SYZYGIES OF SOME GIT QUOTIENTS

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Abstract. Let $X$ be flat scheme over $\mathbb{Z}$ such that its base change, $X_p$, to $\mathbb{F}_p$ is Frobenius split for all primes $p$. Let $G$ be a reductive group scheme over $\mathbb{Z}$ acting on $X$. In this paper, we prove a result on the $N_p$ property for line bundles on GIT quotients of $X_C$ for the action of $G_C$. We apply our result to the special cases of (1) an action of a finite group on the projective space and (2) the action of a maximal torus on the flag variety of type $A_n$.

Keywords: Syzygy, GIT quotient, flag variety

1. Introduction

Syzygies of algebraic varieties have been studied classically since the time of Italian geometers. For instance, the question of projective normality and normal presentation of embeddings of projective varieties in a projective space was studied in depth. The subject has been revived and there is much renewed interest since Green [5, 6] developed a homological framework which encompasses the classical questions. It was noted that projective normality and normal presentation were really properties of a graded free resolution and $N_p$ property was defined as a generalisation of this property.

We briefly review the notion of $N_p$ property.

Let $k$ be an algebraically closed field of characteristic 0. All our varieties are projective, smooth and defined over $k$.

Let $\mathcal{L}$ be a very ample line bundle on a projective variety $X$. Then $\mathcal{L}$ determines an embedding of $X$ into the projective space $\mathbb{P}(H^0(X, \mathcal{L}))$. We denote by $S$ the homogeneous coordinate ring of this projective space. Then the section ring $R(\mathcal{L})$ of $\mathcal{L}$ is defined as $\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^\otimes n)$ and it is a finitely generated graded $S$-module. One looks at the minimal graded free resolution of $R(\mathcal{L})$ over $S$:

$$\cdots \rightarrow E_i \rightarrow \cdots \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \rightarrow R(\mathcal{L}) \rightarrow 0$$

where $E_i = \bigoplus S(-a_{i,j})$ for all $i \geq 0$ and $a_{i,j}$ are some nonnegative integers.

We say that $\mathcal{L}$ has $N_0$ property if $E_0 = S$. This simply means that the embedding determined by $\mathcal{L}$ is projectively normal (or $\mathcal{L}$ is normally generated).

$\mathcal{L}$ is said to have $N_1$ property if $E_0 = S$ and $a_{1,j} = 2$ for all $j$. In this case, we also say that $\mathcal{L}$ is normally presented. Geometrically, this means that the embedding is cut out by quadrics.

For $p \geq 2$, we say that $\mathcal{L}$ has $N_p$ property if $E_0 = S$ and $a_{i,j} = i + 1$ for all $i = 1, \ldots, p$. 


Given a very ample line bundle $\mathcal{L}$, it is an interesting question to ask whether it has $N_p$ property for a given $p$.

There is extensive literature on this question. The following is a sample of results. For line bundles on curves see [7], on surfaces see [3, 4]. For abelian varieties, see [24]. In [2], a general result is proved for very ample line bundles on projective varieties.

In this paper, we are interested in the $N_p$ property for line bundles on GIT quotients. More specifically, we consider varieties defined over $\mathbb{Z}$ and consider the descent of an ample line bundle to a GIT quotient. We obtain a general result on $N_p$ property of this descent (Corollary 4.3) by using a cohomological criterion for $N_p$ property. We prove the required vanishing results using Frobenius splitting methods (Theorem 3.2).

In [15], the authors consider the quotients of a projective space $X$ for the linear action of finite solvable groups and for finite groups acting by pseudo reflections. They prove that the descent of $\mathcal{O}_X(1)^{\otimes |G|}$ is projectively normal. In [10], these results were obtained for every finite group but with a larger power of the descent of $\mathcal{O}_X(1)^{\otimes |G|}$. In this paper, we consider any finite group acting linearly on $X$ and prove a general result on $N_p$ property for the descent of $\mathcal{O}_X(1)^{\otimes |G|}$. For representations $\rho : G \to GL(V)$ such that $\rho(G) \subset SL(V)$, it follows from [28, Theorem 1] that $\mathbb{C}[V]^G$ is Gorenstein (cf [25, Corollary 8.2] also). Thus $G \backslash \mathbb{P}(V)$ is also Gorenstein. In this case, with the further assumption that the order of $G$ divides the dimension of $V$, we prove that the canonical line bundle on $\mathbb{P}(V)$ descends to the quotient $G \backslash \mathbb{P}(V)$ and it is the canonical line bundle.

A question of Fulton concerns the $N_p$ property of line bundles on flag varieties (cf. [2, Problem 4.5]). The special case of flag varieties of type $A_n$ is considered in [18] and a general result is obtained. In line with this, we consider the GIT quotient of a flag variety of type $A_n$ for the action of a maximal torus and we obtain a result on $N_p$ property as an application of our main result.

The organisation of the paper is as follows:

Section 2 consists of preliminaries. Cohomology of line bundles on the quotient variety is studied in Section 3. In section 4, we prove $N_p$ property for GIT quotients of varieties which are defined over $\mathbb{Z}$. We apply these results to the special case of finite group quotients in Section 5 and to the special case of GIT quotient of a flag variety of type $A_n$ for the action of a maximal torus in section 6.

2. Preliminaries

Given a vector bundle $F$ on a projective variety $X$ that is generated by its global sections, we have the canonical surjective map:

\[
H^0(F) \otimes \mathcal{O}_X \to F.
\]

Let $M_F$ be the kernel of this map. We have then the natural exact sequence:

\[
0 \to M_F \to H^0(F) \otimes \mathcal{O}_X \to F \to 0.
\]
Our goal in this paper is to study $N_p$ property of line bundles on GIT quotients of projective varieties with some special property.

**Theorem 2.1.** [2, Lemma 1.6] Let $L$ be a very ample line bundle on a projective variety $X$. Assume that $H^1(L^k) = 0$ for all $k \geq 1$. Then $L$ satisfies $N_p$ property if and only if $H^1(\wedge^m M_L \otimes L^n) = 0$ for all $1 \leq m \leq p + 1$ and $n \geq 1$.

**Remark 2.2.** In characteristic zero, it suffices to prove $H^1(M_L^m \otimes L^n) = 0$ to obtain $N_p$ property as the wedge product $\wedge^m M_L$ is a direct summand of the tensor product $M_L^m$.

**Remark 2.3.** In [2], this theorem was proved only assuming that $L$ is ample and base point free. We will apply this result with only these hypotheses (cf. [19] §1.3, Page 509).

**Lemma 2.4.** Let $E$ and $L_1, L_2, ..., L_r$ be coherent sheaves on a variety $X$. Consider the multiplication maps

\[
\psi : H^0(E) \otimes H^0(L_1) \otimes ... \otimes H^0(L_r) \rightarrow H^0(E \otimes L_1 \otimes ... \otimes L_r),
\]

\[
\alpha_1 : H^0(E) \otimes H^0(L_1) \rightarrow H^0(E \otimes L_1),
\]

\[
\alpha_2 : H^0(E \otimes L_1) \otimes H^0(L_2) \rightarrow H^0(E \otimes L_1 \otimes L_2),
\]

..., 

\[
\alpha_r : H^0(E \otimes L_1 \otimes ... \otimes L_{r-1}) \otimes H^0(L_r) \rightarrow H^0(E \otimes L_1 \otimes ... \otimes L_r).
\]

If $\alpha_1, ..., \alpha_r$ are surjective, then so is $\psi$.

**Proof.** We have the following commutative diagram where $id$ denotes the identity morphism:

\[
\begin{array}{ccc}
H^0(E) \otimes H^0(L_1) \otimes ... \otimes H^0(L_r) & \xrightarrow{\alpha_1 \otimes id} & H^0(E \otimes L_1) \otimes H^0(L_2) \otimes ... \otimes H^0(L_r) \\
\downarrow \phi & & \downarrow \alpha_2 \otimes id \\
H^0(E) \otimes H^0(L_1) \otimes ... \otimes L_r & \xrightarrow{\psi} & H^0(E \otimes L_1 \otimes L_2) \otimes H^0(L_3) \otimes ... \otimes H^0(L_r) \\
\downarrow \psi & & \downarrow \alpha_3 \otimes id \\
H^0(E \otimes L_1 \otimes ... \otimes L_r) & \xrightarrow{\alpha_r} & H^0(E \otimes L_1 \otimes ... \otimes L_{r-1}) \otimes H^0(L_r)
\end{array}
\]

Since $\alpha_1, \alpha_2, ..., \alpha_r$ are surjective and this diagram is commutative, a simple diagram chase shows that $\psi$ is surjective. \hfill $\square$

The following result, known as Castelnuovo - Mumford lemma, will be used often in this paper.

**Lemma 2.5.** [21, Theorem 2] Let $E$ be an ample and base-point free line bundle on a projective variety $X$ and let $F$ be a coherent sheaf on $X$. If $H^i(F \otimes E^{-i}) = 0$ for $i \geq 1$, then the multiplication map

\[
H^0(F \otimes E^i) \otimes H^0(E) \rightarrow H^0(F \otimes E^{i+1})
\]

is surjective for all $i \geq 0$. 

3. Cohomology of the quotient variety

Let $X$ be a flat scheme over $\mathbb{Z}$. Let $p$ be a prime number and let $\bar{F}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. Let $X_p$ denote the $\mathbb{F}_p$-valued points of $X$. Let $X_\mathbb{C}$ denote the $\mathbb{C}$-valued points of $X$. We assume that $X_\mathbb{C}$ is a projective variety over $\mathbb{C}$ and that $X_p$ are projective varieties over $\bar{F}_p$ for all primes.

We assume that there is a sheaf $\mathcal{N}$ on $X$ such that the base change of $\mathcal{N}$ to $X_\mathbb{C}$, $\mathcal{N}_\mathbb{C}$, (respectively, $\mathcal{N}_p$ on $X_p$, for all primes) is an ample line bundle.

Finally assume that $X_p$ is Frobenius split for all primes.

Let $G$ be a reductive (not necessarily connected) algebraic group scheme over $\mathbb{Z}$ acting on $X$ such that the action map $G_\mathbb{C} \times X_\mathbb{C} \rightarrow X_\mathbb{C}$ is a morphism. Assume that every line bundle on $X_\mathbb{C}$ is $G_\mathbb{C}$-linearised and that $(X_\mathbb{C})^G_\mathbb{C}(\mathcal{N}_\mathbb{C})$ is nonempty. We assume that the above hypotheses also hold for base change over $\mathbb{F}_p$ for all but finitely many primes.

Let $Y_\mathbb{C}$ denote the GIT quotient $G_\mathbb{C}\backslash\backslash(X_\mathbb{C})^{ss}(\mathcal{N}_\mathbb{C})$. Similarly let $Y_p$ denote the GIT quotient of $X_p$ with respect to the $G_p$-linearised line bundle $\mathcal{N}_p$. We further assume that $\mathcal{N}_\mathbb{C}$ (respectively, $\mathcal{N}_p$) descends to $Y_\mathbb{C}$ (respectively, $Y_p$ for all primes). Let $\mathcal{L}_\mathbb{C}$ (respectively, $\mathcal{L}_p$) denote the descent of $\mathcal{N}_\mathbb{C}$ to $Y_\mathbb{C}$ (respectively, $\mathcal{N}_p$ to $Y_p$).

For the preliminaries and notion of Geometric Invariant Theory, we refer to [22] and [23]. For the notion of Frobenius splitting, see [20].

**Lemma 3.1.** $\mathcal{L}_\mathbb{C}$ and $\mathcal{L}_p$ are ample line bundles on $Y_\mathbb{C}$ and $Y_p$ respectively.

**Proof.** Let $\phi : (X_\mathbb{C})^{ss}_{G_\mathbb{C}}(\mathcal{N}_\mathbb{C}) \rightarrow Y_\mathbb{C}$ be the natural categorical quotient map and let $\phi^* : \text{Pic}(Y_\mathbb{C}) \rightarrow \text{Pic}((X_\mathbb{C})^{ss}_{G_\mathbb{C}}(\mathcal{N}_\mathbb{C}))$ be the pullback map.

Since $\mathcal{N}_\mathbb{C}$ is a $G_\mathbb{C}$-linearised line bundle on $X_\mathbb{C}$, by [22] Theorem 1.10, Page 38], there is an ample line bundle $\mathcal{M}$ on $Y_\mathbb{C}$ such that the $\phi^*(\mathcal{M}) = \mathcal{N}_\mathbb{C}^\otimes n$ for some $n > 0$.

Since $\phi^*(\mathcal{L}_\mathbb{C}) = \mathcal{N}_\mathbb{C}$, $\mathcal{M} \otimes \mathcal{L}_\mathbb{C}^\otimes n$ is in the kernel of $\phi^*$. Since every line bundle on $X_\mathbb{C}$ is $G_\mathbb{C}$-linearised, $\text{Pic}((X_\mathbb{C})^{ss}_{G_\mathbb{C}}(\mathcal{N}_\mathbb{C})) = \text{Pic}_{G_\mathbb{C}}((X_\mathbb{C})^{ss}_{G_\mathbb{C}}(\mathcal{N}_\mathbb{C}))$. By [10] Proposition 4.2, Page 83], $\phi^*$ is injective. Hence $\mathcal{M} = \mathcal{L}_\mathbb{C}^\otimes n$ and $\mathcal{L}_\mathbb{C}$ is ample.

Proof is similar for $\mathcal{L}_p$. 

**Theorem 3.2.** With the notation as above, the following statements hold.

1. $H^i(Y_\mathbb{C}, \mathcal{L}_\mathbb{C}^e) = 0$ for every $e > 0$ and $i \geq 1$.
2. Assume that $G_p$ is linearly reductive for all but finitely many primes. Then $H^i(Y_p, \mathcal{L}_p^e) = 0$ for every $e < 0$ and $i < d$, where $d$ denotes the dimension of $Y$.

**Proof.** Since $\mathcal{N}_p$ is ample, by Serre’s vanishing theorem, there is a positive integer $r$ such that $H^i(X_p, \mathcal{N}_p^\otimes r^e) = 0$ for $i \geq 1$.

Now we will use the Frobenius splitting property of $X_p$ to prove (1) (cf. [20]). Let $F$ denote the Frobenius morphism corresponding to prime $p$. 

Tensoring the map $O_{X_p} \to F_* O_{X_p}$ by $N_p$ and noting that $N_p \otimes F_* O_{X_p} \cong F_* F^* N_p = F_* N_p^\otimes$ (projection formula) we see that the map $H^i(X_p, N_p) \to H^i(X_p, F_* N_p^\otimes) = H^i(X_p, N_p^\otimes)$ is injective.

Iterating this process, we conclude that the map $H^i(X_p, N_p) \to H^i(X_p, N_p^\otimes)^i$ is injective.

Thus $H^i(X_p, N_p) = 0$ for $i \geq 1$.

Since $X$ is flat over $\mathbb{Z}$, using semicontinuity theorem [8, Theorem III.12.8], we conclude that $H^i(X, N_C) = 0$ for $i \geq 1$ (cf. [1, Proposition 1.6.2]).

Proof of $H^i(X_C, N_C^\otimes) = 0$ for every $i \geq 1$ is similar.

Since $(X_C)^{ss}_G(N_C)$ is nonempty, using [27, Theorem 3.2.a], we have $H^i(Y_C, L_C^e) = H^i(X_C, N_C^\otimes)^G_C$ for every $i > 0$ and $e \geq 0$.

Hence, by the above arguments, $H^i(Y_C, L_C^e) = 0$ for every $i > 0$ and $e \geq 0$. This proves (1).

Since $X_p$ is Frobenius split and $G$ is linearly reductive over $\mathbb{F}_p$, using Reynolds operator, we see that $Y_p$ is also Frobenius split. For a proof, see [14, Theorem 3.7].

It is well known that there is a positive integer $r$ such that $H^j(Y_p, L_p^r) = 0$ for $i \neq d$.

Now the proof of (2) is similar to the proof of (1) using the Frobenius splitting property of $Y_p$.

This completes the proof of theorem. \hfill \Box

4. $N_p$ property

Let $X$ be a flat scheme over $\mathbb{Z}$. We assume that the hypotheses stated at the beginning of Section 3 hold. For simplicity of notation in this section we use letters $X, Y$ and $\mathcal{L}$ to denote $X_C, Y_C$ and $L_C$ respectively.

In this section we prove a result on $N_p$ property for $\mathcal{L}$ using Theorem 2.1. By Remark 2.3, we need the assumption that $\mathcal{L}$ is ample and base point free. By Lemma 3.1, $\mathcal{L}$ is ample.

We assume further that $\mathcal{L}$ is base point free. Let $d = \text{dim}(Y)$.

**Theorem 4.1.** Let $m, a \geq 1$ be a positive integers. Then we have $H^i(Y, M_{\mathcal{L}}^{\otimes m} \otimes \mathcal{L}^b) = 0$ for $i \geq 1$ and $b > md$.

Proof. We proceed by induction on $m$ and use Lemmas 2.4 and 2.5.

Let $m = 1$. Let $a \geq 1$ and $b > d$. We first show that $H^1(Y, M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^b) = 0$.

Consider the sequence 2.2 with $F = \mathcal{L}^{\otimes a}$:

\[(4.1) \quad 0 \to M_{\mathcal{L}^{\otimes a}} \to H^0(\mathcal{L}^{\otimes a}) \otimes \mathcal{O}_Y \to \mathcal{L}^{\otimes a} \to 0.\]

Tensoring with $\mathcal{L}^{\otimes b}$ and taking cohomologies, we get

\[H^0(\mathcal{L}^{\otimes a}) \otimes H^0(\mathcal{L}^{\otimes b}) \cong H^0(\mathcal{L}^{\otimes a+b}) \to H^1(M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^{\otimes b}) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^1(\mathcal{L}^{\otimes b}).\]
Since $H^1(\mathcal{L}^{\otimes b}) = 0$ by Theorem 3.2(1), if the map $\alpha$ is surjective then $H^1(M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^{\otimes b}) = 0$.

To prove surjectivity of $\alpha$, we will use Lemma 2.4 and first prove that the following map is surjective:

$$\alpha_1 : H^0(\mathcal{L}^{\otimes b}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes b+1})$$

For this we use Lemma 2.5. The needed vanishings are $H^i(\mathcal{L}^{b-i}) = 0$, for $i = 1, \ldots, d$. These follow from Theorem 3.2(1).

Similarly, we obtain the surjectivity of the maps:

$$\alpha_2 : H^0(\mathcal{L}^{\otimes b+1}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes b+2}),$$

$$\alpha_3 : H^0(\mathcal{L}^{\otimes b+2}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes b+3}),$$

and so on. By Lemma 2.4, $\alpha$ is surjective.

For $i > 1$, we consider the relevant part of the long exact sequence of (4.1) twisted by $\mathcal{L}^{\otimes b}$:

$$H^{i-1}(\mathcal{L}^{\otimes a+b}) \to H^i(M_{\mathcal{L}^{\otimes a}} \otimes \mathcal{L}^b) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^i(\mathcal{L}^{\otimes b})$$

We get the desired vanishing by Theorem 3.2(1).

Now let $m > 1$ and suppose that the theorem holds for $m - 1$. Let $a \geq 1$ and $b > md$ be given. First let $i = 1$.

Tensor the sequence (4.1) with $M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^b$ and take cohomology:

$$H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^b) \otimes H^0(\mathcal{L}^{\otimes a}) \to H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{\otimes a+b}) \to H^1(M_{\mathcal{L}^{\otimes a}}^{\otimes m} \otimes \mathcal{L}^b) \to H^1(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{\otimes b}).$$

The last term is zero by induction hypothesis. Note that the hypothesis required for $a, b$ hold. Hence it suffices to show that $\alpha$ is surjective.

In order to show that $\alpha$ is surjective we will use Lemma 2.4 and first consider the following

$$\alpha_1 : H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^b) \otimes H^0(\mathcal{L}) \to H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{\otimes b+1}).$$

By Lemma 2.5, this map surjects if $H^i(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{b-i}) = 0$ for $i = 1, \ldots, d$. Since $b > md$, $b - i > (m - 1)d$ for $i \leq d$. Hence the required vanishing is clear from induction hypothesis applied to $m - 1$.

Now consider the map:

$$\alpha_2 : H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{b+1}) \otimes H^0(\mathcal{L}) \to H^0(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{\otimes b+2}).$$

Using Lemma 2.5 as we did for $\alpha_1$, we conclude that $\alpha_2$ is surjective too. Iterating this we obtain surjectivity of $\alpha_i$ for all $i$ and hence $\alpha$ is also surjective.

Finally for $i > 1$, we have:

$$H^{i-1}(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^{\otimes a+b}) \to H^i(M_{\mathcal{L}^{\otimes a}}^{\otimes m} \otimes \mathcal{L}^b) \to H^0(\mathcal{L}^{\otimes a}) \otimes H^i(M_{\mathcal{L}^{\otimes a}}^{\otimes (m-1)} \otimes \mathcal{L}^b)$$

The middle term is zero because the other two are zero by induction.

This completes the proof of the theorem.
Corollary 4.2. Let $X$ be a flat scheme over $\mathbb{Z}$ and let $G$ be a reductive group scheme over $\mathbb{Z}$ acting on $X$. Suppose that all the hypotheses stated at the beginning of Section 3 hold. Let $L$ denote the descent to $Y_C$ of the ample line bundle $N_C$ on $X_C$.

Then $L^\otimes a$ has $N_p$ property for $a > (p+1)d$.

Proof. By Theorem 2.1, $L^\otimes a$ has $N_p$ property if $H^1(M_L^\otimes m \otimes L^\otimes an) = 0$ for all $1 \leq m \leq p+1$ and $n \geq 1$.

We apply the above theorem with $m = 1, \ldots, p+1$ and the required vanishing follows immediately. □

We get a stronger result when we assume that the top cohomology of the structure sheaf vanishes.

Corollary 4.3. Let $X$ be a flat scheme over $\mathbb{Z}$ and let $G$ be a reductive group scheme over $\mathbb{Z}$ acting on $X$. Suppose that all the hypotheses stated at the beginning of Section 3 hold. Let $L$ denote the descent to $Y_C$ of the ample line bundle $N_C$ on $X_C$. Suppose further that $H^d(Y_C, \mathcal{O}_{Y_C}) = 0$.

Then $L^\otimes a$ has $N_p$ property for $a \geq (p+1)d$.

Proof. This follows from the proof of Theorem 2.1. In the base case ($m = 1$), we required vanishing $H^i(L^{b-i}) = 0$, for $i = 1, \ldots, d$. For $i < d$, we apply Theorem 3.2(1). For $i = d$, we use the hypothesis on the structure sheaf. □

Remark 4.4. The results of this section were proved using only Theorem 3.2(1). Thus these $N_p$ results hold for any pair $(Y, L)$, where $Y$ is a complex projective variety and $L$ is an ample and base point free line bundle on $Y$, such that the statement of Theorem 3.2(1) is valid.

5. GIT Quotients for the Action of a Finite Group on a Projective Space

Let $G$ be a finite group of order $n$. Let $\rho : G \to GL(V)$ be a representation of $G$ over $\mathbb{C}$. $G$ operates on the projective space $X = \mathbb{P}(V)$ and every line bundle on $X$ is $G$-linearised. Let $d$ be the dimension of $X$.

Note that for every point $x \in X$, the isotropy subgroup $G_x$ of $x$ in $G$ acts trivially on the fiber of $x$ in $\mathcal{O}_X(n)$. Hence, by [16, Prop 4.2, Page 83], $\mathcal{O}_X(n)$ descends to the GIT quotient $Y = G \backslash X^s_G(\mathcal{O}_X(n))$. Let $L$ denote the descent of $\mathcal{O}_X(n)$ to $Y$.

Let $x \in X$. Since $G$ is finite, there is a $s \in H^0(X, \mathcal{O}_X(1))$ such that $s(gx) \neq 0$ for all $g \in G$. Let $\sigma = \prod_{g \in G} g.s$. Then $\sigma \in H^0(X, \mathcal{O}_X(n))^G$ and $\sigma(x) \neq 0$. Hence $L$ is base point free. Also note that $L$ is ample by Lemma 3.1.

Note that the statement of Theorem 3.2(1) holds in this case. The higher cohomologies of nonnegative powers of $\mathcal{O}_X(n)$ are clearly zero and hence the higher cohomologies of nonnegative powers of $L$ are zero too (cf. [27, Theorem 3.2.a])

Hence, by the Corollary 4.3 we have the following.
Theorem 5.1. $\mathcal{L}^\otimes a$ has $N_p$ property for any $a \geq (p+1)(d-1)$.

For $p = 0$, we deduce the following corollary. This result is new compared to [15] since it works for every group. This result is also new compared to [10] since the bound on degree is small.

Corollary 5.2. $\mathcal{L}^\otimes d-1$ has $N_0$ property.

Note that $\mathcal{O}_X(n)$ is the inverse of the canonical line bundle $K_X$ of $X$. Since $K_X$ descends to $Y$, it is a natural question to ask whether this descent is the canonical line bundle of $Y$, if $Y$ is Gorenstein. We obtain the following result.

Lemma 5.3. Assume that $\rho(G) \subset SL(V)$. Then the descent of the canonical line bundle $K_X$ of $X$ to $Y = G\backslash \mathcal{X}_G^s(K_X^{-1})$ is the canonical line bundle of $Y$.

Proof. Let $\mathcal{L}$ denote the descent of $K_X$ to $Y$. By [28, Theorem 1], $\mathbb{C}[V]^G$ is Gorenstein (cf [25, Corollary 8.2] also). Hence $Y$ is Gorenstein. Hence, the dualising sheaf of $Y$ is the canonical line bundle of $Y$. We denote it by $K_Y$. By [27, Theorem 3.2], there is a positive integer $h$ such that $H^d(Y, \mathcal{L}^\otimes h) = H^d(X, K_X^h).$ By applying Serre duality for the variety $X$, we have $H^d(X, K_X^h)^* = H^0(X, K_X^{(1-h)}).$ By GIT, we have $H^0(Y, \mathcal{L}^\otimes (1-h)) = H^0(X, K_X^{(1-h)}).$ On the other hand, applying Serre duality for the variety $Y$, we have $H^d(Y, \mathcal{L}^h) = H^0(Y, \mathcal{L}^\otimes (-h) \otimes K_Y)^*.$

Summarising, we have $H^0(Y, \mathcal{L}^\otimes (1-h)) = H^0(Y, \mathcal{L}^\otimes (-h) \otimes K_Y).$ By [16, Corollary 4.2, Page 83], $Pic(Y)$ is a subgroup of $Pic(X) = \mathbb{Z}$ and hence it is cyclic, say generated by $\mathcal{L}_0$. Since $\mathcal{L}$ and $K_Y$ are powers of $\mathcal{L}_0$, the above equality of global sections shows that $\mathcal{L} = K_Y$. Hence the descent of $K_X$ to $Y$ is $K_Y$. \hfill \Box

6. GIT quotients for the action of a maximal torus on the flag variety

For the preliminaries on semisimple algebraic groups, semisimple Lie algebras and root systems, we refer to [11, 12]. For the preliminaries on Chevalley groups we refer to [26].

Let $G$ be a semisimple Chevalley group over $\mathbb{C}$ of rank $n$. Let $T$ be a maximal torus of $G$, $B$ a Borel subgroup of $G$ containing $T$, which are defined over $\mathbb{Z}$. Let $N_G(T)$ denote the normaliser of $T$ in $G$. Let $W = N_G(T)/T$ denote the Weyl group of $G$ with respect to $T$.

We note that $G, T, B, W$ are all defined over $\mathbb{Z}$. Hence the flag variety $G/B$ of all Borel subgroups of $G$ and Schubert varieties are also defined over $\mathbb{Z}$ [26, Page 21]. Note that the base change of any Schubert variety to $\overline{\mathbb{F}}_p$ is Frobenius split (cf [20, Theorem 2, Page 38] or [11, Theorem 2.2.5, Page 69]).

We denote by $\mathfrak{g}$ the Lie algebra of $G$. We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the Lie algebra of $T$. Let $R$ denote the roots of $G$ with respect to $T$. Let $R^+ \subseteq R$ be the set of positive roots with respect to $B$. Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq R^+$ denote the set of simple roots with respect to $B$. Let $\langle \cdot, \cdot \rangle$ denote the restriction of the Killing form to $\mathfrak{h}$. Let $\check{\alpha}_i$ denote the coroot corresponding to $\alpha_i$. Let $\varpi_1, \varpi_2, \cdots, \varpi_n$ denote the fundamental weights corresponding to $S$. 


Let $s_i$ denote the simple reflection in $W$ corresponding to the simple root $\alpha_i$. For any subset $J$ of $\{1, 2, \cdots, n\}$, we denote by $W_J$ the subgroup of $W$ generated by $s_j, j \in J$. We denote the complement of $J$ in $\{1, 2, \cdots, n\}$ by $J^c$. For each $w \in W$, we choose an element $n_w$ in $N_G(T)$ such that $n_w T = w$. We denote the parabolic subgroup of $G$ containing $B$ and $\{n_w : w \in W_J\}$ by $P_J$. In particular, we denote the maximal parabolic subgroup of $G$ generated by $B$ and $\{n_s; j \neq i\}$ by $P_i$.

Let $X(B)$ denote the group of characters of $B$ and let $\chi \in X(B)$. Then, we have an action of $B$ on $\mathbb{C}$, namely $b.k = \chi(b^{-1})k$, $b \in B$, $k \in \mathbb{C}$. Consider the equivalence relation $\sim$ on $G \times \mathbb{C}$ defined by $(gb, b.k) \sim (g, k)$, $g \in G, b \in B, k \in \mathbb{C}$. The set of all equivalence classes is the total space of a line bundle over $G/B$. We denote this $G$-linearised line bundle associated to $\chi$ by $\mathcal{L}_\chi$.

Let $G = SL(n + 1, \mathbb{C})$. Let $J$ be a subset of $\{1, 2, \cdots n\}$ and let $P_J$ be the parabolic subgroup of $G$ corresponding to $J$. Since $G$ is simply connected, every line bundle on $G/P_J$ is $G$-linearised (cf. [16], 3.3, Page 82).

Let $W^{J^c}$ be the minimal representatives of elements in $W$ with respect to the subgroup $W_J$. For $w \in W^{J^c}$, let $X(w) = BwP_J/P_J < G/P_J$ be the Schubert variety corresponding to $w$. Note that $X(w)$ is $T$-stable. Hence restriction of any line bundle on $G/P_J$ to $X(w)$ is $T$-linearised.

Let $\chi$ be a dominant character of $T$ which is in the root lattice such that $\langle \chi, \alpha_j \rangle > 0$ for every $j \in J$. Let $w \in W^{J^c}$ be such that $X(w)_{T}^{ss}(\mathcal{L}_\chi)$ is nonempty. By [17], Theorem 3.10.a, Page 758, $\mathcal{L}_\chi$ descends to the GIT quotient $T \backslash X(w)_{T}^{ss}(\mathcal{L}_\chi)$. Let $N_\chi$ denote the descent.

Since $\chi$ is in the root lattice, by [9], Theorem 2.3, for every $x \in X(w)_{T}^{ss}(\mathcal{L}_\chi)$, there is a $T$-invariant section $s$ of $\mathcal{L}_\chi$ such that $s(x) \neq 0$. Hence $N_\chi$ is base point free. Also $N_\chi$ is ample by Lemma [3.1]

**Theorem 6.1.** Let $Y = T \backslash X(w)_{T}^{ss}(\mathcal{L}_\chi)$ be the GIT quotient of $X(w)$ for the $T$-linearised line bundle $\mathcal{L}_\chi$ on $X(w)$. Let $d$ be the dimension of $Y$. Let $N_\chi$ be the descent of $\mathcal{L}_\chi$ to $Y$. Then $N_\chi^{\otimes a}$ has $N_p$ property for $a \geq (p + 1)d$.

**Proof.** This follows from the above discussion and Corollary [4.3].

Now let $X = G/P_J$. We apply this theorem to the inverse of the canonical line bundle $K_X$ of $X$.

Let $R^+_J$ denote the set of all positive roots $\beta$ satisfying $\beta \geq \alpha_j$ for some $j \in J$. Let $\chi_J$ be the sum of all elements in $R^+_J$. Then, by equality (6) in [13], Page 229, we have $K_X^{-1} = \mathcal{L}_{\chi_J}$. Note that $\langle \chi_J, \alpha_j \rangle > 0$ for every $j \in J$. Hence, by [13], Remark 1, Page 232, $K_X^{-1}$ is ample.

By using similar arguments as above, $K_X^{-1}$ descends to the GIT quotient $T \backslash X_T^{ss}(K_X^{-1})$. Let $\mathcal{L}$ denote the descent. Again using similar arguments as above, we see that $\mathcal{L}$ is ample and base point free.

Let $d = dim(X) - dim(T)$. We have

**Corollary 6.2.** $\mathcal{L}^{\otimes a}$ has $N_p$ property for $a \geq (p + 1)d$.
Remark 6.3. For simple algebraic groups $G$ of types different from $A_n$ canonical line bundle of the flag variety $G/B$ does not, in general, descend to the GIT quotient. For instance, if $G$ is of type $B_3$, the coefficient of simple root $\alpha_1$ in the expression of $2\rho$ is 5. By [17, Theorem 3.10.b, Page 758], we see that the canonical line bundle of $G/B$ does not descend to the GIT quotient $T/(G/B)_{T}^{\text{ss}}(L(2\rho))$.

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