1 Introduction and Basic Definitions

There are numerous ways to represent real numbers. We may use, e.g., Cauchy sequences, Dedekind cuts, numerical base-10 expansions, numerical base-2 expansions and continued fractions. If we work with full Turing computability, all these representations yield the same class of real numbers. If we work with some restricted notion of computability, e.g., polynomial time computability or primitive recursive computability, they do not. This phenomenon has been investigated over the last seven decades by Specker \cite{12}, Mostowski \cite{7}, Lehman \cite{9}, Ko \cite{23}, Labhalla & Lombardi \cite{8}, Georgiev \cite{1}, Kristiansen \cite{4,5} and quite a few more.

Irrational numbers can be represented by infinite sums. Sum approximations from below and above were introduced in Kristiansen \cite{4} and studied further in Kristiansen \cite{5}. Every irrational number $\alpha$ between 0 and 1 can be uniquely written as an infinite sum of the form

$$\alpha = 0 + \frac{D_1}{b^k_1} + \frac{D_2}{b^k_2} + \frac{D_3}{b^k_3} + \ldots$$

where

- $b \in \mathbb{N} \setminus \{0, 1\}$ and $D_i \in \{1, \ldots, b - 1\}$ (note that $D_i \neq 0$ for all $i$)

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Let $\hat{A}^b_α(i) = D_ib^{-k_i}$ for $i > 0$ (and let $\hat{A}^b_α(0) = 0$). The rational number $\sum_{i=1}^{∞} A^b_α(i)$ is an approximation of $α$ that lies below $α$, and we will say that the function $A^b_α$ is the base-$b$ sum approximation from below of $α$. The base-$b$ sum approximation from above of $α$ is a symmetric function $\hat{A}^b_α$ such that $1 - \sum_{i=1}^{∞} A^b_α(i)$ is an approximation of $α$ that lies above $α$ (and we have $\sum_{i=1}^{∞} A^b_α(i) + \sum_{i=1}^{∞} A^b_α(i) = 1$). Let $S$ be any class of subrecursive functions which is closed under primitive recursive operations. Furthermore, let $S_{∪Γ}$ denote the set of irrational numbers that have a base-$b$ sum approximation from below in $S$, and let $S_{∩Γ}$ denote the set of irrational numbers that have a base-$b$ sum approximation from above in $S$. It is proved in [4] that $S_{∪Γ}$ and $S_{∩Γ}$ are incomparable classes, that is, $S_{∪Γ} ⊈ S_{∩Γ}$ and $S_{∩Γ} ⊈ S_{∪Γ}$. Another interesting result proved in [5] is that $S_{∩Γ} ⊆ S_{∩Γ}$ iff every prime factor of $b$ is a prime factor of $a$.

In this paper we prove some results on general sum approximations. The general sum approximation from below of $α$ is the function $\hat{G}^α : N × N → Q$ defined by $\hat{G}^α(b, n) = A^b_α(n)$; let $\hat{G}^α(b, n) = 0$ if $b < 2$. The general sum approximation from above of $α$ is the function $\hat{G}^α : N × N → Q$ defined by $\hat{G}^α(b, n) = A^b_α(n)$; let $\hat{G}^α(b, n) = 0$ if $b < 2$. Let $S$ be any class of subrecursive functions which is closed under primitive recursive operations. Furthermore, let $S_{∩Γ}$ denote the set of irrational numbers that have a general sum approximation from below in $S$, and let $S_{∪Γ}$ denote the set of irrational numbers that have a general sum approximation from above in $S$.

It was proved in [4] that $S_{∩Γ} \cap S_{∪Γ}$ contains exactly the irrational numbers that have a continued fraction in the class $S$. In this paper we prove that $S_{∩Γ} \neq S_{∪Γ}$. Moreover, we prove that

$$S_{∪Γ} \neq \bigcap_{b=2}^{∞} S_{∩Γ} \quad \text{and} \quad S_{∩Γ} \neq \bigcap_{b=2}^{∞} S_{∪Γ}.$$ 

Some might find it interesting (at least the authors do) that we manage to complete all our proof without resorting to the standard computability-theoretic machinery involving enumerations, universal functions, diagonalizations, and so on. We prove our results by providing natural irrationals numbers (the numbers are natural in the sense that they have neat and transparent definitions).

2 Preliminaries

We will restrict our attention to real numbers between 0 and 1.

A base is a natural number strictly greater than 1, and a base-$b$ digit is a natural number in the set $\{0, 1, \ldots, b - 1\}$.

Let $b$ be a base, and let $D_1, \ldots, D_n$ be base-$b$ digits. We will use $(0.D_1D_2 \ldots D_n)_b$ to denote the rational number $\sum_{i=1}^{n} D_ib^{-i}$. 
Let $D_1, D_2, \ldots$ be an infinite sequence of base-$b$ digits. We say that $(0.D_1D_2 \ldots)_b$ is the base-$b$ expansion of the real number $\alpha$ if for all $n \geq 1$ we have

$$(0.D_1D_2 \ldots D_n)_b \leq \alpha < (0.D_1D_2 \ldots D_n)_b + b^{-n}.$$ 

Every real number $\alpha$ has a unique base-$b$ expansion (note the strict inequality).

When $\alpha = (0.D_1D_2 \ldots D_n)_b$ for some $n$ with $D_n \neq 0$, we say that $\alpha$ has a finite base-$b$ expansion of length $n$. Otherwise, we say that $\alpha$ has an infinite base-$b$ expansion, and this infinite base-$b$ expansion is periodic iff $\alpha$ is rational. More concretely, if $\alpha = cd^{-1}$ for non-zero relatively prime $c, d \in \mathbb{N}$, then the base-$b$ expansion of $\alpha$ is of the form $D_1 \ldots D_n (D_{n+1} \ldots D_t)\omega$ which we use as shorthand for the infinite sequence $D_1 \ldots D_n D_{n+1} \ldots D_{n+1} \ldots D_{n+1} \ldots D_{t-1} \ldots$. The number $s$ is the largest natural number such that $p^s$ divides $d$ for some prime factor $p$ of $b$. The length of the period $t - s$ is the multiplicative order of $b$ modulo $d_1$ where $d_1$ is the largest divisor of $d$ relatively prime with $b$. It follows straightforwardly that $t < d$. Of course, $\alpha$ has a finite base-$b$ expansion iff $d_1 = 1$, that is, iff every prime factor of $d$ is a prime factor of $b$.

We assume the reader is familiar with subrecursion theory and subrecursive functions. An introduction to the subject can be found in [10] or [11].

A function $\phi$ is elementary in a function $\psi$, written $\phi \leq_E \psi$, if $\phi$ can be generated from the initial functions $\psi$, $2^x$, max, 0, $S$ (successor), $I^n_i$ (projections) by composition and bounded primitive recursion. A function $\phi$ is elementary if $\phi \leq_E 0$. A function $\phi$ is primitive recursive in a function $\psi$, written $\phi \leq_{PR} \psi$, if $\phi$ can be generated from the initial functions by composition and (unbounded) primitive recursion. A function $\phi$ is primitive recursive if $\phi \leq_{PR} 0$.

Subrecursive functions in general, and elementary functions in particular, are formally functions over natural numbers ($\mathbb{N}$). We assume some coding of integers ($\mathbb{Z}$) and rational numbers ($\mathbb{Q}$) into the natural numbers. We consider such a coding to be trivial. Therefore we allow for subrecursive functions from rational numbers into natural numbers, from pairs of rational numbers into rational numbers, etc., with no further comment. Uniform systems for coding finite sequences of natural numbers are available inside the class of elementary functions. Hence, for any reasonable coding, basic operations on rational numbers – like e.g. addition, subtraction and multiplication – will obviously be elementary. It is also obvious that there is an elementary function $\psi(q, i, b)$ that yields the $i^{th}$ digit in the base-$b$ expansion of the rational number $q$.

A function $f : \mathbb{N} \to \mathbb{N}$ is honest if it is monotonically increasing ($f(x) \leq f(x+1)$), dominates $2^x$ ($f(x) \geq 2^x$) and has elementary graph (the relation $f(x) = y$ is elementary).

A class of functions $S$ is subrecursive class if $S$ is an efficiently enumerable class of computable total functions. For any subrecursive class $S$ there exists an honest function $f$ such that $f \not\in S$ (see Section 8 of [4] for more details).

More on elementary functions, primitive recursive functions and honest functions can be found in Section 2 of [4] and in [6].
3 Irrational Numbers with Interesting Properties

Definition 1. Let $P_i$ denote the $i^{th}$ prime ($P_0 = 2, P_1 = 3, \ldots$). We define the auxiliary function $g$ by

$$g(0) = 1 \quad \text{and} \quad g(j + 1) = P_j^{2(j+2)(g(j)+1)^3}.$$ 

For any honest function $f$ and any $n \in \mathbb{N}$, we define the rational number $\alpha_f^n$ and the number $\alpha_f$ by

$$\alpha_f^n = \sum_{i=0}^{n} P_i^{-h(i)} \quad \text{and} \quad \alpha_f = \lim_{n \to \infty} \alpha_f^n$$

where $h(i) = g(f(i) + i)$ (for any $i \in \mathbb{N}$).

It is easy to see that $g$ and $h$ are strictly increasing honest functions. Only the fact that $g$ has elementary graph requires some explanation: for all $x, y \in \mathbb{N}$ the equality $g(x) = y$ holds iff

$$\exists s ((s)_0 = 1 \quad \& \quad \forall j \leq x \quad ((s)_{j+1} = P_j^{2(j+2)((s)_j)^3} \quad \& \quad (s)_x = y).$$

We have that $s$ is the code of a sequence of length $x + 1$ and its $j$-th element $(s)_j$ is equal to $g(j)$ for all $j \leq x$. Since $g$ is increasing and the coding of sequences can be chosen to be monotonic, the existential quantifier on $s$ can be bounded by the code of the sequence $y, \ldots, y$ ($x + 1$ times), which is elementary in $x, y$.

The function $h$ possesses the following growth property, which will need:

$$P_n^{2(n+2)(h(n)+1)^3} < h(n + 1)$$

for any $n \in \mathbb{N}$. The function $f$ itself might not possess it and this explains why we introduce the function $g$ in the definition of $\alpha_f$.

When $f$ is a fixed honest function, we abbreviate $\alpha_f^n$ and $\alpha_f$ to $\alpha_j$ and $\alpha$, respectively.

Lemma 2. The number $\alpha_f$ is irrational.

Proof. Assume that $\alpha_f = \frac{p}{q}$ for some natural numbers $p, q$ with $q$ non-zero. Let $q_i$ be the exponent of $P_i$ in the prime factorization of $q$. We choose $n$, such that $P_n$ is greater than all prime factors of $q$ and $h(n)$ is greater than $q_0 + q_1 + \ldots + q_n$.

It is easy to see that $\alpha_f - \alpha_f^n$ is a non-zero rational number with denominator $P_0^{h(0)} P_1^{h(1)} \ldots P_n^{h(n)}$, where $h(i) = h(i) + q_i$. Therefore,

$$\frac{1}{P_0^{h(0)} P_1^{h(1)} \ldots P_n^{h(n)}} \leq \alpha_f - \alpha_f^n.$$
But we also have
\[ \alpha^f - \alpha_n^f = P_{n+1}^{-h(n+1)} + P_{n+2}^{-h(n+2)} + \ldots \]
and since \( h \) is strictly increasing,
\[ P_{n+1}^{-h(n+1)} + P_{n+2}^{-h(n+2)} + \ldots \leq P_{n+1}^{-h(n+1)} + P_{n+1}^{-h(n+1)} - 1 + \ldots \leq P_{n+1}^{-h(n+1)} + 1. \]
We obtained the inequalities
\[ \frac{1}{P_0^{h(0)}' P_1^{h(1)}' \ldots P_n^{h(n)}'} \leq \alpha^f - \alpha_n^f \leq P_{n+1}^{-h(n+1)} + 1, \]
which imply in turn
\[ P_{n+1}^{h(n+1) - 1} \leq P_0^{h(0)}' P_1^{h(1)}' \ldots P_n^{h(n)}', \]
\[ P_{n+1}^{h(n+1) - 1} \leq P_{n+1}^{h(0)' + h(1)' + \ldots + h(n)'}, \]
\[ h(n+1) \leq (n+1)h(n) + q_0 + \ldots + q_n + 1, \]
\[ h(n+1) \leq (n+2)h(n) + 1. \]
But the last inequality is easily seen to be false by (1). Contradiction. \( \square \)

In fact, it is not hard to show using (1) that \( \alpha^f \) is a Liouville number, therefore it is even transcendental.

**Lemma 3.** For any \( j \in \mathbb{N} \) and any base \( b \), we have

(i) if \( P_i \) divides \( b \) for all \( i \leq j \), then \( \alpha_j \) has a finite base-\( b \) expansion of length \( h(j) \)

(ii) if \( P_i \) does not divide \( b \) for some \( i \leq j \), then \( \alpha_j \) has an infinite (periodic) base-\( b \) expansion.

**Proof.** For any \( j \in \mathbb{N} \) and base \( b \) we have that
\[ \alpha_j = \frac{m}{\prod_{i=0}^{j} P_i^{h(i)'}} , \]
where \( m \) is the sum of \( j + 1 \) summands. Each of these summands is divisible by all \( P_i \) for \( i \leq j \) with the exception of exactly one of them. From this it follows that \( m \) is relatively prime to all \( P_i \) for \( i \leq j \), so the above fraction is in lowest terms. The rest follows easily from the preliminaries on base-\( b \) expansions. \( \square \)

**Lemma 4.** Let

- \( b \) be any base, and let \( j \in \mathbb{N} \) be such that \( P_j > b \)
- \((0.D_1D_2\ldots)_b\) be the base-\( b \) expansion of \( \alpha_j \)
\( (0.\hat{D}_1\hat{D}_2\ldots)_b \) be the base-\( b \) expansion of \( \alpha_{j+1} \)
\( M = M(j) = P_{j+1}^{h(j)} \) and \( M' = M'(j) = h(j+1) \).

Then

(i) there are no more than \( M \) consecutive zeros in the base-\( b \) expansion of \( \alpha_j \),
that is, for any \( k \in \mathbb{N} \setminus \{0\} \) there exists \( i \in \mathbb{N} \) such that
\[ k \leq i < k + M \quad \text{and} \quad D_i \neq 0. \]

(ii) the first \( M' - M \) digits of the base-\( b \) expansions of \( \alpha_j \) and \( \alpha_{j+1} \) coincide,
that is
\[ i \leq M' - M \Rightarrow D_i = \hat{D}_i \]
and moreover, these digits also coincide with the corresponding digits of the base-\( b \) expansion of \( \alpha \).

**Proof.** By Lemma 3 (ii), \( \alpha_j \) has an infinite periodic base-\( b \) expansion of the form
\[ 0.D_1\ldots D_s(D_{s+1}\ldots D_t)^\omega \] with \( s < t \). Using the preliminaries on base-\( b \) expansions we obtain
\[ t - s \leq t < \prod_{i=0}^{j} P_{i}^{h(i)} \leq P_{j+1}^{h(j+1)} = M. \quad (2) \]

Thus (i) holds since every \( M \) consecutive digits of \( \alpha_j \) contain all the digits \( \hat{D}_{s+1}, \ldots, \hat{D}_t \) of at least one period.

We turn to the proof of (ii). We have
\[ \alpha_j < \alpha_{j+1} = \alpha_j + P_{j+1}^{-h(j+1)} \leq \alpha_j + b^{-M'} \quad (3) \]
since \( b^{M'} < P_{j+1}^{M'} = P_{j+1}^{h(j+1)} < P_{j+1}^{h(j+1)} \). At least one digit in the period \( \hat{D}_{s+1}\ldots\hat{D}_t \) is different from \( b-1 \), and the length of the period is \( t-s \). Thus, it follows from (3) that
\[ \hat{D}_i = \hat{D}_i \quad \text{for any} \quad i \leq M' - (t-s). \quad (4) \]

It follows from (2) and (4) that the first \( M' - M \) digits of the base-\( b \) expansions of \( \alpha_j \) and \( \alpha_{j+1} \) coincide. Moreover, since \( M'(j) \) is strictly increasing in \( j \), we have
\[ \alpha_j < \alpha_{j+k} \leq \alpha_j + \sum_{i<k} b^{-M'(j+i)} \leq \alpha_j + b^{-M'(j+1)} \]
for any \( k \geq 1 \). Letting \( k \to \infty \) we obtain as above that the first \( M' - M \) digits of \( \alpha_j \) and \( \alpha \) coincide. \( \square \)
In the proof of the next theorem the following inequality will be needed:

\[ M(j)^2 + M(j) + 1 < M'(j) \]  

(5)

for all \( j \in \mathbb{N} \). To prove it apply (1) together with

\[ P_j^{2(j+1)h(j)} + P_j^{(j+1)h(j)} + 1 < 3P_j^{2(j+1)h(j)} < P_j^{2(j+1)h(j)+2} \]

for all \( j \in \mathbb{N} \).

**Theorem 5.** Let \( f \) be any honest function, and let \( b \) be any base. The function \( \hat{A}_b' \) is elementary.

**Proof.** Fix the least \( m \) such that \( P_m > b \). We will use the functions \( M \) and \( M' \) from Lemma [1]. We will argue that we can compute the rational number \( \hat{A}_b(n) \) elementarily in \( n \) when \( n \geq M(m) \). Note that \( M(m) \) is a fixed number (it does not depend on \( n \)). Thus, we can compute \( \hat{A}_b(n) \) by a trivial algorithm when \( n < M(m) \) (use a huge table).

Assume \( n \geq M(m) \). We will now give an algorithm for computing \( \hat{A}_b(n) \) elementarily in \( n \).

**Step 1 of the algorithm:** Compute (the unique) \( j \) such that

\[ M(j) \leq n < M(j + 1) \]  

(6)

(End of Step 1).

Step 1 is a computation elementary in \( n \) since \( M \) has elementary graph. So is Step 2 as \( M' \) also has elementary graph.

**Step 2 of the algorithm:** Check if the relation

\[ n^2 + 1 < M'(j) - M(j) \]  

(7)

holds. If it holds, carry out step 3A below, otherwise, carry out step 3B (end of Step 2).

**Step 3A of the algorithm:** Compute \( \alpha_j \). Then compute base-\( b \) digits \( D_1, \ldots, D_{n^2+1} \) such that

\[ (0.D_1D_2\ldots D_{n^2+1})_b \leq \alpha_j < (0.D_1D_2\ldots D_{n^2+1})_b + b^{-(n^2+1)} \]

Find \( k \) such that \( D_k \) is the \( n \)th digit different from 0 in the sequence \( D_1, \ldots, D_{n^2+1} \). Give the output \( D_kb^{-k} \) (end of Step 3A).

Recall that \( \alpha_j = \sum_{i=0}^{j} P_i^{-h(i)} \). We can compute \( \alpha_j \) elementarily in \( n \) since \( h(0), h(1), \ldots, h(j) < M(j) \leq n \) and \( h \) is honest. Thus, we can also compute the base-\( b \) digits \( D_1, D_2, \ldots, D_{n^2+1} \) elementarily in \( n \). In order to prove that our algorithm is correct, we must verify that

(A) at least \( n \) of the digits \( D_1, D_2, \ldots, D_{n^2+1} \) are different from 0, and
(B) \( D_1, D_2, \ldots, D_{n^2 + 1} \) coincide with the first \( n^2 + 1 \) digits of \( \alpha \).

By Lemma [4] (i) the sequence \( D_{kM(j)+1}, D_{kM(j)+2}, \ldots, D_{(k+1)M(j)} \) contains at least one non-zero digit (for any \( k \in \mathbb{N} \)). Thus, (A) holds since \( n \geq M(j) \). Using \( \hat{G} \) and Lemma [4] (ii) we see that (B) also holds. This proves that the output \( D_k b^{-k} = A^\alpha_\beta(n) \).

**Step 3B of the algorithm:** Compute \( \alpha_{j+1} \) and \( M(j+1) \). Then proceed as in step 3A with \( \alpha_{j+1} \) in place of \( \alpha_j \) and \( nM(j+1) \) in place of \( n^2 \) (end of Step 3B).

Step 3B is only executed when \( M'(j) - M(j) \leq n^2 + 1 \). Thus, we have \( M'(j) = h(j+1) \leq n^2 + n + 1 \). This entails that we can compute \( h(j+1) \) and also \( \alpha_{j+1} \) and \( M(j+1) \) elementarily in \( n \). The inequality \( \hat{G} \) gives that

\[
M(j + 1)^2 + M(j+1) + 1 < M'(j+1)
\]

which together with \( \hat{G} \) imply

\[
nM(j + 1) + 1 < M'(j + 1) - M(j + 1).
\]

As in step 3A, there will be at least \( n \) non-zero digits among the first \( nM(j + 1) \) digits of \( \alpha_{j+1} \). Moreover, the first \( nM(j + 1) \) digits of \( \alpha_{j+1} \) coincide with the corresponding digits of \( \alpha \).

**Theorem 6.** Let \( f \) be any honest function. We have \( f \leq_{PR} \hat{G}^\alpha \) (\( f \) is primitive recursive in \( \hat{G}^\alpha \)).

**Proof.** Fix \( n \in \mathbb{N} \), and let \( b \) be the base \( b = \prod_{i=0}^n P_i \). By Lemma [4] (i), \( \alpha_n \) has a finite base-\( b \) expansion of length \( h(n) \). By the definition of \( \alpha \), we have

\[
\alpha = \alpha_n + P_{n+1}^{-h(n+1)} + P_{n+2}^{-h(n+2)} + \ldots.
\]

It follows that for any \( j > h(n) \)

\[
\hat{G}^\alpha(b, j) \leq P_{n+1}^{-h(n+1)} + P_{n+2}^{-h(n+2)} + \ldots,
\]

which easily implies \( \hat{G}^\alpha(b, j) \leq P_{n+1}^{-h(n+1)+1} \) (use that \( h \) is strictly increasing). Hence we also have \( (\hat{G}^\alpha(b, j))^{-1} \geq P_{n+1}^{h(n+1)-1} > h(n + 1) - 1 \) for any \( j > h(n) \).

The considerations above show that we can compute \( h(n + 1) \) by the following algorithm:

- assume that \( h(n) \) is computed;
- compute \( b = \prod_{i=0}^n P_i \);
- search for \( y \) such that \( y < (\hat{G}^\alpha(b, h(n) + 1))^{-1} + 1 \) and \( h(n + 1) = y \);
- give the output \( y \).

This algorithm is primitive recursive in \( \hat{G}^\alpha \): The computation of \( b \) is an elementary computation. The relation \( h(x) = y \) is elementary, and thus the search for \( y \) is elementary in \( h(n) \) and \( \hat{G}^\alpha \). This proves that \( h \) is primitive recursive in \( \hat{G}^\alpha \). But then \( f \) will also be primitive recursive in \( \hat{G}^\alpha \) as the graph of \( f \) is elementary and \( f(n) \leq h(n) \) (for any \( n \in \mathbb{N} \)). This proves that \( f \leq_{PR} \hat{G}^\alpha \). \( \square \)
Theorem 7. Let \( f \) be any honest function. There exists an elementary function \( \tilde{T} : \mathbb{Q} \to \mathbb{Q} \) such that (i) \( \tilde{T}(q) = 0 \) if \( q < \alpha^f \) and (ii) \( q > \tilde{T}(q) > \alpha^f \) if \( q > \alpha^f \).

Proof. In addition to the sequence \( \alpha_j \) we need the sequence \( \beta_j \) given by
\[
\beta_0 = P_0^{-h(0)+1} = 2^{-h(0)+1} \quad \text{and} \quad \beta_{j+1} = \alpha_j + P_{j+1}^{-h(j+1)+1}.
\]

Observe that we have \( \alpha < \beta_j \) for all \( j \in \mathbb{N} \), since
\[
\alpha - \alpha_j = P_{j+1}^{-h(j+1)} + P_{j+2}^{-h(j+2)} + \ldots \leq P_{j+1}^{-h(j+1)+1}
\]
for any \( j \in \mathbb{N} \).

Now we will explain an algorithm that computes a function \( \tilde{T} \) with the properties (i) and (ii).

Step 1 of the algorithm: The input is the rational number \( q \). We can w.l.o.g. assume that \( 0 < q < 1 \). Pick any \( m', n \in \mathbb{N} \) such that \( q = m'n^{-1} \) and \( n \geq h(0) \).

Find \( m \in \mathbb{N} \) such that \( q = m(P_0 P_1 \ldots P_n)^{-n} \), and compute the base \( b \) such that \( b = \prod_{i=0}^{n} P_i \) (end of Step 1).

The rational number \( q \) has a finite base-\( b \) expansion of length \( s \) where \( s \leq n \). Moreover, the rational numbers \( \alpha_0, \alpha_1, \ldots, \alpha_n \) and \( \beta_0, \beta_1, \ldots, \beta_n \) also have finite base-\( b \) expansions.

Step 2 of the algorithm: Compute (the unique) natural number \( j < n \) such that
\[
h(j) \leq n < h(j + 1).
\]

Furthermore, compute \( \alpha_0, \alpha_1, \ldots, \alpha_j \) and \( \beta_0, \beta_1, \ldots, \beta_j \) (end of Step 2).

All the numbers \( h(0), h(1), \ldots, h(j) \) are less than or equal to \( n \), and \( h \) has elementary graph. This entails that Step 2 is elementary in \( n \) (and thus also elementary in \( q \)).

Step 3 of the algorithm: If \( q \leq \alpha_k \) for some \( k \leq j \), give the output 0 and terminate. If \( \beta_k < q \) for some \( k \leq j \), give the output \( \beta_k \) and terminate (end of Step 3).

Step 3 obviously gives the correct output. It is also obvious that the step is elementary in \( q \).

If the algorithm has not yet terminated, we have \( \alpha_j < q \leq \beta_j \). Now
\[
q \leq \beta_{j+1} \iff q - \alpha_j \leq P_{j+1}^{-h(j+1)+1} \iff P_{j+1}^{h(j+1)} \leq (q - \alpha_j)^{-1}P_{j+1}.
\]

We have determined \( \alpha_j \), and \( h \) is an honest function. This makes it possible to check elementarily if \( q \leq \beta_{j+1} \): Search for \( y < (q - \alpha_j)^{-1}P_{j+1} \) such that \( h(j + 1) = y \). If no such \( y \) exists, we have \( q > \beta_{j+1} \). If such an \( y \) exists, we have \( q \leq \beta_{j+1} \) if \( P_{j+1}^{y} \leq (q - \alpha_j)^{-1}P_{j+1} \).

Step 4 of the algorithm: Search for \( y < (q - \alpha_j)^{-1}P_{j+1} \) such that \( y = h(j + 1) \).

If the search is successful and \( P_{j+1}^{y} \leq (q - \alpha_j)^{-1}P_{j+1} \), go to Step 5, otherwise go to Step 6B (end of Step 4).
Clearly, Step 4 is elementary in \( q \). If \( q \leq \beta_{j+1} \), the next step is Step 5 (and we have computed \( y = h(j + 1) \)). If \( \beta_{j+1} < q \), the next step is Step 6B.

**Step 5 of the algorithm:** Compute \( \alpha_{j+1} \) and \( \beta_{j+1} \). If \( q \leq \alpha_{j+1} \), give the output 0 and terminate. If \( \alpha_{j+1} < q \), search for \( z < (q - \alpha_{j+1})^{-1}P_{j+2} \) such that \( z = h(j + 2) \). If the search is successful and \( P_{j+2}^z \leq (q - \alpha_{j+1})^{-1}P_{j+2} \), give the output 0 and terminate, otherwise, go to Step 6A (end of Step 5).

Step 5 is elementary in \( q \) since we have computed \( h(j + 1) \) in Step 4. If the algorithm terminates because \( q \leq \alpha_{j+1} \), we obviously have \( q < \alpha \) and the output is correct. If \( q > \alpha_{j+1} \), the algorithm will not proceed to Step 6A iff \( q < \beta_{j+2} \). So assume that \( \alpha_{j+1} < q < \beta_{j+2} \).

It is a well-known fact that for any natural number \( x \geq 2 \) there is a prime number between \( x \) and \( 2x \). It follows by induction that \( P_y \leq 2y + 1 \) for all \( y \in \mathbb{N} \) and therefore \( b = P_0P_1 \ldots P_n \leq 2^{(n+1)^2} \).

Applying \( n < h(j + 1) \) together with inequality (\( \Pi \)) we obtain

\[
y^{h(j+1)+1} \leq (2(n+1)^2)y^{h(j+1)+1} < 2^{h(j+1)+1} < P_j^{h(j+2)-1}
\]

and thus

\[
\frac{1}{P_j^{h(j+2)-1}} < \frac{1}{y^{h(j+1)+1}}.
\]

This entails

\[
\alpha_{j+1} < \alpha < \beta_{j+2} < \alpha_{j+1} + \frac{1}{y^{h(j+1)+1}},
\]

but \( \alpha_{j+1} \) has length \( h(j + 1) \), so the first \( h(j + 1) \) digits of \( \alpha_{j+1}, \alpha, \beta_{j+2} \) coincide. Moreover, \( h(j + 1) > n \geq s \) (recall that \( s \) is the length of the base-\( b \) expansion of \( q \)). Thus, we have \( q < \alpha \), and the algorithm gives the correct output, namely 0. If the algorithm proceeds with Step 6A, we have \( \beta_{j+2} < q \).

**Step 6A of the algorithm:** Compute the least \( t \) such that \( b^t > (q - \alpha_{j+1})^{-1} \). Search for \( u < (q - b^{-t} - \alpha_{j+1})^{-1}P_{j+2} \) such that \( u = h(j + 2) \). If the search is successful and \( P_{j+2}^u < (q - b^{-t} - \alpha_{j+1})^{-1}P_{j+2} \), give the output \( \beta_{j+2} \) and terminate, otherwise, give the output \( q - b^{-t} \) and terminate (end of Step 6A).

It is easy to see that Step 6A is elementary in \( q \); we can compute \( t \) elementarily in \( q, b \) and \( \alpha_{j+1} \) (and we have already computed \( b \) and \( \alpha_{j+1} \) elementarily in \( q \)). When the execution of the step starts, we have \( \beta_{j+2} < q \) (thus, \( \beta_{j+2} \) will be a correct output, but we do not yet know if we will be able to compute \( \beta_{j+2} \)). If the search for \( u \) is successful, we have \( u = h(j + 2) \). Then, we can compute \( \beta_{j+2} \) elementarily in \( u \), and give \( \beta_{j+2} \) as output. We also know that the search for \( u \) is successful iff \( q - b^{-t} < \beta_{j+2} \). Thus, if the search for \( u \) is not successful, we have \( \alpha < \beta_{j+2} \leq q - b^{-t} < q \), and we can give the correct output \( q - b^{-t} \).

**Step 6B of the algorithm:** Exactly the same as 6A, but replace \( j + 1 \) and \( j + 2 \) by \( j \) and \( j + 1 \), respectively (end of Step 6B).

The argument for correctness of Step 6B is the same as for Step 6A, just replace \( j + 1 \) and \( j + 2 \) by \( j \) and \( j + 1 \), respectively, and note that we have \( \beta_{j+1} < q \) when the execution of the step starts. \( \square \)
Definition 8. A function \( D : \mathbb{Q} \to \{0, 1\} \) is a Dedekind cut of the real number \( \beta \) when \( D(q) = 0 \) iff \( q < \beta \).

Corollary 9. Let \( f \) be any honest function. The Dedekind cut of the real number \( \alpha f \) is elementary.

Proof. By Theorem 7 there is an elementary function \( \tilde{T} \) such that \( \tilde{T}(q) = 0 \) iff \( q < \alpha f \). Let \( D(q) = 0 \) if \( \tilde{T}(q) = 0 \), and let \( D(q) = 1 \) if \( \tilde{T}(q) \neq 0 \). The function \( D \) is elementary since \( \tilde{T} \) is elementary. Moreover, \( D \) is the Dedekind cut of \( \alpha f \). \( \square \)

4 Main Results

Theorem 10. For any subrecursive class \( S \) that is closed under primitive recursive operations, we have

\[(i) \ S_{g\downarrow} \subset \bigcap_b S_{b\downarrow} \quad \text{and} \quad (ii) \ S_{g\uparrow} \subset \bigcap_b S_{b\uparrow}. \]

Proof. The inclusion \( S_{g\uparrow} \subseteq \bigcap_b S_{b\uparrow} \) is trivial. Pick an honest function \( f \) such that \( f \not\in S \). By Theorem 5, we have \( \alpha f \in \bigcap_b S_{b\uparrow} \). By Theorem 6, we have \( \alpha f \not\in S_{g\uparrow} \). This proves that \( S_{g\uparrow} \subset \bigcap_b S_{b\uparrow} \). The proof of (i) is symmetric. \( \square \)

Definition 11. A function \( \hat{T} : \mathbb{Q} \to \mathbb{Q} \) is a trace function from below for the irrational number \( \alpha \) when we have \( q < \hat{T}(q) < \alpha \) for any \( q < \alpha \). A function \( \tilde{T} : \mathbb{Q} \to \mathbb{Q} \) is a trace function from above for the irrational number \( \alpha \) when we have \( \alpha < \tilde{T}(q) < q \) for any \( q > \alpha \). A function \( T : \mathbb{Q} \to \mathbb{Q} \) is a trace function for the irrational number \( \alpha \) when we have \( |\alpha - q| > |\alpha - T(q)| \) for any \( q \).

For any subrecursive class \( S \), let \( S_D \) denote the set of irrational numbers that have a Dedekind cut in \( S \); let \( S_{g\uparrow} \) denote the set of irrational numbers that have a trace function from below in \( S \); let \( S_{T\uparrow} \) denote the set of irrational numbers that have a trace function from above in \( S \); let \( S_T \) denote the set of irrational numbers that have a trace function in \( S \); let \( S_{\|} \) denote the set of irrational numbers that have a continued fraction in \( S \).

It is proved in [4] that we have \( S_{g\downarrow} \cap S_{g\uparrow} = S_T = S_{\|} \) for any \( S \) closed under primitive recursive operations. It is conjectured in [4] that \( S_{g\downarrow} \neq S_{g\uparrow} \). Theorem [13] shows that this conjecture holds. The next theorem will be used as a lemma in the proof of Theorem [13].

Theorem 12. For any subrecursive class \( S \) that is closed under primitive recursive operations, we have

\[(i) \ S_{T\uparrow} \cap S_D = S_{g\uparrow} \quad \text{and} \quad (ii) \ S_{T\downarrow} \cap S_D = S_{g\downarrow}. \]
Proof. We prove (i). The proof of (ii) is symmetric. Let $\beta$ be an irrational number in the interval $(0, 1)$, let $q$ be a rational number number in the same interval, and let $m, n \in \mathbb{N}$ be such that $q = mn^{-1}$. We have $q < \beta$ iff $q \leq \hat{G}^\beta(n, 1)$. Let

$$
\hat{T}(q) = \hat{G}^\beta(n, 1) + \hat{G}^\beta(n, 2) \quad \text{and} \quad D(q) = \begin{cases} 0 & \text{if } q \leq \hat{G}^\beta(n, 1) \\ 1 & \text{otherwise}. \end{cases}
$$

Then, $\hat{T}$ is a trace function from below for $\beta$, and $D$ is the Dedekind cut of $\beta$. This proves the inclusion $S_{\hat{T}} \subseteq S_{\hat{T}} \cap S_D$. The inclusion holds for any $S$ closed under elementary operations (and thus also for any $S$ closed under primitive recursive operations).

Let $D$ be the Dedekind cut of an irrational number $\beta$ in the interval $(0, 1)$, and let $\hat{T}$ be a trace function from below for $\beta$. We will now give an algorithm (that uses $D$ and $\hat{T}$) for computing $\hat{G}^\beta(b, n+1)$ (the computation of $\hat{G}^\beta(b, 0)$ is trivial). We leave it to the reader to check that the algorithm is correct.

**Step 1 of the algorithm:** Compute the rational number $q$ such that

$$
q = \hat{T} \left( \sum_{i=0}^{n} \hat{G}^\beta(b, i) \right)
$$

(end of step 1).

The rational number $q$ can be written in the form

$$
q = \sum_{i=0}^{n} \hat{G}^\beta(b, i) + Db^{-k} + \epsilon \quad (8)
$$

for some base-$b$ digit $D > 0$, some $k \in \mathbb{N}$ and some $\epsilon < b^{-k}$.

**Step 2 of the algorithm:** Compute $k$ such that $(8)$ holds. Use the Dedekind cut $D$ to search for $\ell \leq k$ and $D \in \{1, \ldots, b-1\}$ such that

$$
\sum_{i=0}^{n} \hat{G}^\beta(b, i) + Db^{-\ell} < \beta < \sum_{i=0}^{n} \hat{G}^\beta(b, i) + Db^{-\ell} + b^{-\ell}.
$$

Let $\hat{G}^\beta(b, n+1) = Db^{-\ell}$ (end of step 2).

The algorithm above shows that we can compute $\hat{G}^\beta$ when $\hat{T}$ and $D$ are available. The algorithm is not primitive recursive, but it is easy to see that the algorithm can be reduced to a primitive recursive algorithm. (The algorithm uses $\hat{G}^\beta(b, 0), \hat{G}^\beta(b, 1), \ldots, \hat{G}^\beta(b, n)$ in the computation of $\hat{G}^\beta(b, n+1)$. A primitive recursive algorithm is required to only use $\hat{G}^\beta(b, n)$ in the computation of $\hat{G}^\beta(b, n+1)$. Thus we conclude that $S_{\hat{T}} \cap S_D \subseteq S_{\hat{T}}$ when $S$ is closed under primitive recursive operations.

$\square$
Theorem 13. For any subrecursive class $S$ that is closed under primitive recursive operations, there exist irrational numbers $\alpha$ and $\beta$ such that

$(i)$ $\alpha \in S_{g\downarrow} \setminus S_{g\uparrow}$ and $(ii)$ $\beta \in S_{g\uparrow} \setminus S_{g\downarrow}$.

Proof. Pick an honest function $f$ such that $f \notin S$. We have $\alpha f \in S_{T\downarrow}$ by Theorem 7 and we have $\alpha f \in S_D$ by Corollary 9. By Theorem 12 we have $\alpha f \in S_{g\downarrow}$. By Theorem 6 we have $\alpha f \notin S_{g\uparrow}$. This proves (i). The proof of (ii) is symmetric. □

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