A PENCIL OF ENRIQUES SURFACES OF INDEX ONE WITH NO SECTION

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Abstract. Monodromy arguments and deformation-and-specialization are used to prove existence of a pencil of Enriques surfaces with no section and index 1. The same technique completes the strategy from [GHMS05 §7.3] proving the family of witness curves for dimension $d$ depends on the integer $d$.

1. Introduction

This paper uses monodromy and deformation-and-specialization to answer some questions related to [GHMS05]. Theorem 1.3 gives a new, elementary proof of existence of a pencil of Enriques surfaces over $\mathbb{C}$ with no section, which moreover has index 1. Proposition 1.4 completes the strategy from [GHMS05 §7.3] proving the family $H_d$ of witness curves depends on the relative dimension $d$.

The main theorem of [GHS] proves a rationally connected variety defined over the function field of a curve over a characteristic 0 algebraically closed field has a rational point. A converse is proved in [GHMS05]; in particular [GHMS05 Cor. 1.4] proves there is an Enriques surface without a rational point that is defined over the function field of a curve (answering a question of Serre [CS01, p. 153]). Subsequently Lafon [La04] gave an explicit pencil of Enriques surfaces defined over $\mathbb{Z}[1/2]$ whose base-change to any field of characteristic $\neq 2$ has no rational point. Hélène Esnault asked about the index of Enriques surfaces without a rational point.

Definition 1.1. Let $X$ be a finite type scheme, algebraic space, algebraic stack, etc. over a field $K$. The index and the minimal degree are,

$$I(K,X) = \gcd\{[L:K]|X(L) \neq \emptyset\},$$
$$M(K,X) = \min\{[L:K]|X(L) \neq \emptyset\}.$$

Hélène Esnault asked, essentially, what is the possible index of an Enriques surface defined over a function field of a curve. In Lafon’s example, $M(K,X_K) = I(K,X_K) = 2$. In [GHMS05] the index is not computed, but likely there also $I(K,X_K) > 1$.

Question 1.2 (Esnault). If $X$ is an Enriques surface defined over a function field of a curve $K$ with no $K$-point, is $I(K,X) > 1$?

This has to do with whether there is an obstruction to $K$-points in Galois cohomology. If so and if the obstruction is compatible with restriction and corestriction,
the order of the obstruction divides $I(K, X)$. So if there is a cohomological obstruction “explaining” non-existence of $K$-points, then $I(K, X_K) > 1$. The main result proves there is an Enriques surface with no $K$-point whose index is 1.

**Theorem 1.3.** Let $k$ be an algebraically closed field with $\text{char}(k) \neq 2, 3$ that is “sufficiently big”, e.g. uncountable. There exists a flat, projective $k$-morphism $\pi : X \to \mathbb{P}^1_k$ with the following properties,

(i) the geometric generic fiber of $\pi$ is a smooth Enriques surface,
(ii) the invertible sheaf $\pi_*[\omega_{\pi}^{\otimes 2}]$ has degree 6,
(iii) for the function field $K$ of $\mathbb{P}^1_k$ and the generic fiber $X_K$ of $\pi$, $I(K, X_K) = 1$ and $M(K, X_K) = 3$.

Moreover every “very general” Enriques surface over $k$ is a fiber of such a family.

The method is simple. Over $\mathbb{P}^1$ a family of surfaces is given whose monodromy group acts as the full group of symmetries of the dual graph of the geometric generic fiber – which is the 2-skeleton of a cube. There is an action of $\mathbb{Z}/2\mathbb{Z}$ acting fiberwise, and the quotient is a pencil $X/\mathbb{P}^1$ of “Enriques surfaces”. The 8 vertices of the cube give a degree 4 multi-section of the pencil. The 6 faces of the cube give a degree 3 multi-section of the pencil. By monodromy considerations every multi-section of $X$ has degree $\geq 3$. The pencil $X$ together with the degree 3 and degree 4 multi-sections deforms to a pencil whose geometric generic fiber is a smooth Enriques surface. For a general such deformation, $M(K, X_K) = 3$ and $I(K, X_K) = 1$.

The same method gives pencils of degree $d$ hypersurfaces with minimal degree $d$, which is used to complete the argument from [GHMS05, Section 7.3].

**Proposition 1.4.** Let $B$ be a normal, projective variety of dimension $\geq 2$ and let $M$ be an irreducible family of irreducible curves dominating $B$ (i.e., the morphism from the total space of the family of curves to $B$ is dominant). There is an integer $d$ such that $M$ is not a witness family for dimension $d$, i.e., there is a projective, dominant morphism of relative dimension $d$, $\pi : X \to B$, whose restriction to each curve of $M$ has a section, but whose restriction to some smooth curve in $B$ has no section.

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## 2. The construction for hypersurfaces

Let $d, n > 0$ be integers, let $k$ be a field, and let $V$ be a $k$-vector space of dimension $n + 1$. Degree $d$ hypersurfaces in $\mathbb{P}(V)$ are parametrized by the projective space,

$$\mathbb{P}\text{Sym}^d(V) = \text{Proj} \bigoplus_i \text{Sym}^i(\text{Syt}^d(V)),$$

where $\text{Syt}^d(V)$ is the vector space of symmetric tensors in $\otimes^d V$.

Let $B, C$ be $k$-curves isomorphic to $\mathbb{P}^1_k$. There exists a degree $d$, separably-generated $k$-morphism $f : C \to B$ such that $\text{Gal}(k(C)/k(B))$ is the full symmetric group $S_d$.

This is straightforward in every characteristic – in characteristic 0 any morphism with simple branching will do.
Let $g : C \to \mathbb{P}(V^r)$ be a closed immersion whose image is a rational normal curve of degree $n$. Consider the pullback of the tautological surjection, $V \otimes_k \mathcal{O}_C \to g^* \mathcal{O}(1)$. By adjointness, there is a map $\beta : V \otimes_k \mathcal{O}_B \to f_*(g^* \mathcal{O}(1))$. For every locally free $\mathcal{O}_C$-module $\mathcal{E}$ there is the norm sheaf on $B$,

$$\text{Nm}_f(\mathcal{E}) = \text{Hom}_{\mathcal{O}_B}(\bigwedge^d(f_* \mathcal{O}_C), \bigwedge^d(f_* \mathcal{E})),$$

together with the norm map of $\mathcal{O}_B$-modules,

$$\alpha'_x : \bigotimes^d(f_* \mathcal{E}) \to \text{Nm}_f(\mathcal{E}), \quad e_1 \otimes \cdots \otimes e_d \mapsto (e_1 \wedge \cdots \wedge e_d) \mapsto (c_1 \wedge \cdots \wedge (c_1 \cdot e_1) \wedge \cdots \wedge (c_d \cdot e_d)),$$

for $e_1 \otimes \cdots \otimes e_d \in \bigotimes^d(f_* \mathcal{E})$ and $c_1 \wedge \cdots \wedge c_d \in \bigwedge^d(f_* \mathcal{O}_B)$. Only the restriction to the subsheaf of symmetric tensors is needed, $\alpha_x : \text{Sym}^d(f_* \mathcal{E}) \to \text{Nm}_f(\mathcal{E})$. In particular, $\text{Nm}_f(\mathcal{O}_C) = \mathcal{O}_B$ and $\alpha_x(b \otimes \cdots \otimes b) \in \mathcal{O}_B$ is the usual norm of $b \in f_* \mathcal{O}_C$.

Denote by $\gamma$ the composition,

$$\text{Sym}^d(V) \otimes_k \mathcal{O}_B \xrightarrow{\text{Sym}^d(\beta)} \text{Sym}^d(f_* g^* \mathcal{O}(1)) \xrightarrow{\alpha_x \circ \gamma} \text{Nm}_f(g^* \mathcal{O}(1)).$$

Because $\beta$ is surjective, also $\gamma$ is surjective. So there is an induced morphism $h : B \to \mathbb{P}\text{Sym}^d(V^r)$. For every geometric point $b \in B$ whose fiber $f^{-1}(b)$ is a reduced set $\{c_1, \ldots, c_d\}$, $h(b) = \lfloor g(c_1) \times \cdots \times g(c_d) \rfloor$. The degree of $\text{Nm}_f(g^* \mathcal{O}(1))$, and thus the degree of $h$, is $n$.

Denote by $X_h \subset B \times \mathbb{P}(V)$ the preimage under $(h, \text{Id})$ of the universal hypersurface in $\mathbb{P}\text{Sym}^d(V^r) \times \mathbb{P}(V)$, and by $\pi : X_h \to B$ the projection. Let $m = \min(d, n)$ and let $S_{d, n} \subset \mathbb{Z}_{\geq 0}$ denote the additive semigroup generated by $\binom{d}{i}$ for $i = 1, \ldots, m$. Denote $K = k(B)$ and denote by $X_{h, K}$ the generic fiber of $\pi$.

**Proposition 2.1.** Every irreducible multi-section of $\pi$ has degree divisible by $\binom{d}{i}$ for $i = 1, \ldots, m$. The degree of every multi-section is in $S_{d, n}$. In particular, if $d > n$ then $M(K, X_{h, K}) = d$ and $I(K, X_{h, K})$ is divisible by $\text{gcd}(d, \binom{d}{i}, \ldots, \binom{d}{m})$.

**Proof.** Denote by $U \subset B$ the largest open subset over which $f$ is étale and define $W = f^{-1}(U)$. For each $i = 1, \ldots, m$, denote by $W_i/U$ the relative Hilbert scheme $\text{Hilb}^{(d)}_{W/U}$. Because $W$ is étale over $U$, the fiber of $f$ over a geometric point $b$ of $B$ is a set of $d$ distinct points, $f^{-1}(b) = \{c_1, \ldots, c_d\}$, and the fiber of $\text{Hilb}^{(d)}_{W/U}$ is the set of subsets of $f^{-1}(b)$ of size $i$. Every geometric fiber of $X_h \times_B U \to U$ is union of $d$ hyperplanes. Denote by,

$$X_h \times_B U = X_h^1 \sqcup X_h^2 \sqcup \cdots \sqcup X_h^m,$$

the locally closed stratification where $X_h^i$ is the set of points $x$ in precisely $i$ irreducible components of the geometric fiber $X_h \otimes_{\mathcal{O}_B} \mathbb{k}(\pi(x))$. Because every finite subset of distinct closed points on a rational normal curve over an algebraically closed field is in linearly general position, $X_h^i = \emptyset$ for $i > m$; in particular every geometric fiber of $X_h \times_B U \to U$ is a simple normal crossings variety. For each $i = 1, \ldots, m$ the morphism $X_h^i \to U$ factors as an $\mathbb{k}^{n-1}$-bundle over $W_i$ over $U$.

The generic point of every irreducible multi-section is contained in $X_h^i$ for some $i = 1, \ldots, m$. Because $\text{Gal}(k(C)/k(B))$ is $S_d$, $W_i$ is irreducible. Therefore the degree of the multi-section is divisible by $\deg(k(W_i)/k(U)) = \binom{d}{i}$. So the degree of every multi-section, irreducible or not, is in $S_{d, n}$. Moreover, the intersection of $X_{h, K}$ with a general line in $\mathbb{P}(V \otimes_k K)$ is a degree $d$ multi-section, so $M(K, X_{h, K}) = d$. \hfill $\square$
Let $H_n \subset \text{Hom}(B, \mathbb{P}\text{Sym}^d(V^\vee))$ denote the irreducible component of morphisms of degree $n$. Denote by $X \to H_n \times B$ the pullback by the universal morphism of the universal hypersurface in $\mathbb{P}\text{Sym}^d(V^\vee) \times \mathbb{P}(V)$. For every field $k'$ and every $[h] \in H_n(k')$, denote by $X_h$ the restriction of $X$ to Spec $(k') \times B$, by $K'$ the function field $k'(B)$, and by $X_{h,K'}$ the generic fiber of the projection to $B$.

**Corollary 2.2.** Assume $d > n$. In $H_n$ there is a countable intersection of open dense subsets such that for every $[h]$ in this set, $M(K', X_{h,K'}) = d$ and $I(K', X_{h,K'})$ is divisible by $\gcd(d, \ldots, (d^n))$. In particular this holds for the geometric generic point of $H_n$.

**Proof.** The subset $H_n^\text{good} \subset H_n$ where $M(K', X_{h,K'}) \geq d$ and $\gcd(d, \ldots, (d^n)) | I(K', X_{h,K'})$ is a countable intersection of open subsets by standard Hilbert scheme arguments: the complement of this set is the union over the countably many Hilbert polynomials $P(t)$ of multi-sections of degree $< d$ or not divisible by $\gcd(d, \ldots, (d^n))$ of the closed image in $H_n$ of the relative Hilbert scheme $\text{Hilb}^P_{X/H_n}$. By Proposition 2.1 $H_n^\text{good}$ is nonempty, therefore it is a countable intersection of open dense subsets. Of course the intersection of $X_{h,K'}$ with a general line in $\mathbb{P}(V \otimes_k K')$ gives a multi-section of degree $d$, therefore $H_n^\text{good}$ is actually the set where $M(K', X_{h,K'}) = d$ and $\gcd(d, \ldots, (d^n)) | I(K', X_{h,K'})$.

**2.1. Proof of Proposition 1.4** Let $k$ be an uncountable, algebraically closed field. The main case of Proposition 1.4 is $B = \mathbb{P}^1 \times \mathbb{P}^1$ and $M$ is the complete linear system $|\mathcal{O}(a, b)|$. Assume first that one of $a, b = 0$, say $b = 0$. Let $f : Y \to \mathbb{P}^1_k$ be a finite, separably-generated morphism of irreducible curves of degree $> 1$, and let $X = Y \times \mathbb{P}^1$ with projection $\pi = (f, \text{Id})$. Every divisor in $|\mathcal{O}(a, 0)|$ is a union of fibers of $pr_1$, so the restriction of $\pi$ has a section. The restriction of $\pi$ over every fiber of $pr_2$ is just $f$, and so has no rational section. Thus assume $a, b > 0$.

Define $n = 4ab$ and $d = n - 1$. Let $V$ be a $k$-vector space of dimension $n + 1$. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth curve in the linear system $|\mathcal{O}(1, 2b)|$. By Corollary 2.2 there exists a closed immersion of degree $n$, $h : C \to \mathbb{P}\text{Sym}^d(V^\vee)$, such that $M(k(C), X_{h,k(C)}) = d > 1$. Of course $h$ extends to a closed immersion $j : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}\text{Sym}^d(V^\vee)$ such that $j^*\mathcal{O}(1) = \mathcal{O}(2a - 1, 2b)$; after all, $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2a - 1, 2b)) \to H^0(C, \mathcal{O}_C(n))$ is surjective. Define $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ to be the base-change by $j$ of the universal family of degree $d$ hypersurfaces in $\mathbb{P}(V)$. By construction, the restriction over $C$ has no section.

Every divisor in $|\mathcal{O}(a, b)|$ is a curve in $\mathbb{P}\text{Sym}^d(V^\vee)$ of degree $n - b$ whose span is a linear system of hypersurfaces in $\mathbb{P}(V)$ of (projective) dimension $\leq n - b - (a - 1)(b - 1)$. Since $n - b < n$, this linear system has basepoints giving sections of the restriction of $X$ to the divisor. This proves Proposition 1.4 for $B = \mathbb{P}^1 \times \mathbb{P}^1$ and $M = |\mathcal{O}(a, b)|$.

Let $B$ be a normal, projective variety of dimension $\geq 2$ and let $M$ be an irreducible family of irreducible curves dominating $B$. There exists a smooth open subset $U \subset B$ whose complement has codimension $\geq 2$ and a dominant morphism $g : U \to \mathbb{P}^1 \times \mathbb{P}^1$. Intersecting $U$ with general hyperplanes, there exists an irreducible closed subset $Z \subset U$ such that $g|Z : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is generically finite of some degree $e > 0$. For the geometric generic point of $M$, the intersection of the corresponding curve with $U$ is nonempty, and the closure of the image under $f$ is a divisor in the linear
system $|\mathcal{O}(a',b')|$ for some integers $a', b'$. Let $a \geq a'$, and $b \geq b'$ be integers such that $4ab > e + 1$. There exists a projective, dominant morphism $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ whose restriction over every divisor in $|\mathcal{O}(a,b)|$ has a section, but whose restriction over a general divisor in $|\mathcal{O}(1,2b)|$ has minimal degree $4ab - 1$.

Define $X_B \subset B \times X$ to be the closure of $U \times_{\mathbb{P}^1 \times \mathbb{P}^1} X$. Then $\pi_B : X_B \to B$ is a projective dominant morphism. For the geometric generic point of $M$, the restriction of $\pi_B$ to the curve has a section because the restriction of $\pi$ to the image in $\mathbb{P}^1 \times \mathbb{P}^1$ has a section. Let $C_B \subset Z$ be the preimage of a general curve $C$ in $|\mathcal{O}(1,2b)|$. The morphism $C_B \to C$ has degree $e < 4ab - 1$. Because every multi-section of $\pi$ over $C$ has degree $\geq 4ab - 1$, $\pi_B$ has no section over $C_B$.

3. The construction for Enriques surfaces

Let $k$ be a field of characteristic $\neq 2,3$, and let $V_+$ and $V_-$ be 3-dimensional $k$-vector spaces. Denote $V = V_+ \oplus V_-$ and $V' = \text{Sym}^2(V_+) \oplus \text{Sym}^2(V_-)$. Denote $G = \text{Grass}(3,V')$, parametrizing 3-dimensional subspaces of $V'$. This is a parameter space for Enriques surfaces. There are 2 descriptions of the universal family, each useful. First, let $\pi_Z : Z \to \mathbb{P}(V_+) \times \mathbb{P}(V_-)$ be the projective bundle of the locally free sheaf $\mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(-2)$. A general complete intersection of 3 divisors in $|\mathcal{O}_Z(1)|$ is an Enriques surface. Because $H^0(Z,\mathcal{O}_Z(1)) = V'$, the parameter space for these complete intersections is $G$. Second, $G$ parametrizes complete intersections in $\mathbb{P}(V)$ of 3 quadric divisors that are invariant under the involution $\iota$ of $\mathbb{P}(V)$ whose $(-1)$-eigenspace is $V_-$ and whose $(+1)$-eigenspace is $V_+$. A general such complete intersection is a K3 surface on which $\iota$ acts as a fixed-point-free involution; the quotient by $\iota$ is an Enriques surface. The two descriptions are equivalent: the involution extends to an involution $\tilde{\iota}$ on the blowing up $\mathbb{P}(\tilde{V})$ of $\mathbb{P}(V)$ along $\mathbb{P}(V_+) \cup \mathbb{P}(V_-)$ and the quotient is $Z$. Denote by $\mathcal{X} \to G$ the universal family of Enriques surfaces, and denote by $\mathcal{Y} \to G$ the universal family of K3 covers.

Let $B, C, D$ be $k$-curves isomorphic to $\mathbb{P}^1_k$. There exists a degree 2, separably-generated morphism $g : D \to C$ and a degree 3, separably-generated morphism $f : C \to B$ such that $\text{Gal}(k(D)/k(B))$ is the full wreath product $\mathfrak{S}_3 \wr \mathfrak{S}_3$, i.e., the semidirect product $(\mathfrak{S}_3)^3 \rtimes \mathfrak{S}_3$. In characteristic 0, this holds whenever $g$ and $f$ have simple branching and the branch points of $g$ are in distinct, reduced fibers of $f$. There is an involution $\iota_D$ of $D$ commuting with $g$.

Let $j : D \to \mathbb{P}(V')$ be a closed immersion equivariant for $\iota_D$ and $\iota$ whose image is a rational normal curve of degree 5. By the construction in Section 2 there is an associated morphism $i : C \to \text{PSym}^2(V')$. Because $j$ is equivariant, $i$ factors through $\mathbb{P}(V')$. By a straightforward computation, $i^*\mathcal{O}(1) = \text{Nm}_i(j^*\mathcal{O}(1)) \cong \mathcal{O}_C(5)$. The pushforward by $f_*$ of the pullback by $i^*$ of the tautological surjection is a surjection $(V')^\vee \otimes \mathcal{O}_B \to f_*i^*\mathcal{O}(1)$. The sheaf $f_*i^*\mathcal{O}(1)$ is locally free, in fact $f_*i^*\mathcal{O}(1) \cong f_*\mathcal{O}_C(5) \cong \mathcal{O}_B(1)^3$, so there is an induced morphism $h : B \to G$. Denote by $\pi_h : \mathcal{X}_h \to B$ and $\rho_h : \mathcal{Y}_h \to B$ the base-change by $h$ of $\mathcal{X}$ and $\mathcal{Y}$. Denote $K = k(B)$ and denote by $\mathcal{X}_{h,K}$ the generic fiber of $\pi_h$.

**Proposition 3.1.** Every irreducible multi-section of $\pi_h$ has degree divisible by 3 or 4. In particular $M(K, \mathcal{X}_{h,K}) = 3$.

**Proof.** Denote by $U \subset B$ the open set over which $f \circ g$ is étale, and denote by $W \subset D$ the preimage of $U$. Denote by $c : \tilde{W} \to U$ the Galois closure of $W/U$. Then...
For each $p = 1, 2, 3$ and $q = 1, 2$, denote by $j_{p.q} : \tilde{W} \to \mathbb{P}(V^\vee)$ the morphism obtained by composing the idempotent $b_{p.q}$ on $\tilde{W}$ with the basechange of $j$. In particular, $\iota \circ j_{p.1} = j_{p.2}$. Denote by $\Lambda_{p.q} \subset \tilde{W} \times \mathbb{P}(V)$ the pullback by $(j_{p.q}, \mathrm{Id})$ of the universal hyperplane. Denote by $\mathcal{Y}_{\tilde{W}}$ the base-change to $\tilde{W}$ of $\mathcal{Y}_h$. Then,

$$\mathcal{Y}_{\tilde{W}} = \bigcup_{(q_1,q_2,q_3) \in \{1,2\}^3} (\Lambda_{1,q_1} \cap \Lambda_{2,q_2} \cap \Lambda_{3,q_3}).$$

There is a locally closed stratification,

$$\mathcal{Y}_{\tilde{W}} = \mathcal{Y}_{\tilde{W}}^3 \cup \mathcal{Y}_{\tilde{W}}^4 \cup \mathcal{Y}_{\tilde{W}}^5,$$

where $\mathcal{Y}_{\tilde{W}}^l$ is the set of points lying in the intersection of precisely $l$ of the $\Lambda_{p.q}$.

The stratum $\mathcal{Y}_{\tilde{W}}^3$ is the union of 8 connected, open subsets,

$$\Lambda_{(1,q_1,q_2,q_3)} \subset (\Lambda_{1,q_1} \cap \Lambda_{2,q_2} \cap \Lambda_{3,q_3}),$$

for $q_1, q_2, q_3 \in \{1,2\}$. Each connected component is a dense open subset of a $\mathbb{P}^2$-bundle over $\tilde{W}$. The stratum $\mathcal{Y}_{\tilde{W}}^4$ is the union of 12 connected, open subsets,

$$\Lambda_{(*,q_2,q_3)} \subset (\Lambda_{1,1} \cap \Lambda_{1,2}) \cap \Lambda_{2,q_2} \cap \Lambda_{3,q_3},$$

$$\Lambda_{(q_1,*_q),q_3} \subset \Lambda_{1,q_1} \cap (\Lambda_{2,1} \cap \Lambda_{2,2}) \cap \Lambda_{3,q_3},$$

$$\Lambda_{(q_1,q_2,*_q)} \subset \Lambda_{1,q_1} \cap \Lambda_{2,q_2} \cap (\Lambda_{3,1} \cap \Lambda_{3,2})$$

for $q_1, q_2, q_3 \in \{1,2\}$. Each connected component is a dense open subset of a $\mathbb{P}^1$-bundle over $\tilde{W}$. Finally $\mathcal{Y}_{\tilde{W}}^5$ is the union of 6 connected sets,

$$\Lambda_{(*,*,q_3)} = (\Lambda_{1,1} \cap \Lambda_{1,2}) \cap (\Lambda_{2,1} \cap \Lambda_{2,2}) \cap \Lambda_{3,q_3},$$

$$\Lambda_{(*,q_2,*_q)} = (\Lambda_{1,1} \cap \Lambda_{1,2}) \cap \Lambda_{2,q_2} \cap (\Lambda_{3,1} \cap \Lambda_{3,2}),$$

$$\Lambda_{(q_1,*_q,*_q)} = \Lambda_{1,q_1} \cap (\Lambda_{2,1} \cap \Lambda_{2,2}) \cap (\Lambda_{3,1} \cap \Lambda_{3,2})$$

for $q_1, q_2, q_3 \in \{1,2\}$. Each connected component projects isomorphically to $\tilde{W}$.

There is a bijection between multi-sections of $\mathcal{Y}_h$ over $U$ and Galois invariant multi-sections of $\mathcal{Y}_{\tilde{W}}$ over $\tilde{W}$. An irreducible multi-section of $\mathcal{Y}_h$ determines a multi-section of $\mathcal{Y}_{\tilde{W}}$ contained in a single stratum $\mathcal{Y}_{\tilde{W}}^l$. The action of the Galois group $W_{3,2}$ on the connected components of $\mathcal{Y}_{\tilde{W}}^l$ is the obvious one: in particular, it acts transitively on the set of connected components. So every Galois invariant multi-section in $\mathcal{Y}_{\tilde{W}}^3$ has degree divisible by 8, every Galois invariant multi-section in $\mathcal{Y}_{\tilde{W}}^4$ has degree divisible by 12, and every Galois invariant multi-section in $\mathcal{Y}_{\tilde{W}}^5$ has degree divisible by 6. Therefore every irreducible multi-section of $\mathcal{Y}_h$ has degree divisible by 8 or 6. Because $\mathcal{Y}_h$ is a double-cover of $\mathcal{X}_h$, every irreducible multi-section of $\mathcal{X}_h$ has degree divisible by 4 or 3. In particular, the minimal degree of a multi-section of $\mathcal{X}_h$ is 3. □
Because \( f_*i^*\mathcal{O}(1) \cong \mathcal{O}_B(1)^3 \), the scheme \( X_h \subset B \times Z \) is a complete intersection of 3 divisors in the linear system \([\text{pr}_1^*\mathcal{O}_B(1) \otimes \text{pr}_2^*\mathcal{O}_Z(1)]\). A general deformation of this complete intersection is a pencil of Enriques surfaces satisfying Theorem 13 (ii) and (ii) with \( M(K, X_K) \geq 3, I(K, X_K) \mid 4 \) (this is valid so long as \( \text{char}(k) \neq 2 \)). For (iii), it is necessary to deform the pencil together with the degree 3 multi-section. This requires a bit more work, and the hypothesis \( \text{char}(k) \neq 2, 3 \).

The stratum \( Y^3_W \) is Galois invariant and determines a degree 3 multi-section of \( X_h \). As a \( \mathfrak{M}_{3,2} \)-equivariant morphism to \( \tilde{W} \), \( Y^3_W \) is just the base-change of \( D \), and the morphism \( Y^3_W \to \mathbb{P}(V) \) is Galois invariant. By étale descent it is the base-change of a morphism \( j' : D \to \mathbb{P}(V) \). Now \( j' \) induces a morphism to \( \overline{\mathbb{P}(V)} \), the blowing up of \( \mathbb{P}(V) \) along \( \mathbb{P}(V^+) \cup \mathbb{P}(V^-) \). Because \( j' \) is equivariant for \( \iota \) and \( \iota_D \), the quotient morphism \( D \to Z \) factors through \( C \), i.e., there is an induced morphism \( \iota' : C \to Z \). By a straightforward enumerative geometry computation, \( j' \) has degree 5 with respect to \( O_{\mathbb{P}(V)}(1) \). Therefore \( \iota' \) has degree 5 with respect to \( O_Z(1) \). The degree 3 multi-section of \( X_h \) is the image of \( (f, \iota') : C \to B \times Z \).

**Lemma 3.2.** If \( f, g \) and \( j \) are general, then \((\iota')^* : H^0(Z, O_Z(1)) \to H^0(C, O_C(5))\) is surjective.

**Proof.** The condition that \((\iota')^* \) is surjective is an open condition in families, hence it suffices to verify \((\iota')^* \) is surjective for a single choice of \( f, g \) and \( j \) – even one for which \( \text{Gal}(k(D)/k(B)) \) is not \( \mathfrak{M}_{3,2} \). Choose homogeneous coordinates \([S_0, S_1]\) on \( D, [T_0, T_1] \) on \( C \) and \([U_0, U_1]\) on \( B \). Define \( g([S_0, S_1]) = [S_0^2, S_1^2] \) and \( f([T_0, T_1]) = [T_0^3, T_1^3] \). Denote by \( \mu_6 \) the group scheme of 6th roots of unity. There is an action of \( \mu_6 \) on \( D \) by \( \zeta \cdot [S_0, S_1] = [S_0, \zeta S_1] \). This identifies \( \mu_6 \) with \( \text{Gal}(k(D)/k(B)) \).

Let \( e_{+0}, e_{+1}, e_{+2} \) and \( e_{-0}, e_{-1}, e_{-2} \) be ordered bases of \( V_+ \) and \( V_- \) respectively, and let \( X_{+0}, X_{+1}, X_{+2} \) and \( X_{-0}, X_{-1}, X_{-2} \) be the dual ordered bases of \( V'_+ \) and \( V'_- \) respectively. There is an action of \( \mu_6 \) on \( V'_+ \) by,

\[
\zeta \cdot [X_{+0}, X_{+1}, X_{+2}, X_{-0}, X_{-1}, X_{-2}] = [X_{+0}, \zeta X_{+1}, \zeta^2 X_{+2}, \zeta X_{-0}, \zeta^3 X_{-1}, \zeta^3 X_{-2}]
\]

and a dual action on \( V'_- \). Define \( j : D \to \mathbb{P}(V) \) with respect to the ordered basis \( e_{+0}, \ldots, e_{-2} \), to be the \( \mu_6 \)-equivariant morphism,

\[
j([S_0, S_1]) = [S_0^5, S_0^4 S_1, S_0 S_1^4, S_0^3 S_1, S_0^2 S_1^2, S_1^5].
\]

In this case \( U = D_+(U_0 U_1) \subset B \) and \( W = W = D_+(S_0 S_1) \subset C \). It is straightforward to compute \( j' \) with respect to the dual ordered basis \( X_{+0}, \ldots, X_{-2} \),

\[
j'([S_0, S_1]) = [S_1^5, S_0^4 S_1^4, S_0 S_1^4, S_0^3 S_1, S_0^2 S_1^2, S_0^5].
\]

As a double-check, observe this is \( \mu_6 \)-equivariant. The induced map \((j')^* \) is,

\[
\begin{align*}
X_{+0}X_{+0} & \mapsto T_0^3, & X_{+0}X_{+1} & \mapsto T_0 T_1^2, & X_{+0}X_{+2} & \mapsto T_0^2 T_1^3, \\
X_{+1}X_{+1} & \mapsto T_0^2 T_1^3, & X_{+1}X_{+2} & \mapsto T_0^2 T_1^2, & X_{+2}X_{+2} & \mapsto T_0^3 T_1^3, \\
X_{-0}X_{-0} & \mapsto T_0 T_1^3, & X_{-0}X_{-1} & \mapsto T_0^2 T_1^3, & X_{-0}X_{-2} & \mapsto T_0^3 T_1^2, \\
X_{-1}X_{-1} & \mapsto T_0^2 T_1^2, & X_{-1}X_{-2} & \mapsto T_0 T_1, & X_{-2}X_{-2} & \mapsto T_0^5.
\end{align*}
\]

This is surjective by inspection. \( \square \)
3.1. Proof of Theorem 1.3

The subvariety $X_b \subset B \times Z$ is a complete intersection of 3 divisors in the linear system $|\text{pr}_B^*O_B(1) \otimes \text{pr}_Z^*O_Z(1)|$, each containing $(f,i')(C)$. Denote by $I$ the ideal sheaf of $(f,i')(C) \subset B \times Z$, and denote $I = H^0(B \times Z, I \otimes \text{pr}_B^*O_B(1) \otimes \text{pr}_Z^*O_Z(1))$. The projective space of $I$ is the linear system of divisors on $B \times Z$ in the linear system $|\text{pr}_B^*O_B(1) \otimes \text{pr}_Z^*O_Z(1)|$ that contain $(f,i')(C)$. The Grassmannian $G' = \text{Grass}(3,I)$ is the parameter space for deformations of $X_b$ that contain $(f,i')(C)$. For the same reason as in Corollary 2.2 in $G'$ there is a countable intersection of dense open subsets parametrizing subvarieties $X' \subset B \times Z$ with $M(K,X'_b) \geq 3$ and $I(K,X'_b) | 4$. By construction, $X'$ contains the degree 3 multi-section $(f,i')(C)$. Therefore $M(K,X'_b) = 3$ and $I(K,X'_b) = 1$. It is straightforward to compute $\text{pr}_B \ast [\omega_{X'/B}] \cong O_B(6)$. So to prove the theorem, it suffices to prove every “very general” Enriques surface occurs as a fiber of some $X'$, i.e., for a general $[X] \in G$, $X$ occurs as $\text{pr}_Z(X' \cap \pi_B^{-1}(b))$ for some choice of $f,g,i$ and $b \in B$.

A general 0-dimensional, length 3 subscheme of $Z$ occurs as $i'(f^{-1}(b))$ for some choice of $f, g, i$ and $b \in B$. So for a general Enriques surface $[X] \in G$ and a general choice of 0-dimensional, length 3 subscheme of $X$, $X$ is a complete intersection of 3 divisors in the linear system $|O_Z(1)|$ containing $i'(f^{-1}(b))$ for some choice of $f, g, i$ and $b$. To prove that a general $[X] \in G$ is the fiber over $b$ of $X'$ for some $f, g, i$ and $[X'] \in G'$, it suffices to prove every divisor in the linear system $|O_Z(1)|$ containing $i'(f^{-1}(b))$ is the fiber over $b$ of a divisor in the linear system $|I \otimes O_B(1) \otimes O_Z(1)|$.

There is a short exact sequence,

$$0 \rightarrow I \otimes \text{pr}_Z^*O_Z(1) \rightarrow \text{pr}_Z^*O_Z(1) \rightarrow \text{pr}_Z^*O_Z(1)|_C \rightarrow 0,$$

giving a short exact sequence,

$$0 \rightarrow \text{pr}_{B,*}(I \otimes \text{pr}_Z^*O_Z(1)) \rightarrow \text{pr}_{B,*}\text{pr}_Z^*O_Z(1) \rightarrow \text{pr}_{B,*}(\text{pr}_Z^*O_Z(1)|_C) \rightarrow 0.$$  

Because $(i')^*$ is surjective, $\text{pr}_{B,*}(I \otimes \text{pr}_Z^*O_Z(1))$ is a locally free sheaf with $h^1 = 0$. So it is $\cong O_B \oplus O_B(-1)^3$. Twisting by $O_B(1)$, $\text{pr}_{B,*}(I \otimes \text{pr}_Z^*O_B(1) \otimes \text{pr}_Z^*O_Z(1))$ is generated by global sections. Therefore every divisor on $Z$ in the linear system $|O_Z(1)|$ containing the scheme $i'(f^{-1}(b))$ is the fiber over $b$ of a divisor on $B \times Z$ in the linear system $|I \otimes \text{pr}_B^*O_B(1) \otimes \text{pr}_Z^*O_Z(1)|$.

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