ORBITAL STABILITY OF PERIODIC WAVES FOR THE LOG-KDV EQUATION

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ABSTRACT. In this paper we evidence the orbital stability of positive periodic waves related to the logarithmic Korteweg-de Vries equation. Our motivation is inspired in the recent work [8], in which the authors established the well-posedness and the linear stability of Gaussian solitary waves. By using the approach put forward recently in [20] to construct a smooth branch of periodic waves as well as to get spectral properties of the associated linearized operator, we can apply the abstract theory in [16] to deduce the orbital stability in the periodic setting.

1. INTRODUCTION

Results of well-posedness and orbital stability of periodic traveling waves related to the logarithmic Korteweg-de Vries (log-KdV henceforth) equation

\[ u_t + u_{xxx} + 2(\log(|u|)u)_x = 0, \quad (1.1) \]

will be shown in this manuscript. Here, \( u = u(x, t) \) designs a real-valued function of the real variables \( x \) and \( t \). Equation (1.1) is a dispersive equation and it models solitary waves in anharmonic chains with Hertzian interaction forces (see [8], [15], [18], and [21]).

Depending on the boundary conditions imposed on the physical problem, it is natural to consider special kind of solutions called traveling waves, which imply a balance between the effects of the nonlinearity and the frequency dispersion. In our context, such waves are of the form \( u(x, t) = \phi(x - \omega t) \), where \( \omega \in \mathbb{R} \) indicates the wave-speed and \( \phi = \phi_\omega(\xi) \) is a smooth real function. By substituting this kind of solution into (1.1) we obtain, after assuming that the constant of integration is zero, the nonlinear second order differential equation

\[ -\phi_\omega'' + \omega \phi_\omega - 2 \log(|\phi_\omega|)\phi_\omega = 0. \quad (1.2) \]

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It is well known that (1.2) admits a solution given by the Gaussian solitary wave profile (see, for instance, [9] or [12])
\[
\phi_\omega(x) = e^{\frac{1}{2} + \frac{x^2}{\omega^2}}e^{-\frac{x^2}{2}}, \quad \omega \in \mathbb{R}.
\]
(1.3)
The spectral stability related to this solution was studied in [8], where the authors studied the linear operator, arising from the linearization of (1.1) around (1.3), in the space \( L^2(\mathbb{R}) \). In particular, they showed that such an operator has a purely discrete spectrum consisting of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues. In addition, the associated eigenfunctions do not decay like Gaussian functions but have algebraic decay. Also, by using numerical approximations, they also shown that the Gaussian initial data do not spread out and preserve their spatial Gaussian decay in the time evolution of the linearized equation.

It should be noted that the nonlinear orbital stability of (1.3) was also dealt with in [8]. However, in view of the lack of uniqueness and continuous dependence, this is a conditional result. Indeed, the authors establish the orbital stability (in the energy space) provided that uniqueness and continuous dependence upon the data hold in a suitable subspace of \( H^1(\mathbb{R}) \).

Our first concern in this paper is to study the Cauchy problem
\[
\begin{aligned}
& \left\{ \begin{array}{l}
p_t + p_{xxx} + 2(p \log |p|)_x = 0, \\
p(x, 0) = u_0(x),
\end{array} \right.
\end{aligned}
\]
(1.4)
where \( u_0 \) belongs to the periodic Sobolev space \( H^1_{\text{per}}([0, L]) \). Most of our arguments will be based on the approach introduced by Cazenave [12] for the logarithmic Schrödinger equation
\[
iu_t + \Delta u + \log(|u|^2)u = 0,
\]
(1.5)
where \( u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C} \), is a complex-valued function. We point out that, in [8], the authors give us a very simple manner of how to use the arguments in [12] in order to obtain the well-posedness of (1.4) posed in \( Y \) (see 1.10).

The logarithmic nonlinearity in (1.4) brings a rich set of difficulties since the function \( x \in \mathbb{R} \mapsto x \log(|x|) \) is not differentiable at the origin. The lack of smoothness interferes, for instance, in questions concerning the local solvability since it is not possible to apply a contraction argument to deduce existence, uniqueness and continuous dependence upon the data. In order to get a grip on the absence of regularity, the idea is (see [8, 10, 11] and [12]) to solve a regularized or approximated problem, related to the model. Provided we can obtain suitable uniform estimates for the approximate solutions, they converge, in a weak sense, to the solution of the original problem and it gives us the existence of weak solutions in an appropriate Banach space.

Another difficulty coming from the non-smoothness of the nonlinearity, is the strain in establishing the uniqueness of solutions. Indeed, energy methods, as well as, contraction arguments can not be applied in these cases since we need, to this end, to assume that the nonlinearity is, at least, locally Lipschitz. It is clear that the function \( x \in \mathbb{R} \mapsto x \log(|x|) \) does not satisfy such a property at the origin. We emphasize, however, that the uniqueness for the Cauchy problem associated with (1.5) was given in [12] by combining energy estimates with a suitable Gronwall-type inequality.
To begin with our results, let us first observe that (1.1) conserves (at least formally) the energy
\[ E(v) = \frac{1}{2} \int \left( v_x^2 + v^2 - 2v^2 \log(|v|) \right) dx \] (1.6)
and the mass
\[ F(v) = \frac{1}{2} \int v^2 dx. \] (1.7)
The above integrals must be understood on the whole real line or, in the periodic setting, on the interval \([0, L]\).

The next theorem gives a result on the existence of (weak) solutions to (1.4) in the energy space \(X = H^1_{\text{per}}([0, L])\).

**Theorem 1.1.** For any \(u_0 \in X\), there exists a global solution \(u \in L^\infty(\mathbb{R}; X)\) of (1.4) such that
\[ F(u(t)) \leq F(u_0), \quad E(u(t)) \leq E(u_0), \quad \text{for all } t \in \mathbb{R}. \] (1.8)
Moreover, if
\[ \partial_x (\log |u|) \in L^\infty((-t_0, t_0); L^\infty_{\text{per}}([0, L])), \] (1.9)
for some \(t_0 > 0\), then the solution \(u\) exists in \(C((-t_0, t_0); X)\), is unique in the interval \((-t_0, t_0)\) and satisfies \(F(u(t)) = F(u_0)\) and \(E(u(t)) = E(u_0)\) for all \(t \in (-t_0, t_0)\).

Note that when one works with (1.4) on the whole real line, the energy (1.6) makes sense only for functions in the class
\[ Y := \{ u \in H^1(\mathbb{R}); \quad u^2 \log |u| \in L^1(\mathbb{R}) \}. \] (1.10)
This lead the authors in [8] to study (1.4) in \(Y\). On the other hand, in view of the log-Sobolev inequality (see [14, Theorem 4.1])
\[ \int_0^L |v|^2 \log(|v|^2) dx \leq C \left[ \int_0^L v_x^2 dx + \log \left( \frac{1}{L} \int_0^L v^2 dx \right) \int_0^L v^2 dx \right], \] (1.11)
such a restriction is not needed in the periodic framework. Thus the space \(X\) seems to be the natural energy space.

A sketch of the proof of Theorem 1.1 will be presented in next section. However, a few words of explanation are in order. The first one concerns the existence of global weak solutions. This result will follow from an adaptation of the arguments in [8] in the periodic setting. Since the solution \(u\) will be obtained as a weak limit of bounded sequences defined in a reflexive space, one can use Fatou’s Lemma to deduce the “conserved inequalities” in (1.8). The assumption (1.9) then enable us to deduce the uniqueness of local solutions and consequently the existence of the conserved quantities \(E\) and \(F\) defined in (1.6) and (1.7), respectively. Another issue concerns the uniqueness of solutions. The assumption (1.9), is a rather strong requirement. Note, however, that this condition holds if \(u(x, t) = \phi(x - \omega t)\), where \(\phi\) is an \(L\)-periodic and positive function (this is the case of our periodic traveling waves below). Differently, in the non-periodic scenario, if \(\phi\) is as in (1.3) then \(u(x, t) = \phi(x - \omega t)\) does not satisfy (1.9).

Next, we describe a few lines about the orbital stability. As we have mentioned on the top of the introduction, our stability result will be established by using the abstract theory due to Grillakis, Shatah and Strauss in [16]. First of all, let us note that the
functional $E$ is not smooth at the origin. Nevertheless, as we will see below, our periodic waves are strictly positive. Thus, at least in a neighborhood of such waves, $E$ is smooth and this allows us to use the theory in [16].

As is well known in a general setting, one of the key ingredients in the stability theory is the knowledge the spectrum of the linearized operator around the traveling wave in question. Usually, this turns out to be an elliptic operator. Thus, let us consider the Hill operator

$$\mathcal{L}_Q = -\frac{d^2}{dx^2} + Q(x), \quad (1.12)$$

where $Q$ is an $L-$periodic function which is $C^2$ in a convenient open subset. Recently in [20], the authors have presented a new technique based on the classical Floquet theorem to establish a characterization of the first three eigenvalues of $\mathcal{L}_Q$ by knowing one of its eigenfunctions. The key point is that it is not necessary to know an explicit solution of a general nonlinear differential equation of the form

$$-\phi'' + h(\omega, \phi) = 0. \quad (1.13)$$

In addition, it is possible to decide that the eigenvalue zero is simple also without knowing an explicit solution of (1.13). We emphasize that in [1], [2], [3], [4], [5] and references therein, the authors have determined explicit solutions to obtain the behavior of the non-positive spectrum associated with the Hill operator (1.12), as well as, the stability of periodic waves.

Following the arguments in [16], the first requirement for the stability of periodic waves concerns in proving, for a fixed value of $L_0 > 0$, the existence of an open interval $I \subset \mathbb{R}$ and a smooth branch $\omega \in I \mapsto \psi_\omega$, such that $\psi_\omega$ solves (1.2) for all $\omega \in I$. In our case, we will see that $L_0$ belongs to the interval $(\sqrt{2\pi}, +\infty)$. The second main point to obtain the stability consists in analyzing the non-positive spectrum of the linearized operator

$$\mathcal{L} = -\partial_x^2 + \omega - 2 - 2 \log(\psi_\omega). \quad (1.14)$$

The approach in [20] can be used to conclude that the operator in (1.14) has only one negative eigenvalue which is simple and zero is a simple eigenvalue with

$$\ker(\mathcal{L}) = \text{span}\{\psi'_\omega\}.$$

Moreover, the remainder of the spectrum is discrete and bounded away from zero.

Finally, we prove the stability of periodic waves provided that the following condition of positivity holds:

$$d''(\omega) := \frac{d}{d\omega} \int_0^{L_0} \psi_\omega(x)^2 dx > 0. \quad (1.15)$$

For the precise statements we refer the reader to Section 4.

This paper is organized as follows: In Section 2 is proved the well-posedness and the existence of conservation laws related to the model (1.1). Existence of periodic waves is treated in Section 3, whereas the orbital stability of such waves is shown in Section 4.
2. Well Posedness Results - Verbatim of [8]

In this section we sketch the proof of Theorem 1.1 by using the leading arguments in [8] and [9] (see also [10] and [12]). The main different point here is that instead of proving the well-posedness in a class similar to that in (1.10), we establish our result in the whole energy space $X = H^1_{per}([0, L])$. Here and throughout this section, $L > 0$ will be a fixed number.

To begin with, let us recall the following well-posedness result associated with the (generalized) KdV equation in the periodic setting.

**Theorem 2.1.** Consider the initial-value problem

\[
\begin{aligned}
    u_t + u_{xxx} + f'(u)u_x &= 0, & t \in \mathbb{R}, \\
    u(x, 0) &= u_0(x), & x \in \mathbb{R}.
\end{aligned}
\]

(2.16)

Then, (2.16) is locally well-posed provided $f$ is a $C^6$-function and the initial data $u_0$ belongs to $H^s_{per}([0, L])$, $s > 1/2$. More precisely, there exist $T = T(\|u_0\|_{H^s_{per}}) > 0$ and a unique solution, defined in $[-T, T]$, satisfying (2.16) in the sense of the integral equation.

*Proof.* See Theorem 1.3 in [17]. □

In addition, the smoothness of the function $f$ in Theorem 2.1 enable us to establish that

\[
F(u(t)) = F(u_0), \text{ for all } t \in [-T, T],
\]

(2.17)

and

\[
\tilde{E}(u(t)) = \tilde{E}(u_0), \text{ for all } t \in [-T, T],
\]

(2.18)

where $\tilde{E}$ is the modified energy, defined as,

\[
\tilde{E}(v) = \frac{1}{2} \int_0^L u_x^2dx - \int_0^L W(v)dx, \quad W(v) := \int_0^v f(s)ds.
\]

(2.19)

As a consequence of the above conservation laws, we deduce if $u_0$ belongs to $H^1_{per}([0, L])$, then the solution obtained in Theorem 2.1 can be extended globally-in-time.

It is obvious that $f(u) = (\log(|u|))u$ does not satisfy the assumption in Theorem 2.1. The contrivance then is to regularize the nonlinearity. To do so, for any $\varepsilon > 0$, let us define the family of regularized nonlinearities in the form

\[
f_\varepsilon(u) = \begin{cases}
    f(u), & |u| \geq \varepsilon, \\
    p_\varepsilon(u), & |u| < \varepsilon,
\end{cases}
\]

(2.20)

where $f(u) = u \log(|u|)$ and $p_\varepsilon$ is the polynomial of degree 13 defined by

\[
p_\varepsilon(u) := \left(\log(\varepsilon) - \frac{1}{2}\right)u + \sum_{i=1}^6 a_i \varepsilon^{2i+1} u^{2i+1},
\]

with $a_i \in \mathbb{R}$, $1 \leq i \leq 6$, determined by using the equality $\partial^k_u p_\varepsilon(\varepsilon) = \partial^k_u f(\varepsilon)$, for all $0 \leq k \leq 6$.

Next, we consider the approximate Cauchy problem

\[
\begin{aligned}
    u_t^\varepsilon + u_{xxx}^\varepsilon + f'_\varepsilon(u^\varepsilon)u_x^\varepsilon &= 0, & t > 0, \\
    u^\varepsilon(x, 0) &= u_0(x), & x \in \mathbb{R}.
\end{aligned}
\]

(2.21)
and assume that \( u_0 \in H^1_{\text{per}}([0, L]). \) Theorem \[2.1\] implies the existence of global solutions \( u^\varepsilon \) in \( C(\mathbb{R}; H^1_{\text{per}}([0, L])). \) The remainder of the proof follows similarly from the arguments in \[8\]. Indeed, in order to pass the limit in \[2.21\] and proving the existence of weak solutions associated with the original problem \[1.1\], it makes necessary to obtain uniform estimates, independent of \( \varepsilon > 0, \) for the regularized solution \( u^\varepsilon. \) After that, by using some compactness tools, we are in position to obtain the solution \( u \) as a weak limit of the sequence \( u^\varepsilon. \) The uniqueness of solutions is proved once we assume that \( u \) satisfies \( \partial_x (\log |u|) \in L^\infty((-t_0, t_0); L^\infty_{\text{per}}([0, L])). \) Thus, solution \( u \) exists in \( C((-t_0, t_0); X), \) is unique in the interval \((-t_0, t_0)\) and satisfies \( F(u(t)) = F(u_0) \) and \( E(u(t)) = E(u_0) \) for all \( t \in (-t_0, t_0). \) The existence of the conserved quantities can be determined by following the arguments in \[9, \text{Theorem 3.3.9}\] for the general nonlinear Schrödinger equation. Theorem \[1.1\] is thus proved.

3. Existence of periodic traveling waves and spectral analysis - A brief review of \[20\].

3.1. Existence of periodic waves. Our purpose in this subsection is to study the existence of periodic solutions for nonlinear ODE’s written in the general form

\[-\phi''_\omega + g(\omega, \phi_\omega) = 0, \tag{3.22}\]

where \( g \) is a smooth function in all variables. The subject-matter here follows from the approach in \[20\] but, for the sake of completeness, we shall give the main steps.

We assume that the parameter \( \omega \) belong to an open set \( \mathcal{P} \subset \mathbb{R}. \) The equation \[3.22\] is conservative, and thus its solutions are contained in the level curves of the energy

\[\mathcal{E}(\phi, \xi) := -\frac{\varepsilon^2}{2} + G(\omega, \phi),\]

where \( \phi_\omega := \phi, \phi' = \xi, \partial G / \partial \phi = g \) and \( G(\omega, 0) = 0. \)

We assume that:

(c1) For \( \omega \in \mathcal{P}, \) the function \( g(\omega, \cdot) \) has two consecutive zeros \( r_1 \) and \( r_2, \) such that the corresponding equilibrium points \((\phi, \phi') = (r_1, 0)\) and \((\phi, \phi') = (r_2, 0)\) are, respectively, saddle and center.

(c2) The level curve \( \mathcal{E}(\phi, \xi) = \mathcal{E}(r_1, 0) \) contains a simple closed curve \( \Gamma \) that contains \((r_2, 0)\) in its interior.

(c3) For \((\phi, \xi)\) inside \( \Gamma \) and \( \omega \in \mathcal{P}, \) the function \( g(\omega, \phi) \) is of class \( C^1 \) and \( g'(\omega, r_2) < 0, \) where \( g' \) denotes the derivative of \( g \) with respect to \( \phi. \)

All the orbits of \[3.22\] that live inside of \( \Gamma \) are periodic, turn around \((r_2, 0), \) and are contained in the level curves \( \mathcal{E}(\phi, \xi) = B, \) for \( \mathcal{E}(r_1, 0) < B < \mathcal{E}(r_2, 0). \) Moreover, we may suppose, without loss of generality, that the initial condition of such solutions \((\phi(0), \phi'(0)) = (\phi_0, 0)\) is inside \( \Gamma \) and \( \phi_0 > r_2. \) Then, due to the symmetry of the problem, the corresponding solutions are periodic and even.

**Theorem 3.1.** Under the assumptions (c1)-(c3), for every \( \omega \in \mathcal{P} \) there is \( L_\omega \in (\alpha, +\infty) \) such that the equation \[3.22\] has \( L_\omega-\)periodic solutions, where \( \alpha = \alpha(\omega) \) is the period of the solutions of the linearized equation of \[3.22\] at the equilibrium point \((r_2, 0). \) Moreover, the solution \( \phi_\omega \) and the period \( L_\omega \) are continuously differentiable with respect to \( \omega. \)
Proof. As above, (3.22) has periodic solutions $\phi_\omega$ living in the levels of energy $B$, for $E(r_1,0) < B < E(r_2,0)$. The differentiable dependence of such solutions with respect to the parameters is a consequence of the general ODE result about dependence with respect to the parameters. Fix $\omega \in P$, the period $\phi_\omega$ is given by the line integral

$$L = \int_\Lambda 1/|v| \, ds,$$

(3.23)

where $\Lambda$ is the graph of $(\phi, \xi)$ in the energy level $E(\phi, \xi) = B$, $v(\phi, \xi) = (\xi, g(\phi))$ is the vector field of (3.22), and $| \cdot |$ denotes the Euclidean norm. The upper part of $E(\phi, \xi) = B$ can be written as $\xi = \xi(\phi)$, and so

$$L = 2 \int_{b_1}^{b_2} 1/\xi(\phi) \, d\phi = 2 \int_{b_1}^{b_2} 1/\sqrt{2G(\phi) - 2B} \, d\phi,$$

where $b_1, b_2$ are the roots of $E(\phi, 0) = B$. This formula is used to compute the period $L$, but it is not appropriated to study the differentiability of $L$ with respect to $\omega$, since $b_1, b_2$ also depend on $\omega$ and $1/\xi(\phi)$ is singular at the end points. We will look for an appropriate parametrization for $\Lambda$. The linearization equation of (3.22) at the equilibrium point $(r_2, 0)$ is

$$-y'' + g'(r_2) y = 0,$$

where, for simplicity, we are writing $g'(r_2)$ instead of $g'(\omega, r_2)$. All the solutions of this equations are periodic of period

$$\alpha = \frac{2\pi}{\sqrt{-g'(r_2)}},$$

(3.24)

and their graphs are ellipses around the origin:

$$g'(r_2) \frac{y^2}{2} - \frac{y'^2}{2} = D.$$

For $D = -1/2$, this ellipse can be parameterized by the smooth curve $\gamma(t), t \in [0, 2\pi]$, given by

$$\gamma(t) = \left( \frac{1}{\sqrt{-g'(r_2)}} \cos t, \sin t \right).$$

The appropriate parameterization of $\Lambda$ can be obtained through the deformation of the ellipse into the curve $\Gamma$. Consider the system $(F, G, H) = (0, 0, 0)$, where

$$F = \phi - r_2 - \frac{1}{\sqrt{-g'(r_2)}} r \cos t,$$

$$G = \xi - r \sin t,$$

$$H = -\xi^2 + 2G(\omega, \phi) - 2B.$$

The Implicit Function Theorem guarantees that

$$\frac{\partial \phi}{\partial t} = \frac{2\xi r}{D\sqrt{-g'(r_2)}}$$

and

$$\frac{\partial \xi}{\partial t} = \frac{2rg(\phi)}{D\sqrt{-g'(r_2)}}.$$
Therefore, from (3.23) one has that $L$ depends differentially on the parameter $\omega$ and

$$L_\omega = \frac{2}{\sqrt{-g'(r_2)}} \int_0^{2\pi} \frac{-r}{D} \, dt. \quad (3.25)$$

In addition, since the solutions converge uniformly on compact intervals to $\Gamma$, it is easy to see that $L_\omega$ goes to infinity as $B$ goes to $E(r_1, 0)$.

It remains to show that $L_\omega \to \alpha$ as $B \to E(r_2, 0)$. Since

$$g(\phi) = g'(r_2)(\phi - r_2) + O((\phi - r_2)^2)$$

and

$$\phi - r_2 = \frac{1}{\sqrt{-g'(r_2)}} \, r \cos t, \quad \xi = r \sin t,$$

we have that $D$ satisfies

$$D = -2r + O((\phi - r_2)^2).$$

Then, since $\phi \to r_2$ as $B \to E(r_2, 0)$, we have

$$L_\omega = \frac{2}{\sqrt{-g'(r_2)}} \int_0^{2\pi} \frac{-r}{D} \, dt \to \alpha = \frac{2\pi}{\sqrt{-g'(r_2)}},$$

as $B \to E(r_2, 0)$. The proof is thus complete. \hfill \Box

The proof of Theorem 3.1 yields us an alternative formula of how to compute the period of the solutions. In order to apply it, we set

$$\phi = r_2 + \frac{r(t)}{\sqrt{-g'(r_2)}} \cos t,$$

$$\xi = r(t) \sin t, \quad (3.26)$$

$$D = \frac{2g(\phi)}{\sqrt{-g'(r_2)}} \cos t - 2\xi \sin t,$$

with $g(\phi) = g(\omega, \phi), \omega \in \mathcal{P}$, and $r(t)$ the solution of the first order initial-value problem

$$\begin{cases}
Dr' = 2r \left( \frac{g(\phi)}{\sqrt{-g'(r_2)}} \sin t + \xi \cos t \right) \\
r(0) = \sqrt{-g'(r_2)} (\phi_0 - r_2),
\end{cases} \quad (3.27)$$

where $(\phi_0, 0), \phi_0 > r_2$, is the initial condition for $\phi$. Thus, we obtain.

**Corollary 3.1.** Let $r(t)$ and $D(t)$ be defined as above and let $\phi$ be a periodic solution of (3.22) with initial condition $(\phi_0, 0)$. Then, the period of $\phi$ is given by

$$L_\omega = \frac{2}{\sqrt{-g'(r_2)}} \int_0^{2\pi} \frac{-r(t)}{D(t)} \, dt. \quad (3.28)$$

**Corollary 3.2.** The period map $\omega \in \mathcal{P} \mapsto L_\omega \in (\alpha, +\infty)$ obtained in Theorem 3.1 is onto.
Proof. In fact, from Theorem 3.1 one has that $L_\omega$ is a continuously differentiable map with respect to $\omega \in \mathcal{P}$ satisfying $L_\omega \to \alpha$ as $B \to \mathcal{E}(r_2,0)$ and $L_\omega \to +\infty$ as $B \to \mathcal{E}(r_1,0)$. The result is now proved. \hfill \Box

Our next step is to show the existence of a family $\psi_\omega$ where each $\psi_\omega$ has the same fixed period, solves equation (3.22) and smoothly depends on $\omega$, for $\omega$ in a convenient open set $I \subset \mathcal{P}$. Before that, we need some basic concepts. Indeed, let $\phi_\omega$ be a periodic solution of (3.22) with period $L_\omega$ obtained in Theorem 3.1. Let $\mathcal{L}_{\phi_\omega}$ be the linearized operator arising from the linearization of (3.22) at $\phi_\omega$, that is,

$$
\mathcal{L}_{\phi_\omega}(y) := L_\omega(y) = -y'' + g'(\omega, \phi_\omega(x))y, \quad \omega \in \mathcal{P}.
$$

(3.29)

Therefore, $\mathcal{L}_\omega$ is a Hill’s operator, and, according to Floquet’s theory (see e.g., [19]), its spectrum is formed by an unbounded sequence of real numbers

$$
\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots < \lambda_{2n-1} \leq \lambda_{2n} \cdots ,
$$

where equality means that $\lambda_{2n-1} = \lambda_{2n}$ is a double eigenvalue. Moreover, the spectrum of $\mathcal{L}_\omega$ is characterized by the number of zeros of the eigenfunctions: if $p$ is an eigenfunction associated to the eigenvalue $\lambda_{2n-1}$ or $\lambda_{2n}$, then $p$ has exactly $2n$ zeros in the half-open interval $[0, L_\omega)$. We have the following result.

**Theorem 3.2.** Let $L_0 \in (\alpha, +\infty)$ be fixed. Consider $\phi_{\omega_0}, \omega_0 \in \mathcal{P}$, be an even $L_0$–periodic solution of (3.22). Let $\mathcal{L}_{\omega_0} = -\partial_x^2 + g'(\omega_0, \phi_{\omega_0})$ be the linearized operator around the solution $\phi_{\omega_0}$. If $\ker(\mathcal{L}_{\omega_0}) = [\phi_{\omega_0}'\omega_0]$, then there is an open neighborhood $I \subset \mathcal{P}$ of $\omega_0$, and a family $\psi_\omega \in H^2_{\text{per}, e}([0, L_0])$ of $L_0$–periodic solutions of (3.22), which smoothly depends on $\omega \in I$.

**Proof.** First, we denote $H^1_{\text{per}, e}([0, L])$ as the periodic Sobolev space $H^1_{\text{per}}([0, L])$ constituted by even periodic functions. Let $\mathcal{F}$ be the operator given by the equation (3.22) restrict to the even functions, precisely, $\mathcal{F} : \mathcal{P} \times H^2_{\text{per}, e}([0, L_0]) \to L^2_{\text{per}, e}([0, L_0])$ is given as

$$
\mathcal{F}(\omega, \psi) = -\psi'' + g(\omega, \psi).
$$

(3.30)

Since $\phi_{\omega_0}$ is an even periodic solution of (3.22), it is clear that $\mathcal{F}(\omega_0, \phi_{\omega_0}) = 0$. Since $\ker(\mathcal{L}_{\omega_0}) = [\phi_{\omega_0}']$ and $\phi_{\omega_0}'$ is odd, one has that it is not an element of $H^2_{\text{per}, e}([0, L_0])$. As a consequence, $\frac{\partial \mathcal{F}}{\partial \omega}(\omega_0, \phi_{\omega_0}) = \mathcal{L}_{\omega_0} : H^2_{\text{per}, e}([0, L_0]) \to L^2_{\text{per}, e}([0, L_0])$ is invertible and its inverse is bounded. Therefore, the results follow from the Implicit Function Theorem (see e.g., Theorem 15.1 and Corollary 15.1 in [13]). \hfill \Box

3.2. **Spectral Properties.** In order to apply the general theory of orbital stability, [7], [16], and [25], the spectrum of $\mathcal{L}_{\phi_\omega}$ is of main importance and also of major difficulty. In fact, it is necessary to know exactly its non-positive spectrum. In our specific case, we determine the non-positive spectrum by knowing the inertial index $\text{in}(\mathcal{L}_{\phi_\omega})$ of the operator $\mathcal{L}_{\phi_\omega}$. To simplify the notation, we restrict ourselves to the study of the operators treated in our paper. However, it is possible to obtain similar results for a large class of self-adjoint operators and we strongly recommend the reader to [22] and [23] for additional informations. First, we need some basic concepts.
Definition 3.1. Let $L_0 \in (\alpha, +\infty)$ be fixed. Consider $\mathcal{L}_{\omega_0}$ the linearized operator as in (3.29) defined in $L^2_{\text{per}}([0, L_0])$ with domain $D(\mathcal{L}_{\omega_0}) = H^2_{\text{per}}([0, L_0])$. The inertial index of $\mathcal{L}_{\omega_0}$, denoted by $\text{in}(\mathcal{L}_{\omega_0})$, is a pair $(n, z)$, where $n$ is the dimension of the negative subspace of $\mathcal{L}_{\omega_0}$ and $z$ is the dimension of the null subspace of $\mathcal{L}_{\omega_0}$.

Definition 3.2. Let $\psi_\omega$, $\omega \in I$, be the smooth curve of $L_0$-periodic solutions obtained in Theorem 3.3. The family of linear operators $\mathcal{L}_{\psi_\omega} := -\partial_x^2 + g'(\omega, \psi_\omega(x))$, $\omega \in I$, is said to be isoinertial if $\text{in}(\mathcal{L}_{\psi_\omega}) = \text{in}(\mathcal{L}_{\omega_0})$, for all $\omega \in I$.

The next results are based on [22] and [23] and the first one concerns the invariance of the inertial index with respect to the parameters.

Theorem 3.3. Let $\omega \in I \mapsto \psi_\omega$ be the smooth family of $L_0$-periodic solutions given in Theorem 3.3. Then the family of operators $\mathcal{L}_{\psi_\omega}(y) = -y'' + g'(\omega, \psi_\omega)y$, $\omega \in I$, is isoinertial.

Proof. Since $\phi'_\omega$ is an eigenfunction associated with $\lambda = 0$, for every $\omega \in \mathcal{P}$, the result follows from Theorem 3.1 in [22]. □

In view of Theorem 3.3, in order to calculate $\text{in}(\mathcal{L}_{\psi_\omega})$, $\omega \in I$, it suffices to calculate $\text{in}(\mathcal{L}_{\omega_0})$ for a fixed $\omega_0 \in I$. To do this, let $\phi_{\omega_0}$ be an even periodic solution of (3.22) and let $\bar{y}$ be the unique solution of the initial value problem

$$\begin{cases} -\bar{y}'' + g'(\omega_0, \phi_{\omega_0}(x))\bar{y} = 0, \\ \bar{y}(0) = -\phi''_{\omega_0}(0), \\ \bar{y}'(0) = 0. \end{cases} \tag{3.31}$$

Define the constant $\theta$ by

$$\theta = \frac{\bar{y}'(\omega_0)}{\phi''_{\omega_0}(0)}, \tag{3.32}$$

where $L_{\omega_0}$ is the period of $\phi_{\omega_0}$.

From our construction, $\phi'_{\omega_0}$ is an eigenfunction associated with the eigenvalue $\lambda = 0$ and has exactly two zeros in the half-open interval $[0, L_{\omega_0})$. Thus, we have three possibilities:

i) $\lambda_1 = \lambda_2 = 0 \Rightarrow \text{in}(\mathcal{L}_{\omega_0}) = (1, 2),$

ii) $\lambda_1 = 0 < \lambda_2 \Rightarrow \text{in}(\mathcal{L}_{\omega_0}) = (1, 1),$

iii) $\lambda_1 < \lambda_2 = 0 \Rightarrow \text{in}(\mathcal{L}_{\omega_0}) = (2, 1).

The method we use to calculate the inertial index is based on Lemma 2.1 and Theorems 2.2 and 3.1 of [22]. The result can be stated as follows.

Theorem 3.4. Let $\theta$ be the constant given by (3.32), then the eigenvalue $\lambda = 0$ is simple if and only if $\theta \neq 0$. Moreover, if $\theta \neq 0$, then

(i) $\lambda_1 = 0$ if $\theta < 0$,

(ii) $\lambda_2 = 0$ if $\theta > 0$.

Combining Theorems 3.2 and 3.4 one has:

Corollary 3.3. Let $\phi_{\omega_0}, \omega_0 \in \mathcal{P}$, be an even periodic solution of (3.22) with a fixed period $L_{\omega_0} := L_0 \in (\alpha, +\infty)$. If $\theta \neq 0$ then $\ker(\mathcal{L}_{\omega_0}) = [\phi'_{\omega_0}]$ and Theorem 3.2 holds.
The definition of isoinertial operators as above concerns to periodic solutions having the same fixed period $L_0$. Since our intention is to prove orbital stability results of periodic waves with an arbitrary period, we need to introduce the concept of family of linear operators which are isoinertial with respect to the period.

**Definition 3.3.** Let $\phi_\omega$, $\omega \in \mathcal{P}$, be the smooth solution of $L_\omega$-periodic solutions obtained in Theorem 3.1. The family of linear operators $L_\omega$ in (3.29), is said to be isoinertial with respect to the period $L_\omega$, if $\text{in}(L_\omega) = \text{in}(L_\omega)$, for all $\omega_0 \in \mathcal{P}$.

We prove that operator $L_\omega$ is, in fact, isoinertial with respect with the period in next result.

**Theorem 3.5.** If $g$ satisfies the assumptions (c1)-(c3) and the potential of the linearized operator $\varphi(\omega, x) = g'(\omega, \phi_\omega(x))$ is of class $C^1$, then the family of operators $L_\omega(y) = -y'' + g'(\omega, \phi_\omega(x)) y$, $\omega \in \mathcal{P}$ is isoinertial with respect to the period.

**Proof.** Let $L_0 \in (\alpha, +\infty)$ be fixed. Consider $\omega_0 \in \mathcal{P}$ the corresponding parameter obtained in Corollary 3.2 with $L_0$ periodic solution $\phi_{\omega_0}$. Let

$$M_\omega : H^2_{\text{per}}([0, L_0]) \rightarrow L^2_{\text{per}}([0, L_0]),$$

be the operator defined as

$$M_\omega(y) := -y'' + \tau^2 g'(\omega, \phi_\omega(\tau x)) y,$$

where

$$\tau = \frac{L_\omega}{L_0}. \quad (3.33)$$

Let $\eta_\tau$ be the dilatation that maps $L_0$-periodic functions into $L_\omega$-periodic functions

$$\eta_\tau : L^2_{\text{per}}([0, L_0]) \rightarrow L^2_{\text{per}}([0, L_\omega]),$$

$$h(x) \mapsto h\left(\frac{x}{\tau}\right),$$

then, it is easy to see that

$$\eta_\tau^{-1} L_\omega \eta_\tau (y(x)) = \eta_\tau^{-1} L_\omega \left(y\left(\frac{x}{\tau}\right)\right)$$

$$= \eta_\tau^{-1} \left(-\frac{1}{\tau^2} y''\left(\frac{x}{\tau}\right) + g'(\omega, \phi_\omega(x)) y\left(\frac{x}{\tau}\right)\right)$$

$$= \frac{1}{\tau^2} \left(-y''(x) + \tau^2 g'(\omega, \phi_\omega(\tau x)) y(x)\right)$$

$$= \frac{1}{\tau^2} M_\omega(y(x)).$$

Therefore, if $\lambda$ belongs to the resolvent set $\rho(L_\omega)$ of $L_\omega$, then

$$(M_\omega - \tau^2 \lambda I)^{-1} = \left(\frac{1}{\tau^2} M_\omega - \lambda I\right)^{-1} = \frac{1}{\tau^2} \left(\eta_\tau^{-1} L_\omega \eta_\tau - \lambda I\right)^{-1}$$

$$= \frac{1}{\tau^2} \eta_\tau^{-1} (L_\omega - \lambda I)^{-1} \eta_\tau.$$
that is, the resolvent sets of $L_{\omega}$ and $M_{\omega}$ satisfy the relation
\[ \rho(M_{\omega}) = \tau^2 \rho(L_{\omega}), \]
where $\tau$ is given in (3.33). In particular, the operators $L_{\omega}$ and $M_{\omega}$ have the same inertial index. Now, we observe that the potential of the operator $M_{\omega}$ is continuously differentiable in all the variables, and periodic of period $L_0$ for every $\omega \in \mathcal{P}$. Therefore, Theorem 3.1 in [23] (see also Theorem 3.3) implies that $M_{\omega}$ is an isoinertial family of operators. Therefore
\[ \text{in}(L_{\omega}) = \text{in}(M_{\omega}) = \text{in}(M_{\omega_0}) = \text{in}(L_{\omega_0}). \]
The proof of the theorem is now completed. □

**Remark 3.1.** Let $L_0 \in (\alpha, +\infty)$ be a fixed period. Consider $\phi_{\omega_0}$, $\omega_0 \in \mathcal{P}$, the $L_0$—periodic solution of (3.22) given by Theorem 3.1 and Corollary 3.2. If $\theta \neq 0$ we apply Corollary 3.3 to construct a smooth curve of $L_0$—periodic solutions, $\omega \in I \mapsto H^{s}_{\text{per}}([0, L_0])$, $s \gg 1$, associated with (3.22). In addition, Theorem 3.3 enables us to say
\[ \text{in}(L_{\psi_\omega}) = \text{in}(L_{\omega_0}). \]

Next, if one considers $\omega_1 \in \mathcal{P}$, let $\phi_{\omega_1}$ be the corresponding periodic solution of (3.22) with period $L_{\omega_1} \in (\alpha, +\infty)$ obtained in Theorem 3.1. From Theorem 3.5 one has
\[ \text{in}(L_{\omega_1}) = \text{in}(L_{\omega_0}). \]
The fact that $\theta \neq 0$ allows us to construct a smooth curve of periodic solutions
\[ \tilde{\omega} \in \tilde{I} \mapsto \chi_{\tilde{\omega}} \in H^{s}_{\text{per}}([0, L_{\omega_1}]); \]
associated with (3.22) all of them with the same period $L_{\omega_1}$ such that $\chi_{\omega_1} = \phi_{\omega_1}$. Here, $\tilde{I}$ indicates an open interval which contains $\omega_1 \in \mathcal{P}$.

In particular, Theorem 3.3 implies that
\[ \text{in}(L_{\psi_{\omega}}) = \text{in}(L_{\chi_{\tilde{\omega}}}), \text{ for all } \omega \in I, \tilde{\omega} \in \tilde{I}. \]

4. **Stability of Periodic Waves**

In this section, we use the theory put forward in last section in order to establish the existence and orbital stability of periodic traveling waves for (1.1). Let $L > 0$ be fixed. The space we shall be working with is the Hilbert space $X := H^1_{\text{per}}([0, L])$.

We seek for periodic traveling wave solutions of the form $u(x, t) = \phi(x - \omega t)$, $\omega \in \mathbb{R}$. So, by assuming that the first constant of integration is zero, we get that $\dot{\phi} = \phi_\omega$ satisfies the nonlinear equation
\[ -\phi'' + \omega \phi - 2 \log(|\phi|)\phi = 0. \]
(4.34)

Thus, one has that $\phi$ satisfies a general equation as in (3.22), $-\phi'' + g(\omega, \phi) = 0$, with $g(\omega, \phi) = \omega \phi - 2 \log(\phi)\phi$. The consecutive zeros of $g(\omega, \cdot)$ are $r_1 = 0$ and $r_2 = e^{\omega/2}$ and $g$ satisfies the assumptions (c1)-(c3), where the closed curve $\Gamma$ is the graph (in the phase portrait) of the soliton (given in (1.3)), plus the equilibrium point $(r_1, 0) = (0, 0)$. This curve is also given by $\mathcal{E}(\phi, \xi) = \mathcal{E}(r_1, 0)$, where $\mathcal{E}$ is the energy functional
\[ \mathcal{E}(\phi, \xi) = -\frac{\xi^2}{2} + G(\phi) = -\frac{\xi^2}{2} + \frac{\omega + 1}{2} \phi^2 - \frac{1}{2} \log(\phi^2)\phi^2. \]
Since the origin belongs to $\Gamma$, it is clear that $g$ is smooth in the region inside $\Gamma$. Then Theorem 3.1 can be applied to prove the existence of positive periodic solutions. The initial conditions $(\phi(0), \phi'(0)) = (\phi_0, 0)$ that give rise to periodic and even solutions are in the range $r_2 = e^{\omega/2} < \phi_0 < e^{(\omega+1)/2}$, where the constant $e^{(\omega+1)/2}$ is the intersection of $\Gamma$ with the axis $\phi$. The periods of such solutions vary in the range $(\alpha, \infty)$, where $\alpha = \frac{2\pi}{\sqrt{-g'(r_2)}} = \pi \sqrt{2}$ does not depend on $\omega$ (see Theorem 3.1). In addition we have $\mathcal{P} = \mathbb{R}$.

Now let $\phi$ be a fixed solution of (4.34). To prove the orbital stability of $\phi$, our intention is to apply the abstract theory in [16] (see also Bona [7] and [24]) adopted to the periodic case. Before going into details, let us recall the notion of orbital stability.

**Definition 4.1.** We say that a periodic wave $\phi$ is orbitally stable, by the flow of (1.1), in $X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in X$ satisfying $\|u_0 - \phi\|_X < \delta$, the solution of (1.1) with initial data $u_0$ exists globally and satisfies

$$\inf_{y \in \mathbb{R}} \|u(\cdot, t) - \phi(\cdot + y)\|_X < \varepsilon,$$

for all $t \in \mathbb{R}$.

Recall that the quantities $E$ and $F$, defined in (1.6) and (1.7), respectively, are conserved by the flow of (1.1) and are invariant under the action of the group of translations $T(s)f(\cdot) = f(\cdot + s)$, $s \in \mathbb{R}$. Note that functional $E$ is not smooth at the origin on $H^1_\text{per}([0, L])$. Nevertheless, the arguments in Section 3 imply that our periodic wave $\phi$ satisfy $0 < e^{\omega/2} < \phi < e^{(\omega+1)/2}$. Thus, since $\phi$ is strictly positive, we guarantee the smoothness of the functional $E$ around the periodic wave. In order to apply the abstract theory in [16], this is enough, because the stability is determined for initial data sufficiently close to the periodic wave $\phi$.

Define the functional $H = E + \omega F$. Fix $L > \sqrt{2\pi}$. Let $\psi_\omega$, $\omega \in I$ be the smooth curve of $L$--periodic solutions of (3.22). Thus, in a neighborhood of the positive periodic wave $\psi_\omega$, $\omega \in I$, $H$ is smooth. This allows us to calculate the Fréchet derivative of $H$ at $\psi_\omega$ to deduce, from (1.2), that $\psi_\omega$ is a critical point of $H$, that is,

$$H'(\psi_\omega) = (E + \omega F)'(\psi_\omega) = -\psi''_\omega + \omega \psi_\omega - 2\log(|\psi_\omega|)\psi_\omega = 0.$$

Also, in a neighborhood of $\psi_\omega$, we can rewrite equation (1.1) as an abstract Hamiltonian system

$$u_t = JE'(u), \quad (4.35)$$

where $J = \partial_x$.

Next, consider the linearized operator $L := H''(\psi_\omega)$, that is,

$$L(v) = H''(\psi_\omega)v = -v'' + (\omega - 2 - 2\log(\psi_\omega))v. \quad (4.36)$$

One has that $L$ is an unbounded operator defined on $L^2_\text{per}([0, L])$ with domain $H^2_\text{per}([0, L])$.

Finally, we recall if (S1), (S2), and (S3) below hold, then the stability theory presented in [16] states that $\psi_\omega$ is orbitally stable.

(S1) There is an open interval $I \subset \mathbb{R}$ and a smooth branch of periodic solutions, $\omega \in I \subset \mathbb{R} \mapsto \psi_\omega \in H^1_\text{per}([0, L])$. 

(S2) The periods of such solutions vary in the range $(\alpha, \infty)$, where $\alpha = \frac{2\pi}{\sqrt{-g'(r_2)}} = \pi \sqrt{2}$ does not depend on $\omega$ (see Theorem 3.1).

(S3) In addition we have $\mathcal{P} = \mathbb{R}$.
The operator $L$, defined in (4.36), has only one negative eigenvalue which is simple and zero is a simple eigenvalue whose eigenfunction is $T'(0)\psi_\omega = \psi'_\omega$, $\omega \in I$. Moreover, the rest of the spectrum of $L$ is positive and bounded away from zero.

(S3) If $d : I \to \mathbb{R}$ is the function defined as $d(\omega) = H(\psi_\omega)$, then

$$d''(\omega) = \frac{d}{d\omega} F(\psi_\omega) > 0, \quad \text{for all } \omega \in I.$$ 

Theorem 4.1. Suppose that uniqueness and continuous dependence hold accordingly to Theorem 1.1. Let $L \in (\sqrt{2}\pi, +\infty)$ be fixed. Consider a smooth branch $\omega \in I \mapsto \psi_\omega \in H_{\text{per}}^s([0, L]), s \gg 1$, of $L$-periodic waves which solves equation (1.2). Then $\psi_\omega$ in orbitally stable in $X$ by the periodic flow of the equation (1.1).

Proof. As argued above, we need to show that (S1), (S2) and (S3) occur.

In fact, since $L$ is isoinertial with respect to the period (see Remark 3.1), its inertial index can be computed by fixing $\omega_0$. Letting $\omega_0 = 1$ we see that $\phi = \phi_{\omega_0}$ must belong to the interval $(\sqrt{e}, e) \approx (1.65, 2.72)$ and it satisfies the initial-value problem

$$\begin{cases} -\phi'' + \phi - \log(\phi^2)\phi = 0, \\ \phi(0) = 2.5, \\ \phi'(0) = 0. \end{cases}$$

(4.37)

The period $L_0$ of $\phi = \phi_{\omega_0}$ can be determined by (3.28) as $L_0 \approx 4.8$. Solving numerically the IVP (4.37), we can compute the constant $\theta$ given by (3.32) as $\theta \approx -0.725$.

Step 1: (S1) holds. Since $\theta \neq 0$ we obtain from Remark 3.1 that there is an open interval $I$ around $\omega_1$ and a smooth curve $\omega \in I \mapsto \psi_\omega \in H_{\text{per}}^2([0, L])$ of $L$-periodic traveling wave solutions associated with equation (1.2).

Step 2: (S2) holds. The eigenfunction $\phi'_{\omega_0}$ has exactly two zeros in the interval $[0, L_0)$ and $\theta \approx -0.725 < 0$. From Theorem 3.5 and Remark 3.1 we deduce that $\text{in}(L) = (1, 1)$. In addition, because $L$ is a Hill’s operator, the rest of the eigenvalues are strictly positive.

Step 3: (S3) holds. In order to conclude the orbital stability, it remains to prove that

$$d''(\omega) := \frac{1}{2} \frac{d}{d\omega} \int_0^L \psi^2 dx > 0.$$ 

(4.38)

In fact, by differentiating (1.2) with respect to $\omega$, we get

$$-\Phi'' + (\omega - 2 - \log(\psi^2))\Phi + \Phi = 0$$

(4.39)

where $\Phi_\omega = d\psi_\omega/d\omega$. On the other hand, by multiplying (4.39) by $\psi_\omega$ and integrating the first term twice by parts, we obtain

$$\int_0^L (2\Phi_\omega \psi_\omega - \psi^2_\omega) dx = 0,$$ 

(4.40)
that is,
\[
\frac{d}{d\omega} \int_0^L \psi_\omega^2 \, dx = \int_0^L \psi_\omega^2 \, dx > 0.
\]  
(4.41)

This proves (4.38) and concludes the proof of Theorem 4.1. □

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