Convergence of the Approximation scheme to American option pricing via the discrete Morse semiflow

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Abstract. We consider the approximation scheme of the American call option via the discrete Morse semiflow. It is the minimizing scheme of a time-semidiscretized variational functional. In this paper we obtain a rate of convergence of approximate solutions. In addition, the convergence of approximate free boundaries is proved.

1 Introduction

In this paper we consider an approximation scheme to the following obstacle problem:

\begin{align*}
\min \left\{ -C_{\tau} - \frac{\sigma^2}{2} S^2 C_{SS} - (r - q)SC_S + rC, C - \Phi \right\} &= 0 \quad \text{in } (0, T) \times (0, +\infty), \\
C(T, S) &= \Phi(S) := \max(S - K, 0) \quad \text{for } S \in [0, +\infty), \\
C(\tau, 0) &= 0 \quad \text{for } \tau \in (0, T),
\end{align*}

The above equation is called the Black-Scholes equation for the American call option. Here \(C = C(\tau, S)\) is the option price, the positive constants \(\sigma, r, q, K\) denote, respectively, the

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volatility, the interest rate, the dividend and the strike price. Throughout this paper we set \( \sigma = \sqrt{2} \) for simplicity and assume \( r > q \).

It is known that (1.1) has a unique free boundary. Indeed, by van Moerbeke [34], we see that under the assumption \( r > q \), there exists a unique \( S^* \in C^1(0,T) \) such that \( (S^*)' \leq 0 \) and

\[
\{ S \mid C(\tau, S) > \Phi(S) \} = (-\infty, S^*(\tau)), \quad \{ S \mid C(\tau, S) = \Phi(S) \} = [S^*(\tau), +\infty)
\]

for each \( \tau \in (0,T) \),

\[
(1.2) \quad \lim_{\tau \to T} S^*(\tau) = \frac{rK}{q} (> K).
\]

See Wilmott - Howison - Dewynne [35, p.124] for the formal derivation of (1.2). The family \( \{S^*(\tau)\}_{0<\tau<T} \) is called the free boundary or the optimal exercise boundary; for each \( \tau \in (0,T) \), \( S^*(\tau) \) indicates the value of the current stock price under which the holder of the option should (optimally) exercise it.

From the viewpoint of mathematical finance, it would be very convenient to obtain \( \{S^*(\tau)\}_{0<\tau<T} \) explicitly. However, this seems to be very difficult and thus many people have studied numerical schemes for (1.1), especially to approximate the free boundary. Brennan - Schwartz [7] introduced a fully implicit difference scheme for the American put option and obtained a numerical solution. The convergence of their scheme was proved by Jaillet - Lamberton - Lapeyre [13] in the framework of variational inequalities. Lamberton [18] considered the binomial tree method and the finite difference one to approximate \( \{S^*(\tau)\}_{0<\tau<T} \) and showed the convergence of the approximate free boundary by the probabilistic argument and the analytical one. He also obtained in [19, 20] some error estimates for the stochastic approximation to the optimal stopping problems including the American options. Amin - Khanna [2] treated a discrete time model for the American option and proved the convergence of the discrete American option value to the continuous one. Jiang - Dai [14] obtained similar results to those in [18] by the method of viscosity solutions. Omata - Iwasaki - Nakane - Xiong - Sakuma [29] proposed an approximation scheme to (1.1) different from the above ones and obtained a numerical result.

The approximation scheme by [29] is based on the discrete Morse semiflow (DMS), consisting of the minimization of a time-semidiscretized variational functional. The DMS was first used by Rektorys [31] to obtain the solutions of linear parabolic equations. Kikuchi [15, 16] applied the DMS to construct the solutions of parabolic equations associated with a variational functional of a harmonic map type. Besides, in [32, 23, 24, 25] Nagasawa and Tachikawa used the DMS to show the existence and asymptotic behavior of solutions of some semilinear hyperbolic systems. Nagasawa - Omata [22] considered the behavior of the DMS for a free boundary problem. Some applications of the DMS to numerical analysis have been treated in Omata [26, 27], Omata - Okamura - Nakane [30] and Omata - Iwasaki - Kawagoe [28].

The purpose of this paper is to discuss the convergence of the approximation scheme by [29]. Our results are a rate of convergence of the approximate solutions and the convergence of the approximate free boundary. The former result is obtained by applying the rate of convergence of product formula for semigroups by Bentkus - Paulauskas [6] and the precise comparison argument for viscosity solutions by Ishii - Koike [11], in which
they obtained the rate of convergence in elliptic singular perturbations. The latter one is proved by the limit operation of viscosity solutions due to Barles - Perthame [3, 4].

This paper is organized as follows. In Section 2 we introduce the approximation scheme by [29] and state the main results. In Section 3 we discuss the solutions of (2.2) below. In Subsection 3.1 we briefly prove the existence and uniqueness of solutions. In Subsection 3.2 we show some properties of solutions. Section 4 is devoted to the DMS associated with (2.2) (DMS-BS for short) and the free boundary of the DMS-BS. In Subsection 4.1 we derive some estimates for the DMS-BS. Subsection 4.2 is devoted to the existence and uniqueness of the free boundary of the DMS-BS. In Subsection 4.3, we give a proof of Theorem 4.5 in subsection 4.1, an estimate of the difference of the DMS-BS. In Section 5 we prove our main results. Section 6 is the Appendix. In Subsection 6.1 we discuss the formal asymptotic expansion of an ODE related to (4.3) below. This expansion is used to construct sub- and supersolutions of (2.4) and (4.3). In Subsection 6.2 we give an estimate for some coefficients appearing in the estimate of Theorem 4.6.

In the following of this paper, we denote by \( C \) various constants depending only on known ones. The value of \( C \) may vary from line to line.

2 Approximation scheme and Main Results

In this section we state the approximation scheme by [29] and our main results.

We reformulate (1.1) in the following way. Put \( t := T - \tau \), \( x := \log(S/K) \), \( \alpha := (r - q - 1)/2 \) and \( U(t, x) := S^\alpha C(\tau, S)/K^{\alpha+1} \). Then (1.1) turns to

\[
\begin{align*}
\begin{cases}
\min \{ U_t - U_{xx} + \beta U, U - \varphi \} = 0 & \text{in } (0, T) \times \mathbb{R}, \\
U(0, x) = \varphi(x) := e^{\alpha x} \max(e^x - 1, 0) & \text{for } x \in \mathbb{R}, \\
U(t, x) \to 0 & (x \to -\infty) \text{ for } t \in (0, T),
\end{cases}
\end{align*}
\]

where \( \beta := \alpha^2 + r \). From the viewpoint of the numerical analysis, we had better restrict the problem (2.1) on a bounded interval with respect to \( x \). This restriction seems to be reasonable. Because the free boundary \( \{ \log(S(T - t)/K) \}_{0 < t < T} \) for (2.1) is bounded for each \( T > 0 \) and it is easily seen that

\[
U(t, x) = O(e^{\gamma x}) \quad \text{as } x \to -\infty \quad \text{for all } t \in (0, T) \text{ and some } \gamma > 0.
\]

Hence, putting \( \Omega := (-1, 1) \), we consider the following problem instead of (2.1):

\[
\begin{align*}
\begin{cases}
\min \{ u_t - u_{xx} + \beta u, u - \varphi \} = 0 & \text{in } (0, T) \times \Omega, \\
u(0, x) = \varphi(x) & \text{for } x \in \Omega, \\
u(t, \pm 1) = \varphi(\pm 1) & \text{for } t \in (0, T).
\end{cases}
\end{align*}
\]

We assume \( q < r < qe \) and denote by \( \{ x^*(t) \}_{0 < t < T} \) the free boundary for (2.2). Note by [34] that

\[
x^* \in C^1(0, T), \quad (x^*)'(t) \geq 0, \quad \lim_{t \downarrow 0} x^*(t) = x_0 := \log \left( \frac{r}{q} \right) (\in (0, 1)).
\]
The approximation scheme by [29] is stated as follows. Fix a time step \( h > 0 \). Put \( u_0 := \varphi \) and let \( [r] \) be the Gauss symbol for \( r \in \mathbb{R} \). For \( m = 1, 2, \ldots, [T/h] \), we consider the minimization problem of the following functional:

\[
J_m(u) := \frac{1}{2} \int_{\Omega} \left\{ \frac{|u - u_{m-1}|^2}{h} + (u_x)^2 + \beta u^2 \right\} \, dx \quad \text{for } u \in K,
\]

\( K := \{ v \in H^1(\Omega) \mid v - \varphi \in H^1_0(\Omega), \ v \geq \varphi \ \text{a.e. in } \Omega \} \).

We observe by the direct method of calculus of variation that there is a unique minimizer \( u_m \in K \) of \( J_m \). Moreover, \( u_m \) satisfies the elliptic variational inequality:

\[
(2.4) \quad \min \left\{ \frac{u_m - u_{m-1}}{h} - u_{m,xx} + \beta u_m, u_m - \varphi \right\} = 0 \quad \text{in } \Omega, \ u_m(\pm 1) = \varphi(\pm 1).
\]

We call the sequence \( \{u_m\}_{n=0}^{[T/h]} \) the DMS-BS. In addition, there is a unique free boundary \( \{x_m\}_{n=0}^{[T/h]} \) to the DMS-BS, as will be shown in Subsection 4.2 below.

Under these settings, we define \( u^h(t, x) \) and \( x^h(t) \) by

\[
(2.5) \quad u^h(t, x) := u_m(x), \ x^h(t) := x_m
\]

for \( t \in [mh, (m + 1)h), \ x \in \overline{\Omega} \) and \( m = 0, 1, \ldots, [T/h] \).

Then our main results are stated as follows.

**Theorem 2.1** Assume \( q < r < qe \). Then for any \( \delta > 0 \), there exist \( K > 0 \) and \( h_0 > 0 \) such that for all \( h \in (0, h_0) \),

\[
\sup_{(t,x) \in [0,T-\delta] \times \overline{\Omega}} |u^h(t, x) - u(t, x)| \leq K \sqrt{h} \log h.
\]

**Theorem 2.2** Assume \( q < r < qe \) and \( \{x^*(t)\}_{0 < t < T} \subset \Omega \). Then for any \( \delta > 0 \), we have

\[
\lim_{h \to 0} \sup_{t \in [0,T-\delta]} |x^h(t) - x^*(t)| = 0.
\]

**Remark 2.1** (1) The assumptions \( r < qe \) and \( \{x^*(t)\}_{0 < t < T} \subset \Omega \) are technical ones. Since \( \{x^*(t)\}_{0 < t < T} \) is bounded for each \( T > 0 \) (cf. [34]), replacing \( \Omega \) with a larger interval such that \( \{x^*(t)\}_{0 < t < T} \subset \Omega \), we can show Theorems 2.1 and 2.2 assuming only \( r > q \).

(2) The \( |\log h| \) term appear in Theorem 2.1 by some technical reasons.

### 3 Solutions of the problem (2.2)

This section consists of two subsections. In Subsection 3.1, we consider the existence and uniqueness of solutions of (2.2). In Subsection 3.2, we obtain some regularity of solutions of (2.2). To establish the results in these subsections, we use the penalized problem for (2.2):

\[
(3.1) \quad \begin{cases}
  u_\varepsilon - u_\varepsilon_{xx} + \beta u_\varepsilon + \zeta_\varepsilon(u_\varepsilon - \varphi) = 0 & \text{in } (0,T) \times \Omega, \\
  u_\varepsilon(0, x) = \varphi(x) & \text{for } x \in \overline{\Omega}, \\
  u_\varepsilon(t, \pm 1) = \varphi(\pm 1) & \text{for } t \in (0,T),
\end{cases}
\]

where \( \varepsilon > 0 \), \( \zeta_\varepsilon(r) := \zeta(r/\varepsilon) \) and \( \zeta \) is a smooth function such that \( \zeta' \geq 0, \zeta'' \leq 0 \) on \( \mathbb{R} \),
3.1 Existence and uniqueness of solutions

In this subsection we prove the following theorem.

**Theorem 3.1** There exists a unique solution \( u \) of (2.2) in the a.e. sense and in the sense of viscosity solutions such that \( u \in W^{1,2,2}((0, T) \times \Omega) \cap C([0, T) \times \overline{\Omega}) \).

**Remark 3.1** See [8] or [17] for the definition and the theory of viscosity solutions.

By [5, Chapter 2, 3], there is a unique solution \( u^\varepsilon \) of (3.1) in the sense that \( u^\varepsilon - \varphi \in L^2(0, T; H^1_0(\Omega)) \), \( u^\varepsilon_t \in L^2(0, T; L^2(\Omega)) \) and

\[
\int_{\Omega} \{ u^\varepsilon_t \phi + u^\varepsilon_x \phi_x + \beta u^\varepsilon \phi + \zeta(u^\varepsilon - \varphi) \phi \} \, dx = 0 \quad \text{for all } \phi \in H^1_0(\Omega) \text{ and a.e. } t \in (0, T).
\]

In addition, \( u^\varepsilon \in L^2(0, T; H^2(\Omega)) \cap C([0, T); C(\overline{\Omega})) \).

We derive some estimates of \( u^\varepsilon \). By the maximum principle, we get

\[
(3.2) \quad \sup_{\varepsilon > 0} \| u^\varepsilon \|_{C([0,T) \times \overline{\Omega})} \leq \| \varphi \|_{C(\overline{\Omega})}.
\]

By the same arguments as in [5, Chapter 2, Section 2.4] we have

\[
(3.3) \quad \sup_{\varepsilon > 0} (\| u^\varepsilon \|_{L^\infty(0,T;H^1(\Omega))} + \| u^\varepsilon_t \|_{L^2(0,T;L^2(\Omega))}) < +\infty.
\]

To estimate \( \zeta(u^\varepsilon - \varphi) \) and \( u^\varepsilon_{xx} \), we need the following lemma.

**Lemma 3.1** There is \( M_1 > 0 \) such that \( u^\varepsilon \geq \varphi - M_1 \varepsilon \in [0, T) \times \overline{\Omega} \) for all \( \varepsilon > 0 \).

**Proof.** Since \( \varphi \) is Lipschitz continuous and convex in \( \Omega \), we can show that

\[
(3.4) \quad \int_{\Omega} \varphi_x(x) \phi_x(x) \, dx \leq 0 \quad \text{for all } \phi \in H^1_0(\Omega) \text{ satisfying } \phi \geq 0 \text{ in } \Omega.
\]

Set \( \underline{u}(t, x) := \varphi(x) - M_1 \varepsilon \). Then, we use the above inequality to obtain

\[
\int_{\Omega} \{ \underline{u} \phi + \underline{u}_x \phi_x + \beta \underline{u} \phi + \zeta(\underline{u} - \varphi) \phi \} \, dx \leq \int_{\Omega} \{ \beta \varphi + \zeta(-M_1) \} \phi \, dx
\]

for any \( t \in (0, T) \) and \( \phi \in H^1_0(\Omega) \) satisfying \( \phi \geq 0 \) in \( \Omega \). Taking \( M_1 > 0 \) such that \( \beta \| \varphi \|_{C(\overline{\Omega})} + \zeta(-M_1) \leq 0 \), we easily see that \( \underline{u} \) is a weak subsolution of (3.1). Hence we have \( u \leq u^\varepsilon \) in \( [0, T) \times \Omega \) by the maximum principle. Therefore we obtain the result. \( \square \)

Hence from (3.3) and Lemma 3.1, we get

\[
(3.5) \quad \sup_{\varepsilon > 0} \| \zeta(u^\varepsilon - \varphi) \|_{C([0,T) \times \overline{\Omega})} < +\infty, \quad \sup_{\varepsilon > 0} \| u^\varepsilon_{xx} \|_{L^2([0,T) \times \Omega)} < +\infty.
\]

**Proof of Theorem 3.1.** By (3.2), (3.3), (3.5) and Sobolev imbedding, we can extract a subsequence \( \{ \varepsilon_n \}_{n=1}^{+\infty} \) such that for any \( T' < (0, T) \) and \( \lambda \in (0, 1/2) \), as \( n \to +\infty \),

\[
(3.6) \quad u^{\varepsilon_n} \rightharpoonup u \quad \text{in } C^{\lambda,2\lambda}([0,T'] \times \overline{\Omega}),
\]

\[
(3.7) \quad (u^{\varepsilon_n}, u^{\varepsilon_n}_x, u^{\varepsilon_n}_{xx}) \rightarrow (u_t, u_x, u_{xx}) \quad \text{weakly in } L^2((0,T) \times \Omega)^3.
\]

We can see that \( u \) is a unique solution of (2.2) in the a.e. sense and in the viscosity sense (cf. [5] Chapter 3, [8] and [17]). Thus we complete the proof. \( \square \)
3.2 Some properties of solutions

The main results of this subsection are stated as follows. Let $u$ be the solution of (2.2).

**Theorem 3.2** Assume $q < r < qe$ and let $x_0$ be given in (2.3). Then $u \in W^{1,2,\infty}_{loc}((0,T) \times \Omega)$ and it satisfies the following estimates.

1. For any small $x_1 > 0$, there is $L_1 > 0$ such that

$$\|u(t,\cdot)\|_{L^\infty(-x_1,x_1)} \leq \frac{L_1}{\sqrt{t}}, \quad \|u(t,\cdot)\|_{L^\infty(\Omega\setminus(-x_1,x_1))} \leq L_1 \quad \text{for a.e. } t \in (0,T),$$

$$\|u_{xx}(t,\cdot)\|_{L^\infty(-x_1,x_1)} \leq \frac{L_1}{\sqrt{t}}, \quad \|u_{xx}(t,\cdot)\|_{L^\infty(\Omega\setminus(-x_1,x_1))} \leq L_1 \quad \text{for a.e. } t \in (0,T).$$

2. There is $L_2 > 0$ such that

$$\|u(t,x) - u(s,y)\| \leq L_2(|t-s|^{1/2} + |x-y|) \quad \text{for all } (t,x), (s,y) \in [0,T] \times \Omega.$$

**Theorem 3.3** The $u_t$ is nonnegative and continuous in $(0,T) \times \Omega$.

**Remark 3.2** (1) In [13], similar results to Theorem 3.2 are obtained in the case $\Omega = \mathbb{R}^N$.

(2) Theorem 3.3 is similar to [33, Lemma 5] and [9, Corollary 4.2]. As seen in Section 5, it plays an important role to prove Theorem 2.1.

We prepare some pointwise estimates of solutions of (3.1) to prove Theorem 3.2. Let $u^\varepsilon$ be the solution of (3.1).

**Proposition 3.1** We obtain

$$\sup_{\varepsilon > 0} \|u^\varepsilon\|_{L^\infty((0,T) \times \Omega)} < +\infty.$$  

**Proof.** The barrier construction argument yields that $|u^\varepsilon(t,\pm 1)| \leq \|\varphi\|_{L^\infty(\Omega)}$ for all $t \in [0,T)$ and $\varepsilon > 0$. We obtain the result by combining the comparison argument for viscosity solutions (cf. [12, Section 7]) with Lipschitz continuity of $\varphi$ and this estimate. $\square$

**Lemma 3.2** Assume $q < r < qe$ and let $x_0$ be defined by (2.3). Then there exist $x_2 \in (0,x_0)$ such that $u^\varepsilon(t,x) > \varphi(x)$ for all $t \in (0,T)$, $x \in (-1,x_2)$ and $\varepsilon > 0$.

We can formally show this lemma, according to [33, p.124]. Let $x_0$ be defined in (2.3) and assume $x_0 \in (0,1)$. It is seen that for small $\varepsilon > 0$,

$$u^\varepsilon(t,x) \rightarrow u^\varepsilon(0,x_0) = \varphi_{xx}(x) - \beta\varphi(x) \quad \text{for all } x \in (0,1) \text{ as } t \searrow 0.$$

It follows from the definition of $\varphi$ that

$$\varphi_{xx}(x) - \beta\varphi(x) = e^{\alpha x}(-qe^x + r) \begin{cases} > 0 & \text{if } x \in (0,x_0), \\ = 0 & \text{if } x = x_0, \\ < 0 & \text{if } x \in (x_0,1] \end{cases}$$
Thus Lemma 3.2 formally holds with $x_1 = x_0$.

**Proof of Lemma 3.2.** Step 1. Let $u$ be the classical solution of

$$
\begin{cases}
    u_t - uu_{xx} + \beta u = 0 & \text{in } (0, +\infty) \times \Omega, \\
    u(0, x) = \varphi(x) & \text{for } x \in \Omega, \\
    u(t, \pm1) = \varphi(\pm1) & \text{for } t \in (0, +\infty).
\end{cases}
$$

(3.11)

We prove that

$$
u_t(t, x) \geq u(t, x) > \varphi(x) \quad \text{for all } t > 0, \ x \in (-1, 0] \text{ and } \varepsilon > 0.
$$

(3.12)

Since $u$ is a subsolution of (3.1), it follows from the maximum principle that $u \leq u^\varepsilon$ in $[0, T] \times \Omega$. Hence we get (3.12) by $u > 0$ in $(0, +\infty) \times \Omega$ and $\varphi \equiv 0$ on $[-1, 0]$.

Step 2. We show that there exist $t_1, M_2 > 0$ such that

$$
u(t, x) \geq \varphi(x) + M_2 t + \int_0^t e^{-\beta s - x^2/4s} ds \quad \text{for all } t \in (0, t_1] \text{ and } x \in [0, 2x_0/3].
$$

(3.13)

For $a > 0$, define

$$
E^a(t, x, y) := \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{+\infty} \left\{ e^{-(x-y+4an)^2/4t} - e^{-(x+y+4an+2a)^2/4t} \right\},
$$

(3.14)

$$
E_0(t, x, y) := \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}, \quad E^a_1(t, x, y) := E^a(t, x, y) - E_0(t, x, y).
$$

(3.15)

Put $E = E^1$ and $E_1 = E^1_1$ for simplicity. Then $u$ is given by

$$
u(t, x) = e^{-\beta t} \int_0^1 E(t, x, y) \varphi(y) dy - \varphi(1) \int_0^t e^{-\beta(t-s)} E_0(t-s, x, 1) ds.
$$

Differentiating this formula with respect to $t$, we have

$$
u(t, x) = e^{-\beta t} \int_0^1 E(t, x, y) \varphi(y) dy - \beta e^{-\beta t} \int_0^1 E(t, x, y) \varphi(y) dy - e^{-\beta t} \varphi(1) E_0(t, x, 1).
$$

We use the facts $E_t = E_{xx} = E_{yy}$, $\varphi(0) = 0$ and the integration by parts to obtain

$$
u(t, x) = e^{-\beta t} \int_0^1 E(t, x, y) \varphi_{yy}(y) - \beta \varphi(y) dy + e^{-\beta t} E(t, x, 0) \varphi_{y}(0)
=: I_{1,1} + I_{1,2}.
$$

We estimate the right-hand side (RHS for short) of the above formula to have (3.13). Some calculations yield that for small $t > 0$, $x \in [-2x_0/3, 2x_0/3]$ and $y \in (0, 1),

$$
|E(t, x, y) - E_0(t, x, y)| \leq C e^{-x_0^2/64t}.
$$

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We observe from $q < r < qe$ and this estimate that

$$I_{1,1} \geq \left( \int_0^{3x_0/4} + \int_{3x_0/4}^1 \right) E_0(t, x, y) e^{-\beta t + o_y(-q(y - r))dy} - Ce^{-x_0^2/64t}$$

$$\geq \frac{e^{-\beta - |\alpha|}}{\sqrt{4\pi t}} \int_0^{3x_0/4} e^{-(y-x)^2/4t}(-q(y) + r)dy - Ce^{-x_0^2/64t} \left( \frac{1}{\sqrt{4\pi t}} + 1 \right)$$

for small $t > 0$ and $x \in [-2x_0/3, 2x_0/3]$. By (3.10) we have $-q(y) + r \geq M_{2,1}$ for all $y \in [0, 3x_0/4]$ and some $M_{2,1} > 0$. Thus

$$I_{1,1} \geq M_{2,1} \quad \text{for small } t > 0, \ x \in [-2x_0/3, 2x_0/3] \text{ and some } M_2 > 0.$$

Since it is seen that $I_{1,2} \geq e^{-\beta - x^2/4t} / \sqrt{8\pi t}$ for small $t > 0$ and all $x \in \Omega$, we obtain

$$u(t, x) \geq M_2 + \frac{e^{-\beta - x^2/4t}}{\sqrt{8\pi t}} \quad \text{for small } t > 0, x \in [0, 2x_0/3].$$

Therefore, for sufficiently small $t_1 > 0$, we have (3.13) by integrating both sides of this inequality on $[0, t]$ for all $t \in (0, t_1)$.

Step 3. Set $\overline{\tau}(t) := \sup \{y \in [0, 1] \mid u(t, x) \geq \varphi(x) \text{ for all } x \in [0, y) \}$ for each $t \in [t_1, T]$ and define $x_{2,1} := \inf_{t \in [t_1, T]} \overline{\tau}(t)$. We claim $x_{2,1} > 0$.

Suppose $x_{2,1} = 0$. Then for each $n \in \mathbb{N}$, there exists $t_n \in [t_1, T]$ such that $\overline{\tau}(t_n) \leq 1/n$. Extracting a subsequence if necessary, we may assume $t_n \rightarrow t \in [t_1, T]$ as $n \rightarrow +\infty$. Noting that $u(t, \overline{\tau}(t)) = \varphi(\overline{\tau}(t))$, we easily see that

$$\tilde{u}(t, 0) = \lim_{n \rightarrow +\infty} u(t_n, \overline{\tau}(t_n)) = \lim_{n \rightarrow +\infty} \varphi(\overline{\tau}(t_n)) = \varphi(0) = 0.$$ 

This contradicts to (3.12). Hence the claim of this step is proved.

Putting $x_2 := \min \{2x_0/3, x_{2,1} \}$, we obtain the desired result. \(\square\)

**Remark 3.3** It readily follows from Lemma 3.2 that $\zeta_\varepsilon(u^\varepsilon - \varphi) \equiv 0$ in $[0, T) \times (-x_2, x_2)$ for all $\varepsilon > 0$. Hence the boot-strap argument yields that $u^\varepsilon \in C^\infty((0, T) \times \Omega)$.

Based on (3.5) and Lemma 3.2 we prove the following theorem.

**Theorem 3.4** Assume $q < r < qe$. Let $x_0$ be given in (2.2) and $x_2 \in (0, x_0)$ in Lemma 3.2. Then, for each $x_3 \in (0, x_2)$, there exists $L_3 > 0$ such that for any $\varepsilon > 0$,

$$(3.16) \quad \|u^\varepsilon(t, \cdot)\|_{L^\infty(-x_3, x_3)} \leq \frac{L_3}{\sqrt{t}}. \quad \|u^\varepsilon(t, \cdot)\|_{L^\infty(\Omega \setminus (-x_3, x_3))} \leq L_3 \quad \text{for all } t \in (0, T),$$

$$(3.17) \quad \|u^\varepsilon_{xx}(t, \cdot)\|_{L^\infty(-x_3, x_3)} \leq \frac{L_3}{\sqrt{t}}. \quad \|u^\varepsilon_{xx}(t, \cdot)\|_{L^\infty(\Omega \setminus (-x_3, x_3))} \leq L_3 \quad \text{for all } t \in (0, T).$$

**Proof.** The $u^\varepsilon$ is given by

$$u^\varepsilon(t, x) = \overline{u}(t, x) - \int_0^t \int_{\Omega} e^{-\beta(t-s)} E(t-s, x, y) \zeta_\varepsilon(u^\varepsilon(s, y) - \varphi(y))dyds,$$
where $u$ is the solution of \((3.11)\) and $E$ is defined by \((3.14)\). Fix $x_3 \in (0, x_2/2)$. We divide our consideration into three cases.

**Case 1.** $|x| < x_3$.

Differentiating $u^\varepsilon$ with respect to $t$, we get

\[
(3.18) \quad u^\varepsilon_t(t, x) = u(t, x) - \zeta_\varepsilon(u^\varepsilon(t, x) - \varphi(x)) \nonumber \\
- \int_0^t \int_{\Omega} e^{-\beta(t-s)} E(t-s, x, y) \zeta_\varepsilon(u^\varepsilon(s, y) - \varphi(y)) dy ds \\
+ \beta \int_0^t \int_{\Omega} e^{-\beta(t-s)} E(t-s, x, y) \zeta_\varepsilon(u^\varepsilon(s, y) - \varphi(y)) dy ds \\
=: I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}.
\]

It follows from the standard theory for parabolic equations that $|I_{2,1}| \leq C/\sqrt{t}$ for all $t \in (0, T)$ and $x \in \Omega$. Besides, from \((3.5)\) we get $|I_{2,2} + I_{2,4}| \leq C$ for all $t \in (0, T)$ and $x \in \Omega$. As for $I_{2,3}$, noting that $(x-y)^2 \geq x_3^2$ for all $x \in [-x_3, x_3]$ and $y \in [2x_3, 1)$, we observe by \((3.5)\) and Lemma \(3.2\) that

\[
|I_{2,3}| \leq C \left( \left\| \int_0^t \int_{x_3}^{1} E(t-s, x, y) dy \right\| + \left\| \int_0^t \int_{x_3}^{1} E(t-s, x, y) dy \right\| \right) \\
\leq C \left\{ \int_0^t (t-s)^{-3/2} e^{-x_3^2/(4(t-s))} ds + 1 \right\} \leq M_{3,1}.
\]

for all $t \in (0, T)$, $x \in (-x_3, x_3)$ and $\varepsilon > 0$. Here and in the sequel, $M_{3,i}$'s $(i \geq 1)$ are constants depending on $x_3$. Consequently we obtain

\[
\|u^\varepsilon_t(t, \cdot)\|_{L^\infty(-x_3, x_3)} \leq \frac{M_{3,2}}{\sqrt{t}} \quad \text{for all } t \in (0, T) \text{ and small } \varepsilon > 0.
\]

**Case 2.** $x_3 \leq |x| \leq 1$.

Assume that $x_3 \leq x \leq 1$. By using \((3.18)\) and \((3.5)\), it is seen that

\[
|u^\varepsilon_t(t, x_3)| \leq |u(t, x_3)| + C \left\| \int_0^t \int_{x_3}^{1} E(t-s, x, y) dy ds \right\| + C \\
=: I_{3,1} + I_{3,2} + C \quad \text{for all } t \in (0, T) \text{ and small } \varepsilon > 0.
\]

We estimate $I_{3,1}$ and $I_{3,2}$. We observe by the integration by parts that

\[
|I_{3,1}| \leq \left\| \int_0^{x_3/2} E_y(t, x_3, y) \varphi_y(y) dy \right\| + \beta \|\varphi\|_{L^\infty(\Omega)} \\
= \left\| \int_0^{x_3/2} E_y(t, x_3, y) \varphi_y(y) dy \right\| + \left\| \int_{x_3/2}^1 E_y(t, x_3, y) \varphi_y(y) dy \right\| + C \\
=: I_{3,1,1} + I_{3,1,2}.
\]

Using $(x_3-y)^2 \geq x_3^2/4$ for all $y \in (0, x_3/2]$, we can estimate $|I_{3,1,1}| \leq C t^{-3/2} e^{-x_3^2/16t} \leq M_{3,3}$ for all $t \in (0, T)$ and $\varepsilon > 0$. Since $\varphi$ is smooth in $(0, 1)$, it follows from the integration
by parts and the fact \( |E(t, x_3, x_3/2)| + |E(t, x_3, 1)| \leq Ct^{-1/2}e^{-x_3^2/16t} \) that for all \( t \in (0, T) \) and \( \varepsilon > 0 \),

\[ I_{3,1,2} \leq |E(t, x_3, 1)\varphi_y(1) - E(t, x_3, x_3/2)\varphi_y(x_3/2)| + \left| \int_{x_3/2}^t E(t, x_3, y)\varphi_y(y)dy \right| \leq M_{3,4}. \]

Hence \( |I_{3,1}| \leq M_{3,5} \) for some \( M_{3,5} > 0 \). On the other hand, it is observed by the same argument as the estimate for \( I_{2,3} \) in Case 1 that

\[ |I_{3,2}| \leq C \int_0^t (t-s)^{-3/2}e^{-x_3^2/16(t-s)}ds \leq M_{3,6} \quad \text{for all } t \in (0, T) \) and \( \varepsilon > 0 \).

Therefore we conclude that \( |u^\varepsilon(t, x_3)| \leq M_{3,7} \) for all \( t \in (0, T) \) and \( \varepsilon > 0 \).

We provide an estimate for \( \|u^\varepsilon(t, \cdot)\|_{L^\infty(x_3, 1)} \). Differentiating the equation of \((3.1)\) with respect to \( t \), we have

\[
egin{align*}
U^\varepsilon_t - U^\varepsilon_x + \beta U^\varepsilon + \zeta^\varepsilon(u^\varepsilon - \varphi)U^\varepsilon &= 0 \quad \text{in } (0, T) \times (x_3, 1), \\
U^\varepsilon(0, x) &= \varphi_x(x) - \beta \varphi(x) \quad \text{for } x \in (x_3, 1), \\
|U^\varepsilon(t, x_3)| &\leq M_{3,7}, \quad U^\varepsilon(t, 1) = 0 \quad \text{for } t \in (0, T),
\end{align*}
\]

where \( U^\varepsilon := u^\varepsilon_t \). Noting that \( |U^\varepsilon(0, \cdot)| \leq C \) on \( [x_3, 1] \) and that \( \zeta^\varepsilon \geq 0 \), we obtain from the maximum principle \( \|u^\varepsilon_t\|_{L^\infty(x_3, 1)} \leq M_{3,8} \) for all \( t \in (0, T), \) small \( \varepsilon > 0 \).

We can get \( \|u^\varepsilon(t, \cdot)\|_{L^\infty(-\varepsilon, x_3)} \leq M_{3,9} \) for all \( t \in (0, T) \) and small \( \varepsilon > 0 \) by the same way as above.

Consequently, for each \( x_3 \in (0, x_2/2) \), there is \( L_3 \geq \max\{M_{3,2}, M_{3,8}, M_{3,9}\} \) such that \((3.16)\) holds for all \( t \in (0, T) \). The \((3.17)\) follows from \((3.2), (3.5)\) and \((3.16)\).

\section*{Proof of Theorem 3.2}

Proposition 3.1 and Theorem 3.3 yield that there is a subsequence \( \{\varepsilon_n\}_{n=1}^{\infty}, \varepsilon_n \searrow 0 \), such that as \( n \to +\infty \),

\[
(u_{x_{n\prime}}^\varepsilon, \sqrt{t}u_{tt}^\varepsilon, \sqrt{t}u_{xx}^\varepsilon) \to (U_1, U_2, U_3) \quad \text{weakly star in } (L^\infty((0, T) \times \Omega))^3.
\]

By \( L^2((0, T) \times \Omega) \subset L^1(((0, T) \times \Omega) \), we can use \((3.7)\) to have \((U_1, U_2, U_3) = (u_x, \sqrt{t}u_t, \sqrt{t}u_{xx}). \)

Hence the \( u \in W^{1,2}_{loc}((0, T) \times \Omega) \) follows from \((3.2)\) and these convergences.

Set \( x_1 \in (0, x_3) \). The \((3.8)\) and \((3.9)\) are derived from the above convergences and Theorem 3.4. The assertion of \((2)\) follows from \((3.6)\), Theorem 3.3 and Proposition 3.1.

We prepare some estimates for \( u_{x\ell} \) and \( u_{tt} \) to show Theorem 3.3.

\section*{Proposition 3.2}

There exists \( C > 0 \) such that for any small \( \sigma > 0 \),

\[
\sup_{t \in (\sigma, T)} \|u_{x\ell}(t, \cdot)\|_{L^2(\Omega)} + \|u_{tt}\|_{L^2((\sigma, T) \times \Omega)} \leq \frac{C}{\sigma}.
\]

\section*{Proof.}

Let \( u^\varepsilon \) be the solution of \((3.1)\). Then \( U^\varepsilon := u^\varepsilon_t \) satisfies

\[
\begin{cases}
U^\varepsilon_t - U^\varepsilon_x + \beta U^\varepsilon + \zeta^\varepsilon(u^\varepsilon - \varphi)U^\varepsilon = 0 & \text{in } (\sigma, T) \times \Omega, \\
|U^\varepsilon(\sigma, x)| \leq \frac{C}{\sqrt{\sigma}} & \text{for } x \in \overline{\Omega}, \\
U^\varepsilon(t, \pm 1) = 0 & \text{for } t \in (\sigma, T).
\end{cases}
\]

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Here the second inequality follows from (3.8). The same argument as in the proof of [10 Lemma 3.4] yields that
\[
\sup_{t \in (\sigma, T)} \|u^\varepsilon_{xt}(t, \cdot)\|_{L^2(\Omega)} + \|u^\varepsilon_{tt}\|_{L^2((\sigma, T) \times \Omega)} \leq \frac{C}{\sigma}.
\]
Sending \(\varepsilon \to 0\), we have the result. \(\square\)

**Proof of Theorem 3.3.** Step 1. We claim that \(u_t\) is continuous in \((0, T) \times \Omega\).

We observe from the regularity theory for parabolic equations that \(u_t\) is continuous in \(\{(t, x) \mid 0 < t < T, u(t, x) > \varphi(x)\}\). It is obvious that \(u_t\) is so in \(\{(t, x) \mid 0 < t < T, x^*(t) < x < 1\}\). The continuity of \(u_t\) in \(\{(t, x^*(t)) \mid 0 < t < T\}\) can be proved by Proposition 3.2 and the same argument as the proof of [9, Corollary 4.2]. Hence we have the claim.

Step 2. We show \(u_t \geq 0\) in \((0, T) \times \Omega\).

We modify (3.1) as follows. Let \(\{\varphi_\delta\}_{\delta > 0}\) be a sequence of \(C^2\) and convex functions satisfying \(\|\varphi_\delta - \varphi\|_{W^{1, \infty}(\Omega)} \to 0\) as \(\delta \to 0\). We consider the following instead of (3.1).

\[
\begin{align*}
\begin{cases}
u^\delta_{\varepsilon} - \nu^\delta_{xx} + \beta u^\delta_{\varepsilon} + \zeta_\varepsilon(u^\delta_{\varepsilon} - \varphi_\delta - M_4\varepsilon) &= 0 \quad \text{in } (0, T) \times \Omega, \\
u^\delta_{\varepsilon}(0, x) &= \varphi_\delta(x) \quad \text{for } x \in \Omega, \\
u^\delta_{\varepsilon}(t, \pm 1) &= \varphi_\delta(\pm 1) \quad \text{for } t \in (0, T).
\end{cases}
\end{align*}
\]

Here \(M_4 > 0\) is chosen so that \(\beta \sup_{\delta \in [0, 1]} \|\varphi_\delta\|_{L^\infty(\Omega)} + \zeta(-M_4) \leq 0\). Then, there is a unique classical solution \(\nu^\delta_{\varepsilon}\) of (3.19) and it satisfies
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \nu^\delta_{\varepsilon} = u \quad \text{locally uniformly in } [0, T) \times \Omega.
\]
We differentiate (3.19) with respect to \(t\) and set \(U^\delta_{\varepsilon} := u^\delta_{\varepsilon}\). Then we have
\[
\begin{align*}
\begin{cases}
u^\delta_{\varepsilon} - \nu^\delta_{xx} + \beta u^\delta_{\varepsilon} + \zeta_\varepsilon(u^\delta_{\varepsilon} - \varphi_\delta - M_4\varepsilon)U^\delta_{\varepsilon} &= 0 \quad \text{in } (0, T) \times \Omega, \\
u^\delta_{\varepsilon}(0, x) &= \varphi_\delta xx - \beta \varphi_\delta - \zeta(-M_4) \quad \text{for } x \in \Omega, \\
u^\delta_{\varepsilon}(t, \pm 1) &= 0 \quad \text{for } t \in (0, T).
\end{cases}
\end{align*}
\]
The \(U^\delta_{\varepsilon}(0, \cdot) \geq 0\) on \(\Omega\) follows from \(\varphi_\delta xx(x) \geq 0\) and the choice of \(M_4\). Hence we apply the maximum principle to obtain \(U^\delta_{\varepsilon} \geq 0\) in \([0, T) \times \Omega\). The (3.20) and this result yield that \(u(\cdot, x)\) is nondecreasing for each \(x \in \Omega\). Hence we have the result. \(\square\)

4 The discrete Morse semiflow

Fix \(h > 0\). As briefly mentioned in Section 2, for each \(m = 1, 2, \ldots, [T/h]\), there is a unique minimizer \(u_m \in K\) of the functional \(J_m\). Moreover, \(u_m\) satisfies (2.4) in the weak sense (cf. [5, Chapter 3]). We call the sequence \(\{u_m\}_{m=0}^{[T/h]}\) the DMS-BS. This section is devoted to some properties of the DMS-BS and to its free boundary.

In Subsection 4.1, we discuss some properties of the DMS-BS. To prove Theorem 4.3 of this subsection, we need an estimate of the difference \(u_m - u_{m-1}\), Theorem 4.5. Since its proof consists of lengthy and careful calculations, it is given in Subsection 4.3. In Subsection 4.2, we consider the existence and uniqueness of the free boundary of the DMS-BS.
4.1 Some properties of the DMS-BS

First, we have the monotone property of the DMS-BS.

**Theorem 4.1** Let \( \{u_m\}_{m=0}^{[T/h]} \) be the DMS-BS. Then \( u_{m-1} \leq u_m \) on \( \Omega \) for all \( m = 1, 2, \ldots, [T/h] \) and \( h > 0 \).

This theorem can be easily proved by the maximum principle and induction. Hence we omit the proof.

In the following part of this subsection, we show the time-discrete analogues to Theorems 3.1 and 3.2.

**Theorem 4.2** For each \( h > 0 \) and \( m = 1, 2, \ldots, [T/h] \), \( u_m \) is a unique solution of (2.4) in the a.e. sense and in the viscosity sense. In addition, \( \{u_m\}_{m=0}^{[T/h]} \) satisfies

\[
\sup_{h>0} \left( h \sum_{m=1}^{[T/h]} \frac{u_m - u_{m-1}}{h} \right)^2 + \sup_{0 \leq m \leq [T/h]} \|u_m\|_{H^1(\Omega)} + h \sum_{m=1}^{[T/h]} \|u_{m,xx}\|_{L^2(\Omega)} \right) < +\infty.
\]

**Theorem 4.3** Assume \( q < r < q_e \) and let \( x_0 \) be defined in (2.5). Then there exist \( x_4 \in (0, x_0) \) and \( h_1 > 0 \) such that for each \( h \in (0, h_1) \), \( \{u_m\}_{m=0}^{[T/h]} \subset W^{2,\infty}(\Omega) \) and it satisfies the following estimates.

1. There are \( L_4, L_5 > 0 \) depending on \( x_4 \) such that for each \( h \in (0, h_1) \) and \( m = 1, 2, \ldots, [T/h] \),

\[
\frac{|u_m(x) - u_{m-1}(x)|}{h} \leq \begin{cases} 
L_4 \max \left\{ \frac{1}{\sqrt{mh}} \sqrt{\frac{|\log h|}{L_5}} \right\} & \text{for } |x| < x_4, \\
L_4 \sqrt{\frac{\log h}{L_5}} & \text{for } |x| \geq x_4.
\end{cases}
\]

2. There are \( L_6, L_7 > 0 \) depending on \( x_4 \) such that

\[
|u_m(x) - u_n(y)| \leq \begin{cases} 
L_6((m-n)h)^{1/2} + |x-y| & \text{if } m, n \leq L_7(h \log h)^{-1}, \\
L_6((m-n)h)\sqrt{|\log h| + |x-y|} & \text{if } m, n \geq L_7(h \log h)^{-1},
\end{cases}
\]

for all \( m, n = 0, 1, \ldots, [T/h], x, y \in \overline{\Omega} \) and \( h \in (0, h_1) \).

**Remark 4.1** The \( \sqrt{|\log h|} \) appears in Theorem 4.3 by some technical reasons.
To prove Theorems 4.2 and 4.3 we introduce the penalized problem to (2.4): Put $u_0^\varepsilon := \varphi$ and consider
\begin{equation}
(4.3) \quad \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} - u_{m,xx}^\varepsilon + \beta u_{m,x}^\varepsilon + \zeta_{\varepsilon}(u_{m}^\varepsilon - \varphi) = 0 \quad \text{in } \Omega, \ u_{m}^\varepsilon(\pm 1) = \varphi(\pm 1).
\end{equation}

Here $\zeta_{\varepsilon}$ is the same function as in Section 3.

We observe that for each $h > 0$ and $m = 1, 2, \ldots, [T/h]$, there uniquely exists a weak solution $u_m^\varepsilon$ of (4.3) in the sense that $u_m^\varepsilon - \varphi \in H^1_0(\Omega)$ and
\begin{equation}
(4.4) \quad \int_{\Omega} \left\{ \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} \phi + u_{m,x}^\varepsilon \phi_x + \beta u_m^\varepsilon \phi + \zeta_{\varepsilon}(u_{m}^\varepsilon - \varphi) \phi \right\} \, dx = 0 \quad \text{for all } \phi \in H^1_0(\Omega).
\end{equation}

In addition, the regularity theory for elliptic equations yields that $u_m^\varepsilon \in H^2(\Omega) \cap C^2(\Omega)$. Thus $u_m^\varepsilon$ is a classical solution of (4.3).

We derive some uniform estimates of $\{u_m^\varepsilon\}_{m,\varepsilon}$ to prove Theorem 4.2. We get from the maximum principle and induction
\begin{equation}
(4.5) \quad \sup_{h>0,\varepsilon>0} \left( \sup_{0\leq m\leq [T/h]} \|u_m^\varepsilon\|_{C(\overline{\Omega})} \right) \leq \|\varphi\|_{C(\overline{\Omega})}.
\end{equation}

By a similar argument to the proof of Lemma 3.1 and induction, we have

**Lemma 4.1** We have $u_m^\varepsilon \geq \varphi - M_1 \varepsilon$ on $\overline{\Omega}$ for all $\varepsilon > 0$, $m = 1, 2, \ldots, [T/h]$ and $h > 0$. Here $M_1$ is the same constant as in Lemma 3.1.

The following estimate is a time-discrete analogue to (3.3).

**Proposition 4.1** We have
\[ \sup_{h>0,\varepsilon>0} \left( \sum_{m=1}^{[T/h]} \left| \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} \right|_{L^2(\Omega)}^2 + \sup_{0\leq m\leq [T/h]} \|u_m^\varepsilon\|_{H^1(\Omega)} \right) < +\infty. \]

**Proof.** Put $\phi := (u_m^\varepsilon - u_{m-1}^\varepsilon)$ in (4.4). Using $\omega \tilde{\omega} \leq (\omega^2 + \tilde{\omega}^2)/2$ ($(\omega, \tilde{\omega}) = (u_m^\varepsilon, u_{m-1}^\varepsilon)$, $(u_{m,x}^\varepsilon, u_{m-1,x}^\varepsilon)$), we have
\[ \int_{\Omega} \left[ \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} \right]^2 + \frac{(u_{m,x}^\varepsilon)^2 - (u_{m-1,x}^\varepsilon)^2}{2} + \beta \frac{(u_m^\varepsilon)^2 - (u_{m-1}^\varepsilon)^2}{2} + \zeta_{\varepsilon}(u_m^\varepsilon - \varphi)(u_m^\varepsilon - u_{m-1}^\varepsilon) \right] \, dx \leq 0. \]

Since it is easily seen from Lemma 4.1 that
\[ \zeta_{\varepsilon}(u_m^\varepsilon - \varphi)(u_m^\varepsilon - u_{m-1}^\varepsilon) \geq -C|u_m^\varepsilon - u_{m-1}^\varepsilon| \geq - \frac{C h}{2} - \frac{h}{2} \left| \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} \right|^2, \]
we have
\[ \int_{\Omega} \left\{ \frac{h}{2} \left| \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} \right|^2 + \frac{(u_{m,x}^\varepsilon)^2 - (u_{m-1,x}^\varepsilon)^2}{2} + \frac{1}{2} (u_m^\varepsilon)^2 - (u_{m-1}^\varepsilon)^2 \right\} \, dx \leq Ch. \]
for all \( \varepsilon > 0, m = 1, 2, \ldots, [t/h] \) and \( h > 0 \). Summing up these inequalities from \( m = 1 \) to \( m = [t/h] \), we obtain
\[ h \sum_{m=1}^{[t/h]} \left\| \frac{u_m^\varepsilon - u_{m-1}^\varepsilon}{h} \right\|^2_{L^2(\Omega)} + \min\{1, \beta\} \| u_{t/k}^\varepsilon \|^2_{H^1(\Omega)} \leq \max\{1, \beta\} \| \varphi \|^2_{H^1(\Omega)} + C. \]
Since \( t \in (0, T) \) is arbitrary, we have the result. \( \square \)

Proof of Theorem 4.2. From (4.5), Proposition 4.1 and (4.6), we observe that for each \( h > 0 \) and \( m = 1, 2, \ldots, [T/h] \), \( \sup_{\varepsilon > 0} \| u_m^\varepsilon \|_{H^2(\Omega)} \leq C/h \). Hence applying Sobolev imbedding, we can extract a subsequence \( \{ \varepsilon_n \}_{n=1}^\infty \) such that as \( n \to +\infty \),
\[ u_m^{\varepsilon_n} \to \tilde{u}_m \quad \text{in} \ C^1(\overline{\Omega}), \quad u_m^{\varepsilon_n,xx} \to \tilde{u}_m,xx, \quad \text{weakly in} \ L^2(\Omega), \]
for all \( m = 1, 2, \ldots, [T/h] \) and \( h > 0 \). Thus \( \tilde{u}_m \) is a solution of (2.4) in the a.e. sense and in the viscosity sense. The \( \tilde{u}_m = u_m \) follows from the uniqueness of solutions of (2.4). The estimates in Theorem 4.2 follows from Proposition 4.1, (4.6) and (4.7). \( \square \)

We provide some pointwise estimates for \( \{ u_m^{\varepsilon} \}_{m,\varepsilon} \). We can show by a similar argument to the proof of Proposition 3.1 that
\[ \sup_{h > 0, \varepsilon > 0} \left( \sup_{0 \leq m \leq [T/h]} \| u_m^{\varepsilon} \|_{L^\infty(\Omega)} \right) < +\infty. \]
The following theorem plays a crucial role to prove Theorem 4.3 and Theorem 4.7 in Subsection 4.2 below.

Theorem 4.4 Assume \( q < r < qe \) and let \( x_0 \) be given in (2.3). Then there exists \( h_2 > 0 \) such that \( u_m^\varepsilon > \varphi \) in \((-1, x_0 + \sqrt{h}/2)\) for all \( \varepsilon \in (0, h^4), m = 1, 2, \ldots, [T/h] \) and \( h \in (0, h_2) \).

To prove Theorem 4.4 we prepare some lemmas.

Lemma 4.2 Let \( M_1 > 0 \) be given in Lemma 3.7. Then for each \( h > 0, m = 1, 2, \ldots, [T/h] \) and \( \varepsilon \in (0, h^4) \), we have \( u_m^\varepsilon \geq u_{m-1}^\varepsilon - M_1 h^4 \) on \( \overline{\Omega} \).
This lemma is a substitute for Theorem 4.1. Because we do not know such a monotone property for \( \{u_m^n\}_{m=0}^{T/h} \) as Theorem 4.1 holds since it may happen \( u_m^n(x) < \varphi(x) \) in view of Lemma 4.5.

**Proof of Lemma 4.2.** We can prove the case \( m = 1 \) by a similar argument to the proof of Lemma 3.1. Next we consider the case \( m = 2 \). Put \( u_1^\varepsilon := u_1^\varepsilon - M_1 h^4 \). Since \( u_1^\varepsilon \) is a classical solution of (4.3) with \( m = 1 \) and satisfies \( u_1^\varepsilon \geq u_0^\varepsilon - M_1 h^4 \) on \( \Omega \), we observe that

\[
\frac{u_1^\varepsilon - u_1^\varepsilon}{h} - u_1^\varepsilon,xx + \beta u_1^\varepsilon + \zeta u_1^\varepsilon - \varphi \leq \zeta (u_1^\varepsilon - M_1 h^4 - \varphi) - \zeta (u_1^\varepsilon - \varphi) \leq 0 \quad \text{in} \quad \Omega
\]

and \( u_1^\varepsilon(\pm 1) \leq u_1(\pm 1) \) for all \( \varepsilon \in (0, h) \) and \( h > 0 \). Applying the maximum principle, we have \( u_2^\varepsilon \geq u_1^\varepsilon - M_1 h^4 \) on \( \Omega \).

By induction, we obtain the desired result. \( \square \)

**Lemma 4.3** Assume \( q < r < qe \). Then there exists \( h_3 > 0 \) such that \( u_m^\varepsilon > \varphi \) in \((-1, 0] \) and \( u_m^\varepsilon(0) \geq \varphi(0) + \sqrt{h}/4 \) for all \( \varepsilon > 0 \), \( m = 1, 2, \ldots, [T/h] \) and \( h \in (0, h_3) \).

**Proof.** Step 1. We claim that \( u_m^\varepsilon > \varphi \) in \((-1, 0] \) for all \( \varepsilon > 0 \), \( m = 1, 2, \ldots, [T/h] \) and \( h > 0 \).

For each \( m = 1, 2, \ldots, [T/h] \), let \( U_m \) be the solution of

\[
\frac{U_m - U_{m-1}}{h} - U_{m,xx} + \beta U_m = 0 \quad \text{in} \quad \Omega, \quad U_m(\pm 1) = \varphi(\pm 1).
\]

Since \( U_1 \) is a classical subsolution of (4.3) with \( m = 1 \), we have \( u_1^\varepsilon \geq U_1 \) on \( \Omega \) by the maximum principle. We see by induction that \( u_m^\varepsilon \geq U_m \) on \( \Omega \) for all \( \varepsilon > 0 \), \( m = 1, 2, \ldots, [T/h] \) and \( h > 0 \). Therefore, the claim of this step follows from \( U_m > 0 \) in \( \Omega \) for all \( m = 1, 2, \ldots, [T/h] \) and \( \varepsilon \equiv 0 \) on \([-1, 0] \).

Step 2. We prove that there exists \( h_4 > 0 \) such that \( u_m^\varepsilon(0) \geq \varphi(0) + \sqrt{h}/4 \) for all \( \varepsilon > 0 \), \( m = 1, 2, \ldots, [T/h] \) and \( h \in (0, h_4) \).

\( U_1 \) is given by

\[
U_1(x) = \frac{1}{h} \int_{\Omega} G_h(x, y) \varphi(y) dy + \frac{\varphi(1) \sh(z_h(x + 1))}{\sh(2z_h)},
\]

where \( z_h := \sqrt{\beta + 1/h} \), \( \sh(r) := \sinh(r) \) for \( r \in \mathbb{R} \),

\[
(4.8) \quad G_{a,h}(x, y) := \begin{cases} \sh(z_h(a - x))\sh(z_h(a + y)) & (-a < y < x < a), \\ \sh(z_h(a + x))\sh(z_h(a - y)) & (-a < x < y < a), \\ z_h\sh(2a) & \end{cases}
\]

for \( a > 0 \) and \( G_h := G_{1,h} \). To estimate \( U_1(0) \), we directly calculate that for \( -1 \leq x \leq 0 \),

\[
(4.9) \quad U_1(x) = A\sh(z_h(1 + x)) + \frac{h\sh(z_h(1 + x))}{\sh(2z_h)} \left( \frac{q\varepsilon^{a+1}}{1 + qh} - \frac{r\varepsilon^a}{1 + rh} \right),
\]

\[
A := \frac{1}{2hz_h\sh(2z_h)} \left\{ \frac{(z_h - \alpha - 1)(z_h - \alpha)}{(z_h + \alpha + 1)(z_h + \alpha)} - e^{-z_h} \right\}.
\]
Putting \( x = 0 \), we have \( u_1^e(0) \geq U_1(0) \geq \sqrt{h}/3-C\sqrt{h}e^{-x_h} \) for all \( h > 0 \). In view of Lemma 4.2 and \( \varphi(0) = 0 \), selecting \( h_3 > 0 \) sufficiently small, we obtain the desired estimate. □

The following lemma is suggested by the formal asymptotic expansion of solutions of (2.4) (cf. Section 6 below).

**Lemma 4.4** Put \( \mu_h := \sqrt{h} - (\alpha + 1/2)h \) and \( \rho := (x - x_0 - \mu_h)/2\sqrt{h} \). We define

\[
\begin{align*}
\bar{u}(x) &= \begin{cases}
\varphi(x) + e^{\alpha x} \{ h^{3/2} w_3(\rho) + h^2 (w_4(\rho) - M_5) \} & \text{for } x \in [-1, x_0 + \mu_h], \\
\varphi(x) - M_5 h^2 e^{\alpha x} & \text{for } x \in (x_0 + \mu_h, 1]
\end{cases}
\end{align*}
\]

(4.10) \[ w_3(\rho) := r(e^{2\rho} - 1 - 2\rho), \] \[ w_4(\rho) := r \{ e^{2\rho} - (1 + 2\rho + 2\rho^2) + \alpha(e^{2\rho} - 2\rho e^{2\rho} - 1) \}. \]

Then there are large \( M_5 > 0 \) and small \( h_4 > 0 \) such that \( \bar{u} \) is a subsolution of (4.3) with \( m = 1 \) in the a.e. sense satisfying \( \bar{u}(\pm 1) \leq \varphi(\pm 1) \) for all \( \varepsilon \in (0, h^4) \) and \( h \in (0, h_4) \).

**Proof.** Note that \( \bar{u} \in W^{2,\infty}(\Omega) \cap C^2(\Omega \setminus \{ x_0 + \mu_h \}) \) and that \( w_3, w_4 \) satisfy

\[
\begin{align*}
&w_3 - \frac{w_3''}{4} + r(2\rho + 1) = 0, \quad w_4 - \frac{w_4''}{4} - \alpha w_3' + r(2\rho^2 + 2\rho - \alpha) = 0, \\
&\sqrt{h}w_3 + |hw_4| + |\sqrt{h}w_4'| \leq C \quad \text{in } (-1 + x_0 + \mu_h)/2\sqrt{h}, 0].
\end{align*}
\]

We divide our consideration into two cases.

**Case 1.** \( x \in (-1, x_0 + \mu_h] \).

In this case, \( \rho \in (-1 + x_0 + \mu_h)/2\sqrt{h}, 0] \). Using (4.11), we compute that

\[
\frac{\bar{u} - \varphi}{h} - \bar{u}_{xx} + \beta \bar{u} + \zeta(\bar{u} - \varphi) \leq e^{\alpha x} [h\{ -M_5 + r(\sqrt{h}w_3 + hw_4) - \alpha \sqrt{h}w_4' \} + qe^x - r - \sqrt{h}(2\rho + 1) - hr(2\rho^2 + 2\rho - \alpha)]
\]

We see from \( x_0 = \log(r/q) \) and \( x - x_0 = \mu_h + 2\sqrt{h} \rho \) that

\[
qe^x - r = r(e^{\mu_h + 2\sqrt{h}\rho} - 1) \leq r \left\{ (\mu_h + 2\sqrt{h}\rho) + \frac{1}{2!}(\mu_h + 2\sqrt{h}\rho)^2 + h^{3/2} \right\}
\]

\[
\leq r \left\{ \sqrt{h}(2\rho + 1) + h(2\rho^2 + 2\rho - \alpha) \right\} + rh \left[ \left( \alpha + \frac{1}{2} \right) \left( -1 - \sqrt{h}(2\rho + 1) + \frac{h}{2} \left( \alpha + \frac{1}{2} \right) + \sqrt{h} \right) \right].
\]

Here we have used \( \mu_h + 2\sqrt{h}\rho \leq \sqrt{h}/2 \) and the following inequality:

\[
e^\xi - 1 - \xi - \frac{1}{2} \xi^2 \leq \begin{cases}
0 & \text{if } \xi \leq 0, \\
\xi^{3/2} & \text{if } \xi \in [0, \sqrt{h}/2]
\end{cases} \quad \text{for small } h > 0.
\]

By the fact \( -\sqrt{h}(2\rho + 1) \leq x_0 \), we get

\[
qe^x - r - r \left\{ \sqrt{h}(2\rho + 1) + hr(2\rho^2 + 2\rho - \alpha) \right\} \leq C h.
\]
From (4.12) and this estimate, we have
\[
\frac{u - \varphi}{h} - u_{xx} + \beta u + \zeta(x) u - \varphi \leq h e^{\alpha x}(-M_5 + C) \quad \text{in } (-1, x_0 + \mu_h)
\]
for small \( \varepsilon > 0 \) and \( h > 0 \). Taking \( M_5 > 0 \) large enough, we conclude that \( u \) is a classical subsolution of (4.3) with \( m = 1 \) in \((-1, x_0 + \mu_h)\) for small \( \varepsilon > 0 \) and \( h > 0 \).

Case 2. \( x \in (x_0 + \mu_h, 1) \).

Taking \( M_5 \geq 1 \) and small \( h > 0 \), we see that for all \( \varepsilon \in (0, h^4) \) and \( h \in (0, h_5) \),
\[
\frac{u - \varphi}{h} - u_{xx} + \beta u + \zeta(x)(u - \varphi) \leq e^{\alpha x}(q e^x - r) - \zeta \left( -\frac{1}{h^2} \right) \leq 0 \quad \text{in } (x_0 + \mu_h, 1).
\]

Therefore for large \( M_5 > 0 \) and small \( h_4 > 0 \), \( u \) is a subsolution of (4.3) in the a.e. sense for \( \varepsilon \in (0, h^4) \) and \( h \in (0, h_4) \). In view of (4.12), we can get \( u(\pm 1) \leq \varphi(\pm 1) \) by replacing \( h_4 \) with a smaller one if necessary. Thus the proof is completed. \( \square \)

**Proof of Theorem 4.4.** It follows from Lemma 4.4 and the maximum principle that \( u^\varepsilon_1 \geq u \) on \( \overline{\Omega} \) for all \( \varepsilon \in (0, h^4) \) and \( h \in (0, h_5) \). In view of Lemma 4.3, we have only to prove the assertion on \( [0, x_0 + \sqrt{h}/2] \).

First we treat the case \( m = 1 \). Let \( \rho \) and \( \mu_h \) be defined in Lemma 4.4 and set \( \rho_1 := -(x_0 + \mu_h)/2\sqrt{h} \), \( \rho_2 := -1/4 + (2\alpha + 1)\sqrt{h}/4 \). We observe by careful calculations that for small \( h > 0 \),
\[
\frac{d^2}{dp^2} \{ h^{3/2}w_3 + h^2(w_4 - M_5) \} < 0 \quad \text{on } [\rho_1, \rho_2],
\]
\[
h^{3/2}w_3(\rho_1) + h^2(w_4(\rho_1) - M_5) \geq \frac{x_0}{4} h - M_5 h^2,
\]
\[
h^{3/2}w_3(\rho_2) + h^2(w_4(\rho_2) - M_5) \geq \frac{r}{10} h^{3/2} - M_5 h^2.
\]

Hence we have \( h^{3/2}w_3 + h^2(w_4 - M_5) \geq rh^{3/2}/20 \) on \([\rho_1, \rho_2]\) and thus
\[
u^\varepsilon_1 \geq \varphi + \frac{r}{20} h^{3/2} \quad \text{on } [0, x_0 + \sqrt{h}/2] \text{ for any } \varepsilon \in (0, h^4) \text{ and small } h > 0.
\]

In the case \( m \geq 2 \), Lemma 4.2 and the above estimate yield that
\[
u^\varepsilon_m \geq \varphi + \frac{r}{20} h^{3/2} - M_1 T h^3 \geq \varphi + \frac{r}{40} h^{3/2} \quad \text{on } [0, x_0 + \sqrt{h}]
\]
for all \( \varepsilon \in (0, h^4) \), \( m = 1, 2, \ldots, [T/h] \) and small \( h > 0 \).

Hence, selecting \( h_2 > 0 \) sufficiently small, we have \( u^\varepsilon_m \geq \varphi + r h^{3/2}/40 \) on \([0, x_0 + \sqrt{h}/2]\) for all \( \varepsilon \in (0, h^4) \), \( m = 1, 2, \ldots, [T/h] \) and \( h \in (0, h_2) \). Thus we complete the proof. \( \square \)

By Theorem 4.4, we see that for any \( h \in (0, h_2) \), \( m = 1, 2, \ldots, [T/h] \) and \( \varepsilon \in (0, h^4) \), \( u^\varepsilon_m \) satisfies \( u^\varepsilon_m > \varphi \) in \((-x_0, x_0)\) and thus
\[
\frac{u^\varepsilon_m - u^\varepsilon_{m-1}}{h} - u^\varepsilon_{m,xx} + \beta u^\varepsilon_m = 0 \quad \text{in } (-x_0, x_0).
\]
We prove Theorem 4.3 based on this fact. Before doing so, we give some preliminary analysis.

In \((-x_0,x_0)\), \(u_m^\varepsilon\) is given by

\[
u_m^\varepsilon(x) = \frac{1}{h} \int_{\Omega} G_{x_0,h}(x,y) u_{m-1}^\varepsilon(y) dy + \frac{u_m^\varepsilon(-x_0) \text{sh}(z_h(x_0 - x))}{\text{sh}(2x_0 z_h)} + \frac{u_m^\varepsilon(x_0) \text{sh}(z_h(x_0 + x))}{\text{sh}(2x_0 z_h)},
\]

where \(G_{x_0,h}\) is defined by (1.8) with \(a = x_0\). In the sequel we set \(x_0 = 1\) for simplicity.

Define

\[
\mathcal{G}_h[\psi](x) := \frac{1}{h} \int_{-1}^{1} G_h(x,y) \psi(y) dy \quad \text{for} \quad \psi \in C([-1,1]).
\]

For \(-1 \leq x \leq 0\), we get from (4.9)

\[
\mathcal{G}_h[\varphi](x) = \frac{h \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} \left( \frac{g e^{\alpha+1} - r e^\alpha}{1 + qh} - \frac{\varphi(1) \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} \right) + A \text{sh}(z_h(1 + x)).
\]

On the other hand, we observe by tedious calculations that for \(0 < x \leq 1\),

\[
\begin{align*}
\mathcal{G}_h[\varphi](x) = & \varphi(x) + \frac{h \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} \left( \frac{g e^{\alpha+1} - r e^\alpha}{1 + qh} - \frac{\varphi(1) \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} \right) \\
& - h e^{\alpha x} \left( \frac{g e^{\alpha x}}{1 + qh} - \frac{r}{1 + rh} \right) - \frac{\varphi(1) \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} + B \text{sh}(z_h(1 - x)),
\end{align*}
\]

\[
B := \frac{1}{2h z_h \text{sh}(2z_h)} \left\{ e^{z_h} \left( z_h + \alpha + 1 \right) (z_h + \alpha) - e^{-z_h} \left( z_h - \alpha - 1 \right) (z_h - \alpha) \right\}.
\]

Noting that \(\varphi \equiv 0\) on \([-1,0]\) and that \(A > B > 0\), we have the following:

\[
\mathcal{G}_h[\varphi](x) \leq \varphi(x) + R(x) - \frac{\varphi(1) \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} \quad \text{for all} \quad x \in [-1,1],
\]

\[
R(x) := \frac{h \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)} \left( \frac{g e^{\alpha+1} - r e^\alpha}{1 + qh} - \frac{r h e^{\alpha x}}{1 + qh} - \frac{r h e^{\alpha x}}{1 + rh} \right) 1_{\{x > 0\}} + A \text{sh}(z_h(1 - |x|)).
\]

Here \(1_{\{x > 0\}}(x) = 1\) for \(x > 0\) and \(= 0\) for \(x \leq 0\). Recalling \(u_0^\varepsilon = \varphi\), we see that

\[
u_1^\varepsilon(x) = \mathcal{G}_h[\varphi](x) + \frac{u_1^\varepsilon(-1) \text{sh}(z_h(1 - x))}{\text{sh}(2z_h)} + \frac{u_1^\varepsilon(1) \text{sh}(z_h(1 + x))}{\text{sh}(2z_h)}
\]

\[
\leq u_0(x) + R(x) + \frac{u_1^\varepsilon(-1) - u_0^\varepsilon(-1)}{\text{sh}(2z_h)} \text{sh}(z_h(1 - x)) + \frac{u_1^\varepsilon(1) - u_0^\varepsilon(1)}{\text{sh}(2z_h)} \text{sh}(z_h(1 + x)).
\]

We can inductively show that

\[
u_m^\varepsilon(x) \leq u_{m-1}^\varepsilon(x) + \mathcal{G}_h^{m-1}[R](x) + \sum_{k=1}^{m} \frac{u_k^\varepsilon(-1) - u_{k-1}^\varepsilon(-1)}{\text{sh}(2z_h)} \mathcal{G}_h^{m-k}[\text{sh}(z_h(1 - \cdot))](x)
\]

\[
+ \sum_{k=1}^{m} \frac{u_k^\varepsilon(1) - u_{k-1}^\varepsilon(1)}{\text{sh}(2z_h)} \mathcal{G}_h^{m-k}[\text{sh}(z_h(1 + \cdot))](x),
\]

where \(\mathcal{G}_h^k[\psi] := \mathcal{G}_h[\mathcal{G}_h^{k-1}[\psi]]\) and \(\mathcal{G}_h^0[\psi] := \psi\). Thus we need some pointwise estimates for \(\mathcal{G}_h^{m}_{x_0,h}[R]\) and \(\mathcal{G}_h^{m}_{x_0,h}[\text{sh}(z_h(x_0 + \cdot))]\) to prove Theorem 4.3.
Theorem 4.5 \ For h > 0, m = 1, 2, \ldots, [T/h] and x \in [-x_0, x_0], we have

\begin{align}
G_{x_0,h}^m [\text{sh}(z_h (x_0 - | \cdot |))] (x) & \leq \text{sh}(z_h (x_0 - |x|)) \sum_{k=0}^{m} \frac{a_{m,k}}{k!} (z_h |x|)^k + Ch, \\
\end{align}

\begin{align}
G_{x_0,h}^m [\text{sh}(z_h (x_0 \pm \cdot))] (x) & \leq \text{sh}(z_h (x_0 \pm x)) \sum_{k=1}^{m} \frac{k}{2m - k} \frac{a_{m,k}}{k!} (z_h (x_0 \mp x))^k,
\end{align}

where \( a_{m,k} := (2m - k)! \{2^{2m-k} m!(m-k)! \} \).

Theorem 4.6 \ There exist \( x_4 \in (0, x_0) \) and \( L_7, L_8, L_9 > 0 \) and \( h_5 > 0 \) such that for each \( h \in (0, h_5) \),

\begin{align}
G_{x_0,h}^m [R] (x) & \leq \begin{cases} L_7 \sqrt{\frac{h}{m}} & \text{for } x \in (-x_4, x_4), \text{ for all } m = 1, 2, \ldots, [T/h], \\
L_7 h & \text{for } x = \pm x_4,
\end{cases} \\
\sum_{k=1}^{m} \frac{G_{x_0,h}^{m-k} [\text{sh}(z_h (x_0 \pm \cdot))] (x)}{\text{sh}(2z_h x_0)} & \leq L_8 h
\end{align}

for all \( m = 1, 2, \ldots, [L_9/h |\log h|] \) and \( x \in [-x_4, x_4] \).

We admit that Theorem 4.5 holds and prove Theorems 4.6 and 4.3. We give the proof of Theorem 4.5 in Subsection 6.3 below.

**Proof of Theorem 4.6.** \ Set \( x_0 = 1 \) for notational simplicity. Note that

\begin{align}
a_{m,k} \leq \frac{C e^{-k^2/4(2m-k)}}{\sqrt{2m-k}} \quad \text{for } k = 0, 1, \ldots, m.
\end{align}

This will be proved in Subsection 6.2 below.

**Step 1.** \ We show (1.15) for some \( x_{4,1} \in (0, 1) \) and \( L_7 > 0 \).

Since it is easily seen from (1.13) that

\[ G_{x_0,h}^m [R] (x) \leq \sqrt{h} e^{-z_h} \text{sh}(z_h (1 - |x|)) \sum_{k=0}^{m} \frac{a_{m,k}}{k!} (z_h |x|)^k + Ch, \]

we have only to treat the first term of RHS of this inequality. Denote it by \( I_4 (x) \).

Using (4.17) and \( \sum_{k=0}^{m} \frac{(z_h |x|)^k}{k!} \leq e^{z_h|x|} \), we get

\begin{align}
I_4 (x) & \leq C \sqrt{\frac{h}{m}} \quad \text{for all } m = 1, 2, \ldots, [T/h], \text{ } x \in [-1, 1] \text{ and } h > 0.
\end{align}

Fix \( x_{4,1} \in (0, 1) \). We consider only \( I_4 (x_{4,1}) \) since \( I_4 \) is even. We still denote it by \( I_4 \) if no confusion arises. Set \( m_1 := [2z_h x_{4,1}/5] \) and \( m_2 := 3[z_h x_{4,1}] \). Then we have from (4.17)

\[ I_4 \leq C \sqrt{h} e^{-z_h x_{4,1}} \sum_{k=0}^{m_1} \frac{e^{-k^2/4(2m-k)}}{\sqrt{2m-k}} \frac{1}{k!} (z_h x_{4,1})^k. \]
We divide our consideration into three cases. Let $h_2$ be given in Theorem 4.4.

Case 1. $m \leq m_1$.

Using Stirling's formula, we get

$$I_4 \leq C \sqrt{h} e^{-z_h x_{4,1}} I_{4,1}, \quad I_{4,1} := \left\{ 1 + \sum_{k=1}^{m} \frac{1}{\sqrt{k}} \left( \frac{z_h x_{4,1} e}{k} \right)^k \right\}.$$ 

Set $\gamma = k/z_h x_{4,1}$. Then $\gamma \in (1/z_h x_{4,1}, 2/5)$ and $(z_h x_{4,1} e/k)^k = \exp(z_h x_{4,1} (1 - \log \gamma + \gamma))$. Since we see that $-\gamma \log \gamma + \gamma < 4/5$ for all $\gamma \in (1/z_h x_{4,1}, 2/5)$, we have

$$I_{4,1} \leq C e^{4z_h x_{4,1} / 5} \left( 1 + \sum_{k=1}^{m} \frac{1}{\sqrt{k}} \right) \leq C (1 + \sqrt{m_1}) e^{4z_h x_{4,1} / 5} \leq C h^{-1/4} e^{4z_h x_{4,1} / 5}.$$ 

Hence $I_4 \leq C h^{1/4} e^{-z_h x_{4,1} / 5}$ for all $m = 1, 2, \ldots, [T/h]$ and $h \in (0, h_2)$.

Case 2. $m > m_1$.

We may assume $m > m_2$. Similar calculations as in Case 1 yield that

$$I_4 \leq C \sqrt{h} e^{-z_h x_{4,1}} \left( \sum_{k=0}^{m_1} + \sum_{k=m_1+1}^{m_2} - \sum_{k=m_2+1}^{m} \right) \frac{e^{-k^2/4(2m-k)}}{\sqrt{2m-k}} \frac{1}{k!} (z_h x_{4,1})^k$$

$$
\leq C h^{1/4} e^{-z_h x_{4,1} / 5} + C \sqrt{h} e^{-z_h x_{4,1}} \sum_{k=m_2+1}^{m} \frac{e^{-k^2/4(2m-k)}}{\sqrt{2m-k}} \frac{1}{k!} (z_h x_{4,1})^k.
$$

Put $I_{4,2} := \sum_{k=m_2+1}^{m} \frac{e^{-k^2/4(2m-k)}}{\sqrt{2m-k}} \frac{1}{k!} (z_h x_{4,1})^k$.

Setting $m := 1/hs$, we observe that for $k = m_1 + 1, \ldots, m_2$,

$$\frac{e^{-k^2/4(2m-k)}}{\sqrt{2m-k}} \leq C \frac{e^{-m_1 hs/8}}{\sqrt{m}} \leq C \sqrt{hse^{-M_{4,1}s}} \leq M_{4,2} \sqrt{h}.$$ 

Here and in the sequel $M_{4,i}$'s ($i \geq 1$) are positive constants depending on $x_{4,1}$. Hence we have

$$I_{4,2} \leq M_{4,2} \sqrt{h} \sum_{k=m_2+1}^{m} \frac{(z_h x_{4,1})^k}{k!} \leq M_{4,2} \sqrt{h} e^{z_h x_{4,1}}$$

for all $h \in (0, h_2)$.

Consequently we obtain $I_4 \leq M_{4,3} h$ for all $m > m_1$ and $h \in (0, h_2)$.

From Case 1 and 2, choosing $L_7$ large enough, we get (11.15) for all $h \in (0, h_2)$.

Step 3. We show that for any $x_{4,2} \in [0, 3/4)$, there are $h_{5,1} > 0$, $L_8$, $L_9 > 0$ such that

$$I_5(x) \leq 2 e^{-z_h(1-x)}$$

on $[-1, 1]$ to have

$$I_5(x) \leq C e^{-z_h(1-x)} \sum_{k=1}^{m} \frac{\sum_{l=1}^{m-k} \frac{l e^{-l^2/4(2m-k-l)}}{(2(m-k) - l)^{3/2}} \frac{(z_h (1-x))^l}{l!} \right\}}$$

$$\leq C e^{-z_h(1-x)} \sum_{l=1}^{m} \frac{\sum_{k=l}^{m} \frac{l e^{-l^2/4(2m-k-l)}}{(2k-l)^{3/2}} \frac{(z_h (1-x))^l}{l!}}$$

for all $x \in [-1, 1]$. 20
We divide our considerations into two cases.

Case 1. \( m \leq m_1 \).

It is easily observe from the fact \( le^{-t^2/(2k-l)}/\sqrt{2k-l} \leq C \) for all \( k, l \in \mathbb{N} \) that
\[
\sum_{k=1}^{m} \frac{le^{-t^2/(2k-l)}}{(2k-l)^{3/2}} \leq C \sum_{k=1}^{m} \frac{1}{2k-l} \leq C \int_{1}^{T/h} \frac{1}{2r} dr \leq C |\log h| \quad \text{for all } l = 1, 2, \ldots, m.
\]
Hence we use this inequality and the same argument as in Case 1 of Step 1 to obtain
\[
I_5(x) \leq C |\log h| e^{-z_h(1-x)} \sum_{l=1}^{m} \frac{(z_h(1-x))^l}{l!} \leq C |\log h| e^{-z_h(1-x)/5} \quad \text{for all } x \in [-x_{4,2}, x_{4,2}].
\]

Case 2. \( m_1 < m \leq m_3 \).

We may consider \( m > m_2 \). Similar calculations to Case 1 yield that
\[
I_5(x) \leq Ce^{-z_h(1-x)} \left( \sum_{l=1}^{m_1} \sum_{l=m_1+1}^{m_2} \sum_{l=m_2+1}^{m} \frac{le^{-t^2/(2k-l)}}{(2k-l)^{3/2}} \right) \sum_{k=1}^{m} \frac{(z_h(1-x))^l}{l!} \leq C |\log h| e^{-z_h(1-x)/5} + C e^{-z_h(1-x)} \sum_{l=m_1+1}^{m_2} \sum_{l=m_1+1}^{m} \frac{le^{-t^2/(2k-l)}}{(2k-l)^{3/2}} \sum_{k=1}^{m} \frac{(z_h(1-x))^l}{l!}.
\]
Set \( I_{5,1}(x) := \sum_{l=m_1+1}^{m_2} \sum_{k=1}^{m_1} \frac{le^{-t^2/(2k-l)}}{(2k-l)^{3/2}} \frac{(z_h(1-x))^l}{l!} \). From the facts \( l \leq 2k - l \leq 2m_3 \) and \( l > m_1 \), we see that for \( k = l, \ldots, m \) and \( l = m_1 + 1, \ldots, m_2 \),
\[
\frac{le^{-t^2/(2k-l)}}{(2k-l)^{3/2}} \leq \frac{1}{\sqrt{t}} e^{-m_3^2/8m_3} \leq Ch_{1/4} e^{-2|\log h|} = Ch_{9/4}^4.
\]
Thus for \( x \in [-x_{4,2}, x_{4,2}] \) and small \( h > 0 \),
\[
I_{5,1}(x) \leq C \sum_{l=m_1+1}^{m} \sum_{k=1}^{m_1} \frac{h_{9/4}^4 (z_h(1-x))^l}{l!} \leq Ch_{9/4}^4 m_3 \sum_{l=m_1+1}^{m} \frac{(z_h(1-x))^l}{l!} \leq M_{5,1} h_{5/4}^4 |\log h| e^{-z_h(1-x)} \leq M_{5,1} h e^{-z_h(1-x)},
\]
where \( M_{5,1} \) depends on \( 1 - x_{4,2} \). Consequently, we get
\[
I_{5,1}(x) \leq M_{5,1} h \quad \text{for all } x \in [-x_{4,2}, x_{4,2}] \text{ and small } h > 0.
\]

Thus taking large \( L_8 > 0 \), \( L_9 := (1 - x_{4,2})^2/100 \) and \( h_5 := \min\{h_2, h_{5,1}\} \), we obtain \( (4.16) \) for all \( m = 1, 2, \ldots, [T/h], x \in [-x_4, x_4] \) and \( h \in (0, h_5) \).

Setting \( x_4 := \min\{x_{4,1}, x_{4,2}\} \), we complete the proof. \( \square \)

**Proof of Theorem 4.3.** Step 1. We claim that there are \( L_{4,1}, L_{5,1} > 0 \) and \( h_1 > 0 \) such that for all \( h \in (0, h_1) \), \( m = 1, 2, \ldots, [L_{5,1}/h|\log h|] \) and \( x \in \Omega \cap (-x_4, x_4) \).

\[
(4.19) \quad u_m^\varepsilon(x) - u_{m-1}^\varepsilon(x) \leq L_{4,1} h.
\]
Put $h_1 := h_6$. It follows from Theorem 4.6 that

\[(4.20)\quad u_m^\varepsilon(\pm x_4) - u_{m-1}^\varepsilon(\pm x_4) \leq (L_7 + L_8)h\]

for all $m = 1, 2, \ldots, [L_9/h|\log h|]$ and $h \in (0, h_1)$. Choosing $L_4 \geq L_7 + L_8$, we observe that $u_0^\varepsilon + L_4h$ is a classical supersolution of (4.3) in $\Omega \setminus (-x_4, x_4)$ since $u_0^\varepsilon(= \varphi)$ is smooth in this domain. Hence we apply the maximum principle to have (4.19) with $m = 1$. We inductively obtain (4.19) for $m = 2, \ldots, [L_9/h|\log h|]$ and $h \in (0, h_1)$. Putting $L_5 := L_9$, we have the claim.

**Step 2.** We derive the estimates of (1) and (2).

From Theorem 4.6 and (4.19), we obtain

\[(4.21)\quad u_m^\varepsilon(x) - u_{m-1}^\varepsilon(x) \leq \begin{cases} L_4\sqrt{\frac{h}{m}} & \text{if } |x| < x_4, \\ L_4h & \text{if } |x| \geq x_4 \end{cases}\]

for all $m = 1, 2, \ldots, [L_5/h|\log h|]$ and small $h \in (0, h_1)$.

Hence (4.1) holds for all $m = 1, 2, \ldots, [L_5/h|\log h|]$.

In the case $m = [L_5/h|\log h|]$, we have

\[u_m^\varepsilon - u_{m-1}^\varepsilon \leq L_4h\sqrt{\frac{|\log h|}{L_5}} \quad \text{on } \Omega.\]

We can show by the maximum principle and induction that this estimate holds for $m = [L_5/h|\log h|] + 1, \ldots, [T/h]$. Using (4.7) and the above estimates, we have (4.1) for all $m = 1, 2, \ldots, [T/h]$ and $h \in (0, h_1)$.

The (4.2) can be derived from (4.1), (4.5) and (4.6). The estimate of (2) is a consequence of (4.1) and (4.7). □

### 4.2 Free boundary for the DMS-BS

The problem (2.2) has a unique free boundary. However, it does not lead to the existence and uniqueness of that for the DMS-BS. To prove them is the purpose of this subsection.

**Theorem 4.7** Assume $q < r < q_\varepsilon$. Then there is $h_6 > 0$ satisfying the following: For each $h \in (0, h_6)$ and $m = 1, 2, \ldots, [T/h]$, there exists a unique $x_m \in (x_0, 1]$ such that

\[(4.22)\quad \{x \in \Omega \mid u_m(x) > \varphi(x)\} = (-1, x_m), \quad \{x \in \Omega \mid u_m(x) = \varphi(x)\} = [x_m, 1].\]

Moreover, $x_0 < x_1 \leq x_2 \leq \cdots \leq x_m \leq \cdots \leq x_{[T/h]}$ and $x_m \leq \min\{x_0 + \sqrt{mh}, 1\}$ for all $m = 1, 2, \ldots, [T/h]$.

**Proof.** Put $h_6 := \min\{h_1, h_2, h_3\}$. Notice by Theorems 4.1 and 4.4, Lemma 4.3 and (4.7) that $u_m > \varphi$ in $(-1, x_0 + \sqrt{h}/2)$ for all $h \in (0, h_6)$. Since we easily observe by Theorem 4.1 that $x_1 \leq x_2 \leq \cdots \leq x_m \leq \cdots$ if they exist, in the following we show the existence and uniqueness of $x_m$. \[\text{22}\]
Step 1. We treat the case $m = 1$. Set $\rho = (x - x_0 - \sqrt{h})/2\sqrt{h}$ and

$$
\overline{u}_1(x) := \begin{cases} 
\varphi(x) + e^{\alpha x} h^{3/2} w_3(\rho) & \text{for } x \in (0, x_0 + \sqrt{h}), \\
\varphi(x) & \text{for } x \in [x_0 + \sqrt{h}, 1),
\end{cases}
$$

where $w_3$ is defined by (4.10). We show that $\overline{u}_1$ is a supersolution of (2.4) in $(0,1)$ in the a.e. sense.

Note that $\overline{u}_1 \in W^{2,\infty}(\Omega) \cap C^2(\Omega \setminus \{x_0 + \sqrt{h}\})$ and that $\overline{u}_1 \geq \varphi$ on $[0,1]$. We see by (4.11) that in $(0, x_0 + \sqrt{h})$,

$$
\frac{\overline{u}_1 - \varphi}{h} - \overline{u}_{1,xx} + \beta \overline{u}_1 = e^{\alpha x} \{-h \alpha w_3' + rh^{3/2} w_3 + r(\sqrt{h}(2\rho+1) - 1 - \sqrt{h}(2\rho+1))\}.
$$

It follows from the facts $w_3, -w_3' \geq 0$ in $(-\infty,0]$ and $e^y \geq 1 + y$ for all $y \in \mathbb{R}$ that

$$
\overline{u}_1 - \varphi \overline{u}_{1,xx} + \beta \overline{u}_1 \geq 0 \quad \text{in } (0, x_0 + \sqrt{h}).
$$

By (4.10) and this inequality, we see that $\overline{u}_1$ is a supersolution of (2.4) in $(0,1)$ in the a.e. sense.

In view of $\overline{u}_1(0) \leq u_1(0)$, we modify $\overline{u}_1$ to construct a viscosity supersolution of (2.4). Put $\eta := 2(x_0 + \sqrt{h})^2$. Define $\overline{W}_1$ by

$$
\overline{W}_1(x) = \begin{cases} 
-\gamma(x - x_0)^3 e^{-\eta/(x-x_0)^2} & \text{if } 0 \leq x \leq x_0, \\
0 & \text{if } x_0 < x \leq 1,
\end{cases}
$$

where $\gamma > 0$ is selected later. Then $\overline{W}_1 \in C^2(0,1)$, $\overline{W}_1 \geq 0$, $\overline{W}_{1,x} < 0$ in $(0,1)$ and $\overline{W}_1(x_0) = \overline{W}_{1,x}(x_0) = \overline{W}_{1,xx}(x_0) = 0$. Moreover, it is easily observed by the choice of $\eta$ that

$$
(4.23) - \overline{W}_{1,xx} + \beta \overline{W}_1 \geq 0 \quad \text{in } (0,1).
$$

Take $\gamma_1 > 0$ satisfying $\overline{W}_1 > \|\varphi\|_{L^\infty(\Omega)}$ on $[0, x_0/4]$. Since $\overline{W}_2 := \|\varphi\|_{L^\infty(\Omega)}$ is a classical supersolution of (2.4) in $\Omega$, setting

$$
\overline{U}_1(x) := \begin{cases} 
\overline{W}_2 & \text{if } -1 \leq x \leq 0, \\
\min\{\overline{u}_1(x), \overline{W}_1(x), \overline{W}_2\} & \text{if } 0 < x \leq 1,
\end{cases}
$$

we conclude that $\overline{U}_1$ is a viscosity supersolution of (2.4) satisfying $\overline{U}_1(\pm 1) \geq \varphi(\pm 1)$.

We have $u_1 \leq \overline{U}_1$ on $[0,1]$ from the comparison principle for viscosity solutions. Using this inequality, we can obtain a unique $x_1$ satisfying (4.22). Indeed, $\overline{U}_1 = \varphi$ on $[x_0 + \sqrt{h}, 1]$ implies that $u_1 = \varphi$ on $[x_0 + \sqrt{h}, 1]$. Put

$$
x_1 := \inf\{y \in \Omega \mid u_1(x) = \varphi(x) \text{ for all } x \in [x, 1]\}(\leq x_0 + \sqrt{h}).
$$

Clearly $u_1 = \varphi$ on $[x_1, 1]$. To verify $u_1 > \varphi$ in $(x_0 + \sqrt{h}/2, x_1)$, we suppose that there is $\overline{x}_1 \in (x_0 + \sqrt{h}/2, x_1)$ such that $u_1(x) > \varphi(x)$ in $(\overline{x}_1, x_1)$ and $u_1(\overline{x}_1) = \varphi(\overline{x}_1)$. Since $u_1$ and $\varphi$ are solutions of (2.4) with $m = 1$, we get $u_1 = \varphi$ on $[\overline{x}_1, x_1]$ by the uniqueness. This
contradicts to the definition of $x_1$ and hence $u_1 > \varphi$ in $(x_0 + \sqrt{h}/2, x_1)$. This observation also leads to the uniqueness of $x_1$. Therefore we have the desired result of Step 1.

Step 2. We prove the case $m = 2$.

Let $w_3$ be defined by (4.10) and set $\rho_m = (x - x_0 - \sqrt{m\rho_2})/2\sqrt{m\rho}$ for $m = 1, 2, \ldots, [T/h]$. By the facts $w_3' \leq 0$ on $(-\infty, 0]$ and $\rho_m \leq \rho_{m-1} \leq 0$ on $[0, x_0 + \sqrt{(m-1)\rho}]$, we see that

\[ w_3(\rho_m) \geq w_3(\rho_{m-1}) \quad \text{for all } x \in [-1, x_0 + \sqrt{(m-1)\rho}] \text{ and } m = 1, 2, \ldots, [T/h]. \]

Define

\[
\overline{w}_2(x) := \begin{cases} 
\varphi(x) + e^{ax}(2h)^{3/2}w_3(\rho_2) & \text{for } x \in [0, x_0 + \sqrt{2h}], \\
\varphi(x) & \text{for } x \in [x_0 + \sqrt{2h}, 1), 
\end{cases}
\]

\[
\overline{U}_2(x) := \begin{cases} 
\overline{W}_2 & (-1 \leq x \leq 0), \\
\min\{\overline{w}_2(x) + \overline{W}_1(x), \overline{W}_2\} & (0 < x \leq 1), 
\end{cases}
\]

We claim that $\overline{U}_2$ is a viscosity supersolution of (2.4) with $m = 2$. It is observed by (4.23) and $u_1 \leq \overline{w}_1 + \overline{W}_1$ in $(0, x_0 + \sqrt{h}]$ that

\[
\frac{\overline{U}_2 - u_1}{h} - \overline{U}_{2,xx} + \beta \overline{U}_2 \geq \frac{\overline{w}_2 - u_1}{h} - \overline{w}_{2,xx} + \beta \overline{w}_2 
\geq e^{ax}\left[\sqrt{h}(2\sqrt{2} - 1)w_3(\rho_2) - w_3(\rho_1)\right] - 2hw_3(\rho_2)
\]

\[+ r(2h)^{3/2}w_3(\rho_2) + r\left(e^{\sqrt{h}(2\rho_2+1)} - 1 - \sqrt{2h}(2\rho_2+1)\right).\]

in $(0, x_0 + \sqrt{h})$. Using (4.11), (4.24) with $m = 2$ and $e^y \geq 1 + y$ for all $y \in \mathbb{R}$, we get

\[
\frac{\overline{U}_2 - u_1}{h} - \overline{U}_{2,xx} + \beta \overline{U}_2 \geq 0 \quad \text{on } (0, x_0 + \sqrt{h}).
\]

From $\overline{U}_2 \geq \varphi$ on $\overline{T}$, (4.10) and this inequality it follows that $\overline{U}_2$ is a viscosity supersolution of (2.4) with $m = 2$.

Thus we use the comparison principle for viscosity solutions to obtain $u_2 \leq \overline{U}_2$ on $\overline{T}$.

We can show by a similar argument to Step 1 that there exists a unique $x_2$ satisfying (4.22) with $m = 2$ and $x_2 \leq x_0 + \sqrt{2h}$.

Step 3. We consider the case $m \geq 3$.

By induction we assume that there exists a unique $x_{m-1}$ satisfying (4.22) and $x_{m-1} \leq x_0 + \sqrt{(m-1)\rho}$. Then define

\[
\overline{w}_m(x) := \begin{cases} 
\varphi(x) + e^{ax}(mh)^{3/2}w_3(\rho_m) & \text{for } x \in (0, x_0 + \sqrt{mh}], \\
\varphi(x) & \text{for } x \in [x_0 + \sqrt{mh}, 1), 
\end{cases}
\]

\[
\overline{U}_m(x) := \begin{cases} 
\overline{W}_2 & (-1 \leq x \leq 0), \\
\min\{\overline{w}_m(x) + \overline{W}_1, \overline{W}_2\} & (0 < x \leq 1). 
\end{cases}
\]

By a similar argument to Step 2, we can see that $\overline{U}_m$ is a viscosity supersolution of (2.4) and thus $u_m \leq \overline{U}_m$ on $\overline{T}$. Therefore by the same way as in Step 1 we can find a unique $x_m$ satisfying (4.22) and $x_m \leq \min\{x_0 + \sqrt{mh}, 1\}$. □
4.3 Proof of Theorem 4.5.

To prove Theorem 4.5, we prepare some identities. Define

\[ J_k := \int_{-1}^{0} (zh_y)^k \text{ch}(2zh(1+y)) \, dy, \quad J_k := \int_{0}^{1} (zh_y)^k \text{ch}(2zh_y) \, dy, \]

\[ K_k := \int_{x}^{1} (zh_y)^k \text{ch}(2zh(1-y)) \, dy, \quad L_k := \int_{-1}^{x} (zh(1-y))^k \text{ch}(2zh(1+y)) \, dy, \]

\[ M_k := \int_{x}^{1} (zh(1-y))^k \text{ch}(2zh_y) \, dy \quad (k = 1, 2, \ldots, m). \]

Direct calculations yield that

\[ J_0 = \frac{1}{2zh} \text{sh}(2zh), \quad J_1 = \frac{-1}{4zh} (\text{ch}(2zh) - 1), \]

\[ J_0 = \frac{1}{2zh} \text{sh}(2zh_x), \quad J_1 = \frac{1}{2zh} \left\{ P_{1,1} \text{sh}(2zh_x) - P_{2,1} \text{ch}(2zh_x) + \frac{1}{2} \right\}, \]

\[ K_0 = \frac{1}{2zh} \text{sh}(2zh(1-x)), \quad K_1 = \frac{1}{2zh} \left\{ P_{1,1} \text{sh}(2zh(1-x)) + P_{2,1} \text{ch}(2zh(1-x)) - P_{0,1} \right\}, \]

\[ L_0 = \frac{1}{2zh} \text{sh}(2zh(1+x)), \quad L_1 = \frac{1}{2zh} \left\{ Q_{1,1} \text{sh}(2zh(1+x)) + Q_{2,1} \text{ch}(2zh(1+x)) - Q_{0,1} \right\}, \]

\[ M_0 = \frac{1}{2zh} (\text{sh}(2zh) - \text{sh}(2zh_x)), \quad M_1 = \frac{-1}{2zh} \left\{ Q_{1,1} \text{sh}(2zh_x) + Q_{2,1} \text{ch}(2zh_x) - Q_{0,1} \text{ch}(2zh) \right\}, \]

where \( P_{0,1} := 1/2, \quad P_{1,1} := zh_x, \quad P_{0,1} := 1/2, \quad Q_{0,1} := 1/2, \quad Q_{1,1} := zh(1-x), \quad Q_{2,1} := 1/2. \)

For \( k \geq 2, \) the following identities hold.

Lemma 4.5 Let \( J_k, J_k, K_k, L_k, M_k \) be defined as above. For \( k \geq 2, \) we have

\[ (-1)^k J_k = \frac{k!}{2zh} \left\{ -P_{0,k} + \frac{1_{\text{even}}}{2^k} \text{sh}(2zh) + \frac{1_{\text{odd}}}{2^k} \text{ch}(2zh) \right\}, \]

\[ J_k = \frac{k!}{2zh} \left( P_{1,k} \text{sh}(2zh_x) - P_{2,k} \text{ch}(2zh_x) + \frac{1_{\text{odd}}}{2^k} \text{ch}(2zh) \right), \]

\[ K_k = \frac{k!}{2zh} (P_{1,k} \text{sh}(2zh(1-x)) + P_{2,k} \text{ch}(2zh(1-x)) - P_{0,k}), \]

\[ L_k = \frac{k!}{2zh} \left\{ Q_{1,k} \text{sh}(2zh(1+x)) + Q_{2,k} \text{ch}(2zh(1+x)) - Q_{0,k} \right\}, \]

\[ M_k = \frac{-k!}{2zh} \left\{ Q_{1,k} \text{sh}(2zh_x) + Q_{2,k} \text{ch}(2zh_x) - \frac{1_{\text{even}}}{2^k} \text{sh}(2zh) - \frac{1_{\text{odd}}}{2^k} \text{ch}(2zh) \right\}, \]

where

\[ P_{0,k} := \sum_{l=0}^{k} \frac{z_{h-l}}{2^l(k-l)!}, \quad P_{1,k} := \sum_{l=0}^{k} \frac{(zh_x)^{k-l}}{2^l(k-l)!}, \quad P_{2,k} := \sum_{l=0}^{k} \frac{(zh_x)^{k-l}}{2^l(k-l)!}, \]

\[ Q_{0,k} := \sum_{l=0}^{k} \frac{(2zh)^{k-l}}{2^l(k-l)!}, \quad Q_{1,k} := \sum_{l=0}^{k} \frac{(zh(1-x)-l)^{k-l}}{2^l(k-l)!}, \quad Q_{2,k} := \sum_{l=0}^{k} \frac{(zh(1-x))^{k-l}}{2^l(k-l)!}. \]
Proof. Integrating by parts we have, for $k \geq 2$,
\[
(-1)^kJ_k = \frac{1}{2zh} \left\{ -\frac{1}{2}kz_h^{k-1} + \frac{zh}{2}k(k-1)(-1)^{k-2}J_{k-2} \right\},
\]
\[
J_k = \frac{1}{2zh} \left\{ (zh)x^k \text{sh}(2zhx) - \frac{1}{2}k(zh)x^{k-1} \text{ch}(2zhx) + \frac{zh}{2}k(k-1)J_{k-2} \right\},
\]
\[
K_k = \frac{1}{2zh} \left\{ (zh)x^k \text{sh}(2zh(1-x)) + \frac{1}{2}k(zh)x^{k-1} \text{ch}(2zh(1-x)) - \frac{1}{2}k(-zh)^{k-1}
\right.
\]
\[
+ \frac{zh}{2}k(k-1)K_{k-2} \right\},
\]
\[
L_k = \frac{1}{2zh} \left\{ \{zh(1-x)\}^k \text{sh}(zh(1+x)) + \frac{k}{2zh} \{zh(1-x)\}^{k-1} \text{ch}(zh(1+x)) - \frac{k}{2}(2zh)^{k-1}
\right.
\]
\[
+ \frac{zh}{2}k(k-1)L_{k-2} \right\},
\]
\[
M_k = \frac{1}{2zh} \left\{ -(zh(1-x))^k \text{sh}(2zhx) - \frac{k}{2}(zh(1-x))^{k-1} \text{ch}(2zhx) + \frac{zh}{2}k(k-1)M_{k-2} \right\}.\]

Using these recurrence formulae, we obtain the result. □

We separately prove (4.13) and (4.14) of Theorem 4.5. Put $x_0 = 1$ for the sake of simplicity.

Proof of (4.13). Set $G_m(x) := \mathbb{J}_m^m[\text{sh}(zh(1 - |r|))]G_m(x)$ for $x \in [-1, 1]$, $\text{sh}(r) := \sinh(r)$ and $\text{ch}(r) := \cosh(r)$ for $r \in \mathbb{R}$. Note that
\[
G_h > 0 \text{ in } (-1, 1) \times (-1, 1), \quad \int_{-1}^{1} G_h(x, y) dy \leq \frac{1}{hz_h^2}.
\]
In this proof we use the identities in Lemma 4.5 and the following ones.

\[
(4.25) \quad G_h > 0 \text{ in } (-1, 1) \times (-1, 1), \quad \int_{-1}^{1} G_h(x, y) dy \leq \frac{1}{hz_h^2}.
\]

\[
(4.26) \quad \text{sh}^2(zh(1 + y)) = \frac{1}{2}(\text{ch}(2zh(1 + y)) - 1),
\]
\[
(4.27) \quad \text{sh}(zh(1 - y))\text{sh}(zh(1 + y)) = \frac{1}{2}(\text{ch}(2zh) - \text{ch}(2zh y)),
\]
\[
(4.28) \quad -\text{sh}(zh(1 - x))\text{sh}(2zhx) + \text{sh}(zh(1 + x))\text{sh}(2zh(1 - x)) = \text{sh}(2zh)\text{sh}(zh(1 - x)),
\]
\[
(4.29) \quad \text{sh}(zh(1 - x))\text{ch}(2zhx) + \text{sh}(zh(1 + x))\text{ch}(2zh(1 - x)) = \text{sh}(2zh)\text{ch}(zh(1 - x)).
\]

Step 1. We estimate $G_1(x)$. We calculate with using (4.26) and (4.27) to get for $0 \leq x \leq 1$,
\[
G_1(x) = \frac{\text{sh}(zh(1 - x))}{2hz_h \text{sh}(2zh)} \left\{ \frac{1}{2zh} \text{sh}(2zh) - 1 + x\text{ch}(2zh) - \frac{1}{2zh} \text{sh}(2zhx) \right\}
\]
\[
+ \frac{\text{sh}(zh(1 + x))}{2hz_h \text{sh}(2zh)} \frac{1}{2zh} \text{sh}(2zh(1 - x)).
\]
By (4.28), we get
\[ G_1(x) \leq \frac{b}{2hz_h^2}g_1(x) \quad \text{for} \quad x \in [0, 1], \]
where \( g_1(x) := (1 + z_h x) \text{sh}(z_h (1 - x))\) and \( b := \text{ch}(2z_h)/\text{sh}(2z_h)\). By similar calculations we have \( G_1(x) \leq bg_1(-x)/2hz_h^2 \) for \( x \in [-1, 0]\). Hence we obtain
\[ G_1(x) \leq \frac{b}{2hz_h^2}g_1(|x|) \quad \text{for} \quad x \in [-1, 1]. \]

\[ G_1(x) \leq \frac{b}{2hz_h^2}g_1(|x|) \quad \text{for} \quad x \in [-1, 1]. \]

**Step 2.** We consider the case \( m = 2 \).
It directly follows from (4.25) and (4.30) that \( G_2(x) \leq bG_2(x)/2hz_h^2 \) for \( x \in [-1, 1]\).
We observe by (4.26), (4.27) and Lemma 4.5 that for \( 0 \leq x \leq 1 \),
\[ G_2(x) \leq \frac{b}{2hz_h^2}g_2(x) + P_2 \quad \text{for} \quad x \in [-1, 1] \]
we can obtain by the similar way as above \( G_2(x) \leq bG_2(-x)/2hz_h^2 + P_2 \) for \( -1 \leq x \leq 0 \).
Consequently, we have
\[ G_2(x) \leq \frac{b^2}{(2hz_h^2)^2}g_2(|x|) + bP_2 \quad \text{for} \quad x \in [-1, 1]. \]
Step 3. We estimate $G_m(x)$ by induction. We assume that

$$G_{m-1}(x) \leq \frac{b^{m-1}}{(2h z_h^2)^{m-1}g_m(\vert x\vert)} + \sum_{l=2}^{m-1} b^{l-1} \mathcal{P}_l \text{ for } x \in [-1, 1],$$

where $g_{m-1}$ and $\mathcal{P}_l$ are defined by

$$g_{m-1}(x) := \left( \sum_{l=0}^{m-1} \frac{c_{m-1,l}}{l!} (z_h x)^l \right) \text{sh}(z_h(1-x)),$$

$$\mathcal{P}_l := \frac{e^{-z_h}}{4(2h z_h^2)^l \text{sh}(2z_h)} \sum_{p=0}^{l-1} c_{l-1,p} \sum_{q=0}^{p} \frac{z_h^{p-q}}{2^q(p-q)!}.$$

We estimate $G_h[g_{m-1}](x)$ for $0 \leq x \leq 1$. We calculate that

$$G_h[g_{m-1}](x) \leq \left[ \frac{\text{sh}(z_h(1-x))}{2h z_h \text{sh}(2z_h)} \sum_{k=0}^{m-1} \frac{c_{m-1,k}}{k!} \left\{ (-1)^k j_k + \frac{(z_h x)^{k+1}}{z_h(k+1)} \text{ch}(2z_h) - j_k \right\} \right]$$

$$+ \frac{\text{sh}(z_h(1+x))}{2h z_h \text{sh}(2z_h)} \sum_{k=0}^{m-1} \frac{c_{m-1,k}}{k!} \mathcal{K}_k$$

$$+ \sum_{l=2}^{m} D_{l-1} \mathcal{P}_l \text{ for } 0 \leq x \leq 1.$$

It follows from Lemma 4.15 that

$$I_{5,2} := \text{sh}(z_h(1-x)) \{ (-1)^k j_k - j_k \} + \text{sh}(z_h(1+x)) \mathcal{K}_k$$

$$= \frac{k!}{2z_h} \text{sh}(z_h(1-x)) \left\{ -P_{0,k} + \frac{1}{2k} \text{sh}(2z_h) + \frac{1}{2k} \text{ch}(2z_h) \right\}$$

$$- P_{1,k} \text{sh}(2z_h x) + P_{2,k} \text{ch}(2z_h x) - \frac{1}{2k},$$

$$+ \text{sh}(z_h(1+x)) \left\{ P_{1,k} \text{sh}(2z_h(1-x)) + P_{2,k} \text{ch}(2z_h(1-x)) - P_{0,k} \right\}.$$

Using (4.28), (4.29) and $\text{sh}(2z_h) \leq \text{ch}(2z_h)$, we obtain

$$I_{5,2} \leq \frac{k!}{2z_h} \text{sh}(z_h(1-x)) \left\{ P_{1,k} \text{sh}(2z_h) + \frac{1}{2k} \text{ch}(2z_h) \right\} + P_{2,k} \text{ch}(z_h(1-x)) \text{sh}(2z_h)$$

$$- \text{sh}(z_h(1+x)) P_{0,k}.$$

From (4.31) and the fact $P_{2,k} \leq P_{0,k}$ on $[0, 1]$ we get

$$I_{5,2} \leq \frac{k!}{2z_h} \text{sh}(z_h(1-x)) \left\{ (P_{1,k} + P_{2,k}) \text{sh}(2z_h) + \frac{1}{2k} \text{ch}(2z_h) \right\} + e^{-z_h} P_{0,k}.$$

Consequently, we have

$$G_h[g_{m-1}](x) \leq \frac{b \text{sh}(z_h(1-x))}{2h z_h^2} \sum_{k=0}^{m-1} c_{m-1,k} \left\{ \frac{1}{2} \left( \frac{1}{2k} + \sum_{l=0}^{k} \frac{(z_h x)^{k-l}}{2^l(k-l)!} \right) + \sum_{l=0}^{k} \frac{z_h^{k-l}}{2^l(k-l)!} \right\}$$

$$+ e^{-z_h} \sum_{k=0}^{m-1} c_{m-1,k} \sum_{l=0}^{k} \frac{z_h^{k-l}}{2^l(k-l)!} + \sum_{l=2}^{m} b^{l-1} \mathcal{P}_l.$$
Therefore setting

\[(4.32) g_m(x) := \text{sh}(z_h(1 - x)) \sum_{k=0}^{m} \frac{c_{m,k}}{k!} (z_hx)^k \]

we obtain

\[G_m(x) \leq \frac{b^m}{(2hz_h^2)^m} g_m(x) + \sum_{k=2}^{m} t^{k-1}p_k \quad \text{for } x \in [0, 1].\]

Since we see \(G_m(x) \leq \frac{b^m}{(2hz_h^2)^m} g_m(-x) + \sum_{k=2}^{m} t^{k-1}p_k \quad \text{for } -1 \leq x \leq 0\) by the same way as above, we conclude that

\[G_m(x) \leq \frac{b^m}{(2hz_h^2)^m} g_m(|x|) + \sum_{k=2}^{m} t^{k-1}p_k \quad \text{for } x \in [-1, 1].\]

**Step 4.** We determine \(\{c_{m,k}\}_{k=0}^{m}\) for \(m = 1, 2, \ldots, [T/h]\).

From (4.32), we can obtain the following recurrence formulae: for \(m = 2, 3, \ldots, [T/h]\) and \(k = 2, 3, \ldots, m - 1\),

\[c_{1,1} = c_{1,0} = 1, \quad c_{m,m} = c_{m-1,m-1}, \quad c_{m,m-1} = \frac{1}{2} c_{m-1,m-1} + c_{m-1,m-2},\]

\[(4.33) \quad c_{m,m-k} = \sum_{l=0}^{k} \frac{1}{2k-l} c_{m-1,m-1-l}, \quad c_{m,0} = \sum_{l=0}^{m-1} \frac{c_{l-1,l}}{2^l} \cdot\]

First, we easily get

\[c_{m,m} = 1, \quad c_{m,m-1} = \frac{1}{2} (m + 1) \quad \text{for } m = 1, 2, \ldots, [T/h].\]

As for \(c_{m,m-2}\), using (4.33) and these formulae, we have

\[c_{m,m-2} = \frac{1}{2^2 \cdot 2!} (m + 1)(m + 2).\]

We assume by induction that for \(m \geq 3\),

\[c_{m-1,m-1-l} = \frac{1}{2l!} \prod_{p=1}^{l} (m - 1 + p) \quad \text{for } l = 1, 2, \ldots, m - 1.\]

From (4.33) and this equality we compute that for \(k = 2, 3, \ldots, m - 1\),

\[c_{m,m-k} = \frac{1}{2^k} \sum_{l=0}^{k} \frac{l}{l!} \prod_{p=1}^{l} (m - 1 + p) = \frac{1}{2^k} \left\{ \frac{1}{2!} \prod_{p=1}^{k} (m + p) + \sum_{l=3}^{k} \frac{1}{l!} \prod_{p=1}^{l} (m - 1 + p) \right\} \]

\[= \frac{1}{2^k} \left\{ \frac{3}{2!} \prod_{p=1}^{3} (m + p) + \sum_{l=4}^{k} \frac{1}{l!} \prod_{p=1}^{l} (m - 1 + p) \right\} = \frac{1}{2^k k!} \prod_{p=1}^{k} (m + p).\]
Consequently, replacing $k$ with $m - k$, we obtain
\[
c_{m,k} = \frac{(2m - k)!}{2^{m-k}k!(m-k)!} \quad \text{for } k = 0, 1, 2, \ldots, m.
\]

**Step 5.** We derive (4.13).

It is easy to see that for small $h > 0$ and $m = 1, 2, \ldots, [T/h]$, $A \leq (1 + Ce^{-3z_k})/zh^{e^zh}$ and $(b/hz_k)^m \leq 1 + e^{-2zh}$. Besides, since $c_{m,k}/2^m$ is the $m$-th term of the binomial expansion of $(1/2 + 1/2)^{2m-k}$, it is obvious that $c_{m,k}/2^m \leq 1$. Using these facts, we get
\[
\mathcal{G}_h^m[sh(z_h(1 - |\cdot|))](x) \leq \frac{1}{zh^{ez_h}}g_m(|x|) + Czh + \sum_{l=2}^{m} C_l e^zh \leq Czh \leq Ch^{-1}e^{-z_h}
\]

for small $h > 0$. Similarly we observe that
\[
\sum_{l=2}^{m} C_l e^zh \leq \frac{C e^{-zh}}{sh(2zh)} \sum_{l=2}^{m} \frac{1}{2^{l-1,p}} \sum_{p=0}^{l-1} \sum_{q=0}^{p} \frac{(2zh)^q}{q!} \leq C e^{-zh} \sum_{l=2}^{m} \frac{1}{2^l} \leq Ch^{-1}e^{-z_h}
\]

for all $m = 1, 2, \ldots, [T/h]$ and $h > 0$. Setting $a_{m,k} = c_{m,k}/2^m$, we obtain (4.13). □

**Proof of (4.14).** We treat only $\mathcal{G}_h^m[sh(z_h(1 + \cdot))$ because $\mathcal{G}_h^m[sh(z_h(1 - \cdot))]$ can be similarly estimated. Set $H_m(x) := \mathcal{G}_h^m[sh(z_h(1 + \cdot))]$. In this proof, we use the identities in Lemma 4.5, 4.26, 4.27 and the following ones.

(4.34) $\quad sh(z_h(1 - x))sh(2zh(1 + x)) - sh(z_h(1 + x))(sh(2zh) - sh(2zhx)) = 0$,

(4.35) $\quad sh(z_h(1 - x))(ch(2zh(1 + x)) - 1) + sh(z_h(1 + x))(ch(2zhx) - ch(2zh)) = 0$.

**Step 1.** We consider the case $m = 1$.

Using (4.26) and (4.27), we compute that
\[
H_1(x) = \frac{sh(z_h(1 - x))}{2hz_hsh(2zh)} \cdot \frac{1}{2zh}sh(2zh(1 + x)) + \frac{sh(z_h(1 + x))}{2hz_hsh(2zh)} \left\{ (1 - x)ch(2zh) - \frac{1}{2zh}(sh(2zh) - sh(2zhx)) \right\}.
\]

From (4.34) we have
\[
H_1(x) \leq \frac{b}{2hz_h^2}h_1(x) \quad \text{for } x \in [-1, 1], \quad h_1(x) := z_h(1 - x)sh(z_h(1 + x)).
\]

**Step 2.** We estimate the case $m = 2$. 

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It follows from (4.25) and (4.36) that $H_2(x) \leq b \mathcal{G}_h[h_1](x)/2hz_h^2$ for $-1 \leq x \leq 1$. We see from (4.26), (4.27) and Lemma 4.5 that

$$
\mathcal{G}_h[h_1](x) \leq \frac{\text{sh}(z_h(1 - x))}{2hz_h} \left[ \frac{1}{2z_h} \left( (z_h(1 - x)) \text{sh}(2z_h(1 + x)) + \frac{1}{2} (\text{ch}(2z_h(1 + x)) - 1) \right) \right] 
$$

$$
+ \frac{\text{sh}(z_h(1 + x))}{2hz_h} \left[ \frac{1}{2!} z_h(1 - x)^2 \text{ch}(2z_h) + \frac{1}{2z_h} \left\{ z_h(1 - x) \text{sh}(2z_h) + \frac{1}{2} (\text{ch}(2z_h, x) - \text{ch}(2z_h)) \right\} \right].
$$

We use (4.33) and (4.35) to obtain

$$
\mathcal{G}_h[h_1](x) \leq \frac{D}{2hz_h^2} h_2(x), \quad h_2(x) := \left\{ \frac{1}{2!} (z_h(1 - x))^2 + \frac{1}{2 \cdot 1!} (z_h(1 - x)) \right\} \text{sh}(z_h(1 + x)).
$$

Consequently we get

$$
H_2(x) \leq \frac{D^2}{(2hz_h^2)^2} h_2(x) \quad \text{for } x \in [-1, 1].
$$

**Step 3.** We give an estimate for $H_m(x)$ by induction.

Suppose that for $x \in [-1, 1],

$$
H_{m-1}(x) \leq \frac{D^{m-1}}{(2hz_h^2)^{m-1}} h_{m-1}(x), \quad h_{m-1}(x) := \text{sh}(z_h(1 + x)) \sum_{p=1}^{m-1} \frac{d_{m-1,p}(z_h(1 - x))^p}{2k - p!},
$$

It follows from (4.25) and this inequality that $H_m(x) \leq b^{m-1} \mathcal{G}_h[h_{m-1}](x)/(2hz_h^{m-1})$. We easily see by (4.26) and (4.27) that

$$
\mathcal{G}_h[h_{m-1}](x) \leq \frac{\text{sh}(z_h(1 - x))}{2hz_h} \sum_{k=1}^{m-1} \frac{d_{m-1,k}}{2m - 1 - k!} \mathcal{L}_k 
$$

$$
+ \frac{\text{sh}(z_h(1 + x))}{2hz_h} \sum_{k=1}^{m-1} \frac{d_{m-1,k}}{2m - 1 - k!} \left\{ z_h(1 - x)^{k+1} \frac{k^k}{k+1} \text{ch}(2z_h) - \mathcal{M}_k \right\}.
$$

Using Lemma 4.5, 4.34 and 4.35, we have

$$
\text{sh}(z_h(1 - x)) \mathcal{L}_k - \text{sh}(z_h(1 + x)) \mathcal{M}_k 
$$

$$
= \frac{k!}{2z_h} \left\{ \text{sh}(z_h(1 + x)) (Q_{1,k} \text{sh}(2z_h) + Q_{2,k} \text{ch}(2z_h)) + (Q_{2,k} - Q_{0,k}) \text{sh}(z_h(1 - x)) \right\}
$$

$$
=: I_6,
$$

By $Q_{1,k} + Q_{2,k} = \sum_{l=0}^{k} \frac{z_h(1 - x)^{k-l}}{2^l(k - l)!}$ and $Q_{0,k} \leq Q_{2,k}$, we get

$$
I_6 \leq \frac{bk!}{2z_h} \text{sh}(2z_h) \text{sh}(z_h(1 + x)) \sum_{l=0}^{k-1} \frac{(z_h(1 - x)^{k-l}}{2^l(k - l)!}.
$$
Hence we obtain
\[ S_h[h_{m-1}](x) \leq \frac{b\text{sh}(z_h(1 + x))}{2hz_h^2} \sum_{k=1}^{m-1} d_{m-1,k} \left\{ \sum_{l=0}^{k-1} \frac{(z_h(1 - x))^{k-l}}{2^l(k-l)!} + \frac{2(z_h(1 - x))^{k+1}}{(k+1)!} \right\}. \]

Therefore, setting
\[ h_m(x) := \text{sh}(z_h(1 + x)) \sum_{k=1}^{m} \frac{d_{m,k}}{2^{m-k}k!} (z_h(1 - x))^k \]
\[ = \text{sh}(z_h(1 + x)) \sum_{k=1}^{m-1} d_{m-1,k} \left\{ \sum_{l=0}^{k-1} \frac{(z_h(1 - x))^{k-l}}{2^l(k-l)!} + \frac{2(z_h(1 - x))^{k+1}}{(k+1)!} \right\}, \]
we conclude that
\[ H_m(x) \leq \frac{b^m}{(2hz_h)^m} h_m(x) \quad \text{for} \ x \in [-1, 1]. \]

**Step 5.** We determine \( d_{m,k} \)’s.

It follows from the definition of \( h_m \) that for \( m = 1, 2, \ldots, \lfloor T/h \rfloor \) and \( k = 2, 3, \ldots, m-2, \)
\[ d_{m,m} = d_{m-1,m-1} = \cdots = d_{1,1} = 1, \quad d_{2,1} = 1, \]
\[ (4.37) \quad d_{m,m-1} = \sum_{l=1}^{m-1} d_{l,l}, \quad d_{m,m-k} = \sum_{l=1}^{k+1} d_{m-1,m-l}, \quad d_{m,1} = \sum_{l=1}^{m-1} d_{m-1,l}. \]

Here we see that
\[ d_{m,m-1} = m - 1, \quad d_{m,m-2} = \frac{1}{2!}(m - 2)(m + 1). \]

We use these results to obtain \( d_{2,2} = 1, \quad d_{3,1} = d_{3,2} = 2 \) and \( d_{3,3} = 1. \)

By induction we assume that for each \( m \geq 3, \)
\[ d_{m-1,m-1-k} = \frac{1}{k!} (m - 1 - k) \prod_{p=1}^{k-1} (m - 1 + p) \quad \text{for} \ k = 2, 3, \ldots, m - 2. \]

Then we calculate by using (4.37) and this formula that for \( k = 3, 4, \ldots, m - 2, \)
\[ d_{m,m-k} = 1 + (m - 2) + \sum_{p=2}^{k} \frac{(m - 1 - p)}{(p-1)!} \prod_{q=0}^{p-2} (m + q) \]
\[ = \frac{(m - 2)(m + 1)}{2!} + \sum_{p=3}^{k} \frac{(m - 1 - p)}{(p-1)!} \prod_{q=0}^{p-2} (m + q) \]
\[ = \frac{(m - 3)}{3!} \prod_{p=1}^{2} (m + p) + \sum_{p=4}^{k} \frac{(m - 1 - p)}{(p-1)!} \prod_{q=0}^{p-2} (m + q) \]
\[ = \frac{(m - k)}{k!} \prod_{p=1}^{k-1} (m + p) = \frac{(m - k)(m + k - 1)!}{m!k!}. \]
Replacing $k$ with $m - k$, we have
\[ d_{m,k} = \frac{k}{2m - k} \frac{(2m - k)!}{m!(m - k)!} \quad \text{for } k = 2, 3, \ldots, m - 2. \]

This formula clearly holds for $k = 1, m - 1, m$. Thus we obtain (4.14). \hfill \Box

5 Proofs of main results

First we prove Theorem 2.1. Let $x_5 := \min\{x_1, x_4\}$ and $\delta \in (0, T)$. Set $W_1 := [-x_5/2, x_5/2]$, $W_2 := [-x_5, x_5]$, $M_6 := \min\{L_5, x_5^2/16\}$ and
\[ P_h := [0, M_6/|\log h|] \times W_1, \quad Q_{\delta,h} := (\Omega \times [0, T - \delta]) \setminus (\text{int } P_h \cup \{0\} \times W_1). \]

We show that there are $K_1, K_2 > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,
\begin{align*}
\sup_{(t,x) \in P_h} |u(t,x) - u^h(t,x)| &\leq K_1 \sqrt{h}, \\
\sup_{(t,x) \in Q_{\delta,h}} |u(t,x) - u^h(t,x)| &\leq K_2 \sqrt{h}|\log h|.
\end{align*}

Combining these estimates, we obtain the result of Theorem 2.1.

Proof of (5.1). Choose $h_{0,1} > 0$ so small that $h < M_6/|\log h|$ for all $h \in (0, h_{0,1})$. For $t \in [0, h)$, Theorem 3.2 (2) directly yields that
\[ |u(t,x) - u^h(t,x)| = |u(t,x) - \varphi(x)| \leq C\sqrt{h} \quad \text{for all } t \in [0, h), \ x \in \Omega. \]

Hence in the following we consider the case $t \in J_h := [h, M_6/|\log h|]$. The $u$ and $u^h$ are given by, respectively,
\[ u(t,x) = [T(t)u_0](x) - \int_0^t E^{x_5}_y(t-s,x,x_5)u(s,x_5)ds + \int_0^t E^{x_5}_y(t-s,x,-x_5)u(s,-x_5)ds, \]
\[ u^h(t,x) = S^m_{x_5,h}[u_0](x) + \sum_{k=1}^m \frac{u_k(-x_5)}{\text{sh}(2x_5z_h)}S^m_{x_5,h}[-\text{sh}(z_h(x_5 - \cdot))](x) \]
\[ + \sum_{k=1}^m \frac{u_k(x_5)}{\text{sh}(2x_5z_h)}S^m_{x_5,h}[-\text{sh}(z_h(x_5 + \cdot))](x), \]
for $t > 0$, $m = [t/h]$, $x \in W_2$ and $h > 0$. Here the family $\{T(t)\}_{t \geq 0}$ is a contraction and analytic semigroup generated by the operator $Au := -u_{xx} + \beta u$ in $W_2$ and $D(A) = \{u \in C^2(W_2) \mid u(\pm x_1) = 0\}$ (cf. [21 Corollary 3.1.21]). We simply denote $E^{x_5}$, $S_{x_5,h}$ by $S_h$, $E$, respectively if no confusion arises.

Step 1. We estimate $\|T(t)u_0 - S^m_{h}[u_0]\|_{C(W_2)}$.

We use the contraction property of $T(t)$ to have
\[ \|T(t)u_0 - S^m_{h}[u_0]\|_{C(W_2)} \leq \|T(t - mh)u_0 - u_0\|_{C(W_2)} + \|T(h)[u_0] - S_h[u_0]\|_{C(W_2)} \]
\[ + \|T((m - 1)h)[S_h[u_0]] - S^{m-1}_h[u_0]\|_{C(W_2)} =: I_{7,1} + I_{7,2} + I_{7,3}. \]

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Since $u_0(=\varphi)$ is Lipschitz on $\Omega$ and $[T(t-mh)u_0]$ satisfies $u_t-u_{xx}+\beta u=0$, it follows from the theory for parabolic equation that $I_{\tau,1}\leq C\sqrt{h}$ for all $t\in[0,T)$, $m=[t/h]$ and $h>0$. In addition, direct calculations yield that for all $h>0$,

$$I_{\tau,2}\leq \|[T(h)u_0]-u_0\|_{C(W_2)}+\|u_0(x)-G_h[u_0]\|_{C(W_2)}\leq C\sqrt{h}.$$  

As for $I_{\tau,3}$, we notice that $G_h[u_0] \in D(A)$ and $\|G_h[u_0]\|_{C^2(W_2)} \leq C/\sqrt{h}$. Since it follows from [6] Theorem 1.3 that $\|T((m-1)h)-G_h^{m-1}\| \leq Ch$, we get

$$I_{\tau,3}\leq \|[T((m-1)h)-G_h^{m-1}]\|_{C^2(W_2)} \leq C\sqrt{h}$$

for all $m=1,2,\ldots,[T/h]$ and $h>0$. Thus we obtain

$$\sup_{t\in[h,T],m=[t/h]}\|T(t)u_0)-G_h^m[u_0]\|_{C(W_2)} \leq C\sqrt{h} \quad \text{for all } h>0.$$

**Step 2.** We estimate $I_{8,\pm}:=\int_0^t E_y(t-s,x,\pm x_5)u(s,\pm x_5)ds$.

We calculate that

$$|E_y(t-s,x,\pm x_5)-E_{0,y}(t-s,x,\pm x_5)\|_{C(W_2)} \leq Ce^{-x_5^2/16(t-s)}$$

for all $t,s \in (0,T)$ ($t \neq s$) and $x \in W_1$. It is seen by this estimate that for $t \in J_h$ and $x \in W_1$,

$$|I_{8,\pm}| \leq \|u\|_{C([0,T] \times \Omega)}\int_0^t (|E_{0,y}(t-s,x,x_5)|+e^{-x_5^2/16(t-s)})ds$$

$$\leq M_{7,1}\int_0^t \frac{x_5-x}{(t-s)^{3/2}}e^{-(x_5-x)^2/4(t-s)}ds + M_{7,2}h|\log h|^{-1} =: I_{8,1}+M_{7,2}h|\log h|^{-1}.$$  

Here and in the sequel the constants $M_{7,1}$'s $(i \geq 1)$ depend on $x_5$, but not on $h>0$. Setting $r=(x_5-x)/2\sqrt{t-s}$ and using $\int_a^{+\infty}e^{-r^2}dr \leq e^{-a^2}/a$ for $a>0$, we have

$$I_{8,1} \leq M_{7,3}\int_{x_5/4\sqrt{t}}^{+\infty}e^{-r^2}dr \leq \frac{4M_{7,3}\sqrt{t}}{x_5}e^{-x_5^2/16t} \leq M_{7,4}h \quad \text{for all } t \in J_h, x \in W_1 \text{ and } h>0.$$  

Therefore we get $I_{8,\pm} \leq M_{7,4}h$ for all $t \in J_h$. We can get $I_{8,\pm} \leq M_{7,3}h$ for all $t \in J_h$, $x \in W_1$ and small $h>0$ by the same way as above. Hence we have

$$|I_{8,\pm}| \leq M_{7}h \quad \text{for all } t \in J_h, x \in W_1 \text{ and small } h>0.$$  

**Step 3.** We estimate $I_{9,\pm}:=\sum_{k=1}^m \frac{u_k(\pm x_1)}{sh(2x_1\pm y_5)}G_h^{m-k}[\sh(z_h(x_1 \mp \cdot))](x)$.

It directly follows from the proof of Theorem [15] that $|I_{9,\pm}| \leq M_{7,6}h$ for all $m=1,2,\ldots,[M_6/h]\log h|$, $x \in W_1$ and $h>0$.

Therefore we have (5.1) for $h \in (0,h_{0,1})$. □
Next we prove (5.2). The point is to estimate the difference \((u_m - u_n)/h - u_t\). To do so, we use the method similar to \[\text{(11)},\] the precise comparison argument of viscosity solutions.

Before proving (5.2), we recall the definition and some elementary properties of the parabolic 2-jets. Let \(W \subset \mathbb{R}\) be an open interval. For \(u : (0, T) \times W \to \mathbb{R}\), we define \(\mathcal{P}^{2,+} u(t, x)\) and \(\mathcal{P}^{2,-} u(t, x)\) as follows:

\[
\mathcal{P}^{2,+} u(t, x) := \left\{ (a, p, X) \in \mathbb{R}^3 \mid \begin{array}{l}
    u(t + s, x + h) \leq u(t, x) + as + ph + \frac{1}{2}Xh^2 \\
    \quad + o(|s| + |h|^2) \; \text{as} \; (s, h) \to (0, 0)
\end{array} \right\},
\]

\[
\mathcal{P}^{2,-} u(t, x) := \left\{ (a, p, X) \in \mathbb{R}^3 \mid \begin{array}{l}
    \exists \{(t_n, x_n, a_n, p_n, X_n)\}_{n=1}^{\infty} \subset (0, T) \times W \times \mathbb{R}^3 \; \text{such that} \\
    \quad (t_n, x_n, u(t_n, x_n), a_n, p_n, X_n) \longrightarrow (t, x, u(t, x), a, p, X) \\
    \quad \text{as} \; n \to +\infty \; \text{and} \; (a_n, p_n, X_n) \in \mathcal{P}^{2,+} u(t_n, x_n)
\end{array} \right\},
\]

\[
\mathcal{P}^{2,-} u(t, x) := \left\{ (a, p, X) \in \mathbb{R}^3 \mid \begin{array}{l}
    (-a, -p, -X) \in \mathcal{P}^{2,+}(-u(t, x))
\end{array} \right\}.
\]

We use the following lemma to obtain (5.2).

**Lemma 5.1** Let \(u, u_t \in C((0, T) \times W)\). For any \((t, x) \in (0, T) \times W\), if \((a, p, X) \in \mathcal{P}^{2,+} u(t, x)\) (or \(\mathcal{P}^{2,-} u(t, x)\)), then \(a = u_t(t, x)\).

**Proof.** Since \(u\) is differentiable with respect to \(t\), we can easily show that for any \((t, x) \in (0, T) \times W\), if \((a, p, X) \in \mathcal{P}^{2,+} u(t, x)\), then \(a = u_t(t, x)\). The assertion follows from the continuity of \(u_t\) and the definition of \(\mathcal{P}^{2,+} u(t, x)\).

The case \((a, p, X) \in \mathcal{P}^{2,-} u(t, x)\) is proved similarly. \(\square\)

**Proof of (5.2).** First, we show that for any \(\delta > 0\), there exist \(K_{2,1} > 0\) and \(h_{0,2} > 0\) such that

\[
\sup_{(t, x) \in Q_T} (u(t, x) - \overline{u}^h(t, x)) \leq K_{2,1} \sqrt{h} \log h \quad \text{for all } h \in (0, h_{0,2}).
\]

**Step 1.** We define \(\overline{u}^h(t, x)\) by

\[
\overline{u}^h(t, x) := \begin{cases}
    u_0(x) & \text{for } t \in [0, h], \; x \in \overline{\Omega}, \\
    u_m(x) & \text{for } t \in (mh, (m + 1)h], \; m = 1, 2, \ldots, \lfloor T/h \rfloor \text{and } x \in \overline{\Omega}.
\end{cases}
\]

For any \(\delta \in (0, T)\), put \(T_{\delta/2} := T - \delta/2\) and define

\[
\Phi(t, x, s, y) := u(t, x) - \overline{u}^h(t, x) - \frac{1}{2\sqrt{h}}(t - s)^2 - \frac{1}{2\sqrt{h}}(x - y)^2 - \frac{\sqrt{h}}{T_{\delta/2} - t} - \frac{\sqrt{h}}{T_{\delta/2} - s}.
\]

Then \(\Phi\) is upper semicontinuous on \(Q_{\delta/2,h} \times Q_{\delta/2,h}\) and \(\Phi \longrightarrow -\infty\) as \(t, s \nearrow T_{\delta/2}\). Let \((\overline{t}, \overline{x}, \overline{s}, \overline{y})\) be a maximum point of \(\Phi\) on \(Q_{\delta/2,h} \times Q_{\delta/2,h}\). We may consider \(\Phi(\overline{t}, \overline{x}, \overline{s}, \overline{y}) \geq 0\) because if otherwise, we easily get (5.3) with \(K_{2,1} = 4/\delta\).

**Step 2.** We show that there is \(h_{0,3} > 0\) such that

\[
\sup_{(t, \overline{s}) \in Q_T} (u(\overline{t}, \overline{x}) - \overline{u}^h(\overline{s}, \overline{y})) \leq Ch^{1/4} \log h^{1/2} \quad \text{for all } h \in (0, h_{0,3}).
\]
For this purpose, we first study the behavior of \((\ell, \bar{x}, \bar{s}, \bar{y})\). It directly follows from 
\(\Phi(\ell, \bar{x}, \bar{s}, \bar{y}) \geq 0\) that
\[
\frac{1}{2}\sqrt{h}(\ell - \bar{s})^2 + \frac{\sqrt{h}}{T_{\delta/2} - \bar{t}} + \frac{\sqrt{h}}{T_{\delta/2} - \bar{s}} \leq u(\ell, \bar{x}) - u^h(\bar{s}, \bar{y}) \leq C.
\]
Hence we get
\[
(5.6) \quad |\ell - \bar{s}| \leq Ch^{1/4}.
\]
Besides, since it is easily seen from \(\Phi(\ell, \bar{y}, \bar{s}, \bar{y}) \leq \Phi(\ell, \bar{x}, \bar{s}, \bar{y})\) and Theorem 3.2 (2) that
\[
\frac{1}{2}\sqrt{h}(\bar{x} - \bar{y})^2 \leq u(\ell, \bar{x}) - u(\ell, \bar{y}) \leq C|\bar{x} - \bar{y}|,
\]
we have
\[
(5.7) \quad |\bar{x} - \bar{y}| \leq C\sqrt{h}.
\]
To obtain (5.4), we divide our consideration into several cases. Set \(\partial Q_{\delta/2,h} := \partial Q_{\delta/2,h}\setminus\{(T_{\delta/2}) \times \Omega\}\).

Case 1. \((\ell, \bar{x})\) or \((\bar{s}, \bar{y})\) \(\in \partial Q_{\delta/2,h}\).
We may assume \((\ell, \bar{x}) \in \partial Q_{\delta/2,h}\) since the other case can be treated by the same way.

Subcase 1-1. \(\bar{x} \in \partial \Omega\) or \((\ell = 0\) and \(|\bar{x}| > x_1/2\).
Then \(u(\ell, \bar{x}) = \varphi(\bar{x})\) and we get by Theorem 4.3 and (5.7)
\[
u(\ell, \bar{x}) - u^h(\bar{s}, \bar{y}) \leq \varphi(\bar{x}) - \varphi(\bar{y}) \leq C|\bar{x} - \bar{y}| \leq C\sqrt{h}.
\]

Subcase 1-2. \(\ell \in [0, x_0/32] \log h\] and \(|\bar{x}| \leq x_1/2\).
It follows from Theorem 4.3 (2), (5.1), (5.6) and (5.7) that for small \(h > 0\),
\[
u(\ell, \bar{x}) - u^h(\bar{s}, \bar{y}) \leq u(\ell, \bar{x}) - u^h(\ell, \bar{x}) + u^h(\ell, \bar{x}) - u^h(\bar{s}, \bar{y}) 
\leq C\sqrt{h} + C(h^{1/4} \sqrt{\log h} + h^{1/2}) \leq C\sqrt{h}.
\]

Case 2. \((\ell, \bar{x}), (\bar{s}, \bar{y}) \in \text{int} Q_{\delta/2,h}\).
Using the maximum principle for semicontinuous functions, we can find \(a, b, X, Y \in \mathbb{R}\) satisfying
\[
(5.8) \quad \frac{3}{h} I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{3}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad a - b = \frac{\sqrt{h}}{(T_{\delta/2} - \ell)^2} + \frac{\sqrt{h}}{(T_{\delta/2} - \bar{s})^2}.
\]
Set \(m = \lfloor \bar{s}/h \rfloor\). Then \(u^h(\bar{s}, \bar{y}) = u_m(\bar{y})\). In addition, \(((\bar{x} - \bar{y})/\sqrt{h}, Y) \in J^{2,2} - u_m(\bar{y})\) and \(X \leq Y\) (See [8] or [17] for the definitions of \(J^{2,2}\) and the maximum principle for semicontinuous functions).

We estimate the difference \((u_m(\bar{y}) - u_{m-1}(\bar{y}))/h\). It follows from \(\Phi(\ell, \bar{x}, \bar{s}, \bar{y}) = 0\), (5.8), Theorem 3.3 and Lemma 5.1 with \(W = \Omega\) that
\[
(5.9) \quad a = u(\ell, \bar{x}) = \frac{1}{\sqrt{h}}(\ell - \bar{s}) + \frac{\sqrt{h}}{(T_{\delta/2} - \ell)^2}, \quad b = \frac{1}{\sqrt{h}}(\ell - \bar{s}) - \frac{\sqrt{h}}{(T_{\delta/2} - \bar{s})^2}.
\]
Noting $b \geq 0$ by Theorem 4.1, we get $\bar{t} \geq \bar{s}$. We see from (3.8), the first formula of (5.9) and this fact that $|\bar{t} - \bar{s}| \leq C \sqrt{h} |\log h|$. Substituting this into the second formula of (5.9), we have $1/(T_{\delta/2} - \bar{s}) \leq C h^{-1/4} \sqrt{|\log h|}$. Thus $\Phi(\bar{t}, \bar{z}, \bar{s}, \bar{y}) \leq \Phi(\bar{t}, \bar{z}, \bar{s}, \bar{y})$, (5.9) and this estimate yield that

$$
\frac{u_m(\bar{y}) - u_{m-1}(\bar{y})}{h} \leq \frac{1}{2h^{3/2}} \left\{ \frac{(\bar{t} - (\bar{s} - h))^2 - (\bar{t} - \bar{s})^2}{h^{-1/2}} + \frac{h^{-1/2}}{T_{\delta/2} - (\bar{s} - h)} \right\}
\leq b + \frac{1}{2} \sqrt{h} + \frac{h^{3/2}}{(T_{\delta/2} - \bar{s})^3} \leq b + C \sqrt{h}.
$$

By the way, since $u$ is a viscosity subsolution of (2.2) and $u_m$ is a viscosity supersolution of (2.4), we have the following inequalities.

$$
\min \{ a - X + \beta u(\bar{t}, \bar{z}), u(\bar{t}, \bar{z}) - \varphi(\bar{z}) \} \leq 0,
$$

$$
\min \left\{ \frac{u_m(\bar{y}) - u_{m-1}(\bar{y})}{h} - Y + \beta u_m(\bar{y}), u_m(\bar{y}) - \varphi(\bar{y}) \right\} \geq 0.
$$

If $u(\bar{t}, \bar{z}) - \varphi(\bar{z}) \leq 0$ in (5.11), then we easily have by the above inequalities and (5.7)

$$
u(\bar{t}, \bar{z}) - u_m(\bar{y}) \leq \varphi(\bar{z}) - \varphi(\bar{y}) \leq C \sqrt{h}.
$$

Thus, in the sequel we assume $u(\bar{t}, \bar{z}) - \varphi(\bar{z}) > 0$ for small $h > 0$. Then by (5.11),

$$
a - X + \beta u(\bar{t}, \bar{z}) \leq 0 \quad \text{for small } h > 0.
$$

On the other hand, we easily get from (5.10) and (5.12) that

$$
b + C \sqrt{h} - Y + \beta u_m(\bar{y}) \geq 0.
$$

Combining (5.8), (5.13) with this inequality, we have

$$
u(\bar{t}, \bar{z}) - u^h(\bar{s}, \bar{y}) = u(\bar{t}, \bar{z}) - u_m(\bar{y}) \leq C \sqrt{h}.
$$

Taking $h_{0,3} > 0$ small enough, we conclude that (5.4) holds for all $h \in (0, h_{0,3})$.

Step 3. We improve (5.4) and establish (5.3).

Substituting (5.4) into (5.5), we obtain

$$
I_{10} := \frac{1}{T_{\delta/2} - \bar{t}} + \frac{1}{T_{\delta/2} - \bar{s}} \leq Ch^{-1/4} \sqrt{|\log h|}.
$$

We observe from $\Phi(\bar{s}, \bar{z}, \bar{s}, \bar{y}) \leq \Phi(\bar{t}, \bar{z}, \bar{s}, \bar{y})$, Theorem 3.2 (2) and this inequality that

$$
\frac{1}{2 \sqrt{h}} (\bar{t} - \bar{s})^2 \leq C |\log h| |\bar{t} - \bar{s}|.
$$
Hence we get \( |\tilde{r} - \tilde{s}| \leq C\sqrt{h}|\log h| \). By using this estimate, we improve the estimates in Case 1 as follows.

\[
(5.16) \quad u(\tilde{r}, \tilde{s}) - \tilde{u}^h(\tilde{s}, \tilde{y}) \leq C\sqrt{h}|\log h|^{3/2} \quad \text{if } (\tilde{r}, \tilde{s}) \in \partial_p Q_{\delta/2,h}.
\]

Therefore, \((5.4)\) can be improved in the following way:

\[
u(\tilde{r}, \tilde{s}) - \tilde{u}^h(\tilde{s}, \tilde{y}) \leq C\sqrt{h}|\log h|^{3/2}.
\]

Substituting this into \((5.5)\) again, we get \( I_{10} \leq C|\log h|^{3/2} \). Repeating the above argument, we have \( |\tilde{r} - \tilde{s}| \leq C\sqrt{h}|\log h| \) and improve \((5.16)\) as

\[
u(\tilde{r}, \tilde{s}) - \tilde{u}^h(\tilde{s}, \tilde{y}) \leq C\sqrt{h}|\log h| \quad \text{if } (\tilde{r}, \tilde{s}) \in \partial_p Q_{\delta/2,h}.
\]

Consequently, we have from \((5.14)\) and this estimate

\[
\Phi(\tilde{r}, \tilde{s}, \tilde{y}) \leq C\sqrt{h}|\log h| \quad \text{for all } h \in (0, h_{0,3}).
\]

Choosing a large \( K_{2,1} \geq C + 4/\delta \), we obtain \((5.3)\).

Next, we prove that for any \( \delta > 0 \), there are \( K_{2,2} > 0 \) and \( h_{0,4} > 0 \) such that

\[
(5.17) \quad \sup_{(t, x) \in Q_{\delta,h}} (u^h(t, x) - u(t, x)) \leq K_{2,2}\sqrt{h}|\log h| \quad \text{for all } h \in (0, h_{0,4}).
\]

**Step 4.** Let \( u^h \) be defined by \((2.5)\). For any \( \delta \in (0, T) \), define

\[
\Phi(t, x, s, y) := u^h(t, x) - u(s, y) - \frac{1}{2\sqrt{h}}(t - s)^2 - \frac{1}{2\sqrt{h}}|x - y|^2 - \frac{\sqrt{h}}{T_{\delta/2} - t} - \frac{\sqrt{h}}{T_{\delta/2} - s}.
\]

Let \((\tilde{r}, \tilde{s}, \tilde{y}) \in Q_{\delta/2,h} \times Q_{\delta/2,h} \) be a maximum point of \( \Phi \). We may consider \( \Phi(\tilde{r}, \tilde{s}, \tilde{y}) \geq 0 \). Note that \((5.5), (5.6), \) and \((5.7)\) hold.

If \((\tilde{r}, \tilde{s}) \in \partial_p Q_{\delta/2,h} \), then we see by similar arguments to those in Case 1 of Step 2 that

\[
u^h(\tilde{r}, \tilde{s}) - u(\tilde{s}, \tilde{y}) \leq C h^{1/4}|\log h| \quad \text{for small } h > 0.
\]

Thus we may assume \((\tilde{r}, \tilde{s}), (\tilde{s}, \tilde{y}) \in \text{int } Q_{\delta/2,h} \). The maximum principle for semicontinuous functions yields \( a, b, X, Y \in \mathbb{R} \) satisfying

\[
(a, (\overline{x} - \overline{y})/\sqrt{h}, X) \in \mathcal{P}^{2+} u^h(\tilde{r}, \tilde{s}), \quad (b, (\overline{x} - \overline{y})/\sqrt{h}, Y) \in \mathcal{P}^{2-} u(\tilde{s}, \tilde{y}),
\]

\[
-3 \frac{h}{\sqrt{h}} I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3 \frac{h}{\sqrt{h}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad a - b = \frac{\sqrt{h}}{(T_{\delta/2} - t)^2} + \frac{\sqrt{h}}{(T_{\delta/2} - s)^2}.
\]

Let \( m = [\tilde{r}/h] \). Then note that \( u^h(\tilde{r}, \tilde{s}) = u_m(\overline{x}) \) and hence \((\overline{x} - \overline{y})/\sqrt{h}, X) \in \mathcal{P}^{2+} u_m(\overline{x}) \) and \( X \leq Y \).

We estimate \((u_m(\overline{x}) - u_{m-1}(\overline{x}))/h \). We see by \( \Phi_s(\tilde{r}, \tilde{s}, \tilde{y}) = 0 \) and Lemma \((5.1)\) that

\[
(5.18) \quad a = \frac{1}{\sqrt{h}}(\overline{r} - \overline{s}) + \frac{\sqrt{h}}{(T_{\delta/2} - t)^2}, \quad b = u_t(\tilde{s}, \tilde{y}) = \frac{1}{\sqrt{h}}(\overline{r} - \overline{s}) - \frac{\sqrt{h}}{(T_{\delta/2} - s)^2}.
\]
Note that $\bar{t} \geq \bar{s}$ by $u_t(\bar{s}, \bar{y}) \geq 0$. Dividing $\Phi(\bar{t}, \bar{s}, \bar{s}, \bar{y}) \geq \Phi(\bar{t} - h, \bar{s}, \bar{s}, \bar{y})$ by $h$, we observe from this fact that

$$\frac{u_m(\bar{x}) - u_{m-1}(\bar{x})}{h} \geq \frac{1}{2h^{3/2}} \{((\bar{t} - \bar{s})^2 - ((\bar{t} - h) - \bar{s})^2\} - \frac{h^{-1/2}}{T_{\delta/2} - (\bar{t} - h)} + \frac{h^{-1/2}}{T_{\delta/2} - \bar{t}} \geq a - \frac{1}{2} \sqrt{h} - \frac{h^{3/2}}{(T_{\delta/2} - \bar{t})^3}.$$ 

The (4.1), (5.5) and the fact $\bar{t} \geq \bar{s}$ yield that $1/(T_{\delta/2} - \bar{t}) \leq Ch^{-1/4} \sqrt{|\log h|}$ for small $h > 0$. Using this estimate, we have

$$\frac{u_m(\bar{x}) - u_{m-1}(\bar{x})}{h} \geq a - C\sqrt{h} \quad \text{for small } h > 0.$$ 

Since the remainder is totally similar to Step 1, we have

$$u^h(\bar{t}, \bar{x}) - u(\bar{s}, \bar{y}) \leq C\sqrt{h} |\log h|.$$ 

Thus taking $h_{0,4} > 0$ small, we obtain (5.17).

Taking $K_2 := \max\{K_{2,1}, K_{2,2}\} + 4/\delta$ and $h_{0,2} := \min\{h_{0,3}, h_{0,4}\}$, we have the result. □

We establish the result of Theorem 2.1 by choosing $h_0 := \min\{h_{0,1}, h_{0,2}\}$.

The proof of Theorem 2.2 is similar to [14], based on the limit operation of viscosity solutions due to [3, 4].

**Proof of Theorem 2.2.** Recall that $x^h(t)$ is given by (2.5). Define

$$\hat{x}(t) := \limsup_{s \to t, h \to 0} x^h(s), \quad \underline{x}(t) := \liminf_{s \to t, h \to 0} x^h(s).$$

Notice $\hat{x}(0) = \underline{x}(0) = x_0$ by Theorem 4.7.

We show $x^* \leq \underline{x}$ in $[0, T)$. Fix $t \in (0, T)$ and $x > \underline{x}(t)$. Then there exist sequences $\{h_n\}_{n=1}^{+\infty}$ and $\{m_n\}_{n=1}^{+\infty}$ such that as $n \to +\infty$,

$$h_n \to 0, \quad m_n h_n \to t, \quad x^{h_n}(s_n) = x_{m_n} \to \underline{x}(t).$$

Since $x > x_{m_n}$ for large $n \in \mathbb{N}$, we get $u_{m_n}(s_n, x) = u_{m_n}(x) = \varphi(x)$. Letting $n \to +\infty$, we have $u(t, x) = \varphi(x)$ by Theorem 2.1 and thus $x^*(t) \leq \underline{x}(t)$.

To prove $\hat{x} \leq x^*$ in $[0, T)$, we suppose $\hat{x}(t_0) \geq x^*(t_0) + 6\varepsilon_0$ for some $t_0 \in (0, T)$, $\varepsilon_0 > 0$ and get a contradiction. By the continuity of $x^*$ (cf. (2.3)), there exists $\delta > 0$ such that

$$\hat{x}(t_0) > x^*(t_0) + 5\varepsilon_0 \quad \text{for all } t \in (t_0 - 5\delta, t_0 + 5\delta).$$

Choose $\{h_n\}_{n=1}^{+\infty}$ and $\{m_n\}_{n=1}^{+\infty}$ satisfying

$$h_n \to 0, \quad m_n h_n \to t_0, \quad x^{h_n}(m_n h_n) \to \hat{x}(t_0) \quad \text{as } n \to +\infty.$$
Take \( n_0 \in \mathbb{N} \) such that
\[
|m_n h_n - t_0| < \delta, \ |x^{h_n}(m_n h_n) - \tilde{x}(t_0)| < \varepsilon_0 \quad \text{for all} \ n > n_0.
\]
Using (2.3), Theorem 4.7 and these facts, we observe that
\[
x^{h_n}(t) \geq x^{h_n}(m_n h_n) \geq x^*(t) + 4\varepsilon_0 \quad \text{for all} \ t \geq m_n h_n \text{ and } n > n_0.
\]
This implies that for all \( n > n_0 \),
\[
(5.21) \quad u^{h_n} > \varphi \quad \text{in} \ Q := (t_0 + 2\delta, t_0 + 4\delta) \times (x^*(t_0) + \varepsilon_0, x^*(t_0) + 3\varepsilon_0).
\]
On the other hand, we notice the fact that
\[
(5.22) \quad \varphi \quad \text{in} \ Q.
\]
\[
\text{Fix } y_0 \in (x^*(t_0) + 2\varepsilon_0, x^*(t_0) + 4\varepsilon_0). \quad \text{We derive}
\]
\[
(5.23) \quad (t_n, y_n) \rightarrow (t_0 + 3\delta, y_0), \ u^{h_n}(t_n, y_n) \rightarrow u(t_0 + 3\delta, y_0) = \varphi(y_0) \quad \text{as } n \to \infty.
\]
Put \( \tilde{m}_n = [t_n/h_n] \). Then from (5.21), \( u^{\tilde{m}_n}(y_n) = u^{h_n}(t_n, y_n) > \varphi(y_n) \) for \( n > n_0 \). Using the fact that \( u^{\tilde{m}_n} \) is a viscosity subsolution of (2.4) with \( m = \tilde{m}_n \) and Theorem 4.1 we have the following inequality.
\[
-\varphi_{xx}(y_n) + \beta u^{\tilde{m}_n}(y_n) \leq 0.
\]
Letting \( n \to +\infty \), we get (5.22).

However, (5.22) contradicts to (3.10) because of \( y_0 > x^*(t_0) \geq x_0 \). Thus we have \( \tilde{x} \leq x^* \) in \( [0, T] \) and conclude that \( \tilde{x} = x = x^* \) in \( [0, T] \). Applying [8, Section 6], we complete the proof. \( \square \)

6 Appendix

6.1 Formal asymptotic expansion for (2.4)

This subsection is devoted to the formal asymptotic expansion of the solution of (2.4) with \( m = 1 \) near the free boundary as \( h \searrow 0 \).

Let \( x^* > 0 \) be the free boundary of (2.4) with \( m = 1 \). From the facts \( u_1 = \varphi \) on \( [x^*, 1] \) and \( u_1 \in C^1(\Omega) \), it is sufficient to treat the following problem instead of (2.4):

\[
(6.1) \quad \frac{u - \varphi}{h} - u_{xx} + \beta u = 0 \quad \text{for} \ x < x^*, \ u(x^*) = \varphi(x^*), \ u_x(x^*) = \varphi_x(x^*),
\]

We rewrite (6.1). Set \( w(x) := (u(x) - \varphi(x))/e^{\alpha x} \). Then \( w \) satisfies
\[
(6.2) \quad -hw_{xx} - 2\alpha hw_x + (1 + rh)w = h(-q e^x + r) \quad \text{for} \ x < x^*, \ w(x^*) = w_x(x^*) = 0.
\]
The solution $w$ of this problem is given by

\[(6.3) \quad w(x) = k_1e^{\lambda_+(x-x^*)} + k_2e^{\lambda_-(x-x^*)} + h \left( \frac{-qe^x}{1+qh} + \frac{r}{1+rh} \right) \text{ for some } k_1, k_2 \in \mathbb{R},\]

where $\lambda_\pm := -\alpha \pm zh$. Since we see from (6.2) that $w(x) = O(h)$ as $h \to 0$, we have $k_2 = 0$. Moreover, we observe by the conditions for $u$ at $x^*$ and Taylor expansion to log$(1 + s)$ around $s = 0$ that as $h \searrow 0$,

\[(6.4) \quad x^* = x_0 + \log \left\{ \frac{(zh - \alpha)(1 + qh)}{(zh - \alpha - 1)(1 + rh)} \right\} = x_0 + \log \left( 1 + \frac{\sqrt{h}}{\sqrt{1 + \beta h + \alpha \sqrt{h}}} \right) \]

\[= x_0 + \frac{\sqrt{h}}{\sqrt{1 + \beta h + \alpha \sqrt{h}}} - \frac{1}{2} \left( \frac{\sqrt{h}}{\sqrt{1 + \beta h + \alpha \sqrt{h}}} \right)^2 + O(h^{3/2})\]

\[= x_0 + \sqrt{h} - (\alpha + 1/2)h + O(h^{3/2}),\]

\[k_1 = \frac{rh^{3/2}}{1 + qh} + O(h^2).\]

where $x_0$ is given by (2.3). Therefore using (6.3) with these results, we have $u(x) - \varphi(x) = O(h^{3/2})$ near $x < x^*$.

To obtain the asymptotic expansion of $w$ in terms of $h$, from the above estimate, we may assume that $w$ can be expanded as follows:

\[(6.5) \quad w(x) = h^{3/2}w_3(\rho) + h^2w_4(\rho) + O(h^{5/2}).\]

Here $\rho := (x - x^*)/2\sqrt{h}$. We impose the following from the conditions for $w$ at $x^*$ in (6.2):

\[(6.6) \quad w_i(0) = w'_i(0) = 0 \quad \text{for } i = 3, 4.\]

Substituting (6.5) into (6.2), we have by $\beta = \alpha^2 + r$ and Taylor expansion to exp$(x_0 + \sqrt{h} - (\alpha + 1/2)h + O(h^{3/2}))$ as $h \searrow 0$

\[(6.7) \quad h^{3/2} \left( -\frac{w''_3}{4} + w_3 \right) + h^2 \left( -\frac{w''_4}{4} - \alpha w'_4 + w_4 \right)\]

\[= -r \{ h^{3/2}(2\rho + 1) + h^2(2\rho^2 + 2\rho - \alpha) \} + O(h^{5/2}).\]

Here we have neglected $k_1e^{\lambda_{y'}}$ since this is smaller than the last term of (6.3) for $x < x^*$ and $h \searrow 0$.

We determine $w_i$’s ($i = 3, 4$) from the above expansion. Comparing both sides of (6.7), we can derive the following:

\[w_3 - \frac{w''_3}{4} + r(2\rho + 1) = 0, \quad w_4 - \frac{w''_4}{4} - \alpha w'_4 + r \{ 2\rho^2 + 2\rho - \alpha \} = 0.\]

Solving these equations under (6.6) we obtain

\[w_3(\rho) = r(e^{2\rho} - 1 - 2\rho), \quad w_4(\rho) = r \{ e^{2\rho} - (1 + 2\rho + 2\rho^2) + \alpha(e^{2\rho} - 2\rho e^{2\rho} - 1) \}.\]

Therefore we conclude that as $h \to 0$,

\[u(x) = \varphi(x) + e^{\alpha x} \left\{ h^{3/2}w_3 \left( \frac{x - x^*}{\sqrt{h}} \right) + h^2w_4 \left( \frac{x - x^*}{\sqrt{h}} \right) + O(h^{5/2}) \right\} \quad \text{near } x < x^*.\]
6.2 Proof of (4.17)

First, we may assume $m \geq 3$ and consider the case $k = 1, 2, \ldots, m - 1$. Because the case $m = 1, 2$ or $k = 0, m$ is easily proved. We see from Stirling’s formula that for all $p \in \mathbb{N}$,

$$1 + \frac{1}{12p} \leq \frac{p!}{\sqrt{2\pi} p^{p+1/2} e^{-p}} \leq 1 + \frac{1}{12p} + \frac{C}{p^2},$$

where $C$ is independent of $p$. Using this inequality with $p = m, 2m - k, m - k$, we observe that for $k = 1, 2, \ldots, m - 1$,

$$1 + \frac{1}{12p} \leq \frac{2m - k}{2m - k!} \leq 1 + \frac{C}{p^2},$$

Using this inequality with $p = m, 2m - k, m - k$, we observe that for $k = 1, 2, \ldots, m - 1$,

$$(2m - k)! \leq (m - k)! \leq 1 + \frac{C}{m},$$

Set $\delta = \delta(m, k) := k/\sqrt{2m - k}$. Then we have

$$\left(\frac{m}{m - k/2}\right)^{m+1/2} \left(\frac{m - k}{m - k/2}\right)^{-m+1/2} = \left(1 + \frac{\delta}{\sqrt{2m - k}}\right)^{m+1/2} \left(1 - \frac{\delta}{\sqrt{2m - k}}\right)^{-m+1/2}.$$  \hspace{1cm} (6.8)

Denote the RHS of this formula by $I_{11}$. Since it follows from Taylor’s expansion that

$$\log(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{x^4}{4(1 + \theta)^4}, \quad |\theta| < |x| < 1,$$

we obtain for $k = 1, 2, \ldots, m - 1$,

$$\log I_{11} \geq \left(m + \frac{1}{2}\right) \left(\frac{\delta}{\sqrt{2m - k}} - \frac{\delta^2}{2(2m - k)} + \frac{1}{3} \frac{\delta^3}{(2m - k)^{3/2}} - \frac{1}{4} \frac{\delta^4}{(2m - k)^2}\right)$$

$$+ \left(m - k + \frac{1}{2}\right) \left(-\frac{\delta}{\sqrt{2m - k}} - \frac{\delta^2}{2(2m - k)} - \frac{1}{3} \frac{\delta^3}{(2m - k)^{3/2}} - \frac{1}{4} \frac{\delta^4}{(2m - k)^2}\right)$$

$$= \frac{1}{2} \delta^2 - \frac{\delta^2}{2(2m - k)} + \frac{1}{3} \frac{\delta^3}{(2m - k)} - \frac{\delta^4}{4(2m - k)} \left(1 + \frac{1}{2m - k}\right).$$

Since it is easily seen by $m \geq 3$ that

$$\frac{1}{3} - \frac{1}{4} \left(1 + \frac{1}{2m - k}\right) \geq \frac{1}{12} - \frac{1}{4m} \geq 0,$$

Thus we get

$$\log I_{11} \geq \frac{1}{2} \delta^2 - \frac{\delta^2}{2(2m - k)} > \frac{1}{4} \delta^2.$$  \hspace{1cm} (4.17)

Therefore we obtain (4.17). □

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