Abstract. In this paper we study the stability of the replicator dynamics for symmetric games when the strategy set is a separable metric space. In this case the replicator dynamics evolves in a space of measures. We study stability criteria with respect to different topologies and metrics on the space of probability measures. This allows us to establish relations among Nash equilibria (of a certain normal form game) and the stability of the replicator dynamics in different metrics. Some examples illustrate our results.

1. Introduction. In this paper we are interested in symmetric evolutionary games whose strategies’ interactions are modeled with a specific dynamical system known as the replicator dynamics. This type of games models interactions of strategies of a single population, and form part of the so-called population games.

In game theory it is important to consider models with strategies in measurable spaces because this allows us to present in a unified manner essentially all the standard models, namely, with finite strategy sets, compact or unbounded intervals, and so on. This the case for some models in oligopoly theory, international trade theory, war of attrition, and public goods, among others.

This paper presents an evolutionary dynamics for games where the set of strategies is a metric space. We have consequently that the dynamical system lives in a Banach space, which in our case is a space of finite signed measures. In particular, if the strategy set is finite, then the dynamical system lives in $\mathbb{R}^m$, where $m$ is the number of strategies.

Conditions for stability of the replicator dynamics have been developed by several authors. Bomze [5] provides a stability theorem for different topologies. Oechssler and Riedel [23] provide a stability theorem under the total variation norm for pure strategies that are critical points; in [24] the same authors study stability under the Prokhorov metric in a double symmetric game (named a potential game by...
Cressman and Hofbauer [8]) and prove that a strongly uninvadable strategy with dominance in the weak topology satisfies other static evolutionary concepts such as evolutionary stable strategy, continuously stable strategy, and neighborhood invader strategy, which are sufficient to guarantee dynamic stability in the weak topology for the replicator dynamics. Eshel and Sansone [10], Cressman [7], Cressman, Hofbauer and Riedel [9], use these evolutionary concepts and different hypotheses in the payoff function to guarantee dynamic stability. Van Veelen and Spreij [31] also study relationships between static evolutionary concepts and asymptotic evolutionary stability. Cressman and Hofbauer [8] establish general conditions for asymptotic evolutionary stability. Norman [22] establishes dynamic stability in terms of strategies.

The main goal of this paper is to present a general model for evolutionary games and give a stability theorem with respect to the Wasserstein distance. The Wasserstein distance metrizes the weak topology and it is closely related to the variation norm and the Kullback-Leibler distance. This allows us to give better approximations than those in Bomze [4], Theorem 2, for the case where the stability criterion is with respect to the weak topology. We study stability criteria with respect to strong and weak topologies, and with different metrics on a space of probability measures. In particular, we characterize the relation between a Nash equilibrium of a certain normal form game and the stability of the dynamical system (see Corollary 1).

In the theory of evolutionary games there are several interesting dynamics, for instance, the imitation dynamics, the monotone-selection dynamics, the best-response dynamics, the Brown-von Neumann-Nash dynamics, and so forth (see, for instance, Hofbauer and Sigmund [14], [15], Sandholm [29]). Some of these evolutionary dynamics have been extended to games with strategies in a space of probability measures. For instance, Hofbauer, Oechssler and Riedel [13] extend the Brown-von Neumann-Nash dynamics; Lahkar and Riedel extend the logit dynamics [17]. These publications establish conditions for the existence of solutions and the stability of the corresponding dynamical system.

We selected the replicator dynamics partly because it is the most studied dynamics for games with strategies in metric spaces, and partly because it has many interesting properties, as can be seen in Cressman [6], Hofbauer and Weibull [16], and many other references. In particular, with the replicator dynamics it is not difficult to construct a proof of the existence of Nash equilibria and, moreover, when the strategy set is finite, we can give a geometric characterization of the set of Nash equilibria; see Harsanyi [12], Hofbauer and Sigmund [14], Ritzberger [27].

We present results of three types, which we can summarize as follows. We establish relations between three key concepts: the critical points of the replicator dynamics, the Nash equilibrium strategies, and the strongly uninvadable strategies. See, for instance, relations (18) and Propositions 2 and 4. Second, we prove results about stability in different topologies and metrics as, for instance, in Theorem 5.1, relations (30), Proposition 6, and Corollary 1. We also present results about stability for the special case of Dirac measures, as in Theorem 5.2, and Remark 3. Finally, we present two examples, the first one may be applicable to oligopoly models, theory of international trade, and public good models; the second example deals with a graduated risk game.

The paper is organized as follows. Section 2 introduces notation and technical requirements. Section 3 describes the replicator dynamics and its relation to
evolutionary games. Some important technical issues are also summarized. Section 4 establishes the relation between the replicator dynamics and a normal form game, using the concepts of Nash equilibria and strongly uninvadable strategies. Section 5 presents two of our main stability results, Theorems 5.1 and 5.2. Section 6 establishes a relationship between Nash equilibria and the replicator-dynamics stability. Section 7 proposes examples to illustrate our results. We conclude in section 8 with some general comments on possible extensions. An appendix contains results of some technical facts.

2. Technical preliminaries.

2.1. Spaces of signed measures. Consider a separable metric space \((A, \vartheta)\) and its Borel \(\sigma\)-algebra \(\mathcal{B}(A)\). Let \(\mathcal{M}(A)\) be the Banach space of finite signed measures \(\mu\) on \(\mathcal{B}(A)\) endowed with the total variation norm

\[
\|\mu\| := \sup_{\|f\| \leq 1} \left| \int_A f(a) \mu(da) \right|.
\]

(1)

The supremum in (1) is taken over functions in the Banach space \(\mathcal{B}(A)\) of real-valued bounded measurable functions on \(A\), endowed with the supremum norm

\[
\|f\| := \sup_{a \in A} |f(a)|.
\]

(2)

Consider the subset \(\mathcal{C}(A) \subset \mathcal{B}(A)\) of all real-valued continuous and bounded functions on \(A\). Consider the dual pair \((\mathcal{C}(A), \mathcal{M}(A))\) given by the bilinear form \(\langle \cdot, \cdot \rangle : \mathcal{C}(A) \times \mathcal{M}(A) \to \mathbb{R}\)

\[
\langle g, \mu \rangle = \int_A g(a) \mu(da).
\]

(3)

We consider the weak topology on \(\mathcal{M}(A)\) (induced by \(\mathcal{C}(A)\)), i.e., the topology under which all elements of \(\mathcal{C}(A)\), when regarded as linear functionals \(\langle g, \cdot \rangle\) on \(\mathcal{M}(A)\) are continuous. In this topology, a neighborhood of a point \(\mu \in \mathcal{M}(A)\) is of the form

\[
\mathcal{V}_\epsilon^\mathcal{H}(\mu) := \left\{ \nu \in \mathcal{M}(A) : |\langle g, \nu - \mu \rangle| < \epsilon \ \forall g \in \mathcal{H} \right\}
\]

(4)

for \(\epsilon > 0\) and \(\mathcal{H}\) a finite subset of \(\mathcal{C}(A)\).

**Definition 2.1.** A sequence of measures \(\mu_n \in \mathcal{M}(A)\) is said to be weakly convergent if there exists \(\mu \in \mathcal{M}(A)\) such that

\[
\lim_{n \to \infty} \int_A g(a) \mu_n(da) = \int_A g(a) \mu(da)
\]

(5)

for all \(g\) in \(\mathcal{C}(A)\). If \(\mathcal{M}(A)\) is instead the space \(\mathcal{P}(A)\) of probability measures on \(A\), then sometimes we say that \(\mu_n\) converges in distribution to \(\mu\).

2.2. Metrics on \(\mathcal{P}(A)\). There are many metrics that metrize the weak topology. In particular, here we use a Wasserstein metric because of its interesting properties, some of which are described in this section. (For further details see, for instance, Villani [32], chapter 6).

Let \(A\) be a Polish space, that is, a complete separable metric space with a metric \(\vartheta\). Let \(a_0\) be a fixed point in \(A\). For each \(p\) with \(1 \leq p < \infty\), we define the space \(\mathcal{P}_p(A)\) as

\[
\mathcal{P}_p(A) := \left\{ \mu \in \mathcal{P}(A) : \int_A [\vartheta(a, a_0)]^p \mu(da) < \infty \right\}.
\]
Moreover, the metric \( \vartheta \) on \( P \) is defined by

\[
 r_{w_p}(\mu, \nu) := \left[ \inf_{\pi \in \Pi} \int_a^b \vartheta(a, b)^p \pi(da, db) \right]^{\frac{1}{p}},
\]

where \( \Pi \) is the set of probability measures on \( A \times A \) with marginals \( \mu \) and \( \nu \). In particular, when \( p = 1 \) we write the \( L^1 \)-Wasserstein distance \( r_{w_1} \) as \( r_w \). Moreover, the \( L^1 \)-Wasserstein distance coincides with the Kantorovich-Rubinstein metric on \( P(A) \) (see Villani [32]).

Any Wasserstein distance has important properties. For instance, they preserve the metric \( \vartheta \) on \( P(A) \), i.e., for any \( a, b \in A \) and \( p \in [1, \infty) \) the distance between the Dirac measures \( \delta_a \) and \( \delta_b \) is \( r_{w_p}(\delta_a, \delta_b) = \vartheta(a, b) \). This is not true for the total variation norm (1), because, for instance, \( \|\delta_a - \delta_b\| = 2 \) for any \( a, b \in A \) with \( a \neq b \).

Another important property of the Wasserstein distance (6) is its interpretation (see Villani [32]): the distance \( \vartheta(a, b) \) can be seen as the cost for transporting one unit of mass \( \pi(\cdot, b) \) from \( a \) to one unit of mass \( \pi(a, \cdot) \) from \( b \).

**Remark 1.** There exist several metrics that metrize the weak topology of \( P(A) \). Among the most well-known are the Prokhorov metric, the bounded-Lipschitz metric, and the Kantorovich-Rubinstein metric. In the rest of this paper we will denote by \( r_{w} \) any metric that metrizes the weak topology on \( P(A) \) (not to be confused with the notation \( r_{w_p} \) of the \( L^1 \)-Wasserstein distance). Moreover, we denote by \( r \) any metric on \( P(A) \), that includes the total variation norm (1) or any distance that metrizes the weak topology \( \tau_{w_T} \). An open ball in the metric space \( (P(A), r) \) is defined in the classical form

\[
 V^r_\alpha(\mu) := \{ \nu \in P(A) : r(\nu, \mu) < \alpha \}
\]

where \( \alpha > 0 \).

**Remark 2.** a) Let \( A \) be a separable metric space, and \( r_{w_T} \) any distance that metrizes the weak topology \( \tau_{w_T} \) in \( P(A) \). Let \( \mu \) be any measure in \( P(A) \), and consider the family \( V^H(\mu) \) of neighborhoods \( V^H_\alpha(\mu) \) of the form (4). In addition, consider the family \( V^{r_{w_T}}(\mu) \) of the open balls \( V^{r_{w_T}}_\alpha(\mu) \) of the form (7). Both families \( V^H(\mu) \) and \( V^{r_{w_T}}(\mu) \) are neighborhood basis for \( \mu \) in the space \( (P(A), \tau_{w_T}) \). For details see Pedersen [25], chapters I-II.

b) A neighborhood \( V^H_\alpha(\mu) \) of \( \mu \) is contained in some open ball \( V^{r_{w_T}}_\alpha(\mu) \) with center \( \mu \). The converse is also true, i.e., any open ball \( V^{r_{w_T}}_\alpha(\mu) \) is contained in some neighborhood \( V^H_\alpha(\mu) \) (see Munkres [21], Chapter 2, Lemma 13.3).

### 2.3. Differentiability.

**Definition 2.2.** Let \( A \) be a separable metric space. We say that a mapping \( \mu : [0, \infty) \to M(A) \) is strongly differentiable if there exists \( \mu'(t) \in M(A) \) such that, for every \( t > 0 \),

\[
 \lim_{\epsilon \to 0} \left\| \frac{\mu(t + \epsilon) - \mu(t)}{\epsilon} - \mu'(t) \right\| = 0.
\]

### 3. The model.

#### 3.1. Symmetric evolutionary games.

Consider a population of individuals of a single species. Each individual in the population can choose a single element \( a \)
in a set \( A \) of characteristics (the set of pure strategies or pure actions), which is a separable metric space. Let \( \mathbb{P}(A) \) be the set of probability measures on \( A \), also known as the set of mixed strategies. Moreover, consider a payoff function \( J : \mathbb{P}(A) \times \mathbb{P}(A) \to \mathbb{R} \) that explains the interrelation between the population, and which is defined as

\[
J(\mu, \nu) := \int_A \int_A U(a, b)\nu(db)\mu(da)
\]  

(9)

where \( U : A \times A \to \mathbb{R} \) is a given measurable function. If \( \delta_{\{a\}} \) is a probability measure concentrated at \( a \in A \), the vector \((\delta_{\{a\}}, \mu)\) is written as \((a, \mu)\), and (9) becomes

\[
J(\delta_{\{a\}}, \mu) = J(a, \mu).
\]

In particular, (9) yields

\[
J(\mu, \nu) := \int_A J(a, \nu)\mu(da)
\]

(10)

In an evolutionary game, the strategies’ dynamics is determined by a differential equation of the form

\[
\mu'(t) = F(\mu(t)) \quad t \geq 0,
\]

(11)

with some initial condition \( \mu(0) = \mu_0 \). The notation \( \mu'(t) \) represents the strong derivative of \( \mu(t) \) (see Definition 2.2), and \( F(\cdot) \) is a given mapping \( F : \mathbb{P}(A) \to \mathbb{M}(A) \). More explicitly, we write (11) as

\[
\mu'(t, E) = F(\mu(t), E) \quad \forall E \in \mathcal{B}(A),
\]

(12)

where \( \mu'(t, E) \) and \( F(\mu(t), E) \) are the measures \( \mu'(t) \) and \( F(\mu(t)) \) valued at \( E \in \mathcal{B}(A) \).

We shall be working with a special class of so-called symmetric evolutionary games which can be described as a quadruple

\[
\left\{ I, \mathbb{P}(A), J(\cdot), \mu'(t) = F(\mu(t)) \right\},
\]

(13)

where

i) \( I := \{1, 2\} \) is the set of players;

ii) for each player \( i = 1, 2 \) we have a set \( \mathbb{P}(A) \) of mixed actions and a payoff function \( J : \mathbb{P}(A) \times \mathbb{P}(A) \to \mathbb{R} \) (as in (9)); and

iii) the dynamics (11)-(12) is described by the following replicator function (14), for each \( E \in \mathcal{B}(A) \),

\[
F(\mu(t), E) := \int_E \left[ J(a, \mu(t)) - J(\mu(t), \mu(t)) \right] \mu(t, da).
\]

(14)

3.2. Technical issues on the replicator dynamics. For future reference, in the remainder of this section we summarize conditions for the existence of a unique solution to the differential equation (11)-(12) with \( F \) as in (14), and an important property of this solution. These results can be traced back to Bomze [5], Oechssler and Riedel [23]. See also Mendoza-Palacios and Hernández-Lerma [20].

For each \( t \geq 0 \), let

\[
\beta(a|\mu(t)) := J(a, \mu(t)) - J(\mu(t), \mu(t)),
\]

(15)

which is the integrand of (14). Hence, \( \beta(\cdot|\mu(t)) \) is the Radon-Nikodym density of \( F(\mu(t)) \) with respect to \( \mu(t) \), i.e.,

\[
F(\mu(t), E) = \int_E \beta(a|\mu(t))\mu(t, da) \quad \forall E \in \mathcal{B}(A).
\]
Theorem 3.1. Suppose that the function $\beta(\cdot|\mu)$ in (15) satisfies:

i) there exists $C \geq 0$ such that

$$|\beta(a|\mu)| \leq C \quad \forall a \in A \text{ and } \|\mu\| \leq 2,$$

ii) there is a constant $D > 0$ such that

$$\sup_{a \in A} |\beta(a|\eta) - \beta(a|\nu)| \leq D\|\eta - \nu\| \quad \forall \eta, \nu \text{ with } \|\eta\|, \|\nu\| \leq 2.$$

Then there exists a unique solution to the replicator dynamics equation (11)-(12) (with $F(\cdot)$ as in (14)). Moreover, if $\mu(t)$ is such a solution with initial condition $\mu(0)$ in $\mathbb{P}(A)$, then $\mu(0) \ll \mu(t)$ and $\mu(t) \ll \mu(0)$ for all $t > 0$, with Radon-Nikodym density

$$\frac{d\mu(t)}{d\mu(0)}(a) = e^{\int_0^t \beta(a|\mu(s))ds}.$$  \hspace{1cm} (16)

In particular, for every $t > 0$, if $\nu$ is a probability measure satisfying that $\nu \ll \mu(t)$ whenever $\nu \ll \mu(0)$, then

$$\log \frac{d\nu}{d\mu(t)}(a) = \log \frac{d\nu}{d\mu(0)}(a) - \int_0^t \beta(a|\mu(s))ds.$$  \hspace{1cm} (17)

Proposition 1. If the payoff function $U(\cdot)$ in (9) is bounded, then $\beta(\cdot|\mu)$ satisfies the conditions i) and ii) of Theorem 3.1.

4. The replicator dynamics: NESs and SUSs. In this section we consider symmetric evolutionary games as in (13) and compare them with normal-form games (19), below. We study the relation between a Nash equilibrium of a normal-form game and the replicator dynamics (Proposition 2). We also define the important concept of strongly uninvadable strategy (Definition 4.2) and its relation to a Nash equilibrium (Proposition 4). Summarizing, in this section we show that

$$r - SUS \subset N' \subset C.$$ \hspace{1cm} (18)

where $C$ is the set of critical points of the replicator dynamics, $N'$ is the family of Nash equilibrium strategies for (20), $r - SUS$ is the subfamily of $r$-strongly uninvadable strategies for any metric $r$ on $\mathbb{P}(A)$ (see Definition 4.2). This results will be complemented in Section 6 (see Corollary 1).

A normal form game $\Gamma$ (also known as a game in strategic form) can be described as a triplet

$$\Gamma := \left[ I, \left\{ \mathbb{P}(A_i) \right\}_{i \in I}, \left\{ J_i(\cdot) \right\}_{i \in I} \right],$$ \hspace{1cm} (19)

where

i) $I = \{1, 2, ..., N\}$ is the set of players;

ii) for each player $i \in I$, we specify a set of actions (or strategies) $\mathbb{P}(A_i)$; and

iii) a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$.

If $I = \{1, 2\}$, we can obtain from (19) a symmetric normal-form game with two players, and sets of actions and payoff functions as $\mathbb{P}(A) = \mathbb{P}(A_1) = \mathbb{P}(A_2)$ and $J(\mu_1, \mu_2) = J_1(\mu_1, \mu_2) = J_2(\mu_2, \mu_1)$ for all $\mu_1, \mu_2 \in \mathbb{P}(A)$. Hence, we can describe a two-player symmetric normal form game as

$$\Gamma_s := \left[ I = \{1, 2\}, \mathbb{P}(A), J(\cdot) \right].$$ \hspace{1cm} (20)

For symmetric normal form games $\Gamma_s$ we can express a symmetric Nash equilibrium $(\mu^*, \mu^*)$ in terms of the strategy $\mu^* \in \mathbb{P}(A)$, as follows.
Definition 4.1. We say that \( \mu^* \in \mathbb{P}(A) \) is a Nash equilibrium strategy (NES) if the pair \((\mu^*, \mu^*)\) is a Nash equilibrium for \( \Gamma_s \). That is,
\[
J(\mu^*, \mu^*) \geq J(\mu, \mu^*) \quad \forall \mu \in \mathbb{P}(A).
\]

Proposition 2. Let \( \mu^* \) be a NES for \( \Gamma_s \). Then \( \mu^* \) is a critical point of \((11)-(12)\) (i.e., \( F(\mu^*) = 0 \)) when \( F(\cdot) \) is described by the replicator function \((14)\).

Proof. See Mendoza-Palacios and Hernández-Lerma [20], Theorem 5.4. \( \square \)

For a partial converse of Proposition 2 see Proposition 6.

The following definition is a slightly modified version of the strongly uninvadable strategies used in Bomze [5].

Definition 4.2. Let \( r \) be a metric on \( \mathbb{P}(A) \). A measure \( \mu^* \in \mathbb{P}(A) \) is called an \( r \)-strongly uninvadable strategy \((r\text{-SUS})\) if there exists \( \epsilon > 0 \) such that for any \( \mu \) with \( r(\mu, \mu^*) < \epsilon \), it follows that \( J(\mu^*, \mu) > J(\mu, \mu) \). We call \( \epsilon \) the global invasion barrier.

When \( r \) is the Prokhorov metric \( r_p \), Oechssler and Riedel [24] name a \( r_p \text{-SUS} \) an evolutionary robust strategy. If \( r_{\text{w}r} \) is any metric that metrizes the weak topology (recall Remark 1), Cressman and Hofbauer [8] call a \( r_{\text{w}r} \text{-SUS} \) a locally superior strategy.

Proposition 3. Let \( r_{\text{w}r} \) be a distance that metrizes the weak convergence on \( \mathbb{P}(A) \). If a measure \( \mu^* \in \mathbb{P}(A) \) is \( r_{\text{w}r} \text{-SUS} \), then it is \( \| \cdot \| \text{-SUS} \).

Proof. Let \( \mu \) be in the open ball \( \mathcal{V}_\epsilon(\mu^*) \) defined in \((7)\). Then there is some open neighborhood \( \mathcal{V}_\delta(\mu^*) \) of the form \((4)\) such that \( \mu^* \in \mathcal{V}_\delta(\mu^*) \). Then, by Remark 2, there is some open ball \( \mathcal{V}_{\alpha r}(\mu^*) \) such that \( \mu \in \mathcal{V}_{\alpha r}(\mu^*) \) and the proposition follows. \( \square \)

The next lemma is a key fact to provide a general framework for the different stability criteria.

Lemma 4.3. Let \( r_{\text{w}r} \) be a distance that metrizes the weak convergence on \( \mathbb{P}(A) \). For every \( \mu, \nu \in \mathbb{P}(A) \) and \( \epsilon > 0 \), there exist \( \alpha \) and \( \alpha' \) in \((0, 1)\) and \( \eta, \gamma \in \mathbb{P}(A) \) such that
\[
i) \ r_{\text{w}r}(\eta, \mu) < \epsilon \text{ if } \eta = \alpha \nu + (1 - \alpha) \mu, 
\]
\[
ii) \ \| \gamma - \mu \| < \epsilon \text{ if } \gamma = \alpha' \nu + (1 - \alpha') \mu.
\]

Proof. Let \( \alpha_n \) be a sequence in \((0, 1)\) such that \( \alpha_n \to 0 \), and let \( \eta_n := \alpha_n \nu + (1 - \alpha_n) \mu \). If \( f \in \mathcal{C}(A) \) then
\[
\lim_{n \to \infty} \int_A f(a) \eta_n(da) = \lim_{n \to \infty} \alpha_n \int_A f(a) \nu(da) + \lim_{n \to \infty} (1 - \alpha_n) \int_A f(a) \mu(da)
\]
\[
= \int_A f(a) \mu(da).
\]
Hence, by Proposition 7 in the Appendix, part i) follows.

On the other hand, let \( 0 < \alpha' < \frac{\epsilon - \| \gamma - \mu \|}{\| \nu - \mu \|} \). Then
\[
\| \gamma - \mu \| = \| \alpha' \nu + (1 - \alpha') \mu - \mu \| = \alpha' \| \nu - \mu \| < \epsilon,
\]
and ii) holds. \( \square \)

The following proposition shows that a strongly uninvadable strategy is also a Nash equilibrium strategy.
Proposition 4. Let $r$ be a metric on $\mathbb{P}(A)$. If $\mu^*$ is a $r$-SUS, then $\mu^*$ is a NES of $\Gamma_s$.

Proof. Suppose that $\mu^*$ is not a NES of $\Gamma_s$. Then there exists $\nu \in \mathbb{P}(A)$ such that
\[
J(\nu, \mu^*) > J(\mu^*, \mu^*). \tag{22}
\]
By Lemma 4.3, there exists $\eta := \alpha \nu + (1 - \alpha) \mu^*$ for some $\alpha \in (0, 1)$, with $r(\eta, \mu^*) < \epsilon$.
Since $\mu^*$ is $r$-SUS, $J(\mu^*, \eta > J(\eta, \eta)$ and so
\[
\alpha J(\mu^*, \nu) + (1 - \alpha) J(\nu, \mu^*) > \alpha \alpha J(\nu, \nu) + (1 - \alpha) \alpha J(\nu, \mu^*)
+ (1 - \alpha) \alpha J(\mu^*, \nu)
+ (1 - \alpha)(1 - \alpha) J(\mu^*, \mu^*).
\]
Hence
\[
\alpha J(\mu^*, \nu) + (1 - \alpha) J(\mu^*, \mu^*) > \alpha J(\nu, \nu) + (1 - \alpha) J(\nu, \mu^*). \tag{23}
\]
If (22) is true, then there exists $\alpha > 0$ sufficiently small such that (23) is violated. Thus $\mu^*$ is a NES for $\Gamma_s$.

5. Stability. This section presents a review of results on the stability of the replicator dynamics. These results include different stability criteria with respect to various metrics and topologies in the space of probability measures.

Assume that $\nu << \mu$. We define the cross entropy or Kullback-Leibler distance of $\nu$ with respect to $\mu$ as
\[
K(\mu, \nu) := \int_A \log \left( \frac{d\nu}{d\mu}(a) \right) \nu(da). \tag{24}
\]
From Jensen’s inequality it follows that $0 \leq K(\mu, \nu) \leq \infty$ and $K(\mu, \nu) = 0$ if and only if $\mu = \nu$. The Kullback-Leibler distance is not a metric, since it is not symmetric, i.e., $K(\mu, \nu) \neq K(\nu, \mu)$.

Given $\mu^* \in \mathbb{P}(A)$, $\epsilon > 0$, and a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$, we define the set
\[
W_{\varphi(\epsilon)}(\mu^*) := \{ \mu \in \mathbb{P}(A) : K(\mu, \mu^*) < \varphi(\epsilon) \}. \tag{25}
\]

The following theorem characterizes the stability of the replicator dynamics with respect to the $L_1$-Wasserstein metric $r_w$ in (6). This distance metrizes the weak topology and has important relationships with other distances that also metrize the weak topology (see Proposition 7 in the Appendix). Furthermore, the $L_1$-Wasserstein metric is closely related to the variation norm (1) and the Kullback-Leibler distance (24); see Proposition 8. The following two propositions give better approximations than those in Bonze [4], Theorem 2.

Theorem 5.1. Suppose that $A$ is a compact Polish space (with diameter $C > 0$), and the conditions i) and ii) of Theorem 3.1 hold. Let $\mu^*$ be a $r_w$-SUS with global invasion barrier $\epsilon > 0$, and $\mu(\cdot)$ the solution of the replicator dynamics. If $\mu(0) \in W_{\varphi(\epsilon)}(\mu^*)$, with $\varphi(\epsilon) = \left[ \frac{\epsilon}{x^2} \right]^2$, then
i) $\mu(t) \in W_{\varphi(\epsilon)}(\mu^*)$ for all $t \geq 0$;
ii) $||\mu(t) - \mu^*|| < \frac{\epsilon}{x}$ for all $t \geq 0$;
iii) $r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$.
iv) Moreover, if the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in the weak topology, then $r_w(\mu(t), \mu^*) \to 0$ as $t \to \infty$. 

v) Furthermore, parts i) to iv) are also true with the hypothesis that \( \mu^* \) is \( \| \cdot \| \)-SUS, with barrier \( \frac{1}{\epsilon} \).

**Proof.** i) If \( \mu(0) \) is in \( \mathcal{W}_{\epsilon'}(\mu^*) \), then by Theorem 3.1 we know that \( \mu^* \ll \mu(t) \) and so \( K(\mu(t), \mu^*) \) is well defined for all \( t \geq 0 \). Using Theorem 3.1 and Fubini’s theorem,

\[
K(\mu(t), \mu^*) - K(\mu(0), \mu^*) = -\int A \left[ \int_0^t \beta(a|\mu(s))ds \right] \mu^*(da)
\]

\[
= -\int_0^t J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))ds. \quad (26)
\]

By the condition ii) of Theorem 3.1 there exists \( D > 0 \) such that, for any \( a \in A \) and \( \nu, \eta \in \mathbb{P}(A) \)

\[
|\beta(a|\eta) - \beta(a|\nu)| \leq D\|\eta - \nu\|.
\]

So

\[
\left| J(\mu^*, \eta) - J(\eta, \eta) - J(\mu^*, \nu) - J(\nu, \nu) \right| = \left| \int A [\beta(a|\eta) - \beta(a|\nu)] \mu^*(da) \right|
\]

\[
\leq D\|\eta - \nu\|. \quad (27)
\]

By (27) and since \( \mu(s) \) is continuous in \( s \), the map \( s \to [J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))] \) is continuous. Moreover, the time derivative of \( K(\mu(t), \mu^*) \) exists and since \( \mu^* \) is a \( r_w \)-SUS,

\[
\frac{dK(\mu(t), \mu^*)}{dt} = -[J(\mu^*, \mu(t)) - J(\mu(s), \mu(t))] \leq 0. \quad (28)
\]

Hence \( K(\mu(t), \mu^*) \) is nonincreasing in \( t \), and i) holds.

ii), iii) By Proposition 8 and (26),

\[
r_w(\mu(t), \mu^*) \leq C\|\mu(t) - \mu^\| \leq 2C[K(\mu(0), \mu^)]^{\frac{1}{2}} < \epsilon. \quad (29)
\]

Therefore ii) and iii) hold.

iv) Since \( K(\mu(t), \mu^*) \) is a nonincreasing function in \( t \) and, by (28), the map

\[
t \to \int_0^t J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))ds
\]

is increasing and

\[
\lim_{t \to \infty} \int_0^t \left[ J(\mu^*, \mu(s)) - J(\mu(s), \mu(s)) \right] ds < \infty.
\]

Moreover, since the map \( s \to [J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))] \) is continuous, we have \( \lim_{s \to \infty} [J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))] = 0 \). If \( A \) is compact, then the space \( \mathbb{P}(A) \) is compact in the weak topology (see Bobrowski [3]), and the distance \( r_w \) metrizes this topology (see Proposition 7). Suppose now that \( \hat{\mu} \) is an accumulation point of the trajectory \( \mu(\cdot) \). By (29) the \( r_w \)-distance from \( \hat{\mu} \) to \( \mu^* \) is at most \( \epsilon \), and since \( \mu^* \) is a \( r_w \)-SUS, \( J(\mu^*, \hat{\mu}) \geq J(\hat{\mu}, \hat{\mu}) \) if \( \hat{\mu} \neq \mu^* \). By hypothesis, the map \( \mu \to J(\mu^*, \mu) - J(\mu, \mu) \) is weakly continuous. If \( \hat{\mu} \) is such that \( J(\mu^*, \hat{\mu}) - J(\hat{\mu}, \hat{\mu}) = 0 \), then \( \hat{\mu} = \mu^* \), which proves that \( r_w(\mu(t), \mu^*) \to 0 \).

v) Finally if \( \mu^* \) is \( \| \cdot \| \)-SUS with barrier \( \frac{1}{\epsilon} \), then, by (29), parts i) to iv) follow.  

The next theorem characterizes the stability of the replicator dynamics of a SUS that is also a Dirac measure.
Theorem 5.2. Let $A$ be a separable metric space and suppose that the conditions i) and ii) of Theorem 3.1 hold. Let $\delta_m$ be a Dirac measure and $r$ any metric on $P(A)$. If $\delta_m$ is $r$-SUS, $\mu(\cdot)$ is a solution of the replicator dynamics, and $\|\mu_0 - \delta_m\| < \epsilon$ for some small $\epsilon > 0$, then

i) $\|\mu(t) - \delta_m\| < \epsilon$ for all $t \geq 0$;

ii) $\mu(t)$ is in some open ball $V^r_{\alpha,\tau}(\mu^*)$ as in (7) for all $t \geq 0$, where $r_{w_T}$ is a distance that metrizes the weak topology;

iii) if $A$ is a compact Polish space (with diameter $C > 0$), then for all $t \geq 0$, $r_w(\mu(t), \delta_m) < Ce$;

iv) if $A$ is compact (not necessary Polish) and the map $\mu \to J(\delta_m, \mu) - J(\mu, \mu)$ is continuous in the weak topology, then $r_{w_T}(\mu(t), \mu^*) \to 0$, where $r_{w_T}$ is any distance that metrizes the weak topology.

Proof. Parts i), ii) and iv) follow from Proposition 3 and Theorem 6.2 in Mendoza-Palacios and Hernández-Lerma [20]. Part iii) follows from Proposition 8.

Theorem 5.2 is also proved by Oechssler and Riedel [23] with slight changes in the definition of $\| \cdot \|$-SUS.

6. NESs and stability. In this section we are interested in the relation between the stability of the differential equation (11)-(12) with $F$ as in (14), and the static evolutionary concepts NES and SUS.

Let $\mu, \nu \in P(A)$. By Remark 2 and Proposition 8 we know that if $\mu$ and $\nu$ are “close” with respect to the Kullback-Leibler distance $K$, then they are close in the total variation norm $\| \cdot \|$, and consequently they are also “close” in the weak topology. This argument is not true in the opposite direction. Hence we say that the Kullback-Leibler distance is “stronger than” the total variation norm, and that the total variation norm is “stronger than” any distance that metrizes the weak topology.

Definition 6.1. Let $A$ be a separable metric space, and $r_1$ and $r_2$ the Kullback-Leibler distance or some metric on $P(A)$ where $r_1$ is “equal to” or “stronger than” $r_2$. A critical point $\mu^*$ of the replicator dynamics (11)-(12) is said to be

i) $[r_1, r_2]$-stable (in symbols : $[r_1, r_2]$-S) if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $r_1(\mu(0), \mu^*) < \delta$, then $r_2(\mu(t), \mu^*) < \epsilon$ for all $t > 0$. If $r_1 = r_2$ then we only say that $\mu^*$ is $r_1$-stable (in symbols : $r_1$-S).

ii) $[r_1, r_2]$-asymptotically weakly stable if it is $[r_1, r_2]$-stable and $\lim_{t \to \infty} \mu(t) = \mu^*$ in the weak topology.

Consider the Kullback-Leibler distance $K$, the total variation norm $\| \cdot \|$, and any distance $r_{w_T}$ that metrizes the weak topology. The following diagram gives the natural implications between the different concepts of stability.

$$K - S \Rightarrow [K, \| \cdot \|] - S \Rightarrow [K, r_{w_T}] - S \Rightarrow \| \cdot \| - S \Rightarrow [\| \cdot \|, r_{w_T}] - S \Rightarrow r_{w_T} - S$$

These implications are easy to deduce. For example, if the critical point $\mu^*$ is $\| \cdot \|$-S, and the initial condition $\mu_0$ satisfies that $K(\mu_0, \mu^*) < (\xi)^2$ for a small $\epsilon > 0$, then by Proposition 8 $\|\mu_0 - \mu^*\| < \epsilon$: hence $\mu^*$ is also $[K, \| \cdot \|]$-S. On the other
Proposition 6. and so the map \( \mu \) for some \( \kappa > 0 \). Hence \( \mu^* \) is also \( \| \cdot \|_{r_{\text{KL}}}-S \).

The following proposition states the existence of the support of a probability measure on a separable metric space. This concept is used in Proposition 6.

Proposition 5. Let \( A \) be a separable metric space and \( \mu \) in \( \mathbb{P}(A) \). Then there is a unique closed set \( S \subset A \) (called the support of \( \mu \), and denoted \( \text{Supp}(\mu) \)) such that \( \mu(A - S) = 0 \) and \( \mu(O \cap S) > 0 \) for every open set \( O \) for which \( O \cap S \neq \emptyset \).

Proof. See Royden [28], pag. 408.

The following proposition establishes a partial converse of Proposition 2.

Proposition 6. Let \( A \) be a separable metric space, and \( r_1, r_2 \) the Kullback-Leibler distance or some metric in \( \mathbb{P}(A) \) where \( r_1 \) is “equal to” or “stronger than” \( r_2 \). Suppose that the conditions i) and ii) of Theorem 3.1 are satisfied, and let \( \mu^* \) be a critical point of the replicator dynamics. If \( \mu^* \) is \( [r_1, r_2] \)-stable, then \( \mu^* \) is a Nash equilibrium strategy (NES) of \( \Gamma_s \).

Proof. If \( \mu^* \) is a critical point of (11)-(12), then

\[
J(a, \mu^*) - J(\mu^*, \mu^*) = 0 \quad \mu^* \text{-a.s.}
\]

Suppose that \( \mu^* \) is not a NES of \( \Gamma_s \). Then there exist \( a' \) such that it is not in the support of \( \mu^* \) and

\[
J(a', \mu^*) - J(\mu^*, \mu^*) > \kappa > 0,
\]

for some \( \kappa \). By the condition ii) of Theorem 3.1 we have that for any \( \eta, \nu \in \mathbb{P}(A) \)

\[
|\beta(a'| \eta) - \beta(a'| \nu)| \leq D\|\eta - \nu\|
\]

and so the map \( \mu \rightarrow J(a', \mu) - J(\mu, \mu) \) is continuous. Hence, by (31), for any \( \mu \in \mathbb{P}(A) \) near \( \mu^* \) in some \( r_1 \) distance

\[
J(a', \mu) - J(\mu, \mu) > \kappa.
\]

Let \( \epsilon > 0 \) and \( \mu_0 := \lambda c_{a^*} + (1 - \lambda c) \mu^* \) be the condition initial, where \( \lambda c \in (0, 1) \) and \( \mu_0 \in W_r(\mu^*) \), with \( \varphi(c) = c^2 \). The number \( \lambda c \) indeed exists since

\[
K(\mu_0, \mu^*) = \int_{\text{Supp}(\mu^*)} \log \left( \frac{d\mu^*}{d\mu_0} (a) \right) \mu^*(da) = \log \left( \frac{1}{1 - \lambda c} \right),
\]

and the logarithmic function is continuous, and by Remark 2 and Proposition 8, \( \mu_0 \) is near \( \mu^* \) in the \( r_1 \)-distance.

By (32) and Theorem 3.1 we have

\[
\mu(0, \{a'\})e^{\epsilon t} \leq \mu(0, \{a'\})e^{\int_0^t \beta(a'| \mu(s))ds} = \mu(t, \{a'\}),
\]

for all \( t > 0 \). Thus, if the initial condition is \( \mu_0 \), then \( \mu(t, \{a'\}) \) is increasing and the trajectory \( \mu(t) \) is not close to \( \mu^* \) in the \( r_2 \)-distance. So \( \mu^* \) is not \( [r_1, r_2] \)-stable.

Consider the following sets:

i) \( \mathcal{N} := \{ \mu^* \in \mathbb{P}(A) : \mu^* \text{ is a NES of } \Gamma_s \} \),

\[ \mathcal{C} := \{ \mu^* \in \mathbb{P}(A) : \mu^* \text{ is a critical point of } (11) - (12), \text{ with } F \text{ as } (14) \} \).

ii) If \( r \) any metric in \( \mathbb{P}(A) \),

\[ r - \text{SUS} := \{ \mu^* \in \mathbb{P}(A) : \mu^* \text{ is } r - \text{SUS} \}. \]
iii) Let $r_1$ and $r_2$ be the Kullback-Leibler distance or some metric in $\mathbb{P}(A)$, where $r_1$ is “equal to” or “stronger than” $r_2$,
\[ [r_1, r_2] - S := \{ \mu^* \in \mathbb{P}(A) : \mu^* \text{ is } [r_1, r_2] - S \} . \]

The following corollary summarizes our results and complements the diagram (18).

**Corollary 1.** Let $A$ be a compact Polish space, and assume the conditions i) and ii) of Theorem 3.1. Then we have the following relations:

i) $r_w - SUS \subset \| \cdot \| - SUS \subset K - S \subset N \subset C$;

ii) $r_w - SUS \subset \| \cdot \| - SUS \subset [K, \| \cdot \|] - S \subset N \subset C$;

iii) $r_w - SUS \subset \| \cdot \| - SUS \subset [K, r_w] - S \subset N \subset C$.

**Proof.** This is a consequence of Theorem 5.1 and Propositions 2, 3, 6. \qed

**Remark 3.** Let $r_1$ be a metric on $\mathbb{P}(A)$, and let $r_2$ be the total variation norm on $\mathbb{P}(A)$ or some metric equivalent to the weak topology. By Theorem 5.2 and Propositions 2, 6, we have the following implications if a Dirac measure $\delta_a^*$ is a $r_1$-SUS.

\[ \delta_a^* \in r_1 - SUS \Rightarrow \delta_a^* \in [||, r_2] - S \Rightarrow \delta_a^* \in \mathcal{N} \Rightarrow \delta_a^* \in \mathcal{C} . \]

These facts complement Theorem 5.2.

7. **Examples.**

7.1. **A linear-quadratic model.** Let us consider games with two players and payoff function
\[ U(x, y) = -ax^2 - bxy + cy + dy , \]
with $a, b, c > 0$ and $d$ any real number. Let $A = [0, M]$ be the strategy set for $M > 0$ and large enough.

This class of games could represent a Cournot duopoly or models of international trade with linear demand and linear cost (see Bagwell and Wolinsky [1]). It can also represent some models of public good games (see Mas-Colell, Whinston and Green [18]).

If $2c(a - b) > 0$ and $4a^2 - b^2 > 0$, then we have an interior Nash equilibrium strategy (NES)
\[ x^* = \frac{2c(a - b)}{4a^2 - b^2} . \]

For a fixed $y$ the function $U(x, y)$ is concave in $x$ and has the partial derivative $U_x(x, y) = -2ax - by + c$. Let $x(y) := \arg\max U(x, y) = \frac{(c - by)}{2a}$ and note that $x'(y) = -(b/2a) < 0$. Then if $y < x^*$ or $x^* < y$, we have
\[ U(x(y), y) > U(x^*, y) \geq U(y, y) . \]

On the other hand, let $\bar{y}^\mu := \int_A y\mu(dy)$. If $\mu$ is such that $\bar{y}^\mu < x^*$, then by Jensen’s inequality
\[ J(\delta_{x^*}, \mu) = \int_A U(x^*, y)\mu(dy) = U(x^*, \bar{y}^\mu) > U(\bar{y}^\mu, \bar{y}^\mu) \geq J(\mu, \mu) . \]

This is also true if $\bar{y}^\mu > x^*$. Hence, for any metric $r$ on $\mathbb{P}(A)$, the strategy $\delta_{x^*}$ is $r$-SUS. Therefore, by Theorem 5.2, if $\|\mu_0 - \delta_{x^*}\| = 2(1 - \mu_0(\{x^*\})) < \epsilon$, then
\[ \|\mu(t) - \delta_{x^*}\| = 2(1 - \mu(t, \{x^*\})) < \epsilon , \quad r_w(\mu(t), \delta_{x^*}) < M\epsilon \quad \forall t \geq 0 . \]
Moreover, since the payoff function $U(\cdot)$ is continuous and the set $A$ of strategies is compact, we conclude that $\mu(t) \to \delta_{\mu^*}$ in distribution.

7.2. Graduated risk game. A graduated risk game is a symmetric game (proposed by Maynard Smith and Parker [19]), where two players compete for a resource of value $v > 0$. Each player selects her “level of aggression” for the game. This “level of aggression” is captured by a probability distribution on $A := [0, 1]$. In this case, $x \in A$ can be interpreted as the probability that neither player is injured, and $\frac{1}{2}(1-x)$ is the probability that player one (or player two) is injured. If the player is injured its payoff is $v - c$ (with $c > 0$), and hence the expected payoff for the player is

$$U(x, y) = \begin{cases} vy + \frac{v-c}{2}(1-y) & \text{if } y > x \\ \frac{v-c}{2}(1-x) & \text{if } y \leq x \end{cases}$$

where $x$ and $y$ are the “levels of aggression” selected by the player and her opponent, respectively. If $v < c$, this game has a NES with density function

$$\frac{d\mu^*(x)}{dx} = \frac{\alpha - 1}{2} x^{\frac{\alpha - 3}{2}},$$

where $\alpha = \frac{\varphi}{\tau}$. Bishop and Cannings [2] show that if $v < c$, then the NES satisfies that

$$J(\mu^*, \mu) - J(\mu, \mu) > 0 \forall \mu \in \mathbb{P}(A),$$

that is, $\mu^*$ is a $r$-SUS for any metric $r$ in $\mathbb{P}(A)$, with $A = [0, 1]$.

Hence, by Theorem 5.1, if $K(\mu_0, \mu^*) < \varphi'(\epsilon) = (\frac{\varphi}{\tau})^2$, then

i) $\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \geq 0$;

ii) $\|\mu(t) - \mu^*\| < \epsilon$ for all $t \geq 0$;

iii) $r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$.

8. Comments and suggestions for further research. In this paper, we introduce a model of symmetric evolutionary games with strategies in metric spaces. The model can be reduced, of course, to the particular case of evolutionary games with finite strategy sets. We provide a general framework to the replicator dynamics that allows us to analyze different stability criteria. Our main results, in Sections 4, 5, 6 are of three types. The first one concerns the relations between three key concepts: the critical points of the replicator dynamics, the Nash equilibrium strategies and the strongly uninvadable strategies. See, for instance, Propositions 4.2 and 4.6. The second type of results is about stability in different topologies and metrics as, for instance, in Theorem 5.1, relations (30), Proposition 6, and Corollary 1. Finally, the third type of results is about the special case of Dirac measures, as in Theorem 5.2, and Remark 3. Finally, we presented two examples. The first one may be applicable to oligopoly models, theory of international trade, and public good models. The second example deals with a graduated risk game.

There are many questions, however, that remain open. For instance, when the set of pure strategies is finite, Cressman [6] shows that under some conditions the stability of monotone selection dynamics is locally determined by the replicator dynamics. Is this true for games with strategies in the space $\mathbb{P}(A)$ of probability measures? Another important issue would be to obtain a stability theorem for several evolutionary dynamics of games with continuous strategies similar to the result by Hofbauer and Sigmund [15] (Theorem 14) for games with a finite strategy set $A$. 
Appendix: Metrics on \( \mathbb{P}(A) \). In this Appendix we use the notation introduced in Section 2.2.

**Proposition 7.** Let \((A, r)\) be a Polish (that is, a complete, and separable metric) space and \(1 \leq p < \infty\). The \(L^p\)-Wasserstein metric \(r_{wp}\) metrizes the weak convergence on \(\mathbb{P}_p(A)\), i.e., for any sequence \(\{\mu_n\} \subset \mathbb{P}_p(A)\) and \(\{\mu\} \subset \mathbb{P}(A)\), the following conditions are equivalent

i) \(\mu_n\) converges in the weak topology,

ii) \(r_{wp}(\mu_n, \mu) \to 0\).

Moreover, if \(A\) is bounded, then the \(L^p\)-Wasserstein metric \(r_{wp}\), the Prokhorov metric \(r_p\), the bounded Lipschitz metric \(r_{bl}\), and the Kantorovich-Rubinstein metric \(r_{kr}\) all metrize the weak convergence of probability measures in \(\mathbb{P}(A)\). Moreover, if \(p = 1\) then

\[
\frac{1}{3}(r_p(\mu, \nu))^2 \leq r_{bl}(\mu, \nu) \leq r_{kr}(\mu, \nu) = r_{wp}(\mu, \nu) \leq 2[K(\mu, \nu)]^{1/2}.
\]

**Proof.** See Shiryaev [30] chapter 3, and Givens and Shortt [11].

**Proposition 8.** Let \(A\) be a separable metric space. Let \(\mu\) and \(\nu\) be in \(\mathbb{P}(A)\), with \(\nu << \mu\). Then

\[
\|\mu - \nu\| \leq 2[K(\mu, \nu)]^{1/2}.
\]  
(33)

Moreover, if \(A\) is a bounded (with diameter \(C > 0\)) Polish space, then

\[
r_{wp}(\mu, \nu) \leq C\|\mu - \nu\| \leq 2C[K(\mu, \nu)]^{1/2}.
\]  
(34)

**Proof.** See Reiss [26] chapter 3, and Villani [32] chapter 6.

**REFERENCES**

[1] K. Bagwell and A. Wolinsky, Game theory and industrial organization, *Handbook of Game Theory with Economic Applications*, 3 (2002), 1851–1895.

[2] D. Bishop and C. Cannings, A generalized war of attrition, *Journal of Theoretical Biology*, 70 (1978), 85–124.

[3] A. Bobrowski, *Functional Analysis for Probability and Stochastic Processes: An Introduction*, Cambridge University Press, Cambridge, 2005.

[4] I. M. Bomze, Dynamical aspects of evolutionary stability, *Monatshefte für Mathematik*, 110 (1990), 189–206.

[5] I. M. Bomze, Cross entropy minimization in invadable states of complex populations, *Journal of Mathematical Biology*, 30 (1991), 73–87.

[6] R. Cressman, Local stability of smooth selection dynamics for normal form games, *Mathematical Social Sciences*, 34 (1997), 1–19.

[7] R. Cressman, Stability of the replicator equation with continuous strategy space, *Mathematical Social Sciences*, 50 (2005), 127–147.

[8] R. Cressman and J. Hofbauer, Measure dynamics on a one-dimensional continuous trait space: Theoretical foundations for adaptive dynamics, *Theoretical Population Biology*, 67 (2005), 47–59.

[9] R. Cressman, J. Hofbauer and F. Riedel, Stability of the replicator equation for a single species with a multi-dimensional continuous trait space, *Journal of Theoretical Biology*, 239 (2006), 273–288.

[10] I. Eshel and E. Sansone, Evolutionary and dynamic stability in continuous population games, *Journal of Mathematical Biology*, 46 (2003), 445–459.

[11] C. R. Givens and R. M. Shortt, A class of Wasserstein metrics for probability distributions, *The Michigan Mathematical Journal*, 31 (1984), 231–240.

[12] J. C. Harsanyi, Oddness of the number of equilibrium points: A new proof, *International Journal of Game Theory*, 2 (1973), 235–250.

[13] J. Hofbauer, J. Oechssler and F. Riedel, Brown–von Neumann–Nash dynamics: The continuous strategy case, *Games and Economic Behavior*, 65 (2009), 406–429.
[14] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics*, Cambridge University Press, Cambridge, 1998.
[15] J. Hofbauer and K. Sigmund, Evolutionary game dynamics, *Bulletin of the American Mathematical Society*, 40 (2003), 479–519.
[16] J. Hofbauer and J. W. Weibull, Evolutionary selection against dominated strategies, *Journal of Economic Theory*, 71 (1996), 558–573.
[17] R. Lahkar and F. Riedel, The logit dynamic for games with continuous strategy sets, *Games and Economic Behavior*, 91 (2015), 268–282.
[18] A. Mas-Colell, M. D. Whinston and J. R. Green, *Microeconomic Theory*, Oxford university Press, 1995.
[19] J. Maynard Smith and G. A. Parker, The logic of asymmetric contests, *Animal Behaviour*, 24 (1976), 159–175.
[20] S. Mendoza-Palacios and O. Hernández-Lerma, Evolutionary dynamics on measurable strategy spaces: Asymmetric games, *Journal of Differential Equations*, 259 (2015), 5709–5733.
[21] J. R. Munkres, *Topology*, Second edition, Prentice Hall, 2000.
[22] T. W. Norman, Dynamically stable sets in infinite strategy spaces, *Games and Economic Behavior*, 62 (2008), 610–627.
[23] J. Oechssler and F. Riedel, Evolutionary dynamics on infinite strategy spaces, *Economic Theory*, 17 (2001), 141–162.
[24] J. Oechssler and F. Riedel, On the dynamic foundation of evolutionary stability in continuous models, *Journal of Economic Theory*, 107 (2002), 223–252.
[25] G. K. Pedersen, *Analysis Now*, Springer, New York, 1989.
[26] R.-D. Reiss, *Approximate Distributions of Order Statistics*, Springer, New York, 1989.
[27] K. Ritzberger, The theory of normal form games from the differentiable viewpoint, *International Journal of Game Theory*, 23 (1994), 207–236.
[28] H. L. Royden, *Real Analysis*, Third edition, Macmillan, New York, 1988.
[29] W. H. Sandholm, *Population Games and Evolutionary Dynamics*, MIT press, 2010.
[30] A. N. Shiryaev, *Probability*, Springer-Verlag, New York, 1996.
[31] M. Van Veelen and P. Spreij, Evolution in games with a continuous action space, *Economic Theory*, 39 (2009), 355–376.
[32] C. Villani, *Optimal Transport: Old and New*, Springer-Verlag, Berlin, 2009.

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