Time in (2+1)-Dimensional Quantum Gravity

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Abstract

General relativity in three spacetime dimensions is used to explore three approaches to the “problem of time” in quantum gravity: the internal Schrödinger approach with mean extrinsic curvature as a time variable, the Wheeler-DeWitt equation, and covariant canonical quantization with “evolving constants of motion.” (To appear in Proc. of the Lanczos Centenary Conference, Raleigh, NC, December 1993.)
As Karel Kuchař has explained elsewhere in these Proceedings, the nature of time in quantum gravity is at best obscure. Straightforward attempts at quantization lead to such absurdities as vanishing Hamiltonians and undefined inner products. The problems are not merely technical, but reflect an underlying conceptual issue: time translations in general relativity are “gauge symmetries,” and do not have an obvious physical meaning.

It may be that this problem cannot be resolved without a full-fledged quantum theory of gravity. But our confusion about the nature of time is itself an obstacle to the construction of such a theory. It is therefore useful to look for simpler models to explore possible approaches to the role of time in quantum gravity.

General relativity in 2+1 dimensions is one such model. Like realistic (3+1)-dimensional gravity, it is a generally covariant theory of spacetime geometry. But (2+1)-dimensional general relativity has only finitely many physical degrees of freedom, reducing the problem of quantization from quantum field theory to more elementary quantum mechanics.

1 Classical Theory

Let us start by investigating the nature of time in classical (2+1)-dimensional relativity. The basic simplification of 2+1 dimensions is that the full curvature tensor is determined by the Ricci tensor; in particular, any empty space solution of the Einstein equations is flat. For a spacetime with trivial topology, the flat metric is unique up to diffeomorphisms, and no dynamics is left. For a topologically nontrivial spacetime, however, a finite number of global geometric degrees of freedom remain.

We can describe these degrees of freedom in two ways, in which time is treated quite differently:

1. The vanishing of $R_{abcd}$ means that any point in a spacetime $M$ is contained in a coordinate patch isometric to flat Minkowski space with the standard metric $\eta_{ab}$. The transition function between two such patches must be an isometry of $\eta_{ab}$, i.e., an element of the Poincaré group ISO$(2,1)$. One such transition function is required for each noncontractible curve in $M$; in fact, these functions determine a group homomorphism

$$\rho : \pi_1(M) \to \text{ISO}(2,1),$$  \hspace{1cm} (1)

the holonomy, unique up to conjugation by an arbitrary fixed element of ISO$(2,1)$.

Such a construction, in which a manifold is built by gluing together geometric patches with isometries, is known to mathematicians as a geometric structure, and much can be learned from the mathematical literature. In particular, suppose $M \approx \mathbb{R} \times \Sigma$, where $\Sigma$ is a closed surface of genus $g > 0$. Then with appropriate assumptions about causal structure, a homomorphism determines a unique flat spacetime. Moreover, the space of ISO$(2,1)$ holonomies has the structure of a cotangent bundle, whose $(6g - 6)$-dimensional base space consists of the SO$(2,1)$ projections of the homomorphisms. The holonomies thus provide an invariant and time-independent “topological” description of the space of solutions of the field equations, essentially equivalent to that suggested by Witten in the Chern-Simons approach to (2+1)-dimensional gravity.
2. A very different approach to solving the field equations starts with standard metric variables in the ADM formalism. The spatial metric on $\Sigma$ may be decomposed as

$$g_{ij} = e^{2\lambda} f^* \tilde{g}_{ij}(\tau^\alpha)$$  \hspace{1cm} (2)

where $f$ is a spatial diffeomorphism, $e^{2\lambda}$ is an arbitrary conformal factor, and the $\tilde{g}_{ij}$ are a $(6g - 6)$-dimensional family of “standard” metrics of constant negative curvature. A similar decomposition exists for the canonical momentum $\pi^{ij}$. If we choose York’s time slicing, foliating $M$ by surfaces of constant mean extrinsic curvature $\text{Tr}K = T$, the standard constraints of general relativity completely determine all of the variables except the moduli $\tau^\alpha$ and their conjugate momenta $p_\alpha$ \[^5\]. The reduced phase space action becomes

$$S = \int dT \left( p_\alpha \dot{\tau}^\alpha - H(p, \tau, T) \right),$$ \hspace{1cm} (3)

with a calculable effective Hamiltonian $H$.

Although it is far from obvious, this metric picture is equivalent to the description in terms of geometric structures. Given a set of holonomies $\rho$, one may construct the spacetime $M$ with these holonomies, foliate it by surfaces of constant mean curvature, and compute the induced metric on the slice $\text{Tr}K = T$; the resulting moduli $\tau^\alpha$ then obey the equations of motion coming from the action (3). Conversely, starting with a set of moduli and momenta, one may compute the corresponding holonomies — via the Chern-Simons formalism, for example — and show that they give the correct description of the transition functions.

### 2 Quantum Gravity and Time

Let us now turn to the quantum theory. The simplest quantization, a version of the “internal Schrödinger approach,” starts with the reduced phase space action (3). This is the action for an ordinary finite-dimensional quantum mechanical system, and in principle its quantization is straightforward: we simply treat $\tau^\alpha$ and $p_\alpha$ as operators on a Hilbert space of square-integrable functions of the $\tau^\alpha$, with

$$[\hat{\tau}^\alpha, \hat{p}_\beta] = i\hbar \delta^\alpha_\beta.$$  \hspace{1cm} (4)

There are, to be sure, some practical problems — the Hamiltonian $H$ is a nonpolynomial function of the positions and momenta, often known only implicitly, and there are severe operator ordering ambiguities. More interesting is a basic conceptual issue. In this approach to quantization, the choice of time slicing is made classically, and only the reduced phase space variables $\tau$ and $p$ are subject to quantum fluctuations. It is not at all clear that different choices of time slicing will lead to equivalent theories. But (2+1)-dimensional gravity is simple enough that an answer no longer seems out of reach.

A second approach to quantization starts with the Wheeler-DeWitt equation. Rather than solving the Hamiltonian constraint classically, we now impose it as an operator restriction on physical states. In effect, we are permitting quantum fluctuations of the full spatial metric $g_{ij}$ and its conjugate momentum, not just the $\tau^\alpha$ and $p_\alpha$. 

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The resulting expression is surprisingly complex — in particular, it contains nonlocal terms coming from the momentum constraints — and it is hard to find exact solutions. Moreover, the proper choice of an inner product on the space of solutions is not obvious. It may be shown, however, that at least the simplest operator orderings and inner products give results that are not equivalent to those of reduced phase space quantization [6].

Finally, let us return to the idea of classical spacetimes as geometric structures, and use this picture as a starting point for covariant canonical quantization [7, 8], or “quantizing the space of classical solutions.” Recall that the space of holonomies is a cotangent bundle, with a base space consisting of the SO(2,1) holonomies. The quantization of such a bundle is straightforward: wave functions are sections of a line bundle over the base space, and cotangent vectors act as Lie derivatives. But it is not hard to see that the Hamiltonian in this picture is zero, giving us a “frozen time” description of the quantum theory.

In retrospect, this should not have been a surprise: our classical starting point was also time-independent, since the holonomies describe an entire spacetime at once. To see even the classical dynamics, we must pick a time slicing, label slices by a parameter $T$, and compute the moduli $\tau(\rho, T)$ as functions of the holonomies $\rho$ and the “time” $T$. Different slicings give different moduli, but this reflects the fact that genuinely inequivalent geometric questions are being asked.

The implications for quantum dynamics are now clear. We may interpret frozen time quantization as a Heisenberg picture, in which states are indeed time-independent. To construct time-dependent operators, we choose a slicing, solve for the classical moduli and momenta, and rewrite them as functions of operators $\hat{\rho}$, giving what Rovelli calls “evolving constants of motion” [9]. Eigenfunctions of the moduli $\hat{\tau}(\hat{\rho}, T)$ are then ordinary “time”-dependent Schrödinger picture wave functions appropriate for the chosen time slicing.

For the case of a spacetime with topology $\mathbb{R} \times T^2$, this program has been completed [10]. Interestingly, the result is not quite equivalent to reduced phase space quantization. Clearly, much remains to be learned from this model.

Acknowledgements

This research was supported in part by National Science Foundation Young Investigator grant PHY-93-57203 and Department of Energy grant DE-FG03-91ER40674.

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