Producing "new" semi-orthogonal decompositions out of "old" ones in arithmetic geometry

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Abstract

This paper is devoted to constructing new admissible subcategories and semi-orthogonal decompositions of triangulated categories out of old ones. For two triangulated subcategories $T$ and $T'$ of a certain $D$ and a decomposition $(RO, LO)$ of $T$ we look either for a decomposition $(RO', LO')$ of $T'$ such that there are no non-zero $D$-morphisms from $RO$ into $RO'$ and from $LO$ into $LO'$, or for a decomposition $(RO_D, LO_D)$ of $D$ such that $RO_D \cap T = RO$ and $LO_D \cap T = LO$. We prove some general existence statements (that also extend to semi-orthogonal decompositions of arbitrary length) and apply them to various derived categories of coherent sheaves over a scheme $X$ that is proper over the spectrum of a Noetherian ring $R$. This gives a one-to-one correspondence between semi-orthogonal decompositions of $D_{perf}(X)$ and $D^b(coh(X))$; the latter extend to $D^-(coh(X)), D^+_{coh}(Qcoh(X)), D_{coh}(Qcoh(X)), and D(Qcoh(X))$ under very mild assumptions. In particular, we obtain a vast generalization of a theorem of J. Karmazyn, A. Kuznetsov, and E. Shinder.

These applications rely on recent results of Neeman that express $D^b(coh(X))$ and $D^-(coh(X))$ in terms of $D_{perf}(X)$. We also prove and apply a rather similar new theorem that relates $D^+_{coh}(Qcoh(X))$ and $D_{coh}(Qcoh(X))$ (these are certain modifications of the bounded below and the unbounded derived category of coherent sheaves on $X$) to homological functors $D_{perf}(X)^{op} \to R - mod$. Moreover, we discuss an application of this theorem to the construction of certain adjoint functors.

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1 Introduction

This paper is devoted to constructing "new" semi-orthogonal decompositions (see Definitions 2.2.1(1) and 4.2.1(2) below) of certain triangulated categories out of "old" ones; the relevance of this matter is discussed in Remark 1.7(1). We prove some general existence statements and apply them to various triangulated subcategories of \( D(\text{Qcoh}(X)) \) (see Definition 1.1(7)); we always assume \( X \) to be a scheme that is proper over the spectrum of a noetherian ring \( R \). These applications rely on the main results of [Nee18a] and [Nee18b] along with the new Theorem 1.5.

In this introduction we re-formulate some of the results of the main body of the paper in terms of admissible subcategories (cf. Remark 2.2.2(2) below). We suggest the reader to check that these statements can be obtained from their "decomposition" versions (see Theorem 3.2.7) via straightforward applications of Proposition 2.2.3.

We start from some definitions. In this paper all the subcategories we consider will be assumed to be strictly full.

**Definition 1.1.** Let \( D \) be a triangulated category; assume that \( T, T' \), and some \( T_i \) are its (strictly full) triangulated subcategories.

1. We say that \( T \) is left (resp. right) admissible in \( D \) if the embedding \( T \to D \) admits a left (resp. right) adjoint. 

   \( T \) is said to be admissible in \( D \) if it is both left and right admissible in it.

2. We write \( T \cap T' \) for the subcategory of \( D \) whose object class equals \( \text{Obj } T \cap \text{Obj } T' \).

   Moreover, we write \( (T_i \cap T') \) for the family \( (T_i \cap T') \).

3. Given an additive category \( C \) and \( X, Y \in \text{Obj } C \) we will write \( C(X,Y) \) for the set of morphisms from \( X \) to \( Y \) in \( C \).

   Moreover, for \( D, E \subset \text{Obj } D \) we write \( D \perp E \) if \( D(X,Y) = \{0\} \) for all \( X \in D, Y \in E \).
4. We will write $\mathcal{T}^\perp_i$ for the subcategory of $\mathcal{T}'$ whose object class is
\[
\{ Y \in \text{Obj } \mathcal{T}' \colon \{ X \} \perp \{ Y \}, \forall X \in \text{Obj } \mathcal{T} \}.
\]
Dually, we set the object class of the subcategory $\mathcal{T}_i^\perp$ to be
\[
\{ Y \in \text{Obj } \mathcal{T}' \colon \{ Y \} \perp \{ X \}, \forall X \in \text{Obj } \mathcal{T} \}.
\]
Moreover, we will write $(\mathcal{T}_i)^\perp$ for the family $(\mathcal{T}_i^\perp)$.

5. Assume that $\mathcal{D}$ is closed with respect to small coproducts.
Then we will write $\mathcal{T} \mathcal{U}$ for the smallest (strict) triangulated subcategory of $\mathcal{D}$ that is closed with respect to $\mathcal{D}$-coproducts and contains $\mathcal{T}$.
Moreover, we will write $\mathcal{T}_i \mathcal{U}$ (resp. $(\mathcal{T}_i)^\mathcal{U}$) for the category $\mathcal{T} \cap \mathcal{T}'$ (resp. for the families $(\mathcal{T}_i^\mathcal{U})$ and $(\mathcal{T}_i)^\mathcal{U}$).

6. Throughout this paper $R$ will be a commutative unital ring.
We set $R - \text{mod} \subset R - \text{Mod}$ to be the subcategory of finitely generated $R$-modules; $S = \text{Spec } R$.

7. Assume that a scheme $X$ proper over $S$ is fixed. We set $\mathcal{D}_p = D_{\text{perf}}(X) \subset \mathcal{D}^b = D_{\text{coh}}^b(Qcoh(X)) \subset \mathcal{D}^n = D_{\text{coh}}(Qcoh(X)) \subset \mathcal{D}_Q = D(Qcoh(X))$; here $D_{\text{perf}}(X) \subset \mathcal{D}_Q$ is the subcategory of perfect complexes on $X$ (cf. stacks Tag 08CM), and a complex $N$ (in $\mathcal{D}_Q$) belongs to $\mathcal{D}^n$ whenever all its cohomology sheaves $H^i(N)$ are coherent; it also belongs to $\mathcal{D}^b$ (resp. $\mathcal{D}^-$) if we also have $H^i(N) = 0$ for $i \gg 0$ and $i \ll 0$ (resp. for $i \gg 0$ only). Moreover, we consider $\mathcal{D}^+ = D_{\text{coh}}^+(Qcoh(X)) \subset \mathcal{D}^n$ that is defined similarly. We discuss these categories in Remark 1.4 below; cf. also Remark 1.2.3.

8. We will say that $X$ is projective over $S = \text{Spec } R$ if $X$ is a closed subscheme of the projectivization $Y$ of a vector bundle $\mathcal{E}$ over $S$.

9. All $R$-linear categories in this paper will be additive. For two $R$-linear categories $A, B$ we will write $\text{Fun}_R(A, B)$ for the category of $R$-linear functors $A \to B$.

Remark 1.2. Clearly, all the subcategories of $\mathcal{D}$ that we describe in Definition 1.1.2 and 1.5 are triangulated; recall the strictness assumption.

Theorem 1.3. Let $\mathcal{X}$ be a left (resp. right) admissible subcategory of $\mathcal{D}_p$ and $\mathcal{W}$ be a left (resp. right) admissible subcategory of $\mathcal{D}^b$ (see Definition 1.1.7).
1. Then the categories $\mathcal{X}^\perp_{\mathcal{D}_p}$, $\mathcal{X}^\perp_{\mathcal{D}^b}$, and $\mathcal{X}^\perp_{\mathcal{D}_Q}$ are left (resp. right) admissible in $\mathcal{D}^b$, $\mathcal{D}^-$, and $\mathcal{D}_Q$, respectively.

Moreover, $\mathcal{X}^\perp_{\mathcal{D}_Q}$ is left (resp. right) admissible in $\mathcal{D}_Q$ as well, and $\mathcal{X}^\perp_{\mathcal{D}_Q} = (\mathcal{X}^\perp_{\mathcal{D}_p})^\mathcal{U}$.
2. Assume in addition that either $X$ is regular or that regular alteration exist for all integral closed subschemes of $X$. Then $\mathcal{W}$ equals $\mathcal{X}^\perp_{\mathcal{D}_p}$ for some left (resp. right) admissible subcategory $\mathcal{X}'$ of $\mathcal{D}_p$.

Consequently, $\mathcal{W}_{\mathcal{D}^b}$ and $\mathcal{W}_{\mathcal{D}_Q}$ are left (resp. right) admissible in $\mathcal{D}^-$ and $\mathcal{D}_Q$, respectively.

1This assumption is very far from being restrictive; cf. Remark 1.2.3(1) below.
Moreover, the correspondence $\mathcal{E} \mapsto \mathcal{E} \cap \mathcal{D}_p$ gives a one-to-one correspondence between right admissible subcategories of $\mathcal{D}_b$ and left admissible subcategories of $\mathcal{D}_p$.

3. Assume that $X$ is projective over $S = \text{Spec} R$ (see Definition 1.1(8)). Then $\mathcal{X}_D$ is left (resp. right) admissible in $\mathcal{D}_u$ and $\mathcal{X}_D^+$ is left (resp. right) admissible in $\mathcal{D}_u^+$.

4. Assume that $X$ satisfies the assumptions both of assertion 2 and of assertion 3. Then $W_u$ and $W_u^+$ are left (resp. right) admissible in $\mathcal{D}_u$ and $\mathcal{D}_u^+$, respectively.

**Remark 1.4.**

1. A significant part of our arguments can be "axiomatized"; cf. Theorem 3.1.1 below.

2. The "moreover" statement in Theorem 1.3(2) vastly generalizes and extends Theorem A.1 of [KKS22].

3. Furthermore, Corollary 3.2.9 yields that all (right and left) admissible subcategories provided by Theorem 1.3(1) (except $X_{D_Q}$) "restrict" to the intersections of the corresponding subcategories of $\mathcal{D}_D$ with all "support subcategories" of $\mathcal{D}_D$ coming from unions of closed subsets of $S = \text{Spec} R$.

4. Recall that the obvious exact functors $D^{-}(\text{coh}(X)) \rightarrow D_{\text{coh}}(\text{Qcoh}(X))$ and $D^{b}(\text{coh}(X)) \rightarrow D^{b}_{\text{coh}}(\text{Qcoh}(X))$ are equivalences of categories; see stacks Tag [FDA].

On the other hand, a similar functor $D(\text{coh}(X)) \rightarrow D_{\text{coh}}(\text{Qcoh}(X))$ is not necessarily an equivalence; see §3 of [PoS21]. However, it is an equivalence if $X$ is regular; see Corollary 5.12 of ibid.

The theorem was inspired by "duality between weight and $t$-structure" statements that were studied by the author in several papers starting from [Bon10a]; see §4.3 below for more detail. Another recent ingredient are the descriptions of some of our categories as certain categories of functors from $\mathcal{D}_p$ and $\mathcal{D}_b$. These are provided by Theorem 0.2 of [Nee18b] (cf. Remark 1.1.6(1) below) together with the following theorem.

**Theorem 1.5.**

1. Assume that $X$ is projective over $S$ (in the sense of Definition 1.1(8)).

Then for an object $N$ of $\mathcal{D}_Q$ we have $\mathcal{D}_Q(M, N) \in R - \mod$ for all $M \in \text{Obj} \mathcal{D}_p$ if and only if $N \in \text{Obj} \mathcal{D}^u$ (see Definition 1.1(7,9)).

Moreover, for $N \in \text{Obj} \mathcal{D}^u$ we have $N \in \text{Obj} \mathcal{D}^-$ (resp. $N \in \text{Obj} \mathcal{D}^+$) if and only if for any $M \in \text{Obj} \mathcal{D}_p$ we have $\{M[i]\} \downarrow \{N\}$ if $i \leq 0$ (resp. $i > 0$).

2. The restricted Yoneda functor $\mathcal{Y} : \mathcal{D}_u \rightarrow \text{Fun}_R(\mathcal{D}_p^\circ, R - \mod)$ is full.

3. Assume that the ring $R$ is either countable or self-injective, that is, injective as a module over itself. Then any $R$-linear homological functor $\mathcal{D}_p^\circ \rightarrow R - \mod$ is represented by an object of $\mathcal{D}^u$.

**Remark 1.6.**

1. Clearly, Theorem 1.5(1) also yields a similar characterization for $\mathcal{D}_p^b = \mathcal{D}^- \cap \mathcal{D}^+$.

2. This theorem substantially extends Corollary 0.5 of [Nee18a], where only $\mathcal{D}_b$ and $\mathcal{D}^-$ were considered. Respectively, loc. cit. hints that it suffices to assume that $X$ is proper over $R$ in this theorem.

Recall also that ibid. was inspired by the question of existence of certain adjoint functors; see Remark 0.7 of ibid. We prove a nice general result of this sort in Corollary 3.1.3 below and combine it with Theorem 1.5(1) in Remark
3.2.6(1). Possibly, these statements are more "practical" than the corresponding Corollary 0.4 of ibid.

3. We will not apply Theorem 1.5(3) in this paper (yet cf. Remark 4.1.6(1); we mention adjoint functors there as well). Still it is worth noting that combining it with the first part of the theorem one obtains the following (if $X$ is projective over $S$ and $R$ is either countable or self-injective): the essential image of $Y$ consists of all $R$-linear homological functors $D_{R}^{op} \to R \text{-mod}$, and the image of its restriction to $D^{+}$ consists of all those homological functors $H : D_{R}^{op} \to R \text{-mod}$ such that for every $M \in \text{Obj} D_{p}$ we have $H(M[i]) = \{0\}$ if $i \gg 0$. Clearly, the $D^{-}$ and $D^{b}$-versions of this observation follow from Theorem 1.5 as well; yet recall that Corollary 0.5 of [Nee18a] gives these statements without any extra assumptions on on $R$ (and $X$).

4. The proof of Theorem 1.5(1) and of the self-injective version of part 3 originates from the proof of [BVd03] Theorem A.1].

We also prove some more statements of this sort; see Theorem 3.2.5 below.

Remark 1.7. 1. Recall that semi-orthogonal decompositions of certain derived categories of (quasi)coherent sheaves are important for non-commutative geometry.

Note also that Theorem 1.3(1,2) gives more non-trivial statements in the case where $D_{perf}(X) \neq D^{b}(X)$, that is, if $X$ is singular; cf. Remark 3.3.3(2) below. This case is somewhat less popular than the regular one. Yet some non-trivial semi-orthogonal decompositions of $D^{b}(X)$ for a possibly singular $X$ are provided by Theorem 6.7 and Corollary 6.10 of [BeS20]. Moreover, semi-orthogonal decompositions in the case where $X$ is a singular surface are discussed in detail in [KKSS22].

Thus the "geometric" results of the current paper appear to be relevant.

2. An alternative version of this paper is available as [Bon23]. It is more self-contained than the current version and some of the proofs are more detailed; yet several remarks are omitted.

Let us now describe the contents of the paper. Some more information of this sort may be found in the beginnings of sections.

In §2 we give some basic definitions (mostly) related to semi-orthogonal decompositions of triangulated categories, and prove several simple properties of these decompositions.

In §3 we prove an abstract Theorem 3.1.1 on the existence of certain semi-orthogonal decompositions. We use it to prove Theorem 1.3 (see Theorem 3.2.7 that is formulated in the language of "binary" semi-orthogonal decompositions), to deduce the easy Corollary 3.1.3 on the existence of adjoint functors (cf. Remark 3.2.6)), and to study certain "support subcategories" of $D_{Q}$. Moreover, we prove Theorem 1.5(1).

In §4 we finish the proof of Theorem 1.5. We deduce its second part from a general theorem on Neeman-type approximability in triangulated categories. Next we study semi-orthogonal decompositions that (may) consist of more than two components. We also discuss the relation of our arguments to (adjacent) weight structures and $t$-structures; those were studied in several preceding papers of the author.

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2 Preliminaries

In this section we discuss simple notions related to triangulated categories and semi-orthogonal decompositions.

In §2.1 we recall some definitions and statements related to triangulated categories.

In §2.2 we define and study ("binary") semi-orthogonal decompositions.

2.1 A few definitions and statements

• For categories $C', C$ we write $C' \subset C$ if $C'$ is a subcategory of $C$; recall that we only consider strictly full subcategories in this paper.

• The symbol $T$ below will always denote some triangulated category. The symbols $D, LO, RO$ (possibly, endowed with indices) will also be used for triangulated categories only.

• For a class $P \subset \text{Obj} T$ we call a class $P' \subset \text{Obj} T$ the extension-closure of $P$ if $P'$ is the smallest class of objects of $T$ that contains $P \cup \{0\}$ and such that for a $T$-distinguished triangle $A \to C \to B \to A[1]$ we have $C \in P'$ whenever $A, B \in P'$.

• Given a distinguished triangle $X \to Y \to Z$ we will write $Z = \text{Cone}(f)$; recall that $Z$ is determined by $f$ up to a non-canonical isomorphism.

• All coproducts in this paper will be small.

We will also need the following well-known definitions.

Definition 2.1.1. Let $D$ be a triangulated category closed with respect to (small) coproducts.

1. An object $M$ of $D$ is said to be compact (in $D$) if the functor $D(M, \cdot) : D \to \text{Ab}$ respects coproducts.

2. We will say that a (triangulated) subcategory $T$ of $D$ compactly generates $D$ whenever $T$ is essentially small, its objects are compact in $D$, and $D = T^\perp$ (see Definition 1.1(5)).

The following statements are mostly simple and well-known.

Lemma 2.1.2. Let $LO$ and $RO$ be (strictly full) triangulated subcategories of $T$. Take $C$ to be the class of those $M \in \text{Obj} T$ such that there exists a distinguished triangle $L \to M \to R \to L[1]$ with $L \in LO$ and $R \in RO$.

1. If $LO \perp RO$ then $C$ gives a triangulated subcategory of $T$ as well.

2. If $T$ contains all (small) coproducts of its objects, and $LO$ and $RO$ are closed with respect to $T$-coproducts then $C$ is closed with respect to $T$-coproducts as well.

3. If $T$ is closed with respect to coproducts then $LO^T = (LO^T)^T$.

4. If $T$ is closed with respect to coproducts, $LO$ is essentially small and consists of compact objects, and $LO^T = \{0\}$, then $LO$ compactly generates $T$. 

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Proof. Assertions 1 and 2 easily follow from Proposition 2.1.1(1,2) of [BoS19]; note (for assertion 1) that $C[1] = C$.

Assertion 3 is very easy; note that for any object $N$ of $\mathcal{T}$ the class $\perp_\mathcal{T}\{N\}$ is closed with respect to $\mathcal{T}$-coproducts.

Assertion 4 is given by Proposition 8.4.1 of [Nee01]. □

2.2 Semi-orthogonal decompositions: basics

Let us give some more of our central definitions.

Definition 2.2.1. Assume that $\mathcal{T}$ and $\mathcal{T}'$ are triangulated subcategories of a triangulated category $\mathcal{D}$.

1. Let $\mathcal{LO}$ and $\mathcal{RO}$ be (strictly full) triangulated subcategories of $\mathcal{T}$.

We will say that the couple $D = (\mathcal{LO}, \mathcal{RO})$ is a *semi-orthogonal decomposition* of $\mathcal{T}$ (or just gives a decomposition of $\mathcal{T}$) if $\text{Obj} \mathcal{LO} \perp \text{Obj} \mathcal{RO}$ and for any $M \in \text{Obj} \mathcal{T}$ there exists a distinguished triangle

$$L \to M \to R \to L[1] \ (2.2.1)$$

with $L \in \mathcal{LO}$ and $R \in \mathcal{RO}$.

2. The couple $D' = D_{\mathcal{D}'} = (\mathcal{LO}_{\mathcal{D}'}, \mathcal{RO}_{\mathcal{D}'})$ (see Definition 1.4(2)) is said to be $\mathcal{D}$-adjacent to $D$ if $D'$ is a decomposition of $\mathcal{T}'$.

3. If $D_D = (\mathcal{LO}_D, \mathcal{RO}_D)$ is a semi-orthogonal decomposition of $\mathcal{D}$ and $D_D \cap \mathcal{T}' = (\mathcal{LO}_D \cap \mathcal{T}', \mathcal{RO}_D \cap \mathcal{T}')$ (see Definition 1.1(2)) is a semi-orthogonal decomposition of $\mathcal{T}'$ then we say that $D_D$ restricts to $\mathcal{T}'$, $D_D \cap \mathcal{T}'$ is the corresponding restriction, and $D_D$ is an extension of $D_D \cap \mathcal{T}'$ to $\mathcal{D}$.

4. We will write $D_1 \leq D_2$ if $D_i = (\mathcal{LO}_i, \mathcal{RO}_i) \ (i = 1, 2)$ are semi-orthogonal decompositions of $\mathcal{T}$ and $\mathcal{LO}_1 \subset \mathcal{LO}_2$.

Remark 2.2.2. 1. Clearly, $D'$ is $\mathcal{D}$-adjacent to $D$ if and only if it is $\mathcal{D}'$-adjacent to it, where $\mathcal{D}'$ is any triangulated subcategory of $\mathcal{D}$ that contains both $\mathcal{T}$ and $\mathcal{T}'$.

2. Semi-orthogonal decompositions described in Definition 2.2.1(1) may be called the "binary" ones. We postpone the more general "multiple" decompositions and their properties till §4.2. The reason for this is that these more general decompositions do not help in proving anything like Theorem 1.3.

On the other hand, binary semi-orthogonal decompositions are important for our proofs even though Theorems 2.2.1 below contains just a little more information than the corresponding "admissible statements" in Theorem 1.3.

3. We will discuss some predecessors of Definition 2.2.1(2) in Remark 1.3(1) below.

Proposition 2.2.3. 1. Assume that $D = (\mathcal{LO}, \mathcal{RO})$ is a semi-orthogonal decomposition of $\mathcal{T}$.

Then $\mathcal{LO}_{\mathcal{T}} = \mathcal{RO}$, $\mathcal{LO} = \mathcal{T}\mathcal{RO}$, and there exists an exact right adjoint $R_D$ to the embedding $\mathcal{LO} \to \mathcal{T}$ and a left adjoint $L_D$ to the embedding $\mathcal{RO} \to \mathcal{T}$.

Moreover, the triangle (2.2.1) is functorially determined by $M$, and the arrows $L \to M \to R$ in it come from the transformations corresponding to the aforementioned adjunctions.

2. Respectively, the correspondence $D \mapsto \mathcal{RO}$ (resp. $D \mapsto \mathcal{LO}$) gives a bijection of between the class of semi-orthogonal decompositions of $\mathcal{T}$ and that of left (resp. right) admissible subcategories of $\mathcal{T}$; see Definition 1.1(1).
Proof. These are well-known statements. They are mostly contained in Lemmata 2.5 and 2.3 of [Kuz11], yet invoke Proposition 1.3.3. of [BBD82] (along with its proof; the relation of $t$-structures mentioned loc. cit. to semi-orthogonal decompositions is discussed in §4.3 below) for the calculation of the arrows in (2.2.1).

Let us prove some more properties of our notions.

Proposition 2.2.4. Assume $\mathcal{T}, \mathcal{T}' \subset \mathcal{D}$, and $D$ (resp. $D'$) is a semi-orthogonal decomposition of $\mathcal{T}$ (resp. $\mathcal{T}'$).

I. If $D' = D_2$, (see Definition 1.1(1)) and $\mathcal{D}$ is $R$-linear (see Definition 1.1(6)) then the following bi-functors $\mathcal{T}^{op} \times \mathcal{T}' \to R$–$\text{Mod}$ are canonically isomorphic: $\mathcal{D}(L_D(-), -) \cong \mathcal{D}(\mathcal{D}(R_D(-)), -) \cong \mathcal{D}(\mathcal{D}(R_D(-)), -)$.

II. Assume that $D'_1 = D_1$ and $D'_2 = D_2$, where $D_i = (\mathcal{LO}_i, \mathcal{RO}_i)$ ($i = 1, 2$) are semi-orthogonal decompositions of $\mathcal{T}$.

1. Then $D_1 \leq \mathcal{LO} D_2$ if and only if $\mathcal{RO}_2 \subset \mathcal{RO}_1$. Moreover, if these conditions are fulfilled then $D'_2 \leq \mathcal{LO} D'_1$.

2. Suppose that $\mathcal{T} \subset \mathcal{T}'$. Then all the conditions in assertion II.1 are equivalent.

Proof. I. This statement easily follows from Proposition 2.5.4(1) of [Bon10b]; see Remark 4.3.1(2) below for more detail.

II.1. Obvious from our definitions along with Proposition 2.2.3(1).

2. We assume $D'_1 \leq \mathcal{LO} D'_1$; it suffices to prove that $\mathcal{RO}_2 \subset \mathcal{RO}_1$. Now, $\mathcal{LO}'_1 = (\mathcal{LO}_1)_D \cap \mathcal{T}'$ and $\mathcal{RO}_i = (\mathcal{LO}_1)_D \cap \mathcal{T}$ (for $i = 1, 2$; see Proposition 2.2.3(1) once again); hence $\mathcal{RO}_1 = \mathcal{LO}'_1 \cap \mathcal{T}$. Thus $\mathcal{RO}_2 \subset \mathcal{RO}_1$ indeed.

Remark 2.2.5. In all the examples that we consider in this paper the decompositions mentioned in Proposition 2.2.3(1) extend to two adjacent decomposition of a ("big enough") category $\mathcal{D}$; see Theorem 3.1.1(2) below (and Remark 2.2.4)). Now, if $\mathcal{T} = \mathcal{T}' = \mathcal{D}$ then for $M, N \in \text{Obj} \mathcal{D}$ we have bi-functorial isomorphisms $\mathcal{D}(L_D(M), N) \cong \mathcal{D}(L_D(M), R_{D'}(N)) \cong \mathcal{D}(M, R_{D'}(N))$ that come from the corresponding adjunctions.

3 Main results

In §3.1 we prove our main abstract Theorem 3.1.1 on the existence of certain $\mathcal{D}$-adjacent semi-orthogonal decompositions. We also deduce a simple Corollary 3.1.3 on the existence of adjoint functors.

In §3.2 we apply our general results to semi-orthogonal decompositions of various subcategories of $D(\text{Qcoh}(X))$ (where $X$ is proper over the spectrum of a ring $R$); this yields a "geometric" Theorem 3.2.7 on semi-orthogonal decompositions in $D_{\text{perf}}(X), D'_\text{coh}(X), D_{\text{perf}}(X), D'_\text{coh}(\text{Qcoh}(X))$, and $D_{\text{coh}}(\text{Qcoh}(X))$. Moreover, Corollary 3.2.3 says that these $(D(\text{Qcoh}(X))$-adjacent) decompositions can be restricted to certain "support subcategories" of the corresponding categories (that correspond to unions of closed subsets of $S = \text{Spec } R$).

In §3.3 we prove Theorem 1.5(1). We also apply Grothendieck duality arguments to establish the "regular" case of Theorems 1.3 and 3.2.7 and relate semi-orthogonal decompositions to duality; see Corollary 3.3.2.
3.1 Abstract decomposition statements

Now we study certain decompositions coming from semi-orthogonal decompositions of categories of compact objects. We will use much of Definitions 1.1 and 2.1.1

**Theorem 3.1.1.** Assume that $\mathcal{D} = \mathcal{T}^\perp$, where $\mathcal{T} \subset \mathcal{D}$ is a triangulated subcategory whose objects are $\mathcal{D}$-compact, and $\mathcal{D} = (\mathcal{LO}, \mathcal{RO})$ is a semi-orthogonal decomposition of $\mathcal{T}$.

1. Then the couple $D^\perp$ is a semi-orthogonal decomposition of $\mathcal{D}$.
2. Assume in addition that $\mathcal{T}$ is essentially small (respectively, $\mathcal{D}$ is compactly generated by it). Then $D^\perp \mathcal{D}$ is a semi-orthogonal decomposition of $\mathcal{D}$ as well. Moreover, $D^\perp \mathcal{D}$ is also $\mathcal{D}$-adjacent to $D^\perp \mathcal{D}$, and $LO^\perp \mathcal{D} = RO^\perp \mathcal{D}$.
3. Assume that $T_0 \mathcal{D}$ is a subcategory of $\mathcal{D}$ such that $D^\perp \mathcal{D}$ restricts to a semi-orthogonal decomposition $D_0$ on it (see Definition 2.2.1(3)).

Then $D^\perp \mathcal{D}$ restricts to the category $T^0 \mathcal{D}$ as well, and this restriction equals $D_0^\perp \mathcal{D}$.

II. Assume that $R$ is a commutative unitial ring, $\mathcal{D}$ is $R$-linear, and $A$ is an exact abelian subcategory of the category $R$–$\text{Mod}^\mathbb{Z}$ of $\mathbb{Z}$-graded $R$-modules that is stable with respect to obvious shifts on this category (that is, $M = \bigoplus M^i$ belongs to $A$ if and only if the module $M[1] = \bigoplus M^{i+1}$ does). Take the following (full) subcategory $T^0_A$ of $\mathcal{D}$: $N \in \text{Obj} \mathcal{D}$ is an object of $T^0_A$ whenever for any $M \in \text{Obj} \mathcal{T}$ the graded module $S_N(M) = \bigoplus_i \mathcal{D}(M[-j], N)$ belongs to $A$.

1. Then $T^0_A$ is triangulated and the functors $L_{D^\perp}$ and $R_{D^\perp}$ corresponding to $D_{\mathcal{D}}$ (see Proposition 2.2.3(1) and assertion I.2) send $T^0_A$ into itself.
2. Consequently, the couple $D^\perp_A$ is a decomposition of $T^0_A$.

**Proof.** I.1. Since objects of $\mathcal{LO}$ are compact in $\mathcal{D}$, the class $\mathcal{LO}^\perp \mathcal{D}$ is closed with respect to coproducts. Since it contains $\mathcal{RO}$, $\mathcal{LO} \perp \mathcal{RO} \mathcal{D}$ as well. Applying Lemma 2.1.2(3) we obtain that $\mathcal{LO} \mathcal{D} \perp \mathcal{RO} \mathcal{D}$.

It remains to prove the existence of decompositions of the type (2.2.1). Take the set $E$ of those $M \in \text{Obj} \mathcal{D}$ such that there exists a distinguished triangle $L \to M \to R \to M[1]$ with $L \in \mathcal{LO}^\perp \mathcal{D}$ and $R \in \mathcal{RO} \mathcal{D}$. Clearly contains $\text{Obj} \mathcal{T}$ and applying Lemma 2.1.2(1,2) we obtain $E = \text{Obj} \mathcal{D}$.

2. Corollary 2.4 of [NIS09] easily implies that $D^\perp$ is a semi-orthogonal decomposition of $\mathcal{D}$ indeed. Is is obviously $\mathcal{D}$-adjacent both to $\mathcal{D}$ and to $D^\perp \mathcal{D}$.

3. Since all objects of $\mathcal{LO}$ and $\mathcal{RO}$ are compact, both $\mathcal{LO}_{D^\perp}$ and $\mathcal{RO}_{D^\perp}$ are closed with respect to $\mathcal{D}$-coproducts. Hence if $D_0 = (\mathcal{LO}_0, \mathcal{RO}_0)$ then $\mathcal{LO}^\perp \mathcal{D} \perp \mathcal{RO}^\perp \mathcal{D}$.

Consequently, it suffices to verify that the class $C_{0}$ of extensions of elements of $\mathcal{RO}^\perp \mathcal{D}$ by those of $\mathcal{LO}^\perp \mathcal{D}$ coincides with $\text{Obj} \mathcal{T}_{0}^\perp \mathcal{D}$; see Proposition 2.2.3(1). The latter statement easily follows from Lemma 2.1.2(1,2) similarly to the proof of assertion I.1.

II.1. Since the functor $\mathcal{D}(-, N) (N \in \text{Obj} \mathcal{D})$ sends $\mathcal{T}$-distinguished triangles into long exact sequences (one may also say that $S_N$ sends distinguished triangles into triangles of a certain sort), $T^0_A$ is triangulated. Next, Proposition 2.2.4(1) implies that for any $M \in \text{Obj} \mathcal{T}$ we have $S_{L_{D^\perp}(N)}(M) \simeq S_{N}(R_{D}(M))$ and
Since the objects $R_D(M)$ and $L_D(M)$ belong to $\mathcal{T}$, we obtain that $L_{D^\perp}(N)$ and $R_{D^\perp}(N)$ belong to $\mathcal{T}_A$ whenever $N$ does.

2. We should check that any object $M$ of $\mathcal{T}_A$ possesses a $D^\perp$-decomposition (2.2.1) inside $\mathcal{T}_A$. This statement follows from assertion II.1 according to Proposition 2.2.3(1).

\[ \tag*{\square} \]

Remark 3.1.2. 1. Below we will apply our theorem in the following setting only: $R$ is a (commutative unital) Noetherian ring and $\mathcal{A}$ consists of those modules whose components are finitely generated and satisfy some boundedness condition; see Definition 3.2.3.

Note however that one can take finitely presented modules over a coherent ring instead; see Definition 2.1, Theorem 2.4, Corollary 2.7, and Lemma 2.8 of [Swa19]. In particular, it appears that the arguments below that we use for the proof of Theorem 1.5(1) generalize to this setting without much difficulty.

2. Another possible generalization here is to fix an infinite cardinal $\aleph_\bullet$ and take $\mathcal{A}$ that consists of those $M = \bigoplus M^i$ such that each $M^i$ has less than $\aleph_\bullet$ generators over $R$. This gives a certain "smallness" filtration on $\mathcal{D}$, which is clearly exhaustive (since $\mathcal{T}$ is essentially small).

We will also consider certain type of $\mathcal{A}$ related to support sets $T \subset S = \text{Spec } R$ in Corollary 3.2.9 below.

3. It is also easily seen that for any $N$ as above the semi-orthogonal decomposition $D^\perp$ restricts to the smallest triangulated subcategory of $\mathcal{D}$ that it closed with respect to coproducts of less than $\aleph_\bullet$ objects and contains $\mathcal{T}$. Yet the corresponding decompositions seem to be less interesting in the "geometric" setting that we will consider below. Moreover, the author does not know of any arguments that would allow to combine these cardinality restrictions with any bounds on the degree.

4. In our theorem we send an object $N$ of $\mathcal{D}$ into the class $\{S_N(M)\}$ of graded $R$-modules (where $M$ runs through objects of $\mathcal{T}$). Clearly, it is possible to get "more information" from the functor represented by $N$, and this may give descriptions of a larger class of triangulated subcategories of $\mathcal{D}$ that it closed with respect to coproducts of less than $N$ objects and contains $\mathcal{T}$. Yet the corresponding decompositions seem to be less interesting in the "geometric" setting that we will consider below. Moreover, the author does not know of any arguments that would allow to combine these cardinality restrictions with any bounds on the degree.

5. Assertion I.1 is possibly well-known. Note also that in the "geometric" setting that we will study below this statement essentially coincides with Proposition 4.2 of [Kuz11].

Theorem 3.1.1 easily yields the existence of certain adjoint functors.

Corollary 3.1.3. Adopt the assumptions of Theorem 3.1.1(1). Assume that $F : \mathcal{D} \rightarrow \mathcal{D}'$ is an exact functor that respects coproducts and suppose that $\mathcal{T} \subset \mathcal{T}_A$.

Let $T$ be a subcategory of $\mathcal{D}'$ that contains the essential image $F(\mathcal{T}_A)$. For any objects $M$ of $\mathcal{D}$ and $N$ of $\mathcal{D}'$ endow the abelian group $\mathcal{D}'(F(M),N)$ with the structure of $R$-module as follows: define the multiplication by $r \in R$ on $\mathcal{D}'(F(M),N)$ by means of composing its elements with $F(r \text{id}_M)$.
Then the restriction of $F$ to a functor $F_T : \mathcal{T}_A \to T$ possesses a right adjoint if and only if for any $M \in \text{Obj } \mathcal{T}$ and $N \in \text{Obj } \mathcal{T}$ the graded module

$$S'_N(F(M)) = \bigoplus_{i \in \mathbb{Z}} \mathcal{D}'(F(M[-i]), N)$$

belongs to $\mathcal{A}$. Moreover, this adjoint is exact if $T$ is a triangulated subcategory of $\mathcal{D}'$.

**Proof.** If $F_T$ possesses an adjoint functor $G_T$ then for any $M \in \text{Obj } \mathcal{T}_A$ and $N \in \text{Obj } T$ we have $S'_N(F(M)) \cong \bigoplus_{i \in \mathbb{Z}} S_{G_T(N)}(M)$ (cf. Theorem 3.1.1(I.1)), and this isomorphism is clearly an isomorphism of $R$-modules. Since $T \subset \mathcal{T}_A$ we obtain that the graded module $S'_N(F(M))$ belongs to $\mathcal{A}$ whenever $M \in \text{Obj } \mathcal{T}$.

Let us prove the converse implication. Since $\mathcal{D}$ is compactly generated, the functor $F$ is well known to possess an exact right adjoint $G$; see Theorems 8.3.3 and 8.4.4 and Lemma 5.3.6 of [Nee01]. Thus it suffices to verify that $G$ sends $T$ inside $\mathcal{T}_A$. Now, for any $N \in \text{Obj } T$ if $M$ belongs to $T$ then $S_{G_T(N)}(M) \cong S'_N(F(M))$; hence $G(N)$ belongs to $\mathcal{T}_A$ indeed.

We also describe an argument that can be used to restrict semi-orthogonal decompositions of Theorem 3.1.1(I) to "large enough" subcategories of $\mathcal{T}_A$. We will not apply it in this paper.

**Proposition 3.1.4.** Adopt the assumptions and notation of Theorem 3.1.1(I).

Moreover, suppose that $\mathcal{T}'$ is a triangulated subcategory of $\mathcal{T}_A$ such that the restricted Yoneda functor $\mathcal{Y}_{\mathcal{T}'} : \mathcal{T}' \to \text{Fun}_R(\mathcal{T}, R - \text{Mod})$ (see Definition 1.1(9)) that sends $N \in \text{Obj } \mathcal{T}'$ into the restriction of $\mathcal{D}(\cdot, N)$ to $\mathcal{T}$ satisfies the following conditions: it is full and its essential image coincides with the image of the (similarly defined) restricted Yoneda functor $\mathcal{Y}_{\mathcal{T}_A} : \mathcal{T}_A \to \text{Fun}_R(\mathcal{T}, R - \text{Mod})$.

Then $\mathcal{D}_{\mathcal{T}_A}$ is a decomposition of $\mathcal{T}'$.

**Proof.** For an object $N$ of $\mathcal{T}' \subset \mathcal{T}_A$, we take its $D_{\mathcal{T}_A}$-decomposition triangle (see 2.2.1 and Theorem 3.1.1(I.2)): $L \xrightarrow{h} N \to R \to L[1]$.

According to our assumptions, we can choose $L' \in \text{Obj } \mathcal{T}'$ and $h' \in \mathcal{T}'(L', N)$ such that $\mathcal{Y}_{\mathcal{T}'}(h') \cong \mathcal{Y}_{\mathcal{T}_A}(h)$ (in the category of objects over $\mathcal{Y}_{\mathcal{T}_A}(N)$). Clearly, $L' \in \mathcal{L}O_{\mathcal{T}_A}$. It remains to verify that $\text{Cone}(h') \in \mathcal{RO}_{\mathcal{T}_A}$.

If $M \in \mathcal{R}O$ then $M[i] \perp R$ for any $i \in \mathbb{Z}$; hence the homomorphisms $\mathcal{D}(M[i], h) \cong \mathcal{D}(M[i], h')$ are bijective. Looking at the exact sequence

$$\mathcal{D}(M, L') \to \mathcal{D}(M, N) \to \mathcal{D}(M, \text{Cone}(h')) \to \mathcal{D}(M[1], L') \to \mathcal{D}(M[1], N)$$

we obtain that $M \perp \text{Cone}(h')$. Thus $h$ can be completed to a $D_{\mathcal{T}_A}$-decomposition triangle for $N$.

**Remark 3.1.5.** This statement may be combined with Corollary 0.5 of [Nee18a] to obtain a "substitute" for Theorem 4.1.1 below (cf. Remark 4.1.2) that would be sufficient to establish Theorem 3.2.7(I.1).

However, the author does not have any "interesting" examples for this proposition, that is, with $\mathcal{T}'$ distinct from $\mathcal{T}_A$ (recall that $\mathcal{T}'$ is a strict subcategory).
3.2 Main geometric applications

Till the end of the paper we will always assume that the following condition is fulfilled.

**Assumption 3.2.1.** $R$ is a commutative unital Noetherian ring and $X$ is a scheme proper over $S = \text{Spec } R$.

In some of the statements we will also need the following "very common" condition on $X$.

**Assumption 3.2.2.** Regular alterations (see Remark 3.2.8(1) below) exist for all integral closed subschemes of $X$.

**Definition 3.2.3.** We will write $\mathcal{A}^u$ (resp. $\mathcal{A}^+$) for the following subcategories of $\mathcal{A} = \mathcal{R}\text{-mod}$ (see Definition 1.1(6)):

\[ \mathcal{M} = \bigoplus_{M_i \in \text{Obj } \mathcal{A}} \text{ for } i \gg 0 \text{ (resp. for } i \ll 0). \]

Moreover, $\mathcal{A}^b$ is the subcategory corresponding to $\text{Obj } \mathcal{A}^u \cap \text{Obj } \mathcal{A}^+$.

Let us describe some examples for the assumptions of Theorem 3.1.1. We should recollect some statements from stacks that allow us to apply the results of ibid. to various categories of coherent sheaves.

**Remark 3.2.4.** The "main" derived categories of stacks are the derived categories of $\mathcal{O}_X$-modules. However, $D(\mathcal{O}_X)$ contains a full triangulated subcategory $D_{\text{Qcoh}}(\mathcal{O}_X)$ consisting of those complexes whose cohomology is quasi-coherent.

Now, if $X$ is Noetherian then the obvious functor $D_\mathcal{O} = D(\mathcal{O}_X) \to D_{\text{Qcoh}}(\mathcal{O}_X)$ is an equivalence; see [stacks, Tag 09T4]. It clearly follows that $D_{\text{coh}}(\mathcal{O}_X) \cong D_{\text{coh}}(\mathcal{O}_X)$ (cf. Definition 1.1(7) or stacks, Tag 06UP for categorical notation of this sort).

Moreover, the direct and inverse image functors (that is, $f_* : D(\mathcal{O}_X) \cong D(\mathcal{O}_Y) : f^*$ for a quasi-separated and quasi-compact morphism $f : X \to Y$ of schemes) and tensor products respect the subcategories $D_{\text{Qcoh}}(\mathcal{O}_-) \subset D(\mathcal{O}_-)$; see stacks Tags 08DW, 08D5 08DX. These observations allow us to apply results of ibid. to the categories $D(\mathcal{O}_X)$ and their subcategories mentioned in Definition 1.1(7) instead of $D_{\text{Qcoh}}(\mathcal{O}_-) \subset D(\mathcal{O}_-)$ and the corresponding triangulated subcategories that are defined in terms of cohomology of complexes of sheaves of modules (similarly to Definition 1.1(7)).

**Theorem 3.2.5.** I. Take $\mathcal{T} = \mathcal{D}_p$ and $\mathcal{D} = \mathcal{D}_Q$ (see Definition 1.1(7)).

1. Then $\mathcal{D}$ is compactly generated by $\mathcal{T}$.
2. Let the symbol $s$ be equal to $u, +, -,$ or $b$. Then we have $\mathcal{D}^s \subset \mathcal{T}^{\Delta^s}$.
3. Moreover, this inclusion is an equality if either $X$ is projective over $S$ (in the "weak" sense specified in Theorem 1.5(1)) or if $s \in \{b, -\}$.

II. Take $\mathcal{D}$ to be the mock homotopy category $K_m(\text{Proj } X)$ of projectives over $X$ as defined in [Mur07, Definition 3.3], and $\mathcal{T}$ to be the essential image of $\mathcal{D}^{\text{op}}$ under the functor $(-)^\circ U_\lambda$.

1. Then $\mathcal{D}$ is compactly generated by $\mathcal{T}$ and the restriction of $(-)^\circ U_\lambda$ to $\mathcal{D}^b$ is a full embedding.
2. If $X$ satisfies Assumption 3.2.2 then the corresponding category $\mathcal{T}^{\Delta^s}$ equals the essential image $(-)^\circ U_\lambda(\mathcal{D}_p^{\text{op}})$. 12
Proof. I.1. Since \( R \) is noetherian, \( X \) is a noetherian separated scheme; thus the compact generation statement is well-known (see \[stacks\] Tags 09M1, 09I[S] along with Remark \[3.2.3\] (or Theorem 3.1.1 of \[BVd03\]) and Lemma \[2.1.2\].)

2. This statement is easy and probably well-known; our argument in the proof Theorem \[1.5\] (I.1) (in \[3.3\] below) actually establishes it in this generality as well.

3. In the case where \( p \) is projective the assertion is just a re-formulation of Theorem \[1.5\] (I.1).

In the cases \( s = b \) and \( s = - \) Corollary 0.5 of \[Nee18a\] implies the following: for any \( N \in \text{Obj} \mathcal{T}_A \) there exists \( N' \in \text{Obj} \mathcal{D}' \) such that \( \mathcal{Y}(N) \cong \mathcal{Y}(N') \); here we use the notation of Theorem \[1.5\] (II). To prove that \( N \cong N' \) one can either apply some more results of ibid. (see Remark \[4.1.2\] below) or use Theorem \[4.1.1\].

II.1. These statements are given by Theorems 4.10 and 7.4 of \[Mur07\]; see also Proposition 7.2 of ibid. for the notation. Also Proposition 7.2 of ibid. for the notation.

Remark 3.2.6. 1. Clearly, one can combine Corollary \[3.1.3\] with Theorem \[3.2.5\] (I.3) to obtain an if and only if criterion for the existence of a right adjoint to the corresponding restriction \( F_T : \mathcal{D}' \to \mathcal{T} \), where \( F : \mathcal{D}' \to \mathcal{D} \) is an exact functor that respects coproducts, \( F(\mathcal{D}') \subset \mathcal{T} \), and \( (X, s) \) is any couple that satisfies the assumptions of Theorem \[3.2.5\] (I.3).

2. One can also construct a category \( \mathcal{D} \) that satisfies the assumptions of Theorem \[3.2.5\] (II) using certain abstract nonsense; see Corollary 3.7 of \[Kel06\] and Proposition 4.2.5 of \[Bon23\]. However, the author conjectures that all possible choices of \( \mathcal{D} \) that satisfy the conditions of our theorem are equivalent.

Now we pass to semi-orthogonal decompositions; see Definitions \[2.2.1\] and \[1.1\].

**Theorem 3.2.7.** I.1. For any semi-orthogonal decomposition \( \mathcal{D} \) of \( \mathcal{D}_p \) the couples \( D_{\mathcal{D}_p}, D_{\mathcal{D}_{\mathcal{D}+}}, \) and \( D_{\mathcal{D}_Q} \) give semi-orthogonal decompositions of \( \mathcal{D}'_p, \mathcal{D}'_{\mathcal{D}+}, \) and \( \mathcal{D}'_Q \), respectively.

Moreover, \( D_{\mathcal{D}_Q}^\perp = (D_{\mathcal{D}_Q})^\perp \); consequently, \( D_{\mathcal{D}^+}^\perp = (D_{\mathcal{D}^+})^\perp \).

Furthermore, if \( X \) is projective over \( S \) (see Definition \[1.13\]) then \( D_{\mathcal{D}_p}^\perp \) is a decomposition of \( \mathcal{D}'^\perp \) and \( D_{\mathcal{D}^+}^\perp \) is a decomposition of \( \mathcal{D}'^+ \).

2. The maps \( D \to D_{\mathcal{D}_p}^\perp, D \to D_{\mathcal{D}^+}^\perp, D \to D_{\mathcal{D}_Q}^\perp \) are injective, and the following assumptions on semi-orthogonal decompositions \( D_1 \) and \( D_2 \) of \( \mathcal{D}_p \) are equivalent:

\( a \) \( D_1 \leq \mathcal{L} \cap D_2; \)
\( b \) \( D_2^\perp \leq \mathcal{L} \cap (D_1)^\perp; \)
\( c \) \( D_2^\perp \leq \mathcal{L} \cap (D_1)^\perp; \)
\( d \) \( (D_2)^\perp \leq \mathcal{L} \cap (D_1)^\perp. \)

II. Suppose in addition that \( X \) is either regular or satisfies Assumption 3.2.2.
1. Then the map \( D \mapsto D^b_{\mathcal{P}} \) gives all semi-orthogonal decompositions of \( D^b \), and the inverse correspondence is of the form \( E \mapsto \frac{1}{D} E \).

Consequently, the couple \( E^{\mathcal{U}} \) gives a semi-orthogonal decomposition of \( D_Q \) that coincides with \( (\downarrow E)^{\frac{1}{D_Q}} \), and this decomposition restricts to the semi-orthogonal decomposition \( (\downarrow E)^{\frac{1}{D}} \) of \( D^\perp \).

Moreover, if \( X \) is projective over \( S \) then \( E^{\mathcal{U}} \) restricts to \( D^\perp \) and \( D^\perp \).

2. The map \( \mathcal{E} \mapsto \mathcal{E} \cap \mathcal{D}_p \) gives a one-to-one correspondence between right admissible subcategories of \( D^b \) and left admissible subcategories of \( \mathcal{D}_p \).

**Proof.** I.1. Combining Theorem 3.1.1(I,II.2) with Theorem 3.2.5 we immediately obtain that \( D^b_{\mathcal{P}} \), \( D^b_{\mathcal{P}} \), and \( D^b_{\mathcal{Q}} \) give semi-orthogonal decompositions of the corresponding categories, indeed. This is also true for \( D^b_{\mathcal{P}} \) and \( D^b_{\mathcal{P}} \), whenever \( X \) is projective over \( S \).

Next, recall that \( D_p \subset D^b \subset D_Q \). Hence \( D^b_{\mathcal{Q}} = D^b_{\mathcal{Q}} \) (see Theorem 3.2.5(I.1)); thus \( D^b_{\mathcal{Q}} = (D^b_{\mathcal{Q}})^{\mathcal{U}} \) by Theorem 3.1.1(I.3). It clearly follows that \( D^b_{\mathcal{Q}} = (D^b_{\mathcal{Q}})^{\mathcal{U}} \).

2. Immediate from Proposition 2.2.4(II).

II.1. We claim that for any semi-orthogonal decomposition \( E \) of \( D^b \), the couple \( \mathcal{E} (\mathcal{E} \cap \mathcal{D}_p ) \) (resp. \( \mathcal{E} (\mathcal{E} \cap \mathcal{D}_p ) \)) gives a bijection between the class of right admissible subcategories of \( D_p \) and the class of left admissible subcategories of \( \mathcal{D}_p \) (resp. of right admissible subcategories of \( D^b \)). We conclude the proof by noting that \( \mathcal{E} (\mathcal{E} \cap \mathcal{D}_p ) = \mathcal{E} \cap \mathcal{D}_p \).

**Remark 3.2.8.** 1. Recall that alterations were introduced in [dJo96]; regular alterations generalize Hironaka’s resolutions of singularities. Being more precise, a regular alteration for a scheme \( Z \) is a proper surjective morphism \( Y \rightarrow Z \) that is generically finite and such that \( Y \) is regular and finite dimensional.

Since resolutions of singularities exist for arbitrary quasi-excellent \( \text{Spec} \mathbb{Q} \)-schemes according to Theorem 1.1 of [Tem08], part II of our proposition can be applied whenever \( R \) is a quasi-excellent noetherian \( \mathbb{Q} \)-algebra. Moreover, Assumption 3.2.7 is fulfilled whenever \( X \) is of finite type over a scheme \( B \) that is quasi-excellent of dimension at most 3; see Theorem 1.2.5 of [Tem17].

2. It appears to be no easy way to prove that semi-orthogonal decompositions of \( D^b \) extend to \( D^\perp \subset D^\perp \) (cf. part II.1 of our theorem).

Moreover, the only non-trivial (cf. Remark 3.3.3(2) below) extension statement of this sort known to the author is Theorem 6.2 of [HoS20] which relies on certain "geometric" assumptions on the initial decomposition.

3. Recall also that Corollary 1.12 of [Oh06] treats decompositions of \( D^b \) whose components are admissible in the sense of Definition 3.1.1. This additional restriction allows to apply arguments (in the proof of Proposition 1.10 of ibid.) that are rather similar to our ones (but avoid "auxiliary" categories)
to obtain that decompositions of this sort restrict to $\mathcal{D}_p$. It is worth noting that Proposition 1.10 and Corollary 1.12 of ibid. extend to our $R$-linear setting without any difficulty.

Note also that the arguments of the current paper were not inspired by ibid.; cf. §4.3 below for more detail on this matter.

Let us now consider certain support subcategories.

**Corollary 3.2.9.** Let $\mathcal{B}$ be an exact abelian subcategory of $\mathcal{R} - \text{Mod}$; adopt the notation and assumptions of Theorem 3.2.5(I.3).

1. Then for any decomposition $D$ of $\mathcal{D}_p$ the couple $D^s_{\mathcal{B}^s}$ gives a decomposition of the corresponding category $\mathcal{T}_{\mathcal{B}^s}$, and this decomposition restricts to $\mathcal{T}_{A^s} \cap \mathcal{T}_{\mathcal{B}^s}$ for any $s$ as in our theorem.

2. Assume that $\mathcal{B}$ consists of $R$-modules supported on $T$, where $T$ is a union of closed subsets of $S = \text{Spec } R$ (see tags 00L1, 01AT). Then $\mathcal{T}_{\mathcal{B}^s}$ consists of all those objects of $\mathcal{D}_Q$ that are supported on $p^{-1}(T)$, where $p : X \to S$ is the structure morphism.

**Proof.** 1. Theorem 3.2.5(I.3) easily implies that both $\mathcal{B}^s$ and all $\mathcal{B}^s \cap A^s$ satisfy the assumptions on $A$ in Theorem 3.1.11. Now, the latter theorem yields the result immediately.

2. An object $C$ of $\mathcal{D}_Q$ belongs to $\mathcal{T}_{\mathcal{B}^s}$ if and only if for any object $M$ of $\mathcal{D}_p$ and (a Zariski point) $s_0 \in S \setminus T$ we have $\mathcal{D}_Q(M, C) \otimes_R R_{s_0} = \{0\}$. Now,

$$\mathcal{D}_Q(M, C) \otimes_R R_{s_0} \cong \mathcal{D}_Q(M, C) \otimes_R R_{s_0}.$$ Combining Theorem 3.2.5(I.1) with Lemma 2.2.3 we obtain that $C$ belongs to $\mathcal{T}_{\mathcal{B}^s}$ if and only if $C \otimes_R R_{s_0} = 0$ for all $s_0 \in S \setminus T$.

Now, for any $x \in X$ that lies over $s_0$ the stalk $C_x$ of $C$ at $x$ is $R_{s_0}$-linear. Thus $C$ belongs to $\mathcal{T}_{\mathcal{B}^s}$ if and only if all these $C_x$ vanish, and this is precisely what we want to prove.

**Remark 3.2.10.** 1. Clearly, all the categories $\mathcal{T}_{A^s} \cap \mathcal{T}_{\mathcal{B}^s}$ as in Corollary 3.2.9(1) depend on the category $\mathcal{B} \cap R - \text{mod}$ only. Now, all intersections of this sort consist of finitely generated $R$-modules supported at some $T$ as in Corollary 3.2.9(2); see Theorem A of [Tak05] (along with Definition 2.3(1) of ibid.).

2. The functoriality of the decomposition triangles (2.2.11) provided by Proposition 2.2.3(1) implies that $r \text{id}_L = 0 = r \text{id}_M$ whenever $r \in R$ and $r \text{id}_M = 0$. Possibly, this observation can be used to obtain some result related to Corollary 3.2.9(2).

### 3.3 The proof Theorem 1.5(1) and some duality arguments

**Proof.** Denote the projection $X \to S = \text{Spec } R$ by $p$.

Recall that an object $N$ of $\mathcal{D}_Q$ belongs to $\mathcal{D}^+$ if and only if all its cohomology sheaves $H^i(N)$ are coherent. Moreover, $N$ belongs to $\mathcal{D}^-$ (resp. $\mathcal{D}^+$) whenever we also have $H^i(N) = 0$ for $i \gg 0$ (resp. $i \ll 0$). Firstly we discuss the following easy part of Theorem 1.5(1): for any $M \in \text{Obj } \mathcal{D}_p$ and $N \in \text{Obj } \mathcal{D}^a$ we have $\mathcal{D}_Q(M, N) \in R - \text{mod}$, and that $\mathcal{D}_Q(M[n], N) = \{0\}$ whenever $N \in \text{Obj } \mathcal{D}^-$ (resp. $N \in \text{Obj } \mathcal{D}^+$) and $n$ is small (resp. large) enough. Recall that $M$ is
dualizable, its dual is perfect as well, and perfect complexes have finite Tor-amplitude. Hence it suffices to note that the functor $R\pi_* : D_Q \to D(R)$ sends $D^\alpha$ into $D(R - \text{mod})$ and has finite cohomological amplitude.

Now we verify the converse implications. We will ignore the case of $D^-$ for the reasons described in the proof of Theorem \[3.2.4(1,3)]; yet note that the corresponding version of our argument works without any difficulty.

We argue similarly to the proof of \[BV03\] Theorem A.1; recall that $X$ is closed subscheme of the projectivization $Y$ of a vector bundle over $S$.

Let us reduce the latter statement to the case $X = Y$. For any $M \in D_{\text{perf}}(Y)$ we have $D_Q(Li^* M, N) \cong D(\text{Qcoh}(\text{Y}))(M, i_* N)$, where $i$ is the embedding $X \to Y$. Since $Li^* M \in D_p$, the functor represented by the object $i_* N$ fulfills the corresponding assumptions, and it remains to note that $N$ belongs to $D^\alpha$ (resp. $D^+$) if and only if $i_* N$ belongs to $D_{\text{coh}}(\text{Qcoh}(\text{Y}))$ (resp. to $D^\alpha_{\text{coh}}(\text{Qcoh}(\text{Y}))$); see \[stacks\] Tags 01QY, 087T (along with Remark 3.2.4).

Now we assume $X = Y$, and $X$ is of dimension $d \geq 0$ over $S$. We apply Theorem 6.7 of \[BeS20\]. It gives fully faithful functors $\Phi_j : D(R) \to D_Q$: $F \mapsto p^* F(j)$, for $j \in \mathbb{Z}$; here we identify $D(\text{Qcoh}(S))$ with $D(R)$. Moreover, it gives a "multiple semi-orthogonal decomposition" of $D_Q$ into the essential images $\text{Im} \Phi_j$ for $0 \leq j \leq d$; see Definition 4.2.1(2) below (or Definition 4.3.5 of ibid.).

Now we verify the converse implications. We will ignore the case of $D^\alpha$ (resp. to $D^+$) if we assume in addition that $N$ belongs to the extension-closure of $\cup_{0 \leq j \leq m} \text{Im} \Phi_j$; we will write $D_{Q \leq m}$ for the corresponding full triangulated subcategory of $D_Q$ (see Lemma \[2.1.2\] or Proposition \[4.2.2\]) below). This statement is vacuous if $m = -1$.

Suppose that the inductive assertion is fulfilled for $m = m_0 - 1$ (where $0 \leq m_0 \leq d$) and $N \in \text{Obj} D_{Q \leq m_0}$. Now the subcategories $\text{Im} \Phi_{m_0}$ and $D_{Q \leq m_0 - 1}$ give a semi-orthogonal decomposition of $D_{Q \leq m_0}$; see Proposition \[4.2.2\] below or Definition 2.2 of \[Kuz11\]. Hence there exists a distinguished triangle

$$N' \to N \to N'' \to N'[1] \quad (3.3.1)$$

with $N' \in \text{Im} \Phi_{m_0}$ and $N'' \in \text{Obj} D_{Q \leq m_0 - 1}$, and $N' \cong \Phi_{m_0} \circ \Phi_{m_0}'(N)$; here $\Phi_{m_0}'$ is the right adjoint to the functor $\Phi_{m_0} : D(R) \to D_{Q \leq m_0}$. Now, the cohomology of the complex $\Phi_{m_0}'(N)$ is given by $D_Q(p^* R(m_0), N[i])$ for $i \in \mathbb{Z}$ (here $R$ is the tensor unit object of $D(R) \cong D(\text{Qcoh}(S))$). Hence $\Phi_{m_0}'(N)$ belongs to $D_{\text{coh}}(\text{Qcoh}(S)) \subset D(R - \text{Mod})$ (resp. to $D_{\text{coh}}^+$ (resp. $D^+$). Moreover, applying \[3.3.1\] to functors corepresented by objects of $D_p$ we obtain that $D_Q(M, N'')$ belongs to $R - \text{mod}$ for any $M \in \text{Obj} D_p$ (and we also have $D_Q(M[-i], N'') = \{0\}$ for $i \leq 0$ and the $D^+$-version of the argument). Applying the inductive assumption we deduce that $N''$ is an object of $D^\alpha$ (resp. $D^+$) as well; hence the same is valid for $N$ itself.

Lastly, the category $D_{Q < d}$ equals $D_Q$; see Proposition \[13.2.1\] below or combine Definition 5.3 of \[BeS20\] with Lemma \[2.1.2\].

Now let us pass to Grothendieck duality arguments; see Definition \[1.1.7\] for the notation.

**Proposition 3.3.1.** Assume that $X$ admits a dualizing complex in the sense of \[stacks\] Tag 0A87.
1. Then an exact Grothendieck duality functor $D_X : \mathcal{D}^u \to \mathcal{D}^{u\text{op}}$ is defined (uniquely up to an equivalence). The functor $D_X^p = D_X$ is isomorphic to the identity; respectively, $D_X$ is an equivalence (and an involution).

2. $D_X$ switches $\mathcal{D}^-$ and $\mathcal{D}^+$ and fixes $\mathcal{D}^b$.

3. If $Y'$ is a scheme of finite type over a Gorenstein scheme $Y$ (see [stacks, Tag 0AWW]) then $Y'$ admits a dualizing complex. Moreover, if $X$ is Gorenstein then $D_X$ also restricts to an equivalence $\mathcal{D}_p \to \mathcal{D}_p^{op}$. In particular, this is the case if $X$ is regular.

4. If $D_0 = (\mathcal{L}\mathcal{O}_0, \mathcal{R}\mathcal{O}_0)$ is a semi-orthogonal decomposition of a triangulated subcategory $\mathcal{T}_0$ of $\mathcal{D}^b$ then the couple $D_X(D_0) = (D_X(\mathcal{R}\mathcal{O}_0), D_X(\mathcal{L}\mathcal{O}_0))$ is a semi-orthogonal decomposition of the subcategory $D_X(\mathcal{T}_0^{op}) \subset \mathcal{D}^b$.

**Proof.** All statements in assertions 1–3 easily follow from the properties of Grothendieck duality listed in [stacks, Tags 0AU3, 0DWG, 0BFQ part 2].

Assertion 4 is an easy consequence of our definitions; recall that $D_X$ is fully faithful and essentially surjective.

**Corollary 3.3.2.** Assume that $X$ admits a dualizing complex and $E$ is a semi-orthogonal decomposition of $\mathcal{D}^b$.

1. Assume that $X$ is regular.

Then $\mathcal{D}_p = \mathcal{D}^b$, and there exist a unique semi-orthogonal decomposition $\perp E$ of $\mathcal{D}_p$ such that $E = (\bigoplus E)^\perp_{D_p}$. Moreover, $E = D_X(D_X(E))^\perp_{D_p}$.

II. Suppose that $X$ is either regular or satisfies Assumption 3.2.2.

1. Then $E$ extends (see Definition 2.2.13)) to a decomposition of $\mathcal{D}^+$ of the following form: $E^+ = D_X((D_X(E))^{\perp_{D_p}})$.

2. Assume in addition that $X$ is projective over $S$ in the (weak) sense specified in Theorem 1.5.1. Then the decomposition $E^{\perp_{D_p}}$ of $\mathcal{D}^u$ (see Theorem 3.2.7(I.1)) equals $E^u = D_X(D_X(E))^{\perp_{D_p}}$.

Consequently, the aforementioned $E^+$ equals $E^{\perp_{D_p}}$.

III. Assume that $X$ is a Gorenstein scheme (cf. Proposition 3.3.13) and $D = (\mathcal{L}\mathcal{O}, \mathcal{R}\mathcal{O})$ is a semi-orthogonal decomposition of $\mathcal{D}_p$.

Then $D_{\mathcal{D}}^p D = (D_{\mathcal{D}}^p \mathcal{L}\mathcal{O}, D_{\mathcal{D}}^p \mathcal{R}\mathcal{O})$ is a semi-orthogonal decomposition of $\mathcal{D}^b$ and $D$ is a semi-orthogonal decomposition of $\mathcal{D}^*$. Moreover, if $X$ is projective over $S$ then $D_{\mathcal{D}}^p D$ is a semi-orthogonal decomposition of $\mathcal{D}^u$.

**Proof.** I. It is well known that $\mathcal{D}_p = \mathcal{D}^b$ in this case. Hence $D_X(D_X(E))^{\perp_{D_p}}$ is a decomposition of $\mathcal{D}_p$; see Theorem 3.2.7(1.1) and Proposition 3.3.12.4.

Since $D_X$ gives an equivalence $\mathcal{D}_p \to \mathcal{D}_p^{op}$, $(D_X(D_X(E))^{\perp_{D_p}})_{\mathcal{D}_p} = E$ indeed; see Proposition 2.2.3(II.1.1) and the proof of Theorem 3.2.7(II.1).

II.1. According to Proposition 3.3.1.2.4, $D_X(E)$ is a semi-orthogonal decomposition of $\mathcal{D}^b$ as well. By Theorem 3.2.7(1.1), $(D_X(E))^{\perp_{D_p}}$ is a decomposition of $\mathcal{D}^-$ that restricts to $D_X(E)$ on $\mathcal{D}^b$. Applying $D_X$ once again we obtain that $E^+$ is a decomposition of $\mathcal{D}^+$ that restricts to $E$ on $\mathcal{D}^b$.

2. We similarly obtain that $E^u$ is a decomposition of $\mathcal{D}^u$ that extends both $E^+$ and $E = (\mathcal{L}\mathcal{O}_E, \mathcal{R}\mathcal{O}_E)$.

Next, for $D = (D_{\mathcal{D}}^p(D_X(E)))$ we have $D_{\mathcal{D}_p} = D_X(E)$ and $D_{\mathcal{D}_p}^u = (D_X(E))^{\perp_{D_p}}$; see Theorem 3.2.7(II.1). Consequently, if $D = (\mathcal{L}\mathcal{O}, \mathcal{R}\mathcal{O})$ then

\[E^u = (D_{\mathcal{D}}^p(D_X(\mathcal{L}\mathcal{O})), D_{\mathcal{D}}^p(D_X(\mathcal{R}\mathcal{O})))\].

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Since the decomposition \( E^u = (L\mathcal{O}^u, R\mathcal{O}^u) \) extends \( E \) and representable functors convert coproducts into products, it follows that \( L\mathcal{O}_{E_D} \subset L\mathcal{O}^u \) and \( R\mathcal{O}_{E_D} \subset R\mathcal{O}^u \). Hence comparing \( E^u \) with the decomposition \( E^\|_{D^u} \) of \( D^u \) we obtain that \( E^\|_{D^u} = E^u \) indeed; see Theorem 3.2.7(II.1) and Proposition 2.2.6(II.1).

Since \( E^+ = E^u \cap D^+ \), we conclude that \( E^+ \) equals \( E^\|_{D^u} \).

III. Proposition 3.3.1(3) allows us to deduce all these statements from Theorem 3.2.7(I.1) easily.

Remark 3.3.3. 1. Note that people are usually interested in schemes that are of finite type over regular ones (say, over spectra of fields or Dedekind domains) only. In this case \( X \) admits a dualizing complex automatically; see Proposition 3.3.1(3).

2. Certainly, the case where the scheme \( X \) is regular itself is quite important. Note however that in this case the category \( D_p = D^b \) is \( R \)-saturated; see Definition 4.1.1 of [Bon19] that originates from Definition 2.5 of [BoK89]. In this case the existence of the \( D^b \)-adjacent semi-orthogonal decomposition (see Theorem 3.2.7(I.1)) can also be proved similarly to the rather easy Proposition 2.6 of ibid.

3. Clearly, Grothendieck duality arguments can also be combined with the aforementioned Neeman’s Corollary 0.5 of [Nee18a] and Theorem 0.2 of [Nee18b] to yield certain duals of these statements.

4 Supplements and remarks

In §4.1 we prove an abstract Theorem 4.1.1 closely related to [Nee18a]. We use it to prove Theorem 1.5(2); next we prove part 3 of that theorem. We also prove that the aforementioned Theorem 0.2 of [Nee18a] gives certain "almost decompositions" of \( D^- \), and compare two methods for constructing semi-orthogonal decompositions and adjoint functors.

In §4.2 we describe the "multiple" versions of definitions and main properties of semi-orthogonal decompositions. They imply the corresponding generalizations of our central results.

In §4.3 we discuss the relation of our arguments to (adjacent) weight structures and \( t \)-structures.

4.1 More statements related to Neeman’s results

Now we prove a general theorem that yields Theorem 1.5(2). We use some definitions and notation from [Nee18a]; our arguments are also related to ibid.

**Theorem 4.1.1.** Assume that \( \mathcal{T} \) is compactly generated by its subcategory \( \mathcal{T}_c \), and \( F \) is an object of \( \mathcal{T} \). Denote the corresponding Yoneda functor

\[ \mathcal{T} \to \text{Fun}_Z(\mathcal{T}_c^{op}, \text{Ab}) \]

(cf. Theorem 1.5) by \( \mathcal{Y} \).

I.1. Assume that \( F \) is \( \mathcal{T}_c \)-approximable in the following sense: there exists an (infinite) chain of morphisms \( E_0 \to E_1 \to \ldots \) and \( E_i \in \text{Obj} \mathcal{T}_c \) are equipped with compatible morphisms \( s_i : E_i \to F \) that yield an isomorphism \( \lim_i \mathcal{Y}(E_i) \cong \)
\( \mathcal{Y}(F) \). Then for any \( G \in \text{Obj } \mathcal{T} \) any \( \text{Fun}_{\mathcal{T}_e^{op}, \text{Ab}} \)-morphism \( \mathcal{Y}(F) \to \mathcal{Y}(G) \) is induced by some morphism \( F \to G \).

Consequently, the restriction of the functor \( \mathcal{Y} \) to the subcategory \( \mathcal{T}_a \subset \mathcal{T} \) of \( \mathcal{T}_e \)-approximable objects is full, and if \( \mathcal{Y}(G) \cong \mathcal{Y}(F) \) for \( F \in \text{Obj } \mathcal{T}_a \) then \( G \cong F \).

2. Suppose that there exists a chain of morphisms \( F'_0 \to F'_1 \to \ldots \) and \( F'_i \) are equipped with compatible morphisms \( t_i \) into \( F \) that yield an isomorphism \( \lim \mathcal{Y}(F'_i) \cong \mathcal{Y}(F) \). Moreover, assume that there exist morphisms \( c_i : E'_i \to F'_i \) such that \( E'_i \in \text{Obj } \mathcal{T}_e \) and for any \( T \in \text{Obj } \mathcal{T}_e \) there exists \( N_T \geq 0 \) such that \( \{ T \} \perp \{ \text{Cone}(c_i) \} \) for all \( i > N_T \). Then one can choose a subsequence \( E_i \) of \( E'_i \) along with some connecting morphisms between them and compatible maps \( E_i \to F \) such that \( \lim \mathcal{Y}(E_i) \cong \mathcal{Y}(F) \) (as in assertion I.1).

II. Assume that \( \mathcal{T} \) is generated by a single \( G \in \text{Obj } \mathcal{T}_e \), that is, \( \{ G[i], i \in \mathbb{Z} \}^\times = \{0\} \). Then \( F \) is approximable whenever one of the following assumptions is fulfilled.

1. There exist \( c_i : E'_i \to F'_i \) and \( t_i \) as in assertion I.2 such that \( \{ G[i], -j \leq i \leq j \in \mathbb{Z} \} \perp \{ \text{Cone}(c_j), \text{Cone}(t_j) \} \) for all \( j \geq 0 \).

2. There exist a \( t \)-structure on \( \mathcal{T} \) and \( N \in \mathbb{Z} \) such that \( G \in \mathcal{T}^{\leq N} \) and \( \{ G \} \perp \mathcal{T}^{\leq -N} \) (see Definition 1.3.1 of [BD82]), and for any \( i \geq 0 \) there exists a morphism \( c'_i : E'_i \to F'^{\leq i} \) such that \( E'_i \in \text{Obj } \mathcal{T}_e \) and \( \text{Cone}(c'_i) \in \mathcal{T}^{\leq -i} \).

III. One can take \( \mathcal{T} = \mathcal{D}_Q, \mathcal{T}_e = \mathcal{D}_p \), and \( G \) to be any compact generator of \( \mathcal{D}_Q \) (see Example 3.4 of [Nee18a]) in assertion II. Moreover, all the assumptions of assertion II.2 are fulfilled whenever \( t \) is the canonical \( t \)-structure on \( \mathcal{D} \) and \( F \in \text{Obj } \mathcal{D}^m \).

Proof. I.1. The proof relies on rather easy and well-known properties of "triangulated" countable homotopy colimits. We will not define these colimits and only recall the facts we need.

So, there exists an object \( F' = \text{hocolim } E_i \) along with compatible morphisms \( s'_i : E_i \to F' \). Next, for any object \( T \) of \( \mathcal{T} \) the corresponding homomorphism \( \mathcal{T}(F', T) \to \lim \mathcal{T}(E_i, T) \) is surjective, and if \( T \in \text{Obj } \mathcal{T}_e \) then we obtain an isomorphism \( \lim \mathcal{T}(T_e, E_i) \to \mathcal{T}(T_e, F') \); see Lemma 2.1.3(2–4) of [Bon22].

Now we argue somewhat similarly to Lemma 5.8 of [Nee18a]. Taking \( T = F \) and the sequence \( (s_i) \in \lim \mathcal{T}(E_i, T) \) in the first property of \( F' \) we obtain the existence of \( s : F' \to \overline{F} \) such that \( s \circ s'_i = s_i \) for each \( i \geq 0 \). Applying our assumptions on \( F \) along with the second property of \( F' \) we deduce that the homomorphism \( \mathcal{T}(T_e, s) \) is bijective for any \( T_e \in \text{Obj } \mathcal{T}_e \). Consequently, \( \mathcal{T}_e \perp \{ \text{Cone}(s) \} \); hence Lemma 2.1.2(3) implies \( \text{Cone}(s) = 0 \).

Thus \( F \cong F' \). Taking \( G = T \) in the first property of \( F' \) we obtain the surjectivity statement in question. Clearly, it implies that the restriction of \( \mathcal{Y} \) to \( \mathcal{T}_a \) is full.

Lastly, note that our assumptions on \( F \) depend on \( \mathcal{Y}(F) \) only. Thus if \( \mathcal{Y}(G) \cong \mathcal{Y}(F) \) for \( F \in \text{Obj } \mathcal{T}_a \) then \( G \) belongs to \( \mathcal{T}_a \) as well. Applying the fullness statement that we had just proved we obtain that it remains to prove the following: if \( \mathcal{Y}(e) \) is invertible for a \( \mathcal{T}_e \)-endomorphism \( e \) then \( e \) is invertible itself. Now, we obtain \( \mathcal{T}_e \perp \{ \text{Cone}(e) \} \) in this case; once again, it follows that \( \text{Cone}(e) = 0 \).

2. Let us prove that for any \( j \geq 0 \) there exists \( l > j \) along with a morphism
$E_j' \to E_i'$ such that the square

$$
\begin{array}{ccc}
E_j' & \longrightarrow & E_i' \\
\downarrow_{c_j} & & \downarrow_{c_i} \\
F_j' & \longrightarrow & F_i'
\end{array}
$$

is commutative. As follows from the well-known and easy Proposition 1.1.9 of [BBD82], for this purpose it suffices to choose $l$ such that $\{E_j'\} \perp \{\text{Cone}(c_j)\}$. Thus we can take any $l > \text{max}(N_{E_j'}, j)$.

Next, we apply this statement repetitively starting from $n_0 = 0$ to obtain an infinite commutative diagram

$$
\begin{array}{ccc}
E_{n_0}' & \longrightarrow & E_{n_1}' & \longrightarrow & E_{n_2}' & \longrightarrow & E_{n_3}' & \longrightarrow & \ldots \\
\downarrow_{c_{n_0}} & & \downarrow_{c_{n_1}} & & \downarrow_{c_{n_2}} & & \downarrow & & \\
F_{n_0}' & \longrightarrow & F_{n_1}' & \longrightarrow & F_{n_2}' & \longrightarrow & F_{n_3}' & \longrightarrow & \ldots
\end{array}
$$

Composing the compatible morphisms $F_{n_i}' \to F$ with $c_{n_i}$ we clearly obtain compatible morphisms $E_{n_i}' \to F$. We set $E_i = E_{n_i}'$.

Lastly, for any object $T$ of $\mathcal{T}_c$ we have $\lim_{n \geq 0} T(T, E_i) \cong \lim_{n > N \oplus \tau(i)} T(T, E_i) \cong \lim_{n \geq 0} T(T, F_{n_i}') \cong T(T, F)$.

II.1. It clearly suffices to note that any object $T$ of $\mathcal{T}_c$ is a direct summand of $T'$ such that $T'$ belongs to the extension-closure of $\{G[i] : -N \leq i \leq N\}$ for some $N > 0$; see Example 0.13 and Remark 0.15 of [Nee18a] or Proposition 4.4.1 of [Nee11].

2. Recall that $T^{\leq s} = T^{\leq 0}[-s]$, $\text{Cone}(F^{\leq s} \to F) \in T^{\geq s+1}$ and $T^{\leq s} \perp T^{\geq s+1}$ for any $s \in \mathbb{Z}$; see Definition 1.3.1 of [BBD82] (once again). It easily follows that $c_i = c'_{i+N+1}$ along with the canonical morphisms $t_i : F^{\leq N+i+1} \to F$ fulfill the assumptions of the previous assertion.

III. All the statements in question except the $T_c$-approximability of $F$ are provided by Example 3.4 of [Nee18a]. Next, an object $L$ of $\mathcal{D}_Q$ belongs to $\mathcal{D}^n$ if and only if all of its canonical $t$-truncations $L^{\leq t}$ belong to $\mathcal{D}^n$. Thus is remains to note that $\mathcal{D}^n$ in this case equals the corresponding subcategory $\mathcal{T}_{c^n}$ of $\mathcal{T}_c$; see Definition 0.16 of ibid. \qed

Remark 4.1.2. 1. Combining parts I.1 and III of our theorem we immediately obtain Theorem 1.5 2. Moreover, this theorem also extends to the case where $X$ (is proper but) is not necessarily projective over $S$.

2. The arguments used in [Nee18a] to establish the fullness statement similar to Theorem 1.1.1(I.1) require some additional assumptions (see Lemma 7.5 of ibid.). This restricts their "geometric" applications to the category $\mathcal{D}^n$ (instead of $\mathcal{D}^n$ in our Theorem 1.1.1(III)). Note however that the essential uniqueness for the objects that $\mathcal{D}_Q$-represent those functors that correspond to $\mathcal{T}_c^-$ (see Theorem 3.2.5 1.3) or 1.5 (1)) can be easily "extracted" from ibid.; cf. Lemma 5.8 of [Nee18a].

3. One may say that approximations used in ibid. come from certain "truncations from the left", whereas the approximations in Theorem 4.1.1(III) come from some "two-sided truncations".

To prove Theorem 1.5 2 we need a simple lemma.
Lemma 4.1.3. For any \( R \)-linear (additive) category \( B \) the category \( \text{Fun}_R(B, R\text{-Mod}) \) is equivalent to \( \text{Fun}_\mathbb{Z}(B, \text{Ab}) \).

Proof. Any additive functor \( F : B \to \text{Ab} \) naturally becomes an \( R \)-linear one if we define the multiplication by \( r \in R \) on \( F(B) \) for \( B \in \text{Obj } B \) by means of composing with \( F( r \text{id}_B ) \).

The proof of Theorem 1.5(3). First assume that \( R \) is countable. Then for any \( Y \) that is of finite type over \( S \) (that equals the spectrum of a countable ring) the category \( \text{D}_{\text{perf}}(Y) \) is countable, that is, the set of isomorphism classes of objects and all its morphism sets are countable. Indeed, this statement is trivial if \( Y \) is affine, and the general case can be reduced to this one; see also [stacks, Tag 0G0W]. Thus we can apply Theorem 5.1 of [Nee97] to obtain that all homological functors \( \text{D}_{\text{op}} \to \text{Ab} \) are represented by objects of \( \text{D}_\mathbb{Q} \). Thus all homological functors \( \text{D}_{\text{op}} \to R\text{-Mod} \) are representable as well; see Lemma 4.1.3.

Next assume that \( R \) is self-injective. Similarly to the proof of [BV d03, Theorem A.1], we apply a double duality argument. The idea is to extend a homological functor \( \hat{H} : \text{D}_{\text{op}} \to R\text{-mod} \) to a "nice" functor \( \text{D}_{\text{op}} \to R\text{-Mod} \).

Take the functor \( \hat{H} : \text{D}_{\text{op}} \to R\text{-mod}, M \mapsto \text{Hom}_R(H(M), R) \). Since \( R \) is an injective \( R \)-module, \( \hat{H} \) is homological. Next, it extends to a homological functor \( H' : \text{D}_\mathbb{Q} \to R\text{-Mod} \) that respects coproducts; see Proposition 2.3 of [Kra00]. Now we take \( H' : \text{D}_{\text{op}} \to R\text{-Mod}, M \mapsto \text{Hom}_R(H'(M), R) \). This functor is clearly cohomological and respects products. Consequently, \( H' \) is representable; see the well-known Theorem 8.3.3 of [Nee01] along with Lemmata 2.1.2(4) and 4.1.3.

It remains to prove that \( H' \) restricts to \( H \) on \( \text{D}_p \). We note that \( R \) is a quasi-Frobenius ring; see Theorem 15.1 of [Lam12]. Hence the ("double duality") statement is question is provided by Theorem 15.11 of ibid. □

Remark 4.1.4. 1. Since \( R \) is commutative, it is also a Frobenius ring whenever it is self-injective. Moreover, rings of this sort can be described more or less explicitly; see Theorem 15.27 of [Lam12].

Note however that the most important ("from the algebraic geometry point of view") Frobenius rings are fields; this is the only case mentioned in [BV d03].

2. Once again, Corollary 0.5 of [Nee18a] suggests that the (additional) assumptions on \( R \) in Theorem 1.5(3) are not necessary.

Now we deduce one more consequence from Theorem 0.2 of [Nee18b]. We argue somewhat similarly to Proposition 3.1.4.

Proposition 4.1.5. Assume that \( X \) satisfies Assumption 3.2.2 and let \( D = (\mathcal{LO}, \mathcal{RO}) \) be a semi-orthogonal decomposition of \( D^b \).

Then for any \( M \in \text{Obj } D^+ \) there exists a distinguished triangle

\[
L \to M \to R \to L[1]
\]

with \( L \in \perp_+ \mathcal{LO} \) and \( R \in \perp_+ \mathcal{RO} \).

Proof. We fix \( M \). Let \( H_L : D^b \to R\text{-Mod} \) be the functor \( N \mapsto D^-(M, R^L(N)) \) (see Proposition 2.2.3(1)); it is obviously homological. Now recall that the Yoneda-type functor \( D^{\text{op}} \to \text{Fun}_R(D^b, R\text{-Mod}) \) is full, and its essential image
consists of all homological functors $\mathcal{D}^b \to R\text{-mod}$; see Theorem 0.2 of [Nee18b]. Hence all values of $H_L$ belong to $R\text{-mod}$, and applying loc. cit. once again we obtain that $H_L$ is isomorphic to the functor $N \mapsto \mathcal{D}^-(L,N)$ for some $L \in \text{Obj} \mathcal{D}^-$. Since $R_D$ annihilates $\mathcal{L} \mathcal{O}$, $L$ belongs to $\perp_{\mathcal{D}^-} \mathcal{L} \mathcal{O}$.

Next, set $H_M : \mathcal{D}^b \to R\text{-Mod}$ to be the functor $N \mapsto \mathcal{D}^-(L,N)$; take $\Phi : H_L \to H_M$ to be the transformation induced by the morphisms $N \to R_D(N)$ in (2.2.1). Then loc. cit. also implies that this transformation comes from a (possibly, non-unique) morphism $f : L \to M$.

Lastly, if $N \in RO$ then $N \sim R_D(N)$; hence in the exact sequence

$$\mathcal{D}^-(M[1],N) \to \mathcal{D}^-(L[1],N) \to \mathcal{D}^-(\text{Cone}(f),N) \to \mathcal{D}^-(M,N) \to \mathcal{D}^-(L,N)$$

the first and the last maps are isomorphisms, and we obtain $\text{Cone}(f) \in \perp_{\mathcal{D}^-} \mathcal{R} \mathcal{O}$.

**Remark 4.1.6.** 1. Clearly, this argument can be axiomatized similarly to Theorem 3.1.1(II). In particular, one can combine it with Theorem 1.5(3) (see Remark 1.6(3)) to obtain a weaker version of Theorem 3.2.7(I.1).

Moreover, note that the corresponding Yoneda-type functor $\mathcal{Y}_{\mathcal{D}^p} : \mathcal{D}^p \to \text{Fun}_R(\mathcal{D}^b, R\text{-Mod})$ is fully faithful; this statement (that is contained in Theorem 0.2 of [Nee18b] as well) is obvious since $D_p \subset \mathcal{D}^b$. Loc. cit. also says that the image of $\mathcal{Y}_{\mathcal{D}^p}$ consists of all finite homological functors (cf. Theorem 3.2.5(II.2)). Consequently, the corresponding version of the argument in the proof of Proposition 4.1.5 yields an alternative proof of Theorem 3.2.7(II.1) in the case where Assumption 3.2.2 is fulfilled, and it does not require any mock projectives (cf. Theorem 3.2.5(II)).

However, it appears that it makes sense to construct adjacent decompositions using Theorem 3.1.1(II.2) (possibly combining it with Proposition 3.1.4) since this method requires less knowledge on the relation between $\mathcal{T}$ and $\mathcal{T}'$; cf. Theorem 1.5(1). We have to pay the price of specifying certain $\mathcal{D}$ which can be "not that interesting" (cf. Theorem 3.2.5(II) and Remark 2.2.2). Moreover, Theorem 0.3 of [Nee18b] and other results of Neeman may help in constructing "interesting" couples $(\mathcal{T}, \mathcal{T}')$ that satisfy the assumptions of Proposition 3.1.4.

3. Proposition 2.5.4(1) of [Bon10b] easily implies that for $R = \text{Cone}(f)$ the functor $N \mapsto \mathcal{T}'(M, R)$ is isomorphic to $N \mapsto \mathcal{T}'(M, R_D(N))$; cf. Remark 1.3.1. Moreover, we obtain a complete identification of the transformations between the corresponding Yoneda-type functor from part 2 of that proposition.

### 4.2 On "multiple" semi-orthogonal decompositions

Now we generalize Definition 2.2.1(1,3).

**Definition 4.2.1.** Let $n \geq 1$ and assume that $\mathcal{T}_i$, $0 \leq i \leq n$, are (strictly full) triangulated subcategories of $\mathcal{T}$.
1. Then for any \( j, \ -1 \leq j \leq n \) we will write \( \mathcal{T}_{\leq j} \) (resp. \( \mathcal{T}_{\geq n-j} \)) for the smallest (strictly full) triangulated subcategory of \( \mathcal{T} \) that contains \( \mathcal{T}_i \) for all \( i \leq j \) (resp. \( i \geq n-j \)).

2. We will say that the family \( (\mathcal{T}_i) \) gives a \((\text{length } n)\) semi-orthogonal decomposition of \( \mathcal{T} \) (or just a decomposition of \( \mathcal{T} \)) if \( \mathcal{T}_j \perp \mathcal{T}_i \) whenever \( 0 \leq i < j \leq n \), and \( \mathcal{T}_{\leq n} = \mathcal{T} \).

3. Let \( \mathcal{T}' \) be a triangulated subcategory of \( \mathcal{T} \). We will say that a (semi-orthogonal) decomposition \( (\mathcal{T}_i) \) of \( \mathcal{T} \) restricts to \( \mathcal{T}' \) whenever the family \( (\mathcal{T}_i) \cap \mathcal{T}' \) (see Definition 1.1(2)) gives a decomposition of \( \mathcal{T}' \).

**Proposition 4.2.2.** 1. If \( (\mathcal{T}_i) \) is a semi-orthogonal decomposition of \( \mathcal{T} \) then for any \( j, \ 1 \leq j \leq n \), the couple \( (\mathcal{T}_j, \mathcal{T}_{\leq j-1}) \) gives a decomposition of \( \mathcal{T}_{\leq j} \) in the sense of Definition 2.2.1(1), \( (\mathcal{T}_{\geq j}, \mathcal{T}_{j-1}) \) is decomposition of \( \mathcal{T}_{\geq j-1} \), and \( (\mathcal{T}_{\geq j}, \mathcal{T}_{j-1}) \) is a semi-orthogonal decomposition of \( \mathcal{T} \).

2. The correspondence \((\mathcal{T}_i) \mapsto (\mathcal{T}_{\leq i-1}, 1 \leq i \leq n)\) (resp. \((\mathcal{T}_i) \mapsto (\mathcal{T}_{n+1-i}, 1 \leq i \leq n)\) ) gives a bijection between the class of all length \( n \) semi-orthogonal decompositions of \( \mathcal{T} \) and the class of ascending chains \( \{\mathcal{RO}_i\} \), \( 1 \leq i \leq n \), of left (resp. \( \{\mathcal{LO}_i\} \), \( 1 \leq i \leq n \), of right) admissible subcategories of \( \mathcal{T} \).

Moreover, the inverse map is given by sending \( \{\mathcal{RO}_i\} \) into \( \{\mathcal{RO}_{n+1-i}\} \) (resp. \( \{\mathcal{LO}_i\} \) into \( \{\mathcal{LO}_{n+1-i}\} \)) for \( 0 \leq i \leq n \); here we expand the chains \( \{\mathcal{LO}_i\} \) and \( \{\mathcal{RO}_i\} \) by setting \( \mathcal{LO}_0 = \{0\} = \mathcal{RO}_0 \) and \( \mathcal{LO}_{n+1} = \mathcal{T} = \mathcal{RO}_{n+1} \). Furthermore, the corresponding bijection between (length \( n \) ascending) "right admissible chains" and "left admissible chains" in \( \mathcal{T} \) is given by sending \( \{\mathcal{LO}_i\} \) into \( \{\mathcal{LO}_{n+1-i}\} \).

3. Let \( \mathcal{T}' \) be a triangulated subcategory of \( \mathcal{T} \). Then the following conditions for a decomposition \( (\mathcal{T}_i) \) of \( \mathcal{T} \) are equivalent.

(a) \( (\mathcal{T}_i) \) restricts to \( \mathcal{T}' \).

(b) The smallest triangulated subcategory of \( \mathcal{T}' \) that contains all \( \mathcal{T}_i \cap \mathcal{T}' \) is \( \mathcal{T}' \) itself.

(c) \( (\mathcal{T}_{\geq j+1}, \mathcal{T}_j) \) restricts to a semi-orthogonal decomposition of \( \mathcal{T}_{\geq j} \cap \mathcal{T}' \) whenever \( 0 \leq j \leq n-1 \).

(d) \( (\mathcal{T}_{\geq j+1}, \mathcal{T}_{j+1}) \) restricts to a semi-orthogonal decomposition of \( \mathcal{T}' \) for \( 0 \leq j \leq n-1 \).

**Proof.** 1. Clearly, \( \mathcal{T}_j \perp \mathcal{T}_{\leq j-1} \). The existence of decompositions of the type \( (2.2.1) \) for all objects of \( \mathcal{T}_{\leq j} \) easily follows from Lemma 2.1.2(1); cf. the proof of Theorem 1.1(1).

The statements that \( (\mathcal{T}_{\geq j}, \mathcal{T}_{j-1}) \) is decomposition of \( \mathcal{T}_{\geq j-1} \) and \( (\mathcal{T}_{\geq j}, \mathcal{T}_{\leq j-1}) \) is a decomposition of \( \mathcal{T} \) are proved similarly.

2. Clearly, it suffices to study the first correspondence since the second one is essentially its categorical dual.

Applying assertion 1 along with Proposition 2.2.3 we obtain that \( \mathcal{T}_{\leq i-1} \) is left admissible in \( \mathcal{T}_{\leq i} \) for \( 1 \leq i \leq n \). Since \( \mathcal{T}_{\leq n} = \mathcal{T} \), we obtain that all \( \mathcal{T}_{\leq i} \) are left admissible in \( \mathcal{T} \).

Conversely, if one starts from for \( \{\mathcal{RO}_i\} \) as above then she clearly obtains \( \mathcal{T}_j \perp \mathcal{T}_i \) whenever \( 0 \leq i < j \leq n \). Applying easy induction along with Proposition 2.2.3(2) we also obtain that \( \mathcal{T}_{\leq n} = \mathcal{T} \) indeed.

Now let us calculate the compositions of our maps. If we start from \( (\mathcal{T}_i) \) then Proposition 2.2.3(1) easily implies that the corresponding orthogonals equal

\[ 2 \text{ Respectively, } \mathcal{T}_{\leq i-1} = \mathcal{T}_{\geq n+1} = \{0\}. \]
If we start from $(\mathcal{RO}_i)$ then the existence of the triangles of the type $(2.2.1)$ provided by Proposition 2.2.3(2) yields that the corresponding smallest triangulated subcategories of $\mathcal{T}$ are $\mathcal{RO}_i$ indeed.

Lastly, if we start from a right admissible chain $\mathcal{T}_{\leq i}$ that corresponds to $\mathcal{T}_i$ then the corresponding left admissible chain consists of $\mathcal{T}_{\geq n+1-i} = \mathcal{T}_{\leq n-i}^{\perp}$; here we apply assertion 1.

3. The equivalence of conditions (a) and (b) is obvious. Next, applying assertion 1 (along with Proposition 2.2.3(1)) we easily obtain that (b) is equivalent to (c) and (d).

Now we are able to generalize Theorem 3.1.1

**Theorem 4.2.3.** Assume that $\mathcal{D} = \mathcal{T}_{1}^{\perp}$, where $\mathcal{T} \subset \mathcal{D}$ is a triangulated subcategory whose objects are $\mathcal{D}$-compact, and $(\mathcal{T}_i), \ 0 \leq i \leq n$, is a semi-orthogonal decomposition of $\mathcal{T}$.

I.1. Then the family $(\mathcal{T}_{1}^{\perp})$ (see Definition 1.1(5)) is a semi-orthogonal decomposition of $\mathcal{D}$.

2. Assume in addition that $\mathcal{T}$ is essentially small (consequently, $\mathcal{D}$ is compactly generated by it). Then the family $(\mathcal{T}_i) = (\mathcal{T}_{\geq n+1-i}^{\perp} \mathcal{T}_{\leq i}^{\perp})$ is a semi-orthogonal decomposition of $\mathcal{D}$ as well; here we take $0 \leq i \leq n$ (note that $\mathcal{T}_{\geq n+1} = \{0\}$).

Moreover, the corresponding ascending chain of left (resp. right) admissible subcategories of $\mathcal{D}$ (see Proposition 4.2.2(2)) equals $(\mathcal{T}_{\leq n-i}^{\perp})$ (resp. $(\mathcal{T}_{\leq 1}^{\perp})$; here $1 \leq i \leq n$).

3. Assume that $\mathcal{T}_0$ is a subcategory of $\mathcal{D}$ such that the family $(\mathcal{T}_i)$ restricts to a semi-orthogonal decomposition $\mathcal{D}_0$ on it (see Definition 4.2.1(3)).

Then the family $(\mathcal{T}_i)$ also restricts to the category $\mathcal{T}_0^{\perp}$ as well, and this restriction equals $\mathcal{D}_0^{\perp}$ (see Definition 1.1(5) once again).

II. Assume that $\mathcal{R}, \mathcal{A}$, and $\mathcal{T}_A$ are as in Theorem 3.1.1(II).

Then the family $(\mathcal{T}_i)$ (see assertion I.2) restricts to a decomposition of the category $\mathcal{T}_A$ (which is triangulated according to Theorem 3.1.1(II.1)).

**Proof.** I.1. The proof of Theorem 3.1.1(I.1) extends to this setting straightforwardly.

2. Combining the previous assertion with Proposition 4.2.2(1) we obtain that $(\mathcal{T}_{\leq 1}^{\perp}, \mathcal{T}_{\leq 1}^{\perp})$ is a semi-orthogonal decomposition of $\mathcal{D}$ whenever $1 \leq i \leq n$. Combing Theorem 3.1.1(I.2) with Proposition 2.2.3 we obtain that the subcategory $(\mathcal{T}_{\leq 1}^{\perp})_{\mathcal{D}} = \mathcal{T}_{\leq 1}^{\perp}$ is left admissible in $\mathcal{D}$. Applying Proposition 4.2.2(2) we deduce that the family $((\mathcal{T}_{\geq n+1-i}^{\perp})_{\mathcal{D}} = (\mathcal{T}_i)$ gives a semi-orthogonal decomposition of $\mathcal{D}$ indeed.

Lastly, the corresponding "left admissible chain" equals $(\mathcal{T}_{\leq n-i}^{\perp})_{\mathcal{D}} = ((\mathcal{T}_{\leq n-i}^{\perp})_{\mathcal{D}}$ (see Proposition 4.2.2(2)).

3. The proof of Theorem 3.1.1(I.1) carries over to this setting easily (as well) if one applies Proposition 1.2.2(3).

II. According to Proposition 1.2.2(3), it suffices to verify that the corresponding decompositions $(\mathcal{T}_{\leq 1}^{\perp}, \mathcal{T}_{\leq 1}^{\perp})_{\mathcal{D}}$ of $\mathcal{D}$ (see assertion I.2) restrict to $\mathcal{T}_A$ for $1 \leq i \leq n$. The latter statement immediately follows from Theorem 3.1.1(II.2).
Corollary 4.2.5. The easily formulated "multiple decomposition" versions of Theorem 3.2.7 and Corollary 3.3.2 (one should just use the correspondence \((\mathcal{T}_i) \mapsto (\mathcal{T}_i)\) instead of \(D \mapsto D^0\) in them) are valid.

Proof. Given Theorem 3.2.7 and Proposition 3.2.1 there all the arguments used for the proof of Theorem 3.2.7 and Corollary 3.3.2 carry over to the "multiple decomposition context" without any difficulty.

4.3 On the relation to weight and \(t\)-structures

Now we recall that a semi-orthogonal decomposition couple gives both a weight structure and a \(t\)-structure; see Proposition 3.4(4) and Remark 3.5(2) of [BoV19].

The (main) difference between the latter notions and Definition 2.2.1(2) is that we do not require the corresponding \(\mathcal{LO}\) and \(\mathcal{RO}\) to be the object classes of triangulated subcategories of \(\mathcal{T}\). Instead, we only demand that \(\mathcal{LO} \subset \mathcal{LO}[1]\) and \(\mathcal{RO}[1] \subset \mathcal{RO}\) for weight structures and vice versa for \(t\)-structures (see Definition 3.1 and Proposition 3.2(1,2) of ibid.).

If one uses the so-called homological conventions for weight and \(t\)-structures (see Definitions 1.1.1 and 1.2.1, and Remarks 1.1.3(4) and 1.2.3(3) of [Bon19]) then one usually passes to the couples \(w = (\mathcal{T}_{w \leq 0}, \mathcal{T}_{w \geq 0}) = (\mathcal{LO}, \mathcal{RO}[-1])\) and \(t = (\mathcal{T}_{t \leq 0}, \mathcal{T}_{t \geq 0}) = (\mathcal{RO}[1], \mathcal{LO})\), respectively. Consequently, for any object \(M\) of \(\mathcal{T}\) and \(n \in \mathbb{Z}\) there exists an \(n\)-weight decomposition triangle

\[
L^w_n M \to M \to R^w_{n+1} M \to L^w_n M[1]
\]

with \(L^w_n M \in \mathcal{T}_{w \leq 0}[n]\) and \(R^w_{n+1} M \in \mathcal{T}_{w \geq 0}[n+1]\) and and \(n\)-\(t\)-decomposition triangle \(L^t_n M \to M \to R^t_{n-1} M \to L^t_n M[1]\) with \(L^t_n M \in \mathcal{T}_{t \leq 0}[n]\) and \(R^t_{n-1} M \in \mathcal{T}_{t \geq 0}[n-1]\); see Remarks 1.1.3(1) and 1.2.3(2) of ibid.

If \(w\) (resp. \(t\)) comes from a semi-orthogonal decomposition \(D\) then Proposition 2.2.3 gives canonical isomorphisms \(R_D M \to L^w_n M\) and \(L_D M \to R^w_{n+1} M\) (resp. \(R_D M \to L^t_n M\) and \(L_D M \to R^t_{n-1} M\)) for any \(n \in \mathbb{Z}\).

Remark 4.3.1. Now let us recall the predecessors of our Definition 2.2.1(3).

The notion of (left or right) adjacent weight and \(t\)-structures was introduced in [Bon10a]: a weight structure on \(\mathcal{T}\) was said to be left adjacent to \(t\) on \(\mathcal{T}\) if \(\mathcal{T}_{t \geq 0} = \mathcal{T}_{t \geq 0}\) (if one uses the aforementioned homological conventions). If this is the case then \(\mathcal{T}_{t \leq -1} = \mathcal{T}_{w \geq 0}^{op}\) and \(\mathcal{T}_{t \geq 1} = \mathcal{T}_{w \leq 0}^{op}\); cf. our Definition 2.2.1(3).

Next, in [Bon10b] weight and \(t\)-structures on (distinct) triangulated categories \(\mathcal{T}\) and \(\mathcal{T}'\) endowed with a so-called duality bi-functor \(\mathcal{T}^{op} \times \mathcal{T}' \to \mathcal{Ab}\) were considered (moreover, one can replace \(\mathcal{Ab}\) by an arbitrary abelian category here). In this setting a weight structure \(w\) on \(\mathcal{T}\) was said to be (left) orthogonal to \(t\)-structure \(t\) on \(\mathcal{T}'\) whenever \(\Phi(X,Y) = 0\) if \(X \in \mathcal{T}_{w \leq 0}\) and \(Y \in \mathcal{T}'_{t \geq 1}\) or if \(X \in \mathcal{T}_{w \geq 0}\) and \(Y \in \mathcal{T}'_{t \leq -1}\). A simple example of a duality is given by the corresponding restriction of the bi-functor \(\mathcal{D}(-,-)\) whenever \(\mathcal{T}\) and \(\mathcal{T}'\) are triangulated subcategories of \(\mathcal{D}\); see Remark 5.2.2(1) of [Bon19].

\*Note that \((\mathcal{T}_{t \leq 0}, \mathcal{T}_{t \geq 0})\) corresponds to the couple \((\mathcal{T}^{op}, \mathcal{T}^{op})\) in the "cohomological" notation that was used in [BHOS2] and [Nec18a].
Clearly, this orthogonality condition is fulfilled (for this choice of $\Phi$) in the setting of our Definition 2.2.1(3).

2. Proposition 2.5.4(1) of [Bon10b] in our setting (and notation) says

$$\mathcal{D}(M, L_n^t N) \cong \text{Im}(\mathcal{D}(R_n^w M, N) \to \mathcal{D}(R_{n-1}^w M, N))$$

and

$$\mathcal{D}(M, R_n^t N) \cong \text{Im}(\mathcal{D}(L_n^w M) \to \mathcal{D}(L_{n+1}^w M))$$

for any $n \in \mathbb{Z}$; cf. Definition 2.1.1(1) and §2.2 of [Bon19]. If $w$ and $t$ come from $D$ and $D'$ in the setting of Proposition 2.2.4(I) then the connecting morphisms $R_{n-1}^w M \to R_n^w M$ and $L_n^w M \to L_{n+1}^w M$ in (4.3.1) are just $\text{id}_{R(M)}$ and $\text{id}_{L(M)}$, respectively. This concludes the proof of Proposition 2.2.4(I).

Now we explain the relation of the current paper to (§§4–5 of) [Bon19] whose main results inspired the current texts. In §4 of ibid. it was demonstrated that in the settings closely related to the ones above one can construct (left or right) orthogonal $t$-structures on $T'$ from certain weight structures on $T$; these arguments are closer to that of the current paper than the "converse ones" of [Bon19, §5] (see below).

However, the results of [Bon19] §4 suffer from two disadvantages. Firstly, the author does not know of any "general" methods for constructing weight structures on $T$ of this sort (in contrast to the "smashing setting" treated in §3 of ibid.; yet cf. Remark 4.2.2(5) of ibid.).

Secondly, to compute $\mathcal{D}(M[m], L_n^t N)$ using (4.3.1) for all $m \in \mathbb{Z}$ one needs certain values of $\mathcal{D}(L_j^y M, \ldots)$ for all $j \in \mathbb{Z}$. For this reason one requires a certain "stabilization" of $L_j^y M$ (which is equivalent to the stabilization of $R_j^y M$) for $j \ll 0$ or $j \gg 0$; cf. Theorem 4.1.2(I.1, II.1). Note here that in loc. cit. it was assumed that $w$ is bounded (above, below, or both; see Definition 1.2.2(7) of ibid.); this corresponds to the vanishing of certain "candidates for"; see Remark 1.2.3(2) of ibid.) $L_j^y M$ and $R_j^y M$. It appears that these conditions can be weakened. Instead, it suffices to have certain functors $M \mapsto "L_{\leq 0} M"$ and $M \mapsto "R_{\geq 0} M"$ that correspond to certain semi-orthogonal decompositions; thus, our Theorem 3.1.1 can probably be "mixed" with Theorem 4.1.2 of ibid. Possibly, the author will study this question in more detail later. However, currently there is little hope to obtain "interesting geometric" examples for it that do not correspond to semi-orthogonal decompositions (and that are not treated in this paper, respectively).

Next, the "inverse correspondence $t \to w$" was studied in Theorem 5.3.1(II,IV) of ibid. The main disadvantage of all results of this sort is that they require enough projectives in the abelian category $\mathcal{H}: T_{\leq 0} \cap T_{\geq 0}$ (at least, in the case $T \subset T'$); certain boundedness assumptions are also needed. This makes the construction of examples quite difficult (as well).

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