Wess-Zumino Terms for Reducible Anomalous Gauge Theories

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Abstract

Reducible off-shell anomalous gauge theories are studied in the framework of an extended Field-Antifield formalism by introducing new variables associated with the anomalous gauge degrees of freedom. The Wess-Zumino term for these theories is constructed and new gauge invariances appear. The quantum effects due to the extra variables are considered.
1 Introduction

Anomalous gauge theories are characterized by the breakdown of its classical BRST symmetry [1][2] due to quantum corrections. For these theories the Slavnov-Taylor identities [3] are not verified. The obstruction to fulfill these identities, in the Field-Antifield (FA) formalism [1][4][5][6], is the violation of the Quantum Master Equation (QME). Gauge anomalies appear as the obstruction to satisfy the QME in a local way [7]. As a consequence of that, in anomalous gauge theories, some classical gauge degrees of freedom become propagating at quantum level [8]. These new degrees of freedom are introduced covariantly in the FA formalism. This has been done for irreducible theories with closed gauge algebras in ref. [9], where the Wess-Zumino term [10] at one loop has been constructed in terms of the anomalies and the extra variables. A regularization procedure for the new variables was also considered [11].

In this paper we consider the quantization of reducible off-shell anomalous gauge theories with closed algebras. In order to construct the extended formalism and the Wess-Zumino term, we analyze the action on a manifold of a Lie group of transformations which is locally described by a redundant set of parameters. From this analysis we determine what are the new classical degrees of freedom that are propagating at quantum level. The Wess-Zumino term is constructed in terms of the anomalies and the finite gauge transformations. For genuine anomalous theories the integration of the extra variables gives a non local counterterm. Instead, for a non anomalous theory one obtains a local counterterm that restores the BRST invariance at one loop [7][12]. A new characteristic feature of this term with respect to the irreducible case is the appearence of new gauge invariances due to the reducible character of the anomalies. A new kind of ghosts (and ghosts for ghost for non first reducible theories) appear in the formalism because of these gauge transformations. We consider a PV regularization scheme to take into account the quantum effects of all the extra variables introduced. A non-standard aspect of these theories is the appearence of background terms in the action, i.e., terms with $\sqrt{\hbar}$. Unfortunately only a certain restricted set of theories seems to admit the perturbative description developed here. This fact may indicate that a quantum treatment of the Wess-Zumino term together with the original action goes beyond the scope of the usual $\hbar$ perturbative expansion.

Along the paper we consider in detail closed first step reducible gauge theories, where all the relevant new aspects of the formalism already appear. Some aspects for the general
The case of $L$-step reducible theories are studied in an appendix.

The organization of the paper is the following: In section 2 we analyze the structure of reducible anomalous gauge theories. In section 3 we construct a solution of the classical master equation within the extended FA formalism. Section 4 deals with the quantization of the theory, the construction of the Wess-Zumino term and the regularization of the extended theory. Section 5 is devoted to an example to illustrate the formalism. Section 6 ends with some conclusions. The study of a reducible Lie group of transformations is done in the appendix A. Finally, the extension of the formalism to $L$-step reducible gauge theories is considered in appendix B.

## 2 Reducible Gauge Anomalies

Consider a classical action $S_0(\phi)$ which is invariant under the gauge symmetries

$$\delta \phi^i = R_{\alpha}^i(\phi) \varepsilon^\alpha, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m_0$$

and assume that our theory is closed first step reducible off-shell. The minimal proper solution of the classical master equation $(S, S) = 0$ that reproduces the complete gauge structure is, in the classical basis of fields and antifields,

$$S(\Phi, \Phi^*) = S_0(\phi) + \phi^*_i R_{\alpha}^i c^\alpha + c^*_\alpha (-\frac{1}{2} T_{\beta\gamma}^\alpha c^\beta c^\gamma + Z_{\alpha}^a \eta^a)$$

$$+ \eta^*_a (A_{\beta\gamma}^a c^\beta c^\gamma - \frac{1}{3!} F_{\alpha\beta\gamma}^a c^\alpha c^\beta c^\gamma),$$

where $c^\alpha, \eta^a$ are the ghosts and ghost for ghosts respectively; $\phi^*_i, c^*_\alpha, \eta^*_a$ are the antifields of the theory and $R_{\alpha}^i(\phi), T_{\beta\gamma}^\alpha, A_{\beta\gamma}^a, F_{\alpha\beta\gamma}^a$ are algebraic constant quantities that completely characterize the gauge structure of the theory. This structure is obtained by expanding the classical master equation in antifields:

$$\left\{ \frac{\partial S_0}{\partial \phi^i} R_{\alpha}^i(\phi) \right\} c^\alpha = 0 \quad (2.2)$$

$$\left\{ R_{\alpha}^i(\phi) \frac{\partial R_{\beta}^j(\phi)}{\partial \phi^i} - \frac{1}{2} T_{\alpha\beta\gamma} R_{\gamma}^i(\phi) \right\} c^\beta c^\alpha = 0 \quad (2.3)$$

$$\left\{ R_{\alpha}^i(\phi) Z_{\alpha}^a \right\} \eta^a = 0 \quad a = 1, \ldots, m_1 \quad (2.4)$$

1For simplicity we will restrict to the bosonic case $\epsilon(\phi^i) = 0$ and $\epsilon(c^\alpha) = 0$
\[ \left\{ T_{\beta \gamma}^\alpha Z_a^\beta - Z_b^\alpha A_{a \beta}^b \right\} c^\gamma \eta^a = 0 \]  \hspace{1cm} (2.5)

\[ \left\{ \frac{1}{2} T_{\alpha \beta}^\mu T_{\mu \gamma}^\nu - \frac{1}{3!} F_{\alpha \beta \gamma}^a Z_a^\nu \right\} c^\gamma c^\beta c^\alpha = 0 \]  \hspace{1cm} (2.6)

\[ \left\{ Z_a^\beta A_{b \beta}^c \right\} \eta^b \eta^a = 0 \]  \hspace{1cm} (2.7)

\[ \left\{ \frac{1}{2} A_{a \beta}^\sigma T_{\beta \gamma}^\sigma + \frac{1}{2} F_{a \beta \gamma}^a Z_b^\sigma + A_{c \beta}^a A_{b \gamma}^c \right\} c^\gamma c^\beta = 0 \]  \hspace{1cm} (2.8)

\[ \left\{ \frac{1}{(2)!} T_{\sigma \alpha}^\rho F_{\sigma \beta \gamma}^a + \frac{1}{3!} A_{b a}^c F_{a \beta \gamma}^b \right\} c^\gamma c^\beta c^\alpha c^\sigma = 0. \]  \hspace{1cm} (2.9)

The role of the ghosts in these expressions is simply to account for the appropriate antisymmetrization in a compact form. In appendix A we will recover (2.3)-(2.9) from the study of the finite group structure for first reducible off-shell parametrizations of a Lie Group.

At quantum level, and when performing perturbative calculations, it proves convenient to change from the classical basis \((\Phi; \Phi^*) = (\phi^i, c^a, \eta^a; \phi^*_i, c^*_a, \eta^*_a)\) to a gauge fixed one \((\Phi; K)[6]\). The gauge fixed basis, which is necessary in order to have well defined propagators, is implemented by a canonical transformation in the antibracket sense.

The quantization procedure may spoil the classical BRST structure due to quantum corrections. The quantum action is given by

\[ W(\Phi, K) = S(\Phi, K) + \sum_{p=1}^{\infty} \hbar^p M_p(\Phi, K), \]  \hspace{1cm} (2.10)

where the local counterterms should guarantee the finiteness of the theory while preserving the BRST structure at quantum level if that is possible. The possible breakdown of the classical BRST symmetry is reflected in the (potentially anomalous) BRST Ward identities

\[ \frac{1}{2}(\Gamma, \Gamma) = < -i \hbar \Delta W + \frac{1}{2}(W, W) > = -i \hbar A \cdot \Gamma, \]  \hspace{1cm} (2.11)

where \(\Gamma\) is the effective action, \(\Delta\) is defined by \(\Delta \equiv (-1)^A \frac{\partial}{\partial \Phi^*} \frac{\partial}{\partial K_A}\) and \(A \cdot \Gamma\) is the generating functional of the 1PI Green functions with insertion of the composite operator \(A \equiv -i \hbar \Delta W + \frac{1}{2}(W, W)\), which parametrizes possible departures from the classical BRST structure. Quantum BRST invariance will thus hold if \(A\) vanishes, i.e., upon fulfillement through a local object \(W\) of the QME

\[ -i \hbar \Delta W + \frac{1}{2}(W, W) = 0. \]  \hspace{1cm} (2.12)
The potential anomaly at one loop is given by
\[(\Delta S)_{\text{reg}} + i(M_1, S) \equiv \mathcal{A}(\Phi, K),\] (2.13)
where \((\Delta S)_{\text{reg}}\) shows the non BRST invariance of the measure at one loop. In particular, using PV regularization \[7\] we obtain \[11\]
\[(\Delta S)_{\text{reg}} = \lim_{M \to \infty} \text{tr} \left\{ -\frac{1}{2} (\mathcal{R}^{-1} \delta \mathcal{R}) \left( \frac{1}{1 - \frac{1}{M}} \right) \right\} = \lim_{M \to \infty} \delta \left\{ -\frac{1}{2} \text{tr} \ln \left[ \frac{\mathcal{R}}{M - \mathcal{R}} \right] \right\},\] (2.14)
where \(\mathcal{R}(\Phi, K)\) is the PV regulator, \(M\) the mass regulator parameter and \(\delta = (-, S)\) is the BRST transformation generated by \(S\) through the antibracket structure. The anomaly verifies the Wess-Zumino consistency conditions \(\delta \mathcal{A} = 0\) \[10\] because it is BRST exact and \(\delta^2 = 0\).

It can be proved that for closed and reducible off-shell algebras the antifield independent part of the anomaly is BRST closed separately \[11\]. Here we will consider this part. Its most general form is
\[\mathcal{A}(\Phi_{\text{min}}) = \mathcal{A}_a(\phi)c^a.\] (2.15)

Applying the Wess-Zumino consistency conditions, \((\mathcal{A}, S) = 0\), we get, for a reducible gauge anomaly
\[R_\alpha^i \frac{\partial \mathcal{A}_\beta}{\partial \phi^i} - R_\beta^i \frac{\partial \mathcal{A}_\alpha}{\partial \phi^i} = T_{\alpha\beta}^i \mathcal{A}_\gamma,\] (2.16)
\[\mathcal{A}_a Z_a^\alpha = 0.\] (2.17)

Eq. (2.17) is a direct consequence of the reducibility condition (2.4). Notice that \(\text{rank}(\frac{\partial \mathcal{A}_\alpha}{\partial \phi^i}) \leq m_0 - m_1\). From now on, we will consider theories with
\[\text{rank}(\frac{\partial \mathcal{A}_\alpha}{\partial \phi^i}) = m_0 - m_1.\] (2.18)

3 Classical Aspects of the Extended Formalism for Reducible Theories

Anomalous gauge theories are such that some classical gauge degrees of freedom become propagating at quantum level. These new degrees of freedom can be introduced
covariantly within the FA formalism. Here we will consider theories where the whole reducible gauge group is anomalous, i.e., no gauge symmetries survive quantization.

In order to construct the extended FA formalism we first enlarge the space of classical fields \( \{ \phi^i(x) \} \) with a set of \( m_0 \) new fields \( \{ \theta^\alpha(x) \} \) that correspond to the redundant parameters of the anomalous gauge group. The second step is to find the transformation properties of these new variables. As it is done in the irreducible case [9], to determine them we require: i) The gauge invariance of the classical action \( S_0(\phi) \); ii) The gauge invariance of the finite gauge transformations \( \phi^i = F^i(\phi, \theta) \).

The invariance of \( S_0(\phi) \) leads to

\[
\delta \phi^i = R^i_\alpha(\phi) \varepsilon^\alpha, \tag{3.1}
\]

\[
\delta \theta^\alpha = d^\alpha, \tag{3.2}
\]

where \( d^\alpha \) are, as of now, completely arbitrary, i.e.: \( \theta^\alpha(x) \) are pure gauge.

Gauge invariance of \( F^i(\phi, \theta) \) will restrict the \( d^\alpha \) parameters. In fact, under the requirement

\[
\delta F^i(\phi, \theta) = \frac{\partial F^i(\phi, \theta)}{\partial \phi^j} \delta \phi^j + \frac{\partial F^i(\phi, \theta)}{\partial \theta^\alpha} \delta \theta^\alpha = 0 \tag{3.3}
\]

and the use of (A.4) we get

\[
\left. \frac{\partial F^i(\phi, \theta)}{\partial \theta^\beta} \right|_{\theta=0} = \left. \frac{\partial F^i(\phi, \theta)}{\partial \phi^j} \right|_{\theta=0} R^j_\beta(\phi) = \left. \frac{\partial F^i(\phi, \varphi(\theta, \theta'))}{\partial \theta^\beta} \right|_{\theta=0} \left. \frac{\partial F^i(\phi, \theta')}{\partial \theta^\alpha} U^\alpha_\beta(\theta'), \tag{3.4}
\right.

with

\[
U^\alpha_\beta(\theta') \equiv \left. \frac{\partial \varphi^\alpha(\theta, \theta')}{\partial \theta^\beta} \right|_{\theta=0}, \tag{3.5}
\]

from which we rewrite equation (3.3) as

\[
\delta F^i = \frac{\partial F^i(\phi, \theta)}{\partial \theta^\beta} (U^\alpha_\beta(\theta) \varepsilon^\alpha + \delta \theta^\beta) = 0. \tag{3.6}
\]

Now, using (A.29), we get the extended gauge transformations for \( \theta^\alpha(x) \)

\[
\delta \theta^\beta = -U^\beta_\alpha(\theta) \varepsilon^\alpha + Z^\beta_a(\theta) \varepsilon^a, \tag{3.7}
\]

where a new set of gauge parameters, \( \varepsilon^a \), besides the original ones \( \varepsilon^\alpha \), has appeared. Observe that \( U^\beta_\alpha(\theta) \) are the components of our left-invariant vector fields \( U_\alpha(\theta) \) (see [32]), whereas \( Z^\beta_a(\theta) \) are the components of the vector fields \( Z_a(\theta) \) tangent to the reducible orbits.
Let us remark that maybe at this point we loose locality in the extended theory because of a non local \( (3.7) \).

Using the freedom of reparametrization of the orbits \( \theta^\alpha \), we can select an \( \epsilon^a \)-parametrization (see appendix A) such that
\[
Z_a(\theta) = Z_a^\alpha U_\alpha(\theta). \tag{3.8}
\]

In the following we will use such \( \epsilon^a \)-parametrization.

It is useful to introduce a compact notation
\[
\psi^I = (\phi^i, \theta^\sigma) \quad I = 1, \ldots, n + m_0, \tag{3.9}
\]\[
\epsilon^A = (\epsilon^\alpha, \tilde{\epsilon}^a) \quad A = 1, \ldots, m_0 + m_1, \tag{3.10}
\]
in terms of which the gauge transformations are written as
\[
\delta \psi^I = \left( \frac{\delta \phi^i}{\delta \theta^\sigma} \right) = \left( \begin{array}{cc}
R^i_\alpha(\phi) & 0 \\
\tilde{U}_\sigma^\alpha(\theta) & Z^\sigma_a(\theta)
\end{array} \right) \left( \begin{array}{c}
\epsilon^\alpha \\
\tilde{\epsilon}^a
\end{array} \right) = V^I_A(\Phi) \epsilon^A. \tag{3.11}
\]

Now we are going to build the \( (m_0 + m_1) \)-dimensional extended algebra. Consider the \( m_0 + m_1 \) vector fields of \( n + m_0 \) components, \( V_A(\psi) = \left. \frac{\partial}{\partial \psi^I} \right|_{\psi = \phi, \theta} = \{ V_\alpha(\psi), \tilde{V}_a(\psi) \} \), where
\[
V^I_\alpha(\phi, \theta) = \left( \begin{array}{c}
R^i_\alpha(\phi) \\
\tilde{U}_\sigma^\alpha(\theta)
\end{array} \right) \quad V^I_a(\theta) = \left( \begin{array}{c}
0 \\
Z^\sigma_a(\theta)
\end{array} \right). \tag{3.12}
\]

These vectors define the new structure functions \( T^C_{AB} \) according to
\[
[V_A(\psi), V_B(\psi)] = T^C_{AB}(\psi) V_C(\psi). \tag{3.13}
\]

Explicitly we have
\[
[V_\alpha(\phi, \theta), V_\beta(\phi, \theta)] = T^\gamma_{\alpha\beta} V_\gamma(\phi, \theta) + S^c_{\alpha\beta}(\theta) V_c(\theta)
\]
\[
[V_\alpha(\phi, \theta), V_b(\theta)] = C^c_{ab}(\theta) V_c(\theta)
\]
\[
[V_\alpha(\phi, \theta), V_\alpha(\theta)] = B^b_{a\alpha}(\theta) V_b(\theta), \tag{3.14}
\]
where \( S^c_{\alpha\beta}(\theta), C^c_{ab}(\theta) \) and \( B^b_{a\alpha}(\theta) \) are new structure functions, which can be obtained from the functions appearing in the reducible Lie group of transformations (see appendix A).

Observe that a quasigroup structure \([14]\) arises, i.e., we get a “soft” algebra (structure functions) instead of a Lie one (structure constants).
It is worth noting that this extension conserves the same type of reducibility. The $m_0 + m_1$ gauge transformations (3.1) and (3.7) have $m_1$ null vectors

$$
\mathbf{Z}_b = \bar{\mathbf{Z}}_b^A \frac{\partial}{\partial \psi^A}, \quad \bar{\mathbf{Z}}_b^A = \left( \frac{Z_b^a}{\delta_b^a} \right)
$$

which give the $m_1$ dependence relations

$$
V^I_A \bar{Z}_b^A = 0.
$$

As in any reducible theory, we know that there exist quantities $\bar{A}_d^b$ such that, analogously to (2.5),

$$
\bar{T}^A_{BC} \bar{Z}_d^A = \bar{A}_d^b \bar{Z}_b^A ;
$$

it turns out that they are

$$
\bar{A}_d^a = A_d^a
$$

$$
\bar{A}_c^a = 0.
$$

Also, generalization of (2.6) tells that there must be quantities $\bar{F}^{a}_{BCD}$ such that

$$
\sum_{\text{Cyclic}[BCD]} (T^A_{BC} \bar{T}^E_{AD} - V^I_A \bar{T}^E_{CD, I}) = \bar{F}^{a}_{BCD} \bar{Z}_b^E ;
$$

these quantities are

$$
\bar{F}^{a}_{\beta\gamma\delta} = F^{a}_{\beta\gamma\delta}
$$

$$
\bar{F}^{a}_{b\gamma\delta} = 0
$$

$$
\bar{F}^{a}_{b\gamma\delta} = 0
$$

$$
\bar{F}^{a}_{bc\delta} = 0.
$$

Summing up, we have obtained all the algebraic structure functions that characterize the extended gauge algebra $\{V_\alpha, V_a\}$ of the extended classical field space $\{\phi^i(x), \theta^a(x)\}$. Now we ask for a solution of the classical master equation which generates this extended classical gauge structure just derived. It is

$$
\hat{S}(z^a) = S_0 + \Phi_I V^I_A c^A + c^*_A (-\frac{1}{2} T^A_{BC} c^B + \bar{Z}_b^A \eta^b) + \eta^*_a (\bar{A}_d^b c^C \eta^b - \frac{1}{3!} \bar{F}^{a}_{BCD} c^B c^C c^D)
$$

$$
= S_0(\phi) + \phi^*_i R^i_\alpha (\phi) c^\alpha + \theta^*_a (-U^a_\alpha (\theta) c^a + Z^a_\alpha (\theta) v^a) + c^*_a (-\frac{1}{2} T^a_{\beta\gamma} c^\beta c^\gamma + Z^a_\beta c^\gamma c^\beta)
$$

$$
+ v^*_a (-\frac{1}{2} S^a_{\beta\gamma} (\theta) c^\gamma c^\beta - B^a_{\alpha\beta} (\theta) c^\alpha v^\beta - \frac{1}{2} C^a_{\beta\gamma} (\theta) v^\beta c^\gamma + \eta^a)
$$

$$
+ \eta^*_a (\bar{A}_d^a c^\beta \eta^d - \frac{1}{3!} \bar{F}^{a}_{\beta\gamma\delta} c^\beta c^\gamma),
$$

(3.25)
where $\tilde{z}^a$ stands for all the fields and antifields of the extended theory. We have $N = n + 2m_0 + 2m_1$ fields with ghost numbers: $\text{gh}(\phi^i) = 0$, $\text{gh}(\theta^\alpha) = 0$, $\text{gh}(c^\alpha) = 1$, $\text{gh}(v^a) = 1$, $\text{gh}(\eta^a) = 2$.

Note that (3.25) is a non-proper solution of the classical master equation. This comes from the fact that we have not considered all the full set of gauge transformations, (3.1) and (3.2), of the classical action $S_0(\phi)$, to construct the extended gauge algebra, but only a subgroup of them, given by (3.1) and (3.7). So, the rank on shell of the hessian is less than the number of fields,

$$\text{rank} \left( \frac{\partial^2 \tilde{S}}{\partial \tilde{z}^a \partial \tilde{z}^b} \right)_{\text{on-shell}} = \text{rank} \left( \frac{\partial^2 S_0}{\partial \phi^i \partial \phi^j} \right)_{\text{on-shell}} + 2 \text{rank}(V^I_A) + 2 \text{rank}(\tilde{Z}^A_b) = N - (m_0 - m_1).$$

But if we make the partial gauge fixing

$$\theta^*_\alpha = v^*_a = 0$$

we get the closed first step reducible proper solution (2.1) for a classical space of fields $\{\phi^i(x)\}$.

4 Extended Quantized Theory. The Wess-Zumino Term

Let us now consider the quantum aspects of the extended formalism at one loop. To this end we consider the quantum action $\tilde{W} = \tilde{S} + \hbar M_1$. $\tilde{S}$ is non-proper and there is no a kinetic term in $\tilde{S}$ for the new quantum anomalous degrees of freedom. In order to have well defined propagators for these variables we need $\tilde{W}$ to have rank $N$. Therefore $M_1$ should give the rank that is missing in $\tilde{S}$. $M_1$ will contain the WZ term as well as some other local counterterms. Here we will only consider the contribution of the WZ term in $M_1$.

The WZ term can be understood as the local counterterm that relates the antifield independent part of the anomaly which is computed in a BRST non-invariant regularization ($\mathcal{R}$ such that $\delta \mathcal{R} \neq 0$, which gives the anomaly $\mathcal{A} = \lim_{\mathcal{M} \to \infty} \text{tr} \left\{ -\frac{1}{2} (\mathcal{R}^{-1} \delta \mathcal{R})(\frac{1}{\mathcal{M}}) \right\} \neq 0$) with the one computed with an invariant regulator ($\mathcal{R}'$ such that $\delta \mathcal{R}' = 0$, which gives the
anomaly $\mathcal{A}' = 0$). Since we are interested in the antifield independent part of the anomaly, we will focus in the antifield independent part of the regulator which in general is going to be a functional of the classical fields $\mathcal{R}(\phi)$. In the extended theory there exists such a BRST invariant regularization: it is $\mathcal{R}' = \mathcal{R}(F(\phi, \theta))$. The two regularizations can be connected by means of a continuous interpolation $\mathcal{R}(t) = \mathcal{R}(F(\phi, t\theta))$, $t \in [0, 1]$. Then, the counterterm that relates $\mathcal{R}$ to $\mathcal{R}'$ is such that

$$M_1 = \int_0^1 dt \frac{\partial}{\partial t} \left\{ -\frac{1}{2} \text{tr} \ln \left[ \frac{\mathcal{R}(1)}{\mathcal{M} - \mathcal{R}(1)} \right] - \frac{1}{2} \text{tr} \ln \left[ \frac{\mathcal{R}(0)}{\mathcal{M} - \mathcal{R}(0)} \right] \right\},$$

(4.2)

Using the Lie equations (A.54), we can write (4.2) in terms of the anomaly $\mathcal{A}(t)$ of the original theory resulting from the regularization $\mathcal{R}(t)$. We get

$$M_1(\phi, \theta) = -i \int_0^1 dt \mathcal{A}_\alpha(F(\phi, t\theta)) \theta^\alpha,$$

(4.3)

where we have chosen the normal group parametrization such that $\mu^\alpha_\beta(\theta) \theta^\beta = \theta^\alpha$ (see (A.33)). If the integration of the $\theta^\alpha$ variables gives a local expression for the original fields we deal with a non anomalous gauge theory, obtaining the counterterm that restores the BRST invariance at one loop. If we get a non local expression we are working with a genuine anomalous gauge theory.

Now we can check that $\tilde{W}$ has the appropriate rank:

$$\text{rank} \left( \frac{\partial^2 \tilde{W}}{\partial \tilde{z}^a \partial \tilde{z}^b} \right)_{\text{on-shell}} = \text{rank} \left( \frac{\partial^2 \tilde{S}}{\partial \tilde{z}^a \partial \tilde{z}^b} \right)_{\text{on-shell}} + \text{rank} \left( \frac{\partial^2 M_1}{\partial \phi^i \partial \theta^\alpha} \right)_{\text{on-shell}} = N - (m_0 - m_1) + \text{rank} \left( \frac{\partial A_\alpha}{\partial \phi^i} \right)_{\text{on-shell}} = N.$$

(4.4)

Notice that the rank of $M_1$ is not $m_0$, the number of $\theta^\alpha$ parameters. This means that there are new gauge transformations. Its associated BRST parameters are the ghosts $v^\alpha$.

During the above derivation we have not checked whether $\tilde{W}$ satisfies the antifield independent part of the QME of the extended theory, i.e., we should verify that the WZ term of eq. (4.3) really cancels the anomaly of the extended theory. We have implicitly assumed that the regularized computation of the anomaly $\mathcal{A}$ gives the same result as in

2 lim $M \to \infty$ is understood.
the non-extended theory. But now the theory contains new dynamical fields, $\theta^\alpha(x)$ and $v^a(x)$ and maybe they contribute to a new value of the anomaly $\tilde{A} \neq A$.

In our regularization scheme this means that we have to introduce PV fields also for the fields $\theta^\alpha(x)$, the ghosts $v^a(x)$ and the antighosts $\bar{v}^a(x)$. The fields $\theta^\alpha(x)$ only appear into the WZ term. But since this term is of order $O(h)$, the usual $\bar{h}$ perturbative expansion will be spoiled. To circumvent this problem, we should get from $\tilde{W}$ a “classical” part $\tilde{W}_0$ with the usual requirements:

\begin{align}
\text{i) } & \text{rank} \left( \frac{\partial^2 \tilde{W}_0}{\partial \tilde{z}^a \partial \tilde{z}^b} \right)_{\text{on-shell}} = N, \\
\text{ii) } & (\tilde{W}_0, \tilde{W}_0) = 0.
\end{align}

In order to obtain the “classical” part $\tilde{W}_0$ of the action, it is useful to expand $\tilde{W}$ in powers of $\theta^\alpha$. This means to expand the WZ term

$$
\hbar M_1(\phi, \theta) = -i\hbar \left[ A_\alpha(\phi) \theta^\alpha + \frac{1}{2} \theta^\alpha D_{\alpha \beta}(\phi) \theta^\beta + \frac{1}{(3!)} \theta^\alpha \theta^\beta \theta^\gamma (\Gamma_{\alpha \beta \gamma})(\phi) \\
+ ... + \frac{1}{(n!)} \theta^\alpha ... \theta^n (\Gamma_{\alpha_1 ... \alpha_n \beta} D_{\alpha_1 ... \alpha_n})(\phi) + ... \right],
$$

with $\Gamma_{\alpha}$ and $D_{\alpha \beta}$ defined by

$$
\Gamma_{\alpha} = R^i_{\alpha} \frac{\partial}{\partial \phi^i}, \quad D_{\alpha \beta} = \Gamma_{\beta \gamma} A_{\alpha \gamma} = \left( \frac{\partial A_{\alpha}}{\partial \phi^i} R^i_{\beta} \right)
$$

and also to expand the functionals

$$
U^a_{\beta}(\theta) = \delta^a_{\beta} + \frac{1}{2} T^a_{\beta \gamma} \theta^\gamma + O(\theta^2)
$$

$$
S^a_{\alpha \beta \gamma}(\theta) = S^a_{\alpha \beta \gamma}(0) \theta^\gamma + O(\theta^2)
$$

$$
B^a_{bc}(\theta) = -A^a_{bc} + Z^a_{b} S^a_{c \alpha \beta}(0) \theta^\beta + O(\theta^2)
$$

$$
C^a_{bc}(\theta) = C^a_{bc}(0) + Z^a_{b} S^a_{c \beta \gamma}(0) Z^\beta \theta^\gamma + O(\theta^2),
$$

where we have used \((A.48)\) and \((A.49)\). Extracting the quadratic part $3$ in $\theta^\alpha$ of (4.7) we can see that its kinetic term is of order $\hbar$. In order to get standard propagators we make a canonical transformation (in the antibracket sense) \([\Omega]\) \([\Pi]\) in $\theta^\alpha$-sector:

$$
\theta^{\alpha} = \sqrt{\hbar} \, \theta^{\alpha}, \quad \theta^{\alpha*} = \frac{\theta^{\alpha}}{\sqrt{\hbar}}
$$

\[3\]We are going to consider here that rank($D_{\alpha \beta}$) = $m_0 - m_1$
But such canonical transformation introduces inverse powers of $\sqrt{\hbar}$ in the quantized action coming from the pieces of order $\mathcal{O}(\theta^3)$ of the WZ term. They vanish if the kinetic term for the $\theta^\alpha$ fields is gauge invariant,

$$\Gamma_\gamma(D_{\alpha\beta}(\phi)) = 0. \quad (4.14)$$

Note that $S^a_{\alpha\beta}(\theta)$ is $\mathcal{O}(\theta)$, then the transformation (4.13) introduces factors $\frac{1}{\sqrt{\hbar}}$ into the sources of the BRST transformations of the ghosts $v^\alpha$. To avoid this fact we implement the same canonical transformation for them:

$$v'^\alpha = \sqrt{\hbar} v^\alpha, \quad v'^*_a = \frac{v^*_a}{\sqrt{\hbar}}. \quad (4.15)$$

If we impose the absence of terms $\frac{1}{\sqrt{\hbar}}$ in $\tilde{W}$, we have the following restrictions of the structure functions:

$$U^a_{\alpha \beta}(\theta) = \delta^a_{\beta}$$
$$S^a_{\alpha \beta}(\theta) = S^a_{\alpha \beta \gamma}(0) \theta^\gamma$$
$$B^a_{\alpha \beta}(\theta) = 0$$
$$C^a_{\alpha \beta \gamma}(\theta) = 0. \quad (4.16)$$

Notice that only a very restrictive set of theories, those with abelian gauge algebras, satisfy our requirements. This set of theories can be enlarged if we consider gauge theories whose anomalous part is not the whole group, but a proper subgroup.

In our case, the non-anomalous extended quantized reducible proper solution at one loop is (after dropping primes)

$$\tilde{W} = \tilde{W}_0 + \sqrt{\hbar} M_{\frac{1}{2}} \quad (4.17)$$
$$\tilde{W}_0 \equiv S(\Phi, \Phi^*) - \frac{i}{2} \theta^\alpha D_{\alpha\beta}(\phi) \theta^\beta + \theta^*_\alpha Z^a_\alpha v^a - \frac{1}{2} v^*_a S^a_{\alpha \beta \gamma}(0) \theta^\gamma c^\beta c^\alpha$$
$$M_{\frac{1}{2}} \equiv -i A_\alpha \theta^\alpha - \theta^*_\alpha c^\alpha, \quad (4.18)$$

where $S(\Phi, \Phi^*)$ is the reducible proper solution of the non-extended theory. We have that $\tilde{W}$ verifies the antifield independent part of the QME

$$(\tilde{W}_0 + \sqrt{\hbar} M_{\frac{1}{2}}, \tilde{W}_0 + \sqrt{\hbar} M_{\frac{1}{2}}) = 2i\hbar A_\alpha(\phi)c^\alpha. \quad (4.19)$$
Then, arranging in powers of $\hbar$, we get

\begin{align}
(\tilde{W}_0, \tilde{W}_0) &= 0 \quad (4.20) \\
(\tilde{W}_0, M_2^1) &= 0 \quad (4.21) \\
(M_2^1, M_2^1) &= 2iA_\alpha(\phi)c^\alpha. \quad (4.22)
\end{align}

These equations show how the potential anomaly is cancelled by the $M_2^1$ background term \[16\] \[17\]. Observe that (4.20) implies the Noether identities $D_\alpha Z_\beta = 0$, so an additional gauge fixing is needed for the part $\theta^\alpha$.

Once we have the classical part $\tilde{W}_0$ of the action, we can find the possible new value of the anomaly $\tilde{A}$. In general we will have for the antifield independent part of the anomaly

$$\tilde{A} = \tilde{A}_a(\phi, \theta) c^\alpha + \tilde{A}_a(\phi, \theta) v^a = \tilde{A}(\phi) + O(\phi, \theta). \quad (4.23)$$

The contribution $O(\theta)$ is only relevant at higher order in $\hbar$, because $\theta^\alpha = \sqrt{\hbar}\theta^\alpha$. Then we can say that, at lowest order, the part of the extended anomaly which depends on the ghosts directions $c^\alpha$ is

$$\tilde{A}_a c^\alpha = \tilde{a}_k A^{(k)}_a(\phi) c^\alpha, \quad (4.24)$$

where $\{A^{(k)}_a(\phi) c^\alpha\}_k$ is the basis of the $\phi^i(x)$-space functionals of non-trivial cocycles at ghost number one of the theory and $\tilde{a}_k$ are some unknown coefficients. If we now use the Wess-Zumino consistency conditions (2.17) for the extended anomaly (4.23), $\tilde{A}_A\Z^A = 0$, and consider the fact that for the non-extended reducible anomalies $A^{(k)}_a(\phi) Z^A = 0$, we obtain to lowest order

$$\tilde{A}_a(\phi) = 0. \quad (4.25)$$

So, eventually, the functional expression to order $\hbar$ of the extended reducible anomaly is the same as for the non-extended case, but with different values for the coefficients of the anomaly which get renormalized from $a_k$ (those computed without taking into account the new degrees of freedom) to $\tilde{a}_k$,

$$\tilde{A} = \tilde{a}_k A^{(k)}(\phi) c^\alpha. \quad (4.26)$$

And finally, the local proper quantized action which satisfies the antifield independent part of the QME of the extended theory to order $\hbar$ is

$$\tilde{W} = \tilde{S} + \hbar\tilde{a}_k M_1^{(k)} \quad (4.27)$$
with
\[
M^{(k)}_1(\phi, \theta) \equiv -i \int_0^1 A^{(k)}_a(F^i(\phi, t\theta)) \theta^a dt.
\] (4.28)

It still remains to obtain the coefficients \( \tilde{a}_k \). They have to be computed perturbatively. To do so, we have to go to a gauge fixed basis \((\tilde{\Phi}, \tilde{K})\), where \( \tilde{\Phi} \) represents all the fields of the extended theory, via the usual canonical transformation implemented by a gauge fixing fermion, which we take as
\[
\tilde{\Psi} = \bar{c}_A \chi^A(\phi, \theta) + \bar{\eta}_a w^a_A c^A + \bar{e}_A \sigma^A_a \eta^a + \frac{1}{2} \bar{e}_A K^{AB} B_B + \frac{1}{2} \bar{\eta}_a M^a_b B^b,
\] (4.29)
where \( \bar{c}_A = \{\bar{c}_\alpha, \bar{v}_a\} \) are the antighosts; \( \bar{\eta}_a \) are the antighosts for ghost; \( \eta^a, B_B \) and \( B^b \) are Lagrange multipliers which will be integrated out; and \( K^{AB}, M^a_b \) are invertible matrices. Just for simplicity we choose the decoupled gauge fixing conditions
\[
\chi^\alpha = \chi^\alpha(\phi), \quad \chi^a = \chi^a(\theta)
\] (4.30)
\[
w^a_b = \sigma^a_b = K^{a\beta} = K^{\beta a} = 0
\] (4.31)
\[
\text{rank}(\frac{\partial \chi^a}{\partial \phi^i} R^i_\beta) = m_0 - m_1, \quad \text{rank}(\frac{\partial \chi^a}{\partial \theta^\alpha} Z^a_\beta) = m_1
\] (4.32)
\[
\text{rank}(w^a_a) = \text{rank}(\sigma^a_a) = m_1
\] (4.33)
and we get the gauge fixed action
\[
\tilde{W} = \tilde{W}_0 + \sqrt{\tilde{h}} \tilde{M}_\perp
\] (4.34)
\[
\tilde{W}_0 \equiv S(\Phi, K) - \frac{i}{2} \tilde{a}_k \theta^a D^{(k)}_{\alpha\beta}(\phi) \theta^\beta \chi^\alpha(\theta) K_{ab} \chi^b(\theta)
\]
\[
+ \bar{v}_a \left( \frac{\partial \chi^a}{\partial \theta^\alpha} Z^\alpha_b \right) v^b + \theta^a S^a_{b\gamma}(0) \theta^\gamma c^b c^a \quad (4.35)
\]
\[
\tilde{M}_\perp \equiv -i \tilde{a}_k A^{(k)}_a \theta^a - \bar{v}_a \frac{\partial \chi^a}{\partial \theta^\alpha} c^a - \theta^a c^a,
\] (4.36)
where \( S(\Phi, K) \) is the gauge fixed reducible proper solution of the non-extended theory.

It only remains to choose a regularization of the theory and compute the coefficients of the extended reducible anomaly at one loop level. We can follow the usual PV procedure [7] just by recalling that now we have to introduce PV fields for \( \theta^a(x), v^a(x) \) and \( \bar{v}_a(x) \) and their corresponding PV regulator mass terms. Their computation will show whether the coefficients of the WZ term get renormalized or not. If they do so, it will reflect the fact that the measure \( D\theta \ Dv \ D\bar{v} \) is non BRST invariant.
5 Example: Abelian Topological Yang-Mills

Consider the abelian topological Yang-Mills action [18]

\[ S_0 = \int d^4x \, F^{\alpha\beta} \ast F_{\alpha\beta}, \]  

where

\[ F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}; \quad \ast F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}, \]  

with \( \epsilon_{0123} = 1 \). The action has the reducible gauge symmetries

\[ \delta A_{\mu} = \lambda_{\mu} + \partial_{\mu} \Lambda. \]  

The reducibility comes out taking \( \lambda_{\mu} = \partial_{\mu} \xi \) and \( \Lambda = -\xi \) which give \( \delta A_{\mu} = 0 \). Therefore the theory is 1-step reducible off-shell.

\( \mathcal{G} = \{ \lambda^\alpha, \Lambda \} \) is the manifold spanned by our reducible parametrization of the group which acts on the manifold of classical fields \( M = \{ A_{\mu} \} \) by the finite transformations

\[ A'^{\mu} = F^{\mu}_{\nu}(A^{\nu}, \lambda^\alpha, \Lambda) = A_{\mu} + \lambda_{\mu} + \partial_{\mu} \Lambda. \]  

There is an equivalence relation of the group parameters, \( (\lambda^{\mu}, \Lambda) \sim (\lambda'^{\mu}, \Lambda') \), according to \( (A.8) \)

\[ (\lambda'^{\mu}, \Lambda') = (f^{\mu}(\lambda, \epsilon), f(\Lambda, \epsilon)) = (\lambda^{\mu} + \partial^{\mu} \epsilon, \Lambda - \epsilon). \]  

This leads to an identification of the coefficients of the null vector in the parameter space \( (A.26) \) (which generates no transformations in the space of fields) as

\[ Z = (\partial^{\mu}, -1). \]  

From \( (5.4) \) and \( (A.4) \) it is easy to check that \( \mathcal{G} \) is an abelian group,

\[ (\varphi^{\mu}(\lambda_1, \lambda_2), \varphi(\Lambda_1, \Lambda_2)) = (\lambda_1^{\mu} + \lambda_2^{\mu}, \Lambda_1 + \Lambda_2). \]  

The functions \( \Sigma \) and \( \Sigma' \) defined in \( (A.12) \) and \( (A.13) \) are readily computed:

\[ ((\lambda^{\mu} + \partial^{\mu} \epsilon) + \lambda'^{\mu}, (\Lambda - \epsilon) + \Lambda') = ((\lambda^{\mu} + \lambda'^{\mu}) + \partial^{\mu} \Sigma, (\Lambda + \Lambda') - \Sigma) \implies \Sigma = \epsilon, \]  

and, similarly, \( \Sigma' = \epsilon. \)
Taking (5.7) into account, it is obvious that associativity (A.18) is satisfied, i.e.,

\[ \eta = 0. \]  

(5.9)

With this set of functions, the algebraic structure is immediately displayed:

\[ U_\beta^\alpha = \delta_\beta^\alpha, T_\beta^\alpha = 0, A^a_\alpha = 0, F^a_\alpha = 0, S^a_\alpha = 0, B^a_\alpha = 0, C^a_{bc} = 0, \]  

(5.10)

where \( \alpha, a \) refer here to collective indices in the sense of appendix A. Observe we are already in the suitable parametrization such that

\[ Z(\lambda, \Lambda) = Z^\mu U_\mu(\lambda, \Lambda). \]  

(5.11)

The minimal proper solution is

\[ S = \int d^4 x \left\{ F^a_\alpha F^a_\alpha + A^a_\mu (c^\mu + \partial^\mu c) + c^a_\mu \partial^\mu \eta - c^* \eta \right\}, \]  

(5.12)

with \( c^\mu, c, \eta \) having ghost numbers 1, 1, 2 respectively.

If there was a potential anomaly at one loop, i.e., if the measure was not BRST invariant, \( (\Delta S)_{\text{reg}} \neq 0 \), then the Wess-Zumino consistency conditions (2.16)(2.17) requires

\[ \delta (\Delta S)_{\text{reg}} = 0 \]  

(5.13)

A possible solution of these conditions is

\[ (\Delta S)_{\text{reg}} = i \int d^4 x \ F^a_\alpha \partial_\alpha c_\beta = A^a_\alpha c_\alpha. \]  

(5.14)

Here we don’t analyze the existence of a regularization giving this potential anomaly. Instead we will use this solution of the WZ conditions to show some aspects of the extended formalism for reducible theories.

Introduce the new fields \( \theta_\alpha(x) \) and \( \theta(x) \). Their gauge transformations are obtained by demanding the invariance of the finite transformation \( A_\alpha \rightarrow A_\alpha + \theta_\alpha + \partial_\alpha \theta \). We get

\[ \delta \theta_\alpha = -\epsilon_\alpha + \partial_\alpha \xi \]  

(5.15)

\[ \delta \theta = -\Lambda + \xi, \]  

(5.16)

where \( \xi \) is a new gauge parameter of the extended theory, see (3.7).
The WZ term (4.3) is

\[ M_1(A_\alpha, \theta_\beta) = \int_0^1 dt \int d^4x \{ \partial^\alpha (A^\beta + t\theta^\beta) - \partial^\beta (A^\alpha + t\theta^\alpha) \} \partial_\alpha \theta_\beta \tag{5.17} \]

\[ = \int d^4x \{ F^{\alpha \beta} \partial_\alpha \theta_\beta + \frac{1}{2} (\partial^\alpha \theta^\beta - \partial^\beta \theta^\alpha) \partial_\alpha \theta_\beta \} \]

\[ = \frac{1}{4} \int d^4x \{ 2 F^{\alpha \beta} H_{\alpha \beta} + H^{\alpha \beta} H_{\alpha \beta} \} \tag{5.18} \]

where \( H_{\alpha \beta} \equiv \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha \).

Observe that the WZ term has the new gauge invariance \( \delta \theta_\alpha = \partial_\alpha \xi \). It is a general feature of the reducible extended formalism, the WZ term become gauge invariant under transformations induced by the new gauge parameters.

The reducible extended formalism has provided us with a method to obtain a suitable counterterm that would cancel the non BRST invariance of the measure. If the integration of the extra variables gives a non local counterterm \( M'_1(A_\alpha) \), the theory would be genuinely anomalous. Due to the gauge invariance \( \delta \theta_\alpha = \partial_\alpha \xi \) in the WZ term, in general it will be necessary to take a gauge fixing of the extra variables, as for instance \( \partial_\alpha \theta^\alpha = 0 \).

In our case \( \theta^\alpha \) integration gives the local counterterm

\[ M'_1(A_\alpha) = -\frac{1}{4} \int d^4x \ F^{\alpha \beta} F_{\alpha \beta} , \tag{5.19} \]

confirming that abelian topological Yang-Mills is not anomalous.

6 Conclusions

In this paper we have considered closed reducible off-shell potential anomalous gauge theories in an extended Field-Antifield formalism. The pure gauge anomalous degrees of freedom that become propagating at quantum level have been introduced in a covariant way.

We have constructed the Wess-Zumino term in terms of the anomalies and the extra variables. This term has gauge invariances, and one needs to introduce a new ghost structure associated with these new gauge symmetries. If the integration of the extra variables gives a local counterterm we have restored the BRST invariance of the theory at one loop. On the other hand, if the result of the integration gives a non local counterterm the
theory is genuinely anomalous and the price to maintain locality is to keep the extended variables.

We have also considered the quantum one loop effects of the extra variables, which leads to a finite renormalization of the anomalies. Only a certain type of theories seems to admit the perturbative description we present, indicating that maybe a quantum treatment of the Wess-Zumino terms goes beyond the scope of the usual \(\hbar\) perturbative expansion.

The analysis has been done for off-shell algebras. Discussion on on-shell algebras is under investigation and will be published elsewhere.

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A Action of a Lie Group with Reducible Parametrization

Here we will consider the action on a manifold \(\mathcal{M}\), parametrized by the classical fields \(\phi^i(x)\) \((i = 1, ..., n)\), of a Lie group \(G\) which is locally described by a redundant set of parameters. We denote these parameters as \(\theta^\alpha\) \((\alpha = 1, ..., m_0)\), and they identify a point in some space \(\mathcal{G}\). The action of \(\mathcal{G}\) on \(\mathcal{M}\) is then given by

\[
F : \mathcal{M} \times \mathcal{G} \to \mathcal{M} \\
(\phi^i, \theta^\alpha) \mapsto F^i(\phi, \theta) \quad \text{(A.1)}
\]
with the usual requirement that the zero value for the parameters corresponds to the neutral element of the group,
\[ F^i(\phi, 0) = \phi^i. \]  

(A.2)

Since the action of \( G \) on \( M \) is that of a group, there exists, in the space \( \mathcal{G} \) of parameters, a class of structure functions. We take as a representative
\[ \varphi : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \]
\[ (\theta^\alpha, \theta'^\beta) \mapsto \varphi^\alpha(\theta, \theta'), \]  

(A.3)
such that:

i) Satisfies the composition law
\[ F^i(F(\phi, \theta), \theta') = F^i(\phi, \varphi(\theta, \theta')). \]  

(A.4)

ii) Has \( \theta = 0 \) as the neutral element,
\[ \varphi^\alpha(0, \theta) = \varphi^\alpha(\theta, 0) = \theta^\alpha. \]  

(A.5)

iii) For a given \( \theta^\alpha \in \mathcal{G} \), there always exists \( \bar{\theta}^\alpha \in \mathcal{G} \) such that
\[ \varphi^\alpha(\theta, \bar{\theta}) = \varphi^\alpha(\bar{\theta}, \theta) = 0. \]  

(A.6)

Since we are considering a redundant action of \( \mathcal{G} \) on \( M \), an equivalence relation is defined on the elements of \( \mathcal{G} \) by
\[ \theta' \sim \theta \iff F^i(\phi, \theta) = F^i(\phi, \theta'). \]

(A.7)

Thus \( \mathcal{G} \) is split into orbits \([\theta]\). Each orbit represents an element of the group \( G = \mathcal{G}/\sim \).

A.1 The reducible finite structure

To parametrize each orbit, we will use \( m_1 \) parameters, \( \epsilon^a, \quad a = 1, \ldots, m_1 \), where \( m_1 \) is the dimensionality of the orbits, \( m_1 < m_0 \). Then, the equivalence relation (A.7) can be described by \( m_0 \) functions \( f^\alpha(\phi, \epsilon), \quad \alpha = 1, \ldots, m_0 \) such that
\[ \theta \sim \theta' \quad \Rightarrow \quad \text{There are } \epsilon^a \text{ such that } \theta'^\alpha = f^\alpha(\theta, \epsilon) \]  

(A.8)

with the convention
\[ f^\alpha(\theta, 0) = \theta^\alpha. \]  

(A.9)
We will not allow redundancy in the parametrization of the orbits. This means that the
theories we are studying are first step reducible, i.e.,

\[ f^\alpha(\theta, \epsilon) = f^\alpha(\theta, \epsilon') \Rightarrow \epsilon^\alpha = \epsilon'^\alpha. \] \hspace{1cm} (A.10)

The generalization to higher step reducibility is studied in appendix B.

Owing to (A.8), there must exist \( m_1 \) functions \( \Omega^a(\epsilon, \epsilon', \theta) \) such that

\[ f^\alpha(f(\theta, \epsilon), \epsilon') = f^\alpha(f(\theta, \Omega(\epsilon, \epsilon', \theta))). \] \hspace{1cm} (A.11)

Note that (A.9), (A.10) and (A.11) show that each orbit \([\theta]\) has an irreducible quasigroup
structure given by the composition law \( \Omega(\epsilon, \epsilon', \theta) \).

If we change the parameter \( \theta \) or \( \theta' \) by other representatives of the same orbit in (A.4),
two new functions \( \Sigma^a(\epsilon, \theta, \theta') \) and \( \Sigma'^a(\epsilon, \theta, \theta') \) appear, such that

\[ \varphi^\alpha(f(\theta, \epsilon), \theta') = f^\alpha(\varphi(\theta, \epsilon), \Sigma(\epsilon, \theta, \theta')) \] \hspace{1cm} (A.12)

\[ \varphi^\alpha(\theta, f(\theta', \epsilon)) = f^\alpha(\varphi(\theta, \theta'), \Sigma'(\epsilon, \theta, \theta')). \] \hspace{1cm} (A.13)

Use of (A.9) gives the condition

\[ \Sigma^a(0, \theta, \theta') = \Sigma'^a(0, \theta, \theta') = 0, \] \hspace{1cm} (A.14)

whereas (A.5) gives

\[ \Sigma^a(\epsilon, \theta, 0) = \epsilon^a \] \hspace{1cm} (A.15)

\[ \Sigma'^a(\epsilon, 0, \theta') = \epsilon^a. \] \hspace{1cm} (A.16)

Associativity may not hold for the redundant parametrization. In fact, from

\[ F^i(\phi, \varphi(\varphi(\theta, \theta'), \theta'')) = F^i(\phi, \varphi(\phi(\theta, \varphi(\theta', \theta'')))), \] \hspace{1cm} (A.17)

we can only conclude, using (A.8), that there exists a unique function \( \eta^a(\theta, \theta', \theta'') \) such that

\[ \varphi^a(\varphi(\theta, \theta'), \theta'') = f^a(\varphi(\theta, \varphi(\theta', \theta'')), \eta(\theta, \theta', \theta'')). \] \hspace{1cm} (A.18)

Observe that, from (A.3),

\[ \eta^a(\theta, \theta', \theta'') \neq 0 \Rightarrow \theta, \theta', \theta'' \neq 0. \] \hspace{1cm} (A.19)

\[ \text{4Unique because we are in a first step reducible case.} \]
For first step reducible parametrizations, the functions defined in (A.1), (A.3), (A.8), (A.11), (A.12), (A.13) and (A.18) form the complete set of structure functions. It is interesting, for later purposes, to consider some relations among these functions.

If we change \( \theta' \to f(\theta', \epsilon') \) in (A.12) we obtain

\[
\Omega^a (\Sigma' \{\epsilon, \theta, \theta'\}, \Sigma \{\epsilon, \theta, f(\theta', \epsilon')\}, \varphi \{\theta, \theta'\}) = \Omega^a (\Sigma \{\epsilon, \theta, \theta'\}, \Sigma' \{\epsilon', f(\theta, \epsilon), \theta'\}, \varphi \{\theta, \theta'\}).
\]  
(A.20)

From the modified associative law (A.18), if we change \( \theta'' \to f(\theta'', \epsilon) \) and put \( \theta'' = 0 \) we obtain

\[
\Sigma'' (\epsilon, \varphi(\theta, \theta'), 0) = \Omega^a (\Sigma' \{\epsilon', \theta, 0\}, \theta, \theta') \cdot \eta \{\theta, \theta', f(0, \epsilon)\} \cdot \varphi(\theta, \varphi(\theta', f(0, \epsilon)))
\]  
(A.21)

Finally, from the triple composition \( \varphi^a(\varphi(\varphi(\theta_1, \theta_2), \theta_3), \theta_4) \) we get the relation

\[
\begin{align*}
\Omega^a (\eta \{\theta_1, \theta_2, \varphi(\theta_3, \theta_4)\}, \eta \{\varphi(\theta_1, \theta_2), \varphi(\theta_3, \theta_4)\}, \varphi(\theta_1, \varphi(\theta_2, \varphi(\theta_3, \theta_4)))\}) &= \\
\Omega^a (\Sigma' \{\eta \{\theta_1, \theta_2, \theta_3, \theta_4\}, \theta, \varphi(\theta_2, \varphi(\theta_3, \theta_4))\}, \Sigma \{\eta \{\eta \{\varphi(\theta_1, \theta_2, \theta_3), \varphi(\theta_1, \varphi(\theta_2, \theta_3))\} \cdot \theta_4\}, \varphi(\theta_1, \varphi(\theta_2, \varphi(\theta_3, \theta_4)))\}) = \\
\Sigma [\eta \{\theta_1, \theta_2, \theta_3\}, \varphi(\theta_1, \varphi(\theta_2, \theta_3)) \cdot \theta_4], \varphi(\theta_1, \varphi(\theta_2, \theta_3)) \cdot \theta_4] \cdot \varphi(\theta_1, \varphi(\theta_2, \varphi(\theta_3, \theta_4))).
\end{align*}
\]  
(A.22)

### A.2 The algebraic structure

When a Lie group of transformations is described with an irreducible set of parameters, the fundamental functions \( F^i, \varphi^a \) completely determine, through some differentiation processes, all the algebraic relations we need. However, as we have just seen, when the action is reducible there appear new functions \( f^a, \Omega^a, \Sigma^a \) and \( \eta^a \) that will give us new relations for the group algebra. According to the parametrization provided by \( G \), the symmetry generators are given by the fundamental vector fields

\[
R_\alpha(\phi) = R^i_\alpha(\phi) \frac{\partial}{\partial \phi^i} \equiv \frac{\partial F^i(\phi, \theta)}{\partial \theta^\alpha} \bigg|_{\theta=0} \frac{\partial}{\partial \phi^i}.
\]  
(A.23)

Then, applying the operator \( \left( \frac{\partial}{\partial \theta^\alpha} - (\beta \leftrightarrow \alpha) \right)_{\theta=\theta'=0} \) to (A.14), we get the commutation laws

\[
[R_\alpha(\phi), R_\beta(\phi)] = T^\gamma_{\alpha\beta} R_\gamma(\phi),
\]  
(A.24)

with the structure constants

\[
T^\gamma_{\alpha\beta} \equiv \left( \frac{\partial^2 \varphi^\gamma(\theta, \theta')}{\partial \theta^\alpha \partial \theta'^\beta} - (\beta \leftrightarrow \alpha) \right)_{\theta=\theta'=0}.
\]  
(A.25)
The vector fields that generate motions tangent to the orbits $[\theta]$ are

$$\mathbf{Z}_a(\theta) = Z_a^\alpha(\theta) \frac{\partial}{\partial \theta^\alpha} \equiv \frac{\partial f^\alpha(\theta, \epsilon)}{\partial \epsilon^a} \bigg|_{\epsilon=0} \frac{\partial}{\partial \theta^\alpha}$$

(A.26)

and become null vectors when realized on $\mathcal{M}$. Vectors $\mathbf{Z}_a(\theta)$ close among themselves:

$$[\mathbf{Z}_a(\theta), \mathbf{Z}_b(\theta)] = C_{ab}^c(\theta)\mathbf{Z}_c(\theta)$$

(A.27)

with

$$C_{ab}^c(\theta) \equiv \left( \frac{\partial \Omega^c(\epsilon, \epsilon', \theta)}{\partial \epsilon^a \partial \epsilon^b} - (b \leftrightarrow a) \right)_{\epsilon = \epsilon' = 0}$$

(A.28)

The reducibility of the generators $\mathbf{R}_\alpha(\phi)$ is obvious from (A.8):

$$\left. \frac{\partial F^i(\phi, f(\theta, \epsilon))}{\partial \epsilon^a} \right|_{\epsilon = 0} = 0 \quad \Rightarrow \quad \left. \frac{\partial F^i(\phi, \theta)}{\partial \theta^\alpha} Z_\alpha^a(\theta) \right|_{\epsilon = \epsilon' = 0} = 0,$$

(A.29)

which for $\theta = 0$ gives

$$R_\alpha^i(\phi) Z_\alpha^a = 0,$$

(A.30)

where we have defined

$$Z_\alpha^a \equiv Z_\alpha^a(0).$$

(A.31)

If we look at (A.3) as a left action of the “reducible group” $\mathcal{G}$ on itself, we get the generators

$$\mathbf{U}_\alpha(\theta) = U_\alpha^\beta(\theta) \frac{\partial}{\partial \theta^\beta} \equiv \frac{\partial \varphi^\beta(\theta', \theta)}{\partial \theta^\alpha} \bigg|_{\theta' = 0} \frac{\partial}{\partial \theta^\beta},$$

(A.32)

and its algebra is obtained by applying the operator $\left( \frac{\partial}{\partial \theta^\alpha \partial \theta^\beta} - (\gamma \leftrightarrow \beta) \right)_{\theta = \theta' = 0}$ on the modified associative law (A.18). We obtain

$$[\mathbf{U}_\alpha(\theta), \mathbf{U}_\beta(\theta)] = -T_{\alpha\beta}^\gamma \mathbf{U}_\gamma(\theta) + S_{\alpha\beta}^a(\theta)\mathbf{Z}_a(\theta)$$

(A.33)

with

$$S_{\alpha\beta}^a(\theta) \equiv \left( \frac{\partial^2 f^a(\theta', \theta'' \theta)}{\partial \theta'^\alpha \partial \theta''^\beta} - (\beta \leftrightarrow \alpha) \right)_{\theta' = \theta'' = 0}.$$

(A.34)

It is easy to verify that $\{\mathbf{Z}_a(\theta)\}$ generate an ideal of $\{\mathbf{U}_\alpha(\theta)\}$. Derivation of (A.13) with respect $\theta^\alpha$ and $\epsilon^a$ gives

$$[\mathbf{Z}_a(\theta), \mathbf{U}_\alpha(\theta)] = B_{aa}^b(\theta) \mathbf{Z}_b(\theta)$$

(A.35)
with
\[ B^b_{a\alpha}(\theta) \equiv \left. \frac{\partial^2 \Sigma^b_{\alpha}(\epsilon, \theta', \theta)}{\partial \epsilon^a \partial \theta^\alpha} \right|_{\epsilon = \theta' = 0}. \] (A.36)

The effect of shifting the representatives in (A.12) and (A.13) is manifested at algebraic level by some new relations of dependence between the structure constants \( T^\gamma_{\alpha\beta} \). If we consider
\[
\left[ \frac{\partial^2}{\partial \epsilon^a \partial \theta^\alpha} \varphi^\gamma(f(\theta, \epsilon), \theta') - \frac{\partial^2}{\partial \theta^\beta \partial \epsilon^a} \varphi^\gamma(\theta, f'(\theta', \epsilon)) \right]_{\theta = \theta' = \epsilon = 0} = Z^\alpha_{a} T^\gamma_{\alpha\beta} \] (A.37)
and
\[
\left[ \frac{\partial^2}{\partial \epsilon^a \partial \theta^\beta} f^\gamma(\varphi(\theta, \theta'), \Sigma) - \frac{\partial^2}{\partial \theta^\beta \partial \epsilon^a} f^\gamma(\varphi(\theta, \theta'), \Sigma') \right]_{\theta = \theta' = \epsilon = 0} = A^b_{a\beta} Z^\gamma_{b}, \] (A.38)
where we define the new algebraic structure constants
\[
A^b_{a\beta} \equiv \left( \frac{\partial^2 \Sigma^b_{\alpha}(\epsilon, \theta', \theta')}{\partial \epsilon^a \partial \theta^\alpha} - \frac{\partial^2 \Sigma^b_{\alpha}(\epsilon, \theta', \theta')}{\partial \theta^\beta \partial \epsilon^a} \right)_{\theta = \theta' = \epsilon = 0}, \] (A.39)
we get, using (A.12) and (A.13), that
\[
Z^\alpha_{a} T^\gamma_{\alpha\beta} = A^b_{a\beta} Z^\gamma_{b}. \] (A.40)

From the modified associative law we expect a modification of the Jacobi identity. If we apply the operator \( \sum_{P \in \text{Perm}[\alpha\beta\gamma]} (-1)^{P} \frac{\partial^3}{\partial \epsilon^\alpha \partial \theta^\beta \partial \theta^\gamma} \) to (A.18) we get new quantities \( F^a_{\alpha\beta\gamma} \), defined by
\[
F^a_{\alpha\beta\gamma} \equiv \sum_{P \in \text{Perm}[\alpha\beta\gamma]} (-1)^{P} \left( \frac{\partial^3 \eta^a(\theta, \theta', \theta'')}{\partial \epsilon^\alpha \partial \theta^\beta \partial \theta^\gamma} \right)_{\theta = \theta' = \theta'' = 0}, \] (A.41)
and such that they satisfy
\[
\sum_{\text{Cyclic}[\alpha\beta\gamma]} (T^\mu_{\alpha\beta} T^\nu_{\mu\gamma}) = F^a_{\alpha\beta\gamma} Z^\nu_{a}. \] (A.42)
This is the expression of the Jacobi identity in our case.

There is a convenient choice of the parametrization \( \epsilon^a \) such that some calculations become simpler. If we derive (A.12) with respect \( \epsilon^a \) and we put \( \theta^a = \epsilon = 0 \) we get
\[
U^a_{\alpha}(\theta) Z^\beta_{a} = Z^\alpha(\theta)_{b} \Pi^b_{a}(\theta), \] (A.43)
where $\Pi^b_a(\theta) = \left(\frac{\partial \Sigma^b}{\partial \epsilon^a}\right)_{(0,0,0)}$. A suitable parametrization is such that $\bar{\Pi}^b_a(\theta) = \delta^b_a$.

Consider a new parametrization $\bar{f}$ of the orbits which parameters $\lambda^a$, such that it is related to the former parametrization by $\bar{f}(\theta, \lambda) = f(\theta, \epsilon(\lambda, \theta))$, where $\epsilon^a(\lambda, \theta)$ are functions to be determined. The null vectors $\bar{Z}^a_a(\theta) = \left(\frac{\partial \bar{f}_a}{\partial \lambda^b}\right)_{(0,0,\theta)}$ are related with those of the $\epsilon^a$ parametrization, $Z^a_a(\theta)$, by

$$
\bar{Z}^a_a(\theta) = Z^b_a(\theta) \Upsilon^a_{b}(\theta), \tag{A.44}
$$

where $\Upsilon^b_a(\theta) = \left(\frac{\partial \epsilon^b}{\partial \lambda^a}\right)_{(0,\theta)}$. Using (A.43) we have that $\bar{\Pi}^a_d(\theta) = (\Upsilon^{-1})^a_b (\theta) \Pi^b_d(\theta) \Upsilon^d_c(0)$. And the requirement $\bar{\Pi}^a_d(\theta) = \delta^a_d$ is equivalent to the following differential equation:

$$
\left(\frac{\partial \epsilon^a}{\partial \lambda^b}\right)_{(0,\theta)} = \Pi^a_c(\theta) \left(\frac{\partial \epsilon^c}{\partial \lambda^b}\right)_{(0,0)}, \tag{A.45}
$$

As initial condition on $\theta$ we can take, for instance, $\left(\frac{\partial \epsilon^c}{\partial \lambda^b}\right)_{(0,0)} = \delta^c_b$. A solution of (A.45) is

$$
\epsilon^a(\lambda, \theta) = \Pi^a_b(\theta) \lambda^b, \tag{A.46}
$$

where we have

$$
\bar{Z}^a_a(\theta) = Z^a_b(\theta) \Upsilon^a_{b}(\theta). \tag{A.47}
$$

From now we are going to work with such parametrization. In this case, from the commutators (A.27), (A.33) and (A.35), we have that

$$
B^a_{\alpha a}(\theta) = -A^a_{\alpha a} + Z^\gamma_{\beta a} S^a_{\gamma \alpha}(\theta) \tag{A.48}
$$

and all the dependence $\theta^a$ is included into $S^a_{\alpha \beta}(\theta)$, which has the relation

$$
F^a_{\alpha \beta \gamma} = \sum_{\text{Cyclic}[^{\alpha \beta \gamma}]} S^a_{\alpha \beta \gamma}(0). \tag{A.50}
$$

For first step reducible algebras there are no additional algebraic quantities, but there are three relations between them that give some constraints and that correspond to their integrability conditions. These relations can also be obtained from finite relations.

The first one can be obtained by applying $\left(\frac{\partial^2}{\partial \epsilon^a \partial \epsilon^b} + \frac{\partial^2}{\partial \epsilon^a \partial \epsilon^b}\right)_{\epsilon = \epsilon' = \theta = \theta' = 0}$ to the relation (A.20). We have

$$
A^c_{\beta \gamma} Z^\gamma_a + A^c_{\alpha \gamma} Z^\gamma_a = 0. \tag{A.51}
$$
The second one comes from applying the operator \( \left( \frac{\partial^4}{\partial \epsilon^4} \gamma \theta^\epsilon \theta^\alpha \theta^\beta \theta^\gamma \theta^\rho \right) \epsilon = \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0 \) on the relation (A.21) to get

\[
A^a_{\beta\sigma} T^\beta_{\gamma\sigma} + F^a_{\beta\sigma} Z^\sigma_b + A^a_{d\beta} A^d_{\rho\gamma} - A^a_{d\gamma} A^d_{\beta\rho} = 0. \tag{A.52}
\]

And if we apply \( \sum_{P \in \text{Perm}[\alpha\beta\gamma\sigma]} (-1)^P \left( \frac{\partial^4}{\partial \theta^\alpha_1 \partial \theta^\beta_2 \partial \theta^\gamma_3 \partial \theta^\sigma_4} \right) \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0 \) to the relation (A.22) we obtain the third one

\[
2 \left( T^p_{\sigma\alpha} T^a_{\rho\beta \gamma} - T^p_{\beta\alpha} T^a_{\rho\sigma \gamma} - T^p_{\gamma\alpha} T^a_{\rho\beta \sigma} - T^p_{\sigma\gamma} T^a_{\rho\beta \alpha} \right) + 3 \left( A^a_{\beta\alpha} F^b_{\rho\beta \gamma} - A^a_{\alpha\beta} F^b_{\rho\gamma \alpha} - A^a_{\beta\gamma} F^b_{\rho\alpha \alpha} - A^a_{\rho\beta \gamma} F^b_{\alpha\alpha} \right) = 0. \tag{A.53}
\]

Observe that (A.24), (A.30), (A.40), (A.42), (A.51), (A.52) and (A.53) correspond to the classical gauge structure (2.3)-(2.9) of a closed first step reducible off-shell gauge theory.

### A.3 Lie equations

Finally, let us consider the Lie equations for our case of redundant parametrization of the Lie group. Exposing equation (A.4) to the action of \( \frac{\partial}{\partial \theta^\alpha}|_{\theta' = 0} \) we get the Lie equations on \( \mathcal{M} \),

\[
\frac{\partial F^i(\phi, \theta)}{\partial \theta^\alpha} = R^i_j(F(\phi, \theta)) \lambda^\beta_\alpha(\theta), \tag{A.54}
\]

where \( \lambda^\beta_\alpha(\theta) \) is the inverse matrix of

\[
\mu^\beta_\alpha(\theta) = \left. \frac{\partial \phi^\alpha(\theta, \theta')}{\partial \theta^\beta} \right|_{\theta' = 0}. \tag{A.55}
\]

(A.54) has the same form as in ordinary (irreducible) Lie group action. But if we consider the action of \( \mathcal{G} \) on itself given by \( \phi^\alpha(\theta, \theta') \), we get the modified Lie equations

\[
\frac{\partial^4 \phi^\alpha(\theta, \theta')}{\partial \theta^\beta \partial \theta^\gamma \partial \theta^\rho \partial \theta^\sigma} = \mu^\alpha_\gamma(\phi(\theta, \theta')) \lambda^\beta_\sigma(\theta') - Z^\alpha_\gamma(\phi(\theta, \theta')) \left( \frac{\partial \eta^\alpha(\theta, \theta', \theta'')}{\partial \theta^\rho} \right)_{\theta'' = 0} \lambda^\beta_\sigma(\theta'). \tag{A.56}
\]
B Generalization to a $L$-step Reducible Lie Group

All the treatment of the paper for a first reducible Lie group can be repeated for a more complicated reducible gauge theory. Here we are going to sketch the general framework for a closed $L$-step reducible off-shell theory.

B.1 The proper solution

Consider we have a closed $L$-step reducible off-shell theory. The field content of the quantized theory is going to be a $L+2$ tower of fields $C_{s}^{\alpha}(x)$ for $s = -1, 0, ..., L$ and with respective ghost number $s+1$ (for instance: $C_{-1}^{\alpha}(x) \equiv \phi^{i}(x)$, classical fields; $C_{0}^{\alpha}(x) \equiv \epsilon^{\alpha}$, ghosts; $C_{1}^{\alpha}(x) \equiv \eta^{\alpha}(x)$, ghosts for ghosts; etc.). The general proper solution is

$$S(\Phi, \Phi^{*}) = S_{0}(\phi) + \sum_{s=-1}^{L} C_{s,\alpha} \left( \sum_{n=1}^{s+2} F_{\beta_{1},...\beta_{n}}^{\alpha} (\phi) C_{\beta_{1}...\beta_{n}}^{\beta_{1}...\beta_{n}} \right), \quad (B.1)$$

with $0 \leq i_{k} \leq i_{k-1} \leq L$.

$F_{\beta_{1},...\beta_{n}}^{\alpha}(\phi)$ are the algebraic structure constants that, with $(S, S) = 0$ characterize the classical gauge structure. Their number is fixed by the constraint that $g_{h}(S) = 0$. This restriction gives, for a given $s$ and a given ordered set $(i_{1}, ..., i_{n})$, the condition

$$\sum_{k=1}^{n} i_{k} = s + 2 - n \equiv t. \quad (B.2)$$

The number $N(n, t)$ of functions $F_{\beta_{1},...\beta_{n}}^{\alpha}(\phi)$ for $n$ and $t$ fixed, can be determined by the recursion formula

$$N(n, t) = \sum_{k=0}^{\lfloor t/n \rfloor} N(n-1, t-nk), \quad (B.3)$$

with $N(1, t) = 1$.

B.2 Lie group description

The general equation of the reducibility $(A.8)$ is now enlarged to a set of $L$ equations

$$f_{s-2}^{\alpha_{s}} (\epsilon_{s-2}^{\beta}, \epsilon_{s-1}^{\gamma}) = f_{s-2}^{\alpha_{s}} (\epsilon_{s-2}^{\beta}, \epsilon_{s-1}^{\gamma}) \quad \Rightarrow \quad \epsilon_{s-1}^{\beta} = f_{s-1}^{\beta} (\epsilon_{s-1}^{\gamma}, \epsilon_{s}^{\alpha}), \quad (B.4)$$

with $\alpha_{s} = 1, ..., m_{s}$ ($m_{s+1} \leq m_{s}$), $s=1, ..., L$ (in our previous notation for the case of a first step reducible group we had $\epsilon_{-1}^{\alpha} = \phi^{i}, \epsilon_{0}^{\alpha} = \theta^{a}, \epsilon_{1}^{\alpha} = \epsilon^{a}, f_{-1}^{\alpha} (\epsilon, \epsilon_{0}) = F^{i}(\phi, \theta)$ and...
\( f_0^{\alpha_0}(\epsilon_0, \epsilon_1) = f^{\alpha}(\theta, \epsilon) \). Note that in general a reducible function \( f_s^{\alpha_s} \) can depend on all \( \epsilon_t^{\alpha_t} \) with \( t \leq s+1 \). For simplicity we assume that it depends only on \( \epsilon_s^{\alpha_s} \) and \( \epsilon_{s+1}^{\alpha_{s+1}} \).

From (B.3) we have the general relation

\[
\frac{\partial f_{s-2}^{\alpha_{s-2}}(\epsilon_{s-2}, \epsilon_{s-1})}{\partial \epsilon_{s-1}^{\alpha_{s-1}}} R_{s,\alpha_s}^{\gamma_{s-1}}(\epsilon_{s-1}) = 0 \quad \text{for} \quad s = 1, \ldots, L ,
\]

where

\[
R_{s,\alpha_s}^{\gamma_{s-1}}(\epsilon_{s-1}) \equiv \frac{\partial f_{s-1}^{\gamma_{s-1}}}{\partial \epsilon_s^{\alpha_s}} \bigg|_{\epsilon_s^{\beta_s}=0}.
\]

If we put \( \epsilon_s^{\alpha_s} = 0 \) for \( s \geq 0 \) we get

\[
R_{s-1,\alpha_0}^{\alpha_1}(\epsilon_{s-1}) Z_{1,\alpha_1}^{\alpha_0} = 0 \quad \iff \quad (R_{s}^{\alpha_s}(\phi) Z_{s}^{\alpha_s} = 0) \quad \text{B.7}
\]

\[
Z_{s-1,\alpha_{s-1}}^{\alpha_s} Z_{s,\alpha_s}^{\alpha_{s-1}} = 0 \quad s = 2, \ldots, L ,
\]

with \( Z_{s,\alpha_s}^{\alpha_{s-1}} \equiv R_{s,\alpha_s}^{\alpha_{s-1}}(0) \).

The parameters \( \epsilon_s^{\alpha_s} \) \( (s = 0, \ldots, L) \) belong to a manifold \( \mathcal{G}_s \) which is redundantly parametrized except for \( s=L \) \( (\mathcal{G}_L = \mathcal{G}_L) \). Each manifold \( \mathcal{G}_s \) has a structure function

\[
\varphi_s^{\alpha_s} : \quad \mathcal{G}_s \times \mathcal{G}_s \rightarrow \mathcal{G}_s
\]

\[
(\epsilon_s^{\alpha_s}, \epsilon_s^{\beta_s}) \mapsto \varphi_s^{\alpha_s}(\epsilon_s, \epsilon_s')
\]

such that

\[
f_{s-1}^{\alpha_{s-1}}(f_{s-1}^{\gamma_{s-1}}(\epsilon_{s-1}, \epsilon_s'), \epsilon_s') = f_{s-1}^{\alpha_{s-1}}(\varphi_{s-1}(\epsilon_{s-1}, \epsilon_s'), \epsilon_s).
\]

Similarly to the first reducible case, the reducible parametrizations of the manifolds \( \mathcal{G}_s \) give new structure functions. Some of these already appear in the first reducible case. This is the case of the functions \( \Sigma_{s}^{\alpha_s}(\epsilon_s, \epsilon_{s-1}, \epsilon_{s-1}') \) such that

\[
\varphi_{s-1}^{\alpha_{s-1}}(f_{s-1}(\epsilon_{s-1}, \epsilon_s), \epsilon_{s-1}') = f_{s-1}^{\alpha_{s-1}}(\varphi_{s-1}(\epsilon_{s-1}, \epsilon_{s-1}'), \Sigma_{s}(\epsilon_s, \epsilon_{s-1}, \epsilon_{s-1}')).
\]

But if \( L > 1 \) we have a richer structure. For instance, in that case, functions \( \Sigma_{s}^{\alpha_s} \) are not unique except for \( s = L \), and there are new functions \( \Pi_{s+1}^{\alpha_{s+1}} \) such that

\[
\Sigma_{s}^{\alpha_s}(f_{s}(\epsilon_s, \epsilon_{s+1}), \epsilon_{s-1}, \epsilon_{s-1}') = f_{s}^{\alpha_s}(\Sigma_{s}(\epsilon_s, \epsilon_{s-1}, \epsilon_{s-1}'), \Pi_{s+1}(\epsilon_{s+1}, \epsilon_s, \epsilon_{s-1}, \epsilon_{s-1}')).
\]

In the general case, other finite structure functions appear. Once they are found, we will get the algebraic relations of a closed \( L \)-step off-shell reducible gauge theory by differentiation. This will give us all the classical algebraic gauge structure.
B.3 Extended formalism

Consider now a closed $L$-step reducible off-shell theory described by the classical action $S_0(\phi)$ invariant under the gauge transformations $\delta \phi^i = R^i_{\alpha_0}(\phi) \varepsilon^{\alpha_0}$, $i = 1, \ldots, n$, $\alpha_0 = 1, \ldots, m_0$. The reducibility relations are given by (B.7) and (B.8).

We enlarge the theory by adding to it the gauge group parameters $\theta^{\alpha_0}$ in the classical field space. Introducing the compact notation:

$$\psi^I = (\phi^i, \varepsilon_0^\sigma) \quad I = 1, \ldots, n + m_0,$$
$$\varepsilon^{A_0} = (\varepsilon^{\alpha_0}, \varepsilon^{\alpha_1}) \quad A_0 = 1, \ldots, m_0 + m_1,$$

we can write the gauge transformations that keep $S_0(\phi)$ and $F^i(\phi, \theta)$ invariant as

$$\delta \psi^I = V^I_{A_0}(\psi) \varepsilon^{A_0},$$

with the vector fields

$$V^I_{A_0}(\psi) = \left\{ V^I_{\alpha_0} = \begin{pmatrix} R^i_{\alpha_0}(\phi) \\ -U^{\sigma}_{\alpha_0}(\theta) \end{pmatrix}, \quad V^I_{\alpha_1}(\phi, \theta) = \begin{pmatrix} 0 \\ Z^{\alpha_1}_{1,\alpha_1}(\theta) \end{pmatrix} \right\}.$$  

It is worth noting that this extension conserves the $L$-step reducible caracter of the theory. We can define the collective indices $A_s = (\alpha_s, \alpha_{s+1}) = 1, \ldots, m_s + m_{s+1}$, $s = 1, \ldots, L - 1$. Then, the $m_0 + m_1$ gauge transformations (B.12) have $m_1 + m_2$ null vectors

$$\tilde{Z}^{A_0}_{1,B_1} = \left\{ \tilde{Z}^{A_0}_{1,\beta_1} = \begin{pmatrix} Z^{\beta_1}_{1,\beta_1} \\ \delta^{\beta_1}_{\beta_1} \end{pmatrix}, \quad \tilde{Z}^{A_0}_{1,\beta_2} = \begin{pmatrix} 0 \\ Z^{\alpha_1}_{2,\alpha_2} \end{pmatrix} \right\}.$$ 

which give for the gauge generators the $m_1 + m_2$ relations of dependence

$$V^I_{A_0} \tilde{Z}^{A_0}_{1,B_1} = 0 ;$$

and also the relations

$$\tilde{Z}^{A_{s-1},A_{s-1}}_{s-1,A_s} = 0$$

among the null vectors

$$\tilde{Z}^{A_{s-1}}_{s,A_s} = \left\{ \tilde{Z}^{A_{s-1}}_{s,\alpha_s} = \begin{pmatrix} Z^{\beta_{s-1}}_{s,\alpha_s} \\ (-1)^{s-1}\delta^{\beta_{s-1}}_{\alpha_s} \end{pmatrix}, \quad \tilde{Z}^{A_{s-1}}_{s,\alpha_{s+1}} = \begin{pmatrix} 0 \\ Z^{\beta_{s-1}}_{s+1,\alpha_{s+1}} \end{pmatrix} \right\},$$

with $s = 1, \ldots, L (Z^{\beta_{L+1}}_{L+1,\alpha_{L+1}} = 0)$. 

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Finally, when we quantize the extended theory, which has the anomalous degrees of freedom as new dynamical fields, we obtain a complete new ghost structure. The whole field content of the extended theory, compared with those of the original theory, is shown in the following table:

| Ghost number | Original theory | Extended theory |
|--------------|-----------------|-----------------|
| 0            | $\phi^i$        | $\phi^i, \theta^{\alpha_0}$ |
| 1            | $c^{\alpha_0}$  | $c^{\alpha_0}, \psi^{\alpha_1}$ |
| 2            | $\eta_{1}^{\alpha_1}$ | $\eta_{1}^{\alpha_1}, \zeta_{1}^{\alpha_2}$ |
| .            | .               | .               |
| .            | .               | .               |
| .            | .               | .               |
| $s + 1$      | $\eta_{s}^{\alpha_s}$ | $\eta_{s}^{\alpha_s}, \zeta_{s}^{\alpha_{s+1}}$ |
| .            | .               | .               |
| .            | .               | .               |
| $L$          | $\eta_{L-1}^{\alpha_{L-1}}$ | $\eta_{L-1}^{\alpha_{L-1}}, \zeta_{L-1}^{\alpha_{L}}$ |
| $L + 1$      | $\eta_{L}^{\alpha_{L}}$ | $\eta_{L}^{\alpha_{L}}$ |
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