The distribution of rational points on conics

by

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In fond memory of Professor Andrzej Schinzel

1. Introduction. Let \( q(x_1, x_2, x_3) = q(x) \in \mathbb{Z}[x_1, x_2, x_3] \) be a quadratic form, and write

\[
\mathbb{Z}^n_{\text{prim}} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : \text{g.c.d.}(x_1, \ldots, x_n) = 1\}.
\]

The purpose of this paper is to investigate the behaviour as \( B \to \infty \) of the counting function

\[
N(B) = N(B; q) = \# \{ x \in \mathbb{Z}^3_{\text{prim}} : q(x) = 0, \max(|x_1|, |x_2|, |x_3|) \leq B \},
\]

and of its weighted form

\[
N(B, w) = N(B, w; q) = \sum_{\substack{x \in \mathbb{Z}^3_{\text{prim}} \atop q(x) = 0}} w(B^{-1}x).
\]

Here we take \( w : \mathbb{R}^3 \to \mathbb{R} \) to be infinitely differentiable, with compact support. With this notation, the conic \( q = 0 \) has \( \frac{1}{2} N(B) \) rational points of height at most \( B \).

Provided \( q \) is isotropic over \( \mathbb{Q} \) (in other words, if \( q(x) = 0 \) has at least one non-zero integral solution), one has

\[
N(B) \sim \frac{1}{2} \sigma_{\infty} \mathcal{S}(q) B \quad \text{as} \ B \to \infty
\]

where \( \sigma_{\infty} > 0 \) is the real density of solutions, and \( \mathcal{S}(q) > 0 \) may be given explicitly in terms of the usual product of local densities. (The factor \( \frac{1}{2} \) is the “alpha constant” in Peyre’s terminology [4].) Indeed, one has

\[
N(B, w) = \frac{1}{2} \sigma_{\infty} (q; w) \mathcal{S}(q) B + O_{q,w}(B \exp\{-c\sqrt{\log B}\})
\]

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for some absolute constant $c > 0$. These results follow from work of the
author \cite{3} Corollary 2. We stress that the error term in (1) contains an
unspecified dependence on $q$. Our main aim in this paper is to obtain a good
explicit dependence, so as to show how large $B$ has to be, in terms of $q,$
before one sees the true asymptotics for $N(B)$.

In order to see the phenomena that $N(B)$ can display we present a nu-
merical example. Let $q_0$ be the form

$$q_0(x) = -61x_1^2 - 22x_1x_3 - 38x_2^2 + 99x_2x_3 + 39x_3^2.$$  

Then the following graph shows values of $N(B; q_0)$ for $B \leq 10000$.

The graph appears linear from about $B = 6000$ onwards, but there is a
surprising kink around $B = 3500$. Indeed, for $B \leq 2500$ the graph seems
linear, but with a smaller gradient than for the range $B \geq 6000$. It is this
strange behaviour that we aim to explain — see the discussion after Theo-
rem 5.

We begin by introducing some notation and terminology. In general we
will want to allow our form $q$ to have odd cross-terms. We therefore write it
in the asymmetric shape

$$q(x) = \sum_{1 \leq i \leq j \leq 3} q_{ij}x_i x_j,$$

and associate with $q$ the matrix

$$Q = \begin{pmatrix}
2q_{11} & q_{12} & q_{13} \\
q_{12} & 2q_{22} & q_{23} \\
q_{13} & q_{23} & 2q_{33}
\end{pmatrix}.$$  

Moreover, we define the determinant of $q$, somewhat unconventionally, by

$$\Delta = \Delta(q) = \frac{1}{2} \det(Q).$$
Thus $\Delta \in \mathbb{Z}$ for any integral form, and
\[ \Delta(q(Mx)) = \det(M)^2 \Delta(q) \]
for any $3 \times 3$ matrix $M$. By changing the sign of $q$ if necessary we can arrange that $\Delta(q) \geq 0$. We recall that $q$ is said to be primitive if the coefficients $q_{ij}$ have no common factor. With this notation our first result is the following.

**Theorem 1.** Let $q$ be a primitive integral isotropic form with $\Delta > 0$. Then there is a positive integer $K \leq \tau(\Delta)$ and there are non-singular $3 \times 3$ integer matrices $M_1, \ldots, M_K$ having the following properties:

(i) If $\Delta$ is square-free then $K = \tau(\Delta)$.

(ii) The determinant $\det(M_k)$ is a positive divisor of $\Delta$.

(iii) For any primitive integral solution $x$ of the equation $q(x) = 0$, there is a unique index $k$ such that $x \in M_k(\mathbb{Z}^3)$.

(iv) For each $k$ there is a corresponding $D_k \in \mathbb{N}$ such that
\[ q(M_kx) = D_k(x_1x_3 - x_2^2), \]
identically in $x$.

(v) We have
\[ \Delta \det(M_k)^2 = D_k^3, \]
so that $D_k | \Delta$ and $\det(M_k) | D_k$. Moreover, $D_k | \det(M_k)^2$.

(vi) A prime $p$ can divide $\Delta \det(M_k)^{-1}$ only if $v_p(\Delta) \geq 4$.

(vii) If $\Delta$ is cube-free then for every index $k \leq \tau(\Delta)$ the set $M_k(\mathbb{Z}^3)$ contains a primitive zero of $q$.

Here $\tau(\ldots)$ is the usual divisor function, and $v_p(\cdot)$ is the $p$-adic valuation. In addition to this notation we will find it convenient to write $J(x)$ for the quadratic form $x_1x_3 - x_2^2$, so that $q(M_kx) = D_kJ(x)$.

The theorem shows that we can partition the primitive integer zeros of $q$ into $K$ classes $C_1, \ldots, C_K$, corresponding to the different matrices $M_k$. Specifically, we define
\[ C_k = \{ x \in \mathbb{Z}_\text{prim}^3 \cap M_k(\mathbb{Z}^3) : q(x) = 0 \}. \]

Moreover, since the primitive integer zeros of $J$ are given exactly twice each by $\pm(u_1^2, u_1u_2, u_2^2)$, the theorem shows that we can produce the primitive integer solutions of $q(x) = 0$ exactly twice each as
\[ x = \pm M_k(u_1^2, u_1u_2, u_2^2). \]

Here $(u_1, u_2)$ must be primitive if $x$ is, but unfortunately it is not true that $x$ is primitive whenever $u_1$ and $u_2$ are coprime.

Part (vi) of the theorem shows that if $\Delta$ has no fourth-power divisors then $\det(M_k) = D_k = \Delta$ for every index $k$. In what follows, it may help the reader if they first restrict attention to this simplified case.
Part (vii) of the theorem shows that if \( \Delta \) is cube-free then each of the classes \( C_k \) is non-empty. For other values of \( \Delta \) we may discard any values of \( k \) for which \( C_k \) is empty, without affecting the claims in the theorem. Thus we will suppose in what follows that each class \( C_k \) is non-empty.

The form \( q_0 \) given by \( (2) \) has \( \Delta(q_0) = 977861 = p_0 \), say, which is prime. Hence part (i) of the theorem shows that \( K = 2 \), and parts (v) and (vi) yield
\[
\det(M_1) = D_1 = \det(M_2) = D_2 = p_0.
\]

In fact, we may take
\[
M_1 = \begin{pmatrix} 1 & -45 & 3426 \\ 0 & 100 & 3339 \\ -1 & -54 & 3047 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 39 & -21 & -98 \\ 0 & -100 & -1 \\ 61 & 122 & 99 \end{pmatrix}.
\]
Indeed, \( q_0(x) = L_1(x)L_2(x) - p_0L_3(x)^2 \) with
\[
L_1(x) = 100x_1 + 99x_2 + 100x_3, \quad L_2(x) = 9778x_1 + 9877x_2 + 9779x_3,
\]
\( L_3(x) = x_1 + x_2 + x_3, \)
and it turns out that the two classes are
\[
C_1 = \{ x \in \mathbb{Z}^3_{\text{prim}} : q(x) = 0 \text{ and } p_0 \mid L_1(x) \},
\]
\[
C_2 = \{ x \in \mathbb{Z}^3_{\text{prim}} : q(x) = 0 \text{ and } p_0 \mid L_2(x) \}.
\]

In order to use Theorem 1 for quantitative results we will need information on the size of the entries in \( M_k \). However, for each \( k \) there are infinitely many choices for \( M_k \), since the automorphism group
\[
\text{Aut}(J) = \text{Aut}_\mathbb{Z}(J) = \{ U \in M_3(\mathbb{Z}) : J(Mx) = J(x) \}
\]
is infinite. (Note that \( \det(U) = \pm 1 \) for every \( U \in \text{Aut}(J) \). Thus \( U^{-1} \) will automatically be integral.) Our next result shows that we can always make a good choice for \( M_k \). We will write \( \|x\| \) for the \( L^2 \)-norm of the vector \( x \), and \( \|q\| \) for the \( L^2 \)-norm of the coefficients of the matrix \( Q \) of \( q \), as given by (3). Specifically, we have
\[
\|q\| = \|Q\| = \{ 4q_{11}^2 + 2q_{12}^2 + 2q_{13}^2 + 4q_{22}^2 + 2q_{23}^2 + 4q_{33}^2 \}^{1/2}.
\]

**Theorem 2.** In Theorem 1 we may choose \( M_k \) so that if \( M_k^{-1} \) has rows \( r_1, r_2, r_3 \), then
\[
\|r_1\| \cdot \|r_3\| \leq 9D_k^{-1}\|q\| \quad \text{and} \quad \|r_2\|^2 \leq 10D_k^{-1}\|q\|.
\]
Moreover, if \( M_k \) has columns \( c_1, c_2, c_3 \) we will have \( \|c_1\| \leq \|c_3\| \) and
\[
\|c_1\| \cdot \|c_3\| \leq 90\det(M_k)^2D_k^{-2}\|q\|^2 \leq 90\|q\|^2,
\]
\[
\|c_2\| \leq 9\det(M_k)D_k^{-1}\|q\| \leq 9\|q\|.
\]

The constants can certainly be improved, but for our purposes it suffices to know that there is at least one set of numerical values that is valid. From
now on we will assume that the matrices $M_k$ are as described in Theorem 2. For the form (2) we have

$$\|q_0\| = \sqrt{39872} = 199.679\ldots,$$

and one sees that the columns of the matrices (6) amply fulfil the conditions above.

Since $q(c_1) = q(c_3) = 0$, we find that for any isotropic form $q$ and any index $k$ there are two linearly independent zeros $c_1, c_3 \in M_k(\mathbb{Z}^3)$ with $\|c_1\| \cdot \|c_3\| \leq 90\|q\|^2$; in particular, there is at least one vector in $C_k$ of length at most 10\|q\|. (The reader should note that $c_1$ and $c_3$ need not be primitive, while $C_k$ is defined as the set of primitive zeros in $M_k(\mathbb{Z}^3)$.) Thus Theorem 2 recovers (in the case of ternary forms) the results of Davenport [2, Theorem 1] and Cassels [1], which were weaker inasmuch as they referred only to the complete set of zeros of $q$, rather than individual classes $C_k$.

Our next result, which is rather easy, explains how $c_1$ and $c_3$ are related to the smallest and second smallest zeros of $q$ in $C_k$. It is phrased in terms of the quantity

$$\rho = \rho(q) = \|q\|^3/\Delta,$$

which we will refer to as the “aspect ratio” of $q$. We will see that $\rho \geq 2$ in all cases, and we may expect that $\rho(q) \approx 1$ for “typical” forms $q$. We therefore think of forms with large aspect ratio as having untypically small determinant. For the form (2) we have $\rho(q_0) = 8.141\ldots$.

**Theorem 3.** We have $\rho(q) \geq 2$ for any $q$.

Let $z_1$ be an element of $C_k$ of minimal length, and let $z_2$ be an element of $C_k$ of minimal length subject to the condition that $z_2 \neq \pm z_1$. Then

$$\|z_1\| \cdot \|z_2\| \geq D_k/\|q\|.$$

If

$$\|c_1\| < \rho^{-1/2} \det(M_k) D_k^{-1}\|q\|,$$

then $c_1$ must be a scalar multiple of the shortest vector $z_1$. In general,

$$\|c_1\| \leq 90\rho\|z_1\| \quad \text{and} \quad \|c_3\| \leq 90\rho\|z_2\|.$$

When $\rho \ll 1$, as we usually expect, we may interpret Theorem 3 as saying that the lengths of $c_1$ and $c_3$ are within a constant factor of the shortest possible lengths, namely \|z_1\| and \|z_2\|. Moreover, suppose we write $\ell$ for the constant $c = (91\rho)^{-1/2}$ and

$$\ell = \sqrt{90} \det(M_k) D_k^{-1}\|q\|$$

for the maximum length for $c_1$ permitted by (8). Then whenever $c_1$ has length at most $c\ell$, the vector $c_1$ must actually be the minimal zero $z_1$, or a scalar multiple of it. For the form $q_0$, the first column is a minimal zero, so that any other zero in $C_1$ must have length at least $2^{-1/2}p_0/\|q_0\|$.
The third column of $M_1$ gives a zero of length $5671.913 \ldots$. In contrast, the first and third columns of $M_2$ give relatively small zeros in $C_2$.

In fact, Theorem 2 follows from the following more general result.

**Theorem 4.** Suppose $A$ is a $3 \times 3$ integer matrix, and $J(Ax) = q(x)$. Then there is a matrix $U \in \text{Aut}(J)$ such that the rows $a, b, c$ of $UA$ satisfy
\[
\|a\| \cdot \|c\| \leq 9\|q\| \quad \text{and} \quad \|b\|^2 \leq 10\|q\|.
\]

We may use the previous theorems to count primitive zeros of $q(x)$. Our eventual aim is to give a sharp explicit version of the asymptotic formula (1). We begin by estimating the number of zeros in each of the classes $C_k$ with height at most $B$, using the counting function
\[
N(B; C_k) = \sum_{x \in C_k} w(B^{-1}x).
\]

Previously we had said that $w : \mathbb{R}^3 \to \mathbb{R}$ should be infinitely differentiable, with compact support. We shall now be more specific and require that $w(x) = 0$ whenever $\|x\| > 1$. Since it is possible that $w(x)$ might vanish on the zero locus of $q$, we introduce a second weight function $w_0(x)$ defined as
\[
w_0(x) = \begin{cases} 
\exp\{-\frac{1}{1-\|x\|^2}\}, & \|x\| < 1, \\
0, & \text{otherwise}.
\end{cases}
\]

This has the properties required for $w$ itself, and its support includes non-trivial points of the conic $q(x) = 0$. We now define the real density of points on the conic $q = 0$ relative to the weight $w$ by setting
\[
\sigma_\infty(q; w) = \lim_{T \to \infty} \frac{1}{T^3} \int_{\mathbb{R}^3} w(y)K_T(q(y)) \, dy_1 \, dy_2 \, dy_3
\]
with
\[
K_T(t) = T \max(1 - T|t|, 0).
\]

This coincides with the constant occurring in (1); see [3, Theorem 3] which has a mild variant of (12). We shall show in Lemma 10 that the above limit does indeed exist. The following asymptotic formula for $N(B; C_k)$ then holds.

**Theorem 5.** For each class $C_k$ there is a square-free divisor $\Delta_1 \Delta_2$ of $\Delta$ such that $p | \Delta_1$ for every prime for which $p \| \Delta$, and such that
\[
N(B; C_k) = \frac{\Delta_1^{1/2}}{2D_k^{1/2}} \sigma_\infty(q; w) \kappa B \left\{ 1 + O_w\left( \psi(B)\left( \frac{\|z_2\|}{B} \right)^{1/4} \right) \right\} + O_w(1),
\]

\[{}^\text{Note: The symbol } D_k \text{ represents a specific constant related to the class } C_k, \text{ which is not defined in the text.} \]
with
\[ \kappa = \frac{6}{\pi^2} \prod_{p|\Delta} \frac{1}{1 + p^{-1}} \prod_{p|\Delta_2} \frac{1 - p^{-1}}{1 + p^{-1}} \]
and
\[ \psi(B) = 4^{\omega(\Delta)} \rho \frac{\sigma_\infty(q; w_0)}{\sigma_\infty(q; w)} \log B. \]

Here \( z_2 \) is the second smallest element of \( C_k \), as described precisely in Theorem 3.

A number of comments should be made here. Firstly, in interpreting the theorem one should think of the factor \( \psi(B) \) as being roughly of order 1, or more generally as not being too large. We will see in Lemma 11 that \( \sigma_\infty(q; w) \ll \sigma_\infty(q; w_0) \) when \( \text{sup} |w| \leq 1. \) However, we have no estimate in the reverse direction since it is possible that the zero locus of \( q \) only just enters the support of \( w \), making \( \sigma_\infty(q; w) \) small. Nonetheless it is reasonable to think that \( \sigma_\infty(q; w_0) \ll \sigma_\infty(q; w) \) in most cases of interest.

Viewing \( \psi(B) \) as being small we may interpret the theorem as giving a linear asymptotic formula for \( N(B; C_k) \) which takes effect when \( B \) is not much larger than \( ||z_2|| \). Indeed, one can easily show that the error term \( O_w(1) \) is insignificant when \( B \geq \rho ||z_2|| \). Of course, when \( B < ||z_2|| \) the function \( N(B; C_k) \) counts at most the zeros \( \pm z_1 \). Thus \( N(B; C_k) \) is \( O_w(1) \) from \( B = 1 \) to \( B = ||z_2|| \), and then begins to display its typical linear growth.

When \( \Delta \) is square-free we have \( D_k = \Delta \) for every \( k \), by parts (v) and (vi) of Theorem 1. Moreover, we will have \( \Delta_1 = \Delta \) and \( \Delta_2 = 1 \), so that
\[ \kappa = \frac{6}{\pi^2} \prod_{p|\Delta} \frac{1}{1 + p^{-1}} \]
for each index \( k \). Thus when \( \Delta \) is square-free the leading constant in Theorem 3 is the same for each value of \( k \), but the point at which linear growth begins is potentially different.

We are now in a position to explain the observed kink in our graph of \( N(B) \) for the quadratic \( q_0 \). The correspondence between \( N(B) \) and the counting functions \( N(B; C_k) \) is not precise since the former is defined using the condition \( ||x||_\infty \leq B \) while the latter use \( ||x|| = ||x||_2 \). For the class \( C_1 \) we may take \( z_1 = (1, 0, -1) \). The zero of second smallest sup-norm in \( C_1 \) is \((3426, 3339, 3047)\), whence \( N(B; C_1) = 2 \) for \( 1 \leq B < 3426 \). However, as soon as \( B \) is somewhat larger than 3500 we will have \( N(B; C_1) \sim cB \) for a certain constant \( c > 0 \). For \( C_2 \) the two zeros of smallest sup-norm are \((39, 0, 61)\) and \((-98, -1, 99)\) (or \((-38, -99, 38)\), which has the same sup-norm) so that we will have \( N(B; C_2) \sim cB \) as soon as \( B \) is somewhat larger than a few hundred, with the same constant \( c \). Thus the initial section of the graph for \( N(B; q_0) \), up to \( B = 3500 \) or so, reflects the range in which \( N(B; C_1) = 2 \).
but $N(B; C_2)$ is already growing like $cB$, and the later values of $B$ are in the range where both $N(B; C_1)$ and $N(B; C_2)$ are growing like $cB$.

Some remarks on the shape of $\psi(B)$ are also in order. It would be interesting to know to what extent the various factors involved could be reduced, or indeed removed. Although this seems possible to some extent, we hope that the present form of $\psi(B)$ will be sufficient for applications.

We can produce an asymptotic formula for $N(B, w; q)$ by summing up the formulae for $N(B; C_k)$. Since $\|z_2\| \leq \|c_3\|$ and $\|z_1\| \leq \|c_1\|$ for each index $k$ inequality (8) yields

$$\|z_2\| \leq \|c_3\| \ll \|q\|^2/\|c_1\| \leq \|q\|^2/\|z_1\|.$$  

Thus Theorem 5 has the following immediate corollary, in light of part (i) of Theorem 1.

**Theorem 6.** Let $z_0$ be a non-trivial integer zero of $q$ with $\|z_0\|$ minimal. Then

$$N(B, w; q) = \frac{1}{2}\sigma_\infty(q; w)\mathcal{G}(q)B\left\{1 + O_w\left(\psi(B)\left(\frac{\|q\|^2}{\|z_0\|B}\right)^{1/4}\right)\right\} + O_w(\tau(\Delta))$$

with $\psi(B)$ as in Theorem 5

This is the promised improvement of (1), with a good explicit dependence on $q$. It produces a linear asymptotic growth as soon as $B$ is a little larger than $\|q\|^2/\|z_0\|$. Since $\|z_0\|$ is typically of order around $\|q\|$ this is essentially best possible. We should also comment on the quality of the error term, which has a power saving in $B$. In (1) there is a saving of order $\exp\{-c\sqrt{\log B}\}$, which has its origins in the error term for the Prime Number Theorem. Thus one could replace $\sqrt{\log B}$ in the exponent by some slightly larger power of $\log B$, but one cannot hope to establish (1) with a power saving in $B$ by the methods of [3].

The reader may compare our work with that of Sofos [5]. The latter gives an asymptotic formula for an unweighted counting function, and has an error term which has a better dependence on $B$ (of order $B^{1/2} \log B$) and an explicit dependence on $q$, though a much weaker one.

In future work we plan to apply Theorem 6 to count rational points on certain varieties that can be fibred into conics. Indeed, such applications provide the natural motivation for the present paper. In work in preparation (jointly with Dan Loughran) we look at the counting function for Del Pezzo surfaces of degree 5, in the case where there is a conic fibration. Another example, which we plan to examine in due course, is the variety $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ cut out by the equation

$$X_0Y_0^2 + X_1Y_1^2 + X_2Y_2^2 = 0,$$

in which a rational point $P$ represented by a pair of primitive integer vectors
The distribution of rational points on conics

(x, y) has height \( h(P) = \|x\|_\infty \|y\|_\infty \). Both these examples require the full strength of Theorem 6.

2. Proof of Theorem 1. We begin with a result that will allow us to work with matrices over \( \mathbb{Z}/m\mathbb{Z} \), rather than \( \mathbb{Z} \).

Lemma 1. Let \( M \) be an \( n \times n \) integer matrix, with determinant coprime to some positive integer \( r \). Then there is a matrix \( M' \equiv M \pmod{r} \) with prime determinant. Moreover, if \( \det(M) \equiv 1 \pmod{r} \) there is an \( M'' \equiv M \pmod{r} \) in \( \text{SL}_n(\mathbb{Z}) \).

Proof. We can write \( M \) in Smith normal form as \( M = UDV \) with \( U, V \in \text{SL}_n(\mathbb{Z}) \) and \( D \) diagonal. One then sees that it suffices to prove the lemma when \( M \) is diagonal, which we do by induction on \( n \). The case \( n = 1 \) is immediate, by Dirichlet’s Theorem. If the result is true for matrices of size \( n - 1 \), and

\[
M = \text{Diag}(m_1, \ldots, m_n) = \begin{pmatrix} M_0 & 0 \\ 0 & m_n \end{pmatrix},
\]

say, then \( \det(M_0) \) will be coprime to \( r \) so that \( M_0 \equiv M_0' \pmod{r} \) with \( \det(M_0') \) prime. It follows that we may write \( M_0' \) in Smith normal form as \( U_0D_0V_0 \), with \( D_0 = \text{Diag}(1, \ldots, 1, p) \) say. Thus \( M \equiv M_1 \pmod{r} \) with

\[
M_1 = \begin{pmatrix} M_0' & 0 \\ 0 & m_n \end{pmatrix} = U_1 \text{Diag}(1, \ldots, 1, p, m_n)V_1,
\]

where

\[
U_1 = \begin{pmatrix} U_0 & 0 \\ 0 & 1 \end{pmatrix},
\]

and similarly for \( V_1 \). To complete the induction step it remains to show that the lemma holds for the matrix \( \text{Diag}(p, m_n) \). However,

\[
\text{Diag}(p, m_n) \equiv p \begin{pmatrix} 0 & sr \\ r & m_n + tr \end{pmatrix} \pmod{r}
\]

and the matrix on the right has determinant \( pm_n + tpr - sr^2 \). Since \( pm_n \) will be coprime to \( r \), we can make this determinant prime by taking \( t = 0 \) and choosing \( s \) suitably. Moreover, if \( pm_n = 1 + kr \), we can make the determinant equal to 1 by choosing \( s \) and \( t \) so that \( sr - tp = k \). This completes the induction argument.

Our next result describes the reduction of ternary forms modulo a prime \( p \) and its powers. We do not assume that \( p \) is odd.
LEMMA 2. Let $p$ be prime and let $q(x)$ be an integral ternary quadratic form, not divisible by $p$ but with $p^e \mid \Delta(q)$ for some exponent $e \geq 1$. Then there is a matrix $M \in \text{SL}_3(\mathbb{Z})$ such that one of the following holds:

(i) $q(Mx) \equiv \kappa x_3^2 \pmod{p}$ with $p \nmid \kappa$;
(ii) $q(Mx) \equiv x_1x_2 + \kappa p^e x_3^2 \pmod{p^{e+1}}$ for some integer $\kappa$ coprime to $p$;
(iii) $q(Mx) \equiv q_1(x_1, x_2) + \kappa p^e x_3^2 \pmod{p^{e+1}}$ for some integer $\kappa$ coprime to $p$, with $q_1$ irreducible modulo $p$.

Proof. In view of Lemma 1 it suffices to find a suitable $p$-adic matrix $M \in \text{SL}_3(\mathbb{Z}_p)$. When $p$ is odd we can diagonalize over $\mathbb{Z}_p$ to get $Ax_1^2 + Bx_2^2 + Cx_3^2$, say. Since $q$ has determinant divisible by $p$, but does not vanish modulo $p$, we see that either case (i) of the lemma holds, or we may take $p \nmid AB$ and $p^e \mid C$. We then have case (ii) if $-AB$ is a quadratic residue of $p$, and case (iii) otherwise.

For $p = 2$ we consider the reduction of $q$ over $\mathbb{F}_2$. Since $2 \mid \Delta$ we find that $q(x)$ is equivalent to one of $x_3^2$, or $x_1x_2$, or $x_1^2 + x_1x_2 + x_2^2$ over $\mathbb{F}_2$, via a matrix in $\text{SL}_3(\mathbb{F}_2)$. (This can be shown by considering all possible forms $q$ modulo 2, if necessary.) The first case leads immediately to case (i) of the lemma. In the remaining cases, Lemma 1 shows that $q$ is equivalent to $\tilde{q}(x_1, x_2) + \ell(x_1, x_2)x_3 + \mu x_3^2$ over $\mathbb{Z}_2$, where $\ell(x_1, x_2)$ is a linear form, and $\tilde{q}(x_1, x_2) \equiv x_1x_2$ or $x_1^2 + x_1x_2 + x_2^2 \pmod{2}$. Replacing $x_1$ and $x_2$ by $x_1 - \xi_1 x_3$ and $x_2 - \xi_2 x_3$ respectively eliminates the term $\ell(x_1, x_2)x_3$ provided that

$$
\ell(x_1, x_2) = \xi_1 \frac{\partial \tilde{q}(x_1, x_2)}{\partial x_1} + \xi_2 \frac{\partial \tilde{q}(x_1, x_2)}{\partial x_2}.
$$

Suitable $\xi_1, \xi_2 \in \mathbb{Z}_2$ can always be found, since the linear forms $\partial \tilde{q}/\partial x_1$ and $\partial \tilde{q}/\partial x_2$ are congruent modulo 2 to $x_2$ and $x_1$ respectively. We then conclude that $q$ is equivalent to $\tilde{q}(x_1, x_2) + \mu' x_3^2$ over $\mathbb{Z}_2$. Computing the determinant of this we find that $2^e \mid \mu'$. When

$$
\tilde{q}(x_1, x_2) \equiv x_1^2 + x_1x_2 + x_2^2 \pmod{2}
$$

we obtain case (iii) of the lemma. Finally, if $\tilde{q}(x_1, x_2) \equiv x_1x_2 \pmod{2}$ we see from Hensel’s Lemma that $\tilde{q}(x_1, x_2)$ must factor over $\mathbb{Z}_2$, and a further unimodular change of variables leads to case (ii) of the lemma.

We next have the following lemma, which shows how we remove powers of $p$ from $\Delta(q)$.

LEMMA 3. Suppose that $q(x)$ is an integral isotropic ternary quadratic form, not necessarily primitive, and that $p^e \mid \Delta(q)$, $p \neq 0$. Then there is a positive integer $K \leq e + 1$ such that $K = 2$ when $e = 1$, and there are $3 \times 3$ integer matrices $R_1, \ldots, R_K$ with determinants $\det(R_k) = p^{\mu_k}$, such that the following properties hold. Firstly, $\mu_k \leq e$ is a non-negative integer
with $\mu_k \equiv e \pmod{3}$ for each $k \leq K$. Secondly, the form

$$p^{-(e+2\mu_k)/3} q(R_k x)$$

has integer coefficients and has determinant $p^{-e} \Delta(q)$. Thirdly, if $q(x)$ vanishes for some primitive $x \in \mathbb{Z}^3$, then there is exactly one index $k \leq K$ for which $R_k^{-1}x \in \mathbb{Z}^3$.

Proof. Clearly the form (14) has determinant $p^{-e} \Delta(q)$. The proof of the lemma will be by induction on $e$. When $e = 0$ we have $K = 1$ and $R_1$ can be taken to be the identity. To handle the induction step we assume that the lemma holds for exponents strictly less than $e$. Suppose firstly that the form $q$ is identically divisible by $p$, so that $e \geq 3$. Write $q'(x) = p^{-1} q(x)$, whence $p^{e-3} \parallel \Delta(q')$. By the induction assumption we have matrices $R_1', \ldots, R_J'$ with $J \leq (e-3) + 1 = e - 2$, and exponents $\mu_k' \leq e - 3 \leq e$. We now claim that we can take $K = J \leq e + 1$ and $R_k = R_k'$ for every index $k$, so that $\mu_k = \mu_k'$. In the first place we have

$$\mu_k = \mu_k' \equiv e - 3 \equiv e \pmod{3}.$$  

Secondly,

$$p^{-(e+2\mu_k)} q(R_k x) = p^{-(e-3+2\mu_k')/3} q'(R_k' x),$$

which is an integral form. Thirdly, if $q(x) = 0$ for some primitive $x \in \mathbb{Z}^3$, then $q'(x) = 0$, whence there is exactly one index for which $R_k^{-1}x$ is integral. Thus there is exactly one index for which $R_k^{-1}x$ is integral.

When $q(x)$ is not identically divisible by $p$ we apply Lemma 2 and consider separately the three possible cases. Suppose firstly that $q(Mx) \equiv \kappa x_3^2 \pmod{p}$ with $p \nmid \kappa$. In this case we must have $e \geq 2$. Then if $M' = \text{Diag}(1,1,p)$ the form $q'(x) = p^{-1} q(MM'x)$ will be integral, with determinant $p^{-e} \Delta(q)$.

Moreover, it is still isotropic, so that we may apply the induction hypothesis to $q'$, with $p^{e-1} \parallel \Delta(q')$. This produces matrices $R_1', \ldots, R_J'$ with $J \leq e$, and exponents $\mu_k' \leq e - 1$ such that $\det(R_k') = p^{\mu_k}$. We now claim that we can take $K = J$ and $R_k = MM'R_k'$ in the lemma. This will have determinant $p^{\mu_k}$ with $\mu_k = 1 + \mu_k' \equiv 1 + (e - 1) = e \pmod{3}$, as required. Moreover,

$$p^{-(e+2\mu_k)/3} q(R_k x) = p^{-(e+2\mu_k)/3} pq'(R_k' x) = p^{-(e-1+2\mu_k')/3} q'(R_k' x),$$

which is an integral form, by the induction hypothesis. Finally, when $q(x) = 0$ with a primitive $x \in \mathbb{Z}^3$, we set $y = M^{-1}x$, so that

$$0 = q(My) \equiv \kappa y_3^2 \pmod{p}$$

with $p \nmid \kappa$. Then $p \mid y_3$, whence $MM'y \in \mathbb{Z}^3$. It follows that the vector $z = (MM')^{-1}x$ is integral, and is primitive since $x = MM'z$ is primitive. Moreover, $q'(z) = p^{-1} q(x) = 0$, whence the induction hypothesis shows that there is exactly one $R_k'$ for which $R_k^{-1}z \in \mathbb{Z}^3$. Hence there is exactly one
index $k$ such that $R_k^{-1}x \in \mathbb{Z}^3$. This completes the proof of Lemma 3 when we are in case (i) of Lemma 2.

We turn next to case (ii) of Lemma 2, in which

$$q(Mx) \equiv x_1x_2 + \kappa p^e x_3^2 \pmod{p^{e+1}}$$

for some integer $\kappa$ coprime to $p$. We claim that we may take $K = e + 1$ in Lemma 3, and

$$R_k = M \text{Diag}(p^{k-1}, p^{e+1-k}, 1) \quad \text{for } 1 \leq k \leq K,$$

so that $\mu_k = e$ for every $k$. With this choice we have

$$q(R_kx) \equiv p^e(x_1x_2 + \kappa x_3^2) \pmod{p^{e+1}},$$

so that $p^{-e}q(R_kx)$ is integral. Suppose now that $q(x) = 0$ with $x$ primitive, and write $y = M^{-1}x$, so that

$$(15) \quad 0 = q(My) \equiv y_1y_2 + \kappa p^e y_3^2 \pmod{p^{e+1}}.$$ 

It follows that $p^e \mid y_1y_2$, whence there is a positive integer $k \leq e + 1$ such that $p^{k-1} \mid y_1$ and $p^{e+1-k} \mid y_2$. We then see that $y$ lies in the image of $\text{Diag}(p^{k-1}, p^{e+1-k}, 1)$, so that $x \in R_k(\mathbb{Z}^3)$. Finally, if we also have $x \in R_j(\mathbb{Z}^3)$ for some $j > k$ then $p^{j-1} \mid y_1$, whence $p^k \mid y_1$. Since $p^{e+1-k} \mid y_2$ it would follow firstly that $p^{e+1} \mid y_1y_2$, and secondly that $p \mid y_2$, since $e + 1 - k > e + 1 - j \geq 0$. However, when $p^{e+1} \mid y_1y_2$ the congruence (15) shows that $p \mid y_3$. We then reach a contradiction, since $p$ cannot divide $y$ when $x$ is primitive. This completes the proof of Lemma 3 when we are in case (ii) of Lemma 2.

Finally, we examine case (iii) of Lemma 2, in which

$$q(Mx) \equiv q_1(x_1, x_2) + \kappa p^e x_3^2 \pmod{p^{e+1}}$$

for some integer $\kappa$ coprime to $p$, with $q_1$ irreducible modulo $p$. One sees that if $q$ is isotropic we must have $e \geq 2$. The argument is now similar to that for case (i). Let $M' = \text{Diag}(p, p, 1)$. Then the form $p^{-e}q(MM'x)$ will be integral with determinant $p^{-2}\Delta(q)$. This will have corresponding matrices $R'_k$ with determinant $p^{\mu'_k}$, and we may take $R_k = MM'R'_k$ with corresponding value $\mu_k = \mu'_k + 2$. We leave the reader to verify that these fulfil the conditions for Lemma 3. This completes the proof of the argument. $\blacksquare$

Proof of Theorem 1. We will use induction on the number of distinct prime divisors of $\Delta$. We therefore begin by considering the case in which $\Delta = 1$. Here we will have $K = 1$, and we claim that we may take $D_1 = 1$. Since $q(x)$ is isotropic there is a primitive integer vector $z$ such that $q(z) = 0$. We may then construct a unimodular integer matrix $M$ with first column $z$. This produces a form $q(Mx)$ taking the shape $x_1(ax_2 + bx_3) + q_1(x_2, x_3)$. The coefficients $a$ and $b$ must be coprime, since $\Delta = 1$. A further unimodular transformation involving $x_2$ and $x_3$ produces $x_1x_3 + q_2(x_2, x_3)$, say. Now we replace $x_1$ by $x_1 + Ax_2 + Bx_3$ for suitable integers $A, B$ to obtain a form
\(x_1x_3 + \lambda x_2^2\). Since the determinant is still \(\Delta(q) = 1\) we see that \(\lambda = -1\). Thus \(q\) is transformed into \(x_1x_3 - x_2^2\) by a unimodular integer matrix, as required.

Now suppose that \(p^e \parallel \Delta\). Lemma \[3\] produces matrices \(R_1, \ldots, R_K\) with corresponding exponents \(\mu_k\) such that the forms

\[q_k(x) := p^{-(e+2\mu_k)/3} q(R_kx)\]

have determinant \(p^{-e}\Delta\). Our induction hypothesis, applied to \(q_k\), now produces further matrices \(M_{1,k}, \ldots, M_{J,k}\) with \(J = J(k) \leq \tau(p^{-e}\Delta)\). Since the index \(k\) runs up to \(e+1\) at most, there are at most

\[(e+1)\tau(p^{-e}\Delta) = \tau(\Delta)\]

matrices in total. Moreover, if \(e = 1\) the index \(k\) takes exactly the two values 1 and 2. We now claim that the matrices \(R_kM_{j,k}\) have the required properties. Firstly, \(K\) is increased by a factor 2 for each prime factor \(p \parallel \Delta\), so that \(K = \tau(\Delta)\) if \(\Delta\) is square-free. Secondly, \(\det(M_{j,k})\) divides \(p^{-e}\Delta\) by the induction hypothesis, and since \(\det(R_k) \mid p^e\) it follows that \(\det(R_kM_{j,k}) \mid \Delta\), as required. Thirdly, we observe that

\[q(R_kM_{j,k}x) = p^{(e+2\mu_k)/3} q_k(M_{j,k}x) = D_{j,k}J(x)\]

for a suitable integer \(D_{j,k}\), as required for part (iv).

For part (iii), let \(q(x) = 0\) for some primitive \(x \in \mathbb{Z}^3\). Then, according to Lemma \[3\] there is an index \(k\) for which \(R_k^{-1}x \in \mathbb{Z}^3\). Moreover, if we write \(y = R_k^{-1}x\) then \(y\) must be a primitive integer vector, and \(q_k(y) = 0\). Then, by the induction assumption there is a choice of \(j\) such that \(M_{j,k}^{-1}y \in \mathbb{Z}^3\). Thus \((R_kM_{j,k})^{-1}x \in \mathbb{Z}^3\). Finally, if we also have \((R_hM_{i,h})^{-1}x \in \mathbb{Z}^3\) we may write \((R_kM_{j,k})^{-1}x = u \in \mathbb{Z}^3\) and \((R_hM_{i,h})^{-1}x = v \in \mathbb{Z}^3\). Then \(R_k^{-1}x = M_{j,k}u\) and \(R_h^{-1}x = M_{i,h}v\) are both integral. According to Lemma \[3\] we must therefore have \(h = k\). Thus \(R_h^{-1}x = R_k^{-1}x = y\), and both \(M_{i,h}^{-1}y = M_{i,k}^{-1}y = v\) and \(M_{j,k}^{-1}y = u\) are integral. Our induction hypothesis then shows that we must have \(i = j\), so that there is exactly one choice of \(k\) and \(j\) for which \((R_kM_{j,k})^{-1}x\) lies in \(\mathbb{Z}^3\).

To handle the remaining claims of the theorem we do not use the induction argument. Given \[4\], we obtain the relation

\[\Delta \det(M_k)^2 = D_k^3\]

by taking determinants. Since \(\det(M_k) \mid \Delta\) we have

\[\Delta \det(M_k)^2 \mid \Delta^3 \quad \text{and} \quad \det(M_k)^3 \mid \Delta \det(M_k)^2,\]

so that \(D_k^3 \mid \Delta^3\) and \(\det(M_k)^3 \mid D_k^3\). Next we write \(x = \text{Adj}(M_k)y\) in \[4\] and note that \(\text{Adj}(M_k) = \det(M_k)M_k^{-1}\). This yields

\[\det(M_k)^2q(y) = D_kJ(\text{Adj}(M_k)y),\]
whence $D_k \mid \det(M_k)^2$, since the form $q$ was assumed to be primitive. This establishes part (v).

For part (vi) we see that if $p \mid \Delta$ then $p \mid D_k$, and hence $p \mid \det(M_k)$. Moreover, if $p^e \mid \Delta$ and $p^f \mid \det(M_k)$, then $3 \mid e + 2f$, since $\Delta \det(M_k)^2$ is a cube.

We then see that we must $e = f$ whenever $e \leq 3$. Finally, if $\Delta$ is cube-free then so is $\det(M_k)$, whence the entries of $M_k$ can have no common factor. Any vector $M_k(u^2, uv, v^2)$ will be a zero of $q$, so we need to find integers $u, v$ for which $M_k(u^2, uv, v^2)$ is primitive. If $p$ is a prime not dividing $\Delta$ then $M_k$ is invertible modulo $p$ so that $p \nmid M_k(u^2, uv, v^2)$ whenever $p \nmid (u, v)$.

Otherwise $p$ can divide at most two columns of $M_k$. If $p$ does not divide the first column then $p \nmid M_k(1, 0, 0)$. Similarly if $p$ does not divide the third column of $M_k$ then $p \nmid M_k(0, 0, 1)$. Finally, if $p$ divides the first and third columns but not the second, then $p \nmid M_k(1, 1, 1)$.

It follows, via the Chinese Remainder Theorem, that if $(u, v)$ lies in a suitable residue class modulo $\Delta$ then the vector $M_k(u^2, uv, v^2)$ will be coprime to $\Delta$. One can now show via the standard arguments that the set of integer pairs $u, v$ in such a residue class for which $u$ and $v$ are coprime will have positive density, given by

$$\Delta^{-2} \prod_{p \mid \Delta} (1 - p^{-2}).$$

We therefore obtain infinitely many pairs $u, v$ for which $M_k(u^2, uv, v^2)$ is primitive.

This completes the proof of the theorem. ■

3. Proof of Theorem 4

We begin with the following informal observation. If the coefficients of $A$ are very large compared with those of $q$, then $J(Ax) = q(x)$ has coefficients which are much smaller than they might be, so that $J(Ax)$ “nearly vanishes”. If $A$ has rows $a, b, c$, then

$$J(Ax) = (a.x)(c.x) - (b.x)^2,$$

so that $(a.x)(c.x)$ is approximately equal to $(b.x)^2$. If in fact they were identically equal, the linear forms $a.x$, $b.x$ and $c.x$ would have to be proportional, and so the vectors $a$, $b$ and $c$ would also be proportional.

Our next lemma confirms this, in a quantitative way.

**Lemma 4.** Suppose $A$ has rows $a, b, c$, and $\|a\| \leq \|c\|$. Write $a = \lambda c + d$ and $b = \mu c + e$, where $d$ and $e$ are orthogonal to $c$. Then if $J(Ax) = q(x)$ we have

(i) $\|e\| \leq 2^{-1/2}\|q\|^{1/2}$;
(ii) $|\lambda - \mu^2| \leq \|q\|/(2\|c\|^2)$;
(iii) $\|d - 2\mu e\| \leq \|q\|/\|c\|$
(iv) $|\lambda| \leq 1$. 
The reader should note that the bounds (i)–(iii) above imply that

$$|q(x)| = |(\mathbf{a}.x)(\mathbf{c}.x) - (\mathbf{b}.x)^2| \leq 2\|q\| \cdot \|x\|^2.$$ 

Thus they ensure that $q(x)$ has the expected order of magnitude, irrespective of the size of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

**Proof of Lemma 4.** We begin by observing that in general

$$\|Mx\| \leq \|M\| \cdot \|x\|$$

for any $3 \times 3$ matrix $M$, whence

(16)  

$$|q(x)| \leq \frac{1}{2}\|x\| \cdot \|Qx\| \leq \frac{1}{2}\|Q\| \cdot \|x\|^2 = \frac{1}{2}\|q\| \cdot \|x\|^2.$$ 

Taking $x = \mathbf{e}$ we have $|q(\mathbf{e})| \leq \frac{1}{2}\|q\| \cdot \|\mathbf{e}\|^2$. Moreover,

$$q(\mathbf{e}) = (\mathbf{a}.\mathbf{e})(\mathbf{c}.\mathbf{e}) - (\mathbf{b}.\mathbf{e})^2 = -(\mathbf{b}.\mathbf{e})^2,$$

since $\mathbf{e}$ and $\mathbf{c}$ are orthogonal. However, $\mathbf{b}.\mathbf{e} = \|\mathbf{e}\|^2$, again since $\mathbf{e}$ and $\mathbf{c}$ are orthogonal. It follows that

$$\|\mathbf{e}\|^4 = |q(\mathbf{e})| \leq \frac{1}{2}\|q\| \cdot \|\mathbf{e}\|^2,$$

and hence $\|\mathbf{e}\|^2 \leq \frac{1}{2}\|q\|$. The first claim of the lemma then follows.

Alternatively we may take $x = \mathbf{c}$ in (16). Here we have

$$q(\mathbf{c}) = (\mathbf{a}.\mathbf{c})(\mathbf{c}.\mathbf{c}) - (\mathbf{b}.\mathbf{c})^2 = \lambda\|\mathbf{c}\|^4 - \mu^2\|\mathbf{c}\|^4,$$

whence (16) yields

$$|\lambda - \mu^2| \cdot \|\mathbf{c}\|^4 \leq \frac{1}{2}\|q\| \cdot \|\mathbf{c}\|^2.$$ 

This gives us the second assertion of the lemma.

Thirdly we consider $\mathbf{c}Qf^T$, where $f = \mathbf{d} - 2\mu \mathbf{e}$. Recalling that $\mathbf{a}$ etc. are row vectors, we have

$$\mathbf{c}Qf^T = \mathbf{c}A^T \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{array} \right) Af^T = (\mathbf{c}.\mathbf{c})(\mathbf{a}.f) - 2(\mathbf{b}.\mathbf{c})(\mathbf{b}.f) + (\mathbf{a}.\mathbf{c})(\mathbf{c}.f).$$

However, $\mathbf{b}.\mathbf{c} = \mu\|\mathbf{c}\|^2$, and since $\mathbf{f}$ is orthogonal to $\mathbf{c}$ we have $\mathbf{a}.\mathbf{f} = \mathbf{d}.\mathbf{f}$, $\mathbf{b}.\mathbf{f} = \mathbf{e}.\mathbf{f}$ and $\mathbf{c}.\mathbf{f} = 0$, so that

$$\mathbf{c}Qf^T = \{(\mathbf{d}.\mathbf{f}) - 2\mu(\mathbf{e}.\mathbf{f})\}\|\mathbf{c}\|^2 = \|\mathbf{f}\|^2\|\mathbf{c}\|^2.$$ 

On the other hand,

$$|\mathbf{c}Qf^T| \leq \|\mathbf{c}\| \cdot \|\mathbf{Q}f^T\| \leq \|\mathbf{c}\| \cdot \|\mathbf{f}\| \cdot \|Q\| = \|\mathbf{c}\| \cdot \|\mathbf{f}\| \cdot \|q\|.$$ 

Thus $\|\mathbf{f}\| \leq \|q\|/\|\mathbf{c}\|$ as in the third claim of the lemma.

The final part is merely a trivial consequence of our initial assumption that $\|\mathbf{a}\| \leq \|\mathbf{c}\|$.

We are now ready to prove Theorem 4. Suppose we have found a matrix $UA$ with rows $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $\|\mathbf{a}\| \cdot \|\mathbf{c}\|$ is minimal. Since $J(UA\mathbf{x}) = \mathbf{d}$.
$J(Ax) = q(x)$ we may apply Lemma 4 to $UA$. Premultiplying $UA$ by

$$U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \text{Aut}(J)$$

if necessary we may assume that $\|a\| \leq \|c\|$. Similarly, premultiplying by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Aut}(J)$$

if necessary, we may assume that $\mu \geq 0$, in the notation of Lemma 4.

We begin the proof by observing that it suffices to show that we have $\|a\| \cdot \|c\| \leq 9\|q\|$. To see this we note that the choice $x = b$ in (16) yields

$$|(a \cdot b)(c \cdot b) - \|b\|^4| \leq \frac{1}{2}\|q\| \cdot \|b\|^2,$$

whence

$$\|b\|^4 \leq \frac{1}{2}\|q\| \cdot \|b\|^2 + |(a \cdot b)(c \cdot b)| \leq \left\{ \frac{1}{2}\|q\| + \|a\| \cdot \|c\| \right\} \|b\|^2 \leq 10\|q\| \cdot \|b\|^2,$$

given that $\|a\| \cdot \|c\| \leq 9\|q\|$. This gives us the required second bound $\|b\|^2 \leq 10\|q\|$.

We also note that if $\|c\| \leq 3\sqrt{\|q\|}$ then

$$\|a\| \cdot \|c\| \leq \|c\|^2 \leq 9\|q\|,$$

since we are assuming that $\|a\| \leq \|c\|$. Thus we may suppose that

$$\|c\| \geq 3\sqrt{\|q\|}$$

for the remainder of the proof.

We now consider $U_1UA$ where

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$  

Then $U_1 \in \text{Aut}(J)$ and $U_1UA$ has rows $a, -a + b, a - 2b + c$. Since $UA$ was chosen with $\|a\| \cdot \|c\|$ minimal, we conclude that

$$\|a\| \cdot \|a - 2b + c\| \geq \|a\| \cdot \|c\|,$$

and hence

$$\|a - 2b + c\| \geq \|c\|.$$  

We now substitute $a = \lambda c + d$ and $b = \mu c + e$, yielding

$$\|((\lambda - 2\mu + 1)c + d - 2e) \geq \|c\|.$$
The distribution of rational points on conics

Thus parts (ii) and (iii) of Lemma 4 yield
\[ \|c\| \leq |\lambda - 2\mu + 1| \cdot \|c\| + \|d - 2e\| \]
\[ \leq |\mu^2 - 2\mu + 1| \cdot \|c\| + \|q\| \]
However, parts (ii) and (iv) of the lemma, along with our assumption (18), show that
\[ \mu^2 \leq |\lambda| + \|q\|/(2\|c\|^2) \leq \frac{19}{18}. \]
Since we are assuming that \( \mu \geq 0 \) we conclude that
\[ 0 \leq \mu \leq \frac{37}{36} \]
by part (i) of Lemma 4. It now follows that
\[ \mu(2 - \mu) \leq \frac{2\sqrt{\|q\|}}{\|c\|}, \]
and since \( 0 \leq \mu \leq \frac{37}{36} \) we deduce that
\[ \frac{35}{36}\mu \leq \frac{2\sqrt{\|q\|}}{\|c\|}, \quad \text{whence} \quad 0 \leq \mu \leq \frac{72}{35} \frac{\sqrt{\|q\|}}{\|c\|}. \]
From Lemma 4(ii) we now have
\[ |\lambda| \leq \frac{\|q\|^2}{2\|c\|^2} + \left( \frac{72}{35} \right)^2 \frac{\|q\|}{\|c\|^2} \leq \frac{5\|q\|}{\|c\|^2}. \]
Moreover, parts (i) and (iii) yield
\[ \|d\| \leq \|q\|/\|c\| + \sqrt{2}\mu\sqrt{\|q\|} \leq \left( 1 + \frac{72\sqrt{2}}{35} \right) \frac{\|q\|}{\|c\|} \leq 4\frac{\|q\|}{\|c\|}. \]
We therefore conclude that
\[ \|a\| \leq |\lambda| \cdot \|c\| + \|d\| \leq 9\frac{\|q\|}{\|c\|}, \]
which suffices for the theorem.

4. Deduction of Theorem 2. Theorem 2 will follow from Theorem 4. We have
\[ M_k^T Q M_k = D_k \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]
In general, \( M \text{Adj}(M) = \det(M)I \), so that
\[
\det(M_k)^2 Q = D_k A^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} A,
\]
with \( A = \text{Adj}(M_k) \). It follows that
\[
J(Ax) = \det(M_k)^2 D_k^{-1} q(x),
\]
where
\[
(20) \quad AM_k = \det(M_k)I.
\]

We may now apply Theorem 4, which provides a matrix \( U \in \text{Aut}(J) \) such that the rows \( a, b, c \) of \( U A \) satisfy
\[
\|a\| \cdot \|c\| \leq 9 \det(M_k)^2 D_k^{-1} \|q\|, \tag{21}
\]
\[
\|b\|^2 \leq 10 \det(M_k)^2 D_k^{-1} \|q\|. \tag{22}
\]

If the columns of \( M_k U^{-1} \) are \( c_1, c_2, c_3 \), and \( U_2 \) is given by (17), then the columns of \( M_k U^{-1} U_2^{-1} \) are \( c_3, -c_2, c_1 \), while the rows of \( U_2 U A \) are \( c, -b, a \). Thus we are free to replace \( U \) by \( U_2 U \) if we wish. We may therefore suppose without loss of generality that the columns of \( M_k U^{-1} \) have \( \|c_1\| \leq \|c_3\| \). We may also replace \( U \) by \( -U \), which will not affect the properties (21) and (22) or the lengths \( \|c_1\| \) and \( \|c_3\| \). Thus we may also suppose without loss of generality that \( \det(U) = +1 \).

Having suitably modified \( U \) we still have
\[
J(UAx) = \det(M_k)^2 D_k^{-1} q(x),
\]
so that
\[
(UA)^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} UA = \det(M_k)^2 D_k^{-1} Q.
\]

We now claim that we may replace \( M_k \) by \( N_k = M_k U^{-1} \) in Theorem 1. Part (i) of the theorem obviously remains true. Since \( \det(N_k) = \det(M_k) \) the second and sixth assertions of Theorem 1 are immediate. Moreover,
\[
q(N_k x) = q(M_k(U^{-1} x)) = D_k J(U^{-1} x) = D_k J(x),
\]
giving us the fourth assertion, and also the fifth since the value of \( D_k \) is the same for \( N_k \) as it was for \( M_k \). Finally, \( U^{-1}(Z^3) = Z^3 \) since \( \det(U) = 1 \), whence \( N_k(Z^3) = M_k(Z^3) \). This suffices for the third assertion of the theorem.

We proceed to consider the rows of \( N_k^{-1} \). Since \( N_k = M_k U^{-1} \), we have
\[
\text{Adj}(N_k) = \text{Adj}(U^{-1}) \text{Adj}(M_k) = UA.
\]
Thus
\[ N_k^{-1} = \text{det}(N_k)^{-1} \text{Adj}(N_k) = \text{det}(N_k)^{-1}UA. \]

Since \( \text{det}(N_k) = \text{det}(M_k) \), the first pair of inequalities in Theorem 2 now follow from (21) and (22).

To handle the columns \( c_i \) of \( N_k \) we begin with the observation that
\[ N_k = \text{det}(N_k) \text{Adj}(N_k^{-1}). \]

If the rows of \( N_k^{-1} \) are \( r_1 = u, \ r_2 = v \) and \( r_3 = w \), then the first column of \( \text{Adj}(N_k^{-1}) \) will be
\[ (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)^T, \]
and hence will have Euclidean length at most \( \|v\| \cdot \|w\|. \) It follows that
\[ \|c_1\| \leq \text{det}(N_k)\|r_2\| \cdot \|r_3\|, \]
and similarly
\[ \|c_2\| \leq \text{det}(N_k)\|r_1\| \cdot \|r_3\| \quad \text{and} \quad \|c_3\| \leq \text{det}(N_k)\|r_1\| \cdot \|r_2\|. \]
Thus (23) and the second part of (24) yield
\[ \|c_1\| \cdot \|c_3\| \leq \text{det}(N_k)^2\|r_1\| \cdot \|r_2\|. \]
so that the first inequality of (8) follows from (7). The second part of (8) is then a consequence of the fact that \( \text{det}(N_k) \parallel D_k \), as noted in Theorem 1.

To establish (9) we merely combine the first part of (24) with (7), and again use the fact that \( \text{det}(N_k) \parallel D_k \). This completes the proof of Theorem 2.

5. Proof of Theorem 3. The matrix (3) has three real eigenvalues, whose product is \( \text{det}(Q) = 2\Delta \). For any vector \( x \) we have \( \|Qx\| \leq \|Q\| \cdot \|x\| \), so that if \( \lambda \) is an eigenvalue we must have \( |\lambda| \leq \|Q\| \). Since \( \|q\| \) is defined to be \( \|Q\| \) we conclude that \( 2\Delta \leq \|q\|^3 \), giving us the required bound \( \rho(q) \geq 2 \).

If \( z_1, z_2 \in \mathbb{Z}^3 \) are linearly independent zeros of \( q \) from the same class \( C_k \), then \( q(z_1 + z_2) \) cannot vanish, since a non-singular conic cannot have three collinear zeros. However, \( z_1 + z_2 \) will be in \( M_k(\mathbb{Z}^3) \) so that we must have \( D_k \parallel q(z_1 + z_2) \) by (4). It follows that \( q(z_1 + z_2) - q(z_1) - q(z_2) \) is a non-zero multiple of \( D_k \). Recalling the definition (3) of the matrix \( Q \) of \( q \) we see that \( q(z_1 + z_2) - q(z_1) - q(z_2) = z_1^TQz_2 \). We therefore find that
\[ D_k \leq \|q(z_1 + z_2) - q(z_1) - q(z_2)\| = |z_1^TQz_2| \leq \|z_1\| \cdot \|Q\| \cdot \|z_2\|, \]
and hence \( \|z_1\| \cdot \|z_2\| \geq D_k/\|q\| \). This gives us the second assertion of the theorem.

Next, if \( c_1 \) is not a scalar multiple of \( z_1 \) we will have
\[ \|c_1\| \geq \|z_2\| \geq \|z_1\|. \]
This would lead to the inequalities
\[ D_k/\|q\| \leq \|z_1\| \cdot \|z_2\| \leq \|c_1\|^2 < \rho^{-1} \det(M_k)^2 D_k^{-2} \|q\|^2. \]
We then have a contradiction, by virtue of (5). This establishes the third
claim of the theorem.

Finally, since \( \|c_3\| \geq \|z_2\| \) we have
\[ \|c_1\| \leq 90 \frac{\det(M_k)^2 D_k^{-2} \|q\|^2}{\|c_3\|} \leq 90 \frac{\det(M_k)^2 D_k^{-2} \|q\|^2}{\|z_2\|} \]
\[ \leq 90 \frac{\det(M_k)^2 D_k^{-2} \|q\|^2}{D_k/\|q\|} \|z_1\| = 90 \frac{\|q\|^3}{\Delta} \|z_1\|, \]
by (8), (10) and (5). Similarly, since \( \|c_1\| \geq \|z_1\| \) we have
\[ \|c_3\| \leq 90 \frac{\det(M_k)^2 D_k^{-2} \|q\|^2}{\|c_1\|} \leq 90 \frac{\det(M_k)^2 D_k^{-2} \|q\|^2}{\|z_1\|} \]
\[ \leq 90 \frac{\det(M_k)^2 D_k^{-2} \|q\|^2}{D_k/\|q\|} \|z_2\| = 90 \frac{\|q\|^3}{\Delta} \|z_2\|. \]
This completes our proof of Theorem 3.

6. Preliminaries for the proof of Theorem 5. If \( M = M_k \) we see
from Theorem 1 that
\[ N(B; C_k) = \frac{1}{2} \sum_{u \in \mathbb{Z}^2} \text{primal} \ \{ w(B^{-1}Mu^2) + w(-B^{-1}Mu^2) \}, \]
where we write
\[ Mu^2 = M \begin{pmatrix} u_1^2 \\ u_1u_2 \\ u_2^2 \end{pmatrix} \quad \text{when} \quad u = (u_1, u_2), \]
for notational convenience, and we set \( w_+(x) = w(x) + w(-x) \) so that \( w_+ \)
is an even function, supported on the set \( \|x\| \leq 1 \). We then have
\[ N(B; C_k) = \frac{1}{2} \sum_{u \in \mathbb{Z}^2} \text{primal} \ w_+(B^{-1}Mu^2). \]

We begin by considering the condition that \( Mu^2 \) should be primitive. Our
goal is the following result.

**Lemma 5.** The set of primes may be partitioned into sets \( P_0, P_1, P_2 \) with
the following properties. Firstly, if \( p \nmid \Delta \) then \( p \in P_0 \). Secondly, if \( p \in P_0 \)
then \( p | Mu^2 \) if and only if \( p | u \). Thirdly, if \( p \in P_n \) for \( n = 1 \) or \( 2 \) then there
are distinct lattices \( \Lambda_i(p) \subseteq \mathbb{Z}^2 \) for \( 1 \leq i \leq n \) having determinant \( p \), and
such that $p \mid Mu^2$ if and only if $u$ lies in one of the lattices $\Lambda_i(p)$. Finally, if $p \parallel \Delta$ then $p \in P_1$.

Proof. If $p$ is a prime not dividing $\det(M)$ (and in particular for any prime not dividing $\Delta$) the matrix $M$ will be invertible modulo $p$, so that $p \mid Mu^2$ if and only if $p \mid u^2$. In this case the condition that $p \mid Mu^2$ is equivalent to $p \mid u$, and $p$ will be in $P_0$. On the other hand, if $p \mid \det(M)$ then $M$ is singular modulo $p$. It cannot vanish modulo $p$, since the class $C_k$ corresponding to $M = M_k$ is assumed to be non-empty, whence $M$ has rank 1 or 2 modulo $p$.

Suppose firstly that $M$ has rank 1 over $\mathbb{F}_p$, with a non-zero row $(A, B, C)$ say. If the quadratic form $Au^2 + Buv + Cv^2$ is irreducible modulo $p$ then $p \mid Mu^2$ implies $p \mid u$. In this case $p$ will be in $P_0$. If the form $Au^2 + Buv + Cv^2$ splits into distinct factors as $L_1(u, v)L_2(u, v)$ then $p \mid Mu^2$ if and only if $u$ lies in one or both of the lattices $\Lambda_1, \Lambda_2$ given by $p \mid L_i(u, v)$. In this case $p$ will be in $P_2$. On the other hand, if $Au^2 + Buv + Cv^2$ has a repeated factor $L(u, v)^2$, then one has $p \mid Mu^2$ if and only if $u$ lies in the lattice $\Lambda$ given by $p \mid L(u, v)$, so that $p \in P_1$.

A similar analysis applies when $M$ has rank 2 over $\mathbb{F}_p$, showing in this case that the condition $p \mid Mu^2$ is either equivalent to $u \in \Lambda$ for some lattice $\Lambda \subset \mathbb{Z}^2$ of determinant $p$, or is equivalent to $p \mid u$.

Finally, suppose that $p \parallel \Delta$. Then $p \parallel \det(M)$, by part (iii) of Theorem 1, so that $M$ has rank 2 over $\mathbb{F}_p$. Using row operations one sees that there is a matrix $U \in \text{GL}_3(\mathbb{Z}_p)$ such that $UM = R$ takes one of the forms

$$R_1 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & p \end{pmatrix} \quad \text{or} \quad R_2 = \begin{pmatrix} 1 & a & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad R_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Then if $M = M_k$ and $D = D_k$ we have $p \parallel D$ by [5], and

$$J(\text{Adj}(R)x) = \det(M)^2 J(R^{-1}x) = \det(M)^2 D^{-1}q(MR^{-1}x) = \det(M)^2 D^{-1}q(U^{-1}x).$$

Since $U$ is invertible modulo $p$ we conclude that $J(\text{Adj}(R)x)$ vanishes modulo $p$. When $R = R_1$ we have

$$\text{Adj}(R_1) = \begin{pmatrix} p & 0 & -a \\ 0 & p & -b \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$J(\text{Adj}(R_1)x) = (px_1 - ax_3)x_3 - (px_2 - bx_3)^2 \equiv -(a + b^2)x_3^2 \pmod{p}.$$  

In this case we conclude that $p \mid a + b^2$. Since $U$ is invertible modulo $p$ the
condition $p \mid M\mathbf{u}^2$ is equivalent to $p \mid R\mathbf{u}^2$, and for $R = R_1$ this becomes

$$u_1^2 + au_2^2 \equiv u_1u_2 + bu_2^2 \equiv 0 \pmod{p}.$$ 

Since $a \equiv -b^2 \pmod{p}$, this holds precisely when $p \mid u_1 + bu_2$. Thus for $R = R_1$ there is a single lattice condition.

For $R = R_2$ we calculate that

$$J(\text{Adj}(R_2)x) = (px_1 - ax_2)px_3 - x_2^2,$$

which cannot vanish identically modulo $p$. This case is therefore forbidden.

When $R = R_3$ we see that $p \mid R\mathbf{u}^2$ if and only if $p \mid u_2$, which again gives us a single lattice condition. Thus whenever $p \parallel \Delta$ the condition $p \mid M\mathbf{u}^2$ gives us a single lattice condition with determinant $p$. 

Lemma 5 allows us to handle the primitiveness condition in the definition of the sum $N(B; C_k)$ as follows.

**Lemma 6.** Suppose that $w(x)$ is supported on the disc $\|x\| \leq 1$. Then there is a square-free divisor $\Delta_1\Delta_2$ of $\Delta$, and lattices $\Lambda^{(1)}, \ldots, \Lambda^{(J)}$, where

$$J = 2^{\omega(\Delta_1)}3^{\omega(\Delta_2)},$$

with the following properties. Firstly, if $p \parallel \Delta$ then $p \mid \Delta_1$; secondly, the determinant $d(\Lambda^{(j)})$ divides $\Delta_1\Delta_2^2$ for every index $j$; thirdly,

$$N(B; C_k) = \frac{1}{2} \sum_{j=1}^J \lambda(d(\Lambda^{(j)})) \sum_{d=1}^\infty \mu(d) \sum_{\substack{u \in \Lambda^{(j)} - \{0\}}} w_+(d^2B^{-1}M\mathbf{u}^2),$$

where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function; and fourthly,

$$\sum_{j=1}^J \frac{\lambda(d(\Lambda^{(j)}))}{d(\Lambda^{(j)})} \sum_{d=1}^\infty \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} \prod_{p \parallel \Delta_1} \frac{1}{1 + p^{-1}} \prod_{p \parallel \Delta_2} \frac{1 - p^{-1}}{1 + p^{-1}}.$$

**Proof.** For the proof we use the notation $1(A)$ for the characteristic function for the property $A$. We begin by observing that

$$1(p \mid M\mathbf{u}^2) = 1 - 1(p \mid \mathbf{u}), \quad p \in \mathcal{P}_0,$$

$$1(p \mid M\mathbf{u}^2) = 1 - 1(\mathbf{u} \in A_1(p)), \quad p \in \mathcal{P}_1,$$

and finally,

$$1(p \mid M\mathbf{u}^2) = 1 - 1(\mathbf{u} \in A_1(p)) - 1(\mathbf{u} \in A_2(p)) + 1(\mathbf{u} \in A_1(p) \cap A_2(p))$$

when $p \in \mathcal{P}_2$. We now take $\Delta_1$ to be the product of the primes in $\mathcal{P}_1$ and $\Delta_2$ to be the product of the primes in $\mathcal{P}_2$, so that $\Delta_1\Delta_2 \mid \Delta$. Let $\Lambda^{(j)}$ (for $1 \leq j \leq J$) run over all lattices formed by the intersection of none, some, or all, of the lattices $\Lambda_i(p)$ (for $i = 1$ or 2 and $p \in \mathcal{P}_1 \cup \mathcal{P}_2$). Then $J = 2^{\omega(\Delta_1)}3^{\omega(\Delta_2)}$, and
The distribution of rational points on conics

$$1((\Delta_1 \Delta_2, M \mathbf{u}^2) = 1) = \sum_{j=1}^{J} \lambda(d(A^{(j)})) 1(\mathbf{u} \in A^{(j)}),$$

since $A_1(p) \cap A_2(p)$ will have determinant $p^2$ when $p | \Delta_2$. The conditions $p \nmid M \mathbf{u}^2$ for primes $p \in \mathcal{P}_0$ are produced by

$$\sum_{d | \mathbf{u}} \mu(d),$$

so that

$$N(B; C_k) = \frac{1}{2} \sum_{\mathbf{u} \in \mathbb{Z}^2, M \mathbf{u}^2 \text{ primitive}} w_+ \left( \frac{M}{B} \mathbf{u}^2 \right) = \frac{1}{2} \sum_{\mathbf{u} \in \mathbb{Z}^2 - \{0\}, M \mathbf{u}^2 \text{ primitive}} w_+ \left( \frac{M}{B} \mathbf{u}^2 \right)$$

$$= \frac{1}{2} \sum_{j=1}^{J} \lambda(d(A^{(j)})) \sum_{d=1}^{\infty} \mu(d) \sum_{\mathbf{u} \in A^{(j)} - \{0\}} w_+ \left( \frac{M}{B} \mathbf{u}^2 \right)$$

$$= \frac{1}{2} \sum_{d | \mathbf{u}} \mu(d) \sum_{\mathbf{u} \in A^{(j)} - \{0\}} d \left( \frac{M}{B} \mathbf{u}^2 \right)$$

as required. Here we should note that the $d$-summation is finite for all relevant $\mathbf{u}$.

The final part is clear, by multiplicativity. 

In light of Lemma 6 our focus moves to sums of the shape

$$S(\Lambda, B, M_k) = \sum_{\mathbf{x} \in \Lambda - \{0\}} w_+(B^{-1}M_k \mathbf{x}^2),$$

where $\Lambda$ is an integer lattice, $w_+$ is an even weight supported in the disc $||\mathbf{x}|| \leq 1$, and $M_k$ is an integer matrix of the shape described in Theorems 1 and 2. We first need to understand the range of summation in $S(\Lambda, B, M_k)$.

**Lemma 7.** Let $M$ be one of the matrices $M_k$, as described in Theorem 2. Let $X_1 = \sqrt{B||\mathbf{r}_1||}$ and $X_2 = \sqrt{B||\mathbf{r}_3||}$, so that

$$X_1 X_2 \leq 3BD_k^{-1/2}||q||^{1/2}.$$  

Then if $w_+(B^{-1}M \mathbf{x}^2) \neq 0$ with $\mathbf{x} \in \mathbb{R}^2$ we have both $|x_1| \leq X_1$ and $|x_2| \leq X_2$. Moreover, if $w_+(B^{-1}M \mathbf{x}^2) \neq 0$ with $\mathbf{x} \in \mathbb{Z}^2$ we have $B \geq 1$.

**Proof.** We set $B^{-1}M \mathbf{x}^2 = \mathbf{y}$, so that $||\mathbf{y}|| \leq 1$ if $w_+(B^{-1}M \mathbf{x}^2) \neq 0$. If $M^{-1}$ has rows $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, as in Theorem 2 then

$$|(M^{-1} \mathbf{y})_1| = |\mathbf{r}_1^T \mathbf{y}| \leq ||\mathbf{r}_1|| \cdot ||\mathbf{y}|| \leq ||\mathbf{r}_1||,$$
so that $x_1^2 \leq B\|r_1\|$. Similarly $x_2^2 \leq B\|r_3\|$, and the first result follows. If $\mathbf{x} \in \mathbb{Z}^2 - \{\mathbf{0}\}$ with $w_+(B^{-1}M\mathbf{x}^2) \neq 0$ we have $B^{-1}\|M\mathbf{x}^2\| \leq 1$. Since $\mathbf{x}^2$ does not vanish we see that $M\mathbf{x}^2$ must be a non-zero integer vector, since $M$ is non-singular. It follows that $\|M\mathbf{x}^2\| \geq 1$, whence $B \geq 1$ as claimed. ■

We now give a crude bound for $S(A, B, M_k)$.

**Lemma 8.** We have $S(A, B, M_k) = 0$ if $B < 1$, and otherwise

$$S(A, B, M_k) \ll_w D_k^{-1/2} \|q\|^{1/2} \left\{ \frac{B}{d(A)} + B^{1/2}\|c_3\|^{1/2} \right\}.$$  

*Proof.* The first claim is obvious, given Lemma 7. Generally,

$$S(A, B, M_k) \ll_w \# \{ \mathbf{x} \in A : |x_1| \leq X_1, |x_2| \leq X_2 \}.$$  

If we set

$$A_0 = \{ (x_1/X_1, x_2/X_2) : (x_1, x_2) \in A \},$$

then $d(A_0) = d(A)/X_1X_2$, and

$$S(A, B, M_k) \ll_w \# \{ \mathbf{y} \in A_0 : \|y\|_\infty \leq 1 \}.$$  

Thus

$$S(A, B, M_k) \ll_w d(A_0)^{-1} + \lambda_1^{-1} + 1 \ll X_1X_2d(A)^{-1} + \lambda_1^{-1} + 1,$$

where $\lambda_1$ is the length of the shortest non-zero vector in $A_0$. However, one has $\|\mathbf{x}\| \geq 1$ for every non-zero vector in $A$, and hence

$$\lambda_1 \geq \text{max}(X_1, X_2)^{-1}.$$  

We therefore obtain the bound

$$S(A, B, M_k) \ll_w \frac{X_1X_2}{d(A)} + \text{max}(X_1, X_2) + 1.$$  

If $\text{max}(X_1, X_2) \leq \frac{1}{2}$ and $\mathbf{x}$ is an integer vector for which $w_+(B^{-1}M\mathbf{x}^2)$ is non-zero, then we must have $\mathbf{x} = \mathbf{0}$, which is excluded from the sum $S(A, B, M_k)$. Thus $S(A, B, M_k) = 0$ when $\text{max}(X_1, X_2) \leq \frac{1}{2}$. It therefore follows that

$$S(A, B, M_k) \ll_w \frac{X_1X_2}{d(A)} + \text{max}(X_1, X_2)$$

$$\ll_w D_k^{-1/2} \|q\|^{1/2} \left\{ \frac{B}{d(A)} + B^{1/2}\max(\|r_1\|, \|r_3\|)^{1/2} \right\}.$$  

We now claim that

$$\|r_1\| \ll \frac{\|q\|}{D_k}\|c_3\| \quad \text{and} \quad \|r_3\| \ll \frac{\|q\|}{D_k}\|c_1\|.$$  

Clearly this will suffice for the lemma, since we have chosen $M_k$ so that $\|c_1\| \leq \|c_3\|$.  

\[ (25) \]
To prove (25) we begin with the observation that the scalar product $r_1 \cdot c_1$ takes the value 1, since $M^{-1}M = I$. Similarly we have $r_3 \cdot c_3 = 1$. It follows that
\[(26) \quad \|r_1\| \cdot \|c_1\| \geq 1 \quad \text{and} \quad \|r_3\| \cdot \|c_3\| \geq 1.\]
Thus
\[\|r_1\| \leq \|r_1\| \cdot \|r_3\| \cdot \|c_3\|,\]
so that the first part of (25) follows from (7). The second part may be proved entirely analogously.

7. Theorem 5 The leading term. To estimate $S(\Lambda, B, M_k)$ we will use the following form of the Poisson summation formula.

**Lemma 9.** Let $N \in \text{GL}_2(\mathbb{R})$, so that $\Lambda = N(\mathbb{Z}^2)$ is a two-dimensional lattice. Then
\[S(\Lambda, B, M_k) + w_+(0) = d(A)^{-1} \sum_{a \in \mathbb{Z}^2} I(a, M, N)\]
with
\[I(a, M, N) = \int_{\mathbb{R}^2} w_+(B^{-1}Mx^2)e(-a^T N^{-T}x) \, dx_1 \, dx_2.\]

**Proof.** Writing $\varpi(x) = w_+(B^{-1}Mx^2)$, the Poisson summation formula yields
\[S(\Lambda, B, M_k) + w_+(0) = \sum_{y \in \mathbb{Z}^2} \varpi(Ny) = \sum_{a \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \varpi(Nz)e(-a^T z) \, dz_1 \, dz_2.\]
If we substitute $x = Nz$ we have $a^T z = a^T N^{-T}x$, and the result follows since $|\det(N)| = d(A)$.

The main term in Theorem 5 will come from the integral with $a = 0$.

**Lemma 10.** Define $K_T(t)$ as in (13). Then if $q(Mx) = DJ(x)$ as in Theorem 1, we have
\[\int_{\mathbb{R}^3} w(y)K_T(q(y)) \, dy_1 \, dy_2 \, dy_3 \to \frac{D^{1/2}}{\Delta^{1/2}} \int_{\mathbb{R}^2} w_+(Mx^2) \, dx_1 \, dx_2\]
as $T \to \infty$, and hence
\[\int_{\mathbb{R}^2} w_+(B^{-1}Mx^2) \, dx_1 \, dx_2 = \sigma_\infty(q; w) \frac{\Delta^{1/2}}{D^{1/2}} B,\]
where $\sigma_\infty(q; w)$ is given by (12).

**Proof.** Since
\[\int_{\mathbb{R}^3} w(y)K_T(q(y)) \, dy_1 \, dy_2 \, dy_3 = \int_{\mathbb{R}^3} w(-y)K_T(q(y)) \, dy_1 \, dy_2 \, dy_3\]
we have
\[ \sigma_{\infty}(q; w) = \frac{1}{2} \lim_{T \to \infty} \int_{\mathbb{R}^3} w_+(y) K_T(q(y)) \, dy_1 \, dy_2 \, dy_3. \]

Then, writing \( \tilde{w}(x) = w_+(D^{-1/2}Mx) \) we have
\[
\int_{\mathbb{R}^3} w_+(y) K_T(q(y)) \, dy_1 \, dy_2 \, dy_3 = \det(M) \int_{\mathbb{R}^3} w_+(Mz) K_T(q(Mz)) \, dz_1 \, dz_2 \, dz_3
= \det(M) \int_{\mathbb{R}^3} w_+(Mz) K_T(DJ(z)) \, dz_1 \, dz_2 \, dz_3
= \frac{\det(M)}{D^{3/2}} \int_{\mathbb{R}^3} \tilde{w}(x) K_T(J(x)) \, dx_1 \, dx_2 \, dx_3
= \Delta^{-1/2} \int_{\mathbb{R}^3} \tilde{w}(x) K_T(J(x)) \, dx_1 \, dx_2 \, dx_3.
\]
The function \( \tilde{w} \) will be even, so that the above becomes
\[ \frac{2}{\Delta^{1/2}} \int_0^{\infty} \int_{\mathbb{R}^2} \tilde{w}(x) K_T(J(x)) \, dx_1 \, dx_2 \, dx_3. \]

Since \( w_+(y) \) is supported on the set \( ||y|| \leq 1 \) we see that \( \tilde{w}(z) \) is supported on a subset of \( [-C, C]^3 \) for some \( C = C(M, D) > 0 \). If we write \( x_0 = x_2^2/x_3 \) and \( x_1 = x_0 + u \), then both \( \tilde{w}(x) \) and \( \tilde{w}(x_0, x_2, x_3) \) vanish unless \( |x_2| \leq C \) and \( x_3 \leq C \). If \( \phi \) is the function
\[ \phi(u, x_2, x_3) = \tilde{w}(x_0 + u, x_2, x_3) - \tilde{w}(x_0, x_2, x_3) \]
it follows that
\[
\int_0^{\infty} \int_{\mathbb{R}^2} \{ \tilde{w}(x) - \tilde{w}(x_0, x_2, x_3) \} K_T(J(x)) \, dx_1 \, dx_2 \, dx_3
= \int_0^{\infty} \int_{-C}^{C} \int_{-\infty}^{\infty} \phi(u, x_2, x_3) K_T(x_3 u) \, du \, dx_2 \, dx_3
\ll T \int_0^{\infty} \int_{-C}^{C} \int_{|u| \leq 1/T x_3} |\phi(u, x_2, x_3)| \, du \, dx_2 \, dx_3.
\]
We may assume that \( \tilde{w}(x_0 + u, x_2, x_3) \) and \( \tilde{w}(x_0, x_2, x_3) \) do not both vanish, and hence that either \( |x_0| \leq C \) or \( |x_0 + u| \leq C \) (or both). In this case
\[ \phi(u, x_2, x_3) \ll_{w, M} \min(1, |u|). \]

We proceed to consider separately the ranges \( |x_0| \geq 2C \) and \( |x_0| \leq 2C \). When \( |x_0| \geq 2C \) the bound \( |x_0 + u| \leq C \) implies that \( |u| \geq x_0/2 \), and
The distribution of rational points on conics

since $|u| \leq 1/(Tx_3)$ we conclude that $x_2^2 \leq 2/T$. The bound $|x_0| \geq 2C$ then shows that $x_3 \leq x_2^2/(2C) \leq 1/(CT)$. Moreover, $u$ is restricted to a range $|x_0 + u| \leq C$ of length $O_M(1)$. The estimate in (29) is $O_{w,M}(1)$, so that the corresponding contribution to (28) is

$$\ll_{w,M} T \int_0^{1/(CT)} dx_2 dx_3 \ll_{w,M} T^{-1/2}. $$

On the other hand, when $|x_0| \leq 2C$ we have $x_2^2 \leq 2Cx_3$ and

$$\int_{|u| \leq 1/(Tx_3)} \min(1, |u|) du \ll (Tx_3)^{-1} \min(1, 1/(Tx_3)),$$

so that the corresponding contribution to (28) is

$$\ll_{w,M} \int_0^{C} x_3^{-1/2} \min(1, 1/(Tx_3)) dx_3 \ll_{w,M} T^{-1/2}. $$

We therefore conclude that

$$\int_{\mathbb{R}^3} w_+(y)K_T(q(y))dy_1 dy_2 dy_3 = \frac{2}{\Delta^{1/2}} \int_0^\infty \int_{\mathbb{R}^2} \hat{w}(x_0, x_2, x_3)K_T(J(x))dx_1 dx_2 dx_3 + O_{w,M}(T^{-1/2}). $$

The function $\hat{w}(x_0, x_2, x_3)$ is independent of $x_1$ and

$$\int_{\mathbb{R}} K_T(x_1 x_3 - x_2^2) dx_1 = x_3^{-1}. $$

Moreover, with the substitutions $x_3 = u_2^2$ and $x_2 = u_1 u_2$ we have

$$\int_0^\infty \int_{\mathbb{R}} \hat{w}(x_2^2/x_3, x_2, x_3)x_3^{-1} dx_2 dx_3 = 2 \int_0^\infty \int_{\mathbb{R}^2} \hat{w}(u_1^2, u_1 u_2, u_2^2)du_1 du_2 = \int_{\mathbb{R}^2} \hat{w}(u_1^2, u_1 u_2, u_2^2)du_1 du_2 = D^{1/2} \int_{\mathbb{R}^2} w_+(Mu^2)du_1 du_2,$$

and the lemma follows. ■

Our next result tells us about the size of $\sigma_\infty(q; w)$. Recall that $q$ is isotropic, and hence indefinite, with $\Delta > 0$, so that the matrix $Q$ of $q$ has one positive eigenvalue $\lambda$ say, and two negative ones $-\mu$ and $-\nu$ say. We may assume that $\mu \geq \nu (> 0)$. With this notation we have $\Delta = \frac{1}{2} \lambda \mu \nu$. We remind the reader of the notation $f \asymp g$, meaning that both $f \ll g$ and $g \ll f$ hold.
In our context the two implied constants will be absolute. Thus we will have
\[ \max(\lambda, \mu, \nu) \precsim \|q\|, \]
for example.

**Lemma 11.** If \( \sup |w(x)| \leq 1 \) we have
\[
|\sigma_\infty(q; w)| \leq 2e^{4/3}\sigma_\infty(q; w_0),
\]
where the weight \( w_0 \) is given by \([11]\). Moreover,
\[
\sigma_\infty(q; w_0) \precsim \frac{\min(\lambda, \mu, \nu)^{1/2}}{\Delta^{1/2}} \log(2\mu/\nu)
\]
when \( \lambda \geq \mu \geq \nu \),
\[
\sigma_\infty(q; w_0) \precsim \frac{\min(\lambda, \mu, \nu)^{1/2}}{\Delta^{1/2}} \log(2\lambda/\nu)
\]
when \( \mu \geq \lambda \geq \nu \), and
\[
\sigma_\infty(q; w_0) \precsim \frac{\min(\lambda, \mu, \nu)^{1/2}}{\Delta^{1/2}}
\]
when \( \mu \geq \nu \geq \lambda \).

Thus
\[
\frac{1}{\|q\|} \ll \sigma_\infty(q; w_0) \ll \rho^{1/4}/\|q\|
\]
in every case.

**Proof.** According to Lemma 10 we have
\[
|\sigma_\infty(q; w)| = \left| \frac{\Delta^{1/2}}{D^{1/2}} \int_{\mathbb{R}^2} w_+(Mx^2) \, dx_1 \, dx_2 \right| \leq 2\frac{\Delta^{1/2}}{D^{1/2}} \int_{|Mx^2| \leq 1} dx_1 \, dx_2
\]
\[
\leq 2e^{4/3}\frac{\Delta^{1/2}}{D^{1/2}} \int_{|Mx^2| \leq 1} \exp\{-1/(1-\|Mx\|^2/4)\} \, dx_1 \, dx_2
\]
\[
\leq 2e^{4/3}\frac{\Delta^{1/2}}{D^{1/2}} \int_{\mathbb{R}^2} w_0(\frac{1}{2}Mx^2) \, dx_1 \, dx_2
\]
\[
= 4e^{4/3}\frac{\Delta^{1/2}}{D^{1/2}} \int_{\mathbb{R}^2} w_0(Mx^2) \, dx_1 \, dx_2.
\]
On the other hand, applying Lemma 10 to the weight \( w_0 \) we find that
\[
\sigma_\infty(q; w_0) = 2\frac{\Delta^{1/2}}{D^{1/2}} \int_{\mathbb{R}^2} w_0(Mx^2) \, dx_1 \, dx_2.
\]
The first claim of the lemma then follows.
The distribution of rational points on conics

For the remainder of the proof it will be convenient to write

$$I(T) = \int_{\mathbb{R}^3} w_0(y) K_T(q(y)) dy_1 dy_2 dy_3.$$ 

Now let $U$ be a real orthogonal matrix diagonalising $q(x)$, so that $q(Ux) = \text{Diag}(\lambda, -\mu, -\nu)$ say. Substituting $Uy$ in place of $y$, and noting that the weight $w_0$ is invariant under rotations, we deduce that

$$I(T) = \int_{\mathbb{R}^3} w_0(y) K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) dy_1 dy_2 dy_3,$$

whence

$$I(T) \leq \int_{[-1,1]^3} K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) dy_1 dy_2 dy_3$$

and

$$I(T) \gg \int_{[-1/2,1/2]^3} K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) dy_1 dy_2 dy_3.$$

We now consider three cases. Firstly, suppose that $\lambda \geq \mu$. Then if $y$ is in $[-1,1]^3$ and $K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) \neq 0$, we have

$$\lambda y_1^2 \leq T^{-1} + \mu y_2^3 + \nu y_3^2 \leq T^{-1} + 2\mu.$$ 

It follows that $|y_1| \leq 2\sqrt{\mu/\lambda}$ as soon as $T \geq (2\mu)^{-1}$. Writing $\xi = \min(1, 2\sqrt{\mu/\lambda})$ we then deduce that

$$I(T) \leq \int_{[-1,1]^2} \int_{-\xi}^\xi K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) dy_1 dy_2 dy_3$$

= $\sqrt{\mu/\lambda} \int_{[-1,1]^2} \int_{-\xi\sqrt{\lambda/\mu}}^{\xi\sqrt{\lambda/\mu}} K_T(\mu y^2 - \mu y_2^3 - \nu y_3^2) dy dy_2 dy_3,$

on substituting $y_1 = \sqrt{\mu/\lambda} y$. Since $\xi\sqrt{\lambda/\mu} \leq 2$ we obtain

$$I(T) \leq \sqrt{\mu/\lambda} \int_{[-2,2]^3} K_T(\mu y^2 - \mu y_2^3 - \nu y_3^2) dy dy_2 dy_3$$

= $8\sqrt{\mu/\lambda} \int_{[-1,1]^3} K_T(4\mu z_1^2 - 4\mu z_2^3 - 4\nu z_3^2) dz_1 dz_2 dz_3$

= $8\sqrt{\mu/\lambda} (4\mu)^{-1} \int_{[-1,1]^3} K_{4\mu T}(z_1^2 - z_2^3 - \nu \mu^{-1} z_3^2) dz_1 dz_2 dz_3,$

for $T$ sufficiently large.
Similarly when $\lambda \geq \mu$ we have

$$I(T) \gg \int_{[-1/2,1/2]^3} K_T(\lambda y_1^2 - \mu y_2^2 - \nu y_3^2) \, dy_1 \, dy_2 \, dy_3$$

$$= \sqrt{\mu/\lambda} \int_{[-1/2,1/2]^3} \left[ \int_{-\xi \sqrt{\lambda/\mu}}^{\xi \sqrt{\lambda/\mu}} K_T(\mu y^2 - \mu y_2^2 - \nu y_3^2) \, dy \right] \, dy_2 \, dy_3.$$  

Since $\xi \sqrt{\lambda/\mu} \geq 1$ we obtain

$$I(T) \gg \int_{[-1,1]^3} K_T(\mu y^2 - \mu y_2^2 - \nu y_3^2) \, dy \, dy_2 \, dy_3$$

$$= \frac{1}{8} \sqrt{\mu/\lambda} \int_{[-1,1]^3} K_T\left(\frac{1}{4} \mu z_1^2 - \frac{1}{4} \mu z_2^3 - \frac{1}{4} \nu z_3^2\right) \, dz_1 \, dz_2 \, dz_3$$

$$= 8 \sqrt{\mu/\lambda} \left(\frac{1}{4} \mu\right)^{-1} \int_{[-1,1]^3} K_T\left(z_1^2 - z_2^3 - \frac{1}{4} \nu z_3^2\right) \, dz_1 \, dz_2 \, dz_3.$$  

Hence if we write

$$J(T; \delta) = \int_{[-1,1]^3} K_T(z_1^2 - z_2^3 - \delta z_3^2) \, dz_1 \, dz_2 \, dz_3$$

we have

$$(\lambda \mu)^{-1/2} J\left(\frac{1}{4} \mu T; \nu \mu^{-1}\right) \ll I(T) \ll (\lambda \mu)^{-1/2} J(4\mu T; \nu \mu^{-1}).$$

On taking the $\limsup$ as $T \to \infty$ this yields

$$(30) \quad \lim_{T \to \infty} I(T) \asymp (\lambda \mu)^{-1/2} \lim_{T \to \infty} \sup J(T; \nu \mu^{-1})$$

when $\lambda \geq \mu$.

Suppose next that $\mu \geq \lambda \geq \nu$. Then if $K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) \neq 0$ with $y \in [-1,1]^3$, we have

$$\mu y_2^2 \leq \mu y_2^2 + \nu y_3^2 \leq T^{-1} + \lambda y_1^2 \leq T^{-1} + \lambda.$$  

It follows that $|y_2| \leq 2 \sqrt{\lambda/\mu}$ as soon as $T \geq \lambda^{-1}$. We may then replace the range $[-1,1]$ for $y_2$ by $[-\xi, \xi]$ where $\xi = \min(1, 2 \sqrt{\lambda/\mu})$ this time. Proceeding much as before we find that

$$I(T) \leq \sqrt{\lambda/\mu} \int_{[-2,2]^3} K_T(\lambda y_1^2 - \lambda y^3 - \nu y_3^2) \, dy_1 \, dy \, dy_3$$

$$= 8 \sqrt{\lambda/\mu} (4\lambda)^{-1} \int_{[-1,1]^3} K_{4\lambda T}(z_1^2 - z_2^3 - \nu \lambda^{-1} z_3^2) \, dz_1 \, dz_2 \, dz_3.$$
The distribution of rational points on conics

and
\[ I(T) \gg (\lambda \mu)^{-1/2} \int_{[-1,1]^3} K_{\frac{1}{4} \lambda T}(z_1^2 - z_2^3 - \nu \lambda^{-1} z_3^2) \, dz_1 \, dz_2 \, dz_3. \]

Thus
\[ \lim_{T \to \infty} I(T) \asymp (\lambda \mu)^{-1/2} \limsup_{T \to \infty} J(T; \nu \lambda^{-1}) \]
when \( \mu \geq \lambda \geq \nu \).

Thirdly, we suppose that \( \lambda \leq \nu \). In this case if \( K_T(\lambda y_1^2 - \mu y_2^3 - \nu y_3^2) \neq 0 \) and \( y \in [-1,1]^3 \), we have \( |y_2| \leq 2 \sqrt{\lambda/\mu} \) and \( |y_3| \leq 2 \sqrt{\lambda/\nu} \) when \( T \geq \lambda^{-1} \).

We then replace the ranges for \( y_2 \) and \( y_3 \) by \([\xi_2, \xi_2] \) and \([\xi_3, \xi_3] \) respectively, where \( \xi_2 = \min(1, 2 \sqrt{\lambda/\mu}) \) and \( \xi_3 = \min(1, 2 \sqrt{\lambda/\nu}) \). A similar argument to the one before then shows that
\[ \lim_{T \to \infty} I(T) \asymp (\mu \nu)^{-1/2} \limsup_{T \to \infty} J(T; 1) \]
when \( \nu \geq \lambda \).

It remains to consider \( J(T; \delta) \), where \( 0 < \delta \leq 1 \). To obtain a lower bound we restrict the variables \( z_1, z_2 \) to the square given by \( |z_1 + z_2| \leq 1 \) and \( |z_1 - z_2| \leq 1 \), which lies inside the region \([-1,1]^2 \). A suitable change of variable then shows that
\[ J(T; \delta) \geq \frac{1}{2} \int_{[-1,1]^3} K_T(u_1 u_2 - \delta z_3^2) \, du_1 \, du_2 \, dz_3. \]

We now restrict \( u_1, u_2 \) further, so that \( u_1, u_2 \geq 0 \) and \( \delta/4 \leq u_1 u_2 \leq \delta/2 \). For any such \( u_1, u_2 \) and any \( T \geq 4 \delta^{-1} \) the inequality
\[ |u_1 u_2 - \delta z^2| \leq 1/(2T) \]
implies
\[ \delta z^2 \leq u_1 u_2 + (2T)^{-1} \leq \delta, \]
whence one automatically has \( z \in [-1,1] \). Moreover, it also implies
\[ \delta z^2 \geq u_1 u_2 - (2T)^{-1} \geq \delta/8, \]
whence \( |z| \geq \frac{1}{3} \), say. It follows that
\[ |u_1 u_2 - \delta z_3^2| \leq 1/(2T) \]
for an admissible set of values for \( z_3 \) of measure \( \gg (\delta T)^{-1} \). We therefore conclude that
\[ J(T; \delta) \gg \delta^{-1} \text{Meas } \{(u_1, u_2) \in [0,1]^2 : \delta/4 \leq u_1 u_2 \leq \delta/2\} \]
\[ \gg \delta^{-1} \int_{\delta/2}^{1} \frac{\delta}{4u_1} \, du_1 \gg \log(2/\delta). \]
To obtain an upper bound for $J(T; \delta)$ we extend the range of $z_1, z_2$ to the square given by $|z_1 + z_2| \leq 2$ and $|z_1 - z_2| \leq 2$. A suitable change of variable now shows that

$$J(T; \delta) \leq \frac{1}{2} \int_{[-2,2]^3} K_T(u_1 u_2 - \delta z_3^2) \, du_1 \, du_2 \, dz_3.$$ 

When $|u_1 u_2| \leq 2/T$ the integrand is only non-zero when $z_3^2 \leq 3/(\delta T)$, so that this range contributes

$$\ll T \int_{-2}^{2} \min \left(1, \frac{1}{|u_2|T} \right) (\delta T)^{-1/2} \, du_2 \ll \delta^{-1/2} \frac{\log T}{\sqrt{T}}.$$ 

The range $|u_1 u_2| \leq 2/T$ therefore makes no contribution when we let $T$ go to infinity. When $|u_1 u_2| \geq 2/T$ the integrand is only non-zero when $\sqrt{|u_1 u_2|/(2\delta)} \leq |z_3| \leq 2$. Thus the set of values for $z_3$ for which we have $|u_1 u_2 - \delta z_3^2| \leq T^{-1}$ consists of at most two intervals, having total length $O(T^{-1}(|u_1 u_2| \delta)^{-1/2})$. Moreover, the set is empty unless $|u_1 u_2| \leq 8\delta$. It follows that the corresponding contribution to $J(T; \delta)$ is

$$\ll T \int_{-2}^{2} \min \left(1, \frac{1}{|u_2|T} \right) (\delta T)^{-1/2} \, du_2 \ll \delta^{-1/2} \frac{\log T}{\sqrt{T}}.$$ 

In view of the lower bound (33) we therefore have $J(T; \delta) \asymp \log(2/\delta)$, and the second claim of Lemma 11 then follows from (30)–(32).

For the third claim of the lemma we note that

$$\min(\lambda, \mu, \nu) \geq \frac{\lambda \mu \nu}{\max(\lambda, \mu, \nu)^2} \gg \Delta \|q\|^2.$$ 

This produces the lower bound for $\sigma_{\infty}(q; w_0)$. For the upper bound we observe for example that when $\lambda \geq \mu \geq \nu$ we have

$$\min(\lambda, \mu, \nu)^{1/2} \log(2\mu/\nu) = \nu^{1/2} \log(2\mu/\nu) \ll \nu^{1/2} (\mu/\nu)^{1/4}$$

and that

$$(\mu \nu)^{1/4} \ll \Delta^{1/4} \lambda^{-1/4} \ll \Delta^{1/4} \|q\|^{-1/4}.$$ 

Thus when $\lambda \geq \mu \geq \nu$ we have

$$\sigma_{\infty}(q; w_0) \ll \Delta^{-1/4} \|q\|^{-1/4} = \rho^{1/4} \|q\|^{-1}.$$ 

When $\mu \geq \lambda \nu$ or $\mu \geq \nu \geq \lambda$ we may argue similarly. This suffices to complete the proof of the lemma. \hfill \blacksquare
8. Theorem 5 The error term. In Lemma 9 the contribution from $a \neq 0$ will produce an error term, as we now show.

Lemma 12. Let $\Lambda \subseteq \mathbb{Z}^2$ be a 2-dimensional lattice, and let $n_1, n_2$ be a basis for $\Lambda$ chosen so that $||n_1|| \cdot ||n_2|| \ll d(\Lambda)$. Then if $N = (n_1|n_2)$ we have $\Lambda = N(\mathbb{Z}^2)$. Moreover, for any integer $K \geq 0$ we have

$$I(a, M_k, N) \ll_{w, K} BD_k^{-1/2} ||q||^{1/2} \left( \frac{B_0}{B} \right)^{K/2} ||a||^{-K}$$

with

$$B_0 = \rho d(\Lambda)^2 ||c_3||.$$

Proof. Let $N^{-1}a = (b_1, b_2)^T$ and suppose that $|b_1| \geq |b_2|$, say. If we integrate by parts $K$ times with respect to $x_1$ we find that

$$I(a, M_k, N) \ll_K |b_1|^{-K} \int_{\mathbb{R}^2} \left| \frac{\partial^K}{\partial x_1^K} w(B^{-1}M_kx^2) \right| dx_1 dx_2.$$

Since $a = N(b_1, b_2)^T$ we have

$$||a|| \ll \max(||n_1||, ||n_2||) \max(|b_1|, |b_2|) \ll d(\Lambda)|b_1|,$$

whence

$$I(a, M_k, N) \ll_K ||a||^{-K} d(\Lambda)^K \int_{\mathbb{R}^2} \left| \frac{\partial^K}{\partial x_1^K} w(B^{-1}M_kx^2) \right| dx_1 dx_2.$$

We will write the components of $B^{-1}M_kx^2$ as $f_1(x_1), f_2(x_1), f_3(x_1)$, where the $f_i$ are quadratic polynomials which also involve $x_2$. Then the $K$th order partial derivatives of $w(B^{-1}M_kx^2)$ with respect to $x_1$ will be sums of various terms $T_n$. Each $T_n$ will be a product containing a single partial derivative of $w$, of order at most $K$, along with various first and second derivatives of the $f_i$. If there are $r$ first derivatives and $s$ second derivatives then $r + 2s = K$. It therefore follows that

$$\frac{\partial^K}{\partial x_1^K} w(B^{-1}M_kx^2) \ll_{w, K} F_1^r F_2^s$$

for some exponents $r, s$ with $r + 2s = K$, where

$$F_1 = \sup \{ ||f_1''(x_1)|| : w(B^{-1}M_kx^2) \neq 0 \},$$

and $F_2$ is the maximum of $|f_1'''|, |f_2'''|$ and $|f_3'''|$. The leading coefficient of $Bf_i$ will be the $i$th entry in the first column of $M_k$, so that its modulus will be at most $||c_1||$, in the notation of Theorem 2. It follows that $F_2 \leq 2||c_1||/B$. Similarly the coefficient of $x_1$ in $Bf_i$ will have modulus at most $||c_2||$, so that

$$F_1 \leq X_1||c_1||/B + X_2||c_2||/B.$$
It follows via Lemma 7 that
\[
\max_{r+2s=K} F^r F^s \leq F^K + F^{K/2}
\]
\[\ll_K (X_1\|c_1\|/B)^K + (X_2\|c_2\|/B)^K + (\|c_1\|/B)^{K/2}\]
\[\ll_K B^{-K/2}\{\|r_1\|^{K/2}\|c_1\|^{K} + \|r_3\|^{K/2}\|c_2\|^{K} + \|c_1\|^{K/2}\}\]
\[\ll_K B^{-K/2}E^{K/2}\]
with
\[
E = \|r_1\| \cdot \|c_1\|^2 + \|r_3\| \cdot \|c_2\|^2 + \|c_1\|.
\]
Lemma 7 shows that the support of \(w(B^{-1}Mx^2)\) is included in a rectangle of area \(O(BD_k^{-1/2}\|q\|^{1/2})\), and we therefore conclude that
\[
(34) \quad I(a, M_k, N) \ll_{w,K} B\|q\|^{1/2} / D_k^{1/2} \{d(A)^2B^{-1}E\}^{K/2}\|a\|^{-K}.
\]
We now claim that
\[
(35) \quad E \ll \rho\|c_1\| \ll \rho\|c_3\|
\]
when \(|b_1| \geq |b_2|\), as we are currently supposing. In the alternative case the argument is completely analogous, leading to exactly the same bound. To establish our claim we use (23), (7) and (5) to show that
\[
\|r_1\| \cdot \|c_1\|^2 \leq \det(M_k)\|r_1\| \cdot \|r_2\| \cdot \|r_3\| \cdot \|c_1\|
\]
\[\leq 9\sqrt{10}\det(M_k)\|q\|^3/2 D_k^{-3/2}\|c_1\| \ll \rho^{1/2}\|c_1\|.
\]
This is sufficient for the term \(\|r_1\| \cdot \|c_1\|^2\), since \(\rho \geq 2\) by Theorem 3. Secondly, the bounds (26), (7), (9) and (5) yield
\[
\|r_3\| \cdot \|c_2\|^2 \leq \|r_1\| \cdot \|c_1\| \cdot \|r_3\| \cdot \|c_2\|^2 \ll D_k^{-1}\|q\| \cdot \|c_1\| \cdot \|c_2\|^2
\]
\[\ll \det(M_k)^2 D_k^{-3}\|q\|^3\|c_1\| \ll \rho\|c_1\|,
\]
which is sufficient for the term \(\|r_3\| \cdot \|c_2\|^2\). Finally, since \(\rho \geq 2\) we have \(\|c_1\| \leq \rho\|c_1\|\). This gives us the required estimate (35) for \(E\), whence (34) produces the bound
\[
I(a, M_k, N) \ll_{w,K} \|a\|^{-K} B\|q\|^{1/2} / D_k^{1/2} \{d(A)^2B^{-1}\rho\|c_3\|\}^{K/2}.
\]
The lemma now follows. \(\blacksquare\)

We can now summarize the results of our analysis of \(S(A, B, M_k)\).

**Lemma 13.** We have \(S(A, B, M_k) = 0\) if \(B < 1\), and otherwise
\[
S(A, B, M_k) \ll_{w} D_k^{-1/2}\|q\|^{1/2}\left\{\frac{B}{d(A)} + B^{1/2}\|c_3\|^{1/2}\right\}.
\]
Moreover,

\begin{equation}
S(A, B, M_k) = \sigma_\infty(q; w) \frac{\Delta^{1/2}}{D_k^{1/2} d(A)} B + O_w(1)
\end{equation}

\begin{equation}
+ O_w K (D_k^{-1/2} ||q||^{1/2} B (B_0/B)^K)
\end{equation}

for any integer \( K \geq 2 \) with

\[ B_0 = \rho d(A)^2 ||c_3||. \]

**Proof.** The first half is the content of Lemma 8, while the second follows from Lemma 9, together with Lemma 10 for the term \( a = 0 \), and Lemma 12 for \( a \neq 0 \). Here we replace \( K \) by \( 2K \) and observe that

\[ \sum_{a \in \mathbb{Z}^2 \setminus \{0\}} ||a||^{-2K} \ll_K 1 \]

for \( K \geq 2 \). ■

**9. Completing the proof of Theorem 5.** To prove Theorem 5 we will apply Lemma 6, using the crude upper bound from Lemma 13 when \( d(\Lambda^{(j)}) \) or \( d \) is large, and the asymptotic estimate (36) otherwise. We therefore begin by choosing a real parameter \( d_0 \geq 1 \), which we will specify later, and noting that we can restrict attention to the range \( d \leq \sqrt{B} \), by virtue of the first clause of Lemma 13.

The contribution to \( N(B; C_k) \) from terms with \( d_0 \leq d \leq \sqrt{B} \), summed over all the lattices \( \Lambda^{(j)} \), will be

\[ \ll \omega(\Delta) \frac{||q||^{1/2}}{D_k^{1/2}} \sum_{d_0 \leq d \leq \sqrt{B}} \left\{ B d^{-2} + B^{1/2} d^{-1} ||c_3||^{1/2} \right\} \]

\[ \ll \omega(\Delta) \frac{||q||^{1/2}}{D_k^{1/2}} \left\{ B d_0^{-1} + B^{1/2} (\log B) ||c_3||^{1/2} \right\}. \]

Similarly, the contribution from terms with \( d(\Lambda^{(j)}) \geq d_0 \) and \( d \leq d_0 \) will be

\[ \ll \omega(\Delta) \frac{||q||^{1/2}}{D_k^{1/2}} \sum_{j=1}^{J} \sum_{d \leq d_0} \left\{ \frac{B d^{-2}}{d(\Lambda^{(j)})} + B^{1/2} d^{-1} ||c_3||^{1/2} \right\} \]

\[ \ll \omega(\Delta) \frac{||q||^{1/2}}{D_k^{1/2}} \left\{ B d_0^{-1} + B^{1/2} (\log B) ||c_3||^{1/2} \right\}. \]

We now examine the terms for which both \( d \leq d_0 \) and \( d(\Lambda^{(j)}) \leq d_0 \). We have
\[
\sum_{j=1}^{J} \frac{1}{d(A(j))} \sum_{d=1}^{\infty} \frac{1}{d^2} \ll 3^{\omega(\Delta)} d_0^{-1}
\]
and
\[
\sum_{j=1}^{J} \frac{1}{d(A(j))} \sum_{d \geq d_0} \frac{1}{d^2} \ll 3^{\omega(\Delta)} d_0^{-1}.
\]

Hence, when we use the asymptotic formula (36) for terms in which both \(d\) and \(d(A(j))\) are at most \(d_0\), the main term contributes
\[
\frac{1}{2} \sigma_{\infty}(q; w) \frac{\Delta^{1/2}}{D_k^{1/2}} B \{ \kappa + O(3^{\omega(\Delta)} d_0^{-1}) \},
\]
where
\[
\kappa = \sum_{j=1}^{J} \lambda(d(A(j))) \sum_{\substack{d=1 \\
(d, \Delta_1, \Delta_2) = 1}} \mu(d) d^{-2} = \frac{6}{\pi^2} \prod_{p|\Delta_1} \frac{1}{1 + p^{-1}} \prod_{p|\Delta_2} \frac{1}{1 + p^{-1}};
\]
by Lemma 6. Using the estimate from Lemma 11 we see that the \(O\)-term above contributes
\[
\ll w \ 3^{\omega(\Delta)} \rho \frac{1}{\|q\|} \frac{\Delta^{1/2}}{D_k^{1/2}} B \leq 3^{\omega(\Delta)} \rho^{-1/4} \frac{\|q\|^{1/2}}{D_k^{1/2}} \leq 3^{\omega(\Delta)} \frac{\|q\|^{1/2}}{D_k^{1/2}} B \frac{D_k^{1/2}}{d_0}.
\]

On the other hand, the error term \(O_w(1)\) in (36) contributes \(O_w(d_0^2)\), while the second error term contributes
\[
\ll w, K \ \frac{\|q\|^{1/2}}{D_k^{1/2}} B(B_1/B)^K \sum_{j=1}^{J} \sum_{\substack{d \leq d_0 \\
(d(A(j)) \leq d_0}} d(A(j))^{2K} d^{2K-2}
\]
\[
\ll w, K \ \frac{\|q\|^{1/2}}{D_k^{1/2}} B(B_1/B)^K 3^{\omega(\Delta)} d_0^{4K-1}
\]
with
\[
B_1 = \rho \|c_3\|.
\]
Thus, if we assume that \(B \geq B_1\) we may take \(d_0 = B^{1/4} B_1^{-1/4}\) and \(K = 2\), so that the total of all the above error terms is
\[
\ll w \ d_0^2 + 3^{\omega(\Delta)} \|q\|^{1/2} \left\{ \frac{B}{D_k^{1/2}} + B^{1/2}(\log B) \|c_3\|^{1/2} \right\}.
\]
In the notation of Theorem 3 we have
\[
(37) \quad \|c_3\|^2 \geq \|z_1\| \cdot \|z_2\| \geq D_k/\|q\|,
\]
whence
\[ \frac{d_0^2}{B_1^{1/2}} \ll \frac{B_1^{1/2}}{\|c_3\|^{1/2}} \ll \frac{\|q\|^{1/2}}{D_k^{1/2}} B_1^{1/2} \|c_3\|^{1/2}. \]

Thus the error term \(d_0^2\) above is dominated by the final term. Moreover,
\[ \frac{B}{d_0} = B^{3/4} B_1^{1/4} \gg B^{1/2} \max(\|c_1\|, \|c_3\|)^{1/2} \]
when \(B \geq B_1\). We therefore deduce that
\[ N(B; C_k) = \frac{1}{2} \sigma_\infty(q; w) \kappa \frac{\Delta^{1/2}}{D_k^{1/2}} B + O_w \left( 3^{\omega(\Delta)} \frac{\|q\|^{1/2}}{D_k^{1/2}} B_1^{1/4} B^{3/4} \log B \right) \]
for \(B \geq B_1\).

When \(\rho^{-1}\|c_3\| \ll B \leq B_1\) we argue as above with \(d_0 = 1\), showing that
\[ N(B; C_k) \ll_w 1 + 3^{\omega(\Delta)} \frac{\|q\|^{1/2}}{D_k^{1/2}} \{ B + B^{1/2}(\log B)\|c_3\|^{1/2} \}. \]

However, for \(B \leq B_1\) we have
\[ \sigma_\infty(q; w) \kappa \frac{\Delta^{1/2}}{D_k^{1/2}} B \ll 3^{\omega(\Delta)} \frac{\|q\|^{1/2}}{D_k^{1/2}} B_1^{1/4} B^{3/4} \]
by Lemma 11 and \(B \ll B_1^{1/4} B^{3/4}\). Moreover, \(B^{1/2}\|c_3\|^{1/2} \ll B_1^{1/4} B^{3/4}\) when \(B \gg \rho^{-1}\|c_3\|\). It follows that
\[
N(B; C_k) = \frac{1}{2} \sigma_\infty(q; w) \kappa \frac{\Delta^{1/2}}{D_k^{1/2}} B + O_w(1)
+ O_w \left( 3^{\omega(\Delta)} \frac{\|q\|^{1/2}}{D_k^{1/2}} B_1^{1/4} B^{3/4} \log B \right)
\]
in the range \(\rho^{-1}\|c_3\| \ll B \leq B_1\). On the other hand, if \(B < (90\rho)^{-1}\|c_3\|\) then \(\|x\| \leq B\) implies \(\|x\| < \|z_2\|\), in the notation of Theorem 3, so that \(N(B; C_k)\) counts at most the points ±\(z_1\). In this case we will have \(N(B; C_k) \ll_w 1\), and hence (38) holds for \(B < (90\rho)^{-1}\|c_3\|\) too. To complete the proof of Theorem 5 it remains to observe that \(\kappa \gg (3/4)^{\omega(\Delta)}\), that \(\sigma_\infty(q; w) \gg \|q\|^{-1}\), by Lemma 11 and that \(\|c_3\| \ll \rho\|z_2\|\), by Theorem 3.

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