INSTANTON BUNDLES ON THE FLAG VARIETY F(0,1,2)

F. MALASPINA, S. MARCHESI AND J. PONS-LLOPIS

ABSTRACT. Instanton bundles on \(\mathbb{P}^3\) have been at the core of the research in Algebraic Geometry during the last thirty years. Motivated by the recent extension of their definition to other Fano threefolds of Picard number one, we develop the theory of instanton bundles on the complete flag variety \(F := F(0,1,2)\) of point-lines on \(\mathbb{P}^2\). After giving for them two different monadic presentations, we use it to show that the moduli space \(M_{I_F}(k)\) of instanton bundles of charge \(k\) is a geometric GIT quotient with a generically smooth component of dimension \(8k - 3\). Finally we study their locus of jumping conics.

INTRODUCTION

One of the classes of vector bundles which has raised most of the attention among algebraic geometers is the class of instanton bundles. They were defined as the algebraic counterpart that permitted to solve a central problem in Yang-Mills theory, a gauge theory for non-abelian groups whose aim is to provide an explanation of weak and strong interactions. It tries to extend the results of Quantum Electrodynamics (QED), which is a gauge theory for abelian groups describing electromagnetic interactions. In the original Yang-Mills theory, the term instanton or pseudo-particle denoted the minimum action solutions of \(SU(2)\) Yang-Mills fields in the 4-sphere \(S^4\). This problem could be rephrased in terms of differential geometry as follows: find all possible connections with self-dual curvature on a smooth \(SU(2)\)-bundle \(\mathcal{E}\) over the 4-sphere \(S^4\). Instantons come with a 'topological quantum number' that can be interpreted as the second Chern class \(c_2(\mathcal{E})\) of the \(SU(2)\)-bundle on \(S^4\) and it is known as the 'charge' of \(\mathcal{E}\). A major input to obtain the classification of the set of solutions of the Yang-Mills field equations was given by twistor theory, as it was developed by R. Penrose. The essence of the twistor programme is to encode the differential geometry of a manifold by holomorphic data on some auxiliary complex space (a twistor space). In this setting, it was realized that the original Yang-Mills problem on (anti)self-dual \(SU(2)\)-connections on a bundle \(\mathcal{E}\) on \(S^4\) could be restated in terms of the possible holomorphic structures on the pull-back bundle \(\pi^* \mathcal{E}\) on \(\mathbb{P}^3(\mathbb{C})\), where \(\pi : \mathbb{P}^3(\mathbb{C}) \to S^4\) is the projection constructed through twistor theory. This, jointly with the characterization of holomorphic vector bundles on \(\mathbb{P}^3\) as a certain kind of monads due to Horrocks, led to a complete classification of instantons on \(S^4\) (cf. [2], [1]).

Motivated by the previous theorem, the notion of a mathematical instanton bundle \(\mathcal{E}\) with charge \(c_2(\mathcal{E}) = k\) was defined on \(\mathbb{P}^3\) and more recently on an arbitrary Fano threefold with Picard number one (cf. [13] and [17]). The existence of instanton bundles have been proved for almost all Fano threefolds of Picard number one. It became a central problem to understand the geometrical properties of their moduli spaces (smoothness, irreducibility, rationality...). These objects are important on their own as well as for the fact that physicists...
have interpreted various moduli spaces as solution spaces to physically interesting differential equations.

In this paper we propose to extend the study of instanton bundles on other Fano threefolds of higher Picard rank. In particular, we focus our attention to the case of the Flag variety $F := F(0, 1, 2) \subset \mathbb{P}^7$ of pairs point–line in $\mathbb{P}^2$. The main reason for this choice is the following: in [16], Hitchin showed that the only twistor spaces of four dimensional (real) differential varieties which are Kähler (and a fortiori, projective) are $\mathbb{P}^3$ and the flag variety $F(0, 1, 2)$, which is the twistor space of $\mathbb{P}^2$. Indeed, mathematical instanton bundles with charge one on $F(0, 1, 2)$ were studied in [6] and [11].

A instanton bundle $E$ on $F$ with charge $k$ will be defined as a rank 2 $\mu$-semistable vector bundle with Chern classes $c_1(E) = 0$ and $c_2(E) = kh_1h_2$ verifying the "instantonic condition" $H^1(E(-1, -1)) = 0$ (see Definition (3.3)).

Instanton bundles on $F$ will turn out to be Gieseker semistable. Therefore it makes sense to study the (coarse) moduli space of instanton bundles $MI_F(k)$ of charge $k$. For this, we are going to rely on their presentation as the cohomology of a certain monad: recall that, from [13], when the derived category of coherent sheaves on the Fano threefold $X$ has a full exceptional collection, instanton bundles on $X$ have a particular nice presentation as the cohomology of a certain monad. Since this is also the case for the Flag variety $F$ we are dealing with, we give two such presentations (linked by a commutative diagram, see Theorems 5.1 and 5.2 and compare with [6]):

**Theorem 1.** Let $E$ be an instanton bundle with charge $k$ on $F$, then, up to permutation, $E$ is the cohomology of the monad

$$ (1) \quad 0 \to O_F(-1, 0)^{\oplus k} \oplus O_F(0, -1)^{\oplus k} \xrightarrow{\alpha} G_1(-1, 0)^{\oplus k} \oplus G_2(0, -1)^{\oplus k} \xrightarrow{\beta} O_F^{\oplus 2k-2} \to 0. $$

where $G_i$ is the pull-back of the twisted cotangent bundle $\Omega_{\mathbb{P}^2}(2)$ from the two natural projections $p_i : F \subset \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$. Conversely, the cohomology of any such monad gives a $k$-instanton.

To present the second monad let us introduce some notation. Let us denote by $U = H^0(O_F(1, 0))$ the 3-dimensional vector space of global sections and observe $U^* \cong H^0(O_F(0, 1))$. Let $W$, $I_1$ and $I_2$ be vector spaces of dimension $4k+2$, $k$ and $k$ respectively. Let $J : W \to W^*$ be a nondegenerate skew-symmetric form and denote by $Sp(W, J)$ the symplectic group. Associated to any map of vector bundles $A : W \otimes O_F \to (I_1 \otimes O_F(1, 0)) \oplus (I_2 \otimes O_F(0, 1))$ we have a matrix $A \in W^* \otimes ((I_1 \otimes U) \oplus (I_2 \otimes U^*))$. Let us define:

$$ D_k := \{ A \in W^* \otimes ((I_1 \otimes U) \oplus (I_2 \otimes U^*)) \mid AJA^t = 0 \}, $$

and

$$ D_k^0 := \{ A \in D_k \mid A : W \otimes O_F \to (I_1 \otimes O_F(1, 0)) \oplus (I_2 \otimes O_F(0, 1)) \text{ is surjective} \} $$

The group $G_k := Sp(W, J) \times GL(I_1) \times GL(I_2)$ acts on $D_k$ as follows: $(\eta, g_1, g_2)A = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} A \eta^t$ and this action clearly descends to $D_k^0$. Then we have the following:

**Theorem 2.** Let $E$ be a $k$-instanton bundle on the flag $F$. Then, $E$ is the cohomology of a self-dual monad.
(2) \[ 0 \to (I_1^* \otimes O_F(-1,0)) \oplus (I_2^* \otimes O_F(0,-1)) \xrightarrow{JA'} W \otimes O_F \xrightarrow{A} (I_1 \otimes O_F(1,0)) \oplus (I_2 \otimes O_F(0,1)) \to 0. \]
and conversely, the cohomology of such a monad is a \( k \)-instanton. Therefore, the moduli space \( M I_F(k) \) is the geometric quotient \( \mathcal{D}_k^0 / G_k \).

Concerning non-emptiness and main properties we managed to prove by induction the following result (see Theorem 6.1):

**Theorem 3.** Let \( F \subset \mathbb{P}^7 \) be the flag variety. The moduli space of \( k \)-instanton bundles \( M I_F(k) \) has a generically smooth irreducible component of dimension \( 8k - 3 \). Moreover, these instanton bundles have trivial splitting type on the generic conic and line (from both families).

For an instanton bundle \( \mathcal{E} \) on the projective space \( \mathbb{P}^3 \), the study of the behavior of the restriction of \( \mathcal{E} \) to the lines has played a crucial role. On the Flag threefold \( F \), we have realized that an analogous attention should be devoted to the study of the restriction of an instanton bundle to the conics. Therefore, we give a proof of the fact that the Hilbert scheme \( H := Hilb^{2d+1}(F) \) parameterizing rational curves of genus 0 and degree 2 is again isomorphic to \( \mathbb{P}^2 \times \mathbb{P}^2 \). Under this setting, if we denote by \( \mathcal{D}_E \) the set of curves on \( H \) for which the restriction of \( \mathcal{E} \) is not trivial (the "jumping conics"), we obtain the following result (see Prop. 7.2):

**Theorem 4.** Let \( \mathcal{E} \) be a \( k \)-instanton on \( F \). Then \( \mathcal{D}_E \) is a divisor of type \((k, k)\) equipped with a torsion-free sheaf \( G \) fitting into

\[ 0 \to O_H(-1,-1)^{\oplus k} \oplus O_H(-1,0)^{\oplus k} \to O_H^{\oplus k} \oplus O_H(-1,0)^{\oplus k} \to i_*G \to 0. \]

This paper is organized in the following way: in the next section, we recall the definition and basic facts of the flag variety \( F := F(0,1,2) \). In Section 2, we introduce the notion of instanton bundle on \( F \). In Sections 3 and 4, we study the derived category \( D^b(F) \) of coherent sheaves on \( F \) and use it to give a monadic presentation of the instanton bundles. The central part of this paper is Section 5, where we prove the existence of a suitable family of instantons as stated in Theorem 1. Finally, in Section 6, we deal with the notion of jumping conic on the flag variety.

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1. **Preliminaries**

We will work over an algebraically closed field \( K \) of characteristic zero.
Let $F \subseteq \mathbb{P}^7$ be the del Pezzo threefold of degree 6, we can also construct $F$ as the general hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. The projections $\pi_i$ induce maps $p_i : F \to \mathbb{P}^2$ by restriction, $i = 1, 2$ and such maps are isomorphic to the canonical map $\mathbb{P}(\Omega_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. Thinking of the second copy of $\mathbb{P}^2$ as the dual of the first one, then $F$ can also be viewed naturally as the flag variety of pairs point–line in $\mathbb{P}^2$. Let $h_i$, $i = 1, 2$, be the classes of $p^*_i \mathcal{O}_{\mathbb{P}^2}(1)$ in $A^1(F)$ respectively. The class of the hyperplane divisor on $F$ is $h = h_1 + h_2$. For any coherent sheaf $\mathcal{E}$ on $F$ we are going to denote the twisted sheaf $\mathcal{E} \otimes \mathcal{O}_F(\alpha h_1 + \beta h_2)$ by $\mathcal{E}(\alpha h_1 + \beta h_2)$ or by $\mathcal{E}(\alpha, \beta)$. As usual, $H^i(X, \mathcal{E})$ stands for the cohomology groups, $h^i(X, \mathcal{E})$ for their dimension and analogously ext will denote the dimension of the considered Ext group.

The above discussion proves the isomorphisms

$$A(F) \cong A(\mathbb{P}^2)[h_1]/(h_1^2 - h_1 h_2 + h_2^2) \cong \mathbb{Z}[h_1, h_2]/(h_1^2 - h_1 h_2 + h_2^2, h_1^3, h_2^3).$$

In particular, $\text{Pic}(F) \cong \mathbb{Z}^{\oplus 2}$ with generators $h_1$ and $h_2$. We will now explicit some computations that will be useful throughout the paper. We will always denote the Chern polynomial, for a given sheaf, that we will want to compute with

$$1 + \alpha_1 h_1 + \alpha_2 h_2 + \beta_1 h_1^2 + \beta_1 h_2^2 + \gamma h_1^2 h_2$$

In order to compute the Chern classes of the tautological sheaf of the conic on the flag variety, we consider its defining exact sequence

$$0 \to \mathcal{O}_F(-1, 0) \to \mathcal{O}_F(-1, -1) \oplus \mathcal{O}_F \to \mathcal{O}_F \to \mathcal{O}_C \to 0$$

Let us split the previous one in short exact sequences and denote by $K$ the only kernel appearing.

We first compute $c_t(K)$ through the relation

$$c_t(K) \cdot c_t(\mathcal{O}_F(-1, -1)) = c_t(\mathcal{O}_F(-1, 0) \oplus \mathcal{O}_F(0, -1))$$

Explicitly

$$(1 + \alpha_1 h_1 + \alpha_2 h_2 + \beta_1 h_1^2 + \beta_1 h_2^2 + \gamma h_1^2 h_2) \cdot (1 - (h_1 + h_2)) = (1 - (h_1 + h_2) + h_1 h_2)$$

and we get

$$\alpha_1 h_1 + \alpha_2 h_2 - h_1 - h_2 = -h_1 - h_2 \quad \to \quad \alpha_1 = \alpha_2 = 0$$

$$\beta_1 h_1^2 + \beta_2 h_2^2 = h_1 h_2 = h_1^2 + h_2^2 \quad \to \quad \beta_1 = \beta_2 = 1$$

$$\gamma h_1^2 h_2 - h_1^2 h_2 = 0 \quad \to \quad \gamma = 2$$

and now we use

$$c_t(K) \cdot c_t(\mathcal{O}_C) = c_t(\mathcal{O}_F)$$

Explicitly

$$(1 + \alpha_1 h_1 + \alpha_2 h_2 + \beta_1 h_1^2 + \beta_1 h_2^2 + \gamma h_1^2 h_2) \cdot (1 + h_1^2 + h_2^2 + 2 h_1^2 h_2) = 1$$

and we get

$$\alpha_1 h_1 + \alpha_2 h_2 = 0 \quad \to \quad \alpha_1 = \alpha_2 = 0$$

$$\beta_1 h_1^2 + \beta_2 h_2^2 + h_1^2 + h_2^2 = 0 \quad \to \quad \beta_1 = \beta_2 = -1$$

$$\gamma h_1^2 h_2 + 2 h_1^2 h_2 = 0 \quad \to \quad \gamma = -2$$
After that, we consider the sheaf $\mathcal{O}_C(1)$, which will be the right candidate to use the induction process explained in Section 6.3 which will ensure the existence of instanton bundles on the flag variety for each charge. Its Chern polynomial is

$$c_t(\mathcal{O}_C(1)) = 1 - h_1 h_2$$

and we can compute it in two ways. First, we can compute the Chern classes of the twisted sheaf

$$c_3(\mathcal{O}_C(1)) = c_3(\mathcal{O}_C \otimes \mathcal{O}_F(1, 0)) = c_3(\mathcal{O}_C) - 2h_1 c_2(\mathcal{O}_C) = -2h_1^2 h_2 - 2h_1(-h_1 h_2) = -2h_1^2 h_2 + 2h_1^2 h_2 = 0$$

and nothing else being $c_1(\mathcal{O}_C) = 0$.

We could also consider directly its resolution

$$\mathcal{O}_F(0, 0) \rightarrow \mathcal{O}_F(0, -1) \rightarrow \cdots \rightarrow \mathcal{O}_F(1, 0) \rightarrow \mathcal{O}_C(1) \rightarrow 0$$

and obtain the same result.

We now compute the Hirzebruch-Riemann-Roch formula for the flag variety. Recall, that denoting by $T_F$ the tangent bundle of the flag, we have

$$c_1(T_F) = 2(h_1 + h_2), \quad c_2(T_F) = 6h_1 h_2$$

and we compute the Todd polynomial

$$td(T_F) = 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24} c_1 c_2 = 1 + (h_1 + h_2) + \frac{3}{2} h_1 h_2 + \frac{1}{=h_1^2 h_2}$$

Remembering that the Chern character is given by

$$ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1 c_2 + 3c_3)$$

we can finally obtain

$$\chi(\mathcal{E}) = r + \frac{3}{2} c_1 h_1 h_2 + \frac{1}{2}(c_1^3 - 2c_2)(h_1 + h_2) + \frac{1}{6}(c_1^3 - 3c_1 c_2 + 3c_3)$$

(4)

We will use throughout this article the following rank two ACM bundles:

$$\mathcal{G}_1 = p_1^* \Omega_{\mathbb{P}^2}^1(2h_1) \quad \mathcal{G}_2 = p_2^* \Omega_{\mathbb{P}^2}^1(2h_2).$$

We will now recall how to compute the cohomology of the line bundles on $F$ (see [9] Proposition 2.5):

**Proposition 1.1.** For each $\alpha_1, \alpha_2 \in \mathbb{Z}$ with $\alpha_1 \leq \alpha_2$, we have

$$h^i(F, \mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)) \neq 0$$

if and only if

- $i = 0$ and $\alpha_1 \geq 0$;
- $i = 1$ and $\alpha_1 \leq -2, \alpha_1 + \alpha_2 + 1 \geq 0$;
- $i = 2$ and $\alpha_2 \geq 0, \alpha_1 + \alpha_2 + 3 \leq 0$;
- $i = 3$ and $\alpha_2 \leq -2$. 
In all these cases
\[ h^i(F, \mathcal{O}_F(\alpha_1 h_1 + \alpha_2 h_2)) = (-1)^i \frac{(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_1 + \alpha_2 + 2)}{2}. \]

Since
\[ \dim(\text{Ext}^1(\mathcal{O}_F(0, -1), \mathcal{G}_1(2, -2))) = \dim(\text{Ext}^1(\mathcal{O}_F(-1, 0), \mathcal{G}_2(0, -2))) = 3, \]
we denote by \( \mathcal{G}_3 \) and \( \mathcal{G}_4 \) the rank 5 vector bundles arising from the extensions
\begin{align*}
(5) & \quad 0 \to \mathcal{G}_1(2, -2) \to \mathcal{G}_3 \to \mathcal{O}_F(0, -1)^{\oplus 3} \to 0 \\
(6) & \quad 0 \to \mathcal{G}_2(0, -2) \to \mathcal{G}_4 \to \mathcal{O}_F(-1, 0)^{\oplus 3} \to 0.
\end{align*}

The latter extensions will be used in describing the bundle as the cohomology of a monad, indeed they will appear in the exceptional collections used in the spectral sequence (see Sections 4 and 5).

2. Hilbert scheme of lines and curves on the Flag variety

It could be thought that the study of the geometry of lines of the \( F \) will be important to define and understand instanton bundles (like in \( \mathbb{P}^3 \)). Recall that \( F \) contains two families of lines \( \Lambda_1, \Lambda_2 \), each isomorphic to \( \mathbb{P}^2 \). Their representatives in the Chern ring are \( h_1^2, h_2^2 \). Notice that if we look at \( F \) as the projective bundle \( \mathbb{P}(\Omega_{\mathbb{P}^2}^1(2)) \to \mathbb{P}^2 \), these families correspond to the fibers. We have a geometrical description (using the notion of flag variety): given \( p \in \mathbb{P}^2 \), \( \lambda_p := \{ L \in \mathbb{P}^2 | p \in L \} \in \Lambda_1 \). Analogously, given line \( L \subset \mathbb{P}^2 \), \( \lambda_L := \{ x \in \mathbb{P}^2 | x \in L \} \in \Lambda_2 \). Notice \( \lambda_x \cap \lambda_y = \emptyset \) if \( x \neq y \) (clear from cohomological product \( h_1^2 h_2^2 = 0 \)) and \( \lambda_x \cap \lambda_L = \emptyset \) (resp. \( \{ x, L \} \)) if \( x \in L \) (resp. \( x \notin L \)).

Nevertheless, in the case of the flag variety, the main kind of rational curve we are interested in is the conic. In fact, through the Ward correspondence, instanton bundles on \( F \) have trivial splitting on ”real” conics (this is explained in [6] and [11] without explicitly mentioning the degree) and therefore, by semicontinuity, on the general element of \( \text{Hilb}^{2t+1}(F) \).

Lemma 2.1. The Hilbert scheme of rational curves of degree two \( M := \text{Hilb}^{2t+1}(F) \) is isomorphic to \( \mathbb{P}^2 \times \mathbb{P}^2 \). The open set \( \mathbb{P}^2 \times \mathbb{P}^2 \setminus F \) corresponds to smooth conics. Moreover, the canonical map \( p : C \to F \) from the universal conic \( C \) to \( F \) endows \( C \) with the structure of a quadric bundle of relative dimension 2 over \( F \).

Proof. Quadric surfaces \( Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \) are parameterized again by a \( \mathbb{P}^2 \times \mathbb{P}^2 \). Indeed, in order to define a quadric, we need to choose a pair of lines \( (L_1, L_2) \in \mathbb{P}^2 \times \mathbb{P}^2 \) is the associated quadric. Now, it is immediate to see that any quadric in \( \mathbb{P}^2 \times \mathbb{P}^2 \) defines uniquely a conic \( C_{(L_1, L_2)} := F \cap Q_{(L_1, L_2)} \) in the hyperplane section \( F \). Moreover, one can check that \( Q_{(L_1, L_2)} \) is tangent to \( F \) (and therefore \( C_{(L_1, L_2)} \) is singular) if and only if \( (L_1, L_2) \in F \subset \mathbb{P}^2 \times \mathbb{P}^2 \).

Remark 2.2. Indeed, it is known (see for instance [13] Lemma 2.1.1) that any subscheme of \( F \) with Hilbert polynomial \( 2t + 2 \) will be a smooth conic, a pair of distinct lines intersecting on a point, or a line with a double structure. In order to see that there is no such non-reduced
subscheme on $F$ we should observe that for any line $L$ on $F$, we have $N_{L|F} \cong O_L^2$. Therefore there is no surjective map $N_{L|F} \to O_L(-1)$ and we conclude again by [18, Lemma 2.1.1].

**Remark 2.3.** An explicit parametrization of the conics is as follows, for $(p, L) \in \mathbb{P}^2 \times \mathbb{P}^2 \setminus F$, the subscheme of $F$:

$$C_{(p, L)} := \{(q, S) \mid L(q) = S(p) = 0\}$$

is a smooth conic in $F$. Its class in the Chow ring is $h_1 h_2$. On the other hand, if we choose a point $(p, l) \in F$ the associated curve $C_{(p, l)}$ will be the union of the lines $L_p := \{(p, S) \mid s(p) = 0\} \in h_1$ and $L_l := \{(q, l) \mid l(q) = 0\} \in h_2$ intersecting at the point $(p, l)$.

One more argument to explain why the role of the line is taken over by conic in the flag is the following

**Proposition 2.4.** Given any two points of $F$, there exists exactly one smooth conic passing through them.

**Proof.** Given two points $(p_i, L_i) \in F, i = 1, 2$, if we define $q := L_1 \cap L_2, S := \overline{p_1, p_2}$, the conic $C_{(q, S)}$ meets the requirements. \qed

**Remark 2.5.** Notice that if we allow singular conics, two more cases appear.

### 3. Instantons. Definition and First Properties

In this section we will define what we will call an instanton bundle on the flag. Such definition is given through requirements of cohomology vanishings, trivial splitting and fixed Chern classes. As we will see in Section 5 such characterization does not lead to a linear monad, i.e. a monad involving only the trivial, tautological and hyperplane bundles of the flag.

**Definition 3.1.** For any integer $k \geq 1$ we will call real instanton bundle with charge $k$ a rank two vector bundle $E$ on $F$ with $H^1(E(-1, -1)) = 0, c_2(E) = kh_1 h_2$ and such that for any real conic $C \in \text{Hilb}_2(F)$,

$$E_C \cong O_C \oplus O_C.$$

**Remark 3.2.** As in the case of $\mathbb{P}^3$, real instanton bundles have been shown to be in one-to-one correspondence with the antidual solutions of the YM equations over $\mathbb{P}^2(\mathbb{C})$ (see [6, 11]).

**Definition 3.3.** For any integer $k \geq 1$ we will call mathematical instanton bundle with charge $k$ (or, for short, a $k$-instanton) a rank two $\mu$-semistable bundle $E$ on $F$ with $c_1(E) = (0, 0), c_2(E) = kh_1 h_2$ and $H^1(E(-1, -1)) = 0$. Its Hilbert polynomial is

$$\chi(E(t)) = (t + 1)(2t^2 + 4t + 2(1 - k)).$$

**Remark 3.4.**

- For Fano threefolds of Picard number one an analogous definition, extending the case of the projective space $\mathbb{P}^3$, was given in [13] and [17]. In the case of the flag variety $F$, however, we have to allow proper $\mu$-semistable instanton bundles to deal with the higher Picard number. We are going to characterize in a moment the strictly $\mu$-semistable instantons on $F$. 

• The presentation we gave of the Hilbert polynomial of \( \mathcal{E} \) shows that \( \chi(\mathcal{E}(-1, -1)) = 0 \). Indeed, using Serre’s duality, it holds \( H^i(F, \mathcal{E}(-1, -1)) = 0 \) for \( i = 0, \ldots, 3 \).

• An easy but tedious computation shows that for rank two bundles on \( F \) with the same Chern classes as the instanton bundles the notions of Gieseker (semi)stability and Mumford-Takemoto (semi)stability coincide, so we are going to use both of them indistinctly.

In order to check the (semi)-stability of rank 2 real instanton bundles, we are going to use the following version of the Hoppe’s criterium (see [19]):

**Lemma 3.5** (Hoppe’s criterium). A rank 2 vector bundle \( \mathcal{E} \) on \( F \) with first Chern class \( c_1(\mathcal{E}) = 0 \) is (semi)stable if and only if \( H^0(F, \mathcal{E}(p, q)) = 0 \) for any \( p, q \in \mathbb{Z} \) such that \( p + q \leq (\leq)0 \).

**Lemma 3.6.** A real instanton bundle on \( F \) is a mathematical instanton.

**Proof.** Let \( \mathcal{E} \) be real instanton bundle on \( F \). Since the condition about \( c_1(\mathcal{E}) = 0 \) is clearly satisfied, let us show that \( \mathcal{E} \) is \( \mu \)-semistable, using the Hoppe’s criterium: if \( s \in H^0(F, \mathcal{E}(p, q)) \) with \( p + q < 0 \), for \( C \subseteq F \) an arbitrary real conic, \( s|_C \) is a global section of \( \mathcal{E}|_C(p, q) \cong \mathcal{O}_C(p + q)^2 \), i.e., \( s|_C = 0 \). Since \( F \) is covered by real conics we see that \( s \) should be the zero section and therefore we are done.

Hence, we get the following characterization of the instanton bundles which are strictly semistable.

**Proposition 3.7.** Let \( \mathcal{E} \) be a \( k \)-instanton on \( F \). Then, if \( \mathcal{E} \) is not \( \mu \)-stable, then \( k = l^2 \) for some \( l \in \mathbb{Z}, l \neq 0 \) and it can be constructed as an extension \( \Lambda_1 \) of the form

\[
(7) \quad 0 \to \mathcal{O}_F(l, -l) \to \mathcal{E} \to \mathcal{O}_F(-l, l) \to 0.
\]

In particular for any \( k \)-instanton \( \mathcal{E} \), \( H^0(\mathcal{E}) = 0 \). Finally, the only common element of the two families of extensions \( \Lambda_1 \) and \( \Lambda_{-1} \) is the decomposable bundle \( \mathcal{O}_F(l, -l) \oplus \mathcal{O}_F(-l, l) \).

**Proof.** Suppose that \( H^0(F, \mathcal{E}(l, -l)) \neq 0 \) for \( l \in \mathbb{Z} \). Therefore \( \mathcal{E} \) lies on an exact sequence of the form

\[
0 \to \mathcal{O}_F \to \mathcal{E}(l, -l) \to \mathcal{I}_Y(2l, -2l) \to 0.
\]

where \( Y \subseteq F \). Given that \( H^0(F, \mathcal{E}(l - 1, -l)) = H^0(F, \mathcal{E}(l, -l - 1)) = 0 \), \( Y \) is purely 2-codimensional or empty. In order to see that the we are actually dealing with the latter case, notice that, since \( \mathcal{E} \) is Gieseker semistable, we have

\[
p_{\mathcal{O}_F}(l) \leq p_{\mathcal{E}(l, -l)}(l) \leq p_{\mathcal{I}_Y(2l, -2l)}(t) = p_{\mathcal{O}_F(2l, -2l)}(t) - p_{\mathcal{I}_Y(2l, -2l)}(t)
\]

where \( p(t) \) stands for the Hilbert polynomial. But then

\[
p_{\mathcal{I}_Y(2l, -2l)}(t) \leq p_{\mathcal{O}_F(2l, -2l)}(t) - p_{\mathcal{O}_F}(t) = (t+1)(t+1+2l)(t+1-2l)-(t+1)^3 = -4l^2(t+1) < 0 \quad \text{for } t >> 0,
\]

contradicting Serre’s theorem. Therefore \( Y = \emptyset \), \( l \neq 0 \) and \( \mathcal{E} \) is an extension of the desired form.
To prove the last assertion, let us suppose that there exists a \(l^2\)-instanton \(E\) that fits at the same time in the extensions \(\Lambda_l\) and \(\Lambda_{-l}\). Then easily we could construct maps:

\[
O(-l,l) \oplus O(l,-l) \xrightarrow{(\alpha \beta)} E \xrightarrow{\delta_i} O(-l,l) \oplus O(l,-l)
\]

such that \((\delta_i)_{(\alpha \beta)} = (\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix})\) and therefore \(E \cong \mathcal{O}_F(-l,-l) \oplus \mathcal{O}_F(-l,l)\).

\[\square\]

**Lemma 3.8.** Any \(k\)-instanton \(E\) vector bundle on \(F\) distinct from \(\mathcal{O}_F(-l,l) \oplus \mathcal{O}_F(-l,l)\) is simple, namely, \(\text{End}(E) = \mathbb{C}\). In particular, they carry an unique symplectic structure \(\phi : E \to E^\vee\), \(\phi^\vee = -\phi\).

**Proof.** Since it is well-known that stable vector bundles are simple, let \(E\) be an strictly semistable instanton, \(E \cong \mathcal{O}_F(-l,-l) \oplus \mathcal{O}_F(-l,l)\). Then it fits in an exact sequence of the form \(\Lambda_l\) for a unique \(0 \neq l \in \mathbb{Z}\). Tensoring it with \(E\) and using that we have just seen that \(h^0(F,E(-l,l)) = 1\) and \(h^0(F,E(l,-l)) = 0\) we get, since \(E \cong E^\vee\), \(\text{End}(E) = H^0(F,E \otimes E) = \mathbb{C}\). For the last statement, any nonzero two-form \(\omega \in \bigwedge^2 E \cong \mathcal{O}\) defines equivalent symplectic structures on \(E\).

\[\square\]

We are going to conclude this section showing that instanton bundles have trivial splitting type on conics:

**Proposition 3.9.** Let \(E\) be a \(k\)-instanton on \(F\). Then for a general conic \(C \in \text{Hilb}^{2+1}(F)\), \(E_C \cong \mathcal{O}_C^2\).

**Proof.** We are going to use the main result from [15]. Since the three hypothesis of the main Theorem from that paper are satisfied, we can conclude that for a general (smooth) conic \(C \in \text{Hilb}^{2+1}(F)\), if \(E_C \cong \mathcal{O}_C(-i) \oplus \mathcal{O}_C(i)\) then \(2i \leq -\mu_{\text{min}}(\mathcal{T}_C\big|_F \mid C)\) where \(\mathcal{T}_C\big|_F\) is the relative tangent bundle of the canonical projection \(p : C \to F\) and \(\mu_{\text{min}}\) is the minimal slope of the Harder-Narasimhan filtration of its restriction to \(C\). This bundle fits into the exact sequence:

\[
0 \longrightarrow \mathcal{T}_C\big|_F \longrightarrow \mathcal{T}_C \longrightarrow p^* \mathcal{T}_F \longrightarrow 0.
\]

On the other hand, the exact sequence defining the normal bundle \(\mathcal{N}_{C\mid(F \times M)}\) restricted to \(C\):

\[
0 \longrightarrow \mathcal{T}_C \big|_C \longrightarrow \mathcal{T}_{(F \times M)} \big|_C \cong \mathcal{O}_C^4 \oplus \mathcal{O}_C(1)^2 \mathcal{O}_C(2) \longrightarrow \mathcal{N}_{C\mid(F \times M)} \big|_C \cong \mathcal{O}_C(1)^2 \longrightarrow 0.
\]

provides \(\mathcal{T}_C \big|_C \cong \mathcal{O}_C^4 \oplus \mathcal{O}(2)\). Moreover, \(\mathcal{T}_F \big|_C \cong \mathcal{O}_C(1)^2 \oplus \mathcal{O}_C(2)\). If we plug this information into the restriction to \(C\) of the exact sequence (3), we obtain \(\mathcal{T}_{C\big|_F} \big|_C \cong \mathcal{O}_C(-1)^2\), namely \(i = 0\) and \(C\) has trivial splitting type.

\[\square\]

4. The derived category of the flag variety \(F = F(0,1,2)\)

In this section we will recall how to mutate exceptional collections and the Beilinson results, both needed to see the instanton bundles as cohomology of a monad.

Given a smooth projective variety \(X\), let \(D^b(X)\) be the the bounded derived category of coherent sheaves on \(X\). An object \(E \in D^b(X)\) is called exceptional if \(\text{Ext}^i(E,E) = \mathbb{C}\). A set of exceptional objects \(E_1, \ldots, E_n\) is called an exceptional collection if \(\text{Ext}^i(E_i,E_j) = 0\) for any distinct \(i \neq j\).
for $i > j$. An exceptional collection is full when $\text{Ext}^i(E_i, A) = 0$ for all $i$ implies $A = 0$, or equivalently when $\text{Ext}^i(A, E_i) = 0$ for all $i$ does.

**Definition 4.1.** Let $E$ be an exceptional object in $D^b(X)$. Then there are functors $\mathbb{L}_E$ and $R_E$ fitting in distinguished triangles

$$
\mathbb{L}_E(T) \to \text{Ext}^\bullet(E, T) \otimes E \to T \to \mathbb{L}_E(T)[1]
$$

$$
R_E(T)[-1] \to T \to \text{Ext}^\bullet(T, E)^* \otimes E \to R_E(T)
$$

The functors $\mathbb{L}_E$ and $R_E$ are called respectively the left and right mutation functor.

The collections given by

$$
E_i^\vee = \mathbb{L}_E E_0 \mathbb{L}_E E_1 \cdots \mathbb{L}_E E_{n-1} E_{n-i};
$$

$$
\vee E_i = R_E R_{E_{n-1}} \cdots R_{E_{n-i+1}} E_{n-i},
$$

are again full and exceptional and are called the right and left dual collections. The dual collections are characterized by the following property; see [14, Section 2.6].

$$
\text{Ext}^k(\vee E_i, E_j) = \text{Ext}^k(E_i, E_j^\vee) = \begin{cases} C & \text{if } i + j = n \text{ and } i = k \\ 0 & \text{otherwise} \end{cases}
$$

**Theorem 4.2** (Beilinson spectral sequence). Let $X$ be a smooth projective variety and with a full exceptional collection $\langle E_0, \ldots, E_n \rangle$ of objects for $D^b(X)$. Then for any $A$ in $D^b(X)$ there is a spectral sequence with the $E_1$-term

$$
E_1^{p,q} = \bigoplus_{r+s=q} \text{Ext}^{n+r}(E_{n-p}, A) \otimes \mathcal{H}^s(E_p^\vee)
$$

which is functorial in $A$ and converges to $\mathcal{H}^{p+q}(A)$.

The statement and proof of Theorem 4.2 can be found both in [22, Corollary 3.3.2], in [14, Section 2.7.3] and in [4, Theorem 2.1.14].

Let us assume next that the full exceptional collection $\langle E_0, \ldots, E_n \rangle$ contains only pure objects of type $E_i = \mathcal{E}_i^*[-k_i]$ with $\mathcal{E}_i$ a vector bundle for each $i$, and moreover the right dual collection $\langle E_0^\vee, \ldots, E_n^\vee \rangle$ consists of coherent sheaves. Then the Beilinson spectral sequence is much simpler since

$$
E_1^{p,q} = \text{Ext}^{n+q}(E_{n-p}, A) \otimes E_p^\vee = H^{n+q+k_{n-p}}(\mathcal{E}_{n-p} \otimes A) \otimes E_p^\vee.
$$

Note however that the grading in this spectral sequence applied for the projective space is slightly different from the grading of the usual Beilinson spectral sequence, due to the existence of shifts by $n$ in the index $p, q$. Indeed, the $E_1$-terms of the usual spectral sequence are $H^q(A(p)) \otimes \Omega^{-p}(-p)$ which are zero for positive $p$. To restore the order, one needs to change slightly the gradings of the spectral sequence from Theorem 4.2. If we replace, in the expression

$$
E_1^{u,v} = \text{Ext}^u(E_{-u}, A) \otimes E_{n-u}^\vee = H^{u+k_{-u}}(\mathcal{E}_{-u} \otimes A) \otimes F_{-u}
$$

$u = -n + p$ and $v = n + q$ so that the fourth quadrant is mapped to the second quadrant, we obtain the following version of the Beilinson spectral sequence:
Theorem 4.3. Let $X$ be a smooth projective variety with a full exceptional collection $(E_0, \ldots, E_n)$ where $E_i = \mathcal{E}_i^{[*-k_i]}$ with each $E_i$ a vector bundle and $(k_0, \ldots, k_n) \in \mathbb{Z}^{\oplus n+1}$ such that there exists a sequence $(F_n = F_{n+1}, \ldots, F_0 = F_0)$ of vector bundles satisfying

$$\text{Ext}^k(E_i, F_j) = H^{k+k_i}(\mathcal{E}_i \otimes \mathcal{F}_j) = \begin{cases} \mathbb{C} & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

i.e. the collection $(F_n, \ldots, F_0)$ labelled in the reverse order is the right dual collection of $(E_0, \ldots, E_n)$. Then for any coherent sheaf $A$ on $X$ there is a spectral sequence in the square $-n \leq p \leq 0$, $0 \leq q \leq n$ with the $E_1$-term

$$E_1^{p,q} = \text{Ext}^q(E_{-p}, A) \otimes F_{-p} = H^{q+k-p}(\mathcal{E}_{-p} \otimes A) \otimes \mathcal{F}_{-p}$$

which is functorial in $A$ and converges to

$$E_\infty^{p,q} = \begin{cases} A & \text{if } p + q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us focus on our case. We can define the flag as $F = \mathbb{P}(\Omega_{\mathbb{P}^2}(1))$ and we also may consider $F$ as hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. Consider the exceptional collection

$$\{\mathcal{O}_F(-2, -2), \mathcal{O}_F(-2, -1), \mathcal{O}_F(-1, -2), \mathcal{O}_F(-2, 0), \mathcal{O}_F(0, -2), \mathcal{O}_F(0, -1), \mathcal{O}_F(1), \mathcal{O}_F(0, -1), \mathcal{O}_F\}$$

generating the derived category of $\mathbb{P}^2 \times \mathbb{P}^2$, therefore, the derived category of the flag variety can be generated by the exceptional collection

$$\{\mathcal{O}_F(-2, 0), \mathcal{O}_F(0, -2), \mathcal{O}_F(-1, -1), \mathcal{O}_F(-1, 0), \mathcal{O}_F(0, -1), \mathcal{O}_F\}$$

After two left mutations, we can obtain

$$\{\mathcal{O}_F(-1, -1)[-2], \mathcal{O}_F(-2, 0), \mathcal{O}_F(0, -2), \mathcal{O}_F(-1, 0), \mathcal{O}_F(0, -1), \mathcal{O}_F\}$$

After a few other mutations we get

$$\{\mathcal{O}_F(-1, -1)[-2], \mathcal{G}_2(-1, -1)[-2], \mathcal{G}_1(-1, -1)[-1], \mathcal{O}_F(-1, 0)[-1], \mathcal{O}_F(0, -1), \mathcal{O}_F\}$$

and by another left mutation we obtain

$$\{\mathcal{O}_F(-1, -1)[-2], \mathcal{G}_2(-1, -1)[-2], \mathcal{G}_1(-1, -1)[-1], \mathcal{O}_F(-1, 0)[-1], \mathcal{G}_2(0, -2), \mathcal{O}_F(0, -1).\}$$

Now we consider the pair $\mathcal{O}_F(-1, 0)[-1], \mathcal{G}_2(0, -2)$, from

$$\mathcal{L}_{\mathcal{O}_F(-1, 0)[-1]}(\mathcal{G}_2(0, -2)) \rightarrow \text{Ext}^1(\mathcal{O}_F(-1, 0)[1], \mathcal{G}_2(0, -2)) \otimes \mathcal{O}_F(-1, 0)[-1] \rightarrow \mathcal{G}_2(0, -2) \rightarrow \mathcal{L}_{\mathcal{O}_F(-1, 0)[-1]}(\mathcal{G}_2(0, -2)) [1]$$

we get

$$0 \rightarrow \mathcal{G}_2(0, -2) \rightarrow \mathcal{L}_{\mathcal{O}_F(-1, 0)[-1]}(\mathcal{G}_2(0, -2)) [1] \rightarrow \text{Ext}^1(\mathcal{O}_F(-1, 0), \mathcal{G}_2(0, -2)) \otimes \mathcal{O}_F(-1, 0) \rightarrow 0$$

and from (5) we obtain

$$\mathcal{L}_{\mathcal{O}_F(-1, 0)[-1]}(\mathcal{G}_2(0, -2)) = \mathcal{G}_4[-1].$$

Hence we have the following exceptional collection

$$\{\mathcal{O}_F(-1, -1)[-2], \mathcal{G}_2(-1, -1)[-2], \mathcal{G}_1(-1, -1)[-1], \mathcal{G}_4[-1], \mathcal{O}_F(-1, 0)[-1], \mathcal{O}_F(0, -1)\}$$

which will be the one considered in Theorem 5.2.
5. Monads on the Flag Variety \( F = F(0,1,2) \)

In this section we use the Beilinson Theorem in order to characterize the instanton bundles as the cohomology of two different monads, describe the relation between them and use it to give a presentation of the moduli space \( MI_F(k) \) as a GIT-quotient.

**Theorem 5.1.** Let \( \mathcal{E} \) be an instanton bundle with charge \( k \) on \( F \), then, up to permutation, \( \mathcal{E} \) is the homology of a monad

\[
0 \to \mathcal{O}_F(-1,0)^{\oplus k} \oplus \mathcal{O}_F(0, -1)^{\oplus k} \xrightarrow{\alpha} \mathcal{G}_1(-1,0)^{\oplus k} \oplus \mathcal{G}_2(0, -1)^{\oplus k} \xrightarrow{\beta} \mathcal{O}_F^{\oplus 2k-2} \to 0, \tag{14}
\]

**Proof.** We construct a Beilinson complex quasi-isomorphic to \( \mathcal{E} \) as in Theorem 4.3 from (12) by calculating \( H^i(\mathcal{E} \otimes \mathcal{E}_j) \otimes F_j \), with \( i,j \in \{0,1,2,3,4,5\} \) with

\[
E_5 = \mathcal{O}_F(-1,-1)[-2], E_4 = \mathcal{G}_2(-1,-1)[-2], E_3 = \mathcal{G}_1(-1,-1)[-1],
\]
\[
E_2 = \mathcal{O}_F(-1,0)[-1], E_1 = \mathcal{O}_F(0,-1), E_0 = \mathcal{O}_F
\]

and

\[
F_0 = \mathcal{O}_F, F_1 = \mathcal{G}_2(0,-1), F_2 = \mathcal{G}_1(-1,0), F_3 = \mathcal{O}_F(0,-1), F_4 = \mathcal{O}_F(0,-1), F_5 = \mathcal{O}_F(-1,-1)
\]

Let us notice that the cohomological properties in (10) are satisfied, therefore, our goal is to find all possible information on the following table

| \( H^3 \) | \( H^3 \) | 0 | 0 | 0 | 0 |
| --- | --- | --- | --- | --- | --- |
| \( H^2 \) | \( H^2 \) | \( H^3 \) | \( H^3 \) | \( H^3 \) | \( H^3 \) |
| \( H^1 \) | \( H^1 \) | \( H^2 \) | \( H^2 \) | \( H^2 \) | \( H^2 \) |
| \( H^0 \) | \( H^0 \) | \( H^1 \) | \( H^1 \) | \( H^1 \) | \( H^1 \) |
| 0 | 0 | \( H^0 \) | \( H^0 \) | \( H^0 \) | \( H^0 \) |
| 0 | 0 | 0 | 0 | 0 | 0 |

| \( \mathcal{E}(-1,-1) \) | \( \mathcal{E} \otimes \mathcal{G}_2(-1,-1) \) | \( \mathcal{E} \otimes \mathcal{G}_1(-1,-1) \) | \( \mathcal{E}(-1,0) \) | \( \mathcal{E}(0,-1) \) | \( \mathcal{E} \) |

From the cohomological hypothesis it is straightforward to compute that \( H^i(\mathcal{E}(-1,-1)) = 0 \), for \( i = 0,1,2,3 \).

Considering the short exact sequences

\[
0 \to \mathcal{E} \otimes \mathcal{G}_1(-1,-1) \to \mathcal{E}(0,-1)^3 \to \mathcal{E}(1,-1) \to 0
\]

and

\[
0 \to \mathcal{E}(-2,-1) \to \mathcal{E}(-1,-1)^3 \to \mathcal{E} \otimes \mathcal{G}_1(-1,-1) \to 0
\]

we obtain directly

\[
H^0(\mathcal{E} \otimes \mathcal{G}_1(-1,-1)) = H^3(\mathcal{E} \otimes \mathcal{G}_1(-1,-1)) = 0,
\]

moreover, \( H^2(\mathcal{E} \otimes \mathcal{G}_1(-1,-1)) \simeq H^3(\mathcal{E}(-2,-1)) \simeq H^0(\mathcal{E}(0,-1)) = 0 \), hence we only need to compute

\[
H^1(\mathcal{E} \otimes \mathcal{G}_1(-1,-1)) \simeq H^2(\mathcal{E}(-2,-1)) \simeq H^1(\mathcal{E}(0,-1)).
\]

Using (4) to compute the Euler characteristic, we get

\[
-h^1(\mathcal{E}(0,-1)) = \chi(\mathcal{E}(0,-1)) = 2 + \frac{1}{6}(6k) + \frac{1}{2}(2 - 4k) + \frac{1}{12}(-36) = -k.
\]
In the same way, we obtain that
\[ h^i(\mathcal{E} \otimes \mathcal{G}_2(-1,-1)) = \begin{cases} k & \text{if } i = 1, \\ 0 & \text{if } i = 0,2,3. \end{cases} \]

Using the previous computation, we also get that
\[ h^i(\mathcal{E}(0,-1)) = h^i(\mathcal{E}(-1,0)) = \begin{cases} k & \text{if } i = 1, \\ 0 & \text{if } i = 0,2,3. \end{cases} \]

Considering the short exact sequence
\[ 0 \rightarrow \mathcal{E}(-2,-2) \rightarrow \mathcal{E} \otimes \mathcal{G}_1(-3,-1) \rightarrow \mathcal{E}(-3,0) \rightarrow 0 \]
and
\[ 0 \rightarrow \mathcal{E} \otimes \mathcal{G}_1(-3,-1) \rightarrow \mathcal{E}(-2,-1) \rightarrow \mathcal{E}(-1,-1) \rightarrow 0 \]
we can calculate \( H^1(\mathcal{E} \otimes \mathcal{G}_1(-3,-1)) \simeq H^1(\mathcal{E}(-2,-2)) \simeq H^2(\mathcal{E}) = 0 \). Moreover
\[ -h^1(\mathcal{E}) = \chi(\mathcal{E}) = 2 - 2k. \]

Finally, the vanishing \( H^0(\mathcal{E}) = 0 \) implies \( H^3(\mathcal{E}) = 0 \).

Due to all previous computations, the cohomology table becomes

| \( \mathcal{E}(-1,-1) \) | \( \mathcal{E} \otimes \mathcal{G}_2(-1,-1) \) | \( \mathcal{E} \otimes \mathcal{G}_1(-1,-1) \) | \( \mathcal{E}(-1,0) \) | \( \mathcal{E}(0,-1) \) | \( \mathcal{E} \) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | \( k \) | 0 | 0 | 0 | 0 |
| 0 | 0 | \( k \) | 0 | 0 | 0 |
| 0 | 0 | 0 | \( k \) | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 2k - 2 |

proving the result. □

Let us give a second description on an instanton bundles through a monad which indeed, will be used for the presentation of the moduli space \( MI_F(k) \). Later, we will explicit the relation between the two monads.

**Theorem 5.2.** Let \( \mathcal{E} \) instanton bundle with charge \( k \) on \( X \), then, up to permutation, \( \mathcal{E} \) is the cohomology of a monad

\[ (15) \quad 0 \rightarrow \mathcal{O}_F(0,-1)^{\oplus k} \oplus \mathcal{O}_F(-1,0)^{\oplus k} \xrightarrow{\alpha} \mathcal{O}_F(1,0)^{\oplus k} \oplus \mathcal{O}_F(0,1)^{\oplus k} \rightarrow 0. \]

Moreover, the monad obtained is self-dual, i.e. it is possible to find a non degenerate symplectic form \( q : W \rightarrow W^* \), with \( W \) a \((4k + 2)\)-dimensional vector space describing the copies of the trivial bundle in the monad, such that \( \beta = \alpha^\vee \circ (q \otimes \text{id}_{\mathcal{O}_F}) \).

**Proof.** We construct a Beilinson complex quasi-isomorphic to \( \mathcal{E} \), by calculating \( H^i(\mathcal{E} \otimes E_j) \otimes F_j \) as in Theorem 4.3 from [10], with \( i,j \in \{0,1,2,3,4,5\} \), where

\[ E_5 = \mathcal{O}_F(-1,-1)[-2], E_4 = \mathcal{G}_2(-1,-1)[-2], E_3 = \mathcal{G}_1(-1,-1)[-1], \]
\[ E_2 = \mathcal{G}_4[-1], E_1 = \mathcal{O}_F(0,-1)[-1], E_0 = \mathcal{O}_F(-1,0), \]

and

\[ F_0 = \mathcal{O}_F(1,0), F_1 = \mathcal{O}_F(0,1), F_2 = \mathcal{O}_F, F_3 = \mathcal{O}_F(0,-1), F_4 = \mathcal{O}_F(-1,0), F_5 = \mathcal{O}_F(-1,-1). \]
We have computed, from the previous result, all the entries of the cohomology table except one column, hence we already have the following:

|       | $E(-1,-1)$ | $E \otimes G_2(-1,-1)$ | $E \otimes G_1(-1,-1)$ | $E \otimes G_4$ | $E(0,-1)$ | $E(-1,0)$ |
|-------|------------|-------------------------|-------------------------|----------------|-----------|-----------|
| $0$   | $0$        | $0$                     | $0$                     | $0$            | $0$       | $0$       |
| $0$   | $0$        | $0$                     | $H^3$                   | $0$            | $0$       | $0$       |
| $0$   | $k$        | $0$                     | $H^2$                   | $0$            | $0$       | $0$       |
| $0$   | $0$        | $k$                     | $H^1$                   | $k$            | $0$       |           |
| $0$   | $0$        | $0$                     | $H^0$                   | $0$            | $k$       |           |
| $0$   | $0$        | $0$                     | $0$                     | $0$            | $0$       | $0$       |

Since

$$\chi(E \otimes G_2(0,-2)) = \chi(E(0,-1)^{\otimes 3}) - \chi(E) = -3k + 2k - 2 = -2 - k$$

and by the sequence (9) tensored by $E$ we have

$$\chi(E \otimes G_4) = \chi(E(-1,0)^{\otimes 3}) + \chi(E \otimes G_2(0,-2)) = -3k - 2 - k = -4k - 2.$$ 

This means that $H^i(E \otimes G_4) = 0$ for $i \neq 1$ and we obtain

|       | $E(-1,-1)$ | $E \otimes G_2(-1,-1)$ | $E \otimes G_1(-1,-1)$ | $E \otimes G_4$ | $E(0,-1)$ | $E(-1,0)$ |
|-------|------------|-------------------------|-------------------------|----------------|-----------|-----------|
| $0$   | $0$        | $0$                     | $0$                     | $0$            | $0$       | $0$       |
| $0$   | $0$        | $0$                     | $0$                     | $0$            | $0$       | $0$       |
| $0$   | $k$        | $0$                     | $0$                     | $0$            | $0$       | $0$       |
| $0$   | $0$        | $k$                     | $4k + 2$                | $k$            | $0$       |           |
| $0$   | $0$        | $0$                     | $0$                     | $0$            | $k$       |           |
| $0$   | $0$        | $0$                     | $0$                     | $0$            | $0$       | $0$       |

getting the desired monad.

From previous computations in cohomology, we know that $H^1(E \otimes G_1(-1,-1)) \cong H^1(E(0,-1))^*$ and analogously $H^1(E \otimes G_2(-1,-1)) \cong H^1(E(-1,0))^*$. Denote by $I_1$ the vector space of the first isomorphism and by $I_2$ the vector space of the second one, and denote by $W$ the cohomology group $H^1(E \otimes G_4)$.

The vector bundles in the obtained monad satisfy the required cohomological conditions in order to have a bijection between homomorphism of monads and the induced homomorphism in the cohomology bundles (see for example Lemma 4.1.3 in [21]). Recalling the $E$ carries a unique symplectic structure, a consequence of the cited result gives us two isomorphisms

$$q : W \otimes O_F \rightarrow W^* \otimes O_F$$

and

$$h : (I_1 \otimes O_F(1,0)) \oplus (I_2 \otimes O_F(0,1)) \rightarrow (I_1 \otimes O_F(1,0)) \oplus (I_2 \otimes O_F(0,1))$$

such that

$$q^\vee = -q \text{ and } h \circ \beta = \alpha^\vee \circ q.$$

This implies that the monad (15) is self-dual, therefore we have that $\beta = \alpha^\vee \circ (q \otimes id_{O_F})$.

**Remark 5.3.** If $k = 1$ the instanton bundle $E$ twisted by $O_F(1,1)$ is an Ulrich bundle with $c_1(E(1,1)) = (2,2)$ and $c_2(E(1,1)) = 4h_1h_2$ which is $\mu$-stable unless $E$ arises (up to permutation) from an extension.
0 \rightarrow \mathcal{O}_F(1, -1) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_F(-1, 1) \rightarrow 0,

see \cite{9}. Moreover since \( \mathcal{E}(1, 1) \) is Ulrich on a Del Pezzo Threefold, hence, by \cite{8}, we get

\[ \text{Ext}^2(\mathcal{E}, \mathcal{E}) = 0. \]

**Remark 5.4.** As for the instanton bundles on the projective space, the two monads defining instanton bundles on the flag variety are extremely related. Indeed, let us describe the monad defined in Theorem 5.2 with the short exact sequences

\[ 0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}^{4k+2} \rightarrow \mathcal{O}(1, 0)^{\oplus k} \oplus \mathcal{O}(0, 1)^{\oplus k} \rightarrow 0 \]

(16)

\[ 0 \rightarrow \mathcal{O}(-1, 0)^{k} \oplus \mathcal{O}(0, -1)^{k} \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0 \]

The first sequence fits in the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{O}^{\oplus 4k+2} & \xrightarrow{\beta} & \mathcal{O}(1, 0)^{\oplus k} \oplus \mathcal{O}(0, 1)^{\oplus k} \\
\downarrow & & \downarrow \\
\mathcal{O}(1, 0)^{\oplus k} \oplus \mathcal{O}(0, 1)^{\oplus k} & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{O}(1, 0)^{\oplus k} & \xrightarrow{\beta'} & \mathcal{O}(0, 1)^{\oplus k} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

Indeed, we can complete the matrix representing \( \beta \) with columns of the required forms of degree 1 in either the \( x_i \)'s or \( y_i \)'s, in order to keep \( \ker \beta' \) with no global sections. Starting from this diagram, it is possible to induce the following one

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{O}(-1, 0)^{k} \oplus \mathcal{O}(0, -1)^{k} \\
\downarrow & & \downarrow \\
\mathcal{O}(-1, 0)^{k} \oplus \mathcal{O}(0, -1)^{k} & \rightarrow & \mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{K} & \rightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathcal{E} & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{G}_1(-1, 0)^{k} \oplus \mathcal{G}_2(0, -1)^{k} \\
\downarrow & & \downarrow \\
\mathcal{G}_1(-1, 0)^{k} \oplus \mathcal{G}_2(0, -1)^{k} & \rightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{O}^{\oplus 2k-2} \\
\downarrow & & \downarrow \\
\mathcal{O}^{\oplus 2k-2} & \rightarrow & 0 \\
\end{array}
\]

which can be seen as the display of the monad defined in Theorem 5.1.
Remark 5.5. The definition of the instanton bundle through the presented monads implies the cohomological conditions asked for a mathematical instanton bundle, i.e. the two definition are equivalent. Indeed, considering the display of the monad described in (11) and the cohomology group dimensions given in Proposition 1.1, we compute $H^0(O_F(0, -1)) = H^0(O_F(0, -1)) = H^1(O_F(-1, -1))$, which imply the vanishing of $H^1(K(-1, -1))$, that, combined with $H^2(O_F(-2, -1)) = H^2(O_F(-2, -1)) = 0$, gives us that $H^1(E(-1, -1)) = 0$. Moreover, for each pair $(p, q)$ such that $p + q < 0$, we get $H^0(O_F(p, q)) = H^1(O_F(p - 1, q)) = H^1(O_F(p, q - 1)) = 0$, which implies $H^0(E(p, q)) = 0$, hence the semistability of the bundle $E$.

The previous presentation allows us to prove the following result (compare with [10, Section 2]

**Theorem 5.6.** The moduli space $MI_F(k)$ is the GIT quotient $\mathfrak{D}_k^0 / G_k$. It is a geometric quotient.

**Proof.** We just saw that isomorphic classes of instanton bundles are in one-to-one correspondence with $G_k$-orbits in $\mathfrak{D}_k^0$. Moreover, their isotropic group is $\Lambda := \pm(id_{Sp(W,F)}, id_{GL(I_1)}, id_{GL(I_2)})$. Therefore, $G_k/\Lambda$ acts freely on $\mathfrak{D}_k^0$ and in particular all orbits are closed: $\mathfrak{D}_k^0 / (G_k/\Lambda)$ is a geometric quotient. $\blacksquare$

In general, it is a challenging question to know whether a given moduli space is an affine variety. For the case of instanton bundles on $\mathbb{P}^2n+1$ a positive answer was given in [10].

Following [10], we are going to call a matrix $A^t \in Hom((U \otimes I_1) \oplus (U^* \otimes I_2), W)$ degenerate if there exists $u_1 \in U, u_2 \in U^*$, such that $w_2(u_1) = 0$ and such that the induced linear map $A^t((- \otimes u_1) \oplus (- \otimes u_2)) : I_1^\ast \oplus I_2 \rightarrow W$ is non-injective. It is straightforward to show that $A \in W^* \otimes ((I_1 \otimes U) \oplus (I_2 \otimes U^*))$ defines a surjective map if and only if $A^t$ is nondegenerate. Therefore we can conclude that $\mathfrak{D}_k^0 = \{ A \in \mathfrak{D}_k \mid A^t \text{ is nondegenerate} \}$. With these ingredients we are able to prove:

**Proposition 5.7.** $MI_F(1)$ is an affine variety.

**Proof.** Notice that in this case, the map $A : W \otimes O_F \rightarrow (I_1 \otimes O_F(1, 0)) \oplus (I_2 \otimes O_F(0, 1))$ becomes, at the level of global sections, a square $(6 \times 6)$-matrix associated to the map $H^0(A) : W \rightarrow U \oplus U^\ast$. Let us denote by $D(A)$ the usual determinant of this map. Notice that $D$ is $G_k \times GL(U)$-invariant. Let us show that, for $A \in \mathfrak{D}_1, A \in \mathfrak{D}_1^0$ if and only if $D(A) \neq 0$.

First, obviously if $A$ is degenerate, then $D(A) = 0$. Reciprocally, if $A \in \mathfrak{D}_1^0 A$ is defining an instanton bundle $E$ with associated first exact sequence at the display

$$0 \rightarrow O_F(-1, 0) \oplus O_F(0, -1) \rightarrow E \rightarrow K \rightarrow 0,$$

we obtain that $H^0(K) = H^0(E) = 0$ and therefore, taking global sections at the second exact sequence

$$0 \rightarrow H^0(K) = 0 \rightarrow H^0(O_F^0) \cong \mathbb{C}^6 \overset{H^0(A)}{\longrightarrow} H^0(O_F(1, 0) \oplus O_F(0, 1)) \cong \mathbb{C}^6 \rightarrow 0,$$

we obtain $H^0(A)$ injective and $D(A) \neq 0$.

Therefore, we obtain that $\mathfrak{D}_1^0 = \{ A \in \mathfrak{D}_1 \mid D(A) \neq 0 \}$ is an affine variety $Spec(A)$. Therefore, by a Theorem by Hilbert-Nagata, $MI_F(1) = \mathfrak{D}_1^0 / G_k = Spec(A^{G_k})$ is also affine.

$\blacksquare$
**Conjecture 5.8.** $\text{MI}_F(k)$ is affine for any $k$.

6. Construction of instantons

In this section we will construct, through an induction process, stable $k$-instanton bundles on the flag variety for each charge $k$. More concretely, we are going to prove the following

**Theorem 6.1.** Let $F \subset \mathbb{P}^7$ be the flag variety. The moduli space of stable $k$-instanton bundles $\text{MI}_F(k)$ has a generically smooth irreducible component of dimension $8k - 3$. Moreover, these instanton bundles have trivial splitting type on the generic conic and line (from both families).

We will start describing completely the case of charge 1 that is the base case for the induction. We give two approaches to this case: first we give a complete explicit characterization in terms of the monad defining them. Next, we give a more abstract construction using Serre correspondence. Finally, we take care of the induction step.

6.1. Base case of induction. Let us consider the case of charge 1, i.e. a vector bundle which is cohomology of the monad

\[ O_F(-1,0) \oplus O_F(0,-1) \xrightarrow{A} O_F^6 \xrightarrow{B} O_F(1,0) \oplus O_F(0,1) \]

The matrix $B$, up to a change of coordinates and action of the linear groups involved in the monad, is of the form

\[
\begin{bmatrix}
  x_0 & x_1 & x_2 & 0 & 0 & 0 \\
  0 & 0 & 0 & y_0 & y_1 & y_2 \\
\end{bmatrix}
\]

where we can use the coordinates $(x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)$ of the product $\mathbb{P}^2 \times \mathbb{P}^2$. Notice that we can think the entries of the matrix in such coordinates, we just need to consider that they satisfy the equation defining the flag variety, which we can suppose to be of the form $x_0y_0 + x_1y_1 + x_2y_2 = 0$.

This means that $\ker B \simeq G_1(-1,0) \oplus G_2(0,-1)$, and necessarily, the other matrix is of the form

\[
A = \begin{bmatrix}
  f_1 & \gamma y_0 \\
  f_2 & \gamma y_1 \\
  f_3 & \gamma y_2 \\
  \delta x_0 & g_1 \\
  \delta x_1 & g_2 \\
  \delta x_2 & g_3 \\
\end{bmatrix}
\]

where $(f_1 \ f_2 \ f_3)^t$ and $(g_1 \ g_2 \ g_3)^t$ are syzygies of, respectively, $(x_0 \ x_1 \ x_2)$ and $(y_0 \ y_1 \ y_2)$, and $\gamma, \delta \in \mathbb{C}$. Notice that it completely agrees with the fact that the family of charge 1 instanton is 5-dimensional; indeed, each of the two syzygies is 3-dimensional as vector space, and we still have to apply the action of the 2-dimensional linear group of the automorphisms of $O_F(-1,0) \oplus O_F(0,-1)$.

The charge 1 instanton bundles $\mathcal{E}$ restrict trivially on the generic conic and for the generic line for each of the two families. Moreover, we have $\text{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$, because they are Ulrich if stable. Therefore, we obtained the perfect candidate to start the induction process described in the next section. Furthermore, the following result characterizes the semistable case.
Proposition 6.2. Let $\mathcal{E}$ be a charge 1 instanton on the flag variety. Then the bundle is semistable if and only if at least one of two syzygies is zero. Moreover, it splits as $\mathcal{E} \cong \mathcal{O}_F(-1,1) \oplus \mathcal{O}_F(1,-1)$ if and only if both $f_i$’s and $g_i$’s are zero.

Remark 6.3. Vanishing either the $f_i$’s or the $g_i$’s corresponds with the choice of one of two possible families given by the extension, i.e.

$$0 \longrightarrow \mathcal{O}_F(1,-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(-1,1) \longrightarrow 0$$

(17)

$$0 \longrightarrow \mathcal{O}_F(-1,1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(1,-1) \longrightarrow 0$$

Proof. Let us recall the short exact sequences

$$0 \longrightarrow \mathcal{O}_F(-1,0) \longrightarrow \mathcal{G}_2(0,-1) \longrightarrow \mathcal{O}_F(1,-1) \longrightarrow 0$$

(18)

$$0 \longrightarrow \mathcal{O}_F(0,-1) \longrightarrow \mathcal{G}_1(-1,0) \longrightarrow \mathcal{O}_F(-1,1) \longrightarrow 0$$

Also recall that we have already proven that $\mathcal{E}$ is semistable if only if it is one of the extensions described in (17). Consider the first one; it occurs if and only $\mathcal{E}$ fits in the following diagram

We have the previous commutative diagram if and only if the syzygy given by $(g_1 \ g_2 \ g_3)$ is equal to zero.

Analogously, the other extension appear if and only if the syzygy given by $(f_1 \ f_2 \ f_3)$ is equal to zero.

Finally, it comes directly from the previous diagram that $\mathcal{E}$ splits as $\mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1)$ if and only if both syzygies vanish.

Example 6.4. We show a specific example of instanton of charge 2, which we will know to be stable because its second Chern class is not a perfect square (see Proposition 7). Indeed, consider the monad

$$\mathcal{O}_F(-1,0)^{\oplus 2} \oplus \mathcal{O}_F(0,-1)^{\oplus 2} \xrightarrow{A} \mathcal{O}_F^{10} \xrightarrow{B} \mathcal{O}_F(1,0)^{\oplus 2} \oplus \mathcal{O}_F(0,1)^{\oplus 2}$$
with the matrices defining the maps are given by

\[
A = \begin{bmatrix}
-y_0 & 0 & -x_0 - x_2 & 0 \\
-y_1 & 0 & -x_2 & -x_0 \\
0 & 0 & 0 & -x_2 \\
y_0 & y_2 & 0 & -x_1 \\
y_2 & y_0 & 0 & 0 \\
y_0 + y_1 & y_2 & 0 & -x_1 \\
0 & y_1 & 0 & 0 \\
y_1 & 0 & 0 & x_0 \\
0 & 0 & x_0 & x_1 \\
y_2 & 0 & x_0 + x_1 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
x_0 & x_1 & 0 & 0 & 0 & 0 & 0 & x_0 & x_2 \\
0 & x_1 & 0 & 0 & 0 & 0 & x_0 & x_2 & 0 \\
0 & 0 & -y_2 & -y_1 & 0 & 0 & y_2 & y_0 & 0 \\
0 & 0 & 0 & -y_2 & -y_1 & y_2 & y_0 & 0 & 0
\end{bmatrix}
\]

As in the previous case, we denote by \((x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)\) the coordinates of the product \(\mathbb{P}^2 \times \mathbb{P}^2\), keeping in mind that they satisfy the equation \(x_0y_0 + x_1y_1 + x_2y_2 = 0\) of the flag variety.

### 6.2. Alternative proof of the existence of 1-instantons

In this section, using Serre correspondence, we are going to show the existence of stable 1-instanton bundles with trivial splitting type on the general line from both families. The proof will be very similar to the one given in [13].

**Theorem 6.5.** Let \(F\) be the flag variety. Then there exists a family of dimension 5 of stable 1-instantons \(\mathcal{E}\) on \(F\). Moreover, they have trivial splitting type on the generic conic and line (from both families). Indeed, \(\mathcal{E}(1)\) are Ulrich bundles on \(F\).

**Proof.** Let us consider \(S\) a hyperplane section of \(F\). \(S\) is a del Pezzo surface of degree 6 embedded by the anticanonical line bundle \(H_{F,S} := -K_S\). Therefore, \(S\) can be seen as the blow-up at three generic points of \(\mathbb{P}^2\). Using standard terminology, \(-K_S = 3h - e_1 - e_2 - e_3\) where \(l\) denotes the pullback of the class of a line in \(\mathbb{P}^2\) and \(e_i\) are the exceptional divisors. Let us consider a general curve \(C\) of class \(3l - e_1\). It is a smooth elliptic curve of degree 8. Using the short exact sequence that relates the normal bundles:

\[
0 \to \mathcal{N}_{C,S} \to \mathcal{N}_{C,F} \to \mathcal{N}_{S,F|C} \to 0
\]

we obtain

\[
0 \to \mathcal{O}_C(C) \to \mathcal{N}_{C,F} \to \mathcal{O}_C(H_C) \to 0.
\]

By easy Riemann-Roch computations, we obtain \(h^0(\mathcal{N}_{C,F}) = 16\) and \(h^1(\mathcal{N}_{C,F}) = 0\). Deformations theory tells us that the Hilbert scheme \(\text{Hilb}^8(\mathcal{F})\) of curves on \(F\) with Hilbert polynomial \(p(t) := 8t\) is smooth of dimension 16 at the point \([C]\).

Take a general deformation \(D\) of \(C\). It will be a smooth non-degenerate elliptic curve with Chern class \([D] = 4h_1h_2 \in A^1(F)\). Now, a non-zero element of

\[
\text{Ext}^1(\mathcal{I}_{D|F}(2,2), \mathcal{O}_F) = \text{Ext}^1(\mathcal{I}_{D|F}, \omega_F) = H^1(\omega_D) = \mathbb{C}
\]

provides, through Serre correspondence, an unique extension

\[
0 \to \mathcal{O}_F \to \mathcal{G} \to \mathcal{I}_{D|F}(2,2) \to 0.
\]
Let us show that \( \mathcal{G} \) is an Ulrich sheaf, using its characterization by the minimal \( \mathcal{O}_{\mathbb{P}^7} \)-resolution (see [12, Prop. 2.1]). In order to do this, we will use that, being \( F \) and \( D \) quasi-minimal varieties on \( \mathbb{P}^7 \), we know their minimal \( \mathcal{O}_{\mathbb{P}^7} \)-resolution. Indeed,

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^7}(-6) \rightarrow \mathcal{O}_{\mathbb{P}^7}(-4)^9 \rightarrow \mathcal{O}_{\mathbb{P}^7}(-3)^{16} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-2)^9 \rightarrow \mathcal{O}_{\mathbb{P}^7} \rightarrow \mathcal{O}_F \rightarrow 0,
\]

and

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^7}(-8) \rightarrow \mathcal{O}_{\mathbb{P}^7}(-6)^{20} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-5)^{64} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-4)^{90} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-3)^{64} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-2)^{20} \rightarrow \mathcal{O}_{\mathbb{P}^7} \rightarrow \mathcal{O}_D \rightarrow 0.
\]

Applying the Mapping cone Theorem to the short exact sequence (which is exact due to \( F \) being aCM):

\[
0 \rightarrow \mathcal{I}_{F|\mathbb{P}^7} \rightarrow \mathcal{I}_{D|\mathbb{P}^7} \rightarrow \mathcal{I}_{D|F} \rightarrow 0
\]

we obtain the resolution of \( \mathcal{I}_{D|F} \):

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^7}(-8) \rightarrow \mathcal{O}_{\mathbb{P}^7}(-6)^{21} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-5)^{64} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-4)^{81} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-3)^{48} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-2)^{11} \rightarrow \mathcal{I}_{D|F} \rightarrow 0.
\]

Finally, applying the horseshoe Lemma to (19), \( \mathcal{G} \) has a linear \( \mathcal{O}_{\mathbb{P}^7} \)-resolution

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^7}(-4)^{12} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-3)^{48} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-2)^{72} \rightarrow \mathcal{O}_{\mathbb{P}^7}(-1)^{48} \rightarrow \mathcal{O}_{\mathbb{P}^7}^2 \rightarrow \mathcal{G} \rightarrow 0,
\]

namely \( \mathcal{G} \) is an Ulrich rank 2 bundle on \( F \) with Chern classes \( c_1(\mathcal{G}) = 2h_1 + 2h_2 \) and \( c_2(\mathcal{G}) = 4h_1h_2 \). Now, if we define \( \mathcal{E} := \mathcal{G}(-1) \), \( \mathcal{E} \) is an 1-instanton bundle on \( F \). Let us know show that \( \mathcal{E} \) sits on a generically smooth component of \( MI(1) \) of dimension 5. Namely, we compute the dimensions \( \text{ext}^i(\mathcal{E}, \mathcal{E}) (= h^i(\mathcal{E} \otimes \mathcal{E}) \text{thanks to } \mathcal{E}^\vee \cong \mathcal{E}) \). For this, tensoring the exact sequence (19) twisted by \( \mathcal{O}_F(-1, -1) \) with \( \mathcal{E} \) and considering the cohomology vanishings of an Ulrich bundles, we get \( h^i(\mathcal{E} \otimes \mathcal{E}) = h^i(\mathcal{E} \otimes \mathcal{I}_{D|F}(1, 1)) \). On the other hand, from

\[
0 \rightarrow \mathcal{E} \otimes \mathcal{I}_{D|F}(1, 1) \rightarrow \mathcal{E}(1, 1) \rightarrow \mathcal{E} \otimes \mathcal{O}_D(1, 1) \rightarrow 0
\]

we get \( h^2(\mathcal{E} \otimes \mathcal{I}_{D|F}(1, 1)) = h^1(\mathcal{E} \otimes \mathcal{O}_D(1, 1)) \). But \( \mathcal{E} \otimes \mathcal{O}_D(1, 1) \cong \mathcal{N}_{D,F} \) and \( D \), being a general deformation of \( C \), verifies \( h^1(\mathcal{N}_{D,F} = 0 \text{ and } h^0(\mathcal{N}_{D,F}) = 16 \) from where we obtain the result.

Let us observe that, alternatively, the dimension of the component of \( MI(1) \) where \( \mathcal{E} \) sits could have also been computed using that, through Serre correspondence, there is a bijection, at least locally, between pairs \((\mathcal{E}(1), s)\) -where \( \mathcal{E}(1) \) is an Ulrich bundle and \( s \in \mathbb{P}(H^0(\mathcal{E}(1))) \) is a global section- and elliptic normal curves \( D \subset F \). Therefore, since \( h^0(\mathcal{E}(1)) = 12 \) and \( Hilb^{\mathbb{P}^7}(F) \) has dimension 16 at \( D \), we get that the dimension of the considered component of \( MI(1) \) is 5.

We know that Ulrich bundles are semistable (see [7]). Let us now show that the generic Ulrich bundle constructed above is stable: otherwise, by Hoppe’s criterium, it would fit in a short exact sequence of the form:

\[
0 \rightarrow \mathcal{O}_F(l, -l) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_F(-l, l) \rightarrow 0
\]
with \( l \in \mathbb{Z} \). Comparing Chern classes, it turns out that \( l \in \{1, -1\} \). But properly semistable bundles sitting on one of these two exact sequences form families of dimension 2, from the easy calculation \( \text{ext}^1(\mathcal{O}_F(1, -1), \mathcal{O}_F(-1, 1)) = h^1(\mathcal{O}_F(-2, 2)) = 3 \).

It only remains to prove the trivial splitting type with respect to lines and conics on \( F \). Let us start with the conic case. For this, let us consider a generic conic \( A \) with class \( l - e_1 \) on the del Pezzo surface \( S \). It will cut \( C \) on two points \( \{x, y\} \). Tensoring the short exact sequence

\[
0 \rightarrow \mathcal{I}_{C|F}(1, 1) \rightarrow \mathcal{O}_F(1, 1) \rightarrow \mathcal{O}_C(1, 1) \rightarrow 0
\]

by \( \mathcal{O}_A \) we see that \( \mathcal{I}_{C|F}(1, 1) \otimes \mathcal{O}_A \cong \mathcal{O}_A \oplus \mathcal{O}_{\{x,y\}} \). Tensoring again \( \mathcal{I}_{C|F}(1, 1) \otimes \mathcal{O}_A(-1, -1) \) by \( \mathcal{O}_A(-1, -1) \) we get a surjection \( \mathcal{E}|_A \) to \( \mathcal{O}_A \oplus \mathcal{O}_{\{x,y\}} \). Therefore, the only possibility is \( \mathcal{E}|_A \cong \mathcal{O}_A^2 \). So, by semicontinuity, the same will be true for the 1-instanton bundle associated to a generic deformation \( D \) of \( C \) inside \( \text{Hilb}^{8t}(F) \).

For the case of a general line \( L \) on any of the two families \( \mathbb{P}^2 \), we should to change slightly the argument: in this case, we start with a smooth elliptic curve \( C \) of degree 8 on the del Pezzo \( S \) from the linear system \( 5l - 3e_1 - 2e_2 - 2e_3 \). Now, the line \( L \) with class \( l - e_2 - e_3 \) will cut \( C \) on a single point \( x \) and therefore \( \mathcal{I}_{C|F}(1, 1) \otimes \mathcal{O}_L \cong \mathcal{O}_L \oplus \mathcal{O}_{\{x\}} \). Now the previous argument tells us that \( \mathcal{E}|_L \cong \mathcal{O}_L^2 \). Notice that, fixed the line \( L \) on \( S \), we have freedom on the choice of the presentation of \( S \) as a blow-up to assure that \( L \) has class \( l - e_2 - e_3 \). Therefore, this argument covers both families of lines on \( F \).

\[
\square
\]

6.3. The induction step. Let us start considering an instanton bundle \( \mathcal{E} \) on the flag variety with \( c_2(\mathcal{E}) = kh_1h_2 \) and \( c_1(\mathcal{E}) = c_3(\mathcal{E}) = 0 \), stable and satisfying \( h^2(\mathcal{E} \otimes \mathcal{E}) = 0 \). We also suppose it to have trivial restriction on the generic conic and the generic line for each of the two families, i.e.

\[
\mathcal{E}|_C \cong \mathcal{O}_C^{\oplus 2} \quad \mathcal{E}|_{L_1} \cong \mathcal{O}_{L_1}^{\oplus 2} \quad \mathcal{E}|_{L_2} \cong \mathcal{O}_{L_2}^{\oplus 2}
\]

where we denote respectively by \( L_1 \) and \( L_2 \) the generic line of the first and the second family.

Our first step is given considering the following short exact sequence

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{L_1} \rightarrow 0
\]

which gives us a torsion free sheaf \( \mathcal{F} \) with \( c_2(\mathcal{F}) = (k + 1)h_1^2 + kh_2 \), \( c_1(\mathcal{F}) = c_3(\mathcal{F}) = 0 \) and \( H^1(\mathcal{F}(1, -1)) = 0 \). Moreover for the generic conic \( C \) in the flag variety, we have trivial restrictions, i.e. \( \mathcal{F}|_C \cong \mathcal{E}|_C \cong \mathcal{O}_C^{\oplus 2} \). The same happens for the generic lines \( L_1 \) and \( L_2 \).

Applying \( \text{Hom}(\mathcal{F}, \mathcal{F}) \) and \( \text{Hom}(\mathcal{E}, \mathcal{F}) \) to \( \mathcal{F} \), we obtain \( \text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0 \).

We want to prove that \( \mathcal{F} \) can be deformed to a vector bundle. We will denote such deformation by \( \tilde{\mathcal{F}} \). All the required cohomological properties for the subsequent steps will still hold because of the semicontinuity of the Ext sheaves.

Being \( \tilde{\mathcal{F}} \) a torsion free sheaf, we have a canonical short exact sequence

\[
0 \rightarrow \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}^{**} \rightarrow T \rightarrow 0
\]

where \( T \) is a sheaf supported on a scheme of dimension at most 1.

The stability of \( \mathcal{E} \) implies the stability of \( \tilde{\mathcal{F}} \), therefore we get \( h^0(\tilde{\mathcal{F}}^{**}(-1, -1)) = 0 \) and moreover \( h^0(T(-1, -1)) = 0 \), that tells us that the support of \( T \) does not have isolated points, i.e. it is of pure dimension 1. Our next goal is to compute \( c_2(T) \) to get the degree.
of such support. First of all, let us notice that \( \tilde{F}^{**} \) is a 2-rank reflexive sheaf on the flag variety, therefore \( c_3(\tilde{F}^{**}(\alpha_1, \alpha_2)) \) is invariant for each twist by a line bundle \( \mathcal{O}_F(\alpha_1, \alpha_2) \). We will denote such Chern class by \( c \). Moreover, also \( c_2(T(\alpha_1, \alpha_2)) \) is invariant by twist, indeed it describes the degree of the support of the sheaf \( T \), more specifically it is the opposite.

Considering the relation
\[
c_3(T(\alpha_1, \alpha_2)) = c - 2(\alpha_1 h_1 + \alpha_2 h_2)c_2(T)
\]
and using the Hirzebruch-Riemann-Roch formula, we compute
\[
\chi(T(\alpha_1 h_1 + \alpha_2 h_2)) = \frac{1}{2} c - c_2(T)((1 + \alpha_1)h_1 + (1 + \alpha_2)h_2)
\]
Taking every pair of negative \( \alpha_1, \alpha_2 \) such that \( \alpha_1 + \alpha_2 << 0 \) we obtain the following cohomological relations
\[
-\chi(T(\alpha_1, \alpha_2)) = h^1(T(\alpha_1, \alpha_2)) \leq h^2(\tilde{F}(\alpha_1, \alpha_2)) = h_1(\mathcal{O}_{L_1}(\alpha_1, \alpha_2))
\]
where the middle inequality is given because the sheaf \( \tilde{F}^{**} \) is reflexive and the right equality from the semicontinuity of the Ext sheaves. Denoting \( c_2(T) = \beta_1 h_1^2 + \beta_2 h_2^2 \) we thus obtain
\[
-\frac{1}{2} c + (1 + \alpha_1)\beta_2 + (1 + \alpha_2)\beta_1 \leq -\alpha_2 - 1
\]
that gives us \( \beta_1 = -1 \) and \( \beta_2 = 0 \).

Moreover, choosing \( \alpha_1 = \alpha_2 = -1 \) we get
\[
-\frac{1}{2} c = -\chi(T(-1, -1)) = h^1(T(-1, -1)) \geq 0,
\]
hence \( c = 0 \). This means that if \( \tilde{F} \) is not locally free, it would fit in the following short exact sequence
\[
0 \rightarrow \tilde{F} \rightarrow \tilde{F}^{**} \rightarrow \mathcal{O}_{L_1} \rightarrow 0
\]
We would like now to compute
\[
1 - \text{ext}^1(\tilde{F}, \tilde{F}) = \chi(\tilde{F}, \tilde{F}) = \chi(\tilde{F}^{**}, \tilde{F}) - \chi(\mathcal{O}_{L_1}, \tilde{F})
\]
Being \( \tilde{F}^{**} \) locally free (we proved \( c = 0 \)), hence a deformation of an instanton bundle of charge \( k \), and using Hirzebruch-Riemann-Roch we compute \( \chi(\tilde{F}^{**}, \tilde{F}) = 2 - 8k \), and using the resolution of \( \mathcal{O}_{L_1} \) in the flag variety, we compute \( \chi(\mathcal{O}_{L_1}, \tilde{F}) = 2 \). This means that \( \text{ext}^1(\tilde{F}, \tilde{F}) = 8k + 1 \); but the family of sheaves \( \tilde{F} \) defined as in \cite{20} have dimension equals to \( 8k - 3 + 2 + 1 = 8k \), where \( 8k - 3 \) is the dimension of the moduli of instantons of charge \( k \) (this has already been computed in \cite{6}, but it can also be done explicitly using the Hirzebruch-Riemann-Roch described in the preliminary section), \( 2 \) are the lines \( L_1 \) of the first family and \( 1 = \mathbb{P}(H^0(\mathcal{E}_{|L_1}))) \), which represents the choice of one of the two summand of the trivial restriction in order to construct the surjection \( \mathcal{E} \rightarrow \mathcal{O}_{L_1} \). We can conclude that \( \tilde{F} \) can be deformed to a locally free sheaf.

We will exactly take such deformation \( \tilde{F} \) and we basically apply the same technique, using a line \( L_2 \) of the second family. Indeed, consider the short exact sequence
\[
(21) \quad 0 \rightarrow \mathcal{H} \rightarrow \tilde{F} \rightarrow \mathcal{O}_{L_2} \rightarrow 0
\]
which gives us a torsion free sheaf \( \mathcal{H} \) with \( c_2(H) = (k + 1)h_1 h_2, c_1(F) = c_3(F) = 0 \) and \( H^1(F(-1, -1)) = 0 \). As before, for the generic conic \( C \) in the flag variety, we have trivial
restrictions, i.e. $F|_C \cong E|_C \cong \mathcal{O}_C^2$ and the same happens for the generic lines $L_1$ and $L_2$. Applying $\text{Hom}(-, \mathcal{H})$ and $\text{Hom}(\tilde{F}, -)$ to (21), we get $\text{Ext}^2(\mathcal{H}, \mathcal{H}) = 0$.

Once again we want to prove that $\mathcal{H}$ can be deformed to a vector bundle. We will denote such deformation by $\tilde{\mathcal{H}}$. All the required cohomological properties for the subsequent steps will still hold because of the semicontinuity of the $\text{Ext}$ sheaves.

Being $\tilde{\mathcal{H}}$ a torsion free sheaf, we have a canonical short exact sequence
\[ 0 \to \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^{**} \to T \to 0 \]
where $T$ is a sheaf supported on a scheme of dimension at most 1.

The stability of $\tilde{F}$ implies the stability of $\tilde{\mathcal{H}}$, therefore we get $h^0(\tilde{\mathcal{H}}^{**}(-1, -1)) = 0$ and moreover $h^0(T(-1, -1)) = 0$, that tells us that the support of $T$ does not have isolated points, i.e. it is of pure dimension 1. As before, we want to compute $c_2(T)$ and $c_3(\tilde{\mathcal{H}}^{**}(\alpha_1, \alpha_2))$ is invariant for each twist by a line bundle $\mathcal{O}_F(\alpha_1, \alpha_2)$. We will denote such Chern class by $c$. Moreover, also $c_2(T(\alpha_1, \alpha_2))$ is invariant by twist.

In this case, taking every pair of negative $\alpha_1, \alpha_2$ such that $\alpha_1 + \alpha_2 << 0$ we obtain the following cohomological relations
\[ -\chi(T(\alpha_1, \alpha_2)) = h^1(T(\alpha_1, \alpha_2)) \leq h^2(\tilde{\mathcal{H}}(\alpha_1, \alpha_2)) = h_1(\mathcal{O}_{L_2}(\alpha_1, \alpha_2)). \]

Denoting $c_2(T) = \beta_1 h_1^2 + \beta_2 h_2^2$ we thus obtain
\[ -\frac{1}{2} c + (1 + \alpha_1)\beta_2 + (1 + \alpha_2)\beta_1 \leq -\alpha_1 - 1 \]
that gives us $\beta_1 = 0$ and $\beta_2 = -1$.

Moreover, choosing $\alpha_1 = \alpha_2 = -1$ we get
\[ -\frac{1}{2} c = -\chi(T(-1, -1)) = h^1(T(-1, -1)) \geq 0, \]
hence $c = 0$. This means that if $\tilde{\mathcal{H}}$ is not locally free, it would fit in the following short exact sequence
\[ 0 \to \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^{**} \to \mathcal{O}_{L_2} \to 0 \]
Here again, we compute $\text{ext}^1(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}) = 8k + 5$; but the family of sheaves $\mathcal{F}$ defined as in (20) have dimension equals to $8k + 1 + 2 + 1 = 8k + 4$. Again, we conclude that $\mathcal{H}$ can be deformed to a locally free sheaf, which in this case is an instanton bundle of charge $k + 1$ on the flag variety.

7. Jumping rational curves

Now we are going to define the notion of jumping conic. For instanton bundles on $\mathbb{P}^3$, jumping lines have been thoroughly studied. Here we propose an analogous definition for conics on the flag variety that deals at once with the irreducible and reducible case:

**Definition 7.1.** Let $\mathcal{E}$ be an instanton bundle on the flag variety $F$. A conic $C \subset F$ (irreducible or not) is a jumping conic of order $(a, b)$ if satisfies $H^1(E|_C(-1,0)) = a$ and $H^1(E|_C(0,-1)) = b$. $C$ is said to have trivial splitting type when it has order $(0,0)$.
Let us give some insight to our definition. Indeed, suppose first that \( C \subseteq F \) is an irreducible conic, \( C \cong \mathbb{P}^1 \). In that case, \( \mathcal{O}_F(-1,0)|_C = \mathcal{O}_F(0,-1)|_C = \mathcal{O}_C(-1) \) and for an instanton bundle \( \mathcal{E} \) we have \( \mathcal{E}|_C \cong \mathcal{O}_F(-a) \oplus \mathcal{O}_F(a) \) if and only if \( H^1(E|_C(-1,0)) = H^1(E|_C(0,-1)) = a \) if and only if it is a jumping conic of type \((a,a)\).

On the other hand, for a reducible conic \( C = L_1 \cup L_2 \) for lines \( L_i \) intersecting transversely on a single point. In this case, it is well-known that \( \text{Pic}(C) \cong \mathbb{Z}^2 \), where the isomorphism is given by \( \mathcal{L} \mapsto (\text{deg}_{L_1}(\mathcal{L}), \text{deg}_{L_2}(\mathcal{L})) \). Therefore, for an instanton \( \mathcal{E} \) on \( F \) the restriction to \( C \) is of the form \( \mathcal{E}_C \cong \mathcal{O}_C(a,b) \oplus \mathcal{O}_C(-a,-b) \) if and only if it is a jumping conic of type \((a,b)\).

Let us consider the exact sequence associated to a conic \( C \)

\[
0 \to \mathcal{O}_F(-1,-1) \to \mathcal{O}_F(0,-1) \oplus \mathcal{O}_F(-1,0) \to \mathcal{O}_F \to \mathcal{O}_C \to 0
\]

Writing the above sequence in families with respect to global sections of \( \mathcal{O}_F(0,-1) \oplus \mathcal{O}_F(-1,0) \), we get the description of the universal conic \( C \subseteq F \times \mathcal{H} \)

\[
0 \to \mathcal{O}_F(-1,-1) \boxtimes \mathcal{O}_H(-1,-1) \to \mathcal{O}_F(0,-1) \boxtimes \mathcal{O}_H(0,-1) \oplus \mathcal{O}_F(-1,0) \boxtimes \mathcal{O}_H(-1,0) \to \mathcal{O}_F \boxtimes \mathcal{H} \to \mathcal{O}_C \to 0.
\]

We denote by \( \mathcal{D}_\mathcal{E} \) the locus of jumping conics of an instanton \( \mathcal{E} \), and by \( i \) its embedding in \( \mathcal{H} \).

**Proposition 7.2.** Let \( \mathcal{E} \) be a generic \( k \)-instanton on \( F \). Then \( \mathcal{D}_\mathcal{E} \) is a divisor of type \((k,k)\) equipped with a sheaf \( G \) fitting into

\[
0 \to \mathcal{O}_H(-1,1)^{a \oplus k} \oplus \mathcal{O}_H(-1,0)^{a \oplus k} \to \mathcal{O}_H(-1,1)^{\oplus k} \oplus \mathcal{O}_H(-1,0)^{\oplus k} \to i_* G \to 0.
\]

**Proof.** A conic \( C \) is jumping for \( \mathcal{E} \) if and only if the point of \( \mathcal{H} \) corresponding to \( C \) lies in the support of \( R^i q_*(p^*(\mathcal{E}(-1,0))) \).

Let us consider the Fourier-Mukai trasform \( \Phi = q_*(p^*(- \otimes \mathcal{O}_F(-1,0))) \). Let us apply \( \Phi \) to the terms of the monad \((14)\).

\[
R^i q_*(p^*(\mathcal{O}_F(-1,0) \otimes \mathcal{O}_F(-1,0))) \cong R^i q_*(((\mathcal{O}_F(-2,0) \boxtimes \mathcal{O}_H(0,0)))
\]

By \((22)\) tensored by \( \mathcal{O}_F(2,0) \boxtimes \mathcal{O}_H(0,0) \), since the only non zero cohomology on \( F \) is \( h^2(\mathcal{O}_F(-3,0)) = 1 \) we get \( R^i q_*(p^*(\mathcal{O}_F(-1,0) \otimes \mathcal{O}_F(-1,0))) = 0 \) for \( i \neq 1 \) and

\[
R^1 q_*(p^*(\mathcal{O}_F(-1,0) \otimes \mathcal{O}_F(-1,0))) \cong \mathcal{O}_H(-1,0).
\]

\[
R^i q_*(p^*(\mathcal{O}_F(0,-1) \otimes \mathcal{O}_F(-1,0))) \cong R^i q_*(((\mathcal{O}_F(-1,1) \boxtimes \mathcal{O}_H(0,0)))
\]

By \((22)\) tensored by \( \mathcal{O}_F(-1,1) \boxtimes \mathcal{O}_H(0,0) \), since the only non zero cohomology on \( F \) is \( h^3(\mathcal{O}_F(-2,-2)) = 1 \) we get \( R^i q_*(p^*(\mathcal{O}_F(0,-1) \otimes \mathcal{O}_F(-1,0))) = 0 \) for \( i \neq 1 \) and

\[
R^1 q_*(p^*(\mathcal{O}_F(0,-1) \otimes \mathcal{O}_F(-1,0))) \cong \mathcal{O}_H(-1,-1).
\]

\[
R^i q_*(p^*(\mathcal{O}_F \otimes \mathcal{O}_F(-1,0))) \cong R^i q_*(((\mathcal{O}_F(-1,0) \boxtimes \mathcal{O}_H(0,0)))
\]

By \((22)\) tensored by \( \mathcal{O}_F(-1,0) \boxtimes \mathcal{O}_H(0,0) \), since the cohomology on \( F \) is all zero we get for any \( i \)

\[
R^i q_*(p^*(\mathcal{O}_F \otimes \mathcal{O}_F(-1,0))) \cong 0.
\]
\[ R^i \kappa_*((p^* G_1(-1,0) \otimes O_F(-1,0))) \cong R^i \kappa_*((G_1(-2,0) \boxtimes O_H(0,0))). \]

By \cite{22} tensored by \( G_1(-2,0) \boxtimes O_H(0,0) \), since the only non zero cohomology on \( F \) is \( h^1(G_1(-2,0)) = 1 \) we get \( R^i \kappa_*((p^* G_1(-1,0) \otimes O_F(-1,0))) = 0 \) for \( i \neq 1 \) and
\[ R^1 \kappa_*((p^* G_1(-1,0) \otimes O_F(-1,0))) \cong O_H. \]

By \cite{22} tensored by \( G_2(-1,-1) \boxtimes O_H(0,0) \), since the only non zero cohomology on \( F \) is \( h^2(G_2(-2,-1)) = 1 \) we get \( R^i \kappa_*((p^* (O_F(0,1) \otimes O_F(-1,0))) = 0 \) for \( i \neq 1 \) and
\[ R^i \kappa_*((p^* G_2(0,-1) \otimes O_F(-1,0))) \cong O_H(-1,0). \]

Now if we apply \( \Phi \) to the sequence
\[ 0 \to K \to G_1(-1,0)^{\oplus k} \oplus G_2(0,-1)^{\oplus k} \xrightarrow{\beta} \mathcal{O}_F^{\oplus 2k-2} \to 0 \]
we get \( R^i \kappa_*((p^* (K \otimes O_F(-1,0))) = 0 \) for \( i \neq 1 \) and
\[ R^1 \kappa_*((p^* (K \otimes O_F(-1,0)))) \cong O_H(-1,0) \oplus O_H. \]

From
\[ 0 \to O_F(0,-1)^{\oplus k} \oplus O_F(-1,0)^{\oplus k} \to K \to E \to 0 \]
we get
\[ 0 \to R^0 \kappa_*((p^* (\mathcal{E} \otimes O_F(-1,0)))) \to O_H(-1,0)^{\oplus k} \oplus O_H(-1,-1)^{\oplus k} \xrightarrow{\gamma} O_H(-1,0)^{\oplus k} \oplus O_H^{k}. \]

So \( \gamma \) is a \((2k) \times (2k)\) matrix made by a \((k) \times (k)\) matrix linear in the first variables, a \((k) \times (k)\) linear in the second variables a \((k) \times (k)\) matrix of degree 0 and a \((k) \times (k)\) matrix of bidegree \((1,1)\). Hence \( \ker(\gamma) \) is zero and \( \text{coker}(\gamma) \), which is \( R^1 \kappa_*((p^* (\mathcal{E} \otimes O_F(-1,0)))) \), is an extension to \( H \) of a rank 1 sheaf, which we call \( \mathcal{G} \), on \( D_E \), that is a divisor of type \((k,k)\) given by the vanishing of the determinant of \( \gamma \).

\[ \square \]

For strictly semistable instanton bundles we can say much more about the divisor of jumping conics. For this recall that such an instanton fits on a short exact sequence \[7] and in particular has charge \( k = l^2 \). We are going to see that the support of \( D_E \) is exactly the set of reducible conics, namely conics of the form \( C = L_1 \cup L_2 \) for lines \( L_i \) intersecting transversely on a single point:

**Proposition 7.3.** Let \( \mathcal{E} \) be a strictly semistable \( k \)-instanton on \( F \), with \( k = l^2 \). Then \( C \in D_E \) if and only if \( C \) is reducible: \( C = L_1 \cup L_2 \). In particular, if \( \mathcal{E} \) is defined as the extension
\[ 0 \to O_F(l,-l) \to \mathcal{E} \to O_F(-l,l) \to 0 \]
the possible splitting types are given by
\[ \mathcal{E}|_C \cong O_C(l,-a) \oplus O_C(-l,a) \]
with $0 \leq a \leq l$.
Moreover, it is possible to find strictly semistable instanton bundles $E$ on the flag and $l+1$ different reducible conics $C_j$ on $F$, such that
\[ E_{|C_j} \cong O_{C_j}(l,-j) \oplus O_C(-l,j), \]
for $j = 0, \ldots, l$, i.e., we have all the possible splittings.

The family of strictly semistable bundles given by the other extension behave analogously.

Proof. Restricting the short exact sequence (7) that defines $E$ to a smooth conic $C$, we see that $E_{|C} \cong O_C^2$. On the other hand, suppose that $C = L_1 \cup L_2$ with each of the $L_i$ belonging to one of the two families of lines on $F$. Then for the suitable $L_i$ we have
\[ 0 \to O_{L_i}(l) \to E_{|L_i} \to O_{L_i}(-l) \to 0. \]
Then the sequence should split and $E_{|L_i} \cong O_{L_i}(l) \oplus O_{L_i}(-l)$. For the other line, the corresponding exact sequence can have central term $E_{|L_j} \cong O_{L_j}(-a) \oplus O_{L_j}(a)$ for any $0 \leq a \leq l$.

To conclude the proof, consider the semistable family, the other one is studied in the same way, given as extensions of type
\[ 0 \to O_F(l,-l) \to E \to O_F(-l,l) \to 0. \]
Restricting at the lines of second family, we have that $E_{|L_2} \cong O_{L_2}(l) \oplus O_{L_2}(-l)$. For the other family of lines we have
\[ 0 \to O_{L_1}(-l) \to E_{|L_1} \to O_{L_1}(l) \to 0, \]
hence $E_{|L_1} \in \text{Ext}^1(O_{L_1}(l), O_{L_1}(-l)) = H^1(O_{L_1}(-2l))$ and $E \in \text{Ext}^1(O_F(l,-l), O_F(-l,-l)) = H^1(O_F(2l,-2l))$. The two extensions are related by the following exact sequence
\[ 0 \to O_F(2l-2,-2l) \to O_F(2l-1,-2l) \to O_F(2l,-2l) \to O_{L_1}(-2l) \to 0. \]
From Proposition [11] we can compute that $H^1(O_F(2l-2,-2l)) = 0$ for $i = 0, 1, 2, 3$ and $H^i(O_F(2l-1,-2l)) = H^i(O_F(2l,-2l)) = 0$ for $i = 0, 2, 3$, we therefore have a surjective linear map
\[ \varphi : H^1(O_F(2l,-2l)) \to H^1(O_{L_1}(-2l)) \]
between vector spaces of respective dimensions $(2l-1)(2l+1)$ and $2l-1$ and let us consider the associated linear projective map $\tilde{\varphi}$ between the associated projective spaces (of dimension one lower). Take lines $L_j$, $j = 0, \ldots, l$ from this family and points $\alpha_i \in \mathbb{P}(H^1(O_{L_i}(-2l)))$ parameterizing the extension of type $O_{L_i}(-i) \oplus O_{L_i}(i)$ (recall that the isomorphism class of an extension is classified by this projective space). Then $A_i := \tilde{\varphi}^{-1}(\alpha_i)$ will be a linear projective subspace of $\mathbb{P}(H^1(O_F(2l,-2l)))$ of codimension $2l - 1$. The intersection $\cap_{i=0}^l A_i$ will have codimension $(2l-1)(l+1) \leq (2l-1)(2l+1) - 1$ and therefore will be nonempty. The elements of the intersection will parameterize $l^2$-instanton bundles with the desired restriction to the reducible conics $L_i \cup L'_i$ ($L'_i$ is the unique line from the other family that intersects $L_i$).

Nevertheless, by the following result, we observe that for the general case we have that the family jumping conics of type at least(-1,1), which we now know form a divisor, do not
coincide with the family of type at least $(-2,2)$. Hence, we have proper jumping conics of type $(-1,1)$. This result should be compared to [5].

We denote by $S_{E}$ the locus of jumping conics of type at least $(-2,2)$ of an instanton $E$, and by $i$ its embedding in $\mathcal{H}$.

**Proposition 7.4.** Let $E$ be a generic $k$-instanton on $F$. Then $S_{E}$ is a Cohen Macaulay curve equipped with a torsion-free sheaf $G$ fitting into

$$0 \to B \to \mathcal{O}_{\mathcal{H}}(-1,0)^{\oplus k} \oplus \mathcal{O}_{\mathcal{H}}(0,-1)^{\oplus k} \to \mathcal{O}_{\mathcal{H}}^{\oplus 2k-2} \to i_{*}G \to 0.$$  

where $B$ is a rank two torsion free sheaf on $\mathcal{H}$.

**Proof.** A conic $C$ belong to $S_{E}$ if and only if the point of $\mathcal{H}$ corresponding to $C$ lies in the support of $R^{1}q_{*}p^{*}(E)$.

Let us consider the Fourier-Mukai trasform $\Phi = q_{*}(p^{*}(\cdot))$. Let us apply $\Phi$ to the terms of the monad [14]. By (22) tensored by $\mathcal{O}_{F}(-1,0) \boxtimes \mathcal{O}_{\mathcal{H}}(0,0)$, since the cohomology on $F$ is all zero we get for any $i$

$$R^{i}q_{*}(p^{*}(\mathcal{O}_{F})) \cong 0.$$

By (22) tensored by $\mathcal{O}_{F}(0,-1) \boxtimes \mathcal{O}_{\mathcal{H}}(0,0)$, since the cohomology on $F$ is all zero we get for any $i$

$$R^{i}q_{*}(p^{*}(\mathcal{O}_{F} \boxtimes \mathcal{O}_{F}(0,-1))) \cong 0.$$

By (22) tensored by $\mathcal{O}_{F} \boxtimes \mathcal{O}_{\mathcal{H}}(0,0)$, since the only non zero cohomology on $F$ is $h^{0}(\mathcal{O}_{F}) = 1$ we get $R^{i}q_{*}(p^{*}(\mathcal{O}_{F})) = 0$ for $i \neq 0$ and

$$R^{0}q_{*}(p^{*}(\mathcal{O}_{F})) \cong \mathcal{O}_{\mathcal{H}}.$$

By (22) tensored by $\mathcal{G}_{1}(-1,0) \boxtimes \mathcal{O}_{\mathcal{H}}(0,0)$, since the only non zero cohomology on $F$ is $h^{1}(\mathcal{G}_{1}(-2,0)) = 1$ we get $R^{i}q_{*}(p^{*}(\mathcal{G}_{1}(-1,0))) = 0$ for $i \neq 0$ and

$$R^{0}q_{*}(p^{*}(\mathcal{G}_{1}(-1,0))) \cong \mathcal{O}_{\mathcal{H}}(-1,0).$$

By (22) tensored by $\mathcal{G}_{2}(0,-1) \boxtimes \mathcal{O}_{\mathcal{H}}(0,0)$, since the only non zero cohomology on $F$ is $h^{2}(\mathcal{G}_{2}(0,-2)) = 1$ we get $R^{i}q_{*}(p^{*}(\mathcal{G}_{2}(0,1))) = 0$ for $i \neq 0$ and

$$R^{0}q_{*}(p^{*}(\mathcal{G}_{2}(0,-1))) \cong \mathcal{O}_{\mathcal{H}}(0,-1).$$

Now if we apply $\Phi$ to the sequence

$$0 \to \mathcal{O}(0,-1)^{\oplus k} \oplus \mathcal{O}(-1,0)^{\oplus k} \to K \to E \to 0$$

we get $R^{i}q_{*}(p^{*}(E)) \cong R^{i}q_{*}(p^{*}(K))$ for any $i$. From

$$0 \to K \to \mathcal{G}_{1}(-1,0)^{\oplus k} \oplus \mathcal{G}_{2}(0,-1)^{\oplus k} \oplus \mathcal{O}(\oplus 2k-2) \to 0$$

we get

$$0 \to R^{0}q_{*}(p^{*}(E)) \to \mathcal{O}_{\mathcal{H}}(-1,0)^{\oplus k} \oplus \mathcal{O}_{\mathcal{H}}(0,-1)^{\oplus k} \to \mathcal{O}_{\mathcal{H}}^{\oplus 2k+2}.$$  

So $\gamma$ is a $(2k + 2) \times (2k)$ matrix made by a $(k + 1) \times (k)$ matrix linear in the first variables, a $(k + 1) \times (k)$ linear in the seconds variables. Hence ker($\gamma$) is a rank two bundle and coker($\gamma$), which is $R^{1}q_{*}(p^{*}(E))$, is an extension to $\mathcal{H}$ of a torsion free sheaf supported on the curve $S_{E}$.

$\square$
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