Integral representation of martingales motivated by the problem of endogenous completeness in financial economics

Dmitry Kramkov∗
Carnegie Mellon University and University of Oxford,
Department of Mathematical Sciences,
5000 Forbes Avenue, Pittsburgh, PA, 15213-3890, USA

Silviu Predoiu†
Citigroup, New York, USA

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Abstract

Let $Q$ and $P$ be equivalent probability measures and let $\psi$ be a $J$-dimensional vector of random variables such that $dQ$ and $\psi$ are defined in terms of a weak solution $X$ to a $d$-dimensional stochastic differential equation. Motivated by the problem of endogenous completeness in financial economics we present conditions which guarantee that every local martingale under $Q$ is a stochastic integral with respect to the $J$-dimensional martingale $S_t \equiv E^Q[\psi|F_t]$. While the drift $b = b(t,x)$ and the volatility $\sigma = \sigma(t,x)$ coefficients for $X$ need to have only minimal regularity properties with respect to $x$, they are assumed to be analytic functions with respect to $t$. We provide a counter-example showing that this $t$-analyticity assumption for $\sigma$ cannot be removed.

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1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete filtered probability space, \(\mathbb{Q}\) be an equivalent probability measure, and \(S = (S_t^j)\) be a \(J\)-dimensional martingale under \(\mathbb{Q}\). It is often important to know whether every local martingale \(M = (M_t)\) under \(\mathbb{Q}\) admits an integral representation with respect to \(S\), that is,

\[
M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0, 1],
\]

(1.1)

for some predictable \(S\)-integrable process \(H = (H^j_t)\). For instance, in mathematical finance, which is the topic of a particular interest to us, the existence of such a martingale representation corresponds to the \textit{completeness} of the market model driven by stock prices \(S\), see Harrison and Pliska [7].

A general answer is given in Jacod [9, Section XI.1(a)]. Jacod’s theorem states that the integral representation property holds if and only if \(\mathbb{Q}\) is the unique equivalent martingale measure for \(S\). In mathematical finance this result is sometimes referred to as the 2nd fundamental theorem of asset pricing.

In many applications, the process \(S\) is defined in a \textit{forward form}, in terms of its predictable characteristics under \(\mathbb{P}\). The density process \(Z\) of a martingale measure \(\mathbb{Q}\) for \(S\) is then constructed through the use of the Girsanov theorem and its generalizations, see Jacod and Shiryaev [10]. The verification of the existence of integral representations for all \(\mathbb{Q}\)-martingales under \(S\) is often straightforward. For example, if \(S\) is a diffusion process under \(\mathbb{P}\) with the drift vector-process \(b = (b_t)\) and the volatility matrix-process \(\sigma = (\sigma_t)\), then such a representation exists if and only if \(\sigma\) has full rank \(d \mathbb{P} \times dt\) almost surely.

In this paper we assume that both \(S\) and \(Z\) are described in a \textit{backward form}, through their terminal values. Given random variables \(\xi > 0\) and \(\psi = (\psi^j)_{j=1,\ldots,J}\) they are defined as

\[
Z_1 \triangleq \frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\xi}{\mathbb{E}[\xi]},
\]

\[
S_t \triangleq \mathbb{E}^\mathbb{Q}[\psi|\mathcal{F}_t], \quad t \in [0, 1].
\]

We are looking for (easily verifiable) conditions on \(\xi\) and \(\psi\) guaranteeing the integral representation of all \(\mathbb{Q}\)-martingales with respect to \(S\).

Our work is motivated by the problem of \textit{endogenous completeness} in continuous-time financial economics which naturally arises in the construction of Radner equilibrium, see Anderson and Raimondo [1], Hugonnier,
Malamud, and Trubowitz [8], and Riedel and Herzberg [18], and in the study of the equilibrium-based price impact models, see Bank and Kramkov [2] and German [6]. Here $\xi$ is an equilibrium state price density, usually defined implicitly by a fixed point argument, and $\psi = (\psi^j)$ is the random vector of the cumulative discounted dividends for traded stocks. The term “endogenous” is used because the stock prices $S$ are computed as an output of equilibrium. A similar problem also arises in the verification of the completeness of markets where, in addition to stocks, one can also trade options, see Davis and Oblój [5].

We focus on the case when $\xi$ and $\psi$ are defined in terms of a weak solution $X$ to a $d$-dimensional stochastic differential equation. With respect to $x$ the coefficients of this equation satisfy classical conditions guaranteeing weak existence and uniqueness: the drift vector $b(t, \cdot)$ is measurable and bounded and the volatility matrix $\sigma(t, \cdot)$ is uniformly continuous and bounded and has a bounded inverse. With respect to $t$ our assumptions are more stringent: $b(\cdot, x)$ and $\sigma(\cdot, x)$ are required to be analytic functions on $(0, 1)$. We give an example showing that this $t$-analyticity assumption on $\sigma$ cannot be removed.

Our results complement those in [1], [8], and [18]. The primary focus of these papers is the existence of Radner equilibrium. The “backward” martingale representation is used implicitly as a link between static and dynamic equilibrium.

In the pioneering paper [1], $X$ is a Brownian motion. The proofs in this paper rely on non-standard analysis. In [8] the conditions are imposed on the diffusion coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$ and on the transition density $p = p(t, x, s, y)$. In the main body of [8], it is assumed that $b$, $\sigma$, and $p$ are analytic functions with respect to all their arguments. In the technical appendix to [8], these functions are required to be analytic with respect to $t$ and $s$ and 5-times (7-times for $p$) continuously differentiable with respect to $x$ and $y$. In [18] the diffusion coefficients $b$ and $\sigma$ do not depend on $t$, the matrix $\sigma$ is invertible and $b = b(x)$, $\sigma = \sigma(x)$, and $\sigma^{-1} = \sigma^{-1}(x)$ and bounded and analytic functions.

In one important aspect, the assumptions in [1] and [18] and in the main body of [8] are less restrictive than those in this paper. If $\psi = F(X_1)$, where $F = F^j(x)$ is a $J$-dimensional vector-function on $\mathbb{R}^d$, then these papers require the Jacobian matrix of $F$ to have rank $d$ only on some open set. In our framework and also in the setup of the technical appendix to [8], this property needs to hold almost everywhere on $\mathbb{R}^d$. We provide an example showing that in the absence of the $x$-analyticity assumption on $b = b(t, x)$ and $\sigma = \sigma(t, x)$ this stronger condition cannot be relaxed.

From the point of view of applications, the most severe constraint of
our setup is the boundedness assumption on the diffusion coefficients. This condition facilitates references to the results from elliptic PDEs commonly stated for bounded coefficients; our main source is Krylov \[15\]. At the same time, it excludes some popular financial modes such as those driven by an Ornstein-Uhlenbeck process. We leave the extension to unbounded coefficients for future research.

In \[12\] the results of this paper are used to obtain criteria for the existence of dynamic Radner equilibrium.

2 Main results

Let \(X\) be a Banach space with the norm \(\| \cdot \|_X\). We shall often use maps \(f : [0, 1] \to X\) which are analytic on \((0, 1)\) and Hölder continuous on \([0, 1]\), that is, for every \(t \in (0, 1)\) there exist a number \(\epsilon(t) > 0\) and a sequence \((A_n(t))_{n \geq 0}\) in \(X\) such that

\[
 f(s) = \sum_{n=0}^{\infty} A_n(t)(s - t)^n, \quad s \in (0, 1), |s - t| < \epsilon(t),
\]

and there are constants \(N > 0\) and \(\delta > 0\) such that

\[
 \|f(t) - f(s)\| \leq N|t - s|^\delta, \quad s, t \in [0, 1].
\]

In the statements of our main results, \(X\) is one of the following spaces:

\(L_\infty = L_\infty(\mathbb{R}^d, dx)\): the Lebesgue space of bounded real-valued functions \(f\) on \(\mathbb{R}^d\) with the norm \(\|f\|_{L_\infty} \triangleq \text{ess sup}_{x \in \mathbb{R}^d} |f(x)|\).

\(C = C(\mathbb{R}^d)\): the Banach space of bounded and continuous real-valued functions \(f\) on \(\mathbb{R}^d\) with the norm \(\|f\|_C \triangleq \sup_{x \in \mathbb{R}^d} |f(x)|\).

We use standard notations of linear algebra. If \(x\) and \(y\) are vectors in \(\mathbb{R}^n\), then \(xy\) denotes the scalar product and \(|x| \triangleq \sqrt{xx}\). If \(a \in \mathbb{R}^{m \times n}\) is a matrix with \(m\) rows and \(n\) columns, then \(ax\) denotes its product on the (column-)vector \(x\), \(a^*\) stands for the transpose, and \(|a| \triangleq \sqrt{\text{trace}(aa^*)}\).

Let \(\mathbb{R}^d\) be an Euclidean space and the functions \(b = b(t, x) : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d\) and \(\sigma = \sigma(t, x) : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}\) be such that for all \(i, j = 1, \ldots, d\):

(A1) the maps \(t \mapsto b^i(t, \cdot)\) of \([0, 1]\) to \(L_\infty\) and \(t \mapsto \sigma^{ij}(t, \cdot)\) of \([0, 1]\) to \(C\) are analytic on \((0, 1)\) and Hölder continuous on \([0, 1]\). For \(t \in [0, 1]\) and
\[
  x \in \mathbb{R}^d \text{ the matrix } \sigma(t, x) \text{ has the inverse } \sigma^{-1}(t, x) \text{ and there exists a constant } N > 0, \text{ same for all } t \text{ and } x, \text{ such that }
  \]
\[
  |\sigma^{-1}(t, x)| \leq N. \quad (2.1)
\]

Moreover, there exists a strictly increasing function \( \omega = (\omega(\epsilon))_{\epsilon > 0} \) such that \( \omega(\epsilon) \to 0 \) as \( \epsilon \downarrow 0 \) and, for all \( t \in [0, 1] \) and all \( x, y \in \mathbb{R}^d \),
\[
  |\sigma(t, x) - \sigma(t, y)| \leq \omega(|x - y|).
\]

Note that (2.1) is equivalent to the uniform ellipticity of the covariance matrix-function \( a \triangleq \sigma \sigma^* \): for all \( y \in \mathbb{R}^d \) and \( (t, x) \in [0, 1] \times \mathbb{R}^d \),
\[
  ya(t, x)y = |\sigma(t, x)y|^2 \geq \frac{1}{N^2} |y|^2.
\]

Let \( X_0 \in \mathbb{R}^d \). The classical results of Stroock and Varadhan [19, Theorem 7.2.1] and Krylov [16, 13] imply that under (A1) there exist a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a Brownian motion \( W \), and a stochastic process \( X \), both with values in \( \mathbb{R}^d \), such that
\[
  X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, 1], \quad (2.2)
\]
and, moreover, all finite dimensional distributions of \( X \) are defined uniquely. In view of (2.1), we can (and will) assume that the filtration \( \mathcal{F} \) is generated by \( X \):
\[
  \mathcal{F} = \mathcal{F}^X \triangleq (\mathcal{F}^X_t)_{t \in [0, 1]}, \quad (2.3)
\]
where, as usual, \( \mathcal{F}^X_t \) denotes the \( \sigma \)-field generated by \( (X_s)_{s \leq t} \) and complemented with \( \mathbb{P} \)-null sets. In this case, \( \mathbb{P} \) is defined uniquely in the sense that if \( \mathbb{Q} \sim \mathbb{P} \) is an equivalent probability measure on \((\Omega, \mathcal{F}^1) = (\Omega, \mathcal{F}^X)\) such that
\[
  W_t = \int_0^t \sigma^{-1}(s, X_s)(dX_s - b(s, X_s)ds), \quad t \in [0, 1],
\]
is a Brownian motion under \( \mathbb{Q} \), then \( \mathbb{Q} = \mathbb{P} \). Note that the filtration \( \mathcal{F}^X \) is (left- and right-) continuous because every \( \mathcal{F}^X \)-martingale is continuous, see Remark 2.2.

Remark 2.1. With respect to \( x \), the conditions in (A1) are, essentially, the minimal classical assumptions guaranteeing the existence and the uniqueness of the weak solution to (2.2). This weak solution is also well-defined when \( b \) and \( \sigma \) are only measurable functions with respect to \( t \). As we shall see in Example 2.6, the requirement on \( \sigma \) to be \( t \)-analytic is, however, essential for the validity of our main Theorem 2.3.

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Remark 2.2. It is well-known that a local martingale $M$ adapted to the filtration $\mathbf{F}^W$, generated by the Brownian motion $W$, is a stochastic integral with respect to $W$, that is, there exists an $\mathbf{F}^W$-predictable process $H$ with values in $\mathbb{R}^d$ such that

$$M_t = M_0 + \int_0^t H_u dW_u \triangleq M_0 + \sum_{i=1}^d \int_0^t H^i_u dW^i_u, \quad t \in [0, 1]. \tag{2.4}$$

The example in Barlow [3] shows that under (A1) the filtration $\mathbf{F}^W$ may be strictly smaller than $\mathbf{F} = \mathbf{F}^X$. Nevertheless, for a local martingale $M$ adapted to $\mathbf{F}$ the integral representation (2.4) still holds with some $\mathbf{F}$-predictable $H$. This follows from the mentioned fact that a probability measure $Q \sim P$ such that $W$ is a $Q$-local martingale (equivalently, a $Q$-Brownian motion) coincides with $P$ and the integral representation theorems in Jacod [9, Section XI.1(a)].

Recall that a locally integrable function $f$ on $(\mathbb{R}^d, dx)$ is weakly differentiable if for every index $i = 1, \ldots, d$ there is a locally integrable function $g^i$ such that the identity

$$\int_{\mathbb{R}^d} g^i(x) h(x) dx = -\int_{\mathbb{R}^d} f(x) \frac{\partial h}{\partial x^i}(x) dx$$

holds for every function $h \in C^\infty$ with compact support, where $C^\infty$ is the space of infinitely many times differentiable functions. In this case, we set $\frac{\partial f}{\partial x^i} \triangleq g^i$. The weak derivatives of higher orders are defined recursively.

Let $J \geq d$ be an integer and the functions $F^j, G : \mathbb{R}^d \to \mathbb{R}$ and $f^j, g^j, \alpha^j, \beta, \gamma^j : [0, 1] \times \mathbb{R}^d \to \mathbb{R}, j = 1, \ldots, J, i = 1, \ldots, d, n$ be such that for some constant $N > 0$

(A2) The functions $F^j$ and $G$ are weakly differentiable, $G$ is strictly positive, the Jacobian matrix $\left( \frac{\partial F^{j_{i_{\ldots j}}}}{\partial x^i} \right)_{i=1,\ldots,d,j=1,\ldots,J}$ has rank $d$ almost surely under the Lebesgue measure on $\mathbb{R}^d$, and

$$\left| \frac{\partial F^j}{\partial x^i} \right| + \left| \frac{\partial G}{\partial x^i} \right| \leq e^{N(1+|x|)}, \quad x \in \mathbb{R}^d.$$

(A3) The maps $t \mapsto e^{-N|x|} f^j(t, \cdot) \triangleq \left( e^{-N|x|} f^j(t, x) \right)_{x \in \mathbb{R}^d}, t \mapsto e^{-N|x|} g^j(t, \cdot)$ and $t \mapsto \alpha^j(t, \cdot), t \mapsto \beta(t, \cdot), t \mapsto \gamma^j(t, \cdot)$ of $[0, 1]$ to $L_\infty$ are analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$. 
Using these functions we define the random variable
\[ \xi \triangleq G(X_1)e^{\int_0^1 \beta(t,X_t)dt}, \]
the equivalent probability measure \( \tilde{\mathbb{P}} \) and the \( \mathbb{P} \)-martingale \( Y \) by
\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \triangleq \exp \left( \int_0^1 \gamma(s,X_s)dW_s - \frac{1}{2} \int_0^1 |\gamma(s,X_s)|^2 ds \right)
\]
\[ Y_t \triangleq \mathbb{E}[\xi|\mathcal{F}_t], \quad t \in [0, 1], \]
and the random variables
\[
\psi^j \triangleq F^j(X_1)e^{\int_0^1 \alpha^j(t,X_t)dt} + \int_0^1 f^j(t,X_t)e^{\int_0^t \alpha^j(s,X_s)ds} dt
\]
\[ + \int_0^1 g^j(t,X_t)e^{\int_0^t (\alpha^j(s,X_s)+\beta(s,X_s))ds} dt, \quad j = 1, \ldots, J. \]

This is the main result of the paper.

**Theorem 2.3.** Suppose that (2.3) and (A1), (A2), and (A3) hold. Then
the equivalent probability measure \( \mathbb{Q} \) with the density under \( \mathbb{P} \)
\[
\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\xi}{\mathbb{E}[\xi]} = \frac{Y_1}{Y_0},
\]
and the \( \mathbb{Q} \)-martingale
\[ S_t \triangleq \mathbb{E}^\mathbb{Q}[\psi|\mathcal{F}_t], \quad t \in [0, 1], \]
with values in \( \mathbb{R}^J \) are well-defined and every local martingale \( M \) under \( \mathbb{Q} \) is a stochastic integral with respect to \( S \), that is, (1.1) holds.

The proof of Theorem 2.3 is given in Section 4 and relies on the study of parabolic equations in Section 3.

**Remark 2.4.** The construction of \( \mathbb{Q} \) and \( S \) is done with a view to accommodate applications to financial economics, see [12], and to allow for a use of PDE techniques in the proof. The \( t \)-analyticity condition on \( f^j \) and \( g^j \) in (A3) cannot be relaxed even if \( X \) is a one-dimensional Brownian motion, see Example 2.7 below. By contrast, the \( x \)-regularity assumptions on the functions \( F^j \), \( G \), \( f^j \), and \( g^j \) in (A2) and (A3) admit weaker formulations where the \( L_\infty \) space can be replaced by appropriate \( L_p \) spaces. This generalization leads, however, to slightly more delicate and longer proofs and, probably, is not interesting for applications.
Remark 2.5. We stress that the $t$-analyticity conditions in (A1) and (A3) are stated for maps of $(0,1)$ to $L_\infty$ which is more than just boundedness and pointwise analyticity. For example, the function $h(t,x) \triangleq \sin(tx)$, which is bounded and analytic on $\mathbb{R} \times \mathbb{R}$, does not define even a continuous map $t \mapsto h(t,\cdot)$ of $(0,1)$ to $L_\infty$:

$$\|h(t,\cdot) - h(s,\cdot)\|_{L_\infty} \triangleq \sup_{x \in \mathbb{R}}|\sin(tx) - \sin(sx)| \geq 1, \quad t \neq s.$$ 

The use of maps is essential for our proof of the theorem based on analytic semigroups.

We conclude with a few counter-examples illustrating the sharpness of the conditions of the theorem. Our first two examples show that the time analyticity assumptions on the volatility coefficient $\sigma = \sigma(t,x)$ and on the functions $f_j = f_j(t,x)$ and $g_j = g_j(t,x)$ cannot be relaxed. In both cases, we take $b_j(t,x) = \alpha_j(t,x) = \beta(t,x) = \gamma_j(t,x) = 0$ and $G(x) = 1$; in particular, $Q = \tilde{P} = P$.

Example 2.6. We show that the assertion of Theorem 2.3 can fail to hold when all its conditions are satisfied except the $t$-analyticity of the volatility matrix $\sigma$. In our construction, $d = J = 2$ and both $\sigma$ and its inverse $\sigma^{-1}$ are $C^\infty$-matrices on $[0,1] \times \mathbb{R}^2$ which are bounded with all their derivatives and have analytic restrictions to $[0,\frac{1}{2}) \times \mathbb{R}^2$ and $(\frac{1}{2},1] \times \mathbb{R}^2$. Moreover, the maps $t \mapsto \sigma^{ij}(t,\cdot)$ of $[0,1]$ to $C(\mathbb{R}^2)$ belong to $C^\infty$ and have analytic restrictions to $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$.

Let $g = g(t)$ be a $C^\infty$-function on $[0,1]$ which equals 0 on $[0,\frac{1}{2}]$ and is analytic and strictly positive on $(\frac{1}{2},1]$. Let $h = h(t,y)$ be an analytic function on $[0,1] \times \mathbb{R}$ such that $0 \leq h \leq 1$, the function $h(1,\cdot)$ is non-constant, the map $t \mapsto h(t,\cdot)$ of $[0,1]$ to $C$ is analytic, and

$$\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial^2 h}{\partial y^2} = 0.$$ 

For instance, we can take

$$h(t,y) = \frac{1}{2}(1 + e^{\frac{t-1}{2}} \sin y).$$ 

Define a 2-dimensional diffusion $(X,Y)$ on $[0,1]$ by

$$X_t \triangleq \int_0^t \sqrt{1 + g(s)h(s,Y_s)}dB_s,$$

$$Y_t \triangleq W_t,$$
where $B$ and $W$ are independent Brownian motions. Clearly, the volatility matrix
\[
\sigma(t,x,y) = \begin{pmatrix}
\sqrt{1 + g(t)h(t,y)} & 0 \\
0 & 1
\end{pmatrix}
\]
has the announced properties and coincides with the identity matrix for $t \in [0,\frac{1}{2}]$.

Define the functions $F = F(x,y)$ and $H = H(x,y)$ on $\mathbb{R}^2$ as
\[
F(x,y) \equiv x,
\quad H(x,y) \equiv x^2 - 1 - h(1,y) \int_0^1 g(t)dt.
\]
As $h(1,\cdot)$ is non-constant and analytic, the set of zeros for $\frac{\partial h}{\partial y}(1,\cdot)$ is at most countable. Since the determinant of the Jacobian matrix for $(F,H)$ is given by
\[
\frac{\partial F}{\partial x} \frac{\partial H}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial H}{\partial x} = -\frac{\partial h}{\partial y}(1,y) \int_0^1 g(t)dt,
\]
it follows that this Jacobian matrix has full rank almost surely.

Observe now that
\[
S_t \triangleq \mathbb{E}[F(X_1,Y_1)|\mathcal{F}_t] = X_t,
\quad R_t \triangleq \mathbb{E}[H(X_1,Y_1)|\mathcal{F}_t] = X_t^2 - t - h(t,Y_t) \int_0^t g(s)ds,
\]
which can be verified by Ito’s formula. As $g(t) = 0$ for $t \in [0,\frac{1}{2}]$, it follows that $S_t = B_t$ and $R_t = B_t^2 - t$ on $[0,\frac{1}{2}]$. Hence, the Brownian motion $Y = W$ cannot be written as a stochastic integral with respect to $(S,R)$.

**Example 2.7.** This counter-example shows the necessity of the $t$-analyticity assumption on $f^j = f^j(t,x)$ and $g^j = g^j(t,x)$ in (A3). Let $h = h(t)$ be a $C^\infty$-function on $[0,1]$ which equals 0 on $[0,\frac{1}{2}]$, is analytic on $(\frac{1}{2},1]$, and is such that $h(1) \neq 0$. For the functions
\[
F(x) \triangleq e^{h(1)x},
\quad f(t,x) \triangleq -(h'(t)x + \frac{1}{2}h^2(t))e^{h(t)x},
\]
the conditions (A2) and (A3) hold except the time analyticity of the map $t \to e^{-N|\cdot|}f(t,\cdot)$ of $[0,1]$ to $L_\infty$. This map belongs instead to $C^\infty$ and has analytic restrictions to $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$. 

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Take $X$ to be a one-dimensional Brownian motion:

$$X_t \triangleq W_t, \quad t \in [0, 1],$$

and observe that, by Ito’s formula,

$$S_t \triangleq \mathbb{E}[\psi|F_t] = e^{h(t)W_t} - \int_0^t \left( h'(s)W_s + \frac{1}{2} h'^2(s) \right) e^{h(s)W_s} ds,$$

where

$$\psi = F(X_1) + \int_0^1 f(t, X_t) dt.$$

For $t \in [0, \frac{1}{2}]$ we have $h(t) = 0$ and, therefore, $S_t = 1$. Hence, a local martingale $M$ which is non-constant on $[0, \frac{1}{2}]$ cannot be a stochastic integral with respect to $S$.

When the diffusion coefficients $\sigma^{ij}$ and $b^i$ and the functions $f^j$, $g^j$, $\alpha^j$, $\beta$, and $\gamma^j$ in (A3) are also analytic with respect to the state variable $x$, the results in [8] and [18] show that in (A2) it is sufficient to require the Jacobian matrix of $F = F(x)$ to have rank $d$ only on an open set. The following example shows that in the case of $C^\infty$ functions this simplification is not possible anymore.

**Example 2.8.** Let $d = J = 2$ and let $h : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $h(x) = 0$ for $x \leq 0$, while $h'(x) > 0$ and $h''(x)$ is bounded for $x > 0$. For instance, take $h(x) = 1_{\{x > 0\}} e^{-1/x}$.

Define the diffusion processes $X$ and $Y$ on $[0, 1]$ by

$$X_t = B_t,$$

$$Y_t = \int_0^t h''(X_s) ds + W_t,$$

where $B$ and $W$ are independent Brownian motions. Clearly, the diffusion coefficients of $(X, Y)$ satisfy (A1).

Define the functions $F = F(x, y)$ and $H = H(x, y)$ on $\mathbb{R}^2$ as

$$F(x, y) = y,$$

$$H(x, y) = y - 2h(x),$$

and the function $f = f(t, x, y)$ on $[0, 1] \times \mathbb{R}^2$ as

$$f(t, x, y) = -h''(x).$$
Observe that the determinant of the Jacobian matrix for \((F, H)\) is given by
\[
\frac{\partial F}{\partial x} \frac{\partial H}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial H}{\partial x} = 2h'(x),
\]
and, hence, this Jacobian matrix has full rank on the set \((0, \infty) \times \mathbb{R}\).

A simple application of Ito’s formula yields
\[
S_t \triangleq \mathbb{E}[F(X_1, Y_1)] + \int_0^t f(s, X_s, Y_s) ds |\mathcal{F}_t] = W_t,
\]
\[
R_t \triangleq \mathbb{E}[H(X_1, Y_1)|\mathcal{F}_t] = W_t - 2 \int_0^t h'(X_s) dB_s.
\]

Hence, any martingale in the form
\[
M_t = \int_0^t \phi(X_s) dB_s,
\]
where the function \(\phi = \phi(x)\) is different from zero for \(x \leq 0\), cannot be written as a stochastic integral with respect to \((S, R)\).

## 3 A time analytic solution of a parabolic equation

The proof of Theorem 2.3 relies on the study of a parabolic equation in Theorem 3.5 below.

For reader’s convenience, recall the definition of the classical Sobolev spaces \(W^m_p\) on \(\mathbb{R}^d\) where \(m \in \{0, 1, \ldots\}\) and \(p \geq 1\). When \(m = 0\) we get the classical Lebesgue spaces \(L^p = L^p(\mathbb{R}^d, dx)\) of real-valued functions \(f\) on \(\mathbb{R}^d\) with the norm
\[
\|f\|_{L^p} \triangleq \left( \int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}.
\]
When \(m \in \{1, \ldots\}\) the Sobolev space \(W^m_p\) consists of all \(m\)-times weakly differentiable functions \(f\) such that
\[
\|f\|_{W^m_p} \triangleq \|f\|_{L^p} + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha f\|_{L^p} < \infty
\]
and is a Banach space with such a norm. The summation is taken with respect to multi-indexes \(\alpha = (\alpha_1, \ldots, \alpha_d)\) of non-negative integers, \(|\alpha| \triangleq \sum_{i=1}^d \alpha_i\) and
\[
D^\alpha \triangleq \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}.
\]
For $t \in [0, 1]$ and $x \in \mathbb{R}^d$ consider an elliptic operator

$$A(t) \triangleq \sum_{i,j=1}^{d} a^{ij}(t,x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x) \frac{\partial}{\partial x^i} + c(t,x),$$

where $a^{ij}$, $b^i$, and $c$ are measurable functions on $[0, 1] \times \mathbb{R}^d$ such that

(B1) the maps $t \mapsto a^{ij}(t, \cdot)$ of $[0, 1]$ to $C$ and $t \mapsto b^i(t, \cdot)$ and $t \mapsto c(t, \cdot)$ of $[0, 1]$ to $L_\infty$ are analytic on $(0, 1)$ and H"older continuous on $[0, 1]$.

The matrix $a$ is symmetric: $a^{ij} = a^{ji}$, uniformly elliptic: there exists $N > 0$ such that

$$ya(t,x)y \geq \frac{1}{N^2} |y|^2, \quad (t,x) \in [0,1] \times \mathbb{R}^d, \quad y \in \mathbb{R}^d,$$

and uniformly continuous with respect to $x$: there exists a decreasing function $\omega = (\omega(\epsilon))_{\epsilon > 0}$ such that $\omega(\epsilon) \to 0$ as $\epsilon \downarrow 0$ and for all $t \in [0, 1]$ and $y, z \in \mathbb{R}^d$

$$|a^{ij}(t,y) - a^{ij}(t,z)| \leq \omega(|y - z|).$$

Let $g = g(x) : \mathbb{R}^d \to \mathbb{R}$ and $f = f(t,x) : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that for some $p > 1$

(B2) the function $g$ belongs to $W^1_p$ and the map $t \mapsto f(t, \cdot)$ from $[0, 1]$ to $L_p$ is analytic on $(0, 1)$ and H"older continuous on $[0, 1]$.

**Theorem 3.1.** Let $p > 1$ and suppose the conditions (B1) and (B2) hold. Then there exists a unique measurable function $u = u(t,x)$ on $[0, 1] \times \mathbb{R}^d$ such that

1. $t \mapsto u(t, \cdot)$ is a H"older continuous map of $[0, 1]$ to $L_p$ whose restriction on $(0, 1)$ is continuously differentiable,
2. $t \mapsto u(t, \cdot)$ is a continuous map of $[0, 1]$ to $W^1_p$,
3. $t \mapsto u(t, \cdot)$ is a continuous map of $(0, 1)$ to $W^2_p$ whose restriction on $(0, 1)$ is analytic,

and such that $u = u(t,x)$ solves the parabolic equation:

$$\frac{\partial u}{\partial t} = A(t)u + f, \quad t \in (0,1], \quad (3.1)$$

$$u(0, \cdot) = g. \quad (3.2)$$
The proof relies on results from the theory of analytic semigroups where our main references are Pazy [17] and Yagi [20]. We first introduce some notations and state a few lemmas.

Let $X$ and $D$ be Banach spaces. By $\mathcal{L}(X,D)$ we denote the Banach space of bounded linear operators $T : X \to D$ endowed with the operator norm. A shorter notation $\mathcal{L}(X)$ is used for $\mathcal{L}(X,X)$. We shall write $D \subset X$ if $D$ is continuously embedded into $X$, that is, the elements of $D$ form a subset of $X$ and there is a constant $N > 0$ such that $\|x\|_X \leq N\|x\|_D$, $x \in D$. We shall write $D = X$ if $D \subset X$ and $X \subset D$.

Let $D \subset X$. A Banach space $E$ is called an interpolation space between $D$ and $X$ if $D \subset E \subset X$ and any linear operator $T \in \mathcal{L}(X)$ whose restriction to $D$ belongs to $\mathcal{L}(D)$ also has its restriction to $E$ in $\mathcal{L}(E)$; see Bergh and Löfström [4, Section 2.4].

The following lemma is used in the proof of item 2 of the theorem.

**Lemma 3.2.** Let $D$, $E$, and $X$ be Banach spaces such that $D \subset X$, $E$ is an interpolation space between $D$ and $X$, and $D$ is dense in $E$. Let $(T_n)_{n \geq 1}$ be a sequence of linear operators in $\mathcal{L}(X)$ such that $\lim_{n \to \infty} \|T_n x\|_X = 0$ for every $x \in X$ and $\lim_{n \to \infty} \|T_n x\|_D = 0$ for every $x \in D$. Then $\lim_{n \to \infty} \|T_n x\|_E = 0$ for every $x \in E$.

**Proof.** The uniform boundedness theorem implies that the sequence $(T_n)_{n \geq 1}$ is bounded both in $\mathcal{L}(X)$ and $\mathcal{L}(D)$. Due to the Banach property, $E$ is a uniform interpolation space between $D$ and $X$, that is, there is a constant $M > 0$ such that

$$\|T\|_{\mathcal{L}(E)} \leq M \max(\|T\|_{\mathcal{L}(X)}, \|T\|_{\mathcal{L}(D)})$$

for any $T \in \mathcal{L}(X) \cap \mathcal{L}(D)$; see Theorem 2.4.2 in [4]. Hence, $(T_n)_{n \geq 1}$ is also bounded in $\mathcal{L}(E)$. The density of $D$ in $E$ then yields the result. \(\square\)

Let $A$ be an (unbounded) closed linear operator on $X$. We denote by $D(A)$ the domain of $A$ and assume that it is endowed with the graph norm of $A$:

$$\|x\|_{D(A)} \triangleq \|Ax\|_X + \|x\|_X.$$

Then $D(A)$ is a Banach space. Recall that the resolvent set $\rho(A)$ of $A$ consists of complex numbers $\lambda$ for which the operator $\lambda I - A : D(A) \to X$, where $I$ is the identity operator, is invertible; the inverse operator is called the resolvent and is denoted by $R(\lambda, A)$. The bounded inverse theorem implies that $R(\lambda, A) \in \mathcal{L}(X, D(A))$ and, in particular, $R(\lambda, A) \in \mathcal{L}(X)$. 13
The operator \( A \) is called \textit{sectorial} if there are constants \( M > 0, r \in \mathbb{R}, \) and \( \theta \in (0, \pi/2) \) such that the sector
\[
S_{r,\theta} \triangleq \{ \lambda \in \mathbb{C} : \lambda \neq r \text{ and } |\arg(\lambda - r)| \leq \pi - \theta \} \tag{3.3}
\]
of the complex plane \( \mathbb{C} \) is a subset of \( \rho(A) \) and
\[
\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in S_{r,\theta}. \tag{3.4}
\]
The set of such sectorial operators will be denoted by \( S(M, r, \theta) \). Sectorial operators whose domains are dense in \( X \) coincide with the generators of analytic semi-groups, see Pazy [17, Section 2.5].

The following two lemmas are used in the proof of item 3 of the theorem.

**Lemma 3.3.** Let \( X \) and \( D \) be Banach spaces such that \( D \subset X \) and let \( A = (A(t))_{t \in [0,1]} \) be closed linear operators on \( X \) such that \( D(A(t)) = D \) for all \( t \in [0,1] \). Suppose \( A : [0,1] \to \mathcal{L}(D, X) \) is a continuous map, and there are \( M > 0, r < 0, \) and \( \theta \in (0, \pi/2) \) such that \( A(t) \in S(M, r, \theta) \) for all \( t \in [0,1] \).

Then for every \( \lambda \in S_{r,\theta} \) the map \( t \mapsto R(\lambda, A(t)) \) of \( [0,1] \) to \( \mathcal{L}(X, D) \) is continuous and there is \( N > 0 \) such that
\[
\|R(\lambda, A(t))\|_{\mathcal{L}(X, D)} \leq N, \quad \lambda \in S_{r,\theta}, \ t \in [0,1]. \tag{3.5}
\]

**Proof.** If \( A \in S(M, r, \theta) \), then for \( \lambda \in S_{r,\theta} \)
\[
\|R(\lambda, A)\|_{\mathcal{L}(X, D(A))} \leq \|R(\lambda, A)\|_{\mathcal{L}(X)} + \|AR(\lambda, A)\|_{\mathcal{L}(X)} \leq M + 1, \tag{3.6}
\]
where we used (3.4) and the identity \( AR(\lambda, A) = \lambda R(\lambda, A) - I \). As \( A : [0,1] \to \mathcal{L}(D, X) \) is a continuous function and \( D(A(t)) = D \), the Banach spaces \( D \) and \( D(A(t)) \), \( t \in [0,1] \), are uniformly equivalent, that is, there is \( L > 0 \) such that
\[
\frac{1}{L} \|x\|_{D(A(t))} \leq \|x\|_{D} \leq L \|x\|_{D(A(t))}, \quad t \in [0,1], \ x \in D. \tag{3.7}
\]
From (3.6) and (3.7) we obtain (3.5) with \( N = L(M + 1) \). As, for \( s, t \in [0,1] \) and \( \lambda \in S_{r,\theta} \),
\[
R(\lambda, A(t)) - R(\lambda, A(s)) = R(\lambda, A(t))(A(t) - A(s))R(\lambda, A(s)),
\]
we deduce from (3.5) that
\[
\|R(\lambda, A(t)) - R(\lambda, A(s))\|_{\mathcal{L}(X, D)} \leq N^2 \|(A(t) - A(s))\|_{\mathcal{L}(D, X)}.
\]
The desired continuity of \( t \mapsto R(\lambda, A(t)) \) in \( \mathcal{L}(X, D) \) follows now from the continuity of \( t \mapsto A(t) \) in \( \mathcal{L}(D, X) \). \( \square \)
Lemma 3.4. Let $X$ and $D$ be Banach spaces such that $D \subset X$ and let $A = (A(t))_{t \in [0,1]}$ be closed linear operators on $X$ such that $D(A(t)) = D$ for all $t \in [0,1]$. Suppose $A : [0,1] \to \mathcal{L}(D, X)$ is an analytic map, and there are $M > 0$, $r < 0$, and $\theta \in (0, \frac{\pi}{2})$ such that $A(t) \in \mathcal{S}(M, r, \theta)$ for all $t \in [0,1]$.

Then there exist a convex open set $U$ in $\mathbb{C}$ containing $[0,1]$ and an analytic extension of $A$ to $U$ such that $A(z) \in \mathcal{S}(2M, r, \theta)$ for all $z \in U$ and the function $A^{-1} : [0,1] \to \mathcal{L}(X, D)$ is analytic.

Proof. Since $r < 0$, the operator $A(t)$ is invertible for every $t \in [0,1]$. As $A : [0,1] \to \mathcal{L}(D, X)$ is analytic, the inverse function $B = A^{-1} : [0,1] \to \mathcal{L}(X, D)$ is well-defined and analytic. Clearly, there is an open convex set $U$ in $\mathbb{C}$ containing $[0,1]$ on which both $A$ and $B$ can be analytically extended. Then $B = A^{-1}$ on $U$, as $AB$ is an analytic function on $U$ with values in $\mathcal{L}(X)$ which on $[0,1]$ equals the identity operator. By Lemma 3.3 there is a constant $N > 0$ such that the inequality (3.5) holds. Of course, we can choose $U$ so that for every $z \in U$ there is $t \in [0,1]$ such that

$$\|A(z) - A(t)\|_{\mathcal{L}(D, X)} \leq \frac{1}{2N}. \quad (3.8)$$

Fix $\lambda \in S_{r, \theta}$ and take $t \in [0,1]$ and $z \in U$ satisfying (3.8). Then

$$\|(A(z) - A(t))R(\lambda, A(t))\|_{\mathcal{L}(X)} \leq \frac{1}{2}$$

and, hence, the operator $I - (A(z) - A(t))R(\lambda, A(t))$ in $\mathcal{L}(X)$ is invertible and the norm of its inverse is not greater than 2. Since

$$\lambda I - A(z) = (I - (A(z) - A(t))R(\lambda, A(t)))(\lambda I - A(t)),$$

we deduce that the resolvent $R(\lambda, A(z))$ is well-defined and

$$\|R(\lambda, A(z))\|_{\mathcal{L}(X)} \leq \frac{2M}{1 + |\lambda|}.$$ 

This completes the proof. \qed

Proof of Theorem 3.1. Under (B1) for every $t \in [0,1]$ the operator $A(t)$ is closed in $L_p$ and has $W^2_p$ as its domain:

$$D(A(t)) = W^2_p. \quad (3.9)$$
Moreover, the operators \((A(t))_{t \in [0,1]}\) are sectorial with the same constants \(M > 0, r \in \mathbb{R},\) and \(\theta \in (0, \pi/2)\):

\[
A(t) \in S(M,r,\theta), \quad t \in [0,1].
\]  (3.10)

These results can be found, for example, in Krylov [15], see Section 13.4 and Exercise 13.5.1.

To insure the existence of fractional powers for the operators \(-A(t)\) it is convenient for us to assume that the sector \(S_{r,\theta}\) defined in \((3.3)\) contains 0 or, equivalently, that \(r < 0\). This condition does not restrict any generality as for \(s \in \mathbb{R}\) the substitution \(u(t,x) \rightarrow e^{st}u(t,x)\) in \((3.1)\) corresponds to the shift \(A(t) \rightarrow A(t) + s\) in the operators \(A(t)\).

From \((B1)\) we deduce the existence of \(M > 0\) and \(\delta > 0\) such that for \(v \in W^2_p\)

\[
\|(A(t) - A(s))v\|_{L^p} \leq M|t - s|^\delta\|v\|_{W^2_p}, \quad s,t \in [0,1].
\]  (3.11)

Conditions \((3.9),\) \((3.10)\), and \((3.11)\) for the operators \(A = A(t)\), the Hölder continuity of \(f = f(t)\) in \((B2)\), and the fact that \(g \in L^p\) imply the existence and uniqueness of the classical solution \(u = u(t,x)\) to the initial value problem \((3.1)-(3.2)\) in \(L^p\); see Theorem 7.1 in Section 5.7 of [17] or Theorem 3.9 in Section 3.6 of [20]. Recall that \(u = u(t,x)\) is the classical solution to \((3.1)\) and \((3.2)\) if \(u(t,\cdot) \in W^2_p\) for \(t \in (0,1]\), the map \(t \mapsto u(t,\cdot)\) of \([0,1]\) to \(L^p\) is continuous, the restriction of this map to \((0,1]\) is continuously differentiable, and the equations \((3.1)\) and \((3.2)\) hold.

The continuity of the map \(t \mapsto u(t,\cdot)\) of \((0,1]\) to \(W^2_p\) follows from the continuity of the map \(t \mapsto A(t)u(t,\cdot) = \frac{\partial u}{\partial t}(t,\cdot) - f(t,\cdot)\) of \((0,1]\) to \(L^p\) and the continuity of the map \(t \mapsto A^{-1}(t)\) of \([0,1]\) to \(\mathcal{L}(L^p, W^2_p)\), which holds by Lemma 3.3.

To verify item 1 we still need to check the Hölder continuity of the map \(t \mapsto u(t,\cdot)\) of \([0,1]\) to \(L^p\). We use Theorem 3.10 in Section 3.8.2 of Yagi [20] dealing with maximal regularity of solutions to evolution equations. This theorem implies the existence of constants \(\delta > 0\) and \(M > 0\) such that

\[
\left\| \frac{\partial u}{\partial t}(t,\cdot) \right\|_{L^p} \leq Mt^{\delta-1}, \quad t \in (0,1],
\]  (3.12)

provided that the operators \(A = A(t)\) satisfy \((3.9),\) \((3.10)\), and \((3.11)\), the function \(f\) is Hölder continuous as in \((B2)\), and

\[
g \in D((-A(0))^\gamma) \quad \text{for some} \quad \gamma > 0,
\]  (3.13)
where $D((-A(0))^{\gamma})$ is the domain of the fractional power $\gamma$ of the operator $-A(0)$ acting in $L_p$. The inequality (3.12) clearly implies the Hölder continuity of $u(t, \cdot) : [0, 1] \to L_p$ and, hence, to complete the proof of item 1 we only need to verify (3.13).

Since $g \in W^1_p$, we obtain (3.13) if

$$W^1_p \subset D((-A(0))^{\gamma}), \quad \gamma \in (0, \frac{1}{2}).$$

(3.14)

Denote by $\Delta \triangleq \sum_i \partial^2_{x_i^2}$ the Laplace operator and recall that $1 - \Delta$ is a sectorial operator and

$$W^1_p = D((1 - \Delta)^{1/2}), \quad D(-A(0)) = W^2_p = D((-A)^{1/2}),$$

see e.g. [15, Theorem 13.3.12]. The embedding (3.14) now follows from the fact that for constants $0 < \alpha < \beta < 1$ and sectorial operators $A$ and $B$ such that $D(B) \subset D(A)$ and such that the fractional powers $(-A)^{\alpha}$ and $(-B)^{\beta}$ are well-defined we have $D((-B)^{\beta}) \subset D((-A)^{\alpha})$, see [20, Theorem 2.25]. This finishes the proof of item 1.

Another consequence of the maximal regularity properties of $u$ given in [20, Theorem 3.10] is that the map $u(t, \cdot) : [0, 1] \to W^2_p$ is continuous if $g \in W^2_p = D(A(0))$. We shall apply this result shortly to prove item 2.

For $t \in [0, 1]$ define a linear operator $T(t)$ on $L_p$ such that for $h \in L_p$ the function $v = v(t, x)$ given by $v(t, \cdot) = T(t)h$ is the unique classical solution in $L_p$ of the homogeneous problem:

$$\frac{\partial v}{\partial t} = A(t)v, \quad v(0, \cdot) = h.$$  

(3.15)

Actually, $T(t) = U(t, 0)$, where $U = (U(t, s))_{0 \leq s \leq t \leq 1}$ is the evolution system for $A = A(t)$; see Pazy [17, Chapter 5], but we shall not use this relation. Of course, the properties established above for $u = u(t, x)$ also hold for the solution $v = v(t, x)$ to (3.15). It follows that for $h \in L_p$ the map $t \mapsto T(t)h$ of $[0, 1]$ to $L_p$ is well-defined and continuous and if $h \in W^2_p$ then the same map is also continuous in $W^2_p$. Recall now that $W^1_p$ is an interpolation space between $L^p$ and $W^2_p$, more precisely, a midpoint in complex interpolation, see [4, Theorem 6.4.5]. Since $W^2_p$ is dense in $W^1_p$, Lemma 3.2 yields the continuity of the map $t \mapsto T(t)h$ of $[0, 1]$ to $W^1_p$.

Observe now that $u = u(t, x)$ can be decomposed as

$$u(t, \cdot) = T(t)g + w(t, \cdot),$$

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where $w(t, \cdot)$ is the unique classical solution in $L_p$ of the inhomogeneous problem:

$$\frac{\partial w}{\partial t} = A(t)w + f, \quad w(0, \cdot) = 0.$$ 

Since $w$ coincides with $u$ in the special case $g = 0$, the map $t \mapsto w(t, \cdot)$ is continuous in $W^2_p$ and, hence, also continuous in $W^1_p$. This completes the proof of item 2.

In item 3 it only remains to verify the analyticity of the map $t \mapsto u(t, \cdot)$ of $(0, 1)$ to $W^2_p$. Fix $0 < \epsilon < 1/2$. The condition (B1) implies the analyticity of the function $A = A(t) : [\epsilon, 1-\epsilon] \to \mathcal{L}(W^2_p, L_p)$. Let $U$ be an open convex set in $\mathbb{C}$ containing $[\epsilon, 1-\epsilon]$ on which there is an analytic extension of $A$ satisfying the assertions of Lemma 3.4. We choose $U$ so that $f = f(t, \cdot) : [\epsilon, 1-\epsilon] \to L_p$ can also be analytically extended on $U$. Theorem 2 in Kato and Tanabe [11] now implies the analyticity of the map $t \mapsto u(t, \cdot)$ of $[\epsilon, 1-\epsilon]$ to $L_p$. However, as $u(t, \cdot) = (A(t))^{-1} (\frac{\partial u}{\partial t} - f(t, \cdot))$, and since, by Lemma 3.4, the $\mathcal{L}(L_p, W^2_p)$-valued function $(A(t))^{-1}$ on $[\epsilon, 1-\epsilon]$ is analytic, the map $t \mapsto u(t, \cdot)$ on $[\epsilon, 1-\epsilon]$ is also analytic in $W^2_p$. As $\epsilon > 0$ is any small number we obtain the result.

In the proof of our main Theorem 2.3 we actually use Theorem 3.5 below, which is a corollary of Theorem 3.1. Instead of (B2) we assume that the measurable functions $g = g(x) : \mathbb{R}^d \to \mathbb{R}$ and $f = f(t, x) : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ have the following properties:

(B3) There is a constant $N \geq 0$ such that $e^{-N|\cdot|} \frac{\partial g}{\partial x}(\cdot) \in L_\infty$ and for every $p \geq 1$ the map $t \mapsto e^{-N|\cdot|} f(t, \cdot)$ from $[0, 1]$ to $L_p$ is analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$.

Fix a function $\phi = \phi(x)$ such that

$$\phi \in C^\infty(\mathbb{R}^d) \quad \text{and} \quad \phi(x) = |x| \text{ when } |x| \geq 1. \tag{3.16}$$

**Theorem 3.5.** Suppose the conditions (B1) and (B3) hold. Let $\phi = \phi(x)$ be as in (3.16). Then there exists a unique continuous function $u = u(t, x)$ on $[0, 1] \times \mathbb{R}^d$ and a constant $N \geq 0$ such that for every $p \geq 1$

1. $t \mapsto e^{-N\phi} u(t, \cdot)$ is a Hölder continuous map of $[0, 1]$ to $L_p$ whose restriction on $(0, 1]$ is continuously differentiable,

2. $t \mapsto e^{-N\phi} u(t, \cdot)$ is a continuous map of $[0, 1]$ to $W^1_p$. 

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3. $t \mapsto e^{-N\phi}u(t, \cdot)$ is a continuous map of $(0, 1]$ to $W^2_p$ whose restriction on $(0, 1)$ is analytic, and such that $u = u(t, x)$ solves the Cauchy problem (3.1) and (3.2).

Proof. From (B3) we deduce the existence of $M > 0$ such that

$$|\partial g/\partial x_i(x)| \leq M e^{M|x|}, \quad x \in \mathbb{R}^d,$$

and, therefore, such that

$$|g(x) - g(0)| \leq M|x| e^{M|x|}, \quad x \in \mathbb{R}^d.$$

Hence, for $N > M$ and a function $\phi = \phi(x)$ as in (3.16)

$$e^{-N\phi} g \in W^1_p, \quad p \geq 1.$$

Hereafter, we choose the constant $N \geq 0$ so that in addition to (B3) it also has the property above.

Define the functions $\tilde{b}^i = \tilde{b}^i(t, x)$ and $\tilde{c} = \tilde{c}(t, x)$ so that for $t \in [0, 1]$ and $v \in C^\infty$

$$\tilde{A}(t)(e^{-N\phi} v) = e^{-N\phi} A(t)v,$$

where

$$\tilde{A}(t) \triangleq \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d \tilde{b}^i(t, x) \frac{\partial}{\partial x^i} + \tilde{c}(t, x).$$

Direct computations show that $\tilde{b}^i$ and $\tilde{c}$ satisfy same conditions as $b^i$ and $c$ in (B1). From Theorem 3.1 we deduce the existence of a measurable function $\tilde{u} = \tilde{u}(t, x)$ which for $p > 1$ complies with the items $1–3$ of Theorem 3.1 and solves the Cauchy problem:

$$\frac{\partial \tilde{u}}{\partial t} = \tilde{A}(t)\tilde{u} + e^{-N\phi} f, \quad \tilde{u}(0, \cdot) = e^{-N\phi} g.$$

For $p > d$, by the classical Sobolev’s embedding, the continuity of the map $t \mapsto \tilde{u}(t, \cdot)$ in $W^1_p$ implies its continuity in $C$. In particular, we obtain that the function $\tilde{u} = \tilde{u}(t, x)$ is continuous on $[0, 1] \times \mathbb{R}^d$.

To conclude the proof it only remains to observe that $u = u(t, x)$ complies with the assertions of the theorem for $p > 1$ if and only if $\tilde{u} \triangleq e^{-N\phi} u$ has the properties just established. The case $p = 1$ follows trivially from the case $p > 1$ by taking $N$ slightly larger. \qed
4 Proof of Theorem 2.3

We assume the conditions and notations of Theorem 2.3. Observe that without any loss in generality we can take
\[ \gamma^i(t, x) = 0 \quad \text{and, hence,} \quad \tilde{P} = P. \] (4.1)

Indeed, by Girsanov’s theorem,
\[ \tilde{W}_t \triangleq W_t - \int_0^t \gamma(s, X_s)ds \]
is a Brownian motion under \( \tilde{P} \). After this substitution the equation (2.2) becomes
\[ dX_t = (b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t))dt + \sigma(t, X_t)d\tilde{W}_t, \quad X_0 = x, \]
and the argument follows from the fact, that, like \( b \), each component of \( \tilde{b} \triangleq b + \sigma\gamma \) defines a map of \([0, 1]\) to \( L_\infty \) which is analytic on \((0, 1)\) and Hölder continuous on \([0, 1]\).

Hereafter we assume (4.1). We fix a function \( \phi \) satisfying (3.16). We also denote by \( L(t) \) the infinitesimal generator of \( X \) at \( t \in [0, 1] \):
\[ L(t) = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(t, x) \frac{\partial}{\partial x^i}, \]
where \( a \triangleq \sigma \sigma^* \) is the covariation matrix of \( X \).

**Lemma 4.1.** There exist unique continuous functions \( u = u(t, x) \) and \( v^j = v^j(t, x), \ j = 1, \ldots, J, \) on \([0, 1] \times \mathbb{R}^d\) and a constant \( N \geq 0 \) such that

1. For \( p \geq 1 \) the maps \( t \mapsto e^{-N\phi}u(t, \cdot) \) and \( t \mapsto e^{-N\phi}v^j(t, \cdot) \) are
   (a) Hölder continuous maps of \([0, 1]\) to \( L_p \) whose restrictions on \([0, 1]\) are continuously differentiable,
   (b) continuous maps of \([0, 1]\) to \( W^1_p \),
   (c) continuous maps of \([0, 1]\) to \( W^2_p \) whose restrictions on \((0, 1)\) are analytic.

2. The function \( u = u(t, x) \) solves the Cauchy problem:
\[
\frac{\partial u}{\partial t} + (L(t) + \beta)u = 0, \quad t \in [0, 1),
\]
\[ u(1, \cdot) = G, \]

(4.2) (4.3)
3. The function $v^j = v^j(t, x)$ solves the Cauchy problem:

$$
\frac{\partial v^j}{\partial t} + (L(t) + \alpha^j + \beta)v^j + uf^j + g^j = 0, \quad t \in [0, 1), \quad (4.4)
$$

$$
v^j(1, \cdot) = F^j G. \quad (4.5)
$$

Proof. Observe first that (A1) on $\sigma = \sigma(t, x)$ implies (B1) on the covariation matrix $a = a(t, x)$. The assertions for $u = u(t, x)$ and, then, for $v^j = v^j(t, x)$, $j = 1, \ldots, J$, follow now directly from Theorem 3.5, where we need to make the time change $t \to 1 - t$.

Hereafter, we denote by $u = u(t, x)$ and $v^j = v^j(t, x)$, $j = 1, \ldots, J$, the functions defined in Lemma 4.1.

Lemma 4.2. The matrix-function $w = w(t, x)$, with $d$ rows and $J$ columns, given by

$$
w^{ij}(t, x) \triangleq \left( u \frac{\partial v^j}{\partial x^i} - v^j \frac{\partial u}{\partial x^i} \right)(t, x), \quad i = 1, \ldots, d, \quad j = 1, \ldots, J, \quad (4.6)
$$

has rank $d$ almost surely with respect to the Lebesgue measure on $[0, 1] \times \mathbb{R}^d$.

Proof. Denote

$$
g(t, x) \triangleq \det(ww^*)(t, x), \quad (t, x) \in [0, 1] \times \mathbb{R}^d,
$$

the determinant of the product of $w$ on its transpose, and observe that the result holds if and only if the set

$$
A \triangleq \{ (t, x) \in [0, 1] \times \mathbb{R}^d : g(t, x) = 0 \}
$$

has the Lebesgue measure zero on $[0, 1] \times \mathbb{R}^d$ or, equivalently, the set

$$
B \triangleq \{ x \in \mathbb{R}^d : \int_0^1 1_A(t, x)dt > 0 \}
$$

has the Lebesgue measure zero on $\mathbb{R}^d$.

From Lemma 4.1 we deduce that the existence of a constant $N \geq 0$ such that for $p \geq 1$ the map $t \mapsto e^{-N\phi}g(t, \cdot)$ from $(0, 1)$ to $W_p^1$ is analytic and the same map of $[0, 1]$ to $L_p$ is continuous. Taking $p > d$, we deduce from the embedding of $W_p^1$ into $C$ that this map is also analytic from $(0, 1)$ to $C$. It follows that if $x \in B$ then $g(t, x) = 0$ for all $t \in (0, 1)$ and, in particular,

$$
\lim_{t \uparrow 1} g(t, x) = 0, \quad x \in B.
$$
Since
\[ \|g(t, \cdot) - g(1, \cdot)\|_{L^p} = \|g(t, \cdot) - \det(w(ww^*)(1, \cdot))\|_{L^p} \rightarrow 0, \quad t \uparrow 1, \]
the Lebesgue measure of B is zero if the matrix-function w(1, ·) has rank d almost surely. This follows from the expression for w(1, ·):
\[ w^{ij}(1, \cdot) = G \frac{\partial (F^j G)}{\partial x^i} - F^j G \frac{\partial G}{\partial x^i} = G^2 \frac{\partial F^j}{\partial x^i}, \]
and the assumption (A2) on \( F = (F^j) \) and \( G \).  

Recall that in addition to the conditions of the theorem we also assume (4.1).

**Lemma 4.3.** The martingales
\[ Y_t \triangleq \mathbb{E}[\xi | \mathcal{F}_t], \quad R^j_t \triangleq \mathbb{E}[\xi \psi^j | \mathcal{F}_t], \quad j = 1, \ldots, J. \]

are well-defined and have the representations
\[ Y_t = u(t, X_t)e^{\int_0^t \beta(s, X_s)ds}, \quad (4.7) \]
\[ R^j_t = v^j(t, X_t)e^{\int_0^t (\alpha^j + \beta)(s, X_s)ds} + Y_t A^j_t, \quad (4.8) \]
where
\[ A^j_t \triangleq \int_0^t \left( f^j(s, X_s) + g^j(s, X_s) \frac{\partial u}{\partial x^i} \sigma^{ik} \right) Y_s e^{\int_0^s \alpha(r, X_r)dr} ds. \]

Moreover, for \( t \in (0, 1) \),
\[ dY_t = \sum_{i,k=1}^d e^{\int_0^t \beta(s, X_s)ds} \left( \frac{\partial u}{\partial x^i} \sigma^{ik} \right) (t, X_t) dW_t^k, \quad (4.9) \]
\[ dR^j_t = \sum_{i,k=1}^d e^{\int_0^t (\alpha^j + \beta)(s, X_s)ds} \left( \frac{\partial v^j}{\partial x^i} \sigma^{ik} \right) (t, X_t) dW_t^k + A^j_t dY_t. \quad (4.10) \]

**Proof.** Assume that \( Y \) and \( R^j \) are actually defined by (4.7) and (4.8). From the continuity of \( u \) and \( v^j \) on \([0, 1] \times \mathbb{R}^d \) we obtain that such \( Y \) and \( R^j \) are continuous processes on \([0, 1] \) and from the expressions (4.3) and (4.5) for \( u(1, \cdot) \) and \( v^j(1, \cdot) \) that \( Y_1 = \xi \) and \( R^j_1 = \xi \psi^j \). Hence, to complete the proof we only have to show that \( Y \) and \( R^j \) given by (4.7) and (4.8) are martingales.
Let $N \geq 0$ be the constant in Lemma 4.1. Choosing $p = d + 1$ in Lemma 4.1 we deduce that the maps $t \mapsto e^{-N\phi}u(t, \cdot)$ and $t \mapsto e^{-N\phi}v^j(t, \cdot)$ of $(0, 1)$ to $W^{2}_{d+1}$ are analytic and, in particular, continuously differentiable. This enables us to use a variant of the Ito formula due to Krylov, see [14, Section 2.10, Theorem 1]. Direct computations, where we account for (4.2) and (4.4), then yield (4.9) and (4.10).

In particular, we have shown that $Y$ and $R^j$ are continuous local martingales. It only remains to verify their uniform integrability. By Sobolev’s embeddings, since $t \mapsto e^{-N\phi}u(t, \cdot)$ and $t \mapsto e^{-N\phi}v^j(t, \cdot)$ are continuous maps of $[0, 1]$ to $W^{1}_{d+1}$, they are also continuous maps to $C$. In particular, $|u(t, x)| + |v^j(t, x)| \leq e^{N(1+|x|)}$.

Accounting for the growth properties of $f^j$, $\alpha^j$, and $\beta$ and denoting

$$B_t^j \triangleq \int_0^t \frac{g^j(s, X_s)}{Y_s} e^{\int_0^s (\alpha^j(r, X_r) + \beta(r, X_r)) dr} ds,$$

we deduce the existence of a constant $N > 0$ such that

$$\sup_{t \in [0, 1]} (|Y_t| + |R^j_t - Y_t B_t^j|) \leq e^{N(1+\sup_{t \in [0, 1]} |X_t|)}.$$

As $\sup_{t \in [0, 1]} |X_t|$ has all exponential moments we obtain the martingale property for $Y$. The martingale property for $R^j$ follows as soon as we verify the uniform integrability of $(Y_t B_t^j)_{t \in [0, 1]}$.

From the growth properties of $g^j$, $\alpha^j$, and $\beta$ we deduce the existence of a constant $N > 0$ such that

$$Y_t |B_t^j| \leq Y_t \int_0^t \frac{1}{Y_s} e^{N(1+|X_s|)} ds = E[Y_1 |F_t] \int_0^t \frac{1}{Y_s} e^{N(1+|X_s|)} ds$$

$$\leq E[Y_1 \int_0^1 \frac{1}{Y_s} e^{N(1+|X_s|)} |F_t}, \quad t \in [0, 1],$$

which implies the the uniform integrability of $(Y_t B_t^j)_{t \in [0, 1]}$ because

$$E[Y_1 \int_0^1 \frac{1}{Y_s} e^{N(1+|X_s|)}] = E[\int_0^1 e^{N(1+|X_s|)}] < \infty.$$
We are ready to complete the proof of the theorem. Lemma 4.3 implies, in particular, that
\[ \mathbb{E}[|\xi| + \sum_{j=1}^{J} |\xi \psi^j|] < \infty, \]
and, hence, the probability measure $Q$ and the $Q$-martingale $S = (S^j)$ are well-defined. Since $\xi > 0$, the measure $Q$ is equivalent to $P$ and the martingale $Y$ is strictly positive. Observe that
\[ S_t \triangleq \mathbb{E}[\psi | \mathcal{F}_t] = \frac{\mathbb{E}[\xi \psi | \mathcal{F}_t]}{\mathbb{E}[\xi | \mathcal{F}_t]} = \frac{R_t}{Y_t}, \quad t \in [0,1]. \]

From (4.9) and (4.10) we deduce, after some computations, that
\[ dS_t = d\frac{R_t}{Y_t} = e^{\int_0^t \alpha^j(s,X_s)ds} \frac{1}{u^2(t,X_t)} \sum_{i,k=1}^{d} (w^{ij} \sigma^{ik})(t,X_t) dW^Q_{t,k}, \tag{4.11} \]
where the matrix-function $w = w(t,x)$ is defined in (4.6) and
\[ W^Q_{t,k} \triangleq W^k_t - \sum_{l=1}^{d} \int_0^t \left( \frac{1}{u} \frac{\partial u}{\partial x^k} \sigma^{lk} \right)(t,X_t) dt, \quad k = 1, \ldots, d, \quad t \in [0,1]. \]

By Girsanov’s theorem, $W^Q$ is a Brownian motion under $Q$. Note that the division on $u(t,X_t)$ is safe as the process $u(t,X_t) = Y_t e^{-\int_0^t \beta(s,X_s)ds}$, $t \in [0,1]$, is strictly positive.

As we have already observed in Remark 2.2, every $P$-local martingale is a stochastic integral with respect to $W$. This readily implies that every $Q$-local martingale $M$ is a stochastic integral with respect to $W^Q$. Indeed, since $L \triangleq Y M$ is a local martingale under $P$, there is a predictable process $\zeta$ with values in $\mathbb{R}^d$ such that
\[ L_t = L_0 + \int_0^t \zeta_u dW_u \triangleq L_0 + \sum_{i=1}^{d} \int_0^t \zeta^i_u dW^i_u \]
and then
\[ dM_t = d\frac{L_t}{Y_t} = \frac{1}{Y_t} \sum_{i=1}^{d} \left( \zeta^i_t - L_t \sum_{k=1}^{d} \left( \frac{1}{u} \frac{\partial u}{\partial x^k} \sigma^{ki} \right)(t,X_t) \right) dW^Q_{t,i}. \]

In view of (4.11), to conclude the proof we only have to show that the matrix-process $((w^* \sigma)(t,X_t))_{t \in [0,1]}$ has rank $d$ on $\Omega \times [0,1]$ almost surely under the product measure $dt \times d\mathbb{P}$. Observe first that by (2.1) and Lemma 4.2
the matrix-function \( w^* \sigma = (w^* \sigma)(t, x) \) has rank \( d \) almost surely under the Lebesgue measure on \([0, 1] \times \mathbb{R}^d\). The result now follows from the well-known fact that under (A1) the distribution of \( X_t \) has a density under the Lebesgue measure on \( \mathbb{R}^d \), see [19, Theorem 9.1.9].

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References

[1] Robert M. Anderson and Roberto C. Raimondo. Equilibrium in continuous-time financial markets: endogenously dynamically complete markets. *Econometrica*, 76(4):841–907, 2008. ISSN 0012-9682.

[2] Peter Bank and Dmitry Kramkov. A model for a large investor trading at market indifference prices. II: continuous-time case. arXiv:1110.3229v2, October 2011. URL http://arxiv.org/abs/1110.3229v2.

[3] M. T. Barlow. One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc. (2)*, 26(2):335–347, 1982. ISSN 0024-6107.

[4] Jörn Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] Mark Davis and Jan Oblój. Market completion using options. In *Advances in mathematics of finance*, volume 83 of *Banach Center Publ.*, pages 49–60. Polish Acad. Sci. Inst. Math., Warsaw, 2008. doi: 10.4064/bc83-0-4. URL http://dx.doi.org/10.4064/bc83-0-4.

[6] David German. Pricing in an equilibrium based model for a large investor. *Math. Financ. Econ.*, 4(4):287–297, 2011. ISSN 1862-9679. doi: 10.1007/s11579-011-0041-6. URL http://dx.doi.org/10.1007/s11579-011-0041-6.
[7] J. Michael Harrison and Stanley R. Pliska. A stochastic calculus model of continuous trading: complete markets. *Stochastic Process. Appl.*, 15 (3):313–316, 1983. ISSN 0304-4149.

[8] J. Hugonnier, S. Malamud, and E. Trubowitz. Endogenous completeness of diffusion driven equilibrium markets. *Econometrica*, 80(3):1249–1270, 2012. ISSN 1468-0262. doi: 10.3982/ECTA8783. URL http://dx.doi.org/10.3982/ECTA8783.

[9] Jean Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979. ISBN 3-540-09253-6.

[10] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003. ISBN 3-540-43932-3.

[11] Tosio Kato and Hiroki Tanabe. On the analyticity of solution of evolution equations. *Osaka J. Math.*, 4:1–4, 1967. ISSN 0030-6126.

[12] Dmitry Kramkov. Existence of endogenously complete equilibrium driven by diffusion. In preparation.

[13] N. V. Krylov. Addendum: On Ito’s Stochastic Integral Equations. *Theory of Probability and its Applications*, 17(2):373–374, 1973. doi: 10.1137/1117046.

[14] N. V. Krylov. *Controlled diffusion processes*, volume 14 of *Applications of Mathematics*. Springer-Verlag, New York, 1980. ISBN 0-387-90461-1.

[15] N. V. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces*, volume 96 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. ISBN 978-0-8218-4684-1.

[16] N.V. Krylov. On Ito’s Stochastic Integral Equations. *Theory of Probability and its Applications*, 14(2):330–336, 1969. doi: 10.1137/1114042.

[17] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983. ISBN 0-387-90845-5.

[18] Frank Riedel and Frederik Herzberg. Existence of financial equilibria in continuous time with potentially complete markets. arXiv:1207.2010v1, July 2012. URL http://arxiv.org/abs/1207.2010v1.
[19] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. ISBN 978-3-540-28998-2; 3-540-28998-4. Reprint of the 1997 edition.

[20] Atsushi Yagi. *Abstract parabolic evolution equations and their applications*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. ISBN 978-3-642-04630-8. URL http://dx.doi.org/10.1007/978-3-642-04631-5.