Subaging in the transport on a corrugated ratchet potential
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We study far from equilibrium transport of periodically driven overdamped Brownian particles moving in a rocking ratchet potential with quenched correlated disorder. Anomalous transport in its both forms, subdiffusion and superdiffusion has been observed at long times. The origin of these anomalies are due to how the particles explore the landscape. When most of the particles explore every landscape detail, the movement is subdiffusive. However, when most particles lose the landscape details and just some particles get trapped in traps imposed by the quenched disorder, the collective movement tends to disperse and the motion is superdiffusive. Weak breaking of ergodicity and subaging are obtained in forms of anomalous transport. Then, the ergodicity is recovered, for normal movements. Thus, our findings provide a link between aging and transport in periodic potentials with quenched disorder.

I. INTRODUCTION

Disorder may lead to anomalous behaviors characterized by significant variations of the observed dynamical property from sample to sample, for finite time scales. These variations produce systematic deviations of the dynamical property mean value which are usually much larger than statistical errors.

A particular class of systems where anomalous behaviors of this type occur are the so-called glass-forming systems. These systems include, for example, disordered spin systems close to the spin-glass transition, supercooled molecular liquids and jamming colloidal solutions. The physics of a glass-forming liquid is such that when rapidly undercooled under its melting temperature, it loses its ability to flow on experimental time-scales and freezes in an amorphous state with times required to restore equilibrium than can grow enormously or become infinite. This is the characteristic sign of aging systems, systems whose properties depend on the age of the system that can be quantified through the so-called relaxation or waiting time, which for glass-forming liquid, turns out to be proportional to the viscosity of the fluid.

Anomalous behavior due to disorder may occur in particle motion on rough potentials leading to important changes in transport and diffusive behavior. Among the more impressive effects are anomalous diffusion and the several orders of magnitude increase in the diffusion coefficient in correlated disordered tilted periodic potentials. These anomalous behaviors were observed experimentally in the motion on corrugated optical vortices. In general, the dynamics of these systems is extremely sensitive to changes in external parameters as happens with glass-forming systems.

II. MODEL AND SIMULATION METHOD

The model is defined by considering the overdamped motion of identical non-interacting Brownian particles in a rocking ratchet potential with a quenched disorder. The stochastic differential equation (Langevin equation)
for such particles is given by

$$\gamma \dot{z} = F_0 \sin(\omega t) + \xi(t) - dU(x)/dx, \quad (1)$$

where the left-hand side describes a frictional force experienced by the particles when they move relative to their environment, with \( \gamma \) being the drag coefficient. The first term of the right-hand side is an external sinusoidal force with a frequency \( \omega \) and amplitude \( F_0 \) that pushes the particles left and right periodically (rocking ratchet). The second term is Gaussian thermal noise at temperature \( T \). The correlation function of the noise obeys the fluctuation-dissipation relation 

$$\langle \xi(t)\xi(t') \rangle = 2\gamma k_BT \delta(t-t').$$

Ultimately, the last term of the right-hand side in Eq. (1) is the force due to the ratchet potential \( V_p(x) \) and the quenched spatial disorder \( V_r(x) \). They contribute to the total potential \( U(x) \) through the parameter \( \sigma = [0, 1] \) according to

$$U(x) = (1-\sigma)V_p(x) + \sigma V_r(x). \quad (2)$$

Here \( V_p(x) \) is the archetypal ratchet potential of double-well form

$$V_p(x) = -V_0 \sin(2\pi x/\lambda_p) + \mu/2 \sin(4\pi x/\lambda_p), \quad (3)$$

where \( V_0, \lambda_p \) and \( \mu = [0, 0.5] \) are the amplitude, the spatial period and the asymmetry, respectively and \( V_r(x) \) is a Gaussian spatially random potential with correlation function

$$g_r(x) = \langle V_r(x)V_r(0) \rangle = g_0 \exp\left(-2\pi^2 (x/l_r)^2 \right). \quad (4)$$

where \( g_0 \) is the amplitude and \( l_r \) is the ratchet potential roughness. Indeed, smaller values of \( l_r \) lead to a greater number and height of the barriers on the potential \( V_p(x) \).

In Fig. 1(a), an example of the contribution of disorder to the ratchet potential is shown, set for one realization of the potential in one spatial period \( \lambda_p = 2\pi \), asymmetry \( \mu = 0.1 \) and disorder \( l_r = 0.095\lambda_p \). The heights and locations of the barriers are random. Adding this disorder (part (a) of Fig. 1) to the contribution of the ratchet potential (part (b) of the same Fig. 1) results in a disorder level perceptible in the scale of the total potential (part (c)). Note also that the number of barriers is indeed significant in one period \( \lambda_p \).

In addition, in order to ensure that the amplitude of the total potential is of the order of \( V_0 \) independently of \( \sigma \), we choose correlations of both ratchet and random potentials to be the same at \( x = 0 \). The correlation function for the ratchet potential is

$$g_p(x) = V_0^2/2 \left[ \cos(2\pi x/\lambda_p) + (\mu/2)^2 \cos(4\pi x/\lambda_p) \right].$$

Thus, from Eqs. (4) and (5) and matching \( g_r(0) = g_p(0) \), the amplitude of the correlation function (4) is \( g_0 = 17V_0^2/32 \).

The equation of motion (1) can be reduced into a dimensionless form in terms of the rescaled spatial and temporal quantities \( z = 2\pi x/\lambda_p \) and \( \tau = [(2\pi)^2 V_0/\gamma \lambda_p^2] t \) as

$$\dot{z} = - (1-\sigma)f_p(z) - \sigma/\lambda f_r(z/\lambda) + \tilde{F}_0 \sin(\Omega \tau) + \eta(\tau), \quad (6)$$

where \( f_p \) and \( f_r \) are the dimensionless forces arising from the ratchet potential \( V_p \) and random potential \( V_r \), respectively. \( \Omega \), \( \tilde{F}_0 \) and \( f_r \) are the frequency and amplitude of the applied dimensionless sinusoidal force, and \( \eta(\tau) \) is the dimensionless noise. The dimensionless parameters

$$\lambda = l_r/\lambda_p, \quad \tilde{F}_0 = F_0 \lambda_p/2\pi V_0, \quad \text{and} \quad \tilde{T} = k_BT/V_0 \quad (7)$$

combined with \( \sigma = [0, 1] \) and \( \Omega = \omega (2\pi)^2 V_0/\gamma \lambda_p^2 \) define the motion of particles subject to Eq. (6). By using Eqs. (7), the dimensionless noise correlation function is

$$\langle \eta(\tau)\eta(\tau') \rangle = 2T \delta(\tau - \tau').$$

Note that, in Eq. (6), a decrease of \( \lambda \) even for fixed \( \sigma \) leads to an increase in the relative contribution of the random force. Throughout this work we set \( \sigma = 0.025 \) and \( \lambda = 0.095 \), associated with a rough total potential. As we shall see in the next section, increasing \( \lambda \) diminishes the effects of the disorder and loses variants of anomalous transport. In addition, \( \tilde{\Omega} \) and \( \lambda_p \) are also set to 0.1 and 2\pi, respectively. Variations in \( \Omega \) do not lead to any additional phenomenology, while the specific choice of \( \lambda_p \) is only important as a reference value.

We have carried out numerical simulations of the Eq. (6) over a large number \( (10^3 - 10^4) \) of particle trajectories, each one in a different random potential. The method to generate the random force values is described in [13] in more detail. In order to characterize the transport, we measured the mean-square displacement \( MSD \) and the
mean velocity \( v \) of the particles, which are given by

\[
MSD(\tau) = \left\langle (z(\tau) - \langle z(\tau) \rangle)^2 \right\rangle, \\
v(\tau) = \frac{\left\langle z(\tau + \Delta\tau) - z(\tau) \right\rangle}{\Delta\tau},
\]

where \( \langle \ldots \rangle \) indicates the average taken over the many trajectories and \( \Delta\tau \) is chosen equal to \( 2\pi/\Omega \) (see next section). We also computed the two-time velocity correlation

\[
C_v(\tau, \tau_w) = \frac{\left\langle v(\tau) v(\tau_w) \right\rangle - \left\langle v(\tau) \right\rangle \left\langle v(\tau_w) \right\rangle}{\left\langle v(\tau_w)^2 \right\rangle - \left\langle v(\tau_w) \right\rangle^2}
\]

to analyze the ergodicity breaking, with \( \tau \geq \tau_w \). Note that, the velocity correlation \( C_v(\tau, \tau_w) = 1 \), for \( \tau = \tau_w \), and it decreases as \( \tau \) moves away from \( \tau_w \).

### III. RESULTS AND DISCUSSION

#### A. Anomalous transport on the corrugated ratchet potential

The quenched noise can have two kinds of effects on diffusion properties: It may only affect the value of the transport coefficients (velocity, diffusion constant, etc.) as compared to the ordered system or it may alter in various ways the diffusion behavior \([17]\). We are mainly concerned here with anomalous diffusion phenomena, where the second kind of effect takes place. The first effect occurs when the external force amplitude is less than the amplitude of the ratchet in which, due to the quenched disorder, a transient subdiffusive behavior is found and then the particles achieve a constant mean velocity with normal diffusion. The second kind of effect is obtained when the external force amplitude is greater than the amplitude of the ratchet in which a variety of anomalous behaviors are obtained. Subdiffusive and superdiffusive transports are shown in Fig. 2(a) and (b), respectively, in which \( MSD \) versus \( \tau \) is plotted for \( \mu = 0.1 \), \( \lambda = 0.095 \), \( \tilde{F}_0 = 1.47 \) and different temperatures. The black curve in part (a), corresponds to \( \tilde{T} = 0 \), in which, after a superdiffusive transient all the particles are stopped due to the barriers imposed by quenched disorder \([13]\). Without thermal noise, the movement has just a transient nature. In contrast, as \( \tilde{T} \) increases, anomalous diffusion is reached, at long times. The \( MSD \) scales with time as \( MSD(\tau) \sim \tau^\beta \), where the exponent \( \beta \) characterizes the different regimes observed: \( \beta < 1 \) corresponds to subdiffusion, \( \beta > 1 \) to superdiffusion and \( \beta = 1 \) to normal diffusion. The exponent \( \beta \) for different temperatures is shown in the second column of table \([4]\). The motion tends to be subdiffusive at low temperatures, superdiffusive at intermediate temperatures and Brownian at high temperatures. The different regimes are related to how the particles explore the landscape. At low temperatures, most of the particles explore every landscape detail and the movement is subdiffusive. The thermal noise makes the particles jump over many disorder potential peaks and less particles are trapped. In this case, the collective movement tends to disperse and it is superdiffusive. Finally, normal diffusion appears when high enough temperatures dominate particle motion.

The spatial disorder slows down the particle collective motion and the mean velocity decreases over time (subtransport) or at most, it can be constant. This is clearly observed in Fig. 3, where we show the mean velocity as a function of \( \tau \), for the same values of thermal noise as those corresponding to the previous figure. The mean velocity is constant or tends to decrease as \( v \sim \tau^{-\alpha} \), with \( \alpha > 0 \) and for \( \tau >> 1 \).
Following the reasoning in 7 applied to anomalous diffusion, we establish a relation between the exponents \( \alpha \) and \( \beta \) of the velocity and the mean-square displacement, respectively. As mentioned previously, the quenched noise can alter the very elementary properties of Brownian motion. For Brownian motion, the \( \text{MSD} \) depends linearly on time as

\[
\text{MSD}(\tau) = \lambda_p^2 \frac{\langle \tau_p^2 \rangle - \langle \tau_p \rangle^2}{\langle \tau_p \rangle^3} \tau
\]

(10)

provided the first two moments \( \langle \tau_p \rangle \) and \( \langle \tau_p^2 \rangle \) of the distribution \( P(\tau_p) \) are finite, where \( \tau_p \) is the time that it takes a single particle to cover the spatial period of the ratchet potential \( \lambda_p \) over the course of its trajectory for long enough time. In contrast, when any of the first two moments of \( P(\tau_p) \) diverges, anomalous diffusion can be obtained.

According to our numerical results, the asymptotic behavior of the mean velocity can be indistinctly calculated from Eq. (8), as \( v = \langle \tau \rangle / \tau \) or also as \( v = \lambda_p / \langle \tau_p \rangle \). When the velocity decreases as \( v \sim \tau^{-\alpha} \), \( \langle \tau_p \rangle \) behaves as \( \tau^\alpha \) and therefore the distribution \( P(\tau_p) \) has a tail that decreases as power-law for long stretches of time. Thus, in this regime, \( \langle \tau_p^2 \rangle \sim \tau^{\alpha+2} \) and from Eq. (10), we arrive to the expression

\[
\beta = 2(1 - \alpha),
\]

(11)

for \( \tau \gg 1 \). When \( 0 < \alpha < 1/2 \), the movement of the particles is superdiffusive and when \( 1/2 < \alpha < 1 \) the movement is subdiffusive. The values of \( \beta \) obtained from Eq. (11) are listed in the 4th column of Table I.

Finally, when the mean velocity is constant, \( \langle \tau_p \rangle = \lambda_p / v \) is finite and \( \langle \tau_p^2 \rangle \) can be finite or diverge over time. In this case, the movement of particles is Brownian or superdiffusive (see curves in black and in green color in Figs. 2b) and 3).

| \( \mu \)          | \( \tilde{T} \) | \( \beta \) | \( \alpha \) | \( \beta = \text{Eq. (11)} \) |
|-------------------|---------------|------------|------------|-----------------|
| \( 5 \times 10^{-4} \) | 0.26         | 0.86       | 0.28       |
| \( 10^{-3} \)       | 0.67         | 0.71       | 0.58       |
| 0.1                |               |            |            |                  |
| \( 5 \times 10^{-3} \) | 2.00         | 0.05       | 1.90       |
| \( 10^{-2} \)       | 1.17         | 0.00       | —          |
| \( 10^{-1} \)       | 1.00         | 0.00       | —          |
| \( 3 \times 10^{-4} \) | 0.34         | 0.81       | 0.38       |
| 0.5                |               |            |            |                  |
| \( 10^{-3} \)       | 2.27         | 0.00       | —          |

In brief, when \( P(\tau_p) \) decreases as a power-law, anomalous diffusion with asymptotic behaviors of \( v \) and the \( \text{MSD} \) is obtained. Moreover, as we shall see in more detail in the next section, the fat-tailed time distribution promotes aging effects in which most physical properties strongly depend on the history of the sample. The \( \tau_p \) correlation is directly related to the spatial correlation, that decays abruptly with \( l_r \) (see Eq. (4)). Therefore, \( \tau_p \) is not correlated because we assume \( \lambda_p \gg l_r \), i.e. \( \lambda \ll 1 \) (see Eq. (7)).

The effect of the ratchet potential asymmetry on the movement of the particles plays a similar role to the temperature of the system. Indeed, when the asymmetry \( \mu \) increases, superdiffusive behaviors are obtained at lower temperatures (see Table I in which exponents for two values of temperature and \( \mu = 0.5 \) are shown for comparison purposes).

In order to complete this section, it should be mentioned that, for disorder correlation length much larger than the period of the ratchet potential, that is, negligible randomness, normal and superdiffusive behaviors are reached, but the subdiffusive regimes are lost. This has been previously studied in 8 for particles driven by a constant external force over a landscape consisting of a symmetric periodic potential with quenched disorder identical to Eq. (4).
B. Aging in the transport on the corrugated ratchet potential

In equilibrated states and in steady states, physical properties do not change with time. In contrast, in aging systems, the time translation invariance is broken leading to time-dependent dynamics. The aging of the transport on the corrugated ratchet potential can be measured through the two-time velocity correlation function $C_v(\tau, \tau_w)$ given by Eq. (9), with $\tau_w$ the observation time or waiting time. Fig. 4 shows $C_v$ as a function of $(\tau - \tau_w)$, corresponding to a asymptotic superdiffusive regime, for different $\tau_w$. The velocity correlation depends on the observation time $\tau_w$, which is the only relevant time scale for the dynamics of this transport. In addition, Fig. 4 shows that the older systems (circle and square symbols) relax in a slower manner than younger ones (triangle and cross symbols). This can be quantified, assuming that $C_v$ depends on $\tau/\tau_w^\chi$, with $0 < \chi \leq 1$. Thus, we propose the scaling on the x-axis given by $(\tau - \tau_w)/\tau_w^\chi$. Figs. 5 shows that the data from Fig. 4 can be fully collapsed by the previous scaling by taking $\chi = 0.8$.

In particular, the cases $\chi < 1$ have been called subaging because the effective relaxation time grows more slowly than the age of the system [20, 21]. Indeed, the correlation function $C_v$ is linearly related to the response to an external perturbation through generalized Fluctuation-Dissipation theorem [22]. According to this theorem, when the system is disturbed at time $\tau_w$ by a sudden force, the relaxation time is $\tau_w^\chi$. Therefore, the system forgets the perturbation, after a time $\tau_w^\chi$ less than its age $\tau_w$, provided $\chi < 1$.

The aging exponent depends on the way that particles explore the corrugated ratchet potential. Fig. 6 shows the two-time velocity correlation function $C_v$ for an example of subdiffusive transport. Here the aging effect is less than that found for superdiffusive transport. In fact, a good collapse of data of Fig. 6 is obtained in Fig. 7 in which the x-axis is scaling as $(\tau - \tau_w)/\tau_w^\chi$, with $\chi = 0.1$. Regarding scaled figures 5 and 7, we want to point out that the quality of the scaled fits appears to be better for larger values of $\tau_w$.

Finally, $\chi = 0$ is obtained for normal diffusion. Here $C_v$ depends only on the time difference $(\tau - \tau_w)$ and the system is not aging. In such case, the particles explore the whole phase space because of the presence of thermal fluctuations and the system is ergodic. For $0 < \chi \leq 1$
the system is also ergodic but exhibits an extremely slow relaxation weakly breaking the ergodicity. For superfundusive transports, the number of trapped particles that remain in that state grows slowly and this is related to collective motion. In contrast, for subfundusive transports, most of the particles explore the traps and the system tends to ergodicity. This is clearly observed in Fig. 8, where the fraction of particles trapped in a potential well for a time period $\tau_p$ as a function of $\tau$ are shown for superfundusive and subfundusive transport respectively. Thus, aging related exponent $\chi$ depends on the diffusion of particles on the corrugated ratchet potential.

IV. CONCLUSIONS

Anomalous diffusion and subaging appear in the motion of particles on a ratchet potential with both, correlated weak spatial disorder and thermal noise for different temperatures. The correlation of spatial disorder allows to change the rugosity of the ratchet potential and different kinds of diffusion are obtained according to the temperature of the system. For highly corrugated ratchet potentials and large enough times, normal diffusion is obtained for high temperatures, superfundusion for intermediate temperatures, and for low temperatures, subdiffusion. In all cases, the mean velocity of the particles decreases with time or is at most constant, due to the interplay between the thermal noise and the landscape generated by the spatial disorder. The asymptotic decrease of mean velocity is a power function of time related to the kind of diffusion. According to the way that particles explore the landscape, a relationship between the asymptotic exponents of the mean velocity of the particles and those of the mean-square displacement may be derived (see Eq. (11)). In addition, according to the movement of the particles, the system exhibits behavior reminiscent of subaging and of weak ergodicity breaking. When most of the particles lose the landscape details, while a slowly growing fraction of particles remain trapped for long time, the collective movement tends to disperse and becomes superfundusive. The transport weakly breaks the ergodicity because the time to cover the whole phase space turns out to be enormously long [23]. Subaging is obtained for superfundusive transport. On the other hand, the system tends to recover its ergodicity in the subfundusive case. These movements were found after a superfundusive transient due to a high quenched disorder and for an amplitude of the external force higher than the amplitude of the ratchet force. In contrast, a regular ratchet movement is found for an amplitude of the external force less than the amplitude of the ratchet force after a subfundusive transient. In this case, a net motion is obtained for high enough temperatures that combined with the external force let the particle overcome both the ratchet potential and the quenched disorder. In summary, a set of dramatic anomalous behaviors as diverse as subtransport, subdiffusion, and superfundusion with subaging appear when the periodic ratchet potential is modified with a small amount of correlated weak disorder and thermal noise.

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