Density of monochromatic infinite subgraphs II

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Abstract

In 1967, Gerencsér and Gyárfás proved the following seminal result in graph-Ramsey theory: every 2-colored $K_n$ contains a monochromatic path on $\lceil (2n+1)/3 \rceil$ vertices, and this is best possible. In 1993, Erdős and Galvin started the investigation of the analogous problem in countably infinite graphs. After a series of improvements, this problem was recently settled: in every 2-coloring of $K_N$ there is a monochromatic infinite path with upper density at least $(12 + \sqrt{8})/17$, and there exists a 2-coloring which shows this is best possible.

Since 1967, there have been hundreds of papers on finite graph-Ramsey theory with many of the most important results being motivated by a series of conjectures of Burr and Erdős about graphs with linear Ramsey numbers. In a sense, this paper begins a systematic study of infinite graph-Ramsey theory, focusing on infinite analogues of these conjectures. The following are some of our main results.

(i) Let $G$ be a countably infinite, (one-way) locally finite graph with chromatic number $\chi < \infty$. Every 2-colored $K_N$ contains a monochromatic copy of $G$ with upper density at least $\frac{1}{2(\chi - 1)}$.

(ii) Let $G$ be a countably infinite graph having the property that there exists a finite set $X \subseteq V(G)$ such that $G - X$ has no finite dominating set (in particular, graphs with bounded degeneracy have this property, as does the infinite random graph). Every finitely-colored $K_N$ contains a monochromatic copy of $G$ with positive upper density.

(iii) Let $T$ be a countably infinite tree. Every 2-colored $K_N$ contains a monochromatic copy of $T$ of upper density at least $1/2$. In particular, this is best possible for $T_\infty$, the tree in which every vertex has infinite degree.

(iv) Surprisingly, there exist connected graphs $G$ such that every 2-colored $K_N$ contains a monochromatic copy of $G$ which covers all but finitely many vertices of $\mathbb{N}$. In fact, we classify all forests with this property.

1 Introduction

It was proven by Ramsey [22] that for every graph $G$ and every positive integer $r$, there exists a positive integer $N$ such that every $r$-coloring of $E(K_N)$ contains a monochromatic copy of $G$. The smallest possible choice for $N$ is called the $r$-color Ramsey number and is denoted by $R_r(G)$. Determining Ramsey numbers of different (families of) graphs is one of the central topics in combinatorics. In this paper, we are interested in similar problems for countably infinite graphs.

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graphs. (We will not consider uncountably infinite graphs and always mean countable when we write infinite from now on.)

Let $K_N$ be the graph on vertex set $\mathbb{N}$ with edge set $\binom{\mathbb{N}}{2}$ (typically $\mathbb{N}$ denotes the set of positive integers and we typically begin counting at 1; however, there are certain situations where it is convenient to let $\mathbb{N}$ denote the non-negative integers or to start counting at 0, but this distinction will never have an impact on the results). Ramsey [22] also proved that in every $r$-coloring of $K_N$, there is a monochromatic copy of $K_N$. Thus in order to make the problem quantitative, we will thus consider the density of the monochromatic graphs we are looking for.

The upper density of a graph $G$ with $V(G) \subseteq \mathbb{N}$ is defined as

$$d(G) = \limsup_{t \to \infty} \frac{|V(G) \cap \{0, 1, 2, \ldots, t\}|}{t}.$$ 

The lower density, denoted $d(G)$, is defined similarly in terms of the infimum and we speak of the density, whenever lower and upper density coincide.

Erdős and Galvin [11] described a 2-coloring of $K_N$ in which every graph having finitely many isolated vertices and bounded maximum degree has lower density 0, thus we typically restrict our attention to upper densities. However, this does raise the question of whether there is any graph $G$ (with finitely many isolated vertices) having the property that in every 2-coloring of $K_N$ there is a monochromatic copy of $G$ with positive lower density. We will return to this question later and prove that there are such graphs in a strong sense.

Given an $r$-coloring of the edges of $K_N$ and a graph $G$, the Ramsey upper density of $G$ with respect to $\varphi$, denoted $\overline{d}_\varphi(G)$, is the supremum of $d(G)$ over all monochromatic copies of $G$ in the coloring $\varphi$ of $K_N$. The $r$-color Ramsey upper density of $G$, denoted $\overline{d}_r(G)$, is the infimum of $\overline{d}_\varphi(G)$ over all $r$-colorings $\varphi$ of the edges of $K_N$. If $r = 2$, we drop the subscript.

Possibly the first such (implicitly) quantitative result is due to Rado [21] who proved that every $r$-edge-colored $K_N$ contains $r$ vertex-disjoint monochromatic infinite paths which together cover all vertices. In particular, one of them must have upper density at least $1/r$ and hence $\overline{d}_r(P_\infty) \geq 1/r$, where $P_\infty$ is the (one-way) infinite path. For two colors, this was improved by Erdős and Galvin [11] who proved that $2/3 \leq \overline{d}_2(P_\infty) \leq 8/9$. More recently, DeBiasio and McKenney [8] improved the lower bound to $3/4$ and conjectured the correct value to be $8/9$. Progress towards this conjecture was made by Lo, Sanhueza-Matamala and Wang [19], who raised the lower bound to $(9 + \sqrt{17})/16 \approx 0.82019$. Corsten, DeBiasio, Lamaison and Lang [7] finally proved that $\overline{d}_2(P_\infty) = (12 + \sqrt{8})/17 \approx 0.87226$, thereby settling the problem for two colors. In this paper, we initiate a systematic study of Ramsey densities for other infinite graphs. An independent systematic study was undertaken by Lamaison [16], who fortunately focuses on a different aspect of the general problem (locally-finite graphs) and thus the two papers have very little overlap.

### 1.1 Graphs with positive Ramsey upper density

The problem of estimating the Ramsey numbers of sparse finite graphs has received a lot of attention. The problem was motivated by a series of conjectures proposed by Burr and Erdős [2, 3], starting with graphs of bounded maximum degree.

**Conjecture 1.1 (Burr–Erdős [2]).** For all $\Delta \in \mathbb{N}$, there exists some $c = c(\Delta) > 0$ such that every 2-edge-colored $K_n$ contains a monochromatic copy of every graph $G$ with at most $cn$ vertices and $\Delta(G) \leq \Delta$. 


Conjecture 1.1 was solved by Chvátal, Rödl, Szemerédi, Trotter [5] in an early application of the regularity lemma. Since then, there has been many improvements to the constant $c(\Delta)$ (see [6] for a more detailed history). Allen, Brightwell and Skokan [1] proved that this constant can be significantly improved to $c = 1/(2\chi(G) + 4) \geq 1/(2\Delta + 6)$ for graphs of small bandwidth (see [1] for the precise statement of their result), where $\chi(G)$ denotes the chromatic number of $G$.

Our first theorem proves an analog of this for infinite graphs. It turns out that much weaker conditions on the degrees suffice. Given $k \geq 2$, we say that a graph $G$ is one-way $k$-locally finite if there exists a partition of $V(G)$ into $k$ independent sets $V_1, \ldots, V_k$ with $|V_1| \geq \ldots \geq |V_k|$ such that for all $1 \leq i < j \leq k$ and all $v \in V_j$, $d(v, V_i) < \infty$. Note that every vertex in $V_k$ has finite degree, but it is possible for any vertex in $V_1 \cup \cdots \cup V_{k-1}$ to have infinite degree. A good example of a one-way 2-locally finite graph exhibiting this property is the infinite bipartite half graph, which is the graph on $\mathbb{N} = A \cup B$, where $A$ is the set of positive odd integers and $B$ is the set of positive even integers and $uv$ is an edge if and only if $u < v$ and $u$ is odd and $v$ is even. Further note that one-way $k$-locally finite graphs have chromatic number at most $k$ and, if $G$ is locally finite (that is every vertex has finite degree) with $\chi(G) < \infty$, then $G$ is one-way $\chi(G)$-locally finite.

**Theorem 1.2.** Let $k, r \in \mathbb{N}$ and let $G$ be an infinite one-way $k$-locally finite graph.

(i) If $k = 2$, then $\overline{\text{RD}}_r(G) \geq 1/r$.

(ii) If $k \geq 2$, then $\overline{\text{RD}}_r(G) \geq \frac{1}{2(k-1)}$.

(iii) If $r, k \geq 3$, then $\overline{\text{RD}}_r(G) \geq \frac{1}{\sum_{i=0}^{k-2}2^{r+1}(r-1)^i} \geq (1 + o(1))/r^{(k-2)r+1}$.

Since graphs with $\Delta(G) = \Delta < \infty$ have $\chi(G) \leq \Delta + 1$, we get that $\overline{\text{RD}}_r(G) \geq \frac{1}{\Delta}$ and $\overline{\text{RD}}_r(G) \geq 1/r^{\Delta r}$ for every $r \geq 3$ (which answers a question from [8]). However, we are able to prove a slightly stronger result for 2 colors.

**Corollary 1.3.** If $G$ is an infinite graph with $\Delta(G) = \Delta < \infty$, then $\overline{\text{RD}}_2(G) \geq \frac{1}{\Delta}$.

A graph $G$ is $d$-degenerate if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for all $i \geq 1$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq d$. The degeneracy of $G$, denoted $\text{degen}(G)$, is the smallest non-negative integer $d$ such that $G$ is $d$-degenerate; if no such integer exists, say $\text{degen}(G) = \infty$. Note that if $G$ is $d$-degenerate, then $\chi(G) \leq d + 1 \leq \Delta(G) + 1$. Also note that a graph can have finite degeneracy, but infinite maximum degree.

**Conjecture 1.4** (Burr–Erdős [2]). For all $d \in \mathbb{N}$, there exists some $c = c(d) > 0$ such that every 2-edge colored $K_n$ contains a copy of every $d$-degenerate graph on at most $cn$ vertices.

Conjecture 1.4 was recently confirmed by Lee [18]. It would be very interesting to prove an analog of this for infinite graphs.

**Problem 1.5.** For all $d \in \mathbb{N}$, does there exist some $c = c(d) > 0$ such that $\overline{\text{RD}}_2(G) \geq c$ for every infinite graph $G$ with degeneracy at most $d$? A weaker version of this question is for all infinite graphs $G$ with finite degeneracy, does there exist some $c = c(G) > 0$ such that $\overline{\text{RD}}_2(G) \geq c$?

As we will discuss in the next section, we obtain a positive answer to a weaker version of this question.
1.2 Ramsey-dense graphs

We say that an infinite graph $G$ is $r$-Ramsey-dense if in every $r$-coloring of $K_N$ there is a monochromatic copy of $G$ with positive upper density. If $r = 2$, we drop the prefix and just say $G$ is Ramsey-dense. Note that if $G$ is Ramsey-dense, this does not necessarily imply that $\overline{Rd}(G) > 0$ as there are infinitely many colorings, so the infimum of the upper densities over all colorings can be 0. Indeed, we shall see below that the so called Rado graph $\mathcal{R}$ is an example of an infinite graph that is Ramsey-dense yet $\overline{Rd}(\mathcal{R}) = 0$. On the other hand, every infinite graph $G$ with $\overline{Rd}(G) > 0$ is Ramsey-dense.

Ramsey-dense graphs are another natural analog of graphs with linear Ramsey number. We will describe a simple property guaranteeing that a graph is Ramsey-dense and then show that every Ramsey-dense graph is not far from having this property.

A set $X \subseteq V(G)$ is called dominating if every vertex $v \in V(G) \setminus X$ has a neighbor in $X$. We call a set $X \subseteq V(G)$ ruling if $X$ is finite and all but finitely many vertices $v \in V(G) \setminus X$ have a neighbor in $X$. We say that an infinite graph $G$ is $t$-ruled if there are at most $t$ disjoint minimal ruling sets. The ruling number of a graph $G$, denoted by $\text{rul}(G)$, is the smallest $t \in \mathbb{N}$ such that $G$ is $t$-ruled; if no such $t$ exists, we say $G$ is infinitely ruled, or $\text{rul}(G) = \infty$. Equivalently, $\text{rul}(G)$ is the matching number of the hypergraph whose edges are all minimal ruling sets. Note that a graph $G$ is 0-ruled if and only if there is no finite dominating set and finitely-ruled (i.e. $t$-ruled for some $t \in \mathbb{N}$) if and only if there is a finite set $S \subseteq V(G)$ such that $G[S^c]$ has no finite dominating sets.

**Theorem 1.6.** If $G$ is an infinite graph with $\text{rul}(G) < \infty$, then $G$ is $r$-Ramsey-dense for all $r \in \mathbb{N}$.

This has a few interesting corollaries. Since locally finite graphs have ruling number 0, we immediately get the following.

**Corollary 1.7.** If $G$ is a locally finite infinite graph, then $G$ is $r$-Ramsey-dense for all $r \in \mathbb{N}$.

The Rado graph is the graph $\mathcal{R}$ with vertex-set $\mathbb{N}$ defined by placing an edge between $m < n$ if and only if the $m$th digit in the binary expansion of $n$ is 1. The Rado graph has many interesting properties, for example it is isomorphic to the infinite random graph (that is the graph on $\mathbb{N}$ in which every edge is present independently with probability $1/2$) with probability 1. It is easy to verify that the Rado graph does not have any finite dominating sets and hence $\text{rul}(\mathcal{R}) = 0$.

**Corollary 1.8.** The Rado graph $\mathcal{R}$ is $r$-Ramsey-dense for all $r \in \mathbb{N}$.

On the other hand, we will show that $\overline{Rd}(\mathcal{R}) = 0$ (see Corollary 2.4). Another corollary asserts that graphs with bounded degeneracy are Ramsey-dense.

**Corollary 1.9.** If $G$ is an infinite graph with bounded degeneracy, then $G$ is $r$-Ramsey-dense for all $r \in \mathbb{N}$.

By Theorem 1.6, it suffices to show that every $d$-degenerate infinite graph $G$ is $d$-ruled.

**Fact 1.10.** Let $d \in \mathbb{N}$. If $G$ is $d$-degenerate, then $\text{rul}(G) \leq d$.

**Proof.** Suppose for contradiction, there is a $d$-degenerate infinite graph $G$ with $\text{rul}(G) > d$ for some $d \in \mathbb{N}$. Let $S_1, \ldots, S_{d+1}$ be disjoint minimal ruling sets and let $S_0 \subseteq V(G) \setminus (S_1 \cup \ldots \cup S_{d+1})$
be the set of vertices which do not have a neighbor in some \( S_i \). Note that \( S := S_0 \cup S_1 \cup \ldots \cup S_{d+1} \) is finite. Therefore, there is a vertex \( u \in \mathbb{N} \setminus S \) which comes after all vertices in \( S \) in a \( d \)-degenerate ordering of \( V(G) \) and hence \( \deg(u, S) \leq d \). However, by construction, \( u \) has a neighbor in each of \( S_1, \ldots, S_{d+1} \), a contradiction.

**Problem 1.11.** Is there a Ramsey-dense graph \( G \) with \( \text{rul}(G) = \infty \)?

If the answer is no, then together with Theorem 1.6, this would give a complete characterization of Ramsey-dense graphs. We will give a partial answer to the question by showing that if \( \text{rul}(G) = \infty \) and additionally the sizes of the minimal ruling sets do not grow too fast, then \( G \) is not Ramsey-dense (see Theorem 2.14).

### 1.3 Trees

Another famous conjecture of Burr and Erdős [3] concerns the Ramsey number of trees. A graph is acyclic if it contains no finite cycles, a forest is an acyclic graph, and a tree is a connected acyclic graph.

**Conjecture 1.12** (Burr–Erdős [3]). Let \( n \in \mathbb{N} \) and let \( T \) be a tree on at most \( n^2 + 1 \) vertices. Every 2-edge-colored \( K_n \) contains a monochromatic copy of \( T \).

Conjecture 1.12 was solved for large \( n \) by Zhao [25]. The following result provides an analog of this in infinite graphs and can be seen to be best possible. Note that Theorem 1.2 already implies that \( \overline{\text{Rd}}(T) \geq 1/2 \) for every infinite locally finite forest \( T \).

**Theorem 1.13.** \( \overline{\text{Rd}}(T) \geq 1/2 \) for every infinite forest \( T \).

We further show that \( \overline{\text{Rd}}(T_\infty) = 1/2 \) where \( T_\infty \) is the infinite tree in which every vertex has infinite degree and there are also infinite locally finite trees \( T \) with \( \overline{\text{Rd}}(T) = 1/2 \) (see Example 2.2).

Erdős, Faudree, Rousseau, and Schelp [12] showed that if \( T \) is a tree on more than \( [3n/4] \) vertices, then there exists a 2-coloring of \( K_n \) which contains no monochromatic copy of \( T \). Furthermore they showed that this bound can be acheived by certain trees such as the tree obtained by joining the center of \( K_{1,n/4} \) with a path on \( n/2 - 1 \) vertices (see also [24]). In other words, 3/4 is the largest proportion of vertices that a single connected graph can cover in an arbitrary 2-coloring of \( K_n \). We now consider an analogous question for infinite graphs.

Say that a graph \( G \) is Ramsey-cofinite if in every 2-coloring of \( K_\infty \) there exists a monochromatic copy of \( G \) such that \( V(G) \) is cofinite. It is clear that any graph \( G \) with infinitely many isolated vertices is Ramsey-cofinite. Say that a graph \( G \) is Ramsey-lower-dense if in every 2-coloring of \( K_\infty \) there is a monochromatic copy of \( G \) with positive lower density. As mentioned earlier, Erdős and Galvin proved that for any graph \( G \) with finitely many isolated vertices and bounded maximum degree then \( G \) is not Ramsey-lower-dense, and thus \( G \) is not Ramsey-cofinite.

Surprisingly, we show that there exist connected graphs which are Ramsey-cofinite. In fact, we are able to completely characterize all acyclic graphs which are Ramsey-cofinite. Say that a graph \( G \) is weakly expanding if for all \( k \in \mathbb{N} \), there exists \( \ell \in \mathbb{N} \) such that for all independent sets \( A \) in \( G \) with \( |A| \geq \ell \) we have \( |N(A)| > k \). Say that a graph \( G \) is strongly contracting if there exists \( k \in \mathbb{N} \) such that for all \( \ell \in \mathbb{N} \) there exists an independent set \( A \) in \( G \) with \( |A| \geq \ell \) such that \( |N(A)| \leq k \). Note that every infinite graph is either strongly contracting or weakly expanding. Finally, we define a family of forests which will form an exception to the rule. Let
\( \mathcal{T}^* \) be the family of forests \( T \) having one vertex \( t \) of infinite degree such that \( t \) is adjacent to infinitely many leaves and infinitely many non-leaves, every other vertex has degree at most \( d \) for some \( d \in \mathbb{N} \), and cofinitely many vertices of \( T \) have distance at most 2 to \( t \) (so if \( T \) is not connected, then \( T \) has one infinite component and finitely many finite components).

**Theorem 1.14.** Let \( T \) be a forest.

(i) If \( T \) is strongly contracting, has no finite dominating set, and \( T \not\in \mathcal{T}^* \), then \( T \) is Ramsey-cofinite.

(ii) If \( T \) is weakly expanding, has a finite dominating set, or \( T \in \mathcal{T}^* \), then \( T \) is not Ramsey-lower-dense (and thus \( T \) is not Ramsey-cofinite).

To get a better sense of what Theorem 1.14 says in terms of trees, say that a graph \( G \) has unbounded leaf degree if for every \( \ell \in \mathbb{N} \), there exists \( v \in V(G) \) such that \( v \) is adjacent to at least \( \ell \) leaves; otherwise, say that \( G \) has bounded leaf degree. A tree is strongly contracting if and only if it has unbounded leaf degree, and a tree is weakly expanding if and only if it has bounded leaf degree.

In light of Theorem 1.14 it would be natural to ask if there is any connected graph \( T \) such that there is a spanning monochromatic copy of \( T \) in every 2-coloring of \( K_N \); however, this is not possible. Clearly if \( T \) is an infinite star it does not have this property, so suppose \( T \) is not an infinite star and 2-color the edges of \( K_N \) by fixing a vertex \( v \), coloring all edges incident with \( v \) red, and coloring all other edges blue. Every monochromatic copy of \( T \) must be blue and therefore not be spanning.

Completely characterizing all graphs which are Ramsey-cofinite is still an open question and is discussed in Section 8.4.

### 1.4 Bipartite Ramsey densities

Gyárfás and Lehel [14] and independently Faudree and Schelp [13] proved that every 2-colored \( K_{n,n} \) contains a monochromatic path with at least \( \lceil n/2 \rceil \) vertices (that is, roughly half the vertices of the graph). They further proved that this is best possible. We will prove an analog of this for infinite graphs. Here, \( K_{N,N} \) is the infinite complete bipartite graph with one part being all even positive integers and the other part being all odd positive integers.

**Theorem 1.15.** Every 2-colored \( K_{N,N} \) contains a monochromatic path of upper density at least \( 1/2 \).

Pokrovskiy [20] proved that the vertices of every 2-colored complete bipartite graph \( K_{n,n} \) can be partitioned into three monochromatic paths. Soukup [23] proved an analog of this for infinite graphs which holds for multiple colors: The vertices of every \( r \)-colored \( K_{N,N} \) can be partitioned into \( 2r - 1 \) monochromatic paths. He also presents an example where this is best possible. However, in his example all but finitely many vertices can be covered by \( r \) monochromatic paths. Our next result shows that this is always possible for two colors.

**Theorem 1.16.** The vertices of every 2-colored \( K_{N,N} \) can be partitioned into a finite set and at most two monochromatic paths.

Theorem 1.15 is an immediate consequence of Theorem 1.16. We will provide an example which demonstrates that Theorems 1.15 and 1.16 are best possible (see Example 2.5). We believe that a similar statement is true for multiple colors.
Conjecture 1.17. Let $r \in \mathbb{N}$. The vertices of every $r$-colored $K_{N,N}$ can be partitioned into a finite set and at most $r$ monochromatic paths.

Example 2.5 also shows that Conjecture 1.17 is best possible, if true.

1.5 Overview

In Section 2 we collect a variety of examples which are used to for instance obtain upper bounds on the upper Ramsey density of certain graphs. In Section 3 we discuss ultrafilters and a general embedding strategy that we will use to prove our results about one-way locally finite graphs in Section 4 and graphs of bounded ruling number in Section 5. In Section 6 we prove some additional results about graphs with bounded degeneracy. In Section 7 we prove Theorem 1.16. In Section 8 we prove Theorems 1.13 and 1.14 together with a variety of supporting results which may be of independent interest. In Section 9 we discuss a more general extension of the notion of a graph being Ramsey-dense. Finally we end with some open problems in Section 10.

1.6 Notation

For a positive integer $n$, we let $[n] = \{1, 2, \ldots, n\}$.

A subset $X$ of an infinite set $Y$ is called cofinite in $Y$ if $Y \setminus X$ is finite. If $Y$ is clear from context we will call $X$ cofinite and write $X^c = Y \setminus X$. We write $A \subseteq^* B$ to mean that $A \setminus B$ is finite.

Given an edge-colored graph $G$ and a color $c$, we write $G^c$ for the spanning subgraph of $G$ with all edges of color $c$. Given a vertex $v \in V(G)$, we define $N(v)$ to be the set of neighbors of $v$ and, given a color $c$, we define $N_c(v) \subseteq N(v)$ to be the set of vertices which are adjacent to $v$ via an edge of color $c$. Given $S \subseteq V(G)$, we write $N(S) = \bigcup_{v \in S} N(v)$ and $N_c(S) = \bigcap_{v \in S} N(v)$ and, given a color $c$, we define $N_c(S) = \bigcup_{v \in S} N_c(v)$ and $N_c^c(S) = \bigcap_{v \in S} N_c(v)$.

The following well-known fact follows from the definition of upper- and lower- density. For disjoint sets $A, B \subseteq \mathbb{N}$, we have

$$d(A) + d(B) \leq d(A \cup B) \leq d(A) + d(B) \leq d(A \cup B) \leq d(A) + d(B).$$

2 Examples

2.1 Basics

First we present some examples to get a better understanding how the different parameters discussed in this paper are related.

The infinite half graph is the graph on $\mathbb{N}$ such that $uv$ is an edge if and only if $u < v$ and $v$ is even. Given a complete bipartite graph $G$ between two disjoint infinite sets $A$ and $B$, the half graph coloring of $G$ is obtained by taking a bijection $f$ from $A$ to the odd integers and a bijection $g$ from $B$ to the even integers and coloring an edge $uv$ with $u \in A$ and $v \in B$ red if $g(u) < f(v)$ and blue otherwise. Note that in this coloring both the red and the blue graph are isomorphic to the infinite bipartite half graph.

The bipartite Rado graph is the graph $\mathcal{R}_2$ with vertex-set $\mathbb{N} \setminus \{1\}$ defined by placing an edge between $m < n$ if and only if the $m$th digit in the binary expansion of $n$ is 1 and $m$ and $n$ differ in the first bit (i.e. $m$ and $n$ have different parity).

Example 2.1.
(i) There is a graph $G$ with $\text{rul}(G) = 0$, but $\chi(G) = \infty$ and thus $\text{degen}(G) = \infty$ (half graph, Rado graph, infinitely many disjoint $K_N$’s).

(ii) There is a graph $G$ with $\chi(G) = 2$, but $\text{rul}(G) = \infty$ and thus $\text{degen}(G) = \infty$ ($K_{N,N}$).

(iii) There is a graph $G$ with $\text{rul}(G) = 0$ and $\chi(G) = 2$, but $\text{degen}(G) = \infty$ (bipartite Rado graph).

(iv) There is a one-way $2$-locally finite graph $G$ (with $\text{rul}(G) = 0$ and $\chi(G) = 2$), but $\text{degen}(G) = \infty$ (bipartite half graph).

(v) There is a locally finite graph $G$ with $\text{rul}(G) = 0$ but $\chi(G) = \infty$ and thus $\text{degen}(G) = \infty$ (infinite collection of disjoint finite cliques of increasing size).

(vi) There is a graph which is $d$-degenerate (and $d$-ruled) but not one-way $k$-locally-finite for any $k$ ($K_{d,N}, T_\infty$)

### 2.2 Upper bounds on upper densities

**Example 2.2.** Let $r \in \mathbb{N}$.

(i) Let $D \geq 2$. If $T$ is an infinite $D$-ary tree, then $\overline{\text{Rd}}_r(T) \leq \frac{1}{r}(1 + \frac{1}{r})$.

(ii) $\overline{\text{Rd}}_r(T_\infty) \leq 1/r$. (Recall that $T_\infty$ is the tree in which every vertex has infinite degree.)

(iii) There exists an infinite tree $T$ such that $T$ is locally finite and $\overline{\text{Rd}}_r(T) \leq 1/r$.

**Proof.** Partition $\mathbb{N}$ by residues mod $r$, that is $\mathbb{N} = R_1 \cup \ldots \cup R_r$ where $R_i$ is the set of all $n \in \mathbb{N}$ with $n \equiv i \pmod{r}$. We define an $r$-edge-coloring as follows: if $m \in R_i$ and $n > m$, color the edge $mn$ with color $i$. Note that if $n \not\equiv i \pmod{r}$, then $n$ has exactly $\lfloor (n-1)/r \rfloor$ neighbours of color $i$.

(i) Let $T$ be an infinite $D$-ary tree and suppose we have a copy of $T$ of color $i$. For all $n \in \mathbb{N}$, let $V'_n$ be the set of vertices in $V(T) \cap [n]$ which are not congruent to $i \pmod{r}$ and let $t_n = |V'_n|$. Since any vertex $m \in V'_n$ can only have neighbours (of color $i$) in $R_i \cap [n-1]$, we must have
D \cdot t_n \leq (n - 1)/r. So
\[
\frac{|V(T) \cap [n]|}{n} \leq \frac{n}{r} + \frac{t_n}{n} \leq \frac{n}{r} + \frac{n-1}{rD} \to n \to \infty \frac{1}{r} (1 + \frac{1}{D}).
\]

(ii) Suppose $U \subseteq \mathbb{N}$ is the vertex-set of a copy of $T_\infty$ in color $i \in [r]$. Since every vertex in $\mathbb{N} \setminus R_i$ has only finitely many neighbours in color $i$, we have $U \subseteq R_i$ and thus $d(U) \leq 1/r$.

(iii) Let $0 < d_1 < d_2 < \ldots$ be an increasing sequence. Let $T$ be a tree in which every vertex on level $i$ has degree $d_i$. We can repeat the argument from case (i), except now we have $t_n/n \to 0$ as $n \to \infty$.

**Example 2.3.** For every connected graph $G$, we have $\overline{\text{Rd}}(G) \leq \frac{1}{\chi(G) - 1}$ if $\chi(G)$ is finite and $\overline{\text{Rd}}(G) = 0$ if $\chi(G)$ is infinite.

**Proof.** Assume first that $\chi(G) < \infty$ and partition $\mathbb{N}$ by the residues mod $\chi(G) - 1$. Color all edges inside the sets red and all edges between the sets blue. There is no blue copy of $G$, so every copy of $G$ lies entirely inside one of the sets, all of which have density $\frac{1}{\chi(G) - 1}$. If $\chi(G) = \infty$, this construction shows that $\overline{\text{Rd}}(G) \leq 1/(k - 1)$ for every $k \geq 2$ and therefore $\overline{\text{Rd}}(G) = 0$.

**Corollary 2.4.**

(i) $\overline{\text{Rd}}(\mathcal{R}) = 0$ (where $\mathcal{R}$ is the Rado graph).

(ii) There exists a locally finite graph $G$ such that $\overline{\text{Rd}}(G) = 0$.

**Proof.** (i) Since $\mathcal{R}$ contains an infinite clique, we have $\chi(\mathcal{R}) = \infty$ and thus the result follows from Example 2.3.

(ii) Let $G$ be a graph on vertex set $\mathbb{N}$ where $[n(n+1)]^2$ induces a clique for all $n \in \mathbb{N}$. $G$ is locally finite, connected, and contains a clique of order $n$ for all $n \in \mathbb{N}$. So $\chi(G) = \infty$ and thus the result follows from Example 2.3.

**Example 2.5.** There is an $r$-coloring of $K_{N,N}$ in which every monochromatic path has upper density at most $1/r$. In particular, it is not possible to cover all but finitely many vertices with less than $r$ monochromatic paths.

**Proof.** Let $A$ and $B$ be the parts of $K_{N,N}$ and partition both of them into $r$ parts $A_1, \ldots, A_r$ and $B_1, \ldots, B_r$, each of density $1/(2r)$. For all $i, j \in [r]$, color every edge between $A_i$ and $B_j$ by $(i - j) \mod r$. It is easy to see that every part is incident to exactly one other part of each color and therefore, every monochromatic path can cover at most two parts, finishing the proof.

**2.3 Lower density**

As mentioned in the introduction, Erdős and Galvin proved that for all positive integers $\Delta$, there exists a 2-coloring of $K_N$ such that if $G$ is a graph with maximum degree at most $\Delta$ and finitely many isolated vertices, then every monochromatic copy of $G$ has lower density 0. We now show that a broader class of graphs has this property.

Recall that a graph $G$ is weakly expanding if for all $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for all independent sets $A$ in $G$ with $|A| \geq \ell$ we have $|N(A)| > k$. Note that if $G$ is weakly expanding, then there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, if $A$ is an independent set in $G$ with $|A| \geq f(k)$, then $|N(A)| > k$. Also note that if $G$ is weakly expanding, then $G$ has finitely many isolated vertices.
Fact 2.6. $G$ is weakly expanding if 
(i) $G$ has finite independence number, or 
(ii) $G$ has finite maximum degree and finitely many isolated vertices, or 
(iii) $G$ is a tree with bounded leaf degree, or 
(iv) for all $n \in \mathbb{N}$, $G$ has finitely many vertices of degree $n$.

The following is a modification of the example used by Erdős and Galvin [11] to prove the result mentioned about about graphs with bounded maximum degree and finitely many isolated vertices.

Example 2.7 (Forward interval coloring). Let $G$ be a graph. If $G$ is weakly expanding, then $G$ is not Ramsey-lower-dense.

Proof. Suppose $G$ is weakly expanding and let $f$ be the function guaranteed by the definition.

Let $a_n$ be an increasing sequence of natural numbers such that $a_0 = 1$ and for all $k \geq 1$,

$$a_k > k(a_{k-1} + f(a_{k-1})).$$

(2)

For all $u, v \in \mathbb{N}$ with $u < v$, color the edge $uv$ red if $u \in [a_{2n-1}, a_{2n})$ and blue if $u \in [a_{2n}, a_{2n+1})$.

Suppose there is a, say, blue copy of $G$ in this 2-coloring with vertex set $U$. We must have that $U \cap [a_{2n-1}, a_{2n})$ induces an independent set and because of the coloring we have $N_B(U \cap [a_{2n-1}, a_{2n})) \subseteq [0, a_{2n-1})$. Thus by the definition of weakly expanding, $|U \cap [a_{2n-1}, a_{2n})| < f(a_{2n-1})$. We conclude that

$$|U \cap [0, a_{2n})| < a_{2n-1} + f(a_{2n-1}) < \frac{1}{2a_{2n}} a_{2n}$$

and thus $d(U) = 0$.

We conclude with two more examples.

Example 2.8 (Backward interval coloring). Let $G$ be a graph. If $G$ has a finite dominating set (i.e. rul($G$) > 0), then $G$ is not Ramsey-lower-dense.

Proof. Let $a_n$ be an increasing sequence of natural numbers and let $A_i = [a_i, a_{i+1})$ for all $i \in \mathbb{N}$. For all $u \in A_i$ and $v \in A_j$ with $u < v$, color the edge $uv$ red if $j$ is odd and blue if $j$ is even.

Let $A^0$ be the union of all even indexed intervals and let $A^1$ be the union of all odd indexed intervals. We note that every vertex in $A^0$ has finite blue degree to $A^1$ and every vertex in $A^1$ has finite red degree to $A^0$.

Let $D$ be a finite dominating set in $G$ and suppose there is a monochromatic, say, blue copy of $G$ with vertex set $V$. Since $D$ is finite, there exists an index $t$ such that $D \subseteq A_1 \cup A_2 \cup \cdots \cup A_t$. Now for all $i$ such that $2i + 1 > t$, there are no blue edges from $A_{2i+1}$ contradicting the fact that $D$ is a dominating set. So $G$ has finite intersection with say $A^1$ and thus if $a_n$ is increasing fast enough, $G$ has lower density 0.

Example 2.9. If $G$ is a connected graph with $\chi(G) \geq 3$, then $G$ is not Ramsey-lower-dense.

Proof. Let $a_n$ be an increasing sequence of natural numbers and let $A_i = [a_i, a_{i+1})$ for all $i \in \mathbb{N}$. For all $u \in A_i$ and $v \in A_j$, color $uv$ blue if $i$ and $j$ have the same parity. Otherwise, color $uv$ red. Note that the red graph induces a copy of $K_{N,N}$ and thus any monochromatic copy of $G$ must be blue. The blue graph induces two disjoint copies of $K_N$; however, if $a_n$ is increasing fast enough, then the lower density of each such copy is 0 and thus any monochromatic copy of $G$ will have lower density 0.
2.4 The Rado graph, 0-ruled and 0-coruled graphs

If $G$ and $H$ are two graphs, then we write $G \preceq H$ if $G$ is isomorphic to a spanning subgraph of $H$. Clearly, $\preceq$ is reflexive and transitive.

We say that an infinite graph $G$ has the extension property if for every pair of disjoint finite sets $F, F' \subseteq V(G)$, there is a vertex $v \in V(G) \setminus (F \cup F')$ such that $v$ is adjacent to every $w \in F$ and not adjacent to any $w' \in F'$. The following well-known theorem (see [4]) shows why this property is useful.

**Theorem 2.10.** Any two infinite graphs satisfying the extension property are isomorphic.

Furthermore, it is not hard to see that the Rado graph $\mathcal{R}$ and (with probability 1) the infinite random graph (every edge is present independently with probability $1/2$) both satisfy the extension property. Hence, with probability 1, the infinite random graph is isomorphic to the Rado graph.

Observe that $G$ is 0-ruled if and only if $G$ satisfies the “non-adjacency” half of the extension property above, i.e. if for every finite $F' \subseteq V(G)$ there is a vertex $v \in V(G) \setminus F'$ such that $v$ is not adjacent to any $w' \in F'$. We will call $G$ 0-coruled if $G$ satisfies only the “adjacency” half of extension property, i.e. for every finite $F \subseteq V(G)$ there is a $v \in V(G) \setminus F$ such that $v$ is adjacent to every $w \in F$. Using this, it is easy (and very similar to the proof of Theorem 2.10) to prove the following proposition.

**Proposition 2.11.** $G$ is 0-ruled if and only if $G \preceq \mathcal{R}$. On the other hand, $G$ is 0-coruled if and only if $\mathcal{R} \preceq G$.

Note that for finite graphs $G \preceq H$ and $H \preceq G$ implies $G \cong H$, and thus $\preceq$ is a partial order (on isomorphism classes of graphs), but this is not the case for infinite graphs. A simple example is letting $G$ be an infinite clique together with infinitely many disjoint copies of some finite graph $F$ and letting $H$ be two disjoint infinite cliques together with infinitely many disjoint copies of some finite graph $F$). Another example comes from the fact that the infinite half graph is both 0-ruled and 0-coruled, but is not isomorphic to $\mathcal{R}$. We ask the following question out of curiosity.

**Problem 2.12.** Under what conditions on $G$ and $H$ does $G \preceq H$ and $H \preceq G$ imply that $G \cong H$?

The Rado coloring of $E(K_N)$ is the 2-coloring $\rho$ defined by setting $\rho(\{s, t\})$ to be the $s$th bit in the binary expansion of $t$ for all $s, t \in \mathbb{N}$ with $s < t$. For instance $\rho(\{2, 14\}) = 1$ since the 2nd bit (reading right to left) in the binary expansion of 14 is 1, and $\rho(\{5, 14\}) = 0$ since the 5th bit (reading right to left, and appending extra 0's to the left as necessary) in the binary expansion of 14 is 0. Also note that the Rado coloring can be described by coloring all of the edges of the Rado graph with color 1 and coloring all of the edges in the complement of the Rado graph with color 0.

The key property of the Rado coloring is that for any $F \subseteq \mathbb{N}$ and $i \in \{0, 1\}$, we have

$$d \left( \bigcap_{v \in F} N_i(v) \right) = 2^{-|F|}. \quad (3)$$

We first use the Rado coloring to make the following observation about complete multipartite graphs.
Proposition 2.13. Let $K$ be an infinite complete multipartite graph and let $n \in \mathbb{N}$.

(i) If $K$ has at least two infinite parts, or infinitely many vertices in finite parts, then $K$ is not Ramsey-dense.

(ii) If $K$ has exactly one infinite part and exactly $n$ vertices in finite parts, then

$$\frac{1}{2^{2n-1}} \leq \text{Rd}(K) \leq \frac{1}{2^n}.$$ 

Proof. Take the Rado coloring of $K_N$.

If $K$ has at least two infinite parts, or infinitely many vertices in finite parts, then $K$ contains a spanning copy of $K_{N,N}$; let $(A,B)$ be such a spanning copy of $K_{N,N}$. Let $a_1, a_2, \ldots$ be the elements of $A$. Then $B$ is contained in the neighborhood of $a_1, \ldots, a_n$, and hence has density $\leq 2^{-n}$, for each $n$, by (3). Hence $B$ must have density 0. The same goes for $A$.

Now suppose $K$ has exactly one infinite part and exactly $n$ vertices in finite parts. Then by (3), we have $\text{Rd}(K) \leq \frac{1}{2^n}$.

To see $\text{Rd}(K) \geq \frac{1}{2^{2n-1}}$, we are given an arbitrary 2-coloring of $K_N$ and we choose an arbitrary vertex $v$. Either $\overline{\text{d}}(N_R(v)) \geq 1/2$ or $\overline{\text{d}}(N_B(v)) \geq 1/2$. We repeat this process inside the chosen neighborhood for $2n - 1$ steps until we have the desired monochromatic subgraph. \qed

The following result suggests that the question of whether $G$ is Ramsey-dense or not may depend on the rate of growth of the ruling sets in $G$.

Theorem 2.14. Let $G$ be an infinite graph. If $G$ has pairwise-disjoint ruling sets $F_n$ ($n \in \mathbb{N}$) satisfying $|F_n| \leq \log_2(n)$ for all sufficiently large $n$, then $G$ is not Ramsey-dense.

Proof. Consider the Rado coloring of $K_N$. Suppose now that $V$ is the vertex set of a monochromatic copy of $G$, say with color $i$. Then for each $N$, we have

$$V \subseteq (\bigcap_{n=1}^N \bigcup_{v \in F_n} N_i(v)).$$

Note that

$$d\left(\bigcap_{n=1}^N \bigcup_{v \in F_n} N_i(v)\right) = \prod_{n=1}^N (1 - 2^{-|F_n|})$$

Hence

$$\overline{\text{d}}(V) \leq \prod_{n=1}^\infty (1 - 2^{-|F_n|}).$$

It is well-known that an infinite product $\prod_{n=1}^\infty \alpha_n$, with $\alpha_n \in (0,1)$, converges to 0 if and only if

$$\sum_{n=1}^\infty \log(\alpha_n) = -\infty.$$ 

In our case we have $|F_n| \leq \log_2(n)$ for all sufficiently large $n$, so

$$\log(1 - 2^{-|F_n|}) \leq \log \left(1 - \frac{1}{n}\right) \leq -1/n.$$ 

By the limit comparison test and the divergence of the harmonic series, we have $d(V) = 0$. \qed
3 Ultrafilters and embedding

The concept of ultrafilters will play an important role in this paper.

Definition 3.1. Given a set $X$, a set system $\mathcal{U} \subseteq 2^X$ is called an ultrafilter if

(i) $X \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$,

(ii) If $A \in \mathcal{U}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{U}$,

(iii) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$ and

(iv) For all $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$, or

(iv)$'$ $\mathcal{U}$ is maximal among all families satisfying (i) - (iii).

A family satisfying (i)-(iii) is called a filter. Conditions (iv) and (iv)$'$ are equivalent for filters (see [15, Chapter 11, Lemma 2.3]) and we will make use whichever is more convenient for the current application. Let us list some additional properties of ultrafilters.

Proposition 3.2. If $\mathcal{U}$ is an ultrafilter on $X$, we have

(i) If $A_1, \ldots, A_n \in \mathcal{U}$, then $A_1 \cap \ldots \cap A_n \in \mathcal{U}$.

(ii) If $A_1, \ldots, A_n$ are pairwise disjoint and $A_1 \cup \ldots \cup A_n \in \mathcal{U}$, then there is exactly one $i \in [n]$ with $A_i \in \mathcal{U}$.

Informally, we think of sets $A \in \mathcal{U}$ as “large” sets. A common example of an ultrafilter are the so called trivial ultrafilters $\mathcal{U}_x := \{A \subseteq X : x \in A\}$ for $x \in X$. It is not hard to see that an ultrafilter is trivial if and only if it contains a finite set.

We say that an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is positive if every set $A \in \mathcal{U}$ has positive upper density in $\mathbb{N}$. Positive ultrafilters play a crucial role in the proof of Theorem 1.6.

Proposition 3.3. If $X \subseteq \mathbb{N}$ is infinite, then there exists a non-trivial ultrafilter $\mathcal{U}$ on $X$. There exists a positive ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Proof. To prove the first part of the theorem apply Zorn’s lemma to

$$\{\mathcal{F} \subseteq 2^\mathbb{N} : \mathcal{F} \text{ contains all cofinite sets and satisfies (i) - (iii) in Definition 3.1}\}$$

to get a maximal such family $\mathcal{U}$, which must be an ultrafilter. Finally, if $A$ is finite, $\mathcal{U}$ contains the cofinite set $A^c$ and hence $A \notin \mathcal{U}$.

To prove the second part, apply Zorn’s lemma to

$$\{\mathcal{F} \subseteq 2^\mathbb{N} : \mathcal{F} \text{ contains all sets of lower density 1 and satisfies (i) - (iii) in Definition 3.1}\}$$

to get a maximal such family $\mathcal{U}$, which must be an ultrafilter. Furthermore, if $A \subseteq \mathbb{N}$ has upper density 0, then $\mathbb{N} \setminus A$ has lower density 1 (see (1)) and consequently $A \notin \mathcal{U}$. $\square$

Definition 3.4 (Vertex-coloring induced by $\mathcal{U}$). Let $r \geq 2$ be an integer and suppose the edges of an infinite graph $G$ are colored with $r$ colors. Let $\mathcal{U}$ be a non-trivial ultrafilter on $V(G)$. Define a coloring $c_\mathcal{U} : V(G) \to [r]$ where $c_\mathcal{U}(v) = i$ if and only if $N_i(v) \in \mathcal{U}$. Since $V(G) \setminus \{v\} \in \mathcal{U}$ for all $v \in V(G)$, it follows from Proposition 3.2 (ii) that $c_\mathcal{U}$ is well defined. We call $c_\mathcal{U}$ the vertex-coloring induced by $\mathcal{U}$.

The following two propositions allow us to use ultrafilters to embed the desired subgraphs in the proof of Theorem 1.2 and Theorem 1.6.
Proposition 3.5. Let \( k \geq 2 \) be an integer, let \( G \) be a one-way \( k \)-locally finite graph and let \( H \) be a graph such that \( \{U_1, \ldots, U_k\} \) is a partition of \( V(H) \) with \( |U_1| = \cdots = |U_k| = \infty \) and for all \( i \in [k] \) and any finite subset \( W \subseteq U_1 \cup \cdots \cup U_{i-1} \), the set of common neighbors of \( W \) in \( U_i \) is infinite. Then, there is an embedding \( f \) of \( G \) into \( H \) such that \( U_1 \subseteq \text{ran } f \).

Given a \( k \)-partite graph \( G \) with parts \( V_1, \ldots, V_k \) and a set \( S \subseteq V(G) \), the left neighborhood cascade of \( S \) is the tuple \( (S_1, \ldots, S_k) \), where \( S_k = S \cap V_k \), and for all \( 1 \leq i \leq k - 1 \), \( S_i = \left( S \cup \bigcup_{j=i+1}^{k} N(S_j) \right) \cap V_i \).

Proof. Let \( V_1 \cup V_2 \cup \cdots \cup V_k \) be a partition of \( V(G) \) into independent sets which witness the fact that \( G \) is one-way \( k \)-locally-finite (in particular \( V_1 \) is infinite). We will construct an embedding \( f \) iteratively in finite pieces. Initially, \( f \) is the empty embedding. Then, for each \( n \in \mathbb{N} \), we will proceed as follows: let

\[
S_n = \{ \text{min}(V_i \setminus \text{dom } f) : i \in [k] \text{ with } V_i \setminus \text{dom } f \neq \emptyset \}.
\]

That is, \( S_n \) contains the smallest not yet embedded vertex of each \( V_i \) which is not completely embedded yet. Let \( (T_{1,n}, \ldots, T_{k,n}) \) be left neighborhood cascade of \( S_n \) in \( G \). We will now extend \( f \) to cover \( \bigcup_{i \in [k]} T_{i,n} \). Observe that \( T_{i,n} \) is disjoint from \( \text{dom } f \) for all \( i \in [k] \) since we embedded the whole left neighborhood cascade in every previous step. Since \( V_1 \) is infinite, \( T_{1,n} \) is non-empty. Let \( T_{1,n} \subseteq U_1 \setminus \text{ran } f \) be the set of \( \{T_{1,n}\} \text{ smallest vertices in } U_1 \setminus \text{ran } f \) and extend \( f \) by embedding \( T_{1,n} \) into \( T'_{1,n} \) arbitrarily. By assumption \( T'_{1,n} \) has infinitely many common neighbours in \( U_2 \). Since \( \text{ran } f \) is finite, we can select a set \( T_{2,n} \subseteq (U_2 \cap N(T'_{1,n})) \setminus \text{ran } f \) of size \( |T_{2,n}| \). Extend \( f \) by embedding \( T_{2,n} \) into \( T'_{2,n} \) arbitrarily. Similarly, we can extend \( f \) by embedding \( T_{i,n} \) into appropriate sets \( T'_{i,n} \) for all \( i = 3, \ldots, k \).

Since we maintain a partial embedding of \( G \) into \( H \) throughout the process and every vertex of \( G \) will eventually be embedded (by choice of \( S_n \) which contains the smallest not yet embedded vertex of \( V(G) \)), the resulting function \( f \) defines an embedding of \( G \) into \( H \). Since we cover the smallest not-yet covered vertex of \( U_1 \) in each step, we further have \( U_1 \subseteq \text{ran } f \). \( \square \)

Proposition 3.6. Let \( G \) be an infinite 0-ruled graph and let \( H \) be a graph having the property that for every finite set of vertices \( W \subseteq V(H) \), the set of common neighbors of \( W \) is infinite. Then, there is an embedding \( f \) of \( G \) into \( H \) such that \( \text{ran } f = V(H) \).

Proof. Let \( v_1, v_2, \ldots \) be an enumeration of \( V(G) \) and let \( u_1, u_2, \ldots \) be an enumeration of \( V(H) \). Let \( f(v_1) = u_1 \). Now suppose \( \text{dom } f = \{v_1, \ldots, v_n\} \) for some \( n \in \mathbb{N} \). Let \( u_{i,n} \) be the vertex of smallest index in \( V(H) \setminus \text{ran } f \). Since \( G \) is 0-ruled, there exists a vertex \( v_p \) with \( p > n \) such that \( v_p \) has no neighbors in \( \{v_1, \ldots, v_n\} \). We set \( f(v_p) = u_{n,n} \) and if \( p > n + 1 \), we do the following for all \( n + 1 \leq i \leq p - 1 \): since \( \{f(v_1), \ldots, f(v_{i-1}), f(v_p)\} \) has infinitely many common neighbors, we may choose a vertex \( u \in V(H) \setminus \text{ran } f \) which is adjacent to every vertex in \( \{f(v_1), \ldots, f(v_{i-1}), f(v_p)\} \) and set \( f(v_i) = u \). Continuing in this way, we obtain an embedding of \( G \) into \( H \). Since on each step, the vertex of smallest index in \( V(H) \setminus \text{ran } f \) becomes part of the range of \( f \), the embedding is surjective. \( \square \)

4 Graphs of bounded chromatic number

In this section we will prove Theorem 1.2. First note that if \( G \) is one-way \( k \)-locally-finite, then \( G \) is 0-ruled.
Proof of Theorem 1.2. (i) We are given an infinite one-way 2-locally-finite graph $G$ and an $r$-coloring of the edges of $K_N$. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. Let $c_{\mathcal{U}}$ be the vertex-coloring induced by $\mathcal{U}$ and for all $i \in [r]$, let $A_i$ be the set of vertices receiving color $i$. We may suppose without loss of generality that $d(A_1) \geq 1/r$ (see (1)). If $A_1 \not\in \mathcal{U}$, then the set of common neighbors of $S$ in $A_1$ of color 1 is infinite for every finite set $S \subseteq A_1$. Thus, we can apply Proposition 3.6 to embed $G$ in color 1 in such a way that $A_1$ is covered. If $A_1 \not\in \mathcal{U}$, then every finite set $S \subseteq A_1$ has infinitely many common neighbors of colour 1 in $A_1^c$. Hence, by applying Proposition 3.5, we can find a monochromatic copy of $G$ in color 1 such that $A_1$ is covered. Either way, we have a monochromatic copy of $G$ of upper density at least $d(A_1) \geq 1/r$.

![Diagram](image1.png)

Figure 2: An example of the proof of Theorem 1.2. (ii). In this example, $G$ will be embedded in blue into $W_4 \cup A_{3,i_3} \cup A_{1,i_1}$ such that $W_4 \subseteq V(G)$.

(ii) We are given an infinite one-way $k$-locally-finite graph $G$ and an 2-coloring of the edges of $K_N$. Let $\mathcal{U}_1$ be a non-trivial ultrafilter on $\mathbb{N}$. Let $c_{\mathcal{U}_1}$ be the vertex-coloring induced by $\mathcal{U}_1$ and for all $i \in [2]$, let $A_{1,i}$ be the set of vertices receiving color $i$. Choose $i_1 \in [2]$ so that $A_{1,i_1} \in \mathcal{U}_1$ and let $i'_1 = 3 - i_1$. Now let $\mathcal{U}_2$ be a non-trivial ultrafilter on $W_2 = A_{1,i_1}$ and let $c_{\mathcal{U}_2}$ be the vertex-coloring of $W_2$ induced by $\mathcal{U}_2$. For all $i \in [2]$, let $A_{2,i}$ be the set of vertices receiving color $i$. Choose $i_2$ so that $A_{2,i_2} \in \mathcal{U}_2$ and let $i'_2 = 3 - i_2$. Let $W_3 := A_{2,i_2}$ and continue in this manner until the point at which there exists $t$ and $j \in [2]$ such that there exists a set $I \subseteq [t]$ where $|I| = k - 1$ and $A_{i,j} \in \mathcal{U}_t$ for all $i \in I$. Note that by pigeonhole, $t \leq 2k - 3$ and suppose without loss of generality that $j = 1$. Set $W_{t+1} := W_t \setminus A_{t,1}$. One of the sets $A_{1,i_1}, A_{2,i_2}, \ldots, A_{t,i_t}, W_{t+1}$ has upper density at least $\frac{1}{2(k-1)}$ (see (1)). If, say, $d(A_{\ell,i_\ell}) \geq \frac{1}{2(k-1)}$ for some $\ell \in [t]$, then applying Proposition 3.6 with color $i_\ell$ gives the desired monochromatic copy of $G$ covering $A_{\ell}$; otherwise $d(W_{t+1}) \geq \frac{1}{2(k-1)}$ and applying Proposition 3.5 with color 2 gives the desired monochromatic copy of $G$ covering $W_{t+1}$.

(iii) The process is very similar to (ii), in that we repeatedly choose ultrafilters until we are guaranteed that some color appears $k - 1$ times from every set at the end of the process (see Fig. 3). However, the formal proof is a bit more technical.

We will use the following notation. Given $i_1, i_2 \in \mathbb{N}$, and $L_1 \in \mathbb{N}^1$ and $L_2 \in \mathbb{N}^2$, we write $L_1 < L_2$ if $L_1$ is an initial segment of $L_2$. Furthermore, given $L = (j_1, \ldots, j_k) \in \mathbb{N}^k$ for some $i \in \mathbb{N}$, we define $L^- := (j_1, \ldots, j_{i-1})$.

Suppose the edges of $K_N$ are colored with $r$ colors and let $q = (k - 2)r + 1$. We will define sets $A_L$ for $L \in \bigcup_{i=0}^q [r-1]^i$ and colorings $\chi_1 : \{A_L : L \in \bigcup_{i=0}^q [r-1]^i\} \to [r]$ and $\chi_2 : \bigcup_{i=0}^q [r-1]^i \to [r]$ with the following properties.

(a) The sets $A_L$, $L \in \bigcup_{i=0}^q [r-1]^i$, are pairwise disjoint and their union is cofinite.
(b) For every $L \in \bigcup_{i=1}^q [r-1]^i$, $A_L$ is empty or every finite set $S \subseteq A_L$ has infinitely many
Figure 3: An example of the proof of Theorem 1.2.(iii) with $r = 3$ and $k = 3$. Here we have highlighted the sequence $A_{(1,2,1,2)}, A_{(1,2,1)}, A_{(1,2)}, A_{(1)}, A_3$ and note that some color, in this case red, must appear at least twice, which means we can embed $G$ into $A_{(1,2,1,2)} \cup A_{(1,2,1)} \cup A_\emptyset$ in such a way that $A_{(1,2,1,2)}$ is covered.

common neighbors of color $\chi(A_L)$ in $A_L$.

(c) For every $L \in \bigcup_{i=0}^q [r-1]^i$, $A_L$ is empty or every finite set $S \subseteq \bigcup_{L < L'} A_{L'}$ has infinitely many common neighbors of color $\chi_2(L)$ in $A_{L-}$.

We will construct these sets and colorings recursively. In the process, we will also construct sets $B_L$ and ultrafilters $\mathcal{U}_L$ on $B_L$ for every $L \in \bigcup_{i=0}^q [r-1]^i$.

Let $B_0 = \mathbb{N}$ and let $\mathcal{U}_0$ be a non-trivial ultrafilter on $B_0$, where () denotes the empty sequence. Let $c_{\mathcal{U}_0}$ be the vertex-coloring induced by $\mathcal{U}_0$. Let $c$ be the color so that $A_0$, the set of vertices of color $c$, is in $\mathcal{U}_0$ and let $\chi_1(A_0) = c$. Let $[r] \setminus \{c\} = \{j_1, \ldots, j_{r-1}\}$ and, for $i \in [r-1]$, let $B(i)$ be the set of vertices receiving color $j_i$ and let $\chi_2((i)) = j_i$.

In the next step, we proceed as follows for every $i_0 \in [r-1]$. If $B(i_0)$ is finite, let $A(i_0) = B(i_0, i) = \emptyset$ for every $i \in [r-1]$. Otherwise, let $\mathcal{U}_{(i_0)}$ be a non-trivial ultrafilter on $B(i_0)$ and let $c_{\mathcal{U}_{(i_0)}}$ be the vertex-coloring induced by $\mathcal{U}_{(i_0)}$. Let $c$ be the color so that $A(i_0)$, the set of vertices of color $c$, is in $\mathcal{U}_{(i_0)}$ and let $\chi_1(A(i_0)) = c$. Let $[r] \setminus \{c\} = \{j_1, \ldots, j_{r-1}\}$ and, for $i \in [r-1]$, let $B(i_0, i)$ be the set of vertices receiving color $j_i$ and let $\chi_2((i_0, i)) = j_i$.

We proceed like this until we defined the sets $B_L$ for every $L \in [r-1]^q$ and let $A_L := B_L$ for all $L \in [r-1]^q$. It is easy to see from the ultrafilter properties that the above properties hold.

Therefore, for every $L \in \bigcup_{i=0}^q [r-1]^i$, $A_L$ is empty or can be covered by a monochromatic copy of $G$ by Proposition 3.6. Furthermore, for every $L \in [r-1]^q$ for which $A_L$ is non-empty, we find $k-1$ sets $L_1 \prec \ldots \prec L_{k-1} \prec L$ of the same color w.r.t. $\chi_2$ by the pigeonhole principle. Therefore, applying Proposition 3.5 to $U_k := A_{L_1, \ldots, U_2 := A_{L_{k-1}}, U_1 := A_{L}}$, we find a monochromatic copy of $G$ covering $A_L$. Since, there are $C := q \sum_{i=0}^q (r-1)^i$ sets $A_L$, one of them has upper density at least $1/C$.

Let $G$ be a graph with $\Delta := \Delta(G) < \infty$. Since $\chi(G) \leq \Delta(G) + 1$, we immediately obtain as a corollary that $\overline{\text{Rd}}(G) \geq \frac{1}{2\Delta-1}$. However, with a bit more work we obtain the following corollary.

**Corollary 4.1.** Let $G$ be an infinite graph. If $2 \leq \Delta := \Delta(G) < \infty$, then $\overline{\text{Rd}}(G) \geq \frac{1}{2(\Delta-1)}$.

First we note the following fact which also appears in [18, Theorem 1.(i)].

**Proposition 4.2.** For all $r \in \mathbb{N}$, if $G$ has infinitely many components, then $\overline{\text{Rd}}_r(G) \geq 1/r$.

**Proof.** By Ramsey’s theorem, it is possible to partition any $r$-colored $K_\mathbb{N}$ into monochromatic infinite cliques and a finite set. Indeed, greedily take disjoint monochromatic copies of $K_\mathbb{N}$ in
which the smallest vertex is minimal. Either the process ends with a finite set of uncovered vertices, or the process continues for infinitely many steps and the union misses infinitely many vertices. However, now there is a monochromatic copy of $K_N$ whose minimal vertex must be smaller than one of the monochromatic cliques in our collection, a contradiction.

Without loss of generality, suppose the cliques of color 1 have upper density at least $1/r$. Since $G$ has infinitely many components, $G$ can be surjectively embedded into the cliques of color 1 (by merging components if necessary, we may assume that all components are infinite).

Proof of Corollary 4.1. First note that if $G$ has infinitely many components, then we are done by Proposition 4.2. If $\chi(G) \leq \Delta$, then we are done by Theorem 1.2; so suppose that $G$ has finitely many components and $\chi(G) = \Delta + 1$. Now by Brooks theorem, either $\Delta = 2$ and $G$ contains finitely many components which are odd cycles, or $\Delta \geq 3$ and $G$ contains finitely many components which are cliques on $\Delta+1$ vertices. Note that in either case, every infinite component of $G$ (of which there is at least one), has chromatic number at most $\Delta$. Let $V_2 \subseteq V(G)$ be the vertex-set of the finitely many components which are odd cycles or cliques of size $\Delta+1$, and let $V_1 = V(G) \setminus V_2$.

We are given a 2-coloring of $K_N$. If there is a red clique $R$ and a blue clique $B$ each of size $|V_2|$, we can apply Theorem 1.2 to $G[V_1]$ (which is one-way $\Delta$-locally finite) and $K_N \cap [(R \cup B)^c]$ to get a monochromatic copy of $G[V_1]$ of upper density at least $\frac{1}{2(\Delta+1)}$. Together with either $R$ or $B$, this gives the desired copy of $G$.

So suppose that there is no, say, red clique of order $|V_2|$. If $\Delta \geq 3$, we repeat the proof of Theorem 1.2(ii); however, in each iteration $i_j = 1$ (here, blue is 1 and red is 2), otherwise there would be an infinite red clique. Thus we can stop when $t = \chi(G) - 1 \leq \Delta$ and get a monochromatic copy of $G$ of upper density at least $\frac{1}{\Delta+1} \geq \frac{1}{2(\Delta+1)}$. Finally, if $\Delta = 2$, we repeat the proof of Theorem 1.2(ii), but after the first step, we have $A_{1,1} \in \mathcal{F}_1$ and $W_2 = A_{1,2}$. If $d(A_{1,1}) \geq 1/2$, then we are done as usual. So suppose $d(A_{1,2}) \geq 1/2$. If there is an infinite red matching in $A_{1,2}$, then these edges can be used to make the odd cycles comprising $V_2$ and then $V_1$ can be embedded as usual. Otherwise $A_{1,2}$ does not contain an infinite red matching and thus there is a cofinite subset of $A_{1,2}$ which induces a blue clique into which we can embed $G$.

Finally we note the following strengthening of Theorem 1.2 which generalizes a result of Elekes, D. Soukup, L. Soukup, and Szentmiklissy [10] who proved a similar statement for powers of cycles..

**Theorem 4.3.** Let $k, r \in \mathbb{N}$ and let $G$ be a one-way $k$-locally finite graph. In every $r$-coloring of the edges of $K_N$, there exists a collection of

$$f(r, k) = \begin{cases} r & \text{if } k = 2 \\ \sum_{i=0}^{r-2} (r-1)^i & \text{if } k \geq 3 \end{cases}$$

vertex-disjoint, monochromatic copies of $G$ whose union covers all but finitely many vertices.

**Remark 4.4.** The proof of Theorem 1.2 immediately shows that for every one-way $k$-locally finite graph $G$ and every $r$-edge-colored $K_N$, there is a collection of at most $f(r, k)$ monochromatic copies of $G$ covering a cofinite subset of $\mathbb{N}$, where $f(r, k)$ is as in the statement of Theorem 4.3. In order to obtain a partition as required by Theorem 4.3, we need to guarantee that these copies can be chosen to be disjoint. To do so, instead of applying Propositions 3.5 and 3.6, we will
embed the graphs simultaneously doing one step of the embedding algorithms of Propositions 3.5 and 3.6 at a time always making sure not to repeat vertices (which is possible since we have infinitely many choices in every step but only finitely many embedded vertices). Otherwise, the proof is exactly the same and therefore we will omit it.

5 Graphs of bounded ruling number

In this section, we will prove Theorem 1.6.

Proof of Theorem 1.6. Let $G$ be a finitely ruled graph and suppose $K_r$ is colored with $r$-colors for some $r \in \mathbb{N}$. Let $\mathcal{U}$ be a positive ultrafilter on $\mathbb{N}$ and denote by $V_i$ the set of vertices of color $i$ in the vertex-coloring induced by $\mathcal{U}$. Suppose without loss of generality that $V_1 \in \mathcal{U}$. Since $G$ is finitely ruled, there is a finite set $S$ such that $G[S]$ does not have any finite dominating set and in particular $G[S]$ is 0-ruled.

We will now construct the embedding $f : V(G) \rightarrow \mathbb{N}$. First embed $S$ into an arbitrary clique of color 1 in $V_1$ of size $|S|$ (such a clique can be found be iteratively applying the ultrafilter property). Let $V_1^0 = N_1^0(f(S)) \cap V_1$ and note that $V_1^0 \in \mathcal{U}$ and hence satisfies the assumptions of Proposition 3.6. Therefore, $G[S]$ can be surjectively embedded into $V_1^0$, and we can extend $f$ to an embedding of $G$. Since $V_1^0 \subseteq f(V(G))$ has positive upper density, we are done. \hfill \Box

6 Graphs of bounded degeneracy

Given $k \in \mathbb{N}$ and a graph $G$, we say that $X \subseteq V(G)$ is $k$-wise intersecting if for all $S \subseteq X$ with $|S| \leq k$, $N^0(S)$ is infinite. We say that $X \subseteq V(G)$ is $k$-wise self-intersecting if for all $S \subseteq X$ with $|S| \leq k$, $S \cap N^0(S)$ is infinite. We say that a graph $G$ is $k$-wise intersecting if $V(G)$ is $k$-wise intersecting (and consequently $k$-wise self-intersecting). Finally, if $G$ is an $r$-colored graph for some $r \in \mathbb{N}$, we say that $X \subseteq V(G)$ is $k$-wise (self-)intersecting in color $i$ if $X$ is $k$-wise (self-)intersecting in $G_i$.

The following is related to Proposition 3.6.

Proposition 6.1. Let $d \in \mathbb{N}$ and let $G$ be an infinite, 0-ruled, $d$-degenerate graph. If $H$ is a $(d + 1)$-wise intersecting graph, then we can surjectively embed $G$ into $H$.

Proof. Do the same as in the proof of Proposition 3.6, but since $G$ is $d$-degenerate, when we get to the second phase of the embedding step, where we embed all vertices from $\{v_{n+1}, \ldots, v_{p-1}\}$ into $H$ one at a time, we note that each vertex $v_i$ is adjacent to at most $d + 1$ vertices in $\{v_1, \ldots, v_{i-1}\} \cup \{v_p\}$, so it is possible to choose an image for $v_i$ in $H$. \hfill \Box

In the proofs of Theorem 1.2 and Theorem 1.6, we implicitly proved the following.

Proposition 6.2. Let $k \in \mathbb{N}$. For every 2-coloring of $K_\mathbb{N}$, there is a set $X$ with upper density at least $1/2$ and a color $i \in [2]$ such that for every $k$, $X$ is $k$-wise intersecting in color $i$. Moreover there is a set $Y$ with positive upper density and a color $i \in [2]$ such that for every $k$, $Y$ is $k$-wise self-intersecting in color $i$.

The idea is that by Proposition 6.1, for the purposes of embedding 0-ruled, $d$-degenerate graphs we don’t need the set $Y$ described above to be $k$-wise self-intersecting. Thus we can ask if it is possible to find a set $Y$ which is $(d + 1)$-wise self-intersecting and has upper density

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bounded below by some function of $d$. While we haven’t been able to address this question, we now give an example which provides an upper bound on the upper density of such a set. This example is due to Chris Lambie-Hanson [17].

**Proposition 6.3.** For all $k \in \mathbb{N}$, there exists a 2-coloring of $K_n$ such that every monochromatic $k$-wise self-intersecting set has upper density at most $1/2k$.

**Proof.** Let $k \in \mathbb{N}$ and partition $\mathbb{N}$ into sets $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ of equal asymptotic density $1/2k$. Let $A = A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_k$. The coloring is as follows. Given $a \in A$ and $b \in B$, we color $\{a, b\}$ red if $a < b$ and blue otherwise. Given $a, a' \in A$, we color $\{a, a'\}$ red if $a$ and $a'$ are in the same set $A_i$, and blue otherwise. Given $b, b' \in B$, we color $\{b, b'\}$ blue if $b, b'$ are in the same set $B_i$, and red otherwise.

![Figure 4: An example of the coloring from Proposition 6.3 in the case when $k = 2$. The shaded areas denote cliques of the respective colors and a blue/solid (red/dashed) arrow from one part to another indicates that vertices in the first part have cofinitely many blue (red) neighbors in the second part.](image)

The colors are clearly symmetric so it suffices to consider a red $k$-wise self-intersecting set $X$. We claim that $X$ is contained in a single $A_i$.

Note that for any $b \in B_i$, $N_R^\cap(b) \cap A$ is finite and $N_R^\cap(b) \cap B_i = \emptyset$. Thus if $X \cap B_i \neq \emptyset$, $X \cap B_j \neq \emptyset$ for some $j \neq i$. Applying the same argument with elements of $B_i$ and $B_j$, we see that $X \cap B_h \neq \emptyset$ for some $h \neq i, j$, and continuing we get $X \cap B_\ell \neq \emptyset$ for all $\ell = 1, \ldots, k$. But then taking $F$ to be a subset of $X$ consisting of one vertex from each $B_\ell$, we see that $N_R^\cap(F)$ is finite, a contradiction.

So we must have $X \subseteq A$. But note that $N_R^\cap(a) \cap A \subseteq A_i$ for each $a \in A_i$. Hence $X$ must be contained in $A_i$ for some $i$. \hfill \Box

It is not immediately clear that there exists a $d$-wise intersecting graph with bounded degeneracy. So we now give a construction of a family of $d$-wise intersecting graphs which are $d$-degenerate (and 0-ruled).

**Proposition 6.4.** For every $d \in \mathbb{N}$, there is an infinite graph $H_d$ which is $d$-wise intersecting, $d$-degenerate, and 0-ruled.

**Proof.** Let $n_0 = d$. For all $i \geq 0$, let $S_1, S_2, \ldots, S_{\binom{n_2}{d}}$ be an enumeration of all the $d$-element
subsets of \([n_i]\), let \(n_{i+1} = n_i + \binom{n_i}{d}\), and let
\[
E_{i+1} = \bigcup_{1 \leq j \leq \binom{n_i}{d}} \{n_i + j, v\} : v \in S_j\).
\]

Let \(H_d\) be the graph on vertex \(\mathbb{N}\) with edge set \(\bigcup_{j \in \mathbb{N}} E_j\).

By the construction it is clear that \(H_d\) is \(d\)-wise intersecting and \(d\)-degenerate. To see that \(G\) is 0-ruled, note that for any finite set \(X \subseteq \mathbb{N}\) and any \(d\)-element set \(Y \subseteq \mathbb{N} \setminus X\), there are infinitely many vertices which are adjacent to every vertex in \(Y\) and none of the vertices in \(X\). Thus \(G\) cannot have a finite dominating set.

Note that, in particular, \(H_d\) contains a spanning copy of every \((d - 1)\)-degenerate 0-ruled graph. Denote by \(\rho(d)\) the smallest Ramsey upper density of a \(d\)-degenerate infinite graph and by \(\tau(d)\) the largest \(\tau \geq 0\) such that every 2-colored complete graph contains a monochromatic \(d\)-wise self-intersecting subgraph of density at least \(d\). The above propositions imply
\[
\tau(d - 1) \geq \rho(d - 1) \geq \tau(d) \geq \rho(d)
\]
for every \(d \geq 2\). In particular, we have \(\tau(d) > 0\) for every \(d \in \mathbb{N}\) if and only if \(\rho(d) > 0\) for every \(d \in \mathbb{N}\). So in order to answer Problem 1.5 positively for 0-ruled graphs, it would suffice to answer Problem 1.5 positively for \(H_d\) for all \(d\). Note that \(H_1 = T_\infty\) and thus Theorem 1.13 gives a positive answer for the case \(d = 1\).

We conclude this section with a few comments about Problem 1.5.

In light of Theorem 1.2, if there were a function \(f : \mathbb{N} \rightarrow \mathbb{N}\) such that for all \(d \in \mathbb{N}\), every \(d\)-degenerate graph is one-way \(f(d)\)-locally finite, then we would have a positive answer to Problem 1.5; however, this is not the case as there are \(d\)-degenerate graphs which are not one-way \(k\)-locally-finite for any \(k\). For instance, the graph \(H_d\) constructed above is \(d\)-degenerate, but since every vertex has infinite degree, is not one-way \(k\)-locally-finite for any \(k\). Also \(K_{d,\mathbb{N}}\) is \(d\)-degenerate but not one-way \(k\)-locally-finite for any \(k\) (although in this case, we know \(R_d(K_{d,\mathbb{N}}) \geq \frac{1}{2d - 1}\)).

Problem 1.5 is about all \(d\)-degenerate graphs. However, the discussion in this section is about 0-ruled, \(d\)-degenerate graphs. It seems possible that answering Problem 1.5 positively for 0-ruled, \(d\)-degenerate graphs could imply a positive answer for all \(d\)-degenerate graphs (c.f. the proof of Theorem 1.6).

**Problem 6.5.** If Problem 1.5 were true for all 0-ruled \(d\)-degenerate graphs (i.e. \(H_d\)), would this imply that Problem 1.5 was true for all \(d\)-degenerate graphs?

Say that a digraph \(D\) is 2-directed if for all distinct \(u, v \in V(D)\), there exists \(w \in V(D)\) (where it is possible for \(w = u\) or \(w = v\)), such that \((u, w) \in E(D)\) and \((v, w) \in E(D)\). For example, the digraph \(D = (\{a, b, c, d\}, \{(a, d), (b, d), (c, d), (d, d)\})\) is 2-directed.

In order to get a monochromatic \(d\)-wise self-intersecting set with upper density at least some fixed amount in an arbitrary 2-coloring of \(K_{\mathbb{N}}\), we likely have to solve the following problem.

**Problem 6.6.** Given a 2-coloring of the edges of a complete (finite) digraph \(K\) (including loops), is it possible to cover \(V(K)\) with at most four monochromatic 2-directed graphs? (if not four, some other fixed number?)
The reason is that given any 2-coloring of a complete digraph $K$ (plus loops), we can create a corresponding 2-coloring of $K_N$ as follows. Split $\mathbb{N}$ into infinite sets $A_i$, one for each vertex $i$ of $K$. Color the edges inside $A_i$ according to the color of the loop on $i$. Now if both directed edges $(i, j)$ and $(j, i)$ are the same color, give all edges between $A_i$ and $A_j$ that color; if not, then color the bipartite graph between $A_i$ and $A_j$ with the bipartite half graph coloring. Then any 2-wise self-intersecting set $B$ must be the union of some collection of $A_i$’s whose corresponding vertices $i$ make up a monochromatic 2-directed set in $K$.

7 Bipartite Ramsey densities

In this section we prove Theorem 1.16. An infinite graph $G$ is said to be \textit{infinitely connected} if $G$ remains connected after removing any finite set of vertices. Note that every vertex of an infinitely connected graph has infinite degree. Given some set of vertices $S \subseteq V(G)$, we say that $S$ is \textit{infinitely connected} if $G[S]$ is infinitely connected. Similarly, we call a set $S \subseteq V(G)$ \textit{infinitely linked} if for all distinct $u, v \in S$, there are infinitely many internally vertex-disjoint paths in $G$ from $u$ to $v$ (note that the internal vertices of these paths need not be contained in the set $S$). Note that every infinitely connected set is also infinitely linked but the converse is not true (for example, both parts of $K_{N, N}$ are infinitely linked but not connected). Further note that if $S_1, \ldots, S_k$ are sets, each of which is infinitely linked, then there are disjoint paths $P_1, \ldots, P_k$ such that $P_1 \cup \cdots \cup P_k$ covers $S_1 \cup \cdots \cup S_k$.

If $G$ is a colored graph and $c$ is a color, we say that $G$ is \textit{infinitely connected in color} $c$ if $G_c$ (the spanning subgraph of $G$ with all edges of color $c$) is infinitely connected. A set $S \subseteq V(G)$ is \textit{infinitely connected in color} $c$ (infinitely linked in color $c$) if $S$ is infinitely connected (infinitely linked) when restricted to $G_c$. $S$ is called monochromatic infinitely connected (infinitely linked) if it is infinitely connected in some color $c$.

The following proposition directly implies Theorem 1.16 which implies Theorem 1.15.

\textbf{Proposition 7.1.} Every 2-colored $K_{N, N}$ can be partitioned into a finite set and two monochromatic infinitely linked sets $X$ and $Y$.

\textit{Proof.} Let $V_1, V_2$ be the parts of the bipartite graph and let $\mathcal{U}_1, \mathcal{U}_2$ be non-trivial ultrafilters on $V_1$ and $V_2$. For $i = 1, 2$, let $B_i \subseteq V_i$ be the blue vertices in the induced vertex-coloring and let $R_i = V_i \setminus B_i$ be the red vertices.

\textbf{Case 1} ($|R_1| = |R_2| = |B_1| = |B_2| = \infty$). If there are infinitely many disjoint red paths between $R_1$ and $R_2$, then $X := R_1 \cup R_2$ is infinitely linked in red. Indeed, if $v_1, v_2 \in R_1$ or $v_1, v_2 \in R_2$, then they have infinitely many common red neighbors (by the properties of the ultrafilter). If $v_1 \in R_1$ and $v_2 \in R_2$, we will construct infinitely many internally disjoint paths between $x_0 := v_1$ and $x_5 := v_2$ as follows: let $P = x_2 \ldots x_3$ be a red path so that $x_2 \in R_1$ and $x_3 \in R_2$, and let $x_1$ be a common red neighbor of $x_0$ and $x_2$ (of which we have infinitely many as above) and $x_4$ be a common neighbor of $x_3$ and $x_5$. It is clear that $x_0x_1x_2 \ldots x_3x_4x_5$ defines a red path and that we can construct infinitely many internally disjoint paths like this. If there are only finitely many disjoint red paths between $R_1$ and $R_2$, then there is a finite set $S$ so that, in particular, $X := (R_1 \cup R_2) \setminus S$ induces a complete blue bipartite graph with parts of infinite size and hence is infinitely linked in blue. Similarly, there is a set $Y \subseteq B_1 \cup B_2$ which is cofinite in $B_1 \cup B_2$ and infinitely linked in red or infinitely linked in blue.

\textbf{Case 2.} Suppose without loss of generality that $R_1$ is finite. It is easy to verify that $X = B_1 \cup B_2$ is infinitely linked in blue and $Y := R_2$ is infinitely linked in red. \qed
Recall that $P_\infty$ is the one-way infinite path and the value of $\overline{\text{rd}}_3(P_\infty)$ was determined in [7]. It is known that $1/3 \leq \overline{\text{rd}}_3(P_\infty) \leq 1/2$, and the above result has an interesting consequence in determining the value of $\overline{\text{rd}}_3(P_\infty)$. That is, we can now restrict our attention to 3-colorings such that for every color $\alpha \in [3]$, there are infinitely many vertices which have finite degree in color $\alpha$.

**Corollary 7.2.** If we are given a 3-coloring of $K_n$ such that there exists a color $\alpha \in [3]$ in which cofinitely many vertices have infinite degree in color $\alpha$, then there exists a monochromatic copy of $P_\infty$ with $\overline{\text{d}}(P_\infty) \geq 1/2$.

**Proof of Corollary 7.2.** Let $G_\alpha$ be the spanning subgraph induced by all edges of color $\alpha$. By deleting vertices if necessary we may assume that every vertex in $G_\alpha$ has infinite degree. If $G_\alpha$ is infinitely connected, then the result follows from Lemma 8.1 below, so suppose $G_\alpha$ is not infinitely connected. Then, there is a finite set $S$ so that $G_\alpha[\mathbb{N} \setminus S]$ is disconnected. Let $V_1$ be one component and $V_2 := \mathbb{N} \setminus (S \cup V_1)$. Then $V_1$ and $V_2$ induce a 2-edge-colored bipartite graph and both $V_1$ and $V_2$ are infinite since every vertex has infinite degree in $G_\alpha$. Thus the result follows from Theorem 1.15.

8 Trees

8.1 General embedding results

Given $k \in \mathbb{N}$, we say that a connected graph $T$ has radius at most $k$ if there exists $u \in V(T)$ such that for all $v \in V(T)$, there is a path of length at most $k$ from $u$ to $v$; if no such $k$ exists we say that $T$ has unbounded radius.

**Lemma 8.1.** Let $T$ be a graph. A spanning copy of $T$ can be found in every infinitely connected graph $H$ if and only if $T$ is a forest and (i) $T$ has a component of unbounded radius or (ii) $T$ has infinitely many components.

In order to prove Lemma 8.1 we first prove the following structural result about trees with unbounded radius. An increasing star is a tree obtained by taking an infinite collection of disjoint finite paths of unbounded length and joining one endpoint of each of the paths to a new vertex $v$. Note that an increasing star has unbounded radius, no infinite path, and exactly one vertex of infinite degree (which is called the center). Also note that an increasing star has distinct vertices $v_0, v_1, v_2, \ldots$ and internally disjoint paths $P_1, P_2, \ldots$ where for all $i \geq 1$, $P_i$ is a path from $v_0$ to $v_i$ and the length of $P_{i+1}$ is greater than the length of $P_i$.

**Fact 8.2.** Let $T$ be a tree of unbounded radius. Either for all $v \in V(T)$, there is an infinite path in $T$ starting with $v$ or there exists $v_0 \in V(T)$ such that $T$ contains an increasing star having $v_0$ as the center.

**Proof.** Let $T$ be a tree and suppose $T$ does not contain an infinite path. Let $v$ be a vertex in $T$ and note that since $T$ has unbounded radius, we can do the following: let $Q_1$ be a path from $v$ to a leaf $u_1$, which has some length $k_1$, now there must exist a path $Q_2$ of length $k_2 > k_1$ from $v$ to a leaf $u_2$, and so on. This process gives an infinite set of leaves $U$ and an increasing sequence $k_1, k_2, \ldots$ such that there is a path from $v$ to $u_i$ of length $k_i$. Now we apply the Star-Comb lemma [9, Lemma 8.2] to the set $U$. Since $T$ has no infinite path, there must exist a subdivision of an infinite star with center $v_0$ such that all the leaves, call them $U'$, are in $U$. We claim that
for all $k$ there exists a path from $v_0$ to $U'$ which has length greater than $k$, which would prove the lemma. If not, then there exists $k$ such that every path from $v_0$ to $U'$ has length at most $k$. However, this would imply, since there is a path from $v$ to $v_0$, that there exists a $k'$ such that every path from $v$ to $U'$ has length at most $k'$. But this contradicts the fact that the lengths of the paths from $v$ to $U'$ form an increasing sequence.

Proof of Lemma 8.1. First suppose that a spanning copy of $T$ can be found in every infinitely connected graph $H$. It is known that there exist infinitely connected graphs with arbitrarily high girth (see [9, Chapter 8, Exercise 7]); for instance, let $H_0$ be a cycle of length $k$, and for all $i \geq 1$, let $H_i$ be the graph obtained by adding a vertex $x_i$ and internally disjoint paths of length $[k/2]$ from $x_i$ to every vertex in $H_{i-1}$, then let $H = \cup_{i \geq 0} H_i$. So $H$ is infinitely connected and has girth $k$. This proves that $T$ cannot have a cycle because there exists an infinitely connected graph in which every cycle is longer than the shortest cycle in $T$.

Another infinitely connected graph, say $H$, is the infinite blow-up of a one-way infinite path (i.e. replace each vertex with an infinite independent set and each edge with a complete bipartite graph). Clearly every spanning subgraph of $H$ has unbounded radius or infinitely many components. Thus $T$ must be a forest with unbounded radius or infinitely many components.

Next suppose $T$ is a forest with a component of unbounded radius or infinitely many components. If $T$ has infinitely many components $T_1, T_2, \ldots$, we may select for all $i \geq 1$, $t_i \in V(T_i)$ and add the edge $t_it_{i+1}$ for all $i \geq 1$ to get a tree with unbounded radius which contains $T$ as a spanning subgraph. So $T$ has finitely many components $T_1, \ldots, T_k$, at least one of which has unbounded radius. In this case, we may for all $i \in [k - 1]$ add an edge from $t_i \in V(T_i)$ to $t_{i+1} \in V(T_{i+1})$ to get a tree with unbounded radius. Thus we may suppose for the rest of the proof that $T$ is a tree with unbounded radius. By Fact 8.2, there exists a vertex $t_0$ such that either there is an infinite path starting with $t_0$ (in which case we say $T$ is of Type 1), or an increasing star having $t_0$ as the center (in which case we say $T$ is of Type 2). Now starting with $t_0$, fix an enumeration of $V(T) = \{t_0, t_1, t_2 \ldots\}$ such that for all $i \geq 1$, $T[\{t_0, \ldots, t_i\}]$ is connected (in fact, for all $i \geq 1, t_i$ has exactly one neighbor in $\{t_0, \ldots, t_{i-1}\}$). Also fix an enumeration of $V(H) = \{v_0, v_1, v_2, \ldots\}$. We will build an embedding $f$ of $T$ into $H$ recursively, in finite pieces, at each stage ensuring that we add the first vertices of $V(T) \setminus \text{dom } f$ and $V(H) \setminus \text{ran } f$ into the domain and range of $f$ respectively.

Initially, let $f(t_0) = v_0$ (we think of $t_0$ as being the root of the tree and $v_0$ as the embedding of the root in $H$) and let $t_{last} := t_0$ and $v_{last} := v_0$. We now show that Algorithm 1 gives the desired embedding.

Note that if $T$ is of Type 2, then $t_{last} = t_0$ and $v_{last} = v_0$ throughout the process.

To see that $f$ is a well defined surjective embedding of $T$ into $H$, first note that we can always follow lines 4 and 12 of Algorithm 1 since $H$ is infinitely connected and in particular every vertex has infinite degree. Line 5 is always possible since there is either an infinite path starting at $v_0$ or an increasing star having $v_0$ as the center. Line 11 is always possible by the enumeration of $V(T)$. So $f$ is well defined.

We alternate between embedding the vertex $t$ of smallest index from $T$ which has not yet been embedded into an available vertex from $H$ in such that way that the parent $t'$ of $t$ has already been embedded and $f(t)$ is adjacent to $f(t')$, and embedding a path $t_0, t_1, \ldots, t_k$ to a vertex such that $f(t_0)f(t_1)\ldots f(t_k)$ is a path in $H$ and $f(t_k)$ is the vertex of smallest index from $V(H)$ which has yet to be mapped to. So $f$ will be a surjective embedding of $T$.

Now we prove another useful lemma.
Algorithm 1

1: while True do
2:   if $V(H) \setminus \text{ran } f \neq \emptyset$ then
3:     Let next be the smallest index such that $v_{\text{next}} \in V(H) \setminus f$.
4:     Let $P_{\text{next}} \subseteq H$ be a finite path from $v_{\text{next}}$ to $v_{\text{last}}$ which is internally disjoint from
5:     Let $V_{\text{next}}$ be a set of $|V(P_{\text{next}})| - 1$ vertices in $V(T) \setminus \text{dom } f$ such that $\{t_{\text{last}}\} \cup V_{\text{next}}$
6:     Extend $f$ by embedding $V_{\text{next}}$ into $V(P_{\text{next}}) \setminus \{v_{\text{last}}\}$.
7:     if $T$ is of Type 1 then
8:       Set $t_{\text{last}} := f^{-1}(v_{\text{next}})$ and $v_{\text{last}} := v_{\text{next}}$.
9:     if $V(T) \setminus \text{dom } f \neq \emptyset$ then
10:    Let next be the smallest index such that $t_{\text{next}} \in V(T) \setminus \text{dom } f$.
11:   Let back $< next$ be the unique index such that $t_{\text{back}}$ is adjacent to $t_{\text{next}}$.
12:   Embed $t_{\text{next}}$ into an arbitrary vertex in $N_H(f(t_{\text{back}})) \setminus \text{ran } f$.
13: if $T$ is of Type 1 and $t_{\text{back}} = t_{\text{last}}$ then
14:   Set $t_{\text{last}} := t_{\text{next}}$

Lemma 8.3. Let $T$ be a tree with at least one vertex of infinite degree. If $H$ is a graph in which
every vertex has infinite degree, then for all $v \in V(H)$, $H$ contains a copy of $T$ covering $N_H(v)$.

Proof. Let $t_1 \in V(T)$ be a vertex of infinite degree and let $v_1 = v$ from the statement of the
theorem (again we think of $t_1$ as being the root of the tree and $v_1$ as the embedding of the root in
$H$). We will build an embedding $f$ of $T$ into $H$ recursively, in finite pieces, at each stage adding
one more child of every previously embedded $t \in T$ (unless all children have been embedded
already). The embedding strategy is very similar to that in the proof of Lemma 8.1. Initially,
let $f(t_1) = v_1$. We will use the following Algorithm 2.

Algorithm 2

1: while True do
2:    for $t \in \text{dom } f$ do
3:      if $S := N_T(t) \setminus \text{dom } f$ is non-empty then
4:        Embed $\min(S)$ into $\min(N_H(f(t)) \setminus \text{ran } f)$.

First, note that we can always follow line 4 of Algorithm 1 since every vertex in $H$ has infinite
degree. Let $f : V(T) \to V(H)$ be the function produced by Algorithm 2. We need to prove that
$f$ is well-defined, an embedding of $T$ and that $N_H(v) \subseteq \text{dom } f$.

Since we always embed the smallest not yet embedded neighbor of every previously embedded
$t \in V(T)$ in line 4, every other vertex will be embedded eventually as well. Therefore, $f$ is well
defined. Furthermore, by construction of $f$, it defines a proper embedding (whenever a new vertex $t \in T$ is embedded, its parent $t'$ is already embedded and we make sure that $f(t)$ is
adjacent to $f(t')$). Finally note that we are infinitely often in line 4 when $t = t_1$ since $N_T(t_1)$
is infinite. Since we always choose the smallest available vertex in $N_H(v) \setminus \text{ran } f$, it follows that
$N_H(v) \subseteq \text{ran } f$. \qed
8.2 Upper density of monochromatic trees

In this section we will deduce Theorem 1.13 from Lemma 8.1, Lemma 8.3, and the following two lemmas.

**Lemma 8.4.** For any 2-coloring of $K_\mathbb{N}$, there are sets $R$ and $S$ such that

(i) $R \cup S$ is cofinite,

(ii) if $R$ is infinite, then it is infinitely connected in red, and

(iii) if $S$ is infinite, then it is infinitely connected in one of the colors.

**Lemma 8.5.** Let $H$ be a 2-colored $K_\mathbb{N}$. There exists a set $A \subseteq \mathbb{N}$, a vertex $v \in A$, and a color $c$ such that every vertex in $F := H[c][A]$ has infinite degree and $\overline{d}(N_F(v)) \geq 1/2$.

It is now easy to prove Theorem 1.13.

**Proof of Theorem 1.13.** It clearly suffices to prove the result for trees, so let $T$ be an infinite tree and suppose the edges of $K_\mathbb{N}$ are colored with two colors. If $T$ does not have an infinite path, it must have at least one vertex of infinite degree and therefore the theorem follows immediately from Lemmas 8.3 and 8.5. So suppose $T$ has an infinite path. By Lemma 8.4, there is an infinite set $A$ with $\overline{d}(A) \geq 1/2$ and a color $c$, so that the induced subgraph on $A$ is infinitely connected in $c$. By Lemma 8.1, there is a monochromatic copy of $T$ spanning $A$ and we are done. \qed

It remains to prove the two lemmas.

**Proof of Lemma 8.4.** Fix a 2-coloring of $K_\mathbb{N}$. We define a sequence of sets $R_\alpha, S_\alpha$, for all ordinals $\alpha$, as follows. Let $S_0 = \mathbb{N}$. For each $\alpha$, we define $R_\alpha$ to be the set of vertices in $S_\alpha$ whose blue neighborhood has finite intersection with $S_\alpha$, and we set $S_{\alpha+1} = S_\alpha \setminus R_\alpha$. If $\lambda$ is a limit ordinal, then we define $S_\lambda$ to be the intersection of the sets $S_\alpha$, for $\alpha < \lambda$.

Note that the sets $R_\alpha$ are pairwise disjoint, and hence there is some countable ordinal $\gamma$ such that $R_\alpha = \emptyset$ for all $\alpha \geq \gamma$. Let $\gamma^*$ be the minimal ordinal such that $R_{\gamma^*}$ is finite; it follows then that $R_\beta = \emptyset$ for all $\beta > \gamma^*$. Set

$$ R = \bigcup \{ R_\alpha \mid \alpha < \gamma^* \}. $$

(Note that $\gamma^*$ may be 0, in which case $R = \emptyset$.)

Suppose that $R$ is infinite. Then $\gamma^* > 0$ and $R_\alpha$ is infinite for all $\alpha < \gamma^*$. Let $u, v \in R$ with $u \in R_\alpha$ and $v \in R_\beta$ for some $\alpha \leq \beta < \gamma^*$. It follows that the red neighborhoods of both $u$ and $v$ are cofinite in $R_\beta$. Since $R_\beta$ is infinite, this implies that there is a red path of length 2 connecting $u$ and $v$, even after removing a finite set of vertices. Hence $R$ is infinitely connected in red.

Set $S = S_{\gamma^*+1}$. Then $R \cup S = \mathbb{N} \setminus R_{\gamma^*}$, so $R \cup S$ is cofinite. Moreover, since $R_{\gamma^*+1} = \emptyset$, it follows that for every $v \in S$, the blue neighborhood of $v$ has infinite intersection with $S$. Now suppose that $S$ is not infinitely connected in blue. Then there is a finite set $F \subseteq S$ and a partition $S \setminus F = X \cup Y$ such that $X$ and $Y$ are both nonempty, and every edge between $X$ and $Y$ is red. Note that $X$ and $Y$ must both be infinite, since if $x_0 \in X$ and $y_0 \in Y$ then $X \cup F$ and $Y \cup F$ must contain the blue neighborhoods of $x_0$ and $y_0$ (both of which are infinite) respectively. But then the red graph restricted to $X \cup Y = S \setminus F$ is infinitely connected. \qed

**Proof of Lemma 8.5.** Fix a 2-coloring of $K_\mathbb{N}$. Similarly as in the proof of Lemma 8.4 we will construct sets $R_\alpha, B_\alpha, S_\alpha$ for all ordinals $\alpha$ with the following properties.
(i) There is a unique ordinal $\gamma^*$ such that $R_\alpha \cup B_\alpha$ is infinite for all $\alpha < \gamma^*$, finite for $\alpha = \gamma^*$ and empty for all $\alpha > \gamma^*$. We denote $R = \bigcup_{\alpha < \gamma^*} R_\alpha$ and $B = \bigcup_{\alpha < \gamma^*} B_\alpha$.

(ii) $S_\alpha = S_\alpha'$ for all ordinals $\alpha, \alpha' > \gamma^*$. We denote $S = S_{\gamma^*+1}$.

(iii) $R, B, S$ are pairwise disjoint and $R \cup B \cup S$ is cofinite.

(iv) If $v \in R_\gamma$ for some ordinal $\gamma$, then $v$ has finitely many blue neighbors in $S \cup \bigcup_{\alpha \geq \gamma} R_\alpha$.

(v) If $v \in B_\gamma$ for some ordinal $\gamma$, then $v$ has finitely many red neighbors in $S \cup \bigcup_{\alpha \geq \gamma} B_\alpha$.

(vi) Every $v \in S$ has infinitely many neighbors of both colors in $S$.

If $R \cup B$ is empty, then let $A = S$ and choose an arbitrary vertex $v \in S$. Since $A$ is cofinite in $N$, either the blue or the red neighborhood of $v$ in $A$ has upper density at least $1/2$. Since every vertex in $A$ has infinite degree in both colors, we are done.

If $R \cup B$ is non-empty it must be infinite (by the way $R$ and $B$ are defined). Since $R \cup B \cup S = (R \cup S) \cup (B \cup S)$ is cofinite, we may assume without loss of generality that $R$ is non-empty and $d(R \cup S) \geq 1/2$. Let $A = R \cup S$ and let $v \in R_0$ be arbitrary (if $R$ is non-empty, then $R_0$ must be infinite). Clearly every vertex in $A$ has infinite red degree in $A$ and since $v$ has only finitely many blue neighbors in $A$, we are done. \qed

8.3 Ramsey-cofinite forests

In this section we will prove Theorem 1.14.

We already know from Examples 2.7 and 2.8 that if $T$ is weakly expanding or has a finite dominating set, then $T$ is not Ramsey-lower-dense (and thus $T$ is not Ramsey-confinite). So all that remains to prove Theorem 1.14(ii) is to show that every forest in $T^*$ is not Ramsey-lower-dense.

**Proposition 8.6.** If $T \in T^*$, then $T$ is not Ramsey-lower-dense.

**Proof.** Let $T \in T^*$, let $t$ be the vertex of infinite degree in $T$, and let $d$ be the maximum degree of $T - \{t\}$ as guaranteed by the definition.

We begin by partitioning $N$ into intervals $A_0, A_1, A_2, \ldots$ as follows: Let $a_0 = 1$ and for all $i \geq 1$, let $a_i = i \cdot d \cdot a_{i-1}$. Then for all $i \geq 0$, let $A_i = [a_i, a_{i+1})$. Now for all $r \in \{0, 1, 2, 3\}$, let $V_r = A_r \cup A_{r+1} \cup A_{r+2} \cup \ldots$.

Now color all edges which are inside $V_0$ or $V_1$ red, and all edges inside $V_2$ or $V_3$ blue. Color all edges between $V_0$ and $V_1$ blue and all edges between $V_2$ and $V_3$ red. Finally, for all $i \in \{0, 1\}$, $j \in \{2, 3\}$ we color the complete bipartite graphs $K(V_i, V_j)$ according to Fig. 5 as follows: Suppose first that there is a red arrow from $V_i$ to $V_j$ (and thus a blue arrow from $V_j$ to $V_i$). Let $A_s \subseteq V_i$ and $A_t \subseteq V_j$. If $s < t$, then color all edges between $A_s$ and $A_t$ red. If $t < s$, then color all edges between $A_t$ and $A_s$ blue. If there is a blue arrow from $V_i$ to $V_j$ (and thus a red arrow from $V_j$ to $V_i$), we do the opposite.

Assume there is a monochromatic copy $T'$ of $T$, let $f$ be the corresponding embedding, and let $V_i' = f(T) \cap V_i$ for all $i \in \{0, 1, 2, 3\}$. By symmetry, we may assume that $t$ is embedded in $V_0$; and let $v = f(t)$.

Suppose first that $T$ is embedded in the blue subgraph. Since $v$ has finitely many blue neighbors in $V_0$, all but finitely many vertices of $V_0'$ are neighbors of vertices in $V_0'$ in $f(T)$. So for all sufficiently large $q$ (i.e. large enough so that $N(v) \cap V_2 \subseteq A_{4q-2}$), we have

$$|V_2' \cap A_{4q+2}| \leq d|V_1' \cap (A_1 \cup A_5 \cup \cdots \cup A_{4q+1})| \leq d \cdot a_{4q+1} \leq \frac{a_{4q+2}}{4q+2},$$
Figure 5: The shaded areas denote cliques of the respective colors and a blue/solid (red/dashed) arrow from one part to another indicates that vertices in the first part have cofinitely many blue (red) neighbors in the second part. On the right we have an example of the relevant edges in the case where we are embedding a blue of $T$ with $t$ in $V_0$.

and thus

$$d(\text{ran } f) \leq \lim_{q \to \infty} \frac{|\text{ran } f \cap (A_1 \cap \ldots \cap A_{4q+2})|}{(a_1 + \ldots + a_{4q+2})} \leq \lim_{q \to \infty} \frac{(a_1 + \ldots + a_{4q+1} + \frac{a_{4q+2}}{4q+2})}{(a_1 + \ldots + a_{4q+2})} = 0.$$ 

Suppose next that $T$ is embedded in the red subgraph. Since $v$ has finitely many red neighbors in $V_1$, all but finitely many vertices of $V_3'$ are neighbors of vertices in $V_2'$ in $f(T)$. So for all sufficiently large $q$, we have

$$|V_3' \cap A_{4q+3}| \leq d|V_2' \cap (A_2 \cup A_6 \cup \ldots A_{4q+2})| \leq d \cdot a_{4q+2} \leq \frac{a_{4q+3}}{4q+3},$$

and thus $d(f(T)) = 0$. 

We now turn to the part (i) of Theorem 1.14; that is, if $T$ is a forest which is strongly contracting, has no finite dominating set, and $T \notin T^*$, then $T$ is Ramsey-cofinite. We begin with a lemma which helps us to embed forests which are strongly contracting into graphs with infinitely many vertices of cofinite degree. Here, it is not important that we are embedding forests and we will state and prove the lemma more generally. Recall that a graph $F$ is strongly contracting if there exists $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}$ there exists an independent set $A$ in $F$ with $|A| \geq \ell$ such that $|N(A)| \leq k$, and that a forest is strongly contracting if and only if it has finitely many components and unbounded leaf degree.

**Lemma 8.7.** Let $F$ be a graph. A cofinite copy of $F$ can be found in every graph $H$ having infinitely many vertices of cofinite degree if and only if $F$ is strongly contracting.

We delay the proof for now, but note that Lemma 8.7 allows us to focus on colorings in which all but finitely many vertices have infinite degree in both colors. Given such a coloring, we will need to separate into a few cases depending on the structure of $T$.

**Fact 8.8.** Let $T$ be an infinite tree with unbounded leaf degree and no finite dominating set. Then $T \in T^*$, or $T$ has unbounded radius, or there exists a vertex $t \in V(T)$ such that $t$ is adjacent to infinitely many non-leaves and

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(i) there are infinitely many paths of length 3 starting at $t$ which are pairwise vertex-disjoint apart from $t$, or

(ii) there is a vertex in $T - \{t\}$ of infinite degree, or

(iii) the neighbors of $t$ have unbounded degrees.

Proof. Suppose that $T$ has unbounded leaf degree, no finite dominating set, bounded radius and $T \not\in \mathcal{T}^*$. We first show that there is a vertex $t \in V(T)$ which is adjacent to infinitely many non-leaves. Let $s_0 \in V(T)$ and think of it as the root. If $s_0$ is adjacent to infinitely many non-leaves, we are done; so assume it is adjacent to finitely many non-leaves $S_1$. Since $T$ has no finite dominating set, $V(T) \setminus N(s_0)$ is infinite. In particular, some $s_1 \in S_1$ has an infinite subtree. If $s_1$ is adjacent to infinitely many non-leaves, we are done; so assume it is adjacent to finitely many non-leaves $S_2$ other than $s_1$. Since there is no finite dominating set, some $s_2 \in S_2$ has an infinite subtree. We keep iterating until we find a vertex $s_i$ adjacent to infinitely many non-leaves. Since $T$ has bounded radius, this process must finish eventually.

Let $t \in V(T)$ which is adjacent to infinitely many non-leaves and let $S$ be the non-leaves adjacent to $t$. Assume that there are only finitely many paths of length 3 starting at $t$ which are pairwise vertex-disjoint apart from $t$ and that there is no $t' \in V(T) \setminus \{t\}$ of infinite degree (otherwise we are done). Let $S' \subseteq S$ be those vertices whose only children are leaves and let $S'' = S \setminus S'$. By the assumption, we have that $S''$ is finite. We claim that for each $i \in \mathbb{N}$, there is a vertex $t' \in S'$ of degree at least $i$. Assume for contradiction this is not the case and let $d := \max_{s \in S'} \deg(s)$. Note that, since $T$ has bounded radius and no vertex of infinite degree other than $t$, there are only finitely many vertices which are successors of vertices in $S''$. Thus, since $T$ has unbounded leaf degree, $t$ must be adjacent to infinitely many leaves. Therefore $T \in \mathcal{T}^*$, a contradiction.

If $T$ is a tree of unbounded radius (and one of the colors induces an infinitely connected graph), we can use Lemma 8.1 to get a cofinite embedding of $T$. So the main difficulty will be dealing with trees of bounded radius. We already know that trees in $\mathcal{T}^*$ are not Ramsey cofinite and the following lemma deals with the remaining case.

**Lemma 8.9.** Let $H$ be a 2-edge-colored $K_N$ in which every vertex has infinite degree in both colors. Let $T$ be a tree of bounded radius containing a vertex $t$ such that $t$ is adjacent to infinitely many non-leaves and

(i) there are infinitely many paths of length 3 starting at $t$ which are pairwise vertex-disjoint apart from $t$, or

(ii) there is a vertex in $T - \{t\}$ of infinite degree, or

(iii) the neighbors of $t$ have unbounded degrees.

Then there is a cofinite monochromatic copy of $T$ in $H$.

Given Lemmas 8.7 and 8.9 we now prove Theorem 1.14.

**Proof of Theorem 1.14.** Part (ii) follows from Examples 2.7 and 2.8 and Proposition 8.6.

So let $T$ be a forest which is strongly contracting, has no finite dominating set, and $T \not\in \mathcal{T}^*$. Let $H$ be a 2-edge-colored $K_N$, and assume the colors are red and blue. If there are infinitely many vertices of cofinite red degree or infinitely many vertices of cofinite blue degree, then we are done by Lemma 8.7; so (by removing finitely many vertices) we may assume every vertex in $H$ has infinite red degree and infinite blue degree.
First, suppose that $T$ has a component of unbounded radius or infinitely many components. If the blue subgraph $H_B$ is infinitely connected, we can find a monochromatic spanning copy of $T$ in $H_B$ by Lemma 8.1. If $H_B$ is not infinitely connected, there exists a finite set $X$ such that $H_B - X$ is not connected. So there exists a partition \{Y, Z\} of $\mathbb{N} - X$ such that all edges between $Y$ and $Z$ are red. Note that for all $v \in Y$, $N_B(v) \subseteq Y \cup X$ and for all $v \in Z$, $N_B(v) \subseteq Z \cup X$. Since all vertices have infinite blue degree and $X$ is finite, this implies that both $Y$ and $Z$ are infinite. So $Y \cup Z$ is cofinite and $H_R[Y, Z]$ induces a red copy of $K_{N, N}$. Since $T$ has no finite dominating set, both parts of its bipartition are infinite, and we can surjectively embed $T$ into $H_R[Y, Z]$.

Finally, suppose that $T$ has finitely many components $T_1, \ldots, T_k$ all of which have bounded radius. For all $i \in [k]$, let $t_i \in V(T_i)$ and for all $2 \leq i \leq k$, add the edge $t_1t_k$ to get a tree $T'$ which has bounded radius, is strongly contracting, has no finite dominating set, $T' \not\subseteq T^*$, and $T'$ contains $T$ as a spanning subgraph. Therefore, by Fact 8.8, $T'$ satisfies the hypotheses of Lemma 8.9 and thus we can find a monochromatic copy of $T'$ (and consequently $T$) which spans cofinitely many vertices of $H$.

It remains to prove Lemmas 8.7 and 8.9.

**Proof of Lemma 8.7.** Note that in Example 2.7, both the red graph and the blue graph have the property that there are infinitely many vertices of cofinite degree. So if a cofinite copy of $F$ can be found in every graph $H$ having infinitely many vertices of cofinite degree, then $F$ is not weakly expanding; i.e. $F$ is strongly contracting.

Now suppose $F$ is strongly contracting and let $k$ be given as in the definition. We claim that there exists an infinite independent set $A$, a set $B \subseteq V(F) \setminus A$ such that $V(F) \setminus (A \cup B)$ is infinite, a partition \{A_0, A_1, \ldots\} of $A$ into finite sets of increasing size, and a cover \{B_0, B_1, \ldots\} of $B$ with sets of order $k$ such that $N(A_i) \subseteq B_i$ for all $i \in \mathbb{N}$. Indeed, since $F$ is strongly contracting, there exists an independent set $A_0$ such that $|A_0| \geq 1$ and $|N(A_0)| \leq k$; choose $B_0$ so that $N(A_0) \subseteq B_0$. Now suppose we have chosen disjoint independent sets $A_0, \ldots, A_{n-1}$ and sets $B_0, \ldots, B_{n-1}$ such that $N(A_i) \subseteq B_i$ for all $0 \leq i \leq n-1$, and $A = \bigcup_{i=0}^{n-1} A_i$ is disjoint from $B = \bigcup_{i=0}^{n-1} B_i$. Since $F$ is strongly contracting, there exists an independent set $S$ of size at least $|A \cup B| + n$ with at most $k$ neighbors. Let $A_n \subseteq S \setminus (A \cup B)$ with $|A_n| = n$ and let $B_n \subseteq V(F)$ with $|B_n| = k$ such that $N(A_n) \subseteq B_n$. Note that $B_n \cap A = \emptyset$ because, by the construction, $N(A) \subseteq B$. This gives an infinite independent set $A := \bigcup_{i \geq 0} A_i$ and a set $B := \bigcup_{i \geq 0} B_i$ such that $B \subseteq V(F) \setminus A$. Finally, by passing to a subsequence of $A_0, A_1, \ldots$ if necessary, we can ensure that $V(F) \setminus (A \cup B)$ is infinite.

Since there are infinitely many vertices $X$ in $H$ with cofinite degree, we may choose an infinite clique $K \subseteq H$ such that $V(K) \subseteq X$. If $V(H) \setminus V(K)$ is finite, then we are done (since it is clear that we can surjectively embed $F$ into the clique $K$), so suppose not. Let $K' \subseteq K$ such that $V(K')$ and $V(K) \setminus V(K')$ are both infinite. Let $y_1, y_2, \ldots$ be an enumeration of $Y := \mathbb{N} \setminus V(K)$.

**Claim 8.10.** There is a partial embedding $f$ of $F$ into $H$ so that $Y \setminus \text{ran } f$ is finite, $N_F(f^{-1}(Y \setminus \text{ran } f)) \subseteq \text{dom } f$ and both $V(F) \setminus \text{dom } f$ and $V(H) \setminus \text{ran } f$ are infinite.

**Proof.** For all $x \in V(K')$ there exists $\phi(x) \in \mathbb{N}$ such that $y_{\phi(x) - 1} \not\in N(x)$ and $y_j \in N(x)$ for all $j \geq \phi(x)$ (if $Y \subseteq N(x)$, then $\phi(x) = 1$). For all $i \geq 1$, let $K^i = \{x \in K' : \phi(x) = i\}$. Note that if $|K^i| = \infty$ for some $i \in \mathbb{N}$, then we have a complete bipartite graph from $K^i$ to a cofinite subset $Y'$ of $Y$ and we can embed $A$ into $Y'$ and $N(A)$ into the clique $K^i$, thus getting a partial embedding $f$ as desired. Hence, we may assume $K^i$ is finite for all $i$ and thus there
is an enumeration \(x_1, x_2, \ldots \) of \(V(K')\) such that \(\phi(x_i) \leq \phi(x_j)\) whenever \(i \leq j\). For all positive integers \(q\), let \(m_q = \phi(x_{k(q+1)}) - \phi(x_{kq})\). Note that \(m_q \geq 0\) for all \(i\). Let \(n_1, n_2, \ldots\) be a subsequence such that \(|A_{n_i}| \geq m_i\) for all \(i \geq 1\), which is possible since \(|A_0| < |A_1| < |A_2| < \ldots\).

We will surjectively embed \(A_{n_1} \cup A_{n_2} \cup \ldots\) to \(Y \setminus \{y_1, \ldots, y_{\phi(k)-1}\}\) and \(B_{n_1} \cup B_{n_2} \cup \ldots\) to \(K'\). We start by embedding the vertices of \(A_{n_1}\) to \(\{y_{\phi(k)}, \ldots, y_{\phi(k)+m_1}\}\) and embedding the vertices from \(N(A_{n_1})\) to a subset of \(\{x_1, \ldots, x_k\}\). For all \(i \geq 2\), suppose that we have embedded \(A_{n_1}, \ldots, A_{n_{i-1}}\) to \(\{y_{\phi(k)}, \ldots, y_{\phi(k)+m_{i-1}+\ldots+m_1}\}\) and embedded \(N(A_{n_1} \cup \ldots \cup A_{n_{i-1}})\) to a subset of \(\{x_1, \ldots, x_{(i-1)k}\}\). We now embed \(A_{n_i}\) to \(\{y_{\phi(k)+m_{i-1}+\ldots+m_1}, \ldots, y_{\phi(k)+m_1+\ldots+m_1}\}\) and since \(|N(A_{n_i})| \leq k\) and \(|\{x_{(i-1)k+1}, \ldots, x_{ik}\}| = k\), we can embed \(N(A_{n_i})\) into a subset of \(\{x_1, \ldots, x_{ik}\}\). This defines a partial embedding \(f\) with the desired properties.

Since \(K - \text{ran } f\) is infinite and \(F - \text{dom } f\) is infinite, since \(K\) is a clique, and since we have already embedded all neighbors of vertices \(u \in F\) with \(f(u) \not\in K\), we can extend \(f\) to an embedding of \(F\) into \(H\) which covers \(K\), finishing the proof.

**Proof of Lemma 8.9.** Let \(T\) and \(t\) be as in the statement and let \(H\) be a 2-edge-colored \(K_3\) which is as in the statement. Note that \(\text{deg}(t) = \infty\) and there are infinitely many paths of length 2 starting at \(t\) which are vertex disjoint apart from \(t\) \((*)\). Let \(v \in \mathbb{N}\) be an arbitrary vertex. Let \(B\) be the set of blue neighbors of \(v\) and let \(R\) be the set of red neighbors of \(v\). Furthermore, let \(B'\) be the set of vertices in \(B\) with finitely many red neighbors in \(R\) and let \(R'\) be the set of vertices in \(R\) with finitely many blue neighbors in \(B\). Let \(R'' = R \setminus R'\) and \(B'' = B \setminus B'\).

**Case 1** \((B'\) or \(R'\) is finite.)

Suppose without loss of generality that \(R'\) is finite. We will find an embedding \(f\) of \(T\) into the blue subgraph \(H_B\) covering the cofinite set \(\{v\} \cup B \cup R'\). We build \(f\) iteratively in finite pieces maintaining a partial embedding of \(T\) whose domain is connected. Note that, by keeping \(\text{dom } f\) connected, we ensure that every not yet embedded vertex is adjacent to at most one vertex in \(\text{dom } f\). Initially, we set \(f(t) = v\). Then we repeatedly follow the following two steps.

**Step 1.** Let \(s \in V(T) \setminus \text{dom } f\) be the smallest not yet embedded vertex. Since every vertex has infinite blue degree, it is easy to extend \(f\) so that \(\text{dom } f\) remains connected and \(s \in \text{dom } f\) (by adding a path to \(s\)).

**Step 2.** Let \(u \in B \cup R'' \setminus \text{ran } f\) be the smallest not yet covered vertex. If \(u \in B\) we can simply choose a not-yet embedded neighbor of \(t\) and embed \(u\) into it (since \(t\) has infinite degree). If \(u \in R''\), it has infinitely many blue neighbors in \(B\) (by definition of \(R'\)) and therefore there is a blue path \(vru\) of length 2 for some not yet covered vertex \(x \in B\). By \((*)\) we can extend \(f\) to cover \(x\) and \(u\) so that \(\text{dom } f\) remains connected.

Routinely, this defines an embedding of \(T\) into \(H_B\) covering \(B \cup R'\).

**Case 2** \((B'\) and \(R'\) are infinite.)

We further split into subcases depending on the structure of \(T\).

**Case 2.1** \((T\) has infinitely many paths of length 3 starting at \(t\) which are pairwise vertex-disjoint apart from \(t\)): We will find an embedding \(f\) of \(T\) into the blue subgraph \(H_B\) covering \(N\). We build \(f\) iteratively in finite pieces maintaining a partial embedding of \(T\) whose domain is connected. Initially, we set \(f(t) = v\). Then we repeat the following two steps.

**Step 1.** We extend \(f\) to include the smallest not yet embedded vertex as above.

**Step 2.** Let \(u \in \mathbb{N} \setminus \text{ran } f\) be the smallest not yet covered vertex. If \(u \in B \cup R''\) we proceed as above; so suppose \(u \in R'\). Note that \(u\) (as every other vertex) has infinitely many blue neighbors, and by definition of \(R'\) only finitely many of those can lie outside \(R\). Furthermore,
Let vertices in \( \{N_i\} \) extend in case). Set \( f \) and remove the finite set of vertices in \( t \geq j \). Let \( t \) to leaves of distance at least 3.

Furthermore, we may assume we are not in Case 2.1 and thus cofinitely many vertices of \( S \) in \( v \) so that \( P \setminus \{v\} \cap \text{ran } f = \emptyset \). By the case assumption we can extend \( f \) to cover this path.

Routinely, the resulting function \( f \) is an embedding of \( T \) into \( \mathbb{N} \). Observe that the only difference to the previous case is how we deal with vertices in \( R' \). This will be similar in the following cases and we therefore skip some details.

**Case 2.2** (\( T \) has a vertex \( t' \) in \( V(T) \setminus \{t\} \) of infinite degree): Given some integer \( d \geq 1 \), we say that a path \( P \) is \( d \)-good in blue (red) if it has length \( d \), starts at \( v \) and ends at some \( v' \in \mathbb{N} \setminus \{v\} \), is monochromatic in blue (red) and \( v' \) has only finitely many red (blue) neighbors in \( R' (B') \). We will first show that (i) for every positive integer \( d \neq 2 \), there is a red \( d \)-good path and a blue \( d \)-good path, and (ii) there is a red 2-good path or a blue 2-good path.

If \( d = 1 \), any vertex in \( B' \) forms a blue \( d \)-good path with \( v \). Since any two vertices in \( B' \) have infinitely many common blue neighbors in \( R \), we can extend this path to a 3-good path. We can proceed like this to get a \( d \)-good path in blue for any odd \( d \). If \( d \geq 4 \) is even, we start by building a blue path of length 2 to some \( u \in R' \) and take some \( v' \in B' \) not yet in the path. Since \( u \) has infinitely many blue neighbors in \( R \) and every vertex in \( B' \) has only finitely many red neighbors in \( R \), \( u \) and \( v' \) have a common blue neighbor not yet in the path, giving us a \( d \)-good path in blue. We can extend this path now as before to any even length \( d \geq 4 \). We can proceed similarly for red paths. Finally suppose \( d = 2 \). If there are \( u_1 \in R \) and \( u_2 \in R' \) such that \( u_1u_2 \) is red, then \( vu_1u_2 \) is 2-good in red and we are done. Otherwise every \( u \in R \) has only blue neighbors in \( R' \). We can thus form a blue path from \( v \) to \( R \) of length 2, which is 2-good in blue.

Let now \( d \) be the distance from \( t \) to \( t' \) in \( T \) and let \( P \) be a \( d \)-good path (say in blue). Embed \( t \) into \( v \) and the unique path from \( t \) to \( t' \) into \( P \) (and call this partial embedding \( f \)). Let \( v' = f(t') \) and remove the finite set of vertices in \( R' \) which is not in the blue neighborhood of \( v' \). We then extend \( f \) in finite pieces exactly as in the previous case apart from when \( u \in R' \), where we simply embed an available neighbor of \( t' \) into \( u \).

**Case 2.3** (for all \( i \in \mathbb{N} \), there is a vertex \( t' \in N_T(t) \) of degree at least \( i \)): We may assume we are not in Case 2.2 and thus there is an infinite set \( S \subseteq N_T(t) \) of vertices with distinct degrees. Furthermore, we may assume we are not in Case 2.1 and thus cofinitely many vertices of \( T \) have distance at most 2 to \( t \). Let \( T_0 \) be the finite subtree rooted at \( t \) which consists of all paths from \( t \) to leaves of distance at least 3.

Let \( u_1, u_2, \ldots \) be an enumeration of \( N_T(t) \). Let \( y_1, y_2, \ldots \) be an enumeration of \( R \). Let \( B_1 \subseteq B' \) such that \( B_1 \) is infinite and \( B' \setminus B_1 \) is infinite, then set \( B_2 = B \setminus B_1 \).

For all \( x \in B_1 \) there exists \( \phi(x) \in \mathbb{N} \) such that \( y_{\phi(x)-1} \notin N_B(x) \) and \( y_j \in N_B(x) \) for all \( j \geq \phi(x) \) (if \( Y \subseteq N_B(x) \), then \( \phi(x) = 1 \)). For all \( i \geq 1 \), let \( X^i = \{x \in B_1 : \phi(x) = i\} \). If \( |X^i| = \infty \) for some \( i \in \mathbb{N} \), then reset \( B_1 := X^i \) and \( B_2 := B \setminus B_1 \), enumerate \( B_1 \) as \( x_1, x_2, \ldots \). Otherwise \( X^i \) is finite for all \( i \) and thus there is a natural enumeration \( x_1, x_2, \ldots \) of \( B_1 \) such that \( \phi(x_i) \leq \phi(x_j) \) whenever \( i \leq j \).

Initially we set \( f(t) = v \) and we embed \( T_0 \) in \( H_B \) using the fact that every vertex in \( H \) has infinite blue degree. Now every vertex in \( V(T) \setminus \text{dom } f \) has distance at most 2 to \( r \). Now we repeat the following two steps.

Step 1. Let \( x_i \) and \( x_j \) (with \( i < j \)) be the two smallest vertices in \( B_1 \setminus \text{ran } f \) and let \( u_{n_i} \in N_T(t) \setminus \text{dom } f \) such that \( u_{n_i} \) has at least \( \phi(x_j) - \phi(x_i) \) children in \( T \) (which is possible by the case). Set \( f(u_{n_i}) = x_i \) and embed the (finitely many) vertices in \( N_T(u_{n_i}) \setminus \{t\} \) to the smallest vertices in \( \{y_{\phi(x_i)}, y_{\phi(x_i)+1}, \ldots \} \setminus \text{ran } f \).
Step 2. Injectively embed all vertices from \( \{ u_1, u_2, \ldots, u_{n_i+1} \} \setminus \text{dom } f \) (which is non-empty since \( u_{n_i+1} \notin \text{dom } f \)) to the smallest vertices in \( B_2 \setminus \text{ran } f \). Now, using the fact that every vertex in \( H \) has infinite blue degree, iteratively embed the children of each vertex \( u_{n_i} \in \{ u_{n_i-1+1}, \ldots, u_{n_i-1} \} \) anywhere in \( N_B(\text{ran } f) \). Now move to Step 1 (and notice that \( x_j \) will become the smallest vertex in \( B_1 \setminus \text{ran } f \)).

The resulting function \( f \) is an embedding of \( T \) into \( H \) covering a cofinite set.

8.4 General graphs

In the previous section, we completely characterize forests which are Ramsey-cofinite. We know that if a graph \( F \) is Ramsey-cofinite, then \( F \) is bipartite, strongly contracting, and has no finite dominating set (by Examples 2.7 to 2.9). On the other hand, from the proof in the previous section we know that if \( G \) is bipartite, strongly contracting, and has no finite dominating set and we are given a 2-coloring of \( K_N \) such that one of the colors is not infinitely connected, then there is a cofinite monochromatic copy of \( G \). So this raises the question of completely characterizing all graphs which are Ramsey-cofinite. However, given the information from the previous sections, we can narrow this down to a much more specific question.

**Problem 8.11.** Characterize the graphs \( G \) which are bipartite, strongly contracting, and have no finite dominating set such that there exists a cofinite monochromatic copy in every 2-coloring of \( K_N \) in which both colors are infinitely connected.

The following is an easy to state sufficient condition (we are aware of a more general sufficient condition which contains Theorem 1.14(i), but as we don’t believe the more general condition is necessary, we go for simplicity instead).

**Theorem 8.12.** If \( G \) is bipartite, strongly contracting, and has arbitrarily long paths whose internal vertices have degree 2, then \( G \) is Ramsey-cofinite.

This follows because if we are given a 2-coloring of \( K_N \) in which both colors are infinitely connected, then at least one of those colors contains an infinite clique. So we can use the following lemma.

**Lemma 8.13.** Let \( F \) be a connected graph. A spanning copy of \( F \) can be found in every infinitely connected graph \( H \) with an infinite clique if \( F \) has arbitrarily long paths whose internal vertices have degree 2.

**Proof.** Assume that \( F \) has arbitrarily long induced paths and that \( H \) is infinitely connected with an infinite clique \( K \subseteq H \). We will construct an embedding \( f \) of \( F \) into \( H \) iteratively in finite pieces. For each \( i \in \mathbb{N} \), we will do the following two steps: First, let \( t \in V(T) \setminus \text{dom } f \) be the smallest not-yet embedded vertex and embed it into an arbitrary vertex \( u \in V(K) \setminus \text{ran } f \). Second, let \( v \in V(H) \setminus \text{ran } f \) be the smallest not-yet covered vertex and let \( P \) be a finite path in \( H \) which starts and ends in \( K \), contains \( v \) and avoids \( \text{ran } f \) (such a path exist since \( H \) is infinitely connected). Let \( P' \) be an induced path in \( T \) of the same length as \( P \) which avoids \( \text{dom } f \) (such a path exists since \( T \) has arbitrary long induced paths). Extend \( f \) by embedding \( P' \) into \( P \). Note that all neighbors of the internal vertices of \( P' \) will be embedded, and the endpoints of \( P' \) are in \( K \). Therefore, we maintain a partial embedding throughout the process. Since we eventually embed every \( t \in V(T) \), the resulting function \( f \) is an embedding of \( T \) into \( H \). Since we eventually cover every \( v \in V(H) \), this embedding is surjective.
9 Ramseyness of coideals

9.1 Ideals and coideals

An ideal on a set $X$ is a collection $I$ of subsets of $X$ such that (1) for any $B \in I$ and $A \subseteq B$, we have $A \in I$, and (2) for any $A, B \in I$, we have $A \cup B \in I$. We call an ideal $I$ on $X$ proper if $X \not\in I$. If $I$ is an ideal, then we write $I^+$ for its complement $P(X) \setminus I$, and we call $I^+$ a coideal.

In this section we will primarily be concerned with ideals on countable sets, and in particular ideals on $\mathbb{N}$. Some commonly used examples of ideals on $\mathbb{N}$ are

(i) $\text{fin} = \{ A \subseteq \mathbb{N} \mid |A| < \infty \}$,
(ii) $Z_0 = \{ A \subseteq \mathbb{N} \mid d(A) = 0 \}$,
(iii) $I_{1/n} = \{ A \subseteq \mathbb{N} \mid \sum_{n \in A} 1/n < \infty \}$.

In general, we view an ideal $I$ on $X$ as a way of measuring which subsets of $X$ are “small”. In this light, we only consider an ideal $I$ to be nontrivial if $I$ is proper and contains the finite subsets of $X$, since at the very least, the finite subsets of $X$ should be “small”, and $X$ itself should not be “small”.

Let $G$ be a graph and $I^+$ be a coideal on $\mathbb{N}$. We say that $G$ is $I^+$-Ramsey if, for every finite coloring of $K_{\mathbb{N}}$, there is a monochromatic copy of $G$ whose vertex set is in $I^+$.

The first thing to note is that we may reexpress one of the central notions of this paper using the above terminology; namely, a graph $G$ is Ramsey-dense if and only if $G$ is $Z_0^+$-Ramsey. For another example, Ramsey’s theorem says that $K_{\mathbb{N}}$ is $\text{fin}^+$-Ramsey. In the remainder of this section we investigate the relationship between coideals $I^+$ and graphs $G$ such that $G$ is $I^+$-Ramsey. We hope that the results to follow will help the reader to better understand some of the characteristics of Ramsey-dense graphs, while simultaneously establishing a more general setting for the kind of questions we are interested in, where different notions of “small” other than “asymptotic density zero” are considered. Before continuing we note three easy observations.

**Fact 9.1.** Let $I$ and $J$ be ideals on $\mathbb{N}$ and suppose $I \subseteq J$. For any graph $G$, if $G$ is $J^+$-Ramsey, then $G$ is $I^+$-Ramsey.

**Fact 9.2.** Let $I$ be an ideal on $\mathbb{N}$ and let $A \in I^+$. For any partition $\{A_1, \ldots, A_k\}$ of $A$, there exists $i \in [k]$ such that $A_i \in I^+$.

**Fact 9.3.** For every nontrivial ideal $I$, there is an ultrafilter $\mathcal{U} \subseteq I^+$.

**Proof.** Let $\mathcal{F} = \{ I^c : I \in I \}$ and observe that $\mathcal{F}$ satisfies (i) – (iii) in Definition 3.1 (such a family is called a filter). Hence we can apply Zorn’s lemma as in Proposition 3.3 to get an ultrafilter $\mathcal{U}$ containing $\mathcal{F}$. Then, for every $I \in I$, we have $I^c \in \mathcal{F} \subseteq \mathcal{U}$ and thus $I \not\in \mathcal{U}$; hence $\mathcal{U} \subseteq I^+$.

9.2 Finitely-ruled graphs and the ideal $\text{nwd}$

Recall that one of the motivating problems of this paper is to characterize the Ramsey-dense graphs, or in other words the graphs $G$ such that $G$ is $Z_0^+$-Ramsey. In Theorem 9.4 we provide a characterization of those graphs $G$ for which $G$ is $I^+$-Ramsey for every nontrivial ideal $I$ on $\mathbb{N}$. Interestingly, this characterization reduces to one particular ideal, and one particular 2-coloring, both of which we will describe now.

Note that every positive integer $n$ has a binary expansion in which the leftmost digit is a 1, the truncated binary expansion of $n$ is what remains after removing the leftmost digit from the
binary expansion (for instance, the truncated binary expansion of 19 is 0011). Given \( s, t \in \mathbb{N} \), we say that \( t \) extends \( s \), if \( s \leq t \) and the truncated binary expansion of \( t \) contains the truncated binary expansion of \( s \) as its initial segment (for instance 19 extends 7 since 0011 contains 11 as its initial segment, reading from right to left). Given \( s \in \mathbb{N} \), we write \( \langle s \rangle \) for the set of \( t \in \mathbb{N} \) which extend \( s \) (for instance, \( \langle 1 \rangle = \mathbb{N} \), \( \langle 2 \rangle \) is the positive even integers, and \( \langle 3 \rangle \) is the positive odd integers). The ideal \( \mathrm{nwd} \) consists of all sets \( A \subseteq \mathbb{N} \) such that for every \( s \in \mathbb{N} \) there exists an extension \( t \) of \( s \) such that \( A \cap \langle t \rangle = \emptyset \).\(^1\) It is straightforward to check that \( \mathrm{nwd} \) is a non-trivial ideal (that is, a proper ideal containing all of the finite subsets of \( \mathbb{N} \)).

Recall that the Rado coloring was defined in Section 2.4.

**Theorem 9.4.** The following are equivalent for any countably infinite graph \( G \).

(i) For every non-trivial ideal \( \mathcal{I} \) on \( \mathbb{N} \), \( G \) is \( \mathcal{I}^+ \)-Ramsey.

(ii) In the Rado coloring of \( K_\mathbb{N} \), there is a monochromatic copy of \( G \) such that \( V(G) \in \mathrm{nwd}^+ \).

(iii) \( G \) is finitely-ruled.

**Proof of Theorem 9.4.** (i) \( \implies \) (ii) \( \mathrm{nwd} \) is a non-trivial ideal on \( \mathbb{N} \), so in every 2-coloring of the edges of \( K_\mathbb{N} \) (and in particular, the Rado coloring), there is a monochromatic copy of \( G \) with \( V(G) \in \mathrm{nwd}^+ \).

(ii) \( \implies \) (iii) Suppose \( G \) is infinitely-ruled and there exists a monochromatic copy of \( G \) in the Rado coloring of \( K_\mathbb{N} \) with color \( i \in \{0, 1\} \). We show that \( V(G) \in \mathrm{nwd} \).

Let \( F_n \) (\( n \in \mathbb{N} \)) be pairwise-disjoint, finite ruling sets in \( G \), and fix \( s \in \mathbb{N} \). Then there is some \( n \) such that for all \( t \in F_n, t > s \). Let \( u \in \mathbb{N} \) with \( u > \max \{ t \mid t \in F_n \} \) such that \( u \) extends \( s \) and the \( t \)th bit of \( u \) is \( 1 - i \) for all \( t \in F_n \). This means that no vertex in \( \langle u \rangle \) is adjacent to any vertex in \( F_n \) in color \( i \). Since \( F_n \) is a ruling set, this implies that \( V(G) \cap \langle u \rangle \) is finite. Since \( V(G) \cap \langle u \rangle \) is finite, there exists \( u' > u \) such that \( u' \) extends \( u \) and \( V(G) \cap \langle u' \rangle = \emptyset \) and thus \( V(G) \in \mathrm{nwd} \).

(iii) \( \implies \) (i) Let \( F \) be a finite subset of \( V(G) \) for which \( G \setminus F \) is 0-ruled, and let \( n = \vert F \vert \). Fix a proper ideal \( \mathcal{I} \) on \( X \) containing the finite subsets of \( X \), and let \( \mathcal{U} \) be an ultrafilter contained in \( \mathcal{I}^+ \) (which exists by Fact 9.3). Now we proceed exactly as in the proof of Theorem 1.6. \( \Box \)

We see that Theorem 1.6 is a special case of Theorem 9.4 since, in particular, Theorem 9.4 shows that every finitely-ruled graph \( G \) is \( Z_0^+ \)-Ramsey. Problem 1.11 asks whether the converse is true; that is, if \( G \) is \( Z_0^+ \)-Ramsey, is \( G \) finitely-ruled? Theorem 9.4 might be viewed as evidence towards this conclusion, since it shows that this is true at least for the coideal \( \mathrm{nwd}^+ \) in place of \( Z_0^+ \). On the other hand, we might view Theorem 9.4 as evidence in the opposite direction, since one would expect there to be some infinite graph \( G \) which distinguishes the coideals \( \mathrm{nwd}^+ \) and \( Z_0^+ \) as \( \mathrm{nwd} \) and \( \mathcal{Z}_0 \) have very different properties as ideals. Of course, this is all just speculation.

### 9.3 Infinitely ruled graphs and relative density zero ideals

Let \( f: \mathbb{N} \to \mathbb{N} \) be a function. The ideal \( \mathcal{Z}_f \) is defined to be the set of all \( A \subseteq \mathbb{N} \) such that \( \vert A \cap \{1, \ldots, n\} \vert / f(n) \to 0 \) as \( n \to \infty \). The ideal \( \mathcal{Z}_0 \) is one example, where we take \( f \) to be the

\(^1\)The notation \( \mathrm{nwd} \) stands for “nowhere dense” and typically the ideal \( \mathrm{nwd} \) is studied on the set \( \mathbb{Q} \); however, for consistency with the rest of the paper, we state all of the results in terms of the set \( \mathbb{N} \). That being said, it is possible to show that the set \( \mathbb{N} \), when given the topology generated by the sets \( \langle s \rangle \), is homeomorphic to the space \( \mathbb{Q} \cap \langle 0, 1 \rangle \), and under this homeomorphism the sets in \( \mathrm{nwd} \) correspond to those subsets of \( \mathbb{Q} \cap \langle 0, 1 \rangle \) which are not dense in any subinterval of \( \langle 0, 1 \rangle \).
identity function. The reader may check that $Z_f \subseteq Z_g$ whenever $f \leq g$, though of course the converse does not hold.

In Theorem 2.14 we showed that if $G$ is infinitely ruled and the ruling sets grow slowly enough, then $G$ is not $Z^+_0$-Ramsey (i.e. $G$ is not Ramsey dense). In this section we will give an example of a family $\mathcal{G}$ of infinitely ruled graphs (where the size of the ruling sets may go to infinity at any prescribed rate, no matter how slowly) such that for all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n)/n \to 0$ and for all $G \in \mathcal{G}$, $G$ is $Z^+_f$-Ramsey.

Let $T$ be a tree with a fixed root $r$. Given vertices $u, v \in V(T)$, we say that $v$ is an extension of $u$ if $u$ lies on the unique path from $r$ to $v$. We say that two vertices in $T$ are compatible if one is an extension of the other. The compatibility graph of $(T, r)$, $C_{T, r}$, is the graph with vertex set $V(T)$ and edges $\{u, v\}$ for all compatible vertices $u$ and $v$. (This is sometimes called the downward closure of $(T, r)$.)

An antichain in $T$ is a set of pairwise-incompatible vertices, and we call an antichain $A$ maximal if there is no antichain $B$ such that $A$ is a proper subset of $B$. Note that every finite maximal antichain in $T$ is a ruling set in $C_{T, r}$; in particular, if $T$ is locally finite, then the sets $R_n = \{v \in V(T) \mid d_T(v, r) = n\}$ form finite ruling sets in $C_{T, r}$.

We say that $T$ is perfect if every vertex has at least two incompatible extensions. If $T$ is locally finite and perfect, then $|R_n| \to \infty$ as $n \to \infty$, but the growth of this sequence may be arbitrarily slow.

**Theorem 9.5.** Let $f : \mathbb{N} \to \mathbb{N}$ be any function satisfying $f(n)/n \to 0$, and let $T$ be any locally finite, perfect tree with fixed root $r$. Then $C_{T, r}$ is $Z^+_f$-Ramsey.

Theorem 9.5 is immediately implied by the following two results.

**Proposition 9.6.** Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n)/n \to 0$. Then for any finite edge-coloring of $K_\mathbb{N}$, there is a monochromatic complete multipartite subgraph, with finite parts, whose vertex set is in $Z^+_f$.

**Proposition 9.7.** Let $T$ be a perfect tree with a fixed root $r$, and let $M$ be a complete, infinite multipartite graph, with finite parts. Then there is a copy of $C_{T, r}$ in $M$ which spans all but finitely many of the parts of $M$.

**Proof of Proposition 9.6.** Consider the ideal $\mathcal{I}_f$ consisting of all sets $A \subseteq \mathbb{N}$ satisfying

$$\sup_n |A \cap \{1, \ldots, n\}|/f(n) < \infty$$

(Note that the assumption on $f$ implies that $\mathcal{I}_f$ is proper.) Clearly $Z_f \subseteq \mathcal{I}_f$. We moreover note that if $A_1, A_2, \ldots$ is a countable sequence from $\mathcal{I}_f^+$ satisfying $A_1 \supseteq A_2 \supseteq \cdots$, then there is a set $A \in \mathcal{I}_f^+$ satisfying $|A \setminus A_n| < \infty$ for all $n$. (For instance, $A$ may be constructed by letting $A = \bigcup_n A_n \cap [k_n, k_{n+1}]$, where $k_n$ is chosen recursively to satisfy $|A_n \cap [k_n, k_{n+1}]|/f(k_{n+1}) \geq n.$)

Now fix an $r$-edge-coloring $\chi$ of $K_\mathbb{N}$. We will recursively construct sets $A_1, A_2, \ldots \in \mathcal{I}_f^+$, and an $r$-coloring $\rho$ of $\mathbb{N}$, such that $A_1 \supseteq A_2 \supseteq \cdots$ and for all $n$ and $m \in A_n$, $\chi\{\{n, m\}\} = \rho(m)$. This construction goes as follows. First we set $A_1 = \mathbb{N}$. Now, given $A_n$, note that the sets $A_n \cap N_i(n)$, for $i = 1, \ldots, r$, partition $A_n \setminus \{n\}$, and hence at least one must be in $\mathcal{I}_f^+$ (by Fact 9.2). We choose one to be $A_{n+1}$ and define $\rho(n)$ to be the associated color $i$.

Now by the above-mentioned property of $\mathcal{I}_f^+$, we may find a single set $A' \in \mathcal{I}_f^+$ such that $|A' \setminus A_n| < \infty$ for all $n$. Now let $A = A' \cap \rho^{-1}(i)$ for some $i \in [r]$ such that $A \in \mathcal{I}_f^+$. Hence,
for all \( n \in A \) there are only finitely-many \( m \in A \) for which \( \chi(\{n, m\}) \neq i \). In other words, the graph \( K_i[A] \) consisting of edges of color \( i \) induced on vertex set \( A \) has the property that every vertex has cofinite degree. This allows us to construct, recursively, a sequence \( b_0 < b_1 < \cdots \) such that for all \( n, m \in A \) with \( n \leq b_k \) and \( m \geq b_{k+1} \), \( \chi(\{n, m\}) = i \). Let

\[
A_0^r = A \cap \bigcup_{k=0}^{\infty} [b_{2k}, b_{2k+1})
\]

and \( A_0^* = A \setminus A_0 \). Then each of \( A_0^r \) and \( A_0^* \) is the vertex set of a monochromatic multipartite graph with finite parts, and moreover at least one is in \( \mathcal{I}_f^+ \), and hence \( \mathcal{Z}_f^+ \) (by Fact 9.1).

**Proof of Proposition 9.7.** Let \( r_1 := r \) and let \( P = r_1r_2 \ldots r_k \) be the shortest path from \( r_1 \) to a vertex \( r_k \) such that \( r_k \) has at least two successors (if \( r_1 \) itself has two successors, then \( r_k = r_1 \) and \( P = r_1 \) is just a trivial path). Note that every vertex \( r_i \) on the path \( P \) is a maximal antichain (and thus a ruling set in \( C_{T,r} \)). Also every vertex \( v \in V(C_{T,r}) \setminus V(P) \) is part of a maximal infinite independent set which we denote \( I(v) \).

Let \( K := K_{1,\ldots,1,\ldots} \) be the complete multipartite graph with \( k \) parts of order 1 and infinitely many infinite parts. Clearly \( K \) can be embedded into \( \mathcal{I}_f \) in such that way that \( K \) spans all but finitely many parts of \( M \). We will show that \( C_{T,r} \) can be surjectively embedded into \( K \) which will then complete the proof.

First we embed the path \( P = r_1 \ldots r_k \) into the parts of order 1. Let \( v_1, v_2, \ldots \) be an enumeration of the remaining vertices of \( K \). Note that this ordering induces an ordering \( V^1, V^2, \ldots \) of the infinite parts themselves (meaning that if \( v_n \in V^j \), then \( \{v_1, v_2, \ldots, v_n\} \subseteq \bigcup_{i=1}^n V_i \) and an ordering \( v^1_i, v^2_i \) of each \( V^i \). Finally, let \( u_1, u_2, \ldots \) be an enumeration of \( V(T) \setminus V(P) \) such that \( T[\{r_1, \ldots, r_k, u_1, \ldots, u_\ell\}] \) is connected for all \( \ell \geq 1 \).

Initially we set \( f(u_1) = v_1 \) (where \( v_1 \in V^1 \)) and then we repeat the following two steps.

**Step 1.** Let \( v^j_i \in V^i \) be the smallest vertex in \( V(K) \setminus \text{ran} f \). If \( j = 1 \) (i.e. \( V^i \cap \text{ran} f = \emptyset \), move to Step 2. Otherwise, let \( u \in \text{dom} f \) such that \( f(u) \in V^i \). Now let \( u' \in I(u) \setminus \text{dom} f \) and set \( f(u') = v^j_i \).

**Step 2.** Let \( m \) be the largest index such that \( u_{m-1} \in \text{dom} f \) and let \( U' = \{u'_1, u'_2, \ldots, u'_\ell\} := \{u_1, \ldots, u_m\} \setminus \text{dom} f \). Let \( n \) be the largest index such that \( V^n \cap \text{ran} f \neq \emptyset \). Now for all \( i \in [\ell] \), set \( f(u'_i) = v^{n+i}_1 \) (where \( v^{n+i}_1 \) is the first vertex in \( V^{n+i} \)). Note that for all \( i \geq 1 \), \( v^{n+i}_1 \) is adjacent to every vertex in \( \text{ran} f \).

At the end of each instance of Step 1 and Step 2, we have covered the first available vertex in \( V(K) \setminus \text{ran} f \) and we have embedded an entire interval \( \{u_1, u_2, \ldots, u_m\} \) in the ordering of \( V(T) \), including the first available vertex in \( V(T) \setminus \text{dom} f \). Thus we have defined a surjective embedding of \( C_{T,r} \) into \( K \), which completes the proof.

In the proof of Proposition 9.6, we have a graph with vertex set \( A \in \mathcal{I}_f^+ \) in which every vertex has cofinite degree. We use this to show that \( A \) can be partitioned into two infinite complete multipartite graphs with all parts finite, one of which, call it \( M \), must have vertex set in \( \mathcal{I}_f^+ \). We then show that if \( T \) is perfect tree with fixed root \( r \), then \( C_{T,r} \) can be embedded into \( M \). This raises the following two questions.

**Problem 9.8.**

(i) Characterize all graphs which can be cofinitely embedded into every graph in which every vertex has cofinite degree.
(ii) Characterize all graphs which can be cofinitely embedded into every infinite complete multipartite graph with finite parts.

10 Conclusion and open problems

10.1 Graphs of bounded chromatic number/maximum degree/degeneracy

Let $G$ be a graph with $\Delta := \Delta(G) \geq 2$. We know that $2 \leq \chi(G) \leq \Delta(G) + 1$ and we proved that $\frac{1}{\Delta - 1} \leq \overline{Rd}(G) \leq \frac{1}{\chi(G) - 1}$. It would be interesting to know whether these bounds can be improved in general. More generally, we know from Example 2.2.(iii) that the bound in Theorem 1.2.(i) cannot be improved.

Problem 10.1. If possible, improve the bounds in Theorem 1.2.(ii),(iii).

Let $H_d$ be the graph defined in Proposition 6.4.

Problem 10.2. For all $d \in \mathbb{N}$, does there exist $c = c(d) > 0$ such that $\overline{Rd}(H_d) \geq c$? More weakly, is $\overline{Rd}(H_d) > 0$?

We know this is true for $d = 1$, in which case $c = c(1) = 1/2$. Even answering the question for $d = 2$ would be a big step forward. See also Problem 6.5 and Problem 6.6.

10.2 Ramsey-dense graphs and graphs with positive upper Ramsey density

We know that every 0-ruled graph is Ramsey-dense and we know that there exist 0-ruled graphs $G$ with $\overline{Rd}(G) = 0$, but all such graphs $G$ that we know of have $\chi(G) = \infty$ (see Corollary 2.4). So we ask the following question.

Problem 10.3. Does there exist a graph $G$ which is 0-ruled and $\chi(G) < \infty$, but $\overline{Rd}(G) = 0$? For instance, is $\overline{Rd}(R_2) = 0$? (where $R_2$ is the bipartite Rado graph)

In Problem 1.11 we ask if there are Ramsey-dense graphs with $\text{rul}(G) = \infty$. A more general series of questions is the following.

Problem 10.4.

(i) Characterize all Ramsey-dense graphs.

(ii) Characterize graphs $G$ with $\overline{Rd}(G) > 0$.

(iii) Characterize graphs $G$ which are Ramsey-dense, but $\overline{Rd}(G) = 0$.

We proved that a certain class of graphs with infinite ruling number is $Z_f^+$-Ramsey. So we ask the following problem. Also see Problem 9.8.

Problem 10.5. Characterize all graphs $G$ having the property that for all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n)/n \to 0$, $G$ is $Z_f^+$-Ramsey.
10.3 Ramsey-lower-dense and Ramsey-cofinite graphs

One of the main results in the paper is a characterization of all Ramsey-cofinite forests. The most interesting open problem here is the following (c.f. Problem 8.11).

**Problem 10.6.** Characterize all Ramsey-cofinite graphs.

In Theorem 1.14, we proved that every forest is either Ramsey-cofinite or is not Ramsey-lower-dense. We suspect that this is true of every graph (in [8, Problem 8.10] we asked the weaker question of whether $Rd(G) < 1$ implies that $G$ has Ramsey-lower-density 0).

**Conjecture 10.7.** For every graph $G$, if $G$ is not Ramsey-cofinite, then $G$ is not Ramsey-lower-dense.

10.4 Ramsey-upper-density of trees

We showed that $Rd(T) \geq 1/2$ for every infinite tree and we showed that this result is tight for some trees such as $T_{\infty}$. Lamaison [16] obtained sharp results on Ramsey upper densities of locally finite trees.

It would be interesting to extend some of these results to more colors. We know from Example 2.2.(ii), that $Rd_r(T_{\infty}) \leq 1/r$.

**Problem 10.8.** For all integers $r \geq 3$, determine a lower bound on $Rd_r(T_{\infty})$.

10.5 Bipartite graphs

We conjectured (Conjecture 1.17) that the vertices of every $r$-colored $K_{N,N}$ can be partitioned into a finite set and at most $r$ monochromatic paths. We proved this for $r = 2$ and we showed how this has some consequences for the problem of determining $Rd_r(P_{\infty})$, which is an open problem for $r \geq 3$.

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