ON A CONJECTURE OF IIZUKA

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Abstract. For a given odd positive integer $n$ and an odd prime $p$, we construct an infinite family of quadruples of imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$, $\mathbb{Q}(\sqrt{d+1})$, $\mathbb{Q}(\sqrt{d+4})$ and $\mathbb{Q}(\sqrt{d+4p^2})$ with $d \in \mathbb{Z}$ such that the class number of each of them is divisible by $n$. Subsequently, we show that there is an infinite family of quintuples of imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$, $\mathbb{Q}(\sqrt{d+1})$, $\mathbb{Q}(\sqrt{d+4})$, $\mathbb{Q}(\sqrt{d+36})$ and $\mathbb{Q}(\sqrt{d+100})$ with $d \in \mathbb{Z}$ whose class numbers are all divisible by $n$. Our results provide a complete proof of Iizuka’s conjecture (in fact a generalization of it) for the case $m = 1$. Our results also affirmatively answer a weaker version of (a generalization of) Iizuka’s conjecture for $m \geq 4$.

1. Introduction

It has been proved that there are infinitely many real (resp. imaginary) quadratic fields with class numbers divisible by a given positive integer (see [1, 3, 21, 24]). An analogous problem for tuples of quadratic fields arises from Scholz’s Spiegelungssatz [19]. In [13], Komatsu studied this problem for a pair of quadratic fields and proved that there are infinitely many pairs of quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{md})$ with $m, d \in \mathbb{Z}$ whose class numbers are divisible by 3. Later, he generalized this result in [14] to $n$-divisibility of the class numbers of pairs of imaginary quadratic fields. On the other hand, Iizuka [11] studied a slight variant of this problem and construct an infinite family of pairs of imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{d+1})$ with $d \in \mathbb{Z}$ whose class numbers are divisible by 3. Further, he posed the following conjecture in the same paper.

Conjecture 1.1. For any prime number $p$ and any positive integer $m$, there is an infinite family of $m+1$ successive real (or imaginary) quadratic fields $\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}), \cdots, \mathbb{Q}(\sqrt{d+m})$ with $d \in \mathbb{Z}$ whose class numbers are divisible by $p$. 

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In [4], Chattopadhyay and Muthukrishnan extended Iizuka’s result from pairs to certain triples of imaginary quadratic fields following the methods used in [11]. In other words, they gave an affirmative answer of a weaker version of Conjecture 1.1 for \( p = 3 \). It follows from a recent result of Iizuka, Konomi and Nakano [12] that Conjecture 1.1 is true for \( m = 1 \) when \( p = 3, 5, 7 \).

Very recently, Krishnamoorthy and Pasupulati [15] cleverly used [16, Theorem 1] and an extended version of [9, Theorem 3.2] to settled Conjecture 1.1 for \( m = 1 \).

In this paper, for a given odd prime \( p \) we construct an infinite family of quadruples of imaginary quadratic fields \( \mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}), \mathbb{Q}(\sqrt{d+4}) \) and \( \mathbb{Q}(\sqrt{d+4p^2}) \) with \( d \in \mathbb{Z} \) whose class numbers are all divisible by a given odd integer \( n \geq 3 \). This extends the results of [4, 12, 15] in both directions; from pairs to quadruples/quintuples of fields and from primes to odd integers. It also gives a proof of a weaker version of Conjecture 1.1 for any prime \( p \geq 3 \) (in fact for any odd integer \( n \geq 3 \)). The precise statement of our first result is the following:

**Theorem 1.1.** For any odd positive integer \( n \) and any odd prime \( p \), there are infinitely many quadruples of imaginary quadratic fields \( \mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}), \mathbb{Q}(\sqrt{d+4}) \) and \( \mathbb{Q}(\sqrt{d+4p^2}) \) whose class numbers are all divisible by \( n \).

Note that we can construct an infinite family of quintuples or higher tuples of imaginary quadratic fields whose class numbers are all divisible by \( n \) by choosing different values for the prime \( p \). For instance, utilizing Corollary 3.1 and following the proof of Theorem 1.1, we get the following:

**Theorem 1.2.** For a given odd integer \( n \geq 1 \), there are infinitely many quintuples of imaginary quadratic fields

\[
\left(\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}), \mathbb{Q}(\sqrt{d+4}), \mathbb{Q}(\sqrt{d+36}), \mathbb{Q}(\sqrt{d+100})\right)
\]

with \( d \in \mathbb{Z} \) whose class numbers are all divisible by \( n \).

Our method relies on the prominent result of Bilu, Hanrot and Voutier [2] concerning the primitive divisors of Lehmer numbers as well as on the solvability of certain Lebesgue-Ramanujan-Nagell type equations considered in [8]. This method does not allow us to include an imaginary quadratic field of the form \( \mathbb{Q}(\sqrt{d+m}) \) in the tuple, when \( m \) is non-square integer.
2. \(n\)-divisibility of the class-numbers of \(\mathbb{Q}(\sqrt{1-4U^n})\) and \(\mathbb{Q}(\sqrt{1-V^n})\)

Here, we recall some results concerning the \(n\)-divisibility of class numbers of the imaginary quadratic fields \(\mathbb{Q}(\sqrt{1-4U^n})\) and \(\mathbb{Q}(\sqrt{1-V^n})\). These results will be used in the proof of Theorem \ref{theorem1}.1

Theorem A. If \(n \geq 3\) is an odd integer, then for any integer \(U \geq 2\) the ideal class groups of the imaginary quadratic fields \(\mathbb{Q}(\sqrt{1-4U^n})\) contain an element of order \(n\).

In 1978, Gross and Rohrlich gave the outline of a proof of this theorem (see \cite[Theorem 5.3 and Remark 2]{Gross}). Their method of proof was based upon the affine points on the Fermat curve \(x^p + y^p = 1\) over the imaginary quadratic field \(\mathbb{Q}(\sqrt{1-4U^n})\). Later, Louboutin \cite{Louboutin} gave a complete proof of this theorem using number theoretic technique. It follows from Siegel’s theorem (see \cite{Siegel}) that for each integer \(d > 1\) there are at most finitely many positive integers \(U\) such that \(1-4U^n = -dX^2\). This ensures the infinitude of the above family of imaginary quadratic fields.

The \(n\)-divisibility of the class numbers of the family of imaginary quadratic fields \(\mathbb{Q}(\sqrt{1-V^n})\) was studied by Nagell \cite[Theorem 25]{Nagell} for any odd integers \(V \geq 3\) and \(n \geq 3\). Later, Murty \cite[Theorems 1 and 2]{Murty} proved that the class group of the imaginary quadratic field \(\mathbb{Q}(\sqrt{1-V^n})\) has an element of order \(n\) when either \(V^n - 1\) is square-free or its square part is \(< V^{n/2}/8\) for any odd integers \(V \geq 5\) and \(n \geq 3\). However, it follows from the fact \(DX^2 + 1 = V^n\) has no integer solution when both \(V\) and \(n\) are odd, except for \((V, n) = (5, 3), (7, 3), (13, 3)\) (see \cite{Cohn}), that the above conditions are no longer required. This fact also confirms that there are infinitely many such imaginary quadratic fields. Finally, this was elucidated by Cohn in \cite[Corollary 1]{Cohn} as follows:

Theorem B. Assume that \(n \geq 3\) and \(V \geq 3\) are odd integers. Then the class number of the imaginary quadratic field \(\mathbb{Q}(\sqrt{1-V^n})\) is divisible by \(n\), except for \((n, V) = (5, 3)\).

3. The divisibility of the class number of \(\mathbb{Q}(\sqrt{p^2-\ell^n})\)

Many special cases of the divisibility of the class number of the imaginary quadratic field \(\mathbb{Q}(\sqrt{x^2-y^n})\) have been studied with some restrictions on \(x, y\) and \(n\). One of such restrictions is that \(y\) is an odd prime (see \cite{Goldfeld} and the references therein), and hence none of the known results can be used to complete the proof of Theorem \ref{theorem1}.1. Thus, we consider a family of imaginary quadratic fields of the above form where \(y\) is not a prime and it will be a useful ingredient in the proof of Theorem \ref{theorem1}.1. Here, we mainly prove:
Theorem 3.1. Let $\ell > 1$ and $n > 1$ be odd integers, and $p$ an odd prime such that $\ell \equiv 3 \pmod{4}$, $\gcd(\ell, p) = 1$ and $p^2 < \ell^n$. Assume that $-d$ is the square-free part of $p^2 - \ell^n$. If $p \not\equiv \pm 1 \pmod{d}$, then the class number of $\mathbb{Q}(\sqrt{p^2 - \ell^n})$ is divisible by $n$.

Theorem 3.1 extends [3, Theorem 1.1], where the authors assumed that $\ell$ is an odd prime. This primality condition on $\ell$ restricts us to apply [3, Theorem 1.1] in the proof of Theorem 1.1. Further, following the proof of [3, Theorem 1.2], we make the following remark.

Remark 3.1. The family of imaginary quadratic fields discussed in Theorem 3.1 has infinitely many members.

Now for $p = 3, 5$, the condition ‘$p \not\equiv \pm 1 \pmod{d}$’ can be removed by applying 3.7 except for $(\ell, n) = (3, 3)$, and thus we have the following straightforward corollary.

Corollary 3.1. Let $\ell > 1$ and $n > 1$ be as in Theorem 3.1. For $x = 3, 5$ with $\gcd(\ell, x) = 1$, the class number of $\mathbb{Q}(\sqrt{x^2 - \ell^n})$ is divisible by $n$ except the case $(\ell, n) = (3, 3)$.

The proof of Theorem 3.1 relies on the prominent result of Bilu, Hanrot and Voutier [2] on existence of primitive divisors of Lehmer numbers.

3.1. Lehmer numbers and their primitive divisors. A pair $(\alpha, \beta)$ of algebraic integers is said to be a Lehmer pair if $(\alpha + \beta)^2$ and $\alpha \beta$ are two non-zero coprime rational integers, and $\alpha/\beta$ is not a root of unity. For a given positive integer $n$, the Lehmer numbers correspond to the pair $(\alpha, \beta)$ are defined as

$$L_n(\alpha, \beta) = \begin{cases} \alpha^n - \beta^n, & \text{if } n \text{ is odd}, \\ \frac{\alpha^n - \beta^n}{\alpha - \beta}, & \text{if } n \text{ is even}. \end{cases}$$

It is known that all Lehmer numbers are non-zero rational integers. Two Lehmer pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ are said to be equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 \in \{\pm 1, \pm \sqrt{-1}\}$. A prime divisor $p$ of $L_n(\alpha, \beta)$ is said to be primitive if $p \nmid (\alpha^2 - \beta^2)^2 L_1(\alpha, \beta) L_2(\alpha, \beta) \cdots L_{n-1}(\alpha, \beta)$.

The following classical result was proved in [2, Theorem 1.4].

Theorem C. The Lehmer number $L_n(\alpha, \beta)$ has primitive divisors for any integer $n > 30$.

Given a Lehmer pair $(\alpha, \beta)$, let $a = (\alpha + \beta)^2$ and $b = (\alpha - \beta)^2$. Then $\alpha = (\sqrt{a} \pm \sqrt{b})/2$ and $\beta = (\sqrt{a} \mp \sqrt{b})/2$. The pair $(a, b)$ is called the parameters corresponding to the Lehmer pair $(\alpha, \beta)$. The following lemma is extracted from [22, Theorem 1].
Lemma 3.1. Let $t$ be an odd integer such that $7 \leq t \leq 29$. If the Lehmer numbers $L_t(\alpha, \beta)$ have no primitive divisor, then up to equivalence, the parameters $(a, b)$ of the corresponding pair $(\alpha, \beta)$ are as follows:

(i) $(a, b) = (1, -7), (1, -19), (3, -5), (5, -7), (13, -3), (14, -22)$, when $t = 7$;
(ii) $(a, b) = (5, -3), (7, -1), (7, -5)$, when $t = 9$;
(iii) $(a, b) = (1, -7)$, when $t = 13$;
(iv) $(a, b) = (7, -1), (10, -2)$, when $t = 15$.

Let $F_k$ (resp. $L_k$) denote the $k$-th term in the Fibonacci (resp. Lucas) sequence defined by $F_0 = 0, F_1 = 1$, and $F_{k+2} = F_k + F_{k+1}$ (resp. $L_0 = 2, L_1 = 1$, and $L_{k+2} = L_k + L_{k+1}$), where $k \geq 0$ is an integer. The following lemma is a part of [2] Theorem 1.3.

Lemma 3.2. For $p = 3, 5$, let the Lehmer numbers $L_p(\alpha, \beta)$ have no primitive divisor. Then up to equivalence, the parameters $(a, b)$ of the corresponding pair $(\alpha, \beta)$ are:

(i) For $p = 3$, $(a, b) = \begin{cases} (1 + t, 1 - 3t) \text{ with } t \neq 1, \\ (3^k + t, 3^k - 3t) \text{ with } t \equiv 0 \pmod{3}, (k, t) \neq (1, 1); \end{cases}$

(ii) For $p = 5$, $(a, b) = \begin{cases} (F_{k-2}, F_{k-2} - 4F_k) \text{ with } k \geq 3, \\ (L_{k-2}, L_{k-2} - 4L_k) \text{ with } k \neq 1; \end{cases}$

where $t \neq 0$ and $k \geq 0$ are any integers and $\varepsilon = \pm 1$.

3.2. Two important lemmas. Given an integer $D \equiv 0, 1 \pmod{4}$, assume that $h^*(D)$ is the class number of binary quadratic primitive forms with discriminant $D$. Also for a square-free integer $d$, let $h(d)$ denote the class number of $\mathbb{Q}(\sqrt{d})$. Then we have (cf. [10, §16.13; p. 444]) the following:

Lemma 3.3. Let $d \equiv 2 \pmod{4}$ be a square-free positive integer. Then $h(-d) = h^*(-4d)$.

The following lemma is a special case of [8] Theorem 6.2 when $(D_1, D_2) = (1, -d)$.

Lemma 3.4. Let $d > 3$ and $\ell > 1$ be integers such that $\gcd(\ell, 2d) = 1$. If the equation $x^2 + dy^2 = \ell^e$, $x, y, z \in \mathbb{N}$, $\gcd(x, y) = 1$ has a solution, then all the solutions $(x, y, z)$ of this equation can be expressed as $x + y\sqrt{-d} = \varepsilon(a + \mu b\sqrt{-d})^t$, $z = st$, where $a, b, s, t$ are positive integers satisfying $a^2 + db^2 = \ell^e$, $\gcd(a, b) = 1$ and $s \mid h^*(-4d)$.
and \( \varepsilon, \mu \in \{-1, 1\} \).

3.3. **Proof of Theorem 3.1.** Let \( d \) be the square-free part of \( \ell^n - p^2 \). Then \( p^2 - \ell^n = -dr^2 \) for some \( r \in \mathbb{N} \), and thus \((x, y, z) = (p, r, n)\) is a positive integer solution of the equation
\[
x^2 + dy^2 = \ell^z, \quad \gcd(x, y) = 1.
\]

Thus by Lemma 3.4 we get
\[
p + r \sqrt{-d} = \varepsilon(a + \mu b \sqrt{-d})^t, \quad \varepsilon, \mu \in \{-1, 1\}
\]
with
\[
n = st, \quad s, t \in \mathbb{N},
\]
where \( a \) and \( b \) are positive integers satisfying
\[
a^2 + db^2 = \ell^s, \quad \gcd(a, b) = 1
\]
and
\[
s \mid h^{(4s)}(-4d).
\]

Since \( \ell \equiv 3 \pmod{4} \) and \( n \) is odd, so that \( p^2 - \ell^n = -dr^2 \) gives \( d \equiv 2 \pmod{4} \) and \( r \) is odd. Also both \( s \) and \( t \) are odd as \( n \) is odd. Further reading (3.3) modulo 4, we get \( a^2 + 2b^2 \equiv 3 \pmod{4} \) as \( \ell \equiv 3 \pmod{4} \) and \( s \) is odd, which ensures that both \( a \) and \( b \) are odd.

We now equate the real parts from both sides in (3.1) to get
\[
p = \varepsilon a \sum_{j=0}^{t-2j-1} \binom{t}{2j} (-db^2)^j.
\]

This implies \( a \mid p \) and thus \( a = 1, p \). If \( a = 1 \), then it becomes
\[
\sum_{j=0}^{t-2j-1} \binom{t}{2j} (-db^2)^j = p \varepsilon = \pm p.
\]

Reading (3.6) modulo \( d \), we get \( p \equiv \pm 1 \pmod{d} \), which contradicts to the assumption. Therefore \( a = p \) and thus (3.5) becomes
\[
\sum_{j=0}^{t-2j-1} \binom{t}{2j} p^{t-2j-1} (-db^2)^j = \varepsilon = \pm 1.
\]

As \( a = p \), so that (3.1) reduces to
\[
p + r \sqrt{-d} = \varepsilon(p + \mu b \sqrt{-d})^t, \quad \varepsilon, \mu \in \{-1, 1\}.
\]
We now assume that $\alpha = \mu b\sqrt{-d} + p$ and $\beta = \mu b\sqrt{-d} - p$. Then both $\alpha$ and $\beta$ are algebraic integers. Clearly, $(\alpha + \beta)^2 = -4db^2$ and $\alpha\beta = -p^2 - db^2 = -\ell^s$ (by (3.3)) are coprime rational integers. Furthermore, it follows from the following identity

$$\frac{4db^2}{\ell^s} = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2$$

that

$$\ell^s \left(\frac{\alpha}{\beta}\right)^2 + 2(\ell^s - 2db^2)\frac{\alpha}{\beta} + \ell^s = 0.$$ 

Since $\ell > 1$ and $\gcd(\ell^s, 2(\ell^s - 2db^2)) = \gcd(\ell^s, 4db^2) = \gcd(p^2 + db^2, 4db^2) = 1$, so that the last equation shows that $\frac{\alpha}{\beta}$ is not a root of unity. Therefore $(\alpha, \beta)$ is a Lehmer pair with parameters $(-4db^2, 4p^2)$ and thus the corresponding Lehmer number for $t$ is

$$\mathcal{L}_t(\alpha, \beta) = \frac{\alpha^t - \beta^t}{\alpha - \beta}$$

as $t$ is odd. Utilizing (3.8), we get

$$|\mathcal{L}_t(\alpha, \beta)| = 1.$$

This confirms that the Lehmer number $\mathcal{L}_t(\alpha, \beta)$ has no primitive divisor, and hence Theorem C and Lemma 3.1 (utilizing the fact that $(-4db^2, 4p^2)$ is the parameters) ensure that $t \in \{1, 3, 5\}$.

In case of $t = 5$, we get by Lemma 3.2 that $-4db^2 = F_{k-2e}$ or $-4db^2 = L_{k-2e}$. Clearly, none of these is possible.

Finally for $t = 3$, (3.7) implies that $p^2 - 3db^2 = \pm 1$. Reading it modulo 4, we see that ‘+’ sign is not possible, and thus $p^2 - 3db^2 = -1$. This is not possible by reading it modulo 3.

Therefore $t = 1$, and thus (3.2) and (3.4) together imply that $n \mid h^s(-4d)$. Thus, we complete the proof by Lemma 3.3.

4. PROOF OF THEOREM 1.1

We first fix an odd integer $n \geq 3$. We now define the set

$$\mathcal{N}_n = \{ k \in \mathbb{N} : n \mid h(1 - 4k^n) \}.$$ 

Then by Theorem A the set $\mathcal{N}_n$ is infinite.

Now for any $k \in \mathcal{N}_n$, we set $d = 4(1 - 4k^n)^n$. Then $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{1 - 4k^n})$ as $n$ is odd. Thus, by Theorem A there are infinitely many such $d$ satisfying $n \mid h(d)$. In other words, $\mathcal{F}(\mathcal{N}_n) = \{ \mathbb{Q}(\sqrt{1 - 4k^n}) : k \in \mathcal{N}_n \}$ is an infinite set.
Now we assume that \( U = 4k^n - 1 \) with \( k \in \mathcal{N}_n \). Then \( 1 - 4U^n = 1 - 4(4k^n - 1)^n = 4(1 - 4k^n)^n + 1 = d + 1 \), and thus by Theorem [A] we have \( n \mid h(d+1) \).

Again for \( k \in \mathcal{N}_n \), let us assume that \( V = 4k^n - 1 \). Then \( V \geq 3 \) and is odd, and thus by Theorem [B] we get \( n \mid h(1-V^n) \). Since \( 4(1-V^n) = 4 - 4(4k^n - 1)^n = 4 + 4(1 - 4k^n)^n = d+4 \) and \( \mathbb{Q}(\sqrt{4(1-V^n)}) = \mathbb{Q}(\sqrt{4(1-V^n)}) \), so that \( n \mid h(d+4) \).

Finally for any \( k \in \mathcal{N}_n \), let \( \ell = 4k^n - 1 \). Then \( \ell \equiv 3 \pmod{4} \) and hence by utilizing Theorem 3.1 we have \( n \mid h(p^2 - \ell^n) \) for any odd prime \( p \) satisfying \( p \not\equiv \pm 1 \pmod{d} \). Here, \( d \) is the square-free part of \( \ell^n - p^2 \). Now \( 4(p^2 - \ell^n) = 4p^2 - 4\ell^n = 4p^2 - 4(4k^n - 1)^n = d + 4p^2 \), which implies that \( \mathbb{Q}(\sqrt{d + 4p^2}) = \mathbb{Q}(\sqrt{4(p^2 - \ell^n)}) \), and thus \( n \mid h(d+4p^2) \). This completes the proof of Theorem 1.1.

5. CONCLUDING REMARKS

In [23], Xie and Chao studied Conjecture 1.1 and proved the following result using Yamamoto’s [24] construction.

**Theorem D.** For any odd positive integer \( n \) and any positive integer \( m \), there are infinitely many pairs of imaginary fields \( \mathbb{Q}(\sqrt{d}) \) and \( \mathbb{Q}(\sqrt{d+m}) \) whose class groups have an element of order \( n \) respectively.

Theorem D can be viewed as a weaker variant of a generalization of Conjecture 1.1. For \( m = 1 \), it provides a generalization of the main result of [15] though [23] appeared before [15]. In other words, Theorem D gives a complete proof of the following generalization of Conjecture 1.1 for \( m = 1 \) and a proof of a weaker version of the same for \( m \geq 2 \).

**Conjecture 5.1.** For any odd integer \( n \geq 3 \) and any integer \( m \geq 1 \), there is an infinite family of \( m+1 \) successive imaginary (or real) quadratic fields

\[
\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d+1}), \ldots, \mathbb{Q}(\sqrt{d+m})
\]

with \( d \in \mathbb{Z} \) whose class numbers are all divisible by \( n \).

Theorem 1.1 offers a constructive proof of Conjecture 5.1 for \( m = 1 \). This theorem also offers a proof of a weaker version of Conjecture 5.1 for \( m = 4 \), which has missed the families of imaginary quadratic fields \( \mathbb{Q}(\sqrt{d+2}) \) and \( \mathbb{Q}(\sqrt{d+3}) \) from the complete proof. When \( m = 4p^2 \) with \( p \) an odd prime, Theorem 1.1 presents a proof of a weaker version of Conjecture 5.1. We complete this paper by the following remark.

**Remark 5.1.** For a given positive integer \( m \), let \( p_m \) denote the largest prime less than or equal to \( m \) and \( \pi(m) \) the prime-counting function. Then for a given positive odd integer
n, our construction gives an infinite family of at least \((\pi(m) + 2)\)-tuples of imaginary quadratic fields,
\[
(Q(\sqrt{d}), Q(\sqrt{d + 1}), Q(\sqrt{d + 4}), Q(\sqrt{d + 36}), \ldots, Q(\sqrt{d + 4p_m^2}))
\]
with \(d \in \mathbb{Z}\) whose class numbers are all divisible by \(n\).

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