Uniform pointwise asymptotics of solutions to quasi-geostrophic equation∗

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Received 21 March 2019, revised 28 December 2019
Accepted for publication 13 February 2020
Published 14 April 2020

Abstract
We provide two-sided pointwise estimates and uniform asymptotics of the solutions to the subcritical quasi-geostrophic equation with initial data in $L^2((\alpha-1)/2\mathbb{R}^2)$, $\alpha \in (1, 2)$. Furthermore, we give an upper bound of a similar type for any derivative of the solutions. Initial data in $L^p(\mathbb{R}^2)$, $p > 2/(\alpha - 1)$, are also discussed.

Keywords: fractional Laplacian, quasi-geostrophic equation, pointwise estimates, asymptotics
Mathematics Subject Classification numbers: 35B40, 35K55, 35S10.

1. Introduction

In this paper we study the two-dimensional dissipative quasi-geostrophic equation

$$\begin{cases}
\theta_t + R^+ \theta \cdot \nabla \theta + (-\Delta)^{\alpha/2} \theta = 0, \\
\theta(0, x) = \theta_0(x),
\end{cases}
$$

(1)

in the subcritical case $\alpha \in (1, 2)$. Here, $R^+ = (-R_2, R_1)$, where $R = (R_1, R_2)$ is the two-dimensional Riesz transform given by $R_i \theta = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2} \theta$, $i \in \{1, 2\}$. Throughout the paper we assume $\alpha \in (1, 2)$ and $\theta$ is a mild solution to the initial value problem (1), that is $\theta$ satisfies the following equation

$$\theta(t, x) = P_t \theta_0(x) + \int_0^t \int_{\mathbb{R}^2} \nabla \rho_s(t-s, x-y) \cdot R^+ \theta(s, y) \theta(s, y) dy \, ds,$$

(2)

∗The paper is partially supported by the NCN grant 2015/18/E/ST1/00239
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where $P_t = e^{-t\Delta^{n/2}}$ and $p_\alpha(t,x)$ are the semigroup and the heat kernel (the fundamental solution), respectively, related to the operator $-(\Delta)^{n/2}$.

Solutions to the two-dimensional dissipative quasi-geostrophic equation model several phenomena (see [1, 2]) and have been intensively studied for more than the last two decades. In 1995, Resnick [3] proved existence of strong solutions for $\theta_0 \in L^2(\mathbb{R}^2)$ as well as the maximum principle

$$\|\theta(t,\cdot)\|_p \leq \|\theta_0\|_p,$$

where $t \geq 0$ and $1 < p \leq \infty$. This inequality has been improved in several directions by deriving a precise decay rate of $\|\theta(t,\cdot)\|_p$, see e.g. [4–10]. In [4] authors considered the initial condition $\theta_0 \in L^p(\mathbb{R}^2)$ with $p \geq \frac{2}{n+2}$ and obtained many interesting bounds for $L^p$ norms, where $q \geq p$, of mild solutions to (1). In particular, they showed that for $\theta_0 \in L_\infty^\infty(\mathbb{R}^2)$ and any multi-index $k = (k_1, k_2) \in \mathbb{N}^2$ (with $|k| := k_1 + k_2$) the derivatives $\nabla^k \theta = \frac{\partial^{|k|}}{\partial x_1^{k_1} x_2^{k_2}} \theta$ admit the following limit

$$\lim_{t \to \infty} t^{\frac{|k|}{2} - \frac{d}{4} \left(\frac{1}{n} + \frac{k}{4}\right)} \|\nabla^k \theta(t,\cdot)\|_q = 0. \quad (4)$$

Under additional assumption $\theta_0 \in L^1(\mathbb{R}^2)$, for every $\beta \in [0, \frac{1}{n})$ there is $C > 0$ such that

$$\|\nabla^k \theta(t,\cdot) - \nabla^k (P_t \theta_0)\|_q \leq C t^{-\frac{d}{2} \left(\frac{1}{n} - \frac{k}{4}\right) - \beta}. \quad (5)$$

Although all of the aforementioned results provide precise bounds for $L^p$ norms of the solutions, they do not say much about pointwise behaviour of these solutions. In particular, there are no known results on the lower bounds. In fact, this is rather a common problem in the theory of nonlinear differential equations. Nevertheless, in this paper, we solve it in the case the dissipative quasi-geostrophic equation with nonnegative $\theta_0 \in L_\infty^\infty$ by giving two-sided pointwise estimates as well as some uniform asymptotics of mild solutions. The main results of the paper are stated in the following theorems.

**Theorem 1.1.** Let $\theta_0 \in L_\infty^\infty(\mathbb{R}^2)$ be nonnegative. There is a constant $C = C(\theta_0, \alpha) > 1$ such that

$$\frac{1}{C} P_t \theta_0(x) \leq \theta(t,x) \leq C P_t \theta_0(x), \quad t > 0, \ x \in \mathbb{R}^2.$$

If we remove the nonnegativity condition, the upper bound $\theta(t,x) \leq C P_t |\theta_0|$ holds (see theorem 1.3). Note that the semigroup $P_t$ and its kernel $p_\alpha(t,x)$ are well known objects (see section 2.2 for the details).

**Theorem 1.2.** For nonnegative $\theta_0 \in L_\infty^\infty(\mathbb{R}^2)$, we have

$$\lim_{t \to 0} \frac{\|\theta(t,\cdot) - \theta_0\|_\infty}{P_t \theta_0 - 1} = \lim_{t \to \infty} \frac{\|\theta(t,\cdot) - \theta_0\|_\infty}{P_t \theta_0 - 1} = \lim_{|x| \to \infty} \sup_{t \to 0} \frac{\|\theta(t,\cdot) - \theta_0\|_\infty}{P_t \theta_0(x) - 1} = 0. \quad (5)$$

Finally, we complete these results by establishing upper bounds for derivatives of the solutions:

**Theorem 1.3.** For $\theta_0 \in L_\infty^\infty(\mathbb{R}^2)$ and any multi-index $k \in \mathbb{N} \times \mathbb{N}$, there is $C = C(\theta_0, k, \alpha) > 0$ such that

$$\|\nabla^k \theta(t,\cdot)\|_q \leq C t^{-\frac{d}{2} \left(\frac{1}{n} - \frac{k}{4}\right) - \beta}. \quad (5)$$
\[ |\nabla^k \theta(t, x)| \leq C t^{-|k|/\alpha} P_t |\theta_0|(x), \quad t > 0, \ x \in \mathbb{R}^2. \]  

Note that \( \nabla^k P_t \theta_0 \) admits the same estimate [see (9)]. It turns out that the power \( p = \frac{2}{\alpha - 1} \) in the initial condition \( \theta_0 \in L^p \) is critical in some sense. One could observe this phenomenon already in the paper [4]. Depending on whether \( p \) is greater or less than \( \frac{2}{\alpha - 1} \), different difficulties occur and different behaviour of solutions is expected. Similar situation appears in the fractal Burgers equation, which has been studied by the authors in [11, 12] in the case of (not only) critical power of the nonlinear drift term. The methods developed there have been improved and adapted to the quasi-geostrophic equation. Nevertheless, some ideas come from theory of linear perturbations of fractional Laplacian (see e.g. [13, 14]). In fact, the upper bound in (3) is concluded from [15], where also linear equations have been considered.

The paper is organised as follows. Section 2 begins with the introduction of notation used in the paper. Then, we gather some properties of the semigroup kernel \( p_{\alpha}(t, x) \) generated by \(-\Delta^{\alpha/2}\) as well as some basic facts and initial results for Riesz transform. Section 3 is devoted to estimates and asymptotics of solutions to (1), while in section 4 we prove the bounds for their derivatives.

2. Preliminaries

2.1. Notation

Throughout the paper we consider \( \alpha \in (1, 2) \). Let
\[
\nu(z) = \frac{\alpha 2^{\alpha-1} \Gamma \left(1 + \frac{\alpha}{2}\right)}{\pi \Gamma \left(1 - \frac{\alpha}{2}\right)} |z|^{-2 - \alpha}, \quad z \in \mathbb{R}^2.
\]  

For (smooth and compactly supported) test functions \( \varphi \in C^\infty_c(\mathbb{R}^2) \), we define the fractional Laplacian by
\[
\Delta^{\alpha/2} \varphi(x) := -(-\Delta)^{\alpha/2} \varphi(x) = \lim_{\varepsilon \to 0} \int_{|x|>|x|+\varepsilon} [\varphi(x+z) - \varphi(x)] \nu(z) \, dz, \quad x \in \mathbb{R}^2.
\]

In terms of the Fourier transform, \( \hat{\Delta}^{\alpha/2} \varphi(\xi) = -|\xi|^\alpha \hat{\varphi}(\xi) \). Denote by \( p_{\alpha}(t, x) \) the fundamental solution to the equation \( \partial_t u = \Delta^{\alpha/2} u \), that is \( p_{\alpha}(t, x) \) solves
\[
\begin{cases}
\partial_t u = \Delta^{\alpha/2} u, & t > 0, \ x \in \mathbb{R}^2, \\
u(0, x) = \delta_0(x), & x \in \mathbb{R}^2.
\end{cases}
\]

By \( P_t \) we denote the stable semigroup operator,
\[
P_t f(x) = \left( e^{t \Delta^{\alpha/2} } \right)(x) = \int_{\mathbb{R}^2} p_{\alpha}(t, x - y) f(y) \, dy, \quad t > 0, \ x \in \mathbb{R}^2.
\]

The name ‘stable’ comes from the \( \alpha \)-stable process, which is generated by \( \Delta^{\alpha/2} \) and the semigroup \( P_t \) describes its transition probabilities (see, e.g. [16, 17]).
We write $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$ for the standard two-dimensional gradient operator. Furthermore, for any multi-index $k = (k_1, k_2) \in \mathbb{N}^2$ we denote

$$\nabla^k f(x) = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2}} f(x), \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where $|k| = k_1 + k_2$.

By $B(x, r)$ we denote the open ball with centre $x \in \mathbb{R}^d$ and radius $r > 0$. Also, we follow the notation $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$.

We write $f \approx g$ ($f \lesssim g$ respectively) for $f, g \geq 0$ whenever there is a constant $c = c(\alpha, \theta_0) \geq 1$ such that $c^{-1} f \leq g \leq c f$ ($f \leq cg$ respectively) on their common domain. The constants $c, C, c_i$, whose exact values are unimportant, may change in each statement and proof.

Finally, we write $B(a, b)$ for the classical beta function, i.e.

$$B(a, b) = \int_0^1 u^{a-1} (1 - u)^{b-1} du, \quad a, b > 0.$$

### 2.2. Stable semigroup

In this section we recall some results on the stable semigroup $P_t$, and derive some new properties that are needed in the sequel. It is well known that the semigroup kernel $p_\alpha(\cdot, \cdot) \in C^\infty((0, \infty) \times \mathbb{R}^2)$ and it is radial in space, i.e. $p_\alpha(t, x) = p_\alpha(t, y)$ for any $t > 0$ and $x, y \in \mathbb{R}^2$ such that $|x| = |y|$. It also enjoys the following scaling and semigroup properties

$$p_\alpha(t, x) = t^{-2/\alpha} p_\alpha(1, t^{-1/\alpha} x), \quad t > 0, \quad x \in \mathbb{R}^2,$$

$$p_\alpha(t, x) = \int_{\mathbb{R}^2} p_\alpha(t - s, x - z)p_\alpha(s, z) dz, \quad t > s > 0, \quad x \in \mathbb{R}^2,$$

as well as pointwise estimates

$$p_\alpha(t, x) \approx \frac{t}{(t^{1/\alpha} + |x|)^{2+\alpha}} \approx t^{-2/\alpha} \wedge \frac{t}{|x|^{2+\alpha}}, \quad t > 0, \quad x \in \mathbb{R}^2. \quad (8)$$

By scaling property and ([18], lemma 3.1) (see also [19, 20] for more general setting),

$$|\nabla^k p_\alpha(t, x)| \leq c_k t^{-\frac{k}{\alpha}} p_\alpha(t, x), \quad t > 0, \quad x \in \mathbb{R}^2. \quad (9)$$

From (9), we easily get the $L^p$-estimates:

$$\|\nabla^k p_\alpha(t, \cdot)\|_p \lesssim t^{-\frac{k}{\alpha}} \wedge \frac{t^{\frac{1}{\alpha}}}{|x|^\frac{2+\alpha}{\alpha}}. \quad (10)$$

Furthermore, for $f \in L^{\frac{2+\alpha}{\alpha}}(\mathbb{R}^2)$ and $p \in \left[ \frac{2}{\alpha+1}, \infty \right]$, the following estimate for stable semigroup holds ([21]),

$$\|P_t f\|_p \lesssim t^{-\frac{(\alpha-1)}{\alpha}} + \frac{1}{p} \|f\|_p \quad (11)$$

In the lemma below, we note some additional decay properties.
**Lemma 2.1.** For $f \in L^{\frac{2}{1+\alpha}}(\mathbb{R}^2)$, we have

$$\lim_{t \to 0} \|t^{\frac{1}{\alpha - 1}} P_t f\|_\infty = 0, \quad (12)$$

$$\lim_{t \to \infty} \|t^{\frac{1}{\alpha - 1}} P_t f\|_\infty = 0, \quad (13)$$

$$\lim_{|x| \to \infty} \sup_{t > 0} \left| t^{\frac{1}{\alpha - 1}} P_t f(x) \right| = 0. \quad (14)$$

**Proof.** The limit (12) follows from [14], (2.2). Next, for every $\epsilon > 0$ there is $R > 0$ such that $\|f 1_{B(0,R)}\|_\infty < \epsilon$. By Young inequality and (9),

$$\|P_t(f 1_{B(0,R')})\|_\infty \leq \|p_\alpha(t, \cdot)\|_2 \|f 1_{B(0,R)}\|_2 \leq c_1 t^{1 - \frac{1}{\alpha}} \epsilon. \quad (10)$$

Hence,

$$\|t^{\frac{1}{\alpha - 1}} P_t f\|_\infty \leq t^{\frac{1}{\alpha - 1}} \left(\|P_t(f 1_{B(0,R)})\|_\infty + \|P_t(f 1_{B(0,R')})\|_\infty\right) \leq c_1 t^{\frac{1}{\alpha - 1}} \left(\|p_\alpha(t, \cdot)\|_2 \|f\|_2 + t^{\frac{1}{\alpha - 1}} \epsilon\right) \leq c_2 \left(t^{\frac{1}{\alpha - 1}} + \epsilon\right),$$

which yields (13). Finally, for $|x| > 2R$ and $|y| < R$, by (8), we have $p_\alpha(t, x - y) \lesssim t^{1 - \frac{1}{\alpha}} |x - y|^{\alpha - 3} \lesssim t^{1 - \frac{1}{\alpha}} |x|^{\alpha - 3}$. Therefore, for $|x| > 2R$,

$$\sup_{t > 0} \left| t^{\frac{1}{\alpha - 1}} P_t f(x) \right| \leq \sup_{t > 0} t^{\frac{1}{\alpha - 1}} \left(\|P_t(f 1_{B(0,R)})\|_\infty + \|P_t(f 1_{B(0,R')})\|_\infty\right) \lesssim |x|^{\alpha - 3} \|f 1_{B(0,R)}\|_1 + \epsilon \lesssim |x|^{\alpha - 3} \|f\|_\infty + \epsilon$$

and (14) holds. $\square$

Finally, we show that if $t$ is bounded and separated from zero, than $P_t f(x)$ admits the same lower bound as $p_\alpha(t, x)$.

**Lemma 2.2.** Let $0 < t_1 < t_2 < \infty$ and $f \in L^{\frac{2}{1+\alpha}}(\mathbb{R}^2)$. If $\|f\|_\infty > 0$, then there exists a constant $C = C(t_1, t_2, \theta_0)$ such that

$$P_t f(x) \geq \frac{C}{(1 + |x|)^{2+\alpha}}, \quad t_2 > t_1, \ x \in \mathbb{R}^2.$$

**Proof.** Since $f \in L^{\frac{2}{1+\alpha}}$, then $f \in L^1_{loc}$. Hence, there is $R > 0$ such that $C < \int_{B(0,R)} |f(y)| dy < \infty$ for some $c > 0$. Consequently, using (8), we get...
\[
P_t |f(x)| \geq \int_{B(0,R)} p_t(t,x-y)|f(y)|dy \geq c_1 \frac{t}{(t^{4/\alpha} + 2R + |x|)^{2+\alpha}} \int_{B(0,R)} |f(y)|dy \\
\geq c c_1 \frac{t}{(2t^{4/\alpha} + 2R + |x|)^{2+\alpha}} \\
\geq C (1 + |x|)^{2+\alpha}.
\]

\[\square\]

### 2.3. Riesz transform

Let \( R = (R_1, R_2) \) be the two-dimensional Riesz transform, i.e.

\[
R_i f(x) = c \text{ P.V.} \int_{\mathbb{R}^2} \frac{y_i}{|y|^3} f(x-y) dy, \quad x = (x_1, x_2) \in \mathbb{R}^2,
\]

where \( c \) is some constant and P.V. denotes the principal value of the integral. Next, we denote \( R^\perp = (-R_2, R_1) \). It is clear that \( |R^\perp f| = |Rf| \). It is well known that the Riesz transform is continuous on \( L^p \) for \( p \in (1, \infty) \), i.e. for \( f \in L^p \) we have [see e.g. ([22], corollary 4.8)]

\[
\|Rf\|_p \leq c_p \|f\|_p.
\]

(15)

In particular, taking \( \nabla^k p_\alpha(t, \cdot) \) as \( f \), (10) gives us

\[
\|R^\perp \nabla^k p_\alpha(t, \cdot)\|_p \leq c_{p,k} t^{\frac{\alpha}{4} \left(1 - \frac{3}{2}\right)}, \quad 1 < p < \infty, \ t > 0.
\]

(16)

The next proposition not only shows that the above bound holds for \( p = \infty \), but also improves it by providing a pointwise estimate with some dependence on the space argument.

**Proposition 2.3.** For any multi-index \( k \in \mathbb{N} \times \mathbb{N} \) there is a constant \( C > 0 \) such that

\[
|R^\perp \nabla^k p_\alpha(t, x)| \leq C t^{-\frac{3k}{2}} \frac{1}{(t^{1/\alpha} + |x|)^2}, \quad t > 0, \ x \in \mathbb{R}^2.
\]

(17)

**Proof.** It is easy to see that both sides of (17) admit the scaling property \( f(t,x) = t^{-2+3k/\alpha} f(1, t^{-1/\alpha}x) \). Hence, it is enough to consider \( t = 1 \). First, let us write

\[
|R^\perp \nabla^k p_\alpha(1, x)| = c \left| \text{ P.V.} \int_{\mathbb{R}^2} \frac{y_i}{|y|^3} \nabla^k p_\alpha(1, x-y)dy \right| \\
\leq c \left| \text{ P.V.} \int_{|y| \leq 1} \frac{y_i}{|y|^3} \nabla^k p_\alpha(1, x-y)dy \right| + c \int_{|y| > 1} \frac{1}{|y|^2} |\nabla^k p_\alpha(1, x-y)| dy.
\]

It follows from (9) that
\[
\sup_{|w| \leq 1} \left| \nabla \left( \nabla^k p_n(1, x + w) \right) \right| \lesssim t^{-(k+1)/\alpha} p_n(t, x).
\]

Hence, since
\[
P. V. \int_{|y| \leq 1} \frac{y}{|y|^2} \nabla^k p_n(1, x) \, dy = 0,
\]
the mean value theorem gives us
\[
\left| P. V. \int_{|y| \leq 1} \frac{y}{|y|^2} \nabla^k p_n(1, x - y) \, dy \right| = \left| P. V. \int_{|y| \leq 1} \frac{y}{|y|^2} \left( \nabla^k p_n(1, x - y) - \nabla^k p_n(1, x) \right) \, dy \right|
\leq \int_{|y| \leq 1} \frac{|y|^2}{|y|^4} \sup_{|y| \leq 1} \left| \nabla^k p_n(1, x + w) \right| \, dy
\leq p_n(1, x) \lesssim \frac{1}{x^2 + 1}.
\]

Next,
\[
\int_{|y| > |x|} \frac{1}{|y|^2} \left| \nabla^k p_n(1, x - y) \right| \, dy \leq \int_{|y| > |x|} \frac{1}{1 + |x|^2} p_n(1, x - y) \, dy \leq \frac{1}{1 + |x|^2},
\]
which gives (17) for $|x| \leq 1$. Finally, for $1 < |y| \leq |x|$, we have $\frac{1 + |y|^{2 + n}}{|y|^2} \leq 2 |y|^n \leq 2 |x|^n \leq 2^{2 + |x|^2 / |y|^2}$, which yields $\frac{1}{|y|^2} \lesssim \frac{p_n(1, y)}{p_n(2, x)}$. Thus,
\[
\int_{1 < |y| \leq |x|} \frac{1}{|y|^2} \left| \nabla^k p_n(1, x - y) \right| \, dy \lesssim \int_{1 < |y| \leq |x|} \frac{1}{1 + |x|^2} \frac{p_n(1, y) p_n(1, x - y)}{p_n(2, x)} \, dy = \frac{1}{1 + |x|^2}.
\]

**Proposition 2.4.** For every $k \in \mathbb{N}^2$ there is a constant $C_k > 0$ such that for all $t > 0$,
\[
\| \nabla^k R^t P_t \varphi \|_\infty \leq C_k t^{-\frac{k + 1}{n}} \| \varphi \|_{L_x^2}^2, \quad \varphi \in L_x^2(\mathbb{R}^2).
\]

Furthermore,
\[
\lim_{t \to 0} \| R^t P_t \varphi \|_\infty = \lim_{t \to \infty} \| R^t P_t \varphi \|_\infty = \lim_{|x| \to \infty} \sup_{r > 0} \| R^t P_t \varphi(x) \|_\infty = 0.
\]

**Proof.** First, (16) gives us for $i \in \{1, 2\}$
\[ |\nabla^k R_i P_t \phi(x)| = \left| \nabla^k \int_{\mathbb{R}^2} p_o(t,x-y) R_i \phi(y) \, dy \right| \]
\[ \leq \int_{\mathbb{R}^2} |\nabla^k p_o(t,x-y) R_i \phi| \, dy \]
\[ \leq \| \nabla^k p_o(t,\cdot) \| \frac{1}{\sqrt{t}} \| R_i \phi \| \frac{1}{\sqrt{t}} \leq t^{-\frac{(K+\alpha-1)}{\alpha}}, \]
which implies the inequality (18). Let us fix \( \varepsilon > 0 \). There are \( M_\varepsilon > 0 \) and \( R_\varepsilon \) such that \( \| \phi 1_{\{|x| > M_\varepsilon\}} \| \frac{1}{\sqrt{t}} \leq \varepsilon \) and \( \| \phi 1_{R_0 R_0,0} \| \frac{1}{\sqrt{t}} \leq \varepsilon \). Hence, by (16), we get
\[ \int_{|x| > M_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy \leq \| p_o(t,\cdot) \| \frac{1}{\sqrt{t}} \left( \int_{|x| > M_\varepsilon} \| \phi(y) \| \frac{1}{\sqrt{t}} \, dy \right)^\frac{\alpha}{\alpha-1} \leq \varepsilon t^{-\frac{\alpha}{\alpha-1}}, \quad (19) \]
\[ \int_{|x| > R_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy \leq \| p_o(t,\cdot) \| \frac{1}{\sqrt{t}} \left( \int_{|x| > R_\varepsilon} \| \phi(y) \| \frac{1}{\sqrt{t}} \, dy \right)^\frac{\alpha}{\alpha-1} \leq \varepsilon t^{-\frac{\alpha}{\alpha-1}}. \quad (20) \]
Next, using (19) and (16), we obtain
\[ |R_i P_t \phi(x)| = \left| \int_{\mathbb{R}^2} R_i p_o(t,x-y) \phi(y) \, dy \right| \]
\[ \leq \sqrt{M_\varepsilon} \left( \int_{|x| \leq M_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy + \int_{|x| > M_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy \right) \]
\[ \leq \sqrt{M_\varepsilon} |R_i p_o(t,\cdot) \| \frac{1}{\sqrt{t}} \| \phi \| \frac{1}{\sqrt{t}} + \varepsilon t^{-\frac{\alpha}{\alpha-1}} \]
\[ \leq \sqrt{M_\varepsilon} t^{-\frac{\alpha}{\alpha-1}} + \varepsilon t^{-\frac{\alpha}{\alpha-1}}, \]
and consequently \( \| t^{\frac{\alpha}{\alpha-1}} R_i P_t \phi \| \leq \sqrt{M_\varepsilon} t^{-\frac{\alpha}{\alpha-1}} + \varepsilon \), which proves the first limit from the assertion.

Next, combining (10), (17), (19) and (20), we get
\[ |R_i P_t \phi| = \left| \int_{\mathbb{R}^2} R_i p_o(t,x-y) \phi(y) \, dy \right| \]
\[ \leq M_\varepsilon \int_{|x| \leq M_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy + \int_{|x| > M_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy \]
\[ + \int_{|x| > R_\varepsilon} |R_i p_o(t,x-y) \phi(y)| \, dy \]
\[ \leq M_\varepsilon R_\varepsilon^2 t^{-2/\alpha} + \varepsilon t^{-\frac{\alpha}{\alpha-1}}, \]
which lets us conclude \( \lim_{t \to \infty} \| t^{\frac{\alpha}{\alpha-1}} R_i P_t \phi \| = 0 \). By virtue of the previous two limits, it is enough to prove that for any \( 0 < t_1 < t_2 < \infty \)
\[ \lim_{|x| \to \infty} \text{sup}_{t \in (t_1, t_2)} |R_i P_t \phi(x)| = 0. \]

By (15), (17), (20) and Hölder inequality, we get for \(|x| > R_\varepsilon \) and \( t \in (t_1, t_2) \),
\[ |R_p \varphi(x)| = \left| \int_{\mathbb{R}^2} R_p \varphi(t, x - y) \varphi(y) \, dy \right| \]
\[ \lesssim \int_{|y| > R_c} |R_p \varphi(t, x - y) \varphi(y)| \, dy + \int_{|y| \leq R_c} \frac{1}{(1 + |x - y|)^2} \varphi(y) \, dy \]
\[ \lesssim e t^{\frac{\alpha - 1}{\alpha} + \frac{1}{2}} + \frac{1}{(t \theta - |x|)^2} \left( \int_{\mathbb{R}^2} \mathbb{1}_{|y| \leq R_c} \, dy \right)^{\frac{1}{2}} \| \varphi \|_{L^\infty}^{\frac{1}{2}} \]
which is arbitrarily small for large \(|x|\). This proves the last assertion. \(\square\)

3. Asymptotics and estimates of solutions

First, we recall some results from [4] concerning \(L^p\) estimates of the solutions to (1). We assume below that \(\theta_0 \in L^{\frac{2}{\alpha - 1}}\). For \(p \in \left[ \frac{2}{\alpha - 1}, \infty \right)\), we have [see ([4], proposition 3.2)]

\[ i^{\frac{\alpha - 1}{2} + \frac{1}{2}} \varphi \nabla^k \theta \in C_0((0, \infty), L^p(\mathbb{R}^2)), \quad (21) \]

where \(C_0((0, \infty), L^p(\mathbb{R}^2))\) denotes the space of bounded and continuous functions from the half-line \((0, \infty)\) into the space \(L^p(\mathbb{R}^2)\). In particular, for \(p \in \left[ \frac{2}{\alpha - 1}, \infty \right)\),

\[ \| \theta(t, \cdot) \|_p \lesssim t^{-\frac{\alpha - 1}{2} + \frac{1}{2p}}, \quad t > 0. \quad (22) \]

Combining this with (15), for \(p \in \left[ \frac{2}{\alpha - 1}, \infty \right)\), we get

\[ \| R^+ \theta(t, \cdot) \|_p \lesssim t^{-\frac{\alpha - 1}{2} + \frac{1}{2p}}, \quad t > 0. \quad (23) \]

The following technical lemma will be needed in the sequel.

**Lemma 3.1.** Let \(p \geq \frac{2}{\alpha - 1}\) and \(q = \frac{p}{\alpha - 1}\). Assume that \(f(s, \cdot) \in L^q(\mathbb{R}^2)\) and \(g(s, \cdot) \in L^q(\mathbb{R}^2)\) satisfy

\[ \| f(s, \cdot) \|_q \leq c_1 s^{-\frac{1}{2} + \frac{1}{2p}}, \quad \| g(s, \cdot) \|_p \leq c_2 s^{-\frac{1}{2} + \frac{1}{2p}}. \]

Then there is a constant \(C\) such that for \(p \geq \frac{2}{\alpha - 1}\)

\[ \int_{\mathbb{R}^2} |f(t - s, x - y)||g(s, y)| \, dy \leq C(t - s)^{-\left(\frac{1}{\alpha} + \frac{1}{p}\right)} s^{-\frac{\alpha - 1}{\alpha} + \frac{1}{2p}}, \quad 0 < s < t, x \in \mathbb{R}^2. \quad (24) \]

Furthermore, for \(t > 0, x \in \mathbb{R}^2\) and \(p > \frac{2}{\alpha - 1}\), we have

\[ \int_{0}^{t} \int_{\mathbb{R}^2} |f(t - s, x - y)||g(s, y)| s^{-\frac{\alpha - 1}{2}} \, dy \, ds \leq C B \left( \frac{p(\alpha - 1) - 2}{p \alpha}, \frac{p(2 - \alpha) + 2 + p}{\alpha p} \right) t^{-\frac{\alpha - 1}{\alpha}}. \quad (25) \]
Proof. By Hölder inequality,
\[ \int_{\mathbb{R}^2} |f(t-s, x-y)||g(s,y)| \, dy \leq \| f(t-s, \cdot) \|_{p^*} \| g(s, \cdot) \|_p \leq c_1 c_2 (t-s)^{\frac{1}{q}-\frac{1}{mp} s^{-\frac{a-1}{\alpha}} + \frac{2}{p}}, \]
which gives (24). Furthermore, this implies
\[ \int_0^t \int_{\mathbb{R}^2} |f(t-s, x-y)||g(s,y)| \, dy \, s^{-\frac{a-1}{\alpha}} \, ds \leq c \int_0^t (t-s)^{\frac{1}{q}-\frac{1}{mp} s^{-\frac{2a-2}{p}\alpha} + \frac{2}{p}} \, ds 
= c t^{\frac{a-1}{\alpha}} \int_0^1 (1-u)^{\frac{1}{q}-\frac{1}{mp} u^{-\frac{2a-2}{p}\alpha} + \frac{2}{p}} \, du 
= c B \left( p(\alpha-1) - 2, \frac{2(\alpha-1)}{p}; \frac{2}{p} \right) t^{\frac{a-1}{\alpha}}. \]

The following corollary is an immediate consequence of lemma 3.1.

Corollary 3.2. Let \( \theta \) be a solution to (1) with \( \theta_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R}^2) \). For every \( t > 0 \), we have
\[ \int_0^t \int_{\mathbb{R}^2} |R^\perp \nabla \theta_p(t-s, x-y)| \, |R^\perp \theta(s,y)| \, s^{-\frac{a-1}{\alpha}} \, dy \, ds \leq Ct^{\frac{a-1}{\alpha}}. \]  

Proof. Both of the bounds follow from (10), (15), (21) and (23) applied to (25).

In the subsequent proposition we show that the range of \( p \) in estimate (23) may be extended to \((1, \infty)\).

Proposition 3.3. Assume \( \theta_0 \in L^{\frac{1}{\alpha-1}}(\mathbb{R}^2) \). There is a constant \( C > 0 \) such that
\[ \| R^\perp \theta(t, \cdot) \|_\infty \leq Ct^{\frac{a-1}{\alpha}}. \]

Proof. For \( i = 1, 2 \) we rewrite \( R_i \theta(t,x) \) using (2) as
\[ R_i \theta(t,x) = R_i P_i \theta_0(x) + \int_0^t \int_{\mathbb{R}^2} R_i \nabla \theta_p(t-s, x-y) \cdot R^\perp \theta(s,y) \theta(s,y) \, dy \, ds. \]
By proposition 2.4, we have \( \| R_i P_i \theta_0 \|_\infty \leq c t^{\frac{a-1}{\alpha}} \) and the assertion follows from (22) and (26).

Now, we pass to the proof of pointwise upper bounds for solutions to (1). First, let us introduce several function spaces that will appear in the proof of the next theorem. By \( L^{p^\lambda}(\mathbb{R}^2) \) we denote the Morrey space, i.e.
\[ L^{p^\lambda}(\mathbb{R}^2) = \left\{ f \in L^p(\mathbb{R}^2); \| f \|_{L^p^{\lambda}} := \sup_{\lambda > 0} \sup_{x \in \mathbb{R}^2} \int_{B(x, r \lambda)} |f(z)|^p \, dz < \infty \right\}. \]
The Morrey space is a Banach space with the norm \( \| f \|_{L^p^{\lambda}} \). For any Banach space \( X \) equipped with the norm \( \| \cdot \|_X \) we denote by \( L^{p^\lambda}(0, \infty); X \) the space of functions \( f: (0, \infty) \to X \) such that
Finally, we define 
\[ \|f\|_{L^p,(0,\infty)} := \sup_{0<s<t<\infty} \left( \frac{1}{t-s} \int_s^t |f(r)|^p \, dr \right)^{1/p} < \infty. \]

It is also a Banach space with the norm \( \|f\|_{L^p,(0,\infty)} \). As a space \( X \) we will be considering the Campanato space \( L^{p,\lambda}(\mathbb{R}^2) \) defined by

\[ L^{p,\lambda}(\mathbb{R}^2) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^2): \right\}
\]

\[ \|f\|_{L^p,\lambda}(\mathbb{R}^2) := \sup_{x \in \mathbb{R}^2} \left( \frac{1}{|B(x,\lambda)|} \int_{B(x,\lambda)} |f(y) - f(x)|^p \, dy \right)^{1/p} < \infty \}

as well as the space

\[ L^{1,\text{loc}}(\mathbb{R}^2) := \left\{ f \in L^1(\mathbb{R}^2): \right\}
\]

\[ \|f\|_{L^1,\text{loc}} := \sup_{x \in \mathbb{R}^2} \int_{B(x,1)} |f(z)| \, dz < \infty \}

Finally, we define

\[ L^\infty_{\text{loc}}((0,\infty);L^{1,\text{loc}}(\mathbb{R}^2)) := \left\{ f \in L^1_{\text{loc}}(0,\infty) \times \mathbb{R}^2): \sup_{t \in (0,R)} \int_{B(0,t^2)} |f(z)| \, dz < \infty \quad \text{for all} \quad R > 0 \right\} \]

**Lemma 3.4.** Let \( \theta_0 \in L^{\frac{\lambda}{n}}(\mathbb{R}^2) \). There is a constant \( C > 0 \) such that for all \( t > 0 \) and \( x \in \mathbb{R}^2 \), we have

\[ \theta(t,x) \leq C \theta_0(x). \]  

**Proof.** Let \( v = R^{-\theta} \) and consider the linear equation

\[ \partial_t u = \Delta^{\alpha/2} u + v \cdot \nabla u. \]

By ([15], corollary 1.4), the fundamental solution \( \bar{p}(t,x,y) \) of (29) is bounded by \( p_\alpha(t,x-y) \), that is

\[ \bar{p}(t,x,y) \leq c p_\alpha(t,x-y), \quad t > 0, x, y \in \mathbb{R}^2. \]

Indeed, taking \( \lambda = \frac{2\alpha}{n} \) and \( q = \infty \) in ([15], corollary 1.4), we only need to show that all required assumptions are satisfied, i.e. \( \nabla v = 0 \) and

\[ v \in L^{\frac{\lambda}{2}+\frac{1}{2}}((0,\infty);L^{\frac{\lambda}{2}-\frac{1}{2}}(\mathbb{R}^2)), \]

\[ v \in L^\infty((0,\infty);L^{1,\text{loc}}(\mathbb{R}^2)), \]

\[ v \in L^{1,\text{loc}}((0,\infty);L^{1,\text{loc}}(\mathbb{R}^2)). \]

Since \( \lambda = \frac{2\alpha}{n} < 2 \) for \( \alpha > 1 \), the Campanato space \( L^{\frac{\lambda}{2}-\frac{1}{2}}(\mathbb{R}^2) \) reduces to the Morrey space \( L^{\frac{\lambda}{2}}(\mathbb{R}^2) \), see, e.g. [23]. Clearly, we have \( \nabla v = 0 \). Furthermore, by (15), (22) and Hölder inequality,
\[ \| R^+ \theta(u, \cdot) \|_{L^2_{t\leq \theta_0}(\mathbb{R}^2)} \leq \sup_{x \in \mathbb{R}^2 \setminus \mathcal{J}^c \setminus 0} \left( \frac{2^{2j}}{\alpha} \int_{B_{2R}(x)} |R\theta(t, z)|^{\frac{2}{\alpha}} \, dz \right)^{\frac{1}{2}} \]
\[ \leq \sup_{x \in \mathbb{R}^2 \setminus \mathcal{J}^c \setminus 0} \left( \frac{2^{2j}}{\alpha} \left( \int_{B_{2R}(x)} |R\theta(t, z)| \, dz \right)^{\frac{2}{\alpha}} \left( \int_{\mathbb{R}^2} |R\theta(t, z)|^{\frac{2}{\alpha - 1}} \, dz \right)^{\frac{2}{\alpha}} \right)^{\frac{1}{2}} \]
\[ \leq c \| \theta_0 \|^{-\frac{1}{\alpha}}. \]

Hence,
\[ \| v \|_{L^2_{t \leq \frac{T}{2}}(0, \infty)}^{\frac{1}{\alpha}} = \sup_{0 < s < t} \left( (t - s)^{-1/\alpha} \int_s^t \| R^+ \theta(u, \cdot) \|_{L^2_{t \leq \theta_0}(R^2)}^2 \, du \right)^{\frac{1}{2}} \leq c \| \theta_0 \|^{-\frac{1}{\alpha}}, \]

which gives (31). Next, (32) is an immediate consequence of (27). Finally, also by (27), we have
\[ \| R^+ \theta(t, \cdot) \|_{L^2_{t \leq \theta_0}(\mathbb{R}^2)} = \sup_{x \in \mathbb{R}^2} \int_{B_{2R}(x)} |R^+ \theta(t, y)| \, dy \lesssim \| R^+ \theta \|_{\infty} \lesssim t^{-\alpha/\alpha}. \]

Consequently,
\[ \sup_{t > 0} \sup_{0 < s < t} \left( (t - s)^{-1/\alpha} \int_s^t \| R^+ \theta(u, \cdot) \|_{L^2_{t \leq \theta_0}(R^2)}^2 \, du \right) \]
\[ \leq \sup_{t > 0} \sup_{0 < s < t} \left( (t - s)^{-1/\alpha} \int_0^{t - s} u^{-\alpha/\alpha} \, du \right) \leq \alpha, \]

which yields (33). Now consider (29) with initial condition \( u_0 = \theta_0 \). Clearly,
\[ \theta(t, x) = \int_{\mathbb{R}^2} \tilde{p}(t, x, y)\theta_0(y) \, dy \]
is a solution to this problem and (30) gives us
\[ |\theta(t, x)| \leq \int_{\mathbb{R}^2} \tilde{p}(t, x, y)|\theta_0(y)| \, dy \leq c \int_{\mathbb{R}^2} p_n(t, x - y)|\theta_0(y)| \, dy = c P_t|\theta_0|(x). \]
The proof is complete. \( \square \)

**Proposition 3.5.** Assume \( \theta_0 \in L^{\frac{2}{2\alpha - 1}}(\mathbb{R}^2) \). We have
\[ \lim_{t \to 0} \| R^\dagger \theta(t, \cdot) \|_{\infty} = \lim_{t \to \infty} \| R^\dagger \theta(t, \cdot) \|_{\infty} = \lim_{|x| \to \infty} \sup_{t \geq 0} \| R^\dagger \theta(t, x) \| = 0. \]

**Proof.** We will use the integral form of the solution from (2). The required results for the term \( R^P \theta_0(x) \) have been provided in proposition 2.4, so what has left is to deal with the integral term. Formulas (28) and (12) ensure that for every \( \delta > 0 \) there are \( t_{0\delta}, T_{0\delta} > 0 \) such that \( \| \theta(s, \cdot) \|_{\infty} < \delta^{(\alpha - 1)/\alpha} \) for \( s < t_{0\delta} \) or \( s > T_{0\delta} \). We fix some \( p > \frac{2}{\alpha - 1} \).
Consequently, by (26),
\[
\left| \int_0^t \int_{\mathbb{R}^2} R_i \nabla p_n(t-s,x-y) \cdot R^i \theta_\epsilon(s,y) \theta(s,y) dy ds \right| \leq \delta c t^{-\frac{n+\epsilon}{n}}, \quad x \in \mathbb{R}^2, \ t \leq t_\delta,
\]
which gives the first limit in (34). Now, let \( t > 2T_\delta \). By (26), we get
\[
\left| \int_{T_\delta}^t \int_{\mathbb{R}^2} R_i \nabla p_n(t-s,x-y) \cdot R^i \theta_\epsilon(s,y) \theta(s,y) dy ds \right| \leq \delta c t^{-\frac{n+\epsilon}{n}}, \quad x \in \mathbb{R}^2, \ t > 2T_\delta.
\]
Next, by (24) (with \( f = R_i \nabla p \) and \( g = \theta R^i \theta \)) and (22),
\[
\left| \int_0^{T_\delta} \int_{\mathbb{R}^2} R_i \nabla p_n(t-s,x-y) \cdot R^i \theta_\epsilon(s,y) \theta(s,y) dy ds \right| \lesssim \int_0^{T_\delta} (t-s)^{-\frac{1}{n} - \frac{\epsilon}{n} + \frac{2}{np}} s^{-\frac{2}{np} + \frac{1}{np}} ds \\
\lesssim t^{-\frac{1}{n} - \frac{\epsilon}{n} + \frac{2}{np}} \int_0^{T_\delta} s^{-\frac{2}{np} + \frac{1}{np}} ds \\
= ce t^{-\frac{n+\epsilon}{n}}.
\]
This proves the second limit in (34). Finally, we deal with \( \lim_{|x| \to \infty} \sup_{t \geq 0} |\theta_\epsilon R(t,x)| = 0 \).

By (28) and (14), for every \( \epsilon \in (0,1) \) there exists \( r_\epsilon \) such that \( \sup_{t \geq 0} |\theta_\epsilon R(s,y)| < \epsilon \) for \( |y| > r_\epsilon \). Then, by (26),
\[
\left| \int_0^t \int_{B(0,r_\epsilon)^c} R_i \nabla p_n(t-s,x-y) \cdot R^i \theta_\epsilon(s,y) \theta(s,y) dy ds \right| \leq \epsilon c t^{-\frac{n+\epsilon}{n}}.
\]
Furthermore, by (17),
\[
|R_i \nabla p_n(t-s,x-y)| \leq c(t-s)^{-\frac{1}{n}} |x-y|^{-2} < \epsilon r_\epsilon^{-2}(t-s)^{-\frac{1}{2}}
\]
for \( y \in B(0,r_\epsilon) \) and \( |x| \) sufficiently large. Hence, by (22) and (27), we get
\[
\left| \int_0^t \int_{B(0,r_\epsilon)} R_i \nabla p_n(t-s,x-y) \cdot R^i \theta_\epsilon(s,y) \theta(s,y) dy ds \right| \leq \epsilon r_\epsilon^{-2} \int_0^t \int_{B(0,r_\epsilon)} (t-s)^{-\frac{1}{n} + \frac{2}{np}} s^{-\frac{n+\epsilon}{np}} dy ds \\
\leq c \epsilon c t^{-\frac{n+\epsilon}{n}}.
\]
which ends the proof.

**Proof of Theorem 1.2.** First, observe that by (28) and semigroup property of \( p_n(t,x) \),
\[
\left| \int_0^t \int_{\mathbb{R}^2} \nabla p_n(t-s,x-y) \cdot R_i \theta_\epsilon(s,y) \theta(s,y) dy ds \right| \\
\lesssim \int_0^t (t-s)^{-\frac{1}{n}} \| R\theta(s,\cdot) \|_\infty \int_{\mathbb{R}^2} p_n(t-s,x-y) P_e \theta_\epsilon(y) dy ds \\
= P_e \theta_\epsilon(x) \int_0^t (t-s)^{-\frac{1}{n}} \| R\theta(s,\cdot) \|_\infty ds.
\]

By virtue of proposition 3.5, for every $\epsilon > 0$ there are $t_\varepsilon > 0$ and $T_\varepsilon$ such that $\|R^1 \theta(t, \cdot)\|_\infty \leq \varepsilon t^{-(\alpha-1)/\alpha}$ for $t < t_\varepsilon$ or $t > T_\varepsilon$. Hence, by (36), for $t < t_\varepsilon$, we have

$$\left| \int_0^t \int_{\mathbb{R}^2} \nabla p_\alpha(t-s, x-y) \cdot R^1 \theta(s, y) \theta(s, y) dy \, ds \right| \leq \varepsilon P(t_\varepsilon) \left( \int_0^t (t-s)^{-\frac{\alpha-1}{\alpha}} ds \right)$$

$$= \varepsilon P(t_\varepsilon) \mathcal{B} \left( \frac{\alpha-1}{\alpha}, \frac{1}{\alpha} \right).$$

Thus, (2) gives us

$$\frac{\theta(t, x)}{P(t_\varepsilon)} = 1 \leq \varepsilon,$$

which proves the first limit in (5). Similarly, we get for $t > 2T_\varepsilon$,

$$\left| \int_0^t \int_{\mathbb{R}^2} \nabla p_\alpha(t-s, x-y) \cdot R\theta(s, y) \theta(s, y) dy \, ds \right| \leq P(t_\varepsilon) \left( \varepsilon t^{-(\alpha-1)/\alpha} + \varepsilon \int_0^t (t-s)^{-\frac{\alpha-1}{\alpha}} ds \right)$$

$$\leq P(t_\varepsilon) \left( \varepsilon t^{-(\alpha-1)/\alpha} \right),$$

which is less than $2\varepsilon \mathcal{B} \left( \frac{\alpha-1}{\alpha}, \frac{1}{\alpha} \right) P(t_\varepsilon)$ for $t$ large enough. Hence, we obtain the second limit in (5). In particular, it allows us to prove the last limit, i.e. $\lim_{|x| \to \infty} \sup_{0 < t < T} \left| \frac{\theta(t, x)}{P(t_\varepsilon)} - 1 \right| = 0$, by showing that

$$\lim_{|x| \to \infty} \sup_{0 < t < T} \left| \frac{\theta(t, x)}{P(t_\varepsilon)} - 1 \right| = 0$$

holds for any $T > 0$. By (34), for every $\varepsilon > 0$ there is $M > 0$ such that $|t^{\frac{\alpha-1}{\alpha}} R^1 \theta(t, x)| < \varepsilon$ for $|x| > M$. Hence, by (9) and (28), we obtain

$$\left| \int_0^t \int_{|y| > M} \nabla p_\alpha(t-s, x-y) \cdot R\theta(s, y) \theta(s, y) dy \, ds \right| \leq \varepsilon \int_0^t (t-s)^{-\frac{\alpha-1}{\alpha}} \int_{|y| > M} p_\alpha(t-s, x-y) P(t_\varepsilon) \theta(s, y) dy \, ds$$

$$= \varepsilon \mathcal{B} \left( \frac{\alpha-1}{\alpha}, \frac{1}{\alpha} \right) P(t_\varepsilon).$$

Next, by (9), for $|x| > 2M$ and $t < T$, we get

$$\left| \int_0^t \int_{|y| \leq M} \nabla p_\alpha(t-s, x-y) \cdot R\theta(s, y) \theta(s, y) dy \, ds \right| \leq \frac{\varepsilon T_\varepsilon^{\frac{\alpha-1}{\alpha}}}{|x|^{\frac{\alpha}{\alpha-1}}} \int_{|y| \leq M} \frac{1}{|x|^\frac{\alpha}{\alpha-1}} P(t-s, x-y) P(t_\varepsilon) \theta(s, y) dy \, ds$$

This ends the proof.
Proof of Theorem 1.1. The upper bound follows from lemma 3.4. To prove the lower one, note that theorem 1.2 implies that \( \theta(t,x) \gtrsim P_t \theta_0(x) \) whenever \( t \in (0,h_0) \cup (T,\infty) \) or \( |x| > R \) for some \( h_0, R > 0 \). Since both \( \theta(t,x) \) and \( P_t \theta_0(x) \) are continuous, they are comparable on \( [0,T] \times \mathbb{B}(0,R) \) as well. \( \square \)

In the last part of this section, we consider the case \( \theta_0 \in L^p \) with \( p > \frac{2}{\alpha -1} \). As a result, we obtain the local in time analogue of theorem 1.1. Note that by remark 3.3 in [4], for \( p > \frac{2}{\alpha -1} \), we have

\[
\|\theta(t, \cdot)\|_q \lesssim t^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right)}, \quad p \leq q \leq \infty.
\]

Proposition 3.6. For nonnegative \( \theta_0 \in L^p(\mathbb{R}^2) \), \( p > \frac{2}{\alpha -1} \) and \( T > 0 \) there are constants \( C_1 \) and \( C_2 \) (depending on \( T \) and \( \theta_0 \)) such that

\[
C_1 P_t \theta_0(x) \leq \theta(t,x) \leq C_2 P_t \theta_0(x), \quad x \in \mathbb{R}^2, \; 0 < t \leq T.
\]

Proof. Let \( T > 0 \). Let us consider the equation

\[
\begin{align*}
\partial_t u &= \Delta^{\alpha/2} u + b \cdot \nabla u, \\
u(0,x) &= \theta_0(x),
\end{align*}
\]

where \( b = b(t,x) = (R^2 \theta)(t,x) \). Of course \( u(t,x) = \theta(t,x) \) is a solution to the above equation. Furthermore, the continuity of the Riesz transform (15) gives us

\[
\|b(t, \cdot)\|_p \leq c \|\theta(t, \cdot)\|_p \leq c \|\theta_0\|_p, \quad 1 < p < \infty.
\]

By Hölder inequality, we get

\[
\int_S \int_{\mathbb{R}^2} p_o(u - s, z - x) |b(u, z)| dx dz du \leq \int_S \int_{\mathbb{R}^2} \frac{1}{(u - s)^{\alpha}} \|p_o(u - s, \cdot)\|_{\infty} \|b(u, \cdot)\|_p du \leq c \int_S \int_{\mathbb{R}^2} \frac{1}{(u - s)^{\alpha}} (u - s)^{\frac{3 \alpha}{2} - \frac{1}{p} - 1} du = c \int_S (u - s)^{\frac{3 \alpha}{2} - \frac{1}{p}} du = c_1 (t-s)^{1 - \frac{2 \alpha p}{2 \alpha p - \beta}}.
\]

In the same way, one may obtain

\[
\int_S \int_{\mathbb{R}^2} p_o(t - s, z - x) |b(u, z)| dx dz du \leq c_2 (t-s)^{1 - \frac{2 \alpha p}{2 \alpha p + \beta}}.
\]

Note that \( \frac{2 \alpha p}{2 \alpha p - \beta} < 1 \), and consequently \( c(t-s)^{1 - \frac{2 \alpha p}{2 \alpha p - \beta}} \leq \eta + \beta(t-s) \) for arbitrary small \( \eta \) and some \( \beta > 0 \). Hence, we may apply ([14], theorems 2 and 3) and conclude that the fundamental solution of the equation \( \partial_t u = \Delta^{\alpha/2} u + b \cdot \nabla u \) is locally in time comparable with \( p_o(t,x) \) and we get the assertion of the proposition. \( \square \)

4. Gradient estimates

In this section we derive the pointwise estimates for \( \nabla^k \theta \). Recall that for a multi-index \( k = (k_1, k_2) \in \mathbb{N}^2 \) we put \( |k| = k_1 + k_2 \). Note that
\[
\n\nabla^k (fg) = \sum_{m+n=k} c_{m,n} \nabla^m f \nabla^n g,
\]

where the sum is taken over all multi-indices \(m, n \in \mathbb{N}^2\) such that \(m + n = k\).

As a first step, we provide initial estimates with bound depending only on the time variable.

**Lemma 4.1.** For \(\theta_0 \in L^{2/\alpha} (\mathbb{R}^2)\), we have

\[
\|\nabla^k R \theta(t, \cdot)\|_\infty \lesssim t^{\frac{k+1}{\alpha}}, \quad i = 1, 2.
\]

**Proof.** Let us rewrite (2) as follows,

\[
\theta(t, x) = \int_{\mathbb{R}^2} p_0(t, x - y) \theta_0(y) dy + \int_0^{t/2} \int_{\mathbb{R}^2} \nabla p_0(t, x - y) \cdot R^1 \theta(s, y) \theta(s, y) dy ds
\]

\[
+ \int_{t/2}^t \int_{\mathbb{R}^2} \nabla p_0(t, x - y) \cdot R^1 \theta(s, y) \theta(s, y) dy ds.
\]

Since the Riesz transform commutes with derivatives, by (39) and (37), we get

\[
\nabla^k R \theta(t, x) = \int_{\mathbb{R}^2} R \nabla^k p_0(t, x - y) \theta_0(y) dy
\]

\[
+ \int_0^{t/2} \int_{\mathbb{R}^2} R_k \left( \nabla^k \nabla p_0(t, x - y) \right) \cdot R^1 \theta(s, y) \theta(s, y) dy ds
\]

\[
+ \sum_{k_1 + k_2 = k} c_{k_1, k_2} \int_{t/2}^t \int_{\mathbb{R}^2} R_{k_1} \left( \nabla p_0(t, x - y) \right) \cdot R^1 \left( \nabla^{k_2} \theta(s, y) \right) dy ds,
\]

where \(k_1, k_2 \in \mathbb{N}^2\). Hence, by Hölder inequality, (22), (10), (15) and (21), for \(p > \frac{2}{\alpha - 1}\), we obtain

\[
\|\nabla^k R \theta(t, \cdot)\|_\infty \lesssim \|\nabla^k p_0(t, \cdot)\|_{2/\alpha} \|\theta_0\|_{2/\alpha}
\]

\[
+ \int_0^{t/2} s^{-\frac{1}{\alpha - 1}} \|\nabla^k \nabla p_0(t, \cdot)\| \|\theta(s, \cdot)\|_{2/\alpha} ds
\]

\[
+ \sum_{k_1 + k_2 = k} c_{k_1, k_2} \int_{t/2}^t \|\nabla p_0(t, \cdot)\| \|\left( R^1 \nabla^{k_1} \theta(s, \cdot) \right) \|_p \|\nabla^{k_2} \theta(s, \cdot)\|_\infty ds
\]

\[
\lesssim t^{\frac{k+1}{\alpha}} + \int_0^{t/2} s^{-\frac{1}{\alpha - 1}} t^{\frac{k+1}{\alpha}} ds
\]

\[
+ \sum_{k_1 + k_2 = k} c_{k_1, k_2} \int_{t/2}^t (t - s)^{-\frac{1}{\alpha}} t^{\frac{k_1}{\alpha}} + \frac{k_2}{\alpha - 1} + \frac{k_2}{\alpha - 1} ds
\]

\[
\lesssim t^{\frac{k+1}{\alpha}},
\]

as required. \qed
Next, we present a series of auxiliary lemmas that are used in the proof of theorem 1.3.

**Lemma 4.2.** Let \( \theta_0 \in L^{\frac{1}{2}}(\mathbb{R}^2) \). Let \( 0 < t_1 < t_2 < \infty \). There exists a constant \( C \) depending on \( t_1, t_2, R \) and \( \theta_0 \) such that for \( x \in \mathbb{R}^2 \), we have

\[
\int_{D_t} \int_{B(0,R)} (t - s)^{-1/\alpha} p_\alpha(t - s, x - y)s^{-(\alpha - 1)/\alpha} |\nabla^k \theta(s, y)| dy ds \leq C t^{-k/\alpha} p_\alpha|\theta_0|(x),
\]

where \( D_t = (t_1, t_2) \cap (t/2, t) \).

**Proof.** Let us observe that \( D_t = \emptyset \) for \( t \notin (t_1, 2t_2) \), hence, it suffices to consider only \( t_1 < t < 2t_2 \). By (21),

\[
\int_{D_t} \int_{B(0,R)} (t - s)^{\frac{1}{\alpha}} p_\alpha(t - s, x - y)s^{-(\alpha - 1)/\alpha} |\nabla^k \theta(s, y)| dy ds
\]

\[
\leq c_1 \int_{D_t} \int_{B(0,R)} (t - s)^{\frac{1}{\alpha}} p_\alpha(t - s, x - y)s^{-\frac{a_1}{\alpha}} \frac{a_1}{\alpha} dy ds
\]

\[
\leq c t_t^\frac{2^{\alpha + 1}}{\alpha} t^{-1/\alpha} p_\alpha(t, x) ds =: f(t, x).
\]

Note that \( p_\alpha(s, y) \geq \frac{1}{c_1} > 0 \) for \( (s, y) \in (t_1, t_2) \times B(0, R) \). Thus,

\[
P_{\frac{1}{2} - 1} B(0,R)(x) \leq c_1 \int_{\mathbb{R}^2} p_\alpha(t - s, x - y)p_\alpha(s, y) dy = c_1 p_\alpha(t, x) \leq \frac{c_2}{(1 + |x|)^{2 + \alpha}}.
\]

Consequently, by lemma 2.2,

\[
f(t, x) \leq c_3 t^\frac{-k_1}{\alpha} \int_{t_1}^t (t - s)^{-\frac{a_1}{\alpha}} \frac{1}{(1 + |x|)^{2 + \alpha}} ds \leq c_4 t^\frac{k}{\alpha} p_\alpha|\theta_0|(x).
\]

This ends the proof. \( \square \)

**Lemma 4.3.** Let \( \beta > 0 \) be fixed. For any \( v \in (0, 1) \), we have

\[
\int_v^1 r^{-\beta}(1 - r^\alpha)^{-1/\alpha}(r^\alpha - v^\alpha)^{-1/\alpha} dr \approx v^{-\beta}(1 - v)^{1 - 2/\alpha}
\]

with comparability constants depending only on \( \alpha \) and \( \beta \).

**Proof.** Denote the above integral by \( I(v) \). Since \( a^\gamma - b^\gamma \approx (a - b)a^{\gamma - 1} \) for \( a > b > 0 \) and \( \gamma > 0 \) (see e.g. lemma 4 in [24]), we have \( 1 - r^\alpha \approx 1 - r \) and \( r^\alpha - v^\alpha \approx (r - v)r^{\alpha - 1} \). Hence,

\[
I(v) \approx \int_v^1 r^{1/\alpha - 1 - \beta}(1 - r)^{-1/\alpha}(r - v)^{-1/\alpha} dr.
\]

For \( v \geq 1/4 \), we estimate \( r^{1/\alpha - 1 - \beta} \approx 1 \) and substitute \( r = 1 - u(1 - v) \), which gives us
\[ I(v) \approx (1 - v)^{1 - 2/\alpha} \int_0^1 u^{-1/\alpha}(1 - u)^{-1/\alpha} du = c(1 - v)^{1 - 2/\alpha}. \]

In the case \( v < 1/4 \), we split the integral into \( \int_v^{1/2} + \int_{1/2}^1 \) and obtain
\[
I(v) \approx \int_v^{1/2} r^{1/\alpha - 1/\beta}(r - v)^{-1/\alpha} dr + \int_{1/2}^1 (1 - r)^{-1/\alpha} dr = v^{-\beta} \int_1^{(1/2\alpha)} u^{1/\alpha - 1/\beta}(u - 1)^{-1/\alpha} du + \frac{\alpha 2^{(\alpha - 1)/\alpha}}{\alpha - 1} \approx v^{-\beta} + 1 \approx v^{-\beta},
\]
which is equivalent to the required formula under current assumptions.

Since \( \alpha > 1 \), we immediately obtain the following

**Corollary 4.4.** Let \( \beta > 0 \) be fixed. There is a constant \( C_\beta \) such that for \( v \in (0, 1) \), we have
\[
\int_v^1 r^{-\beta}(1 - r^\alpha)^{-1/\alpha} dr \leq C_\beta v^{-\beta}(1 - v)^{-1/\alpha}.
\]

**Lemma 4.5.** Fix \( \gamma \in (0, \frac{1}{\alpha}) \). For any measurable function \( f: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \), define the operator
\[ T_\gamma f(t, x) = \Gamma \int_0^t s^{-\gamma - \frac{\alpha - 1}{\alpha}} (t - s)^{-\frac{1}{\alpha}} P_{t-s}|f|(s, x) ds. \quad (41) \]

Suppose \( T_\gamma f(t, x) < \infty \) and \( f \) satisfies the inequality
\[ f(t, x) \leq C P_1 |\theta_0(x)| + \eta T_\gamma f(t, x), \quad t > 0, \ x \in \mathbb{R}^2, \quad (42) \]
for some constants \( C, \eta > 0 \). If \( \eta \) is sufficiently small, then there exists a constant \( M > 0 \) such that
\[ f(t, x) \leq M P_1 |\theta_0(x)|, \quad t > 0, \ x \in \mathbb{R}^2. \]

**Proof.** Applying estimate (42) of \( f \) to (41), we get
\[
T_\gamma f(t, x) \leq \Gamma \int_0^t s^{-\gamma - (\alpha - 1)/\alpha} (t - s)^{-1/\alpha} \int_{\mathbb{R}^2} P_0(t - s, x - y) \times \left( C P_1 |\theta_0(y)| + \eta \int_0^t u^{-\alpha - 1/\alpha} (s - u)^{-1/\alpha} P_{t-u}|f|(u, y) du \right) dy ds
\]
\[
= CB \left( 1 - \gamma - \frac{\alpha - 1}{\alpha}, 1 - \frac{1}{\alpha} \right) P_1 |\theta_0(x)|
\]
\[
+ \eta \int_0^t \int_0^s s^{-\gamma} (su)^{-\alpha - 1/\alpha} [(t - s)(s - u)]^{-1/\alpha} P_{t-u}|f|(u, x) du ds
\]
\[
= CB \left( 1 - \gamma - \frac{\alpha - 1}{\alpha}, 1 - \frac{1}{\alpha} \right) P_1 |\theta_0(x)|
\]
\[
+ \eta \int_0^t u^{-\alpha - 1/\alpha} P_{t-u}|f|(u, x) \int_u^t s^{-\gamma} [(t - s)(s - u)]^{-1/\alpha} ds du, \quad (43)
\]
where $\mathcal{B}$ is the beta function. Using corollary 4.4 with $\beta = \gamma \alpha$ and $v = (u/t)^{1/\alpha}$, we estimate the last inner integral in (43) as follows

\[
\int_{u}^{t} s^{-\gamma-(\alpha-1)/\alpha}[(t-s)(s-u)]^{-1/\alpha} \, dx = t^{-\gamma-1/\alpha} \int_{u}^{t} s^{-\gamma-(\alpha-1)/\alpha} \left[ (1-s) \left( s - \frac{u}{t} \right) \right]^{-1/\alpha} \, ds
\]

\[
= t^{-\gamma-1/\alpha} \int_{u}^{t} \left( r^{-\gamma} \right) \left( r^\alpha - \frac{u}{t} \right) \left( r^\alpha - \frac{u}{t} \right) \, dr
\]

\[
\leq c_t u^{-\gamma(t-u)^{-1/\alpha}}.
\]

This yields $T_s f(t,x) \leq CB \left( 1 - \gamma - \frac{u-1}{\alpha} - 1 - \frac{u}{t} \right) P_t |\theta_0(x)| + \eta c_T, T_s f(t,x)$. Now, for $\eta < \frac{1}{c_t}$, we get

\[
T_s f(t,x) \leq \frac{CB \left( 1 - \gamma - \frac{u-1}{\alpha} - 1 - \frac{u}{t} \right) P_t |\theta_0(x)|}{1 - \eta c_T},
\]

which ends the proof.

**Proof of Theorem 1.3.** We will use induction with respect to $|k|$. For $|k| = 0$ the assertion is true due to lemma 3.4. Assume now that (6) holds for all multi-indices $k'$ such that $|k'| \leq |k| - 1$ for some multi-index $k$, $|k| \geq 1$. We use (39) and, analogously as in (40), we obtain

\[
\nabla^k \theta(t,x) = \nabla^k P_t \theta_0(x) + \int_{0}^{t/2} \int_{\mathbb{R}^2} \left( \nabla^k \nabla \theta(t-x-y) \right) \cdot R^+ \theta(s,y) \theta(s,y) \, dy \, ds
\]

\[
+ \sum_{k_1+k_2=k} c_{k_1,k_2} \int_{t/2}^{t} \int_{\mathbb{R}^2} \left( \nabla \theta(t-x-y) \right) \cdot R^+ \left( \nabla \theta(s,y) \right) \nabla \theta(s,y) \, dy \, ds.
\]

As mentioned in Introduction, (9) implies

\[
|\nabla^k P_t \theta_0(x)| \leq \int_{\mathbb{R}^2} |\nabla \theta(t-x-y) \theta_0(y)| \, dy \lesssim t^{-\frac{k}{\alpha}} P_t |\theta_0(x)|.
\]

Next, by (9), proposition 3.3, lemma 3.4 and semigroup property, we get

\[
\left| \int_{0}^{t/2} \int_{\mathbb{R}^2} \left( \nabla^k \nabla \theta(t-x-y) \right) \cdot R^+ \theta(s,y) \theta(s,y) \, dy \, ds \right|
\]

\[
\lesssim t^{-\frac{|k|+1}{\alpha}} \int_{0}^{t/2} \int_{\mathbb{R}^2} \theta(t-x-y) P_t |\theta_0(y)| \, dy \, ds
\]

\[
= c_t \frac{|k|}{\alpha} P_t |\theta_0(x)|.
\]

Hence, using the induction assumption for $|k_2| \leq |k| - 1$ together with (9), (28), (38) and semigroup property of $p_t(t,x)$, we conclude
\[
|\nabla^k \theta(t, x)| \leq t^{\frac{k}{\alpha}} P_1 |\theta_0|(x) + \sum_{k_1 + k_2 = k \atop |k_1| \leq |k| - 1} c_{k_1k_2} t^{- \frac{(|k|+\alpha-1)}{\alpha}} \int_{t/2}^t (t - s)^{-1/\alpha} \\
\times \int_{\mathbb{R}^2} p_\alpha(t - s, x - y) P_1 |\theta_0|(y) dy ds \\
+ t^{\frac{k}{\alpha}} \int_{t/2}^t \int_{\mathbb{R}^2} |\nabla p_\alpha(t - s, x - y) \cdot R^k \theta(s, y) \nabla^k \theta(s, y)| dy ds \\
\leq t^{\frac{k}{\alpha}} P_1 |\theta_0|(x) + \sum_{k_1 + k_2 = k \atop |k_1| \leq |k| - 1} c_{k_1k_2} t^{- \frac{(|k|+\alpha-1)}{\alpha}} \int_{t/2}^t p_\alpha(t - s, x - y)|R^k \theta(s, y)||\nabla^k \theta(s, y)|dy ds.
\]

(44)

Let \( \varepsilon > 0 \) be a constant to be fixed later. By (34), there are \( t_1, t_2, R > 0 \) such that 
\(|s^{(\alpha-1)/\alpha} R^k \theta(s, y)| < \varepsilon \) for \((s, y) \notin D = (t_1, t_2) \times B(0, R) \). Thus

\[
|\nabla^k \theta(t, x)| \leq ct^{-\frac{|k|}{\alpha}} P_1 |\theta_0|(x) + \varepsilon \int_{t/2}^t (t - s)^{-1/\alpha} \int_{\mathbb{R}^2} p_\alpha(t - s, x - y)s^{-(\alpha-1)/\alpha} |\nabla^k \theta(s, y)|dy ds \\
+ \int_{t_1/2}^{t_1/2} \int_{B(0, R)} (t - s)^{-1/\alpha} p_\alpha(t - s, x - y)s^{-(\alpha-1)/\alpha} |\nabla^k \theta(s, y)|dy ds.
\]

By lemma 4.2, the last integral is bounded by \( t^{-|k|/\alpha} P_1 |\theta_0|(x) \). This gives us

\[
|\nabla^k \theta(t, x)| \leq ct^{-\frac{|k|}{\alpha}} P_1 |\theta_0|(x) + \varepsilon \int_{t/2}^t (t - s)^{-1/\alpha} \int_{\mathbb{R}^2} p_\alpha(t - s, x - y)s^{-(\alpha-1)/\alpha} |\nabla^k \theta(s, y)|dy ds.
\]

Now, denote \( f_k(t, x) = t^{-\frac{|k|}{\alpha}} |\nabla^k \theta(t, x)| \). Then, for any \( \gamma \in (0, 1/\alpha) \),

\[
f_k(t, x) \leq c P_1 |\theta_0|(x) + \varepsilon \int_{t/2}^t (t - s)^{-1/\alpha} \int_{\mathbb{R}^2} p_\alpha(t - s, x - y)s^{-(\alpha-1)/\alpha} t^{-|k|/\alpha} |\nabla^k \theta(s, y)|dy ds \\
\leq c P_1 |\theta_0|(x) + \varepsilon 2^{\frac{|k|}{\alpha}} \int_{t/2}^t (t - s)^{-1/\alpha} s^{-(\alpha-1)/\alpha} P_{t-s} f_k(s, x) ds \\
\leq c P_1 |\theta_0|(x) + \varepsilon 2^{\frac{|k|}{\alpha}} T_\gamma f_k(t, x),
\]

where \( T_\gamma \) is defined in lemma 4.5. Since \( \varepsilon \) may be arbitrary small, by lemma 4.5,

\[
|\nabla^k \theta(t, x)| \leq M t^{-\frac{|k|}{\alpha}} P_1 |\theta_0|(x).
\]

The proof is complete. \( \square \)

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