Toward a solution of the coincidence problem

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Abstract

The coincidence problem of late cosmic acceleration constitutes a serious riddle with regard to our understanding of the evolution of the Universe. Here we argue that this problem may someday be solved -or better understood- by expressing the Hubble expansion rate as a function of the ratio of densities (dark matter/dark energy) and observationally determining the said rate in terms of the redshift.

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I. INTRODUCTION

The coincidence problem of late cosmic acceleration, i.e., the fact that the density values of dark matter, $\rho_m$, and dark energy, $\rho_\phi$, are of the same order precisely today, constitutes a serious challenge to our understanding of the evolution of the Universe. The otherwise rather successful vacuum-cold dark matter ($\Lambda$CDM) model offers no explanation. In that model, the cosmological constant must be fine-tuned many orders of magnitude for the ratio $r \equiv \rho_m/\rho_\phi$ to be of order unity nowadays. This partly explains why many authors turned to models in which the dark energy density is no longer a constant but evolves with expansion -see e.g. [1] for references.

One way to alleviate the coincidence problem is to provide a mechanism by virtue of which $r$ tends to a constant at late times [2,3] or at least makes it vary slower than the scale factor at present time [4]. This can be achieved by introducing an interaction between dark matter and dark energy regardless if the latter obeys the dominant energy condition or not [5].

Most cosmological models assume, for the sake of simplicity, that matter and dark energy interact only gravitationally. In the absence of an underlying symmetry that would suppress a matter-dark energy coupling (interaction) there is no a priori reason to dismiss it. Further, the coupling is not only likely but inevitable [6] and its introduction is not more arbitrary than assuming it to vanish. Moreover, it has been forcefully argued that the interaction reveals itself when one applies the Layzer-Irvine equation [7] to galaxy clusters [8]. Likewise, the cross-correlation of galaxy catalogs with cosmic microwave background (CMB) maps seems to observationally favor the interaction [9]. Nevertheless, it still remains hypothetical and more abundant, varied, and accurate measurements are needed before its existence (or non-existence) can be established observationally.

Cosmological models where dark matter (DM) and dark energy (DE) do not evolve separately but interact with each other were first introduced to justify the small value of the cosmological constant [10] and currently there is a growing body of literature on the subject -see, e.g. [11] and references therein. Further, in some holographic models of dark energy the accelerated expansion can be traced to the interaction [12]. Recently, various proposals at the fundamental level, including field Lagrangians, have been advanced to account for the coupling [13].
Obviously, a mechanism that makes $r$ tend to a constant today or decrease its rate to a lower value than the scale factor expansion rate ameliorates the coincidence problem significantly, but it does not solve it in full. To do so the said mechanism must also achieve $r_0 \sim O(1)$. We are not aware of any mechanism able to fulfill the latter.

It is the view of the present authors that, on the one hand, $r_0$ ought to be understood -at least, for the time being- as an input parameter much in the same way as other key observational quantities -say, the present value of the cosmic background radiation temperature, the cosmological constant $H_0$, the age of the Universe $t_0$, or the ratio between the number of baryons and photons. And, on the other hand, its value must be closely related to $H_0$ or, for that matter, to $t_0$. Accordingly, if one successfully express $r$ in terms of Hubble rate, $H$, the problem of explaining the $r_0$ value reduces to the problem of explaining the present value of $H$. This is interesting because while we are still lacking model-independent data of $r$ at different redshifts, two sets of observational data values -albeit scarce- of $H$ at various redshifts are now available. The first data set was obtained by computing differential ages of passively evolving galaxies in the redshift range $0.1 < z < 1.8$ [14]. The other data set is based on measurements of 192 supernovae type Ia (SN Ia) and 30 radio galaxies up to redshift 1.2 [15].

Before proceeding, we wish to emphasize that, to the best of our knowledge, no model-independent data of the evolution of the energy densities of matter and dark energy do exist. To obtain such evolution (e.g., with respect to redshift) one must specify some cosmological model. It is to say, the field equations together with the equation of state of dark energy and some assumption about whether the components interact or not. If they are assumed to interact, one must also provide an expression for the interaction.

II. BASIC RELATIONS

Let us consider a spatially flat Friedmann-Lemaitre-Robertson-Walker universe dominated by a two-component system, namely, pressureless dark matter and dark energy, such that the components do not conserve separately but interact with each other in a manner to be specified below. The energy density and pressure of the dark energy, assuming it is a quintessence field, are given by $\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi)$, and $P_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)$, respectively. If the dark energy is a phantom field we have instead, $\rho_\phi = -\frac{1}{2} \dot{\phi}^2 + V(\phi)$, and $P_\phi = -\frac{1}{2} \dot{\phi}^2 - V(\phi)$. 
The upper-dot stands for derivative with respect to the cosmic time and $V(\phi)$ denotes both the quintessence field potential and phantom potential. As is usually done, we postulate that the dark energy component (either quintessence or phantom) obeys a barotropic equation of state, i.e., $P_\phi = w \rho_\phi$ with $w$ a negative constant of order unity and lower than $-1/3$ (a distinguishing feature of dark energy fields is a high negative pressure).

We assume that the dark matter and dark energy components are coupled through a source (loss) term (say, $Q$) whence the energy balances take the form

$$\dot{\rho}_m + 3H \rho_m = Q,$$
$$\dot{\rho}_\phi + 3H (\rho_\phi + P_\phi) = -Q. \quad (1)$$

In what follows we shall consider $Q > 0$. On the one hand, this choice ensures that $r$ decreases monotonously with cosmic expansion (as expected in usual models of cosmic structure formation) and that around present time it varies very slowly, i.e., slower than the scale factor. On the other hand, it shows compatibility with the second law of thermodynamics \[16\]. We note, parenthetically, that if $Q$ were negative (i.e., if dark matter decayed into dark energy), it would exacerbate the coincidence problem since the ratio $r$ would decrease faster than in the $\Lambda$CDM model -see Eq. \((3)\) below.

In virtue of the above expressions and Friedmann’s equation,

$$3H^2 = \kappa^2 (\rho_m + \rho_\phi) \quad (\kappa^2 \equiv 8\pi G), \quad (2)$$

the time evolution of $r$ can be written as

$$\dot{r} = 3H r \left[ w + \frac{\kappa^2 Q (r + 1)^2}{9H^3 r} \right]. \quad (3)$$

Using the relationship $\dot{r} = \dot{H} dr/dH$, where

$$\dot{H} = -\frac{\kappa^2}{2} (\rho_m + \rho_\phi + P_\phi) = -\frac{3}{2} \frac{1 + w + r}{1 + r} H^2,$$

Eq. \((3)\) becomes

$$\frac{dr}{dH} = \frac{3}{H}. \quad (4)$$
Here

$$\bar{z} = -2r \frac{(1+r)}{(1+w+r)} \left[ w + \frac{\kappa^2 Q (r+1)^2}{9H^3} \right].$$  \hspace{1cm} (5)$$

Clearly, Eq.(4) can be integrated whenever an expression for $Q$ in terms of $H$ and $r$ is given.

Notice that to have $dr/dH > 0$ (a very reasonable assumption since one expects that $H(t < t_0) > H_0$) the relationship

$$|w| > \frac{\kappa^2 Q (r+1)^2}{9H^3}$$  \hspace{1cm} (6)$$

must be fulfilled.

We note in passing that for the ΛCDM model (i.e., $w = -1$ and $Q = 0$) one follows

$$H = H_0 [(1+r)/(1+r_0)]^{1/2}.$$  

III. TWO CHOICES FOR $Q$

In the following, we focus on two phenomenological expressions for $Q$ previously considered in the literature. As we have carefully checked, both models fit very well the observational data of Daly et al. obtained from selected sets of SN Ia and 30 radio galaxies [15].

In particular, the maximum variation of $P_\phi$ does not exceed 23% in the redshift interval $0 \leq z \leq 1$, thus showing compatibility with the data analysis of Daly et al. [15].

(i) $Q = 3\alpha H (\rho_\phi + \rho_m)$ with $\alpha$ a positive-definite, dimensionless constant.

This choice has been studied in detail [2]-[5],[9],[17]. As depicted in Fig.1, it leads to a constant but unstable ratio $r$ at early times (high redshifts) and a lower, also constant but stable (attractor) ratio at late times. When the dark energy is of phantom type (not shown in the figure), the evolution of $r$ is also similar. Further, it fits rather well data from SN Ia, cosmic microwave background radiation (Wilkinson Microwave Anisotropy Probe 3 yr), and structure formation -namely, Sloan Digital Sky Survey, and Two Degree Field Redshift Survey- provided $\alpha < 2.3 \times 10^{-3}$ [17]. Likewise, it shows compatibility with cross correlations of galaxy catalogs with CMB maps provided $\alpha < 0.1$ [9]. In this case Eq.(5) reduces to
\[ r = \frac{\rho_m}{\rho_\phi} \]

FIG. 1: Evolution of the ratio \( r = \rho_m/\rho_\phi \) with redshift for different values of the parameter \( \alpha \). In drawing the curves we have fixed \( r_0 = 3/7 \) and \( w = -0.9. \)

\[ \Im = -2r \frac{1 + r}{1 + w + r} \left[ w + \alpha \frac{(r + 1)^2}{r} \right], \quad (7) \]

where we used Eq. (2). The condition given by Eq. (6) leads to restriction \(|w| > \alpha (1+r)^2/r\).

Using last equation in (4) and integrating, the Hubble function is found to be

\[ H = H_0 \exp [I(r) - I(r_0)], \quad (8) \]

where \( I(r) \) stands for the real part of \( \Im(r) \) with

\[ \Im(r) = \frac{1}{2} \ln(1+r) - \frac{1}{4} \ln[-w r - \alpha (1+r)^2] - \frac{1}{2} \frac{w + 2}{\sqrt{-w(4\alpha + w)}} \tan^{-1} \left[ \frac{2\alpha(1 + r) + w}{\sqrt{-w(4\alpha w)}} \right]. \quad (9) \]

Figure 2 shows the dependence of the Hubble function on the densities ratio for \( \alpha = 10^{-4} \) and two values of the equation of state parameter, \( w \). Likewise, we have plotted the prediction of the \( \Lambda \)CDM model. In all the cases \( r_0 = 3/7 \). For other \( \alpha \) values compatible
with the restriction \( \alpha < 2.3 \times 10^{-3} \) set by the Wilkinson Microwave Anisotropy Probe 3 yr experiment the corresponding graphs (not shown) are rather similar.

FIG. 2: Evolution \( H \) vs the ratio \( r = \rho_m/\rho_\phi \) as given by Eq.(8) with \( \alpha = 10^{-4} \) for \( w = -0.9 \) (quintessence, top line) and \( w = -1.1 \) (phantom, bottom line). Also shown is the prediction of the \( \Lambda \)CDM model (middle line). In drawing all the curves we have fixed \( r_0 = 3/7 \).

Unfortunately, as said above, no model-independent data of \( r \) at different values of \( H \) (or \( z \)) exists whereby, for the time being, we cannot directly contrast this model with observation. This is why we turn to determine the dependence of the Hubble factor with redshift and compare it with the two available observational data sets \( H \) vs \( z \). The first model-independent data set was obtained from the study of the differential ages of 32 -carefully selected- passively evolving galaxies in the redshift range \( 0.1 < z < 1.8 \) [14]. The age of each galaxy was found by constraining the age of its older stars with the use of synthetic stellar population models. The differential ages roughly yields \( dz/dt \), then \( H(z) \) is given by
\[ H = -\frac{dz}{(1+z)\,dt} \] -see right panel of Fig. 1 in Ref. [14]. The second model-independent data set (see lower panels in Fig. 9 of Daly et al. [15]) was obtained by applying the model-independent analysis method of Ref. [18] to the coordinate distances of 192 SN Ia of Davis et al. [19] and 30 radio galaxies of Ref. [20].

To obtain the expression for \( H(z) \) of this model we combine Eq. (8) with the integral of (3) in terms of \( z \) which is

\[ r(z) = \text{Re} \left\{ \frac{1}{2\alpha} \left[ -w + \gamma \tan \left( A - \frac{3}{2} \gamma \ln(1+z) \right) \right] \right\} - 1, \tag{10} \]

where \( \text{Re} \) specifies the real part of the corresponding quantity, \( A = \tan^{-1}[(w+2\alpha(1+r_0))/\gamma] \), and \( \gamma = \sqrt{-w(w+4\alpha)} \).

Figures 3 and 4 show the dependence of the Hubble expansion rate on redshift predicted by the model for \( \alpha = 10^{-4} \) and two values of the equation of state parameter, \( w \). For comparison, we have also plotted the prediction of the \( \Lambda \)CDM model. The observational data in Fig. 3 are taken from Simon, Verde, and Jiménez [14] (full circles) and Daly et al. (full diamonds), and in Fig. 4 from Daly et al. (cf. lower panels of figure 9 in Ref. [15]).

As can be seen in both figures, the fit to data does not vary significantly between graphs. Given the scarcity of data it is not worthwhile to compare the \( \chi^2 \) of the different curves.

(ii) \( Q = 3 \beta H \rho_m \) with \( \beta \) a dimensionless positive-definite small constant.

Upon this choice, considered in Ref. [4], the ratio \( r \) does not tend to a constant (i.e., unlike the previous case no attractor exists [21]) but, as Fig. 4 shows, it varies very slowly at late times -by very slowly we mean that \( |(\dot{r}/r)| \lesssim H_0 \), “soft coincidence”- whereby the coincidence problem gets also significantly alleviated.

Likewise, in this instance, Eq.(5) reduces to

\[ \exists = -2r \frac{1+r}{1+w+r} [w + \beta (r+1)] , \tag{11} \]

where we used Eq.(2) to obtain \( Q = 9\beta r H^3/[\kappa^2 (1+r)] \). Now, the condition set by Eq.(6) boils down to \( |w| > \beta (1+r) \).

With the help of Eqs.(4) and (11) the Hubble function can be cast as
FIG. 3: Evolution $H$ vs $z$ with $\alpha = 10^{-4}$ for $w = -0.9$ (quintessence) and $w = -1.1$ (phantom). Also shown is the prediction of the $\Lambda$CDM model (solid line). In all the cases we have fixed $r_0 = 3/7$ and $H_0 = 71$ km/s/Mpc. The data points with their $1\sigma$ error bars are borrowed from Simon, Verde, and Jiménez, Ref. [14] (full circles) and table 2 of Ref. [15] (full diamonds).

\[ H = H_0 \frac{I(r)}{I(r_0)}, \]  

where
FIG. 4: Same as Fig. 3. The upper panel shows the combined sample of 132 SN Ia of Ref. [19] and 30 radio galaxies of Ref. [20]. The bottom panel shows the 30 radio galaxies only. The data points with their 1σ error intervals and the best-fit curve (big solid line) in both panels are borrowed from Daly et al., Ref. [15].

\[
I(r) = \sqrt{(1 + r) \ r^{[-w-1]/[2(\beta+w)]} \ [\beta(1 + r) + w]^{(1-\beta)/[2(\beta+w)]}}.
\] (13)

Numerically, the dependence of the Hubble expansion rate upon \( r \) is very close to the previous case (i) for \( \beta \) values similar to \( \alpha \). This is why we do not show it here.

The next step is to compute the dependence of the Hubble expansion rate on redshift. This can be done by combining Eq. (12) with

\[
r(z) = \frac{\xi r_0}{\beta r_0 - (1 + z)^{-3 \xi} [\beta(1 + r_0) + w]},
\] (14)

where \( \xi = -w - \beta > 0 \). The latter expression follows from integrating Eq. (3) in terms of \( z \).

Again, the corresponding figures for \( H(z) \) is rather similar to Figs. 3 and 4 whereby we do not depict them here.
FIG. 5: Evolution of the ratio $r = \rho_m/\rho_\phi$ with redshift for different values of the parameter $\beta$. In drawing both curves we have fixed $r_0 = 3/7$ and $w = -0.9$. As it is apparent, $\dot{r}/r \simeq 0$ at late times.

IV. DISCUSSION

The current value of the ratio $r = \rho_m/\rho_\phi$ is of order unity (more precisely, about $3/7$ -see e.g. [22]). This represents the “coincidence problem” that the ΛCDM model seems unable to account for. Nowadays, $r_0$ cannot be derived from first principles and like some other key cosmic quantities it has to be considered an input parameter. It seems obvious, nevertheless, that in Einstein relativity it must be closely tied to the present value of the Hubble expansion rate, $H_0$, another quantity whose value we cannot explain but that contrary to $r_0$ it does not mean any puzzle. Therefore, a possible avenue to someday solve the coincidence problem
may well be to express $r$ as a function of $H$ (or vice versa) in models such that $r$ at late times either tends to constant or varies slowly enough (i.e., slower than in the $\Lambda$CDM model). Thus, the problem of explaining $r_0$ reduces to the less acute problem of explaining $H_0$.

In this paper we have taken a first step in that direction. Specifically, we have considered two phenomenological models, previously studied in the literature, in which dark energy and dark matter do not evolve separately but interact with each other in such a way that either $r$ stays constant at late times (case (i)) or varies very slowly (case (ii)) and, in both instances, we have related $H$ to $r$. Since model-independent data relating these two quantities are not currently available we have expressed $r$ in terms of $z$ [Eqs. (10) and (13), respectively] and resorted to the observational data of Refs. [14] and [15] that link $H$ to $z$. Regrettably, given the scarcity and dispersion of the data (cf. Figs. 3 and 4) they are unable to discriminate between models.

Nevertheless, it is fair to say that in order not to spoil the well established primeval nucleosynthesis scenario, $\Omega_\phi$ should not exceed 5% at that early epoch -see Bean, Hansen, and Melchiorri [23]. At any rate, as Figs. 1 and 5 show, the models considered in this work are consistent with this constraint which, in any case, was determined using specific non-interacting dark energy models.

It is to be hoped that in the not far future abundant and accurate observational data, in an extended redshift range, may help determine $H$ in terms of $z$ and solve (or, at least, help to better understand) the coincidence problem of late acceleration. Yet, it may well be that the $\Lambda$CDM model comes to fit the data better than any interacting model. In such a case, we should conclude that the coincidence problem is just that: a coincidence and, therefore, no longer a problem.

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