HOW TO RENORMALIZE A QUANTUM GAUGE FIELD THEORY
WITH CHIRAL FERMIONS

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Abstract

We propose using the method of subtraction to renormalize quantum gauge theories with chiral fermions and with spontaneous symmetry breaking. The Ward-Takahashi identities derived from the BRST invariance in these theories are complex and rich in content. We demonstrate how to use these identities to determine relationships among renormalization constants of the theory and obtain the subtraction constants needed for the renormalization procedure. We have found it particularly convenient to adopt the Landau gauge throughout the scheme. The method of renormalization by subtraction enables one to calculate physical quantities in the theory in the form of a renormalized perturbation series which is unique and definite. There is no ambiguity in handling the $\gamma_5$ matrix associated with chiral fermions.

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1. Introduction

We present a scheme to renormalize quantum gauge field theories with chiral fermions and with spontaneous symmetry breaking. This scheme is based on the method of subtraction with the aid of the Ward-Takahashi identities. The use of the Landau gauge is particularly helpful.

As we all know, a distinctive feature of quantum field theories with spontaneous symmetry breaking is that the number of renormalized parameters invariably exceeds that of the bare parameters. As a general rule, a quantum field theory with an excessive number of renormalized parameters is likely to be not renormalizable. Take, for example, the theory of scalar QED. It is well-known that, if we follow spinor QED to the letter and introduce only two bare parameters, the bare charge and the bare mass of the scalar meson, the corresponding quantum field theory is not renormalizable. This is because there are three parameters which must be renormalized, the third one being the $|\phi^4|$ coupling constant. In order to make scalar electrodynamics renormalizable, one additional parameter, the unrenormalized $|\phi^4|$ coupling constant, must be introduced. It is notable that the Abelian-Higgs field theory is an exception to the general rule. In this theory, there are four bare constants in the boson sector: the bare coupling of the gauge meson, the bare Yukawa coupling constant of the fermion, and the two bare parameters in the Higgs potential. (There is also a parameter $\theta$ for the mixing of the left-handed and right-handed components of the fermion. This parameter is finite and requires no renormalization.) These four bare constants generate more than four renormalized parameters: the physical masses of the fermion, the gauge meson and the Higgs meson, the renormalized coupling constants of the gauge meson, the renormalized Yukawa coupling constants, and various renormalized 3-point and 4-point coupling constants of the scalar meson. While the number of renormalized parameters exceeds that of the bare parameters, the Abelian-Higgs theory is renormalizable. An important reason for this is that the Ward-Takahashi identities put a constraint among some of these parameters.

The contents of the Ward-Takahashi identities in a gauge field theory with spontaneous symmetry breaking are complicated and require some care to disentangle. In order to sidestep these complications, the method of dimensional regularization has been invented[1,2]. With
the use of this method, the renormalized perturbation series automatically satisfies the Ward-Takahashi identities, and one may be led to believe that there is no more need to pay a great deal of attention to the identities which have already been incorporated.

We consider such a belief misplaced. While the method of dimensional regularization is convenient to use in explicit calculations, there are three shortcomings associated with it. First, the definition of the matrix $\gamma_5$ for a non-integral dimension is subjective and controversial up to now[3]. Therefore, the renormalization theory on the basis of dimensional regularization alone remains incomplete. Second, without explicitly exploring the consequences of the Ward-Takahashi identities, the question of excessive renormalized parameters remains unanswered. Third, and perhaps most important, the Ward-Takahashi identities contain rich and physical implications which should be explored. Thus, ignoring the Ward-Takahashi identities under the auspices of dimensional regularization is not an unmitigated blessing. Indeed, it is not unlike throwing away the water with the baby in it.

We believe that the Ward-Takahashi identities are the centerpieces of renormalization. Their contents should be extracted and utilized. The method of dimensional regularization, on the other hand, should be recognized as it is: a mathematical artifice which is helpful to use in some cases—no more and no less.

We shall demonstrate our method of renormalization by the specific example of the Abelian-Higgs model. The Lagrangian density in the Abelian-Higgs model is given by

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^+(D^\mu \phi) + \bar{\psi}_L (i\partial - g_0 (1 + \theta) Y) \psi_L + \bar{\psi}_R (i\partial - g_0 \theta Y) \psi_R - \sqrt{2} f_0 \bar{\phi} \psi_L - \sqrt{2} f_0 \bar{\phi}^+ \psi_R - \lambda_0 (\phi^+ \phi)^2,$$

with

$$D_\mu \phi \equiv (\partial_\mu + ig_0 V_\mu) \phi,$$

$$\psi_L \equiv \frac{1}{2} (1 + \gamma_5) \psi,$$

$$\psi_R \equiv \frac{1}{2} (1 - \gamma_5) \psi,$$

and

$$F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu.$$
In the above, \( V_\mu, \phi, \) and \( \psi \) are the gauge field, the complex scalar field, and the fermion field, respectively, and \( \theta, g_0, f_0, \mu_0 \) and \( \lambda_0 \) are constants. The subscript of the constants in (1) signifies that these constants are bare. As usual, we shall put

\[
\phi \equiv \frac{v_0 + H + i\phi_2}{\sqrt{2}}
\]

where \( v_0 \equiv \mu_0/\sqrt{\lambda_0} \). We shall also call the bare mass of the gauge meson as \( M_0 \equiv g_0v_0 \), and the bare mass of the fermion as \( m_0 \equiv f_0v_0 \).

We shall add to the Lagrangian (1) a gauge-fixing term and a ghost term. Thus, we consider the effective Lagrangian

\[
L_{\text{eff}} \equiv L - \frac{1}{2\alpha} \ell^2 - i(\partial_\mu \eta)(\partial^\mu \xi) + i\alpha M_0^2 \eta \xi + i\alpha g_0 M_0 \eta \xi H,
\]

where

\[
\ell = \partial_\mu V^\mu - \alpha M_0 \phi_2,
\]

\( \alpha \) is a constant, and \( \xi \) and \( \eta \) are ghost fields.

We shall first discuss the renormalization of the propagator of the chiral fermion. By Lorentz covariance, the 1PI amplitude of this propagator is of the form

\[
\delta m + ma(p^2) + m\gamma_5 b(p^2) + c(p^2)\not{p} + d(p^2)\not{p}\gamma_5
\]

where \( \delta m \equiv m - m_0 \). The unrenormalized propagator \( S(p) \) is therefore given by

\[
S(p) = i \left[ \not{p}(1 - c) - m(1 + a) - (bm + d\not{p})\gamma_5 \right]^{-1}.
\]

The invariant amplitudes \( b \) and \( d \) are absent in QED, hence there are, in the Abelian-Higgs theory with chiral fermions, two more invariant functions which must be rendered finite by renormalization.

Fortunately the Abelian-Higgs theory remains renormalizable in spite of this, as both of these amplitudes are finite. To see this, we note that \( S(p) \) by assumption has a pole at \( p_0 \equiv \sqrt{p^2 + m^2} \). The pole of \( S(p) \) at \( p_0 = E \) comes from the integration of \( <0|T\psi(x)\bar{\psi}(y)|0> \) over very large values of time, as the finite range of \( (x_0 - y_0) \) cannot contribute a value of infinity. By the adiabatic hypothesis, \( \psi(x) (\bar{\psi}(y)) \) turns into the out-field (in-field) as
$x_0 \rightarrow \infty (y_0 \rightarrow -\infty)$. Since the in-field and the out-field are free bispinor fields of mass $m$, it is straightforward to derive
\[ c(m^2) = -a(m^2), \quad (5a) \]
and
\[ b(m^2) = d(m^2) = 0, \quad (5b) \]
which are the subtraction conditions needed for the divergent amplitudes $a, b, c$ and $d$.

From (4), we find that, when $p^2 \approx m^2$ and $q$ is set to $m$,
\[ S(p) \approx \frac{2mi}{p^2 - m^2} Z_\psi(m) \quad (6) \]
where
\[ Z_\psi(m) \equiv \frac{1}{1 - c(m^2) - 2m^2 [c(m^2) + a(m^2)]}. \quad (7) \]
In deriving (6) and (7), we have utilized (5). We shall define the renormalized fermion propagator $S^{(r)}(p) \equiv S(p)/Z_\psi(m)$.

In a perturbative calculation, the 1PI amplitude for the fermion propagator is linearly divergent. As we know, a shift of the momentum variable of a linearly divergent integral gives birth to a finite term. Therefore, we may interpret a Feynman integral of the 1PI amplitude of the fermion propagator as one of symmetric integration, with an unknown additive constant arising from an undetermined amount of shift of the momentum variables. The symmetrically integrated amplitude is logarithmically divergent.

There are two observations: (i) Because the amplitudes $a$ and $b$ in (3) are multiplied by a factor of $m$, by power counting these amplitudes are not linearly divergent. Therefore, they do not contain unknown additive constants, which appear only in the amplitudes $c$ and $d$, (ii) There are no counter terms in the Lagrangian for the amplitudes $d$ and $b$. Since the amplitude $b$ does not contain an unknown additive constant, it must be finite and must satisfy (5b) on its own. The amplitude $d$, on the other hand, is allowed an additive constant contributed by linearly divergent integrals. This constant is determined by (5b).

The renormalized perturbation series for the renormalized fermion self-energy 1PI amplitude can therefore be obtained as follows. For a graph which has no divergent subgraphs, we employ the Feynman rules to obtain perturbatively the amplitudes $a, b, c$, and $d$ with
the coupling constants and masses being the renormalized ones. The divergent integrals are symmetrically integrated after Feynman parameters are introduced. The amplitude \( ma(p^2) + c(p^2) \hat{p} \) remains to be logarithmically divergent, and is replaced by the subtracted amplitude

\[
m[a(p^2) - a(m^2)] + [c(p^2) - c(m^2)] \hat{p} - 2m^2[a'(m^2) + c'(m^2)](\hat{p} - m). \tag{8}
\]

These subtractions are the same as the ones in QED. They are so prescribed that \( S(r)(p) \approx 2m_i p^2 - m_2 \) near the mass shell with \( \hat{p} \) set to \( m \). The amplitude \( b(p^2) \) requires no subtraction, while the amplitude \( d(p^2) \) is replaced by the subtracted amplitude \( d(p^2) - d(m^2) \), so that (5b) is satisfied. For graphs with divergent subgraphs, we use the BPHZ formalism[4]. We shall express the renormalized 1PI amplitude for the fermion propagator as

\[
ma_r(p^2) + m\gamma_5 b_r(p^2) + c_r(p^2)\hat{p} + d_r(p^2)\gamma_5. \tag{9}
\]

Next we turn to the Ward-Takahashi identity for three-point functions involving fermions. We shall adopt the Landau gauge and take the limit \( \alpha \to 0 \). In this gauge, we have

\[
-i\Delta_\mu \Gamma_{\psi\bar{\psi}}^\mu(p', p) - m_0 Z \Gamma_{\phi_2\psi\bar{\psi}}(p', p) = S^{-1}(p') \frac{1 + 2\theta + \gamma_5}{2} - \frac{1 + 2\theta - \gamma_5}{2} S^{-1}(p). \tag{10}
\]

In (10), \( p' \) and \( p \) are the outgoing and incoming momenta for the fermion, respectively, and \( \Delta = p' - p \). Also,

\[
Z \equiv \frac{1}{v_0} < 0| (v_0 + H)|0 > .
\]

The functions \( \Gamma_{\psi\bar{\psi}}^\mu, \Gamma_{\phi_2\psi\bar{\psi}} \) and \( S(p) \) are so defined that their lowest-order terms in the unrenormalized perturbation series are \( \gamma^\mu \left[(1 + \theta) \frac{1 + \gamma_5}{2} + \theta \frac{1 - \gamma_5}{2}\right], -i\gamma_5 \) and \( \frac{i}{\hat{p} - m} \), respectively. If we set \( \Delta = 0 \) in (10), we get

\[
m_0 Z \Gamma_{\phi_2\psi\bar{\psi}}(p, p) = -\frac{1}{2} [\gamma_5 S^{-1}(p) + S^{-1}(p) \gamma_5]. \tag{11}
\]

By Lorentz covariance, \( \Gamma_{\phi_2\psi\bar{\psi}} \) is a superposition of scalar amplitudes and pseudo-scalar amplitudes:

\[
\Gamma_{\phi_2\psi\bar{\psi}}(p', p) \equiv \frac{1}{Z_{\phi_2\psi\bar{\psi}}(m^2, m^2, 0)} \left[-i\gamma_5 F_0 + F_1 + i\gamma_5 \not{p} F_2 + \not{p} F_3 + i\Delta(1 + \theta) \frac{1 + \gamma_5}{2} G_+ + i\Delta \theta \frac{1 - \gamma_5}{2} G_- + (H_+(1 + \theta) \frac{1 + \gamma_5}{2} + H_- \theta \frac{1 - \gamma_5}{2})(\Delta \not{p} - \not{p} \Delta) \right], \tag{12}
\]
where \( F_i, G_\pm, \) and \( H_\pm \) are invariant amplitudes which depend on \( p^2, p'^2 \) and \( \Delta^2 \), and \( P \equiv \frac{1}{2}(p' + p) \). Also, \( F_0(m^2, m^2, 0) = 1 \) by definition. By power counting, \( F_1 \) is logarithmically divergent, while \( F_2, F_3, G_\pm \) and \( H_\pm \) are ultraviolet finite. Substituting (12) into (11), we get

\[
\frac{m_0 Z}{Z_{\phi_2 \bar{\psi}\psi}(m^2, m^2, 0)} = m[1 + a(m^2)], \tag{13a}
\]

and

\[
F_i(m^2, m^2, 0) = 0, \quad i = 1, 2, 3. \tag{13b}
\]

Equation (13b) with \( i = 1 \) insures that the amplitude \( F_1 \) is actually ultraviolet convergent.

Let us define the renormalized coupling constant \( f \) by

\[
f \equiv f_0 \sqrt{\frac{Z_{\phi_2}(0)Z_{\psi}(m)}{Z_{\phi_2 \bar{\psi}\psi}(m^2, m^2, 0)}} = \frac{m}{v}[1 + a(m^2)]Z_{\psi}(m^2), \tag{14}
\]

where the renormalized vacuum expectation value \( v \) is given by\[^5\]

\[
v \equiv v_0 Z/\sqrt{Z_{\phi_2}(0)}.\]

Eq. (14) can be written as

\[
f = \frac{m}{v}[1 + a_r(m^2)]. \tag{15}
\]

Thus the values of \( m \) and \( f \) are related. Hence the renormalized Yukawa coupling constant \( f \) is finite as long as \( m \) is finite, and vice versa. Thus the Ward-Takahashi identities help make the theory renormalizable despite the excessive number of renormalized parameters.

Let us next study the Ward-Takahashi identity (10) in the limit where \( \Delta \) is infinitesimal but not zero. We shall keep track of terms in this identity which are linear in \( \Delta \). We remark that the point of subtraction for \( \Gamma_{\psi\bar{\psi}} \) requires some care. This is because that in the Landau gauge adopted in our formalism, \( \Gamma_{\psi\bar{\psi}} \) is infrared divergent if we set both \( p^2 \) and \( p'^2 \) to \( m^2 \), as \( \phi_2 \) is massless in the Landau gauge. Such a divergence is superficial, as all infrared divergent terms in a physical scattering amplitude must cancel. The point is that there is no infrared divergence for the physical amplitudes evaluated in the \( \alpha \)-gauge with \( \alpha \neq 0 \). Since the on-shell amplitudes are \( \alpha \) independent\[^6\], the on-shell amplitudes in the Landau gauge are infrared finite. More precisely, at \( p^2 = p'^2 = m^2 \), the matrix elements \( \bar{u}(p')\Gamma_{\psi\bar{\psi}}u(p) \) are infrared finite. Nevertheless, some of the matrix elements of \( \Gamma_{\psi\bar{\psi}} \) do have infrared
divergence where \( p^2 \) and \( p'^2 \) are both equal to \( m^2 \). Thus the point of subtraction for \( \Gamma_{V\bar{\psi}\psi} \) will be chosen to be at \( p^2 = p'^2 = \Omega^2 \), where \( \Omega^2 \) is not equal to \( m^2 \). The physical amplitudes are \( \Omega \)-independent. By Lorentz covariance, \( \Gamma'_{V\bar{\psi}\psi} \) has terms proportional to \( \gamma^\mu, P^\mu, \) and \( \Delta^\mu \). We shall ignore terms proportional to the latter two in the Ward-Takahashi identity in the limit under consideration. This is because in this identity \( \Gamma_{V\bar{\psi}\psi} \) is dotted with \( \Delta^\mu \). The terms \( P^\mu \) and \( \Delta^\mu \) in \( \Gamma'_{V\bar{\psi}\psi} \) can be ignored as \( P^\mu \) dotted with \( \Delta^\mu \) is equal to zero while \( \Delta^\mu \Delta^\mu \) is quadratic in \( \Delta \). Thus we shall replace \( \Gamma_{V\bar{\psi}\psi} \) in the Ward-Takahashi identity by

\[
\frac{1}{Z_{V\bar{\psi}\psi}(\Omega^2, \Omega^2, 0)} \left[ (\alpha_+ + 2\beta_+ P)\gamma^\mu (1 + \theta) \frac{1 + \gamma_5}{2} + (\alpha_- + 2\beta_- P)\gamma^\mu \theta \frac{1 - \gamma_5}{2} \right].
\]  

(16)

By power counting, the invariant amplitudes \( \beta_+ \) and \( \beta_- \) are ultraviolet finite. We shall therefore pay attention to \( Z_{V\bar{\psi}\psi} \) and \( \alpha_- \) only. In the lowest-order, \( \alpha_+ \) and \( \alpha_- \) are both equal to 1. We shall define \( \alpha_+ \) to be unity at the subtraction point. Obviously, \( \alpha_- \) is not necessarily equal to unity at the point of subtract. By substituting (16) into (10) and equating the coefficients of \(-i\Delta (1 + \theta) \frac{1 + \gamma_5}{2}\), we get

\[
\frac{Z_{\psi}(m)}{Z_{V\bar{\psi}\psi}(\Omega^2, \Omega^2, 0)} = 1 - c_r(\Omega^2) - d_r(\Omega^2) - m[1 + a_r(m^2)]G_+(\Omega^2, \Omega^2, 0). \tag{17}
\]

Eq. (17) shows that \( Z_{\psi}/Z_{V\bar{\psi}\psi} \) is a finite number — the counterpart of \( Z_2/Z_1 = 1 \) in QED. By equating coefficients of \(-i\Delta \theta \frac{1 - \gamma_5}{2}\) in (10), in the limit \( \Delta \) infinitesimal and \( p^2 = p'^2 = \Omega^2 \), we get

\[
\frac{Z_{\psi}(m)}{Z_{V\bar{\psi}\psi}(\Omega^2, \Omega^2, 0)} \alpha_-(\Omega^2, \Omega^2, 0) = 1 - c_r(\Omega^2) + d_r(\Omega^2) - m[1 + a_r(m^2)]G_-(\Omega^2, \Omega^2, 0), \tag{18}
\]

which shows that \( \alpha_-(\Omega^2, \Omega^2, 0) \) is finite. This also means that the bare parameter \( \theta \) should be chosen finite.

Using the rules of subtractions we have given in this section together with the BPHZ formalism, one may perform renomalized perturbative calculations to all orders. The renormalized parameters in the Feynman rules are required to obey relations such as (15). The presence of \( \gamma_5 \), the meaning of which is controversial in dimensional regularization, presents no difficulty whatsoever in our approach.

One may also make use of the Ward-Takahashi identities for Green’s functions of bosons to show that the physical constants associated with bosons are finite[5]. Generalization to
other gauge field theories such as that of $SU(2) \times U(1) \times SU(3)$ is straightforward. We shall demonstrate in the following letter[7] an application of this method by calculating the next-order triangular anomaly.
1. J. Ashmore, Lett. Nuovo Cimento 4, 289 (1972); C.G. Bollini and J.J. Giambiagi, Phys. Lett 40B, 566 (1972).

2. G. ’t Hooft and M. Veltman, Nucl. Phys. B44, 189, (1972).

3. For a review of this issue, see, for example, Guy Bonneau, International Journal of Modern Physics, Vol. 5, 3831 (1990).

4. See, for example, Sec. 6.4 in S.J. Chang, Introduction to Quantum Field Theory (World Scientific, 1990) for a simple discussion on the BPHZ renormalization and also see the references therein.

5. H. Cheng and S.P. Li, Physical Masses and the Vacuum Expectation Value of the Higgs field (to be published).

6. H. Cheng and E.C. Tsai, Chinese J. of Phys., Vol. 25, No. 1, 95 (1987), Phys. Rev. Lett., 57, 511 (1986); Phys. Rev. D40, 1246 (1989).

7. H. Cheng and S.P. Li, The radiative corrections of the Triangular Anomaly. (submitted for publication)