AUSLANDER-REITEN SEQUENCES AND $t$-STRUCTURES ON THE HOMOTOPY CATEGORY OF AN ABELIAN CATEGORY.

ERIK BACKELIN AND OMAR JARAMILLO

Abstract. We use $t$-structures on the homotopy category $K^b(R\text{-mod})$ for an artin algebra $R$ and Watts’ representability theorem to give an existence proof for Auslander-Reiten sequences of $R$-modules. This framework naturally leads to a notion of generalized (or higher) Auslander-Reiten sequences.

1. Introduction

Krause, [K], [K2], used Brown’s representability theorem for triangulated categories to give a short proof for the existence of Auslander-Reiten (abbreviated AR) triangles. The present article is a variation of that theme. We use $t$-structures to define an abelian category $A$ where AR-sequences naturally occur as simple objects. We apply Watts’ representability theorem to $A$ to reprove the existence of AR-sequences for modules over an artin algebra $R$.

To set up the general constructions $C$ will denote an abelian category; later on $C$ will be $R$-mod.

Let $C^b(C)$ be the category of bounded cochain complexes in $C$ and let $K^b(C)$ be the category of cochain complexes with morphisms modulo homotopy. We consider a $t$-structure $(D^{\leq 0}, D^{\geq 0})$ on $K^b(C)$ which is standard in the sense that the localization functor maps it to the tautological $t$-structure on the bounded derived category $D^b(C)$, see Proposition 2.1. (We briefly investigate other standard $t$-structures on $K^b(C)$ as well.)

Let $A = A(C) = D^{\leq 0} \cap D^{\geq 0}$ be the heart of the $t$-structure. This is an abelian category whose objects are complexes

$$[A \rightarrowtail B \twoheadrightarrow C]$$

such that $f$ is injective and $\text{Ker } g = \text{Im } f$.

We observe that the category $A$ is naturally equivalent to a subcategory of the category $\text{Ab}^{\text{op}}$ of functors from $C$ to the category of abelian groups. Moreover, in this way the functor $C \mapsto P_C := [0 \rightarrow 0 \rightarrow C]$ corresponds to the Yoneda embedding $C \rightarrow \text{Ab}^{\text{op}}$. We prove that $D^b(A) = K^b(C)$ and describe injective objects of $A$.

In the case when $C$ is a finite length category we shall see that simple objects of $A$ - if they exist - are given by Auslander and Reiten’s almost split right maps, see Section 4.1.

The functor category $\text{Ab}^{\text{op}}$ has since Auslander been a central tool in AR-theory. Our $A$ provides merely a different realization of it, but a realization that we prefer because it is more intuitive and from its definition it is clear that it lives naturally inside the triangulated category $K^b(C)$. For instance, Auslander’s defect of a short exact sequence, living in $\text{Ab}^{\text{op}}$, corresponds to the homotopy class of the sequence in $A$.

2000 Mathematics Subject Classification. Primary 16G10, 16G70, 18G99.
Now we specialize to the case $\mathcal{C} = R$-$\text{mod}$. The main ingredient in Krause’s existence proof for AR-triangles in a triangulated category was a Serre duality functor and the existence of this he deduced from Brown’s representability theorem following ideas of Neeman, \cite{N}, \cite{N2}. With this in mind it clear that in our setup the existence of AR-sequences would follow from a Serre duality functor on $K^b(\text{R-mod})$, because one could use the $t$-structure to truncate it down to an Auslander-Reiten type duality between short exact sequences (see corollary 4.7 for the precise meaning of such a duality).

However, we cannot deduce the existence of this Serre duality from Brown’s representability theorem, because it is not known to us whether $K(\text{R-Mod})$ is well-generated (compare with \cite{H-J}). Instead, our approach is the (far more elementary) theory of abelian categories. We use Watts’ theorem to prove that any projective object $P_C$ in the abelian category $\mathcal{A}$ has a “Serre dual” object $SP_C \in \mathcal{A}$; this implies a Serre duality on $K^b(\text{R-mod})$, see Proposition 4.6, and also AR-duality. From the latter we deduce, with a proof similar to the one given in \cite{ARS}, the existence of AR-sequences, Theorem 4.8. We also interpret the AR-sequence with end term $C$ as $\text{Im} \tau$, where $\tau : P_C \rightarrow SP_C$ is a certain minimal map.

In fact, in order to prove these existence and duality theorems for AR-sequences we could have altogether avoided to mention triangulated categories and worked, ad hoc, within the abelian category $\mathcal{A}$. However, the viewpoint of $t$-structures on $K^b(\mathcal{C})$ is very valuable. It seems to be a natural source of the theory and it allows us to rediscover or reinterpret familiar notions in AR-theory, like the dual of the transpose, the defect and projectivization. It also naturally generalizes:

There exists a notion of higher AL-sequences (see \cite{I}). In Section 5 we propose another method to generalize AR-theory. We define a generalized AR-sequence to be a simple objects in the heart of a certain $t$-structure on $K^b(\mathcal{C})$, where $\mathcal{C}$ is an additive (not necessarily abelian) category. It is certain that these generalized AR-sequences and the higher AR-sequences of \cite{I} are intimately relate (perhaps they coincide), but we haven’t worked this out.

There are interesting examples of generalized AR-sequences. For instance, Soergel’s theory of coinvariants, \cite{S}, shows that a block in the BGG category $\mathcal{O}$ of representations of a complex semi-simple Lie algebra $\mathfrak{g}$ is equivalent to a category of generalized AR-sequences in $K^b(\mathcal{C})$, with $\mathcal{C}$ is a certain category of modules over the cohomology ring of the flag manifold of $\mathfrak{g}$. See Section 5.1 for a discussion.

We approach some more themes: We discuss the rather obvious fact why $\mathcal{A}$ fails to be a noetherian category and give a brief discussion of duality on $\mathcal{A}$ in the case when $R$ is a Frobenius algebra.

1.1. Acknowledgements. We thank Chaitanya Gutti and the referee for useful comments.

2. Standard $t$-structures on $K^b(\mathcal{C})$

2.1. Notations. We denote by $\text{Ind}(\Lambda)$ the class of iso-classes of indecomposable objects in a category $\Lambda$. $\text{Proj}(\Lambda)$ and $\text{Inj}(\Lambda)$ are the full subcategories of $\Lambda$ whose objects are projective and injective, respectively. Let $R$ be a ring; $R$-$\text{mod}$, $\text{mod}$-$R$, $R$-$\text{Mod}$ and $\text{Mod}$-$R$, denote the categories of finitely generated left, finitely generated right, all left and all right $R$-modules, respectively.
2.2. A standard $t$-structure on $K^b(\mathcal{C})$ and its heart. Let $\mathcal{C}$ be an abelian category. In this section we investigate a specific standard $t$-structure on $K^b(\mathcal{C})$ and its heart. We briefly discuss two other specific standard $t$-structures and the existence of more. We also give an example from representation theory where standard $t$-structures naturally occur.

We follow the notations of [KS] concerning $t$-structures and triangulated categories. See also [GM] and [N2] for more details. We define a $t$-structure $(D^{\leq 0}, D^{\geq 0})$ on $K^b(\mathcal{C})$ as follows. Put

\[
\tag{2.1} D^{\leq 0} = \{ X \in K^b(\mathcal{C}); X^i = 0 \text{ for } i > 0 \}
\]

Here $X = \{X^i, d^i\}$. (To be more precise, the objects of $D^{\leq 0}$ are complexes homotopic to the right hand side of (2.1) but we omit this kind of linguistic precision.) Put

\[
\tag{2.2} D^{\geq 0} = \{ X \in K^b(\mathcal{C}); X^i = 0 \text{ for } i < -2, H^{-2}(X) = H^{-1}(X) = 0 \}
\]

**Proposition 2.1.** $(D^{\leq 0}, D^{\geq 0})$ is a bounded standard $t$-structures on $K^b(\mathcal{C})$.

**Proof.**

a) For a morphism $f : (X, d_X) \rightarrow (Y, d_Y)$ in $K^b(\mathcal{C})$ the mapping cone $M(f)$ is defined by

\[
\tag{2.3} M(f)^n = X^{n+1} \oplus Y^n
\]

with differential given by $d_{M(f)}(x_{n+1}, y_n) = (-d_X x_{n+1} + f(y_n), d_Y y_n)$. Note that the inclusion $D^{\leq 0} \hookrightarrow K^b(\mathcal{C})$ has as right adjoint the truncation functor $\tau^{\leq 0}$ which is defined by $\tau^{\leq 0}(X) = (X \rightarrow X^{-n+1} \rightarrow \ldots \rightarrow X^{-2} \rightarrow X^{-1} \rightarrow \text{Ker } d^0 \rightarrow 0)

For $X \in K^b(\mathcal{C})$, let $\alpha_X : \tau^{\leq 0}(X) \rightarrow X$ be the natural map.

b) We show that $D^{\geq 0} = \{ M(\alpha_X); X \in K^b(\mathcal{C}) \}$. For $X \in K^b(\mathcal{C})$ we have $M := M(\alpha_X) = \rightarrow X^{-n+1} \oplus X^{-n} \rightarrow \ldots \rightarrow X^{-2} \oplus X^{-3} \rightarrow \text{Ker } d^{-1} \oplus X^{-2} \rightarrow X^{-1} \rightarrow X^0 \rightarrow \ldots$

Hence $M \cong M' \oplus M''$ where $M'$ is the $0$-homotopic complex

\[
\tag{2.6} \ldots \rightarrow X^{-n+1} \oplus X^{-n} \rightarrow \ldots \rightarrow X^{-2} \oplus X^{-3} \rightarrow X^{-2} \rightarrow 0
\]

where all differential are the same as in (2.5) except $d : X^{-2} \oplus X^{-3} \rightarrow X^{-2}$ which is given by $d(x_{-2}, x_{-3}) = x_{-2} - dx_{-3}$. $M''$ is the subcomplex

\[
\tag{2.7} 0 \rightarrow \text{Ker } d^{-1} \rightarrow X^{-1} \rightarrow X^0 \rightarrow \ldots
\]

of $M$. Thus $M \cong M''$ in $K^b(\mathcal{C})$ which proves the statement of b).

c) It follows from b) that any $X \in K^b(\mathcal{C})$ fits into a distinguished triangle $\tau^{\leq 0} X \rightarrow X \rightarrow M^{\perp}$ where $\tau^{\leq 0} X \in D^{<0}$ and $M \in D^{\geq 0}$. It follows from the definitions that $\text{Hom}_{K^b(\mathcal{C})}(X, Y) = 0$ for $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$. Thus $(D^{\leq 0}, D^{\geq 0})$ is a $t$-structure; clearly it is standard. Note also that any object of $K^b(\mathcal{C})$ belongs to $D^{\leq a} \cap D^{\geq b}$ for some $a, b \in \mathbb{Z}$ which means that the $t$-structure is bounded. 

It follows from Proposition 2.1 that the inclusion functor $D^{\geq 0} \hookrightarrow K^b(\mathcal{C})$ has the left adjoint $\tau^{\geq 0}$ where $\tau^{\geq 0}(X)$ is the complex (2.7).

Let $\mathcal{A} = D^{\leq 0} \cap D^{\geq 0}$ be the heart of the $t$-structure. Thus objects of $\mathcal{A}$ are sequences

\[
\tag{2.8} [A \xrightarrow{f} B \xrightarrow{g} C]
\]
such that \( f \) is injective and \( \mathrm{Ker} \, g = \mathrm{Im} \, f \). (We use the square-brackets to stress that we consider an object in the abelian category \( \mathcal{A} \).) Morphisms in \( \mathcal{A} \) are morphisms of complexes up to homotopy.

Notice that \([A \to B \to C] \cong 0 \) iff \( B \to C \) is a split surjection.

We next describe the kernels, images and cokernels in \( \mathcal{A} \). Fix a morphism

\[
\phi = (\phi_A, \phi_B, \phi_C) : [A \to B \xrightarrow{f} C] \to [A' \to B' \xrightarrow{f'} C']
\]

Let \( M = M(\phi) \) be the mapping cone of \( \phi \) (considered as a morphism in \( \mathrm{K}^b(\mathcal{C}) \)). Then we have \( \mathrm{Ker}(\phi) = \tau_{\geq 0}(M[-1]) \) and \( \mathrm{Coker}(\phi) = \tau_{\geq 0}(M) \) and it follows from this that

\[
(2.9) \quad \mathrm{Ker}(\phi) = [\mathrm{Ker} \, \pi \to B \oplus \mathrm{Ker} \, d' \xrightarrow{\pi} C \times_{C'} B']
\]

\[
(2.10) \quad \mathrm{Coker}(\phi) = [\mathrm{Ker} \, \delta \to C \oplus B' \xrightarrow{\delta} C'']
\]

\[
(2.11) \quad \mathrm{Im}(\phi) = [A' \to (C \times_{C'} B') \xrightarrow{\pi} C]
\]

Here, \( C \times_{C'} B' = \{(c, b'); \phi_C(c) = -f'(b')\}, p(b, b') = (-f(b), b' + \phi_B(b)), \delta = \phi_C + f' \) and \( \pi(c, b') = c \). Notice that in the case that \( f' \) is surjective, \( \mathrm{Im} \, \phi \) is just the pull-back of an exact sequence by \( C \).

The localization functor \( \mathrm{L} : \mathrm{K}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{C}) \) induces a functor on hearts

\[
(2.12) \quad \mathrm{L} \mid_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}
\]

given by \( \mathrm{L} \mid_{\mathcal{A}}([A \to B \xrightarrow{d} C]) = \mathrm{Coker} \, d \). \( \mathrm{L} \mid_{\mathcal{A}} \) is exact since \( \mathrm{L} \) is \( t \)-exact.

**Definition 2.2.** Let \( \mathcal{A}^0 \) be the full subcategory of \( \mathcal{A} \) consisting of short exact sequences.

It follows from the exactness of \( \mathrm{L} \mid_{\mathcal{A}} \) that \( \mathcal{A}^0 \) is an exact abelian subcategory of \( \mathcal{A} \).

**Definition 2.3.** Define a fully faithful functor by

\[
P : \mathcal{C} \to \mathcal{A}, \ A \mapsto P_A := [0 \to 0 \to A]
\]

Notice that for any object \([A \to B \to C] \) of \( \mathcal{A} \) there is a canonical exact sequence

\[
(2.13) \quad 0 \to P_A \to P_B \to P_C \to [A \to B \to C] \to 0
\]

This shows that the homological dimension of \( \mathcal{A} \) is \( \leq 2 \); it is easy to see that strict inequality holds iff \( \mathcal{C} \) is semi-simple and in that case \( \mathcal{A} \) is also semi-simple.

Let \( \mathbf{Ab} \) be the category of abelian groups and let \( \mathbf{Ab}^{\mathbf{C}^{op}} \) be the abelian category of additive contravariant functors from \( \mathcal{C} \) to \( \mathbf{Ab} \). Then there is the fully faithful Yoneda embedding \( h : \mathcal{C} \to \mathbf{Ab}^{\mathbf{C}^{op}} \) defined by \( A \mapsto h_A := \mathrm{Hom}_\mathcal{C}(A, \cdot) \). By the Yoneda lemma we have that \( h_A \in \mathrm{Proj}(\mathbf{Ab}^{\mathbf{C}^{op}}) \) for all objects \( A \) of \( \mathcal{C} \) and that \( \mathbf{Ab}^{\mathbf{C}^{op}} \) is generated by the collection \( \{h_A; A \in \mathcal{C}\} \).

**Proposition 2.4.** There is an exact fully faithful functor \( \pi : \mathcal{A} \to \mathbf{Ab}^{\mathbf{C}^{op}} \) such that \( P_A \mapsto h_A \).

**Proof.** Let \( f : A \to B \) be a morphism in \( \mathcal{C} \). Then we must have \( \pi([\mathrm{Ker} \, f \to A \to B]) = \mathrm{Coker} \, h_f \), where \( h_f : h_A \to h_B \) is given by \( f \). By construction \( \pi \) is exact and by the Yoneda lemma we have that

\[
\mathrm{Hom}_\mathcal{A}(P_A, P_B) \cong \mathrm{Hom}_\mathcal{C}(A, B) \cong \mathrm{Hom}_{\mathbf{Ab}^{\mathbf{C}^{op}}}(h_A, h_B)
\]

for all objects \( A, B \) in \( \mathcal{C} \). Since the \( P_A \) generates \( \mathcal{A} \) as a category the full faithfulness now follows from general nonsense. \( \square \)
Corollary 2.5. For each $A \in \mathcal{C}$, $P_A \in \text{Proj}(\mathcal{A})$, and each projective of $\mathcal{A}$ is isomorphic to some $P_A$. Hence, $D^b(\mathcal{A})$ is canonically equivalent to $K^b(\mathcal{C})$.

Proof. It follows from the previous proposition that $P_A$ is projective in $\mathcal{A}$ since $h_A$ is projective in $\mathfrak{A}^{bop}$. Thus $\mathcal{A}$ has enough projectives. Moreover, it is easy to see that each projective in $\mathcal{A}$ must be of the form. Thus, $\text{Proj}(\mathcal{A}) \cong \mathcal{C}$ and hence
\[ D^b(\mathcal{A}) \cong K^b(\text{Proj}(\mathcal{A})) \cong K^b(\mathcal{C}). \]

Example 2.6. Let $F$ be a field, $R = \mathbb{F}[x]/(x^2)$, $\mathcal{C} = \text{R-mod}$ and let $\mathcal{A}$ be the heart of the $t$-structure from Proposition 2.3. Then $\mathcal{A}$ has five indecomposable objects: $P_0$ (projective), $P_R$ (projective and injective), $[0 \to \mathbb{F} \to R]$ (simple), $[\mathbb{F} \to R \to \mathbb{F}]$ (simple) and $[\mathbb{F} \to R \to R]$ (injective).

2.3. Other standard $t$-structures on $K^b(\mathcal{C})$. In general $K^b(\mathcal{C})$ will have infinitely many standard $t$-structures. It would be interesting to classify them (and also to relate them to Bridgeland’s stability theory, [B], that classifies all bounded $t$-structures).

In this section we make no attempt to reach such a classification, but merely observe that besides the one we studied in the previous section there are two other particular evident standard $t$-structures. We denote them by $(D'^{\leq 0}, D'^{\geq 0})$ and $(D'^{\leq 0}, D'^{\geq 0})$ and their hearts by $\mathcal{A}'$ and $\mathcal{A}''$, respectively. They are defined as follows:
\begin{align}
D'^{\leq 0} &= \{ X \in K^b(\mathcal{C}); X^i = 0 \text{ for } i > 1 \text{ and } d^0 \text{ is surjective } \} \\
D'^{\geq 0} &= \{ X \in K^b(\mathcal{C}); X^i = 0 \text{ for } i < -1 \text{ and } d^{-1} \text{ is injective } \}
\end{align}
Its heart is given by complexes
\[ \mathcal{A}' = \{ (A \xrightarrow{g} B \xrightarrow{f} C); g \text{ is injective and } f \text{ is surjective} \} \]
The other one is defined by
\begin{align}
D''^{\leq 0} &= \{ X \in K^b(\mathcal{C}); X^i = 0 \text{ for } i > 2 \text{ and } H^1(X) = H^2(X) = 0 \} \\
D''^{\geq 0} &= \{ X \in K^b(\mathcal{C}); X^i = 0, \text{ for } i < 0 \}
\end{align}
The description of the $t$-structure $(D'^{\leq 0}, D'^{\geq 0})$ is dual to that of $(D'^{\leq 0}, D'^{\geq 0})$: The objects $[A \to 0 \to 0]$ are injective in $\mathcal{A}'$, for $A \in \mathcal{C}$, and the functor $A \mapsto [A \to 0 \to 0]$ corresponds to the Yoneda embedding $\mathcal{C} \to (\mathfrak{A}^{b})^{\text{op}}$ defined by $A \mapsto \text{Hom}_\mathcal{C}(, A)$.

The embedding $\mathcal{C} \to \mathcal{A}'$, $A \mapsto [0 \to A \to 0]$ appears to be a mix of the two Yoneda embeddings. We shall not investigate these two $t$-structures any further in this paper.

We conclude this section with an example that shows there are many standard $t$-structures on $\mathcal{C}$.

Example 2.7. Let $F$ be a field, $R = \mathbb{F}[x]/(x^2)$ and let $\mathcal{C} = \text{R-mod}$. Let $n \geq 2$ and let $V$ denote the acyclic complex
\[ 0 \to \mathbb{F} \leftarrow R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} \ldots \xrightarrow{\varepsilon} R \to \mathbb{F} \to 0 \]
where the component $\mathbb{F}$ occurs in degree 0 and $n$. Let $\Omega$ denote the set of all complexes in $K^b(\text{R-mod})$ concentrated in degrees $\leq 0$ together with the complex $V$. 
It follows e.g. from [Ay], Proposition 2.1.70, that there is a unique $t$-structure $(D_n^{<0}, D_n^{>0})$ on $K^b(R\text{-mod})$ such that $\Omega \subset D_n^{<0}$ and

$$D_n^{<0} = \{ X \in K^b(R\text{-mod}); \text{Hom}_{K^b(R\text{-mod})}(A[i], X) = 0, A \in \Omega, i > 0 \}.$$

One can verify that this gives standard $t$-structures which are different for different values of $n$.

3. Injectives

We describe the injectives in $\mathcal{A}$ and in $\mathcal{A}^0$. Contrary to the case with projectives, in order for $\mathcal{A}$ to have enough injectives we need that $\mathcal{C}$ has enough injectives.

3.1. Injectives in $\mathcal{A}$. Let us start with

**Lemma 3.1.** Let

$$\phi = (\phi_A, \phi_B, \phi_C) : [A \to B \xrightarrow{f} C] \to [A' \to B' \xrightarrow{f'} C']$$

be a morphism in $\mathcal{A}$. i) Assume that $[A' \to B' \xrightarrow{f'} C'] \neq 0$ and that $\text{End}_C(C')$ is a local ring. Then $\phi$ is surjective iff $\phi_C$ is a split surjection. ii) $\phi$ is injective iff $\bar{\phi}_C : C/\text{Im } f \to C'/\text{Im } f'$ is injective and the canonical injection $A \to B \oplus \text{Ker } f'$, $a \to (a, -\phi_B b)$ splits.

**Proof.** i) By 2.10 we have that $\phi$ is surjective iff $\phi_C + f' : C \oplus B' \to C'$ is a split surjection. Since $\text{End}_C(C')$ is a local ring this implies that either $f'$ or $\phi_C$ is a split surjection. By the assumption $f'$ is not a split surjection. Hence $\phi_C$ is a split surjection.

Let us prove ii). By 2.9 we have that $\phi$ is injective iff

$$B \oplus \text{Ker } f' \xrightarrow{\epsilon} C \times_{C'} B'$$

is a split surjection, where $\epsilon(b, v) = (-fb, \phi_B b + v)$, for $(b, v) \in B \oplus \text{Ker } f'$.

**Claim:** $\epsilon$ is surjective iff

$$\bar{\phi}_C : C/\text{Im } f \to C'/\text{Im } f'$$

is injective.

**Proof Claim.** Note that 3.2 is equivalent to

$$\phi_C c \in \text{Im } f' \implies c \in \text{Im } f$$

Denote by $K$ the righthand side of 3.1 and fix $(c, b') \in K$. Thus $\phi_C c = -f' b'$ and so if we assume that $\epsilon$ is surjective we see that 3.3 holds. Conversely, assuming 3.3 we show that $\epsilon$ is surjective. We have $(c, b') = (fb, b')$ for some $b \in B$. Then $f' b' = -\phi_C c = -\phi_C fb = -f' \phi_B b$. Let $v = b' + \phi_B b \in \text{Ker } f'$. Then we see that $(fb, b') = \epsilon(b, v)$. This proves the claim.

Now, for $\epsilon$ surjective, we have that $\epsilon$ splits iff the inclusion

$$\text{Ker } \epsilon = \{(b, -\phi_B b); b \in \text{Ker } f \} \hookrightarrow B \oplus \text{Ker } f'$$

splits which proves ii). □

We can now prove

**Proposition 3.2.** An object $[D \to I \to J]$ in $\mathcal{A}$ is injective if $I, J \in \text{Inj}(\mathcal{C})$. 
Proof. a) Assume that $I \in \text{Inj}(\mathcal{C})$. We first show that $P_I$ is injective in $\mathcal{A}$. Consider a commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
[A \to B \xrightarrow{f} C] \\
\downarrow \\
[A' \to B' \xrightarrow{f'} C']
\end{array}
$$

(3.5)

We must construct a lift $\tilde{\gamma}$. By Lemma 3.1 the map $\overline{\phi}_C : C/\text{Im }f \to C'/\text{Im }f'$ is injective and moreover we see that $\gamma$ factors through $P_{C/\text{Im }f}$. Hence we can fill the dotted arrow and get a commutative diagram

$$
\begin{array}{c}
P_I \\
\downarrow \\
P_{C/\text{Im }f} \\
\downarrow \\
[A \to B \to C] \\
\downarrow \\
[A' \to B' \to C']
\end{array}
$$

where $\text{nat}$ and $\text{nat}'$ are the natural maps. Hence we can take $\tilde{\gamma} = (0, \overline{\phi}_C \circ \text{nat}')$.

b) We have the exact sequence

$$
0 \to P_D \to P_I \to P_J \to [D \to I \to J] \to 0
$$

(3.6)

By a) $P_I$ and $P_J$ are injective. Since the homological dimension of $\mathcal{A}$ is $\leq 2$ it follows that $[D \to I \to J]$ is injective as well.

Corollary 3.3. If $\mathcal{C}$ has enough injectives then also $\mathcal{A}$ has enough injectives.

Proof. Let $[A \to B \xrightarrow{d} C]$ in $\mathcal{A}$ be given. By standard arguments we can find an object $[A' \to I \xrightarrow{\partial} J]$ with $I, J$ injective and a morphism $\phi : [A \to B \xrightarrow{d} C] \to [A' \to I \xrightarrow{\partial} J]$ with the properties that $\phi_B$ and $\phi_C$ are injective, $\phi_B(A) = A'$ and $\phi_C : C/\text{Im }d \to J/\text{Im }\partial$ is injective. Thus by Lemma 3.1 $\phi$ is injective.

The converse of Proposition 3.2 also holds

**Corollary 3.4.** Assume that $\mathcal{C}$ has enough injectives. Each injective object of $\mathcal{A}$ is isomorphic to an object of the form $[A \to I \to J]$, where $I, J \in \text{Inj}(\mathcal{C})$.

Proof. Let $E \in \mathcal{A}$ be injective. In the proof of corollary 3.3 we saw that we can find an object $[A' \to I' \to J']$, with $I'$ and $J'$ injective in $\mathcal{C}$ and an injective morphism $\phi : E \to [A' \to I' \to J']$. Since $E$ is injective $\phi$ splits. This implies that $E$ has the desired form.

Note that corollary 3.4 gives a natural bijection $\text{Ind}(\mathcal{C}) \cong \text{Ind}(\text{Inj}(\mathcal{A}))$: a non-injective $A \in \mathcal{C}$ corresponds to $[A \to I \to J]$, where $A \hookrightarrow I \to J$ is an indecomposable injective resolution of $A$ and an injective $A \in \mathcal{C}$ corresponds to $[0 \to 0 \to A]$. 

3.2. Injectives and projectives in \( A^0 \). Recall that \( A^0 \) denotes the full subcategory of \( A \) whose objects are short exact sequences. The inclusion \( A^0 \to A \) has the right adjoint

\[
q : A \to A^0, \quad q([A \to B \overset{d}{\to} C]) = [A \to B \to \text{Im} \, d]
\]

Assume that \( C \) has enough projectives and injectives and put for \( X \in C \),
\[
P^0_X = [\text{Ker} \, d \to Q \overset{d}{\to} X] \quad \text{where} \quad d : Q \to X \text{ is surjective and } Q \text{ is projective. Also, put } I^0_X = q(I_X).
\]

Then we have

**Lemma 3.5.** For each \( X \in C \), \( P^0_X \) is projective in \( A^0 \) and \( \text{Hom}_{A^0}(P^0_X, V) = \text{Hom}_{A}(P_X, V) \) for \( V \in A^0 \). Any indecomposable projective in \( A^0 \) is (isomorphic to an object) of the form \( P^0_X \) where \( X \) is indecomposable and non-projective in \( C \). Similarly, each \( I^0_X \) is injective in \( A^0 \), \( \text{Hom}_{A^0}(V, I^0_X) = \text{Hom}_{A}(V, I_X) \) and each indecomposable injective in \( A^0 \) is of the form \( I^0_X \) where \( X \) is indecomposable and non-injective in \( C \).

**Proof.** All verifications are left to the reader. For the part which states that all projectives and injectives in \( A^0 \) are isomorphic to objects of the prescribed form just mimic the argument of corollary 3.4. \( \Box \)

### 4. AR-sequences for representations of an Artin algebra

4.1. **AR-sequences are simple objects of \( A \).** In this section we assume that \( C \) is a finite length (abelian) category. We start by briefly recalling AR-theory in \( C \), (see [ARS] for details about the material here and compare with [H] and [K] for the theory of AR-triangles in triangulated categories that is not treated here). Fix a morphism

\[
B \overset{f}{\to} C
\]

in \( C \). \( f \) is called an almost split right map if \( f \) is not a split surjection and any map \( \phi : X \to C \) which is not a split surjection factors through \( f \). Assume from now on that \( f \) is right almost split. It follows that \( C \) is necessarily indecomposable.

Almost split right maps have the following properties:

- If \( C \) is projective it has a unique maximal submodule \( \text{rad} \, C \) and \( f \) is the inclusion \( \text{rad} \, C \hookrightarrow C \).
- If \( C \) is not projective then \( f \) is necessarily surjective.

Dually, there is the notion of an almost split left map \( g : A \to B \). \( g \) is not a split injection and any \( h : A \to Y \) which is not a split injection factors through \( g \).

A short exact sequence

\[
0 \to A \overset{\alpha}{\to} B \overset{f}{\to} C \to 0
\]

is called an almost split exact sequence, or an AR-sequence, if \( g \) is left almost split and \( f \) is right almost split. See [A], [J] and [Sm] for some positive and negative existence results for AR-sequences.

Let \( h : X \to Y \) in \( C \) be a given map in \( C \). Recall that a (right) minimal version of \( h \) is a map \( h_{\text{min}} : X' \to Y \) such that \( h_{\text{min}} \) factors through \( h \) and \( h \) factors through \( h_{\text{min}} \) and \( X' \) has minimal length with this property. \( h_{\text{min}} \) exists and is unique up to isomorphism of maps over \( Y \). The minimal length of \( X' \) is equivalent to require that \( X' \) has no non-zero direct summand mapped to 0 by \( h_{\text{min}} \).

If one assumes that \( h \) is right almost split it follows that \( \text{Ker} \, h_{\text{min}} \) is indecomposable.
It is easy to see that if $B \overset{f}{\rightarrow} C$ and $B' \overset{f'}{\rightarrow} C$ are almost split right maps, then $[\text{Ker } f \rightarrow B \overset{f}{\rightarrow} C] \cong [\text{Ker } f' \rightarrow B' \overset{f'}{\rightarrow} C]$ in $A$. We next show that the almost split right maps are precisely the simple objects of $A$.

**Proposition 4.1.** Let $X = [A \rightarrow B \overset{d}{\rightarrow} C]$ be an object of $A$ and assume that $C$ be indecomposable in $C$. Then $X$ is simple iff $d$ is an almost split right map. If $d$ is almost split we write $L_C := X$. In this case $L_C$ is the unique simple quotient of $P_C$.

*Proof.* Note that by the Krull Schmidt theorem $\text{End}_C(C)$ is a local ring.

Assume that $B \overset{d}{\rightarrow} C$ is almost split. We shall show $X$ is simple. For this it is enough to show the following: Let $\phi : X \rightarrow Y$, with $Y \neq 0$, be a surjective map. Then $\phi$ is injective.

We may assume that $d$ is right minimal. Then any endomorphism $h$ of $B$ over $C$ is an automorphism. Since $C$ is indecomposable we may assume that $Y = [A' \rightarrow B' \overset{d'}{\rightarrow} C]$, where $d'$ is not a split surjection, and that $\phi_C = \text{Id}_C$. Since $d$ is almost split it follows that $d'' = d \circ g$, for some $g : B' \rightarrow B$. Then $g \circ \phi_B$ is an endomorphism of $B$ over $C$ and hence an isomorphism. This means that $\phi \circ (g, \text{Id}_C)$ is an automorphism of $L_C$. Thus $\phi$ is injective.

Conversely, assume that $d$ is not almost split. Then we can find $f : D \rightarrow C$ which is not a split epi such that $f$ does not factor through $d$. Consider the composition

$$P_D \rightarrow X \rightarrow [\text{Ker } (d + f) \rightarrow B \oplus D \overset{d + f}{\rightarrow} C]$$

The third object and the first map are non-zero by assumption. But the composition is zero, so $X$ is not simple.

For the last assertion, we have already proved that $L_C$ is simple and, clearly, $L_C$ is a quotient of $P_C$. Conversely, if $Y$ is a simple quotient of $P_C$ it follows from Lemma 3.1) that we may assume the end-term of $Y$ is $C$. Then by what we just have shown we see that $Y$ is given by an almost split right map with target $C$, i.e. $Y \cong L_C$. \(\square\)

We next reprove the well-known result that right almost split maps fit into AR-sequences:

**Corollary 4.2.** Assume that $C$ has enough injectives. Let $0 \rightarrow A \overset{g}{\rightarrow} B \overset{f}{\rightarrow} C \rightarrow 0$ be a short exact sequence in $C$ such that $f$ is minimal right almost split. Then $g$ is left almost split.

*Proof.* Recall that $A$ is indecomposable by the minimality of $f$. Let $X \in \text{Ob}(C)$ and a non-split injection $h : A \rightarrow X$ be given. We must prove that $h$ factors through $B$. We may assume that $h$ is non-zero. Thus $h$ is a non-split map.

Next, by the assumption that $f$ is right almost split we know that $L_C = [A \rightarrow B \overset{f}{\rightarrow} C]$ is a simple object of $A$. Let

$$0 \rightarrow X \rightarrow I \overset{j}{\rightarrow} J$$

be an injective resolution of $X$. Then there is a map $\tilde{h} : L_C \rightarrow [X \rightarrow I \overset{j}{\rightarrow} J]$ such that $\tilde{h}_A = h$. We claim that $\text{Ker } \tilde{h} \neq 0$: indeed, if $\text{Ker } \tilde{h} = 0$ then by Lemma 3.1 $A \rightarrow B \oplus X$ would be a split injection and this is not the case since $A$ is indecomposable and neither $A \rightarrow B$ nor $h$ is split.

Hence, by simpleness of $L_C$ we have that the natural map $\text{nat} : \text{Ker } \tilde{h} \rightarrow L_C$ is an isomorphism. One sees that the inverse morphism $\text{nat}^{-1}$ provides a map $h' : B \rightarrow X$ such that $h' \circ g = h$. \(\square\)
To end this section let us say that an abelian category has enough simples if each of its objects has a simple quotient object. A noetherian category of course has enough simples. The following example shows however that \( A \) will in general not be noetherian although it (by Auslander and Reiten’s theorem) has enough simples. Better means that guarantee enough simples will be given in the following sections.

**Example 4.3.** Let \( F \) be a field and let \( R = F[x,y]/(x^2, xy, y^2) \) and \( C = R\text{-mod.} \) Let \( m = (x, y) \) be the maximal ideal in \( R \).

For \( i > 0 \) define \( R\text{-modules} \) \( M_i = R/x - iy \) and \( B_i = M_1 \oplus M_2 \oplus \ldots \oplus M_i \). Let \( F = R/m \) and \( B_i \to F \) the sum of the natural projections. Let \( V_i = [\text{Ker}_i \to B_i \to F] \), where \( \text{Ker}_i = \text{Ker}(B_i \to F) \). The inclusions \( B_i \hookrightarrow B_{i+1} \) induce surjections

\[
V_i \to \ldots \to V_i \to V_{i+1} \to \ldots
\]

Since \( \text{Hom}_A(P_{M_{i+1}}, V_i) \neq 0 \) and \( \text{Hom}_A(P_{M_{i+1}}, V_{i+1}) = 0 \) we conclude that non of the maps in \( \boxed{} \) are isomorphisms and hence that d.c.c. doesn’t hold on quotient objects in \( A \), so \( A \) is not noetherian.

4.2. **Serre duality for the category \( A \) in the case of representations of an artin algebra.** Let \( R \) be an artin algebra. Thus, by definition there is a commutative artin ring \( k \subset R \) such that \( R \) is a finitely generated \( k \)-module. From now on we shall exclusively consider the case where \( C = R\text{-mod.} \).

Let \( S \) be the direct sum of the irreducible \( k \)-modules and let \( J \) be an injective hull of \( S \) in \( k\text{-mod.} \). Let

\[
\text{\( k\text{-Mod} \supseteq M \mapsto M^*: = \text{Hom}_k(M, J) \in k\text{-Mod} \)}
\]

be the usual duality functor. Thus \( **|_{k\text{-mod.}} \cong \text{Id}_{k\text{-mod.}} \).

In order to later on apply Watts’ representability theorem (\([R], \text{Theorem 3.36}\)) we need to embed \( A \) into a category of modules over a ring. Some technical difficulties arise from the fact that \( A \) does not have a small projective generator (unless \( R \) has finite representation type). Let \( \tilde{A} := A(\text{R-Mod}) \). Thus, \( \tilde{A} \) is an abelian category closed under coproducts containing \( A \) as a full abelian subcategory. Since \( \text{Ind}(\text{R-mod}) \) is a set we can define \( \tilde{P} := \bigoplus_{D \in \text{Ind}(\text{R-mod})} P_D \in \tilde{A} \) and \( F = \text{End}_{A}(\tilde{P}) \).

Consider the exact functor

\[
\mathbb{V} : \tilde{A} \to \text{Mod-F}, \mathbb{V} M = \text{Hom}_{\tilde{A}}(\tilde{P}, M)
\]

and let \( \mathbb{V}|_A : A \to \text{mod-F} \) be the restriction of \( \mathbb{V} \) to \( A \).

**Lemma 4.4.** \( \mathbb{V}|_A \) is a fully faithful embedding. The essential image of \( \mathbb{V}|_A \) consists of all objects \( Y \) in \( \text{mod-F} \) such that there exists a \( C \in \text{R-mod} \) and a surjection \( \mathbb{V} P_C \to Y \).

**Proof.** Let us say that an object \( M \) in \( \tilde{A} \) is good if the natural map

\[
\text{Hom}_{\tilde{A}}(M, N) \to \text{Hom}_{\text{Mod-F}}(\mathbb{V} M, \mathbb{V} N)
\]

is bijective for all \( N \in A \). We must show that any object \( M \in A \) is good. First take \( M = P_C \) for \( C \in \text{Ind}(\text{R-mod}) \). Since, \( P_C \) is a direct summand in \( \tilde{P} \) we get that \( \mathbb{V} P_C \) is a direct summand in the right \( F \)-module \( F = \mathbb{V} \tilde{P} \). From this it is clear that \( P_C \) is good. Thus \( P_C \) is good for any \( C \in \text{R-mod} \). Since every object \( M \in A \) has a presentation \( P_C \to P_{C'} \to M \) we get by the five lemma that \( M \) is good. This proves the full faithfulness of \( \mathbb{V}|_A \).
Since any object $M$ in $\mathcal{A}$ is a quotient of some $P_C$ it follows that $\forall M$ is a quotient of $\forall P_C$. Conversely, assume that $Y \in \text{mod-}F$ and there is a surjection $\phi : \forall P_C \to Y$. Let $T$ be the $\mathcal{A}$-subobject of $P_C$ defined by
\[ T = \sum_{g \in \forall P_C} \text{Im} \ g \]
We now prove $\ker \phi = \forall T$. Note that if $I$ is a sufficiently large index set we get a surjection $f : \widetilde{P} \to T$ such that each component $f_i : P_i \to P_C$ is in $\ker \phi$. Let $T = [B' \to B'' \to B]$ for some $B', B'' , B \in R$-mod. We may assume that $B$ has no direct summand which is the isomorphic image of a direct summand of $B''$.

Since $f$ is surjective we have that the natural map $B'' \oplus \widetilde{P} \to B$ is a split surjection. If $B$ is indecomposable the Krull Schmidt theorem shows that some component $f_j$ of $f$ is a split surjection. In general, after breaking $B$ into indecomposable pieces we find that there is a finite subset $J = \{1, \ldots, n\} \subset I$ such that $f_j \colonequals f|_{\widetilde{P}} : \widetilde{P} \to B$ is a split surjection.

Since $\widetilde{P}$ is projective in $\mathcal{A}$ we see that any map $g : \widetilde{P} \to T$ factors as $g = f_j \circ h$ where $h = (h_1, \ldots, h_n) \in \text{Hom}_A(\widetilde{P}, \widetilde{P}^I) = \text{Hom}_A(\widetilde{P}, \widetilde{P})^n$. Thus $g = \sum_{i=1}^n f_i h_i \in \ker \phi$. Thus $\ker \phi = \forall T$. Hence we have a short exact sequence
\[ 0 \to \forall T \xrightarrow{\mu} \forall P_C \to Y \to 0 \]
By the full faithfulness already proved we see that $\mu = \forall \nu$ for some $\nu \in \text{Hom}_{\mathcal{A}}(T, P_C)$. Hence, $Y \cong \forall(\text{Coker} \ \nu)$. \qed

**Proposition 4.5.** For any $X \in R$-mod, the contravariant functor
\[ \text{Hom}_A(P_X, \ )^* : \mathcal{A} \to \text{k-mod} \]
is representable by an injective $\mathcal{A}$-object $SP_X$.

We shall refer to $SP_X$ as the Serre dual of $P_X$.

**Proof.** Consider the contravariant functor
\[ \Gamma := \text{Hom}_{\text{mod-}F}(\forall P_X, \ )^* : \text{mod-}F \to \text{k-Mod} \]
Since the right $F$-module $\forall P_X$ is a direct summand in $F$, if $X$ is indecomposable, we conclude that $\forall P_X$ is projective for any $X$ in $R$-mod. Thus $\Gamma$ is exact. In particular $\Gamma$ is left exact and since, moreover, $\Gamma$ transforms coproducts to products, Watts’ theorem shows that $\Gamma$ is represented by $\Gamma(F)$.

By Lemma 4.4 it remains to show that there is an object $SP_X \in \mathcal{A}$ such that $\Gamma(F) = \forall SP_X$. Note that
\[ \Gamma(F) \cong \text{Hom}_A(P_X, \widetilde{P})^* \]
with the right $F$-module structure on $\text{Hom}_A(P_X, \widetilde{P})^*$ induced by the left $F$-module structure on $\text{Hom}_A(P_X, \widetilde{P})$ that is given by composition of maps.

Assume first that $X$ is irreducible and let $I$ be an injective hull of $X$ in $R$-mod. Let $\epsilon_1, \ldots, \epsilon_n$ generate $\text{Hom}_A(P_X, P_I)^*$ as a $\text{k}$-module. Since $P_X$ and $P_I$ are direct summands in $\widetilde{P}$ we can interpret $\text{End}_A(P_X)^*$ and $\text{Hom}_A(P_X, P_I)^*$ as direct summands in $F^*$. With this in mind we record that
\begin{equation}
(4.2) \quad \text{End}_A(P_X)^* = \{ \sum_{i=1}^n \epsilon_i g_i ; g_i \in \text{Hom}_A(P_X, P_I) \} \end{equation}
Similarly, we can consider the $\epsilon_i$’s as elements of $\text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^*$ and get the map
\[
\pi : (\nu P) \rightarrow \text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^*, \quad (f_1, \ldots, f_n) \rightarrow \sum \epsilon_i f_i
\]
Note that $\pi$ is right $F$-linear. We shall prove that $\pi$ is surjective. Let
\[
\nu = \prod \nu D \in \text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^* = \prod_{D \in \text{Ind}(R\text{-mod})} \text{Hom}_{\mathcal{A}}(P_X, P_D)^*
\]
be given.

Fix for now $D \in \text{Ind}(R\text{-mod})$. We can write $\text{soc} D = X^m \oplus K$ where $K$ is a direct sum of simple modules all of them non-isomorphic to $X$, where $\text{soc} D$ is the socle of $D$. Thus, since $X$ is simple, $\text{Hom}_{\mathcal{A}}(P_X, P_D) \cong \text{Hom}_{\mathcal{A}}(P_X, P_X^m)$ and so we have an isomorphism
\[
nat : \text{Hom}_{\mathcal{A}}(P_X, P_D)^* \cong \text{Hom}_{\mathcal{A}}(P_X, P_X^m)^* = (\text{Hom}_{\mathcal{A}}(P_X, P_X)^*)^m
\]
By \[4.2\] we can find $h_{D,1}, \ldots, h_{D,n} \in \text{Hom}_{\mathcal{A}}(P_X, P_I)^m$ such that $\text{nat}(\nu D) = \sum_{i=1}^n \epsilon_i h_{D,i}$. Here we used the notation $h_{D,i} = (h_{D,1}, \ldots, h_{D,im})$ and $\epsilon_i h_{D,i} = (\epsilon_i, h_{D,1}, \ldots, \epsilon_i, h_{D,im})$.

Since $P_I$ is injective we can find $\tilde{h}_{D,i} \in \text{Hom}_{\mathcal{A}}(P_D, P_I)^m$ that extends $h_{D,i}$. Thus $\nu D = \sum_{i=1}^n \epsilon_i \tilde{h}_{D,i}$. If we let $h_D = (\tilde{h}_{D,1}, \ldots, \tilde{h}_{D,n})$ and $h = \prod_{D \in \text{Ind}(R\text{-mod})} h_D$ we see that $\pi(h) = \nu$.

For $X \in R\text{-mod}$ simple we have now constructed a surjection $\nu P_A \rightarrow \text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^*$, with $A = I^n$. If $X$ is not simple, it has a finite filtration with simple subquotients $X_1, \ldots, X_N$. By the procedure above we find $A_i \in R\text{-mod}$ and surjections $\nu P_{A_i} \rightarrow \text{Hom}(P_{X_i}, \tilde{P})^*$. Since each $\nu P_{A_i}$ is projective in mod-$F$ this give rise to a surjection
\[
\oplus_{i=1}^N \nu P_{A_i} \rightarrow \text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^*
\]
Thus, by Lemma \[4.3\] there is an object $SP_X \in \mathcal{A}$ such that $\text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^* \cong \mathcal{V}SP_X$. $\square$

Note that the assignment $P_X \mapsto SP_X$ defines a fully faithful functor $\text{Proj}(\mathcal{A}) \rightarrow \text{Inj}(\mathcal{A})$, because for $P_X, P_Y \in \text{Proj}(\mathcal{A})$ we have isomorphisms
\[
\text{Hom}_{\mathcal{A}}(P_X, P_Y) \rightarrow \text{Hom}_{\mathcal{A}}(P_X, P_Y)^{**} \rightarrow \text{Hom}_{\mathcal{A}}(P_Y, SP_X)^*
\]
Let us explicitly describe $SF : SP_X \rightarrow SP_Y$ corresponding to $f \in \text{Hom}_{\mathcal{A}}(P_X, P_Y)$. $f$ induces a morphism
\[
\text{Hom}_{\mathcal{A}}(f, \tilde{P})^* : \mathcal{V}SP_X \cong \text{Hom}_{\mathcal{A}}(P_X, \tilde{P})^* \rightarrow \text{Hom}_{\mathcal{A}}(P_Y, \tilde{P})^* \cong \mathcal{V}SP_X
\]
Now the full faithfulness of $\mathcal{V}|_{\mathcal{A}}$ gives a morphism $SF : SP_X \rightarrow SP_Y$ such that $\mathcal{V}SF = \text{Hom}_{\mathcal{A}}(f, \tilde{P})^*$.

For exactness reasons there cannot exist a Serre dual object $SA \in \mathcal{A}$ of a non-projective object $A \in \mathcal{A}$. But we have

**Proposition 4.6.** The functor $S : \text{Proj}(\mathcal{A}) \rightarrow \text{Inj}(\mathcal{A})$ induces a triangulated functor $\mathcal{S} : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ satisfying
\[
\text{Hom}_{D^b(\mathcal{A})}(A, B)^* \cong \text{Hom}_{D^b(\mathcal{A})}(B, SA)
\]
for all $A, B \in D^b(\mathcal{A})$. 
Proof. We have that $S$ extends to a functor
$$S : C^b(\text{Proj}(A)) \to C^b(\text{Inj}(A))$$
Clearly, this functor induces a triangulated functor between homotopy categories that we denote by the same symbol
$$S : K^b(\text{Proj}(A)) = D^b(A) \to K^b(\text{Inj}(A)) = D^b(A)$$
For $A, B \in C^b(A)$ there is the homomorphism complex $\mathcal{H}om(A, B)$ defined by
$$\mathcal{H}om(A, B)^n = \prod_{i \in \mathbb{Z}} \text{Hom}_A(A_i, B_{i+n})$$
and differential given by $df = d_B \circ f - (-1)^n f \circ d_A$ for $f \in \mathcal{H}om(A, B)^n$. Using [13] it is easy to verify that for $A, B \in C^b(\text{Proj}(A))$, the already constructed isomorphisms $\text{Hom}_A(A_i, B_j)^* \cong \text{Hom}_A(B_j, SA_i)$, $\forall i, j \in \mathbb{Z}$, defines an isomorphism of homomorphism complexes
$$\mathcal{H}om(A, B)^* \cong \mathcal{H}om(B, SA)$$
Then we get $\text{Hom}_{D^b(A)}(A, B)^* = H^0(\mathcal{H}om(A, B)^*)$, since $A \in C^b(\text{Proj}(A))$, and $\text{Hom}_{D^b(A)}(B, SA) = H^0(\mathcal{H}om(B, SA))$, since $SA \in C^b(\text{Inj}(A))$. This establishes the isomorphism stated in the theorem. \hfill \Box

Let us remark that using the $t$-structure it is easy to see that, conversely, Proposition 4.6 implies Proposition 4.5. The result of Proposition 4.6 will not be used in this paper.

4.3. Existence of AR-sequences for representations of an artin algebra. We now approach the existence problem of AR-sequences. First, it is better to work with the category $\mathcal{A}^0$, since the indecomposable projectives of $\mathcal{A}^0$ correspond to non-projectives of $\text{R-mod}$ and AR-sequences must have non-projective end term. Write
$$SP^0_X := q(SP_X),$$
where $q$ is the functor 3.7. It follows from Proposition 4.5 and Lemma 3.5 that

**Corollary 4.7.** For $V \in \mathcal{A}^0$ we have a natural isomorphism
$$\text{Hom}_{\mathcal{A}^0}(P^0_X, V)^* \cong \text{Hom}_{\mathcal{A}^0}(V, SP^0_X).$$

It seems to be an appropriate way (which also generalizes well) to think of AR-duality like this, as a Serre type of duality betweenProj$(A)$ and Inj$(A)$ (or between Proj$(\mathcal{A}^0)$ and Inj$(\mathcal{A}^0)$). We can now deduce Auslander and Reiten’s famous existence theorem and in addition give a rather explicit form for the AR-sequence with given end term.

**Theorem 4.8.** Let $X \in \text{Ind}(\text{R-mod})$ be non-projective. There exist a non-zero map $\tau : P^0_X \to SP^0_X$ which has the property that any non-surjective map $g : P^0_X \to \text{Im} \tau$ must satisfy $g = 0$. Then Im $\tau$ is an AR-sequence with end term $X$.

**Proof.** We first show the existence of a non-zero map $\tau : P^0_X \to SP^0_X$ such that $\tau \circ f = 0$ for all non-units $f$ in the local artin algebra $\text{End}_{\mathcal{A}^0}(P^0_X) = \text{End}_{\text{R-mod}}(X)$. Since $X$ is non-projective we have $P^0_X \neq 0$ and thus by AR-duality
$$\text{Hom}_{\mathcal{A}^0}(P^0_X, SP^0_X) \cong \text{End}_{\mathcal{A}^0}(P^0_X)^* \neq 0$$
Now $\text{Hom}_{\mathcal{A}^0}(P^0_X, SP^0_X)$ is a finitely generated $k$-module and therefor finitely generated as a right module over the ring $\text{End}_{\mathcal{A}^0}(P^0_X) \cong \text{End}_{\text{R-mod}}(X) \supset k$. We can thus find a nonzero element $\tau$ in the socle of $\text{Hom}_{\mathcal{A}^0}(P^0_X, SP^0_X)$ considered as an $\text{End}_{\mathcal{A}^0}(P^0_X)$-module.
This \( \tau \) will satisfy the hypothesis of the theorem because any map \( g : P^0_X \to \text{Im} \tau \) will factor as \( g = \tau \circ f \), for some \( f \in \text{End}_{A^0}(P^0_X) \), by the projectivity of \( P^0_X \). If \( g \) is non-surjective it is clear that \( f \) can not be a unit. Thus \( g = 0 \) in this case.

We now prove that \( \text{Im} \tau \) is simple. For this purpose it is enough to prove that for any \( D \in R\text{-Mod} \) and any non-surjective map \( h : P^0_D \to \text{Im} \tau \) we must have \( h = 0 \). We have

\[
\text{Hom}_{A^0}(\text{Im} \tau, SP^0_X) = 0 \quad \text{AR-duality} \quad \iff \quad \text{Hom}_{A^0}(P^0_X, \text{Im} \tau) = 0
\]

Let \( g \in \text{Hom}_{A^0}(P^0_X, \text{Im} \tau) \). Let \( i : \text{Im} \tau \to \text{Im} \tau \) be the inclusion. Since \( i \) is not surjective \( i \circ g : P^0_X \to \text{Im} \tau \) is not surjective and hence 0 by the assumption on \( \tau \). Thus, \( g = 0 \).

Finally we observe that \( \text{Im} \tau \) is an AR-sequence that ends with \( X \): Since \( SP^0_X \) is injective in \( A \) we see that \( SP^0_X = [X' \to I \to N] \) where \( I \) is injective in \( R\text{-mod} \). Then we have \( \text{Im} \tau = [X' \to I \times_N X \to X] \). □

To end this section let us deduce the classical formulation of Auslander and Reiten duality involving the dual of the transpose. Let \( X \in R\text{-mod} \). Since \( SP^0_X \) is injective we see that

\[
SP^0_X \cong [X' \to I \xrightarrow{d} J]
\]

where \( X' \in R\text{-mod} \) and \( I, J \in \text{Inj}(R\text{-mod}) \). We assume that \( X' \) is chosen to be minimal in the sense that it has no direct summand \( X'' \) such that the composition \( X'' \to X' \to I \) splits; then \( X' \) is well-defined up to isomorphism.

Then it is easy to see that for any \( V \in A^0 \) one has

\[
\text{Hom}_{A^0}(V, SP^0_X) = \text{Hom}_{K^b(R\text{-mod})}(V, X'[2]).
\]

On the other hand (for any \( V \in A \)) we have

\[
\text{Hom}_{A^0}(P^0_X, V) = \text{Hom}_{K^b(R\text{-mod})}(X[0], V).
\]

Thus we have rediscovered

**Proposition 4.9.** (AR-duality.) There is a natural isomorphism

\[
\text{Hom}_{K^b(R\text{-mod})}(X[0], V)^* \cong \text{Hom}_{K^b(R\text{-mod})}(V, X'[2])
\]

of \( k \)-modules, for \( V \in A^0 \), \( X \in R\text{-mod} \) and \( X' \) defined by \((4.4)\).

In [ARS] it is proved that the formula in Proposition 4.9 holds with \( X' \) replaced by the dual of the transpose of \( X \), \( DT \tau X \). Thus we have proved that \( X' \cong DT \tau X \) and also reestablished the existence of the dual of the transpose.

### 5. Generalized AR-sequences

Here we propose a generalization of AR-sequences to the case of non-abelian categories. Let \( \mathcal{C} \) be an additive Karoubi closed category. Let \( D^{\leq 0} \) be the subcategory of \( K^b(\mathcal{C}) \) defined by \((2.1)\). Let \( \tau^{\leq 0} : K^b(\mathcal{C}) \to D^{\leq 0} \) be the functor given by \((2.3)\). Let \( D^{> 0} \) be the collection of \( M \in K^b(\mathcal{C}) \) such that there is an \( M' \in K^b(\mathcal{C}) \) such that \( M \) is homotopic to the cone of the canonical morphism \( \tau^{> 0} M' \to M' \). We make the rather mild assumption (see [Ay], Proposition 2.1.70, for criteria that this holds) that

\[
(D^{\leq 0}, D^{> 0}) \quad \text{(5.1)}
\]

is a \( t \)-structure. This \( t \)-structure is standard in the same sense as before; note that it is now a harder problem than in the abelian case to describe all possible standard \( t \)-structures on
$K^b(C)$ (compare Section 2.3). Then we define a generalized AR-sequence to be a simple object in the abelian category $\mathcal{A} := D^{\leq 0} \cap D^{\geq 0}$.

Note that if $C$ is abelian to start with a generalized AR-sequence will simply be a usual AR-sequence. We haven’t worked out the details, but generalized AR-sequences will certainly be closely related to higher AR-sequences, whose definition we for the sake of completeness recall here:

A higher AR-sequence, see [I], in a suitable additive category $C$, is a long exact sequence

\[ 0 \to X^{-n} \to X^{-n+1} \to \ldots \to X^0 \to 0 \]

such that each $d^{-i}$ belongs to the radical of $C$, $X^{-n}$ and $X^0$ are indecomposable and the sequence

\[ 0 \to \text{Hom}_C(A, X^{-n}) \to \ldots \to \text{Hom}_C(A, X^1) \to J_{A,X^0} \to 0 \]

is exact for all $A \in \text{Ob}(C)$, where $J_{A,X^0}$ is the radical of $\text{Hom}_C(A, X^0)$. (When $n = 2$ this gives usual AR-sequences.)

5.1. Motivation from representation theory: Category $C$. Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and let $R$ be the cohomology ring of the flag manifold of $\mathfrak{g}$. Let $O_0$ be the principal block of the Bernstein-Gelfand-Gelfand category $O$ of representations of $\mathfrak{g}$. Then by Soergel’s theory, $\text{Proj}(O_0)$ is equivalent to an additive Karoubi closed subcategory $C$ of $R$-mod. ($C$ is not abelian unless $\mathfrak{g} = \mathfrak{sl}_2$.) Thus,

\[ D^b(O_0) \cong K^b(\text{Proj}(O_0)) \cong K^b(C). \]

Hence the tautological $t$-structure on $D^b(O_0)$ corresponds to the $t$-structure on $K^b(C)$ which is given by 5.1.

If $\mathfrak{g} = \mathfrak{sl}_2$, then $R = \mathbb{C}[x]/(x^2)$, $C = R$-mod and the standard $t$-structure on $D^b(O_0)$ corresponds to the $t$-structure from Proposition 2.1 on $K^b(C)$; hence the category $A$ is equivalent to $O_0$ in this case. The usual duality on $O$ is also obtained by a general construction that we give in the next section.

In fact, this procedure may be generalized as follows. Start, say, with an additive Karoubi closed subcategory $C$ of the category of all modules over a Frobenius algebra $R$ and consider the heart $A$ of a $t$-structure on $K^b(C)$ of the form 5.1. One may then ask interesting questions such as: If we assume that $R$ is the cohomology ring $H^*(X)$, for some compact manifold $X$, when can the heart, like category $O$, then be realized as a category of perverse sheaves on $X$? When is the heart Koszul, etc? (Compare with [BGS].)

5.2. Duality over a Frobenius algebra. Let $R$ be a commutative Frobenius algebra. Then the classes of injective and projective $R$-modules coincide and the duality functor

\[ R\text{-mod} \to R\text{-mod}, \quad M \mapsto M^* := \text{Hom}_{R\text{-mod}}(M, R) \]

fixes the projective modules. In this case we can define a duality functor $^*$ on $A = A(R\text{-mod})$ as follows.

First we define the dual $(P_C)^*$ of $P_C$ for $C \in R\text{-mod}$. Pick an injective resolution $0 \to C \to I \xrightarrow{d} I'$ and define

\[ (P_C)^* = \text{Ker} \ d^* \to I'^* \xrightarrow{d^*} I^* \]

where $I'^* := \text{Hom}_{R\text{-mod}}(I', R)$ and $d^* : I'^* \to I^*$ is the map induced by $d$. Then $(P_C)^*$ is the dual of $P_C$ in the category $R\text{-mod}$.
$(PC)^*$ is a well-defined object in $\mathcal{A}$ since different injective resolutions of the same object are homotopic. Next, for a general $\mathcal{A}$-object $X = [A \to B \overset{f}{\to} C]$ we define

$$X^* = \text{Ker}((PC)^* \overset{f'}{\to} (PB)^*)$$

where the map $f'$ is naturally induced by $f$. Then some diagram chasing proves that $X \mapsto X^*$ gives a well defined contravariant functor $^*: \mathcal{A} \to \mathcal{A}$ whose square is equivalent to the identity.

**Example 5.1.** In the notations of example 2.6 we have $P^*_F \cong [F \to R \to R]$, while $P_F, P_R$ and $[0 \to F \to R]$ are selfdual.

**References**

[A] M. Auslander, A survey of existence theorems for almost split sequences, pp. 8189 in “Representations of Algebras”, London Math. Soc. Lecture Note Ser., Vol. 116, Cambridge University Press, Cambridge, (1986).

[ARS] M. Auslander, I. Reiten and S. Smalø, “Representation theory of Artin Algebras”, Cambridge Studies in Mathematics, (1997).

[Ay] J. Ayoub, “Les six opérations de Grothendieck et le formalisme de cycle évanescents dans le monde motivique”, I. Astérisque, Vol. 314 (2007), x+464 pages.

[BGS] A. Beilinson, V. Ginzburg and W. Soergel, “Koszul duality patterns in representation theory”, J. Amer. Math. Soc. 9 (1996), 473–527. MR 96k:17010

[B] T. Bridgeland, “Stability conditions on triangulated categories”, Ann. of Math. (2) 166 (2007), no. 2, 317–345.

[GM] S. Gelfand and Y. Manin “Methods of homological algebra”. Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, (2003). xx+372 pp. ISBN 3-540-43583-2

[H] D. Happel, “Triangulated Categories in the Representation Theory of Finite Dimensional Algebras”, London Mathematical Society Lecture Note Series 119, Cambridge University Press, Cambridge, (1988).

[I] O. Iyama “Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories”, Advances in math. 210, no 1 22-50, (2007)

[J] P. Jørgensen “Auslander-Reiten sequences on schemes”, Ark. Mat. 44 (2006), no. 1, 97–103.

[H-J] H. Holm and P. Jorgensen, “Compactly generated homotopy categories” Homology, Homotopy and Applications, vol. 9(1) (2007) 257-274

[KS] M. Kashiwara and P. Shapira “Sheaves on Manifolds”, Springer-Verlag (1994)

[K] H. Krause “Auslander-Reiten theory via Brown representability”, $K$-Theory 20 (2000), no. 4, 331–344.

[K2] H. Krause “Auslander-Reiten triangles and a theorem of Zimmerman”, Bull. London Math. Soc. 37 (2005) 361372

[N] A. Neeman, “The Grothendieck duality theorem via Bousfields techniques and Brown Representability”, Journal of the AMS, Volume 9, Number 1, January 1996.

[N2] A. Neeman, “Triangulated categories”, Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, (2001) viii+449 pp

[R] Rotman, “An introduction to homological algebra”, Pure and Applied Mathematics, 85. Academic Press, Inc., New York-London, 1979. xi+376 pp.

[S] W. Soergel, “Kategorie O, perverse Garben, und Moduln uber den Koinvarianten zur Weylgruppe”, Journal of the AMS 3, 421-445 (1990)

[Sm] S. Smalø, “Almost split sequences in categories of quivers”, Proceedings of the AMS Volume 129, nr 3, 695-698 (2000)

**Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, Bogotá, COLOMBIA**

*E-mail address: erbackel@uniandes.edu.co*

**Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, Bogotá, COLOMBIA**

*E-mail address: od.jaramillo114@uniandes.edu.co*