Piecewise Polyhedral Formulations for a Multilinear Term

Kaarthik Sundar\textsuperscript{1}, Harsha Nagarajan\textsuperscript{1}, Site Wang\textsuperscript{2}, Jeff Linderoth\textsuperscript{3}, Russell Bent\textsuperscript{1}

Abstract

In this paper, we present a mixed-integer linear programming formulation of a piecewise, polyhedral relaxation (PPR) of a multilinear term using its convex hull representation. Based on the solution of the PPR, we also present a MIP-based piecewise formulation which restricts the solutions to be feasible for the multilinear term. We then present computational results showing the effectiveness of proposed formulations on instances from the standard Mixed-Integer Nonlinear Programming Library (MINLPLib) and compare the proposed formulation with a formulation that is built by recursively relaxing bilinear groupings of the multilinear term, typically applied in the literature.

Keywords: multilinear, piecewise polyhedral relaxations, convex hull, global optimization

1. Introduction

In the global optimization literature, many solution techniques used to solve Mixed-Integer Nonlinear Programs (MINLPs) to optimality rely on convex relaxations \cite{10.1016/j.cor.1991.09.005} and piecewise convex relaxations of non-convex functions \cite{10.1016/0304-3915(80)90029-1, 10.1007/978-3-642-57073-9_15, 10.1007/s10107-007-0179-2, 10.1007/s10107-006-0118-9, 10.1007/s10107-006-0118-9} to obtain tight bounds on the optimal objective values of MINLPs. This letter focuses on developing piecewise, polyhedral relaxations (PPR) of a multilinear term represented by its convex hull and on comparing this relaxation with traditional, recursive piecewise relaxations. Our work generalizes formulations that approximate functions with piecewise, linear functions \cite{10.1016/j.orl.2003.11.007, 10.1007/s00186-004-0268-0, 10.1016/j.ipl.2005.10.003, 10.1016/j.ejor.2008.04.009} to formulations that relaxation functions with piecewise, convex polyhedra. Furthermore, since recovering feasible solutions for MINLPs is of great interest in order to quantify the quality of the solutions of PPR, we propose a piecewise formulation based on the solution from PPR, which restricts the solutions to lie on the multilinear term, thus making it a feasible solution.

Throughout the rest of the letter, boldface is used to denote vectors and we formally define a multilinear term as \( \phi(x) : [\ell, u] \to \mathbb{R} \), where

\[
\phi(x) = \prod_{i \in \mathcal{I}} x_i.
\]  

Here, \( \mathcal{I} \) is an index set for the set of variables, \( x \), and \( [\ell, u] = \{ x \in \mathbb{R}^{\left| \mathcal{I} \right|} : \ell \leq x \leq u \} \). For every \( i \in \mathcal{I} \), we associate a spatial disjunction with variable \( x_i \) defined by discretization points \( S_i = \{ s_{i,1}, s_{i,2}, \ldots, s_{i,n_i} \} \) and associated intervals \( P_i = \{ \delta_i^1, \delta_i^2, \ldots, \delta_i^{n_i-1} \} \) where the intervals form a partition of the domain.
of \( x_i \) given by \([\ell_i, u_i]\). i.e. \( \delta^k_i = [s_i, k, s_i, k+1] \) and \( \ell_i = s_i, 1 < s_i, 2 < \cdots < s_i, n_i = u_i \). Given the above notation, each \( x_i, i \in \mathcal{I} \) is constrained to satisfy
\[
x_i \in [s_i, 1, s_i, 2] \lor [s_i, 2, s_i, 3] \lor \cdots \lor [s_i, n_i - 1, s_i, n_i].
\]
(2)

Given \( \mathcal{P}_i \) and \( \mathcal{S}_i \) for every \( i \in \mathcal{I} \), we let \((n_i - 1)\) and \( n_i \) be the cardinality of the respective sets. For any \( i \in \mathcal{I} \) and \( k \in \{1, \ldots, n_i - 1\} \), the interval \( \delta^k_i \) is said defined to be active when the value of \( x_i \) lies in the interval \( \delta^k_i \). The Eq. (2) then enforces exactly one interval to be active per variable. Finally, the full set of discretization points and the partition sets defined by these points and intervals in the space \([\ell, u]\) are given by the sets \( \mathcal{S} = \times_{i \in \mathcal{I}} \mathcal{S}_i \) and \( \mathcal{P} = \times_{i \in \mathcal{I}} \mathcal{P}_i \), respectively.

Given the multilinear term \( \phi(x) \) and the associated sets \( \mathcal{S} \) and \( \mathcal{P} \), this work presents piecewise, polyhedral relaxations (PPR) for the graph of \( \phi(x) \), given by
\[
X = \{(x, w) \in [\ell, u] \times \mathbb{R} : w = \phi(x) \}.
\]
(3)

An important special case occurs when \( \phi(x) \) is bilinear with a single partition on each variable, i.e., \(|\mathcal{I}| = 2\) and \(|\mathcal{P}| = 1\). Here, the McCormick relaxation, given by the following equations
\[
w \geq u_2 x_1 + u_1 x_2 - u_1 u_2, \quad w \geq \ell_2 x_1 + \ell_1 x_2 - \ell_1 \ell_2, \quad (4a)
\]
\[
w \leq u_2 x_1 + \ell_1 x_2 - u_1 u_2, \quad w \leq \ell_2 x_1 + u_1 x_2 - u_1 \ell_2. \quad (4b)
\]
defines the convex hull \([16, 2]\) of the set \( X \), i.e., \( \text{conv}(X) \). For general multilinear terms i.e., \(|\mathcal{I}| \geq 3\) with \(|\mathcal{P}| = 1\), McCormick described a procedure by recursively applying the McCormick relaxation to products of variables, which we refer to as the “recursive McCormick relaxation”. On general asymmetric bounds on the variables, despite not capturing the convex hull \([15]\), the recursive McCormick relaxation for a multilinear term is the basis for relaxations used in many global optimization solvers such as BARON \([24]\), SCIP \([28]\) and Couenne \([7]\). In contrast, a relaxation based on a vertex representation, for \(|\mathcal{I}| \geq 2\) and \(|\mathcal{P}| = 1\) does characterize \( \text{conv}(X) \) \([10, 23]\) and is given by
\[
\text{Proj}_{x, w} \left\{ (x, w, \lambda) \in [\ell, u] \times \mathbb{R} \times \Delta_{2|\mathcal{I}|} : x = \sum_{s=1}^{2|\mathcal{I}|} \lambda_s \hat{x}_s, \ w = \sum_{s=1}^{2|\mathcal{I}|} \lambda_s \phi(\hat{x}_s) \right\},
\]
(5)

where \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2|\mathcal{I}|} \) is a collection of all the vertices of \([\ell, u]\) given by \( \mathcal{S} \) and \( \Delta_{2|\mathcal{I}|} \) is the \( 2^{2|\mathcal{I}|} \)-dimensional 0-1 simplex.

Note that for the case when \(|\mathcal{I}| = 2\) and \(|\mathcal{P}| = 1\), Eq. (5) and Eq. (4) are equivalent. When \(|\mathcal{I}| \geq 2\) and \(|\mathcal{P}| \geq 2\), the resulting convex relaxation can be expressed as a disjunctive union of \(|\mathcal{P}|\) polytopes where each polytope is obtained by applying Eq. (5) to the domain defined by the corresponding partition set. The convex hull of this disjunctive union is completely characterized by utilizing the theory of disjunctive programming \([3, 13]\) and the introduction of a sufficient number of auxiliary variables; in particular, by utilizing the formulations in \([13]\). The formulations in \([13]\) is an extended formulation that works by formulating the constraints for each partition separately and then aggregating them. In contrast, the approach presented in this letter works directly with the combinatorial structure underlying the shared extreme points and presents non-extended PPR for a multilinear term. Furthermore, disjunctive relaxations for the special case of a bilinear term with a spatial disjunction on one variable is addressed in \([17, 19]\). To the best of our knowledge, the work in \([17, 19]\) is the state-of-the-art for PPR of multilinear term.
The contributions of this letter include (a) an SOS-2-based PPR that uses a vertex representation for a single multilinear term and a spatial disjunction on each variable, and (b) given the PPR’s solution, a piecewise formulation that uses the vertex representation of the convex hull polytope to recover the feasible solution for the multilinear term. This PPR and the feasible solutions obtained are compared with a relaxation and feasible solutions recovered based on recursively relaxing bilinear groups of the multilinear term akin to the recursive McCormick relaxation. We refer to this relaxation as the recursive piecewise polyhedral relaxation (R-PPR). To the best of our knowledge, this is the first work that presents a systematic comparison between an SOS-2-based PPR and a recursive piecewise relaxation for a general multilinear term with spatial disjunctions on every variable.

2. Piecewise Polyhedral Relaxation for a Multilinear Term

In this section, we present the PPR for \( X \) given \( I \) (the index set of variables), \( P \) (the partition set), and \( S \) (the set of discretization points). For notation simplicity, for any \( i \in I \) and \( k \in \{1, \ldots, n_i - 1\} \), we use \( \delta^k_i = s_{i,k} \) and \( \delta^{k+1}_i = s_{i,k+1} \) to denote the lower and upper limits for that interval, respectively. We associate a non-negative multiplier variable, \( \lambda_s \), with each \( \hat{x}_s \in S \). We also define \( y_{i,k} \) as a binary indicator for the \( k \)th interval of variable \( x_i \), \( i \in I \). This variable takes a value of 1 when interval \( \delta^k_i \) is active for variable \( x_i \) and takes a value of 0 otherwise. Also, we let \( \lambda \) and \( y \) denote the vector of continuous multiplier variables and indicator variables, respectively. Next, for each \( i \in I \), we define a set-valued function, \( \mu_i(r) = \{ \hat{x}_s \in S : e_i^T \hat{x}_s = r \} \) where \( e_i \) is a unit vector whose \( i \)th coordinate is equal to 1 and is 0 everywhere else. This function defines the subset of points in \( S \) whose \( i \)th component is equal to \( r \). Finally, we let \( \lambda(S) = \sum_{\hat{x}_s \in S} \lambda_s \) for any \( S \subseteq S \).

Given these notations, the PPR for a multilinear term is given by

\[
\prod_{i \in I} x_i = \left\{ (x, w, \lambda, y) \in [\ell, u] \times \mathbb{R} \times \Delta_{|S|} \times \{0, 1\}^{\sum_{i \in I} (n_i - 1)} \mid (x, w, \lambda, y) \text{ satisfies Eq. (7)} \right\}
\]

(6)

where, Eq. (7) is given by

\[
x = \sum_{\hat{x}_s \in S} \lambda_s \hat{x}_s, \quad w = \sum_{\hat{x}_s \in S} \lambda_s \phi(\hat{x}_s),
\]

(7a)

\[
\sum_{k=1}^{n_i-1} y_{i,k} = 1, \quad \forall i \in I,
\]

(7b)

\[
\lambda(\mu_i(\delta^1_i)) \leq y_{i,1}, \quad \forall i \in I,
\]

(7c)

\[
\lambda(\mu_i(\delta^{k+1}_i)) \leq y_{i,k} + y_{i,k-1}, \quad \forall i \in I, \ k \in 2, \ldots, n_i - 2,
\]

(7d)

\[
\lambda(\mu_i(\delta^{n_i-1}_i)) \leq y_{i,n_i-1}, \quad \forall i \in I,
\]

(7e)

\[
\lambda_s \geq 0, \quad \forall \hat{x}_s \in S, \text{ and},
\]

(7f)

\[
y_{i,k} \in \{0, 1\}, \quad \forall i \in I, \ k \in 1, 2, \ldots, n_i - 1.
\]

(7g)

Constraints in Eq. (7) use the binary variables of each variable partition to activate and deactivate the non-negative multiplier variables \( \lambda_s \), \( s \in S \). In particular, Eq. (7b) ensures that only one partition per variable is active. The constraints in Eq. (7c) – (7e) enforce adjacency conditions on the \( \lambda_s \) variables akin to SOS-2 constraints \([6, 5]\). They ensure that the relaxation is the convex combination of the
We extend that approach to build a R-PPR by recursively grouping bilinear terms and applying the vertex-based formulations in the same variable space or using logarithmic number of binary variables variable sums where SOS-2 constraints are imposed.

For a bilinear term with \( \lambda_i \in I \), \( PPR \) for a bilinear term with \( \lambda_i \) counterpart (in Sec. 3).

For a general multilinear term with \( I \geq 3 \) and no partitions i.e., \( |P| = 1 \), the recursive McCormick relaxation successively uses the McCormick relaxation in Eq. (4) on bilinear combinations of the terms. We extend that approach to build a R-PPR by recursively grouping bilinear terms and applying the PPR, given in Eq. (6), to each bilinear term. For \( |I| \geq 3 \), the R-PPR is succinctly expressed as

### 3. Recursive Piecewise Polyhedral Relaxation for a Multilinear Term with \(|I| \geq 3\)

For a general multilinear term with \(|I| \geq 3\) and no partitions i.e., \(|P| = 1\), the recursive McCormick relaxation successively uses the McCormick relaxation in Eq. (4) on bilinear combinations of the terms. We extend that approach to build a R-PPR by recursively grouping bilinear terms and applying the PPR, given in Eq. (6), to each bilinear term. For \(|I| \geq 3\), the R-PPR is succinctly expressed as
Notice that a different R-PPR can be created by choosing a different order of recursive grouping of variables, leading to different quality of relaxations. However, for simplicity purposes, we choose the grouping as shown in (9). Also, while applying R-PPR, only the original space of variables in the multilinear term are chosen to be partitioned, and not the lifted variables resulting from recursive relaxations.

4. MIP-based Piecewise formulation for Recovering Feasible Solutions

Thus far, the focus of the paper has been to develop piecewise polyhedral relaxations for a multilinear term. The optimal solution to this relaxation provides an active partition for each variable. Since the term of interest is a multilinear term, we know that restricting the solution to lie on the one-dimensional faces of the convex hull relaxation represented by the active partition for each variable is also feasible to the original multilinear term. Hence, in this section, we present an alternate MIP that can recover a feasible solution to the original multilinear term by asking for a solution that lies on the edges of the convex hull relaxation of the multilinear term defined by the active partition.

To that end, let $\mathcal{P}^*$ and $\mathcal{S}^*$ denote the optimal partition and the extreme points of the active partition chosen by the the PPR of the multilinear term, respectively and $[\ell^*, u^*]$ denote the variable bound vector defined by the active partition $\mathcal{P}^*$. Applying the convex hull formulation given in Eq. (5) for $x \in [\ell^*, u^*]$ will yield the same solution produced by the PPR. Additionally, enforcing the solution to lie on the one-dimensional faces of the polytope in Eq. (5) for variable bounds $[\ell^*, u^*]$ yields a feasible solution to the multilinear term. To enforce these additional constraints, we introduce some additional notation. Let $V$ and $E$ denote the vertices and the edges (one-dimensional faces) of the polytope defined by Eq. (5) for $x \in [\ell^*, u^*]$. We know that $|V| = 2^{|I|}$ and $|E| = |I| \cdot 2^{|I|-1}$. Then for every $v \in V$, we define a set function $\gamma(v) = \{e \in E : e$ is incident on $v\}$. Additionally, we introduce a binary variable $z_e$ for $e \in E$ that takes a value 1 if the solution lies on the edge $e$. Given these notations, the piecewise formulation that enforces the solution to lie on one of the edges of the polytope defined by Eq. (5) for $x \in [\ell^*, u^*]$ is given by

$$\left\{ (x, w, \lambda, z) \in [\ell^*, u^*] \times \mathbb{R} \times \Delta_{|V|} \times \{0,1\}^{|E|} \mid (x, w, \lambda, z) \text{ satisfies Eq. (11)} \right\}$$

where, Eq. (11) is given by

$$x = \sum_{s} \lambda_s \hat{x}_s, \quad w = \sum_{s} \lambda_s \phi(\hat{x}_s), \quad \sum_{e \in E} z_e = 1, \quad \lambda_v \leq \sum_{e \in \gamma(v)} z_e \quad \forall v \in V,$n

$$\lambda_s \geq 0, \quad \forall \hat{x}_s \in \mathcal{S}^*, \quad z_e \in \{0,1\} \quad \forall e \in E$$

In the next section, we present extensive computational results that compare PPR with R-PPR, and the quality of recovered feasible solutions.
5. Computational Results

In this section, we present three sets of computational results. The first set of results uses the NLP instance shown in Eq. (12). These results test the computational strength of the two formulations presented in this letter. The second set of results uses 60 nonlinear optimization problems that contain only multilinear terms [4]. These instances are used to show the effectiveness of the proposed relaxation on instances from a standard test-suite in the literature. For all 60 instances, we compare the relaxed solution with the known global solution. The third set of results empirically compares the strength of the PPR with three R-PPRs applied to a single quadrilinear term. In all the computational experiments, we partition the variables of the multilinear terms with uniformly located discretization points. All models were implemented in Julia using JuMP and the experimental results used Gurobi v8.0 [1] with Gurobi’s presolver and heuristics turned off. All nonconvex models were solved using BARON v19.7.13 with CPLEX 12.8 and Ipopt 3.12.8 as BARON’s subsolvers. The computational experiments were performed on an Intel Xeon CPU L5420 @2.50GHz with 64 GB RAM.

The first set of results are on the nonlinear, nonconvex instance described in (12) below:

\[
F = \max \quad (x_1 x_2 x_3 x_4 + x_3 x_4 x_5 x_6 + x_5 x_6 x_7 x_8)
\]

subject to:

\[
100 x_1 - x_2 - x_3 + 833 x_4 + 95 x_5 + x_6 - x_7 + 100 x_8 \leq 50000,
\]

\[
100 \leq x_1 \leq 500, \quad 1000 \leq x_2, x_3 \leq 2000, \quad 10 \leq x_i \leq 100 \quad \forall i \in \{4, \ldots, 8\}.
\]

Table 1 compares PPR with three R-PPRs, each with a different variable grouping\(^4\), on the NLP with the maximization objective in Eq. (12). In this table, UB and LB gaps are given by \(\frac{UB - OPT}{UB} \cdot 100\) and \(\frac{OPT - LB}{LB} \cdot 100\), respectively. Here, OPT corresponds to the global optimum value of (12) which is equal to 3.2642E10. UB is the upper bound obtained by applying PPR (R-PPR) formulation (7) and LB is the lower bound obtained by applying formulation (11) based on the PPR (R-PPR) solutions. It is clear from these results that, in both the upper-bounding and lower-bounding piecewise formulations, PPR outperforms all the R-PPRs in the relaxation strength. For example, with only 4 partitions, PPR finds a relaxed solution (UB) that is better than any R-PPR (up to 12 partitions). It is also clear that the quality of the feasible solutions (LB) recovered based on the active partition of the PPR formulation outperforms R-PPR’s solutions for every number of chosen partitions. Note that, the LB gaps do not necessarily decrease monotonically with the increase in the number of partitions as these gaps correspond to the feasible solutions lying on the edges of the active partition’s convex-hull polytope. In these experiments, the run-times of both PPR and R-PPR formulations were less than a minute and hence we do not report them explicitly. Of course, as expected, as the number of partitions increases the computational performance of PPR degrades, however the strength of PPR’s relaxation remains superior.

\(^4\)The choice of grouping changes the relaxation.
Table 1: Relative optimality gaps of upper and lower bounds in percent for the PPR and the R-PPRs for varying number of partitions per variable. The $\langle \cdot \rangle$ represents the grouping of the different variables into bilinear terms in the R-PPRs.

| # of partitions per variable | 2    | 4    | 6    | 8    | 10   | 12   |
|------------------------------|------|------|------|------|------|------|
| % gaps                       | UB, LB | UB, LB | UB, LB | UB, LB | UB, LB | UB, LB |
| $x_a x_b x_c x_d$           | 23.99, 2.33 | 2.20, 0.15 | 2.98, 1.11 | 0.83, 0.15 | 0.69, 0.00 | 0.43, 0.05 |
| $\langle x_a (x_b x_c) \rangle x_d$ | 65.47, 2.33 | 25.37, 36.74 | 21.73, 2.33 | 14.72, 0.15 | 12.98, 0.00 | 10.34, 2.33 |
| $\langle (x_a x_b) x_c \rangle x_d$ | 65.47, 2.33 | 25.37, 36.74 | 21.73, 2.33 | 14.72, 0.15 | 12.98, 0.00 | 10.34, 2.33 |
| $x_a \langle x_b (x_c x_d) \rangle$ | 47.37, 2.33 | 25.27, 5.73 | 16.78, 1.11 | 12.29, 0.15 | 9.59, 0.00 | 8.19, 1.11 |

BARON’s performance on NLP (12): In order to compare the relative optimality gaps obtained by the PPR formulation in Table 1, we also solved instance (12) using BARON solver. The best optimality gap BARON reports after a run time of 3,600 seconds is 20.98% with 233,324 number of open spatial branch-and-bound nodes, while the PPR reports 3.34% with only four partitions per variable (see table 1). The best gap PPR reports is 0.47% (12 partitions) which is close to global optimum, thus reinforcing the value of the convex hull formulations proposed in this paper.

Our second set of results are based on a collection of multilinear problems discussed in [4]. The bounds on all the variables for every problem instance in [4] are $r_0, 1$ s. To demonstrate the strength of the relaxations discussed in this paper, we create a modified set of instances. For each variable, we uniformly distributed the lower and upper bounds in the interval $[0.1, 0.2]$ and $[0.9, 1]$, respectively. This modification is motivated by the following known result [15], the recursive McCormick relaxation for a multilinear term is known to capture the convex hull of the multilinear term if all the variable bounds are either symmetric about the origin or $[0, 1]$. Though the bounds were modified, the global optimum values for all the instances remained the same as those in [4] when solved using BARON [26]. For each problem instance we relaxed the problems using PPR and a R-PPR with a lexicographic ordering. A time limit of one hour was imposed on every run of the instance. The results of all the runs are summarized using the box-plot shown in Fig. 2.

In Fig. 2, the optimality gap for the PPR in most instances is 0% when the number of partitions per variable is three. This is not the case for R-PPR. However, as expected, Fig. 2(a) suggests that when the bound on every variable in the problem is [0, 1], R-PPR is generally the better alternative. This is not surprising since similar recursive relaxations are known to capture the convex hull of a multilinear term with symmetric variable bounds [15]. Once the variable bounds are randomized and made asymmetric, the PPR formulation outperforms the recursive relaxation (R-PPR) in both optimality gap and computation time (Fig. 2(b)). Though the (bottom right) figure suggests that the computation times of PPR are larger than that of R-PPR, it is worth to note that the number of partitions and the run times necessary for the PPR to achieve 0% optimality gap on all the 60 instances is much superior compared to the R-PPR. This demonstrates the effectiveness of the PPR derived in this letter.

The final set of results empirically compares the strength of the PPR with three R-PPRs applied to a product of 4 variables i.e., $\phi = x_1 x_2 x_3 x_4$ in the domain $[0.1, 0.9]^4$. This experiment randomly generated 5000 points uniformly in the domain $[0.1, 0.9]^4$ and for each point $(x_1^k, x_2^k, x_3^k, x_4^k)$, calculates the difference between the upper and lower bound of $\phi$ as defined by the PPR and the R-PPRs with
(a) Gaps and run times for the instances with variable bounds set to $[0, 1]$. The optimality gaps for the PPRs increase from 3 partitions to 4 partitions because of the computation time limit of 1 hour imposed on every run.

(b) Gaps and run times for the instances with variable bounds randomly chosen. In general, the R-PPR has better computation times because of the lower number of multiplier variables ($\lambda$). Nevertheless, the PPR closes the optimality gap with lower number of partitions despite taking a little more computation time.

Figure 2: The optimality gap in % refers to the relative gap between the objective value of the relaxation of the problem and the global optimum to the problem instance. The computation time is the time taken to solve the PPR and the R-PPR to optimality.
5 partitions per variable; this difference is referred to as “gap” and intuitively measures the range of the relaxation at a particular point. Scatter plots that show the gaps for the different relaxations are shown in Fig. 3. The Fig. 3 also shows the line of unit slope. All the points in all the scatter plots lie above this line indicating that the PPR is always strictly stronger than the R-PPRs when the variable bounds are not symmetric about the origin or not of the form \([0, U_i]\).

![Scatter plots of the relaxation gap produced by the PPR and R-PPRs with 5 partitions per variable.](image.png)

Figure 3: Scatter plots of the relaxation gap produced by the PPR and R-PPRs with 5 partitions per variable.

6. Concluding Remarks

This article presents the first piecewise polyhedral relaxation for a multilinear term with spatial disjunctions on every variable. An SOS-2-based formulation that generalizes similar formulations for piecewise linear approximations of functions is developed on the convex hull of a single multilinear term and extensive computational experiments demonstrate that our proposed formulations have significant computational advantages. A MIP-based piecewise formulation is developed to recover high-quality feasible solutions for problems with multilinear terms. Considering the limitation of the exponential growth of the number of variables in the proposed PPR formulation, future work will focus on developing formulations with subsets of variables to keep the size of the PPR tractable and further add valid inequalities, by generalizing the ideas proposed in \([4]\).
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