Simultaneously determining the frequency sum and time difference of two photons through sum frequency generation

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We propose, analyze and evaluate a technique for the joint measurement of time-frequency entanglement between two photons. In particular, we show that the frequency sum and time difference of two photons could be simultaneously measured through the sum-frequency generation process, without measuring the time or frequency of each individual photon. We demonstrate the usefulness of this technique by using it to design a time-frequency entanglement based continuous variable superdense coding and a quantum illumination protocol. Performance analysis of these two protocols suggests that the joint measurement of strong time-frequency entanglement of non-classical photon pairs can significantly enhance the performance of joint-measurement based quantum communication and metrology protocols.

Time and frequency correlation have been formidable resources in a rich range of applications from metrology and spectroscopy to communication and security\(^{[1–3]}\). In particular, time-frequency entanglement (TFE), that there is substantial and simultaneous correlations between two photons in both time and frequency, enables a plethora of advantages beyond what is achievable by classical correlations in the domains of metrology\(^{[4]}\) and communication\(^{[5]}\) applications. These advantages are brought about due to the continuous variable nature and loss resilient property\(^{[6]}\) of TFE. Recent years have witnessed rapid advances in the generation of TFE with attractive properties\(^{[7–9]}\). Thus far, TFE has already been used for entanglement based protocols such as quantum key distribution\(^{[5, 10]}\) and continuous variable Bell test\(^{[11]}\). Such type of protocols only features separate measurements of the time or frequency degree of freedom of each individual photon. In contrast, many other entanglement based applications require a joint measurement of the two entangled photons, that is, measuring a joint variable of the entangled system without resolving the property of each individual photon. One such example is continuous variable superdense coding\(^{[12]}\), in which the sum of the in-phase amplitude \(x_1, x_2\) and the difference of the out-of-phase amplitude \(p_1 - p_2\) of two beams are simultaneous measured. In principle, such joint measurement based protocols could also be implemented with TFE photon pairs, since the time and frequency operator of a single photon obey the same commutation relation as \(x + p\)\(^{[13]}\).

Despite of the progress on the TFE front, to the best of our knowledge, joint-measurement based protocol has not yet been been implemented with TFE photon pairs. In particular, it has not been clear how to measure the sum of frequencies \(\omega_1 + \omega_2\) and difference of times \(t_1 - t_2\) of two photons simultaneously. The frequency sum (time difference) of two photons has to be measured without measuring the frequency (time) of each individual photon, to avoid altering the subsequent measurement of time difference (frequency sum). If such measurements could be implemented in a practical fashion, it could have significant impact on many entanglement based applications that would have their performance depend on such joint measurements. Those include quantum teleportation\(^{[14]}\), quantum superdense coding\(^{[15]}\), and quantum illumination\(^{[16]}\).

Compared to the joint measurement, the generation of TFE is much more developed. To date, the most widely adopted approach to generating TFE photon pairs is continuous-wave spontaneous parametric down-conversion (SPDC). The TFE of SPDC photon pairs is closely related to the properties of SPDC sources. In particular, it has been shown that the temporal correlation and frequency anti-correlation of TFE photon pairs could be tailored with great flexibility with different designs of the waveguide structure of the photon pair source\(^{[5]}\). Given the close connection between the SPDC process and TFE, a natural line of inquiry would be to utilize the time-reversal of the SPDC process, namely sum frequency generation (SFG), to help obtain an effective route for a TFE joint-measurement based protocol. In this paper, we show through theoretical analysis that the SFG process could be used as the joint measurement of TFE. In particular, we showed that SFG could be used for the joint measurement of superdense coding and quantum illumination. These two examples show the potential of SFG as a measurement technique for practical quantum communication and sensing protocols.

The SFG process in a \(\chi^{(2)}\) nonlinear medium could be modeled as the following evolution operator:

\[
V = I + \epsilon \left( \int d\omega_s d\omega_l d\omega_p f_0(\omega_p, \omega_s, \omega_l) a_p(\omega_p) a_s(\omega_s) a_l(\omega_l) - H.C. \right)
\]

where \(\epsilon\) quantifies the interaction strength and \(a_p(\omega), a_s(\omega), a_l(\omega)\) correspond to the pump, signal and idler light, respectively. The pump frequency \(\omega_p\) is assumed to be around twice the frequency of the signal.

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and idler light ($\omega_s, \omega_i$). The time reversed process of SFG, the SPDC process, can be described by the same evolution operator $V$, too. Therefore given the fact that the SPDC process can create TFE photon pairs, it is natural to ponder whether the SFG process can be used to resolve the TFE between two photons. To investigate this and demonstrate it’s viability, we first approximate the factor $f_0(\omega_p, \omega_s, \omega_i)$ as the following form:

$$f_0(\omega_p, \omega_s, \omega_i) = \delta(\omega_p - \omega_s - \omega_i) f(\frac{\omega_p - \omega_i}{\sqrt{2}})$$  \hspace{1cm} (2)

$$f(\omega) = \sqrt{\frac{1}{2\pi\sigma_-}} \exp(-\frac{\omega^2}{2\sigma_-^2})$$  \hspace{1cm} (3)

where $\sigma_-$ is the bandwidth of the SPDC photons. This approximation is exact if the nonlinear medium has certain dispersion properties\[17\]. Using this detailed form of the SFG evolution operator $V$, the spectral density operator $a_{p,\text{det}}(\omega) a_{p,\text{det}}(\omega)$ of the pump light at the SFG output could be expressed as $[17]$

$$a_{p,\text{det}}(\omega_p) a_{p,\text{det}}(\omega_p) = V a_{p,\text{det}}(\omega_p) a_{p,\text{det}}(\omega_p) V^\dagger = c^2 B^\dagger B + O(a_p(\omega_p), a_p(\omega_p))$$  \hspace{1cm} (4)

$$B = \int d\omega_s d\omega_i f(\omega_s - \omega_i) \delta(\omega_s + \omega_i - \omega_p) a_s(\omega_s) a_i(\omega_i)$$  \hspace{1cm} (6)

where $O(a_p(\omega_p), a_p(\omega_p))$ is the sum of normal ordered operators that are at least linear in $a_p(\omega_p)$ or $a_p(\omega_p)$. This term could be neglected due to the absence of pump photons at the input of the SFG process. It could be shown that in the limit of infinite SPDC photon bandwidth $\sigma_- \rightarrow \infty$:

$$P(\omega_p, t) \propto B^\dagger B$$  \hspace{1cm} (7)

where $P(\omega_p, t)$ is the probability density operator for the photon pair to have frequency the sum $\omega_p + \omega_p$ equals $\omega_p$ and the time difference $t_s - t_i$ equals $t$, simultaneously (the definition and properties of $P(\omega_p, t)$ can be found in [17]). Therefore [7] shows that the detection of the pump photon reveals the simultaneous time and frequency correlation (hence TFE) $\omega_p = \omega_s + \omega_i$, $t_s - t_i = 0$ in the limit of $\sigma_- \rightarrow \infty$. Intuitively, such a joint measurement of TFE could be understood as a quantum interference effect: only photon pairs that have frequency sum $\omega_p = \omega_s + \omega_i$ can possibly generate a pump photon at frequency $\omega_p$. Meanwhile in the time domain, a non-zero time difference between the signal and idler photon will induce different phase shift for different frequency components of the photon pair state, and the corresponding probability amplitude of SFG will interfere destructively, leading to a decreased probability of generating a pump photon.

Having shown that the SFG process could be utilized for the joint measurement of TFE, the question remains now is that whether it could benefit any quantum communication and sensing application that needs joint measurement. An important example is continuous variable superdense coding\[12\], which utilizes entanglement between two particles to surpass the classical limit of channel information capacity. The previous proposal \[12\] and implementation\[18\] of continuous variable superdense coding are based on the joint-measurement of quadrature-phase entanglement, compared to which the joint measurement of TFE is more difficult to implement. However, the utilization of TFE can also provide additional advantages compared to the quadrature-phase entanglement. First, unlike the quadrature-phase entanglement, the strength of TFE (the Schmidt number of the photon pair) is not limited by the source power, which may translate to a larger information capacity enhancement compared to the two-fold enhancement achievable by quadrature-phase entanglement\[12\]. Second, the TFE has been demonstrated to be resilient to channel losses\[6\], which is favourable for practical long haul superdense coding applications.

In what follows, we propose a proof-of-principle protocol of TFE based continuous variable superdense coding (TFE SDC). In particular, we will show that one can encode (decode) arbitrarily large amount of information into (out of) both the time and frequency degree of freedom of the signal photon (that is entangled with the idler photon), simultaneously. The basic steps of the TFE SDC protocol are as follows. First Alice generates an entangled photon pair state $|\text{pair}\rangle$ as the entanglement source:

$$|\text{pair}\rangle = \int_{-\infty}^{\infty} d\omega_s d\omega_i \phi_0(\omega_s, \omega_i) a_s(\omega_s) a_i(\omega_i) |0\rangle$$  \hspace{1cm} (8)

$$\phi_0(\omega_s, \omega_i) = h(\frac{\omega_s + \omega_i}{\sqrt{2}}) f(\frac{\omega_s - \omega_i}{\sqrt{2}})$$  \hspace{1cm} (9)

Such a photon pair could be generated by pumping the $\chi^{(2)}$ medium with a strong coherent beam of light in mode $a_p$ that has (square normalized) complex spectral amplitude $\frac{1}{\sqrt{2}} h(\frac{\omega_p}{\sqrt{2}})$. For simplicity, we assume:

$$\frac{1}{\sqrt{2}} h(\frac{\omega_p}{\sqrt{2}}) = \sqrt{\frac{1}{2\pi\sigma_+}} \exp(-\frac{\omega^2}{2\sigma_+^2})$$  \hspace{1cm} (10)

where the SPDC pump bandwidth is proportional to $\sigma_+$. Alice then stores the idler photon locally and sends the signal photon to Bob. Bob will encode information by shifting both the frequency and time of the signal photon by $\Delta \omega$ and $\Delta t$, which could be done with nonlinear frequency conversion\[19\] and a tunable delay line. The information encoded signal photon is sent back to Alice. Then the information encoded photon pair state for Alice to measure is given by:

$$|\text{coded}\rangle = \int d\omega_s d\omega_i \phi_{\text{coded}}(\omega_s, \omega_i) a_s(\omega_s) a_i(\omega_i) |0\rangle$$  \hspace{1cm} (11)

$$\phi_{\text{coded}} = \phi_0(\omega_s - \Delta \omega, \omega_i) \exp(i \omega_s \Delta t)$$  \hspace{1cm} (12)
Alice will perform SFG with the encoded photon pair. The generated pump photon is sent to a single photon spectrometer. The number operator of the generated pump photon is defined as \( N_{\text{SFG}} = \int d\omega \hat{a}_p^\dagger(\omega_p) \hat{a}_p(\omega_p) \). Then the photon generation probability \( \langle N_{\text{SFG}} \rangle \) is given by [17]:

\[
\langle N_{\text{SFG}} \rangle = \langle \text{coded} \rangle V^\dagger N_{\text{SFG}} V |\text{coded}\rangle
= \frac{e^2}{\sqrt{2}} \exp\left( -\frac{\omega_0^2}{2 \sigma^2_{\text{SFG}}} - \frac{\Delta t^2}{2} \right)
\] (13)

For each generated pump photon, the frequency operator is defined as \( \hat{\omega}_{\text{SFG}} = \int d\omega \hat{a}_p^\dagger(\omega_p) \omega_p \hat{a}_p(\omega_p) \), whose mean value \( \omega_{\text{SFG}} \) and variance \( \Delta^2 \omega_{\text{SFG}} \) are [17]:

\[
\omega_{\text{SFG}} = \omega_0 + \Delta \omega \quad \Delta^2 \omega_{\text{SFG}} = \sigma^2_{\text{SFG}}
\] (14)

As could be seen in (14), the probability \( \langle N_{\text{SFG}} \rangle \) of generating a pump photon decreases rapidly as the signal photon time shift \( \Delta t \) exceeds the inverse SPDC photon bandwidth \( 1/\sigma_\omega \). In contrast, the frequency shift \( \Delta \omega \) does not affect much \( \langle N_{\text{SFG}} \rangle \) as long as \( \Delta \omega \ll \sigma_\omega \). The equation (15) shows that the center frequency and bandwidth of the generated pump photon is identical to that of the SPDC pump light, aside from the frequency shift \( \Delta \omega \).

Based on (14) and (15), the joint measurement scheme of the TFE SDC protocol could be designed as follows. After receiving the signal photon from Bob, Alice first apply an additional time shift \( \Delta t_{\text{extra}} \) to the signal photon and then let the encoded photon pair go through the SFG process. A pump photon will be generated through SFG with non-negligible probability only if the total time shift \( \Delta t + \Delta t_{\text{extra}} \) is close to zero \( (\leq 1/ \sigma_\omega) \). The encoded photon pair will remain unchanged after SFG if no pump photon is generated. In such cases, the encoded photon pairs could be reused and go through the SFG process repeatedly until a pump photon is finally generated (see Fig. 1 for the schematic of the experimental setup). Over this SFG loop, the extra time delay \( \Delta t_{\text{extra}} \) is swept continuously and repeatedly. When this SFG loop terminates (a pump photon is generated) with the extra time delay set to \( \Delta t_{\text{extra}} \), the posterior probability distribution of the time shift \( \Delta t \) is centered around \( -\Delta t_{\text{extra}} \) with variance \( \text{var}\{\Delta t\} = 1/\sigma^2_\omega \). The frequency shift \( \Delta \omega \) is obtained by measuring the frequency of the generated pump photon, with variance \( \text{var}\{\Delta \omega\} = \sigma^2_\omega \). The measurement of \( \Delta t \) and \( \Delta \omega \) could be arbitrarily precise simultaneously:

\[
\text{var}\{\Delta t\} \text{var}\{\Delta \omega\} = \frac{\sigma^2_\omega}{\sigma^2_{\text{SFG}}} \ll 1 \quad \text{which implies that arbitrarily large amount of information could be coded in } \Delta t \text{ and } \Delta \omega \text{ simultaneously}. \] (16)

measuring the time \( \hat{t} \) and frequency \( \hat{\omega} \) of the photon. However, the measurement uncertainty product is limited by the Heisenberg uncertainty principle [13]:

\[
\text{var}\{\Delta t\} \text{var}\{\Delta \omega\} = \frac{\text{var}\{\hat{t}\} \text{var}\{\hat{\omega}\}}{\text{var}\{\Delta t\} \text{var}\{\Delta \omega\}} \geq \left| \langle [\hat{t}, \hat{\omega}] \rangle \right|^2 = 1
\] (17)

As could be seen in (14) and (15), the performance of the TFE SDC protocol depends on the \( \chi^2 \) medium that is used for the SPDC photon pair generation and the SFG measurement: the SPDC photon bandwidth \( \sigma_\omega \) dictates the maximal frequency shift \( \Delta \omega \) that can be encoded such that \( \langle N_{\text{SFG}} \rangle \) is constantly \( e^2/\sqrt{2} \) as well as the read out variance of the time shift \( \Delta t \). The nonlinear conversion efficiency \( e^2 \) determines the number of the SFG loops that is needed for the joint measurement, hence the speed of the communication. This outcome demonstrates how TFE could be used for superdense coding, where it offers advantages over existing superdense proposals and demonstrations in that it pivots in its performance on the TFE of the photon pairs, which is not limited by the power of the source, and is resilient to losses.

![FIG. 1: Top: the setup of the SFG measurement for the TFE SDC protocol. SFG: a \( \chi^2 \) medium where the SFG process takes place; DM: a dichroic mirror to separate the generated pump photon from the photon pair; TTD: tunable time delay \( \Delta t_{\text{extra}} \); SM: switch mirror. After the signal and idler pass by the switch mirror, the SM mirror will flip and form a ring cavity. Bottom: the schematic of the TFE QI experimental setup. SP: short pass filter; MRPD: mode resolved single photon detector.](image-url)
dense coding in that they are both based on the same fashion of dimensional-enhancement that is enabled by quantum entanglement. That is to say that both rely on: manipulating one particle of the entangled two-particle system could transform the joint system from its initial state to one of $D$ orthogonal states, where $D$ is greater than the dimension of the Hilbert space of the manipulated particle. In the superdense coding protocol, the dimensional-enhancement allows one to encode more information than the classical limit by manipulating one of the entangled particles. In the quantum illumination protocol, the dimensional-enhancement increases the distinguishability of the entangled state from the background noise states that are evenly distributed in the joint Hilbert space. The basic steps of the TFE based quantum illumination (TFE QI) protocol are as follows (see Fig. 1): The signal photon of the TFE photon pair is sent to probe the target while the idler photon is stored locally. The signal photon is reflected and collected with total transmission $\eta$ ($\eta = 0$ if the target is absent). Regardless of the presence or absence of the target object, a constant level of environmental noise light is always collected into the detection system. The joint measurement of the collected (signal and noise) photon and the idler photon consists of the SFG process and mode-resolved photon detection of the generated pump photon. The photon detection result is then used to infer the presence or absence of the target object. In order to analyze the TFE QI protocol, a suitable mathematical formalism is needed. The formalism utilized here is chosen as it better highlights the connection between TFE and SFG. In addition, an analysis of the TFE QI protocol that parallels the analysis of the TFE SDC protocol could be found in [17]. We shall start directly from the general form of the evolution operator $V$ [1] without applying any approximation or assumption. In general, the $\chi^{(2)}$ evolution operator $V$ could be expressed as a discrete sum through a two-step Schmidt decomposition process [17]:

$$V = I + \epsilon \sum_m (\lambda_m^{(1)} A_m B_m^\dagger - H.C.)$$

(18)

$$B_m = \sqrt{\lambda_m^{(2)} F_{m,n} G_{m,n}}$$

(19)

The equation (18) is obtained through the Schmidt decomposition between the pump and the “signal-idler” joint system with the singular values given by $\{\lambda_m^{(1)}\}$. The equation (19) is obtained through the Schmidt decomposition of each “signal-idler” joint system with the singular values given by $\{\lambda_m^{(2)}\}$. The operators $\{F_{m,n}\}$, $\{G_{m,n}\}$ (with fixed $m$ and different $n$) and $\{A_m\}$ form complete orthogonal sets of annihilation operators for the pump, signal and idler mode, respectively. The definitions of the mode operators could be found in [17]. The photon pair source of the TFE QI protocol is chosen to be $|\text{pair}\rangle = B_0^\dagger |0\rangle$, which could be approximated by SPDC twin beams generated by coherent pump light in the mode $A_0$ (neglecting the vacuum term and multiple pair terms). The noise and loss of the signal photon in the target detection channel is modeled as mixing with the background noise mode on a virtual beam-splitter with transmission $\eta$ for the signal photons. The evolution operator $U_{\text{loss}}$ of the beam-splitter can be expressed as:

$$U_{\text{loss}} = \prod_n \exp\{i \arccos(\eta)(F_{0,n}^\dagger F_{0,n} + H.C.)\}$$

(20)

where $F_{0,n}^\dagger$ is the discrete mode operator for the noise photon that has the same spectral amplitude as $F_{0,n}$. Equivalence between $U_{\text{loss}}$ and the usual beam-splitter transform is shown in [17]. To avoid technical complexities, we assume that the background noise mode is occupied by a noise state $\rho_b$ that satisfies the following conditions:

$$\text{tr}\{F_{0,n}^{(b)} F_{0,n}^{(b)}\dagger \rho_b\} = \delta_{n,n'} \mu_b \quad \text{tr}\{F_{0,n}^{(b)} \rho_b\} = 0$$

(21)

where $\mu_b$ is the average number of noise photons per mode. The above conditions mean that the noise photons are evenly distributed in every signal mode $F_{0,n}$ with random phases and there is no coherence between each mode. It could be shown that such noise is broadband and continuous-wave white noise [17]. The density operator $\rho_{\text{SFG}}$ of the SFG output can be expressed as:

$$\rho_{\text{SFG}} = V U_{\text{loss}} |\text{pair}\rangle \langle \text{pair}| \otimes \rho_b U_{\text{loss}}^\dagger V^\dagger$$

(22)

In the limit of perfect signal photon transmission ($\eta = 1$), the SFG process can only generate pump photons in mode $A_0$ [17]. Therefore in the following analysis, only the photon detection event in mode $A_0$ is taken into consideration. After some algebraic manipulation [17], it can be shown that the photon detection probability $P_{d,QI}$ on mode $A_0$ is given by:

$$P_{d,QI} = \text{tr}\{A_0^\dagger A_0 \rho_{\text{SFG}}\} = \epsilon^2 \lambda_0^{(1)} (\eta + \mu_b/\text{SN})$$

(23)

where $\text{SN} = 1/\sum_n (\lambda_n^{(2)})^2$ is the Schmidt number of the SPDC photon pair. The Schmidt number being larger than unity is a indication of TFE. It can be shown that the conversion efficiency $\lambda_0^{(1)} \epsilon^2$ is also the SPDC conversion efficiency [17]. For comparison, we shall also consider a classical target detection (CI) protocol where a probe photon in an arbitrary temporal-spectral mode $F$ is sent to probe the target [10]. It is easy to see that the photon detection probability is $P_{d,CI} = \eta + \mu_b$. Comparison between $P_{d,CI}$ and $P_{d,QI}$ shows that the TFE QI protocol is equivalent to a CI protocol with detection efficiency $\epsilon^2 \lambda_0^{(1)}$ and noise photon per mode reduced to $\mu_b/\text{SN}$. As could be seen in [23], the performance of the TFE QI protocol is limited by the nonlinear efficiency $\lambda_0^{(2)} \epsilon^2$ and the Schmidt number SN of the entangled photon pair source. If SPDC twin beams are used as the TFE QI source, the Schmidt number SN could be approximated as the ratio of the SPDC photon bandwidth $\sigma_-$ and the SPDC pump
bandwidth $\sigma_+$. Therefore ideally the $\chi^{(2)}$ medium should have very large phase-matching bandwidth. For bulk $\chi^{(2)}$ crystal there is a trade-off between the SPDC photon bandwidth and the length of the crystal (hence the nonlinear conversion efficiency). Therefore it may not be optimal for the TFE QI protocol. Integrated semiconductor $\chi^{(2)}$ waveguide could be an ideal alternative because it offers high nonlinear conversion efficiency in a compact form factor ($\lambda_0 (1)^2 \simeq 2.1 \times 10^{-8}$ for a 1nm long waveguide). Moreover, semiconductor waveguide can provide very large SPDC photon bandwidth with specific structure designs.

It is important to highlight that the noise reduction being directly proportional to the Schmidt number $SN$ is very similar to that in the first quantum illumination protocol reported in [10]. Therefore the TFE QI protocol proposed here could be considered as an implementation of [10]. However, it could be shown that the TFE QI protocol have large performance enhancement over the coherent light /homodyne detection scheme under high noise condition [17]. This extends the result in [21] that shows that the first quantum illumination protocol [10] cannot outperform coherent detection in the low noise limit $\mu_0 < 1$. In addition, as could be seen in [22], the entanglement enhancement of the TFE QI protocol can effectively reduce the environmental noise power down to zero in the limit of large entanglement $SN \gg 1$. Such a result does not contradict the previous finding [22] that at most 6dB of performance advantage could be achieved by Gaussian state quantum illumination. The is because for TFE QI the photon-pair source is assumed to be non-gaussian $|\text{pair}\rangle = B_0^0 |0\rangle$. However, in practice, the SPDC twin beams are commonly used as an approximation of $|\text{pair}\rangle$ by neglecting the vacuum term and the multiple pair terms. Then, it must be remembered that SPDC twin-beams are in Gaussian state and a TFE QI protocol with SPDC twin-beam source can achieve 6 dB enhancement of target detection performance at most. Lastly, the proposed TFE QI protocol is similar to the SFG quantum illumination protocol [23] but with a simpler setup. However, the discussion of the TFE QI protocol here provides a different perspective of the performance advantage achievable by SFG detection from a TFE stand point.

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