Early collective expansion:
Relativistic hydrodynamics and the transport properties of QCD matter

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1. Introduction

The idea of exploiting the laws of ideal hydrodynamics to describe the expansion of the strongly interacting matter that is formed in high energy hadronic collisions was first formulated by Landau in 1953 [1]. Because of their conceptual beauty and simplicity, models based on hydrodynamic principles have been applied to calculate a large number of observables for various colliding systems and over a broad range of beam energies. However, it is by no means clear that the highly excited, but still small systems produced in those violent collisions satisfy the criteria justifying a dynamical treatment in terms of a macroscopic theory which follows idealized laws. Indeed, the history of using hydrodynamics for high-energy phenomenology is checkered, with qualitative successes overshadowed by quantitative failures. Only recently, with data from the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (c.f. the experimental reviews [2–5]), came striking evidence for a strong collective expansion that is, for the first time, in good quantitative agreement with hydrodynamic predictions, at least for the largest collision systems (e.g. Au+Au) at the highest collision energies ($\sqrt{s} = 200$ GeV per nucleon pair) near midrapidity at small to moderate impact parameters. The long list of qualifiers towards the end of the preceding sentence points to continuing limitations of hydrodynamics, at least in its idealized perfect fluid limit: dissipative effects become increasingly important for smaller collision systems, lower collision energies, larger impact parameters and when one moves away from midrapidity. However, once well-calibrated ideal fluid dynamical benchmarks have been established under appropriate experimental conditions, deviations from perfect fluid behaviour can be used to explore transport properties, such as viscosity and heat conduction, of the QCD matter created in the collisions. Such efforts define the present forefront of research in heavy-ion collision dynamics.

The validity of ideal hydrodynamics requires local relaxation times towards thermal equilibrium that are much shorter than any macroscopic dynamical time scale. The significance and importance of rapid thermalization of the created fireball matter cannot be over-stressed: Only if the system is close to local thermal equilibrium, its thermodynamic properties, such as its pressure, entropy density and temperature, are well defined. And only under these conditions can we pursue to study the equation of state of strongly interacting matter at high temperatures. This is particularly interesting in the light of the expected phase transition of strongly interacting matter which, at a critical energy density of about 1 GeV/fm$^3$, undergoes a transition from a hadron resonance gas to a hot and dense plasma of color deconfined quarks and gluons. Lattice QCD calculations indicate [6–9] that this transition takes place rather rapidly at a critical temperature $T_{\text{crit}}$ somewhere between 150 and 190 MeV.

In this article I review and discuss data and calculations that provide strong evidence that the created fireball matter reaches temperatures above $2T_{\text{crit}}$, and which indicate short thermalization times of order 1–2 fm/c. After a pedagogical introduction into the foundations of relativistic hydrodynamics and of the relativistic fluid dynamic equations for ideal and dissipative fluids, I discuss appropriate initial and final conditions for the hydrodynamic expansion stage. I describe a few important aspects of the fireball evolution in central and non-central heavy-ion collisions and the calculation of final hadron spectra. Here the anisotropy of the final momentum spectra in non-central collisions plays a particularly important role because it provides both evidence for fast thermalization in RHIC collisions and access to transport properties of the quark-gluon matter created early in the collision. Based on a comparison with experimental data, I delineate our present knowledge (and its limits) of the properties of QCD matter created at RHIC, and outline future opportunities for quantitative improvement of our understanding of heavy-ion collision dynamics.
2. The equations of relativistic hydrodynamics

2.1. Ideal fluid dynamics for perfect fluids

Any fluid dynamical approach starts from the local conservation laws for energy-momentum and any conserved charges:

\[ \partial_\mu T^{\mu\nu} = 0, \]

\[ \partial_\mu N_i^\mu = 0, \quad i = 1, \ldots, k. \]  (1)

\[ \partial_\mu N_i^\mu = 0, \quad i = 1, \ldots, k. \]  (2)

For simplicity we restrict ourselves to \( k = 1 \) (say, \( N^\mu = \text{net baryon number current} \)). One must also ensure the second law of thermodynamics

\[ \partial_\mu S^\mu \geq 0, \]  (3)

where \( S^\mu \) is the entropy current. Ideal fluid dynamics follows from these equations under the assumption of local thermal equilibrium, i.e. if the microscopic collision time scale is very much shorter than any macroscopic evolution time scale such that the underlying phase-space distribution \( f(x,p) \) relaxes essentially instantaneously to the local equilibrium form (upper signs for fermions, lower signs for bosons)

\[ f_{eq}(x,p) = \frac{1}{e^{[\mu u(x)+\mu(x)]/T(x)} \pm 1}. \]  (4)

Here \( u^\mu(x) \) is the local fluid velocity at point \( x \), \( \mu(x) \) is the local chemical potential associated with the conserved charge \( N \) (it enters with opposite sign in the distribution \( \bar{f} \) for antiparticles), and \( T(x) \) is the local temperature. Plugging this into the kinetic theory definitions

\[ N_i^\mu(x) = \frac{1}{(2\pi)^3} \sum n_i \int \frac{d^3p}{E} p^\mu f_i(x,p), \]  (5)

\[ T^{\mu\nu}(x) = \frac{1}{(2\pi)^3} \sum \int \frac{d^3p}{E} p^\mu p^\nu f_i(x,p), \]  (6)

\[ S^\mu(x) = -\frac{1}{(2\pi)^3} \sum \int \frac{d^3p}{E} p^\mu \left[ f_i(x,p) \ln f_i(x,p) + (1\mp f_i(x,p)) \ln (1\mp f_i(x,p)) \right], \]  (7)

(where the sum is over all particle species (counting particles and antiparticles separately) and \( n_i \) is the amount of conserved charge \( N \) carried by species \( i \)) leads to the ideal fluid decompositions

\[ N_{eq}^\mu = n u^\mu, \]  (8)

\[ T_{eq}^{\mu\nu} = e u^\mu u^\nu - p \Delta^{\mu\nu} \quad \text{(with} \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu), \]  (9)

\[ S_{eq}^\mu = s u^\mu, \]  (10)

where the local net charge density \( n \), energy density \( e \), pressure \( p \) and entropy density \( s \) are given by the standard integrals over the thermal equilibrium distribution function in the local fluid rest frame. They are related by the fundamental thermodynamic relation

\[ T s = p - \mu n + e. \]  (11)

Inserting Eqs. (8)-(10) into Eqs. (2) and (1) yields the relativistic ideal fluid equations shown in Eqs. (12)-(14) below. Using Eq. (11) together with the Gibbs-Duhem relation \( dp = s dT + n d\mu \), it is easy to prove that, in the absence of shock discontinuities, these equations also conserve entropy, i.e. \( \partial_\mu S^\mu = 0 \).
Note that the validity of the decompositions (3)–(10) only requires local momentum isotropy (i.e. that in the local fluid rest frame the phase-space distribution reduces to a function of energy \(E\) only, \(f(x, p) = f(p \cdot u(x); T(x), \mu(x))\)), but not that the distribution function has the specific exponential form (4) that maximizes entropy. This may have relevance in situations where the time scale for local momentum isotropization is much shorter than for thermalization [10–12] (i.e. it is much easier to change the direction of the particles’ momenta than their energies), with the macroscopic hydrodynamic time scale in between.

In this case the local microscopic states would not maximize entropy, and the relation (11) between the quantities \(e, p, n, s\) defined through Eqs. (5)–(10) would not hold. Still, these quantities would follow ideal fluid dynamical evolution since entropy production by microscopic kinetic energy-shifting processes would only happen on time scales that are large compared to the macroscopic evolution time scales.

The ideal fluid equations read (with \(\varepsilon \equiv \partial \cdot u\) denoting the local expansion rate and \(c_s^2 = \frac{\partial p}{\partial e}\) the squared speed of sound)

\[
\dot{n} = -n \varepsilon, \\
\dot{e} = -(e + p) \varepsilon, \\
\dot{u}^\mu = \frac{\nabla^\mu p}{e + p} = \frac{c_s^2}{1 + c_s^2} \nabla^\mu e.
\]

Here we decomposed the partial derivative \(\partial^\mu = u^\nu \partial_\nu + \nabla^\mu\) into “longitudinal” and “transverse” components \(D = u^\nu \partial_\nu\) and \(\nabla^\mu = \Delta^\mu_\nu \partial_\nu\), which in the local fluid rest frame reduce to the time derivative \(\dot{f} \equiv D f\) and spatial gradient \(\nabla f\). The first two equations describe the dilution of the local baryon and energy density due to the local expansion rate \(\varepsilon\), while the third describes the acceleration of the fluid by the spatial pressure gradients in the local rest frame, with the enthalpy \(e + p\) acting as inertia. The second equality in Eq. (14) exhibits the manifest scale invariance of the ideal fluid dynamical equations (the absolute normalization of the energy density profile drops out) and demonstrates that the dynamical “pushing power” of the medium is related to the “stiffness” \(\frac{\partial p}{\partial e}\) of its Equation of State (EOS) \(p = p(e, n)\), reflected in the (temperature-dependent) speed of sound \(c_s(T)\). Together with the EOS, the 5 equations (12)–(14) form a closed set from which the fields \(n, e, p(n, e)\) and \(u^\mu\) (with \(u^\mu u_\mu = 1\)) can be determined.

### 2.2. Dissipative fluid dynamics for viscous relativistic fluids

As the hydrodynamic evolution changes the local energy and baryon density, microscopic processes attempt to readjust the local phase-space distribution to corresponding new local temperatures and chemical potentials. If this does not happen fast enough, the phase-space distribution will start to deviate from its local equilibrium form (3): \(f(x, p) = f_{eq}(p \cdot u(x); T(x), \mu(x)) + \delta f(x, p)\). The optimal values for the (readjusted) local temperature and chemical potential in the first term are fixed by imposing the “Landau matching conditions” [14]

\[
u_\mu \delta T^{\mu\nu} u_\nu = \int \frac{d^3p}{E} (u \cdot p)^2 \delta f(x, p) = 0, \quad u_\mu \delta N^\mu = \int \frac{d^3p}{E} (u \cdot p) \delta f(x, p) = 0.
\]

*In the absence of such a clear separation of time scales entropy production can not be neglected during the macroscopic evolution, and ideal fluid dynamics must be replaced by dissipative fluid dynamics. Furthermore, rapid longitudinal expansion at early times causes strong viscous effects that act against rapid local isotropization [13] of the momentum distribution. Ideal fluid dynamics becomes valid only after these viscous effects have died away.*
The remaining deviations $\delta f$ from local equilibrium generate additional terms in the decompositions of $N^\mu$, $T^{\mu\nu}$, and $S^\mu$:  
\begin{align}
N^\mu &= N_{\text{eq}}^\mu + \delta N^\mu = n u^\mu + V^\mu, \\
T^{\mu\nu} &= T_{\text{eq}}^{\mu\nu} + \delta T^{\mu\nu} = e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu, \\
S^\mu &= S_{\text{eq}}^\mu + \delta S^\mu = n u^\mu + \Phi^\mu.
\end{align}

The new terms describe a baryon flow $V^\mu = \Delta^{\mu\nu}N_\nu$ in the local rest frame, an energy flow $W^\mu = \frac{e+p}{n}V^\mu + q^\mu$ (where $q^\mu$ is the “heat flow vector”) in the local rest frame, the viscous bulk pressure $\Pi = -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} - p$ (which contributes to the trace of the energy momentum tensor), the traceless viscous shear pressure tensor $\pi^{\mu\nu} = T^{(\mu\nu)} = \frac{1}{2}(\Delta^{\nu\sigma}\Delta^{\mu\tau} + \Delta^{\mu\sigma}\Delta^{\nu\tau}) - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\sigma\tau}T_{\tau\sigma}$ (where the expression $\langle \mu\nu \rangle$ is a shorthand for “traceless and transverse to $u_\mu$ and $u_\nu$”, as defined by the projector in square brackets), and an entropy flow vector $\Phi^\mu$ in the local rest frame.

The matching conditions [15] leave the choice of the local rest frame velocity $u^\mu$ ambiguous. This ambiguity can be used to eliminate either $V^\mu$ from Eq. (16) (“Eckart frame” $u^\mu = N^\mu/\sqrt{N\cdot N} = N^\mu/n$, no baryon flow in the local rest frame [15]), in which case the energy flow reduces to the heat flow vector $W^\mu = q^\mu$, or $W^\mu$ from Eq. (17) (“Landau frame” $u^\mu = T^{\mu\nu}u_\nu/\sqrt{u_\alpha T^{\alpha\beta}T_{\beta\gamma}u^\gamma} = T^{\mu\nu}u_\nu/e$, corresponding to no energy flow in the local rest frame, $u_\mu\delta T^{\mu\nu} = 0$ [14]), in which case there is a non-zero baryon flow $V^\mu = -\frac{n}{e+p}q^\mu$ due to heat conduction in the local rest frame. For systems with vanishing net baryon number (as approximately realized in RHIC collisions) the Eckart frame is ill-defined and heat conduction disappears as an independent transport effect [16], so we will use the Landau frame.

Inserting the decomposition (18) into the conservation law (11) and projecting onto time-like and space-like components yields the non-ideal fluid equations for baryon-free systems in the Landau frame
\begin{align}
\dot{e} &= -(e+p+\Pi)\partial + \pi_{\mu\nu}\nabla^{(\mu}u^{\nu)}, \\
(e+p+\Pi)\dot{u}^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}u^\nu.
\end{align}

The non-equilibrium decompositions (16)-(18) involve 1+3+5=9 additional dynamical quantities, the “dissipative flows” $\Pi$, $q^\mu$, and $\pi^{\mu\nu}$ (the counting reflects their transversality to $u^\mu$ and the tracelessness of $\pi^{\mu\nu}$). This means that we need 9 additional dynamical equations which should be compatible with the underlying transport theory for the non-equilibrium deviation $\delta f(x,p)$. For the baryon-free case without heat conduction, Eqs. (19)-(20), the number of needed additional equations reduces to 6.

### 2.3. Transport equations for the dissipative flows

The key property of the kinetic equation governing the evolution of the phase-space distribution function $f = f_{\text{eq}} + \delta f$ is that the collision term satisfies the second law of thermodynamics [8], i.e. entropy is produced until the system has reached a new state of local thermal equilibrium. Here, we don’t want to solve the kinetic theory; instead, we want to write down a phenomenological macroscopic theory which is consistent with the constraints arising from the underlying kinetic theory, in particular the 2nd law. The macroscopic theory will be constructed from an expansion of the entropy production rate in terms of the dissipative flows which themselves are proportional to the off-equilibrium deviation $\delta f$ of the phase-space distribution [17,18]. Assuming the latter to be small, $|\delta f| \ll |f_{\text{eq}}|$, this expansion will be truncated at some low order in the dissipative flows $\delta N^\mu$, $\delta T^{\mu\nu}$. 
The expansion will involve phenomenological expansion coefficients which, in principle, should be matched to the kinetic theory [18–20]. In practice, they will often be considered as phenomenological parameters to be adjusted to experimental data. In the end, the extracted values must then be checked for consistency with the entire approach, by making sure that the dissipative corrections are indeed sufficiently small to justify truncation of the expansion a posteriori.

The equilibrium identity (11) can be rewritten as

\[ S^\mu_{\text{eq}} = p(\alpha, \beta)\beta^\mu - \alpha N^\mu_{\text{eq}} + \beta_\nu T^\nu_{\text{eq}}, \] (21)

where \( \alpha \equiv \frac{\eta}{T}, \beta \equiv \frac{\lambda}{T}, \) and \( \beta_\nu \equiv \frac{\pi_{\nu\mu}}{T} \). The most general off-equilibrium generalization of this is [18]

\[ S^\mu = S^\mu_{\text{eq}} + \Phi^\mu = p(\alpha, \beta)\beta^\mu - \alpha N^\mu + \beta_\nu T^\nu + Q^\mu(\delta N^\mu, \delta T^\nu), \] (22)

where, in addition to the first order contributions implicit in the second and third terms of the r.h.s., \( Q^\mu \) includes terms which are second and higher order in the dissipative flows \( \delta N^\mu \) and \( \delta T^\nu \).

The form of the expansion (22) is constrained by the 2\textsuperscript{nd} law constraint can be further recast into

\[ T \partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^\mu X_{\nu} + T \delta_\mu Q^\mu \geq 0. \] (25)

2.3.1. Standard dissipative fluid dynamics (first-order or Navier-Stokes theory)

The standard approach [14] neglects the higher order contributions to the entropy current and sets \( Q^\mu = 0 \). The inequality (25) can then always be satisfied by postulating linear relationships between the dissipative flows and the thermodynamic forces ("Navier-Stokes relations"),

\[ \Pi = -\zeta \partial, \quad q^\nu = -\lambda n \frac{T^2}{c + p} \nabla^\nu \left( \frac{\mu}{T} \right), \quad \pi^{\mu\nu} = 2 \eta \nabla \left( \mu u^\nu \right) + \pi_\alpha^\beta \pi_\beta^\alpha, \] (26)

with positive transport coefficients \( \zeta \geq 0 \) (bulk viscosity), \( \lambda \geq 0 \) (heat conductivity), and \( \eta \geq 0 \) (shear viscosity):

\[ T \partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\alpha q_\alpha}{2\lambda T} + \frac{\pi^{\alpha\beta} \pi_{\alpha\beta}}{2\eta} \geq 0. \] (27)

(The minus sign in front of the second term is necessary because \( q^\mu \), being orthogonal to \( u^\mu \), is spacelike, \( q^2 < 0 \).) Equations (26) are the desired 9 equations for the dissipative flows. Note that the entropy production rate (27) is of second order in the dissipative flows.

Unfortunately, using these relations in the hydrodynamic equations (19)-(20) leads to hydrodynamic evolution with acausal signal propagation: if in a given fluid cell at a certain time a thermodynamic force happens to vanish, the corresponding dissipative flow also stops instantaneously.
This contradicts the fact that the flows result from the forces through microscopic scattering which involves relaxation on a finite albeit short kinetic time scale. To avoid this type of acausal behaviour one must keep $Q^\mu$.

2.3.2. Second-order Israel-Stewart theory

A causal theory of dissipative relativistic fluid dynamics is obtained by keeping $Q^\mu$ up to terms which are second order in the irreversible flows [18, 22]. For simplicity we here consider only the baryon-free case $n = q^\mu = 0$; see [18, 23] for a general treatment. One writes

$$Q^\mu = - \left( \beta_0 \Pi^2 + \beta_2 \pi^{\nu\lambda} \pi^{\nu\lambda} \right) \frac{u^\mu}{2T}$$

(with phenomenological expansion coefficients $\beta_0$, $\beta_2$) and computes (after some algebra using similar techniques as before) the entropy production rate as

$$T \dot{S} = \Pi \left[ -\dot{\vartheta} - \beta_0 \dot{\Pi} - \Pi T \partial_\mu \left( \frac{\beta_0 u^\mu}{2T} \right) \right] + \pi^{\alpha\beta} \left[ \sigma^{\alpha\beta} - \beta_2 \dot{\pi}^{\alpha\beta} - \pi^{\alpha\beta} T \partial_\mu \left( \frac{\beta_2 u^\mu}{2T} \right) \right],$$

where $\sigma^{\alpha\beta}$ is the flow shear tensor defined in the last equation [25]. From the expressions in the square brackets we see that the thermodynamic forces $-\dot{\vartheta}$ and $\sigma^{\alpha\beta}$ are now modified by terms including the time derivatives (in the local rest frame) of the irreversible flows $\Pi$, $\pi_{\alpha\beta}$. This leads to dynamical (“transport”) equations for the latter. We can ensure the 2nd law of thermodynamics by again writing the entropy production rate in the form [27] (without the middle term), which amounts to postulating

$$\dot{\Pi} = - \frac{1}{\tau_n} \left[ \Pi + \dot{\vartheta} \right] \Pi T \partial_\mu \left( \frac{\Pi u^\mu}{2T} \right) \approx - \frac{1}{\tau_n} \left[ \Pi + \dot{\vartheta} \right],$$

$$\Delta_{\alpha\beta} \Delta_{\beta\mu} \dot{\pi}^{\mu\nu} = - \frac{1}{\tau_\pi} \left[ \pi^{\alpha\beta} - 2\eta \sigma^{\alpha\beta} + \pi^{\alpha\beta} T \partial_\mu \left( \frac{\tau_\pi u^\mu}{2T} \right) \right] \approx - \frac{1}{\tau_\pi} \left[ \pi^{\alpha\beta} - 2\eta \sigma^{\alpha\beta} \right].$$

Here we replaced the coefficients $\beta_{0,2}$ by the relaxation times $\tau_n \equiv \zeta_0 \beta_0$ and $\tau_\pi \equiv 2\eta_\pi \beta_2$. In principle both ($\zeta$, $\eta$) and ($\tau_n$, $\tau_\pi$) should be calculated from the underlying kinetic theory. We will use them as phenomenological parameters, noting that for consistency the microscopic relaxation rates should be much larger than the local hydrodynamic expansion rate, $\tau_{n,\pi} \dot{\vartheta} \ll 1$.

The approximation in the second equalities in Eqs. (30,31) neglects terms that are of combined second order in dissipative flows and gradients of the zeroth-order hydrodynamic quantities [18]. Generically, it is good at early times $\tau - \tau_0 \ll \tau_\pi$, $\tau_\Pi$. During this time, $\dot{\Pi}$ and $\dot{\pi}^{\mu\nu}$ are of first order in deviations from equilibrium (i.e. of the same order as $\Pi$, $\pi^{\mu\nu}$ themselves as well as $\dot{\vartheta}$ and $\sigma^{\alpha\beta}$), and Eqs. (30,31) describe an exponential relaxation (on time scales $\tau_\pi$, $\tau_\Pi$) of the dissipative flows towards their Navier-Stokes values [26]. (The projectors $\Delta_{\mu\nu}$ on the l.h.s. of Eq. 31 ensure the preservation of tracelessness and transversality to $\vartheta$ of the shear pressure tensor during time evolution and can be rewritten as additional source terms on the r.h.s. of this equation [19].)

Once the differences between the dissipative flows and their Navier-Stokes limits have dropped enough to become comparable in magnitude to the second-order terms in Eqs. (30,31) (i.e. the third terms in the square brackets), $\Pi$ and $\pi^{\mu\nu}$ must be counted as being of second order in deviations from local equilibrium, and their further evolution is essentially affected by the second-order driving terms on the right hand sides of Eqs. (30,31). For $\tau - \tau_0 \gg \tau_\pi$, $\tau_\Pi$, the approximations indicated in the last equalities in Eqs. (30,31) thus break down.
Heavy-ion collisions with longitudinally boost invariant initial conditions present an anomalous situation in that the longitudinal expansion rate diverges like \(1/\tau\) at early times. As a result, the third terms in the square brackets of Eqs. (30,31) cannot even be neglected at early times [24] and must be kept throughout the evolution. If this is not done, serious deviations are observed when comparing Israel-Stewart viscous hydrodynamics with microscopic kinetic simulations [24], and one risks violating the second law of thermodynamics.

To get a feeling for the role played by the second-order terms for the time evolution of the dissipative flows, we rewrite Eqs. (30,31) (including these terms) as modified relaxation equations:

\[
\dot{\Pi} = -\frac{1}{\tau_\Pi} \left[ \Pi + \zeta \dot{\vartheta} + \Pi \zeta \gamma \right] = -\frac{1}{\tau_\Pi} \left[ \Pi + \frac{\zeta}{1 + \gamma_\Pi \zeta} \right] \theta = -\frac{1}{\tau_\Pi} \left[ \Pi + \zeta' \vartheta \right]
\]  
(32)

(where \(\gamma_\Pi = T \partial_\mu \left( \frac{\tau_\pi u^\mu}{2T} \right)\)), and similarly for the shear pressure tensor. One sees that the second-order term in the first square bracket modifies both the kinetic relaxation time and the viscosity, by an amount \(\sim \gamma_\Pi\) that involves the macroscopic expansion rate \(\partial_\mu u^\mu\). In regions of rapid hydrodynamic expansion and/or large shear flow, this effectively lowers both the Navier-Stokes limits of the dissipative flows and the relaxation times for approaching them, thereby effectively limiting excursions of the dissipative flows away from their Navier-Stokes limits [25, 26]. Numerical studies [25] show that this reduces the sensitivity of final physical observables to the choice of \(\tau_\pi, \tau_\Pi\). Equations (30,31), through the introduction of non-zero microscopic relaxation times \(\tau_\pi, \tau_\Pi\), thus resolve the issues with acausal signal propagation and numerical instability of the relativistic Navier-Stokes equations (at least for modes with macroscopic wave lengths \(\lambda > c\tau_\pi\)) while largely preserving their physics content.

In the second order Israel-Stewart formalism, one solves the dissipative hydrodynamic equations (19, 20) simultaneously with kinetic relaxation equations of the type (30,31) for the irreversible flows. The second-order terms displayed on the right hand sides of Eqs. (30,31) do not exhaust all terms that one could write down based on symmetries and tensor structure alone [27]. Indeed, microscopic derivations of the dissipative corrections to the ideal-fluid decomposition \(T^{\mu\nu}\) of the energy-momentum tensor, starting from Boltzmann kinetic theory for the distribution function \(f(x, p)\) and expanding it around the local equilibrium form \(\Pi\), produce many more second-order terms than obtained from the macroscopic approach described here [20, 27, 28]. In Boltzmann theory the coefficients of all second-order terms are found to be proportional to (powers of) the microscopic relaxation time \(\tau_\pi\) [20]. While this is an active area of research, it is expected that within the range of applicability of Israel-Stewart theory the exact values of these coefficients are practically irrelevant, i.e. that physical observables show little sensitivity to the value of \(\tau_\pi\) and to the choice of second-order terms (other than those that can be derived macroscopically) that are included.

3. The beginning and end of the hydrodynamic stage in heavy-ion collisions

Hydrodynamics does not rest on the availability of an underlying kinetic theory in terms of colliding particles, but it does require the system to be close to local thermal equilibrium (a concept that can be formulated even for strongly coupled quantum systems that are too hot and dense to allow for a particle description because large scattering rates never let any of the particles go on-shell). Hydrodynamics can never be expected to describe the earliest stage of the collision, just after nuclear impact, during which a fraction of the energy stored in the initial coherent motion along the beam direction is redirected into the transverse directions and randomized. The results of this initial thermalization process enter the hydrodynamic description through initial conditions for \(T^{\mu\nu}(x)\), i.e.
for the macroscopic density and (dissipative) flow distributions, implemented at a suitable starting time \( \tau_0 \) for the hydrodynamic evolution.

If a microscopic description of the early pre-equilibrium stage based on first principles is available, these initial conditions can be calculated from the pre-equilibrium energy-momentum tensor by matching it to the form \( T_{\mu\nu} \) (with \( W^\mu \equiv 0 \)) through Landau matching conditions, as described in Sec. 2.2. Presently there is no sufficiently mature pre-equilibrium description available, so initial conditions for the ideal fluid components of \( T_{\mu\nu} \) are adjusted to experimental data for final observables in central collisions and then extrapolated to non-central collisions using geometric considerations. Central collisions provide more data than necessary for adjusting the hydrodynamic initial conditions: As we will see in Sec. 3.1, a complete initialization requires the total charged hadron multiplicity density at midrapidity \( (dN_{ch}/dy)(y=0) \), its dependence on centrality, and the shapes of the transverse momentum spectra of two hadron species with very different masses. The spectra of all other hadron species from central collisions, as well as all spectra (including their anisotropies) from non-central collisions can thus be considered as tests for the validity of the hydrodynamic model.

As the fluid evolves hydrodynamically, there is continuous competition between the local rate of expansion, which drives the system away from equilibrium, and microscopic relaxation processes attempting to restore local equilibrium. For longitudinally boost-invariant initial conditions that best reflect our present understanding of the microscopic initial particle production processes at high collision energy, the expansion rate \( \tau_{\exp}^{-1} = \partial_\mu u^\mu \approx 1/\tau \), where \( \tau = \sqrt{t^2 - z^2} \) is the longitudinal proper time after nuclear impact. It is huge at very early times but decreases rapidly. On the other hand, all of the local scattering rates (elastic and inelastic) are proportional to the local temperature \( T(x) \), \( \tau_{\text{scatt}}^{-1} \approx T, \) \( T \) being the only dimensionful quantity in a thermalized system of (approximately) massless quarks and gluons. For boost-invariant longitudinal expansion temperature decreases with time as \( T \sim \tau^{-1/3} \), i.e. more slowly than the expansion rate. Hence, the ratio \( \tau_{\text{scatt}}/\tau_{\exp} \) initially decreases with time, improving the conditions for local thermalization.

As time proceeds, transverse flow is generated and the initially entirely longitudinal expansion eventually turns 3-dimensional. For 3-d expansion, the temperature decreases like \( 1/\tau \) (due to relativistic effects even somewhat faster), thus the scattering rate now decreases in lockstep with the expansion rate. If by then the system has not reached local thermal equilibrium, it never will. Below the quark-hadron phase transition the conditions for local thermalization deteriorate quickly since now, due to finite hadron masses, the density falls exponentially with temperature while, as a result of color confinement, the hadronic scattering cross sections saturate and become temperature independent. Once the mean collision time becomes larger than the local “Hubble time” \( \tau_{\exp} = 1/\partial_\mu u^\mu \), the system quickly falls out of equilibrium [29–32], turning into a gas of free-streaming hadrons soon afterwards. This “decoupling process” defines the end of the hydrodynamic evolution.

In the next two subsections we discuss beginning and end of the hydrodynamic stage in more detail.

### 3.1. Initialization

Lacking a microscopic theory of the early pre-equilibrium evolution, initial profiles for hydrodynamics are usually parametrized geometrically, with normalization parameters adjusted to final observables in central heavy-ion collisions.

Ideal fluid simulations for heavy-ion collisions at RHIC energies have been performed in 2+1 dimensions [33–49] and in 3+1 dimensions [50–58]. (The first number indicates the number of sp-
tial dimensions, the +1 stands for time.) Viscous hydrodynamic simulations [13, 19, 25, 26, 59–72] have up to now been restricted to at most 2+1 dimensions. The (2+1)-d simulations assume longitudinal boost-invariance, i.e. initial density profiles that do not depend on space-time rapidity \( \eta_s = \frac{1}{2} \ln\left(\frac{z+t}{z-t}\right) \) (where \( z \) is the beam direction), whereas the (3+1)-d simulations make no such assumption.

Both types of simulations assume an initial longitudinal expansion velocity profile with boost-invariance, \( y_L = \eta_s \), where \( y_L = \frac{1}{2} \ln\left(\frac{1+v_z}{1-v_z}\right) \) is the fluid rapidity in beam direction. All velocities \( v \) are in units of \( c \). This corresponds to an initial longitudinal flow velocity profile \( v_z = z/t \), as suggested by an initial particle production process that, at infinite collision energy, is independent of longitudinal reference frame and depends only on the longitudinal proper time \( \tau \) (i.e. the time in the particles’ longitudinal rest frame [73]). In the (2+1)-d simulations, the identity \( y_L = \eta_s \) holds for all times \( \tau \), due to boost-invariant initial densities and the resulting absence of longitudinal pressure gradients \( \partial \eta_s \rho \neq 0 \). In the (3+1)-d simulations, non-vanishing longitudinal pressure gradients \( \partial \eta_s \rho \neq 0 \) lead to longitudinal acceleration of the fluid, i.e. the longitudinal density profiles \( e(\tau, x, y, \eta_s) \) etc. broaden with time. Due to the logarithmic nature of the rapidity variable, at large values of \( \eta_s \) even small shifts in rapidity require large changes in longitudinal momentum, so these rapidity-broadening effects are limited and, at RHIC and LHC energies, typically well below one unit of rapidity.

The (3+1)-d simulations require input for the initial space-time rapidity profiles of the energy density \( e \) and baryon density \( n \). They are adjusted to the final rapidity distributions of pions and protons in momentum space. Due to the limited rapidity evolution just mentioned and the assumed initial identity of \( \eta_s \) and \( y_L \), the initial space-time rapidity distributions of \( e \) and \( n \) look very similar to the final momentum-space rapidity distributions of pions (\( \pi^\pm \)) and net protons (\( p-\bar{p} \)), respectively.

The initial space-time rapidity density profiles can be taken independent of transverse position \( r_\perp \) relative to the beam axis [58] or, more realistically, \( r_\perp \)-dependent [57]. This makes little difference for the final charged hadron rapidity distributions, but matters for a correct description of the rapidity distributions of net baryons and elliptic flow as a function of collision centrality.

For computing the initial transverse distributions of energy and baryon density, two leading models are on the market: The Glauber model [74, 75], and the Kharzeev-Levin-Nardi (KLN) model [76–81] based on the Color Glass Condensate (CGC) theory. Since these define (in a sense detailed below) the outer limits of viable initializations, both will be briefly outlined in the following. Although not discussed here in more detail, also other initializations, such as the pQCD + final state saturation model (EKRT) [82, 83], have been applied for hydrodynamics at RHIC and LHC [84–86].

The initial transverse collective flow velocity is typically assumed as zero. This makes sense if the hydrodynamic stage starts early, at times \( \tau_0 \ll 1 \) fm. For later starting times, some pre-equilibrium transverse flow should be allowed for and has been introduced in some simulations to improve the agreement with experimental data [42, 45, 87–89]. However, lacking guidance from \textit{ab initio} pre-equilibrium calculations, it is difficult to accurately determine the initial transverse flow phenomenologically.

3.1.1. Glauber model

The microscopic processes that generate the initial entropy are still poorly understood. Before the advent of the Color Glass Condensate theory (c.f. Ref. [90] and references therein) which describes the initial transverse distribution as a dense gluon system characterized by an \( x_\perp \)-dependent saturation momentum \( Q_s(x_\perp, \eta_s) \) (see Sec. 3.1.2), the only available model was the Glauber model which assumes that initial entropy production is controlled by some combination of wounded nucleon and
binary nucleon-nucleon collision distributions [74]:

\[ s(x, B; b) = \kappa_s(\alpha n_{\text{WN}}(x; b) + (1 - \alpha) n_{\text{BC}}(x; b)) \]  

(33)

One assumes that “soft” processes scale with the number of wounded nucleons per unit transverse area while “hard” processes scale with the areal density of binary collisions. The soft fraction \( \alpha \) and the overall normalization are adjusted such [39, 91] that the experimentally observed rapidity density of charged hadrons at the end of the collision [92, 93] and its dependence on the collision centrality [94, 95] are reproduced [39, 47, 57, 96].

To compute these distributions in the transverse plane one starts from Saxon-Woods profiles describing the density distributions of the colliding nuclei with mass numbers \( A \) and \( B \), respectively,

\[ \rho_A(r) = \frac{\rho_0}{e^{(r - R_A)/\xi} + 1}, \]  

(34)

with nuclear radius \( R_A = (1.12 A^{1/3} - 0.86 A^{-1/3}) \text{ fm} \) and surface diffuseness \( \xi = 0.54 \text{ fm} \) [97]. The nuclear thickness function is defined as the optical path-length through the nucleus along the beam direction:

\[ T_A(x, y) = \int_{-\infty}^{\infty} dz \rho_A(x, y, z). \]  

(35)

The coordinates \( x, y \) parametrize the transverse plane, with \( x \) pointing in the direction of the impact parameter \( b \) (such that \( (x, z) \) span the reaction plane) and \( y \) perpendicular to the reaction plane. For a non-central collision with impact parameter \( b \), the density of binary nucleon-nucleon collisions \( n_{\text{BC}} \) at a point \( (x, y) \) in the transverse plane is proportional to the product of the two nuclear thickness functions, transversally displaced by \( b \):

\[ n_{\text{BC}}(x, y; b) = \sigma_0 T_A(x + b/2, y) T_B(x - b/2, y). \]  

(36)

\( \sigma_0 \) is the total inelastic nucleon-nucleon cross section; it enters here only as a multiplicative factor which is later absorbed in the proportionality constant between \( n_{\text{BC}}(x, y; b) \) and the “hard” component of the initial entropy deposition [39]. Integration over the transverse plane (the \( (x, y) \)-plane) yields the total number of binary collisions

\[ N_{\text{BC}}(b) = \int dx dy n_{\text{BC}}(x, y; b). \]  

(37)

Its impact parameter dependence, as well as that of the maximum density of binary collisions in the center of the reaction zone, \( n_{\text{bc}}(0, 0; b) \), are shown as the dashed lines in Fig. 1.

The “soft” part of the initial entropy deposition is assumed to scale with the density of “wounded nucleons” [75], defined as those nucleons in the projectile and target which participate in the particle production process by suffering at least one collision with a nucleon from the other nucleus. The Glauber model gives the following transverse density distribution of wounded nucleons [75]:

\[ n_{\text{WN}}(x, y; b) = T_A(x + b/2, y) \left( 1 - \left( \frac{\sigma_0 T_B(x - b/2, y)}{B} \right)^A \right) + T_B(x - b/2, y) \left( 1 - \left( \frac{\sigma_0 T_A(x + b/2, y)}{A} \right)^B \right). \]  

(38)

Here the value \( \sigma_0 \) of the total inelastic nucleon-nucleon cross section plays a more important role since it influences the shape of the transverse density distribution \( n_{\text{WN}}(x, y; b) \), and its dependence [98] on the collision energy \( \sqrt{s} \) must be taken into account. The total number of wounded nucleons is
obtained by integrating Eq. (38) over the transverse plane. Its impact parameter dependence, as well as that of the maximum density of wounded nucleons in the center of the reaction zone, $n_{WN}(0, 0; b)$, are shown as the solid lines in Fig. 1.

![Graph](image)

**Fig. 1.** *Left:* Number of wounded nucleons and binary collisions as a function of impact parameter, for Au+Au collisions $\sqrt{s} = 130$ $A$ GeV ($\sigma_0 = 40$ mb). *Right:* Density of wounded nucleons and binary collisions in the center of the collision as a function of impact parameter.

Hydrodynamic calculations with soft fraction $\alpha = 0.75 - 0.85$, i.e. with initial conditions that ascribe between 75 and 85% of the initial entropy production to “soft” processes (scaling with $n_{WN}(x, y; b)$) and 15–25% to “hard” processes (scaling with $n_{BC}(x, y; b)$), were found [39, 57, 91, 96] to give reasonable descriptions of the measured [94, 95] centrality dependence of charged particles produced per participating (“wounded”) nucleon. For simplicity and lack of other information, the initial transverse distributions of baryon and entropy density are often assumed to have the same shape (i.e. the entropy per baryon is constant in the transverse plane), but other calculations simply set the net baryon density proportional to the density of wounded nucleons. Their relative normalization is controlled by the net proton to pion ratio at midrapidity. At midrapidity, the net baryon density is small, and the mentioned differences in the initial transverse baryon density profile do not matter. For (3+1)-d calculations, phenomenology requires that the entropy per baryon decreases at forward rapidities; the $\eta_s$-dependence of $s/n$ is thus another parameter in such simulations that needs adjusting.

Entropy conservation in ideal fluid dynamics allows to fix the normalizations of the initial entropy density profile from measurements of the total charged hadron multiplicity $dN_{ch}/dy$ (which is a measure of the total final entropy per unit rapidity) in central collisions [39]. It is natural to assume that the entropy produced per wounded nucleon or per binary nucleon-nucleon collision depends only on collision energy but not on collision geometry. In this case, once their normalization has been fixed in central collisions, normalization and shape of the initial density distributions in peripheral collisions are predicted without additional parameters.

### 3.1.2. Color Glass Condensate theory and KLN model

The second type of initial conditions described here is based on the CGC model [90]. For simplicity, I describe the original Kharzeev-Levin-Nardi (KLN) approach [57, 76, 77] even though a somewhat improved version has recently been developed [79–81]. In this approach, the energy distribution of
produced gluons with rapidity $y$ is given by the $kT$-factorization formula [99]

$$\frac{dE_T}{d^2x_\perp dy} = \frac{4\pi^2 N_c}{N_c^2 - 1} \int \frac{d^2p_T}{p_T} \int_{p_T}^{p_T} d^2k_T \alpha_s(Q^2) \varphi_A(x_1, k_T^2; x_\perp) \varphi_B(x_2, (p_T - k_T)^2; x_\perp),$$

(39)

where $x_{1,2} = p_T \exp(\pm y)/\sqrt{s}$ are the longitudinal momentum fractions of the fusing gluons from nucleus $A$ and $B$, and $p_T$ is the transverse momentum of the produced gluon. For the unintegrated gluon distribution function one uses

$$\varphi_A(x, k_T^2; x_\perp) = \begin{cases} \frac{2\pi^2 C_F}{x^2 \alpha_s(Q^2)} \frac{Q_s^2}{Q_s^2 + \Lambda^2}, & k_T \leq Q_s, \\ \frac{2\pi^2 C_F}{x^2 \alpha_s(Q^2)} \frac{Q_s^2}{k_T^2 + \Lambda^2}, & k_T > Q_s, \end{cases}$$

(40)

where $C_F = \frac{N_c^2 - 1}{2N_c}$, $Q_s$ denotes the gluon saturation momentum, and $\Lambda = 0.2 \text{GeV}$ is a soft regulator. The overall normalization $\kappa$ is determined by fitting the multiplicity of charged hadrons at midrapidity at $\sqrt{s_{NN}} = 200 \text{GeV}$ for the most central collisions. The saturation momentum $Q_s$ of nucleus $A$ in $A+B$ collisions, needed in the function $\varphi_A$, is obtained by solving the following implicit equation at fixed momentum fraction $x$ and transverse position $x_\perp$:

$$Q_s^2(x, x_\perp) = \frac{\pi^2}{C_F} \alpha_s(Q_s^2) x G(x, Q_s^2) \frac{dN_{\text{part}}^A}{d^2x_\perp}.$$ 

(41)

Here $dN^A_{\text{part}}/d^2x_\perp \equiv n_{\text{wounded}}(x_\perp)$ is the transverse density of wounded nucleons in nucleus $A$, given by the first term in Eq. (40). An analogous equation holds for the saturation momentum of nucleus $B$ in $\varphi_B$. For the gluon distribution function $G$ inside a nucleon one takes the simple ansatz [76]

$$x G(x, Q^2) = K \ln \left( \frac{Q_s^2 + \Lambda^2}{Q_s^2 + Q^2} \right) x^{-\lambda} (1 - x)^4$$

(42)

with $\Lambda = \Lambda_{\text{QCD}} = 0.2 \text{GeV}$. Choosing $K = 0.7$ and $\lambda = 0.2$ ensures that the average saturation momentum in the transverse plane yields $(Q_s^2(x = 0.01)) \sim 2.0 \text{GeV}^2/c^2$ in central 200 $A \text{GeV}$ Au+Au collisions at RHIC. For the running coupling constant $\alpha_s$ in (41) one uses the standard perturbative one-loop formula with an additional cut-off in the infra-red region of small $Q_s$ (i.e. near the surface of the nuclear overlap region where the produced gluon density is low), by limiting the coupling constant to $\alpha_s \leq 0.5$. From Eq. (39) one obtains the energy density distribution at time $t_0$ as $\varepsilon(t_0, x_\perp, \eta_s) = dE_T/(\tau_0 d\eta_s d^2x_\perp)$, where $y$ is identified with $\eta_s$.

The KLN model predicts a centrality dependence of the produced charged hadron multiplicity per wounded nucleon that agrees with RHIC measurements [76, 81]. A similar dependence can be obtained in the Glauber model by judicious choice of the “soft” fraction $\alpha$ (see Eq. (40)). The main prediction of the CGC approach is the near independence of $\alpha$ of the collision energy, which is so far confirmed by experiment.

3.1.3. Non-central collisions and initial fireball eccentricity

A key feature of non-central collisions between large nuclei is that they produce deformed fireballs. This breaks the azimuthal symmetry inherent in central collisions between spherical nuclei. In a strongly interacting fireball, the initial geometric anisotropy of the reaction zone gets transferred onto the final momentum spectra and thus becomes experimentally accessible. As we will see, this provides a window into the very early collision stages that central collisions between spherical nuclei do not provide. Full-overlap collisions between deformed nuclei, such as U, allow to explore the same
physics with better resolution and higher initial energy densities [100], but this requires careful event selection [96].

Fig. 2. *Left:* Density of binary collisions in the transverse plane for a Au+Au collision with impact parameter \( b = 7 \) fm. Shown are contours of constant density together with the projection of the initial nuclei (dashed lines). *Right:* Spatial eccentricity \( \epsilon \) as a function of the impact parameter [57], calculated with Eq. (43) using the initial energy density as weight function, for four different models as described in the text.

The left panel of Fig. 2 shows the distribution of binary collisions in the transverse plane for Au+Au collisions at impact parameter \( b = 7 \) fm. Shown are lines of constant density displaced by \( \pm b/2 \) from the origin. The clearly visible geometric deformation of the overlap region can be quantified by the spatial eccentricity

\[
\epsilon_x(b) = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle},
\]

where the average is to be taken with the energy density as weight function [35]. The initial energy density is obtained from the initial entropy density through the equation of state (EOS, see Sec. 4). The right panel of Fig. 2 shows the initial spatial eccentricity for three models where the initial entropy density is taken proportional to the density of wounded nucleons (\( n_{\text{part}} \), green dotted line), of binary nucleon-nucleon collisions (\( n_{\text{binary}} \), black dash-dotted line), and of a superposition of these two with 85% weight for the “soft” component (BGK, blue dashed line). These are compared with a fourth model (solid red line) that uses directly the initial energy density (39) of gluons from the KLN model. One sees that, at any given impact parameter, the KLN model (“CGC”) predicts almost 50% larger spatial eccentricities than the standard Glauber initialization (“BGK”) [57]. A recently improved version of the model called fKLN [79, 81] produces somewhat smaller eccentricities but even those exceed the Glauber model values by 25–30%.

### 3.2. Decoupling and freeze-out

#### 3.2.1. Two-stage decoupling

As explained in Sec. 3 the hydrodynamic description begins to break down again once the transverse expansion becomes so rapid and the matter density so dilute that local thermal equilibrium can no
longer be maintained. Detailed studies \cite{101, 102} comparing local mean free paths with the overall size of the expanding fireball and the local Hubble radius (inverse expansion rate) have shown that bulk freeze-out happens \textit{dynamically}, i.e. it is driven by the expansion of the fireball and not by its finite size. This is similar to the decoupling of the primordial nuclear abundances and the cosmic microwave background in the early universe which, too, were controlled by the cosmological expansion rate.

The similarities between the “Little Bangs” created in heavy-ion collisions and the Big Bang that created our universe do not end here. Similar to the Big Bang, local thermodynamic equilibrium breaks in two stages: In the early universe, \textit{primordial nucleosynthesis} signals the end of inelastic nuclear reactions that can change its chemical composition; it takes hundreds of millions of years after this point to restart nuclear reactions in the cores of stars formed by gravitational collapse of density inhomogeneities. In heavy-ion collisions, an analogous process of \textit{chemical decoupling} happens once inelastic reaction rates among hadrons become too low to maintain chemical equilibrium among the various hadron species. At RHIC energies, chemical decoupling is observed to happen at a temperature of about 160 MeV and appears to be driven by the hadronization process at the quark-hadron phase transition. The 2.7 K thermal background radiation in our universe reflects its \textit{thermal decoupling} at \( T \sim 3000 \) K, cosmologically redshifted by about a factor 1000. At this temperature ions and electrons combined into neutral atoms and the cosmological photons stopped rescattering, thus freezing in their Bose-Einstein thermal energy distribution. The analogous process in the “Little Bang” is called \textit{kinetic} or \textit{thermal freeze-out} and happens when the matter is so dilute that even elastic collisions cease among hadrons, thereby freezing in their momentum distributions. The kinetic freeze-out temperature in heavy-ion collisions is about 100 MeV.

In the Big Bang, chemical and thermal freeze-out are separated by about 400,000 years. Since the fireballs created in the Little Bangs expand about \( 10^{18} \) times faster than the early universe did at similar temperatures, the time separation between chemical and thermal decoupling shrinks to a few fm/c in heavy-ion collisions. That the two decoupling processes do not happen simultaneously but hierarchically is easily seen from the kinetic decoupling criterium: \cite{29–32, 101–105}

\[
\tau_{\text{exp}}(x) \equiv \frac{1}{\partial \cdot u(x)} = \xi \tau_{\text{scatt}}^{(i)}(x) \equiv \xi \sum_j (\sigma_{ij} v_{ij}) \rho_j(x),
\]

where \( \xi \) is an (unknown) parameter of order 1. Local equilibrium requires the mean free time \( \tau_{\text{scatt}} \) between scatterings to be much shorter than the local “Hubble time” \( \tau_{\text{exp}} \) describing the fireball expansion. Equilibrium breaks when the two time scales become of the same order. The scattering rate involves the product of the scattering cross section with the density of scatterers. Since chemical transformations require inelastic processes which constitute only a small fraction of the total cross section whereas momenta get changed by almost all types of collisions, thermal equilibration is driven by much larger cross sections and happens considerably faster than chemical equilibration. Correspondingly, Eq. (44) tells us that, in a medium with given hydrodynamical expansion rate, chemical freeze-out happens at higher particle densities (and thus higher temperatures) than thermal freeze-out. Furthermore, the equation predicts that in general different particle species freeze out at different temperatures, since scattering cross sections are species-specific.

Equation (44) is a \textit{local} criterium. The set of points \((x, \tau_f(x))\) satisfying Eq. (44) defines the \textit{freeze-out hypersurface} \( \Sigma_t \). It is a 3-dimensional surface imbedded in 4-dimensional space-time. The shapes of these freeze-out surfaces depend on the hydrodynamic expansion rate \( \partial \cdot u(x) \), and their computation thus requires a dynamical simulation. Since the matter near the transverse edge of the fireball is dilute and thus freezes out early, the freeze-out surface typically closes on the initialization
surface where the hydrodynamic evolution is started. (It may even close above the initialization surface, i.e. at times \( \tau > \tau_0 \), if hydrodynamics is initialized too early [105]; since the expansion rate diverges like \( 1/\tau \) at early times, hydrodynamics cannot be started until the longitudinal expansion rate has dropped enough to allow for local thermal equilibrium.) Numerical studies [101–105] show that, except near the transverse edge of the fireball where the expansion rate changes rapidly with position, the kinetic freeze-out surfaces defined by Eq. (44) can be well approximated by surfaces of constant temperature. Making use of the fact that at RHIC energies pions form the most abundant species and their kinetic decoupling thus controls thermal freeze-out of all other hadrons, one can approximate the thermal decoupling of all hadron species by a single surface of temperature \( T_{\text{dec}} \) corresponding to pion freeze-out. Its value can be determined phenomenologically from so-called blast-wave model fits [106–111] to experimental hadron spectra. These models characterize the decoupling medium by an average freeze-out temperature and an average transverse flow velocity. Similarly, one can fit the observed final hadron abundance ratios with a thermal model and extract from this the chemical decoupling temperature \( T_{\text{chem}} \). The result of such an exercise [112–114], performed on a huge set of heavy-ion collision data from SIS to RHIC energies, is shown in Fig. 3. The figure demonstrates a clear separation of chemical from thermal decoupling for collision energies above \( \sqrt{s_{NN}} \sim 5 \) GeV.

The kinetic freeze-out criterium (44) predicts a dependence of the (average) freeze-out temperature on the (average) hydrodynamic expansion rate. The latter changes with system size and collision centrality. The right panel of Fig. 4 shows that the thermal decoupling temperature \( T_{\text{kin}} \equiv T_{\text{dec}} \) in Au+Au collisions at RHIC indeed depends on centrality. This dependence is consistent with hydrodynamic predictions and Eq. (44) [103]: Larger collision systems created in more central collisions

![Fig. 3.](image-url)

Chemical and thermal freeze-out points extracted from heavy-ion collisions at the GSI SIS, BNL AGS, CERN SPS and RHIC. The shaded area indicates the likely location of the quark-hadron phase transition as extracted from lattice QCD and theoretical models. An updated version adding many more chemical freeze-out points can be found in Ref. [114].
Mid-rapidity hadron ratios

200 GeV $^{197}$Au + $^{197}$Au central collision

Fig. 4. Left: Abundance ratios of stable hadrons from central 200 A GeV Au+Au collisions at RHIC [4]. The blue lines show predictions from a thermal model fit with $T_{\text{chem}} = 163 \pm 4 \text{ MeV}$, $\mu_B = 24 \pm 4 \text{ MeV}$, and a strangeness saturation factor $\gamma_s = 0.99 \pm 0.07$ [4]. The inset shows the centrality dependence of $\gamma_s$. Right: Centrality dependence (with centrality measured by charged hadron rapidity density $dN_{\text{ch}}/d\eta$) of (a) the thermal freeze-out temperature $T_{\text{kin}} \equiv T_{\text{dec}}$ (open triangles), the chemical freeze-out temperature $T_{\text{chem}}$ (open circles), and the square root of the transverse areal density of pions ($dN_\pi/d\eta$) (solid stars), and (b) the average transverse flow velocity $\langle \beta \rangle \equiv \langle v_\perp \rangle$ (solid triangles), for the same collision system [116].

cool down further and develop larger radial flow $\langle v_\perp \rangle$ than the smaller fireballs formed in peripheral collisions. In contrast, the chemical decoupling temperature shows no sensitivity whatsoever to collision centrality and the accompanying change in expansion rate. (The excellent quality of the chemical fits is shown in the left panel of Fig. 4.) The baryon chemical potential $\mu_B$ and the strangeness saturation factor $\gamma_s$ (which indicates to what extent strange hadrons are suppressed relative to non-strange hadrons) decrease somewhat in peripheral collisions, but $T_{\text{chem}}$ is completely independent of centrality [4, 115].

Chemical freeze-out at RHIC can therefore not be driven by a local competition between inelastic hadron scattering and hydrodynamic expansion, as described by Eq. (44) [103]. The observed universality of the measured chemical freeze-out temperature and the proximity of the value extracted from experiment to the critical temperature $T_c$ of the quark-hadron phase transition predicted by lattice QCD [7–9] can only be understood if one assumes that the phase transition itself controls the chemical freeze-out process. At $T_c$, hadrons are created from quarks and gluons in a state of maximum entropy, with thermal abundances reflecting a temperature $T_{\text{chem}} \approx T_c$ that characterizes the critical energy density for hadronization [117]. At that point, the fireball is already expanding so rapidly and the hadron resonance gas is so dilute that inelastic hadronic reactions can no longer change its chemical composition. The chemical temperature is thus frozen at $T_c$, allowing us to measure the quark-hadron phase transition temperature directly through hadron abundances.

Between chemical decoupling at $T_c$ and thermal decoupling at $T_{\text{dec}}$, hadrons continue to rescatter quasi-elastically through a rich spectrum of hadronic scattering resonances with large cross sections. Since the resonances typically decay into the same particles from which they were created (up to quark exchange), this does not affect the chemical composition, but it changes the hadrons’ momenta. For a while they can thus maintain approximate thermal equilibrium even though chemical equilibrium is broken. As long as thermal equilibration can be locally maintained, hydrodynamics continues to be applicable. The equation of state through which pressure gradients are evaluated must, however, correctly reflect the non-equilibrium chemical composition of the HRG below $T_{\text{chem}}$. 
[45,55,118–120]. The latter is also essential for the computation of elliptic flow since the distribution of the total momentum-anisotropy of the energy-momentum tensor over the various hadron species depends on their relative abundance, i.e. on the (non-equilibrium) chemical composition at thermal freeze-out [45,55,121,122].

3.2.2. Final hadron momentum spectra

The breakdown of local equilibrium ends the hydrodynamic stage of a heavy-ion collision. A relatively easy way to implement this into hydrodynamics is through the Cooper-Frye prescription [123] which postulates a sudden transition from a thermalized fluid to free-streaming particles on a decoupling surface of, say, temperature $T_{\text{dec}}$. In this subsection we describe how this procedure allows to compute final hadron momentum spectra, multiplicities and elliptic flow, both in ideal and viscous fluid dynamics.

The idealization of a sudden freeze-out has, of course, limitations. Even if one correctly accounts for the non-equilibrium chemical composition in the hadronic phase below $T_c$ through appropriate chemical potentials $\mu_i(T)$, it is not a priori clear that a sudden transition can capture all phenomenologically important aspects of the freeze-out process. Real-life freeze-out happens gradually, is particle specific, and should thus be described in a microscopic kinetic approach. Quantitative model predictions for hadron spectra from heavy-ion collisions will thus eventually require matching the hydrodynamic evolution to a hadronic rescattering cascade that describes the final expansion stage [37,40,41,57,58,124–127]. In this case, the Cooper-Frye prescription discussed here is used at a suitable switching temperature $T_{\text{dec}} < T_{\text{sw}} < T_c$ to generate thermally distributed hadrons in an expanding ensemble, which are then used as discrete input into a hadronic cascade that follows their further evolution until all collisions have ceased. To accumulate enough statistics for the final hadron spectra, the hadronic cascade must be run many times with initial conditions sampled by a Monte-Carlo simulation of the Cooper-Frye spectra at $T_{\text{sw}}$. This is numerically expensive, and therefore not many such calculations from a hydro+cascade hybrid approach are presently available [37,40,41,57,58,124–127]. All of these use ideal fluid dynamics to generate the input for the hadron cascade. No systematic studies exist that show the existence of a window of switching temperatures that produces final results independent of $T_{\text{sw}}$. One may expect that, if it exists, that window will be larger when viscous hydrodynamics (with viscosities matched to those of the hadronic cascade) is used to initialize the late kinetic stage.

The Cooper-Frye formalism is based on the following expression for the final momentum spectrum [123]:

$$E \frac{dN_i}{d^3p} = \frac{dN_i}{dp_T dp_T d\varphi_p} = \frac{g_i}{(2\pi)^3} \int_{\Sigma} p^\mu \sigma_\mu(x) f_i(x,p).$$

(45)

Here $d^3\sigma_\mu(x)$ is the outward normal vector on the freeze-out surface $\Sigma(x)$ such that $p^\mu d^3\sigma_\mu f_i$ is the local flux through this surface of particles of species $i$ with momentum $p$.

In ideal fluid dynamics, the phase-space distribution $f_i$ in this formula is the local equilibrium distribution just before decoupling,

$$f_{i,\text{eq}}(x,p) = \frac{1}{\exp[(p\cdot u(x) - \mu_i(x))/T(x)] + 1},$$

(46)

boosted with the local flow velocity $u^\mu(x)$ to the global reference frame by the substitution $E \rightarrow p\cdot u(x)$. $\mu_i(x)$ and $T(x)$ are the chemical potential of particle species $i$ and the local temperature along
The temperature and chemical potentials on $\Sigma$ are computed from the hydrodynamic output for the energy density $e$, net baryon density $n$ and pressure $p$ with the help of the equation of state [128]. The quantum statistical correction $\pm 1$ in the denominator matters only for pions where Bose corrections can reach 10–20% (depending on the pion chemical potential at freeze-out). For all other hadron spectra the Boltzmann approximation is sufficiently accurate.

In viscous hydrodynamics, the distribution function along the decoupling hypersurface is in general slightly out of equilibrium, by an amount proportional to the dissipative flows $\Pi(x)$, $q^\mu$ and $\pi^{\mu\nu}(x)$ on $\Sigma$. For vanishing bulk viscosity and heat conduction one finds [18, 129]

$$f(x, p) = f_{eq}(x, p) \left[1 + \left(1 \mp f_{eq}(x, p)\right) \frac{c_2 \ p^\mu p^\nu \ \pi_{\mu\nu}(x)}{2 \ T^2(x) \ e(x) + p(x)}\right]$$

$$\approx f_{eq}(x, p) \left[1 + \frac{1}{2 \ T^2(x) \ e(x) + p(x)} \ \pi_{\mu\nu}(x)\right] = f_{eq}(x, p) + \delta f(x, p).$$

Here $c_2 = 1$ in Boltzmann approximation; for massless bosons, $c_2 = 1.04$. For massive bosons, $c_2$ is a temperature dependent function that interpolates between these limits [129]. Replacing in Eq. (47) the factor $1 \mp f_{eq}$ by 1 is an excellent approximation even for pions since it deviates from 1 only at small momenta where the non-equilibrium correction is suppressed by two powers of $p$.

The reader should note that shear viscous pressure effects modify the shape of the local momentum distribution by an amount that increases quadratically with $p$. Even for very small shear viscous pressure at freeze-out, the non-equilibrium correction $\delta f$ of the local distribution function eventually becomes big and comparable with the equilibrium contribution if $p$ gets sufficiently large. At this point, the near-equilibrium expansion breaks down, and the spectrum calculated from (47) can no longer be trusted. This emphasizes the nature of (viscous) hydrodynamics as an effective theory that applies at large distances (low momenta) but breaks down at short distances.

To apply Cooper-Frye freeze-out, one first lets the hydrodynamic code run up to large times, assuming hydrodynamics to be valid everywhere. One then determines the space-time hypersurface $\Sigma(x)$ by identifying which fluid cells satisfy the freeze-out criterium. Back-reaction effects arising from the (in principle) non-hydrodynamic behaviour of the matter outside the decoupling surface on the hydrodynamic evolution inside the thermalized space-time region are ignored.

The Cooper-Frye formalism is used to calculate the momentum distributions of all directly emitted hadrons, stable and unstable. Unstable resonances are then allowed to decay if they do so via strong or electromagnetic interactions, accounting for the appropriate branching ratio of different decay channels [98]. Weakly decaying particles are considered as stable because they are usually reconstructed in the experiments. The stable decay products are added to the thermal momentum spectra of the directly emitted stable hadrons to give the total measured particle spectra [130–134].

4. The nuclear equation of state

As emphasized in Sec. 2.1, the hydrodynamic equations require the input of an equation of state (EOS) $p(e, n)$ for closure, and this EOS, through the speed of sound $c_s^2(T) = \frac{\partial p}{\partial e}$, defines the “pushing power” of the medium, i.e. how strongly the matter accelerates in reaction to pressure gradients.

At RHIC energies the net baryon density $n$ is very small at midrapidity, and the dependence of the EOS $p(e, n)$ on $n$ is weak. For hydrodynamic purposes we can thus use the EOS at $n = 0$ with excellent accuracy near midrapidity. To obtain correctly normalized hadron spectra at freeze-out it is, however, important that the used EOS incorporates all relevant hadronic species with the correct chemical composition. For this, the $n$-dependence of the EOS matters.
Left: The equation of state for baryon-free QCD matter. The upper plot shows the pressure $p$ as a function of energy density $e$ and (in the inset) the squared speed of sound $c_s^2 = \frac{\partial p}{\partial e}$ as a function of temperature $T$. The lower panel shows $c_s^2$ as a function of energy density $e$ [25]. The solid red line (SM-EOS Q) is a slightly smoothed version of EOS Q (green dashed line). Right: Energy density (top) and temperature (bottom) of the central cell as a function of longitudinal proper time from a (2+1)-d ideal fluid dynamical simulation with of Au+Au collisions at RHIC [55], for an EOS with a first-order quark-hadron transition at $T_c = 170$ MeV and three choices of the chemical composition of the HRG below $T_c$: CE (dashed) assumes full hadronic chemical equilibrium at all temperatures (this case corresponds to the green dashed lines in the left panel); CFO (dotted) assumes chemical freeze-out of all hadronic species (stable and unstable) at $T_c$; PCE (solid) makes the realistic assumption that unstable resonances continue to re-equilibrate in the HRG phase via resonance scattering, but that the final yields of all stable decay products remain unchanged below $T_c$. The $p(e)$ curves for all three choices are almost identical [55], resulting in identical time evolutions of the energy density $e(\tau)$.

A simple and in the past very popular procedure to construct an EOS for QCD matter (known as EOS Q [33,35]) is to match a non-interacting massless quark-gluon gas (shown as EOS I in Fig. 5) with adjustable bag constant $B$ to a non-interacting, chemically equilibrated hadron resonance gas that includes all known hadron resonances with their measured masses up to a certain mass cutoff (typically between 1.6 and 2 GeV) [128, 135]. Adjusting $B$ to obtain $T_c = 165$ MeV in accordance with lattice QCD data, this construction results in a first order phase transition with a mixed quark-hadron phase for energy densities $0.45 < e < 1.6$ GeV/fm$^3$. The squared speed of sound for EOS Q is $c_s^2 = \frac{1}{3}$ above $T_c$, $c_s^2 \approx 0.15$ between $T_{dec}$ and $T_c$ [33], and $c_s^2 = 0$ for all energy density values in the mixed phase (green dashed lines in Fig. 5). Some viscous hydrodynamic calculations require a slightly smoothed version of this EOS, called SM-EOS Q (solid red line in the left panel of Fig. 5) for numerical stability.

Modern lattice QCD data [7–9, 136] show that this modelling is unrealistic in two aspects: Lattice QCD shows a continuous cross-over phase transition without phase coexistence, instead of a first-order discontinuity at $T_c$. So the speed of sound, while becoming small and developing a minimum
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(“softest point”) near $T_c$ (dash-dotted curve in the left panel of Fig. 5), never drops to zero as assumed in EOS Q. Above $T_c$, lattice QCD data show clear deviations from an ideal gas of massless quarks and gluons which, in the temperature range explored by heavy-ion collisions at RHIC, reduce the squared speed of sound by a significant (and temperature-dependent) fraction, making the EOS softer than EOS Q. More and more hydrodynamic simulations are therefore now being performed with equations of state that are better matched to lattice QCD data (such as EOS L shown in Fig. 5).

Another unrealistic aspect of EOS Q that turns out to be more difficult to fix is the assumption of chemical equilibrium below $T_c$. As already discussed, the experimental data indicate chemical freeze-out near $T_c$ [113]. This requires the introduction of non-equilibrium chemical potentials for the stable hadron species in the hadron resonance gas (HRG) phase [45, 55, 118, 119] that must be anchored at $T_c$, using the correct non-zero baryon chemical potential at $T_c$. A proper matching to the QGP phase must thus be done at all relevant non-zero values of net baryon density $n$. Lattice QCD data at non-zero $n$ have recently become available (see Refs. [137–140] and references therein), and successful quasiparticle parametrizations of lattice QCD data that allow to extrapolate data at $n = 0$ to nonzero net baryon densities are also available [141–143]. However, a proper smooth matching of these data to a chemically non-equilibrated HRG has not yet been performed.

Existing equations of state that properly describe the non-equilibrium chemical composition of the HRG below $T_c$ match to an ideal massless quark-gluon gas through a first-order transition [45, 55]. As shown in the right panel of Fig. 5, the hydrodynamical evolution with such an EOS is virtually indistinguishable from EOS Q, since the non-equilibrium chemical potentials have only very small effects on the EOS $p(e)$ in the HRG phase below $T_c$. What does change, however, is the relationship between energy density and temperature in the HRG. In the chemically frozen HRG baryons and antibaryons as well as pions are not allowed to annihilate, which is ensured by giving the non-equilibrium chemical potentials that grow as $T$ decreases [45, 55, 118, 119]. This stores more of the energy density in particle rest masses, reducing the thermal energy and temperature corresponding to a given energy density. (Surprisingly, the pressure $p(e)$ does not decrease.) This has obvious consequences for the final hadron spectra: at the same decoupling energy density $e_{dec}$, the hydrodynamic flow is the same as with EOS Q but the temperature $T_{dec}$ is smaller, so the transverse momentum spectra are steeper. As we will see, this also results in a significant reshuffling of the momentum anisotropy in non-central collisions which strongly modifies the elliptic flow coefficients [45, 55, 121, 122]. A given set of experimental spectra thus requires a retuning of hydrodynamic initial conditions, to ensure that more radial flow is generated to compensate for the lower decoupling temperature. The predicted elliptic flow will then be different for the chemically non-equilibrated hadron gas than for the unrealistic EOS Q, with important consequences for the interpretation of the data as to how much room they offer for non-zero viscosity of the expanding QCD matter.

5. Phenomenology of the transverse expansion

In this section we study the transverse fireball expansion at midrapidity as it follows from the hydrodynamic equations of motion (Section 2) with the equation of state described in Section 4 and the initial conditions from Section 3.1. We analyze results from (2+1)-d simulations with longitudinal boost-invariance for which both ideal and viscous fluid dynamical codes are available. Results from (3+1)-d ideal fluid dynamics (with the initial conditions discussed in Sec. 3.1) largely agree at midrapidity with those from (2+1)-d ideal fluid simulations.

In Section 5.1 we begin by discussing azimuthally symmetric radial expansion in central collisions
Both Au+Au and Cu+Cu collisions have been simulated but, as it happens, a detailed comparison between ideal and viscous dynamics with and without transverse expansion has only been done for Cu+Cu collisions. We therefore use these for illustration. The collision energy is reflected in the initial entropy or energy density which is adjusted to the final charged hadron multiplicity as discussed in Sec. 3.1. One usually quotes the peak energy density $e_0$ or peak entropy density $s_0$ in the fireball center for $b=0$ collisions for reference. These values then determine the shapes and normalization of the initial density profiles at all impact parameters. Unless stated otherwise, the simulations were started at $\tau_0 = 0.6 \text{ fm/c}$, and thermal freeze-out was implemented on a hypersurface of constant energy density $e_{\text{dec}} = 0.075 \text{ GeV/fm}^3$. These choices will be motivated in Sec. 6.1.

In Section 5.2 we address non-central collisions and discuss the special opportunities provided by the breaking of azimuthal symmetry in this case. We discuss how the initial spatial deformation transforms rapidly into a momentum space anisotropy which ultimately manifests itself through a dependence of the emitted hadron spectra and their momentum correlations on the azimuthal emission angle relative to the reaction plane (“elliptic flow”).

5.1. Radial expansion in central collisions

Even though for boost-invariant longitudinal expansion there is no longitudinal acceleration in $\eta_s$ direction, the thermodynamic pressure performs longitudinal work on the fluid at the expense of thermal energy. For a fluid that has initially no transverse expansion but features boost-invariant longitudinal flow, the velocity shear tensor $\sigma^{\mu\nu}$ has non-zero diagonal elements that induce a negative shear pressure component in the longitudinal direction and equal positive pressure components of half the size in the two transverse directions [16, 59]. As a result, the fluid does less longitudinal work than in the ideal case, while transverse pressure gradients are increased and transverse expansion is accelerated.

Fig. 6. Left: Time evolution of the entropy density in the central cell of the expanding fireball. Shown as an example are central Cu+Cu collisions with an initial peak energy density $e_0 \equiv e(\pm 0, \tau_0=0.6 \text{ fm/c}) = 30 \text{ GeV/fm}^3$, with SM-EOS Q. The two dotted lines show ideal (light blue) and viscous (magenta) boost-invariant longitudinal expansion without transverse expansion. The lower set of lines include transverse radial expansion for ideal (dashed) and viscous (solid red) fluids, again assuming longitudinal boost invariance. The viscous simulations use $\eta/s = 1/4\pi$ for the specific shear viscosity and zero bulk viscosity. Right: Surfaces of constant temperature $T$ and constant radial flow velocity $v_\perp$ for viscous (left half) and ideal fluid dynamics (right half) in 2+1 dimensions, for the same collision system and EOS. MP indicates the mixed phase between QGP and HRG [13].
This is documented in the left panel of Fig. 6. In the absence of transverse flow (which can be simulated by providing initial conditions that are independent of transverse position \( x_\perp \)), longitudinally boost-invariant ideal fluid dynamics causes the entropy density to decrease like \( 1/\tau \) whereas in the viscous case it decreases more slowly (dotted lines). In the QGP phase \( s \sim T^3 \), so this implies a reduced cooling rate in the viscous case. As the volume increases linearly with \( \tau \) in this 1-dimensional situation, total entropy is conserved in the ideal fluid but increases with time in the viscous fluid.

Transverse expansion leads to additional cooling, but at early times the dominant viscous effect is still a reduced cooling rate due to reduced longitudinal work (dashed and solid lines in the left panel of Fig. 6). At later times, however, the additional radial flow caused by the positive viscous contribution to the transverse pressure gradients increases the cooling rate so much that, at least in the fireball center, the entropy density decreases more rapidly in the viscous than in the ideal fluid. At long times, the expansion becomes fully 3-dimensional, so the volume increases approximately like \( \tau^4 \) and the entropy density in both ideal and viscous hydro decreases like \( \tau^{-3} \). Entropy production ceases at late times because all viscous pressure components become very small [13].

The right panel in Fig. 6 gives a picture of the time evolution of the fireball in the transverse plane. It shows surfaces of constant temperature and lines of constant radial flow velocity, for viscous hydrodynamics (green dots) on the left (and mirrored on the right) and ideal hydrodynamics (blue dots) on the right side of the plot. The most prominent feature of the viscous hypersurfaces is their utter smoothness: For the ideal fluid, the isothermal hypersurfaces feature prominent structures arising from the first-order phase transition which are completely smeared out in viscous hydrodynamics. The reason are large velocity gradients near the QGP-MP and MP-HRG interfaces, caused by the sudden change of the speed of sound at these interfaces. These velocity gradients contribute to the shear flow tensor and generate shear viscous pressure gradients which suppress large velocity gradients and, at the same time, mask the discontinuities in the EOS, turning the first-order phase transition effectively into a smooth crossover [13].

The additional radial flow generated in viscous fluid dynamics by the positive transverse components of the shear pressure tensor generates, for identical initial conditions, flatter transverse momentum spectra than for ideal fluids [13, 19, 62, 66, 129]. This requires a retuning of initial conditions if one attempts to describe a given set of experimental spectra [63]. This will be discussed in more detail in Section 6.

The hydrodynamically generated radial flow rapidity profiles in heavy-ion collisions are typically linear, with a slope that initially increases with time but eventually saturates [47]. (More precisely it is the flow rapidity \( y_\perp = \frac{1}{2} \ln[(1+v_\perp)/(1-v_\perp)] \) –which is not constrained by the speed of light—that is proportional to \( r \).) If the fireball matter passes through a phase transition, the transition generates non-monotonic structures in the radial flow profiles at early times [35] which eventually disappear at late times. The developing flow pattern thus approaches a Hubble form where fluid cells recede from the fireball center with flow rapidities that increase linearly with distance – not only in longitudinal direction where this is imposed by the assumed boost-invariance, but also in the transverse directions. Contrary to our cosmos, however, this expansion is not isotropic: it features different “Hubble constants” in longitudinal and transverse directions, and the latter depend on collision energy and the EOS.

Figure 7 shows that the transverse flow rapidity profile is approximately linear not only at fixed proper time, but also along the decoupling surface. It compares the radial flow rapidity profile \( y_\perp(r) \) for Pb+Pb or Au+Au collisions at SPS and RHIC energies for three different equations of state [40], with LH8 corresponding most closely to EOS Q. Figure 7 provides welcome support for the
Fig. 7. The transverse flow rapidity \( y_\perp = \frac{1}{2} \ln[(1 + v_\perp) / (1 - v_\perp)] \) as a function of radial distance \( r \) along a surface of constant energy density \( e = 0.45 \text{GeV/fm}^3 \), for \( \text{Pb+Pb} \) collisions at the SPS (left) and for \( \text{Au+Au} \) collisions at RHIC (right) [40]. Three different equations of state have been explored in this figure [40], with LH8 corresponding most closely to EOS Q shown in Figure 5. The dashed and solid line segments subdivide the surface into 5 pieces through each of which flow 20% of the entropy.

phenomenologically very successful blast-wave parametrization [106–111] which is usually employed with a linear transverse velocity or rapidity profile, for reasons of simplicity. (Note that for the range of velocities covered in the figure the difference between rapidity \( y_\perp \) and velocity \( v_\perp = \tanh y_\perp \) can be neglected.)

As discussed in Section 3.2.1, particle freeze-out is controlled by the competition between the macroscopic expansion time scale [32, 101] \( \tau_{\exp} = (\partial \cdot u)^{-1} \) and the microscopic scattering time scale \( \tau_{\text{scatt}} = 1 / \sum_j \left( \langle \sigma_{ij} v_{ij} \rangle \rho_j \right) \). Figure 7 shows that the expansion rate \( \partial \cdot u \) changes significantly between SPS and RHIC: For boost-invariant longitudinal flow and a linear transverse flow rapidity profile \( y_\perp = \xi r \) the expansion rate is calculated as [46]

\[
\partial \cdot u = \frac{\cosh(\xi r)}{\tau} + \xi \left( \frac{\cosh(\xi r) + \sinh(\xi r)}{\xi r} \right) \approx \frac{1}{\tau} + 2\xi, \tag{48}
\]

where the approximation [144] holds in the region \( \xi r \ll 1 \). Equation (48) gives \( \tau (\partial \cdot u) = 1 + 2\xi \tau \). From Figure 7 we read off \( \xi = 0.07 \) at RHIC energies, but at SPS energies \( \xi \) is about 30% smaller. At freeze-out (\( \tau_{\text{dec}} \approx 15 - 17 \text{fm}/c \) [35, 40]) the expansion rate at RHIC is thus about 25% larger than at the SPS (\( (\partial \cdot u)_{\text{dec}} \approx 0.21 \text{fm}^{-1} \) for \( \text{Au+Au} \) at \( \sqrt{s} = 130 \text{A GeV} \) vs. \( (\partial \cdot u)_{\text{dec}} \approx 0.16 \text{fm}^{-1} \) for \( \text{Pb+Pb} \) at \( \sqrt{s} = 17 \text{A GeV} \)). The corresponding “Hubble times” at freeze-out are \( \tau_{\text{dec}}(\text{RHIC}) \approx 4.8 \text{fm}/c \) and \( \tau_{\text{dec}}(\text{SPS}) \approx 6.1 \text{fm}/c \). Barring a dramatic change in the scattering mean free times between SPS and RHIC energies that could result from different chemical compositions, one is led to the conclusion that at RHIC freeze-out should happen at somewhat higher decoupling temperatures than at the SPS. Note, however, that Fig. 3 does not support this conclusion.

5.2. Anisotropic flow in non-central collisions

In Section 5.1 we have already addressed some of the great opportunities offered by non-central collisions. The most important ones are related to the broken azimuthal symmetry, introduced through the spatial deformation of the nuclear overlap zone at non-zero impact parameter (see Figure 2). If the system evolves hydrodynamically, driven by its internal pressure gradients, it will...
expand more strongly in its short direction (i.e. into the direction of the impact parameter) than perpendicular to the reaction plane where the pressure gradient is smaller [145]. This is shown in Figure 8 where contours of constant energy density are plotted at times 2, 4, 6 and 8 fm/c after thermalization. The figure illustrates qualitatively that, as the system evolves, it becomes less and less deformed. In addition, some interesting fine structure develops at later times: After about 6 fm/c the energy density distribution along the $x$-axis becomes non-monotonous, forming two fragments of a shell that enclose a little ‘nut’ in the center [34]. Unfortunately, when plotting a cross section of the profiles shown in Figure 8 one realizes that this effect is rather subtle, and it was also found to be fragile, showing a strong sensitivity to details of the initial density profile [35] and to even small amounts of viscosity (see Fig. 6).

A more quantitative characterization of the contour plots in Figure 8 and their evolution with time is provided by defining the spatial eccentricity

$$
\epsilon_x(\tau) = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle},
$$

where the brackets indicate an average over the transverse plane with the local energy density $e(x, y; \tau)$ as weight function, and the momentum anisotropy

$$
\epsilon_p(\tau) = \frac{\int dxdy \left( T^{xx} - T^{yy} \right)}{\int dxdy \left( T^{xx} + T^{yy} \right)}.
$$

Note that with these sign conventions, the spatial eccentricity is positive for out-of-plane elongation (as is the case initially) whereas the momentum anisotropy is positive if the preferred flow direction is into the reaction plane.

Figure 9 shows the time evolution of the spatial and momentum anisotropies for Au+Au collisions at impact parameter $b = 7$ fm, for RHIC initial conditions with a realistic equation of state (EOS Q, solid lines) and for a much higher initial energy density (initial temperature at the fireball center $= 2$ GeV) with a massless ideal gas equation of state (EOS I, dashed lines) [44]. The initial spatial asymmetry at this impact parameter is $\epsilon_x(\tau_{equ}) = 0.27$, and obviously $\epsilon_p(\tau_{equ}) = 0$ since the fluid is initially at rest in the transverse plane. The spatial eccentricity is seen to disappear before the fireball matter freezes out, in particular for the case with the very high initial temperature (dashed lines).
where the source is seen to switch orientation after about 6 fm/c and become in-plane-elongated at late times [44]. One also sees that the momentum anisotropy $\epsilon_p$ saturates at about the same time when the spatial eccentricity $\epsilon_x$ vanishes. All of the momentum anisotropy is built up during the first 6 fm/c.

Near a phase transition (in particular a first order transition) the equation of state becomes very soft, and this inhibits the generation of transverse flow. This also affects the generation of transverse flow anisotropies as seen from the solid curves in Figure 9. The rapid initial rise of $\epsilon_p$ suddenly stops as a significant fraction of the fireball matter enters the mixed phase. It then even decreases somewhat as the system expands radially without further acceleration, thereby becoming more isotropic in both coordinate and momentum space. Only after the phase transition is complete and pressure gradients reappear, the system reacts to the remaining spatial eccentricity by a slight further increase of the momentum anisotropy. The softness of the equation of state near the phase transition thus focusses the generation of anisotropic flow to even earlier times, when the system is still entirely partonic and has not even begun to hadronize. At RHIC energies this means that almost all of the finally observed elliptic flow is created during the first 3-4 fm/c of the collision and reflects the hard QGP equation of state of an ideal gas of massless particles ($c_s^2 = \frac{1}{3}$) [35]. Microscopic kinetic studies of the evolution of elliptic flow lead to similar estimates for this time scale [146–149].

The anisotropic flow effects seen in non-central collisions turn out to be very sensitive to viscosity. All examples shown in this subsection so far assumed a perfect fluid. In Fig. 10 we show the evolution of the spatial eccentricity $\epsilon_x$ (top panel) and of the momentum anisotropies $\epsilon_p$ and $\epsilon'_p$ (bottom panel). Here $\epsilon_p$ is the total momentum anisotropy as defined in Eq. (49), using the complete energy momentum tensor. $\epsilon'_p = \langle T^{xx}_{eq} + T^{yy}_{eq} \rangle / \langle T^{xx}_{eq} + T^{yy}_{eq} \rangle$ is a variant of the momentum anisotropy that includes only the ideal fluid part $T^{\mu\nu}_{eq}$ and thus measures only the anisotropy of the transverse momentum density arising from anisotropies in the collective flow pattern. It ignores contributions to the anisotropy arising from the viscous pressure components which reflect momentum anisotropies of the phase-space distribution in the local fluid rest frame, caused by anisotropic deviations $\delta f$ of that distribution from local equilibrium.

The top panel of Figure 10 shows that the viscous fireball loses its spatial deformation initially faster than if it were a perfect fluid. This results mostly from the faster buildup of radial flow due to initially large viscous transverse pressure gradients – the fact that these gradients are themselves anisotropic plays only a minor role here. Early pressure gradient anisotropies manifest themselves in
the initial growth rate of the flow-induced momentum anisotropy $\epsilon'_p$ which is seen to slightly exceed that observed in the ideal fluid at times up to about 1 fm/c after the beginning of the transverse expansion (bottom panel in Fig. 10).

The dotted line in the bottom panel of Fig. 10 shows that for the viscous expansion the flow anisotropy is less than for the ideal fluid, especially at later times. This causes the spatial eccentricity of the viscous fireball to decrease more slowly than that of the ideal fluid towards the end of the expansion phase (solid line in the top panel).

It is instructive to compare the behaviour of the flow-induced ideal-fluid contribution to the momentum anisotropy, $\epsilon'_p$, with that of the total momentum anisotropy $\epsilon_p$. At early times they are very different, with $\epsilon_p$ being much smaller than $\epsilon'_p$ and even turning slightly negative at very early times (see insets in the lower panel of Fig. 9). This reflects very large negative contributions to the anisotropy of the total energy momentum tensor from the shear viscous pressure whose gradients along the out-of-plane direction $y$ strongly exceed those within the reaction plane along the $x$ direction. At early times this effect almost compensates for the larger in-plane gradient of the thermal pressure. The negative viscous pressure gradient anisotropy is responsible for reducing the growth of flow anisotropies, thereby causing the flow-induced momentum anisotropy $\epsilon'_p$ to significantly lag behind its ideal fluid value at later times. The negative viscous pressure anisotropies responsible for the difference between $\epsilon_p$ and $\epsilon'_p$ disappear at later times, since all viscous pressure components then become very small [13].

Figure 11 shows isothermal hypersurfaces and contours of constant transverse flow velocity for non-central Cu+Cu collisions, computed in ideal and viscous hydrodynamics. We again see the smoothing effects of shear viscosity which smears out all structures related to the assumed first-order phase transition, and the viscous slowdown of the cooling process (which now persists until
freeze-out because in peripheral Cu+Cu collisions enough transverse flow is never generated to overcome the effects of reduced longitudinal cooling).

We close this Section with a beautiful example of elliptic flow from outside the field of heavy-ion physics where the hydrodynamically predicted spatial expansion pattern shown in Figure 8 has for the first time been directly observed experimentally [150]: Figure 12 shows absorption images of an ensemble of about 200,000 $^6$Li atoms which were captured and cooled to ultralow temperatures in a CO$_2$ laser trap and then suddenly released by turning off the laser. The trap is highly anisotropic, creating a pencil-like initial spatial distribution with an aspect ratio of about 29 between the length and diameter of the pencil. The interaction strength among the fermionic atoms can be tuned with an external magnetic field by exploiting a Feshbach resonance. The pictures shown in Figure 12 correspond to the case of very strong interactions. The right panels in Figure 12 show that the fermion gas expands in the initially short ("transverse") direction much more rapidly than along the
axis of the pencil. As argued in the paper [150], the measured expansion rates in either direction are consistent with hydrodynamic calculations [151]. At late times the gas evolves into a pancake oriented perpendicular to the pencil axis. The aspect ratio passes through 1 (i.e. $\epsilon_x = 0$) about 600 $\mu$s after release and continues to follow the hydrodynamic predictions to about 800 $\mu$s after release. At later times it continues to grow, but more slowly than predicted by hydrodynamics, perhaps indicating a gradual breakdown of local thermal equilibrium due to increasing dilution. It is important to note that this phenomenon is only observed if the atoms interact strongly through the Feshbach resonance; off-resonance their interaction is weak, and instead of the pattern shown in

Fig. 12. Left: False color absorption images of a strongly interacting degenerate Fermi gas of ultracold $^6$Li atoms as a function of time after release from a laser trap. Right: Atomic density distributions in the initially shorter (top) and longer (bottom) directions at times 0.4 ms (red, narrowest), 1.0 ms (blue) and 2.0 ms (green, widest) after release from the trap. Reprinted with permission from O’Hara et al. [150] © 2002 AAAS.
Fig. 12 one observes ballistic expansion in all directions, with the cloud becoming spherical at late times. This shows that hydrodynamic behaviour, manifesting itself here in elliptic flow, requires a strongly coupled fluid.

6. Comparison with experiment

In heavy-ion collisions, the small size and short lifetime prohibit a similar direct observation of the spatial evolution of the fireball. Only the momenta of the emitted particles are directly experimentally accessible, and spatial information must be extracted somewhat indirectly using momentum correlations. We here discuss the single-particle hadron momentum spectra measured at RHIC. These test the space-time integrated aspects of collective flow in the dynamical models, through their manifestation in the final momentum distributions, in particular through their dependence on the hadron rest masses. A comprehensive review of two-particle correlations and their implications for the space-time structure of the collision fireball is presented in the accompanying article by Lisa and Pratt [152]. This section consists of three parts. In the first two we discuss 1) the rapidity and azimuthally integrated transverse momentum spectra and 2) the momentum-space anisotropies, in particular of the elliptic flow of various hadron species. In the third part we focus on signs of failure of the ideal fluid dynamical approach and point to evidence for viscous effects. We will concentrate on available comparisons with hydrodynamic model calculations. Many more experimental details and data on momentum anisotropies can be found in the accompanying review by Voloshin, Poskanzer and Snellings [153].

6.1. Azimuthally integrated momentum spectra

The primary single-particle observables in heavy-ion collisions are the triple-differential momentum distributions of identified hadrons $i$ as a function of collision centrality (impact parameter $b$):

$$\frac{dN_i}{p_T dp_T dy d\phi_p}(b) = \frac{1}{2\pi} \frac{dN_i}{p_T dp_T dy}(b) \left(1 + 2 \sum_{n=1}^{\infty} v_n^i(p_T, y; b) \cos(n\phi_p) \right).$$  (51)

We have expanded the dependence on the azimuthal emission angle $\phi_p$ relative to the reaction plane into a Fourier series [154]. Due to reflection symmetry with respect to the reaction plane, only cosine terms appear in the expansion. At midrapidity $y = \ln([E+p_z]/[E-p_z]) = 0$ all odd harmonics (in particular the directed flow coefficient $v_1^i$) vanish in symmetric collisions.

We begin by studying in Sec. 6.1.1 the $p_T$- and $\phi_p$-integrated rapidity distributions $dN_i/dy$. Section 6.1.2 will focus on the $\phi_p$-integrated transverse momentum distributions at midrapidity. In Sec. 6.2.1 finally, we discuss the azimuthal momentum anisotropies, in particular the elliptic flow coefficient $v_2(p_T, y; b)$.

6.1.1. Rapidity distributions

The final hadron rapidity distributions reflect their longitudinal collective dynamics at freeze-out. At high collision energies, the theoretically best justified initial particle production models implement longitudinal boost-invariance which identifies the particles’ initial rapidity $y$ with the space-time rapidity $\eta_s$ of their production point. Even after the particles begin to interact with each other, this imparts on the ensemble of produced particle a collective longitudinal expansion characterized by the identity $y_L = \eta_s$ (where $y_L$ is the average flow rapidity of the particles in a cell located at
space-time rapidity \( \eta_s \)). At high energies, this initial longitudinal collectivity completely dominates the final motion in beam direction. This is true even though the density of produced particles is not boost-invariant (i.e. it depends on rapidity). Hydrodynamic deceleration or acceleration effects due to longitudinal density and pressure gradients are weak to negligible, such that the final rapidity distribution of the particles closely resembles their initial space-time rapidity distribution.

Since collective flow affects hadrons of different masses in characteristic ways, and these masses appear only after hadronization but play no role in the initial particle production at the partonic level, one might hope that by comparing rapidity distributions of different hadron species one could explore the validity of the hydrodynamic picture. However, such mass-dependent flow effects are concentrated at low momenta, i.e. one would have to search for them near midrapidity \( (y=0) \) where all rapidity distributions are flat and thus have essentially the same shape [2]. In any case, no hydrodynamic model comparisons with rapidity spectra of identified hadrons have been done so far.

![Fig. 13. Charged hadron pseudorapidity \( \eta = \frac{1}{2} \ln \left( \frac{p+p_z}{p-p_z} \right) \) distributions from Au+Au collisions at \( \sqrt{s_{NN}} = 130 \) GeV (left) and \( \sqrt{s_{NN}} = 200 \) GeV (right) for 5 different centrality bins. The data are from the PHOBOS experiment [155]. The lines show (3+1)-d ideal fluid dynamical simulations with initial distributions derived from the CGC model (see Eq. (39) [77]).](image)

We conclude that the measured charged hadron rapidity distributions test our ability to predict the initial particle distribution in rapidity space, but not much else. Of course, one can always parametrize the initial space-time rapidity distribution such that the final charged hadron distribution matches experiment, and this was indeed done in many (3+1)-d hydrodynamical studies [50, 53, 54, 58, 156]. Such a procedure provides, however, little predictive power. To reproduce the collision energy dependence one must adjust parameters, and the impact parameter dependence is largely dictated by the overlap geometry in the transverse plane. Still, such studies is have provided one important insight [58]: For the rapidity distributions, it is irrelevant whether one ends the hydrodynamic evolution suddenly via Cooper-Frye freeze-out or allows decoupling to happen gradually by matching the hydrodynamic model to a hadronic cascade for the late hadronic stage. The resulting charged hadron rapidity distributions are identical.

There exists one model that claims to predict the rapidity, beam energy and centrality dependence of the initial particle production: the Color Glass Condensate theory [76]. Figure 13 shows that this claim is well supported by experiment. This is a non-trivial success of the CGC model. The Glauber model and its generalizations to non-zero rapidity [57, 157–159] cannot predict the collision energy
dependence of the rapidity distribution nor of the hard fraction \((1 - \alpha)\) in Eq. (33) that controls the non-linearity of the charged hadron multiplicity as a function of participant number \(N_{\text{part}}\). The CGC model does so successfully.

6.1.2. Transverse momentum and transverse mass distributions at midrapidity

The parameters of the hydrodynamic model are fixed by reproducing the measured centrality dependence of the total charged multiplicity \(dN_{\text{ch}}/dy\) as well as the shape of the pion and proton \(p_T\)-spectra in central collisions at midrapidity. The shapes of other hadron spectra, their centrality dependence and the dependence of the elliptic flow coefficient \(v_2\) on \(p_T\), centrality and hadron species \(i\) are then all parameter free predictions of the model [38]. These predictions will be compared with experiment and used to test the hydrodynamic approach and to extract physical information from its successes and failures.

The free parameters of the hydrodynamic model are the starting (thermalization) time \(\tau_{\text{equ}}\), the entropy and net baryon density in the center of the reaction zone at this time, and the freeze-out energy density \(\epsilon_{\text{dec}}\). The corresponding quantities at other fireball points at \(\tau_{\text{equ}}\) are then determined by the Glauber profiles discussed in Sec. 3.1. The ratio of net baryon to entropy density is fixed by the measured proton/pion ratio. Since the measured chemical composition of the final state at RHIC was found [113] to accurately reflect a hadron resonance gas in chemical equilibrium at the hadronization phase transition, we require the hydrodynamic model to reproduce this \(p/\pi\) ratio on a hypersurface of temperature \(T_{\text{crit}}\). In ideal fluid dynamics, the final total charged multiplicity \(dN_{\text{ch}}/dy\) fixes by entropy conservation the initial product \((s \cdot \tau)_{\text{equ}}\) [33, 73, 145]. The value of \(\tau_{\text{equ}}\) controls how much transverse flow can be generated until freeze-out. Since the thermal motion and radial flow affect light and heavy particles differently at low \(p_T\) [106, 160], a simultaneous fit of the final pion and proton spectra separates the radial flow from the thermal component. The final flow strength then fixes \(\tau_{\text{equ}}\) whereas the freeze-out temperature determines the energy density \(\epsilon_{\text{dec}}\) at decoupling.

The top left panel of Fig. [14] shows the ideal hydrodynamic fit [161] to the transverse momentum spectra of positive pions and antiprotons, as measured by the PHENIX and STAR collaborations in central \((b = 0)\) Au+Au collisions at \(\sqrt{s} = 130\ A\ GeV\) [162–165]. The fit yields an initial central entropy density \(s_{\text{equ}} = 95\ fm^{-3}\) at an equilibration time \(\tau_{\text{equ}} = 0.6\ fm\). This corresponds to an initial temperature of \(T_{\text{equ}} = 340\ MeV\) and an initial energy density \(\epsilon = 25\ GeV/fm^3\) in the fireball center. (Note that these parameters satisfy the “uncertainty relation” \(\tau_{\text{equ}} \cdot T_{\text{equ}} \approx 1\).)

Freeze-out was implemented on a hypersurface of constant energy density with \(\epsilon_{\text{dec}} = 0.075\ GeV/fm^3\).

The fit in the top left panel of Fig. [14] was performed with EOS Q which assumes chemical equilibrium in the HRG phase all the way down to \(T_{\text{dec}}\). The chemical equilibrium value for the \(p/\pi\) ratio at this temperature does not agree with experiment which indicates chemical freeze-out at \(T_c\) (see Fig. [4]). The normalization of the other hadron spectra relative to that of the pions must thus be adjusted by hand. The information that is required to fix the initial and freeze-out conditions resides, however, in the shape (and not in the normalization) of the pion and proton spectra. After these conditions have been fixed, the shapes of other hadron spectra in central collisions are model predic-

\footnote{It should be noted that this determination of \(\tau_{\text{equ}}\) is not very precise since earlier starting times also lead to earlier freeze-out, limiting the buildup of radial flow. One really obtains only an upper limit for \(\tau_{\text{equ}}\) in this way. In viscous hydrodynamics one must additionally reduce the product \((s \cdot \tau)_{\text{equ}}\) when shortening \(\tau_{\text{equ}}\), to account for entropy viscous production [68]. The consequences of this for the final shape of the \(p_T\)-spectra have not yet been fully explored.}
Fig. 14. Identified pion, antiproton and kaon spectra for $\sqrt{s_{NN}} = 130$ GeV from the PHENIX [162, 163] and STAR [164, 165] collaborations in comparison with results from an ideal fluid dynamical calculation [161]. The top left panel shows pion and (anti-)proton spectra from central collisions. Shown in the other panels are spectra of five different centralities: from most central (top) to the most peripheral (bottom). The spectra are successively scaled by a factor 0.1 for clarity.

Table 1. Initial conditions for SPS and RHIC energies used to fit the particle spectra from central Pb+Pb or Au+Au collisions. $s_{\text{equ}}$ and $T_{\text{equ}}$ refer to the maximum values at $\tau_{\text{equ}}$ in the fireball center.

| $\sqrt{s_{NN}}$ (GeV) | SPS | RHIC 1 | RHIC 2 |
|------------------------|-----|--------|--------|
| $s_{\text{equ}}$ (fm$^{-3}$) | 43  | 95    | 110    |
| $T_{\text{equ}}$ (MeV)   | 257 | 340   | 360    |
| $\tau_{\text{equ}}$ (fm/$c$) | 0.8 | 0.6   | 0.6    |

Table 1. Initial conditions for SPS and RHIC energies used to fit the particle spectra from central Pb+Pb or Au+Au collisions. $s_{\text{equ}}$ and $T_{\text{equ}}$ refer to the maximum values at $\tau_{\text{equ}}$ in the fireball center.

The remaining three panels of Fig. 14 show the predicted transverse momentum spectra of pions, kaons and antiprotons in five different centrality bins, compared with measurements by the PHENIX [162, 163] and STAR [164, 165] collaborations. For all centrality classes, except the most peripheral one, the ideal fluid dynamical predictions (solid lines) agree quite well with the data. The kaon spectra are reproduced almost perfectly, but for pions the model consistently underpredicts the data at low $p_T$. This has now been understood to be largely an artifact of having employed in these calculations a chemical equilibrium equation of state all the way down to kinetic freeze-
out. Later calculations [45] with a chemical non-equilibrium HRG equation of state, which will be compared to $\sqrt{s} = 200$ A GeV data below, show that, as the system cools below the chemical freeze-out point $T_{\text{chem}} \approx T_{\text{crit}}$, a significant positive pion chemical potential builds up, emphasizing the concave curvature of the spectrum from Bose effects and increasing the feeddown corrections from heavier resonances at low $p_T$. The inclusion of non-equilibrium baryon chemical potentials to avoid baryon-antibaryon annihilation further amplifies the resonance feeddown for pions.

Significant discrepancies are also seen at large impact parameters and large transverse momenta $p_T \gtrsim 2.5$ GeV/$c$. This is not surprising since high-$p_T$ particles require more rescatterings to thermalize and escape from the fireball before doing so. This is in particular true in more peripheral collisions where the reaction zone is smaller.

For the calculations shown in Fig. 14 the same value $\varepsilon_{\text{dec}}$ was used for all impact parameters. Recent work [103] using the kinetic freeze-out criterium [11] has shown that in peripheral collisions the fireball decouples at somewhat higher temperatures than in central collisions, in agreement with the data shown in Fig. 4. The consequences of this for the centrality dependence of the $p_T$-spectra have not yet been explored within the hydrodynamic model.

Without transverse flow, thermal spectra exhibit $m_T$-scaling [166], i.e. after appropriate rescaling of the yields all spectra collapse onto a single curve. Transverse collective flow breaks this scaling at low $p_T \lesssim m_0$ (i.e. for non-relativistic transverse particle velocities) by an amount which increases with the particle rest mass $m_0$ [107, 160, 167]. When plotting the spectra against $p_T$ instead of $m_T$, any breaking of $m_T$-scaling is at least partially masked by a kinematic effect at low $p_T$ that, unfortunately, again increases with the rest mass $m_0$. To visualize the effects of transverse flow on the spectral shape thus requires plotting the spectrum logarithmically as a function of $m_T - m_0$.

Such plots [168–171] show a clear tendency of the heavier hadron spectra to curve and to begin to develop a shoulder at low transverse kinetic energy $m_T - m_0$, as expected from transverse flow.

One such example is shown in Fig. 15 where $\Omega$ hyperons spectra [171] are compared with hydrodynamical calculations.

Fig. 15. Transverse mass spectrum of $\Omega$ hyperons from central 200 A GeV Au+Au collisions at RHIC [171]. The curves are ideal hydrodynamic calculations with different initial and freeze-out conditions: Solid lines correspond to the default of no initial transverse flow at $\tau_{\text{equ}}$, dashed lines assume a small but non-zero radial flow, $v_r = \tanh(\alpha r)$ with $\alpha = 0.02$ fm$^{-1}$, already at $\tau_{\text{equ}}$. The lower (thin) set of curves assumes $\Omega$-decoupling at $T_{\text{crit}} = 164$ MeV, the upper (thick) set of curves decouples the $\Omega$ together with the pions and protons at $T_{\text{dec}} = 100$ MeV [45].
namic predictions. For this comparison the original calculations for 130 A GeV Au+Au collisions [38] were repeated with RHIC2 initial conditions and a chemical non-equilibrium equation of state in the hadronic phase [45]. The solid lines are based on default parameters (see Table 1) without any initial transverse flow at $\tau_{\text{eqv}}$. (The dashed lines will be discussed further below.) Following a suggestion that $\Omega$ hyperons, being heavy and not having any known strong coupling resonances with pions, should not be able to participate in any increase of the radial flow during the hadronic phase and thus decouple early [172], we show two solid lines, the steeper one corresponding to decoupling at $e_{\text{dec}} = 0.45 \text{GeV/fm}^3$, i.e. directly after hadronization at $T_{\text{crit}}$, whereas the flatter one assumes decoupling together with pions and other hadrons at $e_{\text{dec}} = 0.075 \text{GeV/fm}^3$. The data clearly favor the flatter curve, suggesting intense rescattering of the $\Omega$’s in the hadronic phase. The microscopic mechanism for this rescattering is still unclear. However, without hadronic rescattering the hydrodynamic model, in spite of its perfect local thermalization during the early expansion stages, is unable to generate enough transverse flow to flatten the $\Omega$ spectra as much as required by the data. Partonic hydrodynamic flow alone can not explain the $\Omega$ spectrum.

We now illustrate the effects on the ideal fluid dynamic particle spectra caused by correctly accounting for the non-equilibrium chemistry in the hadronic phase [45, 55, 119–122]. Figure 16 shows a compilation of experimental pion, kaon and (anti-)proton spectra for 200 A GeV Au+Au collisions from the four RHIC collaborations [173–175, 178], compared with results from ideal hydrodynamics. The calculations (shown as thick solid red lines in Fig. 16) use the same decoupling energy density $e_{\text{dec}} = 0.075 \text{GeV/fm}^3$ as before. As discussed in Sec. 4 this corresponds to the same flow strength as with the chemically equilibrated EOS, but a significantly lower freeze-out temperature $T_{\text{dec}} \approx 100 \text{MeV}$ [45, 55, 119]. The thin solid (blue) lines in the Figure, shown for comparison, were calculated by assuming kinetic freeze-out already at hadronization, $T_{\text{crit}} = 165 \text{MeV}$. The data clearly favor the additional radial boost resulting from the continued buildup of radial flow in the hadronic phase. Still, even at $e_{\text{dec}} = 0.075 \text{GeV/fm}^3$, the spectra are still steeper than the data and the previous calculations with a chemical equilibrium equation of state shown in Fig. 14 reflecting the combination of the same flow pattern with a lower freeze-out temperature.

Somewhat unexpectedly, the authors of the study [45] were unable to significantly improve the
situation by reducing $e_{\text{dec}}$ even further: The effects of a larger radial flow at lower $e_{\text{dec}}$ were almost completely compensated by the accompanying lower freeze-out temperature, leading to only modest improvements for kaons and protons and almost none for the pions. The reason for this is the steep drop of $T$ with decreasing $e_{\text{dec}}$ for the chemically non-equilibrated EOS (PCE) shown in the lower right panel of Fig. 5.

This motivated the authors to introduce a small but non-vanishing transverse “seed” velocity already at the beginning of the hydrodynamic stage (see also more recent work [89, 176] and the Appendix of P. Kolb’s thesis [42]). The dashed lines in Fig. 16 (and also earlier in Fig. 15) show hydrodynamic calculations with an initial transverse flow velocity profile given by $v_r(r, \tau_{\text{eq}}) = \tanh(\alpha r)$ with $\alpha = 0.02$ fm$^{-1}$. This initial transverse kick is seen to significantly improve the agreement with the pion, kaon and antiproton data up to $p_T \gtrsim 1.5 - 2$ GeV/c for pions and kaons and up to $p_T \gtrsim 3.5$ GeV/c for (anti)protons [45]. It can be motivated by invoking some collective (although not ideal hydrodynamic) transverse motion of the fireball already during the initial thermalization stage. However, this is not the only possible solution to the problem. As discussed below, viscous effects in the late hadronic stage are strong and contribute significantly to the required flattening of the spectra. Such effects are not captured by an ideal fluid dynamical approach.

A great initial surprise at RHIC was the observation that the antiproton/pion ratio increases with $p_T$ and actually exceeds 1 above $p_T \sim 2 - 2.5$ GeV/c [163, 178]. Thermal momentum distributions boosted by hydrodynamical radial flow, combined with the small baryon chemical potential at RHIC, provide a natural explanation of this so-called “$\bar{p}/\pi^- > 1$ anomaly” [47].

We close this subsection with a brief discussion of the centrality dependence of mean transverse momenta per particle, $\langle p_T \rangle$, and the average transverse energy per charged hadron, $\langle E_T \rangle/N_{\text{ch}}$. Figure 17 shows a comparison of $\langle p_T \rangle$ for identified pions, kaons, protons and antiprotons measured by PHENIX in 200 A GeV Au+Au collisions [178–180] with the hydrodynamic results [45]. The bands
reflect the theoretical variation resulting from possible initial transverse flow already at the beginning of the hydrodynamic expansion stage, as discussed at the end of the previous subsection. The figure shows some discrepancies between hydrodynamics and the data for peripheral collisions (small \( N_{\text{part}} \)) which are strongest for the kaons whose spectra are flatter at large impact parameters than predicted by the model.

The right panel in Figure [17] shows the total transverse energy per emitted charged hadron as a function of collision centrality. Although both the charged particle multiplicity and total transverse energy vary strongly with the number of participating nucleons and collision energy, the transverse energy per particle is essentially independent of these parameters. The superimposed band in Figure [17] reflects ideal hydrodynamic calculations for Au+Au collisions at \( \sqrt{s} = 200 \text{ A GeV} \) with and without initial transverse flow. The slight rise of the theoretical curves with increasing \( N_{\text{part}} \) can be attributed to the larger average transverse flow developing in more central collisions, resulting from the higher initial energy density and the somewhat longer duration of the expansion until freeze-out [39]. Successful reproduction of the data requires a correct treatment of the chemical composition at freeze-out (by using a chemical non-equilibrium hadron equation of state below \( T_{\text{crit}} \)). If one instead assumes chemical equilibrium of the hadron resonance gas down to kinetic freeze-out, ideal fluid dynamics overpredicts the transverse energy per particle by about 15-20% [39].

6.2. Anisotropic transverse momentum spectra from deformed fireballs

6.2.1. Elliptic flow as an early fireball signature

In non-central nuclear collisions, or if the colliding nuclei are deformed, the nuclear overlap region is initially spatially deformed (see Fig. 2). Interactions among the constituents of the matter formed in that zone transfer this spatial deformation onto momentum space. Even if the fireball matter does not interact strongly enough to reach and maintain almost instantaneous local equilibrium, and a hydrodynamic description therefore fails, any kind of re-interaction among the fireball constituents will still be sensitive to the anisotropic density gradients in the reaction zone and thus redirect the momentum flow preferably into the direction of the strongest density gradients (i.e. in the “short” direction) [146–149, 184]. The result is a momentum-space anisotropy, with more momentum flowing into the reaction plane than out of it.

Such a “momentum-space reflection” of the initial spatial deformation is a unique signature for re-interactions in the fireball and, when observed, proves that the fireball matter has undergone significant nontrivial dynamics between creation and freeze-out. Without rescattering, the only other mechanism with the ability to map a spatial deformation onto momentum space is the quantum mechanical uncertainty relation. For matter confined to smaller spatial dimensions in \( x \) than in \( y \) direction it predicts \( \Delta p_x > \Delta p_y \) for the corresponding widths of the momentum distribution. However, any momentum anisotropy resulting from this mechanism is restricted to momenta \( p \sim 1/(\text{size of the overlap zone}) \) which for a typical fireball radius of a few fm translates into a fraction of 200 MeV/c. This is the likely mechanism for the momentum anisotropy observed [185] in calculations of the classical dynamical evolution of a postulated deformed “color glass condensate” created initially in the collision. Unlike the experimental data, this momentum anisotropy is concentrated around relatively low \( p_T \) [185].

Whatever the detailed mechanism responsible for the observed momentum anisotropy, the induced faster motion into the reaction plane than perpendicular to it (“elliptic flow”) rapidly degrades the initial spatial deformation of the matter distribution and thus eliminates the driving force for
any further increase of the anisotropic flow. Elliptic flow is therefore “self-quenching” \([146, 147]\), and any flow anisotropy measured in the final state must have been generated early when the collision fireball was still spatially deformed (see Fig. 9). If elliptic flow does not develop early, it never develops at all. It thus reflects the pressure and stiffness of the equation of state during the earliest collision stages \([33, 35, 146–148]\), but (in contrast to many other early fireball signatures) it can be easily measured with high statistical accuracy since it affects all final state particles.

Microscopic kinetic models show that, for a given initial spatial deformation, the induced momentum space anisotropy is a monotonically rising function of the strength of the interaction among the matter constituents \([148, 149, 184]\). The maximum effect should thus be expected if their mean free path approaches zero, i.e. in the ideal fluid limit \([35, 43]\). Viscous effects associated with finite mean free paths reduce the elliptic flow \([186]\), especially at larger \(p_T\) \([129, 149]\). Within the ideal fluid limit, the magnitude of the elliptic flow shows some sensitivity to the nuclear equation of state in the early collision stage, but the variation is not very large (see Fig. 34 in Ref. \([153]\)). To the extent that the initial spatial fireball deformation is known (see Fig. 2 – the average impact parameter can be determined geometrically from the ratio of the observed multiplicity in the event to the maximum multiplicity from all events), the observed magnitudes of the momentum anisotropies, and in particular their dependence on collision centrality \([184, 190]\), provide valuable measures for the degree of thermalization reached early in the collision.

Experimentally this program was first pursued at the SPS in \(158\) \(A\) GeV \(\text{Pb}+\text{Pb}\) collisions \([187]\). These data still showed significant sensitivity to details of the analysis procedure \([188]\) and thus remained somewhat inconclusive \([36]\). Qualitatively, the SPS data (where the directed and elliptic flow coefficients, \(v_1\) and \(v_2\), can both be measured) confirmed Ollitrault’s 1992 prediction \([145]\) that near midrapidity the preferred flow direction is into the reaction plane, supporting the conclusions from earlier measurements in \(\text{Au}+\text{Au}\) collisions at the AGS \([189]\) where a transition from out-of-plane to in-plane elliptic flow had been found between \(4\) and \(6\) \(A\) GeV beam energy. A comprehensive quantitative discussion of elliptic flow became first possible with RHIC data, because of their better statistics and improved event plane resolution (due to the larger event multiplicities) and also as a result of improved analysis techniques \([153, 190]\). In the meantime the latter have also been re-applied to SPS data and produced very detailed results from \(\text{Pb}+\text{Pb}\) collisions at this lower beam energy \([191–193]\). Ideal fluid dynamical predictions for the spectra and differential elliptic flow \(v_2(p_T)\) of pions and protons are now available for collision energies ranging all the way from the AGS to LHC \([86, 194, 195]\).

### 6.2.2. Elliptic flow at RHIC

The second published and still among the most important results from \(\text{Au}+\text{Au}\) collisions at RHIC was the centrality and \(p_T\) dependence of the elliptic flow coefficient at midrapidity \([196]\). For central to midperipheral collisions and for transverse momenta \(p_T \lesssim 1.5\) GeV/c the data were found to be in stunning agreement with hydrodynamic predictions \([35, 36]\), as seen in Fig. 18. In the left panel, the ratio \(n_{ch}/n_{max}\) of the charged particle multiplicity to the maximum observed value is used to characterize the collision centrality, with the most central collisions towards the right near 1. \(n_{ch}/n_{max} = 0.45\) corresponds to an impact parameter \(b \approx 7\) fm \([197]\). Up to this value the observed elliptic flow \(v_2\) is found to track very well the increasing initial spatial deformation \(\epsilon_x\) of the nuclear overlap zone \([197]\), as predicted by ideal fluid dynamics \([35]\).

An important prediction of the hydrodynamic model is the characteristic dependence of the differential elliptic flow \(v_2(p_T)\) on the particle rest mass, shown in the left panel of Fig. 19 \([38]\). It
Elliptic flow of unidentified charged particles in 130 A GeV Au+Au collisions, integrated over \( p_T \) as function of centrality (left) and for minimum bias collisions as a function of \( p_T \) (right). Both data sets (symbols with error bars) are from the original STAR publication [196]. The vertical bars in the left panel [196] indicate the range of earlier hydrodynamic predictions for a variety of equations of state and initial conditions [35]. The top three curves in the right panel [36] represent hydrodynamic predictions for semiperipheral collisions with initial conditions tuned to the observed [92] total charged multiplicity in central collisions where \( v_2 \) vanishes. Different curves correspond to different equations of state and freeze-out temperatures [36].

arises primarily from the assumption of local thermal equilibrium on which hydrodynamics is based. Thermal hadron spectra exhibit \( m_T \)-scaling which is exact in the absence of flow and slightly broken at low \( p_T \) by radial flow (see discussion in Sec. 6.1.2 [106, 160]). When plotted as a function of \( p_T \), an exponential function in \( m_T \) exhibits a shoulder at low \( p_T \) that becomes broader and flatter with increasing particle rest mass. This flattening of the single-particle spectra at low \( p_T \) is the primary reason [38] for the flattening of \( v_2(p_T) \) at low \( p_T \) with increasing rest mass seen in Fig. 19, left panel. Additional scale-breaking effects from radial flow exist but are of less importance. For this reason, the rest mass dependence of the differential elliptic flow can be eliminated almost completely by reploting \( v_2 \) as a function of the transverse kinetic energy \( K_{ET} = m_T - m_0 \) instead of \( p_T \). This is shown in the right panel of Fig. 19.

**Fig. 18.** Elliptic flow of unidentified charged particles in 130 A GeV Au+Au collisions, integrated over \( p_T \) as function of centrality (left) and for minimum bias collisions as a function of \( p_T \) (right). Both data sets (symbols with error bars) are from the original STAR publication [196]. The vertical bars in the left panel [196] indicate the range of earlier hydrodynamic predictions for a variety of equations of state and initial conditions [35]. The top three curves in the right panel [36] represent hydrodynamic predictions for semiperipheral collisions with initial conditions tuned to the observed [92] total charged multiplicity in central collisions where \( v_2 \) vanishes. Different curves correspond to different equations of state and freeze-out temperatures [36].

**Fig. 19.** Left: Ideal fluid dynamical predictions for the differential elliptic flow \( v_2(p_T) \) in minimum bias Au+Au collisions at \( \sqrt{s} = 130 \) A GeV, for different identified hadron species [38]. The hydrodynamical simulations use EOS Q. Right: The same curves redrawn as functions of the transverse kinetic energy \( K_{ET} = m_T - m_0 \) [198].
Fig. 20.  **Left:** Up to $p_T \sim 1.5$ GeV/c, the differential elliptic flow $v_2(p_T)$ follows the hydrodynamical predictions for an ideal fluid almost perfectly [204]. Note that $> 99\%$ of all final hadrons have $p_T < 1.5$ GeV/c. **Middle:** When plotted against transverse kinetic energy, the differential elliptic flow follows different universal curves for mesons and baryons. **Right:** When scaled by the number of valence quarks, the differential elliptic flow per quark follows the same universal curve for all hadrons and for all values of (scaled) transverse kinetic energy [205].

Figure 20 shows that these predictions of the hydrodynamic model are nicely borne out in the experimental data [199–205]. The left panel shows the differential elliptic flow as a function of $p_T$ for five different hadron species. Up to transverse momenta of $p_T \sim 1.5$ GeV/c the data show a clear tendency of $v_2(p_T)$ to decrease with increasing rest mass, and they agree even quantitatively with the hydrodynamic predictions. (Remember that radial flow plays only a subdominant role in this mass hierarchy, so this should be taken as support for approximate local thermal equilibrium, but not necessary for *ideal* fluid dynamics.) Since the majority of hadrons (> 99%) have transverse momenta below 1.5 GeV/c, the bulk of the fireball is seen to be well described by the hydrodynamic model.

At larger transverse momenta ($p_T \gtrsim 1.5$ GeV/c for mesons, $p_T \gtrsim 2.3$ GeV/c for baryons), the measured elliptic flow lags behind the ideal fluid dynamical predictions. This is expected if one accepts that the ideal fluid assumption of instantaneous thermalization is unrealistic and allows for a finite mean free path of the particles. The latter leads to viscous corrections which manifest themselves more strongly as $p_T$ increases [129, 149] (see Eq. (47)). What is not expected is that, above these breakaway points from ideal hydrodynamics, the elliptic flow curves appear to cluster into two groups which, instead of being arranged by mass, are ordered according to whether the hadron is a meson or a baryon. This is more clearly seen in the middle panel of Fig. 20 where the differential elliptic flow is replotted as a function of transverse kinetic energy. Hydrodynamics predicts that then all curves should approximately collapse onto a single line (right panel of Fig. 19). This indeed happens at low KE_T, where the left panel in Fig. 20 has shown that the data agree with hydrodynamics, but at larger KE_T, where the data break away from the fluid dynamical prediction, $v_2(KE_T)$ splits into two curves for baryons and mesons.

Clearly, hydrodynamics provides no explanation for this behaviour, since the splitting only happens where the hydrodynamic model ceases to be valid. The observations can be explained in a quark coalescence model [206–209] which postulates that at intermediate transverse momenta (i.e. above the point where hydrodynamics breaks down and below the range where hard jet fragmentation dominates the hadron yield) hadron production proceeds through the coalescence of valence quarks. This model predicts a scaling of $v_2$ with the number of valence quarks $n_q$ inside the hadron [207]:

\[ v_2 \propto n_q \]
\[ v_{h}^{2}(p_{T}) = n_{q} v_{v}^{2} \left( \frac{p_{T}}{n_{q}} \right). \]

Where this scaling holds, it should yield a universal curve if one plots \( \frac{v_{h}}{n_{q}} \) against \( \frac{p_{T}}{n_{q}} \). At high \( p_{T} \) where rest masses can be neglected, a plot against \( K_{E}/n_{q} \) should be equally good. At low \( p_{T} \) where \( v_{2} \) agrees with hydrodynamics which predicts a linear dependence of \( v_{2} \) on \( K_{E} \), a rescaling of both axes by \( n \) has no effect on the shape of the curve. Hence, a plot of \( \frac{v_{h}}{n_{q}} \left( \frac{K_{E}}{n_{q}} \right) \) should yield a universal curve both at low \( K_{E} \) where hydro works and at intermediate \( K_{E} \) where the coalescence model applies. As seen in the right panel of Fig. 20, the experimental data confirm this expectation beautifully. This collapse of all hadronic elliptic flow data onto a single universal curve works at both \( \sqrt{s_{NN}} = 63 \) and 200 GeV and at all centralities [210], even though both the slope of the curve at low \( K_{E} \) and the saturation values at intermediate \( K_{E} \) change with collision centrality (due to the changing initial fireball eccentricity).

We should not leave this discussion without pointing out an important caveat: The comparison between experimental data and ideal fluid dynamics shown in the left panel of Fig. 20 is based on simulations with EOS Q which do not implement the correct non-equilibrium chemical composition in the HRG stage. When this deficiency is corrected, the \( p_{T} \)-slope of the pion elliptic flow \( v_{2}^{2}(p_{T}) \) increases by about 30% [45, 55, 122], hence the good agreement between theory and experiment shown in Fig. 20 is a bit deceptive. We will pursue this theme further in Section 6.3.

### 6.2.3. Implications: Rapid thermalization and “strongly coupled quark-gluon plasma” (sQGP)

The apparent success of the ideal fluid dynamical picture in describing bulk hadron emission from relativistic heavy-ion fireballs not just on a superficial qualitative level, but in many aspects even quantitatively had tremendous implications for the heavy-ion community’s view of hot QCD matter. It led to a genuine paradigm shift, away from the idea of the QGP as a weakly interacting gas of quarks and gluons and towards that of a strongly coupled plasma with liquid behaviour [43,211,212].

This shift has both motivated and survived the quantitative refinements of the picture that will be discussed in Section 6.3. It generated strong interest outside the field of nuclear physics, in particular in the area of cold atoms (see discussion around Fig. 12) and in superstring theory. In experiments with cold atoms one has the unique ability to continuously change the interaction strength among the particles by dialling an external magnetic field, moving the atoms onto or away from a Feshbach resonance [150]. In superstring theory, Maldacena’s AdS/CFT correspondence [213] between strongly coupled conformal field theories (CFT) and weakly coupled (classical) gravitational theories in curved 5-dimensional Anti-de-Sitter (AdS) spaces opened a window for performing analytical calculations for quantum field theories in the strong-coupling limit, by solving classical differential equations for strings moving in appropriately curved space-times. Even though QCD is not itself a conformal field theory, such studies have established several “universal” results, such as a lower limit for the shear viscosity to entropy density ratio \( \eta/s \geq \frac{\hbar}{4\pi k_{B}} \) (the so-called “KSS bound” [214,215]) that applies to a large class of conformal field theories including a supersymmetric version of QCD. While it is presently unclear whether the almost perfect liquid behaviour of the QGP, as indicated by the results presented above, really signals a breakdown of perturbative QCD, and one may remain doubtful whether in the end superstring theoretical methods based on the AdS/CFT correspondence will really lead to a more efficient and quantitative understanding of QGP properties than appropriately refined perturbative methods in QCD (resummed pQCD), it is an undeniable fact that this cross-fertilization has generated unprecedented productivity in practically relevant situations within field that has long been plagued by doubts that it can ever be confirmed or falsified experimentally. Furthermore, AdS/CFT–based calculations have permanently reshaped the way theorists look at
strongly coupled field theoretical averaged systems where conventional particle-based pictures break down absolutely.

\[
\begin{align*}
\text{Fig. 21. } & \text{Left: Impact parameter averaged elliptic flow as a function of transverse momentum, for Au+Au collisions at } \sqrt{s_{NN}} = 130 \text{ GeV. Experimental results from STAR [196] are compared with parton cascade calculations [149] based on 2-body collisions with varying transport opacities } \xi. \text{ Right: } p_T\text{-integrated elliptic flow as a function of collision centrality, for Au+Au collisions at } \sqrt{s_{NN}} = 200 \text{ GeV. Experimental results from the STAR [216] and PHOBOS [217] collaborations are compared with parton cascade simulations [218] including both two- and three-body interactions [219] with varying values for the strong coupling constant } \alpha_s. \end{align*}
\]

After the discovery of strong elliptic flow at RHIC it was quickly realized [149] that the measured [196] almost linear rise of the charged particle (i.e. predominantly pionic) elliptic flow with \( p_T \) requires strong rescattering among the fireball constituents. The left panel of Figure 21 shows the results from microscopic simulations which describe the dynamics of the early expansion stage by solving a Boltzmann equation with a 2-body collision term for colliding on-shell partons [149]. The different curves are parametrized by the transport opacity \( \xi = \sigma_0 dN_g / d\eta \) involving the product of the parton rapidity density and cross section in the early collision stage. As the opacity is increased, the elliptic flow is seen to approach the data (and the hydrodynamic limit) monotonically from below. Whereas the hydrodynamic limit predicts a continuous rise of \( v_2(p_T) \), the elliptic flow from the parton cascade saturates at high \( p_T \), as also seen in the data [220]. This is due to incomplete equilibration at high \( p_T \). The critical \( p_T \) at which the cascade results cease to follow the hydrodynamic rise shifts to higher (lower) values as the transport opacity is increased (decreased), corresponding to a decrease (increase) of viscous corrections to the distribution function (see Eq. (17)).

From that figure it was concluded [149] that the measured elliptic flow requires very large transport opacities, at least as long as only two-body collisions are included, exceeding perturbative expectations by a factor 15-30 [149]. Xu and Greiner [219] pointed out that the inclusion of radiative collision processes \( gg \leftrightarrow ggg \) changes this conclusion. The right panel in Fig. 21 shows that these radiative processes strongly accelerate the thermalization process, and that a perturbative description of the experimental data does not appear to be entirely excluded (although \( \alpha_s = 0.6 \) may be an uncomfortably large coupling constant for a perturbative QCD approach). What remains true, aside from all ongoing discussions about details, is that the experimental data require high interaction rates and short scattering time scales, i.e. rapid thermalization. Using the BAMPs cascade [219] Xu and Greiner showed that even a shear viscosity to entropy ratio close to the KSS bound \( \eta/s \geq 1/4\pi \) can be reached for \( \alpha_s = 0.6 \) [218].

These microscopic simulations complement the empirical observations in cold atom systems (see
Fig. 22 in demonstrating the importance of rapid thermalization for a successful description of the elliptic flow data. But there is an additional, quite general argument that further reinforces this point [35, 47]. As mentioned earlier, the hydrodynamically predicted elliptic flow is proportional to the initial spatial eccentricity \( \varepsilon_x(\tau_{\text{eq}}) \) at the beginning of the hydrodynamic evolution. If thermalization is slow, the matter will start to evolve in the transverse directions before \( \tau_{\text{eq}} \) is reached, following its initial locally isotropic transverse momentum distribution. Even if no reinteractions among the produced particles occur, this radial free-streaming motion dilutes the spatial deformation, but without generating any momentum anisotropy. Thus, if thermalization and hydrodynamic behavior set in late, they will be able to build only on a significantly reduced spatial eccentricity \( \varepsilon_x \), and the resulting elliptic flow response will be correspondingly smaller. To reach a certain measured value of \( \varepsilon_2 \) at a given impact parameter thus requires thermalization to set in before free radial motion has reduced the spatial deformation so much that even perfect hydrodynamic motion can no longer produce the measured momentum anisotropy. This consideration yields a rigorous upper limit for the thermalization time \( \tau_{\text{eq}} \).

The dilution of the spatial eccentricity by collisionless radial free-streaming is easily estimated [35, 42], using the analytic solution of the collisionless Boltzmann equation for the distribution function \( f(r, p_T, \tau) \) of initially produced approximately massless partons (we only consider their transverse motion):

\[
f(r, p_T, \tau + \Delta \tau) = f(r - c\Delta\tau e_p, p_T, \tau) .
\]

Here \( e_p \) is a unit vector in direction of \( p_T \). With Eq. (52) it is straightforward to compute the time-dependence of the spatial eccentricity:

\[
\varepsilon_x(\tau_0 + \Delta \tau) = \left[ \frac{dx dy (y^2 - x^2)}{\int dx dy (y^2 + x^2)} \right] \int d^2 p_T f(r - c\Delta \tau e_p, p_T, \tau_0) \]

\[
= \left[ \frac{dx dy \int d\varphi_p [(y + c\Delta \tau \sin \varphi_p)^2 - (x + c\Delta \tau \cos \varphi_p)^2]}{\int dx dy \int d\varphi_p [(y + c\Delta \tau \sin \varphi_p)^2 + (x + c\Delta \tau \cos \varphi_p)^2]} \right] f(r, p_T, \tau_0).
\]

The initial distribution at \( \tau_0 \) is even in \( x \) and \( y \), and the initial transverse momentum distribution can be assumed to be locally isotropic. From this it follows directly that

\[
\frac{\varepsilon_x(\tau_0 + \Delta \tau)}{\varepsilon_x(\tau_0)} = \left[ 1 + \frac{(c\Delta \tau)^2}{\langle r^2 \rangle_{\tau_0}} \right]^{-1},
\]

where \( \langle r^2 \rangle_{\tau_0} \) is the azimuthally averaged initial transverse radius squared of the reaction zone. Inserting typical values for, say, Au+Au collisions at \( b = 7 \) fm one finds that a delay of thermalization by \( \Delta t = 2.5 \) fm/c (3.5 fm/c) leads to a decrease of the spatial eccentricity by 30% (50%), without generating any momentum anisotropy. The elliptic flow signal resulting from subsequent hydrodynamic expansion would then be degraded by a similar percentage. If we assume, for the sake of the argument, that the RHIC data exhaust at least 2/3 of the ideal fluid limit calculated with the full initial eccentricity \( \varepsilon_x(\tau_0) \), the thermalization time \( \tau_{\text{eq}} \) can therefore not be larger than about 2.5 fm/c.

6.3. Signs of viscosity

6.3.1. Spectra and elliptic flow at midrapidity

Figure 22 shows a comparison of experimental transverse momentum spectra and elliptic flow measurements for pions and protons from 200 AGeV Au+Au collisions at RHIC with a compilation of
theoretical predictions based on ideal fluid dynamics [5]. All theoretical curves treat the QGP stage of the expansion as an ideal fluid in thermal and chemical equilibrium (albeit with varying equations of state). The differences between the predictions arise mostly from how they deal with the hadronic phase. Except for the solid red lines, all calculations use sudden Cooper-Frye freeze-out at some decoupling energy density of order $e_{\text{dec}} \approx 0.075 \text{GeV/fm}^3$. For them the fireball matter is assumed to expand as an ideal fluid until it reaches the decoupling point. The solid red lines represent a calculation [40] that couples ideal fluid dynamics for the QGP above $T_c$ to an RQMD hadron cascade below $T_c$. It allows freeze-out to happen gradually, by following the hadronic scattering processes microscopically.

A careful study of the figure shows that none of the purely hydrodynamic simulations yields a good description of all the experimental data. Calculations that do not allow for a phase transition to quark-gluon plasma and treat the matter as a chemically equilibrated hadron gas reproduce the pion elliptic flow but miss that of the protons as well as the shape of either the pion or proton spectrum. Using an EOS featuring a quark-hadron phase transition allows to reproduce the elliptic flow of both pions and protons, but only if the HRG phase is assumed to be in chemical equilibrium all the way down to $e_{\text{dec}}$. As discussed before, this assumption yields an incorrect $p/\pi$ ratio; correspondingly the
relative normalization between pion and proton spectra comes out wrong. The curves labelled “PCE” correct this deficiency by implementing non-equilibrium chemical potentials in the HRG phase that ensure the correct chemical composition in the final state, as measured in experiment. Now the pion and proton spectra are (roughly) correctly normalized, but they are too steep, because with the PCE EOS freeze-out happens at a lower temperature (see Fig. 5) while the radial flow remains the same. Even worse, the $p_T$-dependent pion elliptic flow now has a slope that is about 30% larger than in experiment. The reason is two-fold: (i) To absorb the given total momentum anisotropy that was generated during the early hydrodynamic evolution, steeper single particle spectra require a faster rise of $v_2(p_T)$ with $p_T$. This is seen to affect both pions and protons (blue curves in the upper row of Fig. 22), (ii) In the PCE case pions constitute a smaller fraction of the total because baryon-antibaryon pairs are prohibited from annihilating. Each pion must then carry a larger share of the total hydrodynamic momentum anisotropy. This explains the stronger effect on pions (upper left panel) than protons (upper right).

The only curve that gives a reasonable description of all data simultaneously is Teaney’s hydro+RQMD hybrid calculation [40]. By switching from fluid dynamics to a hadron cascade at $T_c$, it suppresses chemical reactions (which are slow in RQMD) and thus correctly reproduces the observed chemical freeze-out at $T_{chem} \approx T_c$. By allowing the hadrons to rescatter quasi-elastically, it generates additional radial flow below $T_c$ that is not too different from the one generated by hydrodynamics. But viscous effects in the RQMD cascade further flatten the $p_T$ spectra [177], compensating for the lower temperature in the chemically non-equilibrated hadronic environment and bringing the spectra in line with experiment. At the same time they suppress the buildup of elliptic flow in the hadronic stage [57], thereby reducing the slope of $v_2(p_T)$ from the PCE hydrodynamic calculations and bringing it also back in line with the data.

Even though it took several years to fully understand these mechanisms, Teaney’s work [40] was the first to exhibit the important viscous effects inherent in the non-equilibrium hadronic cascade dynamics during the late hadronic stage and freeze-out. Figure 22 shows that hadronic viscosity plays a key role for both transverse momentum spectra and elliptic flow, but its most dramatic effect is the reduction of elliptic flow that it causes.

6.3.2. Centrality and rapidity dependence of elliptic flow

Effects of hadronic viscosity become much more prominent in peripheral collisions and at forward rapidities. As one selects larger impact parameters or moves away from midrapidity, the charged multiplicity density per unit overlap area, $(1/S)(dN_{ch}/dy)$, decreases, corresponding to a decrease of the initial entropy density $s(x_{\perp}, \tau_0)$ [158]. Correspondingly the system reaches the phase transition sooner and spends a larger fraction of its evolution in the viscous hadronic phase.

Figure 23 shows the effect this has on the centrality and rapidity dependence of elliptic flow [57]. The solid lines show hydrodynamic calculations that treat the hadronic phase as a (chemically non-equilibrated) ideal fluid. They overpredict the elliptic flow in peripheral collisions and at forward rapidities. The hydrodynamic elliptic flow generated during the QGP stage (dotted green line in the right panel of Fig. 23) is not enough to explain the data, so some hadronic contribution to $v_2$ is required. But an ideal fluid overdoes it – it produces too much elliptic flow. Replacing hydrodynamics by a viscous hadron cascade below $T_c$ gets the elliptic flow just right – at least for Glauber model initial conditions (see Sec. 3.1.1) which were used in the right panel of Fig. 23 and for the blue (“BGK”) lines in the left panel. For such initial conditions, the assumption of an ideal (zero viscosity) QGP fluid followed by a viscous microscopic hadron cascade works beautifully, at all collision
centralities and all rapidities – in straightforward generalization of the observations made in Fig. 22.

Unfortunately, this is not the happy end of the story. The thin and thick red solid lines in the left panel of Fig. 23 show that this conclusion becomes untenable once one allows for alternate models of the initial state. As shown in the right panel of Fig. 2, the CGC model produces up to 50% larger initial source eccentricities than the Glauber model, which hydrodynamics transforms into correspondingly larger elliptic flow coefficients – see the thin solid line in the left panel of Fig. 23. Hadronic viscosity reduces this, but not enough to agree with the data (thick solid line in the same panel). The measured elliptic flow in peripheral collisions is still overpredicted by about 50%.

Does this imply a large shear viscosity during the early QGP stage? The answer is “No!”. While 50% looks like a large effect, we will see in Sec. 6.3.4 that even a small amount of QGP viscosity can cause $v_2$ to decrease by 30% and thus eliminate the discrepancy. The inconvenient truth is, however, that a 50% uncertainty in the initial source eccentricity translates in some sense into an infinite uncertainty about the specific shear viscosity $\eta/s$: The difference between zero QGP viscosity (seemingly compatible with the data for Glauber initial conditions) and even a small nonzero QGP viscosity (required in the case of CGC initial conditions) cannot be reasonably expressed in percent.

6.3.3. Multiplicity scaling of elliptic flow

Before pursuing the issue of quantifying the QGP viscosity further, let us discuss one more piece of qualitative evidence for the increasing importance of viscous effects as the energy density and temperature of QCD matter decrease.

The left panel of Fig. 24 shows an empirical systematics that has become known as “multiplicity scaling of elliptic flow” [191, 221]. The horizontal axis is the charged hadron multiplicity density per unit rapidity and overlap area, which is proportional to the initial entropy density [158]. The vertical axis shows the $p_T$-integrated elliptic flow normalized by the initial eccentricity. The scale invariance of the ideal fluid dynamic equations implies that this ratio depends only on the squared speed of sound $c_s^2$ (see Eq. (14)) if the elliptic flow is allowed to fully develop and the fireball doesn’t freeze out before. This is reflected in the solid and dashed “HYDRO” lines indicated in the left panel. Except
Fig. 24. Multiplicity scaling of elliptic flow. Shown is the charged hadron elliptic flow normalized by the initial source eccentricity as a function of charged hadron multiplicity per unit rapidity and nuclear overlap area at midrapidity. The left panel shows experimental data from AGS, SPS and RHIC [191, 221] together with lines indicating the results from ideal fluid dynamical calculations with EOS Q and a pure hadron gas equation of state without phase transition (EOS H). The right panel shows results from (2+1)-d viscous hydrodynamics with EOS L (see Fig. 5), for three fixed values for the specific shear viscosity $\eta/s$ as indicated [25]. See text for discussion.

for the steep drop on the left side which is due to premature freeze-out when the initial entropy density is not large enough to let the elliptic flow develop to saturation, these curves vary only because the effective stiffness of the EOS probed during expansion depends on the initial entropy density and temperature [35]. The dip in the HYDRO curve corresponding to EOS Q arises from the dip in the speed of sound near the quark hadron phase transition (see Fig. 5). The left diagram shows that the experimental data approach the ideal fluid dynamic limit at high multiplicity, but stay well below that limit at low multiplicity. They show an almost linear scaling with the charged multiplicity density which differs dramatically from the ideal fluid prediction. In particular, the data show no sign of any structure related to the dip in the speed of sound near the phase transition. The agreement of the experimental data with ideal fluid dynamics at high multiplicities relies on the fact that the measured elliptic flow $v_2$ has been scaled with initial eccentricities calculated from the Glauber model. If the larger eccentricities predicted by the CGC model had been used, the experimental data would stay significantly below the ideal fluid prediction even at the highest multiplicities.

The right panel shows predictions for the eccentricity-scaled elliptic flow from viscous hydrodynamics [25]. The calculations where done with constant specific shear viscosities $\eta/s$, ranging from the minimal value $\eta/s = 1/4\pi = 0.08$ suggested by the KSS bound [215] to three times that value, as indicated. For each of these values, one observes approximate “multiplicity scaling”, just as in the data: to first approximation, all dependence of $v_2/\epsilon$ on system size, collision energy, and impact parameter is through the multiplicity density $(1/S) (dN_{ch}/dy)$ associated with these parameters.

One sees that inclusion of viscous effects brings the theoretical predictions closer to the data on the left than ideal fluid dynamics. But it is also obvious that with a constant ratio $\eta/s$ agreement with experiment cannot be achieved. The data require larger $\eta/s$ values at low multiplicity densities.
(low initial entropy densities) and smaller ratios at higher multiplicity densities. This indicates less specific shear viscosity in the hot QGP phase than at lower temperatures, especially at temperatures below $T_c$. With Glauber eccentricities, as used in both panels of Fig. 21 “minimal” shear viscosity $\eta/s = 0.08$ seems to work fine at the highest multiplicities. However, as explained before, this would be quite different if the CGC model eccentricities were true: They would lower $v_2/\epsilon$ by about 30%, allowing for $\eta/s$ values of up to 3 times the minimal value near the right end of the graph.

The viscous hydrodynamic calculations in the right panel of Fig. 24 were done with EOS Q which assumes chemical equilibrium in the HRG phase. All the earlier caveats about hydrodynamic simulations of elliptic flow that do not correctly account for the non-equilibrium chemical composition in the hadron phase therefore apply. Before this is corrected, the numbers quoted above should not be taken too seriously. They do, however, give a feeling for the rough approximate size of the QGP shear viscosity to be expected from future quantitative comparisons with experiment. Clearly, we are not talking about shear viscosities that exceed the KSS bound by a factor 10 or more. Indeed, it would be surprising if $(\eta/s)_{QGP}$ turned out to be larger than about 3-5 times the KSS value once all physical effects are properly included.

6.3.4. Towards extracting the QGP viscosity

We close this review with a discussion of a recent attempt to estimate the QGP shear viscosity from midrapidity elliptic flow data in 200 $A$ GeV Au+Au collisions, as a function of centrality and $p_T$ [68]. Figure 25 shows (2+1)-d viscous hydrodynamic calculations of charged hadron elliptic flow with Glauber model (left) and CGC (right) initial conditions, for several constant values of $\eta/s$ as indicated. The simulations used a lattice QCD equation of state above $T_c$ matched to a chemically equilibrated hadron gas below $T_c$. For $v_2(p_T)$ two sets of data are shown: the originally published data from the STAR Collaboration [210] and a set of points were all values were reduced by 20% to approximately account for “non-flow” contributions in the data [210].

Depending on the assumed initial eccentricities and which set of data one prefers, the comparison indicates a preferred range of $0 < \eta/s < 0.2$. In particular, for Glauber eccentricities and vanishing non-flow contributions in the data, there seems to be no room left for any non-zero shear viscosity at all (even though the simulation treats even the hadronic phase as an almost ideal fluid, which is known [40, 57] to be incorrect)! This is presumably an artifact of the incorrect chemistry of the hadronic phase assumed here. But even after accounting for this, the window for QGP shear viscosity is not large. Even for the larger CGC eccentricities, $\eta/s$ values larger than about three times the KSS bound quickly become incompatible with the experimental data. This conclusion gets stronger when one allows for additional effects from bulk viscosity which appear to further reduce the hydrodynamically predicted elliptic flow [72].

7. Epilogue

Clearly, this is only the beginning of the story of the QGP shear viscosity, and its ending must be told in a future review. But it is the story of trying to answer a question that 10 years ago we didn’t even know how to ask! As the last century came to a close, the heavy-ion community was focussed on discovering the quark-gluon plasma; now, as we are about to complete the first decade of the 21st century, we are in the middle of a process of quantitatively extracting its transport properties. In this endeavour, relativistic hydrodynamics of viscous fluids plays a key role. With the advent of RHIC, hydrodynamics has found a permanent place in the dynamical modelling of heavy-ion
collisions. For the first time in the history of high-energy physics, it has proven to be able to deliver quantitative explanations for experimental observations. Relativistic fluid dynamics is and will be the workhorse of all future efforts to describe the dynamics of heavy-ion collisions. For precise predictions, it must be carefully stitched together with a reliable dynamical theory of the very early pre-equilibrium stage, covering the first 1 fm/c or so, and a realistic hadronic rescattering cascade for the late hadronic scattering and freeze-out stage, covering the last few fm/c. During the 10-15 fm/c that lie in between these two points, hydrodynamics rules.

Acknowledgments

This work was supported by the U.S. Department of Energy under grant DE-FG02-01ER41190. Fruitful discussions with and valuable comments from K. Dusling, K. Eskola, P. Huovinen, P. Kolb, H. Niemi, S. Pratt, D. Rischke, P. Romatschke and H. Song, as well as the persistent encouragement by Reinhard Stock and the enduring patience of Christiane, Matthias and Michael (my wife and sons) are gratefully acknowledged.
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