2-anti fuzzy domination in anti fuzzy graphs

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Abstract: The aim of this paper is to study the concepts of the concepts of 2- anti fuzzy dominating ($2-\text{AFD}$) set and 2- anti fuzzy domination number ($2-\text{AF}$ domination number) of an anti-fuzzy graph ($G_{\text{AF}}$). We determine the 2- anti fuzzy domination number ($2-\text{AFD}$ number $\gamma_{2\text{AF}}$) for some classes of an anti-fuzzy graphs ($G_{\text{AF}}$) and obtain the bounds on ($2-\text{AF}$ domination number) for the same. The relations between ($2-\text{AF}$ domination number), anti fuzzy domination ($\text{AF}$ domination number) and vertex anti-fuzzy vertex covering number ($\alpha_0$) are discussed and found some result.

Keywords: Anti fuzzy graph, AFD number, $2-\text{AFD}$ number and anti fuzzy vertex covering number $\alpha_0$.

1. Introduction

The idea of domination came through the game of chess, to solve the problem of placing the fewer number of chess pieces to dominate of all square chess board. In 1962, Ore used crisp graphs to study domination theory [1]. The concept was developed by E. J. Cockayne and S. T. Hedetnieme [2]. The notion domination number plays an important role to solve various life problems by various kinds of fields in graph theory as topological graphs [3] and labeled graph [4], [5] and the others. Also, the researcher discussed many parameters of domination as in [6, 7]. In 1965, L. Zadeh [8] published his paper "fuzzy set "in which clarified the uncertainty. In (1975) Rosenfeld [9] introduced the concept of fuzzy graph. In (1975) Kauffmann [10] introduced the fundamental idea of a fuzzy graph. The concept of Domination in fuzzy graphs I, introduced by Somasundaram A. and Somasundaram S. [11, 12] and them using effective edges. The concept of Domination in fuzzy graph, so introduced by Nagoorgani A. and Chandrasekaran V. T. [13] but by using strong edges. Muhammad A. [14] was introduced the concept of Anti fuzzy structures graphs, and discussed the concepts connected anti fuzzy graphs, constant anti fuzzy graphs and other concepts. On anti fuzzy graph, Domination on anti fuzzy graph and connected domination on anti fuzzy are introduced by Muthuraj R. and Sasireka A. [15-17]. In this paper we introduce the concept of 2- anti fuzzy
domination($2 - AFD$) number in anti fuzzy graphs($G_{AF}$) by effective edges. In addition some result and theorems are obtains with suitable examples are given.

2. PRELIMINARIES.

We provide definition of anti fuzzy graph with a set of concepts related to it. Let $E \subseteq V \times V$ where $V$ a non-empty set and finite. Then $G = (\eta, \rho)$ is called anti fuzzy graph such that for every $u_1u_2 \in E$ we have $\rho(u_1, u_2) \geq \eta(u_1) \lor \eta(u_2)$ and denote it by $G_{AF} = (\eta, \rho)$ where $\eta$ is a fuzzy set in $V$, $(\eta: V \rightarrow [0,1]$, $(\rho$ is anti fuzzy relation on $E) \rho: V \times V \rightarrow [0,1]$ and $V$: refer to maximum [14]. $G^*_A = (\eta^*, \rho^*)$ is known as underlying anti crisp graph of $G_{AF}$, Where $\eta^* = \{x \in V/ \eta(x) > 0 \}$ and $\rho^* = \{(x,y) \in V \times V/ \rho(x,y) > 0\}$. The size $S$ and order $\rho$ of $G_{AF} = (\eta, \rho)$ are defined to $S = \sum \rho(x, y) x, y \in E$ and $\rho = \sum \eta(x) x \in V$ respectively [15]. A path $P$ in $G_{AF}$ is a sequence of distinct vertices $x_1, x_2, x_3, \ldots, x_n$ such that $\rho(x_1, x_2, \ldots, x_i) > 0$ and $1 \leq i \leq n$ and if there exist a fuzzy path between any two vertices then $G_{AF}$ is connected [15]. $G_{AF} = (\eta, \rho)$ is a complete if $\rho(u, w) = \max[\eta(u), \eta(w)] \forall u, w \in V$ and denote by $K_{\eta}$ and the edge $e = (u, w)$ is effective in $G_{AF}$ if $\rho(u, w) = \max[\eta(u), \eta(w)]$[16]. $N(w) \{u: (w,u)$ is an effective edge}$ is known as the neighbourhood of $w$ and if $N[w] = N(w) \cup w$ is called closed neighbourhood of $w$ [17]. $\overline{G_{AF}} = (\eta, \rho)$ is defined as a complement of $G_{AF} = (\eta, \rho)$ such that $\overline{\eta} = \eta$ and $\overline{\rho}(u_1, u_2) = 1 - \rho(u_1, u_2) + \max[\eta(u_1), \eta(u_2)]$ for all $u_1u_2 \in E$ [18]. If $\rho(u, w) \neq \max[\eta(u), \eta(w)] \forall u \in N(w)$ then $w$ is called isolated, in particular case if $\rho(u, w) = 0 \forall u \in V - \{w\}$. Let $S \subseteq V(G_{AF})$ such that $\rho(u, w) = \max[\eta(u), \eta(w)] \forall u, w \in S$ is a known as the independent anti-fuzzy set. as well it is called maximal if there is no independent anti-fuzzy set $S^*$ of $G_{AF}$ such that $|S^*| > |S|$ [17]. The independence number $\beta_0$ ($G_{AF}$) is the maximum fuzzy cardinality over all maximal independent anti fuzzy set of $G_{AF}$[19]. An anti-fuzzy graph $G_{AF} = (\eta, \rho)$ is called an uninodal if $\eta(x) = k = \rho(x, y)$, while if $\eta(v) = c, \forall v \in V(G_{AF})$ then $G_{AF}$ is known as a $v$- nodal , where $k, c \in [0,1]$[19]. Let $A \subseteq V(G_{AF})$ is a known as an anti-fuzzy vertex cover set of $G_{AF}$ if for each effective edge $e = (u, w)$ at least on of $u, w$ is in $A$. The maximum fuzzy cardinality for all anti fuzzy vertex cover set with minimum number of vertices is known as anti fuzzy vertex covering number($\alpha_0$)of $G_{AF}$ and is represented by $\alpha_0(G_{AF})$[11]. A vertex $u$ is called end vertex of $G_{AF}$ if it has exactly one effective neighbor in $G_{AF}$ [12]. $D \subseteq V(G_{AF})$ is known as anti fuzzy dominating ($AFD$) set of $G_{AF}$ If for each vertex $u_1 \in V - D$ there exists a vertex $u_2 \in D$ such that $\rho(u_1, u_2) = \eta(u_1) \lor \eta(u_2)$ . The($AFD$) set $D$ of $G_{AF}(\eta, \rho)$ with minimum number of vertices is called minimum anti fuzzy dominating ($MAFD$) set. if no proper $D' \subseteq D$ is $AFD$ set .Then $D$ is called minimal AFD set of $G_{AF}$ and take
maximum cardinality for all MAFD sets is known as an anti fuzzy Domination number (AFD) number of $G_{AF}$ is the and denoted by $\gamma_{2AF}(G_{AF})$ or simply $\gamma_{2AF}$.

### 3.2 – anti fuzzy domination number of $G_{AF}$.

In this section, we introduce the $2–AFD$ set of $G_{AF}$, $2–AF$ domination number, and anti fuzzy vertex covering with appropriate examples and we discuss some properties on $2–AFD$ number of $G_{AF}$ by the effective edge.

**Definition 3.1:** Let $\mathcal{D} \subseteq V(G_{AF})$ is said to be ($2–AFD$) set in $G_{AF}$ if for every vertex $v \in V – \mathcal{D}$ there is at least two vertices $u$ and $w \in \mathcal{D}$ such that $\rho(v,u) = \eta(v) \lor \eta(u)$ and $\rho(v,w) = \eta(v) \lor \eta(w)$.

**Definition 3.2:** A $2–AFD$ set $\mathcal{D}$ of $G_{AF}$ with minimum number of vertices is called a minimum $2–AF$ dominating ($M2–AFD$) set.

**Definition 3.3:** A $2–AFD$ set $\mathcal{D}$ of $G_{AF}$ is said to be minimal $2–AFD$ set in $G_{AF}$ if has no proper $2–AFD$ set of $G_{AF}$.

**Definition 3.4:** A $2–AF$ domination number of $G_{AF}$ is the maximum fuzzy cardinality taken over all $M2–AFD$ sets and denoted by $\gamma_{2AF} G_{AF}$ or simply $\gamma_{2AF}$.

**Example 3.1:** Consider an anti fuzzy graph $G_{AF} = (\eta, \rho)$, which given in figure 1, where we have

- $\mathcal{D}_1 = \{u_2, u_3, u_5\}$,
- $\mathcal{D}_2 = \{u_1, u_3, u_4\}$,
- $\mathcal{D}_3 = \{u_1, u_2, u_3, u_5\}$,
- $\mathcal{D}_4 = \{u_2, u_3, u_4, u_5\}$,
- $\mathcal{D}_5 = \{u_1, u_2, u_3, u_4\}$,
- $\mathcal{D}_6 = \{u_1, u_2, u_3, u_4, u_5\}$

are $2–AFD$ sets of $G_{AF}$.

$\mathcal{D}_1$ and $\mathcal{D}_2$ are $M2–AFD$ sets also are minimal $2–AFD$ sets.

Hence, $\gamma_{2AF} (G_{AF}) = \max [\mid \mathcal{D}_1 \mid , \mid \mathcal{D}_2 \mid ] = \max [1.7, 1.6] = 1.7$

![Figure 1](image)

**Preposition 3.1:** Let $G_{AF} \equiv K_{\eta}$ be a complete anti fuzzy graph with $n \geq 3$. then $\gamma_{2AF} (K_{\eta}) = \max [\eta(x) + \eta(y)] \forall x,y \in V(K_{\eta})$.

**Proof:** Given $G_{AF} \equiv K_{\eta}$ be a complete anti fuzzy graph, then $\rho(x,y) = \max [\eta(x) + \eta(y)] \forall x,y \in V(K_{\eta})$, thus each vertex in $K_{\eta}$ dominates to all other vertices in $K_{\eta}$. Therefore, any set in $K_{\eta}$ contains
two vertices, say \( \{x, y\} \) will be of the form \( M_{2-ADF} \) set of \( K_\eta \). Hence \( \gamma_{2AF}(K_\eta) = \max [\eta(x) + \eta(y)] \forall x, y \in V(K_\eta) \). □

**Preposition 3.2:** If \( G_{AF} \equiv K_{\eta_1, \eta_2} \) is a complete bipartite anti fuzzy graph with \( n, m \) vertices then

\[
\gamma_{2AF}(K_{\eta_1, \eta_2}) = \begin{cases} 
\max |V_1| & \text{if } 2 \leq n \leq 4 \\
\max |V_2| & \text{if } 2 \leq m \leq 4 \\
\max \{\eta(v_i) + \eta(v_j) + \eta(u_i) + \eta(u_j)\} & m, n \geq 4, v_i, v_j \in V_1 \text{ and } u_i, u_j \in V_2
\end{cases}
\]

**Proof.** Given \( G_{AF} \equiv K_{\eta_1, \eta_2} \) be a complete bipartite anti fuzzy graph of two bipartite sets \( V_1 \) of \( n \) vertices and \( V_2 \) of \( m \) vertices. So, there are three cases as follows.

Case 1. If \( m \) or \( n \) less than four, then there are two subcases as follows.

Subcase 1. If \( 2 = n < m < 4 \), then it is clear that \( D = \{v_1, v_2\} \) where \( v_1, v_2 \) belong to the set \( V_1 \) is \( M_{2-ADF} \) set. Thus, \( \gamma_{2AF}(G_{AF}) = |V_1| \), similarly If \( 2 = m < n < 4 \), then \( \gamma_{2AF}(G_{AF}) = |V_2| \).

Subcase 1. If \( n = m < 4 \), then as same manner in subcase 1, \( \gamma_{2AF}(G_{AF}) = \max\{|V_1|, |V_2|\} \).

Case 2. If \( n = 4 \), then \( \gamma_{2AF}(G_{AF}) = \max\{|V_1|, \max\{\eta(v_i) + \eta(v_j) + \eta(u_i) + \eta(u_j)\}, v_i, v_j \in V_1 \text{ and } u_i, u_j \in V_2\} \). Similarly, If \( m = 4 \), then \( \gamma_{2AF}(G_{AF}) = \max\{|V_2|, \max\{\eta(v_i) + \eta(v_j) + \eta(u_i) + \eta(u_j)\}, v_i, v_j \in V_1 \text{ and } u_i, u_j \in V_2\} \).

Case 2. If \( n, m > 4 \), then the \( M_{2-ADF} \) set occurs when take two vertices from each sets \( V_1 \) and \( V_2 \). Thus, \( \gamma_{2AF}(G_{AF}) = \max\{\eta(v_i) + \eta(v_j) + \eta(u_i) + \eta(u_j)\}, v_i, v_j \in V_1 \text{ and } u_i, u_j \in V_2\} \).

Therefore, from each cases above the result is obtained. □

**Preposition 3.3:** Let \( G_{AF} \) be a star anti fuzzy graph then \( \gamma_{2AF}(G_{AF}) = \rho - \eta(u) \), where \( u \) is the root.

**Proof:** Given \( G_{AF} \) be a star anti fuzzy graph and \( v \in V(G_{AF}) \) be a root of \( G_{AF} \) then all vertices in star anti fuzzy graph have only one neighbor except the vertex \( \{v\} \). Thus \( V - \{v\} \) is only \( M_{2-ADF} \) set of \( G_{AF} \), hence \( \gamma_{2AF}(G_{AF}) = |V - \{v\}| = \rho - \eta(v) \), \( v \) is a root vertex. □

**Proposition 3.4:** Let \( G_{AF} = C_n \) where the vertices of cycle are \( \{u_1, u_2, \ldots, u_n\} \) and every edge is effective then \( \gamma_{2AF}(C_n) = \max \left\{ \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \eta(u_{i+2j}) : j = 1, 2, \ldots, n \right\} \).

**Proof:** Let \( C_n = \{u_1, u_2, \ldots, u_n\} \) be a cycle, then we have two cases:

Case 1: Let \( n \) be even and let \( D \) be \( 2-ADF \) set of \( G_{AF} \). Thus, \( \forall u \not\in D, u \) must have two neighbors in \( D \) and this condition satisfied if a distance between any two vertices in \( D \) two edges. Thus, \( D_j = \)
\[ \left\{ v_{2k+j} \mid 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, j = 1, 2, \ldots, n \right\}. \]

It clear that \( \mathcal{D}_j \) are the \( M2 - AFD \) set, and there are only two different which are \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) and the other sets are repeated to \( \mathcal{D}_1 \) or \( \mathcal{D}_2 \). Therefore,

\[ y_{2, AF}(C_n) = \max \left\{ \sum_{i=0}^{n-1} \eta(u_{j+2i}) ; j = 1, 2 \right\}. \]

Case 2: if \( n \) is odd.

As same manner in case 1, every vertex in \( V - D \) has two neighbors, so let \( \mathcal{D}_j = \left\{ v_{2k+j} \mid 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, j = 1, 2, \ldots, n \right\} \). The first vertex in each \( \mathcal{D}_j \) is adjacent to the last vertex, so there are \( n \) different \( 2 - AFD \) sets depend on \( j \); \( j = 1, 2, \ldots, n \). One can easily conclude that all the sets \( \mathcal{D}_j \) are \( M2 - AFD \) sets. Therefore,

\[ y_{2, AF}(C_n) = \max \left\{ \sum_{i=0}^{n-1} \eta(u_{j+2i}) ; j = 1, 2, \ldots, n \right\}. \]

From the two cases above, the proof is done. \( \square \)

**Proposition 3.5:** Let \( G_{AF} \equiv P_n \), where the vertices of path are \( (v_1, v_2, \ldots, v_n) \) and every edge is effective, then

\[ y_{2, AF}(P_n) = \begin{cases} \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \eta(v_{2i-1}), & \text{if } n \text{ is odd} \\ \max \left\{ \eta(v_{2i-1}) + \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \eta(v_{2i+2j}) ; i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor ; \ \text{and } 2i + 2j \equiv t \pmod{(n+1)} \right\}, & \text{if } n \text{ is even} \end{cases}. \]

**Proof.** There are two different cases depend on the order of path \( (n) \) as follows.

Case 1. If \( n \) is odd, then \( \mathcal{D} = \left\{ v_{2i-1}, i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \), it is clear that \( \mathcal{D} \) is \( 2 - ADF \) set. Also, the set \( \mathcal{D} \) is \( M2 - AFD \) set, since if there is a set \( F \) such that the number of vertices less than the number of the set \( \mathcal{D} \), then \( F \) is not \( 2 - AFD \) set. Thus,

\[ y_{2, AF}(G) = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \eta(v_{2i-1}). \]

Case 2. If \( n \) is even, the vertices \( v_1 \) and \( v_2 \) must belong to every \( M2 - AFD \) set. In this case, two adjacent vertices must be included in every \( M2 - AFD \) set and distance two edges for each other consecutive two vertices in \( \mathcal{D} \). Now, if \( v_i \) and \( v_{i+1} \) the two adjacent vertices belong to \( \mathcal{D} \) and \( i \) is even, then the next vertices of the set \( \mathcal{D} \) are \( v_{i+1+2j} \) and it is clear that \( i + 1 + 2j \). (id) must be is odd number, so the last vertex in this context is \( v_{n-1} \). Therefore, this \( \mathcal{D} \) is not \( 2 - AFD \) set, since the vertex \( v_n \) has one neighbor. Thus, \( i \) must be odd, so let \( \mathcal{D}_i = \left\{ v_{2i-1}, v_{2i+2j}, j = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \), \( i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) and \( 2i + 2j \equiv t \pmod{(n+1)} \). It is clear that all the sets \( \mathcal{D}_i \) are \( M2 - AFD \) sets. Thus,

\[ y_{2, AF}(P_n) = \max \left\{ \eta(v_{2i-1}) + \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \eta(v_{2i+2j}) ; i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor ; \ \text{and } 2i + 2j \equiv t \pmod{(n+1)} \right\}. \]

\( \square \)

**Example 3.2:** Let \( G_{AF} \equiv P_n \) which given in figure 2
Figure 2

The $M_2-AFD$ sets are $\mathcal{D}_1 = \{x, y, u, w\}$, $\mathcal{D}_2 = \{x, z, v, w\}$, $\mathcal{D}_3 = \{x, z, u, w\}$

$\gamma_{2AF}(G_{AF}) = \max[|\mathcal{D}_1|, |\mathcal{D}_2|, |\mathcal{D}_3|] = \max[2.4, 1.9, 2.2] = 2.4$

Proposition 3.6: Every $2-AFD$ set of $G_{AF} = (\eta, \rho)$ is $AFD$ set of $G_{AF}$.

Proof: direct from definition $2-AFD$ set. □

Corollary 3.1: Let $G_{AF} = (\eta, \rho)$ be a $v-$nodal anti fuzzy graph then $\gamma_{AF}(G_{AF}) \leq \gamma_{2AF}(G_{AF})$

Further equality hold if $\rho(x, y) \neq \eta(x) \lor \eta(y) \forall x, y \in V(G_{AF})$.

Theorem 3.1: For any $G_{AF} = (\eta, \rho)$, $\sum_{\mathcal{E} \in S} \eta(v_i) \leq \gamma_{2AF} \leq P$. Where $S$ be a set containing all the vertices that have at most one neighbour.

Proof: Let $G_{AF} = (\eta, \rho)$ be any anti fuzzy graph, $\mathcal{D}$ be a $2-AFD$ set of $G_{AF}$ such that $|\mathcal{D}| = \gamma_{2AF}(G_{AF})$ and $V(G_{AF}) = V(S) \cup V(H)$, where $S$ the set containing all the vertices that have at most one neighbour and $H$ be the set containing all other vertices of $V(G_{AF})$. Now for lower bounded we have to show that $S \subseteq \mathcal{D}$.

Consider $u \in S$, then $u$ either an isolated vertex or a vertex has one neighbourhood. In both cases, we conclude that $u$ belong to every $M2-AFD$ set of $G_{AF}$. Hence $S \subseteq \mathcal{D}$. ⇒ $|S| \leq |\mathcal{D}|$

Furthermore $\sum_{\mathcal{E} \in S} \eta(u_i) \leq \gamma_{2AF}$. While for upper bound $\gamma_{2AF} \leq P$ is clearly. Hence, $\sum_{\mathcal{E} \in S} \eta(v_i) \leq \gamma_{2AF} \leq P$. □

Example 3.3: Consider $G_{AF} = (\eta, \rho)$ in figure 3.

The $M2-AFD$ sets are $\mathcal{D}_1 = \{v_1,v_3,v_5,v_6\}$, $\mathcal{D}_2 = \{v_1,v_3,v_5,v_6\}$,

$\mathcal{D}_3 = \{v_1,v_4,v_5,v_6\}$, we note that each vertex has one neighbour belongs to every $M2-AFD$ set.

Hence $\gamma_{2AF}(G_{AF}) = \max \{|\mathcal{D}_1|, |\mathcal{D}_2|, |\mathcal{D}_3|\} = \max \{2.5, 1.8, 2.1\} = 2.5 > |S|$, where $S = \{v_1,v_6\}$. 


Corollary 3.2: \( \gamma_{2AF}(G_{AF}) = P \) if and only if every vertex in \( G_{AF} \) is adjacent to less than two vertices.

Corollary 3.3: If \( G_{AF} \equiv mK_2 \) where \( m \geq 1 \), then \( \gamma_{2AF}(G_{AF}) = P \)

Theorem 3.2: Let \( \mathcal{D} \) be a \( 2 - AFD \) set of \( G_{AF} \), then \( V - \mathcal{D} \) not be a \( 2 - AFD \) set in general of \( G_{AF} \).

Proof: Let \( \mathcal{D} \) be \( 2 - AFD \) set of \( G_{AF} \) and assume that \( u \in V(G_{AF}) \)
Case 1: If \( u \) has at most one neighbor in \( G_{AF} \), therefore \( u \) must be in every \( M2 - AFD \) set of \( G_{AF} \), then \( V - \mathcal{D} \) has either one neighbor of \( u \) or has no any neighbor of \( u \), thus \( V - \mathcal{D} \) is not \( 2 - AFD \) set of \( G_{AF} \).
Case 2: Suppose that every vertex in \( \mathcal{D} \) has at least two neighbors in \( V - \mathcal{D} \) in this case every vertex in \( \mathcal{D} \) is dominated by at least two vertices in \( V - \mathcal{D} \). Thus \( V - \mathcal{D} \) is \( 2 - AFD \) set of \( G_{AF} \). Therefore, the result is obtained. by case1 and case2 if \( \mathcal{D} \) be a \( 2 - AFD \) set of \( G_{AF} \) then, \( V - \mathcal{D} \) need not \( 2 - AFD \) set of \( G_{AF} \).

Proposition 3.7: For any anti fuzzy graph \( G_{AF} = (\eta, \rho) \), \( \gamma_{2AF}(G_{AF}) + \gamma_{AF}(G_{AF}) \leq 2P \) and equality hold if \( \rho(u,v) \neq \eta(u) \lor \eta(v) \forall u,v \in V(G_{AF}) \).

Proposition 3.8: For any anti fuzzy \( G_{AF} = (\eta, \rho) \), \( \gamma_{2AF}(G_{AF}) + \gamma_{2AF}(\overline{G_{AF}}) \leq 2P \) Furthermore equality hold if \( 0 < \eta(x) \lor \eta(y) < \rho(x,y) \forall x,y \in V \) and \( 0 < \rho(x,y) < 1 \). where \( \gamma_{2AF}(\overline{G_{AF}}) \) is \( 2 - AFD \) number of complement \( G_{AF} \).

Proof: The inequality is trivial since \( \gamma_{2AF}(G_{AF}) \leq P \) and \( \gamma_{2AF}(\overline{G_{AF}}) \leq P \)
Now, since \( 0 < \eta(x) \lor \eta(y) < \rho(x,y) \forall x,y \in V \) then \( 0 > \eta(x) \lor \eta(y) - \rho(x,y) \), hence \( 0 < 1 - \rho(x,y) + \eta(x) \lor \eta(y) = \overline{\rho(x,y)} \), thus \( \gamma_{2AF}(G_{AF}) = P \) and \( \gamma_{2AF}(\overline{G_{AF}}) = P \).
Hence, \( \gamma_{2AF}(G_{AF}) + \gamma_{2AF}(\overline{G_{AF}}) = 2P \).
Theorem 3.3: Every $2 - AFD$ set $\mathcal{D}$ of $G_{AF} = (\eta, \rho)$ is a $2$-dominating set of $G_{AF}$ but the converse is not necessary is true.

Proof: Let $G_{AF} = (\eta, \rho)$ be anti fuzzy graph and $\mathcal{D}$ is $2 - AFD$ set of $G_{AF}$, then for each vertex $x \in V - \mathcal{D}$ has at least two neighbors in $\mathcal{D}$, i.e there are $y_1, y_2 \in \mathcal{D}$ such that $\rho(x, y_1) = \eta(x) \lor \eta(y_1) > 0$ and $\rho(x, y_2) = \eta(x) \lor \eta(y_2)$, thus $(x, y_1) > \rho^*$ and $(x, y_2) > \rho^*$. Therefore $x$ has two neighbours in $\mathcal{D}$, Hence $\mathcal{D}$ is $2$- dominating set of $G_{AF}$. The below example shows that the converse theorem is not necessary is true. □

Example 3.4: The converse theorem 3.3 is not true. Consider $G_{AF} = (\eta, \rho)$ in figure (4) and anti crisp graph $G_{AF}^*$ in figure (5).

We see that $\mathcal{D} = \{v_1, v_3, v_5\}$ is $2$- dominating set of $G_{AF}^*$ but it is not $2 - AFD$ set of $G_{AF}$.

**Proposition 3.9:** Let $G_{AF} = (\eta, \rho)$ be anti fuzzy graph, then a $2$-dominating set of $G_{AF}^* = (\eta^*, \rho^*)$ is $2 - AFD$ set of $G_{AF}$ if $\rho(x, y) = \eta(x) \lor \eta(y)$ for all $x, y \in V(G_{AF})$.

Proof: Let $\mathcal{D}$ be a $2$-dominating set of $G_{AF}^*$ such that $|\mathcal{D}| = \gamma_{2AF}(G_{AF}^*)$ then $\forall x \in V - \mathcal{D}$ there exist two vertices $y_1, y_2 \in \mathcal{D}$, such that $(x, y_1) > \rho^*$ and $(x, y_2) > \rho^*$, therefore $\rho(x, y_1) > 0$, $\rho(x, y_2) > 0$, since $\rho(x, y_1) = \eta(x) \lor \eta(y_1)$ and $\rho(x, y_2) = \eta(x) \lor \eta(y_2)$, thus $x$ has two effective neighbors in $\mathcal{D}$. Hence, $\mathcal{D}$ is $2 - AFD$ set of $G_{AF}$. □

Corollary 3.4: Let $G_{AF} = (\eta, \rho)$ be any anti fuzzy graph then $\gamma_{2AF}(G_{AF}) \leq \gamma_{2AF}(G_{AF}^*)$, furthermore equality holds if $\rho(x, y) = 1 = \eta(x) \lor \eta(y)$ for all $x, y \in \rho^*$. 

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Proof: Let $\mathcal{D}$ be $2-AFD$ set of $G_{AF} = (\eta, \rho)$, by Theorem (3.3) then $\mathcal{D}$ is a 2-dominating set of $G_{AF}$ since a crisp anti fuzzy graph is anti fuzzy graph with $\eta(x) = 1$ for all $x \in V$ and $\rho(x, y) = 1$ for all $(x, y) \in E$. Therefore, $\gamma_{2AF}(G_{AF}) \leq \gamma_{2AF}(G_{AF}^*)$. Now, If $\mathcal{D}$ is a 2-dominating set of $G_{AF}$ with the above condition then by proposition (3.8) we get $\mathcal{D}$ is $2-AFD$ set of $G_{AF}$. Hence, $\gamma_{2AF}(G_{AF}) = \gamma_{2AF}(G_{AF}^*)$. □

Theorem 3.4: Let $G_{AF} = (\eta, \rho)$ be an uninodal anti fuzzy graph without isolated, $\mathcal{D}$ be a $\gamma_{2AF}$- set of $G_{AF}$ which is not independent set then $\gamma_{AF} + t \leq \gamma_{2AF}$, where $t = \eta(x), x \in \mathcal{D}$.

Proof: Let $\mathcal{D}$ be $2-AFD$ such that $|\mathcal{D}| = \gamma_{2AF}$ of $G_{AF}$ which, is not independent set, and suppose that $x \in \mathcal{D}$ such that $t = |x|$. We have two cases:

Case 1: If $N(x) \cap (V - \mathcal{D}) = \emptyset$, since $G_{AF}$ has no isolates vertex and $\mathcal{D}$ is not independent then $N(x) \cap \mathcal{D} \neq \emptyset$. Thus $\mathcal{D} - \{x\}$ is $AFD$ set of $G_{AF}$, therefor $\gamma_{AF} \leq |\mathcal{D} - \{x\}| = \gamma_{2AF} - t$

Hence, $\gamma_{AF} + t \leq \gamma_{2AF}$.

Case 2: If $N(x) \cap (V - \mathcal{D}) \neq \emptyset$, then for any vertex $y \in N(x) \cap (V - \mathcal{D})$ since $\mathcal{D}$ is $2-AFD$ there exist a vertex $z \in \mathcal{D}$ such that $\rho(y, z) = \eta(y) \lor \eta(z)$ since $\mathcal{D}$ is $2-AFD$. Also, $\mathcal{D}$ is not independent, thus $z$ has neighbor with some vertices of $\mathcal{D}$, thus $\mathcal{D} - \{z\}$ is also $AFD$ set of $G_{AF}$.

Therefore, $\gamma_{AF} \leq |\mathcal{D} - \{z\}| = \gamma_{2AF} - t$, Hence $\gamma_{AF} + t \leq \gamma_{2AF}$. □

Theorem 3.5: If $G_{AF}$ is any anti fuzzy graph with each vertex has at least two neighbors then every anti fuzzy vertex cover set is a $2-AFD$ set of $G_{AF}$. Further $\gamma_{2AF} \leq \alpha_0$.

Proof: Let $\mathcal{A}$ be a minimum anti fuzzy vertex cover set of $G_{AF}$, and let $y \in V(G_{AF}) - A$. Clearly, $N(y) \subseteq A$. Since each vertex in $G_{AF}$ has at least two neighbors, the vertex $y$ is adjacent to at least two vertices of $A$. This implies that $A$ is $2-AFD$ set of $G_{AF}$. Hence, $\gamma_{2AF} \leq \alpha_0$. □

Corollary 3.5: Let $G_{AF}$ be anti fuzzy graph. If $\gamma_{2AF}(G_{AF}) \neq \alpha_0(G_{AF})$ then $G_{AF}$ contains a vertex has at most one neighbor.

Theorem 3.6: If $G_{AF} = (\eta, \rho)$ is a connected an uninodal anti fuzzy graph and with $\gamma_{AF} = \gamma_{2AF}$ then each vertex has at least two neighbors.

Proof: Assume that there is a vertex $x$ has one neighbor, and let $\mathcal{D}$ be a $M2-AFD$ set of $G_{AF}$ such that $|\mathcal{D}| = \gamma_{2AF}$, then $x \in \mathcal{D}$, if $V - \mathcal{D} = \emptyset$ then $\gamma_{AF} < \gamma_{2AF} < \rho = \gamma_{2AF}$ which is a contradiction,
thus $V - D \neq \emptyset$. Let $y$ be the neighbor of the $x$ if $y \in D$, then $D' = D - \{x\}$ is AFD set of $G_{AF}$ with $|D'| = |D - \{x\}|$ this implies $\gamma_{AF} \neq \gamma_{2AF}$, a contradiction. If $y \in V - D$, since $D$ is $M2 - AFD$ set, then there exists a vertex $z \in N(y) \cap D$ with $z \neq x$ and each vertex in $V - D$ has at least two neighbors in $D$, we note that $H = D - \{z, x\} \cup \{y\}$ is AFD set of $G_{AF}$ with $|H| = |D - \{z, x\} \cup \{y\}|$ thus $\gamma_{AF} \neq \gamma_{2AF}$ which a contradiction. Hence, every vertex in $G_{AF}$ has at least two neighbors when $\gamma_{AF} = \gamma_{2AF}$. □

**Proposition 3.10:** Let $G_{AF} = (\eta, \rho)$ be any anti fuzzy graph with every vertex has neighbors at least two then $\gamma_{2AF} + \beta_0 \leq \rho$.

**Proof:** Let $S$ be a maximum independent anti fuzzy set of $G_{AF}$. Then all neighbors to each vertex of $S$ in $V - S$. Since each vertex has at least two neighbors, it follows that $V - S$ is a $2 - AFD$ set of $G_{AF}$. Thus $\gamma_{2AF} \leq |V - S| = \rho - \beta_0$. Hence, $\gamma_{2AF} + \beta_0 \leq \rho$. □

**Proposition 3.11:** If $G_{AF} = (\eta, \rho)$ be anti fuzzy graph with a unique maximal independent anti fuzzy set, then $\gamma_{2AF} \leq \beta_0$.

**Proof:** Let $S$ be a unique maximal independent anti fuzzy set of $G_{AF}$. assume that there is a vertex $x \in V - S$. if $x$ has not neighbor then $x$ must be in $S$.Thus $x$ has unique neighbor $y \in S$, then $S - \{y\} \cup \{x\}$ it is a second maximal independent anti fuzzy set of $G_{AF}$.This leads to a contradiction with $S$. Hence, $\gamma_{2AF} \leq |S| = \beta_0$

4. **Conclusion.**

In this work, $2 - AFD$ set and $2 - AF$ domination number are defined in $G_{AF}$. The $2 - AF$ domination number is applied on different types of $G_{AF}$ and some relationship between ($2 - AFD$ number) with other parameters are discussed.
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