Abstract—Quantum error-correcting codes are used to protect quantum information from decoherence. A raw state is mapped, by an encoding circuit, to a codeword, where the most likely quantum errors can be removed by a decoding procedure.

A good encoding circuit should have some features such as low depth, few gates, and so on. In this paper, we show how to practically implement an encoding circuit of gate complexity $O(n(n-k+c)/\log n)$ for an $[[n, k, c]]$ quantum stabilizer code with the help of $c$ pairs of maximally-entangled state. For the special case of an $[[n, k]]$ stabilizer code with $c=0$, the encoding complexity is $O(n(n-k)/\log n)$, which improves the previous known result of $O(n^2/\log n)$.

On the other hand, we discuss the decoding procedure of an entanglement-assisted quantum stabilizer code and show that the general decoding problem is NP-hard, which strengthens the foundation of the quantum McEliece cryptosystem.

I. INTRODUCTION

Quantum computers have powerful applications but they are hard to implement because of the vulnerable quantum states and imperfect quantum gates. How to handle errors caused by quantum decoherence has been an important problem and a possible method is to use quantum error-correcting (QEC) codes [1]–[3], in which quantum information is encoded in a codespace so that the most likely errors can be treated.

The class of quantum stabilizer codes have similar features to the classical linear codes and are convenient for practical implementations [4]. An $[[n, k]]$ stabilizer code is a $k$-qubit subspace of the $n$-qubit state space. The mapping from the raw qubit space to the encoded space can be implemented by an encoding circuits consisting of elementary gates. Clearly a low-complexity encoding circuit is desired, since quantum coherence decays with time and quantum gates cannot be implemented perfectly. In addition, the syndrome measurement circuit is also closely related to the encoding circuit [5]. As a consequence, a low complexity encoding circuit can be potentially used in designing fault-tolerant procedures to achieve higher threshold in fault-tolerant quantum computation.

In this paper we study the gate complexity of an encoding circuit. To encode an $[[n, k]]$ stabilizer code, Cleve and Gottesman showed that $O(n(n-k))$ gates are required [6], which is proportional to the dimension of the underlying check matrix. Aaronson and Gottesman argued that $O(n^2/\log n)$ gates are sufficient [7] by using the CNOT circuit decomposition algorithm in [8]. We will further show that this encoding complexity can be reduced to $O(n(n-k)/\log n)$ by looking into the structure of the check matrix and applying variants of the decomposition algorithm [8]. More generally, we show that for an $[[n, k, c]]$ entanglement-assisted (EA) quantum stabilizer code [9], the encoding complexity is $O(n(n-k+c)/\log n)$.

On the other hand, it desires to have a very hard general decoding problem to construct a code-based McEliece cryptosystem [10]. General bounded-distance decoding is shown to be NP-hard in the classical case by Berlekamp, McEliece, and van Tilborg [11], and also in the quantum case by Fujita [12]. Several quantum decoding problems are also shown to be NP-hard [13], [14] or even harder (#P-complete) [15]. Notice that Bernstein showed that the McEliece cryptosystem reveals strong resistance to the attacks performed by quantum computers [16]. In this paper we will discuss the decoding procedure for a general EA stabilizer code and show that the corresponding decoding problem is also NP-hard.

This paper is organized as follows. In Section II we give the notation and the basics of quantum codes. The encoding complexity of $O(n(n-k+c)/\log n)$ is shown in Section III. In Section IV we link an EA stabilizer decoding problem to a classical computational problem and show that it is NP-hard. Then we conclude.

II. THE EAQEC SCHEME AND STANDARD FORM

We first review basics of stabilizer codes. Let $G_n$ be the $n$-fold Pauli group $\{i^c \otimes_{j=1}^{n} M_j : M_j \in \{I, X, Y, Z\}, c \in \{0, 1, 2, 3\}\}$. Let $S \subseteq G_n$ be an Abelian stabilizer group, generated by $n-k$ independent stabilizer generators $Z_1, \ldots, Z_{n-k}$, such that $-I \not\in S$. Then an $[[n, k]]$ stabilizer code defined by $S$ is $C(S) = \{ |v\rangle \in \mathbb{C}^{2^n} : g |v\rangle = |v\rangle \quad \forall g \in S \}$.

Let $Z_i = I^{\otimes (i-1)} \otimes Z \otimes I^\otimes (n-i)$ and similarly for $X_i$. For $c = c_1 \cdots c_k \in \{0, 1\}^n$, consider the following initial state

$$|0\rangle^\otimes n \langle c| = X_{c_1}^{n-k+1} \cdots X_{c_k}^{n} |0\rangle^\otimes n$$

with $k$-qubit logical state $|c\rangle$ and $n-k$ ancillas in $|0\rangle$ before encoding. This state has stabilizers $Z_1, \ldots, Z_{n-k}$.

A unitary encoding circuit $U$ for $C(S)$ maps $|c\rangle$ to $|\bar{c}\rangle = U |\bar{c}\rangle$ with stabilizers $\{Z_i = UZ_iU^\dagger, i = 1, \ldots, n-k\}$. The set of the $2^k$ vectors $\{|\bar{c}\rangle : c \in \{0, 1\}^n\}$ forms a basis of $C(S)$. $|\bar{c}\rangle$ can be written as $X_{c_1}^{n-k+1} \cdots X_{c_k}^{n} |0\rangle$, where $X_j = U X_j U^\dagger$, $j = n-k+1, \ldots, n$ are called seed generators and $|\bar{0}\rangle = U |0\rangle^\otimes n$ is a coded zero, which can be generated.
by $|0\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} |i + Z_i \rangle |0\rangle$ and $|g(0)\rangle = \frac{1}{\sqrt{2^{n-k}}} \sum_{g \in S} g |0\rangle$ such that $g(0) = |0\rangle$ for all $g \in S$.

Define a homomorphism $\varphi : \mathcal{G}_n \to \{0,1\}^{2n}$ by $\varphi(g) \equiv (u_1 \cdots u_n | v_1 \cdots v_n)$ for $g = \sigma_1 \cdots \sigma_n \in \mathcal{G}_n$, where

$$
\begin{array}{c|ccc}
\sigma_j & I & X & Z \\
u_j & 0 & 1 & 1 \\
v_j & 0 & 1 & 1
\end{array}
$$

An $(n-k) \times 2n$ binary check matrix $H = [I \ A \ B \ C]$ is the matrix with rows $\varphi(Z_i), i = 1, \ldots, n-k$. For example, the initial state has a check matrix $[O \ O | I \ O]$, where $O$ is the all-zero matrix of appropriate dimensions. Since $S$ is an Abelian subgroup, $H$ has to satisfy the commutation condition $H \Lambda H^T = O$, where $\Lambda = [O_{n \times n} \ I_{n \times n} | O_{n \times n} \ I_{n \times n}]$. The encoding can be done by the reverse of a series of elementary gates that transform $H$ to $[O \ O | I \ O]$  - [5] - [7].

Entanglement-assisted quantum-error correcting codes are a coding scheme that the sender (Alice) and the receiver (Bob) share some maximally entangled Einstein-Podolsky-Rosen (EPR) pairs [9], [17], [18]. Let $|\Phi^+\rangle_{\text{cAB}}$ be the state of $c$ EPR pairs $(2c$ ebits) pre-shared between Alice and Bob. An initial basic state of the overall dimension is $|0\rangle^{n-c} \otimes |\Phi^+\rangle_{\text{cAB}} \otimes |c_1 \cdots c_k\rangle$, where Bob holds $c$ ebits and Alice holds the remaining $n = s + k + c$ qubits prior to communication. After encoding, Alice sends her $n$ qubits to Bob through a noisy quantum channel and Bob’s $c$ qubits are assumed to be error-free. Every initial basic state is stabilized by a set of operators in $\mathcal{G}_{n+c}$ with corresponding check matrix $H_{\text{raw}}$:

$$
H_{\text{raw}} = \begin{bmatrix}
I_x & O & O & O \\
O & I_{\text{cxc}} & O & O \\
O & O & I_{\text{cxc}} & O \\
O & O & O & I_{\text{cxc}}
\end{bmatrix}.
$$

(1)

Similarly, we can consider a unitary encoding circuit $U$ that maps $H_{\text{raw}}$ to some $H$ with $H \Lambda H^T = O$. Upon reception, Bob does a decoding on the total of $(n + c)$ qubits according to the check matrix $H$. Here Bob performs a regular stabilizer decoding, and the errors in the first $n$ qubits caused by communication could be corrected by the parity-check conditions defined in $H$.

We can consider a simplified check matrix $H$ of $H$ without the columns corresponding to Bob’s qubits:

$$
H_{\text{raw}} = \begin{bmatrix}
I_x & O & O & O \\
O & I_{\text{cxc}} & O & O \\
O & O & I_{\text{cxc}} & O \\
O & O & O & I_{\text{cxc}}
\end{bmatrix},
$$

(2)

corresponding to simplified stabilizers $\{Z_1, \ldots, Z_s, X_{s+1}, \ldots, X_{s+c}, Z_{s+1}, \ldots, Z_{s+c}\}$ in $\mathcal{G}_{s+c}$. After encoding by $U$, we have simplified stabilizer generators $\tilde{Z}_i = U Z_i U^\dagger$ and $\tilde{X}_i = U X_i U^\dagger$ satisfying the following commutation relations

$$
\begin{align*}
[Z_i, \tilde{Z}_j] &= 0, \quad \forall i, j; \\
[X_i, \tilde{X}_j] &= 0, \quad \forall i, j; \\
[\tilde{Z}_i, \tilde{X}_j] &= 0, \quad \forall i \neq j; \\
\{\tilde{Z}_i, \tilde{X}_j\} &= 1, \quad \forall i = s + 1, \ldots, s+ c.
\end{align*}
$$

(3)

Thus we have a non-Abelian subgroup $S' = \langle \tilde{Z}_1, \ldots, \tilde{Z}_s, X_{s+1}, \ldots, X_{s+c}, \tilde{Z}_{s+1}, \ldots, \tilde{Z}_{s+c}\rangle$ with corresponding simplified check matrix

$$
H'_{(s+2c)\times 2n} =
\begin{bmatrix}
I & A & B & C \\
O & M_1 & M_2 & M_3 \\
O & M_3 & M_4 & M_5
\end{bmatrix} =
\begin{bmatrix}
\varphi(Z_1) \\
\varphi(Z_{s+1}) \\
\varphi(Z_{s+c})
\end{bmatrix}.
$$

(4)

By [5] we have

$$
B + CA^T + B^T + AC^T = O
$$

$$
M_5 = M_1 C^T + M_2 A^T
$$

$$
M_6 = M_3 C^T + M_4 A^T
$$

$$
M_1 M_2^T + M_2 M_1^T = O
$$

$$
M_3 M_4^T + M_4 M_3^T = O
$$

$$
M_1 M_4^T + M_2 M_3^T = I.
$$

(5)

Without loss of generality, we assume $M_1 = [M_{11}, M_{12}]$ with non-singular $M_{11}$. Now a standard form of the check matrix (with columns corresponding Bob’s qubits) is

$$
H_{(s+2c)\times 2(n+c)} =
\begin{bmatrix}
I & A & O & B & C & O \\
O & M_1 & I & M_5 & M_2 & O \\
O & M_3 & O & M_6 & M_4 & I
\end{bmatrix}
$$

(6)

which satisfies $H \Lambda H^T = O$.

III. AN ENCODING COMPLEXITY $O(n(n-k+c)/\log n)$

We consider the decomposition of a stabilizer circuit [7] or the encoding circuit of an EA stabilizer code [19] by Clifford gates (CNOT, H, P) and (possibly) some swap operations. In particular we use an additional controlled-Z gate (CZ) in the circuit decomposition. Consider a check matrix of the form $H = [H_X | H_Z]$, and we have the following rules:

1) A CNOT gate from qubit $i$ to qubit $j$ adds column $i$ to column $j$ in $H_X$ and adds column $j$ to column $i$ in $H_Z$.

2) A controlled-phase (CZ) gate from qubit $i$ to qubit $j$ adds column $i$ in $H_X$ to column $j$ in $H_Z$ and adds column $j$ in $H_X$ to column $i$ in $H_Z$.

3) A Hadamard gate on qubit $i$ swaps column $i$ in $H_X$ with column $j$ in $H_Z$.

4) A phase gate on qubit $i$ adds column $i$ in $H_X$ to column $i$ in $H_Z$.

Patel, Markov, and Hayes proposed an efficient elimination algorithm to decompose a CNOT circuit [8]. Motivated by their method, we proposed the following algorithm for the elimination of a matrix of the form $[I \ A | B' \ O]$ for our purpose.

**Lemma 1.** For an $(n-k) \times 2n$ check matrix $H = [I \ A | B' \ O]$, there exists a linear transformation that maps $H$ to $[I \ O | B' \ O]$.
using $O\left((2^m + k) \cdot \frac{n-k}{m}\right)$ CNOT gates for some $m$ smaller than $n - k$.

**Proof.** To simply the discussion, first suppose $n - k$ is divided by some integer $m$. Partition $[I \ A]$ into $(n-k)/m$ sections

\[
\begin{align*}
\text{1st section} & \quad \begin{bmatrix} I_{m \times m} & A_1 \\ \vdots & \vdots \end{bmatrix}, \\
\text{$m$th section} & \quad \begin{bmatrix} \cdots & \cdots & \cdots \\ I_{m \times m} & A_{\frac{n-k}{m}} \end{bmatrix},
\end{align*}
\]

with $m \times k$ binary matrices $A_i$. Begin with $i = 1$, perform appropriate column operations so that the columns of $I_{m \times m}$ in the 1st section will go over all nonzero binary vectors of $m$ bits and transform to a matrix $D_{m \times m}$ with all ones in the upper triangular part. For example, if $m = 3$,

\[
I_{m \times m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = D_{m \times m}.
\]

This requires $2^m - m - 1$ operations per section. During the generating process, if there is any column in $A_1$ duplicating to a generated column, eliminate the duplicate column by the generated column, which needs at most extra $k$ operations to eliminate $A_1$. Next, change $D_{m \times m}$ back to $I_{m \times m}$, which needs $m - 1$ operations. For example, consider $A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

So it takes at most $(2^m - m - 1) + k + (m - 1)) < (2^m + k)$ gates to eliminate one section. By repeating the above process for the $\frac{n-k}{m}$ sections, it requires at most $(2^m + k) \cdot \frac{n-k}{m} = O((2^m + k) \cdot \frac{n-k}{m})$ gates to eliminate $A$.

Similarly we propose the following decomposition algorithm for certain phase and CZ circuits.

**Lemma 2.** For an $(n-k) \times 2n$ check matrix $H = [I \ O] [B’ \ O]$ with symmetric $B’$, there exists a linear transformation that maps $H$ to $[I \ O O O]$ using $O((2^m + n - k) \cdot \frac{n-k}{m})$ gates.

**Proof.** By Rule 4), it needs at most $n - k = O(n - k)$ phase gates to eliminate non-zero diagonal entries in $B’$. After that, $CZ$ gates can further eliminate $H$ as $[I \ O O O]$ by Rule 2). To decompose an efficient $CZ$ circuit, after phase gates, consider to first eliminate $B’$ as first

\[
B_0 = \begin{bmatrix} O_{m \times m} & * & \cdots & * \\ * & O_{m \times m} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & \cdots & O_{m \times m} \end{bmatrix}
\]

1st section

\[
\begin{bmatrix} \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{bmatrix}
\]

$n-k/m$th section.

To obtain $O_{m \times m}$, every section needs fewer than $m^2$ CZ gates. To eliminate the * part, every section needs fewer than $(n - k)$ CZ gates by the same technique as in Lemma 1 (though additional $2^m$ CNOT gates are required). Thus eliminating $B’$ needs at most

\[
O(n - k) + O(2^m \cdot \frac{n-k}{m}) + O((2^m + n - k) \cdot \frac{n-k}{m}) = O((2^m + n - k) \cdot \frac{n-k}{m})
\]

gates.

Using these two lemmas, we have the following theorems.

**Theorem 3.** For an $(n-k) \times 2n$ check matrix $H = [I \ A] [B’ \ C]$ and some $m$ smaller than $n - k$, there exists a linear transformation that maps $H$ to $[I \ O O O]$ with $B’ = B + CA^T$ using $O((2^m + k) \cdot \frac{n-k}{m})$ CNOT gates; and there exists a linear transformation that maps $H$ to $[I \ A B’ \ O]$ with $B’ = B + AC^T$ using $O((2^m + k) \cdot \frac{n-k}{m})$ gates; both $B’$ and $B”$ are symmetric.

**Theorem 4.** An $[n, k]$ stabilizer code has an encoding complexity $O(n(n-k)/\log n)$.

**Proof.** Without loss of generality, let the check matrix of the stabilizer code be $H = [I \ A B’ C]_{[(n-k)+2n]}$, and in general, we need to add $k$ seed generators [4], [23], which can be represented by $[O I C’T’]_{k\times 2n}$. Steps of the elimination are

\[
\begin{bmatrix} n-k \\ k \end{bmatrix} \begin{bmatrix} I & A & B & C \\ O & I & O & O \end{bmatrix} \rightarrow \begin{bmatrix} I & A & B’ & O \\ O & I & O & O \end{bmatrix} \text{ by Theorem 3} \quad \text{C eliminated by} \quad O((2^m + k) \cdot \frac{n-k}{m}) \text{ gates and} \quad B’ = B + AC^T \text{ is symmetric,}
\]

where $C^T$ is also eliminated without extra gates by Rule 2).

\[
\begin{bmatrix} I & O & B’ & O \\ O & I & O & O \end{bmatrix} \text{ by Lemma 1} \quad A \text{ eliminated by} \quad O((2^m + k) \cdot \frac{n-k}{m}) \text{ gates},
\]

\[
\begin{bmatrix} I & O & O & O \\ O & I & O & O \end{bmatrix} \text{ by Lemma 2} \quad \text{symmetric B’ eliminated by} \quad O((2^m + n - k) \cdot \frac{n-k}{m}) \text{ gates},
\]

\[
\begin{bmatrix} O & O & I & O \\ O & I & O & O \end{bmatrix} \quad \text{by (n-k) Hadamard gates, Rule 3).}
\]

We do not need Hadamard gates on last $k$ qubits [21]. The overall elimination can be focused on $H$ without putting on the seed generators. Overall the complexity is

\[
2 \cdot O((2^m + k) \cdot \frac{n-k}{m}) + O((2^m + n - k) \cdot \frac{n-k}{m}) + O(n-k) = O((2^m + n - m) \cdot \frac{n-k}{m}) = O(2^m + n - k).
\]

Similar to [8], taking $m = [\alpha \log_2 n]$ for some constant $\alpha < 1$ will make $O((2^m + n) \cdot \frac{n-k}{m}) = O(n \cdot \frac{n-k}{\log n})$ since $2^m \leq n^\alpha = o(n)$.

Theorem 4 and the previous lemmas/theorems are based on a trick: Originally there are $(n-k)$ rows, each of which costs $O(n)$ gates, but if we group every $m$ rows as a section, then we only have $O((n-k)/m)$ sections and every section needs at most $O(2^m + n) = O(n)$ gates.
Theorem 5. An \([n, k; c]\) EA stabilizer code has an encoding complexity \(O(n(n-k+c)/\log n)\).

Proof. It suffices to show that when eliminating (4) and (2), the required number of gates is proportional to \((n-k+c) \times 2n\) before sectionization. By (3) and the Rules 1) to 4) above, we start to eliminate (4) as (2):

\[
\begin{bmatrix}
I & O & M_1 & M_2 & M_3 & M_2 T + M_3 T & M_4 \\
O & M_3 & M_3 C^T & M_4 & M_3 & M_2 T + M_3 T & M_4 \\
O & M_1 & M_2 & M_2 T + M_3 T & M_3 & M_4 & M_4 \\
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
I & O & M_1 & M_2 & M_2 T + M_3 T & M_4 \\
O & M_3 & M_2 & M_2 T + M_3 T & M_3 & M_4 & M_4 \\
O & M_1 & M_2 & M_2 T + M_3 T & M_3 & M_4 & M_4 \\
\end{bmatrix}
\]

by Theorem 3.

Recall \(M_1 = [M_{11} M_{12}]\) with \(M_{11}\) non-singular, and by (5):

\[
\begin{bmatrix}
M_{11} M_{12}^T & M_{12} M_{12}^T + M_{21} M_{11} T + M_{22} M_{12}^T & = O \\
M_{31} M_{11}^T & M_{32} M_{12}^T + M_{41} M_{11} T + M_{42} M_{12}^T & = O \\
M_{11} M_{11}^T & M_{12} M_{12}^T + M_{21} M_{11} T + M_{22} M_{12}^T & = I, \\
\end{bmatrix}
\]

by Rule 1), where \(M_{11}\) become \(M_{11} M_{11}^{-1} = I\) by CNOT gates, and when the LHS times \(M_{11}^{-1} = (P_1 P_2 \cdots)\) with \(P_i\) elementary column operations, the RHS times

\[
(P_1^T P_2^T \cdots)^T = M_{11}^T = P_i = P_1^{-1},
\]

by Theorem 3.

\[
\begin{bmatrix}
I & O & O & O & O & O & O \\
O & M_{31} M_{11}^{-1} & K_3 & O & K_2 & M_{22} \\
\end{bmatrix}
\]

by Theorem 3.

\[
\begin{bmatrix}
I & O & O & O & O & O & O \\
O & M_{31} M_{11}^{-1} & K_3 & O & L_2 & L_4 \\
\end{bmatrix}
\]

by Theorem 4.

\[
\begin{bmatrix}
I & O & O & O & O & O & O \\
O & M_{31} M_{11}^{-1} & K_3 & O & I & L_4 \\
\end{bmatrix}
\]

by Lemma 3.

M_{22} \text{ eliminated by CZ gates, and}

\[
L_2 = K_4 + K_3 M_{22}^T \quad \text{and} \quad L_4 = M_{22} + M_{31} M_{11}^{-1} M_{22},
\]

the symmetric \(K_2\) eliminated by phase and CZ gates, and \(L_2\) become \(I\) because \(L_2 + M_{31} M_{11}^{-1} K_2 = I\)

by substituting \(K_2 = K_2^T = M_{11} M_{21}^T + M_{12} M_{22}^T\),

\[
\rightarrow
\begin{bmatrix}
I & O & O & O & O & O & O \\
O & O & O & O & I & O \\
O & I & K_3 & O & M_{31} M_{11}^{-1} H & L_4 \\
\end{bmatrix}
\]

by Rule 3), where \(M_{31} M_{11}^{-1} H\) and \([\gamma] \text{ swapped by Hadamard gates},

\[
\rightarrow
\begin{bmatrix}
I & O & O & O & O & O & O \\
O & O & O & O & I & O \\
O & I & O & O & W & L_4 \\
\end{bmatrix}
\]

by Theorem 3.

\(K_3\) eliminated by CNOT gates, and \(W = M_{31} M_{11}^{-1} + L_4 K_3^T\) is symmetric by verifying:

\[
W = M_{31} M_{11}^{-1} (I + M_{22} M_{12}^T) + M_{22} M_{12}^T
\]

and by (2), \((I + M_{22} M_{12}^T) = M_{11} M_{41} + M_{12} M_{42} + M_{21} M_{31}\), and with a bit calculation, only symmetric terms remain in \(W\).

\[
\rightarrow
\begin{bmatrix}
I & O & O & O & O & O & O \\
O & I & O & O & O & O \\
O & I & O & O & O & O \\
\end{bmatrix}
\]

by Theorem 4 and Lemma 2.

\(L_4\) and symmetric \(W\) eliminated by CZ gates,

\[
\rightarrow
\begin{bmatrix}
O & O & O & I & O & O \\
O & I & O & O & O & O \\
O & O & O & I & O \\
\end{bmatrix}
\]

by Rule 3).

Hadamard gates used on qubits 1, \(\cdots, s+c\).

We have shown, to eliminate (4) as (2), the required number of gates is proportional to \((n-k+c) \times 2n\). By a derivation similar to the proof of Theorem 3 (cf. the mentioned trick before this theorem), the overall complexity is \(O(n(n-k+c)/\log n)\). A notice is that, when \(c\) is large, a process like (5) is needed to eliminate \(M_{11}\) as \(I\) efficiently.

\section{NP-Hardness of Decoding EA Stabilizer Codes}

We briefly sketch how a major step of decoding an \([n, k; c]\) EA stabilizer code can be considered as a classical computational problem, and the classical problem is NP-hard. Consider the \(H\) in (6). To decode an \([n, k; c]\) EA stabilizer code, upon reception, Bob puts together the received \(n\) qubits and his \(c\) qubits as like that he is to decode an \([n+c, k]\) stabilizer code by doing the projective syndrome measurements

\[
\{(I+g_i)/2, (I-g_i)/2\}, \quad i = 1, \ldots, n-k+c,
\]

with \(g_i\)’s defined by the \(n-k+c = s+2c\) rows in (6). Assume the error is some \(E = E' \otimes I^{c} \in \mathbb{C}^{s+c}\) for some error \(E' \in \mathbb{C}^n\) caused by the channel. Because the \(H\) in (6) is self-orthogonal, i.e., satisfying \(HAH^T = 0\), we have either \(g_i E = E g_i\) with measurement output \(\beta_i = +1\), or \(g_i E = -E g_i\) with measurement output \(\beta_i = -1\). Mapping \(\beta_i = +1\) to a binary \(y_i = 0\) and \(\beta_i = -1\) to \(y_i = 1\), and collecting \(y_i\)’s as a binary vector \(y = (y_1, \cdots, y_{n-k+c}) \in \mathbb{Z}_2^{s+2c}\), we have the rule of \(y\) as just like a classical syndrome vector generated by \(y = \varphi(E) \Lambda H^T\), where \(\varphi(E)\) is the binary vector form of \(E\) (see the definition of \(\varphi\) in Section 1). More precisely, it is \(y = \varphi(E') \Lambda H^T\) with an \(H'\) as in (4). Given \(y\), the receiver (Bob) should find a proper \(e \in \mathbb{Z}_2^s\) satisfying \(eH'^T = y\) and
Corollary 6. EMLD and DEMLD are NP-hard.

Proof. The deduction is simple. We show the argument for EMLD, and for DEMLD it is similar. A general EAQEC decoder should support $c = 0$ and if there exists an efficient EMLD decoder, then it can solve the QMLD in [14] in polynomial time. By Corollary 4 and Theorem 5 of [14], EMLD and DEMLD are NP-hard.

The major inference of Corollary 6 is constructive, for being able to generalize the quantum McEliece cryptosystem [12] as an EAQEC McEliece cryptosystem and promise its safety over the depolarizing channel. Instead of using the generalized weight, it is possible to use classical Hamming weight to infer another channel model, independent X–Z channel [13], [15]. For decoding stabilizer codes over the independent X–Z channel, Hsieh and Le Gall showed that these two kinds of decoding problems are NP-hard [13], and Iyer and Poulin showed that the degenerate decoding is actually harder, #P-complete [15]. Similar inferences can be considered as true if replacing stabilizer codes by EA stabilizer codes over the independent X–Z channel.

V. Conclusion

We have improved the encoding complexity to $O(n(n − k + c)/\log n)$ for an $[n, k; c]_q$ EA QEC code. On the other hand, we showed that decoding general EA stabilizer codes is NP-hard, no matter degeneracy is considered or not. This potentially leads to the application of a secure McEliece cryptosystem from EA stabilizer codes.

CYL was financially supported from the Young Scholar Fellowship Program by Ministry of Science and Technology (MOST) in Taiwan, under Grant MOST107-2636-E-009-005.

REFERENCES

[1] P. W. Shor, “Scheme for reducing decoherence in quantum computer memory,” Phys. Rev. A, vol. 52, pp. 2493–2496, 1995.
[2] A. R. Calderbank and P. W. Shor, “Good quantum error-correcting codes exist,” Phys. Rev. A, vol. 54, p. 1098, 1996.
[3] A. M. Steane, “Error correcting codes in quantum theory,” Physical Review Letters, vol. 77, p. 793, 1996.
[4] D. Gottesman, “Stabilizer codes and quantum error correction,” Ph.D. dissertation, California Institute of Technology, 1997.
[5] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information. Cambridge University Press, 2000.
[6] R. Cleve and D. Gottesman, “Efficient computations of encodings for quantum error correction,” Phys. Rev. A, vol. 56, pp. 76–82, 1997.
[7] S. Aaronson and D. Gottesman, “Improved simulation of stabilizer circuits,” Phys. Rev. A, vol. 70, p. 052328, 2004.
[8] K. Patel, I. L. Markov, and J. P. Hayes, “Optimal synthesis of linear reversible circuits,” Quantum Information and Computation, vol. 8, pp. 282–294, 2008.
[9] T. Brun, I. Devetak, and M.-H. Hsieh, “Correcting quantum errors with entanglement,” Science, vol. 314, p. 436, 2006.
[10] R. J. McEliece, “A public-key cryptosystem based on algebraic coding theory,” NASA DSN Progress Report 44–44, 1978.
[11] E. Berlekamp, R. McEliece, and H. van Tilborg, “On the inherent intractability of certain coding problems,” IEEE Trans. Inform. Theory, vol. 24, pp. 384–386, 1978.
[12] H. Fujita, “Quantum McEliece public-key cryptosystem,” Quantum Information and Computation, vol. 12, pp. 0181–0202, 2012.
[13] M.-H. Hsieh and F. L. Gall, “NP-hardness of decoding quantum error-correction codes,” Phys. Rev. A, vol. 83, p. 052331, 2011.
[14] K.-Y. Kuo and C.-C. Lu, “On the hardness of several quantum decoding problems,” arXiv preprint arXiv:1306.5173, 2013.
[15] P. Iyer and D. Poulin, “Hardness of decoding quantum stabilizer codes,” IEEE Transactions on Information Theory, vol. 61, no. 9, pp. 5209–5223, 2015.
[16] D. J. Bernstein, “Introduction to post-quantum cryptography,” in Postquantum cryptography. Springer, 2009, pp. 1–14.
[17] T. A. Brun, I. Devetak, and M. Hsieh, “Catalytic quantum error correction,” IEEE Trans. Inf. Theory, vol. 60, no. 6, pp. 3073–3089, June 2014.
[18] C. Y. Lai and A. Ashikhmin, “Linear programming bounds for entanglement-assisted quantum error-correcting codes,” IEEE Trans. Inf. Theory, vol. 64, no. 1, pp. 622–639, Jan 2018.
[19] M. M. Wilde and T. A. Brun, “Entanglement-assisted quantum convolutional coding,” Physical Review A, vol. 81, no. 4, p. 042333, 2010.
[20] K.-Y. Kuo and C.-C. Lu, “A further study on the encoding complexity of quantum stabilizer codes,” in Proc. 2010 International Symposium on Information Theory and its Applications, 2010, pp. 1041–1044.
[21] P. K. Sarvepalli and A. Klappenecker, “Encoding subsystem codes,” arXiv preprint arXiv:0806.4954, 2008.