What is $-Q$ for a poset $Q$?

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Abstract

In the context of combinatorial reciprocity, it is a natural question to ask what "$-Q$" is for a poset $Q$. In a previous work, the definition $-Q := Q \times \mathbb{R}$ with lexicographic order was proposed based on the notion of Euler characteristic of semi-algebraic sets. In fact, by using this definition, Stanley’s reciprocity for order polynomials was generalized to an equality for the Euler characteristics of certain spaces of increasing maps between posets. The purpose of this paper is to refine this result, that is, to show that these spaces are homeomorphic if the topology of $Q$ is metrizable.

1 Introduction: Euler characteristic reciprocity

For posets $P$ and $Q$, the set of increasing maps from $P$ to $Q$, denoted by $\text{Hom}^\leq(P, Q)$, is defined as

\[
\text{Hom}^\leq(P, Q) = \{ \eta : P \to Q \mid p_1 < p_2 \implies \eta(p_1) < \eta(p_2) \}.
\]

The set of weakly increasing maps $\text{Hom}^{\leq}(P, Q)$ is similarly defined. For finite posets $P$ and $Q$, the cardinality $|\text{Hom}^{\leq}(P, Q)|$ is an important object of study in enumerative combinatorics and theory of polytopes ([12]). In particular, the following result by Stanley is one of the early results which leads recent active research on combinatorial reciprocities ([2]).

**Theorem 1.1.** ([10] [11]) Let $P$ be a finite poset and $[n]$ denote the totally ordered set $\{1 < 2 < \cdots < n\}$. Then,

(i) (Order polynomials) there exist polynomials $O^{\leq}(P, t), O^{\leq}(P, t) \in \mathbb{Q}[t]$ that satisfy

\[
O^{\leq}(P, n) = |\text{Hom}^{\leq}(P, [n])|,
\]

\[
O^{\leq}(P, n) = |\text{Hom}^{\leq}(P, [n])|,
\]

for $n \geq 1$.

(ii) (Reciprocity)

\[
O^{\leq}(P, t) = (-1)^{|P|} \cdot O^{\leq}(P, -t).
\]

Let $t = n$ in formula (4). The left-hand side makes sense in terms of the cardinality of $\text{Hom}^{\leq}(P, [n])$. However, the right-hand side, the cardinality of $\text{Hom}^{\leq}(P, [-n])$, is meaningless as

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it is. For this reason, it is a natural question to give a definition of “−Q” for the poset Q and give meaning to the formula of the form
\[
\# \text{Hom}^<(P, Q) = (-1)^{|P|} \cdot \# \text{Hom}^\leq(P, -Q). \tag{5}
\]

The cardinality of a finite set is a non-negative integer, however for our purposes we need an extension of “finite sets” such that it takes whole integers (including negative integers) as “cardinality”. Such a problem has been discussed in [8], and one natural answer is topological spaces (in particular, semialgebraic sets) and their Euler characteristics. In fact, number of generalizations of combinatorial results have been obtained using the Euler characteristic [6, 13].

In [7], the definition “−Q := Q × ℝ with lexicographic order” was proposed for this purpose. Then, based on this definition, the above formula (5) can be formulated as an identity for the Euler characteristics.

In order to state the main result of [7], let us recall the Euler characteristic of a semialgebraic set [1]. Let \( X \subset \mathbb{R}^N \) be a semialgebraic set. Then, there exists a finite partition \( X = \bigcup_{\lambda \in \Lambda} X_\lambda \) into semialgebraic sets \( X_\lambda \) which is semialgebraically homeomorphic to the open simplex \( \sigma_d \), where \( \sigma_d = \{ 0 < x_1 < x_2 < \ldots < x_d < 1 \} \subset \mathbb{R}^d \) is the \( d \)-dimensional open simplex (note that \( \sigma_0 = \{ 0 \} \) is the point). Then the Euler characteristic \( e(X) \) of \( X \) is defined as \( e(X) := \sum_{\lambda \in \Lambda} (-1)^d \). Note that if \( X \) is compact, then \( e(X) \) coincides with the usual Euler characteristic. More generally, if \( X \) is locally compact, then \( e(X) \) is coincides with the Euler characteristic of the Borel-Moore homology group [3].

A poset is called a semialgebraic poset if its ground set is a semialgebraic set and order structure is semialgebraically defined. Finite posets and the real line \( \mathbb{R} \) are semialgebraic posets. The Euler characteristic of a semialgebraic poset is a natural generalization of the cardinality of a finite poset. For example, for a finite poset \( P \), \( e(P) \) is equal to the cardinality \( |P| \). Furthermore, due to the multiplicativity of the Euler characteristic and \( e(\mathbb{R}) = -1 \), for a semialgebraic poset \( Q \), we have
\[
e(-Q) = -e(Q).
\]

**Theorem 1.2.** [7] Let \( P \) be a finite poset and \( Q \) be a semialgebraic poset. Then \( \text{Hom}^<(P, Q) \) and \( \text{Hom}^\leq(P, Q) \) are semialgebraic sets. Furthermore,

(i) the Euler characteristics of these spaces satisfy
\[
e(\text{Hom}^<(P, Q)) = (-1)^{|P|} \cdot e(\text{Hom}^\leq(P, -Q)), \tag{6}
e(\text{Hom}^<(P, -Q)) = (-1)^{|P|} \cdot e(\text{Hom}^\leq(P, Q)). \tag{7}
\]

(ii) If furthermore \( Q \) is totally ordered, then the Euler characteristics of these spaces can be expressed using ordered polynomials as follows.
\[
e(\text{Hom}^<(P, Q)) = O^<(P, e(Q)),
e(\text{Hom}^\leq(P, Q)) = O^\leq(P, e(Q)).
\]

Note that Stanley’s reciprocity Theorem [14] can be obtained by considering the totally ordered set \( Q = [n] \). Moreover, Theorem 1.2 asserts that the reciprocity holds for any finite poset \( Q \), not necessarily for the poset of the form \( Q = [n] \).

**Remark 1.3.** Note that, in Theorem 1.2 (i), since \( -Q \neq Q \), the two formulas (6) and (7) are not equivalent.

This paper is organized as follows. In §2 we discuss the refinement of Theorem 1.2 (i), i.e., whether the claim of the Theorem follows from the homeomorphism of spaces. In §3 we formulate the main result. In §4 we summarize the properties of upper semicontinuous functions needed for the proof, and in §5 we give the proof of the main result.
2 A refinement of Euler characteristic reciprocity

Theorem 1.2 (i) asserts that the Euler characteristics of two spaces are equal up to sign factor. Let us reformulate these formulas: noting that \( e(R|P) = (-1)^{|P|} \), the two formulas of Theorem 1.2 (i) can be rewritten as:

\[
\begin{align*}
e(\text{Hom}^\le(P, Q \times \mathbb{R})) &= e(\text{Hom}^\le(P, Q) \times \mathbb{R}^{|P|}), \\
e(\text{Hom}^\le(P, Q \times \mathbb{R})) &= e(\text{Hom}^\le(P, Q) \times \mathbb{R}^{|P|}).
\end{align*}
\]

(8) \hspace{1cm} (9)

It is a natural question to ask whether the equality between these Euler characteristics can be refined. More precisely, are the spaces in the left-hand sides and the right-hand sides homeomorphic?

The main result of this paper is to prove that the second equality (9) holds at the level of space, that is, there exists a homeomorphism

\[\text{Hom}^\le(P, Q) \simeq \text{Hom}^\le(P, Q) \times \mathbb{R}^{|P|}.\]

(10)

(See Theorem 3.2 and Corollary 5.1 for the precise statement).

Remark 2.1. For the first equality (8), the spaces \( \text{Hom}^\le(P, Q \times \mathbb{R}) \) and \( \text{Hom}^\le(P, Q) \times \mathbb{R}^{|P|} \) are not homeomorphic in general. For example, when \( P = [2] \) and \( Q = [1] \), \( \text{Hom}^\le(P, Q) = \emptyset \), therefore, the space in the right-hand side of (8) is empty, while, the left-hand side is non-empty.

As another example, let us consider the case \( P = Q = [2] \). Then, \( \text{Hom}^\le(P, Q) \) consists of a point and \( \text{Hom}^\le(P, Q) \times \mathbb{R}^{|P|} \) is a connected space. On the other hand, \( \text{Hom}^\le(P, Q \times \mathbb{R}) \) looks like Figure 1, which has three connected components. (The figure is drawn using the identification \( \mathbb{R} \) with the open interval \( (0, 1) \).) It is a natural problem to explore the reasons that lead to the equality (8) of the Euler characteristics even though the spaces are not homeomorphic.

3 Metrizable posets and main result

Let \( P \) and \( Q \) be posets. From the definition of lexicographic order, a pair \((\eta, \theta)\) of maps \( \eta : P \rightarrow Q \) and \( \theta : P \rightarrow \mathbb{R} \) is contained in \( \text{Hom}^\le(P, Q \times \mathbb{R}) \) if and only if for every \( p_1, p_2 \in P \) with \( p_1 < p_2 \), either “\( \eta(p_1) < \eta(p_2) \)” or “\( \eta(p_1) = \eta(p_2) \) and \( \theta(p_1) < \theta(p_2) \)” holds. It follows that \( \eta \in \text{Hom}^\le(P, Q) \). Thus, we obtain the natural projection \( \pi : \text{Hom}^\le(P, Q \times \mathbb{R}) \rightarrow \text{Hom}^\le(P, Q) \) (also similarly \( \pi : \text{Hom}^\le(P, Q \times \mathbb{R}) \rightarrow \text{Hom}^\le(P, Q) \)).

Definition 3.1. A poset \( Q \) is a metrizable poset if its ground set is equipped with metrizable topology.
The main result of this paper is as follows.

**Theorem 3.2.** Let $P$ be a finite poset and $Q$ be a metrizable poset. Then there exists a homeomorphism $\varphi : \text{Hom}^\leq(P, Q \times \mathbb{R}) \xrightarrow{\sim} \text{Hom}^\leq(P, Q) \times \mathbb{R}^{|P|}$ which makes the following diagram commutative:

\[
\begin{array}{ccc}
\text{Hom}^\leq(P, Q \times \mathbb{R}) & \xrightarrow{\varphi} & \text{Hom}^\leq(P, Q) \times \mathbb{R}^{|P|} \\
\downarrow \pi & & \downarrow \pi \\
\text{Hom}^\leq(P, Q) & \xrightarrow{id} & \text{Hom}^\leq(P, Q).
\end{array}
\]

Before giving the proof, let us discuss special cases of this result.

**Example 3.3.** If $Q = [1]$ is the poset with one element, then the result gives a homomorphism $\varphi : \text{Hom}^\leq(P, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}^{|P|}$. Let $P = \{p_1, \ldots, p_n\}$. Then $\text{Hom}^\leq(P, \mathbb{R})$ is expressed as follows.

\[
\text{Hom}^\leq(P, \mathbb{R}) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_i < t_j \text{ if } p_i < p_j \text{ in } P\},
\]

which is a convex open subset of $\mathbb{R}^n$. Hence it is homeomorphic to $\mathbb{R}^n$.

**Example 3.4.** Suppose $P = [2]$ and $Q = \mathbb{R}$. Then we have

\[
\text{Hom}^\leq(P, Q \times \mathbb{R}) = \{(q_1, t_1), (q_2, t_2)) \in (\mathbb{R} \times \mathbb{R})^2 \mid q_1 < q_2, \text{ or } q_1 = q_2 \text{ and } t_1 < t_2\}.
\]

Restricting diagram (11) to $(q_1, t_1) = (0, 0)$, Theorem 3.2 asserts that

\[
\{(q_2, t_2) \in \mathbb{R}^2 \mid q_2 > 0, \text{ or } q_2 = 0, t_2 > 0\}
\]

is homeomorphic to $[0, \infty) \times \mathbb{R}$ (Figure 2).

![Figure 2](image_url)

This example shows that considerations of the upper semicontinuous functions are key to the proof of Theorem 3.2.

### 4 Upper semicontinuous functions on metrizable spaces

Recall that a function $f : X \to \mathbb{R} \cup \{-\infty\}$ on a topological space $X$ is said to be **upper semicontinuous** if for every $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha)) \subseteq X$ is open.

**Lemma 4.1.** For a function $f : X \to \mathbb{R} \cup \{-\infty\}$, let $X_f := \{(x, t) \in X \times \mathbb{R} \mid t > f(x)\}$. If $X$ is metrizable and $f$ is upper semicontinuous, then there exists a homeomorphism $\varphi : X_f \to X \times \mathbb{R}$ that makes the following diagram commutative:

\[
\begin{array}{ccc}
X_f & \xrightarrow{\varphi} & X \times \mathbb{R} \\
\downarrow \pi & & \downarrow \pi \\
X & \xrightarrow{id} & X.
\end{array}
\]
Proof. It is classically known ([5 Chapter 9, §2]) that there exists a sequence \( f_n : X \to \mathbb{R} \cup \{-\infty\} \), \((n \geq 1)\) of continuous functions such that

- for each \( x \in X \), \( f_1(x) > f_2(x) > \cdots > f_n(x) > \ldots \), and
- \( \lim_{n \to \infty} f_n(x) = f(x) \).

Then, define \( \varphi : X_f \to X \times \mathbb{R} \) as

\[
\varphi(x,t) = \begin{cases} (x, t - f_0(x)), & (t \geq f_0(x)) \\ (x, -(i + 1) + \frac{t - f_{i+1}(x)}{f_i(x) - f_{i+1}(x)}), & (f_i(x) \geq t \geq f_{i+1}(x)). \end{cases}
\]

This \( \varphi \) gives a desired homeomorphism. \( \square \)

5 Proof of the main result

We give the proof of Theorem 3.2 in this section. We fix a numbering \( P = \{p_1, \ldots, p_n\} \) in such a way that \( 1 \leq i < j \leq n \) implies \( p_j \not\leq p_i \). Such a numbering can be obtained, for example, by letting \( p_i \) be a minimal element of \( P \) and \( p_i \) be a minimal element of \( P \setminus \{p_1, \ldots, p_{i-1}\} \) for \( i > 1 \).

For \( 1 \leq k \leq n \), let us define the subset \( X_k \subset \text{Hom}^\leq(P, Q) \times \mathbb{R}^n \) as follows.

\[
X_k := \{(q_1, \ldots, q_n, t_1, \ldots, t_n) \in \text{Hom}^\leq(P, Q) \times \mathbb{R}^n | \text{For } 1 \leq i < \forall j \leq k, \text{ if } p_i \not< p_j \text{ and } q_i = q_j \text{, then } t_i < t_j \},
\]

where \((q_1, \ldots, q_n) = (\eta(p_1), \ldots, \eta(p_n))\) for \( \eta \in \text{Hom}^\leq(P, Q) \) and \( t_i \in \mathbb{R} \). Note that \( X_1 = \text{Hom}^\leq(P, Q) \times \mathbb{R}^n \) and \( X_n = \text{Hom}^\leq(P, Q \times \mathbb{R}) \).

Let \( 1 \leq k \leq n-1 \). Define the map \( \pi_k : X_k \to \text{Hom}^\leq(P, Q) \times \mathbb{R}^{n-1} \) by \( \pi_k(q_1, \ldots, q_n, t_1, \ldots, t_k, t_{k+2}, \ldots, t_n) \mapsto (q_1, \ldots, q_n, t_1, \ldots, t_k, t_{k+2}, \ldots, t_n) \), and \( Y_k := \pi_k(X_k) \). It follows from the definition that \( X_k = Y_k \times \mathbb{R} \). Next, for \( 1 \leq j \leq k \leq n-1 \), define the function \( f_{jk} : Y_k \to \mathbb{R} \cup \{-\infty\} \) as follows.

\[
f_{jk}(q_1, \ldots, q_n, t_1, \ldots, t_k, t_{k+2}, \ldots, t_n) = \begin{cases} -\infty & (p_j \not< p_{k+1} \text{ or } q_j < q_{k+1}), \\ t_j & (p_j \leq p_{k+1} \text{ and } q_j = q_{k+1}). \end{cases}
\]

Then \( f_{jk} \) is an upper semicontinuous function. In fact, when \( p_j \not< p_{k+1}, \) \( f_{jk} \) is upper semicontinuous because it is a constant function. When \( p_j \not< p_{k+1}, \) we need to verify \( f_{jk}^{-1}([-\infty, \alpha)) \) is open for \( \forall \alpha \in \mathbb{R} \). Indeed, we have

\[
f_{jk}^{-1}([-\infty, \alpha)) = \begin{cases} \{(q_1, \ldots, q_n, t_1, \ldots, t_k, t_{k+1}, \ldots, t_n) \in Y_k | q_j \not= q_{k+1}\} \\ \cup \{(q_1, \ldots, q_n, t_1, \ldots, t_k, t_{k+1}, \ldots, t_n) \in Y_k | t_j < \alpha\}. \end{cases}
\]

The first set is open because \( Q \) is Hausdorff. The second set is clearly open.

Now we consider the function \( f_k := \max\{f_1, \ldots, f_{kk}\} \) on \( Y_k \). Since the maximum of finitely many upper semicontinuous functions is upper semicontinuous, \( f_k \) is upper semicontinuous. By Lemma 4.1 there exists a homeomorphism \( \varphi \) that makes the following diagram commutative.

\[
\begin{array}{ccc}
Y_k \times \mathbb{R} & \xrightarrow{\varphi_k} & Y_{kf_k} \\
\pi \downarrow & & \pi \downarrow \\
Y_k & \xrightarrow{id} & Y_k
\end{array}
\]

where \( Y_{kf_k} = \{(y, t_{k+1}) \in Y_k \times \mathbb{R} | t_{k+1} > f_k(y)\} \). Furthermore, by definition, we have \( Y_{kf_k} = X_{k+1} \). Hence there exists a homeomorphism \( X_k \simeq X_{k+1} \) which commutes with the projection to \( \text{Hom}^\leq(P, Q) \). In particular, we have \( \text{Hom}^\leq(P, Q) \times \mathbb{R}^n = X_1 \simeq \cdots \simeq X_n = \text{Hom}^\leq(P, -Q) \). This completes the proof of Theorem 3.2.

Since a semialgebraic set is metrizable, we have the following.
Corollary 5.1. Let $P$ be a finite poset and $Q$ be a semialgebraic poset. Then $\text{Hom}^\leq(P, Q) \times \mathbb{R}^{|P|}$ and $\text{Hom}^\leq(P, -Q)$ are homeomorphism.

Remark 5.2. It is known that the Euler characteristics of homeomorphic semialgebraic sets coincide ([3]). Therefore, the equality of Euler characteristics (7) in Theorem 1.2 can be obtained from Corollary 5.1. However, it is not clear whether $\text{Hom}^\leq(P, Q) \times \mathbb{R}^{|P|}$ and $\text{Hom}^\leq(P, -Q)$ are semialgebraically homeomorphic or not, because in Lemma 4.1 we use maps that are not semialgebraic. Note that there exist two semialgebraic sets that are homeomorphic, but not semialgebraically homeomorphic ([9]).

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