Charge quantisation without magnetic monopoles: a topological approach to electromagnetism

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Abstract

The present work provides a theoretical explanation for the quantisation of electric charges, an open problem since Millikan’s oil drop experiment in 1909. This explanation is based solely on Maxwell’s theory, it recasts Electromagnetic theory under the language of complex line bundles; therefore, neither magnetic monopoles nor quantum mechanics are invoked.

1 Introduction

Essentially, the existence of magnetic monopoles was the only theoretical explanation for charge quantisation (e.g. Dirac’s monopole [1]), and there is no experimental data supporting their existence — on the contrary, they have never been observed.

My intention with this note is to clarify some features of the Electromagnetic theory and to champion the idea that Maxwell’s theory is not only intrinsically relativistic, but it naturally accommodates phenomena that are usually associated with its quantum aspects: charge quantisation, the Aharonov-Bohm effect, Dirac’s equation, and photons wave functions. All of this become clear by defining the electromagnetic field (over a spacetime region) as an equivalence class of hermitian line bundles with hermitian connexion (considering the spacetime region of interest as the base manifold).

I shall justify my approach by quoting Dirac [1]: “The most powerful method of advance that can be suggested at present ...”, still is, I think, “... to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities”.

An appendix to this note is included. It supplements the text with definitions and theorems (together with sketch of their proofs) that are known in the literature, providing a place to fix notation and a guide for the reader not familiar with differential topology.

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Throughout this article and otherwise stated, all the objects considered will be $C^\infty$; manifolds are real, Hausdorff, paracompact, connected, and orientable; and physical quantities are measured using the International System of Units.

2 A précis of the Electromagnetic Theory

Einstein’s field equations [2] provide the spacetime $(M, g)$: a lorentzian manifold. Classical electromagnetism over a spacetime $(M, g)$ is described by a differential 2-form, the electromagnetic field tensor $\omega \in \Omega^2(M; \mathbb{R})$, solving Maxwell’s equations,

$$\begin{cases}
  d\omega = *_g \alpha \\
  *_g d *_g \omega = 0,
\end{cases} \quad (1)$$

where $*_g$ is the Hodge star operator of the lorentzian metric $g$, and the differential 1-form $\alpha \in \Omega^1(M; \mathbb{R})$ encodes the information about charges and currents distributions. On the Minkowski spacetime, these equations are equivalent to

$$\begin{align*}
  \nabla \cdot \epsilon_0 \vec{E} &= \rho \\
  \nabla \times \frac{1}{\mu_0 c} \vec{B} - \frac{1}{c} \frac{\partial}{\partial t} \epsilon_0 \vec{E} &= -\frac{1}{c} \vec{J} \\
  \nabla \cdot \frac{1}{\mu_0 c} \vec{B} &= 0 \\
  \nabla \times \epsilon_0 \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \frac{1}{\mu_0 c} \vec{B} &= 0 
\end{align*} \quad (2)$$

with $\rho$ and the vectors $\vec{J}$, $\vec{B}$, and $\vec{E}$ defined by

$$g \left( \frac{\rho}{\partial t} + \sum_{j=1}^{3} J_j \frac{\partial}{\partial x^j}, \cdot \right) = -c \, \alpha(\cdot) \quad , \quad (3)$$

$$g \left( \sum_{j=1}^{3} B_j \frac{\partial}{\partial x^j}, \cdot \right) = -\mu_0 c \, \omega \left( -\frac{1}{c} \frac{\partial}{\partial t}, \cdot \right) \quad , \quad (4)$$

and

$$g \left( \sum_{j=1}^{3} E_j \frac{\partial}{\partial x^j}, \cdot \right) = \frac{1}{\epsilon_0} *_g \omega \left( -\frac{1}{c} \frac{\partial}{\partial t}, \cdot \right) \quad . \quad (5)$$

The classical dynamics of light and charged test particles are both described by hamiltonian flows. Light trajectories are null geodesics of $(M, g)$, whilst the trajectories of a test particle, having mass $m$ and charge $q$, are integral curves of the hamiltonian flow defined by the function

$$T^*M \ni (v^*, x) \mapsto \frac{1}{2m} g^1_x (mv, mv) \in \mathbb{R} \quad , \quad (6)$$

with $v^*(\cdot) := m g^1_x (v, \cdot) \in T^*_x M$, on the cotangent bundle $T^*M$ endowed with the symplectic form $d\lambda - \frac{1}{\epsilon_0 c^2} Pr^*(\omega)$, where $d\lambda$ is the canonical symplectic form of $T^*M$.

\textsuperscript{1}My choices are such that the electromagnetic field tensor has units of charge.
induced by the projection $\Pr : T^*M \rightarrow M$. The Lorentz force law,

$$\vec{F} = \frac{q}{\epsilon_0} \left( \epsilon_0 \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right),$$

(7)
can be deduced from this symplectic picture [3].

### 3 The topological approach

I shall consider only globally hyperbolic 4-dimensional lorentzian manifolds with finite first and second Betti numbers to model spacetimes; in particular, a spacetime $M$ is diffeomorphic to $\mathbb{R} \times \mathcal{M}$, and there exists a vector field $X \in \mathfrak{X}(M; \mathbb{R})$ which is the gradient (with respect to the lorentzian metric) of a global time function on $M$ (vide [3] and [2]).

It is not possible to probe inside regions occupied by charges or currents; thus, outside these regions the electromagnetic field tensor satisfies the source free Maxwell’s equations,

$$\begin{align*}
\frac{d}{dt} \omega &= 0 \\
*_{g} d *_{g} \omega &= 0.
\end{align*}$$

(8)

One first eliminates worldlines corresponding to particle (charge) distributions and the worldsheets corresponding to wire (current) distributions from the spacetime region of interest; then, one topologically introduces the source term back on Maxwell’s equation. Let $J \subset M$ stand for the disjoint union of worldlines and worldsheets corresponding to particle and wire distributions, and $M_J := M - J$ the complement of $J$. If the class $[N]$ is a generator of the second singular homology group $H_2(M_J; \mathbb{Z})$, $\int_N \omega$ is the charge inside a spacetime region bounded by any smooth representative $N$; likewise, if the class $[\gamma]$ is a generator of the first singular homology group $H_1(M_J; \mathbb{Z})$, the current passing through a spacetime region bounded by a smooth representative $\gamma$ is $c \int_\gamma i_X \omega$. It is important to mention that $i_X \omega$ is not closed unless $L_X(\omega) = 0$ (static fields), and this means that: if $\gamma t \in [\gamma]$, then $c \int_{\gamma t} i_X \omega = c \int_{\gamma} i_X \omega + c \int_{N} L_X(\omega)$ with $\partial N = \gamma - \gamma t$, and $c \int_{N} L_X(\omega)$ is related to displacement currents.

I am now in a position to provide a precise definition for the electromagnetic field over a spacetime region that is compatible with Maxwell’s theory, free of magnetic monopoles, and able to explain charge quantisation.

Given charge and current distributions $J \subset M$, the electromagnetic field over a spacetime $(M, g)$ is an equivalence class of hermitian line bundles with hermitian connexion over $M_J$, $[(L, \langle \cdot, \cdot \rangle, \nabla^\omega)]$, whose hermitian connexion of any representative satisfies

$$\begin{align*}
\text{curv}(\nabla^\omega) &= \frac{-i}{c} \omega \\
*_{g} d *_{g} \omega &= 0.
\end{align*}$$

(9)

2Intricate circuits and collision between charged particles are not allowed, for the sake of simplicity.
The Chern class \( c_1([L]) \in \hat{H}^2(M_J;\mathbb{Z}) \) of \([L,\langle \cdot,\cdot \rangle,\nabla^\omega]\) is fixed by knowing the charges and quantal phases\(^3\). The periods of the curvature provide, via de Rham’s theorem (cf. theorem [A.3]), an element of the second de Rham cohomology group \([\omega] \in H^2_{dR}(M_J;\mathbb{R})\) lying in the image of the homomorphism between \(\hat{H}^2(M_J;\mathbb{Z})\), the second Cech cohomology group with coefficients in the constant sheaf \(\mathbb{Z}\), and \(H^2_{dR}(M_J;\mathbb{R})\), induced by the inclusion \(\mathbb{Z} \ni k \mapsto 2\pi k \in \mathbb{R}\). Quantal phases provide a homomorphism between the fundamental group \(\pi_1(M_J)\) and the unitary circle \(S^1\), which defines an element on the kernel of the aforementioned homomorphism \(\hat{H}^2(M_J;\mathbb{Z}) \rightarrow H^2_{dR}(M_J;\mathbb{R})\) (cf. lemma [A.1] and theorem [A.2]).

The curvature is fixed by selecting a representative of \([\omega] \in H^2_{dR}(M_J;\mathbb{R})\) which satisfies \(\ast_g \ast_g \omega = 0\) and matches the currents, and possible initial (e.g. background electromagnetic radiation) or boundary (e.g. the vanishing of the electromagnetic field tensor at some regions) conditions.

One can conclude from the above that magnetic effects are a manifestation of flat hermitian line bundles with nonflat connexions, whilst the Aharonov-Bohm effect is due to flat connexions; ergo, topologically, it is impossible to choose a different reference frame where a truly electric effect—which is a manifestation of nonflat hermitian line bundles—is seen as a purely magnetic effect. Furthermore, what is usually associated with the quantum aspects of electromagnetism is encoded in \(\hat{H}^2(M_J;\mathbb{Z})\) (Chern classes, quantisation of charge) and \(\pi_1(M_J)\) (holonomy of flat connexions, Aharonov-Bohm effect), classical properties are due to \(\Omega^2(M_J;\mathbb{R})\) (curvature, electromagnetic field tensor).

### 3.1 Topological charges and currents

In order to demonstrate that this definition implies quantisation of charges (cf. proposition [A.4]) and does not support magnetic monopoles, I shall investigate the sources of electromagnetic fields on Minkowski spacetime.

A point charge has its worldline removed from Minkowski spacetime; therefore, \(M_J \cong \mathbb{R} \times (\mathbb{R}^3 - \{\text{point}\})\) which is homotopic equivalent to \(\mathbb{R}^3 - \{\text{point}\}\), yielding \(\pi_1(M_J) = \{1\}\), \(H_1(M_J) = \{0\}\), and \(\hat{H}^2(M_J;\mathbb{Z}) \cong \mathbb{Z} \cong H^2(M_J;\mathbb{Z})\). If the electromagnetic field has its Chern class determined by the integer \(k \in \mathbb{Z}\), then the electric charge contained in a smooth representative of the generator of \(H^2(M_J;\mathbb{Z})\) determined by the integer 1 is exactly \(ke\), and there is no other source for this electromagnetic field.

An open wire carrying current has its worldsheet removed from Minkowski spacetime; hence, \(M_J \cong \mathbb{R} \times (\mathbb{R}^3 - \{\text{line}\})\) which is homotopic equivalent to \(\mathbb{R}^3 - \{\text{point}\}\), and \(\pi_1(M_J) \cong \mathbb{Z} \cong H_1(M_J)\) and \(\hat{H}^2(M_J;\mathbb{Z}) = \{0\} \cong H^2(M_J;\mathbb{Z})\). A closed wire carrying current has its worldsheet removed from Minkowski spacetime; consequently, \(M_J \cong \mathbb{R} \times (\mathbb{R}^3 - \{\text{circle}\})\) which is homotopic equivalent to \(\mathbb{R}^3 - \{\text{circle}\}\), and as a result \(\pi_1(M_J) \cong \mathbb{Z} \cong H_1(M_J)\) and \(\hat{H}^2(M_J;\mathbb{Z}) = \{0\} \cong H^2(M_J;\mathbb{Z})\). In both situations the electromagnetic field is represented by flat hermitian line bundles with trivial holonomy group—for \(\pi_1(M) = \{1\}\), there cannot be Aharonov-Bohm effects—and a smooth representative of the generator of \(H^1(M_J)\) determined by the integer 1 is exactly a closed curve around the wire: there is no other source for the electromagnetic field.

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\(^3\)If the spacetime region has nontrivial fundamental group before taking out the worldsheets associated to current distributions, then one also needs to take this into account: besides the currents, one must know if there are electromagnetic fields inside the “spacetime holes”.
4 Local features

Let \((M, g) = (\mathbb{R}^4, -dx^0 \otimes dx^0 + \sum_{j=1}^{3} dx^j \otimes dx^j)\) be the Minkowski space with co-ordinates \(x^0 = ct, x^1, x^2, x^3\). Both \(\pi_1(M)\) and \(\hat{H}^2(M; \mathbb{Z})\) are trivial; ergo, the trivial complex line bundle \(L = \mathbb{C} \times M\) is a representative for the source free electromagnetic field in vacuum.

This section is devoted to unravel information contained in the sections of \(L\).

4.1 A wave function of a photon

The trivial complex line bundle admits a flat connexion \(\nabla^0\) satisfying

\[
\nabla^0 f s = df \otimes s \in \Omega^1(M; \mathbb{R}) \otimes_{C^\infty(M; \mathbb{C})} \Gamma(L),
\]

for a unitary section of the complex line bundle \(s \in \Gamma(L)\) and any complex-valued function \(f \in C^\infty(M; \mathbb{C})\).

There are two linear maps, defined via the flat connexion, from \(\mathfrak{x}(M; \mathbb{R})\) to the endomorphisms of \(\Gamma(L) \oplus \Gamma(L) \oplus \Gamma(L)\) to be considered. Taking \(\omega_1, \omega_2, \omega_3 \in \Gamma(L)\), \(X = \frac{\partial}{\partial x_0} \in \mathfrak{x}(M; \mathbb{R})\), and \(Y_j = \frac{\partial}{\partial x_j} \in \mathfrak{x}(M; \mathbb{R})\), the first map is given by:

\[
\begin{align*}
X \mapsto -i\hbar x_0, & \quad x_0(\omega_1 + \omega_2 + \omega_3) = -((\nabla^0_{\partial_0} \omega_1 + \nabla^0_{\partial_0} \omega_2 + \nabla^0_{\partial_0} \omega_3); \\
Y_1 \mapsto i\hbar y_1, & \quad y_1(\omega_1 + \omega_2 + \omega_3) = 0 \oplus i\nabla^0_{\partial_1} \omega_2 \oplus -i\nabla^0_{\partial_1} \omega_3; \\
Y_2 \mapsto i\hbar y_2, & \quad y_2(\omega_1 + \omega_2 + \omega_3) = -i\nabla^0_{\partial_2} \omega_3 \oplus 0 \oplus i\nabla^0_{\partial_2} \omega_1; \\
Y_3 \mapsto i\hbar y_3, & \quad y_3(\omega_1 + \omega_2 + \omega_3) = i\nabla^0_{\partial_3} \omega_2 \oplus -i\nabla^0_{\partial_3} \omega_1 \oplus 0.
\end{align*}
\]

The second map is defined by

\[
\begin{align*}
X \mapsto -i\hbar x_0, & \quad x_0(\omega_1 + \omega_2 + \omega_3) = 0; \\
Y_j \mapsto i\hbar y_j, & \quad y_j(\omega_1 + \omega_2 + \omega_3) = \nabla^0_{\partial_j} \omega_1 \oplus \nabla^0_{\partial_j} \omega_2 \oplus \nabla^0_{\partial_j} \omega_3.
\end{align*}
\]

Let \(L^{-1}\) denote the dual bundle of \(L\). A hermitian structure \(< \cdot, \cdot >\) provides an equivalence between the dual bundle and the complex conjugate bundle \(\bar{L}\) (defined by the complex conjugate of the transition functions of \(L\)). In the particular case of \(M = \mathbb{R}^4\), \(L \cong L^{-1}\).

Wave functions for photons are sections of \(\Gamma(L) \oplus \Gamma(L) \oplus \Gamma(L) \oplus \bar{\Gamma(L)} \oplus \bar{\Gamma(L)} \oplus \bar{\Gamma(L)}\) of the form \(\omega_1^+ \oplus \omega_2^+ \oplus \omega_3^+ \oplus \omega_2^+ \oplus \omega_3^+ \oplus \omega_3^+\), with \(\omega_1^+ \oplus \omega_2^+ \oplus \omega_3^+ \in \Gamma(L) \oplus \Gamma(L) \oplus \Gamma(L)\) lying in the intersection of the kernels of the endomorphisms induced by \(X + \sum_{j=1}^{3} Y_j\),

\[
\begin{align*}
\left\{ \begin{array}{l}
\hbar \left( -x + \sum_{j=1}^{3} y_j \right) \omega_1^+ \oplus \omega_2^+ \oplus \omega_3^+ = 0 \\
\hbar \left( -x_0 + \sum_{j=1}^{3} x_j \right) \omega_1^+ \oplus \omega_2^+ \oplus \omega_3^+ = 0
\end{array} \right.,
\end{align*}
\]

\[
\text{(17)}
\]
and \( \omega^-_1 \oplus \omega^-_2 \oplus \omega^-_3 \in \Gamma(\tilde{L}) \oplus \Gamma(\tilde{L}) \oplus \Gamma(\tilde{L}) \), under the identification \( L \cong \tilde{L} \), satisfying

\[
\begin{align*}
- \hbar \left[ -x - \sum_{j=1}^{3} y_j \right] \omega^-_1 & \oplus \omega^-_2 \oplus \omega^-_3 = 0 \\
- \hbar \left[ -x_0 + \sum_{j=1}^{3} x_j \right] \omega^+_1 & \oplus \omega^+_2 \oplus \omega^+_3 = 0,
\end{align*}
\]

(18)

together with a compatibility condition, considering \( L \cong L^{-1} \),

\[
\langle \omega^-_j, \cdot \rangle = \omega^+_j(\cdot) .
\]

(19)

Using the unitary section \( s \), sections of the trivial complex line bundle can be identified with complex-valued functions, \( \omega^+_j = \left( \frac{1}{mc} B_j - i e_0 E_j \right) s \), and one recovers a wave function for the photon (having a particular helicity) as in references [5] and [6].

4.2 Dirac’s equation

Considering now a connexion \( \nabla^\omega \) in \( L \) whose curvature is \( -i \omega \), one has a linear map from \( \mathfrak{X}(M; \mathbb{R}) \) to the endomorphisms of \( \Gamma(L) \oplus \Gamma(L) \oplus \Gamma(L) \oplus \Gamma(L) \) defined by \( X \mapsto -i \hbar \mathbf{x}(\omega) \) and \( Y_j \mapsto i \hbar \mathbf{y}_j(\omega) \): if \( \Psi_1, \Psi_2, \Psi_3, \Psi_4 \in \Gamma(L) \), then

\[
\begin{align*}
\mathbf{x}(\omega)(\Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_4) &= -\nabla^\omega_{\partial_0} \Psi_1 \oplus -\nabla^\omega_{\partial_0} \Psi_2 \oplus -\nabla^\omega_{\partial_0} \Psi_3 \oplus -\nabla^\omega_{\partial_0} \Psi_4 ; \\
\mathbf{y}_1(\omega)(\Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_4) &= -i \nabla^\omega_{\partial_1} \Psi_1 \oplus -i \nabla^\omega_{\partial_1} \Psi_2 \oplus -i \nabla^\omega_{\partial_1} \Psi_3 \oplus -i \nabla^\omega_{\partial_1} \Psi_4 ; \\
\mathbf{y}_2(\omega)(\Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_4) &= -i \nabla^\omega_{\partial_2} \Psi_1 \oplus -i \nabla^\omega_{\partial_2} \Psi_2 \oplus -i \nabla^\omega_{\partial_2} \Psi_3 \oplus -i \nabla^\omega_{\partial_2} \Psi_4 ; \\
\mathbf{y}_3(\omega)(\Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_4) &= -i \nabla^\omega_{\partial_3} \Psi_1 \oplus -i \nabla^\omega_{\partial_3} \Psi_2 \oplus -i \nabla^\omega_{\partial_3} \Psi_3 \oplus -i \nabla^\omega_{\partial_3} \Psi_4 .
\end{align*}
\]

(20)

(21)

(22)

(23)

Wave functions describing electrons and positrons of mass \( m \) subjected to an electromagnetic field \( [(L, \langle \cdot, \cdot \rangle, \nabla^\omega)] \) are eigenvectors of the endomorphism induced by \( X + \sum_{j=1}^{3} Y_j \) whose eigenvalue is \( mc \):

\[
- \hbar \left[ -x(\omega) + \sum_{j=1}^{3} y_j(\omega) \right] \Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_4 = mc \Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_4 .
\]

(24)

Again, by using the unitary section \( s \), for each \( \Psi_j \) there exists a complex-valued function \( \psi_j \in C^\infty(M; \mathbb{C}) \) such that \( \Psi_j = \psi_j s \), and one recovers Dirac’s equation for a wave function for the electron and positron [7].

Appendix: Hermitian line bundles

This is a summary of relevant results and definitions needed to understand this note —technicalities will be avoided here, though—, the reader can find them in references [8] [9] [10].
I shall assume that the cohomology and homology groups discussed in this appendix have finite dimension.

**Definition A.1.** A fibre bundle $B$ over $M$ with fibre $F$ is a manifold together with a surjective diffeomorphism $\mathcal{F} : B \to M$ such that: for any point $p \in M$ there exists a neighbourhood $A$ containing it and a diffeomorphism $\varphi : \mathcal{F}^{-1}(A) \to N \times A$.

A neighbourhood such as $A$ is called a trivialising neighbourhood, and, when the surjective map $\mathcal{F}$ is omitted, $B\big|_A := \mathcal{F}^{-1}(A)$ represents the restriction of the bundle $B$ on the neighbourhood $A \subset M$ and $B_p := \mathcal{F}^{-1}(p) \cong F$ stands for the fibre over $p \in M$.

**Definition A.2.** A section of a fibre bundle $\mathcal{F} : B \xrightarrow{\mathcal{F}} M$ is a map $s : M \to B$ satisfying $\mathcal{F} \circ s(p) = p$ for all $p \in M$.

For each open set $A \subset M$, a local section of a fibre bundle is a section of $B\big|_A$. One can, then, define the sheaf of sections of a fibre bundle.

**Definition A.3.** The sheaf of sections of a fibre bundle $B$ is denoted by $\mathcal{S}$, and $\Gamma(B) := \hat{H}^0(M; \mathcal{S})$ is the vector space of global sections of the fibre bundle.

The $k$-th Čech cohomology group of a manifold $M$ with coefficients in a sheaf $\mathcal{Z}$ will be denoted by $\hat{H}^k(M; \mathcal{Z})$.

**Definition A.4.** A hermitian line bundle $L \xrightarrow{\mathcal{C}} M$ is a fibre bundle over $M$ with fibres diffeomorphic to $\mathbb{C}$, together with a hermitian structure $\langle \cdot, \cdot \rangle$ defined on the space of sections $\Gamma(L)$.

**Proposition A.1.** For hermitian line bundles, the existence of a trivialisation is equivalent to the existence of a unitary section.

*Proof:* Let $A_j$ be a trivialising neighbourhood of a hermitian line bundle $(L, \langle \cdot, \cdot \rangle)$. There exists a diffeomorphism $\varphi_j : L\big|_{A_j} \to \mathbb{C} \times A_j$, and if $z \in \mathbb{C}$ belongs to the unitary circle, then one can define a section of $\mathbb{C} \times A_j$ via $A_j \ni p \mapsto (z, p)$. A unitary section $s_j \in \Gamma(L\big|_{A_j})$ is, then, obtained by $s_j(p) = \varphi_j^{-1}(z, p)$.

Conversely, let $s_j \in \Gamma(L\big|_{A_j})$ be a section satisfying $\langle s_j, s_j \rangle(p) = 1$ for all $p \in A_j$. The map given by $\mathbb{C} \times A_j \ni (z, p) \mapsto z \cdot s_j(p) \in L\big|_{A_j}$ is a diffeomorphism. ■

**Proposition A.2.** Sections of $L$ can be represented by complex-valued functions over trivialising neighbourhoods.

*Proof:* Let $A_j$ be a trivialising neighbourhood of $L$ with unitary section $s_j$. The line bundle $L\big|_{A_j} \xrightarrow{\mathcal{C}} A_j$ is diffeomorphic to the trivial one, $\mathbb{C} \times A_j$, and under this diffeomorphism, a section $s \in \Gamma(L\big|_{A_j})$ is uniquely defined by a complex-valued function $f \in C^\infty(A_j; \mathbb{C})$: $s(p) \simeq (f(p), p)$ for each point $p \in A_j$. ■

When there is an identification between a section $s$ of the line bundle and a complex-valued function $f$, the diffeomorphism might be omitted, for the sake of simplicity, and the equality $s = f$ will be used.
Two hermitian line bundles \((L, \langle \cdot, \cdot \rangle)\) and \((L', \langle \cdot, \cdot \rangle')\) over the same manifold \(M\) are equivalent if there exists a diffeomorphism \(\phi : L \to L'\) such that, when restricted to a fibre \(L_p, \phi|_{L_p} : L_p \subset L \to L'\) is an isometry between \(L_p\) and \(L'|_{\phi(p)}\) and, when restricted to the zero section \(M \subset L, \phi|_M : M \subset L \to \phi|_M(M) \subset L'\) is the identity. Such a diffeomorphism will be called a hermitian bundle diffeomorphism.

**Definition A.5.** The set of equivalence classes of hermitian line bundles endowed with a group structure given by tensor products of hermitian line bundles will be denoted by \(\mathcal{L}_h(M)\).

Let \(\mathcal{C}_R^\infty\) and \(\mathcal{C}_S^\infty\) denote the sheaves of \(R\)-valued and \(S^1\)-valued functions over a manifold \(M\): \(\check{H}^0(M; \mathcal{C}_R^\infty) = C^\infty(M; \mathcal{C}_R)\) and \(\check{H}^0(M; \mathcal{C}_S^\infty) = C^\infty(M; S^1)\).

**Lemma A.1.** \(\mathcal{L}_h(M) \cong \check{H}^1(M; \mathcal{C}_S^\infty)\).

*Proof:* Given a contractible open cover \(\mathcal{A} = \{A_j\}_{j \in I}\) of \(M\) (which can always be obtained, e.g. using a convenient cover made of balls with respect to a Riemannian metric) and an element \(\varphi\) of \(\check{H}^1(M; \mathcal{C}_S^\infty)\), one can define a hermitian line bundle \(L\) such that each \(A_j\) is a trivialising neighbourhood of it with transition functions given by \(\varphi(A_j \cap A_k)\).

The construction goes as follows: one can define \(L|_{A_j} := C \times A_j\) with the standard hermitian structure (or any other one) on the fibre \(C\) and consider the equivalence relation \((z, p) \sim (\varphi(A_j \cap A_k)(p) \cdot z, p)\) over \(A_j \cap A_k\); then, the set \(L := \bigsqcup_{j \in I} L|_{A_j}/\sim\) has both a natural smooth structure, making it a complex line bundle over \(M\), and a natural hermitian structure \(\langle \cdot, \cdot \rangle\).

Now, supposing that the hermitian line bundle \((L, \langle \cdot, \cdot \rangle)\) is given together with a contractible open cover \(\mathcal{A} = \{A_j\}_{j \in I}\) of \(M\) where each \(A_j\) is a trivialising neighbourhood, i.e. there exist bundle morphisms respecting the hermitian structure \(\varphi_j : L|_{A_j} \to C \times A_j\), one can define an element of \(\check{H}^1(M; \mathcal{C}_S^\infty)\): the diffeomorphisms

\[
\varphi_k|_{A_j \cap A_k} \circ \varphi_j|_{A_j \cap A_k} : C \times A_j \cap A_k \to C \times A_j \cap A_k \quad (25)
\]

induce maps, the transition functions, \(\varphi(A_j \cap A_k) : A_j \cap A_k \to S^1\), defined by

\[
\varphi_k|_{A_j \cap A_k} \circ \varphi_j|_{A_j \cap A_k}(z, p) = (\varphi(A_j \cap A_k)(p) \cdot z, p), \quad (26)
\]

satisfying the cocycle conditions. 

**Lemma A.2.** \(\check{H}^1(M; \mathcal{C}_S^\infty) \cong \check{H}^2(M; \mathcal{Z})\).

*Proof:* The sequence

\[
0 \to \mathcal{Z} \to \mathcal{C}_R^\infty \to \mathcal{C}_S^\infty \to 1 \quad (27)
\]

\[
0 \to 0 \to 2\pi k \to e^{i\phi} \to 1 \quad (28)
\]

is a short exact sequence of sheaves. Indeed,

\[
\ker\{f \mapsto e^{i\phi}\} = 2\pi \mathcal{Z} = \text{im}\{k \mapsto 2\pi k\}. \quad (29)
\]

Thus, there is a long exact sequence of cohomology groups, and the fine property of the sheaf \(\mathcal{C}_R^\infty\) provides the result.
Theorem A.1. $L_h(M) \cong \hat{H}^2(M; \mathbb{Z})$.

Proof: Lemmata A.1 and A.2.

Proposition A.3. $\hat{H}^1(M; S^1) \hookrightarrow \hat{H}^1(M; C_{S^1})$.

Proof: Each element $\varphi \in \hat{H}^1(M; S^1)$ provides a map

$$A_j \cap A_k \ni p \mapsto \varphi_{jk}(p) := \varphi(A_j \cap A_k) \in S^1$$

satisfying the cocycle conditions.

Under pertinent identifications, hermitian line bundles defined by elements of $\hat{H}^1(M; S^1)$ are called flat.

Theorem A.2. The kernel of the homomorphism $\hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ sits in the following short exact sequence of groups:

$$0 \to \hat{H}^1(M; \mathbb{R}) \to \hat{H}^1(M; \mathbb{Z}) \to \ker\{\hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R})\} \to 0.$$ (31)

Proof: The sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$$ (32)

$$0 \to 0 , \ k \mapsto 2\pi k , \ a \mapsto e^{ia} , \ \varphi \mapsto 1$$ (33)

is a short exact sequence of sheaves:

$$\ker\{a \mapsto e^{ia}\} = 2\pi \mathbb{Z} = \text{im}\{k \mapsto 2\pi k\}.$$ (34)

The long exact sequence of cohomology groups associated with it is

$$0 \to \hat{H}^1(M; \mathbb{Z}) \to \hat{H}^1(M; \mathbb{R}) \to \hat{H}^1(M; S^1) \to \hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R}) ;$$ (35)

then,

$$\text{im}\{\hat{H}^1(M; S^1) \to \hat{H}^2(M; \mathbb{Z})\} \cong \ker\{\hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R})\} ,$$ (36)

and the first isomorphism theorem implies that

$$\ker\{\hat{H}^1(M; S^1) \to \hat{H}^2(M; \mathbb{Z})\} \cong \frac{\hat{H}^1(M; \mathbb{R})}{\hat{H}^1(M; \mathbb{Z})}.$$ (37)

Using an isomorphism between Chech and de Rham cohomology, a class on the image of the homomorphism $\hat{H}^2(M; \mathbb{Z}) \to H^2_{dR}(M; \mathbb{R}) \cong \hat{H}^2(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ are said to be integral.

Lemma A.3. Let $K^2(\mathbb{Z}; \mathbb{R})$ be the kernel and $I^2(\mathbb{Z}; \mathbb{R})$ the image of the homomorphism $\hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. The exact sequence

$$0 \to K^2(\mathbb{Z}; \mathbb{R}) \to \hat{H}^2(M; \mathbb{Z}) \to I^2(\mathbb{Z}; \mathbb{R}) \to 0$$ (38)

splits.
Proof: By the first isomorphism theorem,
\[ \text{im}\{\hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R})\} \cong \frac{\hat{H}^2(M; \mathbb{Z})}{\text{ker}\{\hat{H}^2(M; \mathbb{Z}) \to \hat{H}^2(M; \mathbb{R})\}}, \tag{39} \]
and, under this identification, the map \( \hat{H}^2(M; \mathbb{Z}) \to \hat{I}^2(\mathbb{Z}; \mathbb{R}) \) takes an element of \( \hat{H}^2(M; \mathbb{Z}) \) to its class in \( \frac{\hat{H}^2(M; \mathbb{Z})}{K^2(\mathbb{Z}; \mathbb{R})} \). A right split is defined by taking a basis of \( \frac{\hat{H}^2(M; \mathbb{Z})}{K^2(\mathbb{Z}; \mathbb{R})} \) and choosing an arbitrary representative in \( \hat{H}^2(M; \mathbb{Z}) \) for each basis element. 

Lemma A.4. \( \hat{H}^1(M; S^1) \cong \text{Hom}(\pi_1(M); S^1) \), where \( \text{Hom}(\pi_1(M); S^1) \) denotes the set of group homomorphisms between the fundamental group of \( M \), \( \pi_1(M) \), and the unitary circle \( S^1 \).

Proof: Čech and singular cohomologies are isomorphic under the topological assumptions on \( M \), in particular, \( \hat{H}^1(M; S^1) \cong H^1(M; S^1) \), and it suffices to prove \( H^1(M; S^1) \cong \text{Hom}(\pi_1(M); S^1) \). The isomorphism is provided by the following construction: a representative \( \gamma \) of a generator \( [\gamma] \in \pi_1(M) \) defines an element in the singular homology group \( H_1(M; \mathbb{Z}) \), and a cocycle representing a generator of \( H^1(M; S^1) \) can be seen as a homomorphism from \( H_1(M; \mathbb{Z}) \) to \( S^1 \); hence, the cocycle defines a map in \( \text{Hom}(\pi_1(M); S^1) \).

Theorem A.3. One can uniquely define an equivalence class of hermitian line bundles from a map belonging to \( \text{Hom}(\pi_1(M); S^1) \) and a de Rham class \( [\omega] \in H^2_{\text{dR}}(M; \mathbb{R}) \) with \( [\frac{1}{2\pi i} \omega] \) integral.

Proof: Theorem A.1 and lemma A.3 imply \( \mathcal{L}_h(M) \cong \hat{I}^2(\mathbb{Z}; \mathbb{R}) \oplus K^2(\mathbb{Z}; \mathbb{R}) \), furthermore theorem A.2 and lemma A.4 guarantee that an element of \( K^2(\mathbb{Z}; \mathbb{R}) \) is determined by an element of \( \text{Hom}(\pi_1(M); S^1) \), and the identification \( \hat{H}^2(M; \mathbb{R}) \cong H^2_{\text{dR}}(M; \mathbb{R}) \) guarantees that an integral class \( [\frac{1}{2\pi i} \omega] \) determines an element of \( \hat{I}^2(\mathbb{Z}; \mathbb{R}) \).

Definition A.6. A hermitian connexion on a hermitian line bundle \( (L, \langle \cdot, \cdot \rangle) \) over a manifold \( M \) is a linear map
\[ \nabla : \Gamma(L) \to \Omega^1(M; \mathbb{R}) \otimes_{C^\infty(M; \mathbb{C})} \Gamma(L) \tag{40} \]
satisfying:
\[ \nabla(f s) = df \otimes s + f \nabla s, \tag{41} \]
with \( [\nabla s](X) \) denoted by \( \nabla_X s \), and
\[ X(\langle r, s \rangle) = \langle \nabla_X r, s \rangle + \langle r, \nabla_X s \rangle, \tag{42} \]
for any \( r, s \in \Gamma(L) \), \( f \in C^\infty(M; \mathbb{C}) \) and \( X \in \mathfrak{X}(M; \mathbb{R}) \).
With respect to a unitary section $s_j$, defined over a trivialising neighbourhood $A_j \subset M$, the connexion $\nabla$ can be represented by a potential 1-form $\Theta_j \in \Omega^1(A_j; \mathbb{R})$:

$$\nabla s_j = -i\Theta_j \otimes s_j .$$

(43)

Indeed, since $s_j$ is unitary, $\langle s_j, s_j \rangle = 1$, for any $X \in \mathfrak{x}(A_j; \mathbb{R})$:

$$0 = X(\langle s_j, s_j \rangle) = \langle \nabla_X s_j, s_j \rangle + \langle s_j, \nabla_X s_j \rangle = -i\Theta_j(X)\langle s_j, s_j \rangle + \langle s_j, -i\Theta_j(X)s_j \rangle = -i\Theta_j(X)\langle s_j, s_j \rangle = -i\Theta_j(X) + i\Theta_j(X) .$$

(44)

Lemma A.5. The potential 1-forms, $\Theta_j$ and $\Theta_k$, of $\nabla$ for each unitary section, $s_j$ and $s_k$, defined over an intersection $A_j \cap A_k$ are cohomologous.

Proof: Since $L$ is a hermitian line bundle, there exists a transition function $e^{if_{jk}} \in C^\infty(A_j \cap A_k; S^1)$ relating the two unitary sections $s_j = e^{if_{jk}}s_k$; as a result,

$$\nabla s_j = -i\Theta_j \otimes s_j = -ie^{if_{jk}}\Theta_j \otimes s_k ,$$

(45)

however

$$\nabla s_j = \nabla(e^{if_{jk}}s_k) = (de^{if_{jk}} - ie^{if_{jk}}\Theta_k) \otimes s_k = ie^{if_{jk}}(df_{jk} - \Theta_k) \otimes s_k$$

(46)

and, therefore,

$$-ie^{if_{jk}}\Theta_j \otimes s_k = ie^{if_{jk}}(df_{jk} - \Theta_k) \otimes s_k \Rightarrow$$

$$\Theta_k - \Theta_j = df_{jk} .$$

(47)

(48)

Definition A.7. The curvature of a hermitian connexion $\nabla$ on a hermitian line bundle $(L, \langle \cdot, \cdot \rangle)$ over a manifold $M$ is the imaginary 2-form, $\text{curv}(\nabla) \in \Omega^2(M; i\mathbb{R})$, satisfying

$$\text{curv}(\nabla)(X,Y)s = \nabla_X \circ \nabla_Y s - \nabla_Y \circ \nabla_X s - \nabla_{[X,Y]}s ,$$

for any $s \in \Gamma(L)$ and $X, Y \in \mathfrak{x}(M; \mathbb{R})$.

The next lemma implies, in particular, that the curvature is closed.

Lemma A.6. If $\Theta$ is a potential 1-form for the connexion $\nabla$, over a trivialising neighbourhood $A \subset M$, then the curvature is given by $\text{curv}(\nabla) = -i\Theta$, and it is independent of the choice of trivialisation.

Proof: Assuming that $s$ is the unitary section associated with $\Theta$,

$$\text{curv}(\nabla)(X,Y)s = \nabla_X \circ \nabla_Y s - \nabla_Y \circ \nabla_X s - \nabla_{[X,Y]}s$$

$$= \nabla_X(-i\Theta(Y)s) - \nabla_Y(-i\Theta(X)s) + i\Theta([X,Y])s$$

$$= -i(d(i_Y\Theta)(X) - i\Theta(X)\Theta(Y)$$

$$-d(i_X\Theta)(Y) + i\Theta(Y)\Theta(X) - \Theta([X,Y]))s$$

$$= -i(d(i_Y\Theta)(X) - d(i_X\Theta)(Y) - \Theta([X,Y]))s$$

$$= -i(d\Theta)(X,Y)s ,$$

(50)
for any $X, Y \in \mathfrak{X}(A; \mathbb{R})$. Independence of the choice of trivialisation is provided by lemma A.5.

Two equivalent hermitian line bundles $(L, \langle \cdot, \cdot \rangle)$ and $(L', \langle \cdot, \cdot \rangle')$ over the same manifold $M$ have equivalent hermitian connections $\nabla$ and $\nabla'$ if there exists a hermitian bundle diffeomorphism $\phi : L \to L'$ satisfying $\phi \circ \nabla_X s = \nabla'_X \phi \circ s$ for any section $s \in \Gamma(L)$ and vector field $X \in \mathfrak{X}(M; \mathbb{R})$.

**Lemma A.7.** The curvatures of two equivalent hermitian line bundles with hermitian connections $(L, \langle \cdot, \cdot \rangle, \nabla)$ and $(L', \langle \cdot, \cdot \rangle', \nabla')$ are equal.

**Proof:** Locally, the condition
\[
\phi \circ \nabla_X s = \nabla'_X \phi \circ s
\]
implies that the potential 1-forms are equal.

**Theorem A.4.** For a fixed closed form $\omega \in \Omega^2(M; \mathbb{R})$ whose de Rham class $[\frac{1}{e} \omega]$ is integral, there exists an equivalence class of complex line bundles with hermitian connections $[(L, \langle \cdot, \cdot \rangle, \nabla)]$, such that $\text{curv}(\nabla) = \frac{1}{e} \omega$ for any representative.

**Proof:** Given an integral de Rham class $[\frac{1}{e} \omega]$ and a contractible open cover of $M$, $\mathcal{A} = \{A_j\}_{j \in I}$, on each open set $A_j$ the 2-form $\omega$ is exact (Poincaré lemma): $\omega = d\theta_j$. The 1-forms $\theta_k - \theta_j$ are closed over $A_j \cap A_k$; consequently, $\frac{1}{e} \theta_k - \frac{1}{e} \theta_j = dg_{jk}$, the maps $A_j \cap A_k \ni p \mapsto e^{g_{jk}(p)} \in S^1$ give an element of $\tilde{H}^1(M; S^1)$ (since $[\frac{1}{e} \omega]$ is integral), and the potential 1-forms $\frac{1}{e} \theta_j$ provide the hermitian connection.

There is a converse of the previous statement.

**Proposition A.4.** If the hermitian connection $\nabla^\omega$ of a complex line bundle $(L, \langle \cdot, \cdot \rangle)$ satisfies $\text{curv}(\nabla^\omega) = \frac{1}{e} \omega$, then the de Rham class $[\frac{1}{e} \omega]$ is integral.

**Proof:** Let $\mathcal{A} = \{A_j\}_{j \in I}$ be a contractible open cover of $M$ such that each $A_j$ is a trivialising neighbourhood of $L$ with unitary section $s_j$. On each $A_j$ the formula $\omega = d\theta_j$ holds: $\frac{1}{e} \theta_j$ is the potential 1-form with respect to $s_j$ (lemma A.6). As a consequence, over each intersection $\frac{1}{e} \theta_k - \frac{1}{e} \theta_j = df_{jk}$, where $e^{f_{jk}} \in C^\infty(A_j \cap A_k; S^1)$ are the transition functions (lemma A.5). Now, $d(f_{jk} + f_{kl} - f_{jl}) = 0$ implies that $f_{jk} + f_{kl} - f_{jl} = a_{jkl} \in \mathbb{R}$ on $A_j \cap A_k \cap A_l$. The cocycle conditions of the line bundle, $e^{f_{jk} + f_{kl} - f_{jl}} = e^{f_{jl}}$, imply that $a_{jkl} = 0$; thus, $a_{jkl} \in 2\pi \mathbb{Z}$ and $[\frac{1}{e} \omega]$ is integral.

As a consequence of theorem A.3 and proposition A.4, flat hermitian line bundles are equivalent to hermitian line bundles admitting hermitian connections whose curvatures are identically zero (flat connections).

**Definition A.8.** The period between a closed 2-form $\omega$ and a compact 2-dimensional submanifold (without boundary) $N$ is given by $\text{Per}(\omega, [N]) = f_N(\omega)$.

**Proposition A.5.** The period $\text{Per}(\omega, [N])$ only depends on the de Rham cohomology class $[\omega] \in H^2_{dR}(M; \mathbb{R})$ and on the smooth singular homology class $[N] \in H^2_s(M; \mathbb{Z})$. 

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Proof: This is a straightforward application of Stoke’s theorem. \( N \) is a 2-cycle, \( \partial N = \emptyset \); ergo, if \( \omega = d\theta \),

\[
\int_{N} \omega = \int_{N} d\theta = \int_{\partial N} \theta = 0 ,
\]

and when \( N \) is a 2-boundary, \( N = \partial N \),

\[
\int_{N} \omega = \int_{\partial N} \omega = \int_{N} d\omega = 0 .
\] (53)

For an integral class \( [\frac{1}{e} \omega], \text{Per}(\omega, [N]) = ek([\omega], [N]) \), where \( k([\omega], [N]) \in \mathbb{Z} \).

**Theorem A.5.** If \( [N_1], \ldots, [N_{b_2(M)}] \in H^2_{\text{dR}}(M; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \) generates the second singular homology group of \( M \), given a set of integers \( \{k_1, \ldots, k_{b_2(M)}\} \subset \mathbb{Z} \), an integral class \( [\frac{1}{e} \omega] \in H^2_{\text{dR}}(M; \mathbb{R}) \) is uniquely defined by \( \text{Per}(\omega, [N_j]) = k_j e \).

**Proof:** This is a particular instance of de Rham’s theorem.

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