ON 3-DISTANCE SPHERICAL 5-DESIGNS

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Abstract. Inspired by a recently formulated conjecture by Bannai et al., we investigate spherical codes which admit exactly three different distances and are spherical 5-designs. Computing and analyzing distance distributions we provide new proof of the fact (due to Levenshtein) that such codes are maximal and rule out certain cases towards a proof of the conjecture.

1. Introduction

Recently, Bannai et al. [3] formulated the following conjecture.

Conjecture 1.1. If \( C \subset \mathbb{S}^{n-1} \) is a spherical 3-distance 5-design, then one of the following holds.

1. \( n = 2 \) and \( C \) is a regular hexagon or a regular heptagon.
2. \( C \) is a tight spherical 5-design.
3. \( C \) is derived from a tight spherical 7-design.

In fact, in [3] Conjecture 1.1 was referred to as "Many people have conjectured, but not written down, the classification of 3-distance 5-designs". Indeed, it probably goes back to 1977 when Delsarte, Goethals and Seidel [8] introduced spherical designs in a seminal paper. The first author was among these who considered similar questions in 1990’s (see [5, 6]) and is aware about the conjecture since then.

It is known that the cardinality of any spherical 3-distance 5-design \( C \subset \mathbb{S}^{n-1} \) satisfies

\[
n(n + 1) \leq |C| \leq \binom{n + 2}{3} + \binom{n + 1}{2} = \frac{n(n + 1)(n + 5)}{6}.
\]

Both bounds are due to Delsarte-Goethals-Seidel [8]. The lower bound follows since \( C \) is a spherical 5-design, the upper bound – since \( C \) is a spherical 3-distance set (for upper bounds for few distance sets see also [1, 2, 10, 11, 14]). The main result in Levenshtein [12] (see also [13, Section 5]) implies that any spherical 3-distance 5-design is a maximal code.

In this paper we present new proof of the Levenshtein’s result from [12] and derive necessary conditions which underline a possible way to the complete proof of Conjecture 1.1. To this end we compute and carefully investigate the distance distributions of spherical 3-distance 5-designs. We prove that twice the cardinality is divisible by the dimension \( n \), thus reducing by a factor of \( n \) the number of cases under suspicion.

In Section 2 we prove that every spherical 3-distance 5-design needs to have its three inner products exactly equal to the zeros of certain polynomial used by Levenshtein for derivation of...
his bounds. Section 3 is devoted to computing the distance distribution of corresponding codes and different presentations of the results. In Section 4 we obtain divisibility conditions which rule out significant amount of cases. Section 5 describes derived codes as a possible continuation of our investigation. In Section 6 we explain our computer investigation which verifies Conjecture 1.1 in all dimensions $n \leq 1000$.

2. Inner products

Let $C \subset \mathbb{S}^{n-1}$ be a spherical 3-distance 5-design of cardinality $|C| = M$ and inner products $a < b < c$. The case $n = 2$ is elementary. If $n \geq 3$ and $C$ is a tight 5-design, then $a = -1$, $c = -b = 1/m$, where $m$ is odd positive integer, such that $n = m^2 - 2$ or $m = 1/\sqrt{5}$ and $C$ is the icosahedron in three dimensions. Examples are known for $m = 3$ and 5 only, and another folklore conjecture says that no other tight 5-designs exist. Thus, in what follows we focus in dimensions $n \geq 3$ and assume that $C$ is not a tight spherical 5-designs. Conjecture 1.1 is now equivalent to prove that always (3) happens. In this case, $C$ has dimension $n = 3m^2 - 5$, cardinality $M = m^4(3m^2 - 5)/2$ and inner products $-1/(m - 1)$, $-1/(m^2 - 1)$, and $1/(m + 1)$, where $m \geq 2$ is a positive integer. Examples are known for $m = 2$ and 3 only.

The main result from [12] implies that $|a| > |c| > |b| > 0$ (see [5, Corollary 3.9] for detailed proof of these inequalities), and $a$ and $b$ are the roots of the quadratic equation

$$(n + 2)((n + 2)c^2 + 2c - 1)t^2 + 2c(c + 1)(n + 2)t + 3 - (n + 2)c^2 = 0,$$

where $c$ satisfies the equation

$$M = \frac{n((n + 2)(n + 3)c^2 + 4(n + 2)c - n + 1)(1 - c)}{2c(3 - (n + 2)c^2)},$$

In the terminology of [12], one has $a = \alpha_0$, $b = \alpha_1$, $c = \alpha_2 = s$, and $\alpha_0$, $\alpha_1$, and $\alpha_2 = s$ are the zeros of the third degree polynomial

$$P_3(t)P_2(s) - P_3(s)P_2(t) = 0,$$

where $P_i(t) = P_i^{(n-1,n-2)}(t)$ is a Jacobi polynomial normalized for $P_i(1) = 1$, and $s$ is determined as the maximal root of the equation $M = L_5(n, s)$ (see Theorem 4.1 in [12]).

We present new proof of the above facts by using a slight generalization of a result from [7] on the structure of spherical designs. We use the following definition for a spherical design [9]: a code $C \subset \mathbb{S}^{n-1}$ is a spherical $\tau$-design if and only if for any point $y \in \mathbb{S}^{n-1}$ and any real polynomial $f(t)$ of degree at most $\tau$, the equality

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0|C|$$

holds, where $f_0$ is the first coefficient in the Gegenbauer expansion $f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t)$. Another useful tool is the Levenshtein quadrature [12, Theorems 4.1, 4.2], a particular case of which says that

$$f_0 = \frac{f(1)}{M} + \rho_0 f(\alpha_0) + \rho_1 f(\alpha_1) + \rho_2 f(\alpha_2)$$

for every polynomial $f$ of degree at most 5. Here the weights $\rho_i$, $i = 0, 1, 2$, are positive.
Theorem 2.1. If $C \subset \mathbb{S}^{n-1}$ is a spherical 3-distance 5-design, then its inner products are exactly as described above.

Proof. Using [4] and suitable polynomials in [3] Boyvalenkov and Nikova [7, Corollary 2.3] proved a general result which implies in our context that all four intervals $[-1, \alpha_0]$, $[\alpha_0, \alpha_1]$, $[\alpha_1, \alpha_2]$, and $[\alpha_2, 1]$ contain an inner product of $C$. This means that at least one of these intervals does not contain an inner product as interior point, implying as in [7] that all intervals do so, i.e. $a = \alpha_0$, $b = \alpha_1$, and $c = \alpha_2$.

Remark 2.2. The generalization of Corollary 2.3 from [7] mentioned above consists of the fact that it is valid not only for $s \in (\xi_{k-1}, \eta_k)$ as stated in [7] (we use here the notations from that paper) but for every $s \in (\xi_k, t_k)$, where $t_k > \eta_k$ is the largest zero of the Gegenbauer polynomial $P_k^{(n)}(t)$ (see also the comment after Theorem 2.3 in [4]). In fact, this observation allows any cardinalities $M \geq 2$ (though we need only $M \leq n(n+1)(n+5)/6$).

The proof of Theorem 2.1 is naturally (and obviously) extended to the the following general result.

Theorem 2.3. If $C \subset \mathbb{S}^{n-1}$ is a spherical $k$-distance $(2k - 1)$-design, then its inner products are exactly the roots of the Levenshtein polynomial

$$P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0,$$

where $P_i(t) = P_i^{(n-1, n-3)}(t)$ is a Jacobi polynomial normalized for $P_i(1) = 1$ and $s$ is determined as the maximal root of the equation

$$|C| = L_{2k-1}(n, s),$$

$L_{2k-1}(n, s)$ is the Levenshtein bound.

Proof. Similarly to above, if follows from [7, Corollary 2.3] that each of the intervals $[-1, \alpha_0]$, $[\alpha_0, \alpha_1]$, ..., $[\alpha_{k-1}, 1]$ contains an inner product of $C$. The pigeonhole principle now gives that one of these intervals does not have an inner product of $C$ as its interior point. Therefore, again by [7, Corollary 2.3], we conclude that the inner products are exactly $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ and $|C| = L_{2k-1}(n, s)$.

3. Distance distributions

Denote by

$$A_t(x) := |\{y \in C : \langle x, y \rangle \}|$$

the number of the points of $C$ having inner product $t$ with $x$. Then the system of numbers $(A_t(x) : t \in [-1, 1])$ is called distance distribution of $C$ with respect of $x$.

For $x \in C$ let $(X, Y, Z) = (A_a(x), A_b(x), A_c(x))$ be the distance distribution of $C$ with respect of $x$. Applying the quadrature formula [4] with the polynomials $f(t) = t^i$, $i = 0, 1, \ldots, 5$, we obtain that the seven numbers $a, b, c, X, Y, Z$ and $M = |C|$ satisfy the following system of six equations

$$a^i X + b^i Y + c^i Z = f_i M - 1, \quad i = 0, 1, \ldots, 5,$$

where $f_i = 0$ for odd $i$, $f_0 = 1$, $f_2 = 1/n$ and $f_4 = 3/n(n + 2)$. 


Taking the equations with odd $i$
\[aX + bY + cZ = a^3X + b^3Y + c^3Z = a^5X + b^5Y + c^5Z = -1.\]
and assuming that $a + b$, $b + c$, $c + a$ are all nonzero, we resolve with respect to $X$, $Y$ and $Z$ (see [6, Theorem 3.4]; this is a Vandermonde-like system) to obtain
\[
X = -\frac{(1 - b^2)(1 - c^2)}{a(a^2 - b^2)(a^2 - c^2)},
\]
\[
Y = -\frac{(1 - c^2)(1 - a^2)}{b(b^2 - c^2)(b^2 - a^2)},
\]
\[
Z = -\frac{(1 - a^2)(1 - b^2)}{c(c^2 - a^2)(c^2 - b^2)}.
\]

This implies, in particular, that the distance distribution of $C$ does not depend on the choice of the point $x$ and we therefore will omit $x$ in what follows.

The computation of the distance distribution of the codes in Conjecture 1.1 (3) gives
\[
X = \frac{m(m^2 - 2)(m - 1)}{4},
\]
\[
Y = (m^2 - 1)^3,
\]
\[
Z = \frac{m(m^2 - 2)(m + 1)}{4}.
\]

Further algebraic manipulations with the system (6) lead to another proof of Theorem 2.1. In particular, we obtain that $a$, $b$, and $c$ are the roots of the equation
\[(n + 2)[n(n + 3) - 2M]t^3 - n(n + 2)(n - 1)t^2 + (6M - 5n^2 - 7n)t + n(n - 1) = 0\]
(compare to (11)). We use (7) to compute the elementary symmetric polynomials $a + b + c$, $ab + bc + ca$, and $abc$ as rational functions of $n$ and $M$. It is clear that further advances in this direction can be based only on the investigation of the integrality conditions for the distance distribution of $C$. We focus on this in the next section.

4. Investigation of $XYZ$

Since $X, Y, Z$ are not symmetric in $a, b, c$, we need to produce and consider symmetric expressions in order to express then as functions of $n$ and $M$. We present the investigation of the product $XYZ$. It follows from the explicit formulas from the previous section that
\[(8)\quad XYZ = \frac{1}{abc} \left( \frac{(1 - a^2)(1 - b^2)(1 - c^2)}{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)} \right)^2.
\]

Using Vieta formulas and elementary symmetric polynomials from (7) we consecutively compute
\[
\frac{(1 - a^2)(1 - b^2)(1 - c^2)}{(a + b)(b + c)(c + a)} = \frac{M(n - 1)}{n(n + 2)}.
\]
\[
abc = \frac{n(n - 1)}{(n + 2)(2M - n(n + 3))},
\]
and (the most complicated)
\[
(a - b)^2(b - c)^2(c - a)^2 = \frac{R_1(n, M)}{(n + 2)^3(2M - n(n + 3))^4},
\]
where
\[
R_1(n, M) = 1728M^4 - 5184M^3n^2 - 8640M^2n^5 + 5544M^2n^4 + 18936M^2n^3
+ 16632M^2n^2 - 528Mn^7 - 3424Mn^6 - 13056Mn^5 - 23712Mn^4 - 14576Mn^3
+ 4n^{10} + 178n^9 + 1086n^8 + 3428n^7 + 7856n^6 + 10218n^5 + 4878n^4.
\]
Therefore,
\[
(9) \quad XYZ = \frac{M^2(n - 1)(n + 2)^2(2M - n(n + 3))^5}{R_1(n, M)n^3}.
\]

We proceed with derivation of divisibility conditions from (9). As usually, we denote by \(v_p(A)\) the largest power of the prime number \(p\) which divides an integer \(A\).

**Theorem 4.1.** If \(C \subset S^{n-1}\) is a spherical 3-distance 5-design, then \(n\) divides \(2M\).

**Proof.** The cases \(n = 2\) and \(M = n(n + 1)\) (i.e. \(C\) is a tight spherical 5-design) are trivial. So we may continue in the above context.

First we see that \(n\) divides \(M^2(2M)^5\). Hence, all odd primes divisors of \(n\) divide \(2M\) as well. For any odd prime divisor \(p\) of \(n\) we assume that \(v_p(M) < v_p(n)\) and compute the power of \(p\) in the numerator as
\[
v_p(M^2(2M - n(n + 3))^5) = 7v_p(M)
\]
and the denominator as
\[
v_p(n^3R_1(n, M)) = 3v_p(n) + 4v_p(M)
\]
(the case \(p = 3\) needs separate consideration but leads to the same conclusion). Hence, \(7v_p(M) \geq 3v_p(n) + 4v_p(M)\), i.e. \(v_p(M) \geq v_p(n)\), a contradiction.

Further, denote \(v_2(M) = y, v_2(n) = x\) and assume that \(x > 1 + y \geq 2\) for a contradiction (the cases \(y = 0\) and \(1\) are easily ruled out). Then the powers of 2 in the denominator and the numerator are
\[
v_2(n^3R_1(n, M)) \geq 3x + v_2(4M^4) = 3x + 4y + 2,
\]
\[
v_2(M^2(n - 1)(n + 2)^2(2M - n(n + 3))^5) = 2y + 5(y + 1) + 2 = 7y + 7,
\]
respectively. Therefore
\[
3y + 5 \geq 3x \geq 3(y + 2) = 3y + 6,
\]
a contradiction. This completes the proof. \(\square\)

**Remark 4.2.** Note that the conclusion of Theorem 4.1 holds true in the cases (1) and (2) of Conjecture 1.1.
Let $2M = Tn$ for some positive integer $T$. After cancelation, we arrive at

$$XYZ = \frac{T^2 (n-1)(n+2)^2(T-n-3)^5}{4R_2(n,T)},$$

where

$$R_2(n,T) = 108T^4 - 648T^3n^2 - 1080T^3n + 90T^2n^5 + 1386T^2n^4 + 4734T^2n^3 + 4158T^2n^2 - 264Tn^7 - 1712Tn^6 - 6528Tn^5 - 11856Tn^4 - 7288Tn^3 + 2(n+1)(n+3)(2n^4 + 81n^3 + 213n^2 + 619n + 813).$$

We were unable to continue with significant divisibility conclusions from (10).

In the end of this section we express either in terms of $m$ and the dimension $n$ the above parameters in the case (3) of Conjecture 1.1. Taking the values of $X$, $Y$ and $Z$ from Section 3, we obtain

$$XY Z = m^2(m^2 - 2)^2(m^2 - 1)^3 = \frac{(n+5)(n-1)^2(n+2)^3}{243^6}.$$

Theorem 4.1 gives $T = m^4 = (n+5)^2/9$.

5. Derived codes

We describe a construction from \cite{8} defining derived codes (sections) which produces good codes in lower dimensions provided the original codes are good.

Following \cite{8, Section 8} we consider the three derived codes, $C_a$, $C_b$, and $C_c$, of $C$ with respect to the three inner products, respectively. These codes are spherical 3-designs \cite{8, Theorem 8.2} and admit at most three inner products. Therefore their distance distributions satisfy a system of 4 equations, which could be investigated.

For fixed point $x \in C$, the set

$$C_a(x) := \{ y \in C : \langle x, y \rangle = a \}$$

defines, after rescaling on $S^{n-2}$, a spherical 3-design which we denote by $C_a$. The cardinality of $C_a$ is obviously equal to $X$. The inner products of $C_a$ are found by the Cosine Law to be these among

$$\frac{a - a^2}{1 - a^2}, \frac{b - a^2}{1 - a^2}, \frac{c - a^2}{1 - a^2}$$

belonging to $[-1,1]$.

The distance distribution of $C_a$ does not depend on the choice of the point and will be denoted by $(X_a, Y_a, Z_a)$. It satisfies the system

$$(11) \quad \left( \frac{a}{1 + a} \right)^i X_a + \left( \frac{b - a^2}{1 - a^2} \right)^i Y_a + \left( \frac{c - a^2}{1 - a^2} \right)^i Z_a = f_i X - 1, \quad i = 0, 1, 2, 3,$$

where $f_i$ are defined as in \cite{3} (note that $n$ is replaced by $n-1$). The codes $C_b$ and $C_c$ are defined analogously.
6. Computer investigation of the distance distributions

The conditions from Sections 2 and 3 allow easy computational investigation of particular cases of Conjecture 1.1. For fixed dimension $n$, we consider all feasible cardinalities

$$M \in \left( n(n+1), \frac{n(n+1)(n+5)}{6} \right),$$

satisfying the condition $2M = nT$ from Theorem 4.1. We compute $X$, $Y$ and $Z$ in each case and check if they are nonnegative integers. It is clear that enough precision ensures the correctness of the results.

We implemented this algorithm in all dimensions $n \leq 1000$. This computation confirms Conjecture 1.1 in all cases except for the pairs

$$(n, T) = (341, 3744), (638, 7011),$$

where analysis of the derived codes is needed. In both cases, the distance distribution of the derived codes gives a contradiction.

For $(n, T) = (341, 3744)$ we find from (7) that $(a, b, c) = (-\frac{1}{7}, -\frac{1}{35}, \frac{1}{114})$. It follows that $X = 23205$. Further, resolving (11) for $i = 0, 1, 2$, we find $(X_a, Y_a, Z_a) = (\frac{1872}{7}, \frac{57132}{49}, \frac{552500}{49})$, a contradiction. Analogously, in the case $(n, T) = (638, 7011)$ we find $(a, b, c) = (-\frac{1}{8}, -\frac{1}{40}, \frac{1}{20})$, whence $X = 40508$ and, finally, $(X_a, Y_a, Z_a) = (\frac{52193}{224}, \frac{1245375}{56}, \frac{577125}{32})$, a contradiction.

We note that these two exceptional solutions show that the proof cannot be continued by proving that all solutions satisfy $n + 5|9T$, as the expected solution suggests.

Summarizing, we have verified the following theorem.

**Theorem 6.1.** Conjecture 1.1 is true in all dimensions $n \leq 1000$.

7. Conclusions

We have proved that all spherical 3-distance 5-designs have inner products exactly as coming from the Levenshtein framework \[12, 5\] and generalize this result. We prove that in all cases twice the cardinality $2M$ is divisible by the dimension $n$. Using a computer we have verified Conjecture 1.1 in all dimensions $n \leq 1000$.

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