Curse of Dimensionality in Unconstrained Private Convex ERM

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Abstract

We consider the lower bounds of differentially private empirical risk minimization for general convex functions in this paper. For convex generalized linear models (GLMs), the well-known tight bound of DP-ERM in the constrained case is \( \tilde{\Theta}(\sqrt{\frac{p}{\epsilon n}}) \), while recently, Song et al. [2021] find the tight bound of DP-ERM in the unconstrained case is \( \tilde{\Theta}(\sqrt{\frac{\text{rank}}{\epsilon n}}) \) where \( p \) is the dimension, \( n \) is the sample size and rank is the rank of the feature matrix of the GLM objective function. As rank \( \leq \min\{n, p\} \), a natural and important question arises that whether we can evade the curse of dimensionality for over-parameterized models where \( n \ll p \), for more general convex functions beyond GLM. We answer this question negatively by giving the first and tight lower bound of unconstrained private ERM for the general convex function, matching the current upper bound \( \tilde{O}(\sqrt{\frac{p}{n \epsilon}}) \) for unconstrained private ERM. We also give an \( \Omega(\frac{p}{n \epsilon}) \) lower bound for unconstrained pure-DP ERM which recovers the result in the constrained case.

1 Introduction

Since the seminal work of Dwork et al. [2006], differential privacy (DP) has become the standard and rigorous notion of privacy guarantee for machine learning algorithms, among which many fundamental ones are based on empirical risk minimization (ERM). Motivated by this, private ERM becomes one of the most well-studied problems in the DP literature, e.g. Chaudhuri and Monteleoni [2008], Rubinstein et al. [2009], Chaudhuri et al. [2011], Kifer et al. [2012], Song et al. [2013], Jain and Thakurta [2014], Bassily et al. [2014], Talwar et al. [2015], Kasiviswanathan and Jin [2016], Fukuchi et al. [2017], Wu et al. [2017], Zhang et al. [2017], Wang et al. [2017], Iyengar et al. [2019], Bassily et al. [2020], Kulkarni et al. [2021], Asi et al. [2021], Bassily et al. [2021], Wang et al. [2021].

Roughly speaking, in the ERM setting, we are given a convex function family defined on \( \mathcal{C} \subseteq \mathbb{R}^p \) and a sample set \( \mathcal{D} = \{d_1, \cdots, d_n\} \) drawn i.i.d from some unknown distribution \( P \) with the objective to minimize the loss function

\[
L(\theta; \mathcal{D}) = \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; d_i).
\]

The value \( L(\theta; \mathcal{D}) - \min_{\theta \in \mathcal{C}} L(\theta; \mathcal{D}) \) is called the excess empirical loss with respect to solution \( \theta \), measuring how it compares with the best solution in \( \mathcal{C} \).

Private ERM in the constrained case is well studied. More specifically, the constrained case considers convex loss functions defined on a bounded convex set \( \mathcal{C} \subseteq \mathbb{R}^p \). Assuming the functions are 1-Lipschitz over the convex set of diameter 1, the lower bound of private ERM, \( \Omega(\frac{p}{n \epsilon}) \) is given by

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The other direction is to relax the constraints on the loss function. The (arguably) simpler unconstrained GLM is
\[ \Theta\left(\sqrt{\frac{\text{rank}}{\varepsilon n}}\right) \]
and any private algorithm cannot get meaningful bounds on the excess empirical loss. This motivates people to find solutions to remedy this curse.

Until the lower bounds in Bassily et al. (2014), potential methods to evade the curse of dimensionality were two-folds; one is to consider loss functions with simpler forms. GLM has already been extensively studied due to its simplicity and importance, while surprisingly the lower bound based on the simple GLM in Bassily et al. (2014) negates this way as it does not make sense consider simpler functions.

The other direction is to relax the constraints on the loss function. The (arguably) simpler unconstrained case when \( C = \mathbb{R}^d \) usually allows optimization algorithms with better guarantees, yet hasn’t drawn much attention from people until recently. To the best of our knowledge, the first result on the unconstrained case was given by Jain and Thakurta (2014), providing an \( O\left(\frac{n}{\varepsilon^d}\right) \) excess empirical risk for convex GLM, which was improved to \( O\left(\frac{\text{rank}}{\varepsilon n}\right) \) by Song et al. (2021), where rank denotes the rank of the feature matrix of GLM. Song et al. (2021) also constructed a lower bound of \( \Omega\left(\frac{\sqrt{\text{rank}}}{\varepsilon n}\right) \) for unconstrained private GLM.

Recall that the tight bound of private constrained GLM is \( \Theta\left(\sqrt{\frac{d}{\varepsilon n}}\right) \) while the tight bound of private unconstrained GLM is \( \Theta\left(\frac{\text{rank}}{\varepsilon n}\right) \). As we know \( \text{rank} \leq \min\{p, n\} \), though Song et al. (2021) successfully evade the curse of dimensionality by combining GLM with the unconstrained setting, there is still an indispensable gap between constrained and unconstrained cases for over-parameterized models in which \( p \gg n \). A natural and interesting question arises that whether the curse of dimensionality can be evaded and dimension-independent bounds can be derived for general convex functions.

This question was answered partly by Kairouz et al. (2020) and Zhou et al. (2020). They show that with some public dataset to identify the subspace of the gradients of the objective function, the excess loss can depend on the rank of the subspace rather than the ambient dimension. However, whether dimension-independent upper bounds for general convex functions can be achieved in the unconstrained case remains unknown.

In this paper, we answer this question thoroughly by showing that the explicit polynomial dependence on dimension is necessary for general convex losses, even in the (arguably simpler) unconstrained setting, thus closing the gap in understanding when the curse of dimensionality can be avoided.

Our constructions are based on the simple \( \ell_1 \) norm function where \( \ell(\theta; d) = \|\theta - d\|_1 \). We give an \( \Omega\left(\frac{n}{\varepsilon^d}\right) \) lower bound which implies it is impossible to get dimension-independent bound for general convex functions and the current upper bound \( O\left(\sqrt{\frac{n}{\varepsilon^d}}\right) \) in the unconstrained case (such as Theorem 5.1 in Kairouz et al. (2020)) is already tight. moreover, we prove an \( \Omega\left(\frac{n}{\varepsilon^d}\right) \) lower bound for unconstrained pure-DP ERM which recovers the result in the constrained case (Theorem 5.2 in Bassily et al. (2014)), utilizing an \( \ell_2 \) norm loss function.

1.1 Our contributions

We summarize our main results as follows:

- We prove an \( \Omega\left(\min\{1, \sqrt{\frac{p \log(1/\delta)}{n\varepsilon}}\}\right) \) tight lower bound for the excess risk of unconstrained 1-Lipschitz convex loss functions for approximate differentially private algorithm. This bound improves Bassily et al. (2014) by a \( \log(1/\delta) \) factor and matches the upper bound in Kairouz et al. (2020).
- We also prove an \( \Omega\left(\min\{1, \frac{p}{n\varepsilon}\}\right) \) lower bound for the excess risk of unconstrained 1-Lipschitz convex loss functions for any pure differential privacy algorithm.

Note that our main results for unconstrained case can be extended to constrained case directly, thus our lower bound for approximate private algorithm is \( \sqrt{\log(1/\delta)} \) multiplicative better than the well-known bound in Bassily et al. (2014) with the help of group privacy technique in Steinke and Ullman (2015). The construction of lower bounds is based on the Fingerprinting Codes first studied by Bassily et al. (2014), even for (special and simple) generalized linear model (GLM). Though the optimal excess empirical loss is achieved by Bassily et al. (2014), Wang et al. (2017), Asi et al. (2021), Kulkarni et al. (2021) with fewer and fewer gradient queries, all these algorithms suffer from the curse of dimensionality in the constrained case. For example, when \( p = \Omega(n^2) \), the lower bound is \( \Omega(1) \) and any private algorithm cannot get meaningful bounds on the excess empirical loss. This

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- We also prove an \( \Omega\left(\min\{1, \frac{p}{n\varepsilon}\}\right) \) lower bound for the excess risk of unconstrained 1-Lipschitz convex loss functions for any pure differential privacy algorithm.

Note that our main results for unconstrained case can be extended to constrained case directly, thus our lower bound for approximate private algorithm is \( \sqrt{\log(1/\delta)} \) multiplicative better than the well-known bound in Bassily et al. (2014) with the help of group privacy technique in Steinke and Ullman (2015). The construction of lower bounds is based on the Fingerprinting Codes first studied by Bassily et al. (2014), even for (special and simple) generalized linear model (GLM). Though the optimal excess empirical loss is achieved by Bassily et al. (2014), Wang et al. (2017), Asi et al. (2021), Kulkarni et al. (2021) with fewer and fewer gradient queries, all these algorithms suffer from the curse of dimensionality in the constrained case. For example, when \( p = \Omega(n^2) \), the lower bound is \( \Omega(1) \) and any private algorithm cannot get meaningful bounds on the excess empirical loss. This
and developed by Boneh and Shaw [1998], Tardos [2008], which we will briefly introduce below.

### 1.2 Fingerprinting codes

**Definition 1.1** (Fingerprinting codes). We are given \( n, p \in \mathbb{N}, \xi \in (0, 1] \). A pair of (random) algorithms \((\text{Gen}, \text{Trace})\) is called an \((n,p)\)-fingerprinting code with security \( \xi \in (0, 1]\) if Gen outputs a codebook \( C \in \{0,1\}^{n \times p} \) and for any (possibly randomized) adversary \( A_{FP} \) and any subset \( S \subseteq [n] \), if we set \( c \leftarrow_R A_{FP}(C_S) \), then

- \( \Pr[c \in F(C_S) \setminus \text{Trace}(C,c) = \bot] \leq \xi \)
- \( \Pr[\text{Trace}(C,c) \in [n] \setminus S] \leq \xi \)

where \( F(C_S) = \{c \in \{0,1\}^d \mid \forall j \in [d], \exists i \in S, c_j = c_{ij} \} \), and the probability is taken over the coins of Gen, Trace and \( A_{FP} \).

There is a very good motivation behind the fingerprinting codes. For example, a software distributor adds a fingerprint to each copy of her software to protect the I P . A coalition of malicious users can compare their copies and find the digits that differ which belong to the fingerprint. For other locations they can’t decide and won’t change them, which is called the marking condition. This is the reason that we requires \( c \in F(C_S) \).

The two properties of fingerprinting codes demonstrate that one can identify at least one malicious user among all with high probability. Bun et al. [2018] extends the definition that the codes can tolerate a small fraction of errors in the marking condition. We further modify this definition, requiring the codes to have biased means, see Definition 4.1

### 1.3 Dimensionality dependence

As discussed before, algorithms for the problem of private convex ERM usually suffer from the curse of dimensionality, i.e. the excess risk bound relies on the ambient dimension \( p \), which is unsatisfactory in the common over-parameterized setting \( p \gg n \).

The only exception (without public data) is the setting of unconstrained GLM considered in Song et al. [2021], where an \( O(\sqrt[3]{\text{rank}}) \) upper bound is given for minimizing the excess risk of GLMs in the unconstrained case. As \( \text{rank} \leq \min\{n,p\} \), this bound is dimension-independent. One may wonder which condition contributes to the evasion of curse, the unconstrained setting, the GLM loss function, or both?

It has been known that the unconstrained condition is necessary for dimension independence, as pointed out in Bassily et al. [2014] where they prove an \( \Omega(\sqrt[3]{\text{rank}}) \) lower bound even for minimizing constrained GLMs even for the case when \( \text{rank} \leq n \ll p \).

We try to discuss the necessity of the unconstrained condition to get rank-dependent bound. The unconstrained GLM can be viewed as a rank-dimensional problem, as the noise added in the null space of the feature matrix will not affect the excess empirical loss. However, this does not hold in the constrained case. Take the dimension-independent algorithm in Song et al. [2021] which is based on SGD as an example. The pitfall for the dimension-independent algorithm lies in projection if SGD is modified to projected-SGD for constrained case, that running SGD in the constrained setting requires projection which might “increase rank”.

It’s left unknown whether GLM is necessary. Kairouz et al. [2020], Zhou et al. [2020] consider general functions but with public data, which serve to identify a low-rank subspace similar to Song et al. [2021] in spirit.

To sum up, it appears that an objective function with low-rank structure in the unconstrained setting is crucial for evading the curse of dimensionality, while whether we can derive similar upper bounds for more general loss functions remains an important open question.

In this paper we prove an \( \Omega(\sqrt{\frac{n}{\text{rank}}}) \) lower bound in the unconstrained case for general loss functions, showing that it’s impossible to evade the curse of dimensionality even in the simple unconstrained setting. Our results imply the necessity of both unconstrained setting and the low-rank structure assumptions for achieving dimension-independent upper bounds for general functions, and the only meaningful question left is how to design algorithms without the help of public data, filling the
Table 1: Comparison on lower bounds for private convex ERM. Our lower bounds can be extended to constrained case easily. The lower bound of Song et al. [2021] is weaker than ours in the important $p \gg n$ setting.

| Article           | Constrained? | Loss Function | Pure DP | Approximate DP |
|-------------------|--------------|---------------|---------|----------------|
| Bassily et al. [2014] | constrained  | GLM           | $\Omega(\frac{p}{n})$ | $\Omega(\frac{\sqrt{p}}{n})$ |
| Song et al. [2021] | unconstrained| GLM           | N/A     | $\Omega(\sqrt{\text{rank}})$ |
| Ours              | both         | general       | $\Omega(\frac{p}{n})$ | $\Omega(\sqrt{p \log(1/\delta)})$ |

1.4 Other related work

The existing lower bounds of excess empirical loss, i.e. the constrained case in Bassily et al. [2014] and the unconstrained case in Song et al. [2021], are all using GLM functions. The objective function used in Bassily et al. [2014] is $\ell(\theta; d) = \langle \theta, d \rangle$ which can’t be applied in the unconstrained case, otherwise the loss value would be infinite. Considering this limitation, Song et al. [2021] adopts $\ell(\theta; d) = |\langle \theta, x \rangle - y|$. They transfer the problem of minimizing GLM to estimating one-way marginals, and then get the lower bound by properties in the definition of the Fingerprinting Codes.

As mentioned before, our lower bound are based on $\ell_1$ norms, thus we can not transfer to one-way marginals directly. Merely using the properties in the definition of Fingerprinting Codes is not enough for a good lower bound. Instead, we need to make full use of the concrete structure of the codes.

As for the upper bounds, the private ERM [Wang et al. [2017]] and private Stochastic Convex Optimization (SCO) [Feldman et al. [2020]] for convex and smooth functions are extensively studied, where the objective is to minimize the function $\mathbb{E}_{d \sim p}[\ell(\theta; d)]$ in the SCO and people only need (nearly) linear gradient queries to get optimal excess loss. But for convex functions without any smoothness assumption, the current best algorithms [Kulkarni et al. [2021], Asi et al. [2021]] will need much more queries and it is interesting to see if we can use only linear queries. Most of the previous works are considering problems in $\ell_2$ norm, and there are some recent results [Bassily et al. [2021], Asi et al. [2021]] studying the $\ell_p$ norm.

1.5 Roadmap

In section 2 we introduce background knowledge needed in the rest of the paper. In section 3 we prove an $\Omega(\min(1, \frac{p}{n}))$ lower bound for the excess risk of pure DP algorithms for minimizing any unconstrained 1-Lipschitz convex loss function. In section 4 we prove the main result of this paper, an $\Omega(\min(1, \sqrt{\frac{p \log(1/\delta)}{n \epsilon}}))$ lower bound for approximate DP-ERM in the unconstrained case. Section 5 concludes this paper. All missing (technical) proofs can be found in the appendix.

2 Preliminary

We consider minimizing the excess risk of unconstrained Lipschitz convex function with DP algorithms in this paper, where we let $n$ denote the sample size and $p$ be the dimension of a sample. In this section, we will introduce main background knowledge required in the rest of the paper. Additional background knowledge such as the definition of GLM can be found in appendix.

**Definition 2.1** (Differential privacy). A randomized mechanism $M$ is $(\epsilon, \delta)$-differentially private if for any event $O \in \text{Range}(M)$ and for any neighboring databases $D$ and $D'$ that differ in a single data element, one has

$$\Pr[M(D) \in O] \leq \exp(\epsilon) \Pr[M(D') \in O] + \delta.$$  

When $\delta > 0$, we refer to the above condition as approximate differential privacy. The special case when $\delta = 0$ is called pure differential privacy.

**Definition 2.2** (Empirical Risk Minimization). Given a family of convex loss functions $\{\ell(\theta, d)\}_{d \in D}$ of $\theta$ over $K \subseteq \mathbb{R}^p$ and a set of samples $D = \{d_1, \ldots, d_n\}$ over the universe $D$,
the objective of Empirical Risk Minimization (ERM) is to minimize
\[ L(\theta; D) = \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; d_i). \]

The excess empirical loss with respect to a solution \( \theta \) is defined by
\[ L(\theta; D) - L(\theta^*; D) \]
where \( \theta^* \in \arg\min_{\theta \in \mathcal{K}} L(\theta; D) \), measuring the performance of the solution \( \theta \) compared with the best solution in \( \mathcal{K} \).

**Definition 2.3** (G-Lipschitz Continuity). A function \( f : \mathbb{R}^p \to \mathbb{R} \) is G-Lipschitz continuous with respect to \( \ell_2 \) norm if and only if there exists an index \( i \in \mathbb{R}^p \) that satisfies our modified definition of fingerprinting code as well.

**Definition 3.1** (Coordinate dictionary order). A point \( x \) is said to be larger than \( y \) in coordinate dictionary order if and only if there exists an index \( i \in [n] \) such that \( x_i > y_i \), and for any \( j < i \) we have that \( x_j = y_j \).

It’s straightforward to verify that CDO (coordinate dictionary order) is a well-order. Next we use CDO to select a unique member from the set \( P(D) \) of all Fermat points.

**Definition 3.2** (Ordered Fermat point). The Ordered Fermat point \( q(D) \) of a dataset \( D = \{d_1, ..., d_n\} \) is defined as:
\[ q(D) = \arg\max_{x \in P(D)} \text{CDO}(x) \]

In this section, we give a lower bound for \( \epsilon \)-(pure) differentially private algorithms for minimizing unconstrained convex Lipschitz loss function \( L(\theta; D) \). In the construction of lower bounds for constrained DP-ERM (Bassily et al. [2014]), they chose linear function \( \ell(\theta; d) = \langle \theta, d \rangle \) as their objective function which isn’t applicable in the unconstrained setting because it could decrease to negative infinity. We consider the following \( \ell_2 \) norm loss function instead. Define
\[ \ell(\theta; d) = ||\theta - d||_2, \theta, d \in \mathbb{R}^p \]  

For any dataset \( D = \{d_1, ..., d_n\} \), we define \( L(\theta; D) = \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; d_i) \). Clearly, both \( \ell \) and \( L \) are convex and 1-Lipschitz. The structure of the proof is similar to the one in Bassily et al. [2014], while technical details are quite different as we need to handle a non-linear objective function. Different from the simple average of points in Bassily et al. [2014], we need to consider the Fermat point instead, which is the minimizer of the \( \ell_2 \) norm loss function.

3 Pure DP

In this section, we give a lower bound for \( \epsilon \)-pure differentially private algorithms for minimizing unconstrained convex Lipschitz loss function \( L(\theta; D) \). In the construction of lower bounds for constrained DP-ERM (Bassily et al. [2014]), they chose linear function \( \ell(\theta; d) = \langle \theta, d \rangle \) as their objective function which isn’t applicable in the unconstrained setting because it could decrease to negative infinity. We consider the following \( \ell_2 \) norm loss function instead. Define
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3.1 Fermat point

**Definition 3.1** (Fermat point). The set of Fermat points \( P(D) \) of a dataset \( D = \{d_1, ..., d_n\} \) contains points minimizing its \( \ell_2 \) distance to all points in \( D \):
\[ P(D) = \{ \arg\min_{x \in \mathbb{R}^p} \sum_{i=1}^{n} ||x - d_i||_2 \} \]  

One obstacle of using \( \ell_2 \) norm as our loss is that Fermat points aren’t unique in the worst case. Given a (finite) dataset \( D \), we can easily see that \( P(D) \) is a compact subset of the convex hull of \( D \), which encourages us to define a unique “maximum” element in \( P(D) \). To do so, we introduce the following well-order on \( \mathbb{R}^p \).

**Definition 3.2** (Coordinate dictionary order). A point \( x \) is said to be larger than \( y \) in coordinate dictionary order if and only if there exists an index \( i \in [n] \) such that \( x_i > y_i \), and for any \( j < i \) we have that \( x_j = y_j \).

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Such \(q(D)\) must exist for a finite dataset as long as \(P(D)\) is compact and non-empty, because there can’t be an ordered infinite sequence with its limit outside of \(P(D)\) which contradicts compactness. The technical proof of the following proposition is deferred to appendix.

**Proposition 3.1.** \(q(D)\) always exists for a finite dataset \(D\).

Note that \(q(D)\) is unique by definition and is always a minimizer of \(L(\theta; D)\) over \(\mathbb{R}^p\). In the following subsection we are going to show that any pure DP algorithm can’t estimate \(q(D)\) with good accuracy, then prove that a large error in estimating \(q(D)\) will lead to large error in the excess risk of \(\ell_2\) norm loss as well, establishing the main lower bound of this section.

### 3.2 Lower bound

In this subsection, we prove a lower bound on the excess risk incurred by any \(\epsilon\)-differentially private algorithm whose output is denoted by \(\theta^{priv} \in \mathbb{R}^p\). We first introduce the following lemma showing that it’s impossible to find the location of the ordered Fermat point \(q(D)\) with good accuracy using a pure DP algorithm.

The proof follows the spirit of Bassily et al. [2014], constructing datasets ‘far away’ from each other such that the events of estimating the Fermat point of each dataset accurately are mutually disjoint. Then by differential privacy as long as one can estimate one dataset accurately, one can estimate any other one with certain probability as well. The sum of all these probabilities is no more than 1 due to the disjointness, which leads to the desired bound.

We denote \(e_1 \triangleq (1, 0, ..., 0)^\top\) and let \(\oplus\) denote the direct sum of vectors, i.e. \(\alpha \oplus \beta = (\alpha, \beta)\) where \(\alpha \in \mathbb{R}^a, \beta \in \mathbb{R}^b\) are both vectors. For a vector \(\alpha\) and a set \(S\), we denote \(\alpha \oplus S = \{(\alpha, \beta) : \beta \in S\}\).

**Lemma 3.4.** Let \(n, p \geq 6\) and \(\epsilon > 0\). There is a number \(M = \Omega(\min(n, \frac{p}{\epsilon}))\) such that for any \(\epsilon\)-differentially private algorithm \(A\), there is a dataset \(D = \{d_1, ..., d_n\} \subset \left(0 \oplus \left(\frac{1}{\sqrt{p-1}}, -\frac{1}{\sqrt{p-1}}\right)^{p-1}\right) \cup \{e_1, -e_1, 0\}\) with \(|\sum_{i=1}^{n} d_i||_{2} \leq M\) such that, with probability at least \(1/2\) (taken over the algorithm random coins), we have

\[
||A(D) - q(D)||_2 = \Omega(\min(n, \frac{p}{n\epsilon}))
\]

**Lemma 3.4** implies that it’s impossible to estimate the ordered Fermat point with good accuracy using a pure DP algorithm. In the following theorem we are going to show that a bad estimate on the ordered Fermat point leads to higher \(\ell_2\) norm loss. As the Fermat point is a minimizer of \(\ell_2\) norm loss, we can naturally translate the discrepancy in estimating \(q(D)\) to the excess risk.

**Theorem 3.5 (Lower bound for \(\epsilon\)-differentially private algorithms).** Let \(n, p \geq 6\) and \(\epsilon > 0\). For every \(\epsilon\)-differentially private algorithm with output \(\theta^{priv} \in \mathbb{R}^p\), there is a dataset \(D = \{d_1, ..., d_n\} \subset \left(0 \oplus \left(\frac{1}{\sqrt{p-1}}, -\frac{1}{\sqrt{p-1}}\right)^{p-1}\right) \cup \{e_1, -e_1, 0\}\) such that, with probability at least \(1/2\) (over the algorithm random coins), we must have that

\[
L(\theta^{priv}; D) - \min_{\theta} L(\theta; D) = \Omega(\min(n, \frac{p}{n\epsilon}))
\]

### 4 Approximate DP

In this section, we consider the lower bound for approximate differential privacy where \(2^{-O(n)} < \delta < o(1/n)\). Such assumption on \(\delta\) is common in literature, for example in Steinke and Ullman [2015]. We utilize an \(\ell_1\) loss and use the fingerprinting code in Bun et al. [2018] as our ‘hard case’. We modify the definition of fingerprinting code instead for our analysis.

**Definition 4.1** (Error Robust Biased Mean Fingerprinting Codes). Given \(n, p \in \mathbb{N}, \xi, \beta, \alpha_1, \alpha_2, \alpha_3 \in (0, 1)\). We say a pair of (random) algorithms \((Gen, Trace)\) is an \((n, p)\)-fingerprinting code with security \(\xi\) and \((\alpha_1, \alpha_2, \alpha_3)\)-biased mean, robust to a \(\beta\) fraction of errors if \(Gen\) outputs a codebook \(C \in \{0, 1\}^{n \times p}\) and for any (possibly randomized) adversary \(A_{FP}\) and any coalition \(S \subseteq [n]\), if we set \(c \leftarrow R_A(C|S)\), then

- \(\Pr[c \in F_\beta(C|S) \cap \text{Trace}(C, c) = \bot] \leq \xi\)
- \(\Pr[\text{Trace}(C, c) \in [n] \setminus S] \leq \xi\)
- \(\Pr[Gen_\alpha(C) \geq (1 - \alpha_2)] \leq \alpha_3\)
where \( F_\beta(C_S) = \{ c \in \{0,1\}^p \mid Pr_j, n,p \}[\exists i \in S, c_j = c_{ij} \geq 1 - \beta \} \), \( G_\alpha(C_S) = [\{ j : |\sum_{i \in S} c_{ij} / |S| - 1/2 | \leq \alpha \} \) is the fraction of slightly biased columns in \( C_S \) and the probability is taken over the coins of \( Gen, Trace \) and \( A_{FP} \).

We use the fingerprinting code in Bun et al. [2018] for the construction of our lower bound, see Algorithm 1 in the appendix. To proceed, we first introduce a few lemmas which would be of use later. Similar to Bun et al. [2018], we have the following standard lemma which allows us to reduce any \( \epsilon < 1 \) to \( \epsilon = 1 \) case without loss of generality, using the well-known ’secrecy of the sample’ lemma from Kasiviswanathan et al. [2011].

**Lemma 4.1.** A condition \( Q \) has sample complexity \( n^* \) for algorithms with \( (1, o(1/n)) \)-differential privacy \( (n^* \) is the smallest sample size that there exists an \( (1, o(1/n)) \)-differentially private algorithm \( A \) which satisfies \( Q \), if and only if it also has sample complexity \( \Theta(n^*/\epsilon) \) for algorithms with \( (\epsilon, o(1/n)) \)-differential privacy.

Notice that Lemma 4.1 discusses the sample complexity of the algorithm, therefore is independent of the \( (\alpha_1, \alpha_2, \alpha_3) \)-biased mean appeared in the above definition which only concerns the construction of the fingerprinting code. The following lemma verifies that the fingerprinting code Algorithm 1 indeed has biased mean as in definition 4.1. The proof is straightforward by using the Chernoff bound multiple times.

**Lemma 4.2.** Algorithm 1 (the fingerprinting code) has \( (1/100, 999/1000, \exp(-\Omega(p))) \)-biased mean.

Directly combining Lemma 4.2 and Theorem 3.4 from Bun et al. [2018], we have the following lemma, which states that for the fingerprinting code Algorithm 1 which we will use in proving our main theorem to satisfy the error robust biased mean property in definition 4.1 one needs roughly \( \tilde{\Omega}(\sqrt{p}) \) samples.

**Lemma 4.3.** For every \( p \in \mathbb{N} \) and \( \xi \in (0, 1] \), there exists an \( (n, p) \)-fingerprinting code (Algorithm 1) with security \( \xi \) and \( (1/100, 999/1000, \exp(-\Omega(p))) \)-biased mean, robust to a \( 1/75 \) fraction of error for

\[
n = n(p, \xi) = \tilde{\Omega}(\sqrt{p}/\log(1/\xi)).
\]

We are ready to prove the main result of this section by using Lemma 4.3 to reach a contradiction. Consider the following \( \ell_1 \) norm loss function. Define

\[
\ell(\theta; d) = ||\theta - d||_1, \theta, d \in \mathbb{R}^p
\]

For any dataset \( D = \{d_1, ..., d_n\} \), we define \( L(\theta; D) = \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; d_i) \).

**Theorem 4.4** (Lower bound for \( (\epsilon, \delta) \)-differentially private algorithms). Let \( n, p \) be large enough and \( 1 \geq \epsilon > 0, 0.2^{-O(n)} < \delta < o(1/n) \). For every \( (\epsilon, \delta) \)-differentially private algorithm with output \( \theta^{priv} \in \mathbb{R}^p \), there is a dataset \( D = \{d_1, ..., d_n\} \subset \{0, 1\}^p \cup \{ \frac{1}{2} \}^p \) such that

\[
\mathbb{E}[L(\theta^{priv}; D) - L(\theta^*; D)] = \Omega(min(1, \frac{\sqrt{p \log(1/\delta)}}{ne}))(GC)
\]

where \( \ell \) is \( G \)-Lipschitz, \( \theta^* \) is a minimizer of \( L(\theta; D) \), and \( C \) is the diameter of the set \( \{ \arg \min_\theta L(\theta; D) \mid D \subset \{0, 1\}^{n \times p} \} \), which contains all possible true minimizers.

The dependence on the diameter \( C \) makes sense as one can minimize a substitute loss function \( \ell'(x) = \ell(ax) \) where \( a \in (0, 1) \) is a constant instead, which decreases Lipschitz constant \( G \) but increases the diameter \( C \). Note also that \( C > 0 \) whenever all possible \( D \) don’t share the same minimizer of \( L \), which is often the case. This bound improves a log factor over Bassily et al. [2014] by combining the the group privacy technique in Steinke and Ullman [2015] and our modified definition of fingerprinting code.

**Proof.** Let \( (\alpha_1, \alpha_2, \alpha_3) = (1/100, 999/1000, \exp(-\Omega(p))) \) be the parameters in the statement of Lemma 4.3. Let \( k = \Theta(\log(1/\delta)) \) be a parameter to be determined later and \( n_k = \lfloor n/k \rfloor \).

Consider the case when \( p \geq p_{n_k} \) first, where \( p_{n_k} = O(\epsilon^2 n_k^2 \log(1/\delta)) \). Without loss of generality, we assume \( \epsilon = 1 \) first, and \( p_{n_k} = O(n_k^2 \log(1/\delta)) \) corresponds to the number in Lemma 4.3, where
we set $\xi = \delta$. We will use contradiction to prove that for any $(\epsilon, \delta)$-differentially private mechanism $\mathcal{M}$, there exists some $D \in \{0, 1\}^{n \times p}$ with $G_{\alpha_1 - 1/k}(D) \leq 1 - \alpha_2$ such that

$$E[L(\mathcal{M}(D); D) - L(\theta^*; D)] \geq \Omega(p)$$  \hspace{1cm} (10)

Assume for contradiction that $\mathcal{M} : \{0, 1\}^{n \times p} \rightarrow [0, 1]^{n \times p}$ is a (randomized) $(\epsilon, \delta)$-differentially private mechanism such that

$$E[L(\mathcal{M}(D); D) - L(\theta^*; D)] < \frac{\alpha_1 \alpha_2 p}{1000}$$

for all $D \in \{0, 1\}^{n \times p}$ with $G_{\alpha_1 - 1/k}(D) \leq 1 - \alpha_2$. We then construct a mechanism $\mathcal{M}_k = \{0, 1\}^{n_k \times p}$ with respect to $\mathcal{M}$ as follows: with input $D^k \in \{0, 1\}^{n_k \times p}$, $\mathcal{M}_k$ will copy $D^k$ for $k$ times and append enough 0’s to get a dataset $D \in \{0, 1\}^{n \times p}$. The output is $\mathcal{M}_k(D^k) = \mathcal{M}(D)$. $\mathcal{M}_k$ is $(k, \frac{\epsilon}{k+1})$-differentially private by the group privacy. According to the construction above, we know that if $G_{\alpha_1}(D^k) < 1 - \alpha_2$, then $G_{\alpha_1 - 1/k}(D) < 1 - \alpha_2$ as well.

We consider algorithm $A_{FP}$ to be the adversarial algorithm in the fingerprinting codes, which rounds the the output $\mathcal{M}_k(D^k)$ to the binary vector, i.e. rounding those coordinates with values no less than 1/2 to 1 and the remaining 0, and let $c = A_{FP}(\mathcal{M}(D))$ be the vector after rounding. As $\mathcal{M}_k$ is $(k, \frac{\epsilon}{k+1})$-differentially private, $A_{FP}$ is also $(k, \frac{\epsilon}{k+1})$-differentially private.

If for some $D^k \in \{0, 1\}^{n_k \times p}$ with $G_{\alpha_1}(D^k) \leq 1 - \alpha_2$, $D$ (constructed from $D^k$ as above) further satisfies

$$E[L(\mathcal{M}(D); D) - L(\theta^*; D)] < \frac{\alpha_1 \alpha_2 p}{1000},$$

then because appending 0’s can’t change the prediction of the ‘$\alpha_1$-biased’ columns which take at least $\alpha_2$ fraction of all, we have that

$$E[L(\mathcal{M}_k(D^k); D^k) - L(\theta^*; D^k)] < E[L(\mathcal{M}(D); D) - L(\theta^*; D)] + (1 - \alpha_2)p < \frac{p}{900}.$$  

By Markov Inequality we know that

$$\Pr[L(\mathcal{M}_k(D^k); D^k) - L(\theta^*; D^k)] \geq \frac{p}{180} \leq 1/5.$$  

If $c \notin F_\beta(D^k)$, we have that $L(\mathcal{M}_k(D^k); D^k) - L(\theta^*; D^k) \geq \beta p / 2 = p / 150 > p / 180$ for the $D^k$ by Lemma 4.3 implying

$$\Pr[c \in F_\beta(D^k)] \geq 4/5.$$  \hspace{1cm} (11)

By the first property of the codes, one also has

$$\Pr[L(\mathcal{M}_k(D^k); D^k) - L(\theta^*; D^k) \leq p / 180 \wedge \text{Trace}(D^k, c) = \perp] \leq \Pr[c \in F_\beta(D^k) \wedge \text{Trace}(D^k, c) = \perp] \leq \delta.$$  

Recall that the arguments above are for those $D^k \in \{0, 1\}^{n_k \times p}$ with $G_{\alpha_1}(D^k) \leq 1 - \alpha_2$, which happens with probability at least $1 - \alpha_3$ by the third property of fingerprinting codes. By union bound, we can upper bound the probability $\Pr[\text{Trace}(D^k, c) = \perp] \leq 1/5 + \delta + \alpha_3 \leq 1/2$. As a result, there exists $i^* \in [n_k]$ such that

$$\Pr[i^* \in \text{Trace}(D^k, c)] \geq 1/(2n_k).$$  \hspace{1cm} (12)

Consider the database with $i^*$ removed, denoted by $D^k_{i^*}$. Let $c' = A_{FP}(\mathcal{M}(D^k_{i^*}))$ denote the vector after rounding. By the second property of fingerprinting codes, we have that

$$\Pr[i^* \in \text{Trace}(D^k_{i^*}, c')] \leq \delta.$$  

By the differential privacy and post-processing property of $\mathcal{M}$,

$$\Pr[i^* \in \text{Trace}(D^k, c)] \leq e^k \Pr[i^* \in \text{Trace}(D^k_{i^*}, c')] + \frac{e^k - 1}{e - 1} \delta.$$  

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which implies that
\[
\frac{1}{2nk} \leq e^{k+1}\delta.
\]

Recall that \(2^{-O(n)} < \delta < o(1/n)\), we can set \(k\) as large as \(\Theta(\log(1/\delta))\) to improve a log factor in the bound and reach a contradiction to the inequality above at the same time. As a result, there exists some \(D \in \{0, 1\}^{n \times p}\) with \(G_{\alpha_1-1/k}(D) \geq (1 - \alpha_2)\) subject to
\[
\mathbb{E}[L(\mathcal{M}(D); D) - L(\theta^*; D)] \leq \frac{\alpha_1\alpha_2p}{1000} = \Omega(p).
\]

For the \((\epsilon, \delta)\)-differential privacy case, setting \(Q\) to be the condition
\[
\mathbb{E}[L(\mathcal{M}(D); D) - L(\theta^*; D)] = O(p).
\]

in Lemma 4.1, we have that any \((\epsilon, \delta)\)-differentially private algorithm \(\mathcal{M}\) which satisfies \(Q\) for all \(D \in \{0, 1\}^{n \times p}\) with \(G_{\alpha_1-1/k}(D) \geq 1 - \alpha_2\) must have \(n \geq \Omega(\sqrt{p \log(1/n)/\epsilon})\).

Now we consider the case when \(p < p_{\text{aux}}\), i.e. when \(n > n^* \equiv \Omega(\sqrt{p \log(1/\delta)/\epsilon})\). Given any dataset \(D \in \{0, 1\}^{n \times p}\) with \(G_{\alpha_1-1/k}(D) \geq 1 - \alpha_2\), we will construct a new dataset \(D'\) based on \(D\) by appending dummy points to \(D\) like in Lemma 4.1. Specifically, if \(n - n^*\) is even, we append \(n - n^*\) rows among which half are 0 and half are \(\{1\}^p\). If \(n - n^*\) is odd, we append \(\frac{n-n^*+1}{2}\) points \(0, \frac{n-n^*+1}{2}\) points \(\{1\}^p\) and one point \(\{1/2\}^p\).

Denote the new dataset after appending by \(D'\), we will draw contradiction if there is an \((\epsilon, \delta)\)-differentially private algorithm \(\mathcal{M}'\) such that \(\mathbb{E}[L(\mathcal{M}(D'); D') - L(\theta^*; D')] = o(n^*/n)\) for all \(D'\), by reducing \(\mathcal{M}\) to an \((\epsilon, \delta)\)-differentially private algorithm \(\mathcal{M}\) which satisfies \(\mathbb{E}[L(\mathcal{M}(D); D) - L(\theta^*; D)] = o(p)\) for all \(D\) with \(G_{\alpha_1-1/k}(D) \geq 1 - \alpha_2\).

We construct \(\mathcal{M}\) by first constructing \(\mathcal{M}'\), and then use \(\mathcal{M}'\) as a black box to get \(\mathcal{M}(D) = \mathcal{M}'(D')\). It’s clear that such algorithm for \(D\) preserves \((\epsilon, \delta)\)-differential privacy. It suffices to show that if
\[
\mathbb{E}[L(\mathcal{M}'(D'); D') - L(\theta^*; D')] = o(n^*/n),
\]
then \(L(\mathcal{M}(D); D) - L(\theta^*; D) = o(p)\), which contradicts the previous conclusion for the case \(n \leq n^*\). Specifically, if \(n - n^*\) is even, we have that
\[
n^*\mathbb{E}[L(\mathcal{M}(D); D) - L(\theta^*; D)] = n\mathbb{E}[L(\mathcal{M}'(D'); D') - L(\theta^*; D')].
\]

and if \(n - n^*\) is odd we have that
\[
n^*\mathbb{E}[L(\mathcal{M}(D); D) - L(\theta^*; D)] \leq n\mathbb{E}[L(\mathcal{M}'(D'); D') - L(\theta^*; D')] + p/2.
\]

both leads to the desired reduction. Combining results for both cases we have the following:
\[
\mathbb{E}[L(\theta^{pris}; D) - L(\theta^*; D)] = \Omega(min(p, \frac{pm^*}{n}))
\]

To conclude, observe that \(G = \sqrt{p}\) and \(C = \sqrt{n}\). In particular, let \(D = (d, \ldots, d) \in \{0, 1\}^{n \times p}\) contain \(n\) identical copies of rows \(d \in \{0, 1\}^p\), \(\theta^* = d\). Going over all such \(D\), we find that the set \(\{\arg\min_\theta L(\theta; D) | D \in \{0, 1\}^{n \times p}\}\) contains \(\{0, 1\}^p\), with diameter at least \(\sqrt{n}\). Meanwhile, its diameter can’t exceed \(\sqrt{p}\) obviously.

We leave several remarks discussing slight generalizations of Theorem 4.4.

**Remark 4.5.** Our lower bound can be directly extended to the constrained setting, by setting the constrained domain to be \([0, 1]^{n \times p}\) which contains the convex hull of all possible minimizers \(\{\arg\min_\theta L(\theta; D) | D \in \{0, 1\}^{n \times p}\}\).

**Remark 4.6.** Similarly, we can derive an \(\Omega(1, \sqrt{\text{rank} \log(1/\epsilon)})\) lower bound when we additionally assume the rank of gradient subspace. The analysis remains the same except we first apply orthogonal transformation then set the complement of the gradient subspace to be all 0’s in \(D\).

**Remark 4.7.** The third property of definition 4.1 serves the group privacy analysis to further improve a \(\log(1/\delta)\) term over [Bassily et al. 2014]. One can simplify the proof by setting \(k = 1\) and borrow the lower bound for 1-way marginals from [Bun et al. 2018], at the cost of losing this \(\log(1/\delta)\) term. See appendix for details.
5 Conclusion

In this paper, we study unconstrained private convex ERM and give the first lower bound of unconstrained private ERM for the general convex function. We also give an $\Omega(\frac{p}{n\epsilon})$ lower bound for unconstrained pure-DP ERM which recovers the result in the constrained case. Our results demonstrate that it’s impossible to evade the curse of dimensionality for general loss functions even in the simpler unconstrained case, closing the gap in understanding when dimension-independent bounds could be achieved. However, our contributions are limited to pure theoretical aspects. Designing new algorithms for general unconstrained DP-ERM based on our insights would be a interesting and meaningful direction, which we leave as future work.

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A Additional background knowledge

A.1 Generalized Linear Model (GLM)

The generalized linear model (GLM) is a flexible generalization of ordinary linear regression that allows for response variables that have error distribution models other than a normal distribution. To be specific,

**Definition A.1 (Generalized linear model (GLM)).** The generalized linear model (GLM) is a special class of ERM problems where the loss function $\ell(\theta, d)$ takes the following inner-product form:

$$\ell(\theta; d) = \ell(\langle \theta, x \rangle; y)$$

(15)

for $d = (x, y)$. Here, $x \in \mathbb{R}^p$ is usually called the feature vector and $y \in \mathbb{R}$ is called the response.

A.2 Properties of differential privacy

In this subsection we introduce several very basic properties of differential privacy without proving them (refer Dwork et al. [2014] for details). Readers familiar with the field of differential privacy can feel free to skip this section.

**Proposition A.1** (Group privacy). If $\mathcal{M}: X^n \rightarrow Y$ is $(\epsilon, \delta)$-differentially private mechanism, then for all pairs of datasets $x, x' \in X^n$, then $\mathcal{M}(x), \mathcal{M}(x')$ are $(k\epsilon, k\delta e^\epsilon)$-indistinguishable when $x, x'$ differs on exact $k$ locations.

**Proposition A.2** (Post processing). If $\mathcal{M}: X^n \rightarrow Y$ is $(\epsilon, \delta)$-differentially private and $A: Y \rightarrow Z$ is any randomized function, then $A \circ \mathcal{M}: X^n \rightarrow Z$ is also $(\epsilon, \delta)$-differentially private.

**Proposition A.3** (Composition). Let $\mathcal{M}_i$ be an $(\epsilon_i, \delta_i)$-differentially private mechanism for all $i \in [k]$. If $\mathcal{M}_{[k]}$ is defined to be

$$\mathcal{M}_{[k]}(x) = (\mathcal{M}_1(x), ..., \mathcal{M}_k(x))$$

(16)

then $\mathcal{M}_{[k]}$ is $(\sum_{i=1}^k \epsilon_i, \sum_{i=1}^k \delta_i)$-differentially private.

B Fingerprinting code

In this section we briefly introduce the mechanism of the fingerprinting code Algorithm 1. The sub-procedure part is the original fingerprinting code in Tardos [2008], with a pair of randomized algorithms (Gen, Trace). The code generator Gen outputs a codebook $C \in \{0, 1\}^{n \times p}$. The $ith$ row of $C$ is the codeword of user $i$. The parameter $p$ is called the length of the fingerprinting code.

The security property of fingerprinting codes asserts that any codeword can be “traced” to a user $i$. Moreover, we require that the fingerprinting code can find one of the malicious users even when they get together and combine their codewords in any way that respects the marking condition. That is, there is a tracing algorithm Trace that takes as inputs the codebook $C$ and the combined codeword $c'$ and outputs one of the malicious users with high probability.

The sub-procedure Gen’ first uses a $\sin^2 x$ like distribution to generate a parameter $p_1$ (the mean) for each column $j$ independently, then generates $C$ randomly by setting each element to be $1$ with probability $p_1$ according to its location. The sub-procedure Trace’ computes a threshold value $Z$ and a ‘score function’ $S_i(c')$ for each user $i$, then report $i$ when its score is higher than the threshold.

The main-procedure was introduced in Bun et al. [2018], where Gen adds dummy columns to the original fingerprinting code and applies a random permutation. Trace can first ‘undo’ the permutation and remove the dummy columns, then use Trace’ as a black box. This procedure makes the fingerprinting code more robust in that it tolerates a small fraction of errors to the marking condition.

C Omitted proofs

C.1 Proof of Proposition [3.1]

Proof. We assume $P(D) \neq \emptyset$ without loss of generality. To verify $P(D)$ is compact, we first observe that $P(D)$ is bounded. To prove $P(D)$ is closed, notice that when $P(D) \neq \emptyset$, the function $f(x) = \sum_{i=1}^n \|x - d_i\|_2$ is continuous and non negative, which implies its image is of the form $[a, \infty)$. Therefore the pre-image of the open set $(a, \infty)$ is also open, whose complement is exactly $P(D)$. 

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To find the ordered Fermat point, we reduce the dimension of $P(\mathcal{D})$ one after another. Because $P(\mathcal{D})$ is compact, the largest value of the first coordinate $a_1 \triangleq \arg\max_{x \in P(\mathcal{D})} x_1$ exists, and the ordered Fermat point must lie on the restriction of $P(\mathcal{D})$ on $\{x | x_1 = a_1\}$ which is also compact and non-empty. We continue this process until all dimensions are peeled and there is one point left because the only non-empty set with zero dimension is a single point. \hfill \square

C.2 Proof of Lemma 3.4

Proof. By using a standard packing argument we can construct $K = 2^{\frac{n^*}{\sqrt{n^*}-1}}$ points $d^{(1)}, \ldots, d^{(K)}$ in $0 \oplus \{ \frac{1}{\sqrt{n^*}-1}, -\frac{1}{\sqrt{n^*}-1}\}^{n^*-1}$ such that for every distinct pair $d^{(i)}, d^{(j)}$ of these points, we have

$$||d^{(i)} - d^{(j)}||_2 \geq \frac{1}{8}$$

(17)

It is easy to show the existence of such set of points using the probabilistic method (for example, the Gilbert-Varshamov construction of a linear random binary code).

Fix $\epsilon > 0$ and define $n^* = \frac{1}{16\epsilon^2}$. Let’s first consider the case where $n \leq 4n^*$. We construct $K$ datasets $\mathcal{D}^{(1)}, \ldots, \mathcal{D}^{(K)}$ where for each $i \in [K]$, $\mathcal{D}^{(i)}$ contains $n$ copies of $d^{(i)}$. Note that $q(\mathcal{D}^{(i)}) = d^{(i)}$, we have that for all $i \neq j$,

$$||q(\mathcal{D}^{(i)}) - q(\mathcal{D}^{(j)})||_2 \geq \frac{1}{8}$$

(18)

Let $A$ be any $\epsilon$-differentially private algorithm. Suppose that for every $\mathcal{D}^{(i)}, i \in [K]$, with probability at least $1/2$, $||A(\mathcal{D}^{(i)}) - q(\mathcal{D}^{(i)})||_2 < \frac{1}{16} \text{i.e., Pr}[A(\mathcal{D}^{(i)}) \in B(\mathcal{D}^{(i)})] \geq \frac{1}{2}$ where for any dataset $\mathcal{D}$,
\[ B(\mathcal{D}) = \{ x \in \mathbb{R}^p : \|x - q(\mathcal{D})\|_2 < \frac{1}{16} \} \]  

(19)

Note that for all \( i \neq j \), \( \mathcal{D}^{(i)} \) and \( \mathcal{D}^{(j)} \) differs in all their \( n \) entries. Since \( A \) is \( \epsilon \)-differentially private, for all \( i \in [K] \), we have \( \Pr[A(\mathcal{D}^{(i)}) \in B(\mathcal{D}^{(i)})] \geq \frac{1}{2} e^{-\epsilon n}. \) Since all \( B(\mathcal{D}^{(i)}) \) are mutually disjoint, then

\[ \frac{K}{2} e^{-\epsilon n} \leq \sum_{i=1}^{K} \Pr[A(\mathcal{D}^{(i)}) \in B(\mathcal{D}^{(i)})] \leq 1 \]  

(20)

which implies that \( n > 4n^* \) for sufficiently large \( p \), contradicting the fact that \( n \leq 4n^* \). Hence, there must exist a dataset \( \mathcal{D}^{(i)} \) on which \( A \) makes an \( \ell_2 \)-error on estimating \( q(\mathcal{D}) \) which is at least 1/16 with probability at least 1/2. Note also that the \( \ell_2 \) norm of the sum of the entries of such \( \mathcal{D}^{(i)} \) is \( n \).

Next, we consider the case where \( n > 4n^* \). As before, we construct \( K = 2^{n^*} \) datasets \( \tilde{\mathcal{D}}^{(1)}, \ldots, \tilde{\mathcal{D}}^{(K)} \) of size \( n \) where for every \( i \in [K] \), the first \( n^* \) entries of each dataset \( \mathcal{D}^{(i)} \) are the same as dataset \( \mathcal{D}^{(i)} \) from before whereas the remaining \( n - n^* \) entries are constructed as follows. The first \( \lfloor \frac{n-n^*}{2} \rfloor \) of those entries are all copies of \( e_1 \) whereas the following \( \lceil \frac{n-n^*}{2} \rceil \) are copies of \( -e_1 \). The last entry is set to be 0 when \( n - n^* \) is odd.

Note that any two distinct datasets \( \tilde{\mathcal{D}}^{(i)}, \tilde{\mathcal{D}}^{(j)} \) in this collection differ in exactly \( n^* \) entries. Let \( A \) be any \( \epsilon \)-differentially private algorithm for answering \( q \). Suppose that for every \( i \in [K] \), with probability at least 1/2, we have that

\[ \|A(\tilde{\mathcal{D}}^{(i)}) - q(\tilde{\mathcal{D}}^{(i)})\|_2 < \frac{n^*}{32n} \]  

(21)

Note that for all \( i \in [K] \), we have that \( q(\tilde{\mathcal{D}}^{(i)}) = \lambda q(\mathcal{D}^{(i)}) \) where \( \lambda = \frac{n^*}{\sqrt{n^* - 2n^*}} \) if \( n - n^* \) is even and

\[ \lambda = \frac{n^* - 1}{\sqrt{4\left(\frac{n-n^*}{2} - (n^* - 1)^2 \right)}} \]  

(22)

if \( n - n^* \) is odd. We notice that \( \frac{n^*}{n} \leq \lambda \leq \frac{2n^*}{n} \), and is independent of the choice of \( i \). Now, we define an algorithm \( \tilde{A} \) for answering \( q \) on datasets \( \mathcal{D} \) of size \( n^* \) as follows. First, \( \tilde{A} \) computes \( \lambda \) and appends \( e_1, -e_1, 0 \) as above to get a dataset \( \tilde{\mathcal{D}} \) of size \( n \). Then, it runs \( A \) on \( \tilde{\mathcal{D}} \) and outputs \( A(\tilde{\mathcal{D}}) \). Hence, by the post-processing property of differential privacy, \( \tilde{A} \) is \( \epsilon \)-differentially private since \( A \) is \( \epsilon \)-differentially private. Thus for every \( i \in [K] \), with probability at least 1/2, we have that \( \|A(\tilde{\mathcal{D}}^{(i)}) - q(\tilde{\mathcal{D}}^{(i)})\|_2 < \frac{1}{16} \). However, this contradicts our result in the first part of the proof. Therefore, there must exist a dataset \( \tilde{\mathcal{D}}^{(i)} \) in the above collection such that, with probability at least 1/2,

\[ \|A(\tilde{\mathcal{D}}^{(i)}) - q(\tilde{\mathcal{D}}^{(i)})\|_2 \geq \frac{n^*}{32n} \geq \frac{p}{5120\epsilon n} \]  

(23)

Note that the \( \ell_2 \) norm of the sum of entries of such \( \tilde{\mathcal{D}}^{(i)} \) is always \( n^* \).

\[ \square \]

C.3 Proof of Theorem 3.5

\textbf{Proof.} Let \( A \) be an \( \epsilon \)-differentially private algorithm for minimizing \( L \) and let \( \theta^{pr} \) denote its output. We choose the dataset \( \mathcal{D} \) (with corresponding \( d_i \)) constructed in Lemma 3.3. When \( n \leq 4n^* \), \( \mathcal{D} \) contains only identical elements \( d_i \) so that min \( L(\theta; \mathcal{D}) = 0 \), and

\[ L(\theta^{pr}; \mathcal{D}) - \min_{\theta} L(\theta; \mathcal{D}) = L(\theta^{pr}; \mathcal{D}) = \|\theta^{pr} - \theta(\mathcal{D})\|_2 = \Omega(\min(1, \frac{p}{n \epsilon n})) \]  

(24)

by Lemma 3.3. When \( n > 4n^* \), we denote \( r \triangleq \|\theta^{pr} - q(\mathcal{D})\|_2 \geq \Omega(\min(1, \frac{p}{n \epsilon n})) \). Notice that \( d_i, e_1, -e_1, 0 \) all lie in a 2-dimensional subspace and \( \|d_i\|_2 = 1 \) is perpendicular to \( e_1 \), we may assume \( d_i = e_2 \) without loss of generality. Because \( q(\mathcal{D}) = \lambda e_2 \), we parameterize \( \theta^{pr} \) as follows

\[ \theta^{pr} = (x_1, \lambda + x_2, x_3, \ldots, x_p) \]  

(25)
where $\sum_{i=1}^{p} x_i^2 = r^2$. Now the excess loss satisfies

$$L(\theta^{priv}; D) - \min_{\theta} L(\theta; D)$$

$$\geq (n^* \sqrt{r^2 + (\lambda - 1)^2} + 2(\lambda - 1)x_2 + \left\lfloor \frac{n - n^*}{2} \right\rfloor \sqrt{r^2 + 1 + \lambda^2 + 2\lambda x_2 + 2x_1}$$

$$+ \left\lfloor \frac{n - n^*}{2} \right\rfloor \sqrt{r^2 + 1 + \lambda^2 + 2\lambda x_2 - 2x_1}) / n$$

(opening up the expression)

$$\geq (\left\lfloor \frac{n - n^*}{2} \right\rfloor \sqrt{r^2 + (\lambda - 1)^2} + 2x_1 + \left\lfloor \frac{n - n^*}{2} \right\rfloor \sqrt{r^2 + (\lambda - 1)^2 - 2x_1}) / n$$

(dropping the first term)

$$\geq (\left\lfloor \frac{n - n^*}{2} \right\rfloor \sqrt{r^2 + (\lambda - 1)^2}) / n$$

(max$\{x_1, -x_1\} \geq 0$)

$$\geq \left\lfloor \frac{n - n^*}{2} \right\rfloor \cdot \frac{r}{n} = \Omega(\min(1, \frac{p}{n_\varepsilon}))$$

$(n > 4n^*, r = \Omega(\min(1, \frac{p}{n_\varepsilon})))$

\[\square\]

### C.4 Proof of Lemma 4.1

**Proof.** The proof uses a black-box reduction, therefore doesn’t depend on $Q$. The direction that $O(n^*/\varepsilon)$ samples are sufficient is equal to proving the assertion that given a $(1, o(1/n))$-differentially private algorithm $A$, we can get a new algorithm $A'$ with $(\varepsilon, o(1/n))$-differential privacy at the cost of shrinking the size of the dataset by a factor of $\varepsilon$.

Given input $\varepsilon$ and a dataset $X$, we construct $A'$ to first generate a new dataset $T$ by selecting each element of $X$ with probability $\varepsilon$ independently, then feed $T$ to $A$. Fix an event $S$ and two neighboring datasets $X_1, X_2$ that differ by a single element $i$. Consider running $A$ on $X_1$. If $i$ is not included in the sample $T$, then the output is distributed the same as a run on $X_2$. On the other hand, if $i$ is included in the sample $T$, then the behavior of $A$ on $T$ is only a factor of $\varepsilon$ off from the behavior of $A$ on $T \setminus \{i\}$. Again, because of independence, the distribution of $T \setminus \{i\}$ is the same as the distribution of $T$ conditioned on the omission of $i$.

For a set $X$, let $p_X$ denote the distribution of $A(X)$, we have that for any event $S$,

$$p_{X_1}(S) = (1 - \varepsilon)p_{X_1}(S| i \notin T) + \varepsilon p_{X_1}(S| i \in T)$$

$$\leq (1 - \varepsilon)p_{X_2}(S) + \varepsilon(p_{X_2}(S) + \delta)$$

$$\leq \exp(2\varepsilon)p_{X_2}(S) + \varepsilon\delta$$

A lower bound of $p_{X_1}(S) \geq \exp(-\varepsilon)p_{X_2}(S) - \varepsilon\delta/\varepsilon$ can be obtained similarly. To conclude, since $\varepsilon\delta = o(1/n)$ as the sample size $n$ decreases by a factor of $\varepsilon$, $A'$ has $(2\varepsilon, o(1/n))$-differential privacy. The size of $X$ is roughly $1/\varepsilon$ times larger than $T$, combined with the fact that $A$ has sample complexity $n^*$ and $T$ is fed to $A, A'$ has sample complexity at least $\Theta(n^*/\varepsilon)$.

For the other direction, simply using the composability of differential privacy yields the desired result. In particular, by the $k$-fold adaptive composition theorem in [Dwork et al., 2006], we can combine $1/\varepsilon$ independent copies of $(\varepsilon, \delta)$-differentially private algorithms to get an $(1, \delta/\varepsilon)$ one and notice that if $\delta = o(1/n)$, then $\delta/\varepsilon = o(1/n)$ as well because the sample size $n$ is scaled by a factor of $\varepsilon$ at the same time, offsetting the increase in $\delta$. \[\square\]

### C.5 Proof of Lemma 3.2

**Proof.** In line 5 of algorithm [1] every column $j$ is assigned a probability $p_j$ independently where

$$Pr[|p_j - \frac{1}{2}| < 0.002] < \frac{1}{400}$$

(26)

by straightforward calculation. By the Chernoff bound (with $u < p/400, \delta = 1$), with probability at least

$$1 - 2\exp(-p/800)$$

(27)
at least $1 - \frac{1}{200}$ fraction of the columns have $|p_j - \frac{1}{2}| \geq 0.002$. Denote $m_j$ to be the mean of entries of column $j$, then by using the Chernoff bound again (with $\delta = 0.001$), we have that with probability at least

$$1 - 2 \exp(-n/8000000)$$

a column $j$ actually satisfies $|m_j - \frac{1}{2}| \geq 0.001$. Again by the Chernoff bound (with $u \leq 2 \exp(-n/8000000)p$ and $u\delta = 0.01p$) together with the union bound, at least 0.99 fraction of all columns have $|m_j - \frac{1}{2}| \geq 0.001$ with probability at least

$$1 - 2 \exp(-p/800) - 2 \exp(-pe^{n/8000000}/40000) = 1 - O(e^{-\Omega(p)})$$

C.6 Details of Remark 4.7

In Bun et al. [2018] they prove the following lower bound for 1-way marginals:

**Proposition C.1** (Corollary 3.6 in Bun et al. [2018]). The family of 1-way marginals on $\{0, 1\}^d$ has sample complexity at least $\tilde{\Omega}(\sqrt{d})$ for $(1/3, 1/75)$-accuracy and $(O(1), o(1/n))$-differential privacy.

Inspecting the proof we find that the constant $1/3$ in the above proposition is chosen casually, and can be replaced by any constant $c < 1/2$ for free, as the proof only requires $1 - c$ is rounded to 1 and $c$ is rounded to 0 respectively.

The third property of the fingerprinting code implies that with high probability, at most 0.01 fraction of all columns have mean with bias smaller than 0.001. When we assume the opposite for the sake of contradiction, by union bound, at least $1/75 - 1/100 = 1/300$ fraction of columns have both 'large error on 1-way marginal' and 'large bias on mean'.

For any such column $j$, the algorithm is forced to predict wrongly on the question 'Is there more 0's than 1's in column $j$' as the range of prediction is restricted in $[0, 1]$ and choose $c + 0.001 > 1/2$, thus leading to error on $\ell_1$ norm loss.