ALGEBRAIC PROPERTIES OF A HYPERGRAPH LIFTING MAP

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Abstract. Recent work in hypergraph Ramsey theory has involved the introduction of a “lifting map” that associates a certain 3-uniform hypergraph to a given graph, bounding cliques in a predictable way. In this paper, we interpret the lifting map as a linear transformation. This interpretation allows us to use algebraic techniques to prove several structural properties of the lifting map, culminating in new lower bounds for certain 3-uniform hypergraph Ramsey numbers.

1. Introduction

In [1], a lifting map $\varphi : G_2 \rightarrow G_3$ was described that assigned to each graph a unique 3-uniform hypergraph. Here, $G_2$ denotes the set of all graphs of order at least 3 and $G_3$ is the set of all 3-uniform hypergraphs of order at least 3. Both a graph $G$ and its image $\varphi(G)$ share the same vertex set, and an unordered 3-tuple $abc$ forms a hyperedge in $\varphi(G)$ if and only if the subgraph of $G$ induced by $\{a, b, c\}$ contains an odd number of edges.

The lifting $\varphi$ was shown to preserve complements (i.e., $\varphi(\overline{G}) = \overline{\varphi(G)}$) and the way in which $\varphi$ lifted to complete graphs was analyzed. Specifically, it was shown that if $\varphi(G)$ contained a complete hypergraph with vertices $x_1, x_2, \ldots, x_n$, then the subgraph of $G$ induced by $\{x_1, x_2, \ldots, x_n\}$ is the disjoint union of at most 2 complete graphs (including the possibility that it is complete). These properties were then used to provide new lower bounds for certain 3-uniform hypergraph Ramsey numbers.

In [3], a generalization of $\varphi$ was described that allowed graphs to be lifted to $r$-uniform hypergraphs. In this variation, denoted $\varphi^{(r)}$, a hyperedge $x_1x_2\cdots x_r$ is formed in the image of $\varphi^{(r)}$ if and only if the subgraph of $G$ induced by $\{x_1, x_2, \ldots, x_r\}$ is the disjoint union of at most $r - 1$ complete graphs. Like $\varphi$, it was shown that $\varphi^{(r)}$ lifted to complete subhypergraphs in a predictable way, but unfortunately, complements were no longer preserved when $r > 3$ making it ineffective as a tool in Ramsey theory. Rather, [3] included an application involving Turán numbers.

Since $\varphi$ preserved complements, it could be interpreted as describing a way of lifting 2-colorings of the edges of the complete graph of order $n \geq 3$ to 2-colorings of the hyperedges of the complete 3-uniform hypergraph of order $n$. Generalizing this interpretation to 3-colorings was considered in [2], but the problem of deciding how a rainbow triangle should lift led to a focus on Gallai colorings (those lacking rainbow triangles). The present paper sets out to avoid this restriction by recognizing the lifting as a linear transformation between certain vector spaces over a finite field.

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This generalization allows one to extend the lifting map to more than two colors and to consider it as a map between arbitrary uniformities. It also provides an algebraic framework to the lifting map, providing insight into its structure via standard algebraic techniques.

In Section 2, we construct vector spaces of hypergraph edge colorings over finite fields and interpret the lifting map as a linear transformation between such vector spaces. A few general results are proved before focusing our attention on the theory when the field of scalars is $\mathbb{F}_2$ (the finite field of order 2) in Section 3. Finally, in Section 4 we consider the applications to Ramsey theory that follow from our new algebraic description of the lifting map. We are able to prove two new lower bounds for certain 3-color and 5-color 3-uniform hypergraph Ramsey numbers.

2. THE LIFTING MAP AS A LINEAR TRANSFORMATION

In order to establish the lifting map as a linear transformation, we must first formalize the terminology and background surrounding the objects to be studied. An $r$-uniform hypergraph $H = (V(H), E(H))$ consists of a nonempty set of vertices $V(H)$ and a set of hyperedges $E(H)$, whose elements are different $r$-tuples of distinct vertices from $V(H)$. The complete $r$-uniform hypergraph of order $n$ is denoted by $K_n^{(r)}$ and consists of $n$ vertices, every $r$-element subset of which forms a hyperedge. When $r = 2$, we simplify the notation $K_n^{(2)}$ and just write $K_n$.

Let $\mathbb{F}_q$ be the finite field of order $q = p^m$, where $p$ is a prime number and $m \geq 1$ is an integer. An $\mathbb{F}_q$-hyperedge coloring of an $r$-uniform hypergraph $H$ is a map $f : E(H) \to \mathbb{F}_q$. Denote the set of all $\mathbb{F}_q$-hyperedge colorings of $K_n^{(r)}$ by $\mathcal{H}_n^{(r)}(\mathbb{F}_q)$ and observe that it forms a vector space over $\mathbb{F}_q$ under the operations

$$(f + g)(e) = f(e) + g(e) \quad \text{and} \quad (\alpha f)(e) = \alpha f(e),$$

where $e \in E(K_n^{(r)})$, $\alpha \in \mathbb{F}_q$, and $f, g \in \mathcal{H}_n^{(r)}(\mathbb{F}_q)$. A basis for $\mathcal{H}_n^{(r)}(\mathbb{F}_q)$ can be formed using the $\mathbb{F}_q$-hyperedge colorings

$$f_{e'}(e) = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{if } e \neq e', \end{cases}$$

where $e' \in E(K_n^{(r)})$. It follows that $\dim_{\mathbb{F}_q}(\mathcal{H}_n^{(r)}(\mathbb{F}_q)) = \binom{n}{r}$.

If $T$ is a set, then denote by $T^r$ the set of all $r$-element subsets of $T$. For $2 \leq s < r \leq n$, define the lifting $\Psi_{q,n}^{(s,r)} : \mathcal{H}_n^{(s)}(\mathbb{F}_q) \to \mathcal{H}_n^{(r)}(\mathbb{F}_q)$ by

$$(\Psi_{q,n}^{(s,r)} f)(e) = \sum_{e' \in e^{(r)}} f(e'),$$

where $f \in \mathcal{H}_n^{(s)}(\mathbb{F}_q)$ and $e \in E(K_n^{(r)})$. We leave it as an exercise for the reader to check that $\Psi_{q,n}^{(s,r)}$ is a linear transformation and $\Psi_{2,n}^{(2,3)}$ corresponds with the lifting described in $\square$. Realizing this map as a linear transformation elucidates some of its properties, the first of which involves the time required to determine if a given hyperedge coloring is in the image of such a map.

**Proposition 1.** For $g \in \mathcal{H}_n^{(r)}(\mathbb{F}_q)$, there exists a polynomial time algorithm to determine whether or not $g \in \text{Im}(\Psi_{q,n}^{(s,r)})$.

**Proof.** Apply Gaussian Elimination to solve the system $\Psi_{q,n}^{(s,r)}(f) = g$. $\square$
**Theorem 2.** Let \( n \geq r > s \geq 2, \, g \in \text{Im}(\Psi_{q,n}^{(s,r)}) \), and suppose that \( q|\binom{n-s}{r-s} \). Then
\[
\sum_{e \in E(K_n^{(r)})} g(e) = 0_{\mathbb{F}_q}.
\]

**Proof.** Let \( g = \Psi_{q,n}^{(s,r)} f \). By definition, the sum in Theorem 2 becomes
\[
\sum_{e \in E(K_n^{(r)})} g(e) = \sum_{e \in E(K_n^{(r)})} \sum_{e' \in e} f(e').
\]
Within this sum, \( f(e') \) occurs \( \binom{n-s}{r-s} \) times, corresponding to the number of hyperedges \( e \in E(K_n^{(r)}) \) that contain \( e' \). The assumption \( q|\binom{n-s}{r-s} \) implies that the sum is \( 0_{\mathbb{F}_q} \).

In the case where \( s = r - 1 \), we obtain the following theorem.

**Theorem 3.** Let \( r \geq 3 \) and assume that \( \Psi_{q,n}^{(r-1,r)} f = \Psi_{q,n}^{(r-1,r)} g \). If \( f \neq g \), then \( f \) and \( g \) differ by at least \( n - r + 2 \) hyperedge colors.

**Proof.** Assume that \( \Psi_{q,n}^{(r-1,r)} f = \Psi_{q,n}^{(r-1,r)} g \) and \( f \neq g \). Then some hyperedge \( x_1x_2\cdots x_{r-1} \) in \( K_n^{(r-1)} \) receives a different color under \( g \) than it does under \( f \). The hyperedge \( x_1x_2\cdots x_{r-1} \) is contained in exactly \( n - (r - 1) \) \( r \)-tuples, each of which contains a distinct vertex from the set
\[
V(K_n^{(r)}) - \{x_1, x_2, \ldots, x_{r-1}\} = \{y_1, y_2, \ldots, y_{n-(r-1)}\}.
\]
Retaining the colors of these \( r \)-tuples under \( \Psi_{q,n}^{(r-1,r)} \) requires at least \( n - (r - 1) \) additional hyperedges in \( K_n^{(r-1)} \) (each of which includes a single element from \( \{y_1, y_2, \ldots, y_{n-(r-1)}\} \) and some selection of \( r - 2 \) vertices from \( \{x_1, x_2, \ldots, x_{r-1}\} \)) be colored differently under \( g \) than under \( f \). Hence, \( f \) and \( g \) differ by at least \( n - r + 2 \) hyperedge colors.

To see that this theorem is optimal, consider the case of \( \Psi_{2,5}^{(2,3)} \). By Theorem 4 of [H], both a \( \mathbb{F}_2 \)-colored \( K_1 \cup K_4 \) and a \( \mathbb{F}_2 \)-colored \( K_2 \cup K_3 \) map to a \( \mathbb{F}_2 \)-colored \( K_5^{(3)} \) and can be shown to differ by exactly 4 edge colors, the minimum number implied by Theorem 3.

3. **The Case of the Finite Field \( \mathbb{F}_2 \)**

When restricting to the case \( q = 2 \), we have the ability to discuss complements of hyperedge colorings. The preservation of complements is exactly the property that allowed the lifting map to be applied to Ramsey theory in [H]. Let \( H \) be an \( r \)-uniform hypergraph and let \( f : E(H) \rightarrow \mathbb{F}_2 \) be an \( \mathbb{F}_2 \)-hyperedge coloring. Define the complement of \( f \) to be the \( \mathbb{F}_2 \)-hyperedge coloring \( \overline{f} : E(H) \rightarrow \mathbb{F}_2 \) such that for all \( e \in E(H) \),
\[
\overline{f}(e) = \begin{cases} 0_{\mathbb{F}_2} & \text{if } f(e) = 1_{\mathbb{F}_2} \\ 1_{\mathbb{F}_2} & \text{if } f(e) = 0_{\mathbb{F}_2}. \end{cases}
\]

**Theorem 4.** Let \( f \in H_n^{(s)}(\mathbb{F}_2) \). Then
\[
\psi_{n}^{(s,r)}(\overline{f}) = \begin{cases} \psi_{n}^{(s,r)} f & \text{if } \binom{n}{s} \text{ is odd} \\ \psi_{n}^{(s,r)} f & \text{if } \binom{n}{s} \text{ is even}. \end{cases}
\]
Proof. Let \( f \in \mathcal{H}^{(s)}(F_2) \). Note that each \( r \)-uniform hyperedge \( e \in E(K_n^{(r)}) \) corresponds with a selection of \( r \) vertices in \( K_n^{(s)} \). Let \( H \) be the subhypergraph induced by \( e \) in \( K_n^{(s)} \) and observe that \( H \) has \((\binom{s}{r})\) \( s \)-uniform hyperedges. Denote the subhypergraph of \( H \) spanned by all \( 0_2 \)-colored hyperedges in \( f \) by \( H_0 \) and the subhypergraph of \( H \) spanned by all \( 1_2 \)-colored hyperedges in \( f \) by \( H_1 \). Define the subhypergraphs \( H_0 \) and \( H_1 \) similarly under \( f \) and note that \( H_0 \cong H_1 \) and \( H_1 \cong H_0 \). By definition, the \( r \)-uniform hyperedge \( e \) will be \( 1_2 \)-colored in \( \Psi^{(s,r)}_{2,n} f \) if \(|E(H_1)|\) is odd and \( 0_2 \)-colored if \(|E(H_0)|\) is even. In the case where \((\binom{s}{r})\) is odd, exactly 1 of \(|E(H_1)|\) and \(|E(H_0)|\) must be odd. Without loss of generality, assume \(|E(H_1)|\) is odd. This means that \(|E(H_1)|\) is even and while the hyperedge \( e \) is \( 1_2 \)-colored in \( \Psi^{(s,r)}_{2,n} f \), \( e \) is \( 0_2 \)-colored in \( \Psi^{(s,r)}_{2,n} f \). It follows that

\[
\Psi^{(s,r)}_{2,n} f = \Psi^{(s,r)}_{2,n} f
\]

when \((\binom{s}{r})\) is odd. In the case where \((\binom{s}{r})\) is even, either \(|E(H_1)|\) and \(|E(H_0)|\) are both odd or they are both even. Likewise, either \(|E(H_1)|\) and \(|E(H_1)|\) are both odd or they are both even. This means that the hyperedge \( e \) receives the same color in \( \Psi^{(s,r)}_{2,n} f \) as it receives in \( \Psi^{(s,r)}_{2,n} f \), proving that

\[
\Psi^{(s,r)}_{2,n} f = \Psi^{(s,r)}_{2,n} f
\]

whenever \((\binom{s}{r})\) is even. \( \square \)

Next, we consider the sum introduced in Theorem 4 in the case where \( q = 2 \). To simplify the statement of the next theorem, for \( g \in \mathcal{H}^{(r)}(F_2) \), write \( S^{(r)}_{2,n}(g) := \sum_{e \in K_n^{(r)}} g(e) \).

**Theorem 5.** If \( n \geq r > s \geq 2 \) and \( g = \Psi^{(s,r)}_{2,n} f \), then

\[
S^{(r)}_{2,n}(g) = \begin{cases} 
0_2 & \text{if } \binom{n-s}{r-s} \text{ is even} \\
\Psi^{(s)}_{2,n}(f) & \text{if } \binom{n-s}{r-s} \text{ is odd.}
\end{cases}
\]

**Proof.** The first case, where \((\binom{n-s}{r-s})\) is even follows from Theorem 2. In the case where \((\binom{n-s}{r-s})\) is odd, observe that

\[
S^{(r)}_{2,n}(g) = \sum_{e \in K_n^{(r)}} \sum_{e' \in e} f(e'),
\]

with \( f(e') \) occurring \((\binom{n-s}{r-s})\) times. It follows that the sum simplifies to

\[
\sum_{e' \in E(K_n^{(r)})} f(e')
\]

in this case. \( \square \)

Now we consider a couple of theorems in the case where \( s = r - 1 \). The following theorem was motivated by Theorem 2.1 of [3].

**Theorem 6.** Let \( r \geq 3 \) be odd and consider a coloring \( g = \Psi^{(r-1,r)}_{2,n} f \). If the image of a subhypergraph \( K \) of \( K_n^{(r-1)} \) under \( g \) is complete in some color, then \( K \) consists of at most \( r - 1 \) connected components in that color.
proving that $K_r$ is in color $F$ in any hyperedge coloring $\Psi$. Once again, this contradicts the assumption that $K_r$ is even, result in a $0_F$-colored $r$-uniform hyperedge. Once again, this contradicts the assumption that $K$ is monochromatic, proving that $K$ consists of at most $r - 1$ connected components.

The following is a generalization of Theorem 7 of [1].

**Theorem 7.** If $r \geq 3$ is odd, then $K_{n+1}^{(r)} - e$ never occurs as an induced subhypergraph in any hyperedge coloring $\Psi^{(r-1,r)}_{2,n} f$, where $n \geq r + 1$.

**Proof.** Suppose that $K_{n+1}^{(r)} - e$ does occur as an induced subhypergraph in some hyperedge coloring $\Psi^{(r-1,r)}_{2,n} f$. Then the coloring necessarily contains an induced subhypergraph isomorphic to $K_{r+1}^{(r)} - e$. First, consider the case where the $K_{r+1}^{(r)} - e$ is in color $1_F$ (so that $e$ has color $0_F$). There must necessarily be an even number of $(r - 1)$-element subsets of $e$ that are colored $1_F$ in $f$ and an odd number that are colored $0_F$ (since $r$ is odd). For any other hyperedge in the $K_{r+1}^{(r)} - e$ other than $e$, there are an odd number of $(r - 1)$-element subsets that are colored $1_F$ and an even number that are colored $0_F$. Summing over all $r$-element subsets, we obtain an odd number of $(r - 1)$-element subsets in color $1_F$. However, note that each $(r - 1)$-element subset occurs in exactly $2$ $r$-element subsets. It follows that this sum should be $0_F$, giving a contradiction. The case where $K_{r+1}^{(r)} - e$ is in color $0_F$ is the same, but with $0_F$ and $1_F$ switched.

For the remainder of this section, we consider the lifting of graphs to $r$-uniform hypergraphs (the case $s = 2$). Our investigation focuses next on the graphs that lift to complete and empty subhypergraphs (in color $1_F$), so to simplify the statements of our theorems, we introduce some new terminology. Let $G$ be a graph of order $n$ and let $G'$ be the subgraph of $G$ induced by a selection of $r$ vertices such that $2 < r \leq n$. If $G'$ always has an odd number of edges, $G$ is called $r$-complete. If $G'$ always has an even number of edges, $G$ is called $r$-void. If $G$ is neither $r$-complete nor $r$-void, it is called $r$-neutral. Under $\Psi^{(2,r)}_{2,n}$, a $r$-complete graph in color $1_F$ will always lift to a complete $1_F$-colored $r$-uniform hypergraph, while an $r$-void graph will always lift to a complete $0_F$-colored $r$-uniform hypergraph.

**Theorem 8.** Given an odd $r \geq 3$, a complete bipartite graph of order at least $r$ is always $r$-void.

**Proof.** Let $r \geq 3$ be odd and $n \geq r$ such that $K_{s,t}$ is a complete bipartite graph of order $n$. The vertex set of $K_{s,t}$ is the disjoint union of partite sets $V_1$ and $V_2$,
having cardinalities $|V_1| = s$ and $|V_2| = t$, where $n = s + t$. Any selection of $r$ vertices from $K_{s,t}$ results in $u$ vertices selected from $V_1$ and $r - u$ vertices selected from $V_2$. Since $r$ is odd, exactly one of $u$ and $r - u$ must be odd. Without loss of generality, let $u$ be odd and $r - u$ be even. Since $K_{s,t}$ is complete bipartite, each of the $u$ vertices selected from $V_1$ must share an edge with each of the $r - u$ vertices selected from $V_2$. This means that the subgraph induced by these vertices must have $u \cdot (r - u)$ edges. As $r - u$ is even, $u \cdot (r - u)$ must also be even. \hfill \square

**Lemma 9.** Let $r > u$.

1. If $r \equiv 0 \pmod{4}$, then $\binom{u}{2} + \binom{r-u}{2} \equiv u \pmod{2}$.
2. If $r \equiv 2 \pmod{4}$, then $\binom{u}{2} + \binom{r-u}{2} \equiv (u+1) \pmod{2}$.

**Proof.** We run through cases, based on the value of $r$ modulo 4.

Case 1: Suppose that $r \equiv 0 \pmod{4}$. When $u$ is even (i.e., $u \equiv 0, 2 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have the same parity, implying that $\binom{u}{2} + \binom{r-u}{2}$ is even. When $u$ is odd (i.e., $u \equiv 1, 3 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have different parities, implying that $\binom{u}{2} + \binom{r-u}{2}$ is odd.

Case 2: Suppose that $r \equiv 2 \pmod{4}$. When $u$ is even (i.e., $u \equiv 0, 2 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have different parities, implying that $\binom{u}{2} + \binom{r-u}{2}$ is odd. When $u$ is odd (i.e., $u \equiv 1, 3 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have the same parity, implying that $\binom{u}{2} + \binom{r-u}{2}$ is even. \hfill \square

**Theorem 10.** Let $G$ be the disjoint union of 2 complete subgraphs of orders $s$ and $t$ with $r < s + t$. $G$ is $r$-neutral if $r$ is even, $G$ is $r$-void if $r \equiv 1 \pmod{4}$, and $G$ is $r$-complete if $r \equiv 3 \pmod{4}$.

**Proof.** First, consider the case where $r$ is even. We select $u < r$ vertices from the $K_s$ and $r - u$ vertices from the $K_t$. The subgraph induced by these $r$ vertices will contain $\binom{u}{2} + \binom{r-u}{2}$ edges. Next, select $u + 1$ vertices from the $K_s$ and $r - u - 1$ vertices from the $K_t$. The subgraph induced by these $r$ vertices will contain $\binom{u+1}{2} + \binom{r-u-1}{2}$ edges. By Lemma 9, these two selections will yield differing numbers of edges modulo 2. Therefore $G$ must be $r$-neutral. Now consider the case where $r$ is odd. By Theorem 8, $G$ is $r$-void, so by Theorem 9 if $r \equiv 1 \pmod{4}$, $G$ is $r$-void and if $r \equiv 3 \pmod{4}$, $G$ is $r$-complete. \hfill \square

We conclude this section with a result concerning the original lifting map $\Psi_{2,n}^{(2,3)}$, denoted by $\varphi$ in [1].

**Theorem 11.** The lifting $\Psi_{2,n}^{(2,3)}$ is a $2^{n-1}$-to-one mapping.

**Proof.** The fact that every $g \in \text{Im}(\Psi_{2,n}^{(2,3)})$ has the same number $k$ of preimages follows from the linearity of $\Psi_{2,n}^{(2,3)}$. In order to determine the value of $k$, we select an element in the image whose preimages we can easily count. Specifically, consider the element in $\mathcal{H}_{2,n}^{(2)}(\mathbb{F}_2)$ that maps all hyperedges to 0. The induced subhypergraph in color 0 is isomorphic to $K_{n/2}^{(3)}$, and by Theorem 4 of [1], the elements in $\mathcal{H}_{2,n}^{(2)}(\mathbb{F}_2)$ that map to this element are those in which color 0 is given to subgraphs that are complete of order $n$ or are the disjoint union of two complete subgraphs, whose orders add to $n$. When $n$ is even, the possibilities are $K_n$, $K_{1} \cup K_{n-1}$, $K_{2} \cup K_{n-2}$, $\ldots$, $K_{n/2-1} \cup K_{n/2+1}$, $K_{n/2} \cup K_{n/2}$.
with these cases occurring
\[ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n/2 - 1}, \frac{1}{2} \binom{n}{n/2} \]
times, respectively. When \( n \) is odd, the possibilities are
\( K_n, K_1 \cup K_{n-1}, K_2 \cup K_{n-2}, \ldots, K_{(n-1)/2} \cup K_{(n+1)/2}, \)
with these cases occurring
\[ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{(n-1)/2} \]
times, respectively. Applying the identity
\[ \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n, \]
along with the property \( \binom{n}{k} = \binom{n}{n-k} \) to each of these cases, we obtain the statement of the theorem.

\[
\begin{align*}
&4. \text{ Some Applications to Ramsey Theory} \\
&\text{The generalization of the lifting map as a linear transformation allows us to prove a 3-colored Ramsey number bound that does not require the restriction to Gallai colorings, as in [2]. Recall that if } H_1, H_2, \ldots, H_t \text{ are } r\text{-uniform hypergraphs, then the Ramsey number } R(H_1, H_2, \ldots, H_t; r) \text{ is the least positive integer } p \text{ such that every } t\text{-coloring of the hyperedges of } K_p^{(r)} \text{ results in a subhypergraph isomorphic to } H_i \text{ spanned by hyperedges in color } i, \text{ for some } 1 \leq i \leq t. \\
&\textbf{Theorem 12. Let } s_i \geq 3 \text{ for all } 1 \leq i \leq 3. \\
&\quad R(K_{2s_1-1}^{(3)} - e, K_{2s_2-1}^{(3)}, K_{2s_3-1}^{(3)}; 3) \geq R(K_{s_1}, K_{s_2}, K_{s_3}; 2). \\
&\text{Proof. We identify the three colors with the elements in } \mathbb{F}_3. \text{ This proof follows the proof of the analogous Gallai-Ramsey number result in Theorem 3 of [2]. For all 3-tuples of vertices in } K_n \text{ whose induced subgraphs are monochromatic or 2-colored, the image is the expected image in the Gallai-Ramsey case. For rainbow 3-tuples, a single color results, corresponding to the 2-color lifting where the other two colors are identified as being the same. By Theorem 4 of [1], avoiding a monochromatic } K_{s_i} \text{ in color } i \text{ results in an image that avoids a monochromatic copy of } K_{2s_i-1}^{(3)} \text{ in color } i. \text{ For color } 0_{\mathbb{F}_3}, \text{ in particular, the lifting is the same as in the 2-color case. This allows us to further note that no } 0_{\mathbb{F}_3}\text{-colored copy of } K_{2s_1-1}^{(3)} - e \text{ exists by Theorem 7 of [1].} \\
&\text{The following theorem generalizes Theorem 10 of [1] to a 5-color Ramsey-theoretic result.} \\
&\textbf{Theorem 13. Let } q \geq 3 \text{ and } s_i \geq 3 \text{ for all } 1 \leq i \leq 3. \text{ Then} \\
&\quad R(K_{2s_1-1}^{(3)} - e, K_{2s_2-1}^{(3)}, K_{2s_3-1}^{(3)}, K_{5}^{(3)}, K_{q+1}^{(3)}; 3) > q(R(K_{s_1}, K_{s_2}, K_{s_3}; 2) - 1). \\
&\text{Proof. If } p = R(K_{s_1}, K_{s_2}, K_{s_3}; 2), \text{ then by Theorem [12] we can construct a 3-colored } K_p^{(3)} \text{ that avoids a copy of } K_{2s_1-1}^{(3)} - e \text{ in color } 0_{\mathbb{F}_3}, \text{ a copy of } K_{2s_2-1}^{(3)} \text{ in color } 1_{\mathbb{F}_3}, \text{ and a copy of } K_{2s_3-1}^{(3)} \text{ in color } 2_{\mathbb{F}_3}. \text{ Consider the disjoint union of } q \text{ copies of this 3-colored } K_p^{(3)}, \text{ and label the copies } V_1, V_2, \ldots, V_q. \text{ Color the hyperedges that have}
one vertex in some \( V_i \) and the other two vertices in some \( V_j \) \((i \neq j)\) using a fourth color and color the hyperedges that have all vertices coming from distinct \( V_i \) using a fifth color. It is easily confirmed that the largest complete hypergraph in the fifth color uses at most one vertex from any given \( V_i \) and adding in another vertex to such a hypergraph is lacking more than one hyperedge in the fifth color (hence, avoiding a \( K_{q+1}^{(3)} - e \)). The largest complete hypergraph in the fourth color uses at most two vertices from at most two distinct \( V_i \). It follows that no \( K_{3}^{(3)} \) exists in the fourth color and no \( K_{q+1}^{(3)} - e \) exists in the fifth color. \( \square \)

Using the 3-color lower bounds given in Table X of Section 6.1 of Radziszowski's dynamic survey [4], Theorem 13 implies the following lower bounds. Here, we use the usual notation \( R_3(G) \) to denote the 3-color graph Ramsey number \( R(G, G, G; 2) \).

\[
\begin{align*}
R_3(K_3) &= 17 \quad \Rightarrow \quad R(K_3^{(3)} - e, K_5^{(3)}, K_5^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 16q, \\
R_3(K_4) &\geq 128 \quad \Rightarrow \quad R(K_4^{(3)} - e, K_7^{(3)}, K_7^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 127q, \\
R_3(K_5) &\geq 417 \quad \Rightarrow \quad R(K_5^{(3)} - e, K_9^{(3)}, K_9^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 416q, \\
R_3(K_6) &\geq 1070 \quad \Rightarrow \quad R(K_6^{(3)} - e, K_{11}^{(3)}, K_{11}^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 1069q, \\
R_3(K_7) &\geq 3214 \quad \Rightarrow \quad R(K_7^{(3)} - e, K_{13}^{(3)}, K_{13}^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 3213q, \\
R_3(K_8) &\geq 6079 \quad \Rightarrow \quad R(K_8^{(3)} - e, K_{15}^{(3)}, K_{15}^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 6078q, \\
R_3(K_9) &\geq 13761 \quad \Rightarrow \quad R(K_9^{(3)} - e, K_{17}^{(3)}, K_{17}^{(3)}, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 13760q.
\end{align*}
\]

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