Exploiting quantum parallelism of entanglement for a complete experimental quantum characterization of a single qubit device

Francesco De Martini, Andrea Mazzei, and Marco Ricci
Istituto Nazionale di Fisica della Materia, Dipartimento di Fisica, Università “La Sapienza”, Roma, 00185 Italy

Giacomo Mauro D’Ariano
Quantum Optics and Information Group, Istituto Nazionale di Fisica della Materia, Unità di Pavia and Dipartimento di Fisica "A. Volta", via Bassi 6, I-27100 Pavia, Italy

(Dated: October 28, 2018)

We present the first full experimental quantum tomographic characterization of a single-qubit device achieved with a single entangled input state. The entangled input state plays the role of all possible input states in quantum parallel on the tested device. The method can be trivially extended to any n-qubits device by just replicating the whole experimental setup n times.

PACS numbers: 03.65.-w, 03.67.-a, 42.50.-p

The new field of Quantum Information has recently opened the way to realize radically new processing devices, with the possibility of tremendous speedups of complex computational tasks, and of cryptographic communications guaranteed by the laws of physics. Among the many problems posed by the new information technology there is the need of making a complete experimental characterization of the functioning of the new quantum devices. As shown recently in Ref. [2], quantum mechanics provides us with the perfect tool to achieve the task easily and efficiently: this is the so called quantum entanglement, the basis of the quantum parallelism of future computers. In this letter we present the first full experimental quantum characterization of a single-qubit device, based on this method. Since the method can be easily extended to any n-qubit device, the present experiment represents a first test of the feasibility and of the experimental limits of the new general tomographic method.

How do we characterize the operation of a device? In quantum mechanics the evolution of the state is completely described by the so called quantum mechanics the evolution of the state is complete. The normalization constant Tr [E (|n⟩ ⟨n′|)] in Eq. (1) is also the probability of occurrence of the transformation E (e.g. when there are other possible alternatives, such as when we consider the state transformation due to a measuring device for a given outcome). Therefore, apart from a normalization factor, the quantum evolution is always linear, with the quantum operation playing the role of the so-called transfer matrix of the device, a mathematical tool very popular in optics and electrical engineering.

Now the problem is: how to reconstruct the form of E experimentally? Since E is essentially a transfer matrix for a linear system, one would be tempted to adopt the conventional method of running an orthogonal basis |n⟩ of inputs and measuring the corresponding outputs by quantum tomography. However, since states are actually operators—not vectors—as a consequence of the polarization identity in order to get all off-diagonal (complex) matrix elements of the state one actually needs to run not the basis itself, but all the linear combinations of its vectors 2 − 2 |n⟩ ⟨n′| + κ |n⟩ ⟨n′|), with κ = ±1, ±i. In the following we will call such sets of states faithful, since they are sufficient to make a complete tomography of a quantum operation. This method is essentially the quantum process tomography given in Ref. [1], which was experimentally demonstrated in NMR [3] and recently in quantum optics [4]. The main problem with such method is the fact that in most practical situations faithful sets of input states are not feasible in the lab. For example, for continuous-variables process tomography in the Fock representation, the states |n⟩ and |n⟩ would be photon number states, and achieving their superpositions will remain a dream for experimentalists for many years. But here the quantum parallelism of entanglement comes to help us, with a single input entangled state that is equivalent to running all possible input states in parallel [2]. Thus, we don’t need to prepare a complete set of states, but just a single entangled one, a state commonly available in any modern quantum optical laboratory.

In the following we will use the double-ket notation |Ψ⟩ ∈ ℋ ⊗ ℋ to denote bipartite states corresponding to the matrix Ψnn of coefficients on fixed given orthonormal
basis \(|n\rangle \otimes |m\rangle \equiv |nm\rangle\) of \(\mathcal{H} \otimes \mathcal{H}\)

\[
|\Psi\rangle = \sum_{nm} \Psi_{nm} |nm\rangle. \tag{2}
\]

In our optical implementation the entangled systems consist of two single-mode optical beams, and the Hilbert space is two-dimensional, since we will consider only single-photon polarization states. The key feature of the method implies that only one of the two systems is input into the unknown transformation \(E\), whereas the other is left untouched, as in Fig. 1. This setup leads to the output state \(R_{\text{out}}\), which in tensor notation writes as follows

\[
R_{\text{out}} = E \otimes I(|\Psi\rangle \langle \Psi|), \tag{3}
\]

where \(I\) denotes the identical operation. It is a result of linear algebra that \(R_{\text{out}}\) is in one-to-one correspondence with the quantum operation \(E\), as long as the state \(|\Psi\rangle\) is full-rank, i.e., its matrix \(\Psi\) is invertible. This is the case of a so-called maximally entangled state, where the matrix \(\Psi\) is proportional to a unitary one. Full-rank entangled states can be easily generated by spontaneous parametric down conversion of vacuum, as in the experiment reported here. Note that even when a faithful set of input states is available in the lab—which is actually true in our case of single-photon polarization states—nevertheless a single faithful entangled state can be much more efficient and more practical. As a matter of fact, in practice, generation of single-photon polarization states relays anyway on entanglement, and, as we will see in the following, the present method uses all experimental data much more efficiently than the conventional quantum process tomography.

\[
\begin{aligned}
|\Psi\rangle &\xrightarrow{E} Q(k) \quad \xrightarrow{Q(l)} \text{COMPUTER}
\end{aligned}
\]

FIG. 1: General experimental scheme of the method for the tomographic estimation of the quantum operation \(E\) of a single qubit device. Two identical quantum systems—two optical beams in the present experiment—are prepared in an entangled state \(|\Psi\rangle\). One of the two systems undergoes the quantum operation \(E\), whereas the other is left untouched. At the output one makes a quantum tomographic estimation, by measuring jointly two observables (each for each beam) from a quorum \(|Q(l)\rangle\). In the present experiment the quorum is represented by the set of Pauli operators \(\sigma_x, \sigma_y\) and \(\sigma_z\).

Now, how to characterize the entangled state \(R_{\text{out}}\) at the output? For this purpose a technique for the full determination of the quantum state has been introduced and developed since 1994. The method named Quantum Tomography has been initially introduced for the state of a single-mode of radiation, the so called Homodyne Tomography, and thereafter it has been generalized to any quantum system. The basis of the method is the measurement of a suitably complete set of observables called quorum. In our case we need to measure jointly two quora of observables on the entangled qubit, here the quorum being the Pauli matrices \(\sigma_x, \sigma_y, \sigma_z\). The qubit is encoded on polarization of single photons, as follows

\[
|0\rangle \equiv h^\dagger |\Omega\rangle, \quad |1\rangle \equiv v^\dagger |\Omega\rangle, \tag{4}
\]

where \(|\Omega\rangle\) denotes the e.m. vacuum, and \(h, h^\dagger\) and \(v, v^\dagger\) the annihilation and creation operators of the horizontally and vertically polarized modes associated to a fixed wave-vector \(k\), respectively. In synthesis, Eq. (4) means that the "logical zero" is encoded on a single horizontally polarized photon, whereas the "logical one" is encoded on a vertically polarized photon. In the present representation, the Pauli matrices write as follows

\[
\begin{align*}
\sigma_x &= h^\dagger v + v^\dagger h, \\
\sigma_y &= i (h^\dagger v - v^\dagger h), \\
\sigma_z &= h^\dagger h - v^\dagger v. \tag{5}
\end{align*}
\]

According to Eq. (5), the \(\sigma_x\)-photo-detector is simply achieved as in Fig. 2. In order to understand how to design detectors for \(\sigma_y\) and \(\sigma_z\) we still need some simple algebra for wave-plates. The ring of Pauli matrices is completed by including the "identity" \(\sigma_0 = h^\dagger h + v^\dagger v\), corresponding to single-photon states. In the following we use the popular relativistic conventions, denoting by \(\vec{\sigma}\) the three-vector of operators \(\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\) and by \(\sigma\) the tetra-vector \(\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)\), and use Greek indices for three-vectors components \(\alpha = 1, 2, 3\) (or \(\alpha = x, y, z\)), and Latin indices for tetra-vector components \(i = 0, 1, 2, 3\).

A wave-plate changes the two radiation modes according to the matrix transformation

\[
\begin{bmatrix}
h \\
v
\end{bmatrix}
\rightarrow \mathcal{W}_{\phi, \theta}
\begin{bmatrix}
h \\
v
\end{bmatrix}
\]

where the matrix \(\mathcal{W}_{\phi, \theta}\) is given by

\[
\mathcal{W}_{\phi, \theta} = \begin{bmatrix}
z_+ + cz_- & sz_- \\
- sz_+ + cz_- & z_+ - c z_- \end{bmatrix} \tag{6}
\]
where $s=\sin 2\theta$, $c=\cos 2\theta$, $z_\pm = \frac{1}{\sqrt{2}}(1 \pm e^{i\phi})$, $\theta$ is the wave-plate orientation angle around the wave-vector $k$, $\phi = 2\pi\delta/\lambda$, $\lambda$ is the wave-length, and $\delta$ is the length of the optical path through the plate. Special cases are the $\frac{\lambda}{2}$ wave-plate which can be used with $\theta = \pi/4$ to give the right $c_+$ and left $c_-$ circularly polarized modes $c_\pm = e^{i\pi/4} \frac{1}{\sqrt{2}}(\pm h + i v)$, and the $\lambda$ wave-plate which can be used to give the diagonal linearly polarized modes $d_\pm = 2\frac{\lambda}{2}(h \pm v)$. On the Pauli operator vector $\sigma$ the mode transformation due to a wave-plate writes

$$\sigma \rightarrow R_{\phi, \theta} \sigma,$$

with rotation matrix

$$R_{\phi, \theta} = \begin{bmatrix} s^2 + c^2 \cos \phi & c \cos \phi & sc(1 - \cos \phi) \\ c \sin \phi & \cos \phi & -s \sin \phi \\ sc(1 - \cos \phi) & s \sin \phi & c^2 + s^2 \cos \phi \end{bmatrix}. $$

From Eq. (8) we can see that a $\sigma_-$-detector can be obtained by preceding the $\sigma_-$-detector with a $\frac{\lambda}{2}$ wave-plate oriented at $\theta = \pi/8$, whereas a $\sigma_y$-detector is obtained by preceding the $\sigma_z$-detector with a $\frac{\lambda}{2}$ wave-plate oriented at $\theta = \pi/4$. When collecting data at a $\sigma_\alpha$-detector, we will denote by $s_\alpha = \pm 1$ the corresponding random outcome, $s_\alpha = +1$ corresponding to the $h$-detector flashing, and $s_\alpha = -1$ to the $v$-detector flashing. The general experimental setup is then given by two Pauli detectors— for measuring $\sigma_\alpha$ and $\sigma_\beta$ for varying $\alpha$ and $\beta$—at the output of the entangled beams, as in Fig. 1. The experimental data are collected in coincidence, with two of the four photo-detectors firing, one for each Pauli detector, thus guaranteeing that the result will be essentially unaffected by quantum efficiency. The experimental correlations $s_i^{(1)} s_j^{(2)}$ of the random outcomes $s_i^{(n)}$ of the detector at the $n$-th beam ($n = 1, 2$) on the entangled state $|\Psi\rangle$ must coincide with the following theoretical expectations

$$s_i^{(1)} s_j^{(2)} = \langle \Psi | (\sigma_i^{(1)} \otimes \sigma_j^{(2)}) | \Psi \rangle = \text{Tr} \left( \Psi^+ \sigma_i \sigma_j^* \right),$$

and, obviously, $s_i^{(1)} = s_i^{(1)} s_0^{(2)}$ and $s_i^{(2)} = s_0^{(1)} s_i^{(2)}$. For maximally entangled states we have the isotropy condition $s_\alpha^{(1)} = s_\alpha^{(2)} = 0$ for $\alpha = x, y, z$. The four Bell states will correspond to the four Pauli matrices $\sigma_j$ via a state coefficients matrix $\Psi$ given by $\Psi = \sqrt{2} \sigma_j$. On the other hand, when a quantum device performing the unitary transformation $U$ is inserted in one of the two entangled beams as in Fig. 1, the entangled state $|\Psi\rangle$ is changed to $U \otimes |\Psi\rangle$, which corresponds to the new coefficients matrix $\Psi \rightarrow U \Psi$. In our lab we used the “triplet” state corresponding to $\Psi = \sigma_z/\sqrt{2}$, which is generated via spontaneous parametric downconversion by an optical parametric amplifier physically consisting of a nonlinear BBO (b-barium-borate) crystal plate, 2 mm thick, cut for Type II phase matching and excited by a pulsed mode-locked ultraviolet laser UV having pulse duration $\tau = 140$ fsec and wavelength $\lambda_p = 397.5$ nm. The wavelength of the emitted photons is $\lambda = 795$nm. The measurement apparatus consisted of two equal polarizing beam splitters with output modes coupled to four equal Si-avalanche photodetectors SPCM-AQR14 with quantum efficiencies $\text{QE} \simeq 0.42$. The beams exciting the detectors are filtered by equal interference filters within a bandwidth $\Delta \lambda = 6$nm. The detector outputs are finally analyzed by a computer.

With the above apparatus we want now to experimentally determine the matrix elements of the state $|\Psi\rangle$ in Eq. (3). From the trivial identity

$$\langle nm | \Psi \rangle = \Psi_{nm},$$

we obtain the matrix $\Psi_{nm}$ for the input states in terms of the following ensemble averages

$$\Psi_{nm} = e^{i \varphi} \frac{\langle \Psi | 01 \rangle \langle nm | \Psi \rangle}{\sqrt{\langle \Psi | 01 \rangle \langle 01 | \Psi \rangle}},$$

where $\exp(i \varphi) = \Psi_{01}/|\Psi_{01}|$ is an unmeasurable phase factor. The choice of the vector $|01\rangle$ is arbitrary, and it is needed only for the sake of normalization, e. g. we could have used $|10\rangle$ or $|11\rangle$, instead. Using the tomographic expansion over the four Pauli matrices $\Psi$ we see that, via Eq. (12), the matrix element of the input state is obtained from the following experimental averages

$$\Psi_{nm} = \frac{1}{4 \sqrt{p}} \sum_{ij} Q_{nm}^{ij} s_i^{(1)} s_j^{(2)},$$

where

$$p = \langle \Psi | 01 \rangle \langle 01 | \Psi \rangle = \frac{1}{4} (1 + s_3^{(1)})(1 + s_3^{(2)}),$$

is the fraction of events with one $\sigma_-$-detector firing on $h$ and the other on $v$, and the matrix $Q_{nm}^{ij}$ is given by

$$Q_{nm}^{ij} = \langle n | \sigma_i | 0 \rangle \langle m | \sigma_j | 1 \rangle.$$

The unitary matrix $U_{nm}$ of the device is now obtained with the same averaging as above, but now for the state at the output of the device $|U \Psi \rangle = (U \otimes 1 | \Psi \rangle$, namely

$$\langle U \Psi | nm \rangle = e^{i \varphi} \frac{\langle \Psi | U \Psi \rangle \langle nm | U \Psi \rangle}{\sqrt{\langle U \Psi | U \Psi \rangle \langle 01 | U \Psi \rangle \langle U \Psi | 01 \rangle}},$$

where we use again Eqs. (14) and (15), but now the average expressed by Eq. (12) is carried out over the output state $|U \Psi \rangle$. The (complex) matrix elements $U_{nm}$ are obtained from Eq. (16) by matrix inversion. This is of course possible since the matrix $\Psi$ is invertible, due to the maximally entangled character of $|\Psi\rangle$.

An experimental demonstration of the tomographic method is given in Figures 3 and 4, where both the real and imaginary parts of the four measured matrix elements of the unitary operator $U$ of the analyzed device are reported for two different devices, and compared with
FIG. 3: Experimental characterization of a single optical wave-plate with retardation phase $\phi = 0.45\pi$ and orientation angle of the optical axis respect to the laboratory horizontal direction $\theta = -0.138\pi$. The experimental matrix elements $U_{nm}$ of the wave-plate are superimposed to statistical errors for 8000 events, and compared with the theoretical values.

FIG. 4: The same experimental characterization as in Fig. 3, here for a device made of a series of two optical wave-plates: a wave-plate with $\phi = 0.45\pi$ and $\theta = -0.138\pi$ followed by another wave-plate with $\phi = \pi$ and $\theta = +0.29\pi$.

The theoretical values. As one can see, the experimental results are in very good agreement with theory, within experimental errors. As a first experimental demonstration we have considered only unitary devices, however, it is clear that the method works for non unitary devices as well. It is also obvious that the method can be used to characterize $n$-qubit devices—e.g., a controlled-\textsc{not} gate—in which case we just need multiply by $n$ the whole setup, by providing an input entangled state and two Pauli detectors for each qubit of the device, with the full quantum characterization of the the device obtained by a joint tomography on all output entangled pairs. It is clear that the precision of the method will not depend on the particular tested device—whether it is unitary or not—and will also be independent on the number of qubits. What makes the method particularly reliable in the present single-photon polarization encoding is the fact that all measurements are performed in coincidence, making the effect of nonunit quantum efficiency of detectors negligible, and effectively purifying the input entangled state. In a different context—e.g., for continuous variables, such as homodyne tomography of twin-beams [7]—quantum efficiency and entanglement purity will actually affect the final result: however, the quantum tomographic reconstruction can handle all these kinds of detection noises below some thresholds [4], and a mixed input state in place of $|\Psi\rangle$ work well (but less efficiently) as long as the state is \textit{faithful}, namely it is related to a maximally entangled one by an invertible map $\mathcal{E}$. Unfortunately, for the twin-beam homodyne tomography [7], faithfulness requires the knowledge of the phase of the pump relative to the local oscillator—a feasible but difficult experimental task—whereas in the present experiment the form of the entangled input state is completely under control, being determined only by the orientation of the nonlinear crystal with respect to the pump wave-vector and polarization.

In conclusion, we have given the first demonstration of a new tomographic method which allows us to perform a complete characterization of any quantum device, exploiting the intrinsic parallelism of quantum entanglement, with a single entangled state playing the role of all possible input states in quantum parallel. The method works for any generally non unitary multi-qubit device, and is particularly reliable in the present context of single-photon polarization encoding of the qubit.

This work has been supported by the FET European Network on \textsc{qic} (Contract IST-2000-29681-ATESIT) and the INFM PRA 2001 CLON.

[1] I. L. Chuang and M. A. Nielsen, \textit{Quantum Information and Quantum Computation}, (Cambridge Univ. Press, Cambridge, 2000).
[2] G. M. D’Ariano and P. Lo Presti, Phys. Rev. Lett. \textbf{86}, 4195 (2001).
[3] K. Kraus, \textit{States, Effects, and Operations} (Springer-Verlag, Berlin, 1983).
[4] G. M. D’Ariano, \textit{Quantum Tomography}, Proceedings of the...
International School "E.Fermi", Varenna: *Experimental Quantum Computation and Information* (F. De Martini & C. Monroe eds. Editrice Compositori, Bologna, 2002).

[5] A. M. Childs, I. L. Chuang, and D. W. Leung, Phys. Rev. Lett. 64 012314 (2001)

[6] P. Kwiat, in Quantum Communication, Measurements and Computing (Cambridge QCMC’02), ed. by J. Shapiro and O. Hirota, Rinton Press (Paramus NJ 2002) in press

[7] M. Vasilyev, S.-K. Choi, P. Kumar, and G. M. D’Ariano, Phys. Rev. Lett. 84 2354 (2000)

[8] G. M. D’Ariano and P. Lo Presti, unpublished.
