ON FINITELY GENERATED CLOSURES IN
THE THEORY OF CUTTING PLANES

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Abstract
Let $P$ be a rational polyhedron in $\mathbb{R}^d$ and let $\mathcal{L}$ be a class of $d$-dimensional maximal lattice-free rational polyhedra in $\mathbb{R}^d$. For $L \in \mathcal{L}$ by $R_L(P)$ we denote the convex hull of points belonging to $P$ but not to the interior of $L$. Andersen, Louveaux and Weismantel showed that if the so-called max-facet-width of all $L \in \mathcal{L}$ is bounded from above by a constant independent of $L$, then $\bigcap_{L \in \mathcal{L}} R_L(P)$ is a rational polyhedron. We give a short proof of a generalization of this result. We also give a characterization for the boundedness of the max-facet-width on $\mathcal{L}$. The presented results are motivated by applications in cutting-plane theory from mixed-integer optimization.

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1 Introduction
We use standard background from convex geometry; see, for example, [18, Chapter 1] and [20, Part III]. Let $d \in \mathbb{N}$. By $o$ we denote the origin of $\mathbb{R}^d$. The standard scalar product of $\mathbb{R}^d$ is denoted by $\langle \cdot , \cdot \rangle$. For $n \in \mathbb{N}$ we use the notation $[n] := \{1, \ldots , n\}$. Let $L$ be a $d$-dimensional polyhedron in $\mathbb{R}^d$. We introduce the functional $R_L$ by

$$R_L(X) := \text{conv}(X \setminus \text{int}(L)),$$

where ‘conv’ and ‘int’ stand for the convex hull and the interior, respectively, and $X \subseteq \mathbb{R}^d$. Assume that the polyhedron $L$ is rational. If $L \neq \mathbb{R}^d$ and the recession cone of $L$ is a linear space, then by $m(L)$ we denote the minimal value $m \in \mathbb{N}$ such that $L$ can be given by

$$L = \{x \in \mathbb{R}^d : b_i - m \leq \langle a_i , x \rangle \leq b_i \quad \forall i \in [n]\},$$

where

$$n \in \mathbb{N}, \ a_1 , \ldots , a_n \in \mathbb{Z}^d \setminus \{o\}, \ b_1 , \ldots , b_n \in \mathbb{Z}.$$ (1)

If $L = \mathbb{R}^d$ or the recession cone of $L$ is not a linear space, let $m(L) := +\infty$. With some further restrictions on $L$, the authors of [1] use the term max-facet-width to refer to $m(L)$. It is not difficult to show that for $m(L) < +\infty$ the functional $R_L$ maps rational polyhedra to rational polyhedra. For a family $\mathcal{L}$ of $d$-dimensional rational polyhedra in $\mathbb{R}^d$ we define

$$m(\mathcal{L}) := \sup_{L \in \mathcal{L}} m(L).$$ (2)

As an example to [3], consider $\mathcal{L}$ consisting of all split sets $L \subseteq \mathbb{R}^d$, i.e., sets of the form $L = \{x \in \mathbb{R}^d : i - 1 \leq \langle a , x \rangle \leq i\}$ with $a \in \mathbb{Z}^d \setminus \{o\}$ and $i \in \mathbb{Z}$. For such $\mathcal{L}$ one has $m(\mathcal{L}) = 1$. In this note we present two theorems, which are motivated by [1]. Our first theorem is a strengthening of the main result from [1] Theorem 4.3.

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Theorem 1.1. Let $P$ be a rational polyhedron in $\mathbb{R}^d$ and let $\mathcal{L}$ be a family of $d$-dimensional rational polyhedra in $\mathbb{R}^d$ satisfying $m(\mathcal{L}) < +\infty$. Then there exists a finite subfamily $\mathcal{L}'$ of $\mathcal{L}$ such that every $L \in \mathcal{L}$ satisfies $R_{L'}(P) \subseteq R_L(P)$ for some $L' \in \mathcal{L}'$.

Regarding Theorem 1.1 our contribution is not so much the theorem itself as its short self-contained proof. Note that the complete proof of the corresponding Theorem 4.3 from [1, §2-4] occupies nearly 18 pages. Our proof of Theorem 1.1 employs basic facts from convex geometry and the well-known Gordan-Dickson lemma.

In order to explain the relation of Theorem 1.1 to mixed-integer optimization we need several further notions. A subset $L$ of $\mathbb{R}^d$ is called lattice-free if $L$ is a $d$-dimensional closed convex set and $\text{int}(L) \cap \mathbb{Z}^d = \emptyset$. Furthermore, we call $L$ maximal lattice-free if $L$ is a lattice-free set which is not properly contained in another lattice-free set. Given a polyhedron $P$ in $\mathbb{R}^d$ and a family $\mathcal{L}$ of $d$-dimensional polyhedra in $\mathbb{R}^d$, we call a closed halfspace $H$ an $\mathcal{L}$-cut for $P$ if $H \supseteq P \setminus \text{int}(L)$ for some $L \in \mathcal{L}$. If $\mathcal{L}$ consists of lattice-free sets, one obviously has $P \cap \mathbb{Z}^d = P \cap H \cap \mathbb{Z}^d$ for every $\mathcal{L}$-cut $H$ for $P$. The latter property is used by cutting-plane methods for solving integer and mixed-integer programs; for more details see [1, 5, 11, 14].

In contrast to this, for $\mathcal{L}$ consisting of maximal lattice-free sets, one has $P \cap \mathbb{Z}^d = P \cap \mathcal{L} \cap \mathbb{Z}^d$ for every $\mathcal{L}$-cut $\mathcal{L}$ for $P$. The latter property is used by cutting-plane methods for solving integer and mixed-integer programs; for more details see [1, 5, 11, 14].

As a consequence of Theorem 1.1 we obtain.

Corollary 1.2. Let $P$ and $\mathcal{L}$ be as in Theorem 1.1. Then there exists a finite family $\mathcal{H}$ of $\mathcal{L}$-cuts such that each $H \in \mathcal{H}$ is a rational halfspace and $\bigcap_{L \in \mathcal{L}} R_L(P) = \bigcap \mathcal{H}$. In particular, $\bigcap_{L \in \mathcal{L}} R_L(P)$ is a rational polyhedron.

In the terminology of the cutting-plane theory the operation $P \rightarrow \bigcap_{L \in \mathcal{L}} R_L(P)$ from Corollary 1.2 is referred to as the closure (associated to the family of all $\mathcal{L}$-cuts). Direct application of Corollary 1.2 yields the results on polyhedrality of the Chvátal-Gomory closure and the split closure of a rational polyhedron (see [9, 12, 19]). We also refer to two remarkable polyhedrality results of a somewhat different nature: the result from [8] on polyhedrality of the so-called triangle closure and the result from [13] on polyhedrality of the Chvátal-Gomory closure of an arbitrary compact convex set.

Since existing cutting-plane methods are based on lattice-free sets and since maximal lattice-free sets generate the strongest cuts within the family of all lattice-free sets, the study of families $\mathcal{L}$ consisting of maximal lattice-free sets is of particular importance (see also [3, 4, 17] for related recent results). For such families $\mathcal{L}$ in certain situations the assumption of Theorem 1.1 can be reformulated in an equivalent form. This is provided by our next theorem. Two sets $X,Y \subseteq \mathbb{R}^d$ are called $\mathbb{Z}^d$-equivalent if $Y = U(X) + b$, for some $d \times d$ unimodular matrix $U$ and a vector $b \in \mathbb{Z}^d$. A family $\mathcal{X}$ of subsets of $\mathbb{R}^d$ is called finite up to $\mathbb{Z}^d$-equivalence if there exist finitely many sets $X_1, \ldots, X_t$ ($t \in \mathbb{N}$) in $\mathbb{R}^d$ such that each $X \in \mathcal{X}$ is $\mathbb{Z}^d$-equivalent to some $X_i$ for $i \in [t]$.

Theorem 1.3. Let $\mathcal{L}$ be a family of maximal lattice-free rational polyhedra in $\mathbb{R}^d$ such that $\dim(\text{conv}(L \cap \mathbb{Z}^d)) = d$ for every $L \in \mathcal{L}$. Then the following conditions are equivalent:

(i) $m(\mathcal{L}) < +\infty$;

(ii) $\mathcal{L}$ is finite up to $\mathbb{Z}^d$-equivalence.

The implication (ii) $\Rightarrow$ (i) can be verified easily. Thus, (ii) is a ‘simple reason’ of $m(\mathcal{L}) < +\infty$. Theorem 1.3 asserts that, under the given assumptions, (ii) is the ‘only reason’ of $m(\mathcal{L}) < +\infty$. It might seem surprising that the assumption $\dim(\text{conv}(L \cap \mathbb{Z}^d)) = d$ in Theorem 1.3 is relevant for every $d \geq 3$. In fact, for every $d \geq 3$ one can construct a family $\mathcal{L}$ of $d$-dimensional maximal lattice-free rational polyhedra with $\dim(\text{conv}(L \cap \mathbb{Z}^d)) < d$ for every $L \in \mathcal{L}$ and such that, for this family, Condition (i) is fulfilled while Condition (ii) is not fulfilled (see Example 1.4 from Section 3). In contrast to this, for $d = 2$ the assumption $\dim(\text{conv}(L \cap \mathbb{Z}^2)) = d$ in Theorem 1.3 can be omitted. More precisely, as a consequence of Theorem 1.3 and a characterization of maximal lattice-free sets given by Lovász in [16, §3] we obtain the following.
Corollary 1.4. Let \( L \) be a family of maximal lattice-free rational polyhedra in \( \mathbb{R}^2 \). Then the following conditions are equivalent:

(i) \( m(L) < +\infty \);

(ii) \( L \) is finite up to \( \mathbb{Z}^2 \)-equivalence.

2 Proofs of Theorem 1.1 and Corollary 1.2

Given a polyhedron \( P \) in \( \mathbb{R}^d \), by \( \text{vert}(P) \) and \( \text{rec}(P) \) we denote the set of all vertices of \( P \) and the recession cone of \( P \), respectively. If \( P \) is line-free, one has \( P = \text{conv}(\text{vert}(P)) + \text{rec}(P) \). The set \( \gamma \) := \( \{ x + \lambda u : \lambda \geq 0 \} \) with \( x \in \mathbb{R}^d \) and \( u \in \mathbb{R}^d \setminus \{ 0 \} \) is called the ray emanating from \( x \) and having direction \( u \). The well-known Gordan-Dickson lemma (see [15]) states that if \( X \) is a subset of \( \mathbb{N}^d \) then there exists a finite subset \( X' \) of \( X \) such that every \( x \in X \) satisfies \( x' \leq x \) for some \( x' \in X' \) (here \( \leq \) is the standard partial order on \( \mathbb{R}^d \) introduced by comparison of respective components). Apart from ‘conv’ and ‘int’ we also use the abbreviations ‘bd’ and ‘vol’, which stand for the boundary and the volume, respectively. The following proposition has a straightforward proof.

Proposition 2.1. Let \( L \) be a \( d \)-dimensional polyhedron in \( \mathbb{R}^d \) such that \( \text{rec}(L) \) is a linear space. Let \( \gamma \) be a ray in \( \mathbb{R}^d \) with \( \gamma \not\subseteq \text{int}(L) \). Then the set \( \text{int}(L) \cap \gamma \) is bounded.

The following lemma seems to be folklore (see [1] Lemmas 2.3 and 2.4 and [10] Corollary 11.3) for related statements. In order to keep the presentation self-contained, we give a short geometric proof.

Lemma 2.2. Let \( P \) be a line-free polyhedron in \( \mathbb{R}^d \) and let \( L \) be a \( d \)-dimensional polyhedron such that \( \text{rec}(L) \) is a linear space. Then \( R_L(P) \) is a polyhedron. Furthermore, one has

\[
R_L(P) = R_L(S) + \text{rec}(P),
\]

where \( S \) be the union of all edges of \( P \).

Proof. For proving (4) it suffices to verify the following inclusions:

\[
P \setminus \text{int}(L) \subseteq R_L(S) + \text{rec}(P),
\]

\[
S \setminus \text{int}(L) + \text{rec}(P) \subseteq R_L(P).
\]

Consider an arbitrary \( x \in P \setminus \text{int}(L) \). By separation theorems, there exists a hyperplane \( H \) containing \( x \) and disjoint with \( \text{int}(L) \). We have \( x \in P \cap H = \text{conv}(\text{vert}(P \cap H)) + \text{rec}(P \cap H) \), where \( \text{vert}(P \cap H) \subseteq S \cap H \subseteq S \setminus \text{int}(L) \) and \( \text{rec}(P \cap H) \subseteq \text{rec}(P) \). This yields (5). For showing (6) we consider \( x \in S \setminus \text{int}(L) \) and derive \( x + u \in R_L(P) \). For \( u = 0 \), one obviously has \( x + u \in R_L(P) \). Let \( u \neq 0 \). By Proposition 2.1 the intersection of the ray \( \gamma := \{ x + \lambda u : \lambda \geq 0 \} \) with \( \text{int}(L) \) is bounded. Hence \( x + u \in \gamma = R_L(\gamma) \subseteq R_L(P) \). This yields (6).

It remains to show that \( R_L(\gamma) \) is a polyhedron. Let \( E \) be the set of all edges of \( P \). If \( e \in E \) and \( e \setminus \text{int}(L) \) is bounded and nonempty, then \( R_L(e) = \text{conv}(\{u,v\}) \) for some \( u,v \in e \setminus \text{int}(L) \). If \( e \in E \) and \( e \setminus \text{int}(L) \) is unbounded, then \( R_L(e) \subseteq w + \text{rec}(e) \) for some \( w \in e \setminus \text{int}(L) \). Let \( X \) be the finite subset of \( S \setminus \text{int}(L) \) consisting of all \( u,v \) and \( w \) associated to edges \( e \in E \) as above. We have

\[
R_L(S) + \text{rec}(P) = \text{conv} \left( \bigcup_{e \in E} R_L(e) \right) + \text{rec}(P) \subseteq \text{conv}(X) + \text{rec}(P) \subseteq R_L(S) + \text{rec}(P).
\]

Thus, \( \text{conv}(X) + \text{rec}(P) = R_L(S) + \text{rec}(P) = R_L(P) \). Since \( X \) is finite and \( \text{rec}(P) \) is a polyhedral cone, we deduce that \( R_L(P) \) is a polyhedron.

Lemma 2.3. Let \( L \) be a \( d \)-dimensional rational polyhedron in \( \mathbb{R}^d \) such that \( m = m(L) < +\infty \). Let \( \gamma := \{ p + \lambda u : \lambda \geq 0 \} \) be a ray emanating from a point \( p \in \text{int}(L) \) and having direction \( u \in \mathbb{Z}^d \setminus \{ 0 \} \). Assume that \( \text{bd}(L) \) and \( \gamma \) intersect. Assume also that \( p \in \mathbb{Q}^d \), and let \( h \in \mathbb{N} \) be such that \( hp \in \mathbb{Z}^d \).
Then the (unique) intersection point \( q \) of \( \text{bd}(L) \) and \( \gamma \) can be given by \( q = p + \lambda u \), where \( \lambda > 0 \) satisfies

\[
\frac{(hm)!}{\lambda} \in \mathbb{N}.
\]

**Proof.** Let \( L \) be given by \( 1 \)–\( 2 \). Since \( q \in \text{bd}(L) \), there exists \( i \in [n] \) such that \( \langle q, a_i \rangle = b_i - m \) or \( \langle q, a_i \rangle = b_i \). We assume \( \langle q, a_i \rangle = b_i \) (the case \( \langle q, a_i \rangle = b_i - m \) being similar). Since \( p \in \text{int}(L) \), one has \( b_i - m < \langle a_i, p \rangle < b_i \). It follows that \( \langle a_i, u \rangle \neq 0 \) and we can express \( \lambda \) by

\[
\lambda = \frac{b_i - \langle a_i, p \rangle}{\langle a_i, u \rangle} = \frac{hb_i - \langle a_i, hp \rangle}{h \langle a_i, u \rangle},
\]

where \( hb_i - \langle a_i, hp \rangle \) is a natural number not larger than \( hm \). It follows \( \frac{(hm)!}{\lambda} \in \mathbb{N} \). \( \square \)

**Proof of Theorem 1.1.** First we consider the case that \( P \) is line-free. We remark that if \( L \) is represented as a finite union, say \( L = L_1 \cup \cdots \cup L_t \) with \( t \in \mathbb{N} \), then it suffices to verify the assertion for each subfamily \( L_i \) with \( i \in [t] \) in place of \( L \). Let \( E \) be the set of all edges of \( P \). Given \( L \in \mathcal{L} \), we decompose \( E \) into the following three sets:

\[
E^+ = \{ e \in E : R_L(e) = e \}, \quad \text{(the set of edges ‘preserved’ by} L) \),

(7)

\[
E^- = \{ e \in E : R_L(e) = \emptyset \}, \quad \text{(the set of edges ‘removed’ by} L) \),

(8)

\[
E^\pm = \{ e \in E : \emptyset \neq R_L(e) \neq e \}, \quad \text{(the set of edges ‘bisected’ by} L) \).

(9)

In view of the remark on representation of \( \mathcal{L} \) as a finite union, without loss of generality we can assume that \( E^+, E^- \) and \( E^\pm \) do not depend on the choice of \( L \in \mathcal{L} \). That is, we assume that \( E \) can be represented as a union of three sets \( E^+, E^- \) and \( E^\pm \) such that \( 1 \), \( 2 \) and \( 3 \) hold for every \( L \in \mathcal{L} \). The degenerate case \( E^\pm = \emptyset \) can be handled easily using Lemma 2.2. Assume \( E^\pm \neq \emptyset \). Let \( E^\pm = \{ e_1, \ldots, e_s \} \), where \( s \in \mathbb{N} \). Let \( i \in [s] \). By Proposition 2.1, the set int(\( L \)) \( \cap \) \( e_i \) is bounded. Thus, taking into account that \( e_i \in E^\pm \), we see that precisely one endpoint \( p_i \) of \( e_i \) belongs to int(\( L \)). Let us choose any vector \( u_i \in (Z^d \cap \text{rec}(e_i)) \setminus \{ o \} \) if \( e_i \) is unbounded and a vector \( u_i \in Z^d \setminus \{ o \} \) such that \( e_i \subseteq \text{conv}(\{ p_i, p_i + u_i \}) \) if \( e_i \) is bounded. For every \( L \in \mathcal{L} \), the intersection point of \( e_i \) and bd(\( L \)) can be given by \( p_i + \lambda_{i,L} u_i \), where \( \lambda_{i,L} > 0 \). Let us fix \( h \in \mathbb{N} \) such that all vertices of \( hP \) are integer points. By Lemma 2.3, for every \( L \in \mathcal{L} \) one has

\[
y_L := (hm)! \cdot \left( \frac{1}{\lambda_{1,L}}, \ldots, \frac{1}{\lambda_{s,L}} \right)^\top \in \mathbb{N}^s.
\]

By the Gordon-Dickson lemma one can choose a finite subfamily \( \mathcal{L}' \) of \( \mathcal{L} \) such that for every \( L \in \mathcal{L} \) there exists \( L' \in \mathcal{L}' \) with \( y_L' \leq y_L \). For the condition \( y_L' \leq y_L \) we have the following chain of equivalences:

\[
y_L' \leq y_L \iff \lambda_{i,L'} \geq \lambda_{i,L} \quad \forall i \in [s] 
\]

\[
\iff R_L'(e) \subseteq R_L(e) \quad \forall e \in E^\pm 
\]

\[
\iff R_L'(P) \subseteq R_L(P).
\]

The last equivalence in the above chain follows from Lemma 2.2. Thus, \( \mathcal{L}' \) is a subfamily of \( \mathcal{L} \) with the desired properties.

Now, assume \( P \) is not line-free, that is, the linear space \( X := \text{rec}(P) \cap (-\text{rec}(P)) \) is strictly larger than \( \{ o \} \). If \( L \in \mathcal{L} \) and \( X \not\subseteq \text{rec}(L) \), one can choose a line \( \gamma \) through \( o \) which is contained in \( X \) but not in rec(\( L \)). By the choice of \( \gamma \), for every \( x \in P \), the set \( (x + \gamma) \cap \text{int}(L) \) is bounded. Hence \( x + \gamma = R_L(x + \gamma) \subseteq R_L(P) \). This shows the equality \( P = R_L(P) \) for every \( L \in \mathcal{L} \) with \( X \not\subseteq \text{rec}(L) \). Thus, without loss of generality, we can assume \( X \subseteq \text{rec}(L) \) for every \( L \in \mathcal{L} \). The linear space \( X \) is spanned by vectors from \( Z^d \). By this we can choose a basis \( z_1, \ldots, z_k \) of the lattice \( X \cap Z^d \), where \( k \) is the dimension of the linear space \( X \). We extend \( z_1, \ldots, z_d \) to a basis \( z_1, \ldots, z_d \) of the lattice \( Z^d \). After a change of coordinates which transforms the basis \( z_1, \ldots, z_d \) to the standard basis of \( \mathbb{R}^d \) we can assume that \( X = \mathbb{R}^k \times \{ o' \} \), where \( o' \) is the origin of \( \mathbb{R}^{d-k} \). Then \( P \) can be given
by \( P = \mathbb{R}^k \times P' \), where \( P' \) is a line-free rational polyhedron in \( \mathbb{R}^{d-k} \). Furthermore, every \( L \in \mathcal{L} \) can be given by \( L = \mathbb{R}^k \times L' \) for an appropriate rational polyhedron \( L' \), which satisfies \( m(L) = m(L') \).

Taking into account the trivial equality \( R_L(P) = \mathbb{R}^k \times R_{L'}(P') \), we see that, in the case that \( P \) is not line-free, the assertion follows from the assertion for the case of line-free \( P \).

\( \square \)

**Remark 2.4.** The main theorem from [11, Theorem 4.3] asserts that, for \( P \) and \( \mathcal{L} \) as in Theorem 1.1 and under the additional assumption that the elements of \( \mathcal{L} \) are maximal lattice-free sets, the set \( \bigcap_{L \in \mathcal{L}} R_L(P) \) is a rational polyhedron. Thus, in Theorem 1.1 we both relax the assumptions and strengthen the assertion of the main result from [11]. The motivation to relax the assumption is provided by the fact that Theorem 1.1 can be used in the proof of the result on finite convergence of the so-called integral lattice-free closures which was given in [14, Theorem 4]. The authors of [14, §3] indicate that they need to use a modification of the main result of [1] with weaker assumptions. On the other hand, the strengthened assertion gives a more detailed information on the family \( \{ R_L(P) : L \in \mathcal{L} \} \).

**Proof of Corollary 1.2.** The corollary is a straightforward consequence of Theorem 1.1 and Lemma 2.2.  

\( \square \)

### 3 Proofs of Theorem 1.3 and Corollary 1.4

We shall use the following description of maximal lattice-free sets given by Lovász.

**Theorem 3.1.** [16, §3] Let \( L \) be a lattice-free set in \( \mathbb{R}^d \). Then the following statements hold.

I. The set \( L \) is maximal lattice-free if and only if \( L \) is a polyhedron and the relative interior of each facet of \( L \) contains a point of \( \mathbb{Z}^d \).

II. If \( L \) is maximal lattice-free and unbounded, then \( L \) is \( \mathbb{Z}^d \)-equivalent to \( L' \times \mathbb{R}^k \), where \( 0 < k < d \) and \( L' \) is a \((d-k)\)-dimensional maximal lattice-free polytope in \( \mathbb{R}^{d-k} \).

Proofs of Theorem 3.1 can be found in [2, Theorem 1] and [7, Theorem 2.2].

**Lemma 3.2.** Let \( L \) be a \( d \)-dimensional rational polytope in \( \mathbb{R}^d \). Then \( \text{vol}(L) \leq m(L)^d \).

**Proof.** Assume \( m = m(L) < +\infty \), since otherwise the assertion is trivial. Let \( L \) be given by (11)–(12). Since \( L \) is bounded, there exist indices \( 1 \leq j_1, \ldots, j_d \leq n \) such that \( a_{j_1}, \ldots, a_{j_d} \) is a basis of \( \mathbb{R}^d \). Let \( A \) be the matrix with rows \( a_{j_1}, \ldots, a_{j_d} \) (in this sequence) and let \( b := (b_{j_1}, \ldots, b_{j_d})^\top \). We have

\[
L \subseteq \{ x \in \mathbb{R}^d : b - Ax \in [0, m]^d \} = A^{-1}(b - [0, m]^d),
\]

where the matrix \( A \) is integral. Hence \( \text{vol}(L) \leq \frac{1}{|\det A|} m^d \leq m^d \). \( \square \)

**Lemma 3.3.** Let \( P \) be a \( d \)-dimensional integral polytope and let \( m \in \mathbb{N} \). Let \( \mathcal{L} \) be the family of all \( d \)-dimensional rational polytopes \( L \) in \( \mathbb{R}^d \) such that \( L \) is a maximal lattice-free set, \( \text{conv}(L \cap \mathbb{Z}^d) = P \) and \( m(L) = m \). Then \( \mathcal{L} \) is finite.

**Proof.** We shall use the notions ‘distance’ and ‘ball’ in the standard Euclidean sense. By \( \| \cdot \| \) we denote the Euclidean norm of \( \mathbb{R}^d \). Let \( \delta > 0 \) be the least possible distance between a pair of parallel hyperplanes \( H^+ \) and \( H^- \) in \( \mathbb{R}^d \) satisfying \( P \subseteq \text{conv}(H^+ \cup H^-) \). Let us choose \( \rho > 0 \) such that \( P \) is contained in the (closed) ball of radius \( \rho \) with center at \( o \). We consider an arbitrary \( L \in \mathcal{L} \) and assume that \( L \) is given by (11)–(12). For \( i \in [n] \) consider the hyperplanes

\[
H_i^+ := \{ x \in \mathbb{R}^d : \langle a_i, x \rangle = b_i \} \quad \text{ and } \quad H_i^- := \{ x \in \mathbb{R}^d : \langle a_i, x \rangle = b_i - m \}.
\]

If, for some \( i \in [n] \), neither \( L \cap H_i^+ \) nor \( L \cap H_i^- \) is a facet of \( L \), then the corresponding inequalities \( b_i - m \leq \langle a_i, x \rangle \leq b_i \) in (11) are redundant (that is, they can be dropped out without changing \( L \)). Thus, without loss of generality, we can assume that, for every \( i \in [n] \), the set \( L \cap H_i^+ \) or \( L \cap H_i^- \) is a facet of \( L \). Taking into account this assumption and Theorem 3.1 we see that the intersection of
L \cap H_1^+ or L \cap H_1^- contains an integral point. Since all integral points of L lie in P, we deduce that L \cap H_1^+ or L \cap H_1^- contains a point of P.

One has P \subseteq \text{conv}(H_1^+ \cup H_1^-), where H_1^+ and H_1^- are parallel hyperplanes at distance \( \frac{m}{\|a_i\|} \) from each other. From the definition of \( \delta \) we deduce \( \|a_i\| \leq \frac{m}{\delta} \). The hyperplane \( H_i^+ \) is at distance \( \frac{|b_i|}{\|a_i\|} \) from o. Analogously, the hyperplane \( H_i^- \) is at distance \( \frac{|b_i|-m}{\|a_i\|} \) from o. It follows that both \( H_i^+ \) and \( H_i^- \) are at distance at least \( \frac{|b_i|-m}{\|a_i\|} \) from o. On the other hand, the hyperplane \( H_i^+ \) or \( H_i^- \) contains a point of P. Hence, by the choice of \( \rho \), the hyperplane \( H_i^+ \) or \( H_i^- \) is at distance at most \( \rho \) from o. We get \( \frac{|b_i|-m}{\|a_i\|} \leq \rho \), which implies \( |b_i| \leq \rho \|a_i\| + m \leq \frac{m\rho}{\delta} + m \). Thus, for every \( L \in \mathcal{L} \) one can find a representation \( (1) \) such that \( \|a_i\| \leq \frac{m}{\delta} \) and \( |b_i| \leq \frac{m\rho}{\delta} + m \) for every \( i \in [n] \). Since \( \rho \) and \( \delta \) depend only on \( P \), we get the assertion.

**Proof of Theorem 1.3.** Assume (ii) is fulfilled. By Theorem 3.1, for every \( L \in \mathcal{L} \), rec(\( L \)) is a linear space of dimension at most \( d-1 \). Hence \( m(L) < +\infty \) for every \( L \in \mathcal{L} \). Since the parameter \( m(L) \) is invariant with respect to \( \mathbb{Z}^d \)-equivalence, (i) follows immediately.

Now, we assume (i) and show (ii). If \( L \in \mathcal{L} \) is unbounded then, by Theorem 3.1, \( L \) is \( \mathbb{Z}^d \)-equivalent to \( L' \times \mathbb{R}^k \) for some \( 0 < k < d \) and a \((d-k)\)-dimensional maximal lattice-free polytope in \( L' \) in \( \mathbb{R}^{d-k} \). Without loss of generality one can assume \( L = L' \times \mathbb{R}^{d-k} \). Then \( m(L) = m(L') \) and \( \text{conv}(L \cap \mathbb{Z}^d) = \text{conv}(L' \cap \mathbb{Z}^k) \times \mathbb{R}^{d-k} \). In view of the latter relations, we see that it is sufficient to consider the case that \( \mathcal{L} \) consists of bounded polyhedra. By assumption, the family \( \mathcal{P} := \{ \text{conv}(L \cap \mathbb{Z}^d) : L \in \mathcal{L} \} \) consists of \( d \)-dimensional integral polytopes. In view of Lemma 3.2 the volume of each \( P \in \mathcal{P} \) is at most \( m(\mathcal{L})^d \). Having an upper bound on the volume for the class of integral polytopes \( \mathcal{P} \) we deduce that \( \mathcal{P} \) is finite up to \( \mathbb{Z}^d \)-equivalence (this implication is well known; see, for example, (4)). We choose finitely many integral polytopes \( P_1, \ldots, P_t \) \((t \in \mathbb{N})\) such that each \( P \in \mathcal{P} \) is \( \mathbb{Z}^d \)-equivalent to some \( P_i \) for \( i \in [t] \). Then (ii) follows by applying Lemma 3.3 with \( P = P_i \) for each \( i \in [t] \).

**Proof of Corollary 1.4.** In view of Theorem 3.1 II every unbounded element \( L \in \mathcal{L} \) is \( \mathbb{Z}^2 \)-equivalent to \([0,1] \times \mathbb{R} \). By this without loss of generality we can assume that every \( L \in \mathcal{L} \) is bounded. Then \( L \) has at least three edges and, by Theorem 3.1, \( \dim(\text{conv}(L \cap \mathbb{Z}^2)) = 2 \). Thus, the assumptions of Theorem 1.3 are fulfilled and the assertion follows.

**Example 3.4.** As shown by Corollary 1.4, the assumption \( \dim(\text{conv}(L \cap \mathbb{Z}^d)) = d \) in Theorem 1.3 can be omitted for \( d = 2 \). On the other hand, if the dimension \( d \in \mathbb{N} \) is at least 3, then the assumption \( \dim(\text{conv}(L \cap \mathbb{Z}^d)) = d \) cannot be omitted in general. In fact, for every \( d \geq 3 \) we shall construct a family \( \mathcal{L} \) of rational maximal lattice-free polyhedra in \( \mathbb{R}^d \) satisfying \( \dim(\text{conv}(L \cap \mathbb{Z}^d)) < d \) for every \( L \in \mathcal{L} \) and such that \( \mathcal{L} < +\infty \) but \( \mathcal{L} \) is not finite up to \( \mathbb{Z}^d \)-equivalence. Thus, for \( \mathcal{L} \) as above Condition (i) from Theorem 1.3 is fulfilled while Condition (ii) is not. Below, whenever we consider a vector \( x \in \mathbb{R}^d \) and \( i \in [d] \) we denote by \( x_i \) the \( i \)-th component of \( x \). Our construction employs the cross-polytopes \( C_d \) \((d \in \mathbb{N})\) given by

\[
C_d := \{ x \in \mathbb{R}^d : |2x_1 - 1| + \cdots + |2x_d - 1| \leq d \} = \{ x \in \mathbb{R}^d : a_1(2x_1 - 1) + \cdots + a_d(2x_d - 1) \leq d \quad \forall a \in \{-1,1\}^d \}.
\]

It can be shown using Theorem 3.1 II that \( C_d \) is maximal lattice-free. Below we define \( \mathcal{L} \) in such a way that, for every \( L \in \mathcal{L} \), the intersection of \( L \) with the horizontal coordinate hyperplane \( \mathbb{R}^{d-1} \times \{0\} \) coincides with \( C_{d-1} \times \{0\} \) and the transformation \( F \mapsto F \cap (\mathbb{R}^{d-1} \times \{0\}) \) is a bijection from the set of facets of \( L \) to the set of facets of its section \( C_{d-1} \times \{0\} \).

We shall need the following simple observation. If \( A \in \mathbb{Z}^{d \times d} \) is a nonsingular matrix and \( b \in \mathbb{Z}^d \), then for the nonsingular affine transformation \( \phi : \mathbb{R}^d \to \mathbb{R}^d \) given by \( \phi(x) = Ax + b \) \((x \in \mathbb{R}^d)\) and every \( d \)-dimensional rational polyhedron \( P \) in \( \mathbb{R}^d \) the inequality

\[
m(\phi^{-1}(P)) \leq m(P)
\]

holds. This follows directly from the definition of the max-facet-width (see 11 and 24). Consider the set \( A_{d}^+ \) \((d \geq 2)\) of all vectors \( a \in \{-1,1\}^d \) with even (resp. odd) number of entries equal to \(-1\). Every vector \( a \in \{-1,1\}^{d-1} \) \((d \geq 2)\) can be extended to a vector \((a_1, \ldots, a_{d-1}, t) \in A_{d}^+ \), where

\[
\text{...}
\]
Each $k$ we have shown $P\phi$ below as follows: using (11) and (12). In fact, for every $a \vdash A_k$ or $A_k$. The family $\mathcal{L}$ is introduced by applying certain nonsingular affine transformations to the rational polyhedron

$$P := \{ x \in \mathbb{R}^d : \langle a, x \rangle \leq d \ \forall a \in A_k \}.$$ 

We have $\dim(P) = d$ since $o \in \text{int}(P)$. The max-face-width $m(P)$ of $P$ can be bounded from above using (11) and (12). In fact, for every $a \in A_k$ and $x \in P$ one has

$$\langle a, x \rangle = - \sum_{b \in A_k \setminus \{a\}} \langle b, x \rangle \geq - \sum_{b \in A_k \setminus \{a\}} d = -(2^d - 1)d = d - 2^d,$$

which yields

$$m(P) \leq d2^{d-1}.$$ 

For $i \in [d]$ the transformation $a \mapsto (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d)^T$, which discards the $i$-th entry, maps bijectively $\{ a \in A_k : a_i = 1 \}$ onto $A_k^{d-1}$ and $\{ a \in A_k : a_i = -1 \}$ onto $A_k^{d-1}$. Using (11) and (12) the component $x_i$ of every point $x \in P$ (where $d \geq 3$ and $i \in [d]$) can be bounded from above and below as follows:

$$2^{d-2}d \geq \sum_{a \in A_k^d \atop a_i = 1} \langle a, x \rangle = 2^{d-2}x_i \quad \text{and} \quad 2^{d-2}d \geq \sum_{a \in A_k^d \atop a_i = -1} \langle a, x \rangle = -2^{d-2}x_i.$$

We have shown $|x_i| \leq d$ for each $x \in P$ and $i \in [d]$, where $d \geq 3$. That is,

$$P \subseteq [-d, d]^d.$$ 

In particular, $P$ is bounded and thus $\text{vol}(P) < +\infty$. For every $k \in \mathbb{N}$, consider the affine transformation $\phi_k : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\phi_k(x) := (2x_1 - 1, \ldots, 2x_{d-1} - 1, kx_d)^T \ \forall x \in \mathbb{R}^d.$$ 

Each $k \in \mathbb{N}$ determines the rational polyhedron

$$L_k := \phi_k^{-1}(P) = \{ x \in \mathbb{R}^d : \langle a, \phi_k(x) \rangle \leq d \ \forall a \in A_k \}.$$ 

We introduce $\mathcal{L}$ by $\mathcal{L} := \{ L_k : k \in \mathbb{N}, k \geq 2d \}$. By construction $\text{vol}(L_k) = \frac{1}{k2^{d-1}} \text{vol}(P)$. The volume of any two $\mathbb{Z}^d$-equivalent sets is the same. Hence, we deduce that $\mathcal{L}$ is not finite up to $\mathbb{Z}^d$-equivalence. By (12), $m(\mathcal{L}) = \sup_{k \geq 2d} m(L_k) \leq m(P) \leq d2^{d-1} < +\infty$. The bound $k \geq 2d$ on $k$ guarantees that $L_k$ is rather thin in the vertical direction. More precisely, we have

$$L_k = \phi_k^{-1}(P) \subseteq \phi_k^{-1}([-d, d]^d) \subseteq \mathbb{R}^{d-1} \times \left[ -\frac{d}{k}, \frac{d}{k} \right] \subseteq \mathbb{R}^{d-1} \times \left[ -\frac{1}{2}, \frac{1}{2} \right],$$

which implies that all integer points of $L_k$ lie in the coordinate hyperplane $\mathbb{R}^{d-1} \times \{0\}$. By construction $L_k \cap (\mathbb{R}^{d-1} \times \{0\}) = C_{d-1} \times \{0\}$. Thus, $L_k \cap \mathbb{Z}^d = (C_{d-1} \times \{0\}) \cap \mathbb{Z}^d = \{0, 1\}^{d-1} \times \{0\}$. The latter yields $\dim(\text{conv}(L_k \cap \mathbb{Z}^d)) = d - 1 < d$. The relative interior of each facet of $L_k$ contains a point from $\{0, 1\}^{d-1} \times \{0\}$. Consequently, by Theorem [3,1], every polytope from $\mathcal{L}$ is maximal lattice-free.

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