Some properties of evolving wormhole geometries within nonlinear electrodynamics

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In this paper we review some properties for the evolving wormhole solution of Einstein equations coupled with nonlinear electrodynamics. We integrate the geodesic equations in the effective geometry obeyed by photons; we check out the weak field limit and find the traversability conditions. Then we analyze the case when the lagrangian depends on two electromagnetic invariants and it turns out that there is not a more general solution within the assumed geometry.

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I. INTRODUCTION

Recently, the interest in wormholes has increased because of the possibility of interstellar travel or future time travel to past world. Since the formulation of Einstein’s equations, several features of these solutions were addressed, such as Einstein-Rosen bridges [1]; Wheeler [2] analyzed this kind of solutions and coined the name of wormholes. The most important and popular contribution to these solutions was given by Morris and Thorne [3], they were the first who made a complete and detailed investigation in this field. Wormholes are solutions to Einstein’s field equations which represent two connected universes or a connection between two distant regions of universe. For realistic models of traversable wormholes, one should address additional engineering issues such as tidal effects. People is skeptical about them because they contradict some commonly accepted reasonable energy conditions, for example, their structure requires matter with negative energy density as a source, i.e. exotic matter [3]. A full revision on wormholes can be consulted in [4] and [5].

Among an enormous variety of wormhole solutions, there are some kind of spherically symmetric wormholes that evolve with time. This type of wormholes have been extensively analyzed. Roman [6] explored the possibility that inflation might provide a mechanism to enlarge tiny wormholes to macroscopic size and investigated the possibility of avoiding the violation of the energy conditions in the process. Further evolving dynamic wormhole geometries were analyzed, considering specific cases [7]-[9]; and also was considered the evolution of a wormhole model in an FRW background [10].

On the other hand, nonlinear electrodynamics has recently been applied in several branches in physics, namely, as effective theories at different levels of string/M-theory [11], cosmological models [12], black holes [13]-[14] and in wormhole physics [15], among others. In this context, in a recent paper it was shown that (2 + 1) and (3 + 1)-dimensional static spherically symmetric as well as stationary axisymmetric traversable wormholes cannot be supported by nonlinear electrodynamics [16]. However, it was found an evolving wormhole solution within nonlinear electrodynamics [17], where the lagrangian depends in nonlinear way on one electromagnetic invariant.

In this paper we explore some properties of the evolving wormhole geometry in the context of nonlinear electrodynamics. This paper is outlined in the following manner. In section II, we describe briefly (3+1)-dimensional evolving spherically symmetric wormholes coupled with nonlinear electrodynamics. Section III is devoted to the study of wormhole geometry in the aspects of: traversability conditions, linear limit and photon trajectories. In Section IV we analyze the case when the lagrangian depends on the two electromagnetic invariants, attempting to find a more general solution. In section V some conclusions are given. We shall use geometrized units, i.e., $G = c = 1$, throughout this work.

II. (3 + 1)−DIMENSIONAL EVOLVING WORMHOLE SOLUTION

A. Action and spacetime metric

The action of $(3 + 1)$-dimensional general relativity coupled to nonlinear electrodynamics is given by
where \( R \) is the Ricci scalar. \( L(F) \) is the lagrangian, depending in nonlinear form on a single invariant \( F \) given by \( F = F^\mu_\nu F_{\mu\nu}/4 \) [18], where \( F_{\mu\nu} \) is the electromagnetic tensor. Note that in Einstein-Maxwell theory the lagrangian takes the form \( L(F) = -F^2/4\pi \), however, we shall consider more general electromagnetic lagrangians.

Varying the action with respect to the gravitational field provides the Einstein field equations \( G_{\mu\nu} = 8\pi T_{\mu\nu} \), with the stress-energy tensor given by

\[
T_{\mu\nu} = g_{\mu\nu} L(F) - F_{\mu\alpha} F_{\nu}^{\alpha} L_F,
\]

where \( L_F = dL/dF \). The variation with respect to the electromagnetic potential \( A_\mu \), where \( F_{\mu\nu} = A_\mu,_{\nu} - A_\nu,_{\mu} \), yields the electromagnetic field equations

\[
(F^\mu_\nu L_F)_{,\mu} = 0.
\]

The spacetime metric representing a dynamic spherically symmetric \((3+1)\)-dimensional wormhole, which is conformally related to the static wormhole geometry [3], takes the form

\[
ds^2 = \Omega^2(t) \left[ -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

where \( \Phi \) and \( b \) are functions of \( r \), and \( \Omega = \Omega(t) \) is the conformal factor that is finite and positive definite throughout the domain of \( t \). \( \Phi \) is the redshift function, and \( b(r) \) is the shape function [3]. We shall also assume that these functions satisfy all the conditions required for a wormhole solution, namely, \( \Phi(r) \) is finite everywhere in order to avoid the presence of event horizons; \( b(r)/r < 1 \), with \( b(r_0) = r_0 \) at the throat; as well as the fulfilment of the flaring out condition, \( b - b^2 \geq 0 \), with \( b(r_0) < 1 \) at the throat.

Now, taking into account the metric (4) the electromagnetic tensor compatible with the assumed symmetries is given by

\[
F_{\mu\nu} = E(x^\alpha) \left( \delta^t_{\mu} \delta^r_{\nu} - \delta^r_{\mu} \delta^t_{\nu} + B(x^\alpha) \left( \delta^\theta_{\mu} \delta^\phi_{\nu} - \delta^\phi_{\mu} \delta^\theta_{\nu} \right) \right),
\]

being the non-zero components the following: \( F_{tr} = -F_{rt} = E \), the electric field; and \( F_{\theta\phi} = -F_{\phi\theta} = B \), the magnetic field. The invariant \( F \) takes the form

\[
F = -\frac{1}{2\Omega^4} \left[ \left( 1 - \frac{b}{r} \right) e^{-2\Phi} E^2 - \frac{B^2}{r^4 \sin^2 \theta} \right].
\]

**B. Einstein field equations**

We recall that from [17], the solution requirement \( \Phi = 0 \) must be imposed, so using an orthonormal reference frame the non zero components of the Einstein tensor reduce to

\[
G_{tt} = \frac{1}{\Omega^2} \left[ \frac{b'}{r^2} + 3 \left( \frac{\Omega}{\dot{\Omega}} \right)^2 \right],
\]

\[
G_{rr} = \frac{1}{\Omega^2} \left\{ -\frac{b}{r^3} + \left[ \left( \frac{\Omega}{\dot{\Omega}} \right)^2 - 2 \left( \frac{\Omega}{\ddot{\Omega}} \right) \right] \right\},
\]

\[
G_{\theta\theta} = G_{\phi\phi} = \frac{1}{\Omega^2} \left\{ -\frac{b'}{r} - b \left( \frac{\Omega}{\dot{\Omega}} \right)^2 + \left[ \left( \frac{\Omega}{\dot{\Omega}} \right)^2 - \left( \frac{\Omega}{\ddot{\Omega}} \right) \right] \right\},
\]

\[
S = \int \sqrt{-g} \left[ \frac{R}{16\pi} + L(F) \right] d^4x,
\]
where overdot denotes a derivative with respect to the time coordinate, $t$, and the prime a derivative with respect to $r$. Analogously, the nonzero components of the stress energy tensor, from Eq. (2), take the form

$$T_{tt} = -L - \frac{(1 - b/r)}{\Omega^4} E^2 L_F,$$

$$T_{rr} = L + \frac{(1 - b/r)}{\Omega^4} E^2 L_F,$$

$$T_{\theta\phi} = T_{\phi\theta} = L - \frac{1}{\Omega^4 r^4 \sin^2 \theta} B^2 L_F.$$

It is clear that $T_{tt} = -T_{rr}$ and using Einstein’s field equations, the following relation is obtained

$$\frac{b' r - b}{2r^3} = -2\left[ \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - \frac{\ddot{\Omega}}{\Omega} \right].$$

In Eq. (11) each side depends only on one variable, therefore a solution is found by separating variables:

$$b(r) = r \left[ 1 - \alpha^2 (r^2 - r_0^2) \right],$$

$$\Omega(t) = \frac{2\alpha}{C_1 e^{\alpha t} - C_2 e^{-\alpha t}}.$$

where $\alpha$ is the separation constant, $C_1$ and $C_2$ are constants of integration. If $C_1 = C_2$, $\Omega$ is singular at $t = 0$; therefore for $\alpha > 0$ we need to impose $C_1 > C_2$ and if $\alpha < 0$ it is required that $C_1 < C_2$, otherwise the conformal factor becomes singular somewhere in the domain of $t$. The conformal function $\Omega(t) \rightarrow 0$ as $t \rightarrow \infty$, which reflects a contracting wormhole solution. It is important to observe that $\Omega(t) \rightarrow \infty$ as $t \rightarrow t_0 = \alpha^{-1} \log(\pm C_1^{-1} \sqrt{C_1 C_2})$ showing a time singularity that must be avoided. In the spirit of [17] we define the dimensionless parameter $R_0 = \alpha r_0$, so that the shape function is given by

$$b(r) = r \left\{ 1 - R_0^2 \left[ \left( \frac{r}{r_0} \right)^2 - 1 \right] \right\}.$$

A fundamental condition to be a reliable wormhole solution is imposed, that $b(r) > 0$ [19]. Thus, the range of $r$ is $r_0 < r < a = r_0 \sqrt{1 + 1/R_0^2}$; the latter may be arbitrarily large by taking $R_0 \rightarrow 0$. If $a \gg r_0$, i.e., $R_0 \simeq r_0/a \ll 1$, one may have an arbitrarily large wormhole.

C. Electromagnetic field equations

Solving the electromagnetic field Eq. (3), together with

$$\ast F_{\mu\nu} = 0,$$

that can be deduced from Bianchi identities, where the asterisk denotes the Hodge dual [18], we obtain the restrictions that $F_{tr} = -F_{rt} = E(t, r)$, $F_{\theta\phi} = -F_{\phi\theta} = B(\theta)$, $L_F = L_F(t, r)$. Thus the magnetic field is given by

$$B(\theta) = q_m \sin \theta,$$

where $q_m$ is a constant related to magnetic charge.

Furthermore, from Eqs. (8) and (10), we obtain
\[ \frac{\Omega^2}{8\pi} \left( b'r - 3b \right) = \left( 1 - \frac{b}{r} \right) E^2 L_F + \frac{q_m^2}{r^4} L_F. \]  
(17)

The regular solution appears when we set \( E = 0 \). Using Eq. (17) we obtain

\[ L_F = \frac{1}{16\pi q_m^2} \Omega^2 r (b'r - 3b), \]  
(18)

and taking into account Eqs. (14) and (13), the lagrangian is given by

\[ L = -\frac{1}{8\pi \Omega^2} \left[ \frac{b'}{r^2} + 3 \left( \frac{\dot{\Omega}}{\Omega} \right)^2 \right]. \]  
(19)

These equations, together with \( B = q_m \sin \theta, E = 0, F = q_m^2 / (2\Omega^4 r^4) \) and functions (12) and (13) give a wormhole solution well behaved at the throat, with finite fields. This result is in close relationship to the regular magnetic black holes coupled to nonlinear electrodynamics found in [14]. We remind that for static solutions with nonlinear electrodynamics, the null energy condition is not violated, fact that forbids the possibility of wormholes; however for the evolving wormhole under study this does not apply and moreover, the weak energy condition is satisfied.

**III. ANALYSIS OF THE \( E = 0 \) SOLUTION**

In what follows we address mainly four aspects of the solution, namely: flaring out condition and embedded diagram, traversability conditions, the weak field limit and photon trajectories.

**A. Flaring Out Condition and Embedded diagram**

As can be found in [3], [4], an imperative feature to appear in any wormhole solution is the so called flare out condition once the embedded diagram is given. The embedding is obtained when one considers the wormhole metric for an equatorial slice \( \theta = \pi/2 \) and in a fixed time \( t \), embedding then in a flat three dimensional Euclidean space,

\[ ds^2 = d\tau^2 + dr^2 + \tau^2 d\phi^2. \]

In our case the flare out condition requires (17) that

\[ \frac{d^2 \tau (\tau)}{d\tau^2} = \frac{1}{\Omega (t)} \frac{d^2 r (z)}{dz^2} = \frac{1}{\Omega (t)} \frac{b - b'}{2b^2} > 0 \]  
(20)

at or near the throat. Using Eqs. (14) and (13) the flare out condition can be casted as

\[ \frac{d^2 \tau (\tau)}{d\tau^2} = \frac{r}{2} \alpha (C_1 e^{\alpha t} - C_2 e^{-\alpha t}) > 0. \]  
(21)

If \( \alpha > 0 \), then we need to impose \( C_1 e^{\alpha t} > C_2 e^{-\alpha t} \), and if \( \alpha < 0 \) and \( C_1 e^{\alpha t} < C_2 e^{-\alpha t} \), otherwise the conformal factor becomes negative somewhere along the domain of \( t \), violating the condition \( \Omega (t) > 0 \). So in general the flare out condition is fulfilled by the solutions (14) and (13).

Another feature that must be explored for a wormhole solution is the one related to the form of the function \( \tau (\tau) \) from the embedded diagram [3], [4], this function can be obtained integrating the relationship

\[ \frac{d\tau (\tau)}{d\tau} = \frac{dz (r)}{dr} = \pm \left( \frac{r}{b} - 1 \right)^{-1/2}. \]  
(22)

This equation can be rewritten using equation (14) to obtain
\[ \frac{d\tau(\tau)}{d\tau} = \frac{dz(r)}{dr} = \pm \left( \frac{1}{\alpha^2(r^2 - r_0^2)} - 1 \right)^{1/2}, \]  

(23)

from which we would be able to obtain the solution for \( \tau(\tau) \) as

\[ \tau(\tau) = \pm \Omega(t) z(r) = \pm \Omega(t) \int \left( \frac{1}{R_o^2(r^2 - r_0^2)} - 1 \right)^{1/2} dr. \]  

(24)

In Fig. [20] it is shown the embedded wormhole \( z(r) \) for an arbitrary constant time. We can observe that the behavior of the derivative is highly positive near the throat \( r_0 \), as was expected for a wormhole solution, and as \( r \) grows far from the throat, the value of the derivative \( \frac{dz}{dr} \) becomes smaller but positive, indicating that the tangent lines for \( z(r) \) become nearly horizontal. This behavior grant us a shape for the throat very similar to the typical behavior expected for a wormhole solution [3], [4].

![Diagram](image)

**FIG. 1:** The embedded wormhole \( z(r) \) for an arbitrary constant time is shown for distinct values of the parameter \( \alpha \). The continuous curve corresponds to \( \alpha = 0.2 \) while the dotted one is for \( \alpha = 0.3 \).

### B. Traversability Conditions

We explore now the possibility that a human traveler traverses an evolving wormhole like the one presented here. Leaving aside the study of the stability of the solution, we delve into the analysis of the tidal gravitational forces that an infalling radial observer must bear while traversing. We take as our traversability criteria the magnitude of the tidal forces that an observer can stand during the trip: it must not exceed the forces due to Earth gravity, much on the spirit of [21]. To simplify our calculations we shall work in a proper reference frame given by

\[ e_0 = \gamma e_t \mp \beta e_r, \quad e_1 = \mp \gamma e_r + \beta e_t, \quad e_2 = e_\theta, \quad e_3 = e_\phi, \]  

(25)

where \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \) and \( \beta = v/c, v \) is the velocity of the radial motion [9]. Then the traversability criteria can be stated as

\[ |R_{0\hat{t}\hat{t}}| = |C_1 C_2| \leq \frac{g_m}{2m} \approx \frac{1}{(10^{10}\text{cm})^2}, \]  

(26)

\[ |R_{0\hat{r}\hat{r}}| = |R_{\hat{t}\hat{t}}| = \gamma^2 |R_{\hat{t}\hat{t}}| + \gamma^2 \beta^2 |R_{\hat{r}\hat{r}}| = |C_1 C_2| \leq \frac{g_m}{2m} \approx \frac{1}{(10^{10}\text{cm})^2}, \]  

(27)
where \( g_\oplus = 9.81 \text{m/s}^2 \) is Earth gravity; these conditions can be fulfilled with the appropriate selection of the values for \( C_1 \) and \( C_2 \), rendering this solution in a traversable one. We point out the fact that Eq. (27) does not depend on the traverse velocity \( v \), however it is not in contradiction with the results given in [16], Chap. 13.1.1., related to the dependence of tidal forces on the traveler velocity, due to the fact that the condition \( R^\tau_\theta\theta \tau = - R^\tau_\theta\theta \tau \) does not lead to the presence of a horizon, instead, it takes us to equation (11) from which the solution is obtained.

Since in the studied case the redshift has been set equal to zero, the radial tides are zero, then if some danger exists for the traveler it is in the nonzero accelerations while traversing the wormhole. In what follows we show that there is not tidally induced shear.

The \( Q_{IJ} \) matrix that describes tidal forces is defined by

\[
Q_{IJ} = - R_{\mu\alpha\beta\gamma}(\eta_I)^\mu V^\alpha(\eta_J)^\nu V^\beta,
\]

where \( \eta_I, J = 1, 2, 3 \) is an orthonormal triad orthogonal to the four-velocity \( V^\alpha \). Let us consider the traveler’s motion is non radial with a velocity vector given by

\[
V^\alpha = (\gamma, \gamma\beta \cos \psi, 0, \gamma\beta \sin \psi).
\]

In the case we are analyzing, the tidal forces are described with the entrances

\[
Q_{11} = - R^r_r \psi \cos^2 \psi - R^t_t \psi \sin^2 \psi = C_1 C_2,
\]

\[
Q_{22} = - \gamma^2 (R^r_r \psi \sin^2 \psi + R^t_t \psi \cos^2 \psi + \beta^2 R^r_r \psi \sin^2 \psi) = C_1 C_2,
\]

\[
Q_{33} = - \gamma^2 (R^r_r \psi + \beta^2 R^r_r \psi \cos^2 \psi + \beta^2 R^r_r \psi \sin^2 \psi)
\]

\[
= - \gamma^2 [-C_1 C_2 + C_1 C_2 \beta^2 \cos^2 \psi + \beta^2 \sin^2 \psi \left\{ (C_1 e^{\alpha t} + C_2 e^{-\alpha t})^2 \right\}]
\]

\[
\quad \quad \quad \quad + \frac{1}{\alpha^2 r^2} (1 - \alpha^2 (r^2 - r_0^2))(C_1 e^{\alpha t} - C_2 e^{-\alpha t})^2, \]

\[
Q_{12} = - \gamma (R^r_r \psi - R^t_t \psi) \sin \psi \cos \psi = 0,
\]

and

\[
Q_{13} = Q_{23} = 0.
\]

Therefore, the tidal forces do not diverge, furthermore, since \( Q_{12} = 0 \), there is no presence of tidally induced shear, this means that the traveler will pass safe through the wormhole as far as tidal forces concern.

**C. Maxwellian Limit.**

It is important that any nonlinear theory recovers the form and results of the corresponding linear theory, nonlinear electrodynamics is not an exception [18]. However there are cases where the Maxwellian limit is not recovered [22].

To carry on the analysis of the Maxwell limit, it is useful to rewrite the main functions \( F, L \) and \( L_F \) in terms of the coordinates \( r \) and \( t \),

\[
F = \frac{g_\oplus^2}{32\alpha^4} \left( \frac{C_1 e^{\alpha t} - C_2 e^{-\alpha t}}{r} \right)^4,
\]

\[
L = - \frac{1}{32\pi\alpha^2} \left\{ 1 - \alpha^2 (3r^2 - r_0^2) \left[ \frac{C_1 e^{\alpha t} - C_2 e^{-\alpha t}}{r} \right]^2 + 3\alpha^2 (C_1 e^{\alpha t} + C_2 e^{-\alpha t})^2 \right\}.
\]
The result of the analysis of the solution is the absence of a complete Maxwellian limit, i.e., when \( r \to \infty \) we get that \( F \to 0, L \to -\frac{8\sqrt{C_1C_2}}{\pi} \) and \( L_F \to \infty \). The fact that the Lagrangian \( L \) does not go as \( F \), and instead goes to a constant value shows that in this limit the spacetime is not asymptotically flat due to a constant remanent energy. Another impressive result is the corresponding to \( L_F \) which goes to infinity, showing a similar behavior to an ideal conductor [23].

Alternatively, as the solution is only valid for the region \( r_0 < r < a = r_0\sqrt{1 + 1/\beta^2} \), and the Maxwellian limit would be attained as \( r \to \infty \), the requirement of the presence of a region with Maxwellian limit loses strength.

Furthermore, it does make sense the absence of such a limit since for weak fields the energy conditions are always fulfilled, a fact that is contrary to the existence of wormholes.

D. Light rays in the NLED effective geometry

In nonlinear electrodynamics photons do not propagate along null geodesics of the background geometry, instead they propagate along null geodesics of an effective geometry which depends on the nonlinear theory [17], [24]. The discontinuities of the field propagate obeying the equation for the characteristic surfaces \( S \). For a curved spacetime the equation for the characteristic surfaces is

\[
g^{\mu\nu}S_{,\mu}S_{,\nu} = 0 \tag{38}
\]

And when nonlinear electrodynamics is involved, the corresponding “eikonal” equation for the propagation vectors \( k^\mu \) is

\[
(L_F g^{\mu\nu} - 4L_FF\delta^{\mu\alpha}F_\alpha^\nu)k_{,\mu}k_{,\nu} = g^{\mu\nu}k_{,\mu}k_{,\nu} = 0
\]

(39)

In the orthonormal tetrad, the expression of \( g_{\hat{\mu}\hat{\nu}} \) in terms of the electromagnetic field tensor \( T^{\hat{\mu}\hat{\nu}} \) also shows clearly that the modification in the trajectories is due to nonlinear electromagnetic field:

\[
g_{\text{eff}}^\hat{\mu}\hat{\nu} = \left( L_F + \frac{LL_FF}{L_F} \right) \eta^{\hat{\mu}\hat{\nu}} + \frac{L_FF}{L_F} T^{\hat{\mu}\hat{\nu}},
\]

(40)

given in orthonormal coordinates \( \hat{\mu}\hat{\nu} \), with \( \eta^{\hat{\mu}\hat{\nu}} = \text{diag}[-1, 1, 1, 1] \). For the case of the evolving wormhole with nonlinear electrodynamics [17] with (12) and (13), the effective geometry is given by

\[
g_{\text{eff}}^{\hat{r}\hat{t}} = -\left( L_F + 2L_FF/L_F \right) = -g_{\text{eff}}^{\hat{r}\hat{t}}, \tag{41}
\]

\[
g_{\text{eff}}^{\hat{\phi}\hat{\phi}} = L_F + 2L_FF/L_F - 2FL_FF = g_{\text{eff}}^{\hat{\phi}\hat{\phi}}, \tag{42}
\]

where we have taken into account the energy-momentum tensor given by

\[
\text{diag}(T^{\hat{r}\hat{t}}, T^{\hat{r}\hat{r}}, T^{\hat{\phi}\hat{\phi}}, T^{\hat{\phi}\hat{\phi}})
\]

with

\[
T^{\hat{r}\hat{t}} = -L = -T^{\hat{r}\hat{t}}, \tag{43}
\]

\[
T^{\hat{\phi}\hat{\phi}} = L - 2FL_FF = T^{\hat{\phi}\hat{\phi}}, \tag{44}
\]

that are Eqs. (18)-(20) in [17]. Moreover, the expression for the invariant \( F \), if \( E = 0 \) is
\[ F = \frac{1}{2} \frac{g_m}{\Omega^2 r^4} \]  

(45)

Now, in the particular case that \( b(r) = r \left[ 1 - \alpha^2 \left( r^2 - r_0^2 \right) \right] \), we have the relationship

\[ \frac{L_{EF}}{L_F} = -\frac{1}{F}, \]  

(46)

therefore Eqs. (41)-(42) amount to

\[ g_{\alpha\alpha} \frac{\dot{\theta}}{\dot{\phi}} = -\left( \frac{L_F - 2L}{F} \right) = -g_{\alpha\alpha}, \]  

(47)

\[ g_{\theta\theta} = 3L_F - \frac{2L}{F} = g_{\theta\theta}. \]  

(48)

In the effective metric the geodesic equations are

\[
\frac{d^2 t}{d\tau^2} + \left( \frac{\dot{L}_F}{2F(FL_F - 2L)} \right) \left( \left( \frac{dt}{d\tau} \right)^2 - \left( \frac{dr}{d\tau} \right)^2 \right) + \left( \frac{2L_F - 2L\dot{F} - 3L_F^2}{2F(FL_F - 2L)} \right) A^2 = 0 \]  

(49)

\[
\frac{d^2 r}{d\tau^2} - \left( \frac{L_F^2 - 2L' F + 2L'\prime}{2(FL_F - 2L)} \right) \left( \left( \frac{dt}{d\tau} \right)^2 - \left( \frac{dr}{d\tau} \right)^2 \right) + \left( \frac{2L' F + 2L' \prime + 3L_F^2}{2(FL_F - 2L)} \right) A^2 = 0 \]  

(50)

\[
\left( L_F - \frac{2L}{F} \right) \left( \left( \frac{dt}{d\tau} \right)^2 + \left( \frac{dr}{d\tau} \right)^2 \right) + \left( 3L_F - \frac{2L}{F} \right) A^2 = \delta \]  

(51)

\[
\left( \frac{d\theta}{d\tau} \right)^2 + \left( \frac{d\phi}{d\tau} \right)^2 \right) = A^2, \]  

(52)

where \( \tau \) is the affine parameter that generates the geodesic trajectories \( r(\tau), t(\tau), \theta(\tau), \varphi(\tau) \). \( A \) is a conserved quantity related to the existence of Killing vectors. The Eq. (51) is derived from the line element and \( \delta = 1, 0, -1 \) for timelike, null and spacelike geodesics, respectively.

Substituting \((\frac{dt}{d\tau})^2 - (\frac{dr}{d\tau})^2\) from Eq. (51) in Eqs. (49) and (50), for \( \delta = 0 \) it is obtained

\[
\frac{d^2 t}{d\tau^2} + 2 \frac{LL_F \dot{F} + \dot{L} \dot{F} F - L_F \dot{F} F}{(FL_F - 2L)^2} A^2 = 0, \]  

(53)

\[
\frac{d^2 r}{d\tau^2} + 2 \frac{LL_F F' + LL_F' F - L_F L_F'}{(FL_F - 2L)^2} A^2 = 0. \]  

(54)

The constant of motion \( A \) makes easier the integration of the geodesic equations. Considering the expressions for \( L_F, L, F \) in terms of \( r \) and \( t \), straightforward algebra leads to the expressions

\[
0 = \frac{d^2 t}{d\tau^2} + \frac{A^2}{2} \frac{df(r,t)}{dt}, \]  

\[
0 = \frac{d^2 r}{d\tau^2} + \frac{A^2}{2} \frac{dg(r,t)}{dr}, \]  

(55)

with

\[
f(r,t) = \frac{(1 + \alpha^2 r_0^2)}{6r^2} \left[ \frac{(1 + \alpha^2 r_0^2)}{4r^2} - \alpha^2 + \left( \frac{\Omega}{\Omega} \right)^2 \right]^{-1}, \]  

\[
g(r,t) = \frac{2}{3} \left[ \alpha^2 - \left( \frac{\Omega}{\Omega} \right)^2 \right] \left[ \frac{(1 + \alpha^2 r_0^2)}{4r^2} - \alpha^2 + \left( \frac{\Omega}{\Omega} \right)^2 \right]^{-1}. \]  

(56)
Eqs. (55) can be integrated by multiplying the first equation by $2(dt)/(d\tau)$ and the second by $2(dr)/(d\tau)$, so they acquire the form of

$$\frac{d}{d\tau} \left[ \left( \frac{dt}{d\tau} \right)^2 + A^2 f(r,t) \right] = 0,$$

(57)

$$\frac{d}{d\tau} \left[ \left( \frac{dr}{d\tau} \right)^2 + A^2 g(r,t) \right] = 0,$$

(58)

that allows one to obtain the trajectories $r(t)$ of photons as

$$\frac{dr}{dt} = \sqrt{\frac{K_2 - g(r,t)}{K_1 - f(r,t)}},$$

(59)

where $K_1$ and $K_2$ are constants. Substituting $f(r,t)$ and $g(r,t)$ from (56) and simplifying we get

$$\frac{dr}{dt} = \sqrt{\frac{(1 + \alpha^2 r_0^2)K_2(Be^{2\alpha t} - 1)^2 + 16\alpha^2 r^2 B(K_2 + 2/3)e^{2\alpha t}}{(1 + \alpha^2 r_0^2)(K_1 - 2/3)(Be^{2\alpha t} - 1)^2 + 16\alpha^2 r^2 BK_1 e^{2\alpha t}}},$$

(60)

where $B = C_1/C_2$, $B < 1$ for the square root be well defined; $C_1$ and $C_2$ can be adjusted to fulfil the traversability conditions as well.

The graphics of the trajectories $r(t)$ are shown in Fig.2 for different values of $\alpha$, with $K_1 = K_2 = 0$, showing the influence of the magnetic field variation. Fig.3 shows $r(t)$ when $K_1$ and $K_2$ are not zero. It can be observed in both plots the nonvanishing throat, as well as the non-flatness of the spacetime (associated to the non-Maxwellian limit). The effective spacetime as “seen” by photons does not carry as much restrictions as the mere existence of the wormhole.

FIG. 2: Null geodesics $r(t)$ in the effective geometry of the nonlinear electromagnetic field. In these plots $K_1 = K_2 = 0$ and the static case is for $\alpha = 0$. The non-flatness of the spacetime is apparent, as the geodesics do not tend to infinity for large times

IV. EVOLVING WORMHOLE SOLUTION WITH TWO INVARIANTS $F$ AND $G$

The previous analyzed wormhole corresponds to a lagrangian that depends only on one of the two electromagnetic invariants, $F = F^{\mu\nu} F_{\mu\nu}/4$. On the search for a more general solution we explore in this section the case of a lagrangian depending in nonlinear way on both invariants.
FIG. 3: Trajectories of photons $r(t)$ in the effective geometry. The continuous curve corresponds to $K_1 = 1.8, K_2 = 0.5$, while the dotted one is for $K_1 = 0.5, K_2 = -0.66$, in both $\alpha = 0.8$. The nonvanishing of the constants of movement enhances the action of the magnetic field and smooths the non-flatness of the spacetime.

Let us consider the action of $(3 + 1)$-dimensional general relativity coupled to nonlinear electrodynamics given by

$$S = \int \sqrt{-g} \left( \frac{R}{16\pi} + L(F,G) \right) d^4x, \quad (61)$$

where $R$ is the Ricci scalar. $L(F,G)$ is a gauge-invariant electromagnetic lagrangian depending on the invariants $F = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$ and $G = \frac{1}{4} F_{\mu\nu} \tilde{F}_{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic tensor and $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ its dual.

Varying the action with respect to the gravitational field provides the Einstein field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

with the stress-energy tensor given by

$$T_{\mu\nu} = g_{\mu\nu} L - F_{\mu\delta} F^{\delta}_\nu \delta L_F - F_{\mu\delta} \tilde{F}^{\delta}_\nu \delta L_G, \quad (62)$$

where $L_F = \frac{\partial L}{\partial F}$ and $L_G = \frac{\partial L}{\partial G}$ are functions of $r$ and $t$ only, in the same form $L = L(t,r)$. The variation with respect to the electromagnetic potential $A_\mu$, yields the electromagnetic field equations

$$p^{\mu\nu} \equiv (F^{\mu\nu} L_F + \tilde{F}^{\mu\nu} L_G)_{,\nu} = 0. \quad (63)$$

We shall consider a spacetime metric representing a dynamic spherically symmetric $(3 + 1)$-dimensional wormhole given by (4). The non-zero Einstein tensor components are the same that in Eq. (7).

The electromagnetic tensor, compatible with the symmetries of the geometry, is given by (5). The $F$ invariant is given by Eq. (6) while the invariant $G$ is

$$G = \sqrt{1 - \frac{b(r)}{r} E B \Omega^4 r^2 \sin(\theta)}. $$

The components of the stress-energy tensor, Eq. (62), in the orthonormal frame take the following form

$$T_{t\bar{t}} = -T_{r\bar{r}} = -L - \frac{E^2 L_F (1 - \frac{b}{r})}{\Omega^4} + \frac{E B L G \sqrt{1 - \frac{b}{r}}}{\Omega^4 r^2 \sin(\theta)} \Omega^4 L_G + GL_G,$$
\begin{align}
T_{\theta\theta} & = T_{\varphi\varphi} = L - \frac{EBL_G\sqrt{1 - \frac{b}{r}}}{\Omega^4 r^2 \sin(\theta)} - \frac{B^2 L_F}{\Omega^4 r^2 \sin^2(\theta)} \\
& = L - GL_G - \frac{B^2 L_F}{\Omega^4 r^2 \sin^2(\theta)}, \\
T_{\hat{t}\hat{t}} & = T_{\hat{i}\hat{j}} = 0 \text{ (with } i \neq j) \quad \text{(64)}.
\end{align}

As in [17], \( T_{\hat{t}\hat{t}} = -T_{r\varphi} \), so the solution to \( G_{\hat{t}\hat{t}} = -G_{r\varphi} \) can be solved separating variables like in (11):

\[ b(r) = r \left[ 1 - \alpha^2 (r^2 - r_0^2) \right], \quad \text{(65)} \]

\[ \Omega(t) = \frac{2\alpha}{C_1 e^{\alpha t} - C_2 e^{-\alpha t}}, \quad \text{(66)} \]

where \( \alpha \) is a constant, and \( C_1 \) and \( C_2 \) are constants of integration.

The electromagnetic field Eqs. (63) take the form

\[ \sqrt{1 - \frac{b}{r}} \partial_t (EL_F) - \frac{B (\partial_t L_G)}{r^2 \sin(\theta)} = 0, \quad \text{(67)} \]

\[ \partial_r \left( \sqrt{1 - \frac{b}{r}} r^2 EL_F \right) - \frac{B (\partial_r L_G)}{\sin(\theta)} = 0, \quad \text{(68)} \]

\[ \frac{B \cos(\theta) - \partial_\theta B \sin(\theta)}{r^2 \Omega(t)^4 \sin^2(\theta)^3} L_F = 0. \quad \text{(69)} \]

Eq. (67) can be solved for \( EL_F \):

\[ EL_F = \frac{BL_G}{r^2 \sqrt{1 - \frac{b}{r} \sin(\theta)}} + F_1(r). \quad \text{(70)} \]

Now, Eq. (68), when we take into account Eq. (70) give us the expression for \( F_1(r) \)

\[ F_1(r) = \frac{C_E}{r^2 \sqrt{1 - \frac{b}{r}}}, \quad C_E = \text{cte}. \]

Then (70) takes the form

\[ EL_F = \frac{1}{r^2 \sqrt{1 - \frac{b}{r} \sin(\theta)}} \left[ \frac{BL_G}{\sin(\theta)} + C_E \right]. \quad \text{(71)} \]

From this last relation we verify that \( EL_F \) is singular at the throat. Integrating Eq. (69), we obtain \( B = q_m \sin(\theta) \). Of course when \( L_G = 0 \), we recover the expression obtained in [13] for \( EL_F \).

Therefore, when there is dependence on both electromagnetic invariants, \( F \) and \( G \), we recover the solution in [17], except for an additional term \( \kappa G \),

\[ L = L(F) + \kappa G \]

This term breaks the duality rotation symmetry of the electromagnetic field, present for instance in nonlinear electrodynamics of the Born-Infeld type.

We conclude that geometry (4) does not allow nonlinear electromagnetic matter related to a lagrangian \( L(F,G) \). There is open still the question if assuming a less symmetrical spacetime it would admit such kind of matter.

\section*{V. FINAL REMARKS}

In this article we have analyzed the solution representing an evolving wormhole coupled to nonlinear electrodynamics given by Arellano and Lobo in [17]. We presented the integration of the geodesic equations for light rays and the
corresponding graphics that show the influence of the nonlinear field; the traversability conditions as well as the weak field limit were analyzed.

In the conclusions about traversability, we obtained that the tidal gravitational forces are constant in the radial movement, and they can be adjusted to allow a safe human passage.

For the Maxwellian limit we have found that when \( r \to \infty \) the spacetime is asymptotically flat except for a remanent energy, a possible interpretation can be the presence of a magnetic perfect fluid that can be obtained from the stress energy tensor taking that limit. We must stress that the worm hole solution is only valid in a certain region that, even if it can be extended to be large, leaves outside the limit \( r \to \infty \). However, the allowed trajectories for photons show that the NLED effective geometry is less restrictive than the one for the wormhole.

When the dependence on both invariants for the lagrangian is considered, it turns out that the electric-magnetic duality symmetry is lost.

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