DATA REDUCTION IN MARKOV MODEL USING EM ALGORITHM

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Abstract. This paper describes a data reduction technique in case of a markov chain of specified order. Instead of observing all the transitions in a markov chain we record only a few of them and treat the remaining part as missing. The decision about which transitions to be filtered is taken before the observation process starts. Based on the filtered chain we try to estimate the parameters of the markov model using EM algorithm. In the first half of the paper we characterize a class of filtering mechanism for which all the parameters remain identifiable. In the later half we explain methods of estimation and testing about the transition probabilities of the markov chain based on the filtered data. The methods are first developed assuming a simple markov model with each probability of transition positive, but then generalized for models with structural zeroes in the transition probability matrix. Further extension is also done for multiple markov chains. The performance of the developed method of estimation is studied using simulated data along with a real life data.

1. Introduction

Applications of statistical models often encounter with datasets which are too large to store or to gain meaningful interpretation. This motivates the need for a suitable data filtering mechanism which stores only a subset of the available information such that based on the observed data reasonable inferences can be drawn about the parameter. This paper describes such a filtering mechanism in case of discrete markov models. Discrete markov chains are the simplest dependent structure that one can think of and are very useful for modeling a wide range of scientific problems in nature. Some important applications include modeling of dry and wet spells (P. J. Avery and D. A. Henderson (1999)), deoxyribonucleic acid (DNA) sequences (P. J. Avery and D. A. Henderson (1999)), study of chronic diseases (B. A. Craig and A. A. Sendi (2002)).

Any stochastic process $X = \{X_1, X_2, \ldots, X_n\}$ having a finite set $S$ as its state space, is said to be a markov process of order $s$ if

$$P(X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_1 = a_1)$$

$$= P(X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_{n-s} = a_{n-s})$$

For notational convenience let us denote the state space as $S = \{1, 2, \ldots, k\}$. Further we assume that the markov process has stationary transition probabilities, which means

$$P(X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_{n-s} = a_{n-s}) = p_{a_{n-s}, \ldots, a_{n-1}; a_n}$$

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does not depend on \( n \). For \( s = 1 \) we have a simple markov chain with finite state space. Any markov chain of \( s^{th} \) order can be treated as a simple markov chain with suitable parameters. So in this paper we will develop the methods assuming a simple markov chain which will be equally applicable for any markov process of higher order. A markov chain can be completely described by the initial state and the set of transition probabilities. Here we shall consider the initial state of a markov chain to be known and try to make inferences about the transition probabilities based on the observed data. More specifically inferences regarding the transition probability matrix can help us to answer many specific questions regarding the markov process which we usually encounter.

There is an extensive literature available on the statistical inferences of finite markov chains based on complete data. Billingsley(4) gives a good account of the mathematical aspects of different techniques regarding inferences about the transition probabilities which includes Whittle’s formula, maximum likelihood and chi-square methods. Estimation of transition probabilities and testing goodness of fit from a single realisation of a markov chain has been studied by Bartlett(3). Goodman and Anderson(1) derived the estimates of the transition probabilities and their hypothesis when there are more than one realisation of a single markov chain. Their paper also described the asymptotic properties of the methods when the number of realisations increase. All these works assume the observed data to be one or more long, unbroken observations of the chain. In this paper we assume that there is a single realisation of the markov chain which is not completely observed. The observed broken chain which results from the filtering mechanism is therefore not markov.

Based on the filtering mechanism we will observe only certain transitions of a markov chain and treat the remaining part of the chain as missing. From the observed data we will estimate the transition probabilities using EM algorithm. EM algorithm is a standard tool for maximum likelihood estimation in incomplete data problems. Since the missingness in the data occurs due to the filtering process, the data are not missing at random (NMAR). Here the missing mechanism is nonignorable but known. Unlike the conventional uses of EM algorithm where missingness occurs naturally, here we introduce missingness deliberately to reduce the size of the observed data. The E step of the EM algorithm requires to find the conditional expectation of the missing data given the observed data. This is achieved by defining the all possible missing paths for a transition of any order and finding the probability of the same. The standard error of the EM estimate is obtained by the supplemented EM algorithm (SEM) (Meng and Rubin). Usually the standard error of the EM estimate is obtained by inverting the observed information matrix. In our case the observed likelihood cannot be obtained explicitly and hence we avoid the calculation of the observed information matrix. SEM is a technique to calculate asymptotic dispersion matrix of the EM estimate without inverting the observed information matrix.

Section 2 describes the setup of the problem. Section 3 deals with the identifiability issues of the parameter that arise due to filtering of data. In section 2 we assume that the transition probability matrix consists of all positive elements. This assumption is relaxed in section 3 where we allow some structural zeroes in the transition probability matrix. We describe the additional modification we need in the filtering mechanism due to such relaxation. Section 5 describes the methods of
estimation and testing the transition probabilities. In this section we also describe the estimation of standard errors of the estimates by the SEM algorithm. Section 6 describes the generalization of the above methods in case of multiple markov chains. In section 7 we demonstrate the methods developed using simulated data. A real life data analysis is demonstrated in section 8. Section 9 is the appendix which has the proofs of a major theorem of this paper.

2. Setup

Let \( X \) be a simple markov chain with finite state space \( S = \{1, 2, \ldots, k\} \) and transition probability matrix \( P = ((p_{ij}))_{k \times k} \). Let us first assume that \( 0 < p_{ij} < 1 \), \( \forall \, i, j \). We shall relax this assumption later and consider the case where we allow some \( p_{ij} \)'s to be zero. The transition probability matrix \( P \) satisfy the standard condition

\[
P1 = 1 \text{ i.e. } \sum_j p_{ij} = 1 \quad \forall \, i.
\]

Hence we have \( k^2 - k \) independent parameters. We define the vector of the parameters

\[
\theta = (p_{11}, p_{12}, \ldots, p_{1(k-1)}, p_{21}, p_{22}, \ldots, p_{2(k-1)}, \ldots, p_{k1}, p_{k2}, \ldots, p_{k(k-1)})
\]

\[
= (\theta_1, \theta_2, \ldots, \theta_d)
\]

where \( d = k^2 - k \) and the parameter space is

\[
\Theta = \{\theta : \sum_{j=1}^{k-1} p_{ij} < 1, \quad \text{for} \, i = 1, \ldots, k\}
\]

\[
= \{\theta : \sum_{j=1}^{k-1} \theta_j < 1, \quad \sum_{j=0}^{k-1} \theta_{(i-1)k+j} < 1, \quad \text{for} \, i = 2, \ldots, k\}.
\]

Now suppose we consider a single realization \( x \) of the chain and the number of transitions from state \( i \) to state \( j \) in this realization is \( n_{ij} \). We assume that the markov process is continued sufficiently long enough so that the realization \( x \) contains each transition at least once, that is, \( n_{ij} > 0 \) for all \( i \) and \( j \). The matrix of transition count is

\[
N = \begin{bmatrix}
n_{11} & n_{12} & \cdots & \cdots & n_{1k} \\
n_{21} & n_{22} & \cdots & \cdots & n_{2k} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
n_{k-1,1} & n_{k-1,2} & \cdots & \cdots & n_{k-1,k} \\
n_{k1} & n_{k2} & \cdots & \cdots & n_{kk}
\end{bmatrix}.
\]

In this paper we propose a data acquisition protocol which suggests that instead of observing the entire realization \( x \), we record only some of the transitions and treat the remaining part of the chain as missing. The decision about which transitions we record is described in the form of a filter matrix \( F = ((f_{ij}))_{1 \leq i, j \leq k} \) which contains 0 and 1 as elements. In particular we record the transition from state \( i \) to state \( j \) if \( f_{ij} = 1 \). If \( X \) is the complete chain then let \( \phi_F(X) \) denote the chain filtered using \( F \).
Example 1. Consider a three state markov chain $x$ as

$$1123123212331121331$$

Suppose we are given a filter matrix $F$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

Then the transitions we record in $\phi_F(x)$ are

$$1 \rightarrow 1$$

$$2 \rightarrow 2$$

$$3 \rightarrow 1$$

$$3 \rightarrow 2$$

Thus all the transitions in the filtered chain may be classified into one of the three categories:

- directly recorded ($f_{ij} = 1$)
- indirectly recorded ($f_{ij} = 0$ but the transition occurs in the filtered chain, e.g. $2 \rightarrow 3$ in Example 1)
- unobserved ($f_{ij} = 0$ and the transition does not appear in the filtered chain, e.g. $3 \rightarrow 3$ in Example 1)

3. Identifiability

In this section we shall discuss about the identifiability of the parameters based on the filtered chain. We note that our filtered chain no longer possesses the markov property. While applying the filtering mechanism, if we record only a very few transitions then all the parameters of the markov chain may not be identifiable. For example in a markov chain with state space $\{1, 2, \cdots, 10\}$ if we record only the transition $1 \rightarrow 1$ then some parameters, say $p_{55}$, are not identifiable. We need to study how much data we can throw away, so that the problem still remains identifiable. Thus our main aim, in this section, will be to identify a class of filter matrices so that data generated by any filter matrix of that class will retain the identifiability of the parameters. But first we define what is meant by identifiability of parameter on the basis of a random sample.

Definition 2. Let $X$ be a random sample from a distribution characterized by the parameter $\theta$ and $L(\theta, x)$ be the likelihood. Then the parameter $\theta$ is said to be identifiable on the basis of $X$ if for any two distinct values $\theta_0$ and $\theta_1$ in the parameter space

$$L(\theta_1, x) \neq L(\theta_2, x)$$
Suppose \( X \) is a random sample drawn from a population characterized by the parameter \( \theta \). Let \( Y = g(X) \) be function of \( X \). Given \( X \) we can always construct \( Y \) through \( g \). So if \( \theta \) is identifiable on the basis of \( Y \), we can identify \( \theta \) also from \( X \). On the contrary, if \( \theta \) is unidentifiable on the basis of \( X \), then it is also unidentifiable on the basis of \( Y \). This is because, if we assume \( \theta \) to be identifiable on the basis of \( Y \), then given \( X \), we can construct \( Y \) through \( g \) and then \( \theta \) can be identified from \( X \), which is a contradiction. Thus in general we have the following two results:

Claim 3. a) If \( \theta \) is identifiable on the basis of \( Y \), then \( \theta \) is also identifiable on the basis of \( X \).

b) If \( \theta \) is unidentifiable on the basis of \( X \), then \( \theta \) is also unidentifiable on the basis of \( Y \).

In the present situation to prove that the parameters are identifiable it is enough to consider a observed sample \( x \) such that \( \exists t \) such that \( P_{\theta}(\phi_F(x) = t) \neq P_{\theta}(\phi_F(x) = t) \) and prove that any two different values of the parameter \( \theta \) will yield different values of the observed likelihood \( L_{\text{obs}}(\theta, x) \).

Let \( F \) be the class of all \( k \times k \) filter matrices. We call a filter matrix \( F \in F \) identifiable if \( P \) is identifiable with respect to \( \phi_F(X) \). Let \( \mathcal{I}_\theta \subset F \) be the set of all \( k \times k \) filter matrices for which the parameter \( \theta \) is identifiable. Then \( \mathcal{I} = \cap \mathcal{I}_\theta \) is the set of identifiable filter matrices.

With this notation, the general fact stated in claim 3 is also applicable for the data generated by the filter matrices.

**Lemma 4.** For \( H, F \in F \), let \( \phi_H = g \circ \phi_M \) for some function \( g(\cdot) \). Then \( H \in \mathcal{I} \) implies \( M \in \mathcal{I} \) and \( M \in F - \mathcal{I} \) implies \( H \in F - \mathcal{I} \).

**Example 5.** Let

\[
H = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
M = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}
\]

\( M \) is same as \( H \) except that for \( M \) we directly observe one more transition \( 2 \rightarrow 3 \) than \( H \). Then \( \phi_H = g \circ \phi_M \) and hence if \( H \in \mathcal{I} \) then \( M \in \mathcal{I} \).

In general there are \( 2^{k^2} \) possible filter matrices in \( F \). Instead of searching over all possible filter matrices we shall start with some definite structures of filter matrices which are identifiable. The above discussion motivates us to extend the identifiability over a larger class of matrices. This requires some ordering of the filter matrices in terms of the data we store.

**Definition 6.** For filter matrix \( M = ((m_{ij})) \in F \) and \( H = ((h_{ij})) \in F \) we say \( M \succeq H \) if \( \forall i, j \quad h_{ij} = 1 \Rightarrow m_{ij} = 1 \) and \( M \preceq H \) if \( \forall i, j \quad h_{ij} = 0 \Rightarrow m_{ij} = 0 \).

**Lemma 7.** a) If \( H \in \mathcal{I} \) and \( M \succeq H \) then \( M \in \mathcal{I} \).

b) If \( H \in F - \mathcal{I} \) and \( M \preceq H \) then \( M \in F - \mathcal{I} \).

**Proof.** a) \( M \succeq H \) implies \( \phi_M = g(\phi_M) \) for some \( g(\cdot) \). Using Lemma 4 we get \( H \in \mathcal{I} \) implies \( M \in \mathcal{I} \).

b) \( M \preceq B \) implies \( \phi_M = g(\phi_B) \) for some \( g(\cdot) \). Using Lemma 4 we get \( H \in F - \mathcal{I} \) implies \( M \in F - \mathcal{I} \). \( \square \)
Thus if any filter matrix $M$ is identifiable, then all filter matrices which store
more data than $M$ are also identifiable. This fact is also true for any subclass of
filter matrices.

**Definition 8.** If $\mathcal{D} \subseteq \mathcal{F}$, then the closure of $\mathcal{D}$ is defined as
\[
\overline{\mathcal{D}} = \{ F \in \mathcal{F} : F \succeq D \text{ for some } D \in \mathcal{D} \}.
\]

**Lemma 9.** If $\mathcal{D} \subseteq \mathcal{I}$ then $\overline{\mathcal{D}} \subseteq \mathcal{I}$.

**Proof.** Let $M \in \overline{\mathcal{D}}$. Then $M \succeq D$ for some $D \in \mathcal{D}$.
Since $\mathcal{D} \subseteq \mathcal{I}$, we get $D \in \mathcal{I}$. Then Lemma 7 implies $M \in \mathcal{I}$.
This implies $\overline{\mathcal{D}} \subseteq \mathcal{I}$. \qed

Thus given any class of identifiable filter matrices $\mathcal{D}$ we can always extend it to
a larger subclass of identifiable filter matrices.

Our observed chain is a sequence of states and blanks ($\pi$). Given any observed
chain we want to find the condition under which the conditional probability of
a given segment of the observed chain given the initial state in the segment is
identifiable.

**Definition 10.** For any finite sequence $\pi$ of states or blanks ($\pi$) we define
\[
S_\pi = \text{set of all filtered segments where } \pi \text{ occurs in consecutive positions}.
\]
We note that if $\pi_1 \subseteq \pi_2$ then $S_{\pi_1} \supseteq S_{\pi_2}$.

**Lemma 11.** For any filter matrix $F$, if $P(S_\pi) > 0$ then $p_\pi$ is identifiable where
$\pi$ is a sequence of states or blanks which starts and ends with states and $p_\pi$ is the
conditional probability of the sequence $\pi$ given the initial state in $\pi$.

**Proof.** Let $\pi$ start with the state $\alpha$ and end with the state $\beta$.
Let
\[
A = \text{subchains that ends with } \alpha.
\]
\[
B = \text{subchains that ends with the sequence } \pi.
\]
Then $B \subseteq A$. Also $P(S_\pi) > 0$ implies $P(B) > 0$ which implies $P(A) > 0$.
Also from Markov property we get that $P(B|A) = p_\pi$. Thus if $p_\pi$ changes $P(B|A)$
changes. Since the conditional probability of a class of subchains changes, the
joint distribution of the entire filtered chain must also change. Hence two distinct
values of $p_\pi$ will give two distinct values of the observed likelihood. Thus $p_\pi$
is identifiable. \qed

**Corollary 12.** For any filter matrix $F$ the parameter $p_{ij}$ is identifiable if $P(S_{ij}) > 0$.

As mentioned before we want to start with filter matrices of definite structures
which are identifiable and extend them to relatively larger classes. With this view
in mind, we define three classes of filter matrices each of which will be sufficient for
a filter matrix to be identifiable.

**Class 1:** We define $C_1 \subseteq \mathcal{F}$ which consists of all filter matrix $F = ((f_{ij}))$, $1 \leq i, j \leq k$
such that
a) $\exists \alpha$ such that $f_{\alpha j} = 0$, $j = 1, 2, ..., k$ i.e. the $\alpha^{th}$ row of $F$ is zero.
b) $\exists \beta$ such that $f_{i\beta} = 0$, $i = 1, 2, ..., k$ i.e. the $\beta^{th}$ column of $F$ is zero.
c) $f_{pj} = 1$ for exactly one $j$, $1 \leq j \leq k$, $p = 1, 2, ..., k$, $p \neq \alpha$ i.e. except $\alpha$th row every other row must have exactly one element 1.

d) $f_{iq} = 1$ for exactly one $i$, $1 \leq i \leq k$, $q = 1, 2, ..., k$, $q \neq \beta$ i.e. except $\beta$th column every other column must have exactly one element 1.

Class2: We define $\mathcal{C}_2 \subseteq \mathcal{F}$ which consists of all filter matrix $\mathbf{F} = ((f_{ij}))$, $1 \leq i, j \leq k$ such that

a) $\exists \alpha$ and $\beta$ such that $f_{i\alpha} = 0$, $i = 1, 2, ..., k$ and $f_{i\beta} = 0$, $i = 1, 2, ..., k$ i.e. the $\alpha$th and $\beta$th column of $\mathbf{F}$ is zero.

b) $f_{iq} = 1$ for at exactly one $i$, $1 \leq i \leq k$, $q = 1, 2, ..., k$, $q \neq \alpha, \beta$ i.e. except $\alpha$th and $\beta$th column every other column have exactly one element 1.

c) $f_{\alpha j} = f_{\beta j} = 1$, $1 \leq j \leq k$, $j \neq \alpha, \beta$, i.e. except $\alpha$th and $\beta$th column every other element of $\alpha$th and $\beta$th row is 1.

d) $f_{pj} = 1$ for exactly one $j$, $1 \leq j \leq k$, $p = 1, 2, ..., k$, $p \neq \alpha, \beta$, i.e. except $\alpha$th and $\beta$th row every other row have exactly one element 1.

Class3: We define $\mathcal{C}_3 \subseteq \mathcal{F}$ which consists of all filter matrix $\mathbf{F} = ((f_{ij}))$, $1 \leq i, j \leq k$ such that

a) $\exists \alpha$ and $\beta$ such that $f_{i\alpha} = 0$, $i = 1, 2, ..., k$ and $f_{\beta i} = 0$, $i = 1, 2, ..., k$ i.e. the $\alpha$th and the $\beta$th row of $\mathbf{F}$ is zero.

b) $f_{g1} = 1$ for exactly one $i$, $1 \leq i \leq k$, $q = 1, 2, ..., k$, $q \neq \alpha, \beta$ i.e. except $\alpha$th and $\beta$th row every other row have exactly one element 1.

c) $f_{\alpha j} = f_{\beta j} = 1$, $1 \leq j \leq k$, $j \neq \alpha, \beta$, i.e. except $\alpha$th and $\beta$th row every other element of $\alpha$th and $\beta$th column is 1.

d) $f_{jp} = 1$ for exactly one $j$, $1 \leq j \leq k$, $p = 1, 2, ..., k$, $p \neq \alpha, \beta$, i.e. except $\alpha$th and $\beta$th column every other column have exactly one element 1.

The following theorem and its corollary provide sufficient conditions for filter matrices to be identifiable. Any filter matrix which belong to at least one of the three classes is identifiable. The proof of the theorem is given in the appendix.

**Theorem 13.** Consider a simple markov chain $\mathbf{X}$ on finite state space $\{1, 2, ..., k\}$ and transition probabilities $p_{ij}$ where $0 < p_{ij} < 1, i, j = 1, 2, ... k$. Suppose $\mathbf{F}$ be any filter matrix belonging to the class $\mathcal{C}_i = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Then $\mathbf{F}$ must also belong to the class $\mathcal{I}$.

The following corollary to the above theorem is an immediate application of Lemma 9:

**Corollary 14.** $\mathcal{C}_3 \subseteq \mathcal{I}$.

Thus if we start with a definite structure of matrices in $\mathcal{C}_1$ or $\mathcal{C}_2$ or $\mathcal{C}_3$ we get a relatively larger class $\overline{\mathcal{C}}_i$ of identifiable filter matrices. For the rest of the paper we shall be working with filter matrices within this class. We shall find that any filter matrix in this class will provide considerable reduction in data.

4. **Structural zeroes in Transition probability matrix**

In the previous section while obtaining the sufficient conditions for identifiability we assumed $0 < p_{ij} < 1, \forall i, j$. This was a crucial assumption in developing the theory for the sufficient conditions. However in many practical applications this assumption stands out to be too restrictive. For example, while modeling a disease
status the probability of an individual entering from one state to another may be zero (in case of chronic illness, the condition of an individual usually deteriorates). Also the case of structural zeroes in the transition probability matrix will occur later in this paper while dealing with multiple markov chains. In this section we generalize the sufficient conditions for a filter matrix to be identifiable even when some $p_{ij}$'s are zero.

We note that all zeroes (if any) in the transition probability model are structural zeroes, that is, we know the position of the zeroes even before the collection of the data. Also for any $i$, $p_{ij}$ must be positive for at least one $j$ since all the row sums of the transition probability matrix is 1. We further assume

\[(A1) \text{ for any } j, p_{ij} \text{ must be positive for at least one } i.\]

This is a reasonable assumption to make because if such a state $j$ exists we shall ignore that state from our analysis.

As before we have the classes of filter matrix $C_1$, $C_2$ and $C_3$. Further let us define an additional class of filter matrix $R \subseteq F$ as

\[R = \{F \in F : \text{For any } i \in \{1, 2, ..., n\}, f_{ij} = 1 \text{ for at least one } j \in Z\}\]

where $Z = \{j : p_{ij} > 0\}$. This restriction means for every row of a filter matrix, we should observe at least one probable transition. The restriction on the filter matrices is quite justified and does not in any way reduces the applicability of filtering mechanism. The following theorem is a generalization of Theorem 13 in the case where we allow some $p_{ij}$ to be zero.

**Theorem 15.** Consider a simple markov chain $X$ on finite state space $\{1, 2, ..., k\}$ and transition probabilities $p_{ij}$ where $0 \leq p_{ij} \leq 1, i, j = 1, 2, ..., k$. Let $F$ be any filter matrix belonging to the class $S$ where $S = C \cap R$. Then under the assumption $A1$, $F$ must also belong to the class $I$.

The proof of the above theorem is similar to the proof of Theorem 13 because under the assumption $A1$, and for filter matrices within the class $S$, we have $P(S_\pi) > 0$ for all choices of sequences $\pi$, that we require in Theorem 13. Finally application of lemma 9 gives the required result.

### 5. Estimation and Testing

As mentioned earlier, a markov process can be completely characterized by specifying the transition probability matrix. This section deals with drawing inferences regarding the parameters. Instead of recording the entire markov chain $x$, we apply a given filter matrix $F \in F$ to record $\phi_{F}(x)$. $F$ is fixed and does not in any way depend on the data $x$. The choice of $F$ may depend on the availability of the samples, storage facilities or past experience subject to the constraint of identifiability. Based on $\phi_{F}(x)$ we shall find estimates of the transition probabilities and compute the standard error of the estimates. Our main tool for estimation will be EM algorithm. For the computation of the standard error we shall use Supplemented EM algorithm(SEM). The latter half of the section deals with testing of hypothesis regarding the parameters.
5.1. **Estimation of parameters:** In the present situation the complete data is $x$ which is unobserved and the observed data is $\phi_F(x)$. As a natural tool of missing data analysis we will apply EM algorithm for the estimation of the parameter $\theta$. Each iteration of EM algorithm consists of a E step (expectation step) and an M step (maximization step). In the E step of the algorithm we need to find the conditional expectation of the complete data log-likelihood given the observed data and the current iterated value of the parameters. In our case this requires to find the conditional expectation with respect to the conditional distribution of $x$ given $\phi_F(x)$ and the current iterated value $\theta^{(t)}$ of the parameter. The complete data log likelihood is
\[
\ell_{\text{com}}(\theta) \propto \sum_{\alpha, \beta=1}^k n_{\alpha\beta} \log p_{\alpha\beta}.
\]
Since $\ell_{\text{com}}(\theta)$ is linear in $n_{\alpha\beta}$, we need to compute
\[
E(n_{\alpha\beta}|\phi_F(x), \theta^{(t)}).
\]
This conditional distribution cannot be computed directly as the conditional distribution of $x$ given $\phi_F(x)$ cannot be found out explicitly. We shall express $n_{\alpha\beta}$ as a sum of certain indicator variables to evaluate this conditional expectation, the computation of which will be shown in subsection 5.1.2. We will show that this require us to find the following conditional probability:
\[
P(\text{the chain moves from state } \alpha \text{ to state } \beta|\phi_F(x), \theta^{(t)}).
\]
Since the observed chain has runs of missing states, the calculation of the above probability will require us to find the probability of a transition from one state to another in any number of steps such that all the intermediate steps are missing. If the complete chain is available, then the probability of a transition from $a$ to $b$ in $\nu$ steps is the $(a, b)^{th}$ element of $P^\nu$. However we need to find the probability of such transition through some specific ways.

5.1.1. **Defining possible missing paths for a transition:** Consider two states $a$ and $b$. Suppose we are interested in transition from $a$ to $b$ in $\nu$ steps. Each possible way of transition from $a$ to $b$ in $\nu$ steps is called a path of order $\nu$. We call a path of order 1 as edge. Thus a any given path consists of one or more edges. Clearly the transition from $a$ to $b$ in $\nu$ steps can occur through one or more paths. We classify these paths in two categories based on the given filter matrix:
- **observed path**($\mathcal{O}$): whose all edges are observed.
- **unobserved path**($\mathcal{U}$): whose all edges are unobserved.

Clearly the two sets $\mathcal{O}$ and $\mathcal{U}$ are not mutually exhaustive, that is, we cannot classify all paths into any one of these categories.

**Example 16.** Consider a two state markov chain and two filter matrices $F_1$ and $F_2$ such that
\[
F_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
Suppose we consider the *transition from state 1 to state 1 in two steps*. The possible paths are:
\[
w_1 : 1 \rightarrow 1 \rightarrow 1 \quad w_2 : 1 \rightarrow 2 \rightarrow 1
\]
For filter matrix $F_1$, path $w_1 \in O$, i.e. path $w_1$ is observed whereas $U$ is empty, i.e. no paths are unobserved. For filter matrix $F_2$, path $w_1 \in O$ and $w_2 \in U$. If we consider the transition from state 2 to state 2 in two steps, the possible paths are:

$$w_1 : 2 \rightarrow 2 \rightarrow 2 \hspace{1cm} w_2 : 2 \rightarrow 1 \rightarrow 2$$

For filter matrix $F_1$, path $w_1 \in O$, and $U$ is empty whereas for filter matrix $F_2$, path $w_1 \in O$ and $w_2 \in U$.

Now consider the transition probability matrix $P$ of the markov chain. We construct two matrices $P^{[0]} = ([p_{ij}^{[0]}])$ and $P^{[1]} = ([p_{ij}^{[1]}])$ as

$$p_{ij}^{[0]} = \begin{cases} 0 & \text{if } f_{ij} = 1 \\ p_{ij} & \text{if } f_{ij} = 0 \end{cases}$$

and

$$p_{ij}^{[1]} = \begin{cases} 0 & \text{if } f_{ij} = 0 \\ p_{ij} & \text{if } f_{ij} = 1 \end{cases}$$

Then the $(i,j)^{th}$ element of $(P^{[0]})^\nu$ gives the probability of going from state $i$ to state $j$ in $\nu$ steps through unobserved path(s). Also the $(i,j)^{th}$ element of $(P^{[1]})^\nu$ gives the probability of going from state $i$ to state $j$ in $\nu$ steps through observed path(s).

**Example 17.** Returning to the previous example we see that for the filter matrix $F_1$,

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Then

$$(P^{[0]})^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \hspace{1cm} (P^{[1]})^2 = \begin{bmatrix} p_{11}^2 & 0 \\ p_{21}p_{11} + p_{22}p_{21} & p_{22}^2 \end{bmatrix}$$

Thus for filter matrix $F_1$, probability of going from any state $i$ to any state $j$ through the unobserved paths in 2 steps is zero. Also

$$(P^{[0]})^\nu = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \hspace{1cm} \text{for any } \nu$$

which means that the probability of going from any state $i$ to any state $j$ through the unobserved paths in any steps is zero.

Similarly for filter matrix $F_2$,

$$P^{[0]} = \begin{bmatrix} 0 & p_{12} \\ p_{21} & 0 \end{bmatrix}$$

Thus for $F_2$, the probability of going from state 1 to state 1 in 2 steps through the unobserved paths is $p_{12}p_{21}$ and the probability of going from state 2 to state 2 in 2 steps through the unobserved paths is $p_{21}p_{12}$.

Thus given a filter matrix, the probability of going from a state $a$ to a state $b$ in $\nu$ steps through the unobserved paths is the $(a,b)^{th}$ element of $(P^{[0]})^\nu$ which is $p_{ab}^{(\nu)0}$.
5.1.2. **Estimation by EM Algorithm:** For the $i^{th}$ transition, let,

$$Y_{1i} = \text{state from where the transition occurs}$$

$$Y_{2i} = \text{state to where the transition occurs}$$

Thus $Y_{1i}$ and $Y_{2i}$ are two discrete random variables taking values in the state space \{1, 2, \ldots, k\} for all $i$. Let us express the total number of transitions $n_{\alpha\beta}$ from the state $\alpha$ to the state $\beta$ as

$$n_{\alpha\beta} = \sum_{i=1}^{n} I(Y_{1i} = \alpha, Y_{2i} = \beta)$$

where

$$I(Y_{1i} = \alpha, Y_{2i} = \beta) = \begin{cases} 1 & \text{if } Y_{1i} = \alpha, Y_{2i} = \beta \\ 0 & \text{otherwise.} \end{cases}$$

The complete data likelihood then can be written as

$$L_{\text{com}}(p) \propto \prod_{i=1}^{n} f(y_{1i}, y_{2i}|p) = \text{constant} \times \prod_{\alpha, \beta=1}^{k} n_{\alpha\beta}$$

where

$$p_{\alpha k} = 1 - \sum_{j=1}^{k-1} p_{\alpha j} \forall \alpha = 1(1)n.$$  

After $t$ iterations in the EM algorithm we write the E-step and the M-step as:

**E-step:**

Let $P(t) = ((p_{\alpha\beta}(t)))$ be the value of the transition probability matrix after $t$ iterations. The corresponding value of the parameter $\theta$ is $\theta^{(t)}$. The other matrices we construct take the values $P^{0}(t)$ and $(P^{0})^{\nu} = ((p_{\alpha\beta}^{(0)}))$. Then we compute the expected complete data log-likelihood with respect to the conditional distribution of $x|\phi_F(x), \theta^{(t)}$. The complete data log-likelihood is given by

$$\ell_{\text{com}}(\theta) = \text{constant} + \sum_{\alpha, \beta=1}^{k} \left\{ \log p_{\alpha\beta} \times n_{\alpha\beta} \right\}.$$  

We then compute

$$Q(\theta) = E(\ell_{\text{com}}(\theta)|\phi_F(x), \theta^{(t)}).$$

Since $\ell_{\text{com}}(\theta)$ is linear in $n_{\alpha\beta}$, we need to compute

$$E\left(n_{\alpha\beta}|\phi_F(x), \theta^{(t)}\right)$$

$$= \sum_{i=1}^{n} E\left(I(Y_{1i} = \alpha, Y_{2i} = \beta)|\phi_F(x), \theta^{(t)}\right)$$

$$= \sum_{i=1}^{n} P\left(Y_{1i} = \alpha, Y_{2i} = \beta|\phi_F(x), \theta^{(t)}\right).$$

Let us denote $P\left(Y_{1i} = \alpha, Y_{2i} = \beta|\phi_F(x), \theta^{(t)}\right) = P_{\alpha\beta}$. Then for each $i$, $P_{\alpha\beta}$ takes one of the three forms $P^{i(1)}_{\alpha\beta}$, $P^{i(2)}_{\alpha\beta}$ or $P^{i(3)}_{\alpha\beta}$ as follows:
• Case I ($Y_{1i}$ observed): Suppose we have a missing chain of length $\nu - 1$ with the next observed state $b$. Then
\[
P_{i\alpha\beta}^{i(1)} = \begin{cases} 
p_{\alpha\beta} \times p_{\nu-1}^{(1)} & \text{if } Y_{1i} = \alpha \\
0 & \text{if } Y_{1i} \neq \alpha 
\end{cases}
=: P_{i\alpha\beta}^{i(1)}, \text{ say.}
\]

• Case II ($Y_{2i}$ observed): Suppose we have a missing chain of length $\nu - 1$ with the previous observed state $a$. Then
\[
P_{i\alpha\beta}^{i(2)} = \begin{cases} 
p_{a\alpha} \times p_{\nu-1}^{(2)} & \text{if } Y_{2i} = \beta \\
0 & \text{if } Y_{2i} \neq \beta 
\end{cases}
=: P_{i\alpha\beta}^{i(2)}, \text{ say.}
\]

• Case III (Both are not observed): Suppose we have a missing chain of length $\nu - 1$ with the previous observed state $a$ and the next observed state $b$. Then
\[
P_{i\alpha\beta}^{i(3)} = \frac{p_{a\alpha}^{(m)} p_{\nu-1}^{(3)} p_{\beta b}}{\sum_{b} p_{\nu-1}^{(3)} p_{\beta b}} =: P_{i\alpha\beta}^{i(3)}, \text{ say.}
\]

where $m + n = \nu - 1$ and $a = Y_{1(i-m+1)}$ and $b = Y_{2(i+n)}$. If there is no such next observed state (that is, the observed chain ends) then
\[
P_{i\alpha\beta}^{i(3)} = \frac{p_{a\alpha}^{(m)} p_{\nu-1}^{(3)} \sum_{b} p_{\beta b}^{(n)}}{\sum_{b} p_{\nu-1}^{(3)} p_{\beta b}} =: P_{i\alpha\beta}^{i(3)}, \text{ say.}
\]

M-step:
We try to maximize $Q(\theta)$ with respect to $\theta$. Setting $\frac{\partial}{\partial \theta_j} Q(\theta) = 0$ for each $j = 1(1)d$ we get
\[
\theta^{(t+1)} = (\theta_1^{(t+1)}, \theta_2^{(t+1)}, \ldots, \theta_d^{(t+1)})
\]
where
\[
\theta_j^{(t+1)} = \frac{n}{\sum_{l=1}^{k} \sum_{\beta=1}^{n} P_{l\beta}^{1_j}} \text{ for any } j = 1, 2, \ldots, (k-1)
\]
and
\[
\theta_{(i-1)k+j}^{(t+1)} = \frac{n}{\sum_{l=1}^{k} \sum_{\beta=1}^{n} P_{l\beta}^{i_{(i-1)k+j}}} \text{ for any } j = 0, 1, \ldots, (k-1) \text{ and } i = 2, 3, \ldots, k.
\]

5.2. Estimation of Standard Errors: Since EM estimate of the parameters are the maximum likelihood estimate of the observed likelihood, the large sample covariance matrix can be obtained by inverting the observed information matrix. But in our problem the observed likelihood is not known explicitly. An alternative way is using Supplemented EM Algorithm (SEM) which allows us to find the large sample covariance matrix without inverting the estimate of the observed information matrix. SEM algorithm is a procedure of obtaining a numerically stable estimate of
the covariance matrix of the estimated parameters using only the code for the steps in EM algorithm, code for computing the large sample complete data covariance matrix and standard matrix operations.

5.2.1. **Supplemented EM Algorithm**: Since each step of the EM algorithm produces a fresh estimate of the parameter from the previous estimates, EM algorithm can be considered as a mapping \( M \) on the parameter space. The derivative of the EM mapping, which we call \( M(1) \), can be expressed in the form

\[
M(1) = i_{\text{mix}}^{-1} i_{\text{com}}^{-1} = I - i_{\text{obs}}^{-1}.
\]

The above equation implies

\[
i_{\text{obs}}^{-1} = i_{\text{com}}^{-1} (I - M(1))^{-1}
\]

which in turn implies

\[
V_{\text{obs}} = V_{\text{com}} (I - M(1))^{-1}.
\]

Now we note that

\[
V_{\text{obs}} = V_{\text{com}} (I + M(1) - M(1))(I - M(1))^{-1} = V_{\text{com}} + \Delta V
\]

where \( \Delta V = V_{\text{com}} M(1) (I - M(1))^{-1} \) is the increment in variance due to missingness.

5.2.2. **Calculation of** \( V_{\text{com}} \): The complete data log-likelihood is

\[
\ell_{\text{com}}(\theta) = \text{constant} + \sum_{i=1}^{n} n_{ij} \log p_{ij} \quad \text{where } p_{ik} = 1 - \sum_{j=1}^{k-1} p_{ij} \forall i.
\]

\[
\therefore \frac{\partial}{\partial p_{ij}} \ell_{\text{com}} = \frac{n_{ij}}{p_{ij}} - \frac{n_{ik}}{p_{ik}}
\]

Thus the gradient vector is

\[
S = \begin{bmatrix}
\frac{n_{11}}{p_{11}} & \frac{n_{1k}}{p_{1k}} \\
\vdots & \vdots \\
\frac{n_{k(k-1)}}{p_{k(k-1)}} - \frac{n_{kk}}{p_{kk}} & \frac{n_{kk}}{p_{kk}}
\end{bmatrix}
\]

Now for \( i \neq i' \)

\[
\frac{\partial^2}{\partial p_{ij} \partial p_{i'j'}} \ell_{\text{com}} = \begin{cases}
\frac{n_{ij}}{2 p_{ij}^2} & \text{if } j \neq j' \\
\frac{1}{2} \left( \frac{n_{ik}}{p_{ik}} - \frac{n_{ij}}{p_{ij}} \right) & \text{if } j = j'
\end{cases}
\]

Let \( B \) be the matrix of the negatives of the second order derivatives. Then \( B \) is a matrix of order \( k^2 - k \) such that \( B = \text{blockdiagonal}(B_1, B_2, \cdots B_k) \) where \( B_i = ((b_{jj'}))_{k-1} \) where \( b_{jj'} = -\frac{\partial^2}{\partial p_{ij} \partial p_{i'j'}} \ell_{\text{com}} \).

Then the fisher information matrix of the complete data is

\[
i_{\text{com}} = E(B | \theta, \text{data}) = \text{blockdiagonal} \left( E(B_1), E(B_2), \cdots, E(B_k) \right)
\]

where \( E(B_i) = \langle E(b_{jj'}(\theta, \text{data})) \rangle \). Thus the variance-covariance matrix of the complete data is \( V_{\text{com}} = i_{\text{com}}^{-1} \).
Algorithm 1 SEM Algorithm

We take as input $\hat{\theta}$ and $\theta^{(t)}$.

a) Run the usual E step and M steps to get $\theta^{(t+1)}$;
b) Fix $i = 1$. Calculate

$$\theta^{(t)}(i) = (\hat{\theta}_1, \cdots, \hat{\theta}_{i-1}, \hat{\theta}_i^{(t)}, \hat{\theta}_{i+1}, \cdots, \hat{\theta}_d)$$

which is $\hat{\theta}$ except the $i^{th}$ component which equals $\theta_i^{(t)}$.
c) Treating $\theta^{(t)}(i)$ as the current estimate of $\theta$, run one iteration of EM to obtain $\hat{\theta}^{(t+1)}(i)$.
d) Obtain the ratio

$$r_{ij}^{(t)} = \frac{\hat{\theta}_j^{(t+1)}(i) - \hat{\theta}_j}{\hat{\theta}_i^{(t)} - \hat{\theta}_i} \quad \text{for } j = 1, 2, \cdots, d.$$  
e) Repeat steps 2 to 4 for $i = 1, 2, \cdots, d$.

We get as output $\theta^{(t+1)}$ and $\{r_{ij}^{(t)} : i, j = 1, 2, \cdots, d\}$. $M_{(1)}$ is the limiting matrix $\{r_{ij}\}$ as $t \to \infty$.

5.2.3. **Computing $M_{(1)}$ by numerical differentiation:** For our problem the mapping $M = M(\theta_1, \theta_2, \cdots, \theta_d) : \Theta \rightarrow \Theta$ is not known explicitly. The derivative of $M$ at $\hat{\theta}$ is calculated numerically from the output of the forced EM steps. $M_{(1)}$ is the matrix with the $(i,j)^{th}$ element as $\frac{\Delta M_i}{\Delta \theta_j} = \text{change in the } j^{th} \text{ component of } M \text{ due to the change in the } i^{th} \text{ element of } \hat{\theta}$. For this we start with the EM estimate $\hat{\theta}$ and change its $i^{th}$ element $\hat{\theta}_i$ by $\theta_i^{(t)}$. We call this resultant estimate by $\theta^{(t)}(i)$ and run one EM iteration on it to get $\hat{\theta}^{(t+1)}(i)$. Then $\Delta M_j = \hat{\theta}_j^{(t+1)}(i) - \hat{\theta}_j$ and $\Delta \hat{\theta}_i = \theta_i^{(t)} - \hat{\theta}_i$ and so we compute the ratio $r_{ij} = \frac{\Delta M_i}{\Delta \theta_j}$. Thus we run a sequence of SEM iterations, where the $(t+1)^{th}$ iteration is defined as follows:

A difficulty in running the SEM iterations is that while changing the $i^{th}$ element $\hat{\theta}_i$ by $\theta_i^{(t)}$ the resultant estimate $\theta^{(t)}(i)$ may not belong to the parameter space $\Theta$ because the sum of the corresponding row probabilities $\sum_{j=1}^{k-1} p_{ij}$ may be more than 1. Thus theoretically the mapping $M$ may not be defined in such cases. Then we replace $\theta_i^{(t)}$ by $\theta_i^{(t)} - \epsilon$, ($\epsilon > 0$) so that the corresponding sum of probability is less than 1.

5.2.4. **Implementational Issues:** While implementing the SEM algorithm it is always safe to start with the initial values of the original EM algorithm for numerical accuracy. But this may result in too many unnecessary iterations because the initial choice may be too far from the MLE. Hence Meng and Rubin suggested to take the initial choice in SEM as a suitable iterate of the EM algorithm or two complete data standard deviations from the MLE. Computation of $M_{(1)}$ being numerical differentiation is less accurate than evaluating the function $M$ itself. Hence the stopping criterion should be less stringent for SEM algorithm as compared to the original EM algorithm. Meng and Rubin suggested to use square root of the stopping criterion of the original EM as the stopping criterion for SEM.
The observed variance covariance matrix obtained by SEM algorithm should be theoretically a real symmetric positive definite matrix. This provide a diagnostics for programming errors and numerical precision. The numerical symmetry of the final matrix increases with more stringent criterion in the algorithm.

5.3. Testing of Hypothesis. The large sample inferences on the EM estimate can be drawn using the asymptotic distribution

$$\hat{\theta} \sim N(\theta, V_{\text{obs}})$$

Since SEM algorithm helps us to numerically estimate $V_{\text{obs}}$, we can use the above distribution for testing of the parameters and finding confidence intervals.

5.3.1. Testing the transition probability matrix. Suppose we wish to test the hypothesis $H_0 : P = P_0$. Since only $k(k-1)$ parameters of the transition probability matrix are independent, the above hypothesis is equivalent to $H_0 : \theta = \theta_0$. Now

$$(\hat{\theta} - \theta)^\prime V^{-1}_{\text{obs}} (\hat{\theta} - \theta) \sim \chi^2_{k^2}$$

which implies the test statistic for testing $H_0$ is $\chi^2 = (\hat{\theta} - \theta_0)^\prime V^{-1}_{\text{obs}} (\hat{\theta} - \theta_0)$ which has $\chi^2_{k^2}$ distribution under $H_0$. Thus the critical region for testing $H_0$ is $\{x : \chi^2 > \chi^2_{k^2, \alpha}\}$

5.3.2. Test of hypotheses about specific probabilities and confidence regions. First we consider testing the hypothesis that certain transition probabilities $p_{ij}$ have specified values $p^0_{ij}$. Under the null hypothesis $H_{0i} : \theta_i = \theta^0_i$, the statistic $\tau_i = \frac{\hat{\theta}_i - \theta^0_i}{\sqrt{s_{ii}}} \sim N(0, 1)$ distribution. Thus the critical region for testing $H_{0i}$ is $\{x : |\tau_i| > z_{\alpha/2}\}$. The 100(1 - $\alpha$)% confidence interval for $\theta_i$ is $(\hat{\theta}_i - \sqrt{s_{ii}}z_{\alpha/2}, \hat{\theta}_i + \sqrt{s_{ii}}z_{\alpha/2})$.

6. Multiple Markov chains

Let $\{X_n\}$ be a $s^{th}$ order markov chain. In the previous sections we have discussed the case where $s = 1$, that is, simple markov chains. If $s > 1$, then $\{X_n\}$ is called a multiple markov chain of order $s$ with transition probabilities

$$p_{a_1, \ldots, a_{s+1}} = P(X_n = a_{s+1} | X_{n-1} = a_s, X_{n-2} = a_{s-1}, \ldots, X_{n-s} = a_1).$$

Multiple markov chains of any order can be reduced to a simple markov chain by the following technique.

Suppose $\{X_n\}$ is called a markov chain of order $s$ with $k$ states. We define a new stochastic process $\{Y_n, n = 1, 2, \ldots\}$ where $Y_n = (X_n, X_{n+1}, \ldots, X_{n+s-1})$. Then $\{Y_n\}$ is a simple markov chain whose state space has $k^s$ different $s$-tuples. The transition probabilities of the new defined markov process are

$$P(a_1, a_2, \ldots, a_s)(b_1, b_2, \ldots, b_s) = \begin{cases} p_{a_1, a_2, \ldots, a_s, b_s} & \text{if } b_i = a_{i+1}, i = 1, 2, \ldots, s-1 \\ 0 & \text{otherwise.} \end{cases}$$

The number of positive entries in the $k^s \times k^s$ transition probability matrix is $k^{s+1}$. The parameters of interest are the probabilities $p_{a_1, \ldots, a_s, b_{s+1}}$ which requires estimation from the data.

In this situation we apply our filtering technique to the chain $\{y_n\}$. But now the transition probability matrix contains many zero elements and hence the additional restriction described section 4 needs to be applied on the filter matrices. We note that in this case the transition probability matrix satisfies the assumption made
in section 5. The technique of estimation of the parameters from the data $\phi_F(y)$ remains same as in the simple markov chain.

### 7. Simulation Study

For simulation we start with a markov chain with 3 states. A markov chain of length 1000 is being generated with the transition probability matrix

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.8 & 0.1 & 0.1 \\ 0.7 & 0.1 & 0.2 \end{bmatrix}.$$  

The filter matrix for generating the observed chain is

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Clearly the filter matrix used satisfy the sufficient condition for estimability. With this filter matrix we reduce 16% of the data, i.e., from the complete markov chain of length 1000 we do not observe 16% of the data. The precision we use in estimating the parameters through the steps of the EM algorithm is of the order $10^{-12}$ and the precision used in computing the standard error is of the order $10^{-6}$. With this precision the estimated transition probability matrix is

$$\hat{P} = \begin{bmatrix} 0.2411168 & 0.2850831 & 0.4738001 \\ 0.7395851 & 0.1429865 & 0.1174284 \\ 0.7648870 & 0.1067367 & 0.1283763 \end{bmatrix}.$$  

The observed variance covariance matrix $V_{obs}$ as computed by the SEM algorithm is

$$\begin{bmatrix} 8.00 \times 10^{-4} & -3.00 \times 10^{-4} & 2.36 \times 10^{-5} & 4.55 \times 10^{-6} & 7.13 \times 10^{-7} & 7.78 \times 10^{-8} \\ -3.00 \times 10^{-4} & 4.70 \times 10^{-4} & -8.85 \times 10^{-7} & -1.71 \times 10^{-6} & -2.68 \times 10^{-7} & -2.92 \times 10^{-8} \\ 2.36 \times 10^{-5} & -8.84 \times 10^{-6} & 1.01 \times 10^{-3} & -5.10 \times 10^{-4} & -8.05 \times 10^{-5} & -5.30 \times 10^{-6} \\ 4.60 \times 10^{-6} & -1.73 \times 10^{-6} & -5.10 \times 10^{-4} & 6.06 \times 10^{-4} & -5.92 \times 10^{-4} & -1.02 \times 10^{-4} \\ 7.13 \times 10^{-7} & -2.68 \times 10^{-7} & -7.52 \times 10^{-7} & -1.45 \times 10^{-7} & 1.80 \times 10^{-7} & -1.06 \times 10^{-4} \\ 7.78 \times 10^{-8} & -2.92 \times 10^{-5} & -5.30 \times 10^{-5} & -1.02 \times 10^{-5} & -1.06 \times 10^{-4} & 3.92 \times 10^{-4} \end{bmatrix}.$$  

The complete data variance covariance matrix $V_{com}$ is

$$\begin{bmatrix} 0.0003674 & -0.0001380 \\ -0.0001380 & 0.0004092 \\ 0.0009496 & -0.0005214 \\ -0.0005214 & 0.0006041 \\ 0.0006033 & -0.0002739 \\ -0.0002739 & 0.0003199 \end{bmatrix}.$$  

The increase in variance $\Delta V$ is

$$\begin{bmatrix} 4.33 \times 10^{-4} & -1.63 \times 10^{-4} & 2.36 \times 10^{-5} & 4.58 \times 10^{-6} & 7.13 \times 10^{-7} & 7.78 \times 10^{-8} \\ -1.63 \times 10^{-4} & 6.11 \times 10^{-5} & -8.86 \times 10^{-6} & -1.71 \times 10^{-6} & -2.68 \times 10^{-7} & -2.92 \times 10^{-8} \\ 2.36 \times 10^{-5} & -8.84 \times 10^{-6} & 5.75 \times 10^{-5} & 1.11 \times 10^{-5} & -8.06 \times 10^{-7} & -5.30 \times 10^{-8} \\ 4.61 \times 10^{-6} & -1.73 \times 10^{-6} & 1.11 \times 10^{-5} & 2.15 \times 10^{-6} & -5.92 \times 10^{-7} & -1.02 \times 10^{-5} \\ 7.13 \times 10^{-7} & -2.68 \times 10^{-7} & -7.52 \times 10^{-7} & -1.45 \times 10^{-7} & 1.20 \times 10^{-3} & 1.68 \times 10^{-4} \\ 7.78 \times 10^{-8} & -2.92 \times 10^{-5} & -5.30 \times 10^{-5} & -1.02 \times 10^{-5} & 1.68 \times 10^{-4} & 7.22 \times 10^{-5} \end{bmatrix}.$$
8. Practical Example

The data consists of the daily rainfall, measured in millimeters times 10, at Alofi in the Niue Island group. 1096 observations were recorded from 1st January 1987 until 31st December 1989. The data is classified into three states: state 1 which represents “no rain”, state 2 which represents “from non zero until 5mm” and state 3 which represents “more than 5mm” rain. This time series data can be considered as a 3 state Markov chain. P. J. Avery and D. A. Henderson (1999) used this dataset for the fitting of Markov model.

For the generation of the observed data we use the same filter matrix as in case of the simulated data. From 1096 observations we find that this filter matrix leads to a missingsness of 45.35%. While storing only 54.65% of the original data we find the estimate of the transition probability matrix is

\[
\begin{pmatrix}
0.6717154 & 0.2231926 & 0.1050920 \\
0.4585938 & 0.3034812 & 0.2379251 \\
0.2137608 & 0.3447883 & 0.4414509
\end{pmatrix}
\]

We compute the observed variance covariance matrix as

\[
\begin{pmatrix}
4.65 \times 10^{-4} & -3.16 \times 10^{-4} & 1.23 \times 10^{-6} & 8.14 \times 10^{-7} & 1.15 \times 10^{-4} & 1.81 \times 10^{-4} \\
-3.16 \times 10^{-4} & 3.41 \times 10^{-4} & -8.51 \times 10^{-7} & -5.63 \times 10^{-7} & -7.76 \times 10^{-5} & -1.23 \times 10^{-4} \\
1.23 \times 10^{-6} & -8.51 \times 10^{-7} & 9.21 \times 10^{-4} & -4.18 \times 10^{-4} & -4.76 \times 10^{-5} & -3.09 \times 10^{-4} \\
8.14 \times 10^{-7} & -5.63 \times 10^{-7} & -4.18 \times 10^{-4} & 7.49 \times 10^{-4} & -3.15 \times 10^{-5} & -2.05 \times 10^{-4} \\
1.15 \times 10^{-4} & -7.76 \times 10^{-5} & -4.76 \times 10^{-5} & -3.15 \times 10^{-5} & 9.26 \times 10^{-4} & 1.51 \times 10^{-4} \\
1.81 \times 10^{-4} & -1.23 \times 10^{-4} & -3.09 \times 10^{-4} & -2.05 \times 10^{-4} & 1.51 \times 10^{-4} & 0.0026
\end{pmatrix}
\]

The complete data variance covariance matrix \( V_{com} \) is

\[
\begin{pmatrix}
0.000391 & -0.000266 & 0.000837 & -0.000469 & 0.0007185 & -0.000315 \\
-0.000266 & 0.000307 & -0.000469 & 0.0007128 & -0.000315 & 0.0009658
\end{pmatrix}
\]

The increase in variance due to missingsness is

\[
\begin{pmatrix}
7.48 \times 10^{-5} & -5.00 \times 10^{-5} & 1.23 \times 10^{-6} & 8.14 \times 10^{-7} & 1.15 \times 10^{-4} & 1.81 \times 10^{-4} \\
-5.00 \times 10^{-5} & 3.38 \times 10^{-5} & -8.50 \times 10^{-7} & -5.63 \times 10^{-7} & -7.76 \times 10^{-5} & -1.23 \times 10^{-4} \\
1.23 \times 10^{-6} & -8.50 \times 10^{-7} & 8.39 \times 10^{-5} & 5.55 \times 10^{-5} & -4.76 \times 10^{-5} & -3.09 \times 10^{-4} \\
8.14 \times 10^{-7} & -5.63 \times 10^{-7} & 5.55 \times 10^{-5} & 3.68 \times 10^{-5} & -3.15 \times 10^{-5} & -2.05 \times 10^{-4} \\
1.15 \times 10^{-4} & -7.76 \times 10^{-5} & -4.76 \times 10^{-5} & -3.15 \times 10^{-5} & 2.08 \times 10^{-5} & 4.66 \times 10^{-4} \\
1.81 \times 10^{-4} & -1.23 \times 10^{-4} & -3.09 \times 10^{-4} & -2.05 \times 10^{-4} & 4.66 \times 10^{-4} & 0.0016079
\end{pmatrix}
\]

9. Appendix

Proof of Theorem 13:

Proof. We split the proof in three parts. We shall prove \( C_i \subseteq I, i = 1, 2, 3 \). This will imply that \( C_* \subseteq I \).

Part 1:
Suppose a filter matrix \( M \in C_1 \). Then the \( \alpha \)th row and \( \beta \)th column of \( M \) are zero and all other rows and columns of \( M \) have exactly one element nonzero.

Case a: \( \alpha \neq \beta \)
Step 1:
Consider $p_{ij}$, $1 \leq i, j \leq k$, $i, j \neq \alpha, \beta$.
Let the $i^{th}$ column has a element $f_{ai} = 1$ and let the $j^{th}$ row has a element $f_{jb} = 1$.
Then $P(S_{aijb}) > 0$. This implies $P(S_{ij}) > 0$.
Hence corollary 12 implies that $p_{ij}$, $1 \leq i, j \leq k$, $i, j \neq \alpha, \beta$ are estimable.

Step 2:
Next from the $\beta^{th}$ column of the transition probability matrix consider $p_{i\beta}$, $\forall i = 1(1)k$, $i \neq \beta$.
Since $\beta \neq \alpha$, we have a $j$ such that $f_{\beta j} = 1$
Also since $i \neq \beta$ we have a $s$ such that $f_{si} = 1$
Then $P(S_{si\beta j}) > 0$. This implies $P(S_{i\beta}) > 0$.
Again corollary 12 implies that $p_{i\beta}$, $\forall i = 1(1)k$, $i \neq \beta$ are estimable.

Step 3:
Next from the $\alpha^{th}$ row of the transition probability matrix consider $p_{\alpha j}$, $\forall j = 1(1)k$, $j \neq \alpha$.
For $p_{\alpha j}$ choose $i$ and $r$ such that $f_{i\alpha} = 1$, $i \neq \alpha$ and $f_{jr} = 1$
Then $P(S_{i\alpha j r}) > 0$. This implies $P(S_{\alpha j}) > 0$.
From corollary 12 we get $p_{\alpha j}$, $\forall j = 1(1)k$, $j \neq \alpha$ is estimable.

Step 4:
The parameter $p_{\alpha i}$ is estimable from the condition $\sum_{j} p_{\alpha j} = 1$

Step 5:
From the $\beta^{th}$ row of the transition probability matrix consider $p_{\beta j}$, $\forall j = 1(1)k$, $j \neq \alpha$.
If $j$ is such that $f_{\beta j} = 1$ then we get that $p_{\beta j}$ is estimable. Hence we now consider $j$ to be such that $f_{\beta j} = 0$.
For this we now choose any state $a$ and a state $s$ such that $f_{js} = 1$.
Then $P(S_{a s j s}) > 0$ which implies $P(S_{\pi}) > 0$ where $\pi = a s j s$.
Let $C = \{ b : f_{ab} = 1 \land f_{bj} = 0 \}$. We note that $\beta \in C$ and $p_{\pi}$ is of the form
$$p_{\pi} = ( \sum_{b \notin C, b \neq \beta} p_{ab} p_{bj} + p_{a \beta} p_{\beta j} ) \times p_{js}$$
Now since $P(S_{\pi}) > 0$, lemma 11 implies that $p_{\pi}$ is identifiable. Hence
$$p_{\pi} = ( \sum_{b \notin C, b \neq \beta} p_{ab} p_{bj} + p_{a \beta} p_{\beta j} ) \times p_{js} = \text{Known Constant}$$
Since all $p_{ab}$ and $p_{bj}$ and also $p_{a \beta}$ are identifiable, we get $p_{\beta j}$, $\forall j = 1(1)k$, $j \neq \alpha$ are estimable.

Step 6:
From the $\alpha^{th}$ column of the transition probability matrix consider $p_{i\alpha}$, $\forall i = 1(1)k$, $i \neq \beta$.
If $i$ is such that $f_{i\alpha} = 1$ then we get that $p_{i\alpha}$ is estimable. Hence we now consider $i$ to be such that $f_{i\alpha} = 0$.
For this we now choose any state $b$ and a state $r$ such that $f_{ri} = 1$.
Then $P(S_{ri b}) > 0$ which implies $P(S_{\pi}) > 0$ where $\pi = ri b$. 
Let $D = \{a : f_{ab} = 0, f_{ia} = 0\}$. We note that $\alpha \in D$ and $p_\pi$ is of the form

$$p_\pi = p_{ri} \times (\sum_{a \in D, a \neq \alpha} p_{ia} p_{ab} + p_{i\alpha} p_{\alpha b})$$

Now since $P(S_\pi) > 0$, lemma 11 implies that $p_\pi$ is identifiable. Hence

$$p_\pi = p_{ri} \times (\sum_{a \in D, a \neq \alpha} p_{ia} p_{ab} + p_{i\alpha} p_{\alpha b}) = \text{Known Constant}$$

Since all $p_{ab}$ and $p_{ia}$ and also $p_{\alpha b}$ are identifiable, we get $p_{i\alpha}, \forall i = 1(1)k, i \neq \beta$ are estimable.

**Step 7:**

$p_{\beta \alpha}$ is estimable from the condition $\sum_j p_{\beta j} = 1$

**Case b : $\alpha = \beta$**

**Step 1:**

Consider $p_{ij} \quad 1 \leq i, j \leq k, i, j \neq \alpha$.

The estimability of $p_{ij}$ is same as step 1 of case (a).

**Step 2:**

Next from the $\alpha$th column of the transition probability matrix consider $p_{i\alpha}, \forall i = 1(1)k, i \neq \alpha$.

The parameter $p_{i\alpha}$ is identified from the condition $\sum_j p_{ij} = 1$

**Step 3:**

Next from the $\alpha$th row of the transition probability matrix consider $p_{\alpha j}, \forall j = 1(1)k, j \neq \alpha$.

For $p_{\alpha j}$ choose $i$ and $r$ such that $f_{ia} = 0, i \neq \alpha$ and $f_{jr} = 1$.

Then $P(S_{i,j\beta}) > 0$ which implies $P(S_\pi) > 0$ where $\pi = i_{-j\beta}$.

Let $D = \{b : f_{ib} = 0 \quad \text{and} \quad f_{\beta b} = 0\}$. We note that $\alpha \in D$ and $p_\pi$ is of the form

$$p_\pi = (\sum_{b \in D, b \neq \alpha} p_{ib} p_{\beta j} + p_{i\alpha} p_{\alpha \beta}) p_{\beta r}$$

Now since $P(S_\pi) > 0$, lemma 11 implies that $p_\pi$ is identifiable. Hence

$$p_\pi = (\sum_{b \in D, b \neq \alpha} p_{ib} p_{\beta j} + p_{i\alpha} p_{\alpha \beta}) p_{\beta r} = \text{Known Constant}$$

Since each of $p_{ib}, p_{\beta j}, p_{i\alpha}$ in the above equation are already identifiable, we get that $p_{i\alpha}$ can also be identified uniquely.

**Step 4 :**

The parameter $p_{i\alpha}$ is identified from the condition $\sum_j p_{\alpha j} = 1$

Thus all the parameters for $M$ are identifiable. Hence for any matrix $M \in C_1$, we have $M \in F$. Thus $C_1 \subseteq I$.

**Part 2:**

In the next case, suppose a filter matrix $M \in C_2$. Then the $\alpha$th and $\beta$th column of a filter matrix $M$ are zero and all other columns of $M$ have exactly one element nonzero.

**Step 1:**

Consider $p_{ij} \quad 1 \leq i, j \leq k, i \neq \alpha, \beta$. 
Let the \( i \)th column has a element \( f_{ai} = 1 \) and let the \( j \)th row has a element \( f_{jr} = 1 \).
Then \( P(S_{aijr}) > 0 \). This implies \( P(S_{ij}) > 0 \).
Hence applying the corollary 12 \( p_{ij} \) \( \leq \) \( k \), \( i \neq \alpha, \beta \) are estimable.

**Step 2:**
Consider \( p_{\alpha\alpha} \) and \( p_{\beta\alpha} \).
Let the \( \alpha \)th row has a element \( f_{\alpha r} = 1 \), \( r \neq \alpha, \beta \) and we choose a \( i \) such that \( i \neq \alpha, \beta \).
Then \( P(S_{i_\alpha r}) > 0 \) which means \( P(S_{i\pi}) > 0 \) where \( \pi = i_\alpha r \).
Let \( D = \{b : f_{ib} = 0 \text{ and } f_{b\alpha} = 0\} \). Then \( p_\pi \) is of the form
\[
p_\pi = \left( \sum_{b \in D} p_{ib} p_{ba} \right) p_{\alpha r}
\]
Clearly \( \alpha, \beta \in D \) and hence we get
\[
p_\pi = \left( \sum_{b \in D, b \neq \alpha, \beta} p_{ib} p_{ba} + p_{i\alpha p_{\alpha\alpha} + p_{i\beta p_{\beta\alpha}} p_{\alpha r}} \right) p_{\alpha r}
\]
Since \( P(S_\pi) > 0 \), lemma 8 implies that \( p_\pi \) is identifiable. Hence
\[
p_\pi = \left( \sum_{b \in D, b \neq \alpha, \beta} p_{ib} p_{ba} + p_{i\alpha p_{\alpha\alpha} + p_{i\beta p_{\beta\alpha}} p_{\alpha r}} \right) p_{\alpha r} = \text{Known Constant}
\]
Since \( p_{ib}, b \neq \alpha \) and \( p_{ba}, b \neq \alpha, \beta \) and \( p_{\alpha r}, r \neq \alpha, \beta \) are all estimable from the above equation we get an equation of the form
This gives us an equation of the form
\[
C_1 p_{\alpha\alpha} + C_2 p_{\beta\alpha} = K_1
\]
where \( C_i \)'s are constants.
Also we from the condition \( \sum_j p_{ij} = 1 \), since all other parameters are estimable we get an equation of the form
\[
p_{\alpha\alpha} + p_{\beta\alpha} = K_2
\]
These two equations make \( p_{\alpha\alpha} \) and \( p_{\beta\alpha} \) estimable.

**Step 3:**
Consider \( p_{\alpha\beta} \) and \( p_{\beta\beta} \).
Let the \( \beta \)th row has a element \( f_{\beta r} = 1 \), \( r \neq \alpha, \beta \) and we choose a \( i \) such that \( i \neq \alpha, \beta \).
Then \( P(S_{i_\beta r}) > 0 \) which means \( P(S_{i\pi}) > 0 \) where \( \pi = i_\beta r \).
Let \( D = \{b : f_{ib} = 0 \text{ and } f_{b\beta} = 0\} \). Then \( p_\pi \) is of the form
\[
p_\pi = \left( \sum_{b \in D} p_{ib} p_{b\beta} \right) p_{\beta r}
\]
Clearly \( \alpha, \beta \in D \) and hence we get
\[
p_\pi = \left( \sum_{b \in D, b \neq \alpha, \beta} p_{ib} p_{b\beta} + p_{i\alpha p_{\alpha\beta} + p_{i\beta p_{\beta\beta}} p_{\beta r}} \right) p_{\beta r}
\]
Since \( P(S_\pi) > 0 \), lemma 11 implies that \( p_\pi \) is identifiable. Hence
\[
p_\pi = \left( \sum_{b \in D, b \neq \alpha, \beta} p_{ib} p_{b\beta} + p_{i\alpha p_{\alpha\beta} + p_{i\beta p_{\beta\beta}} p_{\beta r}} \right) p_{\beta r} = \text{Known Constant}
\]
Since \( p_{ab}, b \neq \alpha \) and \( p_{b\beta}, b \neq \alpha, \beta \) and \( p_{\beta r}, r \neq \alpha, \beta \) are all estimable from the above equation we get a equation of the form

\[
C_1 p_{\alpha \beta} + C_2 p_{\beta \beta} = K_1
\]

where \( C_1 \) and \( C_2 \) and \( K_1 \) are constants.

Also we from the condition \( \sum_j p_{ij} = 1 \), since all other parameters are estimable we get a equation of the form

\[
p_{\alpha \beta} + p_{\beta \beta} = K_2
\]

These two equations make \( p_{\alpha \beta} \) and \( p_{\beta \beta} \) estimable.

Thus all the parameters for \( M \) are identifiable. Hence for any matrix \( M \in C_2 \), we have \( M \in F \). Thus \( C_2 \subseteq I \).

**Part 3:**

Now suppose a filter matrix \( M \in C_3 \). Then the \( \alpha^{th} \) and \( \beta^{th} \) row of a filter matrix \( M \) are zero and all other rows of \( M \) have exactly one element nonzero.

**Step 1:**

Consider \( p_{ij} \) \( 1 \leq i, j \leq k \), \( j \neq \alpha, \beta \).

Let the \( i^{th} \) column has a element \( f_{i\alpha} = 1 \) and let the \( j^{th} \) row has a element \( f_{jr} = 1 \).

Then \( P(S_{ijr}) > 0 \). This implies \( P(S_{ij}) > 0 \).

Hence applying the corollary 12 \( p_{ij} \) \( 1 \leq i, j \leq k \), \( j \neq \alpha, \beta \) are estimable.

**Step 2:**

Consider \( p_{\alpha j} \) \( j = \alpha, \beta \).

Let the \( \alpha^{th} \) column has a element \( f_{i\alpha} = 1 \), \( i \neq \alpha, \beta \) and we choose a \( r \) such that \( r \neq \alpha, \beta \).

Then \( P(S_{\alpha i r}) > 0 \) which means \( P(S_{\pi}) > 0 \) where \( \pi = i \alpha \_ r \).

Let \( D = \{ b : f_{ab} = 0 \text{ and } f_{br} = 0 \} \). Then \( p_{\pi} \) is of the form

\[
p_{\pi} = p_{i\alpha}(\sum_{b \in D} p_{ab}p_{br})
\]

Clearly \( \alpha, \beta \in D \) and hence we get

\[
p_{\pi} = (\sum_{b \in D, b \neq \alpha, \beta} p_{ab}p_{br} + p_{\alpha \alpha}p_{\alpha r} + p_{\alpha \beta}p_{\beta r})p_{i\alpha}
\]

Since \( P(S_{\pi}) > 0 \), lemma 11 implies that \( p_{\pi} \) is identifiable. Hence

\[
p_{\pi} = (\sum_{b \in D, b \neq \alpha, \beta} p_{ab}p_{br} + p_{\alpha \alpha}p_{\alpha r} + p_{\alpha \beta}p_{\beta r})p_{i\alpha} = \text{Known Constant}
\]

Since \( p_{\alpha i}, i \neq \alpha \) and \( p_{ab}, b \neq \alpha, \beta \) and \( p_{\beta r}, r \neq \alpha, \beta \) and \( p_{ab}, a, b \neq \alpha, \beta \) are all estimable from the above equation we get a equation of the form

\[
C_1 p_{\alpha \alpha} + C_2 p_{\alpha \beta} = K_1
\]

where \( C_i \) and \( K_i \) are constants.

Also from the restriction \( \sum_j p_{\alpha j} = 1 \) we get a equation of the form

\[
p_{\alpha \alpha} + p_{\alpha \beta} = K_2
\]

These two final equations make the parameters \( p_{\alpha \alpha} \) and \( p_{\alpha \beta} \) identifiable.

**Step 3:**
Consider \( p_{\beta j} \ j = \alpha, \beta \).

Let the \( \beta^{th} \) column has an element \( f_{i\beta} = 1, i \neq \alpha, \beta \) and we choose a \( r \) such that \( r \neq \alpha, \beta \).

Then \( P(S_{i\beta \_ r}) > 0 \) which means \( P(S_{\pi}) > 0 \) where \( \pi = i\beta \_ r \).

Let \( D = \{ b : f_{\beta b} = 0 \ \text{and} \ f_{br} = 0 \} \). Then \( p_{\pi} \) is of the form

\[
p_{\pi} = p_{\beta} \left( \sum_{b \in D} p_{\beta b} p_{br} \right)
\]

Clearly \( \alpha, \beta \in D \) and hence we get

\[
p_{\pi} = \left( \sum_{b \in D, b \neq \alpha, \beta} p_{\beta b} p_{br} + p_{\beta \alpha} p_{\alpha r} + p_{\beta \beta} p_{\beta r} \right) p_{i\beta}
\]

Since \( P(S_{\pi}) > 0 \), lemma 11 implies that \( p_{\pi} \) is identifiable. Hence

\[
p_{\pi} = \left( \sum_{b \in D, b \neq \alpha, \beta} p_{\beta b} p_{br} + p_{\beta \alpha} p_{\alpha r} + p_{\beta \beta} p_{\beta r} \right) p_{i\beta} = \text{Known Constant}
\]

Since \( p_{i\beta}, i \neq \alpha \) and \( p_{\beta b}, b \neq \alpha, \beta \) and \( p_{\alpha r}, r \neq \alpha, \beta \) and \( p_{ab}, a, b \neq \alpha, \beta \) are all estimable from the above equation we get a equation of the form

\[
C_1 p_{\beta \alpha} + C_2 p_{\beta \beta} = K_1
\]

where \( C_i \) and \( K_i \) are constants.

Also from the restriction \( \sum_j p_{\beta j} = 1 \) we get a equation of the form

\[
p_{\beta \alpha} + p_{\beta \beta} = K_2
\]

These two final equations make the parameters \( p_{\beta \alpha} \) and \( p_{\beta \beta} \) identifiable. Thus all the parameters for \( M \) are identifiable. Hence for any matrix \( M \in C_3 \), we have \( M \in F \). Thus \( C_3 \subseteq I \).

\[ \square \]

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