KÄHLER METRICS WITH CONSTANT WEIGHTED SCALAR CURVATURE AND WEIGHTED K-STABILITY

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Abstract. We introduce a notion of a Kähler metric with constant weighted scalar curvature on a compact Kähler manifold $X$, depending on a fixed real torus $T$ in the reduced group of automorphisms of $X$, and two smooth (weight) functions $v > 0$ and $w$, defined on the momentum image (with respect to a given Kähler class $\alpha$ on $X$) of $X$ in the dual Lie algebra of $T$. A number of natural problems in Kähler geometry, such as the existence of extremal Kähler metrics and conformally Kähler, Einstein–Maxwell metrics, or prescribing the scalar curvature on a compact toric manifold reduce to the search of Kähler metrics with constant weighted scalar curvature in a given Kähler class $\alpha$, for special choices of the weight functions $v$ and $w$.

We show that a number of known results obstructing the existence of constant scalar curvature Kähler (cscK) metrics can be extended to the weighted setting. In particular, we introduce a functional $M_{v,w}$ on the space of $T$-invariant Kähler metrics in $\alpha$, extending the Mabuchi energy in the cscK case, and show (following the arguments in [60, 75] in the cscK and extremal cases and [55] in the case of conformally Kähler Einstein–Maxwell) that if $\alpha$ is Hodge, then constant weighted scalar curvature metrics in $\alpha$ are minima of $M_{v,w}$. Motivated by the recent work [27, 28, 31, 32] in the cscK and extremal cases, we define a $(v,w)$-weighted Futaki invariant of a $T$-compatible smooth Kähler test configuration associated to $(X, \alpha, T)$, and show that the boundedness from below of the $(v,w)$-weighted Mabuchi functional $M_{v,w}$ implies a suitable notion of a $(v,w)$-weighted K-semistability.

We illustrate our theory with specific computations on smooth toric varieties and on the toric fibre bundles introduced in [5]. As an application, we obtain a Yau–Tian–Donaldson type correspondence for $(v,w)$-extremal Kähler classes on $\mathbb{P}^1$-bundles over products of compact Hodge cscK manifolds, thus extending some of the results in [4, 7] to the $(v,w)$-weighted setting.

1. Introduction

In this paper, we define a notion of a (weighted) $v$-scalar curvature $\text{Scal}_v(\omega)$, associated to a Kähler metric $\omega$ on a smooth compact complex manifold $X$, a real torus $T$ in the reduced group $\text{Aut}_{\text{red}}(X)$ of automorphisms of $X$, and a positive smooth function $v(p)$ defined over the image $P \subset t^*$ of $X$ under the moment map $m_\omega: X \rightarrow t^*$ of $T$ with respect to $\omega$. Here $t^*$ stands for the dual vector space of the Lie algebra $t$ of $T$ and $p$ for a point of $t^*$. Originally, we have identified in [55] $\text{Scal}_v(\omega)$ with a coefficient in the asymptotic expansion of a certain weighted Bergman kernel associated with a function $v$ (see Theorem A.1 for a precise statement) reminiscent to the celebrated results by Catlin [18], Ruan [74], Tian [86] and Zelditch [89] in the case $v = 1$. Our main motivation in this paper for introducing and systematically studying the $v$-scalar curvature is the observation that the problem of finding a $T$-invariant Kähler metric $\omega$ in a given Kähler class $\alpha$ on $X$, for which

$$\text{Scal}_v(\omega) = c_{v,w}(\alpha)w(m_\omega),$$

where $w(p)$ is another given smooth function on $P$ and $c_{v,w}(\alpha)$ is a suitable real constant (depending only on $\alpha$, $P$, $v$ and $w$) englobes a number of problems in Kähler geometry of current interest, including the following well-studied cases:

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We shall refer to the solutions of (1) as constant (v, w)-scalar curvature Kähler metrics or (v, w)-cscK for short and our main thesis in this paper is that most of the known obstructions to the existence of cscK metrics extend naturally to the (v, w)-cscK case. Indeed, as we show in the Appendices A and B to this paper, some of our previous results in [54, 55] regarding (iv) and (v) are just special cases of more general statements concerning (v, w)-cscK metrics, thus providing a more conceptual explanation for the arguments therein. In particular, there is a natural generalization $\mathcal{M}_{v,w}$ of the Mabuchi functional (see e.g. [45, 80]) on the space of $\mathbb{T}$-invariant Kähler metrics in $\alpha$ (which we call the $(v, w)$-Mabuchi energy and define in Section 5 below) and we show that the arguments of [55, Thm. 1] (which in turn build on [60, 75]) yield the following

**Theorem 1.** Let $(X, L)$ be a compact smooth polarized projective variety, $\mathbb{T} \subset \text{Aut}_{\text{red}}(X)$ a real torus, and suppose that $X$ admits a $(v, w)$-cscK metric $\omega$ in $\alpha = 2\pi c_1(L)$ for some smooth functions $v > 0$ and $w$ on the momentum image $P \subset \mathbb{T}^*$ associated to $(\mathbb{T}, \alpha)$. Then, $\omega$ is a global minima of the $(v, w)$-Mabuchi energy $\mathcal{M}_{v,w}$ of $(X, \mathbb{T}, \alpha, P, v, w)$.

Instead of (1), one can more generally consider the condition

$$\text{Scal}_\alpha(\omega) = w(m_\omega)(m_\omega^\xi + c)$$

(2)

for a $\mathbb{T}$-invariant Kähler metric $\omega$ in $\alpha$, where $\xi \in \mathbb{T}$, $c \in \mathbb{R}$ and $m_\omega^\xi := \langle m_\omega, \xi \rangle$ is the Killing potential associated to $\xi$. A $\mathbb{T}$-invariant Kähler metric satisfying (2) generalizes the notion of an extremal Kähler metric (see (1) above), and will be referred to as a $(v, w)$-extremal Kähler metric. As it is apparent from the example (11) above, and as we establish more generally in Section 3 when $w > 0$ the smooth function $(m_\omega^\xi + c)$ in the RHS of (2) must be of the form $w_{\text{ext}}(m_\omega)$ for an affine-linear function $w_{\text{ext}}(p) = \langle \xi, p \rangle + c$ on $\mathbb{T}^*$ defined in terms of $(\mathbb{T}, \alpha, P, v, w)$. Thus, the problem (2) of finding $(v, w)$-extremal Kähler metrics in $\alpha$ reduces to the problem (11) of finding $(v, w_{\text{ext}})$-cscK metrics. Furthermore, as we show in Lemma 6 the corresponding Mabuchi energy $\mathcal{M}_{v,w_{\text{ext}}}$ coincides with the $(v, w)$-weighted relative Mabuchi energy (see e.g. [45, 80] for the
Corollary 1. Let \((X, L)\) be a compact smooth polarized projective variety, \(\mathbb{T} \subset \text{Aut}_{\text{red}}(X)\) a real torus, and suppose that \(X\) admits a \((v, w)\)-extremal metric in \(\alpha = 2\pi c_1(L)\) for some positive smooth functions \(v, w\) defined on the momentum image \(P \subset t^*\) associated to \((\mathbb{T}, \alpha)\). Then, the relative \((v, w)\)-Mabuchi energy \(\mathcal{M}_{v, w}^{\text{rel}}\) of \((X, \mathbb{T}, \alpha, P, v, w)\) is bounded from below.

The above corollary provides a scope of extending [7, Thm. 2] to an obstruction to the existence of \((z + a, p)\)-extremal metrics in the sense of [7], in rational admissible Kähler classes on admissible projective bundles. We explore this ramification in Theorem 8 below.

With the above in mind, the main goal of this paper is to introduce a suitable notion of \((v, w)\)-K-stability associated to \((X, \alpha, \mathbb{T}, P, v, w)\) as above, extending the corresponding notion in the cscK and the extremal cases, introduced by Tian [81, 84], Donaldson [36] and Székelyhidi [77], and extensively studied in recent times. Our inspiration comes mainly from the recent works [28, 27, 31] which, in turn, build on the seminal work of Tian [84, 85] and a key observation by Wang [87] and Odaka [70, 71] that the Donaldson–Futaki invariant of a (suitably compactified) test configuration can be realized as an intersection number defined on the total space of the latter. In the cscK case, it is shown by Dervan–Ross [28, Prop. 2.23] and [31, Prop. 3.12] that in order to test K-stability (or K-semi-stability) of a compact Kähler manifold, it is sufficient to control the sign of the Donaldson–Futaki invariant of test configurations which are smooth and whose central fibre is reduced. This allows one to rewrite the Donaldson–Futaki invariant as a global differential geometric quantity of the test configuration. This is precisely the setting in which we introduce the notion of a \((v, w)\)-Futaki invariant of a smooth Kähler test configuration with reduced central fibre, compatible with \((X, \alpha, \mathbb{T})\), and show in Theorem 7 that it must be non-negative should the \((v, w)\)-Mabuchi energy associated to \((X, \mathbb{T}, \alpha, P, v, w)\) is bounded from below. This, combined with Theorem 1 and Proposition 2 in Section 6 yields our main result, which establishes one direction of a Yau–Tian–Donaldson type correspondence for the existence of \((v, w)\)-cscK metrics.

Theorem 2. Let \((X, L)\) be a compact smooth polarized projective variety, \(\mathbb{T} \subset \text{Aut}_{\text{red}}(X)\) a real torus, and suppose that \(X\) admits a \((v, w)\)-cscK metric in \(\alpha = 2\pi c_1(L)\). Then \(X\) is \((v, w)\)-K-semistable on smooth, \(\mathbb{T}\)-compatible Kähler test configuration with reduced central fibre associated to \((X, \alpha)\), i.e. the \((v, w)\)-Futaki invariant of any such test configuration is non-negative.
clear whether such an algebraic definition of a \((v, w)\)-Donaldson–Futaki invariant can be given for any central fibre \(X_0\), nor that it would agree with our differential geometric definition on the total space of a smooth test configuration. In fact, when \(v\) and \(w\) are not polynomials, the proposed algebraic definition of a \((v, w)\)-Donaldson–Futaki invariant of \(X_0\) involves transcendental quantities leading to difficulties reminiscent (but somewhat more complex) to the ones involved in the definition of the \(L^p\)-norm of a test configuration for positive real values of \(p\), see the discussion at the end of [35]. Nevertheless, we prove that the two approaches give the same invariant in two special cases: when the test configuration is a smooth submersion (Corollary [4]) and when \((X, \alpha, \mathcal{T})\) is a smooth toric variety, and the test configuration is a toric test configuration in the sense of [36] (Proposition [4]).

Using a blowup technique and the gluing theorem of Arezzo–Pacard–Singer [8], Stoppa [79] and Stoppa–Székelyhidi [78] have shown that the existence of a csK or extremal Kähler metric in \(2\pi c_1(L)\) does actually imply \((1, w_{\text{ext}})\)-K-stability relative to \(\mathcal{T}\), i.e. that the corresponding Futaki invariant of a non-product polarized normal test configuration is strictly positive. Similar results hold true for Kähler test configurations by [27] [32]. At this point, it is not clear to us whether or not these techniques can be extended to the \((v, w)\)-case.

Finally, one might hope to extend Theorem 2 beyond the polarized case. Indeed, in the csK and extremal cases such extensions have been found in [27] [28] [32] by using a deep result of Berman–Berndtsson [11] on the convexity and boundedness of the Mabuchi functional. We expect that along the method of [11] (and using Theorem 5 below) similar properties can possibly be established for the \((v, w)\)-Mabuchi functional, but the details go beyond the scope of the present article. We however notice that the arguments in [11] hold true in the case when \(v \equiv 1\) and \(w\) is arbitrary, see Theorem 6. We thus have (by virtue of Theorem 7)

**Theorem 3.** [11] Let \(X\) be a smooth compact Kähler manifold, \(\mathcal{T} \subset \text{Aut}_{\text{red}}(X)\) a real torus, and suppose that \(X\) admits a \((1, w)\)-csK metric \(\omega\) in the Kähler class \(\alpha\) for some smooth function \(w\) on the momentum image \(P \subset \mathfrak{t}^*\) associated to \((\mathcal{T}, \alpha)\). Then, the \((1, w)\)-Mabuchi energy \(\mathcal{M}_{1, w}\) of \((X, \mathcal{T}, \alpha, P, w)\) is bounded from below, and \(X\) is \((1, w)\)-K-semistable on smooth, \(\mathcal{T}\)-compatible Kähler test configuration with reduced central fibre associated to \((X, \alpha)\).

### 1.1. Outline of the paper.

In Section 2 we introduce the weighted v-scalar curvature of a \(\mathcal{T}\)-invariant Kähler metric and the constant \(c_{v, w}(\alpha)\) in [11], in terms of the data \((\alpha, P, v, w)\) on \((X, \mathcal{T})\). As our definitions are new, in Section 3 we describe in some more detail the examples listed in [3] [6] above. In Section 4 we generalize the arguments of Donaldson [23] and Fujiki [41] in the csK case, and of Apostolov–Maschler [6] in the conformally Kähler, Einstein–Maxwell case, thus providing a formal GIT interpretation of the problem of finding solutions of \(1\) within a given Kähler class \(\alpha\) on \(X\). In Section 5.1 we introduce the \((v, w)\)-Mabuchi energy on \((X, \alpha, \mathcal{T})\) associated to [11]. Our main result here is Theorem 5 which extends the Chen–Tian formula for the Mabuchi functional to the general \((v, w)\)-case. In Section 5.2, assuming \(w > 0\), we define the relative \((v, w)\)-Mabuchi energy \(\mathcal{M}_{v, w}\) associated to the problem [2] and show that it is given by the \((v,\omega, \omega_{\text{ext}})\)-Mabuchi energy for a suitable affine linear function \(\omega_{\text{ext}}\) on \(\mathfrak{t}^*\). In Section 5.3 we show the boundedness of the \((1, w)\)-Mabuchi energy. In Section 6 we define the differential-geometric \((v, w)\)-Futaki invariant on \((X, \alpha, \mathcal{T})\) and show in Proposition 2 that it provides a first obstruction of the existence of a solution of \(1\). In the next Section 7 we introduce a global invariant, which we call the \((v, w)\)-Futaki invariant, on a \(\mathcal{T}\)-compatible smooth Kähler test configuration associated to \((X, \mathcal{T}, \alpha)\). We observe in Proposition 3 that by an adaptation of the original arguments
of [30], when the test configuration is a smooth submersion, the corresponding \((v, w)\)-Futaki invariant agrees with the differential-geometric \((v, w)\)-Futaki invariant of \(X_0\). Our main result here is Theorem 7 which shows how in the case when the central fibre is reduced the \((v, w)\)-Futaki invariant of a \(\mathbb{T}\)-compatible Kähler test configuration associated to \((X, \mathbb{T}, \alpha)\) is related to the \((v, w)\)-Mabuchi functional of \((X, \mathbb{T}, \alpha)\). It yields Theorem 2 from the introduction modulo Theorem 1 which we establish in Appendix A. The arguments in the proof of Theorem 7 go back to the foundational works [30, 84] and are very close to the ones in [28, 31]. In Section 8, we discuss the alternative approach to defining a Futaki invariant of a \(\mathbb{T}\)-compatible polarized test configuration in terms of algebraic constructions on the central fibre \(X_0\), as suggested in [6, 7]. In the case when the test configuration is a smooth submersion, by using the asymptotic expansion of the \((v, w)\)-equivariant Bergman kernel, we show in Corollary 4 that two approaches produce the same invariant. Similar result is established in Proposition 3 in the case when \((X, \alpha, \mathbb{T})\) is a smooth toric variety and we consider toric test configurations in the sense of [36]. In Section 10 we consider the case when \((X, \alpha, \mathbb{T})\) is a toric fiber-bundle over the product of cscK smooth projective manifolds, given by the generalized Calabi construction of [5]. We compute the \((v, w)\)-Futaki invariant of certain test configurations of \((X, \alpha, \mathbb{T})\), defined in terms of the toric geometry of the fiber. As an application of our theory, in the case when \(X\) is a \(\mathbb{P}^1\)-bundle over a product of cscK smooth projective manifolds we derive a Yau-Tian-Donaldson type correspondence for \((v, w)\)-extremal Kähler classes in terms of the positivity of a single function of one variable over the interval \((-1, 1)\).

In the Appendices A and B, we extend some of our previous results obtained for special values of \(v\) and \(w\) to the general case, including the proof of Theorem 1 from the introduction, a structure result for the automorphism group of a \((v, w)\)-extremal Kähler metric with \(w > 0\), as well as a stability under deformation of \(v, w\) and \(\alpha\) of the solution of (2) (again assuming \(w > 0\)).

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2. The \(v\)-scalar curvature

Let \(X\) be a compact Kähler manifold of complex dimension \(n \geq 2\). We denote by \(\text{Aut}_{\text{red}}(X)\) the reduced automorphism group of \(X\) whose Lie algebra \(\mathfrak{h}_{\text{red}}\) is given by real holomorphic vector fields with zeros (see [43]). Let \(\mathbb{T}\) be an \(\ell\)-dimensional real torus in \(\text{Aut}_{\text{red}}(X)\) with Lie algebra \(\mathfrak{t}\), and \(\omega\) a \(\mathbb{T}\)-invariant Kähler form on \(X\). We denote by \(\mathcal{K}_\omega^\mathbb{T}\) the space of \(\mathbb{T}\)-invariant Kähler potentials with respect to \(\omega\), and for any \(\phi \in \mathcal{K}_\omega^\mathbb{T}\) by \(\omega^\phi = \omega + dd^c\phi\) the corresponding Kähler form in the Kähler class \(\alpha\). It is well-known that the \(\mathbb{T}\)-action on \(X\) is \(\omega^\phi\)-Hamiltonian (see [43]) and we choose \(m_\phi : X \rightarrow \mathfrak{t}^*\) to be a \(\omega^\phi\)-momentum map of \(\mathbb{T}\). It is also known [41, 48] that \(P_\phi := m_\phi(X)\) is a convex polytope in \(\mathfrak{t}^*\). Furthermore, the following is true

Lemma 1. The following facts are equivalent:

(i) For any \(\phi \in \mathcal{K}_\omega^\mathbb{T}\) we have \(P_\phi = P_\omega\).

(ii) For any \(\phi \in \mathcal{K}_\omega^\mathbb{T}\) we have \(\int_X m_\phi \omega_n^\phi = \int_X m_\omega \omega_n\), where \(\omega_n^\phi := \frac{\omega^n}{n!}\) is the volume form.

(iii) For any \(\xi \in \mathfrak{t}\) and \(\phi \in \mathcal{K}_\omega^\mathbb{T}\) we have \(m_\phi^\xi = m_\omega^\xi + \tilde{d}^c\phi(\xi)\), where \(m_\phi^\xi := \langle m_\phi, \xi \rangle\).
Proof. Presumably, Lemma 1 is well-known, see e.g. [8] Section 4 and [79] Section 3.1 for the case of a single hamiltonian. We include an argument for the sake of completeness. We start by proving that (iii) is equivalent with (iii). By the very definition of the momentum map, Cartan’s formula and the fact that $\xi$ is a real holomorphic vector field we have

$$d(m_\xi^\phi - m_\phi) = -d(d^c \phi(\xi)).$$

Thus, there exist a $\alpha_\phi \in \mathfrak{t}^*$ such that

$$m_\xi^\phi = m_\phi + d^c \phi(\xi) + \alpha_\phi(\xi).$$

Suppose that (ii) holds. Then $\alpha_\phi$ is given by

$$\alpha_\phi(\xi) = \frac{1}{\text{Vol}(X,\alpha)} \left( \int_X m_\xi^\phi \omega^{[n]} - \int_X (m_\phi^\xi + d^c \phi(\xi)) \omega_\phi^{[n]} \right).$$

For a variation $\dot{\phi}$ of $\phi$ in $\mathcal{K}_G^\times$, the corresponding variation of $\alpha_\phi$ is given by

$$-\text{Vol}(X,\alpha) \dot{\phi}_\phi(\xi) = \int_X m_\xi^\phi d^c \dot{\phi} \wedge \omega_\phi^{[n]} + \int_X d^c \phi(\xi) \omega_\phi^{[n]} + \int_X d^c \phi(\xi) d^c \dot{\phi} \wedge \omega_\phi^{[n-1]}$$

$$= \int_X d^c \phi(\xi) d^c \dot{\phi} \wedge \omega_\phi^{[n]} + \int_X dm_\phi^\xi \wedge d^c \phi \wedge \omega_\phi^{[n-1]}$$

$$+ \int_X (-dm_\phi^\xi + d(d^c \phi(\xi))) \wedge \omega_\phi^{[n-1]}$$

$$= \int_X (d(d^c \phi(\xi)) \wedge \omega_\phi^{[n-1]} + \int_X d^c \phi(\xi) d^c \dot{\phi} \wedge \omega_\phi^{[n-1]} = 0,$$

where we have used (3), the fact that $d^c \phi(\xi) \omega_\phi^{[n]} = dm_\phi^\xi \wedge d^c \phi \wedge \omega_\phi^{[n-1]}$, and integration by parts. It follows that $\alpha_\phi = \alpha_\omega = 0$ which gives the implication (iii) $\Rightarrow$ (iii). Conversely if we suppose that (iii) holds, then for any variation $\omega = dd^c \phi$ of $\omega_\phi$ in $\mathcal{K}_G$, we get

$$\frac{d}{dt} \left( \int_X m_\xi^{\phi(t)} \omega_\phi^{[n]} \right) = \int_X -m_\phi^{\xi(t)} d^c \phi \wedge \omega_\phi^{[n]} + d^c \phi(\xi) \omega_\phi^{[n]} = 0.$$

It follows that $\int_X m_\xi^{\phi(t)} \omega_\phi^{[n]} = \int_X m_\xi^\phi \omega_\phi^{[n]}$ for any $\xi \in \mathfrak{t}$, which yields (iii).

Now we prove the equivalence between (i) and (iii). Suppose that (i) is true and let $x \in X$ be a fixed point for the $T$-action on $X$. Then we have

$$m_\phi(x) - m_\omega(x) = (d^c \phi)_x + \alpha_\phi = \alpha_\phi.$$

By a result of Atiyah and Guillemin–Sternberg (see [9] [15]) $P_\phi$ (resp. $P_\omega$) is the convex hull of the image by $m_\phi$ (resp. $m_\omega$) of the fixed points for the $T$-action. It then follows from (b) that $P_\phi = P_\omega + \alpha_\phi$. Using $P_\omega = P_\phi$, we get $\alpha_\phi = 0$ which proves (iii). For the inverse implication if $m_\phi(x) - m_\omega(x) = (d^c \phi)_x$ for any $x \in X$, then $m_\phi(x) = m_\omega(x)$ for any point $x \in X$ fixed by the $T$-action and we have $P_\phi = P_\omega$ by [9] [38].

It follows from Lemma 1 that for each $\phi \in \mathcal{K}_G$ we can normalize $m_\phi$ such that the momentum polytope $P = m_\phi(X) \subset \mathfrak{t}^*$ is $\phi$-independent.

Definition 1. For $\nu \in C^\infty_0(\mathbb{P}_x \mathbb{R}^{>0})$ we define the $\nu$-scalar curvature of the Kähler metric $g_\phi = \omega_\phi(\cdot, \cdot)$ for $\phi \in \mathcal{K}_G$ to be

$$\text{Scal}_\nu(\phi) := \nu(m_\phi) \text{Scal}(g_{\phi}) + 2 \Delta_\phi(\nu(m_\phi)) + \text{Tr}(G_\phi \circ (\text{Hess}(\nu) \circ m_\phi)),$$

where $m_\phi$ is the momentum map of $\omega_\phi$ normalized as in Lemma 1. $\text{Scal}(g_{\phi})$ is the scalar curvature, $\Delta_\phi$ is the Riemannian Laplacian on functions of the Kähler metric $\omega_\phi$ and Hess($\nu$) is the hessian of $\nu$, viewed as bilinear form on $\mathfrak{t}^*$ whereas $G_\phi$ is the bilinear form with smooth coefficients on $\mathfrak{t}$, given by the restriction of the Riemannian metric $g_\phi$ on fundamental vector fields.
In a basis $\xi = (\xi_i)_{i=1,\ldots,\ell}$ of $t$ we have
\[
\text{Tr}(G_\phi \circ (\text{Hess}(v) \circ m_\phi)) := \sum_{1 \leq i, j \leq \ell} v_{ij}(m_\phi)g_\phi(\xi_i, \xi_j),
\]
where $v_{ij}$ stands for the partial derivatives of $v$ with respect to $\xi$.

**Definition 2.** Let $\theta$ be a $T$-invariant closed $(1,1)$-form on $X$. A $\theta$-momentum map for the action of $T$ on $X$ is a smooth $T$-invariant function $m_\theta : X \to t^*$ with the property $\theta(\xi, \cdot) = -dm_\phi^\xi$ for all $\xi \in t$.

**Lemma 2.** Let $\theta$ be a fixed $T$-invariant closed $(1,1)$-form and $m_\theta$ a momentum map for $\theta$. Then with the normalization for $m_\phi$ given by Lemma 4 the following integrals are independent of the choice of $\phi \in K^T_\omega$,
\[
A_\phi(\psi) := \int_X v(m_\phi)\omega^{[n]},
\]
\[
B_\phi^\theta(\psi) := \int_X v(m_\phi)\theta \wedge \omega^{[n-1]} + \langle (dv)(m_\phi), m_\theta \rangle \omega^{[n]},
\]
\[
C_\phi(\psi) := \int_X \text{Scal}_\psi(\phi)\omega^{[n]}.
\]

**Proof.** The fact that $A_\phi(\psi)$ is constant is well known, see e.g. [27, Theorem 3.14]. The constancy of $B_\phi^\theta(\psi)$ can be easily established by a direct computation, but it also follows from the arguments in the proof of Lemma 4 below. Indeed, we note that $B_\phi^\theta(\psi)(1)$ where $B_\phi^\theta$ is the 1-form on $K_\omega^T$ given by (17). By taking $\psi = 1$ in (18) we get $\langle \delta B_\phi^\theta(\psi) \rangle = 0$ where $\psi$ is a $T$-invariant function on $X$ defining a $T$-invariant variation $\omega = dd^c \psi$ of $\omega_\phi$. From this we infer that $B_\phi^\theta(\psi)$ is constant. For the last function $C_\phi(\psi)$, we will calculate its variation $\langle \delta C_\phi \rangle$ with respect to a $T$-invariant variation $\omega = dd^c \phi$ of $\omega_\phi$. For this, we use that the variation of Scal$_\psi(\phi)$ is given by
\[
(\delta \text{Scal}_\psi)(\phi) = -2(D^-d^*)v(m_\phi)(D^-d)\phi + (d\text{Scal}_\psi(\phi), d\phi),
\]
where $D$ is the Levi-Civita connection of $\omega_\phi$, $D^-d$ denotes the $(2,0) + (0,2)$ part of $Dd$ with respect to $J$ and $(D^-d)^*$ is the formal adjoint operator of $(D^-d)$ (see [45, Section 1.23]). Formula (17) will be established in the Appendix B, see (92). By (7), we calculate
\[
\langle \delta C_\phi \rangle(\phi) = \int_X -2(D^-d^*)v(m_\phi)(D^-d)\phi \omega^{[n]} + \int_X d\text{Scal}_\psi(\phi) \wedge d^c \phi \wedge \omega^{[n-1]} - \int_X \text{Scal}_\psi(\phi) dd^c \phi \wedge \omega^{[n-1]}.
\]
Integration by parts yields $\langle \delta C_\phi \rangle(\phi) = 0$. Thus $C_\phi$ does not depend on the choice of $\phi \in K^T_\omega$.

**Definition 3.** Let $(X, \alpha)$ be a compact Kähler manifold, $T \subset \text{Aut}_{red}(X)$ a real torus with momentum image $P \subset t^*$ associated to $\alpha$ as in Lemma 4 and $v \in C^\infty(P, \mathbb{R})$, $w \in C^\infty(P, \mathbb{R})$. The $(v,w)$-slope of $(X, \alpha)$ is the constant given by
\[
(c_{(v,w)}(\alpha)) := \begin{cases} 
\frac{\int_X \text{Scal}_\psi(\omega)\omega^{[n]}}{\int_X w(m_\omega)\omega^{[n]}}, & \text{if } \int_X w(m_\omega)\omega^{[n]} \neq 0 \\
1, & \text{if } \int_X w(m_\omega)\omega^{[n]} = 0,
\end{cases}
\]
which is independent from the choice of $\omega \in \alpha$ by virtue of Lemma 2.

**Remark 1.** If $\phi \in K^T_\omega$ defines a Kähler metric which satisfies Scal$_\psi(\phi) = cw(m_\phi)$ for some real constant $c$ and $\int_X w(m_\omega)\omega^{[n]} \neq 0$, then we must have $c = c_{(v,w)}(\alpha)$ with $c_{v,w}(\alpha)$ given by (8).
Because of Remark 4 above, and to simplify the notation in the case when $\int_X w(m_\omega)\omega^n = 0$, we adopt the following definition

**Definition 4.** Let $(X, \alpha)$ be a compact Kähler manifold, $\mathbb{T} \subset \text{Aut}_{\text{red}}(X)$ a real torus with momentum image $P \subset \mathbb{T}^*$ associated to $\alpha$ as in Lemma 1 and $v \in C^\infty(P, \mathbb{R}_{>0})$, $w \in C^\infty(P, \mathbb{R})$. A $(v, w)$-cscK metric $\omega \in \alpha$ is a $\mathbb{T}$-invariant Kähler metric satisfying (1), where $c_{v, w}(\alpha)$ is given by (3).

3. Examples

We list below some geometrically significant examples of $(v, w)$-cscK metrics, obtained for special values of the weight functions $v, w$.

3.1. Constant scalar curvature and extremal Kähler metrics. When $v \equiv 1$, $\text{Scal}_v(\phi) = \text{Scal}(\phi)$ is the usual scalar curvature of the Kähler metric $\omega_\phi \in K^\mathbb{T}_0$, so letting $w \equiv 1$ the problem (1) reduces to the Calabi problem of finding a cscK metric in the Kähler class $\alpha = [\omega]$. In this case, we can take $\mathbb{T} \subset \text{Aut}_{\text{red}}(X)$ to be a maximal torus by a result of Calabi [17]. More generally, for a fixed maximal torus $\mathbb{T} \subset \text{Aut}_{\text{red}}(X)$ we can consider the more general problem of the existence of an extremal Kähler metric in $K^\mathbb{T}_0$, i.e. a Kähler metric $\omega_\phi$ such that $\text{Scal}(\phi)$ is a Killing potential for $\omega_\phi$. As the Killing vector field $\xi_{\text{ext}}$ generated by $\text{Scal}(\phi)$ is $\mathbb{T}$-invariant, it belongs to the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ (by the maximality of $\mathbb{T}$). More generally, Futaki–Mabuchi [42] observed that for any $\phi \in K^\mathbb{T}_0$, the $L^2$ projection $\text{Scal}(\phi)$ (with respect to the global inner product on smooth functions defined by $\omega_\phi$) of $\text{Scal}(\phi)$ to the sub-space $\{m_\phi^\xi + c, c \in \mathbb{R}\}$ of Killing potentials for $\xi \in \mathfrak{t}$ defines a $\phi$-independent element $\xi_{\text{ext}} \in \mathfrak{t}$, i.e. $\text{Scal}(\phi) = m_\phi^{\xi_{\text{ext}}} + c_\phi$. The vector field $\xi_{\text{ext}}$ is called the extremal vector field of $(X, \alpha, \mathbb{T})$. Furthermore, using the normalization for the moment map $m_\phi$ in Lemma 1, we see that $4\pi c_1(X) \cup \alpha^{[n-1]} = \int_X \text{Scal}(\phi)\omega_\phi^n = \int_X \text{Scal}(\phi)\omega_\phi^n = \int_X m_\phi^{\xi_{\text{ext}}}\omega_\phi^n + c_\phi\text{Vol}(X, \alpha),$ showing that the real constant $c_{\text{ext}} = c_\phi$ is independent of $\omega_\phi$ too. Thus, there exists an affine-linear function $w_{\text{ext}}(p) = \langle \xi_{\text{ext}}, p \rangle + c_{\text{ext}}$ on $\mathfrak{t}$, such that $\omega_\phi \in K^\mathbb{T}_0$ is extremal if and only if $\text{Scal}_v(\phi) = w_{\text{ext}}(m_\phi)$ i.e. if and only if $\omega_\phi$ is $(1, w_{\text{ext}})$-cscK (as $c_1(w_{\text{ext}})(\alpha) = 1$ by definition of $w_{\text{ext}}$).

3.2. $(v, w)$-extremal Kähler metrics. As mentioned in the Introduction (and motivated by the previous example) one can consider instead of (1) the more general problem (2) of finding a $(v, w)$-cscK metric $\omega_\phi \in K^\mathbb{T}_0$ satisfying (2). In other words, if $\phi \in K^\mathbb{T}_0$ is $(v, w)$-extremal then $\omega$ is $(v, w_{\text{ext}})$-cscK. Conversely if $\omega_\phi$ is $(v, w_{\text{ext}})$-cscK, then $\text{Scal}_v(\omega_\phi) = c_{v, w_{\text{ext}}}(\alpha)w(m_\phi)w_{\text{ext}}(m_\phi)$ where $c_{v, w_{\text{ext}}}(\alpha)$ is given by (5). We claim that $c_{w, w_{\text{ext}}}(\alpha) = 1$, which in turn implies
that $\omega_\phi$ is $(v, w)$-extremal. Indeed, if $\int_X w(m_\phi)w_{\text{ext}}(m_\phi)\omega_\phi^{[n]} = 0$, then $c_{v, w\text{ext}}(\alpha) = 1$ by Definition 3. Otherwise, if $\int_X w(m_\phi)w_{\text{ext}}(m_\phi)\omega_\phi^{[n]} \neq 0$, we get

$$c_{v, w\text{ext}}(\alpha) \int_X w(m_\phi)w_{\text{ext}}(m_\phi)\omega_\phi^{[n]} = \int_X (\text{Scal}_v(\phi)/w(m_\phi))w(m_\phi)\omega_\phi^{[n]}$$

$$= \int_X w_{\text{ext}}(m_\phi)w(m_\phi)\omega_\phi^{[n]},$$

showing again that $c_{v, w\text{ext}}(\alpha) = 1$.

3.3. The Kähler-Ricci solitons. This is the case when $X$ is a smooth Fano manifold, $\alpha = 2\pi c_1(X)$ corresponds to the anti-canonical polarization, $T \subset \text{Aut}_{\text{red}}(X)$ is a maximal torus with momentum image $P$, and $v(p) = w(p) = e^{\langle \xi, p \rangle}$ for some $\xi \in t$. It was shown recently in [51] that a solutions of (11) with $w_{\text{ext}}(p) = 2(\langle \xi, p \rangle + c)$ (for some real constant $c$) corresponds to a Kähler metric $\omega \in \alpha$ which is a gradient Kähler–Ricci soliton with respect to $\xi$, i.e. satisfies

$$\text{Ric}(\omega) - \omega = -\frac{1}{2}L_{\xi}^*\omega,$$

where $\text{Ric}(\omega)$ is the Ricci form of $\omega$. Thus, the theory of gradient Kähler-Ricci solitons (see e.g. [12, 10, 82, 84]) fits in to our setting too. Further ramifications of this setting appear in [52].

3.4. Kähler metrics conformal to Einstein–Maxwell metrics. This class of Kähler metrics was first introduced in [56] and more recently studied in [6, 7, 43, 44, 53, 54, 55, 57, 58]. These are $(v, w)$-cscK metrics with

$$v(p) = (\langle \xi, p \rangle + a)^{-2m+1} \quad \text{and} \quad w(p) = (\langle \xi, p \rangle + a)^{-2m-1},$$

where $\langle \xi, p \rangle + a$ is a positive affine-linear function on $P$. In this case, $\text{Scal}_v(\phi)/w(m_\phi)$ equals to the usual scalar curvature of the Hermitian metric $\tilde{g}_\phi = \frac{1}{(m_\phi^2 + a)^2}g_\phi$. Thus, a $(v, w)$-cscK metric $\omega_\phi$ gives rise to a conformally Kähler, Hermitian metric $\tilde{g}_\phi$ which has Hermitian Ricci tensor and constant scalar curvature. The latter include the conformally Kähler, Einstein metrics classified in [23, 26]. In real dimension 4, conformally Kähler, Einstein–Maxwell metrics give rise to analogues, in riemannian signature, of the Einstein–Maxwell field equations with a cosmological constant in general relativity.

3.5. Extremal Sasaki metrics. Following [3], let $(X, L)$ be a smooth compact polarized variety and $\alpha = 2\pi c_1(L)$ the corresponding Kähler class. Recall that for any Kähler metric $\omega \in \alpha$, there exits a unique Hermitian metric $h$ on $L$, whose curvature is $\omega$. We denote by $h^*$ the induced Hermitian metric on the dual line bundle $L^*$. It is well-known (see e.g. [13]) that the principal circle bundle $\pi : S \to X$ of vectors of unit norm of $(L^*, h^*)$ has the structure of a Sasaki manifold, i.e. there exists a contact 1-form $\theta$ on $S$ with $d\theta = \pi^*\omega$, defining a contact distribution $D \subset TS$ and a Reeb vector field $\chi$ given by the generator of the $S^1$-action on the fibres of $S$, and a CR-structure $J$ on $D$ induced from the complex structure of $L^*$. The Sasaki structure $(\theta, \chi, D, J)$ on $S$ in turn defines a transversal Kähler structure $(g_\chi, \omega_\chi)$ on $D$ by letting $\omega_\chi = (d\theta)_D$ and $g_\chi = -((d\theta)_D \circ J)$, where the subscript $D$ denotes restriction to $D \subset TS$; it is a well-known fact that $(g_\chi, \omega_\chi)$ coincides with the restriction to $D$ of the pull-back of the Kähler structure $(g, \omega)$ on $X$ or, equivalently, that $(g_\chi, \omega_\chi)$ induces the initial Kähler structures $(g, \omega)$ on the orbit space $X = S/S^1_\chi$ for the $S^1$-action $S^1_\chi$ generated by $\chi$.

Let $T \subset \text{Aut}_{\text{red}}(X)$ be a maximal torus, with a fixed momentum polytope $P \subset t^*$ associated to the Kähler class $\alpha$ as in Lemma 1. We suppose that $\omega$ is a $T$-invariant Kähler metric in $\alpha$. For any positive affine-linear function $\langle \xi, p \rangle + a$ on $P$, we consider
the corresponding Killing potential \( f = m\xi + a \) of \( \omega \) and define the lift \( \xi_f \) of the Killing vector field \( \xi \in \mathfrak{t} \) on \( X \) to \( S \) by

\[
\xi_f = \xi^D + (\pi^* f) \chi,
\]

where the super-scrip \( D \) stands for the horizontal lift. It is easily checked that \( \xi_f \) preserves the contact distribution \( D \) and the CR-structure \( J \), and defines a new Sasaki structure \( \left((\pi^* f)^{-1}\theta, \xi_f, D, J \right) \) on \( S \). In general, the flow of \( \xi_f \) is not periodic, and the orbit space of \( \xi_f \) is not Hausdorff, but when it is, \( X_f := S/S^1_{\xi_f} \) is a compact complex orbifold endowed with a Kähler structure \((g_f, \omega_f)\). In \([8]\), the triple \((X_f, g_f, \omega_f)\) is referred to as a CR \( f \)-twist of \((X, \omega, g)\) and it is shown there that \((X_f, g_f, \omega_f)\) is an extremal Kähler manifold or orbifold in the sense of Sect. \(3.1\) iff \((X, \omega, g)\) is \((v, w)\)-extremal in the sense of Sect. \(3.2\) with

\[
(10) \quad v(p) = (\langle \xi, p \rangle + a)^{-m-1} \quad \text{and} \quad w(p) = (\langle \xi, p \rangle + a)^{-m-3}.
\]

3.6. The generalized Calabi construction and manifolds without multiplicities. In \([5]\) the authors consider smooth compact manifolds \( X \), which are fibre-bundles over the product of cscK Hodge manifolds \((B, \omega_B) = (B_1, \omega_1) \times \cdots \times (B_N, \omega_N)\) with fibre a smooth \( \ell \)-dimensional compact toric Kähler manifold \((V, \omega_V, \mathbb{T})\). More precisely, \( X \) is a \( V \)-fibre bundle associated to a certain principle \( T \)-bundle over \( B \). They introduce a class of \( T \)-invariant Kähler metrics on \( X \), compatible with the bundle structure, which are parametrized by \( \omega_V \)-compatible toric Kähler metrics on \( V \), and refer to them as Kähler metrics given by the \textit{generalized Calabi construction}. The condition for the metric \( \omega \) on \( X \) to be extremal is computed in \([5]\) and can be re-written in our formulation as (see \([74]\) below)

\[
(11) \quad \text{Scal}_v(g_V) = w(m),
\]

where \( g_V \) is the corresponding toric Kähler metric on \((V, \omega_V)\), with

\[
(12) \quad v(p) = \prod_{j=1}^{N}(\langle \xi_j, p \rangle + c_j)^{d_j},
\]

\[
w(p) = (\langle \xi_0, p \rangle + c_0) \prod_{j=1}^{N}(\langle \xi_j, p \rangle + c_j)^{d_j} - \sum_{j=1}^{N} \text{Scal}_j\left(\prod_{k=1}^{N}(\langle \xi_k, p \rangle + c_k)^{d_j}\right)/\left(\langle \xi_j, p \rangle + c_j\right),
\]

in the above expressions, \( m : V \to \mathfrak{t}^* \) stands for the momentum map of \((V, \omega_V, \mathbb{T})\), \( d_j \) and \( \text{Scal}_j \) denote the complex dimension and (constant) scalar curvature of \((B_j, \omega_j)\), respectively, whereas the affine-linear functions \((\langle \xi_j, p \rangle + c_k), k = 1, \cdots, N \) on \( \mathfrak{t}^* \) are determined by the topology and the Kähler class \( \alpha = [\omega] \) of \( X \), and satisfy \((\langle \xi_j, p \rangle + c_j) > 0 \) for \( j = 1, \cdots, N \) on the Delzant polytope \( P = m(V) \). Thus, a Kähler metric \( \omega \) on \( X \) given by the generalized Calabi ansatz is extremal if and only if the corresponding toric Kähler metric \( g_V \) on \( V \) is \((v, w)\)-extremal for the values of \( v, w \) given in \((12)\). More generally, considering an arbitrary weight function \( w \) in \((11)\) allows one to prescribe the scalar curvature of the Kähler metrics given by the generalized Calabi construction on \( X \). We note that a very similar equation for a toric Kähler metric on \( V \) appears in the construction of Kähler manifolds without multiplicities, see \([39, 72]\). We refer the Reader to \([61, 62, 63]\) for a comprehensive study of the equation \((11)\) on a toric variety, for arbitrary weight functions \( v(p) > 0 \) and \( w(p) \), which is referred to as the \textit{generalized Abreu equation}. 


4. A formal momentum map picture

In this section we extend the momentum map interpretation, originally introduced Donaldson [33] and Fujiki [11] in the cscK case and generalized by Apostolov–Maschler [2] to the case of the Introduction, to arbitrary positive weights v, w on P.

In the notation of Section 2 let $\mathcal{AC}_w^T$ be the space of all $\omega$-compatible, $T$-invariant almost complex structures on $(X, \omega)$ and $C_w^T \subset \mathcal{AC}_w^T$ the subspace of $T$-invariant Kähler structures. We consider the natural action on $\mathcal{AC}_w^T$ of the infinite dimensional group $\text{Ham}^T(X, \omega)$ of $T$-equivariant Hamiltonian transformations of $(X, \omega)$, which preserves $C_w^T$. We identify $\text{Lie} \left( \text{Ham}^T(X, \omega) \right) \cong C^\infty(X, \mathbb{R})^T/\mathbb{R}$ where $C^\infty(X, \mathbb{R})^T/\mathbb{R}$ is endowed with the Poisson bracket.

For any $v \in C^\infty(P, \mathbb{R}_{>0})$, the space $\mathcal{AC}_w^T$ carries a weighted formal Kähler structure $(J, \Omega^v)$ given by (6, 33, 41)

\[
\Omega^v_J(J_1, J_2) := \frac{1}{2} \int_X \text{Tr}(J_1 J_2) v(m_\omega) \omega^n[n],
\]

\[
J_J(J) := J^2,
\]

in which the tangent space of $\mathcal{AC}_w^T$ at $J$ is identified with the space of smooth $T$-invariant sections $\phi$ of $\text{End}(TX)$ satisfying

\[
\phi^2 + J \phi = 0, \quad \omega(\phi, \cdot) + \omega(\cdot, \phi) = 0.
\]

In what follows, we denote by $g_J := \omega(\cdot, J \cdot)$ the almost Kähler metric corresponding to $J \in \mathcal{AC}_w^T$, and index all objects calculated with respect to $J$ similarly. On $C^\infty(X, \mathbb{R})^T$, for $w \in C^\infty(P, \mathbb{R}_{>0})$, we consider the scalar product given by,

\[
\langle \phi, \psi \rangle_w := \int_X \phi \psi w(m_\omega) \omega^n[n],
\]

**Theorem 4.** [6, 33, 11] The action of $\text{Ham}^T(X, \omega)$ on $(\mathcal{AC}_w^T, J, \Omega^v)$ is Hamiltonian whose momentum map at $J \in C_w^T$ is the $(\cdot, \cdot)_w$-dual of $(\frac{\text{Scal}_v(J)}{w(m_\omega)} - c_{v,w}(\omega))$, where $\text{Scal}_v(J)$ is the v-scalar curvature of $g_J$ given by [11] and the real constant $c_{v,w}(\omega)$ is given by [5].

**Proof.** The proof follows from the computation of [6, Theorem 1] and [11, Section 9.6] and will be left to the reader. 

5. A variational setting

5.1. The $(v, w)$-Mabuchi energy. In this section we suppose that $v \in C^\infty(P, \mathbb{R}_{>0})$ and $w \in C^\infty(P, \mathbb{R})$ is an arbitrary smooth function. We consider $K_w^T$ as a Fréchet space with tangent space $T_{\phi}K_w^T = C^\infty(X, \mathbb{R})^T$ the space of $T$-invariant smooth functions $\phi$ on $X$.

**Definition 5.** The $(v, w)$-Mabuchi energy $M_{v, w} : K_w^T \to \mathbb{R}$ is defined by

\[
\begin{cases}
(dM_{v, w})_{\phi}(\phi) = -\int_X \phi (\text{Scal}_v(\phi) - c_{v,w}(\alpha) w(m_\phi) ) \omega^n[n], \\
M_{v, w}(0) = 0,
\end{cases}
\]

for all $\phi \in T_{\phi}K_w^T$, where $c_{v,w}(\alpha)$ is the constant given by [3].

**Remark 2.** The critical points of $M_{v, w}$ are precisely the $T$-invariant Kähler potentials $\phi \in K_w^T$ such that $\omega_\phi$ is a solution to the equation (11).
Lemma 3. The functional $\mathcal{E}_w : \mathcal{K}^T_\omega \rightarrow \mathbb{R}$ given by

\[
(15) \begin{cases}
(d\mathcal{E}_w)_\phi (\dot{\phi}) = \int_X \dot{\phi} w(m_\phi) \omega^{[n]}_\phi, \\
\mathcal{E}_w(0) = 0,
\end{cases}
\]

for any $\dot{\phi} \in T_\phi \mathcal{K}^T_\omega$ is well-defined.

Proof. See [12] Lemma 2.14. \qed

Lemma 4. Let $\theta$ be a fixed $\mathbb{T}$-invariant closed $(1,1)$-form and $m_\theta : X \rightarrow t^*$ a momentum map with respect to $\theta$, see Definition 2. Then the functional $\mathcal{E}_\theta : \mathcal{K}^T_\omega \rightarrow \mathbb{R}$ given by

\[
(16) \begin{cases}
(d\mathcal{E}_\theta)_\phi (\dot{\phi}) = \int_X \dot{\phi} \left[ v(m_\phi) \theta \wedge \omega^{[n-1]}_\phi + \langle (dv)(m_\phi), m_\theta \rangle \omega^{[n]}_\phi \right], \\
\mathcal{E}_\theta(0) = 0,
\end{cases}
\]

for any $\dot{\phi} \in T_\phi \mathcal{K}^T_\omega$ is well-defined.

Proof. As the Frechét space $\mathcal{K}^T_\omega$ is contractible, we have to show that the 1-form on $\mathcal{K}^T_\omega$

\[
(17) \mathcal{B}_\phi(\dot{\phi}) := \int_X \dot{\phi} \left[ v(m_\phi) \theta \wedge \omega^{[n-1]}_\phi + \langle (dv)(m_\phi), m_\theta \rangle \omega^{[n]}_\phi \right],
\]

is closed. For $\dot{\phi}, \dot{\psi} \in T_\phi \mathcal{K}^T_\omega$ we compute

\[
(\delta \mathcal{B}_\phi(\dot{\phi}))(\dot{\psi}) = \int_X \dot{\phi} (d(v(m_\phi)), d\dot{\psi})_\phi \theta \wedge \omega^{[n-1]}_\phi + \int_X \dot{\phi} v(m_\phi) \theta \wedge d\dot{\psi} \wedge \omega^{[n-2]}_\phi
\]

\[
+ \int_X \sum_{j=1}^t \dot{\phi} m_\phi \xi_j d(v, j(m_\phi)) \wedge d\dot{\psi} \wedge \omega^{[n]}_\phi - \int_X \sum_{j=1}^t \dot{\phi} v(m_\phi)_j m_\phi \xi_j d\dot{\psi} \wedge \omega^{[n-1]}_\phi
\]

\[
= \int_X \dot{\phi} d(v(m_\phi)), d\dot{\psi} \theta \wedge \omega^{[n-1]}_\phi + \int_X \dot{\phi} v(m_\phi) \theta \wedge d\dot{\psi} \wedge \omega^{[n-2]}_\phi
\]

\[
- \int_X \sum_{j=1}^t \dot{\phi} v(m_\phi)(dm_\phi \xi_j, d\dot{\psi})_\phi \omega^{[n]}_\phi - \int_X (d\dot{\phi}, d\dot{\psi})_\phi \langle (dv)(m_\phi), m_\theta \rangle \omega^{[n]}_\phi,
\]

where $\xi := (\xi_j)_{j=1,\ldots,t}$ is a basis of $t$. Integrating by parts, we obtain

\[
\int_X \dot{\phi} v(m_\phi) \theta \wedge d\dot{\psi} \wedge \omega^{[n-2]}_\phi
\]

\[
= - \int_X v(m_\phi) \theta \wedge d\dot{\phi} \wedge d\dot{\psi} \wedge \omega^{[n-2]}_\phi - \int_X \dot{\phi} \theta \wedge d(v(m_\phi)) \wedge d\dot{\psi} \wedge \omega^{[n-2]}_\phi
\]

\[
= - \int_X (d\dot{\phi}, d\dot{\psi})_\phi v(m_\phi) \theta \wedge \omega^{[n-1]}_\phi + \int_X (\theta, d\dot{\phi} \wedge d\dot{\psi})_\phi v(m_\phi) \omega^{[n]}_\phi
\]

\[
- \int_X \dot{\phi} (d(v(m_\phi)), d\dot{\psi})_\phi \theta \wedge \omega^{[n-1]}_\phi - \int_X \sum_{j=1}^t \dot{\phi} v(m_\phi)(dm_\phi \xi_j, d\dot{\psi})_\phi \omega^{[n]}_\phi,
\]
where we used that
\[
\theta \wedge d(v(m_\phi)) \wedge \tilde{d}^c \psi \wedge \omega_\phi^{[n-2]} \\
= (d(v(m_\phi)), d\tilde{\psi}_\phi) \theta \wedge \omega_\phi^{[n-1]} - (\theta, d(v(m_\phi)) \wedge \tilde{d}^c \psi)_\phi \omega_\phi^{[n]} \\
= (d(v(m_\phi)), d\tilde{\psi}_\phi) \theta \wedge \omega_\phi^{[n-1]} - \sum_{j=1}^t v_j(m_\phi)(\theta, dm_\phi \wedge \tilde{d}^c \psi)_\phi \omega_\phi^{[n]} \\
= (d(v(m_\phi)), d\tilde{\psi}_\phi) \theta \wedge \omega_\phi^{[n-1]} - \sum_{j=1}^t v_j(m_\phi)(dm_\phi \wedge \tilde{d}^c \psi)_\phi \omega_\phi^{[n]},
\]
showing that
\[
(\delta B_v(\tilde{\phi}))_{\tilde{\phi}} = -\int_X v(m_\phi)(d\tilde{\phi}, d\tilde{\psi}_\phi) \theta \wedge \omega_\phi^{[n-1]} \\
- \int_X (d\tilde{\phi}, d\tilde{\psi}_\phi) (dv(m_\phi), m_\theta) \omega_\phi^n \\
+ \int_X (\theta, d\tilde{\phi} \wedge \tilde{d}^c \psi)_\phi v(m_\phi) \omega_\phi^n,
\]
so that
\[
(dB_v)_{\tilde{\phi}} = (\delta B_v(\tilde{\phi}))_{\tilde{\phi}} - (\delta B_v(\tilde{\psi}))_{\tilde{\phi}} = 0.
\]
Thus, $B_v$ is closed and therefore $E^\theta_v : K^T_\omega \to \mathbb{R}$ is well-defined.

**Definition 6.** We let
\[
H_v(\phi) := \int_X \log \left( \frac{\omega^n_\phi}{\omega^n} \right) v(m_\phi) \omega_\phi^n
\]
be the $v$-entropy functional $H_v : K^T_\omega \to \mathbb{R}$.

**Remark 3.** If $\tilde{\mu}$ is an absolutely continuous measure with respect to $\mu_\omega := \omega_\phi^n$, then the entropy of $\tilde{\mu}$ relatively to $\mu$ is defined by,
\[
\text{Ent}_{\mu_\omega}(\tilde{\mu}) := \int_X \log \left( \frac{d\tilde{\mu}}{d\mu_\omega} \right) d\tilde{\mu}.
\]
The entropy is convex on the space of finite measures $\tilde{\mu}$ endowed with its natural affine structure. In the case when $v \in C^\infty(P, \mathbb{R}_{>0})$, the $v$-entropy functional in Definition 6 is given by
\[
H_v(\phi) = \text{Ent}_\mu \left( v(m_\phi) \omega_\phi^{[n]} \right) + c(\alpha, v)
\]
for all $\phi \in K^T_\omega$, where $c(\alpha, v) = \int_X (v \log ov)(m_\phi) \omega_\phi^{[n]}$ is a constant depending only on $(\alpha, v)$ (see Lemma 2).

**Lemma 5.** (i) For any $T$-invariant Kähler form $\omega$ on $X$, we have
\[
\text{Ric}(\omega)(\xi, \cdot) = -\frac{1}{2} d(\Delta_\omega(m_\omega), \xi).
\]

(ii) For any $\phi \in K^T_\omega$ and $\xi \in \mathfrak{t}$, we have
\[
\text{Ric}(\omega_\phi) = \text{Ric}(\omega) - \frac{1}{2} d^c \Psi_\phi,
\]
\[
m^\xi_{\text{Ric}(\omega_\phi)} = m_{\text{Ric}(\omega)} - \frac{1}{2} (d^c \Psi_\phi)(\xi),
\]
where $m_{\text{Ric}(\omega)} := \frac{1}{2} \Delta_\omega(m_\omega)$ is the Ric(\omega)-momentum map of the action of $T$ on $X$ and $\Psi_\phi = \log \left( \frac{\omega_\phi^n}{\omega^n} \right)$. 

Proof. We give a simple argument for the sake of exposition. The statement (i) is well known (see e.g. \[45, \text{Remark 8.8.2}\] and \[79, \text{Lemma 28}\]). For the statement (ii) let \(\phi \in \mathcal{K}_\omega^T\) and \(\xi \in \mathfrak{t}\). Using that \(\mathcal{L}_{\xi} \omega_\phi = -dd^c m_\xi^\phi\) we obtain

\[\mathcal{L}_{\xi} \omega_\phi^{[n]} = \Delta_\phi (m_\xi^\phi) \omega_\phi^{[n]} \]

It follows that

\[-\frac{1}{2} (d^c \Psi_\phi)(\xi) = \frac{1}{2} \mathcal{L}_{\xi} \Psi_\phi = \frac{1}{2} \mathcal{L}_{\xi} \omega_\phi^{[n]} - \frac{1}{2} \mathcal{L}_{\xi} \omega^{[n]} = m_{\text{Ric}(\omega_\phi)} - m_{\text{Ric}(\omega)}\]

\(\square\)

We now extend a formula obtained in the case \(v = w = 1\) by Chen-Tian (see \[21, \text{[81]}\]) to general values of \(v\) and \(w\).

Theorem 5. We have the following expression for the \((v, w)\)-Mabuchi energy,

\[(19) \quad \mathcal{M}_{v, w} = \mathcal{H}_v - 2\mathcal{E}^{\text{Ric}(\omega)}_v + c_{(v, w)}(\alpha) \mathcal{E}_w.\]

Proof. We compute

\[\begin{align*}
(d\mathcal{H}_v)(\phi) &= -\int_X \phi \Delta_\phi (v(m_\phi)) \omega_\phi^{[n]} - \int_X (d\Psi_\phi, d\phi) \omega_\phi^{[n]} \\
&= -\int_X \sum_{j=1}^\ell \phi v_{ij}(m_\phi) \Delta_\phi (m_\xi^\phi) \omega_\phi^{[n]} + \int_X \sum_{i,j=1}^\ell \phi v_{ij}(m_\phi)(\xi_i, \xi_j) \phi \omega_\phi^{[n]} \\
&\quad + \int_X \sum_{j=1}^\ell \phi (d\Psi_\phi, dm_\xi^\phi) v_{ij}(m_\phi) \omega_\phi^{[n]} - \int_X \phi \Delta_\phi (\Psi_\phi) v(m_\phi) \omega_\phi^{[n]},
\end{align*}\]

where \(\xi := (\xi_j)_{j=1, \ldots, \ell}\) is a basis for \(\mathfrak{t}\). Using Lemma \[8\] and the fact that

\[\Delta_\phi (\Psi_\phi) = -\Lambda_{\omega_\phi} d^c \Psi_\phi = 2\Lambda_{\omega_\phi} (\text{Ric}(\omega_\phi) - \text{Ric}(\omega)) = \text{Scal}_\phi - 2\Lambda_{\omega_\phi} \text{Ric}(\omega),\]

we get

\[\begin{align*}
(d\mathcal{H}_v)(\phi) &= -\int_X \phi \sum_{j=1}^\ell v_{ij}(m_\phi) \Delta_\phi (m_\xi^\phi) \omega_\phi^{[n]} + \int_X \sum_{i,j=1}^\ell \phi v_{ij}(m_\phi)(\xi_i, \xi_j) \phi \omega_\phi^{[n]} \\
&\quad + \int_X \phi \left( \text{Ric}(m_\phi) - \Delta_\phi (m_\xi^\phi) \right) v_{ij}(m_\phi) \omega_\phi^{[n]} \\
&\quad - \int_X \phi (\text{Scal}_\phi - 2\Lambda_{\omega_\phi} \text{Ric}(\omega)) v(m_\phi) \omega_\phi^{[n]}.
\end{align*}\]

It follows that

\[(20) \quad d(\mathcal{H}_v - 2\mathcal{E}^{\text{Ric}(\omega)}_v)(\phi) = -\int_X \phi \text{Scal}_v(\phi) \omega_\phi^{[n]},\]

which yields \[(19)\] via \[(20)\] and \[(15)\]. \(\square\)

By the work of Mabuchi \[67, \text{[68]}\], the space of \(\mathcal{T}\)-invariant Kähler potentials \(\mathcal{K}_\omega^T\) is an infinite dimensional riemannian manifold with a natural riemannian metric, called the Mabuchi metric, defined by

\[\langle \dot{\phi}_1, \dot{\phi}_2 \rangle_\phi = \int_X \dot{\phi}_1 \dot{\phi}_2 \omega_\phi^{[n]},\]

for any \(\dot{\phi}_1, \dot{\phi}_2 \in T_\phi \mathcal{K}_\omega^T\). The equation of a geodesic \((\phi_t)_{t \in [0, 1]} \in \mathcal{K}_\omega^T\) connecting two points \(\phi_0, \phi_1 \in \mathcal{K}_\omega^T\) is given by

\[\ddot{\phi}_t = |d\dot{\phi}_t|^2_{\phi_t}.\]
Proposition 1 \cite{[41,46]}. Let $X$ be a compact Kähler manifold with a fixed Kähler class $\alpha$, $T \subset \text{Aut}_{\text{red}}(X)$ a real torus and suppose that $\omega \in \alpha$ is a $(v, w)$-cscK metric for smooth functions $v \in C^\infty(P, \mathbb{R}_{>0})$, $w \in C^\infty(P, \mathbb{R})$ on the momentum image $P \subset t^*$ associated to $(T, \alpha)$. Then for any $(v, w)$-cscK metric $\omega_\phi \in \alpha$ connected to $\omega$ by a geodesic segment in $\mathcal{K}_T^\omega$, there exists $\Phi \in \text{Aut}_{\text{red}}(X)$ commuting with the action of $T$, such that $\omega_\phi = \Phi^* \omega$.

Proof. By a straightforward calculation using the formula \cite{[92]} in the Appendix B, we obtain the following expression for the second variation of the $(v, w)$-Mabuchi energy along a $T$-invariant segment of Kähler potentials $(\phi_t)_{t \in [0, 1]} \in \mathcal{K}_T^\omega$:

$$
\frac{d^2 M_{v,w}(\phi_t)}{dt^2} = 2 \int_X |D^- d\phi_t|_{\phi_t}^2 v(m_\phi) \omega_{\phi_t}^{[n]} - \int_X (\ddot{\phi} - |d\dot{\phi}|_{\phi_t}^2) (\text{Scal}_v(\phi_t) - w(m_\phi)) \omega_{\phi_t}^{[n]}.
$$

(21)

Suppose now that $\omega_\phi$, $\phi \in \mathcal{K}_T^\omega$ is a $(v, w)$-cscK metric connected to $\omega$ by a smooth geodesic $(\phi_t)_{t \in [0, 1]}$, such that $\phi_0 = 0$ and $\phi_1 = \phi$. Then $\frac{d M_{v,w}(\phi_t)}{dt} \bigg|_{t=0} = \frac{d M_{v,w}(\phi_t)}{dt} \bigg|_{t=1} = 0$, and using (21) we obtain

$$
\frac{d^2 M_{v,w}(\phi_t)}{dt^2} = 2 \int_X |D^- d\phi_t|_{\phi_t}^2 v(m_\phi) \omega_{\phi_t}^{[n]} \geq 0.
$$

It follows that $\frac{d^2 M_{v,w}(\phi_t)}{dt^2} \equiv 0$ and $D^- d\phi_t \equiv 0$. Thus, we have a family of real holomorphic vector vector fields $V_t := -\text{grad}_{\phi_t} \phi_t$, $t \in [0, 1]$. By \cite{[45]} Proposition 4.6.3, $V_t = V_0$ for all $t$, and $\omega_\phi = (\Phi_t^{V_0})^* \omega$ where $\Phi_t^{V_0} \in \text{Aut}_{\text{red}}(X)$ is the flow of the real holomorphic vector field $V_0$.

Remark 4. In general, the space $\mathcal{K}_T^\omega$ is not geodesically connected by smooth geodesics (see \cite{[24]} Theorem 1.2}). However, by a result of Chen \cite{[20]}, the space $\mathcal{K}_T^\omega$ is geodesically connected by $T$-invariant weak $C^{1,1}$-geodesics, i.e. in the space $(\mathcal{K}_T^{1,1})^T$ of $T$-invariant real valued functions $\phi$ such that $\omega + dd^c \phi$ is a positive current with bounded coefficients. Using the formula $m_\phi = m_\omega + dd^c \phi$ and Theorem 5, one can extend the $(v, w)$-Mabuchi energy to a functional $M_{v,w} : (\mathcal{K}_T^{1,1})^T \to \mathbb{R}$. One thus might hope to obtain a uniqueness up to a $T$-equivariant isometry of $(v, w)$-cscK metrics along the lines of the proof of \cite{[11]} Theorem 1.1, but this goes beyond the scope of this article.

5.2. The relative $(v, w)$-Mabuchi energy. In this section we assume that both $v$ and $w$ are positive smooth functions on $P$.

Definition 7. The $(v, w)$-relative Mabuchi energy $M_{v,w}^{\text{rel}} : \mathcal{K}_T^\omega \to \mathbb{R}$ is defined by

$$
\begin{cases}
(d M_{v,w}^{\text{rel}})(\phi) = -\int_X \dot{\phi} (\text{Scal}_v(\phi)/w(m_\phi) - w_{\text{ext}}(m_\phi)) w(m_\phi) \omega_{\phi}^{[n]}, \\
M_{v,w}^{\text{rel}}(0) = 0,
\end{cases}
$$

(22)

for any $\phi \in T_\phi \mathcal{K}_T^\omega$, where $w_{\text{ext}}$ is the affine linear function on $P$ defined in Section 3.2.

Lemma 6. We have $M_{v,w}^{\text{rel}} = M_{v,ww_{\text{ext}}}^{\text{rel}}$.

Proof. In Section 3.2, we showed that $c_{v,ww_{\text{ext}}}(\alpha) = 1$. From the definitions of $M_{v,w}$ and $M_{v,w}^{\text{rel}}$, it then follows that $M_{v,w}^{\text{rel}} = M_{v,ww_{\text{ext}}} + c$ and using $M_{v,w}^{\text{rel}}(0) = M_{v,ww_{\text{ext}}}(0) = 0$ we get $c = 0$. \qed
5.3. Boundedness of the $(1,w)$-Mabuchi energy. Now we show how the results of Berman-Berndtsson in [11] can be extended to the $(1,w)$-cscK metrics.

Theorem 6. Let $X$ be a smooth compact Kähler manifold, $T \subset \text{Aut}_{\text{red}}(X)$ a real torus, and suppose that $X$ admits a $(1,w)$-cscK metric $\omega$ in the the Kähler class $\alpha$ for some smooth function $w$ on the momentum image $P \subset t^*$ associated to $(T, \alpha)$. Then, $\omega$ is a global minima of $\mathcal{M}_{1,w}$.

Proof. We denote by $\mathcal{M}_w$ the $(1,w)$-Mabuchi energy and by $\mathcal{M}$ the $(1,c_{1,w}(\alpha))$-Mabuchi energy. From the definition of the Mabuchi energy we have the following relation

$$\mathcal{M}_w = \mathcal{M} + \mathcal{E}_{\tilde{w}},$$

where $\tilde{w} := c_{1,w}(\alpha)(1-w)$ and $\mathcal{E}_{\tilde{w}}$ is the functional [15]. Let $\phi_0, \phi_1 \in \mathcal{K}_w$ be two smooth Kähler potentials and $\phi_0$ the weak geodesic connecting $\phi_0$ and $\phi_1$ (see [11, 22] and the references therein for the definition of a weak geodesic). By [13, Proposition 10.d] the function $t \mapsto \mathcal{E}_{\tilde{w}}(\phi_t)$ is affine on $[0,1]$, whereas by [11, Theorem 3.4], the function $t \mapsto \mathcal{M}(\phi_t)$ is convex. It follows that $t \mapsto \mathcal{M}_w(\phi_t)$ is convex. By [11, Lemma 3.5] and its proof, we get

$$\lim_{t \to 0^+} \frac{\mathcal{M}_w(\phi_1) - \mathcal{M}_w(\phi_0)}{t} \geq \int_X (\text{Scal}(\phi_0) - c_{1,w}(\alpha) w(m_{\phi_0})) \phi_0^{[n]}.$$

where $\phi := \frac{d\phi_t}{dt}|_{t=0^+}$. Using the sub-slope inequality for convex functions and the Cauchy–Schwarz inequality we get

$$\mathcal{M}_w(\phi_1) - \mathcal{M}_w(\phi_0) \geq \lim_{t \to 0^+} \frac{\mathcal{M}_w(\phi_1) - \mathcal{M}_w(\phi_0)}{t} \geq \int_X (\text{Scal}(\phi_0) - c_{1,w}(\alpha) w(m_{\phi_0})) \phi_0^{[n]}$$

$$\geq - d(\phi_0, \phi_1) \left( \int_X (\text{Scal}(\phi_0) - c_{1,w}(\alpha) w(m_{\phi_0})) \phi_0^{[n]} \right)^{1/2},$$

where $d(\phi_0, \phi_1)^2 = \int_X \hat{\phi}^2 \phi_0^{[n]}$ is the Mabuchi distance between $\phi_0$ and $\phi_1$. In particular, if $\omega_{\phi_0}$ is a $(1,w)$-cscK metric in the Kähler class $\alpha$, then $\mathcal{M}_w(\phi) \geq \mathcal{M}_w(\phi_0)$ for any $\phi \in \mathcal{K}_w^\alpha$. \hfill \Box

6. The $(v,w)$-Futaki invariant for a Kähler class

Let $(X, \alpha)$ be a compact Kähler manifold and $T \subset \text{Aut}_{\text{red}}(X)$ a real torus with momentum polytope $P$ with respect to $\alpha$ as in Lemma 11. For any $\phi \in \mathcal{K}_w^T$ and $V \in h_{\text{red}}^\ast \mathfrak{t}$ in the Lie algebra of the centralizer of $T$ in $\text{Aut}_{\text{red}}(X)$, we denote by $h_\phi^V + \sqrt{-1} f_\phi^V \in C^\infty(X, \mathbb{C})$ the normalized holomorphy potential of $\xi$, i.e. $h_\phi^V$ and $f_\phi^V$ are smooth functions such that

$$V = \text{grad}_{g_\phi} (h_\phi^V) + J \text{grad}_{g_\phi} (f_\phi^V),$$

$$\int_X f_\phi^V \omega_\phi^{[n]} = \int_X h_\phi^V \omega_\phi^{[n]} = 0.$$

Using that the tangent space in $\phi$ of $\mathcal{K}_w^T$ is given by $T_\phi(\mathcal{K}_w^T) \cong C^\infty_{\phi}(X, \mathbb{R})^T \oplus \mathbb{R}$, the vector field $J V$ defines a vector field $J V$ on $\mathcal{K}_w^T$, given by:

$$\phi \mapsto \mathcal{L}_{J V} \omega_\phi = - d\xi J f_\phi^V,$$

so that $J V_\phi = f_\phi^V$. We consider the 1-form $\sigma$ on $\mathcal{K}_w^T$, defined by

$$\sigma_\phi(\hat{\phi}) := (d \mathcal{M}_{v,w})_\phi(\hat{\phi})$$
where $\mathcal{M}_{v,w}$ is the $(v,w)$-Mabuchi energy associated to the smooth functions $v \in C^\infty(P, \mathbb{R}_{>0})$ and $w \in C^\infty(P, \mathbb{R})$ (see (13)). By the invariance of $\sigma$ under the $\text{Aut}_{\text{red}}^\mathbb{T}(X)$-action and Cartan’s formula, we get

$$L_{J\varphi}\sigma = d(\sigma(J\varphi)) = 0.$$ 

Then $\phi \mapsto \sigma_\phi(J\varphi)$ is constant on $K^\mathbb{T}_\omega$, and we define

**Definition 8.** We let $\mathcal{F}_{v,w}(V) := \sigma_\omega(J\varphi) = \int_X (\text{Scal}_\nu(\omega) - c_{(v,w)}(\alpha)w(m_\omega)) f^a_{\omega} \omega^{[n]}$, be the real constant associated to $V \in \mathfrak{h}_{\text{red}}^\mathbb{T}$. We thus get a linear map $\mathcal{F}_{v,w} : \mathfrak{h}_{\text{red}}^\mathbb{T} \to \mathbb{R}$ called the $(v,w)$-Futaki invariant associated to $(\alpha, P, v, w)$.

By its very definition, we have

**Proposition 2.** If $(X, \alpha, \mathbb{T})$ admits a $(v, w)$-cscK metric then

$$\int_X \text{Scal}_\nu(\omega) \omega^{[n]} = c_{(v,w)}(\alpha) \int_X w(m_\omega) \omega^{[n]} \text{ and } \mathcal{F}_{v,w} = 0.$$

**Remark 5.** The first condition in (24) is satisfied when $\int_X w(m_\omega) \omega^{[n]} \neq 0$ by the very definition of $c_{v,w}(\alpha)$ (see Definition 2). Furthermore, in the case of a $(v, w)$-extremal Kähler metric considered in Section 3.1, the both conditions in (24) hold true with respect to the weights $v$ and $ww_{\text{ext}}$, by the very definition of $w_{\text{ext}}$.

7. The $(v, w)$-Futaki invariant of a smooth test configuration

Let $X$ be a compact Kähler manifold endowed with an $\ell$-dimensional real torus $\mathbb{T} \subset \text{Aut}_{\text{red}}(X)$ and a Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$. Following [28, 31, 32] we give the following

**Definition 9.** A smooth $\mathbb{T}$-compatible Kähler test configuration for $(X, \alpha)$ is a compact smooth $(n+1)$-dimensional Kähler manifold $(\mathcal{X}, A)$, endowed with a holomorphic action of a real torus $\hat{T} \subset \text{Aut}_{\text{red}}(\mathcal{X})$ with Lie algebra $\hat{\mathfrak{t}}$ and

- a surjective holomorphic map $\pi : \mathcal{X} \to \mathbb{P}^1$ such that the torus action $\hat{T}$ on $\mathcal{X}$ preserves each fiber $X_\tau := \pi^{-1}(\tau)$ and $(X_1, A|_{X_1}, \hat{T}) \cong (X, \alpha, \mathbb{T})$,
- a $\mathbb{C}^*$-action $\rho$ on $\mathcal{X}$ commuting with $\hat{T}$ and covering the usual $\mathbb{C}^*$-action on $\mathbb{P}^1$,
- a biholomorphism

$$\lambda : \mathcal{X} \setminus X_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\}),$$

which is equivariant with respect to the actions of $\hat{\mathcal{G}} := \hat{T} \times S^1_\rho$ on $\mathcal{X} \setminus X_0$ and the action of $G := \mathbb{T} \times S^1$ on $X \times (\mathbb{P}^1 \setminus \{0\})$.

In what follows we shall tacitly identify $\hat{T}$ with $\mathbb{T}$ and $\hat{\mathcal{G}}$ with $G$.

**Definition 10.** A smooth $\mathbb{T}$-compatible Kähler test configuration $(\mathcal{X}, A, \rho, \mathbb{T})$ for $(X, \alpha, \mathbb{T})$ is called

- **trivial** if it is given by $(\mathcal{X}_0 = X \times \mathbb{P}^1, A_0 = \pi_X^*\alpha + \pi_{\mathbb{P}_1}^*[\omega_{FS}], T)$ and $\mathbb{C}^*$-action $\rho(\tau)(x, z) = (x, \tau z)$ for any $\tau \in \mathbb{C}^*$ and $(x, z) \in X \times \mathbb{P}^1$.
- **product** if it is given by $(\mathcal{X}_{\text{prod}}, A_{\text{prod}}, \rho_{\text{prod}}, \mathbb{T})$ where $\mathcal{X}_{\text{prod}}$ is the compactification (in the sense of [70, 87], see also [14, Example 2.8] and [69, p. 12-13]) of $X \times \mathbb{C}$ with $\mathbb{C}^*$-action $\rho_{\text{prod}}(\tau)(x, z) = (\rho_X(\tau)x, \tau z)$ where $\rho_X$ is a $\mathbb{C}^*$-action on $X$ and $A_{\text{prod}}$ is a Kähler class on $\mathcal{X}_{\text{prod}}$ which restricts to $\alpha$ on $X_1 \cong X$. 

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Let \((\mathcal{X},\mathcal{A},\mathbb{T})\) be a smooth \(\mathbb{T}\)-compatible Kähler test configuration for \((X,\alpha,\mathbb{T})\) and \(\Omega \in \mathcal{A}\) a \(\mathbb{G}\)-invariant Kähler form. The action of \(\mathbb{T}\) on \(\mathcal{X}\) is Hamiltonian with \(\Omega\)-momentum map \(m_\Omega : \mathcal{X} \to \mathfrak{t}^*\), normalized by \(m_\Omega(X_1) = P\), where \(P\) is a fixed momentum polytope for the induced \(\mathbb{T}\)-action on \(X_1 \cong X\).

For any \(\tau \in \mathbb{C}^*\), we denote by
\[
(26)\quad \Omega_\tau := \Omega|_{X_\tau}, \quad \Omega_1 := \omega \quad \text{and} \quad \omega_\tau := \rho(\tau)^* \Omega_\tau,
\]
where \(\rho(\tau) : X_1 \xrightarrow{\sim} X_\tau\) is the restriction of \(\rho(\tau) \in \text{Aut}_{\text{red}}(\mathcal{X})\) to \(X_1\). The action of \(\mathbb{T}\) on \(X_\tau\) is Hamiltonian with \(\Omega_\tau\)-momentum map \((m_\Omega)|_{X_\tau}\). Pulling the structure on \(X_\tau\) back to \(X_1\) via \(\rho(\tau)\), we get a \(\omega_\tau\)-momentum map for the \(\mathbb{T}\)-action on \(X_1\), given by
\[
(27)\quad m_\tau = m_\Omega \circ \rho(\tau).
\]

**Lemma 7.** For any \(\tau \in \mathbb{C}^*\), we have
\[
\int_{X_\tau} m_\Omega \Omega_\tau^n = \int_{X_1} m_\tau \omega_\tau^n = \int_{X_1} m_1 \omega_1^n.
\]
It follows that \(P_\tau = P\) for any \(\tau \in \mathbb{C}^*\), where \(P_\tau := m_\Omega(X_\tau) = m_\tau(X_1)\) is the momentum polytope of the induced action of \(\mathbb{T}\) on \(X_\tau\) and \(\Omega_\tau := \Omega|_{X_\tau}\).

**Proof.** Since \(\Omega\) is \(S^1_\rho\)-invariant, the following integral depends only on \(t = -\log |\tau|\):
\[
\int_{X_\tau} m_\Omega \Omega_\tau^n = \int_{X_1} m_\tau \omega_\tau^n = \int_{X_1} m_\tau (\omega + dd^c \phi_\tau)[n].
\]
Let \(V_\rho\) be the generator of the \(S^1_\rho\)-action. By (27) we have
\[
\frac{d}{dt} m_\tau = \frac{d}{dt} (m_\Omega \circ \phi^t_{\mathcal{J}V_\rho}) = (\phi^t_{\mathcal{J}V_\rho})^* (dm_\Omega, \mathcal{J}V_\rho) \Omega = - (\phi^t_{\mathcal{J}V_\rho})^* (dh^\rho, dh^\rho) \Omega,
\]
where \(\mathcal{J}\) denotes the complex structure on \(\mathcal{X}\), \(\phi^t_{\mathcal{J}V_\rho} = e^{-t}\) is the flow of \(\mathcal{J}V_\rho\) and \(h^\rho\) is a \(\Omega\)-Hamiltonian function for \(V_\rho\). On the other hand, we have
\[
\frac{d}{dt} \rho(t)^* \Omega = (\phi^t_{\mathcal{J}V_\rho})^* \mathcal{L}_{\mathcal{J}V_\rho} \Omega = - (\phi^t_{\mathcal{J}V_\rho})^* dd^c h^\rho.
\]
It follows that
\[
\frac{d}{dt} \int_{X_1} m_\tau \omega_\tau^n = \frac{d}{dt} \int_{X_1} m_\tau ((\rho(t)^* \Omega)|_{X_1})[n]
\]
\[
= - \int_{X_\tau} ((dm_\Omega, dh^\rho)\Omega)|_{X_\tau, \Omega_\tau^n} - \int_{X_\tau} m_\Omega, dd^c h^\rho|_{X_\tau} \wedge \Omega_\tau^{n-1}
\]
\[
= - \int_{X_\tau} ((dm_\Omega, dh^\rho)\Omega)|_{X_\tau, \Omega_\tau^n} + \int_{X_\tau} m_\Omega, \Delta h^\rho(X_\tau) \Omega_\tau^n
\]
\[
= - \int_{X_\tau} ((dm_\Omega, dh^\rho)\Omega)|_{X_\tau, \Omega_\tau^n} + \int_{X_\tau} (dm_\Omega, dh^\rho|_{X_\tau}) \Omega_\tau^n = 0,
\]
where we have used that \(((dm_\Omega, dh^\rho)\Omega)|_{X_\tau} = (dm_\Omega, dh^\rho|_{X_\tau}) \Omega_\tau\) since the symplectic gradient of \(m_\Omega : \mathcal{X} \to \mathfrak{t}^*\) is given by the \(t\)-valued fundamental vector field for the \(\mathbb{T}\)-action, and thus is tangent to the fibers. It follows that
\[
\int_{X_1} m_\tau \omega_\tau^n = \int_{X_1} m_1 \omega_1^n.
\]
Since \(m_\Omega : \mathcal{X} \to \mathfrak{t}^*\) is continuous it follows from Lemma 7 that \(m_\Omega(\mathcal{X}) = P\).
Definition 11. Let \((X, \mathcal{A}, T)\) be a smooth \(T\)-compatible Kähler test configuration for the compact Kähler manifold \((X, \alpha)\) and \(v \in C^\infty(P, \mathbb{R}_{>0}), \ w \in C^\infty(P, \mathbb{R})\). The \((v, w)\)-Futaki invariant of \((X, \mathcal{A}, T)\) is defined to be the real number

\[
\mathcal{F}_{v,w}(X, \mathcal{A}) = - \int_X (\text{Scal}_v(\Omega) - c_{(v,w)}(\alpha)w(\mu)) \Omega^{[n+1]} + 2 \int_X v(\mu) \pi^*\omega_{FS} \wedge \Omega^{[n]}
\]

where \(\Omega \in \mathcal{A}\) is a \(T\)-invariant representative of \(\mathcal{A}\), \(\omega_{FS}\) is the Fubini-Study metric on \(\mathbb{P}^1\) with \(\text{Ric}(\omega_{FS}) = \omega_{FS}\), and \(c_{(v,w)}(\alpha)\) is the \((v,w)\)-slope of \((X, \alpha)\) given by \([5]\).

Remark 6. (i) By Lemma \([21, 25]\) \(\mathcal{F}_{v,w}(X, \mathcal{A})\) is independent from the choice of a \(T\)-invariant Kähler form \(\Omega \in \mathcal{A}\). For \(v = w = 1\) we also recover the Futaki invariant of a smooth test configuration introduced in \([28, 31]\). Thus, we obtain the following equivalent expression for the \((v, w)\)-Futaki invariant

\[
\mathcal{F}_{v,w}(X, \mathcal{A}) = - \int_X (\text{Scal}_v(\Omega) - c_{(v,w)}(\alpha)w(\mu)) \Omega^{[n+1]} + 2 \int_X v(\mu) \omega^{[n]}.
\]

(ii) It is easy to show that

\[
2 \int_X v(\mu) \pi^*\omega_{FS} \wedge \Omega^{[n]} = 2 \int_{X \setminus X_0} v(\mu) \pi^*\omega_{FS} \wedge \Omega^{[n]}
\]

\[
= 2 \int_{\mathbb{P}^1 \setminus \{0\}} \left( \int_{X_\tau} v(\mu_\tau) \Omega^{[n]}_\tau \right) \omega_{FS}
\]

\[
= 2 \text{vol}(\mathbb{P}^1) \left( \int_{X_1} v(\mu) \omega^{[n]} \right)
\]

\[
= (8\pi) \int_X v(\mu) \omega^{[n]},
\]

where for passing from the second line to the third line we used that \(\rho(\tau)^*\Omega_\tau\) and \(\omega\) are in the same Kähler class \(\mathcal{A}|_{X_1}\) on \(X_1\), see Lemma \([2]\). Thus, we obtain the following equivalent expression for the \((v, w)\)-Futaki invariant

\[
\mathcal{F}_{v,w}(X, \mathcal{A}) = - \int_X (\text{Scal}_v(\Omega) - c_{(v,w)}(\alpha)w(\mu)) \Omega^{[n+1]} + (8\pi) \int_X v(\mu) \omega^{[n]}
\]

(iii) It is easy to compute the \((v, w)\)-Futaki invariant of the trivial test configuration \((X_0, \mathcal{A}_0)\) (see Definition \([10]\)), using that for a product Kähler form \(\Omega_0 := \pi^*\omega + \pi^*\omega_{FS}\) we have \(\text{Scal}_v(\Omega_0) = \text{Scal}_v(\omega) + 2v(\mu)\), then \([28]\) reduces to

\[
\mathcal{F}_{v,w}(X_0, \mathcal{A}_0) = -4\pi \int_X (\text{Scal}_v(\omega) - c_{(v,w)}(\alpha)w(\mu)) \omega^{[n]}.
\]

Definition 12. \([27, 32]\) We say that \((X, \alpha, T)\) is

(i) \((v, w)\)-K-semistable on smooth Kähler test configurations if \(\mathcal{F}_{v,w}(X, \mathcal{A}) \geq 0\) for any \(T\)-compatible test configuration \((X, \mathcal{A}, T)\) of \((X, \alpha, T)\) and \(\mathcal{F}_{v,w}(X_0, \mathcal{A}_0) = 0\) for the trivial test configuration \((X_0, \mathcal{A}_0)\).

(ii) \((v, w)\)-K-stable on smooth Kähler test configurations if it is \((v, w)\)-K-semistable and \(\mathcal{F}_{v,w}(X, \mathcal{A}) = 0\) if and only if \((X, \mathcal{A}) = (\mathcal{X}_{\text{prod}}, \mathcal{A}_{\text{prod}})\) is a product in the sense of Definition \([10]\).

Following \([25, 31]\), there is a family of \(T\)-invariant Kähler potentials \(\phi_\tau \in \mathcal{K}_T^\infty(X_1), \ \tau \in \mathbb{C}^* \subset \mathbb{P}^1\) given by the following Lemma.

Lemma 8. Let \(\Omega \in \mathcal{A}\) be a \(\mathbb{G}\)-invariant Kähler form on \(X\).

(i) On \(\mathcal{X}^* := X \setminus X_0\) we have

\[
\Omega = \dot{\omega} + dd^c\Phi
\]
where \( \mathbf{w} := (\pi_X \circ \lambda)^* \omega \) with \( \lambda \) the map given by (25) and \( \pi_X \) is the projection on the first factor of \( X \times (\mathbb{P}^1 \setminus \{0\}) \), and \( \Phi \) is a smooth \( \mathbb{G} \)-invariant function on \( \mathcal{X}^* \), such that for any \( \tau \in \mathbb{C}^* \),

\[
\phi_{\tau} := \rho(\tau)^* \Phi|_{\mathcal{X}_{\tau}} \in \mathcal{K}_{\Omega}^T(\mathcal{X}_1),
\]

satisfies

\[
\omega_{\tau} - \omega = dd^c \phi_{\tau},
\]

where we recall that \( \omega_{\tau} \) is defined in (26).

(ii) \( m^\xi_{\omega} := m^\xi_{\tilde{\Omega}} - (d^\xi \Phi)(\xi), \xi \in t \) is a moment map of \( \tilde{\omega} \) restricted to a fiber \( \mathcal{X}_\tau \) for the \( T \)-action on \( \mathcal{X}^* \), satisfying \( m^\xi_{\tilde{\omega}}(\mathcal{X}^*) = P \).

Proof. (i) Using [81, Proposition 3.10] we can find a smooth function \( \Phi \) on \( \mathcal{X}^* \) such that \( \Omega = \tilde{\omega} + dd^c \Phi \) on \( \mathcal{X}^* \). Taking the restriction of the latter equality to \( \mathcal{X}_\tau \) \( (\tau \neq 0) \) we have \( \Omega_{\tau} = \rho(\tau)^{-1} \omega + dd^c(\Phi|_{\mathcal{X}_{\tau}}) \), pulling back by \( \rho(\tau) \) yields \( \omega_{\tau} - \omega = dd^c \phi_{\tau} \).

(ii) By the relation (30) and the fact that the action of \( T \) preserves the fibers we obtain that \( m^\xi_{\omega} := m^\xi_{\tilde{\Omega}} - (d^\xi \Phi)(\xi) \) is a momentum map of \( (\mathcal{X}_\tau, \tilde{\omega}|_{\mathcal{X}_{\tau}}) \). It thus follows from Lemmas [1] and [7] that \( m^\xi_{\tilde{\omega}}(\mathcal{X}^*) = P \).

The main result of this section is the following theorem which extends the results from [28] to arbitrary values of \( v, w \):

**Theorem 7.** Let \( (\mathcal{X}, \mathcal{A}, T) \) be a smooth \( T \)-compatible Kähler test configuration, for a compact Kähler manifold \( (X, \alpha, T) \) and \( v \in C^\infty(P, \mathbb{R}_{>0}) \), \( w \in C^\infty(P, \mathbb{R}) \) are weight functions. If the central fiber \( X_0 \) is reduced, then

\[
\lim_{t \to +\infty} \frac{M_{v, w}(\phi_t)}{t} = \mathcal{F}_{v, w}(\mathcal{X}, \mathcal{A}),
\]

where \( \phi_t := \phi_{\tau} \) with \( \tau = e^{-\tau + is} \) is given by (31). In particular if \( M_{v, w} \) is bounded from below, then

\[
\mathcal{F}_{v, w}(\mathcal{X}, \mathcal{A}) \geq 0.
\]

Before we give the proof we need a couple of technical lemmas.

**Lemma 9.** Under the hypotheses of Theorem 7 we have,

\[
\lim_{t \to +\infty} \frac{\mathcal{E}_w(\phi_t)}{t} = \int_{\mathcal{X}} w(m_{\Omega}) \Omega^{[n+1]}.
\]

Proof. We will start by showing as in [28] [81] [84] that,

\[
\pi_*(w(m_{\Omega}) \Omega^{[n+1]}) = dd^c \mathcal{E}_w(\phi_{\tau}),
\]

on \( \mathbb{C}^* \subset \mathbb{P}^1 \), in the sens of currents. From the very definition of the functional \( \mathcal{E}_w \) (see (15)) we have

\[
\mathcal{E}_w(\phi_{\tau}) = \int_0^1 \left( \int_X \phi_{\mathcal{X}_\tau} w(m_{\mathcal{X}_\tau}) \omega_{\phi_{\mathcal{X}_\tau}}^{[n]} \right) d\epsilon
\]

\[
= \int_0^1 \left( \int_X \phi_{\mathcal{X}_\tau} w(\epsilon m_{\mathcal{X}_\tau} + (1 - \epsilon) m_{\mathcal{X}_\tau}) (\epsilon \omega_{\mathcal{X}_\tau} + (1 - \epsilon) \omega)^{[n]} \right) d\epsilon
\]

\[
= \int_0^1 \left( \int_{\mathcal{X}_\tau} (\Phi w(m_{\tilde{\Omega}}) \Omega^{[n]})|_{\mathcal{X}_\tau} \right) d\epsilon
\]

where \( \Omega_{\epsilon} := \epsilon \Omega + (1 - \epsilon) \tilde{\omega} \), \( m_{\tilde{\Omega}} := \epsilon m_{\Omega} + (1 - \epsilon) m_{\tilde{\omega}} \), and \( \tilde{\omega} \), \( \Phi \) are given in Lemma 8.

It thus follows that \( \mathcal{E}_w(\phi_{\tau}) \) extends to a smooth function on \( \mathbb{P}^1 \setminus \{0\} \). Let \( f(\tau) \) be a
smooth function with compact support in $\mathbb{C}^* \subset \mathbb{P}^1$. Letting $\hat{f} := \pi^* f$ we have

$$\langle dd^c E_w(\phi_\tau), f \rangle = \int_0^1 \left( \int_{\mathcal{X}^*} dd^c f(\tau) \int_{\mathcal{X}^*} (\Phi w(m_{\Omega}) \Omega_\epsilon^{[n]} \hat{f}) \right) d\epsilon$$

$$= \int_0^1 \left( \int_{\mathcal{X}^*} \Phi w(m_{\Omega}) dd^c \hat{f} \wedge \Omega_\epsilon^{[n]} \right) d\epsilon$$

$$= - \int_0^1 \left( \int_{\mathcal{X}^*} \Phi d\hat{f} \wedge d^c w(m_{\Omega}) \wedge \Omega_\epsilon^{[n]} \right) d\epsilon$$

$$- \int_0^1 \left( \int_{\mathcal{X}^*} w(m_{\Omega}) d\hat{f} \wedge d^c \Phi \wedge \Omega_\epsilon^{[n]} \right) d\epsilon$$

$$= - \int_0^1 \left( \int_{\mathcal{X}^*} \Phi d\hat{f} \wedge d^c w(m_{\Omega}) \wedge \Omega_\epsilon^{[n]} \right) d\epsilon$$

$$+ \int_0^1 \left( \int_{\mathcal{X}^*} \hat{f} w(m_{\Omega}) dd^c \Phi \wedge \Omega_\epsilon^{[n]} \right) d\epsilon$$

$$+ \int_0^1 \left( \int_{\mathcal{X}^*} \hat{f} dw(m_{\Omega}) \wedge d^c \Phi \wedge \Omega_\epsilon^{[n]} \right) d\epsilon,$$

(34)

The first integral in the last equality vanishes. Indeed, for a basis $(\xi_i)_{i=1,\ldots,\ell}$ of $t$ we have

$$d\hat{f} \wedge d^c w(m_{\Omega}) \wedge \Omega_\epsilon^{[n]} = \sum_{i=1}^{\ell} w_i(m_{\Omega}) (df)(\pi^* \xi_i) \Omega_\epsilon^{[n+1]} = 0,$$

since the action of $\mathbb{T}$ preserves the fibers of $\mathcal{X} \to \mathbb{P}^1$. For the remaining integrals in the last equality in (34), integration by parts in the variable $\epsilon$ gives

$$\int_0^1 \left( \int_{\mathcal{X}^*} \hat{f} w(m_{\Omega}) dd^c \Phi \wedge \Omega_\epsilon^{[n]} \right) d\epsilon$$

$$= \int_0^1 \left( \int_{\mathcal{X}^*} \hat{f} w(m_{\Omega}) \frac{d}{d\epsilon} \Omega_\epsilon^{[n+1]} \right) d\epsilon \quad \text{(since $\Omega_\epsilon := \hat{\omega} + cdd^c \Phi$)}$$

$$= \int_{\mathcal{X}^*} \hat{f} w(m_{\Omega}) \Omega_\epsilon^{[n+1]} - \int_0^1 \left( \int_{\mathcal{X}^*} \hat{f} \left( \frac{d}{d\epsilon} w(m_{\Omega}) \right) \Omega_\epsilon^{[n+1]} \right) d\epsilon$$

$$= \int_{\mathcal{X}^*} \hat{f} w(m_{\Omega}) \Omega_\epsilon^{[n+1]} - \int_0^1 \left( \int_{\mathcal{X}^*} \hat{f} dw(m_{\Omega}) \wedge d^c \Phi \wedge \Omega_\epsilon^{[n]} \right) d\epsilon,$$

where for passing from the third line to the last line we used the following

$$\left( \frac{d}{d\epsilon} w(m_{\Omega}) \right) \Omega_\epsilon^{[n+1]} = \sum_{i=1}^{\ell} w_i(m_{\Omega}) d^c \Phi(\xi_i) \Omega_\epsilon^{[n+1]}$$

$$= \sum_{i=1}^{\ell} w_i(m_{\Omega}) dm_{\Omega}^{\xi_i} \wedge d^c \Phi \wedge \Omega_\epsilon^{[n]}$$

$$= w(m_{\Omega}) \wedge d^c \Phi \wedge \Omega_\epsilon^{[n]}.$$

By substituting (35) in (34) we get (33).

Now we establish (32) using (33), following the proof [31, Theorem 4.9]. Let $D_\epsilon \subset \mathbb{C}$ be the disc of center 0 and radius $\epsilon > 0$. Using the change of coordinates $(t, s)$ given
by $\tau = e^{-t+i\theta} \in \mathbb{C}$ and the $S^1$-invariance of $E_w(\phi \tau)$ we calculate

$$
\int_{X} w(m_\Omega) \Omega^{[n+1]} = \lim_{\epsilon \to 0} \int_{X \setminus \pi^{-1}(D_\epsilon)} w(m_\Omega) \Omega^{[n+1]}
$$

$$
= \lim_{\epsilon \to 0} \int_{\mathbb{P}^1 \setminus \mathbb{D}_\epsilon} \pi_* w(m_\Omega) \Omega^{[n+1]}
$$

$$
= \lim_{\epsilon \to 0} \int_{\mathbb{P}^1 \setminus \mathbb{D}_\epsilon} \frac{d}{dt} \mathcal{E}_w(\phi \tau)
$$

by (33)

$$
= \lim_{\epsilon \to 0} \frac{d}{dt} \mathcal{E}_w(\phi \tau) = \lim_{t \to +\infty} \frac{\mathcal{E}_w(\phi \tau)}{t}.
$$

□

Let $\Omega \in \mathcal{A}$ be $G$-invariant Kähler form. We consider the Kähler metric on $X^*$ given by $\hat{\omega} + \pi^* \omega_{FS} = \lambda^* (\pi_X^* \omega + \pi_{P1}^* \omega_{FS})$ (by the equivariance of $\lambda$), where $\hat{\omega} := (\pi_X \circ \lambda)^* \omega$ with $\lambda$ the map given by (25) and $\pi_X, \pi_{P1}$ denote the projections on the factors of $X \times (\mathbb{P}^1 \setminus \{0\})$. Then we have on $X^*$

(36) \[ \text{Ric}(\Omega) - \pi^* \omega_{FS} - \widehat{\text{Ric}(\omega)} = \frac{1}{2} \frac{d}{dt} \Psi, \]

where $\Psi = \log \left( \frac{\Omega^{n+1}}{\omega^n \wedge \pi^* \omega_{FS}} \right)$ and $\widehat{\text{Ric}(\omega)} := (\pi_X \circ \lambda)^* \text{Ric}(\omega)$. Using (36) and Lemma 5 (ii) we obtain on $X^*$

(37) \[ m_{\text{Ric}(\omega)}^\xi = m_{\text{Ric}(\Omega)}^\xi + \frac{1}{2} (d^* \Psi)(\xi), \]

for any $\xi \in t$, where $m_{\text{Ric}(\Omega)}^\xi := (\pi_X \circ \lambda)^* m_{\text{Ric}(\omega)}^\xi$.

**Lemma 10.** Under the hypotheses of Theorem 7, we have

(38) \[ dd^c \mathcal{E}_v^{\text{Ric}(\omega)}(\phi \tau) = \pi_* \left( v(m_\Omega) \text{Ric}(\omega) \wedge \Omega^{[n]} + ((dv)(m_\Omega), m_{\text{Ric}(\omega)}^\xi \Omega^{[n+1]}) \right). \]

**Proof.** From the very definition of $\mathcal{E}_v^{\text{Ric}(\omega)}$ (see (10)) we have

$$
\mathcal{E}_v^{\text{Ric}(\omega)}(\phi \tau) = \int_0^1 \left( \int_{X_{\tau}} [\Phi(v(m_\Omega) \text{Ric}(\omega) \wedge \Omega^{[n-1]} + ((dv)(m_\Omega), m_{\text{Ric}(\omega)}^\xi \Omega^{[n]})] |X_{\tau} | \right) d\tau,
$$

where $X_{\tau} := \epsilon \Omega + (1 - \epsilon) \hat{\omega}$, $m_{\Omega_{\tau}} := \epsilon m_\Omega + (1 - \epsilon) m_{\hat{\omega}}$, and $\hat{\omega}, \Phi$ are given in Lemma 8. As in the proof of Lemma 9 we see that $\mathcal{E}_v^{\text{Ric}(\omega)}(\phi \tau)$ extends to a smooth function on $\mathbb{P}^1 \setminus \{0\}$. Furthermore, for any smooth function $f(\tau)$ with compact support in $C^* \subset \mathbb{P}^1$,
we have
\[
\langle dd^c \mathcal{E}^{\text{Ric}}(\omega)(\phi_\tau), f \rangle = \int_{\mathcal{X}_\tau} \mathcal{E}^{\text{Ric}}(\omega)(\phi_\tau) dd^c f =
\]
\[
= \int_0^1 \left( \int_{\mathcal{X}_\tau} \Phi \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right) \right)_{|_{\mathcal{X}_\tau}} \, \text{de}
\]
\[
= \int_0^1 \left( \int_{\mathcal{X}_\tau} \Phi \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right) \right) \, \text{de}
\]
\[
= - \int_0^1 \left( \int_{\mathcal{X}_\tau} \Phi \left[ d \Phi \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right) \right] \, \text{de}
\]
\[
+ \int_0^1 \left( \int_{\mathcal{X}_\tau} \Phi \left[ d \Phi \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right) \right] \, \text{de}
\]
\[
+ \int_0^1 \left( \int_{\mathcal{X}_\tau} \Phi \left[ d \Phi \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right) \right] \, \text{de}
\]

where \( I_1, I_2, I_3 \) respectively denote the integrals on the first, second and third lines of the last equality. Now we compute each integral individually. We have
\[
d \hat{f} \wedge d \hat{\Phi} \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right) + d \hat{f} \wedge d \hat{\Phi} \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right)
\]
\[
= \sum_{i,j} v_{i,j}(m_{\Omega_\epsilon}) (d \hat{f})(\xi_i) m_{\text{Ric}(\omega)}^{\xi_j} \Omega_{\epsilon}^{[n+1]} + \sum_{i} v_{,i}(m_{\Omega_\epsilon}) d \hat{f} \wedge d \hat{\Phi} \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right)
\]
\[
+ \sum_{i} v_{,i}(m_{\Omega_\epsilon}) (d \hat{f})(\xi_i) (\Lambda_{\Omega_\epsilon} \text{Ric}(\omega)) \Omega_{\epsilon}^{[n+1]} - (d \hat{f} \wedge d \hat{\Phi} \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right)) \Omega_{\epsilon}^{[n+1]}
\]
\[
= \sum_{i} v_{,i}(m_{\Omega_\epsilon}) (d \hat{f} \wedge d \hat{\Phi} \left( \langle (dv(m_{\Omega_\epsilon}), m_{\text{Ric}(\omega)}) \rangle_{\Omega_\epsilon}^{[n]} \right)) \Omega_{\epsilon}^{[n+1]} = 0,
\]
where \( \xi = (\xi_i)_{i=1, \ldots, \ell} \) is a basis of \( t \). It follows that \( I_1 = 0 \). For the integral \( I_2 \), a similar calculation gives
\[
I_2 = \int_0^1 \left( \int_{\mathcal{X}_\tau} \hat{f} \left[ \sum_{i} v_{,i}(m_{\Omega_\epsilon}) (d \hat{f})(\xi_i) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n]} + \sum_{i,j} v_{i,j}(m_{\Omega_\epsilon}) m_{\text{Ric}(\omega)}^{\xi_j} (d \hat{\Phi})(\xi_i) \Omega_{\epsilon}^{[n]} \right] \right) \, \text{de}
\]
Now we consider the integral \( I_3 \). Using the fact that \( \Omega_\epsilon = \hat{\omega} + \epsilon dd^c \Phi \), an integration by parts with respect to \( \epsilon \) gives
\[
I_3 = \int_0^1 \left( \int_{\mathcal{X}_\tau} \hat{f} \left[ (d \hat{f})(v(m_{\Omega_\epsilon})) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n]} + (dd^c \hat{f})(v(m_{\Omega_\epsilon})) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n+1]} \right] \right) \, \text{de}
\]
\[
= \int_{\mathcal{X}_\tau} \hat{f} \left[ (d \hat{f})(v(m_{\Omega_\epsilon})) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n]} + (dd^c \hat{f})(v(m_{\Omega_\epsilon})) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n+1]} \right] \, \text{de}
\]
\[
= \int_0^1 \left( \int_{\mathcal{X}_\tau} \hat{f} \left[ (d \hat{f})(v(m_{\Omega_\epsilon})) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n]} + (dd^c \hat{f})(v(m_{\Omega_\epsilon})) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n+1]} \right] \right) \, \text{de}
\]
By Lemma \( \text{[\text{S}][\text{II}]} \), the second integral on the second equality is given by
\[
= \int_0^1 \left( \int_{\mathcal{X}_\tau} \hat{f} \left[ \sum_{i,j} v_{i,j}(m_{\Omega_\epsilon}) (d \hat{\Phi})(\xi_i) \text{Ric}(\omega) \wedge \Omega_{\epsilon}^{[n]} + \sum_{i,j} v_{i,j}(m_{\Omega_\epsilon}) m_{\text{Ric}(\omega)}^{\xi_j} (d \hat{\Phi})(\xi_i) \Omega_{\epsilon}^{[n]} \right] \right) \, \text{de}
\]
\[
= I_2.
\]
It follows that
\[ I_1 + I_2 + I_3 = \int_{X^*} \hat{f} [v(m_{\Omega})\widehat{\operatorname{Ric}(\omega)} \wedge \Omega^{[n]} + \langle (dv)(m_{\Omega}), m_{\widehat{\operatorname{Ric}(\omega)}} \rangle \Omega^{[n+1]}]. \]
This completes the proof. \hfill \Box

**Lemma 11.** Under the hypotheses of Theorem 7,
\[ \lim_{t \to +\infty} \frac{1}{t} \left( \int_{X^*} v(m_{\phi_{\psi}}) \omega^{[n]}_{\phi_{\psi}} - 2E_{\psi}^V(\phi_{\psi}) \right) \]
\[ = -2 \int_{X^*} v(m_{\Omega})(\operatorname{Ric}(\Omega) - \pi^* \omega_{\text{FS}}) \wedge \Omega^{[n]} + \langle (dv)(m_{\Omega}), m_{\widehat{\operatorname{Ric}(\omega)}} \rangle \Omega^{[n+1]} \]
where \( \phi_{\psi} \) is given by (31) and \( \psi_{\tau} = \psi_{\tau} \) with \( \tau = e^{-t+is} \) is given by
\[ \psi_{\tau} := \rho(\tau)^s(\Psi|\tau_{\tau}) \in C^\infty(X_1, \mathbb{R})^T. \]

**Proof.** We define on \( \mathbb{C}^* \) the function \( H(\tau) := \int_X \psi_{\tau} v(m_{\tau}) \omega^{[n]}_{\tau} \). Let \( f(\tau) \) be a test function with support in \( \mathbb{C}^* \subset \mathbb{P}^1 \) and \( \hat{f} := \pi^* f \). We have
\[ \langle dd^c H, f \rangle = \int_{C^*} dd^c f \int_{X^*} (\Psi v(m_{\Omega}) \Omega^{[n]}), X^* \]
\[ = \int_{X^*} \Psi v(m_{\Omega}) dd^c \hat{f} \wedge \Omega^{[n]} \]
\[ = \int_{X^*} \Psi d(v(m_{\Omega})) \wedge dd^c \hat{f} \wedge \Omega^{[n]} - \int_{X^*} v(m_{\Omega}) d\Psi \wedge dd^c \hat{f} \wedge \Omega^{[n]} \]
Notice that \( d(v(m_{\Omega})) \wedge dd^c \hat{f} \wedge \Omega^{[n]} = 0 \) since the 1-form \( dd^c \hat{f} \) is zero on the fundamental vector fields of the \( T \)-action. Integration by parts gives
\[ \langle dd^c H, f \rangle = \int_{X^*} \hat{f} d\Psi \wedge dd^c v(m_{\Omega}) \wedge \Omega^{[n]} + \int_{X^*} \hat{f} v(m_{\Omega}) dd^c \Psi \wedge \Omega^{[n]} \]
Using the equations (35) and (37) we obtain
\[ \langle dd^c H, f \rangle = -2 \int_{X^*} \hat{f} \langle (dv)(m_{\Omega}), m_{\widehat{\operatorname{Ric}(\omega)}} \rangle \Omega^{[n+1]} \]
\[ - \int_{X^*} \hat{f} v(m_{\Omega})(\operatorname{Ric}(\Omega) - 2\pi^* \omega_{\text{FS}} - \operatorname{Ric}(\omega)) \wedge \Omega^{[n]} \]
Combining (48) and 11 gives
\[ dd^c (H(\tau) - 2E_{\psi}^V(\phi_{\psi})) \]
\[ = -2\pi^* \left( v(m_{\Omega})(\operatorname{Ric}(\Omega) - \pi^* \omega_{\text{FS}}) \wedge \Omega^{[n]} + \langle (dv)(m_{\Omega}), m_{\widehat{\operatorname{Ric}(\omega)}} \rangle \Omega^{[n+1]} \right). \]
We conclude in the same way as in the proof of Lemma 9. \hfill \Box

We consider the following function on \( \mathbb{C}^* \):
\[ \mathcal{M}_{v,w}^\Psi(\phi_{\tau}) := \int_X \psi_{\tau} v(m_{\phi_{\tau}}) \omega^{[n]}_{\phi_{\tau}} - 2E_{\psi}^V(\phi_{\psi}) + c_{(v,w)(\alpha)} E_{\psi}^V(\phi_{\tau}), \]
where \( \phi_{\tau} \) and \( \psi_{\tau} \) are given by (31) and (41) respectively. From the definition of \( \mathcal{M}_{v,w}^\Psi(\phi_{\tau}) \) and Lemmas 9 and 11 we see that
\[ \lim_{t \to +\infty} \frac{\mathcal{M}_{v,w}^\Psi(\phi_{\tau})}{t} = F_{v,w}(X, A). \]
Lemma 12. If the central fiber $X_0$ is reduced, then the integral
\[ \Upsilon(\tau) := \int_{X_{\tau}} \log \left( \frac{\Omega^n \wedge \pi^* \omega_{FS}}{\Omega^{n+1}} \right) v(m_{\Omega}) \Omega^{|n|}_{\tau}, \]
is bounded on $\mathbb{C}^*$.

Proof. The integral $\Upsilon(\tau)$ is bounded from above since $Z(\hat{\tau}) = \frac{\Omega^n \wedge \pi^* \omega_{FS}}{\Omega^{n+1}}$ is a non-negative smooth function on $X$ and the integral $\int_{X_{\tau}} v(m_{\Omega}) \Omega^{|n|}_{\tau}$ is independent from $\tau$ (see Lemma 12). Notice that $\Upsilon(\tau)$ is bounded if and only if $\int_{X_{\tau}} |\log(Z)| v(m_{\Omega}) \Omega^{|n|}_{\tau}$ is bounded. Indeed, if $\Upsilon(\tau) = O(1)$ then
\[ \int_{X_{\tau}} |\log(Z)| v(m_{\Omega}) \Omega^{|n|}_{\tau} = \int_{X_{\tau}} (\log(Z) + |\log(Z)|) v(m_{\Omega}) \Omega^{|n|}_{\tau} - \Upsilon(\tau) = O(1). \]
It follows that $\int_{X_{\tau}} |\log(Z)| v(m_{\Omega}) \Omega^{|n|}_{\tau} = O(1)$. The converse follows from
\[ |\Upsilon(\tau)| \leq \int_{X_{\tau}} |\log(Z)| v(m_{\Omega}) \Omega^{|n|}_{\tau}. \]

Using that $v(m_{\Omega})$ is a smooth function on $X$ we see that $\int_{X_{\tau}} |\log(Z)| v(m_{\Omega}) \Omega^{|n|}_{\tau} = O(1)$ if and only if $\int_{X_{\tau}} |\log(Z)| \Omega^{|n|}_{\tau} = O(1)$, which is also equivalent to $\int_{X_{\tau}} \log(Z) \Omega^{|n|}_{\tau} = O(1)$. By [28, Remark 4.12], if the central fiber $X_0$ is reduced then $\int_{X_{\tau}} \log(Z) \Omega^{|n|}_{\tau} = O(1)$ which implies that $\Upsilon(\tau) = O(1)$. \hfill ☐

Now we are in position to give a proof for Theorem 7.

Proof of Theorem 7. From the modified Chen-Tian formula in Theorem 5 and by Lemma 12 we get
\[ \mathcal{M}_{v,w}(\phi_{\tau}) - \mathcal{M}_{v,w}^{\Psi}(\phi_{\tau}) = \int_{X} \left( \log \left( \frac{\omega^n}{\omega^m} \right) - \psi_{\tau} \right) v(m_{\tau}) \omega^{|n|}_{\tau} \]
\[ = \int_{X_{\tau}} \log \left( \frac{\Omega^n \wedge \pi^* \omega_{FS}}{\omega^n \wedge \pi^* \omega_{FS}} \right) v(m_{\Omega}) \Omega^{|n|}_{\tau} \]
\[ = \int_{X_{\tau}} \log \left( \frac{\Omega^n \wedge \pi^* \omega_{FS}}{\Omega^{n+1}} \right) v(m_{\Omega}) \Omega^{|n|}_{\tau} = O(1). \]
Dividing by $t$ (where we recall $\tau = e^{-t+i\mathcal{S}}$) and passing to the limit when $t$ goes to infinity concludes the proof. \hfill ☐

Proof of Theorems 2 and 3. These are direct corollaries of Theorems 1 and 9 respectively, together with Theorem 7 and Proposition 2. \hfill ☐

Proposition 3. If $(X, \mathcal{A}, \mathbb{T})$ is a Kähler test configuration of $(X, \alpha, \mathbb{T})$ such that $\pi: X \to \mathbb{P}^1$ is a smooth submersion then
\[ \mathcal{F}_{v,w}(X, \mathcal{A}) = \mathcal{F}_{v,w}^\alpha(V_{\rho}) - \frac{\text{Vol}(X, \mathcal{A})}{\text{Vol}(X, \alpha)} \int_{X} \left( \text{Scal}_v(\omega) - c_{v,w}(\alpha) w(m_{\omega}) \right) \omega^{|n|}, \]
where $V_{\rho}$ is the generator of the $S^1_\rho$-action on $X_0$, and $\mathcal{F}_{v,w}^\alpha(V_{\rho})$ is the $(v, w)$-Futaki invariant of the smooth central fibre $(X_0, \alpha)$ introduced in Definition 8. In particular if $(X, \alpha, \mathbb{T})$ is $(v, w)$-semistable on smooth test configurations, then
\[ \int_{X} \text{Scal}_v(\omega) \omega^{|n|} = c_{(v,w)}(\alpha) \int_{X} w(m_{\omega}) \omega^{|n|} \text{ and } \mathcal{F}_{v,w}^\alpha \equiv 0. \]
Proof. We just adapt the arguments from [30] to our weighted setting. From Definition [31] we have
\[
\frac{d}{dt} \mathcal{M}_{v,w}(\phi_t) = - \int_{X_1} \dot{\phi}_t (\text{Scal}_v(\omega_t) - c_{v,w}(\alpha)w(m_{\Omega_t})) \omega_t^{[n]},
\]
(44)
where \( t = -\log |\tau| \), \( \dot{\phi}_t = \frac{d\phi_t}{dt} \) and \( \omega_t, \phi_t, m_t \) are given by (31) and (27). Note that the flow of the vector field \( J \) is given by (45) \[ \rho(t)^{-1} \phi_t(\text{Scal}_v(\Omega_t) - c_{v,w}(\alpha)w(m_{\Omega_t}))\Omega_t^{[n]}, \]
and \( \lim_{t \to \infty} a(t) \) is a constant depending on \( \tau \in \mathbb{C}^* \). By (45) and Lemma [31] we have
\[
a(\tau) = \frac{1}{\text{Vol}(X, \alpha)} \left( \int_{X_t} h^\rho \Omega_t^{[n]} + \frac{d\phi_t}{dt} \right).
\]
Using that \( \pi : \mathcal{X} \to \mathbb{P}^1 \) is a smooth submersion and Lemma [32] we get
\[
\lim_{t \to \infty} a(\tau) = \frac{1}{\text{Vol}(X, \alpha)} \left( \int_{X_t} h^\rho \Omega_t^{[n]} + \text{Vol}(\mathcal{X}, A) \right).
\]
Substituting (46) in (14), we obtain
\[
\frac{d}{dt} \mathcal{M}_{v,w}(\phi_t) = \int_{X_t} (\text{Scal}_v(\Omega_t) - c_{v,w}(\alpha)w(m_{\Omega_t})) h^\rho \Omega_t^{[n]}
\]
(47)
- \( a(\tau) \int_{X_t} (\text{Scal}_v(\Omega_t) - c_{v,w}(\Omega_t)w(m_{\Omega_t})) \Omega_t^{[n]} \).
Passing to the limit when \( t \to \infty \) in (47) and using Theorem [7] we obtain
\[
\mathcal{F}_{v,w}(\mathcal{X}, A) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{M}_{v,w}(\phi_t)
\]
(48)
\[
= \int_{X_0} (\text{Scal}_v(\Omega_0) - c_{v,w}(\alpha)w(m_{\Omega_0})) h^\rho \Omega_0^{[n]}
\]
- \( \frac{1}{\text{Vol}(X, \alpha)} \left( \int_{X_0} h^\rho \Omega_0^{[n]} + \text{Vol}(\mathcal{X}, A) \right) \int_{X_0} (\text{Scal}_v(\Omega_0) - c_{v,w}(\Omega_0)w(m_{\Omega_0})) \Omega_0^{[n]} \)
(49)
- \( \frac{\text{Vol}(\mathcal{X}, A)}{\text{Vol}(X, \alpha)} \int_{X_0} (\text{Scal}_v(\Omega_0) - c_{v,w}(\Omega_0)w(m_{\Omega_0})) \Omega_0^{[n]} \)
\[
= \mathcal{F}_{v,w}^\alpha(V_\rho) - \frac{\text{Vol}(\mathcal{X}, A)}{\text{Vol}(X, \alpha)} \int_X (\text{Scal}_v(\omega) - c_{v,w}(\alpha)w(m_{\omega})) \omega^{[n]}.
\]
where \( \Omega_0 = \Omega|_{X_0} \in \mathcal{A}_W(X_0) \), and we have used in the last equality that for any \( \tau \in \mathbb{C}^* \) we have
\[
\int_{X_\tau} \text{Scal}_v(\Omega_\tau) \Omega_\tau^{[n]} = \int_{X_1} \text{Scal}_v(\omega_\tau) \omega_\tau^{[n]} = \int_X \text{Scal}_v(\omega) \omega^{[n]},
\]
(50)
\[
\int_{X_\tau} w(m_{\Omega_\tau}) \Omega_\tau^{[n]} = \int_{X_1} w(m_{\omega_\tau}) \omega_\tau^{[n]} = \int_X w(m_{\omega}) \omega^{[n]},
\]
see Lemma [2]
For the second statement, as \( \int_X (\text{Scal}_{\omega} - c_{v,w}(\alpha) w(m_\omega)) \Omega^{[n]} = 0 \) by the definition of semi-stability, we consider the product test configurations associated to \( V \) and \(-V\) for any \( V \in h_{\text{red}} \), we obtain \( \mathcal{F}^a_{\text{v,w}}(V) = - \mathcal{F}^a_{\text{v,w}}(V) \geq 0 \) i.e. \( \mathcal{F}^a_{\text{v,w}} \equiv 0 \). \( \square \)

**Remark 7.** In [27], Dervan defines a T-relative Donaldson–Futaki invariant \( \text{DF}_T(X,A) \) for a smooth T-compatible Kähler test configuration \( X \) as follows

\[
\text{DF}_T(X,A) := \mathcal{F}_{1,1}(X,A) - \sum_{i=1}^\ell \langle h_{\rho_i}, \xi_i \rangle_{X_0} \mathcal{F}^a_{1,1}(\xi_i),
\]

where \( \xi := (\xi_i)_{i=1,\ldots,\ell} \) is a basis of \( T \) with corresponding Killing potentials \( h_i = f_i(m_\Omega) = \langle m_\Omega, \xi_i \rangle + \lambda_i \), such that \( \langle h_i, h_j \rangle_{X_0} = \int_{X_0} h_i h_j \Omega^n = 0 \) for \( i \neq j \) and \( \int_{X_0} h_i \Omega^n = 0 \), where the integration on \( X_0 \) is defined by \( \int_{X_0} := \sum_i m_i \int_{(X_0)^{\text{reg}}} \) with \( [X_0] = \sum_i m_i X_0^{(i)} \) being the analytic cycle associated to \( X_0 \) and \( (X_0)^{\text{reg}} \) standing for the regular part of the irreducible component \( X_0^{(i)} \) of \( X_0 \). Using Lemma \( \text{[4]} \) we have

\[
\mathcal{F}^a_{1,1}(\xi_i) = \langle \text{wext}(m_\omega) \rangle_{X_0} = \langle \text{wext}(m_\omega), h_i \rangle_{X_0} = \langle \text{wext}(m_\Omega), h_i \rangle_{X_0},
\]

for any \( \tau \in \mathbb{C}^* \subset \mathbb{P}^1 \). As the family \( \pi : X \to \mathbb{P}^1 \) is proper and flat, the current of integration along the fibers \( X_\tau \) is continuous and converges to the integration over the analytic cycle of the central fiber \( [X_0] \) (see [10]). Passing to the limit when \( \tau \to 0 \) in \( \text{[48]} \), we thus obtain \( \int_X \text{wext}(m_\omega) \Omega^{[n]} = \int_{X_0} \text{wext}(m_\Omega) \Omega^{[n]} \) and \( \mathcal{F}_{1,1}(\xi_i) = \langle \text{wext}(m_\Omega), h_{\rho_i}, \Omega \rangle_{X_0} \). Thus,

\[
\text{DF}_T(X,A) = \mathcal{F}_{1,1}(X,A) - \langle \text{wext}(m_\Omega), h_\rho \rangle_{X_0}.
\]

On the other hand, the \((1, \text{wext})\)-Futaki invariant of \( \langle X,A \rangle \) is given by

\[
\mathcal{F}_{1,\text{wext}}(X,A) = - \int_X \text{Scal}(\Omega) \Omega^{[n+1]} + 2 \int_X \pi^* \omega_{FS} \wedge \Omega^{[n]} + \int_X \text{wext}(m_\Omega) \Omega^{[n+1]}.
\]

(Recall that \( c_{1,\text{wext}}(\alpha) = 1 \), see Section [3.2]). From \( \text{[49]} \) and \( \text{[50]} \), we infer

\[
\mathcal{F}_{1,\text{wext}}(X,A) - \text{DF}_T(X,A) = \langle \text{wext}(m_\Omega), h_\rho \rangle_{X_0} + \int_X \langle \text{wext}(m_\Omega) - c_{1,1}(\alpha) \rangle \Omega^{[n+1]}
\]

\[
= \langle \text{wext}(m_\Omega), h_\rho \rangle_{X_0} + \lim_{t \to \infty} \frac{d\mathcal{E}^a_{\text{wext}}(\phi_\tau)}{dt}
\]

\[
= \langle \text{wext}(m_\Omega), h_\rho \rangle_{X_0} + \lim_{t \to \infty} \left( \int_{X_1} \phi_\tau \hat{w}_{\text{wext}}(m_\tau) \omega^{[n]}_{\tau} \right)
\]

\[
= \langle \text{wext}(m_\Omega), h_\rho \rangle_{X_0} - \lim_{t \to \infty} \left( \int_{X_0} h_\rho \hat{w}_{\text{wext}}(m_\Omega) \Omega^{[n]} \right)
\]

\[
= \langle \text{wext}(m_\Omega), h_\rho \rangle_{X_0} - \int_{X_0} h_\rho \hat{w}_{\text{wext}}(m_\Omega) \Omega^{[n]} = 0,
\]

where in the second equality we used Lemma \( \text{[3]} \) for

\[
\hat{w}_{\text{wext}} = \text{wext} - c_{1,1}(\alpha) = \text{wext} - \frac{1}{\text{Vol}(X,\alpha)} \int_{X_0} \text{wext}(m_\Omega) \Omega^{[n]},
\]

for any \( \tau \in \mathbb{C}^* \) and in the fourth equality we used \( \text{[44]} \). It follows that

\[
\mathcal{F}_{1,\text{wext}}(X,A) = \text{DF}_T(X,A).
\]
8. Algebraic definition of a \((v, w)\)-Donaldson-Futaki invariant

8.1. The \((v, w)\)-Donaldson-Futaki invariant of a smooth polarized variety. Let \((X, L)\) be a smooth compact polarized projective manifold, where \(L\) is an ample holomorphic line bundle on \(X\) and \(\mathbb{T} \subset \text{Aut}(X, L)\) is an \(\ell\)-dimensional real torus on the total space of \(L\), which covers a torus action (still denoted by \(\mathbb{T}\)) in \(\text{Aut}_{\text{red}}(X) \cong \text{Aut}(X, L)/\mathbb{C}^*\). Let \(\xi = (\xi_1, \ldots, \xi_\ell) \in \mathfrak{t}\) be a basis of \(\mathbb{S}^1\)-generators of \(\mathbb{T}\) and \(A_{\xi}^{(k)} := (A_{\xi_1}^{(k)}, \ldots, A_{\xi_\ell}^{(k)})\) the induced infinitesimal actions of \(\xi_i\) on the finite dimensional space \(\mathcal{H}_k := H^0(X, L^k)\) of global holomorphic sections of \(L^k\) for \(k \gg 1\). For a \(\mathbb{T}\)-invariant Hermitian metric \(h\) on \(L\) with curvature two form \(\omega \in 2\pi c_1(L)\) we have (see e.g. \cite[Proposition 8.8.2]{[45]})

\[
A_{\xi_i}^{(k)} + \sqrt{-1} \nabla_{\xi_i} = km^\xi_i \text{Id}_{\mathcal{H}_k},
\]

where \(\nabla\) is the Chern connection of \(h^k := h \otimes h\) and \(m^\xi_i\) is a \(\omega\)-Hamiltonian function of \(\xi_i\). Using the basis \(\xi\) we identify \(t \cong \mathbb{R}^\ell\) and we get a momentum map \(m_\omega := (m_{\xi_1}^\xi, \ldots, m_{\xi_\ell}^\xi) : \mathbb{R}^\ell \to \mathbb{R}^\ell\) for the action of \(\mathbb{T}\) on \(X\) with momentum image \(P := m_\omega(X)\). Notice that if \(h_\phi := e^{-2\phi}h\) is another \(\mathbb{T}\)-invariant Hermitian metric on \(L\) with positive curvature \(\omega_\phi > 0\), the corresponding momentum map satisfies \(m_\omega^\xi = m_{\omega_\phi}^\xi + (d^c \phi)(\xi_i)\), thus showing, by virtue of Lemma~\cite[III]{[12]} that the image \(m_\omega(X) = P\) is independent of the metric \(h_\phi\). We thus have a polytope \(P \subset \mathfrak{t}^*\) associated to the polarization \((X, L, \mathbb{T})\).

The spectrum of \(k^{-1}A_{\xi_j}^{(k)}\) is given by \(\{\lambda_i^{(k)}(\xi_j), i = 1, \ldots, N_k\} \subset \Lambda^*\) is the finite set of weights of the complexified action of \(\mathbb{T}\) on \(\mathcal{H}_k\) and \(\Lambda^*\) is the dual of the lattice \(\Lambda \subset \mathfrak{t}\) of circle subgroups of \(\mathbb{T}\) (see e.g. \cite{[6], [12]}).

**Lemma 13.** The set of weights \(W_k\) is contained in the momentum polytope \(P\) of the action of \(\mathbb{T}\) on \((X, L, \mathbb{T})\).

**Proof.** This Lemma is well known (see e.g. \cite[Section 5]{[6]}), we give the proof for the sake of clarity. Let \(\lambda_i^{(k)} \in W_k, \xi_j \in \xi\) an \(\mathbb{S}^1\)-generator for the \(\mathbb{T}\)-action on \(X\), and \(s_{j,i}^{(k)} \in \mathcal{H}_k\) an eigensection associated to the eigenvalue \(\lambda_i^{(k)}(\xi_j)\) of \(k^{-1}A_{\xi_j}^{(k)}\). Using (51), we have

\[
(\lambda_i^{(k)}(\xi_j) - m_{\omega}^{\xi_j})|s_{j,i}^{(k)}|_{h^k}^2 = (k^{-1}A_{\xi_j}^{(k)} s_{j,i}, s_{j,i})_{h^k} - m_{\omega}^{\xi_j}|s_{j,i}^{(k)}|_{h^k}^2 = -\frac{\sqrt{-1}}{2} (d|s_{j,i}^{(k)}|_{h^k}^2)(\xi_j) - \frac{1}{2} (d|s_{j,i}^{(k)}|_{h^k}^2)(J\xi_j).
\]

At a point of global maximum \(x_0\) of the smooth function \(|s_{j,i}^{(k)}|_{h^k}^2\) on \(X\), we obtain

\[
\lambda_i^{(k)}(\xi_j) = m_{\omega}^{\xi_j}(x_0) \in P.
\]

It follows that \(W_k \subset P\).

Using the weight decomposition of \(\mathcal{H}_k\)

\[
\mathcal{H}_k = \bigoplus_{\lambda_i^{(k)} \in W_k} \mathcal{H}(\lambda_i^{(k)}),
\]

and Lemma~\cite[13]{[12]} for any smooth function \(v \in C^\infty(P, \mathbb{R})\) we can define the operator \(v(k^{-1}A_{\xi}^{(k)}): \mathcal{H}_k \to \mathcal{H}_k\) by

\[
v(k^{-1}A_{\xi}^{(k)}) |_{\mathcal{H}(\lambda_i^{(k)})} := v(k^{-1}\lambda_i^{(k)}) \text{Id}_{\mathcal{H}(\lambda_i^{(k)})}.
\]

**Definition 13.** We define the \(v\)-weight of the action of \(\mathbb{T}\) on \((X, L, \mathbb{T})\) by

\[
W_v(L^k) := \text{Tr}(v(k^{-1}A_{\xi}^{(k)})).
\]
Lemma 14. The \( v \)-weight of the action of \( T \) on \((X, L)\) admits the following asymptotic expansion

\[
(2\pi)^n W_v(L^k) = k^n \int_X v(m_\omega)\omega^{[n]} + \frac{k^{n-1}}{4} \int_X \text{Scal}_v(\omega)\omega^{[n]} + \mathcal{O}(k^{n-2}).
\]

for any smooth function \( v \) with compact support containing \( P \).

Proof. This is a direct consequence of Theorem 14.3 below, by letting \( w = v \) in (54), and integrating in both sides over \( X \).

The following result is a straightforward consequence of Lemma 14.7

Corollary 2. Let \((X, L)\) be a smooth polarized projective variety endowed with a torus action \( T \subset \text{Aut}(X, L) \) and \( v, w \in C^\infty(P, \mathbb{R}) \) smooth functions on the corresponding polytope \( P \subset t^* \). For any \( \mathbb{C}^* \)-action \( \rho \) commuting with \( T \) and a family \( \xi \) of \( S^1 \)-generators of \( T \), we consider the weight

\[
W_v^{(k)}(\xi, \rho) := \text{Tr} \left(v(k^{-1} A^{(k)}_\xi) \cdot k^{-1} A^{(k)}_\rho\right),
\]

where \( A^{(k)}_\rho \) is the induced infinitesimal action of \( \rho \) on \( \mathcal{H}_k \). Then, \( W_w(\xi, \rho) \) admits an asymptotic expansion

\[
W_v^{(k)}(\xi, \rho) = a_v^{(0)}(\xi, \rho)k^n + a_v^{(1)}(\xi, \rho)k^{n-1} + \mathcal{O}(k^{n-2}),
\]

and the \((v, w)\)-Donaldson-Futaki invariant introduced in Definition 8 with respect to the Kähler class \( \alpha := 2\pi c_1(L) \) satisfies

\[
\frac{1}{4(2\pi)^n} F^{\alpha}_{v,w}(V_\rho) = a_v^{(1)}(\xi, \rho) - \frac{c_{v,w}(L)}{4} a_w^{(0)}(\xi, \rho),
\]

where \( V_\rho \) is the generator of the \( S^1_\rho \)-action on \( X \), and \( c_{v,w}(L) \) is the \((v, w)\)-slope of \((X, 2\pi c_1(L))\) defined in (8).

8.2. The \((v, w)\)-Donaldson-Futaki invariant of a polarized test configuration. Following [30], we consider a (possibly singular) polarized test configuration of exponent \( r \in \mathbb{N} \), compatible with \((X, L, T)\), defined as follows:

Definition 14. A \( T \)-compatible polarized test configuration \((\mathcal{X}, \mathcal{L})\) of exponent \( r \in \mathbb{N} \) associated to the smooth polarized variety \((X, L)\) is a normal polarized variety \((\mathcal{X}, \mathcal{L}, \hat{T})\) endowed with a torus \( \hat{T} \subset \text{Aut}(\mathcal{X}, \mathcal{L}) \)

- a flat morphism \( \pi : \mathcal{X} \to \mathbb{P}^1 \) such that the torus action \( \hat{T} \) on \( \mathcal{X} \) preserves each fiber \( \mathcal{X}_r := \pi^{-1}(r) \), and \((\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1}, \hat{T})\) is equivariantly isomorphic to \((X, L^r, T)\);
- a \( \mathbb{C}^* \)-action \( \rho \) on \( \mathcal{X} \) commuting with \( \hat{T} \) and covering the usual \( \mathbb{C}^* \)-action on \( \mathbb{P}^1 \);
- an isomorphism

\[
\lambda : (\mathcal{X} \times (\mathbb{P}^1 \setminus \{0\}), L^r \otimes \mathcal{O}_{\mathbb{P}^1}(r)) \cong (\mathcal{X} \setminus X_0, \mathcal{L}),
\]

which is equivariant with respect to the actions of \( G := \hat{T} \times S^1_\rho \) on \( \mathcal{X} \setminus X_0 \) and the action of \( T \times S^1 \) on \( X \times (\mathbb{P}^1 \setminus \{0\}) \).

To simplify the discussion, we shall assume in the sequel that \( r = 1 \) and that \( L \) is a very ample polarization on \( X \).

By the consideration in Section 8.1, for each \( \tau \neq 0 \), \((X_\tau, \mathcal{L}|_{X_\tau}, \hat{T})\) gives rise to a momentum polytope \( P_\tau \subset t^* \). Using the biholomorphism [55], we know that \((X_\tau, \mathcal{L}|_{X_\tau}, \hat{T})\) and \((\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1}, \hat{T})\) are equivariantly isomorphic polarized varieties, and thus \( P_\tau = P_1 = P \) for all \( \tau \neq 0 \).

For any \( \tau \in \mathbb{P}^1 \), following Section 8.1 we let \( A^{(k)}_\xi(\tau) := (A^{(k)}_{\xi^1}(\tau), \ldots, A^{(k)}_{\xi_r}(\tau)) \) be infinitesimal generators of the \( S^1 \)-actions on \( \mathcal{H}_k(\tau) := H^0(X_\tau, \mathcal{L}_{X_\tau}^{(k)}) \), induced by the
\(S^1\)-generators \(\xi = (\xi_1, \cdots, \xi_)\) for the \(\hat{T}\)-action, on the fiber \((X_+, L_{|X_+})\). We claim that the spectrum of the operators \(A_i^{(k)}(\tau)\) is independent of \(\tau \in \mathbb{P}^1\), and is contained in \(P\). To see this, we can use the observation from [36, Sect. 2.3] which associates to any \(T\)-compatible polarized test configuration \((X, L, \hat{T})\) a continuous family \(V_k(\tau) \subset \text{Sym}^k(\mathbb{C}^{N+1})\) of \(m\)-planes in the Grassmanian \(\text{Gr}_m(\text{Sym}^k(\mathbb{C}^{N+1}))\), where \(\text{Sym}^k\) denotes the vector space of symmetric homogeneous polynomials in \(N + 1\) complex variables. In this picture, \((X_+, L_{|X_+})\) is seen as a polarized subvariety of \((\mathbb{P}^N, \mathcal{O}(1))\), and the space of sections \(H_k(\tau) := H^0(X_+, (L_{|X_+})^k)\) is identified to \(\text{Sym}^k(\mathbb{C}^{N+1})/V_k(\tau)\). We can further assume that the action of \(\hat{T}\) on \((X_+, L_{|X_+})\) comes from the restriction to \(X_+\) of a subtorus of \(\hat{T} \subset \text{SL}(N + 1, \mathbb{C})\), and thus \(\hat{T}\) also acts on \(\text{Sym}^k(\mathbb{C}^{N+1})\); furthermore, writing \(\hat{A}_i^{(k)} := (\hat{A}_i^{(k)}(0), \cdots, \hat{A}_i^{(k)}(0))\), where \(\hat{A}_i^{(k)}\) is the infinitesimal generator of the circle action \(S^1\) on \(\text{Sym}^k(\mathbb{C}^{N+1})\), the operators

\[
\hat{A}_i^{(k)} : \text{Sym}^k(\mathbb{C}^{N+1}) \to \text{Sym}^k(\mathbb{C}^{N+1}),
\]

must preserve the \(m\)-planes \(V_k(\tau)\) (as the action preserves each \(X_+\) viewed as the subspace of common zeroes of elements in \(V_k(\tau)\)), and thus

\[
A_i^{(k)}(\tau) : \text{Sym}^k(\mathbb{C}^{N+1})/V_k(\tau) \to \text{Sym}^k(\mathbb{C}^{N+1})/V_k(\tau)
\]

are the linear maps induced by \(\hat{A}_i^{(k)}\) on the quotient spaces \(H_k(\tau)\). Introducing a \(\hat{T}\)-invariant Hermitian product on \(\text{Sym}^k(\mathbb{C}^{N+1})\), we thus obtain a continuous \(A_i^{(k)}\)-invariant decomposition

\[
\text{Sym}^k(\mathbb{C}^{N+1}) = V_k(\tau) \oplus V_k^\perp(\tau),
\]

and the spectrum of \(A_i^{(k)}(\tau)\) is nothing but the spectrum of \(\hat{A}_i^{(k)}\) restricted to \(V_k^\perp(\tau)\). Using that \(V_k^\perp(\tau)\) vary continuously in the Grasmannian, we conclude that the spectrum of \(\hat{A}_i^{(k)}\) restricted to \(V_k^\perp(\tau)\) is constant. It is contained in \(P\) by Lemma 13.

It follows that for any \(v \in C^\infty(\mathbb{P}, \mathbb{R})\), we can define \(v(\tau^{-1}A_i^{(k)}(0))\), where \(A_i^{(k)}(0) = (A_i^{(k)}(0), \cdots, A_i^{(k)}(0))\) denote the the generators of circle actions corresponding to the central fibre \((X_0, L_{|X_0}, \hat{T})\). Thus, for \(v \in C^\infty(\mathbb{P}, \mathbb{R})\) we can consider the following \(v\)-weight

\[
W_v^{(k)}(\xi, \rho) := \text{Tr} \left(v(\tau A_i^{(k)}(0)) \cdot k^{-1} A_i^{(k)}\right).
\]

**Definition 15.** Let \(v \in C^\infty(\mathbb{P}, \mathbb{R}_{>0})\) and \(w \in C^\infty(\mathbb{P}, \mathbb{R})\), and suppose that we have the following asymptotic expansions on the central fiber \((X_0, L_0)\)

\[
W_v^{(k)}(\xi, \rho) = a_v^{(0)}(\xi, \rho)k^n + \mathcal{O}(k^{n-1}),
\]

\[
W_v^{(k)}(\xi, \rho) = a_v^{(0)}(\xi, \rho)k^n + a_v^{(1)}(\xi, \rho)k^{n-1} + \mathcal{O}(k^{n-2}).
\]

Then we define the \((v, w)\)-Donaldson-Futaki invariant of the normal \(T\)-compatible polarized test configuration \((X, L)\) to be the number

\[
\text{DF}_{v,w}(X, L) := a_v^{(1)}(\xi, \rho) - \frac{c_{v,w}(L)}{4} a_v^{(0)}(\xi, \rho),
\]

where \(c_{v,w}(L)\) is the \((v, w)\)-slope of \((X, 2\pi c_1(L))\) given by \([\xi]\).

Using Lemma 14 we have the following
Corollary 3. If $(\mathcal{X}, \mathcal{L})$ is a $\mathbb{T}$-compatible polarized test configuration with smooth central fiber, then the expansions \((57)\) hold, and
\[
(2\pi)^n W_w^{(k)}(\xi, \rho) = k^n \int_{X_0} h_\rho w(m_{\omega_0}) \Omega_0^{[n]} + O(k^{n-1}),
\]
\[
(2\pi)^n W_v^{(k)}(\xi, \rho) = k^n \int_{X_0} h_\rho v(m_{\omega_0}) \Omega_0^{[n]} + \frac{k^{n-1}}{4} \int_{X_0} h_\rho \text{Scal}_v(\Omega_0) \Omega_0^{[n]} + O(k^{n-2}),
\]
where $h_\rho$ is the $\Omega$-Hamiltonian of the generator $V_\rho$ of the action $\mathbb{S}^1$ on $X_0$ with respect to a $\mathbb{G}$ invariant Kähler metric $\Omega \in 2\pi c_1(\mathcal{L})$ and $\Omega_0 := \Omega|_{X_0}$. In particular, the $(v, w)$-Donaldson-Futaki invariant \((58)\) of $(\mathcal{X}, \mathcal{L})$ is given by
\[
\text{DF}_{v, w}(\mathcal{X}, \mathcal{L}) = \frac{1}{4(2\pi)^n} F_{v, w}(V_\rho),
\]
where $F_{v, w}(V_\rho)$ the Futaki invariant of the class $\alpha := 2\pi c_1(L)$, introduced in Definition \(\text{[1]}\).

We deduce from Corollary \(\text{[2]}\) and Proposition \(\text{[2]}\).

Corollary 4. If $(\mathcal{X}, \mathcal{L})$ is a smooth $\mathbb{T}$-compatible polarized test configuration such that $\pi: \mathcal{X} \to \mathbb{P}^1$ is a smooth submersion, then
\[
\text{DF}_{v, w}(\mathcal{X}, \mathcal{L}) = \frac{1}{4(2\pi)^n} F_{v, w}(\mathcal{X}, 2\pi c_1(\mathcal{L})),
\]
where $F_{v, w}(\mathcal{X}, 2\pi c_1(\mathcal{L}))$ is the $(v, w)$-Futaki invariant of the $\mathbb{T}$-compatible Kähler test configuration $(\mathcal{X}, 2\pi c_1(\mathcal{L}))$ introduced in Definition \(\text{[1]}\).

9. The $(v, w)$-Futaki invariant of a toric test configurations

In this section we consider the special case when $X$ is a smooth toric variety i.e. $\mathbb{T} \subset \text{Aut}_{\text{red}}(X)$ with $\dim \mathbb{T} = \dim X = n$. Let $\omega \in \alpha$ be a fixed $\mathbb{T}$-invariant Kähler form, $m_\omega: X \to \mathfrak{t}^*$ a corresponding momentum map, and $P = m_\omega(X)$ the corresponding momentum polytope. By Delzant Theorem \(\text{[25]}\), $(X, \alpha)$ can be recovered from the labelled integral Delzant polytope $(P, L)$ where $L = (L_j)_{j=1,d}$ is the collection of non-negative defining affine-linear functions for $P$, with $dL_j$ being primitive elements of the lattice $\Lambda$ of circle subgroups of $\mathbb{T}$. We denote by $P_0$ the interior of $P$ and by $X_0 := m_\omega^{-1}(P_0)$ the dense open set of $X$ of points with principle $\mathbb{T}$ orbits. Let us consider the momentum/angle coordinates $(p, t) \in P_0 \times \mathbb{T}$ with respect to the Kähler metric $(g, J, \omega)$. By a result of Guillemin (see \(\text{[1]}\)),
\[
g = \langle dp, G^u \rangle + \langle dt, H^u \rangle dt,
\]
\[
J dt = -\langle G^u, dp \rangle,
\]
\[
\omega = \langle dp \wedge dt \rangle,
\]
on $X_0$, where $u$ is a smooth, strictly convex function called the symplectic potential of $(\omega, J)$, $G^u: P_0 \to S^2 t$ is the Hessian of $u$, $H^u: P_0 \to S^2 t^*$ is its point-wise inverse and $\langle \cdot, \cdot \rangle$ denote the contraction $\mathfrak{t}^* \times S^2 t \times \mathfrak{t}^* \to \mathbb{R}$ or the dual one. Conversely if $u$ is a strictly convex smooth function on $P_0$, \(\text{[2]}\) defines a Kähler structure on $X_0$ which extends to a global $\mathbb{T}$-invariant Kähler structure on $X$ iff $u$ satisfies the boundary conditions of Abreu (see \(\text{[2]}\)). We denote by $S(P, L)$ the set of smooth strictly convex functions on $P_0$ satisfying these boundary conditions. For $u \in S(P, L)$, we have the following expression for the scalar curvature of $(g, J)$ (see \(\text{[1]}\)),
\[
\text{Scal}(g) = -\sum_{i,j=1}^n H^u_{ij,ij}.
\]
where $H^u = (H^u_{ij})$ in a basis of $t$. Let $v \in C^\infty(P, \mathbb{R}_{>0})$. By the calculations in [6, Section 3], the following expression for the $v$-scalar curvature of $(g, J)$ is straightforward

\begin{equation}
\text{Scal}_v(g) = - \sum_{i,j=1}^{n} \left( vH^u_{ij} \right)_{ij}.
\end{equation}

We recall that by the maximality of $T$, any $T$-invariant Killing potential of $\mathbb{H}_\omega$ is the pull-back by $m_\omega$ of an affine-linear function on $P$.

**Lemma 15.** Let $v \in C^\infty(P, \mathbb{R}_{>0})$ and $w \in C^\infty(P, \mathbb{R})$. For any affine-linear function $f$ on $P$, the $(v, w)$-Futaki invariant corresponding to the $T$-invariant Hamiltonian Killing vector field $\xi := df$ is given by

\begin{equation}
(2\pi)^{-n} F^\alpha_{v,w}(\xi) = 2 \int_{\partial P} f v d\sigma - c_{(v,w)}(\alpha) \int_P f w dp,
\end{equation}

where $dp$ is a Lebesgue measure on $t'$, $d\sigma$ is the induced measure on each face $F_i \subset \partial P$ by letting $dL_i \wedge d\sigma = -dp$ and the constant $c_{(v,w)}(\alpha)$ is given by

\begin{equation}
c_{(v,w)}(\alpha) = 2 \left( \frac{\int_{\partial P} v d\sigma}{\int_P w dp} \right).
\end{equation}

**Proof.** Let $u \in S(P, L)$ and $(g, J)$ be the corresponding $\omega$-compatible Kähler structure $X$ given by [69]. The $(v, w)$-Futaki invariant of the Kähler class $\alpha = [\omega]$ is given by

$$F^\alpha_{v,w}(\xi) = \int_X \text{Scal}_v(g) f(m_\omega)\omega^n - c_{(v,w)}(\alpha) \int_X f(m_\omega) w(m_\omega) \omega^n,$$

where $f$ is an affine linear function on $t'$ with $\xi = df \in t$. In the momentum-action coordinates $(p, t) \in P^0 \times T$ we have $\omega^n = \langle dp \wedge dt \rangle^n = dp_1 \wedge dt_1 \wedge \cdots \wedge dp_n \wedge dt_n$. Then, using [60] and [6, Lemma 2], we get

\begin{align*}
(2\pi)^{-n} F^\alpha_{v,w}(\xi) &= - \int_P \sum_{i,j=1}^{n} (vH^u_{ij})_{ij} f dp - c_{(v,w)}(\alpha) \int_P f w dp \\
&= 2 \int_{\partial P} f v d\sigma - c_{(v,w)}(\alpha) \int_P f w dp.
\end{align*}

Similarly we deduce [62].

For any $f \in C^0(P, \mathbb{R})$ we define

\begin{equation}
F^P_{v,w}(f) := 2 \int_{\partial P} f v d\sigma - c_{(v,w)}(\alpha) \int_P f w dp.
\end{equation}

Using again [6, Lemma 2] we obtain

\begin{equation}
(2\pi)^{-n} \int_X (\text{Scal}_v(g_u) - c_{v,w}(w(m_\omega)) f) \omega^n = F^P_{v,w}(f) - \int_P \left( \sum_{i,j=1}^{n} H_{ij} f_{ij} \right) v dp,
\end{equation}

for any $u \in S(P, L)$ and $f \in C^\infty(P, \mathbb{R})$. It follows that

**Lemma 16.** [6, 66] If there exist $u \in S(P, L)$ such that the corresponding $\omega$-compatible Kähler structure $(g, J)$ solves $\text{Scal}_v(g) = c_{(v,w)}(\alpha) w(m_\omega)$ then $F^P_{v,w}(f) \geq 0$ for any smooth convex function $f$ on $P$. 

9.1. Toric test configuration. We start by recalling the construction of toric test
configurations introduced by Donaldson in [36, Section 4]. Let \((X, L)\) be a smooth
polarized toric manifold with integral momentum polytope \(P \subset \mathbb{T}^* \cong \mathbb{R}^n\) with respect
to the lattice \(\mathbb{Z}^n \subset \mathbb{R}^n\) and
\[
(65) f := \max(f_1, \cdots, f_r),
\]
a convex piece-wise affine-linear function with integer coefficients, i.e. we assume that each \(f_j\) in \(65\)
is an affine-linear function \(f_j(p) := \langle v_j, p \rangle + \lambda_j\) with \(v_j \in \mathbb{Z}^n\) and \(\lambda_j \in \mathbb{Z}\).
We also assume that the polytope \(Q\) defined by
\[
(66) Q = \{(p, p') \in P \times \mathbb{R} : 0 \leq p' \leq R - f(p)\},
\]
has integral vertices in \(\mathbb{Z}^{n+1}\), where \(R\) is an integer such that \(f \leq R\) on \(P\). By Proposition 4.1.1[36]
there exist an \((n + 1)\)-dimensional projective toric variety \((X_Q, \mathcal{G})\) and a polarization \(\mathcal{L}_Q \to X_Q\) corresponding to the labelled integral Delzant polytope \(Q \subset \mathbb{R}^{n+1}\) and the lattice \(\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}\). In general, \(X_Q\) is a compact toric orbifold
(see [54]), but \(X_Q\) can be smooth for a suitable choice of \(f\). There is an embedding
\(\iota : X \hookrightarrow X_Q\) such that \(\iota(X)\) is the pre-image of the face \(P = Q \cap (\mathbb{R}^n \times \{0\})\) of
\(Q\), and the restriction of \(\mathcal{L}_Q\) to \(\iota(X)\) is isomorphic to \(L\). Notice that by the Delzant
Theorem [25, 61] the stabilizer of \(\iota(X) \subset X_Q\) in \(\mathcal{G}\) is \(S^1_p = S^1_{(n+1)}\), where \(S^1_{(n+1)}\) is
the \((n + 1)\)-th factor of \(G = \mathbb{R}^{n+1}/2\pi \mathbb{Z}^{n+1}\) so that \(G/S^1_p\) is identified with the torus
action \(\mathbb{T} = \mathbb{R}^n/2\pi \mathbb{Z}^n\) on \(X\). Furthermore, Donaldson shows in [36] that there exist a
\(\mathbb{C}^*\)-equivariant map \(\pi : X_Q \to P^1\) such that \((X_Q, S^1_p, \mathcal{L}_Q)\) is a \(\mathbb{T}\)-compatible polarized
test configuration. We consider the Futaki-invariant \(F_{v, w}(X_Q, 2\pi c_1(\mathcal{L}_Q))\) given by (28)
corresponding to \((X_Q, 2\pi c_1(\mathcal{L}_Q))\), and notice that it makes sense even when \(X_Q\) is an
orbifold.

**Proposition 4.** Let \(f = \max(f_1, \cdots, f_r)\) be a convex piece-wise linear function on \(P\),
with integer coefficients and \(X_Q\) the toric test configuration constructed as above. Then
the \((v, w)\)-Futaki invariant (28) of \((X_Q, 2\pi c_1(\mathcal{L}_Q))\) is given by
\[
(67) F_{v, w}(X_Q, 2\pi c_1(\mathcal{L}_Q)) = (2\pi)^{n+1} F_{v, w}^P(f),
\]
where \(F_{v, w}^P(f)\) is the integral defined in (63). Furthermore, the \((v, w)\)-Donaldson-Futaki
invariant (28) corresponding to \((X_Q, \mathcal{L}_Q)\) is well-defined, and is given by
\[
(68) DF_{v, w}(X_Q, \mathcal{L}_Q) = 4F_{v, w}^P(f).
\]

**Proof.** We start by proving the first claim (67). Let \(\Omega \in 2\pi c_1(\mathcal{L}_Q)\) be a \(G\)-invariant
Kähler form on \(X_Q\) and \(\omega \in 2\pi c_1(L)\) be the induced \(\mathbb{T}\)-invariant Kähler form on \(\iota(X) \subset X_Q\).
We have by Remark 8 [41]
\[
F_{v, w}(\mathcal{X}, 2\pi c_1(\mathcal{L}_Q)) = - \int_{\mathcal{X}} (\text{Scal}_v(\Omega) - c_{(v, w)}(2\pi c_1(L)) w(m_\Omega)) \Omega^{[n+1]} \]
\[
+ (8\pi) \int_X v(m_\omega) \omega^n.
\]
Let \((p, p', t, t') \in Q \times \mathbb{T} \times S^1_p\) be the momentum/angular coordinates on \(X_Q^0\) such that
\((p, t) \in P \times \mathbb{T}\) are the momentum/angular coordinates on \(X^0\). Then,
\[
(70) (8\pi) \int_X v(m_\omega) \omega^n = 4(2\pi)^{n+1} \int_P v(p) dp.
\]
and
\[
(71) \int_{X_Q} w(m_\Omega) \Omega^{[n+1]} = (2\pi)^{n+1} \int_Q w(p) dp \wedge dp' = (2\pi)^{n+1} \int_P w(p)(R - f(p)) dp.
\]
For the remaining term in (69), using (64) we have

\[
(2\pi)^{-(n+1)} \int_{\mathcal{X}_Q} \text{Scal}_v(\Omega)\Omega^{[n+1]} = 2 \int_{\partial Q} v d\sigma_Q
\]

(72)

\[
= 2 \int_{\mathcal{P}} v dp + 2 \int_{(R-f)(\mathcal{P})} v d\mu_{(R-f)(\mathcal{P})} + 2 \int_{\mathcal{P}} (R-f)v d\sigma_{\mathcal{P}}
\]

\[
= 4 \int_{\mathcal{P}} v dp + 2 \int_{\partial \mathcal{P}} (R-f)v d\sigma_{\mathcal{P}},
\]

where the measure \(d\mu_{(R-f)(\mathcal{P})}\) is defined by \(df \land d\mu_{(R-f)(\mathcal{P})} = dp \land dp'\). Substituting (70)–(72) into (69) yields

\[
(2\pi)^{-(n+1)} \mathcal{F}_{v,w}(\mathcal{X}_Q, 2\pi c_1(\mathcal{L}_Q)) = -2 \int_{\partial \mathcal{P}} (R-f)v d\sigma_{\mathcal{P}} + \mathcal{C}_v, w(\alpha) \int_{\mathcal{P}} (R-f)wdp
\]

\[
= \mathcal{F}_{v,w}(f).
\]

Now we give the proof of the second claim (68). The central fiber \(X_0\) is the reduced divisor on \(\mathcal{X}_Q\) associated to the preimage of the union of facets of \(Q\) corresponding to the graph of \(R-f\). By a well-known fact in toric geometry (see e.g. [36]) the set of weights for the complexified torus \(\mathbb{C}^*\) on \(H^0(\mathcal{X}, \mathcal{L}_Q)\) is \(kQ \cap \mathbb{Z}^{n+1}\). It thus follows that the weights for the \(\mathbb{C}^*_\rho\)-action on \(H^0(X_0, L_0^k)\) are \(k(R-f)(kP) \cap \mathbb{Z}\). We conclude that

\[
W_{v}^{(k)}(\xi, \rho) = \sum_{\lambda \in kP \cap \mathbb{Z}^n} (R-f)\left(\frac{\lambda}{k}\right)v(\frac{\lambda}{k}),
\]

where \(W_{v}^{(k)}(\xi, \rho)\) is the \(v\)-weight defined by (63). By [49, 51], for any smooth function \(\Phi\) on \(t^*\) and \(k\) large enough we have

\[
\sum_{\lambda \in kP \cap \mathbb{Z}^n} \Phi \left(\frac{\lambda}{k}\right) = k^n \int_{\mathcal{P}} \Phi dp + \frac{k^{n-2}}{2} \int_{\partial \mathcal{P}} \Phi d\sigma_{\mathcal{P}} + O(k^{n-2}).
\]

Taking \(\Phi := (R-f)v\) and using the above formula for any affine-linear piece of \(\Phi\), we get

\[
W_{v}^{(k)}(\xi, \rho) = k^n \int_{\mathcal{P}} (R-f)vd\sigma_{\mathcal{P}} + \frac{k^{n-2}}{2} \int_{\partial \mathcal{P}} (R-f)v d\sigma_{\mathcal{P}} + O(k^{n-2}).
\]

Analogously, for \(W_{w}^{(k)}(\xi, \rho)\) we obtain

\[
W_{w}^{(k)}(\xi, \rho) = k^n \int_{\mathcal{P}} (R-f)wd\sigma_{\mathcal{P}} + O(k^{n-1}).
\]

Using (68), it follows that

\[
\mathcal{D}_{v,w}(\mathcal{X}_Q, \mathcal{L}_Q) = 4\mathcal{F}_{v,w}(f).
\]

Remark 8. Instead of a convex piece-wise affine-linear function \(f\) with integer coefficients we can take a convex piece-wise affine-linear functions with rational differentials, i.e. assuming that each \(f_j\) in (65) is of the form with \(f_j(p) = \langle v_j, p \rangle + \lambda_j\) with \(v_j \in \mathbb{Q}^n\). The polytope \(Q\) such a function defines is not longer with rational vertices, but still defines a toric Kähler orbifold \((X_Q, A_Q)\), see [61]. This gives rise to a toric Kähler test configuration compatible with \(T\) and the formula (67) in Proposition 4 computes the corresponding \((v, w)\)-Futaki invariant of \((X_Q, A_Q)\).
10. The \((v, w)\)-Futaki invariant of rigid semisimple toric fibrations

This is the case relevant to the example (iv) from the Introduction. Following [5], we consider \(X = V \times_T K \to B\) to be the total space of a fibre-bundle associated to a principle \(T\)-bundle \(K \to B\) over the product \(B = \prod_{j=1}^{N} (B_j, \omega_j, g_j)\) of compact cscK manifolds \((B_j, \omega_j, g_j)\) of complex dimension \(d_j\), satisfying the Hodge condition \([\omega_j/2\pi] \in H^2(B_j, \mathbb{Z})\), and a compact \(2\ell\)-dimensional toric Kähler manifold \((V, \omega_V, g_V, J_V, T)\) corresponding to a labelled Delzant polytope \((P, \mathbb{L})\) in \(t^*\). We assume that \(K\) is endowed with a connection 1-form \(\theta \in \Omega^1(K, t)\) satisfying

\[
d\theta = \sum_{j=1}^{N} \xi_j \otimes \omega_j, \quad \xi_j \in t, \quad j = 1, \ldots, N.
\]

and that the toric Kähler metric \((g_V, \omega_V, J_V)\) on \(V\) is given by (59) for a symplectic potential \(u \in \mathcal{S}(P, \mathbb{L})\). As shown in [5], \(X\) admits a bundle-adapted Kähler metric \((g, \omega)\) which, on the open dense subset \(X^0 = K \times P^0 \subset X\), takes the form

\[
g = \sum_{j=1}^{N} \left(\langle \xi_j, p \rangle + c_j\right) \pi^* g_j + \langle dp, G^u, dp \rangle + \langle \theta, H^u, \theta \rangle, \tag{73}
\]

\[
\omega = \sum_{j=1}^{N} \left(\langle \xi_j, p \rangle + c_j\right) \pi^* \omega_j + \langle dp \wedge \theta \rangle,
\]

where \(p \in P^0\) and \(c_j\) are real constants such that \(\langle \xi_j, p \rangle + c_j > 0\) on \(P\). Such Kähler metrics, parametrized by \(u \in \mathcal{S}(P, \mathbb{L})\) and the real constants \(c_j\), are referred to in [5] as given by the generalized Calabi ansatz in reference to the well-known construction of Calabi [17] of extremal Kähler metrics on \(\mathbb{P}^1\)-bundles.

We notice that the Kähler manifold \((X, \omega, g)\) is invariant under the \(T\)-action with momentum map identified with \(p \in P\). Furthermore, it is shown in [5, (7)] that the scalar curvature of (73) is given by

\[
\text{Scal}(g) = \sum_{j=1}^{N} \frac{\text{Scal}_j}{\langle \xi_j, p \rangle + c_j} - \frac{1}{u(p)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial p_r \partial p_s} \left( u(p) H^u_{rs} \right)
\]

\[
= \sum_{j=1}^{N} \frac{\text{Scal}_j}{\langle \xi_j, p \rangle + c_j} + \frac{1}{u(p)} \text{Scal}_u(g_V),
\]

where we have put \(u(p) := \prod_{j=1}^{N} (\langle \xi_j, p \rangle + c_j)^{d_j}\) and we have used (60) for passing from the first line to the second. Similarly, by [5, equation (12)], the \(g\)-Laplacian of (the pull-back to \(X\) of) a smooth function \(f(p)\) on \(P\) is given by

\[
\Delta_g f = -\frac{1}{u(p)} \sum_{r,s=1}^{\ell} \frac{\partial}{\partial p_r} \left( u(p) \frac{\partial f}{\partial p_s} H^u_{rs} \right).
\]

Using the above formulae, we check by a direct computation that for any positive smooth function \(v\) on \(P\) we have

\[
\text{Scal}_v(g) = v(p) \left( \sum_{j=1}^{N} \frac{\text{Scal}_j}{\langle \xi_j, p \rangle + c_j} + \frac{1}{u(p)} \text{Scal}_u(g_V) \right) \tag{74}
\]

Using that the volume form of (73) is

\[
\omega^{[n]} = u(p) \left( \bigwedge_{j=1}^{N} \omega_j^{[d_j]} \right) \wedge \langle dp \wedge \theta \rangle^{[\ell]},
\]
and the integration by parts formula [6, Lemma 2], we compute that the $(v,w)$-Futaki invariant on $X$ acts on a vector field $\xi \in \mathfrak{t}$ by
\begin{equation}
\frac{\mathcal{F}_{v,w}^{(\omega)}(\xi)}{(2\pi)^{\ell}(\prod_{j=1}^{N} \text{Vol}(B_j, [\omega_j]))} = 2 \int_{\partial \mathcal{P}} f v u d\sigma + \int_{\mathcal{P}} \left( \sum_{j=1}^{N} \frac{\text{Scal}_j}{(\xi_j, p) + c_j} \right) f v u d\sigma
\end{equation}

where $f = (\xi, p) + \lambda$ is a Killing potential of $\xi$.

As in Section 9.1, we can construct a $\mathbb{T}$-compatible smooth Kähler test configuration associated to $X$ defined by a convex piece-wise linear function $f = \max(f_1, \ldots, f_k)$ on $\mathfrak{t}$ such that the polytope $Q \subset \mathbb{R}^{\ell+1}$ given by (60) is Delzant with respect to the the lattice $\mathbb{Z}^{\ell+1}$. Denote by $\mathcal{V}_Q, \mathcal{A}_Q$ the corresponding smooth toric variety, and by $K = K \times S^1 \to B$ the principal $\mathbb{T}^{\ell+1}$-bundle over $B$ with trivial $(\ell + 1)$-factor, and let $X = X \times_{\mathbb{T}^{\ell+1}} K \to B$ be the resulting $\mathcal{V}$-bundle over $B$. We can now consider a Kähler form $\Omega$ on $X$ obtained by the generalized Calabi ansatz (73); as the connection 1-form on $K$ has a curvature $\sum_{j=1}^{N} \xi_j \otimes \omega_j$ with $\xi_j \in \mathbb{t} = \text{Lie}(\mathbb{T}^\ell) \subset \text{Lie}(\mathbb{T}^{\ell+1})$, $\Omega$ induces on the pre-image $X \subset X$ of the facet $P \subset Q$ a Kähler form $\omega$ given by (72) with the same affine linear functions $(\xi_j, p) + c_j$. A similar computation to (75), performed on the total space $(X_Q, \Omega)$ by using Definition 11 (see also the proof of Lemma 15 above) leads to the expression (75) for the $(v,w)$-Futaki invariant associated to $(X_Q, \mathcal{A}_Q)$ with $f$ being the piece-wise linear convex function defining $Q$.

Let us now suppose that $X = \mathbb{P}(O \oplus L) \to B$ with $B$ as above, where $O$ stands for the trivial holomorphic line bundle over $B$ and $L$ is a holomorphic line bundle of the form $L = \bigotimes_{j=1}^{N} L_j$ for $L_j$ being the pull-back to $B$ of a holomorphic line bundle over $B_j$ with $c_1(L_j) = \xi_j[\omega_j]/2\pi$, $\xi_j \in \mathbb{Z}$. This is the so-called admissible setting (without blow-downs) of [4], pioneered in [17] and studied in many works. In our setting above, such an $X$ is a $\mathbb{P}^1$-bundle obtained from the principle $\mathbb{S}^1$-bundle over $B$ associated to $L^{-1}$. We can take $P = [-1, 1] \subset \mathbb{R}$, and suppose that $v(z) > 0$ and $w(z)$ are smooth functions defined over $[-1, 1]$. A Kähler metric $(\omega, g)$ on $X$ of the form (73) can be equivalently written as
\begin{equation}
g = \sum_{j=1}^{N} (\xi_j z + c_j) \pi^* g_j + \frac{dz^2}{\Theta(z)} + \Theta(z) d\theta^2
\end{equation}

\begin{equation}
\omega = \sum_{j=1}^{N} (\xi_j z + c_j) \pi^* \omega_j + dz \wedge d\theta, \quad d\theta = \sum_{j=1}^{N} \xi_j \pi^* \omega_j,
\end{equation}

for positive affine-linear functions $\xi_j z + c_j$ on $[-1, 1]$. This is the more familiar Calabi ansatz, written in terms of the profile function $\Theta(z)$ (see e.g. [54]) which must be smooth on $[-1, 1]$ and satisfy
\begin{equation}
\Theta(\pm 1) = 0, \quad \Theta'(\pm 1) = \mp 2,
\end{equation}

and
\begin{equation}
\Theta(z) > 0 \text{ on } (-1, 1),
\end{equation}

for (76) to define a smooth Kähler metric on $X$. We let $u(z) = \prod_{j=1}^{N} (\xi_j z + c_j)^{d_j}$ be the corresponding polynomial in $z$.

We now take $Q$ be the chopped rectangle with base $P$, corresponding to the convex piece-wise affine linear function $f_{z_0}(z) = \max(z + 1 - z_0, 1)$ where $z_0 \in (-1, 1)$ is a given point. We can construct as above an $\mathbb{S}^1$-compatible Kähler test configuration $(X_Q, \mathcal{A}_Q)$ associated to $(X, [\omega], \mathbb{S}^1)$. It is not difficult to see that the complex manifold $X_Q$ is the
Substituting in the RHS of (79) and integrating by parts over the intervals $[z_0, 1]$, in the case when $w = \text{ext}$ following [7], in the case when $w$ is Kähler class $\omega$ of the quantity $w$-Futaki invariant of $(\mathcal{X}_Q, \mathcal{A}_Q)$ is a positive multiple of the quantity

\[
F(z_0) := 2 \left( f_{z_0}(1)v(1)u(1) - f_{z_0}(-1)v(-1)u(-1) \right) + \int_{-1}^{1} f_{z_0}(z) \left( v(z)u(z) \left( \sum_{j=1}^{N} \frac{\text{Scal}_j}{\xi_j z + c_j} \right) - c_{v,w}(\omega)w(z)u(z) \right) dz.
\]

Let us now assume that there exists a smooth function $\Theta(z)$ on $[-1, 1]$, which satisfies

\[
\left( vu\Theta \right)''(z) = v(z)u(z) \left( \sum_{j=1}^{N} \frac{\text{Scal}_j}{\xi_j z + c_j} \right) - c_{v,w}(\omega)w(z)u(z).
\]

Substituting in the RHS of (77) and integrating by parts over the intervals $[-1, z_0]$ and $[z_0, 1]$ gives

\[
F(z_0) = (v(z_0)u(z_0)\Theta(z_0).
\]

As $v(z)$ and $u(z)$ are positive functions on $[-1, 1]$, we conclude that if $(X, [\omega], S^1)$ is $(v, w)$-K-stable on smooth $S^1$-compatible Kähler test configurations with reduced central fibre, then $\Theta(z)$ must also satisfy (78). By the formula (74), the corresponding Kähler metric $\omega_{\text{ext}}$ will be then $(v, w)$-cscK.

The existence of a solution of (80) satisfying (77) is in general overdetermined. Following [7], in the case when $w(z) > 0$ on $[-1, 1]$ one can resolve the over-determinacy by introducing an affine-linear function $w_{\text{ext}}(z) = A_1 z + A_2$, such that

\[
\left( vu\Theta \right)''(z) = v(z)u(z) \left( \sum_{j=1}^{N} \frac{\text{Scal}_j}{\xi_j z + c_j} \right) - w(z)w_{\text{ext}}(z)u(z)
\]

admits a unique solution $\Theta_{\text{ext}}^{v,w}(z)$ satisfying (77): the coefficients $A_1$ and $A_2$, as well as two constants of integration in (80), are then uniquely determined from the four boundary conditions in (77). Furthermore, a straightforward generalization of [2, Lemma 2.4] shows that $w_{\text{ext}}(z)$ corresponds to the affine-linear function introduced in Section 3.2, i.e. solutions of (82) introduce $(v, w_{\text{ext}})$-cscK metrics of Calabi type, which are equivalently $(v, w)$-extremal. The methods of this article allow us to obtain the following generalization of [7, Theorem 3].

**Theorem 8.** Let $X = P(\mathcal{O} \oplus L) \to B$ be a projective $\mathbb{P}^1$-bundle as above, endowed with the $S^1$-action by multiplication on $\mathcal{O}$, and $\alpha = [\omega/2\pi]$ be the Kähler class of a Kähler metric in the form (70). We let $P = [-1, 1]$ be the momentum polytope of $(X, \alpha, S^1)$, $v, w$ be smooth positive functions on $[-1, 1]$ and $\Theta_{\text{ext}}^{v,w}(z)$ the unique solution of (82) satisfying (77). Then,

- If $(X, \alpha, S^1)$ is $(v, w_{\text{ext}})$-K-stable on $S^1$-compatible smooth Kähler test configurations with reduced central fibre, then $\Theta_{\text{ext}}^{v,w}(z) > 0$ on $(-1, 1)$ and $\alpha$ admits a $(v, w)$-extremal Kähler metric of the form (70) with $\Theta = \Theta_{\text{ext}}^{v,w}$.
- If $(X, \alpha, S^1)$ admits a $(v, w)$-extremal Kähler metric, then $(X, \alpha, S^1)$ is $(v, w_{\text{ext}})$-K-semistable on $S^1$-compatible smooth Kähler test configurations with reduced central fibre, and $\Theta_{\text{ext}}^{v,w}(z) \geq 0$.

**Proof.** The first part follows from the identity (81) which shows that $\Theta_{\text{ext}}^{v,w}$ must satisfy both (77) and (78). The second part follows from formula (81) and Theorem 2 if the constants $(c_1, \ldots, c_N)$ in (70) are rational as in this case the corresponding Kähler class $\alpha$ is rational. To treat the case when $(c_1, \ldots, c_N)$ are not necessarily rational, we can
use Theorem 2 below (with fixed $v, w$ and varying the constants $c_i$). Accordingly, for any rational constants $(\tilde{c}_1, \ldots, \tilde{c}_N)$ sufficiently close to $(c_1, \ldots, c_N)$ the corresponding Kähler class $\tilde{\alpha}$ will admit a $(v, w)$ extremal Kähler metric, and hence the corresponding function $\Theta^v_w(z)$ will be non-negative on $(-1, 1)$ by virtue of Theorem 2. As $\Theta^v_w(z)$ depends smoothly on $(c_1, \ldots, c_N)$, it follows that $\Theta^v_w(z) \geq 0$ too.

**Remark 9.** (i) As already mentioned in the Introduction, we expect that Theorem 2 can be improved by showing that the existence of $(v, w)$-cscK metric in $\alpha$ implies $(v, w)$-K-stability, not only $(v, w)$-K-semi-stability. Accordingly, we expect Theorem 8 to be improved to a complete Yau–Tian–Donaldson type correspondence between $(v, w)$-stable and $(v, w)$-extremal Kähler classes on $X$ of the form (84), in which either notion corresponds to the positivity condition (86) for $\Theta^v_w(z)$.

(ii) In [7], the analogous statement of Theorem 8 is achieved by considering polarized test configuration $(\mathcal{X}_Q, \mathcal{L}_Q)$ as above (corresponding to rational values of $z_0$), and computing the relative version of the algebraic $(v, w)$-Donaldson–Futaki invariant $\mathcal{D}_v,w(\mathcal{X}_Q, \mathcal{L}_Q)$. This provides a yet another instance where the differential-geometric definition coincides with the algebraic definition of the $(v, w)$-Futaki invariant.

**Appendix A. The $(v, w)$-equivariant Bergman kernels and boundedness of the $(v, w)$-Mabuchi energy**

Let $(X, L)$ be a polarized manifold, $\alpha = 2\pi c_1(L)$ the corresponding Kähler class, and $T \subset \text{Aut}(X, L)$ a real $\ell$-dimensional torus with momentum polytope $P$ as in Section 8.1. Let $h$ be a $T$-invariant Hermitian metric on $L$ with curvature 2-form $\omega \in \alpha$. We identify the space of $T$-invariant Hermitian metrics $h_{\phi} := e^{-2\phi}h$ with positive curvature forms $\omega_{\phi}$ with the space $K^T_\infty$ of $T$-invariant Kähler potentials $\phi$ on $X$.

Let $\xi := (\xi_1, \ldots, \xi_\ell)$ be a chosen basis of $S^1$-generators of $T$ and $A^{(k)}_\xi := (A^{(k)}_{\xi_1}, \ldots, A^{(k)}_{\xi_\ell})$ the induced infinitesimal actions of $\xi_i$ on the space $\mathcal{H}_k$ given by (80). For $v \in C^\infty(P, \mathbb{R}_{>0})$ we consider the following weighted $L^2$-inner product on $C^\infty(X, L^K)$

$$\langle s, s' \rangle_{v, k, \phi} := k^n \int_X \langle s, s' \rangle_{k, \phi} v(m_{\phi}) \omega_{\phi}^{[n]}.$$  

(83)

where $\langle s, s' \rangle_{k, \phi} := h_{\phi}^k(s, s')$. The operators $(A^{(k)}_{\xi_j})_{j=1, \ldots, \ell}$ are Hermitian with respect to $\langle \cdot, \cdot \rangle_{v, k, \phi}$, with spectrum contained in the momentum polytope $P$ (see lemma 13).

**Definition A.1.** [12] [80] [91] Let $\phi \in K^T_\infty$, $\{s_i \mid i = 0, \ldots, N_k\}$ be a $\langle \cdot, \cdot \rangle_{v, k, \phi}$-orthonormal basis of $\mathcal{H}$ and $w \in C^\infty(P, \mathbb{R})$. Then the $(v, w)$-equivariant Bergman kernel of the Hermitian metric $h_{\phi}^k$ on $L^K$, is the function defined on $X$ by,

$$B_w(v, k, \phi) := v(m_{\phi}) \sum_{i=0}^{N_k} \left( w(k^{-1}A^{(k)}_{\xi_j})(s_i), s_i \right)_{k, \phi}.$$  

(84)

where $w(k^{-1}A^{(k)}_{\xi_j})$ is given by (52).

Equivalently, $B_w(v, k, \phi)$ is the restriction to the diagonal $\{x = x' \subset X \times X$ of the Schwartz kernel of the operator $w(k^{-1}A^{(k)}_{\xi_j})\Pi_{k, \phi}$, where $\Pi_{k, \phi} : L^2(X, L^K) \to \mathcal{H}_k$ denote the orthogonal projection with respect to the inner product $\langle \cdot, \cdot \rangle_{v, k, \phi}$ (see [91]).

Asymptotic expansions of (83) in $k \gg 1$ are known to exist in many special cases, see e.g. [12] [65] [80] [91]. We need the following general result, which follows essentially from [19], with a ramification from [65].
Proposition A.1. Let $(T_j^{(k)})_{j=1,...,\ell}$ be a family of $\langle \cdot, \cdot \rangle_{\nu,k\phi}$-self adjoint commuting Toeplitz operators, such that the set of joint eigenvalues of $(T_j^{(k)})_{j=1,\ell}$ is contained in P. Suppose that the symbol of $T_j^{(k)}$, $j = 1, \cdot \cdot \cdot, \ell$ is given by

$$\sigma(T_j^{(k)}) := \sum_{i \geq 0} h^i f_i^{(j)} \in C^\infty(X)[[h]].$$

Then for any smooth function $w$ with compact support containing P, the operator $w(T_1^{(k)}, \cdot \cdot \cdot, T_\ell^{(k)})$ is a Toeplitz operator with symbol

$$\sigma(w(T_1^{(k)}, \cdot \cdot \cdot, T_\ell^{(k)})) = s_0(v, w) + s_1(v, w)h + \mathcal{O}(h^2),$$

where $s_0(v, w), s_1(v, w)$ are given by

$$s_0(v, w) = w(f_0^{(1)}, \cdot \cdot \cdot, f_\ell^{(\ell)}),$$

$$s_1(v, w) = w(f_0^{(1)}, \cdot \cdot \cdot, f_\ell^{(\ell)}) S_v(\phi) + \sum_{j=1}^{\ell} w_{,j}(f_0^{(1)}, \cdot \cdot \cdot, f_\ell^{(\ell))})(f_1^{(j)} - f_0^{(j)} S_v(\phi))$$

$$+ \frac{1}{4} \sum_{i,j=1}^{\ell} w_{,ij}(f_0^{(1)}, \cdot \cdot \cdot, f_\ell^{(\ell))})(d_{f_0^{(i)}} d_{f_0^{(j)}} )\phi,$$

with $S_v(\phi) := \frac{1}{4}(\text{Scal}_\phi + 2\Delta_\phi \text{(log}(v(m_\phi))))$.

Proof. In the case of one $\langle \cdot, \cdot \rangle_{\nu,k\phi}$-self adjoint Toeplitz operator $T^{(k)}$ and a smooth function of one variable $w$, the fact that $w(T^{(k)})$ is again a Toeplitz operator is established in [19 Proposition 12]. The proof given in [19] relies on the Helffer-Sjostrand formula, see e.g. [29 Theorem 8.1]. Using its multivariable generalization [29 Equation 8.18] the proof in [19] readily generalizes to show that $w(T_1^{(k)}, \cdot \cdot \cdot, T_\ell^{(k)})$ is a Toeplitz operator for any smooth function on P, and family of $\langle \cdot, \cdot \rangle_{\nu,k\phi}$-self adjoint commuting Toeplitz operators $(T_j^{(k)})_{j=1,...,\ell}$ such that the set of joint eigenvalues of $(T_j^{(k)})_{j=1,\ell}$ is contained in P.

Following [19], one can define a map $\sigma: \mathcal{T} \rightarrow C^\infty(X)[[h]]$ called the full symbol map from the set $\mathcal{T}$ of Toeplitz operators, to the set of formal series $C^\infty(X)[[h]]$, which we introduce as follows. For a general Toeplitz operator $T^{(k)} \in \mathcal{T}$, the restriction to the diagonal $T^{(k)}(x, x)$ of its Schwartz kernel admits an asymptotic expansion in $C^\infty$ of the form,

$$(2\pi)^n T^{(k)}(x, x) = \sum_i k^{-i} a_i(x) + \mathcal{O}(k^{-\infty}),$$

and the symbol of $T^{(k)}$ is given by the formal series

$$\sigma(T^{(k)}) := \sum_i h^i a_i(x).$$

The full symbol map $\sigma$ is a morphism from the the algebra $(\mathcal{T}, \circ)$ with respect to the composition $\circ$ of operators to the algebra $(C^\infty(X)[[h]], \star_v)$ endowed with an associatiave product $\star_v$ called the star product

$$\left( \sum_{j \geq 0} f_j h^j \right) \star_v \left( \sum_{j \geq 0} g_j h^j \right) := \sum_{s \geq 0} \left( \sum_{j=0}^{s} f_j \star_v g_{s-j} \right) h^s,$$

where the product $f \star_v g$ for two smooth functions is also an element of $C^\infty(X)[[h]]$ which we define as follows.

We denote by $\Pi^{k\phi}_v$ the $\langle \cdot, \cdot \rangle_{\nu,k\phi}$-orthogonal projection on $\mathcal{H}_k$ with respect to the weighted $L^2$-inner product [83]. Using [66 Theorem 0.2], for any $f, g \in C^\infty(X, \mathbb{R})$, we
have \( \Pi_{\psi}^{k_0} \Pi_{\phi}^{k_0} g \Pi_{\psi}^{k_0} \in \mathcal{T} \), and the restriction to the diagonal \( \{ x = x' \} \subset X \times X \) of its Schwartz kernel admits a \( C^\infty \)-asymptotic expansion given by

\[
(\Pi_{\psi}^{k_0} \Pi_{\phi}^{k_0} g \Pi_{\psi}^{k_0})(x, x) = f g + \frac{1}{2}(df, dg)_\phi - S_\psi(\phi) f g \| k^{-1} + \mathcal{O}(k^{-2}).
\]

We then let

\[
f \ast_v g := \sigma(\Pi_{\psi}^{k_0} \Pi_{\phi}^{k_0}) = f g + h \left[ \frac{1}{2}(df, dg)_\phi - S_\psi(\phi) f g \right] + \mathcal{O}(h^2)
\]

The star product considered in [19] is \( \ast_1 \) but the theory extends without difficulty for arbitrary weight \( v \), by our definition above. For instance, the unit \( 1_{\ast_v} \) of \( (C^\infty(X)[[h]], \ast_v) \) is identified with the symbol \( 1_{\ast_v} := \sigma(\Pi_{\phi}^{k_0}) \). Let \( \Pi_{\phi}^{k_0}(x, x') \) be the usual Bergman kernel associated to the projection \( \Pi_{\psi}^{k_0} \), i.e.

\[(85) \quad \Pi_{\phi}^{k_0}(x, x) = \sqrt{v(m_\phi(x))} \sum_{i=0}^{N_k} |s_i|_{k_0}^2(x).
\]

We claim that [84] admits the following \( C^\infty \)-asymptotic expansion in \( k \)

\[(86) \quad (2\pi)^v \Pi_{\phi}^{k_0}(x, x) = 1 + \frac{1}{k} S_\psi(\phi) + \mathcal{O} \left( \frac{1}{k^2} \right),
\]

so that \( 1_{\ast_v} = 1 + hS_\psi(\phi) + \mathcal{O}(h^2) \). The expansion [86] follows from [65, Theorem 4.1.1] by taking the twisting bundle \( E := X \times \mathbb{C} \) (in the notation of [65]) being the trivial line bundle over \( X \), endowed with Hermitian metric \( | \cdot |_E := \sqrt{v(m_\phi)} \cdot | \cdot | \), where \( | \cdot | \) is the norm of \( \mathbb{C} \). According to [65, Theorem 4.1.1], [86] admits an asymptotic expansion [80] with coefficient before \( k^{-1} \) being equal to

\[
\frac{1}{4} (\text{Scal}_\phi + 2\Delta_\phi F_E) = \frac{1}{4} (\text{Scal}_\phi + 2\Delta_\phi (\log(v(m_\phi)))) = S_\psi(\phi),
\]

where \( F_E = -dd^c \log(v(m_\phi)) \) is the curvature form of the Chern connection of \( (E, | \cdot |_E) \).

We shall now compute the symbol of the Toeplitz operator \( w(T_1^{(k)}, \ldots, T_\ell^{(k)}) \). Following [19], the full calculus of the symbol of \( w(T_1^{(k)}, \ldots, T_\ell^{(k)}) \) is given by the Taylor series expansion of \( w \) at the point \( a := (f_0^{(1)}(x), \ldots, f_0^{(\ell)}(x)) \), \( a_j := f_0^{(j)}(x) \) as follows:

\[(87) \quad \sigma(w(T_1^{(k)}, \ldots, T_\ell^{(k)})) = w(a) 1_{\ast_v}(x) + \sum_{j=1}^\ell w_{j}(a) \left( \sum_{i=0}^\ell h^i f^{(j)}(y) - a_j 1_{\ast_v}(y) \right)_{|y=x} + \cdots
\]

\[
+ \frac{1}{2!} \sum_{p,q=1}^\ell w_{pq}(a) \left( \sum_{i=0}^\ell h^i f^{(p)}(y) - a_p 1_{\ast_v}(y) \right) \ast_v \left( \sum_{i=0}^\ell h^i f^{(q)}(y) - a_q 1_{\ast_v}(y) \right)_{|y=x} + \cdots
\]

On the other hand, we compute

\[
\left( \sum_{i=0}^\ell h^i f^{(p)}(y) - a_p 1_{\ast_v}(y) \right) \ast_v \left( \sum_{i=0}^\ell h^i f^{(q)}(y) - a_q 1_{\ast_v}(y) \right)_{|y=x}
\]

\[
= \left( (f^{(p)}(y) - a_p) + h(f^{(p)}(y) - S_\psi(y)) \right) \ast_v \left( (f^{(q)}(y) - a_q) + h(f^{(q)}(y) - S_\psi(y)) \right)_{|y=x} + \mathcal{O}(h^2)
\]

\[
= (f^{(p)}(y) - a_p) \ast_v (f^{(q)}(y) - a_q)_{|y=x} + h(f^{(p)}(y) - a_p)(f^{(q)}(y) - S_\psi(y))
\]

\[
+ h(f^{(q)}(y) - a_q)(f^{(p)}(x) - S_\psi(x)) + \mathcal{O}(h^2)
\]

\[
= h \frac{1}{2} (df^{(p)}_0, df^{(q)}_0)_\phi + \mathcal{O}(h^2).
\]
Substituting back in \([87]\), we obtain the claimed formula for the symbol \(\sigma(w(T^{(k)}_1, \ldots, T^{(k)}_\ell))\) up to \(O(h^2)\).

**Theorem A.1.** Let \(w \in C^\infty(P, \mathbb{R})\). The \((v, w)\)-equivariant Bergman kernel of the \(T\)-invariant Hermitian metric \(h^k_\phi\) on \(L^k\) admits an asymptotic expansion when \(k \gg 1\), given by

\[
(2\pi)^n B_w(v, k\phi) = \begin{cases} w(m_\phi) + O(\frac{1}{k}), & \text{if } w = v. \\ v(m_\phi) + \frac{1}{4k} \text{Scal}_v(\phi) + O(\frac{1}{k^2}), \end{cases}
\]

Moreover, the above expansions holds in \(C^\infty\), i.e. for any integer \(\ell \geq 0\) there exist a constant \(C_\ell(v, w) > 0\) such that,

\[
\| (2\pi)^n B_w(v, k\phi) - w(m_\phi) \|_{C^\ell} \leq \frac{C_\ell(v, w)}{k},
\]

\[
\left\| (2\pi)^n B_w(v, k\phi) - v(m_\phi) - \frac{1}{4k} \text{Scal}_v(\phi) \right\|_{C^\ell} \leq \frac{C_\ell(v, v)}{k^2}.
\]

**Proof.** Since the symbol map \(\sigma\) is surjective with kernel given by the ideal of negligible Toeplitz operators \(O(k^{-\infty}) \cap \mathcal{T}\) (see \([19, \text{Proposition 3}]\)), it suffices to calculate \(\sigma(w(k^{-1}A^{(k)}_{\xi})\Pi^{(k)}_\phi)\). We consider the special case of self-adjoint Toeplitz operators \(T^{(k)}_j := k^{-1}A^{(k)}_{\xi}\Pi^{(k)}_\phi\). We have

\[
T^{(k)}_j(x, x) = v(m_\phi) \sum_{i=0}^{N_k} (k^{-1}A^{(k)}_{\xi} s_i, s_i)_{k\phi}.
\]

By a straightforward calculation using [51] the symbol of \(T^{(k)}_j\) is given by

\[
\sigma(T^{(k)}_j) = m^{\xi}_\phi + \left[ m^{\xi}_\phi S_v(\phi) - \frac{1}{2} \sum_{i=1}^\ell (\log \circ v)_i(m_\phi)(\xi_i, \xi_j) \right] h + \cdots
\]

Using Proposition A.1 we get

\[
\sigma(w(A^{(k)}_{\xi})\Pi^{(k)}_\phi) = s_0(v, w) + s_1(v, w) h + \cdots
\]

where

\[
s_0(v, w) = w(m_\phi),
\]

\[
s_1(v, v) = v(m_\phi) S_v(\phi) - \frac{1}{2} \sum_{i,j=1}^\ell \frac{v_i(m_\phi)v_j(m_\phi)}{v(m_\phi)} (\xi_i, \xi_j) + \frac{1}{4} \sum_{i,j=1}^\ell v_{ij}(m_\phi)(\xi_i, \xi_j).
\]

Replacing \(S_v(\phi)\) by its expression in Proposition A.1 we obtain \(s_1(v, v) = \text{Scal}_v(\phi)\). \(\square\)

**A.1. The quantization maps.** Let \(W_k\) denote the set of weights of the complexified action of \(T\), for \(\lambda^{(k)} \in W_k\). We consider the following direct sum decomposition of the space \(\mathcal{B}^T(\mathcal{H}_k)\) of \(T\)-invariant positive definite Hermitian forms on \(\mathcal{H}_k\),

\[
\mathcal{B}^T(\mathcal{H}_k) := \bigoplus_{\lambda^{(k)} \in W_k} \mathcal{B}^T(\mathcal{H}(\lambda^{(k)})),
\]

where \(\mathcal{B}^T(\mathcal{H}(\lambda^{(k)}))\) is the space of \(T\)-invariant positive definite Hermitian forms on \(\mathcal{H}(\lambda^{(k)})\)

**Definition A.2.** Let \(v \in C^\infty(P, \mathbb{R}_{>0})\), \(w \in C^\infty(P, \mathbb{R})\). We introduce the following quantization maps:
Lemma A.1. For \( \omega \) say that a metric \( \omega \) is \((v, w)\)-balanced of order \( k \) if it satisfies:

\[
\text{FS}_{v, w}^k \circ \text{Hilb}_{v, w}^k (\phi) = \phi.
\]

or equivalently

\[
\rho_{v, w}(k\phi) = c_k(v, w)v(m_\phi),
\]

where \( c_k(v, w) \) is given by \( \text{(88)} \).

(i) The \((v, w)\)-Hilbert map \( \text{Hilb}_{v, w}^k : \mathcal{K}_\omega^T \to \mathcal{B}^T(\mathcal{H}_k) \) which associates to every \( T \)-invariant Kähler potential, the \( T \)-invariant Hermitian inner product on \( \mathcal{H}_k \), given by

\[
(\text{Hilb}_{v, w}^k(\phi))(\cdot, \cdot) := \sum_{\lambda_i^{(k)} \in \mathcal{W}_k} \frac{\langle \cdot, \cdot \rangle_{\nu, \kappa}(\lambda_i^{(k)})}{\nu(\lambda_i^{(k)}) - \frac{c_{(\nu, \kappa)}(\alpha)}{4k}w(\lambda_i^{(k)})},
\]

where \( c_{(\nu, \kappa)}(\alpha) \) is given by \( \text{(3)} \).

(ii) The \((v, w)\)-Fubini–Study map \( \text{FS}_{v, w}^k : \mathcal{B}^T(\mathcal{H}_k) \to \mathcal{K}_\omega^T \), given by

\[
\text{FS}_{v, w}^k(H) := \frac{1}{2k} \log \left( \sum_{i=0}^{N_k} |s_i|^2_{k^h} \right) - \frac{\log(c_k(v, w))}{2k},
\]

where \( \{s_i\} \) is an adapted \( H \)-orthonormal basis of \( \mathcal{H}_k \) and \( c_k(v, w) \) is a constant given by:

\[
(88) \quad c_k(v, w) := \frac{1}{k^n \int_X \nu(m_\omega)\omega^n} \left[ W_v(L_k) - \frac{c_{(\nu, \kappa)}(\alpha)}{4k}W_w(L_k) \right],
\]

with \( W_v(L_k) \) the \( v \)-weight of the action of \( T \) on \( L_k \) given by \( \text{(53)} \).

Theorem \( \text{A.1} \) yields

Lemma A.1. For \( \phi \in \mathcal{K}_\omega^T \), the Bergman kernel \( \rho_{v, w}(k\phi) \) of \( \text{Hilb}_{v, w}(k\phi) \) satisfies

\[
\rho_{v, w}(k\phi) = B_v(v, k\phi) - \frac{c_{\nu, w}(\alpha)}{4k}B_w(v, k\phi),
\]

and it has an asymptotic expansion,

\[
(2\pi)^n \rho_{v, w}(k\phi) = v(m_\phi) + \frac{1}{4k} \left( \text{Scal}_v(\phi) - c_{\nu, w}(\alpha)w(m_\phi) \right) + O\left( \frac{1}{k^2} \right).
\]

The above asymptotic expansion holds in \( C^\infty \), i.e. for any integer \( \ell \geq 0 \) we have,

\[
\left\| (2\pi)^n \rho_{v, w}(k\phi) - v(m_\phi) - \frac{1}{4k} (\text{Scal}_v(\phi) - c_{\nu, w}(\alpha)w(m_\phi)) \right\|_{C^\ell} \leq \frac{C_{\ell}(v, w)}{k^2}.
\]

where \( C_{\ell}(v, w) > 0 \).

Following \( \text{[35, 75, 88]} \), we give the following definition

Definition A.3. We say that a metric \( \phi \in \mathcal{K}_\omega^T \) is \((v, w)\)-balanced of order \( k \) if it satisfies:

\[
\text{FS}_{v, w}^k \circ \text{Hilb}_{v, w}^k (\phi) = \phi.
\]

Proposition A.2. Let \( (\phi_j)_{j \geq 0} \) be a sequence in \( \mathcal{K}_\omega^T \) such that every \( \phi_j \) is a \((v, w)\)-balanced metric of order \( j \) and \( \phi_j \) converge in \( C^\infty \) to \( \phi \). Then \( \omega_\phi \) is \((v, w)\)-cscK metric.
A.2. Proof of Theorem \[ \text{[1]} \] Here outline the proof of Theorem \[ \text{[1]} \] from the introduction, which follows from a straightforward extension of the arguments of \[ [37, 55, 60, 75] \] that has been given in the special cases \[ (i), (ii), (iv) \] . We start by defining the finite dimensional analogues of the \((v, w)\)-Mabuchi energy defined on the spaces \( \text{FS}^k_{v,w} \left( B^2(\mathcal{H}_k) \right) \) and \( B^2(\mathcal{H}_k) \) as follows,

\[
\begin{align*}
&\mathcal{L}^k_{v,w} := \mathcal{E}^k_{v,w} \circ \text{Hilb}^k_{v,w} + 2k^{n+1}c_k(v,w)\mathcal{E}_v, \\
&Z^k_{v,w} := 2k^{n+1}c_k(v,w)\mathcal{E}_v \circ \text{FS}^k_{v,w} + \mathcal{E}^k_{v,w},
\end{align*}
\]

where \( \mathcal{E}_v \) is given by \[ (15) \] and \( \mathcal{E}^k_{v,w} : B^2(\mathcal{H}_k) \to \mathbb{R} \) is the function

\[
\mathcal{E}^k_{v,w}(H) = \sum_{\lambda^{(k)} \in W_k} \left( v(\lambda^{(k)}) - \frac{c_{(v,w)}(\alpha)}{4k} w(\lambda^{(k)}) \right) \log \left( \det H_{\lambda^{(k)}} \right).
\]

Using Lemma \[ \text{[A.1]} \] one can show that

\[
\begin{align*}
&\lim_{k \to \infty} \left[ \frac{2}{k^n} \mathcal{L}^k_{v,w} + b_k \right] = \mathcal{M}_{v,w}, \\
&\lim_{k \to \infty} k^{-n} \left[ \mathcal{L}^k_{v,w}(\phi) - Z^k_{v,w} \circ \text{Hilb}^k_{v,w}(\phi) \right] = 0,
\end{align*}
\]

where the convergence holds in the \( C^\infty \)-sense. Suppose that \( K^\mathcal{T}_{\alpha} \) contains a \((v, w)\)-cscK metric \( \phi^* \in K^\mathcal{T}_{\alpha} \). One can show as in \[ [60, 75, 55] \] that the metrics \( \text{Hilb}^k_{v,w}(\phi^*) \) are almost balanced in the sense that there exists a smooth function \( \varepsilon(\phi)(k) \), such that \( \lim_{k \to \infty} \varepsilon(\phi)(k) = 0 \) in \( C^\ell(X, \mathbb{R}) \) and,

\[
\begin{align*}
k^{-n}Z^k_{v,w} \circ \text{Hilb}^k_{v,w}(\phi) &\geq k^{-n}Z^k_{v,w} \circ \text{Hilb}^k_{v,w}(\phi^*) + \varepsilon(\phi)(k),
\end{align*}
\]

for all \( \phi \in K^\mathcal{T}_{\alpha} \). Using \[ (90) \] and \[ (91) \] the proof is identical to the one of \[ [75, \text{Theorem 3.4.1}] \].

A.3. Proof of Corollary \[ \text{[1]} \] This is a direct consequence of Lemma \[ \text{[6]} \] and Theorem \[ \text{[1]} \].

APPENDIX. B. THE STRUCTURE OF THE AUTOMORPHISM GROUP AND STABILITY OF THE \((v, w)\)-CSCK METRICS UNDER DEFORMATIONS

The following result follows from straightforward calculations along the lines of \[ [45, \text{Section 2.5}] \] and is left to the reader.

Lemma \[ \text{[54]} \]. \[ \text{Let} \ (X, \alpha, \mathcal{T}) \ be a compact Kähler manifold with Kähler class \( \alpha \) and \( \mathcal{T} \subset \text{Aut}_{\text{red}}(X) \) a real torus with \( P \) the \( T \)-momentum image of \( X \). Suppose that \( v, w \in C^\infty(\mathbb{P}, \mathbb{R}_{>0}) \) are positive smooth functions on \( P \). Then we have,

\[
(i) \ For \ any \ T\text{-invariant Kähler metric} \ \omega \in \alpha \ and \ any \ variation \ \phi \in T_\phi K^\mathcal{T}_{\alpha} \ we \ have
\]

\[
\delta \left( \frac{\text{Scal}_v(\omega)}{w(\omega)} \right)(\phi) = -2L^\omega_{\phi,w}(\phi) + d\phi(\Xi_{v,w}),
\]

where \( \Xi_{v,w} := J_{\text{grad}} g \left( \frac{\text{Scal}_v(\omega)}{w(\omega)} \right) \) and \( L^\omega_{\phi,w} \) is the elliptic fourth order differential operator given by

\[
L^\omega_{\phi,w} \phi = \frac{\delta \left( \nu(\omega)(D^{-}d)\phi \right)}{w(\omega)},
\]

\( D \) is the Levi-Civita connection of \( \omega \) and \( D^{-}(d\phi) \) is the \( J \)-anti-invariant part of the tensor \( D(d\phi) \).
With these choices, we now suppose that \( v, w \) can choose \( P_t \) the Kähler class \( \alpha \) can choose \( P_t \) the Kähler class

\[
\text{Theorem B.1.} \quad \text{[45, Theorem 3.4.1] and [44].}
\]

The next Theorem is established using Lemma B.1 and the same arguments as in the proof of [45, Theorem 3.4.1] and [44].

**Theorem B.1.** [44, 54] If \( X \) admits a \((v, w)\)-extremal Kähler metric with \( v, w \in C^\infty(P, \mathbb{R}_{>0}) \). Then the complex Lie algebra of \( \mathbb{T} \)-equivariant automorphisms of \( X \) admits the following decomposition

\[
\mathfrak{h}^T = (\mathfrak{a} \oplus \mathfrak{t}^T_{\text{ham}} \oplus J \mathfrak{t}^T_{\text{ham}}) \oplus \bigoplus_{\lambda > 0} \mathfrak{h}^T_{(\lambda)}
\]

where \( \mathfrak{a} \) is the abelian Lie algebra of parallel vector fields, \( \mathfrak{t}^T_{\text{ham}} \) is the real Lie algebra of \( \mathbb{T} \)-equivariant Hamiltonian isometries of \( X \) and \( \mathfrak{h}^T_{(\lambda)} \), \( \lambda > 0 \) denote the subspace of elements \( V \in \mathfrak{h}^T \) such that \( \mathcal{L}_{\Xi, w} V = \lambda JV \). Moreover, the Lie algebra of \( \mathbb{T} \)-equivariant isometries of \( X \) admits the following decomposition

\[
\mathfrak{t}^T = \mathfrak{a} \oplus \mathfrak{t}^T_{\text{ham}}.
\]

Using Theorem B.1 and the arguments in the proof of [45, Theorem 3.5.1] we get the following generalization of the structure theorem for the group of holomorphic automorphisms of a \((v, w)\)-extremal Kähler manifold.

**Corollary B.1.** [17, 44, 54] Let \((X, \omega, g)\) be a compact \((v, w)\)-extremal Kähler manifold \( \omega \) with \( v, w \in C^\infty(P, \mathbb{R}_{>0}) \). Then the group \( \text{Isom}^0_{\mathbb{T}}(X, g) \) of \( \mathbb{T} \)-equivariant isometries of \( X \) is a maximal compact connected subgroup of the identity component of the \( \mathbb{T} \)-equivariant automorphisms \( \text{Aut}^0_{\mathbb{T}}(X) \) of \( X \). In particular, \((g, \omega)\) is invariant under the action of a maximal torus \( T_{\text{max}} \) in \( \text{Aut}_{\text{red}}(X) \). Furthermore, if \((g, \omega)\) is \((v, w)\)-cscK, then \( \text{Aut}^0_{\mathbb{T}}(X) \) is a reductive complex Lie group.

Now we consider the stability of the \((v, w)\)-extremal metrics under deformations of the Kähler class \( \alpha \) and the weight functions \( v, w \in C^\infty(P, \mathbb{R}_{>0}) \).

Let \( X \) be a compact Kähler manifold, \( \alpha \) a Kähler class, \( T_{\text{max}} \subset \text{Aut}_{\text{red}}(X) \) a maximal torus and \( P_\alpha \subset t^* \) a momentum polytope for \( \alpha \) as in Lemma 1. Let \( \beta \in H^{1,1}(X) \) and \( U \) an open subset of \( t^* \) with \( P_\alpha \subset U \). Then there exist \( a > 0 \) such that for any \( |r| < a \) we can choose \( P_{\alpha + r\beta} \subset U \) to be the momentum polytope of \( T_{\text{max}} \) with respect to \( \alpha + r\beta \). With these choices, we now suppose that \( v, w \) are positive smooth functions on \( U \) and \( \tilde{v}, \tilde{w} \) are arbitrary smooth functions on \( U \). We then have

**Theorem B.2.** Suppose that \( \omega \in \alpha \) is a \( T_{\text{max}} \)-invariant \((v, w)\)-extremal Kähler metric associated to \((P_\alpha, v, w)\). Then there exist \( \varepsilon > 0 \), such that for any \( |s| < \varepsilon, |t| < \varepsilon, |r| < \varepsilon \) there exists a \((v + \tilde{v}, w + \tilde{w})\)-extremal Kähler metric in the Kähler class \( \alpha + r\beta \), associated to \((v + \tilde{v}, w + \tilde{w})\) and \( P_{\alpha + r\beta} \subset U \).

The proof follows the lines of that for LeBrun–Simanca stability theorem in [59] (see also [40]).
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