Some Metrics for the Crystal Cubic Carbon $CCC(n)$ and the Layer Cycle Graph $LCG(n, k)$

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Abstract

In a chemical compound, it is very important for chemists to find the size of the smallest set of atoms so that they can identify other atoms relative to that the smallest set of atoms, so chemists require mathematical forms for a set of chemical compound to give distinct representations to distinct compound structures and its corresponding in graph theory is to find the minimal resolving set. The study of the resolving set and its related parameters such as doubly resolving set and strong resolving set are significant, and it is well known that these problems are NP hard. In this article, we define the structure of the crystal cubic carbon $CCC(n)$ by a new method, and we provide a much simpler formula for obtaining its vertices than the formula for calculating its vertices in (Baig et al., 2017). Moreover, we focus on some resolving parameters of two graphs the crystal cubic carbon $CCC(n)$ and the layer cycle graph will be denoted by $LCG(n, k)$.

Keywords: resolving set; doubly resolving; strong resolving.

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1. Introduction

Throughout this article we will considering connected simple graphs. The structure of a graph in graph theory is often considered as a set of vertices and edges. From a chemical graph theory perspective, the molecular graph is a graph consists of atoms called vertices and the chemical bond between atoms called edges. Especially, if we consider a graph as a chemical compound, then by changing the set of atoms and permuting their positions, a collection of compounds is essentially defined that are characterized by the substructure common to them (Chartrand et al., 2000). In a chemical compound, it is very important for chemists to find the size of the smallest set of atoms so that they can identify other atoms relative to that the smallest set of atoms, so chemists require mathematical forms for a set of chemical compound to give distinct representations to distinct compound structures (Baig et al., 2017), and its corresponding in graph theory is to find the minimal resolving set as follows:

Suppose $R = \{r_1, r_2, ..., r_m\}$ is an order subset of vertices belonging to a graph $G$. For each vertex $u$ of $G$, we shall use the notation $r(u|R)$ to denote the representation of $u$ corresponding to $R$ in graph $G$, that is the $m$-tuple $(d(u, r_1), ..., d(u, r_m))$, where $d(u, r_i)$ is the length of geodesic between $u$ and $r_i$, $1 \leq i \leq m$. If the representation of distinct vertices in $V(G) - R$ is distinct, then the order subset $R$ is called a resolving set of graph $G$ [1]. Therefore, it is important to find the smallest resolving set of graph $G$. The cardinality of the smallest resolving set in graph $G$ is called the metric dimension of $G$, and is denoted by $\beta(G)$. The metric dimension and its related parameters has been studied by many researchers over the years, because their remarkable applications in graph theory and other sciences is important. For more specialized topics see [2, 3, 4, 5, 6]. Indeed, the concept and notation of the metric dimension problem, was first introduced by Slater [7] under the term locating set, and the idea of metric dimension of a graph was individually introduced by Harary and Melter in [8].

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One of the more specialized topics related to the metric dimension is a doubly resolving set of graph. Cáceres et al. [9] define the notion of a doubly resolving set. Also, we can verify that an ordered subset $Z = \{z_1, z_2, \ldots, z_6\}$ of vertices of a graph $G$ is called a doubly resolving set for $G$, if every two distinct vertices $u$ and $v$ of $G$ are doubly resolved by some two vertices in the set $Z$, that is, for any two vertices $u, v \in V(G)$ we have $r(u|Z) - r(v|Z) \neq \mu$, where $\mu$ is an integer, and $I$ denotes the unit $n$- vector $(1, \ldots, 1)$. The cardinality of minimum doubly resolving set in graph $G$ is denoted by $\psi(G)$. For more information on the doubly resolving set of graphs see [10, 11, 12, 13].

A vertex $u$ of a graph $G$ is called maximally distant from a vertex $v$ of $G$, if for every $w \in N_G(u)$, we have $d(v, w) \leq d(v, u)$, where $N_G(u)$ denotes the set of neighbors that $u$ has in $G$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then $u$ and $v$ are said to be mutually maximally distant [14]. For vertices $u$ and $v$ of a graph $G$, we use the interval $I_{G}[u, v]$ to denote as the collection of all vertices that belong to a shortest path between $u$ and $v$. A vertex $w$ strongly resolves two vertices $u$ and $v$ if $v$ belongs to $I_{G}[u, w]$ or $u$ belongs to $I_{G}[v, w]$. A set $R = \{r_1, r_2, \ldots, r_k\}$ of vertices of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $R$. The strong metric dimension of a graph $G$ is the cardinality of smallest strong resolving set of $G$ and denoted by $sdim(G)$ see [15, 16, 17].

In this article, we focus on some resolving parameters of two graphs the crystal cubic carbon $CCC(n)$ and the layer cycle graph will be denoted by $LCG(n, k)$. The crystal cubic carbon $CCC(n)$, is defined by (Baig et al., 2017), see [18]. Also, some of the chemical parameters of the crystal cubic carbon $CCC(n)$ have been calculated by other researchers, further details can be given in [19, 20]. In particular the cardinality of minimum resolving set of $CCC(n)$ is computed by (Zhang and Naeem, 2021), see [21]. So, if we label the atoms of a chemical graph, then some of its parameters, including resolving sets, are easier to calculate, because in this case we have a more accurate knowledge of the chemical graph. Hence, we first define the structure of the crystal cubic carbon $CCC(n)$ by a new method, and we provide a much simpler formula for obtaining its vertices than the formula for calculating its vertices in (Baig et al., 2017). For more details of our definition of the crystal cubic carbon $CCC(n)$, see section 3.1. In particular, in section 3.1, we will find the minimal doubly resolving set and the strong metric dimension for the crystal cubic carbon $CCC(n)$. Moreover in section 3.2, we will construct a new class of graphs will be denoted by $LCG(n, k)$ and recall the layer cycle graph with parameters $n$ and $k$. For more details on the definition and structure of the layer cycle graph $LCG(n, k)$ see section 3.2. Also, we will compute some resolving parameters for the layer cycle graph $LCG(n, k)$.

2. Preliminaries

**Definition 2.1.** Consider two graphs $G$ and $H$. If there is a bijection, $\varphi$, say, from $V(G)$ to $V(H)$ so that $u$ is adjacent to $v$ in $G$ if and only if $\varphi(u)$ is adjacent to $\varphi(v)$ in $H$, then we say that $G$ is isomorphic to $H$.

**Theorem 2.1.** [21] Consider the crystal cubic carbon $CCC(n)$. If $n \geq 2$ is an integer, then the cardinality of minimum resolving set of $CCC(n)$ is $16 \times 7^{n-2}$.

**Remark 2.1.** For every pair of mutually maximally distant vertices $u$ and $v$ of a graph $G$ and for every strong resolving set $R$ of $G$, it follows that $u \in R$ or $v \in R$.

**Remark 2.2.** Consider the cycle graph $C_n$. If $n$ is an even integer so that $n \geq 6$, $\psi(G) = 3$ and $sdim(G) = \lceil \frac{n}{2} \rceil$.

**Remark 2.3.** Consider the cycle graph $C_n$. If $n$ is an odd integer so that $n \geq 3$, then $\psi(G) = 2$ and $sdim(G) = \lceil \frac{n}{2} \rceil$.

3. Main results

3.1. Minimal doubly resolving, and strong metric dimension for the Crystal Cubic Carbon $CCC(n)$

Let $n$ be fixed positive integer and $k$ an integer so that $2 \leq k \leq n$. In this section, first we will construct the crystal cubic carbon $CCC(n)$ by a new method. For this purpose, we introduce some notation which is used throughout this section and is related to the crystal cubic carbon $CCC(n)$ as follows. Consider the cartesian product $C_4 \Box P_2$ with vertex set $[1, 2, \ldots, 8]$ on the layers $W_1 = [1, \ldots, 4]$ and $W_2 = [5, \ldots, 8]$ and suppose the edge set $C_4 \Box P_2$ is $E(C_4 \Box P_2) = \{ij | i, j \in W_1, i < j, j-i = 1 \text{or } j-i = 3\} \cup \{ij | i, j \in W_2, i < j, j-i = 1 \text{or } j-i = 3\} \cup \{ij | i \in W_1, j \in W_2, j-i = 4\}$. 

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where $C_4$ and $P_2$ denote the cycle on 4 and the path on 2 vertices, respectively. Also, we shall use the notation $Q_{r}^{(k)}$ to denote a cubic graph of order 8, with vertex set

$$V(Q_{r}^{(k)}) = \{(x_r, 1)^{(k)}, \ldots, (x_r, 8)^{(k)}\},$$

and edge set

$$E(Q_{r}^{(k)}) = \{(x_r, i)^{(k)}(x_r, j)^{(k)} | i, j \in W_1, i < j, j - i = 1 \text{ or } j - i = 3 \} \cup \{(x_r, i)^{(k)}(x_r, j)^{(k)} | i, j \in W_2, i < j, j - i = 1 \text{ or } j - i = 4 \} \cup \{(x_r, i)^{(k)}(x_r, j)^{(k)} | i \in W_1, j \in W_2, j - i = 4 \},$$

for $1 \leq r \leq 8$, and $1 \leq s \leq 7^{k-2}$. We can see that this graph is isomorphic to the cartesian product $C_4 \square P_2$.

Now, suppose $H$ is a graph of order $8 + 642\Sigma_{k=2}^{n}7^{k-2}$ with vertex set

$$V(H) = L_1 \cup L_2 \cup \ldots \cup L_n,$$

where $L_1, L_2, \ldots, L_n$ are called the layers of $H$ such that $L_1 = V(C_4 \square P_2) = \{1, 2, \ldots, 8\}$, and for $k \geq 2$ we have

$$L_k = \{Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_{1 \text{ to } 2}^{(k)} \ldots \ldots Q_{6 \text{ to } 2}^{(k)} \ldots Q_{8 \text{ to } 2}^{(k)}\}.$$

Also, we shall use $Q_{r}^{(k)}$ to denote a cubic of the layer $L_k$, and every $(x_r, 1)^{(k)} \in Q_{r}^{(k)}$ is called head vertex of $Q_{r}^{(k)}$ in the layer $L_k$. Now, let the adjacency relation of graph $H$ given as follows. Suppose that $r$ is an arbitrary vertex in the layer $L_1$, $1 \leq r \leq 8$, and $r$ is adjacent to the head vertex of cubic $Q_{r}^{(k)}$ in the layer $L_2$ by an edge. Also every vertex in cubic $Q_{r}^{(k)} \in L_k (k \geq 2)$, except head vertex $(x_r, 1)^{(k)}$, is adjacent to exactly the head vertex of one cubic in the layer $L_{k-1}$ say $Q_{r}^{(k-1)}$ by an edge, then we can see that the resulting graph is isomorphic to the crystal cubic carbon $CCC(n)$. In particular, we say that two cubes are congruous, if both of them lie in the same layer. It is natural to consider its vertex set of crystal cubic carbon $CCC(n)$ which is defined already. If $n \geq 2$ is an integer, then the cardinality of minimum doubly resolving set of $CCC(n)$ is $24 \times 7^{(n-2)}$.

**Theorem 3.1.** Consider the crystal cubic carbon $CCC(n)$ which is defined already. If $n \geq 2$ is an integer, then the cardinality of minimum doubly resolving set of $CCC(n)$ is $24 \times 7^{(n-2)}$.

**Proof.** Let $V(CCC(n)) = L_1 \cup L_2 \cup \ldots \cup L_n$, be the vertex set of graph $CCC(n)$, where $L_1, L_2, \ldots, L_n$ are the layers of $CCC(n)$, which is defined already. By Theorem 2.1, we know that the cardinality of minimum resolving set of $CCC(n)$ is $16 \times 7^{(n-2)}$. Now, let

$$Z_1 = \{(x_{11}, 2)^{(n)}, \ldots, (x_{11}, 2)^{(n)}; \ldots; (x_{81}, 2)^{(n)}, \ldots, (x_{81}, 2)^{(n)}\}.$$
Thus, the cardinality of minimum doubly resolving set of $Z$ consisting of exactly one adjacent vertex in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$, then the arranged set $Z_3 = Z_1 \cup Z_2$ consisting of exactly two adjacent vertices in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$ is one of minimal resolving sets in $CCC(n)$. Also it is not hard to see that, for every two vertices $u$ and $v$ so that lie in the layers $L_1 \cup L_2 \cup \ldots \cup L_{n-1}$, we have $r(u|Z_1) - r(v|Z_1) \neq \mu I$, where $\mu$ is an integer and $I$ denotes the unit $16 \times 7^{(n-2)}$-vector. In particular, we can show that the set $Z_3$, cannot be doubly resolved all the vertices of each cubic of the layer $L_n$. For this purpose, we consider the cubic $Q_{11}^{(n)}$ in the layer $L_n$ and suppose $x$ is an arbitrarily element of the set $Z_3$ so that $(x_1, 2)^{(n)} \neq x$, $(x_1, 4)^{(n)} \neq x$ and the distance between the head vertex $(x_1, 1)^{(n)}$ and $x$ is a positive integer $c$, that is $r((x_1, 1)^{n}|x) = c$. Now, let $Z = \{(x_1, 2)^{(n)}, (x_1, 4)^{(n)}, x\}$ be a subset of the set $Z_3$, we can view that all the vertices in the cubic $Q_{11}^{(n)}$ cannot be doubly resolved with respect to $Z$. Because, for every $1 \leq i \leq 8$, we have
\[
\begin{align*}
&\ r((x_1, 1)^{n}|Z) = (1, 1, c) \\
&\ r((x_1, 3)^{n}|Z) = (1, 1, c + 2) \\
&\ r((x_1, 5)^{n}|Z) = (2, 2, c + 1) \\
&\ r((x_1, 6)^{n}|Z) = (1, 3, c + 2) \\
&\ r((x_1, 7)^{n}|Z) = (2, 2, c + 3) \\
&\ r((x_1, 8)^{n}|Z) = (3, 1, c + 2),
\end{align*}
\]
and hence $Z_3$, cannot be doubly resolved all the vertices of each cubic of the layer $L_n$, because $x$ is an arbitrarily element of the set $Z_3$. Besides, we can view that every minimal resolving set of $CCC(n)$, consisting of exactly two adjacent vertices in each cubic of the layer $L_n$ with respect to head vertex of each cubic of the layer $L_n$, and hence $\psi(CCC(n))$ must be greater than $16 \times 7^{(n-2)}$. By the discussion above, we deduce that if
\[
Z_4 = \{(x_1, 5)^{(n)}, \ldots, (x_{16-2}, 5)^{(n)}, \ldots, (x_8, 5)^{(n)}, \ldots, (x_{8-2}, 5)^{(n)}\},
\]
consisting of exactly one adjacent vertex in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$, then the arranged set $Z_4 = Z_1 \cup Z_4$, consisting of exactly three adjacent vertices in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$ is one of minimal doubly resolving sets in $CCC(n)$. Thus, the cardinality of minimum doubly resolving set of $CCC(n)$ is $3 \times 8 \times 7^{(n-2)}$.

**Theorem 3.2.** Consider the crystal cubic carbon $CCC(n)$ which is defined already. If $n \geq 2$ is an integer, then the cardinality of minimum strong resolving set of $CCC(n)$ is $32 \times 7^{(n-2)} - 1$.

**Proof.** Let $V(\text{CCC}(n)) = L_1 \cup L_2 \cup \ldots \cup L_n$, be the vertex set of graph $CCC(n)$, where $L_1, L_2, \ldots, L_n$ are the layers of $CCC(n)$, which is defined already. By Theorem 2.1, we know that the cardinality of minimum resolving set of $CCC(n)$ is $16 \times 7^{(n-2)}$. Besides, the arranged set $Z_3 = Z_1 \cup Z_2$, which is defined in previous Theorem, consisting of exactly two adjacent vertices in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$ is one of minimal resolving sets in $CCC(n)$. Also, every two vertices $u$ and $v$ so that lie in the layers $L_1 \cup L_2 \cup \ldots \cup L_{n-1}$, are strongly resolved by an element of $Z_3$. With out loss of generality, if we consider the cubic $Q_{11}^{(n)}$ in the layer $L_n$, then every two vertices of the cubic $Q_{11}^{(n)}$ except two vertices $(x_1, 3)^{(n)}$ and $(x_1, 5)^{(n)}$ are strongly resolved by an element of $Z_3$, and hence if we consider the arranged set $Z_4 = Z_1 \cup Z_4$, which is defined in previous Theorem, consisting of exactly three adjacent vertices in each cubic of the layer $L_n$ with respect to the head vertex of each cubic of the layer $L_n$, then every two vertices so that lie in the one cubic of the layer $L_n$ are strongly resolved by an element of the set $Z_4$, and the number of such vertices is $24 \times 7^{(n-2)}$. Note that, both vertices of $CCC(n)$ so that lie in distinct cubes in the layer $L_n$ and mutually maximally distant, cannot be strongly resolved by an element of $Z_3$ and hence, from both vertices of distinct cubes so that mutually maximally distant, at least one of them must be belongs to the every minimum resolving set of $CCC(n)$. Therefore, in each cube of the layer $L_n$, except one of them, there must be a vertex of that cube that has a maximum distance from the head vertex of that cube in every set of minimum strong resolving set of $CCC(n)$, and hence, the number of such vertices is $8 \times 7^{(n-2)} - 1$. Thus, the cardinality of minimum strong resolving set of $CCC(n)$ must be $32 \times 7^{(n-2)} - 1$. 

\[\square\]
3.2. Minimal resolving, doubly resolving, and strong metric dimension for the layer cycle graph $LCG(n, k)$

Let $n$ and $k$ be fixed positive integers so that $n \geq 3$, $k \geq 2$ and $[n] = \{1, 2, \ldots, n\}$, also, let $p$ be an integer so that $2 \leq p \leq k$. In this section, first we construct a class of graphs of order $n + \sum_{i=2}^{k} n^2(n - 1)^{p-2}$, denoted by $LCG(n, k)$ and recall the layer cycle graph with parameters $n$ and $k$. Moreover, we will compute some metrics for this class of graphs. For this purpose, we introduce some notation which is used throughout this section and is related to the layer cycle graph $LCG(n, k)$ as follows. We shall use the notation $C^{(p)}_{i_r}$ to denote a cycle of order $n$, with vertex set

$$V(C^{(p)}_{i_r}) = \{(x_r, 1)^{(p)}, (x_r, 2)^{(p)}, \ldots, (x_r, n)^{(p)}\},$$

and edge set

$$E(C^{(p)}_{i_r}) = \{(x_r, i)^{(p)}(x_r, j)^{(p)} | i, j \in [n], i < j, j - i = 1 \text{ or } j - i = n - 1\},$$

for $1 \leq r \leq n$, and $1 \leq s \leq (n - 1)^{p-2}$. We can verify that $C^{(p)}_{i_r}$ is isomorphic to the cycle $C_n$, where vertex set of the cycle $C_n$ is $V(C_n) = \{1, 2, \ldots, n\}$ and edge set $E(C_n) = \{ij | i, j \in [n], i < j, j - i = 1 \text{ or } j - i = n - 1\}$. Now, suppose $G$ is a graph with vertex set $V(G) = U_1 \cup U_2 \cup \ldots \cup U_k$, where $U_1, U_2, \ldots, U_k$ are called the layers of $G$ such that $U_1 = V(C_n)$, and for $p \geq 2$ we have

$$U_p = \{C^{(p)}_{1,1}, C^{(p)}_{1,2}, \ldots, C^{(p)}_{1,(n-1)^{p-2}}, C^{(p)}_{2,1}, C^{(p)}_{2,2}, \ldots, C^{(p)}_{2,(n-1)^{p-2}}, \ldots, C^{(p)}_{n_1,1}, C^{(p)}_{n_1,2}, \ldots, C^{(p)}_{n_1,(n-1)^{p-2}}\}.$$

Also, we shall use $C^{(p)}_{i_r}$ to denote a cycle of the layer $U_p$, and every $(x_r, 1)^{(p)} \in C^{(p)}_{i_r}$ is called head vertex of $C^{(p)}_{i_r}$ in the layer $U_p$. Now, let the adjacency relation of graph $G$ given as follows. Suppose that $r$ is an arbitrary vertex in the layer $U_1$, $1 \leq r \leq n$, and $r$ is adjacent to the head vertex of $C^{(p)}_{i_r}$ in the layer $U_2$ by an edge. Also every vertex in cycle $C^{(p)}_{i_r} \in U_p$ ($p \geq 2$), except head vertex $(x_r, 1)^{(p)}$, is adjacent to exactly the head vertex of one cycle in the layer $U_{p+1}$ say $C^{(p+1)}_{i_r}$ by an edge, then the resulting graph is called the layer cycle graph $LCG(n, k)$ with parameters $n, k$. In particular, we say that two cycles are congruous, if both of them lie in the same layer. It is natural to consider its vertex set of layer cycle graph $LCG(n, k)$ as partitioned into $k$ layers. The layers $U_1$ and $U_2$ consisting of the vertices $\{1, 2, \ldots, n\}$ and $\{C^{(2)}_{1,1}, C^{(2)}_{1,2}, \ldots, C^{(2)}_{n_1,1}\}$, respectively. In particular, each layer $U_p$ ($p \geq 2$), consisting of the $n^2(n - 1)^{p-2}$ vertices. The layer cycle graph $LCG(5, 3)$ is depicted in Figure 2.
Theorem 3.3. Consider the layer cycle graph $LCG(n, k)$ which is defined already. If $n, k$ are integers so that $n \geq 3$ and $k \geq 2$, then the metric dimension of $LCG(n, k)$ is $n(n-1)^{k-2}$.

Proof. Let $V(LCG(n, k)) = U_1 \cup U_2 \cup \ldots \cup U_k$ be the vertex set of graph $LCG(n, k)$, where $U_1, U_2, \ldots, U_k$ are the layers of $LCG(n, k)$, which is defined already. If we consider an arranged subset $R_1$ of vertices in the layers $U_1 \cup U_2 \cup \ldots \cup U_{k-1}$, then $R_1$ is not a resolving set for $LCG(n, k)$. In particular, we can express that if $R_2$ is an arranged set, consisting of all the head vertices in the layer $U_k$, then the set $R_2$ is not a resolving set for $LCG(n, k)$, because the degree of each head vertex in the layer $U_k$ is 3, and hence there are two vertices in the cycle $C_{1_{1_k}}^{(k)}$ of $LCG(n, k)$ so that they are adjacent to the head vertex $(x_1, 1)^{(k)}$ in the cycle $C_{1_{1_k}}^{(k)}$ with the same representations. Now, let $R_3 = \{r_1, r_2, \ldots, r_z\}$ be a minimal resolving set of $LCG(n, k)$. We claim that there is exactly one vertex of each cycle in the layer $U_k$ belongs to $R_3$. Suppose for a contradiction that none of vertices of each cycle in the layer $U_k$ belong to $R_3$, and hence with out loss of generality if we consider the head vertex $(x_1, 1)^{(k)}$ in the cycle $C_{1_{1_k}}^{(k)}$, then we can view that the metric representation of two vertices in the cycle $C_{1_{1_k}}^{(k)}$ of $LCG(n, k)$ so that they are adjacent to the head vertex $(x_1, 1)^{(k)}$ is identical with respect $R_3$. Therefore, we deduce that at least one vertex of each cycle in the layer $U_k$ must be belongs in every minimal resolving set of $LCG(n, k)$. Besides, the layer $U_k$ of graph $LCG(n, k)$ consisting of exactly $n(n-1)^{k-2}$ cycles, and hence we deduce that the cardinality of minimum resolving set of $LCG(n, k)$ must be equal or greater than
Consider the layer cycle graph $LCG(n, k)$ which is defined already. If $n \geq 4$ is an even integer and $k$ is an integer so that $k \geq 2$, then the cardinality of minimum doubly resolving set of $LCG(n, k)$ is $2n(n-1)^{k-2}$.

Proof. Let $V(LCG(n, k)) = U_1 \cup U_2 \cup \ldots \cup U_k$, be the vertex set of graph $LCG(n, k)$, where $U_1, U_2, \ldots, U_k$ are the layers of $LCG(n, k)$, which is defined already. From the previous Theorem, the arranged set

$$R_4 = \{(x_1, n)^{(k)}, \ldots, (x_{\frac{n}{2}}, \frac{n}{2})^{(k)}, \ldots, (x_n, n)^{(k)}, \ldots, (x_{\frac{n}{2}+1}, \frac{n}{2})^{(k)}\},$$

is an arranged set, consisting of exactly one adjacent vertex in each cycle of the layer $U_k$ with respect to head vertex of each cycle of the layer $U_k$. We claim that the set $R_4$ is a minimal resolving set for $LCG(n, k)$. Since, each vertex in the layer $U_p$, $1 \leq p < k$ is adjacent to exactly one vertex of the layer $U_{p+1}$ say head vertex, it follows that all the vertices in the layer $U_p$, have different representations with respect to $R_4$. Therefore, it is necessary to show that all the vertices in layer $U_k$ have different representations with respect to $R_4$. Since the layer $U_k$ consisting of all the cycles so that these cycles are congruous, and the set $R_4$ consisting of exactly one adjacent vertex in each cycle of the layer $U_k$ with respect to head vertex of each cycle of the layer $U_k$, it then is sufficient to show that all the vertices in an arbitrarily cycle of the layer $U_k$ have different representations with respect to $R_4$. For this purpose, we consider the cycle $C_{1k}^{(k)}$ in the layer $U_k$ and suppose $x$ is an arbitrarily element of the set $R_4$ so that $(x_1, n)^{(k)} \neq x$, and the distance between the head vertex $(x_1, 1)^{(k)} \in C_{1k}^{(k)}$ and $x$ is a positive integer $c$, that is $r((x_1, 1)^{(k)}|x) = c$. Now, let $R = \{(x_1, n)^{(k)}, x\}$ be a subset of the set $R_4$, we can view that all the vertices in the cycle $C_{1k}^{(k)}$ have different representations with respect to $R$. Because, if $n$ is an even integer then for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have $r((x_1, i)^{(k)}|R) = (i, c + i - 1)$, also, if $\lfloor \frac{n}{2} \rfloor < i \leq n$, then we have $r((x_1, i)^{(k)}|R) = (n - i, n + c + 1 - i)$. Note that, if $n$ is an odd integer then there are two vertices in the cycle $C_{1k}^{(k)}$ with maximum distance from the head vertex $(x_1, 1)^{(k)}$ and hence for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have $r((x_1, i)^{(k)}|R) = (i, c + i - 1)$, also, if $\lfloor \frac{n}{2} \rfloor < i \leq n$, then we have $r((x_1, i)^{(k)}|R) = (n - i, n + c + 1 - i)$. Therefore, by the discussion above we deduce that the metric dimension of $LCG(n, k)$ is $n(n-1)^{k-2}$.

**Theorem 3.4.** Consider the layer cycle graph $LCG(n, k)$ which is defined already. If $n \geq 4$ is an even integer and $k$ is an integer so that $k \geq 2$, then the cardinality of minimum doubly resolving set of $LCG(n, k)$ is $2n(n-1)^{k-2}$.

Proof. Let $V(LCG(n, k)) = U_1 \cup U_2 \cup \ldots \cup U_k$, be the vertex set of graph $LCG(n, k)$, where $U_1, U_2, \ldots, U_k$ are the layers of $LCG(n, k)$, which is defined already. From the previous Theorem, the arranged set

$$R_5 = \{(x_1, \frac{n}{2} + 1)^{(k)}, \ldots, (x_{\frac{n}{2} - \frac{1}{2} + 1}, \frac{n}{2} + 1)^{(k)}, \ldots, (x_n, n)^{(k)}, \ldots, (x_{\frac{n}{2} + \frac{1}{2}}, \frac{n}{2} + 1)^{(k)}\},$$

be an arranged set of vertices in $LCG(n, k)$ and $R_6 = R_4 \cup R_5$. We show that the set $R_6$ is one minimal doubly resolving sets for $LCG(n, k)$. For this purpose we consider the cycle $C_{1k}^{(k)}$ in the layer $U_k$ and suppose $x$ is an arbitrarily element of the set $R_6$ so that $(x_1, n)^{(k)} \neq x$, $(x_1, \lfloor \frac{n}{2} \rfloor + 1)^{(k)} \neq x$ and the distance between the head vertex $(x_1, 1)^{(k)} \in C_{1k}^{(k)}$ and $x$ is a positive integer $c$, that is $r((x_1, 1)^{(k)}|x) = c$. Also, let $R = \{(x_1, n)^{(k)}, x, (x_1, \lfloor \frac{n}{2} \rfloor + 1)^{(k)}\}$ be a subset of the set $R_6$, we can view that all the vertices in the cycle $C_{1k}^{(k)}$ can be doubly resolved by the set $R$. Because, for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have $r((x_1, i)^{(k)}|R) = (i, c + i - 1, \lfloor \frac{n}{2} \rfloor + 1 - i)$, also if $\lfloor \frac{n}{2} \rfloor < i \leq n$, then we have $r((x_1, i)^{(k)}|R) = (n - i, n + c + 1 - i, i - \lfloor \frac{n}{2} \rfloor - 1)$, in particular we can see that every two vertices in distinct cycles of $LCG(n, k)$ can be doubly resolved by the set $R_6$, and hence the set $R_6$ is one of minimal doubly resolving sets for $LCG(n, k)$. By the discussion above, the cardinality of minimum doubly resolving set of $LCG(n, k)$ is $2n(n-1)^{k-2}$.

**Theorem 3.5.** Consider the layer cycle graph $LCG(n, k)$ which is defined already. If $n \geq 5$ is an odd integer and $k$ is an integer so that $k \geq 2$, then the cardinality of minimum doubly resolving set of $LCG(n, k)$ is $2n(n-1)^{k-2}$.
Proof. In a similar way to the previous theorem and a few changes in the proof, we can show that the cardinality of minimum doubly resolving set of $LCG(n,k)$ is $2n(n-1)^{k-2}$. $\square$

Theorem 3.6. Consider the layer cycle graph $LCG(n,k)$ which is defined already. If $n \geq 3$ is an even or odd integer and $k$ is an integer so that $k \geq 2$, then the cardinality of minimum strong resolving set of $LCG(n,k)$ is $\lceil \frac{n}{2} \rceil n(n-1)^{k-2} - 1$.

Proof. Let $V(LCG(n,k)) = U_1 \cup U_2 \cup \ldots \cup U_k$, be the vertex set of graph $LCG(n,k)$, where $U_1, U_2, \ldots, U_k$ are the layers of $LCG(n,k)$, which is defined already. We can view that, the arranged set

$$R_7 = \{(x_1, 2), \ldots, (x_{n\bmod{k}-2}, 2), \ldots, (x_n, 2), \ldots, (x_{n\bmod{k}-2}, 2), \ldots, (x_1, 2)\},$$

consisting of exactly one adjacent vertex in each cycle of the layer $U_k$ with respect to head vertex of each cycle of the layer $U_k$ is one of minimal resolving sets in $LCG(n,k)$. Also every two vertices $u$ and $v$ in the layers $U_1 \cup U_2 \cup \ldots \cup U_{k-1}$, are strongly resolved by an element of $R_7$. In particular, if we consider a cycle and its head vertex in the layer $U_k$, then each vertex in that cycle has the maximum distance from the head vertex is strongly resolved by an element of $R_7$. Note that, the set $R_7$, cannot be strongly resolved other vertices of each cycle of the layer $U_k$, and hence if we consider the arranged set

$$R_8 = \{(x_1, 2), \ldots, (x_{n\bmod{k}-2}, n), \ldots, (x_n, 2), \ldots, (x_{n\bmod{k}-2}, n), \ldots, (x_1, 2)\},$$

consisting of exactly $\lceil \frac{n}{2} \rceil - 1$ elements in each cycle of the layer $U_k$, then we can see that all the vertices in each cycle of the layer $U_k$ are strongly resolved by an element of $R_8$, and the number of such vertices is $n(n-1)^{k-2}((\frac{n}{2}) - 1)$. Note that, both vertices of $LCG(n,k)$ so that lie in distinct cycles in the layer $U_k$ and mutually maximally distant, cannot be strongly resolved by an element of $R_8$, and hence, from both vertices of distinct cycles in the layer $U_k$ so that mutually maximally distant, at least one of them must be belongs to the every minimum resolving set of $LCG(n,k)$. Therefore, in each cycle of the layer $U_k$, except one of them, there must be a vertex of that cycle that has a maximum distance from the head vertex of that cycle in every set of minimum strong resolving set of $LCG(n,k)$, and hence, the number of such vertices is $n(n-1)^{k-2} - 1$. Thus, the cardinality of minimum strong resolving set of $LCG(n,k)$ must be $\lceil \frac{n}{2} \rceil n(n-1)^{k-2} - 1$. $\square$

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