A Generalization of the “Raboter” operation.

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1 Introduction

In a recent talk at Rutgers’ Experimental Math Seminar, Neil Sloane described the “raboter” operation for the base two representation of a number \([1]\). From this representation, one reduces by one the length of each run of consecutive 1s and 0s. Denote this operation by \(r(n)\); so, for example, \(r(12) = 2\) because 12 is represented in binary as 1100, and reducing the length of each run by one yields 10.

Sloane also defined \(L(k) = \sum_{n=2^k}^{2^{k+1}-1} r(n)\) and conjectured that \(L(k) = 2 \cdot 3^{k-1} - 2^{k-1}\), a fact which was quickly proven by Doron Zeilberger \([3]\) and Chai Wah Wu \([2]\).

In Section 2, we generalize this theorem to bases other than 2. Let \(r(b, n)\) be the number whose base-\(b\) representation is generated by taking the base-\(b\) representation of \(n\) and shortening each run of consecutive identical elements by one. Further, let \(L(b, n) = \sum_{n=b^k}^{b^{k+1}-1} r(b, n)\). We will prove that

\[
L(b, k) = \frac{b(b-1)}{2b-1}(2b-1)^k - \frac{b-1}{2}b^k.
\]

In Section 3, we raise \(r(b, n)\) to various powers. Define \(L(p, b, k) = \sum_{n=b^k}^{b^{k+1}-1} r(b, n)^p\); we develop an algorithm in Maple to rigorously compute \(L(p, b, k)\) as an expression in terms of \(k\) for any fixed \(p, b\). In addition, for any fixed \(p\), we can conjecture an expression for \(L(p, b, k)\) in terms of \(b\) and \(k\).

2 More General Bases

Following the example of Zeilberger, we find a recurrence satisfied by \(L(b, k)\) and then find a closed form expression satisfying the same recurrence.

Theorem 2.1. \(L(b, k) = (2b - 1) \cdot L(b, k - 1) + b^{k-1} \frac{(b-1)^2}{2}\) for \(k \geq 2\).

Proof. There are \(b^{k+1} - b^k = b^k(b - 1)\) numbers which contribute to \(L(b, k)\) are exactly those numbers whose base-\(b\) representations use \(k+1\) digits, so each can be written as \(Ab_1b_2\) where \(A \in \{1, \ldots, b - 1\} \times \{0, \ldots, b - 1\}^{k-2}\) and \(b_1, b_2 \in \{0, \ldots, b - 1\}\). If \(b_1 \neq b_2\), then \(b_2\) is a run of just one element, so the raboter operation eliminates it and \(r(b, Ab_1b_2) = r(b, Ab_1)\). Numbers with representations \(Ab_1\) are exactly those which were counted in the calculation of \(L(b, k - 1)\), and each is counted \(b - 1\) times here, once for each \(b_2 \neq b_1\).
If \( b_2 = b_1 \), then the base-\( b \) representation of \( r(b, Ab_1 b_2) \) is the representation of \( r(b, Ab_1) \) with \( b_2 \) appended to the end, and so \( r(b, Ab_1 b_2) = b \cdot r(b, Ab_1) + b_2 \). Thus,

\[
L(b, k) = \sum_A \sum_{b_1} r(b, Ab_1 b_1) + \sum_{b_2 \neq b_1} r(b, Ab_1 b_2)
= \sum_A \sum_{b_1} b \cdot r(b, Ab_1) + b_1 + \sum_{b_2 \neq b_1} r(b, Ab_1)
= \sum_A \sum_{b_1} (2b - 1)r(b, Ab_1) + \sum_A \sum_{b_1} b_1
= (2b - 1)L(b, k - 1) + (b - 1)b^{k-2}\frac{b(b - 1)}{2}
= (2b - 1)L(b, k - 1) + b^{k-1}(b - 1)^2\frac{1}{2}.
\]

Together with initial condition \( L(b, 1) = \frac{b(b - 1)}{2} \), this determines the sequence \( (L(b, k))_{k=1}^{\infty} \). Finding an explicit formula for \( L(b, k) \) is now just a matter of finding a formula which obeys this same recurrence.

**Corollary 2.2.** \( L(b, k) = \frac{b(b-1)}{2b} (2b - 1)^k - \frac{b-1}{2} b^k \).

**Proof.** With some help from Doron Zeilberger’s Maple package Cfinite, we conjecture that the formula for \( L(b, k) \) has the form \( \alpha_1 (2b - 1)^k + \alpha_2 b^k \), so we solve the system of equations

\[
\begin{align*}
\alpha_1 (2b - 1) + \alpha_2 b &= \frac{b(b - 1)}{2} \\
\alpha_1 (2b - 1)^2 + \alpha_2 b^2 &= (2b - 1)\frac{b(b - 1)}{2} + b\frac{(b - 1)^2}{2}
\end{align*}
\]

for \( \alpha_1, \alpha_2 \) and find \( \alpha_1 = \frac{b(b - 1)}{2b - 1} \) and \( \alpha_2 = -\frac{b}{2} + \frac{1}{2} \). Let \( L'(b, k) = \frac{b(b - 1)}{2b - 1} (2b - 1)^k - \frac{b - 1}{2} b^k \). Proving that \( L(b, k) = L'(b, k) \) is simply a matter of verifying that \( L'(b, 1) = \frac{b(b - 1)}{2} \) and \( L'(b, k) = (2b - 1) \cdot L'(b, k - 1) + b^{k-1}(b - 1)^2 \frac{1}{2} \) for \( k \geq 2 \), which can easily be done using Maple or any other computer algebra system.

**3 Higher Moments**

With a formula for \( L(b, k) \) found, we consider the following additional generalization:

\[
L(p, b, k) = \sum_{n=2^k}^{2^{k+1} - 1} r(b, n)^p;
\]

that is the sum of \( r(b, n)^p \) taken over all numbers \( n \) whose base-\( b \) representation has \( k + 1 \)-digits. The trick in this case is to work inductively beginning with the (solved) \( p = 1 \) case,
and, along the way compute \( L(l, p, b, k) \) which we define to be the sum of \( r(b, n)p \) taken over all numbers \( n \) whose base-\( b \) representation has \( k + 1 \)-digits, the last of which is \( l \).

In order to compute \( L(l, p, b, k) \), we use the following recurrence:

**Theorem 3.1.** \( L(l, p, b, k) = (b^p - 1) \cdot L(l, p, b, k - 1) + L(p, b, k - 1) + \sum_{i=1}^{p} \binom{p}{i} b^{p-i} (\binom{p}{i}) L(l, p - i, b, k - 1) \).

**Proof.** The numbers with length-\((k + 1)\) base-\( b \) representations ending in \( l \) are exactly those which can be written as \( Ab_1b_2 \) with \( A \in \{1, \ldots , b - 1\} \times \{0, \ldots , b - 1\}^{k-2}, b_1 \in \{0, \ldots , b - 1\} \), and \( b_2 = l \). Therefore,

\[
L(l, p, b, k) = \sum_{A} \left( \sum_{b_1 \neq l} r(b, Ab_1)p + r(b, All)^p \right)
= \sum_{A} \left( \sum_{b_1 \neq l} r(b, Ab_1)p + (b \cdot r(b, Al) + l)^p \right)
= L(p, b, k - 1) - L(l, p, b, k - 1) + \sum_{A} \sum_{i=0}^{p} \binom{p}{i} b^{p-i}r(b, Al)^{p-i}l^i
= L(p, b, k - 1) - L(l, p, b, k - 1) + b^p L(l, p, b, k - 1)
+ \sum_{i=1}^{p} \binom{p}{i} b^{p-i} L(l, p - i, b, k - 1)^{p-i}l^i
= (b^p - 1) \cdot L(l, p, b, k - 1) + L(p, b, k - 1) + \sum_{i=1}^{p} l^i b^{p-i} \binom{p}{i} L(l, p - i, b, k - 1).
\]

We find a similar recurrence for \( L(p, b, k) \).

**Theorem 3.2.** \( L(p, b, k) = (b^p + b - 1)L(p, b, k - 1) + \sum_{i=0}^{b-1} \sum_{i=1}^{p} b^{p-i} \binom{p}{i} L(l, p - i, b, k - 1) \).

**Proof.** Again, note that the numbers counted by \( L(p, b, k) \) are those which can be written as \( Ab_1b_2 \) with \( A \in \{1, \ldots , b - 1\} \times \{0, \ldots , b - 1\}^{k-2}, b_1 \in \{0, \ldots , b - 1\} \). Therefore, the following equations hold:

\[
L(p, b, k) = \sum_{A} \left( \sum_{b_1 \neq b_2} r(b, Ab_1b_2)p + \sum_{b_1} r(b, Ab_1b_1)p \right)
= \sum_{A} (b - 1) \sum_{b_1} r(b, Ab_1)p + \sum_{b_1} \sum_{b_1} (br(Ab_1) + b_1)^p
= (b - 1) L(p, b, k - 1) + \sum_{A} \sum_{b_1} \sum_{i=0}^{p} \binom{p}{i} b^{p-i} r(\text{Ab}_1)^{p-i}b_1^i
= (b - 1) L(p, b, k - 1) + \sum_{A} \sum_{b_1} b_1^p r(\text{Ab}_1)^p + \sum_{b_1} \sum_{i=0}^{p} \sum_{b_1} \binom{p}{i} b^{p-i} r(\text{Ab}_1)^{p-i}b_1^i
= (b^p + b - 1) L(p, b, k - 1) + \sum_{b_1} \sum_{i=1}^{p} b_1^i b^{p-i} \binom{p}{i} L(b_1, p - i, b, k - 1).
\]
Change the name of \( b_1 \) to \( l \) to maintain consistent notation, and we have derived the claimed equation.

4 Maple Implementation

The Maple package `raboter.txt` available at [http://sites.math.rutgers.edu/~yb165/raboter.txt](http://sites.math.rutgers.edu/~yb165/raboter.txt) contains functions to implement this recurrence. The most important are `SumPowers(b,k,p)` which finds an expression in terms of \( k \) for \( L(p, b, k) \) (for fixed \( b \) and \( p \)) and `GuessGeneralForm(b,n,p)` which conjectures an expression in terms of \( k \) and \( b \) for \( L(p, b, k) \) (for fixed \( p \)).

For example, this package proves that

\[
L(2, 2, k) = \frac{2}{3} 5^k - \frac{1}{6} 2^k - \frac{2}{3} 3^k
\]

and conjectures that

\[
L(2, b, k) = \left(\frac{1}{6} b^2 - \frac{1}{6} b - \frac{1}{3}\right) (b - 1)^k + \left( -\frac{1}{6} b^2 + \frac{1}{3} b - \frac{1}{6}\right) b^k
- \frac{b(b - 1)}{2b - 1} (2b - 1)^k + \frac{2b^3 + 3b^2 - 3b - 2}{6(b^2 + b - 1)} (b^2 + b - 1)^k.
\]

References

[1] N.J.A. Sloane, *Coordination Sequences, Planing Numbers, and Other Recent Sequences*, talk given in Rutgers University Experimental Mathematics Seminar, Nov. 15, 2018. Video part 1: [https://vimeo.com/301216222](https://vimeo.com/301216222); video part 2: [https://vimeo.com/301219515](https://vimeo.com/301219515); slides: [http://sites.math.rutgers.edu/~my237/expmath/EMNov2018.pdf](http://sites.math.rutgers.edu/~my237/expmath/EMNov2018.pdf).

[2] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, Sequence A318921, [http://oeis.org/A318921](http://oeis.org/A318921).

[3] Doron Zeilberger, “Proof of a Conjecture of Neil Sloane Concerning Claude Lenormand’s “Raboter” Operation (OEIS sequence A318921)” *The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger* (2018). [http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/rabot.pdf](http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/rabot.pdf).