Extended sampling method in inverse scattering

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Abstract
A new sampling method for inverse scattering problems is proposed to process far field data of one incident wave. The method sets up ill-posed integral equations and uses the (approximate) solutions to reconstruct the target. In contrast to the classical linear sampling method, the kernels of the associated integral operators are the far field patterns of sound-soft balls. The measured data is moved to right hand sides of the equations, which gives the method the ability to process limited aperture data. Furthermore, a multilevel technique is employed to improve the reconstruction. Numerical examples show that the method can effectively determine the location and approximate the support with little \textit{a priori} information of the unknown target.

Keywords: extended sampling method, limited aperture data, far field equation, inverse scattering

(Some figures may appear in colour only in the online journal)

1. Introduction

The inverse scattering theory has been an active research area for a long time and is still developing [5]. Various sampling methods have been proposed to reconstruct the location and support of the unknown scatterer, e.g. the linear sampling method (LSM), the factorization method, the reciprocity gap method [1, 3, 4, 6, 7, 9, 10, 17, 20, 21]. Using the scattering data (far field pattern or near field data), these methods solve some linear ill-posed integral equations for each point in the sampling domain containing the target. The regularized solutions
are used as indicators for the sampling points and the support of the scatterer is reconstructed accordingly.

Classical sampling methods use full aperture data to set up the linear ill-posed integral equations. For example, the LSM uses the far field patterns of all scattering directions for plane incident waves of all directions. In this paper, we propose a new method, the extended sampling method (ESM), to reconstruct the location and approximate the support of the scatterer using much less data. Similar to the LSM, at each sampling point in a domain containing the target, the ESM solves a linear ill-posed integral equation. However, the kernel of the integral operator is the far field pattern of a sound-soft ball with known center and radius rather than the measured far field pattern of the unknown scatterer. The measured far field pattern is moved to the right hand side of the integral equation. As a consequence, the ESM can treat limited aperture data naturally. Furthermore, the ESM is independent of the wave numbers, i.e. the ESM works for all wavenumbers. In contrast, classical sampling methods need to exclude wavenumbers which are certain eigenvalues related to the scatterer.

In recent years, some direct methods were proposed to reconstruct the scatterer using far field pattern of one incident wave, see, e.g. [8, 12–14, 18, 19]. The ESM is different from these methods in the sense that it is based on the classical sampling method. Furthermore, the behavior of the solutions of the linear ill-posed integral equations can be theoretically justified.

The rest of the paper is arranged as follows. In section 2, we introduce the scattering problems and the far field pattern. In addition, we present the far field pattern of the scattered field due to sound-soft balls, which serves as the kernel of the integral operator. In section 3, the ESM based on a new far field equation is presented. The behavior of the solutions is analyzed. In section 4, numerical examples are provided to show the effectiveness of the proposed method. Finally, we draw some conclusions and discuss future works in section 5. For simplicity, the presentation is restricted to 2D. The extension to 3D is straightforward.

2. Scattering problems and far field pattern

We consider the acoustic obstacle and medium scattering problems in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a bounded domain with a $C^3$ boundary. We denote by $u'(x) := e^{ik \cdot d} x, d \in \mathbb{R}^2$ the incident plane wave, where $|d| = 1$ is the direction and $k > 0$ is the wavenumber. The scattering problem is to find the scattered field $u'$ or the total field $u = u' + u$ such that

\[
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \\
\lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u'}{\partial r} - ik u' \right) &= 0,
\end{aligned}
\]

where $r = |x|$. For the well-posedness of the above direct scattering problem, one needs to impose suitable conditions on $\partial D$, which depend on the physical properties of the scatterer. The total field in (2.1) satisfies

(1) Dirichlet boundary condition

\[ u = 0 \quad \text{on } \partial D \]

for a sound-soft obstacle;

(2) Neumann boundary condition

\[ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \]

for a sound-hard obstacle, where $\nu$ is the unit outward normal to $\partial D$; or
(3) the impedance boundary condition
\[ \frac{\partial u}{\partial \nu}(x) + i\lambda u(x) = 0 \quad \text{on } \partial D \]
for an impedance obstacle with some real-valued parameter $\lambda \geq 0$.

The scattering problem of time-harmonic acoustic waves for an inhomogeneous medium is modeled by
\[ \begin{cases} \Delta u + k^2 n(x) u = 0 & \text{in } \mathbb{R}^2, \\ u = u^i + u^s, \\ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i ku^s \right) = 0, \end{cases} \tag{2.2} \]
where $n(x) = n_1(x) + i n_2(x)$, $n_1(x) > 0$, $n_2(x) \geq 0$ is the refractive index. We assume that $n(x) \neq 1$ in $D$ and $n(x) = 1$ in $\mathbb{R}^2 \setminus D$.

It is known that the scattered fields $u^s$ of the above problems have an asymptotic expansion
\[ u^s(x) = e^{i k r} \left( \sum_{n=1}^{\infty} J_n(k R) H_n^{(1)}(kr) \right) + O \left( \frac{1}{r} \right) \quad \text{as } r = |x| \to \infty \]
uniformly in all directions $\hat{x} = x/|x|$. The function $u_\infty(\hat{x})$ defined on $S := \{ \hat{x} | \hat{x} \in \mathbb{R}^2, |\hat{x}| = 1 \}$ is known as the far-field pattern of $u^s$ due to the incident field $u^i$.

The rest of this section is devoted to the far field pattern of a sound-soft disc, which serves as the kernel of the integral operator in the next section. Let $B \subset \mathbb{R}^2$ be a disc centered at the origin with radius $R$. The far-field pattern corresponding to the incident plane wave with direction $d$ is given by (see, e.g. chapter 3 of [4]):
\[ u^\infty_B(\hat{x}; d) = -e^{-i \pi/4} \sqrt{\frac{2}{\pi k}} \left[ J_0(k R) H_0^{(1)}(kr) + 2 \sum_{n=1}^{\infty} J_n(k R) H_n^{(1)}(kr) \cos(n \theta) \right], \quad \hat{x} \in S, \tag{2.3} \]
where $J_n$ is the Bessel function, $H_n^{(1)}$ is the Hankel function of the first kind of order $n$, $\theta = \angle(\hat{x}, d)$, the angle between $\hat{x}$ and $d$. Let
\[ B_z := \{ x + z; x \in B; z \in \mathbb{R}^2 \} \]
be the shifted disc of $B$ and $u^\infty_B(\hat{x}; d)$ be the far field pattern of $B_z$. Then the translation property holds [11, 15, 23]:
\[ u^\infty_B(\hat{x}; d) = e^{ikz \cdot (d-\hat{x})} u^\infty_B(\hat{x}, d), \quad \hat{x} \in S. \tag{2.4} \]

3. The ESM

The inverse scattering problem of interests is to reconstruct the location and approximate support of the scatterer without knowing the physical properties of the scatterer. We propose a novel method, called the ESM. It can process limited aperture far field pattern, full aperture far field pattern, or far field pattern of multiple frequencies.

In this paper, we mainly focus on the case when the far field pattern is available of all observation directions but only one single incident wave, i.e. $u_\infty(\hat{x}; d_0)$ for all $\hat{x} \in S$ and a fixed $d_0 \in S$. The unique determination of the scatterer by the far field pattern of one single incident wave is a long-standing open problem in the inverse scattering theory. The answer
is only partially known for some special scatterers. Some numerical methods have been proposed in the past few years, e.g. the range test method [19], the no-response method [20], the orthogonality sampling method [18], the direct sampling method [8], and the iterative decomposition method [12].

### 3.1. The far field equation

Let \( u_{\infty}^{B_{z}}(\hat{x}, d), x \in \mathbb{S} \) be the far-field pattern of the sound-soft disc \( B_{z} \) centered at \( z \) with radius \( R \) for plane incident waves of all directions \( d \in \mathbb{S} \). Define a far field operator \( F_{z} : L^{2}(\mathbb{S}) \to L^{2}(\mathbb{S}) \)

\[
F_{z}g(\hat{x}) = \int_{\mathbb{S}} u_{\infty}^{B_{z}}(\hat{x}, d)g(d)ds(d), \quad \hat{x} \in \mathbb{S}. \tag{3.1}
\]

Let \( U^{\prime}(x) \) and \( U_{\infty}(\hat{x}) \) be the scattered field and far field pattern of the scatterer \( D \) due to one incident wave, respectively. Using the far field operator \( F_{z} \), we set up a far field equation

\[
(F_{z}g)(\hat{x}) = U_{\infty}(\hat{x}), \quad \hat{x} \in \mathbb{S}. \tag{3.2}
\]

This integral equation is the main ingredient of the ESM. The advantage of using \( F_{z} \) is that it can be computed easily while the classical far field operator uses full aperture measured far field pattern.

We expect that the (approximate) solution of (3.2) for a sampling point \( z \) would provide useful information for the reconstruction of \( D \). To this end, we first introduce some results that are useful in analyzing the far field equation (3.2) (see corollary 5.31, corollary 5.32 and theorem 3.22 of [4]).

\begin{lemma}
The Herglotz operator \( H : L^{2}(\mathbb{S}) \to H^{1/2}(\partial B_{z}) \) defined by

\[
(Hg)(x) := \int_{\mathbb{S}} e^{ikx \cdot d}g(d)ds(d), \quad x \in \partial B_{z} \tag{3.3}
\]

is injective and has a dense range provided \( k^{2} \) is not a Dirichlet eigenvalue for the negative Laplacian for \( B_{z} \).
\end{lemma}

\begin{lemma}
The operator \( A : H^{1/2}(\partial B_{z}) \to L^{2}(\mathbb{S}) \) which maps the boundary values of radiating solutions \( u \in H_{loc}^{1}(\mathbb{R}^{2} \setminus \overline{B_{z}}) \) of the Helmholtz equation onto the far field pattern \( u_{\infty} \) is bounded, injective and has a dense range.
\end{lemma}

\begin{lemma}
The far field operator \( F_{z} \) is injective and has a dense range provided that \( k^{2} \) is not a Dirichlet eigenvalue for the negative Laplacian for \( B_{z} \).
\end{lemma}

The following theorem is the main result for the far field equation (3.2).

\begin{theorem}
Let \( B_{z} \) be a sound-soft disc centered at \( z \) with radius \( R \). Let \( D \) be an inhomogeneous medium or an obstacle with Dirichlet, Neumann, or the impedance boundary condition. Assume that \( kR \) does not coincide with any zero of the Bessel functions \( J_{n}, n = 0, 1, 2, \cdots \).

Then the following results hold for the far field equation (3.2):

1. If \( D \subset B_{z} \), for a given \( \varepsilon > 0 \), there exists a function \( g_{\varepsilon}^{z} \in L^{2}(\mathbb{S}) \) such that

\[
\left\| \int_{\mathbb{S}} u_{\infty}^{B_{z}}(\hat{x}, d)g_{\varepsilon}^{z}(d)ds(d) - U_{\infty}(\hat{x}) \right\|_{L^{2}(\mathbb{S})} < \varepsilon \tag{3.4}
\]
\end{theorem}

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and the Herglotz wave function \( \psi_{\varepsilon}^z(x) := \int_\Sigma e^{ikz} d g_{\varepsilon}^z(d) ds(d), x \in B_z \) converges to the solution \( w \in H^1(B_z) \) of the Helmholtz equation with \( w = -U \) on \( \partial B_z \) as \( \varepsilon \to 0 \).

2. If \( D \cap B_z = \emptyset \), every \( g_{\varepsilon}^z \in L^2(S) \) that satisfies (3.4) for a given \( \varepsilon > 0 \) is such that

\[
\lim_{\varepsilon \to 0} \| \psi_{\varepsilon}^z \|_{H^1(B_z)} = \infty.
\] (3.5)

3. Let \( D \cap B_z \neq \emptyset \) and \( D \not\subset B_z \). When the scattered solution of \( D \) can be extended from \( \mathbb{R}^2 \setminus D \) into \( \mathbb{R}^2 \setminus (D \cap B_z) \), then the conclusion is the same with case 1; otherwise, the conclusion is the same with case 2.

**Proof.** Since \( kR \) does not coincide with any zero of the Bessel functions \( J_n, n = 0, 1, 2, \cdots \), then \( k^2 \) is not a Dirichlet eigenvalue for the following eigenvalue problem for \( B_z \) [22]:

\[
\begin{cases}
    \Delta u + k^2 u = 0, & \text{in } B_z, \\
    u = 0, & \text{on } \partial B_z.
\end{cases}
\] (3.6)

**Case 1:** Let \( D \subset B_z \). From lemma 3.1, for any \( \varepsilon \), there exists \( g_{\varepsilon}^z \) such that

\[
\| H g_{\varepsilon}^z + U' \|_{L^2(\partial B_z)} \leq \frac{\varepsilon}{\| A \|},
\] (3.7)

where \( A : H^{1/2}(\partial B_z) \to L^2(S) \) is defined in lemma 3.2. Then

\[
A(-H g_{\varepsilon}^z) = \int_S u_{\varepsilon}^z(\cdot, d) g_{\varepsilon}^z(d) ds(d).
\] (3.8)

Since \( D \subset B_z \), we also have

\[
A(U') = U_\infty.
\] (3.9)

Taking the difference of (3.8) and (3.9), and from lemma 3.2 and (3.7), for any \( \varepsilon \), we have \( g_{\varepsilon}^z \) such that

\[
\left\| \int_S u_{\varepsilon}^z(\cdot, d) g_{\varepsilon}^z(d) ds(d) - U_\infty(x) \right\|_{L^2(S)} \\
= \| A(-H g_{\varepsilon}^z) - A(U') \|_{L^2(S)} \\
\leq \| A \| \cdot \| H g_{\varepsilon}^z + U' \|_{L^2(S)} \\
\leq \varepsilon.
\] (3.10)

By (3.7), as \( \varepsilon \to 0 \), the Herglotz wave function \( \psi_{\varepsilon}^z(x) := \int_\Sigma e^{ikz} d g_{\varepsilon}^z(d) ds(d), x \in B_z \) converges to the unique solution \( w \in H^1(B_z) \) of the following well-posed inner Dirichlet problem

\[
\begin{cases}
    \Delta w + k^2 w = 0, & \text{in } B_z, \\
    w = -U', & \text{on } \partial B_z.
\end{cases}
\]

**Case 2:** Let \( D \cap B_z = \emptyset \). From lemma 3.3, for every \( \varepsilon \), there exists \( g_{\varepsilon}^z \in L^2(S) \) such that

\[
\left\| \int_S u_{\varepsilon}^z(\cdot, d) g_{\varepsilon}^z(d) ds(d) - U_\infty(x) \right\|_{L^2(S)} < \varepsilon.
\] (3.11)

Assume to the contrary that there exists a sequence \( \psi_{\varepsilon_n}^z \) such that \( \| \psi_{\varepsilon_n}^z \|_{H^1(B_z)} \) remains bounded as \( \varepsilon_n \to 0, n \to \infty \). Without loss of generality, we assume that \( \psi_{\varepsilon_n}^z \) converges to \( \psi_{\varepsilon_0}^z \in H^1(B_z) \)
weakly as \( n \to \infty \), where \( v_x(x) = \int_{\xi} \varphi \xi \Delta \phi \xi \, ds(d), x \in B \). Let \( V^x \in H^1_0 (\mathbb{R}^2 \setminus \mathbb{B}_x) \) be the unique solution of the following exterior Dirichlet problem

\[
\begin{aligned}
\Delta V^x + k^2 V^x &= 0 \quad \text{in } \mathbb{R}^2 \setminus \mathbb{B}_x, \\
V^x &= -v_x \quad \text{on } \partial \mathbb{B}_x, \\
\lim_{r \to \infty} \sqrt{r} (\partial V^x / \partial r - i k V^x) &= 0.
\end{aligned}
\]

Its far field pattern is \( V_\infty = \int_{\xi} u^\infty_x (x, d) g_z(d) ds(d) \). While from (3.11), as \( \varepsilon \to 0 \), we have

\[
\int_{\xi} u^\infty_x (x, d) g_z(d) ds(d) = U_\infty. 
\] (3.12)

Then by Rellich’s lemma (see lemma 2.12 in [4]), the scattered fields coincide in \( \mathbb{R}^2 \setminus (\mathbb{B}_z \cup \mathbb{D}) \) and we can identify \( U^* : = V^* = U^\infty \) in \( \mathbb{R}^2 \setminus (\mathbb{B}_z \cup \mathbb{D}) \). We have that \( V^* \) has an extension into \( \mathbb{R}^2 \setminus \mathbb{B}_z \) and \( U^\infty \) has an extension into \( \mathbb{R}^2 \setminus \mathbb{D} \). Since \( D \cap B_z = \emptyset \), \( U^\infty \) can be extended from \( \mathbb{R}^2 \setminus (\mathbb{B}_z \cup \mathbb{D}) \) into all of \( \mathbb{R}^2 \), that is, \( U^\infty \) is an entire solution to the Helmholtz equation. Since \( U^\infty \) also satisfies the radiation condition, it must vanish identically in all of \( \mathbb{R}^2 \). This leads to a contradiction since \( U^\infty \) is not a null function.

Case 3: if the scattered field of \( D \) can be extended from \( \mathbb{R}^2 \setminus D \) into \( \mathbb{R}^2 \setminus (D \cap B_z) \), then \( \hat{D} : = D \cap B_z \) has the far field pattern \( U_\infty \). One can simply replace \( D \) with \( \hat{D} \) in case 1 to arrive at the same conclusion. Otherwise, if the scattered solution of \( D \) cannot be extended from \( \mathbb{R}^2 \setminus D \) into \( \mathbb{R}^2 \setminus (D \cap B_z) \), a similar proof to case 2 works.  

**Remark 3.2.** Note that the radius \( R \) of the disc \( B_z \) can be chosen such that \( kR \) is not a Dirichlet eigenvalue for the negative Laplacian for \( B_z \). Hence the ESM is independent of wavenumbers, i.e. the ESM works for all wavenumbers. In contrast, the LSM needs to exclude wavenumbers which are certain eigenvalues related to the scatterer.

### 3.2. The ESM algorithm

Now we are ready to present the ESM to reconstruct the location and approximate the support of a scatterer \( D \) using the far field pattern due to a single incident wave.

Recall that \( u^\infty_x (\cdot, d) \) is the far field pattern of a sound-soft disc \( B_z \) centered at \( z \) with radius \( R \) and \( U_\infty (\hat{z}) \) is the measured far field pattern of an unknown scatterer \( D \) due to an incident plane wave. Let \( \Omega \) be a domain containing \( D \). For a sampling point \( z \in \Omega \), we consider the linear ill-posed integral equation

\[
\int_{\Sigma} u^\infty_x (\hat{z}, d) g_z(d) ds(d) = U_\infty (\hat{z}), \quad \hat{z} \in \Sigma. 
\] (3.13)

Suppose that (3.13) is solved by some regularization scheme, say Tikhonov regularization with parameter \( \alpha \). By theorem 3.1, one expects that the approximate solution \( \| g^\alpha \|_{L^2(\Sigma)} \) is relatively large when \( D \) is not inside \( B_z \) and relatively small when \( D \) is inside \( B_z \). Consequently, an approximation of the location and support of the scatterer \( D \) can be obtained by plotting \( \| g^\alpha \|_{L^2(\Sigma)} \) for all sampling points \( z \in \Omega \).
The algorithm of the ESM is as follows (see figure 1).

3.2.1. The ESM.
1. Generate a set \( T \) of sampling points for \( \Omega \) which contains \( D \).
2. Compute \( u_{\infty}^B(\hat{x},d) \) for all \( \hat{x} \in S \) and \( d \in S \).
3. For each sampling point \( z \in T \),
   a. compute \( u_{\infty}^B(z,d) \) and set up a discrete version of (3.13);
   b. use the Tikhonov regularization to compute an approximate \( g_\alpha^z \) to (3.13) (\( \alpha \) is the regularization parameter).
4. Find the global minimum point \( z^* \in T \) for \( \| g_\alpha^z \|_{L_2} \).
5. Choose \( B_{z^*} \) to be the approximate support for \( D \).

Remark 3.3. Let \( \tilde{D} \) be the smallest subset of \( D \) such that the scattered field for \( D \) can be extended from \( \mathbb{R}^2 \setminus D \) into \( \mathbb{R}^2 \setminus \tilde{D} \). If the radius \( R \) of the sampling disc is greater than the radius of the circumscribe circle of \( \tilde{D} \), then \( B_{z^*} \) found by the ESM should contain \( \tilde{D} \).

3.3. Multilevel ESM

An important step of the ESM is to choose the radius \( R \) of \( B_z \). It would be ideal to choose \( B_z \) to be slightly larger than the scatterer \( D \). However, this is not possible if no a priori information about the scatterer is available. To resolve the difficulty, we propose a multilevel technique to choose \( R \).

3.3.1. Multilevel ESM.
1. Choose the sampling discs with a large radius \( R_0 \). Generate \( T \) such that the distance between sampling points is roughly \( R_0 \). Using the ESM, determine the global minimum point \( z_0 \in T \) for \( \| g_\alpha^{z_0} \|_{L_2} \) and an approximation \( D_0 := B_{z_0}^{R_0} \), i.e. the disc centered at \( z_0 \) with radius \( R_0 \), for \( D \).
2. For \( j = 1, 2, \cdots, J \)
   - Let \( R_j = R_0 / 2^j \) and generate \( T_j \) with the distance between sampling points being roughly \( R_j \).
Employ the ESM using $T_j$ and sampling disc with radius $R_j$.

Find the minimum point $z_j \in T$ and set $D_j := B_{z_j}^R$. If $z_j \notin D_{j-1}$, go to Step 3.

3. Choose $z_{j-1}$ and $D_{j-1}$ to be the location and approximate support of $D$, respectively.

The second step of the Multilevel ESM is a loop. The following is some heuristic argument on the termination of the loop. Domains $D_j$'s in Step 2 should satisfy $D_j \subset D_{j-1}$. If the radius of the sampling disc $R_j$ is too small, the scattered field cannot continuously extend to the complement of any of the sampling discs according to theorem 3.1. Then the minimum point $z_j$ might lie outside $D$. Consequently, $z_j \notin D_{j-1}$ could serve as an indication that the proper radius of the sampling disc is found. We shall see that numerical examples support this argument. The positive integer $J$ is chosen based on some a priori information on the target or chosen to be an integer large enough to guarantee the termination of the loop.

3.4. Relation to other methods The far field equation (3.2) of the ESM looks similar to the LSM proposed by Colton and Kirsch [3]. The far field equation for the LSM is

$$
\int_{S} U_\infty(\hat{x}; d) g_s(d) ds(d) = \Phi_\infty(\hat{x}; z), \quad \hat{x} \in S.
$$

(3.14)

In (3.14), $U_\infty(\hat{x}; d), \hat{x}, d \in S$ is the full-aperture far field pattern of the scatterer $D$ due to the incident plane wave $u^i = \exp(ik \cdot d)$, and $\Phi_\infty(\hat{x}, z)$ is the far field pattern of the fundamental solution of the Helmholtz equation $\Phi(x, z) := \frac{1}{2} H_0^{(1)}(k |x - z|)$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. For each sampling point $z$ in the sampling region $\Omega$ which contains $D$, using some appropriate regularization scheme, one obtains an approximate solution $g_s^d$. In general, the norm $\|g_s^d\|$ is larger for $z \notin D$ and smaller for $z \in D$ such that the support of $D$ can be reconstructed accordingly.

Since $U_\infty(\hat{x}; d)$ is used as the kernel of the integral operator in (3.14), full aperture far field pattern is necessary. Namely, one needs $U_\infty(\hat{x}; d)$ for all $\hat{x} \in S$ and $d \in S$ to set up (3.14). As a consequence, the LSM cannot directly process limited aperture far field pattern, e.g. $U_\infty(\hat{x}; d_0)$ for a single incident direction $d_0$. This also applies to other classical sampling methods such as the Factorization Method and the Reciprocity Gap Method.

In contrast, the kernel of the integral operator $F_\infty$ for the ESM is the full aperture far field pattern $u_{\infty}^d(\hat{x}, d)$ of a sound-soft disc $B_{z_j}$. The measured far field pattern $U_\infty(\hat{x})$ is the right hand side of the far field equation (3.2). Hence, in principle, the ESM can take limited aperture data of any type as input.

Another related method is the range test due to Potthast, Sylvester and Kusiak [19]. Similar to the ESM, it processes $U_\infty(\hat{x})$ for a single incident wave. Let $G$ be a convex test domain. The integral equation in [19] is given by

$$
\int_{\partial G} e^{ik \cdot d} g(d) ds(d) = U_\infty(\hat{x}), \quad \hat{x} \in S.
$$

(3.15)

It has a solution in $L^2(\partial G)$ if and only if the scattered field can be analytically extended up to the boundary $\partial G$. Then a convex support can be obtained numerically and the intersection of these supports provides information to reconstruct $D$.

The ESM uses a slightly different integral equation. More importantly, other than finding a convex domain, the ESM computes an indicator for a sampling point, which makes it possible to process data of multiple directions or even multiple frequencies.
4. Numerical examples

We now present some numerical examples to show the performance of the ESM. In particular, we consider the inverse scattering problems for inhomogeneous media and impenetrable obstacles with Dirichlet, Neumann and impedance boundary conditions. The synthetic far-field data is generated using boundary integral equations [4] for impenetrable obstacles. For inhomogeneous media, a finite element method with perfectly matched layer (PML) [2] is used to compute the scattered data. Then the data are used to obtain the far-field pattern [16].

We consider two obstacles: a triangle whose boundary is given by

\[(1 + 0.15 \cos 3t)(\cos t, \sin t) + (3, 5), \quad t \in [0, 2\pi]\]

and a kite whose boundary is given by

\[(1.5 \sin t, \cos t + 0.65 \cos 2t - 0.65) + (3, 5), \quad t \in [0, 2\pi].\]

In the case of the impedance boundary condition, we set \(\lambda = 2\).

For the inhomogeneous medium, \(D\) is the L-shaped domain given by

\[[-0.9, 1.1] \times [-1.1, 0.9] \setminus [0.1, 1.1] \times [-1.1, -0.1]\]

with the refractive index

\[n(x, y) = \begin{cases} \frac{1}{2} + \cos(2\pi \sqrt{x^2 + y^2}), & (x, y)^T \in D, \\ 1, & \text{otherwise}. \end{cases} \tag{4.1}\]

Using (2.4), we can rewrite the far-field equation (3.13) as

\[\int_{S} e^{ikz \cdot (d - \hat{x})} u_{\infty}^\beta (\hat{x}, d) g_{\infty}(d) ds(d) = U_{\infty}(\hat{x}), \quad \hat{x} \in \hat{S},\]

i.e.

\[\int_{S} e^{ikz \cdot (d - \hat{x})} u_{\infty}^\beta (\hat{x}, d) g_{\infty}(d) ds(d) = e^{ikz \cdot \hat{x}} U_{\infty}(\hat{x}), \quad \hat{x} \in \hat{S}, \tag{4.2}\]

where \(u_{\infty}^\beta (\hat{x}; d)\) is given in (2.3).

The incident plane wave is \(u'(x) = e^{ikx \cdot d}\) with \(d = (1,0)^T\). The wavenumber is \(k = 1\) except figure 7, which demonstrates that the ESM works for all wavenumbers. For all the examples, the synthetic data is a \(52 \times 1\) vector \(F_z\) such that \(F_z_j = e^{ikz \cdot \hat{x}_j} U_{\infty}(\hat{x}_j, d)\), where \(U_{\infty}(\hat{x}_j, d)\) is far field pattern of 52 observation directions \(\hat{x}_j, j = 1, 2, \ldots, 52\), uniformly distributed on the unit circle. Then 3% of uniform random noise is added to \(U_{\infty}(\hat{x}_j, d)\). We use a fixed regularization parameter \(\alpha = 10^{-5}\) for all examples. In fact, we found that the choices of \(\alpha\) do not affect the reconstruction significantly as long as \(\alpha\) is small enough.

4.1. Examples for the ESM

Let \(\Omega = [-10, 10] \times [-10, 10]\) and choose the sampling points to be

\[T := \{(-10 + 0.1m, -10 + 0.1n), \quad m, n = 0, 1, \ldots, 200\}.

The radius of the sampling discs is set to \(R = 1\).

We discretize (4.2) to obtain a linear system \(A^z g_z = F_z\), where \(A^z\) is the matrix given by

\[A^z_{ij} = e^{ikz \cdot (d_i, d_j)}, \quad l, j = 1, 2, \ldots, 52.\]
Then the regularized solution is given by

$$g_\alpha z \approx \left( (A^*)^* A^* + \alpha A^* \right)^{-1} (A^*)^* F^z.$$ 

We plot the contours for the indicator function

$$I_z = \frac{\|g_\alpha^z\|_F}{\max_{z \in T} \|g_\alpha^z\|_F}$$

for all the sampling points $z \in T$.

Figure 2(a) is the triangular scatterer. Figures 2(b)–(d) are the contour plots of $I_z$ for obstacles with Dirichlet, Neumann, and the impedance boundary conditions, respectively. The asterisks are the minimum points of $I_z$.

We plot discs whose centers are $z$’s minimizing $I_z$ in figure 3 for different boundary conditions. These minimization points provide the correct locations of the obstacles.

Figure 4 shows similar results of the EMS for the kite with Dirichlet, Neumann and the impedance boundary conditions. Figure 5 shows the reconstructions of the kite of different boundary conditions. Figure 6 show the results for the L-shape medium.

Next we show the reconstructions of the sound-soft triangle in figure 7 to demonstrate that the ESM works for different wavenumbers. The wavenumbers are $k = 0.2$, $k = 3$ and
Figure 3. Approximate reconstructions for the triangle using sampling discs $R = 1$: (a) Dirichlet BC, (b) Neumann BC, (c) Impedance BC.

Figure 4. The kite (a) and contour plots of $I_z$ for the kite using sampling discs $R = 1$: (b) Dirichlet BC, (c) Neumann BC, (d) Impedance BC.
In particular, \( k = 2.4048 \) is a Dirichlet eigenvalue of the unit disc, which is not covered by theorem 3.1. Nonetheless, ESM reconstructs the target effectively. In fact, the ESM can avoid a particular wavenumber, e.g. a Dirichlet eigenvalue, by choosing a slightly different sampling disc.

### 4.2. Examples for the multilevel ESM

If the size of the scatterer is not known, one needs to decide the proper radius \( R \) of the sampling discs. We start the multilevel ESM using the sampling discs with a large radius \( R = 2.4 \). Then we decrease the radius until a suitable \( R \) is found.

In the following figures, the solid lines are the reconstructions, red dashed lines are the exact scatterers, asterisks are the minimum points of \( I_z \). Figure 8 shows the results of the multilevel ESM for the triangle object with Dirichlet boundary condition. The radius of the sampling discs reduces by half from 2.4 to 0.3. When the radius of the sampling discs is \( R = 0.3 \), the minimum point is not inside the reconstruction for \( R = 0.6 \). Hence the proper
Figure 7. Contour plots of $I_z$ for the sound-soft triangle using different wavenumbers. (a) $k = 0.2$, (b) $k = 3$, (c) $k = 2.4048$.

Figure 8. The multilevel ESM for the triangle with Dirichlet BC: (a) $R = 2.4$, (b) $R = 1.2$, (c) $R = 0.6$, (d) $R = 0.3$. 
Figure 9. The multilevel ESM for the triangle with Dirichlet BC. (a) Contour plot. (b) Reconstruction.

Figure 10. The reconstruction results of the multilevel ESM for (a) the triangle with Neumann BC; (b) the triangle with impedance BC; (c) the kite obstacle with Dirichlet BC; (d) the kite obstacle with Neumann BC; (e) the kite with the impedance BC; and (f) the L-shaped inhomogeneous medium.
radius of the sampling discs should be $R = 0.6$. Consequently, we obtain the estimation of the size of the target.

Using sampling discs with radius 0.6, figure 9(a) shows the contours of $I_z$ on the sampling points $(-10 + 0.1m, -10 + 0.1n), 0 \leq m, n \leq 201$, and figure 9(b) shows the reconstruction result of the multilevel ESM for the triangle with Dirichlet boundary condition.

Using the multilevel ESM, the proper radius $R$ is 0.6 for the triangle and kite with different boundary conditions, and the proper radius $R$ is computed to be 0.3 for the L-shaped medium scatterer. Figure 10 shows the reconstructions of the multilevel ESM for the triangle with Neumann and the impedance boundary condition ($\lambda = 2$), the kite with Dirichlet, Neumann and the impedance boundary condition ($\lambda = 2$), and the L-shape medium.

5. Conclusion

A new sampling method using far field data of one incident wave is proposed. It can effectively obtain the location and approximate support of the scatterer and is wavenumber independent. In this paper, the sampling discs are sound-soft. One can also use sound-hard discs or discs with other boundary conditions. Furthermore, since the kernel of the far field equation is computed, the ESM can process various input data. In future, we plan to study the following variations of the ESM.

• Limited aperture observation data
  Let $S_0$ be a non-trivial proper subset of $S$. By analyticity, $u_\infty (\hat{x})$ for all observation directions $\hat{x} \in S$ can be uniquely determined by $u_\infty (\hat{x})$ for limited observation directions $\hat{x} \in S_0$. It is still possible to employ the ESM by solving the far field equations of limited observation data:

  \[
  \mathcal{F}_z g(\hat{x}) = u_\infty (\hat{x}), \quad \hat{x} \in S_0.
  \] (5.1)

  Then the indicator for the sampling point $z$ is taken as

  \[
  I_z = \|g_z\|_{L^2}, \quad z \in T,
  \] (5.2)

  where $g_z$ is the regularized solution of (5.1) for the partial far field data $u_\infty (\hat{x}), \hat{x} \in S_0$.

• Multiple incident direction data
  For data of multiple incident directions $u_\infty (\hat{x}, d_i), \hat{x} \in S_0, i = 1, \ldots, M$, one can solve the far field equation (5.1) for each incident direction $d_i$. The indicator for a sampling point $z$ can be set as

  \[
  I_z = \sum_{i=1}^{M} I_{z_i} = \sum_{i=1}^{M} \|g_{z_i} (d_i)\|_{L^2}, \quad z \in T,
  \] (5.3)

  where $g_{z_i}$ is the regularized solution of (5.1) for $u_\infty (\hat{x}, d_i), \hat{x} \in S_0$.

• Multiple frequency data
  In a similar way, the ESM can process multiple frequency data

  \[
  u_\infty (\hat{x}, d_i, k_j), \quad \hat{x} \in S_0, i = 1, \ldots, M, j = 1, \ldots, J.
  \]

  Again, one solves the far field equation (5.1) for each incident direction $d_i$ and each frequency $k_j$. The indicator for a sampling point $z$ is then
\[
I_z = \sum_{j=1}^{J} \sum_{i=1}^{M} I_{z,ij} = \sum_{j=1}^{J} \sum_{i=1}^{M} \|g_{\xi}^{\delta,ik}(d_i,k_j)\|_{L^2}, \quad z \in T,
\]

(5.4)

where \(g_{\xi}^{\delta,ik}\) is the regularized solution of (5.1) for \(u_\infty(\hat{x}, d_i,k_j)\), \(\hat{x} \in S_0\).

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