Exact solutions of effective mass Schrödinger equations

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Abstract

We outline a general method of obtaining exact solutions of Schrödinger equations with a position dependent effective mass. Exact solutions of several potentials including the shape invariant potentials have also been obtained.

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1. Apart from being an interesting topic itself Schrödinger equations with a position dependent effective mass have found wide applications in the study of electronic properties of semiconductors [1], quantum dots [2], quantum liquids [3] etc. Although exact solutions of are difficult to obtain, for certain potentials the effective mass Schrödinger equation can be solved [2, 4]. Recently supersymmetric techniques have also been applied to obtain a few exactly solvable potentials [3, 4]. In this article we shall use a simple method

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(which depends on the equivalence of second order differential operators \[8\]) to obtain exact solutions of Schrödinger equation with a position dependent mass. In particular we shall obtain exact solutions of a number of hitherto unknown potentials. To demonstrate the simplicity of the method we shall also rederive the solutions of some of the potentials studied earlier \[6\].

2. When the mass depends on the position the kinetic energy can be defined in several ways. Here we shall be following Lévy-Leblond \[10\] and in this case the Schrödinger equation is given by

\[
-\frac{d}{dx}\left(\frac{1}{2m(x)} \frac{d\psi(x)}{dx}\right) + V(x)\psi(x) = E\psi(x)
\]

The wave function \(\psi(x)\) should be continuous at the mass discontinuity and the derivative of the wave function should satisfy the following condition:

\[
\left.\frac{1}{m(x)} \frac{d\psi(x)}{dx}\right|_- = \left.\frac{1}{m(x)} \frac{d\psi(x)}{dx}\right|_+
\]

We now perform the transformation

\[
\psi(x) = [2m(x)]^{\frac{1}{4}}\phi(x)
\]

on equation (1) and obtain

\[
-\frac{1}{2m(x)} \phi''(x) + \frac{1}{4} \left(\frac{m'(x)}{m^2(x)}\right) \phi'(x) + \frac{4m''(x) - 7m'^2(x)}{32m^3(x)} \phi(x) + V(x)\phi(x) = E\phi(x)
\]

where the prime indicates differentiation with respect to \(x\). Next we make a change of the independent variable defined by

\[
\bar{x} = \int^x \sqrt{2m(y)}dy
\]

Using (3) in equation (4) we get

\[
-\frac{d^2\phi(\bar{x})}{d\bar{x}^2} + \Omega(\bar{x})\phi(\bar{x}) = E\phi(\bar{x})
\]

where for the sake of convenience we have used \(\phi(x)|_{x=\bar{x}} = \phi(\bar{x})\) and \(\Omega(\bar{x})\) is defined by

\[
\Omega(\bar{x}) = V(x = \bar{x}) + \left[\frac{4m''(x) - 7m'^2(x)}{32m^3(x)}\right]_{x=\bar{x}} = V(x = \bar{x}) + V_1(x = \bar{x})
\]
It is important to note that the change of variable in (5) may not always be invertible or at least not easily invertible. But this does not really pose a problem as far as solvability of (1) is concerned. This is because $\bar{x}$ as a function of $x$ is explicitly known from (5) and if we choose $V(x)$ such that

$$V(x) = V_2(\bar{x}) - V_1(\bar{x})$$

where $V_2(\bar{x})$ is a solvable potential then the spectrum of (8) will be known and this in turn will give us the spectrum of (1). The corresponding wave functions can be obtained using (3). In the next section we illustrate the method with a few examples.

3. In order to deal with specific potentials it is necessary to prescribe the mass function $m(x)$. First we consider the choice used in [4]:

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^2, \quad m(0) = \alpha^2, \quad m(\pm\infty) = 1$$

(9)

Then form (5) we get

$$\bar{x} = \sqrt{2}[x + (\alpha - 1)\tan^{-1}x], \quad -\infty < \bar{x} < \infty$$

(10)

and using (10) in (7) we find

$$V_1(x) = \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[-3x^4 + (2\alpha - 4)x^2 + \alpha]$$

(11)

As we mentioned in the last section it is now necessary to choose $V_2(\bar{x})$ to be a solvable potential. Let us first consider the harmonic oscillator:

$$V_2(\bar{x}) = \frac{1}{4}\bar{x}^2$$

(12)

so that

$$E_n = (n + \frac{1}{2})$$

(13)

Then the original potential $V(x)$ is given by

$$V(x) = \frac{1}{2}[x + (\alpha - 1)\tan^{-1}x]^2 + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha]$$

(14)
The potential (14) has harmonic oscillator spectrum given by (13) and using (3) and (5) the wave functions are found to be

\[ \psi_n(x) = N_n \sqrt{\frac{\alpha + x^2}{1 + x^2}} \exp(-\bar{x}^2/2) H_n(2^{-\frac{1}{4}}\bar{x}) \]  

where \( \bar{x} \) is given by (10). We note that the potential (14) is a shape invariant potential [6]. Next we choose \( V_2(\bar{x}) \) as the Morse potential

\[ V_2(\bar{x}) = \lambda^2 \left[ 1 - \exp(-x) \right]^2, \quad E_n = \left[ 2\lambda(n + \frac{1}{2}) - (n + \frac{1}{2})^2 \right], \quad 0 \leq n \leq [\lambda - 1] \]  

(16)

Then for \( V(x) \) we get the following potential which has the same spectrum (16) as the Morse potential:

\[ V(x) = \lambda^2 \left\{ 1 - \exp[-(x + (\alpha - 1)\tan^{-1}x)] \right\}^2 + \frac{(\alpha - 1)}{2(\alpha + x^2)^4} \left[ 3x^4 + (4 - 2\alpha)x^2 - \alpha \right] \]  

(17)

As another example let us consider the soliton potential

\[ V_2(\bar{x}) = -\lambda(\lambda + 1)sech^2\bar{x} \]  

(18)

with energy

\[ E_n = -\left(\lambda - n\right)^2, \quad n < \lambda \]  

(19)

The initial potential is then given by

\[ V(x) = -\lambda(\lambda + 1)sech^2[\sqrt{2}(x + (\alpha - 1)\tan^{-1}x)] + \frac{(\alpha - 1)}{2(\alpha + x^2)^4} \left[ 3x^4 + (4 - 2\alpha)x^2 - \alpha \right] \]  

(20)

with energy given by (19).

Finally we consider an important class of potential, namely, quasi exactly solvable potential [9] for which a part of the spectrum can be determined analytically. A typical representative potential of this class is given by [9]

\[ V(\bar{x}) = \bar{x}^6 - (8j + 3)\bar{x}^2, \quad j = 0, \frac{1}{2}, 1, ... \]  

(21)

and for (21) \( 2j + 1 \) levels can be determined exactly. For example for \( j = \frac{1}{2} \) two levels \( E_{\pm} \) can be found:

\[ E_{\pm} = \mp 2\sqrt{2} \]  

(22)
The quasi exactly solvable potential $V(x)$ corresponding to (1) is given by

$$V(x) = [x+(\alpha-1)\tan^{-1}x]^6 - 7[x+(\alpha-1)\tan^{-1}x]^2 + \frac{(\alpha-1)}{2(\alpha+x^2)^4}[3x^4 + (4-2\alpha)x^2 - \alpha]$$

(23)

and its eigenvalues are given by (22). We note that wave functions corresponding to the potentials (17), (20) and (23) can be obtained using (3) from those of (16), (18) and (21) respectively. It is thus clear that for every potential for which spectral properties of the constant mass Schrödinger equation are known we can obtain a corresponding potential for the effective mass Schrödinger equation with identical spectral properties.

4. The problem of isospectrality of Hamiltonians (when the mass is constant) has been thoroughly investigated [11]. When the mass depends on the space coordinate has also been investigated using supersymmetry [7]. Here we shall demonstrate that two Hamiltonians can indeed be isospectral even when both the mass and the potential are different. To show this we choose a mass of the form

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^4$$

(24)

Then from (3) we get

$$\bar{x} = 2\sqrt{2}[2x + \frac{(\alpha-1)^2x}{(1+x^2)}] + (\alpha-1)(\alpha-3)\tan^{-1}x] , -\infty < \bar{x} < \infty$$

(25)

while (5) gives us

$$V_1(x) = \frac{(\alpha-1)(1+x^2)^2}{(\alpha+x^2)^6}[-3x^4 + (5\alpha - 7)x^2 + \alpha]$$

(26)

Now for $V_2(\bar{x})$ we take

$$V_2(\bar{x}) = \frac{\bar{x}^2}{4}$$

(27)

so that its spectrum is given by (13). In this case $V(x)$ is given by

$$V(x) = \frac{1}{8}[2x + (\alpha-1)^2x(1+x^2) + (\alpha-1)(\alpha-3)\tan^{-1}x]^2 + \frac{(\alpha-1)(1+x^2)^2}{(\alpha+x^2)^6}[3x^4 + (7-5\alpha)x^2 - \alpha]$$

(28)
Since (27) is a harmonic oscillator potential it follows that the effective mass Schrödinger equations for the potentials (14) and (28) with masses given by (1) and (24) respectively are exactly isospectral.

5. In this article we have described a general method of obtaining exact solutions of effective mass Schrödinger equation. In particular the method has been applied to obtain solutions of several potentials which have so far been unknown. It has also been shown that Schrödinger equations with different effective masses as well as different potentials can be isospectral.

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