On prime values of binary quadratic forms with a thin variable

Peter Cho-Ho Lam, Damaris Schindler and Stanley Yao Xiao

Abstract

In this paper, we generalize the result of Fouvry and Iwaniec dealing with prime values of the quadratic form $x^2 + y^2$ with one input restricted to a thin subset of the integers. We prove the same result with an arbitrary primitive positive definite binary quadratic form. In particular, for any positive definite binary quadratic form $F$ and binary linear form $G$, there exist infinitely many $\ell, m \in \mathbb{Z}$ such that both $F(\ell, m)$ and $G(\ell, m)$ are primes as long as there are no local obstructions.

1. Introduction

One of the most difficult problems in number theory concerns finding primes among interesting subsets of the natural numbers. A particular example of such a problem is finding primes among values of a given polynomial. Several famous conjectures belong to this line of investigation, including the Bateman–Horn conjecture.

In the case of polynomials of a single variable, it is unknown whether a given polynomial represents infinitely many primes, except for linear polynomials by the seminal work of Dirichlet. For polynomials in two variables, we have some non-linear examples, including quadratic norm forms, all suitable quadratic polynomials by work of Iwaniec [14], binary cubic forms by work of Heath-Brown [11] and Heath-Brown and Moroz [13], and the polynomial $x^2 + y^4$ due to Friedlander and Iwaniec [6]. One obvious approach is to deduce the analogous results for single variable polynomials from their two-variable counterparts by restricting one variable. Currently, we do not know how to do this, but in some cases we can restrict one of the variables to a sparse subset of the integers. This gives rise to an interesting family of problems.

One particular example that has been considered is the case of the quadratic form $x^2 + y^2$, where $y$ is restricted to a sparse subset of the integers, including the case of $y$ being prime. This was worked out in great detail by Fouvry and Iwaniec [4]. Lam [15, 16] and Pandey [18] studied similar problems for principal forms of certain negative discriminants. In all these cases, the number of admissible $y$ up to size $X$ cannot be less than $O_{\delta}(X^{1-\delta})$ for any positive $\delta$. Friedlander and Iwaniec [6] were able to break this barrier in the special case of restricting $y$ to the set of squares. Heath-Brown and Li [12] later refined their methods to restrict $y$ to the set of prime squares.

In this paper, we further generalize the work of Fouvry and Iwaniec [4] by considering arbitrary primitive positive definite binary quadratic forms. It is worth mentioning that Friedlander and Iwaniec [8] provided a simplified proof of [4] if $y$ is restricted to the set of primes. This was followed by Lam [16] and Pandey [18] in their own works but we decided to follow the original argument.
Let \( F(x, y) \in \mathbb{Z}[x, y] \) be a positive definite and primitive quadratic form (that is, the greatest common divisor of its coefficients is equal to 1). For \( d \in \mathbb{N} \), we set
\[
\rho(d) = \sharp\{\nu \pmod{d} : F(1, \nu) \equiv 0 \pmod{d}\}.
\]
We shall prove the following theorem.

**Theorem 1.1.** Let \( F(x, y) = \alpha x^2 + \beta xy + \gamma y^2 \in \mathbb{Z}[x, y] \) be a primitive positive definite quadratic form and \( X \) be a positive real number. Let \( \lambda(\ell) \) be a sequence of complex numbers supported on the natural numbers which satisfy the bound \(|\lambda(\ell)| \leq C \log^A \ell \) for all \( \ell \in \mathbb{N} \) and some fixed \( A, C > 0 \). Suppose \( q \in \mathbb{N} \) and \( q \leq (\log X)^N \) for some \( N > 0 \). Then for any \( B > 0 \) and \( a \in \mathbb{Z} \) with \( \gcd(a, q) = 1 \), we have
\[
\sum_{F(\ell, m) \leq X \atop F(\ell, m) \equiv a \pmod{q}} \lambda(\ell) \Lambda(F(\ell, m)) = H_{F,q} \sum_{F(\ell, m) \leq X \atop \gcd(\ell, \gamma m) = 1 \atop \ell \equiv \frac{a}{q} \pmod{q}} \lambda(\ell) + O_{A,B,C,F,N}(X(\log X)^{-B}),
\]
where \( \Lambda \) is the von Mangoldt function and
\[
H_{F,q} = \prod_{p \mid q P_F} \left(1 - \frac{\rho(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \mid q P_F} \left(1 - \frac{1}{p}\right)^{-1},
\]
and \( P_F = \prod_{p \leq C_F} p \) with \( C_F \) depending only on \( F \).

Note that \( H_{F,q} \) is positive if \( \rho(2) \neq 2 \). For example, the primitive positive definite binary quadratic form \( 2x^2 + xy + y^2 \) of discriminant \(-7\) cannot represent infinitely many prime values with \( x \) a prime. On the other hand, it exhibits infinitely many prime values with \( y \) prime. The flexibility provided by \( \lambda(\ell) \) and \( F(\ell, m) \equiv a \pmod{q} \) have applications in proving Vinogradov’s three primes theorem with special types of primes; see [9] for details.

The purpose of introducing \( P_F \) in our expression is to remove some small prime factors that prohibit the use of Dirichlet composition law (see Section 2). In practice, this dependence on \( P_F \) can be removed via Möbius inversion. For example, a particularly attractive consequence of Theorem 1.1 is the following:

**Corollary 1.2.** Let \( F \) be a positive definite binary quadratic form and \( G \) be a binary linear form. Assume that for every prime \( p \) there are \( x, y \in \mathbb{Z} \) such that \( p \nmid F(x, y)G(x, y) \). Then there exist infinitely many \( \ell, m \in \mathbb{Z} \) such that both \( F(\ell, m) \) and \( G(\ell, m) \) are primes.

It is also possible to impose the conditions \( F(\ell, m) \equiv a \pmod{q} \) and \( G(\ell, m) \equiv b \pmod{q} \):

**Corollary 1.3.** Let \( F(x, y) \in \mathbb{Z}[x, y] \) be a primitive positive definite quadratic form of discriminant \(-\Delta\). Suppose \( q \in \mathbb{N} \) and \( q \leq (\log X)^N \) for some \( N > 0 \). Then for any \( A > 0 \) and \( a, b \in \mathbb{Z} \) with \( \gcd(ab, q) = 1 \), we have
\[
\sum_{F(\ell, m) \leq X \atop F(\ell, m) \equiv \frac{a}{q} \pmod{q} \atop \ell \equiv \frac{b}{q} \pmod{q}} \Lambda(\ell) \Lambda(F(\ell, m)) = \frac{H_q \rho(\ell; q, b)}{q \phi(q)} \frac{\pi X}{\sqrt{|\Delta|}} + O_{A,N}(X(\log X)^{-A})
\]
where
\[
H_q = \prod_{p \mid q} \left(1 - \frac{\rho(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-1}
\]
and

\[ \rho(d; a, b) = \sharp \{ \nu \pmod{d} : F(b, \nu) \equiv a \pmod{d} \}. \]

One way to phrase Corollary 1.2 is that given the complete norm form \( F(x, y) \) and restricting the first variable \( x \) to primes, the form still represents infinitely many primes. A natural extension of this question is to ask given an arbitrary primitive complete norm form \( N \) in \( n \) variables and restricting a subset of the variables to a special set \( S \), does \( N \) still represent infinitely many primes? In this formulation, Heath-Brown \( [11] \) and Heath-Brown and Moroz \( [13] \) can be viewed as restricting one variable in a complete cubic norm form to be equal to 0. More recently, Maynard \( [17] \) showed that complete norm forms still represent infinitely many primes even with as many as a quarter of the variables are set to zero.

More generally, one expects a polynomial \( G \) with exactly \( r \) factors over \( \mathbb{Q} \) should take values which have exactly \( r \) prime factors if there are no local obstructions. Indeed this is included in Schinzel’s hypothesis. Our corollary 1.2 is a step towards confirming this conjecture for binary cubic forms, following the theorem of Heath-Brown and Moroz \( [13] \), by confirming this for the case when \( F \) has one rational linear factor and negative discriminant.

**Corollary 1.4.** Let \( H(x, y) \in \mathbb{Z}[x, y] \) be a binary cubic form with negative discriminant, that is reducible over \( \mathbb{Q} \). Assume that for every prime \( p \) there are \( x, y \in \mathbb{Z} \) such that \( p \nmid H(x, y) \). Then there exist infinitely many integers \( x, y \) such that \( H(x, y) \) has exactly two prime factors.

In particular, there are infinitely many integers with exactly two prime factors that are sums of two cubes, see also work of Pandey \( [18] \).

To deduce Theorem 1.1 from the work of Fouvry and Iwaniec \( [4] \), we must overcome two difficulties. The first is that the proof of a key lemma which is critical in Fouvry and Iwaniec \( [4] \) fails for a general binary quadratic form. In particular, they obtained an optimal spacing result of roots modulo \( d \) of the congruence \( \nu^2 + 1 \equiv 0 \pmod{d} \). Fortunately an analogous result was developed by Balog, Blomer, Dartyge and Tenenbaum \( [1] \). We will then mimic the argument from \( [7] \) to finish up the proof in Section 4 as the original argument in \( [4] \) is not sufficient for our case. The second issue is that in general the arithmetic over a ring of integers \( \mathcal{O}_K \) with \( K \) a quadratic number field is not quite analogous to the arithmetic over \( \mathbb{Z}[i] \) when \( \mathcal{O}_K \) has a non-trivial class group. To overcome this issue we require several applications of the Dirichlet composition law. We will develop the necessary tools in Section 2 and then employ it in Section 5.

**Notation.** We write \( \tau(n) \) for the divisor function of a natural number \( n \). \( \sum^b \) denotes a sum over positive squarefree integers. For a complex number \( u \), we write \( e(u) \) for \( e^{2\pi i u} \).

### 2. Dirichlet composition

Let \( F(x, y) = ax^2 + \beta xy + \gamma y^2 \) be a primitive positive definite quadratic form of discriminant \(-\Delta\). For a sequence of complex numbers \( \lambda(\ell), \ell \in \mathbb{N} \) and \( N \in \mathbb{N} \), we define a sequence

\[ a_N = \sum_{\substack{F(\ell, m) = N \\ \gcd(\ell, \gamma m) = 1}} \lambda(\ell). \quad (2.1) \]

As \( F(x, y) \) is assumed to be positive definite, this is a finite sum.
In [4], a key component of the bilinear sum estimates is the identity
\[ a_{mn} = \frac{1}{4} \sum_{\|w\|^2 = m} \sum_{\|z\|^2 = n} \lambda(\ell), \]  
(2.2)
where \( \ell = \Re(wz) \), when \( F(x, y) = x^2 + y^2 \) and \( \gcd(m, n) = 1 \). This is based on the classical identity
\[ (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (bc + ad)^2 \]
and the fact that there is one binary quadratic form of discriminant \(-4\) up to (proper) equivalence. We now extend this identity to the case when the class number is not equal to 1. To generalize (2.2), we will use the Dirichlet composition law.

**Definition 2.1 (Dirichlet composition).** Let \( f(x, y) = ax^2 + bxy + cy^2 \) and \( F(x, y) = \alpha x^2 + \beta xy + \gamma y^2 \) be primitive positive definite forms of discriminant \(-\Delta < 0\) which satisfy \( \gcd(a, \alpha, (b + \beta)/2) = 1 \). Then a Dirichlet composition of \( f(x, y) \) and \( F(x, y) \) is of form
\[ h(x, y) = a\alpha x^2 + Bxy + \frac{B^2 + \Delta}{4a\alpha} y^2, \]
where \( B \) is any integer such that
\[ B \equiv b \pmod{2a} \]
\[ B \equiv \beta \pmod{2\alpha} \]
\[ B^2 + \Delta \equiv 0 \pmod{4a\alpha}. \]

See [3] for a good reference on Dirichlet composition, and [2] for a modern perspective. Note that \((b + \beta)/2 \in \mathbb{Z}\) since \( b^2 \equiv \beta^2 \equiv -\Delta \pmod{4} \). This composition makes the equivalence class of binary quadratic of discriminant \( \Delta \) into an abelian group. The term composition is justified by the following identity:
\[ (au^2 + bwv + cv^2)(\alpha X^2 + \beta XY + \gamma Y^2) = a\alpha W^2 + BWZ + \frac{B^2 + \Delta}{4a\alpha} Z^2, \]  
(2.3)
where
\[ W = \left( u - \frac{B - b}{2a} v \right) X - \left( \frac{B - \beta}{2\alpha} u + \frac{(b + \beta)B + \Delta - b\beta}{4a\alpha} v \right) Y \]  
(2.4)
and
\[ Z = \alpha v X + \left( au + \frac{b + \beta}{2} v \right) Y. \]  
(2.5)
It is convenient to have explicit coefficients in the composition for our purposes.

To establish an analogue of (2.2), we need to study the solutions of
\[ mn = F(X, Y) \]
when \( \gcd(m, n) = 1 \). One can show that \( m \) can be represented by a binary quadratic form of the same discriminant, say \( f(x, y) \); and by composing with \( F(x, y) \) we obtain a form that represents \( n \). But to work out the composition explicitly, the condition \( \gcd(a, \alpha, (b + \beta)/2) = 1 \) is needed. This motivates us to construct a set of binary quadratic forms, \( \mathcal{S}_F(t) \), in which this condition is always satisfied.

For any \( F(x, y) = ax^2 + \beta xy + \gamma y^2 \) and any \( t \in \mathbb{Z} \), define \( \mathcal{S}_F(t) \) to be a set of binary quadratic forms of discriminant \(-\Delta\) such that
(1) every primitive binary quadratic form of discriminant $-\Delta$ is properly equivalent to exactly one element in $S_F(t)$;
(2) the principal form is contained in $S_F(t)$; and
(3) the set \{\(f(1,0): f \in S_F(t)\}\) consists of units and distinct primes that do not divide \(t\).

If $S_F(t)$ is the set of primitive reduced forms of discriminant $-\Delta$, then (1) and (2) are satisfied. Since each of them represent infinitely primes, if necessary, we can transform the form so that the coefficient of \(x^2\) is one of these primes and thus it is clear that (3) can be satisfied. Now we put $S_F = S_F(\alpha)$ and define

\[ Q_F = 2\alpha\gamma\Delta \prod_{f \in S_F} f(1,0). \]  

(2.6)

We assume $P_F$ is large enough so that $Q_F|P_F$. We also pick an integer $B$ with the following properties.

(1) $B \equiv b (mod 2a)$ for all $ax^2 + bxy + cy^2 \in S_F$.
(2) $B \equiv \beta (mod 2\alpha)$.
(3) $B^2 + \Delta \equiv 0 (mod 4\alpha)$ for all $ax^2 + bxy + cy^2 \in S_F$.

So $B$ only depends on $F$ and the choice of $S_F$.

**Proposition 2.2.** Let $\Delta$ be a positive integer and $F(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ be a primitive binary quadratic form of discriminant $-\Delta$. Let $m, n$ be positive integers such that $\gcd(mn, P_F) = 1$. If $mn = F(X, Y)$ for some integers $X, Y$ with $\gcd(X, Y) = 1$, then there exists a unique binary quadratic form $f(x, y) = ax^2 + bxy + cy^2 \in S_F$ and integers $u, v, w, z$ such that $\gcd(u, v) = \gcd(w, z) = 1$ and

\[
au^2 + buv + cv^2 = m, \\
aw^2 + Bwz + \frac{B^2 + \Delta}{4\alpha}z^2 = n, \\
\left(au + \frac{b + \beta}{2}v\right)w + \left(\frac{B - \beta}{2\alpha}u + \frac{(b + \beta)B + \Delta - h\beta}{4\alpha}v\right)z = X, \\
-\alpha w + \left(u - \frac{B - b}{2\alpha}v\right)z = Y;
\]

and if $\Delta > 4$, then there is exactly one more tuple, namely $(-u, -v, -w, -z)$, that satisfies the properties. If $\Delta = 3, 4$, we have 6 or 4 solutions, respectively.

**Proof.** Choose an integer $\nu$ such that $\nu \equiv (2\alpha X + \beta Y)Y^{-1} (mod m)$. Then $4m|\nu^2 + \Delta$ and we define the primitive binary quadratic form

\[ M(x, y) = mx^2 + \nu xy + \frac{\nu^2 + \Delta}{4m} y^2. \]

Note that $M$ is primitive, as a common divisor of $m$ and $\nu$ would also be a common divisor of $m$ and $\Delta$, which is a contradiction to $\gcd(m, P_F) = 1$. The form $M(x, y)$ is properly equivalent to some $f(x, y) = ax^2 + bxy + cy^2 \in S_F$. By construction, there exist integers $u, v, r, s$ such that $us - rv = 1$ and

\[ M(x, y) = f(ux + ry, vx + sy). \]  

(2.7)

Therefore, $m = M(1,0) = f(u,v) = au^2 + buv + cv^2$. By comparing the coefficients of $xy$ in (2.7), we deduce that

\[ \nu = 2aur + bus + brv + 2csv. \]
Consequently, we have \( \nu v \equiv -(2au + bv) \pmod{m} \) and therefore
\[
2\alpha X + \beta Y) v \equiv -(2au + bv) Y \pmod{m}.
\]
(2.8)

By \((\nu^2 + \Delta)vY \equiv 0 \pmod{m}\), we obtain
\[
(2au + bv)(2\alpha X + \beta Y) - \Delta vY \equiv 0 \pmod{m}.
\]
(2.9)

Now define \( W, Z \) as in (2.4) and (2.5). By (2.8) and (2.9), they are both divisible by \( m \). Take
\[
w = W/m, z = Z/m.
\]
Then
\[
m^2n = f(u, v)F(X, Y) = a\alpha W^2 + BWZ + \frac{B^2 + \Delta}{4a\alpha} Z^2,
\]
and
\[
n = a\alpha w^2 + Bwz + \frac{B^2 + \Delta}{4a\alpha} z^2.
\]
Solving (2.4) and (2.5) gives
\[
(au + \frac{b + \beta}{2}v)w + \left( B - \frac{\beta}{2\alpha} u + \frac{(b + \beta)B + \Delta - b\beta v}{4a\alpha} \right) z = X,
\]
\[
-\alpha vw + \left( u - \frac{B - b}{2a} v \right) z = Y.
\]
These equations also imply that \( \gcd(u,v) \mid \gcd(X,Y) \) and \( \gcd(z,w) \mid \gcd(X,Y) \), hence \( \gcd(u,v) = \gcd(z,w) = 1 \).

The choice of \( f(x, y) \) is unique since \( f(x, y) \) is properly equivalent to
\[
M(x, y) = mx^2 + \nu xy + \frac{\nu^2 + \Delta}{4m} y^2,
\]
and it can easily be checked that for any \( M'(x, y) \) constructed with a different \( \nu' \equiv \nu \pmod{m} \) and \((\nu')^2 + \Delta \equiv 0 \pmod{4m}\), \( M, M' \) are properly equivalent.

Now suppose \( \Delta > 4 \) and there is another tuple \((u_0, v_0, w_0, z_0)\) that satisfies the requirement.
It is straightforward to verify that
\[
16 = \left( \frac{(2au + bv)(2au_0 + bv_0) + \Delta vv_0}{am} \right)^2 + \Delta \left( \frac{2(uv_0 - uv_0v)}{m} \right)^2.
\]
(2.10)

As \((u, v)\) and \((u_0, v_0)\) both satisfy (2.8), we see that \( uv_0 - uv_0v \equiv 0 \pmod{m} \). It thus follows that
\[
4\Delta > 16 \geq \Delta \left( \frac{2(uv_0 - uv_0v)}{m} \right)^2,
\]
whence \( uv_0 - uv_0v = 0 \). It then follows that \( (u_0, v_0, w_0, z_0) = \pm(u, v, w, z) \).

If \( \Delta = 4 \), we can take \( S_F = \{x^2 + y^2\} \) and \( B = \beta \) (note that \( \beta \) must be even). Then (2.10) becomes
\[
1 = \left( \frac{uu_0 + vv_0}{m} \right)^2 + \left( \frac{uv_0 - uv_0v}{m} \right)^2
\]
and this has four pairs of solutions \((u_0, v_0)\). The case for \( \Delta = 3 \) is similar. \( \square \)

On the other hand, if we have \( f(u, v) = m \) and \( f^*(w, z) = n \), by Dirichlet composition they can produce \( X, Y \in \mathbb{Z} \) such that \( F(X, Y) = mn \) via
\[
(au + \frac{b + \beta}{2}v)w + \left( B - \frac{\beta}{2\alpha} u + \frac{(b + \beta)B + \Delta - b\beta v}{4a\alpha} \right) z = X,
\]
\[
-\alpha vw + \left( u - \frac{B - b}{2a} v \right) z = Y.
\]
However, even if \( \gcd(u, v) = \gcd(w, z) = 1 \), it does not guarantee \( \gcd(X, Y) = 1 \). We are not too far away because by (2.4) and (2.5), we have
\[
\gcd(X, Y) \mid \gcd(mw, mz) = m.
\]
Similarly \( \gcd(X, Y) \mid n \). Hence, if \( \gcd(m, n) = 1 \), we have \( \gcd(X, Y) = 1 \). Furthermore, if \( \gcd(mn, P_F) = 1 \), we also deduce that \( \gcd(X, \gamma) = 1 \) since \( \gamma \mid P_F \).

From Proposition 2.2 and the discussion above, we conclude that

**Proposition 2.3.** If \( \gcd(m, n) = \gcd(mn, P_F) = 1 \), we have
\[
a_{mn} = \frac{1}{2} \sum_{f \in S_F} \sum_{(w, z) \in \mathbb{Z}^2} \sum_{(u, v) \in \mathbb{Z}^2} \lambda(Q_F(u, v; w, z)),
\]
where
\[
f^*(w, z) = a\alpha w^2 + Bwz + \frac{B^2 + \Delta}{4a\alpha} z^2
\]
and
\[
Q_F(u, v; w, z) = \left( au + \frac{b + \beta}{2} v \right) w + \left( B - \beta \frac{u}{2\alpha} + \frac{(b + \beta)B + \Delta - b\beta}{4a\alpha} v \right) z.
\]
Here we set \( \lambda(\ell) = 0 \) if \( \ell < 0 \). If \( \Delta = 3 \) or 4, the constant \( \frac{1}{2} \) before the summation should be \( \frac{1}{6} \) and \( \frac{1}{4} \), respectively.

3. **Setting up a sieve problem**

After the algebraic preparations we now present the general framework of sieving with which we aim to find prime values in the sequence \( F(\ell, m) \) with \( \ell \) restricted to a thin sequence. In order to prove Theorem 1.1, it suffices to consider the sum
\[
P(X; \chi) = \sum_{N \leq X} a_N \chi(N) \Lambda(N),
\]
where \( \Lambda(N) \) is the von Mangoldt function, \( a_N \) is defined as in (2.1) and \( \chi \) is a Dirichlet character modulo \( q \). The character \( \chi \) is present to detect the congruence condition modulo \( q \).

Let \( Y, Z > 1 \) be such that \( X > YZ \). Put
\[
B(X; Y, Z; \chi) := \sum_{bd < X \atop b > Y \atop \gcd(bd, P_F) = 1} \mu(b) \left( \sum_{\substack{c \mid d \atop c > Z}} \Lambda(c) \right) a_{bd} \chi(bd),
\]
and
\[
\delta(N; Y, Z) = \sum_{b > Y} \frac{\mu(b)}{b} \left\{ \rho(b) \log \frac{N}{b} - \sum_{c \leq Z} \frac{\Lambda(c)}{c} \rho(bc) \right\}.
\]
Then we have the analogue of [4, Proposition 9].
Proposition 3.1. Let $Y, Z \geq 1$ and $X > YZ$. Then we have the identity

$$P(X; \chi) = \sum_{N \leq X} \sum_{\substack{\ell \mid N \mod d \\gcd(N, P_F) = 1 \\gcd(\ell, P_F) = 1}} \lambda(\ell; N)(H_{F,q} + \delta(N; Y, Z)) + B(X; Y, Z; \chi) + R(X; Y, Z; \chi) + P(Z; \chi)$$

where

$$H_{F,q} = \prod_{p \nmid qP_F} \left(1 - \frac{\rho(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-1}$$

and $R(X; Y, Z; \chi)$ is given by (3.6).

From Proposition 3.1, we see that Theorem 1.1 follows provided that acceptable estimates for $\delta(N; Y, Z), B(X; Y, Z; \chi), R(X; Y, Z; \chi), P(Z; \chi)$ can be obtained. We will give appropriate bounds for all but $B(X; Y, Z; \chi)$ in this section.

When sieving for prime values of $N$, we will need to study sums of the type

$$A_d(X; \chi) := \sum_{N \leq X} a_N \chi(N) \quad (3.3)$$

for $d$ a positive integer. Note that $A_d(X; \chi) = 0$ if $\gcd(d, qP_F) > 1$. If $N = F(\ell, m) \equiv 0 \pmod{d}$, then from $\gcd(\ell, \gamma m) = 1$ we immediately have $\gcd(\ell, d) = 1$ as well. Hence, we expect that $A_d(X; \chi)$ is approximated by

$$M_d(X; \chi) = \frac{\rho(d)}{d} \sum_{\substack{N \leq X \\gcd(N, P_F) = 1 \\gcd(\ell, N) = 1}} \lambda(\ell; N)$$

when $\gcd(d, qP_F) = 1$, where

$$\lambda(\ell; N) = \chi(N) \sum_{m \in \mathbb{Z}} \lambda(\ell; m)$$

and $M_d(X; \chi) = 0$ otherwise. With this we set

$$R_d(X; \chi) = A_d(X; \chi) - M_d(X; \chi). \quad (3.4)$$

For a parameter $D$, we define the complete remainder term $R(X; D; \chi)$ as

$$R(X; D; \chi) = \sum_{d \leq D} |R_d(X; \chi)|. \quad (3.5)$$

As $F(x, y)$ is assumed to be positive definite, there exists a positive constant $C_1$ depending only on $F$, such that $F(\ell, m) \leq X$ implies $|\ell|, |m| \leq C_1 \sqrt{X}$. Now put

$$R(X; Y, Z; \chi) = \sum_{b \leq Y} \mu(b) \left\{ R_b(X; \chi) \log \frac{X}{b} - \int_1^X R_b(t; \chi) \frac{dt}{t} - \sum_{c \leq Z} \Lambda(c) R_{bc}(X; \chi) \right\}. \quad (3.6)$$
As in [4], we have the bound
\[
|R(X; Y, Z; \chi)| \leq R(X, YZ; \chi) \log X + \int_1^X R(t, Y; \chi) \frac{dt}{t}.
\]  
(3.7)

Proof of Proposition 3.1. Our goal is to derive Proposition 3.1 from [4, Proposition 9]. To do so we must check that condition (7.16) in [4] holds with the function \( \rho(bc) \), with \( c \) a fixed integer. Let \(-D(l)\) denote the discriminant of the quadratic polynomial \( F(x, l) \), and let \(-d = -d(l)\) be the unique fundamental discriminant such that \( \mathbb{Q}(\sqrt{-D(l)}) = \mathbb{Q}(\sqrt{-d(l)}) \). Let \( \rho'_{-d}(n) \) be the multiplicative function
\[
\rho'_{-d}(n) = \sum_{m|n} \left( -\frac{d}{m} \right).
\]
Consider the Dirichlet series
\[
D(s) = \sum_{n=1}^\infty \mu(n) \rho(cn) \frac{n^s}{n^s}.
\]
Then \( D(s) \) differs from the series
\[
E(s) = \sum_{n=1}^\infty \mu(n) \rho'_{-d}(n) \frac{n^s}{n^s} = \prod_p \left( 1 - \frac{\rho'_{-d}(p)}{p^s} \right)
\]
by a holomorphic factor. Here
\[
\rho'_{-d}(p) = \begin{cases} 
1 & \text{if } p|d \\
2 & \text{if } \left( -\frac{d}{p} \right) = 1 \\
0 & \text{if } \left( -\frac{d}{p} \right) = -1.
\end{cases}
\]
Let \( \chi_{-d} \) be the Kronecker symbol. It is then apparent that
\[
E(s) = \frac{\mathcal{G}(s)}{\zeta(s)L(s, \chi_{-d})},
\]
where
\[
\mathcal{G}(s) = \prod_{p: \chi_{-d}(p) = 1} \left( 1 + \frac{1}{p^{2s} - 2p^s} \right)^{-1} \prod_{p: \chi_{-d}(p) = 1} \left( 1 - \frac{1}{p^{2s}} \right)^{-1}.
\]
Plainly, \( \mathcal{G}(s) \) converges and is holomorphic for \( \Re(s) > 1/2 \). We then obtain the bound
\[
\sum_{n \leq X} \mu(n) \rho'_{-d}(n) = O \left( X \exp \left( -c_d \sqrt{\log X} \right) \right)
\]
for some positive number \( c_d \) by standard estimates of the zero-free region of the Dirichlet \( L \)-function \( L(s, \chi_{-d}) \) and the Selberge–Delange method. The desired conclusion then follows from partial summation. \( \square \)

The terms \( B(X; Y, Z, \chi), R(X; Y, Z, \chi) \) in Proposition 3.1 will be controlled by the following lemmas:
Lemma 3.2. Suppose \( q \in \mathbb{N} \) and \( q \leq (\log X)^N \) for some \( N > 0 \). Let \( \varepsilon > 0 \). Assume \( Y, Z > 1 \) and \( YZ < X^{1-\varepsilon} \). Then we have

\[ R(X; Y, Z; \chi) \ll_{\varepsilon} X^{1-\varepsilon/5}. \]

Lemma 3.3. Suppose \( q \in \mathbb{N} \) and \( q \leq (\log X)^N \) for some \( N > 0 \). Let \( \theta_1, \theta_2 \) be two real numbers such that \( 1/2 < \theta_1 < 1 \) and \( 0 < \theta_2 < 1 - \theta_1 \). Then for \( Y = X^{\theta_1} \) and \( Z = X^{\theta_2} \) and any \( B > 0 \),

\[ B(X; Y, Z; \chi) \ll (\log X)^{-B}. \]

We follow a similar strategy to [4] in showing that the term \( R(X; Y, Z; \chi) \) can be bounded by our Type I estimate from Lemma 4.1 and the trivial estimate \( R(t; Y, \chi) \ll t^{1+\varepsilon}(\log Y)^2 \).

With these ingredients we can prove our main Theorem and Corollaries.

Proof of Theorem 1.1. Define \( Y, Z \) as in Lemma 3.3. Together with Lemma 3.2 and Lemma 3.3, we have shown that

\[ \sum_{F(\ell,m) \leq X \atop \gcd(\ell, \gamma m) = 1} \lambda(\ell) \chi(F(\ell, m)) \Lambda(F(\ell, m)) \]

\[ = H_{F,q} \sum_{F(\ell,m) \leq X \atop \gcd(\ell, \gamma m) = 1} \lambda(\ell) \chi(F(\ell, m)) + O_{A,B,C,F,N}(X(\log X)^{-B}). \] (3.8)

The condition \( \gcd(F(\ell, m), P_F) = 1 \) on the left can be removed because of the presence of \( \Lambda(F(\ell, m)) \). Hence, by orthogonality of \( \chi \), it gives

\[ \sum_{F(\ell,m) \leq X \atop F(\ell,m) \equiv a \pmod{q}} \lambda(\ell) \Lambda(F(\ell, m)) = H_{F,q} \sum_{F(\ell,m) \leq X \atop \gcd(\ell, \gamma m) = 1} \lambda(\ell) + O_{A,B,C,F,N}(X(\log X)^{-B}). \]

Finally, we treat the remaining terms in Proposition 3.1. We can use the trivial bound \( P(Z; \chi) \ll_{\varepsilon} Z^{1+\varepsilon} \) for any \( \varepsilon > 0 \). The contribution of the terms with \( \delta(N; Y, Z) \) is negligible as in [4].

4. Level of absolute distribution

In this section, we shall obtain Type I estimates that are needed to prove Lemma 3.2 and the corollaries to Theorem 1.1. The most pressing issue is to control the quantity \( R(X; D, \chi) \) given by (3.5). To this end, we have the following lemma:

Lemma 4.1. For \( 1 \leq D \leq X \), we have the bound

\[ R(X; D, \chi) \ll_{\varepsilon} q^3 D^{1/4} X^{3/4 + \varepsilon}. \]

To prove Lemma 4.1, it is convenient to remove the restrictive condition \( \gcd(N, P_F) = 1 \). We let the scripted letters \( \mathcal{A}, \mathcal{M}, \mathcal{R} \) to denote the analogous quantities \( A, M, R \) which appeared in the previous section, but without the condition \( \gcd(N, P_F) = 1 \). We also set \( \mathcal{M}_d(X; \chi) = 0 \) if \( \gcd(d, q) > 1 \). We then have the following analogue to Lemma 4.1:
Lemma 4.2. For $1 \leq D \leq X$, we have the bound
\[ \mathcal{R}(X, D; \chi) \ll_{\varepsilon} q^{3} D^{1/4} X^{3/4 + \varepsilon}. \]

Lemma 4.1 will be a simple consequence of Lemma 4.2. Furthermore Lemma 4.2 will be used to prove the corollaries from our main theorem. As in [4] we deduce Lemma 4.2 from a version where the $\mathcal{A}_d(X; \chi)$ are smoothed with an auxiliary weight function. Since we need to accommodate the extra assumption that $\gcd(\ell, \gamma m) = 1$, it is convenient to adopt the approach from [7] instead.

Let $\sqrt{X} \leq Y \leq X$ be an additional parameter to be chosen later, and let $w : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function with the following properties:
\[
\begin{align*}
    w(u) &= 0 & \text{if } u \not\in [1, X], \\
    0 &\leq w(u) \leq 1 & \text{if } u \in [1, X], \\
    w(u) &= 1 & \text{if } Y \leq u \leq X - Y, \\
    w^{(j)}(u) &\ll Y^{-j} & \text{for } j = 1, 2.
\end{align*}
\]

For $a, \ell \geq 1$, we define the function
\[ F_{a, \ell}(z) := \int_{\mathbb{R}} w(F(a \ell, at)) e(-zt) \, dt. \]  

Let
\[ \mathcal{A}_d(X; w, \chi) := \sum_{N \equiv 0 \pmod{d}} a_N w(N) \chi(N). \]

When $\gcd(d, q) = 1$, define
\[ \mathcal{M}_d(X; w, \chi) = \frac{\rho(d)}{d} \sum_{\gcd(\ell, \gamma d) = 1} \frac{\lambda(\ell) \phi(\ell)}{\ell} \left( \sum_{k \pmod{q} : \gcd(k, \ell) = 1} \chi(F(\ell, k)) \right) F_{1, \ell}(0) \]  

as well as the smoother remainder term
\[ \mathcal{R}_d(X; w, \chi) := \mathcal{A}_d(X; w, \chi) - \mathcal{M}_d(X; w, \chi). \]

When $\gcd(d, q) > 1$, they are both defined to be 0. We obtain the following lemma.

Lemma 4.3. Let $w$ and $\lambda$ be as above and $1 \leq D \leq X$. Then one has
\[ \sum_{d \leq D} |\mathcal{R}_d(X; w, \chi)| \ll_{\varepsilon} q^{3} D^{1/2} X^{3/2 + \varepsilon}. \]

Proof. 16. Note that
\[ \mathcal{A}_d(X; w, \chi) = \sum_{N \equiv 0 \pmod{d}} \chi(N) w(N) \sum_{F(\ell, m) \equiv N \pmod{d}} \sum_{\gcd(\ell, \gamma m) = 1} \lambda(\ell). \]

We will assume $\gcd(d, q) = 1$ throughout. The conditions $\gcd(\ell, \gamma m) = 1$ and $F(\ell, m) \equiv 0 \pmod{d}$ imply $\gcd(\ell, d) = 1$; hence $\mathcal{A}_d(X; w, \chi)$ can be rewritten as
\[ \mathcal{A}_d(X; w, \chi) = \sum_{\gcd(\ell, \gamma) = 1} \lambda(\ell) \sum_{\nu \pmod{d} : F(1, \nu) \equiv 0 \pmod{d}} \sum_{m \equiv \ell \pmod{d} \gcd(\ell, m) = 1} \chi(F(\ell, m)) w(F(\ell, m)). \]
By Möbius inversion, we can trade the condition $\gcd(\ell, m) = 1$ with
\[
\sum_{\gcd(a, \gamma) = 1} \mu(a) \sum_{\gcd(\ell, \gamma) = 1} \lambda(\ell) \sum_{\nu \pmod d} \sum_{\substack{am \equiv a\ell \nu \pmod d \ \ F(1, \nu) \equiv 0 \pmod d \ \ am \equiv a\ell \nu \pmod d}} \chi(F(\ell, am))w(F(\ell, am))
\]
and $a$ is bounded by $O(\sqrt{X})$. Now the innermost sum can be rewritten as
\[
\sum_{k \pmod q} \chi(F(\ell, ak)) \sum_{\substack{am \equiv a\ell \nu \pmod d \ \ m \equiv k \pmod q \ \ m \equiv \ell \nu \pmod c}} w(F(\ell, am)).
\]
To simplify our notation, let $c = d/\gcd(a, d)$. Then the condition $am \equiv a\ell \nu \pmod d$ is the same as $m \equiv \ell \nu \pmod c$. By Poisson summation formula and Chinese Remainder Theorem,
\[
\sum_{\substack{m \equiv \ell \nu \pmod c \ \ m \equiv k \pmod q}} w(F(\ell, am)) = \frac{1}{cq} \sum_{h \in \mathbb{Z}} e\left(\frac{hk\ell \nu}{q}\right) e\left(\frac{h\ell \nu q}{c}\right) F_{a, \ell}\left(\frac{h}{cq}\right),
\]
where $F_{a, \ell}(z)$ is defined in (4.2). Therefore,

\[
\mathcal{A}_d(X; w, \chi) = \frac{1}{q} \sum_{\gcd(a, \gamma) = 1} \mu(a) \sum_{\gcd(\ell, \gamma) = 1} \lambda(\ell) \sum_{\nu \pmod d} \sum_{k \pmod q} \chi(F(\ell, ak)) \sum_{h \in \mathbb{Z}} e\left(\frac{hk\ell \nu}{q}\right) e\left(\frac{h\ell \nu q}{c}\right) F_{a, \ell}\left(\frac{h}{cq}\right).
\]

Define $\mathcal{M}_d(X; w, \chi)$ to be the summand when $h = 0$, that is, the expression
\[
\frac{\rho(d)}{q} \sum_{\gcd(a, \gamma) = 1} \mu(a) ac \sum_{\gcd(\ell, \gamma) = 1} \lambda(\ell) \sum_{k \pmod q} \chi(F(\ell, ak)) \int_{-\infty}^{\infty} w(F(\ell, t)) dt
\]
and $\mathcal{R}_d(X; w, \chi) = \mathcal{A}_d(X; w, \chi) - \mathcal{M}_d(X; w, \chi)$. This is consistent with (4.3) since

\[
\sum_{a \mid \ell, \gcd(a, \ell) = 1} \frac{\mu(a) \gcd(a, d)}{a} = \prod_{p \nmid \ell, p \nmid q} \left(1 - \frac{\gcd(p, d)}{p}\right)
\]
equals 0 if $\gcd(\ell, d) > 1$ and thus
\[
\mathcal{M}_d(X; w, \chi) = \frac{\rho(d)}{dq} \sum_{\gcd(\ell, \gamma d) = 1} \frac{\lambda(\ell) \phi(\ell)}{\ell} \sum_{k \pmod q} \chi(F(\ell, k)) F_{1, \ell}(0).
\]

Note that for any integer $c$, we have
\[
F_{a, \ell}\left(\frac{h}{cq}\right) = \sqrt{\frac{X}{h}} \int_{-\infty}^{\infty} w\left(a^2 (\alpha \ell^2 + \beta \ell u \sqrt{X/h} + \gamma X u^2 / h^2)\right) e\left(-\frac{\sqrt{X} u}{cq}\right) du.
\]

We wish to sum $\mathcal{R}_d(X; w, \chi)$ dyadically and hence we define
\[
\mathcal{R}(X, D; w, \chi) = \sum_{D < d \leq 2D} |\mathcal{R}_d(X; w, \chi)|.
\]

Substituting $b = d/c = \gcd(a, d)$, each term in the above sum can be bounded by
\[
|\mathcal{R}_d(X; w, \chi)| \ll \frac{1}{dq} \sum_{a \mid b} \sum_{b \mid d} \rho(b) b \sum_{\nu \pmod c} \sum_{k \pmod q} |W_a(c, \nu)|,
\]
where
\[ W_a(c, \nu) = \sum_{h \neq 0 (\nu | m)} \lambda(a_{\ell}) \chi(F(a_{\ell}, ak)) e \left( \frac{hk^2}{q} \right) e \left( \frac{h \ell \nu q}{c} \right) F_{a, \ell} \left( \frac{h}{cq} \right). \]

Hence
\[ R(X, D; w, \chi) \ll \frac{1}{Dq} \sum_{k \equiv 0 (mod q)} a^{\frac{\ell}{a}} \sum_{b | a} \rho(b) b V_a(D/b), \tag{4.8} \]

where
\[ V_a(C) = \sum_{C < c \leq 2C} \sum_{\nu (mod c)} | W_a(c, \nu) |. \]

By dyadic division,
\[ V_a(C) \leq \sum_H \left( V_a^+(C, H) + V_a^-(C, H) \right), \tag{4.9} \]

where \( H \) is a power of 2,
\[ V_a^+(C, H) = \sum_{C < c \leq 2C} \sum_{\nu (mod c)} \left| \sum_{H \leq h < 2H} \sum_{\gcd(\ell, \gamma) = 1} \chi(F(a_{\ell}, ak)) \lambda(a_{\ell}) e \left( \frac{hk^2}{q} \right) e \left( \frac{h \ell \nu q}{c} \right) F_{a, \ell} \left( \frac{h}{cq} \right) \right| \]

and \( V_a^-(C, H) \) is defined similarly for those \( h < 0 \). We only present the argument for \( V_a^+(C, H) \) below for simplicity. For a reduced residue class \( t (mod q) \), we define
\[ \alpha_{h, \ell, t}(u) = \chi(F(a_{\ell}, ak)) \lambda(a_{\ell}) \frac{H}{h} w \left( a^2 \left( \frac{\ell t u \sqrt{X}}{h} \right) + \frac{\gamma X u^2}{h^2} \right) e \left( \frac{hk^2}{q} \right). \]

Then
\[ V_a^+(C, H) \ll \sqrt{X} \int_{H}^{C \cdot H/a} \sum_{(mod q) C < c \leq 2C} \sum_{\nu (mod c)} \left| \sum_{H \leq h < 2H} \sum_{\gcd(\ell, \gamma) = 1} \alpha_{h, \ell, t}(u) e \left( \frac{h \ell \nu q}{c} \right) \right| du. \tag{4.10} \]

The symbol \( \sum^* \) means we are summing over reduced residue classes only. Next, we need to employ Proposition 3 from [1].

**Proposition 4.4.** Let \( F(x, y) = \alpha x^2 + \beta xy + \gamma y^2 \in \mathbb{Z}[x, y] \) be an arbitrary quadratic form whose discriminant is not a perfect square. For any sequence \( \alpha_n \) of complex numbers, positive real numbers \( D, N \), we have
\[ \sum_{D \leq d \leq 2D} \sum_{F(1, \nu) \equiv 0 (mod d)} \left| \sum_{n \leq N} \alpha_n e \left( \frac{\nu n d}{d} \right) \right|^2 \ll \mathcal{F} \left( D + N \right) \sum_{n} |\alpha_n|^2. \]
Note that
\[
\sum_h \sum_\ell \alpha_{h,\ell,t}(u)e\left(\frac{h\ell \nu}{c}\right) = \sum \sum \alpha_{h,\ell,t}(u)e\left(\frac{\nu m}{c}\right)e\left(\frac{\nu h_0 \ell_0}{c}\right),
\]
where \(n = (h\ell - h_0 \ell_0)/q\). Hence, for each fixed pair \(0 \leq h_0, \ell_0 < q\), we only need to estimate
\[
\sum_{C < c \leq 2C} \sum_{F(1,r) \equiv 0 \pmod{c}} \left| \sum_{n \leq N} \alpha_n e\left(\frac{\nu n}{c}\right) \right|,
\]
where
\[
\alpha_n = \sum \sum \alpha_{h,\ell,t}(u)
\]
and
\[
N = q + \frac{1}{q}(2H) \left(\frac{C \sqrt{X}}{a}\right) \ll \frac{H \sqrt{X}}{a}.
\]
Applying this inequality and Cauchy–Schwarz inequality on (4.10), we deduce that
\[
V_a^+(C, H) \ll \frac{\sqrt{X} H q}{H} \left(\frac{C + H \sqrt{X}}{a}\right)^{1/2} \left(C \log C\right)^{1/2}
\]
where
\[
E = \sum_n \left( \sum \sum |\lambda(a\ell)| \right)^2 \ll \frac{H \sqrt{X}}{a} \log^2 A \log^2 (HX).
\]
Hence, we obtain
\[
V_a^+(C, H) \ll \frac{X^{3/4} q C^{1/2} H^{1/2}}{a^{3/2}} \left(\frac{C + H \sqrt{X}}{a}\right)^{1/2} \log^4 (HX).
\]
(4.11)
To develop a similar bound for large values of \(H\), we apply integration by parts twice in (4.7) as in [7], followed with the large sieve type estimate. We arrive at
\[
V_a^+(C, H) \ll \frac{X^{7/4} q^3 C^{5/2} a^{1/2}}{H^{3/2} Y^2} \left(\frac{C + H \sqrt{X}}{a}\right)^{1/2} \log^4 (HX).
\]
(4.12)
When \(H \leq aC \sqrt{Y}^{-1}\), we use (4.11) to deduce that
\[
V_a^+(C, H) \ll \frac{q X^{5/4} C H^{1/2}}{Y^{1/2} a^{3/2}} \log^4 X
\]
and if \(H > aC \sqrt{Y}^{-1}\), we use (4.12) to deduce that
\[
V_a^+(C, H) \ll \frac{q^3 X^2 C^{5/2}}{Y^2 H} \log^4 (HX).
\]
The same estimates hold for $V_a^-(C, H)$ as well. Therefore, by (4.9)
\[ V_a(C) \leq \sum_H (V_a^+(C, H) + V_a^-(C, H)) \ll \frac{q^3 X^{3/2} C^{3/2}}{aY} \log^{A+2} X \tag{4.13} \]
and by (4.8)
\[ R(X, D; w, \chi) \ll \frac{q^3 X^{3/2} \sqrt{D} \log^{A+2} X}{Y} \sum_{a \leq \sqrt{X}} \frac{\tau(a)}{a} \ll \frac{q^3 X^{3/2} \sqrt{D} \log^{A+4} X}{Y}. \]
Finally
\[ \sum_{d \leq D} |R_d(X; w, \chi)| \ll \sum_{D} R(X, D; w, \chi) \ll \frac{q^3 D^{1/2} X^{3/2 + \epsilon}}{Y}. \]

**Proof of Lemma 4.2.** To complete the proof of Lemma 4.2, it suffices to show that the error we made when we replace $A_d(X; \chi)$ with $A_d(X; w, \chi)$ is negligible as well, that is, both $|A_d(X; \chi) - A_d(X; w, \chi)|$ and $|M_d(X; \chi) - M_d(X; w, \chi)|$ are small. Note that
\[ \sum_{d \leq D \text{ gcd}(d, q) = 1} |A_d(X; \chi) - A_d(X; w, \chi)| \ll (\log X)^A \sum_{X - Y < F(\ell, m) \leq X} \tau^2(F(\ell, m)) \ll Y \log^{A+3} X. \]
Here we have used the estimate $a_N \ll C \tau(N)(\log X)^A$. Similarly,
\[ |M_d(X; \chi) - M_d(X; w, \chi)| \ll \frac{\rho(d)}{d} Y \log^{A+1} X. \]
Summing over $d$ and choosing $Y = D^{1/4} X^{3/4}$, we have
\[ R(X, D; \chi) = \sum_{d \leq D} |R_d(X; \chi)| \ll q^3 D^{1/4} X^{3/4 + \epsilon}. \]

**Proof of Lemma 4.1.** Follows from Lemma 4.2 and Mobius inversion.

Finally, we give proofs for the corollaries.

**Proof of Corollaries 1.2 and 1.3.** If $\lambda$ is supported on primes, then starting from (3.8) again, the right-hand side becomes
\[ \sum_{F(\ell, m) \leq X \text{ gcd}(\ell, \gamma m) = 1} \lambda(\ell) \chi(F(\ell, m)) = \sum_{e | P_F} \mu(e) \sum_{F(\ell, m) \leq X \text{ gcd}(\ell, \gamma m) = 1} \lambda(\ell) \chi(F(\ell, m)) \]
\[ = \sum_{e | P_F} \mu(e) M_e(X; \chi) + O \left( \sum_{e | P_F} |R_e(X; \chi)| \right). \]
Therefore, it is also equal to
\[ \sum_{e | P_F} \mu(e) \frac{\rho(e)}{e} \sum_{F(\ell, m) \leq X \text{ gcd}(\ell, \gamma m e) = 1} \lambda(\ell) \chi(F(\ell, m)) + O(R(X, P_F; \chi)). \]
The contribution when $\gcd(\ell, e) > 1$ is negligible. Hence, by Lemma 4.2
\[
\sum_{F(\ell, m) \leq X} \lambda(\ell)\chi(F(\ell, m)) \Lambda(F(\ell, m)) = H_{F,q} \prod_{p \mid F} \left(1 - \frac{\rho(p)}{p}\right) \sum_{\substack{\lambda(\ell) \mid F(\ell, m) \leq X \\gcd(\ell, \gamma m) = 1}} \lambda(\ell)
\]
+ $O_{A,B,C,F,N}(X(\log X)^{-B})$.

The contribution when $\gcd(\ell, \gamma m) > 1$ is also negligible; therefore by orthogonality,
\[
\sum_{F(\ell, m) \leq X} \lambda(\ell)\Lambda(F(\ell, m)) = \frac{H_{\phi(q)}}{q} \sum_{\substack{\lambda(\ell) \mid F(\ell, m) \leq X \\gcd(\ell, a \mod q)}} \lambda(\ell)
\]

Corollary 1.3 follows by taking $\lambda(\ell) = \Lambda(\ell)$ when $\ell \equiv b \pmod{q}$. For Corollary 1.2, let $G(x, y) = mx + ny$ with $\gcd(m, n) = 1$. Then there exist integers $s, t$ such that $ms - nt = 1$. By a change of variables $u = -tx - sy, v = mx + ny$, we obtain
\[
F(x, y) = F(nu + sv, -mu - tv),
\]
which is a binary quadratic form in $u$ and $v$. The result follows from Corollary 1.3 on the pair of forms $F(nu + sv, -mu - tv)$ and $v$ with $q = 1$.

5. Bilinear sums

In this section, we shall estimate $B(X; Y, Z; \chi)$ given in (3.2) by proving Lemma 3.3. For reasons of exposition, we first work under the assumption that $|\lambda(\ell)| \leq 1$ for all $\ell \in \mathbb{N}$. As we save an arbitrary power of $\log X$ in our arguments, the general case can then be obtained by changing the parameter $A$. We proceed as in [4]. First put $\theta = (\log X)^{-A}$ and write
\[
\mathcal{B}(M, N) = \sum_{M < m \leq 2M} \sum_{\substack{\mu(n)\chi(n)a_{mn} \mid M < n < N' \\gcd(m, n) = 1}} \left| \sum_{\substack{\theta x < M < N < X \\gcd(N, p) = 1}} \sum_{\substack{\theta x < M < N < X \\gcd(N, p) = 1}} \mathcal{B}(M, N) + O(\theta X(\log X)^2),
\]

where the error term $O(\theta X(\log X)^2)$ represents a trivial bound for the contribution of $\mu(b)\chi(bd)a_{bd}$ with $bd \leq 2X$ or $e^{-2\theta}X < bd \leq X$, which terms are not covered exactly. As in [4], we need to show that each short sum $\mathcal{B}(M, N)$ satisfies
\[
\mathcal{B}(M, N) \leq \theta^2 X(\log X)^2.
\]

Let $\mathcal{B}_d(M, N)$ denote the sum (5.1) restricted to $\gcd(m, n) = d$. We have
\[
\mathcal{B}(M, N) \leq \sum_{d < \theta^{-1}} \mathcal{B}_d(M, N) + O(\theta^2 X),
\]

where the error term $O(\theta^2 X)$ represents a trivial bound for the contribution of $\mu(n)\chi(mn)a_{mn}$ with $\gcd(m, n) \geq \theta^{-1}$. Note that
\[
\mathcal{B}_d(M, N) \leq B_1(dM, N/d).
Therefore, the proof of Lemma 3.3 is reduced to showing the estimate
\[ B_1(M, N) \ll \theta^3 X (\log X)^2 \] (5.4)
holds for any \( M, N \) with \( M \geq Z, N \geq \theta Y \) and \( \theta X < MN < X \).

Define \( \alpha(n) = \mu(n)\chi(n) \). When applying Proposition 2.3 to decompose \( a_{mn} \) into solutions of
\[ f(u, v) = m, \quad f^*(w, z) = n, \]
in fact later in (5.6) we will decompose the solutions of \( f^*(w, z) = n \) again using the same proposition. We construct \( S_{f^*} \) in the same way we construct \( S_F \) by taking
\[ S_{f^*} = S_{f^*} \left( \prod_{f \in S_F} f(1, 0) \right) \]
and let \( g(x, y) = dx^2 + exy + fy^2 \in S_{f^*} \). We pick an integer \( B \) such that
(1) \( B \equiv b \pmod{2a} \) for all \( ax^2 + bxy + cy^2 \in S_F \);
(2) \( B \equiv e \pmod{2d} \) for all \( dx^2 + exy + fy^2 \in S_{f^*} \);
(3) \( B \equiv \beta \pmod{2a} \); and
(4) \( B^2 + \Delta \equiv 0 \pmod{4a} \) for all \( ax^2 + bxy + cy^2 \in S_F \) and \( dx^2 + exy + fy^2 \in S_{f^*} \).

Such \( B \) always exist since the coefficients of \( x^2 \) of elements in \( S_F \) or \( S_{f^*} \) are distinct primes or units. So \( B \) depends only on \( F \) and the choices of \( S_F \) and \( S_{f^*} \); and hence depends only on \( F \).

In the definition of \( P_F \), we take \( C_F \) large enough so that
\[ \prod_{f \in S_F} Q_f \prod_{p \leq C_F} p = P_F. \]

By Proposition 2.3, we can bound \( B_1(M, N) \) by
\[ B_1(M, N) \leq \sum_{f \in S_F} \sum_{M < f(u, v) \leq 2M} \left| \sum_{\substack{N < f^*(w, z) \leq N' \\gcd(f(u, v), P_F) = 1 \\gcd(f(u, v), f^*(w, z)) = 1}} \alpha(f^*(w, z)) \lambda(Q_F(u, v; w, z)) \right|. \] (5.5)

Proceeding with the argument to relax the condition that \( \gcd(f(u, v), f^*(w, z)) = 1 \), we use the familiar arithmetic identity
\[ \sum_{\gcd(m, n) \mid r} \mu(r) = \begin{cases} 1 & \text{if } \gcd(m, n) = 1 \\ 0 & \text{otherwise} \end{cases} \]
Since \( n \) is squarefree, by Proposition 2.2 we can decompose \( f^*(w, z) = n \) as
\[ g(u_0, v_0) = r, \quad g^*(w_0, z_0) = \frac{n}{r} \] (5.6)
for some \( g \in S_{f^*} \) and we have the relations
\[ \left( du_0 + \frac{e + B}{2} v_0 \right) w_0 + \left( \frac{(B + e)B + \Delta - Be}{4daa} v_0 \right) z_0 = w, \]
\[ -aaw_0 w_0 + \left( u_0 - \frac{B - e}{2d} v_0 \right) z_0 = z. \] (5.7)
We then see that the inner sum of (5.5) becomes
\[
\sum_{g \in S_r} \sum_{g(u_0, v_0) = r} \mu(r) \sum_{N < rg'(w_0, z_0) < N'} \alpha(rg'(w_0, z_0)) \lambda(Q_F(u, v; w, z)).
\]

Now it suffices to evaluate a sum of the shape
\[
\sum_r \sum_{g(u_0, v_0) = r} \sum_{M < f(u, v) \leq 2M} \sum_{N < rg'(w_0, z_0) < N'} \alpha(rg'(w_0, z_0)) \lambda(Q_F(u, v; w, z)) \leq \sum_r \sum_{M < f(u, v) \leq 2M} \sum_{N < rg'(w_0, z_0) < N'} \alpha(rg'(w_0, z_0)) \lambda(Q_F(u, v; w, z)).(5.8)
\]

where \( w \) and \( z \) are determined by (5.7). Now note that
\[
f(u, v)g(u_0, v_0) = adP^2 + BPQ + \frac{B^2 + \Delta}{4ad} Q^2 := H(P, Q),
\]
where
\[
P = \left( u - \frac{B - b}{2a} v \right) u_0 - \left( B - \frac{b + e}{2d} v \right) v_0,
Q = d u v_0 + \left( au + \frac{b + e}{2} v \right) v_0.
\]

When \((P, Q)\) and \((u_0, v_0)\) are fixed, there is at most one pair \((u, v)\) such that (5.9) holds. Also
with (5.7) and (5.9), we deduce that
\[
Q_F(u, v; w, z) = \left( adP + \frac{B + \beta}{2} Q \right) w_0 + \left( B - \frac{\beta}{2\alpha} P + \frac{\Delta + B^2}{4ad\alpha} Q \right) z_0.
\]
To simplify our notations, we define
\[
U = \frac{B - \beta}{2\alpha} P + \frac{\Delta + B^2}{4ad\alpha} Q, \quad V = -adP - \frac{B + \beta}{2} Q.
\]
Then it is not difficult to check that
\[
h(U, V) := adaU^2 + BUV + \frac{\Delta + B^2}{4ad\alpha} V^2 = \frac{\Delta + \frac{\beta^2}{4\alpha}}{H(P, Q)}.
\]

Therefore, the sum (5.8) is less than
\[
\sum_r \rho(r) \sum_{rM < h(P, Q) \leq 2rM} \sum_{N < rg(S_w, z_0) < N'} \alpha(rg'\lambda(S_w, z_0)) \lambda(z_0 P - w_0 Q).
\]

Estimating trivially we find that the terms with \( r \geq \theta^{-2} \), where we take \( \theta = (\log x)^{-A} \) for some large positive number \( A \) as in [4], contribute
\[
O\left( \theta MN \sum_{r > \theta^{-2}} \rho(r)^2 r^{-2} \right) = O(\theta^3 x (\log x)^2).
\]
In the remaining terms, we ignore the conditions \( r^2 | h(P, Q), \gcd(h(P, Q), P_F) = 1 \) and obtain

\[
B_1(M, N) \leq \sum_{r < \theta^{-2}} \rho(r) \sum_{r M < h(P, Q) \leq 2r M} \left| \sum_{N < r g^*(w_0, z_0) < N'} \alpha(r g^*(w_0, z_0)) \lambda(z_0 P - w_0 Q) \right| + O(\theta^3 x (\log x)^2).
\]

Put

\[
C_r(M, N) = \sum_{M < h(P, Q) \leq 2M} \left| \sum_{N < r g^*(w_0, z_0) < N'} \alpha(r g^*(w_0, z_0)) \lambda(z_0 P - w_0 Q) \right|,
\]

We then write

\[
C_{cr}(M, N) = \sum_{M < h(P, Q) \leq 2M} \left| \sum_{N < r g^*(w_0, z_0) < N'} \alpha(r g^*(w_0, z_0)) \lambda(z_0 P - w_0 Q) \right|,
\]

where the asterisk in the sum means that the sum is over primitive pairs. By [4], it then suffices to give a bound of the shape

\[
C_{cr}(M, N) \ll \theta^5 MN
\]

for every \( c, r, M, N \) with \( c < \theta^{-4}, r < \theta^{-2}, M \geq \theta^4 Z, N > \theta^3 Y \), and \( \theta^5 X < MN < X \). Our assumptions in Lemma 3.3 guarantee that \( M, N \) satisfy \( N^\varepsilon < M < N^{1 - \varepsilon} \) for some small \( \varepsilon > 0 \). This assumption will be used in (5.18) and (5.22) and we will give a bound of the form

\[
C_{cr}(M, N) \ll MN(\log N)^{-j}.
\]  

(5.10)

Let \( A = \sqrt{N}/r, B = \sqrt{M} \) and \( \alpha(u, v) = \alpha(r g^*(u, v)) \). Then \( \alpha(u, v) \) is supported in the annulus \( A^2 < g^*(u, v) \leq 4A^2 \). By applying the Cauchy–Schwarz inequality, we obtain

\[
|B(M, N)| \leq \sum_{\ell} \left| \lambda(\ell) \right| \sum_{M < h(w, z) < 2M} \left| \sum_{v w - u z = \ell} \alpha(u, v) \right| \leq A^{1/2} B^{3/2} D(\alpha)^{1/2},
\]

where

\[
D(\alpha) = \sum_{(w, z) \in \mathbb{Z}^2} \psi(w, z) \sum_{\ell} \left| \sum_{Q(u, v; w, z) = \ell} \alpha(u, v) \right|^2
\]

(5.11)

and

\[
Q(u, v; w, z) = v w - u z.
\]

Here \( \psi(w, z) \) can be any non-negative function with \( \psi(w, z) \geq 1 \) if \( B^2 \leq h(w, z) \leq 4B^2 \). We do not need to be specific at this point; nevertheless it will be convenient to assume that \( \psi(w, z) \) takes the form \( \Psi(h(w, z)) \), where

\[
0 \leq \Psi(t) \leq 1, \Psi(t) = 1 \text{ if } B^2 \leq t \leq 4B^2,
\]

\[
\text{supp } \Psi \subset [B^2/4, 9B^2], \Psi^{(j)} \ll B^{-2j}.
\]
Our desired estimate for $D(\alpha)$ is $A^3B$ with a saving of an arbitrary power of $\log N$. Since $\ell$ runs over all integers (without any restriction), after squaring we obtain

$$D(\alpha) = \sum^*_{(w,z) \in \mathbb{Z}^2} \sum_{Q(u,v,w,z)=0} (\alpha \ast \alpha)(u,v), \tag{5.12}$$

where

$$(\alpha \ast \alpha)(u,v) = \sum_{(s_1,t_1)-(s_2,t_2)=(u,v)} \alpha(s_1,t_1)\overline{\alpha}(s_2,t_2).$$

This equality follows because $Q(u,v,w,z)$ is a bilinear form. Note that $(\alpha \ast \alpha)(0,0) \ll A^2$. The orthogonality relation $Q(u,v,w,z)=0$ in (5.12) is equivalent to $(u,v)=(cw,cz)$ for some rational integer $c \in \mathbb{Z}$ since $\gcd(w,z) = 1$. It thus follows that

$$D(\alpha) = \sum_{c \in \mathbb{Z}} \sum^*_{(w,z) \in \mathbb{Z}^2} \psi(w,z)(\alpha \ast \alpha)(cw,cz)$$

$$= D_0(\alpha) + 2D^*(\alpha), \tag{5.13}$$

say, where $D_0(\alpha)$ denotes the contribution of $c=0$ and $D^*(\alpha)$ that of all $|c| > 0$. Thus,

$$D_0(\alpha) = \|\alpha\|^2 \sum^*_{(w,z) \in \mathbb{Z}^2} \psi(w,z) \ll A^2B^2 \tag{5.14}$$

and

$$D^*(\alpha) = \sum_{(s,t) \neq (0,0)} \psi\left(\frac{s}{\gcd(s,t)}, \frac{t}{\gcd(s,t)}\right)(\alpha \ast \alpha)(s,t). \tag{5.15}$$

We trade the primitivity condition for congruence conditions by means of Möbius inversion, getting

$$D^*(\alpha) = \sum_{b,c>0} \mu(b)D(\alpha;b,c), \tag{5.16}$$

where

$$D(\alpha;b,c) = \sum_{(s,t) \equiv (0,0) \pmod{bc}} \psi\left(\frac{s}{c}, \frac{t}{c}\right)(\alpha \ast \alpha)(s,t). \tag{5.17}$$

Note that $g^*(s,t) \leq 2A$ (from the support of $\alpha$) and $cB/2 < g^*(s,t) < 3cB$ (from the support of $\psi$). Observe that these imply that $c < 4AB^{-1}$, otherwise $D(\alpha;b,c)$ is zero. Let $\Xi$ be a parameter such that

$$1 \leq \Xi \leq 4AB^{-1} = C, \tag{5.18}$$

say. We will take $\Xi$ to be a power of $\log N$ at the end and this explains why $N$ needs to be larger than $M$, say $N^{1-\epsilon} > M$. By the trivial bound

$$D(\alpha;b,c) \ll A^2B^2b^{-2},$$

we see that the terms with $b \geq \Xi$ or $c \geq C\Xi^{-1}$ contribute at most $O(A^3B\Xi^{-1})$ to $D^*(\alpha)$ so

$$D^*(\alpha) = \sum_{b \leq \Xi} \mu(b) \sum_{C\Xi^{-1} < c < C} D(\alpha;b,c) + O(A^3B\Xi^{-1}). \tag{5.19}$$
If \( h(w, z) = Dw^2 + Ewz + Fz^2 \) with \( D > 0 \), then
\[
\psi(w, z) = \Psi\left( \frac{(2Dw + Ez)^2 + (4DF - E^2)z^2}{4D} \right)
\]
\[
= \Psi\left( \left( \frac{2Dw + Ez}{2\sqrt{D}} \right)^2 + \left( \frac{\sqrt{|\Delta|} z}{2\sqrt{D}} \right)^2 \right).
\]
Then we can define
\[
\psi_0(w, z) = \psi\left( \frac{\sqrt{|\Delta|} w - Ez}{\sqrt{D|\Delta|}}, \frac{2\sqrt{Dz}}{\sqrt{\Delta}} \right).
\]
Then
\[
\psi(w, z) = \psi_0\left( \frac{2Dw + Ez}{2\sqrt{D}}, \frac{\sqrt{|\Delta|} z}{\sqrt{D}} \right) \quad \text{and} \quad \psi_0(w, z) = \Psi(w^2 + z^2).
\]

We define
\[
\phi_0(x, y) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_0(w, z)e(-xw + yz) \, dw \, dz.
\]
Then \( \phi_0(x, y) \) is radial and we can set \( \Phi(x^2 + y^2) \). By inversion, we obtain
\[
\psi\left( \frac{w}{c}, \frac{z}{c} \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x^2 + y^2)e\left( \frac{2Dw + Ez}{2c\sqrt{D}} + y\frac{\sqrt{|\Delta|} z}{2c\sqrt{D}} \right) \, dx \, dy.
\]
Hence, we have
\[
\psi\left( \frac{w}{c}, \frac{z}{c} \right) = \frac{2c^2}{\sqrt{|\Delta|}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi\left( \frac{4c^2h(-y, x)}{|\Delta|} \right) e(xw + yz) \, dx \, dy.
\]
Therefore,
\[
D(\alpha; b, c) = \frac{2c^2}{\sqrt{|\Delta|}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi\left( \frac{4c^2h(-y, x)}{|\Delta|} \right) S_{bc}(x, y) \, dx \, dy
\]
where
\[
S_d(x, y) = \sum_{(s, t) \equiv (0, 0) \pmod{d}} (\alpha \ast \alpha)(s, t)e(xs + yt)
\]
\[
= \sum_{s_1 \equiv s_2 \pmod{d}} \sum_{t_1 \equiv t_2 \pmod{d}} \alpha(s_1, t_1) \overline{\alpha}(s_2, t_2)e(xs_1 + yt_1)e(-xs_2 - yt_2)
\]
\[
= \sum_{d_1, d_2 \pmod{d}} \left| \sum_{s \equiv d_1 \pmod{d}} \sum_{t \equiv d_2 \pmod{d}} \alpha(s, t)e(xs + yt) \right|^2.
\]
By [4, (9.14)],
\[
c^2 \Phi\left( \frac{4c^2h(-y, x)}{|\Delta|} \right) \ll \frac{c^2B^2}{(1 + c^2B^2h(-y, x))^{3/2} h(-y, x) A^2}.
\]
Hence
\[
D(\alpha; b, c) \ll \Xi A^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) S_{bc}(x, y) \, dx \, dy,
\]
where
\[ H(x, y) = \frac{1}{(1 + h(-y, x)A^2)^{3/2}}. \]

By grouping \( d = bc \) and setting \( D = C\Xi \), we obtain from (5.19)
\[ D^*(\alpha) \ll M\Xi^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) \left( \sum_{d \leq D} d^2 S_d(x, y) \right) \, dx \, dy + A^3 B \Xi^{-1}. \]  

(5.21)

To account for the large \( d \) appearing in the above sum, we need to invoke\[4, \text{Proposition 15}].

**Proposition 5.1.** Suppose \( A \geq D \geq 1 \). Let \( f \) be a complex-valued function on \( \mathbb{Z}[i] \) supported on the disc \( |z| \leq A \). Define
\[ S_f(D) = \sum_{d \leq D} d^2 \sum_{\delta \equiv \delta (\text{mod } d)} \left| \sum_{z \equiv \delta (\text{mod } d)} f(z) \right|^2. \]

Then for any \( G \geq 1 \) we have
\[ S_f(D) \leq 2DS_f(G) + O_\varepsilon(AD(D^{1+\varepsilon} + AG^{\varepsilon-1})||f||^2). \]  

(5.22)

For \( m + ni \in \mathbb{Z}[i] \), we take \( f(m + ni) = \alpha(m, n)e(xm + yn) \). Thus
\[ \sum_{d \leq D} d^2 S_d(x, y) \leq 2D \sum_{d \leq G} d^2 S_d(x, y) + O(A^5 B^{-1} \Xi G^{\varepsilon-1}), \]  

(5.23)

where
\[ D_d(\alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y)S_d(x, y) \, dx \, dy. \]

Similar to \( \Xi \), we expect \( G \) is a power of \( \log N \). To apply (5.22) we need \( DG < A^{1-\varepsilon} \), which is valid if \( B > A^\varepsilon \). By taking \( G = \Xi^6 \) and substituting (5.14), (5.21) and (5.23) into (5.13), we arrive at
\[ D(\alpha) \ll AB\Xi^4 \sum_{d \leq \Xi^6} d^2 D_d(\alpha) + A^2(B^2 + AB\Xi^{-1}), \]

where
\[ D_d(\alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y)S_d(x, y) \, dx \, dy \]
\[ = \sum_{(s, t) \equiv (0, 0) (\text{mod } d)} (\alpha \ast \alpha)(s, t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y)e(xs + yt) \, dx \, dy. \]

Our final obstacle is to develop an estimate of \( D_d(\alpha) \) for small values of \( d \). Here the modulus \( d \) is less than a power of \( \log N \), which is analogous to the classical Siegel–Walfisz theorem. As in (5.20), after some changes of variables the above integral can be expressed as
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y)e(xw + yz) \, dx \, dy = \frac{2\pi}{A^2} \exp \left( -\frac{4\pi \sqrt{h(w, z)}}{A \sqrt{\Delta}} \right). \]
Hence
\[ D_d(\alpha) = 2\pi A^{-2} \sum_{s_1 \equiv s_2 \pmod{d}} \sum_{t_1 \equiv t_2 \pmod{d}} \alpha(s_1, t_1) \bar{\alpha}(s_2, t_2) \exp \left( -\frac{4\pi \sqrt{h(s_1, t_1; s_2, t_2)}}{A \sqrt{|\Delta|}} \right), \]
where
\[ h(s_1, t_1; s_2, t_2) = h(s_1 - s_2, t_1 - t_2). \]
Note that
\[ D_d(\alpha) \ll \max_{d_1, d_2 \pmod{d}} \left| \sum_{N < g^*(s_0, t_0) < N'} \mu(g^*(s, t)) \chi(g^*(s, t)) \exp \left( -\frac{4\pi \sqrt{h(s, t; s_0, t_0)}}{A \sqrt{\Delta}} \right) \right|. \]
Define \( \eta = (\log N)^{-j}. \) It suffices to show that
\[ \sum_{(s, t) \equiv (d_1, d_2) \pmod{d}} \mu(r g^*(s, t)) \chi(r g^*(s, t)) \exp \left( -\frac{4\pi \sqrt{h(s, t; s_0, t_0)}}{A \sqrt{\Delta}} \right) \ll N\eta. \]
We can divide the region \( N < g^*(s, t) < N' \) into non-overlapping sectors of the form
\[ R(Z, \xi) = \{(s, t) \in \mathbb{Z}^2 : Z - \sqrt{N}\eta < g^*(s, t) < Z, \xi < \arg(s + ti) < \xi + \eta \} \]
and there are at most \( \eta^{-2} \) regions. For a fixed \((S, T) \in R(Z, \xi)\) and any \((s, t) \in R(Z, \xi)\), we always have
\[ \exp \left( -\frac{4\pi \sqrt{h(s, t; s_0, t_0)}}{A \sqrt{\Delta}} \right) = \exp \left( -\frac{4\pi \sqrt{h(S, T; s_0, t_0)}}{A \sqrt{\Delta}} \right) + O(\eta). \]
Hence, it suffices to show that
\[ \sum_{(s, t) \in R(Z, \xi) \pmod{d}} \mu(g^*(s, t)) \chi(g^*(s, t)) \ll N\eta^3. \]
This is a special case of [10, Lemma 3.3.6].

**Lemma 5.2.** Let \( Q(x, y) \) be a primitive positive definite quadratic form. Let \( H \leq (\log X)^N \). Then for any \( h_1, h_2 \pmod{H} \), any \( A > 0 \) and sector \( S \subset \mathbb{R}^2 \),
\[ \sum_{Q(x, y) \leq X, x \equiv h_1 \pmod{H}, y \equiv h_2 \pmod{H}, (x, y) \in S} \mu(Q(x, y)) \ll_{A, N} X(\log X)^{-A}. \]
Helfgott proved this for the Liouville function \( \lambda \) but the same proof also works for \( \mu \). This concludes our proof of Lemma 3.3.

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References

1. A. Balog, V. Blomer, C. Dartyge and G. Tenenbaum, ‘Friable values of binary forms’, Comment. Math. Helv. 87 (2012) 639–667.
2. M. Bhargava, ‘Higher composition laws I: a new view on Gauss composition, and quadratic generalizations’, Ann. of Math 159 (2004) 217–250.
3. D. Cox, Primes of the form $p = x^2 + ny^2$, Pure and Applied Mathematics (John Wiley & Sons, Hoboken, NJ, 2013).
4. E. Fouvry and H. Iwaniec, ‘Gaussian primes’, Acta Arith. LXXIX. 79 (1997) 249–287.
5. J. Friedlander and H. Iwaniec, ‘Asymptotic sieve for primes’, Ann. of Math 148 (1998) 1041–1065.
6. J. Friedlander and H. Iwaniec, ‘The polynomial $X^2 + Y^4$ captures its primes’, Ann. of Math 148 (1998) 945–1040.
7. J. Friedlander and H. Iwaniec, ‘Gaussian sequences in arithmetic progressionss’, Funct. Approx. Comment. Math. 37 (2007) 197–203.
8. J. Friedlander and H. Iwaniec, Opera de cribro, American Mathematical Society Colloquium Publications 57 (American Mathematical Society, Providence, RI, 2010).
9. L. Grimmelt, ‘Vinogradov’s theorem with Fouvry-Iwaniec primes’, Preprint, 2018, arXiv:1809.10008 [math.NT].
10. H. Helfgott, ‘Root numbers and the parity problem’, PhD Thesis, Princeton University, 2003.
11. D. R. Heath-Brown, ‘Primes represented by $x^3 + 2y^3$’, Acta Math. 186 (2001) 1–84.
12. D. R. Heath-Brown and X. Li, ‘Prime values of $a^2 + p^4$’, Invent. Math 208 (2017) 441–499.
13. D. R. Heath-Brown and B. Z. Moroz, ‘Primes represented by binary cubic forms’, Proc. Lond. Math. Soc. 84 (2002) 257–288.
14. H. Iwaniec, ‘Primes represented by quadratic polynomials in two variables’, Bull. Acad. Polon. Sci. 20 (1972) 195–202.
15. P. C. H. Lam, ‘Primes of the form $x^2 + Dy^2$’, M. Phil. Thesis, The University of Hong Kong, 2014.
16. P. C. H. Lam, ‘Primes of the form $ax^2 + 3xy + y^2$’, Preprint.
17. J. Maynard, ‘Primes represented by incomplete norm forms’, Preprint, 2015, arXiv:1507.05080.
18. M. Pandey, ‘On Eisenstein primes’, Integers 18 (2018) A59.

Peter Cho-Ho Lam
Department of Mathematics
Simon Fraser University
Burnaby
Canada BC V5A 1S6
cchohol@sfu.ca
http://www.sfu.ca/~cchohol/

Damaris Schindler
Mathematisches Institut
Universitaet Goettingen
Bunsenstrasse 3-5
Goettingen 37073
Germany
damaris.schindler@mathematik.uni-goettingen.de
https://sites.google.com/site/damarishomepage/

Stanley Yao Xiao
Mathematical Institute
University of Oxford
Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road
Oxford OX2 6GG
United Kingdom
stanley.xiao@maths.ox.ac.uk
https://www.maths.ox.ac.uk/people/stanley.xiao

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