MOTS: Minimax Optimal Thompson Sampling

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Abstract

Thompson sampling is one of the most widely used algorithms for many online decision problems, due to its simplicity in implementation and superior empirical performance over other state-of-the-art methods. Despite its popularity and empirical success, it has remained an open problem whether Thompson sampling can achieve the minimax optimal regret $O(\sqrt{KT})$ for $K$-armed bandit problems, where $T$ is the total time horizon. In this paper, we solve this long open problem by proposing a new Thompson sampling algorithm called MOTS that adaptively truncates the sampling result of the chosen arm at each time step. We prove that this simple variant of Thompson sampling achieves the minimax optimal regret bound $O(\sqrt{KT})$ for finite time horizon $T$ and also the asymptotic optimal regret bound when $T$ grows to infinity as well. This is the first time that the minimax optimality of multi-armed bandit problems has been attained by Thompson sampling type of algorithms.

1 Introduction

The Multi-Armed Bandit (MAB) problem models the exploration and exploitation tradeoff in sequential decision processes and is typically described as a game between the agent and the environment with $K$ arms. The game proceeds in $T$ time steps. In each time step $t = 1, \ldots, T$, the agent plays an arm $A_t \in \{1, 2, \ldots, K\}$ based on the observation of the previous $t - 1$ time steps, and then observes a reward $r_t$ that is independently generated from a 1-subGaussian distribution with mean value $\mu_{A_t}$, where $\mu_1, \mu_2, \ldots, \mu_K \in \mathbb{R}$ are unknown. The goal of the agent is to maximize the cumulative reward over $T$ time steps. The performance of a strategy for MAB is measured by the expected cumulative difference over $T$ time steps between playing the best arm and playing the arm according to the strategy, which is also called the regret of a bandit strategy. Formally, the regret $R_\mu(T)$ is defined as follows

$$R_\mu(T) = T \max_{i \in \{1, 2, \ldots, K\}} \mu_i - \mathbb{E}_\mu \left[ \sum_{t=1}^{T} r_t \right]. \quad (1)$$

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For a fixed time horizon $T$, the problem-independent lower bound (Auer et al., 2002b) states that any strategy has at least a regret in the order of $O(\sqrt{KT})$, which is called the minimax-optimal regret or worse case optimal regret. On the other hand, for a fixed model (i.e., $\mu_1, \ldots, \mu_K$ are fixed), Lai and Robbins (1985); Katehakis and Robbins (1995) proved the asymptotically lower bound that any strategy must have at least $C(\mu) \log(T)(1 - o(1))$ regret when the horizon $T$ approaches infinity, where $C(\mu)$ is a constant depending on the model. A strategy with a regret upper-bounded by $C(\mu) \log(T)(1 - o(1))$ is called asymptotically optimal.

In this paper we aim at achieving the asymptotic optimality and minimax optimality for the earliest bandit strategy, Thompson Sampling (TS) (Thompson, 1933). It has been observed in practice that Thompson Sampling can achieve a better performance than many upper confidence bound (UCB)-based algorithms (Chapelle and Li, 2011; Wang and Chen, 2018). In addition, TS is natural, simple and easy to implement. Despite the aforementioned advantages, the theoretical analysis of TS has not been established until the past decade. In particular, Agrawal and Goyal (2012) and Kaufmann et al. (2012) proved the first regret bound of TS and showed that it is asymptotically optimal. Later, Agrawal and Goyal (2017) showed that TS using Beta distribution as the prior achieves $O(\sqrt{KT \log T})$ problem-independent regret bound while maintaining the asymptotic optimality as well. Moreover, Agrawal and Goyal (2017) also proved that TS with Gaussian prior can achieve an improved regret bound $O(\sqrt{KT \log K})$. Meanwhile, Agrawal and Goyal (2017) proved that the vanilla TS strategy with Gaussian prior has a problem-independent bound at least in the order of $O(\sqrt{KT \log K})$.

It remains an open problem (Li and Chapelle, 2012) that whether Thompson Sampling type algorithms can achieve the minimax optimal regret bound $O(\sqrt{KT})$ for MAB problems.

**Main Contributions.** In this paper, we solve this open problem by proposing a new Thompson Sampling algorithm called Minimax Optimal Thompson Sampling (MOTS), which clips the sampling results for each arm based on the history of pulls for the arm. We prove that our proposed MOTS algorithm achieves the asymptotic optimal and minimax optimal regret simultaneously. This is the first TS type algorithm that achieves the minimax optimal regret bound $O(\sqrt{KT})$. Our result also conveys the important message that the lower bound for vanilla TS strategy with Gaussian priors in Agrawal and Goyal (2017) may not hold in more general cases. Our experimental results also demonstrate the superiority of MOTS over the state-of-the-art bandit algorithms such as UCB (Auer et al., 2002a), MOSS (Audibert and Bubeck, 2009) and TS.

**Notations.** A random variable $X$ is said to follow 1-subGaussian distribution, if it holds that $\mathbb{E}_X[\exp(\lambda X - \lambda \mathbb{E}_X[X])] \leq \exp(\lambda^2/2)$ for all $\lambda \in \mathbb{R}$. We reserve the notation $c$ to represent universal positive constants that are independent of problem parameters. The specific value of $c$ can be different line by line. We use $T$ for total number of time steps, $K$ for number of arms and $[K]$ for set $\{1, 2, \ldots, K\}$. Without loss of generality, we assume $\mu_1 = \max_{i \in [K]} \mu_i$ throughout this paper. We use $\Delta_i$ to denote the gap between arm 1 and arm $i$, i.e., $\Delta_i := \mu_1 - \mu_i$, $i \in [K]/\{1\}$. We denote $T_i(t) := \sum_{j=1}^t 1\{A_j = i\}$ as the number of times that arm $i$ has been played at time step $t$ and $\hat{\mu}_i(t) := \sum_{j=1}^t 1\{A_j = i\} \cdot r_j/T_i(t)$ as the average reward for pulling arm $i$ up to time $t$, where $r_j$ is the reward received by the algorithm at time $j$.

$^1O(\cdot)$ notation hides constant factors.
Algorithm 1 Minimax Optimal Thompson Sampling (MOTS)

1: **Initialization:** For each arm $i = 1, 2, \cdots, K$, set $\hat{\mu}_i(1) = 0$ and $T_i(1) = 0$
2: **for** $t = 1, 2, \cdots, T$ **do**
3: For all $i \in [K]$, sample $\theta_i(t)$ independently from $D_i(t)$. Here $D_i(t) = \mathcal{N}^{\text{clip}}(\hat{\mu}_i(t), 1/T_i(t), \delta)$ for $T_i(t) \neq 0$; otherwise $D_i(t) = \mathcal{N}(0, 1)$
4: Play arm $A_t = \arg\max_{i \in [K]} \theta_i(t)$ and observe the reward $r_t$
5: For all $i \in [K]$
   \[ \hat{\mu}_i(t + 1) = \frac{T_i(t)\hat{\mu}_i(t) + r_t \mathbb{1}\{i = A_t\}}{T_i(t) + \mathbb{1}\{i = A_t\}} \]
6: For all $i \in [K]$: $T_i(t + 1) = T_i(t) + \mathbb{1}\{i = A_t\}$
7: **end for**

2 Minimax Optimal Thompson Sampling Algorithm

In this section, we propose a Minimax Optimal Thompson Sampling (MOTS) algorithm, whose details are displayed in Algorithm 1.

Specifically, MOTS maintains a distribution $D_i(t)$ for each arm $i = 1, \ldots, K$ at time step $t = 1, \ldots, T$ during execution, where $D_i(1)$ is initialized as the standard Gaussian distribution. At the $t$-th iteration of Algorithm 1, it samples instances $\theta_i(t)$ independently from distribution $D_i(t)$ for all $i \in [K]$. Then the agent plays the arm $A_t = \arg\max_{i \in [K]} \theta_i(t)$ and receives a reward $r_t$. The average reward $\hat{\mu}_i(t)$ and the number of pulls $T_i(t)$ for each arm are updated accordingly.

The main difference between MOTS and vanilla Thompson Sampling in Agrawal and Goyal (2017) is the choice of distribution $D_i(t)$. In Agrawal and Goyal (2017), $D_i(t)$ is chosen as the Gaussian distribution $\mathcal{N}(\hat{\mu}_i(t), 1/T_i(t))$. In contrast, we define $D_i(t)$ as a clipped Gaussian distribution $D_i(t) = \mathcal{N}^{\text{clip}}(\hat{\mu}_i(t), 1/T_i(t), \delta)$, where $\delta \in (1/2, 1)$ is an arbitrary constant. We describe the detailed procedure of sampling $\theta_i(t)$ from $D_i(t)$ of MOTS as follows.

**Sampling from a clipped Gaussian distribution:** At time step $t$, for all arm $i \in [K]$, we denote the following range

\[
R = \left( -\infty, \hat{\mu}_i(t) + \sqrt{\frac{4}{T_i(t)} \log^+(\frac{T}{KT_i(t)})} \right), \tag{2}
\]

where $\log^+(x)$ is defined as $\max\{0, \log x\}$. For arm $i$, we first sample an instance $\theta$ from Gaussian distribution $\mathcal{N}(\hat{\mu}_i(t), 1/(\delta T_i(t)))$. If $\theta \in R$, then return $\theta_i(t) = \theta$ as a sample from $D_i(t)$; otherwise return $\theta_i(t) = \hat{\mu}_i(t) + \sqrt{4/T_i(t)} \log^+(T/(KT_i(t)))$ as a sample from $D_i(t)$.

**Remark 1.** We would like to point out that the right endpoint in (2) resembles the upper confidence bound in MOSS (Audibert and Bubeck, 2009). Apart from the difference that MOTS is TS-type and MOSS is UCB-type, we claim that they are also very different from a theoretical perspective. Under the definition of $R$ in (2), we will prove in the next section that MOTS is both asymptotically optimal and minimax optimal. However, MOSS is only minimax optimal (Audibert and Bubeck, 2009). The improvement of MOSS to achieve asymptotic optimality is only recently developed in the KL-UCB++ algorithm (Ménard and Garivier, 2017) and the AdaUCB algorithm (Lattimore, 2018),
which can be seen as variants of MOSS. Both KL-UCB++ and AdaUCB need to reduce the constant factor $4$ in the right endpoint of $R$ defined in (2) to 2, which essentially decreases the exploration rate. Moreover, KL-UCB++ utilizes a more complicated upper confidence bound with an additional \( \log^2(T/(KT_i(t))) \) term and AdaUCB only works for Gaussian reward distributions.

In contrast, it is easy to verify that for MOTS the constant $4$ in (2) can be replaced by any constant larger than $4$ while maintaining the asymptotic optimality and minimax optimality. Therefore, MOTS is more robust in the choice of hyperparameter. It will be more suitable to design better algorithms based on MOTS, e.g., achieving instance-dependent optimality (see Lattimore (2018) for detail) while keeping the asymptotic optimality.

## 3 Main Theory

In this section, we present our main theory of MOTS.

**Theorem 1 (Minimax Optimality).** For any fixed $\delta \in (1/2, 1)$, there exists a universal constant $c > 0$ such that the regret of Algorithm 1 with $1$-subGaussian rewards satisfies

\[
R_\mu(T) \leq c\sqrt{KT} + \sum_{i=1}^{K} \Delta_i. \tag{3}
\]

The second term in the right hand side of (3) is due to the fact that we need to pull each arm at least once if $T > K$. Follow the convention in the literature (Audibert and Bubeck, 2009; Agrawal and Goyal, 2017), we only need to consider the case when $\sum_{i=1}^{K} \Delta_i$ is dominated by $O(\sqrt{KT})$.

**Remark 2.** Compared with the results in Agrawal and Goyal (2017), the regret bound of MOTS improves that of TS by a factor of $O(\sqrt{\log T})$ and improves that of TS with Gaussian priors by a factor of $O(\sqrt{\log K})$. This is the first time that a Thompson Sampling type algorithm achieves the minimax optimal regret $O(\sqrt{KT})$ for multi-armed bandit problems (Auer et al., 2002a), which also answers the open problem in Li and Chapelle (2012) where it is conjectured that Thompson samplings regret actually matches the lower bound and is indeed optimal.

**Theorem 2 (Asymptotic Optimality).** For any fixed $\delta \in (1/2, 1)$, the regret of Algorithm 1 with $1$-subGaussian rewards satisfies

\[
\lim_{T \to \infty} \frac{R_\mu(T)}{\log(T)} = \sum_{i: \Delta_i > 0} \frac{2}{\delta \Delta_i}. \tag{4}
\]

**Remark 3.** Theorem 2 indicates that the asymptotic regret rate $\lim_{T \to \infty} R_\mu/\log T$ of MOTS matches the asymptotic optimal rate $\sum_{i: \Delta_i > 0} 2/\Delta_i$ up to a multiplicative factor $1/\delta$, where $\delta \in (1/2, 1)$ is arbitrarily fixed. This is the same as that of vanilla TS in Agrawal and Goyal (2017), where the authors proved an asymptotic regret rate that matches the asymptotic optimal rate by a multiplicative factor $1 + \epsilon$, where $\epsilon \in (0, 1)$ is a fixed constant.

So far, we have assumed the reward follows an unknown subGaussian distribution. In the next theorem, we present an variant of MOTS that achieves the minimax optimality and asymptotic optimality for Gaussian reward distributions.
Theorem 3. If the reward of each arm $i$ follows a Gaussian distribution $N(\mu_i, 1)$, and the right endpoint of range $R$ in (2) is replaced by

$$R = \left( -\infty, \hat{\mu}_i(t) + \sqrt{\frac{2}{T_i(t)} \log^+ \left( \frac{T}{KT_i(t)} \right)} \right), \quad (5)$$

then Theorem 1 and Theorem 2 still hold.

3.1 Proof of the Minimax Optimality

The following lemma will be frequently used throughout our analysis, which characterises the concentration property of subGaussian random variables.

Lemma 1 (Lemma 9.3 in Lattimore and Szepesvári (2020)). Let $X_1, X_2, \cdots$ be independent and 1-subGaussian with zero mean. Denote $\hat{\beta}_t = 1/t \sum_{s=1}^{t} X_s$. Then for any $\Delta > 0$,

$$P\left( \exists \ s \geq 1 : \hat{\beta}_s + \sqrt{\frac{4}{s} \log^+ \left( \frac{T}{sK} \right)} + \Delta \leq 0 \right) \leq cK \Delta^2, \quad (6)$$

where $c > 0$ is a universal constant.

Let $\widehat{\mu}_i$ be the average reward of arm $i$ when it has been played $s$ times. Define

$$\Delta = \mu_1 - \min_{s \leq T} \left\{ \hat{\mu}_i + \sqrt{\frac{4}{s} \log^+ \left( \frac{T}{sK} \right)} \right\}. \quad (7)$$

The regret of Algorithm 1 can be decomposed as follows.

$$R_\mu(T) = \sum_{i: \Delta_i > 0} \Delta_i E[T_i(T)]$$

$$\quad \leq E[2T\Delta] + E \left[ \sum_{i: \Delta_i > 2\Delta} E[\Delta_i T_i(T)] \right]$$

$$\quad \leq E[2T\Delta] + 8\sqrt{KT} + E \left[ \sum_{i: \Delta_i > \max\{2\Delta, 8\sqrt{K/T}\}} \Delta_i T_i(T) \right]. \quad (8)$$

The first term in (8) can be bounded as:

$$E[2T\Delta] = 2T \int_{0}^{\infty} P(\Delta \geq x)dx$$

$$\quad \leq 2T \int_{0}^{\infty} \min \left\{ 1, \frac{cK}{T x^2} \right\} dx$$

$$\quad = 2T \int_{\sqrt{cK/T}}^{\infty} dx + 2cK \int_{\sqrt{cK/T}}^{\infty} \frac{1}{x^2}dx$$

$$\quad = 4\sqrt{cK}, \quad (9)$$
where \( c > 0 \) is a universal constant and the inequality comes from Lemma 1 since
\[
\mathbb{P}\left( \mu_1 - \min_{s \leq T} \left\{ \hat{\mu}_{1s} + \sqrt{\frac{4}{s} \log^+ \left( \frac{T}{sK} \right)} \right\} \geq x \right) \\
= \mathbb{P}\left( \exists 1 \leq s \leq T : \mu_1 - \hat{\mu}_{1s} - \sqrt{\frac{4}{s} \log^+ \left( \frac{T}{sK} \right)} - x \geq 0 \right) \\
\leq \frac{cK}{x^2 T}.
\]

(10)

Now we focus on \( \sum_{i: \Delta_i > \max(2\Delta, 8K/T)} \Delta_i T_i(T) \). Note that the update rules of Algorithm 1 ensure \( D_i(t + 1) = D_i(t) \) whenever \( A_t \neq i \). Hence, we can define \( D_{is} \) as the prior distribution of arm \( i \) when it has been played \( s \) times and obtain the following Lemma.

**Lemma 2** (Theorem 36.2 in Lattimore and Szepesvári (2020)). Let \( 0 < \epsilon \in \mathbb{R} \) be an arbitrary constant. Then the expected number of times that Algorithm 1 plays arm \( i \) is bounded by
\[
\mathbb{E}[T_i(T)] = \mathbb{E}\left[ \sum_{t=1}^{T} \mathbb{1}\{ A_t = i, E_i(t) \} \right] + \mathbb{E}\left[ \sum_{t=1}^{T} \mathbb{1}\{ A_t = i, E_i^c(t) \} \right] \\
\leq 1 + \mathbb{E}\left[ \sum_{s=0}^{T-1} \left( \frac{1}{G_{is}(\epsilon)} - 1 \right) \right] + \mathbb{E}\left[ \sum_{t=0}^{T-1} \mathbb{1}\{ A_t = i, E_i^c(t) \} \right] \\
\leq 1 + \mathbb{E}\left[ \sum_{s=0}^{T-1} \left( \frac{1}{G_{is}(\epsilon)} - 1 \right) \right] + \mathbb{E}\left[ \sum_{s=0}^{T-1} \mathbb{1}\{ G_{is}(\epsilon) > 1/T \} \right],
\]
\]
\[
\text{(11)}
\]
\[
\text{(12)}
\]

where \( G_{is}(\epsilon) = 1 - F_{is}(\mu_1 - \epsilon) \), \( F_{is} \) is the CDF of \( D_{is} \), and \( E_i(t) = \{ \theta_i(t) \leq \mu_1 - \epsilon \} \).

The above lemma is first proved by Agrawal and Goyal (2017) and here we use an improved version presented in Lattimore and Szepesvári (2020). We define
\[
m_{is} = \hat{\mu}_{is} + \sqrt{\frac{4}{s} \log^+ \left( \frac{T}{sK} \right)}.
\]
\[
\text{(13)}
\]

By the definition in (2), we know that \( m_{is} \) is the right endpoint of range \( R \) when \( T_i(t) = s \). From the definition of \( \Delta \) in (7) and note that \( \Delta_i > 2\Delta \), we obtain
\[
m_{1s} = \hat{\mu}_{1s} + \sqrt{\frac{4}{s} \log^+ \left( \frac{T}{sK} \right)} \geq \mu_1 - \Delta \geq \mu_1 - \frac{\Delta_i}{2}.
\]
\[
\text{(14)}
\]

Let \( \theta \) be sampled from the clipped distribution \( D_{is} \). Recall the **clipped sampling** procedure in Section 2. We can first sample \( \theta' \) from distribution \( \mathcal{N}(\hat{\mu}_{1s}, 1/(\delta s)) \). If \( \theta' > m_{is} \), we return \( \theta = m_{is} \); otherwise, we return \( \theta = \theta' \). Combining with (14), we know that \( \mathbb{P}(\theta' \geq \mu_1 - \Delta_i/2) = \mathbb{P}(\theta \geq \mu_1 - \Delta_i - \Delta_i/2) \).

Let \( F'_{is} \) be the CDF of \( \mathcal{N}(\hat{\mu}_{1s}, 1/(\delta s)) \) for \( s > 0 \) and CDF of \( \mathcal{N}(0, 1) \) for \( s = 0 \). Let \( G'_{is}(\epsilon) = 1 - F'_{is}(\mu_1 - \epsilon) \). Thus \( G_{is}(\Delta_i/2) = \mathbb{P}(\theta \geq \mu_1 - \Delta_i/2) = \mathbb{P}(\theta' \geq \mu_1 - \Delta_i/2) = G'_{is}(\Delta_i/2) \). Using (11)
of Lemma 2 and setting $\epsilon = \Delta_i/2$, we have
\[
\mathbb{E}[\Delta_i T_i(T)] \leq \Delta_i + \Delta_i \cdot \mathbb{E} \left[ \sum_{s=0}^{T-1} \left( \frac{1}{G_{1s}((\Delta_i/2)} - 1 \right) \right] + \Delta_i \cdot \mathbb{E} \left[ \sum_{t=0}^{T-1} \mathbb{I}\{A_t = i, E_t^c(t)\} \right]
\]
\[
= \Delta_i + \Delta_i \cdot \mathbb{E} \left[ \sum_{t=0}^{T-1} \mathbb{I}\{A_t = i, E_t^c(t)\} \right] + \Delta_i \cdot \mathbb{E} \left[ \sum_{s=0}^{T-1} \left( \frac{1}{G_{1s}((\Delta_i/2)} - 1 \right) \right].
\]

Bounding term $I_1$: Note that
\[
E_t^c(t) = \left\{ \theta_{t}(t) > \mu_1 - \frac{\Delta_i}{2} \right\} \subseteq \left\{ \hat{\mu}_i(t) + \sqrt{\frac{4}{T_{t}(t)} \log \left( \frac{T}{K T_{t}(t)} \right)} > \mu_1 - \frac{\Delta_i}{2} \right\}.
\]

We define the following notation.
\[
\kappa_i = \sum_{s=1}^{T} \mathbb{I}\left\{ \hat{\mu}_{is} + \sqrt{\frac{4}{s} \log \left( \frac{T}{sK} \right)} > \mu_1 - \mu_i \right\},
\]
which immediately implies
\[
I_1 = \Delta_i \cdot \mathbb{E} \left[ \sum_{t=0}^{T-1} \mathbb{I}\{A_t = i, E_t^c(t)\} \right] \leq \Delta_i \mathbb{E}[\kappa_i].
\]

The following lemma characterizes the bound for $\mathbb{E}[\kappa_i]$.

**Lemma 3.** Let $\omega > 0$ be a constant and $M_1, M_2, \ldots, M_n$ be 1-subGaussian random variables with zero means. Denote $\hat{\mu}_n = \sum_{s=1}^{n} M_s/n$. Then for any $N \leq T$,
\[
\sum_{n=1}^{N} \mathbb{P}\left( \hat{\mu}_n + \sqrt{\frac{4}{n} \log \left( \frac{N}{n} \right)} \geq \omega \right) \leq 1 + 4 \log^+(N\omega^2) + 2 \omega^2 + \sqrt{8\pi \log^+(N\omega^2)}.
\]

Since the proof of Lemma 3 is quite standard, we defer it to the appendix. Now we continue our proof of the minimax optimality of MOTS. Applying Lemma 3 to (17) yields
\[
\Delta_i \mathbb{E}[\kappa_i] = \Delta_i \sum_{s=1}^{T} \mathbb{P}\left\{ \hat{\mu}_{is} + \sqrt{\frac{4}{s} \log \left( \frac{T}{sK} \right)} > \mu_1 - \mu_i \right\}
\]
\[
\leq \Delta_i \sum_{s=1}^{T} \mathbb{P}\left\{ \hat{\mu}_{is} - \mu_i + \sqrt{\frac{4}{s} \log \left( \frac{T}{sK} \right)} > \frac{\Delta_i}{2} \right\}
\]
\[
\leq \Delta_i + \frac{8}{\Delta_i} + \frac{16}{\Delta_i} \left( \log^+ \left( \frac{T \Delta_i^2}{4K} \right) + \sqrt{2\pi \log^+ \left( \frac{T \Delta_i^2}{4K} \right)} \right).
\]

where the first inequality is due to the fact that $\mu_1 > \mu_i$. It is easy to verify that $x \mapsto x^{-1} \log^+(ax^2)$ is monotonically decreasing for $x \geq e/\sqrt{a}$ and any $a > 0$. Since $\Delta_i \geq 8\sqrt{K/T} > e/\sqrt{T/(4K)}$, we
have \( \log^+(T \Delta_2^2/(4K)) \leq 4 \log 2 \). Plugging this fact into (19), we have that \( \mathbb{E}[\Delta_i \kappa_i] \leq \Delta_i + c \sqrt{T/K} \), where \( c > 0 \) is a universal constant.

**Bounding term I_2:** We first prove the following lemma.

**Lemma 4.** There exists a universal constant \( c > 0 \) such that:

\[
\mathbb{E} \left[ \sum_{s=0}^{T-1} \left( \frac{1}{G'_{1s}(\epsilon)} - 1 \right) \right] \leq \frac{c}{\epsilon^2}.
\]  

(20)

**Proof of Lemma 4.** We decompose the proof of Lemma 4 into the proof of the following two statements: (i) there exists a universal constant \( c > 0 \) such that

\[
\mathbb{E} \left[ \frac{1}{G'_{1s}(\epsilon)} - 1 \right] \leq c, \quad \forall s,
\]  

(21)

and (ii) for \( L = \lceil 32/\epsilon^2 \rceil \), it holds that

\[
\mathbb{E} \left[ \sum_{s=L}^{T} \left( \frac{1}{G'_{1s}(\epsilon)} - 1 \right) \right] \leq \frac{4}{\epsilon^2} \left( 1 + \frac{16}{\epsilon^2} \right).
\]  

(22)

For \( s = 0, 1/G'_{1s}(\epsilon) \geq 1/2 \) and \( 1/G'_{1s}(\epsilon) - 1 \leq 1 \). For \( s > 0 \), let \( \Theta_s = \mathcal{N}(\hat{\mu}_{1s}, 1/(\delta s)) \) and \( Y_s \) be the random variable denoting the number of consecutive independent trials until a sample of \( \Theta_j \) becomes greater than \( \mu_1 - \epsilon \). Note that \( G'_{1s}(\epsilon) = \mathbb{P}(\theta \geq \mu_1 - \epsilon) \), where \( \theta \) is sampled from \( \Theta_s \). Hence we have

\[
\mathbb{E} \left[ \frac{1}{G'_{1s}(\epsilon)} - 1 \right] = \mathbb{E}[Y_s].
\]  

(23)

Consider an integer \( r \geq 1 \). Let \( z = \sqrt{2\delta' \log r} \), where \( \delta' \in (\delta, 1) \) and will be determined late. Let random variable \( M_r \) be the maximum of \( r \) independent samples from \( \Theta_s \). Define \( \mathcal{F}_s \) to be the filtration consisting the history of plays of Algorithm 1 up to the \( s \)-th pull of arm 1. Then it holds

\[
\mathbb{P}(Y_s < r) \geq \mathbb{P}(M_r > \mu_1 - \epsilon)
\]

\[
\geq \mathbb{E} \left[ \left( M_r > \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}}, \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}} \geq \mu_1 - \epsilon \right) \left| \mathcal{F}_s \right. \right]
\]

\[
= \mathbb{E} \left[ \mathbb{I} \left\{ \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}} \geq \mu_1 - \epsilon \right\} \cdot \mathbb{P} \left( M_r > \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}} \right) \right].
\]  

(24)

For a random variable \( Z \sim \mathcal{N}(\mu, \sigma^2) \), it holds by Formula 7.1.13 from Abramowitz and Stegun (1965) that

\[
\mathbb{P}(Z > \mu + x\sigma) \geq \frac{1}{\sqrt{2\pi} x^2 + 1} e^{-x^2/2}.
\]  

(25)

Therefore, it holds that

\[
\mathbb{P}(M_r > \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}}) \geq 1 - \left( 1 - \frac{1}{\sqrt{2\pi} \frac{z^2}{x^2 + 1}} e^{-z^2/2} \right)^r
\]

\[
= 1 - \left( 1 - \frac{r^{-\delta'}}{\sqrt{2\pi} \frac{2\delta' \log r}{x^2 + 1}} \right)^r
\]

\[
\geq 1 - \left( 1 - \frac{\sqrt{2\delta'} \log r}{x^2 + 1} \right)^r.
\]
\[ \geq 1 - \exp \left( - \frac{r^{1-\delta'}}{\sqrt{8\pi \log r}} \right), \]  

(26)

where the last inequality is due to \((1 - x)^r \leq e^{-rx}, 2\delta' \log r + 1 \leq 2\sqrt{2}\delta' \log r \) and \(\delta' < 1\). Let \(x = \log r\), then

\[
\exp \left( - \frac{r^{1-\delta'}}{\sqrt{8\pi \log r}} \right) \leq \frac{1}{r^2} \quad \iff \quad \exp((1 - \delta')x) \geq 2\sqrt{8\pi x^2}.
\]

It is easy to verify that for \(x \geq 10/(1 - \delta')^2\), \(\exp((1 - \delta')x) \geq 2\sqrt{8\pi x^2}\). Hence, if \(r \geq e^{10} \cdot \exp[1/(1 - \delta')^2]\), we have \(\exp(-r^{1-\delta'}/(\sqrt{8\pi \log r})) \leq 1/r^2\). Thus, we have

\[
P \left( M_r > \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}} \right) \geq 1 - \frac{1}{r^2}.
\]

(27)

For any \(\epsilon > 0\), it holds that

\[
P \left( \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}} \geq \mu_1 - \epsilon \right) \geq P \left( \hat{\mu}_{1s} + \frac{z}{\sqrt{\delta s}} \geq \mu_1 \right) \geq 1 - \exp(-z^2/(2\delta)) = 1 - \exp(-\delta'/\delta \log r) = 1 - r^{-\delta'/\delta}.
\]

(28)

where the second equality is due to Lemma 8. Therefore, for \(r \geq e^{10} \cdot \exp[1/(1 - \delta')^2]\), substituting (27) and (28) into (24) yields

\[
P(Y_s < r) \geq 1 - r^{-2} - r^{-\frac{\delta'}{\delta}}.
\]

(29)

For any \(\delta' > \delta\), this gives rise to

\[
E[Y_s] = \sum_{r=0}^{T} P(Y_s \geq r) \leq e^{10} \cdot \exp \left[ \frac{1}{(1 - \delta')^2} \right] + \sum_{r \geq 1} \frac{1}{r^2} + \sum_{r \geq 1} r^{-\frac{\delta'}{\delta}}.
\]

\[
\leq e^{10} \cdot \exp \left[ \frac{1}{(1 - \delta')^2} \right] + 2 + 1 + \int_{x=1}^{\infty} x^{-\frac{\delta'}{\delta}} \, dx
\]

\[
\leq 2e^{10} \cdot \exp \left[ \frac{1}{(1 - \delta')^2} \right] + \frac{1}{(1 - \delta) - (1 - \delta')},
\]

where \(c\) is a universal constant. Let \(1 - \delta' = (1 - \delta)/2\). We further obtain

\[
E \left[ \frac{1}{G'_{1s}(\epsilon)} - 1 \right] \leq 2e^{10} \cdot \exp \left[ \frac{4}{(1 - \delta)^2} \right] + \frac{2}{1 - \delta}.
\]

(30)
Since $\delta \in (1/2, 1)$ is fixed, then there exists a universal constant $c > 0$ such that

$$
\mathbb{E} \left[ \frac{1}{G_1'(\epsilon)} - 1 \right] \leq c. \tag{31}
$$

Now, we turn to proving (22). Let $E_s$ be the event that $\hat{\mu}_{1s} \geq \mu_1 - \epsilon/2$. Let $L = \lfloor 32/e^2 \rfloor$. Let $X_{1s} \sim \mathcal{N}(\hat{\mu}_{1s}, 1/(\delta s))$ be a distributed random variable. Under the event $E_s$, using the upper bound of Lemma 7 with $z = \epsilon/(2\sqrt{1/(\delta s)})$, we obtain

$$
\mathbb{P}(X_{1s} > \mu_1 - \epsilon) \geq \mathbb{P}(X_{1s} > \hat{\mu}_{1s} - \epsilon/2) \geq 1 - 1/2 \exp(-s\delta \epsilon^2/8). \tag{32}
$$

Then, we have

$$
\mathbb{E} \left[ \frac{1}{G_1'(\epsilon)} - 1 \right] = \mathbb{E} \left[ \frac{1}{\mathbb{P}(X_{1s} > \mu_1 - \epsilon)} - 1 \right] \\
\leq \mathbb{E} \left[ \frac{1}{\mathbb{P}(X_{1s} > \mu_1 - \epsilon \mid E_s) \cdot \mathbb{P}(E_s)} - 1 \right] \\
\leq \mathbb{E} \left[ \frac{1}{(1 - 1/2 \exp(-s\delta \epsilon^2/8)) \mathbb{P}(E_s)} - 1 \right]. \tag{33}
$$

Applying Lemma 8, we have

$$
\mathbb{P}(E_s) = \mathbb{P}\left(\hat{\mu}_{1s} \geq \mu_1 - \frac{\epsilon}{2}\right) \geq 1 - \exp\left(-\frac{s\epsilon^2}{8}\right) \geq 1 - \exp(-s\delta \epsilon^2/8). \tag{34}
$$

Substituting the above inequality into (33) yields

$$
\mathbb{E} \left[ \sum_{s=L}^{T} \left( \frac{1}{G_1'(\epsilon)} - 1 \right) \right] \leq \sum_{s=L}^{T} \left[ \frac{1}{(1 - \exp(-s\delta \epsilon^2/8))^2} - 1 \right] \\
\leq \sum_{s=L}^{T} 4 \exp\left(-\frac{s\epsilon^2}{16}\right) \\
\leq 4 \int_{L}^{\infty} \exp\left(-\frac{s\epsilon^2}{16}\right) ds + \frac{4}{\epsilon^2} \\
\leq 4 \frac{1}{\epsilon^2} \left( 1 + \frac{16}{\epsilon^2} \right).
$$

The second inequality follows since $1/(1 - x)^2 - 1 \leq 4x$, for $x \leq 1 - \sqrt{2}/2$ and $\exp(-L\delta \epsilon^2/8) \leq 1/e^2$. We complete the proof of Lemma 4 by combining (21) and (22).

From Lemma 4, we immediately obtain

$$
I_2 = \Delta_i \mathbb{E} \left[ \sum_{s=0}^{T-1} \left( \frac{1}{G_1'(\Delta_i/2)} - 1 \right) \right] \leq c\sqrt{\frac{T}{K}}. \tag{35}
$$

Substituting (9), (15), (19) and (35) into (8), we complete the proof of Theorem 1.
3.2 Proof of the Asymptotic Optimality

We first prove the following technical lemma.

**Lemma 5.** For any $\epsilon > 0$ that satisfies $\epsilon + \epsilon_T < \Delta_i$, it holds that

$$
\mathbb{E} \left[ \sum_{s=0}^{T-1} \mathbb{1} \{ G_{is}'(\epsilon) > 1/T \} \right] \leq 1 + \frac{2}{e^2T} + \frac{2 \log T}{\delta(\Delta - \epsilon - \epsilon_T)^2}.
$$

**Proof of Lemma 5.** For sufficiently small $\epsilon_T > 0$ such that $\epsilon_T + \epsilon < \Delta_i$, which also implies $\mu_i + \epsilon_T < \mu_1 - \epsilon$. Applying Lemma 8, we have

$$
\mathbb{P}(\hat{\mu}_{is} > \mu_i + \epsilon_T) \leq \exp(-se^2T/2).
$$

Furthermore,

$$
\sum_{s=1}^{\infty} \exp \left( - \frac{se^2T}{2} \right) \leq \frac{1}{\exp(e^2T/2) - 1} \leq \frac{2}{e^2T}.
$$

(36)

where the last inequality is due to the fact $1 + x \leq e^x$ for all $x$. Define $L_i = 2 \log T/(\delta(\Delta - \epsilon - \epsilon_T)^2)$. For $s \geq L_i$ and $X_{is}$ sampled from $\mathcal{N}(\hat{\mu}_{is}, 1/(\delta s))$, if $\hat{\mu}_{is} \leq \mu_i + \epsilon_T$, then using Gaussian tail bound in Lemma 7, we obtain

$$
\mathbb{P}(X_{is} \geq \mu_1 - \epsilon) \leq \frac{1}{2} \exp \left( - \frac{\delta s(\hat{\mu}_{is} - \mu_1 + \epsilon)^2}{2} \right) \leq \frac{1}{2} \exp \left( - \frac{\delta s(\mu_1 - \epsilon - \mu_i - \epsilon_T)^2}{2} \right) = \frac{1}{2} \exp \left( - \frac{\delta s(\Delta_i - \epsilon - \epsilon_T)^2}{2} \right) \leq \frac{1}{T}.
$$

(37)

Let $Y_i$ be the event that $\hat{\mu}_{is} \leq \mu_i + \epsilon_T$ holds for all $s$. We further obtain

$$
\mathbb{E} \left[ \sum_{s=0}^{T-1} \mathbb{1} \{ G_{is}'(\epsilon) > 1/T \} \right] \leq \mathbb{E} \left[ \sum_{s=0}^{T-1} \mathbb{1} \{ G_{is}'(\epsilon) > 1/T \} \mid Y_i \right] + \mathbb{P}[Y_i]
$$

$$
\leq \sum_{s=[L_i]}^{T} \mathbb{E} \left[ \mathbb{1} \{ \mathbb{P}(X_{is} > \mu_1 - \epsilon) > 1/T \} \mid Y_i \right] + [L_i] + \mathbb{P}[Y_i]
$$

$$
\leq [L_i] + \mathbb{P}[Y_i] \leq 1 + \frac{2}{e^2T} + \frac{2 \log T}{\delta(\Delta - \epsilon - \epsilon_T)^2}.
$$

(38)

where the third inequality is from (37) and the last inequality is from (36).

Now we prove the asymptotic optimality of MOTS.

**Proof of Theorem 2.** Let $Z(\epsilon)$ be the following event

$$
Z(\epsilon) = \left\{ \forall s < T : \hat{\mu}_{is} + \sqrt{\frac{4}{s} \log_+ \left( \frac{T}{sK} \right)} \geq \mu_1 - \epsilon \right\}.
$$
For any arm $i \in [K]$, we have

$$E[T_i(T)] \leq E[T_i(T) | Z(\epsilon)] \mathbb{P}(Z(\epsilon)) + T(1 - \mathbb{P}[Z(\epsilon)])$$

$$\leq 1 + E\left[ \left( \sum_{s=0}^{T-1} \left( \frac{1}{G_{1s}(\epsilon)} - 1 \right) \right) | Z(\epsilon) \right] + T(1 - \mathbb{P}[Z(\epsilon)]) + \mathbb{E} \left[ \sum_{s=0}^{T-1} I\{G_{is}(\epsilon) > 1/T\} \right]$$

$$\leq 1 + E\left[ \sum_{s=0}^{T-1} \left( \frac{1}{G'_{1s}(\epsilon)} - 1 \right) \right] + T(1 - \mathbb{P}[Z(\epsilon)]) + \mathbb{E} \left[ \sum_{s=0}^{T-1} I\{G_{is}(\epsilon) > 1/T\} \right]$$

$$\leq 1 + E\left[ \sum_{s=0}^{T-1} \left( \frac{1}{G'_{1s}(\epsilon)} - 1 \right) \right] + T(1 - \mathbb{P}[Z(\epsilon)]) + \mathbb{E} \left[ \sum_{s=0}^{T-1} I\{G_{is}(\epsilon) > 1/T\} \right],$$

(39)

where the second inequality is due to (12) in Lemma 2, the third inequality is due to the fact that condition on event $Z(\epsilon)$, $G_{1s}(\epsilon) = G'_{1s}(\epsilon)$ and the last inequality is due to the fact that $G_{is}(\epsilon) = G'_{is}(\epsilon)$ for

$$\hat{\mu}_{is} + \sqrt{s \log_2 \left( \frac{T}{sK} \right)} \geq \mu_1 - \epsilon,$$

(40)

and $G_{is}(\epsilon) = 0 \leq G'_{is}(\epsilon)$ for

$$\hat{\mu}_{is} + \sqrt{s \log_2 \left( \frac{T}{sK} \right)} < \mu_1 - \epsilon.$$  

(41)

Let $\epsilon = \epsilon_T = 1/\log \log T$. Applying Lemma 1, we have

$$T(1 - \mathbb{P}[Z(\epsilon)]) \leq T \cdot \frac{cK}{\epsilon_T} \leq cK(\log \log T)^2.$$  

(42)

Using Lemma 4, we have

$$\mathbb{E} \left[ \sum_{s=0}^{T-1} \left( \frac{1}{G'_{1s}(\epsilon)} - 1 \right) \right] \leq c(\log \log T)^2.$$  

(43)

Furthermore using Lemma 5, we obtain

$$\mathbb{E} \left[ \sum_{s=0}^{T-1} I\{G'_{is}(\epsilon) > 1/T\} \right] \leq 1 + 2(\log \log T)^2 + \frac{2 \log T}{\delta(\Delta_i - 2/\log \log T)^2}.$$  

(44)

Combine (39), (42), (43) and (44) together, we finally obtain

$$\lim_{T \to \infty} \frac{\mathbb{E}[\Delta_i T_i(T)]}{\log T} = \frac{2}{\delta \Delta_i}.$$  

(45)

This completes the proof for the asymptotic optimality. 

\[\square\]

3.3 Proof of Theorem 3

The difference between the range defined in (2) and the modified range defined in (5) only influences the concentration inequality used in Lemma 1, which can be replaced by the following lemma for
Gaussian random variables.

Lemma 6 (Lattimore (2018)). Let \(X_1, X_2, \ldots\) be independent Gaussian random variables with zero mean and variance 1. Denote \(\hat{\beta}_t = 1/t \sum_{s=1}^{t} X_s\). Then for any \(\Delta > 0\),

\[
P\left( \exists s \geq 1 : \hat{\beta}_s + \sqrt{\frac{2}{s} \log \left( \frac{T}{sK} \right)} + \Delta \leq 0 \right) \leq cK \frac{T}{T\Delta^2},
\]

where \(c\) is a universal constant.

By part (a) of Lemma 12 in Lattimore (2018) with \(\alpha = T/K, d = 1\) and \(\lambda_1 = \infty\), Lemma 6 follows.

Then the proof of Theorem 3 is the same as previous proofs and thus we omit it for simplicity.

4 Experiments

![Figure 1: The regret for \(K = 10\) and \(\epsilon \in \{0.2, 0.1, 0.05\}\). The experiments are averaged over 2000 repetitions.](image)

In this section, we experimentally compare our algorithm MOTS with existing popular algorithms for MAB including MOSS (Audibert and Bubeck, 2009), the first minimax optimal algorithm, UCB (Auer et al., 2002a), the asymptotically optimal algorithm, and Thompson Sampling (TS for short) (Agrawal and Goyal, 2017). We evaluate the algorithms when the number of arms \(K\) is 10 and 50 respectively. Each arm follows independent Gaussian distributions. The best arm has expected reward mean 1 and variance 1, while the other \(K - 1\) arms have expected reward mean \(1 - \epsilon\) and variance 1. We vary \(\epsilon\) with values 0.2, 0.1, 0.05 in different experiments. The total number of time steps \(T\) is set to \(10^6\). In all experiments, the parameter \(\delta\) in Algorithm 1 is set to 0.9999.

Since we focus on Gaussian rewards, we use the range \(R\) in Theorem 3, i.e.,

\[
R = \left( -\infty, \hat{\mu}_i(t) + \sqrt{\frac{2}{T_i(t)} \log^+ \left( \frac{T}{K T_i(t)} \right)} \right).
\]
When $K = 10$, Figures 1(a), 1(b), and 1(c) report the regrets of all algorithms when $\epsilon$ is 0.2, 0.1, 0.05 respectively. As shown in all the figures, MOTS consistently outperforms the baselines for all $\epsilon$ values, especially when time step $t$ is large. For instance, in Figure 1(a), when time step $t$ is $T = 10^6$, the regret of MOTS is 567, while the regrets of TS, MOSS, and UCB are 730, 912, and 1543 respectively.

When $K = 50$, Figures 2(a), 2(b), and 2(c) report the regrets of MOTS, MOSS, TS, and UCB when $\epsilon$ is 0.2, 0.1, 0.05 respectively. Again, MOTS has the smallest regret in all settings when varying the time step $t$, and consistently outperforms all baselines.

In summary, our algorithm proposed algorithm MOTS consistently outperforms TS, MOSS and UCB when varying $\epsilon$, $K$, and $t$.

5 Related Work

On regret minimization for stochastic bandit problems, there are two types of optimality widely studied in the literature, i.e., asymptotic optimality and minimax optimality. UCB (Garivier and Cappé, 2011; Maillard et al., 2011), Bayes UCB (Kaufmann, 2016), and Thompson Sampling (Kaufmann et al., 2012; Agrawal and Goyal, 2017) are all shown to be asymptotically optimal. In terms of minimax optimality, MOSS (Audibert and Bubeck, 2009) is the first method proved to be minimax optimal. Recently, UCB-based methods AdaUCB (Lattimore, 2018) and KL-UCB++ (Ménard and Garivier, 2017) are also proved to be minimax optimal. Moreover, AdaUCB is also proved to be almost instance-dependent optimal for Gaussian multi-armed bandit problems (Lattimore, 2018).

There are also other studies on regret minimization for stochastic bandits, including explore-then-commit (Auer and Ortner, 2010; Perchet et al., 2016) and $\epsilon$-Greedy (Auer et al., 2002a). However, these studies are proved to be suboptimal (Garivier et al., 2016; Auer et al., 2002a). One exception is the recent proposed double explore-then-commit algorithm (Jin et al., 2020), which is proved to achieve the asymptotic optimality. Another line of works study different variants of problem settings, such as batched bandit problem (Gao et al., 2019) and bandit with delayed feedback (Pike-Burke et al., 2018). We refer interested readers to Lattimore and Szepesvári (2020) for a more comprehensive overview of bandit algorithms.
6 Conclusion and Future Work

We solved a long open problem of the minimax optimality for Thompson Sampling (Li and Chapelle, 2012). We proposed the MOTS algorithm and proved that it achieves the minimax optimal regret $O(\sqrt{KT})$ and the asymptotically optimal regret for $K$-armed bandit problems. Our experiments demonstrate the superior performance of MOTS compared with the state-of-the-art solutions.

In the future, we plan to extend our algorithm to handle exponential distributions. Another interesting direction is to extend MOTS for unknown time horizon $T$, as $T$ is not always available.

A Proof of Lemma 3

Proof of Lemma 3. Let $\gamma = 4 \log(n) + 4 (N\omega^2)/\omega^2$. Note that for $n \geq \gamma$, it holds that $n\omega^2 \geq 4$ and

$$\omega \sqrt{\frac{\gamma}{n}} = \sqrt{\frac{4}{n} \log^+(\frac{n}{N})} \geq \sqrt{\frac{4}{n} \log^+(\frac{N}{n})}. \quad (47)$$

Therefore, we have

$$\sum_{n=1}^{T} P\left(\hat{\mu}_n + \sqrt{\frac{4}{n} \log^+(\frac{n}{N})} \geq \omega \right) \leq \gamma + \sum_{n=\lceil \gamma \rceil}^{T} P\left(\hat{\mu}_n \geq \omega \left(1 - \sqrt{\frac{\gamma}{n}}\right)\right) \leq \gamma + \sum_{n=\lceil \gamma \rceil}^{\infty} \exp\left(-\frac{\omega^2 (\sqrt{n} - \sqrt{\gamma})^2}{2}\right) \leq \gamma + 1 + \int_{\gamma}^{\infty} \exp\left(-\frac{\omega^2 (\sqrt{x} - \sqrt{\gamma})^2}{2}\right) dx \leq \gamma + 1 + \frac{2}{\omega} \int_{0}^{\infty} \left(\frac{y}{\omega} + \sqrt{\gamma}\right) \exp(-y^2/2) dy \leq \gamma + 1 + \frac{2}{\omega^2} + \frac{\sqrt{2\pi\gamma}}{\omega}, \quad (48)$$

where (48) is the result of Lemma 8 and (49) is due to the fact that $\int_{0}^{\infty} y \exp(-y^2/2) dy = 1$ and $\int_{0}^{\infty} \exp(-y^2/2) dy = \sqrt{2\pi}/2$. (49) immediately implies the claim of Lemma 3:

$$\sum_{n=1}^{T} P\left(\hat{\mu}_n + \sqrt{\frac{4}{n} \log^+(\frac{n}{N})} \geq \omega \right) \leq \gamma + \sum_{n=\lceil \gamma \rceil}^{T} P\left(\hat{\mu}_n \geq \omega \left(1 - \sqrt{\frac{\gamma}{n}}\right)\right) \leq \gamma + 1 + \frac{2}{\omega^2} + \frac{\sqrt{2\pi\gamma}}{\omega}. \quad (50)$$

Plugging $\gamma = 4 \log^+(N\omega^2)/\omega^2$ into the above inequality, we obtain

$$\sum_{n=1}^{T} P\left(\hat{\mu}_n + \sqrt{\frac{4}{n} \log^+(\frac{n}{N})} \geq \omega \right) \leq 1 + \frac{4 \log^+(N\omega^2)}{\omega^2} + \frac{2}{\omega^2} + \frac{\sqrt{8\pi \log^+(N\omega^2)}}{\omega^2}, \quad (51)$$

which completes the proof \qed
B Auxiliary Lemmas

In this section, we present auxiliary lemmas used in this paper.

**Lemma 7** (Abramowitz and Stegun (1965)). For a Gaussian distributed random variable $Z$ with mean $m$ and variance $\sigma^2$, for any $z$,

$$
\mathbb{P}( | Z - m | > z\sigma ) \leq \frac{1}{2} \exp \left( -\frac{z^2}{2} \right).
$$

(52)

**Lemma 8** (Lattimore and Szepesvári (2020)). Assume that $X_1, \ldots, X_n$ are independent, $\sigma$-subGaussian random variables centered around $\mu$. Then for any $\epsilon > 0$

$$
\mathbb{P}(\hat{\mu} \geq \mu + \epsilon) \leq \exp \left( -\frac{n\epsilon^2}{2\sigma^2} \right) \quad \text{and} \quad \mathbb{P}(\hat{\mu} \leq \mu - \epsilon) \leq \exp \left( -\frac{n\epsilon^2}{2\sigma^2} \right),
$$

(53)

where $\hat{\mu} = 1/n \sum_{t=1}^n X_t$.

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