SYMPLECTIC GROUPOIDS FOR CLUSTER MANIFOLDS

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Abstract. We construct symplectic groupoids integrating log-canonical Poisson structures on cluster varieties of type $\mathcal{A}$ and $\mathcal{X}$ over both the real and complex numbers. Extensions of these groupoids to the completions of the cluster varieties where cluster variables are allowed to vanish are also considered.

In the real case, we construct source-simply-connected groupoids for the cluster charts via the Poisson spray technique of Crainic and Mărcuț. These groupoid charts and their analogues for the symplectic double and blow-up groupoids are glued by lifting the cluster mutations to groupoid comorphisms whose formulas are motivated by the Hamiltonian perspective of cluster mutations introduced by Fock and Goncharov.

1. Introduction

This is the first of a series of papers whose aim is to implement the Weinstein program of geometric quantization for Poisson manifolds [42] in the case of Poisson structures compatible with a cluster structure. In this first paper, we construct the symplectic groupoids for both the cluster $\mathcal{A}$-varieties and the cluster $\mathcal{X}$-varieties.

The cluster $\mathcal{A}$-varieties are the geometric realization of cluster algebras defined by Fomin and Zelevinsky [20] as the culmination of their study of total positivity and canonical bases for algebraic groups [2, 19]. Many varieties arising in Lie theory, e.g. Grassmannians and double Bruhat cells, are examples of $\mathcal{A}$-varieties [3, 39, 23, 43]. The cluster $\mathcal{A}$-varieties are often endowed with a class of compatible Poisson structures in the sense of Gekhtman, Shapiro, and Vainshtein [24]. In particular, our results immediately give rise to a trio of symplectic/Poisson groupoids integrating a compatible Poisson structure on any cluster $\mathcal{A}$-variety, including those mentioned above.

The cluster $\mathcal{X}$-varieties were introduced by Fock and Goncharov [18] in their study of higher Teichmüller space. An alternative view of double Bruhat cells and Grassmannians reveal cluster $\mathcal{X}$-variety structures as well. A cluster $\mathcal{X}$-variety is always endowed with a canonical Poisson structure. In particular, as with the cluster $\mathcal{A}$-varieties, our results produce symplectic/Poisson groupoids integrating any cluster $\mathcal{X}$-variety. If an $\mathcal{A}$-variety carries a compatible Poisson structure, then the natural map from this $\mathcal{A}$-variety to the corresponding $\mathcal{X}$-variety is Poisson. We provide a lift of this map to comorphisms between analogous groupoids over the $\mathcal{A}$- and $\mathcal{X}$-varieties.

In the case of cluster algebras, the quantum $\mathcal{A}$-varieties [4] and the quantum $\mathcal{X}$-varieties [18] are both concrete examples of deformation quantization of Poisson manifolds. In fact, the $q$-quantization of log-canonical variables can be realized simply as the exponential version of the standard Weyl quantization for canonical variables [16]. Geometric quantization for symplectic manifolds takes one step further by constructing a Hilbert space on which the quantized algebra acts, but this requires that the cohomology class of the symplectic 2-form be integral (see e.g. [1]). The notion of symplectic groupoids was introduced by Weinstein [41], Karasëv [29] and Zakrzewski [44, 45]. Weinstein’s motivation was to find a geometric quantization schema for Poisson manifolds [42], which has only been successfully implemented for a handful of examples [40, 27, 5]. The concrete nature of cluster coordinates provides an ideal testing ground for groupoid quantization. Indeed, this was implicitly implemented by Fock and Goncharov [16] in the case of $\mathcal{X}$-varieties.

In this paper, we take the first step towards the groupoid quantization for both cluster $\mathcal{A}$-varieties and cluster $\mathcal{X}$-varieties by describing their symplectic groupoids. The construction of each is detailed below as we discuss the organization of the paper.

We begin in Section 2 with a general discussion of Poisson manifolds and their associated algebroids and groupoids. In Section 3, we construct three groupoids $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{D}$ for a log-canonical Poisson structure on a vector space $L$, these each possess important properties which justify their consideration. The groupoid $\mathcal{G} \rightarrow L$ is the source-simply-connected symplectic groupoid. The definition of $\mathcal{G} \rightarrow L$ utilizes the construction
of local symplectic groupoids by Poisson sprays [14, 7]. For a log-canonical Poisson structure, there is a natural choice of a Poisson spray which yields $G$ under the spray construction. An important note is that, although this provides the source-simply-connected groupoid, the source and the target maps of $G \rightrightarrows L$ will be transcendental and in particular they do not conform (strictly speaking) to the algebraic nature of cluster theory.

One fix for this is an analogue of the symplectic double introduced by Fock and Goncharov [16]. This groupoid which we denote $D \rightrightarrows L$ is a source-connected Poisson groupoid. Over an orthant $L^X$ of $L$ the groupoid $D \rightrightarrows L$ is actually symplectic and we denote its restriction by $D^X \rightrightarrows L^X$, our construction gives a covering of the symplectic groupoid considered by Fock and Goncharov in the case of $X$-varieties.

The linear degeneracy of the Poisson structure on $D$ over the coordinate hyperplanes is corrected via the blow-up construction introduced in works of the first author [30, 26]. This gives rise to the groupoid $B \rightrightarrows L$ which is source-connected and symplectic. Comparing to $G \rightrightarrows L$, the main advantage of $B \rightrightarrows L$ is that the source and the target maps of $B \rightrightarrows L$ are rational. In this way, it seems likely that the groupoid $B$ will be most useful in the theory of cluster algebras. Observe that both $B \rightrightarrows L$ and $D \rightrightarrows L$ receive maps from the source-simply-connected symplectic groupoid $G$.

The groupoids of the cluster $A$-variety and cluster $X$-variety are not constructed globally but rather are glued from groupoid charts over each cluster chart. In Section 4, we give the explicit formulas for the cluster groupoid mutations $\mu : G \to G'$, $\mu : B \to B'$ and $\mu : D \to D'$. We begin in Section 4.1 in the setting of generalized cluster mutations $\mu : L \to L'$ which we decompose into two maps $\phi^1$ and $\tau$ according to the Hamiltonian perspective of mutations [18, 22]. The first map $\phi^1$ is the time-1 flow of a Hamiltonian vector field defined using the Euler dilogarithm function; the second map $\tau$ is a transformation of the log-canonical coordinates. We then lift $\phi^1$ and $\tau$ to groupoid maps which compose to give the cluster groupoid mutations.

In principle, a Poisson map induces a Lie algebroid comorphism [28], which is then lifted to the groupoid comorphisms [8, 9]. Indeed, the Poisson ensemble map $\rho : L_A \to L_X$ can be lifted only to a groupoid comorphism when the exchange matrix is not square. However, if a Poisson morphism is a diffeomorphism, then the dual bundle map of the induced Lie algebroid comorphism is a Lie algebroid morphism, which then lifts to an honest groupoid morphism. Since both the Hamiltonian flow $\phi^1$ and the transformation $\tau$ are diffeomorphisms on an open dense set, we may lift the cluster mutation $\mu = \tau \circ \phi^1$ to cluster groupoid mutations.

Throughout the paper, we use the following notations:

- We write $\mathbb{R}_x = (0, \infty)$, $\mathbb{R}_x = [0, \infty)$ and $\mathbb{C}_x = \mathbb{C} \setminus \{0\}$.
- For the cluster charts, we use the following notations:
  \[ L = \mathbb{R}^m \text{ or } \mathbb{C}^m, \quad L^X = \mathbb{R}^n \text{ or } \mathbb{C}^n, \quad \bar{L}^X = \mathbb{R}^n. \]
- We denote vectors by boldface, e.g. $\mathbf{x} = (x_1, \ldots, x_m)$.
- The Hadamard product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by $\mathbf{x} \odot \mathbf{y} = (x_1y_1, \ldots, x_my_m)$.
- For a vector $\mathbf{x}$ and a real number $t$, we set $\mathbf{x}^t := (x_1^t, \ldots, x_m^t)$.
- For a smooth (resp. complex) manifold $M$, we denote the space of smooth (resp. holomorphic) functions by $\mathcal{O}_M$.
- For a vector bundle $E$ over $M$, we denote the space of sections by $\Gamma(M, E)$. In particular, we denote the space of vector fields by $\mathcal{T}_M = \Gamma(M, TM)$ and the space of multi-vector fields by $\mathcal{T}^k_M = \Gamma(M, \Lambda^k TM)$. To follow the conventional notation, we denote the space of differential $k$-forms by $\Omega^k(M) = \Gamma(M, \Lambda^k T^* M)$.

Acknowledgements. The authors would like to thank Sam Evens, Rui Loja Fernandes, Michael Gekhtman, Marco Gualtieri, and Alan Weinstein for useful discussions related to this project.

2. Poisson structures and symplectic groupoids

We begin by recalling the equivalent notions of Poisson brackets and Poisson bi-vectors.

Definition 2.1. Let $M$ be either a smooth manifold or a complex manifold. A Poisson structure on $M$ is one of the following two equivalent structures:
(1) a Poisson bracket
\[ \{\cdot, \cdot\} : \mathcal{O}_M \times \mathcal{O}_M \to \mathcal{O}_M \]
which is a Lie bracket satisfying the Leibniz rule
\[ \{f, gh\} = g\{f, h\} + h\{f, g\}; \]

(2) a Poisson bi-vector \( \pi \in \mathcal{T}^2_M \) such that \( [\pi, \pi] = 0 \), where \( [\cdot, \cdot] \) is the Schouten-Nijenhuis bracket. We say \( f \in \mathcal{O}_M \) is a Casimir if \( \{f, g\} = 0 \) for every \( g \in \mathcal{O}_M \).

The two notions are related by the formula: \( \{f, g\} = \pi(df \otimes dg) \) for \( f, g \in \mathcal{O}_M \). The pair \((M, \pi)\), or equivalently \((M, \{\cdot, \cdot\})\), is called a Poisson manifold. The Hamiltonian vector field of \( \pi \in \mathcal{O}_M \) is defined as the contraction \( X_\pi = \iota_\pi \pi \) or equivalently as the vector field naturally associated to the derivation \( \{\cdot, \cdot\} \). A Poisson map from \((M_1, \pi_1)\) to \((M_2, \pi_2)\) is a map \( \varphi : M_1 \to M_2 \) such that \( \varphi_*\pi_1 = \pi_2 \) or equivalently \( \{\varphi^*f, \varphi^*g\} = \varphi^*(\{f, g\}) \) for \( f, g \in \mathcal{O}_M \). For \( f \in \mathcal{O}_M \), the pullback of the Hamiltonian vector field \( X_f \) is \( X_{\varphi^*f} \). A Poisson map \( \varphi : (M_1, \pi_1) \to (M_2, \pi_2) \) is complete if the pullback of a complete Hamiltonian vector field is again a complete vector field.

A bi-vector \( \pi \in \mathcal{T}^2_M \) is called non-degenerate if the bundle map
\[ \pi^\sharp : \Omega^1(M) \to \mathcal{T}_M, \quad \theta \mapsto \iota_\theta \pi \]
is invertible. The inverse of this bundle map then defines a non-degenerate 2-form \( \omega \in \Omega^2(M) \). That is, the bundle map
\[ \omega^\flat : \mathcal{T}_M \to \Omega^1(M), \quad v \mapsto \iota_v \omega \]
is the inverse of \( \pi^\sharp \). The condition \([\pi, \pi] = 0\) is equivalent to \( d\omega = 0 \), so a non-degenerate Poisson bi-vector is the same as a symplectic 2-form. Hence for a non-degenerate Poisson bi-vector \( \pi \), we denote the corresponding symplectic 2-form by \( \pi^{-1} \) and for a symplectic 2-form \( \omega \), we denote the corresponding Poisson bi-vector as \( \omega^{-1} \).

**Definition 2.2.** For a Poisson manifold \((M, \pi)\), a symplectic realization is a symplectic manifold \((S, \omega)\) together with a surjective Poisson map \( \rho : (S, \omega) \to (M, \pi) \).

Of particular importance among all the symplectic realizations is the symplectic groupoid, but first we recall the notion of Lie groupoids and Lie algebroids.

**Definition 2.3.** A groupoid \( \mathcal{G} \rightrightarrows M \) consists of two sets \( \mathcal{G} \) and \( M \) with the following maps:

1. a surjective source map \( \alpha : \mathcal{G} \to M \) and a surjective target map \( \beta : \mathcal{G} \to M \);
2. an injective identity map \( 1 : M \to \mathcal{G}, \ x \mapsto 1_x \);
3. an associative multiplication map \( m : \mathcal{G}_\alpha \times_\beta \mathcal{G} \to \mathcal{G}, \ (g, h) \mapsto gh ; \)
4. and an involutive inversion map \( i : \mathcal{G} \to \mathcal{G}, \ g \mapsto g^{-1} ; \)

which satisfy the following properties:

1. \( \alpha(1_x) = \beta(1_x) = x ; \)
2. \( \alpha(gh) = \alpha(h), \ \beta(gh) = \beta(g) ; \)
3. \( \alpha(g^{-1}) = \beta(g), \ \beta(g^{-1}) = \alpha(g) ; \)
4. \( (1_x)^{-1} = 1_x ; \)
5. \( 1_{\beta(g)}g = g = g1_{\alpha(g)}, \ g^{-1}g = 1_{\alpha(g)}, \ gg^{-1} = 1_{\beta(g)} . \)

A Lie groupoid \( \mathcal{G} \rightrightarrows M \) has the following additional properties:

1. \( \mathcal{G} \) and \( M \) are smooth (or complex) manifolds;
2. the source and target \( \alpha, \beta : \mathcal{G} \to M \) are surjective submersions;
3. the multiplication map \( m : \mathcal{G}_\alpha \times_\beta \mathcal{G} \to \mathcal{G} \) is smooth (or holomorphic);
4. the inversion map \( i : \mathcal{G} \to \mathcal{G} \) is smooth (or holomorphic).

A Lie groupoid \( \mathcal{G} \rightrightarrows M \) is source-connected if the source fiber \( \alpha^{-1}(x) \) is connected for every \( x \in M \); it is source-simply-connected if the source fiber \( \alpha^{-1}(x) \) is connected and simply-connected for every \( x \in M \).

A groupoid is naturally considered as a category with objects the elements of \( M \) and morphisms the elements of \( \mathcal{G} \). Then a morphism of groupoids from \( \mathcal{G}_1 \rightrightarrows M_1 \) to \( \mathcal{G}_2 \rightrightarrows M_2 \) is simply a functor between these categories.

Next we recall the notion of a Lie algebroid, which is the infinitesimal object of a Lie groupoid.
Definition 2.4. For a smooth (or holomorphic) manifold $M$, a Lie algebroid over $M$ is a triple $(A, [\cdot, \cdot], \rho)$, where

1. $A$ is a vector bundle over a $M$;
2. $[\cdot, \cdot]$ is a Lie bracket on the space of sections $\Gamma(M,A)$;
3. $\rho : A \rightarrow TM$ is a bundle morphism preserving the Lie bracket;

with Lie bracket satisfying the Leibniz rule: for sections $X,Y \in \Gamma(M,A)$ and $f \in \mathcal{O}_M$,

$$[X, fY] = f \cdot [X, Y] + (\rho X)(f) \cdot Y.$$ 

There is a Lie functor from the Lie groupoids to the Lie algebroids. For a Lie groupoid $G \rightrightarrows M$, we define its Lie algebroid $A = \text{Lie} G$ as follows. As a vector bundle, we have

$$A = \ker (\alpha_* : T\mathcal{G}|_{1M} \rightarrow TM).$$

The Lie bracket is the bracket of left-invariant vector fields and the anchor map is the restriction of the target map $\beta_* : T\mathcal{G} \rightarrow TM$ to $A$. In this case, we say the Lie groupoid $G$ integrates the Lie algebroid $A$.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, its $k$-nerve,

$$\mathcal{G}^{(k)} := \{ (g_1, g_2, \ldots, g_k) \in G^k \mid \beta(g_{k+1}) = \alpha(g_i) \},$$

is the set of $k$-composable elements. In particular, we have

$$\mathcal{G}^{(2)} = G \times_{\beta G} G, \quad \mathcal{G}^{(1)} = \mathcal{G}, \quad \mathcal{G}^{(0)} = M.$$

The nerve of a Lie groupoid is naturally a simplicial manifold that carries a coboundary operator $\partial : \Omega^*(\mathcal{G}^{(k-1)}) \rightarrow \Omega^*(\mathcal{G}^{(k)})$. The first two operators are given as below:

$$\partial : \Omega^1(\mathcal{G}) \rightarrow \Omega^0(\mathcal{G}), \quad \mu \mapsto \alpha^*(\mu) - \beta^*(\mu);$$

$$\partial : \Omega^0(\mathcal{G}) \rightarrow \Omega^1(\mathcal{G}^{(2)}), \quad \mu \mapsto \rho^1_*(\mu - m^*(\mu)) + \rho^2_*(\mu);$$

where $\rho^1 : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ and $\rho^2 : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ are the first and second projections. A differential form $\mu \in \Omega^*(\mathcal{G})$ is called multiplicative if $d\mu = 0$. Our main interest will be with Lie groupoids equipped with a multiplicative symplectic structure.

Definition 2.5. A symplectic groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a multiplicative symplectic structure $\omega \in \Omega^2(\mathcal{G})$. That is, $\rho^1_*(\omega) + \rho^2_*(\omega) = m^*(\omega)$ or equivalently the graph of the multiplication map $\Gamma_m := \{(g, h, gh) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \}$ is Lagrangian with respect to the symplectic structure $\omega \oplus \omega \oplus -\omega$ on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$.

The source fibers of a symplectic groupoid are symplectic orthogonal to the target fibers. Some important examples of symplectic groupoids include: the Kostant-Kirillov-Souriau Poisson structures $\omega \in \Omega^2(\mathcal{G})$. We note that symplectic groupoids are a special case of Poisson groupoids.

Definition 2.6. A Poisson groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with a multiplicative Poisson structure $\sigma \in T^*_\mathcal{G}$. That is, the graph of the multiplication map $\Gamma_m := \{(g, h, gh) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \}$ is coisotropic with respect to the Poisson structure $\sigma \oplus \sigma \oplus -\sigma$ on $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$.

A Poisson groupoid map from $(\mathcal{G}_1, \sigma_1) \rightrightarrows \mathcal{M}_1$ to $(\mathcal{G}_2, \sigma_2) \rightrightarrows \mathcal{M}_2$ is a Poisson groupoid map $\varphi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ which is Poisson. For a Poisson groupoid $(\mathcal{G}, \sigma) \rightrightarrows \mathcal{M}$, there is a natural Poisson structure $\pi$ on $M$ such that $\alpha : (\mathcal{G}, \sigma) \rightarrow (\mathcal{M}, \pi)$ and $\beta : (\mathcal{G}, \sigma) \rightarrow (\mathcal{M}, -\pi)$ are Poisson maps. For a Poisson manifold $(\mathcal{M}, \pi)$, one can ask if there is a symplectic groupoid $(\mathcal{G}, \omega) \rightrightarrows (\mathcal{M}, \pi)$ such that $\alpha : (\mathcal{G}, \omega) \rightarrow (\mathcal{M}, \pi)$ and $\beta : (\mathcal{G}, \omega) \rightarrow (\mathcal{M}, -\pi)$ are Poisson maps. If the answer is yes, we say that the Poisson manifold $(\mathcal{M}, \pi)$ is integrable. The Lie algebroid of a symplectic groupoid $\mathcal{G} \rightrightarrows \mathcal{M}$ is the cotangent bundle $T^*\mathcal{M}$ [41] with the anchor map

$$\pi^* : T^*\mathcal{M} \rightarrow TM, \quad \theta \mapsto \iota_{\theta}\pi,$$

and the Koszul bracket: for $\theta, \psi \in \Omega^1(\mathcal{M})$,

$$[\theta, \psi] = \mathcal{L}_{\pi(\theta)}\psi - \mathcal{L}_{\pi(\psi)}\theta - d\pi(\theta \otimes \psi).$$

For a Poisson manifold $(\mathcal{M}, \pi)$, we denote its cotangent Lie algebroid by $T^*_\mathcal{M}$.

Just like the infinitesimal object of a Poisson Lie group is a Lie bialgebra, the infinitesimal object of a Poisson groupoid is a Lie bialgebroid [34]. As a Poisson groupoid, the symplectic groupoid $(\mathcal{G}, \omega) \rightrightarrows (\mathcal{M}, \pi)$
integrates the Lie bialgebroid \((T^*_\pi M, TM)\). The integrability of Poisson manifolds, and more generally the integrability of Lie algebroids, is characterized in \([12, 13]\).

In general, a Poisson map does not induce a Lie algebroid morphism and hence does not integrate to a bona fide symplectic groupoid map. Instead a Poisson map naturally induces a Lie algebroid comorphism. Following \([9]\), we outline the procedure to lift Lie algebroid morphisms and comorphisms to Lie groupoid morphisms and comorphisms.

**Definition 2.7.** \([28, 33, 9]\) Let \(A_1\) be a Lie algebroid over \(M_1\) and \(A_2\) be a Lie algebroid over \(M_2\) with anchor maps

\[
\rho_1: A_1 \to TM_1, \quad \rho_2: A_2 \to TM_2.
\]

1. A Lie algebroid morphism is a bundle map \((\varphi, \Phi): (M_1, A_1) \to (M_2, A_2)\) such that the pullback of sections \(\Phi^*: \Gamma(\wedge^\bullet A_2) \to \Gamma(\wedge^\bullet A_1)\) is a chain map of complexes.
2. A Lie algebroid comorphism is a bundle comorphism \((\varphi, \Psi)\) with

\[
\varphi: M_1 \to M_2, \quad \Psi: \varphi^! A_2 = M_1 \times_{M_2} A_2 \to A_1
\]
such that \(\varphi_* \circ \rho_1 \circ \Psi = \rho_2\).

In principle, the graph of a comorphism \((\varphi, \Psi)\) from \(A_1\) to \(A_2\) is a subset of \(\varphi^! A_2 \times A_1 = (A_1 \times_{M_2} A_2) \times A_1\), but we will interpret the graph of \(\Psi\) as a subset of \(A_2 \times A_1\). Hence for a diffeomorphism \(\varphi: M_1 \to M_2\), we have that \((\varphi, \Phi)\) is a Lie algebroid morphism if and only if \((\varphi^{-1}, \Phi^v)\) is a Lie algebroid comorphism where \(\Phi^v\) is the dual bundle map of \(\Phi\).

**Definition 2.8.** \([33, 9]\) Let \(G_1 \rightrightarrows M_1\) and \(G_2 \rightrightarrows M_2\) be Lie groupoids.

1. A Lie groupoid morphism from \(G_1\) to \(G_2\) is a map \((\varphi, \Phi): (M_1, G_1) \to (M_2, G_2)\) that is compatible with groupoid structures.
2. A Lie groupoid comorphism from \(G_1\) to \(G_2\) is a base map \(\varphi: M_1 \to M_2\) together with a map

\[
\Psi: M_1 \times_G G_2 \to G_1
\]

that is compatible with groupoid structures (for details, see p.5 of \([9]\)).

Note the difference in convention: the source and target maps in Definition 2.8 are switched comparing to the convention in \([9]\). As with the Lie algebroids, we will interpret the graph of a Lie groupoid comorphism \(\Psi\) from \(G_1\) to \(G_2\) as a subset of \(G_2 \times G_1\).

In general, a Lie algebroid morphism from \(A_1\) to \(A_2\) integrates to a Lie groupoid morphism from \(G_1\) to \(G_2\) if \(G_1\) is source-simply-connected; a Lie algebroid comorphism from \(A_1\) to \(A_2\) integrates to a Lie groupoid comorphism from \(G_1\) to \(G_2\) if \(G_2\) is source-simply-connected and the algebroid comorphism is complete. Concretely, the graph of a morphism (or a comorphism) from \(A_1\) to \(A_2\) is a Lie subalgebroid of \(A_1 \times A_2\), and the exponential map \(A_1 \times A_2 \to G_1 \times G_2\) integrates the graph to a Lie subgroupoid of \(G_1 \times G_2\) which happens to be the graph of a morphism (or a comorphism) from \(G_1\) to \(G_2\) under the assumptions above.

A Poisson map \(\varphi: (M_1, \pi_1) \to (M_2, \pi_2)\) naturally induces the Lie algebroid comorphism \(\varphi^*: \varphi^T T^*_\pi_2 M_2 \to T^*_\pi_1 M_1\), which means that the following diagram commutes \([28]\).

\[
\begin{array}{ccc}
T^*_\pi_1 M_1 & \xleftarrow{\varphi^*} & T^*_\pi_2 M_2 \\
\downarrow{\pi_1^*} & & \downarrow{\pi_2^*} \\
TM_1 & \xrightarrow{\varphi^*} & TM_2
\end{array}
\]

This comorphism \(\varphi^*\) is complete if for a complete Hamiltonian vector field \(X_{f}\) with \(f \in \mathcal{O}_{M_2}\), the pullback Hamiltonian vector field \(X_{\varphi f}\) is also complete. That is, a Poisson map \(\varphi: (M_1, \pi_1) \to (M_2, \pi_2)\) lifts to a unique symplectic groupoid comorphism from \((G_1, \omega_1)\) to \((G_2, \omega_2)\) if \(G_2\) is source-simply-connected and \(\varphi\) is complete. In fact, the graph of \(\varphi\), which is a coisotropic submanifold of \((M_1 \times M_2, \pi_1 + \pi_2)\), integrates to a Lagrangian subgroupoid of \((G_1 \times G_2, \omega_1 + -\omega_2)\) \([8]\).

If the Poisson map \(\varphi: (M, \pi) \to (M, \pi)\) is a diffeomorphism, then \(\varphi\) induces both a Lie algebroid morphism and a Lie algebroid comorphism, which can be lifted either to a symplectic groupoid morphism or to a symplectic groupoid comorphism. For the periodicity of groupoid mutations Proposition 4.25, we single out the special case when the Poisson map is the identity.
Poisson structure on $\mathbb{R}^n$.

In coordinates, we have

\[ \{x_i, x_j\} = \Omega_{ij} x_i x_j, \quad 1 \leq i, j \leq m \quad \text{or equivalently} \quad \pi = \sum_{j=1}^{m} \Omega_{ij} x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \]

for some skew-symmetric $m \times m$ matrix $\Omega = (\Omega_{ij})$.

Using the results in [14, 7], we construct the source-simply-connected symplectic groupoid of a log-canonical Poisson structure by choosing an appropriate Poisson spray.

**Definition 3.2.** [14] For a Poisson manifold $(M, \pi)$, a Poisson spray is a vector field $X \in T^*M$ such that

1. for $(x, p) \in T^*M$ we have
\[ \tau_x X|_{(x,p)} = \pi^*(p), \]

where $\tau : T^*M \to M$ is the bundle projection;

2. $X$ is homogeneous of degree 1, i.e.
\[ (m_\lambda)_x(X) = \lambda X, \]

where $m_\lambda : T^*M \to T^*M$, $(x, p) \mapsto (x, \lambda p)$ is the fiberwise scaling map.

**Theorem 3.3.** [14, 7] For a smooth Poisson manifold $(M, \pi)$ with a Poisson spray $X \in T^*M$, there exists a neighborhood $U$ of the zero section of $T^*M$ which is a local symplectic groupoid over $(M, \pi)$ with the following structures:

1. the source map $\alpha : U \to M$ is the bundle projection;

2. the target map is
\[ \beta : U \to M, \quad \beta = \tau \circ \varphi^1_X, \]

where $\varphi^1_X : T^*M \to T^*M$ denotes the time-$t$ flow of $X$;

3. the identity map $\mathbf{1} : M \to U$ is the zero section;

4. the inverse map is
\[ \iota : U \to U, \quad \iota = -\varphi^1_X; \]

5. the multiplication $m : U \times \beta U \to U$ is defined as the solution of an ODE (see [7] for details);

6. the symplectic form on $U$ is
\[ \omega = \int_0^1 (\varphi^s_X)^* \omega_0 dt, \]

where $\omega_0$ is the standard symplectic structure on $T^*M$.

**Remark 3.4.** For the standard symplectic structure $\omega_0$ in Theorem 3.3, we use the sign convention that $\omega_0 = -d\theta_0$ for $\theta_0$ the tautological 1-form on $T^*M$. This choice ensures that the source map $\alpha$ is Poisson. In coordinates, we have $\omega_0 = \sum_i dx_i \wedge dp_i$. 

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3. SYMPLECTIC GROUPOIDs OF log-CANONICAL POISSON STRUCTURES

We focus in this paper on the symplectic groupoids of log-canonical Poisson structures.

**Definition 3.1.** Let $L$ be either $\mathbb{R}^n$ or $\mathbb{C}^n$ and write $x = (x_1, \ldots, x_m)$ for a system of coordinates on $L$. A Poisson structure on $L$ is log-canonical if

\[ \{x_i, x_j\} = \Omega_{ij} x_i x_j, \quad 1 \leq i, j \leq m \quad \text{or equivalently} \quad \pi = \sum_{j=1}^{m} \Omega_{ij} x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \]

for some skew-symmetric $m \times m$ matrix $\Omega = (\Omega_{ij})$.

Following [15], we outline a method to lift the Hamiltonian Poisson maps to Hamiltonian symplectic groupoid morphisms. In general, Poisson vector fields are lifted to multiplicative Hamiltonian vector fields. For the Hamiltonian vector field $X_f \in T_M$ of $f \in \mathcal{O}_M$, if $(\mathcal{G}, \omega) \Rightarrow (M, \pi)$ is a symplectic groupoid, then $X_f$ is lifted to the Hamiltonian vector field $X_F \in T_G$, where $F := \alpha^* f - \beta^* f$. Indeed, the function $F \in \mathcal{O}_G$ is multiplicative since $F = \partial f$, where $\partial$ is the groupoid coboundary operator (2.2), and the symplectic form $\omega$ is multiplicative by definition, so the Hamiltonian vector field $X_F = t_{dF} \omega^{-1}$ is multiplicative. It follows that $X_F$ preserves the symplectic groupoid structures of $(\mathcal{G}, \omega)$; and the time-$t$ flow $\varphi^t_F : (M, \pi) \to (M, \pi)$ of $X_F$ is lifted to the time-$t$ flow $\varphi^t_F : (\mathcal{G}, \omega) \to (\mathcal{G}, \omega)$ of $X_F$. 

**Proposition 2.9.** If $\mathcal{G} \Rightarrow M$ is the source-simply-connected symplectic groupoid of $(M, \pi)$, then the identity map $1_M : M \to M$ as a Poisson diffeomorphism lifts to the identity groupoid map $1_G : \mathcal{G} \to \mathcal{G}$. 

If $\mathcal{F} \Rightarrow \mathcal{G}$ is a Poisson map, then $\mathcal{F}$ is lifted to the time-$t$ flow $\varphi^t_F : (\mathcal{G}, \omega) \to (\mathcal{G}, \omega)$ of $X_f$.
Remark 3.5. By a local symplectic groupoid \((\mathcal{G}, \omega) \rightrightarrows (M, \pi)\), we mean that the multiplication \(m : \mathcal{G}_x \times \mathcal{G}_y \to \mathcal{G}\) may not be defined on all of its domain. In general, the local symplectic groupoid structure cannot be extended to the total space \(T^*M\). Indeed: the Poisson spray \(X\) may not be complete; the flow of the Poisson spray \(X\) may contain loops; or the 2-form \(\omega\), though non-degenerate near the zero section of \(T^*M\), may be degenerate globally.

In the next results, we introduce a Poisson spray whose local symplectic groupoid provides an integration of a log-canonical Poisson structure.

Lemma 3.6. For the log-canonical Poisson structure \(\{x_i, x_j\} = \Omega_{ij}x_ix_j\) on \(L = \mathbb{R}^m\), the vector field \(X \in T^*_xL\) given in coordinates \((x, p) = (x_1, \ldots, x_m, p_1, \ldots, p_m)\) on \(T^*L\) by

\[
X = \sum_{1 \leq i, j \leq m} \Omega_{ij}x_ip_j \frac{\partial}{\partial x_j} - \sum_{1 \leq i, j \leq m} \Omega_{ij}x_ip_i \frac{\partial}{\partial p_j}
\]

is a Poisson spray. Its flow is given by

\[
\varphi_t^X : T^*L \to T^*L, \quad (x, p) \mapsto (\alpha^t(x), \alpha^{-t}(p)),
\]

where \(\alpha_j = e^{\sum_i \Omega_{ij} x_i p_i}\). This flow exists for all \(t \in \mathbb{R}\) and contains no loops.

Proof. For the co-vector \(\theta = p_1dx_1 + \ldots + p_mdx_m\) and the point \((x, p)\), we have

\[
\tau_*X|_{(x,p)} = \sum_{i,j} \Omega_{ij}x_ip_j \frac{\partial}{\partial x_j} = \theta \pi.
\]

To find the flow of \(X\), we note that \(x_ip_i\) is a constant under the flow of \(X\):

\[
\frac{d}{dt}(x_ip_j) = \dot{x}_ip_j + x_j\dot{p}_j = \sum_{1 \leq i, j \leq m} \Omega_{ij}x_ip_ix_jp_j - \sum_{1 \leq i, j \leq m} \Omega_{ij}x_ip_ip_jx_j = 0.
\]

Therefore \(\sum_i \Omega_{ij}x_ip_i\) is constant, and

\[
x_j(t) = e^{t\sum_i \Omega_{ij} x_i p_i}(0) = \alpha_j^t x_j(0),
\]

\[
p_j(t) = e^{-t\sum_i \Omega_{ij} x_i p_i}(0) = \alpha_j^{-t} p_j(0).
\]

This Poisson spray \(X\) induces the symplectic groupoid structure below.

Theorem 3.7. For the log-canonical Poisson structure \(\{x_i, x_j\} = \Omega_{ij}x_ix_j\) on \(L\) (which is either \(\mathbb{R}^m\) or \(\mathbb{C}^m\)), there is a source-simply-connected symplectic groupoid \((\mathcal{G}, \omega_\mathcal{G}) \rightrightarrows (L, \pi)\) with the following structures:

1. \(\mathcal{G} \cong T^*L\) has the coordinates \((x, p) = (x_1, \ldots, x_m, p_1, \ldots, p_m)\);
2. the source map is the bundle projection \(\alpha : T^*L \to L, \quad (x, p) \mapsto x\);
3. the target map is \(\beta : T^*L \to L, \quad (x, p) \mapsto a \circ x, \text{ where } a_j = e^{\sum_i \Omega_{ij} x_i p_i}\);
4. the identity map is \(1 : L \to T^*L, \quad x \mapsto (x, 0)\);
5. the inverse map is \(\iota : T^*L \to T^*L, \quad (x, p) \mapsto (a \circ x, -a^{-1} \circ p)\);
6. the multiplication map is \(m : T^*L \times_{\alpha} T^*L \to T^*L, \quad ((a \circ x, p'), (x, p)) \mapsto (x, a \circ p' + p)\);
7. the multiplicative symplectic form \(\omega\) is

\[
\omega_\mathcal{G} = \sum_j dx_j \wedge dp_j + \sum_{i,j} \Omega_{ij}p_ix_jdx_i \wedge dp_j + \sum_{j>i} \Omega_{ij}p_jdx_i \wedge dx_j + \sum_{j<i} \Omega_{ij}x_jdp_i \wedge dp_j,
\]

and equivalently the multiplicative Poisson bi-vector

\[
\sigma_\mathcal{G} = \omega_\mathcal{G}^{-1} = -\sum_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial p_i} - \sum_{i,j} \Omega_{ij}x_ip_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial p_j} + \sum_{j>i} \Omega_{ij}p_j \frac{\partial}{\partial p_j} \wedge \frac{\partial}{\partial p_i} + \sum_{j<i} \Omega_{ij}x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.
\]
Theorem 3.9. Let $\Omega_{ij}$ be the symplectic structure on $L$. For the log-canonical Poisson structure $(\omega, D)$ on $L$, the symplectic double $(\mathcal{D}, \sigma_D) \to (L, \pi)$ is a source-connected Poisson groupoid with the following structures:

1. $\mathcal{D} \cong L \times L^\times$ has the coordinates $(\mathbf{x}, s) = (x_1, \ldots, x_m, s_1, \ldots, s_m)$ (where $L^\times$ is respectively either $\mathbb{R}_+^m$ or $\mathbb{C}_+^m$);
2. the source map is $\alpha : L \times L^\times \to L, (\mathbf{x}, s) \mapsto x$.

Proof. When the underlying field is $\mathbb{R}$, it is straightforward to check that the Poisson spray $(3.1)$ induces the given groupoid structures. To find the sympletic structure $\omega_G$, we have

$$(\varphi_X^t)^* \omega_0 = \sum_j d \left( e^t \sum_i \Omega_{ij} x_i p_i x_j \right) \wedge d \left( e^{-t} \sum_i \Omega_{ij} x_i p_i x_j \right)$$

$$= \sum_j \left( dx_j + t \sum_i \Omega_{ij} x_i x_j dp_i + t \sum_i \Omega_{ij} x_i p_j dx_i \right) \wedge \left( dp_j - t \sum_i \Omega_{ij} x_i p_j dp_i - t \sum_i \Omega_{ij} x_i p_j dx_i \right)$$

$$= \sum_j dx_j \wedge dp_j + 2t \left( \sum_{i,j} \Omega_{ij} x_i x_j dx_i \wedge dp_j + \sum_{i,j} \Omega_{ij} x_i p_j dx_i \wedge dx_j + \sum_{i,j} \Omega_{ij} x_i x_j dp_i \wedge dp_j \right),$$

so it follows that

$$\omega_G = \int_0^1 (\varphi_X^t)^* \omega_0 dt$$

$$= \sum_j dx_j \wedge dp_j + \left( \sum_{i,j} \Omega_{ij} x_i p_j dx_i \wedge dp_j + \sum_{i,j} \Omega_{ij} x_i p_j dx_i \wedge dx_j + \sum_{i,j} \Omega_{ij} x_i x_j dp_i \wedge dp_j \right).$$

Note that $\omega_G$ is non-degenerate since

$$(\omega_G)^m = m! \wedge_{1 \leq j \leq m} dx_j \wedge dp_j$$

is a volume form.

Choosing the standard frames: $\{dx_i, dp_i\}$ for $T^* G$ and $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial p_i}\}$ for $TG$, the bundle map $\sigma_\mathcal{G}^* : T^* G \to TG, \theta \mapsto \iota_\theta \pi_G$ is given by

$$\sigma_\mathcal{G}^* : dx_i \mapsto -\frac{\partial}{\partial p_i} - \sum_j \Omega_{ij} x_i p_j \frac{\partial}{\partial p_j} + \sum_j \Omega_{ij} x_i x_j \frac{\partial}{\partial x_j},$$

$$dp_i \mapsto \frac{\partial}{\partial x_i} - \sum_j \Omega_{ij} p_i x_j \frac{\partial}{\partial x_j} + \sum_j \Omega_{ij} p_i p_j \frac{\partial}{\partial p_j},$$

and the bundle map $\omega_\mathcal{G}^* : TG \to T^* G, v \mapsto \iota_v \omega_G$ is given by

$$\omega_\mathcal{G}^* : \frac{\partial}{\partial x_i} \mapsto dp_i + \sum_j \Omega_{ij} p_i x_j dp_j + \sum_j \Omega_{ij} x_i x_j dx_j,$$

$$\frac{\partial}{\partial p_i} \mapsto -dx_i + \sum_j \Omega_{ij} x_i x_j dx_j + \sum_j \Omega_{ij} x_i x_j dp_j.$$

We leave it to the reader to check that $\sigma_\mathcal{G}^*$ and $\omega_\mathcal{G}^*$ are inverse to each other.

In the case when the underlying field is $\mathbb{C}$, the symplectic groupoid structures can be verified directly. □

Corollary 3.8. For the log-canonical Poisson structure $\{x_i, x_j\} = \Omega_{ij} x_i x_j$ on $L$, the exponential map

$$\exp : T^*_x L \to G, (\mathbf{x}, \mathbf{p}) \mapsto (a \circ \mathbf{x}, a^{-1} \circ \mathbf{p}),$$

where $a_j = e^{\sum_i \Omega_{ij} x_i p_i}$, is a diffeomorphism.

Motivated by [16], we make the following definition of the symplectic double, which is a source-connected Poisson groupoid of the log-canonical Poisson structure on $L$.

Theorem 3.9. [16] For the log-canonical Poisson structure $\{x_i, x_j\} = \Omega_{ij} x_i x_j$ on $L$ (which is either $\mathbb{R}^m$ or $\mathbb{C}^m$), the symplectic double $(\mathcal{D}, \sigma_D) \to (L, \pi)$ is a source-connected Poisson groupoid with the following structures:

1. $\mathcal{D} \cong L \times L^\times$ has the coordinates $(\mathbf{x}, s) = (x_1, \ldots, x_m, s_1, \ldots, s_m)$ (where $L^\times$ is respectively either $\mathbb{R}^m_+$ or $\mathbb{C}^m_+$);
2. the source map is $\alpha : L \times L^\times \to L, (\mathbf{x}, s) \mapsto x$.
(3) the target map is
\[
\beta : L \times L^\times \to L, \quad (x, s) \mapsto \left( x_1 \prod_{i=1}^m s_i^{\Omega_{i1}}, \ldots, x_m \prod_{i=1}^m s_i^{\Omega_{im}} \right);
\]
(4) the identity map is \(1 : L \to L \times L^\times, \quad x \mapsto (x, 1, \ldots, 1);\)
(5) the inverse map is
\[
i : L \times L^\times \to L \times L^\times, \quad (x, s) \mapsto \left( x_1 \prod_{i=1}^m s_i^{\Omega_{i1}}, \ldots, x_m \prod_{i=1}^m s_i^{\Omega_{im}}, \frac{1}{s_1}, \ldots, \frac{1}{s_m} \right);
\]
(6) the multiplication map is
\[
m : (L \times L^\times) \times (L \times L^\times) \to L \times L^\times,
\quad \left( \left( x_1 \prod_{i=1}^m s_i^{\Omega_{i1}}, \ldots, x_m \prod_{i=1}^m s_i^{\Omega_{im}}, s' \right), (x, s) \right) \mapsto (x, s \circ s);
\]
(7) the multiplicative Poisson structure is
\[
\sigma_D = - \sum_i x_i s_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial s_i} + \sum_{j<i} \Omega_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}
\]
and equivalently the singular multiplicative 2-form \(\omega_D\) is
\[
\omega_D = \sigma_D^{-1} = \sum_i \frac{dx_i}{x_i} \wedge \frac{ds_i}{s_i} + \sum_{j<i} \Omega_{ij} \frac{ds_i}{s_i} \wedge \frac{ds_j}{s_j}.
\]

It may be strange that the Poisson groupoid \(D \rightrightarrows L\) is called the **symplectic** double since the Poisson structure drops rank along the coordinate axes. The symplectic double in [16] is defined to be the restriction \(D^\times = \{(x, s) \mid x \in L^\times \} \cong L^\times \times L^\times\) which is genuinely a symplectic groupoid over \(L^\times\). The Poisson groupoid \(D \rightrightarrows L\) may be viewed as the natural extension of the symplectic groupoid \(D^\times \rightrightarrows L^\times\).

**Remark 3.10.** We relate the target map in Theorem 3.9 to the symplectic realization in [22].

With the notation in Theorem 3.9 and the change of variables \(x_i = e^{\chi_i}\) and \(s_i = e^{\xi_i}\), the symplectic double \((D^\times, \omega_D)\) becomes the symplectic vector space \((U, \omega_U)\) with coordinates \((\chi_1, \ldots, \chi_m, \xi_1, \ldots, \xi_m)\), where \(U\) is either \(\mathbb{R}^{2m}\) or \(\mathbb{C}^{2m}\), and
\[
\omega_U = \sum_i \frac{d\chi_i}{\chi_i} \wedge \frac{d\xi_i}{\xi_i} + \sum_{j<i} \Omega_{ij} d\xi_i \wedge d\xi_j.
\]
Although \((U, \omega_U)\) is not a symplectic groupoid integrating \((L, \pi)\) (e.g. the identity map cannot be defined), the maps \(\alpha\) and \(\beta\) are well-defined in these coordinates:
\[
\alpha : U \to L^\times, \quad (\chi_1, \ldots, \chi_m, \xi_1, \ldots, \xi_m) \mapsto (e^{\chi_1}, \ldots, e^{\chi_m});
\]
\[
\beta : U \to L^\times, \quad (\chi_1, \ldots, \chi_m, \xi_1, \ldots, \xi_m) \mapsto (e^{\chi_1 + \sum_i \Omega_{i1} \xi_i}, \ldots, e^{\chi_m + \sum_i \Omega_{im} \xi_i}).
\]
The \(\alpha\)-fibers are symplectic orthogonal to the \(\beta\)-fibers; both \(\alpha : (U, \omega_U) \to (L, \pi)\) and \(\beta : (U, -\omega_U) \to (L, \pi)\) are Poisson maps.

For the standard symplectic structure \(\omega_0 = \sum_i d\chi_i \wedge d\xi_i\) on \(U\), since
\[
\omega_U - \omega_0 = \left( \sum_i \frac{d\chi_i}{\chi_i} \wedge \frac{d\xi_i}{\xi_i} + \sum_{j<i} \Omega_{ij} \frac{d\xi_i}{\xi_i} \wedge \frac{d\xi_j}{\xi_j} \right) - \sum_i \frac{d\chi_i}{\chi_i} \wedge \frac{d\xi_i}{\xi_i} = \sum_{j<i} \Omega_{ij} d\xi_i \wedge d\xi_j
\]
is supported on the \(\alpha\)-fiber, we have that \(\beta_*(\omega_0^{-1}) = \beta_*(-\omega_U^{-1}) = \pi\). The symplectic realization \(\beta : (U, -\omega_0) \to (L^\times, \pi)\) plays an essential role in [22].

Next, we describe the Lie algebroid \(D \rightrightarrows L\). We introduce coordinates on \(A_D \cong L \times L\) with the bundle projection
\[
A_D \cong L \times L \to L, \quad (x, \xi) \mapsto x.
\]
For an \( m \times m \) skew-symmetric matrix \( \Omega \), we define a Lie algebroid structure on \( A_D \) with the anchor map

\[
\rho_D : A_D \cong L \times L \to TL, \quad (x, \xi) \mapsto \sum_{i,j} \Omega_{ij} \xi_j \frac{\partial}{\partial x_j},
\]

where the kernel of \( \rho_D \) has trivial bracket.

**Proposition 3.11.** The Lie algebroid of the Poisson groupoid \( D \lhd L \) in Theorem 3.9 is isomorphic to \( A_D \).

**Proof.** With the notation in Theorem 3.9, if we write \( e^\iota_i = s_i \), then \( \text{Lie} \, D = \ker (\alpha_k : TD|\iota_k \to TL) \) is generated by \( \frac{\partial}{\partial c_i} \), \( i = 1, \ldots, m \). Rewriting the target map \( \beta \) in the \((x, \xi)\) coordinates, we get

\[
\beta : (x, \xi) \mapsto \big(e^{\sum_i \Omega_{ij} \xi_j}, \ldots, e^{\sum_m \Omega_{ij} \xi_m} \big) \big( x_1, \ldots, x_m \big).
\]

Therefore,

\[
(3.2) \quad \beta_* \left( \frac{\partial}{\partial \xi_i} \right) = \sum_j \Omega_{ij} x_j \frac{\partial}{\partial x_j}, \quad \beta_* \left( \sum_i \xi_i \frac{\partial}{\partial \xi_i} \right) = \sum_{i,j} \Omega_{ij} \xi_j \frac{\partial}{\partial x_j}.
\]

This shows that the bundle map

\[
A_D \to \text{Lie} \, D, \quad (x, \xi) \mapsto \sum_i \xi_i \frac{\partial}{\partial \xi_i}
\]

is a Lie algebroid isomorphism. \( \square \)

Fixing an \( m \times m \) skew-symmetric matrix \( \Omega \) and the corresponding log-canonical Poisson space \( L \) as in Definition 3.1, the cotangent Lie algebroid \( T^*_L \) is not isomorphic to the Lie algebroid \( A_D \). In fact, \( T^*_L \) is isomorphic to an iterated elementary modification of \( A_D \) and we obtain a source-connected symplectic groupoid of \((L, \pi)\) via the blow-up construction [26].

**Definition 3.12.** Let \( E \) be a vector bundle over \( M \) and let \( F \) be a subbundle of \( E|_N \) for some hypersurface \( N \subset M \). The elementary (lower) modification of \( E \) along \( F \), denoted by \( [E:F] \), is the vector bundle with the sheaf of sections

\[
\Gamma(M, [E:F]) = \{ s \in \Gamma(M, E) : s|_N \in \Gamma(N, F) \}.
\]

**Proposition 3.13.** [26] Let \( A \) be a Lie algebroid over a manifold \( M \) and let \( B \) be a subbundle of \( A|_N \) for some hypersurface \( N \subset M \) such that \( B \) is also a Lie algebroid over \( N \). Then \( [A:B] \) is a Lie algebroid over \( M \).

Let \( \mathcal{G} \to M \) be a Lie groupoid and let \( \mathcal{H} \to N \) be a Lie subgroupoid over a hypersurface \( N \subset M \). We denote the blow-up of \( \mathcal{G} \) along \( \mathcal{H} \) by \( \text{Bl}(\mathcal{G}, \mathcal{H}) \) with the blow-down map \( \nu : \text{Bl}(\mathcal{G}, \mathcal{H}) \to \mathcal{G} \) and write \([\mathcal{G}:\mathcal{H}] = \text{Bl}(\mathcal{G}, \mathcal{H}) \setminus (S_N \cup T_N)\), where \( S_N \) is the proper transform of \( \alpha^{-1}(N) \) and \( T_N \) is the proper transform of \( \beta^{-1}(N) \).

**Theorem 3.14.** [26] For a Lie groupoid \( \mathcal{G} \to M \) and a Lie subgroupoid \( \mathcal{H} \to N \) for some hypersurface \( N \subset M \), there is a unique Lie groupoid structure \([\mathcal{G}:\mathcal{H}] \to M \) such that the blow-down map \( \nu : [\mathcal{G}:\mathcal{H}] \to \mathcal{G} \) is a groupoid morphism. Moreover, blow-up of Lie groupoids corresponds to elementary modification of Lie algebroids. More precisely, we have

\[
\text{Lie}[\mathcal{G}:\mathcal{H}] = [\text{Lie} \mathcal{G}: \text{Lie} \mathcal{H}].
\]

Our task is now clear. We need to find the Lie subalgebroids of \( A_D \) such that the iterated elementary modifications along these Lie subalgebroids yield the cotangent algebroid \( T^*_L \).

For each \( k = 1, \ldots, m \), we define the Lie subalgebroid \( A_D^k \) of \( A_D \) as follows. Let \( L_k \) be the codimension-1 subspace in \( L \) defined by \( x_k = 0 \). As a vector bundle, \( A_D^k \) is the corank-1 subbundle of \( A_D |_{L_k} \) defined by \( \xi_k = 0 \). That is, the anchor map of \( A_D^k \) is given by

\[
\rho_D^k : A_D^k \cong L_k \times L_k \to TL_k, \quad (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_m, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \mapsto \sum_{1 \leq i,j \leq m, i,j \neq k} \Omega_{ij} \xi_j \frac{\partial}{\partial x_j}.
\]
Proposition 3.15. Fixing an $m \times m$ skew-symmetric matrix $\Omega$ and the corresponding log-canonical Poisson space $L$ as in Definition 3.1, the cotangent algebroid $T^* L$ is isomorphic to the Lie algebroid $[\ldots[[A_D : A^1_B] : A^2_B] : \ldots : A^n_B]]$.

Proof. The Poisson structure $\pi = \sum_{i,j} \Omega_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ gives rise to the Poisson map

$$\pi^B : T^* L \to TL, \quad dx_i \mapsto \sum_j \Omega_{ij} x_i x_j \frac{\partial}{\partial x_j}.$$ (3.3)

The result follows by comparing (3.3) with the anchor map (3.2) for $\text{Lie } D \cong A_D$.

To obtain a source-connected symplectic groupoid of the log-canonical Poisson space $(L, \pi)$ which integrates $T^*_L L$ we need to iteratively blow up the Poisson groupoid $D \cong L$ along the subgroupoids integrating $A^k_D$, for $k = 1, \ldots, m$. Indeed, the Lie subalgebroid $A^k_D$ integrates to the subgroupoid $D_k \cong L_k$ defined by $x_k = 0$ and $s_k = 1$, so we define the Lie groupoid

$$B = [\ldots[[D : D_1] : D_2] : \ldots : D_m]].$$

Using the blow-up coordinates $u_i = \frac{s_i-1}{x_i}$, we have the following symplectic groupoid structure on $B \cong L$.

Theorem 3.16. For the log-canonical Poisson structure $(x_i, x_j) = \Omega_{ij} x_i x_j$ on $L$ (which is either $\mathbb{R}^m$ or $\mathbb{C}^m$), we have that $(B, \omega_B) \cong (L, \pi)$ is a source-connected symplectic groupoid with the following structures:

1. $B \subset L \times L$ has the coordinates $(x, u) = (x_1, \ldots, x_m, u_1, \ldots, u_m)$;
2. $B$ is given by $x_i u_i + 1 > 0$ if $L = \mathbb{R}^m$ and $x_i u_i + 1 \neq 0$ if $L = \mathbb{C}^m$;
3. the source map is $\alpha : B \to L, \quad (x, u) \mapsto x$;
4. the target map is $\beta : B \to L, \quad (x, u) \mapsto \left(x_1 \prod_{i=1}^m (x_i u_i + 1)^{\Omega_{i1}}, \ldots, x_m \prod_{i=1}^m (x_i u_i + 1)^{\Omega_{im}} \right)$;
5. the identity map $1 : L \to B, \quad x \mapsto (x, 0)$;
6. the inverse map is $\iota : B \to B$ sending $(x, u)$ to

$$\left(x_1 \prod_{i=1}^m (x_i u_i + 1)^{\Omega_{i1}}, \ldots, x_m \prod_{i=1}^m (x_i u_i + 1)^{\Omega_{im}}, -\frac{u_1 \prod_{i=1}^m (x_i u_i + 1)^{-\Omega_{i1}}}{x_1 u_1 + 1}, \ldots, -\frac{u_m \prod_{i=1}^m (x_i u_i + 1)^{-\Omega_{im}}}{x_m u_m + 1} \right)$$;
7. the multiplication map is

$$m : B \times_B B \to B,$$

$$\left((x, u), \left(x_1 \prod_{i=1}^m (u_i x_i + 1)^{\Omega_{i1}}, \ldots, x_m \prod_{i=1}^m (u_i x_i + 1)^{\Omega_{im}}, u' \right)\right) \mapsto \left(x, u'' \right),$$

where $u'' = (u'_j(x_j u_j + 1) \prod_{i=1}^m (x_i u_i + 1)^{\Omega_{ij}} + u_j)$;
8. the multiplicative Poisson structure is

$$\sigma_B = -\sum_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial u_i} - \sum_{i,j} \Omega_{ij} x_i u_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial u_j} + \sum_{j<i} \Omega_{ij} u_i u_j \frac{\partial}{\partial u_i} \wedge \frac{\partial}{\partial u_j} + \sum_{j<i} \Omega_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

and equivalently the multiplicative symplectic structure $\omega_B$ is

$$\omega_B = \sum_i \frac{1}{x_i u_i + 1} dx_i \wedge du_i + \sum_{i,j} \frac{\Omega_{ij} u_i x_j}{(x_i u_i + 1)(x_j u_j + 1)} dx_i \wedge du_j + \sum_{j<i} \frac{\Omega_{ij} u_i u_j}{(x_i u_i + 1)(x_j u_j + 1)} dx_i \wedge dx_j + \sum_{j<i} \frac{\Omega_{ij} x_i x_j}{(x_i u_i + 1)(x_j u_j + 1)} du_i \wedge du_j.$$
Proof. These are the same groupoid structures as in Theorem 3.9 with the change of variables $u_i = \frac{x_i - 1}{x_i}$. The difference is precisely that $\omega_B$ is no longer singular. Note that $\omega_B$ is also non-degenerate since

$$(\omega_B)^m = m! \prod_{1 \leq i \leq m} \frac{1}{x_i u_i + 1} \wedge \prod_{1 \leq i \leq m} dx_i \wedge du_i$$

is a volume form. \qed

Remark 3.17. Recall that the infinitesimal object of a Poisson groupoid is a Lie bialgebroid [34]. The Lie bialgebroid of the Poisson groupoid $D \rightrightarrows L$ is isomorphic to $(A_D, TL(-\log D))$, where $TL(-\log D)$ is the log-tangent bundle with respect to the normal crossing divisor $D = L_1 + L_2 + \ldots + L_k$ [26]. That is,

$$TL(-\log D) = [[[TL:TL_1]:TL_2]...:TL_m],$$

or equivalently the sections of $TL(-\log D)$ are the vector fields on $L$ tangent to $L_k$ for $k = 1, \ldots, m$. As a vector bundle, $TL(-\log D)$ is generated by $x_k \frac{\partial}{\partial x_k}$, $k = 1, \ldots, m$, and its dual bundle $T^*L(-\log D)$ is generated by $\frac{dx_k}{x_k}$, $k = 1, \ldots, m$. The bundle map

$$A_D \to T^*L(-\log D), \quad (x, \xi) \mapsto \sum_k \xi_k \frac{dx_k}{x_k}$$

identifies $A_D$ with $T^*L(-\log D)$.

The iterated blow-up construction that takes $D \rightrightarrows L$ to $B \rightrightarrows L$ corresponds to the elementary modification that takes the Lie bialgebroid $(A_D, TL(-\log D))$ to the Lie bialgebroid $(T^*_B L, TL)$, where $T^*_B L$ is an elementary lower modification of $A_D$ and $TL$ is an elementary upper modification of $TL(-\log D)$. See the first author’s thesis for details [30].

For an integrable Lie algebroid, every source-connected groupoid receives a surjective groupoid map from the source-simply-connected groupoid. This means that we have a natural symplectic groupoid morphism $\kappa : \mathcal{G} \to B$. Together with the blow-down map $\nu : B \to D$, the following commutative diagram summarizes the maps among the groupoids $\mathcal{G}$, $B$ and $D$.

\begin{equation}
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\lambda} & D \\
\kappa \downarrow & & \downarrow \nu \\
B & \xrightarrow{\nu} & D
\end{array}
\end{equation}

Explicitly in coordinates, these maps are given by

$$\kappa : \mathcal{G} \to B, \quad (x, p) \mapsto \left(x, e^{x_1 p_1} - 1, \ldots, e^{x_m p_m} - 1\right);$$

$$\nu : B \to D, \quad (x, u) \mapsto \left(x, x_1 u_1 + 1, \ldots, x_m u_m + 1\right);$$

$$\lambda : \mathcal{G} \to D, \quad (x, p) \mapsto \left(x, e^{x_1 p_1}, \ldots, e^{x_m p_m}\right).$$

Remark 3.18. If $L = \mathbb{R}^m$, then $B \rightrightarrows L$ is source-simply-connected and all three groupoid maps are groupoid isomorphisms after restricting the base to $L^x = \mathbb{R}^m_x$, but if $L = \mathbb{C}^m$, then $B$ is not source-simply-connected and $\kappa : \mathcal{G} \to B$ is a covering map for each source fiber.

We summarize the pros and cons of the three groupoids $\mathcal{G} \rightrightarrows L$, $B \rightrightarrows L$ and $D \rightrightarrows L$. The symplectic groupoid $\mathcal{G} \rightrightarrows L$ is source-simply-connected when the underlying field is $\mathbb{C}$, the tradeoff is that its groupoid structure maps are transcendental. On the other hand, the multiplicative Poisson structure of $D \rightrightarrows L$ has a simple expression and its groupoid structure maps are algebraic, but it is only symplectic when restricted to $L^x$. Comparing to $\mathcal{G} \rightrightarrows L$, the symplectic groupoid $B \rightrightarrows L$ has the added bonus that its groupoid structure maps are algebraic, but the tradeoff is that the expressions for its multiplicative Poisson structure and its groupoid structures are more complicated.
4. Hamiltonian Perspective on Mutations

In this section, we introduce mutations of cluster charts from the Hamiltonian viewpoint. This perspective was first given in [16] and is the foundation for the main results of [22]. We demonstrate how the Hamiltonian perspective provides a canonical choice of mutations for groupoid charts which glue to give a symplectic groupoid integrating various log-canonical Poisson structures on cluster varieties. The standard combinatorics of c-vectors, g-vectors, and F-polynomials are then lifted to provide descriptions of iterations of the gluing maps for groupoid charts.

While in Section 3 it was most convenient to use covariant notation for all maps of groupoids, due to the algebraic nature of cluster mutations and also the piecewise nature of the formulas, in this section we use contravariant notation for describing all maps, i.e. we describe them via the pullback of coordinate functions.

Let \( \tilde{B} = (B_{ij}) \) be an \( m \times n \) integer matrix with \( m \geq n \). Write \( B \) for the upper \( n \times n \) submatrix of \( \tilde{B} \) and assume \( B \) is skew-symmetric, i.e. there exists a diagonal integer matrix \( D = \text{diag}(d_1, \ldots, d_n) \) with each \( d_i > 0 \) so that \( DB \) is skew-symmetric. Such an \( m \times n \) matrix \( \tilde{B} \) with skew-symmetrizable principal submatrix \( B \) is called an exchange matrix. We fix a skew-symmetrizing matrix \( D \) and refer to a skew-symmetric \( m \times n \) matrix \( \Omega = (\Omega_{ij}) \) as \( D \)-compatible with \( \tilde{B} \) if \( \tilde{B}^T \Omega = [D \, 0] \), where \( 0 \) denotes an \( n \times (m - n) \) matrix with all zero entries. In this case, we call \((\tilde{B}, \Omega)\) a \( D \)-compatible pair or simply a compatible pair if the symmetrizing matrix \( D \) is understood. Note that the existence of a matrix \( \Omega \) compatible with \( \tilde{B} \) implies \( \tilde{B} \) has rank \( n \), in other words the columns of \( \tilde{B} \) are linearly independent.

4.1. Mutation of Cluster Charts. We begin by recalling the Hamiltonian perspective of mutations for cluster charts. Let \( L_X = \mathbb{R}^n \) and \( L_A = \mathbb{R}^m \) with corresponding orthants \( L_X^+ = \mathbb{R}^n_+ \) and \( L_A^+ = \mathbb{R}^m_+ \). Write \( y = (y_1, \ldots, y_n) \) for a set of coordinates on \( L_X \) and \( x = (x_1, \ldots, x_m) \) for a set of coordinates on \( L_A \); we use the same symbols for their restrictions to \( L_X^+ \) and \( L_A^+ \). Given a \( D \)-compatible pair of matrices \((\tilde{B}, \Omega)\), denote by \( \{\cdot, \cdot\}_{X} \) and \( \{\cdot, \cdot\}_{A} \) the log-canonical Poisson brackets on \( L_X \) and \( L_A \) given by

\[
\{y_i, y_j\}_X = d_k B_{k\ell} y_k y_{\ell} \quad \text{and} \quad \{x_i, x_j\}_A = \Omega_{ij} x_i x_j.
\]

An easy calculation using the compatibility condition for the pair \((\tilde{B}, \Omega)\) shows that there is a Poisson map

\[
\rho : L_A \rightarrow L_X, \quad \rho^*(y_k) = \hat{y}_k := \prod_{i=1}^m x_i^{B_{ik}},
\]

defined on the locus where \( x_i \neq 0 \) if there exists \( k \) with \( B_{ik} < 0 \), i.e. we have \( \{\hat{y}_k, \hat{y}_\ell\}_A = d_k B_{k\ell} \hat{y}_k \hat{y}_\ell \) for \( 1 \leq k, \ell \leq n \). Motivated by the terminology of Fock and Goncharov [18], we will refer to the pair of Poisson manifolds \( L_X \) and \( L_A \) together with the Poisson map \( \rho \) as a Poisson ensemble associated to the compatible pair \((\tilde{B}, \Omega)\).

The Euler dilogarithm is the function of a single real variable defined by

\[
\text{Li}_2(y) = -\int_0^y \log(1 - u) \frac{du}{u}, \quad y < 1.
\]

It will become apparent from the results below that, from the Poisson perspective, the Euler dilogarithm lies at the heart of cluster algebra theory. However, we will work in a slightly more general setting.

Fix a collection of positive integers \( r = (r_1, \ldots, r_n) \in \mathbb{Z}_{>0}^n \) and write \( L_r = \prod_{\ell=1}^n \mathbb{R}^{r_{\ell}-1} \) for the orthant inside \( L_r \). We identify the \( \ell \)-th component of a point in \( L_r \) with the degree \( r_\ell \) monic polynomial \( Z_{\ell}(u) \in \mathbb{R}[u] \) with \( Z_{\ell}(0) = 1 \) given by

\[
Z_{\ell}(u) = 1 + z_{\ell,1} u + \cdots + z_{\ell,r_{\ell}-1} u^{r_{\ell}-1} + u^{r_{\ell}}.
\]

There is a natural star-involution \( z_{\ell,j} \mapsto z_{\ell,j}^* := z_{\ell,r_{\ell}-j} \) on \( L_r \) and we write \( Z_{\ell}^*(u) \) for the image of \( Z_{\ell}(u) \) under this involution, more directly \( Z_{\ell}^*(u) = u^{r_{\ell}} Z_{\ell}(u^{-1}) \) is the the polynomial with reversed coefficients. It will be convenient to introduce the notation \( j^* = r_{\ell} - j \) so that \( z_{\ell,j}^* = z_{\ell,j^*} \). Although \( j^* \) depends on \( \ell \), we suppress it from the notation.

Set

\[
L_{X,r} := L_X \times L_r \quad \text{and} \quad L_{A,r} := L_A \times L_r
\]

with corresponding orthants \( L_{X,r}^+ \) and \( L_{A,r}^+ \). Here we extend the Poisson structures on \( L_X \) and \( L_A \) to \( L_{X,r} \) and \( L_{A,r} \) so that the coordinate functions \( z_{\ell,j} \) on \( L_r \) are Casimirs. Then the map \( \rho : L_A \rightarrow L_X \) extends to a Poisson map \( L_{A,r} \rightarrow L_{X,r} \), which we still denote by \( \rho \), by acting as the identity map on \( L_r \) components.
Given a choice of sign $\varepsilon \in \{\pm 1\}$, define Hamiltonian functions $h^{k,\varepsilon}_{X,r} \in \mathcal{O}_{T^*_X,B}$ and $h^{k,\varepsilon}_{A,r} \in \mathcal{O}_{T^*_X,B}$ for $1 \leq k \leq n$ by
\begin{equation}
(4.4) \quad h^{k,\varepsilon}_{X,r} := -\frac{\varepsilon}{d_k} \int_0^{y_k^*} \log \left( \frac{Z^*_k(u)}{u} \right) \, du \quad \text{and} \quad h^{k,\varepsilon}_{A,r} := -\frac{\varepsilon}{d_k} \int_0^{y_k^*} \log \left( \frac{Z^*_k(u)}{u} \right) \, du,
\end{equation}
where
\[
Z^*_k(u) = \begin{cases}
Z_k(u) & \text{if } \varepsilon = +1; \\
Z^*_k(u) & \text{if } \varepsilon = -1.
\end{cases}
\]
Clearly, there is an equality $\rho^*(h^{k,\varepsilon}_{X,r}) = h^{k,\varepsilon}_{A,r}$, where $\rho$ here denotes the restriction to $L^*_X$.

**Remark 4.1.** Observe that when $r_k = 1$, i.e., $Z_k(u) = 1 + u$, we have $h^{k,\varepsilon}_{X,r} := \frac{\varepsilon}{d_k} \text{Li}_2(-\gamma_k^*)$ and $h^{k,\varepsilon}_{A,r} := \frac{\varepsilon}{d_k} \text{Li}_2(-\gamma_k^*)$. We will refer to this as the “cluster case” since the mutation rules presented below reduce to the classical mutations for cluster algebras when $Z_k(u) = 1 + u$ (c.f. Remark 4.9).

Write $X^{k,\varepsilon}_{X,r} \in T^*_X$ and $X^{k,\varepsilon}_{A,r} \in T^*_X$ for the Hamiltonian vector fields associated to $h^{k,\varepsilon}_{X,r}$ and $h^{k,\varepsilon}_{A,r}$ respectively, i.e. the vector fields naturally associated to the derivations $\{h^{k,\varepsilon}_{X,r}, \cdot \}_X$ and $\{h^{k,\varepsilon}_{A,r}, \cdot \}_A$.

**Lemma 4.2.** For $1 \leq k \leq n$ and $\varepsilon \in \{\pm 1\}$, the vector fields $X^{k,\varepsilon}_{X,r}$ and $X^{k,\varepsilon}_{A,r}$ determine the following dynamics on $L^*_X$ and $L^*_A$:
\begin{align}
(4.5) \quad & \dot{y}_t := \{h^{k,\varepsilon}_{X,r}, y_t\}_X = -B_{kt} \log \left( Z^*_k(y_k^*) \right) y_t \quad \text{for} \quad 1 \leq \ell \leq n; \\
(4.6) \quad & \dot{x}_j := \{h^{k,\varepsilon}_{A,r}, x_j\}_A = -\delta_{jk} \log \left( Z^*_k(y_k^*) \right) x_k \quad \text{for} \quad 1 \leq j \leq m.
\end{align}
Moreover, the points of $L^*_X$ are fixed by the Hamiltonian flows.

**Proof.** Equation (4.5) was essentially proven in [22]. Since equation (4.6) seems to technically be new, we will indicate the key steps in the computation here:
\[
\{h^{k,\varepsilon}_{A,r}, x_j\}_A = \frac{\log \left( Z^*_k(y_k^*) \right)}{d_k y_k} \{\dot{y}_k, x_j\} = -\delta_{jk} \log \left( Z^*_k(y_k^*) \right) x_j.
\]

Since $B_{kk} = 0$, it immediately follows that $y_k$ is a conserved quantity under the flow of $L^*_X$, by the vector field $X^{k,\varepsilon}_{X,r}$. Since the map $\rho$ is Poisson with $\tilde{y}_k = \rho^*(y_k)$ and $h^{k,\varepsilon}_{A,r} = \rho^*(h^{k,\varepsilon}_{X,r})$, we also see that $\tilde{y}_k$ is a conserved quantity under the flow of $L^*_A$, by the vector field $X^{k,\varepsilon}_{A,r}$.

**Corollary 4.3.** For $1 \leq k \leq n$ and $\varepsilon \in \{\pm 1\}$, the time-t flow $\varphi^t_{X,r} : L^*_X \to L^*_X$ of the Hamiltonian vector field $X^{k,\varepsilon}_{X,r}$ is given on coordinates by
\[
(\varphi^t_{X,r})^*(y) = (Z^*_k(y_k^*))^{-tB_{kt}} y_t, \quad (\varphi^t_{X,r})^*(z_t, i) = z_t, i,
\]
and the time-t flow $\varphi^t_{A,r} : L^*_A \to L^*_A$ of the Hamiltonian vector field $X^{k,\varepsilon}_{A,r}$ is given on coordinates by
\[
(\varphi^t_{A,r})^*(x) = (Z^*_k(y_k^*))^{-t\delta_{ik}} x_j, \quad (\varphi^t_{A,r})^*(z_t, i) = z_t, i.
\]
These maps preserve the respective Poisson structures.

**Proof.** The computation of the flows is immediate from the preceding discussion. That the flow of these vector fields preserve the Poisson structures follows from their Hamiltonian definition.

In what follows we use the notation $[b]_+ := \max\{b, 0\}$. For $1 \leq k \leq n$ and a sign $\varepsilon \in \{\pm 1\}$, define the **mutation of the compatible pair** $(\tilde{B}, \Omega)$ in direction $k$ by $\mu_{r,k,\varepsilon}(\tilde{B}, \Omega) = (\tilde{B}', \Omega')$, where
- $\tilde{B}' = (B'_{ij})$ is given by
\begin{equation}
(4.7) \quad B'_{ij} = \begin{cases}
-B_{ij} & \text{if } i = k \text{ or } j = k; \\
B_{ij} + [-\varepsilon B_{ik}r_k]_+ B_{kj} + B_{ik}[\varepsilon r_k B_{kj}]_+ & \text{otherwise};
\end{cases}
\end{equation}
• $\Omega' = E^r_{r,k,\varepsilon} \Omega E_{r,k,\varepsilon}$ for $E_{r,k,\varepsilon}$ the $m \times m$ matrix with entries

$$E_{ij} = \begin{cases} 
\delta_{ij} & \text{if } j \neq k; \\
-1 & \text{if } i = j = k; \\
[-\varepsilon B_{ik} r_k]^+ & \text{if } i \neq j = k.
\end{cases}$$

It is an easy exercise to check that $(\tilde{B}', \Omega')$ is again $D$-compatible and that the mutation of compatible pairs is an involution, more precisely

$$\mu_{r,k,\varepsilon} \mu_{r,k,\varepsilon}' (\tilde{B}, \Omega) = (\tilde{B}, \Omega)$$

for any signs $\varepsilon, \varepsilon' \in \{\pm 1\}$. In particular, the mutation $\mu_{r,k,\varepsilon}$ acting on $D$-compatible pairs is independent of the sign $\varepsilon$ and we simply write $\mu_{r,k}$ for this mutation when the choice of sign can be ignored.

Let $L^{\chi'} = \mathbb{R}^n$ and $L_{A'} = \mathbb{R}^m$ with coordinates $y' = (y'_1, \ldots, y'_n)$ and $x' = (x'_1, \ldots, x'_m)$ respectively and with corresponding orthants $L^{\chi'}_x = \mathbb{R}^n_x$ and $L^{\chi'}_{A'} = \mathbb{R}^m_{A'}$. For $1 \leq k \leq n$, set $(\tilde{B}', \Omega') = \mu_{r,k}(\tilde{B}, \Omega)$ and consider the log-canonical Poisson structures $\{\cdot, \cdot\}_{x'}$ and $\{\cdot, \cdot\}_{A'}$ on $L^{\chi'}$ and $L_{A'}$ given by

$$\{y'_k, y'_l\}_{x'} = d_k B'_k y'_k y'_l \quad \text{and} \quad \{x'_i, x'_j\}_{A'} = \Omega'_{ij} x'_i x'_j.$$ 

Define $L^{\chi'}_{x',r}$ and $L_{A',r}$ as above with corresponding orthants $L^{\chi'}_{x',r}$ and $L_{A',r}$. Write $z'_{i,j}$ for coordinates on the $L_{x,r}$ components.

**Lemma 4.4.** For $1 \leq k \leq n$ and $\varepsilon \in \{\pm 1\}$, there are Poisson maps $\tau^{k,\varepsilon}_{x',r} : L^{\chi}_{x',r} \to L^{\chi}_{x',r}$ and $\tau^{k,\varepsilon}_{A,r} : L^{\chi}_{A,r} \to L^{\chi}_{A,r}$ given on coordinates by

$$\begin{cases} 
(y_k^{-1} y_y y_k^{-1})_{\varepsilon B_{k}} \delta_{k \neq k} & \text{if } \ell = k; \\
y_k^{-1} y_y y_k^{-1} \delta_{k \neq k} & \text{if } \ell \neq k;
\end{cases}$$

$$\begin{cases} 
x_i^{-1} \prod_{i=1}^{m} x_i^{-1} \varepsilon B_{ik} r_k & \text{if } j = k; \\
x_j & \text{if } j \neq k;
\end{cases}$$

$$\begin{cases} 
z_{i,j} & \text{if } \ell = k; \\
z_{i,j} & \text{if } \ell \neq k.
\end{cases}$$

**Proof.** Since the coordinates on $L_{x,r}$ are Casimirs there is nothing to check for these coordinates. By skew-symmetry of the Poisson brackets, there are essentially only two cases to check on $L_{x,r}$ and $L_{A,r}$ but just one of these is non-trivial for each transformation. For $\ell, \ell' \neq k$, we have

$$(\tau^{k,\varepsilon}_{x',r} (y_{\ell}'), (\tau^{k,\varepsilon}_{x',r})^* (y_{\ell'}')) X = \left\{ y_k^{-1} y_y y_k^{-1} \varepsilon B_{k} \delta_{\varepsilon B_{k} \neq k}, y_k^{-1} y_y y_k^{-1} \varepsilon B_{k} \delta_{\varepsilon B_{k} \neq k} \right\}_X$$

$$= (d_k B_{k \ell} + d_k \varepsilon B_{k} \delta_{\varepsilon B_{k} \neq k} + d_k B_{k \ell} \varepsilon B_{k} \delta_{\varepsilon B_{k} \neq k} + y_k^{-1} y_y y_k^{-1} \varepsilon B_{k} \delta_{\varepsilon B_{k} \neq k})$$

$$= d_k \varepsilon B_{k \ell} (\tau^{k,\varepsilon}_{x',r} (y_{\ell}')) (\tau^{k,\varepsilon}_{x',r})^* (y_{\ell'}').$$

The case where $\ell$ or $\ell'$ are equal to $k$ is immediate and we omit the details.

For $\tau^{k,\varepsilon}_{A,r}$, we only check the brackets of $x_k'$ and $x_j'$ for $j \neq k$:

$$\{ (\tau^{k,\varepsilon}_{A,r})^* (x_k'), (\tau^{k,\varepsilon}_{A,r})^* (x_j') \}_A = \left\{ x_k^{-1} \prod_{i=1}^{m} x_i^{-1} \varepsilon B_{ik} r_k, x_j \right\}_A$$

$$= \left( -\Omega_{kj} + \sum_{i=1}^{m} x_i^{-1} \varepsilon B_{ik} r_k \right) x_j$$

$$= \Omega_{kj} (\tau^{k,\varepsilon}_{A,r})^* (x_k') (\tau^{k,\varepsilon}_{A,r})^* (x_j').$$

Since the mutation of $D$-compatible pairs is an involution, we obtain analogous Poisson maps from $L^{\chi'}_{x',r}$ to $L^{\chi}_{x',r}$ and from $L^{\chi'}_{A,r}$ to $L^{\chi}_{A,r}$ which, by a slight abuse of notation, we denote by the same symbols $\tau^{k,\varepsilon}_{x',r}$.
and $\tau_{A,r}^{k,\epsilon}$ respectively. It is important to note that the maps $\tau_{X,r}^{k,\epsilon}$ and $\tau_{A,r}^{k,\epsilon}$ are not independent of the sign $\epsilon$ and, moreover, that they are not involutions, i.e. $(\tau_{X,r}^{k,\epsilon})^2$ and $(\tau_{A,r}^{k,\epsilon})^2$ do not give identity maps on $L_X^{k,\epsilon}$ nor on $L_{A,r}^{k,\epsilon}$. However, an easy calculation shows that $\tau_{X,r}^{k,\epsilon} \tau_{X,r}^{k,-\epsilon}$ and $\tau_{A,r}^{k,\epsilon} \tau_{A,r}^{k,-\epsilon}$ will be identity maps for either choice of sign $\epsilon$.

For $1 \leq k \leq n$ and $\epsilon \in \{-1, 1\}$, define the cluster mutations in direction $k$ by

$$\mu_{X,r}^{k,\epsilon} := \tau_{X,r}^{k,\epsilon} \circ \varphi_{X,r}: L_X^{k,\epsilon} \rightarrow L_{X',r}^{k,\epsilon} \quad \text{and} \quad \mu_{A,r}^{k,\epsilon} := \tau_{A,r}^{k,\epsilon} \circ \varphi_{A,r}: L_A^{k,\epsilon} \rightarrow L_{A',r}^{k,\epsilon}.$$

**Corollary 4.5.** For $1 \leq k \leq n$ and $\epsilon \in \{-1, 1\}$, the cluster mutations provide Poisson morphisms $\mu_{X,r}^{k,\epsilon}: L_X^{k,\epsilon} \rightarrow L_{X',r}^{k,\epsilon}$ and $\mu_{A,r}^{k,\epsilon}: L_A^{k,\epsilon} \rightarrow L_{A',r}^{k,\epsilon}$ given on coordinates by

$$\begin{align*}
(\mu_{X,r}^{k,\epsilon})*&(y_i^\epsilon) = \begin{cases} y_k^{-1} & \text{if } \ell = k; \\
y_h y_k [-\epsilon B_k r_k]_+ & \text{if } \ell \neq k; \end{cases} \\
(\mu_{A,r}^{k,\epsilon})*&(x_j) = \begin{cases} x_k^{-1} \left( \prod_{i=1}^m x_i^{[\epsilon B_k r_k]_+} \right) & \text{if } j = k; \\
x_j & \text{if } j \neq k; \end{cases} \\
(\mu_{\varphi_{A,r}}^{k,\epsilon})*&(z_{\ell,j}) = \begin{cases} z_{k,j} & \text{if } \ell = k; \\
z_{\ell,j} & \text{if } \ell \neq k. \end{cases}
\end{align*}
$$

Moreover, the cluster mutations $\mu_{X,r}^{k,\epsilon}$ and $\mu_{A,r}^{k,\epsilon}$ are involutions which do not depend on the choice of sign $\epsilon$.

**Proof.** The first claim immediately follows by combining Corollary 4.3 and Lemma 4.4. The final claim is a consequence of the identities $B_{k\ell} = -B_{k\ell}$, $B_{\ell k} = -B_{k\ell}$, and $B_{k\ell} = [B_{k\ell}]_+ - [-B_{k\ell}]_+$. \qed

**Remark 4.6.** While we have restricted to considering $L_X^{k,\epsilon}$ and $L_{A,r}^{k,\epsilon}$ to motivate the Hamiltonian nature of the mutations $\mu_{X,r}^{k,\epsilon}$ and $\mu_{A,r}^{k,\epsilon}$, the cluster mutations (4.8) and (4.9) are also well-defined on open dense subsets of $L_X^{k,\epsilon}$ and $L_{A,r}^{k,\epsilon}$. More precisely, the map $\mu_{X,r}^{k,\epsilon}$ is not defined on the locus where $y_k = 0$ nor where $Z_k^\epsilon(y_k) = 0$ if $B_{k\ell} > 0$ for some $\ell$ and that the map $\mu_{A,r}^{k,\epsilon}$ is not defined on the locus where $x_k = 0$ nor where \( \left( \prod_{i=1}^m x_i^{[-\epsilon B_{k\ell} r_k]_+} \right) Z_k^\epsilon(y_k) = 0. \) This holds equally well for the automorphism parts $\varphi_{X,r}^{k,\epsilon}$, $\varphi_{A,r}^{k,\epsilon}$ and the tropical parts $\tau_{X,r}^{k,\epsilon}$, $\tau_{A,r}^{k,\epsilon}$ of the cluster mutations with similar restrictions on the domains of definition.

Similar considerations can be made when $L_X = C^n$, $L_A = C^m$, and $Z_k(u) \in C[u]$ is a monic polynomial with $Z_k(0) = 1$ in which case analogous restrictions as above must be imposed, even when considering mutations restricted to $L_X^\epsilon = C_\epsilon^n$ or $L_A^\epsilon = C_\epsilon^m$. In the cluster case (c.f. Remark 4.1), mutations for $L_X^\epsilon = \mathbb{R}_{\geq 0}$ have also been considered [17] under the name “special completion” for $X$-varieties. See also [6] for similar considerations using cluster charts $L_X = C^n$.

The various Poisson maps are summarized in the following commutative diagram.

$$\begin{align*}
&L_{A,r} \xrightarrow{\varphi_{A,r}} L_{A,r} \xrightarrow{\tau_{A,r}} L_{A',r} \\
&L_{X,r} \xrightarrow{\varphi_{X,r}} L_{X,r} \xrightarrow{\tau_{X,r}} L_{X',r}
\end{align*}$$

To record the iteration of mutations, we introduce the $n$-regular labeled rooted tree $T_n$ with root vertex $t_0$ and with the $n$ edges emanating from each vertex labeled by the set $\{1, \ldots, n\}$. In particular, each vertex $t \in T_n$ is uniquely determined by a sequence of indices specifying the edge labels along the unique path from $t_0$ to $t$.

Fix an initial $m \times n$ exchange matrix $\hat{B}_{0}\nu = (B_{ij}; t_0)$ and assign exchange matrices $\hat{B}_t = (B_{ij}; t)$ to the vertices $t \in T_n$ so that $\hat{B}_t = \mu_{r,k} B_t$ whenever $t$ and $t'$ are joined by an edge labeled by $k$. The collection $\{\hat{B}_t\}_{t \in T_n}$ is called the mutation pattern generated by $\hat{B}_{0}$ and any two exchange matrices $\hat{B}_t, \hat{B}_{t'}$ for $t, t' \in T_n$ are said to be mutation equivalent. Given a skew-symmetric $m \times m$ matrix $\Omega_0 = (\Omega_{ij}; t_0)$ which
is $D$-compatible with $\tilde{B}_t$, we define matrices $\Omega_t = (\Omega_{ij,t})$ compatible with $\tilde{B}_t$ by iterating mutations as above.

It will be convenient to make particular choices of the signs $\varepsilon$ as we perform sequences of mutations for the cluster charts $L_{\tilde{X},t}^X$ and $L_{\tilde{A},t}^X$. To specify these signs, we observe that the mutation pattern gives rise to the following combinatorial construction.

**Definition 4.7.** Given an $n \times n$ skew-symmetric matrix $B$, let $\tilde{B}_{\text{prin},t_0}$ denote the $2n \times n$ exchange matrix with principal submatrix $B$ and lower $n \times n$ submatrix given by the $n \times n$ identity matrix $I_n$. Then, given $\tilde{B}_{\text{prin},t}$ mutation equivalent to $\tilde{B}_{\text{prin},t_0}$, the lower $n \times n$ submatrix $C_t := (C_{ij,t})$ of $\tilde{B}_{\text{prin},t}$ transforms as follows when $t$ and $t'$ are joined by an edge labeled $k$ (note that either choice of sign $\varepsilon$ below will produce the same result):

\[
C_{ij,t'} = \begin{cases} 
-C_{ij,t} & \text{if } j = k; \\
C_{ij,t} + [-\varepsilon C_{ik,t} r_k] + B_{kj,t} + C_{ik,t} [\varepsilon r_k B_{kj,t}] & \text{if } j \neq k.
\end{cases}
\]

(4.11)

It is important to observe that this is a piecewise-linear map on the columns of $C_t$.

The $C$-matrices admit the following remarkable sign-coherence property (c.f. [21, 38, 25, 37]):

- each column of $C_t$, known as a $c$-vector, has either all non-negative entries or all non-positive entries.

Thus each exchange matrix $\tilde{B}_{\text{prin},t}$ mutation equivalent to $\tilde{B}_{\text{prin},t_0}$ admits a collection of tropical signs $\varepsilon_{r,k, t} \in \{\pm 1\}$, where $\varepsilon_{r,k, t} = 1$ if the entries in the $k$-th column of $C_t$ are non-negative and $\varepsilon_{r,k, t} = -1$ if the entries in the $k$-th column of $C_t$ are non-positive. Taking the tropical sign in the mutation formula (4.11) for $c$-vector the transformation gives the following simplified mutation rule:

\[
C_{ij,t'} = \begin{cases} 
-C_{ij,t} & \text{if } j = k; \\
C_{ij,t} + C_{ik,t} [\varepsilon_{r,k, t} r_k B_{kj,t}] & \text{if } j \neq k.
\end{cases}
\]

(4.12)

Observe that the mutation rule has become an honestly linear map on the columns of $C_t$ once we choose to mutate according to the tropical sign.

Given an arbitrary initial exchange matrix $\tilde{B}_{t_0}$, we may construct a corresponding principalized exchange matrix $\tilde{B}_{\text{prin},t_0}$ from the principal submatrix $B$ of $\tilde{B}_{t_0}$. Then for each vertex $t \in \mathbb{T}_n$, we associate the same tropical signs $\varepsilon_{r,k, t}$ as above to the columns of each exchange matrix $\tilde{B}_t$ mutation equivalent of $\tilde{B}_{t_0}$.

We associate a Poisson space $L_{\tilde{X},t}^X$ isomorphic to either $\mathbb{R}_{+}^n$ or $\mathbb{C}_{+}^n$ and a Poisson space $L_{\tilde{A},t}^X$ isomorphic to either $\mathbb{R}_{+}^m$ or $\mathbb{C}_{+}^m$ to each vertex $t \in \mathbb{T}_n$. That is, there is a system of coordinates $x_t = (x_{1,t}, \ldots, x_{m,t})$ on $L_{\tilde{X},t}^X$ and a system of coordinates $y_t = (y_{1,t}, \ldots, y_{n,t})$ on $L_{\tilde{A},t}^X$, respectively.

As above, we also introduce the Poisson spaces $L_{\tilde{X},t}^X$ isomorphic to either $\mathbb{R}_{+}^n \times \prod_{i=1}^n \mathbb{R}_{+}^r$ or to $\mathbb{C}_{+}^n \times \prod_{i=1}^n \mathbb{C}_{+}^r$ and $L_{\tilde{A},t}^X$ isomorphic to either $\mathbb{R}_{+}^m \times \prod_{i=1}^m \mathbb{R}_{+}^r$ or to $\mathbb{C}_{+}^m \times \prod_{i=1}^m \mathbb{C}_{+}^r$. Write $z_{t,ij}$ for coordinates on the $L_t$ components and $Z_{t,ij}(u)$ for the associated exchange polynomials.

**Definition 4.8.** Fix an initial $m \times n$ exchange matrix $\tilde{B}_{t_0}$ and an initial skew-symmetric $m \times m$ matrix $\Omega_{t_0}$ which is $D$-compatible with $\tilde{B}_{t_0}$. Let $\tau = (r_1, \ldots, r_n)$ be a sequence of positive integers. The (generalized) cluster varieties $X_\tau = X_\tau(\tilde{B}_{t_0}, D)$ and $A_\tau = A_\tau(\tilde{B}_{t_0}, \Omega_{t_0})$ are obtained by gluing the cluster charts $L_{\tilde{X},t}^X$, $L_{\tilde{A},t}^X$, $L_{\tilde{X},t}^X$, $L_{\tilde{A},t}^X$ respectively, for $t, t' \in \mathbb{T}_n$ joined by an edge labeled $k$, along the cluster mutations $\mu_{X,\tau}$ and $\mu_{A,\tau}$, where $\varepsilon = \varepsilon_{r,k, t}$. That is,

\[
X_\tau = \bigcup_{t \in \mathbb{T}_n} L_{\tilde{X},t}^X \quad \text{and} \quad A_\tau = \bigcup_{t \in \mathbb{T}_n} L_{\tilde{A},t}^X
\]

with cluster charts glued by the mutations as above.

**Remark 4.9.** When $r_\ell = 1$ for all $\ell$, the construction above gives rise to the ordinary cluster varieties [20, 18]. This justifies Remark 4.1.

For both the cluster $A$-varieties [24] and $X$-varieties [16], a Poisson structure is compatible with mutations if the coordinates of each cluster chart are log-canonical (c.f. equation (4.1)).
Theorem 4.10. [24] The Poisson structures on the cluster charts \( L_{X,t}^X \) and \( L_{A,t}^X \) determined by the D-compatible pairs \((B_t, \Omega_t)\) glue to give global Poisson structures on the cluster varieties \( \mathcal{X}_t \) and \( \mathcal{A}_t \) which are log-canonical over each cluster chart. Moreover, the Poisson morphisms \( \rho_t : L_{A,t}^X \rightarrow L_{X,t}^X \) glue to give a Poisson morphism \( \rho : \mathcal{A}_t \rightarrow \mathcal{X}_t \).

The following result is the famous Laurent phenomenon for (generalized) cluster algebras.

Theorem 4.11. [20, 10] Each cluster coordinate \( x_{i,t} \) on the cluster chart \( L_{A,t}^X \) determines a global function on \( \mathcal{A}_t \) which is given by a Laurent polynomial in the coordinates of any other cluster chart \( L_{X,t'}^X \).

This result can be made more precise as follows, c.f. \([21, 35, 37]\). Given a pair of vertices \( t, t' \in T_n \), the composition of mutations from \( L_{X,t}^X \) to \( L_{X,t'}^X \) along the unique path from \( t \) to \( t' \) in \( T_n \) (always taken with respect to the tropical sign from Definition 4.7) gives rise to birational maps

\[
\mu_{X,t}^{t,t'} : L_{X,t}^X \rightarrow L_{X,t'}^X \quad \text{and} \quad \mu_{A,t}^{t,t'} : L_{A,t}^X \rightarrow L_{A,t'}^X.
\]

To understand these iterated mutations, we introduce the following combinatorial constructions.

**Theorem 4.12.** Since we only perform mutations in directions \( k \) with \( 1 \leq k \leq n \), the definitions immediately imply \( G_{ij;k} = \delta_{ij} \) for \( n + 1 \leq j \leq m \).

Next we introduce the \( F \)-polynomials \( F_{r,j:t} \in \mathbb{C}[y_1, \ldots, y_n] \) of the cluster variables \( x_{i,t} \). These are defined recursively by \( F_{r,j:t} = 1 \) for \( 1 \leq j \leq m \) and via the following recursion when \( t \) and \( t' \) are joined in \( T_n \) by an edge labeled \( k \):

\[
F_{r,j:t'} = \begin{cases} F_{r,k:t}^{-1} \cdot \left( \prod_{\ell=1}^m F_{r,\ell:t}^{-[\varepsilon_{r,k:t} B_{t,\ell} r_k]} \right) \cdot Z_{k:t}^D \left( \prod_{\ell=1}^n u_{t}^{\varepsilon_{r,k:t} C_{k,t} F_{r,\ell,t} B_{t,k}} \right) & \text{if } j = k; \\ F_{r,j:t} & \text{if } j \neq k; \end{cases}
\]

where \( Z_{k,t} \) is the exchange polynomial associated to the \( L_t \) component of the chart \( L_{A,t} \) and

\[
Z_{k,t}^D(u) = \begin{cases} Z_{k,t}(u) & \text{if } \varepsilon_{r,k,t} = +1; \\ Z_{k,t}^*(u) & \text{if } \varepsilon_{r,k,t} = -1. \end{cases}
\]

**Remark 4.13.** Again, since we only perform mutations in directions \( k \) with \( 1 \leq k \leq n \), the definitions immediately imply \( F_{r,j:t} = 1 \) for \( n + 1 \leq j \leq m \).

As promised, the \( c \)-vectors, \( g \)-vectors, and \( F \)-polynomials completely control the iterated mutations via the following formulas.

**Theorem 4.14.** [21, 35, 37] For any vertex \( t \in T_n \), we have

\[
(\mu_{X,t}^{t,t_0})^* (y_{t,t_0}) = \prod_{k=1}^n \left( y_{k,t_0}^{C_{k,t} F_{r,k:t} (y_1, \ldots, y_n, t_0) B_{k,t}} \right);
\]

\[
(\mu_{A,t}^{t,t_0})^* (z_{t,t_0}) = \prod_{i=1}^m \left( z_{i,t_0}^{F_{r,i:t} (y_1, \ldots, y_n, t_0)} \right), \quad y_{k,t_0} = \prod_{i=1}^m B_{i,k,t_0};
\]

(4.17) \[
(\mu_{A,t}^{t,t_0})^* (z_{t,t_0}) = z_{t,t_0}^0,
\]

where \( z_{t,t_0}^0 = \mu_{A,t}^{t,t_0}(z_{t,t_0}) \) if the mutation sequence from \( t_0 \) to \( t \) has an even number of mutations in direction \( \ell \) and \( z_{t,t_0}^0 = \mu_{A,t}^{t,t_0}(z_{t,t_0}) \) if this mutation sequence has an odd number of mutations in direction \( \ell \).

**Remark 4.15.** The separation of additions formula in [21] for the cluster variable \( (\mu_{A,t}^{t,t_0})^* (x_{t,t_0}) \) is usually presented with a denominator computed via a tropical evaluation of the \( F \)-polynomial \( F_{r,j:t} (y_1, \ldots, y_n, t_0) \), this is the auxiliary addition that is usually being “separated” from the ordinary addition in the numerator.
We have bypassed this by using \( g \)-vectors from \( \mathbb{Z}^n \) rather than the standard convention with \( g \)-vectors in \( \mathbb{Z}^n \). It is not hard to see by considering \( x_{m+1:t_0}, \ldots, x_{m:t_0} \) simply as frozen cluster variables that our "separation of additions" formula (4.16) reduces to the standard one in the case of geometric coefficients.

It is often the case that a sequence of cluster mutations is essentially an identity mapping, this is made precise as follows.

**Definition 4.16.** Fix an initial \( m \times n \) exchange matrix \( \tilde{B}_{t_0} \) and an initial skew-symmetric \( m \times m \) matrix \( \Omega_{t_0} \) which is \( D \)-compatible with \( \tilde{B}_{t_0} \). Let \( r = (r_1, \ldots, r_n) \) be a sequence of positive integers. A periodicity of cluster mutations \((t_1, \ldots, t_w; \sigma)\) is a sequence of vertices \( t_1, t_2, \ldots, t_w \in \mathbb{T}_n \) with \( t_i \) joined to \( t_{i+1} \) by an edge labeled \( k_t \) together with a permutation \( \sigma \) of the set \( \{1, \ldots, m\} \) fixing each of \( n+1, \ldots, m \) so that the following equalities hold:

1. \( (\mu_{X,r}^{t_1, t_0})^\gamma(y_{\sigma(t); t_1}) = y_{\sigma(t); t_1} \) for \( 1 \leq \ell \leq n \);
2. \( (\mu_{X,r}^{t_1, t_0})^\gamma(x_{\sigma(j); t_1}) = x_{\sigma(j); t_1} \) for \( 1 \leq j \leq m \);
3. \( (\mu_{X,r}^{t_1, t_0})^\gamma(z_{\sigma(j); t_1}) = z_{\sigma(j); t_1} \) for \( 1 \leq \ell \leq n \) and \( 1 \leq j \leq r_\ell - 1 \);
4. \( \tilde{B}_{ij; t_1} = \tilde{B}_{\sigma(i), \sigma(j); t_1} \) for \( 1 \leq i, j \leq m \);
5. \( \Omega_{ij; t_1} = \Omega_{\sigma(i), \sigma(j); t_1} \) for \( 1 \leq i, j \leq m \).

In other words, a periodicity of cluster mutations determines a gluing of cluster charts which is (up to a permutation) just an identity mapping. We will need the following important consequence of the separation of additions formulas and the sign-coherence of \( e \)-vectors.

**Corollary 4.17.** A periodicity of cluster mutations gives the following equalities:

1. \( C_{ij; t_1} = C_{\sigma(i), \sigma(j); t_1} \) for \( 1 \leq i, j \leq m \);
2. \( G_{ij; t_1} = G_{\sigma(i), \sigma(j); t_1} \) for \( 1 \leq i, j \leq m \);
3. \( F_{r, j; t_1} = F_{r, \sigma(j); t_1} \) for \( 1 \leq j \leq m \).

**Proof.** As in [21, Proposition 5.6], the sign-coherence of \( e \)-vectors implies that all \( F \)-polynomials have constant term 1. Since the columns of \( \tilde{B}_{t_0} \) are linearly independent, the Newton polytope of each evaluation \( F_{r, j; t_1}(\hat{y}_{1:t_0}, \ldots, \hat{y}_{n:t_0}) \) has vertices in the positive cone of the \( n \)-dimensional lattice spanned by the columns of \( \tilde{B}_{t_0} \). Moreover, by the first observation, each of these Newton polytopes will contain the vertex of this cone.

These observations immediately give (2) and (3). Indeed, the second condition of a periodicity is equivalent to the equality \( (\mu_{X,r}^{t_1, t_0})^\gamma(x_{\sigma(j); t_1}) = (\mu_{X,r}^{t_1, t_0})^\gamma(x_{\sigma(j); t_1}) \) which, together with the "separation of additions" formula (4.16), shows that

\[
\frac{F_{r, j; t_1}(\hat{y}_{1:t_0}, \ldots, \hat{y}_{n:t_0})}{F_{r, \sigma(j); t_1}(\hat{y}_{1:t_0}, \ldots, \hat{y}_{n:t_0})} = \prod_{i=1}^{m} x_{\sigma(j); t_1}^{G_{r, \sigma(j); t_1} - G_{r, j; t_1}}
\]

is a single monomial. But then the pointedness of their Newton polytopes implies this monomial must be 1, giving (2) and (3).

Finally, (3) immediately implies (1) by the "separation of additions" formula (4.15) and condition (1) of a periodicity of cluster mutations.

**4.2. Mutation of Groupoid Charts.** We have presented thus far the Hamiltonian viewpoint of mutations for cluster charts. Our goal now is to lift these results to the level of the symplectic groupoids \( G_X, B_X, D_X \) and \( G_A, B_A, D_A \) integrating the log-canonical Poisson structures (4.1) on \( L_X^* \cong \mathbb{R}^n_x \) and \( L_A^* \cong \mathbb{R}^n_a \) respectively as in Section 3. Here is an outline for this section:

1. We lift the Poisson ensemble map \( \rho : L_A \to L_X \) in (4.2) to symplectic groupoid comorhisms. When the exchange matrix is square (and hence invertible), the Poisson ensemble map is a diffeomorphism from its domain onto its image and so the graph of each groupoid comorphism also defines groupoid morphisms over suitable loci.
2. We lift the Poisson maps \( \varphi^{t}_{X, r} \) and \( \varphi^{t}_{A, r} \) in Corollary 4.3 to symplectic groupoid morphisms using their Hamiltonian nature. We also show that the Lie functor maps the graphs of these groupoid morphisms to the graphs of the algebroid comorphisms naturally induced by the Poisson maps \( \varphi^{t}_{X, r} \) and \( \varphi^{t}_{A, r} \).
(3) We lift the Poisson maps $\tau_{X,r}^{k,\varepsilon}$ and $\tau_{A,r}^{k,\varepsilon}$ in Lemma 4.4 to groupoid comorhisms and show that they can also be interpreted as groupoid morphisms.

(4) Combining the last two steps above, we obtain the mutation rule for groupoid charts analogous to the (generalized) cluster mutations in Corollary 4.5. Through analogues of the separation of additions formulas (4.15) and (4.16) (c.f. Theorem 4.30), we show that any periodicity of cluster mutations lifts to a periodicity of groupoid mutations (assuming groupoid mutations are always performed according to the tropical signs of the corresponding c-vectors). In particular, this shows that groupoid charts indeed glue to give symplectic groupoids over the cluster $A$- and $X$-varieties.

As in Section 4.1, to motivate the Hamiltonian natures of the mutations, we begin working with the following groupoids over individual cluster charts with real coordinates:

\[
\mathcal{G}_X \cong \mathbb{R}^n \times L_X, \quad \mathcal{B}_X \cong \mathbb{R}^n \times L_X, \quad \mathcal{D}_X \cong \mathbb{R}^n \times L_X, \\
\mathcal{G}_A \cong \mathbb{R}^m \times L_A, \quad \mathcal{B}_A \cong \mathbb{R}^m \times L_A, \quad \mathcal{D}_A \cong \mathbb{R}^m \times L_A.
\]

We introduce coordinates on these groupoids given as follows:

- $\mathcal{G}_X$ has coordinates $(q, y) = (q_1, \ldots, q_n, y_1, \ldots, y_n)$;
- $B_X$ has coordinates $(v, y) = (v_1, \ldots, v_n, y_1, \ldots, y_n)$;
- $B_X$ has coordinates $(t, y) = (t_1, \ldots, t_n, y_1, \ldots, y_n)$;
- $\mathcal{G}_A$ has coordinates $(p, x) = (p_1, \ldots, p_m, x_1, \ldots, x_m)$;
- $\mathcal{B}_A$ has coordinates $(u, x) = (u_1, \ldots, u_m, x_1, \ldots, x_m)$;
- $\mathcal{D}_A$ has coordinates $(s, x) = (s_1, \ldots, s_m, x_1, \ldots, x_m)$.

The symplectic groupoid structure on $\mathcal{G}_A \rightrightarrows L_A$ is the same as in Theorem 3.7, while the symplectic groupoid structure on $\mathcal{D}_A \rightrightarrows L_A$ is the same as in Theorem 3.9, and the symplectic groupoid structure on $\mathcal{B}_A \rightrightarrows L_A$ is the same as in Theorem 3.16. The symplectic groupoid structures over $L_X$ mimic the structures in the theorems above by replacing $x_i, x_j$ with $y_k, y_\ell$ and $\Omega_{ij}$ with $\delta_{ik} \delta_{\ell j}$ together with the corresponding replacements of coordinates $p_j$ with $q_\ell, u_j$ with $r_t$ and $s_j$ with $t_\ell$. For simplicity of notation, we will write $\alpha$ and $\beta$ for the source and target maps of all groupoids; this slight abuse of notation should not lead to any confusion.

As above, for a sequence $r = (r_1, \ldots, r_n)$ of positive integers, we introduce the symplectic groupoids

\[
\mathcal{G}_{X,r} = \mathcal{G}_X \times T^* L_r, \quad \mathcal{B}_{X,r} = \mathcal{B}_X \times T^* L_r, \quad \mathcal{D}_{X,r} = \mathcal{D}_X \times T^* L_r, \\
\mathcal{G}_{A,r} = \mathcal{G}_A \times T^* L_r, \quad \mathcal{B}_{A,r} = \mathcal{B}_A \times T^* L_r, \quad \mathcal{D}_{A,r} = \mathcal{D}_A \times T^* L_r,
\]

where each cotangent bundle $T^* L_r$ is equipped with its canonical symplectic structure and we write $a_{\ell,j}$ for the cotangent coordinate associated to the coordinate $z_{\ell,j}$ of $L_r$. Translating this symplectic structure into a Poisson bracket on $T^* L_r$, we have

\[
\{z_{k,i}, z_{\ell,j}\}_r = \delta_{k\ell} \delta_{ij}, \quad \{z_{k,i}, a_{\ell,j}\}_r = \delta_{k \ell} \delta_{ij}, \quad \{a_{k,i}, a_{\ell,j}\}_r = 0.
\]

We extend the star-involution on $L_r$ to $T^* L_r$ via $a_{\ell,j} \mapsto a^*_{\ell,j} := a_{\ell, r_{i,j}}$ for the cotangent coordinate associated to the coordinate $z^*_{k,i,j}$.

Define the groupoids

\[
\mathcal{G}^{\times}_{X,r} \cong \mathbb{R}^n \times L_X \times T^* L_r, \quad \mathcal{B}^{\times}_{X,r} \cong \mathbb{R}^n \times L_X \times T^* L_r, \quad \mathcal{D}^{\times}_{X,r} \cong \mathbb{R}^n \times L_X \times T^* L_r, \\
\mathcal{G}^{\times}_{A,r} \cong \mathbb{R}^m \times L_A \times T^* L_r, \quad \mathcal{B}^{\times}_{A,r} \cong \mathbb{R}^m \times L_A \times T^* L_r, \quad \mathcal{D}^{\times}_{A,r} \cong \mathbb{R}^m \times L_A \times T^* L_r,
\]

as the restrictions of the groupoids above to appropriate orthants in $L_{X,r}$ and $L_{A,r}$.

**Proposition 4.18.** The Poisson ensemble map $\rho : L_A \to L_X$ (4.2) lifts to the following comorphisms of symplectic groupoids:

1. $(\rho, P)$ from $\mathcal{G}_A$ to $\mathcal{G}_X$ given on fiber coordinates by

\[
P^*(p_\ell) = \frac{1}{x_i} \sum_{k=1}^{n} B_{ik} y_k q_k;
\]

2. $(\rho, P)$ from $\mathcal{B}_A$ to $\mathcal{B}_X$ given on fiber coordinates by

\[
P^*(u_\ell) = \frac{\prod_{k=1}^{n} (v_k y_k + 1) B_{ik} - 1}{x_i}.
\]
(3) \((\rho, P)\) from \(\mathcal{D}_A\) to \(\mathcal{D}_X\) given on fiber coordinates by

\[
P^*(s_i) = \prod_{k=1}^{n} t_k^{B_{ik}}.
\]

**Proof.** We prove (4.18) over the positive orthants, the formulas for \(\mathcal{B}\) and \(\mathcal{D}\) follow easily using the diagram (3.4). Taking the derivative of \(\rho : L^X_A \to L^X_X\), whose graph is given by the equation

\[
y_k = \prod_{i=1}^{m} x_i^{B_{ik}},
\]

we obtain the pullback map \(\rho^* : \Omega^1(L^X_X) \to \Omega^1(L^X_A)\) given by

\[
dy_k = \sum_{i=1}^{m} \frac{B_{ik}}{x_i} \hat{y}_k dx_i, \quad k = 1, \ldots, n,
\]

\[
\sum_{k=1}^{n} q_k dy_k = \sum_{i=1}^{m} \frac{1}{x_i} \sum_{k=1}^{n} B_{ik} \hat{y}_k q_k dx_i.
\]

We may represent this as the bundle map \(\rho^* : \rho^* T^* \pi_A L^X_A \to T^* \pi_X L^X_X\) whose graph is given by the equation

\[
p_i = \frac{1}{x_i} \sum_{k=1}^{n} B_{ik} \hat{y}_k q_k, \quad i = 1, \ldots, m.
\]

The graph of the Lie algebroid comorphism \(\rho^*\) from \(T^* \pi_A L^X_A\) to \(T^* \pi_X L^X_X\) is defined by

\[
x_i p_i = \sum_{k=1}^{n} B_{ik} y_k q_k, \quad i = 1, \ldots, m.
\]

By Corollary 3.8, both \(x_i p_i\) and \(y_k q_k\) are invariant under the exponential map to the Lie groupoids. But Corollary 3.8 shows that the cotangent Lie algebroids \(T^* \pi_A L^X_A\) and \(T^* \pi_X L^X_X\) are diffeomorphic respectively to the groupoids \(\mathcal{G}_A\) and \(\mathcal{G}_X\), so the associated Lie groupoid comorphism is given by the equivalent formula (4.18). \(\square\)

**Remark 4.19.** When \(m = n\), i.e. the map \(\rho : L_A \to L_X\) is a birational isomorphism, the Poisson ensemble map actually lifts to true morphisms of symplectic groupoids \(\rho : \mathcal{G}_A \to \mathcal{G}_X\), \(\rho : \mathcal{B}_A \to \mathcal{B}_X\), and \(\rho : \mathcal{D}_A \to \mathcal{D}_X\) given respectively on fiber coordinates by

\[
\rho^*(q) = -(d \hat{y}_e)^{-1} \sum_{j=1}^{m} \Omega_{jj} x_j p_j;
\]

\[
\rho^*(v) = \prod_{j=1}^{m} (u_j x_j + 1)^{-d^{-1} \Omega_{jj}};
\]

\[
\rho^*(t) = \prod_{j=1}^{m} s_j^{-d^{-1} \Omega_{jj}}.
\]

Observe that care must be taken when considering the fractional exponents above, however the maps are in fact well-defined. Indeed, a morphism of groupoids must take the identity image to the identity image and thus there is a unique choice of branch to take, in other words the maps \(\rho : \mathcal{B}_A \to \mathcal{B}_X\) and \(\rho : \mathcal{D}_A \to \mathcal{D}_X\) are actually well-defined covering maps on the fibers.

The choice of coordinates used in [16] can be identified with our coordinates \(t_i^{\ell}\), one may motivate such a choice by the desire to avoid the fractional exponents above, however as observed such considerations are unnecessary.

Next, we lift the Hamiltonian Poisson maps \(\varphi^i_{X, r}\) and \(\varphi^i_{A, r}\) in Corollary 4.3. Using appropriate source and target maps, define functions \(H^{k, \varepsilon}_{X, r} := \alpha^*(h^{k, \varepsilon}_{X, r}) - \beta^*(h^{k, \varepsilon}_{X, r})\) on the groupoids \(\mathcal{G}^X_{X, r}, \mathcal{B}^X_{X, r}, \mathcal{D}^X_{X, r}\) and functions
The coordinates $a_{t,j}$ on $T^*L^n$ evolve according to

$$
\dot{a}_{t,j} = \{H_{-t,r}, a_{t,j}\}_- = \delta_{kt} \frac{\varepsilon}{d_k} \int_0^1 u^{\varepsilon j-1} Z_k(u)^{\varepsilon} du,
$$

where the bounds of integration are respectively those from equations (4.19) or (4.20).

Proof. We prove equation (4.21), leaving the other groupoid coordinates as an exercise for the reader. By Theorem 3.7, we have $\{x_i, p_j\}_{\mathcal{G}_A} = -\delta_{ij} - \Omega_{ij} x_i p_j$ and $\{p_i, p_j\}_{\mathcal{G}_A} = \Omega_{ij} p_i p_j$ so that

$$
\{\hat{y}_k, p_j\}_{\mathcal{G}_A} = \sum_{t=1}^m B_{tjk} \frac{\hat{y}_k}{x_i} \{x_i, p_j\}_{\mathcal{G}_A} = -B_{tjk} \frac{\hat{y}_k}{x_i} - \delta_{jk} d_k \hat{y}_k p_k
$$

and

$$
\{e^{-d_k p_k x_k}, p_j\}_{\mathcal{G}_A} = \delta_{jk} d_k p_k e^{-d_k p_k x_k}.
$$
Now equation (4.21) follows by applying the chain rule for the derivation \( \{ \cdot , p_j \} \).

To see equation (4.22), we observe that \( \frac{d \log(Z_k(u))}{dt\epsilon_j} = \delta_{kj} \frac{y_k'}{Z_k(u)} \) and \( \frac{d \log(Z_k(u))}{dt\epsilon_j} = \delta_{kj} \frac{y_k'}{Z_k(u^{-1})} \).

Next we consider the time-\( t \) flows, all denoted \( \varphi_{X,r}^t \), of the vector fields \( X_{X,r}^{k,\varepsilon} \) on the groupoids \( G_{X,r}^\times, B_{X,r}^\times, D_{X,r}^\times \) and the time-\( t \) flows, all denoted \( \varphi_{A,r}^t \), of the vector fields \( X_{A,r}^{k,\varepsilon} \) on the groupoids \( G_{A,r}^\times, B_{A,r}^\times, D_{A,r}^\times \).

**Corollary 4.21.** For \( 1 \leq k \leq n \) and \( \varepsilon \in \{ \pm 1 \} \), the Hamiltonian flow \( \varphi_{X,r}^t : L_{X,r}^\times \rightarrow L_{X,r}^\times \) in Corollary 4.3 is lifted to the multiplicative Hamiltonian flows \( \varphi_{X,r}^t : G_{X,r}^\times \rightarrow G_{X,r}^\times, \varphi_{X,r}^t : B_{X,r}^\times \rightarrow B_{X,r}^\times, \varphi_{X,r}^t : D_{X,r}^\times \rightarrow D_{X,r}^\times \), given on coordinates by

\[
(\varphi_{X,r}^t)^*(qeye) = qeye - \frac{\delta_{kj}}{d\epsilon_k} \log \left( \frac{Z_k^0(y_k') \prod_{j=1}^n (uv_j y_j + 1)^{\delta_{kj}}}{Z_k^0(y_k')} \right);
\]

\[
(\varphi_{X,r}^t)^*(v_\epsilon y\epsilon + 1) = \left( \frac{Z_k^0(y_k') \prod_{j=1}^n (uv_j y_j + 1)^{\delta_{kj}}}{Z_k^0(y_k')} \right)^{-\frac{\delta_{kj}}{d\epsilon_k}} (v_\epsilon y\epsilon + 1);
\]

\[
(\varphi_{X,r}^t)^*(t_\epsilon) = \left( \frac{Z_k^0(y_k') \prod_{j=1}^n (uv_j y_j + 1)^{\delta_{kj}}}{Z_k^0(y_k')} \right)^{-\frac{\delta_{kj}}{d\epsilon_k}} t_\epsilon;
\]

and the Hamiltonian flow \( \varphi_{A,r}^t : L_{A,r}^\times \rightarrow L_{A,r}^\times \) in Corollary 4.3 is lifted to the multiplicative Hamiltonian flows \( \varphi_{A,r}^t : G_{A,r}^\times \rightarrow G_{A,r}^\times, \varphi_{A,r}^t : B_{A,r}^\times \rightarrow B_{A,r}^\times, \varphi_{A,r}^t : D_{A,r}^\times \rightarrow D_{A,r}^\times \), given on coordinates by

\[
(\varphi_{A,r}^t)^*(p_j x_j) = p_j x_j - \frac{B_{jk}}{d\epsilon_k} \log \left( \frac{Z_k^0(y_k') \prod_{j=1}^n (uv_j y_j + 1)^{\delta_{kj}}}{Z_k^0(y_k')} \right);
\]

\[
(\varphi_{A,r}^t)^*(u_j x_j + 1) = \left( \frac{Z_k^0(y_k') (u_j x_j + 1)^{\delta_{kj}}}{Z_k^0(y_k')} \right)^{-\frac{B_{jk}}{d\epsilon_k}} (u_j x_j + 1);
\]

\[
(\varphi_{A,r}^t)^*(s_j) = \left( \frac{Z_k^0(y_k') s_j^{\delta_{kj}}}{Z_k^0(y_k')} \right)^{-\frac{B_{jk}}{d\epsilon_k}} s_j.
\]

Each of the Hamiltonian flows above transform the coordinates \( a_{t,\epsilon} \) on \( T^*L_{r}^\times \) according to

\[
(\varphi_{-r}^t)^*(a_{t,\epsilon}) = a_{t,\epsilon} + t \delta_{\epsilon k} \frac{\varepsilon}{d\epsilon_k} \int_s^{s'} \frac{u^{-1} du}{Z_k(u^{-1})} du
\]

where the bounds of integration are respectively those from equations (4.19) or (4.20).

**Proof.** To see this, it suffices to make the following observations:

- The quantities \( y_k, e^{\varepsilon \sum_{j=1}^n B_{j,k}\epsilon_j y_j}, \prod_{j=1}^n (uv_j y_j + 1)^{\delta_{kj}} B_{j,k} \), and \( \prod_{j=1}^n (uv_j y_j + 1)^{\delta_{kj}} B_{j,k} \) are conserved under the flow of the vector fields \( X_{X,r}^{k,\varepsilon} \). It follows that \( \frac{d}{dt}(v_\epsilon y\epsilon) \) is a constant, \( \frac{d}{dt}(v_\epsilon y\epsilon + 1) \) is a constant multiple of \( v_\epsilon y\epsilon + 1 \), and \( \frac{d}{dt}(t_\epsilon) \) is a constant multiple of \( t_\epsilon \).

- The quantities \( y_k, e^{-\delta_{kj} B_{j,k} x_j}, (u_j x_j + 1)^{\delta_{kj}} B_{j,k} \) are conserved under the flow of the vector fields \( X_{A,r}^{k,\varepsilon} \). It follows that \( \frac{d}{dt}(p_j x_j) \) is a constant, \( \frac{d}{dt}(u_j x_j + 1) \) is a constant multiple of \( u_j x_j + 1 \), and \( \frac{d}{dt}(s_j) \) is a constant multiple of \( s_j \).

- As a result, in the flow of any groupoid, \( \frac{d}{dt}(a_{t,\epsilon}) \) is constant.

In each case, the resulting differential equation is easy to solve and we leave as an exercise for the reader to check the formulas given above.

For the groupoid morphism \( \varphi_{A,r}^t : G_{A,r}^\times \rightarrow G_{A,r}^\times \) in Corollary 4.21, we give an explicit verification that the corresponding Lie algebroid morphism is the dual bundle map of the Lie algebroid comorphism induced by the base Poisson map \( \varphi_{A,r}^t : L_{A,r}^\times \rightarrow L_{A,r}^\times \). A similar story exists for the other groupoid maps in Corollary 4.21.
To ease the notation, we drop the time parameter in the superscript, fix $r$ and use Sans-serif font for the coordinates for the range, e.g. $x$ for the range versus $\tilde{x}$ for the domain. To begin with, we apply the Lie functor, see (2.1) and the ensuing discussion. The source map of $G_A \Rightarrow L_A$ is

$$\alpha : G_A \to L_A, \quad (p, x) \mapsto x,$$

so the kernel of $\alpha_* : T G_A \to TL_A$ is generated by $\frac{\partial}{\partial p_j}$, $j = 1, \ldots, m$. Given the Hamiltonian groupoid morphism

$$\varphi_{A,r}^1 : G_A \to G_A,$$

$$x_j = (Z_k^e(\tilde{y}_k^e))^{-\delta_{jk}} x_j,$$

$$p_j x_j = p_j x_j - \frac{B_{jk}}{d_k} \log \left( \frac{Z_k^e(\tilde{y}_k^e - \varepsilon d_k p_j x_k)}{Z_k^e(\tilde{y}_k^e)} \right),$$

its pushforward is

$$(\varphi_{A,r}^1)_* : \ker(\alpha_* : T G_A \to TL_A) \to \ker(\alpha_* : T G_A \to TL_A),$$

$$\frac{\partial}{\partial p_j} = \frac{\partial}{\partial p_j} + \frac{\varepsilon x_k \tilde{y}_k^e B_{jk}}{x_j} (Z_k^e(\tilde{y}_k^e)) \frac{\partial}{\partial p_k}, \quad \text{if } j \neq k,$$

$$\frac{\partial}{\partial p_k} = \frac{1}{Z_k^e(\tilde{y}_k^e)} \frac{\partial}{\partial p_k}.$$

Restricting to the identity image, which is given by $p_j = 0$, $j = 1, \ldots, m$, we have

$$\text{Lie}(\varphi_{A,r}^1) : \text{Lie}(G_A) \to \text{Lie}(G_A),$$

$$\frac{\partial}{\partial p_j} = \frac{\partial}{\partial p_j} + \frac{\varepsilon x_k \tilde{y}_k^e B_{jk}}{x_j} (Z_k^e(\tilde{y}_k^e)) \frac{\partial}{\partial p_k}, \quad \text{if } j \neq k,$$

$$\frac{\partial}{\partial p_k} = \frac{1}{Z_k^e(\tilde{y}_k^e)} \frac{\partial}{\partial p_k}.$$

On the other hand, taking derivatives of $x_j = (Z_k^e(\tilde{y}_k^e))^{-\delta_{jk}} x_j$, we obtain the pullback map

$$(\varphi_{A,r}^1)^* : \Omega^1(L_A) \to \Omega^1(L_A),$$

$$dx_j = dx_j, \quad \text{if } j \neq k,$$

$$dx_k = \frac{1}{Z_k^e(\tilde{y}_k^e)} dx_k + \sum_{j \neq k} \frac{\varepsilon x_k \tilde{y}_k^e B_{jk}}{x_j} (Z_k^e(\tilde{y}_k^e))^2 dx_j$$

$$= \frac{1}{Z_k^e(\tilde{y}_k^e)} dx_k + \sum_{j \neq k} \frac{\varepsilon x_k \tilde{y}_k^e B_{jk}}{x_j} (Z_k^e(\tilde{y}_k^e))^2 dx_j.$$

If we identify $\text{Lie}(G_A)$ and $T_{\pi A}^* L_A$ via

$$\frac{\partial}{\partial p_j} = dx_j, \quad j = 1, \ldots, m,$$

then $(\varphi_{A,r}^1)^*$ is the dual bundle map of $\text{Lie}(\varphi_{A,r}^1)$. That is, $(\varphi_{A,r}^1)^* : T_{\pi A}^* L_A \to T_{\pi A}^* L_A$ is a Lie algebroid isomorphism and its corresponding Lie algebroid morphism is Lie $(\varphi_{A,r}^1)$, see the discussion preceding Definition 2.8. It follows that $(\varphi_{A,r}^1)^*$ integrates to a groupoid comorphism whose graph coincides with the graph of $\varphi_{A,r}^1 : G_A \to G_A$.

Write $G_{X',r}, B_{X',r}, D_{X',r}$ and $G_{A',r}, B_{A',r}, D_{A',r}$ for the analogous groupoids over the cluster charts $L_{X',r}$ and $L_{A',r}$ with Poisson structures related to those on $L_{X',r}$ and $L_{A',r}$ by mutation in direction $k$.

**Lemma 4.22.** For $1 \leq k \leq n$ and $\varepsilon \in \{\pm 1\}$, there are symplectic groupoid morphisms

$$\tau_{X,r}^{k,\varepsilon} : G_{X,r}^\times \to G_{X',r}^\times,$$

$$\tau_{X,r}^{k,\varepsilon} : B_{X,r}^\times \to B_{X',r}^\times,$$

$$\tau_{X,r}^{k,\varepsilon} : D_{X,r}^\times \to D_{X',r}^\times,$$

$$\tau_{A,r}^{k,\varepsilon} : G_{A,r}^\times \to G_{A',r}^\times,$$

$$\tau_{A,r}^{k,\varepsilon} : B_{A,r}^\times \to B_{A',r}^\times,$$

$$\tau_{A,r}^{k,\varepsilon} : D_{A,r}^\times \to D_{A',r}^\times.$$
lifting respectively the Poisson morphisms \( \tau_{X,r}^{k,e} : L_{X,r}^X \to L_{X',r}^X \) and \( \tau_{A,r}^{k,e} : L_{A,r}^X \to L_{A',r}^X \) of Lemma 4.4. These are given on fiber coordinates by

\[
\tau_{X,r}^{k,e}*(q_{\ell}) = \begin{cases} 
-2kq_k^2 + \sum_{\ell'=1}^n [\varepsilon r_k B_{k\ell'}]+ q_{\ell'} y_{\ell'} y_k & \text{if } \ell = k; \\
y_k \left( v_k y_k + 1 \right) & \text{if } \ell \neq k;
\end{cases}
\]

\[
\tau_{X,r}^{k,e}*(v_{\ell}) = \begin{cases} 
-\varepsilon r_k B_{k\ell} & \text{if } \ell = k; \\
v_k & \text{if } \ell \neq k;
\end{cases}
\]

\[
\tau_{A,r}^{k,e}*(t_{\ell}) = \begin{cases} 
-2p_k x_k^2 \sum_{i=1}^m x_i^{-1} \varepsilon B_{ikr_k} & \text{if } j = k; \\
t_{\ell} & \text{if } \ell \neq k;
\end{cases}
\]

\[
\tau_{A,r}^{k,e}*(p_j) = \begin{cases} 
p_j + \varepsilon B_{jk r_k} & \text{if } j \neq k; \\
p_j & \text{if } j = k;
\end{cases}
\]

\[
\tau_{A,r}^{k,e}*(u_j) = \begin{cases} 
x_k \left( u_k x_k + 1 \right) & \text{if } j = k; \\
x_j^{-1} \left( u_j x_j + 1 \right) & \text{if } j \neq k.
\end{cases}
\]

\[
\tau_{A,r}^{k,e}*(s_j) = \begin{cases} 
s_k^{-1} \varepsilon B_{jk r_k} & \text{if } j = k; \\
s_j s_k^{-1} \varepsilon B_{jk r_k} & \text{if } j \neq k;
\end{cases}
\]

\[
\tau_{A,r}^{k,e}*(a_{i,j}) = \begin{cases} 
a_{k,j} & \text{if } \ell = k; \\
a_{i,j} & \text{if } \ell \neq k.
\end{cases}
\]

**Proof.** We prove (4.23). The other groupoid maps follows from (3.4) and/or Proposition 4.18. To ease the notation, we fix \( r \) and drop all subscripts and superscripts. Taking derivatives of

\[
\tau : L_{A}^X \to L_{A'}^X,
\]

we obtain the pullback map

\[
\tau^* : \Omega^1(L_{A'}^X) \to \Omega^1(L_{A}^X),
\]

\[
dx' = -x_k^{-2} \sum_{i=1}^m x_i^{-1} \varepsilon B_{ik r_k}^+ dx_k + \sum_{j=1}^m \varepsilon B_{jk r_k}^+ x_j x_k \prod_{i=1}^m x_i^{-1} \varepsilon B_{ik r_k}^+ dx_j,
\]

\[
dx' = dx_j & \text{if } j \neq k,
\]

\[
\sum_{j=1}^m p_j' dx_j' = -\frac{p_k'}{x_k^2} \sum_{i=1}^m x_i^{-1} \varepsilon B_{ik r_k}^+ dx_k + \sum_{j \neq k} \left( \frac{p_k' \varepsilon B_{jk r_k}^+}{x_j x_k} \prod_{i=1}^m x_i^{-1} \varepsilon B_{ik r_k}^+ + p_j' \right) dx_j
\]

or as a bundle map

\[
\tau^* : T^* L_{A'}^X \to T^* L_{A}^X,
\]

\[
p_k = \frac{x_k' p_k'}{x_k},
\]

\[
p_j = \frac{x_j' p_j'}{x_j} + p_j' & \text{if } j \neq k.
\]

The graph of the Lie algebroid comorphism \( \tau^* \) from \( T_{\pi A}^* L_A \) to \( T_{\pi A}^* L_{A'} \) is defined by

\[
x_k' p_k = -x_k p_k,
\]

\[
x_j' p_j = x_k p_k [\varepsilon B_{jk r_k}]^+ + x_j p_j & \text{if } j \neq k.
\]
By Lemma 3.6 and Corollary 3.8, both \(x_j p_j\) and \(x_j' p_j'\) are invariant under the exponential map, so (4.23) follows.

**Theorem 4.23.** There are morphisms of Poisson groupoids \(\mu_{X,r}^{k,\varepsilon}\) and \(\mu_{A,r}^{k,\varepsilon}\):

\[
\begin{align*}
\mu_{X,r}^{k,\varepsilon} &: G_{X,r}^\varepsilon \to G_{X,r}^\varepsilon, \\
\mu_{A,r}^{k,\varepsilon} &: G_{A,r}^\varepsilon \to G_{A,r}^\varepsilon,
\end{align*}
\]

defined as the compositions \(\tau_{-,r}^{\varepsilon} \circ \varphi_{-,r}^{\varepsilon}\) given on fiber coordinates by

\[
\begin{align*}
(\mu_{X,r}^{k,\varepsilon})^\ast(q^\ell_k) &= \begin{cases} 
-\eta_k y_k^2 + \sum_{\ell=1}^n \left[\varepsilon e_k B_k r_e^k\right] + q^\ell_k y_k^2 + \frac{\varepsilon}{d_k} \log \left(\frac{Z_k^e\left(y_k^e e^k e^{\ell e} B_k r_e^k\right)}{Z_k^e(y_k^e)}\right) & \text{if } \ell = k; \\
q^\ell_k y_k^{\varepsilon B_k r_e^k} Z_0(y_k^e)_{B_k} & \text{if } \ell \neq k;
\end{cases} \\
(\mu_{X,r}^{k,\varepsilon})^\ast(u^\ell_k) &= \begin{cases} 
y_k \left[\left(u_k y_k + 1\right)^n \prod_{\ell=1}^n \left(\varepsilon e_k B_k r_e^k\right) \right] \left(\frac{Z_k^e(y_k^e e^k e^{\ell e} B_k r_e^k\right)}{Z_k^e(y_k^e)}\right)^{1/d_k} & \text{if } \ell = k; \\
v^\ell_k y_k^{\varepsilon B_k r_e^k} Z_0(y_k^e)_{B_k} & \text{if } \ell \neq k;
\end{cases} \\
(\mu_{X,r}^{k,\varepsilon})^\ast(t^\ell_k) &= \begin{cases} 
\frac{1}{n} \prod_{\ell=1}^n \left[\varepsilon e_k B_k r_e^k\right] + \left(\frac{Z_k^e(y_k^e e^k e^{\ell e} B_k r_e^k\right)}{Z_k^e(y_k^e)}\right)^{1/d_k} & \text{if } \ell = k; \\
t^\ell_k & \text{if } \ell \neq k;
\end{cases} \\
(\mu_{X,r}^{k,\varepsilon})^\ast(p^\ell_k) &= \begin{cases} 
-q_k y_k^2 \left(\prod_{\ell=1}^m \left[\varepsilon e_k B_k r_e^k\right] \right) Z_0^e(y_k^e)^{-1} & \text{if } j = k; \\
p^\ell_k + \left[\varepsilon e_k B_k r_e^k\right] + \frac{B_k}{d_k} \log \left(\frac{Z_k^e(y_k^e e^k e^{\ell e} B_k r_e^k\right)}{Z_k^e(y_k^e)}\right) & \text{if } j \neq k;
\end{cases} \\
(\mu_{X,r}^{k,\varepsilon})^\ast(s^\ell_k) &= \begin{cases} 
s_k^\ell_k & \text{if } \ell = k; \\
s^\ell_k & \text{if } \ell \neq k.
\end{cases}
\end{align*}
\]

The coordinates \(a_{\ell,j}\) on \(T^* L^X_r\) transform according to

\[
\begin{align*}
(\mu_{X,r}^{k,\varepsilon})^\ast(a^\ell_{\ell,j}) &= \begin{cases} 
a^\ell_{\ell,j} + \frac{\varepsilon}{d_k} \int Z_k^e(y_k^e e^k e^{\ell e} B_k r_e^k\right) du & \text{if } \ell = k; \\
\alpha_{\ell,j} & \text{if } \ell \neq k;
\end{cases} \\
(\mu_{A,r}^{k,\varepsilon})^\ast(a^\ell_{\ell,j}) &= \begin{cases} 
a^\ell_{\ell,j} + \frac{\varepsilon}{d_k} \int Z_k^e(y_k^e e^k e^{\ell e} B_k r_e^k\right) du & \text{if } \ell = k; \\
\alpha_{\ell,j} & \text{if } \ell \neq k;
\end{cases}
\end{align*}
\]

where \(\beta\) is the appropriate target map and \(j^* = r_k - j\).

**Proof.** This is immediate from Corollary 4.21 and Lemma 4.22.

The various groupoid maps lifting the Poisson maps in (4.10) are summarized in the following commutative diagram. The groupoid morphisms are represented by solid arrows and the groupoid comorphisms are
represented by dotted arrows.

\[
\begin{array}{cc}
G_A & G'_A \\
\varphi_A & \tau_A \\
\rho & \rho \\
G_X & G'_X \\
\varphi_X & \tau_X \\
\rho & \rho \\
\end{array} =
\begin{array}{cc}
D_A & D'_A \\
\varphi_A & \tau_A \\
\rho & \rho \\
D_X & D'_X \\
\varphi_X & \tau_X \\
\rho & \rho \\
\end{array}
\]

Remark 4.24. While the formulas for the groupoid mutations were only defined over the positive real orthants \( L^+_{X,r} \) and \( L^+_{A,r} \), they can often also be used to glue the various groupoids over larger (sometimes dense) subsets of \( L_{X,r} \) and \( L_{A,r} \) as in Remark 4.6. However, certain additional subtleties must be overcome in the groupoid case:

1. In the real case, the mutation rules for \( q'_\ell \) and \( p'_j \) can only be extended to the source fibers over the coordinate hyperplanes in \( L_{X,r} \) and \( L_{A,r} \) since only the logarithms of positive real numbers are again real. In the complex case, these mutation rules can be extended to open dense subsets of \( G_{X,r} \) and \( G_{A,r} \), namely to the loci where \( Z^c_k(y'_\ell e^{-\sum_{\ell'\neq\ell} e^{d_{\ell'} B_{\ell'} y'_{\ell'}}} / Z^c_k(y'_\ell) \) and \( Z^c_k(q'_\ell e^{-\sum_{k} p_k x_k} / Z^c_k(y'_\ell) \) are defined and nonzero. Indeed, although \( \log(-) \) is multivalued as a function on \( \mathbb{C}_x \), the functions \( (\mu^k)_{X,r} \) and \( (\mu^k)_{A,r} \) are well-defined on these dense subsets of \( G_{X,r} \) and \( G_{A,r} \) respectively. To see this, note that \( \mu^k \) and \( \mu^k_{A,r} \) are groupoid morphisms and hence must carry the identity image to the identity image. This in particular fixes the branch of \( \log(-) \) to choose when computing the values of \( (\mu^k)_{X,r} \) and \( (\mu^k)_{A,r} \) at the origin of each source fiber. Thus, by simple-connectedness, we are able to fix the choice of branch cut to use when computing their values at arbitrary points of the source fibers.

2. Again in the real case, the existence of an even skew-symmetrizer \( d_k \) similarly restricts the extendability of the mutation rules for \( v'_\ell \), \( t'_\ell \) and \( u'_j \), \( s'_j \) to the source fibers over the coordinate hyperplanes in \( L_{X,r} \) and \( L_{A,r} \). In the complex case or when all skew-symmetrizers are odd, we must avoid the loci where \( Z^c_k(y'_\ell \prod_{\ell'=1}^{d_{\ell'} B_{\ell'} y'_{\ell'}} / Z^c_k(y'_\ell) \), \( Z^c_k(y'_\ell \prod_{\ell'=1}^{d_{\ell'} B_{\ell'} y'_{\ell'}} / Z^c_k(y'_\ell) \), \( Z^c_k(y'_\ell \prod_{\ell'=1}^{d_{\ell'} B_{\ell'} y'_{\ell'}} / Z^c_k(y'_\ell) \), and \( Z^c_k(y'_\ell \prod_{\ell'=1}^{d_{\ell'} B_{\ell'} y'_{\ell'}} / Z^c_k(y'_\ell) \) are undefined or zero. Away from these loci in the complex case, the identity image can again be used to define a splitting of the \( d_k \)-fold covering map from \( \mathbb{C}_x \) to \( \mathbb{C}_x \).

3. Finally, the mutations \( \mu^k_{A,r} \) appear to be undefined on \( p'_j \) and \( u'_j \) for \( j \neq k \) when \( x_j = 0 \). However, this is a removable singularity and these maps are actually continuous when appropriate values are assigned. To see this, we first observe that \( -\varepsilon B_{jk} x_k + p_k x_k - B_{jk} \log (Z^c_k(y'_k e^{-\sum_{k} r_{jk} x_k} / Z^c_k(y'_k)) = 0 \) for \( j \neq k \) and \( x_j = 0 \) as follows:

- If \( \varepsilon B_{jk} > 0 \), the first term above is clearly zero and \( x_j \) appears in \( y'_k \) with a positive exponent. Using that \( x_j = 0 \), it follows that the log term reduces to zero as well since \( Z^c_k \) has nonzero constant term.
- When \( B_{jk} = 0 \) there is nothing to show.
- If \( \varepsilon B_{jk} < 0 \), the first term does not vanish but \( x_j \) appears in \( y'_k \) with a negative exponent. Using that \( x_j = 0 \), the term inside the logarithm reduces to \( e^{-\sum_{k} r_{jk} x_k} \) and thus we again get zero.
Therefore, we may take the limit as $x_j \to 0$ in these formulas and set

\[
(\mu_{A, r}^{k, c}|_{x_j = 0})^* (p_j') = p_j + \lim_{x_j \to 0} \frac{-\varepsilon B_k r_k}{d_k} + p_k x_k - \frac{B_k}{d_k} \log \left( \frac{Z_k^x(\tilde{\gamma}_k e^{-d_k p_k x_k})}{Z_k^x(\tilde{\gamma}_k)} \right)
\]

\begin{align*}
&= p_j - \frac{\varepsilon B_k^2}{d_k} \lim_{x_j \to 0} y_k e^{-d_k p_k x_k} (Z_k^x)((y_k e^{-d_k p_k x_k}) Z_k^x(y_k) - y_k Z_k^x(y_k e^{-d_k p_k x_k}))(Z_k^x(y_k)) \\
&= \begin{cases} 
p_j & \text{if } |B_{jk}| \neq 1; \\
p_j - \frac{1}{d_k} \prod_{i \neq j} x_i^{-B_{ik} z_{k, 1} (e^{-d_k p_k x_k} - 1)} & \text{if } B_{jk} = 1; \\
p_j + \frac{1}{d_k} \prod_{i \neq j} x_i^{-B_{ik} z_{k, r_k - 1} (e^{-d_k p_k x_k} - 1)} & \text{if } B_{jk} = -1;
\end{cases}
\end{align*}

where the last equality uses that $aZ_k^x(a)Z_k^x(b) - bZ_k^x(a)Z_k^x(b)$ has lowest degree term $z_k, 1(a - b)$ and highest degree term $z_k, r_k - 1 a^n b^n (b^{-1} - a^{-1})$ while $a(Z_k^x)(a)Z_k^x(b) - bZ_k^x(a)(Z_k^x)'(b)$ has lowest degree term $z_k, r_k - 1 (a - b)$ and highest degree term $z_k, 1 a^n b^n (b^{-1} - a^{-1})$.

Observe that this restricted mutation map is again not defined for $x_i = 0$ when $B_{jk} = 1$ and $B_{ik} < 0$ nor when $B_{jk} = -1$ and $B_{ik} > 0$. Following Remark 4.6, the gluing maps $\mu_{A, r}^{k, c} : L_{A, r} \to L_{A', r}$ are also not defined on these loci since the conditions imply

\[
\left( \prod_{i=1}^m x_i^{[-\varepsilon B_k r_k]} \right) Z_k^x(\tilde{\gamma}_k) = 0,
\]

in particular, the gluing of fibers will not be attempted over these loci.

We leave it as an exercise for the reader to verify the following mutation rule for the coordinates $u_j'$ over the locus where $x_j = 0$:

\[
(\mu_{A, r}^{k, c}|_{x_j = 0})^* (u_j') = \begin{cases} 
u_j & \text{if } |B_{jk}| \neq 1; \\
u_j - \frac{1}{d_k} \prod_{i \neq j} x_i^{-B_{ik} z_{k, 1} ((u_k x_k + 1)^{-d_k} - 1)} & \text{if } B_{jk} = 1; \\
u_j + \frac{1}{d_k} \prod_{i \neq j} x_i^{-B_{ik} z_{k, r_k - 1} ((u_k x_k + 1)^{d_k} - 1)} & \text{if } B_{jk} = -1.
\end{cases}
\]

As in Section 4.1, we record iterated mutations using the $n$-regular rooted tree $T_n$ with root vertex $t_0$. That is, over each cluster chart $L_{X, r; t}$ and $L_{A, r; t}$ we have, respectively, groupoids $G_{X, r; t} : X_{X, r; t}$ and $G_{A, r; t} : X_{A, r; t}$ with all of the structure as above associated to the pairs $(B_t, \Omega_t)$ for each $t \in T_n$. Our goal is to glue the various groupoid charts over the cluster charts to get groupoids over $X$ and $A$.

Given a pair of vertices $t, t' \in T_n$, we will define iterated mutations $\mu_{X, r}^{t, t'}$ and $\mu_{A, r}^{t, t'}$ of the groupoid charts as we did for the cluster charts by composing the maps in Theorem 4.23 along the unique path from $t$ to $t'$ in $T_n$ (with mutations always taken with respect to the tropical sign from Definition 4.7). We have seen that certain sequences of cluster mutations simply permute the coordinates of the corresponding cluster charts (c.f. Definition 4.16), let $(t_1, \ldots, t_w; \sigma)$ be such a periodicity of cluster mutations. These lift to simple permutations of the coordinates on the corresponding groupoid charts.

**Proposition 4.25.** Given any periodicity $(t_1, \ldots, t_w; \sigma)$ of cluster mutations, the following equalities hold:

1. $(\mu_{X, r}^{t_1, t_1})^* (q_{\sigma(t_1); t_1}) = q_{\sigma(t_1); t_1}$ for $1 \leq \ell \leq n$;
2. $(\mu_{X, r}^{t_1, t_1})^* (t_{\sigma(t_1); t_1}) = t_{\sigma(t_1); t_1}$ for $1 \leq \ell \leq n$;
3. $(\mu_{X, r}^{t_1, t_1})^* (v_{\sigma(t_1); t_1}) = v_{\sigma(t_1); t_1}$ for $1 \leq \ell \leq n$;
4. $(\mu_{X, r}^{t_1, t_1})^* (a_{\ell, t_1; t_1}) = a_{\sigma(t_1); t_1}$ for $1 \leq \ell \leq n$ and $1 \leq j \leq r_\ell - 1$.
5. $(\mu_{X, r}^{t_1, t_1})^* (p_{\sigma(t_1); t_1}) = p_{\sigma(t_1); t_1}$ for $1 \leq j \leq m$;
6. $(\mu_{X, r}^{t_1, t_1})^* (s_{\ell, t_1; t_1}) = s_{\ell, t_1; t_1}$ for $1 \leq j \leq m$;
7. $(\mu_{X, r}^{t_1, t_1})^* (u_{\sigma(t_1); t_1}) = u_{\sigma(t_1); t_1}$ for $1 \leq j \leq m$.

**Proof.** Recall that the identity map of a Poisson manifold lifts to the identity map of its source-simply-connected symplectic groupoid (c.f. Proposition 2.9), so parts (1) and (5) immediately follow together with part (4) for the mutations on $G_{X, r}$ and $G_{A, r}$. If we restrict to the case of $\mathbb{R}_+$, then the blow-up groupoid $B$ and the symplectic double $D$ are both source-simply-connected also. Thus parts (2), (3), (6), (7) and (4) follow from Proposition 2.9; the case of $\mathbb{C}_*$ follows when we complexify the mutations. 

Thus it is reasonable to give the following definition.
Definition 4.26. Fix an initial $m \times n$ exchange matrix $\hat{B}_{t_0}$ and an initial skew-symmetric $m \times m$ matrix $\Omega_{t_0}$ which is $D$-compatible with $\hat{B}_{t_0}$. Define groupoids $\mathcal{G}_X$, $\mathcal{B}_X$, $\mathcal{D}_X$ and $\mathcal{G}_A$, $\mathcal{B}_A$, $\mathcal{D}_A$ over the cluster varieties $X_r$ and $A_r$ respectively by gluing the groupoid charts $\mathcal{G}_{X,r,t'}$, $\mathcal{B}_{X,r,t'}$, $\mathcal{D}_{X,r,t'}$ to $\mathcal{G}_{X,r',t''}$, $\mathcal{B}_{X,r',t''}$, $\mathcal{D}_{X,r',t''}$ respectively and by gluing the groupoid charts $\mathcal{G}_{A,r,t'}$, $\mathcal{B}_{A,r,t'}$, $\mathcal{D}_{A,r,t'}$ to $\mathcal{G}_{A,r',t''}$, $\mathcal{B}_{A,r',t''}$, $\mathcal{D}_{A,r',t''}$ respectively, for $t,t' \in \mathbb{T}_n$, along the appropriate groupoid mutations $\mu_{X,t}$ and $\mu_{A,t}$ (where mutations are always performed according to the tropical signs from Definition 4.7).

Remark 4.27. Recall that the symplectic groupoid charts $\mathcal{G}_{X,r,t}$ and $\mathcal{G}_{A,r,t}$ were constructed in Section 3 using Poisson sprays. Ideally these Poisson sprays would be compatible with mutations and thus glue to give a global construction of the source-simply-connected symplectic groupoids over the varieties $X_r$ and $A_r$. Unfortunately, this is not the case and we must make due with the iterative construction presented above.

To understand the iterated mutations $\mu_{t_0}^{t_{t_0}}$, we introduce analogues of the “separation of additions” formulas from Theorem 4.14. As a first step in lifting the separation of additions formulas (4.15) and (4.16) to the level of groupoids we introduce analogues of the $c$-vectors and $g$-vectors which describe iterations of the cluster mutations. To begin let $C_{t_0}^\vee$ be an $m \times m$ identity matrix and let $G_{t_0}^\vee$ be an $n \times n$ identity matrix. We then assign matrices $C_{t}^\vee := (C_{ij,t}^\vee)$ and $G_{t}^\vee := (G_{ij,t}^\vee)$ to the vertices $t \in \mathbb{T}_n$ by the following recursion. For vertices $t, t' \in \mathbb{T}_n$ joined by an edge labeled $k$, the matrices $C_{t}^\vee$ and $C_{t'}^\vee$ are related by

\[ C_{ij,t}^\vee = \begin{cases} C_{ij,t}^\vee & \text{if } i = k; \\ C_{ij,t}^\vee + [-\varepsilon_{r,k,t}B_{tk,t}]C_{kj,t}^\vee & \text{if } i \neq k; \end{cases} \tag{4.25} \]

while the matrices $G_{t}^\vee$ and $G_{t'}^\vee$ are related by

\[ G_{ij,t'}^\vee = \begin{cases} G_{ij,t'}^\vee & \text{if } i = k; \\ G_{ij,t'}^\vee + \sum_{\ell=1}^n [\varepsilon_{r,k,t'}B_{k\ell,t}]G_{\ell j,t}^\vee & \text{if } i \neq k. \end{cases} \tag{4.26} \]

Remark 4.28. An easy induction shows that $C_{ij,t}^\vee$ is independent of $t$ for $1 \leq i \leq m$ and $n + 1 \leq j \leq m$. In particular, $C_{ij,t}^\vee = \delta_{ij}$ for $1 \leq i \leq m$ and $n + 1 \leq j \leq m$.

Lemma 4.29. For $1 \leq k, \ell \leq n$, we have $d_k C_{k\ell,t}^\vee = d_\ell C_{tk,t}$ and $d_k G_{k\ell,t}^\vee = d_\ell G_{tk,t}$ for all $t \in \mathbb{T}_n$.

Proof. This is an easy induction using the recursions (4.12) and (4.25) or (4.13) and (4.26) together with the identities $d_k B_{k\ell,t} = -d_\ell B_{\ell k,t}$ for all $t \in \mathbb{T}_n$. \qed

Using these we have the following analogues of the separation of additions formulas describing the cluster coordinates.
Theorem 4.30. For any vertex $t \in T_n$, we have the following formulas for the iterated mutations $\mu_{t_0, t}$, where each $\beta$ below denotes the target map of the appropriate groupoid:

$$
(\mu_{t, t_0}^*)^\ast(q_{k; t}) = \frac{\sum_{\ell=1}^n G_{k:t_0}^\beta(y_{\ell:t_0}, q_{\ell:t_0}) + \frac{1}{d}\log \left(\frac{\beta(F_{r, t}(y_{t_0}, \ldots, y_{t_0}))}{F_{r, t}(y_{t_0}, \ldots, y_{t_0})}\right)}{\prod_{\ell=1}^n G_{k:t_0}^\beta(y_{\ell:t_0}, q_{\ell:t_0})} ;
$$

$$
(\mu_{t, t_0}^*)^\ast(t_{k; t}) = \left(\prod_{\ell=1}^n G_{k:t_0}^\beta(y_{\ell:t_0} + 1)\right)^{1/d_k} \left(\frac{\beta(F_{r, t}(y_{t_0}, \ldots, y_{t_0}))}{F_{r, t}(y_{t_0}, \ldots, y_{t_0})}\right) - 1 ;
$$

$$
(\mu_{t, t_0}^*)^\ast(v_{k; t}) = \left(\prod_{\ell=1}^n \left(y_{\ell:t_0}\right)^{G_{k:t_0}^\beta(y_{\ell:t_0}, q_{\ell:t_0})} + \frac{1}{d}\log \left(\frac{\beta(F_{r, t}(y_{t_0}, \ldots, y_{t_0}))}{F_{r, t}(y_{t_0}, \ldots, y_{t_0})}\right)\right)^{-1} ;
$$

$$
(\mu_{t, t_0}^*)^\ast(p_{k; t}) = \left(\prod_{j=1}^m C_{j:t_0}^{y_{j:t_0}, t_0} \prod_{\ell=1}^n \left(\frac{\beta(F_{r, t}(y_{t_0}, \ldots, y_{t_0}))}{F_{r, t}(y_{t_0}, \ldots, y_{t_0})}\right)\right)^{B_{t_0}/d_k} ;
$$

$$
(\mu_{t, t_0}^*)^\ast(s_{k; t}) = \left(\prod_{j=1}^m C_{j:t_0}^{y_{j:t_0}, t_0} \prod_{\ell=1}^n \left(\frac{\beta(F_{r, t}(y_{t_0}, \ldots, y_{t_0}))}{F_{r, t}(y_{t_0}, \ldots, y_{t_0})}\right)\right)^{B_{t_0}/d_k} ;
$$

$$
(\mu_{t, t_0}^*)^\ast(u_{k; t}) = \left(\prod_{j=1}^m \left(x_{j:t_0} u_{j:t_0} + 1\right) C_{j:t_0}^{y_{j:t_0}, t_0} \prod_{\ell=1}^n \left(\frac{\beta(F_{r, t}(y_{t_0}, \ldots, y_{t_0}))}{F_{r, t}(y_{t_0}, \ldots, y_{t_0})}\right)\right)^{B_{t_0}/d_k} - 1 .
$$

The coordinates $a_{t, j,t}$ on $T^*L_r$ transform according to

$$
(\mu_{t, t_0}^*)^\ast(a_{t, j,t}) = a_{t, j,t_0}^\ast + \sum_{i, k_i = \ell} \frac{Z_{t_0, t}^\beta(y_{t_0, t_0})}{Z_{t_0, t}^\beta(y_{t_0, t_0})} d\left(\frac{\beta(F_{r, t}(y_{t_0, \ldots, y_{t_0}}))}{F_{r, t}(y_{t_0, \ldots, y_{t_0}})}\right) \left(\frac{Z_{t_0, t}^\beta(y_{t_0, t_0})}{Z_{t_0, t}^\beta(y_{t_0, t_0})}\right)^{-1} d_u ;
$$

$$
(\mu_{t, t_0}^*)^\ast(a_{t, j,t}) = a_{t, j,t_0}^\ast + \sum_{i, k_i = \ell} \frac{Z_{t_0, t}^\beta(y_{t_0, t_0})}{Z_{t_0, t}^\beta(y_{t_0, t_0})} d\left(\frac{\beta(F_{r, t}(y_{t_0, \ldots, y_{t_0}}))}{F_{r, t}(y_{t_0, \ldots, y_{t_0}})}\right) \left(\frac{Z_{t_0, t}^\beta(y_{t_0, t_0})}{Z_{t_0, t}^\beta(y_{t_0, t_0})}\right)^{-1} d_u ;
$$

where

- $t_0$ and $t$ are connected through vertices $t_1, \ldots, t_w$ by mutations in directions $k_0, k_1, \ldots, k_w$;
- we set $a_{t, j,t_0} = a_{t_0, j,t}$ if there are an even number of $i$ so that $k_i = \ell$ while we take $a_{t, j,t_0} = a_{t_0, j,t}$ if there is an odd number of $i$ so that $k_i = \ell$;
- we set $j^\ast = j$ if there are an even number of $i'$ with $i \leq i' \leq w$ so that $k_{i'} = \ell$ while $j^\ast = j^\ast$ if there is an odd number of $i'$ with $i \leq i' \leq w$ so that $k_{i'} = \ell$.

Proof. We prove only equation (4.27), working by induction on the distance from $t_0$ to $t$ inside $T_n$ in this case. First consider when $t$ is joined to $t_0$ by an edge labeled $k$. Then $F_{r, t_0} = Z_{t_0, t_0}^\beta(y_{t_0})$ and $F_{r, t_0} = 1$ for $\ell \neq k$ so that (4.27) just becomes the mutation formula (4.24).
Now suppose $t'$ and $t$ are joined by an edge labeled $k$ with $t$ lying on the unique shortest path from $t_0$ to $t'$. Then $\mu_{A,r}^{t,t_0} = \mu_{A,r}^{t',t_0} \circ \mu_{A,r}^{t,t_0}$ so that
\[
(\mu_{A,r}^{t,t_0})^* (s_{k;t'}) = (\mu_{A,r}^{t,t_0})^* (\mu_{A,r}^{t',t_0})^* (s_{k;t'}) = (\mu_{A,r}^{t,t_0})^* (s_{k;t'}) = (\mu_{A,r}^{t',t_0})^* \left( \prod_{j=1}^{m} \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} \right)^{-B_{k;\ell}/d_\ell} = \left( \prod_{j=1}^{m} \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} \right)^{-B_{k;\ell}/d_\ell},
\]
where the last equality used that $\ell \neq k$ in the product since $B_{kk;\ell} = 0$. On the other hand, writing $\varepsilon = \varepsilon_{r,k;\ell}$ below, for $i \neq k$ we have
\[
(\mu_{A,r}^{t,t_0})^* (s_{i;t'}) = (\mu_{A,r}^{t,t_0})^* (\mu_{A,r}^{t',t_0})^* (s_{i;t'}) = (\mu_{A,r}^{t,t_0})^* \left( s_{i;t} s_{k;\ell}^{-\varepsilon B_{k;\ell,r} r_k} \right) \left( \frac{Z_{k;\ell}^{\varepsilon}(\hat{y}_{k;\ell} s^{-d_\ell}_{k;\ell})}{Z_{k;\ell}^{\varepsilon}(\hat{y}_{k;\ell})} \right)^{-B_{k;\ell}/d_k}.
\]
We handle the “c-vector part” of the above expression first. Applying (4.27) to the first two terms above and recording only the c-vector part gives
\[
\left( \prod_{j=1}^{m} C_{i,j;0}^{t,t_0} \right) \left( \prod_{j=1}^{m} C_{i,j;0}^{t',t_0} \right) \left( [-\varepsilon B_{k;\ell} r_k]_{+} \right) = \left( \prod_{j=1}^{m} s_{j;0}^{-\varepsilon B_{k;\ell} r_k} C_{i,j;0}^{t,t_0} \right) = \prod_{j=1}^{m} s_{j;0}^{-\varepsilon B_{k;\ell} r_k} C_{i,j;0}^{t,t_0},
\]
which is the desired c-vector part of $(\mu_{A,r}^{t',t_0})^* (s_{i;t'})$. Meanwhile the F-polynomial part of this is given by
\[
\prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} = \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})},
\]
where the final exponent above is given by (4.7) and by skew-symmetrizability this may be rewritten as $B_{i;\ell}/d_\ell - B_{k;\ell} [\varepsilon B_{k;\ell} r_k]_{+}/d_k$. On the other hand, considering only $(\mu_{A,r}^{t,t_0})^* (s_{c;0}^{\varepsilon B_{k;\ell} r_k} C_{c;0}^{t,t_0})$ gives
\[
\prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} = \prod_{\ell=1}^{n} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})},
\]
where the last equality used Lemma 4.29. Thus combining the F-polynomial part above with the expression $(\mu_{A,r}^{t,t_0})^* \left( \frac{Z_{k;\ell}^{\varepsilon}(\hat{y}_{k;\ell} s^{-d_\ell}_{k;\ell})}{Z_{k;\ell}^{\varepsilon}(\hat{y}_{k;\ell})} \right)^{-B_{k;\ell}/d_k}$ and applying the F-polynomial recursion (4.14) gives the total F-polynomial part
\[
\prod_{\ell \neq k} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} = \prod_{\ell \neq k} \frac{F_{R,\ell,t}(\hat{y}_{1;0} s^{-d_1}_{1;0}, \ldots, \hat{y}_{n;0} s^{-d_n}_{n;0})}{F_{R,\ell,t}(\hat{y}_{1;0}, \ldots, \hat{y}_{n;0})} 
\]

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But $B_{ik;i'} = -B_{k;i}$ so that this final expression gives the desired $F$-polynomial part of $(\mu^i_A)^* (s_{i;i'})$. □

**Remark 4.31.** Combining the periodicity result from Proposition 4.25 and the iterated mutation formula (4.28) for $T^* L_r$, we obtain the following collection of identities, one for each $1 \leq j \leq r$, and any periodicity $(t_1, \ldots, t_w; \sigma)$ of cluster mutations in which $t_i$ is related to $t_{i+1}$ by mutation in direction $k_i$:

\[
\sum_{i:k_i = \ell} \varepsilon_i \frac{d}{dt} \int_{\mu^i_{X,r}} \left( \begin{array}{c}
\mu^i_{X,r} \\
\mu^i_{X,r}
\end{array} \right)^* \left( \begin{array}{c}
\gamma_{t_i}^j \\
\gamma_{t_i}^j
\end{array} \right) \frac{u^{x_i} i^{j-1}}{Z_{\ell,t_i} (u^{x_i})} \, du = 0;
\]

\[
\sum_{i:k_i = \ell} \varepsilon_i \frac{d}{dt} \int_{\mu^i_{X,r}} \left( \begin{array}{c}
\mu^i_{X,r} \\
\mu^i_{X,r}
\end{array} \right)^* \left( \begin{array}{c}
\gamma_{t_i}^j \\
\gamma_{t_i}^j
\end{array} \right) \frac{u^{x_i} i^{j-1}}{Z_{\ell,t_i} (u^{x_i})} \, du = 0;
\]

where $j^* = j$ if there are an even number of $i'$ with $1 \leq i' \leq i-1$ so that $k_{i'} = \ell$ while $j^* = j$ if there is an odd number of $i'$ with $1 \leq i' \leq i-1$ so that $k_{i'} = \ell$ and $\varepsilon_i$ denotes the tropical sign in the mutation from $t_i$ to $t_{i+1}$ along the unique mutation sequence from $t_1$ to $t_w$.

Similar considerations using charts of the form $L_X \times T^* L_r$ yield the analogous identities:

\[
\sum_{i:k_i = \ell} \varepsilon_i \frac{d}{dt} \int_{\mu^i_{X,r}} \left( \begin{array}{c}
\mu^i_{X,r} \\
\mu^i_{X,r}
\end{array} \right)^* \left( \begin{array}{c}
\gamma_{t_i}^j \\
\gamma_{t_i}^j
\end{array} \right) \frac{u^{x_i} i^{j-1}}{Z_{\ell,t_i} (u^{x_i})} \, du = 0.
\]

These identities are closely related to the connection between periodicities of cluster mutations and Roger’s dilogarithm identities. More precisely, taking the derivative with respect to $x_t$ on the left-hand side of the Roger’s dilogarithm identity [36, Equation (4.28)] results in a number of integral terms coming from taking these derivatives in the Euler dilogarithm terms (c.f. [36, Equation (3.5)]) present in [36, Equation (4.28)]. Equation (4.32) states that the sum of these terms is equal to zero, the remaining collection of logarithm terms also summing to zero by [36, Equation (4.28)]. The vanishing of these logarithm terms seems closely related to the constancy condition [36, Theorem 4.4], it would be interesting to make this connection precise.

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