FLIPS FROM 4-FOLDS WITH ISOLATED COMPLETE INTERSECTION SINGULARITIES WHOSE DOWNSTAIRS HAVE RATIONAL BI-ELEPHANTS

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Abstract. We shall investigate a flipping contraction $g : X \to Y$ from a 4-fold $X$ with at most isolated complete intersection singularities. If $Y$ has an anti-bi-canonical divisor (=bi-elephant) with only rational singularities, then $g$ carries an inductive structure chained up by blow-ups (La Torre Pendente), and in particular the flip exists. This naturally contains Miles Reid’s ‘Pagoda’ as an anti-canonical divisor (=elephant) and its proper transforms.

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We will work over $\mathbb{C}$, the complex number field.
0.1. Minimal Model Conjecture.

Let \( X \) be a smooth projective algebraic variety of dimension \( n \). Then \( X \) is birationally equivalent to a projective variety \( X' \) which has only terminal singularities (Reid [R1]), such that either one of the following holds:

1. \( X' \) is a minimal model, i.e. \( K_{X'} \) is nef, or
2. there exists a surjective morphism \( \varphi : X' \to Y \) with connected fibers such that \( -K_{X'} \) is \( \varphi \)-ample, and \( \dim Y \leq n - 1 \).

For dimension 2, this was classically known to be true by the works of the Italian school in the 19-th century. For dimension 3 or more, however, this has been left to be unknown for a hundred of years.

The attempt overcoming this has been made by the idea of introducing extremal rays (Mori [Mo2]) in early 80’s, and a program has been raised by several people such as Reid [Re1], Kawamata [Kaw1,2], Shokurov [Sh1] and others, how to approach minimal models in terms of extremal rays, which is nowadays known as the so-called Minimal Model Program (see for e.g. [KaMaMa] for introduction). Especially, it has been made it clear that the only obstacle toward the whole problem is concentrated on the appearance of small contractions, i.e. those birational contractions of extremal rays with higher codimensional exceptional loci.

0.2. Definition. Let \( X \) be a projective algebraic variety of dimension \( n \)

(0.2.1) with only \( \mathbb{Q} \)-factorial terminal singularities,

and let \( g : X \to Y \) be the contraction of an extremal ray. Then a priori

(0.2.2) the relative Picard number \( \rho(X/Y) = 1 \), and
(0.2.3) \( -K_X \) is \( g \)-ample.

Then we say that \( g \) is a small contraction, or alternatively a flipping contraction, if

(0.2.4) \( \dim \operatorname{Exc} g \leq n - 2 \).

If assume that there exists a projective variety \( X^+ \) and a birational morphism \( g^+ : X^+ \to Y \) such that

(0.2.1)+ \( X^+ \) has only \( \mathbb{Q} \)-factorial terminal singularities,
(0.2.2)+ \( \rho(X^+/Y) = 1 \),
(0.2.3)+ \( K_{X^+} \) is \( g^+ \)-ample, and
(0.2.4)+ \( \dim \operatorname{Exc} g^+ \leq n - 2 \),

then we call this \( g^+ \) (and also the composite birational map \( \psi := g^+^{-1} \circ g : X \dasharrow X^+ \)) the flip of \( g \).

A concrete example of such kind of transformation is first constructed by P. Francia [Fra].

Now we state the main conjecture, which implies the Minimal Model Conjecture:
0.3. Flip Conjecture.

(1) (Existence) For any \( g : X \to Y \) of Definition 0.1, the flip \( g^+ \) exists.

(2) (Termination) There is no infinite sequence of flips:

\[\not\exists \quad X \to X^+ \to X^{++} \to \ldots\]

Remark. This formulation does work for varieties with a certain class of singularities which is worse than terminal, or for varieties together with divisors, called log-flips (Shokurov [Sh2], Kawamata [Kaw7]). In fact these are harder and more complicated. So it is sometimes necessary to distinguish the above from these, and in such a case we call the above flip specifically terminal flips.

The termination part was first proved in dimension 3 by Shokurov [Sh1]. Here it should be noted that he discovered a numerical invariant coming from the discrepancy, which he called the difficulty [loc.cit], played an essential role. Then the termination was generalized to dimension 4 by Kawamata–Matsuda–Matsuki [KaMaMa]. On the other hand, the existence part was also investigated by several people, first for the semi-stable 3-fold case by Kawamata [Kaw3], Tsunoda, and Shokurov, and later on for the general 3-fold case by Mori [Mo4], applying Kawamata’s method. In this way the Minimal Model Conjecture has been found to be true also in dimension 3.

There are also further developments and generalizations in dimension 3, such as Reid [Re5], Kollár–Mori [KoMo], Shokurov [Sh2] (the existence of log-flips), Kawamata [Kaw7] (the termination of log-flips), [Kaw8], and Corti [Co] (see also [Utah]).

So we are interested in the existence problem of flips in dimension 4. Because the problem is now local on the downstairs \( Y \), we may discuss everything from now on under the following setting (as [Kaw3], [Mo4], [KoMo] did in dimension 3):

0.4. Notation–Assumption. Let \( g : X \to Y \) be a proper bimeromorphic morphism from a 4-dimensional analytic space (4-fold in short) \( X \) with only terminal singularities to a normal 4-fold \( Y \). Assume that the exceptional locus \( E := \text{Exc} g \) is compact, connected, and that \( X \) is a sufficiently small analytic neighborhood of \( E \). Then \( g \) is called an (analytic) flipping contraction if

\[(0.4.1) \quad \dim E \leq 2 \quad \text{and} \quad -K_X \text{ is } g\text{-ample.}\]

We simply write this \( X \supset E \overset{g}{\to} Y \supset g(E) \).

A proper bimeromorphic morphism \( g^+ : X^+ \to Y \) from a 4-fold \( X^+ \) with only terminal singularities is called the flip of \( g \) if

\[(0.4.1)^+ \quad \dim \text{Exc } g^+ \leq 2, \quad \text{and } K_{X^+} \text{ is } g^+\text{-ample.}\]

In this direction, the first achievement has been reached by Kawamata [Kaw4] in the case of smooth 4-folds:

0.5. Theorem. (Kawamata [Kaw4], Structure theorem of flips from smooth 4-folds)
Let $X \supset E \xrightarrow{g} Y \supset g(E)$ be a flipping contraction as in 0.4. Assume that $X$ is a smooth 4-fold. Then $E \simeq \mathbb{P}^2$, $Bs| - K_X | = \emptyset$, and the normal bundle $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

In particular, the flip $g^+ : X^+ \to Y$ of $g$ exists.

This should be considered as the first step of generalizing the Minimal Model Conjecture to dimension 4.

0.6. Remark. In this case the downstairs $Y$ has rational bi-elephants, i.e. $| - 2K_Y |$ has a member with only rational singularities. (cf. [Nak].)

Although the assumption $X$ being smooth seems at a first glance quite casual and collects less attention, the conclusion of the theorem says that it is rather strong in this context, and it in fact determines the structure of $g$ essentially in a unique way. Also in this case the flip can be composed just by a single blow up-and-down, and so we may say this is the simplest flip in dimension 4. Toward the Minimal Model Conjecture, it is necessary however to generalize this to the singular case, i.e. the case that $X$ has terminal singularities.

In his previous paper [Kac2], the author investigated flipping contractions from semi-stable 4-folds whose degenerate fibers satisfy a certain special assumption on singularities. Especially it is found that even we naively generalize Kawamata [Kaw3,7]'s definition of 3-fold semi-stable degenerations to the case of 4-folds, flips may in general destroy this condition, contrary to dimension 3. So we have to look for an appropriate re-definition of semi-stability which is preserved under any flips and divisorial contractions. This might be a problem to confront with in a future.

So we turn back to non-semi-stable flipping contractions. In this paper, we deal with flipping contractions from 4-folds with isolated complete intersection singularities. The following is the object we are going to investigate:

0.7. Assumption A. Let $g : X \to Y$ be as above. Assume

(A-1) (Singularities on $X$)

- $X$ has only isolated terminal complete intersection singularities, and

(A-2) (Existence of a good bi-elephant on $Y$) (cf. 0.6 above)

- $| - 2K_Y |$ has a member with only rational singularities.

A flipping contraction $g : X \to Y$ is said to be of Type $(R)$ if $g$ satisfies both (A-1) and (A-2). $g$ is said to be of Type $(I)$ if $g$ satisfies (A-1) and does not satisfy (A-2).

Then our main result is stated as follows:

0.8. Main Theorem. Let $X \supset E \xrightarrow{g} Y \supset g(E)$ be a 4-fold flipping contraction satisfying the Assumption $A$ above, i.e. of Type $(R)$. Assume $\text{Sing} X(\cap E) \neq \emptyset$. Then

- $E \simeq \mathbb{P}^2$, $Bs| - K_X | = \emptyset$, $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

Moreover, the flip $g^+$ exists.
(As for the description of the flip \( g^+ : X^+ \to Y \), see Corollary 8.11.)

To be more precise, \( g \) is classified in terms of the discrete invariant width \( g \) (see §5), and \( g \) carries an inductive structure chained up by blow-ups with respect to this number (La Torre Pendente), and this gives a complete understanding of the mechanism of flips (see §8).

After the completion of this paper, the author succeeded in proving the implication “(A-1) \( \implies \) (A-2)” so that we obtained the following stronger theorem:

0.9. Theorem. (Existence of flips from 4-folds with isolated complete intersection singularities)

Let \( X \) be a projective 4-fold, with at most isolated complete intersection singularities. Then for any flipping contraction \( g : X \to Y \), the flip \( g^+ \) exists.

The proof will be given in the forthcoming paper.

What should be coming next is to investigate \( g : X \to Y \) where \( X \) has a non-isolated singular locus. For example it is unknown whether the condition (A-1’) \( X \) has terminal complete intersection singularities with \( \dim \text{Sing} \ X = 1 \), implies (A-2), while on the other hand there is an example of a flipping contraction \( g : X \to Y \) satisfying both (A-1’) and (A-2) by M. Gross (which will be appeared in the forthcoming paper).

This paper is organized as follows:

The first couple of sections are preliminary, and the arguments are mostly parallel to Kawamata [Kaw4]’s. (cf. [AW1], [ABW].)

First in §1, we shall prove that the normalization \( \widetilde{E}_i \) of each irreducible component \( E_i \) of \( E \) is isomorphic to \( \mathbb{P}^2 \). We apply Kollár’s theorem [Kol2] on Hilbert schemes parametrizing rational curves on a variety with complete intersection singularities, which is a generalization of Mori [Mo1] in the smooth case. This part we are indebted to Kollár’s idea. All the rest is the reproduction of arguments in [Kaw4]. At the same time, it is proved that \(-K_X\) pulled-back to \( \widetilde{E}_i \)’s are all \( \mathcal{O}_{\mathbb{P}^2}(1) \).

In §2, we shall discuss the freeness of \( |-K_X| \) (Theorem 2.1). Kawamata [Kaw4] first proved this for the smooth case by developing the ‘base-point-free technique’ initiated by himself, and later on Andreatta–Wiśniewski [AW1] extended this to a certain singular case, which is sufficient for what we need. We shall give the proof for readers’ convenience, following Kawamata [Kaw4]’s original argument. Also we shall collect several immediate consequences: The normality of \( E_i \)’s (Corollary 2.8), a cohomological interpretation of the condition (A-2) of 0.7 (Corollary 2.9), and the irreducibility of \( E : E \simeq \mathbb{P}^2 \) for Type (R) contraction (Corollary 2.10).

In §3, we shall introduce an invariant \( \varepsilon_P = \varepsilon_P(Z \supset S) \) for a pair of a germ of an analytic space \( Z = (Z, P) \) and its closed subspace \( S \) passing through \( P \). We give an explicit formula (Theorem 3.1) for computing this \( \varepsilon_P \) in the case that \( Z \) is a complete intersection singularity and \( S \) is a smooth subspace. Especially when \( Z \) is a 3-dimensional terminal Gorenstein singularity (= cDV-singularity) and \( S \) is a smooth
curve, then this coincides exactly with Mori [Mo4]’s invariant $i_P(1)$ (Corollary 3.2), and thus our $\varepsilon_P$ can be thought of as a sort of a higher dimensional analogue of $i_P(1)$.

So far is the preparation.

From §4 on, we concentrate on Type (R) contractions.

In §4, we shall determine the normal bundle $N_{E/X} := (I_E/I_E^2)^\vee$ of $E$ in $X$ for Type (R) contractions. We shall prove that $N_{E/X} \simeq \mathcal{O}_{P^2} \oplus \mathcal{O}_{P^2}(-2)$ (Theorem 4.1). The strategy is first to limit the types of its restrictions to lines (Proposition 4.2), where the assumption (A-2) is essentially used, and then to deduce the uniformity of $N_{E/X}$ (Proposition 4.4) by applying the generalized Namikawa [Nam3]’s local moduli (Theorem 4.5), where we develop the deformation theory for contraction morphisms. (See also Remark 4.9). Then Van de Ven [V]’s characterization of uniform bundles immediately gives us the conclusion.

In §5, we shall introduce an invariant called *widths* for our contractions $g$ (Definition 5.5), after Reid [Re1] for 3-fold *flopping contractions*. Roughly, this measures the infinitesimal length of the embedding $E \subset X$ along a general locus of $E$, while it actually specifies the singularities of $X$ as we will see in §7 (Corollary 7.6). This is thus so to speak a local-to-global invariant, and in fact plays an essential role toward the existence of flips in §8.

In §6, we discuss deformations of a contraction $g : X \to Y$. Since $R^i g_* \mathcal{O}_X = 0$ ($i = 1, 2$), we can talk about deformations of the contraction $g : X \to Y$, rather than that of $X$ itself (see Kollár–Mori [KoMo] §11). We shall prove that any set of local deformations of $X$ along the singular points $(X, P_i)$ ($i = 1, \ldots, r$) (where $\{P_1, \ldots, P_r\} := \text{Sing } X \cap E$) can be patched together to produce a global deformation of $X$, and of $g : X \to Y$ (Theorem 6.1). Especially $X$ is smoothable by a flat deformation. What we actually prove are the following couple of statements:

(a) The *unobstructedness* of deformations (see Ran [Ra2], Kawamata [Kaw6], Namikawa [Nam1,2,3], Gross [G1,2]), and

(b) The surjectivity of $\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\alpha} g_* \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$.

These imply the desired surjectivity of the natural holomorphic map

$$\text{Def } X \longrightarrow \prod_{i=1}^r \text{Def } (X, P_i)$$

between the global and the local *versal deformation spaces* (=Kuranishi spaces). In dimension 3, the corresponding result was proved by Mori [Mo4] §1b in rather complex analytic languages (*L-deformations*), including an essential use of the implicit function theorem. Saying in algebro-geometric words, his argument might correspond to the fact that the obstruction lies in $R^2 g_* T_X$, which automatically vanishes in dimension 3. In dimension 4, however, the proof of $R^2 g_* T_X = 0$ essentially requires the structure of the normal bundle $N_{E/X}$, which we have already determined in §4. This can be viewed as a sort of a relative 4-dimensional analogue of Namikawa [Nam1]’s smoothing theory of Fano varieties. (cf. [Nam2,3], [G1,2].)

Also, for such given family $\mathcal{X} \to \mathcal{Y} = \{X_t \xrightarrow{g_t} Y_t\}_{t \in \Delta}$ we especially focus on the ideal structure of $E$, which is defined naturally as the degeneration of $E_t := \text{Exc } g_t$.
(t \neq 0) (6.7). We shall prove that this subscheme supported on $E$ has no embedded components, and that this has the multiplicity just equal to the number of connected components of $E_t$ ($t \neq 0$) (Proposition 6.8). This observation is crucial toward the subsequent arguments. As opposed to Mori’s L-deformation ([Mo4], [KoMo]), the basic idea of which was to restrict attention to one irreducible component after a deformation and forget everything else, ours is rather to look at all irreducible components simultaneously, and this in fact makes us possible to get desirable informations on the local structure of $X \supset E$ along singularities. For instance, at the final stage of the local classification ($\S 7$), we will show that the singularity $(X, P)$ can be completely determined by the width (Corollary 7.6), which is a primary consequence of this observation. This may thus shed a new light toward the problem of flips.

Based on these preparations, in $\S 7$ we now are going to classify $g$ of Type (R) in $\S 7$. We shall prove that $X$ has exactly one singular point $P$, and that $\varepsilon_P(X \supset E) = 1$ (Theorem 7.1). Hence $g$ is completely classified up to the invariant width $g$, introduced in $\S 5$. We shall make full use of the deformation theory established in $\S 6$. Technically, the hardest part is to show that every singular point of $X$ is a hypersurface singularity (Proposition 7.2). This can be done by giving several different local deformations and looking at the associated subscheme structures of $E$ for every corresponding global deformations.

Finally in $\S 8$ we shall prove the existence of flips for Type (R) contractions. We shall prove this by induction on $m := \text{width } g$. The case $m = 1$ is nothing but Kawamata [Kaw4]’s result (Theorem 0.5). The construction of the flip is as follows: First blow $X$ up with the center $E$: $\varphi : \overline{X} \to X$. Then as we will see in Proposition 8.7, which is the hardest part of this section, there exists a unique flipping contraction $\overline{g}$ from $\overline{X}$ of Type (R), with width $\overline{g} = m - 1$. (This is also a consequence of the deformation theory.) So by the induction hypothesis we are able to flip it: $\overline{X} \dashrightarrow \overline{X}^+$. Then the proper transform of $\text{Exc } \varphi$ in $\overline{X}^+$ can be contracted down; $\overline{X}^+ \to X^+$, to produce the desired flip $X^+ \to Y$. Here we essentially use the structure of Reid [Re1]’s ‘Pagoda’ (see 5.1).

To see the mechanism of this flip operation more concretely, let us follow all the induction steps successively upstreams, then we will eventually arrive $m = 1$, and will get a sequence of blow-ups and blow-downs:

$$
X \leftarrow X^{(1)} \leftarrow \ldots \leftarrow X^{(m-1)} \leftarrow X^{(m)} \rightarrow X^{(m-1)+} \rightarrow \ldots \rightarrow X^{(1)+} \rightarrow X^+,
$$

where $X^{(i)} \dashrightarrow X^{(i)+}$ gives the flip, with the width $m-i$. This should be compared to Reid’s Pagoda, not only since the patterns of the constructions of the flip or the flop look similar, but also since ours in fact contains Pagoda as an anti-canonical divisor (and its proper transforms). On the other hand, though Pagoda was symmetric with respect to the flop (see 5.1), ours is not. So with a great esteem for Reid’s humor of this lovely naming, we name by a special grace our 4-fold flip ‘La Torre Pendente’.

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§1. $\widetilde{E}_i \simeq \mathbb{P}^2$

**Notation 1.0.** Let $X \supset E \xrightarrow{g} Y$ be a flipping contraction satisfying the Assumption (A-1), *i.e.* of Type (R) or (I). Throughout this section, we fix the following notation:

(1.0.1) Let $E = \bigcup_{i=1}^{n} E_i$ be the irreducible decomposition of $E$, and $\nu_i : \widetilde{E}_i \to E_i$ the normalization of $E_i$ ($i = 1, \ldots, n$).

The aim of this section is to prove the following:

**Theorem 1.1.** $\widetilde{E}_i \simeq \mathbb{P}^2$, and $\nu_i^* \mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ ($i = 1, \ldots, n$).

First we shall give the following theorem due to J. Kollár [Kol2], which is a generalization of Mori [Mo1], and is one of the key to our whole argument:

**Theorem 1.2.** (Kollár [Kol2] II 1.3 Theorem)

Let $Z$ be an $n$-dimensional analytic space and $C$ an irreducible rational curve on it. Assume that $Z$ has at most complete intersection singularities, and that $C \not\subset \text{Sing } Z$.

Let $\alpha \colon \widetilde{C} \simeq \mathbb{P}^1 \to C \subset Z$ be the normalization of $C$. Then

$$\dim \text{Hom}_{[\alpha]}(\mathbb{P}^1, Z) \geq n + (-K_Z \cdot C).$$

□

**Remark 1.3.** Kollár’s Theorem 1.2 has many far reaching consequences. For instance, Mori [Mo4]’s result “there is no flipping contraction from a terminal Gorenstein 3-fold” is now an immediate corollary of Theorem 1.2.

Thus we can proceed completely the same argument as in Kawamata [Kaw4] in the smooth case, which is an application of Mori [Mo1], Ionescu [Io] (cf. Wiśniewski [W]) to deduce:

**Proposition 1.4.** (following Kawamata [Kaw4]. See also Andreotti–Ballico–Wiśniewski [ABW].)

(1) $E$ is purely 2-dimensional, and
(2) Each $E_i$ is covered by rational curves $C$ such that $(-K_X \cdot C) = 1$.

**Proof.** First by Kawamata [Kaw5], each $E_i$ is covered by rational curves. Thus by the assumption (0.4.4), (1) is immediate from Theorem 1.2.

Next, let $C$ be a rational curve in $E_i$ which attains the minimum number:

$$r := \text{Min } \{(-K_X \cdot C) \mid C : \text{rational curves in } E_i\}.$$

Let $\alpha : \widetilde{C} \to C$ be the normalization of $C$. Then

$$\dim \text{Hom}_{[\alpha]}(\mathbb{P}^1, X) \geq 4 + r,$$

by Theorem 1.2. Hence there exists a family of curves $p : \mathcal{U} \to H$, with the projection $q : \mathcal{U} \to X$ satisfying $q(\mathcal{U}) = E_i$, which gives the universal deformation of $C$ (with a certain generically reduced closed subscheme structure) inside $X$. By the choice of $C$,
(1.4.2) All fibers of $p$ are irreducible rational curves, and
\[
\dim H = \dim \text{Hom}_{[\alpha]}(\mathbb{P}^1, X) - \dim \text{Aut} \mathbb{P}^1 \
\geq 4 + r - 3 = r + 1.
\]

From now on we shall prove $\dim H = 2$. Let $x$ be a general point of $E_i$, and let
\[
\begin{align*}
\left\{ \begin{array}{l}
H_x := p(q^{-1}(x)), \ U_x := p^{-1}(H_x), \\
p_x := p|_{U_x} : U_x \to H_x, \text{ and } q_x := q|_{U_x} : U_x \to X.
\end{array} \right.
\end{align*}
\]
Then
\[
(1.4.5) \quad q_x(U_x) = E_i,
\]
and
\[
(1.4.6) \quad \dim H_x = \dim q^{-1}(x) = \dim U - \dim E_i = \dim H - 1.
\]

Assume $\dim H \geq 3$ to derive a contradiction, till (1.4.10). Let $y$ be another general point of $E_i$, and let
\[
\begin{align*}
\left\{ \begin{array}{l}
H_{x,y} := p_x(q^{-1}(y)), \ U_{x,y} := p^{-1}_x(H_{x,y}), \\
p_{x,y} := p_x|_{U_{x,y}} : U_{x,y} \to H_{x,y}, \text{ and } q_{x,y} := q_x|_{U_{x,y}} : U_{x,y} \to X.
\end{array} \right.
\end{align*}
\]
Then similarly we have
\[
(1.4.7) \quad \begin{align*}
\left\{ \begin{array}{l}
q_{x,y}(U_{x,y}) = E_i, \text{ and} \\
\dim H_{x,y} \geq \dim H - 2 \ (1.4.6).
\end{array} \right.
\end{align*}
\]
If $\dim H \geq 3$, then this means that
\[
(1.4.8) \quad p_{x,y} : U_{x,y} \to H_{x,y} \text{ admits two disjoint sections}
\]
\[
\begin{align*}
s_x := q_{x,y}^{-1}(x) \quad \text{and} \quad s_y := q_{x,y}^{-1}(y).
\end{align*}
\]

Let $H_1 \subset H_{x,y}$ be any 1-dimensional irreducible closed subset, let $\tilde{H}_1 \to H_1$ be the normalization, and $p_1 : U_{x,y} \times_{H_{x,y}} \tilde{H}_1 \to \tilde{H}_1$ the induced morphism. Then
\[
(1.4.9) \quad p_1 \text{ still admits two disjoint sections } s_{x,1}, s_{y,1} \text{ induced from } s_x, s_y, \text{ respectively, both of which are contractible.}
\]
This is however impossible, since $p_1$ is a $\mathbb{P}^1$-bundle over a smooth complete curve $\tilde{H}_1$ (1.4.2). Hence we must have
\[
(1.4.10) \quad \dim H = 2.
\]
In particular, the inequality (1.4.3) must be the equality:
\[
(1.4.11) \quad r = 1. \quad \Box
\]
1.5. Proof of Theorem 1.1.

By the previous proposition, it is enough to prove \( \tilde{E}_i \cong \mathbb{P}^2 \), so let us return back to the situation (1.4.4) through (1.4.6). Note that \( \dim H_x = 1 \) by (1.4.6) and (1.4.11). Let \( H_0 \) be any 1-dimensional irreducible component of \( H_x \), and \( \tilde{H}_0 \to H_0 \) the normalization. Then the induced 
\[
p_0 : \tilde{U}_0 := U_x \times_{H_x} \tilde{H}_0 \to \tilde{H}_0
\]
is a \( \mathbb{P}^1 \)-bundle over a smooth complete curve \( \tilde{H}_0 \) dominating \( E_i \) through \( q_0 : \tilde{U}_0 \to X \). Moreover, \( p_0 \) has a section \( s_x := q_0^{-1}(x) \), by the construction. Hence the normalization of \( E_i \) must be \( \mathbb{P}^2 \). \( \square \)

**Remark 1.6.** The conclusion of Theorem 1.1 is no longer true if we assume \( X \) merely to be Gorenstein, instead of assuming \( X \) to have complete intersection singularities (see \( \S \) Appendix A.1). In fact there is an example (Example 8.13) due to Mukai, of a flipping contraction \( g : X \to Y \) from a Gorenstein 4-fold which contracts a singular quadric surface.

A ruling \( l \) satisfies \( (-K_X \cdot l) = 1 \), while \( l \) deforms inside \( X \) exactly with 1-dimensional parameter space. So Kollar’s formula (Theorem 1.2) also fails in this case. (For details see 8.13.)

\[\S 2. \text{ Freeness of } | - K_X | \text{ and its consequences.}\]

In this section, we shall overview the proof of the freeness of the anti-canonical linear system \( | - K_X | \) on \( X \) (Theorem 2.1) (without the assumption (A-2)), following Kawamata [Kaw4] and Andreatta–Wiśniewski [AW1]. In the case \( X \) is smooth, this was proved by Kawamata [Kaw4], and was extended later on to a more general situation, i.e. the case that \( X \) has rational Gorenstein singularities, by Andreatta–Wiśniewski [AW1].

This result provides us three corollaries. The first one is the smoothness of each irreducible components of \( E \) (Corollary 2.8). The second is an interpretation of the condition (A-2) in terms of a certain cohomology vanishing (Corollary 2.9). The final one is the irreducibility of \( E \) under the assumption (A-2), i.e. for Type (R) (Corollary 2.10).

**Theorem 2.1.** *(Kawamata [Kaw4], Andreatta–Wiśniewski [AW1].)*

Let \( X \supset E \twoheadrightarrow Y \supset Q \) be a flipping contraction satisfying the assumption (A-1), i.e. of Type (R) or (I). Then \( | - K_X | \) is free, i.e. the homomorphism
\[
 \rho : g^* g_* \mathcal{O}_X(-K_X) \to \mathcal{O}_X(-K_X)
\]
is surjective.

Here we shall adopt Kawamata [Kaw4]’s original proof. We completely follow the argument of [loc.cit] from 2.2 through 2.7 (cf. [Kac1] \( \S 4 \)).

Most of the notations fixed therein (such as \( A, B, C, F_i, G_i, L \) etc.) are valid only for this section.
2.2. Assume $Bs \mid -K_X \mid := \text{Supp Coker } \rho \neq \emptyset$ to derive a contradiction, till 2.7.

Let $\varphi : X' \to X$ be a projective bimeromorphic morphism from a smooth 4-fold $X'$, together with a simple normal crossing divisor $\sum G_i$ on it, such that

\[
\begin{align*}
\varphi^*|-K_X| &= |D'| + \sum_i r_i G_i \text{ with } Bs \mid D'\mid = \emptyset, \\
K_{X'} &= \varphi^* K_X + \sum_i a_i G_i, \quad \text{and} \\
-\varphi^* K_X - \sum_i \delta_i G_i \text{ is } (g \circ \varphi)\text{-ample,}
\end{align*}
\]

for some $r_i \in \mathbb{Z}_{\geq 0}, a_i \in \mathbb{Z},$ and $\delta_i \in \mathbb{Q}_{\geq 0}, 0 < \delta_i < 1$. Since $X$ is assumed to have at most terminal singularities,

\[
a_i \geq 0.
\]

Note that

\[
\varphi(\bigcup_i G_i) = Bs \mid -K_X\mid \cup \text{Sing } X.
\]

Let $c := \min \frac{a_i + 1 - \delta_i}{r_i}$. By shrinking $\delta_i$'s if necessary, we may assume that the minimum $c$ is attained exactly for a single $i$, say $i = 1$. Let

\[
A := \sum_{i \geq 2} (-cr_i + a_i - \delta_i) G_i, \quad \text{and} \quad B := G_1.
\]

Then

\[
\varprojlim A \geq 0 \quad \text{and} \quad \text{Supp } \varprojlim A \subset \text{Exc } \varphi.
\]

Lemma 2.3. Under the notations and the assumption of 2.2,

\[
a_i \geq r_i \quad (\forall i).
\]

Proof. Assume to the contrary, then $c < \frac{a_i + 1}{r_i} \leq 1$, and hence

\[
C := -\varphi^* K_X - K_{X'} + (A - B) \\
= cD' - (2 - c)\varphi^* K_X - \sum_{i \geq 1} \delta_i G_i
\]

is $(g \circ \varphi)$-ample. By the Kawamata-Viehweg vanishing theorem (for e.g. [KaMaMa]), $R^1(g \circ \varphi)_* \mathcal{O}_{X'}(-\varphi^* K_X + \varprojlim A - B) = 0$, and thus

\[
s : (g \circ \varphi)_* \mathcal{O}_{X'}(-\varphi^* K_X + \varprojlim A) \to H^0(B, \mathcal{O}_B(-\varphi^* K_X + \varprojlim A))
\]

is surjective. Here

\[
(2.3.2) \quad (g \circ \varphi)_* \mathcal{O}_{X'}(-\varphi^* K_X + \varprojlim A) \cong g_* \mathcal{O}_X(-K_X)
\]

(2.2.4). On the other hand,
(2.3.3) \[ H^0(B, \mathcal{O}_B(-\varphi^*K_X + \lceil A \rceil)) \neq 0, \]
by the fact that \( \nu^*_i \mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\varphi^2}(1) \) is globally generated (\( \forall i \)). These imply \( \varphi(B) \not\subset \text{Bs} \mid -K_X \mid \), while from (2.2.2) we must have \( \varphi(B) \subset \text{Bs} \mid -K_X \mid \), a contradiction. Hence
\[ a_i \geq r_i \quad (\forall i). \]

**Lemma 2.4.** Let \( D \) be a general member of \( \mid -K_X \mid \). Then \( D \) has at most canonical singularities.

**Proof.** Take a general smooth \( D' \in \mid D \mid \) (2.2.0) and consider \( \varphi|_{D'} : D' \to D \). This gives a resolution of \( D \). By adjunction and (2.2.0),
\[
K_{D'} = K_{X'} + D'|_{D'} = \sum_i (a_i - r_i)(G_i|_{D'}) \\
= (\varphi|_{D'})^*K_D + \sum_i (a_i - r_i)(G_i|_{D'})
\]
(recall \( K_D \sim 0 \)). Since \( a_i - r_i \geq 0 \) (Lemma 2.3), this means that \( D \) has at most canonical singularities. \( \square \)

**2.5.** Let \( D \in \mid -K_X \mid \) be a general member as in Lemma 2.4, and let
\[ h := g|_D : D \to V := g(D). \]

\( h \) is a projective bimeromorphic morphism such that
\[
F := \text{Exc } h \subset E.
\]
Let \( F = \sum_j F_{1,j} + \sum_k F_{2,k} \) be the irreducible decomposition, where
\[
\dim F_{1,j} = 1 \quad \text{and} \quad \dim F_{2,k} = 2.
\]
Moreover let \( F_1 := \sum_j F_{1,j} \) and \( F_2 := \sum_k F_{2,k} \):
\[ F = F_1 + F_2. \]

Note that each \( F_{2,k} \) coincides with some \( E_i \).

**Lemma 2.6.** \( \text{Bs } \mid D \mid = F_2 \).

**Proof.** Let \( L := -K_X|_D \) and consider the exact sequence
\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(-K_X) \to \mathcal{O}_D(L) \to 0. \]
Since \( R^1g_*\mathcal{O}_X = 0 \) [loc.cit],
\[
\text{Bs } \mid L \mid = \text{Bs } \mid -K_X \mid.
\]
Let \( x \in F_1 - F_2 \) be an arbitrary point, then we can find an effective Cartier divisor \( M_x \) on \( D \) such that

\[
(2.6.2) \quad M_x \cap F = \{ x \}, \quad \text{and}
\]

\[
(2.6.3) \quad (M_x, F_{1,j}) = 1 \quad \text{for any } F_{1,j} \text{ containing } x.
\]

In particular, \( L - M_x \) is \( h \)-nef. Again by [loc.cit], \( R^1 h_* \mathcal{O}_D(L - M_x) = 0 \), and

\[
h_* \mathcal{O}_D(L) \to h_* \mathcal{O}_{M_x}(L)
\]

is surjective. Thus \( x \not\in \text{Bs } |L| \). From this and (2.6.1), we have \( x \not\in \text{Bs } |-K_X| \), and we conclude \( \text{Bs } |-K_X| = F_2 \).

\[\square\]

2.7. Proof of Theorem 2.1.

(2.7.0) Let \( H \) be a general very ample divisor on \( D \). By shrinking \( V \) if necessary, \( H \) is decomposed as \( H = H_1 + H_2 \) so that \( H_1 \cap F_1 = H_2 \cap F_2 = \emptyset \), and \( |H_2| \) gives a projective bimeromorphic morphism \( h_1 : D \to \tilde{V} \) such that

\[
\text{Exc } h_1 = F_2.
\]

Let \( F^o \) be a connected component of \( F_2 \), \( D^o \) an analytic neighborhood of \( F^o \) in \( D \), \( V^o := h_1(D^o) \), and

\[
h^o := h_1|D^o : D^o \to V^o.
\]

Then

(2.7.1) \( h^o \) is a projective bimeromorphic morphism which contracts a connected divisor \( F^o \) to a point, and each irreducible component of \( F^o \) coincides with some \( E_i \).

We denote \( L|D^o \) again by \( L \). Then by (2.6.1) and Bertini’s theorem,

(2.7.2) \( \text{Sing } D^o \subset \text{Bs } |L| \).

Let \( h^o(F^o) =: P^o \in V^o \), let \( L_0 \in |L| \) be a general member, \( r \) a sufficiently large integer, and \( L' \) a general hyperplane section of \( (V^o, P^o) \). Let

(2.7.3) \( L_r := L_0 + rh^o_0L' \in |L| \).

Take a projective bimeromorphic morphism \( \psi : D' \to D^o \) from a smooth 3-fold \( D' \), with a simple normal crossing divisor \( \sum_i G'_i \), such that

\[
\psi^* L_r = \sum_i r'_i G'_i,
\]

\[
K_{D'} = \psi^* K_{D^o} + \sum_i a'_i G'_i \sim \sum_i a'_i G'_i, \quad \text{and}
\]

\[
-\psi^* L - \sum_i \delta'_i G'_i \text{ is } (h^o \circ \psi)\text{-ample}
\]

for some \( r'_i, a'_i \in \mathbb{Z} \) with \( r'_i \geq 0 \), and \( \delta'_i \in \mathbb{Q}_{>0} \) (0 < \( \delta'_i < 1 \)).

Note the following three things:

(2.7.5) \( \psi(\bigcup_i G'_i) \subset \text{Bs } |L| \cup \text{Sing } D^o = \text{Bs } |L| = F^o \),

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(2.7.6) \( a'_i \geq 0 \ (\forall i) \), and
(2.7.7) \( r'_i \geq r \ (\forall i) \).
In fact, (2.7.5) comes from Lemma 2.6 (with (2.7.2)), (2.7.6) from Lemma 2.4, and
(2.7.7) from the definition of \( L_r \in |L| \) (2.7.3).

Since \( a'_i \)'s do not depend on \( r \), we can shrink the minimum
\[
c' = c'(r) := \min \frac{a'_i + 1 - \delta'_i}{r'_i}
\]
as little as we need:
(2.7.8) \( 0 < c' \ll 1 \) (as \( r \gg 0 \)).

As before, we may assume that the minimum \( c' \) is attained exactly for a single
\( i = 1 \), say, and let
\[
A' := \sum_{i \geq 2} (-c'_i r'_i + a'_i - \delta'_i)G'_i, \quad \text{and} \quad B' := G'_1.
\]

Then
\[
C' := \psi^*L - K_{D'} + (A' - B') = (1 - c') \left( \psi^*L - \sum \frac{\delta'_i}{1 - c'} G'_i \right)
\]
is \((h^\circ \psi)^{-1}\)-ample (2.7.4), (2.7.8). Thus
\[
R^1(h^\circ \psi)_* \mathcal{O}_{D'}(\psi^*L + \Gamma A' - B') = 0
\]
[loc.cit], and we get the surjection
\[
(h^\circ \psi)_* \mathcal{O}_{D'}(\psi^*L + \Gamma A') = h^\circ \mathcal{O}_{D^\circ}(L) \rightarrow H^0(B', \mathcal{O}_{B'}(\psi^*L + \Gamma A')).
\]
Since \( \psi(B') \subset E \) (2.7.5) and \( \nu^*_i \mathcal{O}_{E_i}(-K_X) \simeq \mathcal{O}_{\mathbb{P}^2}(1) \ (\forall i) \) (Theorem 1.1), the right-hand side does not vanish. Hence \( \psi(B') \not\subset E \), a contradiction. Now the proof of
Theorem 2.1 is completed. \( \square \)

**Corollary 2.8.** \( E_i \simeq \mathbb{P}^2 \ (\forall i) \).

**Proof.** By Theorem 1.1, it is enough to show that the normalization morphism
\[
\mathbb{P}^2 \simeq \tilde{E}_i \xrightarrow{\nu_i} E_i
\]
is an isomorphism. Since \( |-K_X|_{E_i}| \) is free (Theorem 2.1), the associated rational map:
\[
\Phi : E_i \rightarrow \mathbb{P}^{N-1} \quad (N = \dim H^0(E_i, \mathcal{O}_{E_i}(-K_X)))
\]
is actually a morphism. Since \( \mathcal{O}_{E_i}(-K_X) \) is ample,
(2.8.1) \( N \geq 3 \).
Consider the natural inclusion:

\[ H^0(E_i, \mathcal{O}_{E_i}(-K_X)) \hookrightarrow H^0(\tilde{E}_i, \nu^*_i \mathcal{O}_{E_i}(-K_X)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)). \]

Since the right-hand side has dimension just 3, this must be an isomorphism:

\[ (2.8.2) \quad N = 3. \]

Let us consider the composition \( \mathbb{P}^2 \cong \tilde{E}_i \xrightarrow{\nu_i} E_i \xrightarrow{\Phi} \mathbb{P}^2 \). Since both arrows are birational morphisms, so is the composition, and is hence an isomorphism. In particular, \( \Phi \) is set theoretically a bijection, and by Zariski Main Theorem, \( \Phi \) is actually an isomorphism: \( E_i \cong \mathbb{P}^2 \). \( \square \)

**Corollary 2.9.** Let \( X \supset E \twoheadrightarrow Y \ni Q \) be a flipping contraction with the assumption (A-1). Then

\[ (A-2) \quad R^2g_\ast \mathcal{O}_X(2K_X) = 0. \]

**Proof.** Take a general \( L \in |-2K_X| \). By Theorem 2.1 we can choose such \( L \) to be smooth. \( g(L) \in |-2K_Y| \), and \( g|_L : L \to g(L) \) gives a resolution of \( g(L) \). Consider the exact sequence:

\[ 0 \longrightarrow \mathcal{O}_X(2K_X) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_L \longrightarrow 0 \]

Then by \( R^i g_\ast \mathcal{O}_X = 0 \) \((i = 1, 2)\),

\[ R^2 g_\ast \mathcal{O}_X(2K_X) \cong R^1 (g|_L)_\ast \mathcal{O}_L, \]

and hence the result. \( \square \)

**Corollary 2.10.** Let \( X \supset E \twoheadrightarrow Y \ni Q \) be satisfying the Assumption A, i.e. of Type (R). Then \( E \) is irreducible:

\[ E \cong \mathbb{P}^2. \]

**Proof.** Assume that \( E \) has more than one irreducible components. Let \( L \in |-2K_X| \) be a general member, let \( k := g|_L : L \to g(L) \), and consider the exact sequence:

\[ 0 \longrightarrow \mathcal{O}_X(2K_X) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_L \longrightarrow 0 \]

\( R^1 g_\ast \mathcal{O}_X = 0 \) by Grauert–Riemenschneider vanishing theorem, and \( R^2 g_\ast \mathcal{O}_X(2K_X) = 0 \) by Corollary 2.9, hence we have

\[ (2.10.1) \quad R^1 k_\ast \mathcal{O}_L = 0. \]

Let \( C := L \cap E \), then this implies

\[ (2.10.2) \quad H^1(\mathcal{O}_C) = 0. \]

Since \(|-2K_X| \) is also free (Theorem 2.1), and since \( \mathcal{O}_{E_i}(-2K_X) \cong \mathcal{O}_{\mathbb{P}^2}(2) \) (Theorem 1.1), if two of \( \{E_1, \ldots, E_n\} \) intersects with each other along a 1-dimensional subspace,
then $L$ has at least two irreducible components meeting at distinct two points, which contradicts (2.10.2). Thus

(2.10.3) For any two $E_i$ and $E_j$ ($i \neq j$), $E_i \cap E_j$ is at most a discrete set of points.

(2.10.4) In the rest we follow Kawamata [Kaw4]'s argument.

Take in turn a general $D \in |-K_X|$. Then by Theorem 2.1 and by the assumption $\# \text{Sing } X < \infty$, $D$ is smooth. Consider $g|_D : D \to g(D)$, then $-K_D$ is numerically trivial relative to this $g|_D$. Since $(Y, Q)$ is a rational singularity $R^ig_*O_X = 0$ ($i = 1, 2$) and is hence Cohen–Macaulay, $g(D) \in |-K_Y|$ is Gorenstein and is in particular normal. Thus any fiber of $g|_D : D \to g(D)$ must be connected. This, (2.10.3), and Theorem 2.1 imply the irreducibility of $E$. □

§3. The invariant $\varepsilon_P(Z \supset S)$.

In this section we introduce a numerical invariant $\varepsilon_P = \varepsilon_P(Z \supset S)$ for an analytic germ of a singularity $(Z, P)$ together with its closed subspace $S$. This section is logically independent from the former sections.

**Definition 3.0.** Let $(Z, P)$ be an analytic germ of a singularity, and let $S \subset Z$ be an irreducible closed subspace passing through $P$. Assume that

(3.0.1) $S \cap \text{Sing } Z = \{P\}$.

Define:

(3.0.2) $\varepsilon_P = \varepsilon_P(Z \supset S) := \dim_P \mathcal{E}xt^1_S(\Omega^1_Z \otimes O_S, O_S)$.

**Theorem 3.1.** In the above, assume that $Z$ is a complete intersection singularity, and that $S$ is smooth. Then we can write down the set of defining equations of $Z$ inside $(\mathbb{C}^N, 0)$ (where $N$ is the least possible embedding dimension of $(Z, P)$) as follows:

(3.1.1)

\[ S = \{x_{s+1} = \cdots = x_N = 0\} \]

\[ \subset Z = \{f_1(x_1, \ldots, x_N) = \cdots = f_r(x_1, \ldots, x_N) = 0\} \]

\[ \subset V := (\mathbb{C}^N, 0) = \{(x_1, \ldots, x_N)\}, \]

\[ f_i(x_1, \ldots, x_N) = \sum_{j=s+1}^{N} x_j \cdot g_{i,j}(x_1, \ldots, x_s) + h_i(x_1, \ldots, x_N), \]

with $g_{i,j} \in (x_1, \ldots, x_s)$ and $h_i \in (x_{s+1}, \ldots, x_N)^2$.

Then

\[ \mathcal{E}xt^1_S(\Omega^1_Z \otimes O_S, O_S) \cong \bigoplus_{i=1}^{r} O_S/(g_{i,s+1}, \ldots, g_{i,N}) \]

and thus

\[ \varepsilon_P(Z \supset S) = \sum_{i=1}^{r} \text{length } O_S/(g_{i,s+1}, \ldots, g_{i,N}). \]
Proof. Let \( J \) be the ideal sheaf of \( Z \) inside \( V = (\mathbb{C}^N, 0) \). Since \( J/J^2 \) is a free \( \mathcal{O}_Z \)-module,

\[
(3.1.2) \quad 0 \rightarrow J/J^2 \otimes \mathcal{O}_S \rightarrow \Omega^1_V \otimes \mathcal{O}_S \rightarrow \Omega^1_Z \otimes \mathcal{O}_S \rightarrow 0
\]

is exact. Let \( \alpha : \mathcal{T}_V \otimes \mathcal{O}_Z \rightarrow (J/J^2)^\vee := \text{Hom}_Z(J/J^2, \mathcal{O}_Z) \) be the natural \( \mathcal{O}_Z \)-homomorphism:

\[
\frac{\partial}{\partial x_i} \otimes 1 \mapsto \left( [f] := f \mod J^2 \mapsto \frac{\partial f}{\partial x_i} \big|_{Z} \right)
\]

In (3.1.2), \( J/J^2 \otimes \mathcal{O}_S \) and \( \Omega^1_V \otimes \mathcal{O}_S \) are both free \( \mathcal{O}_S \)-modules, so

\[
\mathcal{T}_V \otimes \mathcal{O}_S \xrightarrow{\alpha \otimes \mathcal{O}_S} (J/J^2)^\vee \otimes \mathcal{O}_S \rightarrow \text{Ext}^1_S(\Omega^1_Z \otimes \mathcal{O}_S, \mathcal{O}_S) \rightarrow 0
\]

is exact. In other words,

\[
(3.1.3) \quad \text{Ext}^1_S(\Omega^1_Z \otimes \mathcal{O}_S, \mathcal{O}_S) \cong \text{Coker} (\alpha \otimes \mathcal{O}_S).
\]

Then it is straightforward to see that

Fact. Let \( \varphi : J/J^2 \rightarrow \mathcal{O}_Z \) be an arbitrary \( \mathcal{O}_Z \)-homomorphism. Then \( \varphi \otimes \mathcal{O}_S \in \text{Im} (\alpha \otimes \mathcal{O}_S) \) if and only if there exist \( \xi_1, \ldots, \xi_N \in \mathcal{O}_S \) such that for all \( f \in J \),

\[
\varphi([f])\big|_S = \sum_{i=1}^{N} \xi_i \frac{\partial f}{\partial x_i} \big|_S \quad \text{on } S.
\]

Since \( J/J^2 = \bigoplus_{i=1}^{r} \mathcal{O}_Z \cdot [f_i] \), the condition above is interpreted in terms of the Jacobian matrix as:

\[
(3.1.4) \quad \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} \big|_S & \cdots & \frac{\partial f_1}{\partial x_N} \big|_S \\
\vdots & \ddots & \vdots \\
\frac{\partial f_r}{\partial x_1} \big|_S & \cdots & \frac{\partial f_r}{\partial x_N} \big|_S 
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_N 
\end{pmatrix}
= \begin{pmatrix}
\varphi(f_1)\big|_S \\
\vdots \\
\varphi(f_r)\big|_S 
\end{pmatrix}
\quad (\exists \xi_1, \ldots, \xi_N \in \mathcal{O}_S).
\]

By the expression (3.1.1), the left-hand side can be rewritten as

\[
(3.1.5) \quad \begin{pmatrix}
0 & \cdots & 0 & g_{1,s+1} & \cdots & g_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & g_{r,s+1} & \cdots & g_{r,N} 
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_N 
\end{pmatrix},
\]

so the condition (3.1.4) is further reduced to

\[
(3.1.6) \quad \begin{pmatrix}
g_{1,s+1} & \cdots & g_{1,N} \\
\vdots & \ddots & \vdots \\
g_{r,s+1} & \cdots & g_{r,N} 
\end{pmatrix}
\begin{pmatrix}
\xi_{s+1} \\
\vdots \\
\xi_N 
\end{pmatrix}
= \begin{pmatrix}
\varphi(f_1)\big|_S \\
\vdots \\
\varphi(f_r)\big|_S 
\end{pmatrix}
\]

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$(\exists (\xi_{s+1}, \ldots, \xi_N) \in \mathcal{O}_S^{\oplus (N-s)})$.

Hence

$\text{Im} (\alpha \otimes \mathcal{O}_S) = (g_{1,s+1}, \ldots, g_{1,N}) \oplus \cdots \oplus (g_{r,s+1}, \ldots, g_{r,N})$

as $\mathcal{O}_S$-sub modules of

$$(J/J^2)^{\vee} \otimes \mathcal{O}_S \simeq \mathcal{O}_S \oplus \cdots \oplus \mathcal{O}_S.$$

This means

$$\text{Ext}_S^1(\Omega^1_Z \otimes \mathcal{O}_S, \mathcal{O}_S) \simeq \bigoplus_{i=1}^r \mathcal{O}_S/(g_{i,s+1}, \ldots, g_{i,N}),$$

as required. □

**Corollary 3.2.** Assume furthermore that $Z$ is a hypersurface singularity, and that $S$ is a smooth curve. Then by a suitable biholomorphic change of coordinates, $S \subset Z \subset V = (\mathbb{C}^N, 0)$ is described as:

$$S = \{x_2 = \cdots = x_N = 0\} \subset Z = \{f(x_1, \ldots, x_N) = 0\} \subset V = (\mathbb{C}^N, 0) = \{(x_1, \ldots, x_N)\},$$

$$f(x_1, \ldots, x_N) = x_N \cdot x_1^e + h(x_1, \ldots, x_N), \quad \text{with } h \in (x_2, \ldots, x_N)^2.$$  

Then

$$\varepsilon_P(Z \supset P) = e. \quad \Box$$

**Remark 3.3.** Mori introduced the numerical invariant $i_P(1)$ in [Mo4] §2 in quite a different manner. The above Corollary 3.2 and [loc.cit] ((2.16) lemma) show however that these coincide with each other, when $Z$ is a terminal Gorenstein 3-fold singularity.

**Corollary 3.4.** Let $(Z, P) \supset S$ be as in Theorem 3.1. Then

1. $\varepsilon_P(Z \supset S) \geq \text{emb. codim}(Z, P)$.

In particular, $\varepsilon_P(Z \supset S) = 0$ if and only if $Z$ is smooth.

2. Let $D$ be a Cartier divisor of $Z$ passing through $P$ such that $T := D \cap S$ is again smooth. Then

$$\epsilon_P(Z \supset S) \geq \epsilon_P(D \supset T).$$

**Proof.** Clear. □

For our purpose the case $(\dim Z, \dim Z) = (4, 2)$ is important. Particularly those with $\varepsilon_P(Z \supset S) = 1$ can be completely determined as follows:
Corollary 3.5. Let \((Z, P) \supset S\) be as in Theorem 3.1. Assume that \((Z, P)\) is an isolated singularity, and
\[ \dim X = 4, \quad \dim S = 2, \quad \varepsilon_P(Z \supset S) = 1. \]
Then
\[ (Z, P) \simeq \{ x_1 x_3 + x_2 x_4 + x_5^m = 0 \} \supset S = \{ x_3 = x_4 = x_5 = 0 \} \quad (m \geq 2). \]

Proof. By Corollary 3.4, the assumption \(\varepsilon_P(Z \supset S) = 1\) implies that \((Z, P)\) should be a hypersurface singularity. Let
\[ (Z, P) \simeq \{ f(x_1, \ldots, x_5) = 0 \} \supset S = \{ x_3 = x_4 = x_5 = 0 \}, \]
\[ f(x_1, \ldots, x_5) = g_3(x_1, x_2) \cdot x_3 + g_4(x_1, x_2) \cdot x_4 + g_5(x_1, x_2) \cdot x_5 + h(x_1, \ldots, x_5), \]
\[ (h \in (x_3, x_4, x_5)^2). \]
Then again by the assumption \(\varepsilon_P(Z \supset S) = 1\) and Theorem 3.1, \((g_3, g_4, g_5) = (x_1, x_2)\) in \(\mathbb{C}\{x_1, x_2\}\). So by a suitable biholomorphic change of the variables \(\{x_1, x_2\}\), we may assume
\[ g_3 = x_1, \quad g_4 = x_2, \quad g_5 \in (x_1, x_2) : \text{arbitrary, i.e.} \]
\[ f(x_1, \ldots, x_5) = x_1 x_3 + x_2 x_4 + g_5(x_1, x_2) \cdot x_5 + h(x_1, \ldots, x_5). \]
Then it is easy to see that after a biholomorphic change of \(\{x_3, x_4, x_5\}\),
\[ f(x_1, \ldots, x_5) = x_1 x_3 + x_2 x_4 + c \cdot x_5^m \quad (c \in \mathbb{C}, \ m \geq 2), \]
where \(c \neq 0\) by the assumption that \((Z, P)\) is isolated. Finally, under the above changes of coordinates the ideal \((x_3, x_4, x_5)\) is not changed, hence \(E = \{ x_3 = x_4 = x_5 = 0 \}. \) \(\square\)

§4. The normal bundle \(N_{E/X}\).

Definition 4.0. Assume that \(X \supset E \overset{\varphi}{\longrightarrow} Y \supset Q\) satisfies the Assumption (A-1), i.e. of Type either (R) or (I).

Let \(E_i\) be an irreducible component of \(E\). \(E_i \simeq \mathbb{P}^2\) (Corollary 2.8). Moreover \(E\) is irreducible: \(E = E_i\) for Type (R) (Corollary 2.10). Let \(I_{E_i}\) be the ideal sheaf of \(E_i\) in \(X\). Define:
\[ N_{E_i/X} := \mathcal{H}om(I_{E_i}/I_{E_i}^2, \mathcal{O}_{E_i}). \]
This is a priori a locally free sheaf of rank 2 on \(E_i\). Moreover
\[ c_1(N_{E_i/X}) = -2. \]
In fact take a general line \(l\) (so that \(\text{Sing } X \cap l = \emptyset\)), then \((-K_X \cdot l) = 1\) (Theorem 1.1), i.e. \(c_1(N_{l/X}) = -1\), so
\[ c_1(N_{E_i/X}) = c_1(N_{E_i/X}|_l) = c_1(N_{l/X}) - c_1(N_{l/E_i}) = -2. \]

The main result of this section is:
Theorem 4.1. Let \( X \supset E \cong \mathbb{P}^2 \xrightarrow{g} Y \ni Q \) be a flipping contraction satisfying the Assumptions (A-1), (A-2), i.e. of Type (R). (See Corollary 2.10.) Assume \( \text{Sing} \ X(\cap E) \neq \emptyset \). Then
\[
N_{E/X} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2).
\]

The proof consists of two steps: Proposition 4.2 and Proposition 4.4.

Proposition 4.2. (Step 1)
Let \( l \subset E \) be any line. Then
\[
N_{E/X} \otimes \mathcal{O}_l \cong \begin{cases} 
\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, & \text{or} \\
\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).
\end{cases}
\]

Proof. Take \( D = D_l \in \mid - K_X \mid \) such that \( D \cap E \) coincides with the given \( l \);
\[
D \cap E = l.
\]

Let \( J_l \) be the ideal of \( l \) in \( D \), and let \( J_l^{(m)} \) be the saturation of \( J_l^m \) in \( \mathcal{O}_D \) (see Mori [Mo4]).

\[
(4.2.0) \begin{cases} 
J_l/J_l^{(2)} \cong (J_l/J_l^2)/\text{Tor} & \text{as } \mathcal{O}_l\text{-modules, and} \\
c_1(J_l/J_l^{(2)}) = 2 - \sum_{P \in l \cap \text{Sing} D} \varepsilon_P(D \supset l)
\end{cases}
\]
(see [loc.cit] §2). Let

\[
(4.2.1) \quad J_l/J_l^{(2)} \cong \mathcal{O}_{\mathbb{P}^1}(a_l) \oplus \mathcal{O}_{\mathbb{P}^1}(b_l) \quad \text{with } \ a_l \leq b_l.
\]

First by the assumption (A-2) with Corollary 2.9,
\[
R^2g_*\mathcal{O}_X(2K_X) = 0.
\]

This combined with the exact sequence;
\[
0 \rightarrow \mathcal{O}_X(2K_X) \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_D(K_X) \rightarrow 0
\]

we get

\[
(4.2.2) \quad R^1(g|_D)_*\mathcal{O}_D(K_X) = 0.
\]

This, the formal function theorem, and the exact sequence
\[
0 \rightarrow J_l^{(m)}/J_l^{(m+1)} \otimes \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_D/J_l^{(m+1)} \otimes \mathcal{O}_X(K_X)
\]
\[
\quad \rightarrow \mathcal{O}_D/J_l^{(m)} \otimes \mathcal{O}_X(K_X) \rightarrow 0
\]
yield

\[ H^1(l, J_l/J_l^{(2)} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0. \]

(Recall \( \mathcal{O}_l(K_X) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \).) Hence

\[
\begin{cases}
  a_l \geq 0 & \text{in (4.2.1), and} \\
  \text{in particular} & c_1(J_l/J_l^{(2)}) \geq 0.
\end{cases}
\]

Finally,

(4.2.4) There is a natural injection

\[ J_l/J_l^{(2)} \hookrightarrow (N_{E/X})^\vee \otimes \mathcal{O}_l \]

(which is an isomorphism if \( l \cap \text{Sing} \ X = \emptyset \)).

In fact, let \( \delta_{E/X} : I_E/I_E^2 \rightarrow \Omega^1_X \otimes \mathcal{O}_E \), \( \delta_{l/D} : J_l/J_l^2 \rightarrow \Omega^1_D \otimes \mathcal{O}_l \) be the natural homomorphisms, then \( \text{Im} \, \delta_{l/D} \simeq (\text{Im} \, \delta_{E/X}) \otimes \mathcal{O}_l \), and so

\[
J_l/J_l^{(2)} \simeq (\text{Im} \, \delta_{l/D})^{\vee \vee} \simeq ((\text{Im} \, \delta_{E/X}) \otimes \mathcal{O}_l)^{\vee \vee} \\
\hookrightarrow ((\text{Im} \, \delta_{E/X})^\vee \otimes \mathcal{O}_l)^\vee \simeq (N_{E/X} \otimes \mathcal{O}_l)^\vee \simeq N_{E/X}^\vee \otimes \mathcal{O}_l.
\]

Hence (4.2.4). By (4.0.1), (4.2.3) and (4.2.4),

\[ (N_{E/X})^\vee \otimes \mathcal{O}_l \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \text{ or } \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \]

i.e.

\[ N_{E/X} \otimes \mathcal{O}_l \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \text{ or } \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2). \]

Corollary 4.3. (cf. §Appendix.)

Let \( X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q \) be of Type (R). Then

(1) For each singular point \( P \in \text{Sing} \ X \),

\[ \text{emb. codim}(X, P) \leq 2. \]

(2) If \( \text{emb. codim}(X, P) = 2 \), then there is no other singular point of \( X \) on \( E \):

\[ \text{Sing} \ X = \{ P \}. \]

Proof. (1) Take a general \( D \in |-K_X| \) passing through \( P \), and \( l := D \cap E \), then since \( X \) is Gorenstein i.e. \( D \) is Cartier,
\[ \text{emb.codim} (X, P) = \text{emb.codim} (D, P) \]

\[ \leq \varepsilon_P (D \supset l) \quad \text{(Corollary 3.4)} \]

\[ \leq \sum_{P' \in l \cap \text{Sing} D} \varepsilon_{P'} (D \supset l) \]

\[ = 2 - c_1(J_l/J_l^{(2)}) \quad \text{(4.2.0)} \]

\[ \leq 2. \quad \text{(4.2.3)} \]

2. Assume \( \text{emb.codim} (X, P) = 2 \). Then

(4.3.2) All the inequalities in (4.3.1) must be the equality.

If there exists another singular point \( P' \in \text{Sing} X \), then by taking a general \( D \in |−K_X| \) so that \( D \ni P, P' \), we have \( \varepsilon_{P'} (D \supset l) = 0 \) (4.3.2). This means that \( (D, P') \) is smooth (Corollary 3.4 (2)), hence so is \( (X, P') \), a contradiction. \( \square \)

The second step is:

**Proposition 4.4.** (Step 2)

In Proposition 4.2, \( N_{E/X} \otimes O_t \simeq O_{\mathbb{P}^1} (-1)^{\oplus 2} \) is impossible.

The proof requires the following theorem which is a generalization of Yo. Namikawa’s local moduli [Nam3] (see also Remark 4.9 below):

**Theorem 4.5.** (see also /loc.cit/ §1)

Let \( U \xrightarrow{\varphi} V = \{ U_t \xrightarrow{\varphi_t} V_t \}_{t \in \Delta (t)} \) be a proper bimeromorphic morphism over \( \Delta (t) = \{ t \in \mathbb{C} \mid |t| < 1 \} \) between 4-dimensional normal analytic spaces \( U, V \). Let

\[ C_t := \text{Exc} \varphi_t. \]

Assume the following conditions:

1. \( C_t \simeq \mathbb{P}^1 \) (\( \forall t \in \Delta (t) \)),
2. \( (K_{U_t} \cdot C_t) = 0 \) (\( \forall t \in \Delta (t) \)),
3. \( U \) has only isolated terminal complete intersection singularities such that \( \emptyset \neq \text{Sing} U \subset C_0 \).

Then for \( t \neq 0 \),

\[ N_{C_t/U_t} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} (-2) \text{ or } O_{\mathbb{P}^1} (1) \oplus O_{\mathbb{P}^1} (-3). \]

**Proof.** Assume \( N_{C_t/U_t} \simeq O_{\mathbb{P}^1} (-1)^{\oplus 2} \) to derive a contradiction. (See (5.1.2).)
Since \( R^i \varphi_* \mathcal{O}_U = 0 \) (\( i \geq 1 \)), by Kollár–Mori [KoMo] we can proceed the deformation theory of the morphism \( \varphi : U \to V \) formulated by Z. Ran [Ra1]. (For details see §6, especially 6.2 below.) Since \( R^2 \varphi_* T_U = \text{Ext}^2_{\mathcal{O}_U}(\Omega^1_U, \mathcal{O}_U) = 0 \),

(4.5.0) The natural holomorphic map

\[
\text{Def} U \longrightarrow \prod_{P \in \text{Sing } U} \text{Def}(U, P)
\]

is surjective (see 6.2).

Let

(4.5.1) \( \overline{C^0} := \bigcup_{t \in \Delta(t)} C_t = \text{Exc } \varphi \).

(4.5.2) Pick \( P \in \text{Sing } U \) arbitrarily, and write locally

\[
(U, P) \simeq \{ f_1(x_1, \ldots, x_N) = \ldots = f_{N-4}(x_1, \ldots, x_N) = 0 \} \supset \overline{C^0} = \{ x_3 = \ldots = x_N = 0 \}
\]

in irredundant form \( \text{i.e. } \text{emb.codim } (U, P) = N - 4 \), where \( N = 5 \) or 6 (Corollary 4.3).

(4.5.3) Consider a local deformation \( \Delta(s) \to \text{Def } (U, P) \) of \( (U, P) \) by

\[
s \mapsto \{ f_1(x) + s = f_2(x) = \ldots = f_N(x) = 0 \}.
\]

(4.5.4) Then by (4.5.0) this yields a deformation of \( U : \Delta(s) \to \text{Def } U \). This furthermore produces a deformation of \( \varphi : \)

\[
U \xrightarrow{\psi} V = \{ U^s \xrightarrow{\psi^s} V^s \}_{s \in \Delta(s)}
\]

\( (\psi^0 = \varphi, U^0 = U, V^0 = V) \).

Let us consider the relative Hilbert scheme:

\[
H := \text{Hilb}_{U/V/\Delta(s), [\overline{C^0}]} \xrightarrow{\lambda} \Delta(s)
\]

(see Kollár–Miyaoka–Mori [KoMiMo2,3]).

Claim. \( H \) is irreducible and \( \lambda \) is an isomorphism.

Proof. Let us consider first

\[
H_1 := \text{Hilb}_{U/V/\Delta(s), [C_t]} \xrightarrow{\mu} \Delta(s).
\]

\( \mu^{-1}(0) \) just parametrizes \( \{ C_t \}_{t \in \Delta(t)} \). Since \( (K_U \cdot C_t) = 0 \) (\( \forall t \in \Delta(t) \)) (assumption (2) of the theorem),

(4.5.5) \( \dim H_1, [C_t] \geq \dim U + (-K_U \cdot C_t) - \dim \text{Aut } \mathbb{P}^1 + \dim \Delta(s) = 2 \).
(Theorem 1.2). Moreover since we assumed that \( N_{C_t/U} \cong O_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus O_{\mathbb{P}^1} \) (\( t \neq 0 \)),
\( H^1(N_{C_t/U}) = 0 \). Hence the equality holds in (4.5.5) for those \([C_t]\)'s (\( t \neq 0 \)). In more precise
(4.5.6) There exists a closed analytic subset \( A \subseteq H_1 \) (may contain a whole irreducible component of \( H_1 \)) such that
\[
\begin{cases}
A \cap \mu^{-1}(0) \subset \{[C_0]\}, \\
H_1 - A \text{ is an irreducible smooth surface, and} \\
\mu|_{H_1 - A} : H_1 - A \to \Delta(s) \text{ has connected fibers.}
\end{cases}
\]

We are going to prove that \( H_1 \) is irreducible. So suppose on the contrary that \( H_1 \) is reducible:
\[H_1 = H_1^{pr} \cup H'_1 \cup \ldots\]
where \( H_1^{pr} \) is the irreducible component containing \( H_1 - A \) (4.5.6) as an open dense subset, and \( H'_1 \) one another. Then \( H'_1 \cap \mu^{-1}(0) = \{[C_0]\} \) by (4.5.6), so each \( H'_1 \) must dominate \( \Delta(s) \). By the upper-semi-continuity of fiber dimensions, \( H_1 \) is purely of dimension 1. This is however impossible, since for \([C^s] \in H'_1 \) lying over \( s \in \Delta(s) - \{0\} \), we have \( \dim H_1, [C^s] \geq 2 \) by the same reason as in (4.5.5), a contradiction.

Hence \( H_1 \) must be irreducible. By this, together with (4.5.6), we get
\[
\begin{cases}
H_1 \text{ is an irreducible surface,} \\
\text{Sing } H_1 \cap \mu^{-1}(0) \subset \{[C_0]\}, \text{ and} \\
H_1 \xrightarrow{\mu} \Delta(s) \text{ has connected fibers.}
\end{cases}
\]

Let \( h \) and \( p \) be the universal family over \( H_1 \) and the natural projection, respectively;
\[
\begin{array}{ccc}
\mathcal{H}_1 & \xrightarrow{p} & \mathcal{U} \\
\downarrow h & & \downarrow \\
H_1 & \xrightarrow{\mu} & \Delta(s)
\end{array}
\]
and let
\[\overline{C^s} := p(h^{-1} \mu^{-1}(s)).\]
(Note that \( \overline{C^s} \) coincides with the given \( \overline{C^s} \) (4.5.1) when \( s = 0 \).) Then \([\overline{C^s}] \in H = \text{Hilb}_{\mathcal{U}/\mathcal{V}/\Delta(s), \overline{C^s}}\) and we can define a morphism
\[H_1 \to H \quad \text{(over } \Delta(s)\text{)}\]
by sending the class of a curve (\( \cong \mathbb{P}^1 \)) lying over \( \mu^{-1}(s) \) to \([\overline{C^s}]\). Since \( H_1 \) was an irreducible surface (4.5.7), \( H \) must be an irreducible curve. Moreover, \( H \) birationally dominates \( \Delta(s) \) through \( \lambda \) (4.5.7). Hence \( H \) is smooth and \( \lambda \) is an isomorphism.
(Proof of the Claim completed.)

\((4.5.8)\) Proof of Theorem 4.5 continued.

By the above Claim, \(\overline{C^0}\) forms a flat family where \(\overline{C^0}\) is also reduced. Recall that we gave a deformation \(\mathcal{U}\) of \((U, P)\) \((4.5.3)\) as follows:

\[
\mathcal{U} \ni \{f_1(x_1, \ldots, x_N) + s = f_2(x_1, \ldots, x_N) = \ldots = f_{N-4}(x_1, \ldots, x_N) = 0\}
\]

\[
\supset \overline{C^0} = \{x_3 = \ldots = x_N = s = 0\}.
\]

Let \(I_s\) be the ideal of \(\overline{C^s}\) in \(\mathcal{O}_{U, P} = \mathbb{C}\{x_1, \ldots, x_N, s\}/(f_1 + s, f_2, \ldots, f_{N-4})\) (or rather its lift to \(\mathbb{C}\{x_1, \ldots, x_N, s\}\)). Then \(I_0 = (x_3, \ldots, x_N)\) by the Claim, and hence

\[
I_s = (x_3 + s \cdot e_3(x, s), \ldots, x_N + s \cdot e_N(x, s)) \quad (\exists e_i(x, s) \in \mathbb{C}\{x, s\}).
\]

By the condition \(f_1(x) + s \in I_s\),

\[
f_1(x) + s = \xi_3(x, s)(x_3 + s \cdot e_3(x, s)) + \ldots + \xi_{N-1}(x, s)(x_{N-1} + s \cdot e_{N-1}(x, s)) + \xi_N(x, s)(x_N + s \cdot e_N(x, s))
\]

\[\quad (\exists \xi_i(x, s) \in \mathbb{C}\{x, s\})\]

for at least one \(\xi\), say \(\xi = N, e_N \cdot \xi_N \in \mathbb{C}\{x, s\}^\times\). In particular \(\{f_1(x) = 0\} \subset (\mathbb{C}^N, 0)\) defines a smooth germ, which contradicts the assumption that \((U, P) = \{f_1(x) = \ldots = f_{N-4}(x) = 0\}\) is an irredundant expression. Hence we are done. \(\square\)

\(4.6.\) Proof of Theorem 4.5 \(\Longrightarrow\) Proposition 4.4.

Let \(D_0 \in \mid - K_X\mid\) be a member passing through a singular point, say \(P\), of \(X\), and let \(l_0 := D_0 \cap E\). Now those \(l\)'s such that \(N_{E/X} \otimes \mathcal{O}_l \simeq \mathcal{O}_{\mathbb{P}^1(-1)^{\oplus 2}}\) consist of a Zariski open subset, say \(W\), of \((\mathbb{P}^2)^\vee\) (see [OSS]). Assume \(W \neq \emptyset\), to derive a contradiction.

If we take a smooth conic \(B\) in \(E\) tangent to \(l_0\) general enough (so that \(B \not\subseteq P\)), then

\((4.6.1)\) We can take a family \(\{D_t\}_{t \in \Delta}\) of members of \(\mid - K_X\mid\) tangent to \(B\) such that for each \(l_t := D_t \cap E \quad (t \neq 0)\),

\[
l_t \cap \text{Sing} \; X = \emptyset, \quad \text{and} \; [l_t] \in W.
\]

This provides a simultaneous contraction

\[
U \xrightarrow{\varphi} V = \{D_t \xrightarrow{g|_{D_t}} g(D_t)\}_{t \in \Delta}
\]

satisfying the assumptions of Theorem 4.4. Hence \(N_{l_t/D_t} \simeq N_{E/X} \otimes \mathcal{O}_{l_t}\) should not be \(\mathcal{O}_{\mathbb{P}^1(-1)^{\oplus 2}}\) by Theorem 4.4, which contradicts \([l_t] \in W\) \((4.6.1)\). \(\square\)

\(4.7.\) Proof of Theorem 4.1.
Now the vector bundle $N_{E/X}$ of rank 2 on $E \simeq \mathbb{P}^2$ has the property:

$$N_{E/X} \otimes O_l \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2) \quad (\forall l : \text{line in } E).$$

Hence by Van de Ven’s characterization theorem [V] of uniform bundles,

$$N_{E/X} \simeq O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(-2). \quad \Box$$

Finally we shall give some examples of families as in Theorem 4.5, in the case

$$N_{C_1/U_t} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2).$$

**Example 4.8.** (cf. 5.2 below.)

(1) Case $\#\text{Sing } U = 1$ (Yo. Namikawa)

$$V := \{x_1x_2 + x_3^3 + x_3x_4^2 + t \cdot x_3^2 = 0\} \quad (\subset (\mathbb{C}^5, 0) = \{(x_1, \ldots, x_4, t)\}),$$

$\varphi : U \to V$ is the blow-up with the codimension 1 center:

$$F := \{x_1 = x_3 = 0\} \subset V.$$

$\text{Sing } U_0$ is one point which is an ordinary double point.

(2) Case $\#\text{Sing } U = 2$ (jointly with Yo. Namikawa)

$$V := \{x_1x_2 + x_3(t + x_3)^2 - x_4^{2n} = 0\}.$$

$\varphi : U \to V$ is the blow-up with the center:

$$F := \{x_1 = x_3(t + x_3) + x_4^n = 0\} \subset V.$$

$\text{Sing } U_0$ consists of two points both of which are isomorphic to:

$$\{XY + Z^2 + W^n = 0\}. \quad —$$

**Remark 4.9.** Note that in Theorem 4.5 we did not assume anything about the singularities of the 3-fold $U_0$. What we assumed is just that of the 4-fold $U$ (assumption (3) there).

One of the disadvantages in 4-dimensional birational geometry is that one cannot tell much about the definite structures of terminal singularities, even for hypersurface ones, as opposed to dimension 3. This is considered to be one of the principal reasons that a 4-dimensional algebraic variety seems too complicated to handle with. As an evidence, there in fact exists an example of 4-dimensional terminal hypersurface
singularity whose general hyperplane-section is a ‘K3-singularity’ (M. Reid [Re4]). (See also §Appendix.)

Suppose thus in Theorem 4.5 that \( U_0 \) has only cDV-singularities. Then the conclusion of the theorem has been known as a special case of Namikawa’s local moduli [Nam3], whose proof had been based much on the description of the versal deformations of Du Val singularities (Brieskorn [Br], cf. [Pi1]). In the general case however his method could not be applicable, and what we proceeded instead is the deformation theory for contractions \( \varphi \) (see the proof). Actually, this methodology does not require any particular kind of assumptions on the defining equations of given singularities, but rather makes us possible to advance under enough generality.

This idea indicates a prospect to overcome difficulties arising from the complexity of 4-dimensional singularities.

§5. Widths (after M. Reid).

In this section we extend the notion of widths introduced by M. Reid [Re1] for 3-fold flopping contractions ((5.1.5) for the definition) to our 4-fold flipping contractions (Definition 5.5).

First we recall M. Reid’s Pagoda:

5.1. (Reid’s “Pagoda” [loc.cit])

Let \( U_0 \rightarrow V_0 \) be a proper bimeromorphic morphism from a 3-fold \( U_0 \) with only terminal singularities to a germ of a normal 3-fold singularity \((V_0, Q_0)\), satisfying:

\[
\begin{align*}
\{ C_0 := \text{Exc} \varphi_0 &\cong \mathbb{P}^1 \ (\text{so that } \varphi_0(C_0) = \{Q_0\}), \quad \text{and} \\
(K_{U_0} \cdot C_0) &\equiv 0.
\end{align*}
\]

This is called a 3-fold flopping contraction. Necessarily \((V_0, Q_0)\) is a terminal singularity. Write it simply

\[
U_0 \supset C_0 \cong \mathbb{P}^1 \xrightarrow{\varphi_0} V_0 \ni Q_0.
\]

(5.1.2) If we further assume that \( U_0 \) is smooth, then

\[
N_{C_0/U_0} \cong \begin{cases} 
\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, \\
\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2), \text{ or} \\
\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3).
\end{cases}
\]

(This is just because \( c_1(N_{C_0/U_0}) = 2 \), and \( H^1(N_{C_0/U_0}^\vee) = 0. \))

We call then \( C_0 \) a \((-1, -1)\)-curve, \((0, -2)\)-curve, \((1, -3)\)-curve in \( U_0 \), respectively.

(5.1.3) From now on we specifically consider the case

\[
N_{C_0/U_0} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).
\]
(cf. Laufer [L], Pinkham [Pi1], Katz–Morrison [KaMo] and Kawamata [Kaw9] for
(1, −3)-curve case.)

(5.1.4) (First step, upwards)
Blow up $U_0$ with the center $C_0$;

$$\varphi^{(1)} : U^{(1)} \rightarrow U_0.$$ 

Then $F^{(1)} := \text{Exc } \varphi^{(1)} \simeq \Sigma_2$ (the second Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$), and the negative section $C^{(1)} \subset F^{(1)}$ has the normal bundle either

$$N_{C^{(1)}/U^{(1)}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, \text{ or } \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

In the former case we shall stop the procedure, while in the latter case, blow $U^{(1)}$ up again with the center $C^{(1)}$:

$$\varphi^{(2)} : U^{(2)} \rightarrow U^{(1)}.$$ 

Then $F^{(2)} := \text{Exc } \varphi^{(2)} \simeq \Sigma_2$, and let $C^{(2)}$ be the negative section of it. Then $N_{C^{(2)}/U^{(2)}}$ should again be either $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. In the latter case repeat the process, then after a finitely many steps;

$$\varphi^{(m-1)} : U^{(m-1)} \rightarrow U^{(m-2)} \ (\text{with } F^{(m-1)} := \text{Exc } \varphi^{(m-1)}),$$

we eventually arrive

$$N_{C^{(m-1)}/U^{(m-1)}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}.$$ 

(5.1.5) (Definition of widths [loc.cit])

Define:

width $\varphi_0$ (or width$_{U_0} C_0$) := $m \ (\geq 2).$

(5.1.6) (Second step, the roof)

Blow $U^{(m-1)}$ up once again with the center $C^{(m-1)}$: $\varphi^{(m)} : U^{(m)} \rightarrow U^{(m-1)}$. Then $F^{(m)} := \text{Exc } \varphi^{(m)} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (i.e. no more $\Sigma_2$). Since $N_{C^{(m-1)}/U^{(m-1)}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ is generated by global sections, $U^{(m)}$ has the contraction to the opposite direction, namely, there exists a proper bimeromorphic morphism

$$\varphi^{(m)} + : U^{(m)} \rightarrow U^{(m-1)} +$$

with $F^{(m)} + := \text{Exc } \varphi^{(m)} + = F^{(m)}$, $\varphi^{(m)} + (F^{(m)} +) \simeq \mathbb{P}^1$, and the fibers of $\varphi^{(m)} +$ is not linearly equivalent in $F^{(m)}$ to those of $\varphi^{(m)}$.

(5.1.7) (Third step, downwards)

Then the proper transform $F^{(m-1)} +$ of $\text{Exc } \varphi^{(m-1)}$ to $U^{(m-1)} +$ is still isomorphic to $\Sigma_2$, and is contractible to $\mathbb{P}^1$ (induction on $m$). Contract it down:

$$\varphi^{(m-1)} + : U^{(m-1)} + \rightarrow U^{(m-2)} +.$$
Consider the proper transform $F^{(m-2)} + F^{(m-2)}$ to $U^{(m-2)} +$, and so on.

After $m$ steps of these contraction procedures, we arrive

$$\varphi^{(1)} : U^{(1)} + \rightarrow U_0^+,$$

where $U_0^+$ is an neighborhood of $C_0^+ \simeq \mathbb{P}^1$, with $N_{C_0^+/U_0^+} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Finally this is contractible to the original basement $(V_0, Q_0)$, since everything in the above procedures is defined over $(V_0, Q_0)$:

$$U_0^+ \supset C_0^+ \simeq \mathbb{P}^1 \varphi_0^+ \rightarrow V_0 \ni Q_0.$$

We call this $\varphi_0^+$ the flop of $\varphi_0$. Also we call this whole operation producing $\varphi_0^+$ out of $\varphi_0$ the flop.

The following observation is due also to J. Kollár [Koll]:

**Fact 5.2.** Let $U_0 \supset C_0 \simeq \mathbb{P}^1 \varphi_0 \rightarrow V_0 \ni Q_0$ be as above, with the assumption (5.1.3). Let $m = \text{width } \varphi_0$. Then

$$(V_0, Q_0) \simeq \{xy + z^2 - w^{2m} = 0\} \subset (\mathbb{C}^4, 0) = \{(x, y, z, w)\},$$

and $\varphi_0$, $\varphi_0^+$ is the blow-up of $U_0$, $U_0^+$ with the codimension 1 center

$$\{x = z + w^m = 0\}, \quad \{x = z - w^m = 0\},$$

respectively.

**Remark 5.3.** It is convenient to define the width also for $(-1, -1)$-curves. Namely, for a contraction $U_0 \supset C_0 \simeq \mathbb{P}^1 \varphi_0 \rightarrow V_0 \ni Q_0$ with $N_{C_0/U_0} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$,

$$\text{width } \varphi_0 := 1.$$

This invariant width has an interpretation from quite different point of view:

**Theorem 5.4.** (Yo. Namikawa [Nam3])

(1) Let $U_0 \supset C_0 \simeq \mathbb{P}^1 \varphi_0 \rightarrow V_0 \ni Q_0$ be a contraction as in 5.1. Then there exists an unique number $m$, and a 1-parameter deformation

$$U \varphi \rightarrow V = \{U_t \varphi_t \rightarrow V_t\}_{t \in \Delta}$$

of $U_0 \varphi_0 \rightarrow V_0$ such that for $t \neq 0$, $\varphi_t$ is a contraction of $m$ disjoint $(-1, -1)$-curves.

(2) Moreover, if $N_{C_0/U_0} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, then this number $m$ coincides with the width of $\varphi_0$ (5.1.5):

$$m = \text{width } \varphi_0. \quad \Box$$

Now we are ready to define the width also for our original 4-fold flipping contrac-


Definition 5.5. (widths for dimension 4)
Let \( X \supset E \simeq P^2 \overset{g}{\to} Y \ni Q \) be a flipping contraction with the Assumption A, i.e. of Type (R). Take a general smooth \( D \in | - K_X| \), then by Theorem 4.1 (or Proposition 4.2 and 4.4)
\[
g_D : D \ni l \to g(D) \ni Q
\]
(\( l := D \cap E \)) gives a contraction of the \((0, -2)\)-curve \( l \) (or \((-1, -1)\)-curve only if \( X \) is smooth [Kaw4]). Define
\[
\text{width } g := \text{width}(g_D) \text{ in the sense of (5.1.5)}.
\]

The following is a paraphrase of Proposition 4.4 in terms of widths:

Corollary 5.6. (Characterization of the Kawamata flip [Kaw4])
Let \( X \supset E \simeq P^2 \overset{g}{\to} Y \ni Q \) be a flipping contraction of Type (R). Then
\[
X \text{ is smooth } \iff \text{ width } g = 1. \quad \square
\]

In §4 we determined the normal bundle \( N_{E/X} \) (Theorem 4.1). This provides us the information on the embedding \( E \subset X \) in the first order level, while the width actually measures that in higher orders, and these supplement to each other toward the investigation of the contraction \( g \).

§6. Deformations of contractions \( g : X \to Y \).

This section is devoted to the study of deformations of \( X \supset E \simeq P^2 \overset{g}{\to} Y \ni Q \) of Type (R). The main result of this section is:

Theorem 6.1. (Globalizability of local deformations of \( \text{Sing } X \) for Type (R))
Assume that \( X \supset E \simeq P^2 \overset{g}{\to} Y \ni Q \) is a flipping contraction of Type (R). Let \( \{P_1, \ldots, P_r\} := \text{Sing } X(\neq \emptyset) \), let \( U_i \) be a sufficiently small Stein open neighborhood of \( P_i \) in \( X \), and let \( \{U_i, t\}_{t \in \Delta} \) be any small deformation of \( (U_i, P_i) \) (\( i = 1, \ldots, r \)). Then there exists a 1-parameter deformation
\[
\mathcal{X} \to \mathcal{Y} = \{X_t \overset{g_t}{\to} Y_t\}_{t \in \Delta}
\]
\((X_0 \overset{g_0}{\to} Y_0) = (X \overset{g}{\to} Y)\) satisfying the following conditions:
(1) For sufficiently small Stein open neighborhood \( U_i \) of \( P_i \) in \( \mathcal{X} \), \( U_i \to \Delta \) coincides with the given deformation \( \{U_i, t\}_{t \in \Delta} \) (\( i = 1, \ldots, r \)), and
(2) \( g_t \) is again a flipping contraction of Type (R) (\( \forall t \)) (where \( E_t := \text{Exc } g_t \) may not be irreducible).

6.2. Since \( R^i g_* O_X = 0 \) (\( i = 1, 2 \)), if we give a deformation \( \mathcal{X} \to \Delta \) of \( X \), then by Kollár–Mori ([KoMo] 11.4), there exists a deformation \( \mathcal{Y} \to \Delta \) of \( Y \) and a proper bimeromorphic morphism \( \mathcal{X} \to \mathcal{Y} \) over \( \Delta \) which extends \( g : X \to Y \).
Let \( \mathcal{T}_X := \mathcal{H}om_X(\Omega^1_X, \mathcal{O}_X) \) be the tangent sheaf of \( X \). Note that since \( X \) is a sufficiently small analytic neighborhood of \( E \),

\[
\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) = H^i(\mathbb{R}g_*\mathcal{H}om_X(\Omega^1_X, \mathcal{O}_X)),
\]

which is the abutment of the spectral sequence:

\[
E^{p,q}_2 = R^pg_*\text{Ext}^q_X(\Omega^1_X, \mathcal{O}_X) \implies E^{p+q} = \text{Ext}^{p+q}_X(\Omega^1_X, \mathcal{O}_X).
\]

Let us look at the edge sequence:

\[
(6.2.1) \quad 0 \to R^1g_*\mathcal{T}_X \to \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \to g_*\text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \to R^2g_*\mathcal{T}_X \to \text{Ext}^2_X(\Omega^1_X, \mathcal{O}_X)
\]

By Z. Ran \cite{Ra1} (after the smooth case of Kodaira \cite{Kod} and Horikawa \cite{Ho1,2,3}), (6.2.2) describes the deformation of \( g : X \to Y \) (cf. \cite{Sc}, \cite{Fri}, \cite{Nam1,2,3}, \cite{G}, \cite{NS}).

Let \( \{P_1, \ldots, P_r\} \) be as in Theorem 6.1. Toward the existence of a global deformation \( \mathcal{X} \) of \( X \) as in Theorem 6.1, it is sufficient to prove:

\[
\text{Ext}^2_X(\Omega^1_X, \mathcal{O}_X) = 0, \quad \text{which implies the smoothness of the global deformation space } \text{Def } X, \text{ and}
\]

\[
(6.2.4) \quad \text{The surjectivity of } \text{Ext}^1_X(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\alpha} g_*\text{Ext}_X^1(\Omega^1_X, \mathcal{O}_X),
\]

which (together with (6.2.3)) implies the surjectivity of the natural morphism:

\[
\text{Def } X \to \prod_{i=1}^r \text{Def}(X, P_i).
\]

First we shall give the following:

\textbf{Lemma 6.3.} \quad \text{The exact sequence (6.2.1) can be extended as}

\[
R^2g_*\mathcal{T}_X \to \text{Ext}^2_X(\Omega^1_X, \mathcal{O}_X) \to g_*\text{Ext}^2_X(\Omega^1_X, \mathcal{O}_X) \to 0.
\]

\textbf{Proof.} \quad \text{Let us consider the spectral sequence (6.2.2). If } q \geq 1, \text{ then Supp } \text{Ext}_X^q(\Omega^1_X, \mathcal{O}_X) \subset \text{Sing } X, \text{ so by the assumption dim}(\text{Sing } X \cap E) \leq 0,
\]

\[
E^{p,q}_2 = 0 \quad (\forall p \geq 1, \forall q \geq 1).
\]

On the other hand, since \( g \) has at most a 2-dimensional fiber,

\[
E^{p,0}_2 = 0 \quad (\forall p \geq 3).
\]

Let \( F^r(E^r) \) be the associated filtration of \( E^r \). Recall that the homomorphism \( E^{2,0}_2 \to E^2 \) in the edge sequence is constructed as

\[
E^{2,0}_2 \to E^{2,0}_\infty = F^2(E^2) \subset E^2.
\]

Since \( E^{1,1}_2 = 0 \) (6.3.1), \( E^{1,1}_\infty = \text{Gr}^1(E^2) = 0 \), that is, \( F^2(E^2) = F^1(E^2) \). Thus

\[
\text{Coker } [E^{2,0}_2 \to E^2] \simeq \text{Gr}^0(E^2) = E^{0,2}_\infty \simeq E^{0,2}_3.
\]

Moreover \( E^{2,1}_2 = 0 \) (6.3.1), so \( E^{3,2}_3 \simeq E^{0,2}_2 \):

\[
(6.3.3) \quad \text{Coker } [E^{2,0}_2 \to E^2] \simeq E^{0,2}_2,
\]

which proves our lemma. \( \square \)
Lemma 6.4. \[ \mathcal{E}xt^2_X(\Omega^1_X, \mathcal{O}_X) = 0. \]

Hence by Lemma 6.3, \( R^2 g_* \mathcal{T}_X \rightarrow \mathcal{E}xt^2_X(\Omega^1_X, \mathcal{O}_X) \) is surjective.

Proof. (See also Kollár [Kol2])

Let \( P_i \in S \times X \cap E \) be any point. By the assumption, there exists a local immersion \((X, P_i) \subset (\mathbb{C}^N, 0) =: V\) which is of complete intersection. Let \( J \) be the ideal of \( X \) in \( V \), then both \( \Omega^1_V \otimes \mathcal{O}_X \) and \( J/J^2 \) are free \( \mathcal{O}_{(X, P)} \)-modules, and hence \( \mathcal{E}xt^1_X(J/J^2, \mathcal{O}_X) = \mathcal{E}xt^2_X(\Omega^1_V \otimes \mathcal{O}_X, \mathcal{O}_X) = 0 \). By the exact sequence

\[
0 \rightarrow J/J^2 \rightarrow \Omega^1_V \otimes \mathcal{O}_X \rightarrow \Omega^1_X \rightarrow 0
\]

we have \( \mathcal{E}xt^2_X(\Omega^1_X, \mathcal{O}_X) = 0. \) \( \square \)

Lemma 6.5. For \( g \) of Type \((R)\), \( (6.2.3) \) and \((6.2.4)\) hold.

Proof. By Lemma 6.3 and 6.4, \((6.2.3)\) and \((6.2.4)\) are reduced to

\[
R^2 g_* \mathcal{T}_X = 0. \tag{6.5.1}
\]

By the formal function theorem,

\[
(R^2 g_* \mathcal{T}_X)^\sim = \varprojlim H^2(\mathcal{O}_X/I^n_E \otimes \mathcal{T}_X).
\]

This, combined with;

\[
0 \rightarrow I^n_E/I^{n+1}_E \otimes \mathcal{T}_X \rightarrow \mathcal{O}_X/I^{n+1}_E \otimes \mathcal{T}_X \rightarrow \mathcal{O}_X/I^n_E \otimes \mathcal{T}_X \rightarrow 0
\]

it is enough to show

\[
H^2(I^n_E/I^{n+1}_E \otimes \mathcal{T}_X) = 0 \quad (\forall n \geq 0). \tag{6.5.2}
\]

Since there are homomorphisms

\[
I^n_E/I^{n+1}_E \rightarrow (I^n_E/I^{n+1}_E)^\vee \vee \quad \text{and} \quad S^n(N^V_1) / (I^n_E/I^{n+1}_E)^\vee \vee
\]

which are isomorphisms outside the finite set of points \( \{P_1, \ldots, P_m\} = \text{Sing } X \), \( (6.5.2) \) is further reduced to

\[
H^2(E, S^n(N^1_1) \otimes \mathcal{T}_X) = 0 \quad (\forall n \geq 0). \tag{6.5.3}
\]

The left-hand side is Serre dual to

\[
H^0(E, S^n(N^1_1 \otimes \mathcal{T}_X \otimes \mathcal{O}_E) \otimes \omega_E). \tag{6.5.4}
\]

We claim that this certainly vanishes.

Take a general line \( l \subset E \). Since for any locally free \( \mathcal{O}_E \)-module \( \mathcal{F} \), \( H^0(l, \mathcal{F} \otimes \mathcal{O}_l) = 0 \) implies \( H^0(E, \mathcal{F}) = 0 \), and since

\[
(\mathcal{T}_X \otimes \mathcal{O}_E)^\vee \otimes \mathcal{O}_l \simeq (\Omega^1_X \otimes \mathcal{O}_E)^\vee \otimes \mathcal{O}_l,
\]

it is enough to prove:

\[
H^0(l, S^n(N^1_1 \otimes \mathcal{O}_E)^\vee \otimes \omega_E \otimes \mathcal{O}_l) = 0. \tag{6.5.5}
\]

Since \( N^1_X \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \) (Theorem 4.1), this in fact follows from the exact sequence;

\[
0 \rightarrow N^V_1 \rightarrow (\Omega^1_X \otimes \mathcal{O}_E)^\vee \rightarrow \Omega^1_E \rightarrow 0
\]

tensorized with \( S^n(N^1_1 \otimes \omega_E \otimes \mathcal{O}_l) \simeq S^n(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes \mathcal{O}_{\mathbb{P}^1}(-3). \) \( \square \)
6.6. Proof of Theorem 6.1.

The remaining thing we have to prove is that $g_t : X_t \to Y_t$ again gives a flipping contraction of Type (R).

(6.6.0) To begin with, $Y_t$ has only isolated rational singularities by Elkik [E], and hence is normal.

Let

$$H := \text{Hilb}_{X/Y/\Delta, [E]} \xrightarrow{\lambda} \Delta.$$ (6.6.1)

We claim that $H$ is purely 1-dimensional. First take any line $l \subset E$ and consider

$$H_1 := \text{Hilb}_{X/Y/\Delta, [l]} \xrightarrow{\mu} \Delta.$$ (6.6.2)

and

$$\dim H_1 \geq \dim X + (-K_X \cdot l) - \dim \text{Aut} \mathbb{P}^1 + \dim \Delta = 3.$$ (6.6.3)

Hence for any $t \in \Delta$, $\dim \mu^{-1}(t) \geq 2$. By the upper-semi-continuity of fiber dimensions of $\mu$ and (6.6.2) we must have

$$\dim \mu^{-1}(t) = 2 \quad (\forall t \in \Delta).$$ (6.6.4)

In particular, $g_t$ is not an isomorphism and $\dim E_t \geq 2$, where $E_t := \text{Exc} g_t$. Again by the upper-semi-continuity of fiber dimensions applied in turn to $X \to Y$,

$$\dim E_t = 2, \quad \dim g_t(E_t) = 0.$$ (6.6.5)

(6.6.4) and (6.6.5) imply that $-K_{X_t}$ is $g_t$-ample. Finally by Corollary 2.9 and the upper-semi-continuity,

$$R^2 g_t_* \mathcal{O}_{X_t}(2K_{X_t}) = 0.$$ (6.6.6)

Since $X_t$ has again only terminal complete intersection singularities, $g_t$ is a flipping contraction of Type (R), and we are done. \qed

For later arguments we have to look at the ideal structure of $E$ associated to a global deformation of $g$, determined by the degeneration of $\{E_t\}_{t \in \Delta \setminus \{0\}}$ as $t$ tends to 0.
**Definition 6.7.** \((E_0: \text{The ideal structure of } E \simeq \mathbb{P}^2 \text{ under a given deformation } \mathcal{X} \to \mathcal{Y})\)

Let \(\mathcal{X} \to \mathcal{Y} = \{X_t \to Y_t\}_{t \in \Delta}\) be as in Theorem 6.1. Let \(E_t := \operatorname{Exc} g_t (t \neq 0)\). \(E_t\) is reduced and is a disjoint union of several \(\mathbb{P}^2\)'s. Naturally

\[(6.7.1) \quad \operatorname{Hilb}_{X/\mathcal{Y}/\Delta} [E_t] \to \Delta\]

and so we may regard \(\lambda: H \to \Delta\) (6.6.1) as

\[(6.7.2) \quad \lambda([\text{a connected component of } E_t]) = [E_t].\]

In particular

\[(6.7.3) \quad \deg \lambda = \#(\text{connected components of } E_t) \ (t \neq 0).\]

(6.7.4) Let \(E_0\) be a closed analytic subspace corresponding to \(0 \in \operatorname{Hilb}_{X/\mathcal{Y}/\Delta} [E_t]\) (6.1). Then \(\{E_t\}_{t \in \Delta}\) forms a flat family of closed analytic subspaces of \(\mathcal{X}\). Clearly

\[(6.7.5) \quad \operatorname{red} E_0 = E(\simeq \mathbb{P}^2).\]

(6.7.6) From now on we always distinguish \(E\) and \(E_0\). Note that \(E_0 = E_0^\rho\) depends on the choice of deformations \(\mathcal{X} \to \mathcal{Y}\) of \(g : X \to Y\) i.e. \(\rho : \Delta \to \operatorname{Def} X\). (For example for the trivial deformation; \(\mathcal{X} = X \times \Delta, E_0 = E_0\).) Let \(I_{E_t} = I_{E_0}^\rho\) be the associated ideal sheaf in \(\mathcal{O}_X\) (or in \(\mathcal{O}_{X_t}\)) (also for \(t = 0\)). Define the universal ideal of \(E\) by

\[(6.7.7) \quad I_E^* = I_{E^*} := \bigcap_{\rho: \Delta \to \operatorname{Def} X} I_{E_0}^\rho,\]

and \(E^*\) the associated closed subspace. Denote

\[(6.7.8) \quad \operatorname{mult} E_0 := \text{the multiplicity of } E_0 \text{ at a general point}\]

(or of \(E_0\) as a 2-cycle).

Then the following is the key observation toward the local classification of singularities on \(X\) (§7):

**Proposition 6.8.**

(1) \(\operatorname{mult} E_0 = \deg \lambda\).

(2) \(E_0\) has no embedded components.

**Proof.** (1) \(\operatorname{mult} E_0 \geq \deg \lambda\) is clear. Let \(r := \deg \lambda\). Take a sufficiently large integer \(d\), and a general \(\mathcal{L} \in |-dK_X|\). Let \(L_i := \mathcal{L}|_{X_t}\). Then since for any connected component \(E_{t,i} \simeq \mathbb{P}^2\) of \(E_t\), \(\mathcal{O}_{E_{t,i}}(-K_{X_t}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)\) \((i = 1, \ldots, r)\) (Theorem 1.1),
$C_{t,i} := L_t \cap E_{t,i}$ is a smooth curve of degree $d$ in $E_{t,i} \simeq \mathbb{P}^2$. So on $E = \text{red } E_0 \simeq \mathbb{P}^2$, $L_0 \cap E$ must be a curve of degree

$$d' := \frac{rd}{\text{mult } E_0}.$$ 

If $\text{mult } E_0 > r$, then $d < d'$, so on $E$ we have

$$0 < \deg(-K_X|_E) = \frac{d}{d'} < 1,$$

a contradiction. Hence (1).

(2) comes from (1). $\square$

The following is an immediate corollary of Theorem 6.1:

**Corollary 6.9.** (Existence of a smoothing)

Let $X \supset E \simeq \mathbb{P}^2 \overset{g}{\rightarrow} Y \ni Q$ be of Type (R), with

$$\text{width } g = m(\geq 2).$$

Then

1. There exists a 1-parameter deformation $X \rightarrow \mathcal{Y} = \{X_t \overset{g_t}{\rightarrow} Y_t\}_{t \in \Delta}$ such that for $t \neq 0$, $X_t$ is smooth.

2. Moreover, for any such $X \rightarrow \mathcal{Y}$, $\text{Exc } g_t$ consists of an $m$ disjoint union of $\mathbb{P}^2$’s each of which has the normal bundle $N_{E_{t,i}/X_t} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\otimes 2}$.

In particular,

$$\text{mult } E_0 = m.$$ 

**Proof.** Since $X$ is assumed to have only isolated complete intersection singularities, which are locally smoothable, this follows immediately from Theorem 6.1 and the theorem of Kawamata (Theorem 0.5). $\square$

§7. $\text{Sing } X = \{P\}, \quad \varepsilon_P(X \supset E) = 1.$

In this section, we shall prove the following:

**Theorem 7.1.** Let $X \supset E \simeq \mathbb{P}^2 \overset{g}{\rightarrow} Y \ni Q$ be a flipping contraction of Type (R). Then

$$\# \text{Sing } X = 1.$$ 

So let $\{P\} = \text{Sing } X$. Then

$$\varepsilon_P(X \supset E) = 1.$$ 

The proof will be divided into several steps, and will be completed by Proposition 7.5.

To start with:
**Proposition 7.2.** Let $P \in \text{Sing } X$, then $(X, P)$ is a hypersurface singularity:

$$\text{emb. codim}(X, P) = 1.$$ 

**Proof.** Assume:

(7.2.0) $\text{emb. codim}(X, P) = 2$, and hence $\text{Sing } X = \{P\}$, to derive a contradiction (Corollary 4.3). Then as in (3.1.1)

(7.2.1) $(X, P) \cong \{ f_1(x_1, \ldots, x_6) = f_2(x_1, \ldots, x_6) = 0 \}$

$\implies E = \{x_3 = x_4 = x_5 = x_6 = 0\},$

$$\begin{aligned}
f_1(x_1, \ldots, x_6) &= \sum_{j=3}^{6} g_{1,j}(x_1, x_2) \cdot x_j + h_1(x_1, \ldots, x_6), \\
f_2(x_1, \ldots, x_6) &= \sum_{j=3}^{6} g_{2,j}(x_1, x_2) \cdot x_j + h_2(x_1, \ldots, x_6)
\end{aligned}$$

$(h_1, h_2 \in (x_3, x_4, x_5, x_6)^2).$

(7.2.2) By abuse of notations, we shall denote the lifts of the ideals $I_{E_0, P}, I_{E, P}^* \subset \mathcal{O}_X \cong \mathbb{C}\{x_1, \ldots, x_6\}/(f_1, f_2)$ (Definition 6.7) to $\mathbb{C}\{x_1, \ldots, x_6\}$ by the same notations $I_{E_0, P}$ and $I_{E, P}^*$, respectively:

$$f_1, f_2 \in I_{E, P}^* \subset I_{E_0, P}.$$

Let $A := \mathbb{C}\{x_1, x_2\}$ and $M := A^{\oplus 2}$ regarding as a free $A$-module of rank 2. First we claim:

**Claim 1.** After a suitable permutation of $\{x_3, x_4, x_5, x_6\}$,

$$\begin{pmatrix} g_{1,6} \\ g_{2,6} \end{pmatrix} \in \begin{pmatrix} g_{1,3} \\ g_{2,3} \end{pmatrix} A + \begin{pmatrix} g_{1,4} \\ g_{2,4} \end{pmatrix} A + \begin{pmatrix} g_{1,5} \\ g_{2,5} \end{pmatrix} A.$$

**Proof.** If the Claim 1 is false, then since $I_{E, P}^*$ does not have embedded primary ideals (Proposition 6.8 (2)), and since the radical of $I_{E, P}^*$ is $(x_3, x_4, x_5, x_6)$,

$$I_{E, P}^* = (x_3, x_4, x_5, x_6),$$

a contradiction to Corollary 6.9 (2). Hence the Claim 1.

By the Claim 1, after a suitable biholomorphic change of coordinates we may rewrite (7.2.1) as:

(7.2.3) $$\begin{aligned}
f_1(x_1, \ldots, x_6) &= \sum_{j=3}^{5} (g_{1,j}(x_1, x_2) + h_{1,j}(x_1, \ldots, x_6)) \cdot x_j + c_1 x_6^{n_1}, \\
f_2(x_1, \ldots, x_6) &= \sum_{j=3}^{5} (g_{2,j}(x_1, x_2) + h_{2,j}(x_1, \ldots, x_6)) \cdot x_j + c_2 x_6^{n_2}
\end{aligned}$$

$$\begin{pmatrix} x_j \cdot h_{i,j} \in (x_3, x_4, x_5, x_6)^2, n_1, n_2 \geq 2, c_1, c_2 \in \{0\} \cup \mathbb{C}\{x, s\}^{\times}. \end{pmatrix}$$
Claim 2. In (7.2.3), for any permutation \{x_i, x_j, x_k\} of \{x_3, x_4, x_5\},
\[
\begin{pmatrix}
g_1, k \\
g_2, k
\end{pmatrix} \not\in \begin{pmatrix}
g_1, i \\
g_2, i
\end{pmatrix} A + \begin{pmatrix}
g_1, j \\
g_2, j
\end{pmatrix} A.
\]

Proof. Assume to the contrary that
\[
\begin{pmatrix}
g_1, 5 \\
g_2, 5
\end{pmatrix} \in \begin{pmatrix}
g_1, 3 \\
g_2, 3
\end{pmatrix} A + \begin{pmatrix}
g_1, 4 \\
g_2, 4
\end{pmatrix} A,
\]
say. Then
\[
\left. \frac{\partial f_i}{\partial x_j} \right|_{x_3 = \ldots = x_6 = 0} \sim \begin{pmatrix}
0 & 0 & g_{1,4} & 0 & 0 \\
0 & 0 & 0 & g_{2,4} & 0 & 0
\end{pmatrix},
\]
where \(\sim\) means the equivalence of matrices by the fundamental linear operations of rows, in the matrix ring \(M_{2,6}(A)\). So
\[
\text{Sing } X \supset \left\{ \begin{vmatrix}
g_{1,3}(x_1, x_2) & g_{1,4}(x_1, x_2) \\
g_{2,3}(x_1, x_2) & g_{2,4}(x_1, x_2)
\end{vmatrix} = x_3 = x_4 = x_5 = x_6 = 0 \right\},
\]
which is of dimension 1, a contradiction to the assumption (A-1). Hence the Claim 2.

By the Claim 2 with Proposition 6.8 (2), \(x_3, x_4, x_5 \in I^*_{E,P}\), and hence
\[
(7.2.4) \quad I^*_{E,P} = (x_3, x_4, x_5, x_k^k) \quad (\exists k \geq 2).
\]

Claim 3. \(c_1, c_2 \neq 0\) and \(n_1 = n_2\) in (7.2.3).

Proof. Let
\[
(7.2.5) \quad \{ f_1(x) + t = f_2(x) + \alpha t = 0 \} \quad (\alpha = 0, 1)
\]
be a deformation of \((X, P), \mathcal{X}^{(a)} \to \mathcal{Y}^{(a)}\) its globalization (Theorem 6.1), and \(E_0^{(a)}\) the associated ideal structure (Definition 6.7). Then by \(I_E \subset I^{(a)}_{E_0}\) and Proposition 6.8 (2),
\[
(7.2.6) \quad I^{(a)}_{E_0} = (x_3, x_4, x_5, x_6^{k(a)}) \quad (2 \leq \exists k(a) \leq k),
\]
and hence
\[
(7.2.7) \quad I^{(a)}_E = (x_3 + t \cdot e_3^{(a)}(x, t), x_4 + t \cdot e_4^{(a)}(x, t), x_5 + t \cdot e_5^{(a)}(x, t),
\]
\[
x_6^{k(a)} + t \cdot e_6^{(a)}(x, t))
\]
\[
(\exists e_i^{(a)}(x, t) \in \mathbb{C}\{x, t\}).
\]
Since \( f_1(x) + t \in \mathcal{I}^{(\alpha)}_{E_t} \),
\[
f_1(x) + t = \sum_{j=3}^{5} \xi_j(x,t) \cdot (x_j + t \cdot e_j^{(\alpha)}(x,t)) + \xi_6(x,t) \cdot (x_6^k + t \cdot e_6^{(\alpha)}(x,t))
\]
(\(\exists \xi_j(x,t) \in \mathbb{C}\{x,t\}\)),
in particular,
\[
(7.2.8) \quad \xi_3, \xi_4, \xi_5 \in (x,t)\mathbb{C}\{x,t\} \quad \text{and} \quad \xi_6 \cdot e_6^{(\alpha)} \in \mathbb{C}\{x,t\}^\times.
\]
Hence
\[
(7.2.9) \quad c_1 \neq 0, \quad \text{and} \quad k(\alpha) = n_1 (\alpha = 0, 1) \quad \text{in} \quad (7.2.3), (7.2.6).
\]
By running the same argument through \( f_2 \) with \( \alpha = 1 \), we get \( c_2 \neq 0, k(1) = n_2, \)
and hence the Claim 3.

Let \( n := n_1 = n_2 \). By (7.2.7), (7.2.8) and (7.2.9),
\[
(7.2.10) \quad \mathcal{I}^{(\alpha)}_{E_t} = (x_3 + t \cdot e_3^{(\alpha)}(x,t), x_4 + t \cdot e_4^{(\alpha)}(x,t), x_5 + t \cdot e_5^{(\alpha)}(x,t), x_6^n + ct) (c \in \mathbb{C}\{x,t\}^\times).
\]
Let us put \( \alpha = 0 \) in (7.2.5) in turn, then \( f_2(x) \in \mathcal{I}^{(0)}_{E_t} \), i.e. there exist \( \eta_j(x,t) \in \mathbb{C}\{x,t\} \) \((j = 3, ..., 6)\) such that
\[
(7.2.11) \quad f_2(x) = \sum_{j=3}^{5} \eta_j(x,t) \cdot (x_j + t \cdot e_j^{(0)}(x,t)) + \eta_6(x,t) \cdot (x_6^n + ct) \quad (\forall t \in \Delta),
\]
where \( c := e_6^{(0)} \in \mathbb{C}\{x,t\}^\times \) (7.2.8). As in (7.2.8), we have
\[
(7.2.12) \quad \eta_3, \eta_4, \eta_5 \in (x,t)\mathbb{C}\{x,t\}, \quad \text{and} \quad \eta_6 \in \mathbb{C}\{x,t\}^\times,
\]
by the expression of \( f_2 \) (7.2.3), with \( m = n \) (Claim 3). This is however absurd,
because the left-hand side of (7.2.11) does not depend on \( t \). Hence the assumption
(7.2.0) is false and we are done. \( \square \)

**Lemma 7.3.**

(1) \( (X, P) \simeq \{f(x_1, ..., x_5) = 0\} \supset E = \{x_3 = x_4 = x_5 = 0\}, \)
\[
f(x_1, ..., x_5) = (g_3(x_1, x_2) + h_3(x_1, ..., x_5)) \cdot x_3
+ (g_4(x_1, x_2) + h_4(x_1, ..., x_5)) \cdot x_4 + x_5^k
\]
for some \( k \geq 2, \ g_i \in (x_1, x_2)\mathbb{C}\{x_1, x_2\}, \) and \( h_i \in \mathbb{C}\{x_1, ..., x_5\} \) with \( x_i \cdot h_i \in (x_3, x_4, x_5)^2 \).
Moreover, 
\[ I_E^* = (x_3, x_4, x_5^k). \]

Proof. By Proposition 7.2, \((X, P)\) is a hypersurface singularity, and we may write down the defining equation as:

\[ f(x_1, \ldots, x_5) = g_3(x_1, x_2) \cdot x_3 + g_4(x_1, x_2) \cdot x_4 + g_5(x_1, x_2) \cdot x_5 + h(x_1, \ldots, x_5) \]
\((h \in (x_3, x_4, x_5)^2)\). Let

\[ G := (g_3, g_4, g_5) \subset \mathbb{C}\{x_1, x_2\}. \]  

Then in a similar way to Claim 1 in the proof of Proposition 7.2,

\[ g_5 \in (g_3, g_4), \text{ i.e. } G = (g_3, g_4) \]

(after a suitable permutation of \(\{x_3, x_4, x_5\}\)). Moreover, since

\[ \varepsilon_P(X \supset E) = \dim_P \text{Ext}^1_E(\Omega^1_X \otimes \mathcal{O}_E, \mathcal{O}_E) = \text{length} \mathbb{C}\{x_1, x_2\}/G < \infty \]  

(Theorem 3.1), \(G\) is not a principal ideal. Hence \(I_E^* = (x_3, x_4, x_5^k) \) \((k \geq 2)\), and the rest is clear from the information \(f \in I_E^* \) and (7.3.2). \(\square\)

**Proposition 7.4.** \(\varepsilon_P(X \supset E) = 1.\)

Proof.

(7.4.0) Let \(G = (g_3, g_4)\), where \(g_3, g_4\) are in the expression of \(f\) in Lemma 7.3 (1).

Assume that \(\varepsilon_P(X \supset E) \geq 2.\) Then \((x_1, x_2) \subseteq G\) (7.3.3), so we may assume

\[ x_2 \notin G, \text{ say.} \]

Consider the deformation

\[ \{f(x) + tx_2 = 0\} \]

of \((X, P)\) and let \(E_0\) be as usual. Then by Lemma 7.3 (with Proposition 6.8 (2)),

\[ I_{E_0} = (x_3, x_4, x_5^n) \quad (\exists n \geq 2). \]

Write the condition \(f(x) + tx_2 \in I_{E_1}:\)

\[ f(x) + tx_2 = \xi_3(x, t) \cdot (x_3 + t \cdot e_3(x, t)) + \xi_4(x, t) \cdot (x_4 + t \cdot e_4(x, t)) \]
\[ + \xi_5(x, t) \cdot (x_5^n + t \cdot e_5(x, t)) \]
\((\exists \xi_i(x, s) \in \mathbb{C}\{x, t\})\), where

\[(7.4.4) \quad \xi_3, \xi_4 \in (x, t)\mathbb{C}\{x, t\}\]

as in (7.2.8), (7.2.12). Clearly,

\[(7.4.5) \quad \xi_i(x, t) \cdot e_i(x, t) \text{ contains } c \cdot x_2 \text{ as a monomial } (c \in \mathbb{C}\{x, t\}^\times) \text{ for at least one } i \in \{3, 4, 5\}.
\]

\[(7.4.6) \text{ Assume in } (7.4.5) \quad i = 3 \text{ or } 4, \text{ say } i = 4. \text{ Then } e_4 \in \mathbb{C}\{x, t\}^\times.
\]

Rewrite (7.4.3):

\[(7.4.7) \quad f(x) + tx_2 = \xi_3(x, t) \cdot (x_3 + t \cdot e_3(x, t)) + (c_1 x_2 + \xi'_4(x, t)) \cdot (x_4 + c_2 t)
\]

\[+ \xi_5(x, t) \cdot (x_5^n + t \cdot e_5(x, t)) \quad (c_1, c_2 \in \mathbb{C}\{x, t\}^\times).
\]

Put \(t = 0\), then \(f(x)\) contains \(c_1 c_2 \cdot x_2 x_4\) as a monomial, which contradicts the assumption (7.4.1) and the expression of \(f\) (Lemma 7.3 (1)).

So in (7.4.5) we must have \(i = 5:\)

\[(7.4.8) \quad \xi_5(x, t) \cdot e_5(x, t) = c x_2.
\]

Again by the expression of \(f\) (Lemma 7.3 (1)) and (7.4.3), \(\xi_5 \in \mathbb{C}\{x, t\}^\times, e_5 = c' x_2\) (\(c' \in \mathbb{C}\{x, t\}^\times\)). Hence

\[I_{E_i} = (x_3 + t \cdot e_3(x, t), x_4 + t \cdot e_4(x, t), x_5^n + c' \cdot t x_2).
\]

In particular,

\[(7.4.9) \quad \text{mult } E_0 = n \geq 2, \text{ while } E_i \text{ is irreducible } (t \neq 0), \text{ which contradicts Proposition 6.8 (1)}.
\]

Hence \(\varepsilon_P(X \supset E)\) must be 1. \(\Box\)

**Proposition 7.5.** \(\# \text{ Sing } X = 1.\)

**Proof.** Assume \(P, P' \in \text{ Sing } X\) \((P \neq P')\), say. By Proposition 7.4 with Corollary 3.5,

\[(7.5.1) \quad (X, P) \simeq \{f(x_1, ..., x_5) = 0\} \supset E = \{x_3 = x_4 = x_5 = 0\}, \]

\[(X, P') \simeq \{f'(y_1, ..., y_5) = 0\} \supset E = \{y_3 = y_4 = y_5 = 0\},
\]

\[f(x_1, ..., x_5) = x_1 x_3 + x_2 x_4 + x_5^n,
\]

\[f'(y_1, ..., y_5) = y_1 y_3 + y_2 y_4 + y_5^{n'}.
\]

Consider deformations of \((X, P)\) and of \((X, P')\):

\[(7.5.2) \quad \{f(x) + t = 0\}, \quad \{f'(y) + t^{n'} = 0\}.
\]
These are patched together to give a global deformation $\mathcal{X} \to \mathcal{Y} = \{X_t \xrightarrow{g_t} Y_t\}_{t \in \Delta}$ (Theorem 6.1), and let $\mathcal{E}$ be the exceptional locus of it:

$$\mathcal{E} = \bigcup_{t \in \Delta} E_t.$$ 

In a similar way to the proofs of Propositions 7.2 and 7.3,

$$(7.5.2) \quad (E_t, P) = \{x_3 + t \cdot e_3(x, t) = x_4 + t \cdot e_4(x, t) = x_5^n + t = 0\},$$

in particular

$$(7.5.3) \quad \#(\text{connected components of } E_t) = n (t \neq 0),$$

$$\text{and}$$

$$(7.5.4) \quad (E_t, P') = \{y_3 + t \cdot e_3'(x, t) = y_4 + t \cdot e_4'(x, t) = \prod_{i \in I}(y_5 + \zeta_i^{n'} \cdot t) = 0\}$$

where $\zeta_i^{n'} \in \mathbb{C}$ is a primitive $n'$-th roots of unity, and $I \subset \{1, \ldots, n'\}$ with $\#I = n$ (7.5.3).

So

$$(7.5.5) \quad \mathcal{E} \text{ is smooth in an neighborhood of } P \quad (7.5.2),$$

while $\mathcal{E}$ is a union of $n \geq 2$ irreducible components meeting at the whole $(E, P') = \{y_3 = y_4 = y_5 = 0\}$ (7.5.3), in particular

$$(7.5.6) \quad \text{Sing } \mathcal{E} \cap U' = E \cap U' \quad (\exists U' \ni P' : \text{analytic open set of } \mathcal{X}).$$

(7.5.5) and (7.5.6) contradict to each other, since Sing $X$ is a Zariski closed subset of $E$. Hence $\#\text{Sing } X = 1$. □

Now Theorem 7.1 follows from Proposition 7.4 and 7.5.

**Corollary 7.6.** Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \supset Q$ be of Type (R). By Theorem 7.1 (with Corollary 3.5), Sing $X = \{P\}$, and

$$(X, P) \simeq \{x_1x_3 + x_2x_4 + x_5^m = 0\}.$$ 

Then

$$\text{width } g = m.$$ 

**Proof.** Take a smoothing $\{x_1x_3 + x_2x_4 + x_5^m + t = 0\}$, then

$$\#(\text{connected components of } E_t) = m \ (t \neq 0).$$

The result follows from this, Proposition 6.8 (1), and Corollary 6.9 (2). □
§8. Existence of Flips — La Torre Pendente.

In this section we shall prove the existence of flips for Type (R) contractions.

**Theorem 8.1.** Let $X \supset E \simeq \mathbb{P}^2 \overset{g}{\rightarrow} Y \ni Q$ be a flipping contraction of Type (R). Then the flip $g^+$ exists.

**8.2.** According to the results of §7 (Theorem 7.1 and Corollary 7.6),

(8.2.1) $\text{Sing } X = \{P\}$,

(8.2.2) $(X, P) \simeq \{x_1x_3 + x_2x_4 + x_5^m = 0\} \supset E = \{x_3 = x_4 = x_5 = 0\}$,

where

(8.2.3) $m = \text{width } g$.

(8.2.4) Let $\varphi : \overline{X} \rightarrow X$ be the blow-up of $X$ with the center $E$. Let

$F := \text{Exc } \varphi$,

which is a projective 3-fold.

Since $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, there exists a birational map between $F$ and $\mathbb{P} := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2))$ compatible with their projections:

(8.2.5) $F \xrightarrow{\varphi} \mathbb{P}$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

$E \xrightarrow{\sim} \mathbb{P}^2$

(8.2.6) Let $\overline{E} \subset F$ be the proper transform of the negative section of $\mathbb{P}$, i.e. the section corresponding to $\mathcal{O}_{\mathbb{P}^2}(-1)\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

**Lemma 8.3.** (Local description of $\varphi$ along fibers)

(1) Let $S := \varphi^{-1}(P)$ be the fiber of $\varphi$ at $P$, and $f$ any other fiber of $\varphi$ over $E - \{P\}$. Then

$\varphi|_{F - S} : F - S \rightarrow E - \{P\}$

is a $\mathbb{P}^1$-bundle: $f \simeq \mathbb{P}^1$, while

$S \simeq \mathbb{P}^2$.

(A jumping fiber in the sense of Andreotti–Wiśniewski [AW2].)

(2) $F$ is a Cartier divisor of $\overline{X}$, and $\# \text{Sing } F = 1$.

Let $\{\overline{P}\} := \text{Sing } F$. Then $\overline{P} \in S$, and

$\text{Sing } \overline{X} = \left\{ \begin{array}{ll} \{\overline{P}\} & \text{ (if } m \geq 3) \\ \emptyset & \text{ (if } m = 2) \end{array} \right.$

Moreover,

$(\overline{X}, \overline{P}) \simeq \{y_1y_3 + y_2y_4 + y_5^{m-1} = 0\} \supset F = \{y_5 = 0\}$.

**Proof.** Straightforward. See also [loc.cit], and Beltrametti [Be]. □
Lemma 8.4. *(Global description of $\varphi$ along $F$ -1)*

(1) For any line $l \subset E$ such that $l \not\ni P$,

$$Z_l := \varphi^{-1}(l) \simeq \Sigma_2 z \rightarrow l \simeq \mathbb{P}^1.$$ 

(2) Let $C_l := \overline{E} \cap Z_l$ for such $l$. Then

$$(-K_{\overline{X}}.C_l) = 1.$$ 

Proof. (1) is clear, since $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ (Theorem 4.1).

(2) Denote simply $C_l, Z_l$ by $C, Z$, respectively. Then $\overline{X}$ and $F$ are smooth along $C$ (Lemma 8.3), and

$$C = Z \cap \overline{E} \text{ (meeting transversally)}.$$ 

Hence we have the exact sequence:

$$0 \rightarrow N_{C/F} \rightarrow N_{C/\overline{X}} \rightarrow N_{F/\overline{X}} \otimes \mathcal{O}_C \rightarrow 0$$

with

$$N_{C/F} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

From this, and

$$N_{F/\overline{X}} \otimes \mathcal{O}_C \simeq \mathcal{O}_F(-1N_{E/X}) \otimes \mathcal{O}_C \quad (8.2.5)$$

$$\simeq \mathcal{O}_Z(-1N_{E/X} \otimes \mathcal{O}_l) \otimes \mathcal{O}_C$$

$$\simeq \mathcal{O}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))(-1\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes \mathcal{O}_C$$

$$\simeq \mathcal{O}_{\Sigma_2}(-C - 2f) \otimes \mathcal{O}_C$$

$$\simeq \mathcal{O}_{\mathbb{P}^1},$$

we have

$$c_1(N_{C/\overline{X}}) = c_1(N_{C/F}) + c_1(N_{F/\overline{X}} \otimes \mathcal{O}_C) = -1,$$

i.e. $(-K_{\overline{X}}.C) = 1$. Hence (2). $\square$

Proposition 8.5. *(Global description of $\varphi$ along $F$ -2)*

$$\overline{E} \cap S = \{P\}, \text{ and } \varphi|_{\overline{E}}: \overline{E} \rightarrow E \simeq \mathbb{P}^2 \text{ is an isomorphism.}$$

$C_l$ is a line in $\overline{E} \simeq \mathbb{P}^2$.

Proof.

(8.5.0) Assume dim($\overline{E} \cap S$) = 1, to get a contradiction.
Step 1. First by the construction (8.2.5) and (8.2.6),

\[ \varphi|_E : E \to \mathbb{P}^2 \]

is a birational morphism and

\[(E.f)_F = 1.\]  \hspace{1cm} (8.5.1)

Since \( f \)'s and lines \( n \) on \( S \simeq \mathbb{P}^2 \) belong to the same irreducible component of \( \text{Hilb}_X \) (and also of \( \text{Hilb}_F \)) (see Andreatta–Wiśniewski [AW2]),

\[(E.n)_F = 1 \quad (\forall n \subset S : \text{line}).\]  \hspace{1cm} (8.5.2)

Thus

\[(E \cap S) \text{ is purely 1-dimensional and is a reduced line in } S.\]  \hspace{1cm} (8.5.3)

Step 2. Let \( l_\lambda \subset E \) be any line passing through \( P \) (\( \lambda \in \mathbb{P}^1 \)), and let

\[ m_\lambda := \overline{E \cap \varphi^{-1}(l_\lambda - \{P\})} \]

where \( \overline{\text{—}} \) denotes the closure in \( F \) (or in \( X \)). Then

\[ \varphi|_{m_\lambda} : m_\lambda \to l_\lambda \simeq \mathbb{P}^1 \]

is a birational morphism, and is hence an isomorphism:

\[ m_\lambda \simeq \mathbb{P}^1. \]  \hspace{1cm} (8.5.6)

Step 3.

(8.5.7) Let \( \mu : H \simeq \mathbb{P}^1 \to \text{Hilb}_X \) be a morphism defined by \( \lambda \mapsto [m_\lambda] \), then this is set theoretically an injection.

(8.5.8) Let \( h : \mathcal{H} \to H \) be the family over \( H \) induced from \( \text{Hilb}_X \), with \( \mathcal{H} \) being a normal surface.

Since every fiber is irreducible and \( h \) is a \( \mathbb{P}^1 \)-bundle over a general point on \( H \),

(8.5.9) \( h \) is in fact a \( \mathbb{P}^1 \)-bundle. (see [Mo1], cf. the argument of §1.)

Let

\[ \nu : \mathcal{H} \to \overline{E \subset X} \]

be the projection.

By construction,
(8.5.11) The composition
\[ H \xrightarrow{\nu} E \xrightarrow{\varphi|_E} E \simeq \mathbb{P}^2 \]
is a birational morphism, which is an isomorphism over \( E - \{P\} \).
In particular,
(8.5.12) \( H \simeq \Sigma_1 \).
Moreover, \((\varphi|_E)^{-1}(P) = n_0 \) (8.5.3), so \( \nu \) is finite birational, i.e.
(8.5.13) \( \nu \) is the normalization morphism, which is an isomorphism over \( E - \{P\} \)
(8.5.4). Let \( C_l \) be as in Lemma 8.4 (1), then
(8.5.14) \( \nu^{-1}(C_l) \) is a \((+1)\)-section, and \( \nu^{-1}(n_0) \) is the \((-1)\)-section (negative section), of \( H \simeq \Sigma_1 \).

Summing up, we have:
(8.5.15) Inside \( F \), we have a 1-parameter family \( \{m_\lambda\}_{\lambda \in \Lambda} \) (\( \Lambda \simeq \mathbb{P}^1 \)) such that
\[ m_\lambda \simeq \mathbb{P}^1 \ (\forall \lambda \in \Lambda), \ (-K_X \cdot m_\lambda) = 0, \quad \text{and} \]
\( \Lambda \) forms a whole connected component of \( \text{Hilb} \ X \).
\( (cf. \ Matsuki \ [Ma].) \)

**Step 4.** Finally,
(8.5.16) let us consider a smoothing
\[ \{x_1x_3 + x_2x_4 + x_5^m + t^m = 0\} \]
of \((X, P)\). Then the globalization \( \mathcal{X} \rightarrow \mathcal{Y} \) (Theorem 6.1) satisfies
(8.5.17) \( E_t = \{x_3 = x_4 = \prod_{i \in I} (x_5 + \zeta_m^i t) = 0\} \)
\( (\exists I \subset \{1,...,m\}) \) (after a suitable biholomorphic change of coordinates \( \{(x_1, ..., x_5, t)\} \)), as in the argument of §7. Moreover by Corollary 6.9, \( I = \{1, ..., m\} \):
(8.5.18) \( E_t = \{x_3 = x_4 = x_5^m + t^m = 0\} \).

(8.5.19) Take
\[ E_{t,1} := \{x_3 = x_4 = x_5 + t = 0\}, \quad \text{and} \quad \mathcal{E}_1 := \bigcup_{t \in \Delta} E_{t,1}. \]

(8.5.20) \( \mathcal{E}_1 \simeq \mathbb{P}^2 \times \Delta \), and \( \{E_{t,1}\}_{t \in \Delta} \) forms a flat family, with
\[ E_{t,1} = E \ (\text{reduced}) \ \text{when} \ t = 0. \]
Blow up $\mathcal{X}$ with the center $\mathcal{E}_1$:

$$\overline{\mathcal{X}} \to \mathcal{X} \to \Delta$$

By (8.5.20),

The fiber over $0 \in \Delta$ just coincides with $\varphi : \overline{\mathcal{X}} \to \mathcal{X}$.

By Corollary 6.9, Exc ($\overline{\mathcal{X}}_{t} \xrightarrow{\varphi_t} X_{t}$) consists of a disjoint union of $(m - 1)$ $\mathbb{P}^2$, plus one $\mathbb{P}^2 \times \mathbb{P}^1$. As in Kawamata [Kaw4],

$-K_{\overline{\mathcal{X}}_{t}}$ is $(g_t \circ \varphi_t)$-ample.

Hence $m_{\lambda}$ cannot move outside $\overline{\mathcal{X}}_0$, i.e.

$\Lambda \simeq \mathbb{P}^1$ (8.5.15) is a connected component also of Hilb$\overline{\mathcal{X}}/\mathcal{Y}/\Delta$,

while by Theorem 1.2,

$$\dim \text{Hilb} \overline{\mathcal{X}}/\mathcal{Y}/\Delta, [m_{\lambda}] \geq \dim \overline{\mathcal{X}} + (-K_{\overline{\mathcal{X}}}. m_{\lambda}) - \dim \text{Aut} \mathbb{P}^1 + \dim \Delta$$

$$= 2, \quad (8.5.15)$$

a contradiction to each other.

Hence $\dim (E \cap S) = 0$. Since $E$ and $S$ are surfaces inside the 3-fold $F$,

$$E \cap S \subset \text{Sing } F = \{P\}$$

(Lemma 8.3), and we have done. \(\square\)

**Corollary 8.6.** \(\rho(\mathcal{X}/\mathcal{Y}) = 2, \ -K_{\overline{\mathcal{X}}} \text{ is } (g \circ \varphi)-ample, \text{ and} \)

$$\overline{\text{NE}}(\mathcal{X}/\mathcal{Y}) = \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[C_l]. \quad \square$$

**Proposition 8.7.** The extremal ray $\mathbb{R}_{\geq 0}[C_l]$ determines a flipping contraction $\overline{g} : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ of Type (R), with

$$\text{Exc } \overline{g} = E, \text{ and } \text{width } \overline{g} = m - 1.$$

**Proof.**

**Claim 1.** $\mathbb{R}_{\geq 0}[C_l]$ gives a flipping contraction.

In fact, if $\mathbb{R}_{\geq 0}[C_l]$ defines a divisorial contraction $\overline{g} : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$, then clearly

$$\text{Exc } \overline{g} = F.$$

Since $\varphi(S) = \{P\}$, $\overline{g}$ does not contract $S$, and so

$$\dim \overline{g}(F) = 2.$$
Take a general fiber $\overline{C} \simeq \mathbb{P}^1$ of $\overline{\mathcal{G}}$ over $\overline{\mathcal{G}}(F)$, then

$$(F \cdot \overline{C}) = -1, \quad \text{and} \quad \dim \varphi(\overline{C}) = 1.$$ 

Hence

$$0 < (-K_X \cdot \varphi_*(\overline{C})) = (-\varphi^* K_X \cdot \overline{C})$$

$$= (-K_X + F \cdot \overline{C})$$

$$= 1 - 1 = 0,$$

a contradiction. So $\overline{g}$ must be a flipping contraction.

Since $\text{Sing } \overline{X} \subset \{\overline{P}\}$, and $(\overline{X}, \overline{P})$ is again an isolated hypersurface singularity (Lemma 8.3), $\text{Exc } \overline{g}$ is a union of $\mathbb{P}^2$’s (Corollary 2.8). Moreover,

Claim 2. $\text{Exc } \overline{g} = \overline{E}.$

In fact, let $Z_l \simeq \Sigma_2$ be as in Lemma 8.3. This is a surface contained in the smooth locus of the 3-fold $F$. So if we choose $l$ general enough, and if we assume that $\text{Exc } \overline{g}$ is reducible, then $Z_l \cap \text{Exc } \overline{g}$ is a reducible curve. Consider the birational morphism $\overline{g}|Z_l : \Sigma_2 \simeq Z_l \to \overline{\mathcal{G}}(Z_l)$. Since $C_l$ (Lemma 8.4) is the only contractible irreducible curve on $Z_l$, we get a contradiction. Hence $\text{Exc } \overline{g}$ must be irreducible and $\text{Exc } \overline{g} \cap Z_l = C_l$, thus necessarily $\text{Exc } \overline{g} = \overline{E}$, i.e. the Claim 2.

Finally,

Claim 3. $\overline{g}$ is of Type (R), with $\text{width } \overline{g} = \text{width } g - 1$.

In fact, take a general $D = D_l \in | -K_X|$ with $l := D \cap E$ (so that $D \not\equiv P$), and let $\overline{D}$ be its proper transform in $\overline{X}$, then clearly $\overline{D} \cap \overline{E} = C_l$ (Lemma 8.4), and $\overline{D} \overset{\varphi}{\to} \overline{\mathcal{G}}$ is the blow-up of the $(0, -2)$-curve $l$ in $D$. So by Reid [Re1] (see also §5, particularly 5.1),

$$N_{\overline{E} / \overline{X}}|C_l \simeq N_{C_l / \overline{D}} \simeq \begin{cases} O_{\mathbb{P}^1}(-1)^{\oplus 2} & (m = 2), \\ O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2) & (m \geq 3), \end{cases}$$

In both cases $R^1(\overline{\mathcal{G}}|\overline{D})_* O_{\overline{D}}(K_{\overline{X}}) = 0$, i.e.

$$R^2 \overline{\mathcal{G}}_* O_{\overline{X}}(2K_{\overline{X}}) = 0.$$

Hence by Corollary 2.9, $\overline{g}$ is again of Type (R). Also

$$\text{width } \overline{g} = \text{width}(\overline{\mathcal{G}}|\overline{D}) = m - 1. \quad \square$$

8.8. Set-up of the induction argument toward Theorem 8.1.

We shall prove the following 9 set of statements (8.8.1)$_m$ through (8.8.9)$_m$ by the induction on $m$:

(8.8.1)$_m$ The flip $g^+$ of $g$ exists for any Type (R) contraction $X \supset E \simeq \mathbb{P}^2 \overset{g}{\to} Y \ni Q$ with

$$\text{width } g \leq m.$$
Let $g : X \to Y$ be any such one, with

\[ 2 \leq \text{width } g \leq m. \]

Let $\varphi : \overline{X} \to X$ be the blow-up and $F = \text{Exc } \varphi$, as in 8.2, then as proved in Corollary 8.7, there exists a unique flipping contraction $\overline{g} : \overline{X} \to \overline{Y}$ from $\overline{X}$, which is of Type (R), with width $\overline{g} = \text{width } g - 1$.

Let $\overline{g}^+ : \overline{X}^+ \to \overline{Y}$ be the flip of $\overline{g}$ (where the existence is assured in (8.8.1)$_m$), and let $E^+ := \text{Exc } \overline{g}^+$. Let $F^+$ be the proper transform of $F$ in $\overline{X}^+$. Then

(8.8.2)$_m$ $\overline{X}^+$ is smooth,

(8.8.3)$_m$ $E^+ \simeq \mathbb{P}^1$,

(8.8.4)$_m$ There exists a birational morphism $\varphi^+ : \overline{X}^+ \to X^+$ with $\text{Exc } \varphi^+ = F^+$, $E^+ := \varphi^+ (\text{Exc } \varphi^+) \simeq \mathbb{P}^1$,

(8.8.5)$_m$ $X^+$ is also smooth,

(8.8.6)$_m$ $\varphi^+$ is the blow-up of $X^+$ with the center $E^+$,

(8.8.7)$_m$ $E^+$ is a section of the $\mathbb{P}^2$-bundle $\varphi^+ |_{F^+} : F^+ \to E^+$, and

(8.8.8)$_m$ $N_{E^+/X^+} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

(In particular, $X^+ \supset E^+ \to Y \ni Q$ gives the flip $g^+$ of $g$.)

Finally, let $W_\lambda := \varphi^{-1} (l_\lambda - \{P\})^- \subset F$, where $\{l_\lambda\}_{\lambda \in \Lambda}$ ($\Lambda \simeq \mathbb{P}^1$) is the complete family of lines in $E$ passing through $P$. Then

(8.8.9)$_m$ The fibers of $\varphi^+ |_{F^+}$ ($\simeq \mathbb{P}^2$) are exactly the proper transforms $W^+_\lambda$ of $W_\lambda$’s in $\overline{X}^+$.

To prove Theorem 8.1, it is enough to show the following couple of statements:

(i) (8.8.1)$_2$ through (8.8.9)$_2$, and

(ii) (8.8.1)$_{m-1}$ through (8.8.9)$_{m-1}$ imply (8.8.1)$_m$ through (8.8.9)$_m$.

8.9. [Proof of (8.8.1)$_2$ — (8.8.9)$_2$]

First, (8.8.1)$_2$ is nothing but Kawamata [Kaw4]’s result (Theorem 0.5).

(8.9.0) Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be any flipping contraction of Type (R), with

\[ \text{width } g = 2, \]

and $\overline{g} : \overline{X} \to \overline{Y}$, $\overline{E}$ be as in 8.8. Then since width $\overline{g} = 1$, $\overline{g}$ is a flipping contraction of Kawamata Type [loc.cit] (Corollary 5.6). Thus by [loc.cit],

\[ N_{\overline{E}/\overline{X}} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}, \quad \overline{X}^+ \text{ is smooth, and } \overline{E}^+ \simeq \mathbb{P}^1, \]

i.e. we get (8.8.2)$_2$, (8.8.3)$_2$. 49
Claim 1. Let $\varphi : \overline{X} \to X$ be the blow-up of $X$ with the center $E$, let $\varphi^* := \text{Exc} \, \varphi$, and $F' \subset \overline{X}$ the proper transform of $F \subset X$ in $\overline{X}$. Then $\overline{E} := \varphi^* F' \cap F'$ is a smooth $(1,1)$-divisor in $\overline{F} \simeq \mathbb{P}^2 \times \mathbb{P}^1$. In particular, $\overline{E} \simeq \Sigma_1$.

Proof. Consider $\varphi \circ \varphi : \overline{X} \to X$. The restriction to $\overline{F}$:

$$\overline{F} \simeq \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{\varphi^* \nabla} E \sim E \simeq \mathbb{P}^2$$

coincides with the projection. Moreover, take a general smooth $D \in -K_X$ (so that $D \not
ot \subset P$), and let $l := D \cap E$. Then

$$\varphi^{-1}(l) = \varphi^{-1}(C_1) \cup Z'_1 \subseteq D'$$

where $Z'_1, D'$ is the proper transform of $Z_1(\subset X)$ (Lemma 8.4 (1)), $D(\subset X)$ in $\overline{X}$, respectively, and $C_1$ is as in Lemma 8.4 (2). By Reid [Re1], this forms the Pagoda of width 2 (see 5.1), i.e.

$$\varphi^{-1}(C_1) \simeq \mathbb{P}^1 \times \mathbb{P}^1, \quad Z'_1 \simeq \Sigma_2,$$

and $\varphi^{-1}(C_1) \cap Z'_1$ is an irreducible $(1,1)$-divisor of $\varphi^{-1}(C_1)$. The Claim 1 follows immediately from this.

Claim 2. $F'$ is smooth, and

$$F' \xrightarrow{\varphi|_{F'}} F$$

is a contraction of the $(-1,-1)$-curve $\varphi^{-1}(P) \simeq \mathbb{P}^1$. $S' := \varphi^{-1}(S) \simeq \Sigma_1$.

Proof. $\varphi|_{F'}$ is the blow-up of $F$ with the codimension 1 center $E$. Since

$$\text{Sing } F = \{P\}, \quad (F, \overline{F}) \simeq \{y_1 y_3 + y_2 y_4 = 0\} \cap \overline{E} = \{y_3 = y_4 = 0\}, \quad \cup S = \{y_1 = y_2 = 0\}$$

(Lemma 8.3, Proposition 8.5), this is clear.

Claim 3. $(\varphi \circ \varphi)^{-1}|_{F'} : F' \to E \simeq \mathbb{P}^2$ is factored as

$$F' \xrightarrow{h} \text{Bl}_P E \simeq \Sigma_1 \xrightarrow{\text{bl}_P} E \simeq \mathbb{P}^2,$$

where $\text{bl}_P$ is the blow-up of $E$ with the center $P$. Moreover, $h$ is a $\mathbb{P}^1$-bundle.

Proof. Let $f \simeq \mathbb{P}^1$ be the fiber of the $\mathbb{P}^1$-bundle $\varphi|_{F-S} : F-S \to E-\{P\}$ (Lemma 8.3), and $f'$ its proper transform in $\overline{X}$. Let $H, H'$ be the irreducible component of $\text{Hilb}_{\overline{X}}, \text{Hilb}_{\overline{X}}$, containing the point $[f], \text{[f]}, \text{respectively. Then as in Andreatta–Wiśniewski [AW2], H \simeq Bl_P E \simeq \Sigma_1. Hence by Claim 2,}$

$$H' \simeq H \simeq \Sigma_1.$$
Let
\[ \mathcal{H}' \xrightarrow{p} \overline{X} \]
(8.9.6)
\[ h \]
\[ H' \]
be the standard diagram of the universal family \( h \) over \( H' \), and the projection \( p \).

(8.9.7) \[ p(\mathcal{H}') = F', \]
and \( p \) is set theoretically an injection \([\text{loc.cit.}]\). Since \( F' \) is smooth (Claim 2), \( p \) gives an isomorphism onto \( F' \):

(8.9.8) \[ p : \mathcal{H}' \simto F'. \]
Thus
(8.9.9) \( h \) gives a \( \mathbb{P}^1 \)-bundle over \( \Sigma_1 \):

\[ F' \simto \mathcal{H} \xrightarrow{h} \Sigma_1. \]

Hence the Claim 3.

(8.9.10) Let us consider the composite surjective morphism:

\[ F' \longrightarrow \Sigma_1 \longrightarrow B := \mathbb{P}^1. \]

Clearly \( \rho(F'/B) = 2 \).

(8.9.11) Let \( \varphi^+ : \overline{X} \rightarrow \overline{X}^+ \) be the contraction of \( F \simeq \mathbb{P}^2 \times \mathbb{P}^1 \) to \( \mathbb{P}^1 \) so that \( \overline{X}^+ \) gives the flip of \( \overline{g} : \overline{X} \rightarrow \overline{Y} \) (8.9.0) \([\text{Kaw4}]\). Let

\[ E^+ := \varphi^+(F) \simeq \mathbb{P}^1, \quad F^+ := \varphi^+(F'). \]

Claim 4. \(-K_{F'}\) is relatively ample over \( B \), and the extremal ray, other than that determines \( h \), coincides with

\[ \varphi^+|_{F'} : F' \rightarrow F^+, \]
where \( \varphi^+ \), \( F^+ \) are as in (8.9.11). Moreover, this is a divisorial contraction, with the exceptional divisor

\[ \text{Exc}(\varphi^+|_{F'}) = \overline{E}(\simeq \Sigma_1), \quad \varphi^+(\overline{E}) = \overline{E}^+ \simeq \mathbb{P}^1. \]

\( F^+ \) is smooth, and \( F^+ \rightarrow B = \mathbb{P}^1 \) is a \( \mathbb{P}^2 \)-bundle.
Proof. Let $W_\lambda \subset F$ be as in 8.8, and $W'_\lambda$ its proper transform in $F'$. Then $W_\lambda \cong W'_\lambda \cong \Sigma_1$.

(8.9.12) Let $m_\lambda, m'_\lambda$ be the negative section of $W_\lambda, W'_\lambda$, respectively. Clearly $\{m_\lambda\}_{\lambda \in \Lambda}$ forms the complete family of lines in $\overline{E} \cong \mathbb{P}^2$ passing through $P$. Hence by the Claim 2,

(8.9.13) $\{m'_\lambda\}_{\lambda \in \Lambda}$ forms the complete family of rulings in $\overline{E} \cong \Sigma_1$ (Claim 1).

Since $\overline{\varphi^+} : \overline{X} \to \overline{X}^+$ maps $\overline{F} \cong \mathbb{P}^2 \times \mathbb{P}^1$ to $\mathbb{P}^1$,
(8.9.14) $\overline{\varphi^+}|_{F'} : F' \to F^+$ contracts exactly $m'_\lambda$'s (Claim 1). Hence $\overline{\varphi^+}|_{F'}$ factors through $F' \to B = \mathbb{P}^1$.

Moreover,

(8.9.15) $(-K_{F'} \cdot m'_\lambda) = (-K_{\overline{X}} - F' \cdot m'_\lambda) = 2 - 1 = 1$,

in particular, $-K_{F'}$ is $\overline{\varphi^+}|_{F'}$-ample. Since $\rho(F'/B) = 2$ (8.9.10), necessarily
(8.9.16) $\overline{\varphi^+}|_{F'}$ is the contraction of the extremal ray $\mathbb{R}_{\geq 0}[m'_\lambda]$ of $\overline{NE}(F'/B)$.

(8.9.17) This is furthermore a divisorial contraction from a smooth 3-fold $F'$ (Claim 2), with

$$\text{Exc}(\overline{\varphi^+}|_{F'}) = \overline{E} \cong \Sigma_1, \overline{\varphi^+}(\overline{E}) \cong \mathbb{P}^1.$$ 

Hence by Mori [Mo2], $F^+ = \overline{\varphi^+}(F')$ is smooth. Since $\overline{\varphi^+}|_{F'}$ is defined over $B = \mathbb{P}^1$ (8.9.18),
(8.9.19) There is a surjective morphism $F^+ \to B$.

By construction, the fibers are exactly $\overline{\varphi^+}(W'_\lambda) \cong \mathbb{P}^2$ (8.9.15), so
(8.9.20) $F^+$ is a $\mathbb{P}^2$-bundle over $B = \mathbb{P}^1$.

Hence the Claim 4.

Claim 5.

$$\rho(\overline{X}^+/Y) = 2.$$ 

$$\overline{NE}(\overline{X}^+/Y) = \mathbb{R}_{\geq 0}[\overline{E}^+] + R^+,$$

where $R^+$ is the unique extremal ray. Let $\varphi : \overline{X}^+ \to X^+$ be the associated contraction. Let $\varphi^+(F^+) =: E^+$. Then $X^+$ is smooth, $E^+ \cong \mathbb{P}^1$, and

$$\varphi^+|_{F^+} : F^+ \to E^+ \cong \mathbb{P}^1$$

coincides with the $\mathbb{P}^2$-bundle in the Claim 4.

Proof. $\rho(\overline{X}^+/Y) = 2$ is clear, and

(8.9.22) $$(K_{\overline{X}} \cdot \overline{E}^+) = 1$$

[Kaw4]. So
(8.9.23) $\overline{NE}(X^+/Y)$ has at most one extremal ray.

Let $f^+$ be any line in a fiber of the $\mathbb{P}^2$-bundle $F^+ \to B$ (Claim 4). Then

(8.9.24) \[ (-K_{X^+}.f^+) = (-K_{\overline{X}}.m'_\lambda + f') = (-K_{\overline{X}}.m_\lambda + f) = 2. \]

Hence

(8.9.25) $\overline{NE}(X^+/Y)$ has exactly one extremal ray (8.9.22), (8.9.24). Let $\varphi^+: X^+ \to X^+$ be the associated contraction. Then $\varphi\vert_{F^+}$ coincides with the $\mathbb{P}^2$-bundle $F^+ \to B = \mathbb{P}^1$. Moreover, $X^+$ is smooth along $E^+ := \varphi^+(F^+) = B$, since $N_{W^+_X/\overline{X}} \cong O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(-1)$ (8.9.24). Finally, $\overline{E}^+$ is clearly the section of $\varphi^\vert_{F^+}$. Hence the Claim 5.

Claim 6.

$N_{E^+/X^+} \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-2)$.

$X^+ \to Y$ gives the flip of $g : X \to Y$ in (8.9.0).

Proof. Take $D \in |-K_X|$ general enough.

(8.9.26) In the above procedure

$$ X \leftarrow X \leftarrow \overline{X} \leftarrow \overline{X} \leftarrow X^+ \leftarrow X^+, $$

let $D$, $\overline{D}$, $D^+$, $D^+$ be the proper transforms of $D$ in $X$, $\overline{X}$, $X^+$, $X^+$, respectively. Then

$$ D \leftarrow \overline{D} \leftarrow \overline{D} \to D^+ \to D^+ $$

forms Reid [Re1]'s Pagoda (5.1) of width 2, in particular they are all smooth. So

$$ N_{E^+/D^+} \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2). $$

By this, and the exact sequence:

(8.9.27) \[ 0 \to N_{E^+/D^+} \to N_{E^+/X^+} \to N_{D^+/X^+} \otimes O_{E^+} \to 0 \]

with

(8.9.28) \[ (D^+.E^+)_{X^+} = (-K_{\overline{X}} + 2F' + 4\overline{F}.\overline{f})_{\overline{X}} = 1 + 2 - 4 = -1 \]

(where $\overline{f} \cong \mathbb{P}^1$ is a fiber of $\overline{f}\vert_{\overline{F}} : \overline{F} \to \overline{E}$), we get

$$ N_{E^+/X^+} \cong O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-2). $$

Hence

$$ (K_{X^+}.E^+) = -2 - c_1(N_{E^+/X^+}) = 1, $$

and so $X^+ \to Y$ certainly gives the flip of $g$, i.e. the Claim 6.
Now (8.8.4) through (8.8.9) are consequences of the Claim 5 and 6. Also (8.8.1) follows from the existence of the flip $X^+ \to Y$ (Claim 6).

8.10. [Proof of (8.8.1)$_{m-1}$ — (8.8.9)$_{m-1} \implies (8.8.1)_m — (8.8.9)_m]

Assume (8.8.1)$_{m-1}$ — (8.8.9)$_{m-1}$.

(8.10.0) Let $X \ni E \simeq \mathbb{P}^2 \overset{g}{\to} Y \ni Q$ be any flipping contraction of Type (R) with width $g = m \geq 3$.

(8.10.1) Let $\varphi : X \to X, F, E, \overline{g} : X \to Y$ be as in 8.8.

(8.10.2) Let $\overline{\varphi} : \overline{X} \to X$ be the blow-up of $X$ with the center $E$, let $\overline{F} := \text{Exc} \; \overline{\varphi}, F'$ the proper transform of $F$ in $\overline{X}, E' := F \cap F'$, and $\overline{S} := \overline{\varphi}^{-1}(\overline{P}) \simeq \mathbb{P}^2$.

(8.10.3) Since width $\overline{g} = m - 1 \geq 2$, there exists a flipping contraction $\overline{g} : \overline{X} \to \overline{Y}$ of Type (R) with width $\overline{g} = m - 2$ (Proposition 8.7).

Let $\overline{g}^+ : \overline{X}^+ \to \overline{Y}$ be the flip, where the existence is assured by the induction hypothesis (8.8.1)$_{m-1}$. Let $\overline{E} := \text{Exc} \; \overline{g}, \overline{\psi} := \overline{g}^+ \circ \overline{g} : \overline{X} \to \overline{X}^+, \overline{F}' := \overline{\psi}^*(\overline{F}),$ and $\overline{F}^{+} := \overline{\psi}_*(\overline{F}')$.

Claim 1. $\overline{E} \cap \overline{E}' = \emptyset$, i.e. $\overline{E} \cap F' = \emptyset$.

Hence $\overline{\psi}|_{\overline{F}'} : F' \to F'^+$ is an isomorphism: $F' \simeq F'^+$.

Proof. For any line $l \not\ni P$ in $E$,

\[(\varphi \circ \overline{\varphi})^{-1}(l) \cap \overline{E} \cap \overline{E}' = \emptyset\]  

[Rel]. On the other hand,

\[(\overline{X}, \overline{P}) \simeq \{y_1y_3 + y_2y_4 + y_5^{m-1} = 0\} \]
\[\cap \overline{E} = \{y_3 = y_4 = y_5 = 0\}, \cap S = \{y_1 = y_2 = y_5 = 0\}\]

(Lemma 8.3, Proposition 8.5). Hence on the blown-up $\overline{X}$ (with the center $\overline{E}$), the proper transform $S'$ of $S$ in $\overline{X}$ is away from $\text{Sing} \; \overline{X}$, while again by Proposition 8.5,

$\overline{E} \cap \overline{\varphi}^{-1}(\overline{P}) = \{\overline{P}\}$.

So

\[(\varphi \circ \overline{\varphi})^{-1}(P) \cap \overline{E} \cap \overline{E}' = \emptyset.\]  

(8.10.4) and (8.10.5) show the Claim 1.
By the induction hypothesis (8.8.4)\(_{m-1}\) through (8.8.7)\(_{m-1}\),

There exists a contraction

\[ \varphi^+: \overline{X}^+ \to X^+ \]

with Exc \( \varphi^+ = \overline{F}^+ \), \( \overline{E}^+ := \varphi^+(\overline{F}^+) \simeq \mathbb{P}^1 \), and \( \varphi^+ \) is the blow-up of \( X^+ \) with the center \( \overline{E}^+ \).

Let \( \varphi^+(F'^+) =: F^+ \). By the Claim 1, we have the following claim in a similar way to the proof in 8.9:

**Claim 2.** There exist surjective morphisms \( F' \to \Sigma_1 \) and \( F^+ \to \mathbb{P}^1 \) which make the following diagram commutative:

\[
\begin{array}{ccc}
F' & \simeq & F'^+ \\
\downarrow & & \downarrow \\
\Sigma_1 & \xrightarrow{\mathbb{P}^1\text{-bundle}} & \mathbb{P}^1
\end{array}
\]

(8.10.8)

Let \( \overline{W}_{\lambda}^+ := \varphi^{-1}(\overline{t}_{\lambda} - \{\overline{P}\})^- \) where \( \{\overline{t}_{\lambda}\}_{\lambda \in \Lambda} \ (\Lambda \simeq \mathbb{P}^1) \) is the complete family of lines in \( \overline{E} \) passing through \( \overline{P} \). Let \( \overline{W}_{\lambda}^+ \) be the proper transform of it in \( \overline{X}^+ \). By (8.8.9)\(_{m-1}\),

\( W_{\lambda}^+ \simeq \mathbb{P}^2 \) and \( \varphi^+|_{F'^+} \) contracts exactly with

\[ \overline{W}_{\lambda}^+ \cap W_{\lambda}^+ =: m_{\lambda}^+ \simeq \mathbb{P}^1 \]

(8.10.9)

where \( W_{\lambda}^+ \subset F'+ \) is the proper transform of \( W_{\lambda} \) (8.8.9)\(_{m} \) in \( \overline{X}^+ \). This is the negative section of \( W_{\lambda}^+ \simeq \Sigma_1 \) (Claim 1).

In particular, the morphism \( F^+ \to \mathbb{P}^1 \) in the Claim 2, (8.10.8) is a \( \mathbb{P}^2 \)-bundle.

The rest is exactly the same as in 8.9. \( \square \)

8.8, 8.9 and 8.10 show Theorem 8.1.

**Corollary 8.11.** *(The description of the flip \( g^+ \))*

Let \( X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q \) be a flipping contraction of Type (R), with

width \( g = m(\geq 2) \).

Let \( X^+ \supset E^+ \xrightarrow{g^+} Y \ni Q \) be its flip (Theorem 8.1), where \( E^+ := \text{Exc} \ g^+ \). Then

1. \( X^+ \) is smooth,
2. \( E^+ \simeq \mathbb{P}^1 \),

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(3) \( \text{Bs} | - K_{X+} | = E^+ \),

(4) \( N_{E^+/X^+} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \),

(5) a general member \( D^+ \in | - K_{X^+} | \) is smooth, and

(6) \( E^+ \subset D^+ \) is a \((0, -2)\)-curve, with width \( = \text{width}(g^+|_{D^+}) = m \). □

8.12. (La Torre Pendente)

Follow the induction procedure 8.8, with the proofs 8.9, 8.10, then we get the complete picture of the whole flip operation, starting from the given flipping contraction \( X \supset E \cong \mathbb{P}^2 \xrightarrow{g} Y \ni Q \). Namely, the flip \( X^+ \supset E^+ \xrightarrow{g^+} Y \ni Q \) is obtained by blowing \( X \) up \( m \) times, and then down \( m \) times, like:

\[
X \leftarrow X \leftarrow ... \leftarrow X^{(m-1)} \leftarrow X^{(m)} \rightarrow X^{(m-1)+} \rightarrow ... \rightarrow X^+ \rightarrow X^+.
\]

We name this \text{La Torre Pendente}, after M. Reid’s Pagoda.

Let \( D \) be a general smooth member of \( | - K_X | \), and \( \overline{D}^{(i)}, \overline{D}^{(i)+} \) be its proper transforms in \( X^{(i)}, X^{(i)+} \), respectively, then

\[
D \leftarrow \overline{D} \leftarrow ... \leftarrow \overline{D}^{(m-1)} \leftarrow \overline{D}^{(m)} \rightarrow \overline{D}^{(m-1)+} \rightarrow ... \rightarrow \overline{D}^+ \rightarrow D^+
\]

is nothing but the whole Pagoda diagram of width \( m \) (5.1). So our La Torre Pendente contains Pagoda as an anti-canonical divisor (and its proper transforms).

Here we note that though Pagoda was symmetric with respect to the flop, ours is no more.

At the end we shall give some examples, particularly related with fiber type contractions.

Example 8.13. (S. Mukai)

There exists a projective surjective morphism \( h : X \rightarrow Z \) from a terminal Gorenstein 4-fold \( X \) to a germ of a normal 3-fold singularity \((Z, Q)\) such that \( h|_{X- h^{-1}(Q)} : X - h^{-1}(Q) \rightarrow Z - \{Q\} \) is a \( \mathbb{P}^1 \)-bundle, while the fiber \( F := h^{-1}(Q) \) at \( Q \) is a union of two singular quadric surfaces (in \( \mathbb{P}^3 \)): \( F = F_1 \cup F_2 \), meeting at a point \( P \) which is the vertices of both. \( X \) has a Gorenstein quotient singularity of type \( \frac{1}{2}(1, 1, 1, 1) \) at \( P \) \((i.e. \) the quotient by the involution \((\mathbb{C}^4, 0) \ni z \mapsto -z \in (\mathbb{C}^4, 0))\), and is smooth elsewhere.

\( h \) is factored by a flipping contraction \( g : X \rightarrow Y \) which contracts either one of \( F_1, F_2 \).

So the fact \( \text{Exc} g \cong \mathbb{P}^2 \) is no longer true when \( X \) has worse than complete intersection singularities.
Example 8.14. (A singular analogue of Mukai–Shepherd-Barron–Wiśniewski contraction (MSW))

An example of the following type of contraction is known (Mukai–Shepherd-Barron–Wiśniewski contraction (MSW), see [Kac1]):

A projective surjective morphism \( h : X \to Z \) from a smooth 4-fold \( X \) to a germ of a normal 3-fold singularity \((Z, Q)\), which is a \( \mathbb{P}^1 \)-bundle over \( Z - \{Q\} \), while the central fiber \( F := h^{-1}(Q) \) is a union of two \( \mathbb{P}^2 \)'s: \( F = F_1 \cup F_2 \), meeting at a point \( P \).

\( h \) is factored by a flipping contraction \( g : X \to Y \) contracting either one of \( F_1, F_2 \), which is of Kawamata type [Kaw4]. In this case,

\[
(Z, Q) \simeq \{ z_1 z_2 + z_3^2 + z_4^2 = 0 \}.
\]

Fact. ([Kac1])

This is the only contraction among those \( h : X \to (Z, Q) \) from a smooth 4-fold \( X \) to a 3-fold germ \((Z, Q)\), with \(-K_X\) being \( h \)-ample, such that

1. \( F := \dim h^{-1}(Q) = 2 \),
2. there exists a rational curve \( l \subset F \) with \((-K_X . l) = 1\), and
3. \( h \) is a \( \mathbb{P}^1 \)-bundle over \( Z - \{Q\} \).

What we are going to construct is a similar \( h \) from a singular \( X \) which is factored by a flipping contraction of Type (R).

The following construction is a generalization of Mukai who did in the case \( m = 1 \), that is, the case that \( X \) is smooth.

Construction. Let us start with the 3-fold flopping contraction \( U \supset C \simeq \mathbb{P}^1 \to (Z, Q) \) of a \((0, -2)\)-curve \( C \), with width \( m \geq 2 \). Let \( L \) be an irreducible smooth divisor on \( U \) with \((L . C) = 1\), let \( E := O_U \oplus O_U(-L) \), and let

\[
X^+ := \mathbb{P}(E) \xrightarrow{\psi} U.
\]

This is a \( \mathbb{P}^1 \)-bundle, with the section \( D^+ \) corresponding to \( O_U(-L) \). Then \( E^\prime + := \psi^{-1}(C) \simeq \Sigma \). Let \( E^+ := E^\prime + \cap D^+ \), then since \( D^+ \supset E^+ \) is mapped isomorphically onto \( U \supset C \), \( E^+ \) is a \((0, -2)\)-curve of width \( m \) inside \( D^+ \). Moreover \( E^+ \simeq \mathbb{P}^1 \) is the negative section of \( E^\prime + \), and is a contractible curve in \( X^+ : X^+ \to Y \), with

\[
N_{E^+/X^+} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-2).
\]

Follow the induction argument 8.8 (with 8.9, 8.10) in the reverse direction starting from this \( X^+ \supset E^+ \simeq \mathbb{P}^1 \), then we get a flipping contraction of Type (R), with width \( m \): 

\[
X \supset E \simeq \mathbb{P}^2 \to Y \supset Q
\]
(the inverse flip). In particular there is exactly one singular point $P$ on $X$, which is of form:

$$(X, P) \simeq \{x_1x_3 + x_2x_4 + x_5^m = 0\} \supset E = \{x_3 = x_4 = x_5 = 0\}.$$

Let $E'$ be the proper transform of $E' + X$ in $X$, then $E' \simeq \mathbb{P}^2$, and

$$E \cap E' = \{P\}, \ E' = \{x_1 = x_2 = x_5 = 0\}$$

in the above description.

Since everything in the above procedure is defined over $(Z, Q)$, and since we never touched the part $X^+ - E'^+$, there exists a contraction $h : X \to (Z, Q)$, which is a $\mathbb{P}^1$-bundle over $Z - \{Q\}$. As is easily seen the central fiber is a union of two $\mathbb{P}^2$'s meeting just at $P$: $h^{-1}(Q) = E \cup E'$. Moreover

$$(Z, Q) \simeq \{z_1z_2 + z_3^2 + z_4^{2m} = 0\}.$$ 

This $h$ gives the desired example.

§ Appendix. On terminal complete intersection singularities
— S. Ishii’s theorem

A.0. The following implications hold in arbitrary dimensions:

Hypersurface singularity $\iff$ Complete intersection singularity $\iff$ Gorenstein singularity.

A.1. (Reid [Re1,4])

For 3-dimensional terminal singularities, all of these are equivalent, and general hyperplane sections of those are Du Val singularities. (These are called $cDV$-singularities.)

A.2. In dimension 4, however, the reverse directions of both arrows in A.0 do not hold, even for terminal singularities (Mori [Mo5], Kac–Watanabe [KW]).

Here is a collection of references on terminal singularities in dimension 3 and 4: [D], [Kol1], [KS], [Mo3,4], [MoMoMo], [MS], [Re1,2,4].

After the completion of this paper, the author learned the following theorem, which was originally conjectured independently by Yu. Prokhorov and K.-i. Watanabe, and has been solved by S. Ishii:

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Theorem A.3. \textit{(S. Ishii)}

Let \((Z, P)\) be a germ of an isolated terminal complete intersection singularity, of dimension \(n\). Then

(1) \(\text{emb. codim}(Z, P) \leq n - 2\).

(2) Let

\[(Z, P) \simeq \{ f_1(x_1, \ldots, x_N) = \ldots = f_r(x_1, \ldots, x_N) = 0 \} \subset (\mathbb{C}^N, 0),\]

where \(r := \text{emb. codim}(Z, P)\) and \(N = n + r\) (so that this is an irredundant expression). Then

\[\sum_{i=1}^{r} \text{mult} f_i \leq n + r - 2.\]

(3) In particular if the equality holds in (1) i.e. \(r = n - 2\), then

\[\text{mult} f_i = 2 \ (\forall i = 1, \ldots, n - 2).\]

This reproduces the following, which is a weaker version of the above mentioned theorem (A.1) of Reid [Re1] in the case \(n = 3\):

Corollary A.4. A 3-dimensional terminal complete intersection singularity is a hypersurface double point.

The following is the case \(n = 4\) in Theorem A.3:

Corollary A.5. \textit{(S. Ishii)}

A 4-dimensional isolated terminal complete intersection singularity is either

(a) a hypersurface double or a triple point in \((\mathbb{C}^5, 0)\), or

(b) an intersection of two double hypersurfaces in \((\mathbb{C}^6, 0)\).

(See [Mo5] for some example.)

By virtue of this Corollary A.5, our Corollary 4.3 became unnecessary.
References

[ABW] M. Andreatta, E. Ballico and J. Wiśniewski, Two theorems on elementary contractions, Math. Ann. 297 (1993), 191–198.

[AW1] M. Andreatta and J. Wiśniewski, A note on nonvanishing and applications, Duke Math. J. 72 (1993), 739–755.

[AW2] M. Andreatta and J. Wiśniewski, On good contractions of smooth varieties, Preprint (1996).

[Be] M. Beltrametti, On d-folds whose canonical bundle is not numerically effective, according to Mori and Kawamata, Ann. Mat. Pura. Appl. 147 (1987), 151–172.

[Br] E. Brieskorn, Singular elements of semi-simple algebraic groups, Proc. Int. Cong. Math. Nice 2 (1970), 279–284.

[Co] A. Corti, Semi-stable 3-fold flips, Preprint (1994).

[D] V.I. Danilov, Birational geometry of toric 3-folds, Math. USSR. Izv. 21 (1983), 269–279.

[E] R. Elkik, Singularités rationelles et déformations, Invent. Math. 47 (1978), 139–147.

[Fra] P. Francia, Some remarks on minimal models, Compos. Math. 40 (1980), 301–313.

[Fri] R. Friedman, Simultaneous resolution of threefold double points, Math. Ann. 274 (1986), 671–689.

[G1] M. Gross, Deforming Calabi-Yau threefolds, Preprint (1995).

[G2] M. Gross, Primitive Calabi-Yau threefolds, Preprint (1995).

[H1] E. Horikawa, Deformations of holomorphic maps, I, J. of Math. Soc. Japan 25 (1973), 372–396.

[H2] E. Horikawa, ibid, II, J. of Math. Soc. Japan 26 (1974), 647-667.

[H3] E. Horikawa, ibid, III, Math. Ann. 222 (1976), 275–282.

[Io] P. Ionescu, Generalized adjunction and applications, Math. Proc. Camb. Phil. Soc. 99 (1986), 457–472.

[Is] S. Ishii, On isolated Gorenstein singularities, Math. Ann. 270 (1985), 541–554.

[KW] V. Kac, K.-i. Watanabe, Finite linear groups whose ring of invariants is a complete intersection, Bull. Amer. Math. Soc. 6, No2 (1982), 221–223.

[Kac1] Y. Kachi, Extremal contractions from 4-dimensional manifolds to 3-folds, Ann. di Pisa (To appear).

[Kac2] Y. Kachi, Flips from semi-stable 4-folds whose degenerate fibers are unions of Cartier divisors which are terminal factorial 3-folds, Math. Ann. (To appear).

[Kaw1] Y. Kawamata, Elementary contractions of algebraic 3-folds, Ann. of Math. 119 (1984), 95–110.

[Kaw2] Y. Kawamata, The cone of curves of algebraic varieties, Ann. of Math. 119 (1984), 603–633.

[Kaw3] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. 127 (1988), 93–163.

[Kaw4] Y. Kawamata, Small contractions of four dimensional algebraic manifolds, Math. Ann. 284 (1989), 595–600.

[Kaw5] Y. Kawamata, On the length of an extremal rational curve, Invent. math. 105 (1991), 609–611.

[Kaw6] Y. Kawamata, Unobstructedness deformations – A remark to a paper of Z. Ran, J. of Alg. Geom. 1 (1992), 183–190.
[Kaw7] Y. Kawamata, *Termination of log flips for algebraic 3-folds*, Int. J. of Math. 3 (1992), 653–659.

[Kaw8] Y. Kawamata, *Semistable minimal models of threefolds in positive or mixed characteristic*, J. of Alg. Geom. 3 (1994), 463–491.

[Kaw9] Y. Kawamata, *General hyperplane sections of nonsingular flops in dimension 3*, Math. Res. Lett. 1 (1994), 49–52.

[KaMaMa] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. St. Pure Math. 10 (1987), 283–360.

[KaMo] S. Katz and D. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory of Weyl groups*, J. of Alg. Geom. 1 (1992), 449–530.

[Kod] K. Kodaira, *On stability of compact submanifolds of complex manifolds*, Amer. J. of Math. 85 (1963), 79–84.

[Kol1] J. Kollár, *Flops*, Nagoya Math. J. 113 (1989), 15–36.

[Kol2] J. Kollár, *Rational curves on algebraic varieties*, Univ. of Utah, 1994.

[Kol3] J. Kollár, *Flatness criteria*, Preprint (1994).

[KoMiMo1] J. Kollár, Y. Miyaoka and S. Mori, *Rational curves on Fano varieties*, Preprint, RIMS Kyoto (1991).

[KoMiMo2] J. Kollár, Y. Miyaoka and S. Mori, *Rationally connected varieties*, J. of Alg. Geom. 1 (1992), 429–448.

[KoMiMo3] J. Kollár, Y. Miyaoka and S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. of Diff. Geom. 36 (1992), 765–779.

[KoMo] J. Kollár and S. Mori, *Classification of three dimensional flips*, J. of Amer. Math. Soc. 5 (1992), 533–703.

[KS] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. 91 (1988), 299–338.

[L] H. Laufer, *On \(\mathbb{CP}^1\) as an exceptional set*, in Ann. of Math. Studies, vol. 100, Princeton Univ. Press, 1981, pp. 261–275.

[Ma] K. Matsuki, *Weyl groups and ..., ?? ?? (1995), ??.

[Mo1] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. 110 (1979), 593–606.

[Mo2] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. 116 (1982), 133–176.

[Mo3] S. Mori, *On 3-dimensional terminal singularities*, Nagoya Math. J. 98 (1985), 43–66.

[Mo4] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. of Amer. Math. Soc. 1 (1988), 117–253.

[Mo5] S. Mori, *Dear Miles*, Appendix to [Re5].

[MoMoMo] S. Mori, D. Morrison and I. Morrison, *On four dimensional terminal quotient singularities*, Preprint.

[MS] D. Morrison and G. Stevens, *Terminal quotient singularities in dimension 3 and 4*, Proc. Amer. Math. Soc. 90 (1984), 15–20.

[Nak] N. Nakayama, *On smooth exceptional curves in threefolds*, Thesis, Univ. of Tokyo (1989).

[Nam1] Yo. Namikawa, *Smoothing Fano 3-folds*, Preprint (1994).

[Nam2] Yo. Namikawa, *Deformation theory of Calabi-Yau threefolds and certain invariants of singularities*, Preprint, Max-Planck-Inst. (1995).
[Nam3] Yo. Namikawa, *Stratified local moduli of Calabi-Yau threefolds*, Preprint, Max-Planck-Inst. (1995).

[NS] Yo. Namikawa and J.H.M. Steenbrink, *Global smoothing of Calabi-Yau threefolds*, Preprint, Max-Planck-Inst. (1995).

[OSS] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Progress math., vol. 3, Birkhäuser, 1980.

[Pi1] H. Pinkham, *Résolution simultanée de points doubles rationels*, Lect. Notes Math., vol. 777, Springer, 1980.

[Pi2] H. Pinkham, *Factorization of birational maps in dimension 3*, Proc. Symp. Pure. Math. 40 (1983), 343–371.

[Pr1] Y. Prokhorov, *On Q-Fano fiber spaces with two-dimensional base*, Preprint, Max-Planck-Inst. (1995).

[Pr2] Y. Prokhorov, *On extremal contractions from threefolds to surfaces*, Preprint, Univ. of Warwick. (1995).

[Ra1] Z. Ran, *Deformation of maps*, Algebraic curves and projective geometry, LNM, vol. 1389, Springer, 1989, pp. 246–253.

[Ra2] Z. Ran, *Deformations of manifolds with torsion or negative canonical bundle*, J. of Alg. Geom. 1 (1992), 279–291.

[Re0] M. Reid, *Canonical 3-folds*, in Journée de Géométrie Algébrique d’Angers (A. Beauville, ed.), Sijthoff and Noordhoff, 1980, pp. 273–310.

[Re1] M. Reid, *Minimal models of canonical 3-folds*, Adv. St. Pure Math. 1 (1983), 131–180.

[Re2] M. Reid, *Decomposition of toric morphisms*, in Arithmetic and Geometry II, Progress Math., vol. 36, Birkhaüser, 1983, pp. 395–418.

[Re3] M. Reid, *Projective morphisms according to Kawamata*, Preprint (1983).

[Re4] M. Reid, *Young person’s guide to canonical singularities*, Proc. Symp. Pure Math. 46, Vol. 1 (1987), 345–414.

[Re5] M. Reid, *What is a flip?*, Preprint (1993).

[Sc] M. Schlessinger, *Rigidity of quotient singularities*, Invent. Math. 14 (1971), 17–26.

[Sh1] V.V. Shokurov, *The nonvanishing theorem*, Math. USSR. Izv. 26, No.3 (1986), 591–604.

[Sh2] V.V. Shokurov, *3-fold log flips*, Math. USSR. Izv. 40, No.1 (1993), 95–202.

[Utah] J. Kollár et.al., *Flips and abundance for algebraic threefolds*, Astérisque, vol. 211, Soc. Math. de France, 1992.

[V] A. Van de Ven, *On uniform vector bundles*, Math. Ann. 195 (1972), 245–248.

[W] J. Wiśniewski, *On contraction of extremal rays of Fano manifolds*, J. Reine. Angew. Math. 417 (1991), 141–157.