C*-ALGEBRAS ASSOCIATED WITH ENDMORPHISMS OF GROUPS

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Abstract. In this work we construct a C*-algebra from an injective endomorphisms of some group G, allowing the endomorphism to have infinite cokernel. We generalize results obtained by I. Hirshberg in [12] and by J. Cuntz and A. Vershik in [10]. In good cases we show that the C*-algebra that we study is classifiable by Kirchberg’s classification theorem, with K-groups equal to $K_*(C^*(G))$.

1. Introduction

In [12] Hirshberg defined a C*-algebra associated with endomorphisms of groups with finite cokernel. The obvious sequence of that paper is to construct the same C*-algebra for endomorphisms with infinite cokernel. So in this paper we define and study a universal C*-algebra constructed from an injective endomorphism $\varphi$ with infinite cokernel of a discrete countable group. Thus the biggest difference of this paper with Hirshberg’s ([12]) is that we allow $|G/\varphi(G)| = \infty$.

In order to generalize the constructions, we also associate the C*-algebra with a set $B$ of subgroups of $G$ and call it $U[\varphi, B]$. Their rôle is to implement naturally the multiplication rule inside $U[\varphi, B]$, because here we do not have finitely many projections summing up to one. The relations defining $U[\varphi, B]$ are dictated by the natural representations of $\varphi$, $B$ and $G$ on the Hilbert space $L^2(G)$ of all square summable complex functions on $G$. The unitaries representing the group elements, the projections associated with subgroups of $G$ and the isometry representing $\varphi$ generate a concrete C*-subalgebra of $L(L^2(G))$ which, in good cases, is isomorphic to the C*-algebra $U[\varphi, B]$ and can thus be described by generators and relations.

Beside Hirshberg’s paper, similar constructions have been studied before by various authors [2], [7], [9], [10], [15] and [18]. In particular, somewhat similar C*-algebras have been associated with endomorphisms of abelian groups and also with semigroups. Also the ring C*-algebras studied in [7], [9] arise in a similar way.

It is important to note that even though $\varphi(G)$ is only a subgroup (not necessarily normal, because $G$ can be non-abelian) we are able to count how many elements there are in the above quotient. Also choose some family $B$ of subgroups of $G$, containing $G$.

The group elements give rise to unitary operators $\{U_g\}_{g \in G}$ acting on $L^2(G)$ by left multiplication, and the endomorphism induces an isometry $S$ acting on $L^2(G)$ through $\varphi$: denoting by $\{\xi_h : h \in G\}$ the canonical orthonormal basis of $L^2(G)$, $S$ is defined

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by $S(\xi_h) := \xi_{\varphi(h)}$. With every element $H$ of $B$, consider the projection $E_{[H]}$ with
\begin{equation}
E_{[H]}(\xi_g) = \begin{cases} 
\xi_g, & \text{if } g \in H; \\
0, & \text{otherwise.}
\end{cases}
\end{equation}
The $C^*$-subalgebra of $L(P^2(G))$ generated by the operators above is denoted $C^*_r[\varphi, B]$ (Definition 2.2).
The operators thus defined satisfy some natural relations, and we use these relations to define the universal $C^*$-algebra $U[\varphi, B]$, associated with $\varphi$. Particularly, the sum condition which appears in [10] does not hold in our situation (we would have an inequality of the form $< \infty$). However the projections associated with the subgroups of $G$ in $U[\varphi, B]$ take their place and are crucial to prove the main results.

One of the important ones obtained is that, if $G$ is amenable, if the intersection of the subgroups of $B$ contains some image of $G$ through $\varphi$ and if $\varphi$ is pure, then $U[\varphi, B]$ is a Kirchberg algebra\footnote{The endomorphism $\varphi$ is pure when $\bigcap_{n \in \mathbb{N}} \varphi^n(G) = \{e\}$.}. In particular, this implies that in this case $U[\varphi]$ and $C^*_r[\varphi]$ are isomorphic. This result also extends the ones obtained by Hirshberg in [12] and by Cuntz and Vershik in [10].

To prove the result above, it is crucial to use a semigroup crossed product description of $U[\varphi, B]$. Here we use the definition of a semigroup crossed product presented by Li in Appendix A of [16] using covariant representations. The semigroup implementing the crossed product can be the semidirect product $S := G \rtimes \varphi \mathbb{N}$ or the semigroup of natural numbers $\mathbb{N}$. Such a description allows us to use the six term exact sequence presented by Khoshkam and Skandalis [13] to calculate the K-theory of our $C^*$-algebra, as is done by Cuntz and Vershik in [10].

The semigroup crossed product description above also implies the existence of a (full corner) group crossed product description of $U[\varphi, B]$ (by the minimal automorphic dilation introduced in [5], [6] and generalized later by M. Laca in [15]), using the group of integers $\mathbb{Z}$. This allows one to use the classical Pimsner-Voiculescu exact sequence [19] to calculate their K-groups.

We will see that considering $B = \{G\}$ gives interesting examples, and we then denote the $C^*$-algebra only by $U[\varphi]$. In this case, the isomorphism above is not the only way to represent it as a crossed product: analogously to the work of G. Boava and R. Exel in [11] one can show that $U[\varphi]$ has a partial group crossed product description, which can also be related to an inverse semigroup crossed product by [17]. Apart from giving $U[\varphi]$ another description by an established structure, this result also provides another way to prove the simplicity of $U[\varphi]$ in some cases.

It can be noted that a particular semigroup is very important in our constructions: the semigroup $S = G \rtimes \varphi \mathbb{N}$. We prove that when the group $G$ is amenable and $\varphi$ is pure, the three semigroup $C^*$-algebras defined by Li in [18] - namely $C^*(S)$, $C^*_r(S)$ and $C^*_r(S)$ - associated with the semigroup $S$ are isomorphic to $U[\varphi]$ and also nuclear, simple and purely infinite (Theorem 6.9), answering partially one open question in [18].

To finish, using the semigroup crossed product description of $U[\varphi]$ from Chapter 2, we study its K-theory. Using a natural split exact sequence and the six term exact sequence provided by Khoshkam and Skandalis [13] we easily conclude that the $K$-groups of $U[\varphi]$ are the same as the ones of $C^*(G)$. This implies that, imposing some extra conditions, $U[\varphi]$ is classifiable\footnote{$U[\varphi, B]$ is separable, nuclear, simple and purely infinite.} by Kirchberg’s classification theorem [14].
A weaker version of the result concerning the K-theory of $U[\varphi]$ can be obtained independently using some recent results by Cuntz, Echterhoff and Li in [8]. I would like to thank J. Cuntz for the Ph.D. orientation and the helpful comments and corrections about this paper.

2. Definitions and basic results

In this paper $G$ will always be a discrete countable group with unit $e$ and $\varphi$ an injective endomorphism (monomorphism) of $G$ with infinite cokernel. When necessary, we require the amenability of $G$ or $\varphi$ to be pure. We want to construct a $C^*$-algebra associated with $\varphi$. To generalize Hirshberg’s constructions even more, we also want to associate the $C^*$-algebra with some set $B$ of subgroups of $G$ which contains $G$. We consider it to have a natural behaviour of the multiplication rule inside the $C^*$-algebra.

Now we expand families of operators defined in (1):

**Definition 2.1.** Consider $B$ a family of subgroups of $G$ (containing $G$) defined as above. We denote $C_r^*[\varphi,B]$ the reduced universal $C^*$-algebra generated by the three families of operators defined in [1]:

- a family of projections $\{E_X : X \in C(B)\}$;
- unitaries $\{U_g : g \in G\}$

and the isometry $S$.

Studying the properties of the operators above, it is natural to define its universal version:

**Definition 2.2.** As above choose a set $B$ of subgroups of $G$ (containing $G$) and construct the family $C(B)$. Then $U[\varphi,B]$ is the universal $C^*$-algebra generated by

- a family of projections $\{e_X : X \in C(B)\}$;
- unitaries $\{u_g : g \in G\}$

and one isometry $s$

satisfying:

(i) $u_gs^n u_h s^m = u_{g\varphi^n(h)} s^{n+m}$;
(ii) $u_s e_X s^n u_{g^{-1}} = e_{g\varphi^n(X)}$;
(iii) $e_G = 1$;
(iv) $e_X e_Y = e_{X \cap Y}$ and
(v) $e_X + e_Y = e_{X \cup Y}$.

Since $u_g s^n s^n u_{g^{-1}} = e_{[g\varphi^n(G)]}$, the projections $u_g s^n s^n u_{g^{-1}}$ commute and considering $n \geq m$:

$$u_g s^n s^n u_{g^{-1}} u_h s^m s^m u_h^{-1} = e_{[g\varphi^n(G)]} e_{[h\varphi^m(G)]} = e_{[g\varphi^n(G)] \cap [h\varphi^m(G)]} =
\begin{cases} 
e_{[g\varphi^n(G)]}, & \text{if } h \in g\varphi^m(G); \\ 0, & \text{otherwise,} \end{cases}$$

Since $u_g s^n s^n u_{g^{-1}} = e_{[g\varphi^n(G)]}$, the projections $u_g s^n s^n u_{g^{-1}}$ commute and considering $n \geq m$:

$$u_g s^n s^n u_{g^{-1}} u_h s^m s^m u_h^{-1} = e_{[g\varphi^n(G)]} e_{[h\varphi^m(G)]} = e_{[g\varphi^n(G)] \cap [h\varphi^m(G)]} =
\begin{cases} 
e_{[g\varphi^n(G)]}, & \text{if } h \in g\varphi^m(G); \\ 0, & \text{otherwise,} \end{cases}$$

$$u_g s^n s^n u_{g^{-1}} =
\begin{cases} 
e_{[g\varphi^n(G)]}, & \text{if } h \in g\varphi^m(G); \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 2.3.** It is important to mention that the construction above can be done when $B$ is just a set of subsets of $G$, or even consider some set $C$ in place of $C(B)$, containing any type of set and closed under the regularity conditions.
First of all we will see that only the initial set $B$ is important to generate the $C^*$-algebras above, and the fact that in $C(B)$ some elements are not subgroups of $G$ is not a problem. Note that some elements of $C(B)$ are given by

$$ g \bigcap_{i=1}^{m} \phi^{n_i}(H_i) $$

with $g \in G$, $n_i \in \mathbb{N}$ and $H_i \in B$. In fact we can use these to describe the $*$-algebra $\text{span}(\{e_{[X]} : X \in C(B)\})$.

**Lemma 2.4.** Define

$$ B' := \left\{ \bigcap_{i=1}^{m} \phi^{n_i}(H_i) : H_i \in B, \ n_i \in \mathbb{N} \right\}. $$

Then $\text{span}(\{e_{[X]} : X \in C(B)\}) \cong \text{span}(\{e_{[gH]} : g \in G, H' \in B'\}) =: D'$.

**Proof:**

\begin{itemize}
    \item $\supseteq$: Obvious.
    \item $\subseteq$: Let us call $K' := \{X \subseteq G : e_{[X]} \in D'\}$. It is obvious that $B \subseteq K'$. Moreover $K'$ is closed under:
    \begin{itemize}
        \item $\bigcap_{i=1}^{n} :$ By definition, for $X_1, X_2 \in K'$ it holds that
            $$ e_{[X_1 \cap X_2]} = e_{[X_1]} e_{[X_2]}.$$
        \item Complements: For $X \in K'$:
            $$ e_{[X^c]} = 1 - e_{[X]} = e_{[G]} - e_{[X]} \in D'.$$
        \item $\bigcup_{i=1}^{n} :$ Note that $X \cup Y = [X^c \cap Y^c]^c \in K'$, $\forall X, Y \in K'$.
        \item And if $X \in K'$, $g \in G$ and $n \in \mathbb{N}$, the injectivity of $\varphi$ implies $g \varphi^n(X) \in K'$.
    \end{itemize}

Therefore $K'$ satisfies the regularity conditions, and $C(B) \subseteq K'$, because $C(B)$ is the smallest set containing $B$ satisfying it. Then $\text{span}(\{e_{[X]} : X \in C(B)\}) \subseteq D'$.

This result implies an important and simpler way to describe $U[\varphi, B]$:

**Proposition 2.5.** The universal $C^*$-algebra $U[\varphi, B]$ is generated by

$$ \{e_{[H]}, u_g, s : H \in B, g \in G\}. $$

**Proof:** Due to last lemma we only have to prove that

$$ \text{span}(\{e_{[gH']} : g \in G, H' \in B'\}) \subseteq \text{span}(\{e_{[H]}, u_g, s : H \in B, g \in G\}). $$

But $e_{[gH']} = u_g e_{[H']} u_g^{-1}$ and for $H' = \bigcap_{i=1}^{n} \phi^{n_i}(H_i) \in B'$ with $n_i \in \mathbb{N}$ and $H_i \in B \cup \{G\}$ we have

$$ e_{[H']} = \prod_{i=1}^{n} e_{[\varphi^{n_i}(H_i)]} = \prod_{i=1}^{n} s^{n_i} e_{[H_i]} s^{s^{n_i}}. $$

Therefore $e_{[gH']} \in C^*(\{e_{[H]}, u_g, s : H \in B, g \in G\})$.

**Remark 2.6.** Note that the lemma and the proposition above hold for any choice of $B$ (i.e, even if it doesn’t consists of subgroups).

Another interesting basic result:
Proposition 2.7. Consider $\mathcal{B}$ containing only sets of the form $g_i\varphi^n(H_i)$, with $g_i \in G$ and $H_i$ subsets of $G$. Then

$$U[\varphi, \mathcal{B}] \cong U[\varphi, B]$$

where $B$ contains only the subsets $H_i$.

Proof: By Proposition 2.5 (and the remark above),

$$U[\varphi, B] = C^*(\{e_{[g,H]}, u_g, s : g, H \in \mathcal{B}, g \in G\})$$

But as $e_{[g_i, \varphi^n(H_i)]} = u_{g_i}s^n e_{[H_i]}s^nu_{g_i}^{-1}$

both $C^*$-algebras are isomorphic.

Remark 2.8. If we choose $B = \{G\}$ then $U[\varphi, B]$ is generated only by the unitary elements $\{u_g : g \in G\}$ and the isometry $s$, and it can be viewed as a natural generalization of the constructions in [12] and [10]. This case will be studied with more details in Section 6.

3. Crossed product descriptions

Define

$$D[\varphi, B] := C^*(\{u_gs^n e_{[H]}s^n u_{g^{-1}} : g \in G, n \in \mathbb{N}, H \in \mathcal{B}\})$$

and note that it is a commutative $C^*$-subalgebra of $U[\varphi, B]$ because we have $u_gs^n e_{[H]}s^n u_{g^{-1}} = e_{[g, \varphi^n(H)]}$. We can define an action of the semigroup $S = G \rtimes \varphi \mathbb{N}$ on $D[\varphi, B]$ via

$$\alpha : S \to \text{End}(D[\varphi, B])$$

$$(g, n) \mapsto u_gs^n \cdot s^n u_{g^{-1}}.$$

Proposition 3.1. The $C^*$-algebra $U[\varphi, B]$ is isomorphic to $D[\varphi, B] \rtimes_{\alpha} S$.

Proof: In this proof we use the universality of both $C^*$-algebras to find the desired isomorphism. Remembering the definition from [16], $D[\varphi, B] \rtimes_{\alpha} S$ together with

$$\iota_D : D[\varphi, B] \to D[\varphi, B] \rtimes_{\alpha} S$$

$$x \mapsto \iota_D(x)$$

and

$$\iota_S : S \to \text{Isom}(D[\varphi, B] \rtimes_{\alpha} S)$$

$$(g, n) \mapsto \iota_S(g, n)$$

satisfying

$$\iota_D(u_gs^n x s^n u_{g^{-1}}) = \iota_S(g, n) \iota_D(x) \iota_S(g, n)^*$$

is the semigroup crossed product of the dynamic system $(D[\varphi, B], S, \alpha)$. But note that $U[\varphi, B]$ together with

$$\pi : D[\varphi, B] \to U[\varphi, B]$$

$$x \mapsto x$$

and

$$\rho : S \to \text{Isom}(U[\varphi, B])$$

$$(g, n) \mapsto u_gs^n$$
is a covariant representation of \((D[\varphi, B], S, \alpha)\), since:
\[
\rho(g, n)\pi(x)\rho(g, n)^* = u_g s^n x s^n u_{g^{-1}} = \pi(\alpha_{(g,n)}(x)).
\]
So we conclude that there exists a \(*\)-homomorphism
\[
\Phi : D[\varphi, B] \rtimes_\alpha S \to U[\varphi, B]
\]
such that \(\Phi \circ \iota_D = \pi\) and \(\Phi \circ \iota_S = \rho\).
In the other hand, it is well known that the crossed product \(D[\varphi, B] \rtimes_\alpha S\) is generated as a \(C^*\)-algebra by elements of the form \(\iota_S(g, n)\) and \(\iota_D(e[H])\) with \(H \in B\). Identifying \(\iota_S(g, n)\) with \(u_g s^n\) and \(\iota_D(e[H])\) with \(e[H]\), it is easy to check that they satisfy conditions (i) - (v) of Definition 2.2 which generate \(U[\varphi, B]\).
Therefore we have a \(*\)-homomorphism
\[
\Delta : U[\varphi, B] \to D[\varphi, B] \rtimes_\alpha S
\]
\[
\begin{align*}
\Delta(s) &= \iota_S(0, 1) = \rho(0, 1) = s \\
\Delta(e[H]) &= \iota_D(e[H])
\end{align*}
\]
We now show that (2) and (3) are inverses of each other:
\[
\Phi \circ \Delta(u_g) = \Phi(\iota_S(g, 0)) = \rho(g, 0) = u_g
\]
\[
\Phi \circ \Delta(s) = \Phi(\iota_S(0, 1)) = \rho(0, 1) = s
\]
\[
\Phi \circ \Delta(e[H]) = \Phi(\iota_D(e[H])) = \pi(e[H]) = e[H]
\]
and the other side
\[
\Delta \circ \Phi(\iota_S(g, n)) = \Delta(\rho(g, n)) = \Delta(u_g s^n) = \iota_S(g, n)
\]
\[
\Delta \circ \Phi(\iota_D(e[H])) = \Delta(\pi(e[H])) = \Delta(\iota_D(e[H])) = \iota_D(e[H]).
\]
Thus \(U[\varphi, B]\) and \(D[\varphi, B] \rtimes_\alpha S\) are isomorphic.

\[\square\]

**Remark 3.2.** Note that \(U[\varphi, B]\) is also isomorphic to \((D[\varphi, B] \rtimes_\omega G) \rtimes_\tau N\),
\[
\omega : G \to \text{Aut}(D[\varphi, B])
\]
\[
g \mapsto u_g(\cdot)u_{g^{-1}},
\]
\[
\tau : N \to \text{End}(D[\varphi, B] \rtimes_\omega G)
\]
\[
n \mapsto s^n(\cdot)s^n
\]
where for \(a_g \delta_g\) of \(D[\varphi, B] \rtimes_\omega G\), \(\tau_n(a_g \delta_g) = s^n a_g s^n \delta_{\varphi^n(g)}\).

Using the minimal automorphic dilation presented by Laca in [15] it is possible to see the \(C^*\)-algebra \(U[\varphi, B]\) as a corner of a group crossed product. For this, we need to prove the following.

**Proposition 3.3.** The semidirect product \(S = G \rtimes_\varphi \mathbb{N}\) is an Ore semigroup i.e, it is cancellative and right-reversible.
Proof: Consider \((g_i, n_i) \in S\) for \(i \in \{1, 2, 3\}\). \(S\) is cancellative:
\[
(g_1, n_1)(g_3, n_3) = (g_2, n_2)(g_3, n_3)
\]
\[
\Rightarrow (g_1 \varphi^{n_1}(g_3), n_1 + n_3) = (g_2 \varphi^{n_2}(g_3), n_2 + n_3)
\]
\[
\Rightarrow n_1 = n_2 \text{ and } g_1 \varphi^{n_1}(g_3) = g_2 \varphi^{n_1}(g_3)
\]
\[
\Rightarrow g_1 = g_2
\]
\[
(g_1, n_1)(g_2, n_2) = (g_1, n_1)(g_3, n_3)
\]
\[
\Rightarrow (g_1 \varphi^{n_1}(g_2), n_1 + n_2) = (g_1 \varphi^{n_1}(g_3), n_1 + n_3)
\]
\[
\Rightarrow n_2 = n_3 \text{ and } \varphi^{n_1}(g_2) = \varphi^{n_1}(g_3)
\]
\[
\Rightarrow g_2 = g_3 \text{ as } \varphi \text{ is injective.}
\]
Also any two principal left ideals of \(S\) intersect:
\[
(\varphi^{n_2}(g_1^{-1}), n_2)(g_1, n_1) = (e, n_2 + n_1)
\]
\[
= (\varphi^{n_1}(g_2^{-1}), n_1)(g_2, n_2) \in S(g_1, n_1) \cap S(g_2, n_2).
\]

It follows that the semigroup \(S\) can be embedded in a group, called the enveloping group of \(S\), which we will denote as \(\text{env}(S)\), such that \(S^{-1}S = \text{env}(S)\) (Theorem 1.1.2 [15]). It also implies that \(S\) is a directed set by the relation defined by \((g, n) < (h, m)\) if \((h, m) \in S(g, n)\). Let us define a candidate for \(\text{env}(S)\). Consider
\[
G := \lim \{G_n : \varphi^n\}
\]
(with \(G_n = G\) for all \(n \in \mathbb{N}\)) and with the extended automorphism \(\bar{\varphi}\) of \(G\) construct the group
\[
\overline{S} := G \rtimes \bar{\varphi} \mathbb{Z}.
\]

Proposition 3.4. \(\overline{S} \cong \text{env}(S)\)

Proof: For this we need to show that \(S\) is a subsemigroup of \(\overline{S}\) and \(\overline{S} \subset S^{-1}S\) [4]. First it is obvious that \(S\) is a subsemigroup of the group \(\overline{S}\) considering the inclusion \((g, n) \mapsto (g_0, n)\), where \(g_0 = g \in G = G_0 \hookrightarrow G\).

Without loss of generality take \((g_i, j) \in \overline{S}\) with \(i > |j|\). Then
\[
(g_i, j) = (g_i, -i)(e, j + i) = (g_0, i)^{-1}(e, j + i) \in S^{-1}S.
\]

Now consider the inductive system given by
\[
\overline{D}[\varphi, B] := \lim \{D[\varphi, B]_{(h, m)} : \alpha_{(h, m)}^{(g\varphi^n(h), n+m)}\}
\]
where
\[
D[\varphi, B]_{(h, m)} := D[\varphi, B]
\]
and
\[
\alpha_{(h, m)}^{(g\varphi^n(h), n+m)} : D[\varphi, B]_{(h, m)} \to D[\varphi, B]_{(g, n)(h, m)} = D[\varphi, B]_{(g\varphi^n(h), n+m)}
\]
with \(\alpha_{(h, m)}^{(g\varphi^n(h), n+m)} := \alpha_{(g, n)}\forall (h, m), (g, n) \in S\), where the latter was defined before Proposition 3.1. Then the \(C^*\)-dynamical system \((\overline{D}[\varphi, B], \overline{S}, \overline{\alpha})\) is called the minimal automorphic dilation of \((D[\varphi, B], S, \alpha)\) where:
\[
\bar{\alpha}_{(g, n)} \circ t = t \circ \alpha_{(g, n)}, \forall (g, n) \in G \times \mathbb{N}
\]
with \( \iota : D[\varphi, B] \hookrightarrow D[\varphi, B]_{(e,0)} \to \overline{D[\varphi, B]} \), and
\[
\bigcup_{(g,n) \in S} \overline{\alpha^{-1}(\iota(D[\varphi, B]))} = \overline{D[\varphi, B]}.
\]

Then by Theorem 2.2.1 in [15]:

**Lemma 3.5.** There exists an isomorphism
\[
\Phi : U[\varphi, B] \cong D[\varphi, B] \rtimes_{\iota} S \cong \iota(1)(\overline{D[\varphi, B]} \rtimes_{\iota} S)\iota(1).
\]
\[\square\]

Thus, \( D[\varphi, B] \rtimes_{\alpha} S \) is Morita equivalent to \( \overline{D[\varphi, B]} \rtimes_{\iota} S \), \( \Phi|_{D[\varphi, B]} = \iota \) and also \( \Phi(u_g s^n) = \iota(1)\overline{U_{(g,n)}\iota(1)} \), where \( U : S \to UM(\overline{D[\varphi, B]} \rtimes_{\iota} S) \) (unitary multipliers).

4. Separability, nuclearity and UCT

By Proposition 2.5, we conclude the following.

**Proposition 4.1.** If \( B \) contains countably many subsets of \( G \), then \( U[\varphi, B] \) is separable.

**Proof:** With the condition satisfied we have countably many projections in \( U[\varphi, B] \), and therefore it is generated by countably many elements.
\[\square\]

And the group crossed product description obtained in last section implies two properties:

**Proposition 4.2.** If \( G \) is amenable, \( U[\varphi, B] \) is nuclear.

**Proof:** \( G \) being amenable implies that \( S \) is amenable as well (amenability is closed under direct limits by [23] and also closed under semidirect products). But we know that \( D[\varphi, B] \) is nuclear because it is commutative, therefore \( \overline{D[\varphi, B]} \rtimes_{\iota} S \) is nuclear by Proposition 2.1.2 in [20]. Since hereditary C*-subalgebras of nuclear C*-algebras are nuclear by Corollary 3.3 (4) in [3], we conclude that
\[
U[\varphi, B] \cong D[\varphi, B] \rtimes_{\iota} S \cong \iota(1)(\overline{D[\varphi, B]} \rtimes_{\iota} S)\iota(1)
\]
is nuclear.
\[\square\]

**Proposition 4.3.** If \( G \) is amenable, \( U[\varphi, B] \) satisfies the UCT property.

**Proof:** Since \( D[\varphi, B] \) is commutative, \( \overline{D[\varphi, B]} \rtimes_{\iota} S \) is isomorphic to a groupoid C*-algebra. When the group \( G \) is amenable then \( S \) also is, and the respective groupoid is also amenable. Therefore using a result by Tu ([22] Proposition 10.7), the crossed product satisfies UCT. By Morita equivalence, \( U[\varphi, B] \) also satisfies it.
\[\square\]

5. Purely infinite and simple

To prove that under certain conditions our algebra is purely infinite and simple we use Proposition 5.1 below, which is proven in Proposition 5.2 of [10].

**Proposition 5.1.** Let \( \tilde{A} \) be a dense \( * \)-subalgebra of a unital C*-algebra \( A \). Assume that \( \epsilon \) is a faithful conditional expectation on \( A \) such that for every \( 0 \neq x \in \tilde{A}_+ \) there exist finitely many projections \( f_i \in A \) with

1) \( f_i \perp f_j, \forall i \neq j \),
Using the isomorphism are isomorphic. With this we obtain a canonical conditional expectation of the subalgebra defined above. For this we use the amenability of the group $G$ to find the conditional expectation, it is necessary to suppose in this section that the subalgebra $S$ is amenable. And to prove the main theorem, we suppose that $\varphi$ is pure, i.e:

$$\bigcap_{n \in \mathbb{N}} \varphi^n(G) = \{e\}.$$ 

To start with, the next lemma tells us that

$$S[\varphi, B] := \text{span}\{s^n u_{g^{-1}e[H]} u_{g'} s^m : H \in B', g, g' \in G, n, m \in \mathbb{N}\}$$

is dense in $\mathbb{U}[\varphi, B]$.

**Lemma 5.2.** The $*$-subalgebra of $\mathbb{U}[\varphi, B]$ generated by

$$\{e_{[H]}, u_g, s : H \in B, g \in G\}$$

coincides with $S[\varphi, B]$.

**Proof:** Note that

$$\{e_{[H]}, u_g, s : H \in B, g \in G\} \subseteq S[\varphi, B] \subseteq \text{span}\{e_{[H]}, u_g, s : H \in B, g \in G\}$$

and $S[\varphi, B]$ is closed under multiplication:

$$\begin{align*}
s^n u_{g^{-1}e[H]} u_{g'} s^m u_{h^{-1}e[K]} u_h s^{m'} &= s^n u_{g^{-1}e[H]} u_{g'} s^m s^{m'} (u_{h^{-1}e[K]} u_h) s^{m'} \\
&= s^n u_{g^{-1}e[H]} s^{m'} (s^n e_{[g'H]} s^m) s^n s^{m'} (s^n e_{[h-K]} s^m) s^n u_{h^{-1}h} s^{m'} \\
&= s^{n+m} u_{e^{-1}(g') e_{[g'H]} e_{[h-K]} e_{[h-h']}} u_{e^{-1}(h')} s^{n+m'} \\
&= s^{n+m} u_{e^{-1}(g') e_{[g'H]} e_{[h_h-K]} e_{[h_h]}} u_{e^{-1}(h')} s^{n+m'} \\
&= s^{n+m} u_{e^{-1}(g') e_{[g'H]} e_{[h_h-K]} e_{[h_h]}} u_{e^{-1}(h')} s^{n+m'} \\
&= 0 \in S[\varphi, B] \\
&= s^{n+m} u_{e^{-1}(g') e_{[g'H]} e_{[h_h-K]} e_{[h_h]}} u_{e^{-1}(h')} s^{n+m'} \\
&= s^{n+m} u_{e^{-1}(g') e_{[g'H]} e_{[h_h-K]} e_{[h_h]}} u_{e^{-1}(h')} s^{n+m'} \\
&= s^{n+m} u_{e^{-1}(g') e_{[g'H]} e_{[h_h-K]} e_{[h_h]}} u_{e^{-1}(h')} s^{n+m'} \\
&= S[\varphi, B].
\end{align*}$$

The result follows.

Now we just have to define a conditional expectation to use in Proposition 5.1 with the subalgebra defined above. For this we use the amenability of $G$. Therefore $S$ is amenable, which implies that both the reduced and the full crossed products by $\mathcal{F}$ are isomorphic. With this we obtain a canonical conditional expectation of $\mathbb{U}[\varphi, B]$. Using the isomorphism

$$\Phi : \mathbb{U}[\varphi, B] \cong D[\varphi, B] \times_G S \to \iota(1)(\overline{D}[\varphi, B] \times_\pi \mathcal{F}) \iota(1).$$

obtained in Lemma 3.5, we have the easy-to-prove result below.
Lemma 5.3. There exists a faithful conditional expectation
\[ \theta : U[\varphi, B] \rightarrow \Phi^{-1}(i(1)T[\varphi, B]i(1)) \]
\[ s^{n}u_{g^{-1}}e_{H}u_{g}^{n} \mapsto \begin{cases} 
  s^{n}u_{g^{-1}}e_{H}u_{g}^{n}, & \text{if } n = n' \text{ and } g = g' \\
  0, & \text{otherwise.} 
\end{cases} \]
for all \( H \in B' \), \( g, g' \in G \) and \( n, n' \in \mathbb{N} \).

Now we can prepare to prove that \( U[\varphi, B] \) is simple and purely infinite, upon imposing some conditions. For this aim we follow and adapt the proof of Li [16] (Section 5.2) and use the next lemmas to make the proof of the main theorem cleaner.

Lemma 5.4. Let \( H \) and \( G_{i} \) be distinct subgroups on \( G \) with \( \# \left[ \frac{H}{H \cap G_{i}} \right] = \infty \) for all \( 1 \leq i \leq n \). Then, for all \( h, g_{i} \in G \), we have \( hH \not\subseteq \bigcup_{i=1}^{n} g_{i}(H \cap G_{i}) \).

Proof: By induction. For \( n = 1 \):
\[ hH \subseteq g_{1}(H \cap G_{1}) \Rightarrow H \subseteq h^{-1}g_{1}(H \cap G_{1}) \Rightarrow \frac{H}{H \cap G_{1}} \neq \infty. \]
Assume that the result holds for \( n - 1 \). Let us prove it holds for \( n \). Suppose that
\[ hH \subseteq \bigcup_{i=1}^{n} g_{i}(H \cap G_{i}), \]
for some \( h, g_{i} \in G \), with \( 1 \leq i \leq n \). We can consider two possible cases:
- There exists \( 1 < j \leq n \) with
\[ \# \left[ \frac{H \cap G_{1}}{(H \cap G_{1}) \cap (H \cap G_{j})} \right] < \infty. \]
As
\[ \frac{(H \cap G_{1})(H \cap G_{j})}{H \cap G_{j}} \approx \frac{H \cap G_{1}}{(H \cap G_{1}) \cap (H \cap G_{j})}, \]
it follows that the first one also has cardinality \( < \infty \). But the exact sequence
\[ \frac{(H \cap G_{1})(H \cap G_{j})}{H \cap G_{j}} \rightarrow \frac{H}{H \cap G_{j}} \rightarrow \frac{H}{(H \cap G_{1})(H \cap G_{j})} \]
with \( \# \left[ \frac{(H \cap G_{1})(H \cap G_{j})}{H \cap G_{j}} \right] < \infty \) and \( \# \left[ \frac{H}{H \cap G_{j}} \right] = \infty \) implies that
\[ \# \left[ \frac{H}{(H \cap G_{1})(H \cap G_{j})} \right] = \infty. \]
Define:
\[ \widetilde{G}_{i} := \begin{cases} 
  H \cap G_{i}, & \text{if } G_{i} \neq G_{1} \text{ and } G_{i} \neq G_{j} \\
  (H \cap G_{1})(H \cap G_{j}), & \text{if } G_{i} \in \{G_{1}, G_{j}\}. 
\end{cases} \]
Note that
\[ \# \left[ \frac{H}{H \cap G_{i}} \right] = \infty \]
and
\[ hH \subseteq \bigcup_{i=1}^{n} g_{i}(H \cap G_{i}) \subseteq \bigcup_{i=1}^{n} g_{i}(H \cap \widetilde{G}_{i}), \]
but the latter one contradicts our hypothesis, as \( \# \{\widetilde{G}_{i}\} \leq n - 1 \).
Now suppose that \( 1 < j \leq n \),

\[
\# \left[ \frac{H \cap G_1}{(H \cap G_1)(H \cap G_j)} \right] = \infty.
\]

As \( \# \left[ \frac{H}{H \cap G_1} \right] = \infty \), we have that \( \exists g \in H \) such that \( g(H \cap G_1) \neq g_i(H \cap G_i) \forall 1 \leq i \leq n \). Then:

\[
g(H \cap G_1) = g(H \cap G_1) \cap H \subseteq g(H \cap G_1) \cap \bigcup_{i=1}^{n} g_i(H \cap G_i)
\]

\[
= \bigcup_{g(H \cap G_1) \cap g_i(H \cap G_i) \neq \emptyset} g(H \cap G_1) \cap g_i(H \cap G_i)
\]

and we can conclude that

\[
H \cap G_1 \subseteq \bigcup_{g(H \cap G_1) \cap g_i(H \cap G_i) \neq \emptyset} g^{-1} \tilde{g}_i((H \cap G_1) \cap (H \cap G_i)).
\]

But note that by construction \( g(H \cap G_1) \cap g_i(H \cap G_1) = \emptyset \). So that union has been taken over less than \( n \) elements, what contradicts our claim.

\( \square \)

Let us show that \( U \) and the faithful conditional expectation \( \theta \) taken over less than \( n \) elements, what contradicts our claim.

Now suppose that \( \forall x \in S[\varphi, B]_+ \). As \( \theta(x) \neq 0 \), one has:

\[
\theta(x) = \sum_{(n', X)}^{\text{finite}} \beta_{(n', X)} s^{n'} e_{[X]} s^{n'},
\]

where \( (n', X) \in \mathbb{N} \times C(B) \). Define \( n \) to be the sum of all \( n' \) with

\[
\beta_{(n', X)} s^{n'} e_{[X]} s^{n'} \neq 0.
\]

Then

\[
\theta(x) = s^n \left( \sum_{(n', X)}^{\text{finite}} \beta_{(n', X)} e_{[\varphi^{n-n'}(X)')} \right) s^n.
\]

Moreover using Lemma 2.4, it is possible to write

\[
(4) \quad \theta(x) = s^n \left( \sum_{(g, H)}^{\text{finite}} \beta_{(g, H)} e_{[gH]} \right) s^n,
\]

where the sum is over finitely many \( (g, H) \in G \times B' \).\footnote{Remembering that \( B' = \left\{ \bigcap_{i=1}^{n} \varphi^{n_i}(H_i) : H_i \in B \cup \{G\}, n_i \in \mathbb{N} \right\} \).}

Note that

\[
s^n e_{[gH]} = s^n s^n e_{[gH]} = s^n e_{[\varphi^n(G)]} e_{[gH]} = s^n e_{[\varphi^n(G) \cap gH]},
\]

so we can assume that \( gH \subseteq \varphi^n(G) \), for each \( (g, H) \in G \times B' \).

**Lemma 5.5.** There exist finitely many pairwise orthogonal (nontrivial) projections \( p_i \) in \( \mathbb{Z} \)-span\( (D[\varphi, B]) \) such that \( C^*(\{ e_{[gH]} : \beta_{(g, H)} \neq 0 \}) = C^*(\{ p_i \}). \)

**Proof:** Just orthogonalize the \( e_{[gH]} \). One can arrange the coefficients are integers.

\( \square \)
Thus take some $p \in \{p_i\}$ among the $p_i$’s obtained above. Then

$$p = \sum_j n_j e_{[g_j, H_j]} - \sum_{j' < j} \tilde{n}_{j'} e_{[\tilde{g}_{j'}, \tilde{H}_{j'}]}$$

with finitely many $n_j, \tilde{n}_{j'} \in \mathbb{Z}_{\geq 0}$ and $(g_j, H_j), (\tilde{g}_{j'}, \tilde{H}_{j'}) \in G \times B'$.

**Lemma 5.6.** We can express $p$ as in (3) so that $\forall K, \tilde{K} \in \{H_j, \tilde{H}_j\}$ the cardinality of $\frac{K}{K \cap \tilde{K}}$ is 1 or $\infty$.

**Proof:** By induction. Enumerate $\{H_j, \tilde{H}_j\}$ by $\{K_i\}$. Of course the lemma holds if there is just $K_1$.

Suppose that it holds for $\{K_1, \ldots, K_h\}$. Define $K^{(0)}_{h+1} := K_{h+1}$ and for $j = 1, \ldots, h$

$$K^{(j)}_{h+1} := \begin{cases} K^{(j-1)}_{h+1}, & \text{if } \#(K^{(j-1)}_{h+1}/(K^{(j-1)}_{h+1} \cap K_j)) \in \{1, \infty\}, \\ K^{(j-1)}_{h+1} \cap K_j, & \text{otherwise.} \end{cases}$$

We want to change $K_{h+1}$ successively to $K^{(0)}_{h+1}, K^{(1)}_{h+1}, \ldots$, until $K^{(h)}_{h+1}$.

Suppose that $K^{(j)}_{h+1} = K^{(j-1)}_{h+1} \cap K_j$ as described above in (3). Therefore we have

$$1 < \#(K^{(j-1)}_{h+1}/(K^{(j-1)}_{h+1} \cap K_j)) = M < \infty,$$

and then, $K^{(j-1)}_{h+1} = \bigcup_{i=1}^M g_i(K^{(j-1)}_{h+1} \cap K_j)$.

So we can replace $K_{h+1}$ by $K'_{h+1} := K^{(h)}_{h+1}$, because the projections will still be written using the initial $\{K_i\}$.

**Claim:**

$$\# \left[ \frac{K'_{h+1}}{K_{h+1} \cap K} \right] \in \{1, \infty\}, \forall K \in \{K_1, \ldots, K_h\}.$$

**Proof of claim:** Let us prove by induction on $j$ that $\# \left[ \frac{K^{(j)}_{h+1}}{K^{(j)}_{h+1} \cap K} \right] \in \{1, \infty\},$

for every $K \in \{K_1, \ldots, K_j\}$. By construction it holds for $j = 1$. Suppose it holds for $j - 1$, that is

$$\# \left[ \frac{K^{(j-1)}_{h+1}}{K^{(j-1)}_{h+1} \cap K} \right] \in \{1, \infty\}, \forall K \in \{K_1, \ldots, K_{j-1}\},$$

and let us prove the assertion for $j$. Also by construction $\# \left[ \frac{K^{(j)}_{h+1}}{K^{(j)}_{h+1} \cap K_j} \right]$ belongs to $\{1, \infty\}$. Then, we need to show that

$$\# \left[ \frac{K^{(j)}_{h+1}}{K^{(j)}_{h+1} \cap K_j} \right] \in \{1, \infty\}, \forall K \in \{K_1, \ldots, K_{j-1}\}.$$

If $K^{(j)}_{h+1} = K^{(j-1)}_{h+1}$, then this holds by the induction hypothesis.

But, if $K^{(j)}_{h+1} = K^{(j-1)}_{h+1} \cap K_j$, then $K^{(j)}_{h+1} \subseteq K^{(j-1)}_{h+1}$ and therefore it follows that

$$1 < \# \left[ \frac{K^{(j-1)}_{h+1}}{K^{(j-1)}_{h+1} \cap K_j} \right] < \infty.$$
Now, by our induction hypothesis, we have two possibilities for each $K \in \{K_1, \ldots, K_{j-1}\}$:

- $\# \left[ \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)} \cap K} \right] = 1$: in this case, as $K_{h+1}^{(j)} \subseteq K_{h+1}^{(j-1)} \subseteq K$, it follows that $\# \left[ \frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)} \cap K} \right] = 1$.

- $\# \left[ \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)} \cap K} \right] = \infty$.

Consider the exact sequence:

$$\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)} \cap K} \hookrightarrow \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)} \cap K} \rightarrow \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)} \cap K}.$$

The inclusion $\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)} \cap K} \subseteq \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)} \cap K}$ implies that the second term has size $\infty$. The third term has cardinality $< \infty$ because it is equal to $\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)} \cap K}$. As that sequence is exact, we must have $\# \left[ \frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)} \cap K} \right] = \infty$.

Thus we conclude that $\# \left[ \frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)} \cap K} \right] \in \{1, \infty\}$, $\forall K \in \{K_1, \ldots, K_j\}$. \hfill \Box

Set

$$K'_j := \begin{cases} K_j \cap K_{h+1}', & \text{if } 1 < \# \left[ \frac{K_j}{K_j \cap K_{h+1}'} \right] < \infty, \\
K_j, & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, h$. This gives a new sequence $\{K'_1, \ldots, K'_h\}$. And then it only remains to prove that

$$\# \left[ \frac{K'_j}{K'_j \cap K''_j} \right] \in \{1, \infty\}.$$

Note that, if $j$ or $\tilde{j}$ is equal to $h+1$, this holds by the claim above.

So, suppose that $j$ and $\tilde{j}$ are in $\{1, \ldots, h\}$. Then, by our induction hypothesis, we have two possibilities:

- $\# \left[ \frac{K_j}{K_j \cap K_{\tilde{j}}} \right] = 1$: then $K_j \subseteq K_{\tilde{j}}$, and (7) holds.

If $K'_j = K_j \cap K_{h+1}'$, then $K'_j \subseteq K''_j$, and (7) holds.
Otherwise, $K_j' = K_j \neq K_j \cap K_{h+1}'$, and therefore $\# \left[ \frac{K_j}{K_j \cap K_{h+1}'} \right] = \infty$. Then (as $K_j \subseteq K_j'$) we have the inclusion:

$$\frac{K_j}{K_j \cap K_{h+1}'} \subseteq \frac{K_j}{K_j \cap K_{h+1}} ,$$

which implies $\left[ \frac{K_j}{K_j \cap K_{h+1}} \right] = \infty$. So $K_j' = K_j$ and our claim holds.

- $\# \left[ \frac{K_j}{K_j \cap K_j} \right] = \infty$: As $\frac{K_j}{K_j \cap K_j} \subseteq \frac{K_j}{K_j \cap K_j}$, if $K_j' = K_j$ the claim holds.

Now, if $K_j' = K_j \cap K_{h+1}' \neq K_j$, then we have the exact sequence:

$$\frac{K_j'}{K_j' \cap K_j} \rightarrow \frac{K_j}{K_j' \cap K_j} \rightarrow \frac{K_j}{K_j'} .$$

The set $\frac{K_j}{K_j \cap K_j}$ has size $\infty$ and is contained in the second term, so it has size $\infty$ too.

The third term has size $< \infty$ as $K_j \cap K_{h+1}' \neq K_j$ implies that $\# \left[ \frac{K_j}{K_j \cap K_{h+1}'} \right] < \infty$.

Hence, we conclude that $\left[ \frac{K_j'}{K_j' \cap K_j} \right] = \infty$, proving the lemma.

\[ \square \]

**Lemma 5.7.** There exist finitely many pairwise orthogonal projections $p_i \in U[\varphi, B]$ such that

$$C^*(\{p_i\}) \cong C^*(\{e_{[gH]} : \beta_{(g,H)} \neq 0\}) ,$$

where the $(g, H)$’s come from equation (4). Moreover if exists $m \in \mathbb{N}$ such that $\varphi^m(G) \subseteq \bigcap_{H \in B} H$ then for all $i$, there exists $h_i \in G$ and $m_i \in \mathbb{N}$ with $e_{[h_i \varphi^{m_i}(G)]} \leq p_i$.

**Proof:** We have

$$\theta(x) = s^n \left( \sum_{(g,H)} \beta_{(g,H)} e_{[gH]} \right) s^n , \quad \text{with } (g,H) \in G \times B' ,$$

where we recall that

$$B' = \left\{ \bigcap_{i=1}^m \varphi^{n_i}(H_i) : H_i \in B \cup \{G\}, n_i \in \mathbb{N} \right\} .$$

We can assume that $gH \subset \varphi^m(G)$ and, by Lemma 5.3 we have finitely many pairwise orthogonal projections $p_i$ in $\text{Z-span}(D[\varphi, B])$ with

$$C^*(\{e_{[gH]} : \beta_{(g,H)} \neq 0\}) = C^*(\{p_i\}) .$$

Choose some $p \in \{p_i\}$ and write it as

$$p = \sum_j n_j e_{[g_jH_j]} - \sum_{j'} \tilde{n}_{j'} e_{[g_{j'}H_{j'}]}$$

with finitely many $n_j, \tilde{n}_{j'} \in \mathbb{Z}_{>0}$. We can write $p$ such that each projection $e_{[g,H]}$ appears at most one time and $\# \left[ \frac{K}{K \cap K} \right] \in \{1, \infty\}$ for all $K, \tilde{K} \in \{H_j, H_{j'}\}$ by Lemma 5.6.

\[ \square \]
Choose some maximal $H \in \{H_j, \tilde{H}_j\}$.
Take $g \in G$ and $n \in \mathbb{Z}_{>0}$ so that $ne_{[gH]}$ appears in $p$. Multiplying $p$ with $e_{[gH]}$ gives

$$e_{[gH]}p = ne_{[gH]} + \sum_k n_k e_{[c_k(H \cap H_k)]} - \sum_l \tilde{n}_l e_{[\tilde{c}_l(H \cap \tilde{H}_l)]},$$

for (finitely many) $c_k, \tilde{c}_l \in G$ and $n_k, \tilde{n}_l \in \mathbb{Z}_{>0}$.
Note that we must have $\# \left[ \frac{H}{H \cap H_k} \right] = \infty$ because if $\# \left[ \frac{H}{H \cap H_k} \right] = 1$ then $H_k = H$
would imply $e_{[gH_j]} = e_{[g\tilde{H}_j]}$ for some $j$ and $j'$.
Then, by Lemma 5.4

$$gH \not\subseteq \left[ \bigcup_k c_k(H \cap H_k) \right] \cup \left[ \bigcup_l \tilde{c}_l(H \cap \tilde{H}_l) \right],$$

which allows us to find $r \in gH \setminus \left[ \bigcup_k c_k(H \cap H_k) \right] \cup \left[ \bigcup_l \tilde{c}_l(H \cap \tilde{H}_l) \right]$.
One can conclude that:

$$e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]} \leq e_{[gH]},$$
$$e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]} \perp e_{[c_k(H \cap H_k)]}, \forall k, \text{ and}$$
$$e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]} \perp e_{[\tilde{c}_l(H \cap \tilde{H}_l)]}, \forall l.$$ 
Multiplying the equation above by $e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]}$ gives

$$e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]}p = ne_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]}.$$ 
As the first term is a projection (because it is the product of two commuting projections) we must have $n = 1$. So, $e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]} \leq p$.
If our additional hypothesis is satisfied, we have $\tilde{m} \in \mathbb{N}$ such that

$$e_{[\varphi^{\tilde{m}}(G)]} \leq e_{[r(c_k(H \cap H_k) \cap \tilde{c}_l(H \cap \tilde{H}_l))]} \leq p.$$ 
therefore we just have to denote $h_i = r$ and $m_i = \tilde{m}$. The conclusion holds if this is done for every element of $\{p_i\}$.

\[\square\]

**Remark 5.8.** Note that in last lemma for every $i$ we can choose $m_i$ as big as we want, because $\varphi^{m+1}(G) \subset \varphi^m(G)$.

**Theorem 5.9.** Let $G$ be an amenable group, $B$ some family of subgroups in $G$ containing $G$ and $\varphi$ a pure injective endomorphism of $G$. Also suppose that $\exists k \in \mathbb{N}$ such that $\varphi^k(G) \subseteq \bigcap_{H \in B} H$.
Then the $C^*$-algebra $U[\varphi, B]$ is purely infinite and simple.

**Proof:** We already have the candidates to use with Proposition 5.11 namely

$$\theta : U[\varphi, B] \to \Phi^{-1}(\iota(1)\overline{D}[\varphi, B]_{\iota(1)}),$$

and

$$S[G, B] = \text{span}(\{s^{*n}u_g^{-1}e[I]u_g's^m : I \in B, g, g' \in G, n, m \in \mathbb{N}\}).$$
Take $0 \neq x \in S[G, B]_{sa}$. Then

$$x = \sum_{(g,g',I,J)} \alpha_{(g,g',I,J)} s^{*I}u_g^{-1}e[I]u_g's^{J}.$$ 

\[5\varphi^m(G) \subseteq \bigcap_{H \in B} H \Rightarrow \varphi^{\tilde{m}}(G) \subseteq \bigcap_{H_i \in B'} H_i, \text{ for some } \tilde{m} \text{ bigger than } m.\]
As in previous Lemma 5.7,

$$\theta(x) = s^*\left(\sum_{(g,H)} \beta_{(g,H)} e_{[gH]}\right) s^n,$$

for some $n \in \mathbb{N}$ and $(g, H) \in G \times B'$ with $\beta_{(g,H)} \neq 0$ where $gH \subset \varphi^n(G)$.

By Lemma 5.7 we find finitely many pairwise orthogonal (nontrivial) projections $\{p_i\}$ with $C^*\{(e_{[gH]} : \beta_{(g,H)} \neq 0)\} = C^*\{(p_i)\}$ and there exist $m_i \in \mathbb{N}$ and $h_i \in G$ such that $e_{[h_i\varphi^{m_i}(G)]} \leq p_i \leq e_{[\varphi^n(G)]} \forall i$. Using Remark 5.8 we can suppose that $m_i \geq n \forall i$. Also note that $h_i \in \varphi^n(G)$.

Thus the projections $F_i := s^*e_{[h_i\varphi^{m_i}(G)]} s^n$ satisfy $F_i \leq s^*p_i s^n$ and

$$C^*\{(s^*e_{[h_i\varphi^{m_i}(G)]} s^n : \beta_{(g,H)} \neq 0)\} = C^*\{(s^*p_is^n)\} \rightarrow C^*\{(F_i)\} \\
y \mapsto \sum_i F_i y F_i$$

is an isomorphism that maps $s^*p_i s^n$ to $F_i$.

These projections $F_i$ satisfy only (i) and (ii) of the conditions in Proposition 5.1.

Call $(g, g', l, l', J)$ critical if $\alpha_{(g,g',l,l',J)} s^i u_{g^{-1}e_{lJ}u_{g'}} s'' \neq 0$ and $\delta_{g,g'} \delta_{l,l'} = 0$. Note that

$$x - \theta(x) = \sum_{(g,g',l,l',J) \text{ critical}} s^i u_{g^{-1}e_{lJ}u_{g'}} s''.$$

But for each $i$, it is possible to take some $a_i \in \varphi^{-n}(h_i) \varphi^{m_i - n}(G)$ satisfying

$$\varphi^{l'}(a_i^{-1}) g^{-1} g \varphi^l(a_i) \neq e$$

for all critical $(g, g', l, l', J)$.

Surely, if not then we have $r_1 \neq r_2 \in \varphi^{m_i - n}(G)$ such that

$$\varphi^{l'}(r_1^{-1}) g^{-1} g \varphi^l(r_1) = e \Rightarrow \varphi^{l'}(r_2^{-1}) g^{-1} g \varphi^l(r_2) = e = \varphi^{l'}(r_2^{-1}) g^{-1} g \varphi^l(r_2).$$

If $l = l'$ we have $g \neq g'$ (as $\delta_{g,g'} \delta_{l,l'} = 0$) and then

$$\varphi^l(r_1^{-1}) g^{-1} g \varphi^l(r_1) = e \Rightarrow g^{-1} g = e$$

which contradicts $g \neq g'$.

Suppose now that $l \neq l'$. As $r_1 = r_2 r_2^{-1} r_1$ we get

$$e = \varphi^{l'}((r_2 r_2^{-1} r_1)^{-1}) g^{-1} g \varphi^l(r_2 r_2^{-1} r_1) = \varphi^{l'}(r_1^{-1} r_2) \varphi^l(r_1^{-1} r_1)$$

which implies that $r_1 = r_2$ (because $\varphi$ is pure). This contradicts our assumptions.

Now as our endomorphism $\varphi$ is pure, for all critical $(g, g', l, l', J)$ and for all $i$ there exists $n_{(g,g',l,l',J,i)} \in \mathbb{N}$ (as big as we need) such that $\varphi^{l'}(a_i^{-1}) g^{-1} g \varphi^l(a_i) \notin \varphi^{n_{(g,g',l,l',J,i)}}(G)$.

Let us call

$$b_i := (m_i - n) \prod_{(g,g',l,l',J,i) \text{ critical}} n_{(g,g',l,l',J,i)}.$$

Note that

$$\varphi^{l'}(a_i^{-1}) g^{-1} g \varphi^l(a_i) \notin \varphi^{b_i}(G).$$

Define $f_i := e_{[a_i \varphi^{b_i}(G)]}$. We want to prove that these projections satisfy the conditions of Proposition 5.1 which are:

(i) $f_i \perp f_j, \forall i \neq j$,

(ii) $f_i \sim_{z_i} 1$, via isometries $z_i \in A, \forall i$,
(iii) \( \left\| \sum_i f_i \theta(x) f_i \right\| = \|\theta(x)\| \), and

(iv) \( f_i x f_i = f_i \theta(x) f_i \in C f_i, \forall i. \)

As \( b_i \geq m_i - n \) and \( \varphi^n(a_i) \in h_i \varphi^{m_i} \langle G \rangle \) it follows that 
\( s^n e_{[a_i \varphi^{m_i} \langle G \rangle]} s^n \leq e_{[h_i \varphi^{m_i} \langle G \rangle]} \) and then 
\[
f_i = s^n s e_{[a_i \varphi^{m_i} \langle G \rangle]} s^n s \leq s^n e_{[h_i \varphi^{m_i} \langle G \rangle]} s^n = F_i.
\]

This implies that \( f_i \perp f_j \forall i \neq j \) are pairwise orthogonal and (i) is satisfied. Item (ii) is also easily satisfied, because
\[
f_i = e_{[a_i \varphi^{m_i} \langle G \rangle]} = (u_{a_i} s^h_i)^* (u_{a_i} s^h_i)^* = 1.
\]

As (8) is an isomorphism and \( f_i \leq F_i \) the map
\[
C^* \langle \{ s^n e_{[g,H]} s^n : \beta_{(g,H)} \neq 0 \} \rangle \rightarrow C^* \langle \{ f_i \} \rangle
\]
\[
y \mapsto \sum_i f_i y f_i
\]
is an isomorphism as well. Therefore it is isometric and (iii) is satisfied.

And finally, for the last condition, let us expand \( f_i(x - \theta(x)) f_i \):
\[
f_i(x - \theta(x)) f_i = f_i \left( \sum_{(g,g',l,l',J) \text{ critical}} \beta_{(g,g',l,l',J)} s^l u_g - 1 e_{[l]} u_{g'} s^{l'} \right) f_i
\]
\[
= \sum_{(g,g',l,l',J) \text{critical}} \beta_{(g,g',l,l',J)} s^l u_g - 1 (u_{g'} s^{l'} f s^l u_g - 1) e_{[l]} (u_{g'} s^{l'} f s^l u_g - 1) u_{g'} s^{l'}
\]
\[
= \sum_{(g,g',l,l',J) \text{critical}} \beta_{(g,g',l,l',J)} s^l u_g - 1 e_{[g' \varphi^j (a_i) \varphi^{j+b_i} \langle G \rangle]} e_{[g' \varphi^j (a_i) \varphi^{j+b_i} \langle G \rangle]} e_{[l]} u_{g'} s^{l'}.
\]

Now, note that
\[
[g' \varphi^j (a_i) \varphi^{j+b_i} \langle G \rangle] \cap [g' \varphi^j (a_i) \varphi^{j+b_i} \langle G \rangle] \neq \emptyset \Rightarrow \varphi^j (a_i) \varphi^{j+b_i} \langle G \rangle \in \varphi^{b_i} \langle G \rangle
\]
which is a contradiction with our choice of \( b_i \) by (9). So the intersection above must be empty and then \( f_i x f_i = f_i \theta(x) f_i \in C f_i, \forall i. \)

Therefore, by Proposition 5.4, our \( C^* \)-algebra is simple and purely infinite.

\[\square\]

Corollary 5.10. When satisfied the conditions of the theorem above, the concrete \( C^* \)-algebra \( C^* \varphi, B \) is isomorphic to the universal one \( U \varphi, B \), as defined in Definitions 2.7 and 2.8 respectively.

\[\square\]

Theorem 5.11. If the conditions of the theorem above are satisfied, the universal \( C^* \)-algebra \( U \varphi, B \) is a Kirchberg algebra satisfying the UCT property (Propositions 4.4, 4.5, and 4.9).

\[\square\]
6. THE CASE $B = \{G\}$

In the following chapter we study the particular case when $B$ contains only subgroups of the form $g\varphi^k(G)$ for $k \in \mathbb{N}$ and $g \in G$. It will be now denoted $U[\varphi]$ and its $K$-theory will be calculated using a similar idea as presented in [10] i.e., using the continuity of the functors $K_0$ and $K_1$, and also the Khoshkam-Skandalis sequence [13]. We conclude that $K_1(U[\varphi]) \cong K_1(C^*(G))$ and that when $G$ is amenable, the $C^*$-algebras $U[\varphi]$ are classifiable by Kirchberg’s classification theorem.

To finish we use the recently-introduced semigroup $C^*$-algebras from [17] and [18] and show that $U[\varphi]$ is isomorphic to the full semigroup $C^*$-algebra of the semigroup $S = G \rtimes \varphi \mathbb{N}$. This implies that when the group $G$ is amenable and the endomorphism $\varphi$ is pure the three semigroup $C^*$-algebras (the full one, the reduced one and a third one, given by viewing $S$ as a subsemigroup of the group $S$ defined before Proposition 3.4) defined by Li are isomorphic to $U[\varphi]$ and classifiable by Kirchberg’s classification theorem.

By Proposition 2.7 if we choose $\mathcal{B}$ containing subsets of the form $g\varphi^k(G)$ for $k \in \mathbb{N}$ and $g \in G$, then $U[\varphi, B]$ is isomorphic to the one obtained when we start only with $B = \{G\}$. Therefore:

**Proposition 6.1.** When $B$ contains only subsets of the form $g\varphi^k(G)$, for some fixed $\varphi$, $k \in \mathbb{N}$ and $g \in G$, the $C^*$-algebra $U[\varphi, B]$ is isomorphic to $U[\varphi]$ and can be redefined as the universal $C^*$-algebra generated by

\[
\text{unitaries } \{u_g : g \in G\}
\]

and one isometry $\{s\}$.

satisfying:

(i) \[ u_g s^n u_h s^m = u_{g\varphi^n(h)} s^{n+m}; \]

(ii) \[
\begin{align*}
& u_g s^n s^n u_{g^{-1}} u_h s^m s^m u_{h^{-1}} = u_h s^m s^m u_{h^{-1}} u_g s^n s^m u_{g^{-1}} = \\
& \begin{cases} 
& u_g s^n s^m u_{g^{-1}}, \text{ if } h \in g\varphi^m(G); \\
& 0, \text{ otherwise},
\end{cases}
\end{align*}
\]

for $n \geq m$.

\[ \square \]

A simple use of Propositions 4.1, 4.2 and 4.3, and Theorem 5.9 gives the following.

**Proposition 6.2.** The $C^*$-algebra $U[\varphi]$ is separable. When the group $G$ is amenable, it is also nuclear and satisfies UCT. Furthermore if $G$ is amenable and $\varphi$ is pure, then $U[\varphi]$ is also simple and purely infinite, therefore a Kirchberg algebra satisfying UCT.

\[ \square \]

6.1. K-theory. Using Remark 3.2 we know that

\[ U[\varphi] \cong (D[\varphi] \rtimes_\omega G) \rtimes_\tau \mathbb{N} \]
with
\[ \omega : G \to \text{Aut}(D[\varphi]) \]
\[ g \mapsto u_g(\cdot)u_g^{-1}, \]

\[ \tau : \mathbb{N} \to \text{End}(D[\varphi] \rtimes_\omega G) \]
\[ n \mapsto s^n(\cdot)s^{-n} \]

where for \( a_g \delta_g \in D[\varphi] \rtimes_\omega G \), \( \tau_n(a_g \delta_g) = s^n a_g s^n \delta_{\varphi^n(g)} \). But note that
\[ D[\varphi] \cong \lim_{\longrightarrow} D_n \]

for \( n \in \mathbb{N} \) with
\[ D_n := C^* \left( \left\{ u_g s^k s^k u_{g^{-1}} : 0 \leq k \leq n, \ g \in \frac{G}{\varphi^k(G)} \right\} \right) \]

and the inclusion being the identity. Therefore
\[ D[\varphi] \rtimes_\omega G \cong \lim_{\longrightarrow} (D_n \rtimes_\omega G), \]

where
\[ D_n \rtimes_\omega G \cong C^* \left( \left\{ u_g s^k s^k u_{h^{-1}} : 0 \leq k \leq n, \ g, h \in G \right\} \right). \]

Moreover, for \( k \in \mathbb{N} \), \( A_k := C^* (\left\{ u_g s^k s^k u_{h^{-1}} : g, h \in G \right\}) \) is an ideal of \( D_k \rtimes_\omega G \), because for \( m \leq k \),
\[ u_h s^k s^k u_{h^{-1}} u_g s^m s^m u_{g^{-1}} = \begin{cases} u_h s^k s^k u_{h^{-1}}, & \text{if } g \in h \varphi^m(G) \\ 0, & \text{otherwise}. \end{cases} \]

But note that every element \( u_g s^k s^k u_{h^{-1}} \) in \( A_k \) can be uniquely written as \( u_{g_i} s^k s^k u_{g_j^{-1}} u_{\varphi^k(t)} \), for \( g_i, g_j \in \frac{G}{\varphi^k(G)} \) and \( t \in G \). Therefore, if one defines the correspondence
\[ u_g s^k s^k u_{h^{-1}} = u_{g_i} s^k s^k u_{g_j^{-1}} u_{\varphi^k(t)} \mapsto E_{i,j} \otimes u_{\varphi^k(t)}, \]

where \( \{E_{i,j}\} \) is the family of unit matrices which give rise to the set \( K \) of compact operators, it follows that
\[ A_k \cong K \otimes C^* (\varphi^k(G)) \cong K \otimes C^*(G). \]

So starting with the case \( n = 1 \), we can build the following exact sequence:
\[ 0 \to A_1 \to D_1 \rtimes_\omega G \xrightarrow{\rho} C^*(G) \to 0 \]

where \( \iota \) and \( \rho \) are the canonical inclusion and projection maps respectively. But the sequence above splits if we also consider the canonical inclusion
\[ \gamma : C^*(G) \to D_1 \rtimes_\omega G. \]

This implies that the corresponding exact sequence of \( K \)-groups also splits, which means that (using the Künneth Formula [21])
\[ K_\ast(D_1 \rtimes_\omega G) \cong K_\ast(C^*(G)) \oplus K_\ast(K \otimes C^*(\varphi(G))) \]
\[ \cong K_\ast(C^*(G)) \oplus K_\ast(C^*(G)). \]

Using the same argument repeatedly, it is easy to conclude that
\[ K_\ast(D_n \rtimes_\omega G) = \bigoplus_{i=0}^n K_\ast(C^*(G)) \]
and consequently
\begin{equation}
K_*(D[\varphi] \rtimes_\omega G) = \lim_{n \to \infty} \bigoplus_{i=0}^n K_0(C^*(G)) = \bigoplus_{i=0}^n K_0(C^*(G)),
\end{equation}
where the \( k \)-th group \( K_* (C^*(G)) \) of the direct sum above represents the K-group of \( A_k \).

Applying the Khoshkam-Skandalis sequence for \( \mathbb{N} \)-crossed products [13], we have the following sequence:

\[
\begin{array}{c}
\bigoplus_{\mathbb{N}} K_0(C^*(G)) \xrightarrow{1-K_0(\tau)} \bigoplus_{\mathbb{N}} K_1(C^*(G)) \xleftarrow{1-K_1(\tau)} \bigoplus_{\mathbb{N}} K_0(C^*(G)) \\
\uparrow \qquad \downarrow \\
K_1(U[\varphi]) \xleftarrow{1-\sigma} \bigoplus_{\mathbb{N}} K_0(C^*(G)) \xrightarrow{1-\sigma} K_0(U[\varphi])
\end{array}
\]

where \( \tau_n(u_g) = s^n u_g s^{-n} \). Since \( K_0(K) \) is described only by matrices of the type \( E_{i,i} \), consider some \( u_g, s^n s^{-n} u_{g^{-1}} u_{\varphi^n(t)} \in D[\varphi] \rtimes_\omega G \). Then

\[
K_*(\tau)[u_g, s^n s^{-n} u_{g^{-1}} u_{\varphi^n(t)}] = [u_{\varphi(g)} s^n s^{-n} u_{\varphi(g^{-1})} u_{\varphi^n(t)}],
\]

which implies that \( K_*(\tau) \) corresponds to a shift in \( \bigoplus_{\mathbb{N}} K_* (C^*(G)) \). So denote by \( \sigma \) the shift operator, to see that the six-term sequence above turns into

\[
\begin{array}{c}
\bigoplus_{\mathbb{N}} K_0(C^*(G)) \xrightarrow{1-\sigma} \bigoplus_{\mathbb{N}} K_0(C^*(G)) \xrightarrow{1-\sigma} \bigoplus_{\mathbb{N}} K_1(C^*(G)) \\
\uparrow \quad \downarrow \\
K_1(U[\varphi]) \xleftarrow{1-\sigma} \bigoplus_{\mathbb{N}} K_0(C^*(G)) \xrightarrow{1-\sigma} \bigoplus_{\mathbb{N}} K_1(C^*(G))
\end{array}
\]

But the application \( 1 - \sigma \) has null kernel and \( \text{Im}(1 - \sigma) \) only contains vectors \((x_0, x_1, \ldots, x_n, 0, 0, \ldots)\) whose sum of coordinates equals zero. This together with the direct limit description [13] implies that

\[
\frac{\bigoplus_{\mathbb{N}} K_* (C^*(G))}{\text{Im} (1 - \sigma)} \cong K_*(C^*(G))
\]

via

\[
(x_0, x_1, \ldots, x_n, 0, 0, \ldots) \mapsto \sum_{i=0}^n x_i.
\]

Solving the six-term sequence we get

\[
K_*(U[\varphi]) \cong K_*(C^*(G)).
\]

Therefore, we have the following.

**Theorem 6.3.** Consider \( \varphi \) an injective endomorphism with infinite cokernel of some discrete countable group \( G \), and construct the \( C^* \)-algebra \( U[\varphi] \) as in Proposition 6.1. Then \( K_*(U[\varphi]) \cong K_*(C^*(G)) \).

\[\square\]

**Theorem 6.4.** Consider \( \varphi \) a pure injective endomorphism with infinite cokernel of some discrete countable amenable group \( G \), and construct the \( C^* \)-algebra \( U[\varphi] \) as in Proposition 6.1. Then it is classifiable by Kirchberg’s classification theorem.

\[\square\]

But note that, if we take two different pure injective endomorphisms of some discrete countable amenable group \( G \), both \( C^* \)-algebras will be classifiable by Kirchberg’s theorem, and in both objects \( K_0(U[\varphi]) \ni [1]_0 \mapsto [1]_0 \in K_0(C^*(G)) \). Thus they are isomorphic.

\[\text{The projections and unitaries of } D[\varphi] \rtimes_\omega G \text{ are combinations of elements of that type.}\]
Corollary 6.5. Satisfied the conditions above, for a fixed group $G$ any choice of endomorphism $\varphi$ generates the same $C^*$-algebra $U[\varphi]$.

\[
\square
\]

6.2. Semigroup $C^*$-algebra description of $U[\varphi]$. In [17] and [18] Li introduced and developed the concept of a $C^*$-algebra associated with a semigroup. His definitions are similar to our $C^*$-algebra associated with an endomorphism, and we will prove that when the semigroup is of the form $S = G \rtimes \varphi \mathbb{N}$ i.e., a semidirect product of a group $G$ with $\mathbb{N}$ implemented by an injective endomorphism, the $C^*$-algebra of this semigroup can be viewed as the $C^*$-algebra associated with the endomorphism $\varphi$ defined in this section. This isomorphism together with extra restrictions on our initial data will allow us to conclude similar results concerning the K-theory of $U[\varphi]$ as the one obtained in Theorem 6.3.

The first clue to suggest this isomorphism is that both constructions use a set of isometries indexed by the semigroup to generate the $C^*$-algebras. And the main step in getting the desired isomorphism is to compare the set of projections used in both definitions and, for this purpose, we shall study the sets which index these projections, namely $B'$ in our case (Lemma 2.4) and the set $J$ of constructible right ideals in Li’s case (before Definition 2.2 [18]). Note that both are defined as a certain set of subsets of the given structure, and they are closed with respect to some set operations. The problem is that here $B'$ is a set of subsets of a group and Li defines $J$ containing subsets of a semigroup. However the following holds:

Proposition 6.6. $J = \{(g, n)S : (g, n) \in S\}$.

Proof: One just have to use the fact that sets of the type

\[(g, n)S \cap (h, m)S\]

and

\[(g, n)^{-1}(h, m)S\]

are both of the form $(k, l)S$ or $\emptyset$.

This result is also proved in [8] Lemma 6.3.3.

The result above will allow us to establish the isomorphism between the algebra $U[\varphi]$ defined in this chapter and the full semigroup $C^*$-algebra $C^*(S)$ defined by Li in Definition 2.2 of [18].

Consider an endomorphism $\varphi$ of a group $G$ with $B$ containing only subgroups of the form $\varphi^k(G)$. By Proposition 6.1 the $C^*$-algebra $U[\varphi]$ is the universal one generated by

- unitaries $\{u_g : g \in G\}$
- one isometry $s$

satisfying

\[(i) \quad u_g s^n u_h s^m = u_{g \varphi^n(h)} s^{n+m}.\]

Proposition 6.7. We have

\[U[\varphi] \cong C^*(S),\]

with the latter defined as in [18].
Proof: The C*-algebra \( C^*(S) \) is generated by isometries \( \{ v_{(g,n)} : (g,n) \in S \} \) and projections \( \{ e_X : X \in \mathcal{J} \} \) with \( \mathcal{J} = \{(g,n)S : (g,n) \in S \} \) (by the proposition above).

To prove that the isomorphism holds, first note that the unitaries \( v_{(g,0)} \) and the isometries \( v_{(e,n)} \) satisfy the relation generating \( \mathbb{U}[\varphi] \) ((i) above), so there exists a *-homomorphism

\[
    \Phi : \mathbb{U}[\varphi] \to C^*(S)
    \quad u_g \mapsto v_{(g,0)}; \\
    s^n \mapsto v_{(e,n)}.
\]

For the inverse map, consider the set of isometries \( \{ u_{g}s^n : (g,n) \in S \} \) and the set of projections \( \{ u_{h}s^m : (h,m)S \in \mathcal{J} \} \).

Some calculations show that these two sets satisfy the 5 conditions generating \( C^*(S) \) (ref. [18]). By the universality of this C*-algebra there exists a *-homomorphism

\[
    \Psi : C^*_s(S) \to \mathbb{U}[\varphi]
    \quad v_{(g,n)} \mapsto u_{g}s^n, \text{ and } \\
    e_{((h,m)S)} \mapsto u_hs^m s^m u_{h^{-1}}.
\]

It is easy to see that \( \Phi \) and \( \Psi \) are inverses of each other.

Corollary 6.8. Consider \( \varphi \) an injective endomorphism with infinite cokernel of some discrete countable group \( G \) and the semidirect product semigroup \( S = G \rtimes_\varphi \mathbb{N} \). Then

\[
    K_*(C^*(S)) \cong K_*(C^*(G)),
\]

with \( C^*(S) \) as defined in [18].

There are two more C*-algebras associated with a semigroup \( S \). The first one is the concrete representation of \( S \) called the reduced semigroup C*-algebra of \( S \), denoted by \( C^*_r(S) \) and defined in Definition 2.1 in [18]. It is easy to check that there exists a surjective *-homomorphism

\[
    \lambda : C^*(S) \to C^*_r(S).
\]

For the second one, note that the semigroup \( S \) can be viewed as a subsemigroup of the group \( \mathfrak{N} \) (defined in the beginning of Section 4.1, Chapter 1), and this allows us to define another C*-algebra associated with \( S \), namely \( C^*_s(S) \) (Definition 3.2 of [18]). It has the same generators as \( C^*(S) \) with minor additional relations, so that there is a surjective *-homomorphism

\[
    \pi_s : C^*(S) \to C^*_s(S).
\]

But remember that if \( \varphi \) is pure and \( G \) is amenable the C*-algebra \( \mathbb{U}[\varphi] \) is simple (and purely infinite) by Theorem 5.9 and thus so is \( C^*(S) \). Therefore we have:

Theorem 6.9. For semigroups of the form \( S = G \rtimes_\varphi \mathbb{N} \) with \( G \) an amenable discrete countable group and \( \varphi \) a pure injective endomorphism of \( G \), the C*-algebras \( C^*(S) \), \( C^*_r(S) \) and \( C^*_s(S) \) defined in [18] are isomorphic to \( \mathbb{U}[\varphi] \). By Theorem 6.3, we also conclude that

\[
    K_*(C^*(S)) \cong K_*(C^*(G)).
\]

Moreover by Theorem 6.4, they are classifiable by Kirchberg’s classification theorem [14].
Theorem 6.10. For an amenable group $G$ and a pure injective endomorphism with infinite cokernel $\varphi$ of $G$ consider the semigroup $S = G \rtimes \mathbb{N}$ and choose $B = \{\varphi^k(G)\}$ for some $k \in \mathbb{N}$. Then
\[
K_*(\mathbb{U}[\varphi]) \cong K_*(C^*(G)).
\]
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