REDUCTION OF FREE INDEPENDENCE TO TENSOR INDEPENDENCE

ROMUALD LENCZEWSKI

Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
e-mail lenczew@im.pwr.wroc.pl

Abstract

In the hierarchy of freeness construction, free independence was reduced to tensor independence in the weak sense of convergence of moments. In this paper we show how to reduce free independence to tensor independence in the strong sense. We construct a suitable unital *-algebra of closed operators ‘affiliated’ with a given unital *-algebra and call the associated closure ‘monotone’. Then we prove that monotone closed operators of the form

\[ X' = \sum_{k=1}^{\infty} X(k) \otimes p_k, \quad X'' = \sum_{k=1}^{\infty} p_k \otimes X(k) \]

are free with respect to a tensor product state, where \( X(k) \) are tensor independent copies of a random variable \( X \) and \( (p_k) \) is a sequence of orthogonal projections. For unital free *-algebras, we construct a monotone closed analog of a unital *-bialgebra called a ‘monotone closed quantum semigroup’ which implements the additive free convolution, without using the concept of dual groups.

Mathematics Subject Classification (2000): 46L54, 81R50

\[1\] This work is supported by KBN grant No 2P03A00723 and by the EU Network QP-Applications, Contract No. HPRN-CT-2002-00279
1. Introduction

We have shown in [L1] that free independence can be reduced to tensor independence in the following sense. For a given family of quantum probability spaces \((\mathcal{A}_l, \mu_l)_{l \in L}\), there exists a sequence of quantum probability spaces \((\mathcal{A}^{(m)}, \mu^{(m)})_{m \in \mathbb{N}}\) called the hierarchy of freeness and a sequence of (non-unital) *-homomorphisms

\[
j^{(m)} : \sqcup_{l \in L} \mathcal{A}_l \rightarrow \mathcal{A}^{(m)},
\]

where \(\sqcup_{l \in L} \mathcal{A}_l\) is the free product without identification of units, such that we have convergence of moments

\[
\Phi^{(m)} \circ j^{(m)}(X_1 X_2 \ldots X_n) \rightarrow \ast_{l \in L} \mu_l(X_1 X_2 \ldots X_n)
\]

as \(m \to \infty\), where \(X_1 \in \mathcal{A}_{l(1)}, \ldots, X_n \in \mathcal{A}_{l(n)}\) and \(l(1) \neq l(2) \neq \ldots \neq l(n)\) with \(\ast_{l \in L} \mu_l\) denoting the free product of states \(\mu_l\) in the sense of Avitzour [Av] and Voiculescu [V1]. Moreover, \((\mathcal{A}^{(m)}, \mu^{(m)})_{m \in \mathbb{N}}\) are restrictions of tensor products of unital *-algebras and states on these *-algebras.

The first order approximation corresponding to \(m = 1\) gives the boolean product of states [B]. For simplicity, consider two unital *-algebras \(A\) and \(B\) and extend them freely by projections \(p, p'\) to get \(\tilde{A} = A * \mathbb{C}[p]\) and \(\tilde{B} = B * \mathbb{C}[p']\). Then simple tensors of the form

\[
j^{(1)}(X) = X \otimes p', \quad j^{(1)}(Y) = p \otimes Y
\]

(1.1)

where \(X \in A\) and \(Y \in B\), are boolean independent with respect to the tensor product state \(\mu^{(1)} = \tilde{\mu} \otimes \tilde{\nu}\) on \(\tilde{A} \otimes \tilde{B}\), where \(\tilde{\mu}, \tilde{\nu}\) are the boolean extensions [L1] of \(\mu\) and \(\nu\), respectively.

In the \(m\)-th order approximation, finite sums of simple tensors of the form

\[
j^{(m)}(X) = \sum_{k=1}^{m} X(k) \otimes p'_k, \quad j^{(m)}(Y) = \sum_{k=1}^{m} p_k \otimes Y(k)
\]

(1.2)

give \(m\)-free random variables, whose mixed moments of orders \(\leq 2m\) in the state \(\mu^{(m)}\) agree with moments of free random variables. Here, \(X(k)\)'s and \(Y(k)\)'s are tensor independent copies of \(X\) and \(Y\), respectively and \(p_k\) as well as \(p'_k, k = 1, \ldots, m\), are orthogonal projections.

However, a tensor product representation of free random variables in the strong sense, i.e. as elements of a tensor product *-algebra, is not so simple since it requires us to take infinite series of simple tensors instead of finite sums (see [FLS] for the GNS construction). In this paper we show how this can be done by introducing a suitable closure. Namely, we adapt to our needs the known concept of the algebraic closure of a unital *-algebra [Be1-Be3], which leads us to the notion of the monotone closure. As a result, we show that monotone closed operators of the form

\[
j(X) = \sum_{k=1}^{\infty} X(k) \otimes p_k, \quad j(Y) = \sum_{k=1}^{\infty} p_k \otimes Y(k)
\]

(1.3)
are free with respect to a tensor product of states $\hat{\mu}$ and $\hat{\nu}$, which are, roughly speaking, tensor products of boolean extensions of states $\mu$ and $\nu$ on $\mathcal{A}$. Here, $\otimes$ denotes the \textit{monotone tensor product}, which bears some resemblance to the von Neumann algebra tensor product. One can say that in this ‘quantum orthogonal series’ representation of free random variables, information about freeness is \textit{shifted from states to variables}.

It is worth pointing out that the representations (1.1)-(1.3) enable us to compare free random variables with boolean random variables. In particular, they exhibit in a clear fashion why units are identified in the free product whereas they are not identified in the boolean product (nor the $m$-free products). For finite $m$ we have

$$\sum_{k=1}^{m} p_k \neq 1_A, \quad \sum_{k=1}^{m} p'_k \neq 1_B$$

and that is why $j^{(m)}$ does not map the units in $\mathcal{A}$ and $\mathcal{B}$ onto the unit in the tensor product and thus we cannot identify them. In particular, $j^{(1)}(1_A) = 1_A \otimes p'$ and $j^{(1)}(1_B) = p \otimes 1_B$, respectively. In turn, the construction of the monotone closure is based on the \textit{completness} property

$$\sum_{k=1}^{\infty} p_k = 1_A, \quad \sum_{k=1}^{\infty} p'_k = 1_B$$

of the sequences of orthogonal projections introduced into the model and thus $j(1_A) = j(1_B) = 1$, where $1$ is the unit in the tensor product. Therefore, in this case units can be identified.

Another important point is to show that the monotone closure can also be introduced on the *-bialgebra level since coproduct can be viewed as a mapping which produces \textit{independent} copies of a random variable in a natural and simple fashion. We already know [L1] how to construct *-bialgebras associated with $m$-freeness if $\mathcal{A}$ is a unital free *-algebra generated by a set $\mathcal{G}$. The simplest example of this type is given by the coproduct

$$\Delta(X) = X \otimes p + p \otimes X$$  \hspace{1cm} (1.4)$$

with $\Delta(p) = p \otimes p$, which produces boolean independent copies of $X \in \mathcal{G}$ with respect to the tensor product of extended states, cf. (1.1). Here, $X$ could be called a \textit{pre-boolean random variable} when treated as an element of the associated *-bialgebra equipped with an extended state. In a similar fashion we can produce $m$-free copies of $X$ for all finite $m$.

For $m = \infty$ we introduce the new notion of a \textit{monotone closed quantum semigroup}, which is the algebraic structure with *-bialgebra axioms, in which the algebraic tensor product is replaced by the monotone tensor product. Now, using the monotone closed quantum semigroup structure on some unital *-algebra of monotone closed operators $\mathcal{F}(\mathcal{G})$ associated with $\mathcal{G}$ we show that by applying the coproduct to monotone closed operators written in the form of a series

$$\sum_{k=1}^{\infty} \delta X(k),$$  \hspace{1cm} (1.5)
we obtain, according to (1.3), free copies of \( X \), namely

\[
\Delta \left( \sum_{k=1}^{\infty} \delta X(k) \right) = \sum_{k=1}^{\infty} (X(k) \otimes p_k + p_k \otimes X(k)) \quad (\text{mod ker } \hat{\mu} \otimes \hat{\nu}),
\]

(1.6)

which explains why we call the variables of the form (1.5) \textit{pre-free random variables}.

This also allows us to reproduce the additive free convolution of states on \( \mathcal{A} \) [V2] using ‘quantum groups’ instead of dual groups [V3]. Recall that the usual convolution of measures on a group \( G \) is implemented by a ‘quantum group’ (Hopf-algebra) structure on some commutative algebra \( C(G) \) of functions on \( G \). Namely, if \( \mu, \nu \) are functionals on \( C(G) \) corresponding to measures on \( G \), then their convolution is given by

\[
\mu \ast \nu = (\mu \otimes \nu) \circ \Delta,
\]

where \( \Delta \) is the Hopf-algebra comultiplication on \( C(G) \).

For the additive free convolution [V2], an analogous approach to group-duality was developed by Voiculescu [V3]. Namely, he defined a dual group structure on an algebra \( \mathcal{A} \), with tensor products replaced by free products. Then the free convolution of states \( \mu \) and \( \nu \) on \( \mathcal{A} \) is obtained from the formula

\[
\mu \boxplus \nu = (\mu \ast \nu) \circ \delta
\]

in which the composition of the dual multiplication \( \delta : \mathcal{A} \rightarrow \mathcal{A} \ast \mathcal{A} \) with the free product of states \( \mu \ast \nu \) replaces the composition of the Hopf-algebra comultiplication with the tensor product of states.

Using the monotone closed quantum semigroup structure on \( \mathcal{F}(\mathcal{G}) \), we obtain the additive free convolution \( \mu \boxplus \nu \) of states \( \mu, \nu \) on \( \mathcal{A} \) as a restriction of the quantum semigroup convolution

\[
\hat{\mu} \ast \hat{\nu} := (\hat{\mu} \otimes \hat{\nu}) \circ \Delta
\]

to the \ast\-subalgebra \( \mathcal{F}_{\text{pf}}(\mathcal{G}) \) of \( \mathcal{F}(\mathcal{G}) \) generated by pre-free random variables. Thus one can view the free additive convolution of classical measures as a ‘convolution of quantum measures on a monotone closed quantum semigroup’.

With the results of this paper we complete the program originated in [L1] concerning unification of independence, or reduction of the main types of independence to tensor independence on extended algebras (the more recent notion of monotone independence [M] can be also included). For another general framework, see [L2].

The paper is organized as follows. In Section 2, we introduce the notion of the monotone closure for an increasing sequence of unital *-algebras and show that it has a unital *-algebra structure. In Section 3, we present our main example of a unital *-algebra \( \mathcal{F}_0(\mathcal{G}) \) constructed from copies of a unital free *-algebra \( \mathcal{A} \) generated by the set \( \mathcal{G} \) and we introduce its monotone closure \( \mathcal{F}(\mathcal{G}) \). In Section 4 we show that \( \mathcal{F}(\mathcal{G}) \) can be endowed with a monotone closed quantum semigroup structure. In Section 5, we prove that the associated coproduct produces free random variables which allows us to recover the free additive convolution from the convolution on the monotone closed quantum semigroup. In Section 6 we derive a tensor product representation of free random variables.
2. MONOTONE CLOSED OPERATORS

In this Section we introduce the notion of the monotone closed operators for certain increasing sequences of unital *-algebras. We are guided by the construction of the unital *-algebra of closed operators ‘affiliated’ with a given unital *-algebra [Be1-Be2], on which we model our notation and terminology.

The construction of the closed *-ring (*-algebra) of operators consists in taking all sequences \((x_m, e_m)\), where \(x_m \in B_0\) and \((e_m)\) is a strongly dense domain (SDD), i.e. a sequence of projections such that \(e_m \uparrow 1\) and \(x_n e_m = x_m e_m, x_n^* e_m = x_m^* e_m\) for \(n > m\). On the set of these sequences called operators with closure (OWC) one introduces a suitable equivalence relation, and the set of the corresponding equivalence classes \([x_m, e_m]\) called closed operators (CO) ‘affiliated’ with \(B_0\) denoted by \(B\), can be made into a unital *-ring (*-algebra). The terminology of this theory is motivated by linear operators in Hilbert spaces. Heuristically, the ranges of the \(e_m\) are an increasing sequence of closed linear subspaces whose union is a dense linear subspace. In turn, one can think of \((x_m, e_m)\) as a linear operator whose restriction to the range of \(e_m\) is \(x_m e_m\).

For instance, this can be done if \(B_0\) is a finite Rickart *-ring (*-algebra) or, more generally, a finite Baer *-ring (*-algebra) satisfying \(LP \sim RP\), i.e. the left projection of any \(x \in B_0\) is equivalent to the right projection of \(x\) (equivalence is implemented by a partial isometry from \(B\)). This procedure can also be applied to AW*-algebras, i.e. Baer *-algebras which are \(C^*\)-algebras [Be3]. However, a new type of closure is needed for our purposes since the property \(LP \sim RP\) is not satisfied in the example which is of interest to us, namely that of a unital *-algebra \(F_0(A)\) related to free products.

Consider an increasing sequence of unital *-algebras
\[
B^{(0)} \subset B^{(1)} \subset B^{(2)} \subset \ldots
\] (2.1)
where \(B^{(0)} = \mathbb{C}[p_1, p_2, \ldots]\) is assumed to be the algebra of polynomials in a countable number of orthogonal projections \((p_m)\). Further, we take the union of all algebras \(B^{(m)}\) denoted
\[
B_0 = \bigcup_{m \geq 0} B^{(m)}
\] (2.2)
and assume that the sequence of increasing projections \((q_m)\), where
\[
q_m = p_1 + p_2 + \ldots + p_m, \ m \in \mathbb{N}
\] (2.3)
is an ‘approximate unit’ in \(B_0\), namely \(q_m x = x q_m = x\) for every \(x \in B^{(m-1)}\) and \(m > 1\).

By adding the unit and the zero projection to the sequence \((q_m)\) we obtain a complete lattice
\[
\mathcal{P} = \{q_m; \ 0 \leq m \leq \infty\}
\] (2.4)
where we set \(q_0 = 0\) and \(q_\infty = 1\), which is a sublattice of the lattice of all projections in \(B^{(0)}\) and thus a sublattice of the lattice of all projections in \(B_0\). We have \(q_m \uparrow 1\), both in the sense of lattice supremum in \(\mathcal{P}\) and pointwise in \(B_0\). Note also that all projections in \(\mathcal{P}\) commute and the meet of any two projections \(e, f \in \mathcal{P}\) is given by their product \(e \cap f = \min\{e, f\} = ef\).
Definition 2.1. A monotone strongly dense domain (MSDD) in $\mathcal{B}_0$ is a sequence of projections $(e_m)$, where $e_m \in \mathcal{P}$ and $e_m \uparrow 1$. If $x \in \mathcal{B}$ and $e \in \mathcal{P}$, we write $x^{-1}(e)$ for the largest projection $g \in \mathcal{P}$ such that $exg = xg$. An operator with monotone closure (OWMC) is a sequence $(x_m, e_m)$ with $x_m \in \mathcal{B}^{(m)}$ and $(e_m)$ a MSDD, such that $m < n$ implies $x_n e_m = x_m e_m$ and $x^*_n e_m = x^*_m e_m$.

Example. Of course, $(q_m)$ is a MSDD, but an equally important example for us will be the shifted sequence $(q_m - k) = (e_m)$, where

$$e_m = \begin{cases} 0 & \text{if } m \leq k \\ q_{m-k} & \text{if } m > k \end{cases}$$

i.e. $(q_{m-k})$ is also a MSDD – in the sequel the notation $(q_{m-k})$ will be used with the understanding that the index $m$ is reserved for the running index, whereas $k$ is fixed and is responsible for the shift. Similarly, the sequence $(f_m) = (1_{m-k})$ defined by

$$f_m = \begin{cases} 0 & \text{if } m \leq k \\ 1 & \text{if } m > k \end{cases}$$

is also a MSDD. This MSDD is used to embed $\mathcal{B}_0$ in $\mathcal{B}$. Namely, if $x \in \mathcal{B}^{(k)}$, then the MCO $[x_m, 1_{m-k}]$, where

$$x_m = \begin{cases} 0 & \text{if } m \leq k \\ x & \text{if } m > k \end{cases}$$

can be identified with $x$ (all such embeddings are consistent since they have the same ‘tails’).

Remark. It is sometimes convenient to write OWMC (and MCO) in the form of series

$$(x_m, e_m) = \sum_{m=1}^{\infty} (x_m - x_{m-1})$$

where $x_m \in \mathcal{B}^{(m)}$ and we set $x_0 = 0$ (we then keep in mind the SDD $(e_m)$).

Lemma 2.2. Suppose $(e_m)$ and $(f_m)$ are MSDD. Then $(e_m f_m)$ is an MSDD. Further, if a sequence $(x_m)$, where $x_m \in \mathcal{B}^{(m)}$ for every $m$, satisfies $x_n e_m = x_m e_m$ for every $n > m$, then the sequence $(g_m) = (e_m x^{-1}_m (f_m))$ is a MSDD.

Proof. That $(e_m f_m)$ is a MSDD if $(e_m)$ and $(f_m)$ are MSDD, immediately follows from the definition of MSDD. Let us prove that $(g_m)$ is a MSDD. The proof of monotonicity is the same as in the case of SDD and we quote it after [Bel1-Bel2] only for the reader’s convenience. Denote $h_n = x^{-1}_n(f_n)$. Thus $g_n = e_n h_n$ and $h_n$ is the largest projection from the lattice $\mathcal{P}$ such that

$$(1 - f_n)x_n h_n = 0$$

If $n > m$, then

$$x_n g_m = x_n e_m g_m = x_m e_m g_m = x_m g_m = x_m h_m g_m$$
and thus

\[(1 - f_n)x_ng = (1 - f_n)x_hg = 0\]

which, by maximality of \(h_m\), implies that \(g_m \leq h_m\). Also, \(g_m \leq e_m \leq e_n\) since \((e_m)\) is a MSDD. Thus, \(g_m \leq h_ne_n = g_m\), which ends the proof of monotonicity of \((g_m)\).

Let us show that \(g_m \uparrow 1\). Since \((f_m)\) is a MSDD, we have \(f_m = q_{l(m)}\) where \(0 \leq l(m) \leq \infty\) and \(l(m) \uparrow \infty\). We now use the assumption that each \(x_k \in B^k\), which implies that \(f_m\) acts as an identity when multiplied by \(x_1, x_2, \ldots, x_{l(m) - 1}\). Hence, if \(l(m) > m\), then \((1 - f_m)x_m = 0\), which gives \(h_m = 1\). In turn, if \(l(m) \leq m\), then we write

\[x_m = x_{l(m) - 1} + (x_m - x_{l(m) - 1})\]

and we have

\[(1 - f_m)x_mh_m = (1 - f_m)(x_m - x_{l(m) - 1})h_m, \quad (2.5)\]

since

\[f_mx_m = x_{l(m) - 1} + f_m(x_m - x_{l(m) - 1})\]

but equation

\[(1 - f_m)(x_m - x_{l(m) - 1})h_m = 0\]

is satisfied if we take for \(h_m\) any projection which right-annihilates \((x_m - x_{l(m) - 1})\). Note that \(e_{l(m) - 1}\) is such a projection by assumption. Therefore, \(h_m \geq e_{l(m) - 1}\). Since \(l(m) \rightarrow \infty\) and \((e_m)\) is a MSDD, we obtain \(g_m = h_me_m \uparrow 1\).

**Remark.** One should point out two new features in our definition of OWMC as compared to OWC: we assume that \(x_m \in B^m\) and we take a more restricted family of SDD. Note that in the theory of closed *-rings (*-algebras) we have \(y^{-1}(e) \succeq e\) for any projection \(e\) and any \(y\), where \(\succeq\) is the order implemented by partial isometries. Using this and \(\text{LP} \sim \text{RP}\), one shows that \(e_m \cap x_m^{-1}(f_m)\) is a SDD under assumptions similar to those in Lemma 2.2. However, in the main example studied in this paper, the property \(y^{-1}(e) \succeq e\) does not hold, which is one of the obstacles in applying the usual theory.

**Lemma 2.3.** If \((x_m, e_m)\) and \((y_m, f_m)\) are OWMC and we define

\[k_m = f_my_m^{-1}(e_m)e_m(x_m^*)^{-1}(f_m) \quad (2.6)\]

then \((x^*_m, e_m), (x_m + y_m, e_f m), (\lambda x_m, e_m)\) and \((x_my_m, k_m)\) are OWMC, where \(\lambda \in \mathbb{C}\).

**Proof.** Note that \(x_my_m \in B^m\) since \(x_m, y_m \in B^m\). Then one shows that the sequences \((x_m + y_m, e_f m), (x_my_m, k_m), (\lambda x_m, e_m)\) and \((x^*_m, e_m)\) are OWMC. For instance

\[x_n y_n k_m = x_n y_n e_m f_my_m^{-1}(e_m)(x_m^*)^{-1}(f_m) = x_n y_m e_m f_m(x_m^*)^{-1}(f_m) = x_n e_my_m k_m = x_m e_my_m k_m = x_m y_m k_m\]
where we use the definition of $y_m^{-1}(e_m)$ and the fact that all projections involved commute.

**Definition 2.4.** We say that the OWMC $(x_m, e_m)$, $(y_m, f_m)$ are *equivalent*, written $(x_m, e_m) \equiv (y_m, f_m)$ if there exists an MSDD $(g_m)$ such that $x_m g_m = y_m g_m$ and $x^*_m g_m = y^*_m g_m$ for every $m$. This relation is an equivalence relation and we say that $(g_m)$ *implements* the equivalence. The set of all equivalence classes is denoted by $\mathcal{B}$ and its elements are called *monotone closed operators* (MCO). We denote by $[x_m, e_m]$ the MCO determined by the OWMC $(x_m, e_m)$.

**Theorem 2.5.** If $x = [x_m, e_m], y = [y_m, f_m] \in \mathcal{B}$, then the operations

$$
\begin{align*}
x + y &= [x_m + y_m, e_m f_m] \\
x y &= [x_m y_m, k_m] \\
x^* &= [x^*_m, e_m] \\
\lambda x &= [\lambda x_m, e_m]
\end{align*}
$$

are well-defined and make $\mathcal{B}$ into a unital $\ast$-algebra, where $k_m$ is given by (2.2) and $\lambda \in \mathbb{C}$.

**Proof.** The proof since it is similar to that in the case of closed operators [Be1,Be2]. $\square$

### 3. The unital $\ast$-algebra $\mathcal{F}_0(\mathcal{G})$ and its monotone closure $\mathcal{F}(\mathcal{G})$

In this Section we give a construction of a unital $\ast$-algebra that is related to free products of states in free probability. The main idea of the construction is to extend the free product of a countable number of copies of a given unital $\ast$-algebra by a sequence of projections which will provide us with a ‘nice’ SDD.

Let $\mathcal{A}$ be the unital free $\ast$-algebra generated by the set $\mathcal{G}$. We take countably many copies of this algebra which we label by $\mathcal{A}'(k)$ and $\mathcal{A}''(k)$, where $k \in \mathbb{N}$, with the condition $\mathcal{A}''(1) = \{0\}$. Now, take the free product of all these copies (without identification of units)

$$
\hat{\mathcal{F}}(\mathcal{G}) = \sqcup_{k \in \mathbb{N}} (\mathcal{A}'(k) \sqcup \mathcal{A}''(k))
$$

i.e. the linear span of all words (we treat the unit from $\mathbb{C}$ as the empty word), and extend this product by the $\ast$-algebra of polynomials over the sequence $(p_m)$ of orthogonal projections, namely

$$
\hat{\mathcal{F}}_0(\mathcal{G}) = \hat{\mathcal{F}}(\mathcal{G}) \ast \mathbb{C}[p_1, p_2, p_3, \ldots]
$$

where $p_k p_l = \delta_{k,l} p_k$, $p^*_k = p_k$ and we assume that the unit in $\hat{\mathcal{F}}(\mathcal{G})$ (the empty word) and the unit in $\mathbb{C}[p_1, p_2, p_3, \ldots]$ are identified and denoted by 1.

In other words, $\hat{\mathcal{F}}_0(\mathcal{G})$ can be defined as the linear span of words of the form

$$
s_1 w_1 s_2 w_2 \ldots s_n w_n s_{n+1}
$$

with the juxtaposition product, where $w_1, \ldots, w_n$ are non-empty words from $\hat{\mathcal{F}}(\mathcal{G})$ and $s_1, \ldots, s_{n+1}$ are non-trivial projections from the lattice $\mathcal{P}$ given by (2.4).
For given $X \in \mathcal{G}$, we denote by $X'(k)$ and $X''(k)$ the copies of $X$ in $\mathcal{A}'(k)$ and $\mathcal{A}''(k)$, respectively.

**Definition 3.1.** Let $J$ be the two-sided $*$-ideal in $\hat{\mathcal{F}}_0(\mathcal{G})$ generated by elements of the form

$$(1 - q_m)X'(k) = 0 \text{ for } k < m \quad (3.2)$$
$$(1 - q_m)X''(k) = 0 \text{ for } k < m \quad (3.3)$$
$$q_m(X'(k) - X''(k)) = 0 \text{ for } k > m \quad (3.4)$$

where $X \in \mathcal{G}$. We denote by $\mathcal{F}_0(\mathcal{G}) = \hat{\mathcal{F}}_0(\mathcal{G})/J$ the corresponding quotient algebra.

In other words, we can define $\mathcal{F}_0(\mathcal{G})$ as the linear span of reduced words, i.e. words of the form (3.1) with the minimal number of non-trivial projections from the lattice $\mathcal{P}$ (after (3.2)-(3.4) have been taken into account) with the juxtaposition product inherited from $\mathcal{F}_0(\mathcal{G})$.

If we denote by $\mathcal{F}^{(m)}(\mathcal{G})$ the unital $*$-subalgebra of $\mathcal{F}_0(\mathcal{G})$ spanned by reduced words built from all projections $(q_m)$ and generators from $\mathcal{A}'(k), \mathcal{A}''(k)$ with $1 \leq k \leq m$ and $\mathcal{F}^{(0)}(\mathcal{G}) = \mathbb{C}[p_1, p_2, \ldots]$, then the sequence $(\mathcal{F}^{(m)}(\mathcal{G}))$ is an increasing sequence of unital $*$-algebras (2.1) with union $\mathcal{F}_0(\mathcal{G})$ as in (2.2), for which monotone closure can be constructed along the lines of Section 2. In particular, relations (3.2)-(3.3) make $(q_m)$ into a MSDD in $\mathcal{F}_0(\mathcal{G})$. In turn, relation (3.4) will be needed when introducing certain OWMC associated with free random variables. By $\mathcal{F}(\mathcal{G})$ we denote the monotone closure of $\mathcal{F}_0(\mathcal{G})$ consisting of monotone closed operators $[x_m, e_m]$, where $x_m \in \mathcal{F}_0(\mathcal{G})$ and $(e_m)$ is a MSDD in $\mathcal{F}_0(\mathcal{G})$.

**Definition 3.2.** The monotone closed operators of the form

$$[x_m, q_m] = \sum_{k=1}^{m} \delta X(k), q_m] \quad (3.5)$$

where

$$\delta X(k) = \begin{cases} X'(k) - X''(k) & \text{if } k > 1 \\ X'(1) & \text{if } k = 1 \end{cases} \quad (3.6)$$

and $X \in \mathcal{G}$, will be called pre-free random variables.

Pre-free random variables contain encoded information about freeness. Namely, after applying an appropriate comultiplication to a pre-free random variable we obtain a sum of free random variables with respect to a tensor product of states (see Section 5). Denote by $\mathcal{F}_{pf}(\mathcal{G})$ the (non-unital) $*$-subalgebra of $\mathcal{F}(\mathcal{G})$ generated by pre-free random variables.

To get a glimpse of freeness in this definition, let us now derive the explicit form of the product of pre-free random variables (the remaining algebraic operations are straightforward).
Theorem 3.3. The product of pre-free random variables takes the form
\[
[x_m(1), q_m][x_m(2), q_m] \cdots [x_m(n), q_m] = [x_m(1)x_m(2) \cdots x_m(n), q_{m-n+1}]
\]
where \([x_m(k), q_m] \in F_{pf}(G), \ k = 1, \ldots, n\), are pre-free random variables associated with \(X_1, \ldots, X_n \in G\).

Proof. Instead of giving a formal inductive proof, we prefer to analyze the cases \(n = 2\) and \(n = 3\), which is more intuitive and seems sufficient to see how to proceed in the general case. Let \(X, Y, Z \in G\) be non-zero and let \([x_m(q_m), y_m(q_m), z_m(q_m)]\) be the associated pre-free random variables. We begin with showing that
\[
[x_m(q_m), y_m(q_m)] = [x_m y_m, q_{m-1}].
\]
It can be seen from (2.6) that this boils down to the computation of \(y_m^{-1}(q_m)\). We have to find the largest projection \(g_m \in P\) such that
\[
(1 - q_m) \sum_{k=1}^{m} \delta Y(k) g_m = 0 \tag{3.7}
\]
Note that
\[
(1 - q_m) \sum_{k=1}^{m-1} \delta Y(k) = 0
\]
for any \(m > 1\) in view of (3.2)-(3.3). Hence, (3.7) holds iff
\[
(1 - q_m) \delta Y(m) g_m = 0
\]
and the largest projection from \(P\) which satisfies this equation is the largest projection from \(P\) which right-annihilates \(\delta Y(m)\), namely \(q_{m-1}\). Thus, \(g_m = q_{m-1}\). Similarly, \((x_m)^{-1}(q_m) = q_{m-1}\) and therefore, the \(k_m\) of (2.6) are given by \(k_m = q_{m-1}\). The next step consists in finding the \((k_m)\) in the formula
\[
(z_m, q_m)(x_m y_m, q_{m-1}) = (z_m x_m y_m, k_m),
\]
which is of the form
\[
k_m = q_{m-1}(x_m y_m)^{-1}(q_m) q_m (z_m^*)^{-1}(q_{m-1}).
\]
First, let us find the largest \(g_m \in P\) such that
\[
(1 - q_m)x_m y_m g_m = 0. \tag{3.8}
\]
We claim that if we take \(g_m = q_{m-2}\), then this equation is satisfied. Namely,
\[
x_m y_m q_{m-2} = x_m y_m - 2 q_{m-2} = x_m q_{m-1} y_m - 2 q_{m-2} = x_{m-1} q_{m-1} y_m - 2 q_{m-2}
\]
where we used (3.2)-(3.4) repeatedly, and thus
\[
q_m x_m y_m q_{m-2} = x_{m-1} q_{m-1} y_m - 2 q_{m-2} = x_m y_m q_{m-2}
\]
which implies that \( g \geq q_{m-2} \). We need to show that \( q_{m-2} \) is the largest projection from \( \mathcal{P} \) which satisfies (3.8). Let us check if \( q_{m-1} \) solves (3.8) when substituted for \( g_m \). Using (3.2)-(3.4) we get

\[
(1 - q_m)x_my_{m-1}q_{m-1} = (1 - q_m)x_my_{m-1}q_{m-1} = (1 - q_m)\delta X(m)y_{m-1}q_{m-1} = \sum_{l=1}^{m-1} (1 - q_m)\delta X(m)\delta Y(l)q_{m-1} \neq 0
\]

since we obtained a sum of linearly independent words, each of which is non-zero. In a similar way we show that \( q_m \neq q_{m-1} \) and it is clear that \( g_m \neq q_p \) for \( p > m \) since 

\[
(1 - q_m)x_my_q \neq 0.
\]

This finishes the proof for the product of three OMC. For products of higher order we continue in a similar way. \( \square \)

**Remark.** Equivalently, we could write the product of pre-free random variables in the form

\[
[x^{(1)}_m, q_m][x^{(2)}_m, q_m] \ldots [x^{(n)}_m, q_m] = [x^{(1)}_{m+n-1}x^{(2)}_{m+n-2} \ldots x^{(n)}_m, q_m],
\]

which is not hard to demonstrate. Heuristically, the ranges of products of pre-free random variables form a monotone increasing sequence. Note that the same feature is exhibited by free random variables acting in the free Fock space.

### 4. Quantum semigroup structure on \( \mathcal{F}(\mathcal{G}) \)

In this Section we show that one can endow \( \mathcal{F}(\mathcal{G}) \) with a quantum semigroup (*-bialgebra) structure with the algebraic tensor product \( \otimes \) replaced by an appropriate ‘closure’ \( \overline{\otimes} \).

This is done by first introducing a *-bialgebra structure on \( \mathcal{F}_0(\mathcal{G}) \) and then lifting the comultiplication \( \Delta \) and the counit \( \epsilon \) from \( \mathcal{F}_0(\mathcal{G}) \) to \( \mathcal{F}(\mathcal{G}) \). This can be done once a new type of tensor product, called monotone tensor product,

\[
\mathcal{F}(\mathcal{G}) \overline{\otimes} \mathcal{F}(\mathcal{G}) = \overline{\mathcal{F}_0(\mathcal{G}) \otimes \mathcal{F}_0(\mathcal{G})}
\]

is introduced, where the monotone closure is taken on strongly dense domains \( (r_m) \) implemented by the product lattice \( \mathcal{P}(2) = \Delta(\mathcal{P}) \).

First, let us introduce a *-bialgebra structure on \( \mathcal{F}_0(\mathcal{G}) \).

**Proposition 4.1.** The unital *-algebra \( \mathcal{F}_0(\mathcal{G}) \) becomes a *-bialgebra, when equipped with the comultiplication \( \Delta : \mathcal{F}_0(\mathcal{G}) \to \mathcal{F}_0(\mathcal{G}) \otimes \mathcal{F}_0(\mathcal{G}) \) given by

\[
\begin{align*}
\Delta(X'(k)) &= X'(k) \otimes q_k + q_k \otimes X'(k) \\
\Delta(X''(k)) &= X''(k) \otimes q_{k-1} + q_{k-1} \otimes X''(k) \\
\Delta(q_k) &= q_k \otimes q_k \\
\Delta(1) &= 1 \otimes 1
\end{align*}
\]
and the counit $\epsilon : F_0(G) \rightarrow \mathbb{C}$ given by $\epsilon(X'(k)) = \epsilon(X''(k)) = 0$ and $\epsilon(q_k) = \epsilon(1) = 1$, where $X \in G$.

Proof. It is easy to verify that $\Delta$ is coassociative on generators since $X'(k)$ is $q_k$-primitive, $X''(k)$ is $q_{k-1}$-primitive and $q_k$ as well as the unit $1$ are group-like. Then the coassociativity on all of $F_0(G)$ easily follows. Moreover, $\Delta$ and $\epsilon$ preserve the relations in $F_0(G)$. For instance, if $k < m$, then

$$
\Delta(q_m X'(k)) = (q_m \otimes q_m)(X'(k) \otimes q_k + q_k \otimes X'(k)) \\
= q_m X'(k) \otimes q_k + q_k \otimes q_m X'(k) \\
= X'(k) \otimes q_k + q_k \otimes X'(k) \\
= \Delta(X'(k))
$$

thus (3.2) is preserved (an identical proof holds for (3.3)). In turn, if $k > m$, then

$$
\Delta(q_m (X'(k) - X''(k))) = q_m X'(k) \otimes q_m + q_m \otimes q_m X'(k) \\
- q_m X''(k) \otimes q_m - q_m \otimes q_m X''(k) \\
= 0
$$

an thus (3.4) is preserved. Besides, it is easy to show that $\epsilon$ is a counit. Thus, the triple $(F_0(G), \Delta, \epsilon)$ is a unital *-bialgebra.

In order to lift this structure to the monotone closure $F(G)$, we need to define a new type of tensor product called the ‘monotone tensor product’. Our definition follows the pattern of the von Neumann algebra tensor product.

Let $(B^{(m)})$ and $(C^{(m)})$ be sequences of increasing unital *-algebras (2.1) with unions $B_0$ and $C_0$ (2.2), respectively and let

$$
P = \{b_m; 0 \leq m \leq \infty\}, \quad Q = \{c_m; 0 \leq m \leq \infty\},
$$

where $b_0 = c_0 = 0$ and $b_\infty = 1_{B_0}$, $c_\infty = 1_{C_0}$, be the associated totally ordered lattices (2.4) of projections which generate $B^{(0)}$ and $C^{(0)}$, respectively. The sequence $(B^{(m)} \otimes C^{(m)})$ is then an increasing sequence of unital *-algebras, for which we can construct the monotone closure. Thus, a MSDD in $B_0 \otimes C_0$ is a sequence of projections $(r_m)$ from the lattice

$$
\mathcal{L}^{(2)} = \{b_m \otimes c_m; 0 \leq m \leq \infty\}
$$

such that $r_m \uparrow 1 \otimes 1$. In turn, an OWMC associated with $B_0 \otimes C_0$ is a sequence $(z_m, r_m)$, where $(r_m)$ is a MSDD in $B_0 \otimes C_0$ and

$$
z_m = \sum_k w_m^{(k)} \otimes u_m^{(k)} \in B_0^{(m)} \otimes C_0^{(m)}, \quad m \geq 1
$$

for every $m$ (sums over $k$ are finite) and such that $z_n r_m = z_m r_m$ and $z_n^* r_m = z_m^* r_m$ for all $n > m$. An equivalence relation in the set of all such OWMC is analogous to that of OWMC associated with $B_0$. 

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Definition 4.1. A MCO ‘affiliated’ with \( \mathcal{B}_0 \otimes \mathcal{C}_0 \) is the equivalence class \([z_m, r_m]\) corresponding to OWMC \((z_m, r_m)\). We denote by \( \overline{\mathcal{B}_0 \otimes \mathcal{C}_0} \) the unital *-algebra of MCO ‘affiliated’ with \( \mathcal{B}_0 \otimes \mathcal{C}_0 \). This closure leads to the definition of a monotone tensor product denoted \( \mathcal{B} \boxtimes \mathcal{C} = \overline{\mathcal{B}_0 \otimes \mathcal{C}_0} \).

Remark 1. Note that by setting
\[
[x_m, e_m] \otimes [y_m, f_m] := [x_m \otimes y_m, e_m \otimes f_m]
\]
we obtain a natural *-algebra embedding of \( \mathcal{B} \otimes \mathcal{C} \) in \( \overline{\mathcal{B}_0 \otimes \mathcal{C}_0} \).

Remark 2. For instance, the product in \( \overline{\mathcal{B}_0 \otimes \mathcal{C}_0} \) is given by
\[
[z_m, r_m][w_m, s_m] = [z_m w_m, k_m]
\]
where \( k_m = r_m s_m w_m^{-1}(r_m) (z_m^*)^{-1}(s_m) \) and \( w_m^{-1}(r_m) \) is defined to be the largest projection \( g \in \mathcal{L}^{(2)} \) for which \( r_m w_m g = w_m g \) and thus, it always exists (similarly, \((z_m^*)^{-1}(r_m)\) exists).

Remark 3. One can proceed in a similar manner with monotone tensor products of higher order and this procedure is associative. In particular
\[
\overline{\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})} = (\overline{\mathcal{B} \otimes \mathcal{C}}) \otimes \mathcal{D} = \overline{\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}} = \overline{\mathcal{B}_0 \otimes \mathcal{C}_0 \otimes \mathcal{D}_0},
\]
with the closure taken w.r.t. MSDD from the lattice
\[
\mathcal{L}^{(3)} = \{b_m \otimes c_m \otimes d_m; 0 \leq m \leq \infty\}
\]
where the lattice associated with \( \mathcal{D}_0 \) is generated by an increasing sequence of projections \((d_m)\), with \( d_0 = 0 \) and \( d_\infty = 1 \).

Let us specify now how we can lift unital *-homomorphisms from unital *-algebras to their monotone closures. Namely, let \( \mathcal{B}_0 \) and \( \mathcal{C}_0 \) be unital *-algebras for which the associated monotone closures are \( \mathcal{B} \) and \( \mathcal{C} \), respectively. If \( \tau : \mathcal{B}_0 \rightarrow \mathcal{C}_0 \) is a unital *-homomorphism, then one can lift \( \tau \) to a unital *-homomorphism from \( \mathcal{B} \) to \( \mathcal{C} \) by the formula
\[
\tau [x_m, e_m] = [\tau(x_m), \tau(e_m)],
\]
where we adopt the convention that the extended map is denoted by the same symbol – here, we understand that the monotone closure in \( \mathcal{C} \) is ‘compatible’ with \( \tau \), i.e. is taken w.r.t. MSDD from the lattice \( \tau(\mathcal{P}) \).

In particular, if \( \Delta \) is a comultiplication and \( \epsilon \) is a counit on \( \mathcal{B}_0 \), then we can lift these maps to \( \mathcal{B} \) (in the case of the counit, \( \epsilon(\mathcal{P}) = \{0, 1\} \)). In order to treat the axioms required from the comultiplication and the counit, we need to define a monotone tensor product of maps \( \tau : \mathcal{B}_0 \rightarrow \mathcal{D}_0 \) and \( \sigma : \mathcal{C}_0 \rightarrow \mathcal{E}_0 \), namely
\[
\tau \boxtimes \sigma : \overline{\mathcal{B} \otimes \mathcal{C}} \rightarrow \overline{\mathcal{D} \otimes \mathcal{E}}
\]
\[ \tau \otimes \sigma \left[ \sum_k w_m^{(k)} \otimes u_m^{(k)}, e_m \otimes f_m \right] = \left[ \sum_k \tau(w_m^{(k)}) \otimes \sigma(u_m^{(k)}), \tau(e_m) \otimes \sigma(f_m) \right] \]

where the RHS is a MCO in \( D \otimes E \). In this way we define \( \Delta \otimes \text{id} \) and \( \text{id} \otimes \Delta \) and their compositons with \( \Delta \), \( (\Delta \otimes \text{id}) \circ \Delta \) and \( (\text{id} \otimes \Delta) \circ \Delta \) needed for coassociativity.

Using these preparations, we can lift comultiplications and counits from unital \(*\)-algebras to their monotone closures. The same axioms hold as in the algebraic case except that the algebraic tensor product \( \otimes \) is replaced by the monotone tensor product \( \overline{\otimes} \). Thus we obtain ‘a unital \(*\)-bialgebra with respect to the monotone tensor product’ which we call a **monotone closed quantum semigroup**.

Let us now look at the case of \( F(G) \). Note that since all elements of \( \mathcal{P} \) are in this case group-like w.r.t. \( \Delta \) and \( \mathcal{P}^{(m)} = \Delta^{(m-1)}(\mathcal{P}) \), we can view \( \mathcal{P} \) as a ‘group-like lattice’.

Let us now define for each natural number \( n \) the lattice of projections

\[ \mathcal{P}^{(n)} = \Delta^{(n-1)}(\mathcal{P}) := \{ \Delta^{(n-1)}(p), p \in \mathcal{P} \} \]

where \( \Delta^{(n)} \) is the \( n \)-th iteration of the comultiplication \( \Delta \), namely

\[ \Delta^{(n)} := (\text{id} \otimes \Delta^{(n-1)}) \circ \Delta, \]

which allows us to compare projections from \( \mathcal{P}^{(n)} \) for each \( n \) with the order inherited from \( \mathcal{P} \), i.e. if \( r = \Delta^{(n-1)}(p), s = \Delta^{(n-1)}(q) \), then \( r < s \) iff \( p < q \).

**Theorem 4.2.** The unital \(*\)-algebra \( F(G) \) becomes a monotone closed quantum semigroup when equipped with the comultiplication

\[ \Delta : F(G) \to F(G) \overline{\otimes} F(G), \]

\[ \Delta[x_m, q_m] := [\Delta(x_m), \Delta(q_m)], \]

and the counit \( \epsilon : F(G) \to \mathbb{C} \) given by \( \epsilon([x_m, e_m]) := [\epsilon(x_m), \epsilon(e_m)] \).

**Proof.** Coassociativity of \( \Delta \) on \( F(G) \) follows from the coassociativity of \( \Delta \) on \( F_0(G) \). Verification of this fact and of the axioms on the counit is routine. \( \square \)

We will return to the monotone closed semigroup structure on \( F(G) \) in Section 6. By then it will become clear why this structure can be used to implement the free additive convolution. In the meantime, we will study the tensor product representation of the free product of states in the general case of arbitrary unital \(*\)-algebras.

### 5. Free products

In this section we will concentrate on reconstructing the free product of states from the tensor product of states.

In order to do this we replace the ‘rigid’ requirement for \( \Delta \) to be a homomorphism by the requirement that the ‘half-coproduct’ maps of type (1.1)-(1.3) be homomorphisms. For simplicity, we first establish our result for two unital \(*\)-algebras and then generalize it to the case of an arbitrary family of unital \(*\)-algebras.
Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary unital $*$-algebras. Let us take free products
\[ \hat{\mathcal{H}}(\mathcal{A}) = *_{k \in \mathbb{N}} \mathcal{A}(k), \quad \hat{\mathcal{H}}(\mathcal{B}) = *_{k \in \mathbb{N}} \mathcal{B}(k) \]
of copies of $\mathcal{A}$ and $\mathcal{B}$ (in these free products we identify units), respectively, and extend them by sequences of orthogonal projections
\[ \hat{\mathcal{H}}_0(\mathcal{A}) = \hat{\mathcal{H}}(\mathcal{A}) \ast \mathbb{C}[p_1, p_2, p_3, \ldots] \]
\[ \hat{\mathcal{H}}_0(\mathcal{B}) = \hat{\mathcal{H}}(\mathcal{B}) \ast \mathbb{C}[p'_1, p'_2, p'_3, \ldots] \]
with the associated lattices of increasing projections
\[ P_1 = \{ q_m : 0 \leq m \leq \infty \}, \quad P_2 = \{ q'_m : 0 \leq m \leq \infty \} \]

Next, let $I_1$ and $I_2$ be the two-sided $*$-ideals in $\hat{\mathcal{H}}_0(\mathcal{A})$ and $\hat{\mathcal{H}}_0(\mathcal{B})$ generated by $(1 - q_m)X(k)$ and $(1 - q'_m)Y(k)$ for $k < m$, respectively, where $X(k)$ and $Y(k)$ denote the $k$-th copies of $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Denote by $\mathcal{H}_0(\mathcal{A}) = \hat{\mathcal{H}}_0(\mathcal{A})/I_1$, $\mathcal{H}_0(\mathcal{B}) = \hat{\mathcal{H}}_0(\mathcal{B})/I_2$ the corresponding quotient algebras and by $\mathcal{H}(\mathcal{A})$, $\mathcal{H}(\mathcal{B})$ their monotone closures.

Let us introduce the linear mappings
\[ j_1 : \mathcal{A} \rightarrow \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B}), \quad j_2 : \mathcal{B} \rightarrow \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B}) \]
given by
\[ j_1(X) = \sum_{k=1}^{\infty} X(k) \otimes p'_k \quad (5.1) \]
\[ j_2(Y) = \sum_{k=1}^{\infty} p_k \otimes Y(k) \quad (5.2) \]

where $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, where the associated MSDD is $(q_m \otimes q'_m)$. From now on we adopt the convention that monotone closed operators from the tensor product $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B})$, when written in the form of infinite sums, are taken with MSDD of this form.

**Proposition 5.1.** The mappings $j_1$ and $j_2$ are unital $*$-homomorphisms.

**Proof.** The mappings $j_1$, $j_2$ are $*$-homomorphisms due to orthogonality of the sequences $(p_k)$, $(p'_k)$. Moreover, they are unital since, for instance
\[ j_1(1) = \left[ \sum_{k=1}^{m} 1(k) \otimes p'_k, q_m \otimes q'_m \right] = [1 \otimes q'_m, q_m \otimes q'_m] = [1 \otimes 1', 1 \otimes 1'] \equiv 1 \]
where 1 and $1'$ are units in $\mathcal{H}(\mathcal{A})$ and $\mathcal{H}(\mathcal{B})$, respectively. A similar proof holds for $j_2$. \qed
Heuristically, the lemma below says that a certain ‘conditional expectation’ of alternating mixed moments of \(j_1(X)\)’s and \(j_2(X)\)’s takes values in the two-sided ideal generated by ‘singletons’. We shall use

\[
\mathcal{K} = \langle q_k X(k)q_k, \text{ where } k \geq 1 \text{ and } X \in \mathcal{A} \rangle \\
\mathcal{K}' = \langle q'_k Y(k)q'_k, \text{ where } k \geq 1 \text{ and } Y \in \mathcal{B} \rangle,
\]

the monotone closed two sided ideals in \(\mathcal{H}(\mathcal{A})\) and \(\mathcal{H}(\mathcal{B})\), respectively.

**Lemma 5.2.** Let \(E : \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B}) \to \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B})\) be given by

\[
E(w) = (q_1 \otimes q'_1)w(q_1 \otimes q'_1)
\]

and let \(X_k \in \mathcal{A}_{i_k}\), where \(k = 1, \ldots, n\) and \(i_1 \neq i_2 \neq \ldots \neq i_n\) with \(A_1 = \mathcal{A}, A_2 = \mathcal{B}\). Then

\[
E(j_{i_1}(X_1)j_{i_2}(X_2) \ldots j_{i_n}(X_n)) \subset \mathcal{I}
\]

where \(\mathcal{I} = \mathcal{K} \otimes \mathcal{H}(\mathcal{B}) + \mathcal{H}(\mathcal{A}) \otimes \mathcal{K}'\).

**Proof.** To fix attention, we suppose that \(i_n = 1\) and denote

\[
A_n = j_{i_1}(X_1)j_{i_2}(X_2) \ldots j_{i_n}(X_n).
\]

We claim that

\[
A_n(q_1 \otimes q'_1) = a_n \otimes a'_n
\]

where, if \(n\) is odd,

\[
a_n = X_1(n)X_3(n-2) \ldots X_n(1)q_1 \\
a'_n = X_2(n-1)X_4(n-3) \ldots X_{n-1}(2)q'_1
\]

and, if \(n\) is even,

\[
a_n = X_2(n-1)X_4(n-3) \ldots X_n(1)q_1 \\
a'_n = X_1(n)X_3(n-2) \ldots X_{n-1}(2)q'_1
\]

for all \(X_n, X_{n-2}, \ldots \in \mathcal{A}\) and \(X_{n-1}, X_{n-3}, \ldots \in \mathcal{B}\). We will use induction to prove the claim. For \(n = 1\), the assertion is true since in that case \(A_1(q_1 \otimes q'_1) = X_1(1)q_1 \otimes q'_1\). We suppose now that \(n\) is odd and that the claim holds for the product of \(n - 1\) factors. Then

\[
A_n(q_1 \otimes q'_1) = \sum_{k=1}^{\infty} (X_1(k) \otimes p'_k)(w_1 \otimes w_2)
\]

where

\[
w_1 = X_3(n-2)X_5(n-4) \ldots X_n(1)q_1 \\
w_2 = X_2(n-1)X_4(n-3) \ldots X_{n-1}(2)q'_1.
\]

Let us analyze the contribution from every term indexed by \(k\), namely

\[
B_k = u_k \otimes (v_k - v'_k)
\]
where
\[ u_k = X_1(k)X_3(n-2) \ldots X_n(1)q_1 \]
\[ v_k = q'_kX_2(n-1)X_4(n-3) \ldots X_{n-1}(2)q'_1 \]
\[ v'_k = q'_{k-1}X_2(n-1)X_4(n-3) \ldots X_{n-1}(2)q'_1. \]

\textit{Case 1.} If \( k > n \), then we apply (3.2) and (3.3) to the effect that
\[ q'_kX_2(n-1) = X_2(n-1) \]
\[ q'_{k-1}X_2(n-1) = X_2(n-1) \]
which gives \( v'_k = v'_k \) and thus \( B_k = 0 \).

\textit{Case 2.} In turn, if \( k = n \), then
\[ v_n = X_2(n-1)X_4(n-3) \ldots X_{n-1}(2)q'_1 \]
\[ v'_n = q_{n-1}X_2(n-1)q_{n-1}X_4(n-3) \ldots X_{n-1}(2)q'_1 \]
since
\[ q'_nX_2(n-1) = X_2(n-1) \]
\[ q'_{n-1}X_4(n-3) = X_4(n-3) \]
respectively, which implies that
\[ u_n \otimes v'_n \in \mathcal{J} \]
and \( u_n \otimes v_n \) is of the form given in the claim.

\textit{Case 3.} Finally, if \( k < n \), then
\[ u_k \otimes v_k = u_k \otimes q'_kq'_{n-1}X_2(n-1)q'_{n-1}w \in \mathcal{J} \]
\[ u_k \otimes v'_k = u_k \otimes q'_{k-1}q'_{n-1}X_2(n-1)q'_{n-1}w' \in \mathcal{J} \]
for some words \( w, w' \) which start with \( X_4(n-3) \), using similar arguments as above. This finishes the proof of the claim.

After multiplying (5.3) from the left by \( q_1 \otimes q'_1 \), we can see that
\[ E(A_n) \subset \mathcal{J} \]
since in the expression
\[ q_1X_1(n)X_3(n-2) \ldots X_n(1)q_1, \]
we can write \( q_1 = q_1q_n \) and \( X_3(n-2) = q_nX_3(n-2) \) to produce \( q_nX_1(n)q_n \). This completes the proof.

\[ \square \]
\( \mathcal{A} \star \mathcal{B} \) of unital *-algebras can be represented by the monotone tensor product \( \mu \otimes \nu \) of extended states on \( \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B}) \). For this purpose we use the unital *-homomorphism \( j = j_1 \ast j_2 \) from the free product \( \mathcal{A} \star \mathcal{B} \) with identified units into the monotone tensor product of \( \mathcal{H}(\mathcal{A}) \) and \( \mathcal{H}(\mathcal{B}) \), where \( j_1 \) and \( j_2 \) are given by (5.1)-(5.2).

Roughly speaking, the states \( \hat{\mu} \) and \( \hat{\nu} \) will turn out to be tensor products of boolean extensions of \( \mu \) and \( \nu \), respectively. Let us recall that the boolean extension of a state \( \mu \) on a unital *-algebra \( \mathcal{A} \) to the unital *-algebra \( \tilde{\mathcal{A}} = \mathcal{A} \star \mathbb{C}[P] \) where \( P \) is a projection, is a state \( \tilde{\mu} \) such that

\[
\tilde{\mu}(wPv) = \tilde{\mu}(w)\tilde{\mu}(v), \quad \tilde{\mu}(P) = 1
\]

for any \( w, v \in \tilde{\mathcal{A}} \), and which, when restricted to \( \mathcal{A} \), agrees with \( \mu \).

**Theorem 5.3.** For given states \( \mu \) and \( \nu \) on \( \mathcal{A} \) and \( \mathcal{B} \), respectively, there exist states \( \hat{\mu} \) and \( \hat{\nu} \) on \( \mathcal{H}(\mathcal{A}) \) and \( \mathcal{H}(\mathcal{B}) \) and a unital *-homomorphism

\[
j : \mathcal{A} \star \mathcal{B} \rightarrow \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{B})
\]

such that

\[
\mu \ast \nu = (\mu \otimes \nu) \circ j.
\]

Thus, the unital *-algebras \( j(\mathcal{A}) \) and \( j(\mathcal{B}) \) are free with respect to \( \mu \otimes \nu \).

**Proof.** We divide the proof into 3 steps.

**Step 1.** First, we construct suitable states on \( \mathcal{H}_0(\mathcal{A}) \) and \( \mathcal{H}_0(\mathcal{B}) \) associated with states \( \mu \) on \( \mathcal{A} \) and \( \nu \) on \( \mathcal{B} \). Extend \( \mathcal{A} \) and \( \mathcal{B} \) by projections \( P \) and \( P' \), respectively, namely

\[
\tilde{\mathcal{A}} = \mathcal{A} \star \mathbb{C}[P], \quad \tilde{\mathcal{B}} = \mathcal{B} \star \mathbb{C}[P']
\]

and extend \( \mu \) and \( \nu \) to their boolean extensions \( \tilde{\mu} \) and \( \tilde{\nu} \). Now, take tensor products

\[
\hat{\mathcal{A}} = \tilde{\mathcal{A}} \otimes \infty, \quad \hat{\mathcal{B}} = \tilde{\mathcal{B}} \otimes \infty, \quad \hat{\Phi} = \hat{\tilde{\mu}} \otimes \infty, \quad \hat{\Psi} = \hat{\tilde{\nu}} \otimes \infty
\]

and introduce unital *-homomorphisms

\[
\xi : \mathcal{H}_0(\mathcal{A}) \rightarrow \hat{\mathcal{A}}, \quad \eta : \mathcal{H}_0(\mathcal{B}) \rightarrow \hat{\mathcal{B}}
\]

given by

\[
\xi(X(k)) = 1 \otimes (k-1) \otimes X \otimes 1 \otimes \infty \quad (5.4)
\]

\[
\xi(q_m) = 1 \otimes (m-1) \otimes P \otimes \infty \quad (5.5)
\]

\[
\xi(1) = 1 \otimes \infty \quad (5.6)
\]

with analogous formulas for \( \eta \). Then the linear functionals

\[
\hat{\tilde{\mu}} = \hat{\Phi} \circ \xi, \quad \hat{\tilde{\nu}} = \hat{\Psi} \circ \eta
\]
are states on $\mathcal{H}_0(\mathcal{A})$ and $\mathcal{H}_0(\mathcal{B})$, respectively.

**Step 2.** Let us show now that these states can be extended to states on $\mathcal{H}(\mathcal{A})$ and $\mathcal{H}(\mathcal{B})$, respectively, and their tensor product to $\mathcal{H}(\mathcal{A}) \overline{\otimes} \mathcal{H}(\mathcal{B})$ by taking pointwise limits. Denoting them also by $\tilde{\mu}$ and $\tilde{\nu}$, we set

$$
\tilde{\mu}(z) = \lim_{m \to \infty} \tilde{\mu}(z_m) \tag{5.7}
$$

$$
\tilde{\nu}(u) = \lim_{m \to \infty} \tilde{\nu}(z_m) \tag{5.8}
$$

$$
(\tilde{\mu} \otimes \tilde{\nu})(w) = \lim_{m \to \infty} (\tilde{\mu} \otimes \tilde{\nu})(w_m) \tag{5.9}
$$

where $z = [z_m, e_m] \in \mathcal{H}(\mathcal{A})$, $u = [u_m, f_m] \in \mathcal{H}(\mathcal{B})$ and $w = [w_m, r_m] \in \mathcal{H}(\mathcal{A}) \overline{\otimes} \mathcal{H}(\mathcal{B})$. In fact, let us show that the linear functionals given by (5.7)-(5.9) are well-defined and that they are states. We shall establish existence of the limits on the RHS of (5.7) only since the proofs for (5.8)-(5.9) are similar. We have $e_m = q_{k(m)}$ with $k(m) \uparrow \infty$. Since $[z_m, q_{k(m)}]$ is MCO, we have $z_n q_{k(m)} = z_m q_{k(m)}$ for $n > m$. Since there exists natural $s$ such that $k(m) > 0$ for $m > s$, hence

$$
z_n q_1 = z_n q_{k(m)} q_1 = z_m q_{k(m)} q_1 = z_m q_1
$$

for $n > m > s$. Therefore, there exists $s$ such that

$$
q_1 z_n q_1 = q_1 z_m q_1
$$

for $n > m > s$, which implies that $\lim_{m \to \infty} \tilde{\mu}(z_m)$ exists since

$$
\tilde{\mu}(z_m) = \tilde{\mu}(q_1 z_m q_1) = \tilde{\mu}(q_1 z_n q_1) = \tilde{\mu}(z_n)
$$

for $n > m > s$, where we used $\tilde{\mu}(PuP) = \tilde{\mu}(u)$ and the definition of $\tilde{\mu}$. Moreover, it does not depend on the choice of representatives of MCO. In fact, choose a different representative for $z$, say $(z'_m, q_{l(m)})$ instead of $(z_m, q_{k(m)})$ with the equivalence $(z_m, q_{k(m)}) \equiv (z'_m, q_{l(m)})$ implemented by some MSDD $(e_m) = (q_{r(m)})$, i.e. $z_m e_m = z'_m e_m$ for all $m$. Again, there exists natural $s$ such that $r(m) > 0$ for $m > s$. A similar argument as that given above gives $z_m q_1 = z'_m q_1$ for $m > s$, which gives independence of the limit from the choice of representatives. Finally, positivity and normalization of the extended functional follow from its definition.

**Step 3.** Finally, let us reconstruct the free product of states. Let

$$
 j : \mathcal{A} \ast \mathcal{B} \to \mathcal{H}(\mathcal{A}) \overline{\otimes} \mathcal{H}(\mathcal{B})
$$

be the unital $*$-homomorphism given by

$$
 j(X) = j_1(X), \ j(Y) = j_2(Y)
$$

where $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Use Proposition 5.1 and the fact that $j_1(1_A) = j_2(1_B)$ to conclude that $j$ is well-defined. We will show that $\mu \ast \nu$ agrees with $(\tilde{\mu} \otimes \tilde{\nu}) \circ j$. Note that if $X \in \ker \mu$, where $\mu$ is a state on $\mathcal{A}$, then $PXP \in \ker \tilde{\mu}$, where $\tilde{\mu}$ is the
boolean extension of $\mu$. A similar statement holds for $\nu$. Thus, if $K_{\mu}$ and $K_{\nu}$ denote
the two-sided ideals

\[ K_{\mu} = \langle q_k X(k)q_k, \text{ where } k \in \mathbb{N} \text{ and } X \in A \cap \ker \mu \rangle \]
\[ K_{\nu} = \langle q_k' Y(k)q_k', \text{ where } k \in \mathbb{N} \text{ and } Y \in B \cap \ker \nu \rangle \]
respectively, then

\[ K_{\mu} \boxtimes \mathcal{H}(B) + \mathcal{H}(A) \boxtimes K_{\nu} \subset \ker(\widehat{\mu} \boxtimes \widehat{\nu}). \]
It is now enough to remark that

\[ \widehat{\mu} \boxtimes \widehat{\nu} = (\mu \boxtimes \nu) \circ E \]
since $\tilde{\mu}(P u P) = \tilde{\mu}(u)$ and $\tilde{\nu}(P' w P') = \tilde{\nu}(w)$. Using this and Lemma 5.2, we get

\[ \widehat{\mu} \boxtimes \widehat{\nu}(j_{i_1}(X_1) \ldots j_{i_n}(X_n)) = 0 \]
if $X_1, \ldots, X_n$ are in the kernels of $\mu$ or $\nu$ depending on whether we have the (alternating)
indices $i_1, \ldots, i_n$ equal to 1 or 2, respectively. This completes the proof. \qed

**Corollary 5.4.** For given states $\mu$ and $\nu$ on $A$ and $B$, the random variables
\[ j_1(X) \text{ and } j_2(Y), \text{ where } X \in A \text{ and } Y \in B, \]
have the same distributions as $X$ and $Y$, respectively, and are free with respect to the tensor product state $\widehat{\mu} \boxtimes \widehat{\nu}$.

**Proof.** This is an immediate consequence of Theorem 5.3. \qed

**Remark.** Thus formulas (5.1)-(5.2) give a tensor product representation of free
random variables. Treating states $\widehat{\mu}$ and $\widehat{\nu}$ as canonical extensions of $\mu$ and $\nu$ to $\mathcal{H}(A)$
and $\mathcal{H}(B)$, we can view this representation as the one in which information about
freeness is shifted from states to variables.

The above theorem can be generalized to arbitrary families $(A_l)_{l \in L}$ of unital *-
algebras. Let $\mathcal{H}_0(A_l)$ be the unital *-algebra constructed from copies of $A_l$ as shown in
Section 3 and let $\mathcal{H}(A_l)$ be the associated unital *-algebra of monotone closed operators.
By $(q_{k,l})_{k \in \mathbb{N}}$ for each $l \in L$ we denote the associated increasing sequence of projections.
Finally, let

\[ \bigotimes_{l \in L} \mathcal{H}(A_l) = \bigotimes_{l \in L} \mathcal{H}_0(A_l) \]

where the monotone closure on the right-hand side is taken with respect to the product
lattice

\[ \mathcal{L} = \{ \otimes_{l \in L} q_{m,l} : 0 \leq m \leq \infty \} \]
where we set $q_{0,l} = 0$ and $q_{\infty,l} = 1_l$ for every $l \in L$.

**Theorem 5.5.** For a given family of states $(\mu_l)_{l \in L}$ on unital *-algebras $(A_l)_{l \in L}$
there exist states $(\widehat{\mu})_{l \in L}$ on $(\mathcal{H}(A_l))_{l \in L}$, respectively, and a unital *-homomorphism

\[ j : *_{l \in L} A_l \rightarrow \bigotimes_{l \in L} \mathcal{H}(A_l) \]

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such that
\[ *_{l \in L} \mu_l = (\bigotimes_{l \in L} \hat{\mu}_l) \circ j \]
and thus the unital \(*\)-algebras \((j(A_l))_{l \in L}\) are free with respect to the monotone tensor product state \(\bigotimes_{l \in L} \hat{\mu}_l\).

**Proof.** For \(X \in A_m\) we define \(j(X)\) by the formula
\[ j(X) = \sum_{k=1}^{\infty} X(k) \otimes p_k \] (5.10)
according to the decomposition
\[ \bigotimes_{l \in L} H(A_l) = H(A_m) \otimes \bigotimes_{l \neq m} H(A_l) \]
where \(p_k = q_k - q_{k-1}\) and
\[ q_k = \bigotimes_{l \neq m} q_{k,l}. \]
Then we extend \(j\) to the free product \(*_{l \in L} A_l\) by taking the linear and multiplicative extension of (5.10) for all \(m \in L\). The states \((\hat{\mu}_l)_{l \in L}\) are defined as
\[ \hat{\mu}_l = \Phi_l \circ \xi_l \]
where
\[ \Phi_l = \tilde{\mu}^{\otimes \infty} \]
are states on the tensor product algebras
\[ \hat{A}_l = \tilde{A}_l^{\otimes \infty} \]
where \(\tilde{A}_l = A_l \otimes \mathbb{C}[P_l]\) and \(\xi_l\) is given by the same formula as (5.4)-(5.6) with \(P\) replaced by \(P_l\). The infinite tensor products are understood to be taken with respect to the family \(\{1_l, P_l\}\) [F-L-S], by which we mean that only finitely many sites are occupied by elements different from \(1_l\)’s and \(P_l\)’s. The remaining part of the proof is similar to that in the case of two algebras. \(\blacksquare\).

6. Additive free convolution

We return to the monotone closed semigroup structure on \(F(G)\) in order to show that it implements the additive free convolution. In other words, the comultiplication \(\Delta\) on \(F(G)\) maps pre-free random variables onto the sum of random variables which are free with respect to a tensor product state.

By applying the comultiplication \(\Delta\) to the pre-free random variable associated with \(X \in \mathcal{G}\) one obtains the sum
\[ \Delta(\sum_{k=1}^{\infty} \delta X(k)) = J_1(X) + J_2(X) \]

\[ \Delta(\sum_{k=1}^{\infty} \delta X(k)) = J_1(X) + J_2(X) \]

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of random variables

\[ J_1(X) = \sum_{k=1}^{\infty} (X'(k) \otimes q_k - X''(k) \otimes q_{k-1}) \]  \hspace{1cm} (6.1)

\[ J_2(X) = \sum_{k=1}^{\infty} (q_k \otimes X'(k) - q_{k-1} \otimes X''(k)) \]  \hspace{1cm} (6.2)

from \( F(G) \otimes F(G) \). Although they differ from \( j_1(X) \) and \( j_2(X) \) given by (5.1)-(5.2), this difference is not relevant in the weak sense, namely they turn out to be free with respect to the tensor product state \( \hat{\mu} \otimes \hat{\nu} \) for suitable \( \hat{\mu}, \hat{\nu} \).

The states \( \hat{\mu} \) and \( \hat{\nu} \) will be obtained by lifting the corresponding states on the quotient \( H(G) \) of \( F(G) \) modulo the two-sided ideal generated by \( X'(k) - X''(k) \), \( X \in G \), \( k \in \mathbb{N} \) – by abuse of notation we shall also denote these lifted states by \( \hat{\mu} \) and \( \hat{\nu} \).

**Theorem 6.1.** For given states \( \mu \) and \( \nu \) on \( A \) there exist states \( \hat{\mu} \) and \( \hat{\nu} \) on \( F(G) \) and a unital \(*\)-homomorphism

\[ J : A \ast A \to F(G) \otimes F(G) \]

such that \( \mu \ast \nu \) agrees with \( (\hat{\mu} \ast \hat{\nu}) \circ J \) on \( A \ast A \).

**Proof.** The proof is essentially the same as that of Theorem 5.2. The states \( \hat{\mu} \) and \( \hat{\nu} \) are now defined by

\[ \hat{\mu} = \Phi \circ \xi \circ i, \quad \hat{\nu} = \Psi \circ \eta \circ i \]

where \( i : F(G) \to H(G) \) is the identification map given by \( i(X'(k)) = i(X''(k)) = X(k) \) for every \( X \in G \) and \( i(q_k) = q_k \). Then we set \( J = J_1 \ast J_2 \), i.e. \( J \) is the homomorphic extension of

\[ J(X) = \begin{cases} J_1(X) & \text{if } X \in G_1 \\ J_2(X) & \text{if } X \in G_2 \end{cases} \]

and \( J(1_A) = 1 \otimes 1 \), where \( G_1 \) and \( G_2 \) denote copies of \( G \). The rest of the proof is analogous to that of Theorem 5.2. \( \square \)

**Corollary 6.2.** Let \( \mu \boxplus \nu \) be the free additive convolution of states \( \mu \) and \( \nu \) on \( A \) and let \( \hat{\mu} \ast \hat{\nu} = (\hat{\mu} \otimes \hat{\nu}) \circ \Delta \) be the convolution of states of Theorem 6.1. Then

\[ \mu \boxplus \nu = (\hat{\mu} \ast \hat{\nu}) \circ \tau, \]

where \( \tau : A \to F(G) \) is the unital \(*\)-homomorphism given by

\[ \tau(X) = \sum_{k=1}^{\infty} \delta X(k), \quad \tau(1_A) = 1 \]

where \( X \in G \).

**Proof.** This is an immediate consequence of Theorem 6.1. \( \square \)
Thus we can use a ‘quantum group’ language to speak about the additive free convolution instead of using the dual group language. In particular, let $\mathcal{A} = \mathbb{C}[X], \mathcal{G} = \{X\}$, and let $\mu, \nu$ be states on $\mathbb{C}[X]$ corresponding to measures $\mu, \nu$ on $\mathbb{R}$. Heuristically, the monotone closed quantum semigroup $F(\mathcal{G})$ can be interpreted as a quantum analog of $\mathbb{R}^\infty$ as it is the algebra of polynomials in countably many noncommuting variables. Therefore, the additive free convolution of classical measures $\mu \boxplus \nu$ can be viewed as a restriction of the convolution $\hat{\mu} \ast \hat{\nu}$ of ‘quantum measures’ on ‘quantum $\mathbb{R}^\infty$’ to the ‘quantum free line’ $F_{pt}(\mathcal{G})$.

Acknowledgements

I would like to thank Professor Luigi Accardi for his remarks and suggestions which were very helpful in the preparation of the revised version of the manuscript.

References

[Av] D. Avitzour, “Free products of $C^*$- algebras”, Trans. Amer. Math. Soc. 271 (1982), 423-465.

[B] M. Bozejko, “Uniformly bounded representations of free groups”, J. Reine Angew. Math. 377 (1987), 170-186.

[Be1] S. Berberian, Baer *-Rings, Springer-Verlag, 1972.

[Be2] S. Berberian, “The regular ring of a finite Baer *-ring”, J. Algebra 23 (1972), 35-65.

[Be3] S. Berberian, “The regular ring of a finite AW*- algebra”, Ann. Math. 65 (1957), 224-240.

[F-L-S] U. Franz, R. Lenczewski, M. Schürmann, “The GNS construction for the hierarchy of freeness”, Preprint No. 9/98, Wroclaw University of Technology, 1998.

[L1] R. Lenczewski, “Unification of independence in quantum probability”, Inf. Dim. Anal. Quant. Probab. Rel. Topics 1 (1998), 383-405.

[L2] R. Lenczewski, “Filtered random variables, bialgebras and convolutions”, J. Math. Phys. 42 (2001), 5876-5903.

[M] N. Muraki, “Monotonic independence, monotonic central limit theorem and monotonic law of large numbers”, Inf. Dim. Anal. Quant. Probab. Rel. Topics. 4 (2001), 39-58.

[V1] D. Voiculescu, “Symmetries of some reduced free product $C^*$- algebras”, in Operator Algebras and their Connections with Topology and Ergodic Theory, Lect. Notes Math. 1132, 556-588 (1985).

[V2] D. Voiculescu, “Addition of certain non-commuting random variables”, J. Funct. Anal. 66 (1986), 323-346.

[V3] D. Voiculescu, “Dual algebraic structures on operator algebras related to free products”, J. Operator Theory 17 (1987), 85-98.