Elementary Excitations of One-Dimensional \(t\)-\(J\) Model with Inverse-Square Exchange

Z. N. C. Ha\(^{(1)}\) and F. D. M. Haldane\(^{(2)}\)

\(^{(1)}\)School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540.

\(^{(2)}\)Department of Physics, Princeton University, Princeton, New Jersey 08544.

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Abstract

We identify exact excitation content of the intermediate states for the one-particle Green’s functions, spin-spin and (charge) density-density correlation functions of the periodic one-dimensional \(t\)-\(J\) model with inverse square exchange. The excitations consist of neutral \(S = 1/2\) spinons and spinless (charge \(-e\)) holons with semionic fractional statistics, and bosonic (charge \(+2e\)) “anti-holons” which are excitations of the holon condensate. Due to the supersymmetric Yangian quantum symmetry of this model, only the excited states with \textit{finite} number of elementary excitations contribute to the spectral functions. We find a set of selection rules, and this allows us to map out the regions of non-vanishing spectral weight in the energy-momentum space for the various correlation functions.

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Recently, there have been many developments in understanding the family of Calogero-Sutherland models (CSM) which are identified with their peculiar inverse-square exchange (ISE) \[1-3,6,8-10,17\]. An important feature of these models is that they belong to the same low-energy universality class as the family of Bethe-ansatz solvable models and may provide a new fully soluble paradigm next to the non-interacting models \[2\].

The one-dimensional supersymmetric ISE \(t-J\) model \[3\] represents a fixed point model where the elementary excitations form an ideal gas obeying fractional statistics. In contrast to this model, the NNE \(t-J\) model \[4,5\], which has essentially the same low energy spectra spanned by the same elementary excitations, obscures the simple low energy structure intrinsic to this class of models. We rediscover the spinons, the holons and the antiholons—the elementary excitations of the NNE \(t-J\) model \[5\]—in the context of the supersymmetric Yangian of the ISE model. Furthermore, we find that only the states with finite number of these elementary excitations contribute to the spectral functions of the one-particle Green’s functions \(G^{(1)}\), the charge density-density \(C^{(c)}\) and the spin-spin correlation functions \(C^{(s)}\).

First, we examine the symmetry in the ISE supersymmetric \(t-J\) model. The model with periodic boundary conditions possesses, in addition to the global \(SU(m|n)\) supersymmetry, a hidden dynamical “quantum group” symmetry algebra called the supersymmetric Yangian \[2,6,7\]. This symmetry is responsible for the “supermultiplets” in the energy spectrum and the ideal gas-like features of the elementary excitations and, furthermore, provides us with a simple numerical way to identify the exact content of the elementary excitations relevant for the various correlation functions.

The supersymmetric generalization of the \(SU(n)\) Haldane-Shastry model Hamiltonian \[8-10\] is given by

\[
H = t \sum_{i<j} \frac{P_{ij}}{d^2(n_i - n_j)},
\]

where \(d(x) = (N_a/\pi) \sin(\pi x/N_a)\) and \(N_a\) is the total number of sites. If \(a_{i\alpha}^\dagger\ (a_{i\alpha})\) creates (destroys) a particle of species \(\alpha\) at site \(i\) and satisfies the single occupancy condition,
\[ \sum_{\alpha} a_{i\alpha}^\dagger a_{i\alpha} = 1, \] the exchange operator can be written as \[ P_{ij} = \sum_{\alpha\beta} a_{i\alpha}^\dagger a_{j\beta}^\dagger a_{i\beta} a_{j\alpha}. \] If \( m \) of the species labeled by \( \alpha \) are bosons, and \( n \) are fermions, the model (1) has a global \( SU(m|n) \) supersymmetry with generators given by the traceless part of \( J_0^{\alpha\beta} = \sum_i a_{i\alpha}^\dagger a_{i\beta} \). The Yangian symmetry generator of the periodic ISE model is

\[ J_1^{\alpha\beta} = \sum_{i>j,\gamma} w_{ij} a_{i\alpha}^\dagger a_{j\gamma}^\dagger a_{i\gamma} a_{j\alpha}, \tag{2} \]

where \( w_{ij} = \cot(\pi(i-j)/N_a) \). The higher order generators of the Yangian are obtained recursively from various commutators involving only \( J_0 \) and \( J_1 \) [6,7].

If we specialize to \( SU(1|2) \) supersymmetry, with \( \alpha \in \{0,\uparrow,\downarrow\} \), we can rewrite the Hamiltonian in terms of the \( SU(2) \) fermionic operators \( c_{i\sigma}^\dagger = a_{i\sigma}^\dagger a_{i0} \) as \( PH^0P \), where \( H^0 \) (up to a shift in total energy and in chemical potential) is

\[ -\sum_{i\neq j,\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \sum_{i<j} (J_{ij} S_i \cdot S_j + V_{ij} n_i n_j), \tag{3} \]

where \( t_{ij} = J_{ij}/2 = -2V_{ij} = t/d^2(i-j) \) and \( n_i = n_{i\uparrow} + n_{i\downarrow} \); \( P \) is the projection operator that projects out all states with doubly-occupied sites. The ground state \( |\Psi_0\rangle \) of this model is known [3,9] to be

\[ \sum_{\{x,\sigma\}, j>0} \prod_{i>j} (z_i - z_j)^{\delta_{x_i,x_j}} (i)^{\text{sgn}(\sigma_i - \sigma_j)} \prod_k z_k^{J_0} \prod_j c_{j\sigma_j}^\dagger |0\rangle, \tag{4} \]

where \( z_j = \exp(i2\pi x_j/N_a) \), \( J_0 = -(N/2 - 1)/2 \), \( N \) is the total number of particles, and \( |0\rangle \) the electron vacuum (empty state). In order to have a non-degenerate ground state, we take \( N/2 \) to be odd. Note that this wave function is just the full Gutzwiller projection of a free electron state [11].

A remarkable feature of this model is that the eigenstates of (1) form degenerate “supermultiplets” [8] with multiplicities much higher than those expected from the global supersymmetry. All supermultiplets on the \( SU(m|n) \) model with \( m, n > 0 \) are present (with the same energy and momentum, but multiplicity reduced to 2) in the spinless free fermion \( SU(1|1) \) model [4]. This means that they can be represented by a binary sequence of \( N_a - 1 \) ones and zeroes, representing (in the spinless fermion model) the occupations of Bloch states.
with non-zero momentum (the zero-momentum orbital has zero energy, which is the supersymmetry, and its occupation is not fixed). There are thus $2^{N_a-1}$ distinct supermultiplets.

In the $SU(1|2)$ case, the “occupation number” sequence describes a supermultiplet spanning a large range of possible fermion charges $N$. The state of minimum charge in the supermultiplet is given by the number of zeroes in the sequence; the maximum charge is $N_a$ minus the number of times two consecutive ones occur. The ground state of the model with $t > 0$ has a sequence $111\ldots 111$, so its minimum charge is $N = 0$ and its maximum charge is $N_a - (N_a - 2) = 2$. The multiplet represented by the alternating sequence $1010\ldots 10101$ has a maximum charge state $N = N_a$, which is the spin-singlet ground state of the antiferromagnetic $S = 1/2$ Haldane-Shastry chain, and a minimum charge $(N_a - 2)/2$.

We study the model (I) with $t > 0$ and a chemical potential that maximizes $N$, so the ground-state has $0 < N < N_a$. Then, only intermediate states with the maximum value of charge in their supermultiplet contribute to the thermodynamic limit of the ground-state correlation functions. To determine the excitation content of these maximal charge states, it is convenient to add a zero to each end of the binary sequence, expanding its length to $N_a + 1$. The ground state sequence is then of the form $0101010\ldots 1111111\ldots 0101010$, with a central section of consecutive ones, with equal-length wings of the alternating sequence.

In the limit $N = N_a$, the excitations of the $S = 1/2$ antiferromagnet are neutral spin-$1/2$ spinons \[12\,14\] represented by consecutive zeroes (e.g. $\ldots 01010010101\ldots$) and spinless charge $-e$ holons by consecutive ones (e.g. $\ldots 010101101010\ldots$). At intermediate densities, the central region $\ldots 111111\ldots$ may be considered as a holon condensate or “pseudo-Fermi-sea”. However, the holons and spinons are not fermions, but semions, or particles with “half-fractional” statistics, resulting from the spin-charge separation of a hole, which is a spin-1/2, charge $-e$ fermion. A configuration $\ldots 1111110111111\ldots$ has a “hole in the holon condensate” which we will call an “antiholon”; because of the semionic statistics of the holons, we identify it as a charge $+2e$, spinless boson.

Using concepts from Chern-Simons theory, as applied to the fractional quantum Hall effect \[15\], if condensed particles have charge $q$ and statistics $\Theta = \pi \lambda$, vortices or holes in
the condensate have charge $-q/\lambda$ and statistics $\Theta' = \pi/\lambda$. Here holons have charge $-e$ and $\Theta = \pi/2$ (a semion), so the vortex or hole in the holon condensate (antiholon) then has charge $2e$ and $\Theta = 2\pi$ (a boson). The applicability of such “2D” concepts to 1D ISE-type models has recently been demonstrated: the holon (antiholon) corresponds to particle (hole) excitations of the $\lambda = 1/2$ Calogero-Sutherland model where the particle excitations are semions and the holes $\lambda = 2$ bosons [2,16].

The main results of this paper can be summarized in Table I, which lists all the possible elementary excitations for the corresponding local perturbations of the ground state. The quantum symmetry prevents the injected electron or hole from breaking up into more than a very simple set of elementary excitations consisting of the left (right) spinons ($s_{L(R)}$), holons ($h_{L(R)}$), and antiholons ($\bar{h}$). As a result, the spectral functions of the various dynamical correlation functions vanish except in certain regions of the energy-momentum plane (i.e., has “compact support”).

Figs. 1-3 show the regions of compact support formed by the finite number of elementary excitations contributing to the intermediate states for $G^{(1)}$, $C^{(c)}$, and $C^{(s)}$, respectively. If the correlation functions are given by the following integral,

$$C(x, t) = \int_{(Q,E)\in \sigma} dQ \ dE \ S(Q, E) \ e^{i(Qx - Et)},$$  \hspace{1cm} (5)

the figures show the region $\sigma$ where the spectral function $S(Q, E)$ is non-zero; this is determined by combining the energies and (Bloch) momenta of the finite set of elementary excitations contributing to $S(Q, E)$.

The dispersion relations for the spinon ($E_s$), holon ($E_h$) and antiholon ($E_{\bar{h}}$) in the thermodynamic limit are given by

$$E_{s_{R(L)}}/t = -q(q \mp v_0^s), \ 0 \leq |q| \leq \frac{\pi \bar{n}}{2},$$  \hspace{1cm} (6a)

$$E_{h_{R(L)}}/t = q(q \pm v_0^c), \ 0 \leq |q| \leq \frac{\pi \bar{n}}{2},$$  \hspace{1cm} (6b)

$$E_{\bar{h}}/t = \frac{(v_0^c)^2 - q}{2}, \ -v_c^0 \leq q \leq v_c^0,$$  \hspace{1cm} (6c)
where $v_s^0 = \pi$ (spin-wave velocity), $v_c^0 = \pi(1 - \bar{n})$ (sound velocity) and $\bar{n}$ the density of electrons. The right (left) movers take the upper (lower) signs and are allowed only in $q \geq 0$ ($q \leq 0$) relative to the $Q = 0$ ground state. The curvature of the antiholon dispersion is half that of holon, indicating that $\bar{h}$ is made by destroying two holons. It is then natural to assign charge $C = +2e$ and $S = 0$ to the antiholon while $C = 0$ and $S = \frac{1}{2}$ to the spinon, and $C = -e$ and $S = 0$ to the holon. This assignment is consistent with the results given in Table I and the phase shift calculations. In fact, using this charge conservation argument we were able to identify one extra right holon for the local hole excitation ($\hat{O}_i = c_{i\sigma}$) in Table I, which could not be resolved numerically because of the small system size ($N_a = 12$) studied.

We outline below how to find the regions of support for the various correlation functions. First, we numerically find all the eigenstates having non-zero overlap with the states $c_{i\sigma}|\Psi_0\rangle$ (for $G^{(1)}$), $(n_{i\uparrow} + n_{i\downarrow})|\Psi_0\rangle$ (for $C^{(c)}$) and $(n_{i\uparrow} - n_{i\downarrow})|\Psi_0\rangle$ (for $C^{(s)}$). Second, we identify the excitation content of the states by inspecting the corresponding motifs where the spinons, holons and antiholons can easily be identified (see Table I). We empirically find the following selection rules that the holon ($v_h$), spinon ($v_s$), antiholon ($v_{\bar{h}}$), spin wave ($v_s^0$) and sound ($v_c^0$) velocities always satisfy: (i) $v_c^0 < v_s^0$ (i.e. spin-charge separation), (ii) $v_c^0 \leq |v_h|(|v_s|) \leq v_s^0$, (iii) $|v_{\bar{h}}| \leq v_c^0$, and (iv) for a given spinon-holon pair ($s_R, h_R$), $|v_{s_R}| \geq |v_{h_R}|$. These rules together with the results in Table I and Eqs. (6) allow us to plot the regions of compact support as shown in Figs. 1-3.

Fig. 1 shows the region of support for the one-particle Green’s function where the states $c_{i\sigma}|\Psi_0\rangle$ ($c_{i\sigma}^\dagger|\Psi_0\rangle$) propagate in time with positive (negative) energy with respect to the ground state. The spectral functions should be non-analytic along all the solid lines where the elementary excitations either “touch” the boundaries or the other elementary excitations. When the antiholons are suppressed (i.e. near half filling), the holon is accompanied either by a spinon or by three spinons in $S = 1/2$ state. At $3k_F (2\pi - 3k_F)$, where $k_F = \pi \bar{n}/2$, the left (right) moving spinon is missing from the state $c_{i\sigma}^\dagger|\Psi_0\rangle$ since the charge conservation prevents more than one holon in the presence of one antiholon. Of course, if two antiholons
were allowed then states of the type \((s_L, h_L) + 2\tilde{h} + 2(s_R, h_R)\) would contribute. Our numerical study indicates that states with two antiholons do not contribute. In fact, the observed states listed in Table I are the simplest possible states satisfying the charge (spin) conservation with at most one antiholon.

In Fig. 2, only holon-antiholon branches are present at \(4k_F (2\pi - 4k_F)\) while the spinon-holon branches show up at \(2k_F (2\pi - 2k_F)\). At \(\bar{n} = 0.1\), the spin-charge separation is hardly visible. In Fig. 3 we find that the pure spinon excitations are possible only if they both belong to the same sector, otherwise they are accompanied by two holons and an antiholon. The excitation content we find for \(S_z = (n_{i\uparrow} - n_{i\downarrow})/2\) should be same for \(S_i^{\pm}\) since the ground state is a spin singlet. As \(\bar{n} \to 0\) we recover the two spinon spectrum for the \(S = 1/2\) spin chain.

Finally, we have examined how the ISE results for the charge of the elementary excitations change if we interpolate between the ISE and NNE \(t - J\) models, which are respectively the \(\gamma = 0\) and \(\gamma = \infty\) limits of the integrable family of hyperbolic models with exchange \(\propto 1/\sinh^2 \gamma (i - j)\) \[17\]. Away from the ISE limit, the charge carried by the holon and antiholon excitations vary with their velocity, and become equal in magnitude (and opposite in sign) as the velocities approach the sound velocity \(v_0\). In the ISE limit, however the holon charge (\(|v| > v_0\)) is always \(-e\), and the antiholon charge (\(|v| < v_0\)) is always \(+2e\).

The “dressed charge” carried by the excitations can be calculated using the asymptotic Bethe Ansatz \[18\]. The charge enhancement of the test holon is measured by the difference in the phase shifts of the holon condensate at the pseudo-Fermi points and will in general depend on where the holon is with respect to the condensate. The total charge \(C\) (the bare plus the enhanced) is plotted in Fig. 4 as a function of the momentum of the test holon at a fixed density of electrons (\(\bar{n} = 0.5\)) for various values of \(\gamma\). The pseudo-Fermi points of the condensate for each \(\gamma\) are labeled by “x”. The ISE limit is given by the solid line. The curve with the smallest charge enhancement in the condensate corresponds to the NNE model. In the ISE limit, there is a clear jump in the holon charge from \(-e\) to \(-2e\) at the pseudo-Fermi point \(\pi(1 - \bar{n})\). Therefore, if a holon is taken out the condensate, the hole excitation carries
charge +2e independent of where it is in the condensate. We call this hole an antiholon. For all the other values of \( \gamma \), there is a considerable charge enhancement of the test holon in the condensate, and as \( \gamma \to 0 \) the charge approaches \( -2e \).

In conclusion, we have devised simple rules for constructing the motifs for the excited states of the 1D ISE \( t-J \) model and identified the exact excitation content of the intermediate states for the one-particle Green’s function, the charge density-density and spin-spin correlation functions. We believe that this model is in the same universality class as the NNE model, and that the most relevant states for the ground state correlation functions of the NNE model are also given by Table I. Finally, the presence of spinons, holons, and antiholons in two-dimensional models and their role in the high \( T_c \) superconductivity is an amusing possibility.

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FIGURES

FIG. 1. Compact support for the one-particle Green’s function. The momentum is in units of π and the excitation energy E in \( \pi^2/t \). The contributing elementary excitations to this region are \((h_L, s_L) + \bar{h} + 2(h_R, s_R)\) for the positive energy part (i.e. \( c_{i\sigma} |\Psi_0\rangle \)) and \((s_L, h_L) + \bar{h}\) for the negative part (i.e. \( c_{i\sigma}^\dagger |\Psi_0\rangle \)). Their mirror states (i.e. L and R exchanged) also contribute. The four momenta at which \( E = 0 \) is allowed are \( k_F, 2\pi - 3k_F, 3k_F, \) and \( 2\pi - k_F \) where \( k_F = \pi \bar{n}/2 \).

FIG. 2. Compact support for the density-density correlation function. \((s_L, h_L) + \bar{h} + (s_R, h_R), \bar{h} + 2h_R\) and their mirror states contribute. \( E = 0 \) is allowed at \( 0(2\pi), 2k_F, 2\pi - 4k_F, 4k_F, 2\pi - 2k_F \). Only holon-antiholon branches are present at \( 4k_F (2\pi - 4k_F) \) indicating that \( 4k_F \) is the holon Fermi point.

FIG. 3. Compact support for the spin-spin correlation function. \((s_L, h_L) + \bar{h} + (s_R, h_R), 2s_L\) and their mirror states contribute. \( E = 0 \) allowed at \( 0(2\pi), 2k_F, 2\pi - 2k_F \). This indicates that \( 2k_F \) is the spinon Fermi point.

FIG. 4. Charge of a test holon versus its momentum in the vicinity of the holon condensate for \( \gamma = 0, 0.2, 0.3, 0.5, 1.0, 2.0, \infty \). The charge is in units of \( -e \) where \( e \) is the electron charge and the momentum of the test holon in units of \( \pi \). The pseudo-Fermi points are labeled by “x” for each \( \gamma \). The step function corresponds to the ISE model (\( \gamma = 0 \)). The NNE model has the smallest but still considerable charge enhancement in the condensate (\( \gamma = \infty \)).
TABLE I. List of all the possible excitations from the ground state perturbed by the local operators $c_i\sigma (c_i^\dagger \sigma)$ ($G^{(1)}$), $n_{i\uparrow} + n_{i\downarrow}$ ($C^{(e)}$), and $n_{i\uparrow} - n_{i\downarrow}$ ($C^{(s)}$). The mirror states ($L \leftrightarrow R$) not listed are also allowed. The spinon ($v_s$), holon ($v_h$), antiholon ($\bar{v}_h$), spin-wave ($v_{0s}^c$) and sound ($v_{0c}^s$) velocities always satisfy: (i) $v_{0c}^s < v_{0s}^c$, (ii) $v_{0c}^s \leq |v_h|(|v_s|) \leq v_{0c}^s$, (iii) $|v_h| \leq v_{0c}^c$, and (iv) for a given spinon-holon pair $(s_R, h_R)$, $|v_{s_R}| \geq |v_{h_R}|$.

| Local Operator $\hat{O}_i$ | Excitation contents of $\hat{O}_i|\Psi_0\rangle$ |
|---------------------------|-----------------------------------------------|
| $c_{i\sigma}$             | $(s_L, h_L) + \bar{h} + 2(s_R, h_R)$          |
| $c_{i\sigma}^\dagger$     | $(s_L, h_L) + \bar{h}$                        |
| $n_{i\uparrow} + n_{i\downarrow}$ | $(s_L, h_L) + \bar{h} + (s_R, h_R)$               |
|                           | $\bar{h} + 2h_R$                              |
| $n_{i\uparrow} - n_{i\downarrow}$ | $(s_L, h_L) + \bar{h} + (s_R, h_R)$               |
|                           | $2s_L$                                        |
momentum (in units of $\pi$)

(a) $n=0.1$

(b) $n=0.6$

(c) $n=0.9$

$\frac{1}{2}$ $1$ $\frac{3}{2}$ $2$
momentum (in units of Pi)

(a) $n=0.1$

(b) $n=0.6$

(c) $n=0.9$
momentum (in units of $\pi$)

(a) $n=0.25$

(b) $n=0.6$

(c) $n=0.9$
