Particle Dynamics in a Class of 2-dimensional Gravity Theories

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ABSTRACT

We provide a method to determine the motion of a classical massive particle in a background geometry of 2-dimensional gravity theories, for which the Birkhoff theorem holds. In particular, we get the particle trajectory in a continuous class of 2-dimensional dilaton gravity theories that includes the Callan-Giddings-Harvey-Strominger (CGHS) model, the Jackiw-Teitelboim (JT) model, and the \( d \)-dimensional \( s \)-wave Einstein gravity. The explicit trajectory expressions for these theories are given along with the discussions on the results.
1 Introduction

The complexity of 4-dimensional general relativity encountered in its analytical treatment is considerable. After long and varied attempts, many interesting quantum and classical questions on the gravitation still resist their analytical solutions. One motivation for the development of 2-dimensional gravity theories in recent years is to search for the model theory that captures the essential features of 4-dimensional general relativity and at the same time provides us with a manageable framework of analysis. The big reduction of the degrees of freedom in the gravity sector of a 2-dimensional theory is obvious, while it is plausible that the s-wave sector of a 4-dimensional theory is, at least at a superficial level, 2-dimensional and thereby shares some common physical properties with 2-dimensional gravity theories, such as Jackiw-Teitelboim (JT) model and Callan-Giddings-Harvey-Strominger (CGHS) model. The developments in quantum gravitational issues, including the discussions of the black hole information paradox, afforded by these model theories are by now reported in literature.

The modest aim of this paper is to better understand the classical dynamics of a class of 2-dimensional dilaton gravity theories, which includes many theories of interest as its special cases. In particular, we analytically determine the motion of a point massive particle in the background space-time geometry of the 2-dimensional dilaton gravity theories. The gravity sector action in our consideration is thus given by

$$I_g = \int d^2 x \sqrt{-g} e^{-2\phi} \left[ R + \gamma g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \mu e^{2\lambda \phi} \right], \tag{1}$$

where $R$ and $\phi$ represent the 2-dimensional scalar curvature and the dilaton field, respectively. The parameters $\gamma$, $\mu$ and $\lambda$ are arbitrary real numbers. A specific choice of them corresponds to a particular gravity theory, as shown in the table.

| Theory            | $\gamma$ | $\lambda$ | $a = 2 - \lambda - \gamma/4$ |
|-------------------|----------|-----------|-----------------------------|
| JT                | 0        | 0         | 2                           |
| CGHS              | 4        | 0         | 1                           |
| d-D Einstein ($3 < d < \infty$) | $\frac{d-3}{d-2}$ | $\frac{2}{d-2}$ | $0 < \frac{d-3}{d-2} < 1$ |

Among many theories of the form, three cases have been under the most intensive scrutiny. First, in case of the JT model, one of the earliest models to realize non-trivial gravitational dynamics in 2-dimensional space-time, many properties and reformulations are available in literature. Second, the string-theory-inspired
CGHS model provided us with an analytically tractable framework for the (quantum) study of the gravitation [2]. Another case of importance is the spherically symmetric reduction of the $d$-dimensional Einstein-Hilbert action [6] [7]. In this case, the 2-d dilaton field $\phi$ is directly related to the geometric radius of the $(d-2)$-dimensional sphere in the $d$-dimensional spherically symmetric geometry. To be specific, we write the spherically symmetric $d$-dimensional metric $g^{(d)}_{\mu\nu}$ as the sum of longitudinal and transversal parts

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta - \exp\left(-\frac{4}{d-2}\phi\right)d\Omega^2,$$

where $d\Omega^2$ is the sphere $S^{d-2}$ of unit radius, and we use the metric signature $(+ - \cdots -)$. The $d$-dimensional Einstein-Hilbert action

$$I = \int d^d x \sqrt{-g^{(d)}} R^{(d)},$$

where $R^{(d)}$ is the $d$-dimensional scalar curvature, reduces to (1) with the specified values of parameters in the table (2).

The study of the classical particle trajectory in each model, which is the main focus of this paper, is a good way to (partially) understand the relationship among the theories described by (1) and, eventually, to gain more understandings of the Einstein gravity, its relation to string-motivated gravity, and peculiarities of each theory. To be specific, we will obtain the particle trajectory $x(\tau)$, i.e., the solution of the geodesic equation resulting from the point particle action

$$I_m = -m \int ds = -m \int d^2 x \int d\tau \delta^{(2)}(x-x(\tau)) \left[ g_{\alpha\beta} \dot{x}^\alpha(\tau) \dot{x}^\beta(\tau) \right]^{1/2}$$

in each theory. In case of the JT model, the same problem was solved by the authors of Ref. [8]. The main novelty here is to extend their analysis to the more general 2-dimensional dilaton gravity theories while, at the same time, making it more systematic. Our method relies on the underlying symmetries of the particle motion in a background geometry. In fact, the isometry of the background geometry we utilize in this paper to solve the motion exists for any gravity theory for which the Birkhoff theorem holds, as the consideration presented in section 2 shows. Thus, our method is applicable to a more general class of gravity theories than those considered here. For example, we can replace the dilaton potential in (1) with an arbitrary function of the dilaton field $\phi$ and, likewise, we can add $U(1)$ gauge fields to the action. The Birkhoff theorem still holds under these types of generalizations of (1) [7] [1].
The details of our method are explained in section 2, along with a review of the background geometry of the gravity sector in consideration. A new derivation of the background geometry is presented in Appendix. Although the most of the results in Appendix are already known [10], they are tailored for our current purpose. In section 3, we present the explicit expressions of the particle trajectory for the CGHS model, the JT model, and the 4-dimensional s-wave Einstein theory. We conclude this paper by discussing various aspects of our results in section 4.

2 Background Geometry and Particle Trajectory

In this section, we present the calculations leading to the analytic expression of the particle motion in a 2-dimensional gravity theory, for which the Birkhoff theorem holds. For this purpose, we start by reviewing the background geometry of the gravity sector in conformal gauge. This also serves to fix the notation. The new derivation of the background geometry in conformal gauge, as sketched in Appendix, motivates our calculations in section 2.2 and is instructive in its own right.

2.1 Background Geometry in Conformal Gauge

The equations of motion for the background geometry are given by varying the action $I_g$ in (1) with respect to the metric tensor $g^{\alpha\beta}$ and the dilaton field $\phi$:

$$D_\mu D_\nu \Omega - g_{\mu\nu} D \cdot D\Omega + \gamma \left[ \frac{g_{\mu\nu} (D\Omega)^2}{\Omega} - 2 \frac{D_\mu \Omega D_\nu \Omega}{\Omega} \right] + \mu \frac{2}{2} g_{\mu\nu} \Omega^{1-\lambda} = 0, \quad (6)$$

$$R + \gamma \left[ \frac{(D\Omega)^2}{\Omega^2} - 2 \frac{D \cdot D\Omega}{\Omega} \right] + (1 - \lambda) \mu \Omega^{-\lambda} = 0, \quad (7)$$

where $\Omega \equiv e^{-2\phi}$ and $D$ denotes the covariant derivative. In this paper, we choose to work in the conformal gauge where the metric takes the form

$$ds^2 = -e^{2\rho} dx^+ dx^-, \quad (8)$$

where $x^\pm \equiv x^1 \pm x^0$, and the flat space metric is given by $g_{00} = -g_{11} = 1$, or $g_{+-} = g_{-+} = -1/2$. Under this gauge choice, the equations of motion become

$$\partial_+ \partial_- (\rho + \frac{\gamma}{8} \ln \Omega) + \frac{\gamma}{8} \partial_+ \partial_\Omega + \mu \frac{(1 - \lambda)}{8} \Omega^{-\lambda} e^{2\rho} = 0, \quad (9)$$

*Under our signature choice, $x^0$ is a time-like coordinate and $x^1$ is a space-like coordinate.
\[ \partial_+ \partial_- \Omega + \frac{\mu}{4} \Omega^{1-\lambda} e^{2\rho} = 0. \]  
(10)

We should also impose the accompanying gauge constraints

\[ \frac{\delta I_g}{\delta g_{\pm\pm}} = 0, \]

which we can read off from (9) as

\[ \partial^2 \Omega - 2 \partial_\pm (\rho + \frac{\gamma}{8} \ln \Omega) \partial_\pm \Omega = 0. \]  
(11)

A specific set of solutions \((\rho, \Omega)\), obtained by solving Eqs.(9)-(11), corresponds to a specific background geometry. A method of getting the general solutions of the coupled partial differential equations in a local coordinate neighborhood is given in Appendix.

The general solutions in a local coordinate neighborhood depend on two arbitrary chiral functions \(f_{\pm}(x^\pm)\). In terms of these two chiral functions, we define \(X^\pm(x^\pm)\) via 
\[ dx^\pm/dX^\pm = \exp(f_{\pm}). \]  
We further introduce a variable

\[ x = X^+ + \varepsilon_1 X^- \]

where \(\varepsilon_1 = \pm 1\). We note that if \(\varepsilon_1 = +1\), \(x\) becomes a space-like variable, while \(\varepsilon_1 = -1\) makes \(x\) a time-like variable. The dilaton field \(\Omega = e^{-2\phi}\) depends only on \(x\) and is given implicitly by

\[ \int \frac{d\Omega}{\varepsilon k \Omega^a - M/a} = \int dx, \]  
(12)

where \(M\) is a real parameter and we define \(a \equiv 2 - \lambda - \gamma/4\). Here the constant \(k\) is

\[ k = \left| \frac{\mu e^{2\rho_0}}{4a} \right|^{1/2}. \]  
(13)

where \(\rho_0\) is another real parameter. The constant \(\varepsilon = \pm 1\), depending on other parameters. See Appendix for more explanations on this point. We can write down the conformal factor \(\rho\) as

\[ e^{2\rho} = -\text{sign} \left( \frac{\mu \varepsilon_1}{a} \right) \Omega^{-\gamma/4} e^{-(f_+ + f_-)} \left[ \Omega^a - \frac{M}{\varepsilon ka} \right] e^{2\rho_0}. \]  
(14)

We note that in typical situations, such as the JT model, the CGHS model, and the \(d\)-dimensional \(s\)-wave Einstein theory, \(\mu < 0\). The presence of two arbitrary chiral
fields in the general solutions simply represents the residual gauge symmetry under our gauge choice, namely, the classical conformal symmetry.

In Ref. [7] and [10], the same solutions as the above were obtained for $\varepsilon_1 = +1$ case, i.e., when the dilaton field is space-like, under a different gauge choice. The solutions for the CGHS case represent the dilatonic black hole and the linear dilaton vacuum [11]. In case of the JT theory, the local solutions were used to construct a cylindrical geometry [3]. For $\varepsilon_1 = -1$ case, we have a situation where the dilaton depends only on a time-like variable. Our further analysis is valid for every solution shown in the above, regardless of whether we have a static background geometry or not.

2.2 Particle Motion in a Background Geometry

We now consider the motion of a point particle in a generic background geometry described by (12) and (14). The particle trajectory on a generic background geometry is determined by the geodesic equation from (5)

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau},$$

(15)

where the usual Christoffel symbols are introduced as $\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda})$. As in section 2.1, we choose to describe the particle motion in the conformal gauge. Then, the equations of motion (15) become

$$\frac{d^2 x^\pm}{d\tau^2} = -2\partial_\pm \rho \left( \frac{dx^\pm}{d\tau} \right)^2.$$

(16)

We have a further freedom to choose a particular set of conformal coordinates, thereby fixing the residual conformal symmetry. It turns out that the choice of $(X^+(x^+), X^-(x^-))$ fields as our conformal coordinates provides a simplest description of the particle motion. Thus, we rewrite (16) as

$$\frac{d^2 X^\pm}{d\tau^2} = -2\frac{\partial \tilde{\rho}}{\partial X^\pm} \left( \frac{dX^\pm}{d\tau} \right)^2,$$

(17)

where the new conformal factor can be calculated to be $\tilde{\rho}(x) = \rho + (f_+ + f_-)/2$. The crucial property of this gauge fixing is that the function $\tilde{\rho}$ now depends on $(X^+, X^-)$ only through the combination $x = X^+ + \varepsilon_1 X^-$ (see Eq.(14) and Appendix). This is a consequence of the Birkhoff theorem, which ensures that the general solutions
of our gravity sector look locally isomorphic to static solutions, under a particular coordinate choice. For a gravity theory where the Birkhoff theorem holds, we can straightforwardly determine the particle motion using this property.

To integrate the system described by (17) reducing it to a first-order system, we need two symmetries. One of these symmetries is obvious; our particle action is invariant under the affine transformation of the proper time parameter $\tau$. The translation transformation of the proper time parameter, $\delta \tau = \epsilon_1$, where $\epsilon_1$ is a constant, is thus a symmetry. We can construct its Noether charge

$$c_0 = e^{2\tilde{\rho}} \dot{X}^+ \dot{X}^-,$$

where the overdot represents a differentiation with respect to $\tau$. Since we are interested in the motion of a massive particle, we set $c_0 = -1$. Eq.(18) then becomes the mass-shell condition for a massive relativistic particle. Another symmetry of the particle motion is due to the property of the background geometry. The single variable dependence of the conformal factor $\tilde{\rho}(x)$ permits us to easily find one isometry of the background geometry. Namely, under the transformation

$$\delta X^+ = \epsilon_2, \quad \delta X^- = -\epsilon_1 \epsilon_2,$$

where $\epsilon_2$ is a constant, the background metric is invariant. In case of a flat, static background geometry written in terms of tortoise coordinates, it reduces to the Lorentz transformations. This isometry of the background geometry induces another Noether charge

$$c_1 = e^{2\tilde{\rho}} (\dot{X}^- - \epsilon_1 \dot{X}^+)$$

for the motion of the particle. We have, thus, reduced our problem to a first-order system, Eqs.(18) and (19).

Further integration of the first-order system is also straightforward once we consider the properties of the background geometry. By solving (18) and (19), we have

$$\dot{X}^+ = \frac{-c_1 \pm \sqrt{c_1^2 - 4\epsilon_1 e^{2\tilde{\rho}}}}{2\epsilon_1 e^{2\tilde{\rho}}}, \quad \dot{X}^- = \frac{c_1 \pm \sqrt{c_1^2 - 4\epsilon_1 e^{2\tilde{\rho}}}}{2 e^{2\tilde{\rho}}}.$$

Noting that $x = X^+ + \epsilon_1 X^-$ and $y = X^+ - \epsilon_1 X^-$ are mutually orthogonal coordinates of the background geometry, we can rewrite (20) as

$$\dot{x} = \pm \epsilon_1 e^{-2\tilde{\rho}(x)} \sqrt{c_1^2 - 4\epsilon_1 e^{2\tilde{\rho}(x)}}$$

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Thus, the analysis here applies to more general actions than the class given in (1), as long as the Birkhoff theorem holds. In the context of the CGHS model, for example, we can consider a theory with the loop corrections from string theory.
and

\[ \dot{y} = -c_1 \varepsilon_1 e^{-2\tilde{\rho}(x)}. \]  

(22)

We now have explicitly decoupled differential equations. To get the parametrized form of the particle trajectory, we first solve for \( x(\tau) \) using (21). Then, \( y(\tau) \) can be obtained from (22). The particle trajectory can also be written as

\[ \int dy = \pm \int \frac{c_1 dx}{\sqrt{c_1^2 - 4\varepsilon_1 e^{2\tilde{\rho}(x)}}} \]  

(23)

from (21) and (22).

### 3 Explicit Examples

We evaluate the integrations in (23) for the JT model, the CGHS model and the 4-dimensional s-wave Einstein theory, to get explicit expressions for the particle trajectory. Our JT theory results are compared to those of Ref. \[8\]. We will discuss the CGHS model and the \( d \)-dimensional s-wave Einstein gravity together.

#### 3.1 Jackiw-Teitelboim Model

This model corresponds to the choice of \( \gamma = \lambda = 0 \) and \( \mu = -\Lambda/2 \) in (1). The \( \Omega \) field can be calculated from Eq. (12) to be

\[ \Omega = -\varepsilon \sqrt{\frac{M}{2\varepsilon k}} \tanh \left[ \sqrt{\frac{kM}{2\varepsilon}} (x - x_0) \right] \]  

(24)

where \( k = |\Lambda e^{2\rho_0}/16|^{1/2} \) and \( x_0 \) is the constant of integration. The conformal factor \( \tilde{\rho} \) is given from Eq. (14) as

\[ e^{2\tilde{\rho}} = \frac{|M| e^{2\rho_0} / (2k)}{\cosh^2 \sqrt{\frac{kM}{2\varepsilon}} (x - x_0)}. \]  

(25)

We note that hyperbolic functions in the above Eqs. (24) and (25) become trigonometric functions in case of \( M/\varepsilon < 0 \) using \( \tanh ix = i \tan x \) and \( \cosh ix = \cos x \).

\[ \text{We actually find, in that case, the solutions presented in [8] are not complete, since we find another class of trajectories.} \]
Using $e^{2\rho} = e^{2\tilde{\rho} - f_+ - f_-}$, the metric becomes

$$e^{2\rho} = \text{sign}\left(\frac{\varepsilon \Lambda}{M \varepsilon_1}\right) \frac{2\partial_+ A \partial_- B}{(1 + \Lambda AB/8)^2}$$

(26)

where $A(x^+) = (8/\Lambda)^{1/2} e^{\sqrt{2kM/\varepsilon}(X^+ - x_0/2)}$ and $B(x^-) = (8/\Lambda)^{1/2} e^{\sqrt{2kM/\varepsilon}(X^- - x_0/2)}$. The pre-factor \(\text{sign}(\varepsilon \Lambda/M \varepsilon_1)\) of the right hand side of Eq.(26) can be determined from the table (60) by multiplying the first, the second and the fourth column. We find there are two cases when this pre-factor is positive, i.e., the third and the last row of the table (60). In these cases, however, the range of $\Omega'$, which is given by $\Omega' = -(M/2) \text{sech}^2(\sqrt{kM/2\varepsilon})(x - x_0)$, is inconsistent with the range given in the table (60). For consistency, we have to consider only the case of $\text{sign}(\varepsilon \Lambda/M \varepsilon_1) = -1$. This result is consistent with that of Ref. [8].

When $\Lambda > 0$, $M > 0$ and $\varepsilon_1 = -1$, we get $\varepsilon = +1$ from the table (60). Then, the field $\Omega$ is calculated from Eq. (24) to be

$$\Omega = -\sqrt{M/2k} \tanh \left[\sqrt{\frac{kM}{2}}(x - x_0)\right].$$

(27)

We can easily verify that $\Omega$ satisfies the condition, $-M/2 < \Omega' < 0$, in the table (60). We get $\tilde{\rho}$ from Eq. (25)

$$e^{2\tilde{\rho}} = \frac{M e^{2\rho_0}/(2k)}{\cosh^2 \sqrt{\frac{kM}{2}}(x - x_0)}$$

(28)

and the particle trajectory is given by Eq. (23)

$$\sinh \sqrt{\frac{kM}{2}}(x - x_0) = \pm \sqrt{1 + \frac{2Me^{\rho_0}}{kc_2^2}} \sinh \sqrt{\frac{kM}{2}}(y - y_0).$$

(29)

This is the result obtained in Ref. [8]. Here, we find another class of trajectory not reported there. When $\Lambda > 0$, $M > 0$ and $\varepsilon_1 = +1$, from the table (60), we have $\varepsilon = -1$ and $\Omega' < -M/2$. The results of our calculations in this case are

$$\Omega = -\sqrt{M/2k} \tan \left[\sqrt{\frac{kM}{2}}(x - x_0)\right],$$

(30)

$$e^{2\tilde{\rho}} = \frac{M e^{2\rho_0}/(2k)}{\cos^2 \sqrt{\frac{kM}{2}}(x - x_0)},$$

(31)

and

$$\sin \sqrt{\frac{kM}{2}}(x - x_0) = \pm \sqrt{1 - \frac{2Me^{\rho_0}}{kc_2^2}} \sin \sqrt{\frac{kM}{2}}(y - y_0).$$

(32)
3.2 CGHS and 4-Dimensional Model

We investigate the particle motion in the CGHS model and the spherically symmetric 4-dimensional Einstein gravity. At the classical level, we can treat them together since the CGHS model ($a = 1$) is the same as the $d = \infty$ limit of the spherically symmetric $d$-dimensional gravity where $a = (d-3)/(d-2)$. To simplify the presentation, we regard $\tilde{\rho}$ as a function of $x$ through $\Omega$. We have

$$e^{2\tilde{\rho}} = -\text{sign} \left( \frac{\mu \varepsilon_1}{a} \right) e^{2\rho_0} \left[ 1 - \frac{M}{\varepsilon k a} \Omega^{-a} \right]$$

from Eq.(32). The particle trajectory given by Eq.(23) becomes

$$\int dy = \pm \frac{1}{M} \int dt \frac{\varepsilon k a}{\sqrt{1 - C t}} \left[ \frac{\varepsilon k a M}{M} \left( \text{sign} \left( \frac{\mu}{a} \right) t + 1 \right) \right]^{-1/a},$$

(34)

where $t = -\text{sign}(\mu/a)[1 - M(\varepsilon k a \Omega^a)^{-1}]$ and $C = 4e^{2\rho_0}/c_1^2$. Now we consider the particle trajectory when $M > 0$, $\varepsilon_1 = +1$ and $\mu < 0$, since this choice represents the black hole solutions, as can be seen in Ref. [10]. From the table (60), then, we have $\varepsilon = +1$ and $0 < \Omega'$. Note that the variable $t(\Omega)$ satisfies $0 < t < 1$, which follows from $0 < \Omega'$. We note $t \sim 0$ and $t \sim 1$ represent the space-time region near the black hole horizon and the region close to the asymptotic infinity, respectively.

In case of the CGHS model, we have $a = 1$. From Eq.(12), we get

$$\Omega = \frac{M}{k} + e^{\varepsilon k(x-x_0)},$$

(35)

the dilatonic black hole. The Eq. (14) gives us $\tilde{\rho}$ as

$$\tilde{\rho} = \frac{1}{2} k(x - x_0) - \frac{1}{2} \ln \Omega + \rho_0.$$  

(36)

By integrating Eq.(34), we get the particle trajectory as

$$y - y_0 = \pm \frac{1}{k} \left[ -2 \tanh^{-1} \sqrt{1 - C t} + \frac{2}{\sqrt{1 - C}} \tanh^{-1} \sqrt{\frac{1 - C t}{1 - C}} \right].$$

(37)

In case of the 4-dimensional Einstein theory, we have the parameter of $a = 1/2$. We get from Eq.(12)

$$2\Omega^{1/2} + \frac{4M}{k} \ln \left| \Omega^{1/2} - \frac{2M}{k} \right| = k(x - x_0),$$

(38)
where \( k = |\mu e^{2\alpha_0}/2|^{1/2} \) and \( x_0 \) is a constant of integration. The conformal factor \( \tilde{\rho} \) can be calculated from Eq. (18) and Eq. (14). Using Eq. (14), the particle trajectory can be calculated to be

\[
y - y_0 = \pm \frac{4M}{k^2} \left[ -2 \tanh^{-1} \sqrt{1 - Ct} + \frac{2 - 3C}{(1 - C)^{3/2}} \tanh^{-1} \left( \frac{1 - Ct}{1 - C} \right) + \frac{\sqrt{1 - Ct}}{(1 - C)(1 - t)} \right].
\] (39)

4 Discussions

We presented a method to determine the motion of a test particle in a background geometry of 2-dimensional gravity theories satisfying the Birkhoff theorem. The existence of the isometry of the background geometry that played a crucial role in our method is not specific to a certain choice of the gravity sector, but it is the general property shared by the general 2-dimensional dilaton gravity theory. It is interesting to note that this type of unified description is possible for a large class of theories. We expect the local solutions we get in this paper can be useful building blocks to form non-trivial global structures of the space-time and the resulting particle motion. The study of possible global constructions can further illuminate the similarities and differences among the theories we consider in this paper.

Given our results, there are some further issues that can be addressed. The detailed structure of the isometries of the background geometry is interesting in itself. In flat space-time geometry, our gravity sector’s isometries form the 2-dimensional Poincaré algebra. As the curvature effects creep in, this algebra gets deformed. The isometry we utilized in this work is one part of that algebra. From our solutions, we can possibly uncover the detailed structure of the deformed algebra.

The quantization of a massive particle in a background geometry, including black hole geometries and other non-trivial global geometries, is an issue of great importance. The general solutions for the particle trajectories we obtained in this paper can be a useful starting point for such investigation. By calculating the symplectic structure on the space of all classical solutions, we can see the structure of the quantum phase space and proceed to the quantization of our classical problem. Our work in this direction is in progress, for the purpose of providing a simple quantum mechanical system that can capture essential features of the quantum black hole physics.
Appendix : Derivation of Background Geometry

We obtain the general (local) solution $\rho$ and $\Omega$ of (9) and (10) under the constraints (11). For convenience, we introduce $\bar{\rho} = \rho + (\gamma \ln \Omega) / 8$. By integrating the gauge constraints (11), we get

$$\ln |\partial \pm \Omega| = 2\bar{\rho} + f_\pm(x^\mp),$$  \hspace{1cm} (40)

where $f_+(x^+)$ and $f_-(x^-)$ are arbitrary functions of $x^+$ and $x^-$, respectively. By taking the difference of two equations in (40), we get

$$\frac{\partial \Omega}{\partial X^+} = -\frac{\partial \Omega}{\partial X^-},$$  \hspace{1cm} (41)

where $X^\pm$ is defined by $dx^\pm / dX^\pm \equiv e^{f_\pm}$. This means that $\Omega$ is a function of only one variable $x \equiv X^+ + \varepsilon_1 X^-$ where $\varepsilon_1 = \pm 1$, since (41) implies that $\Omega = \Omega(X^+ \pm X^-)$. Furthermore, since $\partial_+ \partial_- \Omega = \varepsilon_1 e^{-f_+ - f_-} \Omega''$ (where the prime denotes the differentiation with respect to $x$), we find, from (10), that $\hat{\rho}(x) \equiv \bar{\rho} + (f_+ + f_-)/2$ is also a function of only $x$. Using this property we can rewrite the equations of motion (9) and (10) as

$$\hat{\rho}'' + \frac{\mu}{8} \varepsilon_1 (1 - \lambda - \gamma/4) \Omega^{-\lambda-\gamma/4} e^{2\hat{\rho}} = 0,$$  \hspace{1cm} (42)

$$\Omega'' + \frac{\mu}{4} \varepsilon_1 \Omega^{1-\lambda-\gamma/4} e^{2\hat{\rho}} = 0,$$  \hspace{1cm} (43)

and the gauge constraint (11) as

$$\Omega'' - 2\hat{\rho}' \Omega' = 0.$$  \hspace{1cm} (44)

Thus, our system of partial differential equations effectively becomes a system of ordinary differential equations. This provides an alternative proof of the Birkhoff theorem, originally proved for this type of theories in [7] and [9].

General solutions of (42) and (43) under the gauge constraint (44) can be obtained by following the method used in Ref. [10]. First, we find an effective action that produces (42) and (43)

$$I_{\text{eff}} = \int dx \left( \hat{\rho}' \Omega' - \frac{\mu}{8} \varepsilon_1 e^{2\hat{\rho}} \Omega^{1-\lambda-\gamma/4} \right).$$  \hspace{1cm} (45)
Then we find the symmetries of the effective action to construct the Noether charges corresponding to them. We note that the effective action (45) is invariant under

\[
\delta x = \epsilon_1, \\
\delta x = \epsilon_2 x, \quad \delta \Omega = \epsilon_2 \Omega, \quad \delta \hat{\rho} = -\frac{1}{2} \epsilon_2 \left( 2 - \lambda - \frac{\gamma}{4} \right),
\]

producing two Noether charges

\[
q = \hat{\rho}' \Omega' + \frac{\mu}{8} \epsilon_1 e^{2\hat{\rho} \Omega^{1-\lambda-\gamma/4}},
\]

\[
\frac{M}{2} = -q x - \frac{1}{2} \Omega' \left( 2 - \lambda - \frac{\gamma}{4} \right) + \hat{\rho}' \Omega.
\]

We can alternatively integrate the original second-order system, Eqs. (42) and (43), by expressing them as the conservation laws of the Noether charges, to obtain the first-order system shown above. Now, the gauge constraint (44) fixes \(q = 0\). With \(q = 0\), by integrating (49), we get

\[
\hat{\rho} = \frac{M}{2} \int \frac{dx}{\Omega} + \frac{1}{2} a \ln \Omega + \rho_0,
\]

where \(a \equiv 2 - \lambda - \gamma/4\) and \(\rho_0\) is a constant of integration. Plugging (50) into (48), we get a decoupled equation for \(\Omega\) that does not contain \(\hat{\rho}\):

\[
M \Omega' + a (\Omega')^2 = -\frac{\mu}{4} \epsilon_1 \Omega^{2a} \exp \left( M \int \frac{dx}{\Omega} + 2 \rho_0 \right).
\]

We can solve this equation as follows. We differentiate it with respect to \(x\) to get

\[
[M + 2a \Omega'] [\Omega'' - (M + a \Omega') \Omega'/\Omega] = 0.
\]

When the first factor of the left hand side of Eq. (52) vanishes, we have \(\Omega'' = 0\), and then \(\Omega = 0\) from (43). That means \(\phi = \infty\) everywhere, and we discard this case. When the second factor vanishes, we have

\[
\frac{d^2 \Omega}{dx^2} = \left[ M + a \frac{d\Omega}{dx} \right] \frac{d\Omega}{dx} \frac{1}{\Omega},
\]

which is integrated to yield

\[
k \Omega^a = \left| \frac{M}{a} + \frac{d\Omega}{dx} \right| = \varepsilon \left( \frac{M}{a} + \frac{d\Omega}{dx} \right),
\]

13
where $k$ is a positive constant of integration and the constant $\varepsilon = +1$ or $-1$ that comes in when we remove the absolute sign. We must fix the constant $k$ since both (54) and the original equation (51) are first order differential equations. By integrating (53), we get

$$\int \frac{M}{\Omega} \, dx = \ln \left| \frac{d\Omega/dx}{M/a + d\Omega/dx} \right|. \quad (55)$$

By plugging (54) and (55) into (51), we find that $k$ is fixed as

$$k = \left| \frac{\mu e^{2\rho_0}}{4a} \right|^{1/2}. \quad (56)$$

By integrating (54), we obtain the solution for $\Omega$:

$$\int \frac{d\Omega}{\varepsilon k \Omega^a - M/a} = \int dx. \quad (57)$$

From (54) we obtain the solution for $\hat{\rho}$:

$$\hat{\rho} = \frac{1}{2} \ln \left| \varepsilon \Omega^a - \frac{M}{ka} \right| + \rho_0. \quad (58)$$

Using Eqs. (51) and (53), the sign of $\Omega' = \varepsilon k \Omega^a - M/a$ can be calculated as $\text{sign}(\Omega') = -\text{sign}(\mu \varepsilon_1 \varepsilon/a)$, from which we get

$$e^{2\hat{\rho}} = \text{sign} \left( \frac{\mu \varepsilon_1}{a} \right) \left[ \frac{M}{\varepsilon ka} - \Omega^a \right] e^{2\rho_0}. \quad (59)$$

This equation, along with (57), provides us with the desired results, namely, Eqs. (12) and (14). We determine the value of $\varepsilon$ as follows; from the requirement that the exponential function in Eq. (51) should be positive, we determine the range of $\Omega'$. Then, using the range, we choose the value of $\varepsilon$ that makes the right hand side of (54) positive. To summarize, we have

| $\mu \varepsilon_1/a$ | $M/a$ | range of $\Omega'$ | $\varepsilon$ |
|----------------------|-------|---------------------|-------------|
| $+$                  | $+$   | $-M/a < \Omega' < 0$ | $+1$        |
| $+$                  | $-$   | $0 < \Omega' < -M/a$ | $-1$        |
| $-$                  | $+$   | $0 < \Omega'$        | $+1$        |
| $-$                  | $+$   | $\Omega' < -M/a$     | $-1$        |
| $-$                  | $-$   | $-M/a < \Omega'$     | $+1$        |
| $-$                  | $-$   | $\Omega' < 0$        | $-1$        |
where + and − indicate the signs. For example, to have a description of the 4-dimensional Schwarzschild geometry, where \( a = \frac{1}{2}, \mu = -2 \) and the black hole mass \( M > 0 \), we need the first and the third row results. For the inside of the black hole, the dilaton field depends on the time-like variable (therefore \( \varepsilon_1 = -1 \)) and the first row is the case. For the outside of the black hole, the dilaton field depends on the space-like variable (therefore \( \varepsilon_1 = +1 \)) and we have the third row.

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