Cohomology and Deformation of Virasoro Extensions of $q$-Witt Hom-Lie superalgebra

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Abstract

The purpose of this paper is to study Virasoro extensions of the $q$-deformed Witt Hom-Lie superalgebra. Moreover, we provide the cohomology and deformations of the Ramond Hom-superalgebra and special Ramond Hom-superalgebra.

1 Introduction

Various important examples of Lie superalgebras have been constructed starting from the Witt algebra $\mathcal{W}$. It is well-known that $\mathcal{W}$ (up to equivalence and rescaling) has a unique nontrivial one-dimensional central extension, the Virasoro algebra. This is not the case in the superalgebras case, very important examples are the Neveu-Schwarz and the Ramond superalgebras. For further generalizations, we refer to Schlichenmaier’s book [9]. The Neveu-Schwarz and Ramond superalgebras are usually called super-Virasoro algebras since they can be viewed as super-analogs of the Virasoro algebra. Their corresponding second cohomology groups are computed in [4]. One may found the second cohomology group computation of Witt and Virasoro algebras in [6, 7, 8]. The $q$-deformed Witt superalgebra $\mathcal{W}_q$ was defined in [1] as a main example of Hom-Lie superalgebras. The cohomology and deformations of $\mathcal{W}_q$ were studied in [2, 3]. The first and second cohomology groups of the $q$-deformed Heisenberg-Virasoro algebra of Hom-type are computed in [5].

In this paper, we aim to study extensions of Hom-Lie superalgebras and discuss mainly the case of $\mathcal{W}_q$ Hom-superalgebra. We provide a characterization of the Virasoro extensions of the $q$-Witt superalgebra and study their cohomology and deformations. In Section 2, we review the basics about Hom-Lie superalgebras and their cohomology; and in Section 3, we discuss their extensions. In Section 4, we describe the $q$-Witt superalgebra extensions of Virasoro type, we introduce Ramond Hom-superalgebra and special Ramond Hom-superalgebra. Section 5 is dedicated to cohomology and derivations calculations of Virasoro extensions of $q$-Witt superalgebra. In the last section we discuss one-parameter formal deformations of Ramond and special Ramond Hom-Lie superalgebras.

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2 Preliminaries

In this section, we recall definitions of Hom-Lie superalgebras, $q$-deformed Witt superalgebra and some basics about representations and cohomology. For more details we refer to [2].

**Definition 2.1.** A Hom-Lie superalgebra is a triple $(G, [.,.], \alpha)$ consisting of a superspace $G$, an even bilinear map $[.,.] : G \times G \to G$ and an even superspace homomorphism $\alpha : G \to G$ satisfying

$$
[x, y] = -(-1)^{|x||y|}[y, x],
(1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|x|||\alpha(z), [x, y]]} + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0,
$$

for all homogeneous element $x, y, z$ in $G$ and where $|x|$ denotes the degree of the homogeneous element $x$.

### 2.1 A $q$-deformed Witt superalgebra

A $q$-deformed Witt superalgebra $W_q$ can be presented as the $\mathbb{Z}_2$-graded vector space with $\{L_n\}_{n \in \mathbb{Z}}$ as a basis of the even homogeneous part and $\{G_n\}_{n \in \mathbb{Z}}$ as a basis of the odd homogeneous part. It is equipped with the commutator

$$
[L_n, L_m] = (\{m\} - \{n\})L_{n+m}, \tag{2.1}

[L_n, G_m] = (\{m+1\} - \{n\})G_{n+m}, \tag{2.2}
$$

where $\{m\}$ denotes the $q$-number $m$, that is $\{m\} = \frac{1-q^m}{1-q}$. The other brackets are obtained by supersymmetry or are equal to 0. The even linear map $\alpha$ on $W_q$ is defined on the generators by

$$
\alpha(L_n) = (1 + q^n)L_n, \quad \alpha(G_n) = (1 + q^{n+1})G_n.
$$

For more details, we refer to [1].

### 2.2 Cohomology of Hom-Lie superalgebras

Let $(G, [.,.], \alpha)$ be a Hom-Lie superalgebra and $V = V_0 \oplus V_1$ be an arbitrary vector superspace. Let $\beta \in \text{Gl}(V)$ be an arbitrary even linear self-map on $V$ and

$$
[.,.]_V : G \times V \to V, \quad (g, v) \mapsto [g, v]_V
$$

a bilinear map satisfying $[G_i, V_j]_V \subset V_{i+j}$ where $i, j \in \mathbb{Z}_2$.

**Definition 2.2.** The triple $(V, [.,.]_V, \beta)$ is called a representation of the Hom-Lie superalgebra $G = G_0 \oplus G_1$ or $G$-module $V$ if the even bilinear map $[.,.]_V$ satisfies, for $x, y \in G$ and $v \in V$,

$$
[[x, y], \beta(v)]_V = [\alpha(x), [y, v]]_V - (-1)^{|x||y|}[\alpha(y), [x, v]]_V. \tag{2.3}
$$

**Remark 2.3.** When $[.,.]_V$ is the zero-map, we say that the module $V$ is trivial.
**Definition 2.4.** [2] The set $C^k(G, V)$ of $k$-cochains on space $G$ with values in $V$, is the set of $k$-linear maps $f : \otimes^k G \rightarrow V$ satisfying

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_k) = (-1)^{|x_i||x_{i+1}|}f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_k) \text{ for } 1 \leq i \leq k-1.$$ 

For $k = 0$ we have $C^0(G, V) = V$.

Define $\delta^k : C^k(G, V) \rightarrow C^{k+1}(G, V)$ by setting

$$\delta^k(f)(x_0, \ldots, x_k) = \sum_{0 \leq s < t \leq k} (-1)^{s+|x_t|(|x_{s+1}|+\cdots+|x_t|)}f(\alpha(x_0), \ldots, \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), \ldots, \hat{x}_t, \ldots, \alpha(x_k)) + \sum_{s=0}^k (-1)^{s+|x_s|(|f|+|x_0|+\cdots+|x_s|)} \left[ \alpha^{k+1-r-1}(x_s), f\left(x_0, \ldots, \hat{x}_s, \ldots, x_k\right) \right]_V,$$  

(2.4)

where $f \in C^k(G, V)$, $|f|$ is the parity of $f$, $x_0, \ldots, x_k \in G$ and $\hat{x}_i$ means that $x_i$ is omitted.

We assume that the representation $(V, [\cdot, \cdot]_V, \beta)$ of a Hom-Lie superalgebra $(G, [\cdot, \cdot], \alpha)$ is trivial. Since $[\cdot, \cdot]_V = 0$, the operator defined in (2.4) becomes

$$\delta^k_1(f)(x_0, \ldots, x_k) = \sum_{0 \leq s < t \leq k} (-1)^{s+|x_t|(|x_{s+1}|+\cdots+|x_t|)}f(\alpha(x_0), \ldots, \alpha(x_{s-1}), [x_s, x_t], \alpha(x_{s+1}), \ldots, \hat{x}_t, \ldots, \alpha(x_k)).$$  

(2.5)

The pair $(\oplus_{k>0} C_{\alpha, \beta}^k(G, V), \{\delta^k\}_{k>0})$ defines a cohomology complex, that is $\delta^k \circ \delta^{k-1} = 0$.

- The $k$-cochains space is defined as $Z^k(G) = \ker \delta^k$.
- The $k$-coboundary space is defined as $B^k(G) = \text{Im} \delta^{k-1}$.
- The $k^{th}$ cohomology space is the quotient $H^k(G) = Z^k(G)/B^k(G)$. It decomposes as well as even and odd $k^{th}$ cohomology spaces.

Now, we consider the adjoint representation of a Hom-superalgebra and define the first and second coboundary maps. For all $f \in C^1_{\alpha, \alpha}(G, G) = \{g \in C^1(G, G); g \circ \alpha = \alpha \circ g\}$ the operator defined in (2.4) $(r = 0, k \in \{1,2\})$ becomes

$$\delta_0^1(f)(x, y) = -f([x, y]) + (-1)^{|x||f|[x, f(y)]} - (-1)^{|y|(|f|+|x|)[y, f(x)]}$$  

(2.6)

$$\delta_0^2(f)(x, y, z) = -f([x, y], \alpha(z)) + (-1)^{|z||f|[x, [z, \alpha(y)]] + f(\alpha(x), [y, z])] + (-1)^{|z||f|[\alpha(x), f(y, z)]} - (-1)^{|y|(|f|+|z|)[\alpha(y), f(x, z)]} + (-1)^{|z||f|[y, |\alpha(z), f(x, y)]}$$  

(2.7)

Then we have $\delta_0^2 \circ \delta_1^0(f) = 0$, $\forall f \in C^1_{\alpha, \alpha}(G, G)$.

We denote by $H^1(G, G)$ (resp. $H^2(G, G)$) the corresponding 1st and 2nd cohomology groups. An element $f$ of $Z^1(G, G)$ is called derivation of $G$. 

3
3 Extensions of Hom-Lie superalgebras

An extension of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ by a representation $(V, [\cdot, \cdot]_V, \beta)$ is an exact sequence

$$0 \rightarrow (V, \beta) \xrightarrow{i} (K, \gamma) \xrightarrow{\pi} (\mathcal{G}, \alpha) \rightarrow 0$$

satisfying $\gamma \circ i = i \circ \beta$ and $\alpha \circ \pi = \pi \circ \gamma$.

This extension is said to be central if $[K, i(V)]_K = 0$.

In particular, if $K = \mathcal{G} \oplus V$, $i(v) = v$, $\forall v \in V$ and $\pi(x) = x$, $\forall x \in \mathcal{G}$, then we have $\gamma(x, v) = (\alpha(x), \beta(v))$ and we denote

$$0 \rightarrow (V, \beta) \rightarrow (K, \gamma) \rightarrow (\mathcal{G}, \alpha) \rightarrow 0.$$

For convenience, we introduce the following notation for certain cochains spaces on $K = \mathcal{G} \oplus V$, $\mathcal{C}(\mathcal{G}^n, \mathcal{G})$ and $\mathcal{C}(\mathcal{G}^kV^l, V)$, where $\mathcal{G}^kV^l$ is the subspace of $K^{k+l}$ determined by products of $k$ elements from $\mathcal{G}$ and $l$ elements from $V$.

Let $(\phi, \psi) \in \mathcal{C}^2(K, K) \times \mathcal{C}^2(K, K)$. We set, for all $X, Y, Z \in K_0 \cup K_1$,\

$$\phi \circ \psi(X, Y, Z) = (-1)^{|X||Z|} \circ_{X, Y, Z} (-1)^{|X|(|\psi|+|Z|)} \phi(\gamma(X), \psi(Y), Z),$$

and

$$[\phi, \psi] = \phi \circ \psi + (-1)^{|\psi|+|\phi|} \psi \circ \phi.$$

For $f \in \mathcal{C}^2(K, K)$, we set $f = \tilde{f} + \tilde{\alpha} + v + \tilde{v} + \tilde{\nu} + \tilde{\mu}$ where $\tilde{f} \in \mathcal{C}(\mathcal{G}^2, \mathcal{G})$, $\tilde{\nu} \in \mathcal{C}(\mathcal{G}V, \mathcal{G})$, $\tilde{\mu} \in \mathcal{C}(\mathcal{G}V, \mathcal{G})$, $\tilde{v} \in \mathcal{C}(\mathcal{G}V^2, \mathcal{G})$, $\tilde{\nu} \in \mathcal{C}(\mathcal{G}V^2, \mathcal{G})$ and $\tilde{\mu} \in \mathcal{C}(\mathcal{G}V^2, \mathcal{G})$.

Let $d \in \mathcal{C}^2(K, K)$, if $(K, d, \gamma)$ is a Hom-Lie superalgebra and $V$ is an ideal in $K$ (i.e. $d(\mathcal{G}, V) \subset V$), we obtain by using the above notation:

- $\tilde{d} \equiv 0$,
- $\tilde{\nu} \equiv 0$,
- $0 = [d, d](x, y, z) = \left(\tilde{d}, \tilde{d}\right) + 2[v, \tilde{d}] + 2[\tilde{v}, v](x, y, z), \quad \forall x, y, z \in \mathcal{G}$,
- $0 = [d, \tilde{d}](x, y, z) = \left(\tilde{\nu}, \tilde{\nu}\right) + 2[\tilde{d}, \tilde{d}] + 2[\tilde{\nu}, \tilde{v}], \quad \forall x, y, w \in \mathcal{G} \times V$,
- $0 = [d, \tilde{d}](x, y, w) = 2[\tilde{v}, \tilde{v}], \quad \forall (x, y, w) \in \mathcal{G} \times V^2$,
- $0 = [d, d](u, v, w) = \frac{1}{2}[\tilde{\nu}, \tilde{\nu}](u, v, w), \quad \forall (u, v, w) \in V^3$.

We deduce the following theorem

**Theorem 3.1.** The triple $(K, d, \gamma)$ is a Hom-Lie superalgebra if and only if the following conditions are satisfied

- $(\mathcal{G}, \tilde{d}, \alpha)$ is a Hom-Lie superalgebra,
- $[v, \tilde{d}] + [\tilde{v}, v] \equiv 0$, 
\[
\frac{1}{2}[\tilde{v}, \tilde{v}] + [\tilde{\alpha}, \tilde{v}] + [\tilde{\alpha}, v] \equiv 0,
\]
\[
[\tilde{\alpha}, \tilde{v}] \equiv 0,
\]
\[
(V, \tilde{\alpha}, \beta) \text{ is a Hom-Lie superalgebra.}
\]

**Corollary 3.2.** If \( \tilde{\alpha} \equiv 0 \), then the triple \((K, d, \gamma)\) is a Hom-Lie superalgebra if and only if the following conditions are satisfied

- \((G, \tilde{\alpha}, \alpha)\) is a Hom-Lie superalgebra,
- \((V, \tilde{\alpha}, \beta)\) is a representation of \(G\),
- \(v\) is a 2-cocycle on \(V\) (with the cohomology defined by \((G, \tilde{\alpha}, \alpha)\) and \((V, \tilde{\alpha}, \beta)\)).

**Corollary 3.3.** Let

\[
0 \longrightarrow (V, \beta) \longrightarrow (G \oplus V, \tilde{\alpha}) \longrightarrow (G, \alpha) \longrightarrow 0
\]

be an extension of \((G, \tilde{\alpha}, \alpha)\) by a representation \((V, \tilde{\alpha}, \beta)\), where \(\tilde{\alpha}\) is defined by

\[
\tilde{\alpha}(x, v) = (\alpha(x), \beta(v)), \quad \text{for all } x \in G \text{ and } v \in V.
\]

Let \(\varphi \in (C^2(G, V))^j\), \((j \in \mathbb{Z}_2)\). We define a skew-symmetric bilinear bracket operation \(d : \wedge^2(G \oplus V) \rightarrow G \oplus V\) by

\[
d((x, u); (y, v)) = \left( [x, y], [x, v]_V - (-1)^{|u||y|} [y, u]_V + \varphi(x, y) \right) \quad \forall x, y \in G, u, v \in V.
\]

(3.1)

The triple \((G \oplus V, d, \tilde{\alpha})\) is a Hom-Lie superalgebra if and only if \(\varphi\) is a 2-cocycle (i.e. \(\varphi \in Z^2(G, V)\)).

**Proof.** We have \(\tilde{v} \equiv 0, \tilde{d} = [\ldots], v = \varphi, \tilde{\alpha} = [\ldots]_V\) and \(\tilde{\alpha} \equiv 0\). Then, we deduce

- \((G, \tilde{d}, \alpha)\) is a Hom-Lie superalgebra
- \((V, \tilde{\alpha}, \beta)\) is a representation of \(G\).

Therefore, the triple \((G \oplus V, d, \gamma)\) is a Hom-Lie superalgebra if and only if \(\varphi\) is a 2-cocycle (i.e. \(f \in Z^2(G, V)\)).

**Remark 3.4.**

- If \(\varphi\) is even, then \(G_0 \oplus V_0\) is an even homogeneous part and \(G_1 \oplus V_1\) is the odd homogeneous part of \(G \oplus V\).
- If \(\varphi\) is odd, then \(G_0 \oplus V_1\) is an even homogeneous part and \(G_1 \oplus V_0\) is the odd homogeneous part of \(G \oplus V\). The Hom-Lie superalgebra \(G \oplus V\) is called the special extension of \(G\) by \(V\).

**Theorem 3.5.** Let \((V, \tilde{\alpha}, \beta)\) be a representation of a Hom-Lie superalgebra \((G, \tilde{\alpha}, \alpha)\).

The even second cohomology space \(H^2_0(G, V) = Z^2_0(G, V)/B^2_0(G, V)\) is in one-to-one correspondence with the set of the equivalence classes extensions of \((G, \tilde{\alpha}, \alpha)\) by \((V, \tilde{\alpha}, \beta)\).
Definition 3.6. A Hom-Lie superalgebra \((G \oplus V, d, \gamma)\) and \((G \oplus V, d', \gamma)\) be two extensions of \((G, [\, , \,], \alpha)\). So there are two even cocycles \(\phi\) and \(\phi'\) such as \(d((x, u); (y, v)) = [x, y] + [x, v] + [u, v] + \phi(x, y)\) and \(d'((x, u); (y, v)) = [x, y] + [x, v] + [u, v] + \phi'(x, y)\).

If \(\phi - \phi' = \delta^h(x, y)\), where \(h : G \rightarrow V\) is a linear map satisfying \(h \circ \alpha = \beta \circ h\) (i.e. \(\phi - \phi' \in B^2(G, V)\)). Let us define \(\Phi : (G \oplus V, d, \gamma) \rightarrow (G \oplus V, d', \gamma)\) by \(\Phi(x, v) = (x, v - h(x))\). It is clear that \(\Phi\) is bijective. Let us check that \(\Phi\) is a Hom-Lie superalgebras homomorphism. We have

\[
d(\Phi((x, v)), \Phi((y, w))) = d((x, v - h(x)), (y, w - h(y))) = ([x, y], [x, w] + [v, y] - [x, h(y)]) + h(x, y) + \phi(x, y) - h([x, y])) = \Phi((x, [y, [x, y], [x, w] + [v, y] - h([x, y])) = \Phi((x, [y, [x, y], [x, w] + [v, y] - h([x, y])))
\]

\[
d(\Phi((x, v)), \Phi((y, w))) = d((x, v - h(x)), (y, w - h(y))) = ([x, y], [x, w] + [v, y] - h([x, y])) = \Phi((x, [y, [x, y], [x, w] + [v, y] - h([x, y])) = \Phi((x, [y, [x, y], [x, w] + [v, y] - h([x, y])))
\]

Definition 3.7. A Hom-Lie superalgebra \((G, [\, , \,], \alpha)\) is said to be \(Z\)-graded if \(G = \bigoplus_{n \in \mathbb{Z}} G_n\), where \(dim(G_n) < \infty\), \(\alpha(G_n) \subset G_n\) and \([G_n, G_m] \subset G_{n+m}\), for all \(n, m \in \mathbb{Z}\). For an element \(x \in G\), we call \(n\) the degree of \(x\), denoted \(deg(x) = n\), if \(x \in G_n\).

Proposition 3.8. Let \((G, [\, , \,], \alpha)\) be a \(Z\)-graded Hom-Lie superalgebra. We denote \(((G \oplus cC)_0 = G_0 \oplus C\) and \((G \oplus cC)_n = G_n, \forall n \in \mathbb{Z}\). If \(\varphi(G_n, G_m) \subset \delta_{n+m,0}G_{n+m}\) then \((G \oplus cC, d, \gamma)\) is also \(Z\)-graded.

4 Virasoro Hom-superalgebras

In this section, we describe extensions of \(q\)-deformed Witt superalgebra \(W^q\). Let us recall the following result describing its scalar cohomology:

Theorem 4.1. [2]

\[
H^2(W^q, \mathbb{C}) = \mathbb{C}[\varphi_0] \oplus \mathbb{C}[\varphi_1],
\]
where
\[
\varphi_0(x L_n + y G_m, z L_p + t G_k) = x z b_n \delta_{n+p,0},
\]
\[
\varphi_1(x L_n + y G_m, z L_p + t G_k) = x t b_n \delta_{n+k,-1} - y z b_p \delta_{p+m,-1},
\]
with
\[
b_n = \begin{cases} 
\frac{1 + q^2 \{n+1\} \{n\} \{n-1\}}{q^{n-2} \{2\}} \delta_{n,0}, & \text{if } n \geq 0, \\
-b_{-n} & \text{if } n < 0.
\end{cases}
\]

Using the even non trivial 2-cocycle \(\varphi_0\) defined in Theorem 4.1, we can define the followings Virasoro Hom-superalgebras:

- The Neveu-Schwarz Hom-superalgebra can be presented as the \(Z_2\)-graded vector space with \(\{L_n, D\}_{n \in Z}\) as a basis of the even homogeneous part and \(\{F_n\}_{n \in \frac{1}{2} + Z}\) as a basis of the odd homogeneous part. It is equipped with the commutator
\[
[L_n, L_m] = (\{m\} - \{n\}) L_{n+m} + D \frac{1 + q^2 \{n+1\} \{n\} \{n-1\}}{q^{n-2} \{2\}} \delta_{n+m,0}, \quad \forall (n, m) \in (Z_+ \times Z) \cup (Z_- \times Z_-)
\]
\[
[L_n, F_m] = (\{m + \frac{1}{2}\} - \{n\}) F_{n+m}
\]
\[
[F_n, F_m] = [F_n, D] = [L_n, D] = 0.
\]

- The Ramond Hom-superalgebra satisfies the following commutation relations:
\[
[L_n + z, L_m + z'] = (\{m\} - \{n\}) L_{n+m} + c \frac{1 + q^2 \{n+1\} \{n\} \{n-1\}}{q^{n-2} \{2\}} \delta_{n+m,0}, \forall n > 0
\]
\[
[L_n + z, G_m + z'] = (\{m + 1\} - \{n\}) G_{n+m}
\]
\[
[G_n + z, G_m + z'] = 0.
\]

Using the odd non trivial 2-cocycle \(\varphi_1\) defined in Theorem 4.1, we can define the special Ramond Hom-superalgebra. Then it is equipped with the commutator
\[
[L_n + z, L_m + z'] = (\{m\} - \{n\}) L_{n+m}
\]
\[
[L_n + z, G_m + z'] = (\{m + 1\} - \{n\}) G_{n+m} + c \frac{1 + q^2 \{n+1\} \{n\} \{n-1\}}{q^{n-2} \{2\}} \delta_{n+m,0}, \forall n > 0
\]
\[
[G_n + z, G_m + z'] = 0.
\]

In the above cases, the map \(\gamma\) is defined by
\[
\gamma(x, z) = (\alpha(x), z), \quad \forall x \in W^q, \forall z \in C,
\]
where \(\alpha\) is given in Section 2.1.
Remark 4.2. We denote the Ramond Hom-Lie superalgebra by $HR$, the Neveu-Schwarz Hom-Lie superalgebra by $HN$ and the special Ramond Hom-superalgebra by $SHR$. The map $f : HR \to HN$ defined by $f(L_n) = L_n$, $f(G_n) = F_{n+\frac{1}{2}}$ and $f(c) = D$ is a Hom-Lie superalgebras isomorphism.

Hence, Virasoro Hom-superalgebras are characterized in the following Theorem.

**Theorem 4.3.** Every Virasoro Hom-superalgebra is isomorphic to one of the following Virasoro Hom-Lie superalgebras:

- Ramond Hom-superalgebra,
- Special Ramond Hom-superalgebra,
- Trivial Virasoro Hom-superalgebra.

5 Cohomology of Extensions of Hom-Lie superalgebras and Virasoro Hom-superalgebras

Let

\[ 0 \to (V, \beta) \to (K, d, \gamma) \to (G, \alpha) \to 0 \]

be an extension of $(G, \delta, \alpha)$ by a representation $(V, \lambda, \beta)$, where $K = G \oplus V$ and $d = \delta + \lambda + \varphi$ ($\varphi \in Z^2(G, V)$).

5.1 Derivations of Extensions of Hom-Lie superalgebras

If $f \in C^1(K, K)$, we set $f = \tilde{f} + \hat{f} + \varphi \theta$ where $\tilde{f} \in C^1(K, G)$, $\hat{f} \in C^1(V, G)$, $v \in C^1(G, V)$ and $\hat{v} \in C^1(V, V)$. If $(\phi, \psi) \in C^2(K, K) \times C^1(K, K)$, we define

\[ \phi \circ \psi(X, Y) = \phi(\psi(X), \gamma^r(Y), ) - (-1)^{|X||Y|}\phi(\psi(Y), \gamma^r(X), ) \forall X, Y \in K_0 \cup K_1 \]

and

\[ [\phi, \psi] = \phi \circ \psi - (-1)^{|\psi||\phi|}\psi \circ \phi. \]

For $f \in C^1(K, K)$, we have

\[ \delta^1_K(f)((x + u, y + v)) = -f(d(x + u, y + v)) + d(f(x + u), \gamma^r(y + v)) + (-1)^{|f||x+u|}d(\gamma^r(x + u), f(y + v)), \]

which implies $[d, f] = \delta^1_K(f)$.

Then

\[ (f \text{ is a } \alpha^r \text{- derivation of } K) \iff ([d, f] \equiv 0). \quad (5.1) \]

For all $x, y \in G$, $u, v \in V$, we have

\[ [d, f](x, y) = \left( [\delta, \hat{f}] + \delta + \lambda, v \right) + [\varphi, \hat{f} + \tilde{\varphi}])(x, y), \]

\[ [d, f](x, v) = \left( [\delta, \hat{f}] + \lambda, \tilde{\varphi} + \hat{f} + \tilde{\varphi} \right] + [\varphi, \tilde{\varphi}]) \langle x, v \rangle, \]

\[ [d, f](u, v) = 0. \]

Then, we deduce the following result:
Theorem 5.1. For all \(x, y \in G\), \(v \in V\), we have

\[
(f \text{ is a } \alpha^r\text{-derivation of } K) \iff \begin{cases}
\left(\left[\delta, \tilde{f}\right] + \left[\varphi, \hat{f}\right]\right)(x, y) = 0, \\
\left(\left[\varphi, \tilde{f} + \hat{v}\right] + \left[\delta + \lambda, v\right]\right)(x, y) = 0, \\
\left[\delta + \lambda, \tilde{f}\right](x, v) = 0, \\
\left[\lambda, \tilde{f} + \hat{v}\right] + \left[\varphi, \hat{f}\right](x, v) = 0.
\end{cases}
\] (5.2)

5.2 Derivations of Virasoro Hom-superalgebras

Let recall the following result (for the proof see [2]).

Lemma 5.2. The set of \(\alpha^0\)-derivations of the Hom-Lie superalgebra \(W^q\) is

\[
\text{Der}_{\alpha^0}(W^q) = \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle
\]

where \(D_1, D_1, D_2, D_3\) and \(D_4\) are defined, with respect to the basis as

\[
\begin{align*}
D_1(L_n) &= nL_n, & D_1(G_n) &= nG_n, \\
D_2(L_n) &= 0, & D_2(G_n) &= G_n, \\
D_3(L_n) &= nG_{n-1}, & D_3(G_n) &= 0, \\
D_4(L_n) &= 0, & D_4(G_n) &= L_{n+1}.
\end{align*}
\]

Let \((W^q_{\varphi}, d, \gamma)\) be a Hom-Virasoro-superalgebra. Then

\[
W^q_{\varphi} = W^q \oplus \mathbb{C}, \quad d = [\ldots] + \varphi, \quad \text{and } \gamma(x, z) = (\alpha(x), z),
\]

where \(\varphi \in \mathbb{C}\varphi_0 \cup \mathbb{C}\varphi_1\).

Lemma 5.3.

\[
(f \text{ is a } \alpha^r\text{-derivation of } W^q_{\varphi}) \Rightarrow (\hat{f} \equiv 0).
\]

Proof. Let \(f\) be an \(\alpha^r\)-derivation of \(W^q_{\varphi}\).

Using Theorem 5.1 and \(\lambda \equiv 0\), \(\forall z \in \mathbb{C}, \forall n \in \mathbb{Z}\) we have

\[
\begin{align*}
\left[\delta, \tilde{f}\right](L_n, z) &= 0 \\
\Rightarrow &-(-1)^{|x||z|}\delta(\tilde{f}(z), \alpha(L_n)) = 0, \\
\Rightarrow &\hat{f}(z), L_n = 0, \\
\Rightarrow &\hat{f}(z) = 0.
\end{align*}
\]

In the following, we provide the \(\alpha^0\)-derivations of \(W^q_{\varphi}\) explicitly.

Proposition 5.4. The set of \(\alpha^0\)-derivations of the Virasoro Hom-Lie superalgebra \(W^q_{\varphi_i}\) with \(i = 0, 1\), is

\[
\text{Der}_{\alpha^0}(W^q_{\varphi_i}) = \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle.
\]
where $D_1$, $D_1$, $D_2$, $D_3$ and $D_4$ are defined, with respect to the basis, as

\[
D_1(L_n) = nL_n, \quad D_1(G_n) = nG_n, \quad D_1(1) = 0,
\]
\[
D_2(L_n) = 0, \quad D_2(G_n) = G_n, \quad D_2(1) = 0,
\]
\[
D_3(L_n) = nG_{n-1}, \quad D_3(G_n) = 0, \quad D_3(1) = 0,
\]
\[
D_4(L_n) = 0, \quad D_4(G_n) = L_{n+1}, \quad D_4(1) = 0.
\]

**Proof.** Using Lemma 5.2 and the first equation in (5.2), we obtain

\[
\tilde{f} \in \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle.
\]

If $\tilde{f}$ is even, there exist $\lambda_1$ and $\lambda_2$ satisfying $\tilde{f} = \lambda_1 D_1 + \lambda_2 D_2$.

If $\tilde{f}$ is odd, there exist $\lambda_3$ and $\lambda_4$ satisfying $\tilde{f} = \lambda_3 D_3 + \lambda_4 D_4$.

Using (5.2), $\lambda \equiv 0$ and $\tilde{f} \equiv 0$, we obtain

\[
\left( [\varphi, \tilde{f} + \tilde{v}] + [\delta, v] \right)(x, y) = 0
\]
\[
\Rightarrow \varphi(\tilde{f}(x), \alpha(y)) - (-1)^{\|x\|\|y\|} \varphi(\tilde{f}(y), \alpha(x)) - \tilde{v}(\varphi(x, y)) - v(\delta(x, y)) = 0.
\]

\[
\Rightarrow \left\{ \begin{array}{l}
\varphi(\tilde{f}(L_n), \alpha(L_k)) - \varphi(\tilde{f}(L_k), \alpha(L_n)) - \tilde{v}(\varphi(L_n, L_k)) - v(\delta(L_n, L_k)) = 0, \\
\varphi(\tilde{f}(L_k), \alpha(G_n)) - \varphi(\tilde{f}(G_n), \alpha(L_k)) - \tilde{v}(\varphi(L_k, G_n)) - v(\delta(L_k, G_n)) = 0.
\end{array} \right.
\]

With $\varphi \in \{ \varphi_0, \varphi_1 \}$, we have:

\[
\varphi(L_1, x) = 0, \quad \forall x \in \mathcal{W}^q,
\]
\[
\varphi(L_n, L_k) = 0, \quad \forall n + k \neq 0,
\]
\[
\varphi(L_n, G_k) = 0, \quad \forall n + k \neq -1.
\]

Since $\tilde{f} \in \{ \lambda_1 D_1 + \lambda_2 D_2, \lambda_3 D_3 + \lambda_4 D_4 \}$, we obtain

\[
\left\{ \begin{array}{l}
v(\delta(L_n, L_k)) = 0, \\
v(\delta(L_k, G_n)) = 0.
\end{array} \right.
\]

Thus, we can deduce $v \equiv 0$ and $\tilde{v} \equiv 0$.

**5.3 2-cocycles of Extensions of Hom-Lie superalgebras**

Let $f \in C^2(K, K)$, we set $\tilde{f} = \tilde{f} + \tilde{f} + \tilde{v} + \tilde{v} + \tilde{\nu}$ where $\tilde{f} \in C^2(G, \mathcal{G})$, $\tilde{f} \in C^{1,1}(GV, \mathcal{G})$, $\tilde{f} \in C^2(V, G)$, $v \in C^2(G, V)$, $\tilde{v} \in C^{1,1}(GV, V)$ and $\tilde{\nu} \in C^2(V, V)$. In this case, we have $[d, f] = \delta_K$. For all $x, y, z \in \mathcal{G}$, $u, v, w \in V$, we have

\[
[d, f](x, y, z) = \left( [\delta, \tilde{f}] + [\varphi, \tilde{f}] + \lambda + \lambda, v \right) + \left[ \varphi, \tilde{f} + \tilde{v} \right] \right)(x, y, z)
\]
\[
\left( [\delta + \lambda, v] + [\varphi, \tilde{f}] \right)(x, y, z) \in \mathcal{G}
\]
\[
\left( [\delta + \lambda, v] + [\varphi, \tilde{f} + \tilde{v}] \right)(x, y, z) \in V
\]
\[
[d, f](x, y, w) = \left( [\lambda, \delta + \lambda, v] + [\varphi, \tilde{f} + \tilde{v}] + [\lambda, \tilde{f} + \tilde{v} + \tilde{\nu}] \right)(x, y, w)
\]
\[
\left( [\delta, \tilde{f}] + \left( [\lambda, \tilde{f} + \tilde{v} + \tilde{\nu}] + [\varphi, \tilde{f} + \tilde{v}] \right) \right)(x, u, v) = 0
\]
\[
[d, f](u, v, w) = 0.
\]

Therefore, we have the following result
Theorem 5.5. For all $x, y, z \in G$, $u, v, w \in V$, we have

$$f \in Z^2(K, K) \iff \begin{cases} [\delta, \tilde{f}] (x, y, z) = 0, \\ \delta + \lambda, \tilde{v} + [\varphi, f + v] (x, y, z) = 0, \\ \left( \delta + \lambda \right) \tilde{f} + \left[ \varphi, \tilde{f} \right] (x, y, v) = 0, \\ \left( \delta + \lambda \right) \tilde{f} + \left[ \varphi, \tilde{f} + v \right] (x, y, v) = 0, \\ \left( \delta + \lambda \right) \tilde{f} + \left[ \varphi, \tilde{f} + \tilde{f} \right] (x, u, v) = 0, \\ \left( \delta + \lambda \right) \tilde{f} + \left[ \varphi, \tilde{f} + \tilde{f} \right] (x, u, v) = 0, \\ \left( \delta + \lambda \right) \tilde{f} + \left[ \varphi, \tilde{f} + \tilde{f} \right] (x, u, v) = 0. \end{cases}$$ \tag{5.3}

Corollary 5.6. \begin{enumerate} \item If $f = \tilde{f}$, then

$$f \in Z^2(K, K) \iff \begin{cases} [\delta, \tilde{f}] (x, y, z) = 0, \\ \varphi, \tilde{f} (x, y, z) = 0, \\ \lambda, \tilde{f} (x, y, v) = 0. \end{cases}$$ \tag{5.4} \item If $f = v$, then

$$(f \in Z^2(K, K)) \iff (f \in Z^2(G, V))$$ \tag{5.5} \end{enumerate}

Now, we assume that $G$ is $\mathbb{Z}$-graded and $\text{deg}(\varphi) = 0$. Let $s = \text{deg}(\tilde{f}) = \text{deg}(\tilde{f})$. If $x \in G_n$, we set $x_n = x$.

In the first equation in (5.3), we have:

- $\text{deg}([\delta, \tilde{f}] (x_n, x_m, x_p)) = n + m + p + s$,
- $\text{deg}(\tilde{f}(\alpha(x_m), \varphi(x_p, x_n))) = m + s$.

In the third equation in (5.3), we have:

- $\text{deg}([\delta, \tilde{f}]) = n + m + s$,
- $\text{deg}(\tilde{f}(\alpha(x_m), \lambda(x_n, v))) = n + s$.

The other terms are of degree zero. Then, we get

Theorem 5.7. If $f \in Z^2(K, K)$ and $n \neq p$, $n \neq m$, $p \neq m$, we have:

$$\delta \left[ \tilde{f}(x_n, x_m, x_p) \right] = 0; \forall n + m \neq 0, n + p \neq 0, m + p \neq 0,$$ \tag{5.6}

$$\left[ \delta, \tilde{f} \right](x_n, x_m, x_p) + \tilde{f}(\alpha(x_p), \varphi(x_n, x_m)) = 0,$$ \tag{5.7}

$$\tilde{f}(\alpha(x_n), \varphi(x_m, x_p)) = 0, m + p \neq 0, m \neq 0,$$ \tag{5.8}

$$\left[ \delta, \tilde{f} \right](x_n, x_m, u) = 0 \forall n + m \neq 0, n \neq 0, m \neq 0,$$ \tag{5.9}

$$\tilde{f}(\alpha(x_n), \lambda(x_m, u)) = 0 \forall n \neq m, n \neq 0, m \neq 0,$$ \tag{5.10}

$$\left[ \delta, \tilde{f} \right](x_n, x_m, u) + \tilde{f}(\alpha(x_m), \lambda(x_n, u)) = 0, \forall m \neq 0,$$ \tag{5.11}

$$\tilde{f}(\alpha(x_n), \varphi(x_m, x_n)) = 0, \forall n + m \neq 0, n \neq 0, m \neq 0,$$ \tag{5.12}

$$\delta(\alpha(x_n), \tilde{f}(u, v)) = 0, \forall n \neq 0.$$ \tag{5.13}
5.4 Second cohomology of Ramond Hom-superalgebra

Let $f$ be an even 2-cocycle of degree $s$. We can assume that

$$
\tilde{f}(L_n, L_p) = a_{s,n,p}L_{s+n+p}, \tilde{f}(L_n, G_p) = b_{s,n,p}G_{s+n+p}, \tilde{f}(G_n, G_p) = c_{s,n,p}L_{s+n+p},
$$

$$
\tilde{f}(1, L_p) = a'_{s,p}L_{s+p} \text{ and } \tilde{f}(1, G_p) = b'_{s,p}G_{s+p}.
$$

Using (5.6), (5.7) and Lemma 5.8, we obtain

$$
0.
$$

We have $\tilde{f} = 0$. Using (5.9), as above, we obtain $b'_{s,n} = 0$ if $s \neq 0$ and $a'_{0,n} = na'_{0,1}$ if $s = 0$. As above, we obtain $b'_{s,n} = 0$ if $s \neq 0$. $b'_{0,n} = b'_{0,0} + na'_{0,1}$.

Lemma 5.8. If $f$ is a 2-cocycle, we have

$$
\bar{v} \equiv 0, \quad \bar{f} \equiv 0, \quad \bar{f} \equiv 0, \quad \text{and} \quad \bar{v} \equiv 0.
$$

Proof. Since $C$ is continued in even part of $HR$, $\forall z, z' \in C$; we have

$$
\bar{f}(z, z') = zz'\bar{f}(1, 1) = 0 \quad \text{and} \quad \bar{v}(z, z') = zz'\bar{v}(1, 1) = 0.
$$

Let $g = \bar{f}(1, \cdot)$. If we assume $[d, f](x_n, x_m, 1) = 0$, we can deduce

$$
-\alpha(x_n, g(x_n)) + \alpha(x_m, g(x_n)) + g([x_n, x_m]) - \varphi(\alpha(x_n), g(x_n)) + \varphi(\alpha(x_m), g(x_n)) + \bar{v}(1, [x_n, x_m]) = 0.
$$

Since $[\alpha(x_n, g(x_n)) + \alpha(x_m, g(x_n)) + g([x_n, x_m]) \in W^q$ and $-\varphi(\alpha(x_n), g(x_n)) + \varphi(\alpha(x_m), g(x_n)) + \bar{v}(1, [x_n, x_m]) \in C$, we obtain

$$
-\alpha(x_n, g(x_n)) + \alpha(x_m, g(x_n)) + g([x_n, x_m]) = 0.
$$

Then $g$ is an $\alpha$-derivation.

Recall that the set of $\alpha$-derivations is trivial (see [2]). Therefore $g \equiv 0$ and $\bar{v}(1, \cdot) \equiv 0$.

Theorem 5.9. $H^2(HR, HR) = \mathbb{C}[\varphi_1]$.

Proof. We have $H^2(W^q, W^q) = \{0\}$ (see [3]). Then

$$
\left(\delta^2(\bar{f}) \equiv 0\right) \Rightarrow \left(\exists \bar{g} \in C^1_{\alpha,\alpha}(W^q, W^q); \bar{f} = \delta^1(\bar{g})\right).
$$

Using (5.6), (5.7) and Lemma 5.8, we obtain $[\delta, \bar{f}] = 0$. Then $\bar{f} \in Z^2(W^q, W^q)$.

We have $\bar{f} = \delta^1(\bar{g})$, $\bar{f} \equiv 0$, $\bar{f} \equiv 0$, $\bar{v} \equiv 0$ and $\bar{v} \equiv 0$. We deduce, $f = \delta^1(\bar{g}) + v$. Therefore, $f = \delta^1_{HR}(\bar{g}) + w$ where $w(W^q, W^q) \subset C$.

So

$$
\left( f \in Z^2(HR, HR) \right) \Rightarrow \left(\delta^2_{HR}(w) \equiv 0\right) \Rightarrow \left(\delta^2_{HR}(w) \equiv 0\right).
$$

As, $H^2(W^q, \mathbb{C}) = \mathbb{C}[\varphi_0] \oplus \mathbb{C}[\varphi_1]$ and $\varphi_0 \in B^2_{HR}(W^q, \mathbb{C})$, we deduce $w \in \mathbb{C}[\varphi_1]$. 

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5.5 Second cohomology of Special Ramond Hom-superalgebra

Lemma 5.10. If \( f \) is a 2-cocycle, we have
\[
\tilde{v} \equiv 0, \quad \tilde{g} \equiv 0, \quad \bar{f} \equiv 0, \quad \text{and} \quad \hat{v} \equiv 0.
\]

Proof. let \( g = \hat{f}(1,...) \).
\[
\begin{align*}
\{d, f\}(x_n, x_m, 1) = 0 & \Rightarrow \\
- \{\alpha(x_n), g(x_m)\} + \{\alpha(x_m), g(x_n)\} + g([x_n, x_m]) + \hat{f}(1, \varphi(x_n, x_m)) & \\
- \varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}(1, [x_n, x_m]) + \hat{v}(1, \varphi(x_n, x_m)) & .
\end{align*}
\]

Since \( -\{\alpha(x_n), g(x_m)\} + \{\alpha(x_m), g(x_n)\} + g([x_n, x_m]) + \hat{f}(1, \varphi(x_n, x_m)) \in W_q \) and \( -\varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}(1, [x_n, x_m]) + \hat{v}(1, \varphi(x_n, x_m)) \in \mathbb{C} \), we deduce
\[
\begin{align*}
- \{\alpha(x_n), g(x_m)\} + \{\alpha(x_m), g(x_n)\} + g([x_n, x_m]) + \hat{f}(1, \varphi(x_n, x_m)) & = 0. \quad (5.15) \\
- \varphi(\alpha(x_n), g(x_m)) + \varphi(\alpha(x_m), g(x_n)) + \hat{v}(1, [x_n, x_m]) + \hat{v}(1, \varphi(x_n, x_m)) & = 0. \quad (5.16)
\end{align*}
\]

In (5.16), the term \( \hat{f}(1, \varphi(x_n, x_m)) \) is of degree \( s \). The other terms are of degree \( n + m + s \). Then if \( n + m \neq 0 \), we deduce
\[
- \{\alpha(x_n), g(x_m)\} + \{\alpha(x_m), g(x_n)\} + g([x_n, x_m]) = 0.
\]

If \( n + m = 0 \), we have \( \varphi(x_n, x_m) = 0 \). Then
\[
- \{\alpha(x_n), g(x_m)\} + \{\alpha(x_m), g(x_n)\} + g([x_n, x_m]) = 0,
\]
which implies that \( g \) is an \( \alpha \)-derivation. Recall that the set of \( \alpha \)-derivations is trivial (see [2]). We deduce \( g \equiv 0 \) and \( \hat{f}(1,...) \equiv 0 \).

Since \( g \equiv 0 \), the equation (5.16) can be write
\[
\hat{v}(1, [x_n, x_m]) + \hat{v}(1, \varphi(x_n, x_m)) = 0.
\]

If \( n + m = -1 \) and \( s \neq 1 \), with \( \deg(\hat{v}(1, [x_n, x_m])) = n + m + s \) and \( \hat{v}(1, [x_n, x_m]) \in \mathbb{C} \), we obtain \( \hat{v}(1, [x_n, x_m]) = 0 \). Thus \( \hat{v} \equiv 0 \) and \( \hat{v} \equiv 0 \).

Then we obtain the following result about second cohomology of special Ramond Hom-superalgebra.

Theorem 5.11.
\[
H^2(SHR, SHR) = \mathbb{C}[\varphi_0].
\]
6 Deformations of Virasoro Hom-superalgebras

In this section, we discuss deformations of Ramond Hom-superalgebra and special Ramond Hom-superalgebra.

**Definition 6.1.** Let \((G, [., .], 0, \alpha_0)\) be a Hom-Lie superalgebra. A one-parameter formal deformation of \(G\) is given by the \(\mathbb{K}[t]\)-bilinear and the \(\mathbb{K}[t]\)-linear maps \([., .]: G[[t]] \times G[[t]] \to G[[t]], \alpha_t: G[[t]] \to G[[t]]\) of the form
\[
[., .]_t = \sum_{i \geq 0} t^i [., .]_i \quad \text{and} \quad \alpha_t = \sum_{i \geq 0} t^i \alpha_i,
\]
where each \([., .]_i\) is an even \(\mathbb{K}\)-bilinear map \([., .]: G \times G \to G\) (extended to be \(\mathbb{K}[t]\)-bilinear) and each \(\alpha_i\) is an even \(\mathbb{K}\)-linear map \(\alpha_i: G \to G\) (extended to be \(\mathbb{K}[t]\)-linear), and satisfying the following conditions
\[
\begin{align*}
[x, y]_t &= -(-1)^{|x||y|}[y, x]_t, \quad (6.1) \\
\circ_{x,y,z} (-1)^{|x||z|}[\alpha_t(x), [y, z]]_t &= 0. \quad (6.2)
\end{align*}
\]

**Definition 6.2.** Let \((G, [., .], 0, \alpha_0)\) be a Hom-Lie superalgebra. Given two deformations \(G_t = (G, [., .], t, \alpha_t)\) and \(G'_t = (G, [., .], t', \alpha'_t)\) of \(G\), where
\[
[., .]_t = \sum_{i \geq 0} t^i [., .], \quad [., .]'_t = \sum_{i \geq 0} t^i [., .]'_t, \quad \alpha_t = \sum_{i \geq 0} t^i \alpha_i \quad \text{and} \quad \alpha'_t = \sum_{i \geq 0} t^i \alpha'_i.
\]
We say that they are equivalent if there exists a formal automorphism \(\phi_t = \sum_{i \geq 0} t^i \phi_i\), where \(\phi_i \in \left(\text{End}(G)\right)_0\) and \(\phi_0 = id_G\), such that
\[
\phi_t([x, y]_t) = [\phi_t(x), \phi_t(y)]'_t, \quad \forall x, y \in G, \quad (6.3)
\]
and
\[
\phi_t \circ \alpha_t = \alpha'_t \circ \phi_t. \quad (6.4)
\]

A deformation \(G_t\) is said to be trivial if and only if \(G_t\) is equivalent to \(G\) (viewed as a superalgebra on \([G[[t]]]\)).

**Lemma 6.3.** Every deformation \(HR_t\) of Ramond Hom-superalgebra such that \([., .]'_t = [., .]_0 + \sum_{k \geq p} t^k [., .]_k\) and \(\alpha_t = (\sum k \lambda_k t^k)\alpha_0\) is equivalent to a deformation \([., .]_t = [., .]_0 + (\sum k \lambda_k t^k)\varphi_1\).

**Proof.** Let \(HR_t = (HR, [., .]'_t, \alpha_t)\) be a deformation of Ramond Hom-superalgebra \((HR, [., .], 0, \alpha_0)\), where
\[
[., .]'_t = [., .]_0 + \sum_{k \geq p} [., .]'_k t^k \quad \text{and} \quad \alpha_t = (\sum k \lambda_k t^k)\alpha_0.
\]
Condition (6.2) may be written
\[
\circ_{x,y,z} (-1)^{|x||z|} \sum_{s \geq 0} t^s \left(\sum_{k=0}^s \sum_{i=0}^{s-k} [\alpha_i(x), [y, z]'_k]_{s-i-k}\right) = 0. \quad (6.5)
\]
This equation is equivalent to the following infinite system:

\[ \sum_{k=0}^{s} \sum_{i=0}^{s-k} [\alpha_i(x), [y, z]_k]_{s-i-k} = 0, \quad s = 0, 1, \ldots \]  \hspace{1cm} (6.6)

In particular, for \( s = p \),

\[ \sum_{k=0}^{s} \sum_{i=0}^{s-k} [\alpha_0(x), [y, z]_0]_p + \sum_{k=0}^{s} \sum_{i=0}^{s-k} [\alpha_0(x), [y, z]_p]_0 = 0. \]

Therefore \([.,.]_p \in Z^2(HR, HR)\). Since \( H^2(HR, HR) = \mathbb{C}[\varphi_1] \), we deduce

\[ [.,.]_p = -\delta^1(\Phi) + \lambda_p \varphi_1, \quad \text{where} \quad \Phi \in C^1_{\gamma,\gamma}, \quad \text{and} \quad \lambda \in \mathbb{K}. \]

Let \( \Phi_t = \Phi_0 + t \Phi \), then \( \Phi_t^{-1} = \Phi_0 + \sum_{k \geq 1} (-1)^k t^k \Phi^k \).

\[ [x, y]_t = \Phi_t^{-1}(\Phi_t(x), \Phi_t(y)). \]

By a simple identification, it follows that \([x, y]_p = \lambda_p \varphi_1 \).

As well

\[ [.,.]_t = [.,.]_0 + t \lambda_p \varphi_1 + \sum_{k > p} t^k [.,.]_k. \]

Thus, by induction we show that \([.,.]_k = \lambda_k \varphi_1 \quad \forall k \geq p. \quad \blacksquare \]

**Theorem 6.4.** Every deformation \( HR_t \) such that \([.,.]_t' = [.,.]_0 + \sum_{k \geq p} t^k [.,.]_k \) and \( \alpha_t = (\sum_{k \geq 0} a_k t^k) \alpha_0 \) of Ramond Hom-superalgebra is equivalent to a deformation of the form

\[ [.,.]_t = [.,.] + \varphi_0 + t \varphi_1. \]

**Proof.** It follows from

\[ [L_n, G_m]_t = [L_n, G_m] + \sum_{i \geq p} a_i \varphi_1(L_n, G_m)t^i; \]

\[ [L_n, L_m]_t = [L_n, L_m] + \varphi_0(L_n, L_m); \]

\[ [L_n, G_m]'_t = [L_n, G_m] + \varphi_1(L_n, G_m)t; \]

\[ [L_n, L_m]'_t = [L_n, L_m] + \varphi_0(L_n, L_m). \]

\( \Phi_0 = id; \Phi_0(1) = 0, \forall s > 0; \Phi_s(G_m) = \frac{a_s - s 1}{\chi} G_m, \quad \forall s \geq p; \Phi_s(L_n) = 0, \quad \forall s > 0. \)

Thus \([.,.]_t = [.,.] + \varphi_0 + t \lambda \varphi_1 \). By rescaling we obtain the desired result. \quad \blacksquare

**Theorem 6.5.** Every deformation \( SHR_t \) such that \([.,.]_t' = [.,.]_0 + \sum_{k \geq p} t^k [.,.]_k \) and \( \alpha_t = (\sum_{k \geq 0} a_k t^k) \alpha_0 \) of special Ramond Hom-superalgebra is equivalent to a deformation

\[ [.,.]_t = [.,.] + \varphi_1 + t \varphi_0. \]

**Remark 6.6.** Theorem 6.5 can be proved in the same way as Theorem 6.4.
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