SU(2) WZNW model at higher genera from gauge field functional integral

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Abstract
We compute the gauge field functional integral giving the scalar product of the SU(2) Chern-Simons theory states on a Riemann surface of genus > 1. The result allows to express the higher genera partition functions of the SU(2) WZNW conformal field theory by explicit finite dimensional integrals. Our calculation may also shed new light on the functional integral of the Liouville theory.

Dedicated to Ludwig Dmitrievich Faddeev for the 60th birthday

1. Introduction

Since the 1967 seminal paper of Faddeev and Popov [1], the functional integral has become the main tool in the treatment of quantum gauge theories. The main breakthrough which this paper has achieved was the realization that the gauge invariance is not an obstruction but an aid in the treatment of quantum gauge fields. Subsequently, this idea has revealed its full power and the gauge invariance has become the cornerstone of modern theoretical physics.

In the present note, we shall discuss how the same idea allows an explicit solution of the WZNW model of conformal field theory on an arbitrary two-dimensional surface. To make things simpler, we shall limit the discussion to the partition functions of the SU(2) WZNW model on closed compact Riemann surfaces Σ of genus > 1. The correlation functions at genus zero were discussed along similar lines in [2] for the SU(2) case and in [3] for the general compact groups. Higher genera, however, are more difficult and took some time to understand. The aim of this note is to present the main points of the argument leaving the technical details to the forthcoming publication [4]. The complete
work is a small tour de force. This fact was hard to hide even in a softened exposition which may only envy the work [1] its striking simplicity.

Our approach is based on the relation between the WZNW partition functions and the Schrödinger picture states of the Chern-Simons (CS) theory [2][3]. The partition function of the level $k$ ($k = 1, 2, \ldots$) SU(2) WZNW model in an external two-dimensional $su(2)$ gauge field $A \equiv A_\bar{z} dz + A_z d\bar{z}$ is formally given by the functional integral

$$Z(A) = \int e^{-kS(g,A)} \, Dg$$

(1.1)

over $g : \Sigma \to SU(2)$. On the other hand, the SU(2) CS states are holomorphic functionals $\Psi$ of the gauge field $A_\bar{z}$ s.t. for $h : \Sigma \to SL(2, \mathbb{C})$,

$$e^{-kS(h,A_\bar{z})} \Psi(h^{-1}A_\bar{z}) = \Psi(A_\bar{z}) ,$$

(1.2)

where $h^{-1}A_\bar{z} \equiv h^{-1}A_\bar{z}h + h^{-1}\partial h$. Above $S(\cdot,\cdot)$ denotes the WZNW action in the external gauge field [3]. The CS states form a finite-dimensional space with the dimension expressed by the Verlinde formula [7]. Their scalar product is formally given by the functional integral

$$\|\Psi\|^2 = \int |\Psi(A_\bar{z})|^2 e^{\frac{k}{2} \int_\Sigma \text{tr} A_\bar{z} A_z d^2 z} \, DA$$

(1.3)

(with the convention that $A_\bar{z} = -A^\dagger_z$ for an $su(2)$ gauge field $A$). The partition function of the WZNW model is [2]

$$Z(A) = \sum_{a,b} H^{ab} \Psi_a(A_\bar{z}) \overline{\Psi_b(A_\bar{z})} e^{\frac{k}{2} \int_\Sigma \text{tr} A_\bar{z} A_z d^2 z} ,$$

(1.4)

where $(H^{ab})$ is the matrix inverse to that of the scalar products $(\psi_a, \psi_b)$, for an arbitrary basis of the CS states. Hence, in order to find the WZNW partition functions, we should be able to construct CS states and to compute their scalar product. The calculation of the latter is another exercise in the functional integration over gauge fields and may be dealt with similarly as in the original Faddeev-Popov’s geometric argument. The main point is to use the action $A_\bar{z} \mapsto h^{-1}A_\bar{z}$ of the group $G_C$ of the complex gauge transformations in order to reduce the functional integral over the space $A$ of gauge fields to the orbit space $A/G_C$, which is, in fact, finite dimensional. The main aspect differing our situation from that considered by Faddeev and Popov is that neither the integrated function nor the (formal) integration measure $DA$ are invariant under the complex gauge transformations. Nevertheless, both transform in a controllable way. This difference has to be properly accounted for.

The first step in the reduction of the functional integral to the orbit space is to choose a slice $A/G_C \ni n \mapsto A(n) \in A$ which cuts every orbit only once (generically) and to change variables by parametrizing

$$A_\bar{z} = h^{-1}A_\bar{z}(n) .$$

(1.5)
The Jacobian of the change of variables \( \frac{\partial (A)}{\partial (h,n)} \) plays then the role similar to that of the ghost determinant in the approach of [1] where the slice was fixed by constraining functions. The next step is the calculation of the \( h \)-integral. This step is not trivial, in contrast to the gauge invariant situation where the gauge group integral factors out as an overall constant. In fact, in the case at hand, the integration over \( h \), more involved at genera \( g > 1 \) than for \( g = 0 \) and \( g = 1 \), leads to a result which may look surprising at the first sight: it gives not a function but a singular distribution on the orbit space \( \mathcal{A}/G^C \). Its treatment may shed some light on the more complicated case of the Liouville theory functional integral. After the integration over \( h \) is done, one is left with an explicit finite-dimensional distributional integral, essentially over the orbit space \( \mathcal{A}/G^C \), which may be further reduced to the standard integral over the support of the distribution.

The paper is organized as follows. Section 2 is devoted to the description of the slice \( s : \mathcal{A}/G^C \to \mathcal{A} \). The use of \( s \) allows also a more explicit characterization of the CS states. In Section 3, we perform the change of variables (1.5) and study its Jacobian. In Section 4, we describe the calculation of the \( h \)-integral over the \( G^C \)-orbits in \( \mathcal{A} \) which turns out to be iterative Gaussian. Finally, in Section 5 we discuss the resulting finite-dimensional integral representation for the CS scalar product.

2. The slice

We would like to describe a surface \( \{ A(n) \} \) inside \( \mathcal{A} \) which cuts each orbit of \( G^C \) once (or a fixed number of times). The orbit space \( \mathcal{A}/G^C \) (after removal of a small subset of bad orbits) is an object well studied in the mathematical literature under the name of the moduli space of (stable) holomorphic \( SL(2, \mathbb{C}) \)-bundles [8][9]. It has complex dimension \( 3(g - 1) \), where \( g \) denotes the genus of the underlying Riemann surface. This should then also be the dimension of our slice of \( \mathcal{A} \). Let \( L_0 \) be a spin bundle over \( \Sigma \). \( L_0 \) is a holomorphic line bundle with local sections \( (dz)^{1/2} \). \( L_0^{-1} \) will denote its dual bundle. \( L_0^{-1} \oplus L_0 \) is a rank two vector bundle over \( \Sigma \) which, as a smooth bundle, is isomorphic to the trivial bundle \( \Sigma \times \mathbb{C}^2 \). We shall fix a smooth isomorphism

\[
U : L_0^{-1} \oplus L_0 \to \Sigma \times \mathbb{C}^2 .
\]

We may assume that \( U \) preserves the length of the vectors calculated in \( L_0^{-1} \oplus L_0 \) with the help of a fixed Riemannian metric \( \gamma = \gamma_{\bar{z}z} dz d\bar{z} \) on \( \Sigma \). \( U \) may be used to transport the gauge fields \( A_{\bar{z}} \) to the bundle \( L_0^{-1} \oplus L_0 \). More exactly, the relation

\[
B_{\bar{z}} = UA_{\bar{z}}U^{-1} + U\partial_{\bar{z}}U^{-1}
\]

establishes a one to one correspondence between \( A_{\bar{z}} d\bar{z} \) and

\[
B_{\bar{z}} d\bar{z} = \begin{pmatrix} -a & b \\ c & a \end{pmatrix},
\]

where \( a \) is a scalar 01-form on \( \Sigma \) \( (a \in \Lambda^{01}) \), \( b \) is an \( L_0^{-2} \)-valued one \( (b \in \Lambda^{01}(L_0^{-2})) \) and \( c \in \Lambda^{01}(L_0^0) \).
Let us present the surface $\Sigma$ as a polygone with sides $a_1$, $b_1$, $a_1^{-1}$, $b_1^{-1}$, $a_2$, $b_2$, $a_2^{-1}$, $b_2^{-1}$ given by the basic cycles. We shall fix $x_0 \in \Sigma$ in the corner where $b_1^{-1}$ and $a_1$ meet. Let $\omega^j$, $i = 1, \ldots, g$, be the standard basis of holomorphic forms with $\int_{x_0} \omega^j = \delta^j_i$, $\int_{b_1} \omega^j = \tau^j_i$. We shall take a slice of $\mathcal{A}$ formed by the gauge fields $A^{x,b}$ corresponding to

$$B_x d\bar{z} = \begin{pmatrix} -a^x & b_x \\ 0 & a^x \end{pmatrix} \equiv B_{x,b} \, , \quad (2.4)$$

where

$$a^x = \pi \left( \int_{x_0}^{x} \omega^j \right) (1 + i) \bar{\omega}^j \equiv \pi \left( \int_{x_0}^{x} \omega \right) (1 + i) \bar{\omega} \, . \quad (2.5)$$

Let $L_x$ denote the holomorphic line bundle obtained from $L_0$ by replacing its $\bar{\partial}$ operator by $\bar{\partial} + a^x \wedge \equiv \bar{\partial}L_x$. We shall restrict forms $b$ further by taking one representative in each class of $L_0^{-2}$.

Finally, only one $b$ in each complex ray of solutions of $(2.6)$ should be taken since

$$\int_{\Sigma} \left( \nabla + 2a^x \wedge \right) b = 0 \quad (2.6)$$

with $\nabla$ standing for the holomorphic covariant derivative of the sections of $L_0^{-2} = T^0 \Sigma$. Here, we shall content ourselves with the count of dimensions. By the Riemann-Roch theorem, the dimension of $H^1(L_x^{-2})$ is $3(g - 1)$. The projectivization subtracts one dimension which is restored by varying $x \in \Sigma$.

The CS states $\Psi$ admit an explicit representation when restricted to the slice described above. Let us put

$$\psi(x, b) = e^{-\pi \left( \int_{x_0}^{x} \omega \right) (1 + i) \bar{\omega}} e^{\frac{b}{2} \int_{\Sigma} A_0^0 A^{x,b}_0 d^2 z} \Psi(A^{x,b}_0) \, , \quad (2.8)$$

where $A_0^0 \equiv U \nabla_x U^{-1}$. $\psi$ is an analytic function of $x$ and $b$ depending only of the class of $b$ modulo $(\bar{\partial} - 2a^x \wedge)(C^\infty(L_0^{-2}))$

$$\psi(x, b + (\bar{\partial} - 2a^x \wedge) w) = \psi(x, b) \, . \quad (2.9)$$

Under constant complex rescalings of $b$,

$$\psi(x, \lambda b) = \lambda^{k(g-1)} \psi(x, b) \, . \quad (2.10)$$

Under the action of the fundamental group $\pi_1(\Sigma)$,

$$\psi(a_j x, c_{a_j}^2 b) = \nu(a_j)^k \psi(x, b) \, , \quad (2.11)$$

$$\psi(b_j x, c_{b_j}^2 b) = \nu(b_j)^k e^{-2\pi i k \tau^{j,j} - 4\pi i k \int_{x_0}^{x} \omega^j} \psi(x, b) \, , \quad (2.12)$$
where, for each \( p \in \pi_1(\Sigma) \), \( c_p \) is a non-vanishing (univalued) function on \( \Sigma \),
\[
c_p(y) = e^{2\pi i \text{ Im } ((\int_{\Sigma} \omega) (\text{ Im } \tau)^{-1} (\int_{\Sigma} \omega))} \tag{2.13}
\]
and \( \nu \) is a character of \( \pi_1(\Sigma) \),
\[
\nu(p) = e^{-\frac{\pi}{2\nu} \int_{\Sigma} R \ln c_p} \prod_{j=1}^{g} W_{a_j}^{-\frac{1}{\pi} \int_{a_j} c_p^{-1} dc_p} W_{b_j}^{\frac{1}{\pi} \int_{a_j} c_p^{-1} dc_p}. \tag{2.14}
\]

In the last formula, the integral \( \int_{\Sigma} R \ln c_p \) is over the polygon representing the surface, \( R \) is the metric curvature form of \( \Sigma \) normalized so that \( \int_{\Sigma} R = 4\pi i (g-1) \), \( \ln c_p(y) = \int_y^p c_p^{-1} dc_p \) and \( W_p \) stands for the holonomy of \( L_0 \) around the closed curve \( p \). One may identify functions \( \psi \) satisfying relations (2.9), (2.10), (2.11) and (2.12) with sections of the \( k \)-th power of Quillen’s determinant bundle \([10]\) of the holomorphic family \( (\bar{\partial} + B^x, b) \) of operators in \( L_0^{-1} \oplus L_0 \). Holomorphic \( \psi \)'s form a finite dimensional space and the relation \( \Psi \mapsto \psi \) realizes the space of CS states as its, in general proper, subspace.

### 3. The change of variables

We apply the change of variables \([1.3]\) with \( A_z(n) \equiv A_z^{x,b} \) in the functional integral \([1.3]\) giving the scalar product of CS states. The Jacobian is
\[
\frac{\partial (A)}{\partial (h,n)} = \det (D_z(h,n)^\dagger D_z(h,n)) \quad \det (\left( \frac{\partial A_z}{\partial n_\alpha}, \frac{\partial A_z}{\partial n_\beta} \right)) \tag{3.1}
\]
where \( D_z(h,n) = h^{-1}(\partial_z + [A_z(n), \cdot])h \) and \( \left( \frac{\partial A_z}{\partial n_\alpha} \right)^\dagger \) denotes the component of \( h^{-1}\partial_z(n)h \) perpendicular to the image of \( D_z(h,n) \). The first determinant should be regularized (what specific regularization is used does not matter as long as it is insensitive to unitary conjugations of the operator, like the zeta-function regularization). The \( h \)-dependence of the regularized determinants is given by the global version of the non-abelian chiral anomaly formula \([1]\)
\[
\det (D_z(h,n)^\dagger D_z(h,n)) = e^{4S(hh^\dagger, A(n))} \det (D_z(n)^\dagger D_z(n)) \quad \det (\left( \frac{\partial A_z(n)}{\partial n_\alpha}, \frac{\partial A_z(n)}{\partial n_\beta} \right)) \tag{3.2}
\]
where \( D_z(n) \equiv D_z(1,n) \). The covariance property \([1.2]\) implies that
\[
|\Psi(A_z)|^2 e^{\frac{1}{2} \int_{\Sigma} \text{ tr } A_z A_z d^2 z} = |\Psi(A_z(n))|^2 e^{\frac{1}{2} \int_{\Sigma} \text{ tr } A_z(n) A_z(n) d^2 z} e^{kS(hh^\dagger, A(n))}. \tag{3.3}
\]
Hence, after the change of variables, the functional integral \([1.3]\) becomes
\[
\|\Psi\|^2 = \int |\Psi(A_z(n))|^2 e^{\frac{1}{2} \int_{\Sigma} \text{ tr } A_z(n) A_z(n) d^2 z} e^{(k+4)S(hh^\dagger, A(n))} \det (D_z(n)^\dagger D_z(n)) \cdot \det (\left( \frac{\partial A_z(n)}{\partial n_\alpha}, \frac{\partial A_z(n)}{\partial n_\beta} \right)^\dagger D_z(n)^\dagger) \prod d^2 n_\alpha, \tag{3.4}
\]
where we have used the $SU(2)$ gauge invariance of the integral to factor out the integration over the $SU(2)$-valued gauge transformations, like in Faddeev-Popov’s case. There remains, however, the integral over the field $hh$ effectively taking values in the hyperbolic space $\mathcal{H} \equiv SL(2, \mathbb{C})/SU(2)$. $D(hh)$ should be viewed a formal product of the $SL(2, \mathbb{C})$-invariant measures on $\mathcal{H}$.

Working out the explicit form of various terms under the integral in eq. (3.4) for $A(n) \equiv A^{z,b} \equiv A^{z,b}$ is a rather straightforward matter. One obtains

$$|\Psi(A^{z,b})|^2 = e^{\frac{k}{\pi} \int_\Sigma \text{tr} A^z \cdot A^b \, dz} = e^{\frac{k}{\pi} \int_\Sigma \text{tr} A^0 \cdot A^0 \, dz} \cdot e^{-4\pi k (\int_{x_0} (\text{Im} \omega) (\text{Im} \tau)^{-1} (\int_{x_0} (\text{Im} \omega) \cdot \bar{\pi}(\bar{b}, b)) \, |\psi(x, b)|^2. \quad (3.5)$$

(Here and below, $\langle \cdot, \cdot \rangle$ denotes the hermitian structure induced by the Riemannian metric of $\Sigma$ on the powers of its canonical bundle.) The determinant of the operator $D_z(n)^{\dagger} D_z(n)$, unitarily equivalent to the operator $(\bar{\partial}_z + [\bar{B}_z^{x,b}, \cdot])^{\dagger}(\bar{\partial}_z + [\bar{B}_z^{x,b}, \cdot])$ acting on smooth endomorphism of $L_0^{-1} \oplus L_0$, may be found by performing the Gaussian integration

$$\det(D_z(n)^{\dagger} D_z(n)) = e^{-i \int_\Sigma ((\bar{\partial} X + Z)^2 + (\bar{\partial} X + Z) + (\bar{\partial} + 2\bar{a}^x) Y - 2\bar{X} b, \wedge (\bar{\partial} + 2\bar{a}^x) Y - 2\bar{X} b)) \cdot e^{((\bar{\partial} + 2\bar{a}^x) Z, \wedge (\bar{\partial} + 2\bar{a}^x) Z)} \, DY \, DX \, DZ \quad (3.6)$$

over the anticommuting ghost fields: the scalar $X$, the $L_0^{-2}$-valued $Y$ and the $L_0^0$-valued $Z$. Formally, the computation may be done iteratively, first over $Y$, then over $X$ and, at the end, over $Z$. One obtains this way the product of determinants

$$\det(\bar{\partial}_L^{z,2} \bar{\partial}_L^{z,2}) \det'(-\Delta) \det'(\bar{\partial}_L^{z,2} \bar{\partial}_L^{z,2}), \quad (3.7)$$

where $\Delta$ is the scalar Laplacian on $\Sigma$, decorated with zero mode terms. Some care should be taken since the regularization (e.g. by the zeta-function prescription) of the big determinant requires, besides similar regularization of the product determinants, also an additional term which may be found by demanding that the result transforms as in (3.4) under the complex gauge transformations.

The final expression for the measure

$$d\mu(n) = \det(D_z(n)^{\dagger} D_z(n)) \det(\frac{\partial A_z(n)}{\partial n_\alpha}, \frac{\partial A_{\bar{z}}(n)}{\partial n_\beta}) \prod_\alpha d^2 n_\alpha$$

(3.8)

on the slice becomes rather simple. Fix $x \in \Sigma$. Let $(b^\alpha)_{\alpha=1}^{3(\sigma-1)}$ be an orthonormal basis (in the natural $L^2$ scalar product) of $L_0^{-2}$-valued 01-forms $b$ solving eq. (2.6). Similarly, let $(\kappa_\tau)_{\tau=1}^{\gamma-1}$ be an orthonormal basis of sections of $L_0^{-2}$ annihilated by $\nabla \wedge 2\bar{a}^x \wedge$. Set $\mathcal{M}_\alpha^\tau \equiv \int_\Sigma \omega^x(\kappa_\tau, b^\alpha)$. Denote by $(z^\alpha)$ the complex coordinates w.r.t. the basis $(b^\alpha)$ on the space of $b$ satisfying eq. (2.6). One may use $x \in \Sigma$ and the homogeneous coordinates $z^\alpha$ to parametrize the slice of $\mathcal{A}$ and

$$d\mu(x, b) = \text{const.} \frac{\text{area}(\Sigma)}{\det(\text{Im} \tau)} \det(\bar{\partial}_L^{z,2} \bar{\partial}_L^{z,2}) \det'(-\Delta) \det'(\bar{\partial}_L^{z,2} \bar{\partial}_L^{z,2}) \cdot e^{\frac{2}{\pi} \int_\Sigma \omega^x(\bar{b}, b)}$$
\[ \cdot |\epsilon_{\alpha_1, \ldots, \alpha_{3(\chi-1)}} z^{\alpha_1} dz^{\alpha_2} \wedge \ldots \wedge dz^{3(\chi-1)}|^{2} |\sum_{j=1}^{g} \det \left( \sum_{\alpha} M_{\alpha}^{\alpha} z^{\alpha} \right)_{\gamma j} \omega^{j}(x) |^{2}. \quad (3.9) \]

Notice that \( d\mu(n) \) contains as a factor the natural measure on the projectivization of the space of solutions of eq. \((2.6)\).

4. Integral over the gauge orbits

It remains to compute the functional integral

\[ Z_{h}^{(A)}(A, b) \equiv \int e^{(k+4)S(hh^{\dagger}, A^{2}, b)} D(hh^{\dagger}) \quad (4.1) \]

giving the genus \( g \) partition function of a (nonunitary) WZNW model with fields taking values in the hyperbolic space \( H \), first considered in [12]. The calculation of the functional integral \((4.1)\) is the crucial step of the argument. The fields \( hh^{\dagger} \) may be uniquely parametrized by real functions \( \varphi \) and sections \( w \) of \( L_{0}^{-2} = T^{10} \Sigma \) by writing

\[ hh^{\dagger} = U \begin{pmatrix} 1 & e^{\varphi} w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{\varphi} w^{\dagger} & 1 \end{pmatrix} U^{-1} \quad (4.2) \]

where \( w^{\dagger} \) is the section of \( L_{0}^{2} \) obtained by contracting the vector field \( \bar{w} \) with the Riemannian metric. Rewritten in terms of \( \varphi \) and \( w \), the functional integral \((4.1)\) becomes

\[ Z_{h}^{(A)}(A, b) = e^{\frac{k+4}{2n} \int_{\Sigma} (b, \omega b)} \int e^{\frac{k+4}{2n} \int_{\Sigma} [-\varphi (\partial \bar{\varphi} + R) + (e^{-\varphi} b + (\bar{\varphi} + (\partial \varphi)) w, \wedge (e^{-\varphi} b + (\partial \varphi)) w)]} Dw D\varphi, \quad (4.3) \]

where \( \bar{\varphi} \equiv \bar{\varphi}_{L_{2}} \) when acting on \( w \). The \( w \)-integral is Gaussian and may be easily performed:

\[ \mathcal{I}_{w} \equiv \int e^{\frac{k+4}{2n} \int_{\Sigma} (e^{-\varphi} b + (\partial \bar{\varphi})) w, \wedge (e^{-\varphi} b + (\partial \varphi)) w)} Dw = e^{\frac{k+4}{2n} \int_{\Sigma} (P_{\varphi}(e^{-\varphi} b), \wedge P_{\varphi}(e^{-\varphi} b))} \det \left( (\bar{\varphi}_{L_{2}} + (\bar{\varphi}))^{\dagger} (\bar{\varphi}_{L_{2}} + (\bar{\varphi})) \right)^{-1}, \quad (4.4) \]

where \( P_{\varphi} \) denotes the orthogonal projection on the kernel of \( (\bar{\varphi}_{L_{2}} + (\bar{\varphi}))^{\dagger} \). Since the latter is spanned by the vectors \( e^{\varphi} b^{\alpha} \),

\[ i \int E_{\Sigma} (P_{\varphi}(e^{-\varphi} b), \wedge P_{\varphi}(e^{-\varphi} b)) \equiv \| P_{\varphi}(e^{-\varphi} b) \|^{2} = \langle (\bar{b}^{\alpha}, b) (H_{\varphi}^{-1})_{\alpha \beta} (b^{\beta}, b) = z^{\alpha} (H_{\varphi}^{-1})_{\alpha \beta} z^{\beta}, \quad (4.5) \]

where \( H_{\varphi} \) is the matrix of scalar products \( \langle e^{\varphi} b^{\alpha}, e^{\varphi} b^{\beta} \rangle \). The factor \( e^{-\frac{k+4}{2n} \| P_{\varphi}(e^{-\varphi} b) \|^{2}} \) is the classical value of the \( w \)-integral. One may rewrite it as a finite-dimensional integral

\[ e^{-\frac{k+4}{2n} \| P_{\varphi}(e^{-\varphi} b) \|^{2}} = \text{const.} \det (H_{\varphi}) \int e^{-\frac{2n}{k+4} \sum_{\alpha} c_{\alpha} (H_{\varphi})_{\alpha \beta} c_{\beta} + i c_{\alpha} z^{\alpha} + i c_{\alpha} z^{\alpha}} \prod_{\alpha} d^{2} c_{\alpha}. \quad (4.6) \]
By the global version of the abelian chiral anomaly formula,
\[
\det (H_\varphi) \det \left( (\partial_{L_x}^+ + (\partial_\varphi)) (\partial_{L_x}^- + (\partial_\varphi)) \right)^{-1} = e^{- \frac{1}{\pi} \int_\Sigma \varphi R + \frac{3 \pi}{2 \sqrt{\pi}} \int_\Sigma \varphi^2} \det ((\partial_{L_x}^+ \tilde{\partial}_{L_x}^-)^{-1} (4.7)
\]
(recall that $H_0$ is the unit matrix). Gathering eq. (4.4), (4.6) and (4.7), one obtains
\[\mathcal{I}_w = \text{const.} \det ((\partial_{L_x}^+ \tilde{\partial}_{L_x}^-)^{-1} e^{- \frac{1}{\pi} \int_\Sigma \varphi R + \frac{3 \pi}{2 \sqrt{\pi}} \int_\Sigma \varphi^2} \int e^{\frac{k+2}{k+4} \int_\Sigma \varphi R - \frac{2 \pi}{k+4} \epsilon \alpha (H_\varphi)^{\alpha \beta} c_\beta} \prod_\alpha d^2 c_\alpha . \tag{4.8}\]

Note that the right hand side which, together with the $\varphi$-terms left over in (4.3), has to be integrated over $\varphi$ contains a Liouville-type term
\[e^{- \frac{2 \pi}{k+4} \epsilon \alpha (H_\varphi)^{\alpha \beta} c_\beta} = e^{- \frac{2 \pi}{k+4} \int \epsilon^{2\varphi} (c_\alpha b^\alpha, c_\beta b^\beta)} \tag{4.9}\]
notorious for causing problems in the functional integration. Indeed, with the $w$-integral done, the $\varphi$-integral takes the form
\[\int e^{\frac{k+2}{k+4} \int_\Sigma \varphi R - \frac{2 \pi}{k+4} \epsilon \alpha (H_\varphi)^{\alpha \beta} c_\beta} D\varphi \equiv \mathcal{I}_\varphi \tag{4.10}\]
which, unlike at low genera, is of the Liouville theory type, not a Gaussian one. A possible approach to such an integral is to try to get rid of the Liouville term by integrating out the zero mode $\varphi_0 \equiv (\text{area}(\Sigma))^{-\frac{1}{2}} \int_\Sigma \varphi \, da$ of $\varphi$ ($da$ denotes the metric volume measure on $\Sigma$). This was tried in the Liouville theory in \[\text{[13]}\] and, supplemented with rather poorly understood formal tricks, has led in \[\text{[14]}\] to the functional integral calculation of three-point functions for the minimal models coupled to gravity. Multiplying $\mathcal{I}_\varphi$ by
\[1 = (\text{area})^{\frac{1}{2}} \int (\varphi - u (\text{area})^{\frac{1}{2}}) \, du , \]
changing the order of integration, shifting $\varphi_0$ to $\varphi_0 + u$ and setting $M \equiv (k+1)(g-1)$, one obtains
\[\mathcal{I}_\varphi = \text{area}^{1/2} \int \epsilon^{2uM} e^{\frac{k+2}{k+4} \int_\Sigma \varphi R - \frac{2 \pi}{k+4} \epsilon \alpha (H_\varphi)^{\alpha \beta} c_\beta} \int \varphi R \, \delta(\varphi_0) D\varphi \]
\[= \frac{1}{2} (\text{area})^{1/2} \Gamma(-M) (\frac{2\pi}{k+4})^M \int e^{\frac{k+2}{k+4} \int_\Sigma \varphi R - \frac{2 \pi}{k+4} \epsilon \alpha (H_\varphi)^{\alpha \beta} c_\beta} \prod_\alpha d^2 c_\alpha . \tag{4.11}\]
Hence the integration over the zero mode of $\varphi$ diverges but may be easily (multiplicatively) renormalized by removing the overall divergent factor $\Gamma(-M)$. Now, the $c$-integral is easy to perform:
\[\int \epsilon \alpha (H_\varphi)_{\alpha \beta} c_\beta)^M \, e^{i \epsilon \alpha z^\alpha + i c_\alpha z^\alpha} \prod_\alpha d^2 c_\alpha \]
\[= (2\pi)^{(g-1)} \epsilon \alpha z^\alpha \prod_\alpha \delta(z^\alpha) . \tag{4.12}\]
Gathering the above results, we obtain the following “Coulomb gas representation” for the higher genus partition function of the $\mathcal{H}$-valued WZNW model:
\[ \int e^{(k+4)S(hh^1,A^{\alpha,b})} D(hh^1) = \text{const.} \left( \text{area} \right)^{1/2} \det \left( \bar{\partial}_{L_x}^2 \partial_{L_x} \right)^{-1} e^{-\frac{k+6}{2\pi} \int \varphi} \cdot \left( \int e^{\frac{k+4}{2\pi} \int \varphi \wedge \bar{\varphi} - \frac{k+4}{2\pi} \int \varphi R} \left( -(H_{\varphi})^{\alpha \beta} \partial_{\varphi} \partial_{\bar{\varphi}} \right)^M \delta(\varphi_0) D\varphi \right) \prod \delta(z^\alpha) \right. \]

\[ = \text{const.} \left( \text{area} \right)^{1/2} \det \left( \bar{\partial}_{L_x}^2 \partial_{L_x} \right)^{-1} e^{-\frac{k+6}{2\pi} \int \varphi} \cdot \left( \prod_{m} \frac{1}{i} \partial_{z^\alpha m} \partial_{\bar{z}^\beta m} \prod \delta(z^\alpha) \right) \]

\[ \cdot \left( \int \prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+6} G(x_{m_1}, x_{m_2})} \right) \left( \prod_{m} e^{-\frac{4\pi}{k+6} G(x_m, x_m)} \right) \left( \prod_{m} e^{-\frac{4\pi}{k+6} G(x_m, x_m)} \right) \right) , \quad (4.14) \]

where \( m \) runs from 1 to \( M \). The \( \varphi \) integral is now purely Gaussian and easily doable, provided that we Wick order the “screening charge” insertions \( e^{2\varphi(x_m)} \) (this is, again, a multiplicative renormalization). In the end, we obtain

\[ Z_h(A^{\alpha,b}) = \text{const.} \]

\[ \cdot \det \left( \bar{\partial}_{L_x}^2 \partial_{L_x} \right)^{-1} \left( \frac{\det \left( -\Delta \right)}{\text{area}(\Sigma)} \right)^{-1/2} e^{-\frac{k+6}{2\pi} \int \varphi} \cdot \left( \prod_{m} \frac{1}{i} \partial_{z^\alpha m} \partial_{\bar{z}^\beta m} \prod \delta(z^\alpha) \right) \]

\[ \cdot \left( \int \prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+6} G(x_{m_1}, x_{m_2})} \right) \left( \prod_{m} e^{-\frac{4\pi}{k+6} G(x_m, x_m)} \right) \left( \prod_{m} e^{-\frac{4\pi}{k+6} G(x_m, x_m)} \right) , \quad (4.14) \]

where \( G(\cdot, \cdot) \) is the Green function of the scalar Laplacian \( \Delta \) on \( \Sigma \) chosen so that \( \int \varphi R(y) = 0 \). \( G(y, y') \equiv \lim_{\epsilon \rightarrow 0} (G(y, y') - \frac{1}{2\pi} \ln \epsilon) \) where \( \epsilon = d(y, y') \) is the distance between \( y \) and \( y' \).

Formula (4.14) reduces the functional integral over \( hh^1 \) to a finite dimensional integral over positions \( x_m \in \Sigma \) of \( M \) screening charges. The integrand is a smooth function except for \( O(d(x_{m_1}, x_{m_2})^{-\frac{k+6}{4\pi}}) \) singularities at coinciding points. Power counting shows, that the integral converges for \( g = 2 \) but for higher genera it diverge unless special combinations of forms \( \langle b^{\alpha m}, b^{\bar{\beta} m} \rangle \) are integrated. Another feature of the right hand side of eq. (4.14) is even more surprising in the candidate for a partition function: its dependence on the external field \( A^{\alpha,b} \) is not functional but distributional! The entire dependence on \( b \) resides in the term \( \prod_{\alpha} \partial_{z^\alpha m} \partial_{\bar{z}^\beta m} \delta(z^\alpha) \) (recall that \( z^\alpha \equiv (b^\alpha, b) \)). This fact is not so astonishing since the partition function of the \( H \)-valued WZNW model may be expected, by formal arguments similar to the ones used in [15], to be the hermitian square of a holomorphic section of a negative power of the determinant bundle. But there are no such global sections but only distributional solutions of the corresponding Ward identities. The right hand side of (4.14) is one of them.

5. The scalar product formula

In view of the results (3.9) giving the integration measure on the slice and (4.14) computing the integral along the \( G^C \)-orbits, the functional integral expression (4.4) for the scalar
product of the CS states reduces to

$$\|\Psi\|^2 = \text{const. det (Im} \tau)^{-1} \left( \frac{\text{det}(-\Delta)}{\text{area}(\Sigma)} \right)^{1/2} e^{\frac{i}{\hbar} \int_{\Sigma} \text{tr} A^0_v (A^0_v)^{-1} d^2z}
\cdot \int e^{-4\pi k \int_{x_0}^{x} \text{Im} \omega} (\text{Im} \tau)^{-1} (\int_{x_0}^{x} \text{Im} \omega) |\psi(x, b)|^2 |\sum_{j=1}^{g} \text{det} \left( \sum_{\alpha} M^\alpha_{\text{tr}} z^\alpha \right)_{\neq j} \omega^j(x)|^2
\cdot \text{det}'(\bar{\partial}_{L^2} \partial_{L^2} \bar{\partial}) \left( \prod_{m=1}^{1/2} \frac{1}{2} \partial_{z^m} \partial_{\bar{z}^m} \prod_{\alpha} \delta(z^\alpha) \right) \left| e^{\alpha_1, ..., \alpha_{3(g-1)}} z^\alpha \prod_{i=1}^{g} d\omega^i \wedge ... \wedge d\omega^{3(g-1)} \right|^2
\cdot \left( \prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m1}, x_{m2})} \right) \left( \prod_{m} e^{-\frac{4\pi}{k+2} :G(x_{m}, x_{m}):} \langle b^{\alpha_m}, b^{\beta_m} \rangle(x_m) \right). \quad (5.1)$$

The integral is, for fixed $x \in \Sigma$, over the $(3g-4)$-dimensional projective space with homogeneous coordinates $(z^\alpha)$ and over the Cartesian product of $M \equiv (k+1)(g-1)$ copies of $\Sigma$ (positions $x_m$ of the screening charges). Finally, one should integrate over $x \in \Sigma$.

The $(z^\alpha)$-integral should be interpreted as

$$\int_{\mathbb{C}^{3(g-1)}} |P(z)|^2 \left( \prod_{m=1}^{1/2} \frac{1}{2} \partial_{z^m} \partial_{\bar{z}^m} \prod_{\alpha} \delta(z^\alpha) \right) d^{6(g-1)}z = \left. \prod_{m=1}^{1/2} \frac{1}{2} \partial_{z^m} \partial_{\bar{z}^m} \right|_{z=0} |P(z)|^2 \quad (5.2)$$

where $P(z) = \psi(x, b) \sum_{j=1}^{g} \text{det} \left( \sum_{\alpha} M^\alpha_{\text{tr}} z^\alpha \right)_{\neq j} \omega^j(x)$ is a homogeneous polynomial in $(z^\alpha)$ of degree $M$. Formally, the latter integral differs from the one involving the volume form on $\mathbb{P}\mathbb{C}^{3g-4}$ by an infinite constant, which may be interpreted as the factor $\Gamma(-M)$ which appeared in the integral over the zero mode of the field $\varphi$. With the $z$-integration given by eq. (5.2), one obtains the following formula for the scalar product of CS states:

$$\|\Psi\|^2 = \text{const. det (Im} \tau)^{-1} \left( \frac{\text{det}(-\Delta)}{\text{area}(\Sigma)} \right)^{1/2} e^{\frac{i}{\hbar} \int_{\Sigma} \text{tr} A^0_v (A^0_v)^{-1} d^2z}
\cdot \int \prod_{m=1}^{1/2} \frac{1}{2} \partial_{z^m} \partial_{\bar{z}^m} \left| \psi(x, b) \right|^2 \prod_{j=1}^{g} \text{det} \left( \sum_{\alpha} M^\alpha_{\text{tr}} z^\alpha \right)_{\neq j} \omega^j(x)^2
\cdot \left( \prod_{m_1 \neq m_2} e^{-\frac{4\pi}{k+2} G(x_{m1}, x_{m2})} \right) \left( \prod_{m} e^{-\frac{4\pi}{k+2} :G(x_{m}, x_{m}):} \langle b^{\alpha_m}, b^{\beta_m} \rangle(x_m) \right)
\cdot e^{-4\pi k \int_{x_0}^{x} \text{Im} \omega} (\text{Im} \tau)^{-1} (\int_{x_0}^{x} \text{Im} \omega) \text{det}'(\bar{\partial}_{L^2} \partial_{L^2} \bar{\partial}), \quad (5.3)$$

where the numerical constant depends on the genus $g$ and on the level $k$. The integration in (5.3) is over $x_m$, $m = 1, \ldots, (k+1)(g-1)$, and over $x$, all in $\Sigma$. It is not difficult to check that upon the Weyl rescalings $\gamma \mapsto e^{\gamma/\sigma}$ of the Riemannian metric on $\Sigma$, the right hand side of (5.3) (with the zeta-function regularized determinants) changes by the factor $e^{\frac{k}{2} \int_{\Sigma} \frac{1}{2} \partial \sigma \partial \bar{\sigma} + \sigma R}$. This produces the correct value of the Virasoro central charge of the WZNW partition functions given by eq. (1.4).

The arguments which led to eq. (5.3) were clearly formal and the treatment of the Liouville integral might have looked particularly suspicious. Fortunately, one may do
the calculation in a different, more satisfying manner. If the $z$-integral over $PC^{3g-4}$ is done just after the $w$-integration and before the one over $\varphi$ then the final result is exactly as above but no infinite constants (apart from the Wick ordering ones) appear in the intermediate steps. This way, it is rather the convergent integration over the (part of) the modular degrees of freedom, not the divergent $\varphi_0$ integral, which removes the cumbersome Liouville-type terms from the effective action for $\varphi$. This is an important lesson to learn from the above calculation. It is plausible that similar arguments may be used to substantiate the Goulian-Li trick [14] in the gravity case.

Similarly as for the genus zero case discussed in [2], the natural conjecture is that the integral on the right hand side converges if and only if the function $\psi$ defines a global non-singular CS state $\Psi$. It is clear from the form (5.3) of the scalar product that finiteness of the screening charge integral in (5.3) imposes, in general, conditions for the Taylor coefficients of $\psi(x, b)$ at $b = 0$ if $g > 2$. We shall postpone the study of these “fusion rule conditions” to a future work. The case $g = 2$ is specially accessible since there exists a simple global picture of the moduli space of $SL(2, \mathbb{C})$-bundles (it is the projectivization of the 4-dimensional space of degree 2 theta-functions [9] and of the space of CS states (homogeneous polynomials of order $k$ on the same space).

If our conjecture is true, then the formula (5.3) defines a hermitian structure on the holomorphic vector bundle with the fibers given by the spaces of the CS states and the base by the moduli space of complex curves. Such a hermitian structure induces a holomorphic hermitian connection. This connection should coincide with the generalization to higher genera of the Knizhnik-Zamolodchikov connection studied in [16][17][18]. The latter has been constructed in [19] in the geometric terms, and it is challenging to find an interpretation for eq. (5.3) in terms of the moduli space geometry.

References

[1] Faddeev, L. D., Popov, V. N.: *Feynman Diagrams for the Yang-Mills Field*. Phys. Lett. B 26, 29-30 (1967)

[2] Gawędzki, K.: *Quadrature of Conformal Field Theories*. Nucl. Phys. B 328, 733-752 (1989)

[3] Falcoeto, F., Gawędzki, K., Kupiainen, A.: *Scalar Product of Current Blocks in WZW Theory*. Phys. Lett. B 260, 101-108 (1991)

[4] Gawędzki, K.: *Free field representation of the SU(2) WZNW model at higher genera*. In preparation

[5] Witten, E.: *Quantum Field Theory and Jones Polynomial*. Commun. Math. Phys. 121, 351-399 (1989)

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[6] Witten, E.: Non-abelian bosonization in two dimensions. Commun. Math. Phys. 92, 455-472 (1984)

[7] Verlinde, E.: Fusion rules and modular transformations in 2-d conformal field theory. Nucl. Phys. B 300, 360-376 (1988)

[8] Seshadri, C. S.: Space of Unitary Vector Bundles on a Compact Riemann Surface. Ann. of Math. 85, 303-336 (1967)

[9] Narasimhan, M. S., Ramanan, S.: Moduli of vector bundles on a compact Riemann surface. Ann. Math. 89, 15-51 (1969)

[10] Quillen, D.: Determinants of Cauchy-Riemann Operators over a Riemann surface. Funct. Anal. Appl. 19, 31-34 (1985)

[11] Polyakov, A. M., Wiegmann, P. B.: Theory of non-abelian Goldstone bosons in two dimensions. Phys. Lett. B 131, 121-126 (1983)

[12] Gawędzki, K., Kupiainen, A.: Coset Construction from Functional Integrals. Nucl. Phys. B 320, 625-668 (1989)

[13] Gupta, A., Trivedi, S. P., Wise, M. B.: Random Surfaces in Conformal Gauge. Nucl. Phys. B 340, 475-490 (1990)

[14] Goulian, M., Li, M.: Correlation Functions in Liouville Theory. Phys. Rev. Lett. 66, 2051-2055 (1991)

[15] Witten, E.: On Holomorphic Factorization of WZW and Coset Models. Commun. Math. Phys. 144, 189-212 (1992)

[16] Bernard, D.: On the Wess-Zumino-Witten Models on Riemann Surfaces. Nucl. Phys. B 309, 145-174 (1988)

[17] Gawędzki, K.: Constructive Conformal Field Theory. In: Functional Integration, Geometry and Strings, eds. Z. Haba et J. Sobczyk, Birkhäuser, Basel, Boston, Berlin 1989, pp. 277-302

[18] Axelrod, S., Della Pietra, S., Witten, E.: Geometric Quantization of Chern-Simons Gauge Theory. J. Differential Geometry 33, 787-902 (1991)

[19] Hitchin, N.: Flat Connections and Geometric Quantization. Commun. Math. Phys. 131, 347-380 (1990)