Pusz–Woronowicz Functional Calculus and Extended Operator Convex Perspectives

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Abstract. In this article, we first study, in the framework of operator theory, Pusz and Woronowicz’s functional calculus for pairs of bounded positive operators on Hilbert spaces associated with a homogeneous two-variable function on \([0, \infty)^2\). Our construction has special features that functions on \([0, \infty)^2\) are assumed only locally bounded from below and that the functional calculus is allowed to take extended semibounded self-adjoint operators. To analyze convexity properties of the functional calculus, we extend the notion of operator convexity for real functions to that for functions with values in \((-\infty, \infty]\). Based on the first part, we generalize the concept of operator convex perspectives to pairs of (not necessarily invertible) bounded positive operators associated with any operator convex function on \((0, \infty)\). We then develop theory of such operator convex perspectives, regarded as an operator convex counterpart of Kubo and Ando’s theory of operator means. Among other results, integral expressions and axiomatization are discussed for our operator perspectives.

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Contents

1. Introduction 2
2. Preliminaries 5
   2.1. Lower Semibounded Self-Adjoint Operators 5
   2.2. Lower Semibounded Quadratic Forms 6
   2.3. Extended Lower Semibounded Self-Adjoint Part 7
3. Extended Real-Valued Operator Convex Functions 9
4. Pusz–Woronowicz Functional Calculus 14
5. PW-Functional Calculus with Restricted Domain 26
1. Introduction

The concept of operator (convex) perspectives has not been studied so far in full generality. Namely, when given positive operators are not invertible, the existing definition of operator perspectives does not work for many interesting operator convex functions such as $t \log t$ and $t^\alpha$ ($1 < \alpha \leq 2$), because the corresponding (scalar-valued) perspective functions are not locally bounded and moreover take $\infty$ on $[0, \infty)^2$. This difficulty appears even in the finite-dimensional setting, but never does in theory of operator connections/means, viewed as operator perspectives corresponding to operator monotone functions on $[0, \infty)$. The present work attempts to overcome this drawback of the current operator perspective theory, by allowing an operator perspective in question to be unbounded.

Theory of operator connections/means has grown into a significant subject of operator theory. The theory has its origin in a study of parallel sum motivated by electrical networks [2]. Besides parallel sum (a half of harmonic mean), the most interesting and the most studied operator mean is geometric mean, which was introduced in 1975 by Pusz and Woronowicz [46] and further discussed by Ando [5, 6]. In [46] the authors developed a certain type of functional calculus for positive sesquilinear forms on a complex vector space, which we call the Pusz–Woronowicz (PW for short) functional calculus. For two positive sesquilinear forms $\alpha, \beta$ on a complex vector space, the PW-functional calculus determines a new sesquilinear form $\phi(\alpha, \beta)$ on the vector space in a canonical way, associated with a given Borel function $\phi : [0, \infty)^2 \to \mathbb{R}$ that is locally bounded and homogeneous (i.e., $\phi(\lambda x, \lambda y) = \lambda \phi(x, y)$ for $x, y, \lambda \geq 0$). In particular, the geometric mean $\sqrt{\alpha \beta}$ of $\alpha, \beta$ was defined in [46] as $\phi(\alpha, \beta)$ for the function $\phi(x, y) = (xy)^{1/2}$, $x, y \in [0, \infty)$. Furthermore, in [47], Pusz and Woronowicz considered the PW-functional calculus $\phi(\alpha, \beta)$ for more general homogeneous functions $\phi$ on $[0, \infty)^2$ and characterized joint convexity of $\phi(\alpha, \beta)$ in terms of operator convexity properties of $\phi(x, y)$.

In 1980, Kubo and Ando [40] proposed an axiomatic approach (see also the beginning of Sect. 10 of this paper) to a general theory of operator means (and connections) in a close relation to L"owner’s theory [42] on operator monotone functions. In fact, Kubo and Ando’s operator connections $\sigma$ correspond one-to-one to non-negative operator monotone functions $h$ on $[0, \infty)$ in such a way that

$$A\sigma B = A^{1/2}h(A^{-1/2}BA^{-1/2})A^{1/2}$$  \hspace{1cm} (1.1)
for $A, B \in B(H)_+$, the bounded positive operators on a Hilbert space $H$, with $A$ invertible, which is further extended to general $A, B \in B(H)_+$ as

$$A\sigma B = \lim_{\varepsilon \searrow 0} (A + \varepsilon I)\sigma(B + \varepsilon I)$$

(1.2)

in the strong operator topology.

More recently, in [12,13], the notion of operator perspectives associated with real continuous functions $f$ on $(0, \infty)$ was introduced, similarly to (1.1), as

$$P_f(A, B) = B^{1/2}f(B^{-1/2}AB^{-1/2})B^{1/2}$$

(1.3)

for $A, B \in B(H)_+$, the invertible operators in $B(H)_+$, though the roles of $A, B$ are interchanged. It was proved that $P_f(A, B)$ is jointly operator convex on $B(H)_+ \times B(H)_+$ if and only if $f$ is operator convex on $(0, \infty)$. The extension problem of operator perspectives $P_f(A, B)$ to general $A, B \in B(H)_+$ is rather complicated and has not been well studied so far, while a few discussions are in [21, Sect. 2] and [33, Sect. 6]. In fact, when $f$ is operator convex on $(0, \infty)$, the limit of $P_f(A + \varepsilon I, B + \varepsilon I)$ as $\varepsilon \searrow 0$ for $A, B \in B(H)_+$ does not always exist as a bounded operator, unlike (1.2) for operator connections $A\sigma B$.

Recently in [24], Hatano and the second-named author of this paper considered the PW-functional calculus in the framework of operator theory. When positive sesquilinear forms $\alpha, \beta$ on $H$ are defined, for given operators $A, B \in B(H)_+$, by $\alpha(\xi, \eta) := (A\xi, \eta)$ and $\beta(\xi, \eta) := (B\xi, \eta)$ for $\xi, \eta \in H$, the PW-functional calculus $\phi(\alpha, \beta)$ is described in terms of $A, B$ as follows. With a bounded operator $T_{A,B} : H \to H_{A,B} := \overline{\text{ran}}(A + B)$ defined by $T_{A,B}\xi := (A + B)^{1/2}\xi$, $\xi \in H$, we have $R_{A,B}, S_{A,B} \in B(H_{A,B})_+$ such that $R_{A,B} + S_{A,B} = I_{H_{A,B}}$, $A = T^*_{A,B}R_{A,B}T_{A,B}$ and $B = T^*_{A,B}S_{A,B}T_{A,B}$. Then $(T_{A,B} : H \to H_{A,B}, R_{A,B}, S_{A,B})$ is a compatible representation of $(\alpha, \beta)$ in the sense of [46], and $\phi(\alpha, \beta)$ clearly coincides with

$$\phi(A, B) := T_{A,B}^*\phi(R_{A,B}, S_{A,B})T_{A,B}$$

(1.4)

if $\phi$ is a locally bounded and homogeneous Borel function on $[0, \infty)^2$, where $\phi(R_{A,B}, S_{A,B})$ denotes the usual Borel functional calculus of commuting $R_{A,B}, S_{A,B}$. As clarified in [24], Kubo and Ando’s operator connections $A\sigma B$ are captured by the PW-functional calculus, that is, $A\sigma B = \phi(B, A)$ if $\phi$ is the two-variable extension (i.e., the perspective function) of the representing function $h$ in (1.1). Similarly, operator perspectives $P_f(A, B)$ for $A, B \in B(H)_{++}$ are realized as $\phi(A, B)$ with the perspective function $\phi$ of $f$. The convexity criteria given in [47] were also examined in [24, Theorem 9], so that the joint convexity assertion in [12,13] may be considered as a specialized version of the result of [47]. Moreover, it was observed in [24] that operator homogeneity (see Definition 4.1(2) of this paper) generally holds for the PW-functional calculus, while this property was formerly shown in [19] for operator means.

Now we explain our aims of the present article in the following two items:
(1) Assume, for instance, that $\phi$ is the perspective function of a real function $f$ on $(0, \infty)$ and $\phi$ is extended to $[0, \infty)^2$ by continuity, that is, $\phi(x, y) := yf(x/y)$ for $x, y > 0$, $\phi(x, 0) := \alpha x$ for $x \geq 0$ and $\phi(0, y) := \beta y$ for $y \geq 0$, where $\alpha := \lim_{t \to -\infty} f(t)/t$ and $\beta := \lim_{t \to 0^+} f(t)$ (whose limits in $(-\infty, \infty]$ are here assumed to exist). In this situation, $\phi$ is not necessarily $\mathbb{R}$-valued on the boundary of $[0, \infty)^2$, i.e., on $\{0\} \times (0, \infty)$ and $(0, \infty) \times \{0\}$. We are then motivated to extend the PW-functional calculus $\phi(A, B)$ discussed in [24] to locally lower bounded and homogeneous functions $\phi$ on $[0, \infty)^2$ having extended values in $(-\infty, \infty]$. Here we are forced to allow $\phi(A, B)$ to be unbounded. Thus we will formulate the PW-functional calculus associated with $\phi$ as a two-variable mapping from $B(H)_+ \times B(H)_+$ to the extended lower semibounded self-adjoint part $\overline{B(H)}_{lb}$ of $B(H)$, which is slightly bigger than the extended positive part $\overline{B(H)}_+$ of $B(H)$ in the sense of [22]. Our primary aim is to carry out the account of the PW-functional calculus in [24] in this extended setting. The main target here is to obtain a complete set of convexity criteria of $\phi(A, B)$ generalizing those in [24]. This will consequently justify the use of $\overline{B(H)}_{lb}$ in our formulation.

(2) For an operator convex function $f$ on $(0, \infty)$, we can define the operator perspective $\phi_f(A, B)$ (as an element of $\overline{B(H)}_{lb}$) for all $A, B \in B(H)_+$ based on the PW-functional calculus constructed in (1). That is, $\phi_f(A, B)$ is defined to be the PW-functional calculus $\phi(A, B)$ associated with the perspective function $\phi$ of $f$ (extended to $[0, \infty)^2$ by continuity as mentioned in (1)). Then $\phi_f(A, B)$ coincides with $P_f(A, B)$ in (1.3) if $A, B$ are invertible, so the extension problem mentioned after (1.3) is indirectly settled because $\phi_f(A, B)$ is already defined for all $A, B \in B(H)_+$, though the values of $\phi_f(A, B)$ are not necessarily bounded operators. We, instead, have to consider the problem of characterizing when $\phi_f(A, B)$ is bounded. Our second aim is to develop theory of extended operator convex perspectives along the lines of Kubo and Ando’s theory of operator connections. We will consider, for example, their integral expressions and axiomatization of Kubo and Ando’s type.

We end the introduction with a brief summary of contents of the article. Section 2 is a preliminary on the extended lower semibounded self-adjoint part $\overline{B(H)}_{lb}$, and Sect. 3 gives basics of extended $(-\infty, \infty]$-valued operator convex functions for later use. In Sect. 4, extending discussions in [24], we introduce and study the PW-functional calculus $\phi(A, B)$ of $A, B \in B(H)_+$ associated with such a function $\phi$ on $[0, \infty)^2$ as stated in (1) above. The definition of $\phi(A, B)$ is given in an axiomatic fashion with two postulates (see Definition 4.1), while an explicit definition like (1.4) is also possible. The main result (Theorem 4.10) gives characterizations for $\phi(A, B)$ to be jointly convex in $(A, B)$. In Sect. 5 we consider the PW-functional calculus $\phi(A, B)$ associated with a function $\phi$ on $[0, \infty)^2 \setminus (\{0\} \times (0, \infty))$ with a restricted domain of $(A, B) \in B(H)_+ \times B(H)_+$ such that $A \geq \alpha B$ for some $\alpha > 0$. Section 6 establishes the continuity of $\phi(A_n, B_n) \to \phi(A, B)$ in the strong operator topology for decreasing $A_n \searrow A, B_n \searrow B$ in $B(H)_+$ when $\phi$ is $\mathbb{R}$-valued and continuous on $[0, \infty)^2$. We believe that such a general continuity
property for the PW-functional calculus in its original form has not been examined so far.

In the second part, we study operator perspectives $\phi_f(A, B)$ associated with an operator convex function $f$ on $(0, \infty)$, that is, the PW-functional calculus associated with the perspective function of $f$. In Sect. 7 we discuss (semi-)continuity properties of $\phi_f(A, B)$. The main result (Theorem 7.7), in particular, says that the approach taking limit as in (1.2) is also available for $\phi_f(A, B)$. It is also shown (Proposition 7.11) that when $A, B$ are positive trace-class operators, $\text{Tr} \phi_f(A, B)$ is well defined and coincides with the maximal $f$-divergence \cite{27} of $A, B$. In Sect. 8 we examine the cases when $\phi_f(A, B)$ is bounded and when $\phi_f(A, B)$ has a dense domain, i.e., $\phi_f(A, B)$ is a densely-defined self-adjoint operator on $\mathcal{H}$. Interestingly, this problem in the case $f(t) = t^2$ is strongly related to absolute continuity between positive operators \cite{4}. Furthermore, Sect. 9 treats integral expressions and variational expressions of $\phi_f(A, B)$ based on integral expressions of $f$. Finally, Sect. 10 gives some axiomatization results, including a new axiomatization of operator connections different from the familiar one in \cite{40}.

2. Preliminaries

Throughout this article, let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$. We use the notations $B(\mathcal{H})_{\text{sa}}$, $B(\mathcal{H})_+$, and $B(\mathcal{H})_{++}$ for the sets of self-adjoint operators, positive operators, and positive invertible operators in $B(\mathcal{H})$, respectively. Note that $B(\mathcal{H})$ is a von Neumann algebra with the predual $B(\mathcal{H})_* \cong C_1(\mathcal{H})$, the space of trace-class operators on $\mathcal{H}$ with trace-norm. Here we identify $\rho \in C_1(\mathcal{H})$ with a normal functional $\rho(X) = \text{Tr} X \rho$ for $X \in B(\mathcal{H})$, where $\text{Tr}$ is the usual trace on $B(\mathcal{H})$. The positivity $\rho \geq 0$ in the operator sense is equivalent to the positivity of $\rho$ in the functional sense, so we can further identify $B(\mathcal{H})_*^+ = C_1(\mathcal{H})_+$, where $B(\mathcal{H})_*^+$ and $C_1(\mathcal{H})_+$ are the positive parts of $B(\mathcal{H})_*$ and $C_1(\mathcal{H})$, respectively.

In this preliminary section, we briefly describe unbounded objects extending self-adjoint operators for later use. We begin by recalling the notions of lower semibounded self-adjoint operators and lower semibounded quadratic forms (see \cite{35,48} for their general theory).

2.1. Lower Semibounded Self-Adjoint Operators

Let $T$ be a (densely defined) self-adjoint operator on $\mathcal{H}$ with the spectral decomposition $T = \int_{-\infty}^{\infty} t dE_t$. It is said that $T$ is lower semibounded if there exists an $\ell \in \mathbb{R}$ such that $\langle T \xi, \xi \rangle \geq \ell \|\xi\|^2$ for all $\xi \in \mathcal{D}(T)$ (the domain of $T$), or equivalently, $E_t = 0$ for all $t < \ell$. In this case, the largest such lower bound $\ell$ is $\min \sigma(T)$, where $\sigma(T)$ is the spectrum of $T$. 
2.2. Lower Semibounded Quadratic Forms

A positive quadratic form $q$ on $\mathcal{H}$ is a mapping $q : \mathcal{H} \rightarrow [0, \infty]$ such that, for all $\xi, \eta \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$q(\lambda \xi) = |\lambda|^2 q(\xi) \quad \text{ (with the convention } 0 \cdot \infty = 0),$$

$$q(\xi + \eta) + q(\xi - \eta) = 2q(\xi) + 2q(\eta).$$

The domain of $q$ is $\mathcal{D}(q) := \{ \xi \in \mathcal{H} : q(\xi) < \infty \}$, which is obviously a linear subspace of $\mathcal{H}$. We say that $q : \mathcal{H} \rightarrow (-\infty, \infty]$ is a lower semibounded quadratic form if there exists an $\ell \in \mathbb{R}$ such that $q(\xi) - \ell \|\xi\|^2$ ($\xi \in \mathcal{H}$) is a positive quadratic form.

For every positive self-adjoint operator $T$ on a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, a positive quadratic form $q_T$ on $\mathcal{H}$ is defined by

$$q_T(\xi) := \begin{cases} \|T^{1/2}\xi\|^2 & \text{if } \xi \in \mathcal{D}(T^{1/2}), \\ \infty & \text{otherwise}, \end{cases}$$

whose domain $\mathcal{D}(q_T) = \mathcal{D}(T^{1/2})$ is dense in $\mathcal{H}_0$. The defined $q_T$ is lower semicontinuous on $\mathcal{H}$, or equivalently, $q_T|_{\mathcal{D}(q_T)}$ is a closed quadratic form. It is fundamental that the correspondence $T \mapsto q_T$ is bijective between the positive self-adjoint operators on closed subspaces of $\mathcal{H}$ and the lower semicontinuous positive quadratic forms on $\mathcal{H}$. This extends to a bijective correspondence between the lower semibounded self-adjoint operators on closed subspaces of $\mathcal{H}$ and the lower semicontinuous, lower semibounded quadratic forms on $\mathcal{H}$ (see, e.g., [48, Chap. 10]). In fact, for a lower semibounded self-adjoint operator $T$ on a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, the corresponding quadratic form $q_T$ is given as

$$q_T(\xi) = \begin{cases} \|(T - \ell \mathbb{I}_{\mathcal{H}_0})^{1/2}\xi\|^2 + \ell \|\xi\|^2 & \text{if } \xi \in \mathcal{D}((T - \ell \mathbb{I}_{\mathcal{H}_0})^{12}), \\ \infty & \text{otherwise}, \end{cases}$$

(2.1)

with any choice of $\ell \in (-\infty, \min \sigma(T)]$, where $\mathbb{I}_{\mathcal{H}_0}$ is the identity operator on $\mathcal{H}_0$. When $T = \int_{-\infty}^\infty t dE_t$ is the spectral decomposition with the spectral resolution $(E_t)_{t \in \mathbb{R}}$ on $\mathcal{H}_0$, the alternative expression of $q_T$ is

$$q_T(\xi) = \int_{-\infty}^\infty t d\|E_t\xi\|^2 + \infty \cdot \|P_{\mathcal{H}_0}^\perp \xi\|^2, \quad \xi \in \mathcal{H}$$

(2.2)

(with the usual convention $\infty \cdot 0 = 0$), where $P_{\mathcal{H}_0}^\perp = I - P_{\mathcal{H}_0}$ is the projection onto the orthogonal complement $\mathcal{H}_0^\perp$.

Let $T_1, T_2$ be lower semibounded self-adjoint operators on closed subspaces $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{H}$, respectively. The order $T_1 \leq T_2$ (in the form sense) is defined as $\mathcal{D}(q_{T_2}) \subseteq \mathcal{D}(q_{T_1})$ and $q_{T_1}(\xi) \leq q_{T_2}(\xi)$ for all $\xi \in \mathcal{D}(q_{T_2})$. It is known that $T_1 \leq T_2$ holds if and only if $(T_2 - \lambda I)^{-1} \leq (T_1 - \lambda I)^{-1}$ for some (equivalently, for any) $\lambda \in \mathbb{R}$ with $\lambda < \min \sigma(T_1)$ and $\lambda < \min \sigma(T_2)$, where $(T_i - \lambda I)^{-1}$ is understood to be zero on $\mathcal{H}_i^\perp = \mathcal{D}(q_{T_i})^\perp$ ($i = 1, 2$); see [48, Corollary 10.13]. Furthermore, the form sum $T := T_1 + T_2$ is defined in such a way that $\mathcal{D}(q_T) = \mathcal{D}(q_{T_1}) \cap \mathcal{D}(q_{T_2})$ and $q_T(\xi) := q_{T_1}(\xi) + q_{T_2}(\xi)$ for every $\xi \in \mathcal{D}(q_T)$; see [35, Sect. VI.1.6] and [48, Proposition 10.22].
2.3. Extended Lower Semibounded Self-Adjoint Part

The extended positive part $\widehat{B(\mathcal{H})}_+$ of $B(\mathcal{H})$ (in the sense of Haagerup [22]) is the set of mappings $m : B(\mathcal{H})_+^* \rightarrow [0, \infty]$ that satisfy the following:

1. $m(\alpha \rho) = \alpha m(\rho)$ for all $\alpha \geq 0$ and $\rho \in B(\mathcal{H})_+^*$,
2. $m(\rho_1 + \rho_2) = m(\rho_1) + m(\rho_2)$ for all $\rho_1, \rho_2 \in B(\mathcal{H})_+^*$,
3. $m$ is lower semicontinuous on $B(\mathcal{H})_+^*$.

This notion was originally introduced in [22] in a more general setting to study operator valued weights in theory of von Neumann algebras. Recently in [38], the extended positive part $\widehat{B(\mathcal{H})}_+$ was effectively used in a study of parallel sum of unbounded positive operators. For our purpose it is convenient to slightly generalize $\widehat{B(\mathcal{H})}_+$ as follows:

**Definition 2.2.** We define the extended lower semibounded self-adjoint part $\widehat{B(\mathcal{H})}_{\text{lb}}$ of $B(\mathcal{H})$ to be the set of mappings $m : B(\mathcal{H})_+^* \rightarrow (-\infty, \infty]$ that satisfies, in addition to the above (1)–(3), the following:

4. there exists an $\ell \in \mathbb{R}$ such that $m(\rho) \geq \ell \rho(I)$ for all $\rho \in B(\mathcal{H})_+^*$.

The conic and the order structures of $\widehat{B(\mathcal{H})}_{\text{lb}}$ are simply defined as follows. Let $m, m_1, m_2 \in \widehat{B(\mathcal{H})}_{\text{lb}}$. Define $\alpha m$ ($\alpha \geq 0$), $m_1 + m_2 \in \widehat{B(\mathcal{H})}_{\text{lb}}$ by $(\alpha m)(\rho) := \alpha m(\rho)$, $(m_1 + m_2)(\rho) := m_1(\rho) + m_2(\rho)$, and define $m_1 \leq m_2$ if $m_1(\rho) \leq m_2(\rho)$ for all $\rho \in B(\mathcal{H})_+^*$. Clearly, $\widehat{B(\mathcal{H})}_+$ is included in $\widehat{B(\mathcal{H})}_{\text{lb}}$ as a sub-cone (since $\ell = 0$ is available for condition (4) if $m \in \widehat{B(\mathcal{H})}_+$). Each $A \in B(\mathcal{H})_{\text{sa}}$ (resp., $A \in B(\mathcal{H})_+$) is regarded as an element of $\widehat{B(\mathcal{H})}_{\text{lb}}$ (resp., $\widehat{B(\mathcal{H})}_+$) in a natural way that $A(\rho) := \rho(A)$ for $\rho \in B(\mathcal{H})_+^*$.

The next proposition is a slight modification of [22, Theorem 1.5].

**Proposition 2.2.** For every $m \in \widehat{B(\mathcal{H})}_{\text{lb}}$ there exists a spectral resolution $(E_t)_{t \in \mathbb{R}}$ on a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, i.e., a one-parameter family of non-decreasing and right-continuous orthogonal projections $E_t$ ($t \in \mathbb{R}$) with $E_t \not\subset P_{\mathcal{H}_0}$ as $t \rightarrow \infty$, such that $E_0 = 0$ ($t < \ell$) for some $\ell \in \mathbb{R}$ and

$$m(\rho) = \int_{-\infty}^{\infty} t \rho(E_t) + \infty \cdot \rho(P_{\mathcal{H}_0}^\perp), \quad \rho \in B(\mathcal{H})_+^*. \quad (2.3)$$

Furthermore, $\mathcal{H}_0$ and $(E_t)_{t \in \mathbb{R}}$ are uniquely determined by $m$.

**Proof.** This basically follows from [22, Theorem 1.5], so we only sketch it. Let $m \in \widehat{B(\mathcal{H})}_{\text{lb}}$ with $\ell \in \mathbb{R}$ as in (4). By [22, Theorem 1.5] there is a spectral resolution $(F_t)_{t \geq 0}$ for $m - tI \in \widehat{B(\mathcal{H})}_+$ in a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$. Then (2.3) holds with $E_t := 0$ ($t < \ell$), $F_{t-\ell}$ ($t \geq \ell$). The uniqueness of $\mathcal{H}_0$ follows from that for $m - tI$ (see the proof of [22, Lemma 1.4]). Let $T := \int_{-\infty}^{\infty} t dE_t$, a lower semibounded self-adjoint operator on $\mathcal{H}_0$. By (2.3) we have

$$m(\omega_\xi) = \int_{-\infty}^{\infty} t d\|E_t \xi\|^2 + \infty \cdot \|P_{\mathcal{H}_0}^\perp \xi\|^2, \quad \xi \in \mathcal{H},$$
where \( \omega_\xi := \langle \cdot, \xi \rangle \in B(\mathcal{H})^+_\lambda \), a vector functional. By (2.2) this means that
\[
m(\omega_\xi) = q_T(\xi), \quad \xi \in \mathcal{H},
\]
which shows that \( T \) and hence \((E_t)_{t \in \mathbb{R}}\) are uniquely determined by \( m \).

Since any \( \rho \in B(\mathcal{H})^+_\lambda \) is written as \( \rho = \sum_n \omega_{\xi_n} \) for some \( \{\xi_n\} \) in \( \mathcal{H} \) with \( \sum_n \|\xi_n\|^2 < \infty \), an \( m \in \tilde{B}(\mathcal{H})_{lb} \) is uniquely determined by expression (2.4). Summing up the discussions so far, we conclude that there are bijective correspondences \( m \leftrightarrow T \leftrightarrow q \) between the following three objects:

- elements \( m \) of \( \tilde{B}(\mathcal{H})_{lb} \) (Definition 2.1),
- lower semibounded self-adjoint operators \( T \) on closed subspaces of \( \mathcal{H} \),
- lower semicontinuous, lower semibounded quadratic forms \( q \) on \( \mathcal{H} \) (with not necessarily dense domains).

The correspondence \( m \leftrightarrow T \) is determined by (2.3) and (2.4), and \( T \leftrightarrow q = q_T \) is given by (2.1) and (2.2). These correspondences preserve order and sum (described before Proposition 2.2 and in the paragraph after (2.2)). Below we use the symbol \( T \) to denote elements of \( \tilde{B}(\mathcal{H})_{lb} \) with identification between \( m \leftrightarrow T \).

The following wording will be convenient in Sect. 8.

**Definition 2.3.** Let \( T \in \tilde{B}(\mathcal{H})_{lb} \). We call \((E_t)_{t \in \mathbb{R}}\) in Proposition 2.2 (for \( m = T \)) the spectral resolution of \( T \), \( \mathcal{H}_0 \) the essential part of \( T \) and \( \mathcal{H}_0^\infty \) the \( \infty \)-part of \( T \). We briefly say that \( T \) has a dense domain if the \( \infty \)-part of \( T \) is trivial, that is, \( T \) is densely defined on \( \mathcal{H} \) as a lower semibounded self-adjoint operator, or equivalently \( \mathcal{D}(q_T) \) is dense in \( \mathcal{H} \). Also, we say that \( T \) is bounded if it is a bounded self-adjoint operator on \( \mathcal{H} \).

In particular, when \( \mathcal{H} \) is finite-dimensional with \( n = \dim \mathcal{H} \), each \( T \in \tilde{B}(\mathcal{H})_{lb} \) is represented in the form of diagonalization \( T = \sum_{i=1}^n \lambda_i P_{\xi_i} \), with an orthonormal basis \( \{\xi_i\}_{i=1}^n \) of \( \mathcal{H} \) and \( -\infty < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \infty \) (permitting some \( \lambda_i \)'s being \( \infty \)), where \( P_{\xi_i} \) is the rank-one projection onto \( \mathbb{C}\xi_i \). Therefore, considering \( \tilde{B}(\mathcal{H})_{lb} \) is already non-trivial even in the finite-dimensional setting.

We end the section with two simple lemmas, which will be used in subsequent sections. Their proofs are easy to give, and left to the reader.

**Lemma 2.4.** Let \( T \in \tilde{B}(\mathcal{H})_{lb} \) and \( C \) be a bounded operator from another Hilbert space \( \mathcal{K} \) to \( \mathcal{H} \). Then the mapping
\[
\rho \in B(\mathcal{K})^+_* \mapsto T(C^*C^e) \in (-\infty, \infty]
\]
defines an element \( C^*TC \in \tilde{B}(\mathcal{K})_{lb} \), where we use the standard notation \( (C^*C^e)(X) := \rho(C^*XC) \) for \( X \in B(\mathcal{H}) \).

**Lemma 2.5.** Let \( E \) be a spectral measure in \( \mathcal{H} \) on a measurable space \( \Omega \). Let \( f : \Omega \rightarrow (-\infty, \infty] \) be a measurable function, and assume that \( f \) is bounded from below on \( \Omega \). Then the mapping
\[
\rho \in B(\mathcal{H})^+_\lambda \mapsto \int_{\Omega} f(\omega) \mathrm{d}\rho(E(\omega)) \in (-\infty, \infty]
\]
defines an element of $\widehat{B(\mathcal{H})}_{lb}$.

We write $\int_{\Omega} f \, dE$ for the element of $\widehat{B(\mathcal{H})}_{lb}$ given in Lemma 2.5. When $f$ is an $\mathbb{R}$-valued measurable function on $\Omega$ bounded from below, it is immediate to see that $\int_{\Omega} f \, dE$ is a lower semibounded self-adjoint operator defined by the usual spectral integral. In particular, $\int_{\Omega} f \, dE \in B(\mathcal{H})_{sa}$ if $f$ is bounded on $\Omega$.

3. Extended Real-Valued Operator Convex Functions

A notion of operator convex functions with extended real values in $(-\infty, \infty]$ will be essential in our later discussions. In this section we present a brief exposition of such extended operator convex functions, since the subject has been nowhere discussed so far.

Let $J$ be an arbitrary interval in $\mathbb{R}$, either finite or infinite and either closed or open. In this section we consider a Borel function $f : J \to (-\infty, \infty]$, and assume throughout that $f$ is locally bounded from below, i.e., bounded from below on any compact subset of $J$. We write $B(\mathcal{H})_J$ for the set of $A \in B(\mathcal{H})_{sa}$ whose spectrum $\sigma(A)$ is included in $J$. It is clear that $B(\mathcal{H})_J$ is a convex subset of $B(\mathcal{H})_{sa}$.

Let $A \in B(\mathcal{H})_J$ and $A = \int_{\sigma(A)} t \, dE_A(t)$ be the spectral decomposition of $A$ with the spectral measure $E_A$ of $A$ supported on $\sigma(A)$. By Lemma 2.5 we can define $f(A) \in \widehat{B(\mathcal{H})}_{lb}$ by $f(A) := \int_{\sigma(A)} f(t) \, dE_A(t)$, i.e.,

$$f(A)(\rho) := \int_{\sigma(A)} f(t) \, d\rho(E_A(t)) = \int_J f(t) \, d\rho(E_A(t)) \in (-\infty, \infty], \quad \rho \in B(\mathcal{H})^+_s.$$  \hspace{1cm} (3.1)

When $f$ is a continuous $\mathbb{R}$-valued function on $J$, it is clear that $f(A) \in B(\mathcal{H})_{sa}$ is the usual continuous functional calculus of $A$.

Recall that the usual topology on $\mathbb{R}$ is extended to $(-\infty, \infty]$ as is generated by the intervals $(a, b), (a, \infty]$. The continuity of a function $\psi$ from a metric space $\mathcal{X}$ to $(-\infty, \infty]$ is considered against this topology. Namely, if $x_n \to x$ in $\mathcal{X}$, then $\psi(x_n) \to \psi(x)$ holds even when $\psi(x) = \infty$.

Lemma 3.1. Assume that $f$ is continuous on $J$ as a function to $(-\infty, \infty]$. Then for every $\rho \in B(\mathcal{H})^+_s$, the mapping $A \in B(\mathcal{H})_J \mapsto f(A)(\rho) \in (-\infty, \infty]$ is lower semicontinuous in the operator norm. Furthermore, the same holds even in the strong operator topology whenever $\inf_{t \in J} f(t)/(1 + |t|) > -\infty$.

Proof. For each $n \in \mathbb{N}$ set $f_n := f \wedge n$, which is a continuous $\mathbb{R}$-valued function on $J$. For every $A \in B(\mathcal{H})_J$ and $\rho \in B(\mathcal{H})^+_s$, it follows from the monotone convergence theorem that

$$f(A)(\rho) = \sup_n \int_J f_n(t) \, d\rho(E_A(t)) = \sup_n \rho(f_n(A)),$$
where \( f_n(A) \) is the usual continuous functional calculus of \( A \). Since \( A \mapsto f_n(A) \) is continuous on \( B(\mathcal{H})_f \) in the operator norm, the first assertion follows. Next assume that \( \inf_{t \in J} f(t)/(1 + |t|) > -\infty \) and hence \( \sup_{t \in J} |f_n(t)|/(1 + |t|) < \infty \). It follows (see e.g., [49, Appendix A.2]) that \( A \mapsto f_n(A) \) is continuous on \( B(\mathcal{H})_f \) in the strong operator topology for each \( n \in \mathbb{N} \). Hence the latter assertion holds as well. \( \square \)

**Definition 3.2.** (1) We say that \( f : J \to (-\infty, \infty] \) is operator convex if
\[
 f((1 - \lambda)A + \lambda B) \leq (1 - \lambda)f(A) + \lambda f(B) \quad \text{in} \quad \widehat{B(\mathcal{H})}_{lb},
\]
holds for every \( A, B \in B(\mathcal{H})_f \) with an arbitrary Hilbert space \( \mathcal{H} \) and for any \( \lambda \in (0, 1) \).

(2) We say that \( f : J \to (-\infty, \infty] \) is operator monotone (resp., operator monotone decreasing) if \( A \leq B \) implies \( f(A) \leq f(B) \) (resp., \( f(A) \geq f(B) \)) in \( \widehat{B(\mathcal{H})}_{lb} \) for every \( A, B \in B(\mathcal{H})_f \) with any \( \mathcal{H} \).

If \( f \) is an \( \mathbb{R} \)-valued function on \( J \), then the above definitions are obviously the same as the usual operator convexity and the operator monotonicity of \( f \); see [8,25].

Some basic equivalent conditions for \( f \) being operator convex are given in the next proposition, extending the \( \mathbb{R} \)-valued case in [25, Theorem 2.5.7]. Condition (ii) will particularly be useful in later discussions.

**Proposition 3.3.** Let \( f \) be as stated above. Then the following conditions are equivalent, where Hilbert spaces \( \mathcal{H}, \mathcal{H}_i, \mathcal{K} \) are arbitrary and not fixed:

(i) \( f \) is operator convex;

(ii) For every \( A \in B(\mathcal{H})_f \) and every isometry \( V : \mathcal{K} \to \mathcal{H} \),
\[
 f(V^*AV) \leq V^*f(A)V \quad \text{in} \quad \widehat{B(\mathcal{K})}_{lb};
\]

(iii) For every \( A_i \in B(\mathcal{H}_i)_f \) and every bounded operator \( V_i : \mathcal{K} \to \mathcal{H}_i \) for \( 1 \leq i \leq m \) with any \( m \in \mathbb{N} \) such that \( \sum_{i=1}^m V_i^*V_i = I_{\mathcal{K}} \),
\[
 f\left(\sum_{i=1}^m V_i^*A_iV_i\right) \leq \sum_{i=1}^m V_i^*f(A_i)V_i \quad \text{in} \quad \widehat{B(\mathcal{K})}_{lb};
\]

(iv) For every \( A, B \in B(\mathcal{H})_f \) and every projection \( P \in B(\mathcal{H})_f \),
\[
 f(PAP + (I - P)B(I - P)) \leq Pf(A)P + (I - P)f(B)(I - P) \quad \text{in} \quad \widehat{B(\mathcal{H})}_{lb}.
\]

Here note that \( V^*AV \) in (ii) and \( \sum_{i=1}^n V_i^*A_iV_i \) in (iii) are in \( B(\mathcal{K})_f \) automatically. The proof of the proposition is left to the reader. Indeed, the proof is essentially the same as that of [25, Theorem 2.5.7] by taking account of the following basic facts which are immediate from definition (3.1):

(a) \( f(U^*AU) = U^*f(A)U \) in \( \widehat{B(\mathcal{H})}_{lb} \) for every \( A \in B(\mathcal{H})_f \) and any unitary \( U \) on \( \mathcal{H} \),

(b) for \( T \in \widehat{B(\mathcal{K})}_{lb} \) and \( S \in \widehat{B(\mathcal{H})}_{lb} \), \( T \oplus S \in \widehat{B(\mathcal{K} \oplus \mathcal{H})}_{lb} \) is defined by \( (T \oplus S)(\rho) := T(\rho_1) + S(\rho_2) \) for every \( \rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{bmatrix} \in B(\mathcal{K} \oplus \mathcal{H})_+ \),
(c) \( f(A \oplus B) = f(A) \oplus f(B) \) in \( B(K \oplus H)_{lb} \) for every \( A \in B(K)_J \) and \( B \in B(H)_J \).

Remark 3.4. In Definition 3.2 we can fix an infinite-dimensional separable Hilbert space \( H \). This is seen because every \( A, B \in B(H)_{sa} \) can be decomposed into direct sums \( A = \bigoplus_i A_i \) and \( B = \bigoplus_i B_i \) under a direct sum decomposition \( H = \bigoplus_i H_i \) into separable Hilbert spaces \( H_i \).

Remark 3.5. It is clear, from the definition in Lemma 2.5, that \( f(tI) = f(t)I \) holds for any \( t \in J \), where \((\infty \cdot I)(\rho) = \infty \cdot \rho(I)\) (for \( f(t) = \infty \)). This shows that if \( f \) is operator convex (resp., operator monotone), then it is convex (resp., monotone increasing) on \( J \) as a numerical function with values in \((\infty, \infty)\).

Example 3.6. (1) Here we pick out two exceptional examples of operator convex functions \( f : J \to (-\infty, \infty) \). A trivial example is \( f \equiv \infty \). Another particular one is the case when \( f(t_0) < \infty \) for some \( t_0 \in J \) and \( f(t) = \infty \) for all \( t \in J \setminus \{t_0\} \). This case is confirmed as follows. Let \( A, B \in B(H)_J \), \( 0 < \lambda < 1 \) and \( \xi \in H \). If \(((1 - \lambda)A + \lambda B)(\omega \xi) < \infty \), then \( f(A)(\omega \xi) < \infty \) and \( f(B)(\omega \xi) < \infty \).

The following conditions are equivalent:

\((i)\) \( f \) is operator convex on \( J \);
\((ii)\) \( f \) is \( \mathbb{R} \)-valued and operator convex on \( J^\circ \), \( f(a) \geq f(a^+) \) if \( a \in J \), and \( f(b) \geq f(b^-) \) if \( b \in J \).

Theorem 3.7. Assume that \( f(t) < \infty \) at more than one point in \( J \).
Lemma 3.8. Let \(-\infty < \alpha < \gamma < \beta < \infty\).

1. If a function \(f : (\alpha, \beta) \to (-\infty, \infty]\) satisfies either
   \[f(x) < \infty \quad (\alpha < x < \gamma), \quad f(x) = \infty \quad (\gamma < x < \beta),\]
   then \(f(x) = \infty \quad (\alpha < x < \gamma), \quad f(x) < \infty \quad (\gamma < x < \beta),\]
   then \(f(x)\) is not operator convex on \((\alpha, \beta)\).

2. If \(f\) satisfies (3.3) (resp., (3.4)), then \(f\) is not operator monotone (resp.,
   operator monotone decreasing) on \((\alpha, \beta)\).

Proof. By transforming by a linear function, it suffices to show the first case
with \(\alpha = 0 < \gamma = 2 < \beta\) in each of (1) and (2). For \(0 < \delta < \min\{1, \beta - 2\}\)
consider \(A, B \in B(\mathbb{C}^2(\alpha, \beta))\) and \(\xi \in \mathbb{C}^2\) defined to be
\[
A := \begin{bmatrix} 2 - \delta^2 & 0 \\ 0 & 2 + \delta \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 1 - \delta^2 \\ 1 - \delta^2 & 1 \end{bmatrix}, \quad \xi := [1]_0.
\]
Then one has \(f(A)(\omega_\xi) = \langle f(A)\xi, \xi \rangle = f(2 - \delta^2)\). Since the eigenvalues of \(B\)
are \(\delta^2\) and \(2 - \delta^2\), one has \(f(B)(\omega_\xi) = f(\delta^2)|\langle v_1, \xi \rangle|^2 + f(2 - \delta^2)|\langle v_2, \xi \rangle|^2\),
where \(v_1, v_2\) are the unit eigenvectors of \(B\) for \(\delta^2, 2 - \delta^2\) respectively. Therefore,
\[
\frac{1}{2}(f(A) + f(B))(\omega_\xi) = \frac{1}{2}(f(A)(\omega_\xi) + f(B)(\omega_\xi)) < \infty.
\]
On the other hand, a direct calculation shows that two eigenvalues of \(\frac{1}{2}(A + B)\)
are \(\lambda_1(\delta) := 1 + \delta/4 + o(\delta)\) and \(\lambda_2(\delta) := 2 + \delta/4 + o(\delta)\) as \(\delta \downarrow 0\). With the
unit eigenvectors \(u_1(\delta), u_2(\delta)\) corresponding to \(\lambda_1(\delta), \lambda_2(\delta)\) respectively, we have
\[
f\left(\frac{1}{2}(A + B)\right)(\omega_\xi) = f(\lambda_1(\delta))|\langle u_1(\delta), \xi \rangle|^2 + f(\lambda_2(\delta))|\langle u_2(\delta), \xi \rangle|^2 = \infty,
\]
for all sufficiently small \(\delta > 0\), since \(\langle u_2(\delta), \xi \rangle \neq 0\) obviously. Therefore,
\(f\left(\frac{1}{2}(A + B)\right) \leq \frac{1}{2}(f(A) + f(B))\) does not hold, and (1) has been shown.

Next we show (2). Since \(A \geq B\) as immediately verified, one has \((A + B)/2 \leq A\) but
\(f\left(\frac{1}{2}(A + B)\right) \leq f(A)\) does not hold. Hence \(f\) is not operator monotone.

Proof of Theorem 3.7. (1) \(\text{(i)} \implies \text{(ii)}\). Assume item (i); then by Remark 3.5, \(f\)
is numerically convex on \(J\). By assumption on \(f\) (having finite values at more than one point)
there are \(a_0, b_0 \in [a, b]\) with \(a_0 < b_0\) such that \(f(t) = \infty\)
for all \(t \in J \setminus [a_0, b_0]\), \(f|\{a_0, b_0\}\) is an \(\mathbb{R}\)-valued convex function, \(f(a_0) \geq f(a_0^+)\)
if \(a_0 \in J\), and \(f(b_0) \geq f(b_0^-)\) if \(b_0 \in J\), where the limits \(f(a_0^+)\) and \(f(b_0^-)\) exist
in \((\infty, \infty]\) thanks to the numerical convexity of \(f|\{a_0, b_0\}\). If \(a < a_0\) (resp., \(b_0 < b\)),
then we can apply Lemma 3.8(1) with \(a \leq \alpha < \gamma = a_0 < \beta \leq b_0\)
(resp., \(a_0 \leq \alpha < \gamma = b_0 < \beta \leq b\)) to find a contradiction to (i). Hence \(a_0 = a\)
and \( b_0 = b \). Moreover, by applying property (i) to \( A, B \in B(\mathcal{H})_{(a,b)} \) we see that \( f \) is \( \mathbb{R} \)-valued and operator convex on \( J^\circ = (a, b) \).

(ii) \( \implies \) (i). When \( J = (a, b) \), this is obvious. In the following, we will prove (ii) \( \implies \) (i) when \( J = [a, b] \) (so \( -\infty < a < b < \infty \)). The proof is similar when \( J = [a, b] \) or \( J = (a, b] \). Now assume item (ii). First, assume further that \( f \) is continuous at \((a, b)\) (hence on the whole \([a, b]\)) as a function to \((-\infty, \infty]\).

We show property (ii) of Proposition 3.3. Let \( A \in B(\mathcal{H})_{[a, b]}, V : \mathcal{K} \to \mathcal{H} \) be an isometry, and \( \rho \in B(\mathcal{K})^+ \). Choose a sequence \( \delta_n \in (0, (b-a)/2) \) with \( \delta_n \searrow 0 \), and define \( r_n(t) := (t \vee (a + \delta_n)) \wedge (b - \delta_n) \) for \( t \in [a, b] \). Since \( r_n(A) \in B(\mathcal{H})_{(a,b)} \), one has \( f(V^* r_n(A)V) \leq V^* f(r_n(A))V \) in \( B(\mathcal{K})_{\mathbb{R}} \) (thanks to Proposition 3.3 for \( f \text{\scriptsize{\|\(\|_{(a,b)\}}}} \)). Since \( \|V^* r_n(A)V - V^* AV\| \to 0 \), by Lemma 3.1 one has

\[
f(V^* AV)(\rho) \leq \lim_{n \to \infty} \inf_{n \to \infty} f(V^* r_n(A)V)(\rho) \leq \liminf_{n \to \infty} f(V^* f(r_n(A))V)(\rho). \quad (3.5)
\]

Furthermore, note that

\[
(V^* f(r_n(A))V)(\rho) = \int_{[a,b]} f(t) \, d\rho(V^* E_{r_n(A)}(t)V) = \int_{[a,b]} f(r_n(t)) \, d\rho(V^* E_A(t)V). \quad (3.6)
\]

When \( f(a) < \infty \) and \( f(b) < \infty \) (hence \( f(t) < \infty \) for all \( t \in [a, b] \)), it is clear that \( f(r_n(t)) \to f(t) \) uniformly on \([a, b]\). Taking the (numerical) convexity of \( f \) into consideration, we observe the following: When \( f(a) < \infty \) and \( f(b) = \infty \), there are an \( n_0 \) and some \( c \in (a, b) \) such that \( f(r_n(t)) \to f(t) \) uniformly on \([a, c]\) and \( f(r_n(t)) \nearrow f(t) \) for all \( t \in (c, b] \) as \( n_0 \leq n \to \infty \). When \( f(a) = \infty \) and \( f(b) < \infty \), the situation is similar. When \( f(a) = f(b) = \infty \), there is an \( n_0 \) such that \( f(r_n(t)) \nearrow f(t) \) for all \( t \in [a, b] \) as \( n_0 \leq n \to \infty \). Hence, using the bounded and the monotone convergence theorems (after dividing the integration over \([a, b]\) into those over \([a, c]\) and \((c, b]\) if necessary), we have

\[
\lim_{n \to \infty} \int_{[a,b]} f(r_n(t)) \, d\rho(V^* E_A(t)V) = \int_{[a,b]} f(t) \, d\rho(V^* E_A(t)V) = (V^* f(A)V)(\rho). \quad (3.7)
\]

Combining (3.5)–(3.7) gives \( f(V^* AV)(\rho) \leq (V^* f(A)V)(\rho) \), showing that \( f \) is operator convex when \( f \) is continuous on the whole \([a, b]\).

To show (i) without the continuity assumption at \((a, b)\), set \( f_0(t) := f(t) \) for \( t \in (a, b) \), \( f_0(a) := f(a^+) \) and \( f_0(b) := f(b^-) \). Then \( f_0 \) is operator convex as shown above. With the operator convex functions \( \chi_a, \chi_b \) given in Example 3.6(2), one can choose increasing \( \alpha_n, \beta_n \geq 0 \) such that \( f_0 + \alpha_n \chi_a + \beta_n \chi_b \nearrow f \) and hence, by the monotone convergence theorem,

\[
f(A)(\rho) = \lim_n (f_0 + \alpha_n \chi_a + \beta_n \chi_b)(A)(\rho) = \lim_n (f_0(A)(\rho) + \alpha_n \chi_a(A)(\rho) + \beta_n \chi_b(A)(\rho))
\]

for all \( A \in B(\mathcal{H})_{[a,b]} \) and \( \rho \in B(\mathcal{H})^+ \) (and any \( \mathcal{H} \)). Therefore, (i) has been shown.
(2) (i') \implies (ii'). The proof is similar to that of (i) \implies (ii), so we omit the details.

(ii') \implies (i'). As in the proof of (ii) \implies (i), we prove the case \( J = [a, b] \)
and first assume that \( f \) is continuous at \( a, b \). Since \( f(a) \) must be finite, it is clear that \( f \) is operator monotone on \([a, b] \). For the remaining, by transforming \([a, b] \) to \([0, 1] \) by a linear function, we may assume that \([a, b] = [0, 1] \). Let
\[ a, b \in B(H)[0, 1] \] with \( A \leq B \). For any \( r \in (0, 1) \), since \( rA, rB \in B(H)[0, 1] \)
and \( rA \leq rB \), one has \( f(rA) \leq f(rB) \). For every \( \rho \in B(H)^+ \) note that
\[
f(rA)(\rho) = \int_{[0,1]} f(t) d\rho(E_{rA}(t)) = \int_{[0,1]} f(rt) d\rho(E_A(t)) \]
\[
\int_{[0,1]} f(t) d\rho(E_A(t)) = f(A)(\rho)
\]
as \( r \not\rightarrow 1 \) by the monotone convergence theorem. The same holds for \( f(rB)(\rho) \),
so that \( f(A)(\rho) \leq f(B)(\rho) \).

As in the proof of (ii) \implies (i), it remains to show that \( \chi_0 \) is operator monotone decreasing and \( \chi_1 \) is operator monotone on \( B(H)[0, 1] \). Let
\[ A, B \in B(H)[0, 1] \] with \( A \leq B \). Then \( \ker A \supseteq \ker B \) and hence \( \chi_0(A) \geq \chi_0(B) \).
Since \( I - A \geq I - B \), one has \( \ker(I - A) \subseteq \ker(I - B) \), implying \( \chi_1(A) \leq \chi_1(B) \).

\[ \square \]

4. Pusz–Woronowicz Functional Calculus

Throughout this section, we fix an arbitrary Borel function \( \phi : [0, \infty)^2 \rightarrow (-\infty, \infty] \) which is homogeneous and locally bounded from below (i.e., bounded from below on any compact subset of \([0, \infty)^2 \)). Here, \( \phi \) is homogeneous if \( \phi(\lambda x, \lambda y) = \lambda \phi(x, y) \) holds for every \( \lambda, x, y \geq 0 \). Hence \( \phi(0, 0) = 0 \) necessarily holds. Also we remark that \( \phi \) is locally bounded from below if and only if \( \phi(t, 1 - t) \) on \([0, 1] \) is bounded from below. Typical and important examples of such functions are \( \psi(x, y) := x \log(x/y) \) and \( \phi_\alpha(x, y) := y(x/y)^\alpha = x^\alpha y^{1-\alpha} \) (where \( \alpha \geq 0 \)), with conventions \( \psi(0, 0) = \phi_\alpha(0, 0) = 0 \) and \( \psi(x, 0) = \phi_\alpha(x, 0) = \infty \) for \( x > 0 \) and \( \alpha \geq 1 \). Both of them play an important role, for instance, in quantum information theory.

Although the original formalism of Pusz and Woronowicz in \([46, 47] \)
based on positive sesquilinear forms is available even for unbounded functions as above, we will do reformulate their functional calculus in terms of unbounded objects extending self-adjoint operators discussed in Sect. 2. The content of this section is somewhat expository and also may be regarded as an upgrade of the discussions by Hatano and the second-named author in \([24, \text{Sect. 4}] \) to Borel functions locally bounded from below. The approach here is axiomatic unlike \([24] \), though the theory is about a functional calculus still depending on the distinguished function \( \phi \) unlike Kubo and Ando’s axiomatization of operator connections \([40] \). (Axiomatization of Kubo and Ando’s type will be discussed in Sect. 10.)
Associated with the function \( \phi \) above, we introduce the Pusz–Woronowicz functional calculus \( \phi(A, B) \) of pairs \((A, B)\) in \( B(\mathcal{H})_+ \), whose values are elements of the extended lower semibounded self-adjoint part \( \widetilde{B(\mathcal{H})}_{lb} \) introduced in Sect. 2. The definition is given in an axiomatic fashion with two postulates as follows:

**Definition 4.1.** The Pusz–Woronowicz functional calculus (or PW-functional calculus for short) associated with \( \phi \) is an operation giving a two-variable mapping

\[
(A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ \mapsto \phi(A, B) \in \widetilde{B(\mathcal{H})}_{lb}
\]

for each Hilbert space \( \mathcal{H} \) such that the following properties hold:

1. *(Extending the usual functional calculus)* Whenever \((A, B)\) is a commuting pair, \( \phi(A, B) \) is given by the usual functional calculus, that is,

\[
\phi(A, B)(\rho) = \int_{\sigma(A) \times \sigma(B)} \phi(x, y) d\rho(E_{(A,B)}(x, y)), \quad \rho \in B(\mathcal{H})_+^*,
\]

where \( E_{(A,B)} \) denotes the joint spectral measure of the pair \((A, B)\); see, e.g., [10, Exercise 12 in Sect. 10] (also [24, Sect. 2.2] for a handy description of its equivalent form).

2. *(Operator homogeneity)* For any bounded operator \( C : \mathcal{K} \to \mathcal{H} \) from another Hilbert space \( \mathcal{K} \) with \( \text{ran}(A + B) \subseteq \text{ran} C \), the closures of the ranges of \( A + B, C \) respectively, we have

\[
\phi(C^*AC, C^*BC) = C^*\phi(A, B)C,
\]

where the right-hand side is in the sense of Lemma 2.4.

Both items (1) and (2) are quite natural as a functional calculus associated with \( \phi \). The former makes sense thanks to Lemma 2.5. The latter reflects the homogeneity of \( \phi \) and also plays a role of intertwining property built in the recent notion of non-commutative functions; see, e.g., [34] and [1, Part Two].

As item (2) indicates, we will discuss operators on different Hilbert spaces at the same time, and hence we will sometimes denote by \( I_\mathcal{H} \) the identity operator on a Hilbert space \( \mathcal{H} \) to avoid any confusion.

**Remark 4.2.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \). The extended lower semibounded self-adjoint part \( \widetilde{\mathcal{M}}_{lb} \) of \( \mathcal{M} \) is defined by restricting the domain of \( m \) to \( \mathcal{M}_+^+ \) (the positive part of the predual \( \mathcal{M}_+^* \)) in Definition 2.1, as in [22, Definition 1.1]. From [22, Definition 1.8, Proposition 1.9] and Definition 4.1(2) we notice that \( \phi(A, B) \in \widetilde{\mathcal{M}}_{lb} \) whenever \( A, B \in \mathcal{M}_+ \). Thus, our study of the PW-functional calculus can also be done associated with a von Neumann algebra.

The next theorem justifies the above definition.

**Theorem 4.3.** The PW-functional calculus associated with \( \phi \) exists and is uniquely determined.
We will prove this based on the description of PW-functional calculus discussed in [24]. In what follows, we will use the following notations: For a pair \((A, B)\) in \(B(\mathcal{H})_+\), we write
\[
\mathcal{H}_{A,B} := \text{ran}(A + B) = \ker(A + B),
\]
and define a bounded operator \(T_{A,B} : \mathcal{H} \to \mathcal{H}_{A,B}\) by
\[
T_{A,B} \xi := (A + B)^{1/2} \xi \in \mathcal{H}_{A,B}, \quad \xi \in \mathcal{H}.
\]
By construction, \(T_{A,B}\) has a dense range. Remark that \(T_{A,B}\) was given in [24, Sect. 3.1] by the block matrix
\[
T_{A,B} = \begin{bmatrix} 0 & [A]_{A,B} + [B]_{A,B} \end{bmatrix}^{1/2} \quad \text{along} \quad \mathcal{H} = \ker(A + B) \oplus \mathcal{H}_{A,B},
\]
where \([A]_{A,B}, [B]_{A,B}\) are the restrictions to \(\mathcal{H}_{A,B}\) of \(A, B\) respectively.

The next lemma is a collection of technical facts, which will be used successively. Items (1) and (2) are just extracted from [24].

**Lemma 4.4.** With \(\mathcal{H}_{A,B} \) and \(T_{A,B}\) defined above, we have the following facts:

1. For any pair \((A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+\), there exists a unique pair \((R_{A,B}, S_{A,B})\) of positive contractions on \(\mathcal{H}_{A,B}\) such that
   \[
   R_{A,B} + S_{A,B} = I_{\mathcal{H}_{A,B}},
   \]
   \[
   (A, B) = (T_{A,B}^* R_{A,B} T_{A,B}, T_{A,B}^* S_{A,B} T_{A,B}).
   \]
2. For any pair \((A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+\) and any bounded operator \(C : \mathcal{K} \to \mathcal{H}\) with another Hilbert space \(\mathcal{K}\), there exists an isometry \(\hat{C} : \mathcal{K}_{C^*AC,C^*BC} \to \mathcal{H}_{A,B}\) such that
   \[
   \hat{C} T_{C^*AC,C^*BC} = T_{A,B} C,
   \]
   \[
   (R_{C^*AC,C^*BC}, S_{C^*AC,C^*BC}) = (\hat{C}^* R_{A,B} \hat{C}, \hat{C}^* S_{A,B} \hat{C}).
   \]
   Moreover, if \(\text{ran}(A + B) \subseteq \text{ran} C\), then \(\hat{C}\) must be a unitary transform.
3. Let \(R, S\) be a pair of positive contractions on \(\mathcal{H}\) such that \(R + S = I_{\mathcal{H}}\).
   Then we have
   \[
   \phi(R, S) = \phi(R, 1 - R) = \phi(1 - S, S),
   \]
   where \(\phi(R, 1 - R), \phi(1 - S, S) \in B(\mathcal{H})_{1b}\) are defined by
   \[
   \phi(R, 1 - R)(\rho) := \int_{[0,1]} \phi(r, 1 - r) \, d\rho(E_R(r)),
   \]
   \[
   \phi(1 - S, S)(\rho) := \int_{[0,1]} \phi(1 - s, s) \, d\rho(E_S(s))
   \]
   for every \(\rho \in B(\mathcal{H})_+^*\) with the spectral measures \(E_R, E_S\) of \(R, S\) respectively.

**Proof.** Item (1) is in [24, Sect. 3.1]. The first part of (2) is found in the proof of [24, Theorem 9], and the latter part is obvious; see [24, Remark 10].
(3) Note that $f(R,1-R), f(1-S,S)$ are well defined by Lemma 2.5. Let $E$ be the joint spectral measure of $(R,S)$. Observe that for any Borel sets $\Phi, \Psi \subseteq [0,1]$,

$$E(\Phi \times \Psi) = \chi_\Phi(R)\chi_\Psi(I_{\mathcal{H}} - R) = \int_{[0,1]} \chi_\Phi \times \Psi(r, 1-r) dE_R(r).$$

For each $\rho \in B(\mathcal{H})_+^+$, the monotone class theorem yields

$$\rho(E(\Lambda)) = \int_{[0,1]} \chi_\Lambda(r, 1-r) d\rho(E_R(r))$$

for any Borel subset $\Lambda \subseteq [0,1]^2$. This immediately implies, by Definition 4.1(1) and (4.3), that

$$\phi(R,S)(\rho) = \int_{[0,1]^2} \phi(r,s) d\rho(E(r,s)) = \phi(R,1-R)(\rho).$$

The other expression is shown in exactly the same manner. □

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let $(A,B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+$ be an arbitrary pair. Since $R_{A,B} + S_{A,B} = I_{\mathcal{H}_{A,B}}$ by Lemma 4.4(1), we have the joint spectral measure $E$ of $(R_{A,B}, S_{A,B})$. Then we consider $T_{A,B}^* \phi(R_{A,B}, S_{A,B}) T_{A,B}$ in the sense of Lemma 2.4 with

$$\phi(R_{A,B}, S_{A,B}) (\rho) := \int_{[0,1]^2} \phi(r,s) d\rho(E(r,s)), \quad \rho \in B(\mathcal{H}_{A,B})_+^+.$$ 

This procedure is the same as [24, Eq. (1)]. Whenever the desired $\phi(A,B)$ exists, we must have

$$\phi(A,B) = \phi(T_{A,B}^* R_{A,B} T_{A,B}, T_{A,B}^* S_{A,B} T_{A,B}) = T_{A,B}^* \phi(R_{A,B}, S_{A,B}) T_{A,B}$$

by Lemma 4.4(1) again and due to Definition 4.1(2). Since $\phi(R_{A,B}, S_{A,B})$ is uniquely determined due to Definition 4.1(1), this implies the uniqueness of $\phi(A,B)$ (if it exists). Therefore, it suffices to show that $(A,B) \mapsto T_{A,B}^* \phi(R_{A,B}, S_{A,B}) T_{A,B}$ indeed enjoys postulates (1) and (2) of Definition 4.1.

We first confirm item (1). Assume that $(A,B)$ is a commuting pair, and let $A_1, B_1$ be the restrictions to $\mathcal{H}_{A,B}$ of $A, B$ respectively. Consider the Borel functions

$$r(x,y) := \begin{cases} x / (x+y) & (x+y > 0), \\ 0 & (x+y = 0) \end{cases}, \quad s(x,y) := \begin{cases} y / (x+y) & (x+y > 0), \\ 0 & (x+y = 0) \end{cases}.$$

Then $R_{A,B} = r(A_1, B_1)$ and $S_{A,B} = s(A_1, B_1)$ were established in the proof of [24, Proposition 2]. Namely, $R_{A,B}$ and $S_{A,B}$ are given by the restrictions to $\mathcal{H}_{A,B}$ of $r(A,B)$ and $s(A,B)$, respectively, i.e.,

$$R_{A,B} = \int_{\sigma(A) \times \sigma(B)} r(x,y) dE_{(A,B)}(x,y) \bigg|_{\mathcal{H}_{A,B}},$$

$$S_{A,B} = \int_{\sigma(A) \times \sigma(B)} s(x,y) dE_{(A,B)}(x,y) \bigg|_{\mathcal{H}_{A,B}}.$$
Here it should be noticed that the projection from $\mathcal{H}$ onto $\mathcal{H}_{A,B}$ is exactly $I_{\mathcal{H}} - E_{(A,B)}(\{(0,0)\})$, and hence the above restrictions are well defined. For any Borel sets $\Phi, \Psi \subseteq [0, 1]$, we observe that

$$\chi_{\Phi}(R_{A,B})\chi_{\Psi}(S_{A,B}) = \int_{\sigma(A) \times \sigma(B)} \chi_{\Phi}(r(x,y))\chi_{\Psi}(s(x,y)) \, dE_{(A,B)}(x,y) \bigg|_{\mathcal{H}_{A,B}}$$

and hence

$$\langle T_{A,B}^*E_{A,B}(\Phi \times \Psi)T_{A,B}\xi, \xi \rangle$$

$$= \langle \chi_{\Phi}(R_{A,B})\chi_{\Psi}(S_{A,B})(A + B)^{1/2}\xi, (A + B)^{1/2}\xi \rangle$$

$$= \int_{\sigma(A) \times \sigma(B)} \chi_{\Phi}(r(x,y))\chi_{\Psi}(s(x,y))$$

$$d(E_{(A,B)}(x,y)(A + B)^{1/2}\xi, (A + B)^{1/2}\xi)$$

$$= \int_{\sigma(A) \times \sigma(B)} \chi_{\Phi}(r(x,y))\chi_{\Psi}(s(x,y))(x + y) \, d(E_{(A,B)}(x,y)\xi, \xi)$$

for any $\xi \in \mathcal{H}$. Thus, for each $\rho \in B(\mathcal{H})^+_*$, the monotone class theorem ensures that

$$\rho(T_{A,B}^*E_{A,B}(\Lambda)T_{A,B}) = \int_{\sigma(A) \times \sigma(B)} \chi_{\Lambda}(r(x,y), s(x,y))(x + y) \, d\rho(E_{(A,B)}(x,y))$$

for any Borel set $\Lambda \subseteq [0, 1]^2$. Therefore, for every $\rho \in B(\mathcal{H})^+_*$, we have

$$(T_{A,B}^*\phi(R_{A,B}, S_{A,B})T_{A,B})(\rho)$$

$$= \int_{[0,1]^2} \phi(r,s) \, d\rho(T_{A,B}^*E_{r,s}T_{A,B})$$

$$= \int_{\sigma(A) \times \sigma(B)} \phi(r(x,y), s(x,y))(x + y) \, d\rho(E_{(A,B)}(x,y))$$

$$= \int_{\sigma(A) \times \sigma(B)} \phi(x,y) \, d\rho(E_{(A,B)}(x,y))$$

thanks to the homogeneity of $\phi$. This is the identity in item (1).

Next we confirm item (2), i.e., operator homogeneity. Let $C$ be a bounded operator from another Hilbert space $\mathcal{K}$ to $\mathcal{H}$ as in item (2) of Definition 4.1. In what follows, we will use Lemma 4.4(2) freely and write $A' = C^*AC$, $B' = C^*BC$ for short. Let $E$ be the joint spectral measure of $(R_{A,B}, S_{A,B})$; then that of $(R_{A',B'}, S_{A',B'})$ is given by $\widehat{C}E(\cdot)\widehat{C}$ thanks to the simultaneous unitary equivalence of $(R_{A',B'}, S_{A',B'})$ to $(R_{A,B}, S_{A,B})$ with $\widehat{C}$. Hence we have

$$\phi(R_{A',B'}, S_{A',B'})(\rho') = \int_{[0,1]^2} \phi(r,s) \, d\rho'(\widehat{C}E(r,s)\widehat{C})$$

$$= \phi(R_{A,B}, S_{A,B})(\widehat{C}\rho'\widehat{C}^*)$$
for every $\rho' \in B(\mathcal{K}_{A',B'})^+$. Therefore, it follows that

$$
(T_{A',B'}^*(R_{A',B'}, S_{A',B'})T_{A',B'}(\rho) = \phi(R_{A',B'}, S_{A',B'})T_{A',B'}(\rho T_{A',B'})
$$

$$
= \phi(R_{A,B}, S_{A,B})(\tilde{C}T_{A',B'}^*\rho T_{A',B'}^*\tilde{C}^*)
$$

$$
= \phi(R_{A,B}, S_{A,B})(T_{A,B}C\rho C^*T_{A,B})
$$

$$
= (C^*T_{A,B}^*\phi(R_{A,B}, S_{A,B})T_{A,B}C)(\rho)
$$

for every $\rho \in B(\mathcal{K})^+$. This is the required operator homogeneity. \hfill \Box

The proof of Theorem 4.3 (or Definition 4.1 itself) says that an explicit realization of the PW-functional calculus associated with $\phi$ is

$$
\phi(A, B) = T_{A,B}^*\phi(R_{A,B}, S_{A,B})T_{A,B}
$$

(4.4)

for every $A, B \in B(\mathcal{H})_+$. We point out that it is indeed one of the explicit realizations of $\phi(A, B)$ and there are many ways to construct $\phi(A, B)$ corresponding to ‘sections’ in Pusz and Woronowicz’s terminology (see [47]).

Remark that formula (4.4) is clearly rewritten as

$$
\phi(A, B) = (A + B)^{1/2}\phi(R, S)(A + B)^{1/2}
$$

$$
= (A + B)^{1/2}\phi(R, 1 - R)(A + B)^{1/2}
$$

(4.5)

$$
= (A + B)^{1/2}\phi(1 - S, S)(A + B)^{1/2}
$$

with $R := R_{A,B} \oplus 0$ and $S := S_{A,B} \oplus 0$ on $\mathcal{H} = \mathcal{H}_{A,B} \oplus \mathcal{H}_{A,B}^+$, though equality $\phi(R, S) = \phi(R, I - R) = \phi(I - S, S)$ does not hold in general.

Remark 4.5. As explained in [24], $(T_{A,B} : \mathcal{H} \rightarrow \mathcal{H}_{A,B}, R_{A,B}, S_{A,B})$ is a compatible representation (in the sense of [46]) of two positive sesquilinear forms

$$
\alpha(\xi, \eta) := \langle A\xi, \eta \rangle, \quad \beta(\xi, \eta) := \langle B\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.
$$

Then, Pusz and Woronowicz’s original functional calculus $\phi(\alpha, \beta)$ becomes, by their construction in [46, 47], as follows:

$$
\phi(\alpha, \beta)(\xi, \xi) = \langle \phi(R_{A,B}, S_{A,B})T_{A,B}\xi, T_{A,B}\xi \rangle
$$

$$
= (T_{A,B}^*\phi(R_{A,B}, S_{A,B})T_{A,B})(\omega_\xi) = \phi(A, B)(\omega_\xi),
$$

where $\omega_\xi(X) := \langle X\xi, \xi \rangle$ for $\xi \in \mathcal{H}$, $X \in B(\mathcal{H})$. Thus, the formulation of the PW-functional calculus in this section agrees with their original one. Therefore, their technique of obtaining variational integral expressions is available in our setting too; yet we will discuss in Sect. 9 integral expressions in a different manner.

Here is a basic property of the PW-functional calculus, saying that the PW-functional calculus is well behaved with respect to direct sums. This will be used in the proof of Theorem 4.10 below.

**Proposition 4.6.** Let $\phi$ be any Borel function on $[0, \infty)^2$ which is homogeneous and locally bounded from below. If $A, B \in B(\mathcal{H})_+$ be given in direct sums as

$$
\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i, \quad A = \bigoplus_{i \in I} A_i, \quad B = \bigoplus_{i \in I} B_i
$$
with $A_i, B_i \in B(\mathcal{H}_i)_+$, then we have
\[
\phi(A, B)(\rho) = \sum_{i \in I} \phi(A_i, B_i)(P_i\rho P_i)
\]
for all $\rho \in B(\mathcal{H})_+^*$, where $P_i$ is the orthogonal projection $\mathcal{H}$ onto $\mathcal{H}_i$ for each $i \in I$. When $\phi$ is bounded,
\[
\phi(A, B) = \bigoplus_{i \in I} \phi(A_i, B_i)
\]
holds on $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ in the usual sense.

Proof. We easily observe that
\[
\mathcal{H}_{A,B} = \bigoplus_{i \in I} \mathcal{H}_{A_i, B_i}, \quad T_{A,B} = \bigoplus_{i \in I} T_{A_i, B_i},
\]
\[
R_{A,B} = \bigoplus_{i \in I} R_{A_i, B_i}, \quad S_{A,B} = \bigoplus_{i \in I} S_{A_i, B_i}
\]
on $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$. Let $n \in \mathbb{N}$ be fixed for a while, and set $\phi_n(x, y) := \phi(x, y) \wedge n$, a bounded Borel function on $[0, \infty)^2$. Since the usual functional calculus is compatible with direct sum, we observe that
\[
\phi_n(R_{A,B}, S_{A,B}) = \bigoplus_{i \in I} \phi_n(R_{A_i, B_i}, S_{A_i, B_i}),
\]
and hence
\[
T_{A,B}^* \phi_n(R_{A,B}, S_{A,B}) T_{A,B} = \bigoplus_{i \in I} T_{A_i, B_i}^* \phi_n(R_{A_i, B_i}, S_{A_i, B_i}) T_{A_i, B_i}.
\]
It follows that
\[
(T_{A,B} \rho T_{A,B}^*) (\phi_n(R_{A,B}, S_{A,B})) = \rho(T_{A,B}^* \phi_n(R_{A,B}, S_{A,B}) T_{A,B})
\]
\[
= \sum_{i \in I} \rho(P_i T_{A_i, B_i}^* \phi_n(R_{A_i, B_i}, S_{A_i, B_i}) T_{A_i, B_i} P_i)
\]
\[
= \sum_{i \in I} (T_{A_i, B_i}(P_i \rho P_i) T_{A_i, B_i}^* \phi_n(R_{A_i, B_i}, S_{A_i, B_i}))
\]
for any $\rho \in B(\mathcal{H})_+^*$. Since $\phi$ is lower bounded, we can take the limit as $n \to \infty$ (see the proof of Lemma 3.1) to obtain
\[
(T_{A,B}^* \phi(R_{A,B}, S_{A,B}) T_{A,B})(\rho) = \sum_{i \in I} (T_{A_i, B_i}^* \phi(R_{A_i, B_i}, S_{A_i, B_i}) T_{A_i, B_i})(P_i \rho P_i).
\]
The statement when $\phi$ is bounded is clear from the above discussion. □

The following special values of $\phi(A, B)$ will play a role in our further study of the PW-functional calculus.

Lemma 4.7. Let $A \in B(\mathcal{H})_+$ with the spectral measure $E_A$, and $\alpha, \beta \in [0, \infty)$. Then the following hold:
(1) $\hat{\phi}(A, \alpha I) = \phi(A, \alpha)$ and $\phi(\alpha I, A) = \phi(\alpha, A)$, where $\phi(A, \alpha), \phi(\alpha, A) \in B(H)_{lb}$ are defined by

$$\phi(A, \alpha)(\rho) := \int_{\sigma(A)} \phi(t, \alpha) \, d\rho(E_A(t)),$$

$$\phi(\alpha, A)(\rho) := \int_{\sigma(A)} \phi(\alpha, t) \, d\rho(E_A(t))$$

for every $\rho \in B(H)_+^*$. 

(2) $\phi(\alpha A, \beta A) = \phi(\alpha, \beta)A$, where $\infty A$ means $\infty \cdot E_A((0, \infty))$. In particular, $\phi(A, O) = \phi(1, 0)A$ and $\phi(O, A) = \phi(0, A) = \phi(0, 1)A$, where $O$ denotes the zero operator at this moment to distinguish it from the scalar zero.

By Lemma 2.4 note that $\phi(A, \alpha), \phi(\alpha, A) \in B(H)_{lb}$ are well defined. Then (1) is seen by using the monotone class theorem as in the proof of Lemma 4.4(3). The proof of (2) is also easy from $\phi(\alpha I_{H_A}, \beta I_{H_A}) = \phi(\alpha, \beta)I_{H_A}$, where $H_A := \overline{\operatorname{ran}} A$. The details are left to the reader.

The next two theorems are our main observations in this section.

**Theorem 4.8.** (cf. [24, Theorem 4(1)]) For any pair $(A, B)$ in $B(H)_+$ we have

$$\phi(A, B) = \begin{cases} A^{1/2} \phi(1, A^{-1/2}BA^{-1/2})A^{1/2} & \text{if } A \in B(H)_{++}, \\ B^{1/2} \phi(B^{-1/2}AB^{-1/2}, 1)B^{1/2} & \text{if } B \in B(H)_{++}, \end{cases}$$

where $\phi(1, A^{-1/2}BA^{-1/2})$ and $\phi(B^{-1/2}AB^{-1/2}, 1)$ are given as in Lemma 4.7(1).

**Proof.** Assume that $A \in B(H)_{++}$. The operator homogeneity with $C = A^{1/2}$ implies that

$$\phi(A, B) = \phi(A^{1/2}IA^{1/2}, A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2})$$

$$= A^{1/2} \phi(I, A^{-1/2}BA^{-1/2})A^{1/2}$$

$$= A^{1/2} \phi(1, A^{-1/2}BA^{-1/2})A^{1/2}$$

by Lemma 4.7(1). The other case can be confirmed in the same way. \hfill \Box

The discussion above uses operator homogeneity (arising from the homogeneity of $\phi$) explicitly and hence is more conceptual than that of [24, Theorem 4(1)].

**Remark 4.9.** An operator connection (in particular, operator mean) $A\sigma B$ for $A, B \in B(H)_+$ in the Kubo–Ando sense [40] is defined corresponding to an ($\mathbb{R}$-valued) positive operator monotone function $h$ (with $h(1) = 1$ for operator mean) on $[0, \infty)$. As shown in [24], $A\sigma B$ is captured as the PW-functional calculus $\phi(A, B)$ associated with the function $\phi$ defined by

$$\phi(x, y) := xh(y/x) \quad (x > 0, \ y \geq 0), \quad \phi(0, y) := \beta y \quad (y \geq 0),$$

where $\beta := \lim_{t \to \infty} h(t)/t$. In this setting, operator homogeneity in Definition 4.1 was formerly shown in [19, Theorem 3] by appealing to [40, Theorem
3.4], and recently extended to operator connections of positive \( \tau \)-measurable operators (in the von Neumann algebra setting) in [29, Theorem 3.31]. In our general formalism, this property is incorporated into the definition of the PW-functional calculus. When \( A \) is invertible, the first expression of Theorem 4.8 becomes

\[
\phi(A, B) = A^{1/2} h(A^{-1/2} BA^{-1/2}) A^{1/2},
\]

which is a familiar expression of \( A \sigma B \). For instance, for \( \phi_\alpha(x, y) \) corresponding to \( t^\alpha \) with \( 0 \leq \alpha \leq 1 \), we have \( \phi_\alpha(B, A) = A \#_\alpha B \), the \( \alpha \)-weighted geometric mean, which has been the most studied operator mean since the beginning of the subject [5, 6, 46, 47].

The next theorem is concerned with the joint convexity problem for the PW-functional calculus, whose weaker form was given in [24, Theorem 9]. The present statement can be understood as a Hilbert space operator reformulation of Pusz and Woronowicz’s original results in [47] with a thorough proof.

**Theorem 4.10.** (cf. [24, Theorem 9]) The following conditions are equivalent, where Hilbert spaces \( H, K \) are arbitrary:

(i) For every \( A_i, B_i \in B(H)_+ \) \((i = 1, 2)\) and any \( \lambda \in (0, 1) \),

\[
\phi((1 - \lambda)A_1 + \lambda A_2, (1 - \lambda)B_1 + \lambda B_2) \leq (1 - \lambda)\phi(A_1, B_1) + \lambda\phi(A_2, B_2),
\]

or equivalently,

\[
\phi(A_1 + A_2, B_1 + B_2) \leq \phi(A_1, B_1) + \phi(A_2, B_2) \quad \text{(jointly subadditive)};
\]

(ii) For every \( A, B \in B(H)_+ \) and any bounded operator \( C : K \to H \),

\[
\phi(C^* AC, C^* BC) \leq C^* \phi(A, B) C;
\]

(iii) For every \( A, B \in B(H)_+ \) and any isometry \( V : K \to H \),

\[
\phi(V^* AV, V^* BV) \leq V^* \phi(A, B) V;
\]

(iv) \( t \in [0, 1] \mapsto \phi(t, 1-t) \in (-\infty, \infty] \) is operator convex (see Theorem 3.7(1));

(iv') \( t \in [0, 1] \mapsto \phi(1-t, t) \in (-\infty, \infty] \) is operator convex;

(v) \( t \in [0, \infty) \mapsto \phi(t, 1) \in (-\infty, \infty] \) is operator convex and

\[
\phi(1, 0) \geq \lim_{t \to \infty} \frac{\phi(t, 1)}{t}, \quad (4.7)
\]

or equivalently, \( \phi(1, 0) \geq \lim_{t \nearrow 1} \phi(t, 1-t) \);

(v') \( t \in [0, \infty) \mapsto \phi(1-t, t) \in (-\infty, \infty] \) is operator convex and

\[
\phi(0, 1) \geq \lim_{t \to \infty} \frac{\phi(t, 1)}{t},
\]

or equivalently, \( \phi(0, 1) \geq \lim_{t \nearrow 1} \phi(1-t, t) \).

**Proof.** The proof of the equivalence of (i)–(iii) is essentially the same as that of [23, Theorem 2.1] (also [25, Theorem 2.5.2]), so a point here is how the proof of this part goes in the framework of \( \hat{B}(H)_{lb} \). In the proof we repeatedly use Proposition 4.6 (for \( I = \{1, 2\} \)).
(i) \implies (ii). We may assume that $C : K \to \mathcal{H}$ is a contraction, since 
\( \phi(\alpha A, \alpha B) = \alpha \phi(A, B) \) for any $\alpha > 0$ (the special case of operator homogeneity). On the direct sum $K \oplus \mathcal{H}$ define $\tilde{A} := 0 \oplus A$, $\tilde{B} := 0 \oplus B$ and unitaries 
\[
U := \begin{bmatrix} Y & -C^* \\ C & Z \end{bmatrix}, \quad V := \begin{bmatrix} Y & C^* \\ -C & Z \end{bmatrix},
\]
where $Y := (I_K - C^* C)^{1/2}$ and $Z := (I_\mathcal{H} - CC^*)^{1/2}$. Then one can easily verify that 
\[
\phi(C^* AC \oplus Z AZ, C^* BC \oplus ZBZ) = \phi(\frac{1}{2}(U^* \tilde{A}U + V^* \tilde{A}V), \frac{1}{2}(U^* \tilde{B}U + V^* \tilde{B}V)) \leq \frac{1}{2}(\phi(U^* \tilde{A}U, U^* \tilde{A}U) + \phi(V^* \tilde{A}V, V^* \tilde{B}V)) (\text{by assumption (i)}) = \frac{1}{2}(U^* \phi(\tilde{A}, \tilde{B})U + V^* \phi(\tilde{A}, \tilde{B})V) (\text{by operator homogeneity}).
\]
Therefore, for any $\rho \in B(\mathcal{K})^+_+$ we find that 
\[
\phi(C^* AC, C^* BC)(\rho) = \phi(C^* AC \oplus ZAZ, C^* BC \oplus ZBZ)(\rho \oplus 0) (\text{by Proposition 4.6}) \leq \frac{1}{2}(\phi(\tilde{A}, \tilde{B})(U(\rho \oplus 0)U^*) + \phi(\tilde{A}, \tilde{B})(V(\rho \oplus 0)V^*)) = \frac{1}{2}(\phi(A, B)(P_2 U(\rho \oplus 0)U^* P_2) + \phi(A, B)(P_2 V(\rho \oplus 0)V^* P_2)) (\text{by Proposition 4.6}) = \phi(A, B)(C^* \phi(A, B)C)(\rho),
\]
where $P_2$ denotes the orthogonal projection from $\mathcal{K} \oplus \mathcal{H}$ onto the second direct summand.

(ii) \implies (iii) is trivial.

(iii) \implies (i). For any $\lambda \in (0, 1)$ consider an isometry $V \xi := \sqrt{1 - \lambda} \xi \oplus \sqrt{\lambda} \xi$ from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. For every $A_i, B_i \in B(\mathcal{H})_+(i = 1, 2)$ one has 
\[
\phi((1 - \lambda)A_1 + \lambda A_2, (1 - \lambda)B_1 + \lambda B_2) = \phi(V^*(A_1 \oplus A_2)V, V^*(B_1 \oplus B_2)V)) \leq V^*(\phi(A_1 \oplus A_2, B_1 \oplus B_2))V
\]
by assumption (iii). Therefore, for any $\rho \in B(\mathcal{H})^*_+$ one has 
\[
\phi((1 - \lambda)A_1 + \lambda A_2, (1 - \lambda)B_1 + \lambda B_2)(\rho) \leq \phi(A_1 \oplus A_2, B_1 \oplus B_2)(V \rho V^*) = \phi(A_1, B_1)(P_1 V \rho V^* P_1) + \phi(A_2, B_2)(P_2 V \rho V^* P_2) (\text{by Proposition 4.6}) = (1 - \lambda)\phi(A_1, B_1)(\rho) + \lambda \phi(A_2, B_2)(\rho),
\]
where $P_1, P_2$ denote the orthogonal projections from $\mathcal{H} \oplus \mathcal{H}$ onto the first and the second direct summands, respectively. Note also that the equivalence of joint convexity and joint subadditivity in (i) is clear from the scalar homogeneity of $\phi(A, B)$ mentioned at the beginning of the proof of (i) \implies (ii).

(iii) \implies (iv) and (iii) \implies (iv') are obvious by Lemma 4.4(3) (and Proposition 3.3).
(iv) \implies (ii). Here we write $A' = C^*AC$ and $B' = C^*BC$ for short. By Lemma 4.4(2) note that $R_{A',B'} = \hat{C}^*R_{A,B}\hat{C}$ and $\hat{C}T_{A',B'} = T_{A,B}C$ for an isometry $\hat{C}$. We then have

\[
\begin{align*}
\phi(A', B') &= T_{A',B'}^*\phi(R_{A',B'}, S_{A,B})T_{A',B'} \\
&= T_{A',B'}^*\phi(R_{A',B'}, 1 - R_{A',B'})T_{A',B'} \quad \text{(by Lemma 4.4(3))} \\
&= T_{A',B'}^*\phi(\hat{C}^*R_{A,B}\hat{C}, 1 - \hat{C}^*R_{A,B}\hat{C})T_{A',B'} \\
&\leq T_{A',B'}^*\hat{C}^*\phi(R_{A,B}, 1 - R_{A,B})\hat{C}T_{A',B'} \quad \text{(by assumption (iv) and Proposition 3.3)} \\
&= T_{A',B'}^*\hat{C}^*\phi(R_{A,B}, S_{A,B})\hat{C}T_{A',B'} \\
&= C^*T_{A,B}^*\phi(R_{A,B}, S_{A,B})T_{A,B}C \\
&= C^*\phi(A,B)C,
\end{align*}
\]

where the first and the last equalities are due to operator homogeneity (Definition 4.1(2)), and $\phi(R_{A',B'}, 1 - R_{A',B'})$ etc. should be understood as the functional calculus for the single variable function $\phi(t, 1 - t)$. The proof of (iv') \implies (ii) is similar. (Also, (iv) \iff (iv') is obvious.)

(iii) \implies (v). Assume item (iii). Then $\phi(t, 1)$ is operator convex on $[0, \infty)$ by Lemma 4.7(1) (and Proposition 3.3). Since (iii) \implies (iv) (already seen), it follows from Theorem 3.7 that $\phi(1, 0) \geq \lim_{t \to 1} \phi(t, 1 - t)$. Furthermore, since

\[
\frac{\phi(t, 1)}{t} = \frac{t + 1}{t} \phi\left(\frac{t}{t + 1}, 1 - \frac{t}{t + 1}\right), \quad t > 0,
\]

it is clear that the above condition is equivalent to (4.7). The proof of (iii) \implies (v') is similar.

(v) \implies (iv). Assume item (v). Then, as in the proof of [24, Theorem 9, (iii) \implies (ii)], we observe that $t \in [0, 1 - \delta] \mapsto \phi(t, 1 - t) \in (-\infty, \infty]$ is operator convex for any $\delta \in (0, 1)$ and hence so is $t \in [0, 1) \mapsto \phi(t, 1 - t) \in (-\infty, \infty]$. Here let us briefly explain this proof. For each $\delta \in (0, 1)$ we set $c_\delta := (1 - \delta)/\delta$ and define

\[
\psi_\delta(x, y) := \frac{x + y}{c_\delta} \phi\left(\frac{c_\delta x}{x + y}, 1\right) \quad \text{with} \quad \psi_\delta(0, 0) := 0.
\]

Then $\psi_\delta(t, c_\delta - t) = \phi(t, 1)$ holds for $0 \leq t \leq c_\delta$ and $\psi_\delta(t, 1 - t) = \phi(c_\delta t, 1)/c_\delta$ is operator convex on $[0, 1]$ by assumption (v). Applying (iv) \implies (iii) (already established) to $\psi_\delta$, we see that $(A, B) \mapsto \psi_\delta(A, B)$ satisfies the inequality in (iii). Observe that $\phi(t, 1 - t) = \psi_\delta(t, c_\delta(1 - t) - t)$ for every $t \in [0, 1 - \delta]$. Then, for every $A \in B(\mathcal{H})_{[0,1-\delta]}$ and every isometry $V : \mathcal{K} \to \mathcal{H}$ (so $V^*AV \in B(\mathcal{K})_{[0,1-\delta]}$), we have

\[
\begin{align*}
\phi(V^*AV, 1 - V^*AV) &= \psi_\delta(V^*AV, c_\delta(I_{\mathcal{K}} - V^*AV) - V^*AV) \\
&= \psi_\delta(V^*AV, V^*(c_\delta(I_{\mathcal{H}} - A) - A)V) \\
&\leq V^*\psi_\delta(A, c_\delta(I_{\mathcal{H}} - A) - A)V \\
&= V^*\phi(A, 1 - A)V,
\end{align*}
\]
where $\phi(V^*AV, 1 - V^*AV)$ and $\phi(A, 1 - A)$ are the functional calculus for $\phi(t, 1 - t)$ as before, and the first and the last equalities can be confirmed as in the proof of Lemma 4.4(3). Hence we have shown that $\phi(t, 1 - t)$ is operator convex on $[0, 1]$. To prove item (iv), in view of Example 3.6(1) we may assume that $\phi(t, 1 - t) < \infty$ at more than one point in $[0, 1]$. Then, since $\phi(1, 0) \geq \lim_{t \nearrow 1} \phi(t, 1 - t)$, we obtain (iv) by applying Theorem 3.7(1) to $\phi(t, 1 - t)$ on $[0, 1]$.

(v') $\Rightarrow$ (iv') can be shown in exactly the same way as above. \hfill $\square$

Under an additional assumption on the behavior of $\phi(t, 1 - t)$ (or $\phi(1 - t, t)$) as $t \nearrow 1$, we have alternative equivalent conditions to those of Theorem 4.10.

**Theorem 4.11.** Assume that $\lim_{t \nearrow 1} \phi(t, 1 - t) \leq \phi(1, 0) \leq 0$ (resp., $\lim_{t \nearrow 1} \phi(1 - t, t) \leq \phi(0, 1) \leq 0$). Then the conditions of Theorem 4.10 are also equivalent to the following:

(vi) For every $A_1, A_2, B \in B(H)_+$,

$$A_1 \leq A_2 \implies \phi(A_1, B) \geq \phi(A_2, B) \quad (\text{resp., } \phi(B, A_1) \geq \phi(B, A_2));$$

(vii) $t \in [0, \infty) \implies \phi(t, 1) \in (-\infty, \infty)$ (resp., $t \in [0, \infty) \implies \phi(1, t) \in (-\infty, \infty)$) is operator monotone decreasing (see Theorem 3.7(2)).

**Proof.** (vi) $\implies$ (vii). This is immediately seen by letting $B = I$ in (vi) and using Lemma 4.7(1).

(vii) $\implies$ (v). Since $\lim_{t \nearrow 1} \phi(t, 1 - t) \leq 0$ or equivalently $\lim_{t \nearrow \infty} \phi(t, 1)/t \leq 0$, it follows that $\phi(t, 1) < \infty$ for all sufficiently large $t > 0$. Hence by applying Theorem 3.7(2) to $-\phi(t, 1)$, assumption (vii) implies that $\phi(t, 1)$ is $\mathbb{R}$-valued and operator monotone decreasing on $(0, \infty)$ and moreover $\phi(0, 1) \geq \lim_{t \searrow 0} \phi(t, 1)$. Then it is well known that $\phi(t, 1)$ is operator convex on $(0, \infty)$. (Indeed, let $f(t) := \phi(t, 1)$. For every $\varepsilon > 0$, $f(\varepsilon) - f(t + \varepsilon)$ is non-negative and operator monotone on $[0, \infty)$, which is operator concave by [23, Theorem 2.5]. This means that $f$ is operator convex on $(0, \infty)$.) Hence, item (v) holds by Theorem 3.7(1).

(i) $\implies$ (vi). Let $A_1, A_2, B \in B(H)_+$ with $A_1 \leq A_2$. By assumption (i) one has

$$\phi(A_2, B) = \phi((A_1 + (A_2 - A_1), B + 0) \leq \phi(A_1, B) + \phi(A_2 - A_1, 0).$$

Since $\phi(A_2 - A_1, 0) = \phi(1, 0)(A_2 - A_1)$ by Lemma 4.7(2), item (vi) holds thanks to the assumption $\phi(1, 0) \leq 0$. \hfill $\square$

**Example 4.12.** For a positive operator monotone function $h$ on $[0, \infty)$, consider the function $\phi(x, y)$ defined in Remark 4.9. Since $\phi(1, t) = h(t) is operator concave (i.e., $-\phi(1, t)$ is operator convex) and $\lim_{t \to \infty} \phi(1, t)/t = \phi(0, 1)$, Theorem 4.10(i) (applied to $-\phi$) reduces to the familiar joint concavity of an operator connection $A \sigma B$.

**Remark 4.13.** Some remarks related to Theorems 4.10 and 4.11 are in order.

(1) Assume that $\phi(t, 1 - t) < \infty$ at more than one point in $[0, 1]$. If $\phi$ satisfies the convexity properties of Theorem 4.10, then $\phi(x, y)$ becomes
a real analytic function on \((0, \infty)^2\). Indeed, in this case, \(\phi(t, 1)\) is an \(\mathbb{R}\)-valued operator convex (hence real analytic) function on \((0, \infty)\) and \(\phi(x, y) = y\phi(x/y, 1)\).

(2) Theorem 4.10 in particular shows that an \(\mathbb{R}\)-valued function \(f\) on \((0, \infty)\) is operator convex if and only if so is \(g(t) := (1 - t)f(\frac{1}{1-t})\) on \((0, 1)\). Indeed, let \(\phi\) be the perspective function of \(f\), i.e., \(\phi(x, y) := yf(x/y)\) for \((x, y) \in (0, \infty)^2\) (extended to \([0, \infty)^2\) with \(\phi(0, 0) = 0\) and \(\phi(0, 1) = \infty\). Since \(f(t) = \phi(t, 1)\) for \(t > 0\) and \(g(t) = \phi(t, 1 - t)\) for \(t \in (0, 1)\), the assertion follows from Theorem 4.10. For a different approach to this result, see [18] and [28, Theorem C.1].

(3) Assume that \(\phi(t, 1 - t) < \infty\) for all \(t \in (0, 1)\). Then Theorem 4.8 says that \(\phi(A, B)\) for every \(A, B \in B(\mathcal{H})_{++}\) becomes the operator perspective \(B^{1/2}f(B^{-1/2}AB^{-1/2})B^{1/2}\) of \(A, B\) associated with the function \(f(t) := \phi(t, 1)\) on \((0, \infty)\); see Definition 7.1 in Sect. 7. Thus Theorem 4.10 in particular contains the result in [12,13], saying that an \(\mathbb{R}\)-valued function \(f\) on \((0, \infty)\) is operator convex if and only if the operator perspective associated with \(f\) is jointly operator convex on \(B(\mathcal{H})_{++} \times B(\mathcal{H})_{++}\).

The PW-functional calculus \(\phi(A, B)\) enjoys several continuity properties under certain constraints on the given function \(\phi\), as we will discuss in Sects. 6 and 7.

5. PW-Functional Calculus with Restricted Domain

For the convenience of notations, we set
\[
\mathcal{H}_{++} := \{(A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ : A \leq \alpha B \text{ for some } \alpha > 0\};
\]
\[
\mathcal{H}_{++} \times \mathcal{H}_{++} := \{(A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ : A \geq \alpha B \text{ for some } \alpha > 0\};
\]
which are sub-cones of \(B(\mathcal{H})_+ \times B(\mathcal{H})_+\) including \(B(\mathcal{H})_{++} \times B(\mathcal{H})_{++}\).

In the following we continue to use the notations \(\mathcal{H}_{A,B}, T_{A,B}\) and \((R_{A,B}, S_{A,B})\) given in (4.1), (4.2) and Lemma 4.4(1) for \(A, B \in B(\mathcal{H})_+\).

**Lemma 5.1.** For every \(A, B \in B(\mathcal{H})_+\) the following conditions are equivalent:

1. \((A, B) \in (B(\mathcal{H})_+ \times B(\mathcal{H})_+)_{++}\);
2. \(R_{A,B} \geq \alpha S_{A,B}\) for some \(\alpha > 0\);
3. \(R_{A,B} \in B(\mathcal{H}_{A,B})_{++}\).

Furthermore, \(A \geq \alpha B\) is equivalent to \(R_{A,B} \geq \alpha S_{A,B}\).

**Proof.** (1) \(\implies\) (2). Assume that \(A \geq \alpha B\) with \(\alpha > 0\). For every \(\xi \in \mathcal{H}\) one has
\[
\langle R_{A,B}T_{A,B}\xi, T_{A,B}\xi \rangle = \langle A\xi, \xi \rangle \geq \alpha \langle B\xi, \xi \rangle
= \alpha \langle S_{A,B}T_{A,B}\xi, T_{A,B}\xi \rangle
\]
so that \(R_{A,B} \geq \alpha S_{A,B}\).

(2) \(\implies\) (3). Assume (2); then \((\alpha + 1)R_{A,B} \geq \alpha (R_{A,B} + S_{A,B}) = \alpha I_{\mathcal{H}_{A,B}}\).
Let

\[ \text{Theorem 5.2.} \]

replace 'operator concave' (resp., 'operator monotone').

The last assertion is immediate from the above proof. \[ \square \]

In this section we consider a Borel function \( \phi : [0, \infty)^2 \setminus \{0\} \times (0, \infty) \) → \((-\infty, \infty]\) which is homogeneous (i.e., \( \phi(\lambda x, \lambda y) = \lambda \phi(x, y) \) for all \( x > 0, y \geq 0 \) and \( \lambda \geq 0 \)). We assume that \( \phi(t, 1-t) \) on \((0, 1] \) is locally bounded from below (i.e., bounded from below on any compact subset of \((0, 1] \) ). We can define the PW-functional calculus

\[ (A, B) \in (B(H)_+ \times B(H)_+) \geq \phi(A, B) \in B(H)_{lb} \]

by restricting \((A, B)\) to pairs in \((B(H)_+ \times B(H)_+) \geq \) in postulates (1) and (2) of Definition 4.1. Alternatively, by Lemma 5.1 we may define more explicitly as follows:

\[
\phi(A, B) = T^*_{A, B} \phi(R_{A, B}, S_{A, B}) T_{A, B}, \quad (A, B) \in (B(H)_+ \times B(H)_+) \geq. \tag{5.1}
\]

The class of functions \( \phi \) on \([0, \infty)^2 \setminus \{0\} \times (0, \infty) \) for which we can define the PW-functional calculus \( \phi(A, B) \) is somewhat more flexible than those on \([0, \infty)^2 \), although the domain is restricted to \((B(H)_+ \times B(H)_+) \geq \).

For this version of PW-functional calculus \( \phi(A, B) \) with restricted domain, we can rephrase all the results in Sect. 4 by restricting \((A, B)\) to \((B(H)_+ \times B(H)_+) \geq \) with slight necessary modifications. For instance, Theorem 4.8 in this setting is

\[
\phi(A, B) = \begin{cases} A^{1/2} \phi(1, A^{-1/2} B A^{-1/2}) A^{1/2} & \text{for } A \in B(H)_{++}, B \in B(H)_+, \\ B^{1/2} \phi(B^{-1/2} A B^{-1/2}, 1) B^{1/2} & \text{for } A, B \in B(H)_{++}. \end{cases}
\]

In the next theorem, instead of the restricted version of Theorems 4.10 and 4.11, we present its complementary counterpart, where the inequality sign is reversed and 'operator convex' (resp., 'operator monotone decreasing') is replaced with 'operator concave' (resp., 'operator monotone').

**Theorem 5.2.** Let \( \phi : [0, \infty)^2 \setminus \{0\} \times (0, \infty) \) → \((-\infty, \infty]\) be as stated above. Then the following conditions are equivalent, where Hilbert spaces \( \mathcal{H}, \mathcal{K} \) are arbitrary:

(i) For every \( (A_i, B_i) \in (B(H)_+ \times B(H)_+) \geq \) \( (i = 1, 2) \),

\[ \phi(A_1 + A_2, B_1 + B_2) \geq \phi(A_1, B_1) + \phi(A_2, B_2); \]

(ii) For every \( (A, B) \in (B(H)_+ \times B(H)_+) \geq \) and any bounded operator \( C : \mathcal{K} \to \mathcal{H} \),

\[ \phi(C^* AC, C^* BC) \geq C^* \phi(A, B) C; \]

(iii) For every \( (A, B) \in (B(H)_+ \times B(H)_+) \geq \) and any isometry \( V : \mathcal{K} \to \mathcal{H} \),

\[ \phi(V^* AV, V^* BV) \geq V^* \phi(A, B) V; \]

(iv) \( t \in (0, 1] \) → \( \phi(t, 1-t) \) is operator concave;

(iv') \( t \in [0, 1] \) → \( \phi(1-t, t) \) is operator concave;

(v) \( t \in (0, \infty) \) → \( \phi(t, 1) \) is operator concave and \( \phi(1, 0) \leq \lim_{t \to 1} \phi(t, 1-t); \)

(v') \( t \in [0, \infty) \) → \( \phi(1, t) \) is operator concave.
Furthermore, assume that \( \lim_{t \to 1} \phi(t, 1 - t) \geq \phi(1, 0) \geq 0 \). Then the above conditions are also equivalent to the following:

(vi) For every \((A_1, B), (A_2, B) \in (B(H)_+ \times B(H)_+) \geq \),
\[ A_1 \leq A_2 \implies \phi(A_1, B) \leq \phi(A_2, B); \]

(vii) \( \phi(t, 1) \) is operator monotone on \((0, \infty)\).

**Proof.** The proof of the equivalence of (i)–(iii) is the same as that for (i)–(iii) of Theorem 4.10 just by restricting \((A, B)\) to \((B(H)_+ \times B(H)_+) \geq \) and by reversing the inequality sign. The proof for the remaining items can be carried out in a similar way to that of Theorem 4.10 with necessary modifications. Alternatively (and more conveniently), we may first confirm the restricted (but not complementary) version of Theorems 4.10 and 4.11 for \( \phi \) in the present setting, which is stated as above with the reverse inequality sign and with ‘operator convex’ and ‘operator monotone decreasing’ (instead of ‘operator concave’ and ‘operator monotone’). The proof of this version is carried out just by restricting \((A, B)\) etc. to \((B(H)_+ \times B(H)_+) \geq \) in the proofs of Theorems 4.10 and 4.11 (with Lemma 5.1). Next, let us prove the present complementary version. Note that all the conditions hold trivially in the case \( \phi \equiv \infty \). So we may assume that \( \phi \neq \infty \). In this case, it is easy to observe the following:

- \( \phi(t, 1 - t) \) is numerically concave on \((0, 1] \) if one of (i) (hence (ii), (iii)), (iv) and (iv') is satisfied,
- \( \phi(t, 1) \) is numerically concave on \((0, \infty) \) with \( \phi(1, 0) < \infty \) if one of (v), (vi) and (vii) is satisfied,
- \( \phi(1, t) \) is numerically concave on \([0, \infty) \) if (v') is satisfied.

From this observation we see that if any of the conditions of the theorem is satisfied, then \( \phi(t, 1 - t) \) is locally bounded from above (as well as from below) on \((0, 1] \). Hence, to prove the theorem, it suffices to assume that \( \phi(t, 1 - t) \) is locally bounded from above on \((0, 1] \). Thus, we can apply the above-mentioned version of Theorems 4.10 and 4.11 to \(-\phi\), which immediately shows the present complementary version. \( \square \)

A typical example of \( \phi \) to apply Theorem 5.2 is
\[
\phi(x, y) := \begin{cases} 
  y \log(x/y) & (x, y > 0), \\
  0 & (x \geq 0 = y).
\end{cases}
\]

Then \( \phi(t, 1) = \log t \) is operator monotone (and operator concave) on \((0, \infty) \) and \( \lim_{t \to 1} \phi(t, 1 - t) = 0 = \phi(1, 0) \). The function \( \phi(t, 1 - t) \) is locally bounded from below on \((0, 1] \), while it cannot be extended to a function on \([0, 1] \) that is locally bounded from below, since \( \lim_{t \to 0} \phi(t, 1 - t) = -\infty \).

In the next proposition we characterize the case where \( \phi(A, B) \) is bounded for any \((A, B) \in (B(H)_+ \times B(H)_+) \geq \).

**Proposition 5.3.** Let \( \phi \) be as above, and assume that \( H \) is infinite-dimensional. Then the following conditions are equivalent:

1. \( \phi(A, B) \in B(H)_{sa} \) for all \((A, B) \in (B(H)_+ \times B(H)_+) \geq \);
(2) \( \phi(t,1-t) \) is locally bounded on \((0,1]\) (i.e., bounded on \([\delta,1]\) for any \(\delta > 0\));

(3) \( \phi(1,t) \) is locally bounded on \([0,\infty)\).

**Proof.** (2) \( \iff \) (3) is immediately seen from

\[
\phi(1,t) = (1+t)\phi\left(1, \frac{1}{1+t}\right) \quad (t \geq 0),
\]

\[
\phi(t,1-t) = t\phi\left(1, \frac{1-t}{t}\right) \quad (0 < t \leq 1).
\]

(2) \( \implies \) (1) follows from Lemma 5.1 and (5.1).

(1) \( \implies \) (3). We prove by contraposition. Assume that there is a \(c > 0\) such that \(\phi(1,t)\) is unbounded on \([0,c]\). Choose a sequence \(t_n \in [0,c]\) such that \(\phi(1,t_n) \to \infty\). With a sequence \(\{P_n\}\) of orthogonal projections with \(\sum_n P_n = I\), we define \(B := \sum_{n=1}^{\infty} t_n P_n \) in \(B(\mathcal{H})_+\). Then, modifying Lemma 4.7(1) in the present setting, one has \(\phi(I, B) = \phi(1, B) = \sum_{n=1}^{\infty} \phi(1,t_n)P_n\), which is unbounded.

For example, when \(\phi \neq \infty\) satisfies \(v'\) of Theorem 5.2, it is clear that condition (3) of Proposition 5.3 holds. Therefore, we have \(\phi(A,B) \in B(\mathcal{H})_{sa}\) for all \((A,B) \in (B(\mathcal{H})_+ \times B(\mathcal{H})_+)\geq\) whenever one of the conditions of Theorem 5.2 is satisfied and \(\phi \neq \infty\).

**Remark 5.4.** As is immediately seen, Theorem 5.2 and Proposition 5.3 hold also in the situation where \([0,\infty)^2 \setminus \{(0) \times (0,\infty)\}\) and \((B(\mathcal{H})_+ \times B(\mathcal{H})_+)\geq\) are replaced with \([0,\infty)^2 \setminus ((0,\infty) \times \{0\})\) and \((B(\mathcal{H})_+ \times B(\mathcal{H})_+)\leq\), respectively, and the roles of two variables in \(\phi(x,y)\) and \(\phi(A,B)\) are interchanged.

### 6. Upper Continuity for PW-Functional Calculus

Recall [40] that an operator connection \(A\sigma B\) enjoys the upper continuity for decreasing sequences in \(B(\mathcal{H})_+\) in such a way that for \(A,B,A_n,B_n \in B(\mathcal{H})_+\),

\[
A_n \searrow A, \quad B_n \searrow B \quad \Rightarrow \quad A_n\sigma B_n \searrow A\sigma B, \quad (6.1)
\]

where \(A_n \searrow A\) means that \(A_1 \geq A_2 \geq \cdots\) and \(A_n \to A\) in the strong operator topology (SOT for short). (In this section we are concerned only with this continuity property, though the complete definition of operator connections will be given at the beginning of Sect. 10.)

In this section we show the convergence of the PW-functional calculus in SOT for decreasing sequences as above in certain situations. The next theorem vastly improves [24, Theorem 6]. In fact, it is an apparently best possible counterpart of (6.1) for operator connections in the case of the PW-functional calculus, though \(\phi\) is assumed \(\mathbb{R}\)-valued continuous.

**Theorem 6.1.** Let \(\phi\) be a homogeneous and \(\mathbb{R}\)-valued continuous function on \([0,\infty)^2\). Let \(A,B,A_n,B_n \in B(\mathcal{H})_+\) \((n \in \mathbb{N})\) be such that \(A_n \searrow A\) and \(B_n \searrow B\). Then \(\phi(A_n,B_n), \phi(A,B)\) are all bounded and

\[
\phi(A_n,B_n) \longrightarrow \phi(A,B) \quad \text{in SOT.}
\]
Proof. Let $T_n := (A_n + B_n)^{1/2}$ and $T := (A + B)^{1/2}$ in $B(H)$; so $T_n \downarrow T$ by assumption. We have (unique) $R_n, R \in B(H)_+$ such that $0 \leq R_n, R \leq I$, $	ext{ran} R_n \subseteq \text{ran} T_n$, $\text{ran} R \subseteq \text{ran} T$, $A_n = T_n R_n T_n$ and $A = TR T$. Set $f(t) := \phi(t, 1 - t)$ for $t \in [0, 1]$, which is a continuous function on $[0, 1]$ by assumption. As in (4.5) we can write
\[
\phi(A_n, B_n) = T_n f(R_n) T_n, \quad \phi(A, B) = T f(R) T,
\]
where $f(R_n)$ and $f(R)$ are the continuous functional calculus in the present situation. Hence $\phi(A_n, B_n)$ and $\phi(A, B)$ are clearly bounded (as described in [24]). Let $\mathcal{H}_{A,B} := \text{ran} T$ as before. One finds that, for every $\xi, \xi' \in \mathcal{H}$,
\[
|\langle (R_n - R)T \xi, T \xi' \rangle| \leq |\langle R_n (T - T_n) \xi, T \xi' \rangle| + |\langle R_n T_n \xi, (T - T_n) \xi' \rangle| + |\langle R_n T_n \xi, T \xi' \rangle - \langle RT \xi, T \xi' \rangle| \\
\leq \|T \xi\| \|T - T_n \xi\| + \|T_n \xi\| \|T - T_n \xi\| \quad \text{(6.3)}
\]
\[
\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
Consider the function
\[
\psi(x, y) := \begin{cases} \\ x^2 + y^2 = -\frac{xy}{x+y} & \text{if } x + y > 0, \\ 0 & \text{if } x = y = 0. 
\end{cases}
\]
Since $\psi(t, 1 - t) = t^2$ for $t \in [0, 1]$, it follows as (6.2) that
\[
T_n R_n^2 T_n = \psi(A_n, B_n) = A_n - (A_n : B_n), \quad TR^2 T = A - (A : B), \quad \text{(6.4)}
\]
where $A : B$ is the parallel sum of $A, B$, an operator connection having the representing function $t/(t + 1)$; see Remark 4.9. Therefore, one has (thanks to (6.1))
\[
T_n R_n^2 T_n \rightarrow TR^2 T \quad \text{in SOT}.
\]
For every $\xi \in \mathcal{H}$,
\[
\|R_n T \xi\|^2 - \|RT \xi\|^2 \leq |\langle R_n^2 (T - T_n) \xi, T \xi \rangle| + |\langle R_n^2 T_n \xi, (T - T_n) \xi \rangle| + |\langle R_n^2 T_n \xi, (T - T_n) \xi \rangle - \langle RT^2 T_n \xi, (T - T_n) \xi \rangle| \\
\leq (\|T\| + \|T_n\|) \|\xi\| \|T - T_n\| \quad \text{(6.5)}
\]
\[
\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
For every $\eta, \eta' \in \mathcal{H}_{A,B}$, approximating $\eta, \eta'$ with $T \xi, T \xi'$ ($\xi, \xi' \in \mathcal{H}$) and applying the above estimates given in (6.3) and (6.5), one can immediately find that $\langle R_n \eta, \eta' \rangle \rightarrow \langle R \eta, \eta' \rangle$ and $\|R_n \eta\| \rightarrow \|R \eta\|$. By these with $R \eta \in \mathcal{H}_{A,B}$, a standard argument shows that $\|R_n \eta - R \eta\| \rightarrow 0$ as $n \rightarrow \infty$ for all $\eta \in \mathcal{H}_{A,B}$ (though we cannot say that $R_n \rightarrow R$ in SOT on $\mathcal{H}_{A,B}$, because $\mathcal{H}_{A,B}$ is not necessarily invariant for $R_n$’s).

Now, for any $k \in \mathbb{N}$, let us prove by induction that $\|R_n^k \eta - R^k \eta\| \rightarrow 0$ as $n \rightarrow \infty$ for all $\eta \in \mathcal{H}_{A,B}$. Assume that this holds for some $k \in \mathbb{N}$. For every
\( \eta \in \mathcal{H}_{A,B} \), since \( R^k \eta \in \mathcal{H}_{A,B} \), we have
\[
\| (R_n^{k+1} - R^{k+1}) \eta \| \leq \| R_n (R_n^k - R^k) \eta \| + \| (R_n - R) R^k \eta \|
\leq \| (R_n^k - R^k) \eta \| + \| (R_n - R) R^k \eta \|
\longrightarrow 0 \quad \text{as} \quad n \to \infty,
\]
so that the convergence in question holds for \( k + 1 \) too. Applying the Weierstrass approximation theorem to \( f \), we therefore see that \( \| f(R_n) \theta - f(R) \theta \| \longrightarrow 0 \) for all \( \eta \in \mathcal{H}_{A,B} \). For every \( \xi \in \mathcal{H} \) it then follows from (6.2) that
\[
\| \phi(A_n, B_n) \xi - \phi(A, B) \xi \|
\leq \| T_n f(R_n)(T_n - T) \xi \| + \| T_n (f(R_n) - f(R)) T \xi \|
+ \| (T_n - T) f(R) T \xi \|
\leq \| T_1 \| \| f \| \| (T_n - T) \xi \| + \| T_1 \| \| (f(R_n) - f(R)) T \xi \|
+ \| (T_n - T) f(R) T \xi \|
\longrightarrow 0 \quad \text{as} \quad n \to \infty,
\]
where \( \| f \| \infty := \max_{0 \leq t \leq 1} | f(t) | \). Hence the stated assertion follows. \( \square \)

Concerning the above proof, we remark that the expression in (6.4) above was already observed in the proof of [46, Theorem 1.2] and the strong convergence \( T_n R^2 T_n \to TRT \) was essentially derived there. However, the argument after obtaining this convergence has been missing so that no general continuity result has probably been observed so far for the PW-functional calculus.

**Remark 6.2.** (1) It is not essential in the proof of Theorem 6.1 that \( \phi(x, y) \) is \( \mathbb{R} \)-valued on the whole \( [0, \infty)^2 \). In fact, the following extension of Theorem 6.1 holds: Let \( \phi(x, y) \) be a function as in Sect. 4, and assume that \( \phi(x, y) \) is \( \mathbb{R} \)-valued and continuous on the sector \( \{(x, y) \in [0, \infty)^2 : x \geq \alpha y, \beta x \leq y\} \), where \( \alpha, \beta \geq 0 \) with \( \alpha \beta \leq 1 \). Then, for every sequence \( (A_n, B_n) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ \) such that \( A_n \geq \alpha B_n, \beta A_n \leq B_n \) for all \( n \) and \( A_n \not\to \mathbb{A}, B_n \not\to \mathbb{B} \) for some \( (A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ \), we have \( \phi(A_n, B_n), \phi(A, B) \in B(\mathcal{H}) \) and \( \phi(A_n, B_n) \to \phi(A, B) \) in SOT. The proof is essentially same as that of Theorem 6.1, but needs Lemma 5.1.

(2) We have the following modification of Theorem 6.1 to the setting of Sect. 5: Let \( \phi \) be a homogeneous and \( \mathbb{R} \)-valued function on \( [0, \infty)^2 \setminus \{(0) \times (0, \infty)\} \) (resp., on \( [0, \infty)^2 \setminus ((0, \infty) \times \{0\}) \)) such that \( \phi(t, 1 - t) \) is continuous on \( (0, 1) \) (resp., on \( [0, 1] \)). Let \( A, B, A_n, B_n \in B(\mathcal{H})_+ \) (\( n \in \mathbb{N} \)) be such that \( A_n \geq \alpha B_n \) (resp., \( A_n \leq \alpha B_n \)) for all \( n \) with some \( \alpha > 0 \) (independent of \( n \)), \( A_n \not\to A \) and \( B_n \not\to B \). Then \( \phi(A_n, B_n), \phi(A, B) \) are all bounded and \( \phi(A_n, B_n) \to \phi(A, B) \) in SOT. The proof is also similar to that of Theorem 6.1 as item (1) above.

We furthermore include the following corollary of Theorem 6.1 for later use.

**Corollary 6.3.** Let \( \phi : [0, \infty)^2 \to (-\infty, \infty] \) be a function dealt with in Sect. 4, and assume that \( \phi(t, 1 - t) \) is continuous on \( [0, 1] \) as a function to \( (-\infty, \infty] \).
Then for every decreasing sequences $A_n \searrow A$ and $B_n \searrow B$ in $B(H)_+$, we have
\[ \phi(A, B)(\rho) \leq \liminf_{n \to \infty} \phi(A_n, B_n)(\rho), \quad \rho \in B(H)^+. \]

Proof. Let $\varphi_k(t) := \phi(t, 1 - t) \wedge k$ for $t \in [0, 1]$ and $k \in \mathbb{N}$, and $\phi_k$ be the perspective function of $\varphi_k$, i.e., $\phi_k(x, y) = (x + y)\varphi_k(x/(x + y))$ for $x + y > 0$ and $\phi_k(0, 0) = 0$. For each fixed $k$, it follows from Theorem 6.1 that $\phi_k(A_n, B_n) \to \phi_k(A, B)$ in SOT. Since $\phi_k(A_n, B_n) \leq \phi(A_n, B_n)$ as immediately seen from (4.4), one has
\[ \rho(\phi_k(A, B)) = \lim_{n \to \infty} \rho(\phi_k(A_n, B_n)) \leq \liminf_{n \to \infty} \phi(A_n, B_n)(\rho), \quad \rho \in B(H)^+. \]

Hence the assertion follows since $\phi_k(A, B)(\rho) \nearrow \phi(A, B)(\rho)$ as $k \to \infty$ by the monotone convergence theorem. \qed

7. Extended Operator Convex Perspectives

In Sect. 4 we have considered a Borel function $\phi : [0, \infty)^2 \to (-\infty, \infty]$ which is homogeneous and locally bounded below. Let $f(t) := \phi(t, 1)$ for $t \in (0, \infty)$, $\alpha := \phi(1, 0)$ and $\beta := \phi(0, 1)$. As immediately seen, the local boundedness of $\phi$ from below is rephrased as $\alpha, \beta > -\infty$ and $f(t) \geq at + b, t \in (0, \infty)$, for some $a, b \in \mathbb{R}$. In this section we begin with a Borel function $f : (0, \infty) \to (-\infty, \infty]$ such that $f(t) \geq at + b$ for all $t \in (0, \infty)$ with some $a, b \in \mathbb{R}$. With $\alpha, \beta \in (-\infty, \infty]$ given as
\[ \alpha := \limsup_{t \to \infty} \frac{f(t)}{t}, \quad \beta := \limsup_{t \downarrow 0} f(t), \quad (7.1) \]
we define the perspective function $\phi_f : [0, \infty)^2 \to (-\infty, \infty]$ by
\[ \phi_f(x, y) := \begin{cases} yf(x/y) & (x, y > 0), \\ \alpha x & (x \geq 0, \ y = 0), \\ \beta y & (x = 0, \ y \geq 0) \end{cases} \quad (7.2) \]
(with the usual convention $\infty \cdot 0 = 0$). Note that the positions of $x, y$ in the above definition of $\phi_f(x, y)$ are different from those in (4.6) of Remark 4.9. This is because the roles of $A, B$ are reversed between operator connections and operator perspectives of $A, B$ in the literature.

Then $\phi_f$ is homogeneous and locally bounded from below on $[0, \infty)^2$, so that the PW-functional calculus $\phi_f(A, B)$ is defined for any $A, B \in B(H)_+$; see Definition 4.1. Of course, $\alpha, \beta$ can be arbitrary numbers in $(-\infty, \infty]$, but (7.1) is suitable for our discussions below. Also, note that if $f$ is numerically convex (or concave) on $[0, \infty)$, then lim sup’s in (7.1) become lim’s, so that $t \mapsto \phi_f(t, 1 - t)$ is continuous on $[0, 1]$ as a function to $(-\infty, \infty]$. In this case we write $\alpha = f'(\infty)$, as justified since $\lim_{t \to \infty} f(t)/t = \lim_{t \to \infty} f'_+(t)$, and $\beta = f(0^+)$. 

Definition 7.1. Let $f$ and $\phi_f$ be mentioned as above. For every $A, B \in B(H)_+$ we have the PW-functional calculus $\phi_f(A, B) \in \overline{B(H)}_{lb}$ (see Definition 4.1
and Theorem 4.3). We call \( \phi_f(A, B) \) the (extended) operator perspective of \( A, B \) associated with \( f \) as well.

The transpose \( \tilde{f} \) of \( f \) is defined by

\[
\tilde{f}(t) := tf(t^{-1}), \quad t \in (0, \infty),
\]

and set \( \tilde{\alpha} := \lim sup_{t \to 0} \tilde{f}(t)/t, \tilde{\beta} := \lim sup_{t \to 0} \tilde{f}(t) \) as in (7.1). Then \( \tilde{f}(t) \geq bt + a \) for \( t \in (0, \infty) \), and \( \tilde{\alpha} = \beta, \tilde{\beta} = \alpha \). Hence \( \tilde{\phi}_f(x, y) = \phi_f(y, x) \) on \([0, \infty)^2\), so that we have

\[
\phi_f(A, B) = \phi_f(B, A), \quad A, B \in B(\mathcal{H})_+.
\]

(7.3)

Theorem 4.8 is rewritten in the present situation as follows: For any \( A, B \in B(\mathcal{H})_+ \),

\[
\phi_f(A, B) = \begin{cases} 
B^{1/2}f(B^{-1/2}AB^{-1/2})B^{1/2} & \text{if } B \in B(\mathcal{H})_{++}, \\
A^{1/2}\tilde{f}(A^{-1/2}BA^{-1/2})A^{1/2} & \text{if } A \in B(\mathcal{H})_{++}.
\end{cases}
\]

(7.4)

When \( A, B \in B(\mathcal{H})_{++} \), the first expression in (7.4) is the definition of the operator perspective of \( A, B \) in [12–14], where \( f \) is an \( \mathbb{R} \)-valued continuous function on \((0, \infty)\) and \( f(B^{-1/2}AB^{-1/2}) \) is the continuous functional calculus so that \( \phi_f(A, B) \in B(\mathcal{H})_{sa} \). In Definition 7.1 the definition of operator perspectives is made from the beginning for general pairs \((A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ \), extending that in [12–14], while the values are in \( \widehat{B(\mathcal{H})}_{lb} \) rather than \( B(\mathcal{H})_{sa} \).

For convenience we restate main results in Sect. 4 in the next theorem.

**Theorem 7.2.** Let \( f, \tilde{f} \) and \( \phi_f \) be as stated above.

1. Assume that \( f(t) < \infty \) for at least two points in \((0, \infty)\). Then the following conditions are equivalent:
   (i) for all \( A_i, B_i \in B(\mathcal{H})_+ \) (\( i = 1, 2 \)) with any \( \mathcal{H} \),
   \[
   \phi_f(A_1 + A_2, B_1 + B_2) \leq \phi_f(A_1, B_1) + \phi_f(A_2, B_2);
   \]
   (ii) \( f \) is \( \mathbb{R} \)-valued and operator convex on \((0, \infty)\);
   (iii) \( \tilde{f} \) is \( \mathbb{R} \)-valued and operator convex on \((0, \infty)\).

2. Assume that \( f \neq \infty \). Then the following conditions are equivalent:
   (i') for every \( A_1, A_2, B \in B(\mathcal{H})_+ \) with any \( \mathcal{H} \),
   \[
   A_1 \leq A_2 \implies \phi_f(A_1, B) \geq \phi_f(A_2, B);
   \]
   (ii') \( f \) is \( \mathbb{R} \)-valued and operator monotone decreasing on \((0, \infty)\);
   (iii') \( f \) is \( \mathbb{R} \)-valued and operator convex on \((0, \infty)\), and \( \lim_{t \to \infty} f(t)/t \leq 0 \).

**Proof.** (1) In the present setting, note that \( \phi_f(t, 1) = f(t) \) and \( \phi_f(1, t) = \tilde{f}(t) \). If either (ii) or (iii) holds, then we have

\[
\lim_{t \to \infty} \frac{\phi_f(t, 1)}{t} = \alpha = \phi_f(1, 0), \quad \lim_{t \to \infty} \frac{\phi_f(1, t)}{t} = \beta = \phi_f(0, 1)
\]

(see the discussion just above Definition 7.1). Hence (1) is a consequence of Theorem 4.10 (and Theorem 3.7(1)).
(2) Assume item (i'). By Theorem 4.8 (or (7.4)), $f(t) = \phi_f(t, 1)$ is operator monotone decreasing on $(0, \infty)$ and hence operator convex on $(0, \infty)$. Moreover, $\phi_f(1, 0) \leq \phi_f(0, 0) = 0$. Therefore, we have

$$\lim_{t \nearrow 1} \phi_f(t, 1 - t) = \alpha = \phi_f(1, 0) \leq 0, \quad \lim_{t \searrow 0} \phi_f(t, 1 - t) = \beta = \phi_f(0, 1)$$

(see the proof of (iii) $\implies$ (v) of Theorem 4.10). These are also satisfied if either (ii') or (iii') holds, so that the assumption imposed in Theorem 4.11 is automatically satisfied in each case of (i')–(iii'). Hence (2) follows from Theorem 4.11 (and Theorem 3.7(2)).

Remark 7.3. (1) Let $f$ be an operator convex function on $(0, \infty)$. Then it is known (see [32, Proposition 8.4]) that both $\alpha, \beta$ in (7.1) are finite if and only if there are $a, b \in \mathbb{R}$ and an operator monotone function $h \geq 0$ on $(0, \infty)$ such that $f(t) = at + b - h(t)$ for all $t \in (0, \infty)$. In this case, $\phi_f(A, B)$ essentially reduces to the minus sign of the Kubo–Ando operator connection $B\sigma_h A$; more precisely, $\phi_f(A, B) = aA + bB - B\sigma_h A$ for all $A, B \in B(H)_+$. Therefore, the joint convexity in (i) of Theorem 7.2(1) reduces to the joint concavity of the operator connection $\sigma_h$.

(2) Let $f$ be an operator monotone decreasing function on $(0, \infty)$. If $f$ is negative on $(0, \infty)$, then $-\phi_f(A, B)$ is the Kubo–Ando operator connection corresponding to $-f$, so that the inequality in (i') of Theorem 7.2(2) is strengthened to being jointly monotone decreasing. We note that this occurs only when $f$ is negative on $(0, \infty)$. Indeed, if this is the case, then by (7.3) and Theorem 7.2(2) both $f$ and $\tilde{f}$ are operator monotone decreasing on $(0, \infty)$. Therefore, for all $t \in (0, \infty)$ it follows that $0 \geq \tilde{f}'(t) = f(t^{-1}) - t^{-1}f'(t^{-1})$ and so $f(t^{-1}) \leq t^{-1}f'(t^{-1}) \leq 0$. Thus we have seen that any jointly operator monotone operation arising as PW-functional calculus is exactly a Kubo and Ando’s operator connection.

In a conventional approach to Kubo and Ando’s theory, the operator connection $A\sigma B$ defined first for $A, B \in B(H)_{++}$ is extended to general $A, B \in B(H)_+$ as

$$A\sigma B = \lim_{\varepsilon \searrow 0} A_{\varepsilon}\sigma B_{\varepsilon} \quad \text{in SOT}$$

(7.5)

based on the upper continuity in (6.1), where $A_{\varepsilon} := A + \varepsilon I$ and similarly for $B_{\varepsilon}$. A main aim of this section is to show (Theorem 7.7) that a similar approach is available for the operator perspective $\phi_f(A, B)$ when $f$ is an operator convex function on $(0, \infty)$. Before showing this, we will prove (Theorem 7.5) that $\phi_f(A, B)$ enjoys a joint lower semicontinuity property.

From now on, for the convenience of presentation, we will write $OC(0, \infty)$ for the set of all $\mathbb{R}$-valued operator convex functions on $(0, \infty)$. Furthermore, we set

$$OC_0(0, \infty) := \{ f \in OC(0, \infty) : f(1) = 0 \}.$$ 

Lemma 7.4. Let $f \in OC_0(0, \infty)$ and $A, B \in B(H)_+$.

(1) The mapping

$$X \in B(H)_+ \mapsto \phi_f(A + X, B + X) \in \widehat{B(H)}_{lb}$$
is decreasing, that is, for $X, X' \in B(\mathcal{H})_+$,
\[
X \geq X' \implies \phi_f(A + X, B + X) \leq \phi_f(A + X', B + X').
\]
(2) We have
\[
\phi_f(A, B)(\rho) = \sup_{\varepsilon > 0} \rho(\phi_f(A_{\varepsilon}, B_{\varepsilon})), \quad \rho \in B(\mathcal{H})^*_+.
\]

**Proof.** (1) Let $X, X' \in B(\mathcal{H})_+$ and $X \geq X'$. Using Theorem 7.2(1) and Lemma 4.7(2) one has
\[
\phi_f(A + X, B + X) = \phi_f(A + X' + (X - X'), B + X' + (X - X'))
\]
\[
\leq \phi_f(A + X', B + X') + \phi_f(X - X', X - X')
\]
\[
= \phi_f(A + X', B + X') + \phi_f(1, 1)(X - X')
\]
\[
= \phi_f(A + X', B + X')
\]
thanks to $\phi_f(1, 1) = f(1) = 0$.

(2) Item (1) gives
\[
\phi_f(A, B)(\rho) \geq \sup_{\varepsilon > 0} \rho(\phi_f(A_{\varepsilon}, B_{\varepsilon})).
\]
The reverse inequality follows from Corollary 6.3, where the continuity assumption of $\phi_f(t, 1 - t)$ is guaranteed as mentioned in the paragraph before Definition 7.1. □

The next theorem shows a joint lower semicontinuity property of $\phi_f(A, B)$ for SOT-converging sequences, a stronger version of Corollary 6.3 though in the case of operator convex perspectives.

**Theorem 7.5.** Let $f \in \text{OC}(0, \infty)$. If $A, B, A_n, B_n \in B(\mathcal{H})_+$ ($n \in \mathbb{N}$), $A_n \to A$ and $B_n \to B$ in SOT, then
\[
\phi_f(A, B)(\rho) = \lim \inf_{n \to \infty} \phi_f(A_n, B_n)(\rho), \quad \rho \in B(\mathcal{H})^*_+.
\]

**Proof.** Set $f_0(t) := f(t) - f(1)$ for $t \in (0, \infty)$. Then it is obvious that $\phi_f(A, B) = \phi_{f_0}(A, B) + f(1)B$ for all $A, B \in B(\mathcal{H})_+$. Hence we may assume that $f \in \text{OC}_0(0, \infty)$. For any fixed $\varepsilon > 0$, since $(A_n)_\varepsilon \to A_\varepsilon$ and $(B_n)_\varepsilon \to B_\varepsilon$ in SOT, it follows that $(B_n)_\varepsilon^{-1/2}(A_n)_\varepsilon(B_n)_\varepsilon^{-1/2} \to B_\varepsilon^{-1/2}A_\varepsilon B_\varepsilon^{-1/2}$ in SOT. Moreover, $(B_n)_\varepsilon^{-1/2}(A_n)_\varepsilon(B_n)_\varepsilon^{-1/2} \geq \delta I$ ($n \in \mathbb{N}$) for some $\delta > 0$, so that we can use (7.4) to confirm that $\phi_f((A_n)_\varepsilon, (B_n)_\varepsilon) \to \phi_f(A_\varepsilon, B_\varepsilon)$ in SOT as $n \to \infty$, where the SOT-continuity of continuous functional calculus is used as in the proof of Lemma 3.1. (Alternatively, the above SOT-convergence can easily be confirmed by approximating $f$ on a certain $[\delta, k]$ with polynomials.) Therefore, the mapping $(A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ \mapsto \phi_f(A, B)_\varepsilon \in B(\mathcal{H})_{\text{sa}}$ is (sequentially) continuous in SOT. From this and Lemma 7.4(2) the result follows. □

**Remark 7.6.** Let $\sigma$ be any operator connection corresponding to an operator monotone function $h \geq 0$ on $(0, \infty)$. When applied to $f = -h$, Theorem 7.5 says that if $A, B, A_n, B_n \in B(\mathcal{H})_+$ ($n \in \mathbb{N}$), $A_n \to A$ and $B_n \to B$ in SOT, then
\[
\langle (A \sigma B)\xi, \xi \rangle \geq \lim \sup_{n \to \infty} \langle (A_n \sigma B_n)\xi, \xi \rangle, \quad \xi \in \mathcal{H}.
\]
Here note (see [30, Remark A.2]) that there are $A_n, B_n \in B(\mathcal{H})_{++}$ such that $A_n \to I, B_n \to I$ in SOT and $\langle (A_n\sigma B_n)\xi, \xi \rangle \neq \langle \xi, \xi \rangle$ for some operator mean $\sigma$ and some $\xi \in \mathcal{H}$. This fact also says that Theorem 6.1 does not hold for general SOT-converging sequences $A_n \to A$ and $B_n \to B$.

We are now in a position to prove the following:

**Theorem 7.7.** Let $f \in OC(0, \infty)$ and $A, B \in B(\mathcal{H})_+$. If $X_n \in B(\mathcal{H})_+ (n \in \mathbb{N})$ and $X_n \to 0$ in SOT, then

$$\phi_f(A, B)(\rho) = \lim_{n \to \infty} \phi_f(A + X_n, B + X_n)(\rho), \quad \rho \in B(\mathcal{H})^+_\ast.$$  

Furthermore, when $f \in OC_0(0, \infty)$ and $X_n \searrow 0$, the convergence in (7.6) is increasing.

**Proof.** As in the proof of Theorem 7.5, we may assume that $f \in OC_0(0, \infty)$. For every $\rho \in B(\mathcal{H})^+_\ast$ it follows from Lemma 7.4(1) that

$$\phi_f(A, B)(\rho) \geq \sup_n \phi_f(A + X_n, B + X_n)(\rho).$$

On the other hand, Theorem 7.5 gives

$$\phi_f(A, B)(\rho) \leq \liminf_{n \to \infty} \phi_f(A + X_n, B + X_n)(\rho).$$

Hence (7.6) follows, and the latter assertion also follows from Lemma 7.4(1). \qed

The assertion of Theorem 7.7 was dealt with by Fujii and Seo [21, Theorem 2.8] in the case $f(t) = t \log t \in OC(0, \infty)$; see Example 8.12 in Sect. 8 for more about this case.

In particular, when $X_n = \varepsilon_n I$ with $\varepsilon_n \searrow 0$ in Theorem 7.7, expression (7.6) becomes

$$\phi_f(A, B)(\rho) = \lim_{\varepsilon \searrow 0} \rho(\phi_f(A_\varepsilon, B_\varepsilon)) = \lim_{\varepsilon \searrow 0} \rho(B_\varepsilon^{1/2}f(B_\varepsilon^{-1/2}A_\varepsilon B_\varepsilon^{-1/2})B_\varepsilon^{-1/2})$$

thanks to Theorem 4.8 (or (7.4)). This means that the conventional approach based on the limit from $A_\varepsilon, B_\varepsilon$ is also available to define the operator perspective $\phi_f(A, B)$, like (7.5) for operator connections.

**Remark 7.8.** Unlike the convergence property (6.1) for operator connections, it is not possible to strengthen (7.6) to the convergence of $\rho(\phi_f(A_n, B_n))$ when $A_n \searrow A$ and $B_n \searrow B$. Consider $A_n = \alpha_n X$, $B_n = \beta_n X$ with $X \in B(\mathcal{H})_+$ and $\alpha_n, \beta_n > 0$. Then $\phi_f(A_n, B_n) = \phi_f(\alpha_n, \beta_n)X$ thanks to Lemma 4.7(2). For instance, if $f$ is an operator convex function $t^p$ for any $p \in [-1, 0) \cup (1, 2]$, then $\phi_f(\alpha_n, \beta_n)$ can diverge for some $\alpha_n, \beta_n \searrow 0$.

In the rest of the section we apply an approximation procedure given in [26, Lemmas 3.1–3.3] for operator convex functions on $(0, \infty)$. Since the procedure will be useful in Sect. 9 too, we give a brief account on that here. It is known [17, 41] that any $f \in OC(0, \infty)$ admits an integral expression

$$f(t) = a + b(t - 1) + c(t - 1)^2 + d\frac{(t - 1)^2}{t} + \int_{(0, \infty)} \frac{(t - 1)^2}{t + \lambda} \, d\mu(\lambda) \quad (7.7)$$
for all \( t \in (0, \infty) \), where \( a, b \in \mathbb{R}, \) \( c, d \geq 0 \) and \( \mu \) is a positive measure on \( (0, \infty) \) with \( \int_{(0, \infty)} (1 + \lambda)^{-1} d\mu(\lambda) < \infty \) (moreover, \( a, b, c, d \) and \( \mu \) are uniquely determined from \( f \)). As easily verified, the values \( \alpha = f'(\infty) \) and \( \beta = f(0^+) \) are given in terms of the above expression as follows:

\[
\alpha = b + c \cdot \infty + d + \int_{(0, \infty)} d\mu(\lambda), \\
\beta = a - b + c \cdot \infty + \int_{(0, \infty)} \frac{1}{\lambda} d\mu(\lambda).
\] (7.8)

For each \( n \in \mathbb{N} \) we define

\[
f_n(t) := a + b(t - 1) + nc\frac{(t - 1)^2}{t + n} + d\frac{(t - 1)^2}{t + (1/n)} + \int_{[1/n, n]} \frac{(t - 1)^2}{t + \lambda} d\mu(\lambda), \quad t \in (0, \infty).
\] (7.9)

Then it is clear that \( f_n \in \text{OC}(0, \infty) \) (where \( f_n \in \text{OC}_0(0, \infty) \) if \( f \in \text{OC}_0(0, \infty) \)) and \( f_n(t) \nrightarrow f(t) \) for all \( t \in (0, \infty) \) from the monotone convergence theorem. Furthermore, in view of

\[
\frac{(t - 1)^2}{t + \lambda} = t + 1 - \frac{(1 + \lambda)^2}{\lambda} \cdot \frac{t}{t + \lambda}, \quad t, \lambda \in (0, \infty),
\]

we can rewrite \( f_n \) as

\[
f_n(t) = \alpha_n t + \beta_n - \int_{[1/n, n]} \frac{t(1 + \lambda)}{t + \lambda} d\nu_n(\lambda), \quad t \in (0, \infty),
\] (7.10)

where

\[
\alpha_n := f_n'(\infty) = b + nc + d + \int_{[1/n, n]} d\mu(\lambda) < \infty,
\] (7.11)

\[
\beta_n := f_n(0^+) = a - b + c + nd + \int_{[1/n, n]} \frac{1}{\lambda} d\mu(\lambda) < \infty,
\] (7.12)

\[
d\nu_n(\lambda) := (1 + n)c\delta_n + (1 + n)d\delta_{1/n} + \lambda_{[1/n, n]} \frac{1}{\lambda} d\mu(\lambda)
\] (7.13)

with the point masses \( \delta_n \) at \( n \) and \( \delta_{1/n} \) at \( 1/n \). Define an operator monotone function \( h_n \) on \([0, \infty)\) by

\[
h_n(t) := \int_{[1/n, n]} \frac{t(1 + \lambda)}{t + \lambda} d\nu_n(\lambda), \quad t \in (0, \infty),
\] (7.14)

and consider the corresponding operator connection \( \sigma_{h_n} \). Since \( \alpha_n \nrightarrow \alpha, \beta_n \nrightarrow \beta \) and \( f_n(t) \nrightarrow f(t) \) for all \( t \in (0, \infty) \), we have \( \phi_{f_n}(x, y) \nrightarrow \phi_f(x, y) \) for all \( (x, y) \in [0, \infty)^2 \). Then, using the monotone convergence theorem again, we have the following:

**Lemma 7.9.** Let \( f \in \text{OC}(0, \infty) \) and define \( f_n, \alpha_n, \beta_n \) and \( h_n \) \( (n \in \mathbb{N}) \) by (7.9)–(7.14). Then for every \( A, B \in B(\mathcal{H})_+ \) we have

\[
\phi_{f_n}(A, B) = \alpha_n A + \beta_n B - B \sigma_{h_n} A \quad (\in B(\mathcal{H})_{sa})
\]
and
\[ \phi_f(A, B)(\rho) = \lim_{n \to \infty} \rho(\phi_{f_n}(A, B)) \] increasingly, \( \rho \in B(\mathcal{H})^+_\ast \).

Moreover, if \( f \in OC_0(0, \infty) \), then \( f_n \)'s are in \( OC_0(0, \infty) \).

The following is an application of Lemma 7.9. Let \( \Phi : B(\mathcal{H}) \to B(\mathcal{K}) \) be a positive linear map (hence bounded automatically), where \( \mathcal{K} \) is another Hilbert space. Assume that \( \Phi \) is normal in the sense that if \( \{A_i\} \) is a net in \( B(\mathcal{H})_+ \) and \( A_i \nrightarrow A \in B(\mathcal{H})_+ \), then \( \Phi(A_i) \nrightarrow \Phi(A) \); in other words, \( \Phi \) is continuous with respect to the \( \sigma \)-weak topologies on \( B(\mathcal{H}) \) and \( B(\mathcal{K}) \). Then we have the predual map \( \Phi_* : B(\mathcal{K})_+ \to B(\mathcal{H})_+ \) of \( \Phi \), which is a positive linear map such that \( (\Phi_*(\rho))(X) = \rho(\Phi(X)) \) for all \( \rho \in B(\mathcal{K})_+ \) and \( X \in B(\mathcal{H}) \). For every \( T \in \overline{B(\mathcal{H})}_{lb} \) define
\[ \Phi(T)(\rho) := T(\Phi_*(\rho)) = T(\rho \circ \Phi), \quad \rho \in B(\mathcal{K})_+^\ast. \]

Then it is easy to confirm that \( \Phi(T) \in \overline{B(\mathcal{K})}_{lb} \), \( \Phi(\alpha T) = \alpha \Phi(T) \) and \( \Phi(T_1 + T_2) = \Phi(T_1) + \Phi(T_2) \) for all \( \alpha \geq 0 \) and \( T, T_1, T_2 \in \overline{B(\mathcal{H})}_{lb} \). Moreover, the map \( \Phi : B(\mathcal{H})_{lb} \to B(\mathcal{K})_{lb} \) is an extension of \( \Phi : B(\mathcal{H})_+ \to B(\mathcal{K})_+ \).

The next proposition is an extended version of (ii) and (iii) of Theorem 4.10 and also extends [33, Theorem 6.7].

**Proposition 7.10.** Let \( f \in OC(0, \infty) \) and \( \Phi \) be as stated above. Then for every \( A, B \in B(\mathcal{H})_+ \) we have
\[ \phi_f(\Phi(A), \Phi(B)) \leq \Phi(\phi_f(A, B)) \] in \( \overline{B(\mathcal{K})}_{lb} \).

**Proof.** As before it suffices to assume that \( f \in OC_0(0, \infty) \). For every \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})_+^\ast \), by Lemma 7.9 we have
\[ \phi_f(\Phi(A), \Phi(B))(\rho) = \sup_n \rho(\alpha_n \Phi(A) + \beta_n \Phi(B) - \Phi(B)\sigma_{h_n} \Phi(A)). \]

It is well-known that
\[ \Phi(B\sigma_{h_n} A) \leq \Phi(B)\sigma_{h_n} \Phi(A), \] (7.15)
which is due to Ando [6] (though stated in [6] only for geometric mean and parallel sum). Therefore, we have
\[
\rho(\alpha_n \Phi(A) + \beta_n \Phi(B) - \Phi(B)\sigma_{h_n} \Phi(A)) \\
\leq \rho(\Phi(\alpha_n A + \beta_n B - B\sigma_{h_n} A)) = (\Phi_*(\rho))(\phi_{f_n}(A, B)) \\
\leq (\phi_f(A, B))(\Phi_*(\rho)) = \Phi(\phi_f(A, B))(\rho),
\]
showing the result. \( \square \)

Inequality (7.15) holds for a general (not necessarily normal) positive linear map \( \Phi \); however the normality of \( \Phi \) is necessary to define \( \Phi(T) \) for \( T \in \overline{B(\mathcal{H})}_{lb} \).

In particular, by Proposition 7.10 applied to \( \Phi : B(\mathcal{H}) \to \mathbb{C} \) sending \( X \in B(\mathcal{H}) \) to \( \rho(X) \), we have
\[ \phi_f(\rho(A), \rho(B)) \leq \phi_f(A, B)(\rho), \quad \rho \in B(\mathcal{H})_+^\ast, \] (7.16)
which is conceptually similar to the Peierls–Bogoliubov inequality for quantum divergences; see, e.g., [44,45,50] (also [28]).

We end the section with clarifying the relation between the extended operator perspective $\phi_f(A,B)$ and the maximal f-divergence $\hat{S}_f(A\|B)$ for $A,B \in C_1(\mathcal{H})_+$ (identified with $B(\mathcal{H})_+$), i.e., positive trace-class operators on $\mathcal{H}$. Maximal f-divergences are a type of quantum f-divergences, whose details are found in, e.g., [27,31]. Here recall its definition in [27] restricted to the present setting. Let $f \in \text{OC}(0,\infty)$ and $A,B \in C_1(\mathcal{H})_+$. When $A \leq \lambda B$ for some $\lambda > 0$ and $P$ is the support projection of $B$, there exists a unique $X \in B(\mathcal{H})$ such that $XP = X$ and $A^{1/2} = XB^{1/2}$ (hence $A = B^{1/2}X^*XB^{1/2}$). Then $\hat{S}_f(A\|B)$ is defined by $\hat{S}_f(A\|B) := \text{Tr}B^{1/2}f(X^*X)B^{1/2}$, where $f(X^*X)$ is the functional calculus of $X^*X$ as an operator in $B(P\mathcal{H})$. The extension to general $A,B \in C_1(\mathcal{H})$ is given as $\hat{S}_f(A\|B) := \lim_{\epsilon \searrow 0} \hat{S}_f(A + \epsilon(A + B)\|B + \epsilon(A + B))$; see [27, Definitions 2.3, 2.8].

We set

$$\widehat{B(\mathcal{H})}_+ + C_1(\mathcal{H})_{\text{sa}} := \{T + X : T \in \widehat{B(\mathcal{H})}_+, X \in C_1(\mathcal{H})_{\text{sa}}\},$$

which is a sub-cone of $\widehat{B(\mathcal{H})}_{\text{lb}} = \widehat{B(\mathcal{H})}_+ + B(\mathcal{H})_{\text{sa}}$. The trace $\text{Tr}$ on $B(\mathcal{H})_+$ naturally extends to $\widehat{B(\mathcal{H})}_+$ due to [22, Proposition 1.10]. We can further extend $\text{Tr}$ to $\widehat{B(\mathcal{H})}_+ + C_1(\mathcal{H})_{\text{sa}}$ by $\text{Tr}(T + X) := \text{Tr}T + \text{Tr}X$ ($\in (-\infty,\infty]$), and it is easy to confirm that $\text{Tr}$ is positive homogeneous, additive and normal (i.e., $T \not\to T \implies \text{Tr}T \not\to \text{Tr}T$) on $\widehat{B(\mathcal{H})}_+ + C_1(\mathcal{H})_{\text{sa}}$.

**Proposition 7.11.** Let $f \in \text{OC}(0,\infty)$ and $A,B \in C_1(\mathcal{H})_+$. Then $\phi_f(A,B) \in \widehat{B(\mathcal{H})}_+ + C_1(\mathcal{H})_{\text{sa}}$ and

$$\text{Tr} \phi_f(A,B) = \hat{S}_f(A\|B). \quad (7.17)$$

**Proof.** Set $f_0(t) := f(t) - (at + b)$ where $a := f'(1)$ and $b := f(1) - f'(1)$; then $\phi_f(A,B) = \phi_{f_0}(A,B) + (aA + bB)$. Since $\phi_{f_0}(A,B) \in \widehat{B(\mathcal{H})}_+$ thanks to $f_0 \geq 0$, we have $\phi_f(A,B) \in \widehat{B(\mathcal{H})}_+ + C_1(\mathcal{H})_{\text{sa}}$, so that $\text{Tr} \phi_f(A,B)$ is well defined as seen before the proposition.

Let us show equality (7.17). Assume first that $\lambda^{-1}B \leq A \leq \lambda B$ for some $\lambda > 0$, and let $P$ be the support projection of $B$. Then $A = B^{1/2}WB^{1/2}$ for (a unique) invertible $W \in B(P\mathcal{H})_+$. By operator homogeneity (Definition 2.1(2)) and Proposition 4.6 we have

$$\phi_f(A,B) = B^{1/2}\phi_f(W,P)B^{1/2} = B^{1/2}f(W)B^{1/2},$$

where $f(W) \in B(P\mathcal{H})_{\text{sa}}$ is the continuous functional calculus of $W$. Hence $\phi_f(A,B) \in C_1(\mathcal{H})_{\text{sa}}$ and $\text{Tr} \phi_f(A,B) = \text{Tr}B^{1/2}f(W)B^{1/2} = \text{Tr}Bf(W)$; see [43, Lemma 3.4.11] for the last equality. The last term of this is exactly $\hat{S}_f(A\|B)$. For general $A,B \in C_1(\mathcal{H})_+$ with $C := A + B$, the case shown just above gives $\text{Tr} \phi_f(A + \epsilon C,B + \epsilon C) = \hat{S}_f(A + \epsilon C\|B + \epsilon C)$ for any $\epsilon > 0$. 

Furthermore, with $f_0$ as above we have
\[
\text{Tr} \phi_f(A + \varepsilon C, B + \varepsilon C)
\]
\[
= \text{Tr} \phi_{f_0}(A + \varepsilon C, B + \varepsilon C) + \text{Tr}(aA + bB) + \varepsilon(a + b)\text{Tr} C,
\]
which converges to $\text{Tr} \phi_f(A, B)$ as $\varepsilon \downarrow 0$ by Theorem 7.7 and the normality of $\text{Tr}$ (mentioned above). Hence equality (7.17) holds thanks to [27, Lemma 2.6]. (Alternatively, (7.17) can be shown by using [27, Theorem 4.2] which is more directly related to the PW-functional calculus.)

Proposition 7.11 gives a justification for our formulation of extended operator perspectives. Basic properties of maximal $f$-divergences given in [27] are also derived via (7.17) from those of $\phi_f(A, B)$ shown in this section (though in the $B(\mathcal{H})$ setting). For example, when a normal positive map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is trace-preserving, the inequality of Proposition 7.10 yields the monotonicity inequality (called the data-processing inequality) for the maximal $f$-divergence shown in [27, Theorem 2.9].

8. Dense Domain Case and Boundedness

For $f \in \text{OC}(0, \infty)$ we will further discuss the operator perspective $\phi_f(A, B)$ introduced in Definition 7.1 via (7.2) with $\alpha = f'(\infty)$ and $\beta = f(0^+)$. Throughout this section, for any $A, B \in B(\mathcal{H})_+$ we use the notations $\mathcal{H}_{A,B}$, $T_{A,B}$, $R_{A,B}$ and $S_{A,B}$ given in Sect. 4, and let $R_{A,B} = \int_0^1 t \, dE_{R_{A,B}}(t)$ be the spectral decomposition.

Our questions in this section are when $\phi_f(A, B)$ is bounded and when $\phi_f(A, B)$ has a dense domain (i.e., the $\infty$-part is trivial); see Definition 2.3. First we consider the latter question. When both $\alpha$ and $\beta$ are finite, $\phi_f(A, B)$ is bounded for all $A, B \in B(\mathcal{H})_+$; see Remark 7.5(1). Now assume that $\alpha = \infty > \beta$. For every $\xi \in \mathcal{H}$, by (4.4) and Lemma 4.4(3), we find that
\[
\phi_f(A, B)(\omega_\xi) = \phi_f(R, S)(T_{A,B}\omega_\xi T_{A,B}^*)
\]
\[
= \left( \int_{[0,1]} \phi_f(t, 1 - t) \, dE_{R_{A,B}}(t) \right)(\omega_{T_{A,B}})
\]
\[
= \int_{[0,1]} \phi_f(t, 1 - t) \, d\|E_{R_{A,B}}(t)T_{A,B}\xi\|^2
\]
\[
+ \infty \cdot \|E_{R_{A,B}}(\{1\})T_{A,B}\xi\|^2. \tag{8.1}
\]
Write $Q_{A,B} := \int_{[0,1]} (\phi_f(t, 1 - t)\vee 0) \, dE_{R_{A,B}}(t)$, which is a positive self-adjoint operator on $\mathcal{H}_{A,B}$. Then it follows from (8.1) that
\[
\{ \xi \in \mathcal{H} : \phi_f(A, B)(\omega_\xi) < \infty \} = \ker(E_{R_{A,B}}(\{1\})T_{A,B}) \cap \mathcal{D}(Q_{A,B}^{1/2}T_{A,B}).
\]
Therefore, the essential part $\mathcal{H}_0$ (see Proposition 2.2) of $\phi_f(A, B)$ is
\[
\mathcal{H}_0 = \ker(E_{R_{A,B}}((\{1\})T_{A,B}) \cap \mathcal{D}(Q_{A,B}^{1/2}T_{A,B}) \subseteq \ker(E_{R_{A,B}}((\{1\})T_{A,B}), \tag{8.2}
\]
and $\mathcal{H}_0 = \mathcal{H}$ (i.e., $\phi_f(A, B)$ has a dense domain) if and only if $E_{R_{A,B}}((\{1\})T_{A,B})=0$ and $\mathcal{D}(Q_{A,B}^{1/2}T_{A,B})$ is dense in $\mathcal{H}$. Since $\text{ran} \, T_{A,B} = \mathcal{H}_{A,B}$,
For each $t$ below. This function is an extreme case of operator convex power functions

\[ \tilde{\phi}(t) = \phi_f(t, 1-t) \]

clear by applying (1) to Proposition 8.1.

Let $f$ be a function on $\mathbb{R}$.

(1) Assume that $f'(\infty) = \infty > f(0^+)$, and set $Q_{A,B} := \int_{[0,1]} (\phi_f(t, 1-t) \vee 0) \, dE_{R_{A,B}}(t)$ on $\mathcal{H}_{A,B}$. Then $\phi_f(A, B)$ has a dense domain if and only if $\ker S_{A,B} = \{0\}$ and $\mathcal{D}(Q_{A,B}^{1/2} T_{A,B})$ is dense in $\mathcal{H}$.

(2) Assume that $f'(\infty) < \infty = f(0^+)$, and set $Q_{A,B} := \int_{[0,1]} (\phi_f(t, 1-t) \vee 0) \, dE_{R_{A,B}}(t)$ on $\mathcal{H}_{A,B}$. Then $\phi_f(A, B)$ has a dense domain if and only if $\ker R_{A,B} = \{0\}$ and $\mathcal{D}(Q_{A,B}^{1/2} T_{A,B})$ is dense in $\mathcal{H}$.

(3) Assume that $f'(\infty) = f(0^+) = \infty$, and set $Q_{A,B} := \int_{[0,1]} (\phi_f(t, 1-t) \vee 0) \, dE_{R_{A,B}}(t)$ on $\mathcal{H}_{A,B}$. Then $\phi_f(A, B)$ has a dense domain if and only if $\ker R_{A,B} = \ker S_{A,B} = \{0\}$ and $\mathcal{D}(Q_{A,B}^{1/2} T_{A,B})$ is dense in $\mathcal{H}$.

Remark 8.2. In particular, when $\mathcal{H}$ is finite-dimensional, the situation in Proposition 8.1 is much simpler. Note that any densely-defined operator is bounded and $\ker A = \ker R_{A,B} \oplus \mathcal{H}_{A,B}$ in this case. Hence, for instance, item (1) simply says that if $f'(\infty) = \infty > f(0^+)$, then $\phi_f(A, B)$ is bounded if and only if $\ker A \supseteq \ker B$ (or $(A, B) \in (B(\mathcal{H})_+ \times B(\mathcal{H})_+)^\perp$); (2) and (3) are similar.

Concerning the question on the boundedness of $\phi_f(A, B)$, one can state, similarly to Proposition 8.1(1) for example, that if $f'(\infty) = \infty > f(0^+)$, then $\phi_f(A, B)$ is bounded if and only if $\ker S_{A,B} = \{0\}$ and $Q_{A,B}^{1/2} T_{A,B}$ is bounded. But this seems just a restatement of $\phi_f(A, B)$ being bounded, so a more intrinsic condition is desirable. Although the problem seems difficult for general $f \in \mathcal{OC}(0, \infty)$, the special case of $f(t) = t^2$ is tractable as discussed below. This function is an extreme case of operator convex power functions $t^p$ ($p \in [-1, 0] \cup [1, 2]$).

As for the function $t^2$, we begin by setting

\[ g^{(n)}(t) := \frac{n(t-1)^2}{t+n} = nt + 1 - (1 + n)^2 \frac{t}{t+n}, \quad t \in (0, \infty), \]  

for each $n \in \mathbb{N}$. Obviously, $g^{(n)}(t) \nearrow (t-1)^2$ for all $t \in (0, \infty)$. For any $A, B \in B(\mathcal{H})_+$ note that

\[ \phi_{g^{(n)}}(A, B) = nA + B - (1 + n)^2 \phi_{t/(t+n)}(A, B) \]

\[ = nA + B - \frac{(1 + n)^2}{n}(A : nB), \]

where $A : B$ denote the parallel sum of $A, B \in B(\mathcal{H})_+$ (see [2,6]). Moreover, recall a well-known formula

\[ A - (A : B) = A(A + B)^{-1} A, \quad A, B \in B(\mathcal{H})_+, \]
which is more precisely understood as

\[ A - (A : B) = \lim_{\varepsilon \to 0} A(A + B + \varepsilon I)^{-1}A \] in SOT.

(This formula is easy to see. Also, the original definition of parallel sum [2] is \( A : B = A(A + B)^{-1}B \) for matrices, where \((A + B)^{-1}\) is the generalized inverse; see also [9, p. 103].)

**Proposition 8.3.** For every \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})_+^\sigma \), we have

\[
\phi_{t^2}(A, B)(\rho) = \lim_{n \to \infty} n \rho(A - (A : nB)) = \lim_{\varepsilon \to 0} \rho(\varepsilon A + B)^{-1}A
\] (8.6)

increasingly.

**Proof.** Let \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})_+^\sigma \). Since \( \phi_{g(n)}(A, B) \nearrow \phi_{(t-1)^2}(A, B) \), it follows from (8.4) that

\[
\phi_{t^2}(A, B)(\rho) = \phi_{(t-1)^2}(A, B)(\rho) + \phi_{2t-1}(A, B)(\rho)
\]

\[
= \lim_{n \to \infty} \rho \left( nA + B - \frac{(1 + n)^2}{n}(A : nB) \right) + \rho(2A - B)
\]

\[
= \lim_{n \to \infty} \rho \left( \frac{n + 2}{n} n(A - (A : nB)) - \frac{1}{n}(A : nB) \right)
\]

\[
= \lim_{n \to \infty} n \rho(A - (A : nB)).
\]

The above last equality is immediate since \( \frac{1}{n}(A : nB) \leq \frac{1}{n}A \to 0 \). Furthermore, by (8.5) we find that

\[
n(A - (A : nB)) = nA(A + nB)^{-1}A = A(n^{-1}A + B)^{-1}A,
\]

showing the second equality and the limits being increasing. \( \square \)

Here we recall Ando’s work [4] on Lebesgue decomposition of positive operators. For \( B, X \in B(\mathcal{H})_+ \) it is said that \( X \) is \( B \)-absolutely continuous if there is a sequence \( \{X_n\} \in B(\mathcal{H})_+ \) such that \( X_n \not\to X \) and \( X_n \leq \lambda_n B \) for some \( \lambda_n \geq 0 \). Also, \( X \) is said to be \( B \)-singular if \( Y \in B(\mathcal{H})_+ \) satisfying \( Y \leq B \) and \( Y \leq X \) is only \( Y = 0 \). For every \( A, B \in B(\mathcal{H})_+ \) Ando [4] introduced a \( B \)-absolutely continuous part \( [B]A \) of \( A \) by

\[ [B]A := \lim_{n \to \infty} (A : nB) \] increasingly,

which is the maximum of all \( B \)-absolutely continuous \( X \in B(\mathcal{H})_+ \) with \( X \leq A \). Then \( A - [B]A \) is \( B \)-singular and we have a \( B \)-Lebesgue decomposition [4]

\[ A = [B]A + (A - [B]A). \]

**Proposition 8.4.** For every \( A, B \in B(\mathcal{H})_+ \) we have

\[ A - [B]A = T^*_{A,B}E_{R_{A,B}}\{(1)\}T_{A,B} \] (8.7)

Hence \( \ker S_{A,B} = \{0\} \) if and only if \( A \) is \( B \)-absolutely continuous.
Proof. We observe that
\[
A - (A : nB) = T_{A,B}^* (R_{A,B} - (R_{A,B} : n(I_{H,A,B} - R_{A,B}))) T_{A,B}
\]
\[
= T_{A,B}^* \left( \frac{R_{A,B}^2}{n(I_{H,A,B} - R_{A,B}) + R_{A,B}} \right) T_{A,B}
\]
\[
= T_{A,B}^* \left( \int_0^1 \frac{t^2}{n(1-t) + t} dE_{R_{A,B}}(t) \right) T_{A,B},
\]
where the first equality is due to operator homogeneity [19] (see also Definition 4.1(2)) applied to parallel sum. Letting \( n \to \infty \) gives (8.7). From (8.7) and the argument above Proposition 8.1, it follows that \( \ker S_{A,B} = \{0\} \) if and only if \( A - [B]A = 0 \), that is, \( A \) is \( B \)-absolutely continuous. \( \square \)

Proposition 8.4 shows that the (maximal) absolutely continuous part \([B]A\) is expressed as
\[
[B]A = A - T_{A,B}^* E_{R_{A,B}}(\{1\}) T_{A,B} = T_{A,B}^* R_{A,B} E_{R_{A,B}}([0,1]) T_{A,B},
\]
which is somewhat similar to the formula given in [36].

By Propositions 8.1 and 8.4 we have the following necessary condition for \( \phi_f(A,B) \) to be bounded.

Corollary 8.5. Let \( f \in \text{OC}(0, \infty) \) and \( A, B \in B(H)_+ \). Assume that \( \phi_f(A,B) \) is bounded. Then

1. \( A \) is \( B \)-absolutely continuous if \( f'(\infty) = \infty > f(0^+) \),
2. \( B \) is \( A \)-absolutely continuous if \( f'(\infty) < \infty = f(0^+) \),
3. \( A, B \) are mutually absolutely continuous if \( f'(\infty) = f(0^+) = \infty \).

The next theorem gives characterizations for \( \phi_{t^2}(A,B) \) to have a dense domain and to be bounded. The same descriptions hold for \( \phi_{t-1}(A,B) \) with the roles of \( A, B \) exchanged.

Theorem 8.6. For every \( A, B \in B(H)_+ \) set \( Q_{A,B} := \int_{[0,1]} t^2/(1-t) dE_{R_{A,B}}(t) \). Then the following hold:

1. The essential part \( H_0 \) of \( \phi_{t^2}(A,B) \) is
\[
H_0 = \ker(A - [B]A) \cap D(Q_{A,B}^{1/2} T_{A,B}) \subseteq \ker(A - [B]A) \quad (8.8)
\]
2. \( \phi_{t^2}(A,B) \) has a dense domain if and only if \( A \) is \( B \)-absolutely continuous and \( D(Q_{A,B}^{1/2} T_{A,B}) \) is dense in \( H \).
3. \( \phi_{t^2}(A,B) \) is bounded if and only if \( A^2 \leq \lambda B \) for some \( \lambda > 0 \) (which is strictly weaker than \( (A,B) \in (B(H)_+ \times B(H)_+)^< \)). In this case, the operator norm of \( \phi_{t^2}(A,B) \) is
\[
\|\phi_{t^2}(A,B)\| = \min\{\lambda \geq 0 : A^2 \leq \lambda B\}.
\]

Proof. (1) immediately follows from (8.2) and (8.7).

(2) By Proposition 8.4 this is a restatement of Proposition 8.1(1) for \( f(t) = t^2 \).

(3) From (8.6), for any \( \lambda \geq 0 \) we find that \( \phi_{t^2}(A,B) \) is bounded with \( \|\phi_{t^2}(A,B)\| \leq \lambda \) if and only if \( A(\varepsilon A + B + \delta I)^{-1} A \leq \lambda I \) holds for all \( \varepsilon, \delta > 0 \).
Since \( A(\varepsilon A + B + \delta I)^{-1}A \leq \lambda I \) if and only if \( A^2 \leq \lambda(\varepsilon A + B + \delta I) \), the condition is equivalent to \( A^2 \leq \lambda B \). Hence the result follows. \( \square \)

**Example 8.7.** Consider two projections \( P, Q \in B(\mathcal{H}) \). Since \( P : (nQ) = \frac{n}{1+n} (P \wedge Q) \) (see the proof of [40, Theorem 3.7]), we have, for every \( \rho \in B(\mathcal{H}) \),

\[
\phi_{t^2}(P, Q)(\rho) = \lim_{n \to \infty} n\rho \left( P - \frac{n}{1+n} (P \wedge Q) \right) \quad \text{(by (8.6))}
\]

\[
= \lim_{n \to \infty} n\rho \left( P - P \wedge Q + \frac{1}{1+n} (P \wedge Q) \right)
\]

\[
= \rho(P \wedge Q) + \infty \cdot \rho(P - P \wedge Q)
\]

and

\[
P - [Q]P = \lim_{n \to \infty} \left( P - \frac{n}{1+n} (P \wedge Q) \right) = P - P \wedge Q.
\]

Hence the essential part of \( \phi_{t^2}(P, Q) \) is \((P - P \wedge Q)^{-1}\mathcal{H} = \ker(P - [Q]P)\).

**Remark 8.8.** The essential part of \( \phi_{t^2}(A, B) \) is equal to \( \ker(A - [B]A) \), for example, in the finite-dimensional case (see Remark 8.2) and in the two-projection case (Example 8.7). It is also easy to verify that this is the case when \( A, B \) commute. However, this is not true in general. Here we exemplify that the presence of \( D(Q_{A,B}^{1/2}T_{A,B}) \) in (8.8) can make the essential part even trivial while \( A \) is \( B \)-absolutely continuous. Let \( T \) be any non-singular bounded positive operator, and \( Q \) be any non-singular positive self-adjoint operator with the spectral decomposition \( Q = \int_0^\infty \lambda dF_{\lambda} \). With a strictly increasing function \( w : [0, 1) \to [0, \infty) \) given by

\[
w(t) = \frac{t^2}{1-t} \quad (0 \leq t < 1), \quad w^{-1}(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 + 4\lambda}}{2} \quad (0 \leq \lambda < \infty),
\]

we define

\[
R := \int_0^\infty w^{-1}(\lambda) dF_{\lambda} = \int_{[0,1)} t dE_t, \quad (8.9)
\]

where \( E_t := F_{w(t)} \). Then \( 0 \leq R \leq I \), and we further define

\[
A := TRT, \quad B := T(I - R)T. \quad (8.10)
\]

Then we have \( T = (A + B)^{1/2} = T_{A,B} \) (where \( \mathcal{H}_{A,B} = \mathcal{H} \)), \( R = R_{A,B} \), \( E = E_{R_{A,B}} \) and

\[
Q_{A,B} = \int_{[0,1)} w(t) dE_t = \int_0^\infty \lambda dF_{\lambda} = Q,
\]

so that \( T \) and \( Q \) are realized as \( T_{A,B} \) and \( Q_{A,B} \), respectively, in Theorem 8.6 (for \( A, B \) defined by (8.10)). Furthermore, since \( E_{R_{A,B}}(\{1\}) = 0 \) for the spectral measure of \( R = R_{A,B} \) thanks to (8.9), note by (8.7) that \( A = [B]A \) in this case. From a classical result of von Neumann (whose readable account is found in [16]), there are non-singular positive self-adjoint operators \( K, L \)
with bounded inverses such that \( \mathcal{D}(K) \cap \mathcal{D}(L) = \{0\} \). Now consider the above construction with \( T := K^{-1} \) and \( Q := L^2 \). Then

\[
\mathcal{D}(Q_{A,B}^{1/2}(A + B)^{1/2}) = \mathcal{D}(LK^{-1}) = \{ \xi \in \mathcal{H} : K^{-1}\xi \in \mathcal{D}(L) \}
\]

where \( K \) is bounded, and hence Theorem 8.6(3) implies that \( \phi \) is identically \( \infty \) and \( A \) is \( B \)-absolutely continuous.

Based on the integral expression given in (7.7) and Theorem 8.6(3), we can show the boundedness of \( \phi_f(A, B) \) in a more general situation.

**Proposition 8.9.** Let \( f \in \text{OC}(0, \infty) \) and \( A, B \in \mathcal{B}(\mathcal{H})_+ \).

1. Assume that \( f(0^+) < \infty \). If \( A^2 \leq \lambda B \) for some \( \lambda > 0 \), then \( \phi_f(A, B) \) is bounded. In addition, assume that \( c > 0 \) in (7.7) (hence \( f'(\infty) = \infty \)). Then \( \phi_f(A, B) \) is bounded if and only if \( A^2 \leq \lambda B \) for some \( \lambda > 0 \).

2. Assume that \( f'(\infty) < \infty \). If \( B^2 \leq \lambda A \) for some \( \lambda > 0 \), then \( \phi_f(A, B) \) is bounded. In addition, assume that \( d > 0 \) in (7.7) (hence \( f(0^+) = \infty \)). Then \( \phi_f(A, B) \) is bounded if and only if \( B^2 \leq \lambda A \) for some \( \lambda > 0 \).

3. Let \( f \in \text{OC}(0, \infty) \) be arbitrary. If \( A^2 \leq \lambda B \) and \( B^2 \leq \lambda A \) for some \( \lambda > 0 \), then \( \phi_f(A, B) \) is bounded. Moreover, assume that \( c > 0 \) and \( d > 0 \) in (7.7) (hence \( f'(\infty) = f(0^+) = \infty \)). Then \( \phi_f(A, B) \) is bounded if and only if \( A^2 \leq \lambda B \) and \( B^2 \leq \lambda A \) for some \( \lambda > 0 \).

**Proof.** We divide expression (7.7) in two parts as

\[
f_1(t) := c(t - 1)^2 + \int_{[1, \infty)} \frac{(t - 1)^2}{t + \lambda} d\mu(\lambda),
\]

\[
f_2(t) := a + b(t - 1) + d \frac{(t - 1)^2}{t} + \int_{(0, 1)} \frac{(t - 1)^2}{t + \lambda} d\mu(\lambda), \quad t \in (0, \infty).
\]

Of course, we have \( \phi_f(A, B) = \phi_{f_1}(A, B) + \phi_{f_2}(A, B) \). Note that \( f_1(0^+) < \infty \) and \( f_2'(\infty) < \infty \).

1. By (7.8) the assumption forces \( d = 0 \) and \( \int_{(0, \infty)} \lambda^{-1} d\mu(\lambda) < \infty \). Since \( f_2(0^+) < \infty \) as well as \( f_2'(\infty) < \infty \) in this case, \( \phi_{f_2}(A, B) \) is bounded so that the question reduces to the boundedness of \( \phi_{f_1}(A, B) \). Note that

\[
\frac{f_1(t)}{(t + 1)^2} \leq c + \int_{[1, \infty)} \frac{1}{t + \lambda} d\mu(\lambda) \leq k, \quad t \in (0, \infty),
\]

where \( k := c + \int_{[1, \infty)} \lambda^{-1} d\mu(\lambda) < \infty \). Hence one finds that

\[
\phi_{f_1}(A, B) \leq k \phi_{f(t+1)^2}(A, B) = k(\phi_{f_2}(A, B) + 2A + B).
\]

Therefore, the first assertion holds by Theorem 8.6(3). Moreover, assume \( c > 0 \); then \( f_1(t) \geq c(t - 1)^2 \) for all \( t \in (0, \infty) \). If \( \phi_f(A, B) \) is bounded, then so is \( \phi_{f_1}(A, B) \) and hence so is \( \phi_{(t-1)^2}(A, B) \). This means that \( \phi_{f_2}(A, B) \) is bounded, and hence Theorem 8.6(3) implies that \( A^2 \leq \lambda B \) for some \( \lambda > 0 \).

2. is seen by applying item (1) to \( \tilde{f} \) and noting that \( c, d \) are exchanged for \( \tilde{f} \).
(3) The proof is easy by applying items (1) and (2) to \( f_1 \) and \( f_2 \), respectively, given at the beginning of the proof. The details are omitted here.

Remark 8.10. A naive criterion for \( \phi_f(A, B) \) to be bounded is given as follows. When \( f \in \text{OC}_0(0, \infty) \), Theorem 7.7 enables us to see that \( \phi_f(A, B) \) is bounded if and only if there is a \( \lambda > 0 \) such that \( \phi_f(A_\varepsilon, B_\varepsilon) \leq \lambda I \) for all sufficiently small \( \varepsilon > 0 \). This criterion can be extended to any \( f \in \text{OC}(0, \infty) \) by taking \( f_0 \in \text{OC}_0(0, \infty) \) as in the proof of Theorem 7.5. Thus, for any \( f \in \text{OC}(0, \infty) \) we notice that \( \phi_f(A, B) \) is bounded if and only if there is a \( \lambda > 0 \) such that \( f(B_\varepsilon^{-1/2}A_\varepsilon B_\varepsilon^{-1/2}) \leq \lambda B_\varepsilon^{-1} \) for all sufficiently small \( \varepsilon > 0 \). For instance, when \( f(t) = t^2 \), the last condition is rewritten as \( A_\varepsilon B_\varepsilon^{-1} A_\varepsilon \leq \lambda I \) or equivalently \( A_\varepsilon^2 \leq \lambda B_\varepsilon \) (for all small \( \varepsilon > 0 \)), from which one can give an alternative proof of Theorem 8.6(3).

As for \( f(t) = t^\alpha \) with \( 1 < \alpha \leq 2 \), we here collect sufficient or necessary conditions for \( \phi_{t^\alpha}(A, B) \) being bounded as follows.

Corollary 8.11. Let \( 1 < \alpha \leq 2 \). For a pair \( (A, B) \) in \( B(\mathcal{H})_+ \) consider the following conditions:

(a) \( A^2 \leq \lambda B \) for some \( \lambda > 0 \),
(b) \( \phi_{t^\alpha}(A, B) (= \phi_{t^{1-\alpha}}(B, A)) \) is bounded,
(c) \( A^\alpha \leq \lambda B^{\alpha-1} \) for some \( \lambda > 0 \),
(d) there is a \( \lambda > 0 \) such that \( \langle A\xi, \xi \rangle^{\alpha} \leq \lambda \langle B\xi, \xi \rangle^{\alpha-1} \) for all \( \xi \in \mathcal{H}, \|\xi\| = 1 \),
(e) \( A \leq \lambda B^{(\alpha-1)/\alpha} \) for some \( \lambda > 0 \).

Then we have \( (a) \implies (b) \implies (d) \implies (c) \) and \( (a) \implies (c) \implies (d) \).

Proof. We have \( (a) \implies (b) \) by Proposition 8.9(1) and \( (b) \implies (d) \) by (7.16) for \( f(t) := t^\alpha \) and \( \rho := \omega_\xi \). If \( (d) \) holds and \( \|\xi\| = 1 \), then

\[
\langle A\xi, \xi \rangle \leq \lambda^{1/\alpha} \langle B\xi, \xi \rangle^{(\alpha-1)/\alpha} \leq \lambda^{1/\alpha} \langle B^{(\alpha-1)/\alpha} \xi, \xi \rangle.
\]

Hence \( (d) \implies (e) \) holds. Since \( (a) \) gives

\[
A^\alpha \leq \|A\|^{2-\alpha} A^{2(\alpha-1)} \leq \|A\|^{2-\alpha} \lambda^{\alpha-1} B^{\alpha-1},
\]

we have \( (a) \implies (c) \). If \( (c) \) holds and \( \|\xi\| = 1 \), then

\[
\langle A\xi, \xi \rangle^{\alpha} \leq \langle A^\alpha \xi, \xi \rangle \leq \lambda \langle B^{\alpha-1} \xi, \xi \rangle \leq \lambda \langle B\xi, \xi \rangle^{\alpha-1}.
\]

Hence \( (c) \implies (d) \) holds.

Example 8.12. The function \( t \log t \) \( (t > 0) \) in \( \text{OC}(0, \infty) \) with its transpose \(-\log t\) plays a significant role in (quantum) information theory. Similarly to [21, (2.2)], for every \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})_+^\ast \), we have

\[
\phi_{t \log t}(A, B)(\rho) = \lim_{\alpha \downarrow 0} \rho \left( \frac{A - A^\alpha \rho B}{\alpha} \right)
\]

increasingly.

Indeed, since \( (1 - t^\alpha)/\alpha \nrightarrow -\log t \) \( (t > 0) \) as \( \alpha \searrow 0 \), it follows from the monotone convergence theorem that \( \rho (\phi_{(1-t^\alpha)/\alpha}(B, A)) \nrightarrow \phi_{-\log t}(B, A)(\rho) \) as \( \alpha \searrow 0 \). Therefore, when it is bounded, \( \phi_{t \log t}(A, B) \) is the minus sign of the relative operator entropy \( S(A \| B) \) studied in [20, 21]. In the case \( f(t) = t \log t \), Theorem 7.2(2), Proposition 8.9(1) and Corollary 8.5(1) read as follows.
• For $A, B_1, B_2 \in B(\mathcal{H})_+$. $B_1 \leq B_2 \implies \phi_{t \log t}(A, B_1) \geq \phi_{t \log t}(A, B_2)$.
• If $A^2 \leq \lambda B$ for some $\lambda > 0$, then $\phi_{t \log t}(A, B)$ is bounded.
• If $\phi_{t \log t}(A, B)$ is bounded, then $A$ is $B$-absolutely continuous.

These improve the corresponding facts given in [21, Sect. 2]. Here we emphasize that the PW-functional calculus $\phi_{t \log t}(A, B) = \phi_{- \log t}(B, A)$ extends a definition of $-S(A|B)$ to all $A, B \in B(\mathcal{H})_+$, whose value is though admitted to an element of $\overline{B(\mathcal{H})}_B$. This extension is conceptually natural because the original entropy quantity can be $\infty$, and it gives a better understanding of $S(A|B)$ beyond the discussions in [21, Sect. 2].

In the rest of the section we further discuss the question of boundedness of $\phi_f(A, B)$ in connection with AH (Ando–Hiai) inequalities. An operator perspective $\phi_f$ is said to satisfy an AH inequality if we have
\[
\phi_f(A, B) \leq I \implies \phi_f(A^p, B^p) \leq I \tag{8.11}
\]
for all $(A, B)$ in (a certain subset of) $B(\mathcal{H})_+ \times B(\mathcal{H})_+$ and for either all $p \geq 1$ or all $p \in (0, 1]$. Inequalities of this type were first shown in [7] for the weighted geometric means and further studied in, e.g., [33, 51]. A positive $\mathbb{R}$-valued function $f$ on $(0, \infty)$ is said to be power monotone increasing (pmi for short) if $f(t^p) \geq f(t)^p$ for all $t > 0$ and $p \geq 1$. The next proposition is a slight extension of an AH-inequality in [33].

**Proposition 8.13.** Let $f$ be a pmi positive function on $(0, \infty)$. Assume that either $f \in \text{OC}(0, \infty)$ with $f(0^+) = 0$ or $f$ is operator monotone decreasing on $(0, \infty)$. Then for any $A, B \in B(\mathcal{H})_+$, (8.11) holds for all $p \in (0, 1]$ or equivalently,
\[
\|\phi_f(A^p, B^p)\| \leq \|\phi_f(A, B)\|^p, \quad 0 < p \leq 1,
\]
where the operator norm $\|\phi_f(A, B)\|$ is understood to be $\infty$ if $\phi_f(A, B)$ is unbounded. Consequently, if $\phi_f(A, B)$ is bounded, then so is $\phi_f(A^p, B^p)$ for all $p \in (0, 1]$.

**Proof.** Let $A, B \in B(\mathcal{H})_+$ and assume that $\phi_f(A, B) \leq I$. For any $\varepsilon > 0$, Theorem 7.2(1) gives
\[
\phi_f(A_\varepsilon, B_\varepsilon) \leq \phi_f(A, B) + \phi_f(\varepsilon I, \varepsilon I) \leq (1 + \varepsilon f(1))I,
\]
so that
\[
\phi_f\left(\frac{A_\varepsilon}{1 + \varepsilon f(1)}, \frac{B_\varepsilon}{1 + \varepsilon f(1)}\right) \leq I
\]
thanks to the scalar homogeneity of $\phi_f(A, B)$. By [33, Corollary 3.8 or Proposition 6.10] we have, for every $p \in (0, 1]$,
\[
\phi_f\left(\frac{A^p_\varepsilon}{(1 + \varepsilon f(1))^p}, \frac{B^p_\varepsilon}{(1 + \varepsilon f(1))^p}\right) \leq I
\]
and hence $\phi_f(A^p_\varepsilon, B^p_\varepsilon) \leq (1 + \varepsilon f(1))^p I$. Since $A^p \to A^p$ and $B^p \to B^p$ in SOT (even in the operator norm) as $\varepsilon \searrow 0$, Theorem 7.5 implies that $\phi_f(A^p, B^p) \leq I$. Hence the first assertion follows and the remaining are immediate. $\Box$
In particular, when \( f(t) = t^2 \) (or \( f(t) = t^{-1} \)), Proposition 8.13 is an immediate consequence of Theorem 8.6(3) since \( A^2 \leq \lambda B \implies A^{2p} \leq \lambda^p B^p \) for \( 0 < p \leq 1 \).

9. Integral Expressions and Variational Expressions

Integral expression is an important ingredient of theory of operator means and connections in \([40,46]\). The integral expression for operator connections \( \sigma \) in \([40]\) is

\[
A\sigma B = aA + bB + \int_{(0,\infty)} \frac{1+\lambda}{\lambda} \left( (\lambda A) : B \right) d\mu(\lambda), \quad A, B \in B(\mathcal{H})_+, \quad (9.1)
\]

where \( a, b \geq 0 \) and \( \mu \) is a finite positive measure on \((0, \infty)\). The expression is based on the integral representation of operator monotone functions on \([0, \infty)\) (see, e.g., \([8,25]\)). Furthermore, variational expressions for various functional calculi have played an important role in topics related to this paper; see \([3,11,37,39,46,47]\) and so on.

In the first half of this section, for \( f \in \text{OC}(0, \infty) \) we discuss integral expressions of \( \phi_f(A, B) \) in a similar fashion to (9.1). The first result is based on the integral representation (7.7) of general \( f \in \text{OC}(0, \infty) \).

**Theorem 9.1.** Let \( f \in \text{OC}(0, \infty) \) be given in the representation (7.7). Let \( a_0 := b - 2c + d \) and \( b_0 := a - b + c - 2d \). Then for every \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})^+_+ \), we have

\[
\phi_f(A, B)(\rho) = a_0 \rho(A) + b_0 \rho(B) + c\phi_{12}(A, B)(\rho) + d\phi_{12}(B, A)(\rho) + \int_{(0,\infty)} \left[ \rho(A) + \frac{1}{\lambda} \rho(B) - \left( \frac{1+\lambda}{\lambda} \right)^2 \rho(A : (\lambda B)) \right] d\mu(\lambda).
\]

**Proof.** For each \( \lambda \in (0, \infty) \) we set

\[
g_\lambda(t) := \frac{(t - 1)^2}{t + \lambda} = t + \frac{1}{\lambda} - \left( \frac{1+\lambda}{\lambda} \right)^2 \frac{\lambda t}{t + \lambda}, \quad t \in (0, \infty).
\]

We notice that

\[
\phi_{g_\lambda}(A, B) = A + \frac{1}{\lambda} B - \left( \frac{1+\lambda}{\lambda} \right)^2 (A : (\lambda B)) \quad (\in B(\mathcal{H})_+), \quad A, B \in B(\mathcal{H})_+.
\]

(9.3)

For each \( n \in \mathbb{N} \) we define

\[
f_n(t) := a + b(t - 1) + cng_n(t) + dg_{1/n}(t) + \int_{[1/n, n]} g_\lambda(t) d\mu(\lambda), \quad t \in (0, \infty),
\]

which is the same as \( f_n \) given in (7.9). For every \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})^+_+ \), it follows from Lemma 7.9 that

\[
\phi_f(A, B)(\rho) = \lim_{n \to \infty} \rho(\phi_{f_n}(A, B)) = b\rho(A) + (a - b)\rho(B) + \lim_{n \to \infty} \left[ c\rho(\phi_{ng_n}(A, B)) + d\rho(\phi_{g_{1/n}}(A, B)) + \int_{[1/n, n]} \rho(\phi_{g_\lambda}(A, B)) d\mu(\lambda) \right].
\]

(9.4)
Since $ng_n(t) \not\rightarrow (t - 1)^2$ and $g_{1/n}(t) \not\rightarrow (t - 1)^2/t$ for all $t \in (0, \infty)$, by the monotone convergence theorem we have

$$\lim_{n \to \infty} \rho(\phi_{ng_n}(A, B)) = (\phi_{(t-1)^2}(A, B))(\rho) = \phi_{t^2}(A, B)(\rho) - 2\rho(A) + \rho(B),$$

$$\lim_{n \to \infty} \rho(\phi_{g_{1/n}}(A, B)) = (\phi_{(t-1)^2/t}(A, B))(\rho) = (\phi_{(t-1)^2}(B, A))(\rho) \quad \text{(by (7.3))}$$

so that (9.6) immediately follows from a familiar integral expression of

$$\int_{[1/n, n]} \rho(\phi_{g_{\lambda}}(A, B)) d\mu(\lambda)$$

and

$$\int_{(0, \infty)} \rho(\phi_{g_{\lambda}}(A, B)) d\mu(\lambda)$$

$$\int_{(0, \infty)} \left[ \rho(A) + \frac{1}{\lambda} \rho(B) - \left( \frac{1 + \lambda}{\lambda} \right)^2 \rho(A : (\lambda B)) \right] d\mu(\lambda).$$

Hence the asserted expression follows by combining (9.4) and these (increasing) convergences.

We have obtained a handy description of $\phi_{t^2}(A, B)$ in the preceding section. Apart from two $\phi_{t^2}$-terms, the main term of the integral expression (9.2) is (minus) parallel sum with a particular parametrization, though not so simple as (9.1).

Assume that $f(0^+) < \infty$ in Theorem 9.1. Then, since $d = 0$ and $\int_{(0, \infty)} \lambda^{-1} d\mu(\lambda) < \infty$ thanks to (7.8), we can pull $(\int_{(0, \infty)} \lambda^{-1} d\mu(\lambda))\rho(B)$ out of the integral in (9.2). Furthermore, since $b_0 + \int_{(0, \infty)} \lambda^{-1} d\mu(\lambda) = f(0^+)$ by (7.8), we can rewrite (9.2) as

$$\phi_f(A, B)(\rho) = a_0 \rho(A) + f(0^+) \rho(B) + c\phi_{t^2}(A, B)(\rho)$$

$$+ \int_{(0, \infty)} \left[ \rho(A) - \left( \frac{1 + \lambda}{\lambda} \right)^2 \rho(A : (\lambda B)) \right] d\mu(\lambda). \quad (9.5)$$

For any $f \in OC(0, \infty)$ we define

$$f'(0^+) := \lim_{t \uparrow 0^+} f'(t),$$

whose limit exists in $[-\infty, \infty)$ by the numerical convexity of $f$. Obviously, $f'(0^+) > -\infty$ implies $f(0^+) < \infty$. When $f'(0^+) > -\infty$, it is known that $f$ admits, besides expression (7.7), an integral expression

$$f(t) = f(0^+) + f'(0^+)t + ct^2 + \int_{(0, \infty)} \frac{t^2}{t + \lambda} d\nu(\lambda), \quad t \in (0, \infty), \quad (9.6)$$

where $c \geq 0$ and $\nu$ is a positive measure on $(0, \infty)$ with $\int_{(0, \infty)} (1 + \lambda)^{-1} d\nu(\lambda) < \infty$. Indeed, in this case, $h(t) := (f(t) - f(0^+))/t$ with $h(0) := f'(0^+)$ is a non-negative operator monotone function on $[0, \infty)$ (see [23, Theorem 2.4]), so that (9.6) immediately follows from a familiar integral expression of $h$. Since

$$\frac{t^2}{t + \lambda} = t - \frac{\lambda t}{t + \lambda}, \quad t \in (0, \infty),$$
we note as (9.3) that
\[ \phi_{t^2/(t+\lambda)}(A, B) = A - (A : (\lambda B)), \quad A, B \in B(\mathcal{H})_+. \]

Hence the next proposition can be shown, based on (9.6), similarly to Theorem 9.1, whose proof is omitted here.

**Proposition 9.2.** Let \( f \in OC(0, \infty) \) with \( f'(0^+) > -\infty \) so that \( f \) has expression (9.6). Then for every \( A, B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})^{+}_\ast \), we have
\[
\phi_f(A, B)(\rho) = \int_{(0,\infty)} [\rho(A) - \rho(A : (\lambda B))] d\nu(\lambda). \tag{9.7}
\]

The following formula of \( \phi_f(P, Q) \) for two projections \( P, Q \) is similar to [40, Theorem 3.7] for operator connections.

**Proposition 9.3.** (Two projections) For every projections \( P, Q \in B(\mathcal{H}) \) we have
\[
\phi_f(P, Q) = f(1)(P \wedge Q) + f'(\infty)(P - P \wedge Q) + f(0^+)(Q - P \wedge Q). \tag{9.8}
\]

**Proof.** Example 8.7 says that
\[
\phi_{t^2}(P, Q) = P \wedge Q + \infty \cdot (P - P \wedge Q),
\]
and furthermore we have
\[
P + \frac{1}{\lambda} Q - \left( \frac{1 + \lambda}{\lambda} \right)^2 (P : (\lambda Q)) = P + \frac{1}{\lambda} Q - \frac{1 + \lambda}{\lambda} (P \wedge Q)
\]
\[
= (P - P \wedge Q) + \frac{1}{\lambda} (Q - P \wedge Q).
\]

Inserting these into expression (9.2) yields
\[
\phi_f(P, Q) = a_0 P + b_0 Q + (c + d)(P \wedge Q)
\]
\[
+ \left( c \cdot \infty + \int_{(0,\infty)} d\mu(\lambda) \right) (P - P \wedge Q)
\]
\[
+ \left( d \cdot \infty + \int_{(0,\infty)} \frac{1}{\lambda} d\mu(\lambda) \right) (Q - P \wedge Q)
\]
\[
= (a_0 + b_0 + c + d)(P \wedge Q)
\]
\[
+ \left( a_0 + c \cdot \infty + \int_{(0,\infty)} d\mu(\lambda) \right) (P - P \wedge Q)
\]
\[
+ \left( b_0 + d \cdot \infty + \int_{(0,\infty)} \frac{1}{\lambda} d\mu(\lambda) \right) (Q - P \wedge Q),
\]
which is (9.8) because of \( a_0 + b_0 + c + d = a = f(1) \) as well as
\[
a_0 + c \cdot \infty + \int_{(0,\infty)} d\mu(\lambda) = f'(\infty), \quad b_0 + d \cdot \infty + \int_{(0,\infty)} \frac{1}{\lambda} d\mu(\lambda) = f(0^+)
\]
thanks to (7.8). \( \square \)
In the second half of the section, we are concerned with variational expressions of $\phi_f(A, B)$. For any $f \in \text{OC}(0, \infty)$ let $\alpha_n, \beta_n, \nu_n$ and $h_n \ (n \in \mathbb{N})$ be defined by (7.11)–(7.14). For every $A, B \in B(\mathcal{H})_+$ and $\rho \in B(\mathcal{H})_+^*$, by Lemma 7.9 we write

$$
\phi_f(A, B)(\rho) = \sup_n [\alpha_n \rho(A) + \beta_n \rho(B) - \rho(B \sigma_{h_n} A)] \\
= \sup_n \left[ \alpha_n \rho(A) + \beta_n \rho(B) - \int_{[1/n, n]} \frac{1 + \lambda}{\lambda} \rho(A : (\lambda B)) \, d\nu_n(\lambda) \right]
$$

(9.9)

thanks to (9.1).

In the discussions below, for each $\xi \in \mathcal{H}$ and an interval $J \subseteq (0, \infty)$, we will use the notation $\mathcal{P}(\xi; J, \mathcal{H})$ to denote the set of all pairs $(\eta(\cdot), \zeta(\cdot))$ of piecewise constant functions on $J$ with finitely many values in $\mathcal{H}$ such that $\eta(t) + \zeta(t) = \xi$ for all $t \in J$.

The following variational expressions in Theorem 9.4, Proposition 9.7 and Example 9.8 are certainly related to Pusz and Woronowicz’s ones in [46], Sect. 2 (also [11, Sect. 4]), and more directly related to [37] and [26, Sect. III]; see Remark 9.9 below for more specific discussion.

**Theorem 9.4.** Let $f \in \text{OC}(0, \infty)$, and for each $n \in \mathbb{N}$ let $\alpha_n, \beta_n$ and $\nu_n$ be as stated above. Then for every $A, B \in B(\mathcal{H})_+$ and $\xi \in \mathcal{H}$, we have

$$
\phi_f(A, B)(\omega_{\xi}) = \sup_n \sup_{\eta(\cdot), \zeta(\cdot)} \left[ \alpha_n \langle A\xi, \xi \rangle + \beta_n \langle B\xi, \xi \rangle \right]
$$

(9.10)

and

$$
\text{where the second supremum is taken over all pairs } (\eta(\cdot), \zeta(\cdot)) \text{ in } \mathcal{P}(\xi; [1/n, n], \mathcal{H}).
$$

**Proof.** In view of (9.9) (for $\rho = \omega_{\xi}$), it suffices to prove that, for each fixed $n \in \mathbb{N},$

$$
\int_{[1/n, n]} \frac{1 + t}{t} \langle (A : (tB))\xi, \xi \rangle \, d\nu_n(t) = \inf_{\eta(\cdot), \zeta(\cdot)} \int_{[1/n, n]} \frac{1 + t}{t} \left( \langle A\eta(t), \eta(t) \rangle + t \langle B\zeta(t), \zeta(t) \rangle \right) \, d\nu_n(t).
$$

Denote the above left-hand and the right-hand sides by $L_n(A, B, \xi)$ and $R_n(A, B, \xi)$, respectively. In the following, we will crucially use the well-known variational formula for parallel sum due to [3, Theorem 9], saying that

$$
\langle (A : (tB))\xi, \xi \rangle = \inf \left\{ \langle A\eta, \eta \rangle + t \langle B\zeta, \zeta \rangle : \eta, \zeta \in \mathcal{H}, \eta + \zeta = \xi \right\}.
$$

Hence it is clear that $L_n(A, B, \xi) \leq R_n(A, B, \xi)$. (At this point, we remark that the discussion below overlaps with Pusz and Woronowicz’s method of variational expressions based on essentially the same formula [46, p. 161, Lemma] as above.)
Conversely, for any $\delta > 0$ and $s \in [1/n, n]$, one can choose $\eta, \zeta \in \mathcal{H}$ (depending on $\delta$ and $s$) with $\eta + \zeta = \xi$ such that

$$\langle A\eta, \eta \rangle + t\langle B\zeta, \zeta \rangle < \langle (A : (tB))\xi, \xi \rangle + \delta$$

(9.11)

holds for $t = s$. Here we notice that $\langle (A : (tB))\xi, \xi \rangle$ is upper semicontinuous in $t > 0$, because $\langle (A : (tB))\xi, \xi \rangle = \inf_{\varepsilon > 0} \langle (A_{\varepsilon} : (tB_{\varepsilon}))\xi, \xi \rangle$ and $t > 0 \mapsto A_{\varepsilon} : (tB_{\varepsilon})$ is continuous (in the operator norm). Consequently, (9.11) holds for $t$ in an interval $(s - \delta, s + \delta)$. Choosing a finite open covering of $[1/n, n]$ from $\{(s - \delta, s + \delta)\}_{s \in [1/n, n]}$, one can easily define a pair $(\eta(\cdot), \zeta(\cdot))$ as stated in the theorem such that

$$\langle A\eta(t), \eta(t) \rangle + t\langle B\zeta(t), \zeta(t) \rangle < \langle (A : (tB))\xi, \xi \rangle + \delta \quad \text{for all } t \in [1/n, n].$$

This implies that

$$R_n(A, B, \xi) \leq \int_{[1/n, n]} \frac{1 + t}{t} [\langle (A : (tB))\xi, \xi \rangle + \delta] \, d\nu_n(t)$$

$$= \frac{L_n(A, B, \xi) + \delta}{\int_{[1/n, n]} \frac{1 + t}{t} \, d\nu_n(t)}. $$

Since $\delta > 0$ is arbitrary, $R_n(A, B, \xi) \leq L_n(A, B, \xi)$ follows.

**Remark 9.5.** In view of a remark just after Proposition 2.2, note that $\phi_f(A, B)$ is uniquely determined by expression (9.10). Furthermore, we have a variational expression of $\phi_f(A, B)$ directly coupled with $\rho \in B(\mathcal{H})^+_+$ as follows:

$$\phi_f(A, B)(\rho)$$

$$= \sup_n \sup_{X(\cdot), Y(\cdot)} \left[ \alpha_n \rho(A) + \beta_n \rho(B) \right.$$}

$$- \left. \int_{[1/n, n]} \frac{1 + t}{t} \left( \text{Tr}(X(t)X(t)^*A) + t\text{Tr}(Y(t)Y(t)^*B) \right) \, d\nu_n(t) \right],$$

(9.12)

where the second supremum is taken over all pairs $(X(\cdot), Y(\cdot))$ of piecewise constant functions on $[1/n, n]$ with finitely many values in $C_2(\mathcal{H})$ (the Hilbert–Schmidt class) such that $X(t) + Y(t) = \rho^{1/2}$ (the square-root of the density operator of $\rho$) for all $t \in [1/n, n]$. In fact, note that $B(\mathcal{H})$ is standardly represented on the Hilbert space $C_2(\mathcal{H})$ with the inner product $\langle X, Y \rangle = \text{Tr}(Y^*X)$ by left multiplication $\pi(A)X = AX$ for $A \in B(\mathcal{H})$, $X \in C_2(\mathcal{H})$. Hence, expression (9.12) follows from (9.9) in the same way as (9.10). The remark here is also available for all variational expressions given in the rest of the section.

A point of the variational expressions in (9.10) and (9.12) is that the function of $(A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+$ inside the bracket $[\cdots]$ is jointly linear and sequentially continuous in the weak operator topology. Based on this fact, important properties of $\phi_f(A, B)$ such as Theorems 7.2(1), 7.5 and Proposition 7.10 are shown in a straightforward manner. Furthermore, the assumption $A_n \rightarrow A$ and $B_n \rightarrow B$ in SOT in Theorem 7.5 and Remark 7.6 can be relaxed into $A_n \rightarrow A$ and $B_n \rightarrow B$ in the weak operator topology. Here we state this improvement of Theorem 7.5 as a corollary, but we have
shown in §7 a weaker result though in a direct and simpler approach (without variational expression).

**Corollary 9.6.** Let \( f \in \text{OC}(0, \infty) \). If \( A, B, A_n, B_n \in B(\mathcal{H})_+ \quad (n \in \mathbb{N}), \ A_n \to A \text{ and } B_n \to B \) in the weak operator topology, then

\[
\phi_f(A, B)(\rho) \leq \liminf_{n \to \infty} \phi_f(A_n, B_n)(\rho), \quad \rho \in B(\mathcal{H})_*^+.
\]

The above variational expressions (9.10) and (9.12) are presented with the cut-off interval \([1/n, n]\). But it is also possible to provide variational expressions without cut-off, based on the integral expression (9.2) or (9.5) or (9.7). First, by Proposition 8.3 we note that the term \( \phi_{l_2}(A, B) \) enjoys the variational expression

\[
\phi_{l_2}(A, B)(\omega_\xi) = \sup_n \sup_{\eta, \zeta} \langle A \xi, \xi \rangle - \langle A \eta, \eta \rangle - n \langle B \zeta, \zeta \rangle,
\]

where the second supremum is taken over \( \eta, \zeta \in \mathcal{H} \) with \( \eta + \zeta = \xi \). So we may concentrate our consideration to the main integral term of those expressions. As for the main integral term of (9.2) we show the following:

**Proposition 9.7.** In the situation of Theorem 9.1, for every \( A, B \in B(\mathcal{H})_+ \) and \( \xi \in \mathcal{H} \), we have

\[
\int_{(0, \infty)} \left[ \langle A \xi, \xi \rangle + \frac{1}{t} \langle B \xi, \xi \rangle - \left( \frac{1 + t}{t} \right)^2 \langle (A : (tB)) \xi, \xi \rangle \right] d\mu(t)
\]

\[
= \sup_{\eta(\cdot), \zeta(\cdot)} \int_{(0, \infty)} \left[ \langle A \xi, \xi \rangle + \frac{1}{t} \langle B \xi, \xi \rangle - \left( \frac{1 + t}{t} \right)^2 (\langle A \eta(t), \eta(t) \rangle + t \langle B \zeta(t), \zeta(t) \rangle) \right] d\mu(t), \quad (9.13)
\]

where the supremum is taken over all pairs \( (\eta(\cdot), \zeta(\cdot)) \) in \( \mathcal{P}(\xi; (0, \infty), \mathcal{H}) \) such that \( \eta(t) = 0 \) for all sufficiently small \( t > 0 \) and \( \zeta(t) = 0 \) for all sufficiently large \( t > 0 \).

**Proof.** First, we confirm that the function inside the bracket on the right-hand side of (9.13) is \( \mu \)-integrable for any \( (\eta(\cdot), \zeta(\cdot)) \) stated above. For such a pair \( (\eta(\cdot), \zeta(\cdot)) \) we choose an \( r \in (0, 1) \) such that \( \eta(t) = 0 \) for all \( t \in (0, r) \) and \( \zeta(t) = 0 \) for all \( t \in (r^{-1}, \infty) \). Then the function in question is equal to \( k_1(t) \) on \( (0, r) \) and to \( k_2(t) \) on \( (r^{-1}, \infty) \), given by

\[
k_1(t) := \langle A \xi, \xi \rangle - (2 + t) \langle B \xi, \xi \rangle, \quad k_2(t) := -\frac{1 + 2t}{t^2} \langle A \xi, \xi \rangle + \frac{1}{t} \langle B \xi, \xi \rangle.
\]

In view of \( \int_{(0, \infty)} (1 + t)^{-1} d\mu(t) < \infty \), the functions \( k_1(\cdot) \) and \( k_2(\cdot) \) are \( \mu \)-integrable on \( (0, r) \) and \( (r^{-1}, \infty) \), respectively. Also, the function in question is clearly \( \mu \)-integrable on \([r, r^{-1}]\), so the desired \( \mu \)-integrability is verified.

Denote the left-hand and the right-hand sides of (9.13) by \( L(A, B, \xi) \) and \( R(A, B, \xi) \). As in the proof of Theorem 9.4, \( L(A, B, \xi) \geq R(A, B, \xi) \) is immediately seen. For any \( \alpha < L(A, B, \xi) \), by the Lebesgue convergence and
the monotone convergence theorems, we note that

\[
\int_{(0,r)} k_1(t) \, d\mu(t) \\
+ \int_{[r,r^{-1}]} \left[ \langle A\xi, \xi \rangle + \frac{1}{t} \langle B\xi, \xi \rangle - \left( \frac{1+t}{t} \right)^2 \langle (A : (tB))\xi, \xi \rangle \right] \, d\mu(t) \\
+ \int_{(r^{-1},\infty)} k_2(t) \, d\mu(t)
\]

converges to \( L(A, B, \xi) \) as \( r \searrow 0 \). Hence one can choose an \( r \in (0, 1) \) such

that

\[
\int_{[r,r^{-1}]} \left[ \langle A\xi, \xi \rangle + \frac{1}{t} \langle B\xi, \xi \rangle - \left( \frac{1+t}{t} \right)^2 \langle (A : (tB))\xi, \xi \rangle \right] \, d\mu(t) > \beta := \alpha - \int_{(0,r)} k_1(t) \, d\mu(t) - \int_{(r^{-1},\infty)} k_2(t) \, d\mu(t),
\]

that is,

\[
\int_{[r,r^{-1}]} \left( \frac{1+t}{t} \right)^2 \langle (A : (tB))\xi, \xi \rangle \, d\mu(t) < \int_{[r,r^{-1}]} \left( \langle A\xi, \xi \rangle + \frac{1}{t} \langle B\xi, \xi \rangle \right) \, d\mu(t) - \beta.
\]

Now, in a similar way to the proof of Theorem 9.4, one can find a pair \((\eta(\cdot), \zeta(\cdot))\) in \( \mathcal{P}(\xi; [r, r^{-1}], \mathcal{H}) \) such that

\[
\int_{[r,r^{-1}]} \left( \frac{1+t}{t} \right)^2 \langle A\eta(t), \eta(t) \rangle + t \langle B\zeta(t), \zeta(t) \rangle \, d\mu(t) < \int_{[r,r^{-1}]} \left( \langle A\xi, \xi \rangle + \frac{1}{t} \langle B\xi, \xi \rangle \right) \, d\mu(t) - \beta.
\]

Extending \( \eta(t), \zeta(t) \) to \((0, \infty)\) as \( \eta(t) = 0 \) for all \( t \in (0, r) \) and \( \zeta(t) = 0 \) for all \( t \in (r^{-1}, \infty) \), we have

\[
R(A, B, \xi) \geq \int_{(0,r)} k_1(t) \, d\mu(t) + \beta + \int_{(r^{-1},\infty)} k_2(t) \, d\mu(t) = \alpha,
\]

so that \( R(A, B, \xi) \geq L(A, B, \xi) \) follows by letting \( \alpha \nearrow L(A, B, \xi) \). \( \square \)

In the situation where \( f(0^+) < \infty \) or \( f'(0^+) > -\infty \), we have a variational expression for the integral term in (9.5) or (9.7) in a similar manner to Proposition 9.7. Instead of stating these versions, let us give typical examples in the following:

**Example 9.8.** (1) Consider \( t^\alpha \) \((t > 0)\) where \( 1 < \alpha < 2 \), whose familiar expression

\[
t^\alpha = \frac{\sin((\alpha - 1)\pi)}{\pi} \int_{(0,\infty)} \left( \frac{t}{\lambda} - \frac{t}{t+\lambda} \right) \lambda^{\alpha-1} \, d\lambda, \quad t \in (0, \infty),
\]

is a special case of (9.6). The corresponding integral expression (see (9.7)) and the variational expression are given for \( A, B \in B(\mathcal{H})_+ \) and \( \xi \in \mathcal{H} \) as
follows:

\[ \phi_t^\alpha (A, B)(\omega_\xi) \equiv \phi_{t^1-\alpha}(B, A)(\omega_\xi) \]

\[ \equiv \frac{\sin((\alpha - 1)\pi)}{\pi} \int_{(0,\infty)} \left[ \langle A\xi, \xi \rangle - \langle (A : (\lambda B))\xi, \xi \rangle \right] \lambda^{\alpha - 2} d\lambda \]

\[ \equiv \frac{\sin((\alpha - 1)\pi)}{\pi} \sup \left[ \eta(\cdot), \zeta(\cdot) \right] \int_{(0,\infty)} \left[ \langle A\xi, \xi \rangle - \langle A\eta(\lambda), \eta(\lambda) \rangle - \lambda \langle B\zeta(\lambda), \zeta(\lambda) \rangle \right] \lambda^{\alpha - 2} d\lambda, \]

where the supremum is taken over all pairs \((\eta(\cdot), \zeta(\cdot))\) in \(P(\xi; (0, \infty), \mathcal{H})\) such that \(\zeta(\lambda) = 0\) for all sufficiently large \(\lambda > 0\). (The last condition guarantees that the integral against the supremum is well defined.)

(2) Next, consider \(t \log t\) \((t > 0)\), whose expression of the form (7.7) is

\[ t \log t = t - 1 + \int_{(0,\infty)} \frac{(t - 1)^2}{t + \lambda} \cdot \frac{\lambda}{(1 + \lambda)^2} d\lambda. \]

But a better-known formula is

\[ t \log t = \int_{(0,\infty)} \left( \frac{t}{1 + \lambda} - \frac{t}{t + \lambda} \right) d\lambda. \]  \hspace{1cm} (9.14)

Using either expression (though the latter is more convenient), we find the integral expression and the variational expression of \(\phi_t \log t(A, B)\) as follows:

\[ \phi_t \log t(A, B)(\omega_\xi) \equiv \phi_{-\log t}(B, A)(\omega_\xi) \]

\[ \equiv \int_{(0,\infty)} \left[ \frac{1}{1 + \lambda} \langle A\xi, \xi \rangle - \frac{1}{\lambda} \langle (A : (\lambda B))\xi, \xi \rangle \right] d\lambda \]

\[ \equiv \sup \left[ \eta(\cdot), \zeta(\cdot) \right] \int_{(0,\infty)} \left[ \frac{1}{1 + \lambda} \langle A\xi, \xi \rangle - \frac{1}{\lambda} \langle A\eta(\lambda), \eta(\lambda) \rangle - \langle B\zeta(\lambda), \zeta(\lambda) \rangle \right] d\lambda, \]  \hspace{1cm} (9.15)

where the supremum is taken in the same way as in Proposition 9.7.

**Remark 9.9.** Pusz and Woronowicz [47] provided a variational expression for the PW-functional calculus associated with \(\psi(x, y) \equiv x \log(x/y)\) \((x \geq 0, y > 0), \infty (x > 0, y = 0)\) and 0 \((x = y = 0)\) by making use of an integral formula

\[ \psi(x, y) = -\int_{(0,1)} \frac{x(y - x)}{x + (y - x)s} ds. \]  \hspace{1cm} (9.16)

Along the same lines, a variational expression of relative entropy was obtained in [47] and [11, Sect. 4] by applying (9.16) to the PW-functional calculus for two positive quadratic forms suitably induced from two positive linear functionals (on a \(C^*\)-algebra). A different method to variational expression was developed in [37] for relative entropy of normal positive functionals on a von Neumann algebra (whose method was extended in [26] to more general \(f\)-divergences). Here note that the formula (9.16) is essentially the same as (9.14); in fact, a change of variable \(\lambda = s/(1 - s)\) \((0 < s < 1)\) transforms (9.14) into (9.16) for \(y = 1\). Nevertheless, it does not seem possible to directly transform expression (9.15) for \(\psi(A, B) = \phi_{t \log t}(A, B)\) into that given in [47].
10. Axiomatization

Kubo and Ando [40] formulated operator connections $\sigma : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \rightarrow B(\mathcal{H})_+$ with the following postulates:

(I) (Joint operator monotonicity) $A_1 \leq A_2$ and $B_1 \leq B_2$ imply $A_1 \sigma B_1 \leq A_2 \sigma B_2$ for all $A_i, B_i \in B(\mathcal{H})_+$.

(II) (Transformer inequality) $(A \sigma B) C \leq (C A \sigma C) (B \sigma B)$ for $A, B, C \in B(\mathcal{H})_+$.

(III) (Upper continuity) If $A_n, B_n \in B(\mathcal{H})_+ (n \in \mathbb{N})$, $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \downarrow A \sigma B$.

Moreover, operator means $\sigma$ have to satisfy $I \sigma I = I$ in addition.) One of their achievements establishes an order isomorphism between the operator connections $\sigma$ and the non-negative operator monotone functions $h$ on $[0, \infty)$ in such a way that $A \sigma B = A^{1/2} h(A^{-1/2} B A^{-1/2}) A^{1/2}$ (if $A \in B(\mathcal{H})_+$). In this way, they gave an axiomatic formulation of operator connections (and means).

In this section we consider axiomatic formulations of Kubo and Ando’s type for the PW-functional calculus and extended operator convex perspectives. The first theorem is an axiomatization of the general PW-functional calculus though in the bounded situation. As before, we write $A_\varepsilon := A + \varepsilon I$ for $A \in B(\mathcal{H})_+$ and $\varepsilon > 0$ in the following.

Theorem 10.1. An operation $\Phi$ giving, for each Hilbert space $\mathcal{H}$, a mapping

$$\Phi_{\mathcal{H}} : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \rightarrow B(\mathcal{H})_{sa}$$

satisfies

(a) $\Phi$ enjoys operator homogeneity (Definition 4.1(2)),

(b) for each $\mathcal{H}$, $\Phi_{\mathcal{H}}$ is well behaved with respect to direct sums (in the sense of Proposition 4.6),

(c) for each $\mathcal{H}$, if $(A_n, B_n) \rightarrow (A, B)$ in SOT as $n \rightarrow \infty$ and if $A_n + B_n \geq \varepsilon I$ for all $n$ with some $\varepsilon > 0$, then $\Phi_{\mathcal{H}}(A_n, B_n) \rightarrow \Phi_{\mathcal{H}}(A, B)$ in SOT as $n \rightarrow \infty$,

(d) $\Phi_{\mathcal{H}}(A_\varepsilon, B_\varepsilon) \rightarrow \Phi_{\mathcal{H}}(A, B)$ in SOT as $\varepsilon \downarrow 0$ for any $(A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+$ and any $\mathcal{H}$,

if and only if there exists a (unique) $\mathbb{R}$-valued homogeneous and continuous function $\phi$ on $[0, \infty)^2$ such that $\Phi$ coincides with the PW-functional calculus associated with $\phi$.

Proof. Let us first show that the PW-functional calculus associated with a function $\phi$ as above satisfies (a)–(d). Item (a) is in Definition 4.1 and (b) is contained in Proposition 4.6. Item (c) is similar to [24, Proposition 8] and can be confirmed as follows. Since $A_n + B_n \geq \varepsilon I$ for all $n$ as well as $A + B \geq \varepsilon I$, one observes that $T_{A_n, B_n} \rightarrow T_{A, B}$ and $R_{A_n, B_n} \rightarrow R_{A, B}$ in SOT as $n \rightarrow \infty$. Consequently, if $\phi(x, y)$ is $\mathbb{R}$-valued and continuous on $[0, \infty)^2$, then, by operator homogeneity and Lemma 4.4(3), one has

$$\phi(A_n, B_n) = (A_n + B_n)^{1/2} \phi(R_{A_n, B_n}, 1 - R_{A_n, B_n})(A_n + B_n)^{1/2}$$

$$\rightarrow (A + B)^{1/2} \phi(R_{A, B}, 1 - R_{A, B})(A + B)^{1/2} = \phi(A, B)$$

as $n \rightarrow \infty$. Consequently, $\phi$ is continuous on $[0, \infty)^2$. The proof of (d) is similar to the proof of (c).
in SOT as \( n \to \infty \) thanks to the SOT-continuity of single variable continuous functional calculus for \( \phi(t, 1-t) \) on \([0,1]\). For item (d), see Theorem 6.1 (or [24, Theorem 6]).

Next we prove the converse direction. Assume that \( \Phi \) satisfies (a)–(d). Define an \( \mathbb{R} \)-valued function \( \phi \) on \([0, \infty)^2\) by

\[
\phi(x, y) := \Phi_C(x, y) \quad \text{for} \quad x, y \in [0, \infty),
\]

which is homogeneous by condition (a). Then from (c) (for \( \mathcal{H} = \mathbb{C} \)) it is clear that \( \phi \) is continuous on \([0, \infty)^2 \setminus \{(0,0)\} \). From homogeneity this implies that \( \phi \) is continuous at \((0,0)\) too. Now let \( A = \sum_{n=1}^N \alpha_n P_n \) and \( B = \sum_{n=1}^N \beta_n P_n \) with orthogonal projections \( P_n \) where \( \sum_n P_n = I_\mathcal{H} \). Item (b) implies that

\[
\Phi_\mathcal{H}(A, B) = \sum_{i=1}^N \phi_{P_n \mathcal{H}}(\alpha_n P_n, \beta_n P_n).
\]

For each \( n \) fixed, choose an orthonormal basis \( \{\xi_i\}_{i \in I} \) of \( P_n \mathcal{H} \), and consider isometries \( V_i : \mathbb{C} \to \mathcal{H} \) sending any scalar \( \lambda \) to \( \lambda \xi_i \). Then we have

\[
\Phi_{P_n \mathcal{H}}(\alpha_n P_n, \beta_n P_n) = \sum_{i \in I} \Phi_C \xi_i(\alpha_n V_i^* V_i, \beta_n V_i^* V_i) \quad \text{(by (b))}
\]

\[
= \sum_{i \in I} \Phi_C(\alpha_n, \beta_n) V_i^* V_i \quad \text{(by (a))}
\]

\[
= \phi(\alpha_n, \beta_n) P_n.
\]

Therefore, we see that \( \Phi_\mathcal{H}(A, B) = \phi(A, B) \) holds for any pair \((A, B)\) specified as above. Using a standard approximation procedure with condition (c), we have \( \Phi_\mathcal{H}(A_\varepsilon, B_\varepsilon) = \phi(A_\varepsilon, B_\varepsilon) \) for any commuting pair \((A, B) \in B(\mathcal{H})_+ \times B(\mathcal{H})_+ \) and \( \varepsilon > 0 \). Item (d) guarantees that the same identity holds even when \( \varepsilon = 0 \). Thanks to the operator homogeneity of \( \Phi \) (in (a)) and of the PW-functional calculus associated with \( \phi \), we obtain the desired conclusion. \( \Box \)

**Remark 10.2.** In view of Theorem 6.1, items (c) and (d) together can be replaced in Theorem 10.1 with the following single condition:

- if \( A, B, A_n, B_n \in B(\mathcal{H})_+, A_n \searrow A \) and \( B_n \searrow B \), then \( \Phi(A_n, B_n) \to \Phi(A, B) \) in SOT.

To confirm this modification, we only need to show that \( \phi(x, y) := \Phi_C(x, y) \) is continuous on \([0, \infty)^2\). But this can easily be seen from

\[
\phi(x, y) = \begin{cases} 
  xy \phi(1/y, 1/x) & (x, y > 0), \\
  x \phi(1, y/x) & (x > 0), \\
  y \phi(x/y, 1) & (y > 0)
\end{cases}
\]

by homogeneity.

We emphasize that the postulate of ‘extending usual functional calculus’ in Definition 4.1(1) is implemented by three items (b)–(d) in the above theorem. In this way, the \( \mathbb{R} \)-valued continuous PW-functional calculus can be axiomatized like Kubo and Ando’s operator connections, though operator
homogeneity is much stronger than the transformer inequality as a postulate. From this viewpoint, Theorem 10.1 might be understood as a result on non-commutative functions (see [34]).

Throughout the rest of the section, we assume that \( \mathcal{H} \) is a (fixed) infinite-dimensional Hilbert space whenever otherwise stated. Below we will discuss axiomatic characterizations of Kubo and Ando’s type for extended operator perspectives \( \phi_f \) for \( f \in \text{OC}(0,\infty) \). For a map \( \Phi : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \to \hat{B}(\mathcal{H})_{lb} \) we consider the following conditions:

(i) \((\text{Joint operator convexity})\) \( \Phi(A_1+A_2,B_1+B_2) \leq \Phi(A_1,B_1)+\Phi(A_2,B_2) \) for all \( A_i,B_i \in B(\mathcal{H})_+ \).

(ii) \((\text{Transformer inequality})\) \( \Phi(CAC,CBC) \leq C\Phi(A,B)C \) for all \( A,B,C \in B(\mathcal{H})_+ \).

(iii) \((\text{Specialized upper continuity})\) \( \lim_{\varepsilon \searrow 0} \Phi(A_\varepsilon,B_\varepsilon)(\rho) = \Phi(A,B)(\rho) \) for all \( A,B \in B(\mathcal{H})_+ \) and \( \rho \in B(\mathcal{H})^+_* \).

Items (i)–(iii) may be regarded as the counterparts of the above (I)–(III). If \( \Phi \) satisfies (i) and (ii), then letting \( C = 0 \) in (ii) implies that \( \Phi(0,0) \leq 0 \). Since (i) gives \( \Phi(0,0) \leq 2\Phi(0,0) \), we have

\[ \Phi(0,0) = 0. \] (10.1)

Note that if \( C \in B(\mathcal{H})_{++} \) then the transformer inequality in (ii) becomes equality automatically. In particular, (ii) implies the scalar homogeneity

\[ \Phi(\alpha A,\alpha B) = \alpha \Phi(A,B), \quad \alpha \geq 0, \] (10.2)

where the case \( \alpha = 0 \) follows from (10.1). Item (i) is subadditivity to be precise, but with the homogeneity (10.2) this is equivalent to the genuine joint operator convexity.

To prove our axiomatization theorem, we need some more technical conditions as follows:

(iv) \((\text{Special boundedness})\) \( \Phi(tI,I) \in B(\mathcal{H})_{sa} \) for all \( t \in (0,\infty) \).

(v) \((\text{Local upper continuity})\) If \( X_n \in B(\mathcal{H})_+ \) (\( n \in \mathbb{N} \)), \( X_1 \geq X_2 \geq \ldots \) and \( \|X_n\| \to 0 \), then \( \lim_{n \to \infty} \Phi(I+X_n,I)(\rho) = \Phi(I,I)(\rho) \) for all \( \rho \in B(\mathcal{H})^+_* \).

Item (iv) is reasonable for our purpose, and we need (v) to obtain an additional continuity property that cannot be covered by (iii), being not so strong as (III).

We are now in a position to state the main theorem of the section.

**Theorem 10.3.** A map \( \Phi : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \to \hat{B}(\mathcal{H})_{lb} \) satisfies conditions (i)–(v) if and only if there exists a (unique) \( f \in \text{OC}(0,\infty) \) such that \( \Phi(A,B) = \phi_f(A,B) \) for all \( A,B \in B(\mathcal{H})_+ \), where \( \phi_f \) is given in Definitions 4.1 and 7.1.

**Proof.** As for the “if” part, when \( \Phi = \phi_f \) with \( f \) as stated above, conditions (i)–(iii) are guaranteed by Theorems 4.10 (also 7.2(1)) and 7.7, and conditions (iv), (v) are obviously satisfied as well.

To prove the “only if” part, assume that \( \Phi \) satisfies (i)–(v). First let us show that if a projection \( P \in B(\mathcal{H}) \) commutes with \( A,B \in B(\mathcal{H})_+ \), then

\[ P\Phi(A,B)P = P\Phi(AP,BP)P. \] (10.3)
From (i) and (ii) one has
\[ \Phi(A, B) = \Phi(AP + AP^\perp, BP + BP^\perp) \]
\[ \leq \Phi(AP, BP) + \Phi(AP^\perp, BP^\perp) \]
\[ = \Phi(PAP, PBP) + \Phi(P^\perp AP^\perp, P^\perp BP^\perp) \]
\[ \leq P\Phi(A, B)P + P^\perp \Phi(A, B)P^\perp. \]  
(10.4)

Multiplying \( P \) from both sides of the first inequality above gives
\[ P\Phi(A, B)P \leq P\Phi(AP, BP)P + P\Phi(AP^\perp, BP^\perp)P. \]

Moreover, since \( \Phi(AP^\perp, BP^\perp) \leq P^\perp \Phi(A, B)P^\perp \) by (ii), one has
\[ P\Phi(AP^\perp, BP^\perp)P \leq 0 \]
so that
\[ P\Phi(A, B)P \leq P\Phi(AP, BP)P \leq P\Phi(A, B)P. \]

Therefore, (10.3) follows.

Now assume that \( \Phi(A, B) \in B(\mathcal{H})_{sa} \); so \( \Phi(AP, BP) \in B(\mathcal{H})_{sa} \) as well.

For any projection \( P \) commuting with \( A, B \), we write \( \Phi(A, B) = \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \) with \( X \in PB(\mathcal{H}) \) and \( Y \in P^\perp B(\mathcal{H}) P^\perp \). Then (10.4) means that \( \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \leq \begin{bmatrix} 0 & -Z \\ 0 & 0 \end{bmatrix} \), that is, \( \begin{bmatrix} 0 & -Z \\ -Z^* & 0 \end{bmatrix} \geq 0 \), which implies that \( Z = 0 \). Hence \( P \) commutes with \( \Phi(A, B) \) in this case. Similarly \( P \) commutes with \( \Phi(AP, BP) \), so that (10.3) is written as
\[ \Phi(A, B)P = \Phi(AP, BP)P. \]  
(10.5)

In particular, when \( (A, B) = (tI, I) \) for \( t \in (0, \infty) \), from (iv) and the above argument it follows that \( \Phi(tI, I) \) commutes with all projections in \( B(\mathcal{H}) \). Therefore, for each \( t \in (0, \infty) \), \( \Phi(tI, I) \) must be written as \( \Phi(tI, I) = f(t)I \) for some \( f(t) \in \mathbb{R} \). Now, consider \( A \in B(\mathcal{H})_{++} \) whose spectral decomposition is of the form \( A = \sum_{i=1}^n \lambda_i P_i \) where \( \lambda_i > 0 \) and \( \sum_{i=1}^n P_i = I \). By (i) and (ii) we have
\[ \Phi(A, I) \leq \sum_{i=1}^n \Phi(\lambda_i P_i, P_i) \leq \sum_{i=1}^n P_i \Phi(\lambda_i I, I) P_i, \]
so that that \( \Phi(A, I) \in B(\mathcal{H})_{sa} \). Hence we find that \( \Phi(A, I) \) commutes with all \( P_i \) and
\[ \Phi(A, I) = \sum_{i=1}^n \Phi(\lambda_i P_i, P_i) \leq \sum_{i=1}^n P_i \Phi(\lambda_i I, I) P_i, \]  
(10.5)

so that
\[ \Phi(A, I) = \sum_{i=1}^n \Phi(\lambda_i P_i, P_i) P_i = \sum_{i=1}^n \Phi(\lambda_i I, I) P_i (by \ (10.5)) \]
\[ = \sum_{i=1}^n f(\lambda_i) P_i = f(A). \]  
(10.6)
By (i) we notice that the $\mathbb{R}$-valued function $f$ given above is convex on $(0, \infty)$ and hence continuous on $(0, \infty)$, so one can define continuous functional calculus $f(A)$ for all $A \in B(\mathcal{H})_{++}$. Next we show that $\Phi(A, I) = f(A)$ holds for all $A \in B(\mathcal{H})_{++}$. For any such $A$, by approximating the spectral decomposition of $A$, one can choose an increasing sequence $A_n$ and a decreasing sequence $A_n'$ in $B(\mathcal{H})_{++}$ such that

$$A_n = \sum_{i=1}^{m_n} \lambda_{n,i} P_{n,i}, \quad A'_n = \sum_{i=1}^{m_n} \lambda'_{n,i} P_{n,i}$$

where $\sum_{i=1}^{m_n} P_{n,i} = I$,

$$A_n \leq A \leq A'_n, \quad X_n := A - A_n \leq n^{-1}I, \quad X'_n := A'_n - A \leq n^{-1}I.$$

Let $\rho \in B(\mathcal{H})_+^+$ be arbitrary and choose a $\delta \in (0, 1)$ such that $A_n \geq \delta I$ for all $n$. Using (i), (10.2) and (10.6), we have

$$\Phi(A, I)(\rho) = \Phi(A_n + X_n, I)(\rho) = \Phi(A_n - \delta I + X_n + \delta I, (1 - \delta)I + \delta I)(\rho) \leq \Phi(A_n - \delta I, (1 - \delta)I)(\rho) + \Phi(X_n + \delta I, \delta I)(\rho) = (1 - \delta)\Phi\left(\frac{A_n - \delta I}{1 - \delta}, I\right)(\rho) + \delta \Phi(I + \delta^{-1}X_n, I)(\rho) = (1 - \delta)\rho\left(f\left(\frac{A_n - \delta I}{1 - \delta}\right)\right) + \delta \Phi(I + \delta^{-1}X_n, I)(\rho).$$

Note that $X_1 \geq X_2 \geq \cdots$, $\|\delta^{-1}X_n\| \leq \delta^{-1}n^{-1} \to 0$ and

$$\left\|\frac{A_n - \delta I}{1 - \delta} - \frac{A - \delta I}{1 - \delta}\right\| = \frac{\|A_n - A\|}{1 - \delta} \to 0$$

as $n \to \infty$ with $\delta$ fixed. Hence, by (v) we find that

$$\Phi(A, I)(\rho) \leq (1 - \delta)\rho\left(f\left(\frac{A - \delta I}{1 - \delta}\right)\right) + \delta \rho(\Phi(I, I)).$$

Letting $\delta \searrow 0$ gives $\Phi(A, I)(\rho) \leq \rho(f(A))$ for all $\rho \in B(\mathcal{H})_+^+$, so that $\Phi(A, I) \leq f(A)$. On the other hand, we have

$$(1 + \delta)\rho\left(f\left(\frac{A_n' + \delta I}{1 + \delta}\right)\right) = (1 + \delta)\rho\left(\Phi\left(\frac{A_n' + \delta I}{1 + \delta}, I\right)\right) = \rho(\Phi(A_n' + \delta I, (1 + \delta)I)) = \rho(\Phi(A + X'_n + \delta I, I + \delta I)) \leq \Phi(A, I)(\rho) + \Phi(X'_n + \delta I, \delta I)(\rho) = \Phi(A, I)(\rho) + \delta \Phi(I + \delta^{-1}X'_n, I)(\rho).$$

Since $X'_1 \geq X'_2 \geq \cdots$ and $\|\delta^{-1}X'_n\| \to 0$ as $n \to \infty$, we find similarly to the above argument that

$$(1 + \delta)\rho\left(f\left(\frac{A + \delta I}{1 + \delta}\right)\right) \leq \Phi(A, I)(\rho) + \delta \rho(\Phi(I, I)).$$

Letting $\delta \searrow 0$ gives $f(A) \leq \Phi(A, I)$. Hence $\Phi(A, I) = f(A)$ has been shown for all $A \in B(\mathcal{H})_{++}$. Furthermore, this implies by (i) that $f$ belongs to OC($0, \infty$).
For any \( A, B \in B(\mathcal{H})_{++} \), by (ii) and the case proved above, we have
\[
\Phi(A, B) = B^{1/2} \Phi(B^{-1/2}AB^{-1/2}, I)B^{1/2} = B^{1/2} f(B^{-1/2}AB^{-1/2})B^{1/2} = \phi_f(A, B).
\]
Finally, let \( A, B \in B(\mathcal{H})_{+} \) be arbitrary. For every \( \rho \in B(\mathcal{H})_{+} \) we use (iii) to see that
\[
\Phi(A, B)(\rho) = \lim_{\varepsilon \downarrow 0} \Phi(A_{\varepsilon}, B_{\varepsilon})(\rho) = \lim_{\varepsilon \downarrow 0} \phi_f(A_{\varepsilon}, B_{\varepsilon})(\rho) = \phi_f(A, B)(\rho),
\]
where the last equality is due to Theorem 7.7. Hence \( \Phi = \phi_f \) has been shown. The uniqueness of \( f \) is clear from \( \Phi(tI, I) = f(t)I, \, t > 0 \). \( \square \)

**Remark 10.4.** (1) Theorem 10.3 holds also when condition (v) is replaced with the following (vi) (with keeping conditions (i)–(iv)):

(vi) (Local boundedness) For any \( \rho \in B(\mathcal{H})_{+} \), \( X \mapsto \Phi(I + X, I)(\rho) \) is bounded above on some open ball \( U_\varepsilon := \{ X \in B(\mathcal{H})_{sa} : \|X\| < \varepsilon \} \) for some \( \varepsilon \in (0, 1) \).

Indeed, it is clear that (vi) holds when \( \Phi = \phi_f \) with \( f \in OC(0, \infty) \). Conversely, assume that \( \Phi \) satisfies (i)–(iv) and (vi). For any \( \rho \in B(\mathcal{H})_{+} \) it follows from (i) and (vi) that \( X \mapsto \Phi(I + X, I)(\rho) \) is convex and bounded above on some open ball \( U_\varepsilon \). This implies (see, e.g., [15, Proposition I.2.5]) that \( X \mapsto \Phi(I + X, I)(\rho) \) is continuous at \( X = 0 \) in the operator norm. Hence the above proof of Theorem 10.3 can be carried out by using (vi) in place of (v).

(2) Assume that \( \mathcal{H} \) is finite-dimensional with \( n = \dim \mathcal{H} \). We can carry out the above proof of Theorem 10.3 without the approximation procedure in the paragraph after (10.6) (hence without condition (v)). However, in this case, we can only conclude that \( f \) is \( n \)-convex on \((0, \infty)\) (i.e., the operator inequality as in (3.2) holds for \( n \times n \) positive definite matrices \( A, B \)), instead of \( f \in OC(0, \infty) \). Conversely, when \( f \) is only \( n \)-convex on \((0, \infty)\), it does not seem possible to show basic properties (for instance, (ii) and (iii) above) of \( \phi_f \).

Remark 6.2(2) in particular says that if \( f \in OC(0, \infty) \) with \( f'(\infty) < \infty \) (resp., \( f(0^+) < \infty \)), then \( \phi_f(A, B) \) is bounded for all \((A, B) \in (B(\mathcal{H})_{+} \times B(\mathcal{H})_{+})_{\geq} \) (resp., \((A, B) \in (B(\mathcal{H})_{+} \times B(\mathcal{H})_{+})_{\leq} \)). The same holds true when \( f \) is an operator concave function on \((0, \infty)\) with \( f'(\infty) > -\infty \) (resp., \( f(0^+) > -\infty \)). The following proposition is a modification of Theorem 10.3 to the restricted domain case.

**Proposition 10.5.** Let \( \mathcal{B} := (B(\mathcal{H})_{+} \times B(\mathcal{H})_{+})_{\leq} \) (resp., \( \mathcal{B} := (B(\mathcal{H})_{+} \times B(\mathcal{H})_{+})_{\geq} \)). A map \( \Phi : \mathcal{B} \rightarrow B(\mathcal{H})_{sa} \) satisfies

(i') \( \phi(A_1 + A_2, B_1 + B_2) \leq \Phi(A_1, B_1) + \Phi(A_2, B_2) \) for all \((A_i, B_i) \in \mathcal{B}, \)

(ii') \( \Phi(CAC, CBC) \leq C\Phi(A, B)C \) for all \((A, B) \in \mathcal{B} \) and \( C \in B(\mathcal{H})_{+}, \)

(iii') \( \Phi(A_1, B_2) \rightarrow \Phi(A, B) \) in SOT as \( \varepsilon \downarrow 0 \) for all \((A, B) \in \mathcal{B}, \)

(iv') if \( X_n \in B(\mathcal{H})_{++}, \, X_1 \geq X_2 \geq \cdots \) and \( \|X_n\| \rightarrow 0 \), then \( \Phi(X_n, I) \rightarrow \Phi(0, I) \) (resp., \( \Phi(I, X_n) \rightarrow \Phi(I, 0) \)) in SOT,
if and only if there exists a (unique) \( f \in \text{OC}(0, \infty) \) with \( f(0^+) < \infty \) (resp., \( f'(\infty) < \infty \)) such that \( \Phi(A, B) = \phi_f(A, B) \) for all \((A, B) \in \mathcal{B}\).

**Proof.** We may prove the case \( \mathcal{B} := (B(\mathcal{H})_+ \times B(\mathcal{H})_+) \leq \), which implies the other case by considering \( \tilde{\Phi}(A, B) := \Phi(B, A) \) and the transpose \( \tilde{f} \). For the “if” part, items (i') and (ii') are the restrictions of those in Theorem 10.3, (iii') is contained in Remark 6.2(2) (also [33, Theorem 6.2]), and (iv') is obvious since \( \phi_f(X, I) = f(X) \) (where \( f \) is extended to \([0, \infty)\) with \( f(0) = f(0^+) \) for all \( X \in B(\mathcal{H})_+ \).

For the “only if” part, the proof is similar to that of Theorem 10.3 under restricting \((A, B)\) to \( \mathcal{B} \). The only place where we need to modify is the paragraph after (10.6); so only this part will be explained below. Let \( A_n, A'_n, X_n \) and \( X'_n \) be chosen as before. For any \( \delta \in (0, 1) \) fixed, we have

\[
\Phi(A, I) = \Phi(A'_n + X'_n, I) \\
\leq \Phi(A'_n, (1 - \delta)I) + \Phi(X'_n, \delta I) \\
= (1 - \delta)f\left(\frac{A'_n}{1 - \delta}\right) + \delta\Phi(\delta^{-1}X'_n, I) \\
\rightarrow (1 - \delta)f\left(\frac{A}{1 - \delta}\right) + \delta\Phi(0, I)
\]

in SOT as \( n \to \infty \), so that

\[
\Phi(A, I) \leq (1 - \delta)f\left(\frac{A}{1 - \delta}\right) + \delta\Phi(0, I).
\]

Letting \( \delta \downarrow 0 \) gives \( \Phi(A, I) \leq f(A) \). On the other hand, we have

\[
(1 + \delta)f\left(\frac{A_n}{1 + \delta}\right) = (1 + \delta)\Phi\left(\frac{A_n}{1 + \delta}, I\right) = \Phi(A + X_n, (1 + \delta)I) \\
\leq \Phi(A, I) + \Phi(X_n, \delta I) = \Phi(A, I) + \delta\Phi(\delta^{-1}X_n, I).
\]

Letting \( n \to \infty \) and then \( \delta \downarrow 0 \) gives \( f(A) \leq \Phi(A, I) \). Hence \( \Phi(A, I) = f(A) \) follows, so that we have \( \Phi = \phi_f \) for some \( f \in \text{OC}(0, \infty) \) as before. Finally, the additional condition \( f(0^+) < \infty \) is obvious since \( \phi_f(0, I) = f(0^+)I \) is bounded. \( \square \)

**Remark 10.6.** (1) Convergence in SOT in (iii') and (iv') can be replaced with convergence in weak operator topology. Also, condition (v) in Theorem 10.3 is available in Proposition 10.5 in place of (iv').

(2) In view of Remark 6.2(2), items (iii') and (iv') together can be replaced with the following stronger condition:

- if \( A, B, A_n, B_n \in B(\mathcal{H})_+ \), \( A_n \leq \alpha B_n \) (resp., \( A_n \geq \alpha B_n \)) for all \( n \) with some \( \alpha > 0 \) (independent of \( n \)), \( A_n \searrow A \) and \( B_n \searrow B \), then \( \Phi(A_n, B_n) \to \Phi(A, B) \) in SOT.

(3) The operator concavity version of Proposition 10.5 holds too, where the inequality signs in (i') and (ii') are reversed and \( f \in \text{OC}(0, \infty) \) is replaced with an operator concave function \( f \) on \((0, \infty)\). This variant is immediately seen by taking \(-\Phi\) and \(-f\) in Proposition 10.5.
The following variant of Kubo and Ando’s axiomatic characterization of operator connections is worth giving. This is seen from Remark 10.6(2) and (3) because a non-negative function on $(0, \infty)$ is operator monotone if and only if it is operator concave. In $(\text{III}')$ we assume a simple convergence $A_n \sigma B_n \rightarrow A \sigma B$ in SOT (not necessarily decreasing as in (III)), while decreasing convergence holds in $(\text{III}')$ as a consequence.

**Corollary 10.7.** A map $\sigma : B(\mathcal{H})_+ \times B(\mathcal{H})_+ \rightarrow B(\mathcal{H})_+$ is an operator connection if and only if $\sigma$ satisfies the following conditions:

$(\text{I}')$ (Joint operator concavity) $(A_1 + A_2) \sigma (B_1 + B_2) \geq (A_1 \sigma B_1) + (A_2 \sigma B_2)$ for all $A_i, B_i \in B(\mathcal{H})_+$.

$(\text{II}')$ (Transformer inequality) $C(A \sigma B)C \leq (CAC) \sigma (CBC)$ for all $A, B, C \in B(\mathcal{H})_+$.

$(\text{III}')$ (Upper continuity) If $A_n, B_n \in B(\mathcal{H})_+$, $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \sigma B_n \rightarrow A \sigma B$ in SOT.

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