Categorified Young symmetrizers and stable homology of torus links II

Michael Abel 1 · Matthew Hogancamp 2

Abstract We construct complexes $P_{1^n}$ of Soergel bimodules which categorify the Young idempotents corresponding to one-column partitions. A beautiful recent conjecture (Flag Hilbert schemes, colored projectors and Khovanov–Rozansky homology. arXiv:1608.07308, 2016) of Gorsky–Negut–Rasmussen relates the Hochschild homology of categorified Young idempotents with the flag Hilbert scheme. We prove this conjecture for $P_{1^n}$ and its twisted variants. We also show that this homology is also a certain limit of Khovanov–Rozansky homologies of torus links. Along the way we obtain several combinatorial results which could be of independent interest.

Keywords Hochschild cohomology · Young symmetrizers · Categorification · Torus knots

Mathematics Subject Classification Primary 13D03; Secondary 57M27

List of symbols

\((k)\) Grading shift by $k$ (§2.1)
\(\alpha_i\) $x_i - x_{i+1} = q_{i,i+1}$ (§4.4)
Cone Mapping cone (§2.2)
\(\text{deg}_h\) Homological degree (§2.6)

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\(\ell\)  
\(\frac{1}{2}n(n - 1)\)  (§2)

\(\text{End}(M)\)  
Graded endomorphism ring of \(M\)  (§2.2)

\(\text{FT}_n\)  
Rouquier complex of the full-twist braid \(\text{HT}_n\)  (§3.3)

\(\text{gr}\)  
Associated graded  (§2.6)

\(\mathcal{H}_n\)  
Hecke algebra of \(S_n\)  (§2.3)

\(\text{HH}(M)\)  
\(\bigoplus_{k=0}^{\infty} \text{HH}^k(R; M)\)  (§4.1)

\(\text{HH}^k(R; M)\)  
kth Hochschild cohomology of the \((R, R)\)-bimodule \(M\)  (§4.1)

\(\text{HHH}\)  
\(H \circ \text{HH}\)  (§4.1)

\(\text{HHH}^0\)  
\(H \circ \text{HH}^0 = H \circ \text{Hom}\)  (§4.1)

\(\text{hocolim}_k A_k\)  
Homotopy colimit of the direct system \(\{A_k, f_k\}\)  (§3.3)

\(\text{Hom}(C, D)\)  
Bigraded chain complex of bihomogeneous \((R, R)\)-chain maps  (§4.3)

\(\text{Hom}(M, N)\)  
Graded space of homogeneous \((R, R)\)-bimodule morphisms  (§4.3)

\(\text{HT}_n\)  
Rouquier complex of the half-twist braid \(\text{F}_{w_0}\)  (§3.3)

\(\text{HT}_{min}\)  
Minimal complex of \(\text{HT}_n\)  (§3.3)

\(\mathcal{I}\)  
Full subcategory of \(\mathcal{K}^- (\mathbb{S}\text{Bim}_n)\) of chain complexes whose chain bimodules are grading shifts of direct sums of \(B_{w_0}\)  (§2.2)

\(\mathcal{I}^-\)  
All objects \(C \in \mathcal{K}^- (\mathbb{S}\text{Bim}_n)\) such that \(B_{w_0} C \simeq 0\)  (§2.2)

\(\mathcal{J}_k\)  
Full subcategory of \(\mathcal{K}^- (\mathbb{S}\text{Bim}_n)\) of chain complexes whose chain bimodules are shifts of direct sums of \(R \otimes R^k\)  (§2.5)

\(\mathcal{K}(A)\)  
Homotopy category of the additive category \(A\)  (§2)

\(\langle k \rangle\)  
Homological degree shift by \(k\)  (§2.1)

\(P(M)\)  
Poincaré series of \(M\)  (§4.4)

\(\partial_i\)  
Divided difference operator \(\partial_i(f) = (f - s(f))/\alpha_i\)  (§5.1)

\(\partial_{1, 2, \ldots, n-1}\)  
\(\partial_1 \circ \cdots \circ \partial_{n-1}\)  (§5.1)

\(\phi_b, \psi_b\)  
Special quasi-isomorphisms defined in Definition 3.11  (§3.2)

\(P_{1^n}\)  
A resolution of \(R\) by free graded \(R \otimes R^w\) \(R\)-modules  (§2.3)

\(P_{1^n}'\)  
The dual complex of \(P_{1^n}\)  (§2.6)

\(\mathbb{S}\text{Bim}_n\)  
Category of Soergel bimodules  (§2.1)

\(\oplus_{\mathbb{Q}}\)  
(§2.1)

\(\theta_k\)  
An odd variable of degree \(tq^{-2}\)  (§2.4)

\(\tilde{A}_n, \tilde{M}_n\)  
Reduced versions of \(A_n\) and \(M_n\)  (see Remark 2.48)  (§2.6)

\(\text{Tot}(P)\)  
Convolution of a sequence of chain complexes  (§2.5)

\(\xi_k, \xi_k'\)  
Odd variables of degree \(q^{-2i}a\) and \(q^{2i}a^{-1}\) respectively  (§4.1)

\(x\)  
List of variables \(x_1, \ldots, x_n\)  (§2.1)

\(y\)  
List of variables \(y_1, \ldots, y_n\)  (§2.1)

\(\perp \mathcal{I}\)  
All objects \(C \in \mathcal{K}^- (\mathbb{S}\text{Bim}_n)\) such that \(B_{w_0} C \simeq 0\); equivalent to \(\mathcal{I}^-\)  (§2.2)

\(A_n\)  
Superpolynomial \(dg\)-algebra defined in Definition 2.44  (§2.6)

\(a_{ij}\)  
Special polynomials defined in Definition 2.40  (§2.5)

\(AB\)  
\(A \otimes_R B\)  (§2.1)

\(B_x\)  
Soergel bimodule \(R \otimes R^w R(-1)\)  (§2.1)

\(B_{w_0}\)  
Soergel bimodule \(R \otimes R^w R(-\ell)\)  (§2.1)

\(B_{w_1}\)  
\(R \otimes R^{w_{n-1}} R(-\ell + n - 1)\)  (§2.1)

\(C(i, j)\)  
\(C\) shifted up by \(i\) in \(q\)-degree and \(j\) in homological degree  (§4.1)

\(d_A\)  
Differential on \(A_n\) (see Definition 2.44)  (§2.6)
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1 Introduction

The computation of Khovanov–Rozansky homology of torus links is a challenging problem which is made even more interesting by the conjectures in [15, 25] relating these homology groups with certain Hilbert schemes and Cherednik algebras. It seems a more tractable problem to compute the stable limit of the homology of the \((n, m)\) torus link as \(m \to \infty\). The stable limit of homologies of Khovanov–Rozansky homology has a longer history, being conjectured to exist by [7] and proven to exist (in the \(sl_2\) case, that is, the original Khovanov homology) in [32]. Rozansky showed that in the \(sl_2\) case, the stable limit can be computed from a categorified Jones-Wenzl projector [30]. Similar limits were shown to exist by Rose [27] in the \(sl_3\) case and by Cautis [5] in the case of arbitrary \(sl_N\). Recently, the second author [18] proved existence of this stable limit in triply-graded homology, and proved a stable version of conjectures in [15, 25].

In this paper we investigate a second kind of stable limit of Khovanov–Rozansky homology of torus links. In the \(sl_2\) setting, the existence of two different stable limits was due to Rozansky [29]; this second limit computes Hochschild homology of Khovanov’s ring \(H^n\) [21]. In this paper we show that there is a second stable limit in the triply graded setting, which computes a certain kind of Hochschild homology associated to polynomial rings. This is explained below. This stable limit is also explicitly related to Hilbert schemes by a recent conjecture of Gorsky-Rasmussen [14] (see Conjecture 1.8). See §1.1 for a discussion of Young symmetrizers and their categorifications, and see §1.2 for the relation with stable homologies of torus links.

Fix an integer \(n \geq 1\) and let \(R = \mathbb{Q}[x_1, \ldots, x_n]\) be graded by setting \(\deg(x_k) = 2\). Note that \(W := S_n\) acts on \(R\) by permuting variables. Let \(R^W \subset R\) denote the ring of symmetric polynomials. The main results of this paper show that the triply graded homology of the \((n, nk)\)-torus links approach a stable limit as \(k \to \infty\) and that this limit is isomorphic to the Hochschild cohomology of the graded \(R^W\)-algebra \(R\) with coefficients in itself, tensored with an exterior algebra. More precisely, the stable limit is \(\text{Ext}^*_{R^W \otimes_{\mathbb{Q}} R}(R, R) \otimes_{\mathbb{Q}} \Lambda[\xi_1, \ldots, \xi_n]\). In fact, we are able to compute this homology explicitly. The first step is to find an explicit injective resolution of the
graded $R \otimes_{R^w} R$-module $R$; this injective resolution is one of the main subjects of this paper.

**Definition 1.1** Let $R \rightarrow P_1^\vee$ denote an injective resolution of $R$, thought of as a graded $R \otimes_{R^w} R$-module.

We note for future reference that $R \otimes_{R^w} R$ is self-injective. The complex $P_1^\vee$ is graded infinite, being supported in all non-negative homological degrees. This complex categorifies the one-column Young symmetrizer in a sense explained below.

Let $\text{Ch}(R\text{-mod-}R)$ denote the category of complexes of finitely generated, graded $(R, R)$-bimodules. Our first main theorem says that $P_1^\vee$ is related to a certain braid group action on $\text{Ch}(R\text{-mod-}R)$. Associated to each $n$-strand braid, Rouquier [28] defines a complex $F(\beta) \in \text{Ch}(R\text{-mod-}R)$ such that $F(\beta) \otimes RF(\beta') \simeq F(\beta'\beta)$ (actually $F(\beta)$ lies in the full subcategory of complexes of Soergel bimodules; see below). We have:

**Theorem 1.2** Let $F_T \in \text{Ch}(R\text{-mod-}R)$ denote the Rouquier complex associated to the (positive) full twist braid. Then $\{F_T \otimes k\}_{k=0}^\infty$, after the appropriate grading shifts, can be made into a direct system with homotopy colimit $P_1^\vee$.

**Remark 1.3** Dually, one may consider a complex $P_1$ which is defined to be a projective resolution $P_1 \rightarrow R$ viewed as a graded $R \otimes_{R^w} R$-module. Analogously, if we let $F_{-1}$ denote the Rouquier complex of the negative full twist braid then $\{(F_{-1}) \otimes k\}_{k=0}^\infty$ can made into an inverse system with homotopy limit $P_1$, after the appropriate grading shifts. We will prove in §2.6 that the duality functor $(\cdot)^\vee : \text{Ch}(R\text{-mod-}R) \rightarrow \text{Ch}(R\text{-mod-}R)$ sends $P_1$ to $P_1^\vee$.

Our notation prefers complexes which are bounded above, in order to be consistent with [18]. However, in this paper the complex $P_1^\vee$ is often nicer than $P_1$, since $P_1^\vee$ is an algebra in the homotopy category $\mathcal{K}^+(R\text{-mod-}R)$, while $P_1$ is a coalgebra. This is discussed in Remark 1.17. The algebra structure makes writing our following main results easier. See Example 4.13 for a sample computation in the non-dual, $n = 2$ case.

Let us recall the construction of Khovanov and Rozansky’s triply-graded link homology given in [22]. Hochschild cohomology of bimodules defines a functor $\text{HHH}$ from $\text{Ch}(R\text{-mod-}R)$ to the category of triply-graded vector spaces, and $\text{HHH}(F(\beta))$ is a well-defined invariant of the oriented link $\hat{\beta}$, up to isomorphism and overall shift of triply-graded vector spaces. This invariant is called Khovanov–Rozansky homology, hereafter referred to as KR homology.

**Remark 1.4** Actually, in [22] the results are stated in terms of Hochschild homology. In this paper, as in [18], we prefer Hochschild cohomology to Hochschild homology; there is not much difference since for polynomial rings the two are isomorphic up to regrading. Our convention will ensure that $\text{HHH}$ applied to our projector is a graded commutative algebra, rather than an algebra up to regrading.

**Theorem 1.2** allows us to relate KR homology of torus links with $\text{HHH}(P_1^\vee)$:
**Theorem 1.5** The KR homology of the \((n, nk)\) torus links stabilizes as \(k \to \infty\). The stable limit is isomorphic to \(\text{HHH}(P^\vee_{1,n})\).

In order to state our results for the \((n, nk+m)\) torus links, we first introduce an action of the symmetric group. Let \(w \in S_n\), then for any complex \(C \in \text{Ch}(R\text{-mod}-R)\), let \(w(C)\) denote the complex obtained from \(C\) by twisting the right \(R\)-action by \(w^{-1}\), that is, \(f \cdot c \cdot g = fcw^{-1}(g)\). We refer to \(w(P^\vee_{1,n})\) as a twisted projector. The connection between this action of the symmetric group and the action of the braid group by Rouquier complexes is established in §3.2. An elementary property of Hochschild cohomology implies that \(\text{HHH}(w(P^\vee_{1,n}))\) depends only on the conjugacy class (cycle type) of \(w\). Using this, we have:

**Theorem 1.6** Fix integers \(1 \leq m \leq n\), and let \(w \in S_n\) be an \(n\)-cycle. After an appropriate shift in tridegree, the triply-graded KR homology of the \((n, nk+m)\)-torus links have a stable limit as \(k \to \infty\), given by \(\text{HHH}(w^m(P^\vee_{1,n}))\). This limit depends only on the number of components \(\gcd(m, n)\), up to isomorphism of triply graded vector spaces.

Thus, the problem of computing stable homology of torus links reduces to the computation of \(\text{HHH}(w(P^\vee_{1,n}))\) for the appropriate \(w \in S_n\). To compute this we first reduce to the computation of the Hochschild degree zero part \(\text{HHH}^0(P^\vee_{1,n})\). Here and below, we use the same notation for both exterior and polynomial algebras, and will distinguish them by declaring the variables as odd or even, respectively.

**Proposition 1.7** Let \(\xi_k\) denote an odd variable of degree \(q^{-2k}a\), where we use \(a\) to denote Hochschild degree. If \(C \in \text{Ch}(R\text{-mod}-R)\) is a complex whose chain bimodules are direct sums of copies of \(R \otimes_{Rw} R\) with shifts, then \(\text{HHH}(C) = \Lambda[\xi_1, \ldots, \xi_n] \otimes_{\mathbb{Q}} \text{HHH}^0(C)\).

This is proven in §4.2. The following beautiful conjecture was communicated to us by Gorsky et al. [14]:

**Conjecture 1.8** For each \(1 \leq i < j \leq n\), let \(v_{ij}\) denote an even variable of degree \(\deg(v_{ij}) = q^{2i-2j-2t^2}\). Let \(E\) denote the ring \(E = R[v_{ij}]/J\) where \(J\) is the ideal generated by the entries of the commutator of the following matrices:

\[
X = \begin{bmatrix}
x_1 & 1 & 0 & \cdots & 0 & 0 \\
0 & x_2 & 1 & \cdots & 0 & 0 \\
0 & 0 & x_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & x_{n-1} & 1 \\
0 & 0 & 0 & \cdots & 0 & x_n
\end{bmatrix}, \quad V = \begin{bmatrix}
0 & v_{12} & v_{13} & \cdots & v_{1,n-1} & v_{1n} \\
0 & 0 & v_{23} & \cdots & v_{2,n-1} & v_{2n} \\
0 & 0 & 0 & \cdots & v_{3,n-1} & v_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & v_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

Then up to an overall grading shift, \(\text{HHH}^0(w(P^\vee_{1,n}))\) is isomorphic to the quotient of \(E\) in which we identify \(x_{w(i)}\) with \(x_i\) for all \(1 \leq i \leq n\).

Conjecture 1.8 is a special case of a more general conjecture, also due to Gorsky–Negut–Rasmussen. Conjecturally, there exist complexes of Soergel bimodules \(P_T\),...
indexed by standard tableaux, which categorify the Young symmetrizers (see §1.1). For each standard Young tableau $T$, Gorsky et al. [14] define an explicit bigraded algebra $E_T$ and conjecture that $E_T \cong \text{End}(P_T)$, where $\text{End}(P_T)$ denotes the homology of the bigraded complex of endomorphisms. Though it is not strictly necessary in order to understand our work, we outline the geometric origin of the algebras $E_T$ (a special case of which appears in the statement of Conjecture 1.8.

The flag Hilbert scheme $\text{FHilb}^n(\mathbb{C}^2)$ of $n$ points on $\mathbb{C}^2$ is the moduli space of flags

$$\text{FHilb}^n(\mathbb{C}^2) = \{ C[x, y] = I_0 \supset I_1 \supset \cdots \supset I_n | \dim(I_k/I_{k+1}) = 1 \}.$$  

$\text{FHilb}^n(\mathbb{C}^2, \ell)$ is defined as the subscheme of $\text{FHilb}^n(\mathbb{C}^2)$ such that all $I_k$ are set-theoretically supported on the line $\ell = \{ y = 0 \} \subset \mathbb{C}^2$. There is a natural $\mathbb{C}^* \times \mathbb{C}^*$-action on $\text{FHilb}^n(\mathbb{C}^2, \ell)$. The fixed points $z_T$ of this action are parameterized by standard Young tableaux of size $n$, and $E_T$ is the ring of functions on an open chart containing $z_T$.

One of the main contributions of this paper is an exact combinatorial description of the projector $P_1^\lor$ (see §2.6 and also §1.3 of this introduction). Using this description we can prove the above conjecture:

**Theorem 1.9** Conjecture 1.8 is true. Furthermore, there are isomorphisms of bigraded rings

$$\text{HHH}^0(P_1^\lor) \cong E \cong \text{End}(P_1^\lor) \cong R[u_2, \ldots, u_n]/I$$

where $I$ is the ideal generated by the elements $\sum_{i=1}^j u_i a_{i,j}(x, w(x))$ for $1 \leq j \leq n$, $a_{i,j}(x, y)$ are certain polynomials explained below, and $\text{End}(P_1^\lor)$ here denotes the bigraded ring of bihomogeneous chain endomorphisms of $P_1^\lor$, modulo homotopies. The degrees are $\deg(u_k) = t^2q^{-2k}$.

For more of an explanation of the relation between Hochschild cohomology of $P_1^\lor$ and its ring of endomorphisms, see §4.3.

The special polynomials $a_{i,j}(x, y)$ are defined in Definition 2.40, and are essentially the double Schubert polynomials $\mathcal{S}_{w_i}(x, y)$ (See [24] for more information). More precisely, if $w_{i,n}$ denotes the cycle $(n, n-i+1, n-i+2, \ldots, n-1) \in S_n$ and $w_0 \in S_n$ is the longest word, then $a_{i,n} = (-1)^i \mathcal{S}_{w_{i,n}}(w_0(x), y)$. In Proposition 4.25 we explicitly give the relation between $a_{i,j}$ and $v_{i,j}$. We state explicitly the following important special case of torus knots, in which the stable homology is a free superpolynomial algebra:

\footnote{Note added in revision: it has been pointed out that the rings $E_T$ are not always complete intersections, which complicates the conjectural relation between $E_T$ and the complexes $P_T$ somewhat. The conjecture now states that a certain Koszul complex has zeroth homology isomorphic to $E_T$, and total homology isomorphic to $\text{End}(P_T)$. However, if $T$ is the one-row or one-column tableau then the statement of the conjecture remains unchanged.}
Corollary 1.10  If $w \in S_n$ is an $n$-cycle, then

$$\text{HHH} \left( w \left( P^\vee_{1^\ast} \right) \right) \cong R[u_2, \ldots, u_n, \xi_1, \ldots, \xi_n] / I,$$

where $I = I_w$ is the ideal generated by $x_i - x_{i-1}$ ($1 \leq i \leq n - 1$). In particular, this homology is isomorphic to a stable limit of KR homologies of torus knots. The degrees are $\deg(u_k) = t^2 q^{-2k}$ and $\deg(\xi_k) = aq^{-2k}$.

In [1], we use $P_{1^n}$ to give a categorification of the $\Lambda^n$-colored HOMFLYPT polynomial. This polynomial was first categorified by Webster and Williamson using a geometric framework [34]. These two theories are different however, as the homology of the unknot in the theories are quite different.

We now explore the different pieces of this construction in more detail, including a combinatorial description of $P^\vee_{1^n}$. Along the way we compare our results to the results of [18].

1.1 Hecke algebras, Young symmetrizers, and categorification

In this section we discuss our work in the broader context of Hecke algebras, quantum topology, and categorification.

The Hecke algebra $\mathcal{H}_n$ is a $\mathbb{Z}[q, q^{-1}]$-algebra which is a $q$-deformation of the group algebra $S_n$. Inside $\mathbb{Q}[S_n]$ there is a standard collection of primitive idempotents $p_T \in \mathbb{Q}[S_n]$ called Young symmetrizers, indexed by standard tableaux $T$ on $n$ boxes. These idempotents have the property that $\mathbb{Q}[S_n] \cong \bigoplus_T \mathbb{Q}[S_n]p_T$ is a decomposition of the regular representation of $S_n$ into irreducible summands, and $\mathbb{Q}[S_n]p_T$ is isomorphic to the Specht module $S^\lambda$ whenever $T$ has shape $\lambda$. After extending scalars from $\mathbb{Z}[q, q^{-1}]$ to $\mathbb{Q}(q)$, these idempotents admit $q$-analogues in the Hecke algebra [2,17] which play the same role. From now on we let $\mathcal{H}_n$ denote the Hecke algebra with coefficients in $\mathbb{Q}(q)$, where $q$ is a formal indeterminate, and denote the $q$-Young symmetrizers by $p_T \in \mathcal{H}_n$. The $q$-Young symmetrizers play an important role in quantum topology, where “cabling and inserting $p_T$” defines the $\lambda$-colored HOMFLYPT polynomial. One also has the central idempotents $p_\lambda = \sum p_T$, where the sum is over tableaux with shape $\lambda$.

Remark 1.11  For each integer $N \geq 1$, one can consider the quotient of $\mathcal{H}_n$ by the ideal generated by the $p_\lambda$, where $\lambda$ has more than $N$ parts. We call this the $sl_N$ quotient (which we denote by $\mathcal{H}_{n,N}$), since the resulting algebra is isomorphic to $\text{End}_{U_q(sl_N)}(V^\otimes n)$, where $V$ is $q$-deformation of the standard $N$-dimensional representation of $sl_N$. A proof of the preceding fact can given using the quantum version of Schur–Weyl duality (see [20]). In the case $N = 2$, $\mathcal{H}_{n,2}$ is the Temperley–Lieb algebra. In this way, Young symmetrizers act as projection operators on $V^\otimes n$. The dominance order on partitions induces a partial order on the central idempotents $p_\lambda$. The maximal idempotent is the one-row symmetrizer $p_{(n)}$, which acts as the projection of $V^\otimes n$ onto the $q$-symmetric power $\text{Sym}^n(V)$. At the other extreme we have the one-column idempotent $p_{1^n}$, which acts as projection onto the $q$-exterior power $\Lambda^n(V)$, which is zero unless $N \geq n$. Generally, for each $n$, $N$ there is a unique minimal partition $\lambda$ of $n$ such that $p_\lambda$ is
nonzero in $\mathcal{H}_{n,N}$. If $N \geq n$ then $\lambda_{\min}$ is the one-column partition $1^n$, and if $n = Nk + r > N$ then $\lambda_{\min}$ is the partition with column lengths $N$, \ldots, $N, r$. This idempotent projects $V^{\otimes n}$ onto its $\Lambda^r(V)$-isotopic summand. In particular, if $n = kN$ then $\lambda_{\min}$ is the rectangular $k \times N$ partition, and $p_{\lambda_{\min}}$ projects onto the invariants in $V^{\otimes n}$.

The category $\mathcal{Bim}_n$ of Soergel bimodules is a certain category of graded $(R, R)$-bimodules (explained more in §2.1) which categorifies the Hecke algebra $\mathcal{H}_n$. More precisely, associated to an additive category $\mathcal{A}$ we have the split Grothendieck group $K_0(\mathcal{A})$. If $C = (C_\bullet, d)$ is a finite chain complex over $\mathcal{A}$, then there is a well-defined Euler characteristic $[C] := \sum_i (-1)^i [C_i] \in K_0(\mathcal{A})$, which depends only on the chain homotopy type of $C$. If $\mathcal{A}$ is an additive, monoidal category with grading shift (1) and tensor product $1$, then $K_0(\mathcal{A})$ is a $\mathbb{Z}[q, q^{-1}]$-algebra, and one has the following:

1. $[C \oplus D] = [C] + [D]
2. [C \otimes D] = [C][D]
3. [C(i)j] = (-1)^i q^j [C]$, where $i$ and $j$ are homological and $q$-degree degree shifts, respectively.

for all finite complexes $C$, $D$ over $\mathcal{A}$. We will say that a complex $C$ categorifies $[C]$, that a homotopy equivalence $C \simeq D$ categorifies the identity $[C] = [D]$, an additive functor $\mathcal{A} \to \mathcal{B}$ categorifies the induced linear map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$, and so on. Most of the complexes we consider are infinite (but bounded in one direction); nonetheless, their Euler characteristics will be well-defined as formal power series in $q$. We will not make this precise, as the notion of Euler characteristic serves only to motivate our use of the word “categorification.” Such issues are discussed for example in [33].

**Remark 1.12** The $\mathfrak{sl}_N$ quotient $\mathcal{H}_{n,N}$ is categorified by $\mathfrak{sl}_N$ foams [26] or the homotopy category of $\mathfrak{sl}_N$ matrix factorizations $\text{HMF}_{n,N}$ [23], and the quotient $\mathcal{H}_n \to \mathcal{H}_{n,N}$ is categorified by a monoidal functor $F_{n,N} : \mathcal{Bim}_n \to \text{HMF}_{n,N}$ [3]. Conjecturally, there exist complexes $P_T$ of Soergel bimodules, indexed by standard tableaux $T$, which categorify the $q$-Young symmetrizers $p_T$. The images of $P_T$ under the functors $F_{n,N}$ will be categorified $\mathfrak{sl}_N$ idempotents (such as those constructed in [6] for $\mathfrak{sl}_2$). As was mentioned in the first paragraph of the introduction, the maximal idempotent $P_{(n)}$ can be categorified as a limit of full twist complexes. We conjecture that, in $\text{HMF}_{n,N}$, the minimal idempotent $P_{\lambda_{\min}}$ can also be constructed in terms of a limit of full twists. In the case $N = 2$ and $n$ even, such a limit was constructed by Rozansky. In this paper, we accomplish this goal in the case $N \geq n$. The case of general $N$ remains open. Note that our projectors relate to Rozansky’s [29] only in the case that $n = N = 2$.

In the Hecke algebra, the minimal idempotent $p_{1^n}$ can be described as the unique multiple of $b_{w_0}$ which is idempotent. Here, $b_{w_0}$ is the Kazhdan–Lusztig basis element corresponding to the longest word $w_0 \in S_n$ and can be described in terms of the standard basis of $\mathcal{H}_n$ by $b_{w_0} = \sum_{w \in S_n} q^{\ell(w) - \ell(w')} T_w$. Equivalently, $p_{1^n}$ is the unique multiple of $b_{w_0}$ such that $1 - p_{1^n}$ annihilates $b_{w_0}$ from the left and right.

---

2 A construction of $P_T$ in general will be proposed by the second author and Ben Elias in [9].
The indecomposable Soergel bimodules $B_w$ are indexed by permutations $w \in S_n$, and they categorify the Kazhdan–Lusztig basis $\{b_w\} \subset \mathcal{H}_n$. As a special case, the bimodule $B_{w_0} := R \otimes_R \sigma^w R(-\ell)$ categorifies the element $b_{w_0}$. Here $\ell = \frac{1}{2}n(n - 1)$ is the length of the longest element of $S_n$, and $(-\ell)$ is the grading shift which places $1 \otimes 1$ in degree $-\ell$. We will let $\mathcal{K}(\mathbb{Bim}_n)$ denote the homotopy category of chain complexes over $\mathbb{Bim}_n$ which are bounded from above.

**Theorem 1.13** There exists a pair $(P_1, \varepsilon)$ such that $P_1$ is a bounded from above chain complex in $\mathcal{K}(\mathbb{Bim}_n)$ and a map $\varepsilon : P_1 \to R$ such that

1. $P_1$ is homotopy equivalent to a chain complex whose chain bimodules are direct sums of shifted copies of $B_{w_0}$.
2. $\text{Cone}(\varepsilon) \otimes_R B_{w_0} \simeq B_{w_0} \otimes_R \text{Cone}(\varepsilon) \simeq 0$.
3. $P_1 \otimes_R P_1 \simeq P_1$.

If another pair $(P', \varepsilon')$ satisfies conditions (P1) and (P2), then $P'$ is canonically homotopy equivalent to $P_1$; the canonical homotopy equivalence $\Phi : P' \to P_1$ is characterized up to homotopy by $\varepsilon \simeq \Phi \circ \varepsilon'$.

The axioms (P1) and (P2) ensure that $\varepsilon \otimes \text{Id}_{P_1}$ and $\text{Id}_{P_1} \otimes \varepsilon$ are homotopy equivalences, and thus imply (P3). In the language of [19], such an object is called a counital idempotent, and a number of important properties follow immediately from the general theory of such objects (for instance, uniqueness). We prove in §2.3 that the projective resolution definition from Definition 1.1 satisfies (P1) and (P2); this gives the existence part of the proof of the above theorem.

## 1.2 A comparison of stable homologies

Motivated by work of Rozansky [29], one expects that infinite full twists should be intimately related with categorified Young symmetrizers. Applying the functor $\text{HHH}$ then realizes various limits of homologies of torus links and with the homologies of categorified Young symmetrizers. Before making any precise statements, we recall the decategorified situation.

The Hecke algebra is a quotient of the braid group algebra $\mathbb{Q}(q)[\text{Br}_n]$. Multiplication by the full twist braid $f_{n}$ acts diagonally on $\mathcal{H}_n$, and the central idempotents $p_\lambda \in \mathcal{H}_n$ project onto the eigenspace of $f_{n}$ with eigenvalues $q^{2x(\lambda)}$ where $x(\lambda)$ is the content of the partition $\lambda$. The minimal and maximal idempotents therefore project onto the extremal eigenvalues $q^{n(n-1)}$ (respectively $q^{-n(n-1)}$). A straightforward formal argument then shows that after normalizing by an appropriate monomial, powers of $f_{n}$ approach a well-defined limit, and this limit is $p_{1n}$ (respectively $p_{(n)}$). To make sense of these limits, one must first extend coefficients from $\mathbb{Q}(q)$ to all formal power series in positive (respectively negative) powers of $q$.

Now we discuss the categorical analogue of the above. Associated to a braid $\beta$, one has the Rouquier complex $F(\beta)$, which is a complex of Soergel bimodules, well-defined up to homotopy equivalence. Let $c_{n} = \sigma_{n-1} \cdots \sigma_2 \sigma_1$ denote the positive braid lift of the $n$-cycle $w = (n, n-1, \ldots, 2, 1) \in S_n$, so that the closure of $c_{n}^{\pm k}$ is the $(n, \pm k)$ torus link. Set $X_{n} := F(c_{n})$, and let $\text{FT}_{n} = X_{n}^\otimes$ denote the Rouquier complex associated...
to the full twist. When the index $n$ is understood, we will drop the subscript. Suppose we are given a morphism $\alpha : R \rightarrow G(FT)$, where $G$ is a grading shift. Tensoring $\alpha$ with the identity map of $G^k X \otimes^{nk+r}$ gives a map $G^k X \otimes^{nk+r} \rightarrow G^{k+1} X \otimes^{n(k+1)+r}$. In this way, maps $R \rightarrow G(FT)$ yield directed systems

$$X \otimes^r \xrightarrow{\alpha \otimes \Id} GX \otimes^{n+r} \xrightarrow{\FT} \alpha \otimes \Id$$

Given such a direct system, let us denote by $C(r, \alpha)$ the homotopy colimit (see Definition 3.20). This complex exists and is unique up to homotopy equivalence, though it may be necessary to consider bi-infinite complexes whose chain groups are countable direct sums of Soergel bimodules. Note that $C(r, \alpha) \cong C(0, \alpha) \otimes X \otimes^r$, and $C(r, \alpha)$ depends only on the residue of $r$ mod $n$, up to homotopy equivalence and grading shift. We will say that $C(r, \alpha)$ is obtained from $C(0, \alpha)$ by twisting, or that $C(r, \alpha)$ is a twisted version of $C(0, \alpha)$; we temporarily call $r$ the twisting parameter. There are two (somewhat natural) choices for $\alpha$ that we will consider, whose homotopy colimits are especially well behaved:

**Example 1.15** (The one-row map) There is a canonical map $\alpha_{\text{row}} : R \rightarrow FT(n(n-1))(-n(n-1))$ which is the inclusion of the degree zero chain group. In [18], the second author showed that $C(0, \alpha_{\text{row}}) \simeq P_n$ categorifies the one-row Young symmetrizer.

**Example 1.16** (The one-column map) In this paper we construct a map $\alpha_{\text{col}} : R \rightarrow FT(-n(n-1))$ so that the resulting homotopy colimit $C(0, \alpha_{\text{col}})$ categorifies the one-column Young symmetrizer\(^3\). For distinct $r \in \{0, 1, \ldots, n-1\}$, the resulting homotopy colimits $C(r, \alpha_{\text{col}})$ are all distinct, in contrast with the one-row case above.

We illustrate the previous discussion by sketching the case $n = 2$. Let $X = F(\sigma_1)$ denote the Rouquier complex associated to the (positive) elementary braid generator on two strands, so that $FT = X \otimes^2$. A straightforward calculation gives the following, omitting grading shifts for clarity.

$$X \otimes^k \simeq B \xrightarrow{\phi_+} B \xrightarrow{\phi_-} \cdots \xrightarrow{\phi_-} B \rightarrow R.$$

Here $X \otimes^k$ is a complex of length $k + 1$, the sign is $\pm = (-1)^{k-1}$, $B \in \mathcal{S}\text{Bim}_2$ is the nontrivial indecomposable Soergel bimodule, and $\phi_\pm$ are some endomorphisms of $B$. If we shift so that $R$ appears in homological degree zero, then these complexes approach a well defined limit $P_2$ (precisely, a homotopy colimit). If we instead shift $X \otimes^k$ so that the left-most $B$ appears in homological degree zero, then this sequence does not approach a well-defined limit because of the alternating differential $\phi_\pm$. However, there are two “convergent subsequences” which approach the limits (homotopy colimits) $P_1^\vee$ and $s(P_1^\vee)$, corresponding to the cases $r = 0, 1$ in (1.14). Here $\vee$ is the duality functor $\vee : \mathcal{K}^-(\mathcal{S}\text{Bim}_2) \rightarrow \mathcal{K}^+(\mathcal{S}\text{Bim}_2)$, and $s \in S_2$ is the nontrivial permutation.

\(^3\) Actually this limit is the dual of $P_1^\vee$; to obtain $P_1^\vee$ one should instead take homotopy limit of an inverse system involving $FT^{-1}$.
Similarly, for negative powers of $X$, one has

$$X^\otimes - k \simeq R \to B \xrightarrow{\phi_-} B \xrightarrow{\phi_+} \cdots \xrightarrow{\phi_-} B.$$ 

Various homotopy limits of these complexes will produce $P^\vee_2$, $P^\vee_{12}$, and $\pi'(P^\vee_{12})$.

The following table summarizes the basic information regarding the various homotopy (co)limits of Rouquier complexes $X^\otimes nk+r$. The permutation $w \in S_n$ below denotes the $n$-cycle $(n, n-1, \ldots, 2, 1)$.

| Projector | (Co)limit of | Boundedness | Algebra structure |
|-----------|-------------|-------------|------------------|
| $P_n$     | $\text{holim}_k X^k$ | Above | Algebra |
| $P_n^\vee$ | $\text{holim}_k X^{nk}$ | Below | Algebra |
| $w^m(P_{1n}^\vee)$ | $\text{holim}_k X^{nk+r}$ | Below | N/A |
| $P_{1n}$ | $\text{holim}_k X^{-k}$ | Below | Coalgebra |
| $w^m(P_{1n}^\vee)$ | $\text{holim}_k X^{-nk+r}$ | Above | Coalgebra |

Remark 1.17 Note that if $C$ is a homotopy colimit of (1.14) with $r = 0$, then there is a natural map $R \to C$. Thus, one thinks of $C$ as being a “unital object.” In fact, general theory of categorical idempotents [19] implies that, in case $C$ is $P_n$ or $P_{1n}^\vee$, $C$ is actually a unital algebra object in the homotopy category of Soergel bimodules. This induces an algebra structure on $HHH$ of these complexes, which makes computations of these homology groups easier to state. Similarly, the complexes $P_n^\vee$ and $P_{1n}$ are coalgebra objects. This is one reason why Theorems 1.9 and 1.18 below are stated in terms of the positive torus links $T_{n, nk+r}$, rather than $T_{n, -nk-r}$.

Decategorified, there is an automorphism $\mathcal{H}_n \to \mathcal{H}_n$ that sends $q \mapsto q^{-1}$ and $p^\lambda \mapsto p_{\lambda'}$, where $\lambda'$ is the transposed partition. However, the evident asymmetry between the one-row and one-column categorified Young symmetrizers (particularly the dependence on twisting) would seem to destroy hopes of such symmetry manifesting at the categorical level. As further evidence of asymmetry, consider the following computation, which was accomplished by the second author in [18]:

**Theorem 1.18** There is an isomorphism of triply graded algebras

$$HHH(P_n) \simeq \mathbb{Q}[U_1, \ldots, U_n, \Xi_1, \ldots, \Xi_n]$$

where the $U_k$ are even indeterminates of tridegree $t^2 - 2k q^{2k}$ and the $\Xi_k$ are odd indeterminates of tridegree $t^2 - 2k q^{2k-4} a$. This homology is isomorphic to a colimit of KR homologies of the torus links $(n, k)$ as $k \to \infty$. The $a$-degree zero part of this algebra is isomorphic to the bigraded ring of homotopy classes of endomorphisms of $P_n$.

Clearly there is not an isomorphism relating $\text{End}(P_n)$ with $HHH(P_{1n})^\vee$ or $HHH(P_{1n})$. The asymmetry between the one-row and one-column symmetrizers miraculously disappears after twisting by the Coxeter element $X$:

**Remark 1.19** In terms of the notation introduced earlier in this subsection, there is an isomorphism $HHH(C(1, \alpha_{\text{col}})) \cong HHH(C(1, \alpha_{\text{row}}))$ up to a regrading. That is to say,
if \( w \in S_n \) is a cycle, then the triply graded algebras \( \text{HHH}(P_n) \) and \( \text{HHH}(w(P_1^n)) \) in Corollary 1.10 are isomorphic after regrading. To describe the regrading, let us introduce new variables \( q_1 = q^2, t_1 = t^2 q^{-6}, \) and \( a_1 = a q^{-2} \). Then in Theorem 1.18, the even variables have degrees \( q_1 t_1^{1-k} \) and the odd variables have degrees \( a_1 t_1^{1-k} \).

In Corollary 1.10, the even variables have degrees \( t_1 q_1^{1-k} \) and the odd variables have degrees \( a_1 q_1^{1-k} \). Clearly swapping \( t_1 \) with \( q_1 \) exchanges these degrees. This demonstrates a mirror symmetry which is conjectured to exist more generally in [16].

### 1.3 Combinatorial description of \( P_1^n \)

Recall from the definition that \( P_1^n \) is a resolution of \( R \) by free \( R \otimes_R \mathbb{Q} \) \( R \)-modules. In §2.6 we are able to construct an explicit such resolution, which is essential for the computation for \( \text{HHH}(P_1^n) \). We briefly discuss this construction in this section.

Put \( x = \{x_1, \ldots, x_n\} \), \( y = \{y_1, \ldots, y_n\} \) and let \( I_n \subset \mathbb{Q}[x, y] \) denote the ideal generated by \( e_k(x) - e_k(y) \), for \( 1 \leq k \leq n \), where \( e_k \) denotes the \( k \)th elementary symmetric function. It will be more convenient to consider \( (R, R) \)-bimodules as \( \mathbb{Q}[x, y] \)-modules for this section. Note that \( B_{w_0} \simeq \mathbb{Q}[x, y]/I_n \) up to a grading shift of \(-\ell\), where \( \ell = n(n-1)/2 \) is the length of \( w_0 \).

To motivate the construction, we first remark that the decategorified Young symmetrizer \( P_1^n \) equals \( \frac{1}{[n]!} b_{w_0} \), where \([n]! = [n][n-1] \cdots [2] \), and \([k] = \frac{q^k - q^{-k}}{q - q^{-1}} \) is the quantum integer. This is a \( q \)-analogue of a standard identity in the symmetric group. In other words:

\[
P_1^n = q^{-\ell} \frac{(1 - q^{-2})(1 - q^{-2}) \cdots (1 - q^{-2})}{(1 - q^{-2})(1 - q^{-6}) \cdots (1 - q^{-2n})} b_{w_0} \tag{1.20}
\]

To categorify this, we will explicitly construct a complex \( M_n \) in which \( B_{w_0} \) appears with graded multiplicity

\[
q^{-\ell} \frac{(1 + tq^{-2})(1 + tq^{-2}) \cdots (1 + tq^{-2})}{(1 - t^2 q^{-2})(1 - t^2 q^{-3}) \cdots (1 - t^2 q^{-2n})} \tag{1.21}
\]

Factors of the form \((1 - t^2 q^{-2k})^{-1}\) correspond to the action of an even variable \( u_k \) of degree \( t^2 q^{-2k} \), and factors of the form \((1 + tq^{-2})\) are captured by odd variables \( \theta_i \) of degree \( tq^{-2} \). Thus, to start with, we consider the bigraded algebra

\[
A_n := \mathbb{Q}[x, y, u_1, u_2, \ldots, u_n, \theta_1, \theta_2, \ldots \theta_n]
\]

with variables \( u_k, \theta_k \) as above. Then we form the quotient \( M_n = A_n/I_n A_n \), and we shift the bigradings so that \( \bar{1} \in M_n \) lies in \( q \)-degree \(-2\ell \). By construction, the bigraded object \( M_n \) will be a direct sum of infinitely many copies of \( B_{w_0} \), with graded multiplicity (1.21).

Now we put differentials on \( A_n \) and \( M_n \) so that \( A_n \) is a differential bigraded algebra, and \( M_n \) is a differential bigraded \( A_n \)-module. To do this, in Definition 2.40 we introduce
polynomials $a_{ij}(x, y), (1 \leq i \leq j \leq n)$ such that $\sum_{j=1}^{n} a_{ij}(x, y)(y_j - x_j) = 0$ modulo $I_n$. Then we let $d_A$ be the unique $\mathbb{Q}[x, y]$-linear differential on $A_n$ given by

$$d_A(u_k) = 0, \quad d_A(\theta_k) = \sum_{i=1}^{k} a_{ik}(x, y)u_i,$$

(1.22)

together with the graded Leibniz rule. Note that $M_n$ is generated as an $A_n$-module by a single element $\bar{1} \in M_n$, where $\bar{1}$ denotes the image of $1 \in A_n$ under the quotient $A_n \to M_n$. Thus, the differential $d_M$ is uniquely determined by $d_M(\bar{1})$ together with the graded Leibniz rule for dg modules: $d_M(am) = d_A(a)m + adm(m)$, with the sign given by the homological degree of $a$. With this in mind, we let $d_M$ be the unique differential on $M_n$ such that

$$d_M(\bar{1}) = \sum_{i=1}^{n} (y_i - x_i)\theta_i \bar{1}$$

together with the graded Leibniz rule. The proof that $d_M^2 = 0$ is straightforward, and relies on the defining property of the $a_{ij}(x, y)$.

Theorem 1.23 $M_n \cong P_n^\vee$.

To prove this we construct explicit quasi-isomorphisms $\mathbb{Q}[x] \cong M_1 \to M_2 \to \cdots \to M_n$, from which we conclude that $M_n$ is an injective resolution of $\mathbb{Q}[x]$, regarded as a graded $\mathbb{Q}[x, y]/I_n$-module. Then $M_n^\vee \cong P_n$, by Remark 1.3.

The presence of odd variables $\theta_i$ and the structure of the differentials $d_M$ and $d_A$ give the complex various 2-periodicities. This is a common phenomenon in commutative algebra [8], and can be encoded with an object called a matrix factorization. In §4.4 we use this construction to give a proof of Theorem 1.9.

Remark 1.24 There are reduced versions $\tilde{A}_n \cong A_n$ and $\tilde{M}_n \cong M_n$, obtained by setting the variables $u_1, \theta_1$ equal to zero. This has the effect of canceling the $(1 - q^2)$ factors in (1.20) Below, we consider the reduced case.

We now end this section by studying the case $n = 3$. Let $s, t \in S_3$ be the simple transpositions, so that $w_0 = sts$ is the longest word, and set $B := B_{sts}$. For the rest of this discussion let $p_{ij} = y_i - x_j$. Unpacking the definitions, including those from Definition 2.40, we see that the differential on $\tilde{M}_3$ is determined by

$$d_M(\bar{1}) = p_{22}\theta_2 \bar{1} + p_{33}\theta_3 \bar{1}$$

$$d_M(\theta_2 \bar{1}) = p_{12}u_2 \bar{1} - p_{33}\theta_2 \theta_3 \bar{1}$$

$$d_M(\theta_3 \bar{1}) = (p_{23} + p_{12})u_2 \bar{1} + p_{23}p_{13}u_3 \bar{1} + p_{33}\theta_2 \theta_3 \bar{1}$$

$$d_M(\theta_2 \theta_3 \bar{1}) = p_{12}u_2 \theta_3 \bar{1} - p_{23}p_{13}u_3 \theta_2 \bar{1} - (p_{12} + p_{23})u_2 \theta_2 \theta_3 \bar{1}$$

(1.25)

Recall that we have placed $\bar{1}$ in $q$-degree $-6$. Since $B = (\mathbb{Q}[x, y]/I_n)(-3)$, taking into account the degrees of $\theta_i$ and $u_i$, we see that $\tilde{M}_3 \cong P_{13}^\vee$ is the total complex of the below “perturbed” double complex:
The double-peridiocity which is evident in the above diagram is realized by the action of the even variables $u_2, u_3$. The “fundamental region”

\[
\begin{array}{c}
B(-3) \xrightarrow{p_{22}} B(-5) \\
p_{33} \\
B(-5) \xrightarrow{p_{22}} B(-7) \\
p_{23}p_{13} \\
B(-7) \xrightarrow{p_{12}} B(-9) \\
p_{23}p_{13} \\
B(-9) \xrightarrow{p_{22}} B(-11) \\
p_{23}p_{13} \\
B(-11) \xrightarrow{p_{12}} B(-13) \\
p_{23}p_{13} \\
B(-13) \xrightarrow{p_{22}} B(-15) \\
p_{23}p_{13} \\
B(-15) \xrightarrow{p_{12}} B(-17) \\
p_{23}p_{13} \\
\vdots \\
\vdots \\
\end{array}
\]

is spanned by the products of the odd variables. More precisely, this square is the image in $\tilde{M}_3$ of the exterior algebra $\mathbb{Q}[x, y, \theta_2, \theta_3]$, which can also be described as the Koszul complex of the sequence $p_{22}, p_{33}$ acting on $B$. To write down the complex $P_1^n$, we reverse the arrows and change all grading shifts $(-k)$ to $(k)$. The resulting diagram is given explicitly in §2.4.

**Remark 1.26** In this paper we also consider twisted projectors $w(P_1^n)$, which can be described as follows. Let $w \in S_n$ be given. Recall that $P_1^n$ is a complex whose chain bimodules are shifted copies of $\mathbb{Q}[x, y]/I_n$. The differential on $P_1^n$ can be represented by matrices whose entries are polynomials $q(x, y)$. We define $w(P_1^n)$ to be the complex given by the same underlying bimodule, but replacing $q(x, y)$ with $q(x, w(y))$.

1.4 Outline of the paper

Section §2 begins with a recollection of some relevant facts regarding Soergel bimodules. We then give a set of axioms which characterize $P_1^n$, and we construct the dual
complex $P_{1^n}$ as an explicit injective resolution. In §3 we show that $P_{1^n}$ is a homotopy colimit of Rouquier complexes associated to powers of the full-twist. In §4 we compute $\text{HHH}(F \otimes P_{1^n})$ for arbitrary Rouquier complexes $F$, and prove Conjecture 1.8. In §5 we prove some combinatorial results which are likely known to experts but difficult to find in the literature. Their proofs are elementary in any case, so we include them here.

2 A categorified Young symmetrizer

In this section we introduce the category $\mathbb{S}\text{Bim}_n$ of Soergel bimodules and describe an idempotent complex $P_{1^n} \in K^-(\mathbb{S}\text{Bim}_n)$ which categorifies a Young symmetrizer.

Notation 2.1 If $\mathcal{A}$ is an additive category, we let $K(\mathcal{A})$ denote the homotopy category of complexes over $\mathcal{A}$ (with differentials of degree $+1$). We use superscripts $\mathcal{b}$, $\mathcal{+}$, $\mathcal{−}$ to denote the full subcategories of complexes which are bounded, respectively bounded from above, respectively bounded from below. The homological grading shift of complexes is denoted by $\langle k \rangle$, so that $(C \langle k \rangle)_i = C_i - k$. By convention, the differential on $C \langle k \rangle$ is $(-1)^k d_C$.

2.1 The Soergel category

Here we set up some notation which will be used throughout this paper. Fix once and for all an integer $n \geq 1$, and put $W := S_n$. Let $w_0 \in W$ denote the longest word (that is, $w_0(i) = n + 1 - i$) and $\ell = \frac{1}{2}n(n - 1)$ its length. Embed $S_{n−1}$ into $S_n$ in the standard way, as permutations of $\{1, \ldots, n\}$ which fix $n$. Denote the longest word of $S_{n−1} \subset S_n$ by $w_1$. Its length is $\ell - n + 1$.

Set $R := \mathbb{Q}[x_1, \ldots, x_n]$. Regard $R$ as a graded ring, via $\deg(x_i) = 2$ for all $i$. Note that $W$ acts on $R$ by permuting variables. For a simple transposition $s \in W$, let $R^s \subset R$ denote the subalgebra consisting of polynomials $f$ with $s(f) = f$. Define a graded $(R, R)$-bimodule $B_s := R \otimes_{R^s} R(-1)$, where $(k)$ denotes the functor which shifts grading up by $k$. That is $(M(k))_i = M_{i-k}$. Let $\mathbb{S}\text{Bim}_n$ denote the smallest full subcategory of all graded $(R, R)$-bimodules containing $B_s$ and closed under direct sum, direct summands, grading shift, and tensor product $\otimes_R$. Objects of $\mathbb{S}\text{Bim}_n$ are called Soergel bimodules. We will only briefly recall some relevant facts about Soergel bimodules, and we refer the reader to [10] for more details.

The isomorphism classes of indecomposable Soergel bimodules are indexed by $w \in S_n$. In this paper we are interested only in a few special cases of these bimodules, which we now describe. For each parabolic subgroup, that is, the subgroup generated by the simple transpositions $(i, i + 1)$ with $i$ in some subset of $\{1, \ldots, n−1\}$, there exists a unique element of maximal length. We will denote the element as $w'$ in this discussion and denote its length by $\ell(w')$. Let $R' \subset R$ denote the subalgebra consisting of polynomials such that $w(f) = f$ for all $w$ in the parabolic subgroup. It is well known that the indecomposable Soergel bimodule $B_{w'}$ admits the following description:

$$B_{w'} = R \otimes_{R'} R(−\ell(w'))$$  (2.2)
The following bimodules are the most important special cases:

1. \( B_s = R \otimes_R R(-1) \). This corresponds to the case the parabolic subgroup is generated by a single transposition. The longest word \( w' \) is the simple transposition \( s = (i, i + 1) \) which has length 1.

2. \( B_{w_0} = R \otimes_R RW(-\ell) \), where \( RW \subset R \) is the subalgebra of symmetric polynomials. This corresponds to the case we take the parabolic subgroup to be the entire group \( W \).

3. \( B_{w_1} = R \otimes_R R_{n-1}^{-\ell + n - 1} \). This corresponds to the case the parabolic subgroup is generated by the transpositions \( (i, i + 1) \) for all \( i \in \{1, \ldots, n-2\} \).

We will typically denote tensor product over \( R \) simply by juxtaposition: \( AB := A \otimes_R B \). Tensor product over \( Q \) will be denoted by \( \bigoplus \).

**Definition 2.3** Let \( \bigoplus : \mathcal{SBim}_i \times \mathcal{SBim}_j \rightarrow \mathcal{SBim}_{i+j} \) denote the bilinear functor \( M \bigoplus N = M \otimes_R N \). Suppose \( R_i = \mathbb{Q}[x_1, \ldots, x_i] \) and \( R_j = \mathbb{Q}[x_1, \ldots, x_j] \). Note that if \( M \) is a graded \((R_i, R_i)\)-bimodule and \( N \) is a graded \((R_j, R_j)\)-bimodule, then \( M \otimes_R N \) is naturally a graded \((R_i+j, R_i+j)\)-bimodule, since \( R_i \otimes_R R_j \cong R_{i+j} \). Thus, our definition of \( \bigoplus \) makes sense.

For any \( M \in \mathcal{SBim}_k \) with \( 1 \leq k \leq n \), we have \( M \bigoplus R_{n-k} \cong M[x_{k+1}, \ldots, x_n] \). The notational conventions ensure that \( B_w \bigoplus R_{n-k} \cong B_w \in \mathcal{SBim}_n \) for all \( w \in S_k \subset S_n \). Thus, we often abuse notation and regard an object of \( \mathcal{SBim}_k \) as an object of \( \mathcal{SBim}_n \) when there is no possibility of confusion.

**Notation 2.4** For any \( M \in \mathcal{SBim}_k \), we often will denote \( M \bigoplus R_{n-k} \in \mathcal{SBim}_n \) simply by \( M \).

The bimodule \( B_{w_0} \) tends to absorb Soergel bimodules:

**Proposition 2.5** \( B_w B_{w_0} \) and \( B_{w_0} B_w \) are direct sums of shifted copies of \( B_{w_0} \), for each \( w \in S_n \). In the special case where \( w \) is the longest word of \( S_k \subset S_n \), we have

\[
B_w B_{w_0} \cong [k]! B_{w_0} \cong B_{w_0} B_w
\]

Here \( [k]! = [k][k-1] \cdots [2][1] \), and \( [j] = \frac{q^j - q^{-j}}{q - q^{-1}} \). We are employing the convention that for any \( f(q) \in \mathbb{N}[q, q^{-1}] \), \( f(q)M \) denotes the corresponding direct sum of shifted copies of \( M \):

\[
\left( \sum_{a \in \mathbb{Z}} c_a q^a \right) \cdot M := \bigoplus_{a \in \mathbb{Z}} M(a)^{\oplus c_a}.
\]

**Proof** This is proven in [12] and is also a corollary of Theorem 5.1.

We conclude with a notational convention that we will use to shorten many expressions later:
Notation 2.6 We have independent sets of variables $x := \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$. We regard an $(R, R)$-bimodule as a $\mathbb{Q}[x, y]$-module, where $x_k$ acts by left multiplication by $x_k$, and $y_k$ acts via right multiplication by $x_k$. For $1 \leq k \leq n$, let $I_k \subset \mathbb{Q}[x, y]$ denote the ideal generated by elements

$$e_i(y) - e_i(x) \quad (1 \leq i \leq k) \quad \text{together with} \quad y_i - x_j \quad (k + 1 \leq i \leq n)$$

where $e_i$ denotes the elementary symmetric function. We may identify $R \otimes_{R^{\text{gr}}} R$ with $\mathbb{Q}[x, y]/I_k$. In particular $B_{w_0} \cong \mathbb{Q}[x, y]/I_n$ and $B_{w_1} \cong \mathbb{Q}[x, y]/I_{n-1}$, up to grading shifts. Taking the quotient by $y_n - x_n$ defines a canonical map $\mathbb{Q}[x, y]/I_n \twoheadrightarrow \mathbb{Q}[x, y]/I_{n-1}$. Dually, there is a canonical map $\mathbb{Q}[x, y]/I_{n-1} \hookrightarrow \mathbb{Q}[x, y]/I_n$ which sends $1 \mapsto \prod_{i=1}^{n-1}(y_i - x_n)$. This is a well-defined map of $\mathbb{Q}[x, y]$-modules by Proposition 2.30.

### 2.2 The ideal $I \subset \mathcal{K}^-(\text{SBim}_{n})$

Recall that $p_{1^n} \in \mathcal{H}_n$ is the unique multiple of $b_{w_0}$ such that $1 - p_{1^n}$ annihilates $b_{w_0}$. In §2.3 we will give a categorical analogue of this characterization. But first we study the categorical analogue of being a multiple of $b_{w_0}$.

**Definition 2.7** Let $I \subset \mathcal{K}^-(\text{SBim}_{n})$ denote the full subcategory consisting of complexes whose chain bimodules are isomorphic to direct sums of $B_{w_0}$ with shifts.

**Proposition 2.8** The subcategory $I \subset \mathcal{K}^-(\text{SBim}_{n})$ is a two-sided tensor ideal. That is to say, if $I \in I$ and $A \in \mathcal{K}^-(\text{SBim}_{n})$ are arbitrary, then $IA, AI \in I$.

**Proof** Follows trivially from Proposition 2.5. \qed

**Definition 2.9** Let $I^\perp \subset \mathcal{K}^-(\text{SBim}_{n})$ and $\perp I \subset \mathcal{K}^-(\text{SBim}_{n})$ denote the full subcategories consisting of complexes $C$ such that $CB_{w_0} \simeq 0$, respectively $B_{w_0}C \simeq 0$.

**Remark 1.10** An easy limiting argument shows that in fact $C \in I^\perp$ implies that $CD \simeq 0$ for all $D \in I$, and similarly for $\perp I$.

Even though kernels and images don't exist in $\text{SBim}_{n}$, it still makes sense to talk about the homology of $C \in \mathcal{K}(\text{SBim}_{n})$, regarded as a complex of $(R, R)$-bimodules. Our next goal in this section is to show that if $C \in \mathcal{K}^-(\text{SBim}_{n})$ is acyclic (has zero homology), then $C \in I^\perp \cap \perp I$.

**Proposition 2.11** The functors $B_w \otimes (-)$ and $(-) \otimes B_w$ are exact for all $w \in S_n$. In particular, if $C$ is a complex of Soergel bimodules, then

$$H(B_w \otimes_R C) \cong B_w \otimes_R H(C) \quad H(C \otimes_R B_w) \cong H(C) \otimes_R B_w$$

for all $w \in S_n$.

**Proof** Notice that $B_S$ is free as a right (or left) $R$-module. Each $B_w$ is isomorphic to a direct summand of a tensor product of $B_S$'s. Hence each $B_w$ is projective as a right (or left) $R$-module. It follows that tensoring with $B_w$ is exact. \qed
Lemma 2.12 The Soergel bimodule $B_{w_0}$ is free of rank 1 as a graded module over its graded endomorphism ring $\text{End}(B_{w_0}) \cong R \otimes_{R^w} R$.

Proof Obvious. □

Lemma 2.13 Suppose $C \in \mathcal{I}$ is arbitrary. Then $C$ is contractible if and only if $C$ is acyclic.

Proof Certainly $C$ being contractible implies that $C$ is acyclic. To prove the converse, suppose $C \in \mathcal{I}$ is acyclic. Let $C_i$ denote the $i$-th chain bimodule. By definition of $\mathcal{I}$, up to equivalence we may assume that each $C_i$ is a direct sum of shifted copies of of $B_{w_0}$. Then the $R \otimes R$-action on $C_i$ factors through $R \otimes_{R^w} R = \text{End}(B_{w_0})$. So we may regard $C$ as a complex of $R \otimes_{R^w} R$-modules. On the other hand, $B_{w_0} = R \otimes_{R^w} R(\ell)$ is free as a $R \otimes_{R^w} R$-module, hence the acyclic complex $C$ is contractable as a complex of $R \otimes_{R^w} R$-modules by standard arguments. The null-homotopy $h = (h_i : C_i \to C_{i-1})_i$ commutes with the $R \otimes_{R^w} R$-action, hence $h$ also commutes with the $R \otimes R$-action. That is to say, $C$ is contractible as a complex of $(R, R)$-bimodules. □

We have two immediate corollaries. Recall that a chain map $f : C \to D$ is said to be a quasi-isomorphism if $f$ is an isomorphism in homology.

Corollary 2.14 Two complexes $A, B \in \mathcal{I}$ are homotopy equivalent if and only if they are quasi-isomorphic.

Proof Suppose $A, B \in \mathcal{I}$, and let $f : A \to B$ be a chain map. A well known fact states that $f$ is a homotopy equivalence if and only if the mapping cone $\text{Cone}(f)$ is contractible. By Lemma 2.13, this holds if and only if $\text{Cone}(f)$ is acyclic. Another well known fact about mapping cones states that $\text{Cone}(f)$ is acyclic if and only if $f$ is a quasi-isomorphism.

Corollary 2.15 Let $Z \in K^-(\mathbb{S}\text{Bim}_n)$ be a acyclic. Then $Z \in \mathcal{I}^\perp \cap \perp \mathcal{I}$.

Proof Let $Z = (\cdots Z_i \to Z_{i+1} \cdots) \in K^-(\mathbb{S}\text{Bim}_n)$ be acyclic, and let $C \in \mathcal{I}$ be arbitrary. Proposition 2.11 implies that $Z_i C$ and $CZ_i$ are acyclic. Using the Kunneth spectral sequence, we see that both $ZC$ and $CZ$ are also acyclic. Thus Lemma 2.13 implies that $ZC$ and $CZ$ are contractible. □

2.3 Axiomatics

Recall that there is an algebra map $R \otimes_{R^w} R \to R$. The main object in this paper is the following free resolution:

Definition 2.16 Let $P_{1^n} \xrightarrow{\epsilon} R$ denote a resolution of $R$ by free graded $R \otimes_{R^w} R$-modules.

Theorem 2.17 We have

(P1) $P_{1^n}$ is in $\mathcal{I}$.
(P2) $\text{Cone}(P_{1^n} \xrightarrow{\epsilon} R)$ is in $\mathcal{I}^\perp \cap \perp \mathcal{I}$.
Furthermore, the pair \((P_1n, \varepsilon)\) is uniquely characterized by these properties up to canonical equivalence. By this, we mean that if \((Q, \nu)\) is another pair satisfying (P1) and (P2) then there is a unique (up to homotopy) map \(\phi: Q \to P_1n\) such that \(\nu \circ \phi \simeq \varepsilon\); this map is a homotopy equivalence.

Proof By construction, the chain bimodules of \(P_1n\) are direct sums of shifted copies of \(R \otimes_{Rw} R = B_{u0}(\ell)\), so that (P1) clearly holds. Since \(\varepsilon: P_1n \to R\) is by construction a quasi-isomorphism, \(\text{Cone}(\varepsilon)\) is acyclic. Then axiom (P2) holds by Corollary 2.15.

Suppose \(Q \in \mathcal{I}\) is some complex and \(\nu: Q \to R\) is some chain map such that \(\text{Cone}(\nu) \simeq 0\). Then \(\text{Cone}(\nu)P_1n \simeq 0\), since \(P_1n \in \mathcal{I}\). This implies that \(\nu \otimes \text{Id}_{P_1n}\) is a homotopy equivalence \(QP_1n \to P_1n\). A similar argument shows that \(\text{Id}_Q \otimes \varepsilon\) is a homotopy equivalence \(QP_1n \to Q\). Thus, \(Q \simeq P_1n\). Via Corollary 4.29 in [19] one obtains a canonical equivalence. \(\square\)

Remark 2.18 This is the categorical analogue of the characterization of \(p_1n \in \mathcal{H}_n\). The complex \(P_1n\) is a “multiple” of \(B_{u0}\), in the sense that its chain bimodules are sums of \(B_{u0}\) with shifts. The complex \(\text{Cone}(P_1n \to R)\) should be thought of as playing the role of the “difference” of \(R\) and \(P_1n\), and axiom (P2) states that this complex annihilates \(B_{u0}\).

The axioms imply that \(\text{Cone}(\varepsilon)\) annihilates \(P_1n\) from the right and left. Equivalently, \(\varepsilon: P_1n \to R\) becomes an equivalence after tensoring with \(P_1n\) on the right or left. Thus, \((\text{Cone}(\varepsilon), P_1n)\) forms a pair of complementary idempotents in \(\mathcal{K}^{-}(\mathbb{S}\text{Bim}_n)\), in the sense of [19]. Many properties of \(P_1n\) can be deduced immediately from the axioms together with some basic theory of categorical idempotents developed in [19] (also [4]). For instance:

Proposition 2.19 The idempotents \(P_1n \in \mathcal{K}^{-}(\mathbb{S}\text{Bim}_n)\) and \(\text{Cone}(\varepsilon)\) are central. That is, \(A P_1n \simeq P_1n A\) and \(A \text{Cone}(\varepsilon) \simeq \text{Cone}(\varepsilon) A\) for any complex \(A \in \mathcal{K}^{-}(\mathbb{S}\text{Bim}_n)\), and this isomorphism is natural in \(A\).

Proof Follows from the fact that \(\mathcal{I} \subset \mathcal{K}^{-}(\mathbb{S}\text{Bim}_n)\) is a two-sided tensor ideal and Corollary 4.29 in [19]. \(\square\)

2.4 The case \(n = 2\) and \(n = 3\)

Note that \(P_{11} = R\). In this subsection we describe \(P_{12}\) and \(P_{13}\). We generalize these to an explicit combinatorial description of all of the projectors \(P_1n\) in §2.6.

The indecomposable bimodules in \(\mathbb{S}\text{Bim}_2\) are \(R\) and \(B_s\), where \(s \in S_2\) is the nontrivial permutation. Note that \(s\) is the longest word, and has length 1. The projective resolution \(P_{12} \to R\) is given by

\[
\cdots \xrightarrow{y_1-x_2} B_3(7) \xrightarrow{y_2-x_2} B_3(1) \xrightarrow{y_1-x_2} B_3(3) \xrightarrow{y_2-x_2} B_3(5) \xrightarrow{y_1-x_2} B_3(7) \cdots \xrightarrow{1} R
\]

We emphasize that \(P_{12}\) is the complex consisting of terms to the left of the dashed arrow.
Now we give a construction of $P_{13}$. Let $p_{ij} = y_i - x_j$, then $P_{13}$ is the total complex of the following perturbed double complex supported in a single quadrant above the dashed arrows. In $S_3$ there are two simple transpositions, $s, t \in S_3$, where $s$ swaps $1$ and $2$, and $t$ swaps $2$ and $3$. The longest word is $w_0 = sts$, and its length is $3$.

In the above diagram $B = B_{sts}$ is the bimodule associated to the longest word in $S_3$, and $B' = B_s \in \mathcal{K}^{-}(S_{Bim_3})$.

**Remark 2.22** For now, ignore the part of the diagram below the dashed lines. To form the total complex $P_{13}$, first order the rows by $0, 1, 2, \ldots$ starting with the row just above the dashed lines. Then to each horizontal arrow in an odd numbered row place a minus sign. Also number the columns from right to left, starting at zero. The total complex has $-k$-th chain bimodule equal to the direct sum, over $i \in \{0, 1, \ldots, k\}$, of terms on the $i$-th row and the $k-i$-th column. The differential is the (signed) sum of arrows indicated by the diagram.

Now, the part of the diagram below the dashed arrows is simply $P_{12}$, regarded now as an object of $\mathcal{K}^{-}(S_{Bim_3})$. The dashed arrows visibly define a chain map $P_{13} \rightarrow P_{12}$. Note the following properties of $P_{13}$:

- In the horizontal direction, the complex is $2$-periodic and each row looks like $P_{12}$ except replacing all copies of $B_s$ with $B_{sts}$. The extra diagonal arrow is needed to compensate for the fact $p_{22}p_{21} \neq 0$ in $B_{sts}$.
- In the vertical direction, the complex is $2$-periodic. It will be shown that each column of the augmented complex is acyclic, hence the dashed arrows represent a quasi-isomorphism $P_{13} \rightarrow P_{12}$. 

(2.21)
As mentioned in the introduction, we will construct $P_{1^n}$ as an explicit complex in which $B_{w_0}$ appears with graded multiplicity

$$\text{multiplicity of } B_{w_0} \text{ in } P_{1^n} = q^\ell \prod_{i=1}^n \frac{1 + t^{-1}q^2}{1 - t^{-2}q^{2i}}$$  \hspace{1cm} (2.23)

where $t$ represents homological degree. The denominator is realized by an action of the bigraded polynomial ring $\mathbb{Q}[u_1, \ldots, u_n]$ in which $\deg(u_i) = t^{i^2}q^{-2i}$. This action generalizes the multiple periodicity evident in our diagrams (2.20) and (2.21) for $P_{1^2}$ and $P_{1^3}$. The “fundamental domain” for this action is a certain cube-like complex $B_{w_0} \otimes \Lambda[\theta_2, \ldots, \theta_n]$, which contributes the numerator in the Eq. (2.23). The 2-periodicity of our complex is an instance of a common phenomenon in commutative algebra [8], and can be encoded with an object called a matrix factorization. We return to this idea in §4.4.

2.5 An iterated projective resolution construction of $P_{1^n}$

In this section we consider a chain of algebras and algebra maps

$$R \otimes_{R^S_n} R \rightarrow R \otimes_{R^S_{n-1}} R \rightarrow \cdots \rightarrow R \otimes_{R^S_1} R = R$$ \hspace{1cm} (2.24)

defined as follows. It is clear that $R^{S_{k+1}} \subset R^{S_k}$, since if a polynomial $f \in \mathbb{Q}[x]$ is symmetric in $x_1, \ldots, x_{k+1}$, then it is certainly symmetric in $x_1, \ldots, x_k$. The resulting quotient maps $R \otimes_{R^S_{k+1}} R \rightarrow R \otimes_{R^S_k} R$ are the algebra maps above.

By restriction, any graded $R \otimes_{R^S_k} R$-module can also be thought of as a graded $R \otimes_{R^S_{k+1}} R$-module. This gives a useful inductive perspective on the construction of $P_{1^k} (1 \leq k \leq n)$, which by definition is a resolution of $R$ by free graded $R \otimes_{R^S_k} R$-modules.

**Definition 2.25** Let $\mathcal{J}_k \subset \mathcal{K}^- (\mathbb{S}\text{Bim}_n)$ denote the full subcategory consisting of complexes whose chain bimodules are direct sums of shifted copies of $R \otimes_{R^S_k} R$.

**Definition 2.26** For any complex $C \in \mathcal{K}^- (\mathbb{S}\text{Bim}_n)$, a $\mathcal{J}_k$-resolution of $C$ will mean a complex $D \in \mathcal{J}_k$ and a quasi-isomorphism $\phi : D \rightarrow C$. If $C \in \mathcal{J}_k$, then $C$ can be regarded as a complex of graded $R \otimes_{R^S_k} R$-modules, for any $m \geq k$. In this case, a $\mathcal{J}_m$-resolution of of $C$ is the same as a resolution of $C$ by free graded $R \otimes_{R^S_m} R$-modules.

**Remark 2.27** Note that $\mathcal{J}_n = \mathcal{I}$ since $B_{w_0} = R \otimes_{R^S_n} R(-n(n - 1)/2)$, where $w_0$ is the longest word of $S_n$.

Our construction of $P_{1^n}$ is inductive, and is based on the following principle:

1. Assume that we have constructed a $\mathcal{J}_{k-1}$-resolution $\varepsilon_{k-1} : P_{1^{k-1}} \rightarrow R$.
2. Find an explicit $\mathcal{J}_k$-resolution $C \rightarrow R \otimes_{R^S_{k-1}} R$.
3. General theory of projective resolutions states that we may replace each copy of $R \otimes_{R^S_{k-1}} R$ in $P_{1^{k-1}}$ by $C$ up to quasi-isomorphism. The result is a $\mathcal{J}_k$-resolution $\varepsilon_k : P_{1^k} \rightarrow R$. 

In this paper we are interested in explicit formulae, so will spend some time on (2) above. We first address the case when \( k = 2 \). We have the following complex

\[
Z_1 = R(4) \xrightarrow{y_1 - x_2} B_s(3) \xrightarrow{y_2 - x_2} B_s(1) \xrightarrow{1} R.
\]  

(2.28)

As we will see, this complex is acyclic. By “stringing together” infinitely many copies of this complex, we recover the expression (2.20) for \( P_{12} \). More precisely, there exists a map \( \gamma : Z_1 \to Z_1(4(-3)) \) defined as follows:

\[
R(4) \xrightarrow{1} B_s(3) \xrightarrow{y_2 - x_2} B_s(1) \xrightarrow{1} R
\]

(2.29)

By a straightforward short exact sequence argument, \( \text{Cone}(\gamma) \) is acyclic. \( \text{Cone}(\gamma) \) has a contractible direct summand

\[
R(4) \xrightarrow{1} R(4).
\]

Therefore, by Gaussian elimination, \( \text{Cone}(\gamma) \) is homotopy equivalent to the complex \( Z_2 \) pictured below:

\[
R(8) \xrightarrow{y_1 - x_2} B_s(7) \xrightarrow{y_2 - x_2} B_s(5) \xrightarrow{y_1 - x_2} B_s(3) \xrightarrow{y_2 - x_2} B_s(1) \xrightarrow{1} R
\]

Inductively we can define \( Z_{\ell+1} = \text{Cone}(\gamma_\ell : Z_\ell \to Z_1(4\ell(-1 - 2\ell))) \), where \( \gamma_\ell \) is defined in a way similar to \( \gamma \). That is, \( Z_\ell \) has a copy of \( R \) in homological degree \(-1 - 2\ell\), and the map \( \gamma_\ell \) sends this copy of \( R \) by the identity map onto the copy of \( R \) in \( Z_1(4\ell(-1 - 2\ell)) \) in homological degree \(-1 - 2\ell\). This gives a contractible summand of the form

\[
R(4\ell) \xrightarrow{1} R(4\ell).
\]

in \( \text{Cone}(\gamma_\ell) \). There is a natural map \( \pi_\ell : Z_\ell \to Z_{\ell-1} \) coming from the mapping cone construction. The colimit of the inverse system \( \{Z_\ell, \pi_\ell\} \) is an acyclic chain complex since homology commutes with taking colimits. Moreover, after stripping off the contractible summands in each step, we have a complex which has a single copy of \( B_s \) in every homological degree except in homological degree 0 where we have a single copy of \( R \). Therefore, \( Z_\infty = \text{colim}([Z_\ell, \pi_\ell]) \) is an \( R \otimes R^t \)-free resolution of \( R \); in fact we recover exactly the expression of \( P_{12} \) in diagram (2.20).

We now describe in a similar fashion construct a \( J_n \)-resolution of \( B_{w_1} \), where \( w_1 \subset S_{n-1} \subset S_n \) is the longest word. We first ask: how does the acyclic complex (2.28) generalize to arbitrary \( n \)?

**Proposition 2.30** There is an acyclic complex of the form

\[
Z := \left( B_{w_1}(2n) \to B_{w_0}(n + 1) \xrightarrow{y_n - x_n} B_{w_0}(n - 1) \to B_{w_1} \right)
\]

(2.31)
Here, the first and third components of the differential are the bimodule maps

1. \( B_{u_0}(n - 1) \to B_{u_1} \) sending \( 1 \mapsto 1 \).
2. \( B_{u_1}(n - 1) \to B_{u_0} \) sending \( 1 \mapsto \prod_{i=1}^{n-1} (x_i - y_i) \).

Recall that we use \( x_i \) to denote \( x_i \otimes 1 \) and \( y_i \) to denote \( 1 \otimes x_i \) per Notation 2.6. Also recall that \( 1 \in B_{u_0} \) has degree \( \frac{1}{2}n(n - 1) \) and \( 1 \in B_{u_1} \) has degree \( \frac{1}{2}(n - 1)(n - 2) \). The difference of these is \( n - 1 \), which explains the grading shifts above. In proving Proposition 2.30 we recall the definition of a Frobenius algebra.

**Definition 2.32** Let \( k \) be a commutative ring. We call a finite dimensional \( k \)-algebra, \( A \), a Frobenius algebra if there exists a nondegenerate trace \( \partial : A \otimes A \to k \). That is, \( \partial \) is a nondegenerate bilinear map such that \( \partial(xy) = \partial(yx) \) for all \( x, y \in A \).

**Proof of Proposition 2.30** It is known that \( RS^{n-1} \) is a Frobenius algebra over \( RW \), and a pair of dual bases is provided by \( \{e_{n-1-k}(x_1, \ldots, x_{n-1})\}_{k=0}^{n-1} \), where \( e_k \) denotes the elementary symmetric polynomial, and \( \{(-1)^k x_n^k\}_{k=0}^{n-1} \). For a proof of this fact, see Theorem 5.1 in §5.1.

In general, suppose \( k \) is a commutative ring and \( S \) is a Frobenius \( k \)-algebra with trace \( \partial : S \to k \). If \( \{a_i\} \) and \( \{b_i\} \) are dual bases, in the sense that \( \partial(a_i b_j) = \delta_{ij} \), then there is an \((S, S)\)-bilinear map \( \Delta : S \to S \otimes_k S \) sending \( 1 \mapsto \sum_{i=1}^{n} a_i \otimes b_i \). This map is independent of the given pair of dual bases. In our present situation, this gives rise to a map \( \Delta : RS^{n-1} \to R^S_{n-1} \otimes RW \) defined by \( \Delta(1) = \sum_{i=0}^{n-1} (-1)^i x_n^i \otimes e_{n-i-1}(x_1, \ldots, x_{n-1}) = \prod_{i=1}^{n-1} (y_i - x_n) \).

We identify \( R^S_{n-1} \) with a subspace of \( \mathbb{Q}[x] \), and we identify \( R^S_{n-1} \otimes RW \) with a subquotient of \( \mathbb{Q}[x, y] \). Let us abbreviate \( e_k := e_k(y_1, \ldots, y_{n-1}) \) and \( \bar{e}_k := e_k(y_1, \ldots, y_n) \). The above discussion gives us a map \( RS^{n-1} \to R^S_{n-1} \otimes RW \) such that

\[
\Delta(1) = \sum_{i=0}^{n-1} (-1)^i x_n^i e_{n-i-1} = \prod_{i=1}^{n-1} (y_i - x_n)
\]

Consider the following sequence of \((R^S_{n-1}, R^S_{n-1})\)-bimodules and maps of bimodules

\[
R^S_{n-1} \xrightarrow{d_0} R^S_{n-1} \otimes RW R^S_{n-1} \xrightarrow{d_1} R^S_{n-1} \otimes RW R^S_{n-1} \xrightarrow{d_2} R^S_{n-1}
\]

where \( d_0 = \Delta \), \( d_1 \) is multiplication by \( y_n - x_n \), and \( d_2 \) is the map which sends \( 1 \mapsto 1 \) (hence sends \( f(y) - f(x) \mapsto 0 \) for all polynomials \( f \)). We claim that the above is a chain complex. Clearly \( d_2 \circ d_1 = 0 \). The identity \( d_1 \circ d_0 = 0 \) follows since \( \prod_{i=1}^{n} (x_n - y_i) \) is symmetric in \( y_1, \ldots, y_n \), hence equals \( \prod_{i=1}^{n} (x_n - x_i) = 0 \) in \( R^S_{n-1} \otimes RW \).

Now we show that the complex is acyclic. By Theorem 5.1, we have \( R^S_{n-1} \cong \bigoplus_{i=0}^{n-1} (-x_n)^i RW \), hence

\[
R^S_{n-1} \otimes RW R^S_{n-1} \cong \bigoplus_{i=0}^{n-1} (-x_n)^i \otimes R^S_{n-1}
\]
as right $R^{S_{n-1}}$-modules. We may identify \( R^{S_{n-1}} \otimes_{R} R^{S_{n-1}} \) with \((R^{S_{n-1}})^{\otimes n-1}\) as right $R^{S_{n-1}}$-modules. With respect to this decomposition, \(d_0, d_1, d_2\) are represented by the matrices

\[
\begin{bmatrix}
  e_{n-1} \\
  e_{n-2} \\
  \vdots \\
  e_1 \\
  1
\end{bmatrix}, 
\begin{bmatrix}
y_n & 0 & \cdots & 0 & 0 & -\tilde{e}_{n-1} \\
1 & y_n & \cdots & 0 & 0 & -\tilde{e}_{n-2} \\
0 & 1 & \cdots & 0 & 0 & -\tilde{e}_{n-3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & y_n & -\tilde{e}_2 \\
0 & 0 & \cdots & 0 & 1 & y_n - \tilde{e}_1
\end{bmatrix}, \quad \begin{bmatrix}
1 & -y_n & \cdots & (-1)^{n-1} y_{n-1}
\end{bmatrix}
\]

The form of the last column of the middle matrix comes from expressing $x_n^m$ in terms of $\tilde{e}_i$ and $x_j^m$ for $1 \leq j < n$, using the relation 5.14. After Gaussian elimination, this complex is clearly acyclic, in fact contractible as a complex of left $R^{S_{n-1}}$-modules.

Now, to obtain the complex (2.31) from the complex (2.33), apply the functor $R \otimes_{R^{S_{n-1}}} (-) \otimes_{R^{S_{n-1}}} R$. Note that $R$ is free as a right or left $R^{S_{n-1}}$-module, hence this functor sends acyclic to complexes to acyclic complexes. This completes the proof. \(\Box\)

We will refer to the maps $\phi : B_{w_1}(n-1) \to B_{w_0}$ and $\psi : B_{w_0}(n-1) \to B_{w_1}$ as the canonical maps in the sequel. In case $n = 2$ we recover the usual canonical maps (the “dots” in the Elias–Khovanov diagram category \([10]\)) $B_s(1) \to R$ and $R(1) \to B_s$.

This $Z$ forms the building block of a \(J_n\)-resolution of $B_{w_1}$. That is, we consider the following semi-infinite chain complex

\[
\begin{array}{cccccc}
B_{w_1} & \rightarrow & B_{w_0} & \rightarrow & B_{w_1} \\
& & \downarrow{-1} & & \\
B_{w_1} & \rightarrow & B_{w_0} & \rightarrow & B_{w_1} \\
& & \downarrow{-1} & & \\
\cdots & & \downarrow{-1} & & \\
B_{w_0} & \rightarrow & B_{w_1} & & & \\
\end{array}
\]

where we are omitting the degree shifts. The unlabeled arrows are the canonical maps. This complex can be described as a colimit as in the discussion at the beginning of this section, and is acyclic. After Gaussian elimination we obtain:

**Proposition 2.34** Let $w_0 \in S_n$ and $w_1 \in S_{n-1} \subset S_n$ denote the longest words. Let $D_1$ denote the semi-infinite complex

\[
\cdots \xrightarrow{\gamma} B_{w_0}(3n-1) \xrightarrow{\alpha_{3n}} B_{w_0}(n+1) \xrightarrow{\gamma} B_{w_0}(n-1) \xrightarrow{\gamma} B_{w_1} \xrightarrow{0},
\]

and let $D_2$ denote the semi-infinite complex

\[
0 \xrightarrow{0} B_{w_1} \xrightarrow{\gamma} B_{w_0}(1-n) \xrightarrow{\gamma} B_{w_0}(-1-n) \xrightarrow{\alpha_{3n}} B_{w_0}(1-3n) \xrightarrow{\gamma} \cdots
\]
where the unlabeled arrows are the canonical maps, \( a_{nm} = \prod_{i=1}^{n-1} (y_i - x_n) \), and \( z = y_n - x_n \). Then \( D_1 \) and \( D_2 \) are acyclic. In particular \( D_1 \) represents a \( \mathcal{J}_n \)-resolution \( C \xrightarrow{\varepsilon} B_{w_1} \).

**Remark 2.35** Proposition 2.34 also gives us a method to construct \( \mathcal{J}_k \)-resolutions for \( k < n \). In particular, now suppose \( w'_0 \in S_k \) is the longest word in \( S_k \) and \( w'_1 \in S_{k-1} \subset S_k \) is the longest word in \( S_k \). Then we construct a \( \mathcal{J}_{k+1} \)-resolution of \( B_{w'_1} \) in an analogous manner as recorded below:

\[
\cdots \xrightarrow{\varepsilon} B_{w'_0} (3k - 1) \xrightarrow{a_{kk}} B_{w'_0} (k + 1) \xrightarrow{z} B_{w'_0} (k - 1) : B_{w'_1} \to 0,
\]

where \( a_{kk} \) and \( z \) are defined as above.

The polynomial \( a_{nn}(x, y) \) is a special case of Definition 2.40. We want to now iterate the above to construct a resolution of \( R \) by free graded \( R \otimes R^w \)-modules, and thus an explicit construction of \( P_{1n} \). We illustrate this with an example:

**Example 2.36** (Construction of \( P_{13} \)) Let \( n = 3 \). The discussion at beginning of this section gives \( P_{12} \) as a \( \mathcal{J}_2 \)-resolution of \( R \). We now replace each copy of \( B_s \) with its \( \mathcal{J}_3 \)-resolution given by Proposition 2.34. After doing this we get an object that resembles (2.21), but without the diagonal arrows. This is not a double complex, since the horizontal differential does not square to zero. Thus, in forming the “total complex” (precisely: convolution) it is necessary to add extra components to the differential, which in this case are certain diagonal arrows. Doing so gives us the expression for \( P_{13} \) in (2.21).

The existence of the diagonal arrows in the above example is implied by a standard fact from homological algebra:

**Lemma 2.37** (Functorial projective resolutions) Let \( X = (X, d) \) be a chain complex of objects in an abelian category \( C \). Suppose \( P_i \) is a projective resolution of \( X_i \), then we can replace each \( X_i \) with \( P_i \) to form a sequence

\[
P(X) = \cdots \to P_i \xrightarrow{d_i} P_{i-1} \to \cdots
\]

where \( \tilde{d}_i \) is the unique lift (up to homotopy) of \( d_i \). Let \( f_i : P_i \to X_i \) be the augmentation map of the projective resolution \( P_i \) of \( X_i \). There is a unique convolution \( \text{Tot}(P(X)) \) and the maps \( f_i \) determine a quasi-isomorphism \( f : \text{Tot}(P(X)) \to X \). The assignment \( X \mapsto \text{Tot}(P(X)) \) defines a functor \( \mathcal{K}^- (C) \to \mathcal{K}^- (C) \).

This is proven in many standard homological algebra texts such as Weibel (See: Lemma 5.7.2 and Exercise 5.7.3) [35].

**Corollary 2.38** For each \( X \in \mathcal{J}_k \) there is a \( \mathcal{J}_{k+1} \)-resolution of \( X \), canonical up to homotopy, denoted \( P(X) \xrightarrow{\varepsilon_X} X \). The assignment \( X \mapsto \text{Tot}(P(X)) \) defines a functor \( \mathcal{J}_k \to \mathcal{J}_{k+1} \).
Proof Let \( X = (\cdots \to X_i \xrightarrow{d_i} X_{i+1} \to \cdots) \in \mathcal{J}_k \) and let \( R' = R^S_{k+1} \otimes_{\mathbb{Q}[x_{k+2}, \ldots, x_n]} \). Then we can consider \( X \) as a complex of \((R', R')\)-bimodules. A \( \mathcal{J}_{k+1} \)-resolution \( P(X_i) \) of \( X_i \) (as given in Remark 2.35) gives a projective \((R', R')\)-bimodule resolution of \( X_i \), for all \( i \), by construction. Thus Lemma 2.37 gives a sequence

\[
P(X) = (\cdots \to P(X_i) \xrightarrow{d_i} P(X_{i+1}) \to \cdots).
\]

The convolution of this sequence is unique up to homotopy equivalence. Considering \( P(X) \) as an object of \( \mathcal{J}_{k+1} \subset \mathcal{K}^-(\mathcal{S}\text{Bim}_n) \) we receive the desired result. \( \square \)

This result together with Proposition 2.34 gives our construction of \( P_{1^n} \).

Construction 2.39 Consider \( P_{1^{n-1}} \) as a \( \mathcal{J}_{n-1} \)-resolution of \( R \), and let \( \Phi : \mathcal{J}_{n-1} \to \mathcal{J}_n \) denote the functor from Corollary 2.38. Then \( P_{1^n} = \Phi(P_{1^{n-1}}) \). More precisely, each chain module \( X_i \) of \( P_{1^{n-1}} \) is a direct sum of shifted copies of \( B_{w_1} \). Replace each \( X_i \) with the corresponding direct sum of shifted copies of

\[
\cdots \xrightarrow{y_n-x_n} B_{w_0}(3n-1) \xrightarrow{a_{nn}} B_{w_0}(n+1) \xrightarrow{y_n-x_n} B_{w_0}(n-1),
\]

The result is a sequence

\[
P = \cdots \to P_i \to P_{i-1} \to \cdots.
\]

Then \( \text{Tot}(P) \simeq P_{1^n} \) as objects of \( \mathcal{K}^-(\mathcal{S}\text{Bim}_n) \).

It is clear that in the \( P_{1^n} \) just constructed, the bimodule \( B_{w_n} \) appears with graded multiplicity \( \prod_{l=2}^n \frac{q^{-1}+r^{-1}}{q^{-l-1}-r^{-l-2}q^r} \). Substituting \( t = -1 \) yields \( \frac{1}{n!} \) (see also §1.3).

2.6 A combinatorial construction of the projector

Now that we have a rough idea of the structure of \( P_{1^n} \), we are ready to describe \( P_{1^n} \) explicitly. The diagonal arrows (see Diagram 2.21) are described by an explicit family of polynomials \( a_{ij}(x, y) \), which we define first.

Definition 2.40 Let \( a_{ij}(x, y) \) (\( 2 \leq i \leq j \)) denote the polynomials defined by

\[
a_{ij}(x, y) := \sum_{\gamma} \prod_{k=1}^{i-1} (y_{\gamma_k} - x_{\gamma_k+i-k})
\]

where the sum is over decreasing sequences \( \gamma = (\gamma_1, \ldots, \gamma_{i-1}) \) with \( j-1 \geq \gamma_1 > \cdots > \gamma_{i-1} \geq 1 \). Note that the degree of \( a_{ij} \) is \( i-1 \). By convention, we set \( a_{1,j} := 1 \) for all \( j \geq 1 \). We also find it useful to abbreviate \( p_{ij} := y_i - x_j \).

Example 2.41 For \( 1 < i \leq j \leq 4 \), the polynomials \( a_{ij} \) are given below:
(1) \( a_{22} = p_{12} \)
(2) \( a_{23} = p_{12} + p_{23} \).
(3) \( a_{24} = p_{12} + p_{23} + p_{34} \).
(4) \( a_{33} = p_{13}p_{23} \).
(5) \( a_{34} = p_{13}p_{23} + p_{13}p_{34} + p_{24}p_{34} \).
(6) \( a_{44} = p_{14}p_{24}p_{34} \).

Note that \( a_{22}, a_{23}, \) and \( a_{33} \) appear prominently in our diagrams which define \( P_{12} \) and \( P_{13} \) [Diagrams (2.20) and (2.21)]. Also, one of the differentials in the complexes from Proposition 2.34 appears as a special case of the polynomials \( a_{ij} \):  

**Example 2.42** We have \( a_{jj} = p_{1j}p_{2j} \cdots p_{j-1,j} \).

Recall the ideal \( I_n \in \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) generated by \( e_k(x) - e_k(y) \) \( (1 \leq k \leq n) \). In §5.2 we prove the following:  

**Proposition 2.43** The polynomials \( a_{ij}(x, y) \) satisfy \( \sum_{j=k}^{n} a_{i,j}(x, y)(x_j - y_j) = 0 \) modulo \( I_n \). In particular \( \sum_{j=k}^{n} a_{i,j}(x, y)(x_j - y_j) \) acts by zero on \( B_{w_0} \).

Using these polynomials we will define a complex (the equation \( d^2 = 0 \) will follow from Proposition 2.43), and then we will show that this complex is homotopy equivalent to \( P_{1}^\vee \), which is dual to \( P_1^\vee \) (see Definition 2.54 below). The reason that \( P_{1}^\vee \) is easier to describe is that it is naturally a unital algebra in \( \mathcal{K}^+(\text{S} \text{Bim}_n) \), whereas \( P_1^\vee \) is a coalgebra in \( \mathcal{K}^- (\text{S} \text{Bim}_n) \). The language of differential bigraded (dg) algebras and modules will help make the discussion brief, and will make manifest the periodicity in our complexes (evident in our expressions for \( P_{12} \) and \( P_{13} \)).

**Definition 2.44** Let \( u_k \) denote an even formal indeterminate of bidegree \( t^2q^{-2k} \), and let \( \theta_k \) denote an odd formal indeterminate of bidegree \( t^1q^{-2} \). Let \( A_n \) denote the super-polynomial algebra

\[
A_n := \mathbb{Q}[x, y, u_1, \ldots, u_n, \theta_1, \ldots, \theta_n]
\]

with \( \mathbb{Q}[x, y, u_1, \ldots, u_n] \)-linear differential determined by \( d_A(1) = 0 \) and

\[
d_A(\theta_k) = \sum_{i=1}^{k} a_{i,k}(x, y)u_i, \tag{2.45}
\]

for each \( k = 1, \ldots, n \), together with the graded Leibniz rule \( d(ab) = d(a)b + (-1)^{|a|}a d(b) \).

Note that the differential \( d_A \) is defined independently of \( n \). That is, the obvious inclusion \( A_n \rightarrow A_{n+1} \) is a chain map for all \( n \geq 1 \).

**Definition 2.46** Recall the ideal \( I_n \in \mathbb{Q}[x, y] \) from Notation 2.6. We define the \( A_n \)-module \( M_n := A_n/I_nA_n \). Denote the image of \( a \in A_n \) under the quotient map \( A_n \rightarrow M_n \) by \( \bar{a} \). Shift \( M_n \) so that \( \bar{1} \in M_n \) lies in degree \( q^{-2\ell} \) where \( \ell = \binom{n}{2} \) as usual. Define a differential \( d_M \) on \( M_n \) by the rules...
(1) \( d_M(\bar{1}) = \sum_{k=1}^{n} (y_k - x_k)\bar{\theta}_k \)
(2) \( d_M(am) = d_A(a)m + (-1)^{|a|} ad_M(m) \)

for all \( a \in A, m \in M \). Here, \( |a| \) denotes the homological degree of a homogeneous element \( a \in A \).

We check that the above differential is well-defined, hence makes \( M_n \) into a dg \( A_n \)-module. Since \( M \) is generated by \( \bar{1} \in M_n \) as a left \( A_n \)-module, \( d_M \) is unique (if it exists). On the other hand, the formula

\[
d_M(\bar{a}) = d(a\bar{1}) = d_A(a)\bar{1} + (-1)^{|a|} \sum_{k=1}^{n} (y_k - x_k)\bar{\theta}_k
\]

shows that \( d_M \) is a well-defined \( \mathbb{Q}[x, y, u_1, \ldots, u_n] \)-linear map \( M_n \to M_n \). We claim that \( d_M^2 = 0 \). Given the Leibniz rule (2), it suffices to show that \( d_M^2(\bar{1}) = 0 \). Applying \( d_M \) to the the equation \( d_M(\bar{1}) = \sum_{i=1}^{n} (y_i - x_i)\theta_i \bar{1} \) and using the Leibniz rule gives

\[
d_M^2(1) = \sum_{i=1}^{n} (y_i - x_i) d_A(\theta_i)\bar{1} - \sum_{i=1}^{n} (y_i - x_i) \theta_i d_M(\bar{1})
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} (y_i - x_i) a_{ij}(x, y) u_j \bar{1} - \sum_{i=1}^{n} \sum_{j=1}^{n} (y_i - x_i) (y_j - x_j) \theta_i \theta_j
\]

\[
= \sum_{j=1}^{n} u_j \sum_{i=j}^{n} (y_i - x_i) a_{ij}(x, y) + 0
\]

\[
= 0
\]

In the third line we used that the odd variables \( \theta_i \) anti-commute and square to zero. In the fourth line we appeal to Proposition 2.43.

Remark 2.47 We may visualize the complex \( M_n \) in the following way. Set \( B_n := \mathbb{Q}[x, y]/I_n \). Then \( M_n \cong B_n[u_1, \ldots, u_n, \theta_1, \ldots, \theta_n] \) is a direct sum of infinitely many copies of \( B_n \), indexed by the positive orthant \( \mathbb{Z}_+^n \). Multiplication by \( u_k \) acts by translating a distance of two units along the \( k \)-th axis. The summands corresponding to the “unit cube” \( \{0, 1\}^n \) contribute a copy of the exterior algebra \( B_n[\theta_1, \ldots, \theta_n] \subset A_n \), which then generates all of \( A_n \) under the action of the \( u_k \). The differential \( d_M \) makes the the unit cube \( B_n[\theta_1, \ldots, \theta_n] \) into a Koszul complex associated to the sequence \( p_{11}, p_{22}, \ldots, p_{nn} \in B_n \), and the \( \mathbb{Q}[u_1, \ldots, u_n]\)-equivariance ensures that the translates of the unit cube all have the same internal differential.

There are components between different copies of the unit cube which are encoded by the algebra differential \( d_A \); these are defined in terms of the polynomials \( a_{ij}(x, y) \).

Remark 2.48 There are reduced versions of \( A_n \) and \( M_n \) defined as follows. Set \( \tilde{\theta}_k := \theta_k - \theta_1 \). Then set \( \tilde{A}_n = \mathbb{Q}[x, y, u_2, \ldots, u_n, \tilde{\theta}_2, \ldots, \tilde{\theta}_n] \) with differential determined by \( d(\tilde{\theta}_k) = \sum_{i=2}^{k} u_i a_{ik}(x, y) \). The reader may check that \( A_n \cong B \otimes_{\mathbb{Q}} \tilde{A}_n \), where \( B = \mathbb{Q}[u_1, \theta_1] \) with differential \( d(\theta_1) = u_1 \). Clearly \( B \cong \mathbb{Q} \), so that \( A_n \cong \tilde{A}_n \).
Similarly, the reduced version of $M_n$ is $\tilde{M}_n = \tilde{A}_n/I_n\tilde{A}_n$ with differential determined by $d_M(\tilde{1}) = \sum_{i=2}^n p_i\tilde{a}_i$. The reader may check that $M_n \cong B \otimes \tilde{M}_n \cong M_n$.

By $Q[x, y, u_1, \ldots, u_n]$-equivariance, $M_n$ is determined by the $d_M$ applied to a product of $\theta_k$'s. It is instructive to work these out in the cases $n = 2, 3$; The $n = 3$ case is done in §1.3 of the introduction.

After extending scalars, we will regard $A_k$ and $M_k$ as complexes of $Q[x, y]$-modules, for all $1 \leq k \leq n$. For example $A_1 = Q[x, y, u_1, \theta_1]$ with differential $d_A(\theta_1) = u_1$, and $M_1 = Q[x, u_1, \theta_1]$ with differential $d_M(\tilde{1}) = p_{11}\tilde{a}_1$. Clearly $A_1 \cong Q[x, y]$ and $M_1 \cong Q[x]$. We wish to relate $M_{n-1}$ with $M_n$. The following is obvious, since the $d_A(\theta_k)$ does not involve $n$:

**Proposition 2.49** There is a unique map of dg $Q[x, y]$-algebras $A_{n-1} \rightarrow A_n$ sending $\theta_k \rightarrow \theta_k$ for $1 \leq k \leq n-1$.

Thus, by restriction, we regard $M_n$ as a dg $A_{n-1}$-module. We want to construct a canonical map $M_{n-1} \rightarrow M_n$ of dg $A_{n-1}$-modules. First, note that from Proposition 2.30, there is a well defined $Q[x, y]$-module map

$$Q[x, y]/I_{n-1} \rightarrow Q[x, y]/I_n \quad \bar{1} \mapsto \prod_{i=1}^{n-1} (y_i - x_n)\bar{1}$$

(2.50)

**Proposition 2.51** There is a unique map $\phi : M_{n-1} \rightarrow M_n$ of dg $A_{n-1}$-modules sending $\bar{1} \mapsto \prod_{i=1}^{n-1} (y_i - x_n)\bar{1} \in M_n$.

We remark that $\phi$ is analogous to the map (2) in Proposition 2.30. Also note that $\phi$ preserves the bidegrees. This is the reason for choosing $\deg_q(\bar{T}_n) = n - n^2$, where $\bar{T}_n = \bar{1} \in M_n$.

**Proof** Uniqueness is clear, since $M_{n-1}$ is generated by $\bar{1}$ as an $A_{n-1}$-module. The map (2.50) extends to a map $\phi : M_{n-1} \rightarrow M_n$ of $A_{n-1}$-modules. We must check that $\phi$ commutes with the differentials $d_{M_n}$ and $d_{M_{n-1}}$. The computation reduces easily to the computation $\phi(d_{M_{n-1}}(\bar{T})) = d_{M_n}(\phi(\bar{T}))$, which can be checked directly:

$$\phi(d_{M_{n-1}}(\bar{T})) = \phi\left(\sum_{i=1}^{n-1} (y_i - x_i)\phi(\theta_i\bar{1})\right)$$

$$= \sum_{i=1}^n \theta_i(y_i - x_i) \prod_{j=1}^{n-1} (y_j - x_n)\bar{1}$$

$$= \sum_{i=1}^n \theta_i(y_i - x_i) \prod_{j=1}^{n-1} (y_j - x_n)\bar{1}$$

$$= d_{M_n}(\phi(\bar{T}))$$

In the third line we used the fact that $\prod_{i=1}^n (y_i - x_n)$ is zero in $Q[x, y]/I_n$, by Proposition 5.15 (see also Example 5.13).
Lemma 2.52 The map $\phi : M_{n-1} \to M_n$ is a quasi-isomorphism.

Proof We prove this by constructing a filtration on $\text{Cone}(\phi)$ whose subquotients are the acyclic complex $D_2$ from Proposition 2.34.

First, note that there is an obvious isomorphism

$$A_n \cong \mathbb{Q}[x, y, u_n, \theta_n] \otimes_{\mathbb{Q}} \mathbb{Q}[u_1, \ldots, u_{n-1}, \theta_1, \ldots, \theta_{n-1}]$$

as algebras (ignoring the differentials, and also using the usual sign rule for the tensor product of two superalgebras). Taking the quotient by the ideal generated by $e_k(y) - e_k(x)$ gives an isomorphism

$$M_n \cong (\mathbb{Q}[x, y, u_n, \theta_n]/I_n) \otimes_{\mathbb{Q}} \mathbb{Q}[u_1, \ldots, u_{n-1}, \theta_1, \ldots, \theta_{n-1}]$$

Similarly, we have

$$M_{n-1} \cong (\mathbb{Q}[x, y]/I_{n-1}) \otimes \mathbb{Q}[u_1, \ldots, u_{n-1}, \theta_1, \ldots, \theta_{n-1}]$$

We filter each of the above complexes by homological degree on the second tensor factor. We remark that any homogeneous element $c_1 \otimes c_2$ of any of the above complexes satisfies $\deg_h(c_1 \otimes c_2) = \deg_h(c_1) + \deg_h(c_2)$. Since $\deg_h(c_1) \geq 0$, this means that the filtration degree of any homogeneous element is bounded above by its homological degree. We will use this fact at the end of this proof.

The differentials respect the filtrations, as does the map $\phi$. With respect to this filtration we have

1. $d_A(1 \otimes a_2) = 0 + $ (higher) since we are filtering by homological degree on the second factor.
2. $d_A(\theta_n \otimes 1) = a_{nn}(x, y)u_n \otimes 1 + $ (higher).
3. $d_M(\overline{1} \otimes 1) = (y_n - x_n)\theta_n \otimes 1 + $ (higher).

It is now straightforward to check that the induced filtration on $\text{Cone}(\phi)$ has associated graded

$$\text{gr Cone}(\phi) \cong D \otimes_{\mathbb{Q}} \mathbb{Q}[u_1, \ldots, u_{n-1}, \theta_1, \ldots, \theta_{n-1}]$$

where $D_2$ is the complex

$$\mathbb{Q}[x, y]/I_{n-1} \xrightarrow{d_{nn}} \mathbb{Q}[x, y]/I_n \xrightarrow{ym - x_n} \mathbb{Q}[x, y]/I_n \xrightarrow{d_{nn}} \mathbb{Q}[x, y]/I_n \xrightarrow{ym - x_m} \cdots$$

Since $D_2$ is acyclic, it follows that $\text{gr Cone}(\phi)$ is acyclic. The following argument shows that $\text{Cone}(\phi)$ is acyclic, hence $\phi$ is a quasi-isomorphism.

We have a filtration $\text{Cone}(\phi) = F^0 \supset F^1 \supset \cdots$ whose associated graded is acyclic. This implies that any class $z \in H(F^k)$ is homologous to a class in $H(F^{k+1})$, as follows from the long exact sequence in homology associated to the short exact sequence

$$0 \to F^{k+1} \to F^k \to F^k/F^{k+1} \to 0.$$
In particular, any class $z \in H(\text{Cone}(\phi))$ is homologous to a class in $H(F^k)$ for all $k \geq 0$. On the other hand, by the remark above, the filtration degree of any nonzero homogenous element $c \in \text{Cone}(\phi)$ is bounded above by its homological degree. It follows that any class $z \in H(\text{Cone}(\phi))$ is null-homologous. This completes the proof that $\text{Cone}(\phi)$ is acyclic.

Now we show $M_n$ is dual to $P_1^n$.

**Definition 2.54** If $M$ is a graded $(R, R)$-bimodule, set $M^\vee = \text{Hom}_R(M, R)$, where this latter object is the graded space of right $R$-module maps $M \to R$. Since $R$ is commutative, this is a graded $(R, R)$-bimodule via $(\alpha \cdot f \cdot \beta)(m) = f(\beta m \alpha)$ for all $\alpha, \beta \in R$, $m \in M$, $f \in M^\vee$.

It is well known that if $w_I \in S_n$ is the longest word of a parabolic subgroup $W_I \subset S_n$ (see §2.1), then $B_{w_I}^\vee \cong B_{w_I}$. In particular, $B_s^\vee \cong B_s$, so the duality functor sends Soergel bimodules to Soergel bimodules. Extending to complexes gives a contravariant functor $\mathcal{K}^- (S\text{Bim}_n) \leftrightarrow \mathcal{K}^+ (S\text{Bim}_n)$.

**Proposition 2.55** Suppose $C \in \mathcal{K}^+(S\text{Bim}_n)$ is a complex whose chain bimodules are direct sums of copies of $B_{w_0}^\vee$ with grading shifts, and suppose there is a quasi-isomorphism $\nu : R \to C$. Then $C^\vee \cong P_1^n$, with structure map $\nu^\vee : C^\vee \to R$.

**Proof** Since Soergel bimodules are free as right $R$-modules, $(-)^\vee$ is an exact functor. In particular, $H(C) \cong 0$ if and only if $H(C^\vee) \cong 0$, and $\nu : R \to C$ is a quasi-isomorphism if and only if $\nu^\vee$ is a quasi-isomorphism. Therefore, $\text{Cone}(\nu^\vee)$ is an acyclic complex. From the remarks preceding the proposition, the chain bimodules of $C^\vee$ are direct sums of shifted copies of $B_{w_0}^\vee \cong B_{w_0}$, so that $C^\vee \in \mathcal{I}$. By Corollary 2.15, this implies that $\text{Cone}(\nu^\vee) \in \mathcal{I}^\perp \cap \mathcal{I}^\perp$. Thus, $(C^\vee, \nu^\vee)$ satisfies the axioms from Theorem 2.17, so $C^\vee \cong P_1^n$, by uniqueness. □

**Theorem 2.56** $M_n \cong P_1^n$.

**Proof** Composing the quasi-isomorphisms from Lemma 2.52 gives another quasi-isomorphism $\varepsilon : \mathbb{Q}[x] = M_1 \to M_n$. Note that $\varepsilon$ is a map of dg $A_1 = \mathbb{Q}[x, y]$-modules, that is, complexes of $(R, R)$-bimodules. Proposition 2.55 now says that $M_n^\vee \cong P_1^n$, which completes the proof. □

### 3 The projector as an infinite full twist

We introduce the Rouquier complexes, which define an action of the braid group on the homotopy category of complexes of Soergel bimodules. Upon restricting to $\mathcal{I} \in \mathcal{K}^- (S\text{Bim}_n)$ from Definition 2.7, this action factors through the symmetric group. We then use this fact to show that the projector $P_1^n$ is a homotopy colimit of Rouquier complexes associated to powers of the full twist.

#### 3.1 Action of the symmetric group

In this section we construct a categorical action of the symmetric group on the ideal $\mathcal{I} \subset \mathcal{K}^- (S\text{Bim}_n)$ from Definition 2.7.
Given \( w \in W \), let \( R_w \) denote the \((R, R)\)-bimodule which equals \( R \) as a left module, and whose right \( R \)-action is twisted by \( w \). That is, \( f \cdot h \cdot g = fhw(g) \) for all \( f, g \in R, h \in R_w \). The bimodules \( R_w \) are not Soergel bimodules, but are often referred to as standard bimodules.

Note the important difference between superscripts and subscripts: \( R_s \) is the algebra of \( s \)-invariant polynomials, whereas \( R_s \) is the \((R, R)\)-bimodule whose right \( R \)-action is twisted by \( s \).

Proposition 3.2 We have \( B_w R_w \cong B_w \) for every \( w \in S_n \). If \( f = f(x, y) \), thought of as an endomorphism of \( B_w \), then the following square commutes:

\[
\begin{array}{ccc}
B_w R_w & \xrightarrow{\id_{B_w} \otimes f} & B_w R_w \\
\cong & & \cong \\
B_w R_w & \xrightarrow{f(x, w^{-1}(y))} & B_w
\end{array}
\]

There is a similar description of the functor \( R_w \otimes (-) \).

Proof Ignore grading shifts throughout this proof, and identify \( B_w \) with the quotient \( \mathbb{Q}[x, y]/I_n \). By abuse of notation, we will denote elements of \( B_w \) by \( g(x, y) \). Note that \( B_w R_w = \mathbb{Q}[x, y]/I_n \), but with \( \mathbb{Q}[x, y] \)-module structure \( g(x, y) \cdot 1 = g(x, w(y)) \).

With these identifications, it is trivial to check that \( g(x, y) \mapsto g(x, w(y)) \) defines an isomorphism \( B_w \rightarrow B_w R_w \). The statement regarding commutativity of the square is equally straightforward.

Proposition 3.2 implies that the functor \( (-) \otimes R_w \) gives a right action of the symmetric group on \( \mathcal{I} \). We will give a convenient description of this action.

Definition 3.3 Recall that any \( C \in \mathcal{I} \) is homotopy equivalent to a complex whose chain bimodules are direct sums of shifted copies of \( B_w \). Note that each component of the differential can be regarded a matrix with entries in \( \mathbb{Q}[x, y] \). Let \( w(C) \) denote the complex with the same underlying bimodule, but with differential obtained from \( d_C \) by applying \( f(x, y) \mapsto f(x, w(y)) \).

The following is clear:

Proposition 3.4 For each \( C \in \mathcal{I} \) and each \( w \in S_n \), we have \( w(C) \cong CR_{w^{-1}} \).

Definition 3.5 We will refer to the complexes \( w(P_1^n) \) as twisted projectors.

Remark 3.6 The explicit formulae for \( P_1^n \) also apply to \( w(P_1^n) \), by applying the transformation \( f(x, y) \mapsto f(x, w(y)) \) everywhere.

### 3.2 Action of the braid group

In this section we construct an action of the braid group on \( K^-(\mathbb{S}
\text{Bim}_n) \), originally due to Rouquier (see also [28]). We show that when restricted to \( \mathcal{I} \subset K^-(\mathbb{S}
\text{Bim}_n) \),
this action factors through the symmetric group. This fact will be used in our argument that $P_n$ is a limit of Rouquier complexes.

The following is standard in the theory of Soergel bimodules [10]:

**Proposition 3.7** We have short exact sequences

\[
0 \to R_s(1) \to B_s \to R(-1) \to 0 \quad (3.8)
\]
\[
0 \to R(1) \to B_s \to R_s(-1) \to 0 \quad (3.9)
\]

□

The complexes $F_s^{\pm}$ defined below should be thought of as certain kinds of deformations of $R_s$. They are quasi-isomorphic but not chain homotopy equivalent to $R_s$.

**Definition 3.10** Define the following complexes of Soergel bimodules:

\[
F_s := \left( B_s \xrightarrow{1} R(-1) \right)
\]
\[
F_s^{-1} := \left( R(1) \xrightarrow{x_1 - y_2} B_s \right)
\]

where we have underlined the degree zero chain bimodules, and we are using the conventions of Notation 2.6. Suppose $b$ is a braid word, i.e. a word in $(s, \pm)$, where $s$ is a simple transposition. We will denote the corresponding product $F_s^{\pm} \cdots F_s^{\pm}$ simply by $F_b$. We call each $F_b$ a Rouquier complex. If $w$ is a word in the simple transpositions we will denote by $F_w^+$, $F_w^-$ the Rouquier complex associated to the positive (resp. negative) braid lift of $w$.

When more convenient, we may also denote $F_b$ as $F(b)$. If two braid words $b$ and $b'$ represent the same braids, then $F_b \cong F_{b'}$, and the homotopy equivalence is canonical up to homotopy [11].

**Definition 3.11** Let $\phi_s : R_s(1) \to F_s$ denote the chain map induced by the first map in the short exact sequence (3.8). For any positive braid word $\beta$ with length $r$, let $\phi_\beta : R_s(r) \to F(\beta)$ denote the corresponding tensor product of maps $\phi_s$. Similarly, let $\psi_s : F_s^{-1} \to R_s(-1)$ denote the chain map induced by the second map in the short exact sequence (3.9), and let $\psi_\beta : F(\beta) \to R_s(-r)$ denote the corresponding tensor product of maps $\psi_s$, where $\beta$ is a negative braid with length $r$.

**Proposition 3.12** The maps $\phi_\beta : R_w(r) \to F(\beta)$ and $\psi_\beta : F(\beta) \to R_w(-r)$ are quasi-isomorphisms.

**Sketch of proof** First, note that exactness of the sequences (3.8) and (3.9) implies that $\phi_s$ and $\psi_s$ are quasi-isomorphisms. A standard homological algebra fact states that the derived tensor product of two quasi-isomorphisms is a quasi-isomorphism. Soergel bimodules are free as left or right $R$-modules, so for complexes of Soergel bimodules, derived tensor product coincides with ordinary tensor product. □

As a corollary we obtain the following:
Theorem 3.13 The categorical braid group action described above fixes $\mathcal{I}$. This categorical braid group action on $\mathcal{I}$ factors through the symmetric group. More precisely, let $w = (s_1^{v_1}, \ldots, s_r^{v_r})$ denote a braid word ($v_i = \pm 1$), let $w \in S_n$ denote the permutation represented by $w$, and let $e = v_1 + \cdots + v_r$ denote the braid exponent. Then $C \mapsto F_w C \simeq R_e C(e)$ and $C F_w \simeq C R_w(e)$ for all $C \in \mathcal{I}$. This is an isomorphism of functors $\mathcal{I} \to \mathcal{I}$.

Proof First note that $F_s B_{w0} = (B_s B_{w0} \to R(-1) B_{w0})$. It is a well-known fact (see [10]) that $B_s B_{w0} \simeq B_{w0}(-1) \oplus B_{w0}(1)$. This proves that $F_s$ sends objects in $\mathcal{I}$ to objects in $\mathcal{I}$ and this the categorical braid group action described above fixes $\mathcal{I}$.

For the second claim, it suffices to show that tensoring with $R_s(\pm 1)$ and $F_s^{\pm 1}$ give isomorphic functors $\mathcal{I} \to \mathcal{I}$. From Proposition 3.12, $\phi_s : R_s(1) \to F_s$ and $\psi_s : F_s^{-1} \to R(-1)$ are quasi-isomorphisms. By Corollary 2.14, these quasi-isomorphisms become homotopy equivalences after tensoring with any object of $\mathcal{I}$. Thus, $\phi_s$ and $\psi_s$ define natural isomorphisms of endofunctors of $\mathcal{I}$.

3.3 Constructing the projector via infinite full-twists

In this section we show that $P_1^\vee_n$ is a homotopy limit of Rouquier complexes. As a result, $\mathrm{HHH}(P_1^\vee_n)$ is a limit of Khovanov–Rozansky homologies of torus links. We will compute the homology $\mathrm{HHH}(w(P_1^\vee_n))$ for all $w$ in subsequent sections.

Remark 3.14 We focus on the dual projector $P_1^\vee_n$ for the rest of this section. This discussion can easily be altered to work for $P_1$ as well. We make this choice because we focus on $P_1^\vee_n$ in the sequel.

Definition 3.15 Fix an integer $n \geq 1$ and let $H_n T_n = F_{w0}$ denote Rouquier complex associated to the positive lift of the longest word. This is the half-twist on $n$ strands, which has $\ell = \frac{1}{2} n(n - 1)$ crossings. Let $FT_n := H_n T_n$ denote the Rouquier complex associated to the positive full twist on $n$-strands. Let $\phi : (2 \ell) \to FT_n$ denote the quasi-isomorphism from Proposition 3.12. The index $n$ is fixed throughout this section, so we will omit the subscript throughout.

The following is a special case of Theorem 3.13:

Corollary 3.16 For any $C \in \mathcal{I}$, the maps $\text{Id}_C \otimes \phi : C(2\ell) \to C \otimes FT$ and $\phi \otimes \text{Id}_C : C(2\ell) \to FT \otimes C$ are homotopy equivalences. In particular $FT \otimes P_1 \simeq P_1 \otimes (2\ell)$. □

Example 3.17 The full twist on two strands is homotopy equivalent to the complex

$$FT_2 \simeq (B_2(1) \xrightarrow{y_2 - x_2} B_2(-1) \to R(-2))$$

The chain map $\phi : (2) \to FT_2$ has mapping cone

$$\text{Cone}(\phi) \simeq (R(2) \xrightarrow{y_2 - x_2} B_2(1) \xrightarrow{y_1 - x_2} B_2(-1) \to R(-2))$$

This complex is acyclic by Proposition 2.30, hence kills $B_2$ from the left and right by Corollary 2.15. It is a useful exercise to prove this directly.
Lemma 3.18 Let $k \geq 1$ be a given integer. There is a complex $X \simeq \text{HT}$ such that the chain bimodules of $X^\otimes k$ in homological degrees $< k$ are direct sums of $B_{w_0}$ with shifts.

Proof First, note that HT is supported in non-negative homological degrees, since it is the Rouquier complex associated to a positive braid. In the case $k = 1$, we appeal to the fact (see Theorem 6.9 in [13]) that if $F(\beta)$ is the Rouquier complex associated to the positive braid lift of a reduced expression for $w \in S_n$, then stripping off contractible summands yields the minimal complex $(F(\beta))_{\text{min}} \simeq F(\beta)$ whose degree zero chain bimodule is $B_w$. Setting $w = w_0$ proves the $k = 1$ case of the proposition.

Let $X = \text{HT}_{\text{min}}$, so that the degree zero chain bimodule of $X$ is $X_0 = B_{w_0}$. Then the degree $j$ chain bimodule of $X^\otimes k$ is

$$(X^\otimes k)_j = \bigoplus_{i_1 + \cdots + i_k = j} X_{i_1} \cdots X_{i_k}$$

If $j < k$, then this forces at least one of the indices $i_m$ to satisfy $i_m = 0$, hence at least one of the factors above equals $B_{w_0}$. By Proposition 2.5, this forces $(C^\otimes k)_j$ to be a direct sum of $B_{w_0}$ with shifts.

Now basic idea of stablization is this: by the above, the homological degree $< 2k$ chain bimodules of $X^\otimes 2k \simeq \text{FT}^\otimes k$ form an object of $I$. By Corollary 3.16, FT fixes objects of $I$ up to shift, so $\text{FT}^\otimes k+1$ and $\text{FT}^\otimes k$ are homotopy equivalent in homological degrees $< 2k$, up to a grading shift. This is the stablization we seek. We now make these arguments precise. We first define our directed system, and the notion of homotopy colimit.

Definition 3.19 For $k \geq 1$ set $f_k := f_0 \otimes \text{Id}^\otimes k : \text{FT}^\otimes k(-2k\ell) \to \text{FT}^\otimes k+1(-2(k+1)\ell)$.

Definition 3.20 Let $\{A_k, f_k : A_k \to A_{k+1}\}_{k=0}^{\infty}$ be a direct system of chain complexes. The homotopy colimit of $\{A_k, f_k\}$ is by definition the mapping cone

$$\text{hocolim} A_k := \text{Cone} \left( \bigoplus_{k=0}^{\infty} A_k \xrightarrow{\text{Id} - S} \bigoplus_{k=0}^{\infty} A_k \right)$$

Where $S : \bigoplus_{k=0}^{\infty} A_k \to \bigoplus_{k=0}^{\infty} A_k$ is the chain map with components given by the $f_k$.

Theorem 3.21 We have $P^\vee_1 \simeq \text{hocolim} A^\otimes k$, where $\{A^\otimes k, f_k\}$ is the directed system from Definition 3.19.

The proof utilizes several lemmas. First, set $P := P^\vee_1$. Let $\eta = \varepsilon^\vee : R \to P^\vee_1$ be the unit map, and set $C := \text{Cone}(\eta)\langle 1 \rangle$. Note that $P$ and $C$ are supported in non-negative homological degrees. The reader may anticipate that this fact will be used in an essential way in the proof, since otherwise we might work with $P^\vee_1$ instead of $P^\vee_1$. We will utilize the existence of an exact triangle

$$P(1) \to C \to 1 \to P$$

(3.22)
in $\mathcal{K}^+(\mathbb{S}\text{Bim}_n)$, where $\mathbb{I} = R$ is the trivial bimodule, and $\langle 1 \rangle$ denotes the upward shift in homological degree. Also, the properties of $P$ (Theorem 2.17) ensure that

$$PC \simeq CP \simeq 0,$$  \hspace{1cm} (3.23)

In the language of [19], this means that $(P, C)$ is a pair of complementary idempotents in $\mathcal{K}^+(\mathbb{S}\text{Bim}_n)$. An object $A \in \mathcal{K}^+(\mathbb{S}\text{Bim}_n)$ is fixed by $P$ if and only if it is annihilated by the complementary idempotent $C$, and vice versa. In particular, we have:

**Lemma 3.24** $B_{w_0}C \simeq 0 \simeq CB_{w_0}$.

**Proof** The proof of Theorem 2.17 (after applying the duality functor) says that $PB_{w_0} \simeq B_{w_0}$, hence $CB_{w_0} \simeq (CP)B_{w_0} \simeq 0$ by (3.23). A similar argument shows that $B_{w_0}C \simeq 0$. \hfill $\square$

**Lemma 3.25** $HT^\otimes_k C$ is homotopy equivalent to a complex which is supported in homological degrees $\geq k$.

**Proof** Follows easily from Lemmas 3.24 and 3.18. \hfill $\square$

**Lemma 3.26** Suppose $\{N_k, f_k : N_k \to N_{k+1}\}$ is a directed system of complexes, and $N_k$ is homotopy equivalent to a complex which is supported in homological degrees $\leq c_k$, where $c_k \to -\infty$ as $k \to \infty$. Then hocolim $N_k$ is contractible.

**Proof** This is a fairly standard argument. See, for instance, Lemma 2.32 in [18]. \hfill $\square$

**Proof of Theorem 3.21** We first claim that Cone($f_k$) is homotopy equivalent to a complex which is supported in homological degrees $\geq 2k - 1$, hence $f_k$ should be thought of as a homotopy equivalence in degrees $< 2k - 1$. First, observe that by Corollary 3.16, Cone($f_0$)$P \simeq 0$. Tensoring the exact triangle (3.22) on the left with Cone($f_0$) gives an exact triangle

$$0 \to \text{Cone}(f_0)C \to \text{Cone}(f_0) \to 0.$$  

A standard fact about mapping cones now implies that Cone($f_0$)$C \simeq \text{Cone}(f_0)$. For $k \geq 0$, we have $f_k = f_0 \otimes \text{Id}^\otimes k$, so

$$\text{Cone}(f_k) \cong \text{Cone}(f_0) \otimes FT^\otimes k(-2k\ell) \cong \text{Cone}(f_0)C FT^\otimes k(-2k\ell)$$

Cone($f_0$) is supported in degrees $\geq -1$ (because of the shift in degrees in forming the mapping cone), and $C \otimes FT^\otimes k$ is supported in degrees $\geq 2k$ by Lemma 3.25, so Cone($f_k$) is supported in degrees $\geq 2k - 1$. Thus, the directed system $[FT^\otimes k(-2k\ell), f_k]$ is Cauchy in the sense of Rozansky [30], and the homotopy colimit $L := \text{hocolim} FT^\otimes k(-2k\ell)$ is defined in $\mathcal{K}^+(\mathbb{S}\text{Bim}_n)$.

We now claim that $LC \simeq 0$ and $LP \simeq P$. Suppose we have shown this. Tensor the exact triangle (3.22) on the left with $L$. Since $LC \simeq 0$, the result is an exact triangle $LP \langle 1 \rangle \to 0 \to L \to LP$, which implies that $LP \simeq L$. On the other hand $LP \simeq P$, so $P \simeq L$. 
To complete the proof therefore, we must show that $LC \simeq 0$ and $LP \simeq P$. By definition, $L$ is the homotopy colimit of complexes $FT \otimes k(-2k\ell)$, and so $LC$ is the homotopy colimit of complexes $FT \otimes k C(-2k\ell)$, which is contractible by Lemma 3.26 and Lemma 3.25. That is to say, $LC \simeq 0$.

Finally, we will show that $LP \simeq P$. There is a map of directed systems

$$\begin{array}{ccccccccc}
P & P \phi & P \otimes FT & P \phi \otimes FT & P \phi \otimes FT^2 & P \phi \otimes FT^3 & \cdots \\
P & P & P \phi & P \phi \otimes FT & P \phi \otimes FT^2 & P \phi \otimes FT^3 & \cdots \\
P & P & P & P & P & P & \cdots \\
\end{array}$$

(3.27)

where we are abusing notation by denoting $\text{Id}_X$ by $X$, for any chain complex $X$, and we are omitting all explicit grading shifts. The homotopy colimit of the bottom row is simply $P$, and the homotopy colimit of the top row is $PL$. By Corollary 3.16, each vertical arrow is homotopy equivalence. It is a standard fact (and an easy exercise) to deduce from this that the induced map $P \to PL$ is a homotopy equivalence. This completes the proof. \qed

4 Computations of stable homology

In this section we introduce the functor HHH, whose input is a complex of Soergel bimodules and whose output is a triply graded vector space. We introduce matrix factorizations as a way of expressing $P_{1^n}$ (or $P_{1^n}^\vee$) via compact formulae. Using this we compute $\text{HHH}(w(P_{1^n}^\vee))$ for arbitrary permutations $w \in S_n$, proving a conjecture of Gorsky–Rasmussen. In case $w \in S_n$ is an $n$-cycle, $\text{HHH}(w^m(P_{1^n}^\vee))$ is a stable limit of Khovanov–Rozansky homology (or briefly KR homology in the sequel) of the $(n, kn + m)$-torus links as $k \to \infty$.

4.1 Hochschild cohomology

We refer the reader to [22] and references therein for more details on Hochschild (co)homology and the connection to link homology. We review only a few essential facts here.

Let $R^e = R \otimes \mathbb{Q}$. We will regard graded $(R, R)$-bimodules as graded left $R^e$-modules. The Hochschild cohomology of $M$ is defined by

$$\text{HH}^k(R; M) = \bigoplus_{i,j} \text{Ext}^j_{R^e}(R(i), M).$$

We let $\text{HH}(R; M)$ denote the bigraded space $\text{HH}(R; M) = \bigoplus_k \text{HH}^k(R; M)$. When $R$ is understood, we omit it from the notation and write $\text{HH}(M)$. The degree of a homogeneous element $z \in \text{HH}(M)$ will be written multiplicatively, as $\deg(z) = q^i a^j$, here $a$ indicates the Hochschild degree. By definition $\text{HH}^0(M)$ is the graded of graded bimodule maps $R \to M$. This will be denoted $\text{Hom}(R, M)$. 
We may regard $HH$ as a functor from the category of $(R, R)$-bimodules to the category of $R$-modules, where all objects are understood to be graded in our case. The $R$-action on $HH(M)$ is induced by the $R$-action on the first argument of $\text{Ext}_{R^e}(R, M)$.

Since $HH$ is a linear functor, we can extend to complexes. If $C$ is a complex of graded $(R, R)$-bimodules, then $HH(C)$ is the complex such that $HH(C)_k = HH(C_k)$, with differential $d_{HH(C)} = HH(d_C)$. Note that $HH(C)$ is triply graded; the differential has tridegree $(0, 0, 1)$.

**Definition 4.1** If $C$ is a complex of graded $(R, R)$-bimodules, let $HH(C)$ denote the complex obtained by applying $HH$ to the bimodules of $C$. Set $HHH(C) := H(HH(C))$. This is a triply graded vector space. By convention, $z \in HHH_{i,j,k}(C)$ means that $z$ has $q$-degree $i$, and Hochschild degree $j$, and homological degree $k$. In this case we write $\deg(z) = q^i a^j t^k$.

**Remark 4.2** It is possible to show (see [22] and §3 of [18]) that if $\beta$ is an $n$-strand braid and $F(\beta)$ is the Rouquier complex associated to $\beta$, then $HHH(F(\beta))$ depends only on the closure of $\beta$ up to isomorphism of triply graded vector spaces and an overall grading shift. The grading indeterminacy can be fixed by a renormalization; the resulting link invariant is KR homology. Thus, $HHH(P_{1^n}^\vee)$ computes the stable limit of KR homologies of the $(n, nk)$-torus links, and similarly for $HHH(w(P_{1^n}^\vee))$ where $w \in S_n$.

It is a standard property of Hochschild cohomology that $HH(M \otimes_R N) \cong HH(N \otimes_R M)$, and this isomorphism is natural in $M, N$. Naturality of this isomorphism means we can extend to complexes as well, and we obtain:

**Proposition 4.3** Suppose $C$ and $D$ are bounded above complexes of $(R, R)$-bimodules. Then $HH(C \otimes_R D) \cong HH(D \otimes_R C)$. □

We now turn our attention to the computation of $HHH(w(P_{1^n}^\vee))$, for $w \in S_n$. The following simplifies this task:

**Corollary 4.4** Let $w \in S_n$ be given. Then $HHH(w(P_{1^n}^\vee))$ depends only on the conjugacy class (cycle type) of $w$, up to isomorphism.

**Proof** Let $R_w$ be the standard bimodule, so that $w^{-1}(P_{1^n}^\vee) := P_{1^n}^\vee R_w$. Since $P_{1^n}^\vee$ is central, we have

$$P_{1^n}^\vee R_w \cong P_{1^n}^\vee R_{v^{-1}} R_{vw} \cong R_{v^{-1}} P_{1^n}^\vee R_{vw}$$

Applying $HHH(\cdot)$ gives

$$HHH(P_{1^n}^\vee R_w) \cong HHH(R_{v^{-1}} P_{1^n}^\vee R_{vw}) \cong HHH(P_{1^n}^\vee R_{vw} R_{v^{-1}}) \cong HHH(P_{1^n}^\vee R_{vwv^{-1}})$$

This completes the proof. □
Our next simplification says that $\text{HHH}(C)$ is especially simple whenever $C \in \mathcal{I}$, that is, if the chain bimodules of $C$ are direct sums of $B_{w_0}$ with shifts. Recall that $\text{End}(B_{w_0}) = R \otimes_{R^w} R \cong \mathbb{Q}[x, y]/I_n$ (Notation 2.6), so that we can regard the differential $d_k : C_k \rightarrow C_{k+1}$ as a matrix with entries from $\mathbb{Q}[x, y]/I_n$.

**Proposition 4.5** If $C \in \mathcal{I}$, then $\text{HH}(C) \cong \Lambda[\xi_1, \ldots, \xi_n] \otimes \mathbb{Q} \text{HH}^0(C)$, where the $\xi_i$ are odd variables of degree $q^{-2i}a$. Furthermore, $\text{HH}^0$ is the functor that sends $B_{w_0} \mapsto R(\ell)$ and $\text{End}(B_{w_0}) \ni f(x, y) \mapsto f(x, x)$.

The rest of this section is concerned with proving this proposition. As before, let $R = \mathbb{Q}[x]$, $R^e = \mathbb{Q}[x, y]$, $\ell = \frac{1}{2}n(n-1)$. Write $B := B_{w_0} = \mathbb{Q}[x, y]/I_n$, as usual. Let $K$ denote the Koszul resolution of $R$ by free graded $R^e$-modules.

Below, $C$ will denote a complex whose chain bimodules are direct sums of $B$ with grading shifts. We use $C(i, j)$ to denote a grading shift in both $q$ and homological degree:

$$(C(i, j))_k = C_{k-j}(i).$$

**Lemma 4.6** The Hochschild cohomology of $B$ is isomorphic to a shift of a bigraded superpolynomial ring with odd variables $\xi_i$ and even variables $x_i$:

$$\text{HH}(B) \cong \mathbb{Q}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n](\ell),$$

where $\xi_i$ has degree $q^{-2i}a$ and $x_i$ has degree $q^2$.

**Proof** For this proof it is actually much more convenient to work with Hochschild homology $\text{HH}_\bullet$ rather than cohomology $\text{HH}^\bullet$. Hochschild homology is defined by $\text{HH}_{-k}(M) = \text{Tor}^R_k(R, M)$. For later convenience, our grading is such that $\text{HH}_\bullet$ is supported in negative Hochschild degrees. It is well known that for our graded polynomial ring $R$,

$$\text{HH}^\bullet(R; M) = q^{-2n}a^n \text{HH}_\bullet(R; M). \quad (4.7)$$

Throughout this proof, we use variables $q$ and $a$ to denote the grading shifts in those degrees. Now, recall $R^e = R \otimes R \cong \mathbb{Q}[x, y]$ and $q^\ell B \cong \mathbb{Q}[x, y]/I_n$ as in Notation 2.6. It is convenient to compute $\text{HH}_\bullet(q^\ell B)$ using the Koszul resolution:

$$K = \bigotimes_{i=1}^n (\mathbb{Q}[x, y](2i) \xrightarrow{e_i(x) - e_i(y)} \mathbb{Q}[x, y])$$

where the tensor product is over $\mathbb{Q}[x, y]$ and the underlined term is in Hochschild degree 0. We claim that $(e_i(x) - e_i(y))_{i=1}^n$ forms an $\mathbb{Q}[x, y]$-regular sequence. Suppose not, then

$$f_k(x, y)(e_k(x) - e_\gamma(y)) = \sum_{i=1}^{k-1} f_i(x, y)(e_i(x) - e_i(y)),$$

for some $f_i \in \mathbb{Q}[x, y]$, $1 \leq i \leq k$. If we set $y_j = 0$ for all $j = 1, \ldots, n$, then we get the equation
\[ F_k(x)(e_k(x)) = \sum_{i=1}^{k-1} F_i(x)(e_i(x)), \]

where \( F_j(x) \) is the image of \( f_j \in \mathbb{Q}[x] \) under the quotient map setting all \( y_j = 0 \). This implies that \( \{e_i(x)\}_{i=1}^k \) is not a regular sequence. However, it is a well known fact from invariant theory that this is not the case. Therefore, \( \{e_i(x) - e_i(y)\}_{i=1}^n \) forms a regular sequence in \( \mathbb{Q}[x, y] \).

Thus, \( \text{HH}_\bullet(q^\ell B) \) is computed by identifying the left and right actions on \( K \)—that is, setting \( y_i = x_i \) for \( 1 \leq i \leq n \)—and taking homology. Identifying the left and right actions yields

\[ R \otimes_{R^e} K = \bigotimes_{i=1}^n (R(2i) \rightarrow R), \]

If we let \( \xi_i \) be an odd variable of Hochschild degree \(-1\) and \( q \)-degree \( 2i \) then we can rewrite this as

\[ R \otimes_{R^e} K \cong \mathbb{Q}\left[ x, \xi_1^\vee, \ldots, \xi_n^\vee \right] \]

with zero differential. Taking homology gives the computation of \( \text{HH}_\bullet(q^\ell B) = q^\ell \text{HH}_\bullet(B) \). Given Eq. (4.7), we obtain \( \text{HH}^\bullet(B) = q^{-2n-\ell} a^n \mathbb{Q}[x, \xi_1^\vee, \ldots, \xi_n^\vee] \). It is a trivial exercise to check that this is isomorphic to \( q^\ell \mathbb{Q}[x, \xi_1, \ldots, \xi_n] \) as we claimed. The isomorphism is essentially the Hodge star operator, which for instance sends \( \xi_1 \cdots \xi_n \mapsto 1 \) and \( 1 \mapsto \xi_1 \cdots \xi_n \).

**Proof of Proposition 4.5** Suppose \( C \) is a complex whose chain bimodules are direct sums of copies of \( B = B_{w_0} \) with shifts. Lemma 4.6 tells us what happens to the bimodules upon application of the functor \( \text{HH} \). Specifically, \( \text{HH}(B) \cong \text{HH}^0(B) \otimes \mathbb{Q}[\xi_1, \ldots, \xi_n] \), and \( \text{HH}^0(B) \cong R(\ell) \). We need only study how \( \text{HH} \) acts on elements of \( \text{End}(B) \). But every endomorphism of \( B \) is given by multiplication by some \( f(x, y) \in R^e \). Since \( \text{HH} \) identifies the left and right \( R \)-actions, the induced endomorphism \( \text{HH}(f) \) is simply multiplication by \( f(x, x) \) (recall that \( \text{HH}(M) \) is naturally an \( R \)-module). This completes the proof.

**Corollary 4.8** We have \( \text{HH}(w(P_1^\vee)) \cong \Lambda[\xi_1, \ldots, \xi_n] \otimes_{\mathbb{Q}} \text{HH}^0(w(P_1^\vee)), \) where \( \deg(\xi_i) = q^{-2i}a \).

**4.2 Some computations of \( \text{HHH}(w(P_1^\vee)) \)**

In this section we explicitly compute the Hochschild homology of twisted projectors, \( \text{HHH}(w(P_1^\vee)) \), in a few special cases to give motivation for the results of §4.3 and §4.4. We let \( (i, j) = (i)(j) \) denote a shift in \( q \)-degree and homological degree.
Example 4.9 (HHH⁰(₁²²)) By Corollary 4.8, the computation of HHH(₁²²) reduces to the computation of the homology of HHH⁰(₁²²), which we now describe. We replace every copy of $B = B_s$ with $R(1)$ and replace each $y_i$ with $x_i$ in the dual complex of 2.20 to obtain the complex that follows.

$$
\begin{array}{cccccccc}
R & \rightarrow & 0 & R(-2) & x_2 - x_1 & R(-4) & 0 & R(-6) & x_2 - x_1 & \cdots \\
\end{array}
$$

(4.10)

where the underlined term is in homological degree 0. If we let $\alpha = x_2 - x_1$ then we can write $R = \mathbb{Q}[\alpha, x_1]$ and $R/(\alpha) \cong \mathbb{Q}[x_1]$. (4.10) splits into $R$ and the short chain complexes

$$
R(-2 - 4i, 1 + 2i) \xrightarrow{\alpha} R(-4 - 4i, 2 + 2i)
$$

We can reduce each of the above chain complexes, obtaining

$$
H(HH^0(₁²²)) \cong R(-2) \bigoplus \bigoplus_{i \geq 0} R(\alpha)(-4 - 4i, 2 + 2i).
$$

If we let $u_2$ denote the periodicity map described in §2.6 then it is easy to see that

$$
H(HH^0(₁²²)) \cong \mathbb{Q}[x_1, \alpha, u_2]/(\alpha u_2)
$$

Example 4.11 [HHH⁰(s(₁²²))] Let $s \in S_2$ be the nontrivial permutation. We know, by Remark 3.6, that HH₀(s(₁²²)) is the complex

$$
\begin{array}{cccccccc}
R & \xrightarrow{x_2 - x_1} & R(-2) & \rightarrow & 0 & R(-4) & x_2 - x_1 & R(-8) & 0 & \cdots \\
\end{array}
$$

(4.12)

Therefore, HH⁰(s(₁²²)) is a direct sum of complexes of the form

$$
R(-4i, 2i) \xrightarrow{x_2 - x_1} R(-2 - 4i, 2i + 1),
$$

for all $i \geq 0$. The homology of each of these short complexes is $R/(\alpha)(-2 - 4i, 2i + 1)$. Therefore HHH(s(₁²²)) $\cong \Lambda[\xi_1, \xi_2] \otimes_{\mathbb{Q}} \mathbb{Q}[x_1, u_2](-2, 1)$.

Example 4.13 (HHH⁰(₁²²)) Let $S = \mathbb{Q}[x_1, x_2, u_2]$, with zero differential. Consider the dg $S$-module $M_\infty = \mathbb{Q}[x_1, x_2, u_2, \theta_2]$ with differential $d(\theta_2) = (x_1 - x_2)u_2$. Define the following sub and quotient dg $S$-modules:

$$
M_{+} := \mathbb{Q}[x_1, x_2, u_2, \theta_2] \subset M_\infty \quad M_{-} := M_\infty / u_2 M_{+}.
$$

Note that $M_{+}$ and $M_{-}$ are generated by monomials in which $u_2$ appears with non-negative, respectively non-positive, exponent.
In Example 4.9 we saw that $\text{HHH}^0(P_{12}^\vee) \cong H(M_+)$ up to a grading shift. A similar computation shows that $\text{HHH}^0(P_{12}) \cong H(M_-)$ up to a grading shift. We note that $\text{HHH}^0(P_{12})$ is not finitely generated over $\mathbb{Q}[x_1, x_2, u_2]/(x_1u_2 = x_2u_2) \cong \text{End}(P_{12})$. A similar comparison can be done for $\text{HHH}(P_{1n}^\vee)$ and $\text{HHH}(P_{1n})$.

Example 4.14 \cite{HHH^0(w(P_{13}^\vee)), w = (1, 2, 3)} Recall the complex for $P_{13}$ from 2.21. We form the dual complex and twist this complex by the permutation $w = (1, 2, 3)$ (see Remark 3.6) by rewriting each $p_{ij}$ with $p_{w(i)j}$. Signs can be placed in a similar manner to what was done in Remark 2.22.

By Corollary 4.8, when we apply HH to $w(P_{13}^\vee)$, we can factor out a copy of the exterior algebra $\Lambda[\xi_1, \xi_2, \xi_3]$ and we are left with the complex $\text{HH}^0(w(P_{13}^\vee))$. Recall that $\text{HH}^0$ sends $B \mapsto R(3)$ and $f(x, y) \mapsto f(x, x)$. Set $q_{ij} = \text{HH}(p_{ij}) = x_i - x_j$. Note that $q_{ii} = 0$, so $\text{HH}^0(w(P_{13}^\vee))$ is the total complex of...
This complex has split into a complex of the form $\bigoplus_{i, j \geq 0} A(-4i - 6j, 2i + 2j)$, where $A$ is the complex

$$R \xrightarrow{q_{32}} R(-2) \xrightarrow{q_{13}} R(-4) \xrightarrow{q_{32}} R(-6) \xrightarrow{q_{13}} R(-8) \xrightarrow{q_{32}} \cdots$$

Since $R = \mathbb{Q}[x_1, x_2, x_3]$, we have an isomorphism $R \cong \mathbb{Q}[q_{13}, q_{32}, x_1]$. Then $A$ is the Koszul complex associated to the regular sequence $\{q_{13}, q_{32}\}$, and has homology isomorphic to $R/(q_{13}, q_{32})$. Thus, if we let $u_2$ and $u_3$ denote the periodicity maps from §2.6 then we see

$$\text{HHH} \left( w \left( P_{1}^{\nu} \right) \right) \cong \mathbb{Q}[\xi_1, \xi_2, \xi_3, x_1, u_2, u_3](-4, 2).$$

We can extend this pattern to general $n$; we will see in §4.4 that

$$\text{HHH} \left( w \left( P_{n}^{\nu} \right) \right) \cong \mathbb{Q}[\xi_1, \ldots, \xi_n, x_1, u_2, \ldots, u_n](-2n, n),$$

where the $u_i$ are the periodicity maps from §2.6, $\xi_i$ are generators of an exterior algebra with $\deg \xi_i = q^{-2i}a$. 
For general permutations \( w \in S_n \), \( \text{HHH}(w(P_1^\vee)) \) is not a free superpolynomial algebra. However, in Example 4.9 we were able to describe \( \text{HHH}(P_{1^2}^\vee) \) as a free superpolynomial algebra modulo certain relations (after grading shifts). This will be true for all \( n \) and \( w \in S_n \), as proven in §4.4.

4.3 The flag Hilbert scheme and \( \text{End}(P_1^n) \)

The \( q \)-Young symmetrizers are a collection of primitive idempotents \( p_T \in \mathcal{H}_n \) in the Hecke algebra, indexed by Young tableaux. One expects there to be complexes \( P_T \in K^-(\mathbb{Bim}_n) \) which categorify these elements. If \( T \) is the unique one-column tableau, then \( P_T \) is our projector \( P_1^n \), and if \( T \) is the unique one-row tableau, then \( P_T \) is the idempotent complex constructed in [18]. A construction of \( P_T \) for all \( T \) will be proposed by the second author and Elias in [9].

In [14], Gorsky et al. conjecture that \( \text{HHH}(P_T) \) of the idemotents \( P_T \in K^-(\mathbb{Bim}_n) \) can be described in terms of certain rings associated to the flag Hilbert scheme. Our work here seems to suggest that their rings are in fact the homologies of the complexes \( \text{End}(P_T) \). The relationship between \( \text{HHH}(P_T) \) and \( \text{End}(P_T) \) seems to be subtle in general.

In this paper we are concerned with the case where \( T \) is the unique tableau whose shape is the one-column partition \( 1 + \cdots + 1 = n \), which is usually written \( 1^n \). Throughout this section, set \( P := P_1^n \) (\( n \) is understood). The purpose of this section is to prove a conjecture of Gorsky–Rasmussen in this case. Here and below, we use the following:

**Definition 4.18** if \( M \) and \( N \) are graded bimodules then \( \text{Hom}_{R^e}(M, N) \) will denote the graded space of all homogeneous bimodule maps. If \( C \) and \( D \) are complexes of graded bimodules, then \( \text{Hom}_{R^e}(C, D) \) will denote the bigraded chain complex of all bihomogeneous \( R^e \)-linear maps.

Our main theorem in this section is a computation of \( H(\text{Hom}(R, P_1^n)) \), thus a computation of \( \text{HHH}(P_1^n) \). The following ring is defined by Gorsky–Rasmussen in [14].

**Definition 4.19** Let \( x_1, \ldots, x_n \) denote formal (even) indeterminates of bidegree \( q^2 t^{0} \). For all integers \( 1 \leq i < j \leq n \), let \( v_{ij} \) denote formal (even) indeterminates of bidegree \( q^{2(i-j)}-2t^{2} \). Let \( X \) and \( V \) denote the matrices

\[
X = \begin{bmatrix}
x_1 & 1 & 0 & \cdots & 0 & 0 \\
0 & x_2 & 1 & \cdots & 0 & 0 \\
0 & 0 & x_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & x_{n-1} & 1 \\
0 & 0 & 0 & \cdots & 0 & x_n
\end{bmatrix}, \quad V = \begin{bmatrix}
0 & v_{12} & v_{13} & \cdots & v_{1,n-1} & v_{1n} \\
0 & 0 & v_{23} & \cdots & v_{2,n-1} & v_{2n} \\
0 & 0 & 0 & \cdots & v_{3,n-1} & v_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & v_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

(4.20)

Let \( r_{ij} \in \mathbb{Q}[x_k, v_{ij}] \) denote the components of the commutator \( [X, V] \). Then set \( E := \mathbb{Q}[x_k, v_{ij}]/(r_{ij}) \).

Our main theorem in this section is:
Theorem 4.21 We have isomorphisms of algebras

\[ E \cong \HHH^0(P_1) \cong H(\Hom(R, P_1)) \cong H(\End(P)) \cong H(\End(P)) \]

As a consequence,

\[ \HHH(P_1) \cong E \otimes_{\mathbb{Q}} \Lambda[\xi_1, \ldots, \xi_n] \]

where \( \deg(\xi_i) = aq^{2i} \).

The second isomorphism is by definition, the third isomorphism is by general arguments, since \( P_1 \) is a unital idempotent, and the last isomorphism holds by an application of the duality functor. Thus, the essential content is in the first isomorphism. The rest of this section is dedicated to the proof of this theorem. First, we manipulate the definition of \( E \) into a workable form.

Proposition 4.22 Let us introduce the extraneous variables \( v_{i, j} \) for \( 1 \leq i \leq j \leq n \). Then \( E \) is isomorphic to the quotient of \( \mathbb{Q}[x, v_{ij}]_{1 \leq i \leq j \leq n} \) by the relations

\[
\begin{align*}
v_{i, j} &= v_{i-1, j-1} - (x_{i-1} - x_j)v_{i-1, j} \quad (2 \leq i \leq j \leq n) \\
v_{i, i} &= 0 
\end{align*}
\]

(4.23) (4.24)

Proof Straightforward exercise. \( \square \)

The reader may be wondering why we introduce variables \( v_{i, i} \) and then immediately set them equal to zero. The reason for this will become clear shortly. First, note that the relation (4.23) allows us to write \( v_{i, i} \) in terms of \( v_{1, 1}, v_{1, 2}, \ldots, v_{1, n} \). The first few examples are

1. \( v_{2, 2} = v_{1, 1} - q_{12}v_{12} \)
2. \( v_{3, 3} = v_{1, 1} - (q_{12} + q_{23})v_{12} + q_{13}q_{23}v_{13} \)
3. \( v_{4, 4} = v_{1, 1} - (q_{12} + q_{23} + q_{34})v_{12} + (q_{13}q_{23} + q_{13}q_{34} + q_{24}q_{34})v_{13} + q_{14}q_{24}q_{34}v_{14} \)

where \( q_{ij} := x_i - x_j \). The reader may have noticed the similarity between these coefficients and the polynomials \( a_{ij}(x, y) \) from Proposition 2.43. This is no accident.

Proposition 4.25 Using the relation (4.23) we have

\[ v_{kk} = \sum_{i=1}^{k} (-1)^{i-1} a_{ik}(x, x) v_{1i} \]

Proof Fix \( k \in \{2, \ldots, n\} \), and consider the task of simplifying \( v_{k, k} \). Each application of the relation (4.23) reduces the first index by 1, and doubles the number of terms. Thus, \( k - 1 \)-applications of this relation allows us to write \( v_{k, k} \) as a \( \mathbb{Q}[x] \)-linear combination of the variables \( v_{1, i} \), with \( 2^{k-1} \) terms. We may index the terms by sequences \( \Gamma \in \{\cdot L’, \cdot R’\}^{k-1} \), where ‘L’ and ‘R’ stand for the first and second terms of (4.23). The contribution of \( \Gamma \) to \( v_{k, k} \) can easily be deduced from (4.23); it is a product of factors,
where an occurrence of ‘L’ in \( \Gamma \) contributes a factor of 1, and an occurrence of ‘R’ in the \( i \)-th coordinate of \( \Gamma \) contributes a factor of \(-(x_{k-i} - x_{k-i+j+1})\) where \( j \) is the number of occurrences of ‘R’ among the first \( i - 1 \) coordinates. For instance, the word RRLLR (corresponding to \( k = 6 \)) contributes a factor of

\[
(-1)^3(x_5 - x_6)(x_4 - x_6)(x_1 - x_4)
\]

More precisely, let \( i_1 < \cdots < i_j \) denote the indices such that \( \Gamma_{i_j} = \text{‘R’} \), and let \(|\Gamma| = j\) denote the total number of occurrences of ‘R.’ Then define a weight by

\[
\text{wt}(\Gamma) := (-1)^j(x_{k-i_1} - x_{k-i_1+1})(x_{k-i_2} - x_{k-i_2+2}) \cdots (x_{k-i_j} - x_{k-i_j+j})
\]

By construction, we have

\[
v_{kk} = \sum_{j=0}^{k-1} (-1)^j v_{1,j+1} \sum_{|\Gamma| = j} \text{wt}(\Gamma)
\]

Furthermore, comparison with the formula for \( a_{ij}(x, x) \) in Definition 2.40 makes it clear that \( \sum_{|\Gamma| = j} \text{wt}(\Gamma) = (-1)^j a_{j+1,k}(x, y) \). This completes the proof. \( \square \)

**Proof of Theorem 4.21** First, recall the differential bigraded algebra \( A_n \) from Definition 2.44: \( A_n = \mathbb{Q}[x, y, u_1, \ldots, u_n, \theta_1, \ldots, \theta_n] \) with \( \mathbb{Q}[x, y, u_1, \ldots, u_n] \)-linear differential determined by \( d_A(\theta_j) = \sum_{i=1}^{n} u_i a_{ij}(x, y) \), together with the graded Leibniz rule. Then \( P^\mathbb{Q} \cong M_n = A_n/I_n A_n \), with \( \mathbb{Q}[x, y, u_1, \ldots, u_n] \)-linear differential \( d_M \) determined by \( d_M(1) = \sum_{i=1}^{n} (y_i - x_i) \theta_i \) and the graded Leibniz rule \( d_M(a m) = d_A(a) m + (-1)^{|a|} d_M(m) \) for all \( a \in A_n \), \( m \in P^\mathbb{Q} \). This complex is then shifted so that 1 lies in degree \( q^{-2 \ell} t^0 \).

Note that \( d_M(1) \) is zero modulo \( (x_i - y_i) \). By Proposition 4.5, the functor \( \text{Hom}_{\mathbb{Q}[x, y]}(\mathbb{Q}[x], -) \) sends \( \mathbb{Q}[x, y] \to \mathbb{Q}[x](2\ell) \) and \( f(x, y) \to f(x, x) \). The grading shifts cancel, and we see that \( \text{Hom}(\mathbb{Q}[x], M) \) is isomorphic to the superpolynomial ring \( \mathbb{Q}[x, u_1, \ldots, u_n, \theta_1, \ldots, \theta_n] \) with differential determined by

\[
d(u_k) = 0 \quad d(\theta_j) = \sum_{i=1}^{k} u_i a_{ij}(x, x)
\]

This is the Koszul complex associated to the sequence

\[
(u_1), \ (u_1 + u_2 a_{22}(x, x)), \ (u_1 + u_2 a_{23}(x, x) + u_3 a_{33}(x, x)), \ \cdots
\]

in \( \mathbb{Q}[x, u_1, \ldots, u_n] \). It is easy to see that this is a regular sequence since \( u_k \) does not appear until the \( k \)-th term of the sequence, hence the homology is simply the quotient of \( \mathbb{Q}[x, u_1, \ldots, u_n] \) by the ideal generated by the above elements. By Propositions 4.22 and 4.25 it follows that the homology of \( \text{Hom}_{\mathbb{Q}[x, y]}(\mathbb{Q}[x], M_n) \) is isomorphic to \( E \), via an isomorphism which sends \( u_k \to (-1)^{k-1} v_{1,k} \). This proves
that \( H(\text{Hom}(R, P^w_{1n})) \cong E \). The isomorphism \( \text{Hom}(R, P^w_{1n}) \cong H(\text{End}(P_{1r})) \) follows from general theory of categorical idempotents (see §4.2 of [19]). This completes the proof. \( \square \)

### 4.4 Homology of twisted projectors

We first state the conjecture of Gorsky–Negut–Rasmussen for twisted projectors. Let \( x_i \) denote a formal indeterminate of bidegree \( q^2a \) and \( v_{ij} \) denote a formal indeterminate of bidegree \( q^{2(i-j) - 2i^2} \) as before.

**Conjecture 4.26** Recall the ring \( E \) from Definition 4.19. Then for each permutation \( w \in S_n \), the homology \( HH^0(w(P^w_{1n})) \) is isomorphic to \( E/J_w \), where \( J_w \) is the ideal generated by the differences \( x_w(i) - x_i \) for all \( 1 \leq i \leq n \).

The symmetric group acts on \( HH^0(P^w_{1n}) \) by conjugation by Rouquier complexes. More precisely, any chain map \( \phi : R \to P \) gives rise to \( R \simeq F_w F_w^{-1} \to F_w P F_w^{-1} \) for the Rouquier complex \( F_w \) associated to the positive braid lift of \( w \). Since \( P \) absorbs Rouquier complexes we have an equivalence \( F_w P F_w^{-1} \simeq P \). Pre-composing with the map \( F \phi F \) defines \( w(\phi) : R \to P \). This gives rise to an action of \( S_n \) on \( E \cong HH^0(P^w_{1n}) \). If \( w = vwv^{-1} \), then \( \phi \mapsto v(\phi) \) descends to an isomorphism \( E/J_w \to E/J_{vwv^{-1}} \). Thus, both algebras in Conjecture 4.26 depend only on the conjugacy class of \( w \) up to isomorphism.

It will be helpful to encode \( P^w_{1n} \) in the form of a Koszul matrix factorization, the basics of which we now recall. Let \( S \) be a commutative ring and \( z \in S \) a fixed element. For us, a \( z \)-matrix factorization will be a \( \mathbb{Z}/2 \)-graded \( S \)-module \( M = M_0 \oplus M_1 \) together with an \( S \)-linear endomorphism \( d \in \text{End}_S(M) \) such that

1. \( d \) is odd. That is, \( d \) restricts to \( S \)-module maps \( M_0 \to M_1 \) and \( M_1 \to M_0 \).
2. \( d^2 = z \), where \( z \in S \) is regarded as an endomorphism of \( M \).

We call \( z \in S \) the potential.

**Remark 4.27** We regard \( S \) as \( \mathbb{Z}/2 \)-graded, where each element of \( S \) is even. A map of \( \mathbb{Z}/2 \)-graded objects is called even if it preserves parity, and odd if it swaps parity.

A morphism \((M, d) \to (N, d')\) between matrix factorizations with the same potential is an even \( S \)-module map \( f : M \to N \) such that \( d' \circ f = f \circ d \). If \((M, d)\) is a \( z \)-matrix factorization and \((M, d')\) is a \( z' \)-matrix factorization, then the tensor product \( M \otimes_S M' \) is naturally a \((z + z')\)-matrix factorization with \( d_{M \otimes M'} = d \otimes \text{Id}_M + \text{Id}_M \otimes d' \).

Now, let \( a, b \in S \) a pair of elements. We have the basic *Koszul matrix factorization* \( (a|b) := S \begin{array}{c} a \\ b \end{array} S \begin{array}{c} a \\ b \end{array} \)

We regard the left-most \( S \) as sitting in even degree \( 0 \in \mathbb{Z}/2 \), and the right-most \( S \) as sitting in odd degree \( 1 \in \mathbb{Z}/2 \). This is a matrix factorization with potential \( ab \). In general given elements \( a_i, b_i \in S \) with \( 1 \leq i \leq r \), we define
The tensor product here is over $S$. This is a matrix factorization with potential $\sum_{i=1}^{r} a_i b_i \in S$. If $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$, then we also write $(a, b)$ for the corresponding Koszul complex. Usually the ring $S$ will be implicit, but if we wish to include it in the notation, we will do so with a subscript, as in $(a, b)_S$. Similarly, if $M$ is an $S$-module, we denote $(a, b)_M := (a, b) \otimes_S M$.

**Remark 4.28** A Koszul matrix factorization is the superposition of two Koszul complexes. There is the “forward differential” which is the Koszul differential associated to the sequence $a_1, \ldots, a_r$, and there is the “backward differential” which is the Koszul differential associated to $b_1, \ldots, b_r$. In particular, as an $S$-module $(a, b)$ is an exterior $S$-algebra with $r$ generators.

**Remark 4.29** Here is another description of a Koszul matrix factorization. If $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$ are given sequences of elements of $S$, then the Koszul matrix factorization $(0, b)$ is a differential graded algebra

$$(0, b) = \Lambda[\theta_1, \ldots, \theta_r] \quad \text{with differential} \quad d_A(\theta_i) = b_i$$

This is simply the usual Koszul complex associated to $b_1, \ldots, b_r$. Let us call this dg algebra $A$.

Now, $M := (a, b)$ can be thought of as a dg $A$-module generated by one element, with differential $d_M(1) = \sum_{i=1}^{r} a_i \theta_i$. We leave it to the reader to check this.

**Remark 4.30** We will be primarily interested in the case where $S$ is a bigraded ring. The degree of a bihomogeneous element $a \in S$ is written $\deg(a) = q^i t^j$, and $t$ is regarded as the homological degree. We will require our matrix factorizations to be bigraded $S$ modules, with parity given by the mod 2 reduction of homological degree, and we require the differential to satisfy $\deg(d) = q^0 t^1$. If $\deg(b) = q^j t^j$ and $\deg(a) = q^{i} t^{1-j}$, then we incorporate the grading shifts as follows:

$$(a|b) = \left( S \xrightarrow{a} S(i)(j-1) \xleftarrow{b} S \right)_S$$

so that the right-pointing arrow has degree $q^0 t^0$ and the left-pointing arrow has degree $q^0 t^1$. Note that in this case

$$(a|b) \cong q^{-i} t^{1-j} (b|a)$$

Before constructing $P_{1,n}$ as a matrix factorization, we recall two basic operations on Koszul matrix factorizations which preserve the isomorphism type of the matrix factorization. The next two propositions are proven in [23].
Proposition 4.31 Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be sequences of elements in \( S \). Let \( (a, b) \) be the associated Koszul matrix factorization. The change of basis transformation
\[
\begin{pmatrix}
  a_i \\
  a_j \\
  b_i \\
  b_j
\end{pmatrix} \rightarrow \begin{pmatrix}
  a_i \\
  a_j - \lambda a_i \\
  b_i + \lambda b_j \\
  b_j
\end{pmatrix},
\]
where all other rows are fixed, yields isomorphic matrix factorizations for all \( \lambda \in S \). Similarly, if \( \lambda \in S \) is invertible, then \( (a_i|b_i) \sim= (\lambda a_i|\lambda^{-1} b_i) \).

Now assume that \( S \) is a polynomial ring. We can also simplify Koszul matrix factorizations by canceling rows of the form \( (0|b_i) \) or \( (a_i|0) \) (called exclusion of variables), subject to certain restrictions. Precisely, suppose \( (a, b) \) is a Koszul matrix factorization with potential \( z = \sum_{i=1}^{n} (a_i b_i) \). Let \( y \) be a generator of \( S \) and write \( S = S'[y] \). Assume \( z \in S' \) and that one of the rows of \( (a, b) \) has the form \( (0, y - p) \) for \( p \in S' \).

Proposition 4.32 Given the above hypotheses, \( (a, b)_S \) is homotopy equivalent to the matrix factorization of potential \( \sum (a_i b_i) \) over \( S' \) where we remove the row \( (0, y - p) \) and substitute \( p \) for \( y \) everywhere in all other rows.

We have not yet introduced the notion of homotopy equivalence of matrix factorizations. There will be no need to do so, since we are mostly interested in the case when the matrix factorization \( M \) has zero potential (that is, \( M \) is a chain complex), in which case the notion homotopy equivalence is standard. The basic idea of exclusion of a variable is that \( (0|y - p) \) is the complex
\[
0 \rightarrow S' \otimes \mathbb{Q}[y] \xrightarrow{1 \otimes y - p \otimes 1} S' \otimes \mathbb{Q}[y] \rightarrow 0
\]
This complex is homotopy equivalent to \( S' \) via Gaussian elimination. We will now specialize to the case of interest.

Definition 4.33 Fix an integer \( n \). Let \( S = \mathbb{Q}[x, y, u_1, \ldots, u_n] \), bigraded so that \( \deg(u_k) = t^2 q^{-2k} \). Define elements \( b_j \in S \) by \( b_j = \sum_{i=1}^{j} u_i a_{ij}(x, y) \), where the polynomials \( a_{ij}(x, y) \) are as in Definition 2.40. Consider the Koszul matrix factorization
\[
q^{-2\ell} \begin{pmatrix}
  p_{11} & b_1 \\
  p_{22} & b_2 \\
  \vdots & \vdots \\
  p_{nn} & b_n
\end{pmatrix} = q^{-2\ell} \bigotimes_{j=1}^{n} \left( S \xleftarrow{p_{jj}} S(-2l)(1) \right)_S \otimes_{\bar{S}} \bar{S} \quad (4.34)
\]
Here \( \bar{S} \) is the quotient of \( S \) in which we identify \( e_k(x) = e_k(y) \) for all \( k \in \{1, \ldots, n\} \), and \( p_{ij}(x, y) = y_i - x_j \). Note that \( \deg(p_{ij} b_j) = q^0 t^1 \), so that (4.34) is graded as in Remark 4.30.

The following is clear from the definitions.
Proposition 4.35 The matrix factorization defined in Definition 4.33 is equal to $M_n$ (from §2.6) as a bigraded chain complex. Therefore the matrix factorization (4.34) is isomorphic to $P^\wedge_1$ as a bigraded chain complex by Theorem 2.56.

For the grading shift of $q^{-2\ell}$, recall that $B_{w_0} = q^{-\ell}Q[x, y]/I$ and the degree zero chain bimodule of $M_n$ is $q^{-\ell}B_{w_0} = q^{-2\ell}Q[x, y]/I$ (see also Definition 2.46).

Example 4.36 For $n = 2$, the complex looks like

$$M_2 = \begin{pmatrix} p_{11} & u_1 \\ p_{22} & u_1 + p_{12}u_2 \end{pmatrix} \bar{\mathcal{S}}$$

After a change of basis (Proposition 4.31) this is isomorphic to

$$\begin{pmatrix} p_{11} + p_{22} & u_1 \\ p_{22} & u_1 + p_{12}u_2 \end{pmatrix} \bar{\mathcal{S}}$$

Note that $p_{11} + p_{22}$ is zero in $\bar{\mathcal{S}}$, so we obtain

$$M_2 \cong \begin{pmatrix} 0 & u_1 \\ p_{22} & u_1 + p_{12}u_2 \end{pmatrix} \bar{\mathcal{S}} \cong (p_{22}|p_{12})\bar{\mathcal{S}}/(u_1)$$

by exclusion of variables (Proposition 4.32). This is a very compact way of expressing Diagram 2.20 (with the arrows reversed).

Example 4.37 For $n = 3$ we have

$$M_3 = \begin{pmatrix} p_{11} & u_1 \\ p_{22} & u_1 + p_{12}u_2 \\ p_{33} & u_1 + (p_{12} + p_{23})u_2 + p_{13}p_{23}u_3 \end{pmatrix} \bar{\mathcal{S}}$$

After a change of basis we obtain that $M_3$ is isomorphic to

$$\begin{pmatrix} p_{11} + p_{22} + p_{33} & u_1 \\ p_{22} & u_1 + p_{12}u_2 \\ p_{33} & (p_{12} + p_{23})u_2 + p_{13}p_{23}u_3 \end{pmatrix} \bar{\mathcal{S}}$$

An application of the relation $p_{11} + p_{22} + p_{33} = 0$ yields

$$\begin{pmatrix} 0 & u_1 \\ p_{22} & u_1 + p_{12}u_2 \\ p_{33} & (p_{12} + p_{23})u_2 + p_{13}p_{23}u_3 \end{pmatrix} \bar{\mathcal{S}}/(u_1)$$

which by Proposition 4.31 is isomorphic to

$$M_3 \cong \begin{pmatrix} p_{22} & u_1 + p_{12}u_2 \\ p_{33} & (p_{12} + p_{23})u_2 + p_{13}p_{23}u_3 \end{pmatrix} \bar{\mathcal{S}}/(u_1),$$

which is simply the complex from Diagram 2.21, with the arrows reversed.
Now, let \( w \in S_n \) be given. We wish to compute \( H(\HH^0(R, w(P_{1^r}^n))) \). This homology depends only on the conjugacy class of \( w \), so we may as well assume that \( w \) has the special form (4.38).

\[
w = (1, \ldots, m_1)(m_1 + 1, \ldots, m_2) \cdots (m_{r-1} + 1, \ldots, m_r) \quad (4.38)
\]

written in cycle notation, for some integers \( 1 \leq m_1 < m_2 < \cdots < m_r = n \). Applying the permutation \( w \) to \( M_n \) simply permutes the variables \( y_i \) in the formula (4.34). Taking \( \HH^0(R, -) \) sends \( B \mapsto R(\ell) \). Thus \( \HH^0 \) sends \( \mathbb{Q}[x, y]/I \mapsto q^{2\ell}(\mathbb{Q}[x]) \). This cancels the shift of \( q^{-2\ell} \) in (4.34) then identifies \( x \) with \( y \). The result is:

**Lemma 4.39** We have

\[
\HH^0(R, w(M_n)) \cong \left( \begin{array}{cccc}
  x_{w(1)} - x_1 & \bar{b}_1 \\
  x_{w(2)} - x_2 & \bar{b}_2 \\
  \vdots & \vdots \\
  x_{w(n)} - x_n & \bar{b}_n \\
\end{array} \right)_{\mathbb{Q}[x, u_1, \ldots, u_n]}
\]

where \( \bar{b}_j = \sum_{i=1}^{j} u_ia_{ij}(x, w(x)). \) \( \square \)

**Proposition 4.41** If \( j_1 \) and \( j_2 \) are in the same \( w \)-orbit, then

\[
a_{i, j_1}(x, w(x)) = a_{i, j_2}(x, w(x)).
\]

*Proof* We are assuming that \( w \) has a special form, so \( j_1, j_2 \) are in the same \( w \)-orbit if and only if there is a \( 1 \leq k \leq r \) such that \( m_{k-1} < j_1, j_2 \leq m_k \). So without loss of generality, assume that \( j_1 = j_2 + 1 \). From the definition of \( a_{ij}(x, y) \) it is easy to see that \( a_{i, j+1}(x, y) - a_{ij}(x, y) \) is divisible by \( y_j - x_{j+1} \), for all \( 1 \leq i \leq j + 1 \), hence is zero upon specializing \( y_{w(i)} = x_i \). This completes the proof. \( \square \)

We will now simplify the complex (4.40) by focusing on one block at a time. Let \( w \) be as above; for each cycle \( (m, m+1, \ldots, m+j) \) of \( w \) we have the following tensor factor of the right-hand side of (4.40):

\[
\left( \begin{array}{c}
  x_{m+1} - x_m \\
  x_{m+2} - x_{m+1} \\
  \vdots \\
  x_{m+j} - x_{m+j-1} \\
  x_m - x_{m+j} \\
\end{array} \right) \left( \sum_{i=1}^{m} u_ia_{i,m}(x, w(x)) \right)
\]

Set \( \alpha_i = x_i - x_{i+1} \). Then \( x_m - x_{m+j} = \alpha_m + \alpha_{m+1} + \cdots + \alpha_{m+j-1} \). By Proposition 4.41, all of the entries in the right column are equal to one another. Rewriting gives
After a change of basis this is isomorphic to

\[
\begin{pmatrix}
-\alpha_m & \sum_{i=1}^{m+j} u_i a_{i,m+j}(x, w(x)) \\
-\alpha_{m+1} & \sum_{i=1}^{m+j} u_i a_{i,m+j}(x, w(x)) \\
\vdots & \vdots \\
-\alpha_{m+j-1} & \sum_{i=1}^{m+j} u_i a_{i,m+j}(x, w(x)) \\
\alpha_m + \cdots + \alpha_{m+j-1} & \sum_{i=1}^{m+j} u_i a_{i,m+j}(x, w(x))
\end{pmatrix}
\]

Applying such transformations to all of the cycles in \(w\), we obtain the following:

**Proposition 4.42** We have

\[
\text{HH}^0(R, w(M_n)) \cong \begin{pmatrix}
\alpha_1 & 0 \\
\vdots & \vdots \\
\alpha_{m_1-1} & 0 \\
0 & \sum_{i=1}^{m_1} u_i a_{i,m_1}(x, w(x)) \\
\alpha_{m_1+1} & 0 \\
\vdots & \vdots \\
\alpha_{m_2-1} & 0 \\
0 & \sum_{i=1}^{m_2} u_i a_{i,m_2}(x, w(x)) \\
\vdots & \vdots \\
\alpha_{m_r-1+1} & 0 \\
\vdots & \vdots \\
\alpha_{m_r-1} & 0 \\
0 & \sum_{i=1}^{m_r} u_i a_{i,m_r}(x, w(x))
\end{pmatrix}_{\mathbb{Q}[x_1, \ldots, x_{m_r}]}
\]

We are ready to prove our main theorem:
Theorem 4.43 Let \( w \in S_n \) be a permutation with \( r \) cycles. Then \( \text{HHH}^0(w(P_{1,w}^n)) \) is isomorphic to \( E/(x_i - x_{w(i)}) \) shifted so that 1 lies in degree \((q^{-2}t)^{n-r}\), where \( E \) is as in Definition 4.19. That is to say, Conjecture 4.26 is true.

Proof Fix \( w \in S_n \); without loss of generality, we assume that \( w \) has the special form (4.38). Define the following ideals in \( \mathbb{Q}[x, u_1, \ldots, u_n] \):

1. \( I = (\alpha_i) \), where \( i \in \{1, \ldots, n\} \setminus \{m_1, \ldots, m_r\} \).
2. \( I' = (x_i - x_{w(i)}) \) where \( 1 \leq i \leq n \).
3. \( J \) is the ideal generated by \( \sum_{i=1}^{m_k} u_i a_{i,m_k}(x, w(x)) \) where \( 1 \leq k \leq r \).
4. \( J' \) is the ideal generated by \( \sum_{i=1}^{j} u_i a_{i,j}(x, w(x)) \) where \( 1 \leq j \leq n \).
5. \( J'' \) is the ideal generated by \( \sum_{i=1}^{j} u_i a_{i,j}(x, x) \) where \( 1 \leq j \leq n \).

It is clear that \( I = I' \) and \( I' + J' = I' + J'' \). Proposition 4.41 implies that \( J = J' \), so we conclude that \( I + J = I' + J'' \).

The matrix factorization from Proposition 4.42 is clearly isomorphic to the Koszul complex associated to the sequence \( \{\alpha_i : i \in \{1, \ldots, n\} \setminus \{m_1, \ldots, m_r\} \} \cup \{\overline{b}_{m_1}, \ldots, \overline{b}_{m_r}\} \subset R[u_1, \ldots, u_n] \), where \( \overline{b}_j = \sum_{i=1}^{j} u_i a_{i,j}(x, w(x)) \). This sequence is easily seen to be regular, so

\[
H \left( \text{HHH}^0(R, w(M_n)) \right) \cong \mathbb{Q}[x, u_1, \ldots, u_n]/(I + J),
\]

which is isomorphic to \( \mathbb{Q}[x, u_1, \ldots, u_n]/(I' + J'') \) by the remarks above. On the other hand, \( \mathbb{Q}[x, u_1, \ldots, u_n]/J'' \cong E \) by Theorem 4.21, so that \( H(\text{HHH}^0(R, w(M_n)) \) is isomorphic to \( E/(x_i - x_{w(i)}) \), as claimed.

Finally, the overall grading shift is due to the fact (see Remark 4.30) that \((\alpha_i|0) \cong q^{-2}t(0|\alpha_i) \). \( \square \)

Corollary 4.44 Let \( w \) be a permutation with \( r \) cycles. Then \( \text{HHH}(w(P_{1,w}^n)) \) has Poincaré series

\[
\mathcal{P}(q, a, t) = \frac{(q^{-2}t)^{n-r}(1 - t^2 q^{-2})^{r-1} \prod_{i=1}^{n} (1 + a q^{-2})}{(1 - q^2)^r \prod_{j=2}^{n} (1 - t^2 q^{-2})}
\]

Furthermore, \( \mathcal{P}(q, a, -1) \) is the \( 1^n \)-colored HOMFLYPT polynomial of the unknot.

Proof Once again, we fix \( w \in S_n \) and assume \( w \) has the special form (4.19). Also recall that we represent \( \text{deg}(z) \) by a monomial \( q^i a^j t^k \). In the proof of Theorem 4.21, we show that the ideal \( (I + J) \) is generated by a regular sequence. A result of Stanley [31] states the following: If \( A = \mathbb{Q}[z_1, \ldots, z_m] \) is a graded ring and \( I = (a_1, \ldots, a_k) \) is an ideal generated the regular sequence \( \{a_1, \ldots, a_k\} \), then the Poincaré series of \( R/I \) is given by

\[
\mathcal{P}(A/I) = \frac{\prod_{i=1}^{k} (1 - \deg(a_i))}{\prod_{j=1}^{m} (1 - \deg(z_j))}
\]
Therefore, it is easy to see that the denominator of the Poincaré series of 
\( \text{HHH}^0(w(P_{1^n}^\vee)) \) is given by \((1 - q^2)^n \prod_{j=1}^{n} (1 - t^2q^{-2j})\). As for the numerator, the regular sequence generating \( I + J \) is given by \( n - r \) elements of the form \( \alpha_i \) and \( r \) elements of the form \( \bar{b}_{mk} \). Each \( \alpha_i \) has degree \( q^2 \), and each \( \bar{b}_{mk} \) has degree \( t^2q^{-2} \).

Therefore the numerator has the form \((q^{-2}t)^{n-r} (1 - q^2)^{n-r} (1 - t^2q^{-2j})^r \), where \((q^{-2}t)^{n-r} \) takes into account the overall grading shift. The denominator has the form \((1 - q^2)^n \prod_{j=1}^{n} (1 - t^2q^{-2j})\), corresponding to the generators \( x_i, u_i \). The \((1 - t^2q^{-2}) \) factor cancels with one factor from the numerator, so

\[
\mathcal{P} \left( \text{HHH}^0 \left( w \left( P_{1^n}^\vee \right) \right) \right) = \frac{(q^{-2}t)^{n-r} (1 - t^2q^{-2})^{r-1}}{(1 - q^2)^r \prod_{j=2}^{n} (1 - t^2q^{-2j})}.
\]

To compute the Poincaré series for \( \text{HHH}(w(P_{1^n}^\vee)) \) we multiply by \( \prod_{j=1}^{n} (1 + aq^{-2j}) \). This is because \( \text{HHH}(w(P_{1^n}^\vee)) \cong \text{HHH}^0(w(P_{1^n}^\vee)) \otimes \mathbb{Q}[\xi_1, \ldots, \xi_n] \) and that \( \deg(\xi_k) = aq^{-2k} \). We leave checking the second claim to the reader. \( \square \)

One may check that when \( \mathcal{P}(\text{HHH}(w(P_{1^n}^\vee))) \) is expanded into a Laurent series in variables \( q^2 \) and \( t^2q^{-2j} \) for \( j = 2, \ldots, n \) that all coefficients are positive, though this is not obvious upon first glance. This, of course, is expected because the coefficients are dimensions of vector spaces.

5 Combinatorial results

In this section we record some combinatorial results necessary for giving an explicit dg-module construction of \( P_{1^n} \) in §2. These results are known to experts, but we present elementary proofs of these facts for completeness.

5.1 The Frobenius extension \( R^{S_n} \subset R^{S_{n-1}} \)

Recall that \( n \geq 1 \) is fixed, and \( S_{n-1} = S_{n-1} \times S_1 \subset S_n \). In this section we recall the result that \( R^{S_n} \subset R^{S_{n-1}} \) is a Frobenius extension, and we give an explicit dual pair of basis of \( R^{S_{n-1}} \) as a free \( R^{S_n} \)-module. Our main result in this section is the following:

**Theorem 5.1** \( R^{S_{n-1}} \) is free of rank \( n \) over \( R^{S_n} \). There are two natural ordered bases, given by

\[
\left\{ (-1)^i x_n^i \right\}_{i=0}^{n-1} \quad \text{and} \quad \{ e_{n-1-i}(x_1, \ldots, x_{n-1}) \}_{i=0}^{n-1}
\]

These bases are dual with respect to the trace pairing \((f, g) := \partial_{1,2,\ldots,n-1}(fg)\), where \( \partial_{1,2,\ldots,n-1} \) denotes a divided difference operator (Definition 5.2). In particular \( R^{S_n} \subset R^{S_{n-1}} \) is a Frobenius extension.

Iterating this, we recover as a corollary the classical result that \( R \) is free over \( R^{S_n} \) of rank \( n! \). If we consider \( R \) to be graded with \( \deg x_i = q^2 \), and carefully keep track of gradings, then we can recover Proposition 2.5. Our main use for this theorem is in
the proof of Proposition 2.30, which itself is used in the proof of Theorem 2.56. The proof of Theorem 5.1 occupies the remainder of this section. We first introduce the trace $RS_{n-1} \rightarrow RS_n$, which is defined in terms of the divided difference operators.

**Definition 5.2** Let $R := \mathbb{Q}[x_1, \ldots, x_n]$, let $i \in \{1, \ldots, n-1\}$ be given, and let $s = (i, i+1) \in S_n$ denote the simple transposition which swaps $i$ and $i+1$. The divided difference operator $\partial_i : R \rightarrow R$ is defined by

$$\partial_i(f) := \frac{f - s(f)}{x_i - x_{i+1}}$$

for each $f \in R$. We may also write $\partial_s = \partial_i$, by abuse.

Note that $f - s(f)$ is anti-symmetric in $x_i$ and $x_{i+1}$, hence is divisible by $x_i - x_{i+1}$. So $\partial_s(f)$ is indeed a polynomial. The following basic properties are easily checked:

**Proposition 5.3** The divided difference operators satisfy:

1. $\partial_i^2 = 0$
2. $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_i$
3. If $f$ is symmetric in $x_i, x_{i+1}$, then $\partial_i(fg) = f \partial_i(g)$
4. In general, $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$.

*Proof* Straightforward. \(\square\)

The following computation will be very useful:

**Proposition 5.4** Fix an integer $n \geq 1$, and let $R := \mathbb{Q}[x_1, \ldots, x_n]$. Let $\partial_1, \partial_2, \ldots, \partial_{n-1} = \partial_1 \circ \partial_2 \circ \ldots \circ \partial_{n-1}$ denote the composition of divided difference operators. Then

$$\partial_{1,2,\ldots,n-1}(x_n^k) = (-1)^{n-1}h_{k+1-n}(x_1, \ldots, x_n)$$

where $h_k$ denotes the $k$-th complete symmetric function.

*Proof* Induction on $n \geq 1$. For the base case $n = 1$, the composition $\partial_1 \circ \ldots \circ \partial_{n-1}$ is empty, hence $\partial_{1,2,\ldots,n-1}$ is to be interpreted as the identity. Then the base case is trivial: $x_1^k = h_k(x_1)$. Since this case is degenerate, let’s also consider the case $n = 2$, which will be relevant for the inductive step as well. Compute:

$$\partial_1(x_2^k) = -\frac{x_1^k - x_2^k}{x_1 - x_2} = -\left(x_1^{k-1} + x_1^{k-2}x_2 + \ldots + x_2^{k-1}\right)$$

which equals $-h_{k-1}(x_1, x_2)$, as desired.

Assume by induction that we have proven

$$\partial_{1,2,\ldots,n-2}(x_{n-1}^i) = (-1)^{n}h_{i+2-n}(x_1, \ldots, x_{n-1})$$

which equals $-h_{k-2}(x_1, x_2)$, as desired.
Compute:

\[ \partial_1,2,\ldots,n-1 \left( x_n^k \right) = \partial_1,2,\ldots,n-2 \left( \partial_{n-1} \left( x_n^k \right) \right) \]

\[ = - \partial_1,2,\ldots,n-2 \left( \sum_{i+j=k-1} x_{n-1}^i x_n^j \right) \]

\[ = - \sum_{i+j=k-1} \partial_1,2,\ldots,n-2 (x_{n-1}^i x_n^j) \]

\[ = - \sum_{i+j=k-1} (-1)^n h_{i+2-n}(x_1,\ldots,x_{n-1}) x_n^j \]

\[ = (-1)^{n-1} h_{k+1-n}(x_1,\ldots,x_n) \]

In the second line we used the formula (5.5), in the fourth we used the inductive hypothesis, and in the last line we used the well-known recursion satisfied by the complete symmetric functions (see Proposition 5.10). This completes the inductive step. \[\square\]

**Lemma 5.6** Recall the sequence of inclusions of algebras \( R^{S_n} \subset R^{S_{n-1}} \subset R \). As an \( R^{S_n} \)-module, \( R^{S_{n-1}} \) is generated by \( \{1, x_n, \ldots, x_n^{n-1}\} \).

**Proof** Recall that \( R^{S_{n-1}} \subset \mathbb{Q}[x_1,\ldots,x_n] \) is the algebra of polynomials which are symmetric in \( x_1,\ldots,x_{n-1} \). It is well known that \( R^{S_{n-1}} \) is generated, as a unital algebra, by \( x_n \) and the elementary symmetric functions \( e_k(x_1,\ldots,x_{n-1}) \). The recursion from Proposition 5.10 says that

\[ e_k(x_1,\ldots,x_{n-1}) = e_k(x_1,\ldots,x_n) - x_n e_{k-1}(x_1,\ldots,x_{n-1}) \]

Iterating, we can express each partially symmetric function \( e_k(x_1,\ldots,x_{n-1}) \) as

\[ e_k(x_1,\ldots,x_{n-1}) = \sum_{i=0}^k (-1)^{k-i} e_i(x_1,\ldots,x_n) x_n^{k-i} \]

This shows that \( R^{S_{n-1}} \) is generated by the powers of \( x_n \), as an \( R^{S_n} \) algebra. Recall from Example 5.13 that in \( R \) one has the following identity

\[ x_n^n = - \sum_{i=1}^n (-1)^{n-i} e_i(x_1,\ldots,x_n) x_n^{n-i} \]

hence \( 1, x_n,\ldots,x_n^{n-1} \) suffice to generate \( R^{S_{n-1}} \) as an \( R^{S_n} \)-module. \[\square\]

**Proposition 5.7** The divided difference operator \( \partial_1,2,\ldots,n-1 : R \to R \) restricts to an \( R^{S_n} \)-linear map \( R^{S_{n-1}} \to R^{S_n} \).
Proof By part (3) of Proposition 5.3 it is clear that $\partial_{1,2,\ldots,n-1}$ is $R_{S_n}$-linear. Proposition 5.4 shows that $\partial_{1,2,\ldots,n-1}x^k_n$ is an element of $R_{S_n}$, for each $k$. Since the $x^k_n$ generate $R_{S_n-1}$ as an $R_{S_n}$-module (Lemma 5.6), the proposition follows.

We are now ready to prove the main theorem of this section:

Proof of Theorem 5.1 In this proof we will use the brief notation $\partial = \partial_{1,2,\ldots,n-1}$. We will focus on proving that

$$\partial \left( (-1)^i x^i_n e_{n-1-j}(x_1, \ldots, x_{n-1}) \right) = \delta_{ij}, \quad (5.8)$$

where $\delta_{ij}$ is the Kronecker delta. Assume for a moment that we have proven this. A standard linear algebra argument will imply that the set $\beta := \{ (-1)^i x^i_n \}_{j=0}^{n-1} \subset R_{S_n-1}$ is $R_{S_n}$-linearly independent, as is $\beta^* := \{ e_i(x_1, \ldots, x_n) \}_{i=0}^{n-1}$. Lemma 5.6 says that $\beta$ spans $R_{S_n-1}$ as an $R_{S_n}$-module. Thus, $\beta$ forms a basis, and Equation (5.8) implies that $\beta^*$ is the dual basis, up to reordering.

It remains to prove Eq. (5.8). For this, consider the two-variable generating function

$$F(t, u) := \sum_{i,j=0}^{\infty} t^i u^j (-1)^i e_i(x_1, \ldots, x_{n-1})x^j_n$$

Observe:

$$F(t, u) = \frac{\prod_{i=1}^{n-1}(1 + tx_i)}{1 + ux_n} = \frac{\prod_{i=1}^{n-1}(1 + tx_i)}{(1 + ux_n)(1 + tx_n)} = \left( \sum_{i=0}^{n} t^i e_i(x_1, \ldots, x_n) \right) \left( \sum_{j=0}^{\infty} (-1)^j t^j x^j_n \right) \left( \sum_{k=0}^{\infty} (-1)^k u^k x^k_n \right) = \sum_{i,j,k} (-1)^{j+k} t^{i+j} u^k e_i(x_1, \ldots, x_n)x^{j+k}_n$$

Now, taking $\partial$ gives

$$\partial(F(t, u)) = \sum_{i,j,k} (-1)^{j+k} t^{i+j} u^k \partial_{1,2,\ldots,n-1} \left( e_i(x_1, \ldots, x_n)x^{j+k}_n \right) = \sum_{i,j,k} (-1)^{j+k} t^{i+j} u^k e_i(x_1, \ldots, x_n) \partial_{1,2,\ldots,n-1} \left( x^{j+k}_n \right) = \sum_{i,j,k} (-1)^{j+k} t^{i+j} u^k e_i(x_1, \ldots, x_n)(-1)^{n-1} h_{k+j+1-n}(x_1, \ldots, x_n)$$
\[= \sum_{m,k} t^m u^k \sum_{a+b=m+k+1-n} \left(-1\right)^b e_a(x_1, \ldots, x_n) h_b(x_1, \ldots, x_n)\]

In the second line we used the fact that the divided difference operators are \(RS_n\)-linear. In the third line we used Proposition 5.4 to compute \(\partial(x_{n+1}^{i+k})\). In the fourth line we introduced the new indices \(m = i + j\), \(a = i\), and \(b = k + j + 1 - n\). Finally, the last line follows from a well known identity satisfied by the elementary and complete symmetric functions. This computation gives the desired formula. \(\square\)

5.2 An explicit family of relations in \(R \otimes_{RS_n} R\)

Throughout this section we ignore all grading shifts. Let \(I_k \subset \mathbb{Q}[x, y]\) denote the ideal from Notation 2.6, so that \(B_{w_0} \cong \mathbb{Q}[x, y]/I_n\) and \(B_{w_1} \cong \mathbb{Q}[x, y]/I_{n-1}\). In this section we prove Proposition 2.43.

First, recall that the elementary symmetric functions \(e_k(x_1, \ldots, x_n) \in R\) and the complete symmetric functions \(h_k(x_1, \ldots, x_n) \in R\) are defined by their generating functions:

\[
\sum_{k=0}^{n} t^k e_k(x_1, \ldots, x_n) = \prod_{i=1}^{n} (1 + tx_i) \quad \sum_{k=0}^{\infty} t^k h_k(x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^{n} (1 - tx_i)} \tag{5.9}
\]

where \(t\) is a formal indeterminate. These expressions make the following recursion relations obvious:

**Proposition 5.10** We have the following recursive relations, for \(1 \leq k \leq n - 1\):

1. \(e_k(x_1, \ldots, x_{n+1}) = e_k(x_1, \ldots, x_n) + x_n e_{k-1}(x_1, \ldots, x_n)\).
2. \(h_k(x_1, \ldots, x_{n+1}) = \sum_{i=0}^{k} x_i h_{k-i}(x_1, \ldots, x_n)\). \(\square\)

**Proposition 5.11** Fix integers \(1 \leq m \leq n\). The polynomials

\[z_{m,n}(x_1, \ldots, x_n, y_1, \ldots, y_n) := \sum_{i+j=m} (-1)^i e_i(y_1, \ldots, y_n) h_j(x_m, \ldots, x_n) \in \mathbb{Q}[x, y]\]

are zero modulo \(I_n\).

**Proof** Fix an integer \(m \in \{1, \ldots, n\}\), and consider the following formal power series in \(t\):

\[Z(t) := \frac{\prod_{i=1}^{n} (1 + ty_i)}{\prod_{j=1}^{m} (1 + tx_i)} \]
The numerator is completely symmetric in the variables $y_1, \ldots, y_n$ hence each $y_k$ can be replaced by $x_k$ (modulo $I_n$). The result is a polynomial of degree $m - 1$:

$$Z(t) \equiv \prod_{i=1}^{m-1} (1 + tx_i) \pmod{I_n}. \quad (5.12)$$

On the other hand, by (5.9) the generating function $Z(t)$ can be written as

$$Z(t) = \sum_{k=0}^{\infty} t^k \sum_{i+j=k} (-1)^j e_i(y_1, \ldots, y_n) h_j(x_m, \ldots, x_n)$$

By (5.12), we see that the coefficient on $t^k$ vanishes for $k \geq m$, modulo $I_n$. That is to say, for each $k \geq m$ we have

$$\sum_{i+j=k} (-1)^j e_i(y_1, \ldots, y_n) h_j(x_m, \ldots, x_n) \equiv 0 \pmod{I_n}$$

Specializing to $k = m$ gives the identity in the statement. \qed

**Example 5.13** Setting $m = n$ in Proposition 5.11, we obtain that

$$\sum_{i+j=n} (-1)^j e_i(y_1, \ldots, y_n) x_n^j \equiv 0 \pmod{I_n}$$

We can factor this as $\prod_{i=1}^n (y_i - x_n) = 0 \pmod{I_n}$, so we recover the relations from Proposition 5.11 as a special case. Let $I_1 \subset \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the ideal generated by the differences $y_i - x_i$ for $1 \leq i \leq n$. Viewing the above relation in the quotient $R \cong \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I_1$, we find that

$$x_n^m = -e_n(x_1, \ldots, x_n) + x_n e_{n-1}(x_1, \ldots, x_n) - \cdots + (-1)^{n-1} x_n e_1(x_1, \ldots, x_n)$$

in $\mathbb{Q}[x_1, \ldots, x_n]$. \hfill (5.14)

The following gives an explicit formula for the relations $z_{m,n}$ in $R \otimes_{R^G} R$.

**Proposition 5.15** Fix an integer $n \geq 1$. The identity,

$$z_{m,n} = \sum_{\gamma} \prod_{i=1}^{m} (y_{\gamma_i} - x_{\gamma_i + m - i}),$$

holds in $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where the sum is over sequences of integers $\gamma = (\gamma_1, \ldots, \gamma_m)$ with $1 \leq \gamma_1 < \cdots < \gamma_m \leq n$.

Before proving, we give some examples which we hope will illustrate the formula. First:
**Definition 5.16** Set $p_{ij} = y_i - x_j \in \mathbb{Q}[x, y]$. In all occurrences in this paper, the variables will satisfy $1 \leq i \leq j \leq n$.

**Example 5.17** At one extreme, we have $z_{1,n} = p_{11} + p_{22} + \cdots + p_{nn}$, which equals $e_1(y) - e_1(x) \in \mathbb{Q}[x, y]$. At the other extreme we have $z_{n,n} = p_{1n} p_{2n} \cdots p_{nn}$. The relations $z_{mn}$ with $2 \leq m \leq n \leq 4$ are:

1. $z_{22} = p_{12} p_{22}$
2. $z_{23} = p_{12} p_{22} + (p_{12} + p_{23}) p_{33}$
3. $z_{24} = p_{12} p_{22} + (p_{12} + p_{23}) p_{33} + (p_{12} + p_{23} + p_{34}) p_{44}$
4. $z_{33} = p_{13} p_{23} p_{33}$
5. $z_{34} = p_{13} p_{23} p_{33} + (p_{13} p_{23} + p_{13} p_{34} + p_{24} p_{34}) p_{44}$
6. $z_{44} = p_{14} p_{24} p_{34} p_{44}$

**Proof of Proposition 5.15** Observe:

$$
\prod_{1 \leq i \leq \ell \atop m \leq j \leq \ell} \frac{1 + t y_i}{1 + t x_j} - \prod_{1 \leq i \leq \ell - 1 \atop m \leq j \leq \ell - 1} \frac{1 + t y_i}{1 + t x_j} = \prod_{1 \leq i \leq \ell - 1 \atop m \leq j \leq \ell - 1} \frac{1 + t y_i}{1 + t x_j} \left( \frac{1 + t y_\ell}{1 + t x_\ell} - 1 \right)
$$

$$
= \prod_{1 \leq i \leq \ell - 1 \atop m \leq j \leq \ell - 1} \frac{1 + t y_i}{1 + t x_j} \left( \frac{t (y_\ell - x_\ell)}{1 + t x_\ell} \right)
$$

$$
= \prod_{1 \leq i \leq \ell - 1 \atop m \leq j \leq \ell} \frac{1 + t y_i}{1 + t x_j} (y_\ell - x_\ell)
$$

Adding up the resulting identities for $m \leq \ell \leq n$ gives the identity

$$
\prod_{1 \leq i \leq n \atop m \leq j \leq n} \frac{1 + t y_i}{1 + t x_j} - \prod_{1 \leq i \leq m - 1 \atop m \leq j \leq n} (1 + t y_i) = t \sum_{\ell = m}^{n} (y_\ell - x_\ell) \prod_{1 \leq i \leq \ell - 1 \atop m \leq j \leq \ell} \frac{1 + t y_i}{1 + t x_j}
$$

We can rewrite this as

$$
\prod_{1 \leq i \leq n \atop m \leq j \leq n} \frac{1 + t y_i}{1 + t x_j} = t \sum_{\ell = m}^{n} (y_\ell - x_\ell) \prod_{1 \leq i \leq \ell - 1 \atop m - 1 \leq j \leq \ell - 1} \frac{1 + t y_i}{1 + t x_j + 1} + \text{(lower)}
$$

where (lower) denotes terms of $t$-degree $< m$. Iterating this formula, we find that

$$
\prod_{1 \leq i \leq n \atop m \leq j \leq n} \frac{1 + t y_i}{1 + t x_j} = t^m \sum_{k_1 = m}^{n} (y_{k_1} - x_{k_1})
$$

$$
\times \sum_{k_2 = m - 1}^{k_1 - 1} (y_{k_2} - x_{k_2 + 1}) \cdots \sum_{k_m = 1}^{k_{m-1} - 1} (y_{k_m} - x_{k_m + m - 1})
$$
plus terms of $t$-degree $\neq m$. Comparing the coefficients on $t^m$ on either side gives the identity in the statement. □

Recall the polynomials from Definition 2.40:

$$a_{mj}(x, y) := \sum_{\gamma'} \prod_{i=1}^{m-1} (y_{\gamma_i} - x_{\gamma_i + m-i}),$$ (5.18)

where $1 \leq m \leq j \leq n$. The sum is over sequences of integers $\gamma' = (\gamma_1, \ldots, \gamma_{m-1})$ with $1 \leq \gamma_1 < \cdots < \gamma_{m-1} < j$.

**Proposition 2.43** The $a_{ij}(x, y)$ satisfy $\sum_{j=k}^{n} a_{i,j}(x, y)(x_j - y_j) = 0$ modulo $I_n$. In particular $\sum_{j=k}^{n} a_{i,j}(x, y)(x_j - y_j)$ acts by zero on $B_{w_0}$.

**Proof** Observe

$$\sum_{j=m}^{n} a_{mj} \cdot (y_j - x_j) = \sum_{j=m}^{n} \sum_{\gamma} (y_j - x_j) \prod_{i=1}^{m-1} (y_{\gamma_i} - x_{\gamma_i + m-i})$$

$$= \sum_{\gamma} \prod_{i=1}^{m-1} (y_{\gamma_i} - x_{\gamma_i + m-i})$$

$$= z_{m,n}$$

The first equality holds by definition. In the second, we have reindexed by setting $\gamma = (\gamma_1, \ldots, \gamma_{m-1}, j)$. The third equality holds by Proposition 5.15. □

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