On open normal subgroups of parahorics

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Abstract

Let $F$ be a local complete field with discrete valuation, and let $G$ be a quasi-split group over $F$ which splits over some unramified extension of $F$. Let $P$ be a parahoric subgroup of the group $G(F)$ of $F$-points of $G$; the open normal pro-nilpotent subgroups of $P$ can be classified using the standard normal filtration subgroups of Prasad and Raghanathan. More precisely, we show that if $G$ is quasi-simple and satisfies some additional conditions, $H$ is, modulo a subgroup of some maximal torus of $G$, either one of these filtration subgroups or the product of one of them by a standard normal filtration subgroup of $P \cap M$, where $M$ is a proper Levi subgroup of $G$.

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1 Introduction

Let $F$ be a local complete field with discrete valuation; let $\mathcal{O}$ be its ring of integers, $p$ the maximal ideal of $\mathcal{O}$, $K = \mathcal{O}/p$ its residual field. Let $\varpi$ be an uniformizer of $F$, and $v_F$ be the normalized valuation on $F$.

Let $G$ be a connected reductive algebraic group defined over $F$; we’ll assume $G$ is quasisplit, and splits over some unramified extension of $F$; let $G(F)$ be the set of $F$-points of $G$. Let $\Phi$ be the root system of $G(F)$ relatively to some maximal split torus $S$ of $G$; we’ll assume that $\Phi$ satisfies the following condition:
(G1): if the characteristic $p$ of $K$ is 2, $\Phi$ is simply-laced; if $p = 3$, $\Phi$ has no connected component of type $G_2$.

Let $G^0$ be the parahoric component of $G(F)$, that is the subgroup of $G(F)$ generated by all parahoric subgroups of $G(F)$. $G^0$ is an open normal subgroup of $G(F)$, and the quotient $G(F)/G^0$ is abelian; moreover, $G(F)/G^0$ is finite if and only if $G$ is semisimple, and $G^0 = G(F)$ if and only if $G$ is simply-connected.

Let $B$ be the Bruhat-Tits building of $G(F)$, and let $A$ be the apartment of $B$ associated to $S$; $A$ is isomorphic as a $\mathbb{R}$-affine space to $(X_*(S) \otimes \mathbb{R})/(X_*(Z) \otimes \mathbb{R})$, where $Z$ is the split component of the center of $G$ and $X_*(S), X_*(Z)$ are the groups of cocharacters of $S, Z$. The subgroup $Q$ of the group $X^*(S)$ of characters of $S$ generated by $\Phi$ can then be viewed as a group of affine functions on $A$, by the standard duality product.

For every $x \in B$ and every $r \in \mathbb{R}^+$, let $B(x, r)$ be the subset of elements $y$ of $B$ satisfying the following property: for every apartment $A'$ containing both $x$ and $y$ and every root $\alpha$ of $G$ relatively to $S'$, $S'$ being the maximal split torus associated to $A'$, we have $\alpha(y) - \alpha(x) \leq r$. Since $B(x, r)$ is nonempty, the subgroup $G_{x,r}$ of elements of $G^0$ fixing $B(x, r)$ pointwise is an open bounded subgroup of $G(F)$. Moreover, if $G_x$ is the parahoric subgroup of $G(F)$ fixing $x$, $G_{x,r}$ is normal in $G_x$, and $G_{x,s} \subset G_{x,r}$ for $s > r$. We’ll also write:

$$G^+_{x,r} = \bigcup_{s>r} G_{x,s};$$

the group $G^+_{x,r}$ is also normal in $G_x$. Note that since the valuation is discrete, we have $G^+_{x,r} = G_{x,s}$ for any $s > r$ sufficiently close to $r$.

These groups were first introduced in [7] and have been used in [5] and [6] to classify unramified types, in the context of $p$-adic fields, that is fields with finite residual fields. They constitute a standard filtration of parahoric subgroups of $G(F)$ by open normal subgroups, in the following sense: let $A$ be a facet of $B$, and $P_A$ be the parahoric subgroup of $G$ fixing $A$ pointwise. Then for every $x \in A$, $G_{x,0} = G_x = P_A$, and the $G_{x,r}$, with $x \in A$ and $r \geq 0$, are a basis of neighborhoods of unity. Moreover, for every $x \in A$, $G_{x,0}/G^+_{x,0}$ is the group of $K$-points of a reductive $K$-group (which depends only on $A$), and for every $r > 0$, $G_{x,r}/G^+_{x,r}$ is abelian.

We can consider the concave function $f$ on $\Phi$ defined by $f(\alpha) = \alpha(x) + r$ for every $\alpha \in \Phi$. Let $U_{x,r} = U_{G,x,r}$ be the subgroup $U_f$ of $G(F)$ attached to $f$ as in [2, I.6.4]; $U_{x,r}$ is a normal subgroup of $G_x$, and there exists a subgroup $T'$ of $T(F)$, where $T$ is the centralizer of $S$ in $G$, such that $G_{x,r} = T'U_{x,r}$. (For given $x, r$, the subgroups $T''U_{x,r}$, with $T'' \subset T(F)$, which are normal in $G_x$ may be classified with the help of [2, I.6.4.19].)
Ona can naturally ask if these subgroups, for \( r > 0 \), exhaust the open normal pro-nilpotent subgroups of \( G_x \), at least when \( G \) is quasi-simple. The answer is in general no, but they can still be used to classify them.

First we’ll show that to every open bounded subgroup \( H \) of \( G \) normalized by the unique parahoric subgroup \( P_T \) of \( T(F) \), we can attach a concave function \( f_H \) on \( \Phi \). More precisely, we have the following result:

**Proposition 1.1** Assume \( \Phi \) satisfies \((G1)\), and:

- either \( p \neq 2 \) or \( F \) is absolutely unramified;
- \( K \) has at least 4 elements, and if \( K \) has exactly 4 \((\text{resp. 5})\) elements, \( \Phi \) has no component of type \( A_2 \) \((\text{resp. } A_1, C_n \text{ or } BC_n)\).

Let \( H \) be an open bounded pro-nilpotent subgroup of \( G(F) \) normalized by \( P_T \); there exists a bounded subgroup \( T' \) of \( T(F) \) and a concave function \( f_H \) on \( \Phi \) such that \( H = T'U_{f_H} \).

Since every parahoric subgroup of \( G(F) \) contains the parahoric subgroup of some maximal torus of \( G(F) \), this will in particular be true for open normal subgroups of \( G_x \).

We then consider more precisely the concave functions attached to the open normal pro-nilpotent subgroups of parahorics; and more particularly of some given Iwahori subgroup \( I \) of \( G(F) \); we will in fact study the function \( f_e = f - f_I \). By establishing some properties of the elements of a given class of subsets of \( \Phi \), which will be called \( \Delta' \)-complete subsets, where \( \Delta' \) is the extended basis of \( \Phi \) associated to \( I \), we will then determine \( r \in \mathbb{R}_+^* \) and \( x \in A \) which will satisfy the main result of the paper:

**Theorem 1.2** Assume \( G \) is quasi-simple and the conditions of the previous proposition are satisfied. Let \( A \) be a facet of \( B \), \( \overline{A} \) be the closure of \( A \) and \( G_A \) be the parahoric subgroup of \( G \) fixing \( A \); let \( H \) be any open normal pro-nilpotent subgroup of \( G_A \). There exists \( x \in \overline{A} \) and \( r \in \mathbb{R}_+^* \) such that \( H \) satisfies one of the following conditions:

- \( U_{x,r} \subset H \subset G_{x,r} \);
- there exists \( r' > r \) and a proper Levi subgroup \( M \) of \( G \) containing \( T(F) \) and such that \( U_{x,r'}U_{M,x,r} \subset H \subset G_{x,r'}M_{x,r} \).

We conclude with an easy generalization of our main result to non-quasi-simple groups.
2 Generalities

2.1 A few miscellaneous notations

Let $F'$ be a valued field with discrete valuation; we denote by $v_{F'}$ the normalized valuation on $F'$, that is the unique valuation whose image is precisely $\mathbb{Z} \cup \{+\infty\}$. If $E$ is any extension of $F'$, we’ll denote again by $v_{F'}$ the valuation on $E$ whose restriction to $F'$ is $v_{F'}$.

Let $H$ be a group and $H'$ be a subgroup of $H$; we’ll write $N_H(H')$ (resp. $Z_H(H')$) for the normalizer (resp. the centralizer) of $H'$ in $H$.

For every $r \in \mathbb{R}$, we’ll write floor$(r)$ (resp. ceil$(r)$) for the largest integer $\leq r$ (resp. the smallest integer $\geq r$).

2.2 More about roots and affine roots

This section is devoted to some results about root systems which will be useful later. (For the basic results about root systems, see [1]).

Let $V$ be a finite-dimensional $\mathbb{R}$-vector space, and let $\Phi$ be a root system contained in $V$. We’ll assume in this subsection that $\Phi$ is irreducible.

We’ll denote by $W$ the Weyl group of $\Phi$. Let $(\cdot, \cdot)$ be a $W$-invariant scalar product on $V$; we’ll assume $(\cdot, \cdot)$ is chosen such that:

- if $\Phi$ is simply-laced, $(\alpha, \alpha) = 2$ for every $\alpha \in \Phi$;
- if $\Phi$ is not simply-laced, $(\alpha, \alpha) = 2$ if $\alpha$ is a long root in $\Phi$.

If $\Phi$ is reduced, we’ll write $h(\Phi)$ for the Coxeter number of $\Phi$. If $\Phi$ is of type $BC_n$, we’ll set $h(\Phi) = 2n + 1$, that is the Coxeter number of $\Phi$ plus one.

Let $\Phi_{aff}$ be the affine root system associated to $\Phi$, which will be identified with the subset $\Phi \times \mathbb{Z}$ of $X^*(S) \times \mathbb{Z}$. We’ll denote by $W_{aff}$ the affine Weyl group of $\Phi_{aff}$.

Let $\Delta$ be a basis of $\Phi$, and let $\Delta'$ be the extended basis $\Delta \cup \{-\alpha_M\}$ of $\Phi$, where $\alpha_M$ is the largest root of $\Phi$ w.r.t $\Delta$. Set:

$$\Delta_{aff} = \{(\alpha, 0) | \alpha \in \Delta\} \cup \{(-\alpha_M, 1)\};$$

$\Delta_{aff}$ is a basis of $\Phi_{aff}$. 

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Let $\Phi^+$ (resp. $\Phi^+_{aff}$) be the set of positive elements of $\Phi$ (resp. $\Phi_{aff}$) w.r.t $\Delta$ (resp. $\Delta_{aff}$); $\Phi^+_{aff}$ is the set of elements $(\alpha, v)$ in $\Phi_{aff}$ such that $v \geq 0$ if $\alpha \in \Phi^+$ and $v \geq 1$ else. It is well-known (see for example [7, 2]) that every element of $\Phi^+_{aff}$ can be uniquely written as a linear combination with nonnegative integer coefficients of elements of $\Delta_{aff}$; we deduce from this the following result:

**Proposition 2.1** Every element $\alpha$ of $\Phi$ can be written as a linear combination with nonnegative integer coefficients of elements of $\Delta'$; moreover, there is an unique such combination satisfying the following condition: the sum of the coefficients is strictly smaller than $h(\Phi)$.

Consider the projection on $\Phi$ of the decomposition of $(\alpha, v) \in \Phi^+_{aff}$ for any suitable $v$; the unique combination satisfying the last condition is the one obtained with $v = 0$ (resp. $v = 1$) if $\alpha > 0$ (resp. $\alpha < 0$). □

We'll call the height of $(\alpha, v)$ (resp. $\alpha$) relatively to $\Delta_{aff}$ (resp. $\Delta'$) and we'll write $h(\alpha, v)$ (resp. $h(\alpha)$) for the sum of the coefficients of $(\alpha, v)$ (resp. $\alpha$) defined as above.

For every $\alpha \in \Phi$, let $\varepsilon_\alpha$ be the smallest integer such that $(\alpha, \varepsilon_\alpha) \in \Phi^+_{aff}$, that is $\varepsilon_\alpha = 0$ if $\alpha > 0$, $\varepsilon_\alpha = 1$ if $\alpha < 0$. The following results are immediate:

- if $(\alpha, u), (\beta, v)$ are elements of $\Phi^+_{aff}$ such that $(\alpha + \beta, u + v) \in \Phi_{aff}$, then $h(\alpha + \beta, u + v) = h(\alpha, u) + h(\beta, v)$;
- for every $\alpha \in \Phi$, $h(-\alpha, \varepsilon_{-\alpha}) = h(\Phi) - h(\alpha, \varepsilon_\alpha)$.

We obtain the following corresponding assertions about elements of $\Phi$: 

- if $\alpha, \beta$ are elements of $\Phi$ such that $\alpha + \beta \in \Phi$, then $h(\alpha + \beta) \equiv h(\alpha) + h(\beta)$ modulo $h(\Phi)$;
- for every $\alpha \in \Phi$, $h(-\alpha) = h(\Phi) - h(\alpha)$.

In particular, for every $\alpha \in \Phi$, we have $h(\alpha) \in \{1, \ldots, h(\Phi) - 1\}$.

We can define a partial order on $\Phi^+_{aff}$ relatively to $\Delta_{aff}$ as follows: for every $(\alpha, u), (\beta, v) \in \Phi$, we have $\alpha \leq \beta$ if all the coefficients of elements of $\Delta_{aff}$ in the decomposition of $(\alpha, u)$ are smaller than or equal to the corresponding ones in the decomposition of $(\beta, v)$. In particular, if $u < v$, we have $(\alpha, u + \varepsilon_\alpha) < (\beta, v + \varepsilon_\beta)$ for every $\alpha, \beta \in \Phi$.

We deduce from this order a partial order on $\Phi$ as follows: $\alpha \leq \beta$ if and only if $(\alpha, v + \varepsilon_\alpha) \leq (\beta, v + \varepsilon_\beta)$ for every $v$. This partial order is different from the usual one relative to $\Delta$, but it is easy to see that the restrictions of both orders to the subset of positive (resp. negative) elements are the same; the minimal elements (resp. the maximal elements) of our order are the elements of $\Delta'$ (resp. $-\Delta'$), and we easily see that $\alpha \leq \beta$ implies $-\beta \leq -\alpha$. In the
sequel, unless another order on $\Phi$ is explicitly mentioned, we will always refer to this one.

We have the following results:

**Proposition 2.2** Let $\alpha, \beta$ be two elements of $\Phi$ such that $\alpha + \beta \in \Phi$; if $h(\alpha + \beta) = h(\alpha) + h(\beta)$ (resp. if $h(\alpha + \beta) = h(\alpha) + h(\beta) - h(\Phi)$), then $\alpha + \beta$ is greater (resp. lesser) than both $\alpha$ and $\beta$.

The first assertion is obvious. The second one is obtained by simply remarking that if $h(\alpha + \beta) = h(\alpha) + h(\beta) - h(\Phi)$, then we have:

$$h(-\alpha - \beta) = h(\Phi) - h(\alpha + \beta) = 2h(\Phi) - h(\alpha) - h(\beta) = h(-\alpha) + h(-\beta).$$

□

**Proposition 2.3** Let $\alpha, \beta$ be elements of $\Phi$; we have $\alpha \leq \beta$ if and only if there exist elements $\gamma_1, \ldots, \gamma_t$ of $\Delta'$ such that:

- for every $i \in \{1, \ldots, t\}$, $\alpha_i = \alpha + \sum_{j=1}^{i} \gamma_i$ is an element of $\Phi$;
- $\alpha_t = \beta$.

Assume $\alpha \leq \beta$; we’ll show the existence of the $\gamma_i$. Write $\beta = \alpha + \sum_{i=1}^{t} \delta_i$, with $\delta_1, \ldots, \delta_t \in \Delta'$. If $t = 0$, there is nothing to prove; suppose $t > 0$. Since $(\beta, \beta) > 0$, we have either $(\beta, \delta_i) > 0$ for some $i$ or $(\beta, \alpha) > 0$. In the first case, set $\gamma_i = \delta_i$; the result follows from the induction hypothesis applied to $\alpha$ and $\beta - \delta_i$. In the second case, $\beta - \alpha$ is a root. If $(\alpha, \delta_i) < 0$ for some $i$, we can set $\gamma_1 = \delta_i$ and conclude by the induction hypothesis applied to $\alpha + \delta_i$ and $\beta$; if there is no such $i$, we are in one of the following two cases:

- $\Phi$ is of type $B_n$, $C_n$ or $F_4$, and $\alpha$ and $\beta - \alpha$ are orthogonal but not strongly orthogonal;
- $\Phi$ is of type $G_2$, and $\alpha$ and $\beta - \alpha$ are short roots such that $(\alpha, \beta) = 1$.

In both cases, $\beta$ is a long root; the result is then true for $-\beta$ and $-\alpha$, which obviously implies the result for $\alpha$ and $\beta$.

Conversely, assume the $\gamma_i$ do exist. By an easy induction we may assume $t = 1$; we then have either $h(\beta) = h(\alpha) + 1$ or $h(\beta) = h(\alpha) - h(\Phi) + 1$; since the length of any element of $\Phi$ is contained in $\{1, \ldots, h(\Phi) - 1\}$, the second case is impossible and we obtain $\alpha \leq \beta$, as required. □

Let $\Phi$ be the subset $\Phi \cup \{0\}$ of $X^*(T)$. Let $\alpha, \beta$ be two elements of $\Phi$; it is
easy, with the help of [1, 1.3, cor. to th. 1], to check that we always have:

- if \((\alpha, \beta) < 0\), then \(\alpha + \beta \in \Phi\);
- if \((\alpha, \beta) > 0\), then \(\alpha - \beta \in \Phi\).

Set also \(\Phi_{aff} = \Phi \times \mathbb{Z}\). For every \(v > 0\), \((0, v) \in \Phi_{aff}\) can also be uniquely written as a linear combination with nonnegative integer coefficients of elements of \(\Delta_{aff}\); moreover, we have \(h(0, v) = h(\Phi)v\), and the partial order on \(\Phi^+_{aff}\) extends canonically to \(\Phi^+_{aff} = \Phi^+_{aff} \cup \{0\} \times \mathbb{N}^*\) by setting \((\alpha, v - 1 + \varepsilon_\alpha) \leq (0, v) \leq (\beta, v + \varepsilon_\beta)\) for every \(\alpha, \beta\).

**Proposition 2.4** Let \((\alpha_1, c_1 + \varepsilon_{\alpha_1}), \ldots, (\alpha_t, c_t + \varepsilon_{\alpha_t})\) be elements of \(\Phi^+_{aff}\) whose sum is also an element \((\alpha, c + \varepsilon_\alpha)\) of \(\Phi^+_{aff}\). Let \(\beta \in \Phi\) and \(c' \in \{c, c+1\}\) be such that \((\alpha, c + \varepsilon_\alpha) \leq (\beta, c' + \varepsilon_\beta) \leq (0, c + 1)\) (resp. \((0, c) \leq (\beta, c' + \varepsilon_\beta) \leq (\alpha, c + \varepsilon_\alpha)\)); there then exist elements \((\beta_1, c_1' + \varepsilon_{\beta_1}), \ldots, (\beta_t, c_t' + \varepsilon_{\beta_t})\) of \(\Phi^+_{aff}\) whose sum is \((\beta, c + \varepsilon_\beta)\) and such that \((\alpha_i, c_i + \varepsilon_{\alpha_i}) \leq (\beta_i, c_i' + \varepsilon_{\beta_i}) \leq (0, c_i + 1)\) (resp. \((0, c_i) \leq (\beta_i, c_i' + \varepsilon_{\beta_i}) \leq (\alpha_i, c_i + \varepsilon_{\alpha_i})\)) for every \(i\).

We’ll show the proposition with \((\alpha, c + \varepsilon_\alpha) \leq (\beta, c' + \varepsilon_\beta) \leq (0, c + 1)\), the proof for \((0, c) \leq (\beta, c' + \varepsilon_\beta) \leq (\alpha, c + \varepsilon_\alpha)\) being similar. With the previous proposition and an easy induction, we can assume there exists \(\gamma \in \Delta'\) such that \(\beta = \alpha + \gamma\), hence \((\beta, c' + \varepsilon_\beta) = (\alpha, c + \varepsilon_\alpha) + (\gamma, \varepsilon_\gamma)\). If \((\alpha_i, \gamma) > 0\) for some \(i\), then \(\alpha_i + \gamma \in \Phi\), and since \((\beta_i, c_i' + \varepsilon_{\beta_i}) = (\alpha_i, c_i + \varepsilon_{\alpha_i}) + (\gamma, \varepsilon_\gamma) > (\alpha_i, c_i + \varepsilon_{\alpha_i})\), setting \((\beta_j, c_j' + \varepsilon_{\beta_j}) = (\alpha_j, c_j + \varepsilon_{\alpha_j})\) for every \(j \neq i\) proves the proposition. Suppose now \((\alpha, \gamma) \geq 0\) for every \(i\). Since \((\alpha, \alpha) = 0\), there exists an \(i\) such that \((\alpha, \alpha_i) > 0\), hence \(\alpha - \alpha_i \in \Phi\). If \(\alpha = \alpha_i\), then \(\alpha_i + \gamma \in \Phi\) and we conclude as before; we may then suppose \(\alpha - \alpha_i \neq 0\).

Since \((\alpha, \alpha_i) > 0\) and \((\gamma, \alpha_i) \geq 0\), we have \((\beta, \alpha_i) = (\alpha + \gamma, \alpha_i) > 0\), hence \(\beta - \alpha_i \in \Phi\). Moreover, setting \((\alpha', d + \varepsilon_{\alpha'}) = (\alpha, c + \varepsilon_\alpha) - (\alpha_i, c_i + \varepsilon_{\alpha_i})\) and \((\beta', d' + \varepsilon_{\beta'}) = (\beta, c' + \varepsilon_\beta) - (\alpha_i, c_i + \varepsilon_{\alpha_i})\), we must have \(d = d'\) or \(\beta' = 0\), since \(d < d'\) and \(\beta' \neq 0\) would imply \(h(\beta', d' + \varepsilon_{\beta'}) = h(\alpha', d + \varepsilon_{\alpha'}) + 2\), hence \(h(\beta, c + \varepsilon_\beta) \geq h(\alpha, c + \varepsilon_\alpha) + 2\), which contradicts our assumptions; we then have \((\alpha', d + \varepsilon_{\alpha'}) \leq (\beta', d' + \varepsilon_{\beta'}) \leq (0, d+1)\). This way, we can show the result by induction on \(t\); since the case \(t = 1\) is trivial, the proposition is proved. ☐

### 2.3 Valuations

From now on, we’ll assume \(\Phi\) is the root system of \(G\) relatively to some maximal split torus \(S\) of \(G\). We will denote by \(T\) the centralizer of \(S\) in \(G\); since \(G\) is quasisplit, \(T\) is a maximal torus of \(G\), which splits over the same unramified extension of \(F\) as \(G\).

For every \(\alpha \in \Phi\), let \(U_\alpha\) be the root subgroup of \(G\) associated to \(\alpha\). If \(2\alpha\) is
not a root, there exists a finite unramified extension $F_\alpha$ of $F$ such that the
group $U_\alpha(F)$ is isomorphic to $F_\alpha$; for every $x \in F_\alpha$, we’ll write $u_\alpha(x)$ for the
image of $x$ in $U_\alpha(F)$ by some given isomorphism. If $2\alpha \in \Phi$, there exists a
quadratic unramified extension $F_\alpha$ of $F_{2\alpha}$ such that $U_\alpha(F)$ is isomorphic to
the semidirect product $H_\alpha$ of $F_\alpha$ by $F_{2\alpha}$, with the addition law defined as follows:

$$(x, y) + (x', y') = (x + x', y + y' + x\sigma(x)),$$

when $\sigma$ is the nontrivial element of $\text{Gal}(F_\alpha/F_{2\alpha})$; moreover, the image of
$\{(0, y) | y \in F_{2\alpha}\}$ is $U_{2\alpha}(F)$. Let’s choose such an isomorphism; for every $x \in F_\alpha$,
we’ll set $u_\alpha(x)$ to be the image of $(x, 0)$ by that isomorphism.

For every $\alpha$, we’ll write $\mathcal{O}_\alpha$ for the ring of integers of $F_\alpha$, $\mathfrak{p}_\alpha$ for the maximal
ideal of $\mathcal{O}_\alpha$, $K_\alpha$ for the residual field of $F_\alpha$.

Let $(v_\alpha)_{\alpha \in \Phi}$ be a valuation on $(T, (U_\alpha)_{\alpha \in \Phi})$; for every $\alpha$, $v_\alpha$ is an application
from $U_\alpha(F)$ to $\mathbb{R} \cup \{+\infty\}$ satisfying the following conditions:

- for every $\alpha \in \Phi$, there exist constants $b_\alpha \in \mathbb{R}_+, c_\alpha \in \mathbb{R}$ such that for every
  $x \in F_\alpha$, $v_\alpha(u_\alpha(x)) = b_\alpha v_F(x) + c_\alpha$;
- if $2\alpha \in \Phi$, for every $(x, y) \in H_\alpha$, if $u$ is the image of $(x, y)$ in $U_\alpha(F)$,
  $v_\alpha(u) = \text{Inf}(v_\alpha(x, 0), \frac{1}{2}v_\alpha(0, y))$;
- the commutator relations: for every $\alpha \in \Phi$ and every $r \in \mathbb{R}$, let $U_{\alpha, r}$ be the
  subgroup of elements $u$ of $U_\alpha(F)$ such that $v_\alpha(u) \geq r$. If $\alpha, \beta$ are elements
  of $\Phi$ such that $\alpha + \beta \in \Phi$, and if $r, r'$ are elements of $\mathbb{R}$, the commutator
  subgroup $[U_{\alpha, r}, U_{\beta, r'}]$ is contained in $U_{\alpha + \beta, r + r'}$; (and by [3] and [2, Appendix
  A], if (G1) is satisfied, this inclusion is an equality when $r \in \text{Im}(v_\alpha)$ and
  $r' \in \text{Im}(V_\beta)$);
- for every $\alpha \in \Phi$, if $u \in U_\alpha(F)$ and $u' \in U_{-\alpha}(F)$ generate a subgroup of $G$
  which is bounded but not pro-solvable, then $v_\alpha(u) + v_{-\alpha}(u') = 0$.

This is a rewriting of [2, I, definition 6.2.1] in our context (with a discrete
valuation and a group which splits over an unramified extension).

We will also define a valuation on $T(F)$ the following way: for each $r \in \mathbb{R}$, we
define the group $T_r$ as the subgroup of elements $t \in T(F)$ such that for every
$\alpha \in \Phi$ and every $r' \in \mathbb{R}$, we have:

$$\{t, U_{\alpha, r}\} \subset U_{\alpha, r + r'}.$$

For every $t \in T(F)$, we’ll set $v_0(t) = r$ if $t$ belongs to $T_r$ but not to any
$T_{r'}$, $r' > r$. It is easy to see that adding $v_0$ to the family $(v_\alpha)_{\alpha \in \Phi}$ leads to an
extension of the valuation in the sense of [2, I.6.4.38].
We can even define valuations on bounded subgroups of $G(F)$. For every \( \alpha \in \Phi \), set \( U'_\alpha = U_\alpha(F) \) if \( 2\alpha \not\in \Phi \), and if \( 2\alpha \in \Phi \), let \( U'_\alpha \) be the image of the application \( u_\alpha \) (which is not necessarily a subgroup of \( U_\alpha(F) \)); set \( U_{\alpha,r} = U_{\alpha,r} \cap U'_\alpha \) for every \( r \in \mathbb{R}^+ \). Let \( G(F)_r \) be, for a given \( r \), the subgroup of \( G(F) \) generated by \( T_r \) and the \( U'_{\alpha,r} \), \( \alpha \in \Phi \); we’ll also write \( G(F)_r^+ \) for the union of all the \( G(F)_r, r' > r \). The group \( G(F)_0 \) is a parahoric subgroup of \( G(F) \), and \( G(F)_0^+ \) is its pro-nilpotent radical; moreover, the groups \( G(F)_r \) and \( G(F)_r^+ \) are normal subgroups of \( G \).

Of course, these groups depend on the chosen valuation. In fact, we can even show that we can attach to any valuation a point \( x \) of the apartment \( A \) of \( B \) associated to \( T \), and conversely, in such a way that \( G(F)_r = G(x,r) \) for every \( r \), but we will not use this fact; we will only use the following property (see [2, I.7.2]): for every parahoric subgroup \( H \) of \( G(F) \) containing the parahoric subgroup \( P_T \) of \( T(F) \), there exists a valuation such that \( H = G(F)_0 \), which implies by [2, I.3.3.1] that every bounded subgroup of \( G(F) \) is contained in the group \( G(F)_0 \) relative to some valuation.

For every \( g \in G(F)_0 \), we’ll set \( v_{G}(g) = r \) if \( g \) belongs to \( G(F)_r \) but not to \( G(F)_r^+ \).

For convenience, we’ll write \( U_0 = T \), and \( U_{0,r} = T_r \) for every \( r \). The following result follows immediately from the definitions:

**Proposition 2.5** Let \( \alpha \) be any element of \( \Phi \), and let \( u \) be any element of \( U'_\alpha \cap G(F)_0 \). Then \( v_{G}(u) = v_{\alpha}(u) \).

### 2.4 Concave functions

From now on and until the end of the paper, we’ll assume \( \Phi \) satisfies (G1). Moreover, in this subsection, we’ll also assume \( \Phi \) is connected.

Let \( f \) be a map from \( \Phi \) to \( \mathbb{R} \); \( f \) is said to be concave if it satisfies the following conditions:

- for each \( \alpha, \beta \in \Phi \) such that \( \alpha + \beta \in \Phi \), \( f(\alpha + \beta) \leq f(\alpha) + f(\beta) \);
- for every \( \alpha \in \Phi \), \( f(\alpha) + f(-\alpha) \geq 0 \).

Let \( U_f \) be the subgroup of \( G \) generated by the \( U_{\alpha,f(\alpha)}, \alpha \in \Phi \). The following properties are well-known (see [2, I.6.4]):

- if \( f \) is concave, then for every \( \alpha \in \Phi \), \( U_f \cap U_\alpha(F) \) is equal to \( U_{\alpha,f(\alpha)} \) if \( 2\alpha \not\in \Phi \), to \( U_{\alpha,f(\alpha)}U_{2\alpha,f(2\alpha)} \) if \( 2\alpha \in \Phi \);
- since \( \Phi \) satisfies (G1), the converse is true if \( f \) is optimal (i.e. if for every
Moreover, if \( f \) is concave and for every \( \alpha \in \Phi \), \( f(\alpha) + f(-\alpha) > 0 \), we deduce immediately from [2, I.6.4.10] that \( U_f \) is pro-solvable and that we have:

\[
U_f = (T \cap U_f) \prod_{\alpha \in \Phi} (U_\alpha(F) \cap U_f)
\]

the product being taken in any order. Conversely, for any pro-solvable subgroup \( H \) of \( G \) satisfying the above condition and such that for every \( \alpha \in \Phi \), \( U_\alpha(F) \cap H = U_{\alpha,f_{\alpha}} \) for some \( f_{\alpha} \), if moreover \( f_{\alpha} \) is maximal among the elements of \( \mathbb{R} \) satisfying that property, then the map \( f_H : \alpha \mapsto f_{\alpha} \) is concave and optimal, we have \( f_H(\alpha) + f_H(-\alpha) > 0 \) for every \( \alpha \in \Phi \), and there exists a subgroup \( T' \) of \( T(F) \) such that \( H = T'U_{f_H} \).

Assume now the valuation has been chosen in such a way that for every \( \alpha \), \( v_\alpha(U'_\alpha) = c_\alpha + \mathbb{Z} \cup \{ +\infty \} \) for some constant \( c_\alpha \); since \( G \) splits over an unramified extension of \( F \), this is always possible.

**Proposition 2.6** Let \( f \) be a concave function; the group \( P_f = P_T U_f \) is contained in a maximal parahoric subgroup. Moreover, if \( f \) is optimal and for every \( \alpha \in \Phi \), we have \( f(\alpha) + f(-\alpha) \leq 1 \) (resp. \( f(\alpha) + f(-\alpha) = 1 \)), \( P_f \) is a parahoric subgroup (resp. an Iwahori subgroup) of \( G(F) \).

According to [2, I.6.4.9], \( P_f \) is the finite union of the cosets:

\[
P_T \prod_{\alpha \in \Phi} U_{\alpha,f(\alpha)} n,
\]

where \( n \) belongs to a system of representatives of \( (N_G(T(F)) \cap P_f)/P_T \); we deduce then immediately from the definitions ([2, I.3.1.1 and I.8.1.1]) that \( P_f \) is bounded; according to [2, I.3.3.3], there exists a maximal parahoric subgroup \( P \) of \( G \) containing \( P_f \). Let \( P_f^+ \) be its pro-nilpotent radical; since \( P \) contains \( P_T \), there exists a concave function \( f_P \) on \( \Phi \) such that \( P = P_{f_P} \); moreover, we have \( P^+ = U_{f_P'} \), where \( f_P' \) is the concave function on \( \Phi \) defined by \( f_P'(\alpha) = 1 - f_P(-\alpha) \). Since \( P_f \) is contained in \( P \), we have \( f(\alpha) \geq f_P(\alpha) \) for every \( \alpha \), which implies:

\[
f(\alpha) \leq 1 - f(-\alpha) \leq 1 - f_P(-\alpha) = f_P'(\alpha)
\]

for every \( \alpha \); hence \( P^+ \) is contained in \( P_f \). The group \( P_f/P^+ \) is therefore a closed subgroup of the reductive group \( P/P^+ \) containing the maximal torus \( P_T/P^+ \); moreover, if \( \Phi_P \) is the root system of \( P/P^+ \) relatively to \( P_T/P^+ \), viewed as the subsystem of elements \( \alpha \in \Phi \) such that \( f_P(\alpha) + f_P(-\alpha) = 0 \), for every \( \alpha \in \Phi_P \), we have \( f_P'(\alpha) + f_P'(\alpha) = 2 \), hence either \( \alpha \) or \( -\alpha \) is a root of
$P_f/P^+$ relatively to $P_T/P^+$; moreover, we deduce easily from the commutator relations that the set of roots of $P_f/P^+$ is closed in $\Phi_P$. This set is then a parabolic subset of $\Phi_P$, and [1, 1.7, prop. 20] shows that $P_f/P^+$ is a parabolic subgroup of $P/P^+$; hence $P_f$ is a parahoric subgroup of $G(F)$, as required.

Assume now $f(\alpha) + f(-\alpha) = 1$ for every $\alpha \in \Phi$. Then $P_f$ is pro-solvable, and $P_f/P^+$ is solvable, hence is a Borel subgroup of $P/P^+$; $P_f$ is then an Iwahori subgroup of $G(F)$, which shows the proposition. □

3 Proof of the proposition 1.1

Assume now the conditions of the proposition 1.1 are satisfied. Let $H$ be an open bounded pro-nilpotent subgroup of $G(F)$ normalized by $P_T$; we’ll show the existence of $T'$ and $f_H$.

We’ll first show some preliminary results.

Lemma 3.1 It is possible to choose the valuation on $G(F)$ in such a way that $H$ is contained in $P_T G(F)_0^+$.  

Since $H$ is bounded, it is contained in some parahoric subgroup of $G$, hence the set $E_H$ of points of $B$ fixed by $H$ is nonempty. Moreover, this set is bounded since $H$ is open, and stable by the action of $P_T$; hence by [2, I.3.2.4], there exists $x \in E_H$ such that $x$ is fixed by $P_T$, which is tantamount to say that the parahoric subgroup $G_x$ of $G(F)$ contains both $H$ and $P_T$.

Moreover, since $H$ is pro-solvable, the subgroup $H/H \cap G_x^+/G_{x,0}$ is a nilpotent subgroup normalized by the maximal torus $P_T/(P_T \cap G_x^+/G_{x,0})$; it is then contained in the nilpotent radical of some Borel subgroup of $G_x/G_{x,0}$, hence $H$ is contained in the pro-nilpotent radical $I^+$ of the corresponding Iwahori subgroup $I$ of $G(F)$.

We may then assume $H = I^+$. We have $I = P_T U_f$, with $f_I$ being a concave function on $\Phi$ such that $f_I(\alpha) + f_I(-\alpha) = 1$ for every $\alpha \in \Phi$. Let $P_0$ be a special parahoric subgroup containing $I$ and $P_1$ be its pro-nilpotent radical, let $\Delta_0$ be the basis of $\Phi$ associated to the Borel subgroup $I/P_1$ of $P_0/P_1$; assume the valuation on $G(F)$ has been chosen in such a way that we have $f_I(\alpha) = \frac{1}{h(\Phi)}$ for every $\alpha$. Let $\beta$ be any element of $\Phi$; if $\beta$ is positive relatively to $\Delta_0$, we have $U_\beta(F) \cap I = U_\beta(F) \cap P_0$, and we obtain that $f_I(\beta) = \frac{h(\beta)}{h(\Phi)} > 0$ ($h(\beta)$ being taken relatively to $\Delta_0$). If now $\beta < 0$, we have:

$$f_I(\beta) = 1 - f_I(-\beta) = \frac{h(\Phi) - h(\beta)}{h(\Phi)} > 0.$$
Since for every $\beta \in \Phi$, $f_I(\beta) > 0$, $H$ is contained in $P_T G(F)_0^+$, as required. □

For the sake of simplicity of notations, until the end of the proof of the proposition, we will assume $v_G(g) \in \mathbb{Z} \cup \{+\infty\}$ for every $g \in G(F)_0$; this can be done for example by multiplying the valuation previously considered by $h(\Phi)$.

**Proposition 3.2** Assume at least one of the following conditions is fulfilled:

- $K$ has at least 7 elements;
- $K$ has 5 elements and $\Phi$ has no component of type $A_1$, $C_n$ or $B_{C_n}$;
- $K$ has 4 elements and $\Phi$ has no component of type $A_2$.

Let $h$ be any element of $H$; write:

$$h = \prod_{\alpha \in \Phi} h_{\alpha},$$

with $h_{\alpha} \in U'_{\alpha}$ for every $\alpha \in \Phi$. Let $v = v_G(h)$; for every $\alpha \neq 0$ such that $v_{\alpha}(h_{\alpha}) = v$, $H \cap h_{\alpha} G^+_v$ is nonempty.

Let $H_v$ (resp. $H_v^+$) be the subgroup of elements $h \in H$ such that $v(h) \geq v$ (resp. $v(h) > v$); $H_v^+$ is a normal subgroup of $H_v$, and since $v > 0$, the group $H_v/H_v^+$ is abelian, and is a subgroup of $G(F)_v/G(F)_v^+$, which is isomorphic to a direct product of copies of $K$; moreover, $H_v$ and $H_v^+$ are normalized by $P_T$, and if $P_T^+$ is the pro-nilpotent radical of $P_T$, we deduce from the commutator relations that we have:

$$[P_T^+, H_v] \subset H_v^+.$$

The torus $T(K)$ acts then on the abelian group $H_v/H_v^+$. By a well-known result (see for example [4, I.2.11(3)]), $H_v/H_v^+$ is the direct sum of its $T(K)$-weight subgroups; therefore, the assertion of the lemma is true as soon as all the elements of $\Phi$ occurring as weights of $H_v/H_v^+$ are actually distinct characters of $T(K)$.

Let $\alpha \neq \beta$ be two elements of $\Phi$; we’ll show that $\alpha - \beta$ is nontrivial on $T(K)$. This is always the case if $K$ is infinite; if $K$ is finite and $q = \text{card}(K)$, this is true as soon as $\alpha - \beta \not\in (q - 1)X^*(S)$ (if and only if $\alpha - \beta \not\in (q - 1)X^*(S)$ when $G$ is split).

Suppose first $\beta = 0$, and let $\alpha^\vee$ be the coroot associated to $\alpha$; since by definition $\langle \alpha, \alpha^\vee \rangle = 2$, $\alpha$ is nontrivial on $T(K)$ as soon as $q \geq 4$.

Suppose now $\beta = -\alpha$; by the same argument, $\alpha - \beta = 2\alpha$ is nontrivial on $T(K)$ as soon as either $q = 4$ or $q \geq 7$. Moreover, the case $\alpha \in 2X^*(T)$ occurs
only when $\Phi$ has components of type $A_1$, $C_n$ or $BC_n$; in all other cases, there exists $\beta \in \Phi$ such that $\langle \alpha, \beta \rangle = 1$, and $2\alpha$ is nontrivial on $T(K)$ as soon as $q \geq 4$.

Suppose now $\beta = 2\alpha$; we have then $(2\alpha)^\vee = \frac{1}{2} \alpha^\vee$, and $\langle \alpha - \beta, (2\alpha)^\vee \rangle = 1$; $\alpha - \beta = -\alpha$ is then nontrivial on $T(K)$ as soon as $q \geq 3$. By the same reasoning, if $\beta = -2\alpha$, $\alpha - \beta = 3\alpha$ is nontrivial on $T(K)$ as soon as $q \geq 5$.

Suppose next $\alpha$ and $\beta$ are orthogonal. Since $\langle \alpha - \beta, \alpha \rangle = 2$, $\alpha - \beta$ is nontrivial on $T(K)$ as soon as $q \geq 4$.

Suppose finally $\alpha$ and $\beta$ are non-orthogonal and linearly independant. According to [8, 7.5.1], we then have either $\langle \alpha, \beta \rangle = \pm 1$ or $\langle \beta, \alpha \rangle = \pm 1$. Assume for example $\langle \beta, \alpha \rangle = \pm 1$; we obtain $0 < \langle \alpha - \beta, \alpha \rangle \leq 3$, and $\alpha - \beta$ is nontrivial on $T(K)$ as soon as $q \geq 5$; moreover, if $\Phi$ is simply-laced and $\alpha$ and $\beta$ don’t belong to any component of type $A_2$ of $\Phi$, there exists $\gamma \in \Phi$ such that $\langle \alpha - \beta, \gamma \rangle = 1$, and we can replace $q \geq 5$ by $q = 4$. (Remember that the case $\Phi$ not simply-laced and $q = 4$ is excluded by (G1).)

By summarizing the different cases, we get the assertion of the proposition. $\Box$

**Lemma 3.3** Assume $p \neq 2$ or $F$ is absolutely unramified. Let $\alpha$ be an element of $\Phi$, let $v < v'$ be two integers such that there exists $h \in H$ such that $h = h_\alpha h_r$, with $h_\alpha \in U_\alpha(F)$, $v_\alpha(h_\alpha) = v$ and $v(h_r) \geq v'$. Then for any $h'_\alpha \in U_{\alpha,v}$, $H \cap h'_\alpha G_{v'}$ is nonempty.

Assume first $2\alpha \not\in \Phi$. Set $h_\alpha = u_\alpha(x)$, $x \in F_\alpha$; for each $t \in P_T$, we have $th_t t^{-1} \in G_{v'}$, and $u_\alpha(\alpha(t)x)(th_t t^{-1}) \in H$. If there exists $y \in O_\alpha^*$ such that $t = \alpha^\vee(y)$, we obtain, setting $h'_\alpha = th_t t^{-1} \in G_{v'}$:

$$u_\alpha(y^2 x) h'_\alpha \in H$$

for each $y$. Let $X$ be the subgroup of the elements $b$ of $O_\alpha$ such that $u_\alpha(b) G_{v'}$ meets $H$; in order to show that $X = xO_\alpha$, we only have to check that the ring $O_\alpha$ is generated by the squares it contains.

Suppose first the characteristic $p$ of $K$ is odd or zero; we have:

$$z = \frac{1}{2} ((z + 1)^2 - z^2 - 1),$$

which proves the assertion since $\frac{1}{2} = 2(\frac{1}{2})$.

Suppose now $p = 2$. Since $K_\alpha$ is perfect, any element of $K_\alpha^*$ is a square; moreover, we have $2 = 1^2 + 1^2$. We then only have to show that every element of $1 + p_\alpha$ belongs to the subring of $O_\alpha$ generated by the squares, which is simply
done by remarking that if \( z \in 1 + \mathfrak{p}_\alpha \), since \( F_\alpha \) is absolutely unramified, \( \frac{z+1}{2} \) and \( \frac{\alpha}{2} \) belong to \( \mathcal{O}_\alpha \) and \( z = (\frac{z+1}{2})^2 - (\frac{\alpha}{2})^2 \).

(Remark: when \( p = 2 \), the result is false when \( F \) is absolutely ramified. It is easy to check for example that for \( F = F_0[\sqrt{2}] \), where \( F_0 \) is any unramified extension of \( \mathbb{Q}_2 \), and \( G = SL_2 \), we can obtain counterexamples to the assertion of this lemma, and even to the proposition 1.1.)

Assume now \( 2\alpha \in \Phi \); since \( \Phi \) is then not simply-laced, by (G1) we must have \( p \neq 2 \). By an explicit computation in \( SU_3 \), we easily see that for an appropriate choice of the isomorphism between \( H_\alpha \) and \( U_\alpha(F) \), we have, setting again \( t = \alpha^\nu(y) \):

\[
tu_\alpha(x)t^{-1} = u_\alpha(y^2\sigma(y)^{-1}x),
\]

with \( \sigma \) being the nontrivial element of Gal(\( F_\alpha/F_{2\alpha} \)). We claim that the elements \( y^2\sigma(y)^{-1}, y \in \mathcal{O}_\alpha \); generate the ring \( \mathcal{O}_\alpha \); for \( y \in \mathcal{O}_{2\alpha} \), we simply have \( y^2\sigma(y)^{-1} = y \), and for any \( y \in \mathcal{O}_\alpha \) such that \( \sigma(y) = -y \) (such an \( y \) exists because \( F_\alpha/F_{2\alpha} \) is unramified and \( p \neq 2 \)), we have \( y^2\sigma(y)^{-1} = -y \). Since such an element and 1 generate \( \mathcal{O}_\alpha \) as a \( \mathcal{O}_{2\alpha} \)-module, the claim is proved.

On the other hand, we have:

\[
[u_\alpha(x')U_{2\alpha,2\epsilon}G_{\nu'}, u_\alpha(x'')U_{2\alpha,2\epsilon}G_{\nu'}] \subset u_{2\alpha}(\sigma(x')x'' - x'\sigma(x''))G_{\nu'},
\]

where \( \sigma \) is the nontrivial element of Gal(\( F_\alpha/F_{2\alpha} \)); moreover, if \( x', x'' \) are chosen such that the image of \( \frac{x'}{2} \) in \( K_\alpha \) doesn’t belong to \( K \), the valuation of \( \sigma(x')x'' - x'\sigma(x'') \) is exactly \( 2c \); since \( 2(2\alpha) \notin \Phi \), we deduce from the preceding case that for each \( y \in \varpi^{2c}\mathcal{O}_{2\alpha} \), \( H \) meets \( u_{2\alpha}(y)G_{\nu'} \). We combine these two assertions to obtain that \( H \) meets \( h_\alpha G_{\nu'} \) for every \( h_\alpha \in U_\alpha(F) \) such that \( v_\alpha(h_\alpha) \geq c \), as required. \( \Box \)

For every \( \alpha \in \Phi \) and every positive elements \( v \leq c \) of \( \mathbb{Z} \), let \( f_{\alpha,c,v} \) be the function on \( \Phi \) such that \( f_{\alpha,c,v}(\alpha) = v \), \( f_{\alpha,c,v}(2\alpha) = v \) if \( 2\alpha \in \Phi \), and \( f_{\alpha,c,v}(\beta) = c \) for every \( \beta \neq \alpha, 2\alpha \).

**Lemma 3.4** The function \( f_{\alpha,c,v} \) is a concave function; moreover, \( U_{f_{\alpha,c,v}} \) is normal in \( U_{f_{\alpha,c,v}} \) and the quotient \( U_{f_{\alpha,c,v}}/U_{f_{\alpha,c,v}} \) is abelian.

Let \( \beta, \gamma \) be two elements of \( \phi \) such that \( \beta + \gamma \in \phi \). If \( \beta \) and \( \gamma \) are both contained in \( \{\alpha, 2\alpha\} \), then \( \beta = \gamma = \alpha \) and \( \beta + \gamma = 2\alpha \), and \( f_{\alpha,c,v}(\beta + \gamma) = v \leq 2v = f_{\alpha,c,v}(\beta) + f_{\alpha,c,v}(\gamma) \); if now at least one of them is different from \( \alpha \) and \( 2\alpha \), we have:

\[
f_{\alpha,c,v}(\beta) + f_{\alpha,c,v}(\gamma) \geq c + v \geq c \geq f_{\alpha,c,v}(\beta + \gamma).
\]
since it is obvious that $f_{\alpha,c,v}(\beta) + f_{\alpha,c,v}(-\beta) > 0$ for each $\beta \in \phi$, the concavity of $f_{\alpha,c,v}$ is proved; moreover, the above inequalities imply:

$$[U_{\alpha,f_{\alpha,c,v}(\alpha)}, U_{\alpha,f_{\alpha,c,v}(\alpha)}] \subset U_{2\alpha, 2v} \subset U_{2\alpha, f_{\alpha,c+1,v+1}(2\alpha)}$$

if $2\alpha \in \phi$, and:

$$[U_{\beta,f_{\alpha,c,v}(\beta)}, U_{\gamma,f_{\alpha,c,v}(\gamma)}] \subset U_{\beta+\gamma, c+v} \subset U_{\beta+\gamma, f_{\alpha,c+1,v+1}(\beta+\gamma)}$$

for every $\beta, \gamma \in \phi$ such that $\beta + \gamma \in \phi$ and either $\beta$ or $\gamma$ doesn’t belong to $\{\alpha, 2\alpha\}$; moreover, we have, for every $\beta \in \phi$:

$$[U_{\beta,f_{\alpha,c,v}(\beta)}, U_{0,c}] \subset U_{\beta, c+v} \subset U_{\beta, f_{\alpha,c+1,v+1}(\beta)}$$

and, since either $\beta$ or $-\beta$ doesn’t belong to $\{\alpha, 2\alpha\}$:

$$[U_{\beta,f_{\alpha,c,v}(\beta)}, U_{-\beta,f_{\alpha,c,v}(-\beta)}] \subset U_{0,c+v} \subset U_{0,c+1};$$

We deduce from all these inclusions that $U_{f_{\alpha,c+1,v+1}}$ is normal in $U_{f_{\alpha,c,v}}$ and that the quotient $U_{f_{\alpha,c,v}} / U_{f_{\alpha,c+1,v+1}}$ is abelian, as required. \(\square\)

Consider now the group $G(F)_{c,v} = U_{0,v}G(F)_c$; we have the following result:

**Lemma 3.5** The group $G(F)_{c+1,v+1}$ is normal in $G(F)_{c,v}$, and the quotient $G(F)_{c,v} / G(F)_{c+1,v+1}$ is abelian.

The proof is analogous to the proof of the previous lemma. \(\square\)

Now we’ll prove proposition 1.1. According to the remarks made in the previous section and to the lemma 3.3, we only have to show that we have:

$$H = \prod_{\alpha \in \Phi} (H \cap U_{\alpha}(F)),$$

the product being taken in any order.

Let $h = \prod_{\alpha \in \Phi} h_{\alpha}$ be defined as in the proposition 3.2, and set $v = v_G(h)$; assume moreover that the product is chosen according to some order on $\Phi$ satisfying the following conditions:

- if $\alpha$ is any element of $\Phi$, $0 \leq \alpha$;
- if $\alpha, \beta$ are elements of $\Phi$, we have $\alpha < \beta$ if and only if $v(h_{\alpha}) > v(h_{\beta})$.

With the help of the commutator relations, it is easy to check that this is always possible.
Since $H$ is open, there exists an integer $c_0$ such that $G_{c_0} \subset H$. We’ll claim that for every $\alpha \in \Phi$, $H$ contains $h_\alpha$. This may be done by proving that $H$ contains elements of $h_\alpha G(F)_c$, with $c$ arbitrarily large; since this will in particular be true for $c = c_0$, we will obtain that the element $h_\alpha$ itself belongs to $H$. We’ll show the claim by induction on $c_0 - v$, the case $v = c_0$ being trivial.

First assume $\alpha \in \Phi$ and $v_\alpha(h_\alpha) = v$; we’ll proceed by induction on $c$. The case $c = v + 1$ is simply the result of the proposition 3.2; assume now $c > v + 1$, and let $h_{c-1}$ be an element of $h_\alpha G(F)_{c-1} \cap H$; such an element exists by the induction hypothesis. The element $h_{c-1}$ belongs to $U_{f_{\alpha,c-1,v}}$; we deduce then from the lemma 3.4, by using as in the proof of proposition 3.2 the fact that the quotient $U_{f_{\alpha,c-1,v}}/U_{f_{\alpha,c,v+1}}$ can be viewed as a rational representation of $T(K)$, that $h_\alpha U_{f,c+1} \cap H$ is nonempty. Let $h_c$ be any element of this intersection; the lemma 3.3 allows us to choose $h_c$ in $h_\alpha G(F)_c$, as required.

Assume $v_0(h_0) > v$. By an easy induction on $v'$, we see that the element:

$$h' = h (\prod_{\alpha, v_\alpha(h_\alpha) = v} h_\alpha)^{-1} = \prod_{\beta \in \Phi, v_\beta(h_\beta) > v} h_\beta$$

belongs to $H$; our claim follows then from the induction hypothesis applied to $h'$.

Now assume $v = v_0(h_0)$. Since $H$ then meets $G(F)_{v,c_0}$, we deduce from the lemma 3.5 and the proof of the proposition 3.2, as above, that $H$ contains an element of $h_0'(G(F))_{c_0}$, with $h_0'$ being an element of $T(F)$ such that $h_0'^{-1}h \in U_{0,v+1}$; hence $h_0'$ belongs to $H$. Since $h_0'^{-1}h \in H$ and $v_G(h_0'^{-1}h) > v$, we can use the induction hypothesis again.

Since the above claim is true for any $h \in H$, we have just shown the inclusion:

$$H \subset \prod_{\alpha \in \Phi} (H \cap U_\alpha(F));$$

the other inclusion being obvious, the proposition is proved. □

4 Normal functions

From now on and until the end of the paper, $H$ will be a normal open pro-nilpotent subgroup of some parahoric subgroup of $G(F)$. Moreover, until the end of the proof of the theorem 1.2, $G$ will be assumed to be quasi-simple, which amounts to say that $\Phi$ is connected.
Since every parahoric subgroup of $G(F)$ contains an Iwahori subgroup, and all Iwahori subgroups of $G(F)$ are conjugated, we may without loss of generality choose one of them and only consider its normal open pro-nilpotent subgroups; we will then make the following assumptions:

- the (extended) valuation $(v_\alpha)_{\alpha \in \Phi}$ has been chosen in such a way that the subgroup $G(F)_0$ of $G(F)$ is a special parahoric subgroup. Moreover, for every $\alpha \in \Phi$, $v_\alpha(U'_\alpha) = \mathbb{Z} \cup \{+\infty\}$;
- let $B(K)$ be the Borel subgroup of $G(K) \cong G(F)_0/G(F)_0^+$ associated to $\Delta$; $I$ is the inverse image of $B(K)$ in $G(F)_0$.

We then have $I = P_I U_I$, where $f_I$ is the fonction on $\Phi$ defined by $f_I(\alpha) = 0$ (resp. $f_I(\alpha) = 1$) when $\alpha$ is a positive (resp. negative) root relatively to $\Delta$, or equivalently $f_I(\alpha) = \varepsilon_\alpha$ for every $\alpha \in \Phi$.

We may easily check that if $\alpha, \beta$ are elements of $\Phi$ such that $\alpha + \beta \in \Phi$, we have:

- if $\alpha + \beta$ is greater than $\alpha$ and $\beta$, then $f_I(\alpha + \beta) = f_I(\alpha) + f_I(\beta)$;
- if $\alpha + \beta$ is lesser than $\alpha$ and $\beta$, then $f_I(\alpha + \beta) = f_I(\alpha) + f_I(\beta) - 1$.

Write $H = T'U_I$ as in the preceding proposition. According to [2, I.6.4.43], $f_H$ must satisfy:

$$f_H(\alpha + \beta) \leq f_I(\alpha) + f_H(\beta)$$

for every $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$. Conversely, for any $f$ satisfying the above condition, there exists $T' \subset T$ such that $T'U_f$ is normal in $I$. We’ll say $f$ is a normal function for $I$ if it satisfies that condition.

Consider the application $f_\varepsilon = f - f_I$ from $\Phi$ to $\mathbb{Z}$; we deduce from the above inequality that for every $\alpha, \alpha' \in \Phi$ such that $\alpha + \alpha' \in \Phi$ and $\alpha + \alpha' \geq \alpha$, we have:

$$f_\varepsilon(\alpha + \alpha') \leq f_\varepsilon(\alpha).$$

With the proposition 2.3 and an easy induction, we obtain that for every $\alpha, \beta \in \Phi$ such that $\alpha \geq \beta$, $f_\varepsilon(\alpha) \leq f_\varepsilon(\beta)$. Moreover, we have the following results:

**Lemma 4.1** For every $\alpha \in \Phi$, we have $f_\varepsilon(\alpha) \leq f_\varepsilon(-\alpha) + 1$.

Consider the element $u_\alpha(\varpi^{f_I(\alpha)})$ of $I$; assuming $2\alpha \not\in \Phi$, we have:

$$u_\alpha(\varpi^{f_I(\alpha)}) u_{-\alpha}(\varpi^{f_H(-\alpha)}) u_\alpha(-\varpi^{f_I(\alpha)})$$
Lemma 4.2 Let $\alpha$ be an element of $\Phi$; set $v = \text{Sup}(f_\varepsilon(\alpha), f_\varepsilon(-\alpha))$. For every $\beta \in \Phi$, $f_\varepsilon(\beta) \leq v + 1$.

Assume there exists $\beta$ such that $f_\varepsilon(\beta) \geq v + 2$; since we can always replace $\beta$ by a smaller element of $\Phi$, we may assume $\beta \in \Delta'$. Since $(\alpha, \varepsilon_\alpha) + (-\alpha, \varepsilon_{-\alpha}) = (0, 1) \geq (\beta, \varepsilon_{\beta})$, we have either $\beta \leq \alpha$ or $\beta \leq -\alpha$; we may assume for example $\beta \leq \alpha$, which implies $-\alpha \leq -\beta$ and $f_\varepsilon(-\beta) \geq f_\varepsilon(\beta) - 1 \geq v + 1 > f_\varepsilon(-\alpha)$, hence a contradiction. $\square$

By a similar reasoning, we obtain:

Lemma 4.3 Let $\alpha$ be any element of $\Phi$; set $v = \text{Inf}(f_\varepsilon(\alpha), f_\varepsilon(\beta))$. Then for every $\beta \in \Phi$, $f_\varepsilon(\beta) \geq v - 1$.

We deduce from these three lemmas the following corollary:

Corollary 4.4 There exists $v' \in \mathbb{N}$ such that $\text{Im}(f_\varepsilon)$ is contained in $\{v', v' + 1, v' + 2\}$. Moreover, if $\alpha, \beta \in \Phi$ are such that $f_\varepsilon(\alpha) = v'$ and $f_\varepsilon(\beta) = v' + 2$, then $f_\varepsilon(-\alpha) = f_\varepsilon(-\beta) = v' + 1$.

Set $v' = \text{Inf}_{\alpha \in \Phi} f_\varepsilon(\alpha)$; let $\alpha$ be an element of $\Phi$ such that $f_\varepsilon(\alpha) = v'$. Then according to the first lemma, $f_\varepsilon(-\alpha) \leq v' + 1$, and according to the second one, $f_\varepsilon(\beta) \leq v' + 2$ for every $\beta \in \Phi$. The second assertion of the corollary is an immediate consequence of the second and third lemmas. $\square$

In the sequel, we’ll say $f_H$ is:

- of type 1 if $f_\varepsilon$ either is constant or takes only two consecutive different values $v', v' + 1$;
- of type 2 else.

5 $\Delta'$-complete subsets of $\Phi$

Consider the subset $\Psi$ of the elements $\alpha \in \Phi$ such that $f_\varepsilon(\alpha)$ is maximal; we deduce from the previous section that for every $\alpha \in \Psi$ and every $\beta \in \Phi$ such that $\beta \leq \alpha$, $\beta \in \Psi$. We’ll say a subset of $\Phi$ satisfying that condition is
a $\Delta'$-complete subset of $\Phi$. This section will be devoted to the study of some properties of such subsets, which will be useful in the determination of the $x$ and $r$ of the theorem.

Assume for example $f_H$ is of type 1; $x$ and $r$ must satisfy $\alpha(x) - r \leq -v' - \varepsilon_\alpha$ for every $\alpha \in \Psi$, and $\alpha(x) - r \geq -v' - \varepsilon_\alpha$ for every $\alpha \notin \Psi$. Hence if $\alpha_1, \ldots, \alpha_t$ (resp. $\beta_1, \ldots, \beta_s$) are elements of $\Psi$ (resp. $\Phi - \Psi$) whose sum is zero, we must have, setting $z = r - v'$:

$$\sum_{i=1}^{t} \varepsilon_{\alpha_i} \leq tz;$$

$$\sum_{j=1}^{s} \varepsilon_{\beta_j} \geq sz.$$

The purpose of the proposition 5.1 is to prove the existence of some $z$ satisfying the above conditions; when this is done, the proposition 5.9 ensures the existence of a suitable $x$.

Until the end of the section, $\Psi$ will be any $\Delta'$-complete subset of $\Phi$. We have:

**Proposition 5.1** Let $\alpha_1, \ldots, \alpha_t$ (resp. $\beta_1, \ldots, \beta_s$) be elements of $\Psi$ (resp. $\Phi - \Psi$) such that $\sum_{i=1}^{t} (\alpha_i, \varepsilon_{\alpha_i}) = (0, c)$ and $\sum_{j=1}^{s} (\beta_j, \varepsilon_{\beta_j}) = (0, d)$ for some integers $c, d$. Assume $s, t > 0$; then $\frac{c}{t} \leq \frac{d}{s}$, and if $\frac{c}{t} = \frac{d}{s}$, for every $i, j$, $\alpha_i$ and $\beta_j$ are strongly orthogonal.

As a first remark, we see that we can assume $\sum_{i=1}^{t'} \alpha_i \in \Phi$ for every $t' \leq t$; let’s proceed by induction on $t'$, the case $t' = 1$ being obvious. If the sum is zero, then $\sum_{i=1}^{t'} \alpha_i = \alpha_{t'+1} \in \Phi$, as required; if the sum is nonzero, since $\sum_{i=1}^{t'} \alpha_i = -\sum_{i=1}^{t'} \alpha_i$, there exists $i' > t'$ such that $(\sum_{i=1}^{i'+1} \alpha_i, \alpha_{i'}) < 0$, and by rearranging the $\alpha_i$, $i' > t$, we may assume $i' = t + 1$, hence the result. (This is a slight variant of [1, I. proposition 1.19] but not a direct consequence of it.) From now on and unless another rearranging is explicitly mentioned, we will assume the $\alpha_i$ follow that property; we will make the same assumption about the $\beta_j$.

First we’ll prove we can assume $\Phi$ to be simply-laced. (This shouldn’t be really necessary to run the rest of the proof, but it makes it somewhat simpler.) It is well-known (see for example [9, par. 11], and in particular theorem 32) that when $\Phi$ is not simply-laced, there exists a simply-laced root system $(\Phi_{sl}, V_{sl})$ and an automorphism $\sigma$ of this root system, satisfying the following conditions:

- there exists a basis $\Delta_{sl}$ of $\Phi_{sl}$ such that $\sigma(\Delta_{sl}) = \Delta_{sl}$;
- $V$ can be identified to the subspace of the elements of $V_{sl}$ fixed by $\sigma$;
- let $W_{sl}$ be the Weyl group of $\Phi_{sl}$, let $(., .)_{sl}$ be a $W_{sl}$-invariant scalar product,
and let $\pi$ be the orthogonal projection, according to this scalar product, from $V_{sl}$ to $V$. Then $\pi(\Phi_{sl}) = \Phi$ and $\pi(\Delta_{sl}) = \Delta$; moreover, $W$ is the subgroup of the elements of $W_{sl}$ commuting with $\sigma$, and $(..)$ is the restriction of $(..)_{sl}$ to $V$.

Let $\alpha$ be an element of $\Phi$ and let $\alpha_{sl}$ be an element of $\Phi_{sl}$ whose image in $\Phi$ is $\alpha$; we have:

$$\alpha = \frac{1}{d} \sum_{i=0}^{d-1} \sigma^i(\alpha_{sl}),$$

where $d$ is the order of $\sigma$. Moreover, the image of any positive element of $\Phi_{sl}$ is a positive element of $\Phi$, hence $\varepsilon_{\alpha_{sl}} = \varepsilon_{\pi(\alpha_{sl})}$ for every $\alpha_{sl} \in \Phi_{sl}$.

Let $\Phi_{sl, aff}$ be the affine root system associated do $\Phi_{sl}$; the morphism $\pi$ extends canonically to a morphism $\Phi_{sl, aff} \to \Phi_{aff}$. Moreover, if $\Delta'_{sl}$ is the extended simple root system of $\Phi_{sl}$ associated do $\Delta_{sl}$, we have $\pi(\Delta'_{sl}) = \Delta'$, and for every $(\alpha_{sl}, v) \in \Phi_{sl, aff}^+$, $h(\alpha_{sl}, v) = h(\alpha_{sl}, v)$; in particular, $h(\Phi_{sl}) = h(\Phi)$, and $(\alpha_{sl}, v) \leq (\beta_{sl}, v')$ if and only if $\pi(\alpha_{sl}, v) \leq \pi(\beta_{sl}, v')$. We finally have the following results:

**Lemma 5.2** Let $\alpha, \beta$ be elements of $\Phi$ such that $\alpha + \beta \in \Phi$; there exist $\alpha_{sl}, \beta_{sl} \in \Phi_{sl}$ such that $\alpha_{sl} + \beta_{sl} \in \Phi_{sl}$, $\pi(\alpha_{sl}) = \alpha$ and $\pi(\beta_{sl}) = \beta$.

Assume first $(\alpha, \beta) < 0$, and let $\alpha'_{sl}$ (resp. $\beta'_{sl}$) be any element of $\Phi_{sl}$ whose image in $\Phi$ is $\alpha$ (resp. $\beta$). We have:

$$(\sum_{i=0}^{d-1} \sigma^i(\alpha'_{sl}), \sum_{j=0}^{d-1} \sigma^j(\beta'_{sl})) < 0;$$

there exist then $i, j$ such that $(\sigma^i(\alpha'_{sl}), \sigma^j(\beta_{sl})) < 0$, which proves the result. Suppose now $(\alpha, \beta) \geq 0$; we then have $(\alpha + \beta, -\beta) < 0$, and the above reasoning applied to $\alpha + \beta$ and $-\beta$ yields elements $(\alpha + \beta)_{sl}$ and $(-\beta)_{sl}$ of $\Phi$ whose sum is an element $\alpha_{sl}$ such that $\pi(\alpha_{sl}) = \alpha$, hence again the result. \(\square\)

**Corollary 5.3** Let $\alpha_1, \ldots, \alpha_t$ be elements of $\Phi$ such that $(\gamma, v) = \sum_{i=1}^{t}(\alpha_i, \varepsilon_{\alpha_i})$ is an element of $\Phi_{aff}$. There exist $\alpha_{sl, 1}, \ldots, \alpha_{sl, t}, \gamma_{sl} \in \Phi$ such that $\pi(\alpha_{sl,i}) = \alpha_i$ for every $i$, $\pi(\gamma_{sl}) = \gamma$ and $(\gamma_{sl}, v) = \sum_{i=1}^{t}(\alpha_{sl,i}, \varepsilon_{\alpha_{sl,i}})$.\(\square\)

This corollary follows from the previous lemma and an easy induction.

Assume the proposition is true in the simply-laced case. Let $\Psi_{sl}$ be the subset of the elements of $\Phi_{sl}$ whose image belongs to $\Psi$; since $\pi$ preserves the partial order, $\Psi_{sl}$ is a $\Delta'$-complete subset of $\Phi_{sl}$. Let $\alpha_{sl, 1}, \ldots, \alpha_{sl, t}$ (resp. $\beta_{sl, 1}, \ldots, \beta_{sl, s}$) be elements of $\Phi_{sl}$ whose images are the $\alpha_i$ (resp. the $\beta_j$) and which satisfy
the conditions of the previous corollary; the fact that \( \frac{c}{d} \leq \frac{d}{s} \) is simply the first assertion in the simply-laced case; moreover, if for some \( i, j, \alpha_i \) and \( \beta_j \) are not strongly orthogonal, by eventually conjugating all the \( \beta_j \) by \( \sigma^k \) for some \( k \), we can assume \( \alpha_{st,i} \) and \( \beta_{st,i} \) are not strongly orthogonal, and by the simply-laced case, we obtain \( \frac{c}{d} < \frac{d}{s} \), as required.

We will now assume \( \Phi \) is simply-laced. In this case, two elements of \( \Phi \) are strongly orthogonal if and only if they are orthogonal.

First we’ll observe that if there exist families \( (\alpha_1, \ldots, \alpha_t) \) and \( (\beta_1, \ldots, \beta_s) \) such that \( \frac{c}{t} \leq \frac{d}{s} \), then the families consisting of respectively \( d \) copies of the \( \alpha_i \) and \( c \) copies of the \( \beta_j \) satisfy \( t \leq s \). Hence if there don’t exist any families \( (\alpha_1, \ldots, \alpha_t), (\beta_1, \ldots, \beta_s) \) such that \( c = d \) and \( t \leq s \), then for every families, we have \( \frac{c}{t} > \frac{d}{s} \) and the proposition is proved. We’ll then assume such families do actually exist.

Now we’ll make the following claim: if \( (\alpha_1, \ldots, \alpha_t) \) and \( (\beta_1, \ldots, \beta_t) \) satisfy the above condition, and are such that \( c \) is minimal among all families satisfying it, then \( t = s \) and for every \( i, j, \alpha_i \) and \( \beta_j \) are orthogonal. First we’ll show:

**Lemma 5.4** For every proper nonempty subset \( J \) of \( \{1, \ldots, s\} \), \( \sum_{j \in J} \beta_j \neq 0 \).

Assume there exists some \( J \) such that \( \sum_{j \in J} \beta_j = 0 \); let \( s' \) be the cardinal of \( J \), and set \( c' = \sum_{j \in J} \varepsilon_{\beta_j} \); we have \( 0 < c' < c \). Let \( t' \) be an element of \( \{1, \ldots, t\} \) such that \( \sum_{i=1}^{t'} (\alpha_i, \varepsilon_{\alpha_i}) \geq (0, c') \) and \( \sum_{i=1}^{t'-1} (\alpha_i, \varepsilon_{\alpha_i}) < (0, c') \); according to the proposition 2.4, there exist \( \alpha'_1, \ldots, \alpha'_{t''-1} \in \Psi \cup \{0\} \), with \( t'' < t' \), such that:

\[
\sum_{i=1}^{t''} (\alpha'_i, \varepsilon_{\alpha'_i}) = (0, c');
\]

by minimality of \( c \), we must have \( t'' > s' \), hence \( t' \geq s' + 1 \). Moreover, the same argument applied to \( \alpha_{t'}, \ldots, \alpha_t \) yields \( t - t' + 1 \geq s - s' + 1 \); we then obtain \( t + 1 \geq s + 2 \), which contradicts our assumptions. Hence the lemma. \( \square \)

Now we’ll prove the claim. Suppose there exists \( i, j \) such that \( (\alpha_i, \beta_j) > 0 \); we may assume \( i = t \) and \( j = s \). We have \( \beta_s > \alpha_t, \delta_1 = \alpha_t - \beta_s \in \Phi \) and:

\[
\gamma = \sum_{j=1}^{s-1} (\beta_j, \varepsilon_{\beta_j}) = \sum_{i=1}^{t-1} (\alpha_i, \varepsilon_{\alpha_i}) - (\delta_1, \varepsilon_{\delta_1}).
\]

According to the previous lemma, \( \gamma \) is nonzero; there exists then \( j \in \{1, \ldots, s-1\} \) such that \( (\beta_j, \gamma) > 0 \), hence either \( (\beta_j, \alpha_i) > 0 \) for some \( i \) or \( (\beta_j, -\delta_1) > 0 \).
Assume $j = s - 1$; in the first case, assuming $i = t - 1$, we obtain;

$$\sum_{j=1}^{s-2} (\beta_j, \varepsilon_{\beta_j}) = \sum_{i=1}^{t-2} (\alpha_i, \varepsilon_{\alpha_i}) - (\delta_1, \varepsilon_{\delta_1}) - (\delta_2, \varepsilon_{\delta_2}),$$

with $\delta_2 = \beta_{s-1} - \alpha_{t-1}$, and in the second case:

$$\sum_{j=1}^{s-2} (\beta_j, \varepsilon_{\beta_j}) = \sum_{i=1}^{t-1} (\alpha_i, \varepsilon_{\alpha_i}) - (\delta_1, \varepsilon_{\delta_1}) - (\beta_{s-1}, \varepsilon_{\beta_{s-1}}).$$

By iterating the process, we finally obtain, after eventually rearranging the $\beta_j$:

$$0 = \sum_{i=1}^{t'} (\alpha_i, \varepsilon_{\alpha_i}) - \sum_{k=1}^{u} (\delta_k, \varepsilon_{\delta_k}) - \sum_{j=1}^{s'} (\beta_j, \varepsilon_{\beta_j}),$$

with $s' + u = s$ and $t' + u = t$, hence $t' \leq s'$. Consider the equality:

$$\sum_{k=1}^{u} (\delta_k, \varepsilon_{\delta_k}) = \sum_{i=1}^{t'} (\alpha_i, \varepsilon_{\alpha_i}) + \sum_{j=1}^{s'} (-\beta_j, -\varepsilon_{\beta_j}).$$

Although the members of this equality are not necessarily affine roots, we can here use a similar reasoning as in the proof of the proposition 2.4: if the sum $\delta$ of the $\delta_k$ is nonzero, there exists an element $\gamma \in \Delta'$ such that $(\delta, \gamma) > 0$, hence $(\delta_i, \gamma) > 0$ for some $i$; by subtracting $(\gamma, \varepsilon_{\gamma})$ to some appropriate term of the right-hand side and iterating, we finally obtain an equality such as:

$$(0, c') = \sum_{i=1}^{t''} (\alpha'_i, \varepsilon_{\alpha'_i}) - \sum_{j=1}^{s''} (\beta'_j, \varepsilon_{\beta'_j}) - (0, s' - s''),$$

with $t' \leq t$, $s'' \leq s$ and the $\alpha'_i$ (resp. the $\beta'_j$) being elements of $\Psi$ (resp. $\Phi - \Psi$); the last term of the right-hand side corresponds to the $(-\beta_j, -\varepsilon_{\beta_j})$ which are reduced to $(0, -1)$ that way. Hence:

$$\sum_{i=1}^{t''} (\alpha'_i, \varepsilon_{\alpha'_i}) = \sum_{j=1}^{s''} (\beta'_j, \varepsilon_{\beta'_j}) + (0, c' + s' - s'').$$

Moreover, we have $t'' \leq t' \leq s'' + (c' + s' - s'')$. If $(\alpha'_i, \beta'_j) > 0$ for some $i, j$, we can iterate the whole process to get even smaller sums satisfying similar inequalities.
Assume now \((\alpha'_i, \beta'_j) = 0\) for every \(i, j\); we will show by induction on \(c' + s' - s''\) that it leads to a contradiction. First remark that the sum of the \(\alpha'_i\) and the sum of the \(\beta'_j\) must be zero, since they are equal to each other and their product is zero. If \(c' + s' - s'' = 0\), since \(\sum_{i=1}^{t''}(\alpha'_i, \varepsilon_{\alpha'_i}) < (0, c)\), the equality is impossible by minimality of \(c\); if now \(c' + s' - s'' > 0\), by subtracting \((\alpha''_t, \varepsilon_{\alpha''_t})\) to both sides, we obtain:

\[
\sum_{i=1}^{t''-1}(\alpha'_i, \varepsilon_{\alpha'_i}) = \sum_{j=1}^{s''}(\beta'_j, \varepsilon_{\beta'_j}) + (\alpha''_t, c' + s' - s'' - 1 + \varepsilon_{-\alpha''_t}).
\]

after replacing \((\alpha''_t, c' + s' - s'' - 1 + \varepsilon_{-\alpha''_t})\) by \((0, c' + s' - s'' - 1)\) in the right-hand side and applying the proposition 2.4 to the left-hand side, we can use the induction hypothesis to obtain the desired contradiction.

Since assuming \((\alpha_i, \beta_j) > 0\) for some \(i, j\) leads to a contradiction, and since the sum of the \(\alpha_i\) is zero, we have just shown \((\alpha'_i, \beta'_j) = 0\) for every \(i, j\). Let now \(\delta\) be an element of \(\Delta'\) such that \((\alpha_i, \delta) > 0\); since the sum of the \((\beta_j, \varepsilon_{\beta_j})\) is greater than \((\delta, \varepsilon_{\delta})\), there exists \(j\) such that \(\delta \leq \beta_j\). Assuming \(j = s\), we have:

\[
\sum_{j=1}^{s-1}(\beta_j, \varepsilon_{\beta_j}) < \sum_{i=1}^{t-1}(\alpha_i, \varepsilon_{\alpha_i}) + (\alpha_t - \delta),
\]

hence, by applying the proposition 2.4:

\[
\sum_{j=1}^{s-1}(\beta_j, \varepsilon_{\beta_j}) = \sum_{i=1}^{t}(\alpha'_i, \varepsilon_{\alpha'_i}),
\]

the \(\alpha'_i\) being elements of \(\Psi \cup \{0\}\). If \(t < s\), a similar reasoning as above leads to a similar contradiction; hence \(t = s\) and the claim is proved.

Let’s write \(c_0\) for the minimal \(c\) defined as before, and set \(t_0 = \frac{c_0}{\varepsilon(\Psi)}\); for convenience, we will also set \(s_0 = t_0\) and \(d_0 = c_0\). We have the following result:

**Lemma 5.5** The integers \(c_0\) and \(t_0\) are relatively prime.

Assume they are not, and let \(c', t'\) be positive and relatively prime integers such that \(\frac{c'}{t'} = \frac{c_0}{d_0}\). We have either \(\sum_{i=1}^{t'}(\alpha_i, \varepsilon_{\alpha_i}) \geq (0, c')\) or \(\sum_{i=t'+1}^{t_0}(\alpha_{i}, \varepsilon_{\alpha_{i}}) \geq (0, c_0 - c')\). Moreover, \(c_0\) is a multiple of \(c'\); by replacing, in the first case, the family \((\alpha_1, \ldots, \alpha_{t'})\) by a family made of \(\frac{c_0}{c'}\) copies of it, we are reduced to the second case. By applying proposition 2.4, we then obtain \(\sum_{i=t'+1}^{t}(\alpha'_i, \varepsilon_{\alpha'_i}) = (0, c_0 - c')\), the \(\alpha'_i\) being elements of \(\Psi \cup \{0\}\).
Similarly (setting \( s' = t' \) for convenience), either \( \sum_{j=1}^{s''}(\beta_j, \varepsilon_{\beta_j}) \leq (0, c') \) or \( \sum_{j=s''+1}^{s_0}(\beta_j, \varepsilon_{\beta_j}) \leq (0, c_0 - c') \). By the same argument as above, we only have to consider the second case; by applying again proposition 2.4, we obtain as in the proof of the claim an equality such as:

\[
\sum_{j=1}^{s''}(\beta_j, \varepsilon_{\beta_j}) + (0, (s_0 - s') - s'') = (0, c_0 - c'),
\]

with \( t_0 - t' = s_0 - s' \leq s'' + (s_0 - s' - s'') \). We conclude by the same reasoning as in the proof of the claim that it is incompatible with the minimality of \( c_0 \).

\( \square \)

Let \( z(\Psi) \) be the quotient \( \frac{d}{s_0} = \frac{d}{s} \); as a consequence of the claim, it doesn’t depend on the choice of the \( \alpha_i \) and \( \beta_j \).

Now we’ll return to the general case. We’ll show the following result, which will imply the proposition: for every \((\alpha_1, \ldots, \alpha_t)\) and \((\beta_1, \ldots, \beta_s)\), we have \( \frac{c}{t} \leq z(\Psi) \leq \frac{d}{s} \), and both inequalities are equalities only if \( (\alpha_i, \beta_j) = 0 \) for every \( i, j \).

Assume first \( c \) and \( d \) are multiples of \( c_0 \), say \( c = c_1 c_0 \) and \( d = d_1 c_0 \), and \( \frac{c}{t} \geq \frac{d}{s} \); we’ll show we then have \( \frac{c}{t} = z(\Psi) = \frac{d}{s} \), and for every \( i, j \), \( (\alpha_i, \beta_j) = 0 \). We’ll proceed by induction on \( c_1 + d_1 \).

First remark that if \( \Psi \) is \( \Delta' \)-complete, \(- (\Phi - \Psi)\) is \( \Delta' \)-complete too; moreover, by replacing \( \Psi \) by \(- (\Phi - \Psi)\), we replace \( z(\Psi) \) by \( 1 - z(\Psi) \), \( c_0 \) by \( t_0 - c_0 \) and the \( \alpha_i \) (resp. the \( \beta_j \)) by the \(- \beta_j \) (resp. the \(- \alpha_i \)), hence if the result is true for some \( c_1, d_1 \) and for \( \Psi \) and \(- (\Phi - \Psi)\), it is also true with \( c_1 \) and \( d_1 \) switched. Hence we may assume \( \frac{c}{t} \geq z(\Psi) \), if not, then \( \frac{d}{s} \leq z(\Psi) \) and we fall into the symmetrical case.

The case \( c_1 = d_1 = 1 \) is simply the claim. If \( c_1 = 1 \) and \( d_1 > 1 \), then the same claim asserts \( \frac{c}{t} = z(\Psi) \), hence \( \frac{d}{s} \leq z(\Psi) \) and we fall again into the symmetrical case; assume then \( c_1 > 1 \).

Remark: when we’ll apply the induction hypothesis, it will always be to some \( \alpha_1', \ldots, \alpha_{t'} \) and to \( \beta_1, \ldots, \beta_s \), hence the \( \beta_j \) won’t be mentioned. Moreover, since no other elements of \( \Phi - \Psi \) than \( \beta_1, \ldots, \beta_s \) will occur in the proof, we can reduce ourselves to the case where \( \Psi \) is the largest \( \Delta' \)-complete subset of \( \Phi \) not containing them, i.e. that it contains every element of \( \Phi \) which is not greater than any \( \beta_j \).

Let \( t' \in \{1, \ldots, t\} \) be such that \( \sum_{i=1}^{t'}(\alpha_i, \varepsilon_{\alpha_i}) \geq (0, c_0) \) and \( \sum_{i=1}^{t'-1}(\alpha_i, \varepsilon_{\alpha_i}) < (0, c_0) \). By applying proposition 2.4 and the induction hypothesis to \( \sum_{i=1}^{t'}(\alpha_i, \varepsilon_{\alpha_i}) \) (resp. \( \sum_{i=t'}^{t}(\alpha_i, \varepsilon_{\alpha_i}) \)), we obtain \( t' \geq \frac{c}{z(\Psi)} \) and \( t - t' + 1 \geq \frac{c - t' c_0}{z(\Psi)} \). Since \( t \leq \frac{c}{z(\Psi)} \),
at least one of the above inequalities is an equality, and both are equalities if \( t < \frac{c}{z(\Psi)} \).

Assume for example \( t' = \frac{c_0}{z(\Psi)} \), the case \( t - t' + 1 = \frac{c - c_0}{z(\Psi)} \) being symmetrical; we will then show that \((\alpha_i, \beta_j) = 0\) for every \( i, j \).

Set \( \gamma = \sum_{i=1}^{t'} \alpha_i \). If \( \gamma = 0 \), the assertion follows immediately from the induction hypothesis applied first to \((\alpha_1, \ldots, \alpha_{t'})\) and then to \((\alpha_{t'-1}, \ldots, \alpha_t)\); assume now \( \gamma \neq 0 \). The element \( \gamma \) cannot be equal to any \( \alpha_i \), since we would then have \( \sum_{i' \neq i} (\alpha_{i'}, \varepsilon_{\alpha_{i'}}) = (0, c_0) \), which is impossible by the induction hypothesis because \( \frac{c_0}{t'-1} > z(\Psi) \); since \( \Phi \) is simply-laced, we then have \((\gamma, \gamma) = 2\) and \((\alpha_i, \gamma) \leq 1\) for every \( i \), hence there exist at least two different \( i \) such that \((\alpha_i, \gamma) > 0\); let’s call them \( i \) and \( i' \). The character \( \alpha_i - \gamma \) is then an element of \( \Phi \), and \( \alpha_i \geq \gamma \) since the equality \( \sum_{i' \neq i} (\alpha_{i'}, \varepsilon_{\alpha_{i'}}) = (\gamma - \alpha_i, c_0 + \varepsilon_{\gamma-\alpha_i}) \) is impossible (apply proposition 2.4 and the induction hypothesis to check this); we then have:

\[
\sum_{i'' \neq i} (\alpha_{i''}, \varepsilon_{\alpha_{i''}}) + (\alpha_i - \gamma, \varepsilon_{\alpha_i - \gamma}) = (0, c_0).
\]

Since \( t' = \frac{c_0}{z(\Psi)} \) and \( \alpha_i - \gamma \in \Psi \), we deduce from the induction hypothesis that \((\alpha_{i''}, \beta_j) = 0\) for every \( i'' \neq i \) and every \( j \); by replacing \( i \) by \( i' \), we see that it is also true for \( i'' = i \).

We deduce from this that \((\gamma, \beta_j) = 0\) for every \( j \); with the help of the proposition 2.4, we can even replace \( \gamma \) by any smaller element, which way we see that for every element \( \delta \) of \( \Delta' \) occurring in the decomposition of \( \gamma \), \((\delta, \beta_j) = 0\) for every \( j \).

Moreover, we have \( \gamma' = \gamma + \alpha_{t'+1} \in \Phi \); we will show that the above remarks imply \( \gamma' \notin \Psi \). With the proposition 2.3 and an easy induction, we see that we can assume \( \gamma \in \Delta' \). Suppose \( \gamma' \notin \Psi \); by the assumption made on \( \Psi \), we then have \( \gamma' \geq \beta_j \) for some \( j \). Write:

\[
\gamma' = \sum_{\delta \in \Delta'} c_{\delta} \delta;
\]

\[
\beta_j = \sum_{\delta \in \Delta'} b_{\delta} \delta.
\]

Then \( c_{\delta} \geq b_{\delta} \) for every \( \delta \in \Delta' \), and \( c_{\gamma} = b_{\gamma} \) since \( \alpha_{t'+1} \) isn’t greater than \( \beta_j \). Hence:

\[
(\gamma, \beta_j) = 2b_{\gamma} - \sum_{\delta, (\delta, \gamma) = -1} b_{\delta}
\]
\[ \geq 2c_\gamma - \sum_{\delta, (\delta, \gamma) = -1} c_\delta = (\gamma, \gamma') > 0, \]

which leads to a contradiction.

Consider now the equality:

\[ (\gamma', \varepsilon_{\gamma'}) + \sum_{i=t'+2}^{t} (\alpha_i, \varepsilon_{\alpha_i}) = (0, c-c_0), \]

By the induction hypothesis, we have \( t - t' = \frac{c-c_0}{z(\Psi)} \), hence \( t = \frac{c}{z(\Psi)} \), and for every \( j \), \( (\gamma', \beta_j) = 0 \) and \( (\alpha_i, \beta_j) = 0 \) for every \( i \geq t' + 2 \); since we already know this is also true for \( \alpha_i, i \leq t' \), we obtain the desired assertion.

Assume finally \( c \) is not a multiple of \( c_0 \); by considering the equality:

\[ c_0 \left( \sum_{i=1}^{t} (\alpha_i, \varepsilon_{\alpha_i}) \right) = (0, c_0c), \]

we obtain \( \frac{ac}{cd} \leq z(\Psi) \), hence \( t \geq cz(\Psi) \). Since \( cz(\Psi) \) is not an integer, we see the inequality is always strict. The case where \( d \) is not a multiple of \( d_0 \) is treated similarly, and concludes the proof of the proposition. \( \square \)

For every \( \alpha_1, \ldots, \alpha_t \in \Psi \) (resp. \( \beta_1, \ldots, \beta_s \in \Phi - \Psi \)) whose sum is zero, if \( t > 0 \) (resp. \( s > 0 \)), set:

\[ z(\alpha_1, \ldots, \alpha_t) = \frac{1}{t} \sum_{i=1}^{t} \varepsilon_{\alpha_i}, \]

and define \( z(\beta_1, \ldots, \beta_s) \) similarly. Write:

\[ z(\Psi) = \text{Sup}_{(\alpha_1, \ldots, \alpha_t)} z(\alpha_1, \ldots, \alpha_t), \]

\[ z'(\Psi) = \text{Inf}_{(\beta_1, \ldots, \beta_s)} z(\beta_1, \ldots, \beta_s), \]

the upper (resp. lower) bound being taken over all the families \( (\alpha_1, \ldots, \alpha_t) \), \( t > 0 \) (resp. \( (\beta_1, \ldots, \beta_s), s > 0 \)) of elements of \( \Psi \) (resp. \( \Phi - \Psi \)) whose sum is zero; set \( z(\Psi) = 0 \) (resp. \( z'(\Psi) = 1 \)) if there is no such family in \( \Psi \) (resp. \( \Phi - \Psi \)). This definition of \( z(\Psi) \) is clearly consistent with the one used in the proof of the proposition, and we deduce from this same proposition that \( z(\Psi) \leq z'(\Psi) \).

Moreover, we have the following result:
Lemma 5.6 There exist $\alpha_1, \ldots, \alpha_t \in \Psi$ (resp. $\beta_1, \ldots, \beta_s \in \Phi - \Psi$) such that $z(\alpha_1, \ldots, \alpha_t) = z(\Psi)$ (resp. $z(\beta_1, \ldots, \beta_s) = z'(\Psi)$).

We will show the result for $z(\Psi)$, the proof for $z'(\Psi)$ being similar. To prove the desired assertion, we only have to show that the upper bound may be taken on a finite number of families. Let's show the following lemmas:

Lemma 5.7 Assume there exists $t' < t$ such that $\alpha_1 + \ldots + \alpha_{t'} = 0$. Then we have:

$$z(\alpha_1, \ldots, \alpha_t) \leq \text{Sup}(z(\alpha_1, \ldots, \alpha_{t'}), z(\alpha_{t'+1}, \ldots, \alpha_t)).$$

We have $z(\alpha_1, \ldots, \alpha_t) = \frac{t'z(\alpha_1, \ldots, \alpha_{t'}) + (t-t')z(\alpha_{t'+1}, \ldots, \alpha_t)}{t}$, hence the result. □

Lemma 5.8 Assume $t > \text{card}(\Phi) + 1$. Then after eventually rearranging the $\alpha_i$, there exists $t' < t$ such that $\alpha_1 + \ldots + \alpha_{t'} = 0$.

We can assume $\gamma_{t'} = \sum_{i=1}^{t'} \alpha_i$ is an element of $\overline{\Phi}$ for every $t' < t$. If one of them is zero there is nothing to prove; if all of them are nonzero, since $t > \text{card}(\Phi) + 1$, there exist $t' < t''$ such that $\gamma_{t'} = \gamma_{t''}$; we then have $\sum_{i=t'+1}^{t''} \gamma_i = 0$, hence the result. □

According to these two lemmas, we only have to take the upper bound on the set of families $(\alpha_1, \ldots, \alpha_t)$ such that $t \leq \text{card}(\Phi) + 1$; since this set is obviously finite, the lemma 5.6 is proved. □

Now we'll be concerned about the element $x$ of $A$ mentioned in the theorem. Let $A_I$ be the facet of $B$ associated to $I$, and $\overline{A_I}$ its closure; we have:

Proposition 5.9 Let $z$ be any element of $[z(\Psi), z'(\Psi)]$. There exists an element $x$ of $\overline{A_I}$ such that:

- for every $\alpha \in \Psi$, $\alpha(x) \leq z - \varepsilon_\alpha$;
- for every $\alpha \in \Phi - \Psi$, $\alpha(x) \geq z - \varepsilon_\alpha$.

Moreover, if $z(\Psi) < z'(\Psi)$, the set of such elements contains an open subset of $A$.

Let $E_{\Psi,z}$ be the subset of the elements $x$ of $A$ such that:

- for every $\alpha \in \Delta'$, $\alpha(x) \geq -\varepsilon_\alpha$;
- for every maximal element $\alpha$ of $\Psi$, $\alpha(x) \leq z - \varepsilon_\alpha$;
- for every minimal element $\alpha$ of $\Phi - \Psi$, $\alpha(x) \geq z - \varepsilon_\alpha$.

(The first condition simply amounts to say that $x \in \overline{A_I}$.) Let $x$ be any element of $E_{\Psi,z}$; since for every $\alpha \in \Phi$ and every $\beta \in \Delta'$ such that $\alpha + \beta \in \Phi$, we
have \( \varepsilon_{\alpha + \beta} = \varepsilon_{\alpha} + \varepsilon_{\beta} \), we obtain by an easy induction that for every \( \alpha \in \Phi \), \( \alpha(x) \geq -\varepsilon_{\alpha} \); by a similar argument, for every \( \alpha \in \Psi \), \( \alpha(x) \leq z - \varepsilon_{\alpha} \), and for every \( \alpha \in \Phi - \Psi \), \( \alpha(x) \geq z - \varepsilon_{\alpha} \); hence \( x \) satisfies the condition of the lemma.

We then only have to show that \( E_{\Psi, z} \) is nonempty.

Let \( \Phi_m \) be the subset of \( \Phi \) containing \( -\Delta' \), the maximal elements of \( \Psi \) and the opposites of the minimal elements of \( \Phi - \Psi \); set, for every \( \alpha \in \Phi_m \):

- if \( \alpha \in \Psi \), \( f_z(\alpha) = z - \varepsilon_{\alpha} \);
- if \( \alpha \) is the opposite of some minimal element of \( \Phi - \Psi \), \( f_z(\alpha) = -z + \varepsilon_{-\alpha} = 1 - z - \varepsilon_{\alpha} \);
- if \( \alpha \) satisfies more than one of the above conditions, \( f_z \) takes the lowest possible value.

The set \( E_{\Psi, z} \) is nonempty if and only if for every \( \alpha_1, \ldots, \alpha_t \in \Phi_m \) whose sum is zero, \( \sum_{i=1}^t f_z(\alpha_i) \geq 0 \). The first assertion of the proposition follows then from the following result:

**Lemma 5.10** Let \( \alpha_1, \ldots, \alpha_t \) (resp. \( \beta_1, \ldots, \beta_s \)) be elements of \( \Psi \) (resp. \( \Phi - \Psi \)) such that \( \sum_{i=1}^t \alpha_i = \sum_{j=1}^s \beta_j \). Set:

\[
c_0 = \sum_{i=1}^t \varepsilon_{\alpha_i} + \sum_{j=1}^s \varepsilon_{-\beta_j}.
\]

Then \( tz + s(1 - z) \geq c_0 \), and the inequality is strict if \( z(\Psi) < z'(\Psi) \) and \( s, t > 0 \).

Write \( \gamma = \sum_{i=1}^t \alpha_i = \sum_{j=1}^t \beta_j \). Note that \( \gamma \) is not necessarily an element of \( \Phi \).

We'll show the lemma by induction on \( s \). Assume first \( \gamma = 0 \), and let \( c'_0 \) (resp. \( c''_0 \)) be the sum of the \( \varepsilon_{\alpha_i} \) (resp. the \( \varepsilon_{-\beta_j} \)). It follows immediately from the definition of \( z(\Psi) \) and \( z'(\Psi) \) that we have \( tz \geq c'_0 \) and \( sz \leq c''_0 \), and that at least one of these inequalities is strict if \( z(\Psi) < z'(\Psi) \) and \( s, t > 0 \); hence:

\[
tz + s(1 - z) \geq c'_0 + s - c''_0 = c_0,
\]

and the inequality is strict if \( z(\Psi) < z'(\Psi) \) and \( s, t > 0 \).

Moreover, since for each \( j \), \( \varepsilon_{-\beta_j} = 1 - \varepsilon_{\beta_j} \), we obtain:

\[
c'_0 + s - c''_0 = \sum_{i=1}^t \varepsilon_{\alpha_i} + \sum_{j=1}^s \varepsilon_{-\beta_j} = c_0,
\]
which proves $tz + s(1 - z) \geq c_0$ ($> c_0$ if $z(\Psi) < z'(\Psi)$ and $s, t > 0$). Assume now $\gamma \neq 0$; we now have:

$$\left(\sum_{i=1}^{t} \alpha_i, \sum_{j=1}^{s} \beta_j\right) > 0;$$

there exist then $i_0 \in \{1, \ldots, t\}$ and $j_0 \in \{1, \ldots, s\}$ such that $(\alpha_{i_0}, \beta_{j_0}) > 0$; we will assume $i_0 = t$ and $j_0 = s$. We have $\alpha_t - \beta_s \in \Phi$, and since $\alpha_t \in \Psi$ and $\beta_s \notin \Psi$, $\beta_s = \alpha_t + (\beta_s - \alpha_t) > \alpha_t, \beta_s - \alpha_t$, hence $\alpha_t - \beta_s > -\beta_s, \alpha$; we then have $h(\alpha_t) + h(-\beta_s) = h(\alpha_t - \beta_s) < h(\Phi)$. Consider now the equality:

$$\sum_{i=1}^{t-1} (\alpha_i, \varepsilon_{\alpha_i}) + \sum_{j=1}^{s-1} (-\beta_j, \varepsilon_{-\beta_j}) = (-\alpha_t + \beta_s, c_0 - 1).$$

According to the proposition 2.4, there exist $\alpha'_1, \ldots, \alpha'_t \in \Psi \cup \{0\}$ (resp. $\beta'_1, \ldots, \beta'_s \in (\Phi - \Psi) \cup \{0\}$) such that for every $i$ (resp. $j$), $\alpha'_i \leq \alpha'_j$ (resp. $\beta'_i \geq \beta'_j$) and:

$$\sum_{i=1}^{t-1} (\alpha'_i, \varepsilon_{\alpha'_i}) + \sum_{j=1}^{s-1} (-\beta'_j, \varepsilon_{-\beta'_j}) = (0, c_0 - 1),$$

and we deduce from the induction hypothesis:

$$(s - 1)z + (t - 1)(1 - z) \geq c_0 - 1,$$

with a strict inequality if $z(\Psi) < z'(\Psi)$ and $s, t > 0$; hence the result. □

Assume now $z(\Psi) < z'(\Psi)$; we’ll show that $E_{\Psi_z}$ contains an open subset of $A$. Assume this is not the case; since $E_{\Psi_z}$ is obviously convex, it is then contained in the hyperplane defined by some equation $\gamma(y) = \lambda$, with $\gamma \in X^*(T)$ and $\lambda \in \mathbb{R}$, hence:

- there exist $\alpha'_1, \ldots, \alpha'_{t'} \in \Phi_m$ whose sum is $\gamma$ and such that $\sum_{i=1}^{t'} f_z(\alpha'_i) = \lambda$;
- there exist $\beta'_1, \ldots, \beta'_{s'} \in \Phi_m$ whose sum is $-\gamma$ and such that $\sum_{j=1}^{s'} f_z(\beta'_j) = -\lambda$.

We then have $\sum_{i=1}^{t'} \alpha'_i + \sum_{j=1}^{s'} \beta'_j = 0$ and $\sum_{i=1}^{t'} f_z(\alpha'_i) + \sum_{j=1}^{s'} f_z(\beta'_j) = 0$; this is possible only if all the inequalities occuring in the proof of the previous lemma, when applied to those elements, are equalities, and in particular if $tz = c'_0$ and $sz = c''_0$, which implies $z(\Psi) = z = z'(\Psi)$. Hence the result. □
6 Proof of the theorem 1.2

In this section, we’ll prove the main theorem. Let’s begin by the following preliminary result:

**Lemma 6.1** Assume $H = T'U_{x,r}$ for some $T', x, r$. Then $H \subset G_{x,r}$.

Since $T'$ fixes $A$ pointwise, $H$ fixes $B(x, r) \cap A$ pointwise. Moreover, let $y$ be any element of $B(x, r)$, and let $A'$ be an apartment of $B$ containing $A_I$ and $y$ (such an apartment exists by [2, I.2.3.1]). According to [2, I.2.5.8], there exists $g \in G^0$ such that $g(y) \in A$ and $g$ fixes $A_I$ pointwise, hence $g \in I$; $y$ is then fixed by $gHg^{-1} = H$. Hence $H$ fixes $B(x, r)$ pointwise, which amounts to say it is contained in $G_{x,r}$, as required. \(\square\)

Now let’s go on into the proof of the theorem. First we’ll suppose $\Phi$ is of type 1; let $\Psi$ be the subset of the elements $\alpha \in \Phi$ such that $f_{\epsilon}(\alpha) = v' + 1$. We have already seen $\Psi$ is $\Delta'$-complete; let’s show the following result:

**Proposition 6.2** Let $z$ be any element of $[z(\Psi), z'(\Psi)]$; set $r = v' + z$. Let $x$ be any element of $E_{\Psi,z}$; we have:

$$U^+_{x,r} \subset H \subset G_{x,r}.$$  

Let $f_{x,r}$ be the concave function associated to $G_{x,r}$; we have, for every $\alpha \in \Phi$:

$$f_{x,r}(\alpha) = \text{ceil}(-\alpha(x) + r).$$

With the help of the previous lemma, in order to show that $H \subset G_{x,r}$, we only have to prove that $f \geq f_{x,r}$. Let $\alpha$ be any element of $\Phi$; if $\alpha \in \Psi$, we have, using the definition of $E_{\Psi,z}$ and the fact that $f_H(\alpha)$ is an integer:

$$f_H(\alpha) = v' + 1 + \varepsilon_\alpha = r + 1 - (-\varepsilon_\alpha) - z$$

$$\geq \text{ceil}(-\alpha(x) + r - z) \geq f_{x,r}(\alpha),$$

and if $\alpha \not\in \Psi$:

$$f_H(\alpha) = v' + \varepsilon_\alpha = r - (z - \varepsilon_\alpha) \geq \text{ceil}(-\alpha(x) + r),$$

hence the result.
Let now $f_{x,r}^+$ be the concave function associated to $G_{x,r}^+$; we have for every $\alpha \in \Phi$:

\[
f_{x,r}^+(\alpha) = \inf_{r' > r} f_{\alpha,x,r'}(\alpha) = \text{ceil}(\alpha(x) + r') = \text{floor}(\alpha(x) + r + 1);
\]

we’ll show that $f \leq f_{x,r}^+$, which will imply $U_{x,r}^+ \subset H$. For every $\alpha \in \Phi$, we have, when $\alpha \in \Psi$:

\[
f_H(\alpha) = r + 1 - (z - \varepsilon_\alpha) \leq \text{floor}(r + 1 - \alpha(x)),
\]

and when $\alpha \notin \Psi$:

\[
f_H(\alpha) = r + 1 - (1 - \varepsilon_\alpha) - z \\
\leq \text{floor}(r + 1 - \alpha(x) - z) \leq f_{x,r}^+(\alpha),
\]

hence again the result. □

First consider the case $z(\Psi) < z'(\Psi)$; we will show that the groups $H$ such that $U_{x,r} \subset H \subset G_{x,r}$ for some $x, r$ are the ones which satisfy this condition. Since we can choose $z = r - v'$ anywhere in $[z(\Psi), z'(\Psi)]$, there exist $r < r'$ and $x, x'$ such that the previous proposition holds for both $x, r$ and $x', r'$. In particular we obtain:

\[
U_{x,r''} \subset H \subset G_{x',r'}
\]

for any $r'' > r$. Unfortunately this isn’t enough since the inclusion $U_{x,r''} \subset U_{x',r'}$ may very well be strict even if $r'' < r'$; this problem will be solved by checking that we may assume $x = x'$, which is done by the following lemma:

**Lemma 6.3** There exist $z < z' \in [z(\Psi), z'(\Psi)]$ such that $E_{\Psi,z} \cap E_{\Psi,z'} \neq \emptyset$.

Since $z(\Psi) < z'(\Psi)$, for every $z$, $E_{\Psi,z}$ contains an open subset of $A_I$, hence its volume is nonzero. Assume all $E_{\Psi,z}$ are disjoint; then $A_I$ contains an uncountable union of disjoint subsets with positive volume, which is impossible. □

**Corollary 6.4** Set $r' = z' + v'$; we have $U_{x,r'} \subset H \subset G_{x,r'}$.

According to the proposition 6.2 and the previous lemma, we now have $U_{x,r'} \subset U_{x,r}^+ \subset H \subset G_{x,r'}$. □
Hence the theorem is proved for \( f_H \) of type 1 and such that \( z(\Psi) < z'(\Psi) \). In this case, we may show the following converse:

**Proposition 6.5** Assume \( U_{x,r} \subset H \subset G_{x,r} \) for some \( x, r \), and set \( r = z + v' \), with \( z \in [0, 1) \) and \( v' \) being an integer. Then \( f_H \) is of type 1 and \( z(\Psi) < z'(\Psi) \).

For every \( \alpha \in \Phi \), we have:

\[
f_\varepsilon(\alpha) = \text{ceil}(-\alpha(x) + r) - \varepsilon \alpha \in \{\text{ceil}(r - 1), \text{ceil}(r)\},
\]

hence \( f_H \) is of type 1. Moreover, since the valuation is discrete, we have \( H = G_{x,r} \) for every \( r \) in some interval \([r', r'']\), with \( r' < r'' \); we obviously can assume \( \text{floor}(r') = \text{floor}(r'') = v' \). Since \( f_H(x) = \text{ceil}(-\alpha(x) + r) \) for every \( \alpha \in \Phi \) and every \( r \in [r', r''] \), setting \( z' = r' - v' \) and \( z'' = r'' - v' \), we obtain that every element \( \alpha \) of \( \Psi \) (resp. \( \Phi - \Psi \)) must satisfy \( \alpha(x) < z' \) (resp. \( \alpha(x) \geq z'' \)), hence \( z(\Psi) \leq z' < z'' \leq z'(\Psi) \).

Suppose now \( f_H \) is still of type 1, but such that \( z(\Psi) = z'(\Psi) \); according to the previous proposition, \( H \) cannot be equal to any group of the form \( T'U_{x,r} \). Such a case actually occurs: for example, take \( G = SL_4 \), and:

\[
H = \left\{ \begin{pmatrix}
1 + p & p & p & \mathcal{O} \\
p & 1 + p & p & \mathcal{O} \\
p^2 & p & 1 + p & p \\
p^2 & p & p & 1 + p
\end{pmatrix} \right\}.
\]

It is easy to check that this group is a normal subgroup of \( I \) and that the corresponding concave function \( f_H \) is of type 1. Moreover, we have:

\[
\Psi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_3, -\alpha_M, \alpha_3 - \alpha_M\}.
\]

Since \( (\alpha_1 + \alpha_2) + (\alpha_3 - \alpha_M) = 0 \), we have \( z(\Psi) \geq \frac{1}{2} \), and since \( (\alpha_2 + \alpha_3) + (\alpha_1 - \alpha_M) = 0 \), we have \( z'(\Psi) \leq \frac{1}{2} \). Hence \( z(\Psi) = z'(\Psi) = \frac{1}{2} \).

Consider the families \( (\beta_1, \ldots, \beta_s) \) of elements of \( \Phi - \Psi \) such that \( \sum_{j=1}^s (\beta_j, \varepsilon_{\beta_j}) = (0, d) \) with \( \frac{d}{2} = z'(\Psi) \); let's call them \( z'(\Psi) \)-families. Let \( \Phi_0 \) be the root subsystem of elements of \( \Phi \) which are linear combinations of elements of such families; we have:

**Proposition 6.6** The rank of the root subsystem \( \Phi_0 \) of \( \Phi \) is strictly smaller than the rank of \( \Phi \).
Since \( z(\Psi) = z'(\Psi) \), there exist \( \alpha_1, \ldots, \alpha_t \in \Psi \) and an integer \( c \) such that
\[
\sum_{i=1}^{t} (\alpha_i, \varepsilon_{\alpha_i}) = (0, c)
\]
and \( \tilde{z} = z(\Psi) \). According to the proposition 5.1, every element of \( \Phi_0 \) is then strongly orthogonal to all the \( \alpha_i \), and this implies the result. \( \square \)

Moreover, set \( \Psi_0 = \Psi \cap \Phi_0 \); we obviously have \( z'(\Psi_0) = z'(\Psi) \), and since the \( \alpha_i \) of the previous proposition cannot belong to \( \Phi_0 \), \( z(\Psi_0) \) is strictly smaller than \( z(\Psi) \). Let \( M \) be the Levi subgroup of \( G \) generated by \( T \) and the \( U_{\alpha} \), \( \alpha \in \Phi_0 \); \( M \) is proper, and since \( H \cap M \) is obviously normal in the Iwahori subgroup \( I \cap M \) of \( M \), according to the corollary 6.4, there exists an element \( x \) of the apartment \( A_M \) of the Bruhat-Tits building of \( M \) associated to \( S \) and an element \( r \) of \([v', v' + 1] \subset \mathbb{R}^+\) such that \( U_{M,x,r} \subset H \cap M \subset M_{x,r} \); moreover, using the canonical projection \( A \to A_M \), with a slight abuse of notation, setting \( z = r - v' \), we will also call \( E_{\Psi_0,z} \) the convex subset of \( A \) whose image in \( M \) is \( E_{\Psi_0,z} \); we may then consider \( x \) as an element of \( A \).

Now we’ll determine \( r' \). we have the following result:

**Lemma 6.7** Let \( \Psi' \) be the subset of \( \Phi \) which is the union of \( \Psi \) and all \( z'(\Psi) \)-families. Then \( \Psi' \) is \( \Delta' \)-complete, and \( z(\Psi) = z(\Psi') \).

Let’s show the first assertion: we will in fact show the equivalent assertion that \( -(\Phi - \Psi') \) is \( \Delta' \)-complete. Let \( \alpha \) be any element of \( \Phi - \Psi' \). Assume there exists \( \beta_1, \ldots, \beta_t \in \Phi - \Psi \) such that
\[
\sum_{i=1}^{t} (\beta_i, \varepsilon_{\beta_i}) = (0, d'), \quad d' = t z'(\Psi),
\]
and \( \beta_t \geq \alpha \); we then have:
\[
\sum_{i=1}^{t-1} (\beta_i, \varepsilon_{\beta_i}) \leq (-\alpha, d - 1 + \varepsilon_{-\alpha}).
\]

According to the proposition 2.4, there exist then \( \beta_1', \ldots, \beta_{t-1}' \in \Phi - \Psi \) (one may check as in the proof of the proposition 5.1 that assuming that at least one of the \((\beta_i, \varepsilon_{\beta_i})\) is reduced to \((0, 1)\) leads to a contradiction) such that:
\[
\sum_{i=1}^{t-1} (\beta_i', \varepsilon_{\beta_i'}) = (-\alpha, d - 1 + \varepsilon_{-\alpha}),
\]

hence:
\[
\sum_{i=1}^{t-1} (\beta_i', \varepsilon_{\beta_i'}) + (\alpha, \varepsilon_{\alpha}) = (0, d).
\]

Hence the family \((\beta_1', \ldots, \beta_{t-1}', \alpha)\) is a \( z'(\Psi) \)-family. Since \( \alpha \notin \Psi' \), this is impossible; \( \beta_t \) cannot then be greater than \( \alpha \). Since we already know that no
element of Ψ can be greater than α, we have just shown that every element of Φ which is greater than α belongs to Φ − Ψ′; hence −(Φ − Ψ′) is Δ'-complete.

Now we’ll prove that z(Ψ) = z(Ψ′). Let α₁, . . . , αₜ be elements of Ψ′ such that \( \sum_{i=1}^{t'} (\alpha_i, \varepsilon_{\alpha_i}) = (0, c) \), with \( \frac{c}{t'} \geq z(Ψ) \). Assume αᵢ ∈ Ψ for every \( i \leq t' \), and αᵢ belongs to some z(Ψ)-family if \( i > t' \); in the second case, let βᵢ₁, . . . , βᵢₛᵢ₋₁ be the other members of that family. Setting \( d_i = \frac{s_i}{z(Ψ)} \) for every \( i > t' \), we obtain:

\[
\sum_{i=1}^{t'} (\alpha_i, \varepsilon_{\alpha_i}) + \sum_{i=t'+1}^{t} (0, d_i) = \sum_{i=t'+1}^{t} \sum_{j=1}^{s_i-1} (\beta_{i,j}, \varepsilon_{\beta_{i,j}}) + (0, c).
\]

By the same process as in the proof of the proposition 5.1, we can obtain an equality:

\[
0 = \sum_{i=1}^{t''} (\alpha'_i, \varepsilon_{\alpha'_i}) + \sum_{i=t'+1}^{t} (0, d_i) - \sum_{j=1}^{s'} (\beta'_j, \varepsilon_{\beta'_j})
\]

\[
+(0, -c - \sum_{i=t'+1}^{t} (s_i - 1) + s'),
\]

where \( t'' \leq t' \), \( s' \leq \sum_{i=t'+1}^{t} (s_i - 1) \) and the \( \alpha'_i \) (resp. the \( \beta'_j \)) are elements of Ψ (resp. Φ − Ψ) such that \( (\alpha'_i, \beta'_j) = 0 \) for every \( i, j \). The sum of the \( \alpha'_i \) (resp. \( \beta'_j \)) must then be zero; writing \( (0, c') = \sum_{i=1}^{t''} (\alpha'_i, \varepsilon_{\alpha'_i}) \) (resp. \( (0, d') = \sum_{j=1}^{s'} (\beta'_j, \varepsilon_{\beta'_j}) \)), we obtain:

\[
c' + \sum_{i=t'+1}^{t} d_i - d' - c - \sum_{i=t'+1}^{t} (s_i - 1) + s' = 0,
\]

hence, since by definition \( c' \leq t''z(Ψ) \) and \( d' \geq s'z(Ψ) = s'z(Ψ) \), and by assumption \( c \geq tz(Ψ) \):

\[
(t'' + \sum_{i=t'+1}^{t} s_i - s' - t)z(Ψ) - \sum_{i=t'+1}^{t} (s_i - 1) + s \geq 0,
\]

which can be rewritten as:

\[
(t'' - t')z(Ψ) + (\sum_{i=t'+1}^{t} (s_i - 1) - s')(z(Ψ) - 1) \geq 0.
\]

Since \( 0 < z(Ψ) < 1 \), \( t'' \leq t' \) and \( \sum_{i=t'+1}^{t} (s_i - 1) \geq s' \), this is possible only if all the inequalities we have combined to get the above one are equalities, and
in particular if \( v = tz(\Psi) \). Hence \( z(\Psi') \leq z(\Psi) \); since the other inequality is obvious, the lemma is proved. \( \square \)

According to the first lemma, if \( f'_H \) is the concave function on \( \Phi \) associated to \( \Psi_1, U_{f'_H} \) is normal in \( I \); moreover, we obviously have \( z'(\Psi') > z'(\Psi) \) hence \( z(\Psi') < z'(\Psi') \) by the second lemma. There exist then \( x' \in A \) and \( r' > r \in \mathbb{R} \) such that \( U_{f'_H} = U_{x',r'} \).

Moreover, since \( H \cap M \) is contained in \( I \), it normalizes \( U_{f'_H} \). Since for every \( \alpha \in \Phi \), we have \( f'_H(\alpha) \geq f_H(\alpha) \) if \( \alpha \in \Psi_0 \) and \( f'_H(\alpha) = f_H(\alpha) \) if \( \alpha \notin \Psi_0 \), we obtain:

\[
H = (H \cap M)U_{x',r'} = T'U_{M,x,r}U_{x',r'}
\]

for some \( T' \subset P_T \).

It remains to see that we can choose \( x = x' \), or equivalently, that the intersection \( E_{\Psi_0} \cap E_{\Psi',z'} \) is nonempty for some \( z' > z \). Let \( \Psi'' \) be the smallest \( \Delta' \)-complete subset of \( \Phi \) containing \( \Psi_0 \); we have \( z(\Psi'') < z(\Psi) \), and \( z'(\Psi'') = z'(\Psi) \) by a similar argument as in the lemma 6.7. Moreover, we have:

**Lemma 6.8** Let \( \alpha_1, \ldots, \alpha_t \) (resp. \( \beta_1, \ldots, \beta_s \), \( \gamma_1, \ldots, \gamma_r, \delta_1, \ldots, \delta_q \)) be elements of \( \Psi'' \) (resp. \( \Phi - \Psi'', \Psi', \Phi - \Psi' \)) such that \( \sum_{i=1}^t \alpha_i + \sum_{k=1}^r \gamma_k = \sum_{j=1}^s \beta_j + \sum_{l=1}^q \delta_l \). Set:

\[
c_0 = \sum_{i=1}^t \varepsilon_{\alpha_i} + \sum_{j=1}^s \varepsilon_{-\beta_j} + \sum_{k=1}^r \varepsilon_{\gamma_k} + \sum_{l=1}^q \varepsilon_{-\delta_l}.
\]

Then \( tz + s(1-z) + rz' + q(1-z') \geq c_0 \) for \( z' > z(\Psi) > z \) and \( z \) and \( z' \) close enough to each other.

If there exist \( i,j \) such that \( (\alpha_i, \beta_j) > 0 \), we can use the induction hypothesis as in the proof of the lemma 5.10; assume that \( (\alpha_i, \beta_j) \leq 0 \) for every \( i,j \). For the same reason, we can also assume \( (\gamma_k, \delta_l) \leq 0 \) for every \( k,l \), and since \( z' + 1 - z \geq 1 \), \( (\beta_j, \gamma_k) \leq 0 \) for every \( j,k \); moreover, if \( z' \geq 1 - z \) (resp. \( z' \leq 1 - z \)), we have \( z + z' \geq 1 \) (resp. \( 1 - z + (1 - z') \geq 1 \), hence we can assume \( (-\alpha_i, \gamma_k) \leq 0 \) for every \( i,k \) (resp. \( -\beta_j, \delta_l) \leq 0 \) for every \( j,l \)). We will suppose \( z + z' \geq 1 \), the other case being similar.

Consider the equality:

\[
\sum_{k=1}^r \gamma_k = \sum_{j=1}^s \beta_j + \sum_{l=1}^q \delta_l + \sum_{i=1}^t (-\alpha_i).
\]
Since the product of any term of the left-hand side by any term of the right-hand side is smaller than 0, all of them must be zero; in particular, we have $\sum_{k=1}^{r} \gamma_k = 0$. Since we then obtain $s'z' \geq s''z(\Psi) \geq \sum_{k=1}^{r} \varepsilon \gamma_k$, we may assume $r = 0$.

For any $z \in [z(\Psi''), z(\Psi)]$, according to the lemma 5.10 applied to $\Psi''$ and $z$, we have:

$$tz + (s + r)(1 - z) > c_0.$$  

This is in particular true for $z = z(\Psi)$; we easily deduce from this that we have $tz + s(1 - z) + r(1 - z') > c_0$ for $z \leq z(\Psi) < z'$ and $z$ and $z'$ close enough to $z(\Psi)$, as required. □

By the same argument as in the proof of the lemma 5.6, we only have to consider a finite number of families $((\alpha_i), (\beta_j), (\gamma_k), (\delta_l))$; there exist then $z, z'$ such that the above lemma is true for $z, z'$ and all such families. Hence $E(\Psi'', z) \cap E(\Psi', z')$ is nonempty; since $E(\Psi'', z)$ is obviously contained in $E(\Psi_0, z)$, we obtain the desired result.

We'll now turn on to the case when $f_H$ is of type 2. This case actually occurs too: for example, take $G = SL_4$, and:

$$H = \begin{pmatrix} 1 + p & p & p & p \\ p & 1 + p & p & p \\ p^2 & p^2 & 1 + p^2 & p^2 \\ p^2 & p^2 & p^2 & 1 + p^2 \end{pmatrix}.$$

It is easy to check that this group is a normal subgroup of $I$. Moreover, we have $f_\varepsilon(-\alpha_1) = 0$ and $f_\varepsilon(\alpha_3) = 2$, hence $f_H$ is of type 2.

Once again, $h$ is not of the form $T'u_{x,r}$ for any $T', x, r$. Let $\Phi_0$ be the subsystem of $\Phi$ whose elements are the linear combinations of elements $\alpha$ such that $f_\varepsilon(\alpha) = v'$; we have:

**Proposition 6.9** The root subsystem $\Phi_0$ of $\Phi$ is of rank strictly smaller than the rank of $\Phi$. Moreover, it admits a subset of $\Delta'$ as a set of simple roots.

We'll show that if $\alpha, \beta$ are elements of $\Phi$ such that $f_\varepsilon(\alpha) = v'$ and $f_\varepsilon(\beta) = v' + 2$, $\alpha$ and $\beta$ are strongly orthogonal; we'll then obtain the first assertion of the proposition the same way as in the proposition 6.6.
Assume $\alpha + \beta$ belongs to $\Phi$. If $\alpha + \beta$ is greater than $\alpha$ and $\beta$, we have:

$$f_\varepsilon(\beta) \leq f_\varepsilon(\alpha + \beta) + 1 \leq f_\varepsilon(\alpha) + 1 = v' + 1,$$

which is impossible; the case $\alpha + \beta$ lesser than $\alpha$ and $\beta$ is impossible too for similar reasons. Assume now $\alpha - \beta$ belongs to $\Phi$; we have $\beta = (\alpha - \beta) + \alpha$, hence $f_\varepsilon(\beta) \leq f_\varepsilon(\alpha) + 1$, which is again impossible. We then obtain that $\alpha$ and $\beta$ are strongly orthogonal.

Moreover, if $f_\varepsilon(\alpha) = 0$, then $f_\varepsilon(\alpha') = 0$ for every $\alpha' \geq \alpha$; we then obtain that every element of $\Delta'$ occurring in the decomposition of $-\alpha$ belongs to $\Phi_0$, which shows the second assertion. □

Consider the restriction of $f_H$ to $\Phi_0$. The subset $\Phi_0$ doesn’t contain any $\alpha$ such that $f_\varepsilon(\alpha) = v' + 2$, since we have just seen such an $\alpha$ is strongly orthogonal to $\Phi_0$; hence $f_H|_{\Phi_0}$ is of type 1. Moreover, let $\Delta_0'$ be the extended basis of $\Phi_0$ whose elements are the minimal elements of $\Phi_0$, and set $\Psi_0 = \Psi \cap \Phi_0$; $\Psi_0$ is obviously $\Delta_0'$-complete, and we have the following result:

Lemma 6.10 We have $z'(\Psi_0) = 1$.

Consider the extended basis $\Delta_0'$ of $\Phi_0$; this is the union of $\Delta_0 = \Delta' \cap \Phi_0$ with one additional element $\alpha$, which is the inverse of the greatest root of $\Phi_0$ w.r.t $\Delta_0$. Assume $f_H(-\alpha) = v'$; then for every $\beta \in \Phi$ such that $\beta \leq \alpha$, we must then have $f_H(-\beta) = v'$, hence $\beta \in \Phi_0$. By minimality of $\alpha$ in $\Phi_0$, we obtain that no such $\beta$ exists, hence $\alpha \in \Delta'$, which leads to a contradiction. Therefore $\Phi_0 - \Psi_0$ doesn’t contain the whole set $-\Delta_0'$, from which we deduce that $z'(\Psi_0) = 1$. □

Since $z(\Psi_0) < 1$ by definition, we may define $M, x$ and $r$ the same way as in the case $f_H$ of type 1 and $z(\Psi) = z'(\Psi)$; we obtain $r \in [v', v' + 1[$.

Moreover, let $f_H'$ be the function on $\Phi$ defined by $f_H'(\alpha) = f_H(\alpha) + 1$ if $f_\varepsilon(\alpha) = 0$, and $f_H'(\alpha) = f_H(\alpha)$ if $f_\varepsilon(\alpha) > 0$; we have:

Lemma 6.11 The group $U_{f_H'}$ is normal in $I$.

Let $\alpha, \beta$ be elements of $\Phi$ such that $\alpha + \beta \in \Phi$. Suppose that we have:

$$f_H'(\alpha + \beta) > f_H(\alpha) + f_H'(\beta).$$

This is possible only if we have $f_H(\alpha + \beta) = f_H(\alpha) + f_H(\beta), f_H'(\alpha) = f_H(\alpha) - 1$ and $f_H'(\beta) = f_H(\beta)$. The first equality implies:

$$f_\varepsilon(\alpha + \beta) \geq f_\varepsilon(\beta).$$
Since \( f_{\varepsilon}(\alpha + \beta) = 0 \) and \( f_{\varepsilon}(\beta) > 0 \), this is impossible and the lemma is proved. \( \square \)

It is obvious from its definition that \( f^1_H \) is of type 1. Moreover, let \( \Psi' \) be the \( \Delta' \)-complete subset of \( \Phi \) associated to \( f^1_H \); the elements \( \alpha \) of \( \Delta' \) contained in \( \Psi' \) are exactly the ones such that \( f_{\varepsilon}(\alpha) = 2 \). Since, according to the proof of the proposition 6.9, the corresponding subset of \( \Delta' \) is strongly orthogonal to some other nonempty subset of \( \Phi \), \( \Psi' \) cannot contain the whole set \( \Delta' \); hence \( z(\Psi') = 0 \). Since \( z'(\Psi') > 0 \) by definition, we may define \( x' \) and \( r' \) as in the previous case too; we obtain here \( r' \in [v' + 1, v' + 2] \).

We'll again conclude by proving that we may choose \( x = x' \), or equivalently that \( E_{\Psi_{0,z}} \cap E_{\Psi',z'} \) is nonempty for some \( z, z' \). Let \( x \) be an element of \( \mathcal{A} \) satisfying the following conditions:

- \( x \in E_{\Psi_{0,z}} \) for some \( z \geq z(\Psi_0) \), and \( \alpha(x) > 0 \) for every \( \alpha \in \Delta' \cap \Psi_0 \);
- \( x \in E_{\Psi',z'} \) for some \( z' \) such that \( z' < \alpha(x) \) for every \( \alpha \in \Delta' \cap \Psi_0 \).

The first condition is possible because \( E_{\Psi_{0,z}} \) contains an open subset of \( \mathcal{A} \), and the fact that \( \Psi_0 \) and \( \Psi' \) are strongly orthogonal to each other allows us to add the second one; we obtain that way \( \cup_{x,r} U_{x,r} \subset H \subset G_{x,r'} M_{x,r} \), as required.

We'll finally assume \( H \) is a normal subgroup of \( P \), where \( P \) is any parahoric subgroup of \( G \) containing \( I \). Since \( I \) normalizes \( H \), \( H \) is either of the form \( T'U_{x,r} \) for some \( T' \), \( x, r \) or of the form \( T'U_{M,x,r} \) for some \( T' \), \( x, r, M, r' \); it only remains to show that we can choose \( x \) in \( \overline{A}_P \).

Let \( P^+ \) be the pro-nilpotent radical of \( P \); let \( P_S \) be the unique parahoric subgroup of \( S(F) \) and let \( P_S^+ \) be its pro-nilpotent radical. For every \( w \) in the Weyl group \( W_P \) of \( P/P^+ \) relatively to \( P_S/P_S^+ \), \( w \) normalizes \( H \); hence \( H \) is equal to \( w(T')U_{w(x),r} \) (resp. \( w(T')U_{M,w(x),r}U_{w(x),r'} \)). The subset \( B_H \) of \( \overline{A}_I \) containing all elements \( x \) satisfying the condition of the theorem is then stable by \( W_P \); moreover, for any \( x, x' \in B_H \) and every \( t \in [0,1] \), the element \( (1-t)x + tx' \) is well-defined in every apartment of \( B \) containing both \( x \) and \( x' \), and we have:

\[
B(x, r) \cap B(x', r) \subset B((1-t)x + tx', r) \subset E(B(x, r), B(x', r)),
\]

where \( E(B(x, r), B(x', r)) \) is the closure of \( B(x, r) \cup B(x', r) \) in \( B \); hence \( B((1-t)x + tx', r) \) is fixed by \( H \) too. We deduce from this that \( B_H \) is a convex subset of \( \overline{A}_I \); since \( W_P \) is finite, for any \( x \in B_H \), the barycenter of the \( w(x) \), \( w \in W_P \), is both contained in \( B_H \) and fixed by \( W_P \), hence contained in \( \overline{A}_P \). This completes the proof of the theorem. \( \square \)
7 The general case

In this last section, we don’t suppose $G$ to be quasi-simple anymore. Let $\Phi_1, \ldots, \Phi_k$ be the connected components of $\Phi$; for every $i \in \{1, \ldots, k\}$, let $G_i$ be the subgroup of $G(F)$ generated by $T(F)$ and the $U_\alpha(F)$, $\alpha \in \Phi_i$. Let $P$ be a parahoric subgroup containing $P_T$, and let $H$ be a normal subgroup of $P$; an immediate consequence of the proposition 1.1 is that we have:

$$H = \prod_{i=1}^{k} (H \cap G_i).$$

For every $i$, let $\mathcal{B}_i$ be the Bruhat-Tits building of $G_i$, and let $\mathcal{A}_i$ be the apartment of $\mathcal{B}_i$ associated to $S(F)$; it is easy to check that $\mathcal{A}$ is canonically isomorphic to $\prod_{i=1}^{k} \mathcal{A}_i$. Moreover, let $A_P$ be the facet of $\mathcal{A}$ attached to $P$; $A_P$ is canonically isomorphic to $\prod_{i=1}^{k} A_{P_i}$, where for every $i$, $A_{P_i}$ is the facet of $\mathcal{A}_i$ attached to $P_i = P \cap G_i$.

Let $x = (x_1, \ldots, x_k)$ be an element of $\mathcal{A}$, and let $r_1, \ldots, r_k$ be nonnegative real numbers; we’ll set:

$$G_{x,(r_1,\ldots,r_k)} = \prod_{i=1}^{k} (G_i)_{x_i,r_i},$$

where for every $i$, $(G_i)_{x_i,r_i}$ is the standard filtration subgroup of $G_i$ attached to $x_i$ and $r_i$. If $x$ belongs to $A_P$, $G_{x,(r_1,\ldots,r_k)}$ is obviously normal in $P$. We’ll define $U_{x_i,r_i}$ and $U_{x,(r_1,\ldots,r_k)}$ in a similar fashion.

According to theorem 1.2, for every $i$, there exists $x_i \in A_{P_i}$ and $r_i \in \mathbb{R}^+$ such that $H \cap G_i$ satisfies one of the following conditions:

- $U_{x_i,r_i} \subset H \cap G_i \subset (G_i)_{x_i,r_i}$;
- there exists $r_i' > r_i$ and a proper Levi subgroup $M_i$ of $G_i$ containing $T(F)$ and such that $U_{x_i,r_i'} U_{M_i,x_i,r_i} \subset H \cap G_i \subset (G_i)_{x_i,r_i'} (M_i)_{x_i,r_i}$.

(Note that $r_1, \ldots, r_k$ are not related in any way, and can be completely different from each other.)

For every $i$ such that $U_{x_i,r_i}$ is of the first kind, set $M_i = G_i$ and $r_i' = r_i$. Set $M = \prod_{i=1}^{k} M_i$, and $U_{M,x,(r_1,\ldots,r_k)} = \prod_{i=1}^{k} U_{M,x_i,r_i}$; we deduce from above the following generalization of the theorem 1.2:

**Theorem 7.1** Assume the conditions of the proposition 1.1 are satisfied. There exists $x \in A_P$ and $r_1, \ldots, r_k \in \mathbb{R}^+_+$ such that $H$ satisfies one of the following
conditions:

- \( U_{x,(r_1,...,r_k)} \subset H \subset G_{x,(r_1,...,r_k)} \);
- there exist \( r'_1, ..., r'_k \) such that \( r'_i \geq r_i \) for every \( i \) and \( r'_i > r'_i \) for at least one \( i \), and a proper Levi subgroup \( M \) of \( G \) containing \( T \), such that

\[
U_{x,(r'_1,...,r'_k)} U_{M,x,(r_1,...,r_k)} \subset H \subset G_{x,(r'_1,...,r'_k)} M_{x,(r_1,...,r_k)}.
\]

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