Hadronic Contributions to the Muon Lifetime

Timo van Ritbergen and Robin G. Stuart

Randall Physics Laboratory, University of Michigan
Ann Arbor, MI 48109-1120, USA

Abstract

Hadronic corrections to the muon lifetime are calculated in the Fermi theory in the presence of QED using dispersion relations. The result, after convolution of hadron data with the calculated perturbative kernel is

$$\Delta \Gamma_{\text{had}} = -\Gamma_0 \left(\frac{\alpha}{\pi}\right)^2 0.042$$

where $\Gamma_0$ is the tree-level width. The results are also used to obtain the corrections to the muon lifetime coming from virtual muon and tau loops

$$\Delta \Gamma_{\mu\text{on}} = -\Gamma_0 \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{16987}{576} - \frac{85}{36} \zeta(2) - \frac{64}{3} \zeta(3)\right)$$

$$= -\Gamma_0 \left(\frac{\alpha}{\pi}\right)^2 0.0364333$$

$$\Delta \Gamma_{\tau\text{au}} = -\Gamma_0 \left(\frac{\alpha}{\pi}\right)^2 0.00058.$$
1 Introduction

The Fermi coupling constant, $G_F$, with a current error of $\delta G_F/G_F = 1.7 \times 10^{-5}$ is amongst the best measured constants in electroweak physics and, as such, plays a crucial rôle as input into calculations of electroweak observables. Of the quoted error, $0.9 \times 10^{-5}$ is experimental and $1.5 \times 10^{-5}$ is theoretical. The latter is an estimate of the missing and unknown higher order QED corrections to the formula that relates the muon lifetime, $\tau_\mu \equiv \Gamma^{-1}_\mu$, to $G_F$. The calculation of these missing corrections will not only reduce the present error on $G_F$ by half but it also means that future experimental determinations will be unhindered by theoretical limitations down the level of a few parts in $10^8$. Experiments are under consideration both at Brookhaven National Laboratory and the Rutherford-Appleton Laboratory that could lead to a reduction in the experimental error on the muon lifetime of about a factor of 10. At this level, the experimental error on the muon mass, currently $\delta m_\mu/m_\mu = 3.2 \times 10^{-7}$, begins to become a factor in determining the accuracy with which $G_F$ can be extracted. It is likely, however, that this will also undergo sufficient improvement such that the overall error will still be dominated by that of $\tau_\mu$.

Such an improvement in both theory and experiment would be timely since the mass of the $Z^0$ boson $M_Z$ has been determined with an unexpectedly high accuracy of $\delta M_Z/M_Z = 2.2 \times 10^{-5}$ following the LEP runs at the $Z^0$ peak and further improvement might still be possible. This now approaches the accuracy of $G_F$.

The 1-loop QED contributions to the muon lifetime were first calculated by Kinoshita and Sirlin and by Berman. It is known that the Fermi theory in the presence of QED is finite to first order in the Fermi coupling constant, $G_F$, and to all orders in the electromagnetic coupling constant, $\alpha$. This remarkable fact means that $G_F$ can be defined in a physically unambiguous manner at least up to the point where finite $W$ propagator effects begin to appear.

The missing corrections that affect the extraction of $G_F$ from measurements of the muon lifetime are 2-loop QED corrections to the Fermi theory. To these must be added single and double bremsstrahlung contributions in order to produce infrared finite results. Technical developments in the calculation of multiloop diagrams seem to make the calculation of the full set of such corrections feasible.

In the present paper, corrections arising from the hadronic vacuum polarization of the photon are considered. These form an independent subclass of corrections with no associated bremsstrahlung. Hadronic corrections have been calculated for the anomalous magnetic moment of the muon and initial state corrections in $e^+e^- \to \mu^+\mu^-$ at high energies, by convolution of a perturbative kernel with hadronic data. For these processes, however, the fermions to which the virtual photons are attached have fixed 4-momenta. In the case of muon decay, the electron participates in a phase integration and consequently adds significantly to the complexity of the problem.

The contributions of the type that we study in the present paper were discussed in the context of the full electroweak theory by Sirlin who demonstrated that they
do not produce large logarithms other than those that can be incorporated into the renormalization of the electromagnetic coupling constant, $\alpha$.

2 Notation and Conventions

The calculation is performed using the Euclidean metric with time-like momenta squared being negative. The 4-momentum of the initial-state muon will be denoted $p_\mu$ and that of the outgoing electron by $p'_\mu$. These are used to define the 4-momentum $w_\mu = p_\mu - p'_\mu$. The electron mass will be considered as negligible compared to the muon mass, $m_\mu$, and dropped throughout. This amounts to discarding terms that are suppressed by a further factor $m_e^2/m_\mu^2$ compared to the corrections considered here. The application of dispersion relations leads to introduction of an auxiliary mass that will be denoted, $M$. It is convenient to introduce three non-negative real variables, $\rho$, $t$, and $z$, given by

$$\rho = \frac{m_\pi^2}{m_\mu^2} = 1.61395..., \quad t = -\frac{w^2}{m_\mu^2}, \quad z = \frac{M^2}{m_\mu^2}. $$

where $m_\pi$ is the mass of the neutral pion.

$\Gamma_0$ will denote the tree-level inverse muon lifetime in the limit of vanishing electron mass

$$\Gamma_0 = \frac{G_F m_\mu^5}{192\pi^3}.$$  \hspace{1cm} (1)

As usual the Dirac matrices are denoted $\gamma_\mu$ and $\gamma_{L,R} = (1 \pm \gamma_5)/2$ are the left- and right-hand helicity projection operators respectively.
3 Hadronic Corrections

The calculation of the hadronic corrections to muon decay are greatly facilitated by first performing a Fierz rearrangement of the Fermi contact interaction term in the Lagrangian. That being done the effective Feynman diagrams that must be calculated are shown in Fig.1. Diagram (d) represents the insertion the muon mass counterterm. Since the electron is taken to be massless its mass counterterm vanishes. The shaded blob in Fig.1 represents the subtracted photon self-energy that will be denoted

$$\Pi_{\mu\nu}(q^2) = (q^2\delta_{\mu\nu} - q_\mu q_\nu) \left[ \Pi'_{\gamma\gamma}(q^2) - \Pi'_{\gamma\gamma}(0) \right]. \quad (2)$$

The contribution to $\Pi_{\mu\nu}(q^2)$ arising from leptons can be calculated directly in perturbation theory. For hadrons the vacuum polarization can be related via dispersion relations to the hadronic production cross-section $\sigma_{\text{had}} \equiv \sigma(e^+e^- \rightarrow \text{hadrons})$ taken from experiments. In the diagrams of Fig.1 the insertion of the vacuum polarization in the photon propagator amounts to the replacement

$$\frac{\delta_{\mu\nu}}{q^2 - i\epsilon} \rightarrow \frac{\delta_{\mu\sigma}}{q^2 - i\epsilon} (q^2\delta_{\sigma\tau} - q_\sigma q_\tau) \left[ \Pi'_{\gamma\gamma}(q^2) - \Pi'_{\gamma\gamma}(0) \right] \frac{\delta_{\tau\nu}}{q^2 - i\epsilon}. \quad (3)$$

The terms proportional to $q_\mu q_\nu$ cancel amongst the diagrams (a)–(c). Upon substituting of the dispersion integral representation for the photon vacuum polarization one obtains

$$\frac{\delta_{\mu\nu}}{q^2 - i\epsilon} \rightarrow \frac{\alpha}{3\pi} \int_4^{\infty} \frac{dM^2}{M^2} R(M^2) \frac{\delta_{\mu\nu}}{q^2 + M^2 - i\epsilon} \quad (4)$$

where $R(M^2) \equiv \sigma_{\text{had}}/\sigma_{\text{point}}$. Thus the photon is effectively replaced by a massive vector particle whose mass is subsequently integrated over.

Taken together the diagrams of Fig.1 are finite and may be calculated by standard means. This typically involves the reduction of tensor form factors to scalar integrals using the methods of Passarino and Veltman [8] and extensions [9–11]. In practice the reduction was performed using a Mathematica [12] implementation of the program LERG-I [11] that algebraically reduces tensor form factors to expressions containing only scalar integrals. Further details are to be found in the Appendix.

Replacing the photon, as described above, by a massive vector particle leads to an effective interaction vertex for the $\mu$-$e$ current that takes the form

$$V_\mu(w^2, M^2) = \left( \frac{\alpha}{\pi} \right) \left\{ i\gamma_\mu \gamma_L F_L(w^2, M^2) + p_\mu \gamma_R h_R(w^2, M^2) + p'_\mu \gamma_R h'_R(w^2, M^2) \right\}. \quad (5)$$

For a process of the type under scrutiny here the phase space can be decomposed into a sequence of 2-particle final states [13]

$$dPS(\mu^- \rightarrow e^-\nu_\mu\bar{\nu}_e) \sim dw^2 dPS(\mu^- \rightarrow e^-w) dPS(w \rightarrow \nu_\mu\bar{\nu}_e).$$
Squaring the matrix element now becomes straightforward especially since all outgoing fermions are taken to be massless. One obtains that the change in the inverse lifetime of the muon induced by an effective interaction vertex of the form given in Eq. (5) is

$$\Delta \Gamma(M^2) = \Gamma_0 \left( \frac{\alpha}{\pi} \right) \int_0^1 dt \ 4(1-t)^2 \left\{ (2t+1)F_L + m_\mu(1-t)\frac{h_R + h'_R}{2} \right\}$$

in which

$$\int_0^1 dt \ 4(1-t)^2 \left\{ (2t+1)F_L + m_\mu(1-t)\frac{h_R + h'_R}{2} \right\}$$

$$= - \frac{1}{432} (513 + 2540z - 180z^2 - 144z^3)$$

$$- \frac{1}{72} (120 + 172z - 45z^2 - 12z^3) \ln z$$

$$+ \frac{1}{24} (132 - 62z - 11z^2 + 4z^3) \sqrt{\frac{z}{z-4}} \ln \frac{r_1}{r_2}$$

$$- \frac{1}{6} (6 + 16z - 18z^2 + z^4) \left( \frac{\pi^2}{6} + \ln r_1 \ln r_2 \right)$$

$$+ \frac{z}{6} (12 - 2z - z^2) \sqrt{z(z-4)} \left( \text{Li}_2 \frac{1}{r_1} - \text{Li}_2 \frac{1}{r_2} \right)$$

and the quantities $r_1$ and $r_2$ are defined in the Appendix. The factor $\sqrt{z-4}$ in the denominator of the third term comes from the anomalous threshold of a 2-point function that arises in the mass renormalization of the external muon. The accompanying logarithm also vanishes at $z = 4$ and therefore it does not lead to difficulties.

For large $z$ the right hand side of Eq. (7) can be expanded to give the asymptotic form

$$\frac{1}{z} \left( \frac{41}{50} - \frac{2}{5} \ln z \right) + \frac{1}{z^2} \left( \frac{4321}{432} - \frac{10}{3} \zeta(2) - \frac{97}{36} \ln z \right) + O \left( \frac{\ln z}{z^3} \right)$$

and so will converge when incorporated into dispersion integrals. Eq. (8) was also obtained directly using a large mass expansion technique along the lines of Ref. [14] which provides a stringent check on all stages of the foregoing calculation.

The hadronic correction to the muon inverse lifetime may now be obtained by performing the convolution integral indicated in Eq. (4)

$$\Delta \Gamma_{\text{had}} = \frac{\alpha}{3\pi} \int_0^\infty d\rho \frac{dz}{z} R(m_\mu^2 z) \Delta \Gamma(z)$$

The range of integration can be made finite and all radicals eliminated by the substitution $u = \sqrt{1 - 4z}$ which leads to

$$\Delta \Gamma_{\text{had}} = \Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 du \ R \left( \frac{4m_\mu^2}{1-u^2} \right) \ K(u)$$

\[\text{(10)}\]
where

\[ K(u) = \frac{u}{9(1 - u^2)^4} \left\{ \frac{1}{72} (1423 + 18979u^2 - 11699u^4 + 513u^6) \right. \\
- \frac{2}{3} (85 + 127u^2 - 131u^4 + 15u^6) \ln \frac{1 - u^2}{4} \right. \\
- \frac{(9 - 69u^2 - 37u^4 + 33u^6)}{u} \ln \frac{1 + u}{1 - u} \\
- \frac{2(19 + 180u^2 - 30u^4 - 44u^6 + 3u^8)}{(1 - u^2)} \\
\times \left( \frac{\pi^2}{6} + \ln \frac{1 - u}{2} - \ln \frac{1 + u}{2} \right) + O\left(\frac{1}{(1 - u^2)^2}\right) \\
\left. + \frac{64u(3 + 4u^2 - 3u^4)}{(1 - u^2)} \left( Li_2 \frac{1 + u}{2} - Li_2 \frac{1 - u}{2} \right) \right\}. \tag{11} \]

The expression for \( K(u) \) in Eq. (11) obviously suffers from strong numerical cancellations as \( u \to 1 \). In this region \( K(u) \) is best calculated by series expansion

\[ K(u) = \left\{ \frac{41}{300} + \frac{1}{18} \left( \frac{85993}{7200} - 5\zeta(2) \right) (1 - u) + O((1 - u)^2) \right\} \]
\[ + \frac{1}{15} \ln \frac{1 - u}{2} \left\{ 1 + \frac{341}{144} (1 - u) + O((1 - u)^2) \right\}. \tag{12} \]

The convergence of the integral over hadronic data can be improved by writing Eq. (10) as

\[ \Delta \Gamma_{\text{had}} = \Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 R(\infty) \int_1^{1 - \rho^{-1}} K(u) \, du \]
\[ + \Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 \int_1^{1 - \rho^{-1}} \left\{ R \left( \frac{4m_e^2}{1 - u^2} \right) - R(\infty) \right\} K(u) \, du \tag{13} \]

The first integral on the right hand side of Eq. (13) can be solved exactly but the result involves trilogarithms, \( Li_3 \), with arguments containing radicals. However, its numerical value is well-determined and sufficient for practical purposes

\[ \int_1^{1 - \rho^{-1}} K(u) \, du = -0.0316710. \tag{14} \]

The other integral in Eq. (13) can now be evaluated numerically by using a suitable parameterization of hadronic data. The effect of a narrow resonance of mass \( M_R \) and a decay width to \( e^+ e^- \) of \( \Gamma_{e^+ e^-} \) is taken into account by representing it as a suitably
normalized Dirac delta function. A resonance of this type yields a contribution

$$\Delta \Gamma_R = \Gamma_0 \frac{18}{\pi} \frac{m_\mu^2 \Gamma_{e^+ e^-}}{M_R^3 \sqrt{1 - \frac{4m_\mu^2}{M_R^2}}} K \left( \sqrt{1 - \frac{4m_\mu^2}{M_R^2}} \right).$$  \hspace{1cm} (15)$$

The total hadronic contribution to inverse lifetime of the muon \[ \] then is

$$\Delta \Gamma_{\text{had}} = -\Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 0.042 \hspace{1cm} (16)$$

A direct and immediate spin off of Eq. (9) is that the contributions from the diagrams Fig.1 in which the hadronic vacuum polarization is replaced by a muon or tau loop can be easily obtained. Electron loops will not be considered here as they need to be taken together with real \(e^+ e^-\) production in order to yield a physically meaningful result.

In the case of muons one writes

$$R(m_\mu^2 z) = \left( 1 + \frac{2}{z} \right) \sqrt{1 - \frac{4}{z}}$$ \hspace{1cm} (17)

and setting \(\rho = 1\) the integral in Eq. (10) can be performed exactly giving

$$\Delta \Gamma_{\mu\text{on}} = \Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{16987}{576} \frac{85}{36} \zeta(2) - \frac{64}{3} \zeta(3) \right)$$ \hspace{1cm} (18)

$$= -\Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 0.036433.$$ \hspace{1cm} (19)

For tau leptons, the decoupling theorem predicts that their contribution will be suppressed by a factor of \(m_\mu^2/m_\tau^2\). Numerical evaluation of the integral (10) for the case of tau loops and using \(m_\tau^2/m_\mu^2 = 282.9\) yields

$$\Delta \Gamma_{\tau} = -\Gamma_0 \left( \frac{\alpha}{\pi} \right)^2 0.00058.$$ \hspace{1cm} (20)

which, as expected, is significantly smaller and can be discarded for most practical purposes.

4 Summary and Conclusions

The hadronic contributions to the muon lifetime were calculated in the Fermi model using dispersion relations. These form an independent subclass of the 2-loop QED

\footnote{Using Eqs. \[ \] and \[ \] in conjunction with his own parameterization of the hadronic data Swartz \[ \] obtains \(\Delta \Gamma_{\text{had}} = -\Gamma_0(\alpha/\pi)^2(0.0413 \pm 0.0017)\) in good agreement with our result.}
corrections that cannot be evaluated using perturbative methods. In most processes for which the hadronic contributions are known, the momenta of the external fermions are fixed but for muon decay the calculation is complicated by the fact that the outgoing electron participates in the phase space integration.

As a bonus, an exact expression for the contribution due to muon loops in the photon vacuum polarization and a numerical value for tau loops were also obtained. Both of these could be calculated by other means. The present work does not yield the contribution coming of electron loops as these must be taken together with $e^+e^-$ pair creation in order to produce an infrared finite result.

The size of the hadronic contribution is found to be very small compared to the current experimental error and about 1/8 of that anticipated in the next generation of measurements of the muon lifetime. The hadronic uncertainty, coming from the inclusion of hadronic data, is now safely under control. The hadronic and muon loop contributions taken together constitute roughly a quarter of a standard deviation shift in the value that would be extracted for $G_F$. This is small but nonnegligible. The contribution from tau loops, on the other hand, can be safely ignored.

The full set of 2-loop QED corrections, of which the hadronic corrections form a part, when available will immediately halve the overall error on the value of $G_F$ since they will drive the theoretical uncertainty, currently the dominant error, down to a level of a few parts in $10^8$. This is true regardless of overall size of the corrections.

5 Appendix

Following the notation of Passarino and Veltman [8], the general 2-point scalar integral is defined in dimensional regularization to be

$$B_0(p^2; m_1^2, m_2^2) = \int \frac{d^nq}{i\pi^2} \frac{1}{[q^2 + m_1^2][(q + p)^2 + m_2^2]} \tag{21}$$

and the general 3-point scalar integral is

$$C_0(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2) = \int \frac{d^nq}{i\pi^2} \frac{1}{[q^2 + m_1^2][(q + p_1)^2 + m_2^2][(q + p_1 + p_2)^2 + m_3^2]} \tag{22}$$

where $p_5 = p_1 + p_2$.

Evaluating the scalar integrals by of Feynman parameter methods for the parameters that appear here manifests the quadratic

$$x^2 - zx + z - i\epsilon = 0,$$

for which the roots are

$$r_1 = \frac{z}{2} + \frac{1}{2} \sqrt{z(z - 4)} + i\epsilon, \quad r_2 = \frac{z}{2} - \frac{1}{2} \sqrt{z(z - 4)} - i\epsilon.$$
The 2-point scalar integrals that occur here are
\[
B_0(0; m_\mu^2, m_\mu^2) = \Delta - \ln m_\mu^2 \tag{23}
\]
\[
B_0(0; M^2, M^2) = B_0(0; m_\mu^2, m_\mu^2) - \ln z \tag{24}
\]
\[
B_0(w^2; 0, m_\mu^2) = B_0(0; m_\mu^2, m_\mu^2) + \frac{(1 - t)}{t} \ln(1 - t) + 2 \tag{25}
\]
\[
B_0(-m_\mu^2; m_\mu^2, M^2) = B_0(0; m_\mu^2, m_\mu^2) - \frac{z}{2} \ln z + \frac{1}{2} \sqrt{z(z - 4)} \ln \frac{r_1}{r_2} + 2 \tag{26}
\]
in which \(\Delta\) is a logarithmically divergent constant that arises from the use of dimensional regularization.

The 3-point form factor may be written
\[
C_0(-m_\mu^2, w^2, 0; M^2, m_\mu^2, 0) = \int_0^1 dy \frac{m_\mu^2}{m_\mu^2 y[y - (1 - t)]} \times \left\{ \ln[y^2 - zy + z - i\epsilon] - \ln [(1 - t - z)y + z - i\epsilon] \right\}. \tag{27}
\]
As shown Ref. [16], the integral representation takes this particularly simple form because one of the internal masses, that of the electron, is zero. Integrating the right hand side of Eq. (27) by methods given in Ref. [17] leads to a result involving several dilogarithms, \(\text{Li}_2\).

The integration with respect to \(t\) that is required in the phase space integration of Eq. (6) is somewhat arduous but was facilitated by first obtaining the relation
\[
m_\mu^2 \int_0^1 dt (1 - t)^n C_0(-m_\mu^2, w^2, 0; M^2, m_\mu^2, 0) =
- \frac{1}{n} \left\{ - \int_0^1 dt \frac{t^n - 1}{t - 1} \ln t + \frac{1}{n} \ln z 
+ \frac{1}{2} \ln z \int_0^1 dt \left( \frac{t^{n-1} - r_1^{n-1}}{t - r_1} + \frac{t^{n-1} - r_2^{n-1}}{t - r_2} \right) 
+ \int_0^1 dt \left( \frac{t^n - r_1^n}{t - r_1} + \frac{t^n - r_2^n}{t - r_2} \right) \ln \frac{t}{z} 
+ \left( \frac{r_1^n + r_2^n}{2} - 1 \right) \left( \frac{\pi^2}{6} + \ln r_1 \ln r_2 \right) 
- \frac{1}{2} \ln \frac{r_1}{r_2} \int_0^1 dt \left( r_1 \frac{t^{n-1} - r_1^{n-1}}{t - r_1} - r_2 \frac{t^{n-1} - r_2^{n-1}}{t - r_2} \right) 
+ \frac{r_1^n - r_2^n}{2} \left( \text{Li}_2 \frac{1}{r_1} - \text{Li}_2 \frac{1}{r_2} \right) \right\} \tag{28}
\]
valid for integer \(n > 0\). The integrals that remain involve only simple polynomials and logarithms with positive arguments.
6 Acknowledgements

The authors wish to thank H. Burkhardt and B. A. Kniehl for their parameterization and computer code for hadronic data and M. L. Swartz for providing us with his evaluation of the hadronic contribution and uncertainty. Helpful discussions with R. Akhoury and Y.-P. Yao are also gratefully acknowledged. This work was supported in part by the US Department of Energy.

References

[1] LEP Electroweak Working Group, CERN-PPE/97/154.
[2] T. Kinoshita and A. Sirlin, Phys. Rev. 113 (1959) 1652.
[3] S. M. Berman, Phys. Rev. 112 (1958) 267.
[4] S. M. Berman and A. Sirlin, Ann. Phys. 20 (1962) 20.
[5] M. Gourdin and E. de Rafael, Nucl Phys. B 10 (1969) 667.
[6] B. A. Kniehl, M. Krawczyk, J. H. Kühn and R. G. Stuart, Phys. Lett. B 209 (1988) 337.
[7] A. Sirlin, Phys. Rev. D 29 (1984) 89.
[8] G. Passarino and M. Veltman, Nucl. Phys. B 160 (1979) 151.
[9] R. G. Stuart, Comput. Phys. Commun. 48 (1988) 367.
[10] R. G. Stuart and A. Góngora-T., Comput. Phys. Commun. 56 (1990) 337.
[11] R. G. Stuart, Comput. Phys. Commun. 88 (1995) 267; E ibid 88 (1995) 347.
[12] S. Wolfram, The Mathematica Book, 3rd ed., Wolfram Media/Cambridge University Press, (1996).
[13] A. Czarnecki, M. Jeżabek and J. H. Kühn, Phys. Lett. B 346 (1995) 335.
[14] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. B 438 (1995) 278.
[15] M. L. Swartz, Private Communication.
[16] B. Grządkowski, J. H. Kühn, P. Krawczyk and R. G. Stuart, Nucl. Phys. B 281 (1987) 18.
[17] G. ’t Hooft and M. Veltman, Nucl. Phys. B 153 (1979) 365.