Universality and Non-Perturbative Definitions of 2D Quantum Gravity from Matrix Models

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ABSTRACT

The universality of the non-perturbative definition of Hermitian one-matrix models following the quantum, stochastic, or $d = 1$-like stabilization is discussed in comparison with other procedures. We also present another alternative definition, which illustrates the need of new physical input for $d = 0$ matrix models to make contact with 2D quantum gravity at the non-perturbative level.

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1. INTRODUCTION

The double-scaling limit of the matrix models provides a regularized version of the sum over arbitrary surfaces and, consequently, of 2D quantum gravity\(^1,2\). In the case of the Hermitian one-matrix models the result, by now well-known, is that there is an infinite number of local operators, \(\hat{O}_k\), \(k = 0, 1, \ldots\), which realize a KdV flow structure\(^2,3\), and whose simplest operator is the puncture \(O_0 \equiv \hat{P}\). Moreover, the specific heat of the model satisfies a differential equation known as the “string equation”, whose scaling solutions in the cosmological constant are the multicritical models\(^4\). Introducing the couplings of the operators \(\hat{O}_k\), \(t_k\), the string equation is

\[
\sum_{k=1}^{\infty} (2k + 1)t_k R_k[f] = -T, \tag{1.1}
\]

where \(T \equiv t_0\) is the renormalized cosmological constant, \(f = d^2 \ln Z/dT^2 \equiv \langle \hat{P} \hat{P} \rangle\) is the specific heat, \(R_k\) are the Gelfand–Dikii polynomials that define the KdV flows

\[
\frac{\partial f}{\partial t_k} \equiv \langle \hat{P} \hat{P} \hat{O}_k \rangle = R'_k \tag{1.2}
\]

and primes denote derivatives with respect to \(T\). The \(k\)th multicritical model, which is defined by the asymptotic behaviour \(f \approx T^{1/k} + \ldots\) for large positive values of \(T\), corresponds to \(t_j = (2k + 1)^{-1}\delta_{j,k}\). The simplest one realized in the matrix model is pure gravity \((k = 2)\), whose string equation is the Painlevé I equation

\[
f^2 - \frac{1}{3} f'' = T. \tag{1.3}
\]

In any of these models, the contribution of the surfaces with an arbitrary fixed genus to the specific heat is obtained by expanding the solution of the string equation in powers of the string coupling constant, which is related to the cosmological constant, \(g_{\text{str}} \propto T^{-\frac{2k+1}{2k}}\). It was also expected that the string equation, which, strictly speaking, is only valid order by order in the genus expansion, would define 2D quantum gravity at the non-perturbative level as well. This would require to fix the string equation’s boundary conditions. Nevertheless, although it is possible for the odd-\(k\) models\(^5,6\), the even-\(k\) ones present additional complications. In the particular case of pure gravity, it has been argued that no real solution of the Painlevé I equation can be compatible with the loop equations (or Schwinger–Dyson equations) of the matrix model\(^7\). Furthermore, the behaviour of the (KdV) flow between the (well defined) \(k = 3\) and \(k = 2\) models indicates the non-perturbative instability of pure gravity\(^8\).
The definition of the models with even-\( k \) involves matrix models whose potentials are not bounded from below, and this seems to be the origin of the problems\(^5\). Consequently, the two standard procedures to stabilize field theories with unbounded actions have been tried in this context. The first method is the analytic continuation of the dominant term of the potential from positive to negative sign, while a simultaneous deformation of the contour of integration on the matrix eigenvalues is performed\(^9\). Unfortunately, this method leads to a non-perturbative imaginary part in the specific heat, which makes the result unphysical\(^7,10,11\). Using the second generic method, inspired in stochastic quantization and proposed in general by Greensite and Halpern\(^12\), the ill-defined (\( d = 0 \)) matrix model is formulated like a well-defined (\( d = 1 \)) quantum mechanical system\(^13,14,15\). Following this method, the specific heat remains real, in contrast with the result of the analytic continuation.\(^a\)

In addition to its relevance in the context of the matrix models, the study of the stochastic-like or quantum mechanical method is interesting by itself, as a non-trivial example of the general stabilization mechanism proposed in ref.\([12]\). So far, the universality of this regularization has not been fully understood. In fact, all the previous papers concentrate on the definition of pure gravity using the simplest potentials of degree 3 or 4, either in the WKB approximation\(^13,14\), or in numerical computations\(^15,17\). The main reason for this lack of generality is that almost all the calculations rely on the fact that the quantum mechanical system is an ideal Fermi gas for the simplest potentials of degree less than 4. In the next section 2, we show that it is possible to recover the ideal Fermi gas picture for an arbitrary potential through the use of a mean field approximation (à la Hartree–Fock), which agrees with the semiclassical limit of the matrix model given by the loop equations. This result ensures the universality of this regularization procedure for any given multicritical model (not only pure gravity), and provides a connection between the quantum (Fokker–Planck) potential and the semiclassical density of eigenvalues. Using this connection, we show in section 3 that the dominant non-perturbative effects in the quantum mechanical regularization of pure gravity are given by a metastable instanton (bounce) whose lifetime is universal and of the size of the non-perturbative ambiguities in the specific heat. These results ensure, in particular, that the method of Greensite and Halpern is compatible with the double-scaling limit.

\(^a\) In this paper, we shall not be interested in the supersymmetric aspects of the original proposal of Marinari and Parisi\(^13,16\).
The problem of pure gravity and of all the even-$k$ models is that their genus expansions for the specific heat are not Borel-summable; hence, they are not well defined at the non-perturbative level. Therefore, one should not be surprised to have many different “definitions” of pure gravity if the agreement with its genus expansion is the only selecting criterion. In fact, a third consistent definition has been proposed recently by imposing the non-perturbative agreement with the KdV flow structure of eqs. (1.1) and (1.2). This proposal provides a real specific heat too, which is different from that defined by the stochastic-like regularization. Obviously, some new physical insight into non-perturbative 2D quantum gravity is required to select which is the right definition. A good example of a true physical constraint is the unitarity of the $S$-matrix in the $d = 1$ matrix model, where a clear spacetime interpretation is available, in contrast with the $d = 0$ case. In fact, in $d = 0$, the agreement with the KdV flow structure could be a good criterion, but the matrix models provide the KdV structure only at the perturbative level (genus by genus), and it can be broken through non-perturbative contributions. To conclude the paper, we discuss in section 4 the solutions of the Hermitian and anti-Hermitian 1-matrix models with generic potentials that illustrate this last point. Furthermore, we argue that the solution of the anti-Hermitian model is the simplest way to accommodate the complex solutions of the Painlevé equation that have been considered in the analytic continuation method, and provides another consistent definition of pure gravity. This shows, once more, that the real problem is our ignorance about quantum gravity at the non-perturbative level, and that the matrix models alone do not solve these ambiguities.

2. WKB APPROXIMATION AND THE SEMICLASSICAL MATRIX MODEL

In the stochastic-like regularization method, it has been proved that the perturbative expansion of the equivalent quantum mechanical system is the same as that of the original (unstable) model provided that they agree at the first order. In the case of pure gravity, this has already been checked using the lowest-order potentials, where the quantum system is an ideal Fermi gas of $N$ particles. In this section, we show that the quantum mechanical regularization reproduces, in the WKB approximation, the semiclassical limit of the matrix model not only in the case of pure gravity, but for any multicritical model. The main difficulty is the fact that the potentials with degree $\geq 4$ induce interactions between the
fermions, and the Fermi gas is no longer ideal. Nevertheless, a mean field approximation can
be used to decouple the Fermi gas in the WKB approximation.

Let us consider the $d = 0$ Hermitian one-matrix model, whose partition function is

$$Z = \int D\phi e^{-\beta V(\phi)}, \quad (2.1)$$

where $\phi$ is an $N \times N$ Hermitian matrix, and $V(\phi)$ is a generic potential of degree $L$

$$V(\phi) = \sum_{n=2}^{L} g_n \text{Tr}(\phi^n). \quad (2.2)$$

Following refs.[14,15], this model can be formulated like the ground-state of a $d = 1$ quantum mechanical system, whose Hamiltonian is the (positive semi-definite) Fokker–Planck Hamiltonian

$$H_{FP} = \text{Tr}(P^2) + W_{FP} \text{ with } P_{ij} = -i \frac{\partial}{\partial\phi_{ji}} \text{ and } W_{FP} = \frac{\beta^2}{4} \frac{\partial V}{\partial\phi_{ij}} \frac{\partial V}{\partial\phi_{ji}} - \frac{\beta}{2} \frac{\partial^2 V}{\partial\phi_{ij} \partial\phi_{ji}}. \quad (2.3)$$

This means that the expectation value of a generic operator $Q(\phi)$ in the matrix model corresponds to the vacuum-expectation-value (VEV) of the quantum operator $Q(\hat{\phi})$:

$$\langle Q \rangle \equiv \langle 0 | Q(\hat{\phi}) | 0 \rangle = \int D\phi \Psi_0^2(\phi) Q(\phi), \quad H_{FP} \Psi_0(\phi) = E_0 \Psi_0(\phi), \quad (2.4)$$

where $E_0$ and $\Psi_0(\phi)$ are the energy and the wave function of the ground-state. $E_0 = 0$ corresponds to the case of potentials bounded from below, whose matrix models are well defined, and $\Psi_0(\phi) = \exp(-\beta V(\phi)/2)$. Otherwise, if $E_0 > 0$, eq.(2.4) defines $\langle Q \rangle$ in terms of the true ground-state. Notice that the extra dimension introduced is just an auxiliary degree of freedom without interpretation within the matrix model. In fact, the correlators of operators taken at different times in the quantum theory do not have any meaning in terms of the matrix model. At this point, it is worthwhile to mention that the ground-state energy $E_0$ is not the partition function of the $d = 0$ matrix model. In fact, this method only provides closed expressions for the correlators. Nevertheless, it is easy to obtain directly the derivatives of the matrix model partition function with respect to the cosmological constant, i.e., the correlation functions of the puncture operator $\hat{P}$. They are the universal scaling parts of the correlation functions of $\text{Tr}\phi^n$, for any finite value of $n$ (see eq.(2.20)). Therefore, one could add a perturbation to the Fokker–Planck Hamiltonian, $H_{FP} \rightarrow H_{FP} + J\text{Tr}(\phi^n)$, such that the derivative of the perturbed ground-state energy $E_0(J)$ with respect to the source $J$, evaluated at $J = 0$, gives the correlation function of the puncture $17$. 
Following the well-known techniques of ref. [22], it is natural to make a change of variables to the eigenvalues \( \{ \lambda_i \} \) of the matrix \( \phi \), by introducing the effective ground-state wave function

\[
\Psi_{0}^{\text{eff}}(\{ \lambda_i \}) = \prod_{i<j}(\lambda_i - \lambda_j) \Psi_0(\phi),
\]  

(2.5)

which is totally antisymmetric, and describes a gas of \( N \) Fermi particles. In general, the Fokker–Planck potential, \( W_{FP} \), can be splitted into its diagonal \((D)\) and non-diagonal \((ND)\) parts in terms of the eigenvalues

\[
W_{FP} = W_{FP}^{(D)} + W_{FP}^{(ND)}
\]

\[
W_{FP}^{(D)} = \frac{\beta^2}{4} \sum_{i=1}^{N} \left( (V'(\lambda_i))^2 - 4X \left( g_2 + \sum_{n=3}^{L} n g_n \lambda_i^{n-2} \right) \right)
\]

\[
W_{FP}^{(ND)} = -\frac{\beta}{2} \sum_{i,j=1}^{N} \left( \sum_{n=4}^{L} n g_n \sum_{s=0}^{n-4} \lambda_i^{s+1} \lambda_j^{n-3-s} \right),
\]

(2.6)

where \( X = N/\beta = e^{\gamma_0} \) is related to the (bare) 2D cosmological constant in the usual way. Obviously the \( ND \) piece does not vanish if \( L \geq 4 \), and the Fermi gas is not decoupled. Nevertheless, in the semiclassical WKB limit \((\beta \approx \hbar^{-1} \to \infty)\), a mean field approximation \((\text{à la Hartree–Fock})\) may be performed to decouple the system. We show below that

\[
\text{Tr}(\phi^k)\text{Tr}(\phi^p) \approx N \left( \omega_k \text{Tr}(\phi^p) + \omega_p \text{Tr}(\phi^k) - N \omega_k \omega_p \right) + \cdots
\]

(2.7)

where the normalization is fixed by

\[
\langle \text{Tr}(\phi^k)\text{Tr}(\phi^p) \rangle_c \approx N^2 (\omega_k \omega_p + O(1/N)) \quad \text{and} \quad \omega_k = \frac{1}{N} \langle \text{Tr}(\phi^k) \rangle_c,
\]

(2.8)

is consistent with the semiclassical limit of the matrix model. Under this approximation, the quantum mechanical system becomes an ideal Fermi gas of \( N \) particles whose Hamiltonian is

\[
H_{FP} \approx \sum_{i=1}^{N} h_{FP}(\lambda_i), \quad \text{with} \quad h_{FP}(\lambda) = -\frac{\partial^2}{\partial \lambda^2} + \frac{\beta^2}{4} U_{FP}(\lambda), \quad \text{and}
\]

\[
U_{FP}(\lambda) = (V'(\lambda))^2 - 4X \left( \sum_{i=1}^{L-2} \left( \sum_{j=i+2}^{L} j g_j \omega_{j-i-2} \right) \lambda^i + g_2 - \frac{1}{2} \sum_{i=4}^{L} i g_i \sum_{j=2}^{i-2} \omega_{j-1} \omega_{i-j-1} \right)
\]

(2.9)

The effective potential \( U_{FP} \) is bounded from below, and the one-fermion Hamiltonian has a well-defined discrete spectrum, \( h_{FP} \psi_n(\lambda) = \frac{\beta^2}{4} c_n \psi_n(\lambda) \). The ground-state wave function is
the Slater determinant of the first $N$ eigenfunctions, and the vacuum expectation value of a generic operator $Q(\phi)$ is

$$
\langle 0 | Q(\hat{\phi}) | 0 \rangle = \int [d\lambda] \det^2 (\psi_{i-1}(\lambda_j)) Q(\{\lambda\}), \quad i, j = 1, \ldots, N, \tag{2.10}
$$

to be compared with the corresponding expression in the matrix model

$$
\langle Q(\phi) \rangle = \frac{1}{Z} \int [d\lambda] \det^2 \left( e^{-\beta V(\lambda)} \lambda_j^{i-1} \right) Q(\{\lambda\}), \quad i, j = 1, \ldots, N. \tag{2.11}
$$

All the relevant information about the $d = 1$ quantum mechanical system is contained in the particle density, $\rho(\lambda, e) = \langle \lambda | \delta \left( \hbar F P - \frac{\beta^2}{4} e \right) | \lambda \rangle$, whose normalization fixes the Fermi energy $e_F$:

$$
N = \int d\lambda \int_{-\infty}^{e_F} de \rho(\lambda, e). \tag{2.12}
$$

The energy integral of $\rho(\lambda, e)$ provides the quantum-mechanical version of the matrix model semiclassical density of eigenvalues

$$
u(\lambda) = \frac{1}{\beta} \int_{-\infty}^{e_F} de \rho(\lambda, e), \quad X = \frac{N}{\beta} = \int d\lambda \, \nu(\lambda), \tag{2.13}
$$

which, in the WKB approximation, is

$$
u^{WKB}(\lambda) = \frac{1}{2\pi} \sqrt{e_F - U_{FP}(\lambda)} \theta(e_F - U_{FP}(\lambda)) \tag{2.14}
$$

In order to compare the WKB result, eq.(2.14), with the matrix model, we shall use the semiclassical limit of the Schwinger–Dyson loop equations. In the semiclassical (planar) limit, the generating function of monomial expectation values

$$
F(p) = \frac{1}{\beta} \langle \Tr \frac{1}{p - \phi} \rangle_c \tag{2.15}
$$
satisfies the Schwinger–Dyson equation

$$
F(p)^2 - V'(p) F(p) + X \sum_{i=0}^{L-2} \left( \sum_{j=i+2}^{L} jg_j \omega_{j-i-2} \right) p^i = 0, \tag{2.16}
$$

where the “constants of integration” $\omega_k$ have been already defined in eq.(2.8). The solution
of this equation is

\[ F(p) \equiv \int d\lambda \frac{u^{SC}(\lambda)}{p - \lambda} = \frac{1}{2} \left( V'(p) - \sqrt{\Delta(p)} \right) \]

\[ \Delta(p) = (V'(p))^2 - 4X \sum_{i=0}^{L-2} \left( \omega_{j-2} \sum_{j=i+2}^{L} jg_j \omega_{j-2} \right) p^i, \tag{2.17} \]

and, for a given multicritical model, the “constants of integration” are fixed by the condition that the imaginary part of \( F(p) \) defines a proper semiclassical density of eigenvalues\(^7,\)\(^22\). Obviously, \( \Delta(p) \) is a polynomial in \( p \) and all the singularities of \( F(p) \) will be the branch cuts of \( \sqrt{\Delta(p)} \). Therefore, \( \Delta(p) \) has to satisfy the following constraints\(^7,\)\(^22,\)\(^23\): (i) \( \Delta(p) \) must have only real zeros in the complex \( p \)-plane, and (ii) \( \Delta(p) \) cannot have three consecutive odd degree zeros. Under these conditions, the semiclassical density of eigenvalues is

\[ u^{SC}(\lambda) = \frac{1}{\pi} \text{Im} \ F(\lambda) = \frac{1}{2\pi} \sqrt{-\Delta(\lambda)} \theta(-\Delta(\lambda)). \tag{2.18} \]

The branch cuts of \( F(\lambda) \), i.e., the intervals between odd degree (real) zeros of \( \Delta(\lambda) \), are the “bands” on which \( u^{SC}(\lambda) \) has support." \(^b\)

Therefore, under the above mentioned restrictions, the comparison between eqs.(2.17), (2.18) and eqs.(2.9), (2.14) shows that the WKB limit of the \( d = 1 \) Fokker–Planck Hamiltonian, with the mean field approximation of eq.(2.7), agrees with the semiclassical limit of the matrix model if the Fermi energy is

\[ \frac{e_F}{4X} = g_2 + 3g_3 \omega_1 + \sum_{i=4}^{L} \left( \omega_{i-2} + \frac{1}{2} \sum_{j=2}^{i-2} (\omega_{j-1} \omega_{i-j-1}) \right) ig_i. \tag{2.19} \]

Notice, eq.(2.8), that the “constants of integration” \( \omega_k \) are, in fact, correlation functions. Accordingly, the universal piece of the Fermi energy in the double-scaling limit can be expressed in terms of the expectation value of the puncture operator. In the case of even potentials, the precise relation between \( \omega_k \) and the puncture operator is, for the \( k \)th model\(^2\),

\[ N \omega_{2s} = \langle \text{Tr}(\phi^{2s}) \rangle \approx \text{n.u.} + s \binom{2s}{s} \beta^{-\frac{1}{2\pi \beta}} \langle \hat{P} \rangle, \tag{2.20} \]

where n.u. stands for the non-universal part. Therefore, the dominant universal piece of the

\(^b\) All the multicritical models of refs.[1,2] correspond to single-band configurations, which means that \( \Delta(p) \) has exactly two single (odd degree, in general) real zeros in the complex \( p \)-plane.
Fermi energy is

\[ e_F^{(2L)} = n.u. + \left( 4 \sum_{i=1}^{L} \frac{(2i-2)!}{(i-1)!(i-2)!} (2i g_{2i}) \right) \beta^{-\frac{2i+i+2}{2i+1}} \langle \hat{P} \rangle + \cdots \]  

(2.21)

In fact, this is the most direct way to obtain the critical exponents of the multicritical models within the quantum mechanical formalism. Moreover, the scaling factor when \( \beta \to \infty \) will dictate the relevant piece of the Fokker–Planck potential in the double-scaling limit.

Our result agrees with, and generalizes, previous results obtained for pure gravity with the simplest potentials of degree three\(^{13,14,15,17,24} \) and four (even)\(^{25} \), showing that the critical behaviour of the Fokker–Planck Hamiltonian is precisely that of the matrix model. Therefore, in the semiclassical approximation, it is possible to describe any Hermitian one-matrix model as an ideal Fermi gas of \( N \) particles, whose potential is given by the semiclassical density of eigenvalues

\[ U_{FP}(\lambda) = e_F - [2\pi u^{SC}(\lambda)]^2. \]  

(2.22)

Notice that this relationship formally holds only when the above mentioned constraints on the zeros of \( \Delta = U_{FP} - e_F \) are satisfied and \( u^{SC} \) is well defined. Nevertheless, the quantum mechanical system, and \( u^{WKB} \), is defined even when this is not the case. Such quantum mechanical configurations arise when the naïve ground-state wave function, \( \Psi_0 = \exp(-\beta V/2) \), is not normalizable in the WKB approximation either. Therefore, they do not correspond to any semiclassical configuration of the matrix model, and they contain non-perturbative information about the stabilization mechanism. In fact, their study has shown that the quantum mechanical formulation is not compatible with the KdV flows at the non-perturbative level\(^{20} \).

3. UNIVERSAL NON-PERTURBATIVE BEHAVIOUR

In the previous section, we have shown that the WKB limit of the quantum mechanical formulation agrees with the semiclassical limit of the matrix model for an arbitrary potential. If the stabilization mechanism is compatible with the double-scaling limit, this result ensures that the quantum mechanical formalism preserves the universality of the multicritical models. In this section, we show that the non-perturbative contributions are indeed universal in the case of pure gravity, and that they agree qualitatively with the ambiguities expected
in the specific heat. The size of these ambiguities is obtained through the linearization of
the Painlevé equation, eq.(1.3). The result is that the difference between any two func-
tions whose genus expansion is the pure gravity genus expansion should be proportional to
\( T^{-\frac{1}{8}} \exp \left( -\frac{4\sqrt{6}}{3} T^{5/4} \right) \) when \( T \to \infty \). We show below that the typical non-perturbative
exponential suppression corresponds to a universal metastable instanton (bounce) in the
quantum mechanical picture, which is a natural relationship in quantum mechanics when
the perturbative expansion is not Borel-summable.

To identify the universal features of the Fokker–Planck potential, the main tool we shall
use is the connection with the semiclassical density of eigenvalues, eq.(2.22), which allows
the universal behaviour of the potential to be derived from that of the semiclassical density.
From now on, we shall restrict ourselves to the case of the standard multicritical models
defined with even potentials, but the results should be easily extended to the general case.
Using the standard orthogonal polynomial techniques, we can compute the semiclassical
density of eigenvalues directly from the generating function of monomial expectation values,
eqs.(2.16), (2.17), and (2.18):

\[
F(p) = \frac{1}{\beta} \left\langle \text{Tr} \left( \frac{1}{p - \phi} \right) \right\rangle_c = \frac{1}{\beta} \sum_{n=0}^{N-1} \frac{1}{\hbar_n^2} \int d\lambda e^{-\beta V} \frac{1}{p - \lambda} P_n^2(\lambda)
\]

\[
= \frac{1}{\beta} \sum_{n=0}^{N-1} \frac{1}{\sqrt{p^2 - 4R(n)}} \rightarrow \int_0^X dx \frac{1}{\sqrt{p^2 - 4R(x)}},
\]

where \( R_n \) is the coefficient in the recurrence relation of the polynomials, \( \lambda P_n = P_{n+1} + R_n P_{n-1} \), \( x = n/\beta \), and \( R(x) \) is the limit of \( R_n \) when \( \beta \to \infty \). The imaginary part of eq.(3.1)
provides the semiclassical density of eigenvalues

\[
u^{SC}(p) = \frac{1}{\pi} \text{Im} F(p) = \frac{1}{\pi} \int_0^X dx \frac{1}{\sqrt{4R(x) - p^2}} \theta(4R(x) - p^2).
\]

Next, we make the change of variables \( x = W(R(x)) \), with \( W(R) \) defined in terms of the
potential as

\[
W(R) = \int \frac{dz}{2\pi i} V' \left( z + \frac{R}{z} \right),
\]
and the final expression for the semiclassical density is

\[
    u^{SC}(p) = \frac{1}{\pi} \int_{\sqrt{4R-p^2}/4}^{R(X)} dR \frac{W'(R)}{\sqrt{4R-p^2}} \theta(4R(x)-p^2)
\]

\[
    \equiv \frac{1}{2\pi} \sqrt{e_F - U_{FP}(p) \theta(e_F - U_{FP}(p))}.
\]

(3.4)

For the \(k\)th model, the double-scaling limit corresponds to \(\beta \to \infty\), with

\[
    W(R) = 1 - (1 - R)^k, \quad X = 1 - \beta^{-\frac{2k}{2k+1}} T, \quad \text{and} \quad R = 1 - \beta^{-\frac{2}{2k+1}} f(T).
\]

(3.5)

Therefore, the critical behaviour of \(u^{SC}\) is

\[
    u^{SC}(p) = \frac{2k}{\pi 2^{2k}} \int_{0}^{\sqrt{4R(X)-p^2}} dy \left(4 - p^2 - y^2\right)^{k-1}.
\]

(3.6)

Notice that \(U_{FP}(\lambda)\) has two roots of order \(2k-1\), \(\lambda = \pm 2\), at the critical point \(X = R = 1\). Thus, the resulting Fokker–Planck potential has degree \(4k - 2\), and it corresponds to a matrix model whose potential has degree \(L = 2k\), i.e., the canonical representative of the multicritical \(k\)th model.

The Fokker–Planck potential of pure gravity, \(k = 2\), is

\[
    U_{FP}(\lambda) - e_F = \frac{1}{9} \left(\lambda^2 - 4R(X)\right) \left(\lambda^2 - 6 + 2R(X)\right)^2.
\]

(3.7)

This potential has an absolute minimum at \(\lambda = 0\), and two relative minima at \(\lambda^2 = 6 - 2R(X)\); obviously, the Fermi energy is just the value of the potential at these relative minima. The perturbative expansion around the absolute minimum provides the WKB expansion, which reproduces the \(1/N\) expansion of the matrix model\(^{12}\). Nevertheless, we are interested in the double-scaling limit, which corresponds to \(\beta \to \infty\). We have already obtained the scaling behaviour of the Fermi energy in this limit, eq.(2.21), which is like \(\beta^{-6/5}\) for pure gravity. This behaviour corresponds to the scaling of the potential around the secondary minima, i.e., around the Fermi energy, which is\(^d\)

\[
    U_{FP}(\lambda) \approx \pm \beta^{-\frac{6}{5}} \frac{16}{9} \left(4z^3 - 3Tz\right) + \cdots
\]

(3.8)

where \(z = \beta^{\frac{2}{3}}(\lambda \mp 2)\). Notice that, in this limit, the potential becomes unbounded from

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\(^{c}\) As is usual, we have fixed the critical values \(X_c = R_c = 1\), but the result is independent of this choice\(^{28}\).

\(^{d}\) Even though we have obtained this expression starting from even potentials, it can also be derived from the cubic potential\(^{15}\), as expected from its universal character.
below. However, this does not destroy the stabilization mechanism. In fact, the subdominant terms in the $\beta \to \infty$ limit are crucial to ensure that the potential has bound-states, and, in particular, the normalization condition in eq.(2.13). On the other hand, the universal properties correspond to the behaviour of the potential around the Fermi level, and only the energy levels close to the $N^{th}$ one are relevant in this limit. Because $N \to \infty$, these levels do not feel the absolute minimum of the potential, which appear to be unbounded from below. In fact, one can check that the behaviour of the potential around $\lambda = 0$ does depend on the potential one chooses to define pure gravity, and hence it is not universal.

The dominant non-perturbative contributions are related to the classical possibility that the fermions at the Fermi level could be at the secondary minimum of the potential, falling later into the main (unbounded) well due to quantum effects (tunneling). This configuration corresponds to a metastable instanton, also known as a bounce$^{29}$, which decays through barrier penetration effects. The lifetime of the bounce is proportional to the imaginary part of its energy, which can be computed from the Euclidean partition function in the semiclassical approximation$^{29,30}$. The Euclidean partition function corresponding to the Hamiltonian of eq.(2.9), $H^{FP}(\lambda) = -\frac{\partial^2}{\partial \lambda^2} + \frac{\beta^2}{4} U_{FP}(\lambda)$, can be written as the $L \to \infty$ limit of the following functional integral

$$\text{Tr} \left( e^{-LH} \right) = \int_{q(-L/2)=q(+L/2)} [dq(t)] e^{-S[q(t)]},$$

where $S[q(t)]$ is the Euclidean action

$$S[q(t)] = \int_{-L/2}^{+L/2} dt \left( \frac{1}{4} \dot{q}^2(t) + \frac{\beta^2}{4} U_{FP}(q(t)) \right).$$

Tr$(e^{-LH})$ is given by $\sum \exp (-LE_n)$, where the sum extends over the whole spectrum. Consequently, in the $L \to \infty$ limit, the partition function is dominated by the ground-state energy. Nevertheless, the barrier penetration effects induce an imaginary part in the energy value of the bounce, $E_{bc}$, and, then, in Tr$(e^{-LH})$. The dominant contribution can be obtained by computing the imaginary part of the partition function in the semiclassical

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e Notice that this imaginary part cannot be identified with the one obtained in the analytical continuation method, because the ground-state energy of the quantum mechanical system is not the specific heat of pure gravity, which is real within the stochastic-like regularization.
approximation. If \( E_{bc} = E_F + i \Gamma \), where \( E_F = \frac{\beta^2}{4} e_F \), then, in the \( L \to \infty \) limit,

\[
\text{Im} \left( \text{Tr}(e^{-LH}) \right) \propto \text{Im} \left( e^{-L(E_F + i \Gamma)} \right) \approx L \Gamma e^{-LE_F},
\]

(3.11) where we have taken into account the smallness of the non-perturbative imaginary part.

In the semiclassical approximation, the dominant imaginary contribution to \( \text{Tr}(e^{-LH}) \) is given by the bounce, which is a solution of the Euclidean equations of motion, \( \frac{1}{2} \ddot{q}_c = \frac{\beta^2}{4} U'_{FP}(q_c) \), that starts from the secondary minimum, \( q_m \), at Euclidean time \( L = -\infty \), is reflected in the classical turning point, \( q_0 \), and comes back to the minimum at \( L = +\infty \). Therefore, the bounce action is

\[
S[q_c] = \int_{-\infty}^{+\infty} dt \left( \frac{1}{4} \dot{q}_c^2(t) + \frac{\beta^2}{4} U_{FP}(q_c) \right)
= LE_F + \int_{q_m}^{q_0} dq \sqrt{\beta^2 (U_{FP}(q) - e_F)} \equiv L E_F + S_{bc}.
\]

(3.12) The integration around \( q_c \) in the Gaussian approximation provides the bounce contribution to the partition function. This calculation involves a Jacobian factor that becomes complex because of the turning point \( q_0 \), and is proportional to \( L \) because of translation invariance in the Euclidean time. The final result is

\[
\text{Im} \left( \text{Tr}(e^{-LH}) \right) \propto L e^{-S[q_c]} = L e^{-LE_F - S_{bc}} \approx \text{Im} \left( e^{-L(E_F + ie^{S_{bc}})} \right).
\]

(3.13) Therefore, \( \Gamma \propto \exp(-S_{bc}) \) is the imaginary part of the bounce energy, and the lifetime of the bounce is \( \tau_{bc} \propto \Gamma \). In the case of pure gravity, \( S_{bc} \) remains finite in the double-scaling limit\(^{28} \), and its value is

\[
S_{bc} = \frac{4\sqrt{6} T^2}{5}.
\]

(3.14) The lifetime of the bounce is exponentially suppressed for large values of \( T \), and its value is expected to give the characteristic size of non-perturbative effects. Notice that \( \exp(-S_{bc}) \) is, in fact, the size of the ambiguities between functions whose genus expansion is that of the matrix model. Therefore, we conclude that the dominant non-perturbative effects of the quantum mechanics regularization of the matrix model are universal, and consistent with those expected for pure gravity. In fact, this checks, in a non-trivial way, that the stabilization procedure is consistent with the double-scaling limit because it ensures that the genus expansion coming from this method agrees with that of the original matrix model\(^{30} \).
For completeness, let us consider eq. (3.6) for the first well-defined multicritical model, 
\[ k = 3, \]

\[ u_{SC}(p) \propto (4R(X) - p^2)^{1/2} \left( (p^2 + R(X) - 5)^2 + 5(1 - R(X))^2 \right). \] (3.15)

In this case, \( U_{FP}(\lambda) \) has only one absolute minimum at \( \lambda = 0 \). Therefore, the non-perturbative contributions to the \( k = 3 \) model are not related to metastable states. This agrees with the expected relationship between metastability and non-Borel summability of the perturbative series in quantum mechanics. Therefore, we expect that there will be bounces in all the \( k \)-even models, and not in the \( k \)-odd ones. This conclusion gives support to the identification of the instabilities in the matrix model with instanton-like contributions\(^{10,18} \). In fact, it shows that the criterion for having or not instabilities is given by the Fokker–Planck potential \( U_{FP} \) in the double-scaling limit, and not directly by the potential of the matrix model.

To conclude this section, let us estimate the dominant non-perturbative contributions to the specific heat within the quantum mechanical formalism. The matrix model partition function is expressed in terms of the density of eigenvalues as\(^{22} \)

\[ F^{MM} = \beta^2 \int d\lambda u(\lambda) \left( \int d\mu u(\mu) |\lambda - \mu| - V(\lambda) \right). \] (3.16)

The dominant non-perturbative contributions to \( F^{MM} \) will correspond to the dominant non-perturbative modifications of the density of eigenvalues, \( \delta u(\lambda) \), which are related to the bounce and are dominant outside the classical (WKB) range. Therefore, the dominant modification of the matrix model is\(^{18} \)

\[ \delta F^{MM} = \beta^2 \int d\lambda \delta u(\lambda) \left( 2 \int d\mu u^{WKB}(\mu) |\lambda - \mu| - V(\lambda) \right) \]

\[ = -\beta^2 \int d\lambda \delta u(\lambda) \int dq \sqrt{U_{FP}(q) - e_F \theta(q - q_0)}, \] (3.17)

where eq. (2.17) has been used, and \( q_0 \) is the classical turning point at the Fermi energy, \( i.e.\), the limit of the classically allowed range of motion, eq.(3.4).

The dominant contribution to \( \delta u(\lambda) \) is related to the classical possibility that the fermions at the Fermi energy can be sitting at the secondary minimum. Therefore, one should expect \( \delta u(\lambda) \propto N \delta(\lambda - q_m) \), where the normalization is expected to be proportional to \( \beta^{-1} \), because it involves only the fermions at the Fermi energy\(^{18} \), and to \( \exp(-S_{bc}) \), because it is the
expected suppression of the non-perturbative effects.\footnote{This last factor is apparently missing from ref.\[18\].} In fact, we can check this normalization with the results of ref.\[24\], where the non-perturbative effects in the bound-state energies have been computed, and they result to be proportional to $\beta^{-6/5} \exp(-S_{bc})$. Therefore, a similar contribution to the Fermi energy, $e_F$, should be expected. This non-perturbative contribution to the Fermi energy induces a modification in the normalization condition of the semiclassical density of eigenvalues, eq.(2.13), which should be compensated by the normalization of $\delta u$

$$X = \frac{N}{\beta} = \int \frac{d\lambda}{2\pi} \sqrt{e_F - U_{FP}(\lambda)} + N. \quad (3.18)$$

If $e_F - e_{FKB}^F \propto \beta^{-6/5} \exp(-S_{bc})$, then $\delta \left( \int (d\lambda/2\pi) \sqrt{e_F - U_{FP}} \right) \propto \beta^{-1} \exp(-S_{bc})$, and the normalization of $\delta u$ behaves as expected. Therefore, we obtain that $\delta u \propto \beta^{-1} \exp(-S_{bc}) \delta(\lambda - q_m)$, and the dominant non-perturbative effects in the matrix model free-energy, within the quantum mechanical formalism, are

$$\delta F^{MM} \propto \beta^2 N \int_{q_0}^{q_m} dq \sqrt{U_{FP} - e_F} \propto S_{bc} e^{-S_{bc}}, \quad (3.19)$$

which are finite, universal, and consistent with the size of the ambiguities of pure gravity.

4. DISCUSSION AND CONCLUSIONS

As we said in the introduction, we should expect to have many different models with the same genus expansion as pure gravity or any other $k$-even multicritical model, and it is necessary to have a criterion to decide which one is describing 2D quantum gravity at the non-perturbative level. It seems very difficult to obtain such a requirement from the matrix model, where all the results are linked to the double-scaling limit. In fact, even the KdV structure, which is a very strong and important result, holds only genus by genus, and the matrix models themselves provide examples where it is broken at the non-perturbative level. In particular, we shall briefly discuss the solutions of the Hermitian and anti-Hermitian one-matrix models with generic potentials, where this is the case.

The solution of the Hermitian one-matrix models with generic potential, in the one-arc sector, is well-known\[31\]. There exist two functions, $f_H^{(\pm)} = f_H \pm g_H$, that satisfy the string
equation, eq.(1.1), and in terms of which the KdV flows can be constructed. Their sum, \( f_H \), is the specific heat of the matrix model, and \( g_H \) is a non-perturbative function whose normalization should be related to the couplings of the odd powers of the matrix in the potential. Therefore, the KdV flow structure is not realized in terms of the specific heat because of non-perturbative contributions. In the particular case of pure gravity, \( k = 2 \), the two functions \( f_H^{(\pm)} \) satisfy the Painlevé I equation, eq.(1.3), which can be written in terms of the specific heat as

\[
0 = 2f_H g_H - \frac{1}{3} g_H'' \\
T = f_H^2 - \frac{1}{3} f_H' + g_H^2.
\]  

The dominant behaviour for large positive values of \( T \) is \( f_H \approx \sqrt{T} \), as required, and \( g_H \propto T^{-1/8} \exp\left(-\frac{4\sqrt{5}}{3}T^{5/4}\right) \). Therefore, the specific heat satisfies an equation different from the Painlevé I equation, and the difference is non-perturbative. Of course, this provides an additional definition of pure gravity to those described in the introduction, but we are again restricted to the real solutions of the Painlevé I equation, and the problems pointed out by David in ref.[7] remain.

Another interesting example is the solution of the anti-Hermitian models\(^{32} \). In this case, also in the one-arc sector with a generic potential, there is one complex function, \( \chi_A = f_A + ig_A \) satisfying the string equation, and in terms of which the KdV flows can be expressed. The specific heat is given by the real part of this function, and it is obviously real, while the imaginary part is non-perturbative. The correlation functions are given by the real part of the KdV flow equations

\[
\frac{\partial f_A}{\partial t_k} = \langle \hat{P} \hat{P} \hat{O}_k \rangle = Re \left( R_k' \right). \tag{4.2}
\]

Therefore, again, the KdV flow structure is not realized in terms of the specific heat because of non-perturbative contributions. In particular, the equations defining pure gravity are now

\[
0 = 2f_A g_A - \frac{1}{3} g_A'' \\
T = f_A^2 - \frac{1}{3} f_A' + g_A^2. \tag{4.3}
\]

The definition of pure gravity from the anti-Hermitian matrix model has some nice relevant features. It was shown in ref.[7] that no real solution of the Painlevé I equation can be consistent with the loop equations of the matrix model, because all of them have poles
along the real axis. Nevertheless, this is not the case with the complex solutions, and there is a unique one without poles along the real axis and consistent with the required asymptotic behaviour of the specific heat, \( f_A \approx \sqrt{T} \) when \( T \to +\infty \): the “triply truncated solution”\(^7,\text{10}\). In contrast with the case of Hermitian matrix models, the complex solutions of the Painlevé equation are natural in the context of the anti-Hermitian models. Therefore, this definition of pure gravity using the “triply truncated solution” is expected to be consistent with the matrix model loop equations, and with the reality of the specific heat. It is also worth noticing the similarity of this solution to the one proposed in ref.[19] (compare the “Ref” in fig.(1) of ref.[11] with fig.(1) of ref.[19]). In fact, their asymptotic behaviours are the same also for the large negative values of \( T \), where \( f_A \approx 0 \), and it could be interesting to investigate their relationship further.

To summarize, we have shown that the stochastic, quantum, or \( d = 1 \)-like definition of the \( d = 0 \) matrix models preserves universality in the double-scaling limit. Moreover, the dominant non-perturbative contributions of the even-\( k \) models are expected to be related to the existence of bounces, showing their instability. Nevertheless, we would like to finish by just repeating that the essential problem, from the 2D gravity point of view, is how to choose between all the possible definitions compatible with the genus expansion of the matrix models, and that some new information about the non-perturbative behaviour is needed to solve it.

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