Construction of Blow-Up Manifolds to the Equivariant Self-dual Chern–Simons–Schrödinger Equation

Kihyun Kim\(^1\) · Soonsik Kwon\(^2\)

Received: 17 January 2021 / Accepted: 17 February 2023 / Published online: 21 March 2023
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Abstract
We consider the self-dual Chern–Simons–Schrödinger equation (CSS) under equivariance symmetry. Among others, (CSS) has a static solution \(Q\) and the pseudoconformal symmetry. We study the quantitative description of pseudoconformal blow-up solutions \(u\) such that

\[
u(t, r) = \frac{e^{i\gamma \ast}}{T - t} Q\left(\frac{r}{T - t}\right) \to u^\ast \quad \text{as} \quad t \to T^-.
\]

When the equivariance index \(m \geq 1\), we construct a set of initial data (under a codimension one condition) yielding pseudoconformal blow-up solutions. Moreover, when \(m \geq 3\), we establish the codimension one property and Lipschitz regularity of the initial data set, which we call the blow-up manifold. This is a forward construction of blow-up solutions, as opposed to authors’ previous work [25], which is a backward construction of blow-up solutions with prescribed asymptotic profiles. In view of the instability result of [25], the codimension one condition established in this paper is expected to be optimal. We perform the modulation analysis with a robust energy method developed by Merle, Raphaël, Rodnianski, and others. One of our crucial inputs is a remarkable conjugation identity, which (with self-duality) enables the method of supersymmetric conjugates as like Schrödinger maps and wave maps. It suggests how we proceed to higher order derivatives while keeping the Hamiltonian form and construct adapted function spaces with their coercivity relations. More interestingly, it shows a deep connection with the Schrödinger maps at the linearized level and allows us to find a repulsivity structure for higher order derivatives.
Keywords Chern–Simons–Schrödinger equation · equivaraint · self-duality · pseudoconformal blow-up · blow-up manifold · conjugation identity · conditional stability

Mathematics Subject Classification 35B44 · 35Q55

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1 Introduction

In this paper, we study a quantitative description of the blow-up dynamics for the equivariant self-dual Chern–Simons–Schrödinger equation. More precisely, we construct a codimension one manifold of initial data yielding the pseudoconformal blow-up solutions. In contrast to the backward construction in the authors’ previous work [25], here we take an initial value problem point of view and investigate the forward construction of the blow-up dynamics.

1.1 Self-dual Chern–Simons–Schrödinger Equation

We consider the self-dual Chern–Simons–Schrödinger equation

\[
\begin{align*}
D_t \phi &= i D_j D_j \phi + i |\phi|^2 \phi, \\
F_{01} &= -\text{Im}(\overline{\phi} D_2 \phi), \\
F_{02} &= \text{Im}(\overline{\phi} D_1 \phi), \\
F_{12} &= -\frac{1}{2} |\phi|^2,
\end{align*}
\]

where \( \phi : \mathbb{R}^{1+2} \to \mathbb{C} \) is a scalar field, \( D_\alpha := \partial_\alpha + i A_\alpha \) for \( \alpha \in \{0, 1, 2\} \) are the covariant derivatives associated with the real-valued 1-form \( A := A_0 dt + A_1 dx_1 + A_2 dx_2 \), and \( F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha \) are the curvature components. Repeated index \( j \) means that we sum over \( j \in \{1, 2\} \).

The Chern–Simons theory is a gauge theory on a three-dimensional domain (spacetime) and can be used to describe planar physical phenomena, i.e. the dynamics of particles restricted to a plane. Several examples are the Quantum Hall effect and high temperature superconductivity. This is a sharp contrast to the Yang Mills or Maxwell theory formulated on the four-dimensional space-time. In the 1990s, Chern–Simons models were introduced to investigate vortex solutions in the planar quantum electromagnetics. Jackiw and Pi introduced in [21–24] the Chern–Simons–Schrödinger equations as non-relativistic quantum models that describe the dynamics of a large number of interacting charged particles in the plane with electromagnetic gauge field.
Coulomb gauge

(1.1) is gauge-invariant; any solution \((\phi, A)\) to (1.1) and a function \(\chi : \mathbb{R}^{1+2} \rightarrow \mathbb{R}\) give rise to a gauge-equivalent solution \((e^{i\chi} \phi, A - d\chi)\) to (1.1). In order to consider the Cauchy problem of (1.1), we need to fix a gauge. In other words, we choose the representatives of gauge-equivalent classes. In this paper, we fix the gauge by imposing the Coulomb gauge condition:

\[
\partial_1 A_1 + \partial_2 A_2 = 0.
\]

A brief computation yields

\[
A_0 = A_0[\phi] = \epsilon_{jk} \Delta^{-1} \partial_j \text{Im}(\overline{\phi} \mathbf{D}_k \phi),
\]

\[
A_j = A_j[\phi] = \frac{1}{2} \epsilon_{jk} \Delta^{-1} \partial_k |\phi|^2,
\]

where \(\epsilon_{jk}\) is the anti-symmetric tensor with \(\epsilon_{12} = 1\). Now (1.1) can be reduced into a single evolution equation of \(\phi\) by substituting the above formulae. As recognized in [21], (1.1) under the Coulomb gauge has a Hamiltonian structure. We note that the gauge potentials \(A_\alpha\) are of long-range as the convolution kernel of \(\Delta^{-1} \partial_j\) has a nontrivial tail of size \(O(|x|^{-1})\).

Equivariance

Under the Coulomb gauge condition, we focus on equivariant solutions. This means that we work on solutions \(\phi\) of the form

\[
\phi(t, x) = e^{im\theta} u(t, r),
\]

where \((r, \theta)\) denotes the usual polar coordinate of \(\mathbb{R}^2\), and \(m \in \mathbb{Z}\) is called the equivariance index. Under equivariance, it is also convenient to write the connection \(A\) in the polar coordinates, i.e. \(A = A_0 dt + A_r dr + A_\theta d\theta\). It turns out that \(A_0, A_r,\) and \(A_\theta\) are spherically symmetric and their radial parts are given by

\[
A_r = 0, \quad A_\theta = A_\theta[u] = -\frac{1}{2} \int_0^r |u|^2 \frac{r'}{r} dr', \quad A_0 = A_0[u] = -\int_r^\infty (m + A_\theta)|u|^2 \frac{dr'}{r}.
\]

As a consequence, we can write (1.1) in terms of \(u\):

\[
i \partial_t u + (\partial_r r + \frac{1}{r} \partial_r) u - \left(\frac{m + A_\theta}{r}\right)^2 u - A_0 u + |u|^2 u = 0. \quad \text{(CSS)}
\]

This is the main equation that we consider.
Symmetries and conservation laws

(CSS) has various symmetries and associated conservation laws. From the time translation $(u(t, r) \mapsto u(t + t_0, r) \text{ for } t_0 \in \mathbb{R})$ and phase rotation symmetry $(u(t, r) \mapsto e^{i\gamma} u(t, r) \text{ for } \gamma \in \mathbb{R})$, the energy and charge are conserved:

$$E[u] := \int \left( \frac{1}{2} |\partial_r u|^2 + \frac{1}{2} \left( \frac{m + A_0}{r} \right)^2 |u|^2 - \frac{1}{4} |u|^4 \right), \quad \text{(Energy)}$$

$$M[u] := \int |u|^2, \quad \text{(Charge)}$$

where we denoted $\int f := 2\pi \int_{0}^{\infty} f(r) r dr$. There is also the time reversal symmetry $u(t, r) \mapsto \overline{u}(-t, r)$.\(^{1}\)

Of particular importance are the $L^2$-scaling invariance

$$u(t, r) \mapsto \frac{1}{\lambda} u \left( \frac{t}{\lambda}, \frac{r}{\lambda} \right), \quad \lambda \in \mathbb{R}_+,$$

and the pseudoconformal symmetry

$$u(t, r) \mapsto \frac{1}{t} e^{\frac{|x|^2}{4t^2}} u \left( -\frac{1}{t}, \frac{r}{t} \right).$$

The associated algebraic identities are the virial identities

$$\left\{ \begin{array}{ll}
\partial_t \left( \int r^2 |u|^2 \right) = 4 \int \text{Im}(\overline{u} r \partial_r u), \\
\partial_t \int \text{Im}(\overline{u} r \partial_r u) = 4E[u].
\end{array} \right.$$  

Let us finally mention the Hamiltonian structure of (CSS):

$$\partial_t u = -i \frac{\delta E}{\delta u}, \quad \text{(1.2)}$$

where $\frac{\delta}{\delta u}$ is the Fréchet derivative under the real inner product $\langle f, g \rangle_r := \int \text{Re}(\overline{f} g)$.

Self-duality

One of the special features of (CSS) is the self-duality. We introduce the Bogomol’nyi operator

$$\tilde{D}_\pm := D_1 + i D_2.$$  

---

1 In fact, there is the time reversal symmetry for (1.1) without symmetry reductions. It is not simply given by conjugating the scalar field $\phi$; it reads $\phi(t, x_1, x_2) \mapsto \overline{\phi}(-t, x_1, -x_2)$, $A_\alpha(t, x_1, x_2) \mapsto A_\alpha(-t, x_1, -x_2)$ for $\alpha \in \{0, 2\}$, and $A_1(t, x_1, x_2) \mapsto A_1(-t, x_1, -x_2)$. In particular, the time reversal symmetry preserves the equivariance index, and hence (CSS) still enjoys the time reversal symmetry.
Within Coulomb gauge (so that $A_\alpha = A_\alpha[\phi]$), $\tilde{D}_+$ also depends on $\phi$ (i.e., $\tilde{D}_+ = \tilde{D}_+(\phi)$). However, we will suppress this $\phi$-dependence if there is no confusion. We define $D_+ = D_+^{(u)}$ by the radial part of $\tilde{D}_+$:

$$\tilde{D}_+[u(r)e^{im\theta}] = [D_+^{(u)}u](r)e^{i(m+1)\theta}.$$  

(1.3)

We note that $\tilde{D}_+$ shifts the equivariance index by 1.

Now we obtain the self-dual expression of the energy:

$$E[u] = \frac{1}{2} \int |D_+^{(u)}u|^2.$$  

(1.4)

Then the Hamiltonian structure (1.2) yields the self-dual form of (CSS):

$$\partial_t u = -i L^*_u D_+^{(u)} u,$$

(1.5)

where $L^*_u$ is the adjoint of the linearized operator $L_u$ of the expression $D_+^{(u)} u$. We note that this self-dual expression is tied to the fact that the coupling constant of the nonlinearity (i.e. the coefficient of $|u|^2u$) is equal to 1. As a consequence, the energy is always nonnegative. We will later see that this leads to the self-dual factorization of the linearized operator at the static solution.

**Static solution $Q$**

Let us fix $m$ to be a nonnegative integer. It is of great physical interest to study time independent solutions to (CSS), so called vortex (or, static) solutions. To find such solutions, we need to solve time-independent (CSS) (setting $i \partial_t u = 0$). However, the self-dual factorization (1.5) says that it suffices to solve the first-order Bogomol’nyi equation

$$D_+^{(Q)} Q = 0.$$  

(1.6)

Note that $Q$ must have zero energy. In fact, a solution (with enough regularity and decay) has zero energy if and only if it is static; see [20] and also the discussion in [25, Section 1.3].

One can find a solution $Q$ to (1.6) following Jackiw–Pi [21]. In fact it reduces to the Liouville equation, whose solutions are known. Thus (1.6) has the following explicit solution (when $m \geq 0$)

$$Q(r) = \sqrt{8}(m + 1) \frac{r^m}{1 + r^{2(m+1)}}.$$

Moreover, the solution $Q$ to (1.6) is unique up to scaling and phase rotation symmetries; see [7–9] and also the discussion in [25, Section 1.3].

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1.2 Known Results

The equation (CSS) under equivariance is known to be well-posed in the scaling critical space $L^2(\mathbb{R}^2)$. Indeed, Liu–Smith [30, Section 2] showed the small data global well-posedness and large data local well-posedness of (CSS).

We mention some previous works on the covariant Chern–Simons–Schrödinger equation without symmetry (this allows a general coupling constant $g \in \mathbb{R}$ multiplied to the cubic nonlinearity $i|\phi|^2\phi$ in the RHS of (1.1), where $g = 1$ corresponds to the self-dual case). The local well-posedness of (1.1) in the critical space $L^2$ (under whatever gauges) remains open. Under the Coulomb gauge, the local well-posedness in $H^1$ is proved by Lim [29], after the preceding works [3, 19]. If one changes to the heat gauge, Liu–Smith–Tataru [31] were able to lower the regularity assumption and prove the small data local well-posedness in $H^{0+}$.

There are also works in global-in-time behaviors. Bergé–de Bouard–Saut [3] applied Glassey’s convexity argument [13] to the virial identities and obtained a sufficient condition for finite time blow-up. These authors also performed in [4] a formal computation of the log-log blow-up for negative energy solutions. We remark that these two works only apply to the negative energy solutions, which are available only when $g > 1$. On the other hand, Oh–Pusateri [41] showed the global existence and scattering for small data in weighted Sobolev spaces.

We come back to the equivariant self-dual case (CSS). Having local theory at hands, it is natural to study global-in-time behavior of large solutions. One of the central objects in the global-in-time analysis is the ground state, a standing wave solution with minimal charge, in our context. It is believed to be the smallest (in some sense) example exhibiting nonlinear behavior of the equation. In case of (CSS), the static solution $Q$ plays a role of the ground state. Liu–Smith [30] proved the following threshold theorem: any solution having charge less than that of $Q$ is global and scatters. We also remark that the result of [30] applies to the non-self-dual case. In particular, the global well-posedness and scattering for any solutions is proved in the defocusing regime (where $E[u] \geq 0$ and $E[u] = 0$ if and only if $u = 0$) on which there are no ground states.

The next question is the dynamics at and above the threshold. At the threshold charge $M[u] = M[Q]$, there are two fundamental examples of nonscattering solutions. The first one is the static solution $Q$ and the other one is the explicit pseudoconformal blow-up solution obtained by applying the pseudoconformal transform to $Q$ [18, 21]:

$$S(t, r) := \frac{1}{|t|} Q\left(\frac{r}{|t|}\right)e^{-i \frac{r^2}{2|t|}}, \quad t < 0.$$  \hspace{1cm} (1.7)

Recently, the authors [25] studied pseudoconformal blow-up solutions with prescribed asymptotic profiles. Here, by a pseudoconformal blow-up solution, we mean a solution $u$ to (CSS) which decomposes as

$$u(t, r) \approx S(t, r) + z(t, r),$$
where \( z(t, r) \) is regular at and near the blow-up time. Let \( m \geq 1 \). Fix a small smooth \( m \)-equivariant function \( z^*(r) \), called an asymptotic profile, satisfying an additional degeneracy condition \( \partial_r^{(m)} z^*(0) = 0 \). Then, there exists a pseudoconformal blow-up solution \( u \) to (CSS) on \((-\infty, 0)\) such that \( z(0, r) = z^*(r) \). Moreover, they exhibited an instability mechanism, the rotational instability, (see Sect. 3 for more details) of these pseudoconformal blow-up solutions. This result is a backward construction of the blow-up solutions. In the (NLS) context

\[
i \partial_t \psi + \Delta \psi + |\psi|^2 \psi = 0, \quad \psi : I \times \mathbb{R}^2 \to \mathbb{C},
\]

they are referred to as Bourgain–Wang solutions [5]. The instability of Bourgain–Wang solutions in (NLS) is settled by Merle–Raphaël–Szeftel [40].

### 1.3 Main Results

In this paper, we study the forward construction and codimension one property of the pseudoconformal blow-up solutions. Compared to the backward construction of blow-up solutions with prescribed asymptotic profiles, here we investigate a quantitative description of the dynamics starting near \( Q \) and aim to characterize the initial data set yielding the pseudoconformal blow-up.

The first part of our main results is the construction of an initial data set (under a codimension one condition) resulting in pseudoconformal blow-up. The second part is to establish the codimension one property and Lipschitz regularity of the initial data set (the blow-up manifold).

To state our result, we need to introduce some technicalities. To describe the sense of the codimension one manifold, we introduce the relevant data sets and their coordinates.

We first describe the \( H^3_m \)-initial data set \( \mathcal{O}_{\text{init}} \) and the coordinates \((\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0)\). We fix a codimension four linear subspace \( Z^\perp \) of \( H^3_m \) for \( \epsilon_0 \). Codimension four conditions are given by four orthogonality conditions \((4.2)\). Here, \( H^k_m \) denotes the usual Sobolev space \( H^k(\mathbb{R}^2) \) restricted on \( m \)-equivariant functions; see Sect. 2.3 for details. For some \( b^* > 0 \), we define

\[
\tilde{U}_{\text{init}} := \{ (\lambda_0, \gamma_0, b_0, \epsilon_0) \in \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R} \times Z^\perp : b_0 \in (0, b^*), \quad \| \epsilon_0 \|_{H^3_m} < b_0^3 \}.
\]

Next we introduce one more coordinate, \( \eta_0 \), to describe the full open set of initial data. For some large universal constant \( K > 1 \), we define the set of coordinates

\[
\mathcal{U}_{\text{init}} := \{ (\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0) : (\lambda_0, \gamma_0, b_0, \epsilon_0) \in \tilde{U}_{\text{init}}, \quad \eta_0 \in (-\frac{K}{2} b_0^{3/2}, \frac{K}{2} b_0^{3/2}) \}.
\]

The initial data set \( \mathcal{O}_{\text{init}} \) is defined by the set of images:

\[
\mathcal{O}_{\text{init}} := \left\{ e^{i \gamma_0} \frac{P(\cdot; b_0, \eta_0) + \epsilon_0}{\lambda_0} \left( \frac{r}{\lambda_0} \right) : (\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0) \in \mathcal{U}_{\text{init}} \right\} \subseteq H^3_m.
\]
where \( P(y; b_0, \eta_0) \) is the modified profile defined in (3.7). Note that \( \mathcal{O}_{\text{init}} \) is invariant under \( L^2 \)-scalings and phase rotations. It will be shown (see Lemma 4.2) that if \( b^* \) is sufficiently small, then the set \( \mathcal{O}_{\text{init}} \) is open in the \( H^3_m \)-topology and one can view \((\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0)\) as coordinates on \( \mathcal{O}_{\text{init}} \). Note that the static solution \( Q \) does not belong to \( \mathcal{O}_{\text{init}} \) (due to \( Q = P(\cdot; 0, 0) \)), but it lies in the boundary of \( \mathcal{O}_{\text{init}} \).

Starting from some initial data \( u_0 \) in \( \mathcal{O}_{\text{init}} \), we will construct (in a codimension one sense) a so called \( H^3_m \)-trapped solution \( u \), which has the decomposition on its maximal forward lifespan

\[
    u(t, r) = \frac{e^{i\gamma(t)}}{\lambda(t)} \left[ P(\cdot; b(t), \eta(t)) + \epsilon(t, \cdot) \right] \left( \frac{r}{\lambda(t)} \right)
\]

and satisfies \( \epsilon \in \mathcal{Z}^\perp \),

\[
    b \in (0, b^*) \quad \text{and} \quad \eta < Kb^5, \quad \|\epsilon\|_{L^2} < K(b^*)^{\frac{1}{4}}, \quad \|\epsilon_1\|_{L^2} < K, \quad \|\epsilon_3\|_{L^2} < Kb^5.
\]

Here, \( \epsilon_1 = L_Q \epsilon \) and \( \epsilon_3 = A_Q^* A_Q L_Q \epsilon \) are the adapted derivatives of \( \epsilon \) defined in Sect. 2.2. It will be shown that \( H^3_m \)-trapped solutions blow up in finite time with the pseudoconformal blow-up rate.

We can now state the construction part of our main results.

**Theorem 1.1 (Existence of blow-up solutions)** Let \( m \geq 1 \). There exist constants \( K > 1 \) and \( b^* > 0 \) with the following properties:

- (Openness and coordinates) The set \( \mathcal{O}_{\text{init}} \) is open in \( H^3_m \) and \((\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0) \in \tilde{U}_{\text{init}} \) serves as coordinates for \( \mathcal{O}_{\text{init}} \) in the sense of Lemma 4.2.

- (Existence of trapped solutions in codimension one sense) Let \((\lambda_0, \gamma_0, b_0, \epsilon_0) \in \tilde{U}_{\text{init}} \). Then, there exists \( \eta_0 \in (-\frac{K}{2} b_0^{3/2}, \frac{K}{2} b_0^{3/2}) \) such that the solution \( u(t, r) \) starting from the initial data

\[
    u_0(r) = \frac{e^{i\gamma_0}}{\lambda_0} \left[ P(\cdot; b_0, \eta_0) + \epsilon_0 \right] \left( \frac{r}{\lambda_0} \right) \in \mathcal{O}_{\text{init}}
\]

is a \( H^3_m \)-trapped solution.

- (Pseudoconformal blow-up) The \( H^3_m \)-trapped solution \( u \) satisfies:

1. (Finite-time blow-up) \( u \) blows up in finite time \( T = T(u_0) \in (0, \infty) \).

2. (Pseudoconformal blow-up) There exist \( \ell = \ell(u_0) \in (0, \infty) \), \( \gamma^* = \gamma^*(u_0) \in \mathbb{R} \), and \( u^* \in L^2 \) such that

\[
    u(t, r) - \frac{e^{i\gamma^*}}{\ell(T - t)} Q \left( \frac{r}{\ell(T - t)} \right) \to u^* \quad \text{in} \ L^2
\]

as \( t \uparrow T \).

\[\text{In fact, we should add "there exists } M > 1 \text{" in the statement. In the definition (4.2) of } \mathcal{Z}^\perp, \text{ one has to introduce the large parameter } M > 1.\]
3. (Further regularity of the radiation) \( u^* \) has further regularity

\[ u^* \in H^1_m. \]

We note that when \( m \in \{1, 2\} \), the profile \( Q(y)e^{-ib^2/4} \) for the explicit pseudoconformal blow-up solution \( S(t, r) \) does not belong to \( \mathcal{O}_{\text{init}} \), due to \( Q(r)e^{-ib^2/4} \notin H^3_m \). Thus the trapped solutions constructed in Theorem 1.1 are different from the blow-up solutions constructed in [25].

According to the instability result of [25], the above trapped solutions are believed to be non-generic. In the next theorem, we show that the set of initial data yielding trapped solutions forms a codimension one manifold. Indeed, we obtain the uniqueness of \( \eta_0 \) to complement Theorem 1.1. Moreover, we establish the Lipschitz continuity of \( \eta_0 = \eta_0(b_0, \varepsilon_0) \), and hence the Lipschitz regularity of the blow-up manifold.

To achieve this, we further restrict the initial data set and equivariance index:

\[ u_0 \in H^5_m \text{ and } m \geq 3. \]

The restricted initial data set \( \mathcal{O}^5_{\text{init}} \) is described as follows. Let

\[
\tilde{U}^5_{\text{init}} := \{(\lambda_0, \gamma_0, b_0, \varepsilon_0) \in \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R} \times (\mathbb{Z}^* \cap H^5_m) : b_0 \in (0, b^*), \|\varepsilon_0\|_{H^5_m} < b_0^5 \},
\]

\[
U^5_{\text{init}} := \{(\lambda, \gamma, b_0, \eta_0, \varepsilon_0) : (\lambda_0, \gamma_0, b_0, \varepsilon_0) \in \tilde{U}^5_{\text{init}}, \eta_0 \in \left(-\frac{K}{2} b_0^{3/2}, \frac{K}{2} b_0^{3/2}\right)\}.
\]

Next, the initial data set \( \mathcal{O}^5_{\text{init}} \) is given by the set of images:

\[
\mathcal{O}^5_{\text{init}} := \left\{ \frac{e^{i\gamma_0}}{\lambda_0} [P(\cdot; b_0, \eta_0) + \varepsilon_0]\left(\frac{r}{\lambda_0}\right) : (\lambda_0, \gamma_0, b_0, \eta_0, \varepsilon_0) \in U^5_{\text{init}} \right\} \subseteq H^5_m.
\]

(1.12)

Starting from some initial data \( u_0 \) in \( \mathcal{O}^5_{\text{init}} \), we construct \( H^5_m \)-trapped solutions \( u \), which enjoys further smallness in the \( \dot{H}^5_m \)-norm: \( u \) has the decomposition (1.9) on its maximal forward lifespan and satisfies \( \varepsilon \in \mathbb{Z}^* \cap H^5_m \),

\[
b \in (0, b^*), \quad \eta < K b^3, \quad \|\varepsilon\|_{L^2} < K (b^*)^{1/4}, \quad \|\varepsilon_1\|_{L^2} < K b, \quad \|\varepsilon_3\|_{L^2} < K b^3, \quad \|\varepsilon_5\|_{L^2} < K b^{9/2}.
\]

(1.13)

Here, \( \varepsilon_5 = A^*_Q A_Q^* A_Q^* L_Q \varepsilon \) is an adapted derivative of \( \varepsilon \) defined in Sect. 2.2.

We mean by a manifold, a subset of a Banach space, which can be locally represented by a graph on a linear subspace. The regularity of the manifold is described by the regularity of the graphs. Here we use the following definition of a locally Lipschitz codimension one manifold of a Banach space.
Definition 1.2 (Locally Lipschitz codimension one manifold) Let \( M \) be a subset of a Banach space \( X \). We say that \( M \) is a locally Lipschitz codimension one manifold of \( X \) if it satisfies the following properties: let \( p \in M \) be arbitrary.

- There exist codimension one closed subspace \( X_s \) and one-dimensional subspace \( X_u \) of \( X \) such that \( X = X_s \oplus X_u \).
- There exist open neighborhood \( O_{p,s} \) of \( p \) in \( p + X_s \), open neighborhood \( O_p \) of \( p \) in \( X \), and a Lipschitz map \( f : O_{p,s} \to X_u \) such that \([id_{O_{p,s}} \oplus f](O_{p,s}) = M \cap O_p\).

We note that this property is invariant under \( C^1 \)-diffeomorphisms.

There are several works in dispersive equations on establishing regularities of the manifolds of objects exhibiting certain dynamics. To name a few, Collot [11] constructed a Lipschitz manifold of blow-up solutions to the energy-supercritical NLW. There is an extensive literature on the study of similar manifolds, for example, manifolds of global solutions that scatter to solitary waves, or blow-up solutions. We refer to [1, 2, 6, 12, 26, 28, 32] and references therein. The most relevant one to this paper is the work of Collot [11].

Our next result is on the uniqueness of \( \eta_0 \) and the Lipschitz regularity of the blow-up manifold.

Theorem 1.3 (Blow-up manifold when \( m \geq 3 \)) Let \( m \geq 3 \). There exist constants \( K > 0 \) and \( b^* > 0 \) with the following properties.

- (Existence of \( H_5^m \)-trapped solutions) The statements of Theorem 1.1 hold when we replace \( \tilde{U}_{init}, O_{init}, H_5^m \) by \( \tilde{U}_{5,init}, O_{5,init}, H_5^m \). Moreover, the radiation \( u^* \) has further regularity \( u^* \in H_3^m \).
- (Uniqueness of \( \eta_0 \)) Given \( (\lambda_0, \gamma_0, b_0, \epsilon_0) \in \tilde{U}_{5,init} \), \( \eta_0 \) is unique in \((-\frac{K}{2}b_{0}^{3/2}, \frac{K}{2}b_{0}^{3/2})\) such that the solution \( u(t, r) \) starting from the initial data \( u_0(\tau) \) in (1.11) is a \( H_5^m \)-trapped solution.
- (Lipschitz blow-up manifold) Let \( M \) be the set of initial data in \( O_{5,init} \) yielding \( H_5^m \)-trapped solutions. Then, \( M \) is a locally Lipschitz codimension one manifold in \( H_5^m \).

Comments on Theorems 1.1 and 1.3.

1. Codimension one condition. One of the crucial observations in authors’ previous work [25] is the finding of unstable modulation parameter \( \eta \), which leads to the rotational instability. This parameter \( \eta \) is the source of codimension one condition.

2. Comparison with the results of [25]. In the authors’ previous work [25], we constructed and studied an instability mechanism of pseudoconformal blow-up solutions with prescribed asymptotic profiles via the backward construction. In particular, the authors exhibited an unstable direction (or, constructed an one-parameter family of solutions) that prevents the pseudoconformal blow-up. On the contrary, this work uses the forward construction. Here we constructed a set of initial data on that admits blow-up solutions and showed that it is a locally Lipschitz codimension one manifold. In view of [25], this work sheds light on optimal conditional stability (codimension one) of pseudoconformal blow-up solutions.

In [25], the authors imposed one extra degeneracy condition on the asymptotic profiles. However, it is unclear how this degeneracy condition on the asymptotic profiles is related to the set of initial data.
3. The $m = 0$ case. Our proof breaks down in many places when $m = 0$. For example, $Q$ decays slower ($Q \sim (y)^{-2}$). Moreover, the Hardy controls of the adapted norms at $H^1$ and $H^3$ and repulsivity property of $A_Q A_Q^*$ are weakened near the spatial infinity. However, we believe that the essential difficulty for the $m = 0$ case is the construction of the modified profile. In fact, the size of the radiation term (or, the error from the modified profile) becomes critical and seems to require further corrections on the modulation equations. Thus the current profile seems not working. The $m = 0$ case will be treated in a forthcoming work. Experiences of energy-critical problems such as wave maps, Schrödinger maps, or heat flows, suggest further refinement of the modified profiles and logarithmic corrections to the modulation equations, exploiting the resonance of the linearized operator and the tail computations [38, 43, 44].

4. Further regularity. Although we stated $u^* \in H^1_m$ (or $u^* \in H^3_m$) in our main theorems, by inspecting their proofs, for any $k \in \mathbb{N}$ one can prove $u^* \in H^{2k-1}_m$ if $m \geq 2k - 1$ and $O_{init}$ is restricted to $H^{2k+1}_m$-functions.

5. Comparison with Schrödinger maps (SM) and nonlinear Schrödinger equation (NLS). (CSS) shares many features with energy-critical Schrödinger maps (SM) and (NLS), as they are critical Schrödinger-type equations. Earlier works on those equations have become a nice guide for us to study (CSS).

(CSS) and (NLS) share many similarities such as symmetries and conservation laws. Among others, the pseudoconformal symmetry is the most crucial and motivates us to study pseudoconformal blow-up solutions. However, they have essential differences between the linearized operators, yielding completely different instability mechanisms of pseudoconformal blow-up solutions as drawn in [40] and [25]. A construction of pseudoconformal blow-up solutions for one-dimension quintic ($L^2$-critical) NLS in codimension one sense is given in [27].

One drastic difference between (CSS) and (NLS) is that (NLS) allows a stable blow-up regime, namely, the celebrated log-log blow-up studied by Merle and Raphaël [33–37, 42]. As the formal analysis in [4] suggests, it is believed that the focusing non self-dual CSS is similar to (NLS).

(CSS) is also similar to the energy-critical Schrödinger maps (SM). First of all, both of them are self-dual. The linearized operators have self-dual factorizations and the method of supersymmetric conjugates [38, 43, 46] can be performed. A remarkable observation in this paper is that the linearized operator of (CSS) and that of (SM) are the same after we go by one higher adapted derivative, due to the conjugation identities (2.4). See Remark 2.3 for more details.

Furthermore, the modulation equations detected in the 1-equivariant (SM) [38] (without log corrections) are the same as those of (CSS). Therefore the codimension one blow-up with rotational instability are expected in both cases. As seen in (1.3), taking the first adapted derivative shifts the equivariance index by 1. Thus the $m = 0$ case of (CSS) seems to share many features with the 1-equivariant (SM) including the logarithmic corrections to the modulation equations. In a forthcoming work, the conjugation identities (Proposition 2.2) will also be a key algebraic tool to attack the $m = 0$ case.

However, (SM) with equivariance index $k \geq 3$ has no blow-up near the harmonic map, but the asymptotic stability is known [14–16]. This is in contrast to (CSS), where the pseudoconformal blow-up occurs for all $m \geq 1$. 
6. **Assumption $m \geq 3$ for Theorem 1.3.** When establishing the Lipschitz regularity of the blow-up manifold, we need to work with $H^5_m$-trapped solutions (more precisely, the solutions with good $H^5_m$ a priori bounds). When $m \in \{1, 2\}$, current profiles and modulation parameters are insufficient to construct $H^5_m$-trapped solutions. Due to slow decay of $Q$, there appear exotic modes (of two real dimensions) from the coercivity relations. We also note that when $m = 2$, these exotic modes have worst decay and there appear additional complications due to logarithmic weakening of the Hardy controls near the spatial infinity. See Remark 2.8 for more details. Establishing the Lipschitz dependence of blow-up solutions for $m \in \{1, 2\}$ remains as an interesting open problem.

7. **Rotational instability.** Our main theorems construct a codimension one manifold of initial data yielding the pseudoconformal blow-up. A natural question is to ask dynamics of solutions starting from initial data that are close to, but do not belong to the constructed blow-up manifold. In [25], the authors exhibited an instability mechanism (the rotational instability) for solutions that are perturbed in a certain direction ($\rho$) from the pseudoconformal blow-up solutions. In fact, this direction $\rho$ turns out to be transversal to our blow-up manifold. We conjecture that the rotational instability is universal for solutions near the blow-up manifold: if the initial data $u_0$ lies in $\mathcal{O}_{\text{init}} \setminus \mathcal{M}$, then they concentrate up to some small scale $\sim |\eta| > 0$, then stop concentrating but take an abrupt spatial rotation by the angle $\text{sgn}(\eta)(\frac{m+1}{m})\pi$ on the time scale $\sim |\eta|$, and then expand out like $S(-t, r)$ for $t \gtrsim |\eta|$ as observed in [25].

According to Theorem 1.3, if one slightly perturbs $\eta_0$ from $u_0 \in \mathcal{M}$ with coordinates $(\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0)$, then the solution should escape the trapped regime by the uniqueness of $\eta_0$. Our proof moreover shows that this solution exits the trapped regime by $|\eta| \gtrsim o(b)$ at the exit time. The nonlinear rotational instability requires the study of the forward-in-time dynamics after the exit time.

It seems that the rotational instability is not particular for (CSS). For the Landau–Lifschitz–Gilbert equation, which contains harmonic heat flow and Schrödinger maps equations, authors in [47] performed formal computations and provided numerical evidences for a quick spatial rotation by the angle $\pi$, near the blow-up solution.

1.4 **Strategy of the Proof**

Our general scheme of the proof is the forward construction using modulation analysis. As a main step of the proof, we perform a modified energy method in higher derivatives. We found a conjugation identity (Proposition 2.2) which enables us to carry out the rest of the analysis. It also shows a deep connection with the Schrödinger maps and wave maps.

To begin with, we decompose the solution into the blow-up profile and the remainder. The former is a finite-dimensional object, whose dynamics is described by a system of ODEs of modulation parameters. The latter belongs to some finite codimension space described by orthogonality conditions.

We first detect the evolution laws of the modulation parameters exhibiting pseudoconformal blow-up and its rotational instability. Next, we control the remainder part by the robust energy method combined with repulsivity properties observed in higher
Sobolev norms. This strategy was successfully implemented in various contexts. To name a few relevant examples, we refer to Rodnianski–Sterbenz [46] and Raphaël–Rodnianski [43] for energy-critical wave maps, Merle–Raphaël–Rodnianski [38] for energy-critical Schrödinger maps. We also refer to [17, 44, 45] for relevant works in the energy-critical equations. The strategy also extends to energy-supercritical equations, for example in Merle–Raphaël–Rodnianski [39] and Collot [10, 11]. This list is not exhaustive. The works [11, 38, 39] are the most relevant to our work.

1. Setup for the modulation analysis. Let \((\lambda_0, \gamma_0, b_0, \epsilon_0) \in \tilde{U}_{\text{init}}\) be given and let \(\eta_0\) vary. Set our initial data

\[
u_0(r) = \frac{e^{i\gamma_0}}{\lambda_0} [P(\cdot; b_0, \eta_0) + \epsilon_0]\left(\frac{r}{\lambda_0}\right),
\]

where \(P(\cdot; b_0, \eta_0)\) is a modified profile to be used in this paper, which deforms from \(Q\). The construction of \(P\) and the roles of \(b_0\) and \(\eta_0\) will be explained soon. The introduction of the modified profile \(P\) and modulation parameters \(\lambda_0, \gamma_0, b_0, \eta_0\) are motivated from the generalized null space relations of the linearized operator.

Let \(u\) be the forward-in-time evolution of \(u_0\). Requiring certain orthogonality conditions on \(\epsilon\), we can decompose \(u\) as

\[
u(t, r) = \frac{e^{i\gamma}}{\lambda} [P(\cdot; b, \eta) + \epsilon]\left(t, \frac{r}{\lambda}\right)
\]
as long as \(u\) belongs to the trapped regime (1.10) or (1.13), i.e. \(\epsilon\) is kept small and \(|\eta| \ll b\). Here, \(\lambda, \gamma, b, \eta\) are functions of \(t\). This decomposition will be clarified in Lemma 4.2.

The proof of Theorem 1.1 essentially reduces to the following assertions:

- (Main bootstrap) Smallness assumptions on \(\epsilon\) is kept in the trapped regime (Proposition 4.4), and doing so, the formal parameter equation (1.14) approximately holds;
- There exists special \(\eta_0\) such that \(|\eta| \ll b\) holds for the whole lifespan of \(u\) (Proposition 4.5), so \(u\) is a trapped solution;
- \(u\) satisfies the statements of Theorem 1.1.

The proof of Theorem 1.3 additionally requires the difference estimates:

- The difference of the unstable parameter \(\eta\) is controlled by the difference of the stable parameters \(b\) and \(\epsilon\). (Proposition 6.1)

The heart of the proof is the main bootstrap part. Here we focus on the main bootstrap argument.

2. Formal parameter ODEs and rotational instability. Write (CSS) in the self-dual form (1.5):

\[
\partial_t u + iL^a_+ D^{(u)}_+ u = 0.
\]
We renormalize $u$ by introducing the renormalized variables $(s, y)$ by
\[ \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad y = \frac{r}{\lambda}, \quad u^\flat(s, y) := e^{-i\gamma \lambda} u(t, \lambda y). \]

This yields
\[ \left( \frac{\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i}{\lambda} \right) u^\flat + i L_{u^\flat}^* D_{+}^{(u^\flat)} u^\flat = 0, \]
where $\Lambda = r \partial_r + 1$ is the $L^2$-scaling vector field. From the identities (where we temporarily write $f_b(y) := e^{-ib \frac{y^2}{2}} f(y)$)
\[ L_{f_b}^* D_{+}^{(f_b)} f_b = [L_{f}^* D_{+}^{(f)} f] ] b + i b \Lambda f_b - i b^2 \partial_b(f_b) \text{ and } D_{+}^{(Q)} Q = 0, \]
the pseudoconformal blow-up is encoded in the following ODE system of modulation parameters
\[ \frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = 0, \quad b_s + b^2 = 0. \]

If it were true that the decomposition of the form $\frac{\epsilon^y}{\lambda} [Q_b + \epsilon \xi] \xi$ is enough to prove that $\epsilon$ is kept small forward-in-time and the above modulation equations are valid, then we would have a stable blow-up. However, this is not the case we observed in [25], which asserts that pseudoconformal blow-up solutions exhibit rotational instability. This motivates us to further introduce the fourth modulation parameter (denoted by $\eta$) and modify our profile (denoted by $P(\cdot; b, \eta)$). The parameter $\eta$ is chosen to generate the rotational instability. In [25], the authors introduced a fixed small parameter $\eta \geq 0$ to generate the rotational instability:
\[ \frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = \eta \theta_{\eta}, \quad b_s + b^2 + \eta^2 = 0, \]
where $\theta_{\eta} \approx m + 1 \neq 0$.

In this work, we view $\eta$ as a dynamical parameter and write the formal parameter ODE system as
\[ \frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = \eta \theta_{\eta}, \quad b_s + b^2 + \eta^2 = 0, \quad \eta_s = 0. \tag{1.14} \]
A typical example of solutions to this ODE system is
\[ b(t) = |t|, \quad \lambda(t) = (t^2 + \eta^2)^{\frac{1}{2}}, \quad \eta(t) = \eta_0, \]
\[ \gamma(t) = \begin{cases} 0, & \text{if } \eta_0 = 0, \\ \pm \text{sgn}(\eta)(m + 1) \tan^{-1}(\frac{t}{|\eta_0|}), & \text{if } \eta_0 \neq 0. \end{cases} \]
The rotational instability is exhibited when \( \eta_0 \neq 0 \), where \( \gamma(t) \) changes \((m + 1)\pi\) on the time interval \(|t| \ll |\eta|\). When \( \eta_0 \) is small and nonnegative, an exact solution to (CSS) satisfying this law is constructed in [25]. Such solutions blow up only when \( \eta_0 = 0 \). This explains why we can expect at best codimension one blow-up.

In our setting, \( \eta \) is a dynamical parameter with \( \eta_s \approx 0 \). To ensure the pseudoconformal blow-up, we need \(|\eta| \ll b\) for trapped solutions. Due to \( \eta_s \approx 0 \), the condition \(|\eta| \ll b \to 0\) does not propagate forward-in-time generically. In other words, \( \eta \) is an unstable parameter. The \( \eta \)-bound cannot be bootstrapped, and we will show the existence of special \( \eta_0 \) by a soft connectivity argument.

Our next goal is to construct the modified profile \( P \) that admits the formal ODEs (1.14).

3. Construction of the modified profile. As many of the previous works, one may try a Taylor expansion in small parameters \( b \) and \( \eta \) to construct the modified profiles. However, this procedure does not work very well, due to nonlocal nonlinearities; see Remarks 3.1 and 3.4. Instead, we use a nonlinear ansatz introduced in [25]. This is possible due to the explicit pseudoconformal symmetry and self-duality. For the parameter \( b \), we use the pseudoconformal phase \( e^{-ib\frac{y^2}{4}} \). For the parameter \( \eta \), the authors in [25] found a remarkable nonlinear ansatz

\[
D_+^{(Q^{(\eta)})} Q^{(\eta)} = -\eta \frac{y}{2} Q^{(\eta)} \quad \Rightarrow \quad L_{Q^{(\eta)}}^* D_+^{(Q^{(\eta)})} Q^{(\eta)} + \eta \theta Q^{(\eta)} + \eta^2 \frac{y^2}{4} Q^{(\eta)} = 0
\]

(1.15)

motivated from the self-dual form (1.5) and the Bogomol’nyi equation (1.6). The profile \( Q^{(\eta)} \) can be constructed (without divergence at infinity) when \( \eta \geq 0 \). Due to the factor \(-\eta \frac{y}{2} Q^{(\eta)}\), it is easy to expect that \( Q^{(\eta)} \) has an exponential decay \( e^{-\eta \frac{y^2}{4}} \).

In this work, as \( \eta \) is a dynamical parameter, we need to consider both positive and negative \( \eta \). Thus we only use \( Q^{(\eta)}(y) \) in the linearization regime \( y \ll |\eta|^{-\frac{1}{2}} \), i.e. \( e^{-\eta \frac{y^2}{4}} \approx 1 \). As a consequence, we will use the profile

\[
P(y; b, \eta) := e^{-ib\frac{y^2}{4}} Q^{(\eta)}(y) \chi_{b^{-1/2}}(y),
\]

where \( Q^{(\eta)} \) solves (1.15) in the region \( y \ll |\eta|^{-\frac{1}{2}} \) and \( \chi_{b^{-1/2}} \) is a smooth cutoff to the region \( y \lesssim b^{-\frac{1}{2}} \). As we will only consider the regime \(|\eta| \ll b\), this definition makes sense.

4. Propagation of smallness of \( \epsilon \). Having fixed the profile \( P \), we decompose

\[
u(t, r) = e^{i\gamma} \left[ P(\cdot; b, \eta) + \epsilon \right](t, \frac{r}{\lambda})
\]

by the orthogonality conditions adapted to the generalized null space. The equation of \( \epsilon \) is given as

\[
(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i) \epsilon + i L \epsilon = \text{Mod} \cdot v - i R_{L-1} - i R_{NL} - i \Psi,
\]
where $L_Q = L^*_Q L_Q$ is the linearized operator (see Sect. 2.1), $\text{Mod}$, $v$, $R_{L-L}$, and $R_{NL}$ are given in (5.1), and $\Psi$ is given in (3.9). Roughly speaking, $\text{Mod} \cdot v$ contains the modulation equations, $R_{L-L}$ is the difference of the linearized operators at $P$ and $Q$, $R_{NL}$ consists of quadratic and higher order terms in $\epsilon$, and $\Psi$ is the error generated from the profile $P$.

We will perform the energy method in the $H^3$-level of $\epsilon$. To motivate this, the scaling argument says that we can expect $\|\epsilon\|_{\dot{H}^k} \lesssim \lambda^k$ at best. Since $\lambda \sim b$ in the pseudoconformal regime, this reads $\|\epsilon\|_{\dot{H}^k} \lesssim b^k$. The standard modulation estimate says $|bs + b^2| \lesssim \|\epsilon\|_{\dot{H}^k}$, so we need to work with $k \geq 3$ not to disturb the law $bs + b^2 \approx 0$.

When we go up to higher order derivatives of $\epsilon$, we will take adaptive derivatives. In view of the factorization $L_Q = L^*_Q L_Q$, we write the equation of $\epsilon_1 = L_Q \epsilon$:

$$\partial_s \epsilon_1 + L_Q i L^*_Q \epsilon_1 = \cdots$$

Here comes one of the novelties of this work. We observe a conjugation identity (see (2.5))

$$L_Q i L^*_Q = i A^*_Q A_Q.$$  

Using this, we see that $\epsilon_1$ solves a Hamiltonian equation

$$\partial_s \epsilon_1 + i A^*_Q A_Q \epsilon_1 = \cdots$$

with the associated energy $(\epsilon_1, A^*_Q A_Q \epsilon_1)_r = \|A_Q \epsilon_1\|_{L^2}^2$. In contrast to $[L_Q, i] \neq 0$, we have $[A_Q, i] = 0$ so that we can proceed further conjugation by $A_Q$ and $A^*_Q$ alternately. In particular, $\epsilon_2 = A_Q \epsilon_1$, $\epsilon_3 = A^*_Q \epsilon_2$, and so on. This also suggests how to define adapted function spaces for higher derivatives of $\epsilon$, $\mathcal{H}_m^3$ or $\mathcal{H}_m^5$. More remarkably, this $A_Q$ is equal to the linearized Bogomol’nyi operator in the wave maps and Schrödinger maps; see Remark 2.3.

With the above conjugation, the equation for $\epsilon_2 = A_Q L_Q \epsilon$ enjoys a repulsive dynamics. In the linearized dynamics $(\partial_s + i L_Q) \epsilon = 0$, there are four enemies preventing the repulsivity: the generalized kernel elements $\{\Lambda Q, i Q, iy^2 Q, \rho\}$ (See (2.1)). Interestingly enough, $A_Q L_Q$ kills all these elements and there are no nontrivial static solutions to $(\partial_s + i A_Q A^*_Q) \epsilon_2 = 0$. Moreover, we have a formal monotonicity formula from a virial type computation (see (2.8))

$$\frac{1}{2} \partial_s (\epsilon_2, -i \Lambda \epsilon_2)_r \geq (\epsilon_2, A_Q A^*_Q \epsilon_2)_r = \|\epsilon_3\|_{L^2}^2.$$  

We note that this type of repulsivity is already observed in [46]. We will use a truncated version of this monotonicity to define the corrective terms of our modified energy in a similar spirit of [39]. See Sects. 2.2 and 5.4 for more detailed exposition.

On the other hand, it is still required to extract nontrivial contributions from $R_{L-L}$ and $R_{NL}$:

$$R_{L-L} = -\theta_{L-L} P + \tilde{R}_{L-L} \quad \text{and} \quad R_{NL} = -\theta_{NL} P + \tilde{R}_{NL}.$$
The contributions $\theta_{L-L}P$ and $\theta_{NL}P$ are absorbed into the phase corrections. Such corrections arise from the nonlocality of the gauge potential $A_0$. See Sect. 5.2 for more details. A similar idea was presented in [25].

Wrapping up the above strategies, we can close the bootstrap procedure on $\epsilon$ by the modified energy method.

5. Existence of trapped solutions. After closing the bootstrap for $\epsilon$, we show the existence of special $\eta_0$ such that $|\eta| \ll b$ holds on the whole forward-in-time lifespan of $u$. Recall that $\eta$ is an unstable parameter, so the $\eta$-bound cannot be bootstrapped. We show the existence of $\eta_0$ by a standard connectivity argument. This shows the existence of trapped solutions in a codimension one sense. Finally, the sharp asymptotics of the blow-up rate $\lambda(t)$ and convergence of the phase parameter $\gamma(t)$ for trapped solutions easily follow by integrating (1.14). This concludes Theorem 1.1.

6. Lipschitz regularity of the blow-up manifold $\mathcal{M}$. The strategy is quite similar to that of the previous bootstrap argument. This time, we take two trapped solutions (say $u$ and $u'$) from $\mathcal{M}$, and estimate the differences of their modulation parameters and $\epsilon$’s. An important point is how we measure the differences because the blow-up times of $u$ and $u'$ are in general different. For this, we introduce the adapted time (as in Collot [11]) to compare these differences of modulation parameters and $\epsilon$’s (say $\delta\epsilon(s) = \epsilon(s) - \epsilon'(s'(s))$ and similarly for $\delta b$, $\delta \eta$, …). A trade-off to introducing this adapted time is that we need to work with trapped solutions with two higher derivatives (due to the linear term $(ds' - 1)LQ\epsilon'$).

For the stable parameters $b$ and $\epsilon$, we write the equation of $\delta\epsilon$, prove modulation estimates for the difference of modulation parameters, and perform energy estimates for adaptive derivatives of $\delta\epsilon$. This yields forward-in-time controls of $\delta b$ and $\delta\epsilon$. We will control the unstable parameter difference $\delta\eta$ by integrating the modulation estimates backwards in time. As a consequence, we will see that $\delta\eta_0$ is controlled by $\delta b_0$ and $\delta\epsilon_0$ in the Lipschitz sense. This essentially implies Theorem 1.3.

Remark 1.4 [Parameter dependence] To realize the above strategy, we will use a sequence of large (and small) parameters. We will presume the order of choosing large parameters as follows:

$$1 \ll K \ll M \ll M_1 \ll M_2 \ll (b^*)^{-1}.$$

The statements of our lemmas and propositions will omit the dependence of these parameters. More precisely, our statements hold after adding the following: “for sufficiently large $K$, for sufficiently large $M$ (depending on $K$), for sufficiently large $M_1$ (depending on $M$), for sufficiently large $M_2$ (depending on $M_1$), for sufficiently small $b^* > 0$ (depending on $M_2$), the following hold:”

Notation

For nonnegative quantities $A$ and $B$, we write $A \lesssim B$ if $A \leq CB$ holds for some implicit constant $C$. We write $A \gtrsim B$ if $B \lesssim A$; we write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. If $C$ is allowed to depend on some parameters, then we write them as subscripts of
\( \lesssim, \sim, \gtrsim \) to indicate the dependence. We write \( \langle x \rangle := (1 + |x|^2)^{\frac{1}{2}} \). Classical \( L^p \) spaces and Sobolev spaces \( H^s \) are also used on the domain \( \mathbb{R}^2 \).

We use a smooth spherically symmetric cutoff function \( \chi_R \), such that \( \chi_R(y) = \chi(R^{-1}y) \), \( \chi(x) = 1 \) for \( |x| \leq 1 \), and \( \chi(x) = 0 \) for \( |x| \geq 2 \). We also denote the sharp cutoff function on a set \( A \) by \( 1_A \).

For \( x \) in a metric space \( X \) and \( \delta > 0 \), we use the notation \( B_\delta(x) \) to denote the ball of radius \( \delta \) centered at \( x \).

Recall that a function \( f : \mathbb{R}^2 \to \mathbb{C} \) is \( m \)-equivariant if \( f(x) = g(r)e^{im\theta} \) on the polar coordinates \( (r, \theta) \) for some \( g : \mathbb{R}^+ \to \mathbb{C} \). We call \( g \) as the radial part of \( f \).

We use an abuse of notation that \( g \) is often considered as an \( m \)-equivariant function. For example, we say that \( g \) belongs to some \( m \)-equivariant function space if its \( m \)-equivariant extension belongs to that. We will later introduce what function spaces we use in this paper.

We will use a shorthand for the integration of radial functions \( f : (0, \infty) \to \mathbb{C} \) as

\[
\int f := \int_{\mathbb{R}^2} f(|x|)dx = 2\pi \int_0^\infty f(r)rdr.
\]

We also use the real \( L^2 \)-inner product

\[
(f, g)_r := \text{Re} \int f \overline{g}.
\]

For \( s \in \mathbb{R} \), we need the \( \dot{H}^s \)-scaling generator

\[
\Lambda_s f := \frac{d}{d\lambda} \bigg|_{\lambda=1} \lambda^{1-s} f(\lambda \cdot) = \left[ 1 - s + r \partial_r \right] f,
\]

\( \Lambda := \Lambda_0 \).

For \( k \in \mathbb{Z}_{\geq 0} \) and functions \( f : (0, \infty) \to \mathbb{C} \), we use the notation

\[
|f|_k(y) := \sup_{0 \leq \ell \leq k} \left| y^\ell \partial_y^\ell f \right|,
\]

\[
|f|_{-k}(y) := \sup_{0 \leq \ell \leq k} \left| y^{-\ell} \partial_y^{k-\ell} f \right| = y^{-k} |f|_k.
\]

The following Leibniz rules hold:

\[
|fg|_k \lesssim_k |f|_k |g|_k \quad \text{and} \quad |fg|_{-k} \lesssim |f|_{-k} |g|_k.
\]

**Organization of the paper**

In Sect. 2, we review the linearization of (CSS), derive the conjugation identities, and develop the adapted function spaces. In Sect. 3, we introduce the modified profile. In Sect. 4, we specify the decomposition of the solutions and reduce the proof of the existence of trapped solutions to the main bootstrap proposition. In Sect. 5, we close
this bootstrap procedure, and finish the proof of Theorem 1.1 and the first part of Theorem 1.3. Finally in Sect. 6, we establish the Lipschitz regularity of the blow-up manifold, thus finishing the proof of Theorem 1.3.

There is one appendix. In Appendix A, we prove weighted Hardy’s inequality, properties of the adapted function spaces, and (sub-)coercivity estimates of the adaptive derivatives.

2 Conjugation Identities

In this section, we discuss the linearized operators and their adapted function spaces. In Sect. 2.1, we recall some facts on the linearization of (CSS) at $Q$. For instance, the self-dual factorization $iL_Q = iL_Q^\ast L_Q$ and its generalized null space $\{\Lambda_1, iQ, ir^2Q, \rho\}$. In Sect. 2.2, we introduce conjugation identities and adapted derivatives. We observe a new factorization $L_OiL_Q^\ast = iA_Q^\ast A_Q$, which plays a key role in our analysis. This suggests us how we proceed to higher adapted derivatives and enables us to obtain the repulsivity. In Sect. 2.3, we construct adapted function spaces and prove (sub-)coercivity estimates for adapted derivatives.

2.1 Linearization of (CSS)

In this subsection, we briefly recall some facts on linearization made in [25, Section 3]. Motivated from the self-duality (1.4), we first linearize the Bogomol’nyi operator:

$$D_{+}^{(w+\epsilon)}(w + \epsilon) = D_{+}^{(w)}w + L_w\epsilon + N_w(\epsilon),$$

$$L_w\epsilon := D_{+}^{(w)}w + wB_w,$$

$$N_w(\epsilon) := \epsilon B_w\epsilon + \frac{1}{2}wB_\epsilon\epsilon + \frac{1}{2}\epsilon B_\epsilon\epsilon,$$

where

$$B_{fg} := \frac{1}{r} \int_{0}^{r} \text{Re}(\overline{f}g)r'dr'.$$

One can rewrite (CSS) as

$$i\partial_t u = \frac{\delta E}{\delta u} = L_u^\ast D_{+}^{(u)}u,$$

where $L_u^\ast$ denotes the adjoint of $L_u$. Now one can linearize $L_u^\ast D_{+}^{(u)}u$, as

$$L_{w+\epsilon}^\ast D_{+}^{(w+\epsilon)}(w + \epsilon) = L_{w}^\ast D_{+}^{(w)}w + L_w\epsilon + (\text{h.o.t}),$$

$$L_w\epsilon = L_{w}^\ast L_w\epsilon + (B_{w}\epsilon) + B_\epsilon^\ast(\overline{w}\cdot) + B_\epsilon^\ast(\overline{w}\cdot)(D_{+}^{(w)}w),$$
where (h.o.t) denotes quadratic and higher order terms in $\epsilon$. In particular, we observe the self-dual factorization of $L_Q$ from $D^O_+ Q = 0$:

$$L_Q = L^*_Q L_Q.$$  

This identity was first observed by Lawrie, Oh, and Shahshahani in their unpublished note and its derivation can be found in [25]. Then the linearized equation at $Q$ is given as

$$\partial_t \epsilon + i L_Q \epsilon = 0.$$  

Next, we recall the generalized null space relations of $i L_Q$:[25, Proposition 3.4]

$$i L_Q \rho = i Q; \quad i L_Q r^2 Q = 4\Lambda Q; \quad i L_Q i Q = 0; \quad i L_Q \Lambda Q = 0;$$  

where $\rho$ is given in Lemma 2.1 below. We remark that $L_Q$ is only $\mathbb{R}$-linear, not $\mathbb{C}$-linear. Note that except the relation $L_Q \rho = Q$, other relations can be derived either by direct computations or differentiating the phase/scaling/pseudoconformal symmetries applied to the static solution $Q$.

**Lemma 2.1 (The generalized eigenmode $\rho$)** There exists a unique smooth function $\rho : (0, \infty) \to \mathbb{R}$ satisfying the following properties:

1. (Smoothness on the ambient space) The $m$-equivariant extension $\rho(x) := \rho(r) e^{im\theta}, x = re^{i\theta}$, is smooth on $\mathbb{R}^2$.
2. (Equation) $\rho(r)$ satisfies

$$L_Q \rho = \frac{1}{2(m + 1)} r Q \quad \text{and} \quad L_Q \rho = Q.$$  

3. (Pointwise bounds) We have

$$|\rho|_k \lesssim_k r^2 Q, \quad \forall k \in \mathbb{N}.$$  

4. (Nondegeneracy) We have

$$(\rho, Q)_r = \|L_Q \rho\|_L^2 \neq 0.$$  

**Proof** We first recall the results proved in [25, Section 3]. As $L^*_Q r Q = 2(m + 1) Q$ by an explicit computation, it suffices to construct $\rho : (0, \infty) \to \mathbb{R}$ such that $L_Q \rho = \frac{1}{2(m + 1)} r Q$. In the proof of [25, Lemma 3.6], such $\rho$ is constructed by solving the following integral equation for $\tilde{\rho} := Q^{-1} \rho$

$$\partial_r \tilde{\rho} + \frac{1}{r} \int_0^r Q^2 \tilde{\rho} r' dr' = \frac{r}{2(m + 1)}, \quad \forall r \in (0, \infty),$$  

where $L_Q = L^*_Q L_Q$. In particular, we observe the self-dual factorization of $L_Q$ from $D^O_+ Q = 0$:  

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$$\partial_r \tilde{\rho} + \frac{1}{r} \int_0^r Q^2 \tilde{\rho} r' dr' = \frac{r}{2(m + 1)}, \quad \forall r \in (0, \infty),$$  

where $L_Q = L^*_Q L_Q$.
in the class where $|\tilde{\rho}(r)| \lesssim r^2$. As the proof relies on the contraction principle, uniqueness is shown in the same class. Applying the above equation and coming back to $\rho$, we have $\rho \in \tilde{H}_m^1$, $L_Q \rho = \frac{1}{2(m+1)} r Q$, and $|\rho| \lesssim r^2 Q$. The nondegeneracy follows from
\[ (\rho, Q) = (\rho, L_Q^* L_Q \rho) = \|L_Q \rho\|_{L^2}^2 = \frac{1}{4(m+1)^2} \|r Q\|_{L^2}^2 \neq 0. \]

To complete the proof, we first deal with smoothness. Writing the equation $L_Q \rho = r Q$ in the ambient space $\mathbb{R}^2$ (see the proof of Lemma A.5, for instance), utilizing the ellipticity of $\partial_1 + i \partial_2$, and starting from $\rho \in \tilde{H}_m^1$, the $m$-equivariant extension $\rho(x) = \rho(r)e^{im\bar{\theta}}$ is smooth. To show the pointwise estimates, we come back to the radial part and view $L_Q \rho = \frac{1}{2(m+1)} r Q$ of the form $r \partial_r \rho = -Q \int_0^r Q \rho r' dr' + \frac{1}{2(m+1)} r^2 Q$. This shows for any $k \geq 0$ the recursive estimate
\[ |\rho|_{k+1} \lesssim |Q|_{k+1} \int_0^r Q \rho r' dr' + |r^2 Q^2 \rho|_k + r^2 Q \lesssim r^2 Q + |\rho|_k. \]

Starting from the initial bound $|\rho| \lesssim r^2 Q$, we get $|\rho|_k \lesssim_k r^2 Q$ as desired. $\square$

In the linearized equation
\[ \partial_t \epsilon + i L_Q \epsilon = 0, \]
there are two invariant subspaces: the generalized null space $N_g(i L_Q) \subset \mathbb{R}^2$ of $i L_Q$
\[ N_g(i L_Q) = \text{span}_\mathbb{R}\{L Q, i Q, i r^2 Q, \rho\} \]
and the orthogonal complement of the generalized null space of the adjoint $L_Q^i$
\[ N_g(L_Q^i) := \{i \rho, r^2 Q, Q, i L Q\}^\perp. \]

Since the $4 \times 4$ matrix formed by taking the inner products of $L Q, i Q, i r^2 Q$ has nonzero determinant (c.f. (4.10)), $N_g(i L_Q)$ and $N_g(L_Q^i) \perp$ are transversal. In other words, $N_g(i L_Q) \oplus N_g(L_Q^i) \perp$ is equal to the whole space. The linearized evolution is decoupled into its restriction on the subspaces $N_g(i L_Q)$ and $N_g(L_Q^i) \perp$. This motivates how we decompose the solution. When we write
\[ u = e^{i\gamma} \left[ P(\cdot; b, \eta) + \epsilon \left( \frac{r}{\lambda} \right) \right], \]
we introduce parameters for scaling $\lambda$, phase rotation $\gamma$, pseudoconformal phase $b$, and the additional parameter $\eta$ associated with $\rho$ responsible for the rotational instability. Then, our modulated profile $P(\cdot; b, \eta)$ (applied with phase rotations $\gamma$ and scalings

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3 When $m = 1$, $i r^2 Q$ and $\rho$ do not belong to $L^2$. We mean by $i r^2 Q, \rho \in N_g(i L_Q)$ the algebraic relations (2.1).

4 The inner products may not well-defined when $m = 1$ due to the slow decay of $r^2 Q$ and $\rho$. We will resolve this technical issue with introducing cutoffs. See Sect.4.1.
\( \lambda \) is tangent to \( N_g(i L_Q) \). On the other hand, we require orthogonality conditions for \( \epsilon \) to lie in \( N_g(L_Qi)^\perp \).

### 2.2 Conjugation Identities

In view of modulation analysis, one of the main ingredients of the proof of our main theorems is the energy estimates of \( \epsilon \). More precisely, we need energy estimates for higher Sobolev norms of \( \epsilon \). Due to scaling considerations (\( \lambda \to 0 \) as approaching to the blow-up time), we expect that higher order derivatives of \( \epsilon \) become smaller. For further discussions, see Sect. 5.4.

To perform an energy estimate for higher order derivatives, let us start from the Hamiltonian flow

\[
\partial_t \epsilon + i H \epsilon = 0,
\]

for a symmetric differential operator \( H \). The Hamiltonian structure suggests the energy functional \((\epsilon, H \epsilon)_r\). To control the higher derivatives, commuting \( \partial_r \) (or the gradients \( \nabla \)) will not work very well, because it simply does not commute with \( H \) and loses the Hamiltonian structure. In order to keep the Hamiltonian structure, a natural conjugation is through \((i H)^k\):

\[
(\partial_t + i H)(i H)^k \epsilon = 0.
\]

The associated higher order energy is \( ((i H)^k \epsilon, H(i H)^k \epsilon)_r \).

As in our case, when \( H \) enjoys a special factorization \( L_Q = L_Q^* L_Q \), the energy functional at the \( H^1 \)-level can be written as \( \| L_Q \epsilon \|_{L^2}^2 \). To control the higher derivatives, one can try the conjugation by one derivative (or, half of \( H \)) to get

\[
\partial_t \epsilon_1 + L_Q i L_Q^* \epsilon_1 = 0, \quad \epsilon_1 := L_Q \epsilon. \tag{2.2}
\]

Due to \([L_Q, i] \neq 0\), this does not seem to be of Hamiltonian form. However, surprisingly enough, it is of Hamiltonian form.

**Proposition 2.2** (Conjugation identity) Let \( \phi \) solve (1.1). Then, \( \widetilde{D}_+ \phi \) solves

\[
(i D_t + \frac{1}{2} |\phi|^2) \widetilde{D}_+ \phi - \widetilde{D}_+^* \widetilde{D}_+ \widetilde{D}_+ \phi = 0. \tag{2.3}
\]

At the linearized level, we have the conjugation identity

\[
L_Q i L_Q^* = i A_Q^* A_Q, \tag{2.4}
\]

where \( A_Q \) is the radial part of \( \widetilde{D}_+^{(Q)} \) acting on \( m + 1 \) equivariant functions, i.e.

\[
\widetilde{D}_+^{(Q)} (f(r) e^{i(m+1)\theta}) = [A_Q f](r) e^{i(m+2)\theta},
\]
\[ A_Q := \partial_r - \frac{m+1+A_\theta[Q]}{r}. \] (2.5)

Let us postpone the proof for a moment.

This new factorization is one of our novelties of this work. One advantage of (2.4) is that, as alluded to above, (2.2) is now written as a Hamiltonian form. There is another advantage, which is more important. We will observe repulsivity in the variable \( \epsilon_2 := A_Q L Q \epsilon \), which ultimately relies on the fact that \( A_Q L Q \) kills all the elements of the generalized null space.

Using (2.4), we rewrite the \( \epsilon_1 \)-equation as
\[
\partial_t \epsilon_1 + iA^*_Q A_Q \epsilon_1 = 0. \quad (2.6)
\]
The associated energy functional is \( \| A_Q \epsilon_1 \|^2_{L^2} \) at the \( \dot{H}^2 \)-level. Moreover, in contrast to \([L_Q, i] \neq 0\), we have
\[
[A_Q, i] = 0.
\]
Thus we can proceed by the method of supersymmetric conjugates (i.e. alternating the conjugations \( A^*_Q \) and \( A_Q \)) to keep the Hamiltonian form \((\partial_t + iH)\epsilon_1 = 0\). For example, \( \epsilon_1 = L_Q \epsilon, \epsilon_2 = A_Q \epsilon_1, \epsilon_3 = A^*_Q \epsilon_2, \epsilon_4 = A_Q \epsilon_3, \epsilon_5 = A^*_Q \epsilon_4 \), and so on:
\[
(\partial_t + iA^*_Q A_Q)\epsilon_2 = 0, \\
(\partial_t + iA^*_Q A_Q)\epsilon_3 = 0. \quad (2.7)
\]
The associated energy functional at the \( \dot{H}^3 \)-level is \( \| \epsilon_3 \|^2_{L^2} \). In this fashion, we can naturally define associated higher order energies.

More remarkably, the conjugation identity (2.4) is crucial to obtain repulsivity properties. In earlier works of wave maps and Schrödinger maps, which also enjoy self-duality, the authors in [38, 43, 46] used the method of supersymmetric conjugates to obtain the repulsivity in the conjugated dynamics. We will later use repulsivity as a corrective term for the energy identity, in the the main bootstrap argument. See Sect. 5.4.

In order to obtain repulsivity properties of the linearized equation \( \partial_t \epsilon + iL_Q \epsilon = 0 \), we need to restrict ourselves on \( N_g(L_Q i)^\perp \) as the elements of \( N_g(iL_Q) \) do not decay in time. However, it is not clear that one can obtain repulsivity working directly on \( \partial_t \epsilon + iL_Q \epsilon = 0 \), as one has to use somehow the relation \( \epsilon \in N_g(L_Q i)^\perp \).

Motivated from the works [38, 43, 46], we conjugate \( L_Q \) to the linearized equation get
\[
\partial_t \epsilon_1 + L_Q iL^*_Q \epsilon_1 = 0.
\]
Unlike the wave and Schrödinger maps, taking \( L_Q \) does not kill all the elements of \( N_g(iL_Q) \), so the repulsivity of (2.2) will not hold for general \( \epsilon_1 \). For example, \( \epsilon_1(t, r) = rQ = 2(m+1)L_Q \rho \) is a static solution to (2.2). Thus it is still not sufficient to obtain repulsivity directly from (2.2).
Perhaps it would be natural to further conjugate by $iL_Q^*$. However, the resulting equation is merely the original linearized equation $(\partial_t + iL_Q^*L_Q)\tilde{\epsilon}_2 = (\partial_t + iL_Q)\tilde{\epsilon}_2 = 0$, where $\tilde{\epsilon}_2 = iL_Q^*\epsilon_1$.

Rather than conjugating $iL_Q^*$, we conjugate $A_Q$, which is naturally suggested from the new factorization of $L_QiL_Q^* = iA_Q^*A_Q$. A further conjugation by $A_Q$ (and using $[A_Q, i] = 0$) to (2.6) yields

$$(\partial_t + iA_QA_Q^*)\epsilon_2 = 0,$$

$$\epsilon_2 = A_Q\epsilon_1 = A_QL_Q\epsilon.$$

The main enemy to (2.2), the static solution $rQ$, is now ruled out in the $\epsilon_2$-equation, thanks to the identity

$$A_Q(rQ) = 0.$$  

To put it differently, $A_QL_Q$ kills all the elements of $N_g(iL_Q)$, which are the main enemies for the repulsivity. A crucial observation now is that $A_QA_Q^*$ has a repulsive potential. More precisely, we can write

$$A_QA_Q^* = -\partial_{rr} - \frac{1}{r} \partial_r + \frac{\tilde{V}}{r^2},$$

$$\tilde{V} := (m + 2 + A_\theta[Q])^2 + \frac{1}{2} r^2 Q^2$$

satisfying

$$\tilde{V} \gtrsim 1 \quad \text{and} \quad r \partial_r \tilde{V} = -r^2 Q^2 \leq 0.$$  

The repulsivity yields monotonicity by the virial-type computation:

$$\frac{1}{2} \partial_t (\epsilon_2, -iA\epsilon_2)_r = (\epsilon_2, A_QA_Q^*\epsilon_2)_r + (\epsilon_2, -\frac{\partial_r \tilde{V}}{2r} \epsilon_2)_r$$

$$\geq (\epsilon_2, A_QA_Q^*\epsilon_2)_r = \|\epsilon_3\|_{L^2}^2.$$  

In Sect. 5.4, we will use a localized version of this monotonicity as a corrective term for the energy identity.

**Remark 2.3** [Connection with wave maps and Schrödinger maps] It is remarkable that (CSS) has a connection with the wave and Schrödinger maps. It turns out that $A_Q$ is identical to the linearized Bogomol’nyi operator appearing in the wave and Schrödinger maps. A direct computation using

$$A_\theta[Q] = -2(m + 1) \frac{r^{2m+2}}{1 + r^{2m+2}}$$

shows that

$$A_Q = \partial_r - \frac{m + 1 + A_\theta[Q]}{r} = \partial_r - \frac{m + 1}{r} - \frac{r^{2m+2}}{1 + r^{2m+2}}.$$
This is equal to the linearized Bogomol’nyi operator in the wave and Schrödinger maps with equivariance index $k = m + 1$. (See [46, (34) and (47)] for the wave maps and [38, (2.11)] for the Schrödinger maps with $k = 1$.) It tells us that at least in the linearized level, (CSS)-dynamics for $\varepsilon_1 = L_Q \varepsilon$ (2.6) is closely related to the dynamics for wave and Schrödinger maps. The shift of equivariance from $m$ to $m + 1$ comes from (1.3) and the fact that our variable $\varepsilon_1 = L_Q \varepsilon$ is obtained from linearizing it. Thus $\varepsilon_1$ is regarded as the radial part of an $(m + 1)$-equivariant function.

**Proof of Proposition 2.2** Note that (2.4) can be proved by direct computations. However, we will derive (2.4) by linearizing (2.3).

We first prove (2.3). We start from writing (1.1) as

$$(iD_t + \frac{1}{2}|\phi|^2)\phi - \tilde{D}^*_+ \tilde{D}^+ \phi = 0.$$  \hspace{1cm} (2.9)

This form is adapted to the expression $\phi(t, x) = [Q + \epsilon(t, \cdot)](r)e^{im\theta}$. Indeed, if $\epsilon$ is small, then $i \partial_t \phi \approx 0$ and $\tilde{D}^*_+ \phi \approx 0$ so we can expect that $iD_t + \frac{1}{2}|\phi|^2 \approx i \partial_t$. Now we conjugate $\tilde{D}^*_+$ to the equation (2.9). From the formulae

$$[\tilde{D}^*_+, iD_t + \frac{1}{2}|\phi|^2] = i[\tilde{D}^*_+, D_t] + (\partial_1 + i \partial_2)(\frac{1}{2}|\phi|^2)$$

$$= (F_{01} + iF_{02}) + (Re(\phi D_1) + iRe(\tilde{\phi} D_2)) = \tilde{\phi} \tilde{D}^+ \phi,$$

$$[\tilde{D}^*_+, \tilde{D}^+] = 2i[D_2, D_1] = 2F_{12} = -|\phi|^2,$$

we get

$$0 = \tilde{D}^+(iD_t + \frac{1}{2}|\phi|^2)\phi - \tilde{D}^*_+ \tilde{D}^+ \phi$$

$$= (iD_t + \frac{1}{2}|\phi|^2)\tilde{D}^+ \phi + (|\phi|^2 - \tilde{D}^*_+ \tilde{D}^+ \phi) \tilde{D}^+ \phi$$

$$= (iD_t + \frac{1}{2}|\phi|^2)\tilde{D}^+ \phi - \tilde{D}^*_+ \tilde{D}^+ \tilde{D}^+ \phi,$$

which shows (2.3).

We will write (2.3) under the Coulomb gauge with equivariance, $\phi(t, x) = u(t, r)e^{im\theta}$. For notational convenience, let $q = D_1^{(u)}u$ so that $[\tilde{D}^*_+, \phi](t, x) = q(t, r)e^{i(m+1)\theta}$, where we recall by (1.3) that $\tilde{D}^+ \phi$ is $(m + 1)$-equivariant. We need to know how $iD_t + \frac{1}{2}|\phi|^2$ and $\tilde{D}^*_+ \tilde{D}^+$ acts on $(m + 1)$-equivariant functions. First, the potential part of $iD_t + \frac{1}{2}|\phi|^2$ is $\partial_t - A_0 + \frac{1}{2}|\phi|^2$ is spherically symmetric and has the formula

$$[-A_0 + \frac{1}{2}|\phi|^2](x) = -\int_{|x|}^{\infty} \text{Re}(\overline{u} D_1^{(u)}u)dr.$$  

In particular, $iD_t + \frac{1}{2}|\phi|^2$ is merely a small perturbation of $\partial_t$, provided that $D_1^{(u)}u$ is small as observed above. We thus have

$$(iD_t + \frac{1}{2}|\phi|^2)\tilde{D}^+ \phi = \left[i \partial_t q - \left(\int_{r}^{\infty} \text{Re}(\overline{u} q)dr'\right)\right]e^{i(m+1)\theta}.$$  

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Next, if we define $A_u$ by the radial part of $\tilde{D}^+_\phi$ acting on $(m + 1)$-equivariant functions, i.e.,

$$\tilde{D}^+_\phi (f(r)e^{i(m+1)\theta}) = [A_u f](r)e^{i(m+2)\theta},$$

$A_u := \partial_r - \frac{m+1 + A_0[u]}{r}$,

then we have

$$\tilde{D}^+_\phi \tilde{D}^+_\phi \phi = \tilde{D}^+_\phi(q(r)e^{i(m+1)\theta}) = [A_u^* A_u q]e^{i(m+1)\theta}.$$

Combining altogether, (2.3) reads

$$i \partial_t q - \left( \int_r^\infty \text{Re}(\overline{u}q)dr \right) q - A_u^* A_u q = 0. \quad (2.10)$$

Now we consider the linearization

$$u = Q + \epsilon, \quad q = L_Q \epsilon + O(\epsilon^2), \quad A_u = A_Q + O(\epsilon).$$

Substituting these into (2.10) yields

$$(i \partial_t - A_Q^* A_Q) L_Q \epsilon = O(\epsilon^2).$$

Comparing this with (2.2) explains why the conjugation identity (2.4) should hold. □

### 2.3 Adapted Function Spaces

So far, our discussion was purely algebraic. In this subsection, we investigate the function spaces adapted to the aforementioned operators like $L_Q$, $A_Q$, and $A_Q^*$. Our function spaces will be defined for equivariant functions on $\mathbb{R}^2$, but by an abuse of notations, we will identify equivariant functions (defined on $\mathbb{R}^2$) with their radial parts (defined on $(0, \infty)$). Strictly speaking, $L_Q$, $A_Q$, and $A_Q^*$ act on radial parts. With this abuse of notations, we will regard $D^{(u)}_+$ sending $m$-equivariant functions to $m+1$-equivariant functions by (1.3). Similarly, $L_Q$, $A_Q$, $A_Q^*$ send $m, m+1, m+2$-equivariant functions to $m+1, m+2, m+1$-equivariant functions, respectively. (c.f. (2.5))

The operators will be shown to be well-defined on adapted function spaces. Moreover, we study the (sub-)coercivity estimates, which allow us to estimate $\epsilon$ from its adapted derivatives $\epsilon_1 = L_Q \epsilon$, $\epsilon_2 = A_Q \epsilon_1$, or $\epsilon_3 = A_Q^* \epsilon_2$ and motivate the choice of our function spaces. Of course, because $A_Q$ and $L_Q$ have nontrivial kernels, coercivity

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5 Note that $\tilde{D}^+_\phi$ shifts the equivariance index from $m + 1$ to $m + 2$. Similarly $\tilde{D}^+_\phi$ shifts the equivariance index from $m + 2$ to $m + 1$. The following display shows how $\tilde{D}^+_\phi$ acts on the radial parts of $m + 1$ equivariant functions, which can be seen as a variant of (1.3).
estimates are obtained after imposing suitable orthogonality conditions. Note that we need to work at least in the $\dot{H}^3$-level, as explained in Sect. 1.4. The choices of these adapted derivatives are motivated in Sect. 2.2.

Moreover, we will need to take into account the fact that $\epsilon, \epsilon_1 = L_Q \epsilon, \epsilon_2 = A_Q \epsilon_1, \epsilon_3 = A_Q^* \epsilon_2$ correspond to the radial parts of $m, m+1, m+2, m+1$ equivariant functions on $\mathbb{R}^2$. Thus we need to develop our function spaces with various regularities and equivariance indices.

Equivariant Sobolev spaces

For $s \geq 0$, we denote by $H^s_m$ the restriction of the usual Sobolev space $H^s(\mathbb{R}^2)$ on $m$-equivariant functions. The set of $m$-equivariant Schwartz functions, denoted by $S_m$, is dense in $H^s_m$. We use the $H^s_m$-norms and $\dot{H}^s_m$-norms to mean the usual $H^s$-norms and $\dot{H}^s$-norms, but the subscript $m$ is used to emphasize that we are applying these norms to $m$-equivariant functions.

One of the advantages of using equivariant Sobolev spaces is the generalized Hardy’s inequality [25, Lemma A.7]: whenever $0 \leq k \leq m$, we have

$$\| |f| - k \|_{L^2} \sim \| f \|_{\dot{H}^k_m}, \quad \forall f \in S_m.$$  (2.11)

In particular, when $0 \leq k \leq m$, we can define the homogeneous equivariant Sobolev space $\dot{H}^k_m$ by taking the completion of $S_m$ under the $\dot{H}^k_m$-norm and have the embeddings

$$S_m \hookrightarrow H^k_m \hookrightarrow \dot{H}^k_m \hookrightarrow L^2_{\text{loc}}.$$  

Specializing this to $k = 1$ and applying the fundamental theorem of calculus to $\partial_r |f|^2 = 2\text{Re}(\overline{f} \partial_r f)$, we have the Hardy-Sobolev inequality [25, Lemma A.6]: whenever $m \geq 1$, we have

$$\| r^{-1} f \|_{L^2} + \| f \|_{L^\infty} \lesssim \| f \|_{\dot{H}^1_m}.$$  (2.12)

The estimates (2.11) and (2.12) also hold for negative $m$, requiring either $0 \leq k \leq |m|$ or $|m| \geq 1$. Note that in general $H^1 \hookrightarrow L^\infty$ is false on $\mathbb{R}^2$.

Adapted function space at $\dot{H}^1$-level

Here, we investigate the relation $\epsilon_1 = L_Q \epsilon$ such that $\epsilon_1$ lies in $L^2$. Analyzing at this level is well-suited for the original energy functional $E[u].$

**Lemma 2.4** (Boundedness and subcoercivity for $L_Q$ on $\dot{H}^1_m$) For $v \in \dot{H}^1_m$, we have

$$\| L_Q v \|_{L^2} + \| Q v \|_{L^2} \sim \| v \|_{\dot{H}^1_m}.$$  

Moreover, the kernel of $L_Q : \dot{H}^1_m \rightarrow L^2$ is span$_{\mathbb{R}} \{ A_Q, iQ \}$.  

\[ Springer \]
The equivalence in Lemma 2.4 explains why the space \( \dot{H}^1_m \) is the right function space at \( \dot{H}^1 \)-level. The main tool is the weighted Hardy’s inequality adapted to the operator \( L_Q \), written in the form of Corollary A.3. We postpone the proof in Appendix A. Here we explain the heuristics of the subcoercivity estimate \( (\gtrsim) \) and how it suggests the right function space.

First, the \( QB_Q \)-contribution of \( L_Q \) is perturbative. We focus on the \( D^{(Q)}_+ \)-part of \( L_Q \). Notice that \( D^{(Q)}_+ \approx \partial_r - \frac{m^2}{r} \) when \( r \) is small and \( D^{(Q)}_+ \approx \partial_r + \frac{m+2}{r} \) when \( r \) is large. When \( r \) is small, an application of the noncritical case of Corollary A.3 with \( \ell = m \) and \( k = 0 \) says that \( \|1_{r \leq 1} D^{(Q)}_+ f \|_{L^2} \) plus a boundary term at \( r \sim 1 \) has a lower bound \( \|1_{r \leq 1} r^{-1} f \|_{L^2} \). Note that the assumption \( m \geq 1 \) is necessary to control \( r^{-1} f \) in the small \( r \) regime. Similarly, when \( r \) is large, Corollary A.3 with \( \ell = -m - 2 \) and \( k = 0 \) says that \( \|1_{r \geq 1} D^{(Q)}_+ f \|_{L^2} \) plus a boundary term at \( r \sim 1 \) has a lower bound \( \|1_{r \geq 1} r^{-1} f \|_{L^2} \).

The \( Qv \) term of Lemma 2.4 can be safely deleted after ruling out the kernel elements \( \Lambda Q, i Q \) of \( L_Q \).

**Lemma 2.5** (Coercivity for \( L_Q \) on \( \dot{H}^1 \)) Let \( \psi_1 \) and \( \psi_2 \) be elements of \( (\dot{H}^1_m)^* \), which is the dual space of \( \dot{H}^1_m \). If the \( 2 \times 2 \) matrix \((a_{ij})\) defined by \( a_{i1} = (\psi_1, \Lambda Q)_r \) and \( a_{i2} = (\psi_i, i Q)_r \) has nonzero determinant, then we have a coercivity estimate

\[
\|v\|_{\dot{H}^1_m} \gtrsim \|L_Q v\|_{L^2} \gtrsim_{\psi_1, \psi_2} \|v\|_{\dot{H}^1_m}, \quad \forall v \in \dot{H}^1_m \cap \{\psi_1, \psi_2\}^\perp.
\]

We postpone the proof in Appendix A.

**Adapted function space at \( \dot{H}^3 \)-level**

We will perform an energy method for \( \epsilon_3 = \Lambda^* \epsilon_2 = \Lambda^* A_Q L_Q \epsilon \). This is due to a scaling consideration, which is motivated in Sect. 1.4 and also detailed in Sect. 5.4. Henceforward, we look for adapted function space for \( \epsilon \) such that \( \epsilon_3 = \Lambda^* A_Q L_Q \epsilon \) lies in \( L^2 \).

It turns out that the space \( \dot{H}^3_m \) is not the right choice for \( m \in \{1, 2\} \). We need to introduce an adapted function space \( \dot{H}^3_m \). It turns out that \( \dot{H}^3_m \) is slightly smaller than \( \dot{H}^3_m \) when \( m \in \{1, 2\} \). For a domain \( \Omega \subseteq \mathbb{R}^2 \), we define the (semi-)norm

\[
\|f\|_{\dot{H}^3_m(\Omega)} := \|\partial_+ f\|_{L^2(\Omega)} + \|\partial_{rr} f\|_{L^2(\Omega)} + \|r^{-1} (\log_r r)^{-1} |f|_{L^2(\Omega)}.
\]

We will use \( \Omega \in \{\mathbb{R}^2, B_R(0), B_{2R}(0) \setminus B_R(0)\} \) and denote

\[
\|f\|_{\dot{H}^3_m} := \|f\|_{\dot{H}^3_m(\mathbb{R}^2)},
\]

\[
\|f\|_{\dot{H}^3_m, \lesssim_r} := \|f\|_{\dot{H}^3_m(B_R(0))}.
\]
Define the space $\mathring{H}^3_m$ by taking the completion of $S_m$ under the $\mathring{H}^3_m$-norm. Here, we recall that $\partial_+=\partial_1+i\partial_2$, which acts on $m$-equivariant functions by $\partial_r-\frac{m}{r}$ and shifts the equivariance index from $m$ to $m+1$. Note by (2.11) that $\|\partial_+f|_{-2}\|_{L^2}$ is equivalent to $\|\partial_+f\|_{\mathring{H}^2_{m+1}}$. The terms $\partial_{rr}f$ for $m=2$ and $\partial_{rr}f|_{-1}$ for $m=1$ are in fact redundant, because $|\partial_{rr}f|\lesssim|\partial_r f|_{-2}$ when $m=2$ and $|\partial_{rr}f|_{-1}\lesssim|\partial_r f|_{-2}$ when $m=1$. For convenience in referring, we keep them in the definition. When $f$ is supported in $r\geq 1$, we indeed have

$$\|f\|_{\mathring{H}^3_m}\sim\|f|_{-3}\|_{L^2}.$$  

We first compare $\mathring{H}^3_m$ with $\mathring{H}^3_m$. As the Laplacian $\Delta$ on $\mathbb{R}^2$ admits the decomposition $\Delta=\partial_-\partial_+$ and $\partial_+f$ is now $m+1$-equivariant (with $m+1\geq 2$), one has

$$\|f\|_{\mathring{H}^3_m}\sim\|\partial_+f\|_{\mathring{H}^2_{m+1}}, \quad \forall f\in S_m.$$  

Thus the $\mathring{H}^3_m$-norm is stronger than (or equal to) the $\mathring{H}^3_m$-norm. When $m\geq 3$, these are same, thanks to (2.11). When $m\in\{1,2\}$, however, the $\mathring{H}^3_m$-norm turns out to be strictly stronger than the $\mathring{H}^3_m$-norm. Despite this fact, we have

$$L^2\cap \mathring{H}^3_m = \mathring{H}^3_m.$$  

See Lemma A.7 for details.

We now motivate the definition of $\mathring{H}^3_m$. We again use the weighted Hardy’s inequality, but we will use both the noncritical and critical case of Corollary A.3.

Now recall that $\epsilon, \epsilon_1=L_Q\epsilon, \epsilon_2=A_Q\epsilon_1$, and $\epsilon_3=A_Q^*\epsilon_2$ correspond to $m, m+1, m+2$, and $m+1$ equivariant functions. We first consider $\epsilon_3 = A_Q^*\epsilon_2$ with $\epsilon_3\in L^2$. Thanks to the positivity of $A_QA_Q^*$, we have

$$\|A_Q^*v\|_{L^2}\sim\|\hat{v}^1_{m+2}\|.$$  

(2.14)

Thus the natural function space for $\epsilon_2$ is $\mathring{H}^1_{m+2}$. Next, we consider $\epsilon_2 = A_Q\epsilon_1$ with $\epsilon_2 \in \mathring{H}^1_{m+2}$. We will see that the natural function space for $\epsilon_1$ is $\mathring{H}^2_{m+1}$. Indeed, as $m+1\geq 2$, we can apply (2.11) to see that $A_Q: \mathring{H}^2_{m+1} \to \mathring{H}^1_{m+2}$ bounded. On the other hand, $A_Q\approx\partial_r-\frac{m}{r}$ for small $r$ and $A_Q\approx\partial_r+\frac{m}{r}$ for large $r$, by a similar application of the noncritical case of Corollary A.3 with $\ell=\pm(m+1)$ and $k=1$ says that $\|r^{-1}A_Qf\|_{L^2}$ plus a boundary term localized at $r\sim 1$ has a lower bound $\|r^{-2}f\|_{L^2}$. Here, the assumption $m\geq 1$ is necessary to control $r^{-2}f$ in the small $r$ regime.

Finally, we consider $\epsilon_1 = L_Q\epsilon$ with $\epsilon_1 \in \mathring{H}^2_{m+1}$. Treating $QB_Q$ term perturbatively, it suffices to study the subcoercivity property of $\|D^Q_+v\|_{\mathring{H}^2_{m+1}}$. Recall that $D^Q_+\approx\partial_r-\frac{m}{r}$ for small $r$ and $D^Q_+\approx\partial_r+\frac{m+2}{r}$ for large $r$. When $r$ is large, $\|D^Q_+v\|_{\mathring{H}^2_{m+1}}$
plus a boundary term at \( r \sim 1 \) can control \( \|1_{r \geq 1} r^{-3} v\|_{L^2} \) by Corollary A.3 with \( \ell = -m - 2 \) and \( k = 2 \), for all \( m \geq 1 \). In view of

\[
| (\partial_r + \frac{m+2}{r^2}) v |_{-2} + r^{-3} |v| \sim |v|_{-3} \gtrsim |\partial_r v|_{-2},
\]

we conclude that \( \|D^{(Q)}_+ v\|_{\tilde{H}^3_{m+1}} \) plus a perturbative term at \( r \sim 1 \) can control the \( r \geq 1 \) portion of (2.13).

However, when \( r \) is small, the situation is delicate and we should introduce logarithmic terms in (2.13) when \( m \in \{1, 2\} \). The main source of these logarithmic terms is the logarithmic Hardy’s inequality, the critical case of Corollary A.3. Note that the lower bound \( \|\partial_+ f\|_{\tilde{H}^2_1} \) in (2.13) comes from \( D^{(Q)}_+ \approx \partial_+ = \partial_r - \frac{m}{r} \) for small \( r \).

We also saw before that the special terms \( \partial_{rrr} f \) (when \( m = 2 \)) and \( |\partial_{rr} f|_{-1} \) (when \( m = 1 \)) can be absorbed into \( |\partial_r f|_{-2} \). Henceforth, we focus on deriving the Hardy terms of (2.13):

\[
\begin{align*}
\|1_{r \leq 1} r^{-3} f\|_{L^2} & \quad \text{if } m \geq 3, \\
\|1_{r \leq 1} r^{-3} (\log r)^{-1} f\|_{L^2} & \quad \text{if } m = 2, \\
\|1_{r \leq 1} r^{-2} (\log r)^{-1} f\|_{L^2} & \quad \text{if } m = 1.
\end{align*}
\]

When \( m \geq 3 \), we simply apply the noncritical case of Corollary A.3 with \( \ell = m \) and \( k = 2 \). When \( m = 2 \), the critical case of Corollary A.3 with \( \ell = k = 2 \) applies and yields a logarithmic loss. When \( m = 1, \ell = 1 \) and \( k = 2 \) will belong to the noncritical case, but \( \ell < k \) does not give the boundary term localized at \( r \sim 1 \). Instead, we get a weaker control by applying \( \ell = 1 \) and \( k = 1 \). This is how we construct the adapted function space \( \tilde{H}^3_m \), which gives a subcoercivity estimate for \( A^*_Q A_Q L_Q \).

The precise version of the subcoercivity estimate of \( A^*_Q A_Q L_Q \) is as follows.

**Lemma 2.6** (Boundedness and subcoercivity for \( A^*_Q A_Q L_Q \) on \( \tilde{H}^3_m \)) For \( v \in \tilde{H}^3_m \), we have

\[
\|A^*_Q A_Q L_Q v\|_{L^2} + \|Q v|_{-2}\|_{L^2} \sim \|v\|_{\tilde{H}^3_m}.
\]

Moreover, the kernel of \( A^*_Q A_Q L_Q : \tilde{H}^3_m \to L^2 \) is span\( \mathbb{R} \{\Lambda Q, i Q, \rho, i r^2 Q\} \).

Ruling out the elements \( \Lambda Q, i Q, \rho, i r^2 Q \) of the kernel, the coercivity estimate follows.

**Lemma 2.7** (Coercivity for \( A^*_Q A_Q L_Q \) on \( \tilde{H}^3_m \)) Let \( \psi_1, \psi_2, \psi_3, \psi_4 \) be elements of \( (\tilde{H}^3_m)^* \), which is the dual space of \( \tilde{H}^3_m \). If the \( 4 \times 4 \) matrix \((a_{ij})\) defined by \( a_{i1} = (\psi_i, \Lambda Q)_r, a_{i2} = (\psi_i, i Q)_r, a_{i3} = (\psi_i, ir^2 Q)_r, \) and \( a_{i4} = (\psi_i, \rho)_r \) has nonzero determinant, then we have a coercivity estimate

\[
\|v\|_{\tilde{H}^3_m} \gtrsim \|A^*_Q A_Q L_Q v\|_{L^2} \gtrsim_{\psi_1, \psi_2, \psi_3, \psi_4} \|v\|_{\tilde{H}^3_m}, \quad \forall v \in \tilde{H}^3_m \cap \{\psi_1, \psi_2, \psi_3, \psi_4\}^\perp.
\]

We postpone the proofs of the above lemmas to Appendix A.
Adapted Function Spaces at $\dot{H}^5$ when $m \geq 3$

From now on, we assume $m \geq 3$ until the end of this section.

For the difference estimate of trapped solutions, we will need not only the $H^3_m$-controls, but also the $\dot{H}^5$-controls. More precisely, to control the difference at the $H^3_m$-level, we will need a priori bounds of trapped solutions at the $\dot{H}^5$-level. For this purpose, we will perform the construction analysis and develop the adapted function spaces at the $\dot{H}^5$-level.

To construct the adapted function space at the $\dot{H}^5$-level, say $\dot{H}^5_m$, we need to prove subcoercivity estimates for $A_Q^*A_QA_Q^*A_QL_Q : \dot{H}^5_m \rightarrow L^2$. We follow a similar analysis as in $H^3_m$. However, we restrict to $m \geq 3$. See Remark 2.8.

For a domain $\Omega \subseteq \mathbb{R}^2$, we define the (semi-)norm

$$\|f\|_{\dot{H}^5_m(\Omega)} := \|\hat{\partial}_+ f\|_{-4} \|L^2(\Omega)}$$

$$+ \left\{ \begin{array}{ll}
\|\hat{\partial}_- r^{m} f\|_{L^2(\Omega)} + \|r^{-1} \log^{-1} f\|_{-4} \|L^2(\Omega)} & \text{if } m \geq 5,
\|\hat{\partial}_- r^{m} f\|_{L^2(\Omega)} + \|r^{-1} \log^{-1} f\|_{-4} \|L^2(\Omega)} & \text{if } m = 4(2.15)
\|\hat{\partial}_- r^{m} f\|_{L^2(\Omega)} + \|r^{-1} \log^{-1} f\|_{-4} \|L^2(\Omega)} & \text{if } m = 3.
\end{array} \right.$$}

We define $\|f\|_{\dot{H}^5_m}, \|f\|_{\dot{H}^5_m, x}$, and $\|f\|_{\dot{H}^5_m, x}$ as in $H^3_m$. Define the space $\dot{H}^5_m$ by taking the completion of $S_m$ under the $\dot{H}^5_m$-norm.

As like (2.14), we will have boundedness and positivity of $A_Q^*A_QA_Q^* : \dot{H}^3_{m+2} \rightarrow L^2$ when $m \geq 3$:

$$\|A_Q^*A_QA_Q^* v\|_{L^2} \sim \|v\|_{\dot{H}^3_{m+2}}.$$  \hspace{1cm} (2.16)

Thanks to (2.16), the construction of $\dot{H}^5_m$ and (sub-)coercivity estimates for $A_Q^*A_QA_Q^*A_QL_Q$ is very similar to those in the $\dot{H}^3_m$ case. When $m \in \{1, 2\}$, it turns out that (2.16) becomes false. This not only weakens the subcoercivity estimate, but also seems to prevent modulation analysis with current profiles and four modulation parameters. This is the main reason for the restriction $m \geq 3$.

**Remark 2.8** [Discussion when $m \in \{1, 2\}$] When $m \in \{1, 2\}$, $\|A_Q^*A_QA_Q^* v\|_{L^2} \lesssim \|v\|_{\dot{H}^3_{m+2}}$ is still true, but the ($\lesssim$)-direction becomes false. To see why, let us note that $A_Q^*A_QA_Q^* h = 0$ for

$$h(r) = \frac{1}{r^2 Q} \int_0^r (r')^3 Q^2 \, dr' \sim \begin{cases} r^{m+2} & \text{if } r \leq 1, \\
 r^m & \text{if } r \geq 1. \end{cases}$$

When $m = 1$, we have $h \in \dot{H}^3_3$ due to its slow growth $r$. In other words, $A_Q^*A_QA_Q^* : \dot{H}^3_3 \rightarrow L^2$ has the kernel spanned by $h$. One can have (A.7) only after ruling out this kernel.

When $m = 2$, $h$ does not belong to $\dot{H}^3_4$ due to the growth $r^2$. However, we cannot expect that $\|A_Q^*A_QA_Q^* v\|_{L^2}$ controls $\|v\|_{\dot{H}^3_3}$ near the infinity, due to the example...
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\( v(x) = (1 - \chi)(x_1 + i x_2)^2 \). In other words, \( \hat{H}_3^4 \) is not the right choice for the subcoercivity estimates of \( A_Q^* A_Q A_Q^* \). To remedy this, we need to introduce a new adapted function space by *logarithmically weakening* the \( \hat{H}_4^3 \)-norm (or, enlarge the function space) near the infinity to have subcoercivity estimates of \( A_Q^* A_Q A_Q^* \). Then the problem is that \( h \) then belongs to this enlarged space, saying that \( A_Q^* A_Q A_Q^* \) has the kernel spanned by \( \{ h \} \).

Overall, \( A_Q^* A_Q A_Q^* A_Q L_Q \) has the kernel of six (real) dimensions. Thus four orthogonality conditions are not sufficient to have coercivity of \( A_Q^* A_Q A_Q^* A_Q L_Q \). In the modulation analysis, this requires us to add two more modulation parameters and change profiles.

Thanks to (2.16), we obtain the following subcoercivity estimate of \( A_Q^* A_Q A_Q^* A_Q L_Q \) on \( \mathcal{H}_m^5 \): \( \| A_Q^* A_Q A_Q^* A_Q L_Q v \|_{L^2} + \| Q \| v \|_{-4} \to L^2 \sim \| v \|_{\mathcal{H}_m^5} \).

Moreover, the kernel of \( A_Q^* A_Q A_Q^* A_Q L_Q : \mathcal{H}_m^5 \to L^2 \) is spanned by \( \{ \Lambda Q, i Q, \rho, i r^2 Q \} \).

Ruling out the elements \( \Lambda Q, i Q, \rho, i r^2 Q \) of the kernel, the coercivity estimate follows.

**Lemma 2.9** (Boundedness and subcoercivity for \( A_Q^* A_Q A_Q^* A_Q L_Q \) on \( \mathcal{H}_m^5 \))

For \( v \in \mathcal{H}_m^5 \), we have

\[ \| A_Q^* A_Q A_Q^* A_Q L_Q v \|_{L^2} + \| Q \| v \|_{-4} \to L^2 \sim \| v \|_{\mathcal{H}_m^5}. \]

For details, see Appendix A.

**3 Modified Profile**

Our main goal is to study the pseudoconformal blow-up using the modulation analysis. Namely, we decompose a blow-up solution into modulated blow-up profile \( P \) and the error \( \epsilon \):

\[ u(t, r) = e^{i \gamma(t)} [P(\cdot; b(t), \eta(t)) + \epsilon(t, \cdot)] \left( \frac{r}{\lambda(t)} \right). \]

Here, \( P(\cdot; b, \eta) \) is a deformation of the static solution \( Q \). This section aims to construct \( P \) and derive evolution equations of the parameters \( \lambda, \gamma, b, \eta \) such that \( \frac{\partial}{\partial \lambda} P(\cdot; b, \eta)(\xi) \)
becomes an approximate solution to (CSS). Moreover, we aim to capture the pseudoconformal blow-up (and its instability) using the evolution equations of these parameters.

Introduction of $\lambda$ and $\gamma$ renormalizes our solution $u$ to lie near $Q$. These parameters $\lambda$ and $\gamma$ correspond to the kernel elements of $iL_Q$, which are $\Lambda Q$ and $iQ$. In other words, the $\lambda$-curve $\frac{1}{\lambda} P(\frac{1}{\lambda})$ has tangent vector $\Lambda Q$ and the $\gamma$-curve $e^{iy}P$ has tangent vector $iQ$.

There are two more elements, $\mathbb{r} Q$ and $\rho$, in the generalized null space $N_g(iL_Q)$ of $iL_Q$. Turning on these elements triggers the growth in $\Lambda Q$ and $iQ$ directions in the linearized equation. At the nonlinear level, this changes the scaling and phase parameters.

The pseudoconformal blow-up occurs by turning on the $\mathbb{r} Q$ direction. Due to $iL_Q \mathbb{r} Q = 4\Lambda Q$, it corresponds to the change in scales. At the nonlinear level, this can be done by introducing the pseudoconformal phase $e^{-ib\frac{r^2}{4}}$, parametrized by $b$ as in the (NLS) setting [33, 40]. To see how the pseudoconformal phase leads to blow-up, we consider $P = Q_b$ with

$$Q_b(y) := Q(y)e^{-ib\frac{r^2}{4}}.$$  

Note that $\partial_y Q_b = -i\frac{r^2}{4}Q$. Substituting this $P$ into (CSS) written in the renormalized variables $(s, y)$, we get

$$(i\partial_s - i\frac{\lambda_s}{\lambda} \Lambda - \gamma_s) Q_b - L^*_Q D(Q_b) Q_b = 0.$$  

From the identity [25, Lemma 4.2]

$$L^*_Q D(Q_b) Q_b = [L^* Q D(Q) + ib\Lambda Q + b^2 \frac{r^2}{4} Q] e^{-ib\frac{r^2}{4}} = ib\Lambda Q_b - b^2 \frac{r^2}{4} Q_b,$$

we are led to

$$-i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda Q_b - \gamma_s Q_b + (b_s + b^2) \frac{r^2}{4} Q_b = 0.$$  

The pseudoconformal blow-up is derived by

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = 0, \quad b_s + b^2 = 0.$$  

The explicit pseudoconformal blow-up solution $S(t, r)$ can be expressed as

$$S(t, r) = \frac{e^{iy(t)}}{\lambda(t)} Q_b(t) \left(\frac{r}{\lambda(t)}\right).$$
with
\[ \lambda(t) = |t|, \quad \gamma(t) = 0, \quad b(t) = |t|. \]

Let us remark that the presence of \( b^2 \) in the \( b \)-equation is a consequence of nonlinear algebras, not detected in the linearized dynamics itself. Without \( b^2 \)-term, it would be the self-similar blow-up. The pseudoconformal blow-up rate is derived from the \( b^2 \)-term.

The other element in the generalized null space, \( \rho \), leads to the change in phase parameter. At the linearized level, this is due to the relation \( i L \mathcal{Q} \rho = i \mathcal{Q} \). This motivates us to introduce the parameter \( \eta \) for \( \rho \) and set
\[ \gamma_s = \eta. \]

At the nonlinear level, introducing the parameter \( \eta \) is responsible for the rotational instability of pseudoconformal blow-up solutions. This is observed in authors’ previous work [25].

Indeed, we add one more degree of freedom to \( Q_b \). We assume \( Q_b^{(\eta)} = Q^{(\eta)} e^{-ib^2 \frac{y^2}{4}} \) is a modified profile satisfying \( \partial_\eta Q^{(\eta)} = -\rho \). We again start from the (CSS) on the renormalized variables
\[
(i \partial_s - i \frac{\lambda^s}{\lambda} \Lambda - \gamma_s) Q_b^{(\eta)} - L^*_{Q_b^{(\eta)}} D^{(Q^{(\eta)} b)} Q^{(\eta)} = 0.
\]

From the conjugation properties by the pseudoconformal phase \( e^{-ib^2 \frac{y^2}{4}} \), this reduces to
\[
-i \left( \frac{\lambda^s}{\lambda} + b \right) \Lambda Q_b^{(\eta)} - \gamma_s Q_b^{(\eta)} + (b_s + b^2) \frac{y^2}{4} Q_b^{(\eta)}
- [L^*_{Q^{(\eta)} b} D^{(Q^{(\eta)}) b} Q^{(\eta)}] e^{-ib^2 \frac{y^2}{4}} + i \eta_s \partial_\eta Q^{(\eta)} = 0.
\]

As before, we set \( \frac{\lambda^s}{\lambda} + b = 0 \). As we assumed \( \partial_\eta Q^{(\eta)} = -\rho \), this leads us to set \( \gamma_s = \eta \). It is also possible to set \( \eta_s = 0 \), by forcing \( Q^{(\eta)}(y) \) to be real-valued. Thus we may assume \( \eta \) is a nonzero constant. Now we are led to
\[
L^*_{Q^{(\eta)} b} D^{(Q^{(\eta)}) b} Q^{(\eta)} + \eta Q^{(\eta)} - (b_s + b^2) \frac{y^2}{4} Q^{(\eta)} = 0. \tag{3.1}
\]

One of the crucial observations in [25, Section 4.2] is that, in the nonlinear equation (3.1), \( b_s + b^2 \) has a nontrivial \( \eta^2 \)-order term as
\[ b_s + b^2 = -c \eta^2, \quad c = \frac{1}{(m+1)^2} + o_{\eta \to 0}(1). \]
This was derived by the Pohozaev-type identity, i.e. taking the inner product of (3.1) with $\Lambda Q^{(\eta)}$. Thus our modulation equation becomes

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = \eta, \quad b_s + b^2 + c\eta^2 = 0, \quad \eta_s = 0.$$ 

A typical example of the solutions to this system is (when $\eta \neq 0$)

$$\begin{cases}
\lambda(t) = \sqrt{t^2 + c\eta^2}, \\
\gamma(t) = \frac{1}{\sqrt{c}} \tan^{-1}\left(\frac{t}{\sqrt{c} \eta}\right), & \forall t \in \mathbb{R} \\
b(t) = -t
\end{cases}$$

For fixed $\eta \neq 0$, it not only says that the phase rotation takes place by turning on the $\rho$-direction of $N_k(iL_Q)$, but the solution does not blow up. In fact, it is global and scattering. It moreover turns out that the phase rotation takes place at the fixed amount of angle, which is $\text{sgn}(\eta)(m+1)\pi$. For $m$-equivariant functions, the phase rotation by $\pm(m+1)\pi$ reflects a spatial rotation by $\pm(m+1)\pi$. This is the main mechanism for instability of pseudoconformal blow-up solutions in [25]. This is the reason why we can expect at most the codimension-one blow-up.

Then the remaining question is to construct $Q^{(\eta)}$ of

$$L^*_{Q^{(\eta)}} D_+ (Q^{(\eta)}) Q^{(\eta)} + \eta Q^{(\eta)} + c\eta^2 \frac{y^2}{4} Q^{(\eta)} = 0 \quad (3.2)$$

such that $Q^{(\eta)}$ deforms from $Q$ with a small parameter $\eta$. We note that (3.2) is a nonlocal second-order ODE.

**Remark 3.1** A standard approach to construct $Q^{(\eta)}$ solving (3.2) would be to make a Taylor expansion $Q^{(\eta)} = Q + \sum_j \eta^j T_j$ with $T_1 = -\rho$ and solve for $T_j$ iteratively, by inverting the linearized operator $L_Q$. There are several technical difficulties for doing this. First, in order to invert the operator $L_Q$, we need to check the solvability condition: $L_Q T_j = \varphi$ would be solvable when $\varphi$ is orthogonal to the kernel elements of $L_Q$. To check this solvability condition, one needs to develop a generalization of the Pohozaev-type identity, for instance as in [43]. Second, the nonlocal terms from the gauge potential $A_0$ contain $\int r^{-\infty}$-integral. As one usually loses $y^2$-decay in the inversion procedure, the $\int r^{-\infty}$-integral may not be defined for higher order terms. Adding cutoffs in each of the inversion step will complicate the analysis.

In [25, Section 4.3], the authors get around the technical difficulties in Remark 3.1 by introducing a simple but remarkable nonlinear ansatz for $Q^{(\eta)}$. Motivated from the self-duality, or the Bogomol’nyi equation (1.6), solving (3.2) can be reduced to solving a first order (nonlocal) differential equation. More precisely, if $Q^{(\eta)}$ is a solution to the modified Bogomol’nyi equation

$$D^{(Q^{(\eta)})}_+ Q^{(\eta)} = -\eta \frac{y}{2} Q^{(\eta)}, \quad (3.3)$$
then it is a solution to
\[
L^*_Q(\eta) D_+^{(Q(\eta))} Q(\eta) + \eta \theta_\eta Q(\eta) + \eta^2 \frac{\gamma^2}{4} Q(\eta) = 0,
\]
\[
\theta_\eta = \frac{1}{4\pi} \int |Q(\eta)|^2 - (m + 1) \approx m + 1.
\]

With this new \(Q(\eta)\),\(^6\) the formal parameter ODEs become
\[
\lambda_s + b = 0, \quad \gamma_s - \eta \theta_\eta = 0, \quad b_s + b^2 + \eta^2 = 0, \quad \eta_s = 0.
\]

The authors solved the modified Bogomol’nyi equation for \(\eta \geq 0\) in [25, Proposition 4.4]. In the construction of global decaying \(Q(\eta), \eta \geq 0\) is necessary. Indeed, \(Q(\eta) \approx e^{-\eta \frac{\gamma^2}{4}}\) is expected due to the factor \(-\eta \frac{\gamma^2}{4}\) of (3.3). When \(\eta\) is negative, there is no decaying solution \(Q(\eta)\). Indeed, if \(Q(\eta)\) were to decay, then \(\frac{1}{2} (m + A_\theta[Q(\eta)])\) would be dominated by the factor \(-\eta \frac{\gamma^2}{4}\). This yields a contradicting fact: exponential growth of \(Q(\eta)\) for \(y \gg |\eta|^{-\frac{1}{2}}\).

In contrast to the previous work [25], in this paper we will use the parameter \(\eta\) as a modulation parameter that can vary in time. Moreover, we need to allow \(\eta\) to be either positive or negative. In particular, we need to construct \(Q(\eta)(y)\) even for \(\eta < 0\). This motivates us to truncate the profile in the region \(y \ll |\eta|^{-\frac{1}{2}}\), which is the regime where the exponential factor \(e^{-\eta \frac{\gamma^2}{4}} \lesssim 1\).

When we construct pseudoconformal blow-up dynamics, in view of
\[
\frac{\lambda_s}{\lambda} + b = 0 \quad \text{and} \quad b_s + b^2 + \eta^2 = 0,
\]
we can expect that the pseudoconformal blow-up occurs when \(\eta \to 0\) and \(|\eta| \ll b\), to guarantee \(b_s + b^2 \approx 0\).

We recall the construction of the modified profile \(Q(\eta)\). This is originally done in [25, Proposition 4.4] with \(\eta \geq 0\), but here we will truncate \(Q(\eta)\) for \(y \ll |\eta|^{-\frac{1}{2}}\) to allow \(\eta < 0\).\(^7\) We record basic properties of \(Q(\eta)\).

**Lemma 3.2** (Modified profile in the linearization regime) There exist universal constants \(0 < \delta < 1\) and \(\eta^* > 0\) and unique one-parameter family \(\{Q(\eta)\}_{\eta \in [-\eta^*, \eta^*]}\) of smooth real-valued functions on \((0, \delta|\eta|^{-\frac{1}{2}})\) satisfying the following.

1. (Smoothness on \(\mathbb{R}^2\)) The \(m\)-equivariant extension \(Q(\eta)(x) := Q(\eta)(y)e^{im\theta}, x = ye^{i\theta},\) defined on \(\{x \in \mathbb{R}^2 : |x| < \delta|\eta|^{-\frac{1}{2}}\}\) is smooth.

2. (Equation) We have
\[
D_+^{(Q(\eta))} Q(\eta) = -\eta \frac{\gamma^2}{2} Q(\eta), \quad \forall y \in (0, \delta|\eta|^{-\frac{1}{2}}).
\]

---

\(^6\) We naturally redefine \(Q(\eta)\) under \(\eta \leftrightarrow \eta \theta_\eta\).

\(^7\) In this paper, our \(Q(\eta)\) for \(\eta \geq 0\) is equal to that in [25, Proposition 4.4] on the region \(y < \delta \eta^{-\frac{1}{2}}\).
3. (Uniform bounds) For each $k \in \mathbb{N}$, we have

$$|Q^{(n)}|_k \lesssim_k Q.$$ 

4. (Differentiability in $\eta$) $Q^{(n)}(y)$ for $y \in (0, \delta|\eta|^{-\frac{1}{2}})$ is differentiable with respect to $\eta$. Moreover, the $\rho$-equivariant extension of $\partial_\eta Q^{(n)}$ defined on $\{x \in \mathbb{R}^2 : |x| < \delta|\eta|^{-\frac{1}{2}}\}$ is smooth. For each $k \in \mathbb{N}$, it also satisfies the pointwise estimates

$$|\partial_\eta Q^{(n)} + (m + 1)\rho|_k \lesssim_k |\eta|^y Q,$$

$$|\partial_\eta^2 Q^{(n)}|_k \lesssim_k y^4 Q.$$ 

In particular,

$$|Q^{(n)} + \eta(m + 1)\rho|_k \lesssim_k |\eta|^2 y^4 Q,$$

$$|\partial_\eta Q^{(n)}|_k \lesssim_k y^2 Q.$$

**Proof** Most of the assertions are proved in [25]. The construction of $Q^{(n)}$ (for $\eta \geq 0$) is already done in [25, Proposition 4.4], but in the regime $y \ll |\eta|^{-\frac{1}{2}}$, the sign of $\eta$ is irrelevant. Here, we focus on deriving the estimates both on $Q^{(n)}$ and $\partial_\eta Q^{(n)}$. For this purpose, we further look at $\eta$-variated equation. Nevertheless, the proof will be very similar to [25, Proposition 4.4]. We only sketch the proof and refer [25] for details.

Formally differentiating the equation in $\eta$, we get

$$D_+^{Q^{(n)}} Q^{(n)} = -\eta^y Q^{(n)},$$

$$L Q^{(n)} \partial_\eta Q^{(n)} = -\eta^y Q^{(n)} - \eta^2 \partial_\eta^2 Q^{(n)}.$$ 

Substituting $\eta = 0$ suggests that $\partial_{\eta=0} Q^{(n)} = -(m + 1)\rho$.

Motivated from this, we first introduce the unknown $v_1$ such that

$$Q^{(n)} = Q - \eta(m + 1)\rho + \eta^2 Q v_1,$$ 

(3.4)

and write the system of integral equations for $v_1, v_2$. The equation for $v_1$ is derived in [25, Proposition 4.4] and given as

$$v_1(y) = \int_0^y (m+1) \cdot \frac{\gamma^y}{Q} - (m + 1)^2 \frac{\rho}{Q} B_Q(\rho)dy' + \int_0^y B_Q(\eta Q v_1)dy'$$

$$+ \eta \int_0^y (-\frac{1}{2}y' + (m + 1)B_Q(\rho))v_1dy' + \eta^2 \int_0^y (\frac{1}{2}B_Q(\eta Q v_1))v_1dy'.$$ 

(3.5)

Following the proof of [25, Proposition 4.4], there exist $\delta > 0, \eta^* > 0$ such that one can construct $v_1$ for $|\eta| \leq \eta^*$ in the region $y < \delta|\eta|^{-\frac{1}{2}}$. Note that the sign of $\eta$ is irrelevant. Moreover, one has the estimate $|v_1| \lesssim y^4$ uniformly in $\eta$ (and region $y < \delta|\eta|^{-\frac{1}{2}}$). Iterating the integral equation (3.5) gives $|v_1| \lesssim_k y^4$ for any $k \in \mathbb{N}$. ❇️ Springer
Next, we want to know differentiability in $\eta$. For this purpose, we formally differentiate (3.4) to have

$$\partial_\eta Q^{(n)} = -(m+1)\rho + 2\eta Q v_1 + \eta^2 Q \partial_\eta v_1.$$  

Of course, this makes sense if we knew that $v_1$ is $\eta$-differentiable. To ensure that $Q^{(n)}$ is differentiable in $\eta$, we construct $\partial_\eta v_1$ similarly as $v_1$. Now we set $v_2 = \partial_\eta v_1$ as unknown, and derive the equation for $v_2$. Formally taking $\partial_\eta$ to (3.5) yields the $v_2$-equation:

$$v_2 = \int_0^\gamma B_Q(Q v_2) dy' + \eta \int_0^\gamma (-\frac{1}{2} y' + (m+1) B_Q \rho) v_2 dy' + \int_0^\gamma \eta (1 - \frac{1}{2} B_Q(Q v_2)) v_1 dy' + \eta^2 \int_0^\gamma \frac{1}{2} B_Q(Q v_1) v_2 dy' + 2\eta \int_0^\gamma \frac{1}{2} B_Q(Q v_1) v_1 dy'. \tag{3.6}$$

This is similar to (3.5) with some additional terms. Possibly shrinking $\delta$ and $\eta^*$, one can construct $v_2$ for $|\eta| \leq \eta^*$ in the regime $y < \delta|\eta|^{-\frac{1}{2}}$, by the mimicking the proof of [25, Proposition 4.4]. Moreover, the pointwise estimates $|v_2|_k \lesssim_k y^6$ hold uniformly in $\eta$ and in the region $y < \delta|\eta|^{-\frac{1}{2}}$.

To ensure that $Q^{(n)}$ is twice differentiable in $\eta$, we set $v_3 = \partial_\eta v_2$ as unknown, and derive the equation for $v_3$ by formally taking $\partial_\eta$ to (3.6). Similarly proceeding as before, the pointwise estimates $|v_3|_k \lesssim_k y^8$ hold uniformly in $\eta$ and in the region $y < \delta|\eta|^{-\frac{1}{2}}$. We omit the details.

Now define $Q^{(n)} = Q - \eta(m+1)\rho + \eta^2 Q v_1$. Due to the derivation of integral equations, $Q^{(n)}$ satisfies $D_+^{(Q^{(n)})} Q^{(n)} = -\eta^2 Q^{(n)}$. By the definition of $v_2$, $Q + \int_0^\eta (-\eta \rho + 2\eta Q v_1 + \eta^2 Q v_2) dy'$ also satisfies the same equation and pointwise estimates on $(0, \delta|\eta|^{-\frac{1}{2}})$, thus by uniqueness it must be equal to $Q^{(n)}$. In particular, $\partial_\eta Q^{(n)} = -(m+1)\rho + 2\eta Q v_1 + \eta^2 Q v_2$. From the pointwise bound $|v_1|_k + |v_2|_k \lesssim_k y^4$, the bounds for $Q^{(n)}$ and $\partial_\eta Q^{(n)}$ follow from those of $v_1, v_2$, and Lemma 2.1 (bounds for $\rho$). A similar argument using $v_3$ shows that twice differentiability of $Q^{(n)}$ the bounds for $\partial_\eta Q^{(n)}$ follow.

The smoothness of $Q^{(n)}(x)$ and $\partial_\eta Q^{(n)}(x)$ can be shown by writing the operators $D_+^{(Q^{(n)})}$ and $L_{Q^{(n)}}$ on the ambient space $\mathbb{R}^2$ and utilizing the ellipticity $\partial_1 + i \partial_2$; see for instance the proof of Lemma A.5. This completes the proof.\qed

Let $\delta$ and $\eta^*$ be fixed universal constants, given in the above lemma. In the sequel, we will assume $0 < b < b^*$ and $|\eta| < (\frac{1}{2})^2 b$. Here, $b^*$ is some small constant to be chosen later, but at this moment, let us only require $b^* < \eta^*$. We define the modified profile $P(y) = P(y; b, \eta)$ by applying the pseudoconformal phase to $Q^{(n)}$ and make a cutoff at $y \lesssim b^{-\frac{1}{2}}$:

$$P(y) := P(y; b, \eta) = \chi_{b^{-\frac{1}{2}}}(y) Q^{(n)}(y) e^{-ib^2 y^2} \tag{3.7}.$$
We may also define \( P(y; 0, 0) := Q(y) \). Here, we recall the notation that \( \chi_{b^{-\frac{1}{2}}} (y) \) is a smooth cutoff function supported in the region \( y \leq 2b^{-\frac{1}{2}} \).

**Proposition 3.3** (Properties of the modified profile) Let \( 0 < b < b^* \) and \( |\eta| < (\frac{\delta}{2})^2 b \). The modified profile \( P(y) = P(y; b, \eta) \) satisfies the following properties.

1. (Bounds for \( y \leq b^{-\frac{1}{2}} \)) For \( y \leq b^{-\frac{1}{2}} \), we have

\[
|P - Q|_5 + |\Delta P - \Delta Q|_5 \lesssim b y^2 Q,
\]
\[
|\partial_b P + i \frac{y^2}{4} Q|_5 + |\partial_\eta P + (m + 1) \rho|_5 \lesssim b y^4 Q.
\]

2. (Global bounds) We have

\[
|P|_5 + |\Delta P|_5 \lesssim 1_{y \leq 2b^{-1/2}} Q,
\]
\[
|\partial_b P|_5 + |\partial_\eta P|_5 \lesssim 1_{y \leq 2b^{-1/2}} y^2 Q,
\]
\[
|\partial_{bb} P|_5 + |\partial_b \eta P|_5 + |\partial_\eta \eta P|_5 \lesssim 1_{y \leq 2b^{-1/2}} y^4 Q.
\]

3. (Approximate generalized null space)

\[
|||A Q L Q v|_{-1}||_{L^2} + |||A Q L Q v|_{-3}||_{L^2} \lesssim b, \quad \forall v \in \{ \Delta P, i P, \partial_b P, \partial_\eta P \}(3.8)
\]

4. (Approximate pseudoconformal profile) One can write

\[
L_P^* D^{(P)}_+ P - ib \Delta P + (\eta \theta_\eta + \theta_\psi) P + (b^2 + \eta^2) i \partial_b P = \Psi,
\]

where the scalars \( \theta_\eta, \theta_\psi, \) and function \( \Psi(y) = \Psi(y; b, \eta) \) satisfy

\[
|\theta_\eta - (m + 1)| \lesssim |\eta| + b^{m+1},
\]
\[
|\theta_\psi| \lesssim b^{m+2},
\]
\[
|\Psi|_5 \lesssim b^{\frac{m}{2} + 1} 1_{b^{-1/2} \leq y \leq 2b^{-1/2}}.
\]

Moreover, we have the charge estimate

\[
\| P \|_{L^2}^2 = \| Q \|_{L^2}^2 + O(|\eta| + b^{m+1}).
\]

5. (Difference estimate) We have

\[
|\partial_b \theta_\eta| \lesssim b^m,
\]
\[
|\partial_\eta \theta_\eta| \lesssim 1,
\]
\[
|\partial_b \theta_\psi| + |\partial_\eta \theta_\psi| \lesssim b^{m+1},
\]
\[
|\partial_b \Psi|_5 + |\partial_\eta \Psi|_5 \lesssim b^{\frac{m}{2} + 1} 1_{b^{-1/2} \leq y \leq 2b^{-1/2}}.
\]

8 Note that \( \theta_\eta \) also depends on \( b \) (the cutoff distance \( b^{-1/2} \), in fact).
6. (Size of $\Psi$) We have

\[ \| \Psi \|_{H^3} \lesssim b^\frac{m}{2} + 1, \tag{3.17} \]
\[ \| \Psi \|_{H^5} \lesssim b^\frac{m}{2} + 3, \quad \text{if } m \geq 3. \tag{3.18} \]

**Remark 3.4** This nonlinear approach for the construction of modified profile is much simpler than the linear expansion such as $P = Q - ib^\frac{y^2}{4} Q - (m + 1) \eta \rho + (\text{h.o.t.)}. But it relies on a very special algebraic property, the self-duality. In view of the above discussions, we are led to solve (3.2) for $Q^{(n)}$. However, it is a nonlocal second-order elliptic equation. We used the self-duality to reduce it to a first-order nonlocal equation (3.3). This reduction is crucial in our analysis. We are not sure if it is possible to solve (3.2) without this reduction. There are similar nonlinear approaches for construction of modified profiles in the context of mass-critical NLS [33, 40]. There, the nonlinear terms are local and it is possible to solve it by variational methods.

**Remark 3.5** There seems to be a slight flexibility for the cutoff radius $b^{-\frac{1}{2}}$. We expect that one can use $b^{-\frac{1}{2}} \pm$. Here we chose $b^{-\frac{1}{2}}$ because it matches the radius where $|e^{-ib^\frac{y^2}{4}}|_k \sim k$ and $|e^{-\eta^\frac{y^2}{4}}|_k \ll k$.

**Proof of Proposition 3.3** For simplicity in notations, let us write

\[ B := b^{-\frac{1}{2}}. \]

Note that $\partial_y \chi_B(y) = \frac{1}{2} b^{-\frac{1}{2}} y[\partial_y \chi](b^{\frac{1}{2}} y)$ so $|\partial_y \chi_B|_4 \lesssim y^2 \chi_B \leq B \leq y \lesssim B$.

(1) Since $\Lambda = y \partial_y + 1$ and the cutoff function $\chi_B$ can be ignored for $y \leq B$, we have

\[ |P - Q|_5 + |\Lambda P - \Lambda Q|_5 \lesssim |e^{-ib^\frac{y^2}{4}} Q^{(n)} - Q|_6 \]
\[ \lesssim |e^{-ib^\frac{y^2}{4}} - 1|_6 |Q^{(n)}|_6 + |Q^{(n)} - Q|_6 \lesssim by^2 Q. \]

Similarly,

\[ |\partial_y P + i^\frac{y^2}{4} Q|_5 + |\partial_\eta P + (m + 1) \rho|_5 \]
\[ = | - i^\frac{y^2}{4} e^{-ib^\frac{y^2}{4}} Q^{(n)} + i^\frac{y^2}{4} Q|_5 + |e^{-ib^\frac{y^2}{4}} \partial_\eta Q^{(n)} + (m + 1) \rho|_5 \]
\[ \lesssim |e^{-ib^\frac{y^2}{4}} - 1|_5(|y^2 Q^{(n)})_5 + |\partial_\eta Q^{(n)}|_5 + y^2 (Q^{(n)} - Q)_5 + |\partial_\eta Q^{(n)} + (m + 1) \rho|_5 \]
\[ \lesssim by^4 Q. \]

(2) This easily follows from the arguments in (1), uniform bounds on $Q^{(n)}$ in Lemma 3.2, $|\partial_y \chi_B|_6 \lesssim y^2 \chi_B \leq 2B$, and $|\partial_\eta \chi_B|_6 \lesssim y^4 \chi_B \leq 2B$. 

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(3) Using $L_Q \Lambda Q = 0$, boundedness of $A_Q L_Q : \hat{H}^3_m \to \hat{H}^1_{m+2}$ (Lemmas A.10 and A.13), (1) for $y \leq B$, and (2) for $y > B$, we have

$$
\| A_Q L_Q \Lambda P \|_{L^2} = \| A_Q L_Q(\Lambda P - \Lambda Q) \|_{L^2} \\
\lesssim \| \Lambda P - \Lambda Q \|_{L^2} \\
\lesssim \| 1_{y \leq B} b y Q + 1_{y > B} y^{-3} Q \|_{L^2} \lesssim b.
$$

A similar argument can be done for $i P$. Next, using $A_Q L_Q \rho = 0$, and similarly as before, we have

$$
\| A_Q L_Q \partial_i P \|_{L^2} = \| A_Q L_Q(\partial_i P + (m + 1) \rho) \|_{L^2} \\
\lesssim \| \partial_i P + (m + 1) \rho \|_{L^2} \\
\lesssim \| 1_{y \leq B} b y Q + 1_{y > B} y^{-3} Q \|_{L^2} \lesssim b.
$$

A similar argument can be done for $\partial b P$. Estimates of $\| A_Q L_Q v \|_{L^2}$ can be shown in a similar way using boundedness of $A_Q L_Q : \hat{H}^5_m \to \hat{H}^3_{m+2}$ (Lemmas A.20 and A.22).

(4, 5) By adding the cutoff $\chi_B$ to $Q^{(\eta)}$, we have

$$
D_+^{(\chi_B Q^{(\eta)})} [\chi_B Q^{(\eta)}] = -\eta \chi_B Q^{(\eta)} + \phi Q^{(\eta)}, \\
\phi := \partial_y \chi_B + \frac{1}{y} (A_\theta [Q^{(\eta)}] - A_\theta [\chi_B Q^{(\eta)}]) \chi_B.
$$

We now take $L^*_{\chi_B Q^{(\eta)}}$. First we recall the computation from [25]

$$
L^*_{\chi_B Q^{(\eta)}} (-\frac{y}{2} \chi_B Q^{(\eta)}) = \frac{1}{2} D_+^{(\chi_B Q^{(\eta)})} \chi_B Q^{(\eta)} + (m + 1 - \frac{1}{4\pi} \int |\chi_B Q^{(\eta)}|^2) \chi_B Q^{(\eta)}.
$$

Next we write

$$
L^*_{\chi_B Q^{(\eta)}} [\phi Q^{(\eta)}] = D_+^{(\chi_B Q^{(\eta)})} [\phi Q^{(\eta)}] \\
+ (\int_0^{\infty} \chi_B \phi |Q^{(\eta)}|^2 dy') \chi_B Q^{(\eta)} - (\int_0^{y} \chi_B \phi |Q^{(\eta)}|^2 dy') \chi_B Q^{(\eta)}.
$$

Summing the above two displays, we have

$$
L^*_{\chi_B Q^{(\eta)}} D_+^{(\chi_B Q^{(\eta)})} \chi_B Q^{(\eta)} + (\eta \partial_\eta + \theta_\psi) \chi_B Q^{(\eta)} + \eta^2 \frac{y^2}{4\pi} \chi_B Q^{(\eta)} = \tilde{\Psi},
$$

where

$$
\theta_\eta = \frac{1}{4\pi} \int |\chi_B Q^{(\eta)}|^2 - (m + 1), \\
\theta_\psi = -\int_0^{\infty} \chi_B \phi |Q^{(\eta)}|^2 dy, \\
\tilde{\Psi} = (\eta \frac{y}{2} + D_+^{(\chi_B Q^{(\eta)})}) \phi Q^{(\eta)} - (\int_0^{y} \chi_B \phi |Q^{(\eta)}|^2 dy') \chi_B Q^{(\eta)}.
$$
Now we conjugate the pseudoconformal phase to get

$$L_p^* D_+^{(P)} P - i b \Lambda P + (\eta \theta_\eta + \theta_\Psi) P + (b^2 + \eta^2) \frac{y^2}{\tau} P = \tilde{\Psi} e^{-ib \frac{y^2}{\tau}}.$$  

The proof of (3.9) follows by setting

$$\Psi = \tilde{\Psi} e^{-ib \frac{y^2}{\tau}} + (b^2 + \eta^2)(i \partial_b P - \frac{y^2}{\tau} P)$$

$$= [\tilde{\Psi} + i(b^2 + \eta^2)(\partial_b \chi) Q^{(n)}] e^{-ib \frac{y^2}{\tau}}.$$  

We turn to show $\theta_\eta$ estimates. The estimate (3.10) follows from

$$\left| \int \chi B Q^{(n)} - \int_{1 \leq y \leq B} |Q^{(n)}|^2 \right| \leq \int_{1 \leq y \leq 2B} |Q^{(n)}|^2 \lesssim b^{m+1},$$

$$\left| \int_{1 \leq B} |Q^{(n)}|^2 - \int_{1 \leq B} |Q|^2 \right| \lesssim \int_{1 \leq B} |Q^{(n)} - Q| \lesssim \int_{1 \leq B} |\eta| y^2 Q^2 \lesssim |\eta|,$$

$$\int_{1 \leq B} |Q^2 - \int_{1 \leq B} Q^2 = 8\pi (m + 1) - o(b^{m+1}).$$

The above display also shows the charge estimate. Next, (3.13) follows from

$$\left| \partial_b \theta_\eta \right| \lesssim \int |\partial_b \chi B| Q^2 \lesssim b^m,$$

$$\left| \partial_\eta \theta_\eta \right| \lesssim \int \chi B Q \cdot y^2 Q \lesssim 1.$$  

We turn to show $\theta_\Psi$ estimates. We note that $\phi$ is supported in the region $B \leq y \leq 2B$ and satisfies

$$|\phi|_6 \lesssim y^{-1} 1_{B \leq y \leq 2B} \quad \text{and} \quad |\partial_b \phi|_6 \lesssim y 1_{B \leq y \leq 2B}.$$  

Thus we have (3.11) and (3.15):

$$|\theta_\Psi| \lesssim \int_0^\infty 1_{y \sim B} y^{-1} Q^2 dy \lesssim b^{m+2},$$

$$|\partial_b \theta_\Psi| + |\partial_\eta \theta_\Psi| \lesssim \int_0^\infty 1_{y \sim B} y Q^2 dy \lesssim b^{m+1}.$$  

Next, we show the $\Psi$ estimates. We note that $\tilde{\Psi}$ is supported in the region $B \leq y \leq 2B$. We also note that

$$|A_\theta [\chi B Q^{(n)}]|_5 + |\partial_b A_\theta [\chi B Q^{(n)}]|_5 + |\partial_\eta A_\theta [\chi B Q^{(n)}]|_5 \lesssim 1,$$

thus

$$\left| D_+^{(\chi B Q^{(n)})} f \right|_5 + |\partial_b D_+^{(\chi B Q^{(n)})} f|_5 + |\partial_\eta D_+^{(\chi B Q^{(n)})} f|_5 \lesssim y^{-1} |f|_5.$$  

Using $|\eta| \leq b$ and the above $|\phi|_6$-estimates, we have

$$\left| \tilde{\Psi} \right|_5 \lesssim 1_{B \leq y \leq 2B} y^{-2} Q \lesssim b^{\frac{m}{2}+2} 1_{B \leq y \leq 2B},$$

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On the other hand, we have

\[
|\partial_b \tilde{\Psi}|_5 + |\partial_\eta \tilde{\Psi}|_5 \lesssim 1_{B \leq y \leq 2B} Q \lesssim b^{-\frac{m}{2}} 1_{B \leq y \leq 2B}.
\]

Adding the pseudoconformal phase completes the proof of (3.12) and (3.16).

(6) By the definitions (2.13) and (2.15), and pointwise estimates (3.12), we have

\[
\|\tilde{\Psi}\|_{\dot{H}^3} \lesssim \|\tilde{\Psi}\|_{L^2} \lesssim b^{\frac{m}{2} + 3},
\]

\[
\|\tilde{\Psi}\|_{\dot{H}^5} \lesssim \|\tilde{\Psi}\|_{L^2} \lesssim b^{\frac{m}{2} + 4}, \quad \text{if } m \geq 3.
\]

\[
\square
\]

4 Trapped Solutions

So far, we have constructed the modified profile \(P\), which formally suggested the pseudoconformal blow-up. In this section, we present how we decompose our solution \(u\) into the blow-up part \(P\) and the error \(\epsilon\). The dynamics of the blow-up part \(P\) will be governed by four modulation parameters \(\lambda, \gamma, b, \eta\). As illustrated above, we will fix the modulation parameters through orthogonality conditions on \(\epsilon\). The choice of orthogonality conditions is motivated to almost decouple the dynamics of \(P\) and \(\epsilon\). It will be shown that we can decompose our solution \(u\) as long as \(u\) stays in a soliton tube \(Q\) with \(|\eta| \ll b\). This is done in Lemma 4.2.

Having fixed the decomposition, we discuss how to reduce the existence part of our main theorems into Proposition 4.4 (main bootstrap) and Proposition 4.5 (existence of special \(\eta_0\)). Here we motivate the notion of trapped solutions defined in Introduction. In fact the conditions of trapped solutions are exactly the bootstrap hypotheses.

4.1 Decomposition of Solutions

We will decompose our solution of the form

\[
u(t, r) = e^{i\gamma(t)} \frac{r}{\lambda(t)} [P(\cdot; b(t), \eta(t)) + \epsilon(t, \cdot)](\frac{r}{\lambda(t)}).
\]

(4.1)

For \(P\) and \(\epsilon\), we denote the rescaled spatial variable by \(y = \frac{r}{\lambda(t)}\). To get an idea of this decomposition, let us recall the discussion in Sect. 2.1. The linearized evolution has two invariant subspaces \(N_g(i\mathcal{L}_Q)\) and \(N_g(\mathcal{L}_Q i)^\perp\), and the whole space decomposes as \(N_g(i\mathcal{L}_Q) \oplus N_g(\mathcal{L}_Q i)^\perp\). Note that this decomposition is not orthogonal, but is possible because \((4.14)\) is nonsingular. This fact will be a key to our decomposition (4.1)
(Lemma 4.2). Roughly speaking, $\frac{\epsilon^{ij}}{\Lambda} P(\partial_i \rho, b, \eta)$ corresponds to the $N_g(iLQ)$ part and $\epsilon$ corresponds to $N_g(L_Qi)_{\perp}$ part of the solution.

For the $N_g(iLQ)$ part, we introduce four modulation parameters $\lambda(t)$, $\gamma(t)$, $b(t)$, and $\eta(t)$ corresponding to the generalized null space $N_g(iLQ) = \text{span}_\mathbb{R}\{\Lambda Q, iQ, iy^2Q, \rho\}$. The scaling $\lambda(t)$ and phase rotation $\gamma(t)$ correspond to the kernel elements $\Lambda Q$ and $iQ$. As (CSS) has explicit scaling and phase rotation symmetries, we use $\lambda(t)$ and $\gamma(t)$ to renormalize our solution. More precisely, this renormalization enables us to transform the blow-up dynamics to a near-$Q$ dynamics. The pseudoconformal phase $b(t)$ and an additional parameter $\eta(t)$ correspond to the generalized null space elements $iy^2Q$ and $\rho$. These two parameters are used in the modified profile in the previous section. Due to the generalized null space relations $iLQi y^2Q = 4\Lambda Q$ and $iLQ\rho = iQ$, turning on the parameters $b(t)$ and $\eta(t)$ change the scaling $\lambda(t)$ and phase rotation $\gamma(t)$. It is observed in [25] that the $\eta$-parameter accounts for the instability of pseudoconformal blow-up. In [25] we fixed $\eta$ as a small constant in time, but in this paper we allow $\eta(t)$ to vary in time, to take advantages of additional degree of freedom for choosing the modulation parameters. This extra degree of freedom will sharpen our modulation analysis.

For the $N_g(L_Qi)_{\perp}$ part, we impose four orthogonality conditions on $\epsilon$. Since we will carry out energy estimates for $\epsilon$ in $H^3_m$ and $H^5_m$ levels, we will regard $\epsilon$ as an element of the adapted function space $\mathcal{H}^3_m$ and $\mathcal{H}^5_m$, respectively (see Sect. 2.3). However, if $m$ is small, $N_g(L_Qi)_{\perp} = \{i\Lambda Q, Q, y^2Q, i\rho\}_{\perp}$ is not well-defined in the adapted function spaces, due to lack of decay $|y^2Q|, |\rho| \sim y^{-m}$ for $y \gg 1$. Thus we will use localized versions of these orthogonality conditions defined as follows.

Define the codimension-four linear subspace $\mathcal{Z} \subset H^3_m$ by

$$\mathcal{Z}_{\perp} := \{\epsilon \in H^3_m : (\epsilon, Z_k)_r = 0 \text{ for all } k \in \{1, 2, 3, 4\}\},$$

where

$$Z_1 := y^2Q\chi_M - \frac{(\rho, y^2Q\chi_M)}{(\rho, \rho\chi_M)}L_Q(\rho\chi_M),$$

$$Z_2 := i\rho\chi_M - \frac{(y^2Q, \rho\chi_M)}{4(\Lambda Q, y^2Q\chi_M)}L_Q(iy^2Q\chi_M),$$

$$Z_3 := L_QiZ_1,$$

$$Z_4 := L_QiZ_2.$$

Thus in the decomposition (4.1), we will require $\epsilon \in \mathcal{Z}_{\perp}$. For $H^5_m$ solutions, we will require $\epsilon \in \mathcal{Z}_{\perp} \cap H^5_m$. Note that $\mathcal{Z}_k$’s deform from the basis elements of the generalized null space $N_g(L_Qi)$ by some large cutoff parameter $M$. The additional factors in $Z_1$ and $Z_2$ help us to see transversality of $\mathcal{Z}_{\perp}$ and $N_g(iLQ)$. More precisely, this results in that the inner product matrix of $iQ, \Lambda Q, \rho, iy^2Q$ (which correspond to

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9 We note that (CSS) has explicit pseudoconformal symmetry (in continuous form), but we will not apply it directly to our solution since it will require us to work with the weighted Sobolev spaces. Instead, we apply the pseudoconformal phase to our modified profile. The parameter $\eta(t)$ does not seem to correspond to an explicit symmetry.
tangent vectors to the manifold of our modified profiles) and $Z_1$, $Z_2$, $Z_3$, $Z_4$ becomes diagonal; see (4.10). In other words, we can impose the orthogonality conditions (4.2).

**Lemma 4.1** (Estimates of $Z_k$) We have the following estimates of $Z_k$:

1. (Pointwise estimates) We have

   \[
   |Z_1| + |Z_2| \lesssim 1_{y \leq 2M} (y^2 + \log M) Q + 1_{y \geq 2M} (\log M) y^{-2} Q, \\
   |Z_3| + |Z_4| \lesssim 1_{y \leq 2M} (1 + (y)^{-2} \log M) Q + 1_{y \geq 2M} y^{-2} Q, 
   \]

   (4.4)

2. (Sobolev estimates) There exists a universal constant $C$ such that

   \[
   \sup_{k \in \{1, 2, 3, 4\}} \|Z_k\|_{L^2} \lesssim \log M, \\
   \sup_{k \in \{1, 2, 3, 4\}} \|Z_k\|_{(H^3_m)^*} \lesssim M^C, \\
   \sup_{k \in \{1, 2, 3, 4\}} \|Z_k\|_{(H^m_m)^*} \lesssim M^C, \quad \text{if } m \geq 3. 
   \]

   (4.5) (4.6) (4.7)

3. (Approximate null space relations) We have

   \[
   \sup_{k \in \{1, 2\}} \|y A_Q L_Q i Z_k\|_{L^2} \lesssim M^{-1}, \\
   \sup_{k \in \{1, 2\}} \|y^3 A_Q L_Q i Z_k\|_{L^2} \lesssim M^{-1}, \quad \text{if } m \geq 3. 
   \]

   (4.8) (4.9)

4. (Transversality) We have

   \[
   (\Lambda Q, Z_k)_r = (-\|y Q\|_{L^2}^2 + o_{M \to \infty}(1)) \delta_{1k}, \\
   (iQ, Z_k)_r = (\|L_Q \rho\|_{L^2}^2 + o_{M \to \infty}(1)) \delta_{2k}, \\
   (iy^2 Q, Z_k)_r = (4\|y Q\|_{L^2}^2 + o_{M \to \infty}(1)) \delta_{3k}, \\
   (\rho, Z_k)_r = (-\|L_Q \rho\|_{L^2}^2 + o_{M \to \infty}(1)) \delta_{4k}, 
   \]

   (4.10)

   where $\delta_{jk}$ is the Kronecker-delta symbol.

**Proof** (1) We first show pointwise estimates on $Z_k$. We note

\[
|\rho \chi M|_4 + |y^2 Q \chi M|_4 \lesssim 1_{y \leq 2M} y^2 Q, \\
|L_Q (\rho \chi M)|_3 + |L_Q (iy^2 Q \chi M)|_3 \lesssim 1_{y \leq 2M} y Q + 1_{y \geq 2M} y^{-1} Q, \\
|L_Q (\rho \chi M)|_2 + |L_Q (iy^2 Q \chi M)|_2 \lesssim 1_{y \leq 2M} Q + 1_{y \geq 2M} y^{-2} Q. 
\]
Using $|y^2Q, \rho Q_M| \leq |ho, y^2Q_M| \lesssim \log M$, we get the first estimate of (4.4). Next, we take $L_QiL^*Q = iA^*QA_Q$ to have

$$|L_QiL_Q(\rho Q_M)||1 + |L_QiL_Q(iy^2Q_M)| \lesssim y|L_Q(\rho Q_M)| - 3 + |L_Q(iy^2Q_M)| - 3 \lesssim 1_{y \leq 2M}y^{-2}Q + 1_{y \geq 2M}y^{-3}Q,$$

$$|L_QiL_Q(\rho Q_M)| + |L_QiL_Q(iy^2Q_M)| \lesssim 1_{y \leq 2M}y^{-2}Q + 1_{y \geq 2M}y^{-3}Q.$$

Combining the above estimates completes the proof of (4.4).

(2) The Sobolev estimates on $Z_k$ follow from applying the pointwise estimates to

$$\|Z_k\|_{(H^\ell)^*} \lesssim \|y\|_\ell Z_k\|_{L^2}, \quad \forall \ell \in \{0, 3, 5\}.$$

(3) We turn to (4.8) and (4.9). From $A_QL_Q\rho = 0$, we have

$$|A_QL_Q(\rho Q_M)| \leq |A_QL_Q(\rho - \rho Q_M)| \lesssim |L_Q(\rho - \rho Q_M)| - 1 \lesssim 1_{y \geq M}Q,$$

$$|A_QL_Q(iy^2Q_M)| \lesssim |y^2Q - y^2Q_M| - 2 \lesssim 1_{y \geq M}Q.$$

Further utilizing $L_QiL^*Q = iA^*QA_Q$, we have

$$|A_QL_QiL_Q(\rho Q_M)| \lesssim |L_Q(\rho - \rho Q_M)| - 3 \lesssim 1_{y \geq M}y^{-2}Q,$$

$$|A_QL_QiL_Q(iy^2Q - iy^2Q_M)| \lesssim |y^2Q - y^2Q_M| - 4 \lesssim 1_{y \geq M}y^{-2}Q.$$

Thus

$$\sup_{k \in [1.2]} |A_QL_QiZ_k| \lesssim 1_{y \geq M}Q.$$

This pointwise estimate immediately implies (4.8) and (4.9).

(4) Finally, we verify the transversality condition. First,

$$(\Lambda Q, Z_k)_r = (\Lambda Q, y^2Q_M)_r \delta_{1k},$$

$$(iQ, Z_k)_r = (Q, \rho Q_M)_r \delta_{2k},$$

because $Z_1$, $Z_4$ are real, $Z_2$, $Z_3$ are imaginary, and $L_Q\Lambda Q = L_QiQ = 0$. Next, due to the additional terms in the definition of $Z_1$ and $Z_2$, and the algebras $L_QiQy^2Q = -4i\Lambda Q$ and $L_Q\rho = Q$, we have

$$(iy^2Q, Z_k)_r = (-4\Lambda Q, y^2Q_M)_r \delta_{3k},$$

$$(\rho, Z_k)_r = (Q, -\rho Q_M)_r \delta_{4k}.$$
Now the computations
\[ (\Lambda Q, y^2 Q \chi_M)_r = (\Lambda Q, y^2 Q)_r + O(M^{-2}) = -\|yQ\|_{L^2}^2 + O(M^{-2}), \]
\[ (Q, \rho \chi_M)_r = (Q, \rho)_r + O(M^{-2}) = \|L \rho\|_{L^2}^2 + O(M^{-2}), \]
complete the proof. \( \square \)

In the sequel, we will perform a bootstrap analysis. In the decomposition (4.1), we will assume \(|\eta| \ll b\) and \(\epsilon\) stay small to guarantee the pseudoconformal blow-up. A quantitative version of this smallness will be given as our bootstrap assumptions and in fact used in the definition of trapped solutions.

Having fixed \(\mathbb{Z}^\perp (4.2)\), let \(\mathcal{U}\) be a subset of \(\mathbb{R}_+ \times \mathbb{R} / 2\pi \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{Z}^\perp\) consisting of \((\lambda, \gamma, b, \eta, \epsilon)\) satisfying
\[ b \in (0, b^*), \ |\eta| < Kb^{3/2}, \ |\epsilon|_{L^2} < K(b^*)^{1/2}, \ |\varepsilon_1|_{L^2} < Kb, \ |\varepsilon_3|_{L^2} < Kb^{5/2}(4.11) \]
(4.11) will be used as a bootstrap hypothesis. The largeness of \(K\) and smallness of \(b^*\) will be chosen in the proof. See also Remark 1.4 for the parameter dependence. We now define \(\mathcal{O} \subseteq H_m^3\) by the set of images
\[ u(r) = \frac{e^{i\gamma}}{\lambda} \left[P(\cdot; b, \eta) + \epsilon \right] \left(\frac{r}{\lambda}\right) \in H_m^3. \quad (4.12) \]
By the definition (1.8), \(\mathcal{O}\) contains \(\mathcal{O}_{\text{init}}\). Moreover, by the definition (1.10), the \(H_m^3\)-trapped solutions lies in \(\mathcal{O}\) in their forward lifespan. Next, we define \(\overline{\mathcal{U}}\) and \(\overline{\mathcal{O}}\) by the closure of \(\mathcal{U}\) and \(\mathcal{O}\), respectively. Note that the case \(b = \eta = 0\) belongs to \(\overline{\mathcal{U}}\), and we will use \(P(\cdot; 0, 0) = Q\). As Lemma 4.2 shows, the set \(\mathcal{O}\) is open and \((\lambda, \gamma, b, \eta, \epsilon)\) can be considered as coordinates of \(u\).

Similarly, we can describe \(H_m^5\)-bootstrap assumptions when \(m \geq 3\). Let \(\mathcal{U}^5\) be a subset of \(\mathbb{R}_+ \times \mathbb{R} / 2\pi \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times (\mathbb{Z}^\perp \cap H_m^5)\) consisting of \((\lambda, \gamma, b, \eta, \epsilon)\) satisfying
\[ b \in (0, b^*), \ |\eta| < Kb^{3/2}, \ |\epsilon|_{L^2} < K(b^*)^{1/2}, \ |\varepsilon_1|_{L^2} < Kb, \]
\[ |\varepsilon_3|_{L^2} < Kb^{5/2}, \ |\varepsilon_5|_{L^2} < Kb^{9/2}. \quad (4.13) \]
Define \(\mathcal{O}^5 \subseteq H_m^5\) by the set of images according to the formula (4.12). Thus \(H_m^5\)-trapped solutions lies in \(\mathcal{O}^5\) in their forward lifespan.

**Lemma 4.2** (Decomposition) Consider
\[ \Phi : \overline{\mathcal{U}} \to H_m^3 \]
that maps \((\lambda, \gamma, b, \eta, \epsilon) \in \overline{\mathcal{U}}\) to \(u\) via (4.12). Then, \(\Phi(\overline{\mathcal{U}}) = \overline{\mathcal{O}}\) and \(\Phi\) is a homeomorphism onto \(\overline{\mathcal{O}}\). The subset \(\mathcal{O}\) is open in \(H_m^3\). Moreover, \(\Phi|_{\mathcal{U}}\) is \(C^1\) in \((\gamma, b, \eta, \epsilon)\) and continuous in \(\lambda\). The \((\lambda, \gamma, b, \eta, \epsilon)\)-components of \(\Phi^{-1}|_{\mathcal{O}}\) is \(C^1\) and \(\epsilon\)-component is continuous. The analogous statements also hold when \(\mathcal{U}, \mathcal{O}, H_m^3\) are replaced by \(\mathcal{U}^5, \mathcal{O}^5, H_m^5\).
Proof Let us introduce some notations that will be used only in this proof. For $\lambda \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}/2\pi \mathbb{Z}$, let us denote

$$f_{\lambda, \gamma}(y) := \frac{e^{iy}}{\lambda} f\left(\frac{y}{\lambda}\right).$$

We equip $\mathbb{R}_+$ with the metric $\text{dist}(\lambda_1, \lambda_2) := |\log(\lambda_1/\lambda_2)|$, and equip $\mathbb{R}/2\pi \mathbb{Z}$ with the induced metric from $\mathbb{R}$. We will use small parameters $\delta_1, \delta_2 > 0$ (to be chosen later in this proof) such that $0 < \delta_2 \ll \delta_1 \ll \min\{\delta, \eta^*, M^{-1}\}.\text{11}

Now we turn to $P(y; b, \eta)$. Notice that the profile $P(y; b, \eta)$ is only defined for $(b, \eta)$ with $|\eta| \ll b$, not for $(b, \eta) \ll 1$. We extend it for $b < 0$ with $|\eta| \leq (\frac{1}{2})^2|b|$ as $P(y; b, \eta) := \chi_{[b]^{-}\frac{2}{5}}(y) Q^{(\eta)}(y) e^{-ib \frac{2}{5} \eta^2}$. We hope to apply the implicit function theorem near $Q = P(\cdot; 0, 0)$, so we will define an artificial extension of the profile $P(y; b, \eta)$ for $(b, \eta)$ in a neighborhood of $(0, 0)$. We introduce a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(\tilde{\eta}) = \tilde{\eta}$ for $|\tilde{\eta}| \leq 1$ and $\sup(|\psi| + |\tilde{\eta}\psi'|) \leq 2$. For $|b| < \delta_1$, we define $\psi(b) = |b| \frac{5}{4} \psi(|b|^{-\frac{5}{4}} \tilde{\eta})$ if $b \neq 0$ and $\psi(0) = 0$. Thus $\tilde{\partial}_{b} \psi(b) = -\text{sgn}(b) |b|^{\frac{5}{4}} [\lambda_2 \psi(|b|^{-\frac{5}{4}} \tilde{\eta})]$ if $b \neq 0$ and $\tilde{\partial}_{b=0} \psi(b) = 0$. Finally, we define an artificial extension $\tilde{P}(\cdot; b, \eta)$ of $P(y; b, \eta)$ by

$$\tilde{P}(\cdot; b, \eta) := P(\cdot; b, \psi(b)(\eta)) - (m+1)(\eta - \psi(b)(\eta)) \rho \chi_{2M}$$

for $|\eta|, |b| < \delta_1.\text{12}$ Thus $\tilde{P}(\cdot; b, \eta) = P(\cdot; b, \eta)$ for $|\eta| \leq |b|^{\frac{4}{5}}$ and hence for $(b, \eta)$ in our bootstrap hypothesis (4.11).

We now come to our main part of the proof. We hope to apply the implicit function theorem to the map $F = (F_1, F_2, F_3, F_4)^{T} : \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_{\delta_1}(0) \times B_{\delta_1}(0) \times L^2 \to \mathbb{R}^4$ defined by

$$F_k(\lambda, \gamma, b, \eta, u) := (\epsilon, Z_k)_{r}, \quad \forall k \in \{1, 2, 3, 4\}$$

$$\epsilon = u_{\lambda^{-1}, \gamma, b, \eta}(y) - \tilde{P}(y; b, \eta).$$

We claim that $F$ is $C^1$ and $\partial_{\lambda, \gamma, b, \eta} F$ is invertible at $(\lambda, \gamma, b, \eta, u) = (1, 0, 0, 0, Q)$. To see this, we compute using (4.10)

$$\partial_{\lambda} F_k = [(\Lambda Q)_{\lambda^{-1}}, \gamma, Z_k]_{r} - (u - Q, [\Lambda Z_k]_{\lambda, \gamma})_{r}$$

$$= -\delta_1 k\|Q\|_{L^2}^2 + M C O(\text{dist}(\lambda, \gamma), (1, 0)) + \|u - Q\|_{L^2} + o_{M \to \infty}(1),$$

$$\partial_{\gamma} F_k = -([i Q]_{\lambda^{-1}, \gamma, Z_k})_{r} + (u - Q, [i Z_k]_{\lambda, \gamma})_{r}$$

$$= -\delta_2 k\|L Q\rho\|_{L^2}^2 + M C O(\text{dist}(\lambda, \gamma), (1, 0)) + \|u - Q\|_{L^2} + o_{M \to \infty}(1).$$

\text{11} Recall from Remark 1.4 that this implies $\delta_1 \leq M^{-C}$.

\text{12} Note here that smallness condition on $\delta$ and $\eta^*$ (defined in Lemma 3.2) is necessary to define the object $\tilde{P}(\cdot; b, \eta)$. 

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Next, by pointwise estimates of Proposition 3.3 and using \( \| \partial_b \psi_b \|_{L^\infty} \lesssim |b|^{\frac{1}{2}} \), we have\(^\text{13}\)

\[
|\partial_b \widetilde{P}(0; b, \eta) + i \frac{\gamma^2}{4} Q| = \left| (\partial_b P(\cdot; b, \tilde{\eta})|_{\tilde{\eta}=\psi_b(\eta)} + i \frac{\gamma^2}{4} Q) \right|
\]
\[
+ \partial_b \psi_b(\eta) \cdot \partial_{\tilde{\eta}=\psi_b(\eta)} P(\cdot; b, \tilde{\eta}) + (m + 1) \partial_b \psi_b(\eta) \rho \chi_{2M} |\lesssim 1_{y \leq 2|b|^{-1/2}} (|b|Q^4 + |b|^2 y^2 Q) + 1_{y \geq 2|b|^{-1/2}} y^2 Q.
\]

Combining this with the pointwise estimates on \( \mathcal{Z}_k \) and (4.10), we have

\[
\partial_b F_k = (i \frac{\gamma^2}{4} Q, \mathcal{Z}_k)_r + O(M^C |b|^{\frac{1}{2}}) = \delta_{3k} \| yQ \|_{L^2}^2 + O(M^C |b|^{\frac{1}{2}}) + O_{M \to \infty}(1).
\]

Next, again by pointwise estimates of Proposition 3.3

\[
|\partial_{\eta} \widetilde{P}(\cdot; b, \eta) + (m + 1) \rho| = |\psi_b'(\eta)(\partial_{\tilde{\eta}=\psi_b(\eta)} P(\cdot; b, \tilde{\eta}) + (m + 1) \rho) + (1 - \psi_b'(\eta))(m + 1)(\rho - \rho \chi_{2M})| \lesssim 1_{y \leq 2|b|^{-1/2}} |b|Q^4 + 1_{y \geq 2M} y^2 Q.
\]

Combining this with the above pointwise estimates on \( \mathcal{Z}_k \) and (4.10), we have

\[
\partial_{\eta} F_k = (m + 1)(\rho, \mathcal{Z}_k)_r + O(M^C |b| + M^{-4} \log M)
= -\delta_{4k}(m + 1) \| LQ \rho \|_{L^2}^2 + O(M^C |b|) + O_{M \to \infty}(1).
\]

Finally, we have

\[
\frac{\delta F_k}{\delta u} = (\mathcal{Z}_k)_{\lambda, \gamma} \in L^2.
\]

Gathering the above computations, we have \( F \in C^1 \) and moreover

\[
\partial_{\lambda, \gamma, b, \eta} F_{(1,0,0,0,Q)} + o_{M \to \infty}(1)
= \left( \begin{array}{cccc}
-\| yQ \|_{L^2}^2 & 0 & 0 & 0 \\
0 & -\| LQ \rho \|_{L^2}^2 & 0 & 0 \\
0 & 0 & \| yQ \|_{L^2}^2 & 0 \\
0 & 0 & 0 & -(m + 1)\| LQ \rho \|_{L^2}^2
\end{array} \right).
\tag{4.14}
\]

\(^\text{13}\) Note that Proposition 3.3 holds for \( 0 < |b| < b^* \). When \( b = 0 \), one directly computes the limit of \( b^{-1}(P(y; b, \eta) - P(y; 0, \eta)) \) as \( b \to 0 \), using \( P(y; 0, \eta) = Q - (m + 1) \eta \rho \chi_M \). As a result, one gets \( \partial_{b=0} \widetilde{P}(0; b, \eta) + i \frac{\gamma^2}{4} Q = 0 \) pointwisely and also in \( L^2_{\text{loc}} \). In particular, \( \partial_{b=0} F_k = (i \frac{\gamma^2}{4} Q, \mathcal{Z}_k)_r = \delta_{4k}(\Lambda Q, y^2 Q \chi_M)_r \).
Therefore, by the implicit function theorem, provided that $M \gg 1$, there exist
\[0 < \delta_2 \ll \delta_1\] and $C^1$-map $G^{1,0}: B_{\delta_2}(Q) \to B_{\delta_1}(1, 0, 0, 0)$ such that for given
\[u \in B_{\delta_2}(Q) \subseteq L^2, G^{1,0}(u)\] is a unique solution to $F(G^{1,0}(u), u) = 0$ in $B_{\delta_1}(1, 0, 0, 0)$. In particular, we have a Lipschitz bound
\[
\text{dist}(G^{1,0}(u), (1, 0, 0, 0)) \lesssim \|u - Q\|_{L^2}.
\]

We now use scale/phase invariances to cover the $\delta_2$-neighborhood of $\{Q_{\lambda, \gamma} : \lambda \in \mathbb{R}_+, \gamma \in \mathbb{R}/2\pi \mathbb{Z}\}$ in $L^2$. For $\lambda \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}/2\pi \mathbb{Z}$, define $G^{\lambda, \gamma}: B_{\delta_2}(Q_{\lambda, \gamma}) \to B_{\delta_1}(\lambda, \gamma, 0, 0)$ in an obvious way, by applying the scale/phase invariances to $G^{1,0}$. Thus uniqueness property of $G^{\lambda, \gamma}$ holds for values in $B_{\delta_1}(\lambda, \gamma, 0, 0)$.

We now claim that the family $(G^{\lambda, \gamma})_{\lambda, \gamma}$ is compatible, i.e.
\[
G := \bigcup_{\lambda, \gamma} G^{\lambda, \gamma} : \bigcup_{\lambda, \gamma} B_{\delta_2}(Q_{\lambda, \gamma}) \to \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_{\delta_1}(0) \times B_{\delta_1}(0)
\]
is well-defined. Indeed, if $u \in B_{\delta_2}(Q_{\lambda_1, \gamma_1}) \cap B_{\delta_2}(Q_{\lambda_2, \gamma_2})$, then
\[
\text{dist}((\lambda_1, \gamma_1), (\lambda_2, \gamma_2)) \lesssim \delta_2\] thus $\text{dist}(G^{\lambda_2, \gamma_2}(u), (\lambda_1, \gamma_1, 0, 0)) \lesssim \delta_2 \ll \delta_1$. Since $G^{\lambda_2, \gamma_2}(u)$ satisfies the equation $F(G^{\lambda_2, \gamma_2}(u), u) = 0$, the uniqueness property of $G^{\lambda_1, \gamma_1}(u)$ in $B_{\delta_1}(\lambda_1, \gamma_1, 0, 0)$ gives $G^{\lambda_2, \gamma_2}(u) = G^{\lambda_1, \gamma_1}(u)$, showing the claim.

We further need the following uniqueness property of $G$: given $u \in B_{\delta_2}(Q) \subseteq L^2$, $G(u)$ is a unique solution to $F(G(u), u) = 0$ in $\mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_{\delta_1}(0) \times B_{\delta_1}(0)$ with $\|\epsilon\|_{L^2} \lesssim \delta_2$. Indeed, suppose that $G' = (\lambda', \gamma', b', \eta')$ is a solution to $F(G', u) = 0$ such that $\epsilon' = u_{(\lambda', 1) - \gamma', -\eta'} - P(\cdot; b', \eta')$ satisfies $\|\epsilon'\|_{L^2} \lesssim \delta_2$. When dist$(G', G(u)) < \delta_1$, then the uniqueness of $G^{\lambda', \gamma'}$ yields $G' = G(u)$. If dist$(G', G(u)) > \delta_1$, then $\|\tilde{P}(y; b', \eta')_{\lambda', \gamma'} - \tilde{P}(y; b, \eta)_{\lambda, \gamma}\|_{L^2} \gtrsim \delta_1$ but $\|\epsilon'\|_{L^2}, \|\epsilon\|_{L^2} \lesssim \delta_2$. This contradicts to the identity $[\tilde{P}(y; b', \eta') + \epsilon']_{\lambda', \gamma'} = u = [P(\cdot; b, \eta) + \epsilon]_{\lambda, \gamma}$.

We now conclude the proof of this lemma on the $H^3_\mathcal{O}$-topology. Notice that the domain of $G$ contains $\overline{\mathcal{O}}$. As $G$ is $C^1$ with respect to the $L^2$-topology on $u$, it is also $C^1$ with respect to the $H^3_\mathcal{O}$-topology on $u$. Note that the map $u \mapsto (G(u), u, u) \mapsto \epsilon$ is only continuous. Thus the map $(G, \epsilon)$ restricted on $\overline{\mathcal{O}}$ is continuous. By the definition of $\epsilon$, $(G, \epsilon)$ is the right inverse of $\Phi$. By the uniqueness property of $G$, one also has that $(G, \epsilon)$ is the left inverse of $\Phi$. These facts show that $\Phi(\mathcal{U}) = \overline{\mathcal{O}}$ and $\Phi$ is a homeomorphism with the inverse $(G, \epsilon)$. Openness of $\mathcal{O}$ now follows from openness of $\mathcal{U}$. Finally, when we restrict $\Phi$ on $\mathcal{U}$, then it is $C^1$ in $(\gamma, b, \eta, \epsilon)$ and continuous in $\lambda$.

The argument of the previous paragraph is still valid when $\mathcal{U}, \mathcal{O}, H^3_\mathcal{O}$ are replaced by $\mathcal{U}^3, \mathcal{O}^5, H^5_\mathcal{O}$. This finishes the proof for the $H^3_\mathcal{O}$-topology.

\begin{remark}
Since the curve $\lambda \mapsto \lambda u(\lambda \cdot)$ is continuous but not $C^1$, $\Phi$ is merely continuous in $\lambda$.
\end{remark}

4.2 Existence of Trapped Solutions

In this section, we reduce the existence of trapped solutions in Theorems 1.1 and 1.3 into two propositions. The heart of the proof is the main bootstrap argument, provided in Proposition 4.4. It roughly says that all the assumptions except the $\eta$-bound of
Proof of the existence of trapped solutions, assuming Propositions 4.4 and 4.5 Here we prove the existence of $H^3_m$-trapped solutions in Theorem 1.1 and $H^5_m$-trapped solutions in Theorem 1.3. Proofs of those are soft arguments and very similar, so we only prove the former.

Let $(\lambda_0, \gamma_0, b_0, \epsilon_0) \in \tilde{U}_{\text{init}}$ and consider $\eta_0$ which varies in the range $(-\frac{K}{2}b_0^{3/2}, \frac{K}{2}b_0^{3/2})$. Define $u_0 \in \mathcal{O}_{\text{init}}$ via (1.11) and let $u$ be the forward-in-time maximal solution to (CSS) with the initial data $u_0$. Let $[0, T)$ with $0 < T \leq +\infty$ be the lifespan of $u$.

We claim that $u(t) \in \mathcal{O}$ for all $t \in [0, T)$ for a well-chosen $\eta_0$. As $u(0) = u_0 \in \mathcal{O}$ initially and $\mathcal{O}$ is open, we can define the exit time

$$T_{\text{exit}} := \sup\{\tau \in [0, T) : u(\tau') \in \mathcal{O} \text{ for } \tau' \in [0, \tau] \} \in (0, T].$$

If $T_{\text{exit}} = T$ for some $\eta_0$, then we are done. Now assume $T_{\text{exit}} < T$ for all $\eta_0$. Note that $u(T_{\text{exit}}) \in \overline{\mathcal{O}} \setminus \mathcal{O}$ due to openness of $\mathcal{O}$ and maximality of $T_{\text{exit}}$. We associate the modulation parameters and remainder $(\lambda, \gamma, b, \eta, \epsilon)$ with $u$ for each time $t \in [0, T_{\text{exit}}]$ according to Lemma 4.2. The following proposition is shown in Sect. 5, and is the heart of the proof of Theorem 1.1:

**Proposition 4.4** (Main bootstrap)

- Let $m \geq 1$; let $u$ be a solution to (CSS) with initial data $u_0 \in \mathcal{O}_{\text{init}}$. If $u(t) \in \mathcal{O}$ (i.e., the $H^3$-bootstrap hypothesis (4.11) is satisfied) for $t \in [0, \tau_*]$ for some $\tau_* > 0$, then the bootstrap conclusion holds for $t \in [0, \tau_*]$:

$$b \in (0, b_0], \quad \|\epsilon\|_{L^2} < \frac{K}{2}(b^*)^\frac{1}{8}, \quad \|\epsilon_1\|_{L^2} < \frac{K}{2}b, \quad \|\epsilon_3\|_{L^2} < \frac{K}{2}b^\frac{5}{2}. $$

- Let $m \geq 3$; let $u$ be a solution to (CSS) with initial data $u_0 \in \mathcal{O}_{\text{init}}^5$. If $u(t) \in \mathcal{O}^5$ (i.e., the $H^5$-bootstrap hypothesis (4.13) is satisfied) for $t \in [0, \tau_*]$ for some $\tau_* > 0$, then the bootstrap conclusion holds for $t \in [0, \tau_*]$:

$$b \in (0, b_0], \quad \|\epsilon\|_{L^2} < \frac{K}{2}(b^*)^\frac{1}{4}, \quad \|\epsilon_1\|_{L^2} < \frac{K}{2}b, \quad \|\epsilon_3\|_{L^2} < \frac{K}{2}b^3, \quad \|\epsilon_5\|_{L^2} < \frac{K}{2}b^\frac{9}{2}. $$

Compared to (4.11), all the assumptions except the $\eta$-bound are improved. It is indeed impossible to improve the $\eta$-bound, because the $\eta$-parameter accounts of the instability.

Back to the proof, $u(T_{\text{exit}}) \in \overline{\mathcal{O}} \setminus \mathcal{O}$ and Proposition 4.4 says that either $b = 0$ or $|\eta| = K b^3/2$ at $t = T_{\text{exit}}$. It is easy to exclude the case $b = 0$. If $b = 0$ at $t = T_{\text{exit}}$, then $u(T_{\text{exit}})$ must be a scaled $Q$, which is a static solution. Hence $u_0 = u(T_{\text{exit}})$ is a scaled $Q$, which does not belong to $\mathcal{O}_{\text{init}}$, yielding a contradiction.

Thus $|\eta| = K b^3/2$ at $t = T_{\text{exit}}$. To derive a contradiction, we use a basic connectivity argument. Let $\mathcal{I}_{\pm}$ be the set of $\eta_0$ such that $\eta(T_{\text{exit}}) = \pm K b^3/2(T_{\text{exit}})$. Note that $\mathcal{I}_{\pm}$ partitions $(-\frac{K}{2}b_0^{3/2}, \frac{K}{2}b_0^{3/2})$. The following proposition is shown in Sect. 5.7.

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Proposition 4.5 (The sets $I_{\pm}$) The sets $I_{\pm}$ are nonempty and open.

We have a contradiction from connectivity of $(-\frac{K}{2}, b_0^{3/2}, \frac{K}{2}, b_0^{3/2})$. Thus our claim is proved: $u(t) \in \mathcal{O}$ for all $t \in [0, T)$, for a well-chosen $\eta_0$. □

It remains to show that the trapped solutions are pseudoconformal blow-up solutions satisfying the statements of Theorem 1.1 (and those of Theorem 1.3). We will complete the proof in Sect. 5.8.

5 Main Bootstrap

In this section, we perform modulation analysis to prove Theorem 1.1 (and the first part of Theorem 1.3). The heart of the proof is the main bootstrap procedure (Proposition 4.4). The proof of Proposition 4.4 will be finished in Sect. 5.6. In Sects. 5.7 and 5.8, we finish the proof of Theorem 1.1 (and the first part of Theorem 1.3).

In Sect. 4.1, we decomposed our solution $u$ at each time $t$ of the form

$$u(t, r) = e^{i\gamma(t)}\frac{1}{\lambda(t)}[P(\cdot; b(t), \eta(t)) + \epsilon(t, \cdot)]\left(\frac{r}{\lambda(t)}\right).$$

The main goal of this section is to study the dynamics of $\epsilon$ and show that $\epsilon$ satisfies the bootstrap conclusions.

5.1 Equation of $\epsilon$

Here we derive the equation of $\epsilon$. As usual, we want to view the dynamics of $u$ as a near soliton evolution. For this purpose, we renormalize $u$ by setting the renormalized variables $(s, y)$

$$y = \frac{r}{\lambda} \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda^2},$$

where $\frac{1}{\lambda^2}$ is motivated by the scaling symmetry of (CSS). Under this relation, we will freely change the variables either by $s = s(t)$ or $t = t(s)$, and by abuse of notations, we denote functions of $t$ or $s$ identically. For example, we write $\lambda(s) = \lambda(t(s))$, $\epsilon(s, y) = \epsilon(t(s), y)$, and so on. We define

$$u^b(s, y) = e^{-iy}\lambda u(t, \lambda y) = P(y; b, \eta) + \epsilon(s, y).$$

Then $u^b$ satisfies\(^{14}\)

$$(\partial_s - \frac{\lambda s}{\lambda} \Lambda + \gamma s i)u^b + iL^w_P D^{(w)} u^b = 0.$$\(^{14}\) In [25], we intensively used $\sharp/\flat$ notations when switching the dynamical variables $(t, x) \leftrightarrow (s, y)$. See [25] for more details. In this paper, we will not heavily rely on $\sharp/\flat$ notations.

\[\text{Springer}\]
On the other hand, by (3.9), we have

\[ \partial_s P + i L_p^* D_+^{(P)} P = -b \lambda P - (\eta \theta_\eta + \theta \psi) i P + (b_s + b^2 + \eta^2) \partial_\eta P + \eta_s \partial_\eta P + i \Psi. \]

Here, \( \theta_\eta, \theta \psi, \Psi \) are defined in Proposition 3.3. Subtracting the second from the first, we arrive at the preliminary version of \( \epsilon \)-equation:

\[ (\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i) \epsilon + i L_Q \epsilon = \text{Mod} \cdot \mathbf{v} - i R_{L-L} - i R_{NL} - i \Psi, \tag{5.1} \]

where

\[
\begin{align*}
\text{Mod} &:= (\frac{\lambda_s}{\lambda} + b, \gamma_s - \eta \theta_\eta - \theta \psi, b_s + b^2 + \eta^2, \eta_s)' , \\
\mathbf{v} &:= (\Lambda P, -i P, -\partial_\eta P, -\partial_\eta P)' , \\
R_{L-L} &:= (L_P - L_Q) \epsilon , \\
R_{NL} &:= L_p^* D_+^{(P')} u^b - L_p^* D_+^{(P)} P - L_P \epsilon .
\end{align*}
\]

The terms on the RHS of (5.1) are viewed as the remainder terms. The last term \( i \Psi \) is the error from the modified profile and is independent of \( \epsilon \). It is already estimated in Proposition 3.3 and determines the optimal bootstrap assumptions on \( \epsilon \). The term \( i R_{L-L} \) is the difference of two linearized operators \( i L_P \epsilon \) and \( i L_Q \epsilon \). It is linear in \( \epsilon \) and enjoys further degeneracy from \( P - Q \), so roughly speaking it is \( O((P - Q) \epsilon) = O(be) \). The term \( i R_{NL} \) collects the quadratic and higher terms in \( \epsilon \) of the expression \( i L_p^* D_+^{(P + \epsilon)} (P + \epsilon) \), and is roughly of size \( O(\epsilon^2) \). Lastly, the smallness of \( \text{Mod} \cdot \mathbf{v} \) indicates the formal evolution of the modulation parameters (1.14).

In fact, we will reorganize the RHS of (5.1) by moving the phase correction terms from \( i R_{L-L} \) and \( i R_{NL} \) into \( \text{Mod} \cdot \mathbf{v} \). They will be written by \( i \tilde{R}_{L-L}, i \tilde{R}_{NL} \), and \( \tilde{\text{Mod}} \cdot \mathbf{v} \); see (5.26). These remainder terms are estimated in the following sections.

### 5.2 Estimates of Remainders

In this subsection, we provide estimates of remainders \( R_{L-L} \) and \( R_{NL} \). The term \( R_{L-L} \) contains \( \epsilon \) and also \( P - Q \) part from \( L_P - L_Q \). The term \( R_{NL} \) is quadratic or higher order in \( \epsilon \). These estimates are crucial ingredients of the modulation estimates and energy estimates. We treat \( R_{L-L} \) and \( R_{NL} \) in Lemmas 5.1 and 5.6, respectively.

Let us introduce some notations to be used only in this subsection. We recall the nonlinearity of (CSS)

\[ \mathcal{N}(u) = -|u|^2 u + \frac{2m}{\gamma^2} A_\theta [u] + \frac{1}{\gamma^2} A_0^2 [u] u + A_0 [u] u . \]

Terms in \( R_{L-L} \) and \( R_{NL} \) are multilinear terms as in \( \mathcal{N}(u) \). We recognize them as a form of \( V \psi \), where \( \psi \in \{ \epsilon, P, P - Q \} \) and \( V \) is a (nonlinear) potential part of \( \mathcal{N} \).
We define the potential part of the cubic nonlinearity by
\[ V_3[\psi_1, \psi_2] := -\text{Re}(\overline{\psi_1}\psi_2) + \frac{2m}{y^2} A_\theta[\psi_1, \psi_2] + A_{0,3}[\psi_1, \psi_2] \]
and that of the quintic nonlinearity by
\[ V_5[\psi_1, \psi_2, \psi_3, \psi_4] = \frac{1}{y^2} A_\theta[\psi_1, \psi_2] A_\theta[\psi_3, \psi_4] + A_{0,5}[\psi_1, \psi_2, \psi_3, \psi_4]. \]

Here,
\[ A_\theta[\psi_1, \psi_2] := -\frac{1}{2} \int_0^y \text{Re}(\overline{\psi_1}\psi_2) y' dy', \]
\[ A_{0,3}[\psi_1, \psi_2] := -m \int_y^\infty \text{Re}(\overline{\psi_1}\psi_2) \frac{dy'}{y'}, \]
\[ A_{0,5}[\psi_1, \psi_2, \psi_3, \psi_4] := -\int_y^\infty A_\theta[\psi_1, \psi_2] \text{Re}(\overline{\psi}_3\psi_4) \frac{dy'}{y'} \]

We further let
\[ V_3[\psi] := V_3[\psi, \psi] \quad \text{and} \quad V_5[\psi] := V_5[\psi, \psi, \psi, \psi]. \]

Thus
\[ L_u^* D_+^{(u)} u = -\Delta_m u + N(u) = -\Delta_m u + V_3[u]u + V_5[u]u. \quad (5.2) \]

We start with estimates for \( i R_{L-L}. \) (CSS) contains two types of long-range interactions, one from \( A_\theta \) and the other from \( A_0. \) In this work, we need to recognize the long-range effect of \( A_0, \) which contains \( \int_y^\infty \text{-integral. It turns out that this effect can be identified with the phase correction to our solution. For instance, } i R_{L-L} \text{ contains the term} \]
\[ -\left( \int_y^\infty m\text{Re}((P-Q)e) \frac{dy'}{y'} \right) i P. \]

Due to the weak Hardy control of \( \hat{H}_m^2 \) (or \( \hat{H}_m^5 \)) at spatial infinity (see (2.13) and (2.15)), it is more efficient that we pull out the \( \int_0^\infty \text{-integral and rewrite the above as} \]
\[ \left( \int_0^y m\text{Re}((P-Q)e) \frac{dy'}{y'} \right) i P - \left( \int_0^\infty m\text{Re}((P-Q)e) \frac{dy}{y} \right) i P. \]

For the first term, we can use the decay property of \( i P \) outside the integral. The latter term is of the form \( -\theta \cdot i P, \) whose role is the phase correction. One may compare with \( \theta_{\text{cor}} \) in [25]. This phase correction also arises in \( R_{NL}. \)
For $k \in \{3, 5\}$, we further introduce
\[
\tilde{A}_{0,k}(y) := A_{0,k}(y) - A_{0,k}(y = 0)
\]
and define $\tilde{V}_k$ by replacing $A_{0,k}$-part of $V_k$ by $\tilde{A}_{0,k}$.

Lemma 5.1 (Various estimates for degenerate linear terms) We decompose $R_{L-L}$ as
\[
R_{L-L} = -\theta_{L-L} P + \tilde{R}_{L-L}
\]
such that the following estimates hold.

- (Estimate of $\tilde{\theta}_{L-L}$) We have
  \[
  |\tilde{\theta}_{L-L}| \lesssim b^{\frac{1}{2}} \|\hat{\epsilon}\|_{\dot{H}^{3}_m}.
  \]
  \[
  (5.3)
  \]

- ($\dot{H}^{3}_m$-estimate of $\tilde{R}_{L-L}$) We have
  \[
  \|\tilde{R}_{L-L}\|_{\dot{H}^{3}_m} \lesssim b(\|\epsilon\|_{\dot{H}^{3}_m, \leq M_2}^3 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{3}_m}).
  \]
  \[
  (5.4)
  \]
  In particular,
  \[
  \|\tilde{R}_{L-L}\|_{\dot{H}^{3}_m} \lesssim b(\|\epsilon\|_{\dot{H}^{3}_m, \leq M_2}^3 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{3}_m}).
  \]
  \[
  (5.5)
  \]
  where the implicit constant $M_C$ comes from the bounds of $Z_k$ (4.6).

In addition, if $m \geq 3$, we have estimates in terms of the higher Sobolev norm $\dot{H}^{5}_m$:

- ($\dot{H}^{5}_m$-estimate of $\tilde{R}_{L-L}$) We have
  \[
  \|\tilde{R}_{L-L}\|_{\dot{H}^{5}_m} \lesssim b(\|\epsilon\|_{\dot{H}^{5}_m, \leq M_2}^5 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{5}_m}).
  \]
  \[
  (5.6)
  \]

- ($\dot{H}^{3}_m$-estimate of $\tilde{R}_{L-L}$) We have
  \[
  \|\tilde{R}_{L-L}\|_{\dot{H}^{3}_m} \lesssim b(\|\epsilon\|_{\dot{H}^{3}_m, \leq M_2}^5 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{3}_m}).
  \]
  \[
  (5.7)
  \]

- ($\dot{H}^{5}_m$-estimate of $\tilde{R}_{L-L}$) We have
  \[
  \|\tilde{R}_{L-L}\|_{\dot{H}^{5}_m} \lesssim b(\|\epsilon\|_{\dot{H}^{5}_m, \leq M_2}^5 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{5}_m}).
  \]
  \[
  (5.8)
  \]

- ($\dot{H}^{3}_m$-estimate of $\tilde{R}_{L-L}$) We have
  \[
  \|\tilde{R}_{L-L}\|_{\dot{H}^{3}_m} \lesssim b(\|\epsilon\|_{\dot{H}^{3}_m, \leq M_2}^5 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{3}_m}).
  \]
  \[
  (5.9)
  \]

In particular,
\[
\|A_Q L Q i \tilde{R}_{L-L} \|_{L^2} \lesssim b(\|\epsilon\|_{\dot{H}^{5}_m, \leq M_2}^5 + o_{M_2 \to \infty}(1)\|\hat{\epsilon}\|_{\dot{H}^{5}_m}).
\]
\[
(5.10)
\]
Remark 5.2 According to the parameter dependence (Remark 1.4), more precise statement of (5.4) is as follows. For all sufficiently large \( M_2 \), (5.4) holds for all sufficiently small \( b < b^* (M_2) \). As mentioned in the beginning of this section, we are also assuming the bootstrap hypothesis, and in particular \( |\eta| \lesssim b^2 \). The same applies for (5.10). As explained in Introduction (Sect. 1.4), having a smallness factor \( o_{M_2 \to \infty} (1) \) is crucial in our argument.

Remark 5.3 In fact, (5.6) also holds for \( i R_{L-L} \) instead of \( i \tilde{R}_{L-L} \), due to (5.3) and (3.8) (which relies heavily on the fact \( L_Q i P = O(b) \)). However, (5.4) does not hold for \( i R_{L-L} \).

Remark 5.4 One can apply the bootstrap hypotheses (4.11) or (4.13) to write the bounds for \( \tilde{R}_{L-L} \) in terms of \( b \). The upper bounds are written of the form \( O(b \epsilon) \) in order to keep the original form of \( R_{L-L} \). A similar remark applies for \( i R_{NL} \) below.

Remark 5.5 Instead of bounds of the form \( b \epsilon \| \epsilon \|_{H^3_{\{\eta\}}} \), it is crucial to have improved upper bounds of the form \( b (\| \epsilon \|_{H^3_{\{\eta\}}} + o_{M_2 \to \infty} (1) \| \epsilon \|_{H^3_{\{\eta\}}} ) \). This will be detailed in Sect. 5.4.

Proof Step 1. Decomposition \( R_{L-L} = -\theta_{L-L} P + \tilde{R}_{L-L} \).

In view of (5.2), we can write \( \mathcal{L}_P \epsilon \) as

\[
\mathcal{L}_P \epsilon = -\Delta_m \epsilon + V_3[P] \epsilon + 2V_3[P, \epsilon]P
+ V_5[P] \epsilon + 2V_5[P, \epsilon, P, P]P + 2V_5[P, P, P, \epsilon]P
\]

and similarly for \( \mathcal{L}_Q \epsilon \). Thus \( R_{L-L} = \mathcal{L}_P \epsilon - \mathcal{L}_Q \epsilon \) decomposes as

\[
R_{L-L} = (V_3[P] - V_3[Q] + V_5[P] - V_5[Q]) \epsilon
+ 2(V_3[P, \epsilon]P - V_3[Q, \epsilon]Q)
+ 2(V_5[P, \epsilon, P, P]P - V_5[Q, \epsilon, Q, Q]Q)
+ 2(V_5[P, P, P, \epsilon]P - V_5[Q, Q, Q, \epsilon]Q).
\]

We rearrange the above as

\[
R_{L-L} = (V_3[P] - V_3[Q] + V_5[P] - V_5[Q]) \epsilon
+ 2(V_3[Q, \epsilon]P + V_5[Q, \epsilon, Q, Q] + V_5[Q, Q, Q, \epsilon]) (P - Q)
+ 2V_3[P - Q, \epsilon]P + 2(V_5[P, \epsilon, P, P] - V_5[Q, \epsilon, Q, Q]) P
+ 2V_5[P, P, P, \epsilon]P - V_5[Q, Q, Q, \epsilon]P)
\]

We focus on the terms that end in \( P \). As illustrated above, we capture the phase correction term \( \theta_{L-L} P \). We replace the outermost integral \( \int_y^\infty \) of \( A_0 \) by \( \int_y^0 \) at cost of introducing \( \theta_{L-L} \).
\[ \tilde{R}_{L-L} := R_{L-L} + \theta_{L-L} P \]
\[ = (V_3[P] - V_3[Q])\epsilon + (V_3[P] - V_3[Q])\epsilon + 2\tilde{V}_3[P - Q, \epsilon] P \]
\[ + 2V_3[Q, \epsilon](P - Q) \]
\[ + 2(\tilde{V}_3[Q, \epsilon, P, \epsilon]) P + 2V_3[Q, \epsilon, Q, Q](P - Q) \]
\[ + 2(\tilde{V}_3[Q, P, P, \epsilon] - \tilde{V}_3[Q, Q, Q, \epsilon]) P + 2V_3[Q, Q, \epsilon, \epsilon](P - Q), \]

where

\[ \theta_{L-L} := \int_0^\infty 2(m + A_\theta[P])Re((P - Q)\epsilon) \frac{dy}{y} \]
\[ + \int_0^\infty 2(A_\theta[P] - A_\theta[Q])Re(Q\epsilon) \frac{dy}{y} \]
\[ + \int_0^\infty 2(A_\theta[\epsilon]|P|^2 - A_\theta[Q, \epsilon]Q^2) \frac{dy}{y}. \]

**Step 2.** The \( \theta_{L-L} \)-estimate (5.3).

We use \( |Q - P| \lesssim \min\{by^2, 1\} Q \) (see Proposition 3.3), \( \frac{dy}{y} = y^{-2} \cdot ydy \), and

\[ \int_0^y \min\{b(y')^2, 1\} f(y')|dy'| \lesssim \min\{by^2, 1\} \int_0^y |f(y')|dy' \]

to reduce

\[ \|y^{-2} \min\{by^2, 1\} A_\theta[Q, \epsilon]Q^2\|_{L^1} \lesssim b^{\frac{1}{2}} \|\epsilon\|_{\mathcal{H}^3_m}, \]
\[ \|y^{-2} \min\{by^2, 1\} A_\theta[Q]Q\epsilon\|_{L^1} \lesssim b^{\frac{1}{2}} \|\epsilon\|_{\mathcal{H}^3_m}. \]

The first estimate follows from

\[ \|y^{-2} A_\theta[Q, \epsilon]\|_{L^2} \lesssim \|Q\epsilon\|_{L^2} \lesssim \|\epsilon\|_{\mathcal{H}^3_m}, \quad \|\min\{by^2, 1\} Q^2\|_{L^2} \lesssim b. \]

The second estimate follows from

\[ \|y^{-2} \min\{by^2, 1\} A_\theta[Q]\|_{L^2} \lesssim b^{\frac{1}{2}}, \quad \|Q\epsilon\|_{L^2} \lesssim \|\epsilon\|_{\mathcal{H}^3_m}. \]

This completes the proof of (5.3).

**Step 3.** The estimates of \( \tilde{R}_{L-L} \).

We turn to the estimates of \( \tilde{R}_{L-L} \). Note that (5.5) follows from (5.4) and (4.6). Next, positivity of \( A_Q A_Q^* \) and boundedness of \( A_Q^* A_Q L_Q : \mathcal{H}^3_m \to L^2 \) (Lemma 2.6) say that (5.6) follows from (5.4). Similarly, (5.8) and (5.9) follow from (5.7). Finally, using positivity of \( (A_Q A_Q^*)^3 \) (Lemma A.19) and boundedness of \( A_Q^* A_Q A_Q L_Q : \mathcal{H}^3_m \to L^2 \) (Lemma 2.9), (5.11) follows from (5.10).

Henceforth, we focus on (5.4), (5.7), and (5.10). For simplicity of notations, we recognize each term of (5.12) as \( V\psi \), where \( V \) is the potential part and \( \psi \in \{\epsilon, P, P - \)

\( \tilde{R}_{L-L} \).
As one can see in (2.13), when $m \in \{1, 2\}$, $\|y^{-1}|\psi|_{-2}\|_{L^2}$ is not controlled by $\|\psi\|_{\dot{H}^3_m}$ due to the singularity at the origin. Thus we use $y(y)^{-2}|\psi|_{-2}$ instead of $y^{-1}|\psi|_{-2}$. The price to pay is that we need to control $y^{-1}(y)^2|\partial_\gamma V|_2$, which is more singular than $y|\partial_\gamma V|_2$ at the origin.

Therefore, the first term of (5.12) corresponds to $V = V_3[P] - V_3[Q]$ and $\psi = \epsilon$, the seventh term of (5.12) corresponds to $V = 2(\tilde{V}_5[P, P, P, \epsilon] - \tilde{V}_5[Q, Q, Q, \epsilon])$ and $\psi = P$, and so on.

With these notations, we claim

$$\|y^{-3}1_{y \geq 1} V \psi\|_{L^2} + \|\|V \partial_+ |\psi|_{-2}\|_{L^2} + \|\|\partial_\gamma (V \psi)|_{-2}\|_{L^2} \lesssim b(\|\epsilon\|_{\dot{H}^3_m} + o_{M_2} \rightarrow \infty(1)\|\epsilon\|_{\dot{H}^3_m}) \quad \text{if } m \geq 1,$$

and

$$\|y^{-5}1_{y \geq 1} V \psi\|_{L^2} + \|\|V \partial_+ |\psi|_{-4}\|_{L^2} + \|\|\partial_\gamma (V \psi)|_{-4}\|_{L^2} \lesssim b(\|\epsilon\|_{\dot{H}^5_m} + o_{M_2} \rightarrow \infty(1)\|\epsilon\|_{\dot{H}^5_m}) \quad \text{if } m \geq 3.$$

Let us finish the proof of (5.4), (5.7), and (5.10) assuming the claims (5.13) and (5.14). As $P, Q, \epsilon \in H^3_m$ (for each fixed time), we have $V \psi \in H^3_m$. Thus we can apply Lemma A.7 to get

$$\|V \psi\|_{\dot{H}^3_m} \lesssim \|y^{-3}1_{y \geq 1} V \psi\|_{L^2} + \|\|\partial_+ (V \psi)|_{-2}\|_{L^2} \lesssim \|y^{-3}1_{y \geq 1} V \psi\|_{L^2} + \|\|V \partial_+ |\psi|_{-2}\|_{L^2} + \|\|\partial_\gamma (V \psi)|_{-2}\|_{L^2}.$$

Thus (5.4) and (5.7) follow from the claim (5.13). Similarly, using Lemma A.17, we see that (5.10) follows from the claim (5.14).

It remains to show the claims (5.13) and (5.14). We divide into three cases: when $\psi = \epsilon$, $\psi = P - Q$, or $\psi = P$.

Case A: $\psi = \epsilon$.

In this case, $V = V_3[P] - V_3[Q]$ or $V = V_5[P] - V_5[Q]$.

From the pointwise estimates

$$|y^{-3}1_{y \geq 1} V \epsilon| \lesssim |V| \cdot y^{-3}1_{y \geq 1} \epsilon,$$

$$|V \partial_+ \epsilon|_{-2} \lesssim |V|_2 |\partial_+ \epsilon|_{-2} \lesssim (|V| + y^{-1}(y)^2|\partial_\gamma V|_1) \cdot |\partial_+ \epsilon|_{-2},$$

$$|(\partial_\gamma V) \epsilon|_{-2} \lesssim |\partial_\gamma V|_2 \cdot |\epsilon|_{-2} \lesssim y^{-1}(y)^2|\partial_\gamma V|_2 \cdot y(y)^{-2}|\epsilon|_{-2},$$

definition of the $\dot{H}^3_m$-norm (2.13) and $\dot{H}^5_m$-norm (2.15)\(^{16}\)

$$\|\langle y \rangle^{-2} \cdot y(y)^{-2}|\epsilon|_{-2}\|_{L^2} + \|\langle y \rangle^{-2} \cdot |\partial_+ \epsilon|_{-2}\|_{L^2}$$

\(^{15}\)One can show that the nonlinearities of (CSS) for $H^3_m$-solutions also belong to $CT H^3_m$ using Littlewood-Paley decomposition. See for example [25, Appendix B].

\(^{16}\)As one can see in (2.13), when $m \in \{1, 2\}$, $\|y^{-1}|\epsilon|_{-2}\|_{L^2}$ is not controlled by $\|\epsilon\|_{\dot{H}^3_m}$ due to the singularity at the origin. Thus we use $y(y)^{-2}|\epsilon|_{-2}$ instead of $y^{-1}|\epsilon|_{-2}$. The price to pay is that we need to control $y^{-1}(y)^2|\partial_\gamma V|_2$, which is more singular than $y|\partial_\gamma V|_2$ at the origin.
Thus (5.13) would follow from the pointwise estimates
\[
|V| + y^{-1} (y)^2 |\partial_y V|_2 \lesssim b(y)^{-2}.
\]
Similarly, (5.14) would follow from the pointwise estimates
\[
|V| + y^{-1} (y)^2 |\partial_y V|_4 \lesssim b(y)^{-2}.
\]
Thus it suffices to show the latter.
Starting from the pointwise bound \(|P - Q|_5 \lesssim by^2 Q\), we have
\[
|P^2 - Q^2|_5 \lesssim by^2 Q^2 \lesssim by^2 \langle y \rangle^{-4}.
\]
Thus
\[
\begin{align*}
|V_3[P] - V_3[Q]| + y^{-1} (y)^2 |\partial_y (V_3[P] - V_3[Q])|_4 \\
\lesssim y^{-2} (y)^2 (|P^2 - Q^2|_5 + \frac{1}{y} \int_0^y |P^2 - Q^2|_5 dy') + \int_y^\infty |P^2 - Q^2|_5 dy' \\
\lesssim b(y)^{-2}.
\end{align*}
\]
Estimates of \(V_3[P] - V_3[Q]\) follow from combining the bounds
\[
|A_\theta[P]|_5 + |A_\theta[Q]|_5 + |y^2 P^2|_5 + |y^2 Q^2|_5 \lesssim 1
\]
and the previous estimates of \(V_3[P] - V_3[Q]\) with the help of Leibniz’s rule.

**Case B:** \(\psi = P - Q\).

According to (5.12), \(V\) is either \(2V_3[Q, \epsilon]\), \(2V_5[Q, \epsilon, Q, Q]\), or \(2V_5[Q, Q, Q, \epsilon]\). Substituting the pointwise bounds \(|P - Q|_5 \lesssim by^2 Q\) and \(Q \lesssim y^m \langle y \rangle^{-(2m+2)}\) into \(\psi\) of (5.13) and (5.14), it suffices to show
\[
\|y^{-4} V\|_{L^2 \leq M_2} \lesssim \|\epsilon\|\mathcal{H}^{\delta}_{m, \leq M_2} + o_{M_2 \to \infty} (1) \|\epsilon\|\mathcal{H}^{\delta}_{m} \quad \text{if } m \geq 1,
\]
\[
\|y^{-4} V\|_{L^2 \leq M_2} \lesssim \|\epsilon\|\mathcal{H}^{\delta}_{m, \leq M_2} + o_{M_2 \to \infty} (1) \|\epsilon\|\mathcal{H}^{\delta}_{m} \quad \text{if } m \geq 3.
\]

\[\text{Indeed, one can easily improve the factor } b \text{ by } |\eta| + h^2. \text{ Notice that the pseudoconformal phase } e^{-ib\frac{y^2}{4}} \text{ plays no role in the gauge potential, so } V_k(P) = V_k(Q_{\eta})_{x_{b-1/2}} \text{ for } k \in \{3, 5\}. \text{ Thus one can use } |Q_{\eta}^{(k)} x_{b-1/2} - Q|_3 \lesssim (|\eta|)^2 y_{b-1/2} + 1_{y_{b-1/2}} Q \text{ instead of the cruder estimate } |P - Q|_3 \lesssim by^2 Q. \text{ However, we chose to use the crude bound } |P - Q|_3 \lesssim by^2 Q \text{ because it is better suited for the difference estimate; see the proof of Lemma 6.7.}\]
We first estimate $V_3$:

\[
\| (y)^{-4} | V_3[Q, \epsilon] | s \| L^2 \\
\lesssim \| (y)^{-4} (Q| \epsilon|_3 + y^{-2} A_0[Q, \epsilon]|_3 + | A_{0,3}[Q, \epsilon]|_3) \| L^2 \\
\lesssim \| (y)^{-4} | Q| \epsilon|_3 \| L^2 + \| y^{-2} Q| \epsilon|_1 \| L^1 \\
\lesssim \| y^{-2} (y)^{\frac{3}{2}} | Q| \epsilon|_3 \| L^2 \lesssim \| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^3 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^3,
\]

where in the second inequality we put $(y)^{-4}$ inside of $A_0$ and use $\| y^{-2} \int_0^y y y' dy \| L^2 \lesssim \| f \| L^2$ for $A_0$-term; we put $(y)^{-4} \in L^2$ and $\| \int_y^{\infty} f \frac{dy'}{y} \| L^\infty \lesssim \| y^{-2} f \| L^1$ for $A_{0,3}$-term. Similarly,

\[
\| (y)^{-4} | V_3[Q, \epsilon] | s \| L^2 \\
\lesssim \| (y)^{-4} | Q| \epsilon|_5 \| L^2 + \| y^{-2} Q| \epsilon|_1 \| L^1 \\
\lesssim \| y^{-2} (y)^{\frac{3}{2}} | Q| \epsilon|_5 \| L^2 \lesssim \| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^5 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^5 \text{ if } m \geq 3.
\]

We turn to $V_5$. For the $y^{-2} A_0^2$-type, we use $| A_0[Q]|_k \lesssim 1$ and the above $(y)^{-4} | y^{-2} A_0[Q, \epsilon]|_k$ bound for $k \in \{3, 5\}$ to get

\[
\| (y)^{-4} | y^{-2} A_0[Q, \epsilon] A_0[Q]|_3 \| L^2 \lesssim \| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^3 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^3 \text{ if } m \geq 1,
\]

\[
\| (y)^{-4} | y^{-2} A_0[Q, \epsilon] A_0[Q]|_5 \| L^2 \lesssim \| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^5 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^5 \text{ if } m \geq 3.
\]

For the $A_{0,5}$-type,

\[
\| (y)^{-4} | A_{0,5}[Q, \epsilon, Q, Q] + A_{0,5}[Q, Q, Q, \epsilon]|_3 \| L^2 \\
\lesssim \| y^{-2} (A_0[Q, \epsilon] Q^2 + A_0[Q|Q| \epsilon]) \| L^1 + \| (y)^{-4} | A_0[Q, \epsilon] Q^2 + A_0[Q|Q| \epsilon]|_2 \| L^2 \\
\lesssim \| (y)^{-\frac{1}{2}} |Q| \epsilon|_2 \| L^2 \lesssim \| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^3 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^3 \text{ if } m \geq 1.
\]

and similarly

\[
\| (y)^{-4} | A_{0,5}[Q, \epsilon, Q, Q] + A_{0,5}[Q, Q, Q, \epsilon]|_5 \| L^2 \\
\lesssim \| (y)^{-\frac{1}{2}} |Q| \epsilon|_4 \| L^2 \lesssim \| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^5 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^5 \text{ if } m \geq 3.
\]

**Case C**: $\psi = P$.

In this case, $V$ contains $P - Q$-part and $\epsilon$. According to (5.12), $V$ is either $2 \tilde{V}_3[Q - Q, \epsilon]$, $2(\tilde{V}_5[P, \epsilon, P, P] - \tilde{V}_5[Q, \epsilon, Q, Q])$, or $2(\tilde{V}_5[P, P, P, \epsilon] - \tilde{V}_5[Q, Q, \epsilon, \epsilon])$. Notice that $A_{0,k}$ is replaced by $\tilde{A}_{0,k}$ for $k \in \{3, 5\}$. Using $| P|_3 \lesssim Q$, it suffices to show

\[
\| y^{-2} (y)^{-4} | V|_3 \| L^2 \lesssim b(\| \epsilon \| \dot{\mathcal{H}}_{m, \leq M_2}^3 + o_{M_2 \to \infty}(1) \| \epsilon \| \dot{\mathcal{H}}_{m}^3) \text{ if } m \geq 1,
\]
\[ \| y^{-2} \langle y \rangle^{-4} |V||s| \|_{L^2} \lesssim b(\| \varepsilon \|_{\dot{H}_m^5} + o_{M_2 \rightarrow \infty}(1) \| \varepsilon \|_{\dot{H}_m^5}) \quad \text{if } m \geq 3. \]

One can observe that there is an extra factor \( y^{-2} \) in the LHS compared to that of Case B. Since \( V \) contains \( P - Q \)-parts, we can exploit the degeneracy \( |P - Q|_3 \lesssim b y^2 Q \) to handle the \( y^{-2} \) factor. We can estimate \( \tilde{V}_3 \) by

\[
y^{-2} \langle y \rangle^{-4} |\tilde{V}_3[ P - Q, \varepsilon ]|_3 
\lesssim y^{-2} \langle y \rangle^{-4} ((|P - Q|_3 + |y^{-2} A_0| P - Q, \varepsilon ) + |\tilde{A}_{0,3}| P - Q, \varepsilon ))
\lesssim b \cdot \langle y \rangle^{-4} (|Q\varepsilon|_3 + \frac{1}{y^2} \int_0^y Q|\varepsilon| y'dy' + \int_0^y Q|\varepsilon| \frac{dy'}{y}).
\]

In the last inequality, we pulled out the \( b y^2 \) factor from \( \int_0^y \)-integrals in \( A_0 \) and \( \tilde{A}_{0,3} \). This was possible since we replaced \( A_{0,3} \) by \( \tilde{A}_{0,3} \). The last line is essentially same as \( b(y^{-4})|V_3(Q, \varepsilon)|_3 \), though we replaced \( A_{0,3} \) by \( \tilde{A}_{0,3} \). Now we can apply the arguments in Case B to estimate \( L^2 \)-norm of the last line. A similar argument applies to \( \tilde{V}_5 \) and also \( |V|_3 \).

We now turn to \( R_{NL} \).

**Lemma 5.6** (Various estimates for nonlinear terms) We decompose \( R_{NL} \) as

\[ R_{NL} = -\theta_{NL} P + \tilde{R}_{NL} \]

such that the following estimates hold.

- *(Estimate of \( \theta_{NL} \)) We have
  \[ |\theta_{NL}| \lesssim \| \varepsilon \|_{\dot{H}_m^1}^2. \quad (5.15) \]

- *(\( \dot{H}_m^3 \)-estimate of \( \tilde{R}_{NL} \)) We have
  \[ \| \tilde{R}_{NL} \|_{\dot{H}_m^3} \lesssim \| \varepsilon \|_{\dot{H}_m^3}^2 + \| \varepsilon \|_{\dot{H}_m^1}^2 \| \varepsilon \|_{\dot{H}_m^3} + \| \varepsilon \|_{\dot{H}_m^1}^5. \quad (5.16) \]

In particular,

\[ \sup_{k \in \{1,2,3,4\}} |(i \tilde{R}_{NL}, Z_k)|_r \lesssim M^C (\| \varepsilon \|_{\dot{H}_m^3}^2 + \| \varepsilon \|_{\dot{H}_m^1}^2 \| \varepsilon \|_{\dot{H}_m^3} + \| \varepsilon \|_{\dot{H}_m^1}^5), \quad (5.17) \]

\[ \| A_0 L Q \tilde{R}_{NL} \|_{-1} \|_{L^2} \lesssim \| \varepsilon \|_{\dot{H}_m^3}^2 + \| \varepsilon \|_{\dot{H}_m^1}^2 \| \varepsilon \|_{\dot{H}_m^3} + \| \varepsilon \|_{\dot{H}_m^1}^5, \quad (5.18) \]

where the implicit constant \( M^C \) comes from the bounds of \( Z_k \) (4.6).

In addition, if \( m \geq 3 \), we have estimates in terms of the higher Sobolev norm \( \dot{H}_m^5 \):

- *(\( \dot{H}_m^5 \)-estimate of \( \tilde{R}_{NL} \)) We have
  \[ \| \tilde{R}_{NL} \|_{\dot{H}_m^5} \lesssim \| \varepsilon \|_{\dot{H}_m^5} (\| \varepsilon \|_{\dot{H}_m^3}^2 + \| \varepsilon \|_{\dot{H}_m^1}^2 + \| \varepsilon \|_{\dot{H}_m^3} + \| \varepsilon \|_{\dot{H}_m^1}^4). \quad (5.19) \]
In particular,

\[
\sup_{k \in \{1, 2, 3, 4\}} |(i \tilde{R}_\text{NL}, Z_k)_r| \lesssim M^C \left( \| \epsilon \|_{H^2}^2 \left( \| \epsilon \|_{H^3}^2 + \| \epsilon \|_{H^4}^2 \right) + \| \epsilon \|_{H^3} \| \epsilon \|_{H^4} \right),
\]

and

\[
\| AQAQLQ_i \tilde{R}_\text{NL}|_{-3} \|_{L^2} \lesssim \| \epsilon \|_{H^3} \left( \| \epsilon \|_{H^3} + \| \epsilon \|_{H^4} \right) + \| \epsilon \|_{H^3} \| \epsilon \|_{H^4}.
\]

(5.20)

Remark 5.7 As will be detailed in Sect. 5.4, we need bounds of the form \( \| AQAQLQ_i \tilde{R}_\text{NL}|_{-1} \|_{L^2} \ll b \| \epsilon \|_{L^2} \) after applying the bootstrap hypotheses on \( \epsilon \). Similarly, we need \( \| AQAQA_AQLQ_i \tilde{R}_\text{NL}|_{-1} \|_{L^2} \ll b \| \epsilon \|_{L^2} \) to close our bootstrap for \( \| \epsilon \|_{L^2} \).

Remark 5.8 In contrast to \( iR_{L-L} \), the estimates (3.8) and (5.15) do not imply that (5.18) for \( iR_{NL} \) instead of \( i\tilde{R}_{NL} \) holds. In other words, it seems necessary to extract the phase correction term \( \theta_{NL} P \) from \( iR_{NL} \) to have (5.18).

**Proof** Step 1. Decomposition \( R_{NL} = -\theta_{NL} P + \tilde{R}_{NL} \).

To derive the decomposition, we proceed similarly as before. As \( R_{NL} \) collects the quadratic and higher terms in \( \epsilon \) of the nonlinearity \( \hat{N}(P + \epsilon) \), we have

\[
R_{NL} = \sum_{k \in \{3, 5\}} \sum_{\{j: \psi_j = \epsilon\} \geq 2} V_k[\psi_1, \ldots, \psi_{k-1}] \psi_k
\]

\[
= \sum_{k \in \{3, 5\}} \sum_{\{j: \psi_j = \epsilon\} \geq 1} V_k[\psi_1, \ldots, \psi_{k-1}] \epsilon
\]

\[
+ \sum_{k \in \{3, 5\}} \sum_{\{j: \psi_j = \epsilon\} \geq 2} V_k[\psi_1, \ldots, \psi_{k-1}] P.
\]

We focus on the term that ends in \( P \). As in \( R_{L-L} \), we capture the phase correction term \( \theta_{NL} P \). We replace the outermost integral \( \int_0^\infty \) of \( A_0 \) by \( \int_0^\infty \) at cost of introducing \( \theta_{NL} \):

\[
R_{NL} + \theta_{NL} P = \sum_{k \in \{3, 5\}} \sum_{\{j: \psi_j = \epsilon\} \geq 1} V_k[\psi_1, \ldots, \psi_{k-1}] \epsilon
\]

\[
+ \sum_{k \in \{3, 5\}} \sum_{\{j: \psi_j = \epsilon\} \geq 2} \tilde{V}_k[\psi_1, \ldots, \psi_{k-1}] P,
\]

where

\[
\theta_{NL} := \int_0^\infty m [\epsilon]_2 \frac{dy}{y} + \sum_{\{j: \psi_j = \epsilon\} \geq 2} \int_0^\infty A_0[\psi_1, \psi_2] \text{Re}(\overline{\psi}_3 \psi_4) \frac{dy}{y}
\]
\[ = \int_0^\infty (m + A_\theta[P + \epsilon])|\epsilon|^2 \frac{dy}{y} + 2\int_0^\infty (A_\theta[P + \epsilon] - A_\theta[P])\text{Re}(\Phi\epsilon) \frac{dy}{y} + \int_0^\infty A_\theta[\epsilon]||P||^2 \frac{dy}{y}. \]

**Step 2.** The \(\theta_{\text{NL}}\)-estimate (5.15).

\[
|\int_0^\infty (m + A_\theta[P + \epsilon])|\epsilon|^2 \frac{dy}{y}| \lesssim (1 + \|P + \epsilon\|_L^2)\|\frac{1}{y}\epsilon\|_L^2 \lesssim \|\epsilon\|^2_{\tilde{H}_m^L},
\]

\[
|\int_0^\infty (A_\theta[P + \epsilon] - A_\theta[P])\text{Re}(\Phi\epsilon) \frac{dy}{y}| \lesssim \|\frac{1}{y} (A_\theta[P + \epsilon] - A_\theta[P])\|_L \|P\|_L \lesssim \|\epsilon\|^2_{\tilde{H}_m^L},
\]

\[
|\int_0^\infty A_\theta[\epsilon]||P||^2 \frac{dy}{y}| \lesssim |\int_0^\infty A_\theta[\frac{1}{y}\epsilon]Q^2 ydy| \lesssim \|\epsilon\|^2_{\tilde{H}_m^L}.
\]

This completes the proof of (5.15).

**Step 3.** The estimates of \(\tilde{R}_{\text{NL}}\).

For the \(\tilde{H}_m^L\)-estimate, as in the proof of Lemma 5.1, we need to show

\[
\|y^{-3}1_{y \geq 1} V \psi \|_{L^2} + \|V \partial_+ \psi \|_{L^2} + \|(|\partial_y V| \psi) \|_{L^2} \leq \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} \tag{5.22}
\]

In what follows, we will show a stronger estimate

\[
\|V \psi \|_{L^2} \lesssim \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} \tag{5.23}
\]

when \(V\) is not of \(A_0\)-type, i.e. \(V\) is of \(|\phi|^2\), \(\frac{1}{y^2} A_\theta\), and \(\frac{1}{y^2} A_\theta^2\). For \(V\) of \(A_0\)-type, we will directly show (5.22).\(^{18}\)

Similarly, for the \(\tilde{H}_m^5\)-estimate when \(m \geq 3\), we need to show

\[
\|y^{-3}1_{y \geq 1} V \psi \|_{L^2} + \|V \partial_+ \psi \|_{L^2} + \|(|\partial_y V| \psi) \|_{L^2} \leq \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} \tag{5.24}
\]

For \(|\phi|^2\), \(\frac{1}{y^2} A_\theta\), and \(\frac{1}{y^2} A_\theta^2\)-types, we will show a stronger estimate

\[
\|V \psi \|_{L^2} \lesssim \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} + \|\epsilon\|^2_{\tilde{H}_m^L} \tag{5.25}
\]

In the sequel, we prove the estimates (5.22)-(5.25). We proceed by types of non-linearity. In order to reduce redundant expressions, we use index \(\ell \in \{3, 5\}\) to denote the regularity of \(\tilde{H}_m^\ell\). When \(\ell = 5\), we further assume \(m \geq 3\).

\(^{18}\) In this case, we cannot hope to estimate \(\|A_{0,3}\phi|-3\|_{L^2}\). Indeed, \(A_{0,3}\) can have nonzero value at the origin, so we can only expect \(|A_{0,3}\phi| \lesssim y^m\) near the origin, even if \(\phi\) is smooth \(m\)-equivariant. Thus \(y^{-3}A_{0,3}\phi\) may not belong to \(L^2\) when \(m \in \{1, 2\}\), due to the singularity at the origin. The same discussion applies for \(R_{L,-1}\) (5.13).
Case A: $|\phi|^2\phi$-type.
It suffices to show (5.23) and (5.25), i.e.

$$\|P \epsilon_\cdot \|_{L^2} + \|\epsilon^3\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^1_m} (\|\epsilon\|_{\hat{H}^3_m} + \|\epsilon^2\|_{\hat{H}^1_m}) .$$

For $P \epsilon^2$, using $|P| \lesssim Q$,

$$|P \epsilon^2|_{-3} \lesssim Q |\partial_{yyy} \epsilon| |\epsilon| + |\epsilon|_{-2} \cdot y^{-1} Q |\epsilon|_1,$$

$$|P \epsilon^2|_{-5} \lesssim Q |\partial_{yyyyy} \epsilon||\epsilon| + |\epsilon|_{-4} \cdot y^{-1} Q |\epsilon|_2,$$

thus

$$\|P \epsilon^2\|_{L^2} \lesssim (\|\partial_{\cdot \cdot} \epsilon\|_{L^2} + \|y \gamma^{-2} \|_{\gamma^{-1} L^2}) \|y^{-2} \gamma^2 Q |\epsilon|_2 \|_{L^\infty}$$

$$\lesssim \|\epsilon\|_{\hat{H}^1_m} \|\epsilon\|_{\hat{H}^3_m} ,$$

where we estimated the $L^\infty$-term using (A.5).

For $\epsilon^3$, we have

$$|\epsilon^3|_{-3} \lesssim |\partial_{yyy} \epsilon| |\epsilon^2| + |\partial_{yy} \epsilon| |\epsilon|_{-1} |\epsilon| + |\epsilon|_{-3},$$

$$|\epsilon^3|_{-5} \lesssim |\partial_{yyyyy} \epsilon||\epsilon^2| + |\partial_{yyyy} \epsilon||\epsilon|_{-1} |\epsilon| + |\epsilon|_{-3}(|\epsilon|_{-2} |\epsilon| + |\epsilon|_{-1}^2).$$

For $|\epsilon^3|_{-3}$, we estimate

$$\|\epsilon^3\|_{L^2} \lesssim \|\partial_{yyy} \epsilon\|_{L^2} \|\epsilon\|_{L^\infty}^2 + \|\partial_{yy} \epsilon\|_{L^2} \|\epsilon|_{-1} \|_{L^\infty} \|\epsilon\|_{L^\infty} + \|\epsilon|_{-3} \|_{L^\infty} \|\epsilon|_{-1} \|_{L^2}$$

$$\lesssim \|\epsilon\|_{\hat{H}^3_m} \|\epsilon\|_{\hat{H}^1_m}^2 ,$$

where in the last inequality we used (2.12) and (A.4). For $|\epsilon^3|_{-5}$, we estimate

$$\|\epsilon^3\|_{L^2} \lesssim \|\partial_{yyyyy} \epsilon\|_{L^2} \|\epsilon\|_{L^\infty}^2 + \|\partial_{yyyy} \epsilon\|_{L^2} \|\epsilon|_{-1} \|_{L^\infty} \|\epsilon\|_{L^\infty}$$

$$+ \|\epsilon|_{-3} \|_{L^\infty} \|\epsilon|_{-2} \|_{L^\infty} \|\epsilon|_{-1} \|_{L^\infty} \lesssim \|\epsilon\|_{\hat{H}^5_m} \|\epsilon\|_{\hat{H}^1_m}^2 ,$$

where in the last inequality we used (A.4), (A.9), and the interpolation of $\hat{H}^3_m$ by $\hat{H}^1_m$ and $\hat{H}^5_m$.

Case B: $\frac{A_0}{y^2} \phi$-type.
It suffices to show (5.23) and (5.25)

$$\|\frac{A_0}{y^2} \phi\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^1_m} (\|\epsilon\|_{\hat{H}^3_m} + \|\epsilon^2\|_{\hat{H}^1_m} ) .$$

Since

$$\frac{1}{y^2} A_0 [\psi_1, \psi_2, \psi_3 - \ell \lesssim \frac{1}{y^2} A_0 [\psi_1, \psi_2] [\psi_3 - \ell \lesssim \frac{1}{y} |\psi_1 \psi_2 \psi_3|_{-1} .$$
and \( \frac{1}{y} |\psi_1 \psi_2 \psi_3|^{-(\ell-1)} \lesssim |\psi_1 \psi_2 \psi_3|^{-\ell} \) is treated in Case A, it suffices to show

\[
\| \frac{1}{y} A_\theta [\psi_1, \psi_2] |\psi_3|^{1-\ell} \|_{L^2} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 (|\epsilon| \| \hat{\mathcal{H}}_m^1 \|_1^2 ) ,
\]

\[
\| \frac{1}{y^2} A_\theta [\psi_1, \psi_2] |\psi_3|^{1-\ell} \|_{L^2} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 (|\epsilon| \| \hat{\mathcal{H}}_m^1 \|_1^2 ) .
\]

If \( \psi_3 = \epsilon \), then by (A.5) and (2.12) we have

\[
\| \frac{1}{y^2} A_\theta [\psi_1, \psi_2] \epsilon |^{-\ell} \|_{L^2} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 (|\epsilon| \| \hat{\mathcal{H}}_m^1 \|_1^2 ) .
\]

Note that we used \( m \geq 3 \) when \( \ell = 5 \) to get \( \| y^2 (y)^{-2} |\epsilon|^{-5} \|_{L^2} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 \). If \( \psi_3 = \epsilon \), then by \( |P|^{-\ell} \lesssim y^{-2} (y)^{-(1+\ell)} \) (here we also used \( m \geq 3 \) when \( \ell = 5 \) at the origin) and (A.5), we have

\[
\| \frac{1}{y} A_\theta [\epsilon] |P|^{-\ell} \|_{L^2} \lesssim \| y^{-2} (y)^{-(1+\ell)} \|_{L^2} \lesssim \| y^{-1} (y)^{-\ell} \|_{L^2} \| y^{-1} (y)^{-1} \|_{L^\infty} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 \| \hat{\mathcal{H}}_m^1 .
\]

Case C: \( A_{0,3} \phi \)-type.

Here, when \( \psi = P \), by subtracting \( \theta_{NL} P \), \( A_{0,3} \) is replaced by \( \tilde{A}_{0,3} \). In other words, \( V \psi \) is of the form \( A_{0,3}[\psi_1, \psi_2] \epsilon \) or \( \tilde{A}_{0,3}[\epsilon] P \).

We directly show (5.22) and (5.24). We note that if \( \partial_y \) hits \( A_{0,3} \) (or \( \tilde{A}_{0,3} \)), then it reduces to the \( \frac{1}{y} |\phi|^2 \)-type, which is already estimated in Case A. Thus we may assume that \( \partial_y \) never hits \( A_{0,3} \) (or \( \tilde{A}_{0,3} \)). It suffices to show

\[
\| A_{0,3}[\psi_1, \psi_2] (|\partial_y \epsilon|^{-\ell-1} + 1_{y \geq 1} \frac{1}{y^2} \epsilon) \|_{L^2} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 (|\epsilon| \| \hat{\mathcal{H}}_m^1 \|_1^2 ) ,
\]

\[
\| \tilde{A}_{0,3}[\epsilon] |P|^{-\ell} \|_{L^2} \lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 \| \hat{\mathcal{H}}_m^1 .
\]

For the first one, we estimate by

\[
\| A_{0,3}[\psi_1, \psi_2] \|_{L^\infty} \| \epsilon \| \hat{\mathcal{H}}_m^3 \lesssim \| y^{-2} \psi_1 \psi_2 \|_{L^1} \| \epsilon \| \hat{\mathcal{H}}_m^3
\]

\[
\lesssim \| \epsilon \| \hat{\mathcal{H}}_m^3 \cdot \begin{cases} \| y^{-2} Q \epsilon \|_{L^1} & \text{if } \{ \psi_1, \psi_2 \} = \{ P, \epsilon \}, \\ \| y^{-2} \epsilon^2 \|_{L^1} & \text{if } \psi_1 = \psi_2 = \epsilon, \\ \| \epsilon \| \hat{\mathcal{H}}_m^3 (|\epsilon| \| \hat{\mathcal{H}}_m^1 \|_1^2 ) .
\end{cases}
\]
For the second one, we simply use $|P|_{-\ell} \lesssim y^{-2}(y)^{-(1+\ell)}$ and (A.5) to estimate

$$
\|A_{0,3}[\epsilon]|P|_{-\ell}\|_{L^2} \lesssim \|y^{-2}(y)^{-(1+\ell)}\epsilon^2\|_{L^2} \lesssim \|y^{-1}(y)^{-\ell}\epsilon\|_{L^2}\|y^{-1}(y)^{-1}\epsilon\|_{L^\infty} \lesssim \|\epsilon\|_{\hat{H}^\ell_m}\|\epsilon\|_{\hat{H}^\ell_1}.
$$

Case D: $\frac{1}{\gamma^2}A_\theta^2\phi$-type.

It suffices to show (5.23) and (5.25)

$$
\|\frac{1}{\gamma^2}A_\theta[\psi_1, \psi_2]A_\theta[\psi_3, \psi_4]\psi|_{-3}\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^\ell_m}(\|\epsilon\|_{\hat{H}^\ell_m} + \|\epsilon\|_{\hat{H}^\ell_1}^2) + \|\epsilon\|_{\hat{H}^\ell_1}^5,
$$

$$
\|\frac{1}{\gamma^2}A_\theta[\psi_1, \psi_2]A_\theta[\psi_3, \psi_4]\psi|_{-5}\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^\ell_m}(\|\epsilon\|_{\hat{H}^\ell_m} + \|\epsilon\|_{\hat{H}^\ell_1}^2) + \|\epsilon\|_{\hat{H}^\ell_1}^4\|\epsilon\|_{\hat{H}^\ell_1}^4.
$$

From the pointwise bound

$$
\|\frac{1}{\gamma^2}A_\theta^2\psi|_{-\ell} \lesssim \frac{1}{\gamma^2}A_\theta^2\|\psi|_{-\ell} + \frac{1}{\gamma^2}A_\theta\psi_2\psi_1|_{-(\ell-1)} + \frac{1}{\gamma^2}A_\theta\psi_3\psi_4|_{-(\ell-1)},
$$

it suffices to show

$$
\|\frac{1}{\gamma^2}A_\theta^2|\psi|_{-\ell}\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^\ell_m}(\|\epsilon\|_{\hat{H}^\ell_m} + \|\epsilon\|_{\hat{H}^\ell_1}^2),
$$

$$
\|\frac{1}{\gamma^2}A_\theta\psi_3\psi_4|_{-(\ell-1)}\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^\ell_m}(\|\epsilon\|_{\hat{H}^\ell_m} + \|\epsilon\|_{\hat{H}^\ell_1}^2) + \|\epsilon\|_{\hat{H}^\ell_m}^3\|\epsilon\|_{\hat{H}^\ell_1}^4.
$$

We first estimate $\frac{1}{\gamma^2}A_\theta^2|\psi|_{-\ell}$. By symmetry, it suffices to estimate

$$
\|\frac{1}{\gamma^2}A_\theta[\epsilon, \psi_2]A_\theta[\psi_3, \psi_4]\epsilon|_{-\ell}\|_{L^2},
$$

$$
\|\frac{1}{\gamma^2}A_\theta[\epsilon]A_\theta[\psi_3, \psi_4]|P|_{-\ell}\|_{L^2},
$$

$$
\|\frac{1}{\gamma^2}A_\theta^2[P, \epsilon]|P|_{-\ell}\|_{L^2}.
$$

For the the first two terms, we merely estimate $|A_\theta[\psi_3, \psi_4]| \lesssim 1$ and use the estimate in Case B. For the last term, we estimate

$$
\|\frac{1}{\gamma^2}A_\theta[P, \epsilon]A_\theta[P, \epsilon]|P|_{-\ell}\|_{L^2} \lesssim \|P\epsilon\|_{L^\infty}\|y^2|P|_{-\ell}\|_{L^2} \lesssim \|\epsilon\|_{\hat{H}^\ell_m}\|\epsilon\|_{\hat{H}^\ell_m},
$$

where in the last inequality we used (A.5) for $\ell = 3$ and (A.9) for $\ell = 5$.

Next, we estimate $|\frac{1}{\gamma}A_\theta\psi_3\psi_4\psi_5|_{-(\ell-1)}$. From the pointwise estimate

$$
|\frac{1}{\gamma}A_\theta\psi_3\psi_4\psi_5|_{-(\ell-1)} \lesssim |A_\theta[\psi_1, \psi_2]| \cdot \frac{1}{\gamma} |\psi_3\psi_4\psi_5|_{-(\ell-1)} + |\psi_1\psi_2\psi_3\psi_4\psi_5|_{-(\ell-2)},
$$

we estimate these two terms.

We estimate $|A_\theta[\psi_1, \psi_2]| \cdot \frac{1}{\gamma} |\psi_3\psi_4\psi_5|_{-(\ell-1)}$. If $\psi_3, \psi_4, \psi_5$ contain at least two $\epsilon$, then we bound $|A_\theta[\psi_1, \psi_2]| \lesssim 1$ and use the $L^2$-estimate of $|\frac{1}{\gamma}\psi_3\psi_4\psi_5|_{-(\ell-1)}$ done
in Case A. Otherwise, there are \( j, k \in \{3, 4, 5\} \) such that \( j \neq k \), \( \psi_1, \psi_2, \psi_j \) contain at least two \( \epsilon \), and \( \psi_k = P \). Say \( j = 3 \) and \( k = 4 \). We estimate as

\[
\|A_\theta[\psi_1, \psi_2] \frac{1}{y} |\psi_3 P\psi_5| - (\ell - 1)\|_{L^2} \lesssim \| \frac{1}{y^2} A_\theta[\psi_1, \psi_2] \frac{1}{y} |\psi_3| - (\ell - 1)\|_{L^2} \| y^2 Q |\psi_5| \ell - 1\|_{L^\infty} \lesssim \|\epsilon\|_{H^m_\ell} (\|\epsilon\|_{H^m_3} + \|\epsilon\|_{H^m_1}^2),
\]

where in the last inequality we used \( \|y^2 Q |\psi_5| \ell - 1\|_{L^\infty} \lesssim 1 \) and the \( L^2 \)-estimate of

\[
\frac{1}{y^2} A_\theta[\psi_1, \psi_2] ||\psi_3| - \epsilon\|_\ell \text{ done in Case B.}
\]

Finally, we estimate \( |\psi_1 \psi_2 \psi_3 \psi_4 \psi_5| - (\ell - 2) \). If \( \psi_1 = \cdots = \psi_5 = \epsilon \), then we have

\[
|\epsilon^5|_{-1} \lesssim |\epsilon|_{-1}|\epsilon|^4,
|\epsilon^3|_{-3} \lesssim (|\partial_{yyy} \epsilon||\epsilon|^2 + |\partial_{yy} \epsilon||\epsilon|_{-1}|\epsilon| + |\epsilon|^3_{-1})|\epsilon|^2.
\]

Thus as in Case A

\[
|||\epsilon^5|_{-1}||_{L^2} \lesssim ||\epsilon||^4_{L^\infty} |||\epsilon|_{-1}||_{L^2} \lesssim ||\epsilon||^5_{H^m_1},
|||\epsilon^3|_{-3}||_{L^2} \lesssim ||\partial_{yy} \epsilon||\epsilon|^2 + ||\partial_{yy} \epsilon||\epsilon|_{-1}|\epsilon| + ||\epsilon|^3_{-1}||_{L^2} ||\epsilon||^2_{L^\infty} \lesssim ||\epsilon||_{H^m_3} ||\epsilon||^4_{H^m_1}.
\]

Otherwise, there exist at least one \( P \) and two \( \epsilon \)'s. When \( \ell = 3 \), we estimate this contribution by

\[
||Q \epsilon|_{-1}||_{L^\infty} ||Q^2 + Q|\epsilon| + |\epsilon|^2||_{L^2} \lesssim \|\gamma^{-1}|\epsilon|_{-1}||_{L^\infty} ||\gamma^{-2}|\epsilon||_{L^\infty} \lesssim ||\epsilon||_{H^m_3}^2,
\]

where we bounded the \( L^2 \)-term by 1 and used (A.5). When \( \ell = 5 \), we further subdivide the case. If there is exactly one \( P \), then we estimate this contribution using \( |||\epsilon^3|_{-3}||_{L^2} \) estimate of Case A and (A.10) by

\[
|||P \epsilon|_{3}||_{L^\infty} |||\epsilon^3|_{-3}||_{L^2} \lesssim ||\epsilon||_{H^m_5} ||\epsilon||_{H^m_3} ||\epsilon||_{H^m_1}^2.
\]

Otherwise, there exist at least two \( P \)'s and two \( \epsilon \)'s, say \( \psi_4 = \psi_5 = P \). We estimate this contribution using \( |||\psi_1 \psi_2 \psi_3 |_{-5}||_{L^2} \) estimate of Case A by

\[
||y^2 P^2|_{3}||_{L^\infty} \|y^{-2} \psi_1 \psi_2 \psi_3 |_{-3}||_{L^2} \lesssim ||\epsilon||_{H^m_5} ||\epsilon||_{H^m_3} ||\epsilon||_{H^m_1}^2.
\]

**Case E:** \( A_{0.5} \phi \)-type.

Here, when \( \psi = P \), by subtracting \( \theta_{NL}P \), \( A_{0.5} \) is replaced by \( \tilde{A}_{0.5} \). In other words, \( V \psi \) is of the form \( A_{0.5}[\psi_1, \psi_2, \psi_3, \psi_4] \epsilon \) or \( \tilde{A}_{0.5}[\psi_1, \psi_2, \psi_3, \psi_4] P \).

We directly show (5.22) and (5.24). We note that if \( \partial_\gamma \) hits \( A_{0.5} \) (or \( \tilde{A}_{0.5} \)), then it reduces to the \( \frac{1}{y^2} A_\theta|||\phi||_2 \phi \)-type, which is estimated in Case D. Thus we may assume that \( \partial_\gamma \) never hits \( A_{0.5} \) (or \( \tilde{A}_{0.5} \)). It suffices to show

\[
||A_{0.5}[\psi_1, \psi_2, \psi_3, \psi_4] (||\partial_+ \epsilon|_{-(\ell-1)} + 1_{y \geq 1} \frac{1}{y^\ell} \epsilon)||_{L^2} \lesssim ||\epsilon||_{H^m_5} (||\epsilon||_{H^m_3} + ||\epsilon||_{H^m_1}^2),
\]

\[\square\] Springer
\[\|A_{0,5}[^{1}[\psi_1, \psi_2, \psi_3, \psi_4]] P|_{-\ell}\|_{L^2} \lesssim \|\epsilon\|_{H^3_0}^2 (\|\epsilon\|_{H^3_0}^2 + \|\epsilon\|_{H^1_0}^2).\]

For the first one, it suffices to show (as in Case C)
\[\|y^{-2} A_\theta[^{1}[\psi_1, \psi_2]]\psi_3 \psi_4\|_{L^1} \lesssim \|\epsilon\|_{H^3_0}^2 + \|\epsilon\|_{H^1_0}^2.\]

If \(\epsilon \in \{\psi_3, \psi_4\}\), then
\[\|y^{-2} A_\theta[^{1}[\psi_1, \psi_2]]\psi_3 \psi_4\|_{L^1} \lesssim \|y^{-2} \psi_3 \psi_4\|_{L^1} \lesssim \|\epsilon\|_{H^3_0}^2 + \|\epsilon\|_{H^1_0}^2,\]
where we used \(\|A_\theta[^{1}[\psi_1, \psi_2]\|_{L^\infty} \lesssim 1\) and the \(\|y^{-2} \psi_3 \psi_4\|_{L^1}\)-estimate done in Case C. Otherwise, \(\psi_3 = \psi_4 = P\) and \(\epsilon \in \{\psi_1, \psi_2\}\). We estimate
\[\|y^{-2} A_\theta[^{1}[\psi_1, \psi_2]] P^2\|_{L^1} \lesssim \|A_\theta[^{1}[y^{-2} \psi_1, \psi_2]\|_{L^\infty} \|P^2\|_{L^1} \lesssim \|y^{-2} \psi_1 \psi_2\|_{L^1},\]
where we used the \(\|y^{-2} \psi_1 \psi_2\|_{L^1}\)-estimate done in Case C. For the second one, using \(|P|_{-\ell} \lesssim y^{-2} \langle y \rangle^{-(1+\ell)}\), it suffices to show
\[\|y^{-2} \langle y \rangle^{-(1+\ell)} A_\theta[^{1}[\psi_1, \psi_2]\psi_3 \psi_4\|_{L^2} \lesssim \|\epsilon\|_{H^3_0}^2 (\|\epsilon\|_{H^3_0}^2 + \|\epsilon\|_{H^1_0}^2).\]
If \(\psi_3 = \psi_4 = \epsilon\), then we simply bound \(\|A_\theta[^{1}[\psi_1, \psi_2]\|_{L^\infty} \lesssim 1\) to estimate
\[\|y^{-2} \langle y \rangle^{-(1+\ell)} A_\theta[^{1}[\psi_1, \psi_2]\epsilon^2\|_{L^2} \lesssim \|y^{-1} \langle y \rangle^{-(\ell-1)} \epsilon\|_{L^2} \|y^{-1} \langle y \rangle^{-2} \epsilon\|_{L^\infty} \lesssim \|\epsilon\|_{H^3_0}^2 \|\epsilon\|_{H^3_0}^2.\]
If \(P \in \{\psi_3, \psi_4\}\), say \(\psi_4 = P\), then we estimate
\[\|y^{-2} \langle y \rangle^{-(1+\ell)} A_\theta[^{1}[\psi_1, \psi_2]\psi_3 P\|_{L^2} \lesssim \|y^{-2} \langle y \rangle^{-(1+\ell)} A_\theta[^{1}[\psi_1, \psi_2]\psi_3\|_{L^2} \lesssim \|\epsilon\|_{H^3_0}^2 (\|\epsilon\|_{H^3_0}^2 + \|\epsilon\|_{H^1_0}^2),\]
where in the last inequality we used \(\|\frac{1}{y^2} A_\theta[^{1}[\psi_1, \psi_2]\|_{L^\infty} \lesssim \|y \cdot L\|_{L^2}\)-estimate of Case B. \(\square\)

### 5.3 Modulation Estimates

As discussed in Sect. 5.2, we reorganize the remainder terms of (5.1). Our \(\epsilon\)-equation becomes
\[(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i)\epsilon + i L_Q \epsilon = \widetilde{\text{Mod}} \cdot \mathbf{v} - i \tilde{\Lambda}_{\text{NL}} - i \tilde{\Psi},\quad (5.26)\]
where
\[\widetilde{\text{Mod}} := \left(\frac{\lambda_s}{\lambda} + b, \gamma_s - \eta \theta \eta - \theta \psi - \theta \Lambda - \theta_{\text{NL}}, b_s + b^2 + \eta^2, \eta_s\right).\]
In this subsection, we estimate $\widetilde{\text{Mod}}$. This says that our modulation parameters follow the formal parameter ODEs (1.14).

**Lemma 5.9** (Modulation estimates) We have

$$|\widetilde{\text{Mod}}| \lesssim \begin{cases} \omega M \rightarrow \infty (1) \| \epsilon_3 \|_{L^2} + M^C (b \| \sqrt{\epsilon} \|_{H^m} + b^4) & \text{if } m \geq 1, \\ \omega M \rightarrow \infty (1) \| \epsilon_5 \|_{L^2} + M^C (b \| \sqrt{\epsilon} \|_{H^m} + b^6) & \text{if } m \geq 3. \end{cases}$$

In particular, $\lambda$ and $b$ are decreasing, and

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim b, \quad |\gamma_s| \lesssim |\eta| + b^{\frac{3}{2}}.$$

**Remark 5.10** As will be detailed in Sect. 5.4, we need the bound $|\widetilde{\text{Mod}}| \ll \| \epsilon_3 \|_{L^2}$ (or $\| \epsilon_5 \|_{L^2}$) to close the bootstrap.

**Proof** We differentiate the orthogonality conditions in the renormalized time variable $s$. In other words, we take the inner product of (5.26) with each $Z_k$, $k \in \{1, 2, 3, 4\}$ to get

$$\frac{\lambda_s}{\lambda} (\epsilon, \Lambda Z_k)_r - \gamma_s (\epsilon, i Z_k)_r - (\epsilon, L_Q i Z_k)_r = \widetilde{\text{Mod}} \cdot (v, Z_k)_r - (i \tilde{R}_{L-L}, Z_k)_r - (i \tilde{R}_{NL}, Z_k)_r - (i \Psi_{1}^{(n)}, Z_k)_r.$$

Using $\frac{\lambda_s}{\lambda} = \widetilde{\text{Mod}}_1 - b$ and $\gamma_s = \widetilde{\text{Mod}}_2 + \eta \theta_{\eta} + \theta_{\Psi} + \theta_{L-L} + \theta_{NL}$, we can rewrite the above as

$$\sum_{j=1}^{4} \left\{ (v_j, Z_k)_r - \delta_j 1 (\epsilon, \Lambda Z_k)_r + \delta_j 2 (\epsilon, i Z_k)_r \right\} \widetilde{\text{Mod}}_j$$

$$= -(\epsilon, L_Q i Z_k)_r - b (\epsilon, \Lambda Z_k)_r - (\eta \theta_{\eta} + \theta_{\Psi} + \theta_{L-L} + \theta_{NL}) (\epsilon, i Z_k)_r + (i \tilde{R}_{L-L}, Z_k)_r + (i \tilde{R}_{NL}, Z_k)_r + (i \Psi, Z_k)_r,$$

where $\delta_{jk}$ denotes the Kronecker-delta symbol. By the choice of $Z_k$, the matrix

$$( (v_j, Z_k)_r - \delta_{j1} (\epsilon, \Lambda Z_k)_r + \delta_{j2} (\epsilon, i Z_k)_r )_{1 \leq j, k \leq 4}$$

is invertible with uniformly bounded inverse (cf. (4.14)). It now suffices to estimate all terms of the RHS of (5.27).

The main contribution comes from the linear part $(\epsilon, L_Q i Z_k)_r$. By the definition of $Z_3$ and $Z_4$, we have $(\epsilon, L_Q i Z_k)_r = 0$ for $k \in \{1, 2\}$. For $k \in \{3, 4\}$, $(\epsilon, L_Q i Z_k)_r$ does not necessarily vanish, but we can exploit the degeneracy of $L_Q i Z_k$ (4.8) or (4.9). We start with

$$(\epsilon, L_Q i Z_k)_r = (\epsilon, L_Q i L_Q i Z_{k-2})_r$$
\[ (\epsilon, L_Q^* i A_Q^* A_Q L_Q i \mathcal{Z}_{k-2}) = (\epsilon_2, i A_Q L_Q i \mathcal{Z}_{k-2}) \]

Applying (4.8) and positivity of \( A_Q^* A_Q \) (2.14), we have

\[ |(\epsilon, L_Q i \mathcal{Z}_k)_r| \lesssim M^{-1} \frac{1}{\gamma} \epsilon_2 \| L^2 \| \lesssim M^{-1} \| \epsilon_3 \| L^2 = o_{M \to \infty}(1) \| \epsilon_3 \| L^2. \]

Similarly applying (4.9) and the positivity Lemma A.19, we have

\[ |(\epsilon, L_Q i \mathcal{Z}_k)_r| = o_{M \to \infty}(1) \| \epsilon_5 \| L^2 \quad \text{if } m \geq 3. \]

Next, as \( \mathcal{Z}_k \in (\dot{\mathcal{H}}^3_m)^* \) and \( \mathcal{Z}_k \in (\dot{\mathcal{H}}^5_m)^* \) if \( m \geq 3 \) (see either (4.6) or (4.7)), we have

\[ |b(\epsilon, \Lambda \mathcal{Z}_k)_r| + |(\eta \theta_\eta + \theta \Psi + \theta_{NL-L} + \theta_{NL})(\epsilon, i \mathcal{Z}_k)_r| \lesssim b M^C \| \epsilon \| _{\dot{\mathcal{H}}^3_m}. \]

The remaining contributions are already estimated in Sect. 5.2 and Proposition 3.3. Indeed, the contribution from \( i \dot{\mathcal{R}}_{L-L} \) is estimated in (5.5) (or (5.8) if \( m \geq 3 \)). The contribution from \( i \dot{\mathcal{R}}_{NL} \) can be estimated by (5.17) (or (5.20) if \( m \geq 3 \)), substituting the bootstrap hypotheses, and using the parameter dependence (Remark 1.4). Finally, the contribution from \( i \Psi \) can be estimated using pointwise estimates of \( i \Psi \) (3.12) and \( \mathcal{Z}_k \) (4.4), yielding the bound \((\log M) b^{m+3}\). \( \square \)

### 5.4 Local Virial Control and Modified Energy Inequality

In this subsection, we prove monotonicity of \( \epsilon \), which enables us to control \( \epsilon \) forward in time. The main idea is to prove a modified energy inequality in higher Sobolev norms. As explained in Sect. 2.2, we are able to take Hamiltonian equations of higher order adapted derivatives. Recall that adapted derivatives are \( \epsilon_1 = L_Q \epsilon \), \( \epsilon_2 = A_Q^* \epsilon_1 \), \( \epsilon_3 = A_Q^* \epsilon_2 \), \( \epsilon_4 = A_Q^* \epsilon_3 \), \( \epsilon_5 = A_Q^* \epsilon_4 \), and so on. We also observed repulsivity in the equations of \( \epsilon_2 \) (and \( \epsilon_4 \)). With this repulsivity, we will be able to obtain monotonicity for the modified energies \( \mathcal{F}_3 \) and \( \mathcal{F}_5 \) at the \( \dot{\mathcal{H}}^3 \) and \( \dot{\mathcal{H}}^5 \)-levels, i.e.

\[ \mathcal{F}_3 \approx \| \epsilon_3 \| _{L^2}^2 = \| A_Q^* \epsilon_2 \| _{L^2}^2, \]

\[ \mathcal{F}_5 \approx \| \epsilon_5 \| _{L^2}^2 = \| A_Q^* \epsilon_4 \| _{L^2}^2. \]

Let us explain why we work at least in the \( \dot{\mathcal{H}}^3 \)-level. Due to scaling considerations, the optimal bound (what we can expect) for \( \| \epsilon_k \| _{L^2} \) is \( O(\lambda^k) \). In the pseudoconformal blow-up regime, \( \lambda \sim b \) is expected so this bound reads \( \| \epsilon_k \| _{L^2} \lesssim b^k \). We now recall that the main contribution to the modulation estimate \( b_\gamma + b^2 \) was the linear term, say \( o_{M \to \infty}(1) \| \epsilon_k \| _{L^2} \). This says that in order to justify \( b_\gamma + b^2 \approx 0 \), we need to get \( \| \epsilon_k \| _{L^2} \lesssim b^{2+} \). In other words, we need to work at least with \( k > 2 \).\( ^{19} \)

\( ^{19} \) If \( m \) is large, one may use untruncated orthogonality conditions (i.e., putting \( \epsilon \in N_\gamma(L_Q i)^ \perp \)) to improve the modulation estimate for \( b_\gamma + b^2 \) (say, by the factor of \( b \)). Then, working at the \( \dot{\mathcal{H}}^{1/2} \)-level would be sufficient. However, we do not know how to perform an energy estimate at the \( \dot{\mathcal{H}}^{1/2} \)-level due to the lack of the repulsivity structure (the truncated virial functional in this paper is defined for the \( \epsilon_2 \)-variable and would require at least \( \dot{\mathcal{H}}^{1/2} \)-regularity of \( \epsilon_2 \)).
Next, we explain how we choose the powers of $b$ in the bootstrap hypothesis. One of the restrictions comes from the above scaling considerations $\|\varepsilon_k\|_{L^2} \lesssim b^k$. There is the other source of the restrictions, the error $i\Psi$ from the modified profile. To see this, let us consider the toy model

$$\frac{1}{2} (\partial_s - 2k \frac{\lambda}{\chi}) \|\varepsilon_k\|_{L^2}^2 = (\varepsilon_k, -(i\Psi)_k)_r \lesssim \|\varepsilon_k\|_{L^2} \|i\Psi\|_{L^2},$$

where $(i\Psi)_k$ is the $k$-th adapted derivative of $i\Psi$. Let us formally rewrite this as

$$(\partial_s - k \frac{\lambda}{\chi}) \|\varepsilon_k\|_{L^2} \lesssim \|i\Psi\|_{L^2}.$$  

Assuming the size $\|i\Psi\|_{L^2} \lesssim b^p$ and using $\partial_t = \lambda^{-2} \partial_s$, we integrate $\lambda^{-k} \|\varepsilon_k\|_{L^2}$ as

$$\frac{\|\varepsilon_k(t)\|_{L^2}}{\lambda^k(t)} = \frac{\|\varepsilon_k(0)\|_{L^2}}{\lambda^k(0)} + O \left( \int_0^t \frac{b^p(\tau)}{\lambda^{k+2}(\tau)} d\tau \right).$$

Using the pseudoconformal regime $\lambda \sim b$ and the ansatz $b = -\frac{\lambda s}{\lambda} = -\lambda \lambda_1$, we can integrate

$$\int_0^t \frac{b^p}{\lambda^{k+2}} d\tau \sim \int_0^t \frac{-\lambda_1}{\lambda^{k-p+2}} d\tau \sim \left( \frac{1}{k-p+1} \right) \left( \frac{1}{\lambda^{k-p+1}} \right) \bigg|_0^t \sim \begin{cases} b^{p-1-k}(t) & \text{if } k > p - 1, \\ b^{p-1-k}(0) & \text{if } k < p - 1. \end{cases}$$

Thus

$$\|\varepsilon_k(t)\|_{L^2} \lesssim \begin{cases} (\lambda^{-k}(0) \|\varepsilon_k(0)\|_{L^2}) b^k(t) + b^{p-1}(t) & \text{if } k > p - 1, \\ (\lambda^{-k}(0) \|\varepsilon_k(0)\|_{L^2} + b^{p-1-k}(0)) b^k(t) & \text{if } k < p - 1. \end{cases}$$

In other words, we can only expect

$$\|\varepsilon_k(t)\|_{L^2} \lesssim b^\min[k, p-1](t).$$

Recalling that we need to get $\|\varepsilon_k\|_{L^2} \lesssim b^{2+}$, the error $i\Psi$ from the modified profile should satisfy $\|(i\Psi)_k\|_{L^2} \lesssim b^{3+}$. Our choice of the modified profile satisfies this bound, see (3.17) when $m \geq 1$ and $k = 3$. Our bootstrap bounds (4.11) for $m \geq 1$ and (4.13) for $m \geq 3$ are motivated from the $(i\Psi)_k$ bounds (3.17) and (3.18).

When we compute $\frac{1}{2} (\partial_s - 6 \frac{\lambda}{\chi}) \|\varepsilon_3\|_{L^2}^2$ (and similarly for $\varepsilon_5$), we will meet the error terms coming from the $e_3$-equation. We hope that such error terms do not disturb our bootstrap procedure. We show some heuristics to determine how much errors we can allow. We claim that $Cb\|\varepsilon_3\|_{L^2}^2$ error (with possibly large constant $C \gtrsim 1$) is not perturbative. Indeed, the previous computations say that we roughly lose one $b$ when
we integrate in time. In other words,
\[ \frac{1}{2} (\partial_s - 6\frac{\lambda}{\Lambda}) \| \epsilon_3 \|_{L^2}^2 \leq C b \| \epsilon_3 \|_{L^2}^2 + C_2 b^p \]
would yield
\[ \| \epsilon_3 \|_{L^2}^2 \lesssim C \| \epsilon_3 \|_{L^2}^2 + C_2 b^{\min\{p-1,3\}}. \]
This says that we may not close the bootstrap if \( C \) is large.

Such error terms of size \( C b \| \epsilon_3 \|_{L^2} \) can appear from \( \langle \epsilon_3, A_Q^* A_Q L \tilde{Q} i \tilde{R}_{L-L} \rangle \) because \( i \tilde{R}_{L-L} \) is roughly \( O(b\varepsilon) \)-like term (see (5.6)). (In fact, there is another such term induced from scalings.) From the above heuristics, we know that using a cruder bound \( b \| \epsilon \| \dot{H}_m \) for \( \tilde{Q}_{\lambda} \) may not be sufficient.

To get around this difficulty, we need to use correction terms for the energy. We now recall that the repulsive nature of \( \epsilon_2 \)-equation allows us to control the energy \( \| \epsilon_3 \|_{L^2}^2 = \| A_Q^* \epsilon_2 \|_{L^2}^2 \) via the virial identity (2.8). So it would be natural to use \( M_1 b(\epsilon_2, -i \Lambda \epsilon_2) \) with large \( M_1 \) to dominate the error \( C b \| \epsilon_3 \|_{L^2}^2 \). However, the virial functional is not bounded on our function spaces. Thus we need to localize it, say \( M_1 b(\epsilon_2, -i \Lambda M_2 \epsilon_2) \), and guarantee that at least the local portion \( (y \leq M_2) \) of \( C b \| \epsilon_3 \|_{L^2}^2 \) can be dominated. This motivates the localized form of the estimates (5.6). Such idea was used in [11, 39].

Finally, to deal with the technical error coming from localizing virial functionals, we use an averaging argument over the parameter \( M_2 \) as in [25].

From now on, we prove a modified energy inequality for \( \dot{H}_m^3 \) and \( \dot{H}_m^5 \). We start from the equation of \( \epsilon_2 \):
\[
(\partial_s - \frac{\lambda}{\Lambda} \Lambda_{-2} + \gamma_3 i) \epsilon_2 + i A_Q A_Q^* \epsilon_2 = \tilde{\text{Mod}} \cdot A_Q L \tilde{Q} v + \frac{\lambda}{\Lambda} \partial_\lambda (A_Q^* L \tilde{Q}_\lambda) \epsilon - \gamma_3 A_Q [L, i] \epsilon
\]
\[
= \tilde{\text{Mod}} \cdot A_Q A_Q^* A_Q L \tilde{Q} v + \frac{\lambda}{\Lambda} \partial_\lambda (A_Q^* A_Q^* L \tilde{Q}_\lambda) \epsilon - \gamma_3 A_Q A_Q^* A_Q [L, i] \epsilon
\]
\[
= - A_Q^* A_Q L \tilde{Q} \tilde{R}_{L-L} - A_Q L \tilde{Q} \tilde{R}_{NL} - A_Q L \tilde{Q} i \Psi,
\]
where we denote \( Q_{\lambda}(y) = \frac{1}{\lambda} Q(\frac{y}{\lambda}) \) and used the computation (at \( \lambda = 1 \))
\[
\Lambda_{-2} A_Q L \tilde{Q} \epsilon = -\partial_\lambda A_Q L \tilde{Q}_\lambda \epsilon = -\partial_\lambda (A_Q L \tilde{Q}_\lambda) \epsilon + A_Q L \Lambda \epsilon,
\]
\[
[A_Q L \tilde{Q}, i] \epsilon = A_Q [L, i] \epsilon.
\]
Similarly, we have the equation of \( \epsilon_4 \):
\[
(\partial_s - \frac{\lambda}{\Lambda} \Lambda_{-4} + \gamma_4 i) \epsilon_4 + i A_Q A_Q^* \epsilon_4 = \tilde{\text{Mod}} \cdot A_Q^* A_Q A_Q^* L \tilde{Q} v + \frac{\lambda}{\Lambda} \partial_\lambda (A_Q^* A_Q^* A_Q^* L \tilde{Q}_\lambda) \epsilon - \gamma_3 A_Q A_Q^* A_Q [L, i] \epsilon
\]
\[
= - A_Q A_Q^* A_Q L \tilde{Q} \tilde{R}_{L-L} - A_Q A_Q^* A_Q L \tilde{Q} i \tilde{R}_{NL} - A_Q A_Q^* A_Q L \tilde{Q} i \Psi.
\]
We first estimate the commutator terms. From $\frac{\lambda}{\lambda} \partial_{\lambda} (A_{Q^\lambda} L_{Q^\lambda}) \epsilon$ and $\frac{\lambda}{\lambda} \partial_{\lambda} (A_Q^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon$ are of $O(b \epsilon)$-like terms. Thus we aim to estimate these by local $\tilde{H}_m^{\lambda}$ or $\tilde{H}_m^{\lambda}$ norms, as in $\tilde{R}_{L-Q}$.

**Lemma 5.11 (Commutator terms)** We have

$$
\| \partial_{\lambda=1} (A_{Q^\lambda}^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| y^{-1} \partial_{\lambda=1} (A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| \left[ L_Q, i \right] \epsilon \|_{-2} \|_{L^2} \lesssim \begin{cases} 
\| \epsilon \|_{\tilde{H}_m^{\lambda}} + o_{M^2 \to \infty}(1) \| \epsilon \|_{\tilde{H}_m^{\lambda}}, & \text{if } m \geq 1, \\
\| \epsilon \|_{\tilde{H}_m^{\lambda}}, & \text{if } m \geq 3,
\end{cases}
$$

and when $m \geq 3$

$$
\| \partial_{\lambda=1} (A_{Q^\lambda}^* A_{Q^\lambda} A_{Q^\lambda}^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| y^{-1} \partial_{\lambda=1} (A_{Q^\lambda} A_{Q}^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| \left[ L_Q, i \right] \epsilon \|_{-4} \|_{L^2} \lesssim \| \epsilon \|_{\tilde{H}_m^{\lambda}} + o_{M^2 \to \infty}(1) \| \epsilon \|_{\tilde{H}_m^{\lambda}}.
$$

**Proof** Note the computations

$$
\partial_{\lambda=1} A_{Q^\lambda} = -\frac{1}{2} y Q^2,
$$

$$
\partial_{\lambda=1} (A_{Q^\lambda}^* A_{Q^\lambda}) = 2 (m + 1 + A \theta [Q]) Q^2,
$$

$$
\partial_{\lambda=1} L_{Q^\lambda} = -\frac{1}{2} y Q^2 - \Lambda Q B_Q - QB_{Q^\lambda}.
$$

Moreover,

$$
\left[ L_Q, i \right] f = -i Q B_Q (\text{Re}(f)) - Q B_Q (\text{Im}(f)).
$$

Here, the operators $\Lambda Q B_Q$ and $Q B_{Q^\lambda}$ are amenable to Lemma A.11, so we may regard them as $Q B_Q$. Thus

$$
\| \partial_{\lambda=1} (A_{Q^\lambda}^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| y^{-1} \partial_{\lambda=1} (A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| \left[ L_Q, i \right] \epsilon \|_{-2} \|_{L^2} \lesssim \| \epsilon \|_{\tilde{H}_m^{\lambda}} + o_{M^2 \to \infty}(1) \| \epsilon \|_{\tilde{H}_m^{\lambda}} \quad \text{if } m \geq 1,
$$

$$
\lesssim \begin{cases} 
\| \epsilon \|_{\tilde{H}_m^{\lambda}} + o_{M^2 \to \infty}(1) \| \epsilon \|_{\tilde{H}_m^{\lambda}}, & \text{if } m \geq 3,
\end{cases}
$$

as desired. Similarly, we use Lemma A.21 to get

$$
\| \partial_{\lambda=1} (A_{Q^\lambda}^* A_{Q^\lambda} A_{Q^\lambda}^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| y^{-1} \partial_{\lambda=1} (A_{Q^\lambda} A_{Q^\lambda}^* A_{Q^\lambda} L_{Q^\lambda}) \epsilon \|_{L^2} + \| \left[ L_Q, i \right] \epsilon \|_{-4} \|_{L^2} \lesssim \| \epsilon \|_{\tilde{H}_m^{\lambda}} + o_{M^2 \to \infty}(1) \| \epsilon \|_{\tilde{H}_m^{\lambda}}.
$$

$$
\lesssim \| \epsilon \|_{\tilde{H}_m^{\lambda}} + o_{M^2 \to \infty}(1) \| \epsilon \|_{\tilde{H}_m^{\lambda}}.
$$
\[ \lesssim \| \epsilon \| _{ \dot{H}^5_{m, \leq M_2}^3 } + o_{M_2 \to \infty}(1) \| \epsilon \| _{ \dot{H}^5_{m}^3 } . \]

This completes the proof. \(\square\)

First, we prove preliminary energy estimates. We will compute

\[ \left( \partial_s - \frac{6 \lambda s}{\lambda} \right) \| \epsilon_3 \| _{L^2}^2 \quad \text{and} \quad \left( \partial_s - 10 \frac{\lambda s}{\lambda} \right) \| \epsilon_5 \| _{L^2}^2 . \]

**Lemma 5.12** (Energy identity for \( \dot{H}^m_3 \)) We have

\[ \left| \left( \partial_s - \frac{6 \lambda s}{\lambda} \right) \| \epsilon_3 \| _{L^2}^2 \right| \lesssim b \| \epsilon_3 \| _{L^2} \left( o_{M \to \infty}(1) \| \epsilon_3 \| _{L^2} + \| \epsilon \| _{ \dot{H}^5_{m, \leq M_2}^3 } + o_{M_2 \to \infty}(1) \| \epsilon \| _{ \dot{H}^5_{m}^3 } + b^5 \right) \tag{5.30} \]

If \( m \geq 3 \), we further have

\[ \left| \left( \partial_s - \frac{6 \lambda s}{\lambda} \right) \| \epsilon_3 \| _{L^2}^2 \right| \lesssim b \| \epsilon_3 \| _{L^2} \cdot b^7 . \tag{5.31} \]

One can observe that the contributions of \( O(b^e) \)-like terms, \( \tilde{R}_{L-L} \) and scaling induced terms, are estimated by

\[ C b \| \epsilon_3 \| _{L^2} ( \| \epsilon \| _{ \dot{H}^5_{m, \leq M_2}^3 } + o_{M_2 \to \infty}(1) \| \epsilon \| _{ \dot{H}^5_{m}^3 } ) . \tag{5.32} \]

Later this will be dominated by adding a correction term into the energy.

In the display (5.30), the terms \( o_{M \to \infty}(1) \| \epsilon_3 \| _{L^2} \) and \( o_{M_2 \to \infty}(1) \| \epsilon \| _{ \dot{H}^5_{m}^3 } \) can be absorbed by \( b^5 \) using the bootstrap hypothesis (4.11) and the linear coercivity Lemma 2.7. However, we chose to keep those terms, which motivate our choice of parameters as in Remark 1.4.

**Proof** Taking \( A^*_Q \) to (5.28), we get the equation of \( \epsilon_3 \):

\[
\begin{align*}
(\partial_s - \frac{\lambda s}{\lambda}) A_{-3} + \gamma s i) \epsilon_3 + i A^*_Q A_Q \epsilon_3 \\
&= \tilde{\text{Mod}} \cdot A^*_Q A_Q L Q v + \frac{\lambda s}{\lambda} \partial_s (A^*_Q, A_Q, L Q.) \epsilon - \gamma s A^*_Q A_Q [L Q, i] \epsilon \\
&\quad - A^*_Q A_Q L Q i \tilde{R}_{L-L} - A^*_Q A_Q L Q i \tilde{R}_{NL} - A^*_Q A_Q L Q i \Psi.
\end{align*}
\]

We take the inner product of (5.33) with \( \epsilon_3 \) to get

\[
\left| \frac{1}{2} \left( \partial_s - \frac{6 \lambda s}{\lambda} \right) \| \epsilon_3 \| _{L^2}^2 \right| = \left| (\epsilon_3, \text{RHS of (5.33)})_r \right| \lesssim \| \epsilon_3 \| _{L^2} \| \text{RHS of (5.33)} \| _{L^2} .
\]

To show (5.30), we keep \( \| \epsilon_3 \| _{L^2} \) and estimate \( \| \text{RHS of (5.33)} \| _{L^2} \). To show (5.31) when \( m \geq 3 \), it is enough to show that \( \| \text{RHS of (5.33)} \| _{L^2} \lesssim b^7 \).
Henceforth, we estimate \( \| \text{RHS of (5.33)} \|_{L^2} \). By Lemma 5.9 and (3.8), we have
\[
\| \widetilde{\text{Mod}} \cdot A_Q^* A_Q L_Q \psi \|_{L^2} \lesssim \begin{cases} 
b(b_{M \to \infty}(1) \| \epsilon_3 \|_{L^2} + M^C(b \| \epsilon \|_{\tilde{H}^3_m} + b^4)) & \text{if } m \geq 1, \\
b(b_{M \to \infty}(1) \| \epsilon_5 \|_{L^2} + M^C(b \| \epsilon \|_{\tilde{H}^5_m} + b^6)) & \text{if } m \geq 3,
\end{cases}
\]
which are enough in view of the bootstrap hypotheses (4.11) and (4.13) and the coercivity Lemmas 2.7 and 2.10. By Lemmas 5.9 and 5.11, we have
\[
\| \frac{\lambda_s}{\lambda} \partial_s (A_{Q, \lambda} A_{Q, \lambda} L_{Q, \lambda}) \epsilon \|_{L^2} + \| \gamma_s A_Q^* A_Q [L_Q, i] \epsilon \|_{L^2},
\]
which is enough. The contribution from \( \tilde{R}_{L-L} \) is estimated by (5.6) if \( m \geq 1 \) and (5.9) if \( m \geq 3 \). The contribution from \( \tilde{R}_{NL} \) is estimated in (5.18). Finally, the contribution from \( \Psi \) is estimated in (3.17) (take \( m = 1 \) for \( m \geq 1 \) and \( m = 3 \) for \( m \geq 3 \)). \( \square \)

When \( \epsilon \) is a \( H^3_m \)-solution (i.e. \( \epsilon \in \mathcal{C}^3 \)), then the energy estimate (5.31) for \( \epsilon_3 \) suffices.

No virial corrections are needed for \( \| \epsilon_3 \|_{L^2} \). In fact, such a good estimate can be obtained because we were able to use higher Sobolev norm \( \tilde{H}^5_m \) (thus having more power of \( b \)).

Similarly to (5.30), we need an energy estimate for \( \epsilon_5 \).

**Lemma 5.13** (Energy identity for \( \tilde{H}^5_m \)) Let \( m \geq 3 \). We have
\[
\left| \left( \partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-5} + \gamma_s i \right) \| \epsilon_5 \|_{L^2} \right| 
\leq b \| \epsilon_5 \|_{L^2} \left( o_{M \to \infty}(1) \| \epsilon_5 \|_{L^2} + \| \epsilon \|_{\tilde{H}^5_{m, \leq M_2}} + o_{M_2 \to \infty}(1) \| \epsilon \|_{\tilde{H}^5_m} + b^9 \right).
\]
(5.34)

As above, the terms \( o_{M \to \infty}(1) \| \epsilon_5 \|_{L^2} \) and \( o_{M_2 \to \infty}(1) \| \epsilon \|_{\tilde{H}^5_m} \) can be absorbed into the term \( b^9 \).

**Proof** One proceeds similarly as in the proof of (5.30). Taking \( A_Q^* \) to the equation, we get the equation of \( \epsilon_5 \):
\[
(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-5} + \gamma_s i) \epsilon_5 + i A_Q^* A_Q \epsilon_5 
= \widetilde{\text{Mod}} \cdot A_Q^* A_Q A_Q^* A_Q L_Q \psi + \frac{\lambda_s}{\lambda} \partial_s (A_Q^* A_Q, A_Q^* A_Q L_Q) \epsilon \
- \gamma_s A_Q^* A_Q A_Q^* A_Q [L_Q, i] \epsilon - A_Q^* A_Q A_Q^* A_Q L_Q (i \tilde{R}_{L-L} + i \tilde{R}_{NL} + i \Psi).
\]
(5.35)

As before, it suffices to estimate \( \| \text{RHS of (5.35)} \|_{L^2} \). For the modulation term, use \( m \geq 3 \) case of Lemma 5.9. For the commutator terms, use the last estimate of Lemma 5.11. For \( \tilde{R}_{L-L}, \tilde{R}_{NL} \), and \( \Psi \), one may use (5.11), (5.21), and (3.18), respectively. \( \square \)
For the highest Sobolev norms, the estimates (5.30) and (5.34) seem to be optimal.
As explained above, the local $\mathcal{H}_m^3$ and $\mathcal{H}_m^5$ terms (e.g. (5.32)) can be controlled by the
monotonicity, namely the repulsivity from the virial functional.

Let us recall the formal computation (2.8)

$$
\frac{1}{2} \partial_t (\varepsilon_2, -i \Lambda \varepsilon_2)_r \approx (\varepsilon_2, A_0 A'_Q \varepsilon_2)_r + (\varepsilon_2, -\frac{\partial_y \tilde{V}}{2y} \varepsilon_2)_r \geq (\varepsilon_2, A_0 A'_Q \varepsilon_2)_r = \|\varepsilon_3\|_{L^2}^2.
$$

However, the virial functional $(\varepsilon_2, i \Lambda \varepsilon_2)_r$ is unbounded, so we will truncate this.
In order to mimic the original virial functional in a large compact region and keep
symmetricity of $i \Lambda$, we truncate as follows.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\phi'(y) = y$ for $|y| \leq 1$ and
$\phi'(y) = 0$ for $y \geq 2$. For $A \geq 1$, define the smooth radial function $\phi_A : \mathbb{R}^2 \to \mathbb{R}$ such that $\phi_A(x) = \frac{\phi(|x|)}{A}$. Denote $\phi_A := \partial_r \phi_A$. The deformed $L^2$-scaling vector field is
defined by

$$
\Lambda_A := \phi'_A \partial_r + \frac{\Delta \phi_A}{2},
$$

where $\Delta$ is the Laplacian on $\mathbb{R}^2$ (thus $\Delta \phi_A = (\partial_{yy} + \frac{1}{y} \partial_y) \phi_A$). We note that $i \Lambda_A = i \Lambda$
in the region $y \leq A$ and it is symmetric.

Now we prove a localized version of the repulsivity.

**Lemma 5.14** (Localized repulsivity) We have

$$
(A_0 A'_Q \varepsilon_2, \Lambda_M \varepsilon_2)_r \geq c_M \|\varepsilon\|_{H^3_{m, \leq M_2}}^2 - O\left(\|1_{y \sim M_2} |\varepsilon|_3 \|_{L^2}^2\right) - o_{M_2 \to \infty}(1) \|\varepsilon\|_{H^3_{m_2}}^2.
$$

(5.36)

If $m \geq 3$, we have

$$
(A_0 A'_Q \varepsilon_4, \Lambda_M \varepsilon_4)_r \geq c_M \|\varepsilon\|_{H^5_{m, \leq M_2}}^2 - O\left(\|1_{y \sim M_2} |\varepsilon|_5 \|_{L^2}^2\right) - o_{M_2 \to \infty}(1) \|\varepsilon\|_{H^5_{m_2}}^2.
$$

(5.37)

**Remark 5.15** The error term in the region $y \sim M_2$ can be easily managed using an
averaging argument in $M_2$, see Lemma 5.16 below.

**Proof** By a direct computation, we have for $\ell \in \{2, 4\}$ that

$$
(-\partial_{yy} \varepsilon_\ell - \frac{1}{y} \partial_y \varepsilon_\ell, \Lambda_M \varepsilon_\ell)_r = \int \frac{\phi''_{M_2}}{y^2} |\partial_y \varepsilon_\ell|^2 - \int \frac{\Delta^2 \phi_{M_2}}{4} |\varepsilon_\ell|^2,
$$

$$
\left(\frac{\tilde{V}}{y^2} \varepsilon_\ell, \Lambda_M \varepsilon_\ell\right)_r = \int \frac{\phi'_{M_2}}{y^2} |\varepsilon_\ell|^2 - \int \frac{\phi'_{M_2}}{2y} \frac{\partial_y \tilde{V}}{y^2} |\varepsilon_\ell|^2.
$$

Using $-y \partial_y \tilde{V} \geq 0$ and $|\Delta^2 \phi_{M_2}| \lesssim \frac{1}{y^2} 1_{y \sim M_2}$, we get

$$
(A_0 A'_Q \varepsilon_\ell, \Lambda_M \varepsilon_\ell)_r \geq \int \phi''_{M_2} |\partial_y \varepsilon_\ell|^2 + \int \frac{\phi'_{M_2}}{y^2} \frac{\tilde{V}}{y^2} |\varepsilon_\ell|^2 + O\left(\|1_{y \sim M_2} \frac{1}{y} \varepsilon_\ell\|_{L^2}^2\right).
$$
If one formally substitutes $M_2 = \infty$, one would get the full monotonicity as (2.8).

We now propagate the above lower bound on $\epsilon$ to that in terms of $\epsilon$. Using $\phi''_{M_2} = \frac{1}{\ell} \phi'_{M_2}$ on the region $\{y \leq M_2\}$ and $\phi'_{M_2} = 0$ on the region $\{y \geq 2M_2\}$, we have (when $\ell = 2$)

$$\int \phi''_{M_2} |\partial_y \epsilon|^2 + \int \frac{\phi'_{M_2}}{y} \cdot \frac{\epsilon}{y^2} |\epsilon|^2$$

$$= \int |\partial_y A_Q(\phi''_{M_2} \epsilon)|^2 + \int \frac{\epsilon}{y^2} |A_Q(\phi''_{M_2} \epsilon)|^2 + O(\|1_{y \sim M_2}|\epsilon|^{-2}\|_{L^2})$$

$$= \int |A^*_Q A_Q(\phi''_{M_2} \epsilon)|^2 + O(\|1_{y \sim M_2}|\epsilon|^{-2}\|_{L^2})$$

and (when $\ell = 4$)

$$\int \phi''_{M_2} |\partial_y \epsilon|^2 + \int \frac{\phi'_{M_2}}{y} \cdot \frac{\epsilon}{y^2} |\epsilon|^2$$

$$= \int |\partial_y A_Q A^*_Q A_Q(\phi''_{M_2} \epsilon)|^2 + \int \frac{\epsilon}{y^2} |A^*_Q A_Q A_Q(\phi''_{M_2} \epsilon)|^2 + O(\|1_{y \sim M_2}|\epsilon|^{-4}\|_{L^2})$$

$$= \int |A^*_Q A_Q A_Q A_Q(\phi''_{M_2} \epsilon)|^2 + O(\|1_{y \sim M_2}|\epsilon|^{-4}\|_{L^2}).$$

We now write $\epsilon_1 = L_Q \epsilon$ and commute $\phi''_{M_2}$ and $L_Q$. Recall $L_Q = D_{+}^{(Q)} + QB_Q$. Because of the local nature of $D_{+}^{(Q)}$, we can easily commute $\phi''_{M_2}$ and $D_{+}^{(Q)}$ with an error localized in $\{y \sim M_2\}$. Thus

$$\|\|\phi''_{M_2}, D_{+}^{(Q)}\|\epsilon|\epsilon\|_{L^2} \lesssim \|1_{y \sim M_2}|\epsilon|^{-1}\|_{L^2}.$$

When we commute $\phi''_{M_2}$ and $QB_Q$, the error is not necessarily supported in $\{y \sim M_2\}$. However, $[\phi''_{M_2}, QB_Q] \epsilon$ only uses the information of $\epsilon$ in $\{y \sim M_2\}$. Combining this observation with either Lemma A.11 or A.21, we have

$$\|\|\phi''_{M_2}, QB_Q\|\epsilon|\epsilon\|_{L^2} \lesssim \|1_{y \sim M_2}|\epsilon|^{-1}\|_{L^2}.$$

Thus we have

$$(A_Q A^*_Q \epsilon_2, A_M \epsilon_2)_r \geq \|A^*_Q A_Q L_Q(\phi''_{M_2} \epsilon)\|_{L^2}^2 - O(\|1_{y \sim M_2}|\epsilon|^{-3}\|_{L^2})$$

and when $m \geq 3$

$$(A_Q A^*_Q \epsilon_4, A_M \epsilon_4)_r \geq \|A^*_Q A_Q A^*_Q A_Q L_Q(\phi''_{M_2} \epsilon)\|_{L^2}^2 - O(\|1_{y \sim M_2}|\epsilon|^{-5}\|_{L^2}).$$

To complete the proof, it suffices to derive a lower bound for the adapted derivatives of $\phi''_{M_2} \epsilon$. The function $\phi''_{M_2} \epsilon$ does not necessarily satisfy the orthogonality conditions (4.2), but it almost satisfies them in the sense that

$$|(\phi''_{M_2} \epsilon, Z_k)_r| = |(\epsilon, (\phi''_{M_2} - 1) Z_k)_r|.$$
for each $k \in \{1, 2, 3, 4\}$. Thus we can apply the coercivity estimates (Lemma 2.7 or 2.10) with an additional error either $o_{M_2 \to \infty}(1)\|\epsilon\|_{\dot{H}^3_m}$ or $o_{M_2 \to \infty}(1)\|\epsilon\|_{\dot{H}^5_m}$. This finishes the proof.

In view of (5.36) and (5.37), $\partial_s \{b(\epsilon_2, i\Lambda M_2 \epsilon_2)\}$ will have an error of the form

$$b \cdot O(\|1_{y \sim M_2}|\epsilon| - 3\|^2_{L^2}).$$

If one crudely discards the localization $1_{y \sim M_2}$, this error is on the borderline of the acceptable errors. In order to make use of this localization, we use an averaging argument over $M_2$, as in [25].

**Lemma 5.16** (Local virial control) We have

$$\left| \frac{b}{\log M_2} \int_{M_2}^2 (\epsilon_2, -i \Lambda M_2 \epsilon_2) \frac{dM_2'}{M_2'} \right| \lesssim b M_2^C \|\epsilon_3\|^2_{L^2}, \quad (5.38)$$

$$\left( \partial_s - \frac{6\lambda_s}{\lambda} \right) \left[ \frac{b}{\log M_2} \int_{M_2}^2 (\epsilon_2, -i \Lambda M_2 \epsilon_2) \frac{dM_2'}{M_2'} \right] \geq b \left( c_M \|\epsilon\|^2_{\dot{H}^3_m, \leq M_2} - o_{M_2 \to \infty}(1)\|\epsilon\|_{\dot{H}^3_m}^2 - M_2^C \|\epsilon_3\|_{L^2} (b\|\epsilon\|_{\dot{H}^3_m} + b^\frac{7}{2}) \right). \quad (5.39)$$

If $m \geq 3$, then

$$\left| \frac{b}{\log M_2} \int_{M_2}^2 (\epsilon_4, -i \Lambda M_2 \epsilon_4) \frac{dM_2'}{M_2'} \right| \lesssim b M_2^C \|\epsilon_5\|^2_{L^2}, \quad (5.40)$$

$$\left( \partial_s - 10\frac{\lambda_s}{\lambda} \right) \left[ \frac{b}{\log M_2} \int_{M_2}^2 (\epsilon_4, -i \Lambda M_2 \epsilon_4) \frac{dM_2'}{M_2'} \right] \geq b \left( c_M \|\epsilon\|^2_{\dot{H}^5_m, \leq M_2} - o_{M_2 \to \infty}(1)\|\epsilon\|_{\dot{H}^5_m}^2 - M_2^C \|\epsilon_5\|_{L^2} (b\|\epsilon\|_{\dot{H}^5_m} + b^\frac{11}{2}) \right). \quad (5.41)$$

**Proof** Here we only show how we can get (5.38) and (5.39) from the computations in Lemma 5.12, (5.36), and the averaging argument. The proof of (5.40) and (5.41) would follow from the same argument using the computations in Lemma 5.13 and (5.37) instead of Lemma 5.12 and (5.36).

We claim the unaveraged version (and without $b$-factor) of the estimates

$$|(\epsilon_2, -i \Lambda M_2 \epsilon_2)| \lesssim M_2^2 \|\epsilon_3\|^2_{L^2},$$

$$\left( \partial_s - \frac{6\lambda_s}{\lambda} \right) (\epsilon_2, -i \Lambda M_2 \epsilon_2) \geq c_M \|\epsilon\|^2_{\dot{H}^3_m, \leq M_2} - O(\|1_{y \sim M_2}|\epsilon| - 3\|^2_{L^2})$$

$$- o_{M_2 \to \infty}(1)\|\epsilon\|_{\dot{H}^3_m}^2 - M_2 \|\epsilon_3\|_{L^2} \cdot O(b\|\epsilon\|_{\dot{H}^3_m} + b^\frac{7}{2})$$
Assuming the claim, (5.38) follows from
\[
\left| \frac{b}{\log M_2} \int_{M_2}^{M_2^3} (\epsilon_2, -i \Lambda M_2^3 \epsilon_2) \frac{dM_2'}{M_2'} \right| \lesssim \frac{b}{\log M_2} \int_{M_2}^{M_2^3} (M_2')^2 \|\epsilon_3\|^2_{L^2} \frac{dM_2'}{M_2'} \lesssim M_2^C \|\epsilon_3\|^2_{L^2}.
\]
Next, (5.39) follows from the identities
\[
\left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) \left( b(\epsilon_2, -i \Lambda M_2^3 \epsilon_2)_r \right) = b \left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) (\epsilon_2, -i \Lambda M_2^3 \epsilon_2)_r + b_s (\epsilon_2, -i \Lambda M_2^3 \epsilon_2)_r,
\]
applying \(|b_s| \lesssim b^2\), and the averaging argument (using Fubini):
\[
\frac{1}{\log M_2} \int_{M_2}^{M_2^3} \frac{dM_2'}{M_2'} = 1,
\]
\[
\frac{1}{\log M_2} \int_{M_2}^{M_2^3} \left( \int 1_{y \sim M_2} |\epsilon|^2 \right) \frac{dM_2'}{M_2'} \lesssim \frac{1}{\log M_2} \int 1_{M_2 \leq y \leq M_2^2} |\epsilon|^2 \lesssim \frac{1}{\log M_2} \int 1_{M_2 \leq y \leq M_2^2} |\epsilon|^2 = o_{M_2 \to \infty}(1) \|\epsilon\|^2_{H^3}. \]

From now on, we prove the above claim. We first note
\[
\|y \Lambda M_2 \epsilon_2\|_{L^2} + \|\Lambda M_2, \Lambda \|\epsilon_2\|_{L^2} \lesssim M_2^2 \|\epsilon_3\|_{L^2},
\]
which is a consequence of the crude estimate \(|y \Lambda M_2 \epsilon_2| + \|\Lambda M_2, \Lambda \|\epsilon_2\| \lesssim L_{y \leq M_2} y^2 |\epsilon_2| - 1\) and positivity of \(A^*_Q A_Q\). In particular, the boundedness property of the local virial functional is clear from
\[
| (\epsilon_2, -i \Lambda M_2 \epsilon_2)_r | \lesssim \| \frac{1}{y^2} \epsilon_2 \|_{L^2} \| y \Lambda M_2 \epsilon_2 \|_{L^2} \lesssim M_2^2 \|\epsilon_3\|^2_{L^2}.
\]

We now turn to the monotonicity estimate. As \(-i \Lambda M_2\) is symmetric, we have
\[
\frac{1}{2} \left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) (\epsilon_2, -i \Lambda M_2 \epsilon_2)_r = (\partial_s \epsilon_2, -i \Lambda M_2 \epsilon_2)_r - 3 \frac{\lambda_s}{\lambda} (\epsilon_2, -i \Lambda M_2 \epsilon_2)_r
\]
\[
= (A^*_Q A_Q \epsilon_2, \Lambda M_2 \epsilon_2)_r + \frac{\lambda_s}{\lambda} (y \partial_y \epsilon_2, -i \Lambda M_2 \epsilon_2)_r + \text{RHS of (5.28), } -i \Lambda M_2 \epsilon_2)_r
\]

As seen in Lemma 5.14, the monotonicity comes from the first term with acceptable errors. We now show that the remaining terms can be treated as errors. First, from the estimate
\[
(y \partial_y \epsilon_2, -i \Lambda M_2 \epsilon_2)_r = (\Lambda \epsilon_2, -i \Lambda M_2 \epsilon_2)_r - (\epsilon_2, -i \Lambda M_2 \epsilon_2)_r
\]
\[
= \frac{1}{2} \{ -i \Lambda M_2, \Lambda \} \epsilon_2, \epsilon_2)_r - (\epsilon_2, -i \Lambda M_2 \epsilon_2)_r = O(M_2^2 \|\epsilon_3\|^2_{L^2}),
\]
we have
\[
\left| \frac{\lambda_s}{\lambda} (y \partial_y \epsilon_2, -i \Lambda M_2 \epsilon_2)_r \right| \lesssim b(M_2)^2 \|\epsilon_3\|^2_{L^2}.
\]

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Next, proceeding as in the proof of Lemma 5.12, we have

$$\| y^{-1} \text{(RHS of (5.28))} \|_{L^2} \lesssim b \| \epsilon \|_{\mathcal{H}_m^3} + b^7. $$

Thus

$$\left| (\text{RHS of (5.28)}, -i \Lambda M \epsilon_2) \right| \lesssim \| y^{-1} \text{(RHS of (5.28))} \|_{L^2} \cdot M_2 \| \epsilon_3 \|_{L^2} \lesssim M_2 \| \epsilon_3 \|_{L^2} (b \| \epsilon \|_{\mathcal{H}_m^3} + b^7). $$

Summing up the above estimates, we obtain the monotonicity. $\square$

As explained above, the lower bounds

$$c_M b \| \epsilon \|_{L^2 \mathcal{H}_m^3} \leq M_2 \| \epsilon_3 \|_{H^1_{m \leq M_2}} \quad \text{and} \quad c_M b \| \epsilon \|_{L^2 \mathcal{H}_m^5} \leq M_2 \| \epsilon_5 \|_{H^1_{m \leq M_2}}$$

from the local virial controls dominate the dangerous contributions

$$O(b \| \epsilon_3 \|_{L^2 \mathcal{H}_m^3}) \quad \text{and} \quad O(b \| \epsilon_5 \|_{L^2 \mathcal{H}_m^5})$$

of the preliminary energy estimates (5.30) and (5.34). For this purpose, let us define the modified energies by

$$F_3 := \| \epsilon_3 \|_{L^2}^2 - M_1 \frac{b}{\log M_2} \int_{M_2}^{M_2^2} (\epsilon_2, -i \Lambda M' \epsilon_2) \frac{dM'}{M_2'},$$

$$F_5 := \| \epsilon_5 \|_{L^2}^2 - M_1 \frac{b}{\log M_2} \int_{M_2}^{M_2^2} (\epsilon_4, -i \Lambda M' \epsilon_4) \frac{dM'}{M_2'}.$$  

**Proposition 5.17** (Modified energy inequality for $\mathcal{H}_m^3$ and $\mathcal{H}_m^5$) There exists some universal constant $C$ such that the following estimates hold.

1. (Modified energy inequality for $\mathcal{H}_m^3$)

$$| F_3 - \| \epsilon_3 \|_{L^2}^2 | \leq \frac{1}{100} \| \epsilon_3 \|_{L^2}^2, \quad (5.42)$$

$$\left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) F_3 \leq b \left( \frac{1}{100} \| \epsilon_3 \|_{L^2}^2 + C b^5 \right). \quad (5.43)$$

2. (Energy identity for $\mathcal{H}_m^3$ when $m \geq 3$)

$$\left| \left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) \| \epsilon_3 \|_{L^2}^2 \right| \leq C b \| \epsilon_3 \|_{L^2} \cdot b^7 \cdot \frac{1}{M_2}. \quad (5.44)$$
3. (Modified energy inequality for $\dot{H}^5_{m}$ when $m \geq 3$)

\[
|F_5 - \|\epsilon_5\|_{L^2}^2| \leq \frac{1}{100} \|\epsilon_5\|_{L^2}^2,
\]

(5.45)

\[
(\partial_s - 10 \frac{\lambda_s}{\lambda}) F_5 \leq b \left( \frac{1}{100} \|\epsilon_5\|_{L^2}^2 + Cb^9 \right).
\]

(5.46)

**Remark 5.18** The number $\frac{1}{100}$ can be replaced by any small number.

**Remark 5.19** In the proof, we will see why we require $M \ll M_1 \ll M_2$ for the large parameters $M, M_1, M_2$. (Remark 1.4)

**Proof** (1) (5.42) follows from $MC^2b \ll 1$. We now show (5.43). Using (5.30), (5.39), and parameter dependence (Remark 1.4), we have

\[
(\partial_s - 6 \frac{\lambda_s}{\lambda}) F_3 \leq b \left( C - M_1 \epsilon_M \right) \|\epsilon\|_{\dot{H}^3_{m, \leq M_2}}^2
\]

\[
+ b \left( \frac{1}{200} + o_{M \to \infty}(1) \right) \|\epsilon_3\|_{L^2}^2 + M_1 o_{M_2 \to \infty}(1) \|\epsilon\|_{\dot{H}^3_{m}}^2 + Cb^5 \right),
\]

where $C$ is some universal constant. Notice that the first term of the RHS becomes negative due to $M \ll M_1$. The proof is now finished by applying the coercivity estimate (Lemma 2.7) and parameter dependence (Remark 1.4) for the remaining terms.

(2) This is merely a restatement of (5.31).

(3) This follows as the proof of (1) using (5.34), (5.41), and Lemma 2.10 instead of their $\dot{H}^3_{m}$-versions. We omit the proof. $\square$

### 5.5 Energy Estimate for $L^2$

In this subsection, we consider the time-variation of $\|\epsilon\|_{L^2}^2$. To close the bootstrap hypothesis for $\|\epsilon\|_{L^2}$, we use the energy method for the $\epsilon$-equation. Thanks to the mass conservation, one can simply close the bootstrap for $\|\epsilon\|_{L^2}$. But here we provide the proof not relying on the mass conservation. There are two reasons for this. First, in Sect. 6, we will need $L^2$-estimates of the difference of $\epsilon$’s (say $\delta \epsilon = \epsilon - \epsilon'$), where the conservation laws cannot be applied. Second, the argument in this subsection will in fact estimate $\partial_t \epsilon^5$ (i.e. the time-variation of $\epsilon$ in the original scalings $(t, r)$). This will be used in Sect. 5.8. Henceforth, we start with measuring the time-variation of $\|\epsilon\|_{L^2}$.

Recall the equation (5.26) of $\epsilon$. In contrast to the equations for $\epsilon_1, \ldots, \epsilon_5$, the linear part $i \mathcal{G}_Q \epsilon$ is not anti-symmetric due to $[\mathcal{G}_Q, i] \neq 0$. We will merely replace the linear part of $\epsilon$-equation by $-i \Delta_m \epsilon$, where $\Delta_m = \partial_{yy} + \frac{1}{y} \partial_y - \frac{m^2}{y^2}$ is the Laplacian acting on $m$-equivariant functions. The resulting error $i (\mathcal{G}_Q + \Delta_m) \epsilon$ is linear in $\epsilon$ so it should be worse than $R_{L-1}$ or $R_{NL}$. However, we have a good higher Sobolev control so we can roughly estimate this by $\|\epsilon\|_{\dot{H}^5_{m}} \lesssim b^{\frac{7}{2}}$ (by interpolation). Since $b^{\frac{7}{2}} \ll b$, this is safe.
In this regard, we will look at a rougher version of the $\epsilon$-equation:

\[(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \chi_s i)\epsilon - i \Delta_m \epsilon = \mathbf{Mod} \cdot \nu - i[N(P + \epsilon) - N(P)] - i\Psi. \quad (5.47)\]

We estimate the RHS of (5.47).

**Lemma 5.20** ($L^2$-estimate for the remainder terms) We have

\[
\| \mathbf{Mod} \cdot \nu \|_{L^2} \lesssim |\log b|\frac{1}{2} (\| \epsilon \|_{\dot{H}^1_m} + \| \epsilon \|_{\dot{H}^2_m}^2),
\]

\[
\| N(P + \epsilon) - N(P) \|_{L^2} \lesssim \| \epsilon \|_{\dot{H}^1_m} \| \epsilon \|_{\dot{H}^2_m}^2 + \| \epsilon \|_{\dot{H}^2_m}^3,
\]

\[
\| \Psi \|_{L^2} \lesssim b^{\frac{n}{2} + \frac{3}{2}}.
\]

**Remark 5.21** It is only important to have bounds of the form $b^{1+}$.

**Proof** The estimate (5.48) follows from

\[|\mathbf{Mod}| \lesssim |\dot{\mathbf{Mod}}| + |\partial_L \theta_L + \partial_{NL}| \lesssim \| \epsilon \|_{\dot{H}^1_m} + \| \epsilon \|_{\dot{H}^2_m}^2,
\]

\[\| \nu \|_{L^2} \lesssim |\log b|\frac{1}{2}.
\]

The estimate (5.50) follows from the pointwise estimates of $\Psi$ (3.12).

It remains to show (5.49). View

\[N(P + \epsilon) - N(P) = \sum_{\psi_1, \psi_2, \psi_3 \in \{P, \epsilon\}} N_3(\psi_1, \psi_2, \psi_3) + \sum_{\psi_1, \ldots, \psi_5 \in \{P, \epsilon\}} N_5(\psi_1, \ldots, \psi_5).
\]

Next, we recall the result of [25, Lemma 2.3]: the $L^2$-norm of $N$ can be estimated by the product of $\dot{H}^1_m$-norms of any two of its arguments and $L^2$-norms for the remaining arguments. In particular, the nonlinear terms with $\#\{j : \psi_j = \epsilon\} \geq 2$ can be estimated by $\| \epsilon \|_{\dot{H}^1_m}^2$.

The argument of the previous paragraph does not apply for the terms with $\#\{j : \psi_j = \epsilon\} = 1$. In this case, we view $N$ as a $V\psi$ form as in Sect. 5.2. If $\psi = \epsilon$, then $V$ is either $V_3[P]$ or $V_5[P]$, which are estimated by $|V| \lesssim \langle y \rangle^{-2}$. Therefore,

\[\| V \epsilon \|_{L^2} \lesssim \| \langle y \rangle^{-3} \epsilon \|_{L^2} \| \langle y \rangle^{-1} \epsilon \|_{L^2} \lesssim \| \epsilon \|_{\dot{H}^1_m} \| \epsilon \|_{\dot{H}^1_m}.
\]

If $\psi = P$, then $V$ is either $V_3[P, \epsilon]$, $V_5[P, P, P, \epsilon]$, or $V_5[P, \epsilon, P, P, P]$, where we always find the product $\text{Re}(P\epsilon)$ in the expression of $V$. Thus we estimate using (A.4)

\[\| VP \|_{L^2} \lesssim \| V \|_{L^\infty} \| P \|_{L^\infty} \| y^{-2} P \|_{L^1} \lesssim \| \epsilon \|_{\dot{H}^1_m} \| \epsilon \|_{\dot{H}^1_m}.
\]

This completes the proof.
Lemma 5.22 \textit{(L}$^2$\text{-energy estimate)} We have
\[
\partial_s \|\epsilon\|_{L^2}^2 \lesssim b^3.
\]

\textbf{Proof} Note that
\[
\frac{1}{2} \partial_s \|\epsilon\|_{L^2}^2 = (\epsilon, \text{RHS of (5.47)})_r \lesssim \|\epsilon\|_{L^2} \|\text{RHS of (5.47)}\|_{L^2}.
\]
Applying Lemma 5.20, substituting the bootstrap hypotheses, and using $K \lesssim b^{0-}$, the proof follows. \qed

5.6 Closing the Bootstrap

Here we finish the proof of Proposition 4.4 by gathering the modulation estimates and (modified) energy estimates for adaptive derivatives of $\epsilon$.

Lemma 5.23 \textit{(Consequences of modulation estimates)} We have
\[
\int_0^t \frac{b^3}{\lambda^2} \cdot \frac{b^5}{\lambda^6} d\tau \leq \left(1 + O((b^*)^{\frac{1}{2}})\right) b^5(t) \frac{\lambda^5(t)}{\lambda^6(t)},
\]
(5.52)
\[
\frac{\lambda^5(t)}{b(t)} \leq \frac{\lambda^5_0}{b_0},
\]
(5.53)
\[
\int_0^t \frac{b^3}{\lambda^2} d\tau \leq 2 \left(1 + O((b^*)^{\frac{1}{2}})\right) b^5_0 b^0(t),
\]
(5.54)
\[
\frac{b(t)}{\lambda(t)} = \left(1 + O((b^*)^{\frac{1}{2}})\right) \frac{b_0}{\lambda_0}.
\]
(5.55)

\textbf{Proof} The assertions will be obtained by integrating in time using the modulation estimates
\[
b = -\frac{\lambda_s}{\lambda} + O(b^3) \quad \text{and} \quad b_s = -b^2 + O(b^3),
\]
which are obtained by Lemma 5.9 and applying the bootstrap hypothesis with $K \ll M$.

Now (5.52) follows from $\frac{b}{\lambda^2} = -\frac{\lambda_s}{\lambda} + O(\frac{b^{3/2}}{\lambda^2})$ and integration by parts:
\[
\int_0^t \frac{b^3}{\lambda^2} \cdot \frac{b^5}{\lambda^6} d\tau = \frac{1}{6} \left[ \frac{b^5}{\lambda^6} \right]_0^t - \frac{5}{6} \int_0^t \frac{b^3 b^4}{\lambda^5} d\tau + O\left( b^5 \int_0^t \frac{b b^5}{\lambda^2} \cdot \frac{b^5}{\lambda^6} d\tau \right)
\]
\[
= \frac{1}{6} \left[ \frac{b^5}{\lambda^6} \right]_0^t + \frac{5}{6} \int_0^t \frac{b b^5}{\lambda^2} \cdot \frac{b^5}{\lambda^6} d\tau + O\left( b \int_0^t \frac{b}{\lambda^2} \cdot \frac{b^5}{\lambda^6} d\tau \right).
\]
(5.53) follows from
\[
\partial_s \log \left( \frac{\lambda^5}{b} \right) = \frac{\lambda_s}{5\lambda} + \left( \frac{\lambda}{\lambda} + b \right) - \left( \frac{b_s + b^2}{b} \right) = -\frac{b}{5} + O(b^3) \leq 0.
\]
\[\square\]
\[ \int_0^t \frac{b^2}{\lambda^2} d\tau = \int_0^t \frac{1}{b^2} \left( -b_t + \frac{b_s + b^2}{\lambda^2} \right) d\tau = -2[b^{\frac{1}{2}}]_0 + O\left((b^*)^{\frac{1}{2}} \int_0^t \frac{b^3}{\lambda^2} d\tau\right). \]

(5.54) follows from

\[ \int_0^t \frac{b^2}{\lambda^2} d\tau = \int_0^t \frac{1}{b^2} \left( -b_t + \frac{b_s + b^2}{\lambda^2} \right) d\tau = -2[b^{\frac{1}{2}}]_0 + O\left((b^*)^{\frac{1}{2}} \int_0^t \frac{b^3}{\lambda^2} d\tau\right). \]

(5.55) follows from integrating

\[ \left| \partial_t \log \left( \frac{b}{\lambda} \right) \right| = \frac{1}{\lambda^2} \left| \left( \frac{b_s + b^2}{b} \right) - \left( \frac{\lambda_s}{\lambda} + b \right) \right| \lesssim \frac{b^2}{\lambda^2} \quad (5.56) \]

using (5.54).

We finish the proof of Proposition 4.4.

**End of the proof of Proposition 4.4** We note that \( b \leq b_0 \) is clear from \( b_s = -b^2 + O(b^5) \). Thus we focus on the estimates on \( \epsilon \).

We first close the \( \|\epsilon_3\|_{L^2} \)-bound for \( H_m^3 \)-solutions. By the modified energy inequality (5.43), bootstrap hypothesis (4.11), and (5.52), we have

\[ \mathcal{F}_3(t) - \mathcal{F}_3(0) \leq \left( \frac{b(t)}{b_0} \right)^5 \|\epsilon_3(0)\|_{L^2}^2 + \left( 1 + O((b^*)^{\frac{1}{2}}) \right) \left( \frac{K^2}{100} + C \right) b^5. \]

Applying (5.42) and (5.53), we have

\[ \frac{99}{100} \|\epsilon_3(t)\|_{L^2}^2 \leq \frac{101}{100} \left( \frac{b(t)}{b_0} \right)^5 \|\epsilon_3(0)\|_{L^2}^2 + (1 + O((b^*)^{\frac{1}{2}})) \left( \frac{K^2}{100} + C \right) b^5. \]

Using the initial bound \( \|\epsilon_3(0)\|_{L^2} \leq b_0^3 \), this closes the \( \|\epsilon_3\|_{L^2} \)-bound for \( H_m^3 \)-solutions, due to \( K \gg 1 \).

Next, we close the \( \|\epsilon_3\|_{L^2} \)-bound for \( H_m^5 \)-solutions when \( m \geq 3 \). Here, we merely apply the energy identity (5.44) and the claims (5.55) and (5.54):

\[ \left| \frac{\|\epsilon_3(t)\|_{L^2}^2}{\lambda^6(t)} - \frac{\|\epsilon_3(0)\|_{L^2}^2}{\lambda^6(0)} \right| \lesssim K \int_0^t \frac{b}{\lambda^8} \cdot b^{^6+\frac{1}{2}} d\tau \lesssim \left( \frac{b_0}{\lambda_0} \right)^6 \int_0^t \frac{b^3}{\lambda^2} d\tau \lesssim K \left( \frac{b_0}{\lambda_0} \right)^6 b^3. \]

Again by (5.55), we get

\[ \|\epsilon_3(t)\|_{L^2} \lesssim \left( \frac{b(t)}{b_0} \right)^6 \|\epsilon_3(0)\|_{L^2}^2 + K b_0^{\frac{1}{2}} \cdot b^6(t). \]

Using the initial bound \( \|\epsilon_3(0)\|_{L^2} \leq b_0^3 \), this closes the \( \|\epsilon_3\|_{L^2} \)-bound for \( H_m^5 \)-solutions when \( m \geq 3 \), due to \( K \gg 1 \).
Next, we close the $\|\epsilon_5\|_{L^2}$-bound for $H^5_m$-solutions when $m \geq 3$. Applying the modified energy inequality (5.43) and analogues of the claims (5.52) and (5.53) (replace $b_0^5$ by $b_0^g$), we have

$$\frac{99}{100} \|\epsilon_5(t)\|_{L^2}^2 \leq \frac{101}{100} \left( \frac{b(t)}{b_0} \right)^9 \|\epsilon_5(0)\|_{L^2}^2 + (1 + O((b^*)^{\frac{1}{2}})) \left( \frac{K^2}{100} + C \right) b^9.$$  

Using the initial bound $\|\epsilon_5(0)\|_{L^2} \leq b_0^5$, this closes the $\|\epsilon_3\|_{L^2}$-bound for $H^3_m$-solutions, due to $K \gg 1$.

Next, we close the $\|\epsilon_1\|_{L^2}$-bound. We use the energy conservation

$$\frac{1}{\lambda_0^2} \int |D^{(P+;\eta_0)+\epsilon_0}_+(P(\cdot; b_0, \eta_0) + \epsilon_0)|^2 = \frac{1}{\lambda^2} \int |D^{(P+\epsilon)}_+(P + \epsilon)|^2.$$

We write

$$D^{(P+\epsilon)}_+(P + \epsilon) = D^{(P)}_+ P + L_Q \epsilon + (L_P \epsilon - L_Q \epsilon) + N_P(\epsilon).$$

Note that $\|D^{(P)}_+ P\|_{L^2} \lesssim b$. We note that the last two terms are linear combinations of the expression

$$\frac{1}{y} (\int_0^y \Re \overline{\psi_1} \psi_2 y'dy') \psi_3,$$

whose $L^2$-norm can be bounded by (see for instance [25, Lemma 2.3])

$$\|\psi_{j_1}\|_{L^2} \|\psi_{j_2}\|_{L^2} \|\psi_{j_3}\|_{\dot{H}^{m}_1},$$

where $\{j_1, j_2, j_3\} = \{1, 2, 3\}$ can be arbitrarily chosen. We note that each term of $L_P \epsilon - L_Q \epsilon$ is of the form (5.57) with $\psi_{j_1} = P - Q$, $\psi_{j_2} \in \{P, Q\}$, $\psi_{j_3} = \epsilon$, so its contribution is bounded by $b |\log b|^{\frac{1}{2}} \|\epsilon\|_{\dot{H}^{m}_1} = O(b)$. Similarly, each term of $N_P(\epsilon)$ is of the form (5.57) with $\psi_{j_1} \in \{P, \epsilon\}$ and $\psi_{j_2} = \psi_{j_3} = \epsilon$, so its contribution is bounded by $\|\epsilon\|_{\dot{H}^{m}_1} \|\epsilon\|_{L^2} \lesssim K(b^*)^{\frac{1}{2}} \|\epsilon\|_{\dot{H}^{m}_1} = O(b)$. Therefore, we have

$$\lambda \|\epsilon_1\|_{L^2} = \|D^{(P+\epsilon)}_+(P + \epsilon)\|_{L^2} + O(b) = \|D^{(P(\cdot; b_0, \eta_0)+\epsilon_0)}_+(P(\cdot; b_0, \eta_0) + \epsilon_0)\|_{L^2} + O(b_0) = \lambda_0 \|\epsilon_1(0)\|_{L^2} + O(b_0).$$

Applying (5.55) yields

$$\|\epsilon_1\|_{L^2} \lesssim b.$$

As $K \gg 1$, this closes the $\|\epsilon_1\|_{L^2}$-bound.
Finally, we close the \( \| \epsilon \|_{L^2} \)-bound. One can use the mass conservation (similarly as in the \( H^1_m \)-estimate above), but as mentioned in Sect. 5.5, we will rely on \( \partial_t \| \epsilon \|_{L^2}^2 \).

By Lemma 5.22, we have

\[ \partial_t \| \epsilon(t) \|_{L^2}^2 \lesssim \lambda^{-2} b^{3 \frac{1}{2}}. \]

Integrating this using (5.54) and the initial bound yields

\[ \| \epsilon(t) \|_{L^2}^2 \lesssim \| \epsilon_0 \|_{L^2}^2 + \int_0^t \frac{b^3}{\lambda^2} b^2 \tau \lesssim b_0^\frac{1}{2}. \]

This closes the \( \| \epsilon \|_{L^2} \)-bound due to \( K \gg 1 \).

5.7 Existence of Special \( \eta_0 \)

Here we prove Proposition 4.5.

Proof of Proposition 4.5 To show nonempty-ness of \( \mathcal{I}_\pm \), we show \( \pm \frac{K^2}{10} b_0^{3/2} \in \mathcal{I}_\pm \). We compute the variation of the ratio \( \frac{\eta}{b^{3/2}} \):

\[ \partial_s \left( \frac{\eta}{b^{3/2}} \right) = \frac{3}{2} \frac{\eta}{b^{3/2}} \left( b - \frac{b_s + b_b^2}{\lambda^2} \right) + \frac{\eta_s}{b^{3/2}} = \frac{3}{2} \left( \frac{\eta}{b^{3/2}} \right) b (1 + O(b^{1/2})) + O(b). \]

Thus if \( | \frac{\eta}{b^{3/2}} | \geq \frac{K}{10} \) (recall that \( K \gg 1 \)) holds at some time, then \( | \frac{\eta}{b^{3/2}} | \) starts to increase. In particular, if \( \eta_0 = \pm \frac{K}{10} b_0^{3/2} \), then \( \eta(T_{\text{exit}}) \) must have same sign with \( \eta_0 \), saying that \( \pm \frac{K}{10} b_0^{3/2} \in \mathcal{I}_\pm \).

We now show that \( \mathcal{I}_\pm \) is open. Fix \( \eta_0 \in \mathcal{I}_\pm \) and let us denote by \( u(\eta_0) \) the corresponding evolution. Let us add a superscript (\( \eta_0 \)) for clarification. Since \( \eta_0 \in \mathcal{I}_\pm \), there exists \( T(\eta_0) \in [0, T_{\text{exit}}(\eta_0)) \) such that \( \pm \eta(\eta_0) (T(\eta_0)) > \frac{K}{10} b^{3/2} (T(\eta_0)) \). Now, by continuous dependence, (obtained by combining the local well-posedness and Lemma 4.2) we should have \( \eta(\eta_0) (T(\eta_0)) > \frac{K}{2} b^{3/2} (T(\eta_0)) \) for all \( \eta_0 \) near \( \eta_0 \). Such \( \eta_0 \) belongs to \( \mathcal{I}_\pm \) due to the argument in the previous paragraph. This completes the proof.

5.8 Pseudoconformal Blow-Up of Trapped Solutions

Here, we conclude the proof of Theorem 1.1 and the first part of Theorem 1.3. By the reduction in Sect. 4.2, it only remains to show that trapped solutions are pseudoconformal blow-up solutions. Both the proofs for \( H^3_m \)-trapped solutions and \( H^5_m \)-trapped solutions are quite similar. The only difference for \( H^3_m \)-trapped solutions and \( H^5_m \)-trapped solutions is the regularity of the radiation \( u^* \), i.e. \( u^* \in H^1_m \) or \( u^* \in H^3_m \). Here we prove it for \( H^3_m \)-trapped solutions. For \( H^5_m \)-solutions, see Remark 5.24.
End of the proof of Theorem 1.1 By the claim (5.55), we have
\[
\partial_t \lambda = -\frac{b}{\lambda} + \frac{1}{\lambda} \left( \frac{\lambda_s}{\lambda} + b \right) = -\frac{b}{\lambda} (1 + O((b^*)^{1/2})) = -\frac{b_0}{\lambda_0} (1 + O((b^*)^{1/2})) < -\frac{b_0}{2\lambda_0}.
\]
This implies \( T < +\infty \). By the standard blow-up criterion, we have \( \lambda(T) := \lim_{t \uparrow T} \lambda(t) = 0 \).

We now rewrite the claim (5.55) as
\[
\frac{b}{\lambda} = \ell (1 + O(b^{1/2})), \quad \ell := \lim_{t \uparrow T} \frac{b(t)}{\lambda(t)} \in (0, \infty).
\]
(5.58)

The existence of \( \ell \in (0, \infty) \) follows from (5.56) and (5.54). The error bound \( O(b^{1/2}) \) follows from integrating (5.56) on \([t, T]\) instead of \([0, t]\). In particular, \( b(T) := \lim_{t \uparrow T} b(t) = 0 \).

We now derive the asymptotics of the modulation parameters. We again compute \( \partial_t \lambda \) and \( \partial_t b \), but with the help of (5.58):
\[
\partial_t \lambda = -\frac{b}{\lambda} (1 + O(b^{1/2})) = -\ell (1 + O(b^{1/2})),
\]
\[
\partial_t b = \frac{b_s + b^2}{\lambda^2} - \frac{b^2}{\lambda^2} = -\frac{b^2 (1 + O(b^{1/2}))}{\lambda^2} = -\ell^2 (1 + O(b^{1/2})).
\]

Integrating the above relations from backward in time shows
\[
\lim_{t \uparrow T} \frac{\lambda(t)}{T - t} = \ell, \quad \lim_{t \uparrow T} \frac{b(t)}{T - t} = \ell^2.
\]

From the modulation estimate (Lemma 5.9) and the estimates of \( \theta_\eta, \theta_\psi, \theta_{\Pi-1}, \) and \( \theta_{NL} \) estimates (see (3.10), (3.11), (5.3), and (5.15)), we have
\[
|\partial_t \gamma| \lesssim \lambda^{-2} (K b^3 + K^2 b^2) \lesssim \lambda^{-2} K b^3 \lesssim K \ell (T - t)^{-1/2}
\]
for \( t \) near \( T \). This shows that \( \gamma(t) \) converges as \( t \to T \), say \( \gamma^* \). Finally, we note that \( \eta(t) \to 0 \) as \( t \to T \) due to \( b(t) \to 0 \) and the definition of \( \mathcal{O}_{trap} \).

It only remains to show that \( u \) decomposes as in Theorem 1.1. Let us define
\[
\epsilon_\tau(t, r) := \frac{e^{i\gamma(t)}}{\lambda(t)} \epsilon \left( t, \frac{r}{\lambda(t)} \right).
\]
We should show that
\[
\tilde{\epsilon}_\tau(t) := \left\{ \frac{e^{i\gamma(t)}}{\lambda(t)} P \left( \frac{r}{\lambda(t)} ; b(t), \eta(t) \right) - \frac{e^{i\gamma^*}}{\ell (T - t)} Q \left( \frac{r}{\ell (T - t)} \right) \right\} + \epsilon^*(t)
\]

\footnote{The standard Cauchy theory of (CSS) says that the solution blows up at finite time \( T < +\infty \) if and only if \( \lim_{t \uparrow T} ||u(t)||_{H^1} = \infty \).}
converges in $L^2$ as $t \to T$ and the limit belongs to $H^1_m$. Since $(\gamma, b, \eta) \to (\gamma^*, 0, 0)$ and $\frac{\ell(T-t)}{\lambda(t)} \to 1$ as $t \to T$, the first term of the above display converges to 0 in $L^2$. The second term $\epsilon^u(t)$ is uniformly bounded in $H^1_m$, thanks to the boundedness of $\frac{b}{\lambda}$ shown in (5.55) and $\|\epsilon^u\|_{H^1_m} = \lambda^{-1} \|\epsilon\|_{H^1_m}$. Thus it only remains to show that $\{\epsilon^u(t)\}_{t \to T}$ is Cauchy in $L^2$.

To show that $\{\epsilon^u(t)\}_{t \to T}$ is Cauchy in $L^2$, it suffices to show that $\|\partial_t \epsilon^u\|_{L^2}$ is integrable in time. From the $\epsilon$-equation (5.47), we have

$$\partial_t \epsilon^u = \frac{1}{\lambda^2} \cdot e^{i\gamma(t)} \frac{\epsilon}{\lambda(t)} \left[ i \Delta_m \epsilon + \text{Mod} \cdot v - i (\mathcal{N}(u) - \mathcal{N}(P)) - i \Psi \right] \left( t, \frac{r}{\lambda(t)} \right).$$

By Lemma 5.20, we have

$$\|\partial_t \epsilon^u\|_{L^2} \lesssim \lambda^{-2} \left( \|\Delta_m \epsilon\|_{L^2} + \|\text{RHS of (5.47)}\|_{L^2} \right) \lesssim \lambda^{-2} \left( \|\epsilon\|_{H^1_m} + \|\epsilon\|_{H^3_m} + b^2 \right) \lesssim \ell(T - t)^{-\frac{1}{2}}.$$

Thus $\|\partial_t \epsilon^u\|_{L^2}$ is integrable in time. This ends the proof of Theorem 1.1. \qed

**Remark 5.24** For $H^5_m$-trapped solutions, thanks to the improved bound for $\epsilon_3$, we see that $\epsilon^u(t)$ is uniformly bounded in $H^3_m$. This says that the radiation enjoys further regularity $H^3_m$.

### 6 Lipschitz Blow-Up Manifold

Throughout this section, we assume

$$m \geq 3$$

and only deal with $H^5_m$-trapped solutions.

So far, we constructed trapped solutions. Due to a soft connectivity argument, we were only able to guarantee the existence of pseudoconformal blow-up solutions, with a weak codimension one condition. Recall that $\mathcal{M}$ is the set of initial data in $O^5_{\text{init}}$ yielding $H^5_m$-trapped solutions. Hence $\mathcal{M}$ contains the blow-up solutions constructed in the first part of Theorem 1.3. In this section we aim to prove more quantitative information on $\mathcal{M}$. First, it includes the uniqueness statement saying that $\mathcal{M}$ is equal to the solutions constructed by the proof of the first part of Theorem 1.3. It also includes the Lipschitz dependence on initial data. These will also finish the proof of Theorem 1.3.

Among many works on establishing the regularity of blow-up manifolds (or, stable/unstable manifolds), the most relevant one to this paper is the work of Collot [11]. Our proof uses the idea of [11].

Recall from the proof of Theorem 1.1 that the codimension one condition stems from the *unstable parameter* $\eta$. In order to finish the proof of Theorem 1.3, we have to
ensure uniqueness of \( \eta_0 \) yielding trapped solutions and also the Lipschitz dependence on (the stable modes of) initial data. Thus the heart of the proof will be to control the difference of unstable parameter \( \eta \) by the difference of stable parameters \( (b \text{ and } \epsilon) \) if we modulo out the scaling and phase rotation symmetries.

### 6.1 Reduction of Theorem 1.3

In this subsection, we reduce the proof of Theorem 1.3 into Proposition 6.1, which is the control of the difference of unstable parameters by the difference of stable parameters.

Let \( u \) and \( u' \) be two \( H^5_m \)-trapped solutions. By Lemma 4.2, \( u \) has associated modulation parameters \( \lambda, \gamma, b, \eta \) and the error part \( \epsilon \). Similarly, \( u' \) has parameters \( \lambda', \gamma', b', \eta' \) and the error \( \epsilon' \). At the initial time \( t = 0 \), we add a subscript 0 to these: \( b(0) = b_0, \eta'(0) = \eta_0, \) and so on.

**Proposition 6.1** (Lipschitz estimate modulo scaling/phase invariances) Let \( u \) and \( u' \) be two \( H^5_m \)-trapped solutions. If \( b'_0 \) is sufficiently close to \( b_0 \), then

\[
|\eta_0 - \eta'_0| \lesssim_{b_0} |b_0 - b'_0| + \|\epsilon_0 - \epsilon'_0\|_{H^3_m}.
\]

In particular, for given \( b_0 \in (0, b^*) \) and \( \|\epsilon_0\|_{H^5_m} < b^5_0 \), there exists unique \( \tilde{\eta}_{\text{prem}} = \tilde{\eta}_{\text{prem}}(b_0, \epsilon_0) \) such that \( P(\cdot; \tilde{\eta}_{\text{prem}} + \epsilon_0) \in \mathcal{M} \). Moreover, \( \tilde{\eta} \) is locally Lipschitz in \( b_0, \epsilon_0 \).

We remark that Proposition 6.1 implies the uniqueness of \( \eta_0 \) (for given \( b_0 \) and \( \epsilon_0 \)) of Theorem 1.3. Hence \( \mathcal{M} \) is equal to the solutions constructed by the proof of the first part of Theorem 1.3. Due to the scaling and phase rotation symmetries, once \( P(\cdot; b_0, \eta_0) + \epsilon_0 \in \mathcal{M} \), we have \( e^{i\gamma_0/\lambda_0} [P(\cdot; b_0, \eta_0) + \epsilon_0(\cdot)](\cdot/\lambda_0) \in \mathcal{M} \). Thus \( \eta_0 \) does not depend on the scaling and phase parameters.

Notice also that the statement of Proposition 6.1 is insensitive to the scales and phases of \( u \) and \( u' \). Thus this measures the difference of solutions modulo scaling/phase symmetries.

If one fixes the scale 1 and phase 0, then one can construct a codimension three blow-up manifold, say \( \mathcal{M}_{1,0} \). A naive try to recover the codimension one manifold \( \mathcal{M} \) is to apply scaling and phase rotation symmetries to \( \mathcal{M}_{1,0} \) since \( \mathcal{M} = \bigcup_{\lambda, \gamma} \mathcal{M}_{\lambda, \gamma} \), where \( \mathcal{M}_{\lambda, \gamma} = \{ e^{i\gamma/\lambda} u_0(\cdot/\lambda) : u_0 \in \mathcal{M}_{1,0} \} \). However, the scaling symmetry is not Lipschitz continuous on any Sobolev spaces. More precisely, the map

\[
(\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0) \mapsto e^{i\gamma_0/\lambda_0} [P(\cdot; b_0, \eta_0) + \epsilon_0(\cdot)](\cdot/\lambda_0)
\]

is not Lipschitz continuous in \( \lambda_0 \), due to the \( \frac{1}{\lambda_0} \epsilon_0(\cdot/\lambda_0) \)-part. This says that Lipschitz property of \( \mathcal{M} \) does not immediately follow from applying the scaling and phase rotation symmetries to \( \mathcal{M}_{1,0} \).
The remedy is to consider the decomposition formula

$$(\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0^\#) \mapsto \frac{e^{iy_0}}{\lambda_0} P\left(\frac{\gamma}{\lambda_0}; b_0, \eta_0\right) + \epsilon_0^\#$$

instead, where we denoted $\epsilon_0^\#(r) := \frac{e^{iy_0}}{\lambda_0} \epsilon_0(\frac{r}{\lambda_0})$. Once we use $\epsilon_0^\#$ as an independent variable instead of $\epsilon_0$, the above formula is Lipschitz continuous in terms of $(\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0^\#)$. Viewing $\epsilon_0 = e^{-iy_0} \lambda_0 \epsilon_0^\#(\lambda_0 \cdot)$, we can write $\tilde{\eta}_{prem}$ as a function of $\lambda_0, \gamma_0, b_0, \epsilon_0^\#$, say $\tilde{\eta}_{prem}(b_0, \epsilon_0) = \tilde{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\#)$. Then the Lipschitz regularity of $\mathcal{M}$ would follow if we show that $\tilde{\eta} = \tilde{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\#)$ is Lipschitz continuous.

From this, we need a variant of Proposition 6.1, where we measure the difference in the original variables.

**Proposition 6.2** (Lipschitz estimate in the original variables) Let $u$ and $u'$ two $H_m^5$-trapped solutions. If $b_0'$ is sufficiently close to $b_0$, and $\lambda_0$ and $\lambda_0'$ are sufficiently close to 1, then

$$|\eta_0 - \eta_0'| \lesssim_{b_0} |\lambda_0 - \lambda_0'| + |\gamma_0 - \gamma_0'| + |b_0 - b_0'| + \|\epsilon_0^\# - (\epsilon_0')^\#\|_{H_m^3},$$

where $\epsilon_0^\# = \frac{e^{iy_0}}{\lambda_0} \epsilon_0(\frac{\gamma}{\lambda_0})$ and $(\epsilon_0')^\# = \frac{e^{iy_0}}{\lambda_0} \epsilon_0'(\frac{\gamma}{\lambda_0})$. In particular, for given $(\lambda_0, \gamma_0, b_0, \eta_0) \in \tilde{U}_{init}^5$, there exists unique $\tilde{\eta} = \tilde{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\#)$\(^{21}\) such that $\frac{e^{iy_0}}{\lambda_0} P\left(\frac{\gamma}{\lambda_0}; b_0, \tilde{\eta}\right) + \epsilon_0^\# \in \mathcal{M}$. Moreover, $\tilde{\eta}$ is a locally Lipschitz function of $\lambda_0, \gamma_0, b_0, \epsilon_0^\#$.

Let us show how one can prove Theorem 1.3 assuming Proposition 6.2.

**Proof of Theorem 1.3 assuming Proposition 6.2** We remind the reader that the existence of $H_m^5$-trapped solutions is proved in Sect. 4.2. The uniqueness of $\eta_0$ is an immediate consequence of Proposition 6.1. Henceforth, we show that $\mathcal{M}$ has the Lipschitz regularity.

Let $\widehat{u} \in \mathcal{M}$ be a reference element. It suffices to show that a neighborhood of $\widehat{u}$ in $\mathcal{M}$ can be expressed as a Lipschitz graph of some codimension one subspace of $H_m^5$.

Denote by $\tilde{\lambda}, \tilde{\gamma}, \tilde{b}, \tilde{\eta}, \tilde{\epsilon}$ the associated parameters and error part. Denote by $\tilde{\epsilon}^\# := \frac{\epsilon_0^\#}{\lambda_0} \epsilon(\frac{\gamma}{\lambda_0})$. By scaling and phase rotation symmetries, we may assume that $\tilde{\lambda} = 1$ and $\tilde{\gamma} = 0$.\(^{22}\) Denote by $\tilde{P} = P(\cdot; \tilde{b}, \tilde{\eta}), \partial_{\tilde{b}} \tilde{P} = [\partial_{\tilde{b}} P](\cdot; \tilde{b}, \tilde{\eta})$, and $\partial_{\tilde{\eta}} \tilde{P} = [\partial_{\tilde{\eta}} P](\cdot; \tilde{b}, \tilde{\eta})$. Similarly denote the modulation vector at this reference point by $\tilde{v} = (\Lambda \tilde{P}, -i \tilde{P}, -\partial_{\tilde{b}} \tilde{P}, -\partial_{\tilde{\eta}} \tilde{P})^\dagger$. Define the stable/unstable subspaces

$$X_s := \text{span}_{\mathbb{R}} \{\Lambda \tilde{P}, i \tilde{P}, \partial_{\tilde{b}} \tilde{P}\} \oplus (Z^\perp \cap H_m^5),$$

$$X_u := \text{span}_{\mathbb{R}} \{\partial_{\tilde{\eta}} \tilde{P}\},$$

\(^{21}\) In fact, $\tilde{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\#) = \tilde{\eta}_{prem}(b_0, \epsilon_0)$ under the relation $\epsilon_0^\# = \frac{e^{iy_0}}{\lambda_0} \epsilon_0(\frac{\gamma}{\lambda_0})$.

\(^{22}\) Thus $\tilde{\epsilon}^\# = \tilde{\epsilon}$, but we will use the notation $\tilde{\epsilon}^\#$ to emphasize that we are working with the original variable $r$. 

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such that

\[ H^5_m = X_s \oplus X_u. \]

In the proof we also use the notation

\[ f_{\lambda, \gamma} := \frac{e^{i\gamma}}{\lambda} f \left( \frac{\cdot}{\lambda} \right). \]

On a small neighborhood of \( \hat{u} \) in \( \hat{u} + X_s \), define the map \( h \) taking values in \( M \) by

\[ \hat{u} + (\delta \lambda) \Lambda \hat{P} + (\delta \gamma) i \hat{P} + (\delta b) \partial_b \hat{P} + (\delta \epsilon^\perp_0) \mapsto P (\cdot; b_0, \eta_0)_{\lambda_0, \gamma_0} + \epsilon^\perp_0, \]

where \( \delta \lambda, \delta \gamma, \delta b, \delta \epsilon^\perp_0 \) are considered as independent variables and

\[
\begin{align*}
\lambda_0 &:= 1 + \delta \lambda, \quad \gamma_0 := \delta \gamma, \\
b_0 &:= \hat{b} + \delta b, \quad \eta_0 := \hat{\eta}(\lambda_0, \gamma_0, b_0, \epsilon^\perp_0), \\
\epsilon^\perp_0 &:= \hat{\epsilon}^\perp + \delta \epsilon^\perp_0 - \sum_{j,k} (A^{-1})_{jk} (\hat{\epsilon}^\perp + \delta \epsilon^\perp_0, [Z_k]_{\lambda_0, \gamma_0}),
\end{align*}
\]

Here, \( A^{-1} \) is the inverse matrix of \( A \) whose components are defined by \( A_{jk} := (Z_j, \hat{v}_k)_r \), and \( \hat{\eta}(\lambda_0, \gamma_0, b_0, \epsilon^\perp_0) \) is defined in Proposition 6.2. The (seemingly complicated) additional term in the definition (6.1) is designed to guarantee that \( \epsilon^\perp_0 \) satisfies the orthogonality conditions at the scale \( \lambda_0 \) and phase \( \gamma_0 \), i.e.

\[
(\epsilon^\perp_0, [Z_k]_{\lambda_0, \gamma_0})_r = 0 \quad \forall k \in \{1, 2, 3, 4\}.
\]

By the definition of \( \hat{\eta} \), we verify that \( h \) maps into \( M \) and \( h(\hat{u}) = \hat{u} \).

On a small neighborhood of \( \hat{u} \) in \( H^5_m = (\hat{u} + X_s) \oplus X_u \), we define a \( C^1 \)-diffeomorphism \( H \) by

\[ \hat{u} + (\delta \lambda) \Lambda \hat{P} + (\delta \gamma) i \hat{P} + (\delta b) \partial_b \hat{P} + (\delta \eta) \partial_\eta \hat{P} + (\delta \epsilon^\perp_0) \mapsto P (\cdot; b_0, \eta_0)_{\lambda_0, \gamma_0} + \epsilon^\perp_0, \]

where \( b_0, \lambda_0, \gamma_0, \epsilon^\perp_0 \) are as in the definition of \( h \) and \( \eta_0 := \hat{\eta} + \delta \eta \). Note that \( H(\hat{u}) = \hat{u} \). To see the \( C^1 \)-diffeomorphism property, we observe the following. The partial derivatives of \( H \) along the \( \delta \lambda_0, \delta \gamma_0, \delta b_0, \delta \eta_0 \) directions are \(-\frac{1}{\lambda_0} [\Lambda P]_{\lambda_0, \gamma_0}, i P_{\lambda_0, \gamma_0}, [\partial_b P]_{\lambda_0, \gamma_0}, [\partial_\eta P]_{\lambda_0, \gamma_0} \) evaluated at \((b_0, \eta_0)\) plus some \( O(\epsilon^\perp_0) \) error. Next, the functional derivative of \( H \) along the \( \delta \epsilon^\perp_0 \) is \( \text{id} (\perp H^5_m) \) plus a rank four operator whose range lies in the span of \([\Lambda P]_{\lambda_0, \gamma_0}, i P_{\lambda_0, \gamma_0}, [\partial_b P]_{\lambda_0, \gamma_0}, [\partial_\eta P]_{\lambda_0, \gamma_0} \). In summary,

\[
\frac{\delta H}{\delta ((\delta \lambda), (\delta \gamma), (\delta b), (\delta \eta), (\delta \epsilon^\perp_0))}
\]

\[ A \] is an almost diagonal matrix, as seen in (4.14).

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\[
\begin{pmatrix}
-1 & 0 & 0 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & \text{id}_{(Z^1 \cap H^3_m)}
\end{pmatrix}
+ O_{\mathcal{L}(H^5_m, H^3_m)}(|\delta \lambda| + \cdots + |\delta \eta| + \|\epsilon_0^\varepsilon\|_{L^2}).
\]

where the first four rows are written in the directions $\Lambda \hat{P}$, $i \hat{P}$, $\partial_\eta \hat{P}$, $\partial_\gamma \hat{P}$. Applying the inverse function theorem at $\hat{u}$ says that $H$ is a $C^1$-diffeomorphism on a neighborhood of $\hat{u}$ in $H^5_m$.

Therefore, it suffices to show that there is an open neighborhood $\hat{O}$ of $\hat{u}$ such that $H^{-1}(\mathcal{M} \cap \hat{O})$ is a locally Lipschitz codimension one manifold. We note that $H^{-1} \circ h$ is given by

\[
\hat{u} + (\delta \lambda) \Lambda \hat{P} + (\delta \gamma) i \hat{P} + (\delta b) \partial_\gamma \hat{P} + (\delta \epsilon_0^\varepsilon)
\]

where $\hat{\eta} = \hat{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\varepsilon)$. Notice that this looks like the form $\text{id}_{\hat{u} + X_s} \oplus f$, where $f = (\hat{\eta} - \hat{\eta}) \partial_\eta \hat{P}$. According to Definition 1.2, it suffices to show that (1) there are small neighborhoods $\hat{O}$ of $\hat{u}$ in $H^5_m$ and $\hat{O}$ of $\hat{u}$ in $\hat{u} + X_s$ such that $\mathcal{M} \cap \hat{O} = h(\hat{O})$, and (2) the map $(\delta \lambda, \delta \gamma, \delta b, \delta \epsilon_0^\varepsilon) \mapsto (\hat{\eta} - \hat{\eta}) \partial_\eta \hat{P}$ is Lipschitz continuous.

(1) For $\hat{O}$ to be chosen small, let $u_0 \in \mathcal{M} \cap \hat{O}$. Because $u_0$ is near $\hat{u}$, we can perform the decomposition according to Lemma 4.2 and the associated $\lambda_0, \gamma_0, \eta_0, b_0, \epsilon_0^\varepsilon$ are all close to the reference data $\hat{\lambda}, \ldots, \hat{\epsilon}$. Since $u_0 \in \mathcal{M}$ and $u_0$ is near $\hat{u}$, the uniqueness statement of Proposition 6.2 says that $\eta_0 = \hat{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\varepsilon)$. We then define $\delta \epsilon_0^\varepsilon$ by inverting the formula (6.1). Note that the inversion is possible because the summation part is small, thanks to the orthogonality condition. If $\hat{O}$ is sufficiently small, then this $(\lambda_0, \gamma_0, b_0, \delta \epsilon_0^\varepsilon)$ lies in the domain of $h$. In other words, $u_0$ lies in the image of $h$. Thus $\mathcal{M} \cap \hat{O}$ is contained in the image of $h$. Finally set $\hat{O} := h^{-1}(\hat{O})$.

(2) It suffices to show that $(\delta \lambda, \delta \gamma, \delta b, \delta \epsilon_0^\varepsilon) \mapsto \hat{\eta}(\lambda_0, \gamma_0, b_0, \epsilon_0^\varepsilon)$ is Lipschitz continuous. This follows from Proposition 6.2 and the $\epsilon_0^\varepsilon$-formula (6.1).

This ends the proof of Theorem 1.3. \qed

One can also deduce Proposition 6.2 from Proposition 6.1.

**Proof of Proposition 6.2 assuming Proposition 6.1** Without loss of generality, we may assume that $\lambda_0 \geq \lambda_0'$. As $\lambda(t)$ is strictly decreasing to 0 as $t$ goes to the blow-up time, there exists unique $t_0$ such that $\lambda(t_0) = \lambda_0'$. Now $u(t_0)$ and $u_0'$ are located at the same scale $\lambda_0' \approx 1$ so that

\[
\|\epsilon_0^\varepsilon(t_0)\|_{H^3_m} \lesssim \|\epsilon_0^\varepsilon(t_0) - (\epsilon_0')^\varepsilon\|_{H^3_m} + |\gamma_0 - \gamma_0'|.
\]

Applying Proposition 6.1 for $u(t_0)$ and $u_0'$ yields

\[
|\eta(t_0) - \eta_0'| \lesssim_{b_0} |b(t_0) - b_0'| + |\gamma_0 - \gamma_0'| + \|\epsilon_0^\varepsilon(t_0) - (\epsilon_0')^\varepsilon\|_{H^3_m},
\]
provided that $b_0(\tilde{t}_0)$ is near $b_0$.

We claim that
\[
|\eta(\tilde{t}_0) - \eta_0| + |b(\tilde{t}_0) - b_0| + \|\varepsilon^\sharp(\tilde{t}_0) - \varepsilon^\sharp_0\|_{H^3_m} \lesssim \lambda_0 - \lambda'_0. \tag{6.3}
\]

Let us assume the claim (6.3) and finish the proof. Note that the claim guarantees $b_0(\tilde{t}_0) \approx b_0$, because we assumed that $\lambda_0$ and $\lambda'_0$ are sufficiently close. The proof is completed by (6.2) and the claim (6.3):

\[
|\eta_0 - \eta'_0| \leq |\eta_0 - \eta(\tilde{t}_0)| + |\eta(\tilde{t}_0) - \eta'_0| \\
\lesssim b_0 |\eta(\tilde{t}_0) - \eta_0| + |b(\tilde{t}_0) - b'_0| + |\gamma_0 - \gamma'_0| + \|\varepsilon^\sharp(\tilde{t}_0) - (\varepsilon^\sharp_0)^\sharp\|_{H^3_m} \\
\lesssim b_0 |\eta(\tilde{t}_0) - \eta_0| + |b(\tilde{t}_0) - b_0| + \|\varepsilon^\sharp(\tilde{t}_0) - (\varepsilon^\sharp_0)^\sharp\|_{H^3_m} \\
+ |b_0 - b'_0| + |\gamma_0 - \gamma'_0| + \|\varepsilon^\sharp_0 - (\varepsilon^\sharp_0)^\sharp\|_{H^3_m} \\
\lesssim b_0 |\lambda_0 - \lambda'_0| + |b_0 - b'_0| + |\gamma_0 - \gamma'_0| + \|\varepsilon^\sharp_0 - (\varepsilon^\sharp_0)^\sharp\|_{H^3_m}.
\]

It now remains to prove the claim (6.3).

We first show $|\eta(\tilde{t}_0) - \eta_0| + |b(\tilde{t}_0) - b_0| \lesssim \lambda_0 - \lambda'_0$ by a bootstrap argument. From the modulation estimate, we have
\[
(\lambda^2)^r = -(2 + O(b^2))b.
\]

Since both $\lambda_0$ and $\lambda'_0$ are close to 1, we have $\lambda^2_0 - (\lambda'_0)^2 \approx 2(\lambda_0 - \lambda'_0)$. Thus
\[
\tilde{t}_0 \approx b_0^{-1}(\lambda_0 - \lambda'_0),
\]

provided that $b(t) \approx b_0$ on $[0, \tilde{t}_0]$. Then the modulation estimates $|\eta_t| \lesssim b^\frac{5}{2}$ and $|b_t| \lesssim b^2$ (recall that $\lambda \approx 1$ on $[0, \tilde{t}_0]$) say that
\[
|\eta(\tilde{t}_0) - \eta_0| + \sup_{t \in [0, \tilde{t}_0]} |b(t) - b_0| \lesssim b^2\tilde{t}_0 \lesssim \lambda_0 - \lambda'_0,
\]

provided that $b(t) \approx b_0$ on $[0, \tilde{t}_0]$. By a standard bootstrap argument, we obtain the following: if $\lambda_0$ and $\lambda'_0$ are close to 1, then $|\eta(\tilde{t}_0) - \eta_0| + |b(\tilde{t}_0) - b_0| \lesssim \lambda_0 - \lambda'_0$,

$b(t) \approx b_0$ on $[0, \tilde{t}_0]$, and $\tilde{t}_0 \approx b_0^{-1}(\lambda_0 - \lambda'_0)$.

Finally, we need to measure the difference of $\varepsilon_0^\sharp$ and $\varepsilon^\sharp(\tilde{t}_0)$. For this, we will rewrite the equations of $\epsilon$ and $\epsilon_3$ in the $(t, r)$-variables. Let
\[
\varepsilon^\sharp(t, r) := \frac{\epsilon^{ij}(r)}{\lambda(t)}\epsilon\left(t, \frac{r}{\lambda(t)}\right) \quad \text{and} \quad \varepsilon_{3}^{\sharp} (t, r) := \frac{\epsilon^{ij}(r)}{\lambda^3(t)}\epsilon_3\left(t, \frac{r}{\lambda(t)}\right),
\]

where $\sharp^{\sharp}$ means the $H^{-3}$-scaling. Then we have
\[
\partial_t \varepsilon^\sharp = \frac{1}{\lambda^3}\left(i\Delta_m \epsilon + \text{RHS of (5.47)}\right)^\sharp,
\]
\[\square\] Springer
\[
\partial_t \epsilon_3^{\#-3} = \frac{1}{\lambda^2} \left( -i A_Q^* A Q \epsilon_3 + \text{RHS of (5.33)} \right)^{\#-3}.
\]

Using \( \lambda \approx 1 \), we get

\[
\| \partial_t \epsilon \|_{L^2} \lesssim \| \Delta_m \epsilon \|_{L^2} + \| \text{RHS of (5.47)} \|_{L^2},
\]
\[
\| \partial_t \epsilon_3^{\#-3} \|_{L^2} \lesssim \| \epsilon_3 \|_{L^2} + \| \text{RHS of (5.33)} \|_{L^2}.
\]

For the first term of each RHS, we can apply a priori \( H^5_m \)-estimates of \( \epsilon \). For the second term of each RHS, we recall that these are estimated in Lemma 5.22 and 5.12, respectively. As a result,

\[
\| \partial_t \epsilon \|_{L^2} + \| \partial_t \epsilon_3^{\#-3} \|_{L^2} \lesssim b_3^\frac{3}{2}.
\]

Integrating this on \([0, \tilde{t}_0]\) and applying the coercivity Lemma 2.7 for \( \epsilon_3 \), we finally get

\[
\| \epsilon_3(\tilde{t}_0) - \epsilon_0 \|_{H^3_m} \lesssim \tilde{t}_0 b \lesssim \lambda_0 - \lambda'_{0}.
\]

This completes the proof of (6.3). \( \square \)

Therefore, it only remains to show Proposition 6.1.

### 6.2 Further Reduction of Proposition 6.1

So far, we have seen how the Lipschitz control on the difference of unstable parameter \( \eta \) by that of stable parameters \( b \) and \( \epsilon \) yields Lipschitz regularity of the manifold \( \mathcal{M} \). From now on, we focus on proving this Lipschitz control, Proposition 6.1.

Roughly speaking, the difference \(|\eta_0 - \eta'_0|\) can be controlled backwards in time because \( \eta \) is an unstable parameter, and \( \eta \) and \( \eta' \) have zero limit (with sufficient decay) at their blow-up times. Thus \(|\eta_0 - \eta'_0|\) will be controlled by the differences of all parameters in the future times. The differences of \( b \) and \( \epsilon \) can be controlled forwards in time, because they are stable parameters. Thus they will be controlled by \(|b_0 - b'_0|\) and \( \|\epsilon_0 - \epsilon'_0\|_{H^3_m} \) (propagation of the initial data difference), and the differences of all parameters in the past times. Combining the above controls, one may expect that \(|\eta_0 - \eta'_0|\) can be controlled by \(|b_0 - b'_0|\) and \( \|\epsilon_0 - \epsilon'_0\|_{H^3_m} \). This idea is formulated in Proposition 6.4. In this subsection, we reduce Proposition 6.1 to Proposition 6.4.

In order to realize this idea, an important issue is the meaning of the difference. Given any two \( H^5_m \)-trapped solutions \( u \) and \( u' \), their blow-up times are in general different. Thus taking the difference in the original time variables \( t \) will not work. However, as observed in [11], there is a natural choice of the time for measuring the difference of \( u \) and \( u' \).

To be more precise, we first apply the dynamic rescaling for \( u \) and \( u' \). First, fix any \( s_0 \in \mathbb{R} \) and define \( s = s(t) : [0, T+(u)) \rightarrow [s_0, \infty) \) such that \( \frac{ds}{dt} = \frac{1}{\lambda^2(t)} \) and \( s(0) = s_0 \). Write \( \lambda, \gamma, b, \eta, \epsilon \) as functions of the renormalized time \( s \in [s_0, \infty) \). Then,
\( \epsilon = \epsilon(s, y) : [s_0, \infty) \times (0, \infty) \rightarrow \mathbb{C} \) satisfies the equation

\[
\partial_s \epsilon - \frac{\lambda' \Lambda}{\lambda} \Lambda \epsilon = -i \mathcal{L}_Q \epsilon + \cdots.
\]

Similarly, fix any \( s'_0 \) and define \( s' = s'(t) : [0, T_+(u')) \rightarrow [s'_0, \infty) \) such that \( \frac{ds'}{dt} = \frac{1}{(\lambda')^2(t)} \) and \( s'(0) = s'_0 \). Write \( \lambda', y', b', \eta', \epsilon' \) as functions of \( s' \in [s'_0, \infty) \). Then, \( \epsilon' = \epsilon'(s', y) : [s'_0, \infty) \times (0, \infty) \rightarrow \mathbb{C} \) satisfies the equation

\[
\partial_{s'} \epsilon' - \frac{\lambda' \Lambda}{\lambda'} \Lambda \epsilon' = -i \mathcal{L}_Q \epsilon' + \cdots.
\]

We now forget the original time variable \( t \) and regard \( s \) and \( s' \) as independent time variables. A priori \( s \in [s_0, \infty) \) and \( s' \in [s'_0, \infty) \) live in different spaces. In order to compare \( \epsilon \) and \( \epsilon' \), we introduce a transform \( s' = s'(s) : [s_0, \infty) \rightarrow [s'_0, \infty) \) with \( s'(s_0) = s'_0 \) to put \( \epsilon \) and \( \epsilon' \) on the same time domain \( [s_0, \infty) \) and measure their differences by \( \epsilon(s) - \epsilon'(s'(s)) \). To choose a right transform \( s' = s'(s) \), let us rewrite the equation of \( \epsilon' \) in the \( s \)-variable (abusing the notation \( \epsilon'(s, y) = \epsilon'(s'(s), y) \)) using

\[
\partial_s \epsilon' = ds' \frac{\lambda' \Lambda}{\lambda'} \Lambda \epsilon' = ds' \frac{ds'}{ds} (-i \mathcal{L}_Q \epsilon' + \cdots).
\]

When we write the equation for the difference \( \delta \epsilon(s) = \epsilon(s) - \epsilon'(s'(s)) \), the difference of \( \frac{\lambda' \Lambda}{\lambda} \Lambda \epsilon \) and \( \frac{ds'}{ds} \frac{\lambda' \Lambda}{\lambda'} \Lambda \epsilon' \) will contain the term \( \frac{\lambda' \Lambda}{\lambda} - \frac{ds'}{ds} \frac{\lambda' \Lambda}{\lambda'} \Lambda \epsilon \). As the scaling vector field \( \Lambda \) is unbounded, we want to delete this term. This motivates us to choose \( s' = s'(s) \) such that

\[
\frac{ds'}{ds} \frac{\lambda' \Lambda}{\lambda'} = \frac{\lambda_s}{\lambda}.
\]

Integrating this, we have

\[
\frac{\lambda'(s'(s))}{\lambda(s)} = \frac{\lambda'_0}{\lambda_0}.
\]

This says that we choose \( s' = s'(s) \) such that the ratio between \( \lambda'(s') \) and \( \lambda(s) \) remains constant. The price to pay is that the difference equation has the term \( (\frac{ds'}{ds} - 1)i \mathcal{L}_Q \epsilon' \). In the energy estimate of \( \delta \epsilon \), this term requires a priori control of \( i \mathcal{L}_Q \epsilon' \) having two more derivatives.

From now on, we reduce Proposition 6.1 to Proposition 6.4. Set

\[ s_0 = s'_0 = b_0^{-1}. \]
As we have set of Proposition 6.1 is invariant under scalings, we may assume that
\[ \lambda_0 = \lambda_0' = s_0^{-1}. \]

As motivated from the previous paragraph, we define a strictly increasing function \( s' = s'(s) : [s_0, \infty) \to [s_0', \infty) \) such that \( s'(s_0) = s_0 \) and
\[ \lambda'(s') = \lambda(s). \]

Differentiating this relation, we have
\[ \frac{ds'}{ds} = \frac{\lambda_s}{\lambda_s'} = \left( \frac{\lambda_s'}{\lambda'} \right)^{-1} \left( \frac{\lambda_s}{\lambda} \right). \]

**Lemma 6.3** (Rough asymptotics) *For sufficiently large \( s_0 \), we have for \( s \in [s_0, \infty) \)
\[ s'(s) = s(1 + O(s_0^{-1})) \]
\[ b(s) = s^{-1}(1 + O(s_0^{-1})) = b'(s'(s)), \]
\[ \lambda(s) = s^{-1}(1 + O(s_0^{-1})) = \lambda'(s'(s)). \]

**Proof** We first prove asymptotics of \( b(s) \) and \( \lambda(s) \). We start from writing the modulation equation \( b_s + b^2 = O(b^\frac{3}{2}) \) in the form
\[ |(b^{-1})_s - 1| \lesssim b^{\frac{1}{2}}. \]

Using \( b^{\frac{1}{2}} = -b^{-\frac{3}{2}}b_s + O(b) \) and integration by parts, we have \( \int_{s_0}^{s} b^{\frac{3}{2}}(\sigma)d\sigma \lesssim b^{\frac{1}{2}}(s) + O((b^*)^{\frac{3}{2}}) \int_{s_0}^{s} b^{\frac{1}{2}}(\sigma)d\sigma. \) Thus \( \int_{s_0}^{s} b^{\frac{3}{2}}(\sigma)d\sigma \lesssim b^{-\frac{1}{2}}(s) \) so \( |b^{-1}(s) - s| \lesssim b^{-\frac{1}{2}}(s) \). In other words, \( b(s) = s^{-1}(1 + O(s^{-\frac{1}{2}})) \). Next, from
\[ |(\log(s\lambda))_s| = \left| \left( \frac{1}{s} - b \right) + \left( \frac{\lambda_s}{\lambda} + b \right) \right| \lesssim s^{-\frac{3}{2}}, \]
we get \( |s\lambda(s) - 1| \lesssim s_0^{-\frac{1}{2}}. \)

We now focus on \( s'(s) \) and \( b'(s'(s)) \). Applying the argument in the previous paragraph for \( b'(s') \), we have \( b'(s') = (s')^{-1}(1 + O((s')^{-\frac{1}{2}})) \) and \( |s'(s') - 1| \lesssim s_0^{-\frac{1}{2}}. \)

Combining these with \( |s\lambda(s) - 1| \lesssim s_0^{-\frac{1}{2}} \) and \( \lambda'(s'(s)) = \lambda(s) \), we have
\[ |s'(s) - s|\lambda(s) \leq |s'(s)\lambda(s) - 1| + |s\lambda(s) - 1| \]
\[ = |s'(s)\lambda'(s'(s)) - 1| + |s\lambda(s) - 1| \lesssim s_0^{-\frac{1}{2}}. \]

\[ ^{24} \] As we have set \( s_0 = b_0^{-1} \), this is equivalent to saying “for sufficiently large \( b_0^{-1} > (b^*)^{-1} \).”
Since \( \lambda(s) \approx s^{-1} \), we get \(|s'(s) - s| \lesssim s_0^{-\frac{1}{2}} s \). Substituting this time difference estimate into \( b'(s') = (s')^{-1}(1 + O((s')^{-\frac{1}{2}})) \), we get \( b'(s(s)) = s^{-1}(1 + O(s_0^{-\frac{1}{2}})) \) as desired. \( \square \)

For \( s \in [s_0, \infty) \), let

\[
\delta b(s) := b(s) - b'(s(s)), \\
\delta \eta(s) := \eta(s) - \eta'(s(s)), \\
\delta \epsilon(s) := \epsilon(s) - \epsilon'(s(s)).
\]

The adapted derivatives are defined similarly. We note that \((\delta \epsilon)_k = \delta(\epsilon_k)\).

We define the distances of stable/unstable parameters

\[
D^s(s) := s|\delta b| + \|\delta \epsilon\|_{L^2} + s^\frac{5}{2} \|\delta \epsilon_3\|_{L^2}, \\
D^u(s) := s^{\frac{3}{2}} |\delta \eta|, \\
D(s) := D^s(s) + D^u(s).
\]

Taking the supremum over \([s_0, \infty)\), we define

\[
D^s(s_0, \infty) := \sup_{s \in [s_0, \infty)} D^s(s), \\
D^u(s_0, \infty) := \sup_{s \in [s_0, \infty)} D^u(s).
\]

We note that \(D^s(s_0, \infty)\) and \(D^u(s_0, \infty)\) are finite due to (1.13). We now motivate the weights in front of the differences \(|\delta b|, |\delta \eta|, \|\delta \epsilon\|_{L^2}, \|\delta \epsilon_3\|_{L^2}\). Indeed, it is designed to have an \(s\)-independent bound

\[
D^s(s) + D^u(s) \lesssim D^s(s_0).
\]

First, due to scalings, \(\|\delta \epsilon\|_{L^2} \lesssim \|\epsilon_0 - \epsilon'_{0}\|_{L^2}\) and \(\|\delta \epsilon_3\|_{L^2} \lesssim (\frac{s_0}{s})^3 \|\epsilon_3(s_0) - \epsilon'_3(s_0)\|_{L^2}\) are expected at best. However, due to a limitation of the method, we will only have \(\|\delta \epsilon_3\|_{L^2} \lesssim s^{-3-\frac{1}{2}} \|\epsilon_3(0) - \epsilon'_3(0)\|_{L^2}\), so we choose \(s^{-\frac{3}{2}}\) to work with. Next, it will turn out that \((\delta b)_s + b(\delta b) \approx 0\) so \((\delta b) \approx s^{-1}(b_0 - b'_0)\) using \(b(s) \approx s^{-1}\). Finally, since \(|(\delta \eta)_s| \lesssim \|\delta \epsilon_3\|_{L^2} + error\), and \(\eta(s)\) and \(\eta'(s(s))\) go to zero as \(s \to \infty\), integrating backwards in time roughly yields \(|\delta \eta(s)| \lesssim s \|\delta \epsilon_3\|_{L^2} \lesssim s^{-\frac{3}{2}}\).

**Proposition 6.4** (Forward/backward controls) If \(s_0 = b_0^{-1}\) is chosen sufficiently large, then we have

1. (Forward-in-time control for stable modes)

\[
D^s(s_0, \infty) \lesssim D^s(s_0) + s_0^{-\frac{1}{2}} D(s_0, \infty).
\]
2. (Backward-in-time control for unstable modes)

\[ D^u(s_0, \infty) \lesssim D^s(s_0, \infty) + s_0^{-\frac{1}{2}} D(s_0, \infty). \tag{6.5} \]

In particular, we have

\[ D(s_0, \infty) \lesssim D^s(s_0). \]

**Remark 6.5** In fact, we can show

\[ D^s(s_0, s) \lesssim D^s(s_0) + s_0^{-\frac{1}{4}} D(s_0, s), \]
\[ D^u(s, \infty) \lesssim D^s(s, \infty) + s^{-\frac{1}{2}} D(s, \infty), \]

where we denoted \( D^s(I) := \sup_{s \in I} D^s(s) \) and \( D^u(I) := \sup_{s \in I} D^u(s) \) for an interval \( I \subseteq [s_0, \infty) \). The first estimate is obtained by the forward-in-time integration on \([s_0, s]\). The second estimate is obtained by the backward-in-time integration on \([s, \infty)\) with \( \delta \eta(\infty) = 0 \). We note that it suffices to have negative powers for \( s_0 \) in the second term (time integral term) of each RHS. These terms are perturbative.

Proposition 6.4 implies the conclusion of Proposition 6.1 from the inequality

\[ |\delta \eta(s_0)| \lesssim s_0^{-\frac{1}{2}} |\delta b(s_0)| + s_0 \| \delta \epsilon(s_0) \|_{H^3_{\delta}} \lesssim b_0 |\delta b(s_0)| + \| \delta \epsilon(s_0) \|_{H^3_{\delta}}. \]

Therefore, it remains to prove Proposition 6.4. The strategy is quite similar to that of Proposition 4.4. For the forward-in-time control, we write the equation of \( \delta \epsilon \), prove modulation estimates for the difference of modulation parameters, and perform energy estimates for adaptive derivatives of \( \delta \epsilon \). The control of \( \delta \eta \) will follow from integrating the modulation estimates backwards in time.

**6.3 Equation of \( \delta \epsilon \)**

We start by writing the equation for \( \delta \epsilon(s) = \epsilon(s) - \epsilon'(s'(s)) \). We recall the equation of \( \epsilon \):

\[ (\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i) \epsilon + i \mathcal{L}_Q \epsilon = \mathcal{Mod} \cdot v - i \widetilde{R}_{L-L} - i \widetilde{R}_{NL} - i \Psi. \]

Next, we write the equation of \( \epsilon' \) in the \( s \)-variable using \( \partial_s = \frac{ds'}{ds} \partial_{s'} \):

\[ (\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma'_s i) \epsilon' + \frac{ds'}{ds} \cdot i \mathcal{L}_Q \epsilon' = \frac{ds'}{ds} \left( \mathcal{Mod}' \cdot v' - i \widetilde{R}'_{L-L} - i \widetilde{R}'_{NL} - i \Psi' \right). \]

Here, we use prime notations canonically; for an expression \( f = f(b, \eta, \epsilon, \ldots) \) for \( u \), we denote the corresponding object for \( u' \) by \( f' := f(b', \eta', \epsilon', \ldots) \) evaluated at time...
We begin the difference estimates of the RHS of (6.6).

Lemma 6.6 (Difference estimates for profiles, modulation vectors, radiation terms)
We have

\[ |\delta P_{\text{1}}| \lesssim (|\delta b| + |\delta \eta|) y^2 Q_{\text{1}} y \lesssim b^{-1/2}, \]

\[ |\delta \nu_{\text{1}}| \lesssim (|\delta b| + |\delta \eta|) y^4 Q_{\text{1}} y \lesssim b^{-1/2}, \]

\[ |\delta (\eta \theta_{\eta})| \lesssim |\eta b^m (\delta b)| + |\delta \eta| \lesssim s^{-2} D, \]

\[ |\delta \theta_{\psi}| \lesssim b^{m+1} (|\delta b| + |\delta \eta|), \]

\[ |\delta \Psi_{\text{1}}| \lesssim b^{m+1} (|\delta b| + |\delta \eta|) \leq b^{1/2}. \]
Proof For each of the above quantities, we use

$$|\delta f| \lesssim |\delta b| |\partial_b f| + |\delta \eta| |\partial_\eta f|.$$  

Thus the estimate for $|\delta P|_3$ and $|\delta \psi|_3$ follow from global bounds of Proposition 3.3. The estimates for $\delta(\eta \partial_\eta)$, $\delta \theta \psi$, and $|\delta \psi|_3$ follow from (3.13), (3.14), (3.15), and (3.16).

\[\square\]

Lemma 6.7 (Difference estimate for $R_{L-L}$) We have

$$|\delta \theta_{L-L}| \lesssim s^{-3} D, \quad (6.7)$$

$$\|\delta \tilde{R}_{L-L}\|_{\dot{H}_m^3} \lesssim b \left( \|\delta \varepsilon\|_{\dot{H}_m^3, \leq M_2} + o_{M_2} \rightarrow \infty (1) \|\delta \varepsilon\|_{\dot{H}_m^3} + s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D) \right). \quad (6.8)$$

Remark 6.8 Indeed $s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D) = s^{-3} D$. But we want to emphasize that $s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D)$ is perturbative relative to $\|\delta \varepsilon\|_{\dot{H}_m^3}$-part ($\sim s^{-\frac{5}{2}} D$).

Proof Lemma 6.7 is an analogue of Lemma 5.1 for the difference. The proof relies on that of Lemma 5.1.

We start with $\delta \theta_{L-L}$-estimate. In the proof of the $\theta_{L-L}$ estimate (5.3), we viewed $\theta_{L-L}$ as a linear combination of

$$\int_0^\infty m \text{Re}(\bar{\psi}_1 \psi_2) \frac{dy}{y} \text{ and } \int_0^\infty A_\theta [\psi_1, \psi_2] \text{Re}(\bar{\psi}_3 \psi_4) \frac{dy}{y},$$

where $\psi_j = \varepsilon$ and $\psi_k = P - Q$ for some $j$ and $k$, and $\psi_\ell \in \{P, Q\}$ for $\ell \neq j, k$. To show the $\delta \theta_{L-L}$ estimate, we view $\delta \theta_{L-L}$ as the same linear combination, where

- $\psi_j \in \{\varepsilon, \varepsilon', \delta \varepsilon\}$ and $\psi_k \in \{P - Q, P' - Q, \delta P\}$ for some $j$ and $k$,
- $\psi_\ell \in \{P, P', Q, \delta P\}$ for $\ell \neq j, k$,
- $\delta$ should appear exactly once among $\psi_i$’s.

If $\delta$ hits $\varepsilon$, i.e. $\psi_j = \delta \varepsilon$, then the proof of (5.3) works by replacing $\varepsilon$ by $\delta \varepsilon$ and yields the bound $b^2 \|\delta \varepsilon\|_{\dot{H}_m^3}$. If $\delta$ hits $P - Q$, i.e. $\psi_k = \delta P$, then $|\delta P| \lesssim (|\delta b| + |\delta \eta|)^2 Q_{1, y \leq b^{-1/2}} \lesssim b^{-1}(|\delta b| + |\delta \eta|) \min\{b y^2, 1\} Q$. Compared to $|P - Q| \lesssim \min\{b y^2, 1\} Q$ used in the proof of (5.3), it suffices to multiply $b^{-1}(|\delta b| + |\delta \eta|)$ to the bound (5.3). This yields the bound $b^{-1}(|\delta b| + |\delta \eta|)(\|\varepsilon\|_{\dot{H}_m^3} + \|\varepsilon'\|_{\dot{H}_m^3})$. If $\delta$ hits $P$, i.e. $\psi_\ell = \delta P$, then $|\delta P| \lesssim b^{-1}(|\delta b| + |\delta \eta|) Q$. Thus we need to multiply $b^{-1}(|\delta b| + |\delta \eta|)$ to the bound obtained in (5.3). As a result,

$$|\delta \theta_{L-L}| \lesssim b^2 \|\delta \varepsilon\|_{\dot{H}_m^3} + b^{-\frac{1}{2}} (|\delta b| + |\delta \eta|)(\|\varepsilon\|_{\dot{H}_m^3} + \|\varepsilon'\|_{\dot{H}_m^3}).$$

Substituting the a priori $\dot{H}_m^3$-bound for $\varepsilon$ and $\varepsilon'$ (4.13) and using the definition of $D$ yield the $\delta \theta_{L-L}$ bound.

For the $\delta \tilde{R}_{L-L}$-estimate, we similarly argue as in the $\delta \theta_{L-L}$ estimate. We view $\delta \tilde{R}_{L-L}$ as a linear combination of

$$V_3[\psi_1, \psi_2] \psi_3 \text{ and } V_5[\psi_1, \psi_2, \psi_3, \psi_4] \psi_5,$$
(replace $V_3$ by $\tilde{V}_3$ if $\psi_3 \in \{P, P'\}$ and similarly for $V_5$) where $\psi_j \in \{\epsilon, \epsilon', \delta \epsilon\}$ and $\psi_k \in \{P - Q, P' - Q, \delta P\}$ for some $j$ and $k$, $\psi_\ell \in \{P, P', Q, \delta P\}$ for $\ell \neq j, k$, and $\delta$ should appear exactly once among $\psi_i$'s. If $\psi_j = \delta \epsilon$, the proof of (5.4) works by replacing $\epsilon$ by $\delta \epsilon$. If $\delta$ hits $\psi_k$ or $\psi_\ell$, we need to multiply $b^{-1}(|\delta b| + |\delta \eta|)$ to the bound (5.4).

As a result,

$$\|\delta \tilde{R}_{L - L}\|_{\dot{H}^3_m} \lesssim b(\|\delta \epsilon\|_{\dot{H}^3_{m, \leq M^2}} + o_{M^2 \rightarrow \infty}(1))\|\delta \epsilon\|_{\dot{H}^3_m} + (|\delta b| + |\delta \eta|)(\|\epsilon\|_{\dot{H}^3_m} + \|\epsilon'|_{\dot{H}^3_m}).$$

Substituting the a priori $\dot{H}^3_m$-bound (4.13) and using the definition of $D$, we get the conclusion.$\square$

**Lemma 6.9** (Difference estimate for $R_{NL}$) We have

\begin{align*}
|\delta \theta_{NL}| & \lesssim s^{-\frac{3}{2}} D, \\
\|\delta \tilde{R}_{NL}\|_{\dot{H}^3_m} & \lesssim bs^{-\frac{1}{2}} (s^{-\frac{5}{2}} D).
\end{align*}

**Proof** We argue as in the proof of Lemma 6.7. We view $\delta \theta_{NL}$ as a linear combination of

$$f_0^\infty m \text{Re} (\bar{\psi}_1 \psi_2) \frac{dy}{y} \quad \text{and} \quad f_0^\infty A_\theta [\psi_1, \psi_2] \text{Re}(\bar{\psi}_3 \psi_4) \frac{dy}{y},$$

where at least two $\psi_j$'s belong to $\{\epsilon, \epsilon', \delta \epsilon\}$, remaining $\psi_\ell$'s are filled with $P, P', \delta P$, and $\delta$ should appear exactly once among all $\psi_i$'s.

Let us rewrite the bound (5.15) of $\theta_{NL}$ as

$$|\theta_{NL}| \lesssim \|\epsilon\|^2_{\dot{H}^1_m} (1 + \|\epsilon\|_{L^2})^2,$$

by inspecting the proof of (5.15) without assuming $\|\epsilon\|_{L^2} \leq 1$. If $\delta$ hits $\epsilon$, i.e. one of $\psi_j$ is $\delta \epsilon$, the proof of (5.15) works by replacing one $\epsilon$ by $\delta \epsilon$. Thus the resulting bound is

$$\|\epsilon\|_{\dot{H}^1_m} \|\delta \epsilon\|_{\dot{H}^3_m} (1 + \|\epsilon\|_{L^2})^2 + \|\epsilon\|^2_{\dot{H}^1_m} \|\delta \epsilon\|_{L^2} (1 + \|\epsilon\|_{L^2}).$$

Applying the interpolation and coercivity (Lemma 2.7), we estimate $\|\delta \epsilon\|_{\dot{H}^1_m} \lesssim \|\delta \epsilon\|^2_{L^2} \|\delta \epsilon\|^2_{\dot{H}^3_m} \lesssim M s^{-\frac{5}{2}} D$. To remove the $M$-dependence, we lose a small power of $s$. Applying a priori bounds of $\epsilon$, the above bound is dominated by $s^{-\frac{3}{2}} D$, as desired. If $\delta$ hits $P$, i.e. one of $\psi_\ell$ is $\delta P$, then we lose $b^{-1}(|\delta b| + |\delta \eta|)$ to the bound (5.15). This yields $s^{-2} D$.

We turn to $\delta \tilde{R}_{NL}$.

We view $\delta \tilde{R}_{NL}$ as a linear combination of

$$V_3[\psi_1, \psi_2] \psi_3 \quad \text{and} \quad V_5[\psi_1, \psi_2, \psi_3, \psi_4] \psi_5,$$

\[25\] In fact, the upper bound should contain the contributions of $\epsilon'$. We omit them for readability. The same remark applies for $\delta \tilde{R}_{NL}$.
Let us denote by $\theta_{\text{cor}} := - (\theta_\Psi + \theta_{L-L} + \theta_{NL})$. Then, we have
\[
(\delta \gamma)_s = \left( \Mod_2 - \frac{ds'}{ds} \Mod_2' \right) + \delta (\eta \theta_\eta + \theta_{\text{cor}}) - \left( \frac{ds'}{ds} - 1 \right) (\eta' \theta_\eta' + \theta_{\text{cor}}'),
\]
\[
\frac{ds'}{ds} - 1 = \frac{1}{b'} (\delta b - \left( \Mod_1 - \frac{ds'}{ds} \Mod_1' \right)).
\]

Thus (6.12) and (6.13) follow from (6.11), the estimates for $\delta (\eta \theta_\eta)$ and $\delta \theta_{\text{cor}}$ (Lemma 6.6, (6.7), and (6.9)), and the estimates for $\eta \theta_\eta$ and $\theta_{\text{cor}}$ ((3.10), (3.11), (5.3), and (5.15)).
Henceforth, we focus on the proof of (6.11). We rewrite (6.6) as

\[
\begin{align*}
\tilde{(\mathbf{Mod} - \frac{ds'}{ds}) \cdot (\mathbf{v}' - \frac{e_1}{b'}((\eta' \theta_{\eta'} + \theta_{cor}')i \epsilon') + i \mathcal{L}_Q \epsilon' + i \tilde{R}_{L-L}' + i \tilde{R}_{NL}' + i \Psi}')}
& - e_2(i \epsilon') \tag{6.14}
\end{align*}
\]

As in the proof of Lemma 5.9, we take the inner product of (6.14) and \(Z_k\).

We first consider the inner product of \(Z_k\) and the LHS of (6.14). By (4.14), the inner product coming from \(v'\) becomes the main term. The remaining contributions are treated as errors: (see the proof of Lemma 5.9)

\[
\begin{align*}
(b')^{-1}|(\eta' \theta_{\eta'} + \theta_{cor}')i \epsilon'|, (\mathcal{L}_Q i Z_k)_r & \lesssim b^2 M^C \| \epsilon' \|_{H^3_m}, \\
(b')^{-1}|(i \mathcal{L}_Q \epsilon', Z_k)_r & \lesssim b^{-1} \| \epsilon'_3 \|_{L^2}, \\
(b')^{-1}|(i \tilde{R}_{L-L}' + i \tilde{R}_{NL}' + i \Psi', Z_k)_r & \lesssim b^{-1} M^C \| \tilde{R}_{L-L}' + \tilde{R}_{NL}' + \Psi' \|_{H^3_m}, \\
| (i \epsilon', Z_k)_r & \lesssim M^C \| \epsilon' \|_{H^3_m}.
\end{align*}
\]

These are all bounded by \(b^2\). Therefore, the matrix formed by taking the inner product of

\[
(v' - \frac{e_1}{b'}((\eta' \theta_{\eta'} + \theta_{cor}')i \epsilon' + i \mathcal{L}_Q \epsilon' + i \tilde{R}_{L-L}' + i \tilde{R}_{NL}' + i \Psi') - e_2(i \epsilon'))
\]

and \(Z_k\) has uniformly bounded inverse.

Henceforth, it suffices to consider the inner product of \(Z_k\) and the RHS of (6.14). As in the proof of Lemma 5.9, the leading term comes from \((\delta \epsilon, \mathcal{L}_Q i Z_k)_r\):

\[
| (\delta \epsilon, \mathcal{L}_Q i Z_k)_r | \lesssim o_{M \to \infty}(1) \| \delta \epsilon_3 \|_{L^2}.
\]

All the remaining terms will be considered as errors. We have

\[
| \partial_s \delta \epsilon, Z_k \rangle_r = \partial_s (\delta \epsilon, Z_k)_r = 0, \\
| \frac{\lambda}{\Lambda} (\delta \epsilon, \Lambda Z_k)_r | + | \gamma s (\delta \epsilon, i Z_k)_r | \lesssim b \cdot M^C \| \delta \epsilon \|_{H^3_m} \lesssim s^{-3} D.
\]

Next, we have

\[
| (\delta (\eta \theta_{\eta} + \theta_{cor})i \epsilon', Z_k)_r | \lesssim | \delta (\eta \theta_{\eta} + \theta_{cor}) | \cdot M^C \| \epsilon' \|_{H^3_m} \lesssim s^{-3} D.
\]
Next, by the previous paragraph, we have

\[ |\delta b| b'((\eta'\theta' + \theta'_{cor})i\epsilon' + iL\epsilon' + i\tilde{R}_{L-L} + i\tilde{R}_{NL} + i\Psi', Z_k)_{r}| \lesssim b^2|\delta b| \lesssim s^{-3}D. \]

Next, by the modulation estimate (Lemma 5.9), estimates of \( \delta v \) (Lemma 6.6), and pointwise bounds of \( Z_k \) (4.4), we have

\[ |\tilde{\text{Mod}}||\delta v, Z_k)_{r}| \lesssim b^\frac{5}{2} \cdot M^C (|\delta b| + |\delta \eta|) \lesssim s^{-3}D. \]

Finally, by (6.8), (6.10), and Lemma 6.6, we have

\[ \left| (i\delta \tilde{R}_{L-L} + i\delta \tilde{R}_{NL} + i\delta \Psi, Z_k)_{r} \right| \lesssim M^C \|\delta \tilde{R}_{L-L} + \delta \tilde{R}_{NL} + \delta \Psi\|_{H^3_m}
\lesssim M^C b (\|\delta \epsilon\|_{H^3_m} + s^{-\frac{1}{2}}(s^{-\frac{5}{2}}D)) \lesssim s^{-3}D, \]

which is absorbed into \( s^{-3}D \). This completes the proof of (6.11).

\[ \blacksquare \]

### 6.6 Energy Estimate in \( \dot{H}^3_m \)

In this subsection we propagate \( \|\delta \epsilon_3\|_{L^2} \) forwards in time. The argument is similar to Sect. 5.4. We again use local virial corrections to derive a modified energy inequality.

We start with the equation of \( \delta \epsilon_2 \) obtained by taking \( A_QL_Q \) to (6.6):

\[
(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-2} + \nu_s i) \delta \epsilon_2 + iA_QA_Q^* \delta \epsilon_2
= -(\delta \gamma)_s i\epsilon_4 + \left( \frac{ds'}{ds} - 1 \right) i\epsilon_4' + \frac{\lambda_s}{\lambda} \partial_\nu (A_Q, L, Q) \delta \epsilon - \nu_s A_Q[L, Q, i] \delta \epsilon
\]

\[ + \tilde{\text{Mod}} \cdot A_QL_Q(\delta \nu) + (\tilde{\text{Mod}} - \frac{ds'}{ds} \tilde{\text{Mod}}') \cdot A_QL_Q \nu'
- A_QL_Q(i\delta \tilde{R}_{L-L} + i\delta \tilde{R}_{NL} + i\delta \Psi)
\]

\[ + \left( \frac{ds'}{ds} - 1 \right) A_QL_Q(i\tilde{R}'_{L-L} + i\tilde{R}'_{NL} + i\Psi'). \]

Similarly to Lemma 5.12, we prove the following.

**Lemma 6.11** (Energy identity for \( \dot{H}^3_m \)) We have

\[
\left| \left( \partial_s - \frac{6\lambda_s}{\lambda} \right) \|\delta \epsilon_3\|_{L^2} \right|^2 \lesssim b \|\delta \epsilon_3\|_{L^2}
\times \left( o_{M \to \infty}(1) \|\delta \epsilon_3\|_{L^2} + \|\delta \epsilon\|_{H^3_m \geq M^2} + o_{M^2 \to \infty}(1) \|\delta \epsilon\|_{H^3_m} + s^{-\frac{1}{2}}(s^{-\frac{5}{2}}D) \right). \]

**Remark 6.12** Here the reader can observe why we need a priori \( H^5 \)-control of \( \epsilon \). In (6.16), the term \( \left( \frac{ds'}{ds} - 1 \right) i\epsilon_4' \) appears. Thus we need to perform an energy estimate even for \( \epsilon_5 \) and hence obtain \( H^5_m \)-trapped solutions.
\textbf{Proof} Taking $A^*_Q$ to (6.15), we get the equation of $\delta \epsilon_3$:

\[
\begin{align*}
&\left(\partial_s - \frac{\lambda_s}{\lambda} \Delta_{-3} + \gamma_s i\right) \delta \epsilon_3 + i A^*_Q A_Q \delta \epsilon_3 \\
&= -(\delta \gamma)_s i \epsilon_3' + \left(\frac{ds'}{ds} - 1\right) i \epsilon_5' + \frac{\lambda_s}{\lambda} \partial_s \left(A^*_Q A_Q L_Q, i\right) \delta \epsilon - \gamma_s A^*_Q A_Q [L_Q, i] \delta \epsilon \\
&\quad + \overline{\text{Mod}} \cdot A^*_Q A_Q L_Q (\delta v) + \left(\overline{\text{Mod}} - \frac{ds'}{ds} \overline{\text{Mod}}'\right) \cdot A^*_Q A_Q L_Q v' \\
&\quad - A^*_Q A_Q L_Q (i \delta \tilde{R}_{L-L} + i \delta \tilde{R}_{NL} + i \delta \Psi) \\
&\quad + \left(\frac{ds'}{ds} - 1\right) A^*_Q A_Q L_Q (i \tilde{R}'_{L-L} + i \tilde{R}'_{NL} + i \Psi').
\end{align*}
\] (6.16)

We take the inner product of (6.16) with $\delta \epsilon_3$ to get

\[
\left\| \frac{1}{2} \left(\partial_s - \frac{\lambda_s}{\lambda}\right) \| \delta \epsilon_3 \|_{L^2}^2 \right\| = \left\| (\delta \epsilon_3, \text{RHS of (6.16)}) \right\| \lesssim \| \delta \epsilon_3 \|_{L^2} \| \text{RHS of (6.16)} \|_{L^2}.
\]

It now suffices to estimate $\| \text{RHS of (6.16)} \|_{L^2}$. By (6.13) and a priori $\epsilon_3$ bound, we have

\[
\| (\delta \gamma)_s i \epsilon_3' \|_{L^2} \lesssim bs^{-3} D.
\]

By (6.12) and \textit{a priori} $\epsilon_5$ bound, we have

\[
\left\| \frac{ds'}{ds} - 1 \right\| i \epsilon_5' \|_{L^2} \lesssim bs^{-3} D.
\]

By Lemma 5.11, we have

\[
\| \frac{\lambda_s}{\lambda} \partial_s \left(A^*_Q A_Q L_Q, i\right) \delta \epsilon \|_{L^2} \lesssim b(\| \delta \epsilon \|_{\dot{H}^3_{m, \leq M_2}} + o_{M_2 \to \infty}(1) \| \delta \epsilon \|_{\dot{H}^3_m}).
\]

By Lemma 5.9 and Lemma 6.6, we have

\[
\| \overline{\text{Mod}} \cdot A^*_Q A_Q L_Q (\delta v) \|_{L^2} \lesssim bs^{-3} D.
\]

By Lemma 6.10 and (3.8), we have

\[
\left\| \left(\overline{\text{Mod}} - \frac{ds'}{ds} \overline{\text{Mod}}'\right) \cdot A^*_Q A_Q L_Q v' \right\|_{L^2} \lesssim b(o_{M \to \infty}(1) \| \delta \epsilon_3 \|_{L^2} + s^{-\frac{1}{2}}(s^{-\frac{5}{2}} D)).
\]

Next, by (6.8), (6.10), and Lemma 6.6, we have

\[
\| A^*_Q A_Q L_Q (i \delta \tilde{R}_{L-L} + i \delta \tilde{R}_{NL} + i \delta \Psi) \|_{L^2}
\]
Finally, by (6.12), (5.9), (5.18), and (3.17), we have

\[
\| (d/\partial s' - 1) A_Q^* A_Q L_Q (i \tilde{R}_{L-L} + i \tilde{R}_{N-L} + i \Psi) \|_{L^2} \lesssim b s^{-3} D.
\]

This completes the proof.

\section*{Lemma 6.13 (Local virial control)}

We have

\[
\left| \frac{b}{\log M_2} \int_{M_2}^2 (\partial s_2, -i \Lambda_{M_2} \delta \epsilon_2) dM_2 \right| \lesssim M_2^c b \| \delta \epsilon \|_{L^2}^2,
\]

(6.17)

\[
\left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) \left| \frac{b}{\log M_2} \int_{M_2}^2 (\partial \epsilon_2, -i \Lambda_{M_2} \delta \epsilon_2) dM_2 \right| \]

(6.18)

\[
\geq b \left( c_M \| \delta \epsilon \|_{H^3_{m, \lesssim M_2}}^2 - o_{M_2 \to \infty} (1) \| \delta \epsilon \|_{H^3_{m}}^2 - \| \delta \epsilon \|_{L^2}^2 \right) - O \left( M_2^2 b \| \delta \epsilon \|_{H^3_{m}}^2 + M_2^2 b s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D) \right).
\]

\textbf{Proof} We note that (6.17) is same as (5.38), after replacing \( \epsilon_2 \) by \( \delta \epsilon_2 \). Henceforth, we focus on (6.18). By the averaging argument (see the proof of Lemma 5.16), it suffices to show the unaveraged version of (6.18):

\[
\left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) (\partial s_2, -i \Lambda_{M_2} \delta \epsilon_2)_r \geq c_M \| \delta \epsilon \|_{H^3_{m, \lesssim M_2}}^2 - O \left( \int 1_{y \sim M_2} | \delta \epsilon |_{L^2}^2 \right)
\]

\[
- o_{M_2 \to \infty} (1) \| \delta \epsilon \|_{H^3_{m}}^2 - \| \delta \epsilon \|_{L^2}^2 \cdot O \left( M_2^2 b \| \delta \epsilon \|_{H^3_{m}}^2 + M_2^2 b s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D) \right).
\]

To show this, we start with

\[
\frac{1}{2} \left( \partial_s - 6 \frac{\lambda_s}{\lambda} \right) (\partial s_2, -i \Lambda_{M_2} \delta \epsilon_2)_r = (\partial_s \delta \epsilon_2, -i \Lambda_{M_2} \delta \epsilon_2)_r
\]

\[
= (A_Q^* A_Q \delta \epsilon_2, \Lambda_{M_2} \delta \epsilon_2)_r + \frac{\lambda_s}{\lambda} (y \partial_y \delta \epsilon_2, -i \Lambda_{M_2} \delta \epsilon_2)_r + \text{(RHS of (6.15), } -i \Lambda_{M_2} \delta \epsilon_2)_r
\]

The first term of the RHS is treated in Lemma 5.14, after replacing \( \epsilon_2 \) by \( \delta \epsilon_2 \). The second term of the RHS is bounded by

\[
\left| \frac{\lambda_s}{\lambda} (y \partial_y \delta \epsilon_2, -i \Lambda_{M_2} \delta \epsilon_2)_r \right| \lesssim b (M_2^2 \| \delta \epsilon \|_{L^2}^2).
\]

Next, proceeding as in the proof of Lemma 6.11, we have

\[
\| y^{-1} \text{(RHS of (6.15))} \|_{L^2} \lesssim b \| \delta \epsilon \|_{H^3_{m}} + b s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D).
\]
Thus

\[
|\text{RHS of (6.15), } -i \Lambda M_2 \delta \epsilon_2_r| \lesssim \|y^{-1}(\text{RHS of (6.15)})\|_{L^2} \cdot M_2^2 \|\delta \epsilon_3\|_{L^2}
\]

\[
\lesssim \|\delta \epsilon_3\|_{L^2} (M_2^2 b \|\delta \epsilon\|_{H_\alpha'} + M_2^2 b s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D^2)).
\]

Summing up the above estimates completes the proof. 

Define the modified energy by

\[
F_3[\delta \epsilon] := \|\delta \epsilon_3\|_{L^2}^2 - M_1 \frac{b}{\log M_2} \int_{M_2}^{M_2^2} (\delta \epsilon_2, -i \Lambda M_2 \delta \epsilon_2) r \frac{dM_2'}{M_2^2}.
\]

**Proposition 6.14** (Modified energy inequality for \(\dot{H}_m^3\)) We have

\[
|F_3[\delta \epsilon] - \|\delta \epsilon_3\|_{L^2}^2| \leq \frac{1}{100} \|\delta \epsilon_3\|_{L^2}^2, \quad (6.19)
\]

\[
(\partial_s - 6 \frac{\lambda_s}{\lambda}) F_3[\delta \epsilon] \leq b \left( \frac{1}{100} \|\delta \epsilon_3\|_{L^2}^2 + Cs^{-1}(s^{-5} D^2) \right). \quad (6.20)
\]

**Proof** This follows as the same fashion as the proof of Proposition 5.17. 

### 6.7 Energy Estimate in \(L^2\)

In this subsection, we propagate \(\|\delta \epsilon\|_{L^2}\). Similarly to Sect. 5.5, we replace the linear part \(i \mathcal{L}_Q \delta \epsilon\) by \(-i \Delta_m \delta \epsilon\), as the \(i \mathcal{L}_Q\)-flow does not conserve \(L^2\)-norm. Thus we rewrite the equation of \(\delta \epsilon\) as

\[
(\partial_s - 6 \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i) \delta \epsilon - i \Delta_m \delta \epsilon = -(\delta \gamma)_s i \epsilon' - \left( \frac{ds'}{ds} - 1 \right) i \Delta_m \epsilon' + \text{Mod} \cdot (\delta v) + \left( \text{Mod} - \frac{ds'}{ds} \text{Mod}' \right) \cdot v'
\]

\[
- i \delta [N(P + \epsilon) - N(P')] - i \delta \Psi + \left( \frac{ds'}{ds} - 1 \right) (i [N'(P' + \epsilon') - N'(P')] + i \Psi').
\]

By an analogue of Lemma 5.20, we record the following.

**Lemma 6.15** (Some remainder estimates for \(L^2\) difference) We have

\[
\|\text{Mod} - \frac{ds'}{ds} \text{Mod}'\|_{L^2} \lesssim s^{-2} D, \quad (6.22)
\]

\[
\|\delta [N(P + \epsilon) - N(P')]\|_{L^2} \lesssim M s^{-\frac{3}{2}} D. \quad (6.23)
\]

**Proof** First, (6.22) follows from writing

\[
|\text{Mod} - \frac{ds'}{ds} \text{Mod}'| \lesssim |\text{Mod} - \frac{ds'}{ds} \text{Mod} | + |\delta \theta_{L-L} + \delta \theta_{NL}| + |\frac{ds'}{ds} - 1||\theta_{L-L} + \theta_{NL}|
\]
and applying (6.11), (6.7), (6.9), and (6.12), and \( \|v'\|_{L^2} \lesssim 1 \) (as \( m \geq 3 \)).

It remains to show (6.23). Rewrite the bound of (5.49) as

\[
\|N(P + \epsilon) - N(P)\|_{L^2} \lesssim \|\epsilon\|_{H^1}^2 \|\epsilon\|_{H^3}^2 + \|\epsilon\|_{H^m}^2 (1 + \|\epsilon\|_{L^2})^3.
\]

Here, the first term corresponds to the estimates for a multilinear expression of \( N(P + \epsilon) - N(P) \) having exactly one \( \epsilon \); the second term corresponds to those having two or more \( \epsilon \)’s.

We proceed similarly to earlier proofs of difference estimates (Lemmas 6.7 and 6.9). If \( \delta \) hits \( P \), then we lose \( b^{-1}(|\delta b| + |\delta \eta|) \), yielding the bound

\[
b^{-1}(|\delta b| + |\delta \eta|)(|\|\epsilon\|_{H^1}^2 \|\epsilon\|_{H^3}^2 + \|\epsilon\|_{H^m}^2) \lesssim M s^{-2} D.
\]

If \( \delta \) hits \( \epsilon \), then we replace one \( \epsilon \) by \( \delta \epsilon \), yielding the bound (use \( \|\delta \epsilon\|_{H^1}^2 \|\delta \epsilon\|_{H^3}^2 \lesssim M s^{-\frac{5}{3}} D \))

\[
|\|\epsilon\|_{H^1}^2 \|\epsilon\|_{H^3}^2 + |\|\epsilon\|_{H^m}^2| \lesssim M s^{-\frac{5}{3}} D.
\]

This shows (6.23). \( \square \)

**Lemma 6.16** (Energy identity for \( L^2 \)) We have

\[
\partial_s \|\delta \epsilon\|_{L^2}^2 \lesssim b s^{-\frac{1}{2}} D^2.
\]

**Proof** We have

\[
\frac{1}{2} \partial_s \|\delta \epsilon\|_{L^2}^2 = (\delta \epsilon, \text{RHS of (6.21)}) \lesssim \|\delta \epsilon\|_{L^2} \|\text{RHS of (6.21)}\|_{L^2}.
\]

Thus it suffices to show

\[
\|\text{RHS of (6.21)}\|_{L^2} \lesssim s^{-\frac{3}{2}} D.
\]

First, by (6.13), we have

\[
\|(\delta \gamma)s i \epsilon'\|_{L^2} \lesssim s^{-\frac{3}{2}} D.
\]

By (6.12) and a priori bound of \( \epsilon' \), we have

\[
\|(\frac{d}{ds} - 1)i \Delta_m \epsilon'\|_{L^2} \lesssim D \cdot \|\epsilon'\|_{H^1}^\frac{1}{2} \|\epsilon'\|_{H^3}^\frac{1}{2} \lesssim M s^{-2} D \lesssim s^{-\frac{3}{2}} D.
\]

By (5.51) and Lemma 6.6,

\[
\|\text{Mod} \cdot (\delta v)\|_{L^2} \lesssim_M s^{-2} \cdot |\log s|^{\frac{3}{2}} (|\delta b| + |\delta \eta|) \lesssim s^{-\frac{3}{2}} D.
\]
By Lemma 6.15, we have
\[ \|(\text{Mod} - \frac{ds'}{ds} \text{Mod}') \cdot v'\|_{L^2} + \|i\delta[N(P + \epsilon) - N(P)]\|_{L^2} \lesssim s^{-\frac{3}{2}} D. \]

By Lemma 6.6,
\[ \|i\delta \Psi\|_{L^2} \lesssim s^{-3} D. \]

Finally, by (6.12), (5.49), and (5.50), we have
\[ \|(\frac{ds'}{ds} - 1)(i[N(P' + \epsilon') - N(P')] + i\Psi')\|_{L^2} \lesssim M s^{-2} D \lesssim s^{-\frac{3}{2}} D. \]

This completes the proof. \(\Box\)

### 6.8 Proof of Proposition 6.4

**Proof of Proposition 6.4** We first show the forward-in-time control (6.4). It suffices to show
\[ \sup_{s \in [s_0, \infty)} s^5 \|\delta \epsilon_3(s)\|_{L^2}^2 \lesssim s_0^5 \|\delta \epsilon_3(s_0)\|_{L^2}^2 + \frac{1}{2} \sup_{s \in [s_0, \infty)} D^2(s), \quad (6.25) \]
\[ \sup_{s \in [s_0, \infty)} \|\delta \epsilon(s)\|_{L^2}^2 \lesssim \|\delta \epsilon(s_0)\|_{L^2}^2 + \frac{1}{2} \sup_{s \in [s_0, \infty)} D^2(s), \quad (6.26) \]
\[ \sup_{s \in [s_0, \infty)} s |\delta b|(s) \lesssim s_0 |\delta b|(s_0) + \frac{1}{2} \sup_{s \in [s_0, \infty)} D(s). \quad (6.27) \]

To show (6.25), we integrate (6.20) on \([s_0, s]\) to obtain
\[ \frac{\mathcal{F}_3[\delta \epsilon](s)}{\lambda^6(s)} \leq \frac{\mathcal{F}_3[\delta \epsilon](s_0)}{\lambda^6(s_0)} + \int_{s_0}^{s} b \left( \frac{1}{100} \|\delta \epsilon_3\|_{L^2}^2 + \sigma^{-\frac{1}{2}} (\sigma^{-5} D^2) \right) d\sigma. \]

Applying (6.19) and \(b(\sigma) \approx \lambda(\sigma) \approx \sigma^{-1}\) of Lemma 6.3, we have
\[ \frac{98}{100} s^6 \|\delta \epsilon_3(s)\|_{L^2}^2 \leq \frac{102}{100} s_0^6 \|\delta \epsilon_3(s_0)\|_{L^2}^2 + \frac{2}{100} s \sup_{s \in [s_0, s]} (\sigma^{-5} \|\delta \epsilon_3(\sigma)\|_{L^2}^2) \]
\[ + 3 s^\frac{1}{2} \sup_{s \in [s_0, s]} D^2(\sigma). \]

We divide both sides by \(s\) and take the supremum over \(s \in [s_0, \infty)\) to get the claim (6.25).

To show (6.26), we integrate (6.24) on \([s_0, s]\) and use \(b(\sigma) \approx \sigma^{-1}\) to obtain
\[ \|\delta \epsilon(s)\|_{L^2}^2 \lesssim \|\delta \epsilon(s_0)\|_{L^2}^2 + s_0^{-\frac{1}{2}} \sup_{s \in [s_0, s]} D^2(\sigma). \]
Taking the supremum over $s \in [s_0, \infty)$ yields the claim (6.26).

To show (6.27), we observe that

$$
(\delta b)_s + b(\delta b) = (\delta b)_s + b^2 - \frac{ds'}{ds}(b')^2 - b'(b - \frac{ds'}{ds}b')
$$

$$
= (\tilde{\text{Mod}} - \frac{ds'}{ds}\tilde{\text{Mod}}')_3 - b'(\tilde{\text{Mod}} - \frac{ds'}{ds}\tilde{\text{Mod}}')_1 - (\eta^2 - \frac{ds'}{ds}(\eta')^2).
$$

Applying the modulation estimate (Lemma 6.10) and the definition of $D$, we have

$$
| (\delta b)_s + b(\delta b) | \lesssim s^{-\frac{5}{2}} D.
$$

Notice that $b \approx -\frac{\lambda}{s}$. Combining this with the modulation estimate (Lemma 5.9), we obtain

$$
\lambda \left| \left( \frac{\delta b}{\lambda} \right)_s \right| \lesssim | (\delta b)_s + b(\delta b) | + \left| \left( \frac{\lambda}{\lambda} + b \right)(\delta b) \right| \lesssim s^{-\frac{5}{2}} D.
$$

Integrating this and applying $\lambda(s) \approx s^{-1}$ yield the claim (6.27).

We turn to (6.5). By the modulation estimate (Lemma 6.10), we have

$$
| (\delta \eta)_s | \lesssim s^{-\frac{5}{2}} D^s + s^{-\frac{1}{2}} (s^{-\frac{5}{2}} D).
$$

Integrating this backwards in time yields (6.5).

\[ \square \]

Acknowledgements The authors appreciate Sung-Jin Oh for helpful discussions and encouragement to this work. The authors are partially supported by Samsung Science & Technology Foundation BA1701-01 and NRF-2019R1A5A1028324. Part of this work was done while the first author was visiting Bielefeld University through IRTG 2235. He would like to appreciate its kind hospitality. The authors are grateful to anonymous referees for their careful reading of this manuscript.

\section*{Appendix Hardy Inequalities and Adapted Function Spaces}

In this section, we assume $m \geq 1$ as before. Here we collect results related to Hardy’s inequality for equivariant functions. We also present proofs of the facts regarding to the adapted function spaces introduced in Sect. 2.3. Most importantly, we prove (sub-)coercivity estimates of Lemmas 2.4–2.7.

\subsection*{A.1 Weighted Hardy’s Inequality}

In Sect. 2.3, we motivated the adapted function space $\tilde{\mathcal{H}}^{\lambda}_m$, in spirit that how much weighted Hardy’s inequality we can obtain from the operators $L_Q$, $A_Q$, and $A^*_Q$. We formulate and prove the weighted Hardy’s inequality, which is the main tool in Sect. 2.3.

\begin{lemma} [Weighted Hardy’s inequality for $\partial_r$] \label{lem:weighted Hardy}
Let $0 < r_1 < r_2 < \infty$; let $\varphi : [r_1, r_2] \to \mathbb{R}_+$ be a $C^1$ weight function such that $\partial_r \varphi$ is nonvanishing and $\varphi \lesssim |r\partial_r \varphi|$.
\end{lemma}

\[
\square \ Springer
Then, for smooth \( f : [r_1, r_2] \to \mathbb{C} \), we have

\[
\int_{r_1}^{r_2} \left| \frac{f}{r} \right|^2 r |\partial_r \varphi| r dr \lesssim \int_{r_1}^{r_2} |\partial_r f|^2 \varphi r dr + \left\{ \begin{array}{ll}
\varphi(r_2)|f(r_2)|^2 & \text{if } \partial_r \varphi > 0, \\
\varphi(r_1)|f(r_1)|^2 & \text{if } \partial_r \varphi < 0.
\end{array} \right.
\]

**Proof** We only consider the case with \( \partial_r \varphi > 0 \) and leave the case \( \partial_r \varphi < 0 \) to the readers. We compute

\[
\partial_r (|f|^2) = (\partial_r \varphi) |f|^2 + 2 \varphi \text{Re}(\overline{f} \partial_r f).
\]

Integrating on the interval \([r_1, r_2]\) and using the fundamental theorem of calculus, we have

\[
\varphi(r_2)|f(r_2)|^2 \geq \int_{r_1}^{r_2} (\varphi(r_2) |f(r_2)|^2 + O(\int_{r_1}^{r_2} |\varphi| |\partial_r f| r dr)).
\]

Applying the Cauchy-Schwarz to the last term and using \( \varphi \lesssim |r \partial_r \varphi| \), we obtain

\[
\int_{r_1}^{r_2} \varphi(r_2)|f(r_2)|^2 + \int_{r_1}^{r_2} |\partial_r f|^2 \varphi r dr,
\]

which is the desired estimate. \( \square \)

**Remark A.2** [Monotonicity of \( \partial_r \) in weighted space] Lemma A.1 can be understood as the monotonicity of the operator \( \partial_r \) in various weighted spaces. The LHS, \( r^{-1} f \), is the lower bound obtained from the monotonicity. The first term of the RHS, \( \partial_r f \), is the inhomogeneous term of the equation \( \partial_r f = F \). The last term \( \varphi(r)|f(r)|^2 \) at either \( r = r_1 \) or \( r = r_2 \) are the initial or final data. Roughly speaking, the boundary value controls \( r^{-1} f \) (with an error of inhomogeneous term \( \partial_r f \)) to the left when \( \varphi \) is increasing, and to the right when \( \varphi \) is decreasing.

We then apply the techniques of conjugation by \( r^\ell \) weights and using logarithmically decreasing weight functions to Lemma A.1 to get the following estimates.

**Corollary A.3** (Weighted Hardy’s inequality for \( \partial_r - \frac{\xi}{r^\ell} \)) Let \( \ell, k \in \mathbb{R} \); let \( 0 < r_1 < r_2 < \infty \). Let \( f : [r_1, r_2] \to \mathbb{C} \) be a smooth function.

- **(Noncritical case)** If \( \ell \neq k \), then

\[
\int_{r_1}^{r_2} \left| \frac{f}{r^{k+1}} \right|^2 r dr \lesssim \int_{r_1}^{r_2} (\partial_r - \frac{\xi}{r^\ell}) f^2 r dr + \left\{ \begin{array}{ll}
|(r_2)^{-k} f(r_2)|^2 & \text{if } \ell > k, \\
|(r_1)^{-k} |f(r_1)|^2 & \text{if } \ell < k.
\end{array} \right.
\]

- **(Critical case, or logarithmic Hardy’s inequality)** If \( \ell = k \), then

\[
\int_{r_1}^{r_2} \left| \frac{f}{r^{k+1} (\log r)} \right|^2 r dr \lesssim \int_{r_1}^{r_2} (\partial_r - \frac{\xi}{r^k}) f^2 r dr + \left\{ \begin{array}{ll}
|f(1)|^2 & \text{if } 1 \in [r_1, r_2], \\
|(r_2)^{-k} f(r_2)|^2 & \text{if } r_2 \leq 1, \\
|(r_1)^{-k} f(r_1)|^2 & \text{if } r_1 \geq 1.
\end{array} \right.
\]
**Remark A.4** In Corollary A.3, if \( \ell > k \) and \( f \) is assumed to decay rapidly (this is in general easy to assume in practice, thanks to the density argument), then we may ignore the boundary term. Indeed, we send \( r_2 \to \infty \) and \( r_1 \to 0 \) to obtain

\[
\int_0^\infty \left| \frac{f}{r^{k+1}} \right|^2 r dr \lesssim_{\ell-k} \int_0^\infty \left| \frac{1}{r^k} \left( \partial_r - \frac{\ell}{r} \right) f \right|^2 r dr.
\]

**Proof of Corollary A.3** For the noncritical case, we simply apply the method of conjugation as

\[
\partial_r - \frac{\ell}{r} = r^\ell \partial_r r^{-\ell}.
\]

The noncritical case follows from an application of Lemma A.1 with the weight

\[
\varphi = r^{2\ell-2k}
\]

and substituting \( r^{-\ell} f \) in place of \( f \).

For the critical case, by separating the integral \( \int_{r_1}^{r_2} = \int_{r_1}^1 + \int_1^{r_2} \) if \( 1 \in [r_1, r_2] \), we may assume either \( r_2 \leq 1 \) or \( r_1 \geq 1 \). Let us only consider the case \( r_2 \leq 1 \) and leave the case \( r_1 \geq 1 \) for the readers. Again, by the method of conjugation \( \partial_r - \frac{k}{r} = r^k \partial_r r^{-k} \), it suffices to show

\[
\int_{r_1}^{r_2} \left| \frac{f}{r (\log r)} \right|^2 r dr \lesssim \int_{r_1}^{r_2} \left| \partial_r f \right|^2 r dr + |f(r_2)|^2.
\]

Define the weight \( \varphi : (0, 1] \to \mathbb{R}_+ \) solving the differential equation

\[
r \partial_r \varphi = (\log r)^{-2} \quad \text{with} \quad \lim_{r \to 0^+} \varphi(r) = 0.
\]

Note that \( \varphi(r) \sim (\log r)^{-1} \), so \( \varphi \lesssim r \partial_r \varphi \) does not hold. However, one can follow the proof of Lemma A.1 and uses

\[
\int_{r_1}^{r_2} \varphi \frac{|f|}{r} |\partial_r f| r dr \lesssim \left( \int_{r_1}^{r_2} |\partial_r f|^2 r dr \right)^{1/2} \left( \int_{r_1}^{r_2} \frac{|f|}{(\log r)} \right)^{1/2} r dr
\]

instead to obtain the desired estimate. \( \square \)

**A.2 Equivariant Sobolev Spaces**

Here we recall some basic notations and facts mentioned in Sect. 2.3. For an \( m \)-equivariant function \( f \), its radial part \( g : \mathbb{R}_+ \to \mathbb{C} \) is defined to satisfy \( f(x) = g(r)e^{im\theta} \), under the usual polar coordinates relation \( x_1 + ix_2 = r e^{i\theta} \). We use an abuse of notation that \( g \) is often considered as an \( m \)-equivariant function. For example, we say that \( g \) belongs to some \( m \)-equivariant function space if its \( m \)-equivariant extension belongs to that. Associated function norms are also used.
For $s \geq 0$, denote by $H^s_m$ the set of $m$-equivariant $H^s(\mathbb{R}^2)$ functions. The set of $m$-equivariant Schwartz functions, denoted by $S_m$, is dense in $H^s_m$. The $H^s_m$-norm and $\dot{H}^s_m$-norm mean the usual $H^s(\mathbb{R}^2)$-norm and $\dot{H}^s(\mathbb{R}^2)$-norm, but we use the subscript $m$ to indicate the equivariance index. When $0 \leq k \leq m$, we have generalized Hardy’s inequality [25, Lemma A.7]:

$$\| |f| - k\|_{L^2} \sim_k \|f\|_{\dot{H}^k_m}, \quad \forall f \in S_m.$$  \hfill (A.1)

As a special case, when $m \geq 1$ and $k = 1$, we have the Hardy-Sobolev inequality [25, Lemma A.6]:

$$\|r^{-1} f\|_{L^2} + \|f\|_{L^{\infty}} \lesssim \|f\|_{\dot{H}^1_m}.$$  \hfill (A.2)

The generalized Hardy’s inequality (A.1) allows us define the space $\dot{H}^k_m$ when $0 \leq k \leq m$ by taking the completion of $S_m$ under the $\dot{H}^k_m$-norm, with the embedding properties

$$S_m \hookrightarrow H^k_m \hookrightarrow \dot{H}^k_m \hookrightarrow L^2_{\text{loc}}.$$  

A.3 Adapted Function Space $\dot{H}^1_m$

Lemma A.5 (Boundedness and subcoercivity for $L_Q$ on $\dot{H}^1_m$; Lemma 2.4) For $v \in \dot{H}^1_m$, we have

$$\|L_Qv\|_{L^2} + \|Qv\|_{L^2} \sim \|v\|_{\dot{H}^1_m}.$$  

Moreover, the kernel of $L_Q : \dot{H}^1_m \to L^2$ is $\text{span}_{\mathbb{R}} \{A_Q, iQ\}$.

Proof By density, we may assume $v \in S_m$. We first express $L_Q = D^{(Q)}_+ + QB_Q$ and treat $QB_Q$-term as an error:

$$\|QB_Qv\|_{L^2} \lesssim \|Q\|_{L^{\infty}} \|r^{-1}B_Qv\|_{L^2} \lesssim \|Qv\|_{L^2}.$$  

Thus we have

$$\|L_Qv\|_{L^2} \lesssim \|D^{(Q)}_+v\|_{L^2} + \|Qv\|_{L^2} \lesssim \|D^{(Q)}_+v\|_{L^2} + \|v\|_{\dot{H}^1_m},$$

$$\|L_Qv\|_{L^2} + \|Qv\|_{L^2} \gtrsim \|D^{(Q)}_+v\|_{L^2}.$$  

We now focus on $D^{(Q)}_+v$. Since $D^{(Q)}_+ \approx \partial_r - \frac{m}{r}$ for small $r$ and $D^{(Q)}_+ \approx \partial_r + \frac{m+2}{r}$ for large $r$, the Hardy lower bound $|v|_{-1}$ is expected in view of Corollary A.3 as explained in Sect. 2.3. But there is a simple but different route to achieve this; we merely integrate by parts as

$$\|D^{(Q)}_+v\|_{L^2}^2 = \|\partial_{rr} v\|_{L^2}^2 - 2(\partial_r v, \frac{1}{r} (m + A_\vartheta[Q])v)_r + \frac{1}{r^2} (m + A_\vartheta[Q])v\|_{L^2}^2$$

$$= \|\partial_{rr} v\|_{L^2}^2 + \int \frac{1}{r^2} (m + A_\vartheta[Q])^2 - \frac{1}{2} r^2 Q^2 |v|^2.$$  

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Since both of the limits \(m + A_\theta[Q]\) as \(r \to 0\) and \(r \to \infty\) do not vanish, we get
\[
\|D^{(Q)}_+ v\|_{L^2} \lesssim \|v\|_{-1} \lesssim \|v\|_{\dot{H}^1_m},
\]
\[
\|D^{(Q)}_+ v\|_{L^2} + \|r^2(r)^{-2} Q v\|_{L^2} \gtrsim \|v\|_{\dot{H}^1_m}.
\]

It now suffices to characterize the kernel of \(L_Q : \dot{H}^1_m \to L^2\). Let \(f \in \dot{H}^1_m\) be such that \(L_Q f = 0\). It is shown in [25, Lemma 3.3] that \(f \in \text{span}_{\mathbb{R}}\{Q, iQ\}\) when \(f\) is smooth (on \(\mathbb{R}^2\)). To use this result, we want to apply the elliptic regularity on \(\mathbb{R}^2\), so that \(f \in \dot{H}^1_m\) can be upgraded to a smooth function. We consider the cartesian representation of \(L_Q\); we obtain \(L_Q f\) by taking the linear part of \(\tilde{D}^{(Q+f)}_+ (Q + f)\) and recalling \(\tilde{D}_+ = D_1 + iD_2\) as
\[
L_Q f = \tilde{D}^{(Q)}_+ f + 2i(A_1[Q, f] + iA_2[Q, f])Q,
\]
\[
A_1[f, g] := \frac{1}{2} \epsilon_{jk} \Delta^{-1} \partial_k \text{Re}(\bar{f}g),
\]
where the anti-symmetric tensor \(\epsilon_{jk}\) is defined by \(\epsilon_{12} = 1\). Thus \(L_Q f = 0\) means that
\[
(\partial_1 + i \partial_2) f = \frac{A_1[Q](x)}{|x|^r} (x_1 + i x_2) f - 2i(A_1[Q, f] + iA_2[Q, f])Q.
\]

Note that the radial part of the above display corresponds to (\(\partial_r - \frac{m}{r}\) \(f = \frac{A_1[Q]}{r} f - QB_Q f\)). Starting from \(f \in \dot{H}^1_m\) (thus a priori having \(\nabla f \in L^2, |x|^{-1} f \in L^2,\) and \(f \in L^2_{\text{loc}}\)), iterating the above display concludes that \(f \in H^\infty_{\text{loc}}\). Therefore, \(f\) is indeed a smooth solution to \(L_Q f = 0\). By [25, Lemma 3.3], we finish the proof. \(\Box\)

**Lemma A.6** (Coercivity for \(L_Q\) on \(\dot{H}^1_m\), Lemma 2.5) Let \(\psi_1\) and \(\psi_2\) be elements of \((\dot{H}^1_m)^*\), which is the dual space of \(\dot{H}^1_m\). If the \(2 \times 2\) matrix \((a_{ij})\) defined by \(a_{ij} = (\psi_i, A_1 Q)\) and \(a_{12} = (\psi_i, iQ)\) has nonzero determinant, then we have a coercivity estimate
\[
\|v\|_{\dot{H}^1_m} \gtrsim \|L_Q v\|_{L^2} \gtrsim \psi_1, \psi_2 \|v\|_{\dot{H}^1_m}, \quad \forall v \in \dot{H}^1_m \cap \{\psi_1, \psi_2\}^\perp.
\]

**Proof** This is indeed shown in [25, Lemma 3.9], but let us give a sketch of the proof for convenience.

Suppose not. We can choose a sequence \(\{v_n\}_{n \in \mathbb{N}} \subseteq \dot{H}^1_m\) such that \(\|L_Q v_n\|_{L^2} = \frac{1}{n}\), \(\|v_n\|_{\dot{H}^1_m} = 1\), and \((\psi_1, v_n)_r = (\psi_2, v_n)_r = 0\). In particular, \(\{v_n\}_{n \in \mathbb{N}}\) is bounded in \(\dot{H}^1_m\). After passing to a subsequence, there exists \(v \in \dot{H}^1_m\) such that \(v_n\) converges to \(v_\infty\) weakly in \(\dot{H}^1_m\) and strongly in \(L^2_{\text{loc}}\). By weak convergence, we have \(L_Q v_\infty = 0\) and \((\psi_1, v_\infty)_r = (\psi_2, v_\infty)_r = 0\).

On one hand, \(v_\infty = 0\) by the kernel characterization of Lemma A.5 and the orthogonality conditions. On the other hand, the subcoercivity estimate says that \(\|Q v_n\|_{L^2} \gtrsim 1\) uniformly for all large \(n\), and the strong \(L^2_{\text{loc}}\)-convergence (and \(\dot{H}^1_m\)-boundedness) says that \(\|Q v_n\|_{L^2} \to \|Q v_\infty\|_{L^2} \gtrsim 1\). This yields \(v_\infty \neq 0\), a contradiction. \(\Box\)
A.4 Adapted Function Space $\hat{\mathcal{H}}^3_m$

Recall from Sect. 2.3 that the $\hat{\mathcal{H}}^3_m$-norm is defined by

\[
\| f \|_{\hat{\mathcal{H}}^3_m} := \| \partial_+ f \|_{\hat{\mathcal{H}}^2_{m+1}} + \begin{cases} \| f \|_{L^2} & \text{if } m \geq 3, \\ \| \partial_3 r f \|_{L^2} + \| r^{-1} (\log r)^{-1} | f |_{L^2} & \text{if } m = 2, \\ \| \partial_3 r f \|_{L^2} + \| r^{-1} (\log r)^{-1} | f |_{L^2} & \text{if } m = 1. \end{cases}
\]

(initially for $m$-equivariant Schwartz function $f$). The space $\hat{\mathcal{H}}^3_m$ is obtained by taking the completion of $S_m$ under the $\hat{\mathcal{H}}^3_m$ norm. As all $L^2$-norms are involved, one can equip $\hat{\mathcal{H}}^3_m$ with a Hilbert space structure (by modifying the $\hat{\mathcal{H}}^3_m$-norm by some comparable one). The choice of $\hat{\mathcal{H}}^3_m$ is motivated to have boundedness/subcoercivity estimates of the operator $A^*_Q A Q L Q$.

Comparison of $\hat{\mathcal{H}}^3_m$ and $\hat{\mathcal{H}}^3_m$

One may naturally ask how much the $\hat{\mathcal{H}}^3_m$-norm and $\hat{\mathcal{H}}^3_m$-norm differ. It turns out that they are equivalent when $m \geq 3$, but the $\hat{\mathcal{H}}^3_m$-norm is stronger than the $\hat{\mathcal{H}}^3_m$-norm when $m \in \{1, 2\}$.

**Lemma A.7** (Comparison of $\hat{\mathcal{H}}^3_m$ and $\hat{\mathcal{H}}^3_m$) For $f \in S_m$, we have

\[
\| f \|_{\hat{\mathcal{H}}^3_m} \sim \| \partial_+ f \|_{\hat{\mathcal{H}}^2_{m+1}}
\]

and

\[
\| f \|_{\hat{\mathcal{H}}^3_m} \sim \begin{cases} \| f \|_{\hat{\mathcal{H}}^3_m} & \text{if } m \geq 3, \\ \| f \|_{\hat{\mathcal{H}}^3_m} + \| 1_{r \geq 1} \frac{1}{r^2} \| L^2 & \text{if } m = 2, \\ \| f \|_{\hat{\mathcal{H}}^3_m} + \| 1_{r \sim 1} \| L^2 & \text{if } m = 1. \end{cases}
\]

Due to $\| 1_{r \gtrsim 1} \frac{1}{r^2} f \| L^2 \lesssim \| f \|_{L^2}$, we have

\[
L^2 \cap \hat{\mathcal{H}}^3_m = H^3_m.
\]

**Remark A.8** One cannot replace $\| 1_{r \geq 1} \frac{1}{r^2} \| L^2$ by $\| 1_{r \sim 1} \| L^2$ when $m = 2$; and one cannot eliminate $\| 1_{r \sim 1} \| L^2$ when $m = 1$. When $m = 2$, the estimate $\| f \|_{\hat{\mathcal{H}}^3_2} \lesssim \| 1_{r \gtrsim 0} r^{-3} f \| L^2$ is false for any $r_0 > 0$, which can be seen by considering $f(x) = (x_1 + i x_2)^2 \chi_{\leq R}(|x|)$ and take $R \to \infty$. When $m = 1$, the estimate $\| f \|_{\hat{\mathcal{H}}^3_1} \lesssim \| 1_{r \sim 1} \| L^2$ is false, which can be seen by the example $f(x) = R(x_1 + i x_2) \chi_{\leq R}(|x|)$.

**Proof** We first show for $f \in S_m$:

\[
\| f \|_{\hat{\mathcal{H}}^3_m} \sim \| \partial_+ f \|_{\hat{\mathcal{H}}^2_{m+1}} \lesssim \| f \|_{\hat{\mathcal{H}}^3_m}.
\]
This follows from
\[ \|f\|_{\dot{H}^3_m} \approx \|\nabla f\|_{H^2} \sim \|\partial_+ f\|_{\dot{H}^2_{m+1}} + \|\partial_- f\|_{\dot{H}^2_{m-1}} \sim \|\partial_+ f\|_{\dot{H}^2_{m+1}} + \|\Delta \partial_- f\|_{L^2}. \]
and the observation \(\Delta \partial_- = \partial_- \Delta = \partial_- \partial_- \partial_+\).

We now consider the reverse inequality for \(f \in S_m\). When \(m \geq 3\), we have
\[ \|f\|_{\dot{H}^3_m} \approx \|f\|_{-3} \|L^2 \] by (A.1). Thus \(\|f\|_{\dot{H}^3_m} \gtrsim \|f\|_{\dot{H}^3_m}\) easily follows.

When \(m = 2\), we apply (A.1) to \(\partial_+ f\) \(H^2_m\) and use the pointwise estimate \(|\partial_+ f|_{-2} + \frac{1}{r^2} f| \gtrsim |f|_{-3}\) to get \(\|\partial_+ f\|_{H^2_m} + \|1_{r \geq 1} \frac{1}{r^2} f\|_{L^2} \gtrsim \|1_{r \geq 1} |f|_{-3}\|_{L^2}\). This treats the \(r \geq 1\) part. Thus it suffices to show the \(r \leq 1\) contribution:
\[ \|r^{-1} |\partial_+ f|_{-1}\|_{L^2} + \|1_{r \sim 1} f\|_{L^2} \gtrsim 1_{r \leq 1} r^{-1} (\log r)^{-1} |f|_{-2}\|_{L^2}, \]
\[ \|\partial_+ f\|_{H^2_m} \gtrsim \|1_{r \leq 1} \partial_{rrr} f\|_{L^2}. \]

To show the first assertion, an application of Corollary A.3 with \(\ell = k = 2\) and averaging the boundary term yield
\[ \|r^{-2} \partial_+ f\|_{L^2} + \|1_{r \sim 1} f\|_{L^2} \gtrsim 1_{r \leq 1} r^{-3} (\log r)^{-1} f\|_{L^2}. \]

Combining this with the pointwise estimates
\[ 1_{r \leq 1} r^{-2} (\log r)^{-1} |\partial_r f| \leq 1_{r \leq 1} r^{-2} (\log r)^{-1} |\partial_+ f| + O(1)\|1_{r \leq 1} r^{-3} (\log r)^{-1} |f|\|, \]
\[ 1_{r \leq 1} r^{-1} (\log r)^{-1} |\partial_{rr} f| \leq 1_{r \leq 1} r^{-1} (\log r)^{-1} |\partial_+ f|_{-1} \]
\[ + O(1)\|1_{r \leq 1} r^{-2} (\log r)^{-1} |f|_{-1}\|, \]

we get
\[ \|r^{-1} |\partial_+ f|_{-1}\|_{L^2} + \|1_{r \sim 1} f\|_{L^2} \gtrsim \|1_{r \leq 1} r^{-1} (\log r)^{-1} |f|_{-2}\|_{L^2}, \]
showing the first assertion. The second assertion follows by observing the special algebra
\[ \partial_{rrr} = (\partial_r + \frac{1}{r})^2 (\partial_r - \frac{2}{r}) \]
and noticing that \(\partial_r - \frac{2}{r}\) is the radial part of \(\partial_+\) acting on 2-equivariant functions.

When \(m = 1\), it suffices to show the controls
\[ \|1_{r \leq 2} r^{-1} \partial_+ f\|_{L^2} + \|1_{r < 1} f\|_{L^2} \gtrsim \|1_{r \leq 1} r^{-1} (\log r)^{-1} |f|_{-1}\|_{L^2}, \]
\[ \|1_{r \geq 2} r^{-2} \partial_+ f\|_{L^2} + \|1_{r \sim 1} f\|_{L^2} \gtrsim \|1_{r \geq 1} r^{-2} |f|_{-1}\|_{L^2}, \]
\[ \|\partial_+ f\|_{H^2_m} \gtrsim \|\partial_{rr} f\|_{-1}\|_{L^2}. \]

To show the first assertion, we apply Corollary A.3 with \(\ell = k = 1\) to get
\[ \|1_{r \leq 2} r^{-1} \partial_+ f\|_{L^2} + \|1_{r > 1} f\|_{L^2} \gtrsim \|1_{r \leq 1} r^{-2} (\log r)^{-1} f\|_{L^2}. \]
Combining this with the pointwise estimate
\[ 1_{r \leq 1} r^{-1} (\log r)^{-1} |\partial_r f| \leq 1_{r \leq 1} r^{-1} (\log r)^{-1} |\partial_+ f| + 1_{r \leq 1} r^{-2} (\log r)^{-1} |f| \]
yields
\[ \|1_{r \leq 2} r^{-1} \partial_+ f\|_{L^2} + \|1_{r \sim 1} f\|_{L^2} \gtrsim \|1_{r \leq 1} r^{-1} (\log r)^{-1} |f| - 1\|_{L^2}, \]
showing the first assertion. To show the second assertion, we apply Corollary A.3 with \( \ell = 1 \) and \( k = 2 \). Here notice that we can control the region \( r \geq 1 \) by the values of \( f \) on \( r \sim 1 \). We left this case to the interested readers. Finally, the third assertion follows from observing the special algebra
\[ \partial_{rr} = (\partial_r + \frac{1}{r})(\partial_r - \frac{1}{r}) \]
and noticing that \( \partial_r - \frac{1}{r} \) is the radial part of \( \partial_+ \) acting on 1-equivariant functions. \( \square \)

Subcoercivity estimates for \( A^*_Q A_Q L_Q \)

From now on, we turn to prove (sub-)coercivity estimates of \( A^*_Q A_Q L_Q \) in \( \dot{H}^2_m \); Lemmas 2.6 and 2.7. We start by proving subcoercivity estimates of each differential operator \( A^*_Q, A_Q, \) and \( L_Q \).

**Lemma A.9** (Boundedness and positivity for \( A^*_Q \)) For \( v \in \dot{H}^1_{m+2} \), we have
\[ \|A^*_Q v\|_{L^2} \sim \|v\|_{\dot{H}^1_{m+2}}. \]

**Proof** By density, we may assume \( v \in S_{m+2} \). Thus the integration by parts \( \|A^*_Q v\|_{L^2}^2 = (v, A_Q A^*_Q v)_r \) is justified. As \( A_Q A^*_Q = -\partial_{rr} - \frac{1}{r} \partial_r + \frac{\tilde{V}}{r^2} \) with \( \tilde{V} \sim 1 \) (this property holds only when \( m \geq 1 \)), we conclude
\[ \|A^*_Q v\|_{L^2}^2 = (v, A_Q A^*_Q v)_r \sim \|v|_{-1}\|_{L^2}^2. \]
This completes the proof. \( \square \)

**Lemma A.10** (Boundedness and subcoercivity for \( A_Q \)) For \( v \in \dot{H}^2_{m+1} \), we have
\[ \|A_Q v\|_{\dot{H}^1_{m+2}} + \|r(r)^{-2} Q v\|_{L^2} \sim \|v\|_{\dot{H}^2_{m+1}}. \]
Moreover, the kernel of \( A_Q : \dot{H}^2_{m+1} \to \dot{H}^1_{m+2} \) is \( \text{span}_C \{r Q\} \).

**Proof** By density, we may assume \( v \in S_{m+1} \). Note that \( r^{-1} A_Q v = D_+^{(Q)} (r^{-1} v) \). Thus we can proceed as in the proof of Lemma A.5 to have
\[ \|r^{-1} A_Q v\|_{L^2} \lesssim \|r^{-1} v|_{-1}\|_{L^2}. \]
\[ \| r^{-1} A_Q v \|_{L^2} + \| r \langle r \rangle^{-2} Q v \|_{L^2} \gtrsim \| r^{-1} v \|_{-1} \|_{L^2}. \]

From
\[ \partial_r A_Q v = \partial_{rr} v - \frac{1}{r} (m + A_0[Q]) \partial_r v + \frac{1}{2} Q^2 v = \partial_{rr} v + O(r^{-1} |v|_{-1}), \]
we have
\[ \| \partial_r A_Q v \|_{L^2} \lesssim \| v \|_{-2} \|_{L^2}, \]
\[ \| \partial_r A_Q v \|_{L^2} + \| r^{-1} v \|_{-1} \|_{L^2} \gtrsim \| \partial_{rr} v \|_{L^2}. \]

Summing up the above results, we have
\[ \| A_Q v \|_{\dot{H}_{m+1}^1} \lesssim \| v \|_{\dot{H}_{m+1}^2}, \]
\[ \| A_Q v \|_{\dot{H}_{m+1}^1} + \| r \langle r \rangle^{-2} Q v \|_{L^2} \gtrsim \| v \|_{\dot{H}_{m+1}^2}. \]

Finally, we characterize the kernel of \( A_Q : \dot{H}_{m+1}^2 \to \dot{H}_{m+1}^1 \). If \( A_Q f = 0 \) for some \( f \in \dot{H}_{m+1}^2 \), then \( \partial_r - \frac{m+1+A_0[Q]}{r} f(r) = 0 \) on \( (0, \infty) \). Since \( A_Q \) is a first-order differential operator and \( A_Q(rQ) = 0 \), \( f \in \text{span}_C \{rQ\} \) follows by the ODE uniqueness result.

It remains to obtain the boundedness/subcoercivity properties of \( L_Q \) at the \( \dot{H}_{m}^3 \)-level. We first show that we can always ignore \( QB_Q \) part of \( L_Q \).

**Lemma A.11** (Contribution of \( QB_Q \)) Let \( v \in \dot{H}_{m}^3 \). We have
\[ \| QB_Q v \|_{\dot{H}_{m+1}^2} \lesssim \| r \langle r \rangle^{-2} Q v \|_{-1} \|_{L^2}. \]

**Remark A.12** The proof only uses the property \( |Q|_2 \lesssim Q \). Thus the above boundedness property holds if we replace some \( Q \) in \( QB_Q \) by \( \Lambda Q \), as \( |\Lambda Q|_2 \lesssim Q \) also holds. This fact is used in the proof of Lemma 5.11.

**Proof** Note the pointwise inequality
\[ |QB_Q v|_{-2} \lesssim r^{-2} Q |B_Q v| + |Q^2 v|_{-1} \lesssim r^{-1} \langle r \rangle^{-2} Q \cdot r^{-1} B_Q \langle r \rangle^{-2} v + Q^2 |v|_{-1}. \]

We estimate its \( L^2 \)-norm by
\[ \| QB_Q v \|_{-2} \|_{L^2} \lesssim \| r^{-1} \langle r \rangle^{-2} Q \|_{L^\infty} \| r \langle r \rangle^{-2} Q \|_{-1} \|_{L^2} \lesssim \| r \langle r \rangle^{-2} Q \|_{-1} \|_{L^2}, \]
which is the desired estimate. \( \square \)
Lemma A.13 (Boundedness and subcoercivity for \( L_Q \) on \( \mathcal{H}^3_{m} \)) For \( v \in \mathcal{H}^3_{m} \), we have
\[
\| L_Q v \|_{\mathcal{H}^2_{m+1}} + \| Q |v|_{-2} \|_{L^2} \sim \|v\|_{\mathcal{H}^3_{m}}.
\]
Moreover, the kernel of \( L_Q : \mathcal{H}^3_{m} \to \mathcal{H}^2_{m+1} \) is spanned by \( \Lambda Q, iQ \).

Proof By density, we may assume \( v \in \mathcal{S}_m \). By Lemma A.11, we can remove the \( QB_Q \)-part of \( L_Q \) with an error of the size \( \| r(r)^{-2} Q |v|_{-1} \|_{L^2} \), so it suffices to show the boundedness/subcoercivity estimates for \( D^{(Q)}_+ \).

From now on, we focus on \( \| D^{(Q)}_+ v \|_{\mathcal{H}^2_{m+1}} \). We integrate by parts as
\[
\frac{1}{r^2} \| D^{(Q)}_+ v \|^2_{L^2} = \frac{1}{r^2} \| \partial_+ v \|^2_{L^2} - 2(\frac{1}{r^2} \partial_+ v, \frac{1}{r^2} A_\theta(Q) v) + \frac{1}{r^2} A_\theta(Q) v \|_{L^2}^2
\]
\[
= \frac{1}{r^2} \| \partial_+ v \|^2_{L^2} + \int (2m - 4 + A_\theta(Q) \partial r) A_\theta(Q) - \frac{1}{2} r^2 Q^2 ) |r|^2 v^2.
\]
Using the asymptotics of \( A_\theta(Q) \) as \( r \to 0 \) or \( r \to \infty \), we are led to
\[
\frac{1}{r^2} \| D^{(Q)}_+ v \|_{L^2} + \frac{1}{r^2} Q v \|_{L^2} \sim \frac{1}{r^2} \| \partial_+ v \|_{L^2} + \| 1_{r \geq 1} \frac{1}{r^2} v \|_{L^2}.
\]
Notice the \( \| 1_{r \geq 1} \frac{1}{r^2} v \|_{L^2} \) term, which cannot be obtained for the \( \mathcal{H}^3_{m} \)-norm when \( m = 2 \); see Lemma A.7. Noting brief computations (where we used \( |A_\theta(Q)| \geq Q^2 \))
\[
\frac{1}{r} \partial_\tau D^{(Q)}_+ v = \frac{1}{r} \partial_\tau \partial_+ v + O \left( \frac{|A_\theta(Q)|}{r^2} |v|_{-1} \right),
\]
\[
\partial_\tau \partial_\tau D^{(Q)}_+ v = \partial_\tau \partial_\tau \partial_+ v + O \left( \frac{|A_\theta(Q)|}{r} |v|_{-2} \right),
\]
and (by (A.3))
\[
\| Q |v|_{-2} \|_{L^2} + \frac{A_\theta(Q)}{r} |v|_{-2} \|_{L^2} \lesssim \|v\|_{\mathcal{H}^3_{m}},
\]
we obtain
\[
\| D^{(Q)}_+ v \|_{\mathcal{H}^2_{m+1}} + \| Q |v|_{-2} \|_{L^2} \lesssim \|v\|_{\mathcal{H}^3_{m}},
\]
\[
\| D^{(Q)}_+ v \|_{\mathcal{H}^2_{m+1}} + \| Q |v|_{-2} \|_{L^2} \gtrsim \| \partial_+ v \|_{\mathcal{H}^2_{m+1}} + \| 1_{r \geq 1} |v|_{-3} \|_{L^2} \gtrsim \|v\|_{\mathcal{H}^3_{m}}.
\]
The first estimate establishes the boundedness property of \( D^{(Q)}_+ \). The second estimate establishes the subcoercivity property of \( D^{(Q)}_+ \) by Lemma A.7.

Finally, we characterize the kernel of \( L_Q : \mathcal{H}^3_{m} \to \mathcal{H}^2_{m+1} \). Note the embedding \( \mathcal{H}^3_{m} \hookrightarrow \mathcal{H}^1_{m,\text{loc}} \). We have seen in Lemma A.5 that the kernel of \( L_Q : \mathcal{H}^1_{m} \to L^2 \) is spanned by \( \{ \Lambda Q, iQ \} \). Here we combine a cutoff argument to extend this kernel characterization to \( L_Q : \mathcal{H}^1_{m,\text{loc}} \to L^2_{\text{loc}} \). This is possible as the nonlocal term \( QB_Q \) of \( L_Q \) contains only the \( \int_0^r \)-integral. Let \( f \in \mathcal{H}^1_{m,\text{loc}} \) be such that \( L_Q f = 0 \). Then, for any \( R > 0 \), \( L_Q (\chi_{x \leq R} f)(x) = 0 \) for all \( |x| \leq R \), thanks to the expression...
Lemma A.14 (Boundedness and subcoercivity for $A^*_Q A_Q L_Q$ on $\mathcal{H}_m^3$; Lemma 2.6) For $v \in \mathcal{H}_m^3$, we have

$$\|A^*_Q A_Q L_Q v\|_{L^2} + \|Q|v|^{-2}\|_{L^2} \sim \|v\|_{\mathcal{H}_m^3}.$$ 

Moreover, the kernel of $A^*_Q A_Q L_Q : \mathcal{H}_m^3 \to L^2$ is $\text{span}_\mathbb{R}\{A_Q, i Q, \rho, i r^2 Q\}$.

**Proof** By collecting the boundedness estimates of the previous lemmas, we see that $A^*_Q A_Q L_Q : \mathcal{H}_m^3 \to L^2$ bounded. For the subcoercivity estimate, applying the previous lemmas yields

$$\|A^*_Q A_Q L_Q v\|_{L^2} + \|r(r)^{-2}Q(L_Q v)\|_{L^2} + \|Q|v|^{-2}\|_{L^2} \gtrsim \|v\|_{\mathcal{H}_m^3}.$$ 

It now suffices to show that

$$\|r(r)^{-2}Q(L_Q v)\|_{L^2} \lesssim \|r^2(r)^{-2}Q|v|^{-2}\|_{L^2} \lesssim \|Q|v|^{-2}\|_{L^2}. $$

This is an easy consequence of the following estimates:

$$\|r(r)^{-2}Q(Q B_Q v)\|_{L^2} \lesssim \|r Q\|_{L^\infty} \|r^{-2}Q B_Q v\|_{L^2} \lesssim \|r(r)^{-2}Q|v|^{-1}\|_{L^2},$$

$$\|r(r)^{-2}Q D^Q_{+} v\|_{L^2} \lesssim \|r(r)^{-2}Q|v|^{-1}\|_{L^2}.$$ 

Therefore, the subcoercivity estimate is proved.

We turn to the characterization of the kernel of $A^*_Q A_Q L_Q$ in $\mathcal{H}_m^3$. Let $v$ be an element of the kernel. Since $A_Q L_Q v \in \mathcal{H}_m^{1+2}$, the positivity in Lemma A.9 says that $A_Q L_Q v = 0$. Since $L_Q v \in \mathcal{H}_m^{2+1}$, the kernel characterization of $A_Q : \mathcal{H}_m^{2+1} \to \mathcal{H}_m^{1+2}$ (Lemma A.10) says that $L_Q v \in \text{span}_\mathbb{C}\{r Q\}$. Now, combining the facts $L_Q \rho = \frac{1}{2(m+1)}r Q$, $L_Q i r^2 Q = 2i r Q$, and the kernel characterization of $L_Q$ in $\mathcal{H}_m^3$ completes the proof. 

**Lemma A.15** (Coercivity for $A^*_Q A_Q L_Q$ on $\mathcal{H}_m^3$; Lemma 2.7) Let $\psi_1, \psi_2, \psi_3, \psi_4$ be elements of $(\mathcal{H}_m^3)^*$, which is the dual space of $\mathcal{H}_m^3$. If the $4 \times 4$ matrix $(a_{ij})$ defined by $a_{11} = (\psi_1, A_Q \rho), a_{12} = (\psi_1, i Q), a_{13} = (\psi_1, i r^2 Q), a_{14} = (\psi_1, \rho)$ has nonzero determinant, then we have a coercivity estimate

$$\|v\|_{\mathcal{H}_m^3} \gtrsim \|A^*_Q A_Q L_Q v\|_{L^2} \gtrsim \psi_1, \psi_2, \psi_3, \psi_4 \|v\|_{\mathcal{H}_m^3}, \quad \forall v \in \mathcal{H}_m^3 \cap \{\psi_1, \psi_2, \psi_3, \psi_4\}.$$
Proof Suppose not. Choose a sequence \( \{v_n\}_{n \in \mathbb{N}} \subseteq \tilde{H}^3_m \) such that \( A^*_Q A_Q L_Q v_n \| L^2 = 1/n \), \( \| v_n \| \tilde{H}^3_m = 1 \), and \( (\psi_i, v_n)_r = 0 \) for \( i \in \{1, 2, 3, 4\} \). In particular, \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( \tilde{H}^3_m \). After passing to a subsequence, there exists \( v \in \tilde{H}^3_m \) such that \( v_n \) converges to \( v_\infty \) weakly in \( \tilde{H}^3_m \) and strongly in \( H^3_{\text{loc}} \). By the weak convergence, we have \( A^*_Q A_Q L_Q v_\infty = 0 \) and \( (\psi_i, v_\infty)_r = 0 \) for all \( i \).

On one hand, \( v_\infty = 0 \) by the kernel characterization of Lemma A.14 and orthogonality conditions. On the other hand, the subcoercivity estimate of Lemma A.14 says that \( \| Q |v_n|^{-2} L^2 \| \geq 1 \) uniformly for all large \( n \), and the strong \( H^3_{\text{loc}} \)-convergence says that \( \| Q |v_n|^{-2} L^2 \| \to \| Q |v_\infty|^{-2} L^2 \| \geq 1 \). This yields \( v_\infty \neq 0 \), a contradiction. \( \square \)

Interpolation and \( L^\infty \) estimates

Lemma A.16 For \( v \in H^3_m \), the following estimates hold. When \( m \geq 1 \),

\[
\| |v|^{-1} L^\infty + \| \partial_r v \| L^2 \| \lesssim \| v \|_{H^3_m} \lesssim \| v \|_{\tilde{H}^3_m}, \tag{A.4}
\]

\[
\| (\log^{-1} r) ^{-1} |v|^{-2} L^\infty \| \lesssim \| v \|_{\tilde{H}^3_m}. \tag{A.5}
\]

In fact, we have stronger estimates when \( m \geq 2 \)

\[
\| |v|^{-1} L^\infty + \| |v|^{-2} L^2 \| \lesssim \| v \|_{\tilde{H}^3_m} \lesssim \| v \|_{H^3_m} \lesssim \| v \|_{\tilde{H}^3_m},
\]

\[
\| (\log^{-1} r) ^{-1} |v|^{-2} L^\infty \| \lesssim \| v \|_{\tilde{H}^3_m},
\]

and when \( m \geq 3 \)

\[
\| |v|^{-2} L^\infty \| \lesssim \| v \|_{\tilde{H}^3_m}.
\]

Proof By density, we may assume \( v \in S_m \).

When \( m = 1 \), \( \partial_+ v = (\partial_r - \frac{1}{r}) v \) so

\[
\| \frac{1}{r} v \|_{L^\infty} \lesssim \int_0^\infty |(\frac{1}{r}(\partial_r - \frac{1}{r}) v)(\frac{1}{r}) v)|dr \lesssim \| \frac{1}{r} \partial_+ v \| L^2 \| \frac{1}{r} v \| L^2,
\]

\[
\| \partial_+ v \|_{L^\infty} \lesssim \int_0^\infty |\partial_r \partial_+ v| |\partial_+ v| dr \lesssim \| \frac{1}{r} \partial_+ v \| L^2 \| \partial_+ v \| L^2.
\]

Thus we have

\[
\| |v|^{-1} L^\infty \| \lesssim \| \partial_+ v |^{-2} L^2 \| |v|^{-1} L^2 \| \lesssim \| v \|_{\tilde{H}^1} \| v \|_{\tilde{H}^1}.
\]

Next, we use \( \partial_r v = (\partial_r + \frac{1}{r}) \partial_+ v \), the interpolation \( \tilde{H}^1 \) by \( L^2 \) and \( \tilde{H}^2 \), and the embedding \( \tilde{H}^3 \hookrightarrow \tilde{H}^1 \) (Lemma A.7)

\[
\| \partial_r v \|_{L^2} \lesssim \| \partial_+ v \|_{\tilde{H}^2} \lesssim \| \partial_+ v \|_{L^2} \| \partial_+ v \|_{\tilde{H}^2} \lesssim \| v \|_{\tilde{H}^1} \| v \|_{\tilde{H}^1}.
\]
For \( r(r)^{-1} |v|_{-2} \), we again perform the FTC argument as in \( ||v|_{-1}\|_{L^\infty}^2 \) to get

\[
\| r^{-1} (r)^{-1} v \|_{L^\infty}^2 \lesssim \min \{ \frac{1}{r}, \frac{1}{r^2} \} \| v \|_{L^\infty}^2 \\
\lesssim \| 1_{r \leq 1} \frac{1}{r^2} \partial_+ v \|_{L^2}^2 \| 1_{r \leq 1} \frac{1}{r} v \|_{L^2} + \| 1_{r \geq 1} \frac{1}{r^2} |v|_{-1} \|_{L^2} \lesssim \| v \|_{\dot{H}^3}^2,
\]

and

\[
\| r(r)^{-1} |\partial_+ v|_{-1} \|_{L^\infty}^2 \lesssim \| |\partial_+ v|_{-1} \|_{L^\infty}^2 \lesssim \| |\partial_+ v|_{-2} \|_{L^2} \| r^{-1} |\partial_+ v|_{-1} \|_{L^2} \lesssim \| v \|_{\dot{H}^3}^2.
\]

Thus \( \| r(r)^{-1} |v|_{-2} \|_{L^\infty} \lesssim \| v \|_{\dot{H}^3} \) follows.

When \( m \geq 2 \), we can use (A.2) using \( m - 1 \geq 1 \) to have

\[
\| |v|_{-1} \|_{L^\infty} \lesssim \| \partial_+ v \|_{L^\infty} + \| \partial_- v \|_{L^\infty} \lesssim \| \partial_+ v \|_{\dot{H}^4_{m+1}} + \| \partial_- v \|_{\dot{H}^4_{m-1}} \lesssim \| v \|_{\dot{H}^5_m}
\]

and also by (A.1)

\[
\| |v|_{-2} \|_{L^2} \lesssim \| v \|_{\dot{H}^2_m}.
\]

We again perform the FTC argument to get

\[
\| (\log r)^{-1} |v|_{-2} \|_{L^\infty}^2 \lesssim \| (\log r)^{-1} |v|_{-3} \|_{L^2} \| r^{-1} (\log r)^{-1} |v|_{-2} \|_{L^2} \lesssim \| v \|_{\dot{H}^3_m}^2.
\]

When \( m \geq 3 \), we can perform the FTC argument to get

\[
\| |v|_{-2} \|_{L^\infty}^2 \lesssim \| |v|_{-3} \|_{L^2} \| r^{-1} |v|_{-2} \|_{L^2} \lesssim \| v \|_{\dot{H}^3_m}^2.
\]

This completes the proof.

\[\square\]

A.5 Adapted Function Space \( \dot{H}^5_m \)

In this subsection, we assume \( m \geq 3 \).

Recall from Sect. 2.3 that the \( \dot{H}^5_m \)-norm is defined by

\[
\| f \|_{\dot{H}^5_m} := \| \partial_+ f \|_{\dot{H}^4_{m+1}}
\]

\[
\left\{ \begin{array}{ll}
\| f |_{-5} \|_{L^2} & \text{if } m \geq 5, \\
\| \partial_r r r r r f \|_{L^2} + \| r^{-1} (\log r)^{-1} |f|_{-4} \|_{L^2} & \text{if } m = 4, \\
\| \partial_r r r r f |_{-1} \|_{L^2} + \| r^{-1} (\log r)^{-1} |f|_{-3} \|_{L^2} & \text{if } m = 3,
\end{array} \right. \quad (A.6)
\]

initially for \( m \)-equivariant Schwartz function \( f \). The space \( \dot{H}^5_m \) is obtained by taking the completion of \( S_m \) under the \( \dot{H}^5_m \) norm. The choice of \( \dot{H}^5_m \) is motivated to have boundedness/subcoercivity estimates of the operator \( A^*_Q A_Q A^*_Q A_Q L_Q \).
Most of the results and their proofs in this subsection follow by shifting the equivariance index by 2 (i.e. \( m \geq 1 \) to \( m \geq 3 \)) and the regularity index by 2 (i.e. from 3 to 5). Because the equivariance index and the regularity index are shifted by the same amount 2, weighted Hardy’s inequality (Corollary A.3) applies in the same way. Henceforth, we shall record the facts and sketch (or omit) their proofs.

**Comparison of \( \dot{\mathcal{H}}^5_m \) and \( \dot{\mathcal{H}}^5_{m+2} \)**

We start by comparing the \( \dot{\mathcal{H}}^5_m \)-norm and \( \dot{\mathcal{H}}^5_{m+2} \)-norm. It turns out that they are equivalent when \( m \geq 5 \), but the \( \dot{\mathcal{H}}^5_m \)-norm is stronger than the \( \dot{\mathcal{H}}^5_{m+2} \)-norm when \( m \in \{3, 4\} \).

**Lemma A.17** (Comparison of \( \dot{\mathcal{H}}^5_m \) and \( \dot{\mathcal{H}}^5_{m+2} \)) For \( f \in S_m \), we have

\[
\| f \|_{\dot{\mathcal{H}}^5_m} \sim \| \partial_+ f \|_{\dot{\mathcal{H}}^4_{m+1}}
\]

and

\[
\| f \|_{\dot{\mathcal{H}}^5_{m+2}} \sim \begin{cases} 
\| f \|_{\dot{\mathcal{H}}^5_m} & \text{if } m \geq 5, \\
\| f \|_{\dot{\mathcal{H}}^5_m} + \| 1_{r \geq 1} \frac{1}{r^5} f \|_{L^2} & \text{if } m = 4, \\
\| f \|_{\dot{H}^5_m} + \| 1_{r \sim 1} f \|_{L^2} & \text{if } m = 3.
\end{cases}
\]

Due to \( \| 1_{r \geq 1} \frac{1}{r^5} f \|_{L^2} \lesssim \| f \|_{L^2} \), we have

\[ L^2 \cap \dot{\mathcal{H}}^5_m = H^5_m. \]

**Remark A.18** Similarly as in Remark A.8, one cannot replace \( \| 1_{r \geq 1} \frac{1}{r^5} f \|_{L^2} \) by \( \| 1_{r \sim 1} f \|_{L^2} \) when \( m = 4 \); and one cannot eliminate \( \| 1_{r \sim 1} f \|_{L^2} \) when \( m = 3 \).

**Proof** Similarly proceeding as in the proof of Lemma A.7, we have

\[
\| f \|_{\dot{\mathcal{H}}^5_m} \sim \| \partial_+ f \|_{\dot{\mathcal{H}}^4_{m+1}} \lesssim \| f \|_{\dot{\mathcal{H}}^5_m}.
\]

We now consider the reverse inequality for \( f \in S_m \). When \( m \geq 5 \), we have \( \| f \|_{\dot{\mathcal{H}}^5_m} \sim \| f \|_{-5} \) \( L^2 \) by (A.1). Thus \( \| f \|_{\dot{\mathcal{H}}^5_m} \lesssim \| f \|_{\dot{\mathcal{H}}^5_m} \) easily follows.

When \( m = 4 \), we apply (A.1) to \( \| \partial_+ f \|_{\dot{\mathcal{H}}^5_4} \) to get \( \| \partial_+ f \|_{\dot{H}^4_4} + \| 1_{r \geq 1} \frac{1}{r^5} f \|_{L^2} \gtrsim \| 1_{r \geq 1} f \|_{-5} \) \( L^2 \). This treats the \( r \geq 1 \) part. Thus it suffices to show the \( r \leq 1 \) contribution:

\[
\| r^{-1} \partial_+ f \|_{L^2} + \| 1_{r \sim 1} f \|_{L^2} \gtrsim \| 1_{r \leq 1} \frac{1}{r^5} \|_{L^2}.
\]

To show the first assertion, an application of Corollary A.3 with \( \ell = k = 4 \) and averaging the boundary term yield

\[
\| r^{-4} \partial_+ f \|_{L^2} + \| 1_{r \sim 1} f \|_{L^2} \gtrsim \| 1_{r \leq 1} r^{-5} \|_{L^2}.
\]
Starting from this, one can proceed as in the proof of Lemma A.7 to control higher derivatives, hence getting the first assertion. The second assertion follows by observing the special algebra
\[
\partial_{rrrrr} = (\partial_r + \frac{1}{r})^4(\partial_r - \frac{4}{r}),
\]
and noticing that \(\partial_r - \frac{4}{r}\) is the radial part of \(\partial_+\) acting on 4-equivariant functions.

When \(m = 3\), it suffices to show the controls
\[
\|1_{r \leq 2} r^{-1} |\partial_+ f| -2 \|_{L^2} + \|1_{r \leq 1} r^{-1} (\log_- r)^{-1} |f| -3 \|_{L^2},
\]
\[
\|1_{r \geq 2} r^{-2} |\partial_+ f| -2 \|_{L^2} + \|1_{r \leq 1} r^{-1} |f| -3 \|_{L^2},
\]
\[
\|\partial_+ f \|_{H^4} \sim \|\partial_{rrrr} f \|_{-1} \|_{L^2}.
\]

To show the first assertion, apply Corollary A.3 with \(\ell = k = 3\) and proceed as in the proof of Lemma A.7 to control higher derivatives. To show the second assertion, apply Corollary A.3 with \(\ell = 3\) and \(k = 4\). To show the third assertion, use the special algebra
\[
\partial_{rrrr} = (\partial_r + \frac{1}{r})^3(\partial_r - \frac{3}{r})
\]
and notice that \(\partial_r - \frac{3}{r}\) is the radial part of \(\partial_+\) acting on 3-equivariant functions. \(\square\)

**Subcoercivity estimates for \(A^*_Q A_Q A^*_Q A_Q L_Q\)**

We turn to prove (sub-)coercivity estimates of \(A^*_Q A_Q A^*_Q A_Q L_Q\) in \(\dot{H}^5_m\). As previously, we prove (sub-)coercivity estimates for \(A^*_Q A_Q A^*_Q\), \(A_Q\), and \(L_Q\). The latter two subcoercivity estimates \(A_Q : \dot{H}^4_{m+1} \to \dot{H}^3_{m+2}\) and \(L_Q : \dot{H}^5_m \to \dot{H}^4_{m+1}\) are very similar to \(A_Q : \dot{H}^2_{m+1} \to \dot{H}^1_{m+2}\) and \(L_Q : \dot{H}^3_m \to \dot{H}^2_{m+1}\) in the previous subsection and their proofs are omitted. As like \(A^*_Q\), the operator \(A^*_Q A_Q A^*_Q\) turns out to be positive for \(m \geq 3\). (See also Remark 2.8.)

**Lemma A.19** (Boundedness and positivity for \(A^*_Q A_Q A^*_Q\)) For \(v \in \dot{H}^3_{m+2}\), we have
\[
\|A^*_Q A_Q A^*_Q v\|_{L^2} \sim \|v\|_{\dot{H}^3_{m+2}}. \tag{A.7}
\]

**Proof** We first claim the subcoercivity estimate for \(A^*_Q A_Q A^*_Q\):
\[
\|A^*_Q A_Q A^*_Q v\|_{L^2} + \|r^2 (r)^{-2} Q |v| -2 \|_{L^2} \sim \|v\|_{\dot{H}^3_{m+2}}. \tag{A.8}
\]

The perturbative term \(\|r^2 (r)^{-2} Q |v| -2 \|_{L^2}\) will be deleted at the end of the proof. By density, we may assume \(v \in S_{m+2}\). By Lemmas A.9 and A.10, we have
\[
\|A^*_Q A_Q A^*_Q v\|_{L^2} + \|r (r)^{-2} Q |v| -1 \|_{L^2} \sim \|A^*_Q v\|_{\dot{H}^2_{m+1}}.
\]
It now suffices to invert $A_Q^*$. Note that $-A_Q^* \approx \partial_r - \frac{m}{r}$ near the infinity. As $m \geq 3$, we can apply Corollary A.3 with $\ell = m$ and $k = 2$ (noncritical case) near the infinity. Near the origin, $-A_Q^* \approx \partial_r + \frac{m+2}{r}$ so we can apply Corollary A.3 with $\ell = -m - 2$ and $k = 2$ (noncritical case). Thus one has

$$\left\| \frac{1}{r^2} A_Q^* v \right\|_{L^2} + \left\| (r)^{-2} Q v \right\|_{L^2} \sim \left\| \frac{1}{r^2} |v|^{-1} \right\|_{L^2}.$$ 

Starting from this, one can control higher derivatives as before to get (A.8). We omit the proof.

To delete the perturbative term of (A.8), one can proceed as in the proof of Lemma A.15, but in this case $A_Q^* A_Q A_Q^* : \dot{H}^{3}_{m+2} \rightarrow L^2$ does not have nontrivial kernel; see Remark 2.8. Thus the coercivity estimate follows without any orthogonality conditions. We omit the proof. \(\Box\)

Having settled the coercivity estimates of $A_Q^* A_Q A_Q^*$, it suffices to obtain subcoercivity estimates of $A_Q$ and $L_Q$ at $\dot{H}^4$ and $\dot{H}^5$ levels, respectively. Compared to the previous subsection, both the equivariance index and regularity index are shifted by the same amount 2. Thus the following lemmas are obtained in the same way as for Lemmas A.10–A.15, after adding two more derivatives to the perturbative terms. We omit the proofs.

**Lemma A.20** (Boundedness and subcoercivity for $A_Q$) For $v \in \dot{H}^{4}_{m+1}$, we have

$$\left\| A_Q v \right\|_{\dot{H}^{3}_{m+2}} + \left\| (r)^{-2} Q |v|^{-2} \right\|_{L^2} \sim \left\| v \right\|_{\dot{H}^{4}_{m+1}}.$$ 

Moreover, the kernel of $A_Q : \dot{H}^{4}_{m+1} \rightarrow \dot{H}^{3}_{m+2}$ is $\text{span}_\mathbb{C} \{r Q\}$.

**Lemma A.21** (Contribution of $Q B_Q$) Let $v \in \dot{H}^5_{m}$. We have

$$\left\| Q B_Q v \right\|_{\dot{H}^{4}_{m+1}} \lesssim \left\| (r)^{-2} Q |v|^{-3} \right\|_{L^2}.$$ 

**Lemma A.22** (Boundedness and subcoercivity for $L_Q$ on $\dot{H}^5_{m}$) For $v \in \dot{H}^{5}_{m}$, we have

$$\left\| L_Q v \right\|_{\dot{H}^{4}_{m+1}} + \left\| Q |v|^{-4} \right\|_{L^2} \sim \left\| v \right\|_{\dot{H}^{5}_{m}}.$$ 

Moreover, the kernel of $L_Q : \dot{H}^{5}_{m} \rightarrow \dot{H}^{4}_{m+1}$ is $\text{span}_\mathbb{R} \{\Lambda Q, i Q\}$.

Combining the above subcoercivity lemmas for $A_Q$ and $L_Q$ with the positivity estimates of $A_Q^* A_Q A_Q^*$, we finally obtain the (sub-)coercivity estimates for $A_Q^* A_Q A_Q^* L_Q$.

**Lemma A.23** (Boundedness and subcoercivity for $A_Q^* A_Q A_Q^* A_Q L_Q$ on $\dot{H}^{5}_{m}$; Lemma 2.9) For $v \in \dot{H}^{5}_{m}$, we have

$$\left\| A_Q^* A_Q A_Q^* A_Q L_Q v \right\|_{L^2} + \left\| Q |v|^{-4} \right\|_{L^2} \sim \left\| v \right\|_{\dot{H}^{5}_{m}}.$$ 

Moreover, the kernel of $A_Q^* A_Q A_Q^* A_Q L_Q : \dot{H}^{5}_{m} \rightarrow L^2$ is $\text{span}_\mathbb{R} \{\Lambda Q, i Q, \rho, i r^2 Q\}$.
Lemma A.24 (Coercivity for $A_Q^*A_QA_Q^*A_QL_Q$ on $\mathcal{H}_m^5$; Lemma 2.10) Let $\psi_1, \psi_2, \psi_3, \psi_4$ be elements of $(\mathcal{H}_m^5)^*$, which is the dual space of $\mathcal{H}_m^5$. If the $4 \times 4$ matrix $(a_{ij})$ defined by $a_{i1} = (\psi_i, i\Lambda Q)_r$, $a_{i2} = (\psi_i, iQ)_r$, $a_{i3} = (\psi_i, i^2Q)_r$, and $a_{i4} = (\psi_i, \rho)_r$ has nonzero determinant, then we have a coercivity estimate

$$\|v\|_{\mathcal{H}_m^5} \gtrsim \|A_Q^*A_QA_Q^*A_QL_Qv\|_{L^2} \gtrsim \psi_1, \psi_2, \psi_3, \psi_4 \parallel v\parallel_{\mathcal{H}_m^5},$$

$$\forall v \in \mathcal{H}_m^5 \cap \{\psi_1, \psi_2, \psi_3, \psi_4\}^\perp.$$ 

Interpolation and $L^\infty$ estimates

Lemma A.25 For $v \in H_m^3$, the following estimates hold. When $m \geq 3$,

$$\|(v)_3\|_{L^\infty} + \|\partial_{rrrr}v\|_{L^2} \lesssim \|v\|_{H_m^3}^{1/2} \|v\|_{H_m^5}^{1/2}, \quad (A.9)$$

$$\|r^{-1}(v)_4\|_{L^\infty} \lesssim \|v\|_{H_m^5}. \quad (A.10)$$

In fact, we have stronger estimates when $m \geq 4$

$$\|(v)_3\|_{L^\infty} + \|v\|_{L^2} \lesssim \|v\|_{H_m^4} \lesssim \|v\|_{H_m^3}^{1/2} \|v\|_{H_m^5}^{1/2},$$

$$\|\log^{-1}(v)_4\|_{L^\infty} \lesssim \|v\|_{H_m^5},$$

and when $m \geq 5$

$$\|(v)_4\|_{L^\infty} \lesssim \|v\|_{H_m^5}.$$

Proof By density, we may assume $v \in S_m$. One can proceed as in the proof of Lemma A.16 with suitable modifications. For example, one uses Lemma A.17, (A.1), and the algebras $\partial_+ = \partial_r - \frac{3}{r}$ and $\partial_{rrrr} = (\partial_r + \frac{1}{r})^3 \partial_+$ when $m = 3$. \qed

References

1. Beceanu, M.: A critical center-stable manifold for Schrödinger’s equation in three dimensions. Commun. Pure Appl. Math. 65(4), 431–507 (2012)
2. Bejenaru, I., Krieger, J., Tataru, D.: A codimension-two stable manifold of near soliton equivariant wave maps. Anal. PDE 6(4), 829–857 (2013)
3. Bergé, L., De Bouard, A., Saut, J.-C.: Blowing up time-dependent solutions of the planar, Chern–Simons gauged nonlinear Schrödinger equation. Nonlinearity 8(2), 235–253 (1995)
4. Bergé, L., De Bouard, A., Saut, J.-C.: Collapse of Chern–Simons-gauged matter fields. Phys. Rev. Lett. 74(20), 3907–3911 (1995)
5. Bourgain, J., Wang, W.: Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25(1–2), 197–215 (1998), 1997. Dedicated to Ennio De Giorgi
6. Burzio, S., Krieger, J.: Type II blow up solutions with optimal stability properties for the critical focusing nonlinear wave equation on $\mathbb{R}^{1+1}$. Mem. Am. Math. Soc. 278(1369), iii+75 (2022)
7. Byeon, J., Huh, H., Seok, J.: Standing waves of nonlinear Schrödinger equations with the gauge field. J. Funct. Anal. 263(6), 1575–1608 (2012)
8. Byeon, J., Huh, H., Seok, J.: On standing waves with a vortex point of order $N$ for the nonlinear Chern–Simons–Schrödinger equations. J. Differ. Equ. 261(2), 1285–1316 (2016)

9. Chou, K.S., Wan, T.Y.-H.: Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc. Pac. J. Math. 163(2), 269–276 (1994)

10. Collot, C.: Nonradial type II blow up for the energy-supercritical semilinear heat equation. Anal. PDE 10(1), 127–252 (2017)

11. Collot, C.: Type II blow up manifolds for the energy supercritical semilinear wave equation. Mem. Am. Math. Soc. 252(1205), v+163 (2018)

12. Donninger, R., Krieger, J., Szeftel, J., Wong, W.: Codimension one stability of the catenoid under the vanishing mean curvature flow in Minkowski space. Duke Math. J. 165(4), 723–791 (2016)

13. Glassey, R.T.: On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. J. Math. Phys. 18(9), 1794–1797 (1977)

14. Gustafson, S., Kang, K., Tsai, T.-P.: Schrödinger flow near harmonic maps. Commun. Pure Appl. Math. 60(4), 463–499 (2007)

15. Gustafson, S., Kang, K., Tsai, T.-P.: Asymptotic stability of harmonic maps under the Schrödinger flow. Duke Math. J. 145(3), 537–583 (2008)

16. Gustafson, S., Nakanishi, K., Tsai, T.-P.: Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on $\mathbb{R}^2$. Commun. Math. Phys. 300(1), 205–242 (2010)

17. Hillairet, M., Raphaël, P.: Smooth type II blow-up solutions to the four-dimensional energy-critical wave equation. Anal. PDE 5(4), 777–829 (2012)

18. Huh, H.: Blow-up solutions of the Chern–Simons–Schrödinger equations. Nonlinearity 22(5), 967–974 (2009)

19. Huh, H.: Energy solution to the Chern–Simons–Schrödinger equations. Abstr. Appl. Anal., Art. ID 590653, 7 (2013)

20. Huh, H., Seok, J.: The equivalence of the Chern–Simons–Schrödinger equations and its self-dual system. J. Math. Phys. 54(2), 021502, 5 (2013)

21. Jackiw, R., Pi, S.-Y.: Classical and quantal nonrelativistic Chern–Simons theory. Phys. Rev. D (3) 42(10), 3500–3513 (1990)

22. Jackiw, R., Pi, S.-Y.: Soliton solutions to the gauged nonlinear Schrödinger equation on the plane. Phys. Rev. Lett. 64(25), 2969–2972 (1990)

23. Jackiw, R., Pi, S.-Y.: Time-dependent Chern–Simons solitons and their quantization. Phys. Rev. D (3) 44(8), 2524–2532 (1991)

24. Jackiw, R., Pi, S.-Y.: Self-dual Chern-Simons solitons. Progr. Theoret. Phys. Suppl. 107, 1–40 (1992). Low-dimensional field theories and condensed matter physics (Kyoto, 1991)

25. Kim, K., Kwon, S.: On pseudoconformal blow-up solutions to the self-dual Chern–Simons–Schrödinger equation: existence, uniqueness, and instability. Mem. Am. Math. Soc. 284(1409), v+128 (2023)

26. Krieger, J., Schlag, W.: Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. J. Am. Math. Soc. 19(4), 815–920 (2006)

27. Krieger, J., Schlag, W.: Non-generic blow-up solutions for the critical focusing NLS in 1-D. J. Eur. Math. Soc. (JEMS) 11(1), 1–125 (2009)

28. Krieger, J., Nakanishi, K., Schlag, W.: Center-stable manifold of the ground state in the energy space for the critical wave equation. Math. Ann. 361(1–2), 1–50 (2015)

29. Lim, Z.M.: Large data well-posedness in the energy space of the Chern–Simons–Schrödinger system. J. Differ. Equ. 264(4), 2553–2597 (2018)

30. Liu, B., Smith, P.: Global wellposedness of the equivariant Chern–Simons–Schrödinger equation. Rev. Mat. Iberoam. 32(3), 751–794 (2016)

31. Liu, B., Smith, P., Tataru, D.: Local wellposedness of Chern–Simons–Schrödinger. Int. Math. Res. Not. IMRN 23, 6341–6398 (2014)

32. Martel, Y., Merle, F., Nakanishi, K., Raphaël, P.: Codimension one threshold manifold for the critical gKdV equation. Commun. Math. Phys. 342(3), 1075–1106 (2016)

33. Merle, F., Raphaël, P.: Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. Geom. Funct. Anal. 13(3), 591–642 (2003)

34. Merle, F., Raphaël, P.: On universality of blow-up profile for $L^2$ critical nonlinear Schrödinger equation. Invent. Math. 156(3), 565–672 (2004)
35. Merle, F., Raphaël, P.: The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. Ann. Math. (2) 161(1), 157–222 (2005)
36. Merle, F., Raphaël, P.: Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation. Commun. Math. Phys. 253(3), 675–704 (2005)
37. Merle, F., Raphaël, P.: On a sharp lower bound on the blow-up rate for the $L^2$ critical nonlinear Schrödinger equation. J. Am. Math. Soc. 19(1), 37–90 (2006)
38. Merle, F., Raphaël, P., Rodnianski, I.: Blowup dynamics for smooth data equivariant solutions to the critical Schrödinger map problem. Invent. Math. 193(2), 249–365 (2013)
39. Merle, F., Raphaël, P., Rodnianski, I.: Type II blow up for the energy supercritical NLS. Camb. J. Math. 3(4), 439–617 (2015)
40. Merle, F., Raphaël, P., Szeftel, J.: The instability of Bourgain–Wang solutions for the $L^2$ critical NLS. Am. J. Math. 135(4), 967–1017 (2013)
41. Oh, S.-J., Pusateri, F.: Decay and scattering for the Chern–Simons–Schrödinger equations. Int. Math. Res. Not. IMRN 24, 13122–13147 (2015)
42. Raphaël, P.: Stability of the log–log bound for blow up solutions to the critical non linear Schrödinger equation. Math. Ann. 331(3), 577–609 (2005)
43. Raphaël, P., Rodnianski, I.: Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang–Mills problems. Publ. Math. Inst. Hautes Études Sci. 115, 1–122 (2012)
44. Raphaël, P., Schweyer, R.: Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow. Commun. Pure Appl. Math. 66(3), 414–480 (2013)
45. Raphaël, P., Schweyer, R.: Quantized slow blow-up dynamics for the corotational energy-critical harmonic heat flow. Anal. PDE 7(8), 1713–1805 (2014)
46. Rodnianski, I., Sterbenz, J.: On the formation of singularities in the critical O(3)σ-model. Ann. Math. (2) 172(1), 187–242 (2010)
47. van den Berg, J.B., Williams, J.F.: (In-)stability of singular equivariant solutions to the Landau–Lifshitz–Gilbert equation. Eur. J. Appl. Math. 24(6), 921–948 (2013)

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