ON THE VOLUMES OF LINEAR SUBVARIETIES IN MODULI SPACES OF
PROJECTIVIZED ABELIAN DIFFERENTIALS

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Abstract. For \( k \in \mathbb{Z}_{>0} \), let \( \mathcal{H}_{k,0}^{(q)} \) denote the vector bundle over \( \mathcal{M}_{g,n} \), whose every fiber consists of meromorphic \( k \)-differentials with poles of order at most \( k - 1 \) on a fixed Riemann surface of genus \( g \) with \( n \) marked points (all the poles must be located at the marked points). The bundle \( \mathcal{H}_{k,0}^{(q)} \) and its associated projective bundle \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \) admit natural extensions, denoted by \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \) and \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \) respectively, to the Deligne-Mumford compactification \( \overline{\mathcal{M}}_{g,n} \) of \( \mathcal{M}_{g,n} \). We prove the following statement: let \( M \) be a subvariety of dimension \( d \) of the projective bundle \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \). Denote by \( \mathcal{O}^{(-1)}_{\mathcal{H}_{k,0}^{(q)}} \) the tautological line bundle over \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \). Then the integral of the \( d \)-th power of the curvature form of the Hodge norm on \( \mathcal{O}^{(-1)}_{\mathcal{H}_{k,0}^{(q)}} \) over the smooth part of \( M \) is equal to the intersection number of the \( d \)-th power of the divisor representing \( \mathcal{O}^{(-1)}_{\mathcal{H}_{k,0}^{(q)}} \) and the closure of \( M \) in \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \). As a consequence, if \( M \) is a linear subvariety of the projectivized Hodge bundle \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \) whose local coordinates do not involve relative periods, then the volume of \( M \) can be computed by the self-intersection number of the tautological line bundle on the closure of \( M \) in \( \mathbb{P}\mathcal{H}_{k,0}^{(q)} \).

1. Statements of the main results

1.1. Integration of the curvature form of the Hodge norm on moduli spaces of projectivized \( k \)-differentials. Let \( g, n \) be two non-negative integers such that \( 2g - 2 + n > 0 \). Let \( \overline{\mathcal{M}}_{g,n} \) denote the moduli space of complex \( n \)-pointed stable curves of genus \( g \), and \( \pi_{g,n} : \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n} \) the universal curve over \( \overline{\mathcal{M}}_{g,n} \). As usual, \( \mathcal{M}_{g,n} \) denotes the subset of \( \overline{\mathcal{M}}_{g,n} \) which parametrizes \( n \)-pointed smooth curves (i.e. compact Riemann surfaces with \( n \) marked points). Let \( K_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}} \) denote the relative canonical line bundle of \( \pi_{g,n} \), and \( D_i \) denote the image of the tautological section \( \sigma_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n} \) associated with the \( i \)-th marked points for \( i = 1, \ldots, n \).

Fix a positive integer \( k \in \mathbb{Z}_{>0} \). Define

\[
\mathcal{K}_{g,n}^{(k)} := K_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sum_{i=1}^{n} (k - 1) \cdot D_i).
\]

For every point \( q \in \overline{\mathcal{M}}_{g,n} \) denote by \( C_q \) the fiber of \( \pi_{g,n} \) over \( q \). Let \( x_i = C_q \cap D_i \) be the \( i \)-th marked point on \( C_q \). Then \( \mathcal{K}_{g,n}^{(k)}(\overline{\mathcal{C}}_{g,n}) \sim \omega_{C_q}^{(k)}(\sum_{i=1}^{n} (k - 1) \cdot x_i) \), where \( \omega_{C_q} \) is the dualizing sheaf of \( C_q \). By Riemann-Roch Theorem, one readily sees that

\[
\dim H^0(C_q, \mathcal{K}_{g,n}^{(k)}(\overline{\mathcal{C}}_{g,n})) = \begin{cases} 
\frac{g}{2} & \text{if } k = 1 \\
(g - 1)(2k - 1) + n(k - 1) & \text{if } k > 1.
\end{cases}
\]
Since \( \dim H^0(C_q, \mathcal{N}^{(k)}_{g,n|C_q}) \) does not depend on \( q \), the direct image \( \pi_{g,n}^{*}\mathcal{N}^{(k)}_{g,n} \) is a (holomorphic) vector bundle \( \mathcal{H}^{(k)}_{g,n} \) over \( \mathfrak{M}_{g,n} \). The fiber of \( \mathcal{H}^{(k)}_{g,n} \) over \( q \in \mathfrak{M}_{g,n} \) is identified with \( H^0(C, \omega_{C_q}^{\otimes k} \sum_{i=1}^{n} (k-1) \cdot x_i) \), that is the space of \( k \)-differentials on \( C \) with poles of order at most \( k-1 \) at the marked points. Let \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) denote the projective bundle associated with \( \mathcal{H}^{(k)}_{g,n} \). Elements of \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) will be called projectivized \( k \)-differentials.

Let \( \mathcal{H}^{(k)}_{g,n} \) and \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) be the restrictions of \( \mathcal{H}^{(k)}_{g,n} \) and \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) to \( \mathfrak{M}_{g,n} \) respectively. In the case \( k = 1 \), \( \mathcal{H}^{(1)}_{g,n} \) is the usual Hodge bundle. For simplicity, in this case we will write \( \mathcal{H}^{(1)}_{g,n}, \mathbb{P}\mathcal{H}^{(1)}_{g,n}, \mathcal{H}^{(2)}_{g,n}, \mathbb{P}\mathcal{H}^{(2)}_{g,n} \) instead of \( \mathcal{H}^{(1)}_{g,n}, \mathbb{P}\mathcal{H}^{(1)}_{g,n}, \mathcal{H}^{(1)}_{g,n}, \mathbb{P}\mathcal{H}^{(1)}_{g,n} \) respectively. Note also that in the case \( k = 2 \), \( \mathcal{H}^{(2)}_{g,n} \) is in fact the cotangent bundle of \( \mathfrak{M}_{g,n} \).

By definition, elements of \( \mathcal{H}^{(k)}_{g,n}\setminus\{0\} \) are tuples \((C, x_1, \ldots, x_n, \eta)\), where \( C \) is a compact Riemann surface of genus \( g \), \( x_1, \ldots, x_n \) are \( n \) marked points on \( C \), and \( \eta \) is a meromorphic \( k \)-differential on \( C \) such that all of its poles have order at most \( k-1 \), and are located in the set \( \{x_1, \ldots, x_n\} \). We define

\[
\|\eta\| := \left( \int_C |\eta|^{2/k} \right)^{1/k}.
\]

Since the poles of \( \eta \) have orders at most \( k-1 \), the integral in the right hand side of (1) is finite. Note that \( \int_C |\eta|^{2/k} \) is the total area of \( C \) with respect to the flat metric defined by \( |\eta|^{2/k} \). For this reason, we call \( \|\cdot\| \) a finite area \( k \)-differential on \( C \). It is not difficult to check that \( \|\cdot\| \) gives a norm on each fiber of the bundle \( \mathcal{H}^{(k)}_{g,n} \), which will be called the Hodge norm. In the case \( k = 1 \), \( \|\cdot\| \) is induced by the usual Hodge metric on \( \mathcal{H}^{(1)}_{g,n} \).

The Hodge norm provides us with a Hermitian metric on \( \mathcal{O}(-1)_{\mathcal{H}^{(k)}_{g,n}} \), the tautological line bundle over \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \). Let \( \Theta \) denote the curvature form of this metric. In this paper we will prove

**Theorem 1.1.** Let \( N \) be a subvariety of \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) of dimension \( d \). Denote by \( N_0 \) the set of regular points of \( N \), and by \( \overline{N} \) the closure of \( N \) in \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \). Then we have

\[
\int_{N_0} \left( \frac{1}{2\pi} \Theta \right)^d = c_1\left( \mathcal{O}(-1)_{\mathbb{P}\mathcal{H}^{(k)}_{g,n}} \right) \cdot [\overline{N}]
\]

where \( \mathcal{O}(-1)_{\mathbb{P}\mathcal{H}^{(k)}_{g,n}} \) is the tautological line bundle over \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \), and \([\overline{N}]\) is the equivalence class of \( \overline{N} \) in \( A_*(\mathbb{P}\mathcal{H}^{(k)}_{g,n}) \).

If \( \overline{N} \) is a smooth subvariety of \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) then (2) trivially holds since the cohomology class of \( \frac{1}{2\pi} \Theta \) is equal to \( c_1(\mathcal{O}(-1)_{\mathbb{P}\mathcal{H}^{(k)}_{g,n}}) \). The content of Theorem 1.1 is that (2) also holds when \( \overline{N} \) is singular and intersects the boundary of \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \). In the case \( N \) is a stratum of \( \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) (see §1.2, Theorem 1.1 has been known by the works [5,10].

An unexpected consequence of Theorem 1.1 is the following

**Corollary 1.2.** Let \( N \subset \mathbb{P}\mathcal{H}^{(k)}_{g,n} \) be a subvariety of dimension \( d \geq 2g \). Then

\[
c_1^d(\mathcal{O}(-1)_{\mathbb{P}\mathcal{H}^{(k)}_{g,n}}) \cdot [\overline{N}] = 0.
\]
This corollary is in fact not new because it can be derived from the fact that $c_1^{2g}(\mathcal{O}(-1)_{\mathbb{P}^k_{g,n}}) = 0 \in H^4_{\ast}(\mathbb{P}^k_{g,n})$, which is a consequence of the Mumford relation $c(E) \cdot c(E^*) = 1$. Nevertheless Corollary 1.2 gives a stark contrast between the cases $k = 1$ and $k > 1$. Namely, for $k > 1$, let $\mathcal{N}$ be the principal stratum of $\mathbb{P}^k_{g,n}$ (that is the space of $k$-differentials which have poles of order $(k - 1)$ at the marked points and only simple zeros elsewhere). In this case $\mathcal{N} = \mathbb{P}^k_{g,n}$, and $(\theta \Theta)^d$ is a volume form on $\mathcal{N}$. In particular, the left hand side of (2) must be non-zero. Therefore we have $c_1^{\dim \mathbb{P}^k_{g,n}}(\mathcal{O}(-1)_{\mathbb{P}^k_{g,n}}) \neq 0 \in H^4(\mathbb{P}^k_{g,n})$ if $k > 1$.

1.2. Volumes of linear subvarieties in the moduli spaces of projectivized Abelian differentials.

A stratum of $\mathcal{H}^{(k)}_{g,n}$ is a subvariety consisting of (meromorphic) $k$-differentials whose number of zeros and poles as well as their order are prescribed. Specifically, let $\kappa = (\kappa_1, \ldots, \kappa_\ell)$, with $\ell \geq n$, be a sequence of integers such that

- $\kappa_i \in \mathbb{Z}_{> 0}$, for $i = 1, \ldots, n$,
- $\kappa_i \in \mathbb{Z}_{> 0}$, for $i = n + 1, \ldots, \ell$,
- $\kappa_1 + \cdots + \kappa_\ell = k(2g - 2)$.

Then the set

$$\mathcal{H}^{(k)}_{g,n}(\kappa) := \{(X, x_1, \ldots, x_n, \eta) \in \mathcal{H}^{(k)}_{g,n}, \ \eta = \kappa_1 \cdot x_1 + \cdots + \kappa_\ell \cdot x_\ell\}$$

where $x_{n+1}, \ldots, x_\ell$ are $\ell - n$ distinct points in $X \setminus \{x_1, \ldots, x_n\}$ is called a stratum of $\mathcal{H}^{(k)}_{g,n}$. By convention, we consider all the points in $\{x_1, \ldots, x_n\}$ as zeros or poles of $\eta$, possibly with order 0. We will call $\{x_1, \ldots, x_\ell\}$ the zero set of $\eta$ and denote it by $Z(\eta)$.

Consider now a stratum $\mathcal{H}^{(k)}_{g,n}(\kappa)$ of the Hodge bundle $\mathcal{H}^{(k)}_{g,n}$. Let $x = (X, x_1, \ldots, x_n, \eta)$ be a point in $\mathcal{H}^{(k)}_{g,n}(\kappa)$. Then via period mappings (see §6.1 for more details), a neighborhood of $x$ in $\mathcal{H}^{(k)}_{g,n}(\kappa)$ is identified with an open subset of $H^1(X, \mathcal{O}(\eta), \mathbb{C})$. Let $\mathcal{p} : H^1(X, \mathcal{O}(\eta), \mathbb{C}) \to H^1(X, \mathbb{C})$ be the natural projection. A linear subvariety of $\mathcal{H}^{(k)}_{g,n}(\kappa)$ is an algebraic subvariety $\Omega M$ which satisfies the following

- if $x = (X, x_1, \ldots, x_n, \eta) \in \Omega M$, then the image under the period mappings of any local branch of $\Omega M$ through $x$ is an open subset of a vector subspace $V$ of $H^1(X, \mathcal{O}(\eta), \mathbb{C})$, and
- the restriction of the intersection form on $H^1(X, \mathbb{C})$ to $\mathcal{p}(V)$ is non-degenerate.

If in addition we have $\ker(\mathcal{p}) \cap V = \{0\}$, then $\Omega M$ is said to be absolutely rigid. We denote by $\mathcal{M}$ the projection of $\Omega M$ in $\mathbb{P}\mathcal{H}^{(k)}_{g,n}$. On an absolutely rigid linear subvariety $\Omega M$ of $\mathcal{H}^{(k)}_{g,n}(\kappa)$, one has a canonical volume form constructed from the intersection form on $H^1(X, \mathbb{C})$. By a standard construction (see for instance [32]), this volume form induces in turn a volume form $d\mu$ on $\mathcal{M}$.

In the case of $k$-differentials with $k \geq 2$, it is a well known fact that any stratum $\mathcal{H}^{(k)}_{g,n}(\kappa)$ can be locally identified with a linear subvariety of some stratum $\mathcal{H}^{(k)}_{g,n}(\hat{\kappa})$ of Abelian differentials via the cyclic cover construction. For strata $\mathcal{H}^{(k)}_{g,n}(\kappa)$ such that none of the entries of $\kappa$ is divisible by $k$, the corresponding linear subvariety is absolutely rigid. Thus, for such strata we also have a canonical volume form $d\mu$ on $\mathbb{P}\mathcal{H}^{(k)}_{g,n}(\kappa)$ (see [27], [26]).

Among linear subvarieties of $\mathcal{H}^{(k)}_{g,n}$, those that are defined locally by linear equations with real coefficients are of particular interest. They are actually orbit closures for an action of $\text{GL}^+(2, \mathbb{R})$ on $\mathcal{H}^{(k)}_{g,n}$, and known as affine invariant submanifolds of $\mathcal{H}^{(k)}_{g,n}$ (see [13], [14], [17]). Their volume is a significant invariant as it allows one to compute other dynamical invariants such as the Lyapunov
exponents of the Teichmüller geodesic flow or the Siegel-Veech constants. It is also well known that the volumes of strata of \(k\)-differentials, with \(k \in \{1, 2, 3, 4, 6\}\), give the asymptotics of the counting of tilings of surfaces by triangles and squares (see for instance \([15, 12, 26]\)).

As an application of Theorem \([1, 2]\) we obtain the following, which is the main motivation of this paper.

**Theorem 1.3.**

a) Let \(\Omega M\) be an absolutely rigid linear subvariety of \(\mathcal{H}_{g,n}\) and \(M\) its projectivization in \(\mathbb{P}\mathcal{H}_{g,n}\). Denote by \(\overline{M}\) the closure of \(M\) is \(\mathbb{P}\mathcal{H}_{g,n}\). Then we have

\[
\mu(M) = (-1)^d \cdot \frac{\pi^{d+1}}{(d+1)!} \cdot c_d((\partial(-1)_{\mathbb{P}\mathcal{H}_{g,n}}) \cdot [\overline{M}]
\]

where \(d = \dim M\), and \([\overline{M}]\) is the class of \(M\) in \(A_*(\mathbb{P}\mathcal{H}_{g,n})\).

b) Let \(\mathcal{H}_{g,n}^{(k)}(\kappa)\) be a stratum of \(\mathcal{H}_{g,n}^{(k)}\) and \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)\) the projectivization of \(\mathcal{H}_{g,n}^{(k)}(\kappa)\) in \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}\) with \(k \geq 2\). Let \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)\) denote the closure of \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)\) in \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}\). Assume that none of the entries of \(\kappa\) is divisible by \(k\). Then we have

\[
\mu(\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)) = (-1)^d \cdot \frac{\pi^{d+1}}{(d+1)!} \cdot c_d((\partial(-1)_{\mathbb{P}\mathcal{H}_{g,n}^{(k)}}) \cdot [\overline{\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)}])
\]

where \(d = \dim \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)\), and \([\overline{\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)}]\) is the class of \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)\) in \(A_*(\mathbb{P}\mathcal{H}_{g,n}^{(k)})\).

**Remark 1.4.** The constant on the right hand side of (4) depends obviously on the volume form we choose on the stratum \(\mathcal{H}_{g,n}^{(k)}(\kappa)\). The values of this constant that correspond to other volume forms on \(\mathcal{H}_{g,n}^{(k)}(\kappa)\) have been computed in the literature (see for instance \([8, \S 2]\) and \([29, \S 5]\)).

1.3. **Context and related works.** Each stratum of the bundles \(\mathcal{H}_{g,n}^{(1)}(\kappa)\) and \(\mathcal{H}_{g,n}^{(2)}(\kappa)\), carries a special volume form called the Masur-Veech measure. The computation of the Masur-Veech volumes of those strata has attracted great attention because of their application to billiards and Teichmüller dynamics. For a thorough introduction to these fascinating fields of research we refer to the excellent surveys \([28, 32, 34]\). Masur-Veech measure can also be defined on strata of \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa)\) for \(k \in \{3, 4, 6\}\) (see for instance \([12]\)). In all cases, the Masur-Veech measure always differs from the volume form \(d\mu\) in \(\mathcal{H}_{g,n}\) by a constant (see \([27]\) for more details on this constant).

It was shown in \([9]\) that the Masur-Veech volumes of the strata of \(\mathbb{P}\mathcal{H}_{g,n}\) can be computed by intersection theory on the projectivized Hodge bundle \(\mathbb{P}\mathcal{H}_{g,n}\). In \([10]\), Costantini, Möller, and Zachhuber showed that \(\Omega M\) is a stratum of \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}\) for all \(k\). Their proof uses in an essential way specific compactifications of strata of \(\mathbb{P}\mathcal{H}_{g,n}^{(k)}\) known as the moduli spaces of multiscale differentials that are constructed for Abelian differentials in \([5]\) and generalized to \(k\)-differentials for all \(k \geq 2\) in \([10]\). The proof we present in this paper does not rely on those constructions.

Linear subvarieties of \(\mathcal{H}_{g,n}\) are generalizations of affine invariant submanifolds introduced in \([13]\) (see also \([2]\) and \([17]\)). The definition of a linear subvariety we use in this paper has been introduced in \([26]\). In \([6]\) and \([7]\), Benirschke, Dozier, and Grushevsky give a similar definition (without the requirement that the restriction of the intersection form to the fiber of the tangent bundle is non-degenerate) and prove some important properties of those subvarieties. One of the advantages of the
Since \([24]\), any power of the curvature form of the Hodge norm on \(\mathcal{M}\) we have a smooth projective variety \(\hat{Y}\) admitting a surjective morphism onto \(\mathcal{M}\), and where \(\partial(-1)\mathcal{H}_{g,n}\) is replaced by some line bundle on \(\hat{Y}\).

In general, there is no Masur-Veech measure on a linear subvariety unless it is defined by linear equations with rational coefficients, in which case it is called \textit{arithmetic} (see \([33]\) for more details on the field of definition of an affine invariant submanifold). The Masur-Veech volume of an arithmetic affine invariant submanifold can be interpreted as the asymptotics of the numbers of square-tiled surfaces contained in this submanifold. Similarly, the Masur-Veech volumes of strata to \(k\)-differentials with \(k \in \{2, 3, 4, 6\}\) can also be interpreted as the asymptotics of the numbers of triangulations or quadrangulations on a given (topological) surface. Theorem 1.3 means that for absolutely rigid linear subvarieties (as well as strata of \(P\)) the Masur-Veech volume can be computed from the self-intersection number of the tautological line bundle modulo the ratio between the Masur-Veech measure and the volume form \(d\mu\).

1.4. \textbf{Outline of the proof of Theorem 1.1}

1.4.1. \textit{Case} \(k = 1\). The main difficulty of Theorem 1.1 in this case is that the Hodge norm does not extend smoothly to the boundary \(\partial\mathcal{P}H_{g,n} = \mathcal{P}H_{g,n} \setminus \mathcal{P}H_{g,n}^0\) and that \(\mathcal{N}\) may acquire singularities at \(\partial\mathcal{P}H_{g,n}\). Recall that \(\mathcal{M}_{g,n}\) carries a Variation of Hodge Structure (VHS) of weight 1 \(\{F^0, F^1\}\), where \(F^0\) is the local system whose fiber over a point \(q \in \mathcal{M}_{g,n}\) is identified with \(H^1(C_q, \mathbb{C})\), and \(F^1 \sim \mathcal{H}_{g,n}\). Note that the local system \(F^0\) has unipotent monodromies about the boundary divisor \(\partial\mathcal{M}_{g,n}\) of \(\mathcal{M}_{g,n}\).

We will show that \(\mathcal{N}\) admits a “resolution” \(\hat{N}\) which is in fact an orbifold such that the inverse image of \(\partial\mathcal{P}H_{g,n}\) in \(\hat{N}\) is a normal crossing divisor about which the pullback of the local system \(F^0\) has unipotent monodromies. A fundamental result on VHS (\([11, 30]\)) then asserts that \(\{F^0, F^1\}\) extends canonically to a filtration of holomorphic vector bundles over \(\hat{N}\). It follows from a result of Kawamata (see \([21\ p.\ 266]\)) that the canonical extension of \(F^1\) is precisely the extended Hodge bundle \(\mathcal{H}_{g,n}\). Since \(\partial(-1)\mathcal{H}_{g,n}\) is a line subbundle of (the pullback of) \(\mathcal{H}_{g,n}\) over \(\mathcal{P}H_{g,n}\), by a deep result by Kollár \([24]\), any power of the curvature form of the Hodge norm on \(\partial(-1)\mathcal{P}H_{g,n}\) is a representative in the sense of currents of the corresponding power of the first Chern class of \(\partial(-1)\mathcal{P}H_{g,n}\) on \(\hat{N}\), and \(2\) follows.

1.4.2. \textit{Case} \(k \geq 2\). We first show that the problem can be reduced to the case where \(N\) is subvariety of a stratum \(\mathcal{P}H_{g,n}(k)\) of \(\mathcal{P}H_{g,n}\), and the (projectivized) \(k\)-differentials in \(N\) are primitive, that is they are not tensor powers of some \(k'\)-differentials with \(k' < k\).

One can embed the stratum \(\mathcal{P}H_{g,n}(k)\) into \(\mathcal{P}H_{\hat{g}}\) for some \(\hat{g}\) determined by \(k\) via the cyclic covering construction. Unfortunately, this embedding does not extend properly to the boundary of \(\mathcal{P}H_{g,n}(k)\). For this reason, we will take a different route using admissible \(U_k\)-covers, where \(U_k \simeq \mathbb{Z}/k\mathbb{Z}\). To define the appropriate \(U_k\)-covers, we need to number all the zeros and poles of the \(k\)-differentials in
\( \mathbb{P}H^{(k)}(\kappa) \). This means that we have to pass from \( H^{(k)}(\kappa) \subset H^{(k)}_{g,r}(\k) \) to \( H^{(k)}(\kappa) \subset H^{(k)}_{g,r} \), with \( \ell = |\k| \). We can then replace \( \mathcal{N} \) by its inverse image in \( \mathbb{P}H_{g,r}(\kappa) \).

Consider now the moduli space \( \overline{\mathcal{M}}_{g,v}^{ac} \) of admissible \( U_k \)-covers associated to the stratum \( \mathcal{H}^{(k)}_{g,r}(\kappa) \). Each element of \( \overline{\mathcal{M}}_{g,v}^{ac} \) is a stable curve together with a distinguished action of the group \( U_k \) such that the quotient is a stable curve in \( \overline{\mathcal{M}}_{g,r}^{\ast} \). We have a finite morphism \( \mathcal{P} : \overline{\mathcal{M}}_{g,v}^{ac} \to \overline{\mathcal{M}}_{g,r}^{\ast} \) which sends a curve \( \hat{C} \in \overline{\mathcal{M}}_{g,v}^{ac} \) to \( \hat{C}/U_k \in \overline{\mathcal{M}}_{g,r}^{\ast} \).

Denote by \( \overline{\mathcal{H}}_{g,v}^{ac,\ell} \) the Hodge bundle over \( \overline{\mathcal{M}}_{g,v}^{ac} \) and by \( \mathbb{P}H_{g,v}^{ac,\ell} \) its associated projective bundle. There is a subbundle \( \overline{\mathcal{H}}_{g,v}^{ac,\ell} \) of \( \overline{\mathcal{H}}_{g,v}^{ac} \), where \( \zeta \) is a primitive \( k \)-th root of unity, together with a finite map \( \hat{\mathcal{P}} : \mathbb{P}H_{g,v}^{ac,\ell} \to \mathbb{P}H_{g,v}^{ac,\ell} \) which covers the map \( \mathcal{P} : \overline{\mathcal{M}}_{g,v}^{ac} \to \overline{\mathcal{M}}_{g,r}^{\ast} \) and satisfies \( \hat{\mathcal{P}}^* O(-1)_{\mathbb{P}H_{g,v}^{ac,\ell} \mathcal{P}^*} \sim O(-1)_{\mathbb{P}H_{g,v}^{ac,\ell}} \), where as usual \( \mathbb{P}H_{g,v}^{ac,\ell} \) denote the projective bundle associated to \( \overline{\mathcal{H}}_{g,v}^{ac,\ell} \). The map \( \hat{\mathcal{P}} \) is not necessarily surjective. However, it image contains the closure of \( \mathbb{P}H_{g,v}^{ac,\ell}(\kappa) \). Let \( \mathcal{M} \) be the inverse image of \( \mathcal{N} \) by \( \hat{\mathcal{P}} \). Then equality (2) for \( \mathcal{N} \) is equivalent to the same formula for \( \mathcal{M} \). Now since \( \mathcal{M} \) is a subvariety of the Hodge bundle \( \mathbb{P}H_{g,v}^{ac,\ell} \), the arguments of the case \( k = 1 \) allow us to conclude.

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### 2. VHS and desingularization

#### 2.1. Extension of the VHS on \( \mathcal{M}_{g,n} \) to \( \overline{\mathcal{M}}_{g,n} \)

Consider a point \( q \in \overline{\mathcal{M}}_{g,n} \). There are an open neighborhood \( \overline{U} \) of \( 0 \) in \( \mathbb{C}^{3g-3} \) (in both Euclidean and Zariski topologies) and a finite group \( G \) acting on \( \overline{U} \) by isomorphisms such that \( \overline{U}/G \) is isomorphic to a neighborhood of \( q \) in \( \overline{\mathcal{M}}_{g,n} \) with \( q \) being identified with 0. Note that in the analytic setting, \( G \) can be assumed to be a subgroup of \( \text{GL}(3g-3, \mathbb{C}) \). Moreover, the preimage of \( \partial \overline{\mathcal{M}}_{g,n} \) in \( \overline{U} \), which will be denoted by \( \partial \overline{U} \), is a simple normal crossing divisor in \( \overline{U} \).

In what follows, we will abusively consider \( \overline{U} \) as a neighborhood of \( q \) in \( \overline{\mathcal{M}}_{g,n} \).

We have a variation of Hodge structure (VHS) of weight 1 \{\( F^0 \), \( F^1 \)\} over \( \overline{U}^0 := \overline{U} \setminus \partial \overline{U} \), where

- \( F^0 \) is the complex vector bundle of rank 2\( g \) associated with the local system whose fiber over a point \( (X, x_1, \ldots, x_n) \in \overline{U}^0 \) is given by \( H^1(X, \mathbb{Z}) \). By definition, the fiber of \( F^0 \) over \( X \) is identified with \( H^1(X, \mathbb{Z}) \otimes \mathbb{C} \equiv H^1(X, \mathbb{C}) \).
- \( F^1 \) is the holomorphic subbundle of \( F^0 \) with fiber over \( (X, x_1, \ldots, x_n) \) being \( H^0(X, K_X) \).

Note that the total space of \( F^1 \) is nothing else but \( \mathcal{H}^{(\kappa)}_{g,0}(\overline{U}^0) \). The VHS \{\( F^0 \), \( F^1 \)\} comes equipped with the Hodge metric whose restriction to \( F^1 \) is precisely \( \|\cdot\| \).

Fix a base point \( q_0 \sim (X_0, x_1^0, \ldots, x_n^0) \in \overline{U}^0 \). Each irreducible component of \( \partial \overline{U} \) is determined by the homotopy class of a simple closed curve on the Riemann surface \( X_0 \) as follows: a generic point in the irreducible component of \( \partial \overline{U} \) represents a (complex) nodal curve obtained by pinching the corresponding closed curve on \( X_0 \). The monodromy of \( F^0_{q_0} \equiv H^1(X_0, \mathbb{C}) \) along a loop about an irreducible component of \( \partial \overline{U} \) is given by a Dehn twist about the corresponding curve on \( X_0 \). It is a well
known fact that the action of a Dehn twist on $H^1(X_0, \mathbb{Z})$ is given by a transvection matrix in $\text{Sp}(2g, \mathbb{Z})$ (see for instance [16] Ch.6). In particular, the associated monodromy is unipotent. It then follows from a result of Schmid [30] that the filtration $\{F^0, F^1\}$ extends canonically to a filtration $\{\overline{F}^0, \overline{F}^1\}$ of holomorphic vector bundles over $\overline{U}$.

Recall that $\pi_{g,n} : \overline{C}_{g,n} \to \overline{M}_{g,n}$ is the universal curve over $\overline{M}_{g,n}$. There is a family $\tilde{\pi}_U : \tilde{C}_U \to \tilde{U}$ of $n$-pointed stable curves of genus $g$ which satisfies

- the action of $G$ on $\tilde{U}$ lifts to an action on $\tilde{C}_U$ by isomorphisms,
- for every point $\tilde{q} \in \tilde{U}$, let $C_{\tilde{q}}$ denote the fiber $\tilde{\pi}_U^{-1}(\{\tilde{q}\})$, then $\text{Stab}_G(\tilde{q}) = \text{Aut}(C_{\tilde{q}})$,
- $\tilde{C}_U/G$ is isomorphic to $\pi_{g,n}^{-1}(U) \subset \overline{C}_{g,n}$, and the restriction of $\pi_{g,n}$ to $\pi_{g,n}^{-1}(U)$ is induced by $\tilde{\pi}_U$.

The open $\tilde{U}$ can be chosen such that $\tilde{C}_U$ is a smooth variety (that is a complex manifold in analytic setting). By definition $\tilde{\pi}_U$ has connected fibers, and the restriction of $\tilde{\pi}_U$ to $\tilde{\pi}_U^{-1}(\overline{U})$ is a smooth morphism. Thus it follows from a result by Kawamata (see [21] Th.5 and §4]) that the canonical extension $\overline{F}^1$ of $F^1$ to $\overline{U}$ is isomorphic to $\tilde{\pi}_U^*K_{\tilde{C}_U/\tilde{U}}$, where $K_{\tilde{C}_U/\tilde{U}}$ is the relative canonical bundle of the projection $\tilde{\pi}_U : \tilde{C}_U \to \tilde{U}$. But by definition, $\tilde{\pi}_U^*K_{\tilde{C}_U/\tilde{U}}$ precisely is the restriction of the Hodge bundle $\overline{H}_{g,n}$ to $\tilde{U}$. Therefore, we have $\overline{F}^1 = \overline{H}_{g,n|\tilde{U}}$. Note that the action of $G$ on $\tilde{U}$ naturally lifts to an action on $\overline{H}_{g,n|\tilde{U}}$ by isomorphisms of vector bundles.

2.2. Orbifold model. Consider now a subvariety $N$ of $\mathbb{P}H_{g,n}$. Denote by $\overline{N}$ its closure in $\mathbb{P}\overline{H}_{g,n}$. Let $h : \mathbb{P}\overline{H}_{g,n} \to \overline{M}_{g,n}$ be the natural projection. Define

$$\partial\mathbb{P}\overline{H}_{g,n} := \mathbb{P}\overline{H}_{g,n} \setminus \mathbb{P}H_{g,n} = h^{-1}(\partial\overline{M}_{g,n}), \text{ and } \partial\overline{N} := \overline{N} \setminus N.$$  

Note also that $\partial\overline{N} = h^{-1}(\partial\overline{M}_{g,n}) \cap \overline{N}$. Our goal now is to show

Lemma 2.1. There exist an algebraic orbifold $\hat{N}$ and a surjective proper birational morphism $\varphi : \hat{N} \to \overline{N}$ such that

(i) $\partial\hat{N} := \varphi^{-1}(\partial\overline{N})$ is a normal crossing divisor in $\hat{N}$.
(ii) $\psi := h \circ \varphi : \hat{N} \to \overline{M}_{g,n}$ is an orbifold morphism.

Remark 2.2.

a) That $\partial\hat{N}$ is a normal crossing divisor in $\hat{N}$ means the following: let $p$ be a point in $\partial\hat{N}$ with a neighborhood $V$ isomorphic to a quotient space $\tilde{V}/H$, where $\tilde{V}$ is a neighborhood of 0 in $\mathbb{C}^{\dim\hat{N}}$ and $H$ is a finite subgroup of $\text{GL}(\dim\hat{N}, \mathbb{C})$. Then the preimage of $\partial\hat{N}$ in $\tilde{V}$ is defined by an equation of the form $s_1 \ldots s_\alpha = 0$, where $s_1, \ldots, s_\alpha$ are some (pairwise distinct) coordinate functions on $\mathbb{C}^{\dim\hat{N}}$.

b) Let $p$ be a point in $\partial\hat{N}$ and $q := \psi(p) \in \overline{M}_{g,n}$. By condition (i) we must have $q \in \partial\overline{M}_{g,n}$. Let $U = \tilde{U}/G$ be a neighborhood of $q$ described above. Condition (ii) means that one can choose $V, U$ such that there exist a group morphism $\rho : H \to G$ and a map $\tilde{\psi}_V : \tilde{V} \to \overline{V}$ satisfying the followings

. $\tilde{\psi}_V(\gamma \cdot \tilde{p}) = \rho(\gamma) \cdot \tilde{\psi}_V(\tilde{p})$, for all $\tilde{p} \in \tilde{V}$ and $\gamma \in H$, and
. the restriction of $\psi$ to $V = \tilde{V}/H$ is induced by $\tilde{\psi}_V$. 
Recall that \( \partial \bar{U} \) is a simple normal crossing divisor in \( \bar{U} \). Thus there exists a finite family of coordinate functions \( t_1, \ldots, t_b \) on \( \bar{U} \) such that \( \partial \bar{U} \) is defined by the equation \( t_1 \cdots t_b = 0 \). Condition (i) then implies that for all \( i \in \{1, \ldots, b\} \) we have

\[
\tilde{\psi}^* t_i = u_i \cdot s_1^{a_{i1}} \cdots s_u^{a_{iu}}, \quad \alpha_{ij} \in \mathbb{Z}_{\geq 0},
\]

where \( u_i \) is a non-vanishing holomorphic function on a neighborhood of \( p \). This property is crucial to our proof of Theorem 1.1. One can think of \( (\hat{N}, \partial \hat{N}) \) as a log resolution of \( (\bar{N}, \partial \bar{N}) \). However, if we take an arbitrary resolution, then a lift of \( \psi|_V \) (that is a map from \( V \) to \( \bar{U} \) with the desired properties) may not exist.

**Proof of Lemma 2.1.** We first remark that if \( \dim N = \dim \mathbb{P}H_{g,n} \), then \( N = \mathbb{P}H_{g,n} \). In this case we can simply take \( \hat{N} = \mathbb{P}H_{g,n} \). Therefore, from now we will suppose that \( N \) is a proper subvariety of \( \mathbb{P}H_{g,n} \).

Consider a point \( \bar{q} \in \bar{N} \), and let \( q \) be its projection in \( \mathbb{P}H_{g,n} \). Assume first that \( q \) is not an orbifold point of \( \mathbb{P}H_{g,n} \). This means that the group \( G = \text{Aut}(C_q) \) is trivial, where \( C_q \) is the stable curve represented by \( q \). We can choose a neighborhood \( U = \bar{U} \) of \( q \) such that \( H_{g,n|U} \) is trivial. Hence \( \hat{U} := h^{-1}(U) \subset \mathbb{P}H_{g,n} \) is isomorphic to \( U \times \mathbb{P}^1 \). Let \( \partial \hat{U} \) denote the intersection \( \hat{U} \cap \partial \mathbb{P}H_{g,n} \). Since \( \partial U := \partial \mathbb{P}H_{g,n} \cup U \) is a simple normal crossing divisor in \( U \), \( \partial \hat{U} \) is also a simple normal crossing divisor in \( \hat{U} \). By definition \( \bar{N}_U := \bar{N} \cap \hat{U} \) is a subvariety of \( \hat{U} \). Let \( I_U \) denote the ideal sheaf of \( \bar{N}_U \).

The key ingredient of the proof is an application of the functoriality of the desingularization process (see [25 Ch. 3]). We handle the cases \( \text{codim} N = 1 \) and \( \text{codim} N \geq 2 \) in different ways.

**Case codim \( N = 1 \):** in this case \( \bar{N}_U \) is a divisor in \( \hat{U} \). Let \( I_U' \) denote the ideal sheaf of the divisor \( \bar{N} \cup \partial \hat{U} \) in \( \hat{U} \). Apply Theorem 3.35 of [25] to the triple \( (X = \hat{U}, I = I_U', E = \emptyset) \), we obtain a sequence of blow-ups

\[
\mathbb{P}(U, I_U', \emptyset) = (\Pi : X_r \xrightarrow{\pi_{r+1}} X_{r-1} \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = \hat{U})
\]

such that the pull-back \( \Pi^* \partial \hat{U} \) is the ideal sheaf of a simple normal crossing divisor \( D \) in \( X_r \). The proper transform \( \bar{N}_U' \) of \( \bar{N}_U \) in \( X_r \) is a finite union \( D_1' \cup \cdots \cup D_k' \), where each \( D_i' \) is an irreducible component of \( D \). By definition, each of the \( D_i' \) is smooth. However two different components \( D_i', D_j' \) may intersect, therefore \( \bar{N}_U' \) is not necessarily smooth. Let \( \hat{N}_U \) denote the normalization of \( \bar{N}_U' \), which is isomorphic to the disjoint union of the divisor \( \{D_i', i = 1, \ldots, k\} \). Alternatively, we can define \( \hat{N}_U \) as the proper transform of \( \bar{N}_U' \) after a sequence of blow-ups of \( X_r \) along the loci of the intersections of \( m \) divisors in the family \( \{D_i', i = 1, \ldots, k\} \), with \( m \) taking values in the sequence \( (k, k-1, \ldots, 2) \). Note that \( \hat{N}_U \) is smooth, and we have naturally a birational map \( \varphi_U : \hat{N}_U \to \bar{N}_U \).

We claim that \( \varphi_U^{-1}(\partial \hat{U}) \) is a simple normal crossing divisor in \( \hat{N}_U \). By definition, \( \Pi^{-1}(\partial \hat{U}) \) is the union of some irreducible components of \( D \), none of which is contained in the family \( \{D_i', i = 1, \ldots, k\} \). Since \( D \) has simple normal crossing, for each \( i \in \{1, \ldots, k\} \), the intersection \( \Pi^{-1}(\partial \hat{U}) \cap D_i' \approx \varphi_U^{-1}(\partial \hat{U}) \cap D_i' \) is a simple normal crossing divisor in \( D_i' \), and the claim follows.

**Case codim \( N \geq 2 \):** in this case, we apply Theorem 3.35 of [25] to the triple

\[
(X = \hat{U}, I = I_U, E = \partial \hat{U}).
\]
We then have a sequence of blow-ups

$$\mathcal{B}\mathcal{P}(\hat{U}, I_U, \partial \hat{U}) = (\Pi : X_r \rightarrow X_{r-1} \rightarrow \ldots \rightarrow X_1 \rightarrow X_0 = \hat{U}),$$

with smooth centers having simple normal crossing with $\partial \hat{U}$ (see [25, Def. 3.25]) such that

a) $\Pi : X_r \rightarrow X_0$ is an isomorphism over $X_0 \setminus \text{cosupp}(l) = \hat{U} \setminus \mathcal{N}_U$.

b) The pull-back $\Pi^* I_\theta$ is the ideal sheaf of a simple normal crossing divisor.

c) $\mathcal{B}\mathcal{P}$ commutes with smooth morphisms.

Since codim$\mathcal{N} \geq 2$, for some $i \in \{1, \ldots, r\}$ we have that the center $Z_j \subset X_j$ of $\pi_j$ contains the proper transform of $\mathcal{N}_U$, but for all $i < j$, the proper transform of $\mathcal{N}_U$ in $X_i$ is not contained in the center of the blow-up $\pi_i$. By assumption, $Z_j$ is smooth. Since $\Pi$ is an isomorphism over $\hat{U} \setminus \mathcal{N}_U$, we must have $\Pi_{j-1} := \pi_0 \circ \cdots \circ \pi_{j-1}$ maps $Z_j$ onto $\mathcal{N}_U$. Hence $\Pi_{j-1}|_{Z_j} : Z_j \rightarrow \mathcal{N}_U$ is birational (if dim$Z_j > \text{dim}\mathcal{N}_U$) then there is a point in $Z_j$ but not in $\Pi_{j-1}(\mathcal{N}_U)$, which means that $\mathcal{N}_U \subset \Pi_{j-1}(Z_j)$ and there is a point $\tilde{q}^i$ in $\hat{U} \setminus \mathcal{N}_U$ such that $\Pi^{-1}(\tilde{q}^i)$ has positive dimension). We claim that $Z_j$ is not contained in $\Pi_{j-1}(\partial \hat{U})$. This is because $\Pi_{j-1}(Z_j) = \mathcal{N}_U$, while $\partial \hat{U} \setminus \mathcal{N}_U$ is a proper subvariety of $\mathcal{N}_U$. Since $Z_j$ has simple normal crossing with the total transform of $\partial \hat{U}$ in $X_j$ by assumption, we see that $\Pi_{j-1}(\partial \hat{U}) \cap Z_j$ is a simple normal crossing divisor in $Z_j$. We can then define $\hat{N}_U := Z_j$.

Assume now that $q$ is an orbifold point of $\mathcal{N}_{g,n}$, that is $G \neq \{1\}$. Recall that a neighborhood of $q$ in $\mathcal{N}_{g,n}$ is identified with $U = \hat{U}/G$. Note that the action of $G$ preserves the preimage $\partial \hat{U}$ of $\partial \mathcal{N}_{g,n}$ in $\hat{U}$ (this is because the curves corresponding to points in a $G$-orbit are all isomorphic). Let $\hat{U}$ denote the pullback of $\mathcal{P}\mathcal{H}_{\hat{g},n}$ to $\hat{U}$. Then the action of $G$ on $\hat{U}$ extends naturally to an action on $\hat{U}$ by bundle isomorphisms. Note that we can choose $U$ such that $\hat{U} = \hat{U} \times \mathbb{P}^{r-1}$, and a neighborhood of $\tilde{q}$ in $\mathcal{P}\mathcal{H}_{\hat{g},n}$ can be identified with $\hat{U}/G$.

Let $\hat{N}_U$ be the preimage of $\mathcal{N}_U$ in $\hat{U}$. Note that $\hat{N}_U$ is a $G$-invariant subvariety of $\hat{U}$. Let $I_{\hat{U}}$ be the ideal sheaf of $\hat{N}_U$, and $I_{\hat{U}}'$ the ideal sheaf of $\mathcal{N}_U \cup \partial \hat{U}$, where $\partial \hat{U} = \hat{U} \cap \partial \mathcal{P}\mathcal{H}_{\hat{g},n}$. If codim$\mathcal{N} = 1$, we apply [25, Th. 3.35] to the triple $(\hat{U}, I_{\hat{U}}, \partial \hat{U})$. Since $\mathcal{N}_U$ and $\partial \hat{U}$ are $G$-invariant, by the functoriality of $\mathcal{B}\mathcal{P}(\cdot)$, the proper transform $\hat{N}_U^\#$ of $\mathcal{N}_U$ is $G$-invariant. Thus we have a $G$-action on the normalization $\hat{N}_U^{\# \text{norm}}$ of $\hat{N}_U$. We then define $\hat{N}_U := \hat{N}_U^{\# \text{norm}}/G$.

If codim$\mathcal{N} \geq 2$, we apply [25 Th.3.35] to the triple $(\hat{U}, I_{\hat{U}}', \partial \hat{U})$. Let $Z_j$ be as above. By the functoriality of the blow-up sequence $\mathcal{B}\mathcal{P}(\cdot)$, every element of $G$ lifts to an automorphism of $X_j$ that preserves $Z_j$. In particular, we get an action of $G$ on $Z_j$. We define $\hat{N}_U$ to be the quotient $Z_j/G$.

By taking a finite cover of $\mathcal{N}$ by opens of the form $\mathcal{N} \cap (\hat{U}/G)$, with $\hat{U}$ and $G$ as above, we get a finite family of algebraic varieties $\hat{N}_U$, each of which is a finite quotient of a smooth one. The functoriality of $\mathcal{B}\mathcal{P}(\cdot)$ then allows one to patch these varieties together to form an algebraic variety $\hat{N}_U$, which is also an orbifold, together with a map $\varphi : \hat{N} \rightarrow \mathcal{N}$ with the desired properties. \qed

Remark 2.3. In the case $\mathcal{N}$ is a stratum of $\mathcal{P}\mathcal{H}_{\hat{g},n}$, the moduli space of (projectivized) multiscale differentials constructed in [3] is an avatar of $\hat{N}_U$. 


3. Proof of Theorem 3.1 for Abelian differentials

We are now in a position to prove the following

**Theorem 3.1.** Let \( N \) be a subvariety of \( \mathbb{P}H_{g,n} \) of dimension \( d \). Denote by \( N_0 \) the set of regular points of \( N \), and by \( \overline{N} \) the closure of \( N \) in \( \mathbb{P}H_{g,n} \). Let \( \Theta \) denote the curvature form of the Hodge norm on \( \mathcal{O}(-1)_{\mathbb{P}H_{g,n}} \). Then we have

\[
\left( \frac{i}{2\pi} \right)^d \int_{N_0} \Theta^d = c_1^\mathbb{C}(\mathcal{O}(-1)_{\mathbb{P}H_{g,n}}) \cdot [\overline{N}]
\]

where \([\overline{N}]\) is the equivalence class of \( \overline{N} \) in \( \text{A}_*(\mathbb{P}H_{g,n}) \).

**Proof.** Let \( \hat{N} \) be the complex orbifold obtained by Lemma 2.1. By construction, \( \hat{N} \) has a finite cover by open subsets of the form \( V_i = \overline{V}_i/G_i, i \in I \), where \( \overline{V}_i \) is a complex manifold and \( G_i \) is a finite group acting on \( V_i \) by isomorphisms. Moreover, we have a surjective birational morphism \( \varphi: \hat{N} \to \overline{N} \) such that the composite map \( \psi = h \circ \varphi: \hat{N} \to \overline{M}_{g,n} \) satisfies the followings: for each \( i \in I \), there is an open subset \( U_i \) of \( \overline{M}_{g,n} \) such that

- \( U_i \cong \overline{U}_i/G_i \), where \( \overline{U}_i \) is an open neighborhood of 0 in \( \mathbb{C}^{3g-3} \) (in the Euclidean topology), and \( G_i \) acts on \( \overline{U}_i \) by restriction of linear isomorphisms,
- there is a morphism \( g_i: G_i \to \text{GL}(g, \mathbb{C}) \) such that \( \overline{H}_{g,n|U_i} \) is isomorphic to \( \overline{U}_i \times \mathbb{C}^g/G_i \), where the action of \( G_i \) on \( \overline{U}_i \times \mathbb{C}^g \) is given by \( \gamma \cdot (q, v) = (\gamma \cdot q, g_i(\gamma) \cdot v) \), for all \( \gamma \in G_i, q \in \overline{U}_i, v \in \mathbb{C}^g \),
- \( \varphi|_{V_i}: V_i \to U_i \) lifts to a map \( \tilde{\psi}_i: \overline{V}_i \to \overline{U}_i \) which is \( G_i \)-equivariant and satisfies \( \partial \overline{V}_i := \tilde{\psi}_i^{-1}(\partial \overline{U}_i) \) is a simple normal crossing divisor in \( \overline{V}_i \), where \( \partial \overline{U}_i \) is the pre-image of \( \partial \overline{M}_{g,n} \) in \( \overline{U}_i \).

Consider a point \( p \in \partial \overline{V}_i \). By assumption there are some local coordinate functions \( s_1, \ldots, s_a \) on a neighborhood of \( p \) such that \( \partial \overline{V}_i \) is locally defined by the equation \( s_1 \cdots s_a = 0 \). Let \( q = \tilde{\psi}_i(p) \). Then there are some local coordinate functions \( t_1, \ldots, t_b \) of \( \overline{U}_i \) in a neighborhood of \( p \) such that \( \partial \overline{U}_i \) is defined by the equation \( t_1 \cdots t_b = 0 \). The condition \( \tilde{\psi}_i^{-1}(\partial \overline{U}_i) = \partial \overline{V}_i \) means that for all \( j \in \{1, \ldots, b\} \) we can write

\[
t_j = u_j \cdot s_1^{\alpha_j_1} \cdots s_a^{\alpha_j_a},
\]

where \( \alpha_{jk} \in \mathbb{Z}_{\geq 0} \) for all \( k = 1, \ldots, a \), and \( u_j \) is a non-vanishing holomorphic function in a neighborhood of \( p \).

Let \( \overline{V}_i^0 := \overline{V}_i \setminus \partial \overline{V}_i \). By definition, we have \( \tilde{\psi}_i(\overline{V}_i^0) \subset \overline{U}_i^0 \). Recall that we have a VHS \( \{F^0, F^1\} \) of weight 1 over \( \overline{U}_i^0 \). Pulling back by \( \tilde{\psi}_i \), we get the VHS \( \{\tilde{\psi}_i^*F^0, \tilde{\psi}_i^*F^1\} \) on \( \overline{V}_i^0 \). Recall that the monodromy of \( F^0 \) along a loop about any irreducible component of \( \partial \overline{U}_i \) is unipotent. It follows that the monodromy of \( \tilde{\psi}_i^*F^0 \) along a loop about any irreducible component of \( \partial \overline{V}_i \) is also given by a unipotent matrix. Thus the filtration \( \{\tilde{\psi}_i^*F^0, \tilde{\psi}_i^*F^1\} \) extends canonically to \( \overline{V}_i \), and this extension is isomorphic to \( \{\tilde{\psi}_i^*F^0, \tilde{\psi}_i^*F^1\} \).

In §2.1 we have seen that \( F^1 = \overline{H}_{g,n|U_i} \). Thus \( \tilde{\psi}_i^*\overline{H}_{g,n} \) is the canonical extension of \( \tilde{\psi}_i^*F^1 \) to \( \overline{V}_i \). By construction, the group \( G_i \) acts on \( F^1, \overline{F}^1 \) by isomorphisms of vector bundles. Since \( \tilde{\psi}_i \) is \( G_i \)-equivariant, \( G_i \) also acts on \( \tilde{\psi}_i^*\overline{F}^1 \) by vector bundle isomorphisms. It follows that the quotient
ψ|F|/G is an orbifold vector bundle over V_i. Gluing the {V_i, i ∈ I} together we obtain an orbifold vector bundle over ˆN which will be denoted by ψ|F|. By construction, we have ψ|F| is isomorphic to ψ|H|.

Consider now the restriction of Ω(-1)|H| to ˆN, which will be denoted by L. By definition, L is a line subbundle of h|H| (recall that h : |H| → |M| is the bundle projection). Thus, ˆL := ϕ*L is a line subbundle of ϕ*|H| on ˆN (in the sense of orbifold vector bundles).

Let ˆN0 = ˆN \ ∂ ˆN. Since ψ(ˆN0) ⊂ |M| and the Hodge norm is well defined on |H| = |H|\|M|, we get a well defined Hermitian metric on ||| on ˆL|N|. Denote by ˆΘ the curvature form of this metric. Since ˆL is a subbundle of ϕ*|H| ∼ ϕ*|F|, by a result of Kollár (see [24, Theorem 5.1 and Remark 5.19]) (2π ˆΘ)m is a representative in the sense of currents of c1(ˆL) on ˆN for all m ∈ Z≥0. In particular, taking m = d = dim ˆN we have

\[ \int_{ˆN0} ˆΘ^d = \left( \frac{1}{2\pi} \right)^d \int_{ˆN} ˆΘ^d = c_1^{d}(ˆL) \cdot [ˆN] ∈ \mathbb{Q}. \]

By definition, we have ˆL = ϕ*Ω(-1)|H| and ˆΘ = ϕ*Θ on ˆN0. Thus

\[ \int_{ˆN0} ˆΘ^d = \int_{N0} Θ^d \]

and

\[ c_1^{d}(ˆL) \cdot [ˆN] = c_1^{d}(Ω(-1)|H|) \cdot [N] \]

(because φ : ˆN → N is birational). Combining (6), (7), and (8) we get the desired conclusion.

4. Strata of k-differentials and moduli spaces of admissible cyclic coverings

To prove Theorem [1.1] in full generality, we will reduce the general case to the case k = 1 and use the arguments of Theorem 3.1 to conclude. The idea is to use canonical cyclic covers of k-differentials. More precisely, we will consider the moduli space of Abelian differentials that are canonical cyclic covers of k-differentials in a fixed connected component of some stratum |H| is contained in the total space of the Hodge bundle over the moduli space of stable curves endowed with an admissible U_k-action, where U_k ≃ Z/kZ. Note that the topology of the curves in the latter space and the U_k-action strongly depend on the stratum |H| is contained in some stratum of |H|. This condition is however not at all restrictive, since for any N we can always find some stratum that intersects N in an open dense subset of N.

4.1. Mark and unmark the zeros. Consider a stratum |H| of k-differentials, where κ = (κ1, ..., κ_ℓ) with ℓ ≥ n. Let (X, x_1, ..., x_n, η) be a point in |H|. By definition, we have

\[ (η) = \sum_{i=1}^{n} κ_i \cdot x_i. \]
Note that the first $n$ points in the support of $(\eta)$, which can be zeros or poles, are the marked points of $X$, hence they are endowed with a fixed labelling. The remaining points in the support of $(\eta)$ are all zeros which are not labelled. To construct the space of admissible covers of $k$-differentials in $H_{g,n}^{(k)}(\kappa)$ it is important to label all the zeros and poles of $\eta$, that is we would like to consider $\eta$ as an element of $H_{g,n}^{(k)}(\kappa)$. Our first task is to show that the problem in $H_{g,n}^{(k)}(\kappa)$ can be transposed without change into $H_{g,\ell}^{(k)}(\kappa)$.

Let $Z(\eta)$ denote the support of $(\eta)$. There is a natural map $\mathcal{F} : H_{g,\ell}^{(k)}(\kappa) \to H_{g,n}^{(k)}(\kappa)$ which consists of forgetting the labeling of the last $(\ell-n)$ zeros in $Z(\eta)$. It is clear that $\mathcal{F}$ is a finite morphism, whose degree is equal to the cardinality of the group of permutations of $\{n+1, \ldots, \ell\}$ preserving the vector $(\kappa_{n+1}, \ldots, \kappa_\ell)$.

Let $\mathbb{P}H_{g,n}^{(k)}(\kappa) \subset \mathbb{P}H_{g,\ell}^{(k)}(\kappa)$ and $\mathbb{P}H_{g,n}^{(k)}(\kappa) \subset \mathbb{P}H_{g,\ell}^{(k)}(\kappa)$ be the projectivizations of $H_{g,n}^{(k)}(\kappa)$ and of $H_{g,\ell}^{(k)}(\kappa)$ respectively. Their closures in $\mathbb{P}H_{g,n}$ and in $\mathbb{P}H_{g,\ell}$ are denoted by $\overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$ and $\overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa)$. The restrictions of the tautological line bundles on $\mathbb{P}H_{g,n}$ and on $\mathbb{P}H_{g,\ell}$ to $\overline{\mathbb{P}H}_{g,n}(\kappa)$ and $\overline{\mathbb{P}H}_{g,\ell}(\kappa)$ will be denoted by $\mathcal{O}(-1)_{\overline{\mathbb{P}H}^{(i)}_{g,n}(\kappa)}$ and $\mathcal{O}(-1)_{\overline{\mathbb{P}H}^{(i)}_{g,\ell}(\kappa)}$. By a slight abuse of notation, we will also denote by $\mathcal{F}$ the induced map from $\overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa)$ onto $\overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$.

**Proposition 4.1.** The map $\mathcal{F}$ admits an extension $\overline{\mathcal{F}} : \overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa) \to \overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$, and we have

$$\overline{\mathcal{F}} \mathcal{O}(-1)_{\overline{\mathbb{P}H}^{(i)}_{g,\ell}(\kappa)} \sim \mathcal{O}(-1)_{\overline{\mathbb{P}H}^{(i)}_{g,n}(\kappa)}.$$ 

**Proof.** Consider a point $(Y', y'_1, \ldots, y'_\ell, \eta') \in \overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa)$. Let $Y'_1, \ldots, Y'_\ell$ be the irreducible components of $Y'$. It follows from the main result of [4] that $\eta'$ corresponds to a collection $(\lambda'_1, \eta'_1, \ldots, \lambda'_\ell, \eta'_\ell)$, where $\lambda'_i \in \mathbb{C}$ and $\eta'_i$ is a $k$-differential of $Y'_i$. For each $i$, the support of $(\eta'_i)$ consists of the marked points that are contained in $Y'_i$, that is $\{y'_1, \ldots, y'_\ell\} \cap Y'_i$, and the nodes of $Y'_i$. Let $(Y, y_1, \ldots, y_n) \in \mathbb{M}_{g,n}$ be the image of $(Y', y'_1, \ldots, y'_\ell)$ by the forgetful map. The curve $Y$ is obtained from $Y'$ by contracting unstable components after removing the marked points $\{y'_{n+1}, \ldots, y'_\ell\}$. In our situation, such a component is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and contains either

(i) one node, one point in $\{y'_1, \ldots, y'_n\}$, and at least one point in $\{y'_{n+1}, \ldots, y'_\ell\}$, or

(ii) two nodes, no point in $\{y'_1, \ldots, y'_n\}$, and at least one point in $\{y'_{n+1}, \ldots, y'_\ell\}$.

Assume for instance that $Y'_1$ is a component that is contracted by the forgetful map. Since the order of $\eta'_1$ at a point in $\{y'_1, \ldots, y'_n\}$ is at least $1 - k$, and at point in $\{y'_{n+1}, \ldots, y'_\ell\}$ is at least $1$, it follows that in both cases, the order of $\eta'_1$ at a node is smaller than $-k$. Therefore we must have $\lambda'_1 = 0$, that is $\eta'$ vanishes identically on $Y'_1$. Thus by assigning to each component of $Y$ the restriction of $\eta'$ to the corresponding component in $Y'$, we get well defined a $k$-differential $\eta$ which belongs to the fiber of $H_{g,n}^{(k)}$ over $(Y, y_1, \ldots, y_n)$. This implies that $\overline{\mathcal{F}} : \overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa) \to \overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$ extends to a map $\overline{\mathcal{F}} : \overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa) \to \overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$. Since $\overline{\mathcal{F}}$ is a proper, and $\overline{\mathcal{F}}(\overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa)) = \overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$, we get that $\overline{\mathcal{F}}(\overline{\mathbb{P}H}_{g,\ell}^{(k)}(\kappa)) = \overline{\mathbb{P}H}_{g,n}^{(k)}(\kappa)$. It is also clear from the definition that $\overline{\mathcal{F}} \mathcal{O}(-1)_{\overline{\mathbb{P}H}^{(i)}_{g,\ell}(\kappa)}$ is isomorphic to $\mathcal{O}(-1)_{\overline{\mathbb{P}H}^{(i)}_{g,n}(\kappa)}$.

**Remark 4.2.**

- Proposition 4.1 does not hold if some of the entries in $\{\kappa_{n+1}, \ldots, \kappa_\ell\}$ are negative.
• Assume that \( \mathcal{N} \) is a subvariety of \( \mathbb{P}\mathcal{H}^{(k)}(k) \). Let \( \mathcal{N}' \) be its pre-image in \( \mathbb{P}\mathcal{H}^{(k)}(k) \). Then it follows from Proposition 4.1 that equality (2) holds for \( \mathcal{N} \) if and only if it holds for \( \mathcal{N}' \).

4.2. **Canonical cyclic covers of \( k \)-differentials.** Let \((X,x_1,\ldots,x_\ell,\eta)\) be a \( k \)-differential in \( \mathcal{H}^{(k)}(k) \). We now review some key properties of the canonical cyclic cover of \((X,x_1,\ldots,x_\ell,\eta)\). Recall that \( Z(\eta) = \{x_1,\ldots,x_\ell\} \). We first observe that \(|\eta|^{2/k}\) defines a flat metric on \( X \) with conical singularities in \( Z(\eta) \). The cone angle at \( x_i \) is equal to \((1 + \frac{2}{k}) \cdot 2\pi \). The holonomies of the metric \(|\eta|^{2/k}\) gives a group morphism

\[
\chi : \pi_1(X \setminus Z(\eta)) \rightarrow \mathbb{U}_k \cong \{ e^{2\pi i j/k}, \quad j = 0, \ldots, k - 1 \}.
\]

The image of \( \chi \) stays unchanged if we replace \((X,x_1,\ldots,x_\ell,\eta)\) by a point nearby in the same stratum. Thus \( \text{Im}(\chi) \) is an invariant of connected components of \( \mathcal{H}_{g,k}(k) \). If \( \text{Im}(\chi) \neq \mathbb{U}_k \), then there exists \( k' \in \mathbb{Z}_{>0}, k' < k \), such that \( \text{Im}(\chi) = \mathbb{U}_{k'} \) (\( \mathbb{U}_{k'} \) is the group of \( k' \)-th roots of unity). In this case there is a \( k' \)-differential \( \eta' \) on \( X \) such that \( \eta = \eta'^{k'/k} \). For this reason, when \( \text{Im}(\chi) = \mathbb{U}_k \), we will call \( \eta \) a **primitive \( k \)-differential**, and the connected component of \( \mathcal{H}^{(k)}(k) \) to which \((X,x_1,\ldots,x_\ell,\eta)\) belongs a **primitive component**. For simplicity, in what follows we will abusively denote by \( \mathcal{H}^{(k)}(k) \) a connected component of the corresponding stratum, which is supposed to be primitive.

Let us fix a primitive \( k \)-th root of unity \( \zeta \). It is a well known fact (see [4, 27] for different proofs) that there are a covering \( f : \hat{X} \rightarrow X \) of degree \( k \) ramified over \( Z(\eta) \), an automorphism \( \tau \in \text{Aut}(\hat{X}) \) of order \( k \), and an Abelian differential \( \hat{\omega} \) on \( \hat{X} \) such that

- \( X \simeq \hat{X}/\langle \tau \rangle \),
- \( \tau^* \hat{\omega} = \zeta \cdot \hat{\omega} \), and
- \( f^* \eta = \hat{\omega}^k \).

Moreover, the triple \((\hat{X}, \hat{\omega}, \tau)\) is unique up to isomorphism. It is called the **canonical cyclic cover** of \((X,x_1,\ldots,x_\ell,\eta)\). In fact if \( Z(\hat{\omega}) \) is the inverse image of \( Z(\eta) \) in \( \hat{X} \), then \( f_1 \pi_1(\hat{X} \setminus Z(\hat{\omega})) = \ker(\chi) \subset \pi_1(X \setminus Z(\eta)) \).

All of the zeros of \( \hat{\omega} \) are contained in \( Z(\hat{\omega}) \), but it may happen that some of the points in \( Z(\hat{\omega}) \) are not zero of \( \hat{\omega} \). We will refer to these points as zeros of order \( 0 \) of \( \hat{\omega} \). By construction, we have a \( \mathbb{U}_k \)-action on \((\hat{X}, Z(\hat{\omega}))\) which is generated by \( \tau \).

4.3. **Admissible \( \mathbb{U}_k \)-covers.** Let \( \hat{x}_1,\ldots,\hat{x}_\ell \) be the points in \( Z(\hat{\omega}) \). Let \((Y,y_1,\ldots,y_\ell)\) be a pointed smooth curve representing a point in \( \mathbb{P}\mathcal{H}_{g,\ell} \). Call \( f_Y : (\hat{Y},\hat{y}_1,\ldots,\hat{y}_\ell) \rightarrow (Y,y_1,\ldots,y_\ell) \) an admissible \( \mathbb{U}_k \)-cover compatible with \( \mathcal{H}^{(k)}(k) \) if there exist two homeomorphisms \( \phi : (X,x_1,\ldots,x_\ell) \rightarrow (Y,y_1,\ldots,y_\ell) \) and \( \hat{\phi} : (\hat{X},\hat{x}_1,\ldots,\hat{x}_\ell) \rightarrow (\hat{Y},\hat{y}_1,\ldots,\hat{y}_\ell) \) such that \( \tau_{\hat{\phi}} := \hat{\phi} \circ \tau \circ \hat{\phi}^{-1} \in \text{Aut}(\hat{Y}) \) and the following diagram is commutative

\[
\begin{array}{ccc}
(\hat{X},\hat{x}_1,\ldots,\hat{x}_\ell) & \xrightarrow{\hat{\phi}} & (\hat{Y},\hat{y}_1,\ldots,\hat{y}_\ell) \\
\downarrow f & & \downarrow f_Y \\
(X,x_1,\ldots,x_\ell) & \xrightarrow{\phi} & (Y,y_1,\ldots,y_\ell)
\end{array}
\]
Two admissible $U_k$-covers $f_Y : (\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell) \to (Y, y_1, \ldots, y_\ell)$ and $f_{Y'} : (\hat{Y}', \hat{y}'_1, \ldots, \hat{y}'_\ell) \to (Y', y'_1, \ldots, y'_\ell)$ are isomorphic if there exist two isomorphisms $\hat{\phi} : \hat{Y} \to \hat{Y}'$ and $\phi : Y \to Y'$ such that $f_{Y'} \circ \hat{\phi} = \phi \circ f_Y$, and the distinguished automorphisms $\tau_{\hat{y}} \in \text{Aut}(\hat{Y}), \tau_{y} \in \text{Aut}(Y')$ are conjugate by $\hat{\phi}$.

Note that $\hat{\phi}$ is not unique because precomposing $\hat{\phi}$ with any automorphism $\tau' \in \text{Aut}(\hat{X}, \hat{x}_1, \ldots, \hat{x}_\ell)$ provides us with another homeomorphism with the same properties. This implies in particular that there is no canonical way to label the points in $\{\hat{y}_1, \ldots, \hat{y}_\ell\}$.

The notion of admissible $U_k$-cover can be generalized to stable curves (see [11 Ch.16, §5]), and there is a (coarse) moduli space of admissible $U_k$-covers compatible with $\tilde{\mathcal{M}}^k_{g,\ell}(\kappa)$, which will be denoted by $\mathcal{M}^\text{ac}_{g,\ell,\kappa}$. Elements of $\mathcal{M}^\text{ac}_{g,\ell,\kappa}$ are tuples $(\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell, \tau_{\hat{y}})$, where $(\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell)$ is a pointed stable curve, and $\tau_{\hat{y}}$ is an automorphism of $\hat{Y}$ of order $k$ such that

- the set $\{\hat{y}_1, \ldots, \hat{y}_\ell\}$ consists of $\ell \langle \tau_{\hat{y}} \rangle$-orbits,
- let $Y := \hat{Y}/\langle \tau_{\hat{y}} \rangle$ and $\{y_1, \ldots, y_\ell\}$ be the image of $\{\hat{y}_1, \ldots, \hat{y}_\ell\}$ in $Y$, then $(Y, y_1, \ldots, y_\ell)$ is a pointed stable curve in $\mathcal{M}_{g,\ell}$, and
- when $\hat{Y}$ is smooth, the natural projection $f_Y : (\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell) \to (Y, y_1, \ldots, y_\ell)$ is an admissible $U_k$-cover compatible with $\tilde{\mathcal{M}}^k_{g,\ell}(\kappa)$.

Note that when $\hat{Y}$ is a nodal curve, the nodes of $\hat{Y}$ are also partitioned into orbits of $U_k$, each orbit maps to a node of $Y$.

### 4.4. Projection from $\mathcal{M}^\text{ac}_{g,\ell,\kappa}$ onto $\mathcal{M}_{g,\ell}$

There is a natural morphism $\mathcal{P} : \mathcal{M}^\text{ac}_{g,\ell,\kappa} \to \mathcal{M}_{g,\ell}$ which maps a tuple $(\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell, \tau_{\hat{y}}) \in \mathcal{M}^\text{ac}_{g,\ell,\kappa}$ to the pointed curve $(Y, y_1, \ldots, y_\ell) \in \mathcal{M}_{g,\ell}$. Let $\mathcal{M}_{g,\ell,\kappa}$ denote the set of $(\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell, \tau_{\hat{y}}) \in \mathcal{M}^\text{ac}_{g,\ell,\kappa}$ where $\hat{Y}$ is smooth. Then $\mathcal{M}_{g,\ell,\kappa}$ is an open dense subset of $\mathcal{M}^\text{ac}_{g,\ell,\kappa}$. Its complement is denoted by $\partial \mathcal{M}^\text{ac}_{g,\ell,\kappa}$.

Given $y = (Y, y_1, \ldots, y_\ell) \in \mathcal{M}_{g,\ell}$, each admissible $U_k$-cover of $y$ is uniquely determined by the group morphism $y \circ \phi^{-1} : \pi_1(Y \setminus \{y_1, \ldots, y_\ell\}) \to U_k$. Since the set of such morphisms is finite (because $U_k$ is finite), we conclude that the set of admissible $U_k$-covers of $y$ is finite. This means that $\mathcal{P}$ has finite degree, and in particular

$$\dim \mathcal{M}^\text{ac}_{g,\ell,\kappa} = \dim \mathcal{M}_{g,\ell} = 3g - 3 + \ell.$$

Let $y = (Y, y_1, \ldots, y_\ell)$ now be point in $\mathcal{M}_{g,\ell}$. A neighborhood of $y$ in $\mathcal{M}_{g,\ell}$ can be identified with a quotient space $B/H$, where $B$ is an open neighborhood of $0$ in $\mathbb{C}^{3g-3+\ell}$ and $H$ a finite group acting on $B$ by isomorphisms. Let $q_1, \ldots, q_\ell$ be the nodes of $Y$. We can choose the local coordinates $(t_1, \ldots, t_{3g-3+\ell})$ on $B$ such that $\partial \mathcal{M}_{g,\ell} \cap B$ is defined by $t_1 \cdots t_\ell = 0$.

Let $\hat{y} = (\hat{Y}, \hat{y}_1, \ldots, \hat{y}_\ell, \tau_{\hat{y}})$ be a point in $\mathcal{P}^{-1}(y)$. A neighborhood of $\hat{y}$ in $\mathcal{M}^\text{ac}_{g,\ell,\kappa}$ can also be identified with a quotient $\tilde{B}/\tilde{H}$, where $\tilde{B}$ is an open subset of $\mathbb{C}^{3g-3+\ell}$, and $\tilde{H}$ is a finite group acting on $\tilde{B}$ by isomorphisms. Shrinking $\tilde{B}$ if necessary, we can assume that $\mathcal{P}(\tilde{B}/\tilde{H}) \subset B/H$. Recall that each node of $Y$ corresponds to a $U_k$-orbit of nodes in $\hat{Y}$. Define $r_j := \frac{k}{\text{card} f^{-1}(q_j)}$. There exists a system of local coordinates $(s_1, \ldots, s_{3g-3+\ell})$ on $B$ such that $\hat{y}$ is identified with $0$, and $B \cap \partial \mathcal{M}^\text{ac}_{g,\ell,\kappa}$ is defined by the equation $s_1 \cdots s_\ell = 0$. In this setting, $\mathcal{P}_{|\tilde{B}/\tilde{H}} : \tilde{B}/\tilde{H} \to B/H$ is induced by a map $\tilde{\mathcal{P}}_{B} : \tilde{B} \to B$ which
This means in particular that the lift of the \( r_j \)-th power of the Dehn twist associated to the node \( q_j \) of \( Y \) is the product of the Dehn twists associated to the nodes of \( \hat{Y} \) in the preimage of \( q_j \).

4.5. Hodge bundle over the space of admissible \( U_k \)-covers. Denote by \( \mathcal{H}^{ac}_{g,f,k} \) vector bundle over \( \mathfrak{M}_{g,f,k} \) whose fiber over \( \hat{Y} \) is identified with \( H^0(\hat{Y}, \omega_\hat{Y}) \). As usual let \( \mathbb{P}\mathcal{H}^{ac}_{g,f,k} \) denote the projectivization of \( \mathcal{H}^{ac}_{g,f,k} \). We will call \( \mathcal{H}^{ac}_{g,f,k} \) (resp. \( \mathbb{P}\mathcal{H}^{ac}_{g,f,k} \)) the Hodge bundle (resp. the projectivized Hodge bundle) over \( \mathfrak{M}_{g,f,k} \).

Let \( \zeta \) be a \( k \)-th root of unity. For every \( \hat{Y} := (\hat{Y}, \hat{Y}_1, \ldots, \hat{Y}_q, \tau_\hat{Y}) \in \mathfrak{M}_{g,f,k} \), denote by \( H^0(\hat{Y}, \omega_\hat{Y})^{\hat{Y}} \) the eigenspace associated with \( \zeta \) of the action of \( \tau_\hat{Y} \) on \( H^0(\hat{Y}, \omega_\hat{Y}) \). We have

**Lemma 4.3.** For all \( \zeta \in U_k \), there is a holomorphic subbundle \( \mathcal{H}^{ac,\zeta}_{g,f,k} \) of \( \mathcal{H}^{ac}_{g,f,k} \) whose fiber over \( \hat{Y} \) is \( H^0(\hat{Y}, \omega_\hat{Y})^{\hat{Y}} \).

**Proof.** We essentially only need to check that \( \dim H^0(\hat{Y}, \omega_\hat{Y})^{\hat{Y}} \) does not depend on \( \hat{Y} \). Let \( r_\zeta(\hat{Y}) := \text{rk}(\tau_\hat{Y} - \zeta \text{id}) \). Then \( r_\zeta \) is a lower continuous function. This means that \( r_\zeta(\hat{Y}) \leq r_\zeta(\hat{Y}') \) for all \( \hat{Y}' \) in a neighborhood of \( \hat{Y} \). Since \( \tau_\hat{Y} \) has order \( k \), we have

\[
\hat{g} = \dim H^0(\hat{Y}, \omega_\hat{Y}) = \sum_{\zeta \in U_k} \dim \ker(\tau_\hat{Y} - \zeta \text{id}) = \sum_{\zeta \in U_k} \dim \ker(\tau_\hat{Y} - \zeta \text{id})
\]

for all \( \hat{Y}' := (\hat{Y}', \hat{Y}_1', \ldots, \hat{Y}_q', \tau_{\hat{Y}}') \) in a neighborhood of \( \hat{Y} \). Since \( \dim \ker(\tau_\hat{Y} - \zeta \text{id}) \geq \dim \ker(\tau_\hat{Y} - \zeta \text{id}) \) for all \( \zeta \in U_k \), actually we must have \( \dim \ker(\tau_\hat{Y} - \zeta \text{id}) = \dim \ker(\tau_\hat{Y} - \zeta \text{id}) \) for all \( \zeta \in U_k \). \( \square \)

We denote by \( \mathbb{P}\mathcal{H}^{ac,\zeta}_{g,f,k} \) the projectivization of \( \mathcal{H}^{ac,\zeta}_{g,f,k} \). For every \( \xi \in H^0(\hat{Y}, \omega_\hat{Y})^{\hat{Y}} \), we have \( \tau_\hat{Y}^{(k)} \xi = \xi^{\otimes k} \). This means that \( \xi^{\otimes k} \) is the pullback of some \( k \)-differential \( \eta \) on \( Y = \hat{Y}/U_k \). Taking quotient by the \( \mathbb{C}^* \)-actions, we get a map \( \mathcal{P} : \mathbb{P}\mathcal{H}^{ac,\zeta}_{g,f,k} \to \mathbb{P}\mathcal{H}^{(k)}_{g,f,k} \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{P}\mathcal{H}^{ac,\zeta}_{g,f,k} & \xrightarrow{\mathcal{P}} & \mathbb{P}\mathcal{H}^{(k)}_{g,f,k} \\
\mathfrak{M}_{g,f,k} & \xrightarrow{\mathcal{P}} & \mathfrak{M}_{g,f,k}
\end{array}
\]

where the vertical arrows are bundle projections. Note that even though \( \mathcal{P} \) is surjective, \( \mathcal{P} \) is not necessarily surjective because there are \( k \)-differentials on \( Y \) that do not admit \( k \)-th root on \( \hat{Y} \). Nevertheless, by the existence of the canonical cyclic cover, for some value of \( \zeta \in U_k \) (which depends on the choice of the automorphisms \( \tau_\hat{Y} \)), the image of \( \mathcal{P} \) does contain \( \mathbb{P}\mathcal{H}^{(k)}_{g,f,k} \). From now on we will fix such a value of \( \zeta \). Since \( \mathcal{P} \) is a finite morphism, so is \( \mathcal{P} \).
Recall that $\mathcal{W}_{g,f,k}$ is the set of $\mathcal{F} = (\hat{Y}, \hat{y}_1, \ldots, \hat{y}_r, \tau_{\hat{y}})$ where $\hat{Y}$ is smooth. Let $\mathcal{H}_{g,f,k}^{ac,\ell}$ denote the restriction of $\mathcal{H}_{g,f,k}^{ac,\ell}$ to $\mathcal{W}_{g,f,k}$, and by $\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}$ its projectivization. By construction, the Hodge norms on $\mathcal{H}_{g,f,k}^{(k)}$ and on $\mathcal{H}_{g,f,k}^{ac,\ell}$ satisfies the following: if $\xi \in \mathcal{H}_{g,f,k}^{ac,\ell}$ is a $k$-th root of $\eta \in \mathcal{H}_{g,f,k}^{(k)}$, then we have 
\[ ||\eta|| = ||\xi||^k. \]

The following proposition is straightforward from the construction

**Proposition 4.4.** We have 
\[ \mathcal{D}^* \mathcal{O}(-1)_{\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}} \sim \mathcal{O}(-1)_{\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}}, \]
where $\mathcal{O}(-1)_{\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}}$ is the tautological line bundle over $\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}$. Denote by $\Theta$ and $\overline{\Theta}$ the curvature forms of the Hodge norms on $\mathcal{O}(-1)_{\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}}$ and $\mathcal{O}(-1)_{\mathbb{P}\mathcal{H}_{g,f,k}^{ac,\ell}}$ respectively. Then we have 
\[ \mathcal{D}^* \Theta = k \cdot \overline{\Theta}. \]

4.6. **Proof of Theorem 1.1**

**Proof.** In what follows, we will call a connected component of a stratum of $\mathcal{H}_{g,n}^{(k)}$ or of $\mathbb{P}\mathcal{H}_{g,n}^{(k)}$ simply a stratum. Given a subset $A$ of $\mathbb{P}\mathcal{H}_{g,n}^{(k)}$, we denote by $A^0$ its intersection with $\mathbb{P}\mathcal{H}_{g,n}^{(k)}$. Without loss of generality, we can assume that $\mathcal{N}$ is a closed subvariety of $\mathbb{P}\mathcal{H}_{g,n}^{(k)}$, that is $\overline{\mathcal{N}} = \mathcal{N}$, and $\mathcal{N}$ is irreducible.

**Claim 4.5.** There is a stratum $\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$ such that $\mathcal{N} \cap \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$ is an open dense subset of $\mathcal{N}$.

**Proof.** Consider a stratum of highest dimension such that $\mathcal{N} \cap \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N}) \neq \emptyset$. We claim that $\mathcal{N} \subset \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})^0$. Indeed, let $m = \dim \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$. Consider $A = \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})^0$, and 
\[ B = \bigcup_{\dim \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})^0 \leq m, \mathcal{N} \neq \emptyset} \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})^0. \]

Observe that $A$ and $B$ are closed subsets of $\mathbb{P}\mathcal{H}_{g,n}^{(k)}$, and $A \cup B$ is the union of all strata of dimension at most $m$. Since $\mathcal{N}$ does not intersect any stratum of dimension greater than $m$, we have $\mathcal{N} \subset A \cup B$. Thus $\mathcal{N} = (\mathcal{N} \cap A) \cup (\mathcal{N} \cap B)$. Since the closure of a stratum cannot intersect another stratum of the same dimension or higher, $B$ does not intersect $\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$. Therefore we have $\mathcal{N} \cap B \subseteq \mathcal{N}$ (since $\mathcal{N} \cap \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$ is not empty). If $\mathcal{N} \cap A \subseteq \mathcal{N}$, then we have a contradiction with the hypothesis that $\mathcal{N}$ is irreducible. Therefore, we must have $\mathcal{N} \cap A = \mathcal{N}$. That is $\mathcal{N} \subset \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})^0$. Since $\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$ is an open dense subset of $\mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})^0$, $\mathcal{N} \cap \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\mathcal{N})$ is an open dense subset in $\mathcal{N}$. \[ \square \]

Let $\ell := |\mathcal{N}|$. We now consider the stratum $\mathcal{H}_{g,f,k}^{(k)}(\mathcal{N}) \subset \mathcal{H}_{g,f,k}^{(k)}$. Recall that $\mathcal{H}_{g,f,k}^{(k)}$ contains the same $k$-differentials as $\mathcal{H}_{g,f,n}^{(k)}$, but as elements of $\mathcal{H}_{g,f,k}^{(k)}$ all the zeros and poles of those differentials are labelled. In §[4][4], we have shown that the natural forgetful map $\mathcal{F} : \mathbb{P}\mathcal{H}_{g,f,k}^{(k)} \to \mathbb{P}\mathcal{H}_{g,n}^{(k)}$ extends to
a map \( \overline{\mathcal{F}} : \mathbb{P}H_{g,d}^{(k)}(k) \to \mathbb{P}H_{g,d}^{(k)}(k) \) which satisfies \( \overline{\mathcal{F}}^* \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \sim \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \). Note that the map \( \mathcal{F} \) is finite-to-one, but over the boundary of \( \mathbb{P}H_{g,d}^{(k)}(k) \) fibers of \( \mathcal{F} \) may have positive dimension.

Since \( \mathcal{N} \subset \mathbb{P}H_{g,d}^{(k)}(k) \), we have \( \overline{\mathcal{N}} \subset \mathbb{P}H_{g,d}^{(k)}(k) \). Let \( \mathcal{N}', \overline{\mathcal{N}} \) be respectively the preimages of \( \mathcal{N}, \overline{\mathcal{N}} \) in \( \mathbb{P}H_{g,d}^{(k)}(k) \). Note that we have \( \mathcal{N}' = \overline{\mathcal{N}}' \cap \mathbb{P}H_{g,d}^{(k)}(k) \). We abusively denote by \( \Theta \) the curvature forms of the Hodge norms on both \( \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \) and \( \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \).

**Claim 4.6.** Let \( N'_0 \) denote the set of regular points of \( N' \). Then equality (2) is equivalent to

\[
\left( \frac{1}{2\pi} \right)^d \int_{N'_0} \Theta^d = c_1^d(\mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)}) \cdot [\overline{\mathcal{N}}].
\]

**Proof.** Since \( \overline{\mathcal{F}} = \mathcal{F}_{|\mathbb{P}H_{g,d}^{(k)}(k)} \) is a finite map, so is the restriction \( \overline{\mathcal{F}}_{|\mathcal{N}' \cap \mathbb{P}H_{g,d}^{(k)}(k)} : \mathcal{N}' \cap \mathbb{P}H_{g,d}^{(k)}(k) \to \mathcal{N} \cap \mathbb{P}H_{g,d}^{(k)}(k) \). It follows that \( \overline{\mathcal{F}}_{|\mathcal{N}} : \overline{\mathcal{N}} \to \overline{\mathcal{N}} \) has finite degree.

It is clear from the definition that the forgetful map \( \mathcal{H}_{g,d}^{(k)}(k) \to \mathcal{H}_{g,d}^{(k)}(k) \) preserves the Hodge norm. Therefore, we have \( \mathcal{F}^* \Theta = \Theta \). It follows that

\[
\int_{N_0} \Theta^d = \int_{\mathcal{N}_0} \mathcal{F}^* \Theta^d = \delta \int_{N_0} \Theta^d,
\]

where \( \delta \) is the degree of the map \( \mathcal{F}_{|\mathcal{N}} \). On the other hand, we have

\[
c_1^d(\mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)}) \cdot [\overline{\mathcal{N}}] = \mathcal{F}^* c_1^d(\mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)}) \cdot [\overline{\mathcal{N}}]
\]

\[
= \delta \cdot c_1^d(\mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)}) \cdot [\overline{\mathcal{N}}]
\]

and the claim follows. \( \square \)

Claim 4.6 means that to prove Theorem 1.1 it suffices to consider the case where \( \mathcal{N} \) is contained in a stratum with all the zeros and poles of the \( k \)-differentials being labelled. For this reason, from now on we will assume that \( \mathcal{N} \) is a subvariety of some stratum \( \mathbb{P}H_{g,d}^{(k)}(k) \), where \( \ell = |k| \), which is primitive.

Consider the moduli space \( \overline{\mathbb{H}}_{g,d,k}^{\text{ad}} \) of admissible \( \text{U}_k \)-covers compatible with \( \mathcal{H}_{g,d}^{(k)}(k) \). In §4.5 we have seen that there is a subbundle \( \overline{\mathbb{H}}_{g,d,k}^{\text{ad},\zeta} \) of the Hodge bundle \( \overline{\mathcal{H}}_{g,d,k}^{\text{ad}} \) over \( \overline{\mathbb{H}}_{g,d,k}^{\text{ad}} \), for some primitive \( k \)-th root of unity \( \zeta \), together with a map \( \overline{\mathcal{F}} : \overline{\mathbb{H}}_{g,d,k}^{\text{ad},\zeta} \to \mathbb{P}H_{g,d}^{(k)}(k) \) which satisfies

- \( \overline{\mathcal{F}} \) is a finite morphism,
- \( \mathbb{P}H_{g,d}^{(k)}(k) \subset \overline{\mathcal{F}}(\overline{\mathbb{H}}_{g,d,k}^{\text{ad},\zeta}) \),
- \( \overline{\mathcal{F}}^* \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \sim \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \),
- \( \overline{\mathcal{F}}^* \Theta = k \cdot \Theta \), where \( \Theta \) and \( \Theta \) are the curvature forms of the Hodge norms on \( \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \) and \( \mathcal{O}(-1)_{\mathbb{P}H_{g,d}^{(k)}(k)} \) respectively.
Since \( \hat{\mathcal{P}} \) is proper, we also have \( \mathbb{P}\mathcal{H}_{g,n}^{(c)}(\kappa) \subset \hat{\mathcal{P}}(\mathbb{P}\mathcal{H}_{g,n}^{ac,f,c}) \). Let \( \mathcal{M} \) and \( \mathcal{M} \) be the preimages of \( \mathcal{N} \) and \( \overline{\mathcal{N}} \) in \( \mathbb{P}\mathcal{H}_{g,n}^{ac,f,c} \). By the same arguments as Claim 4.6, we have

**Claim 4.7.** Let \( \mathcal{M}_0 \) denote the set of regular points in \( \mathcal{M} \). Then equality \( (2) \) is equivalent to

\[
\left( \frac{t}{2\pi} \right)^d \int_{\mathcal{M}_0} \overline{\Theta}^d = c_1^d(\Theta(-1)\overline{\mathcal{M}}) \cdot [\overline{\mathcal{M}}].
\]

Now, \( \mathcal{M} \) is a subvariety of \( \mathbb{P}\mathcal{H}_{g,n}^{ac,f,c} \) and \( \overline{\mathcal{M}} \) is its closure in \( \mathbb{P}\mathcal{H}_{g,n}^{ac,f,c} \). Thus \( (10) \) follows from the same arguments as the proof of Theorem 3.1. Theorem 1.1 is then proved. \( \square \)

5. **Proof of Corollary 1.2**

**Proof.** We will actually show that \( \Theta^d = 0 \) everywhere in \( \mathbb{P}\mathcal{H}_{g,n} \) for all \( d \geq 2g \) and use \( (2) \) to conclude. Let \( x_0 \) be a point in \( \mathbb{P}\mathcal{H}_{g,n} \). By definition, \( x_0 \) is a tuple \((x_0, x_0^1, \ldots, x_0^n, [\omega_0])\), where \( x_0 \) is a compact Riemann surface of genus \( g \), \( x_0^1, \ldots, x_0^n \) are \( n \) marked points on \( x_0 \), and \([\omega_0] = \mathbb{C} \cdot \omega_0\) is the complex line generated by a holomorphic 1-form \( \omega_0 \in H^0(X, K_X) \).

Let \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) be a symplectic basis of \( H_1(X, \mathbb{Z}) \). Without loss of generality, we can assume that \( \int_{\partial_1} \omega_0 \neq 0 \). For all \( x = (X, x_1, \ldots, x_n, [\omega]) \) in a neighborhood \( U \) of \( x_0 \) in \( \mathbb{P}\mathcal{H}_{g,n} \), we can also consider \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) as a symplectic basis of \( H_1(X, \mathbb{Z}) \). Since \( \int_{\partial_1} \omega_0 \neq 0 \), we also have \( \int_{\partial_1} \omega = 0 \). Thus we can normalize \( \omega \) by setting \( \int_{\partial_1} \omega = 1 \) for all \( x \in U \). The map \( \sigma : x \mapsto \omega \) is then a section of \( \Theta(-1)\mathbb{P}\mathcal{H}_{g,n} \) over \( U \). By definition, we have

\[ \Theta = -\partial \bar{\partial} \log ||\sigma||. \]

Let us write

\[ z_i := \int_{\partial_i} \omega, \quad i = 2, \ldots, g, \quad \text{and} \quad w_j := \int_{\partial j} \omega, \quad j = 1, \ldots, g. \]

Then \( z_i \)'s and \( w_j \)'s are holomorphic functions on \( U \). Now, we have

\[
||\sigma(x)||^2 = ||\omega||^2 = \frac{1}{2} \cdot \int_X \omega \wedge \overline{\omega} = \frac{1}{2} \cdot \sum_{i=1}^g \left( \int_{\partial_i} \omega \int_{\partial_i} \overline{\omega} - \int_{\partial_i} \omega \int_{\partial j} \omega + \int_{\partial j} \omega \int_{\partial i} \omega \right) = \frac{1}{2} \cdot (w_1 - w_1) + \frac{1}{2} \cdot \sum_{i,j=1}^g (z_i w_j - \bar{z}_i \bar{w}_j).
\]

Let \( E_x \) denote the complex vector subspace generated by \( \{dz_i, d\bar{z}_i, \quad i = 2, \ldots, g\} \) and \( \{d\bar{w}_j, d\bar{w}_j, \quad j = 1, \ldots, g\} \) in \( T_x^c \mathbb{P}\mathcal{H}_{g,n} \otimes \mathbb{C} \). Then \( \Theta(x) \in \Lambda^{1,1} E_x \), and hence \( \Theta^d(x) \in \Lambda^{d,d} E_x \). But since \( \dim_{\mathbb{C}} E_x = 2(2g - 1) = 4g - 2 \), we have \( \Lambda^{d,d} E_x = \{0\} \) if \( d > 2g - 1 \). Therefore \( \Theta^{2g} \) vanishes identically. It follows from \( (2) \) that

\[ c_1^d(\Theta(-1)\overline{\mathcal{M}}) \cdot [\overline{\mathcal{M}}] = 0 \]

whenever \( d = \dim \mathcal{N} \geq 2g \). \( \square \)
6. Volumes of absolutely rigid linear subvarieties

6.1. Volume forms on absolutely rigid linear subvariety of \( \mathcal{H}_{g,n} \). Let \( x := (X, x_1, \ldots, x_n, \omega) \) be a point in some stratum \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) of \( \mathcal{H}_{g,n} \). For every point \( x' = (X', x'_1, \ldots, x'_n, \omega') \) close enough to \( x \) in \( \mathcal{H}_{g,n}^{(k)}(\kappa) \), one can identify \( H^1(X', Z(\omega'), \mathbb{C}) \) with \( H^1(X, Z(\omega), \mathbb{C}) \) via a distinguished homeomorphism between the pairs \( (X, Z(\omega)) \) and \( (X', Z(\omega')) \). Thus one can associate to \( x' \) the cohomology class of \( \omega' \) in \( H^1(X', Z(\omega'), \mathbb{C}) \) \( \cong H^1(X, Z(\omega), \mathbb{C}) \). This correspondence is called period mapping. A classical result due to H. Masur and W. Veech asserts that period mappings form an atlas of \( \mathcal{H}_{g,n}^{(k)} \) with transition maps given by matrices in \( \text{GL}(2g - 1 + \ell, \mathbb{Z}) \), where \( \ell = |Z(\omega)| \).

Recall that an absolutely rigid linear subvariety of \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) is an algebraic subvariety \( \Omega M \) which satisfies the followings: let \( \phi \) be a period mapping defined on a neighborhood \( \mathcal{V} \) of \( x \):

- i) the image of any irreducible component of \( \Omega M \cap \mathcal{V} \) through \( x \) by \( \phi \) is an open subset of a linear subspace \( V \) in \( H^1(X, Z(\omega), \mathbb{C}) \),
- ii) the restriction of the intersection form on \( H^1(X, \mathbb{C}) \) to \( p(V) \) is non-degenerate, where \( p : H^1(X, Z(\omega), \mathbb{C}) \rightarrow H^1(X, \mathbb{C}) \) is the natural projection, and
- iii) \( V \cap \ker p = \{0\} \) (or equivalently, \( p_V : V \rightarrow p(V) \) is an isomorphism).

On such a subvariety \( \Omega M \) we have a natural volume form defined as follows: let \( \theta \) denote the skew-symmetric bilinear form on \( H^1(X, \mathbb{C}) \) that is the imaginary part of the intersection form. Since \( p_V : V \rightarrow p(V) \) is an isomorphism, and the restriction of the intersection form to \( p(V) \) is non-degenerate by assumption, the restriction of \( \theta \) to \( V \) is a symplectic form. It follows that \( \frac{1}{\dim(V)} (\theta)_{\dim(V)} \) gives a volume form on \( V \), which does not depend on the choice of the period mapping. Therefore, we get a well-defined volume form \( d\mu \) on \( \Omega M \) (see [27] for more details).

Let \( M \subset \mathbb{P}\mathcal{H}_{g,n}(\kappa) \) be the projectivization of \( \Omega M \). The volume form \( d\mu \) induces a volume form \( d\mu_1 \) on \( M \) by the following well known process: define

\[ \Omega_1 M := \{(X, x_1, \ldots, x_n, \omega) \in \Omega M, ||\omega|| < 1 \}. \]

Denote by \( d\mu_1 \) the restriction of \( d\mu \) to \( \Omega_1 M \). The volume form \( d\mu \) is then defined to be the pushforward of \( d\mu_1 \) under the projection \( \Omega_1 M \rightarrow M \).

Consider now a \( k \)-differential form \( x := (X, x_1, \ldots, x_n, \eta) \) in some stratum \( \mathcal{H}_{g,n}^{(k)}(\kappa) \), with \( k \geq 2 \). We assume that \( \eta \) is primitive. Let \( (\hat{X}, \hat{x}_1, \ldots, \hat{x}_n, \hat{\omega}, \tau) \) be the canonical cyclic cover of \( (X, x_1, \ldots, x_n, \eta) \). The automorphism \( \tau \) acts naturally on the space \( H^1(\hat{X}, Z(\hat{\omega}), \mathbb{C}) \) of relative cohomology, where \( Z(\hat{\omega}) \) is the inverse image of \( \{x_1, \ldots, x_n\} \) in \( \hat{X} \). There is a primitive \( k \)-th root of unity \( \zeta \) such that a neighborhood of \( x \) in \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) is identified with an open subset of \( V_{\zeta} := \ker(\text{Id} - \zeta \tau) \subset H^1(\hat{X}, Z(\hat{\omega}), \mathbb{C}) \) (see [4, 27]). Let \( p : H^1(\hat{X}, Z(\hat{\omega}), \mathbb{C}) \rightarrow H^1(\hat{X}, \mathbb{C}) \) be the natural projection. We then have (see [27])

\[ a) \dim(V_{\zeta} \cap \ker(p)) \text{ equals the number of entries in } \kappa \text{ that are divisible by } k, \text{ and} \]
\[ b) \text{the restriction of the intersection form on } H^1(\hat{X}, \mathbb{C}) \text{ to } p(V_{\zeta}) \text{ is non-degenerate.}\]

It follows in particular that \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) can be considered locally as a linear subvariety in some stratum \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) of Abelian differentials. This linear subvariety is absolutely rigid of none of the entries of \( \kappa \) is divisible by \( k \). Thus, in the case where \( k \) does not divide any entry of \( \kappa \), the spaces \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) and \( \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa) \) come equipped with the volume forms \( d\mu \) and \( d\mu_1 \) defined above.

For \( k \in \{2, 3, 4, 6\} \), we have other natural volume forms on \( \mathcal{H}_{g,n}^{(k)}(\kappa) \) (resp. on \( \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa) \)) known as the Masur-Veech volumes (see for instance [12, 26]). The total volume of \( \mathbb{P}\mathcal{H}_{g,n}^{(k)}(\kappa) \) with respect to the
Masur-Veech volume is an important invariant in numerous problems (enumerating tilings of surfaces by triangles or squares, Teichmuller dynamics in moduli spaces, etc...). It was shown in [27] that the Masur-Veech volume form on $\mathcal{H}_{g,n}(k)$ (resp. on $\mathbb{P}\mathcal{H}_{g,n}(k)$) always differs from $d\text{vol}$ (resp. from $d\mu$) by a rational constant.

For the proof of Theorem 1.3, we first need to determine the ratio $\frac{d\mu}{d\text{vol}}$ in the case $\mathcal{M}$ is absolutely rigid. To this purpose, consider a $\mathbb{C}$-vector space $V$ of dimension $m + 1$, which is endowed with a Hermitian form $h$ of signature $(p, q)$, where $p \geq 1$ and $p + q = m + 1$. Since $h$ is non-degenerate, the imaginary part $\theta$ of $h$ is a symplectic form on $V$. Denote by $d\text{vol}$ the volume form $\frac{\omega^{m+1}}{(m+1)!}$ on $V$.

Let $V^+$ denote the cone of positive vectors in $V$, that is $V^+ = \{v \in V, \ h(v, v) > 0\}$, and $\mathbb{P}V^+ := V^+/\mathbb{C}^* \subset \mathbb{P}V$ the projectivization of $V^+$. Let $V_1 = \{v \in V, \ 0 < h(v, v) < 1\} \subset V^+$ and denote by $d\text{vol}_1$ the restriction of $d\text{vol}$ to $V_1$. We define $\mu$ to be the measure on $\mathbb{P}V^+$ which is the pushforward of $d\text{vol}_1$ by the projection $pr : V_1 \to \mathbb{P}V^+$. This means that for every open subset $B \subset \mathbb{P}V^+$, $\mu(B) = \text{vol}(C(B) \cap V_1)$, where $C(B)$ is the cone over $B$ in $V^+$. The Hermitian form $h$ provides us with a metric on the tautological line bundle $\mathcal{O}(-1)_{\mathbb{P}V}$ over $\mathbb{P}V^+$. Let $\Theta$ denote the curvature form of this metric.

**Lemma 6.1.** The measure $\mu$ on $\mathbb{P}V^+$ is defined by a volume form $d\mu$ which satisfies

$$d\mu = (-1)^m \cdot \frac{2\pi}{2^{m+1}(m+1)!} (\mu \Theta)^m.$$  \hspace{1cm} (11)

**Sketch of proof.** We can identify $V$ with $\mathbb{C}^{m+1}$ and assume that $h$ is given by the matrix $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Let $(z_0, \ldots, z_m)$ be the coordinates on $\mathbb{C}^{m+1}$. We have

$$\theta = \frac{1}{2} \left( \sum_{i=0}^{p-1} dz_i \wedge d\bar{z}_i - \sum_{i=p}^{m} dz_i \wedge d\bar{z}_i \right).$$

and

$$d\text{vol} = \frac{\omega^{m+1}}{(m+1)!} = (-1)^q \cdot \left( \frac{1}{2} \right)^{m+1} dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m.$$

Let $x$ be a point in $\mathbb{P}V^+$, since the group $U(p, q)$ acts transitively on $\mathbb{P}V^+$, we can assume that $x = [1 : 0 : \cdots : 0]$. We identify a neighborhood $W$ of $x$ in $\mathbb{P}V^+$ with a neighborhood of 0 in $\mathbb{C}^m$ via the map $w := (w_1, \ldots, w_m) \mapsto [1 : w_1 : \cdots : w_m]$. For all $w = (w_1, \ldots, w_m) \in W$, define $\sigma(w) := (1, w_1, \ldots, w_m)$, and

$$\psi : [0, 2\pi] \times \mathbb{R}_{>0} \times W \to V^+ \\
(\theta, t, w) \mapsto e^{\theta \cdot t} \cdot \sigma(w).$$

Observe that $\sigma$ is a section of $\mathcal{O}(-1)_{\mathbb{P}V}$ over $W$, and $\psi([0, 2\pi] \times \mathbb{R}_{>0} \times W)$ is the cone $C(W)$ over $W$. Let

$$f(w) := h(\sigma(w), \sigma(w)) = 1 + \sum_{i=1}^{p-1} |w_i|^2 - \sum_{i=p}^{m} |w_i|^2.$$ 

Consider a measurable set $B \subset W$. By definition, we have

$$C(B) \cap V_1 = \psi((\theta, t, w) \in [0, 2\pi] \times \mathbb{R}_{>0} \times B, \ h(\psi(\theta, t, w), \psi(\theta, t, w)) < 1))$$

$$= \psi((\theta, t, w) \in [0, 2\pi] \times \mathbb{R}_{>0} \times B, \ t^2 < \frac{1}{f(w)}).$$
A quick computation shows \( \psi^* dz_0 dz_1 \ldots dz_m = -2 \frac{t^{2m+1}}{f(w)^m} dt dw_m dw_m \). Therefore

\[
\mu(B) = \text{vol}(C(B) \cap V_1) = (-1)^q \cdot \left(\frac{t}{2}\right)^m \int_B \left(\int_0^{\sqrt{\frac{1}{f(w)}}} \frac{1}{t^{2m+1}} dt\right) dw_1 dw_m \ldots dw_m
\]

which implies that \( \mu \) is induced by the volume form

\[
d\mu = (-1)^q \cdot \left(\frac{1}{f(w)^m} \right) \left(\frac{t}{2}\right)^m dw_1 dw_m \ldots dw_m.
\]

Now, by definition,

\[
\Theta = -\partial \overline{\partial} \log(f(w)) = -\sum_{i=1}^{m-1} dw_i \wedge d\bar{w}_i - \sum_{i=p}^{m} dw_i \wedge d\bar{w}_i + \frac{\partial f(w) \wedge \overline{\partial f(w)}}{f^2(w)}
\]

hence

\[
\Theta^m = \frac{(-1)^m}{f^m(w)} \left( \sum_{i=1}^{m-1} dw_i \wedge d\bar{w}_i - \sum_{i=p}^{m} dw_i \wedge d\bar{w}_i \right)^m + \frac{m (-1)^{m-1} m!}{f^{m+1}(w)} \left( \sum_{i=1}^{m-1} dw_i \wedge d\bar{w}_i - \sum_{i=p}^{m} dw_i \wedge d\bar{w}_i \right)^{m-1} \wedge \partial f(w) \wedge \overline{\partial f(w)}
\]

\[
= \frac{(-1)^m m!}{f^{m+1}(w)} \left( f(w) - \sum_{i=1}^{m-1} |w_i|^2 + \sum_{i=p}^{m} |w_i|^2 \right) \cdot (-1)^q \cdot dw_1 \wedge d\bar{w}_1 \ldots dw_m \wedge d\bar{w}_m.
\]

It follows that

\[
\frac{d\mu}{(i\Theta)^m} = (-1)^m \cdot \frac{2\pi}{2^{m+1}(m+1)!}
\]

and the lemma is proved. \( \square \)

**Remark 6.2.** The power of \(-1\) on the right hand side of (11) is not the same as in the formula in [26, Lem. 3.2]. This is because our definition of the volume form \( d\text{vol} \) on \( V \) is different from the one in [26]. In fact the two volume forms only agree up to sign. The advantage of the choice of \( d\text{vol} \) in this paper (that is \( d\text{vol} = \frac{\partial (1)}{(m+1)!} \)) is that the constant on the right hand side of (11) does not depend on the signature of \( h \).

6.2. **Proof of Theorem 1.3**

**Proof.** Let \( \Omega M \) be an absolutely rigid linear subvariety of some stratum \( \mathcal{H}_{r,n}(k) \subset \mathcal{H}_{r,n} \), and \( \mathcal{M} \) its projectivization in \( \mathbb{P}\mathcal{H}_{r,n} \). Let \( \Theta \) be the curvature form of Hodge norm on \( \mathcal{O}(-1)_{\mathcal{P}\mathcal{H}_{r,n}} \). Recall that \( d = \dim \mathcal{M} \). Denote by \( \mathcal{L} \) the restriction of \( \mathcal{O}(-1)_{\mathcal{P}\mathcal{H}_{r,n}} \) to \( \mathcal{M} \). By Lemma 5.1 we have

\[
d\mu = (-1)^d \frac{2\pi}{2^{d+1}(d+1)!} (i\Theta)^d.
\]
Therefore (3) is equivalent to
\[
\left(\frac{l}{2\pi}\right)^d \int_M \Theta^d = c^d(L) \cdot [M]
\]
which is an immediate consequence of Theorem 1.1.

In the case \(M = \mathbb{P}H^{(k)}_{g,\nu}(\kappa)\), where none of the entries of \(\kappa\) is divisible by \(k\), recall from [4.5] we have a morphism \(\mathcal{F} \circ \hat{\mathcal{P}} : \mathbb{P}H^{(k)}_{g,\nu}(\kappa) \to \mathbb{P}H^{(k)}_{g,\nu}(\kappa)\) such that \((\mathcal{F} \circ \hat{\mathcal{P}})^* \Theta = k \cdot \tilde{\Theta}\), where \(\Theta\) and \(\tilde{\Theta}\) are respectively the curvature forms of the Hodge norms on \(\mathcal{O}(-1)_{\mathbb{P}H^{(k)}_{g,\nu}(\kappa)}\) and \(\mathcal{O}(-1)_{\mathbb{P}H^{(k)}_{g,\nu}(\kappa)}\) (see Proposition 4.4). Thus, on \(\mathbb{P}H^{(k)}_{g,\nu}(\kappa)\) we have
\[
d\mu = \frac{(-1)^d}{k^d} \cdot \frac{2\pi}{2d+1(d+1)!} (\mathcal{F} \circ \hat{\mathcal{P}})^* \Theta^d.
\]
and (4) also follows from Theorem 1.1. □

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