A NOTE ON THE COMPLEX AND BICOMPLEX VALUED NEURAL NETWORKS

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Abstract. In this paper we first write a proof of the perceptron convergence algorithm for the complex multivalued neural networks (CMVNNs). Our primary goal is to formulate and prove the perceptron convergence algorithm for the bicomplex multivalued neural networks (BMVNNs) and other important results in the theory of neural networks based on a bicomplex algebra.

1. Introduction

The perceptron was created by a psychologist (Frank Rosenblatt, circa 1958), based on the model of the neuron developed by McCulloch and Pitts [23], and with the aim to define the Hebb model of learning (see [18] or, more precisely [11, p. 514]). The perceptron is a linear classifier, with a non-linear activation function, and an important example of a supervised learning algorithm.

Note that related works in theoretical mathematics, (in the setting of \(\mathbb{R}^N\), see [26, p. 93]), were also done earlier in 1954 by Agmon (see [1]), and by Motzkin and Schoenberg (see [25]). A review of proofs and applications of the perceptron theorem in the real setting may be found in [26, pp. 92-93] and we mention in particular the classical works of [11, 24, 27, 30, 33], as well as more recent ones [2, 36]. We also give a quick reminder of these concepts in Subsection 1.1.

The perceptron is also the smallest unit capable of learning a classification problem between two classes and its ability to do so is proven in the perceptron convergence theorem, a recursive procedure which allows one to find a separating hyperplane for a separable family of vectors in \(\mathbb{R}^N\). In this present work we extend this important result and procedure to the bicomplex setting, which we introduce in Section 2.

There has been a revived interest in the subject of perceptrons due to the needs of quantum computing and we refer the reader to [31, 21], where a quantum perceptron model is introduced. We hope that our present work will open new avenues in this direction as well.

We will first extend the perceptron convergence algorithm to the complex case and rewrite a proof here. We then set up the basis of the perceptron algorithm in the bicomplex
case and prove its convergence. A proof of the convergence algorithm in the complex case exists in [16] (as well as in [3]). However, for completeness, we include our own proof in this case, as it sheds light on the bicomplex perceptron theorem, which is the main aim of our paper.

Just as in the case of complex numbers, where the individual components of the complex number can be treated independently as two real numbers, the bicomplex space can be treated as two complex or four real numbers. In [19], it has been shown that the operation of complex multiplication limits the degree of freedom of the complex valued neural network at the synaptic weighting, therefore a CVNN is not quite equivalent to a two-dimensional real-valued neural network. In the same way a BVNN is not quite equivalent to a four-dimensional real-valued neural network, due to the bicomplex multiplication rules. This being said, complex-valued neural network research in signal processing applications include channel equalization [20, 35], satellite communication equalization [10] and, from a biological perspective, the complex-valued representation has been used in [29].

We start with short review of the regular perceptron algorithm in the real case, then set up the complex perceptron environment. The bicomplex approach gives a better way of combining two-valued complex networks compared to the usual two-valued approach in the literature [3], as we can treat the theory as a single variable one due to its algebra structure. In our approach the structure of the bicomplex algebra is essential in providing a convergent bicomplex perceptron algorithm as seen in Section 4.

This work is part of a general hypercomplex setting and we refer the reader to [12] and [] for other examples, such as the Clifford algebra.

1.1. Preliminaries: Classical Perceptron Algorithm. The setting is a (possibly infinite) family $C$ of vectors in the feature space $\mathbb{R}^N$ (for some fixed $N \in \mathbb{N}$), strictly divided into two classes $C_+$ and $C_-$ via an hyperplane: more precisely, we assume that there exists a unit norm vector $a \in \mathbb{R}^N$ such that

$$x \in C_+ \iff a^t x > 0 \quad \text{and} \quad x \in C_- \iff a^t x < 0,$$

and

$$\exists \delta > 0 \text{ such that } \inf_{x \in C} |a^t x| \geq \delta.$$ 

Under these hypothesis, $0 \not\in C$, and the hyperplane will not be unique. This condition is called separability.

The perceptron convergence theorem gives an iterative way to compute an hyperplane which separates the two classes after a finite number of steps. The coefficients of the equation of an hyperplane solving the problem (the weights) are learned via an algorithm, which is the content of the perceptron convergence theorem. A proof of this algorithm can be found in [17].

**Theorem 1.1.** Under the separability hypothesis, let $C$ be a possibly not countable family of non-zero vectors in $\mathbb{R}^N$, and let $x^{(1)}, x^{(2)}, \ldots$ be a countable family of elements of $C$, each element often appearing infinitely many times. Then the sequence defined by $a^{(0)} = x^{(1)}$
and

\[
 a^{(n+1)} = \begin{cases} 
 a^{(n)} & \text{if } x^{(n)} \in \mathcal{C}_+ \text{ and } (a^{(n)})^t x^{(n)} > 0, \\
 a^{(n)} & \text{if } x^{(n)} \in \mathcal{C}_- \text{ and } (a^{(n)})^t x^{(n)} < 0, \\
 a^{(n)} + \frac{x^{(n)}}{\|x^{(n)}\|} & \text{if } x^{(n)} \in \mathcal{C}_+ \text{ and } (a^{(n)})^t x^{(n)} \leq 0, \\
 a^{(n)} - \frac{x^{(n)}}{\|x^{(n)}\|} & \text{if } x^{(n)} \in \mathcal{C}_- \text{ and } (a^{(n)})^t x^{(n)} \geq 0, 
\end{cases}
\]

is stationary after a finite number of steps, (i.e. \(a^{(n)} = a^{(M)}\) for any \(n \leq M\), with \(M + 1 \leq \frac{1}{\delta^2}\).

We denote by \(M\) the number of times the sequence \(a^{(0)}, a^{(1)}, \ldots\) changes from one index to the next one. Recall also that one assumes that there exists \(a \in \mathbb{R}^N\) (but, as already mentioned, \(a\) is not unique, and unknown) which answers the problem. Following the book [24], the algorithm can be divided into four steps and the convergence follows.

**Remark 1.2.** The proof holds also in the case of a finite sequence. The result is then of interest when \(\frac{1}{\delta^2}\) is much smaller than the cardinal of \(\mathcal{C}\). The proof also holds when the vectors belong to a general Hilbert space.

The proof of the convergence algorithm relies heavily on the Cauchy-Schwarz Inequality, as well as the triangle inequality for the Euclidean metric on \(\mathbb{R}^N\). Variations of the algorithms are possible; see for instance [15, p. 180].

\[
 a^{(n+1)} = \begin{cases} 
 a^{(n)} & \text{if } x^{(n)} \in \mathcal{C}_+ \text{ and } (a^{(n)})^t x^{(n)} > 0, \\
 a^{(n)} & \text{if } x^{(n)} \in \mathcal{C}_- \text{ and } (a^{(n)})^t x^{(n)} < 0, \\
 a^{(n)} + e(n) \frac{x^{(n)}}{\|x^{(n)}\|} & \text{if } x^{(n)} \in \mathcal{C}_+ \text{ and } (a^{(n)})^t x^{(n)} \leq 0, \\
 a^{(n)} - e(n) \frac{x^{(n)}}{\|x^{(n)}\|} & \text{if } x^{(n)} \in \mathcal{C}_- \text{ and } (a^{(n)})^t x^{(n)} \geq 0, 
\end{cases}
\]

where \(e(n) > 0\). The sequence is stationary after a finite number of iterations when the weights sequence satisfies

\[
 \lim_{M \to \infty} \frac{\sum_{m=1}^{M} e(m)^2}{\left(\sum_{m=1}^{M} e(m)\right)^2} = 0.
\]

In our paper we will generalize the classical perceptron algorithm to the set of bicomplex numbers and prove its convergence.

### 1.2. Overview of our results.

We start with a short review of the bicomplex algebra in Section 2 where we re-introduce a hyperbolic valued modulus that will be used in the proof of the bicomplex algorithm. In Section 3 we re-write a complete proof of the convergence of the perceptron algorithm in the complex case. The last part of the paper, in the section 4 is dedicated to the proof of analogues of the complex perceptron theorems in the bicomplex case.
2. Introduction to Bicomplex Numbers

The algebra of bicomplex numbers was first introduced by Segre in [32]. During the past decades, a few isolated works analyzed either the properties of bicomplex numbers, or the properties of holomorphic functions defined on bicomplex numbers, and, without pretense of completeness, we direct the attention of the reader first to the book of Price, [28], where a full foundation of the theory of multicomplex numbers was given, then to some of the works describing some analytic properties of functions in the field [9, 13, 14, 37]. Applications of bicomplex (and other hypercomplex) numbers can be also found in the works of Alfsmann, Sangwine, Glöcker, and Ell [7, 8].

We now introduce, in the same fashion as [13, 22, 28], the key definitions and results for the case of holomorphic functions of complex variables. The algebra of bicomplex numbers is generated by two commuting imaginary units \( i \) and \( j \) and we will denote the bicomplex space by \( \mathbb{BC} \). The product of the two commuting units \( i \) and \( j \) is denoted by \( k := ij \) and we note that \( k \) is a hyperbolic unit, i.e. it is a unit which squares to 1. Because of these various units in \( \mathbb{BC} \), there are several different conjugations that can be defined naturally. We will make use of these appropriate conjugations in this paper, and we refer the reader to [22, 37] for more information on bicomplex and multicomplex analysis.

2.1. Properties of the bicomplex algebra. The bicomplex space, \( \mathbb{BC} \), is not a division algebra, and it has two distinguished zero divisors, \( e_1 \) and \( e_2 \), which are idempotent, linearly independent over the reals, and mutually annihilating with respect to the bicomplex multiplication:

\[
\begin{align*}
  e_1 &:= \frac{1+k}{2}, \quad e_2 := \frac{1-k}{2}, \\
  e_1 \cdot e_2 &= 0, \quad e_1^2 = e_1, \quad e_2^2 = e_2, \\
  e_1 + e_2 &= 1, \quad e_1 - e_2 = k.
\end{align*}
\]

Just like \{1, j\}, they form a basis of the complex algebra \( \mathbb{BC} \), which is called the idempotent basis. If we define the following complex variables in \( \mathbb{C}(i) \):

\[
\beta_1 := z_1 - iz_2, \quad \beta_2 := z_1 + iz_2,
\]

the \( \mathbb{C}(i) \)-idempotent representation for \( Z = z_1 + jz_2 \) is given by

\[
Z = \beta_1 e_1 + \beta_2 e_2.
\]

The \( \mathbb{C}(i) \)-idempotent is the only representation for which multiplication is component-wise, as shown in the next lemma.

**Remark 2.1.** The addition and multiplication of bicomplex numbers can be realized component-wise in the idempotent representation above. Specifically, if \( Z = a_1 e_2 + a_2 e_2 \) and \( W = b_1 e_1 + b_2 e_2 \) are two bicomplex numbers, where \( a_1, a_2, b_1, b_2 \in \mathbb{C}(i) \), then

\[
\begin{align*}
  Z + W &= (a_1 + b_1) e_1 + (a_2 + b_2) e_2, \\
  Z \cdot W &= (a_1 b_1) e_1 + (a_2 b_2) e_2, \\
  Z^n &= a_1^n e_1 + a_2^n e_2.
\end{align*}
\]
Moreover, the inverse of an invertible bicomplex number \( Z = a_1 e_1 + a_2 e_2 \) (in this case \( a_1, a_2 \neq 0 \)) is given by

\[
Z^{-1} = a_1^{-1} e_1 + a_2^{-1} e_2,
\]

where \( a_1^{-1} \) and \( a_2^{-1} \) are the complex multiplicative inverses of \( a_1 \) and \( a_2 \), respectively.

One can see this also by computing directly which product on the bicomplex numbers of the form

\[
x + i x_2 + j x_3 + k x_4, \quad x_1, x_2, x_3, x_4 \in \mathbb{R}
\]
is component wise, and one finds that the only one with this property is given by the mapping:

\[
x + i x_2 + j x_3 + k x_4 \mapsto ((x_1 + x_4) + i(x_2 - x_3), (x_1 - x_4) + i(x_2 + x_3)),
\]

which corresponds exactly with the idempotent decomposition

\[
Z = z_1 + j z_2 = (z_1 - i z_2)e_1 + (z_1 + i z_2)e_2,
\]

where \( z_1 = x_1 + i x_2 \) and \( z_2 = x_3 + i x_4 \).

**Remark 2.2.** These split the bicomplex space in \( \mathbb{B} \mathbb{C} = \mathbb{C} e_1 \oplus \mathbb{C} e_2 \), as:

\[
Z = z_1 + j z_2 = (z_1 - i z_2)e_1 + (z_1 + i z_2)e_2 = \lambda_1 e_1 + \lambda_2 e_2.
\]

Simple algebra yields:

\[
\begin{align*}
z_1 &= \frac{\lambda_1 + \lambda_2}{2} \\
z_2 &= \frac{i(\lambda_1 - \lambda_2)}{2}.
\end{align*}
\]

Because of these various units in \( \mathbb{B} \mathbb{C} \), there are several different conjugations that can be defined naturally and we will now define the conjugates in the bicomplex setting, as in [13, 22]

**Definition 2.3.** For any \( Z \in \mathbb{B} \mathbb{C} \) we have the following three conjugates:

\[
\begin{align*}
\overline{Z} &= \overline{z_1} + j \overline{z_2} \\
Z^\dagger &= z_1 - j z_2 \\
Z^* &= \overline{Z^\dagger} = \overline{z_1} - j \overline{z_2}.
\end{align*}
\]

We refer the reader to [22] for more details.

### 2.2. Hyperbolic subalgebra and the hyperbolic-valued modulus.

A special subalgebra of \( \mathbb{B} \mathbb{C} \) is the set of hyperbolic numbers, denoted by \( \mathbb{D} \). This algebra and the analysis of hyperbolic numbers have been studied, for example, in [9, 22, 34] and we summarize below only the notions relevant for our results. A hyperbolic number can be defined independently of \( \mathbb{B} \mathbb{C} \), by \( \mathfrak{h} = x + k y \), with \( x, y, k \in \mathbb{R} \), \( k \notin \mathbb{R} \), \( k^2 = 1 \), and we denote by \( \mathbb{D} \) the algebra of hyperbolic numbers with the usual component–wise addition and multiplication. The hyperbolic conjugate of \( \mathfrak{h} \) is defined by \( \mathfrak{h}^\circ := x - k y \), and note that:

\[
\mathfrak{h} \cdot \mathfrak{h}^\circ = x^2 - y^2 \in \mathbb{R},
\]

which yields the notion of the square of the modulus of a hyperbolic number \( \mathfrak{h} \), defined by

\[
|\mathfrak{h}|_D := \mathfrak{h} \cdot \mathfrak{h}^\circ.
\]
**Remark 2.4.** It is worth noting that both \( Z \) and \( Z^\dagger \) reduce to \( z^\diamond \) when \( Z = z \). In particular \( e_2 = e_1^* = e_1^\dagger \).

Similar to the bicomplex case, hyperbolic numbers have a unique idempotent representation with real coefficients:

\[
Z = se_1 + te_2,
\]

where, just as in the bicomplex case, \( e_1 = \frac{1}{2}(1 + k) \), \( e_2 = \frac{1}{2}(1 - k) \), and \( s := x + y \) and \( t := x - y \). Note that \( e_1^* = e_2 \) if we consider \( \mathbb{D} \) as a subset of \( \mathbb{H}\mathbb{C} \), as briefly explained in the remark above. We also observe that

\[
|Z|^2 = x^2 - y^2 = (x + y)(x - y) = st.
\]

The hyperbolic algebra \( \mathbb{D} \) is a subalgebra of the bicomplex numbers \( \mathbb{H}\mathbb{C} \) (see [22] for details). Actually \( \mathbb{H}\mathbb{C} \) is the algebraic closure of \( \mathbb{D} \), and it can also be seen as the complexification of \( \mathbb{D} \) by using either of the imaginary unit \( i \) or the unit \( j \).

**Definition 2.5.** Define the set \( \mathbb{D}^+ \) of non-negative hyperbolic numbers by:

\[
\mathbb{D}^+ = \left\{ x + ky \left| x^2 - y^2 \geq 0, x \geq 0 \right. \right\} = \left\{ se_1 + te_2 \left| s, t \geq 0 \right. \right\}.
\]

**Remark 2.6.** As studied extensively in [9], one can define a partial order relation defined on \( \mathbb{D} \) by:

\[
Z_1 \preceq Z_2 \text{ if and only if } Z_2 - Z_1 \in \mathbb{D}^+,
\]

and we will use this partial order to study the hyperbolic-valued norm, which was first introduced and studied in [9].

The Euclidean norm \( \|Z\| \) on \( \mathbb{H}\mathbb{C} \), when it is seen as \( \mathbb{C}^2(i), \mathbb{C}^2(j) \) or \( \mathbb{R}^4 \) is:

\[
\|Z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\text{Re}(|Z|^2)} = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}.
\]

As studied in detail in [22], in idempotent coordinates \( Z = \lambda_1 e_1 + \lambda_2 e_2 \), the Euclidean norm becomes:

\[
\|Z\| = \frac{1}{\sqrt{2}} \sqrt{\lambda_1^2 + \lambda_2^2}.
\]

It is easy to prove that

\[
\|Z \cdot W\| \leq \sqrt{2} (\|Z\| \cdot \|W\|),
\]

and we note that this inequality is sharp since if \( Z = W = e \), one has:

\[
\|e_1 \cdot e_1\| = \|e_1\| = \frac{1}{\sqrt{2}} = \sqrt{2} \|e_1\| \cdot \|e_1\|,
\]

and similarly for \( e_2 \).

**Definition 2.7.** One can define a hyperbolic-valued norm for \( Z = z_1 + jz_2 = \lambda_1 e_1 + \lambda_2 e_2 \) by:

\[
\|Z\|_{\mathbb{D}^+} := \lambda_1 |e_1| + \lambda_2 |e_2| \in \mathbb{D}^+.
\]

It is shown in [9] that this definition obeys the corresponding properties of a norm, i.e. \( \|Z\|_{\mathbb{D}^+} = 0 \) if and only if \( Z = 0 \), it is multiplicative, and it respects the triangle inequality with respect to the order introduced above.
2.3. Hyperbolic-valued modulus of vectors in $\mathbb{B}\mathbb{C}$. The previous norm can be generalized to the space of $\mathbb{B}\mathbb{C}$ vectors, i.e. elements of $\mathbb{B}\mathbb{C}^n$, and we will also define an inner product on the space of vectors in $\mathbb{B}\mathbb{C}$. Let $\langle X, Y \rangle$ be the usual Hermitian inner product on $\mathbb{C}^n$, then we have the following:

**Definition 2.8.** For any $X, Y \in \mathbb{B}\mathbb{C}^n$, we have the following $\mathbb{D}$—valued inner product

\[(2.12) \langle X, Y \rangle_{\mathbb{D}} = \langle X_1, Y_1 \rangle e_1 + \langle X_2, Y_2 \rangle e_2,
\]

where $X = X_1 e_1 + X_2 e_2$ and $Y = Y_1 e_1 + Y_2 e_2$, and $X_l, Y_l \in \mathbb{C}^n$ for $l = 1, 2$.

This inner product yields the hyperbolic-valued modulus of a vector $X = X_1 e_1 + X_2 e_2$ as:

\[\|X\|_{\mathbb{D}+} = \|X_1\| e_1 + \|X_2\| e_2.\]

3. The convergence of the perceptron algorithm in the complex case

In this section we will give a proof of the complex perceptron algorithm and pave the way to the bicomplex case. One of the first works on complex activation function was done by Naum Aizenberg [5, 6], where a multi-valued neuron (MVN) is a neural element with $n$ complex inputs and one complex output on the unit circle, with complex-valued weights.

**Definition 3.1.** Let $\varepsilon = \exp \left(\frac{2\pi i}{k}\right)$ be the root of unity of order $k$. For $T \subset \mathbb{C}^n$ one can define the following activation function $P$ by $P(z) = \varepsilon^l$, whenever $\frac{2\pi i l}{k} \leq \text{Arg}(z) < \frac{2\pi i (l+1)}{k}$.

Using this activation function, following [3], one can define the concept of a threshold function:

**Definition 3.2.** Let $n \geq 1$ and $T \subset \mathbb{C}^n$. Then, a complex valued function $f : T \rightarrow \mathbb{C}$ is called a threshold function if there exists a weighting vector $W = (w_0, w_1, \ldots, w_n) \in \mathbb{C}^{n+1}$ such that:

\[(3.1) f(x_1, \ldots, x_n) = P(w_0 + \sum_{\ell=1}^{n} w_\ell x_\ell), \quad \text{for any } (x_1, \ldots, x_n) \in T.\]

The activation function divides the complex plain into $k$ equal sectors, and implements a mapping of the entire complex plane onto the unit circle. Here, we give a different proof of Theorem 2.1 in [3]:

**Theorem 3.3.** Let $T$ be a bounded domain of $\mathbb{C}^n$, $f : T \rightarrow \mathbb{C}$ a threshold function and $(w_0, 0, \ldots, 0)$ a weighting vector of $f(x_1, \ldots, x_n)$. Then, there exists $w'_0 \in \mathbb{C}$ and $\delta > 0$ such that $(w'_0, w_1, \ldots, w_n)$ is a weighting vector of $f$ for every $w_1, \ldots, w_n$ that satisfy $|w_j| < \delta$ with $j = 1, \ldots, n$.

**Proof.** In the $k$—valued case we have that $\varepsilon = e^{i\frac{2\pi}{k}}$ and we can find $t$ such that $P(w_0) = \varepsilon^t$. We first define $w'_0 = \varepsilon^{t+\frac{1}{2}}$. Moreover, since $T$ is bounded, there exists $M$ such that $|x_j| \leq M, \quad \forall j = 1, \ldots, n.$
Now, we can define $\delta = \frac{1}{nM} \sin\left(\frac{\pi}{k}\right)$. We will show that $w'_0$ and $\delta$ will satisfy the conclusion of the theorem.

For $W = x_1w_1 + ... + x_nw_n$, such that $|w_j| < \delta$ for every $j = 1, \ldots, n$, it is easy to note that we have

$$||W|| < M (|w_1| + ... + |w_n|) < Mn\delta.$$ 

Thus, we have

\[(3.2)\]  

$$||W|| < Mn\delta = \sin\left(\frac{\pi}{k}\right),$$

which yields:

$$||(W + w'_0) - w'_0|| = ||W|| < \sin\left(\frac{\pi}{k}\right).$$

Therefore $W + w'_0$ belongs to the ball $D(w'_0, R)$ centered in $w'_0$ and with radius $R = \sin\left(\frac{\pi}{k}\right)$. This ball is tangent to the rays $\varepsilon^t$ and $\varepsilon^{t+1}$, therefore for every $z$ in $D(w'_0, R)$ we have:

$$\frac{2\pi t}{k} \leq \arg(z) < \frac{2\pi (t + 1)}{k}.$$ 

Hence, in particular:

$$\frac{2\pi t}{k} \leq \arg(w'_0 + W) < \frac{2\pi (t + 1)}{k},$$

for any $W = x_1w_1 + ... + x_nw_n$, such that $|w_j| < \delta$.

We then obtain that:

$$P(w'_0 + \sum_{l=1}^{n} w_lx_l) = P(w_0),$$

which yields:

$$f(x_1, ..., x_n) = P(w'_0 + \sum_{l=1}^{n} w_lx_l), \quad \forall(x_1, ..., x_n) \in T.$$

\[\square\]

The theorem above guarantees the existence of a threshold function and we now write the separability conditions under which the complex algorithm becomes convergent.

**Definition 3.4.** We call the sets $\{A_i\}_{1 \leq i \leq k}$, where $A_i \subset \mathbb{C}^n$, $k$–separable, if and only if there exists $W \in \mathbb{C}^n$ and a permutation $\{\alpha_l\}_{1 \leq i \leq k}$ of $\{1, \ldots, k\}$ such that $P((X, W')) = \varepsilon^{\alpha_l}$ for any $X \in A_l$.

We can now define the MVN complex learning algorithm. Given $k$–separable sets $A_l \subset \mathbb{C}^n$ one can write the following learning rule for the complex perceptron:

\[(3.3)\]  

$$W_{k+1} = W_k + C_k(\varepsilon^{q_k} - \varepsilon^{s_k})X_k,$$

where $X_k \in \bigcup_{l=1}^{k} A_l$, $q_k$ is the desired output and $s_k$ is the actual output. One can see that for vectors already organized in their desired sets the algorithm will stop, This algorithm will find a vector $W$ in a finite number of steps (this vector may not be unique) and in the
following theorem we will write and prove the convergence for the Perceptron Theorem in the complex case.

**Theorem 3.5.** Under the \( k \)-separability condition the MVN learning algorithm converges after a finite number of steps.

*Proof.* We will prove that

\[
O(k^2) \leq ||W_{k+1}||^2 \leq O(k),
\]

which yields convergence. For the first part we will use the \( k \)-separability condition. For the algorithm mentioned above:

\[
W_{k+1} = W_k + C_k(\varepsilon^k - \varepsilon^k)\overline{X_k},
\]

and we can reorganize all the vectors \( X_k \) using the separation hypothesis. Indeed, we write

\[
W_{k+1} = W_k + (1 - \varepsilon^k)\overline{X_k}.
\]

- If \( P(\langle X_k', W_k \rangle) = \varepsilon_0 \) we remove the elements for which we have \( W_{k+1} = W_k \) and denote the remaining ones by \( \sim X_k \).
- If \( P(\langle X_k', W_k \rangle) \neq \varepsilon_0 \) we let

\[
\sim W_{k+1} = (1 - \varepsilon^k)\overline{X_k} + \ldots + (1 - \varepsilon^k)\overline{X_k}.
\]

Then, by hypothesis there exists \( W \) such that

\[
(3.4) \quad P(\langle X', W \rangle) = \varepsilon_0.
\]

Moreover, it holds that

\[
\langle \sim W_{k+1}, W \rangle = \sum_{i=1}^{k} (1 - \varepsilon^k)\langle \sim X_i, W \rangle.
\]

Thus, we get

\[
\langle \sim W_{k+1}, W \rangle = \sum_{i=1}^{k} (1 - \varepsilon^k)\langle \sim X_i, W \rangle.
\]

We have \( \text{Arg}(1 - \varepsilon^k) = \frac{\pi}{2} - \frac{\pi r_l}{k} \) and \( 0 < r_l \leq k - 2 \) where \( r_l = s_l - q_l \mod(k) \). We note that \( \text{Arg}(1 - \varepsilon^k) = -\frac{\pi}{2} + \frac{\pi r_l}{k} \). It holds that

\[
-\frac{\pi}{2} + \frac{\pi r_l}{k} \leq \text{Arg}((1 - \varepsilon^k)\langle \sim X_i, W \rangle) \leq -\frac{\pi}{2} + \frac{\pi r_l + 2}{k}.
\]

Therefore, taking the maximum of \( r_l \) from the right and its minimum from the left we get

\[
-\frac{\pi}{2} < \text{Arg}((1 - \varepsilon^k)\langle \sim X_i, W \rangle) \leq -\frac{\pi}{2} + \frac{k}{2} = \frac{\pi}{2}.
\]

Thus, \( \text{Re} \left( (1 - \varepsilon^k)\langle \sim X_i, W \rangle \right) \geq 0 \). Then, setting \( m = \min_{l=1, \ldots, k} \left( \text{Re} \left( (1 - \varepsilon^k)\langle \sim X_i, W \rangle \right) \right) \)

we obtain

\[
(3.5) \quad |\langle \sim W_{k+1}, W \rangle| \geq km.
\]

However, using the Cauchy Schwarz inequality we know also that
\[ |\langle \tilde{W}_{k+1}, W \rangle| \leq ||\tilde{W}_{k+1}|| \cdot ||W||. \]

In particular, if we combine the previous inequality with (3.5) we obtain

(3.6) \[ \frac{m^2}{||W||^2} k^2 \leq ||\tilde{W}_{k+1}||^2. \]

For the second part, since \( W_{k+1} = W_k + C_k(\varepsilon^{q_k} - \varepsilon^{s_k})X_k \), we have

\[ ||W_{k+1}||^2 = ||W_k||^2 + C_k^2||\varepsilon^{q_k} - \varepsilon^{s_k}||^2||X_k||^2 + 2C_k \text{Re}[\overline{W_kX_k}(\varepsilon^{q_k} - \varepsilon^{s_k})] \]

Thus,

(3.7) \[ ||W_{k+1}||^2 = ||W_k||^2 + C_k^2||\varepsilon^{q_k} - \varepsilon^{s_k}||^2||X_k||^2 + 2C_k \text{Re}[\overline{W_kX_k}(\varepsilon^{q_k}(1 - \varepsilon^{s_k-s_k}))] \]

We note also that

\[ \frac{2\pi(n - s_k - 1)}{n} \leq \arg(W_kX_k) \leq \frac{2\pi(n - s_k)}{n} \]

Then, using trigonometric identities we observe that:

\[
\varepsilon^{q_k} - \varepsilon^{s_k} = \varepsilon^{q_k}(1 - \varepsilon^{s_k-q_k})
\]
\[
= \varepsilon^{q_k} \left(1 - \cos \left(\frac{2\pi(s_k - q_k)}{n}\right) - i \sin \left(\frac{2\pi(s_k - q_k)}{n}\right)\right)
\]
\[
= \varepsilon^{q_k} \left(2 \sin^2 \left(\frac{2\pi(s_k - q_k)}{2n}\right) - 2i \sin \left(\frac{2\pi(s_k - q_k)}{2n}\right) \cos \left(\frac{2\pi(s_k - q_k)}{2n}\right)\right)
\]
\[
= 2(-i)\varepsilon^{q_k} \sin \left(\frac{2\pi(s_k - q_k)}{2n}\right) \left(\cos \left(\frac{2\pi(s_k - q_k)}{2n}\right) + i \sin \left(\frac{2\pi(s_k - q_k)}{2n}\right)\right)
\]
\[
= 2e^{\pi i (q_k+s_k)/n} \sin \left(\frac{2\pi(s_k - q_k)}{2n}\right)
\]
\[
= 2 \left(\cos \left(\frac{\pi(q_k + s_k)}{n} - \frac{\pi}{2}\right) + i \sin \left(\frac{\pi(q_k + s_k)}{n} - \frac{\pi}{2}\right)\right) \sin \left(\frac{2\pi(s_k - q_k)}{2n}\right).
\]

Therefore, we obtain

\[
\varepsilon^{q_k}(1 - \varepsilon^{s_k-q_k}) = 2 \left(\cos \left(\frac{\pi(q_k + s_k)}{n} - \frac{\pi}{2}\right) + i \sin \left(\frac{\pi(q_k + s_k)}{n} - \frac{\pi}{2}\right)\right) \sin \left(\frac{2\pi(s_k - q_k)}{2n}\right).
\]

Setting \( r = ||X_kW_k|| \) and \( \mu = \arg(W_kX_k) \) we have
Re\(W_k X_k(\varepsilon^{q_k} - \varepsilon^{s_k})\) = Re\(W_k X_k\varepsilon^{q_k}(1 - \varepsilon^{s_k-q_k})\)

= 2Re\(re^{\mu i} \left( \cos \left( \frac{\pi(q_k + s_k)}{n} - \frac{\pi}{2} \right) + i \sin \left( \frac{\pi(q_k + s_k)}{n} - \frac{\pi}{2} \right) \right) \sin \left( \frac{2\pi(s_k - q_k)}{2n} \right) \)

= 2Re\(re^{\mu i} e^{\frac{\pi(q_k + s_k)}{n} i} e^{-\frac{\pi}{2} i} \sin \left( \frac{2\pi(s_k - q_k)}{2n} \right) \)

= 2r Re\((-ie^{(\mu + \frac{\pi(q_k + s_k)}{n}) i}) \sin \left( \frac{\pi(s_k - q_k)}{n} \right) \)

= 2r \sin\(\mu + \frac{\pi(s_k + q_k)}{n} \sin \left( \frac{\pi(s_k - q_k)}{n} \right) \)

We note that for \(2\pi - \frac{2\pi(s_k + 1)}{n} \leq \mu < 2\pi - \frac{2\pi s_k}{n}\) we have

\[2\pi + 2\pi \frac{(q_k - s_k)}{2n} - \frac{2\pi}{n} < \mu + \frac{\pi(s_k + q_k)}{n} < 2\pi + \frac{2\pi(q_k - s_k)}{2n} .\]

Hence, we have

\[\text{Re}(W_k X_k(\varepsilon^{q_k} - \varepsilon^{s_k})) = 2r \sin \left( \mu + \frac{\pi(s_k + q_k)}{n} \right) \sin \left( \frac{\pi(s_k - q_k)}{n} \right) \leq 0.\]

Then, inserting the previous inequality in (3.7) we obtain

\[||W_{k+1}||^2 \leq ||W_k||^2 + 2C^2||X_k||^2 \leq ||W_k||^2 + M,\]

with \(M > 0\). Hence, by iteration we obtain

\[||W_{k+1}||^2 \leq kM.\]

Therefore, combining both the first and second parts we obtain:

\[\frac{m^2}{||W||^2}k^2 \leq ||W_{k+1}||^2 \leq kM,\]

which completes the proof of the convergence algorithm in the complex case. ∎

**Remark 3.6.** A similar proof was written in [16], we are including our own version for completion. An interested reader may also consult the books of I. Aizenberg and N. Aizenberg [3, 4].

4. **The convergence of the perceptron algorithm in the bicomplex case**

In order to consider the bicomplex convergence algorithm we explore first the existence of a bicomplex threshold function, and the definition of separability in this context, using the hyperbolic valued norm as in Section 2.3 an extension to the norm described in [9].

4.1. **Existence of activation functions.** We have the following definition of the activation function \(\mathcal{P}\) in the bicomplex case:

**Definition 4.1.** For \(w_0 = w_{0_1}e_1 + w_{0_2}e_2, w_\ell = w_{\ell_1}e_1 + w_{\ell_2}e_2\) and \(x_\ell = x_{\ell_1}e_1 + x_{\ell_2}e_2\) in \(\mathbb{BC}\) with \(\ell = 1, \ldots, n\). We define the bicomplex activation function by

\[\mathcal{P}(w_0 + \sum_{\ell=1}^n w_\ell x_\ell) := P(w_{0_1} + \sum_{\ell=1}^n w_{\ell_1}x_{\ell_1})e_1 + P(w_{0_2} + \sum_{\ell=1}^n w_{\ell_2}x_{\ell_2})e_2,\]
where $P$ is the complex activation function given by Definition 3.1.

We can now generalize the notion of threshold function as follows:

**Definition 4.2.** Let $n \geq 1$ and $T \subset \mathbb{BC}^n$. Then, a complex valued function $f : T \rightarrow \mathbb{BC}$ is called a $\mathbb{BC}$-threshold function if there exists a weighting vector $W = (w_0, w_1, ..., w_n) \in \mathbb{BC}^{n+1}$ such that:

$$f(x_1, ..., x_n) = P(w_0 + \sum_{\ell=1}^{n} w_\ell x_\ell), \quad \forall (x_1, ..., x_n) \in T.$$  

**Theorem 4.3.** Let $T$ be a bounded domain of $\mathbb{BC}^n$ (i.e: there exist $T_1, T_2 \subset \mathbb{C}^n$ which are bounded such that we have $T = T_1 e_1 + T_2 e_2$). Let $f : T \rightarrow \mathbb{BC}$ a bicomplex threshold function and $(w_0, 0, ..., 0)$ a weighting vector of $f(x_1, ..., x_n)$. Then, there exists $w_0^* \in \mathbb{BC}$ and $\delta > 0$ such that $(w_0', w_1, ..., w_n)$ is a weighting vector of $f$ for every $w_1, ..., w_n$ that satisfy $|w_l| \leq \delta e_1 + \delta e_2$ (i.e: $|w_l| < \delta$ and $|w_l| < \delta$).

**Proof.** We take $\delta = \min(\delta_1, \delta_2)$ and define $w_0' = \varepsilon^{t_1} + \varepsilon^{t_2} e_1 + \varepsilon^{t_2} e_2$. Thus, we have

$$w_0' = \left(\varepsilon^{t_1} + \varepsilon^{t_2}\right)\left(\varepsilon^{t_1} - \varepsilon^{t_2}\right)e_j.$$

\[ \square \]

### 4.2. Bicomplex Perceptron Theorem

In order to write the bicomplex perceptron algorithm, we first introduce the notion of separability for bicomplex sets.

**Definition 4.4.** Let us consider the sets of $\mathbb{BC}^n$ given by:

$$A_l = A_{l_1} e_1 + A_{l_2} e_2,$$

where $A_{l_1}$ and $A_{l_2}$ are subsets of $\mathbb{C}^n$. We say that the sets $(A_l)_{1 \leq l \leq k}$ are $k$-separable if and only if there exists $W \in \mathbb{BC}^n$ such that if $W = \omega_1 e_1 + \omega_2 e_2$ we have that $(A_{l_1})_{1 \leq l \leq k}$ and $(A_{l_2})_{1 \leq l \leq k}$ are $k$-separable with respect to $\omega_1$ and $\omega_2$. In other words, there exist two permutations $(\pi_{l_1})_{1 \leq l \leq k}$ and $(\pi_{l_2})_{1 \leq l \leq k}$ such that

$$P((X_1, \omega_1)) = \varepsilon^{|\pi_{l_1}|}, \quad \forall X_1 \in A_{l_1}$$

and

$$P((X_2, \omega_2)) = \varepsilon^{|\pi_{l_2}|}, \quad \forall X_2 \in A_{l_2},$$

where $\varepsilon = \exp\left(\frac{2\pi i}{k}\right)$ and $P$ is the complex activation function used in Section 3.

**Theorem 4.5.** Under the $k$-separability condition as in Definition 4.4, the bicomplex perceptron algorithm converges after a finite number of steps given by $n = \max(n_1, n_2)$ where $n_1$ and $n_2$ are the number of steps given by the two respective complex perceptron algorithms from the decomposition.

**Proof.** We consider the following learning rule in the bicomplex setting:

$$W_{k+1} = W_k + C \xi_k X_k^*,$$

where $C \in \mathbb{R}$ and

$$\xi_k = (\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) e_1 + (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) e_2.$$
We should use the representation in \( Z = \lambda_1 e_1 + \lambda_2 e_2 \) in order to prove the results then translate everything in terms of \( Z = z_1 + z_2 j \). We will use the following notations for each term in (4.5):

\[
W_k = W_{k,1} + jW_{k,2} = w_{k,1}e_1 + w_{k,2}e_2,
\]

and

\[
X_k = X_{k,1} + jX_{k,2} = x_{k,1}e_1 + x_{k,2}e_2.
\]

The conjugate considered in (4.5) is the one defined by \( X_k^* = \overline{x_{k,1}}e_1 + \overline{x_{k,2}}e_2 \), and \( \rho_l \) in each component of \( \xi \) corresponds to a segment on the respective unit disk as in [16]. Thus, using the relations of \( e_1 \) and \( e_2 \) we have

\[
W_{k+1} = (w_{k,1} + C(\varepsilon^{q_1,n} - \varepsilon^{s_1,n})\overline{x_{k,1}})e_1 + (w_{k,2} + C(\varepsilon^{q_2,n} - \varepsilon^{s_2,n})\overline{x_{k,2}})e_2
\]

Hence, we obtain

\[
\|W_{k+1}\|_{D^+} = \|w_{k,1} + C(\varepsilon^{q_1,n} - \varepsilon^{s_1,n})\overline{x_{k,1}}\|e_1 + \|w_{k,2} + C(\varepsilon^{q_2,n} - \varepsilon^{s_2,n})\overline{x_{k,2}}\|e_2
\]

It follows that:

\[
\|W_{k+1}\|_{D^+} = \|w_{k+1,1}\|e_1 + \|w_{k+1,2}\|e_2,
\]

with the complex learning rules on each component can be written as:

\[
w_{k+1,1} = w_{k,1} + C(\varepsilon^{q_1,n} - \varepsilon^{s_1,n})\overline{x_{k,1}},
\]

and

\[
w_{k+1,2} = w_{k,2} + C(\varepsilon^{q_2,n} - \varepsilon^{s_2,n})\overline{x_{k,2}}.
\]

Thus, using the complex perceptron convergence theorem proved in Theorem 3.5, we know that there exists \( w_1 \) and \( w_2 \) such that

\[
\frac{k^2}{\|w_1\|^2} \leq \|w_{k+1,1}\| \leq kM,
\]

and

\[
\frac{k^2}{\|w_2\|^2} \leq \|w_{k+1,2}\| \leq kM.
\]

Now we can consider the bicomplex number given by \( W = w_1e_1 + w_2e_2 \), we have

\[
\|W\|_{D^+} = \|w_1\|e_1 + \|w_2\|e_2,
\]

and

\[
\frac{1}{\|W\|_{D^+}} = \frac{1}{\|w_1\|}e_1 + \frac{1}{\|w_2\|}e_2.
\]

Hence, we deduce that

\[
\frac{k^2}{\|W\|_{D^+}} \leq \|W_{k+1}\|_{D^+} \leq kMe_1 + kMe_2 = kM
\]

This ends the proof of the convergence of the bicomplex perceptron algorithm.

We will now return to the bicomplex setting expressed in terms of the complex units \( i, j \) and re-write the algorithm in this case, for ease of computation and to highlight the bicomplex structure in a way that will be used in other generalizations in the future.
We have

\[ W_{k+1} = w_{k+1,1} e_1 + w_{k+1,2} e_2 = (w_{k,1} + C(\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) \overline{x}_{k,1}) e_1 + (w_{k,2} + C(\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) \overline{x}_{k,2}) e_2 = W_{k+1,1} + jW_{k+1,2}. \]

**Corollary 4.6.** The bicomplex algorithm can be re-written as:

(4.10) \[ W_{k+1} = W_k + (1 + ij) \frac{C}{2} ((\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) x_k^* + (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) \overline{x}_k). \]

**Proof.** We have:

\[ W_{k+1,1} = W_{k,1} + \frac{C}{2} (\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) (X_{k,1} - iX_{k,2}) + (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) (\overline{X}_{k,1} + i\overline{X}_{k,2}) \]

\[ W_{k+1,2} = W_{k,2} + \frac{C}{2} (\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) (X_{k,1} - iX_{k,2}) - (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) (\overline{X}_{k,1} + i\overline{X}_{k,2}) \]

We recall that \( X_k = X_{k,1} + jX_{k,2} \) and use the following bicomplex conjugates

\[ x_k^* = \overline{X}_{k,1} - j\overline{X}_{k,2}, \quad \overline{x}_k = \overline{X}_{k,1} + j\overline{X}_{k,2}. \]

Then, we easily observe that

\[ \overline{X}_{k,1} = \frac{1}{2} (x_k^* + \overline{x}_k), \quad \overline{X}_{k,2} = j\frac{1}{2} (x_k^* - \overline{x}_k). \]

Thus, we have

\[ \overline{X}_{k,1} - i\overline{X}_{k,2} = \frac{1}{2} ((1 + ij)x_k^* + (1 - ij)\overline{x}_k), \]

and

\[ \overline{X}_{k,1} + i\overline{X}_{k,2} = \frac{1}{2} ((1 - ij)x_k^* + (1 + ij)\overline{x}_k). \]

Hence, we obtain

\[ W_{k+1,1} = W_{k,1} + \frac{C}{4} [(\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) ((1 + ij)x_k^* + (1 - ij)\overline{x}_k) + (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) ((1 - ij)x_k^* + (1 + ij)\overline{x}_k)], \]

and

\[ W_{k+1,2} = W_{k,2} + \frac{C}{4} [(\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) ((i - j)x_k^* + (j + i)\overline{x}_k) + (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) ((j + i)x_k^* + (i - j)\overline{x}_k)]. \]

Finally, we obtain

(4.11) \[ W_{k+1} = W_k + (1 + ij) \frac{C}{2} ((\varepsilon^{q_1,n} - \varepsilon^{s_1,n}) x_k^* + (\varepsilon^{q_2,n} - \varepsilon^{s_2,n}) \overline{x}_k). \]

This completes the convergence algorithm in the standard bicomplex form. □
5. Conclusions and Future Work

The authors are working to establish other perceptron theorems in different hypercomplex settings, as well as different algorithms in the bicomplex case. Interesting future work will include the quaternionic case, the hyperbolic case, as well as higher dimension hypercomplex algebras, commutative or not. These algorithms can simplify and reduce errors and be quite useful in their implementation. Hypercomplex valued neural networks have the ability of accumulating several complex variables into a single variable theory that can ease calculations and improve convergence.

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