The chirality-flow formalism for standard model calculations

Joakim Alnefjord, Andrew Lifson, Christian Reuschle and Malin Sjödahl

Abstract Scattering amplitudes are often split up into their color ($\mathfrak{su}(N)$) and kinematic components. Since the $\mathfrak{su}(N)$ gauge part can be described using flows of color, one may anticipate that the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ kinematic part can be described in terms of flows of chirality. In two recent papers we showed that this is indeed the case, introducing the chirality-flow formalism for standard model calculations. Using the chirality-flow method — which builds on and further simplifies the spinor-helicity formalism — Feynman diagrams can be directly written down in terms of Lorentz-invariant spinor inner products, allowing the simplest and most direct path from a Feynman diagram to a complex number. In this presentation, we introduce this method and show some examples.

1 Introduction

Since a few decades it is known that calculations in SU(3) color space can be elegantly simplified using a flow picture for color [1, 2]. In this talk we ask the question if we can similarly simplify the Lorentz structure, which at the algebra level is associated with a left and a right chiral $\mathfrak{su}(2)$.

More specifically, bearing in mind that for color, one can formulate color-flow Feynman rules, we ask whether we can analogously formulate a set of chirality-flow Feynman rules to simplify calculations of Lorentz structure. In this presentation we will answer this question affirmatively and show how Feynman rules can be recast into chirality flows and that this beautifully simplifies calculations [3–6].

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On the QCD side, we can translate every color structure to flows of color using the su(N) Fierz identity to remove adjoint indices

$$i \quad \vdots \quad j \quad \quad k \quad \vdots \quad l \quad = \quad i \quad \vdots \quad j \quad - \frac{1}{N} \quad k \quad \vdots \quad l \quad \delta_{il} \delta_{kj} \quad \delta_{ij} \delta_{kl}.$$  (1)

Similarly, external gluons can be rewritten in terms of color-anticolor pairs (with a color suppressed “U(1)” gluon contribution), and the color structure of triple-gluon vertices can be expressed in terms of traces, such that in the end, every amplitude is a linear combination of products of Kronecker deltas in color space [1, 2].

Before attempting the same procedure for the Lorentz structure, we recall that at the level of the (complexified) algebra, the Lorentz group consists of two copies of su(2), su(2)_{left} \oplus su(2)_{right}, and that the Dirac spinor structure transforms under the direct sum representation \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\). In the chiral (or Weyl) basis we have (for some conventions)

\[ (u^L, u^R) \rightarrow \begin{pmatrix} e^{-i\theta \cdot \bar{\sigma} \cdot \bar{\eta} \cdot \bar{\sigma} \cdot \frac{1}{2}} & 0 \\ 0 & e^{-i\bar{\theta} \cdot \bar{\eta} \cdot \bar{\sigma} \cdot \frac{1}{2}} \end{pmatrix} (u^L, u^R), \]  (2)

i.e. we actually have two copies of SL(2, C), generated by the complexified su(2) algebra.

We will build heavily on the chiral representation and the spinor-helicity formalism [7–16], and start with considering the massless case, for which

\[ u^+ (p) = \begin{pmatrix} 0 \\ p \end{pmatrix}, \quad u^- (p) = \begin{pmatrix} p \\ 0 \end{pmatrix}, \quad \bar{u}^+ (p) = \begin{pmatrix} p \rangle \\ \langle p \end{pmatrix}, \quad \bar{u}^- (p) = \begin{pmatrix} \langle p \\ 0 \end{pmatrix}. \]  (3)

From the spinor-helicity formalism we also borrow the expressions for the polarization vectors [13, 16], expressed in terms of the physical momentum \(p\), and a reference momentum \(r\)

\[ \epsilon^\mu_L (p, r) \rightarrow \frac{\langle p |}{(rp)} \quad \text{or} \quad \frac{|p \rangle}{(rp)} \quad \text{or} \quad \frac{|p \rangle}{(pr)}, \quad \epsilon^\mu_R (p, r) \rightarrow \frac{|p \rangle}{(pr)} \quad \text{or} \quad \frac{|pr \rangle}{(pr)}, \]  (4)

where \(\epsilon^\mu_L\) is for incoming negative helicity or outgoing positive helicity and \(\epsilon^\mu_R\) is for incoming positive helicity or outgoing negative helicity.

To construct Lorentz invariant amplitudes we build invariant spinor inner products using the only SL(2, C) invariant tensor, \(\epsilon^{\alpha\beta} \quad (\epsilon^{12} = -\epsilon^{21} = \epsilon^{12} = -\epsilon^{12} = 1)\). With \(\langle i | = \langle p_i |\) etc., we have

\[ \epsilon^{\alpha\beta}_{\alpha\beta} |i\rangle |j\rangle \alpha = \langle i |^{\alpha} |j\rangle_{\alpha} = \langle ij \rangle, \quad \epsilon^{\alpha\beta}_{\dot{\alpha}\dot{\beta}} |\dot{i}\rangle |\dot{j}\rangle \dot{\alpha} = [\dot{i}]_{\dot{\alpha}} |\dot{j}\rangle \dot{\alpha} = [ij]. \]  (5)
Amplitudes are thus built up out of contractions of the form \( \langle ij \rangle \), \( [ij] \sim \sqrt{s_{ij}} \), and if we manage to create a flow picture, the “flow” must contract left (dotted) and right (undotted) indices separately.

2 Towards chirality flow

For the Lorentz structure, a fermion-photon vertex is associated with a factor
\[
\gamma^\mu = \sqrt{2} \begin{pmatrix} 0 & \tau^\mu \\ \tau_\mu & 0 \end{pmatrix} \]
\( (\tau^\mu = \sigma^\mu/\sqrt{2} \) normalized in analogy with eq. (1)). This can be split into two terms, and when a \( \tau^\mu \) from one vertex is contracted with a \( \bar{\tau}^\mu \) from another other vertex, we have (always reading indices along arrows),
\[
\alpha \beta \gamma \eta \sim \delta^\alpha_\alpha \delta^\beta_\beta
\]
(6)

We note that due to the presence of \( \tau^0 \) there is no \( 1/N \)-suppressed term. In this sense chirality flow is even simpler than color flow.

When a \( \tau(\bar{\tau}) \) is contracted with a \( \tau(\bar{\tau}) \) from the other vertex, the situation is more subtle, and we have to apply charge conjugation at the level of expressions contracted with spinors before removing the vector index
\[
\begin{align*}
\langle 1 | 3 \rangle = & \langle (1 | \alpha \bar{\tau}^\mu \alpha \beta \rangle (3 | \gamma \bar{\tau}^\mu \gamma \eta \rangle | 4 \rangle & \\
\langle 4 | \beta \rangle = & \langle (4 | \eta \bar{\tau}^\mu \gamma \eta \rangle | 3 \rangle & \text{charge conjugated}
\end{align*}
\]
were we have implicitly used the identification of spinors and their graphical representation
\[
| j \rangle = \bullet \rightarrow \bullet, \quad [j] = \bullet \rightarrow \bullet, \quad | i \rangle = \bullet \rightarrow \bullet, \quad [i] = \bullet \rightarrow \bullet
\]

In a similar way, charge conjugation can be applied when additional photons are attached to a quark-line \( \bullet \). A consistent arrow direction with opposing arrows for spin-1 particles can therefore always be chosen \( \bullet \), and the fermion-photon vertex can be translated to
We also need to recast Fermion propagators to the flow picture. To this end, we split \( p_\mu \gamma^\mu = p_\mu \sqrt{2} \left( \begin{array}{c} 0 \\ \bar{\tau}^\mu \end{array} \right) \) into two terms

\[
\psi \equiv \sqrt{2} p_\mu \tau_{\mu \beta} = \cdots p \cdots, \quad \bar{\psi} \equiv \sqrt{2} p_\mu \bar{\tau}_{\mu \alpha} = \cdots p \cdots,
\]

where we have introduced a graphical “momentum-dot” notation for momenta slashed with \( \sigma \) or \( \bar{\sigma} \).

We further note that for massless momenta we have

\[
\sqrt{2} p_\mu \tau_{\mu \alpha} = |p\rangle \langle p|, \quad \sqrt{2} p_\mu \bar{\tau}_{\mu \beta} = |p\rangle \langle p|.
\]

Thus any sum of light-like momenta, \( p_\mu = \sum_i p_{\mu i}, \ p_i^2 = 0 \), can be written

\[
\psi = \sum_i \langle i | \tau^\alpha \langle i | \beta \rangle, \quad \bar{\psi} = \sum_i \langle i | \alpha \langle i | \beta \rangle \text{ for } p_i^2 = 0.
\]

In particular, this gives for the fermion propagator

\[
\frac{p}{p^2} \quad \rightarrow \quad \frac{i}{p^2} \quad \text{or} \quad \frac{i}{p^2}
\]

where the momentum is read along the fermion arrow. (It may be aligned or anti-aligned with the chirality-flow arrows, any arrow assignment with opposing gauge boson arrows will do for massless tree-level QED and QCD since there is always an even number of spinor contractions [4]).

For the photon propagator we have [4]

\[
\mu \nu \rightarrow -\frac{i}{p^2} \quad \text{or} \quad -\frac{i}{p^2}.
\]

Finally, it is straightforward to translate the spinor structure of external gauge bosons to the flow picture, for example

\[
\epsilon_L^\mu(p, r) \rightarrow \frac{1}{\langle rp \rangle} \quad \text{or} \quad \frac{1}{\langle rp \rangle}
\]

In a similar way, Feynman rules can be written down for massless QCD. The main complication is the introduction of a momentum-dot in the triple-gluon vertex, whereas the four-gluon vertex is just a linear combination of chirality-flows with one dotted and one undotted line for each metric factor [4].
3 Examples

Equipped with the Feynman rules for QED, we consider the standard example of $e^+e^- \rightarrow \mu^+\mu^-$. For assigned helicities, it is not hard to calculate this amplitude within the spinor helicity formalism,

$$\frac{2ie^2}{s_{ee}} \left[ 2 [\bar{\alpha} \tau^\alpha \beta] [1] \beta \right] \left( \frac{4 [\bar{\alpha} \tau^\mu [3] \beta]}{\alpha \beta} [1] \beta \right) = \frac{2ie^2}{s_{ee}} \left[ 2 [\bar{\alpha} [3] [4] \beta] [1] \beta \right] = \frac{2ie^2}{s_{ee}} \left[ 23 \right] \left[ 41 \right]$$

but with chirality flow the answer can directly be drawn

Similarly, the value of even a very complicated massless tree-level diagram can just be written down, for example (for reference vectors $r_8$ and $r_9$)

$$\times [15] [4] [10] \left[ (r_9 g_9)^0 + (r_9 10) [10 r_9] \right] \left[ (33) [37] + [34] [47] + [36] [67] \right] \left( - (89) [91] [12] - (89) [95] [52] - (810) [10 1] [12] - (810) [10 5] [52] \right) \times \left( \frac{1}{[8 r_8] [r_9 g_9]} \right)$$

4 Massive chirality flow

To treat mass, we first note that a massive momentum $p$ always can be written as a linear combination of two lightlike momenta, $p^\mu$ and $q$, $p^\mu = p^\mu + \alpha q^\mu$ where $\alpha = \frac{s_{ee}}{2p q}$. This decomposition can be achieved in infinitely many ways — as is obvious from considering the system in its rest frame, where the mo-
menta can be taken to have any opposing direction. Different decompositions correspond to different directions of measuring the momentum \[5, 17, 18\], and in general the spin is measured along

\[
s^{\mu} = \frac{1}{m} (p^{\mu} - \alpha q^{\mu}) = \frac{1}{m} (p^{\mu} - 2 \alpha q^{\mu})
\]

(15)

for, for example, a \(u^+\) spinor of the form \[5\]

\[
u^+(p) = \begin{pmatrix} m \\ \[q \bar{p}] \\ p^\prime \end{pmatrix}.
\]

(16)

The presence of Kronecker delta functions may give rise to an odd number of spinor inner products, implying that signs will have to be carefully tracked in the massive case. We also need to treat the third polarization degree of freedom for a massive vector boson, but other than that, the massive case follows quite straightforwardly from the massless case and using the above decompositions, all Feynman rules of the standard model can be written down \[5\].

5 Conclusion

Splitting Lorentz structure into \(su(2)_{\text{left}}\) and \(su(2)_{\text{right}}\), we have been able to recast all standard model Feynman rules to chirality-flow rules, giving a transparent and intuitive way of understanding the Lorentz inner products appearing in amplitudes.

If the ordinary spinor helicity method takes us from \(4 \times 4\) Dirac matrices to \(2 \times 2\) Pauli matrices, the chirality-flow method takes us from Pauli matrices to scalars. This significantly simplifies calculations with Feynman diagrams. Many processes are within range of quick pen and paper calculations, often without intermediate steps and the final result is transparent and intuitive.

More practically, we expect our method to be useful for event simulations with Monte Carlo event generators, in particular when sampling over helicity. Work towards consistent loop calculations is ongoing.
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