties. All invariants are maintained.

Operation \texttt{Flush-Up} allows for accessing the representative elements with smallest represented priorities in \( D \) by only accessing its front top buffer. Invariants 2 and 3 imply that the next-larger represented priorities with respect to the front top buffer’s maximum represented priority are stored in the upper subtrees’ front top buffers and in turn the next-larger ones are stored in the front middle buffer. Similarly, this holds between the middle buffer with respect to the lower subtrees and the bottom buffer. The subroutine \texttt{Flush-Up}(\( D,b \)) respects this sequence when it moves elements with minimum represented priorities from front buffers to the front buffer of \( b \). Specifically, Step 1 ensures that the representative elements in \( b \) with priority smaller than the \( \left( \frac{1}{4}|b| \right) \)-th largest priority in \( b \) are stored in its front buffer. It also ensures this recursively for the front top buffers of the subtrees immediately below \( b \). If the front buffer of \( b \) contains less than \( \frac{1}{4}|b| \) such elements, Step 2 attempts to move into a temporary array enough elements from below \( b \) in \( D \). The temporary priority queue is used (at Steps 2 and 2.1) to ensure that indeed the smallest-priority representative elements are moved, first from the subtrees (Step 2.2) and, if not enough elements have been moved, from the buffer immediately below \( b \) in \( D \) (at Step 2.3). At most two key-sorted runs are created in the temporary array (one from the subtrees and one from the buffer immediately below \( b \)) which are merged to the front buffer of \( b \) by Step 3 while maintaining Invariants 1 and 4. Step 4 revokes the effects of the temporary priority queue, allowing it to be discarded at Step 5 which also accounts for the moved elements. It also removes any empty subtrees, which may violate Invariant 5. Step 6 restores Invariant 5.

Operation \texttt{Flush-Down} leaves the bottom rear buffer of \( D \) empty and the bottom front buffer with at most \( \frac{1}{2} (D.x)^{1+\alpha} \) elements. By Invariants 2 and 3, the largest priority elements of \( D \) are in the bottom rear buffer and they are removed at Steps 1. However, if they don’t account for a constant fraction of \( D \)’s size, Step 2 removes such a fraction from the bottom front buffer, which contains the next-smaller elements. Invariant 1 is maintained. Step 4 accounts for the removed elements.

Operation \texttt{Split} leaves all (front and rear) buffers of \( D \) half-full. Since it operates on \( x\)-treaps that are more than half-full, whose bottom buffers contain a constant fraction of the total number of elements in the \( x\)-treap, the execution of Steps 1, 2 and 4 moves at most \( \frac{1}{2} (D.x)^{1+\alpha} \) elements to the temporary array. Step 3 accounts for the removed elements. All
invariants are maintained.

A.2 Interface operations

Operation \textsc{Batched-Insert} accommodates the insertion of at most $\frac{1}{3}|b|$ elements by allocating recursively extra space within $D$. Step 1 considers the recursive structure. Specifically, Step 1.1 allows for moving representative elements from $b$ and inserted elements by resolving $b$ with respect to the temporary array. Step 1.2 identifies the subtreaps immediately below $b$ (repeatedly in increasing key-order) whose associated key range contains too many keys (stored in $b$ and the temporary array, but not in the considered subtreap) and attempts to move the largest-priority elements within this key range into the subtreap. Step 1.2.1 identifies the at most $\frac{1}{3}|b|\frac{1}{2}$ elements to be moved and Step 1.2.2 recursively inserts them to the subtreap. However if the subtreap is full, Step 1.2.3 splits it into two subtreaps with enough space. Nonetheless if the level cannot contain a new subtreap, Step 1.2.4 essentially moves all elements in $b$ and in all subtreaps immediately below $b$, to the buffer $b'$ immediately below $b$. Step 1.3 accounts for the number of inserted elements and the changes in number of subtreaps. Step 2 allows for recursing down to the base case.

Operation \textsc{Batched-ExtractMin} considers only the top buffer of $D$. Step 1.1 ensures that there are enough minimum-priority representative elements in the front top buffer of $D$ to be extracted by Step 1.2. Step 1.3 accounts from the extracted elements. Step 2 allows for recursing down to the base case. All Invariants are maintained.

B Proof of Lemma 2

\textit{Proof.} We number the levels of the structure sequentially, following the defined “above/below” order from top to bottom, where a base case structure counts for one level. Hence, the total number of levels is given by $L(D.x) = 3 + 2L \left( \left( D.x \right)^{\frac{1}{2}} \right)$ with $L \left( cM^{\frac{1}{1+\alpha}} \right) = 1$, which solves to the stated bound. Since, \textsc{Resolve} leaves in the operated buffer at most one element with a given key, the space bound follows. \hfill \Box

C Proof of Lemma 4

\textit{Proof.} All steps of operation \textsc{Resolve} are implemented by a constant number of scans over buffers of size at most $|b_i|$. Since elements can only be added to the front buffer and only be removed from the rear buffer, the maximum difference in potential occurs, when $\frac{|b_i| - |b_i|}{3} = \frac{|b_i|}{6}$ elements are moved from a full rear buffer to a front buffer that contains $\frac{|b_i|}{3}$ elements. We have that:

$$\Delta \Phi(b_i) \leq \Delta \Phi_f(b_i) + \Delta \Phi_r(b_i) \leq \frac{c_0}{B} \cdot \frac{|b_i|}{6} \cdot i - \frac{2c_0}{B} \cdot \frac{|b_i|}{6} \cdot i \leq \frac{c_0}{B} \cdot \frac{|b_i|}{6} \cdot i \leq 0,$$

for $|b_i|, i \geq 0$. \hfill \Box
D  Proof of Lemma 5

Proof. Excluding all recursive calls (to BATCHED-INSERT at Steps 1.2.2, 1.2.3 and 1.2.4 and to INITIALIZE at Step 1.2.3), the worst-case cost of BATCHED-INSERT on a buffer \( b_i \) is \( \text{Scan} (|b_i|^{1+\alpha}) + O(1) \) I/Os by Lemmata 3 and 4 by [3] and because this is also the worst-case I/O-cost of FLUSH-DOWN and INITIALIZE. The base case (Step 2) charges an extra \( O\left(\frac{1}{B}\right) \) I/Os per element by Lemma 1.

In Step 1.2.2, at most \( \frac{1}{3} |b_i| + 1 \) elements are moved from a buffer \( b_i \) at level \( i \) to a buffer \( b_{i+1} \) at level \( i + 1 \), where \( |b_i| = |b_{i+1}|^{1+\frac{\alpha}{2}} \). The maximum difference in potential occurs when all these elements are moved from a full rear buffer of \( b_i \) to a rear buffer of \( b_{i+1} \). We have that:

\[
\sum_{j=i}^{i+1} \Delta \Phi_r(b_j) \leq -\frac{2c_0}{B} \frac{|b_{i+1}|}{3} (h - i) + 2 \frac{c_0}{B} \frac{|b_{i+1}|}{3} (h - i - 1) \leq -\frac{2c_0}{3B} |b_{i+1}| \leq 0
\]

for \( |b_{i+1}| \geq 0 \). If the same scenario occurs between front buffers, we have that:

\[
\sum_{j=i}^{i+1} \Delta \Phi_f(b_j) \leq -\frac{1c_0}{3B} |b_{i+1}| \leq 0
\]

for \( |b_{i+1}| \geq 0 \). Hence \( \sum_{j=i}^{i+1} \Delta \Phi(b_j) \leq 0 \), given every newly inserted element is charged with an \( O(h) = O(\log_M D.x) \) initial amount of potential.

In Step 1.2.3, operation SPLIT removes at most \( \frac{1}{2} (D.x)^{1+\alpha} \) from a subtreap \( D \), whose bottom buffer \( b_i \) we assume to be at level \( i \). Operation INITIALIZE will add these elements to a new subtreap, whose bottom buffer is also at level \( i \). Without loss of generality, we focus only on these bottom buffers, since \( D \) is more than half-full when SPLIT is called on it and since the bottom buffer’s size is constant fraction of the subtreap’s total size. The maximum difference in potential occurs when \( \frac{1}{4} |b_i| \) elements are removed from a full bottom front/rear buffer of \( b_i \) and added to an empty bottom front/rear buffer, respectively. We have that:

\[
\Delta \Phi_f(b_i) \leq -\frac{c_0}{B} \frac{|b_i|}{6} (h - i) - \frac{c_0}{B} M^\varepsilon \frac{|b_i|}{6} (h - i) \leq 0
\]

for \( |b_{i+1}| \geq 0 \).

\[
\Delta \Phi_r(b_i) \leq -\frac{2c_0}{B} \frac{|b_i|}{2} (h - i) + 2 \frac{c_0}{B} \frac{|b_i|}{2} (h - i) = 0
\]

\( \Delta \Phi(b_i) \leq 0 \), given that subtreaps created by INITIALIZE are charged with an \( O((D.x)^{1+\alpha} \cdot h) = O((D.x)^{1+\alpha} \log_M D.x) \) initial amount of potential. This extra charge does not exceed by more than a constant factor, the initial potential charged to every newly inserted element, and is amortized over the \( O((D.x)^{1+\alpha}) \) elements that are inserted to the created subtreap.

In Step 1.2.4, operation FLUSH-DOWN removes elements from a bottom buffer \( b_i \) at level \( i \) and inserts them to a middle or bottom buffer \( b_{i+1} \) at level \( i + 1 \), where \( |b_{i+1}| = |b_i|^{1+\frac{\alpha}{2}} \).
The maximum difference in potential occurs when \( \frac{1}{2}|b_i| \) elements are removed from a full bottom rear buffer of \( b_i \) and added to a rear buffer at level \( i + 1 \). We have that:

\[
\sum_{j=i}^{i+1} \Delta \Phi_r(b_j) \leq -2 \frac{c_0}{B} \frac{|b_i|}{2} (h - i) + 2 \frac{c_0}{B} \frac{|b_i|}{2} (h - i - 1) \leq -\frac{c_0}{B} |b_i| \leq 0
\]

for \( |b_{i+1}| \geq 0 \). For the case where Step 2 is executed, in the worst case at most \( \frac{1}{6}|b_i| \) elements are removed from a full bottom front buffer of \( b_i \). By Invariants 2 and 3, operation RESOLVE at Step 1.1 of the recursive call to BATCHED-INSERT will move these elements to the middle/bottom front buffer at level \( i + 1 \). We have that:

\[
\sum_{j=i}^{i+1} \Delta \Phi_f(b_j) \leq -\frac{1}{6} \frac{c_0}{B} |b_i| \leq 0
\]

for \( |b_i| \geq 0 \). Hence \( \sum_{j=i}^{i+1} \Delta \Phi(b_j) \leq 0 \).

\[\square\]

### E Proof of Lemma 6

**Proof.** The cost of BATCHED-EXTRACTMIN is dominated by the call to FLUSH-UP on a buffer \( b_i \), where \( |b_i| = O(D.x) \). Steps 1,2 (extraction) and 2 (base case) cost an extra \( O \left( \frac{M^{\frac{1}{\alpha}}}{B} \right) \) amortized I/Os per element, due to the potential’s definition and by Lemma 1, respectively.

Excluding all recursive calls (to FLUSH-UP at Steps 1, 2.2 and 2.3 and to INITIALIZE at Step 6), the worst-case cost of FLUSH-UP on a buffer \( b_i \) is \( O(|b_i|) + O(1) \) I/Os. Specifically, FLUSH-UP executes \( O(|b_i|) \) INSERTS and EXTRACTMINS to the temporary priority queue (that does not support DECREASEKEY) and a constant number of scans and merges. The priority queue operations \[16\] and Step 3 take \( O(\text{Sort}(|b_i|)) \) I/Os, which dominates the cost of incurred random accesses and of calls to RESOLVE (Lemmata 3 and 4, respectively). The I/O-cost is amortized over the \( \Theta(|b_i|) \) elements returned by BATCHED-EXTRACTMIN, resulting in \( O \left( \frac{1}{B} \log \frac{M}{B} |b_i| \right) \) amortized I/Os per element.

To prove the negative cost of the recursive calls (Steps 1 and 2.3), it suffices to argue that there is a release in potential when buffers in consecutive levels are being processed. Without loss of generality, we assume that elements are moved from the middle front buffer at level \( i \) to the top front buffer at level \( i - 2 \) (\( |b_i| = |b_{i-2}|^{1+\frac{\alpha}{2}} \)), where the upper level subtreaps at level \( i - 1 \) are base case structures (hence they do not affect the potential). This assumption charges an extra \( O \left( \frac{M^{\frac{1}{\alpha}}}{B} \right) \) I/Os per element by Lemma 1. The case between the bottom and middle buffers is analogous. The maximum difference in potential occurs when \( \frac{|b_{i-2}|}{4} \) elements are removed from a front middle buffer at level \( i \) with less than \( \frac{|b_i|}{4} \) elements, and added to an empty front top buffer at level \( i - 2 \). Since the rear buffers do not change, we have that:

\[
\sum_{j=i-2}^{i} \Delta \Phi(b_j) \leq \sum_{j=i-2}^{i} (\Delta \Phi_f(b_j) + \Delta \Phi_r(b_j)) \leq \Delta \Phi_f(b_{i-2}) + \Delta \Phi_f(b_i) \leq 0
\]
\[- \frac{c_0}{B} M^\varepsilon |b_i| \left( h - i + 2 \right) + \frac{c_0}{B} M^\varepsilon |b_i| \left( h - i \right) \leq - \frac{c_0}{B} M^\varepsilon |b_i| \left\lfloor \frac{2}{2} \right\rfloor \leq 0, \]

for $|b_i| \geq 0$.

\[\Box\]

**F Proof of Theorem 2**

**Proof.** To each element in $D_i$ that has been updated since the last time that $D_i$ has undergone a Flush-Down operation, we define an update potential of:

\[
\frac{1 + \frac{\alpha}{B} \log \frac{M^\varepsilon}{B}}{2^{(1+\alpha)i}} = \left( 1 + \frac{\alpha}{B} \right)^{i+1} \frac{1}{B \log \frac{M^\varepsilon}{B}} - \frac{1 + \frac{\alpha}{B} \log \frac{M^\varepsilon}{B}}{B}
\]

When $D_i$ reaches capacity and Flush-Down is called on it, the number of elements that have been updated in the priority queue is a constant fraction of the $D_i$'s capacity. This is because the only way element could appear in $D_i$ is from:

- The elements that were already in the priority queue at $D_j$ for $j < i$.
- The elements that were newly inserted.
- The elements inserted from $D_j$ for $j > i$ during ExtractMin process. However, these elements will never bring the number in $D_i$ above a constant fraction of the capacity.

Hence, an Insert operation, before any needed Flush-Down operation, increases the update potential by:

\[
\sum_{i=0}^{\log_\frac{1+\alpha}{B} \log_2 N} \left( \left( 1 + \frac{\alpha}{B} \right)^{i+1} \frac{1}{B \log \frac{M^\varepsilon}{B}} - \frac{1 + \frac{\alpha}{B} \log \frac{M^\varepsilon}{B}}{B} \right) \leq \sum_{i=0}^{\log_\frac{1+\alpha}{B} \log_2 N} \left( 1 + \frac{\alpha}{B} \right)^{i+1} \frac{1}{B \log \frac{M^\varepsilon}{B}}
\]

\[
= O \left( \frac{\log N}{B \log \frac{M^\varepsilon}{B}} \right)
\]

\[
= O \left( \frac{\log \frac{M^\varepsilon}{B} N}{B} \right)
\]

To every key $k$ that has been returned by operation ExtractMin and occurs in $d$ elements in $D$, we define a potential of $\Phi(k) = d \cdot \frac{c_d}{B} M^\varepsilon \cdot \log \frac{M^\varepsilon}{B} D.x$ (for constant $c_d \geq 1$). Insert does not affect this potential by the assumption that an extracted key is not reinserted to the structure.

To $D_i$ we give an extract-min potential of:

\[
M_\frac{1+\alpha}{B} \log \frac{M^\varepsilon}{B} \frac{2^{(1+\alpha)i}}{B} = \left( 1 + \frac{\alpha}{B} \right)^{i+1} \frac{M_\frac{1+\alpha}{B}}{B \log \frac{M^\varepsilon}{B}} - M_\frac{1+\alpha}{B} \frac{1 + \alpha}{B} \log \frac{M^\varepsilon}{B}
\]
for the number of ExtractMin operations performed since the last time a Batched-ExtractMin operation was performed on $D_i$. When $D_i$ gets empty and ExtractMin is needed to fill it, the number of elements ExtractMin’d from the priority queue since the last time that happened is a constant fraction of $D_i$’s capacity. This is true, since the only way elements could be removed from $D_i$ between Batched-ExtractMins is because $D_i$ reached its capacity from Updates and thus Flush-Down needed to be called on $D_{i-1}$.

This, however, only reduces the number of elements to a constant fraction of $D_i$’s capacity. Hence, an ExtractMin operation, before any needed Flush-Down operation, increases the extract-min potential by:

$$\sum_{i=0}^{\log_1(1 + \alpha) \log_2 N} \left( (1 + \alpha)^{i+1} \frac{M^{\alpha}}{B \log \frac{M}{B}} - M^{\alpha} \frac{1 + \alpha}{B} \log \frac{M}{B} \right) \leq \sum_{i=0}^{\log_1(1 + \alpha) \log_2 N} (1 + \alpha)^{i+1} \frac{M^{\alpha}}{B \log \frac{M}{B}}
$$

$$= O \left( \frac{\alpha}{M^{\alpha}} \log \frac{N}{B} \right)
$$

Finally, the potential is constructed to exactly pay for the cost of Flush-Down between $x$-treaps.

The worst-case cost of operation Delete is $O(1)$ I/Os [12]. However, its amortized cost includes the potential necessary to remove all the elements with the deleted key from the structure. We introduce an extra “ghost potential” of $O \left( \frac{\alpha}{M^{\alpha}} \log \frac{N}{B} \right)$ to every key in $X$ that has been Deleted or ExtractMin’d. The stated I/O-cost for Delete follows from the fact that the first time a key is Deleted or ExtractMin’d from the structure $O \left( \log \frac{\alpha}{B} \right)$ elements in the structure get charged by this potential (by Lemma 1). Whenever an ExtractMin returns an element with ghost potential, this potential is released in order to ExtractMin the next element at no extra amortized cost.

The priority queue may contain an $N$-treap that occupies $O \left( \left( \frac{N}{B} \right)^{1+\alpha} \log \frac{M}{B} \log \frac{N}{B} \right)$ blocks of space (by Theorem 1). However, with carefull manipulation of the unaddressed empty space, this space usage (and thus of the whole priority queue) can be reduced back to $O \left( \frac{N}{B} \log \frac{M}{B} \log \frac{N}{B} \right)$ blocks.

## G Proof of Theorem 3

The BRT on $N$ elements consists of $1 + \log_1(1 + \alpha) \log_2 N$ $x$-boxes of doubly increasing size with parameter $\alpha$ being set the same in all of them. We obtain the stated bounds by modifying the proof of the $x$-Dict [11, Theorem 5.1] to account for Lemmata 7 and 8.
Lemma 7. For \( D.x = \Omega \left( M^\frac{1}{1+\alpha} \right) \), an \( x \)-box supports operation \( \text{Batched-Insert} \) in \( O \left( \frac{1+\alpha}{B} \log \frac{M}{B} \frac{D.x}{B} \right) \) amortized I/Os and operation \( \text{Extract} \) on \( K \) extracted elements in \( O \left( (1 + \alpha) \log \frac{M}{B} \frac{D.x}{B} + \frac{K}{B} \right) \) amortized I/Os, using \( O \left( \frac{(D.x)^{1+\alpha}}{B} \right) \) blocks of space.

Proof. Regarding \( \text{Batched-Insert} \) on a cache-aware \( x \)-box, we obtain \( O \left( \frac{1+\alpha}{B} \log \frac{M}{B} \frac{D.x}{B} \right) \) amortized I/Os by modifying the proof of \( \text{Batched-Insert} \) \([4\text{, Theorem 4.1}]\) according the proof of Lemma 5. Specifically, every element is charged \( O \left( \frac{1}{B} \right) \) amortized I/Os, instead of \( O \left( \frac{1}{B^\frac{1}{1+\alpha}} \right) \), and the recursion stops when \( D.x = O \left( M^\frac{1}{1+\alpha} \right) \), instead of \( D.x = O \left( B^\frac{1}{1+\alpha} \right) \).

Regarding \( \text{Searching} \) for the first occurrence of a key in a cache-aware \( x \)-box, we obtain \( O \left( \log \frac{M}{B} \frac{D.x}{B} \right) \) amortized I/Os by modifying the proof of \( \text{Search} \) \([4\text{, Lemma 4.1}]\), such that the recursion stops when \( D.x = O \left( M^\frac{1}{1+\alpha} \right) \), instead of \( D.x = O \left( B^\frac{1}{1+\alpha} \right) \). To \( \text{Extract} \) all \( K \) occurrences of the searched key, we access them by scanning the \( x \)-box and by following fractional cascading pointers, which incurs an extra \( O \left( K/B \right) \) I/Os. \( \square \)

Lemma 8. An \( O \left( M^\frac{1}{1+\alpha} \right) \)-box fits in internal memory and supports operations \( \text{Batched-Insert} \) in \( O \left( 1/B \right) \) amortized I/Os per element and operation \( \text{Extract} \) on \( K \) extracted elements in \( \text{Scan} \left( M^\frac{\alpha}{1+\alpha} \right) \) amortized I/Os per element.

Proof. We allocate an array of size \( O \left( M \right) \) and implement \( \text{Batched-Insert} \) by simply appending the inserted element to the array and \( \text{Extract} \) by scanning the array and removing and returning all occurrences of the searched key. \( \square \)