IMPROVED $A_1 - A_\infty$ AND RELATED ESTIMATES FOR COMMUTATORS OF ROUGH SINGULAR INTEGRALS

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Abstract An $A_1 - A_\infty$ estimate, improving on a previous result for $[b, T_\Omega]$ with $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$ and $b \in \text{BMO}$ is obtained. A new result in terms of the $A_\infty$ constant and the one supremum $A_q - A_\infty^\exp$ constant is also proved, providing a counterpart for commutators of the result obtained by Li. Both of the preceding results rely upon a sparse domination result in terms of bilinear forms, which is established using techniques from Lerner.

Keywords: rough singular integrals; commutators; sparse bounds; weights

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1. Introduction

We recall that a weight $w$, namely a non-negative locally integrable function, belongs to $A_p$ if

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{1/(1-p)} \right)^{p-1} < \infty, \quad 1 < p < \infty$$

or in the case $p = 1$ if

$$[w]_{A_1} = \text{ess sup}_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty.$$

Given $\Omega \in L(\mathbb{S}^{n-1})$ with $\int_{\mathbb{S}^{n-1}} \Omega = 0$, we define the rough singular integral $T_\Omega$ by

$$T_\Omega f(x) = \pv \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y'|^n} f(x - y) \, dy$$

where $y' = y/|y|$.

In recent years, there has been increasing interest in the study of the sharp dependence on the $A_p$ constants of rough singular integrals. In particular, it was established in [10]...
that
\[ \|T_\Omega\|_{L^2(w)} \leq c_n \|\Omega\|_{L^\infty(S^{n-1})}[w]_{A_2}^2. \]

Recently, the following sparse domination (very recently reproved in [13] for the case \( \Omega \in L^\infty(S^{n-1}) \)) was established in [3].

**Theorem.** For all \( 1 < p < \infty \), \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^{p'}(\mathbb{R}^n) \), we have that
\[
\left| \int_{\mathbb{R}^n} T_\Omega(f)g \, dx \right| \leq c_n C_T s' \sup_S \left( \int_Q |f| \left( \frac{1}{|Q|} \int_Q |g|^s \right)^{1/s} \right),
\]
where each \( S \) is a sparse family of a dyadic lattice \( \mathcal{D} \),
\[
\begin{cases}
1 < s < \infty & \text{if } \Omega \in L^\infty(S^{n-1}) \\
q' \leq s < \infty & \text{if } \Omega \in L^{q,1} \log L(S^{n-1})
\end{cases}
\]
and
\[ C_T = \begin{cases}
\|\Omega\|_{L^\infty(S^{n-1})} & \text{if } \Omega \in L^\infty(S^{n-1}) \\
\|\Omega\|_{L^{q,1} \log L(S^{n-1})} & \text{if } \Omega \in L^{q,1} \log L(S^{n-1}).
\end{cases} \]

The preceding sparse domination was widely exploited in [20]. Among other estimates, the following \( A_1 - A_\infty \) estimate was established in that paper (see Lemma 2.2 in §2 for the definition of the \( A_\infty \) constant):
\[
\|T_\Omega\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(S^{n-1})}[w]_{A_1}^{1/p}[w]_{A_\infty}^{1/p'}.
\]
The preceding inequality is an improvement of the following estimate established earlier in [22]:
\[
\|T_\Omega\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(S^{n-1})}[w]_{A_1}^{1/p}[w]_{A_\infty}^{1+(1/p')}. \]

Now we recall that the commutator of an operator \( T \) and a symbol \( b \) is defined as
\[ [b, T]f(x) = T(bf)(x) - b(x)Tf(x). \]
In the case of \( T \) being a Calderón–Zygmund operator, this operator was introduced by Coifman et al. in [2]. They established that \( b \in \text{BMO} \) is a sufficient condition for \([b, T]\) to be bounded on \( L^p \) for every \( 1 < p < \infty \), and also gave a converse result in terms of the Riesz transforms, namely that the boundedness of \([b, R_j]\) on \( L^p \) for some \( 1 < p < \infty \) and for every Riesz transform implies that \( b \in \text{BMO} \).

In [22] the following estimate for commutators of rough singular integrals and a symbol \( b \in \text{BMO} \) was obtained:
\[
\|[b, T_\Omega]\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(S^{n-1})}[w]_{A_1}^{1/p}[w]_{A_\infty}^{2+(1/p')}.
\]
One of the main goals of this paper is to improve the dependence on the \([w]_{A_\infty}\) constant in (1.2). Our result is the following.
Theorem 1.1. Let \( T_\Omega \) be a rough homogeneous singular integral with \( \Omega \in L^\infty(S^{n-1}) \) and let \( b \in \text{BMO} \). For every weight \( w \) we have that
\[
\| [b, T_\Omega] \|_{L^p(M_r(w)) \to L^p(w)} \leq c_n \| \Omega \|_{L^\infty(S^{n-1})} \| b \|_{\text{BMO}(p')} 3^2 (r')^{1+(1/p')}
\] (1.3)
where \( r > 1 \). Assuming additionally that \( w \in A_\infty \),
\[
\| [b, T_\Omega] \|_{L^p(M(w)) \to L^p(w)} \leq c_n \| \Omega \|_{L^\infty(S^{n-1})} \| b \|_{\text{BMO}(p')} 3^2 \| w \|_{A_\infty}^{1+(1/p')}
\]
and, furthermore, if \( w \in A_1 \), then
\[
\| [b, T_\Omega] \|_{L^p(w)} \leq c_n \| \Omega \|_{L^\infty(S^{n-1})} \| b \|_{\text{BMO}(p')} 3^2 \| w \|_{A_1}^{1+(1/p')} \| w \|_{A_\infty}^{1+(1/p')}
\]

Very recently, a conjecture left open by Moen and Lerner in [14] was solved by Li in [19]. Actually, he obtained a more general result.

Theorem. Let \( T \) be a Calderón–Zygmund operator or a rough singular integral with \( \Omega \in L^\infty(S^{n-1}) \). Then for every \( 1 < q < p < \infty \)
\[
\| T \|_{L^p(w)} \leq c_{n,p,q} c_T [w]_{A_q^{1/p}(A_\infty^{1/p'})}
\]
where
\[
[w]_{A_q^{1/p}(A_\infty^{1/p'})} = \sup_Q (w_Q) \langle w^{1/(1-q)} \rangle_Q^{(q-1)/p} \exp(\langle \log w^{-1} \rangle_Q) 1/p'
\]
and
\[
c_T = \begin{cases} \| \Omega \|_{L^\infty(S^{n-1})} & \text{if } T = T_\Omega \text{ with } \Omega \in L^\infty(S^{n-1}), \\
c_K + \| T \|_{L^2} + \| \omega \|_{\text{Dini}} & \text{if } T \text{ is an } \omega\text{-Calderón–Zygmund operator.} \end{cases}
\]

This result can be regarded as an improvement on the linear dependence on the \( A_q \) constant established in [20], which, as was stated there, follows from the linear dependence on the \( A_1 \) constant by [5, Corollary 4.3]. Such an improvement stems from the fact that
\[
[w]_{A_q^{1/p}(A_\infty^{1/p'})} \leq c_n [w]_{A_q},
\]
In the next theorem, we provide a counterpart of the preceding result for commutators.

Theorem 1.2. Let \( T \) be a Calderón–Zygmund operator or a rough singular integral with \( \Omega \in L^\infty(S^{n-1}) \). Then for every \( 1 < q < p < \infty \)
\[
\| [b, T] \|_{L^p(w)} \leq c_{n,p,q} c_T [w]_{A_\infty} [w]_{A_q^{1/p}(A_\infty^{1/p'})}.
\] (1.4)

Recall the following known estimates:
\[
\| [b, T] \|_{L^p(w)} \leq c [w]_{A_q}^2,
\]
\[
\| [b, T_\Omega] \|_{L^p(w)} \leq c [w]_{A_q}^3.
\]
The first of these can be derived as a consequence of the quadratic dependence on the \( A_1 \) constant of \([b, T]\) obtained in [24] combined with [5, Corollary 4.3], while the second
was established in [22]. In both cases, we improve the dependence on the $A_q$ constant, since we are able to prove a mixed $A_\infty - A_1^{1/p} (A_\infty^{\exp})^{1/p'}$ bound and

$$\max \{ [w]_{A_\infty}, [w]_{A_1^{1/p} (A_\infty^{\exp})^{1/p'}} \} \leq c_n [w]_{A_q}.$$  

In order to establish Theorems 1.2 and 1.1, we will rely upon a suitable sparse domination result for $[b, T]\Omega$. This result will be a natural bilinear counterpart of the result obtained in [18] for $[b, T]$ with $T$ a Calderón–Zygmund operator and also of (1.1). The precise statement is the following.

**Theorem 1.3.** Let $T\Omega$ be a rough homogeneous singular integral with $\Omega \in L^\infty(S^{n-1})$. Then, for every compactly supported $f, g \in C_\infty(\mathbb{R}^n)$, every $b \in \text{BMO}$ and $1 < p < \infty$, there exist $3^n$ dyadic lattices $\mathcal{D}_j$ and $3^n$ sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that

$$|\langle [b, T\Omega] f, g \rangle| \leq C_n p' \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty (T_{\mathcal{S}_j, 1, p}(b, f, g) + T_{\mathcal{S}_j, 1, p}^*(b, f, g)) \ (1.5)$$

where

$$T_{\mathcal{S}_j, r, s}(b, f, g) = \sum_{Q \in \mathcal{S}_j} \langle f \rangle_{r, Q} \langle (b - b_Q)g \rangle_{s, Q} |Q|,$$

$$T_{\mathcal{S}_j, r, s}^*(b, f, g) = \sum_{Q \in \mathcal{S}_j} \langle (b - b_Q)f \rangle_{r, Q} \langle g \rangle_{s, Q} |Q|.$$

**Remark 1.4.** In the preceding theorem and throughout the rest of this work, $\langle h \rangle_{\alpha, Q}^w = (1/w(Q)) \int_Q |h|^\alpha w \, dx)^{1/\alpha}$. We may drop $\alpha$ in the case $\alpha = 1$ and $w$ when we consider the Lebesgue measure.

The rest of the paper is organized as follows. We devote §2 to gathering some results and definitions that will be needed to prove the main theorems. §3 is devoted to the proof of Theorem 1.3. In §4 we prove Theorem 1.1. We end this work by providing a proof of Theorem 1.2 in §5.

2. Preliminaries

In this section, we gather some definitions and results that will be necessary for the proofs of the main theorems.

We start by borrowing some definitions and a basic lemma from [15]. Given a cube $Q_0 \subset \mathbb{R}^n$, we denote by $\mathcal{D}(Q_0)$ the family of all dyadic cubes with respect to $Q_0$, namely, the cubes obtained by repeatedly subdividing $Q_0$ and each of its descendants into $2^n$ subcubes of the same side length.

We say that $\mathcal{D}$ is a dyadic lattice if it is a collection of cubes of $\mathbb{R}^n$ such that:

1. if $Q \in \mathcal{D}$, then $\mathcal{D}(Q_0) \subset \mathcal{D}$;

2. for every pair of cubes $Q', Q'' \in \mathcal{D}$ there exists a common ancestor, namely, we can find $Q \in \mathcal{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$;

3. for every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathcal{D}$ such that $K \subset Q$. 


Lemma 2.1 (3\(^n\) dyadic lattices lemma). Given a dyadic lattice \(\mathcal{D}\), there exist 3\(^n\) dyadic lattices \(\mathcal{D}_1, \ldots, \mathcal{D}_{3^n}\) such that

\[
\{3Q : Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j,
\]

and for each cube \(Q \in \mathcal{D}\) and \(j = 1, \ldots, 3^n\), there exists a unique cube \(R \in \mathcal{D}_j\) with side length \(l(R) = 3l(Q)\) containing \(Q\).

Now we gather some results that will be needed to prove Theorem 1.1. The first of these is the so-called reverse Hölder inequality that was proved in [8] (see also [9]).

Lemma 2.2. For every \(w \in A_\infty\), namely, for every weight such that

\[
[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty,
\]

the following estimate holds:

\[
\left( \frac{1}{|Q|} \int_Q w^{r_w} \right)^{1/r_w} \leq 2 \left( \frac{1}{|Q|} \int_Q w \right)
\]

where \(r_w = 1 + (1/(\tau_n[w]_{A_\infty}))\) and \(\tau_n > 0\) is a constant independent of \(w\).

At this point, recall that if \(w \in A_p \subseteq A_\infty\), then \([w]_{A_\infty} \leq c_n[w]_{A_p}\). This fact makes mixed \(A_\infty - A_p\) bounds interesting, since they provide a sharper dependence than \(A_p\) bounds. We also need to borrow the following lemma from [22].

Lemma 2.3. Let \(w \in A_\infty\). Let \(\mathcal{D}\) be a dyadic lattice and \(\mathcal{S} \subset \mathcal{D}\) be an \(\eta\)-sparse family. Let \(\Psi\) be a Young function. Given a measurable function \(f\) on \(\mathbb{R}^n\), define

\[
\mathcal{B}_S f(x) := \sum_{Q \in \mathcal{S}} \|f\|_{\Psi(L),Q} \chi_Q(x).
\]

Then we have

\[
\|\mathcal{B}_S f\|_{L^1(w)} \leq \frac{4}{\eta} [w]_{A_\infty} \|M_{\Psi(L)} f\|_{L^1(w)}.
\]

We recall that \(\Psi : [0, \infty) \to [0, \infty)\) is a Young function if it is a convex, increasing function such that \(\Psi(0) = 0\). We define the local Orlicz norm associated with a Young function \(\Psi\) as

\[
\|f\|_{\Psi(L)(\mu), E} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(E)} \int_E \Psi \left( \frac{|f|}{\lambda} \right) d\mu \leq 1 \right\}
\]

where \(E\) is a set of finite measure. We note that in the case \(\Psi(t) = t^r\) we recover the standard \(L^r\) local norm. We shall drop \(\mu\) from the notation in the case of the Lebesgue measure and write \(w\) instead of \(w \, dx\) for measures that are absolutely continuous with respect to the Lebesgue measure.
Using the preceding definition of a local norm, we can define the maximal function associated with a Young function $\Psi$ in the natural way,

$$M_{\Psi(L)}f(x) = \sup_{x \in Q} \|f\|_{\Psi(L(\mu), Q)}.$$  

We end this section recalling two basic estimates that work for doubling measures. The first is a particular case of the generalized Hölder inequality, and the second can be derived, for example, from [1, Lemma 4.1].

$$\frac{1}{\mu(Q)} \int_Q |f - f_Q| |g| \, d\mu \leq \|f - f_Q\|_{\exp L(\mu), Q} \|g\|_{L \log L(\mu), Q} \tag{2.1}$$

$$\|f\|_{L \log L(\mu), Q} \leq c_n r' \left( \frac{1}{\mu(Q)} \int_Q w^r \, d\mu \right)^{1/r}, \quad r > 1. \tag{2.2}$$

For a detailed account of local Orlicz norms and maximal functions associated with Young functions, we encourage the reader to consult references such as [21, 23, 25] or [4].

3. Proof of Theorem 1.3

The proof of Theorem 1.3 relies upon techniques recently developed by Lerner in [13]. Given an operator $T$ we define the bilinear operator $\mathcal{M}_T$ by

$$\mathcal{M}_T(f, g)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |T(f \chi_{\mathbb{R}^n \setminus 3Q})| |g| \, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$. Our first result provides a sparse domination principle based on that bilinear operator.

**Theorem 3.1.** Let $1 \leq q \leq r$ and $s \geq 1$. Assume that $T$ is a sublinear operator of weak type $(q, q)$, and $\mathcal{M}_T$ maps $L^r \times L^s$ into $L^{r', \infty}$, where $1/\nu = 1/r + 1/s$. Then, for every compactly supported $f, g \in C^\infty(\mathbb{R}^n)$ and every $b \in \text{BMO}$, there exist $3^n$ dyadic lattices $D_j$ and $3^n$ sparse families $S_j \subset D_j$ such that

$$|\langle [b, T]f, g \rangle| \leq K \sum_{j=1}^\infty (T_{S_j, r, s}(b, f, g) + T_{S_j, r, s}^*(b, f, g)) \tag{3.1}$$

where

$$T_{S_j, r, s}(b, f, g) = \sum_{Q \in S_j} \langle f \rangle_{r, Q} \langle (b - b_Q)g \rangle_{s, Q} |Q|,$$

$$T_{S_j, r, s}^*(b, f, g) = \sum_{Q \in S_j} \langle (b - b_Q)f \rangle_{r, Q} \langle g \rangle_{s, Q} |Q|$$

and

$$K = C_n (\|T\|_{L^q \rightarrow L^{q, \infty}} + \|\mathcal{M}_T\|_{L^r \times L^s \rightarrow L^{r', \infty}}).$$
It is possible to relax the condition imposed on $b$ for this result and the subsequent ones, but we restrict ourselves to this choice for the sake of clarity.

**Proof of Theorem 3.1.** By Lemma 2.1, there exist $3^n$ dyadic lattices $D_j$ such that for every $Q \subset \mathbb{R}^n$, there is a cube $R = R_Q \in D_j$ for some $j$, for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$. Let us fix a cube $Q_0 \subset \mathbb{R}^n$. Now we can define a local analogue of $\mathcal{M}_T$ by

$$\mathcal{M}_{T, Q_0}(f, g)(x) = \sup_{Q \ni x, Q \subset Q_0} \frac{1}{|Q|} \int_Q |T(f \chi_{3Q_0 \setminus 3Q})||g| \, dy.$$  

We define the sets $E_i \ i = 1, \ldots, 4$ as follows:

- $E_1 = \{x \in Q_0 : |T(f \chi_{3Q_0})(x)| > A_1(f)_{q,3Q_0}\}$,
- $E_2 = \{x \in Q_0 : \mathcal{M}_{T, Q_0}(f, g(b - b_{R_{Q_0}}))(x) > A_2(f)_{r,3Q_0}g(b - b_{R_{Q_0}})_{s,Q_0}\}$,
- $E_3 = \{x \in Q_0 : |T(f \chi_{3Q_0}(b - b_{R_{Q_0}}))(x)| > A_3(f(b - b_{R_{Q_0}})q,3Q_0)\}$,
- $E_4 = \{x \in Q_0 : \mathcal{M}_{T, Q_0}(f(b - b_{R_{Q_0}}), g)(x) > A_4((b - b_{R_{Q_0}})f)_{r,3Q_0}(g)_{s,Q_0}\}$.

We can choose $A_i$ in such a way that

$$\max(|E_1|, |E_2|, |E_3|, |E_4|) \leq \frac{1}{2n+5}|Q_0|.$$  

Actually, it suffices to take

$$A_1, A_3 = (c_n)^{1/q}||T||_{L^q \rightarrow L^{q,\infty}} \quad \text{and} \quad A_2, A_4 = c_{n,r,\nu}||\mathcal{M}_T||_{L^r \times L^r \rightarrow L^{r,\infty}}$$

with $c_n, c_{n,r,\nu}$ large enough. For this choice of $E_i$, the set $\Omega = \cup_i E_i$ satisfies $|\Omega| \leq (1/(2^{n+2}))|Q_0|$.

Now, applying Calderón–Zygmund decomposition to the function $\chi_\Omega$ on $Q_0$ at height $\lambda = (1/(2^{n+1}))$, we obtain pairwise disjoint cubes $P_j \in D(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and also $|\Omega \setminus \cup_j P_j| = 0$. From the properties of the cubes it readily follows that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and $P_j \cap \Omega^c \neq \emptyset$.

Now, since $|\Omega \setminus \cup_j P_j| = 0$, we have that

$$\int_{Q_0 \setminus \cup_j P_j} |T(f \chi_{3Q_0})(b - b_{R_{Q_0}})g| \leq A_1(f)_{q,3Q_0} \int_{Q_0} |g(b - b_{R_{Q_0}})|,$$

$$\int_{Q_0 \setminus \cup_j P_j} |T(f(b - b_{R_{Q_0}})f \chi_{3Q_0})||g| \leq A_3(f(b - b_{R_{Q_0}})f)_{q,3Q_0} \int_{Q_0} |g|.$$  

Also, since $P_j \cap \Omega^c \neq \emptyset$, we obtain

$$\int_{P_j} |T((b - b_{R_{Q_0}})f \chi_{3Q_0 \setminus 3P_j})||g| \leq A_2((b - b_{R_{Q_0}})f)_{r,3Q_0}(g)_{s,Q_0}|Q_0|,$$

$$\int_{P_j} |T(f \chi_{3Q_0 \setminus 3P_j})(b - b_{R_{Q_0}})g| \leq A_4(f)_{r,3Q_0}((b - b_{R_{Q_0}})g)_{s,Q_0}|Q_0|.$$
Our next step is to observe that for any arbitrary pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \),
\[
\int_{Q_0} |[b,T](f\chi_{3Q_0})||g| \\
= \int_{Q_0 \cup P_j} |[b,T](f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |[b,T](f\chi_{3Q_0})||g| \\
\leq \int_{Q_0 \cup P_j} |[b,T](f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |[b,T](f\chi_{3Q_0 \cup 3P_j})||g| \\
+ \sum_j \int_{P_j} |[b,T](f\chi_{3P_j})||g|.
\]

For the first two terms, using that \([b,T]f = [b - c,T]f\) for any \( c \in \mathbb{R} \), we obtain
\[
\int_{Q_0 \cup P_j} |[b,T](f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |[b,T](f\chi_{3Q_0 \cup 3P_j})||g| \\
\leq \int_{Q_0 \cup P_j} |b - b_{RQ_0}||T(f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |b - b_{RQ_0}||T(f\chi_{3Q_0 \cup 3P_j})||g| \\
+ \int_{Q_0 \cup P_j} |T((b - b_{RQ_0})f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |T((b - b_{RQ_0})f\chi_{3Q_0 \cup 3P_j})||g|.
\]

Therefore, combining all the preceding estimates with Hölder’s inequality (here we take into account \( q \leq r \) and \( s \geq 1 \)) and calling \( A = \sum_i A_i \), we have that
\[
\int_{Q_0} |[b,T](f\chi_{3Q_0})||g| \leq \sum_j \int_{P_j} |[b,T](f\chi_{3P_j})||g| \\
+ A(\langle f \rangle_{r,3Q_0} \langle (b - b_{RQ_0})g \rangle_{s,Q_0}|Q_0| + \langle (b - b_{RQ_0})f \rangle_{r,3Q_0} \langle g \rangle_{s,Q_0}|Q_0|).
\]

Since \( \sum_j |P_j| \leq (1/2)|Q_0| \), iterating the above estimate, we obtain that there is a \((1/2)\)-sparse family \( \mathcal{F} \subset \mathcal{D}(Q_0) \) such that
\[
\int_{Q_0} |[b,T](f\chi_{3Q_0})||g| \leq A \sum_{Q \in \mathcal{F}} \langle (b - b_{RQ})f \rangle_{r,3Q} \langle g \rangle_{s,Q}|Q| \\
+ A \sum_{Q \in \mathcal{F}} \langle f \rangle_{r,3Q} \langle g(b - b_{RQ}) \rangle_{s,Q}|Q|.
\] (3.2)

To end the proof, take now a partition of \( \mathbb{R}^n \) by cubes \( R_j \) such that \( \text{supp} \,(f) \subset 3R_j \) for each \( j \). One way to do this is the following. We take a cube \( Q_0 \) such that \( \text{supp} \,(f) \subset Q_0 \) and cover \( 3Q_0 \setminus Q_0 \) by \( 3^n \) - 1 congruent cubes \( R_j \). Each of them satisfies \( Q_0 \subset 3R_j \). We continue covering \( 9Q_0 \setminus 3Q_0 \), and so on, in the same way. The family of cubes resulting from this process, including \( Q_0 \), satisfies the desired property.
Having such a partition, apply (3.2) to each \( R_j \). We obtain a \((1/2)\)-sparse family \( F_j \subset D(R_j) \) such that

\[
\int_{R_j} |[b,T](f)||g| \leq A \sum_{Q \in F_j} \langle (b - b_{R_Q}) f \rangle_{r,3Q} \langle g \rangle_{s,Q} |Q| \\
+ A \sum_{Q \in F_j} \langle f \rangle_{r,3Q} \langle g(b - b_{R_Q}) \rangle_{s,Q} |Q|.
\]

Therefore, setting \( F = \cup_j F_j \),

\[
\int_{\mathbb{R}^n} |[b,T](f)||g| \leq A \sum_{Q \in F} \langle (b - b_{R_Q}) f \rangle_{r,3Q} \langle g \rangle_{s,Q} |Q| \\
+ A \sum_{Q \in F} \langle f \rangle_{r,3Q} \langle g(b - b_{R_Q}) \rangle_{s,Q} |Q|.
\]

Now, since \( 3Q \subset R_Q \) and \( |R_Q| \leq 3^n |3Q| \), clearly \( \langle b \rangle_{\alpha,3Q} \leq c_n \langle b \rangle_{\alpha,R_Q} \). Further, setting \( S_j = \{ R_Q \in D_j : Q \in F \} \), and using that \( F \) is \((1/2)\)-sparse, we obtain that each family \( S_j \) is \(1/(2 \cdot 9^n)\)-sparse. Hence

\[
\int_{\mathbb{R}^n} |[b,T](f)||g| \leq c_n A \sum_{j=1}^{3^n} \sum_{R \in S_j} \langle (b - b_R) f \rangle_{r,R} \langle g \rangle_{s,R} |R| \\
+ c_n A \sum_{j=1}^{3^n} \sum_{R \in S_j} \langle f \rangle_{r,R} \langle g(b - b_R) \rangle_{s,R} |R|
\]

and (3.1) holds.

Given \( 1 \leq p \leq \infty \), we define the maximal operator \( M_{p,T} \) by

\[
M_{p,T} f(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |T(f\chi_{\mathbb{R}^n \setminus 3Q})|^p \, dy \right)^{1/p}
\]

(in the case \( p = \infty \), we call \( M_{p,T} f(x) = M_T f(x) \)).

Our next step is to provide a suitable version of [13, Corollary 3.2] for the commutator. The result is the following.

**Corollary 3.2.** Let \( 1 \leq q \leq r \) and \( s \geq 1 \). Assume that \( T \) is a sublinear operator of weak type \((q,q)\), and \( M_{s',T} \) is of weak type \((r,r)\). Then, for every compactly supported \( f, g \in C^\infty(\mathbb{R}^n) \) and every \( b \in \text{BMO} \), there exist \( 3^n \) dyadic lattices \( D_j \) and \( 3^n \) sparse families \( S_j \subset D_j \) such that

\[
|\langle [b,T]f, g \rangle| \leq K \sum_{j=1}^{\infty} (T_{S_j,r,s}(b,f,g) + T_{S_j,s,r}(b,f,g))
\]

(3.3)
where
\[ T_{S_j,r,s}(b,f,g) = \sum_{Q \in S_j} \langle f \rangle_{r,Q} \langle (b-b_Q)g \rangle_{s,Q} |Q| \]
\[ T_{S_j,r,s}^*(b,f,g) = \sum_{Q \in S_j} \langle (b-b_Q)f \rangle_{r,Q} \langle g \rangle_{s,Q} |Q| \]
and
\[ K = C_n \left( \|T\|_{L^q \to L^{q,\infty}} + \|M_{s',T}\|_{L^r \to L^{r,\infty}} \right). \]

**Proof.** The proof is the same as that of [13, Corollary 3.2]. It suffices to observe that
\[ \|M_T\|_{L^r \times L^s \to L^{\nu,\infty}} \leq C_n \left( \|M_{s',T}\|_{L^r \to L^{r,\infty}} \right) \frac{1}{\nu} = \left( \frac{1}{r} + \frac{1}{s} \right), \]
and to apply Theorem 3.1. \( \square \)

**Remark 3.3.** At this point, note that if \( T \) is an \( \omega \)-Calderón–Zygmund operator, with \( \omega \) satisfying a Dini condition, since \( M_T \) is of weak-type \((1,1)\) with
\[ \|M_T\|_{L^1 \to L^{1,\infty}} \leq C_n \left( \|T\|_{L^2} + \|\omega\|_{\text{Dini}} \right) \]
(see [12], also for the notation) and we have that
\[ \|T\|_{L^1 \to L^{1,\infty}} \leq C_n \left( \|T\|_{L^2} + \|\omega\|_{\text{Dini}} \right), \]
then from the preceding corollary we recover a bilinear version of the sparse domination established in [18].

In order to use Corollary 3.2 to obtain Theorem 1.3, we need to borrow some results from [13]. Given an operator \( T \), we define the maximal operator \( M_{\lambda,T} \) by
\[ M_{\lambda,T}f(x) = \sup_{Q \ni x} \left( T(f\chi_{\mathbb{R}^n \setminus 3Q})\chi_Q \right)^*(\lambda|Q|), \quad 0 < \lambda < 1. \]
That operator was proved to be of weak type \((1,1)\) in [13], where the following estimate was established.

**Theorem 3.4.** If \( \Omega \in L^\infty(S^{n-1}) \), then
\[ \|M_{\lambda,T}\|_{L^1 \to L^{1,\infty}} \leq C_n \|\Omega\|_{L^\infty(S^{n-1})} \left( 1 + \log \frac{1}{\lambda} \right), \quad 0 < \lambda < 1. \] (3.4)
Also in [13], the following result showing the relationship between the \( L^1 \to L^{1,\infty} \) norms of the operators \( M_{\lambda,T} \) and \( M_{p,T} \) was provided.

**Lemma 3.5.** Let \( 0 < \gamma \leq 1 \) and let \( T \) be a sublinear operator. The following statements are equivalent:
(1) there exists \( C > 0 \) such that for all \( p \geq 1 \),
\[ \|M_{p,T}f\|_{L^1 \to L^{1,\infty}} \leq Cp^\gamma; \]
(2) there exists $C > 0$ such that for all $0 < \lambda < 1$,

$$\| M_{\lambda, T} f \|_{L^1 \to L^{1, \infty}} \leq C \left(1 + \log \frac{1}{\lambda}\right)^\gamma.$$ 

At this point, we are in a position to prove that Theorem 1.3 follows as a corollary from the previous results.

**Proof of Theorem 1.3.** Theorem 3.4 combined with Lemma 3.5 with $\gamma = 1$ yields

$$\| M_{p, T} \Omega \|_{L^1 \to L^{1, \infty}} \leq c_n p \| \Omega \|_{L^{\infty}(\mathbb{S}^{n-1})}$$

with $p \geq 1$. Also, by [26], we have that

$$\| T_{\Omega} \|_{L^1 \to L^{1, \infty}} \leq C_n \| \Omega \|_{L^{\infty}(\mathbb{S}^{n-1})}.$$ 

Hence, by Corollary 3.2 with $q = r = 1$ and $s = p > 1$, there exist $3^n$ dyadic lattices $D_j$ and $3^n$ sparse families $S_j \subset D_j$ such that

$$|\langle [b, T_{\Omega}] f, g \rangle| \leq C_n p' \| \Omega \|_{L^{\infty}(\mathbb{S}^{n-1})} \sum_{j=1}^{3^n} (T_{S_j, 1, p}(b, f, g) + T_{S_j, 1, p}(b, f, g)).$$

4. **Proof of Theorem 1.1**

We start providing a proof for (1.3). We follow some of the key ideas from [16, 17] (see also [22]). By duality, to prove (1.3) it suffices to show that

$$\left\| \frac{[b, T_{\Omega}] f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq c_n \| \Omega \|_{L^{\infty}(\mathbb{S}^{n-1})} \| b \|_{BMO(p')} \| p^2 (r')^{1 + (1/p')} \| \frac{f}{w} \|_{L^{p'}(w)}.$$ 

We can calculate the norm by duality. Then,

$$\left\| \frac{[b, T_{\Omega}] f}{M_r w} \right\|_{L^{p'}(M_r w)} = \sup_{\| h \|_{L^p(M_r w)} = 1} \left| \int_{\mathbb{R}^n} [b, T_{\Omega}] f(x) h(x) \, dx \right|.$$ 

Let us now define a Rubio de Francia algorithm suited for this situation (see [6, Chapter IV.5] and [4] for various applications of the Rubio de Francia algorithm). First, we consider the operator

$$S(f) = \frac{M(f(M_r w)^{1/p})}{(M_r w)^{1/p}}$$

and observe that $S$ is bounded on $L^p(M_r w)$ with norm bounded by a dimensional multiple of $p'$. Relying upon $S$, we define

$$R(h) = \sum_{k=0}^\infty \frac{1}{2^k} \frac{S^k h}{\| S^k \|_{L^p(M_r w)}}.$$
This operator has the following properties:

(a) \(0 \leq h \leq R(h)\);
(b) \(\|Rh\|_{L^p(M_t, w)} \leq 2\|h\|_{L^p(M_t, w)}\);
(c) \(R(h)(M_t w)^{1/p} \in A_1\) with \([R(h)(M_t w)^{1/p}]_{A_1} \leq c_1\). We also note that \([Rh]_{A_3} \leq c_3\).

Using Theorem 1.3 and taking into account (a), we have that,

\[
\left| \int_{\mathbb{R}^n} [b, T_\Omega] f(x) h(x) \, dx \right| \\
\leq C_n s' \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^\infty (T_{S_j,1,s}(b,f,h) + T_{S_j,1,s}^*(b,f,h)) \\
\leq C_n s' \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^\infty (T_{S_j,1,s}(b,f,Rh) + T_{S_j,1,s}^*(b,f,Rh))
\]

and it suffices to obtain estimates for

\[
I := T_{S_j,1,s}(b,f,Rh) \quad \text{and} \quad II := T_{S_j,1,s}^*(b,f,Rh).
\]

First, we focus on \(I\). We choose \(r, s > 1\) such that \(sr = 1 + (1/(\tau_n[Rh]_{A_\infty}))\). For instance, choosing \(r = 1 + (1/(2\tau_n[Rh]_{A_\infty}))\), we have that \(s = 2(1 + \tau_n[Rh]_{A_\infty})/(1 + 2\tau_n[Rh]_{A_\infty})\) and also that \(sr' = 2(1 + \tau_n[Rh]_{A_\infty}) \simeq [Rh]_{A_\infty}\). Now, recall that for every \(0 < t < \infty\) it was established in [7, Corollary 3.1.8] that

\[
\left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^t \, dx \right)^{1/t} \leq (t\Gamma(t))^{1/t} e^{(1/t)} + 2^n \|b\|_{BMO}.
\]

For \(t > 1\), we have that \((t\Gamma(t))^{1/t} e^{(1/t)} + 2^n \leq c_n t\). Taking into account the preceding estimate, the choices for \(r\) and \(s\), the reverse Hölder inequality (Lemma 2.2) and the property (c) above, we have that

\[
I \leq \sum_{Q \in S_j} \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^t |Rh(x)|^s \, dx \right)^{1/s} \int_Q |f| \, dy \\
\leq \sum_{Q \in S_j} \langle b - b_Q \rangle_{s',Q} \langle Rh \rangle_{s',Q} \int_Q |f| \, dy \\
\leq c_n (sr') \|b\|_{BMO} \sum_{Q \in S_j} \left( \frac{1}{|Q|} \int_Q Rh \right) \int_Q |f| \, dy
\]
\[ \leq c_n[Rh]_{A_\infty} \|b\|_{\text{BMO}} \sum_{Q \in S_j} Rh(Q) \frac{1}{|Q|} \int_Q |f| \, dy \]

\[ \leq c_n p' \|b\|_{\text{BMO}} \sum_{Q \in S_j} Rh(Q) \frac{1}{|Q|} \int_Q |f| \, dy. \]

An application of Lemma 2.3 with \( \Psi(t) = t \) yields

\[ \sum_{Q \in S_j} Rh(Q) \frac{1}{|Q|} \int_Q |f| \, dy \leq 8[Rh]_{A_\infty} \|Mf\|_{L^1(Rh)} \leq c_n p' \|Mf\|_{L^1(Rh)}. \]

From here,

\[ \|Mf\|_{L^1(Rh)} \leq \left( \int_{\mathbb{R}^n} |Mf|^{p'} (M_r, w)^{1-p'} \right)^{1/p'} \left( \int_{\mathbb{R}^n} (Rh)^p M_r w \right)^{1/p} \]

\[ \leq 2 \left\| \frac{Mf}{M_r w} \right\|_{L^{p'}(M_r, w)}. \]

Now, by [16, Lemma 3.4] (see also [24, Lemma 2.9]),

\[ \left\| \frac{Mf}{M_r w} \right\|_{L^{p'}(M_r, w)} \leq cp'(r')^{1/p'} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}. \]

Gathering all the preceding estimates, we have that

\[ I \leq c_n \|b\|_{\text{BMO}} p'(p')^{3}(r')^{1/p'} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}. \]

Now we turn our attention to \( II \). Recalling that we have chosen \( rs = 1 + (1/(\tau_n[Rh]_{A_\infty})) \), taking into account the reverse Hölder inequality and also applying (2.1), we have that

\[ II \leq \sum_{Q \in S_j} \left( \frac{1}{|Q|} \int_Q |b(y) - b_Q| f(y) \, dy \right) \langle Rh \rangle_{s,Q} |Q| \]

\[ \leq \sum_{Q \in S_j} \left( \frac{1}{|Q|} \int_Q |b(y) - b_Q| f(y) \, dy \right) \langle Rh \rangle_{rs,Q} |Q| \]

\[ \leq c_n \|b\|_{\text{BMO}} \sum_{Q \in S_j} \|f\|_{L_{\log L,Q} Rh(Q)}. \]

Then, a direct application of Lemma 2.3 with \( \Psi(t) = t \log(e + t) \) yields the following estimate:

\[ \sum_{Q \in S_j} \|f\|_{L_{\log L,Q} Rh(Q)} \leq 8[Rh]_{A_\infty} \|ML_\log L f\|_{L^1(Rh)}. \]
Arguing as in the estimate of $I$,
\[
\| M_{L \log L} f \|_{L^1(Rh)} \leq 2 \left\| \frac{M_{L \log L} f}{M_r w} \right\|_{L^{p'}(M_r w)}.
\]

Now [24, Proposition 3.2] gives
\[
\left\| \frac{M_{L \log L} f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq c_n p^{2}(r')^{1+(1/p')} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.
\]

Combining all the estimates, we have that
\[
II \leq c_n \|b\|_{\text{BMO}(p')}^{2} p^{2}(r')^{1+(1/p')} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.
\]

Finally, collecting the estimates we have obtained for $I$ and $II$, we arrive at the desired bound, namely
\[
\left\| \frac{[b, T_{\Omega}] f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq c_n \|\Omega\|_{L^{\infty}(\mathbb{R}^{n-1})} [b]_{\text{BMO}(p')}^{3} p^{2}(r')^{1+(1/p')} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.
\]

We end the proof by observing that the $A_{\infty}$ and the $A_{1} - A_{\infty}$ results are a direct consequence of the estimate we have just established and of the reverse Hölder inequality (see [8, 16, 17] for this kind of argument).

5. Proof of Theorem 1.2

Let us consider first the case in which $T$ is a Calderón–Zygmund operator. Calculating the norm by duality we have that
\[
\| [b, T] f \|_{L^{p}(w)} = \sup_{\|g\|_{L^{p'}(w)} = 1} \left| \int [b, T] (f) g w \right|.
\]

Now, taking into account Remark 3.3 (or [18]), we have that
\[
\left| \int [b, T] (f) g w \right| \leq c_n c_T \sum_{j=1}^{3^n} (T_{S_{j,1,1}}(b, f, g w) + T_{S_{j,1,1}^{*}}(b, f, g w))
\]
so it suffices to provide estimates for
\[
T_{S_{j,1,1}}(b, f, g w) \quad \text{and} \quad T_{S_{j,1,1}^{*}}(b, f, g w).
\]

First, we work on $T_{S_{j,1,1}}(b, f, g w)$. Following ideas in [19] we have that
\[
\langle w \rangle^{Q} (w_{1/(1-q)}^{-1})^{q-1} = \langle w \rangle^{Q} (\sigma^{1/p'})^{p} A_{Q}
\]
where \( \overline{A}(t) = t^{p/(q-1)} \) and \( \sigma = w^{1-p'} \). Then, choosing \( s < p' \) and taking into account [11, Lemma 6], (2.1) and (2.2),

\[
T_{S,1,1}(b, f, gw) = \sum_{Q \in S} \langle f \rangle_Q \langle g(b - b_Q)w \rangle_Q |Q|
\]

\[
\leq c \sum_{Q \in S} \langle f \rangle_{A, Q} \langle g \rangle_{(1/p)} \langle w \rangle_{A, Q} \langle g \rangle_{s,Q} |Q|
\]

\[
\leq c s' \|b\|_{BMO[w]} A_{\infty} \sum_{Q \in S} \langle f \rangle_{A, Q} \langle g \rangle_{(1/p)} \langle w \rangle_{A, Q} \langle g \rangle_{s,Q} |Q|
\]

\[
\leq c s' \|b\|_{BMO[w]} A_{\infty} \sum_{Q \in S} \langle f \rangle_{A, Q} \langle g \rangle_{s,Q} |Q|
\]

\[
\leq c n \gamma^{-1} \|MA\|_{L^p} \|b\|_{BMO[w]} A_{\infty} \sum_{Q \in S} \langle f \rangle_{A, Q} \langle g \rangle_{Q} |Q|
\]

\[
\leq c \|b\|_{BMO} \sum_{Q \in S} \langle f \rangle_{A, Q} \langle g \rangle_{Q} |Q|
\]

and from this point it suffices to follow the proof of [19, Theorem 3.1] to obtain the following estimate:

\[
T_{S,1,1}^*(b, f, gw) \leq c [w] A_{q'/p} (A_{\infty}^{p'})^{1/p} \|f\|_{L^p(w)} \|g\|_{L^{p'}(w)}
\]

Combining the estimates for \( T_{S,1,1}(b, f, gw) \) and \( T_{S,1,1}^*(b, f, gw) \), we obtain (1.4) in the case of \( T \) being a Calderón–Zygmund operator.

Let us consider now the remaining case. Assume that \( T \) is a rough singular integral with \( \Omega \in L^\infty(S^{n-1}) \). Calculating the norm by duality and denoting by \([b, T]^f \) the adjoint of \([b, T] \), we have that

\[
\|[b, T]f\|_{L^p(w)} = \sup_{\|g\|_{L^{p'}(w)} = 1} \left| \int [b, T](f)gw \right| = \sup_{\|g\|_{L^{p'}(w)} = 1} \left| \int [b, T]^f(gw)f \right|
\]
Taking into account that $[b, T]^t$ is also a commutator, we can use the sparse domination obtained in Theorem 1.3 so we have that

$$\left| \int [b, T]^t (gw)f \right| \leq c_n u' \|\Omega\|_{L^\infty(\mathbb{R}^{n-1})} \sum_{j=1}^{3^n} (T_{S_j, u,1}(b, f, gw) + T_{S_j, u,1}(b, f, gw)), $$

and then the question reduces to control of both

$$T_{S_j, u,1}(b, f, gw) \text{ and } T_{S_j, u,1}(b, f, gw).$$

We begin by observing that, arguing as before, choosing $1 < s < p'$,

$$T_{S_j, u,1}(b, f, gw) = \sum_{Q \in S_j} \langle f \rangle_{u, Q} \langle (b - b_Q)gw \rangle_{1, Q}|Q|$$

$$\leq cs'[w]_{A_\infty} \|b\|_{\text{BMO}} \sum_{Q \in S_j} \langle f \rangle_{u, Q} \langle g \rangle_{s, Q}^w w(Q) = c[w]_{A_\infty} \|b\|_{\text{BMO}} B_1.$$  

On the other hand, we have that for $s_1 > 1$ to be chosen later,

$$T_{S, u,1}(b, f, gw) = \sum_{Q \in S} \langle (b - b_Q)f \rangle_{u, Q} \langle gw \rangle_{1, Q}|Q|$$

$$\leq \sum_{Q \in S} \langle f \rangle_{us_1, Q} \langle b - b_Q \rangle_{us_1, Q} \langle gw \rangle_{1, Q}|Q|$$

$$\leq c \|b\|_{\text{BMO}} \sum_{Q \in S} \langle f \rangle_{us_1, Q} \langle gw \rangle_{1, Q}|Q| = c \|b\|_{\text{BMO}} B_2.$$  

By the Hölder inequality, we have that both $B_1$ and $B_2$ are controlled by

$$\sum_{Q \in S} \langle f \rangle_{us_1, Q} \langle gw \rangle_{s, Q}^w w(Q).$$

Note that we can choose $us_1$ as close to 1 as we want, so let us rename $us_1 = r$. Now, denoting $\overline{B}(t) = t^{p/(r(q-1))}$ and arguing as in [19, Theorem 3.1], we have that

$$\sum_{Q \in S_j} \langle f \rangle_{r, Q} \langle gw \rangle_{s, Q}^w w(Q) \leq [w]_{A_q}^{1/p(r(A_{\infty}^{\exp}))} \left( \sum_{Q \in S} \langle f \rangle_{r, Q}^{p/r} \langle gw \rangle_{s, Q}^p \exp((\log w)Q)|Q| \right)^{1/p}$$

$$\times \left( \sum_{Q \in S} \langle gw \rangle_{s, Q}^p \exp((\log w)Q)|Q| \right)^{1/p'}$$

$$\leq c_n \gamma^{-1} p \|M_B\|_{\text{L}^{p/r}(w)} [w]_{A_q}^{1/p(r(A_{\infty}^{\exp}))} \|f\|_{L^p(w)} \|g\|_{\text{L}^{p'}(w)},$$

where in the last step we have used again the sparsity of $S$ and the Carleson embedding theorem [8, Theorem 4.5]. Collecting all the estimates:

$$\left| \int [b, T]^t (gw)f \right| \leq c_n \|\Omega\|_{L^\infty(\mathbb{R}^{n-1})} [w]_{A_\infty} [w]_{A_q}^{1/p(r(A_{\infty}^{\exp}))} \|f\|_{L^p(w)} \|g\|_{\text{L}^{p'}(w)}.$$  

This ends the proof of Theorem 1.2.  

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References
1. M. Caldarelli, A. K. Lerner and S. Ombrosi, On a counterexample related to weighted weak type estimates for singular integrals, Proc. Amer. Math. Soc. 145(7) (2017), 3005–3012.
2. R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. Math. 103(3) (1976), 611–635.
3. J. M. Conde-Alonso, A. Culiuc, F. Di Plinio and Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE 10(5) (2017), 1255–1284.
4. D. Cruz-Uribe, J. M. Martell and C. Pérez, Weights, extrapolation and the theory of Rubio de Francia, Operator Theory: Advances and Applications, Volume 215 (Birkhäuser/Springer Basel AG, Basel, 2011).
5. J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds, J. Funct. Anal. 260 (2015), 1886–1901.
6. J. García-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North-Holland Mathematics Studies, Volume 116. Notas de Matemática (Mathematical Notes) Volume 104 (North-Holland Publishing, Amsterdam, 1985).
7. L. Grafakos, Modern Fourier analysis, 3rd edn. Graduate Texts in Mathematics, Volume 250 (Springer, New York, 2014).
8. T. P. Hytönen and C. Pérez, Sharp weighted bounds involving $A_\infty$, Anal. PDE 6(4) (2013), 777–818.
9. T. P. Hytönen, C. Pérez and E. Rela, Sharp reverse Hölder property for $A_\infty$ weights on spaces of homogeneous type, J. Funct. Anal. 263(12) (2012), 3883–3899.
10. T. P. Hytönen, L. Roncal and O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals, Israel J. Math. 218(1) (2017), 133–164.
11. G. H. Ibañez-Firnkorn and I. P. Rivera-Ríos, Sparse and weighted estimates for generalized Hörmander operators and commutators, preprint. Available at https://arxiv.org/abs/1704.01018.
12. A. K. Lerner, On pointwise estimates involving sparse operators, N. Y. J. Math. 22 (2016), 341–349.
13. A. K. Lerner, A weak type estimate for rough singular integrals, preprint. Available at https://arxiv.org/abs/1705.07397, to appear in Revista Mat. Iberoamericana.
14. A. K. Lerner and K. Moen, Mixed $A_p - A_\infty$ estimates with one supremum, Studia Math. 219(3) (2013), 247–267.
15. A. K. Lerner and F. Nazarov, Intuitive dyadic calculus: the basics, preprint. Available at http://arxiv.org/abs/1508.05639, to appear in Expo. Math.
16. A. K. Lerner, S. Ombrosi and C. Pérez, Sharp $A_1$ bounds for Calderón–Zygmund operators and the relationship with a problem of Muckenhoupt and Wheeden, Int. Math. Res. Not. IMRN 2008, rnm161.
17. A. K. Lerner, S. Ombrosi and C. Pérez, $A_1$ bounds for Calderón–Zygmund operators related to a problem of Muckenhoupt and Wheeden, Math. Res. Lett. 16(1) (2009), 149–156.
18. A. K. Lerner, S. Ombrosi and I. P. Rivera-Ríos, On pointwise and weighted estimates for commutators of Calderón–Zygmund operators, Adv. Math. 319 (2017), 153–181.
19. K. Li, Sharp weighted estimates involving one supremum, *C. R. Math. Acad. Sci. Paris* **355**(8) (2017), 906–909.

20. K. Li, C. Pérez, I. P. Rivera-Ríos and L. Roncal, Weighted norm inequalities for rough singular integral operators, preprint. Available at https://arxiv.org/abs/1701.05170

21. C. Pérez, On sufficient conditions for the boundedness of the Hardy–Littlewood maximal operator between weighted $L^p$-spaces with different weights, *Proc. London Math. Soc.* (3) **71**(1) (1995), 135–157.

22. C. Pérez, I. P. Rivera-Ríos and L. Roncal, $A_1$ theory of weights for rough homogeneous singular integrals and commutators, preprint. Available at https://arxiv.org/abs/1607.06432, to appear in *Ann. Sci. Scuola Norm. Sup.* (Scienze).

23. L. Pick, A. Kufner, O. John and S. Fučík, *Function spaces*, Second revised and extended edition, Volume 1, De Gruyter Series in Nonlinear Analysis and Applications, Volume 14 (Walter de Gruyter and Co., Berlin, 2013).

24. C. Ortiz-Caraballo, Quadratic $A_1$ bounds for commutators of singular integrals with BMO functions, *Indiana Univ. Math. J.* **60**(6) (2011), 2107–2130.

25. M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, Volume 146 (Marcel Dekker, New York, 1991).

26. A. Seeger, Singular integral operators with rough convolution kernels, *J. Amer. Math. Soc.* **9**(1) (1996), 95–105.