THE SCALAR CURVATURE PROBLEM ON FOUR-DIMENSIONAL MANIFOLDS

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Abstract. We study the problem of existence of conformal metrics with prescribed scalar curvatures on a closed Riemannian 4-manifold not conformally diffeomorphic to the standard sphere $S^4$. Using the critical points at infinity theory of A.Bahri [6] and the positive mass theorem of R.Schoen and S.T.Yau [32], we prove compactness and existence results under the assumption that the prescribed function is flat near its critical points. These are the first results on the prescribed scalar curvature problem where no upper-bound condition on the flatness order is assumed.

1. Introduction and main results. Given a compact Riemannian manifold $(M, g_0)$ of dimension $n \geq 3$, with no boundary and a function $K$ on $M$, we are interested in finding a new metric $g \in [g_0]$; $[g_0]$ is the conformal class of $g_0$, such that the scalar curvature of $g$ is equal to $K$. It is known that if $g = u^{\frac{4}{n-2}} g_0$, (here $u$ is a smooth positive function and the exponent $\frac{4}{n-2}$ is used to make the subsequent equations simpler), then the scalar curvatures $R_{g_0}$ and $R_g$ of the metrics $g_0$ and $g$ respectively, are related by the following equation:

$$-\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = \frac{n-2}{4(n-1)} R_g u^{\frac{n+2}{n-2}},$$

(E)

see for example [2]. Here $-\Delta_{g_0}$ is the Laplace-Beltrami operator associated to metric $g_0$.

According to (E), the problem is then reduced to the resolution of the equation

$$-\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = \frac{n-2}{4(n-1)} K u^{\frac{n+2}{n-2}},$$

(1)

It is known that if $R_{g_0} = 0$, then (1) has a solution if and only if $K$ changes sign and $\int_M K d\nu_{g_0} < 0$ (see [22]). When $R_{g_0} < 0$, then (1) always possesses a solution.

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when $K$ is negative on $M$ (see [21]). In this work, we only consider the case when $R_{g_0} > 0$. Using a direct integration, one deduces that a necessary condition to solve (1) is that $K$ is positive somewhere on $M$. Moreover Kazdan-Warner [24] found a topological obstruction to the resolution of (1) when $M$ is the unit sphere $S^n$.

Equation (1) has an underlying variational formulation on the Sobolev space $H^1(M)$. However, the variational structure presents a loss of compactness since the exponent $\frac{n+2}{n-2}$ is critical and therefore $H^1(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$ is continuous and not compact. Recall that by the regularity theorem of Trudinger [36], a weak solution of (1) is smooth.

The scalar curvature problem has always been one of major subject in differential geometry. Intensive studies were dedicated to this topic, in dimension 3 and 4, see [1, 8, 10, 11, 13, 18, 28, 34] as well as in high dimensions, see [3, 4, 5, 12, 14, 15, 16, 17, 19, 22, 23, 26, 27, 30, 35], and the references therein.

The best existence result of this problem addresses the 3-dimensional compact Riemannian manifold $(M^3, g_0)$, with $R_{g_0} > 0$ and which is not conformally diffeomorphic to $S^3$. In that case Escobar-Schoen [22], proved that the problem of existence of conformal metrics with prescribed scalar curvature on $(M^3, g_0)$ has a solution if and only if the prescribed function is positive somewhere on $M^3$. The situation in dimension $n \geq 4$ is different. Indeed, if we look at all existing results in these dimensions, as well as those that we will prove here, we see that we are far from optimal results. This being true even from the view point of the search of satisfactory topological conditions on the critical points of the prescribed function $K$.

The present paper deals with the 4-dimensional closed Riemannian manifolds $M$ which are not conformally diffeomorphic to $S^4$. Our aim is to prove global compactness and existence results for the problem when the prescribed function $K$ satisfies the so-called "$\beta$-flatness" condition near its critical points. The main novelty of our results is that the flatness order $\beta(y)$ varies in the entire interval $(1, \infty)$ for any critical point $y$ of $K$. Let us point out that existence results for the scalar curvature problem under the "$\beta$-flatness" condition were discussed for $S^n$, $n \geq 3$; in [19] under the assumption that $\beta(y) \in (1, n-2]$, for any critical point $y$ of $K$ and previously in [26] and [27] under the assumption that $\beta(y) \in [n-2, n)$ for any critical point $y$ of $K$.

This paper not only addresses to the case of $\beta(y) \geq n$ but also the mixed case; that is when there exist some critical points $y$ of $K$ such that $\beta(y) < n - 2$ and other critical points $y$ such that $\beta(y) \geq n - 2$. This question has never been handled before in the non-perturbative setting. Let

$$-L_{g_0} := -\Delta_{g_0} + \frac{n-2}{4(n-1)}R_{g_0}$$

be the conformal Laplacian associated to the metric $g_0$. We denote by $G(.,.)$ the Green function of $-L_{g_0}$ on $M$ and $H(.,.)$ its regular part. Following ([25], section 5), see also ([10], section 2), we associate to any $a \in M$ a conformal metric

$$g_a = u_a^{\frac{4}{n-2}} g_0 \text{ on } M,$$

where $u_a$ is a smooth positive function on $M$ ($u_a$ depends on $a$ smoothly and satisfies $u_a(a) = 1$ and $Du_a(a) = 0$). In addition if we denote by $\{x_i\}$ the conformal normal coordinates near $a$ that are the geodesic coordinates for the metric $g_a$ in $B(a, \rho_0)$ with respect to the exponential map ($\rho_0$ is a fixed positive constant independent of
Let $a$, upper bounded by the injective radius of $(M, g_0)$, we have
\[
\det(g_a(x)) = 1 + O(|x-a|^N),
\]
for any $x \in B(a, \rho_0)$. Here $x-a := \{x_i - a_i\}$ and $N \geq 5$. In the sequel, we will identify a point $x \in B(a, \rho_0)$ and its image by the exponential map.

It is proved in ([25], section 6), that if $n = 3, 4, 5$ or $M$ is locally conformally flat, the asymptotic expansion of the Green function $G(a, \cdot)$ in the conformal normal coordinates near $a$ is as follows
\[
G(a, x) = |x-a|^{2-n} + A_a + O(|x-a|), \text{ where } A_a = H(a, a).
\]
The sign of $A_a$ has been established in [31, 32] and [33], where, R.Schoen and S.T. Yau proved that $A_a \geq 0$. Moreover, $A_a > 0$, for any $a \in M$ if and only if $M$ is not conformally diffeomorphic to $S^n$.

For $\lambda > 0$ and $a \in M$, define
\[
\delta_{(a, \lambda)}(x) = \frac{\lambda^{n+2}}{(1 + \lambda^2|x-a|^2)^{n+2}}, \quad x \in B(a, 2\rho_0),
\]
it satisfies
\[
-\Delta \delta_{(a, \lambda)} = n(n-2)\delta_{(a, \lambda)}^{n+2} \quad \text{in } \mathbb{R}^n.
\]
Let $w_a(x)$ be a cut-off function on $M$ defined by
\[
w_a(x) = 1 \quad \text{on } B(a, \rho_0) \quad \text{and } w_a(x) = 0 \quad \text{on } M \setminus B(a, 2\rho_0)
\]
and let
\[
\hat{\delta}_{(a, \lambda)}(x) = w_a(x)\delta_{(a, \lambda)}(x).
\]
We denote by $\varphi_{(a, \lambda)}$ the unique solution of
\[
-L_{g_0} u = n(n-2)\delta_{(a, \lambda)}^{n+2} \quad \text{on } M
\]
and
\[
H_{(a, \lambda)} = \lambda^{n-2}(\varphi_{(a, \lambda)} - \hat{\delta}_{(a, \lambda)}).
\]
Clearly, $H_{(a, \lambda)}$ is smooth on $M$ and satisfies for $n = 3, 4$,
\[
H_{(a, \lambda)}(a) \to A_a, \text{ as } \lambda \to +\infty, \quad (2)
\]
and
\[
H_{(a, \lambda)}(a) \to G(a, x), \text{ in } B(a, \rho_0)^c, \text{ as } \lambda \to +\infty, \quad (3)
\]
see ([10], section 2).

Next, we state the “$\beta$-flatness” condition.

(1) Assume that $K : M \to \mathbb{R}$ is a $C^1$-positive function satisfying at any critical point $y$ of $K$ the following: There exists a real number $\beta = \beta(y)$ such that in the conformal normal coordinates system near $y$, $K$ is expressed as follows:
\[
K(x) = K(y) + \sum_{k=1}^{n} b_k|x_k - y_k|^\beta + o(|x-y|^\beta),
\]
with $\nabla^s o(|x-y|^\beta) = o(|x-y|^\beta-s)$, for any $s = 1, \ldots, [\beta]$. Moreover, $b_k = b_k(y) \neq 0, \forall k = 1, \ldots, n$ if $\beta(y) \leq n-2$, with
\[
\sum_{k=1}^{n} b_k(y) \neq 0, \text{ if } \beta(y) < n-2,
\]
and
\[-n \frac{c_1 \sum_{k=1}^{n} b_k}{w_{n-1} K(y)} - A_y \neq 0, \text{ if } \beta(y) = n - 2,\]
Here \(c_1 = \int_{\mathbb{R}^n} |z|^2 \frac{|z|^2 - 1}{(1+|z|^2)^{n/2}} \, dz\) and \(w_{n-1}\) is the volume of the \((n-1)\)-sphere.

Our first Theorem provides a complete description of the concentration phenomenon of the variational structure associated to (1.1) under \((f)_\beta\)-condition, \(\beta \in (1, \infty)\).

Let \(\mathcal{K}\) be the set of all critical points of \(K\) and let
\[
\mathcal{K}_{<n-2} = \{ y \in \mathcal{K}, \beta(y) < n - 2 \}, \quad \mathcal{K}_{=n-2}^+ = \{ y \in \mathcal{K}_{<n-2}, -\sum_{k=1}^{n} b_k(y) > 0 \},
\]
\[
\mathcal{K}_{=n-2} = \{ y \in \mathcal{K}, \beta(y) = n - 2 \}, \quad \mathcal{K}_{>n-2}^+ = \{ y \in \mathcal{K}_{=n-2}, -\sum_{k=1}^{n} b_k(y) - A_y > 0 \}
\]
and
\[
\mathcal{K}_{>n-2} = \{ y \in \mathcal{K}, \beta(y) > n - 2 \}
\]

For any \(p\)-tuple of distinct points \(\tau_p := (y_1, \ldots, y_p) \in (\mathcal{K}_{>n-2}^+)^p, 1 \leq p \leq |\mathcal{K}|\), we define the following symmetric matrix \(M(\tau_p) = (m_{ij})_{i,j}\) with
\[
m_{ii} = m(y_i, y_i) = \frac{1}{K(y_i)^{n-2}} \left[ -n \frac{c_1 \sum_{k=1}^{n} b_k}{w_{n-1} K(y_i)} - A_{y_i} \right], \quad \forall i = 1, \ldots, p.
\]
\[
m_{ij} = m(y_i, y_j) = -\frac{G(y_i, y_j)}{[K(y_i)K(y_j)]^{n-2}}, \quad 1 \leq i \neq j \leq p.
\]
\((A)\) Assume that the least eigenvalue \(\rho(\tau_p)\) of \(M(\tau_p)\) is non zero, for any \(\tau_p \in (\mathcal{K}_{>n-2}^+)^p\).

We introduce the following sets:
\[
\mathcal{C}_{>n-2} := \{ (y_1, \ldots, y_p) \in (\mathcal{K}_{>n-2}^+)^p, 1 \leq p \leq |\mathcal{K}|, \forall i \neq j \}
\]
\[
\mathcal{C}_{n-2} := \{ (y_1, \ldots, y_p) \in (\mathcal{K}_{>n-2}^+)^p, 1 \leq p \leq |\mathcal{K}|, y_i \neq y_j, \forall i \neq j \text{ and } \rho(y_1, \ldots, y_p) > 0 \}.
\]

**Theorem 1.1.** Let \((M, g_0)\) be a closed Riemannian manifold of dimension \(n = 4\) with a non-negative scalar curvature and not conformally diffeomorphic to the standard sphere \(S^4\). Let \(K\) be a given function on \(M\) satisfying \((A)\) and \((f)_\beta, \beta \in (1, \infty)\).

Under the assumption that (1.1) has no solution, the critical points at infinity of the associated variational problem (see section 2 for more details) are
\[
(y_1, \ldots, y_p)_{\infty} := \sum_{j=1}^{p} \frac{1}{K(y_j)} \varphi(y_j, \infty),
\]
where \((y_1, \ldots, y_p) \in C_{>n-2} \cup C_{n-2} \cup (C_{<n-2} \times C_{n-2})\). The index of such a critical point at infinity is
\[
i(y_1, \ldots, y_p)_{\infty} = p - 1 + \sum_{j=1}^{p} n - \tilde{i}(y_j),
\]
where \(\tilde{i}(y_j) = \sharp\{b_k(y_j), 1 \leq k \leq n, s.t. b_k(y_j) < 0\}\). Its level through \(J\) is
\[
S_n \sum_{j=1}^{p} \frac{1}{K(y_j)} = \sum_{j=1}^{p} \frac{1}{K(y_j)} = S_n \text{ is the best constant of the Sobolev inequality.}
\]

The above characterization of the critical points at infinity enables us to derive the following existence result.
Theorem 1.2. Assume that \((M, g_0)\) is a closed Riemannian manifold of dimension \(n = 4\) with a non-negative scalar curvature and not conformally diffeomorphic to \(S^4\). Let \(K : M \to \mathbb{R}\) be a given function satisfying \((A)\) and \((f_\beta)\), \(\beta \in (1, \infty)\). If
\[
\sum_{\tau_i \in C^\infty_{n-2} \cup C^6_{n-2} \cup (C^\infty_{n-2} \times C^\infty_{n-2})} (-1)^{i(\tau_i)} \neq 1,
\]
then there exists a metric \(g\) conformally equivalent to \(g_0\) with its scalar curvature equal to \(K\).

Our method enables to give another proof to the result of ([22], Theorem 2.1) in dimension 4. We remove here the locally conformally flat assumption on \(M\) form their result and we extend it to any closed 4-dimensional manifold not conformally diffeomorphic to \(S^4\).

Theorem 1.3. Let \((M, g_0)\) be a closed Riemannian manifold of dimension \(n = 4\) with a non-negative scalar curvature and not conformally diffeomorphic to \(S^4\) and let \(K : M \to \mathbb{R}\) be a \(C^1\)-positive function. Denote by \(y_0\) a global maximum of \(K\). If the derivatives of \(K\) up to order \(n-2\) vanish at \(y_0\), then \(K\) is the scalar curvature of a conformal metric \(g\).

Remark 1. The extension of our results in dimension \(n\), \(n = 5\), or \(n \geq 6\) and \(M\) is locally conformally flat, is related to the extension of \((2)\) and \((3)\) to these dimensions. Recall that the boundness of \(H_{(a, \lambda)}\) on \(M\) by a constant independent of \(\lambda\) is an important step to prove \((2)\) and \((3)\). It holds in dimensions 3 and 4. Indeed, following the computation of ([10], page 637) we have
\[
H_{(a, \lambda)}(x) = O(\lambda^{\frac{n-2}{2}} R_{g_a}(x) \delta_{(a, \lambda)}(x)), \quad \text{in } B(a, \rho_0), \quad \forall n \geq 3.
\]
Using the fact that \(R_{g_a}(x) = O(|x - a|^2)\), see ([25], Theorem 5.1) and the fact that \(\lambda^{\frac{n-2}{2}} \delta_{(a, \lambda)}(x) = O(|x - a|^{2-n})\), we get \(H_{(a, \lambda)}(x) = O(|x - a|^{4-n})\).

However our results hold in dimension 3. Theorem 1.1 shows that there is no critical points at infinity of the associated variational structure when \((M, g_0)\) is non-negative and not conformally diffeomorphic to \(S^3\) and Theorem 1.3 shows that the problem has always a solution when \(K > 0\), since each critical point of \(K\) have obviously an order of flatness \(> n-2\).

2. Variational structure. The Sobolev space \(H^1(M)\) is equipped with its norm:
\[
\|u\| = \left(\int_M -L_{g_0} u u dv_{g_0}\right)^\frac{1}{2}
\]
Define \(\Sigma = \{u \in H^1(M), \|u\| = 1\}\) and \(\Sigma^+ = \{u \in \Sigma, u > 0\}\). Let
\[
J(u) = \frac{\|u\|^2}{(n-2)(\int_M K(x) u^{\frac{2n}{n-2}} dv_{g_0})^{\frac{n-2}{n}}} , \quad u \in H^1(M) \setminus \{0\}.
\]
It is known that the solutions of the equation \((1.1)\) correspond to the critical points of \(J\) in \(\Sigma^+\). Since the Sobolev embedding \(H^1(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)\) is not compact, \(J\) fails to satisfy the Palais-Smale condition ((P.S) for short). The sequences which violate the (P.S) condition has been analyzed in [10] using the local conformal normal coordinates system.

For \(p \in \mathbb{N}^+\) and \(\varepsilon > 0\), define
Proposition 1 ([7, 29]). Assume that (1.1) has no solution, for any sequence \((u_k)\) in \(\Sigma^+\) such that \(J(u_k)\) is bounded and \(\partial J(u_k) \to 0\), there exist \(p \in \mathbb{N}^*\), a sequence \((\varepsilon_k)\), \(\varepsilon_k \to 0\) and an extracted sub-sequence of \((u_k)\), denoted again \((u_k)\) such that \(u_k \in V(p, \varepsilon_k)\), \(\forall k\).

The parametrization of \(V(p, \varepsilon)\) is given in the following proposition,

Proposition 2 ([7]). For any \(u \in V(p, \varepsilon)\) the following minimization problem

\[
\min\{\|u - \sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i)\|, \alpha_i > 0, \lambda_i > 0, a_i \in M\}
\]

has a unique solution (up to permutation). Thus we can uniquely write \(u\) as follows

\[
u = \sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) + v,
\]

where \(v \in H^1(M)\) and satisfies

\[
(V_0) : \langle v, \varphi \rangle = 0, \forall \varphi \in \{\varphi(a_i, \lambda_i), \frac{\partial \varphi(a_i, \lambda_i)}{\partial \lambda_i}, \frac{\partial \varphi(a_i, \lambda_i)}{\partial a_i}, \forall i = 1, ..., p\}.
\]

Here \(\langle ., . \rangle\) denotes the inner product in \(H^1(M)\) associated to the norm \(\|.\|\).

The next proposition shows that the \(v\)-part can be negligible with respect to the concentration phenomenon.

Proposition 3 ([10]). For any \(u = \sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) \in V(p, \varepsilon)\), the following minimization problem

\[
\min\{J(\sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) + v), v \in H^1(M)\ \text{satisfying} \ (V_0)\}
\]

has a unique solution \(\varpi = \varpi(\alpha_i, a_i, \lambda_i)\). Moreover there exists a \(C^1\) - change of variables \(v - \varpi \to V\) such that

\[
J(\sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) + v) = J(\sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) + \varpi) + \|V\|^2.
\]

The following estimate of \(\varpi\) was given in ([19]).

Proposition 4. There exists \(c > 0\) such that for any \(u = \sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) \in V(p, \varepsilon)\),

\[
\|\varpi\| \leq c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_i} + \frac{\|K(a_i)\|}{\lambda_i} + \frac{\log(\lambda_i)}{\lambda_i^{\frac{n+2}{n-2}}} + \sum_{i,j} \epsilon_{ij}(\log(\epsilon_{ij}^{-1}))^{\frac{n+2}{2}}\right).
\]

Next, we define the critical points at infinity of \(J\).
Definition 2.1 ([6]). The critical points at infinity of \( J \) are the ends of non-compact flow-lines of the gradient vector field \((-\partial J)\). By Proposition 2, a non compact flow-line \( u(s) \) can be written for \( s \gg 1 \) as \( u(s) = \sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i)(s) + v(s) \).

Denoting by
\[
y_i = \lim a_i(s) \text{ and } \alpha_i = \lim \alpha_i(s),
\]
we then denote
\[
\sum_{i=1}^{p} \alpha_i \varphi(y_i, \infty) \text{ or } (y_1, \ldots, y_p)_\infty
\]
such a critical point at infinity.

3. **Asymptotic expansion of the gradient of \( J \).** In this section we estimate the variation of \( J \) in \( V(p, \varepsilon) \) with respect to \( \lambda_j, j = 1, \ldots, p \) and with respect to \((a_j)_k, j = 1, \ldots, p \) and \( k = 1, \ldots, n \). Here \((a_j)_k\) denotes the \( k\text{th}-\)component of \((a_j)\) in the local normal coordinates system for the metric \( g_0 \). We consider here only the fourth dimensional case. Our aim is to refine the computation of ([10], Lemmas B5 and B6) taken in some non-degeneracy condition and extend it to the \((f)_\beta\)-condition, \( \beta \in (1, \infty) \). We point out that the non-degeneracy condition (see [10], page 634 ) is a particular case of our framework. It corresponds to \((f)_\beta\)-condition with \( \beta = \beta(y) = 2 \) at any critical point \( y \) of \( K \).

To obtain the estimates of the next propositions, we will work with the local conformal normal coordinates in order to reduce the calculation to \( \mathbb{R}^4 \). In the sequel, we will identify a point \( x \in B(a, \rho_0) \) and its image by the exponential map. We will also identify the function \( K \) and its related transformation.

**Proposition 5.** Assume that \( K \) satisfies \((f)_\beta\)-condition, \( \beta \in (1, \infty) \). For any \( u = \sum_{i=1}^{p} \alpha_i \varphi(a_i, \lambda_i) \in V(p, \varepsilon) \) and for any \( j = 1, \ldots, p \) such that \( a_j \in B(y_j, \rho) \), \( y_j \in K \) and \( \rho \leq \rho_0 \), we have the following two expansions.

1) \(< \partial J(u), \alpha_j \lambda_j \frac{\partial \varphi(a_j, \lambda_j)}{\partial \lambda_j} > \)

\[
= -4J(u) \sum_{i \neq j} c_{ij} \alpha_i \alpha_j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + J(u)w_3 \sum_{i \neq j} \alpha_i \alpha_j \frac{H(a_i, \lambda_i)(a_j)}{\lambda_i \lambda_j} \\
+ J(u) \alpha_j^2 w_3 \frac{H(a_j, \lambda_j)(a_j)}{\lambda_j^2} + O\left( \sum_{s=2}^{[\min(3, \beta)]} \frac{|a_j - y_j|^{\beta-s}}{\lambda_j^s} \right) \\
+ O\left( \frac{1}{\lambda_j^{\min(3, \beta)}} \right) + o(\sum_{i \neq j} \varepsilon_{ij}).
\]

2) \(< \partial J(u), \alpha_j \lambda_j \frac{\partial \varphi(a_j, \lambda_j)}{\partial \lambda_j} > \)

\[
= -4J(u) \sum_{i \neq j} c_{ij} \alpha_i \alpha_j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + J(u)w_3 \left[ \sum_{i \neq j} \alpha_i \alpha_j \frac{H(a_i, \lambda_i)(a_j)}{\lambda_i \lambda_j} \\
+ \alpha_j^2 \frac{H(a_j, \lambda_j)(a_j)}{\lambda_j^2} \right] + \left( \frac{4c_1}{w_3 K(a_j)} \alpha_j^2 \sum_{k=1}^{4} b_k \right) \\
+ O(|a_j - y_j|^\beta) + o\left( \frac{1}{\lambda_j^2} \right) + o\left( \sum_{i \neq j} \varepsilon_{ij} \right).
\]
Here
\[ c_1 = \int_{\mathbb{R}^4} |z|^\beta \frac{|z|^2 - 1}{(1 + |z|^2)^5} dz \text{ and } c_{ij} = 1 \text{ if } |a_i - a_j| < \frac{\rho}{2} \text{ and } c_{ij} = 0 \text{ if } |a_i - a_j| \geq 2\rho. \]

**Proof of Proposition 5** Let \( u = \sum_1^p \alpha_i \varphi_{a_i, \lambda_i} \in V(p, \varepsilon). \)

\[ \partial J(u) = \lambda(u)[u + \lambda(u)^2 L_{g_0}(K u^3)], \]

where \( \lambda(u) = 2 J(u). \) Following the computation of ([10], Lemma B6), we have

\[ < \partial J(u), \alpha_j \frac{\partial \varphi_{a_j, \lambda_j}}{\partial \lambda_j} >= -8 \lambda(u) \sum_{i \neq j} \alpha_i \alpha_j \int_M \varphi_{a_i, \lambda_i} \delta^3_{(a_j, \lambda_j)} dv_{g_0} \]

\[ + 2 \lambda(u) \alpha_j^2 w_3 \frac{H(a_j, \lambda_j)(a_j)}{\lambda_j^3} - \lambda(u)^3 \alpha_j^2 \int_M K(x) \delta^3_{(a_j, \lambda_j)} \partial \delta_j(a_j) dv_{g_0} + o(\sum_i \varepsilon_{ij}), \]

\[ = -8 \lambda(u) \sum_{i \neq j} c_{ij} \alpha_i \alpha_j \frac{\partial \varphi_{a_j, \lambda_j}}{\partial \lambda_j} + 2 \lambda(u) w_3 \sum_{i \neq j} \alpha_i \alpha_j \frac{H(a_i, \lambda_j)(a_j)}{\lambda_i \lambda_j} \]

\[ + 2 \lambda(u) \alpha_j^2 w_3 \frac{H(a_j, \lambda_j)(a_j)}{\lambda_j^3} - \lambda(u)^3 \alpha_j^2 \int_M K(x) \delta^3_{(a_j, \lambda_j)} \partial \delta_j(a_j) dv_{g_0} + o(\sum_i \varepsilon_{ij}). \]

A direct calculation shows that the smooth function \( \delta_{(a, \lambda)} \) defined in the first section satisfies

\[ \delta_{(a, \lambda)}^3 \frac{\partial \delta_{(a, \lambda)}(x)}{\partial \lambda} = \lambda^4 \frac{1 - \lambda^2 |x - a|^2}{(1 + \lambda^2 |x - a|^2)^5}, \quad x \in \mathbb{R}^4. \]

Let \( \mu > 0 \) such that \( B(a_j, \mu) \subset B(y_j, \rho). \) After considering the local conformal coordinates system, we have

\[ \int_M K(x) \delta^3_{(a_j, \lambda_j)} \frac{\partial \delta_{(a_j, \lambda_j)}}{\partial \lambda_j} dv_{g_0} = \int_{B(a_j, \mu) \subset \mathbb{R}^4} K(x) \delta^3_{(a_j, \lambda_j)} \delta_{(a, \lambda)} \frac{\partial \delta_j(a_j)}{\partial \lambda_j} dx + O\left(\frac{1}{\lambda_j^4}\right). \]

Setting \( z = \lambda_j (x - a_j) \), we get

\[ \int_{B(a_j, \mu)} K(x) \delta^3_{(a_j, \lambda_j)} \frac{\partial \delta_j(a_j)}{\partial \lambda_j} dx = \int_{B(0, \lambda_j \mu, \mu)} K(a_j + \frac{z}{\lambda_j}) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz. \]

A Taylor expansion of \( K \) around \( a_j \) up to order \( \lfloor \min(3, \beta) \rfloor \); the integer part of \( \min(3, \beta) \), yields

\[ K(a_j + \frac{z}{\lambda_j}) = K(a_j) + \sum_{s=1}^{\lfloor \min(3, \beta) \rfloor} \frac{D^s K(a_j)(\frac{z}{\lambda_j})^s}{s!} + O\left(\frac{z}{\lambda_j}^{\min(3, \beta)}\right). \]

Observe that

\[ \int_{B(0, \lambda_j \mu, \mu)} K(a_j) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = O\left(\frac{1}{\lambda_j^4}\right), \]

and

\[ \int_{B(0, \lambda_j \mu, \mu)} DK(a_j)(z) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = 0. \]

Therefore,

\[ \int_{B(0, \lambda_j \mu, \mu)} K(a_j + \frac{z}{\lambda_j}) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = \sum_{s=2}^{\lfloor \min(3, \beta) \rfloor} \int_{B(0, \lambda_j \mu, \mu)} \frac{D^s K(a_j)(z)^s}{s! \lambda_j^s} \frac{1 - |z|^2}{(1 + |z|^2)^5} dz \]

\[ + \int_{B(0, \lambda_j \mu, \mu)} K(a_j + \frac{z}{\lambda_j}) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = \sum_{s=2}^{\lfloor \min(3, \beta) \rfloor} \int_{B(0, \lambda_j \mu, \mu)} \frac{D^s K(a_j)(z)^s}{s! \lambda_j^s} \frac{1 - |z|^2}{(1 + |z|^2)^5} dz. \]
\[ + O \left( \frac{1}{\lambda_j^{\min(3,\beta)}} \right) \int_{\mathbb{R}^4} |z|^{\min(3,\beta)} \frac{|1 - |z|^2|}{(1 + |z|^2)^5} dz \]
\[ = O \left( \sum_{s=2}^{\min(3,\beta)} \frac{|a_j - y_{ij}|^{\beta-s}}{\lambda_j^s} \right) + O \left( \frac{1}{\lambda_j^{\min(3,\beta)}} \right), \]

since under \((f)_\beta\)-condition we have
\[ |D^s K(a_j)| = O(|a_j - y_{ij}|^{\beta-s}). \]

This concludes the first estimate of proposition 5.

To obtain the estimate 2) of proposition 5, we use the \((f)_\beta\)-expansion around \(y_i\), \(\forall z \in B(0, \lambda_j \mu)\) we have
\[ K(a_j + \frac{z}{\lambda_j}) = K(y_i) + \sum_{k=1}^{4} b_k |\frac{z}{\lambda_j} + (a_j - y_{ij})|^{\beta} + o(|\frac{z}{\lambda_j} + (a_j - y_{ij})|^{\beta}). \]

Therefore,
\[ \int_{B(0, \lambda_j \mu)} K(a_j + \frac{z}{\lambda_j}) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz \]
\[ = \frac{1}{\lambda_j^\beta} \sum_{k=1}^{4} b_k \int_{B(0, \lambda_j \mu)} |z_k + \lambda_j(a_j - y_{ij})|^{\beta} \frac{1 - |z|^2}{(1 + |z|^2)^5} dz \]
\[ + o \left( \frac{1}{\lambda_j^\beta} \right) \int_{B(0, \lambda_j \mu)} |z|^{\beta} \frac{1 - |z|^2}{(1 + |z|^2)^5} dz + o(|a_j - y_{ij}|^{\beta}) \int_{\mathbb{R}^4} |1 - |z|^2| \frac{1}{(1 + |z|^2)^5} dz \]
\[ = \frac{1}{\lambda_j^\beta} \sum_{k=1}^{4} b_k \int_{B(0, \lambda_j \mu)} |z|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^5} dz + o \left( \frac{1}{\lambda_j^\beta} \right) \int_{B(0, \lambda_j \mu)} |z|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^5} dz \]
\[ + O(|a_j - y_{ij}|^{\beta}). \]

Observe that, for \(\beta < 4\)
\[ \int_{B(0, \lambda_j \mu)} |z|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = \int_{\mathbb{R}^4} |z|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^5} dz + O \left( \frac{1}{\lambda_j^{4-\beta}} \right) = -c_1 + O \left( \frac{1}{\lambda_j^{4-\beta}} \right). \]

For \(\beta = 4\)
\[ \int_{B(0, \lambda_j \mu)} |z_k|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = O(\log(\lambda_j)). \]

and for \(\beta > 4\)
\[ \int_{B(0, \lambda_j \mu)} |z|^\beta \frac{1 - |z|^2}{(1 + |z|^2)^5} dz = O \left( \frac{1}{\lambda_j^{\beta-\beta}} \right). \]

Thus,
\[ \int_{B(0, \lambda_j \mu)} K(a_j + \frac{z}{\lambda_j}) \frac{1 - |z|^2}{(1 + |z|^2)^5} dz \]
\[ = -c_1 \sum_{k=1}^{4} \frac{b_k}{\lambda_j^\beta} + o \left( \frac{1}{\lambda_j^{\beta}}, \beta < 4 \right) + O(|a_j - y_{ij}|^{\beta}) + o \left( \frac{1}{\lambda_j^{\beta}} \right). \]

This ends the proof of proposition 5 after recalling that \(\lambda(u)^2 \alpha_j^2 K(a_j) \rightarrow 8\) as \(\varepsilon\) goes to zero. \(\square\)
Proposition 6. Under the assumption that \( K \) satisfies \((f)_{\beta}\)-condition, \( \beta \in (1, \infty) \), for any \( u = \sum_{i=1}^{p} a_i \varphi_i(x_i) \in V(p, \varepsilon) \) and for any \( j = 1, \ldots, p \), such that \( a_j \in B(y_j, \rho) \), \( y_j \in K \), we have the following expansions

\[
a_j \prec \partial J(u), \alpha_j \frac{1}{\lambda_j} \frac{\partial \varphi(a_j, \lambda_j)}{\partial (a_j)_k} \succ = -2J(u)\alpha_j^2 \beta b_k \text{sign}(a_j - y_j)_k \frac{|(a_j - y_j)_k|^{\beta - 1}}{K(a_j)_j} \]

\[
+ O\left( \sum_{s=2}^{\min(4, \beta)} \frac{|a_j - y_j|^{\beta - s}}{\lambda_j^s} \right) + O\left( \frac{1}{\lambda_j^{\min(4, \beta)}} \right) + O\left( \frac{1}{\lambda_j^3} \right) + O\left( \sum_{i \neq j} \frac{1}{\lambda_j} \frac{\partial \varepsilon_{ij}}{\partial a_j} \right),
\]

where \( c_2 = \int_{\mathbb{R}^4} \frac{|z|^2}{(1 + |z|^2)^5} dz \). Moreover, if \( \lambda_j |a_j - y_j| \) is bounded and \( \beta < 5 \), we have

\[
b_j \prec \partial J(u), \alpha_j \frac{1}{\lambda_j} \frac{\partial \varphi(a_j, \lambda_j)}{\partial (a_j)_k} \succ
\]

\[
= -2J(u)\alpha_j^2 \beta b_k \frac{b_k}{K(a_j)_j \lambda_j^2} \int_{\mathbb{R}^4} |z_k + \lambda_j (a_j - y_j)_k|^{\beta - 1} \frac{z_k}{(1 + |z|^2)^5} dz
\]

\[
+ o\left( \frac{1}{\lambda_j^2} \right) + O\left( \sum_{i \neq j} \frac{1}{\lambda_j} \frac{\partial \varepsilon_{ij}}{\partial a_j} \right).
\]

Proof of Proposition 6. Following the computation of ([10], Lemma B5). We have

\[
< \partial J(u), \alpha_j \frac{1}{\lambda_j} \frac{\partial \varphi(a_j, \lambda_j)}{\partial (a_j)_k} >
\]

\[
= -\lambda(u)^3 \alpha_j^4 \int_M K(x)^3 (a_j, \lambda_j) \frac{1}{\lambda_j} \frac{\partial \delta(a_j, \lambda_j)}{\partial (a_j)_k} dv_{g_0} + O\left( \sum_{i \neq j} \frac{1}{\lambda_j} \frac{\partial \varepsilon_{ij}}{\partial a_j} \right).
\]

Observe that

\[
\delta^3(a, \lambda) \frac{1}{\lambda} \frac{\partial \delta(a, \lambda)}{\partial (a)_k} = \frac{\lambda^5 (x - a)_k}{(1 + \lambda^2 |x - a|^2)^5}
\]

Therefore by a Taylor expansion as the one of the proof of proposition 5, we find

\[
\int_M K(x)^3 (a_j, \lambda_j) \frac{1}{\lambda_j} \frac{\partial \delta(a_j, \lambda_j)}{\partial (a_j)_k} dv_{g_0}
\]

\[
= \sum_{s=1}^{\min(4, \beta)} \int_{B(0, \lambda, \mu)} \frac{D^s K(a_j)(z)^s}{s! \lambda_j^s} \left( \frac{z_k}{(1 + |z|^2)^5} \right) dz + O\left( \frac{1}{\lambda_j^{\min(4, \beta)}} \right) + O\left( \frac{1}{\lambda_j^3} \right)
\]

\[
= \int_{B(0, \lambda, \mu)} \frac{D K(a_j)(z)}{\lambda_j} \frac{z_k}{(1 + |z|^2)^5} dz + O\left( \sum_{s=2}^{\min(4, \beta)} \frac{|a_j - y_j|^{\beta - s}}{\lambda_j^s} \right)
\]

\[
+ O\left( \frac{1}{\lambda_j^3} \right) + O\left( \frac{1}{\lambda_j^{\min(4, \beta)}} \right).
\]

Observe that

\[
\int_{B(0, \lambda, \mu)} \frac{D K(a_j)(z) z_k}{(1 + |z|^2)^5} dz = \sum_{i=1}^{4} \frac{\partial K(a_j)}{\partial x_i} \int_{B(0, \lambda, \mu)} \frac{z_k z_i}{(1 + |z|^2)^5} dz
\]

\[
= \frac{\partial K(a_j)}{\partial x_k} \int_{B(0, \lambda, \mu)} \frac{z_k^2}{(1 + |z|^2)^5} dz = \frac{1}{4} \frac{\partial K(a_j)}{\partial x_k} \int_{\mathbb{R}^4} \frac{|z|^2}{(1 + |z|^2)^5} dz + O\left( \frac{1}{\lambda_j^3} \right),
\]
since \( \int_{B(0, \lambda, \mu)} \frac{z_k^2 z_l^2}{(1 + |z|^2)^2} dz = 0, \forall l \neq k. \) Moreover under \((f_\beta)-condition\) we have
\[
\frac{\partial K(a_j)}{\partial z_k} = b_k \beta \text{sign}(a_j - y_{i_j})|(a_j - y_{i_j})_k|^{\beta - 1} + o(|(a_j - y_{i_j})_k|^{\beta - 1}).
\]

This ends the proof of a/ of proposition 5. Concerning b/, it follows from the following estimate
\[
\int_{B(0, \lambda, \mu)} K(a_j + \frac{z_k}{\lambda_j}) \frac{z_k}{(1 + |z|^2)^{\beta}} dz = \frac{b_k}{\lambda_j^3} \int_{\mathbb{R}^4} |z_k + \lambda_j (a_j - y_{i_j})|^{\beta} \frac{z_k}{(1 + |z|^2)^{\beta}} dz + o(\frac{1}{\lambda_j^3}).
\]

4. critical points at infinity. In this section we study the concentration phenomenon of the problem and we provide a complete description of the critical points at infinity of \( J \) under \((f)\beta\)-condition, \( \beta \in (1, \infty). \)

**Theorem 4.1.** Assume that \((M, g_0)\) is a closed 4- Riemannian manifold with a non negative scalar curvature and not conformally diffeomorphic to \( S^4 \). Under the assumptions (A) and \((f_\beta), \beta \in (1, \infty), \) there exists a decreasing pseudo gradient \( W \) in \( V(p, \epsilon) \) satisfying the following:

\[
(i) < \partial J(u), W(u) > \leq -c \left( \sum_{i=1}^{p} \left( \frac{1}{\lambda_{\min(2, \beta)}} + \frac{||\nabla K(a_i)||}{\lambda_i} \right) + \sum_{i \neq j} \epsilon_{ij} \right),
\]

\[
(ii) < \partial J(u + \nu), W(u) + \frac{\partial \nu}{\partial(a_{i}, a_i, \lambda_i)}(W(u)) > \leq -c \left( \sum_{i=1}^{p} \left( \frac{1}{\lambda_{\min(2, \beta)}} + \frac{||\nabla K(a_i)||}{\lambda_i} \right) + \sum_{i \neq j} \epsilon_{ij} \right),
\]

for any \( u = \sum_{i=1}^{p} a_i \varphi_{a_i} \lambda_i \in V(p, \epsilon). \) Here \( c \) is a positive constant independent of \( u. \)

Moreover \( W \) is bounded and the only case where \( \lambda_i(s), i = 1, \ldots, p \) tend to \( \infty \) is when \( a_i(s) \) tends to \( y_i, \forall i = 1, \ldots, p \) such that \( (y_{i_1}, \ldots, y_{i_p}) \in C_{\infty}^2 \cup C_{\infty}^1 \cup (C_{\infty}^2 \times C_{\infty}^1). \)

The proof of Theorem 4.1 is based on the following sequence of Lemmas which describe the concentration phenomenon in particular regions on \( V(p, \epsilon) \) and hint the construction of the required pseudo-gradient \( W. \) For \( \delta > 0 \) small enough, we define

\[
V_{<2}(p, \epsilon) = \{ u = \sum_{i=1}^{p} a_i \varphi_{a_i} \lambda_i + v \in V(p, \epsilon), \text{s.t.} a_i \in B(y_{i_1}, \rho), y_{i_1} \in K_{<2}, \lambda_i|a_i - y_{i_1}| < \delta, \forall i = 1, \ldots, p \text{ and } y_{i_1} \neq y_{i_j}, \forall i \neq j \},
\]

\[
V_2(p, \epsilon) = \{ u = \sum_{i=1}^{p} a_i \varphi_{a_i} \lambda_i + v \in V(p, \epsilon), \text{s.t.} a_i \in B(y_{i_1}, \rho), y_{i_1} \in K_{2}, \lambda_i|a_i - y_{i_1}| < \delta, \forall i = 1, \ldots, p \text{ and } y_{i_1} \neq y_{i_j}, \forall i \neq j \},
\]

\[
V_{\leq 2}(p, \epsilon) = \{ u = \sum_{i=1}^{p} a_i \varphi_{a_i} \lambda_i + v \in V(p, \epsilon), \text{s.t.} a_i \in B(y_{i_1}, \rho), y_{i_1} \in K_{<2} \cup K_2, \lambda_i|a_i - y_{i_1}| < \delta, \forall i = 1, \ldots, p \text{ and } y_{i_1} \neq y_{i_j}, \forall i \neq j \},
\]
and

\[ V_{>1}(p, \epsilon) = \{ u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} + v \in V(p, \epsilon), s.t., a_i \in B(y_i, \rho), y_i \in \mathcal{K}, \lambda_i |a_i - y_i| < \delta, \forall i = 1, ..., p \text{ and } y_i \neq y_j, \forall i \neq j \}. \]

**Lemma 4.2.** There exists a bounded pseudo-gradient \( W_1 \) in \( V_{<2}(p, \epsilon) \) satisfying the inequality (i) of the above Theorem for any \( u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{<2}(p, \epsilon) \). Moreover, the only case where \( \lambda_i(s), i = 1, ..., p \) tend to \( \infty \) is when \( a_i(s) \) tends to \( y_i, \forall i = 1, ..., p \) with \((y_1, ..., y_p) \in C_{<2}.\)

**Proof of Lemma 4.2** To construct \( W_1 \), we divide \( V_{<2}(p, \epsilon) \) into two regions, we construct an appropriate pseudo-gradient in each region and we glue up through a convex combination. Define

\[ R_1 := \{ u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{<2}(p, \epsilon), \text{ s.t., for any } i = 1, ..., p \text{ we have } - \sum_{k=1}^{p} b_k(y_i) > 0 \}. \]

\[ R_2 := \{ u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{<2}(p, \epsilon), \text{ s.t., there exists at least } i_1, 1 \leq i_1 \leq p, \text{ satisfying } - \sum_{k=1}^{p} b_k(y_{i_1}) < 0 \}. \]

**Pseudo-gradient in \( R_1.** We increase in this region all \( \lambda_i \) by setting

\[ \dot{\lambda}_i = \lambda_i, \forall i = 1, ..., p. \]

For any \( u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in R_1 \), we define

\[ W_1^1(u) = \sum_{i=1}^{p} \alpha_i \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}, \dot{\lambda}_i. \]

Using the second expansion of Proposition 5, we find

\[ < \partial J(u), W_1^1(u) > = J(u) w_3 \left[ \sum_{i \neq j}^{p} \alpha_i \alpha_j \frac{H(a_i, \lambda_i)(a_j)}{\lambda_i \lambda_j} + \sum_{j=1}^{p} \alpha_j^2 \frac{H(a_j, \lambda_j)(a_j)}{\lambda_j^2} \right. \]

\[ + \frac{4}{w_3} \sum_{j=1}^{p} \frac{c_1}{K(a_j)} \alpha_j^2 \sum_{k=1}^{4} \frac{b_k}{\lambda_j^2} \left| a_j - y_j \right|^\beta + O(\sum_{j=1}^{p} \frac{1}{\lambda_j^\beta}) \bigg] + O(\sum_{j=1}^{p} \frac{1}{\lambda_j^\beta}). \]

Observe that for any \( i, j = 1, ..., p \), we have

\[ \frac{1}{\lambda_i \lambda_j} = O(\frac{1}{\lambda_i^\beta}) + O(\frac{1}{\lambda_j^\beta}) = o(\frac{1}{\lambda_i^\beta}) + o(\frac{1}{\lambda_j^\beta}), \quad (7) \]

since \( \beta < 2 \). Moreover for \( \delta \) small enough, we have

\[ \left| a_j - y_j \right|^\beta = o(\frac{1}{\lambda_j^\beta}). \quad (8) \]

Therefore,

\[ < \partial J(u), W_1^1(u) > = 4J(u) \sum_{j=1}^{p} \frac{c_1}{K(a_j)} \alpha_j^2 \sum_{k=1}^{4} \frac{b_k}{\lambda_j^2} + o(\sum_{j=1}^{p} \frac{1}{\lambda_j^\beta}) \leq -c \sum_{j=1}^{p} \frac{1}{\lambda_j^\beta}, \quad (9) \]
since each \( \sum_{k=1}^{4} b_k \) is negative in \( R_1 \). Using now the fact that
\[
\epsilon_{ij} \sim \frac{1}{\lambda_i \lambda_j}, \forall i \neq j \quad \text{and} \quad |\nabla K(a_i)| = O(|a_i - y_i|^\beta - 1),
\]
we derive from (7), (8) and (9) that
\[
< \partial J(u), W^1_1 > \leq -c \left( \sum_{j=1}^{p} \frac{1}{\lambda_j^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j} \epsilon_{ij}.
\]

**Pseudo-gradient in \( R_2 \).** Let \( u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in R_2 \). We order all \( \lambda_i^\beta \). Without loss of the generality, we may assume that
\[
\lambda_1^\beta \leq ... \leq \lambda_p^\beta.
\]
Recall that for each index \( i, \beta \) depends to \( y_i \). Let \( i_1 \) be the first index such that
\(- \sum_{k=1}^{4} b_k < 0 \). Define
\[
\tilde{\lambda}_{i_1} = - \lambda_{i_1}
\]
The associated vector field is \( W_{i_1}(u) = - \alpha_{i_1, \lambda_{i_1}} \frac{\partial \varphi_{a_{i_1}, \lambda_{i_1}}}{\partial \lambda_{i_1}} \). Using (7), (8) and the second estimate of proposition 3.1, we have
\[
< \partial J(u), W_{i_1} > \leq -c \frac{1}{\lambda_{i_1}^\beta} + o \left( \sum_{j=1}^{p} \frac{1}{\lambda_j^\beta} \right) \leq -c \frac{1}{\sum_{j=1}^{p} \lambda_j^\beta} + o \left( \sum_{j=1}^{p} \frac{1}{\lambda_j^\beta} \right),
\]
since \( \lambda_j^\beta \geq \lambda_{i_1}^\beta, \forall j = i_1, ..., p \).

Therefore, if \( i_1 = 1 \), we then obtain the desired estimate of Lemma 4.2 from (7), (10) and (11). If \( i_1 \geq 2 \), we write \( u \) as follows
\[
u = \sum_{i=1}^{i_1-1} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{i=i_1}^{p} \alpha_i \varphi_{a_i, \lambda_i} := u_1 + u_2
\]
Observe that \( u_1 \in R_1 \) (here \( R_1 \) is constructed by \( i_1 - 1 \) masses). Setting in this statement
\[
\tilde{W}^1_1(u) = W^1_1(u_1),
\]
where \( W^1_1(u_1) \) is defined in the above region \( R_1 \). It satisfies
\[
< \partial J(u), \tilde{W}^1_1(u) > \leq -c \left( \sum_{i=1}^{i_1-1} \frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{1 \leq i < j \leq i_1-1} \epsilon_{ij} + O \left( \sum_{1 \leq i \leq i_1-1} \frac{1}{\lambda_i} \right).
\]
Using (7), (10) and (11), we get
\[
< \partial J(u), \tilde{W}^1_1(u) + W_{i_1}(u) > \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j} \epsilon_{ij}.
\]
Observe that along the deformation in \( R_2 \), \( \sup_{1 \leq i \leq p} \lambda_i^\beta \) do not change. Therefore \( \sup_{1 \leq i \leq p} \lambda_i \) remains bounded as long as the flow line remains in \( R_2 \). However if the flow line enter in \( R_1 \) all \( \lambda_i, i = 1, ..., p \) increase and go to \( \infty \), (concentration phenomenon). This ends the proof of Lemma 4.2.

The following Lemma is proved in [10].

**Lemma 4.3 ([10]).** There exists a bounded pseudo-gradient \( W_2 \) in \( V_2(p, \epsilon) \) satisfying the inequality (i) of Theorem 4.1 for any \( u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_2(p, \epsilon) \). Moreover the only case where \( \lambda_i(s), i = 1, ..., p \) tends to \( \infty \) is when \( \alpha_i(s) \) tends to \( y_i, \forall i = 1, ..., p \) with \( (y_1, ..., y_p) \in C_2 \).
Lemma 4.4. There exists a bounded pseudo-gradient $W_3$ in $V_{\leq 2}(p, \epsilon)$ satisfying the inequality (i) of Theorem 4.1 for any $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{\leq 2}(p, \epsilon)$. Moreover the only case where $\lambda_i(s)$, $i = 1, ..., p$ tends to $\infty$ is when $\alpha_i(s)$ tends to $y_i$, $\forall i = 1, ..., p$ with $(y_1, ..., y_p) \in C_{\leq 2}^\infty \cup (C_{\leq 2}^\infty \times C_{\leq 2}^\infty)$.

Proof of Lemma 4.4 Let $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{\leq 2}(p, \epsilon)$. Following Lemmas 4.2 and 4.3 the only case that we will consider here is when $u$ can be written as

$$u = \sum_{i=1}^{q} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{i=q+1}^{p} \alpha_i \varphi_{a_i, \lambda_i} := u_1 + u_2,$$

with $1 \leq q < p$, $u_1 \in V_{< 2}(q, \epsilon)$ and $u_2 \in V_2(p-q, \epsilon)$.

There cases may occur.

Case 1.

$$u_1 \in \{u = \sum_{j=1}^{q} \alpha_j \varphi_{a_j, \lambda_j} \in V_{\leq 2}(q, \epsilon) \text{ with } (y_1, ..., y_q) \in C_{\leq 2}^\infty\}$$

and

$$u_2 \in \{u = \sum_{j=q+1}^{p} \alpha_j \varphi_{a_j, \lambda_j} \in V_2(p-q, \epsilon) \text{ with } (y_1, ..., y_{p-q}) \in C_{\leq 2}^\infty\}.$$

We set $\tilde{W}_1(u) = W_1(u_1)$ and $\tilde{W}_2(u) = W_2(u_2)$, where $W_1$ and $W_2$ are the pseudo-gradients defined respectively in lemma 4.2 and lemma 4.3. We have

$$< \partial J(u), \tilde{W}_1(u) > \leq -c(\sum_{i=1}^{q} \frac{1}{\lambda_i^2} + \sum_{1 \leq i \neq j \leq q} \epsilon_{ij}) + O(\sum_{1 \leq i \leq q, \sum_{1 \leq j \leq p} \frac{1}{\lambda_i \lambda_j}).$$

$$< \partial J(u), \tilde{W}_2(u) > \leq -c(\sum_{i=q+1}^{p} \frac{1}{\lambda_i^2} + \sum_{q+1 \leq i \neq j \leq p} \epsilon_{ij}) + O(\sum_{q+1 \leq i \leq p, \sum_{1 \leq j \leq q} \frac{1}{\lambda_i \lambda_j}).$$

Observe that, $\forall i \leq q$ and $\forall q + 1 \leq j \leq p$ we have

$$\frac{1}{\lambda_i \lambda_j} = o(\frac{1}{\lambda_i^2}) + o(\frac{1}{\lambda_j^2}). \quad (12)$$

Indeed, let $M > 1$. If $\lambda_i > M \lambda_j$, then $\frac{1}{\lambda_i \lambda_j} \leq \frac{1}{M \lambda_i^2} = o(\frac{1}{\lambda_j^2})$ by taking $M$ large enough. If $\lambda_i < M \lambda_j$, then $\frac{1}{\lambda_i \lambda_j} \leq \frac{M}{\lambda_i^2} = o(\frac{1}{\lambda_i^2})$, since $\beta < 2$. Using (12) we obtain

$$< J(u), \tilde{W}_1(u) + \tilde{W}_2(u) > \leq -c(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \frac{\nabla K(a_{ij})}{\lambda_i}) + \sum \epsilon_{ij}).$$

Observe that through $\tilde{W}_1 + \tilde{W}_2$, the components $\lambda_i(s)$, $i = 1, ..., p$, of the flow lines satisfy the differential equation $\lambda_i = \lambda_i$, therefore a concentration phenomenon happens in this case.

Case 2:

$$u_1 \notin \{u = \sum_{j=1}^{q} \alpha_j \varphi_{a_j, \lambda_j} \in V_{\leq 2}(q, \epsilon) \text{ with } (y_1, ..., y_q) \in C_{\leq 2}^\infty\}.$$
Let $\tilde{W}_1(u) = W_1(u_1)$, where $W_1$ is defined in Lemma 4.2. It satisfies
\[
< \partial J(u), \tilde{W}_1(u) > \leq c\left(\sum_{i=1}^{q} \left( \frac{1}{\lambda_i^2} + \frac{\|\nabla K(a_i)\|}{\lambda_i} \right) + \sum_{1 \leq i \neq j \leq q} \epsilon_{ij} \right) + O\left( \sum_{q+1 \leq j \leq p, 1 \leq i \leq q} \frac{1}{\lambda_i \lambda_j} \right).
\tag{13}
\]

The second expansion of Proposition 3.1 and the estimate (8) yield
\[
< \partial J(u), Z(u) > = -J(u) w_3 \sum_{j \in I} \left[ \sum_{i \neq j} \alpha_i \alpha_j H(a_j, \lambda_j)(a_j) \lambda_i \lambda_j + \alpha_j^2 H(a_j, \lambda_j)(a_j) \right] + O\left( \sum_{j \in I} \frac{1}{\lambda_j^3} \right).
\]

Define $I = \{i, 1 \leq i \leq p, s.t., \lambda_i^\beta \geq \frac{1}{2} \lambda_{i_0}^\beta \}$. Estimates (10), (12) and (13) yield
\[
< \partial J(u), \tilde{W}_1(u) > \leq c\left(\sum_{i \in I} \left( \frac{1}{\lambda_i^2} + \frac{\|\nabla K(a_i)\|}{\lambda_i} \right) + \sum_{1 \leq i \leq q, i \neq j} \epsilon_{ij} \right) + o\left( \sum_{j \notin I} \frac{1}{\lambda_j^3} \right).
\tag{14}
\]

Our aim now is to make appear $(-\sum_{i \notin I, j \neq i} \epsilon_{ij})$ in the upper bound of the above inequality. For this we decrease all $\lambda_i$, $i \in I$. Setting
\[
Z(u) = -\sum_{i \in I} \alpha_i \lambda_i \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}.
\]

The second expansion of Proposition 3.1 and the estimate (8) yield
\[
< \partial J(u), Z(u) > = -J(u) w_3 \sum_{j \in I} \left[ \sum_{i \neq j} \alpha_i \lambda_j \frac{H(a_j, \lambda_j)(a_j)}{\lambda_i \lambda_j} + \alpha_j \lambda_j \frac{H(a_j, \lambda_j)(a_j)}{\lambda_j^2} \right] + O\left( \sum_{j \in I} \frac{1}{\lambda_j^3} \right).
\]

Observe that $H(a_i, \lambda_i)(a_j) = G(a_i, a_j) + o(1)$, $\forall i \neq j$. Therefore,
\[
< \partial J(u), Z(u) > \leq -c \sum_{j \in I, i \neq j} \frac{1}{\lambda_j^3 \lambda_i} + O\left( \sum_{j \notin I} \frac{1}{\lambda_j^3} \right) \leq -c \sum_{j \in I, i \neq j} \epsilon_{ij} + O\left( \sum_{j \notin I} \frac{1}{\lambda_j^3} \right). \tag{15}
\]

We derive from (14) and (15)
\[
< \partial J(u), \tilde{W}_1(u) + m_1 Z(u) > \leq -c\left(\sum_{i \in I} \left( \frac{1}{\lambda_i^2} + \frac{\|\nabla K(a_i)\|}{\lambda_i} \right) + \sum_{i \in I, i \neq j} \epsilon_{ij} \right) + o\left( \sum_{j \notin I} \frac{1}{\lambda_j^3} \right). \tag{16}
\]

In the above estimate, our upper bound is limited to those indices $i$ such that $i \in I$. To get the left indices, let $J = \{1, ..., p\} \setminus I$ and let $\pi_2 = \sum_{i \in I} \alpha_i \varphi_{a_i, \lambda_i}$. Observe that $\pi_2 \in V_2(\sharp I, \epsilon)$.

Define $\tilde{W}_2(u) = W_2(\pi_2)$ where $W_2(\pi_2)$ is the pseudo-gradient of Lemma 4.3. We have
\[
< \partial J(u), \tilde{W}_2(u) > \leq -c\left(\sum_{i \notin J} \left( \frac{1}{\lambda_i^2} + \frac{\|\nabla K(a_i)\|}{\lambda_i} \right) + \sum_{i \notin J, i \neq j} \epsilon_{ij} \right) + O\left( \sum_{j \notin J, i \notin J} \frac{1}{\lambda_j \lambda_i} \right). \tag{17}
\]

Let $m_2 > 0$ small enough, we derive from (16) and (17) that
\[
< \partial J(u), m_2 \tilde{W}_2(u) + \tilde{W}_1(u) + m_1 Z(u) > \leq -c\left(\sum_{i = 1}^{p} \left( \frac{1}{\lambda_i^2} + \frac{\|\nabla K(a_i)\|}{\lambda_i} \right) + \sum_{i \neq j} \epsilon_{ij} \right).
\]

Observe that along the vector field defined in this case $\sup_{1 \leq i \leq p} \lambda_i$ remains bounded, (a deconcentration phenomenon).
Case 3.

\[ u_2 \notin \{ u = \sum_{i=1}^{p-q} \alpha_i \varphi_{a_i, \lambda_i} \in V_2(p-q, \epsilon) \text{ with } (y_{i_1}, \ldots, y_{p-q}) \in C_2^\infty \}. \]

We argue as the second case. Let \( \widetilde{W}_2(u) = W_2(u_2) \), where \( W_2(u_2) \) is defined in Lemma 4.3. It satisfies

\[
< \partial J(u), \widetilde{W}_2(u) > \leq \epsilon \left( \sum_{i=q+1}^{p} \left( \frac{1}{\lambda_i^q} + \left| \nabla K(a_i) \right| \lambda_i \right) + \sum_{q+1 \leq i \neq j \leq p} \epsilon_{ij} \right) + O\left( \sum_{q+1 \leq i \leq p, 1 \leq j \leq q} \frac{1}{\lambda_j^q \lambda_i} \right).
\]

Let \( i_0 \) be an index in \( \{ q+1, \ldots, p \} \) such that \( \lambda_{i_0} = \min \{ \lambda_i, q+1 \leq i \leq p \} \) and \( L = \{ i, 1 \leq i \leq p, s.t., \lambda_i^q \geq \frac{1}{2} \lambda_{i_0}^2 \} \). Define

\[ Z(u) = - \sum_{i \in L} \alpha_i \lambda_i \frac{\partial \varphi(a_i, \lambda_i)}{\partial \lambda_i} \text{ and } \widetilde{W}_1(u) = W_1(u_1), \]

where \( u_1 = \sum_{i \notin L} \alpha_i \varphi_{a_i, \lambda_i} \) and \( W_1(u_1) \) is defined in Lemma 4.2. We proceed as in case 2, we have

\[
< \partial J(u), \widetilde{W}_2(u) + m_1 Z(u) + m_2 \widetilde{W}_1(u) > \leq -\epsilon \left( \sum_{i=1}^{p} \left( \frac{1}{\lambda_i^q} + \left| \nabla K(a_i) \right| \lambda_i \right) + \sum_{i \neq j} \epsilon_{ij} \right).
\]

Here \( m_1 \) and \( m_2 \) are two positive constants small enough. This complete the proof of Lemma 4.4.

\[ \square \]

**Lemma 4.5.** There exists a bounded pseudo-gradient \( W_4 \) in \( V_{>1}(p, \epsilon) \) satisfying the inequality (i) of Theorem 4.1 for any \( u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{>1}(p, \epsilon) \). Moreover the only case where \( \lambda_i(s), i = 1, \ldots, p \) tend to \( \infty \) is when \( a_i(s) \) tends to \( y_i, \forall i = 1, \ldots, p \), with \( (y_{i_1}, \ldots, y_{i_p}) \in C_{\infty}^2 \cup C_{\infty}^2 \cup (C_{\infty}^2 \times C_{\infty}^2) \).

**Proof of Lemma 4.5** Let \( u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i} \in V_{>1}(p, \epsilon) \) and let

\[ J = \{ i, 1 \leq i \leq p, s.t., y_i \in K_{>2} \}. \]

Following Lemma 4.3, we will only consider the case where \( J \neq \emptyset \). Let \( i_1 \) be an index such that

\[ \lambda_{i_1} = \min \{ \lambda_i, i \in J \}. \]

We decrease \( \lambda_{i_1} \) by setting \( \dot{\lambda}_{i_1} = -\lambda_{i_1} \). The associate vector field is

\[ W_{i_1}(u) = -\alpha_{i_1} \lambda_{i_1} \frac{\partial \varphi_{a_{i_1}, \lambda_{i_1}}}{\partial \lambda_{i_1}}. \]

Using the second expansion of the second expansion of proposition 5, we have

\[
< \partial J(u), W_{i_1}(u) > = -J(u)w_3 \left[ \sum_{i_1 \neq j} \alpha_{i_1} \alpha_j \frac{H(a_{i_1}, \lambda_{i_1})(a_j)}{\lambda_{i_1} \lambda_j} + \alpha_{i_1}^2 \frac{H(a_{i_1}, \lambda_{i_1})(a_{i_1})}{\lambda_{i_1}^2} \right] + O(\frac{1}{\lambda_{i_1}^2}) + o(\sum_{i_1 \neq j} \epsilon_{ij}),
\]

since \( \beta = \beta(y_{i_1}) > 2 \). Recall that by (2) and (3) we have

\[ H(a_{i_1}, \lambda_{i_1})(a_{i_1}) = A_{i_1} + o(1), \text{ as } \lambda \to +\infty \]
and

\[ H(a_{i1}, \lambda_{i1})(a_j) = G(a_{i1}, a_j) + o(1), \text{ as } \lambda \to +\infty, \]

where \( A_{a_{i1}} > 0 \), since \( M \) is not conformally diffeomorphic to \( S^4 \). This with the estimate (8) yield

\[ < \partial J(u), W_i(u) > \leq -c \sum_{j \neq i} \frac{H(a_j, \lambda_j)(a_i)}{\lambda_i \lambda_j} + o(\sum_{j \neq i} \frac{1}{\lambda_{i1}^{\min(2, \beta)}}) \]

where \( I = \{ i, 1 \leq i \leq p, s.t, \lambda_i^{\min(2, \beta)} \geq \frac{1}{2} \lambda_i^2 \} \). Let

\[ Z(u) = -\sum_{i \in I} \alpha_i \lambda_i \frac{\partial \phi_{a_i, \lambda_i}}{\partial \lambda_i}, \]

we have

\[ < \partial J(u), Z(u) > \leq -c \sum_{i \in I} \left( \frac{1}{\lambda_i^{\min(2, \beta)}} + \frac{\nabla K(a_i)}{\lambda_i} \right) \sum_{i \in I} \epsilon_{i}, \]

Thus for \( m_1 > 0 \) and small, we derive from the above two estimates that

\[ < \partial J(u), W_i(u) + m_1 Z(u) > \leq -c \left( \frac{1}{\lambda_i^{\min(2, \beta)}} + \frac{\nabla K(a_i)}{\lambda_i} \right) \sum_{i \in I, j \neq i} \epsilon_{i}, \]

since from (10) and (8), we have \( \frac{\nabla K(a_i)}{\lambda_i} = o\left( \frac{1}{\lambda_i^{\min(2, \beta)}} \right) \).

To get the left indices we set \( J = \{ 1, \ldots, p \} \setminus I \) and \( \bar{\pi} = \sum_{i \in J} \alpha_i \phi(a_i, \lambda_i) \). Define \( \bar{W}_3(u) = W_3(\bar{\pi}) \), where \( W_3(\bar{\pi}) \) is the vector field defined in Lemma 4.4. It satisfies

\[ < \partial J(u), \bar{W}_3(u) > \leq -c \sum_{i \in J} \left( \frac{1}{\lambda_i^2} + \frac{\nabla K(a_i)}{\lambda_i} \right) \sum_{i, j \in J \neq i} \epsilon_{i} + O\left( \sum_{i \in J, j \notin J} \frac{1}{\lambda_i \lambda_j} \right). \]

For \( m_2 > 0 \), small enough, we get

\[ < \partial J(u), W_i(u) + m_1 Z + m_2 \bar{W}_3(u) > \leq -c \sum_{i = 1}^p \left( \frac{1}{\lambda_i^{\min(2, \beta)}} + \frac{\nabla K(a_i)}{\lambda_i} \right) \sum_{j \neq i} \epsilon_{ij} \]

We point out here, that in our construction sup_{1 \leq i \leq p} \lambda_i(s) remains bounded along the flow lines as long as these flow lines are not caught by neighborhoods of \( C_{s}^2 \cup C_{s}^\infty \cup (C_{s}^{2} \times C_{s}^{\infty}) \). This conclude the proof of Lemma 4.5.

**Lemma 4.6.** There exists a bounded pseudo-gradient \( W_5 \) in \( V(p, e) \setminus V_{\geq 1}(p, e) \) satisfying the inequality (i) of Theorem 4.1, for any \( u = \sum_{i=1}^p \alpha_i \phi_{a_i, \lambda_i} \in V(p, e) \setminus V_{\geq 1}(p, e) \), such that \( \lambda_i(s) \) is bounded, for all \( i = 1, \ldots, p \).

**Proof of Lemma 4.6** We decompose \( V(p, e) \setminus V_{\geq 1}(p, e) \) into three regions.

\[ R_1^{'i} = \{ u = \sum_{i=1}^p \alpha_i \phi_{a_i, \lambda_i}, a_i \in B(y_i, \rho), y_i \in K, \forall i = 1, \ldots, p, y_i \neq y_j, \forall i \neq j \}

and \( \exists i_1 \in \{ 1, \ldots, p \}, s.t. \lambda_{i_1} |a_{i_1} - y_{i_1}| > \delta \}, \]

\[ R_2^{'i} = \{ u = \sum_{i=1}^p \alpha_i \phi_{a_i, \lambda_i}, a_i \in B(y_i, \rho), y_i \in K, \forall i = 1, \ldots, p, \]

and there \( \exists i \neq j \) with \( y_i = y_j \),
and
\[ R'_3 = \{ u = \sum_{i=1}^{p} a_i \varphi_{a_i, \lambda_i} \in V(p, \varepsilon), s.t. \exists i \in \{1, \ldots, p\}, \text{ with } a_i \not\in \cup_{y \in K} B(y, \rho) \}. \]

We will give the construction of the pseudo-gradient in \( R'_1 \). The construction in \( R'2 \) and \( R'3 \) proceeds as in ([10], pages 652 – 655).

Let \( u = \sum_{i=1}^{p} a_i \varphi_{a_i, \lambda_i} \in R'_1 \). For any \( i = 1, \ldots, p \), we set
\[ Z_i(u) = -a_i \lambda_i \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}. \]

Let \( I = \{ i, 1 \leq i \leq p, s.t \lambda_i|a_i - y_i| \geq \delta \} \). We claim the following:

(c) For any \( i_1 \in I \) there exists a vector field \( X_{i_1}(u) \) satisfying
\[ < \partial J(u), X_{i_1}(u) >= -c(\frac{1}{\min(2, \beta)} + \frac{|\nabla K(a_i)|}{\lambda_i}) + \sum_{i \not= i_1} \epsilon_{i_1,i} + o(\sum_{j \not= i} \epsilon_{ij}). \]

To prove claim (c), we distinguish for a given index \( i \in I \) four-cases.

**Case 1:** \( \beta(y_i) \leq 2 \) and \( \lambda_i|a_i - y_i| \leq \frac{1}{2} \).

In this case we consider
\[ Y_i(u) = \alpha_i \sum_{k=1}^{4} b_k \int_{\mathbb{R}^4} |z_k + \lambda_i(a_i - y_i)k|^2 \frac{1}{(1 + |z|^2)5} z_k dz \frac{1}{\lambda_i} \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}. \]

The second expansion of proposition 3.2 yields
\[ < \partial J(u), Y_i(u) > = -c(\int_{\mathbb{R}^4} \frac{|z_k + \lambda_i(a_i - y_i)k|}{(1 + |z|^2)5} z_k dz)^2 + o\left( \frac{1}{\lambda_i} \right) + o(\sum_{j \not= i} \epsilon_{ij}), \]

since \( \frac{1}{\lambda_i} \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i} = o(\epsilon_{ij}) \), for any \( i \not= j \) such that \( |a_i - a_j| \geq c \). Here \( k_a \) is an index satisfying \( |(a_i - y_i)k_a| = \max_{1 \leq k \leq 4} |(a_i - y_i)k| \). Using the fact that \( |a_i - y_i| \geq \delta \) and then \( |(a_i - y_i)k_a| \geq \frac{1}{4} \), we get
\[ \left( \int_{\mathbb{R}^4} \frac{|z_k + \lambda_i(a_i - y_i)k|}{(1 + |z|^2)5} z_k dz \right)^2 \geq c > 0. \]

Therefore,
\[ < \partial J(u), Y_i(u) > = -c(\int_{\mathbb{R}^4} \frac{|z_k + \lambda_i(a_i - y_i)k|}{(1 + |z|^2)5} z_k dz)^2 + o\left( \frac{1}{\lambda_i} \right) + o(\sum_{j \not= i} \epsilon_{ij}), \]

since
\[ \frac{|\nabla K(a_i)|}{\lambda_i} = O\left( \frac{|a_i - y_i|^{-1}}{\lambda_i} \right) = O\left( \frac{1}{\lambda_i^2} \right). \]

In order to appear \( -\sum_{j \not= i} \epsilon_{ij} \) in the above upper bound, we apply \( Z_i(u) \). Using the second expansion of proposition 3.1, we have
\[ < \partial J(u), Z_i(u) > \leq -c \sum_{i \not= j} \frac{1}{\lambda_i \lambda_j} + O\left( \frac{1}{\lambda_i^2} \right) + O\left( |a_i - y_i|^2 \right) \leq -c \sum_{i \not= j} \epsilon_{ij} + O\left( \frac{1}{\lambda_i^2} \right). \]

Let for \( m_1 > 0 \) and small, \( X_i(u) = Y_i(u) + m_1 Z_i(u) \). It satisfies
\[ < \partial J(u), X_i(u) > \leq -c(\frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \not= j} \epsilon_{ij}) + o(\sum_{k \not= r} \epsilon_{kr}). \]

**Case 2:** \( \beta(y_i) \leq 2 \) and \( \lambda_i|a_i - y_i| \geq \frac{1}{2} \).
Let in this case
\[ Y_i(u) = \alpha_i \sum_{k=1}^{4} b_k \text{sign}(a_i - y_i) k \frac{1}{\lambda_i} \frac{\partial^2 a_i}{\partial (a_i)_k}. \]

By the first expansion of proposition 6, we have
\[ \langle \partial J(u), Y_i(u) \rangle \leq -c \sum_{k=1}^{4} b_k^2 \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} + O \left( \sum_{s=2}^{|\min(3, \beta)|} \frac{|a_i - y_i|^{\beta - s}}{\lambda_i^s} \right) + O(\frac{1}{\lambda_i^3}) + o(\sum_{s \neq j} \epsilon_{ij}). \]

Observe that
\[ \frac{|a_i - y_i|^{\beta - s}}{\lambda_i^s} = o \left( \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} \right), \forall s \geq 2, \tag{18} \]
and
\[ \frac{1}{\lambda_i^{\beta - 1}} = o \left( \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} \right), \text{ as } \delta \text{ small.} \tag{19} \]

Indeed,
\[ \frac{|a_i - y_i|^{\beta - s}}{\lambda_i^s} \frac{\lambda_i}{|a_i - y_i|^{\beta - 1}} = \frac{1}{(\lambda_i |a_i - y_i|)^{s-1}} < \delta^{s-1}, \]
and
\[ \frac{1}{\lambda_i^{\beta - 1}} \frac{\lambda_i}{|a_i - y_i|^{\beta - 1}} = \frac{1}{(\lambda_i |a_i - y_i|)^{\beta - 1}} < \delta^{\beta - 1}. \]

Therefore,
\[ \langle \partial J(u), Y_i(u) \rangle \leq -c \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} + o(\sum_{i \neq j} \epsilon_{ij}), \tag{20} \]

since \( \sum_{k=1}^{4} |a_i - y_i|^{\beta - 1} \sim |a_i - y_i|^{\beta - 1} \). From another part, by the first expansion of proposition 5, we have
\[ \langle \partial J(u), Z_i(u) \rangle \leq -c \sum_{i \neq j} \frac{1}{\lambda_i \lambda_j} + O \left( \sum_{s=2}^{\min(3, \beta)} \frac{|a_i - y_i|^{\beta - s}}{\lambda_i^s} \right) + O(\frac{1}{\lambda_i^3}) + o(\sum_{i \neq j} \epsilon_{ij}). \]

Using (18) and (19), we obtain
\[ \langle \partial J(u), Z_i(u) \rangle \leq -c \sum_{j \neq i} \epsilon_{ij} + o \left( \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} \right) + o(\sum_{k \neq r} \epsilon_{kr}). \]

This with (20) yield
\[ \langle \partial J(u), Y_i(u) + Z_i(u) \rangle \leq -c \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} + \sum_{i \neq j} \epsilon_{ij} + o(\sum_{k \neq r} \epsilon_{kr}). \]

Using the fact that \( |\nabla K(a_i)| \sim |a_i - y_i|^{\beta - 1} \) and (19) we derive for \( X_i(u) = Y_i(u) + Z_i(u) \),
\[ \langle \partial J(u), X_i(u) \rangle \leq -c \frac{1}{\lambda_i^{\beta - 1}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \neq i} \epsilon_{ij} + o(\sum_{k \neq r} \epsilon_{kr}). \]

**Case 3:** \( \beta(y_i) > 2 \) and \( \lambda_i^{\beta} |a_i - y_i|^{\beta} < \delta \).
We set \( X_i(u) = Z_i(u) \). Using (2) and the fact that \( A_{\alpha} > 0, \forall \alpha \in M \), we obtain from the second expansion of proposition 5

\[
< \partial J(u), X_i(u) > \leq - c \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \epsilon_{ij} \right) + O(|a_i - y_i|^\beta) + o(\sum_{k \neq r} \epsilon_{kr})
\]

\[
\leq - c \left( \frac{1}{\lambda_i^2} + \sum_{j \neq i} \epsilon_{ij} \right) + o(\sum_{k \neq r} \epsilon_{kr}),
\]

since \( |a_i - y_i|^\beta = o(\frac{1}{\lambda_i^2}) \), as \( \delta \) small. Also we have

\[
\frac{|\nabla K(a_i)|}{\lambda_i} = O\left( \frac{|a_i - y_i|^\beta - 1}{\lambda_i} \right) = O\left( \frac{1}{\lambda_i^{1 + \frac{2(\beta - 1)}{\beta}}} \right) = o\left( \frac{1}{\lambda_i^2} \right).
\]

Therefore,

\[
< \partial J(u), X_i(u) > \leq - c \left( \frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \neq i} \epsilon_{ij} \right) + o(\sum_{k \neq r} \epsilon_{kr}).
\]

**Case 4**: If \( \beta(y_i) > 2 \) and \( \lambda_i^2 |a_i - y_i|^\beta > \delta \).

We define

\[
Y_i(u) = \alpha_i \sum_{k=1}^{4} b_k \text{sign}(a_i - y_i) \frac{1}{\lambda_i} \frac{\partial \phi_{a_k, \lambda_i}}{\partial (a_i)_k}.
\]

By the first expansion of proposition 6, we find

\[
< \partial J(u), Y_i(u) > \leq - c \sum_{k=1}^{4} b_k^2 \frac{|(a_i - y_i)_k|^\beta - 1}{\lambda_i} + O\left( \sum_{s=2}^{[\min(3, \beta)]} \frac{|a_i - y_i|^\beta - s}{\lambda_i^s} \right) + O\left( \frac{1}{\lambda_i^{\min(3, \beta)}} \right).
\]

Observe that,

\[
\left| \frac{|a_i - y_i|^\beta - s}{\lambda_i^s} \right| \frac{\lambda_i}{\left| a_i - y_i \right|^\beta - 1} = o\left( \frac{|(a_i - y_i)_k|^\beta - 1}{\lambda_i} \right), \text{ as } \lambda_i \to \infty,
\]

indeed,

\[
\left| \frac{|a_i - y_i|^\beta - s}{\lambda_i^s} \right| \frac{\lambda_i}{\left| a_i - y_i \right|^\beta - 1} = \frac{1}{(\lambda_i |a_i - y_i|)^{s-1}} \leq \left( \frac{1}{\delta} \right)^{\frac{s-1}{s}} \frac{1}{\lambda_i^{(s-1)(1-\frac{1}{s})}}.
\]

Also,

\[
\frac{1}{\lambda_i} = o\left( \frac{|a_i - y_i|^\beta - 1}{\lambda_i} \right)
\]

and

\[
\frac{1}{\lambda_i^\beta} = o\left( \frac{|a_i - y_i|^\beta - 1}{\lambda_i} \right).
\]

Therefore,

\[
< \partial J(u), Y_i(u) > \leq - c \left( \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij} \right).
\]

In order to make appear \( - \sum_{i \neq j} \epsilon_{ij} \), we apply \( Z_i(u) \). Using the first expansion of proposition 3.1 and the estimate (2), we find

\[
< \partial J(u), Z_i(u) > \leq - c \left( \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij} \right) + O\left( \sum_{s=2}^{[\min(4, \beta)]} \frac{|a_i - y_i|^\beta - s}{\lambda_i^s} \right) + O\left( \frac{1}{\lambda_i^{\min(4, \beta)}} \right),
\]
the estimate (21) yields
\[ < \partial J(u), Z_i(u) > \leq -c \left( \frac{1}{\lambda_i} + \sum_{i \neq j} \epsilon_{ij} \right) + o \left( \frac{|a_i - y_i|^{\beta - 1}}{\lambda_i} \right) + o \left( \sum_{k \neq r} \epsilon_{kr} \right). \]

Let in this case \( X_i(u) = Y_i(u) + Z_i(u) \). It satisfies
\[ < \partial J(u), X_i(u) > \leq -c \left( \frac{1}{\lambda_i} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j} \epsilon_{ij} \right) + o \left( \sum_{k \neq r} \epsilon_{kr} \right). \]

This complete the proof of claim (c).

We now denote \( \lambda_{i_0} = \min_{i \in I} \lambda_i \) and define
\[ L = I \cup \{ i, 1 \leq i \leq p, \lambda_i^{\min(2, \beta)} \geq \frac{1}{2} \lambda_{i_0}^{\min(2, \beta)} \}. \]

We have
\[ < \partial J(u), \sum_{i \in I} X_i(u) > \leq -c \left( \sum_{i \in L} \frac{1}{\lambda_i^{\min(2, \beta)}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j, i \in L} \epsilon_{ij} \right) + o \left( \sum_{k \neq r} \epsilon_{kr} \right). \]

By the preceding technic we obtain
\[ < \partial J(u), \sum_{i \in I} X_i(u) + \sum_{i \in L \setminus I} m_1 Z_i(u) > \]
\[ \leq -c \left( \sum_{i \in L} \frac{1}{\lambda_i^{\min(2, \beta)}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{i \neq j, i \in L} \epsilon_{ij} \right) + o \left( \sum_{k \neq r} \epsilon_{kr} \right). \]

Let \( \bar{\pi} = \sum_{i \in L} \alpha_i \varphi_{a_i, \lambda_i} \). Observe that \( \bar{\pi} \in V_{>1}(\xi, \lambda) \). Define \( \bar{W}(u) = W_4(\bar{\pi}) \), where \( W_4(\bar{\pi}) \) is the vector field Lemma 4.5.

Thus, we get for \( W_5 = \sum_{i \in I} X_i(u) + m_1 \sum_{i \in L \setminus I} Z_i(u) + m_2 \bar{W}(u), \)
\[ < \partial J(u), W_5(u) > \leq -c \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^{\min(2, \beta)}} + \frac{\nabla K(a_i)}{\lambda_i} + \sum_{j \neq i} \epsilon_{ij} \right). \]

This complete the proof of Lemma 4.6.

\[ \square \]

**Proof of Theorem 4.1.** The required pseudogradient \( W \) is defined by a convex combination of \( W_1, ..., W_5 \) of Lemmas 4.2, ..., 4.6. It satisfies the inequality (i). Using the fact that the upper bound of (i) is small with respect to \( \|v\| \) (given in proposition 4), the inequality (ii) follows as in \( (10 \text{, Lemma B4}) \). Observe that \( W \) is bounded and the only case where \( \lambda_i(s), i = 1, ..., p \) tend to \( \infty \) is when \( \alpha_i(s) \) tends to \( y_i, \forall i = 1, ..., p \) such that \( (y_1, ..., y_p) \in C_{>2}^{\infty} \cup C_{>2}^{\infty} \cup (C_{<2}^{\infty} \times C_{>2}^{\infty}) \).

\[ \square \]

**Proof of Theorem 1.1.** According to the result of Theorem 4.1, the proof of Theorem 1.1 proceeds exactly as in the proof of Corollary 3.1 of [20].

\[ \square \]

**Proof of Theorem 1.2.** We argue by contradiction. Assume that \( J \) has no critical points in \( \Sigma^+ \). Following Theorem 1.1, the critical points at infinity of \( J \) are in one to one correspondence with the elements \( \tau_p \in C_{>2}^{\infty} \cup C_{>2}^{\infty} \cup (C_{<2}^{\infty} \times C_{>2}^{\infty}) \).

Using Bahri-Rabinowitz deformation Lemma [9], the half sphere \( \Sigma^+ \) retracts by deformation on
\[ \bigcup_{\tau_p \in C_{>2}^{\infty} \cup C_{>2}^{\infty} \cup (C_{<2}^{\infty} \times C_{>2}^{\infty})} W_5^{\infty}(\tau_p)_{\infty}, \]
where \( W_u^\infty (\tau_p) \) is the unstable manifold of the critical point at infinity
\[
(\tau_p)_\infty = \sum_{i=1}^p \frac{1}{K(y_i,\infty)} \varphi(y_i,\infty).
\]
We compute now the Euter-poincaré characteristic of each space. We obtain after recalling that \( \Sigma^+ \) is a contractible space,
\[
1 = \sum_{\tau_p \in C^\infty_{\Sigma^+} \cup C^\infty_{\Sigma^-}} (-1)^i(\tau_p)_\infty,
\]
where \( i(\tau_p)_\infty \) is the index of \( (\tau_p)_\infty \). This is a contradiction with the hypothesis of theorem 1.2.

**Proof of Theorem 1.3** Arguing by contradiction. Assume that (1.1) has no solution. Using the result of Theorem 1.1, the level of any critical point at infinity
\[
(\tau_p)_\infty = (y_1, ..., y_p)_\infty \text{ is } C_\infty(\tau_p) = s^4 \sum_{i=1}^p \frac{1}{K(y_i,\infty)}.\]
Let \( y_0 \) be an absolute maximum of \( K \) on \( M \). As a first step we will assume that \( y_0 \) is the unique absolute maximum of \( K \) on \( M \). Therefore,
\[
C_\infty(\tau_p) > C_\infty(y_0), \text{ for any critical point at infinity } (\tau_p)_\infty \neq (y_0)_\infty.
\]
Let \( \epsilon_0 > 0 \) small enough such that \( c_1 := c_\infty(y_0) + \epsilon_0 < \min_{\tau_p 
eq (y_0)} c_\infty(\tau_p) \). We deform \( J_{c_1} := \{ u \in \Sigma^+, J(u) \leq c_1 \} \). Let \( Y \) be the decreasing pseudogradient defined by a convex combination of \( -\partial J \) outside \( \cup_{p \geq 1} V(p, \frac{\epsilon}{2}) \) and \( W; \) the vector field defined in Theorem 4.1 in \( \cap_{p \geq 1} V(p, \epsilon) \). For \( u \in J_{c_1} \), we denote by \( s \mapsto \eta(s, u_0) \) the one parameter group generated by \( Y \) with initial condition \( (0, u_0) \). Since \( Y \) is bounded, the existence time is infinite. Proposition 1 and Theorem 4.1 show that \( \eta(s, u_0) \) must enter in \( \bar{V}(1, \epsilon) := \{ u = \alpha \varphi_{a_1, \lambda_1} + v, a_1 \in B(y_0, \rho), \lambda_1 >> 1 \} \). In \( \bar{V}(1, \epsilon) \), the vector field \( W \) keeps \( \lambda(s) \) bounded. This is due to the fact that \( \beta(y_0) > 2 \) and \( M \) is not conformally diffeomorphic to \( S^4 \). Thus,
\[
| < \partial J(\eta(s, u_0)), Y(\eta(s, u_0)) > | \geq c > 0, \forall s \geq 0.
\]
Therefore \( J(\eta(s)) \) goes to \( -\infty \) when \( s \) goes to \( +\infty \). This contradicts the fact that \( J \) is lower bounded on \( \Sigma \).

Now if \( y_0 \) is not the unique maximum of \( K \), let \( y_0' \neq y_0 \) such that \( K(y_0) = K(y_0') = \max_M K \). We approximate \( K \) by a family of function \( K_\epsilon, \epsilon > 0 \) and small such that
\[
\begin{cases}
K_\epsilon = K \text{ in } M \setminus B(y_0', \rho), \\
K_\epsilon(y_0') = K(y_0') - \epsilon.
\end{cases}
\]
Observe that for any \( \epsilon > 0 \) small enough, \( K_\epsilon \) have a unique absolute maximum \( y_0 \). Therefore \( J_\epsilon \) possesses a critical point \( w_\epsilon \) in \( \Sigma^+ \). Under our construction \( J_\epsilon \) and \( J \) have the same critical points at infinity in \( V(p, \epsilon), p \geq 1 \), defined in Theorem 1.1. Moreover, Theorem 4.1 shows that at any critical point at infinity, there exists a neighborhood independent of \( \epsilon \) which does not contain any (true) critical point of \( J \) and \( J_\epsilon \). Therefore \( (w_\epsilon)_{\epsilon > 0} \) converges to a (true) critical point of \( J \) as \( \epsilon \to 0 \). This finishes the proof of theorem 1.3.

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