A SHORT PROOF OF THE BLOW-UP LEMMA FOR APPROXIMATE DECOMPOSITIONS

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Abstract. Kim, Kühn, Osthus and Tyomkyn (Trans. Amer. Math. Soc. 371 (2019), 4655–4742) greatly extended the well-known blow-up lemma of Komlós, Sárközy and Szemerédi by proving a ‘blow-up lemma for approximate decompositions’ which states that multipartite quasirandom graphs can be almost decomposed into any collection of bounded degree graphs with the same multipartite structure and slightly fewer edges. This result has already been used by Joos, Kim, Kühn and Osthus to prove the tree packing conjecture due to Gyárfás and Lehel from 1976 and Ringel’s conjecture from 1963 for bounded degree trees as well as implicitly in the recent resolution of the Oberwolfach problem (asked by Ringel in 1967) by Glock, Joos, Kim, Kühn and Osthus.

Here we present a new and significantly shorter proof of the blow-up lemma for approximate decompositions. In fact, we prove a more general theorem that yields packings with stronger quasirandom properties so that it can be combined with Keevash’s results on designs to obtain results of the following form. For all \( \varepsilon > 0 \), \( r \in \mathbb{N} \) and all large \( n \) (such that \( r \) divides \( n - 1 \)), there is a decomposition of \( K_n \) into any collection of \( r \)-regular graphs \( H_1, \ldots, H_{n-1}/r \) on \( n \) vertices provided that \( H_1, \ldots, H_{\varepsilon n} \) contain each at least \( \varepsilon n \) vertices in components of size at most \( \varepsilon^{-1} \).

1. Introduction

The theme of decomposing ‘large’ objects into ‘smaller’ objects or finding a maximal number of specified ‘small’ objects in a ‘larger’ object is among the key topics in mathematics. In discrete mathematics, it appears in Euler’s question from 1782 for which \( n \) there exist pairs of orthogonal Latin squares\(^1\) of order \( n \), in Steiner’s questions for Steiner systems from the 1850s which cumulated in the ‘existence of designs’ question, in Walecki’s theorem on decompositions of complete graphs into edge-disjoint Hamilton cycles from the 1890s, and in Kirkman’s famous ‘school girl problem’.

These questions and results set off an entire branch of combinatorics and design theory. Several decades later in the 1970s, Wilson \([27, 28, 29]\) famously proved that (the edge set of) the complete graph on \( n \) vertices can be decomposed into any fixed graph provided necessary divisibility conditions are satisfied and \( n \) is large, thereby solving the ‘existence of designs’ question for graphs. In 2014, Keevash verified the ‘existence of designs’ for hypergraphs \([14]\). This has been reproved and generalised by Glock, Kühn, Lo and Osthus in \([11, 12]\). Keevash extended his results to a more general framework in \([15]\).

In contrast to these questions and results where we aim to decompose a large graph into graphs of fixed size, one can also ask for decompositions into larger pieces, for example into graphs with the same number of vertices as the host graph. A prime example is the Oberwolfach problem where Ringel asked in 1967 whether one can decompose (the edge set of) \( K_{2n+1} \) into \( n \) copies of any 2-regular graph on \( 2n + 1 \) vertices. This problem received considerable attention and Glock, Joos, Kim, Kühn and Osthus \([10]\) solved it for large \( n \). Possibly equally well-known is Ringel’s conjecture from 1963 stating that \( K_{2n+1} \) can be decomposed into any tree with \( n \) edges, as well as the tree packing conjecture due to Gyárfás and Lehel from 1976 stating that \( K_n \) can be decomposed into any collection of trees \( T_1, \ldots, T_n \), where \( T_i \) has \( i \) vertices. Ringel’s conjecture has been solved by Montgomery, Pokrovskiy and Sudakov \([24, 25]\) and both conjectures have been solved for bounded degree trees by Joos, Kim, Kühn and Osthus \([13]\); Allen, Böttcher, Clemens and Taraz \([1]\) have solved these conjectures for trees with many leaves and maximum degree \( o(n/\log n) \) (in fact, they proved a more general result on degenerate graphs) – all mentioned results apply only when \( n \) is large.
for each collection of vertex partitions \((d_i)\). We also refer to for every \(G\) of multipartite graphs in a host graph \(H\) producing some terminology and then state the blow-up lemma for approximate decompositions. We reduce some terminology and then state the blow-up lemma for approximate decompositions. We call \(H\) a \(\phi\)-packing of \(G\) if there is a function \(\phi : V(H) \to V(G)\) such that \(\phi|_{V(H)}\) is injective and \(\phi\) injectively maps edges onto edges. In such a case, we call \(\phi\) a packing of \(H\) into \(G\). Our general aim is to pack a collection \(H\) of multipartite graphs in a host graph \(G\) having the same multipartite structure which is captured by a so-called ‘reduced graph’ \(R\). To this end, let \((H,G,R,\mathcal{X},\mathcal{V})\) be a blow-up instance if

- \(H, G, R\) are graphs where \(V(R) = [r]\) for some \(r \geq 2\);
- \(\mathcal{X} = (X_i)_{i \in [r]}\) is a vertex partition of \(H\) into independent sets, \(\mathcal{V} = (V_i)_{i \in [r]}\) is a vertex partition of \(G\) such that \(|V_i| = |X_i|\) for all \(i \in [r]\);
- \(H[X_i, X_j]\) is empty whenever \(ij \in \binom{[r]}{2} \setminus E(R)\).

We also refer to \(\mathcal{B} = (H, G, R, \mathcal{X}, \mathcal{V})\) as a blow-up instance if \(H\) is a collection of graphs and \(\mathcal{X}\) is a collection of vertex partitions \((X^H_i)_{i \in [r], H \in \mathcal{H}}\) so that \((H,G,R, (X^H_i)_{i \in [r]}, \mathcal{V})\) is a blow-up instance for every \(H \in \mathcal{H}\).

The quasi-random notion mostly used in this paper coincides with the notion used in Szemerédi’s regularity lemma. For a bipartite graph \(G\) with vertex partition \((V_1, V_2)\), we define the density of \((W_1, W_2)\) with \(W_1 \subseteq V_1\) by \(d_G(W_1, W_2) := e_G(W_1, W_2)/|V_1||V_2|\). We say \(G\) is \((\varepsilon, d)\)-regular if \(d_G(W_1, W_2) = d \pm \varepsilon\) for all \(W_1 \subseteq V_1\) with \(|W_1| \geq \varepsilon|V_1|\), and \(G\) is \((\varepsilon, d)\)-super-regular if in addition \(|N_G(v) \cap V_{3-i}| = (d \pm \varepsilon)|V_{3-i}|\) for each \(i \in \{0, 1\}\) and \(v \in V_i\). The blow-up instance \(\mathcal{B}\) is \((\varepsilon, d)\)-super-regular if \(G[V_i, V_j]\) is \((\varepsilon, d)\)-super-regular for all \(ij \in E(R)\), and \(\mathcal{B}\) is \(\Delta\)-bounded if \(\Delta(R), \Delta(H) \leq \Delta\) for every \(H \in \mathcal{H}\).

We are ready to state the blow-up lemma for approximate decompositions.

**Theorem 1.1** (Kim, Kühn, Osthus, Tyomkyn [18]). For all \(\alpha \in (0, 1)\) and \(r \geq 2\), there exist \(\varepsilon = \varepsilon(\alpha) > 0\) and \(n_0 = n_0(\alpha, r)\) such that the following holds for all \(n \geq n_0\) and \(d \geq \alpha\). Suppose \((\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})\) is an \((\varepsilon, d)\)-super-regular and \(\alpha^{-1}\)-bounded blow-up instance such that \(|V_i| = n\) for all \(i \in [r]\), \(|\mathcal{H}| \leq \alpha^{-1}n\), and \(\sum_{H \in \mathcal{H}} e_H(X^H_i, X^H_j) \leq (1 - \alpha)dn^2\) for all \(ij \in E(R)\). Then there is a packing \(\phi\) of \(\mathcal{H}\) into \(G\) such that \(\phi(X^H_i) = V_i\) for all \(i \in [r]\) and \(H \in \mathcal{H}\).
We remark that there are more general versions of Theorem 1.1 in [18], but omit the more technical statements here. Instead we state our main result and the interested reader can easily check that it generalises\(^2\) the more technical versions in [18].

1.2. Main result. Most blow-up lemmas exhibit their power if they are applied in conjunction with Szemerédi’s regularity lemma. This, however, comes with the expense of a small set of vertices over which we have no control. Consequently, in such a setting, when embedding a graph \(H\) into \(G\), it is often the case that some vertices of \(H\) are already embedded and the blow-up lemma is applied only to some nice part of \(G\). To deal with such scenarios we consider extended blow-up instances. We say \((H,G,R,X,V,\phi_0)\) is an extended blow-up instance if

- \(H,G,R\) are graphs where \(V(R) = \{r\}\) for some \(r \geq 2\);
- \(X = (X_i)_{i \in [r]}\) is a vertex partition of \(H\) into independent sets, \(V = (V_i)_{i \in [r]}\) is a vertex partition of \(G\) such that \(|V_i| = |X_i|\) for all \(i \in [r]\);
- \(H[X_i, X_j]\) is empty whenever \(i \notin (\{r\} \setminus E(R))\);
- \(\phi_0\) is an injective embedding of \(X_0\) into \(V_0\).

This definition also extends as above to the case when \(H\) is replaced by a collection of graphs \(\mathcal{H}\) in the obvious way as before. An extended blow-up instance is \((\varepsilon,d)\)-super-regular if \(G[V_i, V_j]\) is \((\varepsilon,d)\)-super-regular for all \(ij \in E(R)\).

Let \(\mathcal{B} = (\mathcal{H}, G, R, X, V, \phi_0)\) be an extended blow-up instance. We say \(\mathcal{B}\) is \((\varepsilon,\alpha)\)-linked if

- at most \(\varepsilon|X_i^H|\) vertices in \(X_i^H\) have a neighbour in \(X_0^H\) for all \(i \in [r]\), \(H \in \mathcal{H}\);
- \(|\phi_0^{-1}(v)| \leq \varepsilon|\mathcal{H}|\) for all \(v \in V_0\);
- \(\sum_{H \in \mathcal{H}} |N_H(x_0^H) \cap N_H(x_i^H)| \leq \varepsilon |V_i|^2/2\) for all \(i \in [r]\) and distinct \(v_0, v_0' \in V_0\) where \(x_0^H = \phi_0^{-1}(v_0) \cap X_0^H\) and \(x_i^H = \phi_0^{-1}(v_0') \cap X_i^H\) for \(H \in \mathcal{H}\).

One feature of our result is that one can replace ‘blow-up instance’ in Theorem 1.1 by ‘extended blow-up instance that is \((\varepsilon,\alpha)\)-linked’. We remark that the above conditions are easily met in applications known to us and are similar to conditions found elsewhere for this purpose.

Next, we define two types of structures for \(\mathcal{B}\) and our main result yields a packing that behaves as we would expect it from an idealised typical random packing with respect to these structures. We say \((W,Y_1,\ldots,Y_k)\) is an \(\ell\)-set tester for \(\mathcal{B}\) if \(k \leq \ell\) and there exist \(i \in [r]\) and distinct \(H_1,\ldots,H_k \in \mathcal{H}\) such that \(W \subseteq V_i\) and \(Y_j \subseteq X_i^H\) for all \(j \in [k]\). We say \((v,\omega)\) is an \(\ell\)-vertex tester for \(\mathcal{B}\) if \(v \in V_i\) and \(\omega : \bigcup_{H \in \mathcal{H}} X_i^H \to [0,\ell]\) for some \(i \in [r]\). For a weight function \(\omega\) on a finite set \(X\), we define \(X(\omega) := \sum_{x \in X} \omega(x)\) for any \(X \subseteq X\). The following theorem is our main result.

**Theorem 1.2.** For all \(\alpha \in (0,1]\) and \(r \geq 2\), there exist \(\varepsilon = \varepsilon(\alpha) > 0\) and \(n_0 = n_0(\alpha, r)\) such that the following holds for all \(n \geq n_0\) and \(d \geq \alpha\). Suppose \((H,G,R,X,V,\phi_0)\) is an \((\varepsilon,d)\)-super-regular, \(\alpha^{-1}\)-bounded and \((\varepsilon,\alpha)\)-linked extended blow-up instance. \(|V_i| = (1 \pm \varepsilon)n\) for all \(i \in [r]\), \(|\mathcal{H}| \leq \alpha^{-1}n\), and \(\sum_{H \in \mathcal{H}} e(H, X_i^H, X_j^H) \leq (1 - \alpha)dn^2\) for all \(ij \in E(R)\). Suppose \(W_{\text{set}}, W_{\text{ver}}\) are sets of \(\alpha^{-1}\)-set testers and \(\alpha^{-1}\)-vertex testers of size at most \(n^{\log n}\), respectively. Then there is a packing \(\phi\) of \(\mathcal{H}\) into \(G\) which extends \(\phi_0\) such that

(i) \(\phi(X_i^H) = V_i\) for all \(i \in [r]\) and \(H \in \mathcal{H}\);
(ii) \(|W \cap \bigcup_{j \in [r]} \phi(Y_j)| = |W||Y_1| \cdots |Y_i|/n^\ell \pm \alpha n\) for all \((W,Y_1,\ldots,Y_k)\) \(\in W_{\text{set}}\);
(iii) \(\omega(\bigcup_{H \in \mathcal{H}} X_i^H \cap \phi^{-1}(v)) = \omega(\bigcup_{H \in \mathcal{H}} X_i^H)/n^\ell \pm \alpha n\) for all \((v,\omega)\) \(\in W_{\text{ver}}\).

1.3. Applications. The multipartite framework can be used to obtain results for the non-partite setting. The next theorem applies to graphs \(G\) that are \((\varepsilon,d)\)-quasirandom; that is, if \(n\) is the order of \(G\), then \(|N_G(u)| = (d \pm \varepsilon)n\) and \(|N_G(u) \cap N_G(v)| = (d^2 \pm \varepsilon)n\) for all distinct \(u, v \in V(G)\). Given \(G\) and a collection of graphs \(\mathcal{H}\) on at most \(n\) vertices, we say \((W,Y_1,\ldots,Y_k)\) is an \(\ell\)-set tester if \(k \leq \ell\) and there exist distinct \(H_1,\ldots,H_k \in \mathcal{H}\) such that \(W \subseteq V(G)\) and \(Y_i \subseteq V(H_i)\) for all \(i \in [k]\). We say \((v,\omega)\) is an \(\ell\)-vertex tester if \(v \in V(G)\) and \(\omega : \bigcup_{H \in \mathcal{H}} V(H) \to [0,\ell]\).

\(^2\)Observe that we do not allow different densities between the cluster pairs in \(G\). However, this technical complication could very easily be implemented by adding at numerous places extra indices. As this feature has never been used so far in applications, we omitted it for the sake of a clearer presentation.
Theorem 1.3. For all $\alpha > 0$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose $G$ is an $(\varepsilon,d)$-quasirandom graph on $n$ vertices and $\mathcal{H}$ is a collection of graphs on at most $n$ vertices with $|\mathcal{H}| \leq \alpha^{-1}n$ and $\sum_{\mathcal{H} \in \mathcal{H}} e(\mathcal{H}) \leq (1 - \alpha)e(G)$ as well as $\Delta(\mathcal{H}) \leq \alpha^{-1}$ for all $\mathcal{H} \in \mathcal{H}$. Suppose $\mathcal{W}_{\text{set}}, \mathcal{W}_{\text{ver}}$ are sets of $\alpha^{-1}$-set testers and $\alpha$-vertex testers of size at most $n^{\log n}$, respectively. Then there is a packing $\phi$ of $\mathcal{H}$ into $G$ such that

- $|W \cap \bigcap_{i \in [\ell]} \phi(Y_i)| = |W||Y_1| \cdots |Y_\ell|/n^\varepsilon \pm o(n)$ for all $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{\text{set}}$;
- $\omega(\bigcup_{\mathcal{H} \in \mathcal{H}} V(\mathcal{H})) \cap \phi^{-1}(v) = \{\mathcal{H} \in \mathcal{H} \mid V(\mathcal{H}) \cap \{v, \omega\} \neq \emptyset\}$ for all $(v, \omega) \in \mathcal{W}_{\text{ver}}$.

In many scenarios when one applies approximate decomposition results, as for example Theorem 1.3, it is important that the graph $G - \phi(\mathcal{H})$ has ‘small’ maximum degree. Here, this can be easily achieved by utilising vertex testers $(v, \omega)$ where $\omega$ assigns to all $x \in \bigcup_{\mathcal{H} \in \mathcal{H}} V(\mathcal{H})$ the degree of $x$. We remark that set and vertex testers in our main result are very flexible and capture many desirable properties. For example, Theorem 1.3 implies the approximate decomposition result in [1] when restricted to graphs of bounded maximum degree.

In this paper, we give one example how to apply Theorem 1.3. By exploiting set testers, we can combine it with Keevash’s results on hypergraph decompositions to decompose pseudorandom graphs into regular spanning graphs as long as only a few graphs contain a few vertices in components of bounded size. This is stronger as some results in [10] when restricted to graphs of bounded maximum degree. This is stronger as some results in [10] when restricted to graphs of bounded maximum degree.

For this result, we need a stronger notation of pseudorandomness as used by Keevash in [15]. We say a graph $G$ on $n$ vertices is $(\varepsilon,s,d)$-typical if $|\bigcap_{u \in U} N_G(u)| = (1 \pm \varepsilon)d^U/n$ for all sets $U \subseteq V(G)$ with $|U| \leq s$.

Theorem 1.4. For all $\alpha > 0$, there exist $\varepsilon > 0$ and $s, n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose $G$ is a regular $(\varepsilon,s,d)$-typical graph on $n$ vertices and $\mathcal{H}$ is a collection of regular graphs on $n$ vertices with $\sum_{\mathcal{H} \in \mathcal{H}} e(\mathcal{H}) = e(G)$ as well as $\Delta(\mathcal{H}) \leq \alpha^{-1}$ for all $\mathcal{H} \in \mathcal{H}$. Suppose there are at least $\alpha n$ graphs $H \in \mathcal{H}$ such that at least $\alpha n$ vertices in $H$ belong to components of order at most $\alpha^{-1}$. Then there is a decomposition of the edge set of $G$ into $\mathcal{H}$.

Theorem 1.4 makes progress on a conjecture by Glock, Joos, Kim, Kühn and Osthus who conjecture in [10] that $K_n$ can be decomposed into any collection $\mathcal{H}$ of regular bounded degree graphs with $\sum_{\mathcal{H} \in \mathcal{H}} e(\mathcal{H}) = \binom{n}{2}$.

2. Proof sketch

Before we explain our approach, we briefly sketch the approach of Kim, Kühn, Osthus and Tyomkyn in [18]. Their first step is to stack several graphs $H \in \mathcal{H}$ together to a new graph $\tilde{H}$ such that $\tilde{H}[X^H_i, X^H_j]$ is essentially regular for all $ij \in E(R)$. Let $\tilde{\mathcal{H}}$ be the collection of these graphs $\tilde{H}$. They prove that such graphs $\tilde{H}$ can be embedded into $G$ by a probabilistic algorithm in a very uniform way. For some $\gamma \ll \alpha$, they apply this algorithm to $\gamma n$ graphs in $\tilde{\mathcal{H}}$ in turn. Observe that this may cause edge overlaps in $G$. Nevertheless, after embedding $\gamma n$ graphs, they remove all ‘used’ edges from $G$ and repeat. At the end, they eliminate all edge overlaps by unembedding several vertices and complete the packing by utilising a thin edge slice put aside at the beginning.

Our approach is somewhat perpendicular to their approach. We proceed cluster by cluster and find a function $\phi_i$ which maps almost all vertices in $\bigcup_{\mathcal{H} \in \mathcal{H}} X^H_i$ into $V_i$ and which is consistent with our partial packing so far. Our ‘Approximate Packing Lemma’, stated in Section 4, performs one such step using an auxiliary hypergraph where we aim to find a large matching which is pseudorandom with respect to certain weight functions. At the end, we complete the packing by also using a thin edge slice similar to [18]. At the beginning, we partition the clusters of our blow-up instance into many smaller clusters with the only purpose to ensure that $H[X^H_i, X^H_j]$ is a matching (see Section 3.4). This preprocessing is comparably simple and first used in [26].

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1. The results in [10] consider only 2-regular graphs. However, their proof for the part where they consider collections of graphs $\mathcal{H}$ that contain a few graphs with almost all vertices in components of bounded size carries over verbatim to $r$-regular graphs for any $r$ if $n$ is large in terms of $r$.

2. In fact, their main theorem only applies to collections of graphs that are essentially regular and this stacking had to be performed again in [4] and [13] which made the application in both cases technically involved.
Both the approach in [18] and ours draw on ideas from an alternative proof of the blow-up lemma by Rödl and Ruciński [26]. In spirit, our approach is again closer to the procedure in [26] as they also embed the clusters of $H$ in turn. Many generalisations of the original blow-up lemma build on this alternative proof. We hope that our alternative proof of the blow-up lemma for approximate decompositions paths the way for further developments in the field.

Some ideas in this paper are taken from our recent article with Glock [7] on rainbow embeddings in graphs. As a crucial part of our proof, we apply the main result from another paper with Glock [6], where we prove the existence of quasirandom hypergraph matchings in hypergraphs with small codegree. The idea that hypergraph matchings can be used to embed (almost) spanning graphs has been brought to our attention by [17].

3. Preliminaries

3.1. Notation. For $k \in \mathbb{N}$, we write $[k] := [k] \cup \{0\} = \{0, 1, \ldots, k\}$, where $[0] = \emptyset$. For a finite set $S$ and $k \in \mathbb{N}$, we write $\binom{S}{k}$ for the set of all subsets of $S$ of size $k$ and $2^S$ for the powerset of $S$. For a set $\{i, j\}$, we sometimes simply write $ij$. For $a, b, c \in \mathbb{R}$, we write $a = b \pm c$ whenever $a \in [b-c, b+c]$. For $a, b, c \in (0, 1]$, we sometimes write $a \ll b \ll c$ in our statements meaning that there are increasing functions $f, g : (0,1] \to (0,1]$ such that whenever $a \leq f(b)$ and $b \leq g(c)$, then the subsequent result holds. For $a \in [0,1]$ and $b \in (0,1]^k$, we write $a \ll b$ whenever $a \ll b_i$ for all $b_i \in b$, $i \in [k]$. For the sake of a clearer presentation, we avoid roundings whenever it does not affect the argument.

For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively. We say $u \in V(G)$ is a $G$-neighbour of $v \in V(G)$ if $uv \in E(G)$ and let $N_G(u)$ be the set of all $G$-neighbours of $u$ as well as $\deg_G(u) := |N_G(u)|$. As usual, $\Delta(G)$ denotes the maximum degree of $G$. For $u, v \in V(G)$, let $N_G(u, v) := N_G(u) \cap N_G(v)$ denote the common neighbourhood of $u$ and $v$, and let $N_G[u] := N_G(u) \cup \{u\}$. For a set $S$, let $N_G(S) := \bigcup_{v \in S} N_G(v)$. We frequently treat collections of graphs as the graph obtained by taking the disjoint union of all members; for example, for a collection $H$ of graphs, we define $N_H(v) := \bigcup_{H \in H} N_H(v)$ for all $v \in V(H)$ with $H \in H$. For disjoint subsets $A, B \subseteq V(G)$, let $G[A, B]$ denote the bipartite subgraph of $G$ between $A$ and $B$ and $G[A]$ the subgraph of $G$ induced by $A$. For convenience, let $G[A, A] := G[A]$. Let $e(G)$ denote the number of edges of $G$ and let $e_G(A, B)$ denote the number of edges of $G[A, B]$. For $k \in \mathbb{N}$, let $G^k$ denote the $k$-th power of $G$, that is, the graph obtained from $G$ by adding all edges between vertices whose distance in $G$ is at most $k$. For graphs $G, H$, we write $G - H$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

3.2. Probabilistic tools and graph regularity. To verify the existence of subgraphs with certain properties we frequently consider random subgraphs and use McDiarmid’s inequality to verify that specific random variables are highly concentrated around their mean.

**Theorem 3.1** (McDiarmid’s inequality, see [22]). Suppose $X_1, \ldots, X_m$ are independent Bernoulli random variables and suppose $b_1, \ldots, b_m \in [0, B]$. Suppose $X$ is a real-valued random variable determined by $X_1, \ldots, X_m$ such that changing the outcome of $X_i$ changes $X$ by at most $b_i$ for all $i \in [m]$. Then, for all $t > 0$, we have

$$\mathbb{P} [|X - \mathbb{E}[X]| \geq t] \leq 2 \exp \left( -\frac{2t^2}{B\sum_{i=1}^m b_i} \right).$$

We will also need the following two standard results concerning the robustness of $\varepsilon$-regular graphs.

**Fact 3.2.** Suppose $G$ is an $(\varepsilon, d)$-regular bipartite graph with vertex partition $(A, B)$ and $Y \subseteq B$ with $|Y| \geq \varepsilon |B|$. Then all but at most $2\varepsilon |A|$ vertices of $A$ have $(d \pm \varepsilon)|Y|$ neighbours in $Y$.

**Fact 3.3.** Suppose $1/n \ll \varepsilon \ll d$. Suppose $G$ is an $(\varepsilon, d)$-super-regular bipartite graph with vertex partition $(A, B)$, where $\varepsilon^{1/6} n \leq |A|, |B| \leq n$. If $\Delta(H) \leq \varepsilon n$ and $X \subseteq A \cup B$ with $|X| \leq \varepsilon n$, then $G[A \setminus X, B \setminus X] - H$ is $(\varepsilon^{1/3}, d)$-super-regular.

We will also use the next result from [5]. (In [5] it is proved in the case when $|A| = |B|$ with $16 \varepsilon^{1/5}$ instead of $\varepsilon^{1/6}$. The version stated below can be easily derived from this.)
Theorem 3.4. Suppose $1/n \ll \varepsilon \ll \gamma, d$. Suppose $G$ is a bipartite graph with vertex partition $(A, B)$ such that $|A| = n$, $\gamma n \leq |B| \leq \gamma^{-1} n$ and at least $(1 - 5\varepsilon)n^2/2$ pairs $u, v \in A$ satisfy $\deg_G(u), \deg_G(v) \geq (d - \varepsilon)|B|$. Then $G$ is $\varepsilon^{1/6}$-regular.

At the end of our packing algorithm we apply the following version of the blow-up lemma due to Komlós, Sarközy, and Szemerédi.

Theorem 3.5 (Komlós, Sarközy, and Szemerédi [19]). Suppose $1/n \ll \varepsilon \ll 1/\Delta, d$ and $1/n \ll 1/r$. Suppose $(H, G, R, (X_i)_{i \in [r]}, (V_i)_{i \in [r]})$ is an $(\varepsilon, d')$-super-regular and $\Delta$-bounded blow-up instance, with $d' \geq d$ as well as $|V_i| = (1 \pm \varepsilon)n$ for all $i \in [r]$ and $(A_i)_{i \in [r]}$ is a collection of graphs such that $A_i$ is bipartite with vertex partition $(X_i, V_i)$ and $(\varepsilon, d_i)$-super-regular for some $d_i \geq d$. Then there is a packing $\phi$ of $H$ into $G$ such that $\phi(x) \in N_{A_i}(x)$ for all $x \in X_i$ and $i \in [r]$.

3.3. Pseudorandom hypergraph matchings. A key ingredient in the proof of our ‘Approximate Packing Lemma’ in Section 4 is the main result from [6] on pseudorandom hypergraph matchings.

For this we need some more notation. Given a set $X$ and an integer $\ell \in \mathbb{N}$, an $\ell$-tuple weight function on $X$ is a function $\omega: \binom{X}{\ell} \rightarrow \mathbb{R}_{\geq 0}$. For a subset $X' \subseteq X$, we then define $\omega(X') := \sum_{S \in \binom{X'}{\ell}} \omega(S)$. For $k \in [\ell]_0$ and a tuple $T \in \binom{X}{k}$, define $\omega(T) := \sum_{S \supseteq T} \omega(S)$ and let $||\omega||_k := \max_{T \in \binom{X}{k}} \omega(T)$. Suppose $H$ is an $r$-uniform hypergraph and $\omega$ is an $\ell$-tuple weight function on $E(H)$. Clearly, if $M$ is a matching, then a tuple of edges which do not form a matching will never contribute to $\omega(M)$. We thus say that $\omega$ is clean if $\omega(E) = 0$ whenever $E \in \binom{E(H)}{\ell}$ is not a matching.

Theorem 3.6 ([6]). Suppose $1/\Delta \leq \delta, 1/r, 1/L, r \geq 2$, and let $\varepsilon := \delta/50L^{2r^2}$. Let $H$ be an $r$-uniform hypergraph with $\Delta(H) \leq \Delta$ and $\Delta'(H) \leq \Delta^{1-\delta}$ as well as $e(H) \leq \exp(\Delta^2)$. Suppose that for each $\ell \in [L]$, we are given a set $W_{\ell}$ of clean $\ell$-tuple weight functions on $E(H)$ of size at most $\exp(\Delta^2)$, such that $E(H(H)) \geq ||\omega||_k \Delta^{k+\delta}$ for all $\omega \in W_{\ell}$ and $k \in [\ell]$. Then there exists a matching $M$ in $H$ such that $\omega(M) = (1 \pm \Delta^{-\varepsilon})\omega(E(H))/\Delta^\ell$ for all $\ell \in [L]$ and $\omega \in W_{\ell}$.

3.4. Refining partitions. Here, we provide a useful result to refine the vertex partition of a blow-up instance such that every $H \in \mathcal{H}$ only induces a matching between its refined partition classes. While in [26] this procedure was easily obtained by applying the classical Hajnal–Szemerédi Theorem, we perform a random procedure to obtain more control on the mass distribution of a weight function with respect to the refined partition.

Lemma 3.7. Suppose $1/n \ll \beta \ll \alpha$ and $1/n \ll 1/r$. Suppose $\mathcal{H}$ is a collection of at most $\alpha^{-1}n$ graphs, $(X_i^H)_{i \in [r]}$ is a vertex partition of $H$, and $\Delta(H) \leq \alpha^{-1}$ for every $H \in \mathcal{H}$. Suppose $n^{2/3} \leq |X_i^H| = |X_i^{H'}| \leq 2n$ for all $H, H' \in \mathcal{H}$ and $i \in [r]$. Suppose $W$ is a set of weight functions $\omega: \bigcup_{H \in \mathcal{H}, i \in [r]} X_i^H \rightarrow [0, \alpha^{-1}]$ with $|W| \leq \sqrt{n}$. Then for all $H \in \mathcal{H}$ and $i \in [r]$, there exists a partition $(X_{i,j}^H)_{j \in [\beta^{-1}]}$ of $X_i^H$ such that for all $H \in \mathcal{H}, \omega \in W, i, i' \in [r], j, j' \in [\beta^{-1}]$ where $i \neq i'$ or $j \neq j'$, we have that

(i) $X_{i,j}^H$ is independent in $H^2$;
(ii) $|X_{i,1}^H| \leq \ldots \leq |X_{i,\beta^{-1}}^H| \leq |X_{i,1}^H| + 1$;
(iii) $\omega(X_{i,j}^H) = \beta \omega(X_i^H) \pm \beta^{3/2}/n$.
(iv) $\sum_{H \in \mathcal{H}} e_H(X_{i,j}^H) \leq \beta^2 \sum_{H \in \mathcal{H}} e_H(X_i^H, X_i^H) \pm n^{5/3}$.

Proof. Our strategy is as follows. We first consider every $H \in \mathcal{H}$ in turn and construct a partition $(Y_{i,j})_{j \in [\beta^{-1}]}$ that essentially satisfies (i)–(iii) with $Y_{i,j}$ playing the role of $X_{i,j}^H$. Then we perform a vertex swapping procedure to resolve some conflicts in $Y_{i,j}$ and obtain $Z_{i,j}^H$. In the end, we permute the ordering of $(Z_{i,j}^H)_{j \in [\beta^{-1}]}$ for each $H \in \mathcal{H}, i \in [r]$ to also ensure (iv).

To simplify notation, we assume from now on that $|X_i^H|$ is divisible by $\beta^{-1}$ for all $i \in [r]$ and at the end we explain how very minor modifications yield the general case.

Fix some $H \in \mathcal{H}$ and write $X_i$ for $X_i^H$. We claim that there exist partitions $(Y_{i,j})_{j \in [\beta^{-1}]}$ of $X_i$ for each $i \in [r]$ such that for all $i \in [r], j \in [\beta^{-1}]$
(a) \(|Y_{i,j}| = \beta |X_i| \pm \beta^2 n;\)
(b) at most \(2n^{-2}\beta^2 n\) pairs of vertices in \(Y_{i,j}\) are adjacent in \(H^2;\)
(c) \(\omega(Y_{i,j}) = \beta \omega(X_i) \pm \beta^2 n\) for all \(\omega \in \mathcal{W}.\)

Indeed, the existence of such partitions can be seen by assigning every vertex in \(X_i\) uniformly at random to some \(Y_{i,j}\) for \(j \in [\beta^{-1}]\). Together with a union bound and Theorem 3.1 we conclude that (a)–(c) hold simultaneously with positive probability.

Next, we slightly modify these partitions to obtain a new collection of partitions. These modifications can be performed for each \(i \in [r]\) independently. Hence, we fix some \(i \in [r]\). For all \(j \in [\beta^{-1}]\), let \(W_j \subseteq Y_{i,j}\) be such that \(|Y_{i,j} \setminus W_j| = \beta |X_i| - \beta^{5/3} n\) and \(W_j\) contains all vertices in \(Y_{i,j}\) that contain an \(H^2\)-neighbour in \(Y_{i,j}\) (the sets \(W_j\) clearly exist by (a),(b)). Let \(\{w_1, \ldots, w_s\} = W := \bigcup_{j \in [\beta^{-1}]} W_j\) and observe that \(s = |X_i| - \sum_{j \in [\beta^{-1}]} |Y_{i,j} \setminus W_j| = \beta^{2/3} n.\)

Now arbitrarily assign labels in \([\beta^{-1}]\) to the vertices in \(W\) such that each label is exactly \(\beta^{2/3} n\) times. Let \(Z_j(0) := Y_{i,j} \setminus W_j\) for all \(j \in [\beta^{-1}]\). To obtain the desired partitions we perform the following swap procedure for every \(t \in [s]\) in turn. Say \(w_t \in W_j\) and \(w_t\) receives label \(j'\). We select \(j'' \in [\beta^{-1}] \setminus \{j, j'\}\) such that \(w_t\) has no \(H^2\)-neighbour in \(Z_{j''}(t - 1)\) and such that \(Z_{j''}(t - 1)\) contains a vertex \(w\) that has no \(H^2\)-neighbour in \(Z_{j''}(t - 1)\). In such a case we say that \(j''\) is selected in step \(t\). Then we define \(Z_{j''}(t) := (Z_{j''}(t - 1) \cup \{w_{t}\}) \setminus \{w\} \), \(Z_{j''}(t) := Z_{j''}(t) \cup \{w\}\) and \(Z_k(t) := Z_k(t - 1)\) for all \(k \in [\beta^{-1}] \setminus \{j', j''\}\). Note that \(\beta |X_i| - \beta^{5/3} n \leq |Z_k(t)| \leq \beta |X_i|\) for all \(t \in [s], k \in [\beta^{-1}].\) Observe also that we have always at least \(\beta^{-1}/2\) choices to select \(j''\) in step \(t\). As \(s = \beta^{2/3} n\), we can ensure that each \(j'' \in [\beta^{-1}]\) is selected, say, at most \(10\beta^{2/3} n\) times. We write \(Z_{H,i} := Z_{j}(s)\) and then it is straightforward to see that for all \(j \in [\beta^{-1}]\) we have that

(a') \(|Z_{H,i}^j| = \beta |X^H_i|;\)
(b') \(Z_{H,i}^j\) is independent in \(H^2;\)
(c') \(\omega(Z_{H,i}^j) = \beta \omega(X^H_i) \pm \beta^2/2 n\) for all \(\omega \in \mathcal{W}.\)

As \(i \in [r]\) and \(H \in \mathcal{H}\) are chosen arbitrarily, the statements (a')–(c') hold for all \(i \in [r], H \in \mathcal{H}.\)

In the remainder of the proof we show how to find permutations \(\{\pi_{i,j}^H\}_{H \in \mathcal{H}, i \in [r]}\) such that (iv) holds for \(X_{i}^{H,j} := Z_{H,i}^j\). The answer is simple, random permutations yield (iv) with probability, say, at least \(1/2\). To see this, fix \(i, i' \in [r], j, j' \in [\beta^{-1}]\) where \(i \neq i'\) or \(j \neq j'\), and then Theorem 3.1 implies that \(\sum_{H \in \mathcal{H}} e_H(X_{i}^{H,j}, X_{i'}^{H,j'}) = \beta^2 \sum_{H \in \mathcal{H}} e_H(X^H_i, X^H_{i'}) \pm n^{5/3}/2\) with probability at least \(1 - 1/\alpha\). A union bound over all \(i, i', j, j'\) completes the proof.

In the beginning we made the assumption that \(\beta^{-1}\) divides \(X^H_i\). To avoid this assumption, we simply remove a set \(\tilde{X}^H_i\) of size at most \(\beta^{-1} - 1\) from \(X^H_i\) such that \(\beta^{-1}\) divides \(X^H_i \setminus \tilde{X}^H_i\) and perform the entire procedure with \(X^H_i \setminus \tilde{X}^H_i\) instead of \(X^H_i\). That is, for all \(H \in \mathcal{H}\) and \(i \in [r]\), we obtain a partition \((\tilde{X}^H_i)_{i \in [\beta^{-1}]})\) of \(X^H_i \setminus \tilde{X}^H_i\) satisfying (i)–(iv), where \(|\tilde{X}^H_i| = |X^H_i|\) for all \(i \in [r], j, j' \in [\beta^{-1}]\), and (iv) and (iii) hold with error terms \(\pm n^{5/3}/2\) and \(\pm \beta^2/2 n\), respectively. At the very end we add the vertices in \(\tilde{X}^H_i\) to the partition \((\tilde{X}^H_i)_{i \in [\beta^{-1}]})\) while preserving (i) and (ii). We may do so by performing a swap argument as before. Observe that our error bounds give us enough room to spare.

4. Approximate packings

The goal of this section is to provide an ‘Approximate Packing Lemma’ (Lemma 4.3). Given a blow-up instance \((\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})\), it allows us to embed almost all vertices of \(\bigcup_{H \in \mathcal{H}} X^H_i\) into \(V_i\), while maintaining crucial properties for future embedding rounds of other clusters. To describe this setup we define a packing instance and collect some more notation.

4.1. Packing instances. Given a graph \(G\) and a set \(E\), we call \(\psi\) a \([E, G, R, \mathcal{X}, \mathcal{V}]\) an edge set labelling of \(G\). A label \(\alpha \in \mathcal{E}\) appears on an edge \(e\) if \(\alpha \in \psi(e)\). We define the maximum degree \(\Delta_\psi(G)\) of \(\psi\) as the maximum number of edges of \(G\) on which any fixed label appears. We define the maximum codegree \(\Delta_\psi^c(G)\) of \(\psi\) as the maximum number of edges of \(G\) on which any two fixed labels appear together.
Let \( r \in \mathbb{N}_0 \). We say \((\mathcal{H}, G, R, A, \psi)\) is a packing-instance of size \( r \) if

- \( \mathcal{H} \) is a collection of graphs, and \( G \) and \( R \) are graphs, where \( V(R) = [r]_0 \);
- \( A = \bigcup_{H \in \mathcal{H}, i \in [r]_0} A_i^H \) is a union of balanced bipartite graphs \( A_i^H \) with vertex partition \( (X_i^H, V_i) \);
- \( (X_i^H)_{i \in [r]_0} \) is a partition of \( H \) into independent sets for every \( H \in \mathcal{H} \), and \( (V_i)_{i \in [r]_0} \) is a partition of \( V(G) \);
- \( R = R_A \cup R_B \) is the union of two edge-disjoint graphs with \( N_R(0) = [r]_0 \); for all \( H \in \mathcal{H} \), the graph \( H[X_i^H, X_j^H] \) is a matching if \( i,j \in E(R) \) and empty otherwise; in such a case, we write for simplicity \( X \) and their edge set labelling according to the following two definitions.

\[ \text{Definition 4.1 (Updated candidacy graphs). For a conflict-free packing } \sigma: \mathcal{X}_0^r \rightarrow V_0 \text{ in } \mathcal{X}_0 \text{ and all } H \in \mathcal{H}, i \in [r]_0, \text{ let } A_i^H[\sigma] \text{ be the updated candidacy graph (with respect to } \sigma) \text{ which is defined by the spanning subgraph of } A_i^H \text{ that contains precisely those edges } xv \in E(A_i^H) \text{ for which the following holds: if } x \text{ has an } H \text{-neighbour } x_0 \in \mathcal{X}_0^r \text{ (which would be unique), then } \sigma(x_0)v \in E(G[V_0, V_i]). \]

\[ \text{Definition 4.2 (Updated labelling). For a conflict-free packing } \sigma: \mathcal{X}_0^r \rightarrow V_0 \text{ in } \mathcal{X}_0 \text{, let } \psi[\sigma] \text{ be the updated edge set labelling (with respect to } \sigma) \text{ defined as follows: for all } H \in \mathcal{H}, i \in [r]_0 \text{ and } xv \in E(A_i^H[\sigma]), \text{ if } x \text{ has an } H \text{-neighbour } x_0 \in \mathcal{X}_0^r, \text{ then set } \psi[\sigma](xv) := \psi(xv) \cup \{\sigma(x_0), v\}, \text{ and otherwise set } \psi[\sigma](xv):= \psi(xv). \]

In order to be able to analyse our packing process in Section 5, we carefully maintain quasi-random properties of the candidacy graphs throughout the procedure. To this end, we refer to a packing instance \((\mathcal{H}, G, R, A, \psi)\) of size \( r \) as an \((\varepsilon, d)\)-packing-instance, where \( d = (d_A, d_B, d_0, \ldots, d_r) \), if

\[ (P1) \quad G[V_i, V_j] \) is \((\varepsilon, d_1, d_2)\)-super-regular for all \( ij \in E(R_Z), Z \in \{A, B\}; \]
\[ (P2) \quad A_i^H \) is \((\varepsilon, d_1)\)-super-regular for all \( H \in \mathcal{H}, i \in [r]_0; \]
\[ (P3) \quad e_H(N_{A_i^H}(v_i), N_{A_i^H}(v_j)) = (d_2, d_3 \pm \varepsilon) e_H(X_i^H, X_j^H) \) for all \( H \in \mathcal{H}, i,j \in E(R_A), v_i, v_j \in E(G[V_i, V_j]); \]
\[ (P4) \quad \Delta_0(\mathcal{X}_0^r) \leq (1 + \varepsilon)d_i[V_i] \) for all \( i \in [r]_0. \]

Property \( (P4) \) ensures that no edge is a potential candidate for too many graphs in \( \mathcal{H} \) and \( (P3) \) enables us to maintain this property for future embedding rounds (see Lemma 4.3(IV),4.3 below). Let \( \mathcal{P} = (\mathcal{H}, G, R, A, \psi) \) be an \((\varepsilon, d)\)-packing-instance of size \( r \). Similarly as for a blow-up instance, we say \((W, Y_1, \ldots, Y_k)\) is an \( \ell \)-set tester for \( \mathcal{P} \) if \( k \leq \ell \) and there exist distinct \( H_1, \ldots, H_k \in \mathcal{H} \) such that \( W \subseteq V_0 \) and \( Y_j \subseteq X_i^{H_j} \) for all \( j \in [k] \). For \( i \in N_{R_B}(0) \) and \( v \in V_i \), we say \( \omega: E(\mathcal{X}_0^r) \rightarrow [0, \ell] \) is an \( \ell \)-edge tester with centre \( v \) for \( \mathcal{P} \) if \( \omega(x'v') = 0 \) for all \( x'v' \in E(\mathcal{X}_0^r) \) with \( v' \in V_i, v' \neq v \). We
say \( \omega : E(\mathcal{A}_0^i) \to [0, \ell] \) is an \( \ell \)-edge tester with centres in \( \mathcal{X}_0^i \) if there exist vertices \( \{x_H^i\}_{H \in \mathcal{H}} \) with \( x_H^i \in X_H^i \) for each \( H \in \mathcal{H} \) such that \( \omega(x_H^i) = 0 \) for all \( x_H' \in E(\mathcal{A}_0^i) \) with \( x_H' \notin \{x_H^i\}_{H \in \mathcal{H}} \). Further, let \( \dim(\omega) \) be the dimension of \( \omega \) defined as

\[
\dim(\omega) = \begin{cases} 
1 & \text{if } \omega(E(\mathcal{A}_0^i)) = \omega(E(A_H^i)) \text{ for some } H \in \mathcal{H}, \\
2 & \text{otherwise.}
\end{cases}
\]

Moreover, for every \( H \in \mathcal{H} \), let \( H_+ \) be an auxiliary supergraph of \( H \) that is obtained by adding a maximal number of edges between \( X_H^i \) and \( X_H^j \) for every \( i \in [r] \) subject to \( H_+|X_H^i, X_H^j \) being a matching. We call \( \mathcal{H}_+ := \bigcup_{H \in \mathcal{H}} H_+ \) an enlarged graph of \( \mathcal{H} \). We say that \( \mathcal{P} \) is nice (with respect to \( \mathcal{H}_+ \)) if

\[
\begin{align*}
(N1) & \quad |N_{A_H^i}(x_i) \cap N_G(v_j)| = (d, dZ \pm \varepsilon)|V_j| \text{ for all } x_i v_j \in E(A_H^i) \text{ whenever } \{x_i\} = N_{H_+}(x_i) \cap X_H^i, \\
(N2) & \quad H_+ \in \mathcal{H}, i j \in E(RZ), Z \subseteq \{A, B\};
\end{align*}
\]

Using standard regularity methods (see Facts 3.2 and 3.3), it is straightforward to verify the following:

\[(3) \quad \forall \varepsilon \ll \varepsilon' \ll d, 1/r, 1/s. \quad \text{Suppose } (\mathcal{H}, G, R, A, \psi) \text{ is an } (\varepsilon, \mathcal{A}) \text{-packing-instance of size } r \text{ and every enlarged graph } \mathcal{H}_+. \text{ Then exist spanning subgraphs } G' \subseteq G \text{ and } A' \subseteq A \text{ such that } \psi(\mathcal{H}, G', R, A', \psi) \text{ is a nice } (\varepsilon', \mathcal{A}') \text{-packing-instance of size } r \text{ with respect to } \mathcal{H}_+ \text{ for some } \varepsilon' \text{ with } \varepsilon \ll \varepsilon' \ll 1/r.\]

### 4.2. Approximate Packing Lemma

We now state our Approximate Packing Lemma. Roughly speaking it states that given a packing instance, we can find a conflict-free packing such that the updated candidacy graphs are still super-regular, albeit with a smaller density. Moreover, with respect to certain weight functions on the candidacy graphs, the updated candidacy graphs behave as we would expect this by a random and independent deletion of the edges.

**Lemma 4.3 (Approximate Packing Lemma).** Let \( 1/n \ll \varepsilon \ll \varepsilon' \ll d, 1/r, 1/s. \) Suppose \((\mathcal{H}, G, R, A, \psi)\) is an \((\varepsilon, \mathcal{A})\)-packing-instance of size \( r \) and every enlarged graph \( \mathcal{H}_+ \) of \( \mathcal{H} \), there exist spanning subgraphs \( G' \subseteq G \) and \( A' \subseteq A \) such that \((\mathcal{H}, G', R, A', \psi)\) is a nice \((\varepsilon', \mathcal{A}')\)-packing-instance of size \( r \) with respect to \( \mathcal{H}_+ \) for some \( \varepsilon' \) with \( \varepsilon \ll \varepsilon' \ll 1/r \).

There then is a conflict-free packing \( \sigma : X_0^i \to V_0^i \) in \( \mathcal{X}_0^i \) such that for all \( H \in \mathcal{H} \), we have \( |X_0^i \cap X_H^i| \geq (1 - \varepsilon)n \) and for all \( i \in [r] \) there exists a spanning subgraph \( A_{i,\text{new}}^H \) of the updated candidacy graph \( A_H^i[\sigma] \) (where \( \mathcal{A}_{\text{new}} := \bigcup_{H \in \mathcal{H}} \mathcal{A}_{i,\text{new}}^H \)) with

\[(I)_{4.3} A_{i,\text{new}}^H \text{ is } (\varepsilon', d, dZ)^-\text{-super-regular for all } i \in N_{R_2}(0), Z \subseteq \{A, B\};
\]

\[(II)_{4.3} e_H(N_{A_{i,\text{new}}^H}(v_i), N_{A_{i,\text{new}}^H}(v_j)) = (d, d, d' \pm \varepsilon')|V_H| \quad \text{for all } i j \in E(R - \{0\}) \text{ and } v_i v_j \in E(G[v_i, v_j]);
\]

\[(III)_{4.3} \omega(E(\mathcal{A}_{\text{new}}^i)) = (1 + \varepsilon^2)d_A\omega(E(\mathcal{A}_i)) \pm \varepsilon^2n^{\dim(\mathcal{A})} \quad \text{for all } \omega \in \mathcal{W}_\text{edge} \text{ and } i \in N_{R_2}(0);
\]

\[(IV)_{4.3} \Delta(\mathcal{A}_{\text{new}}^i) \leq (1 + \varepsilon')d_A|V_i| \quad \text{for all } i \in N_{R_2}(0);
\]

\[(V)_{4.3} \Delta(\mathcal{A}_{\text{new}}^i) \leq \sqrt{n} \quad \text{for all } i \in N_{R_2}(0);
\]

\[(VI)_{4.3} W \cap \bigcap_{i \in [r]} \sigma(Y_i \cap \mathcal{X}_0^i) = |W|Y_i| \cdots Y_i|/n^t \pm \varepsilon' \quad \text{for all } W, Y_1, \ldots, Y_t \in \mathcal{W}_\text{set};
\]

\[(VII)_{4.3} \omega(M(\sigma)) = (1 + \varepsilon')\omega(E(\mathcal{A}_i))/d_0 \pm \varepsilon' \quad \text{for all } \omega \in \mathcal{W}_\text{edge} \text{ with centre } V_0 \text{ or centres in } \mathcal{X}_0^i.
\]

Properties \((I)_{4.3}, (II)_{4.3}\) and \((IV)_{4.3}\) ensure that \((P2)-(P4)\) are also satisfied for the updated candidacy graphs \( \mathcal{A}_{\text{new}}^i \), respectively, and \((V)_{4.3}\) ensures that the codegree of the updated labelling \( \psi(\sigma) \) is still small on \( \mathcal{A}_{\text{new}}^i \). Property \((III)_{4.3}\) states that the weight of the edge testers on the updated candidacy graphs \( \mathcal{A}_{\text{new}}^i \) is what we would expect by a random sparsification of the edges in \( \mathcal{A}_i \), and \((VI)_{4.3}\) and \((VII)_{4.3}\) guarantee that \( \sigma \) behaves like a random packing with respect to the set and edge testers.

We split the proof into two steps. In Step 1, we construct an auxiliary hypergraph and apply Theorem 3.6 to obtain the required conflict-free packing \( \sigma \). By defining suitable weight functions in Step 2, we employ the conclusions of Theorem 3.6 to establish \((I)_{4.3}-(VII)_{4.3}\).
Proof. Let $\mathcal{H}_+$ be an enlarged graph of $\mathcal{H}$, and for $i \in [\gamma]$, let $\mathcal{A}_i^{\text{bad}}$ and $\mathcal{A}_i^{\text{good}}$ be spanning subgraphs of $\mathcal{A}_i$ such that $\mathcal{A}_i^{\text{bad}}$ contains precisely those edges $xv \in E(\mathcal{A}_i)$ where $N_{\mathcal{H}_+}(x) \cap \mathcal{Z}_0 = \emptyset$, and $E(\mathcal{A}_i^{\text{good}}) := E(\mathcal{A}_i) \setminus E(\mathcal{A}_i^{\text{bad}})$. We may assume that $|\mathcal{H}| = sn$ and $\sum_{H \in \mathcal{H}} |E(X_H^0, X_H^J)| \leq (d_\Delta + \epsilon s^3/2)n^2$ for all $i \in N_{\mathcal{H}_+}(0)$, where the last inequality will be only used in (4.31). (Otherwise we artificially add some graphs to $\mathcal{H}$ subject to the condition that still $e_H(X_H^0, X_H^J) \geq \epsilon^2 n$ for all $H \in \mathcal{H}$, $ij \in E(R)$, and accordingly we add some graphs to $\mathcal{A}$ satisfying (P1)–(P4).) We may also assume that $\psi$: $A(\mathcal{A}) \to 2^E$ is such that $|\psi(e)| = s$ for all $e \in E(\mathcal{A}_0)$ (otherwise we add artificial labels that we delete at the end again), and $(\mathcal{H}, G, R, \mathcal{A}, \psi)$ is a nice $(\varepsilon, d)$-packing-instance with respect to $\mathcal{H}_+$ (otherwise we may employ (4.3) and replace $\varepsilon$ by some $\tilde{\varepsilon}$, where $\varepsilon \ll \tilde{\varepsilon} \ll \varepsilon$; observe also that this does not cause problems with the weight of the edge testers in (III)4.3 and (VII)4.3, as the operation in (4.3) only deletes few edges of $\mathcal{A}$ incident to every vertex).

Step 1. Constructing an auxiliary hypergraph

We want to use Theorem 3.6 to find the required conflict-free packing $\sigma$ in $\mathcal{A}_0$. To this end, let $(V_0^H)_{H \in \mathcal{H}}$ be disjoint copies of $V_0$, and for $H \in \mathcal{H}$ and $e = x_0v_0 \in E(A_0^H)$, let $e^H := x_0v_0^H$ where $v_0^H$ is the copy of $v_0$ in $V_0^H$. Let $f_e := e^H \cup \psi(e)$ for each $e \in E(A_0^H)$, $H \in \mathcal{H}$ and let $H^{\text{aux}}$ be the $(s + 2)$-uniform hypergraph with vertex set $\bigcup_{H \in \mathcal{H}} (X_0^H \cup V_0^H) \cup \mathcal{E}$ and edge set $\{f_e : e \in E(\mathcal{A}_0^H)\}$. A key property of the construction of $H^{\text{aux}}$ is a bijection between conflict-free packings $\sigma$ in $\mathcal{A}_0$ and matchings $\mathcal{M}$ in $H^{\text{aux}}$ by assigning $\sigma$ to $\mathcal{M} = \{f_e : e \in M(\sigma)\}$. (Recall that $M = M(\sigma)$ is the edge set corresponding to $\sigma$.)

It is easy to estimate $\Delta(H^{\text{aux}})$ and $\Delta^c(H^{\text{aux}})$ in order to apply Theorem 3.6. Since $A_0^H$ is $(\varepsilon, d_0)$-super-regular for each $H \in \mathcal{H}$, $|X_0^H| = |V_0| = (1 \pm \varepsilon)n$, and $\Delta(\mathcal{A}_0^H) \leq (1 + \varepsilon)d_0|V_0|$, we conclude that

\[(4.4) \quad \Delta(H^{\text{aux}}) \leq (d_0 + 3\varepsilon)n =: \Delta.\]

Note that the codegree in $H^{\text{aux}}$ of two vertices in $\bigcup_{H \in \mathcal{H}} (X_0^H \cup V_0^H)$ is at most 1, and similarly, the codegree in $H^{\text{aux}}$ of a vertex in $\bigcup_{H \in \mathcal{H}} (X_0^H \cup V_0^H)$ and a label in $\mathcal{E}$ is at most 1 because $\Delta(\mathcal{A}_0^H) \leq 1$ for all $H \in \mathcal{H}$. By assumption, $\Delta^c(\mathcal{A}_0^H) \leq \sqrt{n}$. Altogether, this implies that

\[(4.5) \quad \Delta^c(H^{\text{aux}}) \leq \sqrt{n} \leq \Delta^{1-\varepsilon^2}.\]

Suppose $W = \bigcup_{\ell \in [s]} W_\ell$ is a set of size at most $n^{4\log n}$ of given weight functions $\omega \in W_{\ell}$ for $\ell \in [s]$ with $\omega_e : (E(\mathcal{A}_0^H)) \to [0, s]$. Note that every weight function $\omega : (E(\mathcal{A}_0^H)) \to [0, s]$ naturally corresponds to a weight function $\omega^{\text{aux}} : (E(H^{\text{aux}})) \to [0, s]$ by defining $\omega^{\text{aux}}(\{f_{e_1}, \ldots, f_{e_k}\}) := \omega(\{e_1, \ldots, e_k\})$. We will explicitly specify $W$ in Step 2 and it is simple to check that for each $\omega \in W$ the corresponding weight function $\omega^{\text{aux}}$ will be clean. Our main idea is to find a hypergraph matching in $H^{\text{aux}}$ that behaves like a typical random matching with respect to $\{\omega^{\text{aux}} : \omega \in W\}$ in order to establish (I)4.3–(VII)4.3.

Suppose $\ell \in [s]$ and $\omega \in W_{\ell}$. If $\omega(\mathcal{A}_0^H) \geq n^{1+\varepsilon^2}/2$ or $\ell \geq 2$, define $\overline{\omega} := \omega$. Otherwise, choose $\overline{\omega} : E(\mathcal{A}_0^H) \to [0, s]$ such that $\omega \leq \overline{\omega}$ and $\overline{\omega}(\mathcal{A}_0^H) = n^{1+\varepsilon^2}$. By (4.4) and (4.5), we can apply Theorem 3.6 (with $(d_0 + 3\varepsilon)n, \varepsilon^2, 2, s + 2, s, \{\omega^{\text{aux}} : \omega \in W_{\ell}\}$) playing the roles of $\Delta, \delta, r, L, W_{\ell}$ to obtain a matching $\mathcal{M}$ in $H^{\text{aux}}$ that corresponds to a conflict-free packing $\sigma$: $\mathcal{X}_0^\sigma \to V_0$ in $\mathcal{A}_0$ with its corresponding edge set $M = M(\sigma)$ that satisfies the following properties (where $\hat{\varepsilon} := \varepsilon^{1/2}$):

\[(4.6) \quad \omega(M) = (1 \pm \hat{\varepsilon}) \frac{\omega(\mathcal{A}_0^H)}{d_0n} \text{ for } \omega \in W_{\ell}, \ell \in [s] \text{ where } \omega(\mathcal{A}_0^H) \geq ||\omega||_k \Delta^{k+\varepsilon^2} \text{ for all } k \in [\ell];\]

\[(4.7) \quad \omega(M) \leq \max\left\{ (1 + \hat{\varepsilon}) \frac{\omega(\mathcal{A}_0^H)}{d_0n}, n^\varepsilon \right\} \text{ for all } \omega \in W_{\ell}.\]

One way to exploit (4.6) is to control the number of edges in $M$ between sufficiently large sets of vertices. To this end, for subsets $S \subseteq X_0^H$ and $T \subseteq V_0$ for some $H \in \mathcal{H}$ with $|S|, |T| \geq 2\varepsilon n$, we define a weight function $\omega_{S,T} : E(A_0^H) \to \{0, 1\}$ with

\[(4.8) \quad \omega_{S,T}(e) := 1_{\{e \in E(A_0^H(S,T))\}}.\]
That is, \( \omega_{S,T}(M) \) counts the number of edges in \( A^H_0 \) between \( S \) and \( T \) that lie in \( M \). Since \( A^H_0 \) is \((\varepsilon, d_0)\)-super-regular we have that \( e_{A^H_0}(S,T) = (d_0 \pm \varepsilon)|S||T| \geq \varepsilon^3 n^2 \) which implies together with (4.6) that whenever \( \omega_{S,T} \in \mathcal{W} \), then \( \sigma \) is chosen such that

\[
|\sigma(S \cap \mathcal{P}^\sigma_0) \cap T| = \omega_{S,T}(M) = (1 \pm 2\varepsilon)\frac{|S||T|}{n}.
\]

Step 2. Employing weight functions to conclude (I)_{4.3}–(VII)_{4.3}

By Step 1, we may assume that (4.6) and (4.7) hold for a set of weight functions \( \mathcal{W} \) that we will define during this step. We will show that for this choice of \( \mathcal{W} \) the conflict-free packing \( \sigma: \mathcal{P}^\sigma_0 \to V_0 \) as obtained in Step 1 satisfies (I)_{4.3}–(VII)_{4.3}. Similarly as in Definition 4.1 (here, \( \mathcal{H} \) is replaced by \( \mathcal{H}_{+} \)), we define subgraphs \( A^H_{i,*} \) of \( A^H_i \) as follows.

For all \( H \in \mathcal{H}_+ \), \( i \in [r] \), let \( A^H_{i,*} \) be the spanning subgraph of \( A^H_i \) containing precisely

\[
\text{those edges } xz \in E(A^H_i) \text{ for which the following holds: if } x_0 = N_{H_{+}}(x) \cap \mathcal{P}^\sigma_0, \text{ then } \sigma(x_0) \in E(G[V_0, \mathcal{V}]).
\]

Observe that \( A^H_{i,*} \) is a spanning subgraph of the updated candidacy graph \( A^H[H] \) as in Definition 4.1. By taking a suitable subgraph of \( A^H_{i,*} \), we will in the end obtain the required candidacy graph \( A^H_i \) new.

First, we show that \( |\mathcal{P}^\sigma_0 \cap X^H_i| \geq (1 - 3\varepsilon) n \) for each \( H \in \mathcal{H} \). Adding \( \omega_{X^H_i, V_0} \) as defined in (4.8) for every \( H \in \mathcal{H} \) to \( \mathcal{W} \) and using (4.9) yields

\[
|\mathcal{P}^\sigma_0 \cap X^H_i| = \omega_{X^H_i, V_0}(M) \geq (1 - 3\varepsilon) n.
\]

Step 2.1. Checking (I)_{4.3}

For all \( H \in \mathcal{H} \) and \( i \in N_{R_2}(0), Z \in \{A, B\} \) we proceed as follows. Let \( Y^H_i := N_{H_{+}}(\mathcal{P}^\sigma_0) \cap X^H_i \). We first show that \( A^H_{i,*}[Y^H_i, V_i] \) is \((\varepsilon^{1/18}, d_i d_2 Z)\)-super-regular (see (4.15)). We do so by showing that every vertex in \( Y^H_i \cup V_i \) has the appropriate degree, and that the common neighbourhood of most pairs of vertices in \( V_i \) have the correct size such that we can employ Theorem 3.4 to guarantee the super-regularity of \( A^H_{i,*}[Y^H_i, V_i] \).

Note that \( |Y^H_i| \geq |\mathcal{P}^\sigma_0 \cap X^H_i| - 2\varepsilon n \geq (1 - 4\varepsilon) n \) by (4.11). For every vertex \( x \in Y^H_i \) with \( \{x_0\} = N_{H_{+}}(x) \cap \mathcal{P}^\sigma_0 \), we have \( \deg_{A^H_{i,*}}(x) = |N_{H_{+}}(x) \cap N_{G}(\sigma(x_0))| \). Since our packing-instance is nice, (N1) implies that \( \deg_{A^H_{i,*}}(x) = (d_i d_2 Z \pm \varepsilon)|V_i| \). For \( v \in V_i \), let \( N_{v} := N_{A^H_{i,*}}(v) \). Observe that

\[
\deg_{A^H_{i,*}[Y^H_i, V_i]}(v) = |\sigma(N_{H_{+}}(v) \cap \mathcal{P}^\sigma_0) \cap N_{G}(v)|,
\]

and \( |N_{H_{+}}(v) \cap X^H_i| = |N_{v}| \pm 2\varepsilon n = (d_i \pm 5\varepsilon)n \), and \( |N_{G}(v) \cap V_0| = (d_Z \pm 3\varepsilon)n \). Adding for every vertex \( v \in V_i \), the weight function \( \omega_{S,T} \) as defined in (4.8) for \( S := N_{H_{+}}(v) \cap X^H_i \) and \( T := N_{G}(v) \cap V_0 \), we obtain by (4.9) and (4.12) that

\[
\deg_{A^H_{i,*}[Y^H_i, V_i]}(v) = (1 \pm 2\varepsilon)|N_{H_{+}}(v) \cap X^H_i||N_{G}(v) \cap V_0|n^{-1} = (d_i d_2 Z \pm \varepsilon^{1/2})|Y^H_i|.
\]

Next, we use Theorem 3.4 to show that \( A^H_{i,*}[Y^H_i, V_i] \) is \((\varepsilon^{1/18}, d_i d_2 Z)\)-super-regular. We call a pair of vertices \( u, v \) in \( V_i \) good if \( |N_{A^H_{i,*}}(u, v)| = (d_i \pm \varepsilon^2)|X^H_i| \), and \( |N_{G}(u, v) \cap \forall_{0}| = (d_2 \pm \varepsilon^2)|V_i| \). By the \( \varepsilon \)-regularity of \( A^H_{i,*} \) and \( G[V_0, V_i] \), there are at most \( 2\varepsilon |V_i|^2 \) pairs \( u, v \in V_i \) which are not good.

For all good pairs \( u, v \in V_i \), let \( S_{u,v} := N_{H_{+}}(N_{A^H_{i,*}}(u, v) \cap X^H_i) \) and \( T_{u,v} := N_{G}(u, v) \cap V_0 \). We add the weight function \( \omega_{S,T} \) as defined in (4.8) to \( \mathcal{W} \). Observe that \( |S_{u,v}| = |N_{A^H_{i,*}}(u, v)| \pm 2\varepsilon n = (d_i \pm \varepsilon^{1/2})^2n \) and \( |T_{u,v}| = (d_Z \pm \varepsilon^{1/2})^2n \). By (4.9), we obtain for all good pairs \( u, v \in V_i \) that

\[
|N_{A^H_{i,*}[Y^H_i, V_i]}(u,v)| = |\sigma(S_{u,v} \cap \mathcal{P}^\sigma_0) \cap T_{u,v}| = (1 \pm 2\varepsilon)|S_{u,v}||T_{u,v}|n^{-1} \leq (d_i d_2 Z \pm \varepsilon^{1/3})|Y^H_i|.
\]
Now, by (4.13) and (4.14), we can apply Theorem 3.4, and obtain that
\[
A_i^{H,*}[Y_i^H, V_i] \text{ is } (\varepsilon^{1/18}, d_i d_Z)\text{-super-regular.}
\]

In order to complete the proof of (I)\textsubscript{4.3}, we show that we can find a spanning subgraph \(A_i^{H,\text{new}}\) of \(A_i^{H,*}\) that is \((\varepsilon', d_i d_Z)\)-super-regular. Let
\[
E(A_i^{H,\text{new}}[Y_i^H, V_i]) := E(A_i^{H,*}[Y_i^H, V_i]).
\]
For every vertex \(x \in X_i^H \setminus Y_i^H\), we have that \(d_{A_i^{H,*}}(x) = (d_i \pm \varepsilon)|V_i|\) because \(A_i^H\) is \((\varepsilon, d_i)\)-super-regular. Suppose \(W^{\text{bad}}\) is a collection of at most \(n^{14\log n}\) weight functions \(\omega^{\text{bad}}: E(A_i^{H,*}) \to [0, s]\); we will specify \(W^{\text{bad}}\) explicitly when we establish (III)\textsubscript{4.3}. We claim that we can delete for every vertex \(x \in X_i^H \setminus Y_i^H\) some incident edges in \(A_i^{H,*}\) and obtain a subgraph \(A_i^{H,\text{new}}\) such that
\[
\deg_{A_i^{H,\text{new}}}(x) = (d_i d_Z \pm 2\varepsilon)|V_i| \text{ for every } x \in X_i^H \setminus Y_i^H;
\]
\[
\omega^{\text{bad}}(E(x|x'|^{\text{new}})) = (1 \pm \varepsilon)d_Z \omega^{\text{bad}}(E(x|x'|^{\text{bad}})) \pm 6\varepsilon n \text{ for every } \omega^{\text{bad}} \in W^{\text{bad}}.
\]
This can be easily seen by a probabilistic argument: For all \(H \in \mathcal{H}\) and \(x \in X_i^H \setminus Y_i^H\), we keep each edge incident to \(x\) in \(A_i^H\) independently at random with probability \(d_Z\). Then, McDiarmid’s inequality (Theorem 3.1) together with a union bound yields that (4.17) and (4.18) hold simultaneously with probability at least, say, 1/2.

Since \(|X_i^H| = (1 \pm \varepsilon)n\), we have that \(|X_i^H \setminus Y_i^H| \leq 4\varepsilon n\) by (4.11). Hence, (4.15) implies together with (4.17) that \(A_i^{H,\text{new}}\) is \((\varepsilon', d_i d_Z)\)-super-regular, which establishes (I)\textsubscript{4.3}.

Step 2.2. Checking (II)\textsubscript{4.3}

For all \(H \in \mathcal{H}, i,j \in E(R_H \setminus \{0\})\), and \(v_i v_j \in E(G[V_i, V_j])\) we proceed as follows. Let \(\tilde{E} := E(H[N_{A_i^H}(v_i), N_{A_j^H}(v_j)])\) and
\[
S := \{x', x_j \} \subseteq X_0^H: x_i x', x_j x_j' \in E(H), x_i, x_j \in \tilde{E},
\]
\[
S_1 := \{S' \in S: |S'| = 1\}, \quad \text{and} \quad S_2 := \{S' \in S: |S'| = 2\},
\]
\[
E_1 := \{x v \in E(A_j^H): x \in S_1, v \in N_G(v_i, v_j)\},
\]
\[
E_2 := \{x v, x' v' \in (E(A_i^H)/2): x, x' \in S_2, v, v' \in N_G(v_i, v_j), v \neq v'\}.
\]
By assumption (see (P3)), we have that \(|\tilde{E}| = (d_i d_j \pm \varepsilon)|v_i, v_j|\). Since \(|v_i, v_j| \geq \varepsilon^2 n\), we conclude that
\[
|S| = |S_1| + |S_2| = (d_i d_j \pm \varepsilon)|v_i, v_j| \pm 4\varepsilon n = (d_i d_j \pm \varepsilon)|v_i, v_j|.
\]
Note that the term of ‘\pm 4\varepsilon n’ in (4.19) accounts for possible vertices \(x_i \in N_{A_i^H}(v_i)\) and \(x_j \in N_{A_j^H}(v_j)\) that do not have an \(H_+\)-neighbour in \(X_0^H\).

We define the following weight functions \(\omega_1: E(A_i^H) \to \{0, 1\}\) and \(\omega_2: (E(A_i^H)/2) \to \{0, 1\}\) by setting \(\omega_1(e) := 1_{\{|e| \in E_1\}}\) and \(\omega_2(\{e_i, e_j\}) := 1_{\{|e_i, e_j| \in E_2\}}\) and add them to \(\mathcal{W}\). By the definition of \(A_i^{H,\text{new}}\) (recall (4.10) and (4.16)), we crucially observe that
\[
e_H(N_{A_i^{H,\text{new}}}(v_i), N_{A_j^{H,\text{new}}}(v_j)) = \omega_1(M) + \omega_2(M) \geq 5\varepsilon n.
\]
Note that the term of ‘\pm 5\varepsilon n’ in (4.20) accounts for possible vertices \(x_i \in N_{A_i^H}(v_i)\) and \(x_j \in N_{A_j^H}(v_j)\) that do not have an \(H_+\)-neighbour in \(X_0^H\) (at most \(4\varepsilon n\), and possible vertices in \(S\) that are left unembedded (at most \(4\varepsilon n\) by (4.11)).

Let us for the moment assume that \(|S_1|, |S_2| \geq \varepsilon^3 n\) (otherwise the claimed estimations in (4.23) and (4.24) below are trivially true). Since \(A_0^H\) is \((\varepsilon, d_0)\)-super-regular and \(|N_G(v_i, v_j) \cap V_0| = (d_0^2 \pm 3\varepsilon)n\) by (N2), we obtain that
\[
\omega_1(E(A_i^H)) = |E_1| = (d_0 \pm \varepsilon)|S_1| N_G(v_i, v_j) \cap V_0| = (d_0 d_0^2 \pm \varepsilon)|S_1| n.
\]
By Fact 3.2, all but at most \(6\varepsilon n\) elements \(\{x', x_j'\} \in S_2\) are such that \(x_i'\) has \((d_0 \pm \varepsilon)|N_G(v_k) \cap V_0|\) neighbours in \(N_G(v_k) \cap V_0\) for both \(k \in \{i, j\}\) because \(A_0^H\) is \((\varepsilon, d_0)\)-super-regular and \(G[V_0, V_k] \in \mathcal{W}\)....
is \( (\varepsilon, d_A) \)-super-regular. Each of these 6\( \varepsilon n \) exceptional elements contributes at most \(|N_G(v_i) \cap V_0||N_G(v_j) \cap V_0| \leq 3n^2\) to \( \omega_2(E(A_H^H)) \). This implies that

\[
\omega_2(E(A_0^H)) = |E_2| = (d_0 \pm 2\varepsilon n^2)|N_G(v_i) \cap V_0||N_G(v_j) \cap V_0| \pm 18\varepsilon n^3 \label{4.22}
\]

For all \( e_1 \in E(A_H^0) \), the number of edges \( e_2 \) for which \( \{e_1, e_2\} \in E_2 \) is at most 2\( n \), implying \( \|\omega_2\|_1 \Delta^{1+\varepsilon^2} \leq 2n \Delta^{1+\varepsilon^2} \leq \omega_2(E(A_H^H)) \), and clearly, \( \|\omega_2\|_2 \Delta^{1+\varepsilon^2} \leq \Delta^{1+\varepsilon^2} \leq \omega_1(E(A_H^H)) \). (Recall that \( \Delta = (d_0 + 3\varepsilon n) \)) Hence, by (4.6), we conclude that

\[
\omega_1(M) = (1 + \varepsilon^2)\frac{\omega_1(E(A_H^H))}{d_0 n} \tag{4.23}
\]

\[
\omega_2(M) = (1 + \varepsilon^2)\frac{\omega_2(E(A_H^H))}{(d_0 n)\varepsilon} \tag{4.24}
\]

Clearly, the final equalities in (4.23) and (4.24) are also true if \(|S_1|, |S_2| < \varepsilon^5 n \) because \( \varepsilon_H(\chi^H_1, \chi^H_2) \geq \varepsilon^2 n \). Now, together with (4.19) and (4.20) this implies that

\[
e_H(N_{A_H^H}(v_i), N_{A_H^H}(v_j)) = (d_1 d_2 d_A^2 \pm \varepsilon) e_H(\chi^H_1, \chi^H_2)
\]

which establishes (II)4.3.

**Step 2.3. Checking (III)4.3**

We will even show that (III)4.3 holds for all \( \omega \in W_{edge} \cup W'_{edge} \) with \( \omega: E(\mathcal{A}_i) \rightarrow [0, s] \) and centre \( v \in V_i, i \in N_{R_i}(0) \), where \( W_{edge} \) is a set of edge testers that we will explicitly specify in Step 2.4 when establishing (IV)4.3. For all \( \omega \in W_{edge} \cup W'_{edge} \) with centre \( v \in V_i, i \in N_{R_i}(0) \) we define a weight function \( \omega_0: E(\mathcal{A}_0) \rightarrow [0, s] \) by

\[
\omega_0(x_0 v_0) := \begin{cases} \omega(x_0 v) & \text{if } \{x_0\} = N_{\mathcal{H}_+}(x_0) \cap \mathcal{X}_i, x_0 v \in E(\mathcal{A}_i^\text{good}) \text{ and } v_0 \in E(G), \\ 0 & \text{otherwise}, \end{cases}
\]

and we add \( \omega_0 \) to \( \mathcal{W} \). (Recall that \( \mathcal{A}_i^\text{good} \) is the spanning subgraph of \( \mathcal{A}_i \) containing precisely those edges \( x_i v_i \in E(\mathcal{A}_i) \), where \( N_{\mathcal{H}_+}(x_i) \cap \mathcal{X}_0 \neq \emptyset \).)

For every edge \( x_i v \in E(\mathcal{A}_i^\text{good}) \) with \( \{x_0\} = N_{\mathcal{H}_+}(x_0) \cap \mathcal{X}_0 \), property (N1) yields that

\[
|N_{\mathcal{H}_0}(x_0) \cap N_G(v)| = (d_0 d_A \pm 3\varepsilon)n
\]

Hence, every edge \( x_i v \in E(\mathcal{A}_i^\text{good}) \) contributes weight \( \omega(x_i v) \cdot (d_0 d_A \pm 3\varepsilon)n \) to \( \omega_0(E(\mathcal{A}_0)) \), and we obtain

\[
\omega_0(E(\mathcal{A}_0)) = \omega(E(\mathcal{A}_i^\text{good}))(d_0 d_A \pm 3\varepsilon)n
\]

By the definition of \( \mathcal{A}_i^\text{new} \) (recall (4.10) and (4.16)), if \( \sigma(x_0) \in N_G(v) \) for \( \{x_0\} = N_{\mathcal{H}_+}(x_i) \cap \mathcal{X}_0 \), then the edge \( x_i v \in E(\mathcal{A}_i^\text{good}) \) is in \( E(\mathcal{A}_i^\text{new}) \). Hence, if \( x_0 v_0 \in M(\sigma) = M \), then this contributes weight \( \omega_0(x_0 v_0) \) to \( \omega(E(\mathcal{A}_i^\text{new})) \). If \( \omega(E(\mathcal{A}_i^\text{good})) \geq \varepsilon n \), then \( \omega_0(E(\mathcal{A}_0)) \geq n^{1+\varepsilon} \geq s \Delta^{1+\varepsilon^2} \geq \|\omega_0\|_1 \Delta^{1+\varepsilon^2} \), and thus (4.6) implies that

\[
\omega_0(M) = (1 + \varepsilon)\frac{\omega_0(E(\mathcal{A}_0))}{d_0 n} = (1 + 2\varepsilon d_4 \omega_0(E(\mathcal{A}_i^\text{good})) \pm \varepsilon n.
\]

If \( \omega(E(\mathcal{A}_i^\text{good})) < \varepsilon n \), then (4.7) implies that

\[
\omega_0(M) \leq \max \{(1 + \varepsilon)\frac{\omega_0(E(\mathcal{A}_0))}{d_0 n}, n\varepsilon\} \leq \varepsilon n,
\]

and hence, (4.25) also holds in this case.

We now make a key observation:

\[
\omega(E(\mathcal{A}_i^\text{new})) = \omega_0(M) + \omega(\Lambda) \pm \omega(\Gamma),
\]

for \( \Gamma := \{x_i v \in E(\mathcal{A}_i^\text{good}) \cap N_{\mathcal{H}_+}(x_i) \cap \mathcal{X}_0^\text{bad} = \emptyset \} \) and \( \Lambda := E(\mathcal{A}_i^\text{bad}) \cap E(\mathcal{A}_i^\text{new}) \). Next, we want to control \( \omega(\Gamma) \) and \( \omega(\Lambda) \).
In order to bound \( \omega(\Gamma) \), we define a weight function \( \omega_T: E(\mathcal{A}_i) \rightarrow [0, s] \) by
\[
\omega_T(x_iv) := \begin{cases} 
\omega(x_iv) & \text{if } \{x_i\} = \mathcal{N}_{H_i}(x_0) \cap \mathcal{A}_i, \ x_iv \in E(\mathcal{A}_i^{\text{good}}), \\
0 & \text{otherwise},
\end{cases}
\]
and we add \( \omega_T \) to \( W \). Observe that \( \omega_T(M) \) accounts for the \( \omega \)-weight of edges \( x_iv \in E(\mathcal{A}_i^{\text{good}}) \) such that \( \mathcal{N}_{H_i}(x_0) \cap \mathcal{A}_i \in \mathcal{A}_0^\sigma \) and thus \( x_iv \notin \Gamma \). Hence \( \omega(\Gamma) = \omega(\mathcal{A}_i^{\text{good}})) - \omega_T(M) \). For every vertex \( x_0 \in \mathcal{A}_0^\sigma \), we have \( \deg_{\mathcal{A}_0}(x_0) = (d_0 \pm 3\varepsilon)n \). Hence, every edge \( x_iv \in E(\mathcal{A}_i^{\text{good}}) \) contributes weight \( \omega(x_iv) \cdot (d_0 \pm 3\varepsilon)n \) to \( \omega_T(E(\mathcal{A}_0)) \), and we obtain
\[
\omega_T(E(\mathcal{A}_0)) = \omega(\mathcal{A}_i^{\text{good}})(d_0 \pm 3\varepsilon)n.
\]
If \( \omega(\mathcal{A}_i^{\text{good}}) \geq \varepsilon n \), then \( \omega_T(E(\mathcal{A}_0)) \geq n^{1+\varepsilon} \geq s\Delta^{1+\varepsilon^2} \geq ||\omega||_1\Delta^{1+\varepsilon^2} \), and thus (4.6) implies that
\[
\omega_T(M) = (1 \pm \varepsilon)\frac{\omega_T(E(\mathcal{A}_0))}{d_0n} = (1 \pm 2\varepsilon)\omega(\mathcal{A}_i^{\text{good}})) \pm \varepsilon n.
\]
Again, if \( \omega(\mathcal{A}_i^{\text{good}}) < \varepsilon n \), then (4.7) implies that (4.27) also holds in this case.
Hence, we conclude that
\[
\omega(\Gamma) = \omega(\mathcal{A}_i^{\text{good}})) - \omega_T(M) \leq 2\varepsilon\omega(\mathcal{A}_i^{\text{good}})) + \varepsilon n.
\]
In order to bound \( \omega(\Lambda) \), we use (4.18) and add \( \omega|_{E(\mathcal{A}_i^{\text{bad}})} \) to \( W^{\text{bad}} \). Then (4.18) implies that
\[
\omega(\Lambda) = (1 \pm \varepsilon)d_A\omega(\mathcal{A}_i^{\text{bad}})) \pm \varepsilon n.
\]
Finally, equations (4.25), (4.26), (4.28) and (4.29) yield that
\[
\omega(\mathcal{A}_i^{\text{new}}) = (1 \pm 2\varepsilon)d_A\omega(\mathcal{A}_i^{\text{good}})) + (1 \pm \varepsilon)d_A\omega(\mathcal{A}_i^{\text{bad}})) \pm 2\varepsilon\omega(\mathcal{A}_i^{\text{good}})) \pm 3\varepsilon n
\]
\[
(1 \pm \varepsilon^2)d_A\omega(\mathcal{A}_i^{\text{new}})) \pm \varepsilon^2 n.
\]
This establishes (III)4.3 for all \( \omega \in W_{\text{edge}} \cup W^r_{\text{edge}} \).

Step 2.4. Checking (IV)4.3

We show that for the updated edge set labelling \( \psi[\sigma] \), we have \( \Delta_{\psi[\sigma]}(\mathcal{A}_i^{\text{new}}) \leq (1+\varepsilon')d_A|V_i| \) for every \( i \in N_{R_A}(0) \). Recall that we defined \( \psi[\sigma] \) in Definition 4.2 such that for \( xv \in E(\mathcal{A}_i^{H,\text{new}}) \), we have \( \psi[\sigma](xv) = \psi(xv) \cup \{x(\sigma{x_0})v \}, \) if \( x \) has an \( H \)-neighbour \( x_0 \in \mathcal{A}_0^\sigma \), and otherwise \( \psi[\sigma](xv) = \psi(xv) \). We split the proof of (IV)4.3 into two claims, where Claim 1 bounds the number of edges on which an ‘old’ label of \( \psi \) appears on the updated candidacy graph, and Claim 2 bounds the number of edges on which a ‘new’ label that we additionally added to \( \psi[\sigma] \) appears in the updated candidacy graph. Let \( \psi_i: E(\mathcal{A}_i) \rightarrow 2^{E_i} \) be the (old) edge set labelling \( \psi \) restricted to \( \mathcal{A}_i \) and we may assume that \( |E_i| \leq n^4 \).

Claim 1. We can add at most \( n^5 \) weight functions to \( W^r_{\text{edge}} \) to ensure that \( \Delta_{\psi[\sigma]}(\mathcal{A}_i^{\text{new}}) \leq (1+\varepsilon')d_A|V_i| \) for every \( i \in N_{R_A}(0) \).

Proof of claim: For all \( i \in N_{R_A}(0) \) and \( e \in E_i \), let \( \omega_e: E(\mathcal{A}_i) \rightarrow \{0, 1\} \) be such that \( \omega_e(x_iv_i) := 1\{e \in \psi[\sigma(x_0v_i)]\} \) and we add \( \omega_e \) to \( W^r_{\text{edge}} \). By assumption (see (P4)), we have \( \Delta_{\psi[\sigma]}(\mathcal{A}_i^e) \leq (1+\varepsilon)d_A|V_i| \), which implies that \( \omega_e(\mathcal{A}_i^e) \leq (1+\varepsilon)d_A|V_i| \). Since (4.30) in Step 2.3 is also valid for \( \omega_e \in W^r_{\text{edge}} \), we conclude that \( e \) appears on at most
\[
(1 + \varepsilon^2)d_A(1+\varepsilon)d_A|V_i| + \varepsilon^2 n \leq (1 + \varepsilon')d_A|V_i|
\]
edges of \( \mathcal{A}_i^{\text{new}} \), which completes the proof of Claim 1.
Claim 2. We can add at most \( n^3 \) weight functions to \( W \) to ensure that each \( e \in E(G[V_0, V_i]) \) appears on at most \((1 + \varepsilon')d_i d_A [V_i]\) edges of \( \mathcal{A}_i \) for every \( i \in N_{R_4}(0) \).

Proof of claim: For all \( i \in N_{R_4}(0) \) and \( e = v_0 v_1 \in E(G[V_0, V_i]) \), we proceed as follows. Let \( N := N_{\mathcal{A}_0}(v_0) \cap N_{\mathcal{A}_i}(v_1) \). We define a weight function \( \omega_e : E(\mathcal{A}_0) \to (0, 1) \) by \( \omega_e(xv) := 1_{\{v_0 = v \text{ and } v \in N\}} \) for every \( xv \in E(\mathcal{A}_0) \), and we add \( \omega_e \) to \( W \). Then, \( e \) appears on \( \omega_e(M) \) edges of \( \mathcal{A}_{i \text{ new}} \). Observe that

\[
\omega_e(E(\mathcal{A}_0)) = |N| = \sum_{H \in \mathcal{H}} e_H(N_{A_H}(v_0), N_{A_H}(v_1)) \tag{P3}
\]

\[
(4.31) = \sum_{H \in \mathcal{H}} (d_0 d_i + \varepsilon) e_H(X_0^H, X_i^H) \leq (d_0 d_i d_A + 2\varepsilon'^3/2)n^2,
\]

where the last inequality holds because \( \sum_{H \in \mathcal{H}} e_H(X_0^H, X_i^H) \leq (d_A + \varepsilon'^3/2)n^2 \), by assumption. With (4.7), we obtain that

\[
\omega_e(M) \leq \max \left\{ (1 + \varepsilon) \frac{\omega_e(E(\mathcal{A}_0))}{d_0 n}, n^2 \right\} \leq (1 + \varepsilon')d_i d_A |V_i|,
\]

which completes the proof of Claim 2.

Step 2.5. Checking \((V)_{3.4}\)

Recall that \( \psi_i : E(\mathcal{A}_i) \to 2^{\mathcal{I}_i} \) denotes the edge set labelling \( \psi \) restricted to \( \mathcal{A}_i \). For each \( i \in [r] \), \( e = v_0 v_1 \in E(G[V_0, V_i]) \), and \( f \in \mathcal{I}_i \), we show that \( \{e, f\} \) appears on at most \( \sqrt{n} \) edges of \( \mathcal{A}_{i \text{ new}} \). This will imply \((V)_{3.4}\) because any set \( \{e', f'\} \in (\mathcal{I}_2^2) \) appears also on at most \( \Delta_{\psi}(\mathcal{A}_i) \leq \sqrt{n} \) edges of \( \mathcal{A}_{i \text{ new}} \), and no two edges of \( E(G[V_0, V_i]) \) appear together as a label on an edge of \( \mathcal{A}_{i \text{ new}} \). Let

\[
\mathcal{A}_i^0 := N_{\mathcal{I}_i} \left( \{x_i \in \mathcal{I}_i : v_0 v_1 \in E(\mathcal{A}_i), f \in \psi(x_i v_i) \} \right) \cap \mathcal{A}_0.
\]

We define a weight function \( \omega_{e,f} : E(\mathcal{A}_0) \to (0, 1) \) by \( \omega_{e,f}(xv) := 1_{\{v_0 = v \text{ and } v \in \mathcal{A}_i^0\}} \) for every \( xv \in E(\mathcal{A}_0) \) and add \( \omega_{e,f} \) to \( W \). Since \( \Delta_{\psi}(\mathcal{A}_i) \leq (1 + \varepsilon)d_i |V_i| \) by \((P4)\), we obtain that \( \omega_{e,f}(E(\mathcal{A}_0)) \leq 2n \). Note that \( \{e, f\} \) appears on at most \( \omega_{e,f}(M) \) edges of \( \mathcal{A}_{i \text{ new}} \). Now, (4.7) implies that \( \omega_{e,f}(M) \leq n^2 \), which establishes \((V)_{3.4}\).

Step 2.6. Checking \((VI)_{3.4}\)

For each \((W, Y_1, \ldots, Y_\ell) \in W_{\mathcal{E}} \) with \( H_1, \ldots, H_\ell \in \mathcal{H} \) such that \( Y_j \subseteq X_0^H \), we define

\[
E(W, Y_1, \ldots, Y_\ell) := \left\{ \bigcup_{j \in \ell} \{xy_j \} : xy_j \in E(A_0^H[W, Y_j]) \text{ for all } j \in \ell \right\} \subseteq \left( E(\mathcal{A}_0^\ell) \right)
\]

and a weight function \( \omega(W, Y_1, \ldots, Y_\ell) : E(\mathcal{A}_0^\ell) \to (0, 1) \) by \( \omega(W, Y_1, \ldots, Y_\ell)(\{e_1, \ldots, e_\ell\}) := 1_{\{e_1, \ldots, e_\ell\} \in E(W, Y_1, \ldots, Y_\ell)} \) and we add \( \omega(W, Y_1, \ldots, Y_\ell) \) to \( W \). Observe that

\[
(4.32) \omega(W, Y_1, \ldots, Y_\ell)(M) = |W \cap \bigcup_{j \in \ell} \sigma(Y_j \cap \mathcal{A}_i^0)|.
\]

In view of the statement, we may assume that \( |W|, |Y_j| \geq \varepsilon'^2 n \) for all \( j \in \ell \). Since \( \ell \leq s \) and \( A_0^H \) is \((\varepsilon, d_0)-\text{super-regular}\) for every \( H \in \mathcal{H} \), we obtain with Fact 3.2 that are at most \( \varepsilon'^2 n \) vertices in \( W \) that do not have \((d_0 \pm \varepsilon)|Y_j| \) many neighbours in \( Y_j \) for every \( j \in \ell \). Hence we obtain that

\[
(4.33) \omega(W, Y_1, \ldots, Y_\ell)(E(\mathcal{A}_0^\ell)) = |E(W, Y_1, \ldots, Y_\ell)| = (d_0^\ell \pm \varepsilon'^2)|W||Y_1| \cdots |Y_\ell|.
\]

For \( k \in [\ell] \), any set of \( k \) edges \( \{e_1, \ldots, e_k\} \) is contained in at most \((2n)^{\ell-k} \ell-tuples \) in \( E(W, Y_1, \ldots, Y_\ell) \), which implies that

\[
\|\omega(W, Y_1, \ldots, Y_\ell)\|_K \Delta^{k+\varepsilon^2} \leq (2n)^{\ell-k} \Delta^{k+\varepsilon^2} \leq \omega(W, Y_1, \ldots, Y_\ell)(E(\mathcal{A}_0^\ell)). \tag{4.33}
\]

Hence, by (4.6), we conclude that

\[
\omega(W, Y_1, \ldots, Y_\ell)(M) \leq (1 + \varepsilon') \omega(W, Y_1, \ldots, Y_\ell)(E(\mathcal{A}_0^\ell)) \leq (1 + \varepsilon') \frac{|W||Y_1| \cdots |Y_\ell|}{n^\ell} \leq \omega(W, Y_1, \ldots, Y_\ell)(E(\mathcal{A}_0^\ell)).
\]
which establishes (VI)4.3 by (4.32).

Step 2.7. Checking (VII)4.3

We add $W_{\text{edge}}$ to $W$ and fix some $\omega \in W_{\text{edge}}$. If $\omega(E(\phi_0)) \leq n^{1+\varepsilon/2}$, then we obtain by (4.7) that $\omega(M) \leq n^\varepsilon$ and thus, $\omega(M) = (1 \pm \varepsilon')\omega(E(\phi_0))/d_0n \pm \varepsilon'n$. If $\omega(E(\phi_0)) \geq n^{1+\varepsilon/2}$, then we obtain by (4.6) that $\omega(M) = (1 \pm \varepsilon)\omega(E(\phi_0))/d_0n$. This establishes (VII)4.3 and completes the proof of Lemma 4.3. \qed

5. Proof of the main result

The following lemma is very similar to Theorem 1.2. We only require additionally that all graphs in $H$ only span a matching between two clusters that is either empty or not too small. This reduction has already been used in [26] (and in several other extensions of the blow-up lemma) and it is also not complicated in our framework.

Lemma 5.1. Let $1/n \ll \varepsilon \ll \alpha, d$ and $1/n \ll 1/r$. Suppose $(H, G, R, X, Y, \phi^*)$ is an $(\varepsilon,d)$-super-regular, $\alpha^{-1}$-bounded and $(\varepsilon, \alpha)$-linked extended blow-up instance, $[V_i] = (1 \pm \varepsilon)n$ for all $i \in [r]$, and $|H| \leq \alpha^{-1}n$. Suppose that $\sum_{H \in H} \chi^I_{X_i} \chi^I_{X_j} \chi^L \chi^L \leq (1 - \alpha)\delta n^d$ for all $ij \in E(R)$ and $H[X_i^H, X_j^H]$ is a matching of size at least $\alpha^2n$ if $ij \in E(R)$ and empty if $ij \in (\binom{r'}{2}) \setminus E(R)$ for each $H \in H$. Suppose $W_{\text{set}}, W_{\text{ver}}$ are sets of $\alpha^{-1}$-sets testers and $\alpha^{-1}$-vertex testers of size at most $n^{2\log n}$, respectively. Then there is a packing $\phi$ of $H$ in $G$ which extends $\phi^*$ such that

(i) $\phi(X_i^H) = V_i$ for all $i \in [r']$ and $H \in H$;
(ii) $|W \cap \bigcap_{j \in [r]} \phi(Y_j)| = |W|Y_i \cdots Y_l|/n^\varepsilon \pm \text{on}$ for all $(W, Y_1, \ldots, Y_l) \in W_{\text{set}}$;
(iii) $\omega(\bigcup_{H \in H} \chi^I_{X_i} \phi^I_{\omega}(v)) = \omega(\bigcup_{H \in H} \chi^I_{X_i} \phi^I_{\omega})/n \pm \text{on}$ for all $(v, \omega) \in W_{\text{ver}}$ and $v \in V_i$.

We first prove our main result (Theorem 1.2) assuming Lemma 5.1.

Proof of Theorem 1.2. We choose a new constant $\beta$ such that $\varepsilon \ll \beta \ll \alpha, d$. For each $(W, Y_1, \ldots, Y_l) \in W_{\text{set}}$ with $W \subseteq V_i$, $i \in [r]$ and $k \ell \in [\ell]$, let $\omega_{Y_k}: \bigcup_{H \in H} \chi^I_{X_i} \to \{0, 1\}$ be such that $\omega_{Y_k}(x) = 1$ for all $x \in Y_k$ and let $W_Y$ be the set containing all those weight functions. We delete from every $H \in H$ the set $X_0^H$ and apply Lemma 3.7 to this collection of graphs and the set of weight functions $W^* := \{\omega: (v, \omega) \in W_{\text{ver}}\} \cup W_Y$, which yields a refined partition of $H$; to be more precise, for all $H \in H$ and $i \in [r]$, we obtain a partition $(X_{i,j}^H)_{j \in [r]}$ of $X_i^H$ satisfying (i)–(iv) of Lemma 3.7. Let $X'$ be the collection of vertex partitions of the graphs in $H$ given by $(X_{i,j}^H)_{H \in H}$ and $(X_{i,j}^H)_{H \mid i,j \in [\beta^{-1}]}$. In particular, Lemma 3.7(iii) yields that

$$\omega(X_{i,j}^H) = \beta \omega(X_i^H) \pm \beta^3/2n, \quad \text{for all } H \in H, \omega \in W^*, i \in [r], j \in [\beta^{-1}].$$

Let $R'$ be the graph with vertex set $[r] \times [\beta^{-1}]$ and two vertices $(i, j), (i', j')$ are joined by an edge if $ii' \in E(R)$. Note that $\Delta(R') \leq \alpha^{-1}$ because $\Delta(R) \leq \alpha^{-1}$.

According to the refinement $X'$ of $X$, we claim that there exists a refined partition $V'$ of $V$ consisting of the collection of $V_0$ together with $(V_{i,j})_{i \in [r], j \in [\beta^{-1}]}$, where $(V_{i,j})_{j \in [\beta^{-1}]}$ is a partition of $V_i$ for every $i \in [r]$ such that

(a) $|W \cap V_{i,j}| = \beta|W| \pm \beta^3/2n$ for all $(W, Y_1, \ldots, Y_l) \in W_{\text{set}}$ and $j \in [\beta^{-1}]$ with $W \subseteq V_i$, $\ell \in [\alpha^{-1}]$;
(b) $(H, G, R', X', V', \phi_0)$ is an $(\varepsilon/2, d)$-super-regular, $\beta^{-2}$-bounded and $(\varepsilon/2, \alpha/2)$-linked extended blow-up instance.

Indeed, the existence of $V'$ follows by a simple probabilistic argument. For each $i \in [r]$, let $\tau_i: V_i \to [\beta^{-1}]$ where $\tau_i(v)$ is chosen uniformly at random for every $v \in V_i$, all independently, and let $V_{i,j} := \{v \in V_i: \tau_i(v) = j\}$ for every $j \in [\beta^{-1}]$. Chernoff’s inequality and a union bound imply that (a) holds simultaneously together with the following properties with probability at least $1 - e^{-n^\varepsilon}$:

- $G[V_{i,j}, V_{i,j}']$ is $(\varepsilon/2, d)$-super-regular for all $ii' \in E(R), j, j' \in [\beta^{-1}]$;
- $\bigcap_{x \in X_{i,j}^{\beta^{-1}}} N_G(\phi_0(x_0)) \cap V_{i,j} \geq \alpha/2 |V_{i,j}|$ for all $x \in X_{i,j}^{\beta^{-1}}, i \in [r], j \in [\beta^{-1}], H \in H$. 

Standard properties of the multinomial distribution yield that $|V_{i,j}| = |X_{i,j}^H|$ for all $i \in [r], j \in [\beta^{-1}]$, $H \in \mathcal{H}$ with probability at least $\Omega(n^{-r\beta^{-1}})$. To see in (b) that the instance is $(\varepsilon^{1/2}, \alpha/2)$-linked, observe further that the number of vertices in $X_{i,j}^H$ that have a neighbour in $X_0^H$ is at most $\varepsilon|X_{i,j}^H| \leq \varepsilon^{1/2}|X_{i,j}^H|$ and $\sum_{H \in \mathcal{H}} |N_{H}(\phi_{0}^{-1}(v_0), \phi_{0}^{-1}(v_0')) \cap X_{i,j}^H| \leq \varepsilon |V_i|^{1/2} \leq \varepsilon^{1/2} |V_{i,j}|^{1/2}$ for all $i \in [r], j \in [\beta^{-1}]$ and distinct $v_0, v_0' \in V_0$. Thus, for every $i \in [r]$, there exists a partition $(V_{i,j})_{j \in [\beta^{-1}]}$ of $V_i$ satisfying (a) and (b). Let $n' := \beta n$.

Next we show how to lift the vertex and set testers from the original blow-up instance to the just defined blow-up instance. For each $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{\text{set}}$ and distinct $H_1, \ldots, H_\ell \in \mathcal{H}$ such that $W \subseteq V_i$ for some $i \in [r]$ and $Y_k \subseteq X_{i,j}^H$ for all $k \in [\ell]$, we define $(W, Y_1, \ldots, Y_\ell)$ by setting $W_j := W \cap V_{i,j}$ and $Y_{k,j} := Y_k \cap X_{i,j}^H$ for all $j \in [\beta^{-1}], k \in [\ell]$. By (a), we conclude that $|W_j| = \beta |W| \pm \beta^{3/2} n$, and by (5.1), we have that $|Y_{k,j}| = \omega_{Y_k}(X_{i,j}^H) = \omega_{Y_k}(X_{i,j}^H) \pm \beta^{3/2} n = \beta |Y_k| \pm \beta^{3/2} n$.

Let $\mathcal{W}_{\text{set}} := \{(W, Y_1, \ldots, Y_\ell_j) : j \in [\beta^{-1}], (W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{\text{set}}\}$. For each $(v, \omega) \in \mathcal{W}_{\text{ver}}$ with $v \in V_{i,j}$, let $\omega' := \omega|_{U_{H \in \mathcal{H}} X_{i,j}^H}$ and $\mathcal{W}_{\text{ver}} := \{(v, \omega') : (v, \omega) \in \mathcal{W}_{\text{ver}}\}$.

Next, we add some edges to the graphs in $\mathcal{H}$ ensuring that all matchings between two clusters are either empty or of small linear size. To this end, we add a minimum number of edges to $H[X_{i,j}^H, X_{i',j'}^H]$ for all $\{(i, j), (i', j')\} \in E(R')$ and $H \in \mathcal{H}$ such that the obtained supergraph $H'[X_{i,j}^H, X_{i',j'}^H]$ is a matching of size at least $\beta n$. Note that $\Delta(H') \leq \Delta(R') + \alpha^{-1} \leq \beta^{-2}$. Let $\mathcal{H}'$ be the collection of graphs $H'$ obtained in this manner. Together with Lemma 3.7(iv), we conclude for all $\{(i, j), (i', j')\} \in E(R')$ that

$$\sum_{H' \in \mathcal{H}'} e_{H'}(X_{i,j}^H, X_{i',j'}^H) \leq 2 \beta^4 \cdot n \cdot \alpha^{-1} n + \sum_{H \in \mathcal{H}} e_{H}(X_{i,j}^H, X_{i',j'}^H) \leq \beta^3 n^2 + \beta^2 \sum_{H \in \mathcal{H}} e_{H}(X_{i,j}^H, X_{i',j'}^H) + n^{5/3} \leq (1 - \alpha/2)dn^2.$$ 

Obviously, it suffices to construct a packing of $\mathcal{H}'$ into $G$ which extends $\phi_0$ and satisfies Theorem 1.2(i)–(iii). By (b) and because $\beta \ll \alpha$, also $(\mathcal{H}', G, R', \mathcal{X}', \mathcal{Y}', \phi_0)$ is an $(\varepsilon^{1/2}, d)$-super-regular, $\beta^{-2}$-bounded and $(\varepsilon^{1/2}, \beta^2)$-linked extended blow-up instance, and we can apply Lemma 5.1 to $(\mathcal{H}', G, R', \mathcal{X}', \mathcal{Y}', \phi_0)$ with set testers $\mathcal{W}_{\text{set}}'$ and vertex testers $\mathcal{W}_{\text{ver}}'$ as follows:

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} n' & \varepsilon^{1/2} & \beta^2 & d & r & r^{-1} \\ \hline \end{array}$$

Hence, we obtain a packing $\phi$ of $\mathcal{H}'$ in $G$ which extends $\phi_0$ such that for all $i \in [r], j \in [\beta^{-1}]

(I) $\phi(X_{i,j}^H) \subseteq V_{i,j}$ for all $H \in \mathcal{H}$;

(II) $|W_j \cap \bigcap_{k \in [\ell]} \phi(Y_{k,j})| = |W_j||Y_1| \cdots |Y_\ell|/n^{\ell} \pm \beta^2 n'$ for all $(W_j, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{\text{set}}'$;

(III) $\omega' \left( \bigcup_{H \in \mathcal{H}} X_{i,j}^H \cap \phi^{-1}(v) \right) = \omega' \left( \bigcup_{H \in \mathcal{H}} X_{i,j}^H \right)/n' \pm \beta^2 n'$ for all $(v, \omega') \in \mathcal{W}_{\text{ver}}'$ with $v \in V_{i,j}$.

Observe that (I) establishes Theorem 1.2(i).

For $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_{\text{set}}$, we conclude that

$$\begin{aligned}
\left| W \cap \bigcap_{k \in [\ell]} \phi(Y_k) \right| &= \sum_{j \in [\beta^{-1}]} \left| W_j \cap \bigcap_{k \in [\ell]} \phi(Y_{k,j}) \right| \\
&\underbrace{\sum_{j \in [\beta^{-1}]} \left( \frac{\beta^{\ell+1} \left( |W_j||Y_1| \cdots |Y_\ell| \pm \beta^{1/3} n^{\ell+1} \right) \pm \beta^2 n'}{\beta n^{\ell}} \right)}_{(5.1)} \\
&= |W_j||Y_1| \cdots |Y_\ell|/n^{\ell} \pm \alpha n.
\end{aligned}$$

Hence, Theorem 1.2(ii) holds.

For $(v, \omega) \in \mathcal{W}_{\text{ver}}$ with $v \in V_{i,j}$ and its corresponding tuple $(v, \omega') \in \mathcal{W}_{\text{ver}}'$, we conclude that

$$\omega'(\bigcup_{H \in \mathcal{H}} X_{i,j}^H \cap \phi^{-1}(v)) = \omega'(\bigcup_{H \in \mathcal{H}} X_{i,j}^H)/n' \pm \beta^2 n'$$

$$\text{(5.1)} \quad \underbrace{\omega'(\bigcup_{H \in \mathcal{H}} X_{i,j}^H)}_{\beta n} \pm \beta^{1/3} n^{2} \pm \beta^2 n' = \omega(\bigcup_{H \in \mathcal{H}} X_{i,j}^H)/n \pm \alpha n.$$
This yields Theorem 1.2(iii) and completes the proof. □

Theorem 1.3 can be easily deduced from Theorem 1.2 by randomly partitioning $G$ and applying Lemma 3.7 to $\mathcal{H}$ with $r = 1$. In particular, the proof is very similar to the proof of Theorem 1.2 and therefore omitted. We proceed with the proof of Lemma 5.1.

Proof of Lemma 5.1. We split the proof into four steps. In Step 1, we partition $G$ into two edge-disjoint subgraphs $G_A$ and $G_B$. In Step 2, we define ‘candidacy graphs’ that we track for the partial packing in Step 3, where we iteratively apply Lemma 4.3 to consider the clusters in turn. We only use the edges of $G_A$ for the partial packing in Step 3 such that we can complete the packing in Step 4 using the edges of $G_B$ and the ordinary blow-up lemma.

We will proceed cluster by cluster in Step 3 to find a function that packs almost all vertices of $\mathcal{H}$ into $G$. Since $r$ may be much larger than $\varepsilon^{-1}$, we need to carefully control the growth of the error term. We do so, by considering a proper vertex colouring $c : V(R) \to [T]$ of $R^3$ where $T := \alpha^{-3}$, and choose new constants $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_T, \mu, \gamma$ such that $\varepsilon \ll \varepsilon_0 \ll \varepsilon_1 \ll \cdots \ll \varepsilon_T \ll \mu \ll \gamma \ll \alpha, d$. To obtain the order in which we consider the clusters in turn, we simply relabel the cluster indices such that the colour values are non-decreasing; that is, $c(1) \leq \cdots \leq c(r)$. Note that the sets $(c^{-1}(k))_{k \in [T]}$ are independent in $R^3$. For $i \in [r], t \in [r]_0$, let

$$c(t) := \max\{\{0\} \cup \{c(j) : j \in N_R[i] \cap [t]\}\}, \quad \text{and} \quad m(t) := |N_R(i) \cap [t]|.$$  

That is, if we think of $[t]$ as the indices of clusters that have already been embedded, then $c(t)$ denotes the largest colour of an already embedded cluster in the closed neighbourhood of $i$ in $R$, and $m(t)$ denotes the number of neighbours of $i$ in $R$ that have already been embedded.

For $t \in [r]_0$, let

$$\mathcal{X}_t := \bigcup_{H \in \mathcal{H}} X_t^H, \quad \mathcal{X}_t := \bigcup_{\ell \in [t]_0} \mathcal{X}_t, \quad \mathcal{Y}_t := \bigcup_{\ell \in [t]_0} \mathcal{Y}_t.$$  

For every vertex tester $(v, \omega) \in \mathcal{W}_{ver}$ with $v \in V_i$ for some $i \in [r]$, we define its corresponding function $\omega_v$ on $(x,v_1 \colon x \in \mathcal{X}_i, v_1 \in V_i)$ by setting $\omega_v(x,v_1) := \omega(x)_{\{v_1=v\}}$. Let

$$\mathcal{W}_{edge}^i := \{\omega_v : (v, \omega) \in \mathcal{W}_{ver}, v \in V_i\}.$$  

Step 1. Partitioning the edges of $G$

In order to reserve an exclusive set of edges for the completion in Step 4, we partition the edges of $G$ into two subgraphs $G_A$ and $G_B$. For each edge $e$ of $G$ independently, we add $e$ to $G_B$ with probability $\gamma$ and otherwise to $G_A$. Let $d_A := (1 - \gamma) d$, $d_B := \gamma d$, $\alpha_A := (1 - \gamma)^{-1} \alpha / 2$ and $\alpha_B := \gamma^{-1} \alpha / 2$. Using Chernoff’s inequality, we can easily conclude that with probability at least $1 - 1/n$ we have for all $Z \in \{A, B\}$ that

$$G_Z[V_i, V_j] \text{ is } (2\varepsilon, d_Z)\text{-super-regular for all } ij \in E(R),$$  

$$|V_i \cap \bigcap_{x \in X_i H \cap N_H(x)} N_{G_Z} (\phi^*(x))| \geq \alpha_Z |V_i| \text{ for all } x \in X_i H, i \in [r], H \in \mathcal{H}.$$  

Hence, we may assume that $G$ is partitioned into $G_A$ and $G_B$ such that (5.4) and (5.5) hold.

Step 2. Candidacy graphs

For $t \in [r]_0$, we call $\phi : \bigcup_{H \in \mathcal{H}, i \in [t]_0} \hat{X}_i^H \to \mathcal{X}_t$ a $t$-partial packing if $\hat{X}_i^H \subseteq X_i^{\phi}$, $\phi|_{\hat{X}_i^H} = \phi^*|_{X_i^{\phi}}$, and $\phi(\hat{X}_i^H) \subseteq V_i$ for all $H \in \mathcal{H}, i \in [t]_0$. Note that $\hat{X}_i^H = X_i^H$, and $\phi|_{\hat{X}_i^H}$ is injective for all $H \in \mathcal{H}$ and $i \in [t]_0$. For convenience, we often write

$$\mathcal{X}_i^\phi := \bigcup_{H \in \mathcal{H}, i \in [t]_0} \hat{X}_i^H.$$  

Suppose $t \in [r]_0$ and $\phi_t : \mathcal{X}_i^{\phi_t} \to \mathcal{X}_t$ is a $t$-partial packing. We introduce the notion of candidates (with respect to $\phi_t$) for future packing rounds and track those relations in two kinds of bipartite auxiliary graphs that we call candidacy graphs: A graph $A_i^H(\phi_t)$ with bipartition $(X_i^H, V_i)$, $i \in [r]$ that will be used to extend the $t$-partial packing $\phi_t$ to a $(t + 1)$-partial packing $\phi_{t+1}$ via Lemma 4.3 in Step 3, and a graph $B_i^H(\phi_t)$ that will be used for the completion in Step 4. For convenience, we define $B_i^H(\phi_t)$ on a copy $(X_i^{H,B}, V_i^B)$ of $(X_i^H, V_i)$. That is, for all $H \in \mathcal{H}, i \in [r]$, let $X_i^{H,B}$ and
When we apply Lemma \((C1.5)\) and \((\text{C1.5})\) to its copy in \(U_{H \in \mathcal{H} : i \in [r]} (X^H_i \cup V_i)\), let \(G_+ \) and \(H_+\) be supergraphs of \(G_A\) and \(H \in \mathcal{H}\) with vertex partitions \((V_0, \ldots, V_t, V^1, \ldots, V^r)\) and \((X^H_0, \ldots, X^H_r, X^H_i, X^{H,B}_i, \ldots, X^{H,B}_r)\), respectively, and edge sets

\[
E(G_+) := E(G_A) \cup \{u \pi(v) : uv \in E(G_B)\},
\]

\[
E(H_+) := E(H) \cup \{u \pi(v) : uv \in E(H)\}.
\]

Let \(R_B\) be the graph on \([r] \cup \{1^B, \ldots, r^B\}\) with edge set \(E(R_B) := \{ij^B : ij \in E(R)\}\). By taking copies \((X^{H,B}_i, V^1)\) for all \((X^H_i, V_i)\) and defining the candidacy graphs \(B^H(\phi_t)\) on these copies, and by enlarging \(G, H\) and \(R\) accordingly to \(G_+, H_+\) and \(R \cup R_B\), we will be able to update the candidacy graphs \(A^H(\phi_t)\) and \(B^H(\phi_t)\) simultaneously in Step 3 when we apply Lemma 4.3 in order to extend \(\phi_t\) to a \((t + 1)\)-partial packing \(\phi_{t+1}\).

We now define \(A^H(\phi_t)\) and \(B^H(\phi_t)\). Let \(X^{H,A}_i := X^H_i\) and \(V^A_i := V_i\) for all \(H \in \mathcal{H}, i \in [r]\). For \(Z \in \{A, B\}, H \in \mathcal{H}\) and \(i \in [r]\), we say that \(v \in V^Z_i\) is a candidate for \(x \in X^{H,Z}_i\) given \(\phi_t\) if

\[
\phi_t(N_{H_+}(x) \cap X^{\phi_t}_i) \subseteq N_{G_+}(v).
\]

For all \(Z \in \{A, B\}\), let \(Z^H_i(\phi_t)\) be a bipartite graph with vertex partition \((X^{H,Z}_i, V^Z_i)\) and edge set

\[
E(Z^H_i(\phi_t)) := \{xv : x \in X^{H,Z}_i, v \in V^Z_i, \text{ and } v \text{ is a candidate for } x \text{ given } \phi_t\}.
\]

We call every spanning subgraph of \(Z^H_i(\phi_t)\) a candidacy graph (with respect to \(\phi_t\)).

Furthermore, for all \(H \in \mathcal{H}\) and \(i \in [r]\), we assign to every edge \(xv \in E(A^H_i(\phi_t))\) an edge set labelling \(\psi_t(xv)\) of size at most \(\alpha^{-1}\). This set encodes the edges between \(v\) and \(\phi_t(N_H(x) \cap X^{\phi_t}_i)\) in \(G_A\) that are covered if we embed \(x\) onto \(v\); to be more precise, for all \(H \in \mathcal{H}, i \in [r]\), and every edge \(xv \in E(A^H_i(\phi_t))\), we set

\[
\psi_t(xv) := E(G_A[\phi_t(N_{H_+}(x) \cap X^{\phi_t}_i), \{v\}]).
\]

Tracking this set enables us to extend a \(t\)-partial packing \(\phi_t\) to a \((t + 1)\)-partial packing \(\phi_{t+1}\) by finding a conflict-free embedding (see definition in (4.1)) in \(\bigcup_{H \in \mathcal{H}} A^H_{t+1}(\phi_t)\) via Lemma 4.3. Since \(|N_H(x) \cap X^{\phi_t}_i| \leq \alpha^{-1}\), we have \(|\psi_t(xv)| \leq \alpha^{-1}\).

Before we proceed to Step 3 and extend \(\phi_t\) to \(\phi_{t+1}\), we consider the candidacy graphs and their edge set labelling with respect to \(\phi^*\).

**Claim 1.** For all \(H \in \mathcal{H}, i \in [r], Z \in \{A, B\}\), there exists a candidacy graph \(Z^H_i(\phi_t) \subseteq Z^H_i(\phi^*)\) with respect to \(\phi^*\) (where \(\phi^* := \bigcup_{H \in \mathcal{H}} A^H_t(\phi_t)\) such that

- (C1.1) \(Z^H_i\) is \((\varepsilon_0, \alpha_2)\)-super-regular;
- (C1.2) \(\Delta_{\phi_0}(\phi^*) \leq \varepsilon_0 n^2\);
- (C1.3) \(\Delta_{\phi_0}(\phi^*) \leq \sqrt{n}\);
- (C1.4) \(e_H(N_{A^H}(u_i), N_{A^H}(v_j)) = (\alpha_2^2 \pm \varepsilon_0) e_H(X^H_i, X^H_j)\) for all \(i, j \in [r]\), \(i \in [r]\); and
- (C1.5) \(\omega_v(E(\phi_t)(\phi^*)) = \alpha_2 \omega(\phi^*) \pm \varepsilon_0 n^2\) for all \(v, w \in W_{\text{edge}}\).

**Proof of claim:** We fix \(H \in \mathcal{H}, i \in [r], ij \in E(R), Z \in \{A, B\}, v_i, v_j \in E(G_A[V_i, V_j])\) and \(\omega_v \in W_{\text{edge}}^i\) as defined in (5.3). For each \(k \in \{i, j\}, \) let \(\tilde{X}^H_k\) be the set of vertices in \(X^H_k\) that have a neighbour in \(X^H_i\). Observe that \(Z^H_k(\phi^*)[X^H_k \setminus \tilde{X}^H_k, V_k]\) is a complete bipartite graph for \(k \in \{i, j\}\), and \(e_H(N_{Z^H_k(\phi^*)[X^H_k \setminus \tilde{X}^H_k, V_k]}(v_i), N_{Z^H_k(\phi^*)[X^H_k \setminus \tilde{X}^H_k, V_k]}(v_j)) = e_H(X^H_i, X^H_j) + 4\varepsilon n\) because \(|\tilde{X}^H_k| \leq \varepsilon|X^H_k| \leq 2n\). By (5.5) and the definition of candidates in (5.6), we obtain that \(\deg_{Z^H_k(\phi^*)}(x_i) \geq \alpha_2|V_i|\) for all \(v \in \tilde{X}^H_i\). Note further that \(\omega_v(E(\bigcup_{H \in \mathcal{H}} A^H_t(X^H_i \setminus \tilde{X}^H_i, V_i))) = \omega_v(\phi^*) \pm \varepsilon n^2\). Hence, there exists a subgraph \(Z^H_i \subseteq Z^H_i(\phi^*)\) that satisfies (C1.1), (C1.4) and (C1.5), which can be seen by keeping each edge in \(Z^H_k(\phi^*)[X^H_k \setminus \tilde{X}^H_k, V_k]\) independently at random with probability \(\alpha_2\) and by possibly removing some edges incident to \(x_i \in \tilde{X}^H_i\) in \(Z^H_i(\phi^*)\) deterministically. In order to see (C1.2), note that \(|\psi_t^{-1}(v_0) - v_0| \leq |N_{H_+}(v_0) - v_0| \leq \alpha^{-1}|\phi_0^{-1}(v_0)| \leq \alpha^{-1} \varepsilon|H| \leq \varepsilon_0 n|H| \leq \varepsilon_0 n^2\). Since
the blow-up instance is \((\varepsilon, \alpha)\)-linked, we have \(\sum_{H \in \mathcal{H}} |N_H(\phi^{-1}_0(v_0), \phi^{-1}_0(v'_0)) \cap X^H_i| \leq \varepsilon|V_i|^{1/2}\) for all distinct \(v_0, v'_0 \in E(G[V_0, V_i])\), \(i \in [r]\), which implies (C1.3). This completes the proof of the claim.

---

**Step 3. Induction**

We inductively prove that the following statement \(S(t)\) holds for all \(t \in [r][0], \) which will provide a partial packing of \(\mathcal{H}\) into \(G_A\).

\(S(t)\). For all \(H \in \mathcal{H}\) and \(Z \in \{A, B\}\), there exists a \(t\)-partial packing \(\phi_t: X^\phi_t \rightarrow V_t\) with 
\(|X^\phi_t \cap X^H_i| \geq (1 - \varepsilon_{c_i(t)}|n|)\) for all \(i \in [t]\), and there exists a candidacy graph \(Z^H_i \subseteq Z^H_t(\phi_t)\) (where \(\mathcal{A}_t := \bigcup_{H \in \mathcal{H}} A^H_i\)) such that

\[(a)\] \(Z^H_i = (\varepsilon_{c_i(t)}, \alpha Z_i^m(t))\)-super-regular for all \(i \in [r] \setminus [t]\) if \(Z = A\) and for all \(i \in [r]\) if \(Z = B\); 
\[(b)\] \(\Delta_{\phi_t}(\mathcal{A}_t) \leq (1 + \varepsilon_{c_i(t)})\alpha A_i^m(t)\) for all \(i \in [r] \setminus [t]\); 
\[(c)\] \(\Delta^\phi_{\phi_t}(\mathcal{A}_t) \leq \sqrt{n}\) for all \(i \in [r] \setminus [t]\); 
\[(d)\] \(e_H(N_A^v(v_i), N_A^v(v_j)) = (\alpha A_i d^m_{A_i} + m_j(t)) + \varepsilon_{\max(c_i(t), c_j(t))}\) for all \(H \in \mathcal{H}\), 
\(i, j \in E(R - [t])\) and \(v_i, v_j \in E(G_A[V_i, V_j])\); 
\[(e)\] \(|\phi_t^{-1}(v)| \geq (1 - \varepsilon^{1/2}_{c_i(t)})|H| - \varepsilon_{c_i(t)} n\) and \(|\phi_t^{-1}(v) \cap N_H(X^\phi_t \setminus X^H_i)| \leq \varepsilon^{1/2}_{c_i(t)} n\) for all \(v \in V_i\), 
\(i \in [t]\); 
\[(f)\] \(\omega_2(E(\mathcal{A}_t)) = \alpha A_i d^m_{A_i}(\mathcal{A}_t) \pm \varepsilon_{c_i(t)} n^2\) for all \(\omega_2 \in W^2_{edge}\) and \(i \in [r] \setminus [t]\); 
\[(g)\] \(|W \cap \bigcup_{j \in [t]} \phi_t(Y_j \cap X^\phi_j)| = |W||Y_1| \cdots |Y_t|/n^t \pm \alpha n/2\) for all \((W, Y_1, \ldots, Y_t) \in W_{set}\) with \(W \subseteq V_i\), \(i \in [t]\); 
\[(h)\] \(\omega(\mathcal{A}_t \cap \phi_t^{-1}(v)) = \omega(\mathcal{A}_t)/n \pm \alpha n/2\) for all \((v, \omega) \in W_{ver}\) with \(v \in V_i\), \(i \in [t]\).

Properties \(S(t)(a)\)–\((d)\) will be used particularly to establish \(S(t + 1)\) by applying Lemma 4.3. Property \((f)\) enables us to establish \((h)\), which together with \((g)\) basically implies Lemma 5.1(ii) and (iii) as we merely modify the \(r\)-partial packing \(\phi_r\) for the completion in Step 4 where we exploit \((a)\) for \(Z = B\) and \((e)\).

The statement \(S(0)\) holds for \(\phi_0 = \phi^*\) by Claim 1. Hence, we assume the truth of \(S(t)\) for some \(t \in [r - 1][0]\) and let \(\phi_t: X^\phi_t \rightarrow V_t\) and \(A_t^H\) and \(B_t^H\) be as in \(S(t)\); we set \(\mathcal{A}_t := \bigcup_{H \in \mathcal{H}} A_t^H\) and \(\mathcal{B}_t := \bigcup_{H \in \mathcal{H}} B_t^H\). We will extend \(\phi_t\) such that \(S(t + 1)\) holds. Any function \(\phi: \mathcal{X}_{t+1}^\varphi \rightarrow V_{t+1}\) with \(\mathcal{X}_{t+1}^\varphi \subseteq \mathcal{X}_{t+1}\) extends \(\phi_t\) to a function \(\phi_{t+1}: \mathcal{X}_{t+1}^\varphi \cup \mathcal{X}_{t+1}^\sigma \rightarrow V_{t+1}\) as follows:

\[
\phi_{t+1}(x) := \begin{cases} 
\phi_t(x) & \text{if } x \in \mathcal{X}_{t+1}^\varphi, \\
\sigma(x) & \text{if } x \in \mathcal{X}_{t+1}^\sigma.
\end{cases}
\]

We now make a key observation: By definition of the candidacy graphs \(\mathcal{A}_{t+1}\) and their edge set labellings as in (5.8), if \(\sigma\) is a conflict-free packing in \(\mathcal{A}_{t+1}\) as defined in (4.1), then \(\phi_{t+1}\) is a \((t + 1)\)-partial packing.

We aim to apply Lemma 4.3 in order to obtain a conflict-free packing \(\sigma\) in \(\mathcal{A}_{t+1}\). Let

\[
\mathcal{H}_{t+1} := \bigcup_{H \in \mathcal{H}} H_+ \left[ \bigcup_{i \in N_R[t+1]\setminus[t]} X^H_i \cup \bigcup_{i \in N_R[t+1]} X^H_i B \right],
\]

\[
G_{t+1} := G_+ \left[ \bigcup_{i \in N_R[t+1]\setminus[t]} V_i \cup \bigcup_{i \in N_R[t+1]} V_i B \right],
\]

\[
\mathcal{A}_{t+1} := \bigcup_{i \in N_R[t+1]\setminus[t]} \mathcal{A}_i \cup \bigcup_{i \in N_R[t+1]} \mathcal{B}_i,
\]

\[
R_{t+1} := R[N_R[t+1]\setminus[t]] \cup R[N_R[t+1]]] .
\]

Note that \(\mathcal{P} := (\mathcal{H}_{t+1}, G_{t+1}, \mathcal{A}_{t+1}, \mathcal{B}_{t+1}, \psi|E(\mathcal{A}_{t+1}))\) is a packing instance of size \(\deg_{R_{t+1}}(t + 1)\) with \(t + 1\) playing the role of 0, and we claim that \(\mathcal{P}\) is indeed an \((\varepsilon_{c_{t+1} - 1}[1], d)\)-packing instance, where \(d = (d_A, d_B, (\alpha A_i d^m_{A_i})_{i \in N_R[t+1]\setminus[t]}, (\alpha B_i d^m_{B_i})_{i \in N_R[t+1]}).\) Observe that by definition of \(c_i(t)\) and \(m_i(t)\) in (5.2), we have:

\[
(a)\] \(i \in N_R(t + 1)\), then \(m_i(t + 1) = m_i(t) + 1\), and \(c(t + 1) = c_i(t + 1) > \max\{c_i(t), c_j(t)\}\) for all \(j \in N_R(i)\). If \(i \in [r] \setminus N_R(t + 1)\), then \(m_i(t + 1) = m_i(t)\).
Note that for the inequality in (5.10) we used that no pair of adjacent vertices in $R$ has two neighbours in $R$ that are coloured alike as we have chosen the vertex colouring as a colouring in $R^3$. In particular, we infer from (5.10) that $\varepsilon_{c(i+1)-1} = \varepsilon_{c(i+1)-1} \geq \varepsilon_{c(i)}$ for all $i \in N_R(t+1)$. Therefore, (P1) follows from (5.4), property (P2) follows from $S(t)(a)$, property (P3) follows from $S(t)(d)$ with $R[N_R[t+1] \setminus [t]]$ playing the role of $R_A$, and (P4) follows from $S(t)(b)$. 

Observe further that 

- $\psi_i$ as defined in (5.8) satisfies $\|\psi_i\| \leq \alpha^{-1}$; 
- $\sum_{H \in \mathcal{H}} \varepsilon_i (X^H_i, X^H_j) \leq (1 - \alpha)dn^2 \leq d^An^2$ for all $i \in N_R(t+1) \setminus [t]$; 
- $\Delta^\alpha_{\varepsilon_i} (\mathcal{A}_t) \leq \sqrt{n}$ for all $i \in N_R[t+1] \setminus [t]$ by $S(t)(c)$. 

Hence, we can apply Lemma 4.3 to $\mathcal{P}$ with 

\[
\begin{array}{cccccc}
\frac{n! \varepsilon_{c(i+1)-1}}{n} & \varepsilon \varepsilon' s & \alpha^{-1} \deg_{R_{t+1}}(t+1) & r & R[R_N[t+1] \setminus [t]] \\
& & & & R_A
\end{array}
\]

and with set testers $\mathcal{W}^{t+1}_{\text{set}}$ where we denote by $\mathcal{W}^{t+1}_{\text{set}} \subseteq \mathcal{W}_{\text{set}}$ the set of set testers $(W, Y_1, \ldots, Y_t)$ with $W \subseteq V_{t+1}$, and with edge testers $\mathcal{W}^{t+1}_{\text{edge}} \cup \mathcal{W}^{t+1}_{\text{edge}}$ where we will define the set $\mathcal{W}^{t+1}_{\text{edge}}$ when proving $S(t+1)(d)$ and (e) in Steps 3.3 and 3.4. 

Let $\sigma: \mathcal{P}_{t+1} \rightarrow V_{t+1}$ be the conflict-free packing in $\mathcal{A}_{t+1}$ obtained from Lemma 4.3 with $|\mathcal{P}_{t+1} \cap X^H_{t+1}| \geq (1 - \varepsilon_{c(i+1)-1})n$ for all $H \in \mathcal{H}$, which extends $\phi_t$ to $\phi_{t+1}$ as defined in (5.9). Fix some $H \in \mathcal{H}$. By Definition 4.1, the updated candidacy graphs with respect to $\sigma$ obtained from Lemma 4.3 are also updated candidacy graphs with respect to $\phi_{t+1}$ as defined in (5.7) in Step 2. Hence, the graphs $\tilde{A}^t_i$, $\tilde{A}^t_i$ in Lemma 4.3 correspond to subgraphs $\tilde{A}^t_i \subseteq A^t_i (\phi_{t+1})$ for all $i \in N_R(t+1) \setminus [t]$, and $\tilde{B}^t_i \subseteq B^t_i (\phi_{t+1})$ for all $i \in N_R(t+1)$ that satisfy (I$4.3$)-(VII$4.3$).

Step 3.1. Checking $S(t+1)(a)$

By (I$4.3$) and (5.10), we obtain that $\tilde{A}^t_i$ is $(\varepsilon_{c(i+1)-1}, \alpha A d^A_{A}(t+1))$-super-regular for all $i \in N_R(t+1) \setminus [t]$, and $\tilde{B}^t_i$ is $(\varepsilon_{c(i+1)-1}, \alpha B d^B_{A}(t+1))$-super-regular for all $i \in N_R(t+1)$. Note that for each $i \in [r] \setminus N_R(t+1)$, we have $m_i(t) = m_i(t+1)$ and $A^t_i (\phi_t) = A^t_i (\phi_{t+1})$ and $B^t_i (\phi_t) = B^t_i (\phi_{t+1})$. For $i \in [r] \setminus [t+1]$, let

\[
\tilde{A}^t_i := \begin{cases} \tilde{A}^t_i & \text{if } i \in N_R(t+1) \setminus [t], \\ \hat{A}^t_i & \text{otherwise,} \end{cases}
\]

\[
\tilde{B}^t_i := \begin{cases} \tilde{B}^t_i & \text{if } i \in N_R(t+1), \\ \hat{B}^t_i & \text{otherwise.} \end{cases}
\]

Then the graphs $\tilde{A}^t_i$ and $\tilde{B}^t_i$ are candidacy graphs satisfying $S(t+1)(a)$.

Step 3.2. Checking $S(t+1)(b)$ and $S(t+1)(c)$

The new edge set labelling $\psi_{t+1}$ as defined in (5.8) corresponds to the updated edge set labelling as in Definition 4.2. By (IV$4.3$, S(t)(b) and (5.10), we obtain for every $i \in [r] \setminus [t+1]$ that $\Delta^\alpha_{\varepsilon_{c(i+1)-1}} (\bigcup_{H \in \mathcal{H}} \hat{A}^t_i) \leq (1 + \varepsilon_{c(i+1)-1}) \alpha A d^A_{A}(t+1)[V_i]$. This establishes $S(t+1)(b)$. Similarly, by (V$4.3$, with $R[N_R[t+1] \setminus [t]]$ playing the role of $R_A$ and by $S(t)(c)$, we obtain for every $i \in [r] \setminus [t+1]$ that $\Delta^\alpha_{\varepsilon_{c(i+1)-1}} (\bigcup_{H \in \mathcal{H}} \hat{A}^t_i) \leq \sqrt{n}$, which establishes $S(t+1)(c)$. 

Step 3.3. Checking $S(t+1)(d)$

In order to show $S(t+1)(d)$, fix $H \in \mathcal{H}$, $ij \in E(R - [t+1])$, and $v_i v_j \in E(G[V_i, V_j])$. Observe, that $\{|i,j| \cap N_R(t+1)\} \in \{0, 1, 2\}$. 

If $|\{|i,j| \cap N_R(t+1)\}| = 2$, then this implies together with (5.10) that $c_i(t+1) = c_j(t+1) = c(t+1) \geq \max\{c_i(t), c_j(t)\}$, and $m_i(t+1) = m_i(t+1)$ as well as $m_j(t) = m_j(t+1)$. Hence we obtain by (II$4.3$) with $R[N_R[t+1] \setminus [t]]$ playing the role of $R_A$ and $S(t)(d)$ that

\[
(5.12) \quad e_H(N_{\tilde{A}^t_i}(v_i), N_{\tilde{A}^t_i}(v_j)) = (\alpha A d^A_{A}(t+1) + m_i(t+1) + \varepsilon_{\max(c_i(t+1), c_j(t+1))}) e_H(X^H_j, X^H_i).
\]

If $|\{|i,j| \cap N_R(t+1)\}| = 1$, say $i \in N_R(t+1)$, then this implies together with (5.10) that

\[
(5.13) \quad c_i(t+1) = \max\{c_i(t+1), c_j(t+1)\} = c(t+1) > \max\{c_i(t), c_j(t)\},
\]

and $m_i(t+1) = m_i(t+1)$, $m_j(t) = m_j(t+1)$.
By (5.11), we have that $\tilde{A}_j^H = A_j^H$ because $j \notin N_R(t+1)$. Let $N := N_{A_i^H}^{\epsilon}(v_i) \cap N_H(N_{A_i^H}^{\epsilon}(v_j))$ and we define a weight function $\omega_N : E(\mathcal{A}_i^\epsilon) \to \{0,1\}$ by $\omega_N(xv) := 1_{\{v_i,v_j\} \subseteq X \subseteq N_H}$ and add $\omega_N$ to $W^*_e$. Note that $\dim(\omega_N) = 1$ (with $\dim(\omega_N)$ defined as in (4.2)) and that $\omega_N(E(\mathcal{A}_i^\epsilon)) = |N| = (\alpha_A^2 d_A^{m_1(t)} d_A^{m_1(t)} + \varepsilon_{\text{max}}(c_1(t),c_2(t))) e_H X_i^{H}, X_j^{H})$ by $S(t)(d)$. This implies that

$$e_H(N_{A_i^H}^{\epsilon}(v_i), N_{A_i^H}^{\epsilon}(v_j)) = |N_{A_i^H}^{\epsilon}(v_i) \cap N_H(N_{A_i^H}^{\epsilon}(v_j))| = |N_{A_i^H}^{\epsilon}(v_i) \cap N = \omega_N(E(\tilde{A}_j^H))$$

(III)_{4.3} (1 + \varepsilon_{c(t+1)}^2 d_A \omega_N(E(\mathcal{A}_i^\epsilon)) \pm \varepsilon_{c(t+1)}^2 n)

(5.13) = (\alpha_A^2 d_A^{m_1(t)+m_2(t)} + \varepsilon_{\text{max}}(c_1(t),c_2(t))) e_H(X_i^{H}, X_j^{H}).$

If $|\{i,j\} \cap N_R(t+1)| = 0$, then this implies together with (5.10) and (5.11), that $m_i(t) = m_i(t+1)$, $m_j(t) = m_j(t+1)$, and $\tilde{A}_j^H = A_j^H$. Consequently, (5.12) holds which establishes $S(t+1)(d)$.

**Step 3.4. Checking $S(t+1)(e)$**

In order to establish $S(t+1)(e)$, we first consider $v_i \in V_i$ for $i \in N_R(t+1) \cap [t]$. We define a weight function $\omega_i : E(\mathcal{A}_i^\epsilon) \to \{0,1\}$ by $\omega_i(xv) := 1_{\{v_i \in \mathcal{A}_i^\epsilon\}}$ for $\mathcal{A}_i^\epsilon := N_H(H_i^{-1}) \cap \mathcal{A}_i^{t+1}$ and every $xv \in E(\mathcal{A}_i^\epsilon)$, and we add $\omega_i$ to $W^*_e$. By $S(t)(a)$, we have

$$\omega_i(E(\mathcal{A}_i^\epsilon)) = (\alpha_A^2 d_A^{m_1(t)} + \varepsilon_{c(t+1)}^2 n) |\mathcal{A}_i^\epsilon| n,$$

and by $S(t)(e)$, we have

$$\omega_i(E(\mathcal{A}_i^\epsilon)) = (\alpha_A^2 d_A^{m_1(t)} + \varepsilon_{c(t+1)}^2 n) |\mathcal{A}_i^\epsilon| n,$$

(5.14) $$|\phi_i^{-1}(v_i) \cap N_H(X_i^{t+1} \setminus X_i^{t+1})| \leq \varepsilon_{1/2}^1 n \geq |\mathcal{A}_i^\epsilon| - \varepsilon_{c(t+1)}(M).$$

with $M = M(\sigma)$ being the corresponding edge set to $\sigma$. By (VII)_{4.3}, we obtain that

$$\omega_i(M) = (1 + \varepsilon_{c(t+1)}^2) \omega_i(E(\mathcal{A}_i^\epsilon)) \pm \varepsilon_{c(t+1)}^2 n \geq (1 - \varepsilon_{c(t+1)}^3) |\mathcal{A}_i^\epsilon| - \varepsilon_{c(t+1)}(M).$$

Together with (5.15), this implies that $|\phi_i^{-1}(v_i) \cap N_R(X_i^{t+1} \setminus X_i^{t+1})| \leq \varepsilon_{1/2}^1 n$.

Hence, it now suffices to establish $S(t+1)(e)$ for all $v_i \in V_i$ by $S(t)(e)$. We define weight functions $\omega_{v_i}^{\epsilon}, \omega_{v_i}^{\epsilon} : E(\mathcal{A}_i^\epsilon) \to \{0,1\}$ by $\omega_{v_i}^{\epsilon}(xv) := 1_{\{v_i \in v_{t+1} \cup x \in \mathcal{A}_i^\epsilon\}}$ for $\mathcal{A}_i^{t+1} := N_H(\mathcal{A}_i \setminus \mathcal{A}_i^{t+1}) \cap \mathcal{A}_i^{t+1}$ and every $xv \in E(\mathcal{A}_i^\epsilon)$, and we add $\omega_{v_i}^{\epsilon}$ and $\omega_{v_i}^{\epsilon}$ to $W^*_e$. Observe that $S(t)$ implies that

$$\omega_{v_i}^{\epsilon}(E(\mathcal{A}_i^\epsilon)) = (\alpha_A^2 d_A^{m_1(t)} + \varepsilon_{c(t+1)}(t)) |\mathcal{A}_i^\epsilon| n,$$

(5.16) $$\omega_{v_i}^{\epsilon}(E(\mathcal{A}_i^\epsilon)) = (\alpha_A^2 d_A^{m_1(t)} + \varepsilon_{c(t+1)}(t)) |\mathcal{A}_i^\epsilon| n,$$

and we have that $|\phi_{v_i}^{-1}(v_i) \cap N_R(X_i^{t+1} \setminus X_i^{t+1})| = \varepsilon_{c(t+1)}(M)$ and $|\phi_{v_i}^{-1}(v_i) \cap N_R(X_i^{t+1} \setminus X_i^{t+1})| \leq \varepsilon_{c(t+1)}(M)$. By (VII)_{4.3}, we obtain that

$$\omega_{v_i}^{\epsilon}(M) = (1 + \varepsilon_{c(t+1)}^2) \omega_{v_i}^{\epsilon}(E(\mathcal{A}_i^\epsilon)) \pm \varepsilon_{c(t+1)}^2 n \geq (1 - \varepsilon_{c(t+1)}^3) |\mathcal{A}_i^\epsilon| - \varepsilon_{c(t+1)}(M);$$

$$\omega_{v_i}^{\epsilon}(M) = (1 + \varepsilon_{c(t+1)}^2) \omega_{v_i}^{\epsilon}(E(\mathcal{A}_i^\epsilon)) \pm \varepsilon_{c(t+1)}^2 n \leq \varepsilon_{c(t+1)}^2 n.$$

Note that $\varepsilon_{c(t+1)} = \varepsilon_{c(t+1)}(t)$. Altogether, this establishes $S(t+1)(e)$.

**Step 3.5. Checking $S(t+1)(f)$**

In order to establish $S(t+1)(f)$, consider $\omega_v \in W^*_e$ for $i \in N_R(t+1) \setminus [t+1]$. By (5.10), it holds that $c(t+1) = c_i(t+1)$. With (III)_{4.3} we obtain that

$$\omega_v(E(\bigcup_{H \in H} \tilde{A}_i^H)) = (1 + \varepsilon_{c(t+1)}^2) d_A \omega_v(E(\mathcal{A}_i^\epsilon)) \pm \varepsilon_{c(t+1)}^2 n \omega_{v_i}^{\epsilon}(M) = \alpha_A^2 d_A^{m_1(t)} \omega_{v_i}^{\epsilon}(M) \pm \varepsilon_{c(t+1)}^2 n,$$

which together with $S(t)(f)$ establishes $S(t+1)(f)$.

Next we verify $S(t+1)(g)$. Note that (VI)_{4.3} implies that $|W \cap \bigcup_{H \in H} \sigma(Y_i \cap \mathcal{A}_i^{t+1})| = |W||Y_i| \cdots |Y_t|/n^t \pm \varepsilon_{c(t+1)} n$ for all $(W,Y_1, \ldots, Y_t) \in W^{t+1}_e$, which together with $S(t)(g)$ yields $S(t+1)(g)$. 
In order to establish $S(t + 1)(h)$, let $\mathcal{W}_{\text{ver}}^{t+1} \subseteq \mathcal{W}_{\text{ver}}$ be the set of vertex testers $(v, \omega)$ with $v \in V_{t+1}$. Hence, for all $(v, \omega) \in \mathcal{W}_{\text{ver}}^{t+1}$ and its corresponding edge tester $\omega_v \in \mathcal{W}_{\text{edge}}^{t+1}$ as defined in (5.3), property (VII)$_{4,3}$ implies that
\[
\omega(\mathcal{X}_{t+1} \cap r^{-1}(v)) = \omega_v(M) = \left(1 \pm \varepsilon_{c(t+1)}\right)\frac{\omega_v(E(\mathcal{X}_{t+1}))}{\alpha_A d_{A}^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n
\]
with
\[
S(t)(h) = \left(1 \pm \varepsilon_{c(t+1)}\right)\frac{\omega(\mathcal{X}_{t+1})}{\alpha_A d_{A}^{m_{t+1}(t)} n} \pm \varepsilon_{c(t+1)} n = \frac{\omega(\mathcal{X}_{t+1})}{n} \pm \alpha n/2.
\]
Together with $S(t)(h)$, this yields $S(t + 1)(h)$.

Step 4. Completion

Let $\phi_t : \bigcup_{H \in \mathcal{H}, r \in [r]} \hat{X}_i^H \to V_B$ be an $r$-partial packing satisfying $S(r)$ with $(\varepsilon_T, d_r)$-super-regular candidacy graphs $B_i^H \subseteq B_i^H(\phi_r)$ where $d_r := \alpha_B d_{B}^{\deg_B(i)}$ for all $i \in [r]$. We aim to apply iteratively the ordinary blow-up lemma in order to complete the partial packing $\phi_r$ using the edges in $G_B$. Recall that $\varepsilon_T \ll \mu \ll \gamma \ll \alpha, d$. Our general strategy is as follows. For every $H \in \mathcal{H}$ in turn, we choose a subgraph $X_i \subseteq \hat{X}_i^H$ for all $i \in [r]$ of size roughly $\mu n$ by selecting every vertex uniformly at random with the appropriate probability and adding $X_i \setminus \hat{X}_i^H$ deterministically. Afterwards, we apply the blow-up lemma to embed $H[X_1 \cup \ldots \cup X_r]$ into $G_B$, which together with $\phi_r$ yields a complete embedding of $H$ into $G_A \cup G_B$. Before we proceed with the details of our procedure (see Claim 3), we verify in Claim 2 that we can indeed apply the blow-up lemma to a subgraph of $H \in \mathcal{H}$ provided some easily verifiable conditions are satisfied.

Recall that we have defined the candidacy graph $B_i^H$ on a copy $(X_i^H, B_i^H)$ of the original graph $G_i^H$ via the bijection $\pi$ only to conveniently apply Lemma 4.3 in Step 3. That is, for all $H \in \mathcal{H}, i \in [r]$, we can identify $B_i^H$ with an isomorphic bipartite graph on $(X_i^H, V_i)$ and edge set $\{xv : \pi(x) \pi(v) \in E(B_i^H)\}$. Let $B$ be the union over all $H \in \mathcal{H}, i \in [r]$ of these graphs. For $H \in \mathcal{H}$ and a subgraph $G_r$ of $G_B$, we write $B[G_r]$ for the graph that arises from $B[X_i^H, V_i]$ by deleting every edge $xv$ with $x \in X_i^H, v \in V_i$ for which there exist $x'x \in E(H)$ such that $\phi_r(x')v \in E(G_r)$. (We may think of $E(G_r)$ as the edge set in $G_B$ that we have already used in our completion step for packing some other subgraphs of $H \in \mathcal{H}$ into $G_A \cup G_B$.) For future reference, we observe that
\[
\Delta(B[X_i^H, V_i] - B[G_r][X_i^H, V_i]) \leq \alpha - \Delta(G_r) \quad \text{for all } i \in [r].
\]

Claim 2. Suppose $H \in \mathcal{H}$, $G_r \subseteq G_B$ and $W_i \subseteq V_i$ for all $i \in [r]$ such that the following hold:
(a) $V_i \setminus \phi_r(\hat{X}_i^H) \subseteq W_i \subseteq V_i$ and $|W_i| = (\mu \pm \varepsilon_T^{1/2}) n$ for all $i \in [r]$;
(b) $B_r[X_i, W_i]$ is $(\mu^{1/3}, d_r)$-super-regular for all $i \in E(R)$;
(c) $G_B[W_i, W_j]$ is $(\varepsilon_T^{1/3}, d_r)$-super-regular for all $ij \in E(R)$;
(d) $|N_G(v) \cap W_i| \leq \mu^{1/3} n$ for all $v \in V_j$ and $ij \in E(R)$.

Then there exists an embedding $\phi_r^H$ of $\tilde{H} := H[X_1, \ldots, X_r]$ into $\tilde{G} := G_B[W_1, \ldots, W_r] - G_r$ such that $\phi_r^H := \phi_r^H \cup \phi_r|_{V(H) \setminus \tilde{V}(\tilde{H})}$ is an embedding of $H$ into $G$ where $\phi_r^H(X_i^H) = V_i$ for all $i \in [r]$ and all edges incident to a vertex in $\bigcup_{i \in [r]} \tilde{X}_i$ are embedded on an edge in $G_B - G_r$.

Proof of claim: Note that $|X_i| = |W_i| = (\mu \pm \varepsilon_T^{1/2}) n$ by (a) and (b)) and $\tilde{H}[X_i, X_j]$ is empty whenever $ij \notin E(R)$ and a matching otherwise. Moreover, by (c) and (d), Fact 3.3 yields that $\tilde{G}[W_i, W_j]$ is $(\mu^{1/3}, d_r)$-super-regular for all $ij \in E(R)$ (with room to spare). This shows that $\tilde{H} \cup \tilde{G} \cup \tilde{R}, (X_i)_{i \in [r]}, (W_i)_{i \in [r]}$ is a $(\mu^{1/3}, d_r)$-super-regular blow-up instance. We apply Theorem 3.5 to this blow-up instance (with $(B[G_r][X_i, W_i])_{i \in [r]}$ playing the role of $(A_i)_{i \in [r]}$) and obtain an embedding $\phi_r^H$ of $\tilde{H}$ into $\tilde{G}$. Recall that $\phi_r|_{V(H) \setminus \tilde{V}(\tilde{H})}$ is an embedding of $H - V(\tilde{H})$ into $G_A$.

For $x \in V(\tilde{H})$, $x' \in V(H) \setminus V(\tilde{H})$ and $xx' \in E(H)$, we conclude that $\phi_r^H(x)\phi_r(x') \in E(G_B - G_r)$ by the definition of the candidacy graphs in (5.7) and the definition of $B[G_r]$. --
Claim 3. For all $H \in \mathcal{H}$ and $i \in [r]$, there exist sets $X_i^H \subseteq \widehat{X}_i^H$ with $|X_i^H| \geq (1 - 2\mu)n$ and a packing $\phi$ of $\mathcal{H}$ into $G$ that extends $\phi^*$ such that $\phi_{|X_i^H} = \phi_{|X_i^H}$ as well as $\phi(X_i^H) = V_i$.

Proof of claim: We write $\mathcal{H} = \{H_1, \ldots, H_{|\mathcal{H}|}\}$ and let $\mathcal{H}_h := \{H_1, \ldots, H_h\}$ for all $h \in [|\mathcal{H}|]_0$.
For all $v \in V_i$, $i \in [r]$, let $\tau^h(v) := \{H \in \mathcal{H}_h; v \in V_i \setminus \phi_r(\widehat{X}_i^H)\}$ and $\sigma^h(v) := \{\phi^{-1}_r(v) \cap \bigcup_{H \in \mathcal{H}_h,j \in N_H(i)} N_H(X_i^H \setminus \widehat{X}_i^H)\}$. We inductively prove that the following statement $C(h)$ holds for all $h \in [|\mathcal{H}|]_0$.

$C(h)$. There exists a packing $\phi^h$ of $\mathcal{H}_h$ into $G$ that extends $\phi^*$ such that for $G^h : G_B \cap \phi^h(\mathcal{H}_h)$ we have
(A) $\deg_{\phi^h}(v) \leq \alpha^n h(v) + \sigma^h(v)) + \mu^{2/3}n$ for all $v \in V(G)$;
(B) for all $H \in \mathcal{H}_h$ and $i \in [r]$, there exist sets $X_i^H \subseteq \widehat{X}_i^H$ with $|X_i^H| \geq (1 - 2\mu)n$ such that $\phi^h_{|X_i^H} = \phi_{|X_i^H}$ as well as $\phi^h(X_i^H) = V_i$.

Let $G^0_h$ be the edgeless graph on $V(G)$ and $\phi^0$ be the empty function; then $C(0)$ holds. Hence, we may assume the truth of $C(h)$ for some $h \in [|\mathcal{H}|]_0 - 1$ and let $\phi^h$ and $G^h$ be as in $C(h)$.

By $C(h)$, there are at most $\sum_{H \in \mathcal{H}_h,j \in N_H(i)} |\mathcal{H}_j|/\alpha^n |X_i^H \setminus \widehat{X}_i^H| \leq 5\alpha^{-3/2}n^2$ edges of $G^0_h$ incident to a vertex in $V_i$ for each $i \in [r]$. Hence, there are at most $10\alpha^{-3}n^{1/3}n$ vertices in $V_i$ of degree at least $\mu^{2/3}n/2$ in $G^0_h$. Let $V_i^{high} \subseteq V_i$ be a set of size $\mu^{1/4}n$ that contains all vertices of degree at least $\mu^{2/3}n/2$ in $G^0_h$. For all $i \in [r]$, we select every vertex in $\phi_r(\widehat{X}_i^{H_{h+1}}) \setminus V_i^{high}$ independently with probability $\mu(1 - \mu^{1/3})^{-1}$ and denote by $W_i$ their union together with $V_i \setminus \phi_r(\widehat{X}_i^{H_{h+1}})$; we define $X_i := (X_i^{H_{h+1}} \setminus \phi_r^{-1}(W_i \cap \phi_r(\widehat{X}_i^{H_{h+1}}))) \cup (X_i^{H_{h+1}} \setminus \widehat{X}_i^{H_{h+1}})$. Note that $S(r)$ implies that

\begin{equation}
|X_i^{H_{h+1}} \cup \widehat{X}_i^{H_{h+1}}| = |V_i \setminus \phi_r(\widehat{X}_i^{H_{h+1}})| \leq 2\varepsilon_Tn \text{ for all } i \in [r].
\end{equation}

In the following we will show that the assumptions of Claim 2 are satisfied with probability at least $1/2$, say, for $H_{h+1}$ and $G^0_h$ playing the role of $H$ and $G^*$, respectively. In particular, there is a choice for $W_i$ such that the assumptions of Claim 2 hold.

To obtain (a), we apply Chernoff’s inequality to the sum of indicator variables which indicate whether a vertex in $V_i$ is randomly selected. Together with (5.19), this shows that (a) holds with probability at least $1 - 1/n^3$, say.

By $C(h)(A)$ and $S(r)(e)$, we obtain that $\Delta(G^0_h) \leq 2\mu^{2/3}n$. We exploit (5.18) and conclude that $\Delta[G, X_i^{H_{h+1}}, V_i] - B_{G^0_h}[X_i^{H_{h+1}}, V_i] \leq 2\alpha^{-1}\mu^{2/3}n$ for all $i \in [r]$. Thus Fact 3.3 implies that $B_{G^0_h}[X_i^{H_{h+1}}, V_i] = (\mu^{1/3}, d_i)$-super-regular for all $i \in [r]$. For all $i \in [r]$ and $x \in X_i^{H_{h+1}}$, Chernoff’s inequality implies that $|N_{B_{G^0_h}}(x) \cap W_i| = (d_i \pm 2\mu^{1/2})|W_i|$ and similarly, for all $v \in V$, we have $|N_{B_{G^0_h}}(v) \cap X_i| = (d_i \pm 2\mu^{1/2})|X_i|$. Moreover, for all distinct $v, v'$ with $|N_{B_{G^0_h}}(v, v')| = (d_i \pm 2\mu^{1/2})^2$, (which we call good, and there are at least $(1 - 2\mu^{1/2})(\alpha r)\binom{n}{2}$ good pairs), we also obtain $|N_{B_{G^0_h}}(v, v') \cap X_i| = (d_i \pm 3\mu^{1/2})^2|X_i|$. Observe that Theorem 3.1 implies that there are at least $(1 - 3\mu^{1/3})(\mu r)^2$ good pairs in $W_i$ with probability at least $1 - 1/n^3$. Therefore, we may apply Theorem 3.4 and obtain that $B_{G^0_h}[X_i, W_i]$ is $(\mu^{1/3}, d_i)$-super-regular for all $i \in [r]$ which yields (b) with probability at least $1 - 1/n^2$, say.

To obtain (c), for each $ij \in E(R)$, we proceed as follows. Observe first that $G_B[V_i, V_j]$ is $(2\varepsilon, d_B)$-super-regular by (5.4). Hence $G_B[V_i, V_j]$ is clearly $\varepsilon^{1/3}$-regular as $|V_i|, |V_j| \geq \mu n/2$ by (a). Therefore, we only need to control the degrees of the vertices in $G_B[V_i, V_j]$ which follows directly by Chernoff’s inequality with probability at least $1 - 1/n^3$ and because of (5.19).

Since $\Delta(G^0_h) \leq 2\mu^{2/3}n$ and because of (5.19), we conclude by Chernoff’s inequality that (d) holds with probability at least $1 - 1/n^3$.

Therefore, the assumptions of Claim 2 are achieved by our construction with probability at least $1/2$. Fix such a choice for $W_1, \ldots, W_r$ and apply Claim 2 that returns an embedding $\phi^{H_{h+1}}$ of $H_{h+1} := H_{h+1}[X_1, \ldots, X_r]$ into $G_B[W_1, \ldots, W_r] - G^0_h$ such that $\phi_i := \phi^{H_{h+1}} \cup \phi_{|V(H_{h+1}) \setminus V(H_{h+1})]}$ is an embedding of $H_{h+1}$ into $G$ where $\phi_i(X_i^H) = V_i$ and all edges incident to a vertex in $\bigcup_{i \in [r]} X_i$
are embedded on an edge in $G_B - G_h^c$. We define $\phi^{h+1} := \phi^h \cup \phi'$ and obtain $C(h+1)(B)$ with $X_i^{h+1} := X_i^{h+1} \setminus X_i$. It is straightforward to check that by our construction also $C(h+1)(A)$ holds.

Let $\phi$ be as in Claim 3. This directly implies conclusion (i) of Lemma 5.1. Conclusion (ii) of Lemma 5.1 follows from Claim 3 together with $S(r)(g)$ as we merely modified $\phi_r$ to obtain $\phi$. For a similar reason, Lemma 5.1 (iii) follows from Claim 3 and $S(r)(h)$. This completes the proof. □

6. Applications

In what follows, we provide an illustration for an application of vertex and set testers so that the leftover is suitably well-behaved. For a graph $H$, let $H^+$ arise from $H$ by adding a labelled vertex $x$ and joining $x$ to all vertices of $H$. We call $x$ the apex vertex of $H^+$.

**Theorem 6.1** (Keevash [15, cf. Theorem 7.8], cf. [10, Theorem 3.6 and Corollary 3.7]). Suppose $1/n \ll \varepsilon \ll 1/s \ll d_0, 1/m$. Suppose $H$ is an $r$-regular graph on $m$ vertices. Let $G$ be a graph with vertex partition $(V, W)$ such that $W$ is an independent set, $d_0 n \leq |W| \leq |V| = n$ and $\left| \bigcap_{v \in V \cup W} N_G(v) \right| = (1 + \varepsilon) d_W |W|^2 n$ for all $V' \subseteq V, W' \subseteq W$ with $1 \leq |V' | + |W'| \leq s$ where $d_V = r d_W |W| n$ and $d_W = d$ for some $d \geq d_0$. Suppose that $|N_G(v) \cap V| = r |N_G(v) \cap W|$ for all $v \in V$ and $m$ divides $\deg_G(w)$ for all $w \in W$. Then there is a decomposition of the edge set of $G$ into copies of $H^+$ where the apex vertices are contained in $W$.

**Proof of Theorem 1.4.** Choose $\varepsilon \ll \delta \ll 1/s \ll \beta \ll \alpha$. Among the $\alpha n$ graphs in $\mathcal{H}$ that contain at least $\alpha n$ vertices in components of size at most $\alpha^{-1}$, there is a collection $\mathcal{H}'$ of $\beta n$ $r$-regular graphs that contain each at least $\beta n$ vertices in components all isomorphic to some graph $J$ where $|V(J)| \leq \alpha^{-1}$ and $r \in [\alpha^{-1}]$.

For all $H \in \mathcal{H}'$, let $H^-$ arise from $H$ by deleting $\beta n/|V(J)|$ components isomorphic to $J$. We denote by $I^H$ a set of $\beta n$ isolated vertices disjoint from $V(H^-)$. Let $\tilde{H} := (\mathcal{H}' \setminus \mathcal{H}) \cup \bigcup_{H \in \mathcal{H}} (H \cup I^H \cup I^H)$. Let $G_1$ be a $(2\varepsilon, s, d_1)$-typical subgraph of $G$ where $d_1 := (1 + \varepsilon)(d - \beta^2 r)$ such that $G - G_1$ is $(2\varepsilon, s, d - d_1)$-typical; that is, $\delta(H) \leq (1 - \delta/2)\epsilon(G_1)$. Clearly, $G_1$ exists by considering a random subgraph and then applying Chernoff’s inequality. Now we apply Theorem 1.3 to obtain a packing $\phi$ of $\tilde{H}$ in $G_1$ with $\beta^2$ playing the role of $\alpha$ and sets of set and vertex testers $W_{ver}, W_{ver}$ defined as follows. For all $H_1, \ldots, H_{\ell_1} \in \mathcal{H}'$ and $v_1, \ldots, v_{\ell_1} \in V(G)$ with $1 \leq \ell_1$ and $\ell_1 + \ell_2 \leq s$, we add the set tester $(v^r, I^H_1, \ldots, I^H_{\ell_1})$ to $W_{ver}$ where $V' := V(G) \cap \bigcup_{i \in [\ell_1]} N_{G-G_1}(v_i)$. Then Theorem 1.3 implies that (where $d_2 := d - d_1$)

\[
\left| \bigcap_{i \in [\ell_1]} \phi(I^H_i) \cap \bigcup_{i \in [\ell_2]} N_{G-G_1}(v_i) \right| = (\beta^1 d_2^2 \pm \beta^2) n.
\]

For each $v \in V(G)$, we define a vertex tester $(v, \omega)$ where $\omega$ assigns every vertex in $V(H)$ its degree for all $H \in \tilde{H}$ and add $(v, \omega)$ to $W_{ver}$. Then Theorem 1.3 implies that $\Delta(G_1 - \phi(\tilde{H})) \leq 2\beta n$.

Let $G_2 := G - \phi(\tilde{H})$. Next, we add $\beta n$ vertices $W := \{v_H\}_{H \in \mathcal{H'}}$ to $G_2$ and join $v_H$ to all vertices in $\phi(I^H)$ and denote this new graph by $G_3$. Let $d_V := \beta^2 r$ and $d_W := \beta$. Hence (6.1), the typicality of $G - G_1$, and $\Delta(G_1 - \phi(\tilde{H})) \leq 2\beta n$ imply that for all $w_1, \ldots, w_{\ell_1} \in W$ and $v_1, \ldots, v_{\ell_2} \in V(G)$

\[
\bigcup_{x \in \{w_1, \ldots, w_{\ell_1}, v_1, \ldots, v_{\ell_2}\}} N_{G_3}(x) = (1 \pm \sqrt{\delta}) \beta^1 \beta^2 n = (1 \pm \sqrt{\delta}) \beta^2 d_2 d_2^2 n
\]

whenever $1 \leq \ell_1 + \ell_2 \leq s$. We apply Theorem 6.1 to $G_3$ to obtain a decomposition of $G_3$ into copies of $J^+$ where the apex vertices are contained in $W$. Observe that this yields the desired decomposition of $G$ into $H$. Indeed, for $H \in \mathcal{H}'$, let $J_H$ be the set of all copies of $J^+$ in $G_3$ whose apex vertex is $v_H$; hence $|J_H| = |I^H|/|V(J)| = \beta n/|V(J)|$. We define a packing $\phi'$ of $\tilde{H}$ in $G$ as follows. For all $H \in \mathcal{H}' \setminus \tilde{H}$, let $\phi'(v_H) := \phi(v_H)$. For all $H \in \tilde{H}$, let $\phi'(v_H) := \phi(v_H)$ and each component of $H - V(H^-)$ (which is isomorphic to $J$) is mapped to $C - v_H$ for some $C \in J_H$ such that every $C - v_H$ is the image of exactly one component isomorphic to $J$ of $H - V(H^-)$. □
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