Extended Version of
“The Philosophy of the Trajectory Representation of Quantum Mechanics”

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The philosophy of the trajectory representation differs with Copenhagen and Bohmian philosophies. The trajectory representation is a strongly causal, nonlocal theory of quantum mechanics that is deterministic. It is couched in a generalized Hamilton-Jacobi formulation. For bound states, each particular trajectory determines a unique microstate of the Schrödinger wave function. Hence, the Schrödinger wave function is not an exhaustive description of nonrelativistic quantum phenomenon. A tunnelling example shows that assigning a probability amplitude to the Schrödinger wave function is unnecessary. The trajectory representation in the classical limit (h → 0) manifests a residual indeterminacy where the trajectory representation does not go to classical mechanics. This residual indeterminacy is contrasted to the Heisenberg uncertainty principle and is also compared with ’t Hooft’s information loss. The trajectory representation is contrasted with the Copenhagen and Bohmian representations. For a square well duct, consistent overdetermination of a trajectory by a redundant set of observed constants of the motion are beyond the Copenhagen interpretation. Also, the trajectory representation makes different predictions than the Copenhagen interpretation for impulsive perturbations, even under Copenhagen epistemology. Although the trajectory representation and Bohmian mechanics use the same generalized Hamilton-Jacobi equations, they have different equations of motion.

Prologue: “The Philosophy of the Trajectory Representation of Quantum Mechanics” [1] is an extract of this opus, was presented at the Vigier 2000 Symposium, 21–25 August 2000, in Berkeley, California, and will be published in the Proceedings of the Symposium. The Symposium celebrated Jean Pierre Vigier’s eightieth birthday.

1. Introduction: The seminal work on the trajectory representation was published in 1982 [2]. The trajectory representation sprang from improvements in the WKB approximation [2,3] and in acoustical ray tracing [6,7]. The equations of motion for the trajectories are developed from a quantum Hamilton-Jacobi formulation. These trajectories are deterministic and continuous. Ergo, there is no need by precept for any collapse of the wave function during observation. Early analyses used numerical methods or power-series expansions until exact closed-form solutions were introduced [8].

Recently, Faraggi and Matone [9,10] have independently generated the same quantum Hamilton-Jacobi formulation from an equivalence principle free from any axioms. Faraggi and Matone have shown that although all quantum systems can be connected by an equivalence coordinate transformation (trivializing map), all systems in classical mechanics are not so connected. Some of the goals of their work include synthesis of gravity, mass and quantum mechanics and possible relations to string theory [15,16] and producing an expression for the interaction terms, including gravity, that have a pure quantum origin [17]. The development of the equivalence principle is beyond the scope of this exposition.

We present the philosophical aspects of the trajectory representation of quantum mechanics that distinguish this representation. We exhibit its interpretation, which we contrast to the Copenhagen interpretation and the Bohmian stochastic interpretation. Our findings are presented in closed form in one dimension for the time-independent case whenever one dimension suffices. The work in one dimension for the time-independent case renders a counter example that refutes Born’s postulate of the Copenhagen interpretation attributing a probability amplitude to the Schrödinger wave function, shows that the Heisenberg uncertainty principle is premature, refutes the Copenhagen interpretation that the Schrödinger wave function is
an exhaustive description of nonrelativistic quantum phenomenon, and questions the wave-particle duality of Bohr’s complementarity. Bertoldi, Faraggi and Matone have recently extended the quantum Hamilton-Jacobi formulation to higher dimensions, time dependence and relativistic quantum mechanics [18]. A small amount of work in higher dimensions is presented where necessary to establish our findings.

We explicitly note that the trajectory representation is not just another interpretation of quantum mechanics because it also predicts results that differ with contemporary, orthodox practice (Copenhagen interpretation). Trajectory and Copenhagen analyses predict different results from a perturbing impulse [19]. A test has been proposed to show that consistent overdetermination of a trajectory by a redundant set of observed constants of the motion would be beyond the Copenhagen interpretation [20].

Beyond the philosophical aspects, we refer the interested reader to five other advances of the trajectory representation that have been developed elsewhere but not presented in the Proceedings [1]. First, an initial application of the trajectory representation has been made to relativistic quantum mechanics [21]. Second, the trajectory representation is not a hydrodynamical formulation of wave mechanics as trajectories may cross. Thus, the trajectory representation may manifest caustics as has been presented elsewhere, albeit couched in acoustics [22]. We note that the trajectory representation renders not only all caustics that correspond to the caustics described by classical ray tracing but also additional caustics that are extra to classical geometric acoustics. Third, creation and annihilation of interference patterns are studied [23]. Fourth, trajectory dwell times during tunneling and reflection are examined [20,24]. Fifth, the generalized Hamilton-Jacobi equation and the Schrödinger equation form an Ermakov system which generates an Ermakov invariant [25]. The Ermakov invariant is a constant of the motion for the particular trajectory (microstate).

In Section 2, we present the fundamentals of the trajectory representation from a philosophical aspect. We give references for more detailed development of the trajectory representation for the interested reader. The equations of motion are presented for the trajectory. We present why microstates of the wave function exist for bound states. Much of the philosophy of the trajectory representation is innate in the development of this representation. In Section 3, we present different predictions rendered by trajectories and Copenhagen. We continue to contrast in Section 4 the trajectory representation to the Copenhagen interpretation. In Section 5, we compare the trajectory representation with the Bohmian stochastic representation. In Appendix A, we show that no particular set of independent solutions of the Schrödinger equation are privileged.

2. The Trajectory Representation:

2.1. Equation of Motion: The trajectory representation is based upon a phenomenological, nonlocal generalized Hamilton-Jacobi formulation. The quantum stationary Hamilton-Jacobi equation (QSHJE) is given in one dimension $x$ by [26,27]

\[
\frac{(W')^2}{2m} + V - E = -\frac{\hbar^2}{4m} \langle W;x \rangle \tag{1}
\]

where $W$ is Hamilton’s characteristic function (also known as the reduced action), $W'$ is the momentum conjugate to $x$, $\langle W;x \rangle$ is the Schwarzian derivative of $W$ with respect to $x$, $V$ is the potential, $E$ is energy, $m$ is the mass of the particle, and $\hbar = h/(2\pi)$ where in turn $h$ is Planck’s constant. Explicitly, the Schwarzian derivative raises the QSHJE to a third-order nonlinear differential equation and is given by

\[
\langle W;x \rangle = \frac{W'''}{W'} - \frac{3}{2} \left( \frac{W''}{W'} \right)^2 = \ln [(W')]'' - \frac{1}{2} \{ \ln [(W')]' \}^2.
\]

The left side of Eq. (1) manifests the classical Hamilton-Jacobi equation; the right side, the higher order quantum effects in the Schwarzian derivative. Faraggi and Matone have independently derived the QSHJE from the equivalence principle. We note that $W$ and $W'$ are real even in a classically forbidden region. The general solution for $W'$ is given by [8]

\[
W' = (2m)^{1/2}(a\phi^2 + b\theta^2 + c\phi\theta)^{-1} \tag{2}
\]
where \((a, b, c)\) is a set of real coefficients such that \(a, b > 0\), and \((\phi, \theta)\) is a set of normalized independent solutions of the associated stationary Schrödinger equation, \(-\hbar^2 \psi''/(2m) + (V - E)\psi = 0\). The independent solutions \((\phi, \theta)\) are normalized so that their Wronskian, \(W(\phi, \theta) = \phi\theta' - \phi'\theta\), is scaled to give \(W^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)] > 0\). This ensures that \((ab^2 + b\theta^2 + c\phi\theta) > 0\) and that \(W'\) is real in the classically forbidden regions \((V > E)\). This normalization is determined by the nonlinearity of Eq. (1) rather than by total probability of finding the particle in space being unity as done by the Copenhagen interpretation. A particular set \((\phi, \theta)\) of independent solutions of the Schrödinger equation may be chosen by the superposition principle so that the coefficient \(c\) is zero. The motion in phase space is specified by Eq. (2). This phase-space trajectory is a function of the set of coefficients \((a, b, c)\).

If the Schrödinger equation can be solved in closed form, then the QSHJE may also be solved in closed form for conjugate momentum as Eq. (2) expresses \(W\) trajectory as a function of the set of coefficients \((\phi, \theta)\), which are scaled to give \(W^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)] > 0\). This ensures that \((ab^2 + b\theta^2 + c\phi\theta) > 0\) and that \(W'\) is real in the classically forbidden regions \((V > E)\). This normalization is determined by the nonlinearity of Eq. (1) rather than by total probability of finding the particle in space being unity as done by the Copenhagen interpretation. A particular set \((\phi, \theta)\) of independent solutions of the Schrödinger equation may be chosen by the superposition principle so that the coefficient \(c\) is zero. The motion in phase space is specified by Eq. (2). This phase-space trajectory is a function of the set of coefficients \((a, b, c)\).

In general, the conjugate momentum expressed by Eq. (2) is not the mechanical momentum, i.e., \(W' \neq m\dot{x}\). Actually, \(m\dot{x} = m\partial E/\partial W\).

The solution for the generalized Hamilton's characteristic function, \(W\), is given by

\[ W = \hbar \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) + K \]  

where \(K\) is an integration constant that we may set to zero herein.

Hamilton's characteristic function is a generator of motion. The equation of motion in the domain \([x, t]\) is rendered by the Hamilton-Jacobi transformation equation for constant coordinates (often called Jacobi's theorem). The procedure simplifies for coordinates whose conjugate momenta are separation constants. Carroll has shown that for stationarity Jacobi's theorem is valid for \(W\) is a Legendre transform of Hamilton's principal function \(\psi\).

For stationarity, \(E\) is a separation constant for time. Thus, the equation of motion for the trajectory time, \(t\), relative to its constant coordinate \(\tau\), is given as a function of \(x\) by

\[ t - \tau = \partial W/\partial E \]  

where the trajectory is a function of a set of coefficients \((a, b, c)\) and \(\tau\) specifies the epoch.

The set \((\phi, \theta)\) can only be a set of independent solutions of the Schrödinger equation. Direct substitution of Eq. (2) for \(W'\) into Eq. (2) gives

\[ \frac{a\phi + c\theta/2}{a\phi^2 + b\theta^2 + c\phi\theta} [-\hbar^2/(2m)\phi'' - (E - V)\phi] + \frac{b\theta + c\phi/2}{a\phi^2 + b\theta^2 + c\phi\theta} [-\hbar^2/(2m)\theta'' - (E - V)\theta] \]

\[ - \frac{[W^2\hbar^2(ab - c^2/4)/(2m) - 1]}{(a\phi^2 + b\theta^2 + c\phi\theta)^2} = 0. \]  

For the general solution for \(W'\), the real coefficients \((a, b, c)\) are arbitrary within the limitations that \(a, b > 0\) and from the Wronskian that \(ab - c^2/4 > 0\). Hence, for generality the expressions within each of the three square brackets on the left side of Eq. (3) must vanish identically. The expressions within the first two of these square brackets manifest the Schrödinger equation, so the expressions within these two square brackets are identically zero if and only if \(\phi\) and \(\theta\) are solutions of the Schrödinger equation. The expression within third bracket vanishes identically if and only if the normalization of the Wronskian is such that \(W^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)]\). For \(W(\phi, \theta) \neq 0\), \(\phi\) and \(\theta\) must be independent solutions of the Schrödinger equation. Hence, \(\phi\) and \(\theta\) must form a set of independent solutions of the Schrödinger equation.

Equation (3) is independent of any particular choice of \(\text{ansatz}\). When comparing trajectories to Copenhagen and Bohm, we have broad selection for choosing a convenient \(\text{ansatz}\) to generate the equivalent wave picture (nothing herein implies that the trajectories need waves for completeness; only convenience).
2.2. Tunneling with Certainty: The Hamilton's characteristic function for the trajectory of a particle with sub-barrier energy that tunnels through the barrier with certainty can be established by the continuity conditions of $W$, $W'$ and $W''$ across the barrier interfaces \[24\]. The corresponding Schrödinger wave function for this trajectory that tunnels with certainty was also developed from $W$ and $W'$ \[24\]. We now outline this development.

While Eq. (2) gives the relationship between the conjugate momentum $W'$ and the solution set of independent wave functions $(\phi, \theta)$, an inverse relationship, not necessarily unique, is given by Ref. 8 as

$$
\psi = \frac{\exp(iW/\hbar)}{(W')^{1/2}}. \tag{6}
$$

Let us consider a rectangular barrier whose potential is given by

$$
V(x) = \begin{cases} 
U, & |x| < q \\
0, & |x| \geq q.
\end{cases}
$$

For $x > q$, we specify a transmitted, unmodulated running wave given by

$$
\psi = (\hbar k)^{-1/2} \exp[\mp ik(x-q)], \quad x > q \tag{7}
$$

where $k = (2mE)^{1/2}/\hbar$ and the integration constant, $K$, has been chosen so the phase is zero at the barrier interface $x = q$. In turn, $E$ is positive, sub-barrier, that is $0 < E < U$. For $x > q$, Hamilton’s characteristic function is given by $W = \hbar k(x-q)$. Anywhere that $x > q$, $W = \hbar k(x-q)$ and its first two derivatives render a valid set of initial conditions.

From the continuity of $W$, $W'$ and $W''$, we may now establish $W$ for this tunneling problem to be \[24\]

$$
W = \begin{cases} 
\hbar k(x-q), & x > q \\
\hbar \arctan(\{(k/\kappa)\tanh(\kappa(x-q))\}, & -q \leq x \leq q \\
\hbar \arctan(N/D), & x < q
\end{cases}
$$

where $\kappa = [2m(U-E)]^{1/2}/\hbar$,

$$
N = (k/\kappa) \sinh(-2\kappa q) \cos(k(x+q)) + \cosh(-2\kappa q) \sin[k(x+q)],
$$

and

$$
D = \cosh(-2\kappa q) \cos[k(x+q)] + (k/\kappa) \sinh(-2\kappa q) \sin[k(x+q)].
$$

Note that $W$ monotonically increases everywhere with increasing $x$. While $W$, as given above, resolves tunneling in trajectory representation, we present the more familiar $\psi$ as derived from $W$ and Eq. (3) to give a gentler introduction to the insights of the trajectory representation.

In the classically forbidden region inside the barrier, $-q \leq x \leq q$, and from Eq. (3) the continuity conditions on $W$, $W'$ and $W''$ at $x = q$, the Schrödinger wave function is \[24\]

$$
\psi = \{[(k/\kappa) \cos^2(\kappa x) + (k/\kappa) \sin^2(\kappa x)]/(\hbar \kappa)\}^{1/2} \exp \left[ i \arctan \left( \frac{k}{\kappa} \tanh(\kappa(x-q)) \right) \right], \quad -q \leq x \leq q \tag{8}
$$

where for Eqs. (3) and (3) $\phi = \cosh(\kappa(x-q))$, $\theta = \sinh(\kappa(x-q))$, $a = [(2m)^{1/2}/(\hbar \kappa)](k/\kappa)$, $b = [(2m)^{1/2}/(\hbar \kappa)](k/\kappa)$, and $c = 0$. This Schrödinger wave function represented by Eqs. (3) and (3) has a continuous logarithmic derivative across the barrier interface at $x = q$. The phase of $\psi$ inside the barrier increases monotonically with increasing $x$. As Eq. (3) manifests a spatially compound wave running in the
positive $x$-direction in the classically forbidden region that has a continuous logarithmic derivative at $x = q$ with a wave that is running in the positive $x$-direction in the region $x > q$, there is no reflection at the interface at $x = q$.

In the domain before the barrier, $x < -q$, and from the continuity conditions for $W$, $W'$ and $W''$ at $x = -q$ and from Eq. (6), the Schrödinger wave function is presented as

$$\psi = \left( \frac{\mathcal{A}}{\hbar k} \right)^{1/2} \exp[i \arctan(\mathcal{B})], \quad x < -q \quad (9)$$

where

$$\mathcal{A} = \cosh(2\kappa q) + \frac{1}{2} \left( \frac{\kappa}{k} + \frac{k}{\kappa} \right) \sinh(-4\kappa q) \sin[2k(x + q)]$$

$$+ \sinh^2(2\kappa q) \left[ \left( \frac{k}{\kappa} \sin[k(x + q)] \right)^2 + \left( \frac{k}{\kappa} \cos[k(x + q)] \right)^2 \right] \quad (10)$$

and

$$\mathcal{B} = \frac{k}{2} \sinh(-2\kappa q) \cos[k(x + q)] + \cosh(-2\kappa q) \sin[k(x + q)] \right) \cos(-2\kappa q) \sin[k(x + q)] + \frac{k}{2} \sinh(-2\kappa q) \sin[k(x + q)]. \quad (11)$$

The Schrödinger wave function, as represented by Eqs. (8) and (9), has a continuous logarithmic derivative across the barrier interface at $x = -q$. Similar to the situation at the barrier interface at $x = q$, Eqs. (8) and (9) manifest a wave with compound spatial modulation of phase and amplitude for $x < q$ that progresses in the positive $x$-direction. This wave with compound spatial modulation has a continuous logarithmic derivative at $x = -q$, so there is no reflection of this wave at the barrier interface $x = q$.

The Schrödinger wave function, as represented by Eqs. (7)–(9), manifests a running wave progressing in the positive $x$-direction everywhere. Nowhere is there any reflection of this running wave. This Schrödinger wave function is an eigenfunction with eigenvalue energy $E$ for the given rectangular barrier. Hence, this eigenfunction represents a particle with sub-barrier energy that tunnels through the barrier with certainty.

Only recently did physicists recognize that eigenfunctions for a constant potential could be wave functions with compound spatial modulation in amplitude and wavenumber [23]. However, mathematicians knew it all along, cf. Appendix A. While one could confirm that the wave function represented by Eqs. (7) through (9) is an eigenfunction by brute force by substituting this wave function into the Schrödinger equation, we suggest referring to Ref. [24] where the wave function representations, Eq. (8) has been resolved into its customary hyperbolic components inside the barrier by

$$\psi = \frac{1}{(\hbar k)^{1/2}} \left( \cosh[k(x - q)] + i(k/\kappa) \sinh[k(x - q)] \right), -q \leq x \leq q \quad (12)$$

and where Eq. (9) has been resolved into the customary incident and reflected unmodulated plane-wave components before the barrier by

$$\psi = \left( \frac{\mathcal{A}}{\hbar k} \right)^{1/2} \left[ \cosh(-2\kappa q) + \frac{i}{2} \left( \frac{k}{\kappa} - \frac{k}{\kappa} \right) \sinh(-2\kappa q) \right] \exp[ik(x + q)]$$

$$+ (\hbar k)^{-1/2} \frac{i}{2} \left( \frac{k}{\kappa} - \frac{k}{\kappa} \right) \sinh(-2\kappa q) \exp[-ik(x + q)], \quad x < q. \quad (13)$$
Thus, Eqs. (8) and (9) manifest synthesized waves in and before the barrier respectively.

We note that the synthetic incident wave, Eq. (3), has spectral components traveling in both the positive and negative \( x \) directions. Any concern that the synthetic wave, Eq. (4), would spontaneously split apart is put to rest in Appendix A.

2.3. Bound States: The boundary value problem is not so simple [3,5]. The solutions for boundary value problem, if they exist at all, need not be unique. As is well known for bound states, solutions for the Schrödinger wave function do exist for the energy eigenvalues. Not as well known, solutions for Hamilton’s characteristic function for the trajectory representation of quantum mechanics exist for bound states if the action variable, \( J \), is quantized [29], that is

\[
J = \oint W' \, dx = n\hbar, \quad n = 1, 2, 3, \ldots
\]  

(14)

The action variable is independent of the set of coefficients \((a, b, c)\) by the theory of complex variables [8]. The set of coefficients \((a, b, c)\) only posits the singularities (poles) and terminal points of the Riemann sheets. The set of coefficients \((a, b, c)\) does not effect the number of poles or Riemann sheets.

Specifically, we consider the bound state problem where \( \psi \to 0 \) as \( x \to \pm \infty \). These are the bound state eigenfunctions which are unique. While the Schrödinger wave function is unique for bound states, the conjugate momentum is not [8,27]. In the generalized Hamilton-Jacobi representation of quantum mechanics, the boundary conditions for bound motion manifest a phase-space trajectory with turning points at \( x = \pm \infty \). This is accomplished by \( W' \to 0 \) as \( x \to \pm \infty \). However, the generalized Hamilton-Jacobi equation for the bound states is a nonlinear differential equation that has critical (singular) points at the very location where the boundary values are applied, i.e., \( x = \pm \infty \). By Eq. (2), \( W' \to 0 \) as \( x \to \pm \infty \) because at least one of the independent solutions, \( \phi \) or \( \theta \), of the Schrödinger equation must be unbound as \( x \to \pm \infty \). As the coefficients satisfy \( a, b > 0 \) and \( ab > c^2/4 \), the conjugate momentum exhibits a node as \( x \to \pm \infty \) for all permitted values of \( a, b, \) and \( c \) [8]. Hence, the boundary values, \( W'(x = \pm \infty) = 0 \), for Eq. (4) permit non-unique phase-space trajectories for \( W' \) for energy eigenvalues or quantized action variables. Likewise, the trajectories in configuration space are not unique for the energy eigenvalue as the equation of motion, \( t - \tau = \partial W/\partial E \), specifies a trajectory dependent upon the coefficients \( a, b \) and \( c \).

2.4. Microstates: The non-unique trajectories in phase space and configuration space manifest microstates of the Schrödinger wave function [8,27]. For bound states in one dimension, the time-independent Schrödinger wave function may be real except for an inconsequential phase factor. Bound states have the boundary values that \( \psi(x = \pm \infty) = 0 \). Let us choose \( \phi \) to be the bound solution. Then \( \psi = \alpha \phi + \beta \theta \) where \( \alpha \) and \( \beta \) are coefficients. The Schrödinger wave function for bound states can be represented by [23]

\[
\psi = \frac{(2m)^{1/4} \cos(W/\hbar)}{(W')^{1/2}[a - c^2/(4b)]^{1/2}}
\]

\[
= \frac{(a\phi^2 + b\theta^2 + c\phi \theta)^{1/2}}{[a - c^2/(4b)]^{1/2}} \cos \left[ \arctan \left( \frac{b\theta/\phi + c/2}{(ab - c^2/4)^{1/2}} \right) \right] = \phi.
\]  

(15)

Thus, \( \alpha = 1 \) and \( \beta = 0 \) for all permitted values of the set \((a, b, c)\). Each of these non-unique trajectories of energy \( E \) manifests a microstate of the Schrödinger wave function for the bound state. These microstates of energy \( E \) are specified by the set \((a, b, c)\). See Ref. [27] for an example.

The existence of microstates is a counter-example refuting the assertion of the Copenhagen interpretation that the Schrödinger wave function be the exhaustive description of nonrelativistic quantum phenomena.

Historically, others including Ballinger and March [31], Light and Yuan [31], and Korsch [32] had noted that the bound-state solution to Eq. (1), or its equivalent by transformation, was arbitrary. (There may be others of whom I am unaware.) These investigators enjoyed freedom in choosing the coefficients \((a, b, c)\) or
their equivalents. These investigators choose the particular solution that rendered well behaved results for the density of states close to WKB values \[32\] or gave good fits to extended Thomas-Fermi approximations \[30,31\]. Ballinger and March \[30\] and Korsch \[32\] acknowledge that their choices of the particular solution, while fitting the work at hand, could not be justified from quantum principles.

2.5. Classical Limit, Loss of Information, Heisenberg Uncertainty and Residual Indeterminacy: For the classical limit (\(\hbar \to 0\)), the QSHJE, a third-order non-linear differential equations, reduces to the classical stationary Hamilton-Jacobi equation (CSHJE), a first-order nonlinear differential equation. Reducing the order in turn reduces the set of initial values necessary and sufficient to establish unique solution. Hence, less information is necessary to solve the CSHJE than the QSHJE. For the CSHJE, simultaneous knowledge of momentum and position specifies the energy and the trajectory. While momentum and position form a sufficient set of initial conditions for classical mechanics, quantum mechanics also needs the higher order derivatives \(W''\) and \(W'''\) \[26\]. The Heisenberg uncertainty principle alleges uncertainty in such simultaneous knowledge implying that trajectories do not exist at the quantum level. This is premature as momentum and position form only a subset smaller than the set of initial conditions necessary and sufficient to solve the QSHJE \[33\].

We note that this loss of information differs with the recent proposal of ‘t Hooft \[34\] that quantization results from the loss of information about “primordial” trajectories of continuous energy. No dissipation of information happens in the trajectory representation when going to the classical limit, but rather this loss of information induces an indeterminacy.

As \(\hbar \to 0\), we can test Planck’s correspondence principle as to whether quantum mechanics goes to classical mechanics. In the trajectory representation, the equation of motion for a free particle (i.e., \(V = 0\)) can be expressed as \[33\]

\[
t - t_o = \frac{(ab - c^2/4)^{1/2}(2m/E)^{1/2}x}{a + b + (a^2 - 2ab + b^2 + c^2)^{1/2} \cos \{2(2mE)^{1/2}x/\hbar + \cot^{-1}[c/(a-b)]\}}.
\]

In the limit \(\hbar \to 0\), the cosine term in the denominator of Eq. (16) fluctuates with an infinitesimal short wavelength. For the particular case, \(a = b\) and \(c = 0\), Planck’s correspondence principle holds for Eq. (16). On the other hand for \(a \neq b\) or \(c \neq 0\), the cosine term becomes indefinite in the classical limit. This leads to a residual indeterminacy in the classical limit. Thus, Planck’s correspondence principle does not hold in general. This is consistent with the findings of Faraggi and Matone \[14\] that the equivalence principle does not hold for classical mechanics \[33\]. It has also been shown elsewhere \[33\] that quantum mechanics does not reduce to statistical mechanics for \(\hbar \to 0\).

Note that residual indeterminacy and the Heisenberg uncertainty principle differ: the former exists for \(\hbar \to 0\); the latter, for \(\hbar\) finite \[33\]. Furthermore, Heisenberg uncertainty exists in the \([x, p]\) domain (where \(p\) is momentum) as the Hamiltonian operates in the \([x, p]\) domain. But the trajectory representation, through a canonical transformation to its Hamilton-Jacobi formulation, operates in the \([x, t]\) domain \[33\]. Residual indeterminacy of the trajectory representation is in the \([x, t]\) domain, cf. Eq. (14).

In closing this subsection, we note that \(\hbar\) remains finite and is very small. Here, we treated \(\hbar\) hypothetically as an independent variable to show even in the limit \(\hbar \to 0\), quantum trajectories do not generally reduce to classical trajectories.

2.6. Superluminality: The Aspect experiments deny local reality \[36,37\]. Yet the trajectories for bound states must penetrate infinitely deep into the classically forbidden zone \[12\]. This infinitely long trip must be done in a finite period of time. Hence, superluminality follows. This superluminality is a two-way superluminality. An example that shows this is given by Ref. \[20\].

Let us consider a particle traveling in a two-dimensional square-well duct. The particle has a trajectory down the duct in the axial direction while vertexing at infinite turning points in the transverse direction. The trajectory at these infinite turning points has been shown to be a cusp where velocity increases without.
bound and both legs of the cusp become tangent to the surface of Hamilton’s characteristic function \[20\]. This manifests the extreme example that the trajectory is not generally orthogonal to the W-surface.

Our trajectories incorporate reality by precept. The underlying generalized Hamilton-Jacobi equation is a phenomenological equation. Therefore, we find that since the trajectories have reality inherently, they must describe a nonlocal reality where phenomena violate Einstein separability. Thus, the trajectory representation renders a quantitative phenomenological description that favors choosing quantum mechanics, albeit without the Copenhagen interpretation thereof, in resolving the paradox between quantum mechanics and Einstein separability that exists, for example, in EPR experiments.

3. Different Predictions between Trajectories and Copenhagen:

3.1. Impulsive Perturbations: Floyd \[19\] has shown that the trajectory and Copenhagen representations render different predictions for the first-order change in energy, \(E_1\) due to a small, spatially symmetric perturbing impulse, \(\lambda V(x)\delta(t)\), acting on the ground state of a infinitely deep, symmetric square well. The different predictions are due to the different roles that causality plays in the trajectory and Copenhagen interpretations. In the trajectory representation, \(E_1\) is dependent upon the particular microstate, \((a,b,c)\). This has been investigated under a Copenhagen epistemology even for the trajectory theory, where complete knowledge of the initial conditions for the trajectory as well as knowledge of the particular microstate are not necessary to show differences for an ensemble sufficiently large so that all microstates are individually well represented. In the trajectory representation, the first-order change in energy, \(E_1\), is due to the location of the particle in its trajectory when the impulse occurs. The trajectory representation finds that the perturbing impulse, to first order, is as likely to do work on the particle as the particle is to do work perturbing system, cf. Eqs. (15) and (17)–(20) of Ref. \[19\]. Hence, the trajectory representation evaluates \(\langle E_1\rangle_{\text{average}} = 0\). On the other hand, Copenhagen predicts \(E_1\) to be finite as Copenhagen evaluates \(E_1\) by the trace ground-state matrix element \(\lambda V_{00}\delta(0)\) at the instant of impulse. Due to spatial symmetry of the ground state and \(V(x)\), \(V_{00} \neq 0\).

In an actual test, we do not need perturbing impulses, which were used for mathematical tractability. A rapid perturbation whose duration is much shorter than the period of the unperturbed system would suffice \[19\].

3.2. Overdetermination: For a square well duct, we have proposed a test where consistent overdetermination of the trajectory by a redundant set of observed constants of the motion would be beyond the Copenhagen interpretation \[20\]. The overdetermined set of constants of the motion should have a redundancy that is consistent with the particular trajectory. On the other hand, Copenhagen would predict a complete lack of consistency among these observed constants of the motion as Copenhagen denies the existence of trajectories. Such a test could be designed to be consistent with Copenhagen epistemology.

4. Other Differences between Trajectories and Copenhagen: As the trajectory exists by precept in the trajectory representation, there is no need for Copenhagen’s collapse of the wave function.

The trajectory representation can describe an individual particle. On the other hand, Copenhagen describes an ensemble of particles while only rendering probabilities for individual particles.

The trajectory representation renders microstates of the Schrödinger wave function for the bound state problem. Each microstate by Eq. (15) is sufficient by itself to determine the Schrödinger wave function. Thus, the existence of microstates is a counter example refuting the Copenhagen assertion that the Schrödinger wave function be an exhaustive description of nonrelativistic quantum phenomenon.

The trajectory representation is deterministic. We can now identify a trajectory and its corresponding Schrödinger wave function with sub-barrier energy that tunnels through the barrier with certainty. Hence, tunneling with certainty is a counter example refuting Born’s postulate of the Copenhagen interpretation that attributes a probability amplitude to the Schrödinger wave function.
As the trajectory representation is deterministic and does not need $\psi$, much less to assign a probability amplitude to it, the trajectory representation does not need a wave packet to describe or localize a particle. The equation of motion, Eq. (4) for a particle (monochromatic wave) has been shown to be consistent with the group velocity of the wave packet [23].

Normalization, as previously noted herein, is determined by the nonlinearity of the generalized Hamilton-Jacobi equation for the trajectory representation and for the Copenhagen interpretation by the probability of finding the particle in space being unity.

Though probability is not needed for tunneling through a barrier [24], the trajectory interpretation for tunneling is still consistent with the Schrödinger representation without the Copenhagen interpretation. The incident wave with compound spatial modulation of amplitude and phase for the trajectory representation, Eq. (13), has only two spectral components which are the incident and reflected unmodulated waves of the Schrödinger representation [24].

Trajectories differ with Feynman’s path integrals in three ways. First, trajectories employ a quantum Hamilton’s characteristic function while a path integral is based upon a classical Hamilton’s characteristic function. Second, the quantum Hamilton’s characteristic function is determined uniquely by the initial values of the QSHJE while path integrals are democratic summing over all possible classical paths to determine Feynman’s amplitude. While path integrals need an infinite number of constants of the motion even for a single particle in one dimension, motion in the trajectory representation for a finite number of particles in finite dimensions is always determined by only a finite number of constants of the motion. Third, trajectories are well defined in classically forbidden regions where path integrals are not defined by precept.

As previously noted in Section 2.5, the Heisenberg uncertainty principle shall remain premature as long as Copenhagen uses an insufficient subset of initial conditions ($x, p$) to describe quantum phenomena.

Bohr's complementarity postulates that the wave-particle duality be resolved consistent with the measuring instrument’s specific properties. On the other hand, Faraggi and Matone [14–14] have derived the QSHJE from an equivalence principle without evoking any axiomatic interpretation of the wave function. Furthermore, Floyd [27] and Faraggi and Matone [14–14] have shown that the QSHJE renders additional information beyond what can be gleaned from the Schrödinger wave function alone.

Anonymous referees of the Copenhagen school have had reservations concerning the representation of the incident modulated wave as represented by Eq. (9) before the barrier. They have reported that compoundly modulated wave represented by Eq. (9) is only a clever superposition of the incident and reflected unmodulated plane waves. They have concluded that synthesizing a running wave with compound spatial modulation from its spectral components is nonphysical because it would spontaneously split. We have put these reservations to rest in Appendix A and Ref. [24]. By the superposition principle of linear differential equations, the spectral components may be used to synthesize a new pair of independent solutions with compound modulations running in opposite directions. Likewise, an unmodulated plane wave running in one direction can be synthesized from two waves with compound modulation running in the opposite directions for mappings under the superposition principle are reversible.

5. Trajectories vis-a-vis Bohmian mechanics: The trajectory representation differs with Bohmian representation [38,39] in many ways despite both representations being based on equivalent generalized Hamilton-Jacobi equations. We describe the various differences between the two representations in this section. These differences may not necessarily be independent of each other.

First, the two representations have different equations of motion. The Hamilton-Jacobi transformation equation, Eq. (4), are the equations of motion for the trajectory representation. Meanwhile, Bohmian mechanics eschews solving the Hamilton-Jacobi equation for a generator of the motion, but instead assumes that the conjugate momentum be the mechanical momentum, $\dot{m}x \dot{W}$, which could be integrated to render the trajectory. But the conjugate momentum is not the mechanical momentum as already shown by Floyd [28], Faraggi and Matone [14] and Carroll [28]. Recently, Brown and Hiley [40] had stated that prior associating momentum in Bohmian mechanics with $W'$ by appealing to classical canonical theory was a “backward step” and “totally unnecessary”. Brown and Hiley still do not advocate solving the QSHJE for $W$. Rather, they now advocate that $W'$ be a “beable” momentum and $\dot{x}$ be given by the probability current divided by the
square of the probability amplitude.

Bohmian mechanics considers $\psi$ to form a field, a quantum field that fundamentally effects the quantum particle. The trajectory representation considers the Schrödinger equation to be only a phenomenological equation where $\psi$ does not represent a field. To date, no one has ever measured such a $\psi$-field.

Bohmian mechanics postulates a quantum potential, $Q$, in addition to the standard potential, that renders a quantum force proportional to $-\nabla Q$. (Bohm's quantum potential in one dimension appears in the QSHJE as the negative of the term containing the Schwarzian derivative or the right side of Eq. (1), i.e., $Q = \{h^2/(4m)\}|W; x\}$. But this quantum potential is inherently dependent upon $E$. By the QSHJE, $Q$ is also dependent upon the microstate $(a, b, c)$ of a given eigenvalue energy $E$ because

$$Q = E - V + (a\phi^2 + b\theta^2 + c\phi\theta)^{-2}.$$ 

Therefore, $Q$ as a function is path dependent and cannot be a conservative potential. Consequently, $-\nabla Q$ does not generally render a force. The average energy associated with $Q$ or the Schwarzian derivative term of the QSHJE in the classical limit ($h \to 0$) for the free particle ($V = 0$) is dependent upon the microstate as specified by $(a, b, c)$ and is given by

$$\left\langle \lim_{h \to 0} Q \right\rangle_{\text{average}} = E \left(1 - \frac{(a + b)/2}{(ab - c^2/4)^{1/2}}\right) = -\text{variance of lim}_{h \to 0} W_x \leq 0.$$ (17)

So the average energy, in the classical limit of Bohm's quantum potential, $Q$, is proportional to the negative of the variance of the classical limit of the conjugate momentum. The quantum potential is a function of the particular microstate and may be finite even in the classical limit as shown by Eq. (17). Nothing herein implies that Eq. (17) is general. Others cases have not been examined.

While Bohmian mechanics postulates pilot waves to guide the particle, the trajectory representation does not need any such waves.

Bohmian mechanics uses an ansatz that contains an exponential with imaginary arguments. The Bohmian ansatz in one dimension is $\psi = (W')^{-1/2}\exp(iW/h)$, the same as Eq. (6). Anonymous referees of the Bohm school have expressed reservations regarding the validity of trigonometric ansätze. Herein, we have presented, without using any particular ansatz, the reversible relationship between the generalized Hamilton-Jacobi equation, Eq. (6), to the Schrödinger equation by Eq. (9). As Eq. (9) is valid for any set $(\phi, \theta)$, other ansätze of the form $\psi = (W')^{-1/2}[A\exp(iW/h) + B\exp(-iW/h)]$, where $A, B$ are arbitrary, are acceptable. When $|A| = |B|$, then the ansatz becomes trigonometric. In the past, the trajectory representation had properly used other ansätze that were trigonometric in nature such as Eq. (13). For completeness, Bohm's ansatz has significantly more versatility than first apparent if $|A| \neq |B|$. Consider

$$\psi = (W')^{-1/2} \frac{A + B}{A - B}[A\exp(iW/h) + B\exp(-iW/h)]$$

$$= (W')^{-1/2} \frac{A + B}{A - B}[(A + B)^2 \cos^2(iW/h) + (A - B)^2 \sin^2(iW/h)]^{1/2} \exp \left[\frac{i}{h} \arctan \left(\frac{A - B}{A + B} \tan(W)\right)\right]$$

$$= (\tilde{W}')^{-1/2} \exp(i\tilde{W}/h), \; |A| \neq |B|$$

where

$$\tilde{W} = \arctan \left(\frac{A - B}{A + B} \tan(W)\right).$$

So, we have returned to Bohm’s one-dimensional ansatz with a new Hamilton’s characteristic function $\tilde{W}$ for $|A| \neq |B|$. This ansatz is reminiscent of the modulated wave that we presented in Eqs. (6) and (9).

Bohmian mechanics asserts that particles could never reach a point where the Schrödinger wave function vanishes. On the other hand, trajectories have been shown to pass through nulls of $\psi^2$. Furthermore,
the conjugate momentum is finite at these nulls by Eq. (2) as \(\phi\) and \(\theta\) cannot be both zero at the same point for they are independent solutions of a linear differential equation of second order.

Bohmian mechanics asserts that bound-state particles should have zero velocity because the spatial part of the bound-state wave function can be expressed by a real function. On the other hand, the generalized Hamilton-Jacobi equation, Eq. (1) is still applicable for bound states in the trajectory representation. For bound states, the trajectories form orbits whose action variables are quantized according to Eq. (14).

Bohmian mechanics asserts that a particle should follow a path normal to the surfaces of constant \(W\). On the other hand, our trajectories, when computed in higher dimensions, are not generally normal to the surfaces of constant \(W\) \[20,23\]. In higher dimensions, the trajectories are determined by the Hamilton-Jacobi transformation equations for constant coordinates (Jacobi’s theorem) rather than by \(\nabla W\).

Bohmian mechanics asserts that the possible Bohmian trajectories for a particular particle should not cross. Rather, Bohmian trajectories are channeled and follow hydrodynamic-like flow lines. On the other hand, the trajectory representation describes trajectories that not only can cross but can also form caustics as shown elsewhere in an analogous, but applicable acoustic two-dimensional duct \[22\]. We note that the Schrödinger equasion and the separated acoustic wave equations are both Helmholtz equations.

The two representations differ epistemologically whether probability is needed. The trajectory representation is deterministic. Bohmian mechanics purports to be stochastic and consistent with Born’s probability amplitude \[39\]. In one dimension, Bohmian mechanics introduces stochasticity, by assigning a position, \(\chi\), of the particle as a separate variable from the argument, \(x\), of the Schrödinger wave function, \(\psi\). In other words, Bohmian mechanics introduces stochasticity by assuming different initial positions of the particle within the initial wave packet for the probability amplitude of the particle. The particle position, \(\chi\), would be a stochastic variable. From Bell \[41\], the argument \(x\) of \(\psi\) could be treated as the “hidden” variable instead of \(\chi\). We note that this additional variable, \(\chi\), is extraneous for consistency with the Schrödinger equation \[23\].

Let us consider three dimensions in this paragraph to examine the familiar stationary auxiliary equation

\[
\nabla \cdot (R^2 \nabla W) = 0 \tag{18}
\]

to the three-dimensional QSHJE. Bohm and Hiley \[39\] identify \(R\) as a probability amplitude and Eq. (18) as the continuity equation conserving probability. Bertoldi, Faraggi and Matone \[18\] only require that \(R\) satisfy Eq. (18) nontrivially. Hence, \(R^2 \nabla W\) must be divergenceless. The trajectory representation can now show a non-probabilistic interpretation of \(R^2 \nabla W\). Let us consider a case for which the stationary Bohm’s ansatz, \(\psi = R \exp(iW/\hbar)\), is applicable. Bohm used \[38\]

\[
R^2 = U^2 + V^2 \quad \text{and} \quad W = \hbar \arctan(V/U)
\]

where \(U = \Re(\psi) = R \cos(W/\hbar)\) and \(V = \Im(\psi) = R \sin(W/\hbar)\). Hence, by the superposition principle, \(U\) and \(V\) are a pair of solutions, not necessarily independent, to the stationary Schrödinger equation. (If \(U\) and \(V\) are not independent, then \(W\) is a constant and \(\psi\) is real except for a constant phase factor.) Upon substituting \(U\) and \(V\) into Eq. (18), we get as an intermediate step

\[
R^2 \nabla W = U \nabla V - V \nabla U,
\]

which is like a three-dimensional Wronskian. Again, we do not need this Wronskian analogy to be a constant, just divergenceless. The divergence of \(R^2 \nabla W\) is

\[
\nabla \cdot (R^2 \nabla W) = \nabla U \nabla V (1 - 1) + \frac{2m}{\hbar^2} (E - V) U \nabla (1 - 1) = 0.
\]

Indeed \(R^2 \nabla W\) is divergenceless. Thus, the trajectory representation finds that the auxiliary equation contains a three-dimensional Wronskian analogy that satisfies Eq. (18) without any need for evoking a probability amplitude.
Bohm had expressed concerns regarding the initial distributions of particles. Bohm [38] had alleged that in the duration that nonequilibrium probability densities exist in his stochastic representation, the usual formulation of quantum mechanics would have insoluble difficulties. The trajectory representation has shown that the initial conditions of nonlocal hidden variable may be arbitrary and still be consistent with the Schrödinger representation [26].

Stochastic Bohmian mechanics, like the Copenhagen interpretation, uses a wave packet to describe the motion of the of the associated $\psi$-field. As previously described herein, the deterministic trajectory needs neither waves nor wave packets to describe or localize particles.

Holland [42] reports that the Bohm’s equation for particle motion could be deduced from the Schrödinger equation but the process could not be reversed. On the other hand, the development of Eq. (5) is reversible. The Schrödinger equation and the generalized Hamilton-Jacobi equation mutually imply each other.

In application, the two representations differ regarding tunneling. Dewdney and Hiley [43] have used Bohmian mechanics to investigate tunneling through a rectangular barrier by Gaussian pulses. While Dewdney and Hiley assert consistency with the Schrödinger representation, they do not present any results in closed form. Rather, they present graphically an ensemble of numerically computed trajectories for eye-ball integration to show consistency with the Schrödinger representation. On the other hand, our trajectory representation exhibits in closed form consistency with the Schrödinger representation (the unbound wave function does not have microstates [27]). In addition, we note that every Bohmian trajectory that successfully tunnels slows down while tunneling. Hence, a particle following any one of these Bohmian trajectories would slow down while tunneling even though Steinberg et al [44] have shown that the peak of the associated wave packet speeds up while tunneling. On the other hand Floyd [20,24] has shown that trajectories that successfully tunnel speed up consistent with the findings of Olkhovsky and Racami [13] and Barton [14] and the finding of Hartmann [17] and Fletcher [48] for thick barriers.

Appendix A — Inverse Mapping: In this Appendix we show that no particular set of independent solutions is privileged [24]. The incident wave with compound spatial modulation of amplitude and phase, Eq. (9), can be synthesized under the superposition principle from two spectral components running in opposite directions as shown by Eq. (13). Likewise, an unmodulated plane wave running in one direction can be synthesized from two waves with compound modulation running in opposite directions for mappings under the superposition principle are reversible.

As a heuristic example consider analyzing the unmodulated plane wave (eigenfunction for the free particle with energy $E$) into the solution set $(\zeta_+, \zeta_-)$ where

$\zeta_{\pm} = \left( \frac{A}{\hbar k} \right)^{1/2} \exp[\pm i \arctan(B)]$

and where $A$ and $B$ have already been specified by Eqs. (10) and (11) respectively.

Hence, $\zeta_+$ and $\zeta_-$ are two modulated waves that run in opposite directions as their phases monotonically increase or decrease respectively with increasing $x$. The customary incident and reflected unmodulated plane waves before the barrier are given respectively by [24]

$$(\hbar k)^{-1/2} \left[ \cosh(-2\kappa q) + \frac{i}{2} \left( \frac{k}{\kappa} - \frac{\kappa}{k} \right) \sinh(-2\kappa q) \right] \exp[ik(x + q)]$$

$$= \left[ \cosh(-\kappa q) + \frac{1}{4} \left( \frac{k}{\kappa} - \frac{\kappa}{k} \right)^2 \sinh^2(-2\kappa q) \right] \zeta_+$$

$$- \left[ \cosh(-\kappa q) + \frac{i}{2} \left( \frac{k}{\kappa} - \frac{\kappa}{k} \right) \sinh(-2\kappa q) \right] \left[ \frac{i}{2} \left( \frac{k}{\kappa} - \frac{\kappa}{k} \right) \sinh(-2\kappa q) \right] \zeta_- \quad (A1)$$
\begin{align*}
\frac{i/2}{(\hbar k)^2} \left( \frac{k}{\kappa} + \frac{\kappa}{k} \right) \sinh(-2\kappa q) \\
= -\frac{1}{4} \left( \frac{k}{\kappa} + \frac{\kappa}{k} \right)^2 \sinh^2(-2\kappa q) \zeta_+ \\
+ \left[ \cosh(-\kappa q) + \frac{i}{2} \left( \frac{k}{\kappa} - \frac{\kappa}{k} \right) \sinh(-2\kappa q) \right] \left[ \frac{i}{2} \left( \frac{k}{\kappa} - \frac{\kappa}{k} \right) \sinh(-2\kappa q) \right] \zeta_-.
\end{align*}

Equations (A1) and (A2) respectively map the customary incident unmodulated plane wave and the customary reflected unmodulated plane wave into the set \((\zeta_+, \zeta_-)\). We have synthesized the customary incident and reflected unmodulated plane waves from two modulated waves, \((\zeta_+, \zeta_-)\), travelling in the opposite directions. Hence, the superposition principle and its mappings are reversible. If the customary unmodulated incident and reflected waves do not spontaneously split apart, then neither does the modulated incident wave. If a pulse can be formed with unmodulated plane waves, so can the corresponding pulse be formed with modulated waves. The set of unmodulated plane waves solutions to the time-independent Schrödinger equation for a free particle is not privileged.

We note that Eq. (A1), the customary unmodulated incident plane wave, and Eq. (A2), the customary unmodulated reflected plane wave, sum to \(\zeta_+\), which manifests the incident wave with compound spatial modulation, Eq. (9), as expected.

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