REALISATIONS OF QUANTUM $GL_{p,q}(2)$ AND JORDANIAN $GL_{h,h'}(2)$

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The quantum group $GL_{p,q}(2)$ is known to be related to the Jordanian $GL_{h,h'}(2)$ via a contraction procedure. It can also be realised using the generators of the Hopf algebra $G_{r,s}$. We contract the $G_{r,s}$ quantum group to obtain its Jordanian analogue $G_{m,k}$, which provides a realisation of $GL_{h,h'}(2)$ in a manner similar to the $q$-deformed case.

1 Introduction

Non Standard (or Jordanian) deformations of Lie groups and Lie algebras has been a subject of considerable interest in the mathematical physics community. Jordanian deformations for $GL(2)$ were introduced in [1,2], its two parametric generalisation given in [3] and extended to the supersymmetric case in [4]. Non Standard deformations of $sl(2)$ (i.e. at the algebra level) were first proposed in [5], the universal $R$-matrix presented in [6-8] and irreducible representations studied in [9,10]. A peculiar feature of this deformation (also known as $h$-deformation) is that the corresponding $R$-matrix is triangular. It was shown in [11] that up to isomorphism, $GL_q(2)$ and $GL_h(2)$ are the only possible distinct deformations (with central determinant) of the group $GL(2)$. In [12], an interesting observation was made that the $h$-deformation could be obtained by a singular limit of a similarity transformation from the $q$-deformations of the group $GL(2)$. Given this contraction procedure, it would be useful to look for Jordanian deformations of other $q$-groups.

In the present paper, we focus our attention on a particular two parameter quantum group, denoted $G_{r,s}$, which provides a realisation of the well known $GL_{p,q}(2)$. We investigate the contraction procedure on $G_{r,s}$, in order to obtain its non standard counterpart. The generators of the contracted structure are employed to realise the two parameter non standard $GL_{h,h'}(2)$. This is similar to what happens in the $q$-deformed case.

2 Quantum $G_{r,s}$ and Realisation of $GL_{p,q}(2)$

The two parameter quantum group $G_{r,s}$ is generated by elements $a$, $b$, $c$, $d$, and $f$ satisfying the relations

\[
\begin{align*}
ab &= r^{-1}ba, \\
ac &= r^{-1}ca, \\
bc &= cb, \\
[a,d] &= (r^{-1} - r)bc
\end{align*}
\]
and
\[ af = fa, \quad cf = sfc \]
\[ bf = s^{-1}fb, \quad df = fd \]
Elements \( a, b, c, d \) satisfying the first set of commutation relations form a subalgebra which coincides exactly with \( GL_q(2) \) when \( q = r^{-1} \). The matrix of generators is
\[
T = \begin{pmatrix}
    a & b & 0 \\
    c & d & 0 \\
    0 & 0 & f
\end{pmatrix}
\]
and the Hopf structure is given as
\[
\Delta(T) = T \otimes T \\
\varepsilon(T) = 1
\]
The Casimir operator is defined as \( D = ad - r^{-1}bc \). The inverse is assumed to exist and satisfies \( \Delta(D^{-1}) = D^{-1} \otimes D^{-1}, \varepsilon(D^{-1}) = 1, \ S(D^{-1}) = D, \) which enables determination of the antipode matrix \( S(T) \), as
\[
S \begin{pmatrix}
    a & b & 0 \\
    c & d & 0 \\
    0 & 0 & f
\end{pmatrix} = D^{-1} \begin{pmatrix}
    d & -rc & 0 \\
    -r^{-1}c & a & 0 \\
    0 & 0 & Df^{-1}
\end{pmatrix}
\]
The quantum determinant \( \delta = Df \) is group-like but not central.

The quantum group \( G_{r,s} \) was proposed in \[13\] as a particular quotient of the multiparameter \( q \)-deformation of \( GL(3) \). The structure of \( G_{r,s} \) is interesting because it contains the one parameter \( q \)-deformation of \( GL(2) \) as a Hopf subalgebra and also gives a simple realisation of the quantum group \( GL_{p,q}(2) \) in terms of the generators of \( G_{r,s} \). There is a Hopf algebra morphism \( F \) from \( G_{r,s} \) to \( GL_{p,q}(2) \) given by
\[
F : G_{r,s} \mapsto GL_{p,q}(2)
\]
\[
F \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} \mapsto \begin{pmatrix}
    a' & b' \\
    c' & d'
\end{pmatrix} = f^N \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\]
The elements \( a', b', c' \) and \( d' \) are the generators of \( GL_{p,q}(2) \) and \( N \) is a fixed non-zero integer. The relation between the deformation parameters \( (p, q) \) and \( (r, s) \) is given by
\[
p = r^{-1} s^N, \quad q = r^{-1} s^{-N}
\]
This quantum group can, therefore, be used to realise both \( GL_q(2) \) and \( GL_{p,q}(2) \) quantum groups.
3 R-matrices and Contraction limits

The $R$-matrix of $G_{r,s}$ explicitly reads

$$R(G_{r,s}) = \begin{pmatrix}
    r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & r - r^{-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & s & 0 & 0 & r - r^{-1} & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & r - r^{-1} \\
    0 & 0 & 0 & 0 & 0 & 0 & s^{-1} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & r \\
\end{pmatrix}$$

with entries labelled in the usual numerical order (11), (12), (13), (21), (22), (23), (31), (32), (33). If we reorder the indices of this $R$-matrix with the elements in the order (11), (12), (21), (22), (13), (23), (31), (32), (33), then we obtain a block matrix, say $R_q$ which is similar to the form of the $GL_q(2)$ $R$-matrix with the $q$ in the $R_{11}$ position itself replaced by the $GL_q(2)$ $R$-matrix.

$$R_q = \begin{pmatrix}
    R(GL_r(2)) & 0 & 0 & 0 \\
    0 & S & \lambda I & 0 \\
    0 & 0 & S^{-1} & 0 \\
    0 & 0 & 0 & r \\
\end{pmatrix}$$

where $R(GL_r(2))$ is the $4 \times 4$ $R$-matrix for $GL_q(2)$ with $q = r$, $\lambda = r - r^{-1}$, $I$ is the $2 \times 2$ identity matrix and $S$ is the $2 \times 2$ matrix $S = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ where $r$ and $s$ are the deformation parameters. The zeroes are the zero matrices of appropriate order. The usual block structure of the $R$-matrix is clearly visible in this form.

It is well known [12] that the non standard $R$-matrix $R_h(2)$ can be obtained from the $q$-deformed $R_q(2)$ as a singular limit of a similarity transformation

$$R_h(2) = \lim_{q \to 1} (g^{-1} \otimes g^{-1}) R_q(2)(g \otimes g)$$

where $g = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$. Such a transformation has been generalised to higher dimensions [14] and has also been successfully applied to two parameter quantum groups. Here we apply the above transformation for the $G_{r,s}$ quantum group. Our starting
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point is the block diagonal form of the $G_{r,s} R$-matrix, denoted $R_q$

$$
R_q = \begin{pmatrix}
  r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & r & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & s & 0 & \lambda & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & \lambda & 0 \\
  0 & 0 & 0 & 0 & 0 & s^{-1} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & r \\
\end{pmatrix}
$$

where $\lambda = r - r^{-1}$. We apply to $R_q$ the transformation

$$(G^{-1} \otimes G^{-1})R_q(G \otimes G)$$

Here the transformation matrix $G$ is a $3 \times 3$ matrix and chosen in the block diagonal form

$$
G = \begin{pmatrix}
  g & 0 \\
  0 & 1 \\
\end{pmatrix}
$$

where $g$ is the transformation matrix for the two dimensional case. Substituting $\eta = \frac{m}{r-1}$ and then taking the singular limit $r \to 1, s \to 1$ (such that $\frac{1-s}{1-r} \to \frac{k}{m}$) yields the Jordanian $R$-matrix

$$
R_h = R(G_{m,k}) =
\begin{pmatrix}
  1 & m & -m & m^2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & m & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & -m & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & k & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & -k & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

where the entries are labelled in the block diagonal form (11), (12), (21), (22), (13), (23), (31), (32), (33). It is straightforward to verify that this $R$-matrix is triangular and a solution of the Quantum Yang Baxter Equation

$$
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
$$

It is interesting to note that the block diagonal form of $R(G_{m,k})$ embeds the $R$-matrix for the single parameter deformed $GL_h(2)$ for $m = h$. 

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4 Jordanian $G_{m,k}$ and Realisation of $GL_{h,h'}(2)$

A two parameter Jordanian quantum group, denoted $G_{m,k}$, can be formed by using the contracted $R$-matrix $R(G_{m,k})$ in conjunction with a $T$-matrix of the form

$$T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$$

The $RTT$-relations give the commutation relations between the generators $a$, $b$, $c$, $d$ and $f$.

$$[c,d] = mc^2, \quad [c,b] = m(ac + cd) = m(ca + dc)$$
$$[c,a] = mc^2, \quad [d,a] = m(d - a)c = mc(d - a)$$
$$[d,b] = m(d^2 - D)$$
$$[b,a] = m(D - a^2)$$

and

$$[f,a] = kcf, \quad [f,b] = k(df - fa)$$
$$[f,c] = 0, \quad [f,d] = -kcf$$

The element $D = ad - bc - mac = ad - cb + mcd$ is central in the whole algebra. The coalgebra structure of $G_{m,k}$ can be written as

$$\Delta(T) = T \hat{\otimes} T$$
$$\varepsilon(T) = 1$$

Adjoining the element $D^{-1}$ to the algebra enables determination of the antipode matrix $S(T)$,

$$S \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = D^{-1} \begin{pmatrix} d - mc & -b - m(d - a) + m^2c & 0 \\ -c & a + mc & 0 \\ 0 & 0 & Df^{-1} \end{pmatrix}$$

(The Hopf structure of $D^{-1}$ is $\Delta(D^{-1}) = D^{-1} \otimes D^{-1}, \varepsilon(D^{-1}) = 1, S(D^{-1}) = D_{\cdot}$)

It is evident that the elements $a$, $b$, $c$ and $d$ of $G_{m,k}$ form a Hopf subalgebra which coincides with non standard $GL(2)$ with deformation parameter $m$. This is exactly analogous to the $q$-deformed case where the first four elements of $G_{r,s}$ form the $GL_q(2)$ Hopf subalgebra. Again, the remaining fifth element $f$ generates the $GL(1)$ group, as it did in the $q$-deformed case, and the second parameter appears only through the cross commutation relations between $GL_m(2)$ and $GL(1)$ elements. Therefore, $G_{m,k}$ can also be considered as a two parameter Jordanian deformation of classical $GL(2) \otimes GL(1)$ group.

Now we wish to explore the connection of $G_{m,k}$ with the two parameter Jordanian $GL_{h,h'}(2)$. A Hopf algebra morphism

$$\mathcal{F} : G_{m,k} \rightarrow GL_{h,h'}(2)$$
of exactly the same form as in the $q$-deformed case, exists between the generators of $G_{m,k}$ and $GL_{h,h'}(2)$ provided that the two sets of deformation parameters $(h, h')$ and $(m, k)$ are related via the equation

$$h = m + Nk, \quad h' = m - Nk$$

Note that for vanishing $k$, one gets the one parameter case. In addition, using the above realisation together with the coproduct, counit and antipode axioms for the $G_{m,k}$ algebra and the respective homeomorphism properties, one can easily recover the standard coproduct, counit and antipode for $GL_{h,h'}(2)$. Thus, the non standard $GL_{h,h'}(2)$ group can in fact be reproduced from the newly defined non standard $G_{m,k}$. It is curious to note that if we write $p = e^h, q = e^{h'}, r = e^{-m}$ and $s = e^k$, then the relations between the parameters in the $q$-deformed case and the $h$-deformed case are identical.

5 Conclusions

We have applied the contraction procedure to the $G_{r,s}$ quantum group and obtained a new Jordanian quantum group $G_{m,k}$. The group $G_{m,k}$ has five generators and two deformation parameters and contains the single parameter $GL_h(2)$ as a Hopf subalgebra. Furthermore, we have given a realisation of the two parameter $GL_{h,h'}(2)$ through the generators of $G_{m,k}$ which also reproduces its full Hopf algebra structure. The results match with the $q$-deformed case.

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