A numerical algorithm for solving a two-layered composite beam subjected to vibrational loads

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Abstract. In this work, double-layer beams are being researched using Rzhanitsyn’s theory of composite rods. One of the fundamental propositions of this theory is that multiple layers of such bars are connected by absolutely rigid cross links and elastically-pliable shear links. The concentrated forces and partially-constant distributed loads which are changing in time according to the harmonious law are taken into account as the external influence. The method of direct numerical integration of the equations of motion in time is used to solve the problem of the forced oscillations of the beam. The difference equations of the serial approximations method are used to approximate partial spatial coordinate derivatives. The results of the calculation of the beam under the vibrational influence got with different intervals of spatial and temporal coordinates have been given as the illustration of the suggested method.

Above-stated method might interest the specialists in the field of the analysis of building constructions because of the universality of Rzhanitsyn’s theory of built-up bars which finds application in calculation of multilayer wooden constructions, perforated metal beams, multilayer reinforced-concrete beams with different elasticity modulus, in modelling the crack formation processes, in calculation of multi-storey buildings’ frames.

1. Introduction

Rzhanitsyn’s theory of Composite Rods finds many cases to be functioned to calculate buildings, constructions and building structures for different types of actions. Analysis and calculation method for multi-story skeletons under the static loads was developed in [2]. The opportunity of the theory of Composite Rods to be functioned to calculate frame buildings for seismic actions was reviewed in [3]. Rzhanitsyn’s theory is used to calculate building structures of different construction materials, such as reinforced concrete [4-6], metal [7, 8], wood [9, 10]. Numerical method of the calculation of the multilayer composite rods under the forced oscillations was suggested in [11].

2. Calculation Methodology

The system of solution differential equations can take its dimensionless form as

\[
\frac{\partial^2 m}{\partial \psi^2} = -\left[ p(\psi, \tau) - \bar{\mu} \frac{\partial^2 w}{\partial \tau^2} - \bar{\chi} \frac{\partial w}{\partial \tau} \right] \tag{1}
\]

\[
\frac{\partial^2 w}{\partial \psi^2} = -a_i \left( m - s \right) \tag{2}
\]
\[ \frac{\partial^2 w}{\partial \psi^2} = -(a_2 m - a_3 s) \]  

where  
\[ \psi = \frac{x}{l}; \quad w = \frac{y}{l}; \quad m = \frac{Ml}{EI}; \]  
\[ s = \frac{Tlc}{EI}; \quad \mu = \frac{\mu l^4}{EI\tau^2}; \quad \chi = \frac{\chi l^2}{EIT_0}; \]  
\[ \bar{\tau} = \frac{t}{T_0}; \quad a_1 = \frac{EI}{\sum EI}; \quad a_2 = \frac{l^2 c^2 \gamma}{\sum EI}; \]  
\[ a_3 = l^2 \zeta \gamma; \quad \gamma = \left( \frac{1}{E_1 F_1} + \frac{1}{E_2 F_2} + \frac{c^2}{\sum EI} \right); \]  
\[ p(\psi) = \frac{q(x)l^3}{EI}; \]

\( x \) – Coordinate, measured along the composite rod;  
\( l \) – Rod span;  
\( y \) – Deflection;  
\( M \) – Moment summation over rod’s section, out of a shear connectors;  
\( q(x,t) \) – Time-dependent load distributed according to an arbitrary function  
\( E, I \) – Fixed values for elasticity (young’s) modulus & sections’ moment of inertia, respectively  
\( F_1, F_2 \) – Cross sectional area for sections 1 & 2 respectively;  
\( T \) – Shear force affecting the rod’s connections, to which subjected the studied section;  
\( c \) – Step dimension in-between branches;  
\( \mu \) – Overall rod’s mass;  
\( \chi \) – Energy absorption coefficient;  
\( T_0 \) – time period of 1st oscillation;  
\( t \) – Coordinate, measured along the time axis;  
\( \bar{\tau} \) – Stiffness coefficient as per [1].

Solving the differential equations system (1) - (3) figuring out with the aid of difference equations of successive approximation method [12]. Considering the continuity of \( m, w, \dot{w} = \frac{\partial w}{\partial \tau}, \ddot{w} = \frac{\partial^2 w}{\partial \tau^2} \), difference equation can be approximated to equation (1):  
\[ m_{j-1} - 2m_j + m_{j+1} + h \cdot \Delta m_j = \frac{h^2}{12} \mu \left( \dot{m}_{j-1} + 10 \dot{m}_j + \dot{m}_{j+1} \right) - \frac{h^2}{12} \chi \left( \ddot{m}_{j-1} + 10 \ddot{m}_j + \ddot{m}_{j+1} \right) = \]  
\[ = -\frac{h^2}{12} \left( \Pi p_{j-1} + 10 \Pi p_j + \Pi p_{j+1} \right) + \frac{5}{12} h^2 \Delta p_j + \frac{h^3}{12} \Delta p_j, \]

where \( h \) – dimensionless step of spatial mesh, \( \Delta m_j = ^\wedge \Delta m_j = \Pi m_j - ^\Pi m_j; \; m' = \frac{dm}{dy} \); left superscriptions \( ^\Lambda \) и \( ^\Pi \) – means the functions value on left and right of the studied node. The same wise, other symbols holding similar superscriptions.

Getting to (1) velocity \( ^{(k)} \dot{w} \) & acceleration \( ^{(k)} \ddot{w} \) drifting nodes over the structure \( k \)-time period of a time layer \( ^{(k-1)} \dot{w}, \; ^{(k-1)} \ddot{w} \), for previous \( (k-1) \) layer, in details discussed at [12] parabolic spline for direct integration of that time period.
\((k)\ddot{w}_j = -\frac{2}{\tau^2}((k-1)\dot{w}_j) - \frac{2}{\tau^2}((k-1)w_j - (k)w_j);\)  

\[(5)\]

\((k)\ddot{w}_j = -((k-1)\dot{w}_j) - \frac{2}{\tau^2}((k-1)w_j - (k)w_j),\)

where \(\tau\) – step along the dimensionless time axis.

Proceeding equation (5) u equation (6) for nodes \((j-1)\; & \; (j+1),\) for equation (4). Down written for partially-constant uniformly distributed load case

\[(k)m_{j-1} - 2(k)m_j + (k)m_{j+1} + \Delta(k)m'_j = \frac{h^2}{12}(\frac{2}{\tau} \bar{m} + \bar{K})(((k-1)\dot{w}_{j-1} + 10((k-1)\dot{w}_j + (k-1)\dot{w}_{j+1}) +

+ \frac{h^2}{6\tau}(\tau + \bar{K})(((k-1)w_{j-1} + 10((k-1)w_j + (k-1)w_{j+1} =

\[= -\frac{h^2}{12} \left( P_{j-1} + 10\Lambda(k)p_j + \Lambda(k)p_{j+1} \right) + \frac{5}{12}h^2\Delta(k)p_j, \)

Now, approximating equation (2) & equation (3) for a continues \(w, s\) with their 1st derivatives:

\[(k)w_{j-1} - 2(k)w_j + (k)w_{j+1} =

\[= -\frac{h^2}{12}a_1((k)m_{j-1} + 10(k)m_j + (k)m_{j+1} - (k)s_{j-1} - 10(k)s_j - (k)s_{j+1}) + \frac{h^3}{12}a_1\Delta(k)m'_j; \)

\[= -\frac{h^2}{12} \left[ a_2((k)m_{j-1} + 10(k)m_j + (k)m_{j+1} - h\Delta(k)m'_j) - a_3((k)s_{j-1} + 10(k)s_j + (k)s_{j+1}) \right]. \)

Equations (7), (8) & (9) are solved for inner regular calculation node taking into account boundary conditions. [13] is a good material for detailed boundary conditions difference approximation.  

For \(k = 1\; &\; given\; initial\; conditions (0)w, (0)\dot{w}, \; from\; solving\; equations\; (7), \; (8)\; &\; (9)\; system, \; (1)m, \; (1)w \; \& \; (1)s\; are\; easy\; figured\; out.\; Afterwards,\; using\; equation\; (6)\; to\; calculate\; the\; value\; of \; (1)\dot{w}\; for\; every\; calculation\; node.\; Typically,\; such\; a\; procedure\; is\; repeated\; for \(k = 2, \; \ldots\; up\; till\; a\; certain\; required\; calculation\; time\; period.\)

3. Examples

Example 1.

As a matter of the first calculation example, a simply supported beam shown in (Figure 1) is considered.

Subjected to a mid-span dynamic point load frequent per time as per a harmony \(P(t) = P\sin(\theta t)\) where \(P, \theta\) are the load amplitude value and the frequency. The transition to the dimensionless amplitude value of the concentrated force is performed by \(\Delta m' = P\frac{L^2}{EI} \). In order to define the relation with which the load varies along the dimensionless time axis, \(\theta t = \theta T_0 = \frac{\theta}{\omega_0} 2\pi T\) to be calculated where \(\omega_0\) is the fundamental frequency of the natural oscillations. Let \(\theta = 0.8\omega_0\). Dissipative forces are not taken into the consideration \(\chi = 0\).
As other initial data, consider the beam’s span \( l = 6 \text{ m} \); cross-section of branches and connecting bars refer back to (Figure 1) \( 0.01\times 0.1 \text{ (m}^2) \); \( c = 0.3 \text{ m} \); step of the connecting bars along the axis of the beam \( B = 0.5 \text{ m} \); material of branches and connecting bars is steel, \( E = 2.1\times 10^8 \text{ kPa} \), density \( \rho = 7.85\times 10^3 \text{ kg/m}^3 \).

The shear seam stiffness is determined from [1] as

\[
\xi = \frac{24E}{Bc^2 \left( \frac{2c}{I_1} + \frac{B}{I_2} \right)},
\]

where \( I_1 \) – the connecting bars moment of inertia (vertical elements); \( I_2 \) – the identical branches moment of inertia (horizontal elements)

Considering a free shear by the end of the beam boundary conditions are \( m = w = s = 0 \). Initial conditions are \( (0) \dot{w}_j = (0) w_j = 0 \).

In the methodical purposes we will show the solution with maximal intervals of spatial and temporal coordinates. Further calculation will be made for one inner calculation node (with one breakdown of the beam along its length into two calculated sections), \( h = \frac{0.5l}{l} = \frac{1}{2} \), step along the dimensionless time axis \( \tau = \frac{1}{4} \).

Let the amplitude value of dimensionless load \( \Delta m' = 1 \). In our example this corresponds to \( P = \frac{EI}{l^2} \Delta m' = \frac{2.1\times 10^8 \cdot 8.33\times 10^{-7}}{36} = 4.86 \text{ kN} \).

Dimensionless load for above-mentioned inner calculation node \((j = 2)\) and for the time layer \( k = 1\), \( \bar{t} = \frac{1}{4} \): \( \Delta^{(ij)} m' = 1\sin \left( 0.8\cdot 2\pi \frac{1}{4} \right) = 0.951 \).

Coefficients from the equations (8) & (9) take following values: \( a_1 = 0.5 \), \( a_2 = 785.42 \), \( a_3 = 814.51 \).

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Figure 1. shows a simply supported beam subjected to a mid-span dynamic point load.
Proceeding the solving equations (7), (8) & (9) for node $j = 2$:

- Equation (7): $-2^{(i)} m_2 + \frac{1}{2} 0.951 + \frac{1}{6} 2^{2} 3.877 (-10^{(i)} w_2) = 0$;
- Equation (8): $-2^{(i)} w_2 = -\frac{1}{12} 2^{2} \left(10^{(i)} m_2 - 10^{(i)} s_2\right) + \frac{1}{12} 2^{2} 0.951$;
- Equation (9): $-2^{(i)} s_2 = -\frac{1}{12} 2^{2} \left[785.42 \left(10^{(i)} m_2 - \frac{1}{2} 0.951\right) - 814.51 \left(10^{(i)} s_2\right)\right]$.

Results of the calculation are: $^{(i)} m_2 = 0.2319$, $^{(i)} w_2 = 4.51 \times 10^{-4}$, $^{(i)} s_2 = 0.1757$.

Using the equations (6) get $^{(i)} \dot{w}_2 = -\frac{2}{0.25} \left(0 - 4.51 \times 10^{-4}\right) = 3.608 \times 10^{-3}$.

Then proceed to the next time layer $k = 2$, $\tau = \frac{2}{4}$.

![Graph](image_url)

**Figure 2.** shows graphs of the functions of dimensionless load ($p$) and dimensionless vertical deflection of a mid-span node ($w$)

Graphs of the functions of dimensionless load (values are increased by one order) and dimensionless vertical deflection of a mid-span node (values are increased by the order of four) are shown in (Figure 2).

Results are carried out for $h = \frac{1}{8}$ and $\tau = \frac{1}{16}$.

The values of dimensionless vertical deflection of a mid-span node at time $\tau = \frac{1}{4}$ for several different calculations are shown in Table 1. This value has been carried out on bases of meshes different in dimensions.
Table 1. Dimensionless deflection of the mid-span node (increased by the order of four)

| Calculated Solution No. | Mesh parameters          | Dimensionless deflection |
|------------------------|--------------------------|--------------------------|
| 1                      | $h = \frac{1}{4}; \tau = \frac{1}{8}$ | 5.10                     |
| 2                      | $h = \frac{1}{8}; \tau = \frac{1}{8}$ | 5.15                     |
| 3                      | $h = \frac{1}{8}; \tau = \frac{1}{16}$ | 4.93                     |

The technique procedure for determining forces in a composite rod was described in details in [14].

Next here we show the transition from dimensionless values of deflection to the dimensional ones.

For calculation node ($j = 2$) at time $\bar{t} = \frac{1}{4}$ vertical deflection is $y_{x=3} = w_2 \cdot l = 4.51 \cdot 10^{-4} \cdot 6 \cdot 10^3 = 2.7$ mm.

Example 2.
The same simply supported beam is considered as a matter of the second calculation example. It is shown in (Figure 3). But now it is subjected to a half-span dynamic distributed load

$\bar{P} = \bar{p} \sin \frac{\theta}{\omega} \cdot 2\pi \bar{t}$,

where $\bar{p}$ – amplitude value of the dimensionless distributed load.

Let $\bar{p} = 1$, $\theta = 1.5\omega_0$, $h = \frac{1}{2}$, $\tau = \frac{1}{4}$.

Figure 3. shows a simply supported beam subjected to a half-span dynamic

Distributed load has a finite discontinuity at the node ($j = 2$) $\Lambda \bar{p}_2 = 1$, $\Pi \bar{p}_2 = 0$,

$\Delta \bar{p}_2 = \Lambda \bar{p}_2 - \Pi \bar{p}_2 = 1$.

Dimensionless load for the time layer $k = 1$: $^{(1)}p = 1\sin \frac{3}{4} \pi = 0.707$.

Proceeding the solving equations (7), (8) & (9) for node $j = 2$:

equation (7) $-2^{(i)}m_2 + \frac{1}{6} \frac{1}{2^2} 3.877 \left(-10^{(i)}w_2 \right) = -\frac{1}{12} \frac{1}{2^2} 10 \cdot 0.707 + \frac{5}{12} \frac{1}{2^2} 0.707$. 
equation (8) \( -2^{(i)}w_2 = -\frac{1}{12} \left[ \frac{1}{2^2} \left( 10^{(i)}m_2 - 10^{(i)}s_2 \right) \right]; \)

equation (9) \( -2^{(i)}s_2 = -\frac{1}{12} \left[ \frac{1}{2^2} \left[ 785.42 \left( 10^{(i)}m_2 \right) - 814.51 \left( 10^{(i)}s_2 \right) \right] \right]. \)

Results of the calculation are: \( (i)m_2 = 4.284 \cdot 10^{-2}; \) \( (i)w_2 = 1.048 \cdot 10^{-4}; \) \( (i)s_2 = 4.08 \cdot 10^{-2}. \)

4. Summary

The demonstrated theory shows an effective numerical approach based on the difference equations of successive approximation method for calculating composite rods considering discontinuous load functions.

This technique extremely suits the calculation of structures that lead to the composite rod models subjected to dynamic loads.

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