A major problem for evolutionary theory is understanding the so-called open-ended nature of evolutionary change, from its definition to its origins. Open-ended evolution (OEE) refers to the unbounded increase in complexity that seems to characterise evolution on multiple scales. This property seems to be a characteristic feature of biological and technological evolution and is strongly tied to the generative potential associated with combinatorics, which allows the system to grow and expand their available state spaces. Several theoretical and computational approaches have been developed to properly characterise OEE. Interestingly, many complex systems displaying OEE, from language to proteins, share a common statistical property: the presence of Zipf’s law. Given and inventory of basic items (such as words or protein domains) required to build more complex structures (sentences or proteins) Zipf’s law tells us that most of these elements are rare whereas a few of them are extremely common. Using Algorithmic Information Theory, in this paper we provide a fundamental definition for open-endedness, which can be understood as postulates. Its statistical counterpart, based on standard Shannon Information theory, has the structure of a variational problem which is shown to lead to Zipf’s law as the expected consequence of an evolutionary processes displaying OEE. We further explore the problem of information conservation through an OEE process and we conclude that statistical information (standard Shannon information) is not conserved, resulting into the paradoxical situation in which the increase of information content has the effect of erasing itself. We prove that this paradox is solved if we consider non-statistical forms of information. This last result implies that standard information theory may not be a suitable theoretical framework to explore the persistence and increase of the information content in OEE systems.

I. INTRODUCTION

Life has been evolving on our planet over billions of years, undergoing several major transitions along with multiple events of both slow and rapid change affecting structure and function [1,4]. Life seems to be indefinitely capable of increasing in complexity. This is illustrated, as an instance, by the trend towards larger genomes and diverse cell types exhibited by multicellular organisms. Moreover, the emergence of high neuronal plasticity and complex communication provided the substrate for non-genetic modes of adaptation. A key concept that pervades many of these innovations is the idea that evolution is “open-ended”. Following [5], open ended evolution (OEE) can be defined as follows: “a process in which there is the possibility for an indefinite increase in complexity”. What kind of systems can exhibit such unbounded growth in complexity [6]? What are the conditions under which the complexity—and thus, the information content of the system—can increase and what are the footprints of such an open ended increase of complexity? Which kind of information is encoded in an OEE system? The aim of this paper is to give hints to the these questions, starting from a rigorous and general definition of OEE.

Open ended evolutionary change needs a dynamical behaviour allowing complexity to grow in an unbounded way [5,7]. This requires a very large exploration space but this is only a necessary requirement. As noticed in [8] mathematical models used in population genetics involving infinite alleles (using Markov models) do not display open-ended evolution. Here we concentrate in a different, but seemingly widespread pattern found in many complex systems generated through OEE processes. These include a diverse array of systems, including human language, biological and digital genomes, technological development, molecular networks [13–15] and complex circuits [16] to cite just a few.

Previous attempts to address the problem of OEE involved different approximations and degrees of abstraction. John von Neumann was one of the early contributors to this issue [5,17,18]. In all these studies, some underlying mechanism is assumed to be operating, and arguments are made concerning the presence of self-replication, genotype-phenotype mappings, special classes of material substrates and physico-chemical processes [5,19]. Here we move instead into a different direction, by proving a deep connection between open ended evolution and a widespread law shared by complex systems exhibiting generative rules and recursivity. The best known example is human language. Syntactic rules are able to produce infinite well-formed structures (using recursion) and thereby the number of potential sentences in a given language is unbounded [9]. Additionally, we
should notice that evolution proceeds through tinkering 
[10, 11] thus involving reuse of existing parts, which are first copied and incorporate some degree of redundancy but are later on modified through evolution. Interestingly, despite the obvious differences existing between evolution and engineering [10] this process of tinkering appears to be common too in the evolution of technological systems, thus indicating that copy-and-paste dynamics might be more frequent than expected [12].

These systems are very different in their constitutive components, dynamics, and scale. However, all share the presence of a common statistical pattern linked to their diversity: fat tailed distributions. Three examples are provided in figure 1. In all these cases, the frequency distribution of the basic units decays following Zipf’s law. Zipf’s law was first identified as a prominent statistical regularity widespread across all human languages: the frequency of any word is inversely proportional to its rank [20, 21]. Specifically, if we rank all the occurrences of words in a text from the most common word to the less common one, Zipf’s law states that the probability \( p(s_i) \) that in a random trial we find the \( i \)-th most common word \( s_i \) (with \( i = 1, \ldots, n \)) falls off as

\[
p(s_i) = \frac{1}{Z} i^{-\gamma}
\]

with \( \gamma \approx 1 \) and \( Z \) the normalization constant, i.e., \( Z = \sum_{i \leq n} i^{-\gamma} \). Stated otherwise, the most frequent word will appear twice as often as the second most frequent word, three times as often as the third one, and so on. This law is found in many different contexts and can emerge under different types of dynamical rules (see [21–24] and references therein).

The four examples shown in figure 1 involve: (a) LEGO® models, (b) human language, (c) proteins and (d) evolved electronic circuits. The first example provides an illustration of Zipf’s law in a non-biological setting. This toy allows to exploit the intrinsic combinatorial explosion associated to the multiple ways in which different bricks can be interlinked. In figure 1a, right we plot the number of times that each type of brick occurred within a very large data set of Lego models [25]. The rank plot reveals that some simple bricks (as those shown in figure 1b, right) are extremely common whereas most bricks (having more complex shapes and larger size) are rare. The analysis showed that the statistical distribution can be well fitted using a generalized form of equation (1) known as the Pareto-Zipf distribution. This reads:

\[
p(s_i) = \frac{1}{Z} (i + i_0)^{-\gamma},
\]

where \( Z \) is again the corresponding normalization and \( i_0 \) a new parameter that allows to take into account the curvature for small \( i \)-values. This picture is similar to the one reported from the study of large written corpuses, as illustrated in figure 1; 20.

Our third example is given by so called protein domains, which are considered the building blocks of protein organization and an essential ingredient to understand the large scale evolution of biological complexity [27, 30]. Here each protein domain (or fold) is characterized by its essentially independent potential for folding in a stable way and each protein can be understood as a combination of one, two, or more domains. In figure 4, the rank distribution of observed folds from a large protein database is displayed. Domains define the combinatorial fabric of the protein universe and their number, although finite, has been increasing through evolution [30]. The fourth example gives the frequency of use of 4-element modules within complex circuits [31].

The repertoire of LEGO® bricks, words, and domains provide the raw materials to combinatorial construction; but they also share the underlying presence of a grammar, to be understood here as the compact description of a language. As indicated in [25], if we treat pieces of LEGO® as words and models as utterances, LEGO® appears as a class of artificial language and the resulting structures are passed from generation to generation through cultural transmission. Protein domains too exhibit a grammar in which a set of generative rules for combining the available folds provides an explanatory mechanism for the observed repertoire of protein structures [32, 34]. In summary, these systems (and others like electronic circuits or genomes, and even evolved technology [35]) are characterized by a growth process that is expanding their inventories over time, the presence of generative rules allowing new structures to emerge, and a common statistical pattern described by Zipf’s law.

In this paper we provide a general definition, or postulates of OEE based on Algorithmic Information Theory (AIT), and we show that the common presence of Zipf’s law in these seemingly disparate systems is deeply connected to their shared open ended nature. Furthermore, we explore the consequences that OEE has for the conservation of the information, identifying the information loss paradox in OEE systems. This paradoxical situation, in which the system looses all its past information in the long run, even though the step-by-step information transmission is maximised, is shown to be a problem of the statistical nature of Shannon Information Theory. Indeed, we prove that, in the general setting of Algorithmic Information Theory, information can be conserved and systems can grow unboundedly without removing the traces of its past. Therefore, the general study of OEE systems must be framed in a theoretical construct not based on standard information theory, but in a much more general one, inspired in non-statistical forms of information content. The connection of fundamental results of computation theory and even Gödel’s incompleteness theorem has been taken before in [8, 30].

II. THEORETICAL FRAMEWORK

A natural framework to address the problem of OEE is AIT. AIT provides a powerful framework to deal with liv-
FIG. 1: Zipf’s law distributions are commonly found in very different systems displaying open-ended evolution. Here we show several examples of scaling behavior involving (a) Lego systems, (b) written language and (c) proteins. In (a) we display (in log scale) the probability of finding the $i$-th most abundant type of LEGO brick within a very large number of systems (see details in [25]). In (b) the log-scale rank-size distribution of Herman Melville’s Moby Dick is displayed. The dashed line shows the frequency versus rank for words having length 5, which is the average length of words in this particular book. The linear rank plot displayed in (c) shows the corresponding rank distribution of protein folds in a large protein database (redrawn from [32]). The line is a power law fit. Here the names of some of the domains, which are associated to particular functional traits, are indicated. (d) Zipf’s law in the frequency of logic modules used in evolved complex circuits (adapted from [31]).
ing complexity, which is often associated to information storage in the genome [36-37]. Such information results from the growth of genome complexity through both gene duplication and the interactions with the external world and is (by definition) a path dependent process. Here we will assume that the length of these strings can grow in time, while defining the conditions that allow this process to generate increasing complexity. As we shall see, the derived framework is completely general, and applies to any system susceptible of displaying OEE.

A. Algorithmic complexity

Let us first introduce a key concept required for our analysis. This is the so called Kolmogorov complexity (or algorithmic complexity), independently developed by Kolmogorov [35], Solomonoff [39] and Chaitin [40]. Roughly speaking, if a given process can be described in terms of a string of bits, the complexity of this string can be measured as the shortest computer program capable of generating it [42, 43]. The underlying intuition behind this picture (figure 2) is that simple, predictable strings, such as 10101010101010... can be easily obtained from a small piece of code that essentially says ‘write “10”’ followed by ‘repeat’ as many times as needed. This would correspond to a regular system, such as a pendulum or an electronic oscillator (figure 2a-b) and the simple dynamical pattern is reproduced by a simple program. Instead, a random string generated by means of a coin toss (say 010011001111011101011010...) would only be reproduced by using a program that writes exactly that sequence and is thus as long as the string itself (figure 2d). From this perspective, such a random string would be incompressible. Other stochastic processes generating fluctuations (figure 2e-f) and represented as strings of n bits can be similarly described.

Although this stochasticity invites to think in terms of statistical or information entropy, the Kolmogorov complexity is, conceptually, a more fundamental measure of the complexity of such processes [41]. A rigorous definition follows. Let x and p be finite binary strings of length ℓ(x) and ℓ(p) respectively. Let Tu be a universal Turing machine1. Note that a finite binary string p can define the computations that a universal Turing machine [41] will implement when p is fed as an input – i.e. it can define programs executed by the Turing machine. Let Tu(p) denote the output of the computer Tu when running the program p. Considering now all possible programs p that produce x as an output when fed into Tu, the Kolmogorov Complexity KTu(x) of the string x with respect to the universal computer Tu is defined as:

\[ K_{Tu}(x) = \min_{p: Tu(p) = x} \{ \ell(p) \} \]

This quantity is computer independent up to an additive constant [41] so we will omit the subindex when referring to it. As we can see, if x is a random string, we would have a simple relation:

\[ K(x) = \ell(x) \]

since all ℓ(x) bits need to be included.

In addition, and as it happens with the statistical entropy, one can define the conditional algorithmic complexity as follows: Let x, y, and p be finite binary strings again and let Ty(p) be a universal Turing machine to which a description of y has already been made available. The Kolmogorov complexity of x given y is the length of the shortest program p that, when applied to a universal Turing machine, modifies y to display x as an output:

\[ K(x|y) = \min_{p: Ty(p) = x} \{ \ell(p) \} \]

Notice that even if K(x) is arbitrarily high, we now search for the minimum program that knows the differences between x and y and amends them.

B. General conditions for Open Ended Evolution

We shall concern ourselves with dynamic systems whose description σ can be made in terms of finite binary strings σt at each time step t over evolutionary time. The complexity of such an object over time is given by K(σt). This object shall evolve through intermediate steps in a path dependent manner; thus the quantities K(σt), K(σt+Δt) (with Δt > 0 defining our evolutionary time step), and K(σt+Δt|σt) and the relationships between them will play a paramount role.

A minimum condition of an open-ended evolving system is that its complexity must increase over time:

\[ K(σ_t) \leq K(σ_{t+Δt}) \]

and that this increase must be unbounded, i.e. for any natural number N ∈ ℕ there is a time t such that

\[ K(σ_t) > N. \]

If we consider an evolutionary, path-dependent process, we can assume that the information of the state σt is partly conserved in σt+Δt. Moreover, we need to define the minimal amount of change between strings such that the above conditions hold, allowing for open ended evolution. The underlying assumption is that strings and available states grow through evolution, but minor changes occur most of the time.

In order to address the problem, we map the evolutionary trajectory into an AIT-like least action principle.
The minimisation principle is crucial to approach the evolving system. The role of this minimisation principle is to state that the increase of complexity has been obtained from a previous state and is thus path dependent. In other words, at every evolutionary step there is some information from the recent past that is preserved.

Using this formal framework, a system evolving in time require satisfying equations (6) and (7) as conditions for open ended evolution while the path-dependent character of the process is introduced by (8). Equations (6–8) are, therefore, the postulates of OEE. In a nutshell, our working definition of open endedness given by these three equations states that the size of the algorithm describing the system does not converge in time. Therefore, even if every evolutionary stage accepts a finite algorithm as a description, the object is asymptotically uncomputable. For statistical systems, such scenario implies that i) the system exhibits Zipf’s law as an inevitable consequence of the lack of an algorithm able to describe the whole evolutionary process and ii) the system loses any information of the past in the large. Nevertheless, point ii) can be overcome if we consider non-statistical forms of information, namely, if we take an AIT perspective of the information transfer through evolutionary time steps.

III. STATISTICAL SYSTEMS: A VARIATIONAL APPROACH TO OPEN ENDED EVOLUTION

We will explore now the consequences of the definition stated above for systems that accept a description—possibly partial—in terms of statistical ensembles. The aim is to write the three conditions for OEE in the language of statistical information theory. Note the departure from the finite sequence \( \sigma_t \) that described a system at a stage of its evolution in the previous section. We will assume now that the statistical properties of this very finite string \( \sigma_t \) are themselves accurately accounted for by a random variable \( X_t \). Therefore, the algorithmic complexity \( K(\sigma_t) \) is of the order of the Shannon entropy \( H(X_t) \) associated to the random variable \( X_t \) [41]:

\[
K(\sigma_t) = H(X_t) + O(1).
\]

Recall that this is the minimal information required to describe \( X_t \), not a sequence of trials of the random variable \( X_t \). Assume that we discretise the time, so we use the subscript \( n \) or \( m \) instead of \( t \), and that we label the states \( i = 1, \ldots, n \). Now let us define the following family of nested subsets:

\[
\Omega_1 = \{1\}
\]

\[
\Omega_2 = \Omega_1 \cup \{2\} = \{1, 2\}
\]

\[
\ldots = \ldots
\]

\[
\Omega_{n+1} = \Omega_n \cup \{n+1\} = \{1, \ldots, n + 1\}
\]

---

2 This quantity has been used as a conditional complexity within the context of evolved symbolic sequences [37]. In this case, \( K(s|e) \) referred to the length of the smallest program that gives the string \( s \) from a given environment \( e \), also defined as a string.

3 Rigorously speaking, one should say that, if \( \sigma \) is the description in bits of the outcomes of \( N \) trials of \( X_t \), then \( \frac{K(\sigma)}{N} \rightarrow H(X_t) \).
The open-ended evolutionary process will traverse the above family of nested subsets, adding a new state per evolutionary time step. We now define a sequence of different random variables

\[ X_1, \ldots, X_n \]

such that \( X_k \) takes values over the set \( \Omega_k \) and follows the probability distribution \( p_k(1), \ldots, p_k(k) \), with \( \sum_{i \leq k} p_k(i) = 1 \). Then:

\[ H(X_n) = -\sum_{i \leq n} p_n(i) \log p_n(i). \]  

(11)

The variational principle derived from the path dependent process implies now the minimization of the conditional entropy of the random variable \( X_{n+1} \) given the random variable \( X_n \), namely:

\[ H(X_{n+1}|X_n) = -\sum_{i \leq n} p_n(i) \sum_{k \leq n+1} \mathbb{P}_n(k|i) \log \mathbb{P}_n(k|i) \]

(12)

where \( \mathbb{P}_n(k|i) \equiv \mathbb{P}(X_{n+1} = k|X_n = i) \). We will finally assume (without loss of generality) that the probability distributions \( p_2, \ldots, p_n \) are sorted in decreasing order, i.e.,

\[ p_k(1) > p_k(2) > \ldots > p_k(k). \]  

(13)

In the appendix B1 we discuss the conditions under which the consecutive achievement of ordered probability distributions is possible.

Therefore, for statistical systems, the previous constraints for open-endedness from equations (6) and (7) must now be rewritten as:

\[ H(X_n) \leq H(X_{n+1}) \]  

(14)

and, for any \( N \in \mathbb{N} \), there will be a \( n \) such that

\[ H(X_N) > N; \]  

(15)

In addition, path dependence condition stated in equation (8) implies that:

\[ \text{minimize } H(X_{n+1}|X_n). \]  

(16)

In summary, we took a set of conditions (described by equations (6) and (8)) valid in the general AIT framework, and we have re-written them in terms of statistical entropy functions (see equations (14) and (16)). We finally observe that the condition that the probability distribution must be strictly ordered leads to:

\[ H(X_n) < \log n. \]  

(17)

A. Minimizing the differences between shared states

Condition (16) is hard to handle directly. Nevertheless, it can be approached as follows: Let us define the distribution \( \hat{p}_{n+1} \) as

\[ \hat{p}_{n+1}(k) \equiv p_{n+1}(k| k < n + 1) = \sum_{i \leq n+1} \frac{p_{n+1}(k)}{p_{n}(i)}. \]

\( \hat{p}_{n+1}(k) \) is the probability that \( k < n + 1 \) appears when we draw the random variable \( X_{n+1} \) excluding the outcomes \( = n + 1 \). Clearly, \( \hat{p}_{n+1} \) and \( p_n \) are defined over the set \( \Omega_n \), whereas \( p_{n+1} \) is defined over the set \( \Omega_{n+1} \). Since the support sets for both \( \hat{p}_{n+1} \) and \( p_n \) are the same, one can use the Kullback-Leibler divergence (KL-divergence) defined as the relative entropy (or information gain) between \( p_n \) and \( \hat{p}_{n+1} \):

\[ D(p_n||\hat{p}_{n+1}) = \sum_{k \leq n} p_n(k) \log \frac{p_n(k)}{\hat{p}_{n+1}(k)} \]  

(18)

Now we impose the condition of path dependence as a variational principle over the K-L divergence and then we write the following Lagrangian which defines the evolution of our system:

\[ \mathcal{L}(\hat{p}_{n+1}(1), \ldots, \hat{p}_{n+1}(n); \theta_{n+1}) = \]

\[ = D(p_n||\hat{p}_{n+1}) + \theta_{n+1} \left( \sum_{k \leq n} \hat{p}_{n+1}(k) - 1 \right). \]  

(19)

The minimization of this Lagrangian with respect to the variables upon which it depends imposes that:

\[ \hat{p}_{n+1} = p_n \]  

(20)

This implies that

\[ p_{n+1}(k) = \theta_{n+1} p_n(k) \quad \forall k \leq n \]

\[ p_{n+1}(n + 1) = 1 - \theta_{n+1} \]  

(21)

Under our assumption of entropy increase \([17]\), one can impose \( \theta_{n+1} > 0 \). In the appendix A4 it is proven that, at least, this solution converges to the absolute minimum in the limit of large systems.

So far we have been concerned about the minimisation of the K-L divergence. Condition (16) requires that the functional to be minimised is \( H(X_{n+1}|X_n) \). The first observation is that, under (21),

\[ \begin{cases} (\forall i \leq n) \; & \mathbb{P}_n(X_{n+1} = i|X_n = k) = \delta_{ik} \theta_n \\ (\forall k \leq n) \; & \mathbb{P}_n(X_{n+1} = n + 1|X_n = k) = 1 - \theta_n \end{cases} \]  

(22)

This channel structure leads to:

\[ H(X_{n+1}|X_n) = H(\theta_{n+1}), \]  

(23)

being \( H(\theta_{n+1}) \) the entropy of a Bernoulli process having parameter \( \theta_{n+1} \), i.e:

\[ H(\theta) = -\theta_{n+1} \log \theta_{n+1} - (1 - \theta_{n+1}) \log (1 - \theta_{n+1}). \]

If this is the case, we observe that, from eq. (A2), the relation between the entropies of \( X_n \) and \( X_{n+1} \) satisfies the following Fano’s-like equality:

\[ H(X_{n+1}) = \theta H(X_n) + H(\theta_{n+1}). \]
Finally, from the definition of mutual information between $X_n$ and $X_{n+1}$

$$I(X_{n+1} : X_n) = H(X_{n+1}) - H(X_{n+1} | X_n)$$

and from equations (A2) and (24) we arrive at the amount of information transmitted from the time step $n$ to $n + 1$: 

$$I(X_{n+1} : X_n) = \theta_{n+1} H(X_n).$$

This is a good estimate of the maximum possible information transmitted per evolutionary time step. Nevertheless, even in this case, the statistical information transmitted along time in an open-ended system has to face a paradoxical behaviour: the total loss of any past history in the long run—see section III C.

### B. Zipf’s Law: the footprint of OEE

As discussed at the beginning, a remarkably common feature of several systems known to exhibit OEE is the presence of Zipf’s law. We will rely now on previous results [22, 23] to show that the solution to the problem discussed above is given precisely by Zipf’s law. We first note that, thanks to equation (21), the quotient between probabilities:

$$\frac{p_n(i + j)}{p_n(i)} = f(i, i + j)$$

remains constant for all $n$ as soon as $p_n(i + j) > 0$. From [24] we know that the solution of our problem lies in the range defined by:

$$\left(\frac{i + 1}{i}\right)^{(1-\delta)} > f_n(i, i + 1) > \left(\frac{i + 1}{i}\right)^{(1+\delta)}.$$ 

It can be shown that $\delta \to 0$ if the size of the system is large enough. Therefore:

$$f_n(i, i + 1) = \frac{p_n(i)}{p_n(i + 1)} \approx \frac{i + 1}{i}$$

which leads us to the scaling:

$$p_n(s_i) \propto i^{-1},$$

i.e. Zipf’s law is the only asymptotic solution, which immediately suggests a deep connection between the potential for open-ended evolutionary dynamics and the presence of this particular power law.

### C. The loss of information paradox in OEE

The above description of the evolution of open-ended statistical ensembles leads to an unexpected result: Statistical systems displaying OEE lose any information of the past after a large period of complexity growing. Indeed, in spite information is conserved in a huge fraction step by step, it is not conserved at all if we compare large periods of evolution. Therefore, the capacity to generate an ensemble encoding an unbounded amount of information through evolution results in a total erasure of the past, even if the system displays OEE.

![FIG. 3: The paradox of information loss](image-url)
From which one can prove (see appendix B2) that:

\[ p_m(1) = \frac{1}{C_m}, \]

Thanks to equation (22) we know that \( P(1|1) = 1 \). Then, applying the Bayes’ rule for conditional probabilities and imposing that \( p_n(1) = \frac{1}{C_n} \), we can conclude that:

\[ (\forall n, m)(n > m) I(X_m : X_n) \geq \frac{1}{C_n} \log C_m. \]  \hspace{1cm} (29)

Furthermore, we can generalize equation (24) as follows:

\[ I(X_m : X_n) \leq \theta_{m+1} \cdots \theta_n H(X_m), \]

thereby obtaining the following chain of inequalities:

\[ \frac{1}{C_n} \log C_m < I(X_m : X_n) \]
\[ \leq \prod_{m < i \leq n} \theta_i H(X_n) \]
\[ = \frac{1}{C_n} \prod_{2 \leq k \leq m} \theta_k^{-1} H(X_m) \]
\[ = \frac{C_m}{C_n} H(X_m). \]

The above inequalities have interesting consequences. The first one is that, if \( C_n \to \infty \), then

\[ \lim_{n \to \infty} I(X_m : X_n) \leq \lim_{n \to \infty} \frac{C_m}{C_n} H(X_m) = 0. \]  \hspace{1cm} (30)

In the appendix B2 it is proven that, in OEE statistical systems, indeed we have that \( C_n \to \infty \). Thus \( I(X_m : X_n) \to 0 \): no statistical information is conserved in open-ended systems in the large. Instead, if the system is not open ended, i.e., \( \exists N \in \mathbb{N} : (\forall n)(H(X_n) < N) \), then, by virtue of equation (24) and equation (29), \( C_n \) is bounded and, thus:

\[ I(X_m : X_n) > \frac{1}{C_\infty} \log C_m > 0. \]

In this case, a fraction of the information is conserved, but we lost the OEE property.

D. Solving the paradox: Algorithmic information can be maintained

We have shown above that statistical information cannot be maintained through arbitrarily long evolutionary paths if the evolution is open-ended. The emphasis is put in the word statistical, because, as we shall see with a simple reasoning, other types of information, based on the general setting of AIT, can be maintained. Let \( \sigma_n \) be a description, in bits, of an object at time \( m \) and \( \sigma_N \) its description at time \( N > n \). Let us assume that \( \sigma_N \) can be written as a concatenation of two descriptions, to be indicated with symbol ”\( \oplus \)”: \n
\[ \sigma_N = \sigma_n \oplus \sigma_{N-n} \]

Now assume that \( K(\sigma_N) = \mu N, K(\sigma_n) = \mu n \) and \( K(\sigma_{N-n}) = \mu (N-n) \), with \( 0 < \mu < 1 \). If \( \pi_n \) is the minimal program that prints \( \sigma_n \) and \( \pi_{N-n} \) is the minimal program that prints \( \sigma_{N-n} \). Then, there is a program \( \pi_N \) defined as

\[ \pi_N = \pi_n \oplus \pi_{N-n} \]

such that, when applied to a Universal Turing machine, gives \( \sigma_N \), i.e., \( T_u(\pi_N) = \sigma_N \). If we already know \( \pi_n \), it is clear that

\[ K(\sigma_N|\sigma_n) = K(\sigma_{N-n}) + O(1) \]

We observe that, under the assumptions we made,

\[ |K(\sigma_N) - K(\sigma_n)| = K(\sigma_{N-m}) \]

so \( K(\sigma_N|\sigma_n) \approx |K(\sigma_N) - K(\sigma_n)| \) close to the bound provided by Zurek in [22], already used in equation (9). As we shall see, the immediate consequence of that is that the algorithmic mutual information between \( \sigma_N \) and \( \sigma_n \) does not depend on \( N \). Let \( I(\sigma_N : \sigma_n) \) be the algorithmic mutual information between \( \sigma_N \) and \( \sigma_n \):

\[ I(\sigma_N : \sigma_n) = K(\sigma_N) - K(\sigma_N|\sigma_n) \]

Then, one has that:

\[ I(\sigma_N : \sigma_n) = K(\sigma_N) - K(\sigma_N|\sigma_n) \]
\[ \approx K(\sigma_N) - K(\sigma_{N-m}) \]
\[ \approx K(\sigma_n) \]

we thus have

\[ \lim_{N \to \infty} I(\sigma_N : \sigma_n) \approx K(\sigma_n). \]  \hspace{1cm} (31)

which, within the algorithmic information theory framework, implies that information of previous stages of the evolution can be maintained.

The above reported result has an important consequence: In an OEE system in which information is maintained, the information is encoded in a non-statistical way. Therefore, Shannon information theory is of little use to understand the persistence of the memory of past states in OEE systems.

IV. DISCUSSION

In this paper we have considered a new approach to a key problem within complex systems theory, namely the conditions for open-ended evolution. This refers to an evolutionary process where complexity can be constantly created and increased. Such problem has been specially
addressed within the context of artificial life by considering the possibility of building a system able to evolve under artificial conditions and maintain a constant source of creativity. In this paper we provided a general formalisation, or postulates, given by equations (6–8) of the problem based on the framework of AIT.

Despite the high degree of abstraction of such an approach—which, at the same time, gives to it a great degree of generality—important conclusions can be drawn: (i) In statistically describable systems, Zipf’s law is the expected outcome of Open Ended Evolution. (ii) OEE systems have to face the statistical information loss paradox: Shannon information between different stages of the process tends to zero, and all information of the past is lost in the limit of large time periods. (iii) This paradoxical situation is solved when considering non-statistical forms of information, and we provided an example where algorithmic information between arbitrary time steps is maintained.

Since Zipf’s law is the outcome of a statistical interpretation of the OEE postulates given in equations (6–8), one may be tempted to conclude that information is not conserved in those systems exhibiting Zipf’s law in its statistical patterns. Instead, it is important to stress that the statistical ensemble description can be just a partial picture of the system, and that other mechanisms of information prevalence, not necessarily statistic, are at work. Therefore, if our system exhibits Zipf’s law and we have evidences of information conservation, the statistical pattern should be interpreted as the projection of other types of non-statistical information to the statistical observables.

The importance of open-endedness has been highlighted in many different contexts, including the evolution of artificial systems, from purely virtual to embodied ones. As mentioned above, there is an implicit component common to all systems displaying OEE, namely the presence of a grammar constraining the ways different objects can combine. This paper provides a simple formal approach to a difficult problem that pervades the evolution of complexity. Biological systems exhibit marked capacity for OEE resulting from their potential for growing and exploring new states and achieving novel functionalities. This open-endedness pervades the apparently unbounded exploration of the space of the possible.

The two biological systems cited in the introduction, namely human language and the protein universe, share the presence of an underlying grammar, which both enhances and constrains their combinatorial potential. Models of evolution through gene duplication (or tinkering) revealed that scaling laws and other properties displayed by protein networks emerge from the amplification phenomena introducing both growth through copy-and-paste dynamics. One way of doing this is provided by the tinkered nature of evolutionary change, where systems evolve by means of extensive reuse of previous parts. This mechanism fully matches our assumptions: generative rules that allow a way of expanding the state space, while the redundant nature of the process allows keeping most of the previous structures.

Postulates (6–8) explicitly relate OEE to unpredictability. This, according to classic results like the No free lunch theorem, puts a question mark on the possibility of a theory of evolution in the sense of classical physics. This issue, discussed also in, may exclude the possibility of a predictive theory in terms of the explicit evolutionary innovations that will eventually emerge. Nevertheless, in this paper we prove that this is not an all-or-nothing situation: Interestingly, the postulates of OEE, which rule out the existence of a predictive theory, are precisely the conditions that allow us to identify one of the possible statistical laws—Zipf’s law—governing such systems and thereby make predictions and, eventually, propose physical principles for them. According to that, these principles would predict the statistical observables, but not the specific events that they represent.

Acknowledgments

We thank Jordi Fortuny, Kepa Ruiz-Mirazo and Carlos Rodriguez-Caso and the members of the Complex Systems Lab for useful discussions. This work has been supported by Austrian Science Foundation FWF under grant P29252 (BC-M), the Botin Foundation, by Banco Santander through its Santander Universities Global Division, a MINECO FIS2015-67616 fellowship, the Secretaria d’Universitats i Recerca del Departament d’Economia i Coneixement de la Generalitat de Catalunya and the Santa Fe Institute.

Appendix A: Asymptotic behaviour of the minimisation of the K-L divergence

In this appendix we will prove that the solution provided by the minimisation of the K-L divergence converges to the absolute minimum of in an OEE statistical system. This implies that, even we cannot prove that this is the absolute solution, we can prove that it is arbitrarily close to it.

Let us have the following relation between successive probability distributions:

\[ p_{n+1}(k) = \theta_{n+1} p_n(k) \quad \forall k \leq n \]
\[ p_{n+1}(n+1) = 1 - \theta_{n+1} \]

(A1)

where is the solution of the minimisation of the K-L divergence as shown in section. This leads to an amount of noise:

\[ H(X_{n+1}|X_n) = H(\theta_n), \]

(A2)

being the entropy of a Bernoulli process having parameter , i.e:

\[ H(\theta_n) = -\theta_n \log \theta_n - (1 - \theta_n) \log(1 - \theta_n) \]
Let \( H(X_{n+1}|X_n) \leq H(\theta_n) \) be the absolute minimum of \( H(X_{n+1}|X_n) \) under the conditions of OEE described in equations \([14, 16]\) of the main text. We will show that \((\forall \epsilon > 0) \exists M\) for which, for any \( N > M \):

\[ |\min H(X_{n+1}|H(X_n)) - H(\theta_N)| < \epsilon \]

Indeed, let us suppose that our system is open-ended. This implies that \( 1 > \theta_n \geq (n-1)/n \) see appendix \([B1]\). So, knowing that, by definition \( H(\theta) = H(1-\theta) \) we have that for any \( \epsilon' > 0 \exists M \) such that, for any \( N > M \):

\[ 1 - \theta < \epsilon' \]

This implies that, for any \( \epsilon > 0 \), \( \exists M \) such that, for any \( N > M \):

\[ H(\theta_N) < \epsilon \]

Since \( H(X_{n+1}) > H(X_n) \) by the postulates of OEE, then \( \min H(X_{n+1}|H(X_n)) > 0 \). In addition, we have proven that \( H(\theta_N) < \epsilon \). Therefore, since, by assumption \( \min H(X_{n+1}|H(X_n)) \leq H(\theta_n) \), we demonstrated, taking the \( \epsilon \) above defined, that: \((\forall \epsilon > 0) \exists M \) for which, for any \( N > M \):

\[ |\min H(X_{n+1}|H(X_n)) - H(\theta_N)| < \epsilon \]

Therefore, at least, \( H(\theta_n) \) converges asymptotically the the absolute minimum.

Appendix B: Properties of the probability distribution

1. Conditions emergence of order in the probability distribution

Throughout the text we emphasised that the probability distribution is ordered. It is therefore crucial that such ordering is maintained. Here we state the conditions under which the emerging probability distribution is ordered.

Let us suppose that we have a sequence \( \theta_1, ..., \theta_n \), with \( \theta_1 = 0 \). The solution that minimises the noise has been proven to be

\[ (\forall k \leq n) \quad p_{n+1}(k) = \theta_n p_n(k); \]

if \( k = n + 1; \quad p_{n+1}(k) = 1 - \theta_n \] \( (B1) \)

This implies a successive process of rescaling of the probabilities, as long as the system grows in size. So, one has:

\[
\begin{align*}
p_n(1) &= \theta_2 \cdot \theta_3 \cdot \ldots \cdot \theta_n = \prod_{k \leq n} \theta_k \\
p_n(2) &= (1 - \theta_2) \prod_{2 < k \leq n} \theta_k \\
& \quad \quad \ldots = \ldots \\
p_n(i) &= (1 - \theta_i) \prod_{i < k \leq n} \theta_k \\
& \quad \quad \ldots = \ldots \\
p_n(n) &= 1 - \theta_n
\end{align*}
\]

We observe that we can establish a recurrence relation between probabilities:

\[
p_n(1) = \prod_{k \leq n} \theta_k \\
p_n(k) = a_k p_n(k-1)
\]

with \( a_k \) defined as:

\[
a_k = \frac{1 - \theta_k}{(1 - \theta_{k-1}) \theta_k}
\]

If \( a_k = 1 \), then \((\forall k \leq n) \) \( p_n(k) = \frac{1}{n} \), and \( H(X_n) = \log n \). It is easy to see that

\[
\theta_k = \frac{k - 1}{k} \implies (\forall k \leq n) \ p_n(k) = \frac{1}{n}
\]

and, consistently, \( 1 - \theta_k = \frac{1}{k} \). Now let us suppose that the function \( 1 - \theta_k \) is dominated by \( \frac{1}{k} \), i.e., \( 1 - \theta_k \) decays faster than \( \frac{1}{k} \). Then,

\[
\frac{(1 - \theta_k)}{(1 - \theta_{k-1})} < \frac{k - 1}{k} \quad \text{and} \quad \theta_k > \frac{k - 1}{k}
\]

so

\[
(\forall k \leq n) \ a_k = \frac{(1 - \theta_k)}{(1 - \theta_{k-1}) \theta_k} < 1
\]

The immediate consequence of the above result is that, from equation \((B2)\):

\[
p_n(1) > p_n(2) > \ldots > p_n(n) \quad . \quad (B2)
\]

We observe that if \( H(X_n) < \log n \) then \( \theta_k \neq \frac{k - 1}{k} \). We impose that the solution taken is the one giving

\[ 1 > \theta_k > \frac{k - 1}{k} \]

such that equation \((B2)\) is satisfied.

2. Divergence of the normalization constant

Given the expression of \( p_n(1) \) obtained above:

\[
p_n(1) = \prod_{2 \leq k \leq n} (\theta_k)
\]

one can define a normalisation constant

\[
C_n = \prod_{2 \leq k \leq n} (\theta_k)^{-1} \quad (B3)
\]

We observe that we can rewrite the probability distribution \( p_n \) in the following form:

\[
\frac{1}{C_n} \cdot \frac{1 - \theta_2}{C_n} \cdot \frac{1 - \theta_3}{C_n} \cdot \theta_2 C_n \ldots
\]
following the above series, it is not difficult to see that, \((\forall i)(1 < i \leq n)\):
\[
p_n(i) = \frac{1 - \theta_i}{C_n} \prod_{2 \leq k < i} (\theta_k)^{-1}.
\]

Now we connected the parameters related to the increase of the entropy during the evolutionary path and the normalization constant of the distribution. This normalization constant will be the key of our argument. Indeed, thanks to the properties of the Riemann \(\zeta\) function it is known that, if \((\exists \epsilon > 0)\) : \((\exists m)\) : \((\forall k > m)\)
\[
\left(\frac{p_n(k+1)}{p_n(k)}\right) < \left(\frac{k}{k+1}\right)^{-1+\epsilon},
\]
(i.e. the probability distribution \(p_n\) is dominated by the probability distribution \(q_n(i) \propto i^{-1-\epsilon}\)), then \((\exists N \in \mathbb{N})\) : \((N > C_\infty)\), being
\[
C_\infty = \lim_{n \to \infty} C_n.
\]
This means that,
\[
(\forall n) \ p_n(1) \geq \frac{1}{C_\infty} > 0.
\]

However, in the case of \(p_n\) not being dominated by any \(q_n(i) \propto i^{-(1+\epsilon)}\), things go different, since
\[
\lim_{n \to \infty} C_n = \infty.
\]
The direct consequence for the above consideration is that \((\forall \epsilon > 0)(\exists n)\) : \((p_n(1) < \epsilon)\). The presence of an upper bound in the Shannon entropy is directly related to the divergence or convergence of \(C_n\). If the probability distribution is not dominated by \(q_n(i) \propto i^{-(1+\epsilon)}\) for any \(\epsilon > 0\), then the entropy, as well as \(C_n\), diverges. On the contrary, if \(p_n\) is dominated by \(q_n(i) \propto i^{-(1+\epsilon)}\) for any \(\epsilon > 0\) the entropy converges and so does \(C_n\). In formal terms:
\[
\left(\lim_{n \to \infty} C_n = \infty\right) \Leftrightarrow \left(\lim_{n \to \infty} H(X_n) = \infty\right).
\]

Zipf’s law is thus at the twilight zone between bounded and unbounded complexity.

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