A modified penalty function method for treating multi freedom constraints in finite element analysis of frames

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Abstract. In finite element analysis, treating boundary condition having multi freedom constraints is needed to produce modified system of equation based on master stiffness equation considering multi freedom constraint. Generally the operation of imposing multi freedom constraints can be developed using Master-Slave Elimination, Penalty Augmentation or Lagrange Multiplier Adjunction methods. The master-slave method is useful only for simple cases but exhibits serious shortcomings for treating arbitrary constraints. The Penalty Augmentation method and Lagrange Multiplier Adjunction are better in many applications. But there are not free of disadvantages. The penalty method has difficulty of choice of weight values that balance solution accuracy with the violation of constraint conditions. The multiplier method is sensitive to the degree of linear independence of the constraints, and the bordered stiffness is singular in the case of the dependent constrains. To reduce the disadvantages of these methods, this paper presents a new method for treating multi freedom constraint. Proposed method is similar to penalty function method and it is called “modified penalty function method”. The concept of the modified penalty function method is based on the constructing equivalent solver system of equations from traditional modified system using regulatory parameters. The basic of method for selecting the regulatory parameters is reducing the violation of modified system. For solving equivalent system the authors proposed an iterative algorithm for seeking the solution. The calculation programs are established based on proposed algorithm. The calculation results using the proposed method matched with the calculation results using other methods.

1. Introduction

In finite element method, it is necessary to modify the system of master stiffness equation for imposing multi freedom constrains and getting equation solver. The modified process can be developed using basically there method: Master-Slave Elimination, Penalty Augmentation and Lagrange Multiplier Adjunction [1]. The master-slave method is easy to explain and is useful when a few simple linear constraints are imposed by hand, but exhibits serious shortcomings for treating arbitrary constraints. Therefore the master-slave is not widely utilized in treating multi freedom constants. The penalty augmentation method and Lagrange multiplier adjunction method are better suited to general implementations of the finite element method, whether linear or nonlinear, and both techniques are mostly developed in boundary treatment. But Penalty Augmentation method and Lagrange Multiplier Adjunction method are not free of disadvantages. The main disadvantage of penalty method is difficulty of choice of weight values that balance solution accuracy with the
violation of constraint conditions. The multiplier method is sensitive to the degree of linear independence of the constraints, and the bordered stiffness is singular in the case of the dependent constrains. This paper is intended to introduce a new method for treating multi freedom constraints in static analysis of frame system using finite element method. The proposed method is similar to the penalty function method and is called modified penalty function method.

2. Problem formulation for imposing multi freedom constraints

2.1. Boundary condition and Multi freedom constraints
In finite element structural analysis [1,2], separating known and unknown components is needed for developing equations for a linear solver. The necessary step is applying the physical support conditions as displacement boundary conditions to eliminate rigid body motions and render the system non-singular. The constraint conditions can be single freedom constraints or multi freedom constraints.

Single freedom constraints that are mathematically expressable as constraints on individual degrees of freedom. It can be linear homogeneous \( u=0 \) or linear non-homogeneous \( u=\text{prescribed value} \).

Multi freedom constraints are functional equations that connect two or more displacement component. These can be linear homogeneous, linear non-homogeneous or nonlinear homogeneous.

For frame structures shown in fig.1 the constraint conditions are

- \( u_3=0, u_{10}=0, u_{12}=0 \) - linear homogeneous case of single freedom constraint;
- \( u_{11}=-1 \) - linear non-homogeneous case of single freedom constraint;
- \( u_9 + u_8 = 0 \) - linear homogeneous case of multi freedom constraint;
- \( u_1 + u_2 = -2\sqrt{2} \) - linear non-homogeneous case of multi freedom constraint.

![Figure 1. Frame with single freedom and multi freedom constraints](image)

Using hand (or computer) oriented techniques can incorporate single freedom constraints into the master stiffness equations. In this research the attention is devoted to the linear homogeneous and linear non-homogeneous or nonlinear homogeneous.

2.2. Multi freedom constrains Imposition
The set of multi freedom constraints is expressed as may be collected into single matrix relation

\[
Au = b
\]  
(1)

Where \( u \) contains all degree of freedom and \( A \) is a row vector with same length as \( u \).

Accounting for multi freedom constraints is done by changing the assembled master equations (2) to produce a modified system of equations (3)

\[
Ku = f
\]  
(2)

\[
\text{Applying MFCs} 
\]

\[
\tilde{K}\tilde{u} = \tilde{f}
\]  
(3)

The modified system (3) is incorporating the multi freedom constrains to finite element model. The modified system is that submitted to the equation solver, which returns \( \tilde{u} \) and recover \( u \) in necessary cases.
3. Methods for imposing multi freedom constraints

3.1. Penalty function method
The concept of penalty function method is that each multi freedom constraint is viewed as the presence of a fictitious elastic structural element called penalty element that enforces it approximately. This element is parameterized by a numerical weight \( w \). The exact constraint is recovered if the weight goes to infinity. The multi freedom constraints are imposed by augmenting the finite element model with the penalty elements.

The penalty augmented system can be developed by minimization of the augmented potential energy function and expressed as

\[
(K + K_w)u = (f_p + f_w)
\]

(4)

Where:

\[
K_w = W A^T A; \quad f_w = W A^T b
\]

(5)

\( W \) is a diagonal matrix of penalty weights and \( f_p \) is nodal load vector.

The important advantage of the penalty function method is its lack of sensitivity with respect to whether constraints are linearly dependent. But there is main disadvantage is the choice of weight values \( W \) that balance solution and escape the violation of constraint conditions. The difficult is selecting appropriate weights. It is needed the knowledge of the magnitude of stiffness coefficients and required extensive numerical experimentation.

3.2. Lagrange multiplier method
The concept of this method based on the technique of variational calculus. The potential energy of the unconstrained finite element model is

\[
\Pi = \frac{1}{2} u^T Ku - u^T f_p
\]

(6)

To impose the constraint, adjoin Lagrange multipliers collected in vector \( \lambda \) (unknowns) and form the Lagrangian with condition (1)

\[
L(u, \lambda) = \frac{1}{2} u^T K u - u^T f_p + \lambda^T (Au - b)
\]

(7)

The multiplier-augmented system of equation, getting by extremizing \( L \) with respect to \( u \) and \( \lambda \) yields, is expressed as

\[
\begin{bmatrix} K & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f_p \\ b \end{bmatrix}
\]

(8)

The main advantage of this method is it gives exact solution. But it has the important disadvantage of sensitivity with respect to whether constraints are linearly dependent. In these cases the border stiffness can be singular.

3.3. Modified penalty function method
To escape the disadvantages of penalty function method and the author proposed a modified penalty function method for treating multi freedom constraint solving problems.

To resolve load vector into two components and express the master stiffness equations as written

\[
Ku = f_p + f_r
\]

(9)

Where \( f_p \) is nodal load vector and \( f_r \) is nodal reaction load vector

From the equations (1) and (9), having the system of equations

\[
\begin{align*}
Ku &= f_p + f_r \\
Au &= b
\end{align*}
\]

(9)
The unknowns of (9) are displacement vector \( u \) and nodal reaction load vector \( f_r \).
Where \( f_p \) is nodal load vector and \( f_r \) is nodal reaction load vector.

To multiply both sides of the equations (9.2) by \( \alpha_1 A^T \) and by \( \alpha_2 A^T \), getting

\[
\begin{align*}
\alpha_1 A^T u &= \alpha_1 A^T b \quad (10.1) \\
\alpha_2 A^T u &= \alpha_2 A^T b \quad (10.2)
\end{align*}
\]

Adding (10.1) and (9.1), (10.2) and (9.1), getting

\[
\begin{align*}
(K + \alpha_1 A^T A)u &= f_p + f_r + \alpha_1 A^T b \\
(K + \alpha_2 A^T A)u &= f_p + f_r + \alpha_2 A^T b
\end{align*}
\]

Where \( \alpha_1, \alpha_2 \) are called regulatory parameters with condition \( \alpha_1 \neq \alpha_2 \).

The systems (11) and (9) are equivalent. The following step is seeking the solution of equivalent system (11). For solving linear system is used the iterative method [4]. The iterative algorithm for seeking the solution is proposed as described below.

Consider \( u^* \) and \( f_r^* \) are the roots of the system (11). To construct the iterative sequences \( \{u^k\} \) and \( \{f_r^k\} \) converging to the roots \( u^* \) and \( f_r^* \) when \( k \) goes to infinity.

\[
\text{Figure 2. Iterative procedure}
\]

Using iterative method (shown in fig.2), the system (11) can be written as

\[
\begin{align*}
(K + \alpha_1 A^T A)u^k &= f_p + f_r^k + \alpha_1 A^T b \\
(K + \alpha_2 A^T A)u^k &= f_p + f_r^{k+1} + \alpha_2 A^T b
\end{align*}
\]

(12)

When \( u=u^* \) and \( f_r=f_r^* \) the system (11) is expressed

\[
\begin{align*}
(K + \alpha_1 A^T A)u^* &= f_p + f_r^* + \alpha_1 A^T b \\
(K + \alpha_2 A^T A)u^* &= f_p + f_r^* + \alpha_2 A^T b
\end{align*}
\]

(13)

Taking subtractive operation: (12)-(13), changing the index from “\( k \)” to “\( k+1 \)”, getting
\[
\begin{align*}
(K + \alpha_1 A^T A)(u^{k+1} - u^*) &= f_{r}^{k+1} - f_r^* \\
(K + \alpha_2 A^T A)(u^k - u^*) &= f_r^{k+1} - f_r^* 
\end{align*}
\] (14.1) (14.2)

Setting equation from left-hand sides of (14.1) and (14.2), written as
\[
(K + \alpha_1 A^T A)(u^{k+1} - u^*) = (K + \alpha_2 A^T A)(u^k - u^*)
\] (15)

To multiply both sides of the equations (15) by \(1_2\), getting
\[
\|u^{k+1} - u^*\|_\infty = \|(K + \alpha_1 A^T A)^{-1}(K + \alpha_2 A^T A)(u^k - u^*)\|_\infty
\] (17)

From (17), getting condition below
\[
\|u^{k+1} - u^*\|_\infty \leq \|(K + \alpha_1 A^T A)^{-1}(K + \alpha_2 A^T A)\|_\infty \|u^k - u^*\|_\infty
\] (18)

From the condition (18), it can be seen that, the sequence can be reached to root \(\{u^k\} \rightarrow u^*\) when \(k \rightarrow \infty\), if and only if the selected regulatory parameters \(\alpha_1, \alpha_2\) must be satisfied condition
\[
\|(K + \alpha_1 A^T A)^{-1}(K + \alpha_2 A^T A)\|_\infty < 1
\] (19)

The difficulty is selecting the regulatory parameters. In this research is proposed a new approach. Based on
\[
\frac{\|(K + \alpha_2 A^T A)\|_\infty}{(K + \alpha_1 A^T A)\|_\infty} \leq \|u^{k+1} - u^*\|_\infty \leq \|K + \alpha_1 A^T A\|_\infty \|u^k - u^*\|_\infty
\] (20)

Suppose to select regulatory parameters so that the below conditions are satisfied
\[
\frac{\|(K + \alpha_2 A^T A)\|_\infty}{(K + \alpha_1 A^T A)\|_\infty} < 1
\] (21)

and
\[
\|u^{k+1} - u^*\|_\infty \leq \frac{\|(K + \alpha_2 A^T A)\|_\infty}{(K + \alpha_1 A^T A)\|_\infty} \|u^k - u^*\|_\infty
\] (22)

Getting rule for selecting the regulatory parameters
\[
\begin{cases}
\alpha_2 > 0 \\
\alpha_1 \geq \frac{2 \max |K_{ij}|}{\min \{A^T A\}_{ij}} + \alpha_2, \quad \forall 1 \leq i, j \leq n
\end{cases}
\] (23)

Where “\(n\)” is the number of nodal displacement unknowns.
It can be easy to prove the convergence of proposed method as below.

Setting
\[
q = \frac{\|(K + \alpha_2 A^T A)\|_\infty}{(K + \alpha_1 A^T A)\|_\infty} < 1
\]

From (22) it can be written
\[
\|u^{k+1} - u^*\|_\infty \leq q \|u^0 - u^*\|_\infty
\] (24)
Studying (24), obviously seeing when \( k \to \infty \), then \( u^{k+1} - u^k \to 0 \) or \( \{ u^k \} \to u^* \).

From (14-1), when \( u^{k+1} - u^k \to 0 \) , then \( f_r^{k+1} - f_r^k \to 0 \) or \( \{ f_r^k \} \to f_r^* \).

Based on the proposed above modified penalty function method the block diagram of algorithm is established (shown in fig.3)

**Figure 3.** Block diagram of algorithm using proposed modified penalty function method
4. Examples and results
For purpose of this research the analysis of frame having multi freedom constrains is implemented. The calculation programs are established based on the algorithms developed by penalty function method, Lagrange multiplier method and modified penalty function method.

4.1. Example formulation
The frame is composed of three bars made of the same material and had the same geometrical properties (frame is shown in fig 4). The geometric parameters, material parameters and loading parameters are given

\[ E = 2.10^4 \text{kN/cm}^2, I_x = 2000\text{cm}^4, A = 20\text{cm}^2, L = 420\text{cm}, f_{p1} = f_{p2} = f_{p3} = 0, \]
\[ f_{p4} = 40\text{kN}, f_{p5} = -60\text{kN}, f_{p6} = -1000\text{kNcm}, f_{p7} = -40\text{kN}, f_{p8} = -60\text{kN}, f_{p9} = 1000\text{kNcm}, \]
\[ f_{p10} = 0, f_{p11} = 0, f_{p12} = 0. \]

![Figure 4. The examined frame](image)

Setting
- Nodal displacement vector
  \[ u = \{u_1, u_2, \ldots, u_{12}\} \]
- Nodal load vector
  \[ f_P = \{f_{p1}, f_{p2}, \ldots, f_{p12}\} \]
- Nodal reaction load vector (unknowns)
  \[ f_r = \{f_{r1}, f_{r2}, \ldots, f_{r12}\} \]
- Equation of multi freedom constraints

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\{u_1\} \\
\{u_2\} \\
\ldots \\
\{u_{12}\}
\end{bmatrix} =
\begin{bmatrix}
2\sqrt{2} \\
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]
4.2. Results

The results of frame analysis using penalty function method with variable weight values $W$ are shown in Table 1. The correct solution is written in bold alphanumeric.

**Table 1. Displacement results of frame analysis using penalty function model with variable weight values $W$.**

| $\{u\}$ | $W = 10^9 I$ | $W = 10^6 I$ | $W = 10^4 I$ | $W = 10^2 I$ |
|----------|--------------|--------------|--------------|--------------|
| $u_1$    | 54.3939cm    | 22.1039cm    | 36.0926cm    | 8.7164cm     |
| $u_2$    | -57.2073cm   | -24.9324cm   | -38.921cm    | -11.5448cm   |
| $u_3$    | 0.0923rad    | 1.177*10^-6 rad | 2.162*10^-12 rad | 4.561*10^-14 |
| $u_4$    | 10.7692cm    | 4.3166cm     | 4.6478cm     | 1.5465cm     |
| $u_5$    | -57.1914cm   | -24.9601cm   | -38.9383cm   | -11.5909cm   |
| $u_6$    | 0.1266rad    | 0.0653rad    | 0.1111rad    | 0.0273rad    |
| $u_7$    | 10.743cm     | 4.2468cm     | 4.5506cm     | 1.4947cm     |
| $u_8$    | -10.8181cm   | -4.2468cm    | -4.5506cm    | -1.4947cm    |
| $u_9$    | 0.0391rad    | 0.0087rad    | 0.0206rad    | 0.009rad     |
| $u_{10}$ | -9.6951cm    | -3.1838cm    | -3.4784cm    | -0.4683cm    |
| $u_{11}$ | -10.7551cm   | -4.1838cm    | -4.4784cm    | -1.4683cm    |
| $u_{12}$ | -0.0923rad   | -1.177*10^-6 rad | -1.484*10^-12 rad | -4.388*10^-14 |

The results of frame analysis using Lagrange multiplier method are shown in Table 2.

**Table 2. Displacement results of frame analysis using Lagrange multiplier model.**

| Displacements $\{u\}$ | Lagrange multiplier $\{\lambda\}$ |
|------------------------|-----------------------------------|
| $u_1$                  | $\lambda_1$                      | -26.451288kN |
| $u_2$                  | $\lambda_2$                      | 11769.846977kN |
| $u_3$                  | $\lambda_3$                      | 60kN |
| $u_4$                  | $\lambda_4$                      | -11769.846977kN |
| $u_5$                  | $\lambda_5$                      | -33.548712kN |
| $u_6$                  | 0.065258rad                      | | |
| $u_7$                  | 4.246756cm                       | | |
| $u_8$                  | -4.246756cm                      | | |
The results of frame analysis using proposed modified penalty function method are shown in Table 3&4.

Table 3. Results of the selecting regulatory parameters for modifies penalty function model and number of iterations

| n  | $\alpha_1$ | Number of iterations | $\alpha_1$ | Number of iterations |
|----|------------|----------------------|------------|----------------------|
| 0  | 1.52382*10^6 | 7                    | 2.52381*10^6 | 18                   |
| 1  | 1.52382*10^7 | 4                    | 2.52381*10^7 | 5                    |
| 2  | 1.52382*10^8 | 3                    | 3.52381*10^8 | 3                    |
| 3  | 1.52382*10^9 | 2                    | 2.52381*10^9 | 2                    |
| 4  | 1.52382*10^10| 2                    | 3.52381*10^10| 3                    |
| 5  | 1.52382*10^11| 2                    | 2.52381*10^11| 3                    |

Table 4. Displacement and reaction results using modified penalty function model with convergence tolerance $\varepsilon_u = 10^{-6}$

| Displacements $\{u\}$ | Reactions $\{f_r\}$ |
|------------------------|----------------------|
| $u_1$ 22.10353cm | $f_{r1}$ 26.451kN |
| $u_2$ -24.931957cm | $f_{r2}$ 26.451kN |
| $u_3$ 1.265461*10^-13rad | $f_{r3}$ -1.177*10^4kNcm |
| $u_4$ 4.31653cm | $f_{r4}$ 0 |
| $u_5$ -24.959731cm | $f_{r5}$ 0 |
| $u_6$ 0.065258rad | $f_{r6}$ 0 |
| $u_7$ 4.246756cm | $f_{r7}$ 33.549kN |
| $u_8$ -4.246756cm | $f_{r8}$ 33.549kN |
| $u_9$ 0.008717rad | $f_{r9}$ 0 |
| $u_{10}$ -3.183756cm | $f_{r10}$ -60kN |
| $u_{11}$ -4.183756cm | $f_{r11}$ 60kN |
| $u_{12}$ -6.299199*10^-14rad | $f_{r12}$ 1.177*10^4kNcm |

5. Conclusion
The calculation results of displacements using the proposed modified penalty method matched with the calculation results using penalty function method and LaGrange multiplier method.

As penalty function method, proposed method is lacked in sensitivity with respect to whether constraints are linearly dependent. This is one remarkable advantage in comparison with LaGrange multiplier method.
Using the proposed method, the selecting regulator parameters is required not so much number of iterations and the solution is quickly convergence. It takes important advantage of penalty function method.

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