Abstract: A new Partially Averaged Navier-Stokes (PANS) bridging model is derived from existing \((k-\omega)\) and \((k-\varepsilon)\) PANS formulations. The model behaves like the PANS \((k-\omega)\) model near rigid walls and like the PANS \((k-\varepsilon)\) model away from walls. The new model is tested using well-known benchmark problems; a backward-facing step representing wall-bounded flows, and a circular cylinder representing free shear flows. Our results are compared to existing experimental data and previous simulation results using PANS \((k-\omega)\) and PANS \((k-\varepsilon)\). The comparisons show our model to be superior at predicting velocity profiles in both flows. In addition, Reynolds stress predictions are also shown to improve.

Keywords: hybrid RANS/Large Eddy Simulation (LES); partially averaged Navier Stokes; unsteady turbulent flows; turbulence modeling

1. Introduction

The desire for inexpensive unsteady simulations of turbulent flows has led to the development of numerous modeling methods which aim to give accurate solutions at a reasonable computational cost. One of the earliest methods, the Unsteady Reynolds Averaged Navier-Stokes (URANS) or Very Large Eddy Simulation (VLES), is inexpensive but unable to resolve the random fluctuations associated with turbulent flows [1]. However, when used in free shear flows with good grid resolution, it can provide useful information about the large-scale structures and in particular the shedding frequency. Certainly, the most accurate method for capturing the physics of unsteady flows is Direct Numerical Simulation (DNS) [2]. In DNS, the full Navier-Stokes equations are solved with no simplifying assumptions. However, this comes at a very high computational cost and is limited to low Reynolds number flows. This is a big drawback since most flows of engineering interest have a high Reynolds number. In order to overcome the Reynolds number limitation of DNS, the Kolmogorov hypothesis [3] is used to take advantage of the universality of the small scales of turbulence at high Reynolds numbers. Based on that, Kolmogorov proposed the first two-equation eddy viscosity model [4] and opened the door for many more to come later. The scale separation operation proposed by Kolmogorov [3] is the foundation for a new method called Large Eddy Simulation (LES). In LES the smallest scales of motion are modeled, while the large scales are fully resolved [5–7]. This space filtering operation reduces the computational cost when compared to DNS. However, the computational cost remains high at high Reynolds numbers and hence the need for even cheaper methods.

In recent years, a new family of methods has emerged. These methods known as Hybrid RANS/LES take advantage of the vast knowledge developed in RANS to bridge the gap in the wavenumber space with LES, thus the often-used name of Bridging models [8,9]. One such method is the Detached Eddy Simulation (DES) of Squires et. al. [10]. In DES, the wall region is modeled with RANS while LES is used away from viscous walls. There are several variants of DES in the literature [11–13].
each claiming an improvement over the original model. Another bridging method is the Partially
Averaged Navier-Stokes (PANS) due to Girimaji [14]. PANS can be used to smoothly transition from
URANS to DNS by changing the values of some resolution control parameters and refining the grid.
Two PANS models exist in the literature [15,16], one based on the Jones and Launder RANS \((k - \varepsilon)\)
model [17] and the other based on the Wilcox RANS \((k - \omega)\) model [18]. There are several other Hybrid
RANS/LES models in the literature [19,20], each claiming a specific benefit.

In this paper, we propose a new PANS model based on the \((k - \omega)\) and \((k - \varepsilon)\) versions of PANS
and using the Menter blending idea introduced in RANS [21]. Our goal is to take advantage of the
near-wall benefits of PANS \((k - \omega)\) and the far field benefits of PANS \((k - \varepsilon)\) with the overarching goal
of improving predictions from the resulting model. The remainder of the paper is organized as follows;
the mathematical model is derived in Section 2, followed by the results and discussion in Section 3,
and the concluding remarks are given in Section 4.

2. Mathematical Model and Method of Solution

The unsteady, incompressible forms of the Navier-Stokes equations are given by

\[
\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} + \tau_{ij} \quad (1)
\]

where \(V\) is the instantaneous velocity and \(p\) is the pressure. The Poisson equation relates pressure and
velocity through

\[
-\frac{\partial^2 p}{\partial x_i \partial x_i} = \frac{\partial V_i}{\partial x_j} \quad (2).
\]

The flow variables are filtered or partially averaged, and the resulting resolved variables are
given by

\[
U_i = \langle V_i \rangle \quad p_{\text{U}} = \langle p \rangle \quad (3)
\]

The velocity and pressure fields are related to the resolved and unresolved variables by

\[
V_i = U_i + u_i \quad p = p_{\text{U}} + p' \quad (4)
\]

Introducing Equations (3) and (4) into Equations (1) and (2) leads to the Partially Averaged
Navier-Stokes (PANS) equations

\[
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p_{\text{U}}}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} \quad (5)
\]

\[
-\frac{\partial^2 p_{\text{U}}}{\partial x_i \partial x_i} = \frac{\partial U_i}{\partial x_j} \frac{\partial U_j}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_i} \quad (6)
\]

The filtering process introduces sub-filter scale stresses (SFS) which lead to a closure problem
in the PANS equations. The SFS are given by the generalized central second moment of the
instantaneous velocities

\[
\tau(V_i,V_j) = \langle V_i V_j \rangle - \langle V_i \rangle \langle V_j \rangle \quad (7)
\]

The pressure fluctuation \(p'\) in Equation (4) is solved with the following Poisson Equation [14]

\[
\nabla^2 p' = -2 \frac{\partial U_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \quad (8)
\]

Modeling of the SFS stress is performed by the extension of Reynolds Averaged Navier Stokes
(RANS) turbulence models to PANS. In the PANS formulation, the ratio of the unresolved to the total
turbulent quantities are specified by the PANS resolution control parameters. The subscript \(u\) is used
to signify unresolved quantities

\[
f_\varepsilon = \frac{\varepsilon_u}{\varepsilon} \quad f_k = \frac{k_u}{k} \quad f_\omega = \frac{\omega_u}{\omega} \quad (9)
\]
In Equation (9), $k$ is the turbulent kinetic energy, $\varepsilon$ the turbulent dissipation and $\omega$ the turbulent frequency. The resolution control parameters vary between 0 and 1 depending on the flow conditions and grid resolution. When using RANS, the values of the resolution control parameters are set to 1, whereas in DNS the values should be set to 0; i.e., all scales are resolved. This scale resolution cannot be achieved without an appropriate grid resolution based on the Reynolds number. In this paper, the values used are taken from the literature for comparisons. The grid resolution should be appropriate for the chosen values. The reader is referred to prior work for more details on these parameters.

Starting with the $(k - \varepsilon)$ PANS formulation given by [14]

$$\frac{\partial \varepsilon_u}{\partial t} + U_j \frac{\partial \varepsilon_u}{\partial x_j} = \frac{C_{e1} P_u \varepsilon_u}{k_u} - \frac{C_{e2} \varepsilon_u^2}{k_u} + \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \varepsilon_u}{\partial \varepsilon_u \partial x_j} \right)$$

(10)

$$\frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} = P_u - \varepsilon_u + \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial k_u}{\partial \varepsilon_u \partial x_j} \right)$$

(11)

The turbulent dissipation rate, $\varepsilon_u$, and the turbulent production, $P_u$, are given by

$$\varepsilon_u = \beta^* k_u \omega_u$$

$$P_u = \frac{1}{2} \tau (V_i, V_j) \frac{\partial U_i}{\partial x_j}$$

(12)

The resolution control parameters are used to vary the PANS model constants from the RANS constants. Only one constant must be changed in the $(k - \varepsilon)$ PANS formulation

$$C_{e2}' = C_{e1} + \frac{h}{f} (C_{e2} - C_{e1})$$

(13)

To bridge the $(k - \varepsilon)$ and $(k - \omega)$ models, the $\varepsilon$-equation will be rewritten in terms of $\omega$ using Equation (12). To this end, the time derivative and gradient of the turbulent dissipation can be written in terms of the turbulent kinetic energy and frequency

$$\frac{\partial \varepsilon_u}{\partial t} = \beta^* \left( \alpha_u \frac{\partial \varepsilon_u}{\partial t} + k_u \frac{\partial \omega_u}{\partial t} \right)$$

(14)

$$U_j \frac{\partial \varepsilon_u}{\partial x_j} = U_j \beta^* \left( \alpha_u \frac{\partial \varepsilon_u}{\partial x_j} + k_u \frac{\partial \omega_u}{\partial x_j} \right)$$

(15)

The terms on the right-hand side of the dissipation equation, Equation (10), can also be rewritten as

$$\frac{C_{e1} P_u \alpha_u}{k_u} = C_{e1} P_u \beta^* \omega_u$$

(16)

$$\frac{C_{e2} \varepsilon_u^2}{k_u} = C_{e2} \beta^* k_u \omega_u^2$$

(17)

$$\frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \varepsilon_u}{\partial \varepsilon_u \partial x_j} \right) = \beta^* \frac{k_u}{\varepsilon_u} \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \omega_u}{\partial \varepsilon_u \partial x_j} \right) + \beta^* \frac{\varepsilon_u}{\omega_u} \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \omega_u}{\partial \varepsilon_u \partial x_j} \right) + \frac{2 \nu_u \beta^* \omega_u \frac{\partial \omega_u}{\partial x_j}}{\varepsilon_u \frac{\partial \omega_u}{\partial x_j}}$$

(18)

Rewriting the full transformed equation and rearranging the terms gives

$$\beta^* \omega_u \left( \frac{\partial k_u}{\partial t} + U_j \frac{\partial k_u}{\partial x_j} \right) - P_u + \beta^* k_u \omega_u - \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial k_u}{\partial \varepsilon_u \partial x_j} \right) + \beta^* k_u \left( \frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} \right) =$$

$$(C_{e1} - 1) P_u \beta^* \omega_u - (C_{e2} - 1) \beta^2 k_u \omega_u^2 + \beta^* k_u \frac{\partial}{\partial x_j} \left( \frac{\nu_u \partial \omega_u}{\partial \varepsilon_u \partial x_j} \right) + \frac{2 \nu_u \beta^* \omega_u \frac{\partial \omega_u}{\partial x_j}}{\varepsilon_u \frac{\partial \omega_u}{\partial x_j}}$$

(19)
The first bracketed term on the left-hand side is equal to zero by virtue of Equations (11) and (12), and using
\[
\frac{\nu\omega}{\sigma_{ku}} = \frac{\nu_{k2}\sigma_{u2}}{\sigma_{ku}} = \frac{\nu}{\sigma_{ku}}
\]
leads to the \((k - \omega)\) model obtained from the transformed \((k - \varepsilon)\)
\[
\frac{\partial k_u}{\partial t} + U_i \frac{\partial k_u}{\partial x_j} = P_u - \beta^* k_u \omega_u + \frac{\partial}{\partial x_j} \left[ \frac{\nu}{\sigma_{ku}} \frac{\partial k_u}{\partial x_j} \right]
\]
\[
\frac{\partial \omega_u}{\partial t} + U_j \frac{\partial \omega_u}{\partial x_j} = \gamma u_2 P_u \frac{\omega_u}{\kappa} \beta_2 \omega_u^2 + 2 \frac{\partial^2 \omega_u}{\partial x_j} \frac{\partial \omega_u}{\partial x_j} + \frac{\partial}{\partial x_j} \left[ \frac{\nu}{\sigma_{u2}} \frac{\partial \omega_u}{\partial x_j} \right]
\]
The model constants are given by
\[
\sigma_{u2} = \frac{\beta^* \omega_u^2 \nu^2}{C_{\mu}} = \frac{\nu^2 \omega_u^2}{C_{\mu}} = \frac{\nu^2 \omega_u^2}{C_{\mu}/\kappa}, \quad \gamma u_2 = C_{\varepsilon 1} - 1, \quad \beta u_2 = (C_{\varepsilon 2} - 1) \beta^*, \quad \sigma_{ku} = \frac{\sigma_k^2 \nu^2}{\kappa}, \quad \sigma_{\rho u} = \frac{\sigma_{\rho u}^2 \nu^2}{\kappa}, \quad \sigma_{\rho k} = \sigma_k
\]
where \(\sigma_k = 1.0, \sigma_\varepsilon = 1.3, C_{\varepsilon 1} = 1.42, C_{\varepsilon 2} = 1.92, C_\mu = 0.09, \beta^* = 0.09. The original \((k - \omega)\) PANS is given by [16]
\[
\frac{\partial k_{u1}}{\partial t} + U_i \frac{\partial k_{u1}}{\partial x_j} = \gamma u_1 P_u \frac{\omega_{u1}}{\kappa} \beta_{u1} \omega_{u1}^2 + \frac{\partial}{\partial x_j} (\nu \frac{\partial \omega_{u1}}{\partial x_j})
\]
\[
\frac{\partial \omega_{u1}}{\partial t} + U_j \frac{\partial \omega_{u1}}{\partial x_j} = P_u - \beta^* k_{u1} \omega_{u1} + \frac{\partial}{\partial x_j} (\nu \frac{\partial \omega_{u1}}{\partial x_j})
\]
where the constants are
\[
\gamma u_1 = \alpha, \quad \beta_{u1} = \beta^* = \alpha \left( 1 - \frac{1}{\nu} \right) + \frac{\delta}{\nu_k}, \quad \sigma_{u1} = \frac{\sigma_k^2 \nu^2}{\kappa}, \quad \sigma_{\rho u1} = \sigma_k
\]
where \(\sigma_k = 2.0, \sigma_k = 2.0, \beta^* = 0.09, \beta = 0.075 and \alpha = 5/9. The unresolved eddy viscosity of the \((k - \varepsilon)\) and \((k - \omega)\) PANS models can be related to one another through
\[
(v_u)_{k-\varepsilon} = \frac{C_k}{\beta^*} (v_u)_{k-\omega}
\]
The two \((k - \omega)\) models are bridged by multiplying the original \((k - \omega)\) PANS with a blending function \(F\) and the \((k - \omega)\) PANS obtained from the modified \((k - \varepsilon)\) PANS by \(1 - F\) and adding the two
\[
\frac{\partial k_{u1}}{\partial t} + U_i \frac{\partial k_{u1}}{\partial x_j} = P_u - \beta^* k_{u1} \omega_{u1} + \frac{\partial}{\partial x_j} (\nu \frac{\partial k_{u1}}{\partial x_j})
\]
\[
\frac{\partial \omega_{u1}}{\partial t} + U_j \frac{\partial \omega_{u1}}{\partial x_j} = \gamma P_u \frac{\omega_{u1}}{\kappa} \beta_{u1} \omega_{u1}^2 + \frac{\partial}{\partial x_j} (\nu \frac{\partial \omega_{u1}}{\partial x_j}) + 2(1-F) \sigma_k \frac{\partial \omega_{u1}}{\partial x_j} + \frac{\partial}{\partial x_j} (\nu \frac{\partial \omega_{u1}}{\partial x_j})
\]
The blending function is given by
\[
F = \tanh(\text{arg}^4)
\]
with
\[
\text{arg} = \min \left( \max \left( \frac{\nu_k}{\beta \omega_u y^4}, \frac{500 \nu}{\omega_u^4} \right), \max \left( \frac{8k_k}{\sigma_{u2} \omega_u^2 \nu^2}, \frac{\nu_k}{\sigma_{u2} \omega_u^2}, 4 \times 10^{-10} \nu_k \right) \right)
\]
where \(y\) is the distance from the wall and \(v\) is the kinematic viscosity. The blended constants are given by
\[
\sigma_{k_u} = \frac{\sigma_k \nu^2 \omega_u^2}{\nu_k \rho_2 + (1-\rho_2) \sigma_{u1}}, \quad \sigma_{u2} = \frac{\sigma_2 \nu^2 \omega_u^2}{\nu_k \rho_2 + (1-\rho_2) \sigma_{u1}}, \quad \gamma = F \gamma_{u1} + (1-F) \gamma_{u2}
\]
\[
\beta = F \beta_{u1} + (1-F) \beta_{u2}
\]
The model described above was then implemented in the open-source software known as OpenFOAM [22]. The grid generation was performed using either the built-in tool, blockMesh, within OpenFOAM; or using Gmesh [23] outside of OpenFOAM. All these tools are readily available for download from the web. Depending upon the size of the grid used for a given problem, a single processor or a multi-processor machine was used. For the two problems presented in this paper, computations can take hours to a few days depending on machine availability. The computations presented here are time-dependent and hence require a long integration time to achieve a statistically steady state. The boundary conditions used are the standard inflow-outflow, far field, and symmetry in the streamwise, vertical, and spanwise directions, respectively (see Reference [24] for more details).

3. Results and Discussion

Simulations have been performed on wall-bounded and free shear flows to validate the model. A backward facing step is used for the wall-bounded flow and a circular cylinder is used for the free shear flow. Experimental data [25,26] and previous PANS simulation results [14–16] have been used for comparisons.

3.1. Wall-Bounded Flows: Backward-Facing Step

Despite its simple geometry, a backward-facing step flow presents a challenging problem for turbulence modeling and is considered a standard test case for turbulence model validation. The relevant physics include boundary layer separation from the edge of the step, shear layer reattachment downstream, a recirculation region near the step, and accompanying turbulent motion of the fluid. For the backward-facing step, the size of the computational domain is $10H \times 5H \times 3H$ in the region upstream of the step and $20H \times 6H \times 3H$ in the region downstream of the step, where the step height is $H$ and the dimensions are given in the streamwise, normal, and spanwise directions, respectively. A $6H$ wide computational domain was used to examine the effect of the spanwise dimension on the results. The domain spanning $6H$ has equivalent streamwise and normal dimensions to the domain spanning $3H$. The Reynolds number based on the step height is 37,500. The domain is discretized with a structured mesh of hexahedral elements, as shown in Figure 1. For the $3H$ domain, the inlet region upstream of the step has $92 \times 81 \times 60$ cells. Downstream of the step, the grid is $230 \times 168 \times 60$ in the streamwise, normal, and spanwise directions, respectively. The $6H$ domain has 120 cells in the spanwise direction. The grid resolution gives dimensionless wall-based distances of $y^+ < 1$, and $z^+ < 60$ throughout the whole domain. The streamwise resolution varies from $x^+ < 1$ at the step to a maximum of $x^+ < 140$ at $5H$ and farther downstream.

Figure 1. View of the backward step mesh near the step.
In wall-bounded flows such as the backward-facing step, resolving the turbulence dissipation near the wall becomes critical for accurate prediction of flow quantities. Therefore, the value of $f_c$ should be less than 1 and it is set to 0.667 as in prior references. In addition, the value of $f_k$ is fixed at 0.2 indicating a high resolution of the turbulent kinetic energy. The mean turbulent kinetic energy, Reynolds stress, and velocity profiles are compared with experimental data and previous $(k−\varepsilon)$ and $(k−\omega)$ PANS results at several locations downstream of the step. Following the available experimental data, the mean turbulent kinetic energy neglects the spanwise fluctuation contribution and is calculated using $k = 0.5(\overline{u'^2} + \overline{v'^2})$.

Before comparing the results of our new model to those in the literature and experimental data, we embark on a domain size study, namely the lateral extent of the computational domain. Prior numerical studies [15] used three step heights ($3H$) as the lateral dimension. However, questions arose as to whether the two-point correlation of the fluctuating velocity decayed to zero within the computational domain. To this end, another simulation was carried out with a computational domain size doubled in the lateral direction, i.e., $6H$. Two-point correlations of the velocity fluctuations across the span of the domain were calculated for both the $3H$ and $6H$ cases. The sampling location was at the height of the step ($1H$ from the lower wall) and $1H$ downstream of the step. Figure 2 gives the two-point correlation for the $x$-component of the streamwise velocity fluctuation. The figure shows that the two-point correlation decays rapidly to zero away from the center for both the $3H$ and $6H$ domains. This indicates that the $3H$-wide computational domain is sufficient to capture the relevant physics. To further confirm that, comparisons of the mean velocity profiles at a downstream location of $x/H = 5$ are shown in Figure 3. Little to no difference between the velocity profiles is shown by the figure. Similarly, the mean kinetic energy profile at the same downstream location is shown in Figure 4. In this case, small differences are shown due to the proximity of the reattachment point. Similar results are obtained for the mean Reynolds stress profile, Figure 5. Based on these results, the $3H$-wide computational domain is deemed sufficient and is used for the remainder of this paper.

![Figure 2](image-url)  
**Figure 2.** Two-point correlation of $u'_x$ at 1H downstream of the step ($f_c = 0.667, f_k = 0.2$).
Figure 3. Mean velocity profile at $\frac{x}{H} = 5$ from the step ($f_c = 0.667, f_k = 0.2$).

Figure 4. Mean kinetic energy profiles at $\frac{x}{H} = 5$ from the step ($f_c = 0.667, f_k = 0.2$).
The performance of the proposed model is compared to experimental data [25] and previous \((k - \varepsilon)\) and \((k - \omega)\) PANS results [14–16]. As mentioned in the previous section, our results are obtained using a computational domain spanning \(3H\) in width. The \((k - \omega)\) PANS results [15] were obtained for \(f_k = 0.5\). Figures 6 and 7 show the mean velocity profiles at \(x/H\) of \(5H\) and \(6H\) downstream of the step. The figures show that the new model gives the best agreement with experimental data. In addition, the predictions of the new model fall between the previous \((k - \varepsilon)\) and \((k - \omega)\) PANS models’ results, which is expected. The downstream locations chosen are near the reattachment point.
Figure 7. Mean velocity profiles at $x/H = 6$ from the step ($f_\epsilon = 0.667, f_k = 0.2$).

The mean turbulent kinetic energy profiles at $x/H$ of 2 and 5 are given in Figures 8 and 9. At both locations, our model predictions are in better agreement with the experimental data.

Figure 8. Mean kinetic energy profiles at $x/H = 2$ from the step ($f_\epsilon = 0.667, f_k = 0.2$).
Figure 9. Mean kinetic energy profiles at $x/H = 5$ from the step ($f_c = 0.667, f_k = 0.2$).

The mean Reynolds stress profiles are given in Figures 10 and 11 at the downstream locations of $x/H = 2$ and 5 from the step. Near the step, $x/H = 2$, our model predictions are in better agreement with experimental data. Away from the step, $x/H = 5$, our model follows the experimental data closely near the wall but overpredicts the peak and its location.

Figure 10. Mean Reynolds stress profiles at $x/H = 2$ from the step ($f_c = 0.667, f_k = 0.2$).
where \( \Omega \times A \) coarse mesh and a fine mesh are used for the simulations. The coarse mesh consists of 240 \( \times 320 \times 48 \) cells and the fine mesh consists of 300 \( \times 360 \times 54 \) cells in the radial, angular, and spanwise directions, respectively. A coarse mesh and a fine mesh are used for the simulations. The coarse mesh consists of 240 \( \times 320 \times 48 \) cells and the fine mesh consists of 300 \( \times 360 \times 54 \) cells in the radial, angular, and spanwise directions, respectively.

The flowfield around a circular cylinder is characterized by strong vortex shedding downstream of the cylinder. The blue color indicates low velocities, and the reddish color indicates higher velocities. As expected, the blue color dominates just downstream of the step due to the recirculation zone.

\[
Q = \frac{1}{2} (|\Omega|^2 - |S|^2)
\]  

(33)

where \( \Omega \) is vorticity and \( S \) is the rate of strain.

**Figure 11.** Mean Reynolds stress profiles at \( x/H = 5 \) from the step \( (f_\epsilon = 0.667, f_k = 0.2) \).

Figure 12 shows the Q-criterion iso-contours colored by velocity, which illustrates the turbulent structures shed from the step of Equation (33). The blue color indicates low velocities, and the reddish color indicates higher velocities. As expected, the blue color dominates just downstream of the step due to the recirculation zone.

\[
Q = \frac{1}{2} (|\Omega|^2 - |S|^2)
\]  

(33)

where \( \Omega \) is vorticity and \( S \) is the rate of strain.

**Figure 12.** Q-criterion iso-contours colored by velocity for \( x = 10^5 \) \( (f_\epsilon = 0.667, f_k = 0.2) \).

### 3.2. Free-Shear Flows: Circular Cylinder

The flowfield around a circular cylinder is characterized by strong vortex shedding downstream of the cylinder. The size of the computational domain is \( 30D \times 30D \times 3D \), where \( D \) is the diameter of the cylinder, and the dimensions are in the streamwise, normal, and spanwise directions, respectively. A coarse mesh and a fine mesh are used for the simulations. The coarse mesh consists of 240 \( \times 320 \times 48 \) cells and the fine mesh consists of 300 \( \times 360 \times 54 \) cells in the radial, angular, and spanwise directions, respectively. The coarse grid resolution gives a dimensionless wall-normal distance of \( y^+ < 1 \). The spanwise resolution on the windward side of the cylinder is \( z^+ < 920 \) and on the leeward side is \( z^+ < 550 \). The fine grid resolution gives \( y^+ < 1, z^+ < 760 \) on the windward side and \( z^+ < 500 \) on the leeward side. A coarse mesh is shown in Figure 13.
The flow Reynolds number based on cylinder diameter is $Re_D = 1.4 \times 10^5$. Unlike the backward-facing step flow dominated by the step and the viscous wall, the flow physics in the case of a circular cylinder is dominated by vortex shedding and its subsequent breakdown downstream of the cylinder; accordingly, and because of the absence of walls, the value of $f_\epsilon$ is kept equal to 1.0 for all simulations. In order to compare to previous computational results, two values of $f_k$ were used, 0.5 and 0.7. A fine grid is used with $f_k = 0.5$ and a coarser grid is used for $f_k = 0.7$. Figure 14 shows the mean centerline streamwise velocity profile for $f_k = 0.7$. Our results agree with the $(k-\omega)$ PANS results of Girimaji et al. [16] both near the cylinder and far downstream and are in good agreement with the experimental data [26]. Figure 15 shows the mean centerline velocity profile for $f_k = 0.5$. Our results follow closely those of $(k-\omega)$ PANS across the whole domain and are in reasonable agreement with the experimental data.
Figure 15. The mean centerline streamwise velocity profile \( (f_\epsilon = 1.0, f_k = 0.5) \).

Figure 16 shows the vertical profile of the streamwise velocity at \( x/D = 1 \) and for \( f_k = 0.7 \). Our model shows better agreement with the experimental data near the centerline and follows closely the \( (k-\epsilon) \) PANS away from the centerline. When the grid is refined and \( f_k \) is lowered to 0.5, the results from our model are significantly improved everywhere, as shown in Figure 17. Figure 18 shows the vertical profile of the streamwise velocity at \( x/D = 3 \) for \( f_k = 0.7 \). Though our model predictions are better than that of the \( (k-\epsilon) \) PANS model, the predictions from the \( (k-\omega) \) PANS are in better agreement with the experimental data. When the grid is refined and \( f_k \) is lowered to 0.5, our model’s predictions are better everywhere than those of \( (k-\epsilon) \) and \( (k-\omega) \) PANS, as shown in Figure 19. Our model’s predictions are in excellent agreement with the experimental data across the entire profile.
Figure 17. The mean streamwise velocity profile at $x/D = 1$ ($f_\epsilon = 1.0$, $f_k = 0.5$).

Figure 18. The mean streamwise velocity profile at $x/D = 3$ ($f_\epsilon = 1.0$, $f_k = 0.7$).
Figure 19. The mean streamwise velocity profile at $x/D = 3$ ($f_\varepsilon = 1.0$, $f_k = 0.5$).

The mean vertical velocity profile at $x/D = 1$ for $f_k = 0.7$ is shown in Figure 20. Our results agree well with the experimental data and the results from the $(k - \omega)$ PANS. A refined grid and a lowering of $f_k$ to 0.5 resulted in improvements near the peak velocities and away from the centerline, Figure 21.

Figure 20. The mean vertical velocity profile at $x/D = 1$ ($f_\varepsilon = 1.0$, $f_k = 0.7$).
4. Conclusions

A new Partially Averaged Navier-Stokes model has been derived. The model is based on the existing \((k - \omega)\) and \((k - \epsilon)\) PANS models. Menter’s [21] bridging approach is used. The new model is tested using two benchmark problems; the backward-facing step, representing wall-bounded flows, and flow over a circular cylinder, representing free shear flows. For comparison reasons, the PANS resolution control parameters were taken from the open literature with an appropriate grid resolution. For both benchmark problems, experimental data is used for comparison as well as existing computational results using PANS \((k - \omega)\) or \((k - \epsilon)\).

For the backward-facing step problem, the mean velocity profile downstream of the step obtained by our model is found to be between that of the \((k - \omega)\) and \((k - \epsilon)\) models and in excellent agreement.
with the experimental data [25]. In addition, there was a marked improvement in the turbulent kinetic energy and Reynolds stress profiles predicted by our model.

For the circular cylinder, all mean velocity profiles predicted by our model; centerline streamwise, streamwise, and vertical; are in excellent agreement with the experimental data [26] for $f_k$ of 0.5. Our model agrees well with the $(k\omega)$ PANS model and is everywhere better than the $(k\epsilon)$ PANS model.

In conclusion, the PANS Bridging model proposed in this paper shows significant improvements in predictions over existing PANS models.

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