Spinning test particle in four-dimensional Einstein-Gauss-Bonnet Black Hole

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In this paper, we investigate the motion of a classical spinning test particle orbiting around a static spherically symmetric black hole in a novel four-dimensional Einstein-Gauss-Bonnet gravity [D. Glavan and C. Lin, Phys. Rev. Lett. 124, 081301 (2020)]. We find that the effective potential of a spinning test particle in the background of the black hole has two minima when the Gauss-Bonnet coupling parameter $\alpha$ is nearly in a special range $-6.1 < \alpha/M^2 < -2$ ($M$ is the mass of the black hole), which means such particle can be in two separate orbits with the same spin angular momentum and orbital angular momentum. We also investigate the innermost stable circular orbits of the spinning test particle and find that the effect of the particle spin on the the innermost stable circular is similar to the case of the four-dimensional black hole in general relativity.

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I. INTRODUCTION

As the most successful gravitational theory, general relativity (GR) can explain the relation between geometry and matter. One of the most impressive result derived from GR is black hole solutions. As vacuum solutions of strong gravity systems, black holes have lots of interesting characters, for examples, a binary black hole system can produce gravitational waves [1], and a black hole can act as an accelerator of particles [2, 3]. But we should note that, even GR is so powerful and can be used to explain much phenomena, there are still some problems that can not be interpreted by GR. Therefore, it is believed that there should be more fundamental theory beyond GR.

It is well known that the existence of a singularity locating at the inner of a black hole leads to the geodesics incompleteness \cite{4,5}. To overcome the problem of singularity, several quantum theories of gravity have been proposed, like the superstring/M theory and the extension of such theories. With the help of perturbation approximation of these theories, the Gauss-Bonnet (GB) term was found as the next leading order term \cite{6,7}, and this term is ghost-free combinations and does not add higher derivative terms into the gravitational field equations \cite{8}. The GB term appears in $D$-dimensional spacetime as follows

$$S_{\text{GB}}[g_{\mu\nu}] = \int d^Dx \sqrt{-g} \alpha \mathcal{G},$$

(1)

where $D$ is the number of the spacetime dimensions, $\alpha$ is the GB coupling parameter with mass dimension $D-4$, and the GB invariant $\mathcal{G}$ is defined as

$$\mathcal{G} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2.$$  (2)

Black holes solutions of GB gravity in $D \geq 5$ have been derived, such as the vacuum case \cite{9}, Einstein-Maxwell fields with a GB term \cite{10,11}, and anti-de Sitter (AdS) case \cite{12}. In four-dimensional spacetime, the GB term does not make contributions to the gravitational dynamics, which leads to that the four-dimensional minimally coupled GB gravity is hard to get. However, very recently, D. Glavan and C. Lin \cite{13} proposed a novel four-dimensional Einstein-Gauss-Bonnet (EGB) gravity that bypasses the Lovelock theorem by adopting an artful coupling constant $\alpha \rightarrow \frac{\alpha}{\sqrt{\alpha}}$. In this novel four-dimensional EGB gravity, the GB invariant term does not affect the properties of the massless graviton and a four-dimensional static and spherically symmetric black hole solution was obtained. The stability and shadow of this four-dimensional EGB black hole have been studied in \cite{14}, where the quasinormal modes of a scalar, electromagnetic, and gravitational perturbations were studied. The solutions of charged black hole \cite{15} and spinning black hole \cite{16} were also been obtained, and constraint to the GB parameter $\alpha$ was first given in Ref. \cite{16} in terms of the shadow of the rotating black hole. Inspired by the novel four-dimensional EGB gravity, the novel four-dimensional Einstein-Lovelock gravities are also proposed \cite{17,18}.

To understand the geometry of a black hole, the behaviors of the geodesics of a test particle around a black hole can be used. We know that a massless or massive particle can orbit around a central black hole and the motion is dependent on the geometry of the central black hole. In Ref. \cite{19}, the innermost stable circular orbit (ISCO) of a spinless test particle and shadow in the background of the four-dimensional EGB black hole were studied. The range of the GB coupling parameter for the black hole solution was extended to the range of $-8 \leq \alpha/M^2 \leq 1$, where $M$ is the mass of the black hole. Compared to the
case of a test particle in the background of a Schwarzschild black hole in GR, a positive GB coupling parameter reduces the radii of the ISCOs and a negative one enlarges them. There also appears similar effect for a spinning test particle in the background of a black hole in GR, where the spin-curvature force $-\frac{1}{2}R_{\alpha\beta}^{\mu}u^\nu S^{\alpha\beta}$ also reduces or enlarges the corresponding radii of the ISCOs [20]. Inspired by the effects of the four-dimensional GB term and the non-vanishing spin of a particle on the motion of the test particle, it is necessary to investigate what the total effects of them on the motion of a test particle in the four-dimensional EGB black hole. In this paper, we will investigate the motion of a spinning test particle in the background of the novel four-dimensional EGB black hole. For simplicity, we only consider the motion of a spinning test particle in the equatorial plane.

For a spinning test particle, its motion will not follow the geodesics because of the spin-curvature force $-\frac{1}{2}R_{\alpha\beta}^{\mu}u^\nu S^{\alpha\beta}$. The equations of motion for the spinning test particle are described by the Mathisson-Papapetrou-Dixon (MPD) equations [21–29] under the “pole-dipole” approximation, and the four-velocity $u^\mu$ and the four-momentum $P^\mu$ are not parallel [25, 30] due to the spin-curvature force. The four-momentum $P^\mu$ of a spinning test particle keeps timelike along the trajectory ($P^\mu P_\mu = -m^2$, $m$ is the mass of the test particle), in the contrary, the four-velocity would be superluminal [25, 30] when the spin of the test particle is too large. Actually, this superluminal behavior comes from the ignorance of the “multi-pole” effects. When such effects are considered, the superluminal problem can be avoided [31–35]. For the properties of the spinning test particle in different black hole backgrounds, see Refs. [20, 36–58].

This paper is organized as follows. In Sec. II, we use the MPD equation to obtain the four-momentum and four-velocity of a spinning test particle in the novel four-dimensional EGB black hole background. In Sec. II B, we study the motion of the spinning test particle and give the relations between the motion of the spinning test particle and the properties of the four-dimensional EGB black hole. Finally, a brief summary and conclusion are given in Sec. III.

II. MOTION OF A SPINNING TEST PARTICLE IN FOUR-DIMENSIONAL EGB BLACK HOLE

A. Four-momentum and four-velocity of the spinning test particle

In this part, we will solve the equations of motion for a spinning test particle in the novel four-dimensional EGB black hole background.

The action of the $D$-dimensional EGB gravity is described by

$$S = \int d^Dx \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + \alpha \mathcal{G} \right],$$  \hspace{1cm} (3)

where $\kappa$ is the gravitational constant and will be set as $\kappa^2 = 1/2$ in this paper. The GB term does not contribute to the dynamics of the four-dimensional spacetime because it is a total derivative. Recently, by rescaling the coupling parameter as

$$\alpha \to \frac{\alpha}{D-4},$$  \hspace{1cm} (4)

and taking the limit $D \to 4$, Glaan and Lin [13] obtained the four-dimensional novel EGB gravity. The four-dimensional static spherically symmetric black hole solution was found [13]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \hspace{1cm} (5)$$

$$f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{8\alpha M}{r^3}} \right), \hspace{1cm} (6)$$

where $M$ is the mass of the black hole and the coupling parameter $-8 \leq \frac{\alpha}{M^2} \leq 1$ [19]. Solving $f(r) = 0$, one can get two black hole horizons

$$r_{\pm} = M \pm \sqrt{M^2 - \alpha}. \hspace{1cm} (7)$$

In fact, the above solution (5-7) was also found in gravity with a conformal anomaly in Ref. [59] and was extended to the case with a cosmological Ref. [60].

The motion of a spinning test particle is described by the MPD equations

$$\frac{DP^\mu}{D\lambda} = -\frac{1}{2}R^{\mu}_{\nu\alpha\beta}u^\nu S^{\alpha\beta}, \hspace{1cm} (8)$$

$$\frac{DS^{\mu\nu}}{D\lambda} = P^{\mu}u^{\nu} - u^{\mu}P^{\nu}, \hspace{1cm} (9)$$

where $P^\mu$, $S^{\mu\nu}$, and $u^\mu$ are the four-momentum, spin tensor, and tangent vector of the spinning test particle along the trajectory, respectively. Note that the MPD equations are not uniquely specified and we should use a spin-supplementary condition to determine them. This spin-supplementary condition is related to the center of mass of the spinning test particle with different observers [61–65]. In this paper, we choose the Tulczyjew spin-supplementary condition [66]

$$P_\mu S^{\mu\nu} = 0, \hspace{1cm} (10)$$

and the four-momentum $P^\mu$ satisfies

$$P^\mu P_\mu = -m^2, \hspace{1cm} (11)$$

which makes sure that the spinning test particle keeps timelike along the trajectory.

For the equatorial motion of the spinning test particle with spin-aligned or anti-aligned orbits, the four-momentum and spin tensor should satisfy $P^\theta = 0$ and $S^{\theta\mu} = 0$. The non-vanishing independent variables for the equatorial orbits are $P^t$, $P^r$, $P^\phi$, and $S^{\phi\phi}$. After
adopting the spin-supplementary condition (10), we have [26]

\[ S^{rt} = -S^{r\phi} \frac{P_t}{P_r}, \quad S^{\phi t} = S^{r\phi} \frac{P_r}{P_t}. \]

Solving Eqs. (11), (17), and (18), we get the non-vanishing components of the four-momentum:

\[ P_t = -\frac{m^2 (\alpha (2\phi^3 \Delta + 2jM^2 s) - jM^3 s (\Delta - 1))}{\alpha (2\phi^3 \Delta + 2M^2 \Delta) - r^3 s^2 (\Delta - 1)}, \]

(19)

\[ P_\phi = \frac{2\alpha m^2 r^3 (jM - e \hat{s})}{\alpha (2\phi^3 \Delta + 2M^2 \Delta) - r^3 s^2 (\Delta - 1)}, \]

(20)

and

\[ (P_r)^2 = -\frac{m^2 + g^{\phi t} P_\phi^2 + 2g^{\phi t} P_\phi P_t + g^{tt} P_t^2}{g_{rr}}, \]

(21)

where the function \( \Delta = \sqrt{1 + \frac{2\alpha M}{r^3}} \). We can solve the four-velocity \( u^\mu \) by using the equations of motion (8) and (9) and the components of \( S^{\mu\nu} \) in (15) [56, 67].

\[ DS^{tr}/D\lambda = P_t \ddot{r} - P_r = \frac{\dot{\hat{s}}}{2r} g_{\phi \mu} R_{\alpha \beta \mu \nu} u^{\alpha \beta} - \frac{\dot{\hat{s}}}{r^2} P_\phi \dot{r}, \]

(22)

\[ DS^{t\phi}/D\lambda = P_t \dot{\phi} - P_\phi = \frac{\dot{\hat{s}}}{2r} g_{\mu \nu} R_{\alpha \beta \mu \nu} u^{\alpha \beta} - \frac{\dot{\hat{s}}}{r^2} P_t \ddot{r}. \]

(23)

Finally, the non-vanishing components of the four-velocity are obtained as

\[ \dot{r} = \frac{b_2 c_1 - b_1 c_2}{a_2 b_1 - a_1 b_2}, \]

(24)

\[ \dot{\phi} = \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1}, \]

(25)

where the functions \( a_1, b_1, c_1, a_2, b_2, \) and \( c_2 \) are defined as

\[ a_1 = P^t - \frac{\hat{s}}{r^2} P_\phi + \frac{\hat{s}}{2r} R_{\phi \mu \nu} S^{\mu \nu}, \]

(26)

\[ b_1 = \frac{\hat{s}}{2r} R_{\phi \mu \nu} S^{\mu \nu}, \]

(27)

\[ c_1 = -P^t + \frac{\hat{s}}{2r} R_{\phi \mu \nu} S^{\mu \nu}, \]

(28)

\[ a_2 = \frac{s P_r}{r^2} - \frac{s}{2r} R_{t \mu \nu} S^{\mu \nu}, \]

(29)

\[ b_2 = P^t - \frac{s}{2r} R_{t \mu \nu} S^{\mu \nu}, \]

(30)

\[ c_2 = -P^\phi - \frac{s}{2r} R_{t \mu \nu} S^{\mu \nu}. \]

(31)

We can set the affine parameter \( \lambda \) as coordinate time and choose \( u^t = 1 \) because the trajectories of the test particle are independent of the affine parameter \( \lambda \) [24, 65]. Then the orbital frequency parameter \( \Omega \) of the test particle is

\[ \Omega \equiv \frac{u^\phi}{u^t} = \dot{\phi}. \]

(32)

B. Properties of a spinning particle in circular orbits

The motion of a test particle in a central field can be solved in terms of the radial coordinate in the Newtonian
We can use the same way to simplify the motion of a test particle in the black hole background by using the effective potential method in general relativity. The radial velocity \( u^r \) is parallel to the radial momentum \( P^r \), therefore the effective potential of the spinning test particle can be solved by using the form of \( P^r \) [53]. We decompose the \((P^r)^2 \) [21] as \([42, 53]\)

\[
\frac{(P^r)^2}{m^2} = (A\bar{c}^2 + B\bar{c} + C) \propto \left( \bar{c} - \frac{-B + \sqrt{B^2 - 4AC}}{2A} \right) \left( \bar{c} + \frac{B + \sqrt{B^2 - 4AC}}{2A} \right),
\]

(33)

where the functions \( A, B, \) and \( C \) are

\[
A = 2E^{-1} \alpha m^2 r (8\alpha M + r^3) \left( r^2 \bar{s}^2 (\Delta - 1) + 2\alpha (r^2 - \bar{s}^2) \right),
\]

(34)

\[
B = 8E^{-1} \alpha^2 mj^2 M \bar{s} (-3Mr^2\Delta + 8\alpha M + r^3),
\]

(35)

and

\[
C = -2m^2 (\alpha E)^{-1} \left\{ 16\alpha^4 Mr (j^2 M^2 + r^2) + \alpha^3 \left[ j^2 M^2 (4Mr^3 (1 - \Delta) - M^2 \bar{s}^2 + r^4) + 4Mr^3 \bar{s}^2 (\Delta - 4) + Mr^5 (8 - 8\Delta) + 2M^2 \bar{s}^2 (\bar{s}^2 - 8\bar{s}^2) + 2r^6 \right] + \alpha^3 \left[ j^2 M^2 r^3 (2M \bar{s}^2 (\Delta - 3) + r^3 (1 - \Delta)) - M^2 r^5 \bar{s}^4 (\Delta - 9) - 2Mr^3 \bar{s}^4 (\Delta - 3) + r^6 (1 - \Delta) + 2r^6 \bar{s}^2 (\Delta - 1) + 2Mr^3 \bar{s}^2 (5\Delta - 9) \right] + \alpha r^5 \bar{s}^2 \left[ j^2 M^2 r (\Delta - 1) + r \bar{s}^2 (1 - \Delta) - 4M \bar{s}^2 (\Delta - 2) + r^3 (2\Delta - 2) \right] + r^8 \bar{s}^4 ((1 - \Delta)) \right\},
\]

(36)

where the function \( E \) is

\[
E = [r^3 \bar{s}^2 (\Delta - 1) - 2\alpha (r^3 \Delta + M \bar{s}^2)]^2.
\]

(37)

The effective potential of the test particle is defined by the positive square root of Eq. (33)

\[
V_{\text{eff}}^{\text{spin}} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}
\]

(38)

The positive square root root corresponds to the four-momentum pointing toward future, while the negative one corresponds to the past-pointing four-momentum [70]. When the spin of the test particle is zero, it reduces to

\[
V_{\text{eff}} = \sqrt{1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{8\alpha M}{r^3}} \right)} \sqrt{1 + \frac{j^2}{r^2}}
\]

\[
= \sqrt{f(r) \left( 1 + \frac{j^2}{r^2} \right)}.
\]

(39)

Note that for the four-dimensional Schwarzschild black hole in GR, the function \( f(r) = 1 - \frac{2M}{r} \).

The properties of a test particle in a central field are mainly determined by the effective potential. Thus, the effects on the motion of a spinning test particle can be derived based on how the effective potential depends on the GB coupling parameter \( \alpha \) and the spin angular momentum \( \bar{s} \). We plot some shapes of the effective potential in Fig. 1. We can see that the radii of the extreme points become smaller when the coupling parameter \( \alpha > 0 \) and become larger when the parameter \( \alpha < 0 \). These phenomena mean that a positive GB coupling parameter induces the attractive effect and a negative one results in the repulsive effect on the motion of the test particle.

In addition to the attractive or repulsive effects on the motion of the test particle, some more interesting results are found when we check the shapes of the effective potential in the parameter space \((\bar{s} - j)\). We find that the effective potential has two minima when the GB coupling parameter \( \alpha \) is in a special range. When the test particle move in stable circular orbits [53], the radial velocity...
should be zero
\[ \frac{dr}{d\lambda} = 0, \quad (40) \]
and the radial acceleration vanishes
\[ \frac{d^2r}{d\lambda^2} = 0, \quad \left( \frac{dV_{\text{eff}}}{dr} = 0 \text{ and } \frac{d^2V_{\text{eff}}}{dr^2} > 0 \right). \quad (41) \]

The conditions \( \frac{dV_{\text{eff}}}{dr} = 0 \) and \( \frac{d^2V_{\text{eff}}}{dr^2} > 0 \) mean that the energy of the particle should equal to the minimum of the effective potential.

Therefore, when the effective potential of a spinning test particle has two minima, there will be two stable circular orbits for the particle with a spin angular momentum and an orbital angular momentum. This is a new feature for the motion of a spinning test particle in four-dimensional EGB black hole background. We plot the effective potential with two minima in Fig. 2, where the corresponding two separate orbits of the spinning test particle with \( \bar{s} = 0.3 \) and \( \bar{j} = 5 \) are still given.

The spinning test particle with the same spin \( \bar{s} \) and the same angular momentum can possess two stable circular orbits only happens in the case of \( \alpha < 0 \) with a special range for \( \alpha \). We give the numerical results in Fig. 3 and find that the range of \( \alpha/M^2 \) is nearly in \((-6.1, -2)\).

We have mentioned that the MPD equations of the spinning test particle is obtained under the “pole-dipole” approximation, which will lead to the four-velocity transform from timelike to spacelike if the particle spin is too large. In order to make sure the motion of the spinning test particle is timelike, we adopt the superluminal constraint \[ a_\mu a_\mu = \frac{g_{\mu\nu}}{c^2} \frac{\dot{r}r + \dot{\phi}\phi}{c} + g_{\phi\phi} \left( \frac{\dot{\phi}}{c} \right)^2 + 2g_{\phi r} \dot{r} \dot{\phi} < 0. \quad (42) \]

By using the superluminal constraint and circular orbit conditions \( (40) \) and \( (41) \) for the spinning test particle, we obtain the parameter space \((\bar{s} - \bar{l})\) in Fig. 4, which describes whether the motion on a circular orbit is timelike or spacelike. We can compare the results in Fig. 3 and in Fig. 4 to check whether the motion of the spinning test particle is timelike. We confirm that the motion of the particle with the same spin and the same total angular momentum that can move in two separate orbits is timelike. We have known that the effects from the GB term on the motion of the test particle can be attractive or repulsive, where a positive GB coupling parameter \( \alpha \) leads to an attractive force and a negative one results in a repulsive force. The spin-curvature force can also be attractive or repulsive. The same directions of the spin angular momentum and orbital angular momentum of the spinning test particle will lead to attractive spin-curvature force, while the opposite direction will lead a repulsive force. When the effects induced by the GB term and spin-curvature force exist simultaneously, the total attractive or negative force will be enhanced or weakened. Then it will change the shapes of the regions in \((\bar{l} - \bar{j})\).

Thus, when the GB coupling parameter \( \alpha > 0 \), the attractive effects will lead to disappearance of the parts of the top left and bottom right of the region (II). While for the case of \( \alpha < 0 \), the repulsive effects will result in disappearance of the parts of the top right and bottom left of the region (II).

Next we will investigate the ISCO of the spinning test particle. The ISCO of the test particle locates at the position where the maximum and minimum of the effective potential merge. Thus, the effective potential of the test particle at the ISCO should satisfy
\[ \frac{d^2V_{\text{eff}}}{dr^2} = 0. \quad (43) \]

By using Eqs. \((40)\), \((41)\), and \((43)\), we can derive the ISCO of the test particle. In Ref. \[19\], the authors showed that the radius of the ISCO for a spinless test particle varies in the form of
\[ r_{\text{ISCO}} = 6M - \frac{11}{18} \alpha + O(\alpha). \quad (44) \]

This result was derived under the linear approach with a small \( \alpha \) around 0. Obviously, the ISCO of a spinless test particle can be larger or smaller due to the existence of the GB term. This phenomenon is consistent with the behavior of the effective potential, see the subfigure (a) in Fig. 1.

When the test particle possesses a non-vanishing spin, the contribution of the spin-curvature force should affect the properties of the motion. The relation between the effective potential and the spin of the test particle is still shown in Fig. 1. We give the numerical results of the ISCO in Fig. 5. Note that, there is a jump behavior for the ISCO parameters in the subfigure (e) in Fig. 5, which is induced by the fact that the effective potential has two minima. Because we use the position where the maximum and minimum of the effective potential merge to locate the ISCO and our step length of spin is not small enough to cover the change of the ISCO parameters. We summarize how the ISCO of the spinning test particle depends on the spin \( \bar{s} \) and GB coupling parameter \( \alpha \) as follows:

- For the ISCO of the spinning test particle in four-dimensional EGB black hole, the corresponding radius and angular momentum decrease with the spin \( \bar{s} \) when the GB coupling parameter \( \alpha \) is fixed. And when the effect from the GB term is considered, the spinning test particle can orbit at more smaller radius of the ISCO than the case of the Schwarzschild black hole in GR, and the Gauss-Bonnet term does not change the laws of the ISCO with spin.
- When the spin of the test particle is fixed, the radius and angular momentum of the ISCO decrease with the GB coupling parameter and this behavior is almost the same as the results of the spinless case in Ref. \[19\].
FIG. 1: The effective potential for a spinning test particle in the four-dimensional EGB black hole background. The parameters are set as $M = 1$ and $m = 1$.

FIG. 2: Plots of the effective potential and orbits for a spinning test particle with $\bar{s} = 0.3$, $\bar{j} = 5$, and $\alpha = -6$. The subfigures (b) and (c) are the two minima of the effective potential shown in the subfigure (a). The subfigures (e) and (f) are two separate orbits around the two minima of the effective potential shown in the subfigure (d), and they are related to the effective potential in the subfigures (b) and (c). The values of the red dashed line in the subfigures (b) and (c) stand for the energy of the test particle. And the range of the red dashed in the radial direction stands for the radial range that the test particle can move in, see the corresponding orbits in subfigures (e) and (f). The test particles on the two orbits have the same spin and orbital angular momentum. The parameters are set as $M = 1$ and $m = 1$.

- When the spin of the spinning particle and GB coupling parameter are in the region of $\bar{s} - \bar{j}$ that the
FIG. 3: Plots of the parameter space \((\bar{s} - \bar{j})\) describing whether a spinning test particle can move in two circular orbits with the same spin \(\bar{s}\) and the same total angular momentum \(\bar{j}\). The vertical axis is the spin parameter in the range of \(\bar{s} \in (0, 1.2)\), and the horizontal axis is the total angular momentum \(\bar{j}\) in the range of \(\bar{j} \in (0, 10)\). The other parameters are set as \(M = 1\) and \(m = 1\). In the black and blue regions, the effective potential has two minima (corresponding to two stable circular orbits) and one minimum (corresponding to one stable circular orbit), respectively. In the gray and yellow regions, the test particle has no stable circular orbits.

A spinning test particle can have two separate orbits, the radius and angular momentum of the ISCO will become more smaller than the case of the spinless test particle in the EGB black hole [19] or the case of the spinning test particle in the Schwarzschild black hole in GR [42].

III. SUMMARY AND CONCLUSION

In this paper, we investigated the motion of a spinning test particle in the equatorial plane of the four-dimensional novel EGB black hole. We solved the four-momentum and four-velocity of the particle and investigated how its motion depends on the four-dimensional GB term and particle’s spin. We found that the ISCO of the spinning test particle has the similar behavior as the case of a spinning test particle in general relativity. And the GB term and spin parameter \(\bar{s}\) can make the radii of the ISCO become larger or smaller. The new feature for the motion of the spinning test particle is that it can move at two separate orbits with the same spin \(\bar{s}\) and same total angular momentum \(\bar{j}\) when the GB coupling parameter \(\alpha\) is in a special range of \(-6.1 < \alpha/M^2 < -2\). We also gave the superluminal constraint on the four-velocity of the spinning test particle in the circular orbits and confirmed that the motion of the spinning test particle that can move at two separate orbits is timelike.

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[1] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 061102 (2016); Phys. Rev. Lett. 116, 241103 (2016); Phys. Rev. Lett. 118, 221101 (2017); Phys. Rev. Lett. 119, 141101 (2017); Phys. Rev. Lett. 119, 161101 (2017).

[2] M. Banados, J. Silk, and S. M. West, Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy, Phys. Rev. Lett. 103, 111102 (2009), arXiv:0909.0169 [hep-th].

[3] S.-W. Wei, Y.-X. Liu, H. Guo, and C.-E. Fu, Charged spinning black holes as Particle Accelerators, Phys. Rev.
FIG. 4: Properties of circular orbits for the spinning test particle in the four-dimensional GB black hole background. The vertical axis is the spin parameter in the range of $\tilde{s} \in (-8, 8)$, and the horizontal axis is the orbital angular momentum $\tilde{l} = \tilde{j} - \tilde{s}$ in the range of $\tilde{l} \in (-8, 8)$. The other parameters are set as $M = 1$ and $m = 1$. In the gray region (I) in the $l - s$ space, the test particle can have timelike circular orbits. In the red region (II) and yellow region (III), the test particle does not have stable timelike circular orbits. In the region (III), the motion of the test particle is spacelike and unphysical.

D 82, 103005 (2010), arXiv:1006.1056 [hep-th].

[4] R. Penrose, Gravitational collapse and space-time singularities, Phys. Rev. Lett. 14, 57 (1965).

[5] S. W. Hawking and R. Penrose, The Singularities of gravitational collapse and cosmology, Proc. Roy. Soc. Lond. A 314, 529 (1970).

[6] D. J. Gross and E. Witten, Superstring Modifications of Einstein’s Equations, Nucl. Phys. B 277, 1 (1986); D. J. Gross and J. H. Sloan, The Quartic Effective Action for the Heterotic String, Nucl. Phys. B 291, 41 (1987).

[7] M. C. Bento and O. Bertolami, Maximally Symmetric Cosmological Solutions of higher curvature string effective theories with dilatons, Phys. Lett. B 368, 198 (1996), arXiv:gr-qc/9503057.

[8] B. Zwiebach, Curvature Squared Terms and String Theories, Phys. Lett. B. 156, 315 (1985).

[9] D. G. Boulware and S. Deser, String-generated gravity models, Phys. Rev. Lett. 55, 2656 (1985).

[10] D. L. Wiltshire, Spherically symmetric solutions of Einstein-Maxwell theory with a Gauss-Bonnet term, Physics Letters B, 169, 36 (1986).

[11] D. L. Wiltshire, Black holes in string-generated gravity models, Phys. Rev. D, 38, 2445 (1988).

[12] R.-G. Cai, Gauss-Bonnet black holes in AdS spaces, Phys. Rev. D 65, 084014 (2002), arXiv:hep-th/0109133.

[13] D. Glavan and C. Lin, Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime, Phys. Rev. Lett. 124, 081301 (2020), arXiv:1905.03601 [gr-qc].

[14] R. A. Konoplya and A. F. Zinhailo, Quasinormal modes, stability and shadows of a black hole in the novel 4D Einstein-Gauss-Bonnet gravity, arXiv:2003.01188 [gr-qc].

[15] P. G. S. Fernandes, Charged Black Holes in AdS Spaces in 4D Einstein-Gauss-Bonnet Gravity, arXiv:2003.01188 [gr-qc].

[16] S.-W. Wei and Y.-X. Liu, Testing the nature of Gauss-Bonnet gravity by four-dimensional rotating black hole shadow, arXiv: 2003.07769 [gr-qc].

[17] R. A. Konoplya and A. Zhidenko, Black holes in the four-dimensional Einstein-Lovelock gravity, arXiv:2003.07788 [gr-qc].

[18] A. Casalino, A. Colleaux, M. Rinaldi, and S. Vicentini, Regularized Lovelock gravity, arXiv:2003.07068 [gr-qc].

[19] M.-Y. Guo and P.-C. Li, The innermost stable circular orbit and shadow in the novel 4D Einstein-Gauss-Bonnet gravity, arXiv: 2003.02523 [gr-qc].
FIG. 5: The ISCO parameters of the spinning test particle with different values of $\alpha$. The parameters are set as $M = 1$ and $m = 1$.

of a spinning particle in Kerr space-time, Phys. Rev. D 58, 023005 (1998), arXiv:gr-qc/9712095.
[21] M. Mathisson, New mechanics of material systems, Acta Phys. Pol. 6, 163 (1937).
[22] A. Papapetrou, Spinning test-particles in general relativity. I, Proc. Roy. Soc. Lond. A 209, 248 (1951).
[23] E. Corinaldesi and A. Papapetrou, Spinning test-particles in general relativity. II, Proc. R. Soc. Lond. A 209, 259 (1951).
[24] W. G. Dixon, Dynamics of extended bodies in general relativity II. Moments of the charge-current vector, Proc. R. Soc. Lond. A 314, 499 (1970).
[25] S. A. Hojman, PhD thesis, Princeton University (1975) (unpublished).
[26] R. Hojman and S. Hojman, Spinning Charged Test Particles in a Kerr-Newman Background, Phys. Rev. D 15, 2724 (1977).
[27] B. Mashhoon and D. Singh, Dynamics of Extended Spinning Masses in a Gravitational Field, Phys. Rev. D 74, 124006 (2006), arXiv:astro-ph/0608278.
[28] N. Zalaquett, S. A. Hojman, and F. A. Asenjo, Spinning massive test particles in cosmological and general static spherically symmetric spacetimes, Class. Quant. Grav. 31, 085011 (2014), arXiv:1308.4435 [gr-qc].
[29] R. Uchupol, J. V. Sarah, and A. H. Scott, Gyroscopes orbiting black holes: A frequency-domain approach to precession and spin-curvature coupling for spinning bodies on generic Kerr orbits, Phys. Rev. D 94, 044008 (2016), arXiv:1512.00376 [gr-qc].
[30] C. Armaza, M. Banados, and B. Koch, Collisions of spinning massive particles in a Schwarzschild background, Class. Quant. Grav. 33, 105014 (2016), arXiv:1510.01223 [gr-qc].
[31] A. A. Deriglazov and W. G. Ramírez, Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations in ultrarelativistic regime and gravimagnetic moment, Int. J. Mod. Phys. D 26, 1750047 (2017), arXiv:1511.00645 [gr-qc].
[32] W. G. Ramírez and A. A. Deriglazov, Ultrarelativistic Spinning Particle and a Rotating Body in External Fields, Adv. High Energy Phys. 2016, 1376016 (2016), arXiv:1511.00645 [gr-qc].
[33] A. A. Deriglazov and W. G. Ramírez, Relativistic effects due to gravimagnetic moment of a rotating body, Phys. Rev. D 96, 124013 (2017), arXiv:1709.06894 [gr-qc].
[34] A. A. Deriglazov and W. G. Ramírez, Recent progress on the description of relativistic spin: vector model of spinning particle and rotating body with gravimagnetic moment in General Relativity, Adv. Math. Phys. 2017, 7397159 (2017), arXiv:1710.07135 [gr-qc].
[35] J. Steinhoff and D. Puetzfeld, Multipolar equations of motion for extended test bodies in general relativity, Phys.
W.-B. Han, *Gravitational Radiations from a Spinning Compact Object around a supermassive Kerr black hole in circular orbit*, Phys. Rev. D 81, 044019 (2010), arXiv:0909.3756 [gr-qc].

V. Witzany, J. Steinhoff, and G. Lukes-Gerakopoulos, *Hamiltonians and canonical coordinates for spinning particles in curved space-time*, Class. Quantum Grav. 36 075003 (2019), arXiv:1808.06582 [gr-qc].

P. I. Jefremov, O. Yu. Tsupko, and G. S. Bisnovatyi-Kogan, *Innermost stable circular orbits of spinning test particles in Schwarzschild and Kerr space-times*, Phys. Rev. D 91, 124030 (2015), arXiv:1503.07060 [gr-qc].

B. Toshmatov and D. Malafarina, *Spinning test particle in the \( \gamma \) space-times*, Phys. Rev. D 100, 104052 (2019), arXiv:1910.11565 [gr-qc].

U. Nucamendi, R. Becerril, and P. Sheoran, *Bounds on spinning particles in their innermost stable circular orbits around rotating braneworld black hole*, Eur. Phys. J. C 80, 35 (2020), arXiv:1910.00156 [gr-qc].

Y.-P. Zhang, B.-M. Gu, S.-W. Wei, and J. Yang, and Y.-X. Liu, *Charged spinning black holes as accelerators of spinning particles*, Phys. Rev. D 94, 124017 (2016), arXiv:1608.08705 [gr-qc].

C. Conde and C. Galvis, *Properties of the Innermost Stable Circular Orbit of a spinning particle moving in a rotating Maxwell-dilaton black hole background*, Phys. Rev. D 99, 104059 (2019), arXiv:1905.01323 [gr-qc].

Y.-L. Liu and X.-D. Zhang, *Maximal efficiency of the collisional Penrose process with spinning particles in Kerr-Sen black hole*, Eur. Phys. J. C 80, 31 (2020), arXiv:1910.01872 [gr-qc].

R.-G. Cai, L.-M. Cao, and N. Ohta, *Black Holes in Gravity with Conformal Anomaly and Logarithmic Term in Black Hole Entropy*, JHEP 1004, 082 (2010), arXiv:0911.4379 [hep-th].

R.-G. Cai, *Thermodynamics of Conformal Anomaly Corrected Black Holes in AdS Space*, Phys. Lett. B 733, 183 (2014), arXiv:1405.1246[hep-th].

R. M. Wald, *Gravitational spin interaction*, Phys. Rev. D 6, 406 (1972).

G. Lukes-Gerakopoulos, J. Seyrich, and D. Kunst, *Investigating spinning test particles: spin supplementary conditions and the Hamiltonian formalism*, Phys. Rev. D 90, 104019 (2014), arXiv:1409.4314 [gr-qc].

L. Filipe O. Costa, G. Lukes-Gerakopoulos, and Oldřich Semerák, *On spinning particles in general relativity: momentum-velocity relation for the Mathisson-Pirani spin condition*, Phys. Rev. D 97, 084023 (2018), arXiv:1712.07281 [gr-qc].

G. Lukes-Gerakopoulos, E. Harms, S. Bernuzzi, and Oldřich Semerák, *Dynamics of test bodies orbiting around a Kerr black hole: circular dynamics and gravitational-wave fluxes*, Phys. Rev. D 96, 064051 (2017), arXiv:1707.07537 [gr-qc].

G. Lukes-Gerakopoulos, *Time parameterizations and spin supplementary conditions of the Mathisson-Papapetrou-Dixon equations*, Phys. Rev. D 96, 104023 (2017), arXiv:1709.08942 [gr-qc].

W. Tulczyjew, Acta Phys. Pol. B 18, 393 (1959).

S. A. Hojman and F. A. Aminov, *Can gravitation accelerate neutrinos?*, Class. Quant. Grav. 30, 025008 (2013), arXiv:1203.5008 [physics.gen-ph].

S. A. Kaplan, JETP, 19, 951 (1949).

L. D. Landau, E. M. Lifshitz, The Classical Theory of Fields, Pergamon, Oxford (1993).

C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Freeman, New York, 1973), p. 911.

W. Han, *Gravitational Radiations from a Spinning Compact Object around a supermassive Kerr black hole in circular orbit*, Phys. Rev. D 82, 084013 (2010), arXiv:1008.3324 [gr-qc].

E. Harms, G. Lukes-Gerakopoulos, S. Bernuzzi, and A. Nagar, *Spinning test body orbiting around a Schwarzschild black hole: Circular dynamics and gravitational-wave fluxes*, Phys. Rev. D 94, 104010 (2016), arXiv:1609.00356 [gr-qc].

G. Lukes-Gerakopoulos, E. Harms, S. Bernuzzi, and A. Nagar, *Spinning test-body orbiting around a Kerr black hole: circular dynamics and gravitational-wave fluxes*, Phys. Rev. D 96, 064051 (2017), arXiv:1707.07537 [gr-qc].

S. Mukherjee and K. Rajesh Nayak, *Off-equatorial stable circular orbits for spinning particles*, Phys. Rev. D 98, 084023 (2018), arXiv:1804.06070 [gr-qc].

M. Zhang and W.-B. Liu, *Innermost stable circular orbits of charged spinning test particles*, Phys. Lett. B 789, 393 (2019), arXiv:1812.10115 [gr-qc].

D. Pugliese, H. Quevado, and R. Ruffini, *Equatorial circular orbits of neutral test particles in the Kerr Newton spacetime*, Phys. Rev. D 88, 024042 (2013), arXiv:1303.6250 [gr-qc].

Y.-P. Zhang, S.-W. Wei, W.-D. Guo, T.-T. Sui, and Y.-X. Liu, *Innermost stable circular orbit of spinning particle in charged spinning black hole background*, Phys. Rev. D 97, 084056 (2018), arXiv:1711.09361 [gr-qc].

Z. Stuchlík, *Equilibrium of spinning test particles in the Schwarzschild-de Sitter spacetimes*, Acta Physica Slovaca. 49, 319 (1999).

Z. Stuchlík and J. Ková, *Equilibrium conditions of spinning test particles in Kerr-de Sitter spacetimes*, Class. Quant. Grav. 23, 3935 (2006), arXiv:gr-qc/0611153.

R. Plyatsko, M. Fenyk, and V. Panat, *Highly relativistic spin-gravity-A coupling*, Phys. Rev. D 96, 064038 (2017), arXiv:1711.04270 [gr-qc].

R. Plyatsko, V. Panat, and M. Fenyk, *Nonequatorial circular orbits of spinning particles in the Schwarzschild-de Sitter background*, Gen. Rel. Grav. 50, 150 (2018), arXiv:1811.01391 [gr-qc].

W.-B. Han, *Dynamics of extended bodies with spin-induced quadrupole in Kerr spacetime: generic orbits*, Gen. Rel. Grav. 49, 48 (2017), arXiv:1611.07602 [gr-qc].

N. Warburton, T. Osburn, and C. R. Evans, *Evolution of small-mass-ratio binaries with a spinning secondary*, Phys. Rev. D 96, 084057 (2017), arXiv:1708.03720 [gr-qc].

Y. Liu and W.-B. Liu, *Energy extraction of a spinning particle via the super Penrose process from an extremal Kerr black hole*, Phys. Rev. D 97, 064024 (2018).

S. Mukherjee, *Gravitational Radiations from a Spinning Compact Object around a supermassive Kerr black hole in circular orbit*, Phys. Lett. B 778, 54 (2018).

G. Faye, L. Blanchet, and A. Buonanno, *Higher-order spin effects in the dynamics of compact binaries. I. Equations of motion*, Phys. Rev. D 74, 104033 (2006), arXiv:gr-qc/0605139.

V. Witzany, J. Steinhoff, and G. Lukes-Gerakopoulos, *Hamiltonians and canonical coordinates for spinning particles in curved space-time*, Class. Quantum Grav. 36