Abstract

In this paper, we give a positive answer to the question raised in Kosiński (Complex Anal Oper Theory 9(6):1349–1359, 2015) and Zapałowski (J Math Anal Appl 430(1):126–143, 2015), i.e., we show that the pentablock \( \mathcal{P} \) is a \( \mathbb{C} \)-convex domain.

Contents

1 Introduction ............................................. 1
2 Preliminary Results ......................................... 2
  2.1 Pentablock ............................................ 2
  2.2 Pentablock as a Hartogs Domain ...................... 3
  2.3 Some Useful Results .................................... 3
  2.4 \( \mathbb{C} \)-convex Domain ............................ 4
3 The Set of All Tangent Hyperplanes to \( \mathcal{P} \) at the Non-smooth Part .......... 4
4 Proof of Theorem 1.1 ........................................ 13
References ................................................ 13

1 Introduction

Recently, many authors showed great interest in two domains: the symmetrized bidisc and the tetrablock, arising from the \( \mu \)-synthesis, from the aspect of geometric function theory. Actually, both domains are \( \mathbb{C} \)-convex but non-convex, and they cannot be exhausted by domains biholomorphic to convex ones, with the Lempert’s theorem (see Lempert [13,14]) holding on these two domains, i.e., the Lempert function and the Carathéodory distance coincide on them (see [2,6–8,20]). So from the point of view of the Lempert’s theorem holding, these two domains play an important role in...
the study of a long-standing open problem whether Lempert’s theorem still holds for \( \mathbb{C} \)-convex domain. However, as far as we know, the answer is positive for \( \mathbb{C} \)-convex domain with \( \mathcal{C}^2 \) boundary (see [10]).

In 2015, Agler, Lykova and Young [1] introduced a new bounded domain \( \mathcal{P} \) by

\[
\mathcal{P} := \left\{ (a_{21}, \text{tr} A, \det A) : A = [a_{ij}]_{i,j=1}^2 \in \mathbb{B} \right\},
\]

where

\[
\mathbb{B} := \left\{ A \in \mathbb{C}^{2 \times 2} : ||A|| < 1 \right\}
\]
denotes the open unit ball in the space \( \mathbb{C}^{2 \times 2} \) with the usual operator norm. They called this domain the pentablock as \( \mathcal{P} \cap \mathbb{R}^3 \) is a convex body bounded by five faces, three of which are flat and two are curved (see [1]).

The pentablock \( \mathcal{P} \) is polynomially convex and starlike about the origin, but neither circled nor convex. Moreover, it does not have a \( \mathcal{C}^1 \) boundary (see [1]). This new domain is also arising from the \( \mu \)-synthesis, just like the symmetrized bidisc and the tetrablock. So it is naturally to consider analogous properties of the pentablock, such as the question about \( \mathcal{C} \)-convexity of \( \mathcal{P} \), and Lempert’s theorem on the equality of holomorphically invariant functions and metrics for the pentablock (see [1,12,19]). In this paper, we give a positive answer to the \( \mathcal{C} \)-convexity of \( \mathcal{P} \). More precisely, we obtain the following theorem.

**Theorem 1.1** The pentablock \( \mathcal{P} \) is a \( \mathcal{C} \)-convex domain.

Throughout this paper, \( \mathbb{D} \) denotes the open unit disc in the complex plane, while \( \mathbb{T} \) denotes the unit circle. And other basic notions, definitions, and properties from the theory of invariant functions, linearly convex and \( \mathcal{C} \)-convex domains that we shall use in the paper may be found in [3,9,11].

### 2 Preliminary Results

#### 2.1 Pentablock

We first recall the definition of the pentablock \( \mathcal{P} \).

**Theorem 2.1** [1, Theorem 1.1 and Theorem 5.2] Let

\[
(s, p) = (\lambda_1 + \lambda_2, \lambda_1 \lambda_2),
\]

where \( \lambda_1, \lambda_2 \in \mathbb{D} \). Let \( a \in \mathbb{C} \) and

\[
\beta = \frac{s - \bar{s}p}{1 - |p|^2}.
\]

The following statements are equivalent:
(1) \((a, s, p) \in \mathcal{P}\),
(2) \(|a| < \left| 1 - \frac{\frac{1}{2}s\bar{\beta}}{1 + \sqrt{1 - |\beta|^2}} \right|\),
(3) \(|a| < \frac{1}{2}|1 - \bar{\lambda}_1\lambda_2| + \frac{1}{2}(1 - |\lambda_1|^2)^{\frac{1}{2}}(1 - |\lambda_2|^2)^{\frac{1}{2}}\),
(4) \(\sup_{z \in \mathbb{D}} |\Psi_z(a, s, p)| < 1\), where \(\Psi_z\) is the linear fractional map

\[
\Psi_z(a, s, p) = \frac{a(1 - |z|^2)}{1 - sz +pz^2}.
\]

### 2.2 Pentablock as a Hartogs Domain

Following the description of the pentablock \(\mathcal{P}\), we can learn that the pentablock \(\mathcal{P}\) is closely related to the symmetrized bidisc \(\mathbb{G}_2\), which is defined by

\[
\mathbb{G}_2 = \left\{(s, p) \in \mathbb{C}^2 : |s - \bar{p}| + |p|^2 < 1 \right\}.
\]

In fact, the pentablock \(\mathcal{P}\) can be seen as a Hartogs domain in \(\mathbb{C}^3\) over the symmetrized bidisc \(\mathbb{G}_2\) (see [1]), that is,

\[
\mathcal{P} = \left\{(a, s, p) \in \mathbb{D} \times \mathbb{G}_2 : |a|^2 < e^{-\varphi(s, p)} \right\},
\]

where

\[
\varphi(s, p) = -2 \log \left| 1 - \frac{\frac{1}{2}s\bar{\beta}}{1 + \sqrt{1 - |\beta|^2}} \right|,
\]

\((s, p) \in \mathbb{G}_2\) and \(\beta = s - \bar{p} + |p|^2\).

Hartogs domain is a one of important research object in several complex variable. For the studies on Hartogs domain, please refer to [4,5,16–18]. So considering the pentablock \(\mathcal{P}\) as a Hartogs domain will be great helpful for us to study the convexity of the pentablock \(\mathcal{P}\).

### 2.3 Some Useful Results

In this subsection, we will give some useful results on the symmetrized bidisc \(\mathbb{G}_2\) and the pentablock \(\mathcal{P}\). In order to study the pentablock \(\mathcal{P}\), it is sufficient to learn the \(\mathbb{C}\)-convexity of \(\mathbb{G}_2\).

**Theorem 2.2** [15] The symmetrized bidisc \(\mathbb{G}_2\) is \(\mathbb{C}\)-convex.

Through the study of the boundary of \(\mathcal{P}\), we can learn that there are two main part of the boundary, i.e., the smooth part and the non-smooth part. So it is necessary to study some basic convexity property of \(\mathcal{P}\) to simplify the problem.

**Theorem 2.3** [12, Proposition 9] The pentablock \(\mathcal{P}\) is linearly convex.
In order to study the $\mathbb{C}$-convexity of $\mathcal{P}$ in some simple way, we give the whole holomorphic automorphism group $\text{Aut}(\mathcal{P})$ as follows.

**Theorem 2.4** [12, Theorem 15] All mappings of the form

$$f_{\omega, \nu}(a, \lambda_1 + \lambda_2, \lambda_1 \lambda_2) = \left(\frac{\omega(1-|\alpha|^2)a}{1-\bar{\alpha}(\lambda_1+\lambda_2)+\bar{\alpha}^2\lambda_1\lambda_2}, \nu(\lambda_1) + \nu(\lambda_2), \nu(\lambda_1)\nu(\lambda_2)\right),$$

where $(a, \lambda_1 + \lambda_2, \lambda_1 \lambda_2) \in \mathcal{P}$, $\lambda_1, \lambda_2 \in \mathbb{D}$, $\nu$ is a Möbius function of the form $\nu(\lambda) = \frac{\eta \lambda - a}{1-\bar{\alpha} \lambda}$, and where $\omega, \eta \in \mathbb{T}$, $\alpha \in \mathbb{D}$, form the whole group $\text{Aut}(\mathcal{P})$ of holomorphic automorphisms of the pentablock $\mathcal{P}$.

### 2.4 $\mathbb{C}$-convex Domain

A domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-convex if for any affine complex line $\ell$ such that $\ell \cap D \neq \emptyset$, and the set $\ell \cap D$ is connected and simply connected. For a domain $D \subset \mathbb{C}^n$ and a point $a \in \mathbb{C}^n$, we denote by $\Gamma_D(a)$ the set of all complex hyperplanes $L$ such that $(a + L) \cap D = \emptyset$. Then we have the basic criterion on $\mathbb{C}$-convexity.

**Theorem 2.5** [3, Theorem 2.5.2] The bounded domain $D \subset \mathbb{C}^n$, $n > 1$, is $\mathbb{C}$-convex iff for any boundary point $x \in \partial D$, the set $\Gamma_D(x)$ is non-empty and connected.

**Remark 2.6** By Theorem 2.5, we only need to give a full description of the tangent hyperplanes to the pentablock $\mathcal{P}$. And together with Theorem 2.3, we can only need to consider the non-smooth part of the boundary. Furthermore, through Theorem 2.4 we can simplify the situation into just four different types, i.e., (1) $(a, 1, 0)$ with $|a| \leq \frac{1}{2}$; (2) $(a, 0, -1)$ with $|a| < 1$; (3) $(1, 0, -1)$; (4) $(0, 2, 1)$.

### 3 The Set of All Tangent Hyperplanes to $\mathcal{P}$ at the Non-smooth Part

In this section, we will give a full description of the tangent hyperplanes to the pentablock $\mathcal{P}$ at the non-smooth boundary part. Set $P_0 = (a_0, s_0, p_0)$ be a non-smooth boundary point of the pentablock $\mathcal{P}$, and let $\Gamma_{\mathcal{P}}(P_0)$ denote the set of all tangent hyperplanes to $\mathcal{P}$ at the boundary point $P_0$. Now assume the hyperplane in $\mathbb{C}^3$ that

$$\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : k_1 a + k_2 s + k_3 p = 0\}.$$

Then

$$\mathcal{L} \in \Gamma_{\mathcal{P}}(P_0) \iff P_0 \in \mathcal{L} \text{ and } \mathcal{L} \cap \mathcal{P} = \emptyset.$$

Together with automorphisms of $\mathcal{P}$ by Theorem 2.4, we can only need to consider some special boundary points $P_0$:

(1) If $(s_0, p_0) \in \partial G_2 \setminus \partial \delta G_2$ and $|a_0|^2 \leq e^{-\psi(s_0, p_0)}$.

Actually we can assume that $(s_0, p_0) = (1, 0)$, and then we have $|a_0| \leq \frac{1}{2}$. Now we consider the hyperplane in $\mathbb{C}^3$ passing through the boundary point $P_0$,

$$\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : k_1(a - a_0) + k_2(s - s_0) + k_3(p - p_0) = 0\}. \quad (3.1)$$
If \( k_1 \neq 0 \), then \( L = \{(a, s, p) \in \mathbb{C}^3 : a = a_0 + k_2(1 - s) - k_3 p\} \).

Next suppose that \( L \in \Gamma_\mathcal{P}(P_0) \), so we get \( L \cap \mathcal{P} = \emptyset \). This means that for any \((s, p) \in \mathbb{G}_2\), we have

\[
|a_0 + k_2(1 - s) - k_3 p|^2 \geq e^{-\varphi(s, p)}.
\]

Now set \( p = 0 \), we can learn that for any \( s \in \mathbb{D} \),

\[
\frac{1}{2} + \frac{1}{2}(1 - |s|^2)^\frac{1}{2} \leq |a_0 + k_2(1 - s)| \leq |a_0| + |k_2(1 - s)|.
\]

Together with \( |a_0| \leq \frac{1}{2} \), we obtain that the following inequality holds for all \( s \in \mathbb{D} \).

\[
|k_2| \geq \frac{\frac{1}{2}(1 - |s|^2)^\frac{1}{2}}{|1 - s|}
\]

hence, taking \( s \) tends to 1, we can conclude that such \( k_2 \) does not exist. It follows that if the hyperplane \( L \) in (3.1) belongs to \( \Gamma_\mathcal{P}(P_0) \), then \( k_1 = 0 \).

Moreover, if we have a hyperplane \( L \in \Gamma_\mathcal{P}(P_0) \), then consider the following hyperplane in \( \mathbb{C}^2 \):

\[
L' := \{(s, p) \in \mathbb{C}^2 : (a, s, p) \in L\}.
\]

Easily, we can see that \((s_0, p_0) = (1, 0) \in L' \cap \partial \mathbb{G}_2 \) and \( L' \cap \mathbb{G}_2 = \emptyset \). So

\[
L' \in \Gamma_{\mathbb{G}_2}(s_0, p_0).
\]

This implies that

\[
\Gamma_\mathcal{P}(P_0) \subseteq \mathbb{C} \times \Gamma_{\mathbb{G}_2}(s_0, p_0).
\]

Clearly the other inclusion holds. Therefore, we have

\[
\Gamma_\mathcal{P}(a_0, 1, 0) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(1, 0) \quad \left(|a_0| \leq \frac{1}{2}\right).
\]

(2) If \((s_0, p_0) \in \partial \mathbb{G}_2 \) and \(|a_0|^2 < e^{-\varphi(s_0, p_0)}\).

From the assumption, we can learn that \( s_0^2 \neq 4p_0 \). Otherwise, \( e^{-\varphi(s_0, p_0)} = 0 \). Hence, we can assume that \((s_0, p_0) = (0, -1) \) and \(|a_0| < 1 \).

By the same way, consider the hyperplane passing through the boundary point \( P_0 \) with the assumption \( k_1 \neq 0 \), namely,

\[
L = \{(a, s, p) \in \mathbb{C}^3 : a = a_0 + k_2s + k_3(1 + p)\}.
\]

Then suppose that \( L \in \Gamma_\mathcal{P}(P_0) \), so this leads to \( L \cap \mathcal{P} = \emptyset \). Hence, for any \((s, p) \in \mathbb{G}_2\), we have

\[
|a_0 + k_2s + k_3(1 + p)|^2 \geq e^{-\varphi(s, p)}.
\]
Now set $s = 0$, we obtain that for any $p \in \mathbb{D}$,
\[
1 \leq |a_0 + k_3(1 + p)| \leq |a_0| + |k_3(1 + p)|.
\]

This means that the following inequality
\[
|k_3(1 + p)| \geq 1 - |a_0| > 0
\]
holds for any $p \in \mathbb{D}$. Thus such $k_3$ does not exist. And it follows that $k_1$ must be zero. Therefore, following by the same procedure, we obtain
\[
\Gamma P(a_0, 0, -1) = \mathbb{C} \times \Gamma_{G_2}(0, -1) \quad (|a_0| < 1).
\]

(3) $(s_0, p_0) \in \partial s \setminus \Sigma$ and $|a_0|^2 = e^{-\varphi(s_0, p_0)}$.

By the assumption, we can also see that $s_0^2 \neq 4p_0$. So we can assume $(a_0, s_0, p_0) = (1, 0, -1)$. Then consider the hyperplane
\[
\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2s + k_3(1 + p)\},
\]
and suppose that $\mathcal{L} \in \Gamma P(P_0)$. This implies that $\mathcal{L} \cap P = \emptyset$. Thus for any $(s, p) \in G_2$, we have
\[
|1 + k_2s + k_3(1 + p)|^2 \geq e^{-\varphi(s, p)}.
\]

Let $s = 0$, then for any $p \in \mathbb{D}$, we have
\[
|1 + k_3 + k_3 p| \geq 1.
\]

Now we give the following lemma to help us.

**Lemma 3.1** For any $z \in \mathbb{D}$, the inequality $|1 + k + k z| \geq 1$ holds iff $k \geq 0$.

**Proof** Set $f(z) = 1 + k + k z$, and then we directly learn that $f(z)$ is a holomorphic function on $\mathbb{D}$. Since $|f(z)| \geq 1$, it means that $f(z)$ has no zero in $\mathbb{D}$. Hence, by the maximal principle, $|f(z)| \geq 1$ holds for all $z \in \overline{\mathbb{D}}$. Now let $|z| = 1$, we can obtain
\[
|1 + k + k z| \geq 1,
\]
\[
\Leftrightarrow |1 + k|^2 + |k|^2 - 1 \geq -2 \text{Re} \left( (|k|^2 + k)z \right), \quad (|z| = 1)
\]
\[
\Leftrightarrow 2|k|^2 + 2\text{Re}k \geq 2 \left| k + |k|^2 \right|, \quad \text{(the inequality holds for all } z \in \partial \mathbb{D})
\]
\[
\Rightarrow (\text{Re})^2 \geq |k|^2.
\]

This means that $k$ is real. Now if $k < 0$, then there exists $M > 0$ such that
\[
M|k| < 1,
\]
and we can choose $z_0$ with $-1 < z_0 < 0$ such that

$$1 + z_0 < M.$$  

Hence, we have

$$0 < 1 + Mk < 1 + k(1 + z_0) < 1.$$  

This leads to a contradiction. On the other hand, for $k \geq 0$, the inequality is evidently valid. Therefore, we conclude that $k \geq 0$. \hfill $\Box$

By Lemma 3.1, we have $k_3 \geq 0$. Now we want to prove the inequality (3.2) on the whole $\Gamma_2$.

If there exists $(s_1, p_1) \in \partial \Gamma_2$ such that the inequality (3.2) does not hold. Then set $a_1 = 1 + k_2 s_1 + k_3 (1 + p_1)$, and by the assumption, we have

$$|a_1|^2 < e^{-\varphi(s_1, p_1)}. \quad (3.3)$$

Since $(a_1, s_1, p_1) \in \mathcal{L}$ and $\mathcal{L} \in \Gamma P(a_0, s_0, p_0)$, we can see that

$$\mathcal{L} \in \Gamma P(a_1, s_1, p_1).$$

So together with (3.3), by the case (1) and case (2), we know that such $\mathcal{L}$ does not exist. Hence, the inequality (3.2) also holds for all $(s, p) \in \Gamma_2$.

Back to the inequality (3.2), since $(s, p) \in \Gamma_2$, we can set $s = \lambda_1 + \lambda_2$ and $p = \lambda_1 \lambda_2$. So for any $(\lambda_1, \lambda_2) \in \partial(D \times D)$, we have

$$|1 + k_2 \lambda_1 + k_3 + \lambda_2 (k_2 + k_3 \lambda_1)| \geq \frac{1}{2} |1 - \bar{\lambda}_1 \lambda_2|.$$  

Now let $\lambda_1 = 1$, then for any $\lambda_2 \in \partial D$, we can obtain

$$|1 + k_2 + k_3 + \lambda_2 (k_2 + k_3)| \geq \frac{1}{2} |1 - \lambda_2|,$$

$$\Leftrightarrow |1 + k|^2 + |k|^2 - \frac{1}{2} \geq -\text{Re} \left( \left( 2k + 2|k|^2 + \frac{1}{2} \right) \lambda_2 \right), \quad (k = k_2 + k_3)$$

$$\Leftrightarrow 2|k|^2 + 2\text{Re}k + \frac{1}{2} \geq 2|k|^2 + 2k + \frac{1}{2}, \quad \text{(the inequality holds for all } \lambda_2 \in \partial D)$$

$$\Rightarrow (\text{Re}k)^2 \geq |k|^2.$$  

This implies that $k$ is real. On the other hand, we can learn that if $k$ is real, then we have

$$2|k|^2 + 2\text{Re}k + \frac{1}{2} = 2k^2 + 2k + \frac{1}{2}$$

$$= \frac{1}{2} (2k + 1)^2 \geq 0.$$
Thus we can obtain that for any \( \lambda_2 \in \partial \mathbb{D} \),

\[
|1 + k_2 + k_3 + \lambda_2(k_2 + k_3)| \geq \frac{1}{2}|1 - \lambda_2| \iff k = k_2 + k_3 \text{ is real}.
\]

Since \( k_3 \geq 0 \), \( k_2 \) is real. Following this result, we want to omit the assumption \( \lambda_1 = 1 \). Thus we give the following lemma.

**Lemma 3.2** For any \( (\lambda_1, \lambda_2) \in \partial \mathbb{D} \times \partial \mathbb{D} \), the inequality

\[
|1 + k_2\lambda_1 + k_3 + \lambda_2(k_2 + k_3\lambda_1)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|
\]

holding is equivalent to \( k_3^2 + k_3 \geq k_2^2 \).

**Proof** By direct calculation, we have

\[
|1 + k_2\lambda_1 + k_3 + \lambda_2(k_2 + k_3\lambda_1)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|,
\]

\[
\iff |1 + k_2\lambda_1 + k_3|^2 + |k_2 + k_3\lambda_1|^2 - \frac{1}{2} \geq -\text{Re}\left(\frac{1}{2}(2(k_2 + k_3\lambda_1)(1 + k_2\bar{\lambda}_1 + k_3) + \frac{1}{2}\bar{\lambda}_1)\lambda_2\right),
\]

\[
\iff \left(\frac{1}{2} + 2k_2^2\right) + (2k_3 + 2k_3^2) + (2k_2 + 4k_2k_3)\text{Re}\lambda_1
\]

\[
\geq \left|\left(\frac{1}{2} + 2k_2^2\right)\bar{\lambda}_1 + (2k_3 + 2k_3^2)\lambda_1 + (2k_2 + 4k_2k_3)\right|.
\]

Now set \( a = \frac{1}{2} + 2k_2^2, b = 2k_3 + 2k_3^2 \) and \( c = 2k_2 + 4k_2k_3 \). Thus we see that if

\[
a + b + c\text{Re}\lambda_1 \geq |a\bar{\lambda}_1 + b\lambda_1 + c|,
\]

then we have

\[
(1 - (\text{Re}\lambda_1)^2)(4ab - c^2) \geq 0.
\]

It follows

\[
k_3^2 + k_3 \geq k_2^2.
\]

Notice that \( a > 0 \) and \( b \geq 0 \). So if we want to get the equivalence condition for the inequality (3.4), we only need to consider the following inequality holding for all \( \lambda_1 \in \partial \mathbb{D} \),

\[
a + b + c\text{Re}\lambda_1 \geq 0.
\]

However, it is not hard to see

\[
a + b \geq |c|.
\]

So together with \( a + b \geq 0 \), we obtain that the inequality (3.5) is equivalent to

\[
(a + b)^2 \geq c^2.
\]
In fact, \((a + b)^2 - c^2 \geq (a + b)^2 - 4ab \geq 0\). Hence, this must be an equivalence condition. \(\square\)

Therefore, we have

\[ \Gamma P(1, 0, -1) \subseteq (\mathbb{C} \times \Gamma G_2(0, -1)) \cup \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2s + k_3(1 + p)\}, \]

where \(k_2\) is real, \(k_3 \geq 0\) and \(k_3^2 + k_3 \geq k_2^2\).

Now we want to show the other inclusion. Let \(k_2\) be a real number, and \(k_3 \geq 0\) with \(k_3^2 + k_3 \geq k_2^2\). Then consider the hyperplane

\[ L := \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2s + k_3(1 + p)\}. \]

In order to prove \(L \in \Gamma P(1, 0, -1)\), we only need to show the following inequality

\[ |1 + k_2s + k_3(1 + p)|^2 \geq e^{-\varphi(s, p)} \]

holding for any \((s, p) \in \mathbb{C}^2\).

Define

\[ h(s, p) = 1 + k_2s + k_3(1 + p). \]

If \(k_2 \neq 0\) and \(k_3 \neq 0\), then \(h(s, p)\) is a well-defined holomorphic function on \(\mathbb{C}^2\).

By Lemma 3.2, for any \((s, p) \in \partial \mathbb{C}^2\), we have

\[ |h(s, p)|^2 \geq e^{-\varphi(s, p)}. \]

Now set \(s = \lambda_1 + \lambda_2\) and \(p = \lambda_1 \lambda_2\), and then for any \((\lambda_1, \lambda_2) \in \partial \mathbb{D} \times \partial \mathbb{D}\), we have

\[ |1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1 \lambda_2)| \geq \frac{1}{2} |\lambda_1 - \lambda_2|. \]

(3.7)

Thus, if there exists \((s_1, p_1) \in \partial \mathbb{C}^2\) such that \(h(s_1, p_1) = 0\), then we must have \(s_1^2 = 4p_1\). Hence, we can assume that \(s_1 = 2\lambda_0\) and \(p_1 = \lambda_0^2\) for some \(\lambda_0 \in \partial \mathbb{D}\). So for \(h(s_1, p_1) = 0\), we obtain

\[ 1 + 2k_2\lambda_0 + k_3(1 + \lambda_0^2) = 0. \]

(3.8)

Notice that \(|\lambda_0| = 1\), and then assume \(\lambda_0 = x_0 + y_0i\). Thus we have \(x_0^2 + y_0^2 = 1\). From (3.8), we see that

\[ 1 + 2k_2(x_0 + y_0i) + k_3(2x_0^2 + 2x_0y_0i) = 0. \]

Thus we get

\[ \begin{cases} \k_2y_0 + k_3x_0y_0 = 0, \\ 1 + 2k_2x_0 + 2k_3x_0^2 = 0. \end{cases} \]
If $y_0 = 0$, then $2|k_2| = 2k_3 + 1$, which contradicts to $k_2^2 + k_3 \geq k_2^2$; if $y_0 \neq 0$, then $k_2 = -k_3x_0$. It follows that $1 + 2k_2x_0 + 2k_3x_0^2 = 1 \neq 0$. Hence, such $\lambda_0$ does not exist. This means that for any $(s, p) \in \partial s \cap \mathbb{D}$, we have

$$h(s, p) \neq 0.$$  \hfill (3.9)

Next we want to prove that the inequality (3.7) still holds for any $(\lambda_1, \lambda_2) \in \partial \mathbb{D} \times \mathbb{D}$, i.e.,

$$|1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1\lambda_2)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|, \quad (\lambda_1, \lambda_2) \in \partial \mathbb{D} \times \mathbb{D}.$$  

If there exists $(\lambda_0^1, \lambda_0^2) \in \partial \mathbb{D} \times \mathbb{D}$ such that

$$1 + k_2(\lambda_0^1 + \lambda_0^2) + k_3(1 + \lambda_0^1\lambda_0^2) = 0,$$

then we have

$$1 + k_2\lambda_0^1 + k_3 = -(k_2 + k_3\lambda_0^1)\lambda_0^2.$$  

Thus, if $k_2 + k_3\lambda_0^1 = 0$, then $\lambda_0^2$ is real. So we have

$$0 = 1 + k_2\lambda_0^1 + k_3 = 1 - k_3(\lambda_0^1)^2 + k_3 = 1.$$  

This leads to a contradiction. Hence, since $|\lambda_0^1| < 1$, we can see

$$|1 + k_2\lambda_0^1 + k_3| < |k_2 + k_3\lambda_0^1|,$$

$$\Leftrightarrow 1 + 2k_3 < -2k_2\text{Re}\lambda_0^1, \quad (|\lambda_0^1| = 1)$$

$$\Leftrightarrow 1 + 4k_3 + 4k_3^2 < 4k_2^2(\text{Re}\lambda_0^1)^2 \leq 4k_2^2 \leq 4k_3 + 4k_3^2.$$  

This leads to another contradiction. Thus, define

$$g_{\lambda_1}(\lambda_2) := \frac{\frac{1}{2}(\lambda_1 - \lambda_2)}{1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1\lambda_2)},$$

and then we get that for any fixed $\lambda_1 \in \partial \mathbb{D}$, $g_{\lambda_1}(\lambda_2)$ is a well-defined holomorphic function on $\overline{\mathbb{D}}$. Thus, by the maximal principle, together with (3.7), we have

$$|g_{\lambda_1}(\lambda_2)| \leq 1, \quad \forall \lambda_2 \in \overline{\mathbb{D}}.$$  

So, we can obtain that the inequality

$$|h(\lambda_1, \lambda_2)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|.$$

holds for all \((\lambda_1, \lambda_2) \in \partial \mathbb{D} \times \mathbb{D}\), and also for all \((\lambda_1, \lambda_2) \in \mathbb{D} \times \partial \mathbb{D}\). Thus, together with (3.9), we can conclude that
\[ h(s, p) \neq 0 \quad (s, p) \in \partial \mathbb{G}_2. \]

Notice that \(k_2 \neq 0\) and \(k_3 \neq 0\), so by the Hartogs theorem we have
\[ h(s, p) \neq 0 \quad (s, p) \in \overline{\mathbb{G}_2}. \]

Now set \(s = \beta + \bar{\beta} p\), then if we want to show (3.6), we only need to prove the following inequality
\[ |1 + k_2 \beta + k_3 + (k_2 \bar{\beta} + k_3) p| \geq \left| 1 - \frac{\frac{1}{2}(|\beta|^2 + \bar{\beta}^2 p)}{1 + \sqrt{1 - |\beta|^2}} \right| \] (3.10)
holds for any \((\beta, p) \in \mathbb{D} \times \mathbb{D}\).

Fixed any \(\beta \in \mathbb{D}\), define
\[ f_{\beta}(p) = \frac{1 - \frac{1}{2}(|\beta|^2 + \bar{\beta}^2 p)}{1 + k_2 \beta + k_3 + (k_2 \bar{\beta} + k_3) p}, \]
and then we can see that \(f_{\beta}(p)\) is a well-defined holomorphic function on \(\overline{\mathbb{D}}\). So if we want to show (3.10), we only need to prove that for any fixed \(\beta \in \mathbb{D}\),
\[ |f_{\beta}(p)| \leq 1, \quad \forall p \in \mathbb{D}. \]

With the maximal principle, we just need to show
\[ |f_{\beta}(p)| \leq 1, \quad \forall p \in \partial \mathbb{D} \text{ with any fixed } \beta \in \mathbb{D}. \]

However, for any fixed \(\beta \in \mathbb{D}\) and \(p \in \partial \mathbb{D}\), there exist \(\lambda_1, \lambda_2\) such that
\[ |\lambda_1| = |\lambda_2| = 1, \quad \beta = \frac{\lambda_1 + \lambda_2}{2} \text{ and } p = \lambda_1 \lambda_2. \]

So it suffices to show the following inequality
\[ |1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1 \lambda_2)| \geq \frac{1}{2} |\lambda_1 - \lambda_2| \]
holding for all \((\lambda_1, \lambda_2) \in \partial \mathbb{D} \times \partial \mathbb{D}\). By Lemma 3.2, we can conclude that
\[ \mathcal{L} \in \Gamma_p(1, 0, -1). \]
Now if \( k_3 = 0 \), then \( k_2 = 0 \). Easily, we can see that
\[
\mathcal{L} = \{ a = 1 \} \in \Gamma_{\mathcal{P}}(1, 0, -1);
\]
and if \( k_2 = 0 \), then \( \mathcal{L} = \{ a = 1 + k_3(1 + p) \} \). Notice that
\[
|1 + k_3(1 + p)| \geq 1 + k_3 - k_3|p| \geq 1.
\]
Hence, we can also see that \( \mathcal{L} \in \Gamma_{\mathcal{P}}(1, 0, -1) \).

Therefore, we have
\[
\Gamma_{\mathcal{P}}(1, 0, -1) = (\mathbb{C} \times \Gamma_{\mathcal{G}_2}(0, -1)) \cup \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2s + k_3(1 + p)\},
\]
where \( k_2 \) is real, \( k_3 \geq 0 \) and \( k_3^2 + k_3 \geq k_2^2 \).

(4) \( (s_0, p_0) \in \partial \mathcal{G}_2 \cap \Sigma \) and then \( a_0 = 0 \).

Now suppose that \( (a_0, s_0, p_0) = (0, 2, 1) \), and then by the same way, consider the hyperplane
\[
\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : a = k_2(2 - s) + k_3(1 - p)\} \in \Gamma_{\mathcal{P}}(0, 2, 1).
\]

Using the same argument, we obtain that for any \( (s, p) \in \overline{\mathcal{G}_2} \),
\[
|k_2(2 - s) + k_3(1 - p)|^2 \geq e^{-\varphi(s, p)}.
\]

Now set \( p = 1 \), then we have \( -2 \leq s \leq 2 \). So we obtain
\[
|k_2(2 - s)| \geq \sqrt{1 - \frac{1}{4}s^2}.
\]

Thus, such \( k_2 \) does not exist as \( s \to 2^- \). This leads that \( k_1 \) must be zero. Therefore, we obtain
\[
\Gamma_{\mathcal{P}}(0, 2, 1) = \mathbb{C} \times \Gamma_{\mathcal{G}_2}(2, 1).
\]

In summary, we can give a full description of \( \Gamma_{\mathcal{P}}(P_0) \) as follows.

**Theorem 3.3** We can give the description of the tangent hyperplanes to the pentablock \( \mathcal{P} \) at four different boundary points.

(1) \( \Gamma_{\mathcal{P}}(a_0, 1, 0) = \mathbb{C} \times \Gamma_{\mathcal{G}_2}(1, 0) \), \( |a_0| \leq \frac{1}{2} \);
(2) \( \Gamma_{\mathcal{P}}(a_0, 0, -1) = \mathbb{C} \times \Gamma_{\mathcal{G}_2}(0, -1) \), \( |a_0| < 1 \);
(3) \( \Gamma_{\mathcal{P}}(1, 0, -1) = (\mathbb{C} \times \Gamma_{\mathcal{G}_2}(0, -1)) \cup \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2s + k_3(1 + p)\} \), where \( k_2 \) is real, \( k_3 \geq 0 \) and \( k_3^2 + k_3 \geq k_2^2 \);
(4) \( \Gamma_{\mathcal{P}}(0, 2, 1) = \mathbb{C} \times \Gamma_{\mathcal{G}_2}(2, 1) \).
4 Proof of Theorem 1.1

Proof Linear convexity of \( \mathcal{P} \) implies that in the case of a smooth boundary point \( P_0 \in \partial \mathcal{P} \), the set \( \Gamma_1 P (P_0) \) is a singleton. Consider then the non-smooth point \( P_0 \in \partial \mathcal{P} \). By Theorem 2.4, it is sufficient to consider the only four different cases

\[
P_0 = (a_0,1,0), |a_0| \leq \frac{1}{2}; \\
P_0 = (a_0,0,-1), |a_0| < 1; \\
P_0 = (1,0,-1); \\
P_0 = (0,2,1).
\]

Then Theorems 3.3 and 2.2 imply that \( \Gamma_1 P (P_0) \) is the union of connected sets whose intersection is non-empty for the non-smooth boundary point \( P_0 \), so it is connected. Thus, by Theorem 2.5, we can conclude that the pentablock \( \mathcal{P} \) is \( \mathbb{C} \)-convex. This finishes the proof.

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