JACOBI STABILITY ANALYSIS AND IMPULSIVE CONTROL OF A 5D SELF-EXCITING HOMOPOLAR DISC DYNAMO

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ABSTRACT. In this paper, we make a thorough inquiry about the Jacobi stability of 5D self-exciting homopolar disc dynamo system on the basis of differential geometric methods namely Kosambi-Cartan-Chern theory. The Jacobi stability of the equilibria under specific parameter values are discussed through the characteristic value of the matrix of second KCC invariants. Periodic orbit is proved to be Jacobi unstable. Then we make use of the deviation vector to analyze the trajectories behaviors in the neighborhood of the equilibria. Instability exponent is applicable for predicting the onset of chaos quantitatively. In addition, we also consider impulsive control problem and suppress hidden attractor effectively in the 5D self-exciting homopolar disc dynamo.

1. Introduction. Magnetic field exists widely in various astrophysical bodies, the most typical of which is the earth magnetic field. Over the past few decades, the earth’s magnetic field has always been an important research topic, and nondimensionalized differential equations are often used to describe the phenomenon of reversals in the magnetic field [7, 16, 26, 31].

Bullard [9] first study the stability of a simplest homogeneous disk dynamo under three working conditions. The main results give some hints on the fluid oscillatory behaviors in the magnetic field and some preconditions for the stable operation of the dynamo. Possible couplings and range of temperature fluctuation for a two-disk dynamos were emphasized in [27]. The results point out that even a simple system

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will produce extremely complex oscillations, and there will occur reversal of the electric current, unlike the singular Bullard’s dynamo. Moffatt [23] rectified the inadequacy of description for Bullard’s dynamo system and introduces the segmented disk generator considering azimuthal current, which solved the basic inconsistency in previous treatment. By adding the effect of friction to Moffatt’s dynamo system, it is shown that the equations used to describe the segmented disc dynamo were consistent with the Lorenz equation [17]. Recently, a segmented disc dynamo system with mechanical friction was studied in [22], and the complex dynamics properties, including integrability and Hopf bifurcation, were completely investigated. As far as the hidden chaotic attractors is concerned, literature [33] shows the imperfection of Moffatt’s dynamo system. Besides, intensive studies have been made on segmented disc dynamo system possessing hidden attractors, and electronic circuit application has also been realized [4, 5, 11, 18, 32]. It is worth mentioning that Wei, et.al. [34] studied the following 5D self-exciting homopolar disc dynamo:

\[
\begin{align*}
\dot{x} &= r(y - x) + u, \\
\dot{y} &= -(1 + m)y + xz - v, \\
\dot{z} &= g[1 + mx^2 - (1 + m)xy], \\
\dot{u} &= 2(1 + m)u + xz - k_1x, \\
\dot{v} &= -mv + k_2y.
\end{align*}
\] (1)

The phase diagrams of system (1), showing the hidden hyperchaotic attractor, are depicted in Fig.1 for a set of chosen parameters \(m = 0.04, g = 140.6, r = 7, k_1 = 34, k_2 = 12\) under the initial condition \((0.05, -0.5, 0.1, -1, 2)\). More detailed hidden hyperchaos analysis and the application in electronic circuit of this dynamo system have been fulfilled in [34].

![Figure 1. The phase portraits of hyperchaotic dynamo system with the parameter values for \((m, g, r, k_1, k_2) = (0.04, 140.6, 7, 34, 12)\): (a) x-y plane; (b) time series of \(x(t)\).](image)

On the one hand, the basic notion of stability in the state space is to deal with the asymptotic convergence to equilibrium of trajectories that start off an equilibrium point. However, KCC theory studies the deviation of nearby trajectories through
establishing the one-to-one correspondence between the second-order differential equation and geodesic flow [3]. By virtue of this theory, concept of Jacobi stability for the geodesic equation is proposed, which can be interpreted as the robustness of dynamical system and extended to arbitrary suitable systems [28]. Robustness, as a measure of the insensitivity and adaptability to the changes of system parameters and environment, plays a vital part in qualitative analysis [1]. Indeed, Jacobi stability analysis has a good connection with the robustness of some biological systems. For example, it can reveal the fragility and robustness of a cell model with arrest states [29] and reflect the geometric structure of interactions between the competition and predation [36,37]. Jacobi stability analysis was seen as a powerful method used for detecting a kind of stability artifact [8]. In the study of classical Lorenz system, two important quantitative indicators namely curvature and instability exponent were introduced to reveal the underlying chaotic evolution [13]. Since then, more and more researchers have been devoted to exploring the onset of chaos in different physical and engineering systems [2,10,12,14,15].

On the other hand, the application of Lyapunov function to impulsive dynamic systems allows an improvement of the results from stability analysis and design of impulsive control. Moreover, indefinite Lyapunov function occupies an increasingly important position. As we know, some new sufficient conditions of input-to-state stability have been obtained by employing an indefinite Lyapunov function, such as continuous-time dynamic systems [24] as well as time-varying discrete-time dynamic systems [21]. The results show that it is easier to obtain the corresponding conditions with indefinite Lyapunov function than with negative definite one, and the former allows Lyapunov function to be positive definite in some time period. More precisely, indefinite Lyapunov functions may increase at some impulses or in some continuous portion of the trajectory. Recently, new sufficient conditions concerning asymptotic stability analysis of the origin for a chaotic system have also been proposed via indefinite Lyapunov functions [20].

Based on the numerical method, classical chaotic attractors are found from some unstable equilibria in some nonlinear systems [6,30,35]. These attractors can start from neighborhood of equilibrium and be evolve of local unstable manifold. Study of chaotic system is still of considerable difficulty in the nonlinear dynamical theories, and the hidden attractor is regarded as an remarkable feature of multistability [19,25]. To the best of our knowledge, our study firstly gives an application of KCC theory in five-dimensional dynamo system, which provides an innovative perspective to explore the stability and onset of chaos. Therefore, we will provide some important sources of hidden chaos from the view of Jacobi instability, and design nonlinear impulsive control to suppress the occurrence of hidden chaos via indefinite Lyapunov functions.

The paper is divided into six parts. Some basic definitions and theorems about KCC theory and Jacobi stability are given is Section 2. In section 3, we discussed the Jacobi stability of the equilibria and periodic orbit for a 5D self-exciting homopolar disc dynamo in detail. The dynamic behavior of the deviation vector and instability exponents are also given in section 3. Impulsive control is used to suppress hidden attractor via Lyapunov functions which have a positive jump at all impulses in section 4. The last section is the conclusion.

2. **KCC theory and Jacobi stability.** Before discussing the Jacobi stability of the system (1), some basic knowledge related to the definitions of Jacobi stability are
summarized as follows. Let’s define \( x = (x_1, x_2, \ldots, x_n) \) on an smooth \( n \)-dimensional manifold \( M \) and \( y = (y_1, y_2, \ldots, y_n) \) on the tangent bundle \( TM \) where \( y_i = \frac{dx_i}{dt} \). Assume that there is an open connected subset \( \Omega \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \), which belongs to the \((2n+1)\)-dimensional Euclidian space. Let \((x, y, t) \in \Omega \) and consider the following second-order differential equation (SODE) given by

\[
\frac{d^2 x_i}{dt^2} + 2G_i(x, y, t) = 0, \quad i = 1, 2, \ldots, n, \tag{2}
\]

where \( G_i \) is \( C^\infty \) in a small neighborhood of the initial condition \((x_0, y_0, t_0)\).

By taking the above equation as the geodesic equations, the geometric invariants are constructed for further research. Introducing the nonsingular coordinate transformations

\[
x_i^* = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n, \quad t^* = t,
\]

we firstly define the covariant derivative of a contravariant vector field \( \xi_i(x) \) on the open connected subset \( \Omega \) as

\[
\frac{D \xi_i}{dt} = \frac{d \xi_i}{dt} + N^j_i \xi_j,
\]

where \( N^j_i = \frac{\partial G_i}{\partial y_j} \). Here, we use the Einstein summation convention throughout the paper to express the coefficient of nonlinear connection \( N^j_i \). Let \( \xi_i = y_i \), the equation (3) can be changed to

\[
\frac{D y_i}{dt} = N^j_i y_j - 2G_i = -\epsilon_i. \tag{4}
\]

In general, the first KCC invariant \( \epsilon_i \) represents the external forces in geodesic equations. Next, consider the equation of the trajectories \( x_i \) changing to the nearby one as

\[
x_i^*(t) - x_i(t) = \beta \xi_i(t), \tag{5}
\]

where \( |\beta| \) is sufficiently small. \( \xi_i(t) \) are the components of contravariant vector field along the trajectories \( x_i \). Note that \( x_i^* \) and \( x_i \) are all the solution of Eq.(2), then we get

\[
\beta \ddot{\xi}_i + G(x + \beta \xi, y + \beta \dot{\xi}, t) - G(x, y, t) = 0. \tag{6}
\]

Making use of the mean value theorem and taking the limit \( \beta \to 0 \), we can obtain the variational equations concerning the deviation vector

\[
\frac{d^2 \xi_i}{dt^2} + 2N^j_i \frac{d \xi_j}{dt} + 2 \frac{\partial G_i}{\partial x_j} \xi_j = 0. \tag{7}
\]

Further, taking into account of the KCC-covariant differential equation (3), the Jacobi equations are given by

\[
\frac{D^2 \xi_i}{dt^2} = P^j_i \xi_j, \tag{8}
\]

where

\[
P^j_i = -2 \frac{\partial G_i}{\partial x_j} - 2G_i G^j_l + y_i \frac{\partial N^j_i}{\partial x_l} + N^j_l N^l_j + \frac{\partial N^j_i}{\partial t}. \tag{9}
\]

The coefficient of the Berwald connection is \( G^j_l = \frac{\partial N^j_i}{\partial y_l} \). Then, we get the second KCC invariant \( P^j_i \) named the deviation curvature tensor. Other KCC invariants are not listed here.
Suppose that \( x_i = x_i(t) \) is the curve in the Euclidean space \((\mathbb{R}^n, < \cdot, \cdot >)\), where \(< \cdot, \cdot >\) is the canonical inner product in \(\mathbb{R}^n\). Deviation vector \(\xi\) satisfies \(\xi(0) = O \in \mathbb{R}^n, \xi(0) = W \neq O\) at the initial moment \(t_0 = 0\). To judge the Jacobi stability of system (2), we are devoted to considering the focusing tendency of the trajectories. An adapted inner product is considered to the deviation tensor \(\xi(t)\) by
\[
\langle X, Y \rangle := \frac{1}{<W,W>} <X,Y>,
\]
where the vectors \(X,Y \in \mathbb{R}^n\).

Then, we can have a clear idea about the focusing tendency in a small vicinity of \(t_0\): the tendency of the trajectories are bunching (or dispersing) together namely \(||\xi(t)|| < t^2\) (or \(||\xi(t)|| > t^2\)) if and only if the real part of the eigenvalues of \(P_{ij}\) are strictly negative (or positive), where the norm \(||\cdot||\) induced by a positive definite inner product [28]. The aforementioned discussion gives the definition of Jacobi stability.

**Definition 2.1.** If Eq.(2) satisfies the initial conditions \(||x_i(t_0) - x^*_{i}(t_0)|| = 0\) and \(||\dot{x}_i(t_0) - \dot{x}^*_{i}(t_0)|| \neq 0\), then the trajectories of Eq.(2) are called Jacobi stable if and only if the real part of the eigenvalues of the deviation curvature tensor are strictly negative, and Jacobi unstable otherwise.

3. Jacobi stability of the 5D self-exciting homopolar disc dynamo.

3.1. **The second KCC invariant.** In this section, we will apply KCC-theory to discuss this 5D system. It’s noted that the last equation of system (1) can be rewritten as
\[
y = \frac{1}{k_2}(mv + \dot{v}),
\]
(11)
taking the variable \(y\) into its derivative, we obtain
\[
\ddot{y} = \frac{1+m}{k_2}(mv + \dot{v}) + xz - v.
\]
(12)

Further, one can deduce the second derivative of residual variables of system (1) given by
\[
\ddot{x} = r(xz - v - \dot{x}) + \dot{u} - \frac{r(1+m)(mv + \dot{v})}{k_2},
\]
\[
\ddot{z} = 2mgx\dot{x} - \frac{(1+m)g}{k_2} \dot{x}(mv + \dot{v}) - (1 + m)g[-\frac{(1+m)}{k_2}(mxv + x\dot{v}) + x^2z - xv],
\]
\[
\ddot{u} = 2(1 + m)\dot{u} + \dot{x}z + x\dot{z} - k_1 \dot{x},
\]
\[
\ddot{v} = -mv\dot{v} - (1 + m)(mv + \dot{v}) + k_2(xz - v).
\]
(13)

In accordance with the previous notation, we redefine the variables
\[
x_1 = x, \quad x_2 = z, \quad x_3 = u, \quad x_4 = v, \quad y_1 = \dot{x}, \quad y_2 = \dot{z}, \quad y_3 = \dot{u}, \quad y_4 = \dot{v},
\]
(14)
then we have

\begin{align*}
\ddot{x}_1 &= r(x_1x_2 - x_4 - y_1) + y_3 - \frac{r(1 + m)(mx_4 + y_4)}{k_2}, \\
\ddot{x}_2 &= g[2mx_1y_1 - (1 + m)(x_1^2x_2 - x_1x_4) + \frac{(1 + m)(x_1 + mx_1 - y_1)(mx_4 + y_4)}{k_2}], \\
\ddot{x}_3 &= 2(1 + m)y_3 + y_1x_2 + x_1y_2 - k_1y_1, \\
\ddot{x}_4 &= -my_4 - (1 + m)(mx_4 + y_4) + k_2(x_1x_2 - x_4),
\end{align*}

(15)

which is equivalent to system (1).

The obtained equations are in the form of (2). Thus, the relevant results of the KCC theory can be applied to discuss the dynamo system, and the expression of $G_i$ can be derived directly as

\begin{align*}
G_1 &= -\frac{1}{2}[r(x_1x_2 - x_4 - y_1) + y_3 - \frac{r(1 + m)(mx_4 + y_4)}{k_2}], \\
G_2 &= -\frac{g}{2}[2mx_1y_1 - (1 + m)(x_1^2x_2 - x_1x_4) + \frac{(1 + m)(x_1 + mx_1 - y_1)(mx_4 + y_4)}{k_2}], \\
G_3 &= -(1 + m)y_3 - \frac{1}{2}(y_1x_2 + x_1y_2 - k_1y_1), \\
G_4 &= -\frac{1}{2}(-my_4 - (1 + m)(mx_4 + y_4) + k_2(x_1x_2 - x_4)].
\end{align*}

(16)

Therefore, we can calculate the coefficients of nonlinear connection namely

\begin{align*}
N^1_1 &= \frac{r}{2}, \quad N^1_2 = 0, \quad N^1_3 = -\frac{1}{2}, \quad N^1_4 = \frac{r(1 + m)}{2k_2}, \\
N^2_1 &= -mgx_1 + \frac{g(1 + m)(mx_4 + y_4)}{2k_2}, \quad N^2_2 = 0, \quad N^2_3 = 0, \\
N^2_4 &= \frac{g(1 + m)(y_1 - x_1 - mx_1)}{2k_2}, \\
N^3_1 &= \frac{k_1 - x_2}{2}, \quad N^3_2 = -\frac{x_1}{2}, \quad N^3_3 = -(1 + m), \quad N^3_4 = 0, \\
N^4_1 &= 0, \quad N^4_2 = 0, \quad N^4_3 = 0, \quad N^4_4 = \frac{1 + 2m}{2}.
\end{align*}

(17)

and obtain the coefficients of Berwald connection that is

\begin{align*}
G^2_{14} &= G^2_{41} = \frac{g(1 + m)}{2k_2}. \\
\end{align*}

(18)

Other Berwald connection coefficients unlisted are all equal to zero.
Taking these values into system (8), one can get the most critical factor called the deviation curvature tensor as follows

\[ P_1^1 = rx_2 + \frac{1}{4}(r^2 - k_1 + x_2), \quad P_2^1 = \frac{(4r + 1)x_1}{4}, \quad P_3^1 = \frac{2(1 + m) - r}{4}, \]

\[ P_4^1 = -r + \frac{r(1 + m)(1 + r - 2m)}{4k_2}, \]

\[ P_2^2 = \frac{g}{2}(1 + m)(-3x_1x_2 + x_4) - mx_1 + 2my_1 + \frac{g(1 + m)(2 + 2m + r)(mx_4 + y_4)}{4k_2}, \]

\[ P_2^2 = -g(1 + mx_1), \quad P_3^2 = \frac{1}{2}mgx_1 - \frac{g(1 + m)}{4k_2}(mx_4 + y_4), \]

\[ P_4^2 = (1 + m)gx_1 + \frac{g(1 + m)}{4k_2}[2r(x_1x_2 - x_4 - y_1 - mx_1) + (x_1 + mx_1 - y_1)(2m - 1) \]

\[ - 2(1 + m)y_1 + 2y_1] - \frac{g(1 + m)^2r}{4k_2}(mx_4 + y_4), \]

\[ P_1^3 = \frac{(r - 2 - 2m)(k_1 - x_2)}{4} + \frac{1}{2}(yg_1 + mgx_1) - \frac{gx_1(1 + m)(mx_4 + y_4)}{4k_2}, \]

\[ P_2^3 = \frac{(1 + m)x_1 + y_1}{2}, \quad P_3^3 = (1 + m)^2 + \frac{1}{4}(-k_1 + x_2), \]

\[ P_4^3 = \frac{1 + m}{4k_2}[r(k_1 - x_2) + gx_1(x_1 + mx_1 - y_1)], \]

\[ P_1^4 = k_2x_2, \quad P_2^4 = k_2x_1, \quad P_3^4 = 0, \quad P_4^4 = -k_2 - m(1 + m) + \frac{(1 + 2m)^2}{4}. \]

(19)

As a matter of fact, the matrix of deviation curvature tensor is given by

\[
\begin{pmatrix}
    P_1^1 & P_2^1 & P_3^1 & P_4^1 \\
    P_1^2 & P_2^2 & P_3^2 & P_4^2 \\
    P_1^3 & P_2^3 & P_3^3 & P_4^3 \\
    P_1^4 & P_2^4 & P_3^4 & P_4^4
\end{pmatrix}
\]

whose characteristic equation is written as

\[ \lambda^4 + b_1 \lambda^3 + b_2 \lambda^2 + b_3 \lambda + b_4 = 0. \]

(20)

According to the definition (2.1) as well as the Routh-Hurwitz criteria, we can further determine the Jacobi stability of the dynamo system on the basis of the coefficients of characteristic equation (20). More precisely, the dynamo system is Jacobi stable only when

\[ H_i = \begin{vmatrix}
    b_1 & b_3 & 0 & 0 \\
    1 & b_2 & b_4 & 0 \\
    0 & b_1 & b_3 & 0 \\
    0 & 1 & b_2 & b_4
\end{vmatrix} > 0, \quad i = 1, 2, 3, 4, \]

(21)

or say,

\[ b_1 > 0, \quad b_4 > 0, \quad b_1 b_2 - b_3 > 0, \quad b_1 b_2 b_3 - b_1^2 - b_4 > 0, \]

otherwise it’s Jacobi unstable.

### 3.2. Jacobi stability analysis of the equilibria

By virtue of the symmetry characteristic toward $z$-axis of the dynamo system, we can obtain two equilibria $E_{1,2}$ by choosing parameter values

\[ m = 0.04, \quad g = 140.6, \quad r = 7, \quad k_1 = 34, \quad k_2 = 12, \]

(23)
where

\[
E_1 = (5.71919, 0.388094, 20.428, 37.3177, 116.428), \\
E_2 = (-5.71919, -0.388094, 20.428, -37.3177, -116.428).
\]

It’s shown that both of the equilibria are linear stable [34], for comparison, we need to further explore its Jacobi stability based on KCC theory. By using the results in previous discussion, the eigenvalues of its corresponding characteristic equation (20) at the two equilibria are same, which are calculated as follows

\[
-4646.57, \\ 0.901495, \\ 0.300151 + 3.98468i, \\ 0.300151 - 3.98468i.
\]

Note that eigenvalues have a negative real root, a positive real root as well as a pair of imaginary roots with positive real parts. According to the definition of Jacobi stability, it leads us to obtain the following theorem.

**Theorem 3.1.** Both of the equilibria \(E_{1,2}\) of the dynamo system (1) are linear stable but Jacobi unstable for the parameter values \(m = 0.04, g = 140.6, r = 7, k_1 = 34, k_2 = 12\).

It is noteworthy that Jacobi stability can be regarded as the robustness of the system, essentially different from the intrinsic properties of linear stability. This can help to explain why linear stable equilibria will produce different result of Jacobi stability.

### 3.3. Jacobi stability analysis of the periodic orbit.

We have known that the 5D dynamo system of (1) will occur a periodic orbit by choosing parameters
for $m = 0.04, g = 140.6, r = 3.5, k_1 = 34, k_2 = 12$ under the initial condition $(2, 1, 2, 0, 0)$ [34], which is depicted in Fig.2. On account of the solutions of periodic orbit cannot be explicitly given, there is also no way for us to give the expression of eigenvalues. Thus we need to discuss the Jacobi stability of the periodic orbit through numerical calculation. In reality, in order to illustrate the Jacobi stability of periodic orbit, it is essential for us to verify whether the four conditions in (22) are satisfied. Through observing the phase diagrams given in Fig.3, we can find that negative parts exist in four figures, which implies that we can not guarantee all the real parts of eigenvalues of the periodic orbit are negative. Then this periodic orbit is thought to be Jacobi unstable. Combined with its physical meaning mentioned in the background, it is natural for us to think that the robustness of periodic orbit is weakness.

**Figure 3.** Part of time variation figures for four judgment conditions with $(m, g, r, k_1, k_2) = (0.04, 140.6, 3.5, 34, 12)$.

In this section, we will numerically analyze the dynamic behaviors of the deviation vector, and the quantitative analysis is carried on through instability exponent. The corresponding results indicate the generation of chaos.
3.4. **Dynamic behaviors of deviation vector.** Indeed, that studying dynamic behaviors of deviation vector can help to infer the occurrence of chaos has been affirmed in some three-dimensional dynamic systems [2,10,12,14,15]. But for higher dimensional systems, little research has been done. The discussion to the 5D dynamo system shows its applicability. As a matter of fact, the deviation vector $\xi(t)$ is used to describe the trajectories behaviors nearby the fixed point $x_i(t_0)$. Just as the derivation from (7), we can obtain the variational form of the components of the deviation vector

\[
\begin{align*}
\frac{d^2 \xi_1}{dt^2} &= -r \frac{d \xi_1}{dt} + \frac{(1 + m)r \, d \xi_4}{k_2} + r x_2 \xi_1 + r x_1 \xi_2 - (1 + \frac{m(m + 1)}{k_2})r \xi_4, \\
\frac{d^2 \xi_2}{dt^2} &= [2mgx_1 - \frac{g(1 + m)(mx_4 + y_4)}{k_2}] \frac{d \xi_1}{dt} + \frac{g(1 + m)(x_1 + mx_1 - y_1)}{k_2} \frac{d \xi_4}{dt} \\
&\quad + [2gmy_1 - g(1 + m)(2x_1x_2 - x_4) + \frac{g(1 + m)^2(mx_4 + y_4)}{k_2}] \xi_1 - g(1 + m)x_2^2 \xi_2 \\
&\quad + g(1 + m)(x_1 + \frac{m(x_1 + mx_1 - y_1)}{k_2}) \xi_4, \\
\frac{d^2 \xi_3}{dt^2} &= (-k_1 + x_2) \frac{d \xi_1}{dt} + x_1 \frac{d \xi_2}{dt} + 2(1 + m) \frac{d \xi_3}{dt} + y_2 \xi_1 + y_1 \xi_2, \\
\frac{d^2 \xi_4}{dt^2} &= -(1 + 2m) \frac{d \xi_4}{dt} + k_2 x_2 \xi_1 + k_2 x_1 \xi_2 - [k_2 + m(1 + m)] \xi_4.
\end{align*}
\]  
\] (24)

and the resulting deviation vector can be expressed as

\[
\xi(t) = \sqrt{\xi_1^2(t) + \xi_2^2(t) + \xi_3^2(t) + \xi_4^2(t)}.
\]  
\] (25)

**Figure 4.** Deviation vector near $E_1$ under the initial condition $(\xi_1, \xi_2, \xi_3, \xi_4) = (0, 0, 0, 0)$ and different values for $(\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4)$:

(a) $(\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4) = (10^{-8}, 10^{-8}, 10^{-8}, 10^{-8})$; (b) $(\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4) = (10^{-5}, 10^{-5}, 10^{-5}, 10^{-5})$.

Time variation of the deviation vector near equilibria $E_1$ and $E_2$ are displayed in Fig 4 and Fig.5 respectively. With a slight change to the initial condition,
it can be seen that these presented diagrams show completely different behaviors. More specifically, when we choose the initial condition for \((\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4) = (10^{-8}, 10^{-8}, 10^{-8}, 10^{-8})\), the dynamic behaviors near equilibria is accompanied by a rapid oscillation although its amplitudes tend to be stable. However, when we choose another set of initial condition \((\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3, \dot{\xi}_4) = (10^{-5}, 10^{-5}, 10^{-5}, 10^{-5})\), the behavior of oscillation disappears. This means that the deviation vector is sensitive to the initial conditions. Obviously, such behaviors indicate the onset of chaos in this dynamo system, which gives new idea to predict chaotic behaviors.

3.5. Instability exponents. In the following, we are looking for the quantitative analysis about the onset of chaos. By analogy with Lyapunov exponents, instability exponents are proposed to explain the onset of chaos qualitatively, which is conducive to analyze the rate of divergence nearby the trajectories. Instability exponents and its components are defined by \[26\]

\[
\begin{align*}
\delta_i(S) &= \lim_{t \to \infty} \frac{1}{t} \ln \left[ \frac{\xi_i(t)}{\xi_{i0}} \right], \quad i = 1, 2 \\
\delta(S) &= \lim_{t \to \infty} \frac{1}{t} \ln \left[ \frac{\xi(t)}{\xi_{10}} \right],
\end{align*}
\]

where \(\dot{\xi}_1(0) = \xi_{10}, \dot{\xi}_2(0) = \xi_{20}\).

Phase diagrams of instability exponent near the equilibria are shown in Fig.6, where the deviation vectors are also discussed under different initial conditions. It should be pointed out that once the instability exponent is positive, the trajectories are deemed to be divergent in Jacobi stability space. From what is shown in Fig.6, it is obvious that the instability exponents near two equilibria show different dynamic behaviors, but we can find that the instability exponents tend to small positive values regardless of the initial conditions. Thus, it can be seen as an indication of the generation of chaos.
4. Impulsive control. Now we construct an indefinite Lyapunov function and prove that the equilibrium is asymptotically stable under certain conditions. Moreover, the design of impulse controller makes it possible to control this dynamo system.

Definition 4.1. A function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ is of the class $PCD$ if

(H1) $V$ is continuous on each of the sets $(t_{k-1}, t_k) \times \mathbb{R}^n$, 
\[ \lim_{(t, y) \to (t_k, x)} V(t, y) = V(t_k, x) \] and \[ \lim_{(t, y) \to (t_k, x)} V(t, y) \] exists.

(H2) $V$ is differentiable on each of the sets $(t_{k-1}, t_k) \times \mathbb{R}^n$.

Consider system described by the following equations
\[
\begin{align*}
\dot{x}(t) &= f(x(t)), \quad t \neq t_k \\
x(t) &= g(x(t^-)), \quad t = t_k
\end{align*}
\] (27)
where \( x \in \mathbb{R}^+ \), \( t_0 = 0 < t_1 < \cdots < t_k < \cdots \) ( \( \lim_{k \to +\infty} t_k = +\infty \)), \( t_1 - 0 = t_{k+1} - t_k = h \).

The functions \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) are Lipschitz continuous and meet the conditions \( f(0) = 0, g(0) = 0 \).

**Proposition 1.** Consider system (27) on the set \( D = \{ x \in \mathbb{R}^n | |x| \leq R \} \) (a constant \( R > 0 \)). If there exist a function \( V \in PCD \), functions \( \alpha_1, \alpha_2 \in K_\infty \) and constants \( M > 0, 1 > \xi > 0 \) such that

\[
\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{28}
\]

\[
\dot{V}(x(t)) = \frac{dV}{dx}(x(t))f(x(t)) \leq MV(x(t)), \tag{29}
\]

for \( t \neq t_k, k \in \{1,2,3,\cdots\} \),

\[
V(g(x(t))) \leq \xi V(x(t^-)), \tag{30}
\]

for \( t = t_k, k \in \{1,2,3,\cdots\} \),

\[
\frac{\ln \xi}{h} \leq -M, \tag{C4}
\]

then \( V \) is an indefinite-time Lyapunov function for system (27), and moreover the origin of system (27) is asymptotically stable in \( D \).

**Proof.** The result is deduced from ref. [20]. \( \square \)

![Figure 7](image)

**Figure 7.** (a) Time series of system (31) with hidden hyperchaotic attractor; (b) The trajectory of the indefinite Lyapunov function \( V \) for system (32) with and the resetting time \( t_k = 0.01k, k = 1,2,\cdots \). The initial condition we choose is \((-5.6692,0.1119,-20.3280,-36.3177,-114.4281)\).

We reconsider the system (1) with parameters \( r = 7, m = 0.04, g = 140.6, k_1 = 34, k_2 = 12 \). From Fig.1, we know there is a hidden hyperchaotic attractor with initial condition \((0.05,-0.5,0.1,-1,2)\). After we transfer the equilibrium \( E_1 \) to
D is stable in indefinite Lyapunov function for system (32) and the origin of (32) is asymptotically responding initial condition (8, 2, 2, 1, 1).

Figure 8. (a) Time series of system (31) with hidden hyperchaotic attractor; (b) The trajectory of the indefinite Lyapunov function $V$ for system (32) and the resetting time $t_k = 0.01k, k = 1, 2, \cdots$.

The initial condition we choose is $(0, 2, 2, 1, 1)$.

origin by linear transformation and denote $x_1, x_2, x_3, x_4, x_5$ as $x, y, z, u, v$ in (1), system (1) can be described by

$$
\dot{x}(t) = \begin{cases}
    r(x_2(t) - x_1(t)) + x_4, \\
    -(1 + m)x_2(t) + x_1(t)x_3(t) + 20.428x_1(t) + 5.71919x_3 - x_5, \\
    g(mx_1^2(t) + 0.0539x_1(t) - 5.9480x_2(t) - (1 + m)x_1x_2), \\
    2(1 + m)x_4 + x_1x_3 + (20.408 - k_1)x_1 + 5.71919x_3, \\
    -mx_5 + 12x_2,
\end{cases}
$$

(31)

where the parameters $r = 7, m = 0.04, g = 140.6, k_1 = 34, k_2 = 12.$

To ensure the origin of system (31) in $D = \{x \in \mathbb{R}^5 | |x| \leq 30\}$, the impulsive controller is designed as the following

$$
\begin{cases}
    7x_2(t) - x_1(t) + x_4, \\
    -1.04x_2(t) + x_1(t)x_3(t) + 20.428x_1(t) + 5.71919x_3 - x_5, \\
    5.624x_1^2(t) + 7.58086x_1(t) - 836.283x_2(t) - 146.224x_1x_2, \\
    2.08x_4 + x_1x_3 - 13.572x_1 + 5.71919x_3, \\
    -0.04x_5 + 12x_2,
\end{cases}
$$

(32)

$$
x(t) = 0.01x(t^{-}), t = t_k,
$$

where $t_k = 0.01k, k > 0, k \in \mathbb{Z}^+.$

Let $V(x) = |x|^2$ be an indefinite Lyapunov function candidate for system (32).

It is evident that $h = 0.01, \xi = 0.0001$. We check if there exists a constant $M = 500$ such that $(C_1), (C_2), (C_3)$ of Proposition 5.1 hold. By calculation, we get that

$$
\dot{V}(x(t)) \leq 500V(x(t)),
$$

(33)

for $t \neq t_k, k \in \{1, 2, 3, \cdots\}$, and $\frac{\log k}{h} \leq -M$. Then based on Proposition 1, $V$ is an indefinite Lyapunov function for system (32) and the origin of (32) is asymptotically stable in $D$. From the initial condition $(0.05, -0.5, 0.1, -1, 2)$ in Fig.1, we choose corresponding initial condition $(-5.6692, 0.1119, -20.3286, -36.3177, -114.4281)$ for
system (32) and obtain the result in Fig. 7. In addition, we also set a different initial condition (8, 2, 2, 1, 1) for system (32) and obtain the result in Fig. 8 for better verifying the correctness of the method.

5. Conclusion. The main content of this paper is the analysis of the Jacobi stability and impulsive control of a 5D self-exciting homopolar disc dynamo system. Although the establishment of classical Lyapunov stability is fully recognized, it’s still of great significance for us to explore other method of stability analysis and compare with the classical method. Jacobi stability is such an innovative point, which makes it possible to analyze the stability in terms of differential geometry. The equilibria $E_{1,2}$ of the dynamo system are verified to be linear stable but Jacobi unstable under a set of chaotic parameter values. It needs to be emphasized that Jacobi stability can reflect the robustness of the system, inherently different from the meaning of Lyapunov stability. Somehow this can also explain why linear stable equilibria will produce Jacobi unstable result. Through the analysis of the deviation vector near the equilibria, the onset of chaos is detected by the dynamic behaviors of the deviation vector and the quantitative indicators namely instability exponent. It can contribute a lot to the in-depth discussion of high dimensional dynamo systems and promote the study of the mechanism of hidden chaos for other dynamical systems. We also propose sufficient conditions for asymptotic stability of the equilibrium of nonlinear impulsive self-exciting homopolar disc dynamo system. Moreover, the impulsive controller is designed based on the construction of indefinite Lyapunov functions, and the corresponding numerical results shows effectiveness and advantages of the designed impulsive controller.

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