Simultaneous quantization of edge and bulk Hall conductivity

Hermann Schulz-Baldes, Johannes Kellendonk, Thomas Richter
TU Berlin, FB Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany

Abstract

The edge Hall conductivity is shown to be an integer multiple of $e^2/h$ which is almost surely independent of the choice of the disordered configuration. Its equality to the bulk Hall conductivity given by the Kubo-Chern formula follows from K-theoretic arguments. This leads to quantization of the Hall conductance for any redistribution of the current in the sample. It is argued that in experiments at most a few percent of the total current can be carried by edge states.

Soon after the discovery of the integer quantum Hall effect (QHE) [19], several geometric interpretations of the observed quantization of the Hall conductance of a two dimensional electron gas were put forward in the framework of non-relativistic quantum mechanics. Laughlin proposed an adiabatic Gedankenexperiment in order to calculate the Hall conductance [15], Halperin and later on Büttiker studied the conduction by edge channels [13, 10], while Thouless, Kohmoto, Nightingale and den Nijs investigated the Hall conductivity as given by the Kubo formula [20]. Laughlin’s argument was rigorously analyzed by Avron, Seiler and Simon even for multiparticle Hamiltonians and in presence of a disordered potential [3, 4, 5]. Bellissard, recently joint by van Elst and Schulz-Baldes, generalized the TKN$_2$-work in order to show quantization of the Hall conductivity also in presence of a disordered potential as long as the Fermi level lies in a region of dynamically localized states [6, 7], a result that was also obtained by Aizenman and Graf [1]. All these beautiful mathematical approaches exhibit the Hall conductance and conductivity respectively to have a deep geometrical meaning and allow to calculate them as an index of a certain Fredholm operator. In [20, 6, 7, 1], the edges of the sample play no particular rôle.

Recently there has been a revived interest in edge states of magnetic Schrödinger operators. Hatsugai linked an edge state winding number to the Chern numbers for Harper’s equation with rational flux [14]. Akkermans, Avron, Narevich and Seiler introduced spectral boundary conditions giving rise to a linear dispersion relation for edge channels and a natural setting for the Laughlin wave function as a many body bulk state [2]. The stability of the absolutely continuous spectrum associated to edge states under the perturbation with a random potential was studied by several authors with Mourre’s positive commutator estimates [16, 8, 12].

Our first main result is a rigorous proof of the edge current quantization in the sense of Halperin for a discrete magnetic half-plane operator containing a disordered potential, notably we show quantization of what we call the edge Hall conductivity. Our second mathematical result is its equality to the bulk Hall conductivity as calculated by the Kubo-Chern formula [20, 6, 7]. The proof of this equality unveals a deep connection between the plane and edge geometry as it is based on Bott periodicity, the heart of K-theory [9]. We still need a gap in the spectrum of the plane operator, but a generalization to a region of dynamically localized states is under investigation. Using these results, we reproduce Halperin’s argument explaining why the Hall conductance of a
Hall bar is quantized no matter what proportion of the current is actually carried by the edge or the bulk states respectively. Finally we present a simple theoretical reasoning showing that in a typical experimental situation at most 10% of the current flows by edge states.

For the definition of the edge Hall conductivity, we consider a gas of charged independent particles in the (discrete) upper half plane $\Gamma = \{(x, y) \in \mathbb{Z}^2 | y \geq 0\}$ submitted to a perpendicular magnetic field $B$. Let $\hat{H}$ denote the one-particle Hamiltonian acting on $\ell^2(\Gamma)$. All operators on the half-plane space carry a hat from now on. Typically $\hat{H}$ is the projection onto $\ell^2(\mathbb{Z}^2)$ of an operator $H = H_H + V$ acting on $\ell^2(\mathbb{Z}^2)$ where $H_H$ is Harper’s magnetic Hamiltonian and $V$ is the sum of a periodic and a disordered potential. As the edge of the plane intercepts the cyclotron orbits, there may be a net electric current along the edge. In order to calculate it, let $J$ be a spectral interval lying in a gap of the plane Hamiltonian $H$. Such an interval typically contains extended edge states of $\hat{H}$ [14], even in presence of a weak disordered potential [13, 16, 8, 12]. If $\hat{P}_J$ is the spectral projection of $\hat{H}$ on $J$, then the electric edge current in $x$-direction carried by the eigenstates in $J$ is equal to $q\bar{T}(\hat{P}_J \nabla_x (\hat{H}))/\hbar$. Here $q$ is the charge of the particles, $\nabla_x (\hat{H}) = i [X, \hat{H}]$ is the current operator given by the commutator of the Hamiltonian and the $X$-position operator and finally the trace $\bar{T} = \text{Tr}_y \bar{T}_x$ is the trace per unit volume [3, 4] in the $x$-direction and the usual trace in $y$-direction. Now given an energy $E$ in a gap of extended states of $H$, we define

$$\sigma^e_\perp (E) = \frac{q^2}{\hbar} \lim_{J \rightarrow \{E\}} \frac{1}{|J|} \bar{T}(\hat{P}_J \nabla_x (\hat{H})).$$

Because an infinite half-plane is a typical model for a mesoscopic volume with a boundary, we call $\sigma^e_\perp (E)$ the edge Hall conductivity rather than the edge Hall conductance just as the bulk Hall conductivity are idealized quantities for which clear mathematical results can be obtained. Further we note that one could define the edge Hall conductivity for a strip geometry, but this would not lead to quantization because of backscattering, that is tunneling from upper to lower edge states [10].

Before starting the more mathematical analysis, let us consider the Harper Hamiltonian $H_H$ on $\ell^2(\mathbb{Z}^2)$ in order to familiarize ourselves with the notion of edge Hall conductivity. It is defined by the finite difference equation $(H_H \psi)_{n,m} = \psi_{n+1,m} + \psi_{n-1,m} + e^{2\pi i \varphi} \psi_{n,m+1} + e^{-2\pi i \varphi} \psi_{n,m-1}$ and we suppose here that the magnetic flux per unit cell is rational $\varphi = p/q$. Then the spectrum of $H_H$ is known to be a band spectrum. To analyse the half plane operator $\tilde{H}_H$ on $\ell^2(\Gamma)$, we use the translation invariance in the $x$-direction to make a Bloch decomposition $\tilde{H}_H = \sum_{l=\tau,\pi} \frac{\delta q}{2\pi} \hat{H}_H(k_x)$ where $\hat{H}_H(k_x)$ is a Jacobi matrix on $\ell^2(\mathbb{N})$. The spectrum of each $\hat{H}_H(k_x)$ contains the bands of the corresponding periodic operator $H_H(k_x)$ on $\ell^2(\mathbb{Z})$, but there may now also be a Dirichlet eigenvalue $\tilde{E}_l(k_x)$ in each gap of $H_H(k_x)$ [4]. Upon varying $k_x$, the eigenvalues form a finite number of continuous curves end-points of which touch the adjacent Bloch bands of $H_H$ (see Fig. 1). To each of these so-called edge channels we associate a weight $+1$ (respectively $-1$) if the Dirichlet eigenvalues of the channel vary from the upper towards the lower (respectively lower to upper) adjacent Bloch band as $k_x$ increases. Let $s_n$ be the sum of all these weights in the $n$th gap $G_n$ of $H_H$. Then the edge current carried by the edge states in an interval $J$ contained in $G_n$ is equal to $s_n|J|q^2/\hbar$ because

$$\bar{T}(\hat{P}_J \nabla_x (\hat{H}_L)) = \sum_l \int_{-\pi}^\pi dk_x \chi_J(\tilde{E}_l(k_x)) \frac{d\tilde{E}_l(k_x)}{dk_x}.$$

Here $\chi_J$ denotes the indicator function on $J$. This implies that $\sigma^e_\perp (E) = s_n q^2/\hbar$ for all $E \in G_n$. Hatsugai, in a beautiful paper [14], has shown that $s_n$ is equal to the sum of the Chern numbers.
of the $n$ bands below $G_n$. This sum multiplied by $q^2/h$ is the bulk Hall conductivity $\sigma^b_\perp(E)$ [20]. Hence we obtain $\sigma^e_\perp(E) = \sigma^b_\perp(E)$ for all energies in the gaps of $H_H$, which is a particular case of Theorem 2 below. Note that the equivalent result for the Landau Hamiltonian simply states that there are $n$ edge channels in the gap between the $n$th and $(n+1)$th Landau bands [13].

Now we would like to add a disordered potential $V$. First of all, if $V$ is sufficiently small, sufficiently large gaps of $H_H$ remain open for $H = H_H + V$. It follows further from Mourre estimates on the current operator that the spectrum remains absolutely continuous in the gaps of $H$ for a weak potential whenever the current of the edge states of $\hat H_H$ has a definite sign [8]. Whereas the latter condition is always satisfied for the Landau Hamiltonian, it may not hold in the discrete case (cf. Fig. 1 and the numerical studies in [14] where edge channels having edge states with group velocity both to the left and to the right are exhibited). In this situation, the positive commutator methods cannot be applied. Nevertheless, we shall be able to show that the current remains constant. However, we cannot deduce that the spectrum is still absolutely continuous once a small perturbation is added.

In order to treat the situation with broken translation invariance, we parallel Bellissard’s non-commutative generalization of the TKN$_2$ work [6, 7]. No particular structure of the Hamiltonian $H$ on $\ell^2(\mathbb{Z}^2)$ is needed except for its homogeneity in the sense of [6, 7]. The main mathematical tool in [6, 7] is the $C^*$-algebra $A$ of homogeneous observables in the plane. It has the structure of a crossed product algebra $A = C(\Omega) \times \mathbb{Z}_x \times \mathbb{Z}_y$ associated to the dynamical system given by the magnetic translations $\mathbb{Z}_x$ and $\mathbb{Z}_y$ in the $x$ and $y$-direction respectively acting on the compact space of disorder configurations $\Omega$ which is the hull of $H$. Each such configuration $\omega \in \Omega$ induces a representation $\pi_\omega$ of the observable algebra $A$ on physical Hilbert space $\ell^2(\mathbb{Z}^2)$. There exists an $H \in A$ such that $\pi_\omega(H)$ is precisely the Hamilton operator with disordered configuration $\omega \in \Omega$. We now consider the Toeplitz extension $T(A)$ with respect to the crossed product structure of $\mathbb{Z}_y$ [18]. Its physical representations give operators in the half plane. This naturally gives rise to an exact sequence of $C^*$-algebras [18]

$$0 \to \mathcal{E} \overset{\delta}{\to} T(A) \overset{\pi}{\to} A \to 0.$$  \hspace{1cm} (2)

Here $\mathcal{E}$ is the $C^*$-algebra of observables localized near the edge $y = 0$; it is isomorphic to the $C^*$-tensor product of $C(\Omega) \times \mathbb{Z}_x$ with the compact operators $K$. The exact sequence (2) induces two six-term exact sequences, one for K-theory groups [18, 9] and one for the cyclic cohomology groups [17], and we shall use their duality [17] to prove the equality of bulk and edge Hall conductivities.

Let us illustrate these notions for the Harper Hamiltonian with arbitrary flux $\varphi$, but without a further potential. The $C^*$-algebra $A$ is then the rotation algebra generated by the two magnetic translations $U_x$ and $U_y$ satisfying the commutation relation $U_x U_y = e^{i2\pi \varphi} U_y U_x$. Thus in this case
$C(\Omega) \cong \mathbb{C}$. The Toeplitz extension is generated by $\hat{U}_x$ and $\hat{U}_y$ satisfying the same commutation relation, but, while $\hat{U}_x$ remains unitary, $\hat{U}_y$ is now only an isometry satisfying $\hat{U}_y^*\hat{U}_y = 1 - \Pi_0$ where $\Pi_0$ is the projection on the states supported by the boundary of $\Gamma$. Finally, $\mathcal{E}$ is isomorphic to the tensor product of $C^*(\hat{U}_x) \cong C(S^1)$ with $K$. The maps in (3) are the inclusion $i$ and the projection $\pi$ given by $\pi(\hat{U}_{x,y}) = U_{x,y}$.

The traces $\mathcal{T}_{x,y}$ of physical representations $\pi_\omega$ of an observable are almost surely independent of $\omega$ with respect to any given invariant and ergodic measure $\mathbf{P}$ on $\Omega$. Hence they allow to define traces on the observable algebras $\mathcal{A}$ and $\mathcal{E}$. Now the definition (4) of the edge Hall conductivity remains valid as long as the projections $\hat{P}_J$ are in the Schatten ideal of traceclass operators with respect to $\hat{T}$ for $J$ sufficiently close to $\{E\}$. This is possible even though $\hat{P}_J$ is only an element of the bicommutant $\mathcal{E}''$, the enveloping von Neumann algebra. Now the crucial observation is that the current of the edge states in an interval $J$ lying in a gap $G$ of the spectrum of $H$ can be calculated using Duhamel’s formula and taking into account elementary properties of projections:

$$\hat{T}(\hat{P}_J \nabla_x(\hat{H})) = \frac{|J|}{2\pi i} \hat{T}( (\hat{U}(J)^* - 1) \nabla_x \hat{U}(J)),$$

where

$$\hat{U}(J) = \exp \left( 2\pi i \hat{P}_J \frac{\hat{H} - E'}{|J|} \right), \quad E' = \inf(J). \quad \text{(4)}$$

Although $\hat{U}(J)$ is built out of the operators $\hat{P}_J$ and $\hat{H}$ which are not localized near the boundary and not even in the $C^*$-algebra $T(\mathcal{A})$, we can show that $\hat{U}(J) - 1$ is an element of the edge algebra $\mathcal{E}$ by using the exponential map of the six-term exact sequence of $K$-groups $\mathbf{K}$ associated to the exact sequence $\mathbf{4}$. More precisely, the image under the exponential map of the class $[P_{\mu}]_0 \in K_0(\mathcal{A})$ associated to the Fermi projection $P_{\mu}$ is equal to the class $[\hat{U}(J)]_1 \in K_1(\mathcal{E})$ whenever the Fermi level $\mu$ is in $J$. In fact, $P_{\mu}$ is equal to the continuous function of the Hamiltonian $f(H) = P_{E'} - P_J(H - E')/|J|$. Now a self-adjoint lift of $P_{\mu}$ is given by $f(\hat{H})$. From $[\hat{P}_{E'}, \hat{P}_J] = 0$ thus follows

$$\exp([P_{\mu}]_0) = [\exp(-2\pi i f(\hat{H}))]_1 = [\hat{U}(J)]_1.$$ 

Finally we note that continuously varying the boundaries of $J$ to those of $G$ leads to a homotopy from $\hat{U}(J)$ to $\hat{U}(G).$ Thus (4) actually associates to $G$ a class in the $K$-group $K_1(E)$.

It now follows from Connes’ non-commutative geometry $\mathbf{11}$ that $\frac{1}{i} \hat{T}(i(\hat{U} - 1) \nabla_x \hat{U})$ is an integer for any unitary $\hat{U}$ in (a suitable subalgebra of) $\mathcal{E}$. Actually $\xi_1(\hat{A}, \hat{B}) = \frac{1}{i} \hat{T}((\hat{A} \nabla_x \hat{B}))$ defines a 1-cocycle on $\mathcal{E}$ because $\hat{T}$ is invariant under $\nabla_x$. With some calculatory effort, this cocycle can be linked to the standard 1-cocyle of the Fredholm module $(C_1 \otimes \mathcal{E}_0, \pi_\omega \oplus \pi_\omega, \ell^2(\Gamma) \oplus \ell^2(\sigma_2 \otimes iX/|X|))$ where $\mathcal{E}_0$ is the $\mathcal{E}$ dense $*$-algebra of operators with finite support in $y$-direction and $C_1$ a two-dimensional $\mathbb{Z}_2$ graded Clifford algebra in Mat$(C^2)$, $\pi_\omega \oplus \pi_\omega$ is a doubling of the physical representation on the doubled physical Hilbert space $\ell^2(\Gamma) \oplus \ell^2(\Gamma)$ and the Dirac phase is constructed from the Pauli matrix $\sigma_2$ and the position operator $X$. Hence the odd index theorem $\mathbf{11}$, p. 291, a density and homotopy argument linking $\mathcal{U}(G)$ to an element in $\mathcal{E}_0$ $\mathbf{11}$, p. 249] and a treatment of the disorder configuration along the lines of $\mathbf{4}$ imply the following result.

**Theorem 1** Suppose that $G \subset \mathbb{R}$ is a spectral gap of the plane operator $H$ acting on $\ell^2(\mathbb{Z}^2)$. Let $\Pi$ denote the projection from $\ell^2(\mathbb{Z} \otimes \mathbb{N})$ onto $\ell^2(\mathbb{N} \otimes \mathbb{N})$ and let $\mathcal{U}(G)$ constructed by $\mathbf{11}$ from $\hat{P}_G$. Then for $\mathbf{P}$-almost every $\omega \in \Omega$, the operator $\Pi_\omega(\mathcal{U}(G))\Pi$ is a Fredholm operator on $\ell^2(\mathbb{N} \otimes \mathbb{N})$ with constant index and for all $E \in G$

$$\sigma^\omega_\perp(E) = \frac{q^2}{\hbar} \text{Ind} \left( \Pi_\omega(\mathcal{U}(G))\Pi \right).$$
We remark that the index can also be written as a relative index of a pair of projections as defined by Avron, Seiler and Simon \[5\], notably as the relative index of \(\Pi\) and \(\pi_\omega(\hat{U}(G))\Pi_{\pi_\omega(\hat{U}(G))^*}\).

Using the exact sequence \(2\), we now link this edge theory to the bulk theory as developed in \([4]\). From the above follows that \(\sigma_\perp^b(E)\) actually results from a pairing \([11]\) between \([\hat{U}(G)]_1 \in K_1(\mathcal{E})\) with the odd cyclic cohomology class defined by the 1-cocycle \(\zeta_1\) given above. Similarly, the bulk Hall conductivity \(\sigma_\perp^b(\mu)\) for a Fermi level \(\mu\) in a gap of \(H\) comes from a pairing of the class of the Fermi projection \([P_\mu]_0 \in K_0(\mathcal{A})\) with the 2-cocycle \(\zeta_2\) over \(\mathcal{A}\) defined by \(\zeta_2(\mathcal{A}, B, C) = 2\pi i \mathcal{T}_x \mathcal{T}_y (A\nabla_x B \nabla_y C - A \nabla_y B \nabla_x C)\) \([7]\):

\[
\sigma_\perp^b(\mu) = \langle \zeta_2, [P_\mu]_0 \rangle.
\]

We showed above that \([\hat{U}(G)]_1\) is the image of \([P_\mu]_0\) under the exponential map of K-theory. Next one can verify that the 1-cocycle \(\zeta_1\) over \(\mathcal{E}\) is mapped to the 2-cocycle \(\zeta_2\) over \(\mathcal{A}\) under the mapping \# defined in \([17]\), Sec. 8. For this map, the duality theorem of the pairing holds, notably \(\langle \zeta_1, \exp([P]) \rangle = \langle \# \zeta_1, [P]_0 \rangle\) for any projection \(P \in \mathcal{A}\) \([17]\), Sec. 12]. Hence we obtain:

**Theorem 2** \(\sigma_\perp^b(E) = \sigma_\perp^b(E)\) for all energies \(E\) in a spectral gap of \(H\).

At this point, let us comment on generalizations of these results. Just as one does not need the existence of a gap \(G\) in order to prove the quantization of the bulk Hall conductivity \([3, 4]\), it is likely that Theorems 1 and 2 hold under the weaker hypothesis that the interval \(G\) only contains dynamically localized states of \(H\) in the sense of \([4]\). Furthermore, the whole theory should have a continuous analogon for a disordered Landau Hamiltonian. As both of these results ask for more lengthy and detailed proofs, they will be subject of a forthcoming publication.

We now sketch how the above results lead to the desired explanation of a QH regime measurement in a QH bar. Following Halperin \([13]\), we suppose that the measured Hall voltage \(V_\perp\) across the bar is the sum of the potential drop \(V^b\) due to an electrostatic field and the (relativ) chemical potential difference \(\Delta \mu/q = (\mu_u - \mu_l)/q\) between the upper and the lower edge. Furthermore, let the interval \([\mu_l, \mu_u]\) be contained in a gap \(G\) of \(H\) (the above generalization only needs the weaker condition that \(G\) is dynamically localized). In linear response approximation, the electric field leads to a bulk current \(I^b = \sigma_\perp^b V^b\). Now both the upper and the lower edge may carry a current. In absence of backscattering, we can treat them as two separate half-plane problems. But actually the lower edge can be seen as an upper edge with reversed magnetic field, which is equivalent to a time reversal. This changes the orientation of its current so that the net current carried by both edges comes from the upper edge states with energies in \([\mu_l, \mu_u]\). From the above thus results a net edge current \(I^e = \sigma_\perp^e \Delta \mu/q\). Hence the Hall conductance of the sample given by the quotient of the total current \(I = I^e + I^b\) and the voltage \(V_\perp\) is equal to the integer \(\sigma_\perp^e = \sigma_\perp^b\) for any value of \(V^b/V_\perp\).

An interesting question which has led to considerable theoretical and experimental work (see \([21]\) and references therein) is how much current is carried by either edge or bulk states in a typical QH experiment. Let us argue that at most 10\% of the current is carried by the edge states. This agrees with recent experimental studies \([21]\). For the edge current of \([\mu_l, \mu_u]\) to be equal to an integer times \(\Delta \mu\), the difference of chemical potentials \(\Delta \mu\) clearly has to be smaller than the energetic distance \(\hbar \omega_c (1 - p)\) (here \(\omega_c\) is the cyclotron frequency so that \(\hbar \omega_c\) is the distance between two Landau levels, and \(p\) is the quotient of the energetic width of the plateaux and \(\hbar \omega_c\)). Hence the proportion of edge current has to be smaller than \(\Delta \mu/p V_\perp\). In order to estimate this condition and the temperature corrections below, we use the experimental data from \([19]\), Chapter 2 for the \(\sigma_\perp = 4\) plateau: \(B \approx 6 T, V_\perp \approx 170 mV\) and \(T \approx 1.2 K\) and \(p \approx 0.6\). Using the data for
the effective electron mass \( m_\ast \approx 0.07m_e \) and the electron charge, we obtain \( \hbar \omega_c \approx 48 \text{meV} \) and a maximal proportion of edge currents of 10%.

We acknowledge support by the SFB 288.

References

[1] Aizenman M, Graf G M 1998 J. Phys. A 31 6783
[2] Akkermans E, Avron J E, Narevich R, Seiler R 1998 European Phys. J. B 1 117
[3] Avron J E, Seiler R, Simon B 1983 Phys. Rev. Lett. 51 51
[4] Avron J E, Seiler R 1985 Phys. Rev. Lett. 54 259
[5] Avron J E, Seiler R, Simon B 1994 Commun. Math. Phys. 159 399
[6] Bellissard J in Proc. of the Bad Schandau Conference on Localization, edited by Ziesche & Weller, (Teubner-Verlag, Leipzig, 1987).
[7] Bellissard J, van Elst A, Schulz-Baldes H 1994 J. Math. Phys 35 5373
[8] De Bievre S, Pulé J V 1999 Elect. J. Math. Phys. 5 17 pg.
[9] Blackadar B K-Theory for Operator Algebras (Springer, Berlin, 1986)
[10] Büttiker M 1988 Phys. Rev. B 38 9375
[11] Connes A 1985 Publ. IHES 62 257
[12] Fröhlich J, Graf G M, Walcher J preprint 1999, math-ph/9903014, to appear Annales H. Poincaré.
[13] Halperin B I 1982 Phys. Rev. B 25 2185
[14] Hatsugai Y 1993 Phys. Rev. Lett. 71 3697
[15] Laughlin R B 1981 Phys. Rev. B 23 5632
[16] Macris N, Martin P A, Pulé J V 1999 J. Phys. A 32 1985
[17] Nest R 1988 J. Funct. Ana. 80 235
[18] Pimsner M, Voiculescu D 1980 J. Op. Theory 4 93
[19] Prange R, Girvin S Editors, The Quantum Hall Effect, (Springer-Verlag, Berlin, 1985).
[20] Thouless D J, Kohmoto M, Nightingale M, den Nijs M 1982 Phys. Rev. Lett. 49 405
[21] Yahel E, Tsukernik A, Palevski A, Shtrikman A 1998 Phys. Rev. Lett. 81 5201