On the equivalence between quantum and random walks on finite graphs

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Received: 10 February 2020 / Accepted: 28 October 2020 / Published online: 13 November 2020
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Abstract
Quantum walks on graphs are ubiquitous in quantum computing finding a myriad of applications. Likewise, random walks on graphs are a fundamental building block for a large number of algorithms with diverse applications. While the relationship between quantum and random walks has been recently discussed in specific scenarios, this work establishes a formal equivalence between the processes on arbitrary finite graphs and general conditions for shift and coin operators. It requires empowering random walks with time heterogeneity, where the transition probability of the walker is non-uniform and time dependent. The equivalence is obtained by equating the probability of measuring the quantum walk on a given vertex of the graph and the probability that the random walk is at that same vertex, for all vertices and time steps. The result is given by the construction procedure of a matrix sequence for the random walk that yields the exact same vertex probability distribution sequence of any given quantum walk, including the scenario with multiple interfering walkers. Interestingly, these matrices allow for a different simulation approach for quantum walks where vertex samples respect neighbor locality, and convergence is guaranteed by the law of large numbers, enabling efficient (polynomial) sampling of quantum graph trajectories (paths). Furthermore, the complexity of constructing this sequence of matrices is discussed in the general case.

Keywords Quantum walks · Random walks · Markov chains

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1 Introduction

Quantum walks on graphs are a prominent area of research in quantum computing inspired to be the quantum analogue of classical random walks [1,2]. As with random walks, quantum walks have proven to be an insightful tool for designing quantum algorithms, culminating on efficient solutions for problems such as element distinctness [3], marked-vertex searching [16] and Hamiltonian simulation [4]. Among its marvelous capabilities, quantum walks were shown to perform universal quantum computation for both continuous- [6] and discrete-time [15] models. Extensive surveys covering multiple aspects of quantum walks can be found in the literature [11,20,29].

A few discrete-time models for quantum walks have shown increased community interest over the past years [1,22,28]. The coined model works on an extended Hilbert space which codifies both graph vertices and walker direction and has pioneered discrete-time models [1], where the coin space was introduced to allow unitary evolution. The later Szegedy model [28] performs quantization over a bipartite Markov chain. In this model, a reflection based operator is constructed once the transition probabilities to cross the bipartite sets are defined. The operators of the Szegedy model have a well-described spectra, and its properties are mainly derived from spectral analysis. The staggered model [22] is based on graph tessellations and generalizes the bipartite construction of the Szegedy walk. The partition-based quantum walk framework has recently enabled the analysis of the discrete-time models under the same perspective [13]. This formalism allowed to prove that the two-step coined model, the 2-tessalable staggered model, and the extended Szegedy model for multigraphs are equivalent under unitarity, unifying different models for quantum walks. However, this present work focuses on the coined model.

The case of multiple walkers has also been investigated in different contexts. As with the single quantum walker, the interacting multi-walker model was also shown to be universal for quantum computing [7]. Non-interacting multi-walker models on arbitrary graphs have been treated generically, with proposed physical implementations [23]. The two-walker case was specifically analyzed, leading to interesting results [26,27,31].

In the classical realm, random walks on graphs [14] have been extensively used to drive the design of classical algorithms for a myriad of problems in diverse areas of computing, ranging from sampling [10] to user recommendation [19]. Most applications of random walks assume time homogeneity, which implies that the walker behavior, as it moves on the graph, does not change over time. Time homogeneity favors analytical tractability and important known results have been derived under this restriction, such as the conditions for time convergence of the probability distribution [9]. On the other hand, non-homogeneity, or time-dependence, has been explored on particular niches, such as the celebrated simulated annealing meta-heuristic for optimization problems [12].

The connection between quantum and random walks has been investigated and it is clear that homogeneous random walks cannot match quantum walks on arbitrary graphs. However, it has been shown that the evolution of quantum walks on infinite lines is partially described by time-homogeneous Markovian processes [25]. Its probability evolution can be expressed as a time-independent Markov process with an
additional interference term. This separation method was further used to construct a master equation for the global chirality distribution (GCD) of the quantum walk [24], showing a convergence behavior of homogeneous Markovian processes for the GCD. In addition, a relationship between the walk dimension of both processes was explored through the use of renormalization-group analysis (RG) to evaluate scaling factors of the quantum walk limiting distribution [5]. This analysis allows for the calculation of the walk dimension for quantum walks on some non-trivial graphs and has lead to the conjecture that the number of walk dimensions for the quantum case is half of that of the random walk, a well-known result in the case of homogeneous lattices [5].

In the search for their equivalence, a recent work has shown that non-homogeneous random walks can yield probability sequences identical to the probabilities of quantum walks on the infinite integer line [18]. In this context, an analysis was carried out to generate a given desired distribution sequence over the integers with time- and site-dependent discrete-time coined quantum walks and non-homogeneous random walks. The matching is performed by constructing a random walk with time-varying probabilities that has the same distribution sequence of a Hadamard-coined quantum walk on the infinite line.

A different perspective is the quantum stochastic walk (QSW) model, a generalization of both quantum and random walks which accounts for non-unitary transformations [30]. Using the formalism of density matrices, a super-operator is constructed to perform both Hamiltonian (coherent) and stochastic evolution based on the Kossakowski–Lindblad master equation. The walk behavior over a graph is achieved upon connectivity restrictions on the terms that map the states of the system. Depending on how such terms are chosen, the behavior of both classical and quantum walks can be obtained, as well as the behavior of a more general quantum stochastic process not captured by either of them. However, QSW has no bearing on the equivalence of random and quantum walks.

This article focus on the connection between unitary discrete-time coined quantum walks and random walks on finite graphs, and formally proves that the probability evolution of any quantum walk can be matched exactly by a time-dependent random walk on the same underlying graph. This connection stems from the locality property of both random and quantum walks. Our main contribution is a prescription for the time-dependent matrix that drives the random walk dynamics in order to produce the same probability distribution sequence of any quantum walk. More precisely, when the random walk evolves according to these matrices, its probability distributions over the vertices are identical to that of the quantum walk. While the sequence of matrices describing the random walk clearly depends on the graph and the quantum walk operators, the prescription is very general and requires mild assumptions, such as unitarity.

Furthermore, the equivalence is also established for the case of interacting multiple quantum walkers. The interaction model is taken to be very general, with restrictions solely on the walkers’ movement. The equivalence is provided by equating the evolution of the joint probability distribution of the multiple walkers with the joint distribution of the same number of random walkers. The proof for the single-walker case is gracefully extended to the multiple walkers through arguments of unitarity. As the quantum case, the state representation for the random walk has to increase in order
to accommodate all possible movements. This behavior is captured by constructing a graph in which vertices represent the current position of the walkers. The process can then be viewed as a single random walk on a much larger graph.

A direct consequence of the time-dependent matrices that provide the equivalence is the possibility to simulate a time-dependent random walk on the graph which is equivalent to its quantum walk counterpart. This simulation captures quantum behavior while generating samples that preserve neighbor locality. Differently than the commonly used quantum walk simulation procedure, the samples obtained from the random walk simulation are paths of the graph, allowing trajectories driven by the quantum behavior to be sampled.

It is worth noting that quantum walks on graphs resembles Feynman’s path integral formulation for quantum mechanics [8] in discrete time and space, in the sense that the probability amplitude of a discrete-time walker system at instant $t$ is described by summing up the contributions of all possible paths in the graph with length $t$ connecting the initial and final states. In an essential way, the simulation of trajectories through random walks is a procedure for sampling paths from quantum walks following a trajectory distribution in which, for every instant $t$, the marginal vertex distribution coalesces to the quantum walk vertex distribution. This provides a powerful tool for efficient simulation of quantum walk trajectories on arbitrary graphs.

The remainder of this article is structured as follows. The notation for both quantum and random walks, as well as formal definitions, appears in Sect. 2. The theorem that shows how to construct the equivalent non-homogeneous random walk for any given quantum walk is stated and proved in Sect. 3. In Sect. 4, the results are generalized for the case of multiple walkers. The simulation of graph trajectories from the random walk matrices is treated in Sect. 5. An evaluation of the time complexity of the procedure to construct the transition matrices appears in Sect. 6. Final remarks are drawn in Sect. 7.

2 Quantum and random walks

Let $G = (V, E)$ be a directed graph obtained from an non-directed graph by introducing two directed edges for each initial one, i.e., $(u, v) \in E$ if, and only if $(v, u) \in E$. Let the sets $N^+(v) \subseteq V$ and $N^-(v) \subseteq V$ to denote the sets of outward and inward neighbors of $v$, respectively.

2.1 Quantum walks

A discrete-time coined quantum walk on a graph $G$ is an evolution process of a complex vector in a Hilbert space $\mathcal{H}_w \subseteq \mathcal{H}_v \otimes \mathcal{H}_c$ defined by the graph structure [21]. The vertex space $\mathcal{H}_v$ has dimension $|V|$ and codifies the vertices of the graph, while the coin space $\mathcal{H}_c$ denotes the degrees of freedom of the walker movements, with dimension given by the maximum degree of the graph $D = \{\max\{d(v) : v \in V\}$. Precisely, $\mathcal{H}_w$ is $\mathcal{H}_v \otimes \mathcal{H}_c$ only when $G$ is a regular graph.

Denoting $\{|c\rangle\}$ and $\{|v\rangle\}$, respectively, as the basis for the spaces $\mathcal{H}_c$ and $\mathcal{H}_v$, and $C_v = \{0, \ldots, d(v) - 1\}$ as the integer set for the number of outward edges of a vertex...
\(\mathcal{H}_w\) is \(\{|v, c\} : v \in V, c \in C_v\}\). Assuming \(|\Psi(t)\rangle\) is the walker wavefunction at discrete time instant \(t\), the quantum walk evolution is given by the action of two unitary operators \(S : \mathcal{H}_w \rightarrow \mathcal{H}_w\) and \(W : \mathcal{H}_w \rightarrow \mathcal{H}_w\) on the system state vector as

\[|\Psi(t + 1)\rangle = SW|\Psi(t)\rangle. \tag{1}\]

In this work, we assume that both \(S\) and \(W\) may vary with time, although the dependence will be omitted in order to simplify notation.

### 2.1.1 The coin operator

The coin operator \((W)\) acts on the degrees of freedom of the walker. The most general coin operator is given by

\[W = \sum_{v \in V} |v\rangle \langle v| \otimes W_v, \tag{2}\]

where \(W_v\) is a unitary operator. The coin is responsible for mixing the amplitude of a given state \(|v, c\rangle\) with states \(|v, c'\rangle\) such that \(c, c' \in C_v\), i.e., degrees of freedom of the same vertex, through weights \(w_{v c' c}\). This mixing behavior is enlightened when one observes the action of \(W\) on a generic state vector \(|v, c\rangle\)

\[W |v, c\rangle = \sum_{j \in C_v} |v, j\rangle w_{v j c}. \tag{3}\]

For \(W\) to be unitary, one must impose conditions on the complex values of \(w_{v j c}\). In particular, once the product \(WW^\dagger\) is analyzed, unitarity demands that the operator coefficients obey

\[\sum_{i \in C_v} |w_{vik}|^2 = 1 : v \in V, \quad \text{and} \tag{4}\]

\[\sum_{i \in C_v} \sum_{j \in C_v} \sum_{k \neq j} w^*_{vij} w_{vik} = 0 : v \in V. \tag{5}\]

Two coin operators which will be important further ahead are the Hadamard and the Grover operators. The \(D\)-dimensional Hadamard operator \(H_D\) can be constructed for Hilbert spaces with dimension of the form \(D = 2^k\), for \(k \in \{1, 2, \ldots\}\). Its formal definition is given by

\[H_D = H_2^\otimes H_2 \tag{6}\]

where

\[H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7}\]
On the other hand, taking \( |c \rangle \) to denote the vectors of the computational basis, the Grover operator can be defined for Hilbert spaces with arbitrary dimension \( D \) as

\[
G = \frac{2}{D} \sum_{c=0}^{D-1} \sum_{c'=0}^{D-1} |c \rangle \langle c'| - I.
\]  

(8)

### 2.1.2 The shift operator

The shift, or swap, operator \((S)\) acts by moving the mixed amplitudes created by the operator \( W \) to outward edges. Let \( \eta : V \times C \rightarrow V \) be a mapping between outward edges and outward neighbors of every vertex, \( i.e., u = \eta(v, c) \) is the \( c \)-th outward neighbor of vertex \( v \); let \( \sigma : V \times V \rightarrow C \) be a function that maps an inward edge of a vertex with one of its outward edges, such that \( \sigma(u, v) = c \) associates the inward edge \((u, v)\) of vertex \( v \) with its \( c \)-th outward neighbor; and let \( \sigma^{-1} : V \times V \rightarrow C \) to be the inverse association such that if \( v = \eta(u, c') \) and \( \sigma(u, v) = c \) than \( \sigma^{-1}(u, v) = c' \). The action of the shift operator is formally defined as

\[
|u, c \rangle \rightarrow |\eta(u, c), \sigma(u, \eta(u, c))\rangle.
\]  

(9)

The functions \( \eta \) and \( \sigma \) can be arbitrarily defined as long as the operator remains unitary and the graph edges are respected. In fact, different definitions for \( \eta \) and \( \sigma \) lead to different state amplitude dynamics. To illustrate their generality, the flip flop shift operator can be simply implemented by taking \( \eta(v, c) = u \), \( \eta(u, c') = v \), \( \sigma(u, v) = c \) and \( \sigma(v, u) = c' \) for all \((u, v) \in e\), and an arbitrary choice for orderings the outward neighbors. Furthermore, the action of SW on a generic state vector

\[
|\Psi(t)\rangle = \sum_{v \in V} \sum_{c \in C_v} \Psi(v, c, t) |v, c\rangle
\]

is given by

\[
|\Psi(t+1)\rangle = SW \sum_{v \in V} \sum_{c \in C_v} \Psi(v, c, t) |v, c\rangle,
\]  

(10)

\[
|\Psi(t+1)\rangle = \sum_{v \in V} \sum_{c \in C_v} \sum_{j \in C_v} w_{vjc} \Psi(v, c, t) |\eta(v, j), \sigma(v, \eta(v, j))\rangle,
\]  

(11)

\[
|\Psi(t+1)\rangle = \sum_{v \in V} \sum_{u \in N^{-}(v)} \sum_{j \in C_u} (\Psi(u, j, t) w_{u, \sigma^{-1}(u, v), j}) |v, \sigma(u, v)\rangle.
\]  

(12)

Equation 12 is obtained by noting that each degree of freedom of a given vertex \( v \) corresponds to exactly one outward neighbor of \( v \) and by fixing the vertex element of the basis state vector from the summation through a variable substitution from \( \eta \). Let \( \rho : V \times C \times N \rightarrow [0, 1] \) be the probability density function of measuring the walker at a given state such that \( \rho(v, c, t) = |\Psi(v, c, t)|^2 \). Let \( v : V \times N \rightarrow [0, 1] \) be the
probability density function of measuring vertex \( v \) at a given time instant. Since the walker state forms a basis for the state space, it follows that

\[
\nu(v, t) = \sum_{c \in C_v} |\Psi(v, c, t)|^2,
\]

which combined with Eq. 12 leads to

\[
\nu(v, t) = \sum_{u \in N^-(v)} \left| \sum_{j \in C_u} \Psi(u, j, t)w_{u, \sigma^{-1}(u, v), j} \right|^2.
\]

The probability distribution in Eq. 14 encapsulates the statistical properties of the quantum walk in terms of vertices. An alternative description for the dynamics of coined quantum walks can be found in terms of two partition quantum walks [13]. A common swap operator which will be mentioned on further sections of this article is the moving-shift operator \( M \), which is simply defined by the relationship

\[
M : |u, c\rangle \rightarrow |\eta(u, c), c\rangle.
\]

### 2.2 Non-homogeneous random walks

A non-homogeneous random walk on a directed graph \( G = (V, E) \) is, in essence, a diffusion process of a probability distribution over the vertices of \( V \) through the edges of \( E \) with time-varying transition (conditional) probabilities. Let \( \pi(t) \in \mathbb{R}^{|V|}_+ \) denote a probability vector (or a discrete probability distribution) over the set \( V \) at discrete time instant \( t \). Let \( p_{vu}(t) \in [0, 1] \) be the transition probability for the walker to step from vertex \( u \) to vertex \( v \), for which holds the law of total probability and that \( p_{vu} > 0 \) only if \( (u, v) \in E \). The behavior of the random walk is determined by the evolution of its probability distribution given by

\[
\pi_v(t + 1) = \sum_{u \in N^-(v)} p_{vu}(t)\pi_u(t).
\]

Equation 16 states that the probability of a vertex at instant \( t + 1 \) is given by a convex combination of the probabilities of its inward neighbors, on the previous instant \( t \). From this perspective, the sets of transition probabilities can be defined arbitrarily as long as the law of total probability remains valid, implying that the distributions that can be achieved by time evolution are fundamentally constrained by Eq. 16. This property will be denoted as the local convex evolution of probabilities.

In matrix form, Eq. 16 is represented as

\[
\pi(t + 1) = P(t)\pi(t),
\]

where \( P(t) \) is a stochastic matrix with entries \( p_{vu}(t) \) denoting the transition probability to move from vertex \( u \) to vertex \( v \), at instant \( t \).
Note that when \( \pi_u(t) = 0 \), the values of transition probabilities \( p_{vu}(t) \) do not contribute to the diffusion process at further times, i.e., \( p_{vu}(t) \) does not influence \( \pi(t + k) \) for \( k > 0 \).

In order to illustrate the potential of non-homogeneous random walks in modifying the dynamics of a classic random walk, the following example is provided. Consider a ring (cycle graph) with \( n \) vertices and walker that starts in vertex 0. Let \( D(t) \) denote the hop-distance from vertex 0 at time \( t \), given by \( D(t) = \min(X(t), n - X(t)) \) where \( X(t) \) denotes the vertex of the walker at time \( t \). Let \( \mu_D(t) \) denote its expected value for each time \( t \). Note that the rate at which \( \mu_D(t) \) increases provides information about the spread of the walker in a time regimen of \( t < n/2 \).

Figure 1 shows \( \mu_D(t) \) over time for four different walkers. It is known that a Hadamard-coined walk on the cycle with moving-shift operator and initial condition \( |\Psi(0)\rangle = |0, 0\rangle + |0, 1\rangle \sqrt{2} \) spreads faster than the uniform random walk with \( \pi_0(0) = 1 \), as indicated by the larger increase of \( \mu_D(t) \) in Fig. 1. However, by considering a non-homogeneous walker and increasing the probability that it moves away from the origin with time, it is possible to increase the rate of \( \mu_D(t) \) such that it surpasses the rate of the uniform random walk (linear increase). Moreover, if the increase in probability grows faster with time, the spread of the non-homogeneous random walk surpasses the spread of the Hadamard walk (quadratic increase). Thus, by carefully choosing the time-dependent transition probabilities, it is possible for the non-homogeneous random walk to match the vertex distribution of the quantum walk, and thus, its spreading behavior. Indeed, proving this formally under very mild assumptions is the main contribution of this paper.

### 3 Quantum walks as non-homogeneous random walks

The law of total probability and Eq. 16 provide the starting point to establish the equivalence between quantum and random walks. Within this context, let \( \pi : \mathbb{N} \rightarrow [0, 1]^{\left|\mathbb{V}\right|} \) be the probability vector representing the vertex probability distribution of a quantum walk at given instant such that \( \pi_v(t) = \nu(v, t) \). From this perspective, it is necessary to define the non-homogeneous random walk that matches the evolution of \( \pi(t) \), for all \( t \). A sufficient condition is the construction of the time-dependent transition matrix \( P(t) \) for which \( \pi(t + 1) = P(t)\pi(t) \), for all \( v, t \). The existence of such sequence of matrices implies the principle of local convex evolution, in the sense of Eq. 16, for the full quantum walk operator \( SW \), regardless of initial conditions. Theorem 1 establishes the construction of the random walk matrix sequence.

**Theorem 1** (Quantum walk local convex evolution) For any time instant \( t \), the evolution of the vertex probability of a quantum walk performed by the action of the unitary operator \( SW \) is locally convex and is given by the Markovian matrix

\[
p_{vu}(t) = \begin{cases} 
\frac{\nu(v,c,t+1)}{\nu(u,t)}, & \text{if } \nu(u,t) > 0 \text{ and } (u,v) \in E \\
\frac{1}{d(u)}, & \text{if } \nu(u,t) = 0 \text{ and } (u,v) \in E \\
0, & \text{otherwise}
\end{cases}
\]

(18)
Fig. 1 Expected value of the hop-distance $\mu_D(t)$ over time for four walkers on a cyclic graph with 101 vertices for the first 30 time steps. The homogeneous random walk is the uniform random walk with $p = q = 0.5$ with initial probability distribution $\pi_0(0) = 1$. The quantum walk is the Hadamard walk with initial condition $|0,0\rangle + |0,1\rangle \sqrt{2}$. Both of the non-homogeneous random walks start as the uniform random walk but the probability of moving away from the origin increases with time. Let $p(t)$ denote the probability to move from vertex $i$ to $i + 1$ for $i = 0, \ldots, n/2$, and to move from vertex $i$ to $i - 1$ for $i = n, \ldots, n/2$.

In the linear non-homogeneous random walk $p(t) = \min(0.5, 0.001t, 1)$. In the quadratic case $p(t) = \min(0.5 + 0.001t^2, 1)$

when applied to the vertex probability vector $\pi(t)$, where $c = \sigma(u, v)$, and such that $\pi(t + 1) = P(t)\pi(t)$

**Proof** To completely prove the claim, it is necessary to show that the following three properties hold for $P$:

1. $0 \leq p_{vu}(t) \leq 1$ for every $u, v \in V$;
2. $\sum_{v \in N^+(u)} p_{vu}(t) = 1$ for each $v \in V$;
3. $\nu(v, t + 1) = \sum_{u \in N^-(v)} p_{vu}(t)\nu(u, t)$ for each $v \in V$.

Whenever $\nu(u, t) = 0$, choosing $p_{vu}(t) = \frac{1}{d(u)}$ avoids division by zero and assures the first and the second conditions. Since $p_{vu}(t)\nu(u, t) = 0$ for this particular case, the task is to show that the three conditions hold for $\nu(u, t) > 0$. Note that $p_{vu}(t)$ could be chosen arbitrarily, as long as the $u$-th column of $P$ respected conditions 1 and 2. Uniform weights were chosen for simplicity. Using Eq. 14, and taking $c \in C_v$, $c = \sigma(u, v)$ and $c' = \sigma^{-1}(u, v)$, one has:

$$\rho(v, c, t + 1) = \left| \sum_{j \in C_u} \Psi(u, j, t)w_{uc'j} \right|^2 \quad (19)$$
\[ p_{vu}(t) = \frac{\left| \sum_{j \in C_u} \Psi(u, j, t) w_{uc'j} \right|^2}{v(u, t)}. \]  

(20)

The numerator on the right-hand side of Eq. 20 can be thought of as the result of the inner product between the vectors \( |\Psi^*(u, t)\rangle \) and \( |W_u\rangle \) with \( j \)-th coordinates, respectively, given by \( |\Psi^*(u, t)\rangle_j = \Psi^*(u, j, t) \) and \( |W_u\rangle_j = w_{uc'j}, j \in C_u \). By the Cauchy–Schwarz inequality

\[ \left| \langle \Psi^*(u, t) \mid W_u \rangle \right|^2 \leq \langle \Psi^*(u, t) \mid \Psi^*(u, t) \rangle \langle W_u \mid W_u \rangle. \]  

(21)

Since \( \langle W_u \mid W_u \rangle = 1 \) due to the unitarity of \( W \) (Eq. 4),

\[ \left| \sum_{j \in C_u} \Psi(u, j, t) w_{uc'j} \right|^2 \leq v(u, t) \]  

(22)

implies that \( p_{vu}(t) \leq 1 \). As both the numerator and the denominator of Eq. 20 are positive, \( p_{vu}(t) \geq 0 \), proving property 1.

Furthermore, the numerator of the sum of conditional probabilities

\[ \sum_{v \in N^+(u)} p_{vu}(t) = \sum_{v \in N^+(u)} \frac{\left| \sum_{j \in C_u} \Psi(u, j, t) w_{(u,\sigma^{-1}(u,v),j)} \right|^2}{v(u, t)} \]  

(23)

is exactly the value of the inner product \( \langle \Psi(u, t) \mid W^\dagger W \mid \Psi(u, t) \rangle \), with

\[ |\Psi(u, t)\rangle = \sum_{i \in C_u} \Psi(u, i, t) |u, i\rangle. \]

To see this, note that the correspondence given by the function \( \sigma^{-1}(u, v) \) between degrees of freedom is unique, as well as the correspondence between the degrees of freedom of \( u \) and its neighbors, yielding

\[ \sum_{v \in N^+(u)} \sum_{j \in C_u} \Psi(u, j, t) w_{(u,\sigma^{-1}(u,v),j)} \]  

\[ = \sum_{k \in C_u} \sum_{j \in C_u} \Psi(u, j, t) w_{(u,k,j)} \]  

(24)

Due to the unitarity of \( W \), such inner product is precisely \( v(u, t) \), proving property 2.

Property 3 follows trivially from the definition of the Markovian matrix \( P \) in Eq. 18 and from the orthogonality of the basis states. \( \square \)

Theorem 1 establishes that any discrete-time coined quantum walk with unitary operators \( W \) and \( S \), respectively, described by Eq. 2 and Relation 9, is statistically equivalent, from the perspective of vertex probability evolution, to a non-homogeneous random walk over the same graph. Note that both \( W \) and \( S \) may depend on time, as long as the graph connectivity restrictions remain valid.
4 Generalization for multiple walkers

To extend Theorem 1 for multiple walkers, some additional definitions are needed. In particular, the Hilbert space in which the process unfolds grows to allow for the joint description of the walkers. Let $K$ denote the number of walkers and, again, let $\mathcal{H}_w$ denote the Hilbert space for a single-walker on $G$. The enlarged space for $K$ walkers is $\mathcal{H}_K^w = \bigotimes_{i=1}^K \mathcal{H}_w$. Let $v = (v_1, \ldots, v_k)$ denote an ordered sequence of $K$ vertices and $c = (c_1, \ldots, c_k)$ denotes its associated degrees of freedom such that $c_i \in C_{v_i}$. Let the set $B^K = \{|v, c\rangle\}$ denote a basis for $\mathcal{H}_K^w$ of which elements represents the joint position of the $K$ walkers. Let

$$|\Psi(t)\rangle = \sum_{|v, c\rangle \in B^K} \Psi(v, c, t) |v, c\rangle$$

denote the state of the system at instant $t$ and $\rho : V^K \times C^K \times \mathbb{N} \to [0, 1]$ be the joint probability distribution of states at instant $t$. Similarly, let $\nu : V^K \times \mathbb{N} \to [0, 1]$ denote the joint vertex probability distribution of the $K$ walkers at a given instant. Assuming each walker can behave differently, with specific coins and shift operators, let $W_i$ and $S_i$, respectively, denote the coin and shift operator for the $i$-th walker, implying that the full operators are of the form $S = \bigotimes_{i=1}^K S_i$ and $W = \bigotimes_{i=1}^K W_i$. If there is no interaction among the walkers, the system evolves, in the enlarged space, as in Eq. 1 and the joint distribution of vertices at an instant $t$ is merely

$$\nu(v, t) = \prod_{i=1}^K \nu_i(v_i, t),$$

where $\nu_i : V \times \mathbb{N} \to [0, 1]$ is the marginal vertex distribution of the $i$th walker.

A more interesting scenario appears when the walkers can interact, allowing a dependency among their marginal probability distributions. Let $U : \mathcal{H}_K^w \to \mathcal{H}_K^w$ be a unitary operator defined as

$$U = \sum_{|v, c\rangle \in B^K} \sum_{|v, c'\rangle \in B^K\dagger} \theta(v, c, c') |v, c\rangle |v, c'\rangle$$

which accounts for walker interactions, such that the whole system state evolves as

$$|\Psi(t + 1)\rangle = SWU |\Psi(t)\rangle.$$
phase shifts, amplitude mixing and even amplitude shifts within the degrees of freedom of a walker controlled by the position of the others. Within this framework, the connectivity restrictions of the dispersion of the wavefunction are maintained, since amplitudes can only be transmitted through the edges of the graph. Theorem 2 follows as an extension of Theorem 1 for this broader context, in which the movement of \(K\) quantum walks is shown to be statistically equivalent to that of \(K\) non-homogeneous random walks.

Let \(\pi : \mathbb{N} \to [0, 1]|V|^{K}\) be the joint vertex probability vector of the walkers such that that

\[
\pi_v = \nu(v, t).
\]

**Theorem 2** *(Local convex evolution of multiple interacting walkers)* For any time instant \(t\), the evolution of vertex probabilities for the \(K\) walkers performed by the action of the unitary operator \(\text{SWU}\) is locally convex and is given by the Markovian matrix

\[
p_{vu}(t) = \begin{cases} 
\frac{\nu(v, c, t + 1)}{\nu(u, t)}, & \text{if } \nu(u, t) > 0 \text{ and } (u_i, v_i) \in E \text{ for all } i \\
\frac{1}{d(u)}, & \text{if } \nu(u, t) = 0 \text{ and } (u_i, v_i) \in E \text{ for all } i \\
0, & \text{otherwise}
\end{cases}
\]

when applied to the joint vertex distribution vector \(\pi(t)\), where \(c = \sigma(u, v), i \in \{1, \ldots, K\}\) and such that \(\pi(t + 1) = P(t)\pi(t)\).

**Proof** The random walk dictated by matrix \(P(t)\) accounts for the joint movement of the walkers in the sense that an index \(u\) of \(P\) is a vector of dimension \(K\) and denotes the position of the walkers. To formalize, let \(G' = (V', E')\) denote a graph with \(V' = V^K\) and \(E' = E^K\), such that, for all \(v, u \in V'\) with \(v = (v_1, \ldots, v_K)\) and \(u = (u_1, \ldots, u_K)\), \(e = (v, u) \in E\) if, and only if \((v_i, u_i) \in E\) for all \(i\). Note that \(d(u) = \prod_{i=1}^{K} d(u_i)\). In particular, each vertex of \(G'\) represents the simultaneous position of all walkers and its edges codifies all of their possible combined movements. It must be shown that \(P(t)\) indeed represents a non-homogeneous random walk over \(G'\) and that its vertex probability evolution matches Eq. 26.

The three properties which where shown to hold for Theorem 1 are to be demonstrated for this general case, since the requirements for one walker extend to \(K\) walkers naturally. Let

\[
|\Psi(u, t)\rangle = \sum_{c \in C_u} \Psi(u, c, t) |u, c\rangle
\]

denote the overall state of \(u \in V'\) such that \(|||\Psi(u, t)|||^2 = \nu(u, t)\). Note that \(|\Psi(u, t)\rangle \in \mathcal{H}_{u}^{K}\), that \(c = (c_1, \ldots, c_K)\) is a tuple denoting the degrees of freedom of each walker and that the functions \(\eta\) and \(\sigma\) are now defined for tuples of vertices and degrees of freedom. Assuming that \(v = \eta(u, c')\) and \(\sigma(u, \eta(u, c')) = c\) for a given \(c' \in C_u\), the action of SWU gives

\[
\frac{\rho(v, c, t + 1)}{\nu(u, t)} = \frac{|||\Psi(c', v)\rangle\langle v, c|\text{SWU}|\Psi(u, t)\rangle||^2}{|||\Psi(u, t)|||^2}.
\]
Since SWU is unitary and $\langle v, c|s \rangle \leq 1$ for any unitary $|s \rangle \in H^K_w$, the inequality

$$0 \leq \frac{|||v, c\rangle\langle v, c|\text{SWU}||\Psi(u, t)||^2}{||\Psi(u, t)||^2} \leq 1$$

(30)
demonstrates property 1.

Simultaneously, the action of SWU also implies that the inequality

$$|v, c_v\rangle\langle v, c_v|\text{SWU}||\Psi(u, t)|| \neq 0$$

(31)
is only valid for $v \in V$ and $c_v \in C_v$ if $v = \eta(u, c)$ and $c_v = \sigma(u, c)$ for some $c \in C_u$. Assuming that $v = \eta(u, c)$ and $c_{vu} = \sigma(u, c)$, the last condition gives

$$\sum_{v \in N^+(u)} \frac{|||v, c_{vu}\rangle\langle v, c_{vu}|\text{SWU}||\Psi(u, t)||^2}{||\Psi(u, t)||^2} = 1.$$  

(32)

Due to the fact that SWU is a unitary operator, Eq. 32 and the orthogonality of the basis states lead to properties 2 and 3.

Essentially, Theorem 2 constructs a non-homogeneous random walk on $G'$ that matches the evolution of the joint vertex probability distribution of $K$ walkers induced by SWU and, thus, asserts that the vertex probability distribution of the multiple walker interaction model has a local convex evolution on the vertices of $G'$. Again, it is worth emphasizing that SWU may vary with time as long as unitarity, graph connectivity and the conditions for the interaction operator $U$ remains valid.

5 Simulation of quantum walk trajectories

Theorems 1 and 2 establish, respectively, the construction procedure for a non-homogeneous random walk that is statistically equivalent to practically any single- and multiple-walker quantum walk. This random walk can be simulated to generate paths on the graph that by correspondence also capture the quantum walk statistical dynamics. The simulation of a random walk naturally constructs random paths on a graph: At time $t + 1$, the walker can only be found in an outward neighbor of vertex $v$, given that it was in vertex $v$ at time $t$. Thus, the simulation constructs a sample path that ensures neighbor locality. We denote this sample path by quantum walk trajectory. Note that this procedure is fundamentally different than the usual simulation procedure for quantum walks, where the distribution $\rho$ is sampled independently at each time instant $t$ and graph trajectories are not necessarily constructed. To exemplify, the procedure prescribed in Theorem 1 was used to simulate quantum walks on a 2-D torus with Grover and Hadamard coins, and moving-shift operators, generating quantum walk trajectories (each trajectory is independent of all others). The ensemble of trajectories are depicted in Figs. 2 and 3 respectively. Without loss of generality, the following discussion assumes a single walker.
Fig. 2 Ensemble of 100 trajectories obtained using Theorem 1 for a Grover-coined walk on a 10-by-10 2-D torus with moving shift operators for a localized initial state $|0, 0\rangle$ (left edge of the origin). Each one of the solid lines corresponds to an independent trajectory. The blue dashed line represents the empirical average of the vertex random variable for each time instant, while the orange point-dashed line represents its expected value.

Fig. 3 Ensemble of 100 trajectories obtained using Theorem 1 for a Hadamard-coined walk on a 10-by-10 2-D torus with moving shift operators for a localized initial state $|0, 0\rangle$ (left edge of the origin). Each one of the solid lines corresponds to an independent trajectory. The blue dashed line represents the empirical average of the vertex random variable for each time instant while the orange point-dashed line represents its expected value.
Fig. 4 Analysis of the total variation distance between the empirical distribution obtained from trajectory sampling and the quantum walk distribution of a Grover-coined walk with moving shift on a 10-by-10 2-D torus. The left figure shows the total variation value for the distribution at each instant, with different curves representing different ensemble sizes. The right figure gives the distance value for a fixed time instant when the ensemble size grows.

While one simulated trajectory respects locality, an ensemble of trajectories recover the distribution of the quantum walk for every $t$. In particular, let $\chi = \{\tau_1, \ldots, \tau_M\}$ be an ensemble of $M$ independent trajectories. Let $\tau_i(t)$ denote the vertex visited by the walker at instant $t$ in the $i$-th trajectory. Let $1(\cdot)$ denote an indicator function activated by its argument condition. Let

$$\hat{p}^M_u(t) = \frac{1}{M} \sum_{i=1}^{M} 1(\tau_i(t) = u)$$

(33)

denote the fraction of time vertex $u$ was visited by the walker at instant $t$. Thus, by the law of large numbers, $\hat{p}^M_u(t) \to \nu(u, t)$ as $M \to \infty$ and the trajectories recover the vertex distribution of the quantum walk for all $t$.

Convergence is observed through the decreasing behavior of the total variation distance

$$D_t(p, \nu) = \frac{1}{2} \sum_u \left| \hat{p}^M_u(t) - \nu(v, t) \right|$$

(34)

between the empirical vertex distribution of the trajectory ensemble and the quantum walk vertex distribution, as it can be seen in Figure 4 for a Grover-coined quantum walk on the 2-D torus with moving-shift operator.

The non-homogeneous random walk simulation is a novel perspective for the study of quantum walks as it gives an efficient (polynomial) procedure for sampling trajectories which recover, by the law of large numbers, the vertex probability sequence of any quantum walk. As a matter of fact, measuring quantum walks on a possible physical implementation or independently sampling vertices from the quantum sequence of distributions do not address this question, since samples are obviously independent and there is no guarantee that trajectories (paths) will be generated. An alternative to sample quantum trajectories on a graph is to consider a rejection method.
that accepts only sequences of vertices that correspond to paths in the graph. However, the marginal empirical distributions $\hat{p}_v(t)$ within the accepted trajectories would not necessarily match $\rho(v, t)$. See details in “Appendix A”.

6 Complexity of random walk description and simulation

An interesting question which arises once Theorems 1 and 2 are considered is the computational complexity involved in constructing the corresponding random walk matrices $P(t)$. The information required to compute its entries at time instant $t$ are the probability distributions of the quantum walk at times $t$ and $t+1$. Thus, if the state and the vertex distributions for time $t$ and $t+1$ are known, constructing the matrix $P(t)$ has an intrinsic complexity of $O(|V|^2)$, since each of its entries can be computed in $O(1)$.

In general, however, the matrix can be computed by using Eq. 1 to calculate both $|\psi(t)\rangle$ and $|\psi(t+1)\rangle$. Assuming that both $S$ and $W$ may vary with time, the cost for computing the wave function is $O(t|E|^2)$, since $t$ matrix-by-vector multiplications are performed, each with complexity $O(|E|^2)$. Computing the vertex distribution from the wavefunction has complexity $O(|E|)$, since the probability of each outward edge of a vertex must be considered. Hence, the overall complexity is $O(t|E|^2 + |E| + |V|^2) \in O(t|E|^2)$.

The computation of the quantum walk wavefunction is the general bottleneck for constructing the random walk matrices, unless the probability distributions of the quantum walk can be computed more efficiently. In terms of complexity, the problem of describing the probability evolution of the non-homogeneous random walk is at least as hard as solving the quantum walk distribution. Nonetheless, specific walker systems can have their wavefunctions computed by algorithms that are more efficient than direct matrix multiplication. Walker dynamics with known closed-formula expressions for the wavefunction are an interesting case. For example, a generic coined quantum walk on an infinite line for which the walker moves in a single direction or remains on its position at every instant has known explicit probability distribution for all time $t$ [17].

Alternatively, for particular quantum walks and graphs, the wavefunction and the vertex probability distribution may be computed recursively and more efficiently than the general approach (see “Appendix B” for an example).

7 Conclusions

As the central contribution of this work, Theorems 1 and 2 establish a construction procedure for non-homogeneous random walks that yield the same vertex probability distribution sequence of any single or multiple quantum walk. Besides establishing a formal equivalence between the two processes, this procedure allows for the efficient simulation of quantum walk trajectories, which can be used to investigate quantum walks from the perspective of vertex locality, as opposed to the simulation of independent samples over time. In a nutshell, the theorems establish a formal correspondence between random and quantum walks on the same graph by showing that the vertex distribution of the two processes are identical for all time $t$. Moreover, any statistical
property of a quantum walk can be analyzed through quantum walk trajectories. This concept and its simulation could possibly be exploited to evaluate theoretical properties and concepts of Markov chains which were initially modified to address quantum walks, such as mixing, dispersion and hitting time [1], in their original circumstances.

Due to the universality of quantum walks for quantum computation, both Theorems may have important implications in the development of this larger field. The connections between generic computational processes and time-dependent Markov chains can be explored to guide new interesting research on quantum computing. On the other hand, Theorems 1 and 2 do not provide improvements on the computation of the state probability distribution sequences of quantum walks, since the construction of the non-homogeneous random walk requires solving the quantum problem.

While this work showed that any single or multiple quantum walk has a corresponding random walk, an interesting future consideration would be establishing the reverse correspondence. In particular, answering whether or not any (single and multiple) random walk has a corresponding quantum walk.

Acknowledgements This work was supported in part by research grants from CNPq (Brazil) and FAPERJ (Brazil).

Appendix A Limitations of the rejection method for QWT

The question in hand is to generate samples of quantum walk trajectories. As defined in Sect. 5, quantum walk trajectories are random paths of a graph $G$ for which the empirical distribution $\hat{p}(v, t)$ (fraction of instances that the walker is found in vertex $v$ after $t$ hops) approaches $p(v, t)$, the given vertex distribution of a quantum walk defined on $G$, for all $v \in V$, $t \in \{0, \ldots, L - 1\}$. Essentially, a path of the graph, or a graph trajectory, of length $L$ is a sequence of vertices $\{v_1, \ldots, v_L\}$ such that $(v_i, v_{i+1}) \in E$ for $i \in \{1, \ldots, L - 1\}$.

The usual simulation method for a quantum walk consists in drawing independent samples from distinct time instants. Effectively, assume that one wants to draw a sequence of $L$ samples from a quantum walker system such that the $i$-th sample is drawn according with the distribution $\nu$ at instant $i$. Note that $\nu$ depends on the initial condition. Denoting $V_i = \{v \in V | \nu(v, i) \neq 0\}$ as the set of all vertices that can be the outcome of a measurement of the system at instant $i$, define $\mathcal{X}_L = V_0 \times \ldots \times V_{L-1}$ as the set of all possible sequences of measurements of length $L$ that can be obtained from the quantum walker system. Simply, the set $\mathcal{X}_L$ is the sample space for the usual simulation procedure of a quantum walk.

Analyzing the set $\mathcal{X}_L$ and considering the general case of initial conditions, it is clear that not every sequence $\tau \in \mathcal{X}_L$ is a valid path of the graph. However, every path of the graph that can be obtained as a quantum walk trajectory belongs to $\mathcal{X}_L$. Thus, this method cannot be directly used to sample quantum walk trajectories as some measured sequences would not be proper paths of the graph.

To overcome this limitation, rejection sampling could be applied where a sample is accepted as a quantum walk trajectory only if it is a valid path on $G$. In order to address this matter formally, let $\mathcal{R}_L \subset \mathcal{X}_L$ be the set of all sequences of measurements
that are paths of length $L$ of $G$. Let $p(\tau)$ be the probability of obtaining a sequence $\tau = \{\tau_0, \ldots, \tau_{L-1}\} \in \mathcal{X}$ from the independent sampling procedure. Let $\mathcal{X}_L^v \subset \mathcal{X}_L$ and $\mathcal{T}_L^v \subset \mathcal{T}_L$ denote the set of all sequences and trajectories of length $L$ in which $v$ appears in position $t$. It follows trivially from independence that

$$p(\tau) = \prod_{t=1}^{L} \upsilon(\tau_t, t), \quad (35)$$

where $\tau_t$ denotes the vertex measured at $t$. In this case, the vertex probability at $t$ is simply,

$$\upsilon(v, t) = \sum_{\tau \in \mathcal{X}_L^v, t} p(\tau). \quad (36)$$

However, rejecting non-trajectory samples yields the following estimator $\hat{p}(v, t)$ for the probability of finding vertex $v$ at instant $t$:

$$\hat{p}(v, t) = \frac{\sum_{\tau \in \mathcal{T}_L^v, t} p(\tau)}{\sum_{\tau' \in \mathcal{T}_L} p(\tau')} \quad (37)$$

It is not clear whether Eqs. 36 and 37 are equal for every possible quantum walk, since the ratio between the probability of generating trajectories with vertex $v$ at position $t$ and the probability of generating any trajectory would have to be $\upsilon(v, t)$, for all $v \in V$ and $t \in \{0, \ldots, L - 1\}$.

Additionally, even for the cases where Eqs. 36 and 37 are equal, the expected time to accept a sample in the rejection procedure is precisely the inverse of the probability of generating a trajectory. Although this probability depends on a myriad of factors such as the graph structure, the number of trajectories of a given length $L$ within a graph can be exponentially smaller than the number of possible sequences of measurements. As an example, a $D$-dimensional torus with $V$ vertices would have $V D^{L-1}$ paths of length $L$ and $V^L$ possible sequences of vertices, which for values of $D \in O(1)$ is exponentially larger than the number of trajectories, suggesting that the expected time to generate a trajectory sample would be unfeasible.

In precise terms, Theorem 1 offers a polynomial time procedure to sample trajectories for any quantum walk. Given the transition matrix $P(t)$, a quantum trajectory of length $L$ can be sampled in time $O(L d_{\text{max}})$ where $d_{\text{max}}$ is the maximum degree of the graph. Moreover, if multiple trajectories are to be generated, then the alias method could be used to decide the next neighbor at time instant in which case the amortized time complexity for each trajectory is $O(L)$, and no longer depends on vertex degrees.

### Appendix B Dynamic programming for Grover walk on torus

The Grover-coined (Eq. 8) walk on a $D$-dimensional torus with moving-shift operator (Eq. 15) and purely real initial conditions serves as an example where the probability
distribution can be computed by a dynamic programming algorithm that is more efficient than direct matrix multiplication. The number of degrees of freedom within the $D$-dimensional torus is $2^D$. Analyzing the action of the total walk operator $MG$ on a given state

$$\Psi(u, t) = \sum_{c \in Cu} \Psi(u, c, t) |(u, c)\rangle$$

and having $\eta(u, c) = v$, the probability $\rho(v, c, t + 1)$ is described as

$$\rho(v, c, t + 1) = \left| \sum_{c' \in Cu} \Psi(u, c', t) - \Psi(u, c, t) \right|. \quad (38)$$

Assuming that $\psi(u, c, t) = \sqrt{\rho(u, c, t)} e^{i \cos \theta_{uct}}$ and noting that $\theta_{uct} = 0$ for every $u, c$ and $t$ whenever purely real initial conditions are considered yields

$$\rho(v, c, t + 1) = \left| \sum_{c' \in Cu} \sqrt{\rho(u, c', t)} - \sqrt{\rho(u, c, t)} \right|. \quad (39)$$

From Theorem 1, the entries of the random walk matrices for which $\rho(v, c, t) > 0$ are given by

$$\frac{\rho(v, c, t + 1)}{v(u, t)} = \frac{1}{v(u, t)} \left| \sum_{c' \in Cu} \sqrt{\rho(u, c', t)} - \sqrt{\rho(u, c, t)} \right|. \quad (40)$$

Equation 40 can be solved through a dynamic programming algorithm in which each time instant has complexity $O(|V|^2D)$, implying on an overall procedure of complexity $O(t|V|^2D)$ for all matrices up to time $T$. For $D \in O(1)$, the algorithm has complexity $O(t|V|^2)$, showing a quadratic improvement over the generic procedure, since $O(t|E|^2) \in O(t|V|^4)$.  

**References**

1. Aharonov, D., Ambainis, A., Kempe, J., Vazirani, U.: Quantum walks on graphs. In: Proceedings of the 33rd annual ACM symposium on theory of computing, pp. 50–59. ACM, (2001)
2. Aharonov, Y., Davidovich, L., Zagury, N.: Quantum random walks. Phys. Rev. A 48(2), 1687 (1993)
3. Ambainis, A.: Quantum walk algorithm for element distinctness. SIAM J. Comput. 37(1), 210–239 (2007)

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1 $D$ can be at most $|V|$ for the case when the torus degenerates to the complete graph. The overall procedure for this case is in $O(t|V|^3)$, which still represents a linear improvement from the general procedure of matrix multiplication.
4. Berry, D.W., Childs, A.M., Kothari, R.: Hamiltonian simulation with nearly optimal dependence on all parameters. In: 2015 IEEE 56th annual symposium on foundations of computer science, pp. 792–809. IEEE, (2015)
5. Boettcher, S., Falkner, S., Portugal, R.: Relation between random walks and quantum walks. Phys. Rev. A 91(5), 052330 (2015)
6. Childs, A.M.: Universal computation by quantum walk. Phys. Rev. Lett. 102(18), 180501 (2009)
7. Childs, A.M., Gosset, D., Webb, Z.: Universal computation by multiparticle quantum walk. Science 339(6121), 791–794 (2013)
8. Feynman, R.P., Hibbs, A.R., Styer, D.F.: Quantum Mechanics and Path Integrals. Courier Corporation, North Chelmsford (2010)
9. Häggström, O., et al.: Finite Markov Chains and Algorithmic Applications, vol. 52. Cambridge University Press, Cambridge (2002)
10. Hastings, W.K.: Monte carlo sampling methods using markov chains and their applications. Biometrika 57(1), 97–109 (1970)
11. Kempe, J.: Quantum random walks: an introductory overview. Contemp. Phys. 50(1), 339–359 (2009)
12. Kirkpatrick, S., Gelatt, C.D., Vecchi, M.P.: Optimization by simulated annealing. Science 220(4598), 671–680 (1983)
13. Konno, N., Portugal, R., Sato, I., Segawa, E.: Partition-based discrete-time quantum walks. Quantum Inf. Process. 17(4), 100 (2018)
14. Lovász, L., et al.: Random walks on graphs: a survey. Comb. Paul Erdos Eighty 2(1), 1–46 (1993)
15. Lovett, N.B., Cooper, S., Everitt, M., Trevers, M., Kendon, V.: Universal quantum computation using the discrete-time quantum walk. Phys. Rev. A 81(4), 042330 (2010)
16. Magniez, P., Nayak, A., Roland, J., Santha, M.: Search via quantum walk. SIAM J. Comput. 40, 08 (2006)
17. Montero, M.: Quantum walk with a general coin: exact solution and asymptotic properties. Quantum Inf. Process. 14(3), 839–866 (2015)
18. Montero, M.: Quantum and random walks as universal generators of probability distributions. Phys. Rev. A 95(6), 062326 (2017)
19. Page, L., Brin, S., Motwani, R., Winograd, T.: The pagerank citation ranking: Bringing order to the web. Technical report, Stanford InfoLab, (1999)
20. Portugal, R.: Quantum walks and search algorithms. Springer, Berlin (2013)
21. Portugal, R., Segawa, E.: Coined quantum walks as quantum markov chains. arXiv preprint arXiv:1612.02448, (2016)
22. Portugal, R., Santos, R.A., Fernandes, T.D., Gonçalves, D.N.: The staggered quantum walk model. Quantum Inf. Process. 15(1), 85–101 (2016)
23. Rohde, P.P., Schreiber, A., Štefaňák, M., Jex, I., Silberhorn, C.: Multi-walker discrete time quantum walks on arbitrary graphs, their properties and their photonic implementation. New J. Phys. 13(1), 013001 (2011)
24. Romanelli, A.: Distribution of chirality in the quantum walk: Markov process and entanglement. Phys. Rev. A 81, 062349 (2010)
25. Romanelli, A., Schifino, A.S., Siri, R., Abal, G., Auyuanet, A., Donangelo, R.: Quantum random walk on the line as a markovian process. Phys. A Stat. Mech. Its Appl. 338(3–4), 395–405 (2004)
26. Štefanák, M., Kiss, T., Jex, I., Mohring, B.: The meeting problem in the quantum walk. J. Phys. A Math. Gen. 39(48), 14965 (2006)
27. Szegedy, M.: Quantum speed-up of markov chain based algorithms. In: 45th annual IEEE symposium on foundations of computer science, pp. 32–41, (2004)
28. Venegas-Andraca, S.E.: Quantum walks: a comprehensive review. Quantum Inf. Process. 11(5), 1015–1106 (2012)
29. Whitfield, J.D., Rodríguez-Rosario, C.A., Aspuru-Guzik, A.: Quantum stochastic walks: a generalization of classical random walks and quantum walks. Phys. Rev. A 81(2), 022323 (2010)
30. Xue, P., Sanders, B.C.: Two quantum walkers sharing coins. Phys. Rev. A 85, 022307 (2012)

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