A Unified Framework for the Pareto Law and Matthew Effect using Scale-Free Networks

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Abstract. We investigate the accumulated wealth distribution by adopting evolutionary games taking place on scale-free networks. The system self-organizes to a critical Pareto distribution (1897) of wealth \( P(m) \sim m^{-(v+1)} \) with \( 1.6 < v < 2.0 \) (which is in agreement with that of U.S. or Japan). Particularly, the agent’s personal wealth is proportional to its number of contacts (connectivity), and this leads to the phenomenon that the rich gets richer and the poor gets relatively poorer, which is consistent with the Matthew Effect present in society, economy, science and so on. Though our model is simple, it provides a good representation of cooperation and profit accumulation behavior in economy, and it combines the network theory with econophysics.

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1 Introduction

The interaction of many cooperatively interacting agents in economy has many features in common with the statistical physics of interacting systems. A century ago, Pareto (1897) showed that the probability distribution \( P(W) \) for income or wealth of an individual in the market decreased with the wealth \( W \) according to a power law \[1]:

\[ P(W) \propto W^{-(1+v)} \]  \hspace{1cm} (1)

where the value of \( v \) was found to lie between 1 and 2 \[2,3,4,5\]. Studies on real data show that the high-income group indeed follows the Pareto law, with \( v \) varying from 1.6 for USA \[2\] to 1.8-2.2 in Japan \[3\].

The unique feature of our work is that we adopt the scale-free network to represent the cooperative structure in population and study the wealth increment by using evolutionary games as a paradigm for economic activities.

A wide range of systems in nature and society can be described as complex networks. Since the discovery of small-world phenomena by Watts and Strogatz \[12\] and Scale-free phenomena by Barabási and Albert \[13\], investigation of complex networks has attracted continuous attention from the physics community \[14\].

Network theory provides a natural framework to describe the population structure by representing the agents of a given population with the network vertices, and the contacts between those agents with edges \[15\]. One can easily conclude that well-mixed populations can be represented by complete (fully-connected, regular) networks. Spatially-structured populations are associated with regular networks, exhibiting a degree distribution \( d(k) \) which is sharply peaked at a single value of the connectivity \( k \), since all agents generally have the same averaged connectivity. Recently, much empirical evidence of real-world social networks has revealed that they are associated with a scale-free, power-law degree distribution, \( d(k) \sim k^{-\gamma} \) with \( 2 \leq \gamma \leq 3 \) \[14,15,16,17\]. That is, interactions in real-world...
networks are heterogeneous that different individuals have different numbers of average neighbors whom they interact with. Thus, the classic regular or random networks are not good representations of many real social networks which likely possess the self-organized mechanism. Hence, in this paper, we adopt the scale-free network model to construct the cooperation structure in population.

The evolutionary game theory has been considered to be an important approach for characterizing and understanding the cooperative behavior in systems consisting of selfish individuals \[18,19\]. Since their introduction, the Prisoner’s Dilemma (PD) and the Snowdrift Game (SG) have drawn much attention from scientific communities \[20,21,22,23,24\]. In both games, two players simultaneously decide whether to cooperate (C) or defect (D). Each player will get a payoff based on his and his opponent’s strategy in each step and then the players will choose to change their strategy or to keep their strategy unchanged based on some take-over strategies. One can see that both games’ dynamics are very similar to the cooperation and payoff activities between agents in economy and so they are intrinsically suitable for characterizing the payoff and wealth accumulating behavior in populations.

In this paper, we investigate the wealth accumulation of agents playing evolutionary games on the scale-free network. The simulation results show the Pareto wealth distributions along with some remarkable phenomena including the total wealth variation with game parameters, and the Matthew Effect in economy, science, fame, and so on \[25,26,27,28\].

### 2 Model

In this paper, the simulation starts from establishing the underlying cooperation network structure according to the most general Barabási-Albert (BA) scale-free network model \[18\]. In this model, starting from \( m_0 \) fully connected vertices, one vertex with \( m \leq m_0 \) edges is attached at each time step in such a way that the probability \( \Pi_i \) of being connected to the existing vertex \( i \) is proportional to the degree \( k_i \) of the vertex, i.e. \( \Pi_i = \frac{k_i}{\sum_{j}k_j} \), where \( j \) runs over all existing vertices. Initially, an equal percentage of cooperators or defectors was randomly distributed among the agents (vertices) of the population. At each time step, the agents play the PD or SG with their neighbours and get payoff according to the games’ payoff matrix.

In the Prisoner’s Dilemma, each player can either ‘cooperate’ (invest in a common good) or ‘defect’ (exploit the others investment). Two players both receive \( R \) upon mutual cooperation and \( P \) upon mutual defection. A defector exploiting a cooperator gets an amount \( T \) and the exploited cooperator receives \( S \), such that \( T > R > P > S \). So, ‘defect’ is the best response to any action by the opponent \[24\]. Thus in a single play of the game, each player should defect. In the Snowdrift Game (SG), the order of \( P \) and \( S \) is exchanged, such that \( T > R > S > P \). Comparing with PD, SG is more in favor of cooperation. Following common practice \[20,23\], we firstly rescale the games such that each depends on a single parameter. For the PD, we choose the payoffs to have the values \( T = b > 1, R = 1, \) and \( P = S = 0 \), where \( 1 < b \leq 2 \) represents the advantage of defectors over cooperators. That is, mutual cooperators each gets 1, mutual defectors 0, and D gets \( b \) against C. The parameter \( b \) is the only parameter. For the SG, we make \( T = 1 + \beta, R = 1, S = 1 - \beta, \) and \( P = 0 \) with \( 0 < \beta < 1 \) as the only parameter.

Evolution is carried out by implementing the finite population analogue of replicator dynamics \[18,24\]. In each step, all pairs of directly connected individual \( x \) and \( y \) engage in a single round of a given game. The total payoff of agent \( i \) for the step is stored as \( P_i \). And the accumulative payoff (Wealth) of agent \( i \) since the beginning of simulation is stored as \( W_i \). Then the strategy of each agent (Cooperate or Defect) is updated in parallel according to the following rule: whenever a site \( x \) is updated, a neighbor \( y \) is drawn at random among all \( k_x \) neighbors, and the chosen neighbor takes over site \( x \) with probability:

\[
P_{xy} = \frac{1}{1 + e^{(P_x - P_y)/\gamma}},
\]

where \( \gamma \) characterizes noise introduced to permit irrational choices \[29,30,31\], and we make \( \gamma = 0.1 \) as in \[30,31\].

### 3 Simulation Results

We carry out the simulation for a population of \( N = 10^4 \) agents occupying the vertices of a BA scale-free network. The distributions of wealth, total wealth, and k-wealth relation were obtained after a time period of \( T = 10^5 \) steps.

We first examine the wealth distribution \( P(W) \) of the system. Fig. 1 and Fig. 2 show the \( P(W) \) for PD \( (b = 1.5) \) and SG \( (\beta = 0.5) \) respectively. One can see that both
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tially corresponds to ‘real wealth’ or ‘material wealth’, and
In this sense, the wealth distribution we study here essen-
payoff and wealth accumulating behavior in population.
For different simulations, the exponential factor $v$ varies
between 1.6 and 2.0 that are in agreement with the em-
pirical values observed in economies including that of U.S
(1.60), and Japan (1.80 ~ 2.20). We focus on the
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tially corresponds to ‘real wealth’ or ‘material wealth’, and
not the ‘paper money’ that is generally conserved in the
economic system. We also note that the wealth distribu-
tion is independent of the system size $N$ or the simulation
time $T$. Although the system’s maximum personal wealth
is different for Fig. 1 and Fig. 2 because of the difference
in cooperators' frequency, the power law persists for both
high and low cooperator’s frequency cases. All these fac-
tors indicate the robustness of our model to reproduce the
Pareto Law of economy.

Now we consider the system’s total wealth variation
with the parameter $b$ or $\beta$. Fig. 3 and Fig. 4 show the
variation of total wealth of a $N = 10^4$ agents system play-
ning PD and SG respectively. One can see from Fig. 4 that
the total wealth takes a high value ($\approx 4 \times 10^9$) when $b$
is relatively small ($\leq 1.10$). Then there is a bistable region
($1.12 < b < 1.40$) where the total wealth can be either high
($\approx 4 \times 10^9$) or low ($\approx 5 \times 10^5$). When $b$ is greater than 1.40,
the total wealth remains low ($\approx 5 \times 10^5$). The high value
of the system’s total wealth can be as large as $10^4$ times
of the low value. We note that the total wealth value is
related to the frequency of cooperators such that the sys-
tem’s total wealth is high when the frequency is high, and
a low total wealth shows up when the frequency is low.
For instance, the frequency of cooperators is 0.9999 and
the maximum total wealth is 3996720318 when $b = 1.0$.
However, the frequency of cooperators is only 0.2137 and
the total wealth is only 5461747 when $b = 1.5$. This phe-
nomenon implies that when the advantage of defectors
over cooperators is too high, the system will take the
risk of sharply reducing its total wealth. Thus, a defector-
favored economic rule can prohibit the emergence of coop-
erators and, what is more, greatly reduce the total wealth
of the system.

However, because the SG payoff matrix $T > R > S >
P$ is intrinsically cooperator-favored, the total wealth for
SG fluctuates as the $\beta$ value changes as shown in Fig. 4.

Fig. 3 and Fig. 4 show the relation of personal wealth
$W$ with its connectivity $k$. One can see in both cases (PD
and SG) that the personal wealth is proportional to its
connectivity. Since the number of agents it contacts re-
charts show power-law distribution of personal wealth which
is in agreement with Pareto’s law with $v = 1.90$ and
$v = 1.84$ respectively. We perform different simulations by
altering the values of $b$ and $\beta$, and the results show similar
wealth distributions with extremely robust power law.
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Fig. 2. Wealth distribution $P(W)$ for $N = 10^4$ agents playing
SG game with $\beta = 0.5$ for $10^5$ steps. The frequency of coop-
erators is 0.9999, and the maximum personal wealth is about
$10^7$.

Fig. 3. Total Wealth variation for $N = 10^4$ agents playing
PD game. The arrows with $b_1 = 1.10$ and $b_2 = 1.40$ show
the boundaries of the bistable region. The insert shows the
fluctuation of the total wealth in the high branch of the bistable
region.

Fig. 4. Total Wealth variation for $N = 10^4$ agents playing
SG game.
effects the information resources it has, this model also provides a framework to explain the fact that agents with more information resources can gain more profit in modern society’s economy.

This proportional relation between personal wealth and its connectivity is also a possible mechanism for the emergence of the Matthew Effect in economy. The “Matthew Effect” refers to the idea that in some areas of life (wealth, achievement, fame, success etc), the rich gets richer and the poor gets poorer [25,26,27,28]. The eminent sociologist Robert Merton used the term “Matthew effect” to describe the deplorable practice of scientists giving exclusive credit to the most distinguished one among several equally deserving candidates [25]. The Matthew effect for Countries (MEC) was also discovered [26]. Our simulations capture a possible underlying mechanism for these phenomena. In Fig. 7 and Fig. 8, the wealth variations of two individual agents are compared. One can see that with both PD and SG, the wealth of the agent with more connectivity exceeds the agent with less connectivity. We note that this tendency remains the same when different values of parameter $b$ or $\beta$ are used. And also the tendency is independent of the system size $N$ or the simulation time $T$. Thus, the agents with more cooperation partners will get richer and richer while those with fewer partners will get relatively poorer.

4 Conclusions

In conclusion, we have studied the wealth distribution in economy by calculating the accumulative payoff of agents
involving in revolutionary games on the cooperation network with scale-free property. The simulations confirm Pareto’s power law of wealth distribution. And the values of exponential factor $v$ are in agreement with the empirical observations.

The simulation shows that the system’s total wealth varies with the game parameters. The results of the PD game shows that agents tend to cooperate with a frequency of nearly 1.0 and a high total wealth can be achieved when the advantage of defector over cooperator ($b$) is relatively low. But the total wealth will drop to a very low value when $b$ is high. The total wealth of SG fluctuates as the $\beta$ value changes.

The model also provides a possible explanation for the Matthew Effect from a statistical physics point of view. The simulations show that the agents’ personal wealth is proportional to the number of its contacts (connectivity). This leads to the phenomenon that the rich gets richer and the poor gets poorer (Matthew Effect). Thus, in this sense, one has to increase the number of partners in order to gain more profit in modern society. This also suggests a framework to explain why agents with more information resources can gain more profit in modern society’s economy, since the connectivity is a representation of an agent’s information resource.

It is evident from the above discussions that, our model provides a simple but good approach to study the wealth phenomena in economy, and therefore is worthy of more attention.

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References

1. V. Pareto, Le Cours d’Economique Politique, Macmillan, Lausanne, Paris (1987).
2. A.A. Dragulescu, V.M. Yakovenko, Physica A 299, 213(2001).
3. S. Mass de Oliveira, P.M.C. de Oliveira, D. Stauer, Evolution, Money, War and Computers, B.G. Tuebner, Stuttgart, Leipzig, (1999).
4. Y. Fujiwara, W. Souma, H. Aoyama, T. Kaizoji, M. Aoki, Physica A 321, 508(2003).
5. M. Levy, S. Solomon, Physica A 242, 90(1997).
6. A. Chakraborti, B.K. Chakrabarti, Eur. P. J. B 17, 167(2000).
7. A. Dragulascu, V.M. Yakovenko, Eur. P. J. B 17, 723(2000).
8. R. Fischer, D. Braun, Physica A 321, 605(2003).
9. Y. Wang, N. Ding, L. Zhang, Physica A 324, 665(2003).
10. A. Chatterjee, B. K. Chakrabarti, S.S. Manna, Physica A 335, 155(2004).
11. N. Xi, N. Ding, Y. Wang, Physica A 357, 543(2005).
12. D.J. Watts, S.H. Strogatz, Nature 393, 440(1998).
13. A.L. Barabási, R. Albert, Science 286, 509(1999).
14. R. Albert, A.L. Barabási, Rev. Mod. Phys. 74, 47(2002).
15. F.C. Santos, J.M. Pacheco, Phys. Rev. Lett. 95, 098104(2005).
16. S.N. Dorogotsev, J.F.F. Mendes, Evolution of Networks: From Biological Nets to the Internet and WWW, Oxford University, Oxford (2003).
17. M.E.J. Newman, Phys. Rev. E 64, 016132(2001).
18. H. Gintis, Game Theory Evolving, Princeton University, Princeton, NJ(2000).
19. A.M. Colman, Game Theory and its Applications in the Social and Biological Sciences, Butterworth-Heinemann, Oxford (1995).
20. M. Nowak, K. Sigmund, Nature (London) 355, 250(1992).
21. M. Nowak, K. Sigmund, Nature (London) 364, 1(1993).
22. M. Nowak, R.M. May, Nature (London) 359, 826(1992); Int. J. Bifurcation Chaos Appl. Sci. Eng. 3, 35(1993).
23. C. Hauert, M. Doebeli, Nature 428, 643(2004).
24. J.M. McNamara, Z. Barta, A.I. Houston, Nature 428, 745(2004).
25. R.K. Merton, Science 159, 56(1968); ISIS 79, 606(1988).
26. M. Bonitz, E. Bruckner, A. Scharnhorst, Scientometrics 40(3), 407(1997); M. Bonitz, Scientometrics 64(3), 375(2005).
27. D.F. Brewer, Physics Today 44(10), 154(1991);
28. R.H. Wade, Inter. J. Health Services 35(4), 631(2005).
29. G. Szabó, C. Tóke, Phys. Rev. E 58, 69(1998).
30. G. Szabó, C. Hauert, Phys. Rev. Lett. 89(11), 118101(2002).
31. G. Szabó, J. Vukov, Phys. Rev. E 69, 036107(2004).