On Waylen’s regular axisymmetric similarity solutions

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Abstract
We review the similarity solutions proposed by Waylen for a regular time-dependent axisymmetric vacuum space-time, and show that the key equation introduced to solve the invariant surface conditions is related by a Bäcklund transform to a restriction on the similarity variables. We further show that the vacuum space-times produced via this path automatically possess a (possibly homothetic) Killing vector, which may be time-like.

1 Introduction
Waylen [1] introduced a general solution for a time-dependent, axisymmetric, vacuum gravitational field, taking the form of a convergent radial power series whose coefficients are determined by a single arbitrary generating function with cylindrical coordinate expression $a(t, z)$. Motivated by this work, Waylen [2] then examined the problem of obtaining exact forms of these series solutions, and found that for similarity solutions, the invariants could be written in closed form in terms of a so-called key function $\chi$. A sufficient condition for the existence of such a solution is a third-order quasi-linear partial differential equation for $\chi$, here called Waylen’s equation:

\begin{align}
(\chi_t - 2Kz)(\chi u^2 - \chi_{zz}u_{tz}) &+ (\chi_z + 2Kt)(\chi_{zz}u_{tt} - \chi_{tt}u_{tz}) + 4\chi_{tt}u_{tz}\chi_{zz} = 0, \\
\end{align}

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where $K$ is a constant parameter.

Given a solution $\chi(t, z)$ to equation (1), the similarity variables and forms are then known explicitly. Substituting these into the vacuum field equations yields a reduced set of partial differential equations (whose consistency is guaranteed by equation (1)). These must be solved before the final metric can be written down. Further details of this process can be found in the original papers already cited.

The aim of this note is to study the structure of Waylen’s procedure, focusing on the point symmetries in the similarity solution and the connection with Waylen’s equation.

2 Field equations

In this section, we describe Waylen’s reduced field equations and exact similarity reductions for Einstein’s vacuum equations with axisymmetry, that is, metrics for which there is a single hypersurface-orthogonal Killing vector $\partial_\phi$.

In the 1987 paper [1], Waylen showed that the metric for this case could be chosen in the form

$$ds^2 = A \, dt^2 + 2B \, dt \, dz - C \, dz^2 - (AC + B^2) \, d\rho^2 - \rho^2 \, d\phi^2,$$

(2)

where $A$, $B$ and $C$ depend upon $\rho$, $t$ and $z$ only. For this metric, the fundamental equations to be solved were shown to be

$$A_{\rho \rho} + \rho^{-1} A_\rho = A_\rho E_\rho + AH + DE_{tt} + A_z B_t - A_t B_z,$$

$$B_{\rho \rho} + \rho^{-1} B_\rho = B_\rho E_\rho + BH + DE_{tz} + \frac{1}{2}(A_t C_z - A_z C_t),$$

$$C_{\rho \rho} + \rho^{-1} C_\rho = C_\rho E_\rho + CH - DE_{zz} + C_z B_z - C_z B_t,$$

(3)

in which the abbreviated expressions are

$$D = AC + B^2, \quad E = \log D, \quad \text{and} \quad H = C_{tt} + 2B_{tz} - A_{zz} - D^{-1}(A_\rho C_\rho + B_\rho^2).$$

(4)

The remaining components of Einstein’s equations for solutions analytic on the axis of symmetry were satisfied by imposing the boundary conditions

$$ac = e^{2k}, \quad b = 0,$$

(5)

on the axis of symmetry, where $a = A|_{\rho=0}$, $b = B|_{\rho=0}$, $c = C|_{\rho=0}$ and $k$ is a constant. Using this information, a convergent series expansion of the general solution was given in terms of $a(t, z)$ and $k$. 

2
In his 1993 paper [2], Waylen stated without proof that the exact forms of $A$, $B$ and $C$ for similarity solutions of equations (3) and (5) are given by

$$A = a \left\{ L + (1 - f^2)^{-1}[2fM + (1 + f^2)N] \right\},$$

$$C = a^{-1} \left\{ L - (1 - f^2)^{-1}[2fM + (1 + f^2)N] \right\},$$

$$B = (1 - f^2)^{-1}[(1 + f^2)M + 2fN].$$

The symmetry invariants $L$, $M$ and $N$ are functions of the two similarity variables $u = U(t, z)$ and $v = \rho^2 V(t, z)$, with

$$U(t, z) = [a^{-1}(\chi_t - 2Kz)]_t - [a(\chi_z + 2Kt)]_z,$$

$$V(t, z) = (a^{-1})_t - a_{zz}.$$  

Here $\chi$ is the key function, and $K$ is a dimensionless constant. The remaining element $f(t, z)$ of the exact form (6) is given by

$$f = a(\chi_z + 2Kt)/(\chi_t - 2Kz),$$

while the boundary data $a(t, z)$ is related to $\chi(t, z)$ through

$$a = (-\chi_{tt}/\chi_{zz})^{1/2}.$$  

Inserting the exact form (6) into the field equations (3) yields reduced equations for $L$, $M$ and $N$, whose integrability demands that $\chi$ satisfy Waylen’s equation (1).

In view of the rich structure involved in the procedure outlined above, it seems worthwhile to investigate the main field equations (3) for Lie point symmetries in order to shed some light on the form of the solution described.

### 3 Point Symmetries

The purpose of this section is to analyze the point symmetries of the field equations (3), paying special attention to the boundary conditions (4), and to derive the conditions for the similarity variables. We also examine the implications for further space-time symmetries.

Using Langton’s enhanced version [3] of Kersten’s computer algebra package [4], we find the point symmetries of the main field equations (3) to be generated by the vector field

$$X = (c_1 \rho + c_2 \rho \log \rho) \partial_{\rho} + (\chi_z + 2Kt) \partial_t - (\chi_t - 2Kz) \partial_z$$

$$+ 2([2K - c_1 - c_2(1 + \log \rho) - \chi_{tz}]A + \chi_{tt}B) \partial_A$$

$$- (\chi_{zz}A - 2[2K - c_1 - c_2(1 + \log \rho)]B + \chi_{tt}C) \partial_B$$

$$+ 2(\chi_{zz}B + [2K - c_1 - c_2(1 + \log \rho) - \chi_{tz}]C) \partial_C.$$  

3
where the constants $c_1$, $c_2$ and $K$ take any values, and $\chi(t, z)$ is an arbitrary expression. At this stage, $\chi$ is not restricted to satisfy Waylen’s equation (1). The point symmetries thus form an infinite-dimensional Lie pseudogroup.

For solutions analytic on the axis of symmetry $\rho = 0$, we remove the logarithmic terms by setting $c_2 = 0$, leaving

$$X = c_1\rho \partial_\rho + (\chi_z + 2Kt)\partial_t - (\chi_t - 2Kz)\partial_z$$

$$+ 2(2K - c_1)(A\partial_A + B\partial_B + C\partial_C)$$

$$- 2(\chi_{zz}A + \chi_{tt}B)\partial_A - (\chi_{zz}A + \chi_{tt}C)\partial_B + 2(\chi_{zz}B - \chi_{tz}C)\partial_C.$$  

(12)

The vector (12) generates symmetries of the main field equations (3) alone. These equations are not sufficient by themselves to guarantee a solution of Einstein’s vacuum equations. We must further ensure that the boundary conditions (5) are also preserved under the symmetry transformations, up to a possible change in the constant $k$. Rewriting the boundary conditions as the simultaneous equations

$$\rho = A_t C + A C_t = A_z C + A C_z = B = 0,$$

they will be preserved provided the conditions

$$c_1 = 2K$$

$$\chi_{tt} + a^2 \chi_{zz} = 0$$

are satisfied. The first condition (14) on the constants will re-appear below, while the constraint (15) on the previously arbitrary function $\chi$ is the origin of the relation (10).

Leaving aside condition (14) for a moment, it is interesting to ask whether the symmetry vector (12) of the main field equations (3) induces a symmetry of the metric (2) itself. To this end we compute

$$\mathcal{L}_X g = (X g_{\mu\nu}) dx^\mu \otimes dx^\nu + g_{\mu\nu} d(X x^\mu) \otimes dx^\nu + g_{\mu\nu} dx^\mu \otimes d(X x^\nu),$$

(16)

and find for $g$ given by (2) that

$$\mathcal{L}_X g = 2(4K - c_1)g + 4(2K - c_1)\rho^2 d\phi \otimes d\phi.$$  

(17)

Hence condition (14), necessary for the full Ricci tensor to be preserved, also guarantees that the similarity solution generated by $X$ will have a homothetic Killing vector.

To summarise the considerations so far, point symmetries of the main field equations (3) which preserve the boundary conditions (5) and the metric’s
analyticity on the axis are generated by the vector field

\[ X = 2Kρ \partial_ρ + (\chi_z + 2Kt) \partial_t - (\chi_t - 2Kz) \partial_z \]

\[ - 2(\chi_{tz} - \chi_{tt} B) \partial_A - (\chi_{zz} A + 2Kt) \partial_B + 2(\chi_{zz} B - \chi_{tz} C) \partial_C, \]

(18)

and result in an axisymmetric space-time with an additional homothetic Killing vector obeying

\[ \mathcal{L}_X g = 4Kg. \] (19)

Returning to the similarity reduction of the main field equations (3), we seek functions of the variables \( t, z, ρ, A, B \) and \( C \) which are invariants of the symmetry transformation and are thus annihilated by the vector field \( X \). Since the coefficients of \( \partial_t, \partial_z \) and \( \partial_ρ \) are functions of the independent variables only, it follows that there are two similarity variables \( u \) and \( v \), which may furthermore be taken in the form \( u(t, z, ρ) = U(t, z) \) and \( v(t, z, ρ) = ρ^2V(t, z) \). The conditions \( Xu = Xv = 0 \) for \( u \) and \( v \) to be similarity variables become

\[ (\chi_z + 2Kt)U_t - (\chi_t - 2Kz)U_z = 0 \]

\[ 4KV + (\chi_z + 2Kt)V_t - (\chi_t - 2Kz)V_z = 0. \] (20)

For \( U \) and \( V \) independent, these can be resolved to give

\[ χ_t - 2Kz = \frac{4KVU_t}{U_tV_z - U_zV_t}, \quad χ_z + 2Kt = \frac{4KVU_z}{U_tV_z - U_zV_t}. \] (21)

## 4 Bäcklund transform

Knowing the point symmetries (18) of the field equations and the invariance conditions (20) for similarity variables, we are still a long way from Waylen’s equation and the exact form (5) of the similarity variables. In this section we present the link between the invariance conditions and Waylen’s equation as a Bäcklund map, showing that the latter represents a restriction on the set of possible similarity solutions.

We start from Waylen’s equation (1), and re-introduce the series generator \( a(t, z) \) given by equation (10). Treating \( χ \) and \( a \) as independent, Waylen’s equation becomes a quasilinear system of mixed first and second order

\[ χ_{tt} + a^2 χ_{zz} = 0 \]

\[ (χ_t - 2Kz)a_z - (χ_z + 2Kt)a_t = 2az. \] (22)

This system of equations can be written as a set of 2-forms on an appropriate jet-bundle [5]. Following the techniques developed by Estabrook
and Wahlquist, a complete prolongation structure can be computed, in preparation for applying known solution techniques.

Here we are interested in deriving one particular Bäcklund transform. To this end, it is useful to introduce new variables $c$ and $s$ defined by

$$
\chi_t - 2Kz = \frac{c}{c^2 - s^2}, \quad \chi_z + 2Kt = \frac{s/a}{c^2 - s^2}.
$$

(23)

The transformation from $(\chi_t, \chi_z)$ to $(c, s)$ is a smooth coordinate transformation away from the origin $c = s = 0$. With these variables, equations (22) become

$$
(c^2 + s^2)(c_t - (as)_z) - 2sc(s_t - (ac)_z) = 0
$$

$$
(c^2 + s^2)(s_t - (ac)_z) - 2sc(c_t - (as)_z) = 0.
$$

(24)

Since the origin $c = s = 0$ is excluded, we conclude that Waylen’s equation is equivalent to the remarkably simple system

$$
s_t = (ac)_z, \quad c_t = (as)_z.
$$

(25)

Each of these equations can be expressed in potential form by introducing potentials $x$ and $y$ such that

$$
x_t = ac, \quad x_z = s
$$

$$
y_t = as, \quad y_z = c.
$$

(26)

Equations (26) (together with the transformation (23)) constitute a Bäcklund map between the variables $(a, \chi)$ and $(x, y)$. Inverting the map (26) leads immediately to a first-order equation

$$
x_t x_z - y_t y_z = 0.
$$

(27)

Applying the integrability condition $\chi_{tz} = \chi_{zt}$ gives rise to another equation of second order, which in turn has a first integral

$$
x_t y_z - x_z y_t = e^{4Ky} F(x)
$$

(28)

for an arbitrary function $F$ of one argument. The integrability conditions on the Bäcklund map (26) are now exhausted, so the pair of equations (27), (28) constitutes a Bäcklund transform for Waylen’s equation (22).

To complete the link between Waylen’s equation and the similarity variables discussed in the previous section, we use the Bäcklund map (26) and transformation (23), to write

$$
\chi_t - 2Kz = \frac{y_z}{y_z^2 - x_z^2}, \quad \chi_z + 2Kt = \frac{y_t}{y_t^2 - x_t^2}.
$$

(29)
Making use of the relation (27), these two equations can be written
\[
\chi_t - 2Kz = \frac{x_t}{x_t y_t - x_z y_t}, \quad \chi_z + 2Kt = \frac{x_z}{x_t y_z - x_z y_t}.
\] (30)

Relabelling
\[x = U, \quad y = \frac{1}{4K} \log V,\] (31)
these equations become
\[
\chi_t - 2Kz = \frac{4KVU_t}{U_t V_z - U_z V_t}, \quad \chi_z + 2Kt = \frac{4KVU_z}{U_t V_z - U_z V_t},
\] (32)
which agrees completely with the invariance condition (21).

Using the correspondence (31), we can re-write the first-order Bäcklund transform equation (27) as
\[(4KV)^2 U_t U_z - V_t V_z = 0,\] (33)
while the first integral (28) becomes
\[U_t V_z - U_z V_t = 4KV^2 F(U).\] (34)

The important point here is that (34) is a consequence of the invariance conditions (21) as it can be derived directly using \(\chi_{tz} = \chi_{zt}\). The constraint (33) however cannot be derived from the invariance conditions: it is an additional restriction on the similarity variables. Recalling that this restriction arose from a Bäcklund transform of Waylen’s equation (1), we conclude that the latter selects a restricted class of the possible similarity variables. It follows that the exact forms (6) of the invariants are not valid for all similarity solutions, but only for a subset.

5 Summary

We have described Waylen’s exact similarity solution results for an axisymmetric vacuum space-time. Our analysis shows that Waylen’s key function \(\chi\) and constant \(K\) parameterise the infinite-dimensional Lie pseudogroup of point symmetries for the vacuum field equations. Furthermore, Waylen’s equation (4) for the key function represents a restriction on the class of similarity solutions which is expressed via a Bäcklund transform as a constraint (33) on the similarity variables. Finally, we have shown that all similarity solutions (whether Waylen’s equation is satisfied or not) result in an additional Killing vector (homothetic if \(K \neq 0\)) for the space-time metric.
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References

[1] PC Waylen, “Canonical solution of the equations of axisymmetric gravitation including time dependence”, Proc R Soc Lond A 411(1987)49–57

[2] PC Waylen, “Exact treatment of time-independent axisymmetric gravitation”, Proc R Soc Lond A 440(1993)711–715

[3] BT Langton, “Lie Symmetry Techniques for Exact Interior Solutions of the Einstein Field Equations for Axially Symmetric, Stationary, Rigidly Rotating Perfect Fluids”, PhD Thesis, University of Sydney, 1997

[4] PHM Kersten, “Infinitesimal symmetries: a computational approach”, CWI Tract 34 (Centre for Mathematics and Computer Science, Amsterdam, 1987)

[5] D Hartley, “A Bäcklund transform for Waylen’s equation”, to appear in “Proceedings of the 6th Monash General Relativity Workshop” ed AWC Lun

[6] HD Wahlquist and FB Estabrook, “Prolongation structures of nonlinear evolution equations”, J Math Phys 16(1975)1–7

[7] JD Finley, private communication