A Conditionally Cubic-Gaussian Stochastic Lagrangian Model for Acceleration in Isotropic Turbulence

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The modelling of fluid particle accelerations in homogeneous, isotropic turbulence in terms of second-order stochastic models for the Lagrangian velocity is considered. The basis for the Reynolds model (A. M. Reynolds, Phys. Rev. Lett. 91(8), 084503 (2003)) is reviewed and examined by reference to DNS data. In particular, we show DNS data that support stochastic modelling of the logarithm of pseudo-dissipation as an Ornstein-Uhlenbeck process (Pope and Chen 1990) and reveal non-Gaussianity of the conditional acceleration PDF. The DNS data are used to construct a simple stochastic model that is exactly consistent with Gaussian velocity and conditionally cubic-Gaussian acceleration statistics. This model captures the effects of intermittency of dissipation on acceleration and the conditional dependence of acceleration on pseudo-dissipation (which differs from that predicted by the refined Kolmogorov (1962) hypotheses). Non-Gaussianity of the conditional acceleration PDF is accounted for in terms of model nonlinearity. The
diffusion coefficient for the new model is chosen based on DNS data for conditional two-time velocity statistics. The resulting model predictions for conditional and unconditional velocity statistics and timescales are shown to be in good agreement with DNS data.

1. Introduction

Of late, the statistics of fluid particle acceleration in turbulence have been the subject of many experimental (e.g., Voth et al. (1998); La Porta et al. (2001); Voth et al. (2002); Christensen & Adrian (2002); Gylfason et al. (2004); Mordant et al. (2004)) and numerical (e.g., Yeung (1997); Vedula & Yeung (1999); Biferale et al. (2005); Yeung et al. (2005)) efforts. These investigations have spurred a renewed interest (Pope (2002); Beck (2001, 2002); Reynolds (2003)) in the modelling of conditional and unconditional statistics of velocity and acceleration in terms of second-order Lagrangian stochastic models. Recent work in this area has focused on the construction of stochastic models that are capable of reproducing intermittency and Reynolds number effects in Lagrangian statistics as observed in experiments and DNS. In other words, second-order stochastic models can be formulated in such a way as to incorporate accurate one-time statistics (and their Reynolds-number dependence) from experiments or DNS and be able to reproduce intermittent two-time statistics in good agreement with experiments or DNS.

Reynolds-number effects in Lagrangian stochastic models were first addressed by Sawford (1991). The Sawford (1991) model is exactly consistent with a joint-normal stationary one-time distribution for $Z = [U, A]^T$, where $U(t)$ and $A(t)$ denote (modelled) stochastic processes for one component of the Lagrangian velocity and acceleration. As a result, the
SDEs for the Sawford (1991) model are linear:

\[
dZ = \begin{bmatrix} 0 & 1 \\ -\frac{\sigma_A^2}{\sigma_U^2} & -\frac{b^2}{2\sigma_A^2} \end{bmatrix} Z dt + \begin{bmatrix} 0 \\ b \end{bmatrix} dW,
\]

where \(\sigma_U\) and \(\sigma_A\) denote standard deviations for velocity and acceleration, \(b\) is a diffusion coefficient, and \(W\) is a standard Brownian motion (or Wiener process). Sawford (1991) showed that matching of the second-order Lagrangian velocity structure function \(D_U(s) = \langle (U(t+s) - U(t))^2 \rangle\) with the Kolmogorov (1941) hypotheses for the universal equilibrium range uniquely identifies the diffusion coefficient as

\[
b = \sqrt{\frac{2\sigma_U^2}{C_0\langle \varepsilon \rangle} (T_L^\infty - 1 + t_\eta^{-1}) T_L^\infty - 1 t_\eta^{-1}},
\]

where \(T_L^\infty = \frac{2}{C_0\langle \varepsilon \rangle}\) and \(t_\eta = \frac{C_0^2}{2a_0\langle \varepsilon \rangle}\). Here, \(C_0\) is the Kolmogorov constant for the second-order Lagrangian velocity structure function, \(a_0\) is the acceleration variance normalized by the Kolmogorov scales, \(\langle \varepsilon \rangle\) is the mean dissipation and \(\nu\) is the kinematic viscosity. Sawford (1991) also showed that model predictions are very close to DNS data for unconditional velocity and acceleration autocorrelations at low Reynolds number. However, the Sawford model ignores intermittency of Lagrangian statistics and incorporates a Gaussian Lagrangian acceleration PDF, at variance with the observed non-Gaussianity of acceleration found in experiments (La Porta et al. (2001)) and DNS (Yeung & Pope (1989); Yeung et al. (2005)).

Reynolds (2003) addressed the problem of incorporating a strongly non-Gaussian PDF of acceleration into a Lagrangian stochastic model. Reynolds showed that an improved representation for the Lagrangian acceleration PDF in a second-order stochastic model can be obtained by explicitly accounting for intermittency of dissipation. Specifically, he assumed a log-normal distribution for the dissipation rate, \(\varepsilon\), together with a Gaussian assumption for the conditional PDF of \(A|\varepsilon\). The latter assumption may be restated in terms of the \textit{conditionally standardized acceleration} defined by \(\tilde{A} \equiv \frac{A}{\sigma_A|\varepsilon}\) (which has
zero and unit values for its conditional mean and variance). In the Reynolds model, the conditional distribution ˜A|ε is assumed to be universal and, in particular, standard normal. This may be interpreted to imply that intermittency of dissipation is solely responsible for intermittency in acceleration.

Reynolds also assumed Gaussian velocity statistics and independence of velocity from dissipation and acceleration. In other words, the Reynolds model is (by construction) exactly consistent with a joint-normal stationary one-time distribution of (U, ˜A, ln ε).

To completely specify his model, Reynolds assumed the Kolmogorov (1962) prediction for the conditional acceleration variance,

\[ \sigma^2_{A|\varepsilon}/\sigma^2_\eta = a^*_0 (\varepsilon/\langle \varepsilon \rangle)^{3/2}, \]

where \( a^*_0 \) is a Kolmogorov constant and \( a_\eta = (\langle \varepsilon \rangle^3/\nu)^{1/4} \) is the Kolmogorov acceleration scale. Following Pope & Chen (1990), Reynolds also assumed an Ornstein-Uhlenbeck (OU) process for \( \chi \equiv \ln \varepsilon \). The resulting model can be written as an SDE for \( Z = [U, ˜A, \ln \varepsilon - \langle \ln \varepsilon \rangle]^T \):

\[
\begin{align*}
dZ &= \begin{bmatrix}
0 & \sigma_{A|\varepsilon} & 0 \\
\frac{\sigma_{A|\varepsilon}}{\sigma_U} & -\frac{\kappa^2}{2\sigma_{A|\varepsilon}} & 0 \\
0 & 0 & -T_\chi^{-1}
\end{bmatrix}Zdt + \begin{bmatrix}
0 & 0 \\
0 & b/\sigma_{A|\varepsilon} & 0 \\
0 & 0 & \sqrt{2\sigma_\chi^2/T_\chi}
\end{bmatrix} \begin{bmatrix}
dW \\
dW'
\end{bmatrix}.
\end{align*}
\]

In these equations, \( \sigma_\chi \) and \( T_\chi \) denote the standard deviation and the integral scale for \( \chi \), whereas \( W \) and \( W' \) are independent Wiener processes. The dissipation equation is effectively decoupled from the rest of the system and therefore the Reynolds model is linear in \( U \) and \( ˜A \). Additional assumptions made by Reynolds are: (i) a choice of diffusion coefficient made by analogy with the Sawford (1991) model, i.e.,

\[
b = \sqrt{2\sigma_U^2 (T_{L,\varepsilon}^{-1} + \tau_{\eta,\varepsilon}^{-1}) T_{L,\varepsilon}^{-1} \tau_{\eta,\varepsilon}^{-1}},
\]
where \( T_{L,\varepsilon} = \frac{2}{C_0} \frac{\sigma_U^2}{\varepsilon} \) and \( t_{r,\varepsilon} = \frac{C_0}{2\sigma_U} \sqrt{\varepsilon} \) (\( C_0 \) being a model constant), and (ii) \( T_{\chi}(\varepsilon) / \sigma_U^2 = \text{const} \).

In this paper, we first review the basis for the Reynolds model against DNS. Then, a novel stochastic model is constructed that incorporates one-time information from DNS and yields model predictions for two-time velocity statistics in good agreement with DNS.

The plan for this paper is as follows. In Section 2 we review DNS data (first presented in Yeung et al. (2005)) for intermittency of dissipation, the PDF of conditionally standardized acceleration and the variance of acceleration conditioned on the pseudo-dissipation. In Section 3 a novel stochastic model that incorporates non-Gaussian one-time statistics from DNS is formulated. Non-Gaussianity of the conditionally standardized acceleration PDF is accounted for in terms of nonlinearity in the model. In Section 4 we show a choice of diffusion coefficient based on DNS data for conditional velocity autocorrelations that yields model predictions for conditional and unconditional velocity autocorrelations and timescales in good agreement with DNS. Conclusions for this work are summarized in Section 5.

2. DNS Data for Stochastic Modelling

2.1. Intermittency of Dissipation

Figure 1 shows (one-time) PDFs for \( \ln \varepsilon \), \( \ln \zeta \) and \( \ln \varphi \) for \( R_\lambda \approx 680 \) (\( R_\lambda = \sqrt{\frac{15\sigma_U}{\nu(\varepsilon)}} \) being the Taylor-scale Reynolds number), where \( \varepsilon = 2\nu s_{ij} s_{ij} \) is the dissipation rate, \( \zeta = 2\nu r_{ij} r_{ij} \) is the “enstrophy”, and \( \varphi = \nu u_{i,j} u_{i,j} \) is the pseudo-dissipation \( (s_{ij} \text{ and } r_{ij} \text{ being the strain-rate and rotation-rate tensors, i.e., } u_{i,j} = s_{ij} + r_{ij}) \). This figure suggests that pseudo-dissipation (as opposed to the dissipation rate, or the enstrophy) is closest to log-normal for \( R_\lambda \approx 680 \). In fact, DNS data support this conclusion for
Figure 1. One-time PDFs (solid) for the standardized logarithm of $X$ (with $X = \varepsilon, \zeta, \varphi$) vs. standard normal PDF (dashed).

Reynolds numbers in the range $R_{\lambda} \approx 140 - 680$. In this paper we do not purport to discuss the validity of the log-normal model as opposed to more accurate intermittency models for dissipation. We limit ourselves to the observation that pseudo-dissipation may be approximately described as log-normal for $R_{\lambda} \approx 140 - 680$.

Let us now assume that $\chi \equiv \ln \varphi / \langle \varepsilon \rangle$ evolves by an OU process,

$$d\chi = - \left( \chi + \frac{\sigma^2}{2} T_{\chi}^{-1} \right) dt + \sqrt{\frac{2\sigma^2}{T_{\chi}}} dW'. \tag{2.1}$$

It follows that $\langle X(t)|X(0) = \phi \rangle = \phi e^{-t/T_{\chi}}$, where $X \equiv (\ln \varphi - \langle \ln \varphi \rangle) / \sigma_{\ln \varphi}$. Figure 2 shows (Lagrangian) two-time conditional means of $X$ from DNS (Fig. 3 shows two-time conditional means from DNS with $X$ re-defined in terms of $\varepsilon$). These figures suggest that two-time conditional means of the (standardized) logarithm of pseudo-dissipation are closest to simple exponentials (at least for $R_{\lambda} \approx 140 - 680$, based on DNS at different
Figure 2. DNS data for conditional expectations (solid) with $X = (\ln \varepsilon - \langle \ln \varepsilon \rangle) / \sigma_{\ln \varepsilon}$ vs. $\phi e^{-\tau/T_X}$ (dashed, $T_X$ being the Lagrangian integral timescale for $X$).

Figure 3. DNS data for conditional expectations (solid) with $X = (\ln \varphi - \langle \ln \varphi \rangle) / \sigma_{\ln \varphi}$ vs. $\phi e^{-\tau/T_X}$ (dashed, $T_X$ being the Lagrangian integral timescale for $X$).
Reynolds numbers). Therefore, on the basis of DNS, we argue that, for the purposes of stochastic modelling, pseudo-dissipation is well-approximated by an OU process.

To completely specify the dissipation model (2.1) (where \( \chi \equiv \ln \varphi / \langle \varepsilon \rangle \), \( \sigma_\chi \) and \( T_\chi \) have to be prescribed. As discussed by Yeung et al. (2005), the DNS data support the Kolmogorov (1962) prediction for

\[
\sigma_\chi^2 = A + \frac{3\mu}{2} \ln R_\lambda, \tag{2.2}
\]

with \( \mu \approx 0.25 \), in good agreement with Sreenivasan & Kailasnath (1993) \((A \approx -0.863\) is reported by Yeung et al. (2005)). The integral timescale \( T_\chi \) is chosen to match the DNS data (Table 1).

|  |  |  |  |  |  |
|---|---|---|---|---|
| \( R_\lambda \) | 38 | 140 | 240 | 400 | 680 |
| \( T_\chi/T \) | 0.3317 | 0.1719 | 0.1311 | 0.1056 | 0.0906 |

Table 1. Values of \( T_\chi/T \) (with \( T \equiv 1.5\sigma_\chi^2 / \langle \varepsilon \rangle \)) from DNS of homogeneous turbulence.

2.2. PDF of Conditionally Standardized Acceleration

We now investigate the conditionally Gaussian assumption in the Reynolds model. Figure 4 shows PDFs \( f_{\tilde{A}|\varepsilon}(a|\tilde{\varepsilon}) \) for different values of \( \tilde{\varepsilon} \) for \( R_\lambda \approx 680 \) (a standard normal PDF and the unconditional acceleration PDF for \( R_\lambda \approx 680 \) are also shown). Figure 5 shows analogous information when \( \varphi \) is used in place of \( \varepsilon \). Both sets of conditional PDFs are much less intermittent (with weaker tails at large fluctuations) than the unconditional acceleration PDF. This is particularly true of the PDFs of \( \tilde{\varphi} \) (with \( \tilde{\varphi} \equiv \tilde{A}/\sigma_{\tilde{A} \mid \varphi} \)) which, however, still show significant non-Gaussian behaviour. A remarkable degree of collapse
Figure 4. Standardized conditional PDFs of $\tilde{A}$ given $\varepsilon = \hat{\varepsilon}$ from 2048\textsuperscript{3} DNS at $R_\lambda \approx 680$. Lines A-E are for $\hat{\varepsilon}/\langle \varepsilon \rangle = \{0.0359, 0.136, 0.469, 1.62, 6.05\}$, corresponding to $Z = (\ln \hat{\varepsilon} - \langle \varepsilon \rangle)/\sigma_{\ln \varepsilon} = \{-2.054, -0.994, 0, 0.994, 2.054\}$. Also shown are the unconditional PDF of acceleration (solid unmarked line) and a standard Gaussian PDF (dashed).

of these PDFs for different values of the conditioning variable is notable (except for very small and very large conditional fluctuations). Therefore, to a first approximation, the PDFs $f_{\tilde{A}|\varphi}(\tilde{a}|\varphi)$ may be described as approximately independent of $\varphi$. In fact, based on simulations at different Reynolds numbers (not shown), the conditional PDF of $\tilde{A}|\varphi$ may also be (approximately) described as independent of the Reynolds number.
Figure 5. Conditional PDFs of $\tilde{A}$ given $\varphi = \hat{\varphi}$ from $2048^3$ DNS at $R_\lambda \approx 680$. Lines A-E are for $\hat{\varphi}/\langle \varphi \rangle = \{0.0362, 0.134, 0.458, 1.56, 5.79\}$, corresponding to $Z = (\ln \hat{\varphi} - \langle \varphi \rangle)/\sigma_{\ln \varphi} = \{-2.054, -0.994, 0, 0.994, 2.054\}$. Also shown are the unconditional PDF of acceleration (solid unmarked line) and a standard Gaussian PDF (dashed).

Yeung et al. (2005) suggest that the PDF of $\tilde{A}|\varphi$ can be described (to a very good approximation) as cubic-Gaussian. By definition, a random variable $Z$ is cubic-Gaussian with parameter $p$ (also denoted as $Z \sim G^3(p)$) if

$$Z = C[(1 - p)X + pX^3], \quad (2.3)$$

where $X$ is a standardized Gaussian random variable and $C$ is determined by the standardization condition $\langle Z^2 \rangle = 1$ as $C(p) = (1 + 4p + 10p^2)^{-1/2}$. Figure 6 shows that the
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Figure 6. Standardized conditional PDFs of $\tilde{A}$ given $\varphi = \hat{\varphi}$. The symbols are from the 2048$^3$ DNS; the lines are the cubic-Gaussian PDF with the same kurtosis as the DNS data. The values of the conditioning variable are such that $Z = (\ln \hat{\varphi} - \langle \varphi \rangle) / \sigma_{\ln \varphi} = \{-2.054, -0.994, 0, 0.994, 2.054\}$. In each plot, the lower curve and the y-axis correspond to the lowest conditioning value. The curves for the other conditioning values are successively shifted upwards, by an amount 0.2 in the linear plot (left), and by a factor of 100 in the logarithmic plot (right).

cubic-Gaussian PDF provides a remarkably accurate description of the conditional PDFs $f_{\tilde{A}|\varphi}(\tilde{a}|\hat{\varphi})$. Comparable accuracy to that in Fig. 6 is achieved when the cubic-Gaussian PDF is used to fit DNS data for $f_{\tilde{A}|\varphi}(\tilde{a}|\hat{\varphi})$ at lower Reynolds numbers (not shown, but see Sect. IV of Yeung et al. (2005)). A value of $p \approx 0.1$ results from the observation (based on DNS) that $\mu_4(\tilde{A}|\hat{\varphi}) \approx 8$ (approximately independent of $\hat{\varphi}$ and $R_\lambda$; more details in Yeung et al. (2005)).
Figure 7. Variance of acceleration conditioned on the pseudo-dissipation for different values of $R_\lambda$. The symbols are the DNS data; the lines are the empirical fit Eq. (2.5). The lowest curve and the $y$-axis correspond to $R_\lambda \approx 38$. The other four curves are for $R_\lambda \approx 140, 240, 400, 680$, successively shifted upwards by a factor of $\sqrt{10}$.

2.3. Conditional Acceleration Variance

With a view to the joint (stochastic) modelling of acceleration and pseudo-dissipation, we now investigate the validity of the Kolmogorov (1962) prediction for the conditional acceleration variance,

$$\frac{\sigma_{A|^\varphi}}{a_\eta^2} = a_0^\ast (\varphi/\langle \varepsilon \rangle)^{3/2},$$  \hfill (2.4) 

Figure 4 shows $\varphi$-dependences from DNS for the conditional acceleration variance at different Reynolds numbers. In the same figure, the following expression (first presented in Yeung et al. (2005))

$$\frac{\sigma_{A|^\varphi}}{a_\eta^2} = \frac{1.2}{R_\lambda^{0.2}} \left( \frac{\varphi}{\langle \varepsilon \rangle} \right)^{0.15} + \ln \left( \frac{R_\lambda}{20} \right) \left( \frac{\varphi}{\langle \varepsilon \rangle} \right)^{1.25},$$  \hfill (2.5)
is shown to be an accurate representation (except at the smallest $R_\lambda$) of the DNS data. As may be seen, the low-$\phi$ behaviour for the conditional acceleration variance deviates strongly from that predicted by Eq. (2.4). Also, careful measurement of the slope for the large-$\phi$ portion of the curves in Fig. 7 yields values that are systematically less than 1.5, again at variance with the Kolmogorov (1962) prediction.

Equation (2.5) is most useful for stochastic modelling purposes because it accurately parameterizes the conditional acceleration variance (given $\phi$) in terms of both the value being conditioned upon and the Reynolds number.

3. Conditionally Cubic-Gaussian (CCG) Stochastic Lagrangian Models

Lagrangian statistics for $\varepsilon$, $\zeta$ and $\phi$ from DNS show that stochastic modelling is easiest when pseudo-dissipation is used in place of the dissipation rate or the enstrophy. This is because (i) $\phi$ is closest to log-normal, and (ii) two-time conditional means of $\ln \phi$ are closest to exponential, and (iii) the conditional PDFs of acceleration given $\phi = \hat{\phi}$ collapse best, and with the narrowest tails. Thus, we base the model on pseudo-dissipation $\phi$ and take the OU process Eq. (2.1) as its stochastic model.

Conditioning on pseudo-dissipation is most useful when considering the joint-statistics of acceleration and pseudo-dissipation because the PDF of $\tilde{A}|\phi$ may be described (to a first approximation) as universal and, in particular, cubic-Gaussian. In other words, given $\phi = \hat{\phi}$ and a standardized Gaussian random variable $\bar{A}$, the acceleration $A$ can be modelled as

$$A = \sigma_{A|\hat{\phi}} C[(1-p)\bar{A} + p\bar{A}^3].$$

For given $\phi$, the relation between $A$ and $\tilde{A}$ is invertible (and one-to-one).

The stochastic model is most conveniently expressed in terms of the velocity $U(t)$,
the “Gaussian” acceleration $\ddot{A}(t)$ (related to the acceleration by Eq. (3.1)), and the log-pseudo-dissipation variable $\chi^* \equiv \chi - \langle \chi \rangle = \ln(\varphi/\langle \varepsilon \rangle) - (\ln(\varphi/\langle \varepsilon \rangle)).$

The model is

$$dU = A dt = \sigma_{A|\varphi} C((1 - p)\ddot{A} + p\ddot{A}^3)dt,$$

$$d\ddot{A} = \ddot{\theta} dt + \ddot{b} dW,$$

$$d\chi^* = -\chi^* dt + \sqrt{2 \sigma_{\chi}^2} \frac{\ddot{b}}{T_{\chi}} dW',$$

where $\ddot{\theta}$ and $\ddot{b}$ are drift and diffusion coefficients specified below.

The stationary one-time joint PDF of $U$, $\ddot{A}$ and $\chi^*$ is denoted by $f(v, \ddot{a}, x^*)$, where $v$, $\ddot{a}$ and $x^*$ are sample-space variables corresponding to $U$, $\ddot{A}$ and $\chi^*$. We now assume that this PDF is joint-normal with the variables being uncorrelated with each other (at the same time). Thus, with the assumptions made the joint PDF is

$$f = \frac{1}{\sigma_U \sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma_U^2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\ddot{a}^2}{2}\right) \frac{1}{\sigma_{\chi} \sqrt{2\pi}} \exp\left(-\frac{x^{*2}}{2\sigma_{\chi}^2}\right).$$

The imposition of this PDF leads to a constraint for the drift coefficient $\ddot{\theta}$ in Eq. (3.3), namely,

$$\ddot{\theta}(v, \ddot{a}, \varphi) = -\frac{\sigma_{A|\varphi}}{\sigma_U} C v (1 + p + p\ddot{a}^2) + \frac{\ddot{b}^2}{2} \frac{\partial}{\partial \ddot{a}} \ln \ddot{b}^2 f + \ddot{\theta}^*,$$

where $\ddot{\theta}^*$ is any function such that $\frac{\partial}{\partial \ddot{a}} (\ddot{\theta}^* f) = 0$, which for simplicity we take to be zero.

We also introduce the assumption $\partial \ddot{b}/\partial \ddot{a} = 0$ (which can be supported using an adiabatic elimination argument in the limit $R\lambda \to \infty$). Then, Eq. (3.3) can be rewritten as

$$d\ddot{A} = -\frac{\ddot{b}^2}{2} \ddot{A} dt - \frac{\sigma_{A|\varphi}}{\sigma_U} C \varphi U C (1 + p + p\ddot{A}^2) dt + \ddot{b} dW.$$

This equation (together with Eqs. (3.2) and (3.4)) defines a class of CCG models, i.e., different models with the same stationary distribution correspond to different choices of $\ddot{b}$. Each model captures the conditional dependence of acceleration on pseudo-dissipation based on DNS (Eq. (2.5)) that accounts for deviations from the Kol-
Figure 8. Velocity autocorrelation for CCG model (solid) vs. DNS data (dashed). The thicker, dashed line is for the Reynolds (2003) model.

mogorov (1962) hypotheses. Also, each model is non-linear because it accounts for the non-Gaussianity of the conditionally standardized acceleration PDF.

4. Specification of Diffusion Coefficient

The diffusion coefficient for the Reynolds (2003) model is specified by analogy with the Sawford (1991) model (Eq. (1.5)). This choice is arbitrary and, furthermore, it leads to poor agreement with DNS for unconditional autocorrelations (Fig. 8).

We now investigate the question of how to select $\hat{b}(\chi)$ by reference to two-time conditional velocity statistics from DNS. To this end, the Reynolds model with a frozen dissipation $\chi \equiv \hat{\chi}$ is considered. Then, the first two equations in (1.4) revert to a Sawford ’91 type of model. With the frozen model, we investigate $\chi$-dependences for $b$ such that the frozen model matches the DNS values of $\sigma_{\lambda|x}$ and $\tau_{U|x}$ ($\tau_{U|x}$ being halving times for con-
Results from such calculations for different values of $R_\lambda$ are shown in Fig. 9. This figure would suggest that $\varphi$-dependences for $\dfrac{(\tau_\eta b^2/a_\eta^2)\langle \sigma_{A|x} \rangle}{\sigma_{A|x} u_\eta}$ at different Reynolds numbers may be approximately described in terms of a single function $G(\varphi) = \left( \dfrac{\varphi}{\langle \varepsilon \rangle} \right)^{0.3}$. The diffusion coefficient is then given by

$$\left( \dfrac{b}{\sigma_{A|x}} \right)^2 = G(\varphi) \dfrac{\sigma_{A|x}}{u_\eta},$$

(4.1)

where $u_\eta = (\nu/\langle \varepsilon \rangle)^{1/4}$ is the Kolmogorov velocity scale. We then use this determination to specify $\bar{b}$ in (3.3), i.e. we let

$$\bar{b}^2 = c(R_\lambda) G(\varphi) \dfrac{\sigma_{A|x}}{u_\eta},$$

(4.2)

where $c(R_\lambda)$ is a correction factor (given in Table 2) so selected as to ensure good agree-
Figure 10. Conditional velocity autocorrelations for CCG model (solid) vs. DNS (dashed).

The resulting model predictions for conditional velocity autocorrelations and timescales are shown in Figs. 10 and 11. Model predictions for unconditional and conditional acceleration autocorrelations are shown in Figs. 12 and 13. These plots suggest that the specification (4.2) is capable of reproducing reasonable agreement with DNS data for conditional and unconditional velocity autocorrelations and timescales. Also, Eq. (4.2) yields PDFs of Lagrangian velocity increments that are approximately Gaussian for large time-lags (Fig. 14). These PDFs develop stretched tails as the time-lag decreases and ultimately approach the Lagrangian acceleration PDF for very small time-lags. This behaviour is consistent with recent observations of Lagrangian intermittency in experiments and simulations (Mordant et al. 2004).
5. Conclusions

After a brief review of the basis for the Reynolds (2003) model against DNS, we have shown the formulation of a novel stochastic based on simple data assimilation from DNS. This model is exactly consistent with Gaussian velocity and conditionally cubic-Gaussian acceleration statistics and incorporates a representation for the logarithm of
Figure 12. Acceleration autocorrelation for CCG model (solid) vs. DNS (dashed). The thicker, dashed line is for the Reynolds (2003) model.

Figure 13. Conditional acceleration autocorrelations for CCG model (solid) vs. DNS (dashed).
pseudo-dissipation as an Ornstein-Uhlenbeck process. The new model captures the effects of intermittency of dissipation on acceleration and the deviations from the Kolmogorov (1962) hypotheses (based on DNS) in the conditional dependence of acceleration on pseudo-dissipation. Further, non-Gaussianity of the conditionally standardized acceleration PDF (as observed in DNS) is captured in terms of model nonlinearity. An empirical specification of diffusion coefficient based on DNS data for conditional velocity timescales yields reasonable agreement with DNS data for conditional and unconditional velocity autocorrelations and timescales.

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