Hybrid multibody system method for the dynamic analysis of an ultra-precision fly-cutting machine tool

Hanjing Lu\textsuperscript{1,2,3} | Xiaoting Rui\textsuperscript{1,2} | Ziyao Ma\textsuperscript{1,3} | Yuanyuan Ding\textsuperscript{2} | Yiheng Chen\textsuperscript{2} | Yu Chang\textsuperscript{4} | Xuping Zhang\textsuperscript{3}

\textsuperscript{1}College of Engineering, Peking University, Beijing, China
\textsuperscript{2}Institute of Launch Dynamics, Nanjing University of Science & Technology, Nanjing, China
\textsuperscript{3}Department of Engineering, Aarhus University, Aarhus, Denmark
\textsuperscript{4}School of Science, Nanjing University of Science and Technology, Nanjing, China

Correspondence
Prof. Xiaoting Rui, Institute of Launch Dynamics, Nanjing University of Science & Technology, Nanjing, China.
Email: ruxt@163.net

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Abstract
The dynamics of an ultra-precision machine tool determines the precision of the machined surface. This study aims to propose an effective method to model and analyze the dynamics of an ultra-precision fly-cutting machine tool. First, the dynamic model of the machine tool considering the deformations of the cutter head and the lathe head is developed. Then, the mechanical elements are classified into M subsystems and F subsystems according to their properties and connections. The M-subsystem equations are formulated using the transfer matrix method for multibody systems (MSTMM), and the F-subsystem equations are analyzed using the finite element method and the Craig–Bampton reduction method. Furthermore, all the subsystems are assembled by combining the restriction equations at connection points among the subsystems to obtain the overall transfer equation of the machine tool system. Finally, the vibration characteristics of the machine tool are evaluated numerically and are validated experimentally. The proposed modeling and analysis method preserves the advantages of the MSTMM, such as high computational efficiency, low computational load, systematic reduction of the overall transfer equation, and generalization of its computational capability to general flexible-body elements. In addition, this study provides theoretical insights and guidance for the design of ultra-precision machine tools.

KEYWORDS
transfer matrix method for multibody systems, finite element method, Craig–Bampton reduction method, ultra-precision fly-cutting machine tool

1 | INTRODUCTION
The degree of reliance on ultra-precision machining is a measure of the manufacturing technology level achieved. Ultra-precision machining implies that the machined surface waviness error without any post-polishing must be less than one sub-micrometer or several nanometers.\textsuperscript{3} The ultra-precision machining is becoming increasingly important for modern industrial technology and high-performance products.\textsuperscript{2} The ultra-precision machine tool, whose dynamic characteristics have a significant effect on the machining quality, is the foundation for ultra-precision machining. It is critically important but challenging to develop an effective method to model and analyze the
dynamics of the ultra-precision fly-cutting machine tool to improve the machining precision.

In the current literature, the finite element method (FEM) is the most widely used method for the dynamic analysis of machine tools. Liang et al.\textsuperscript{3,4} investigated the relations between the machined surface and the dynamics of an ultra-precision machine tool by combining ABAQUS with MATLAB. Based on FEM, Liang et al.\textsuperscript{5} also proposed a state-space model of an ultra-precision machine tool and calculated the machined waviness along the cutting direction and feed direction excited by the cutting force. Zhang\textsuperscript{6} and Yao et al.\textsuperscript{7} combined ANSYS with the dynamic simulation software ADAMS to simulate the dynamic and static characteristics of a machine tool system to identify the weak parts. Chen et al.\textsuperscript{8} proposed a simplified method based on FEM to study the interaction between the machining process and the quality of the machined surface to obtain the interaction relationships. An et al.\textsuperscript{9} integrated the virtual material method with FEM to investigate the dynamic performance of an ultra-precision fly-cutting machine tool. However, in all the FEM-based dynamic analysis of machine tool systems, the orders of the global dynamic equations are so high that deriving and solving the equations are computationally expensive.\textsuperscript{10}

Research efforts have been devoted to the dynamic modeling of the machine tools, mostly focusing on the single part of the machine tools due to the lack of efficient dynamic modeling methods. An et al.\textsuperscript{11} utilized the rigid-body dynamic method to predict the motion of the spindle part in an ultra-precision fly-cutting machine tool. An et al.\textsuperscript{12} observed a low-frequency vibration signal (about 1/6 of the spindle rotation frequency), which is considered to be the source of the medium-frequency ripple on the surface. Chen et al.\textsuperscript{13} developed a complete degree-of-freedom (DOF) dynamic model of the bolt coupling system to analyze the effect of misalignment coupling faults. Yang et al.\textsuperscript{14} developed a surface topography model of the whole machine tool and analyzed the air spindle using ANSYS Workbench. Badiger et al.\textsuperscript{15-18} studied the characterization and tribological behavior of coatings and found that the surface roughness of the workpiece and the wear of the tool could be significantly improved by using coated tools.

To enhance the computational efficiency, the classical transfer matrix method (TMM) was used for the dynamic analysis of the spindle component in the machine tool.\textsuperscript{19} While the theory of TMM is suitable for a single element, challenges are encountered when using TMM for multibody systems (MSTMM), referred to as the "Rui Method."\textsuperscript{10,20,21} Compared to conventional dynamic methods, MSTMM has several advantages, such as replacing the global dynamic equations with low-order transfer equations, high computational speed, and simplified computational implementation.\textsuperscript{22} In our previous studies,\textsuperscript{23-25} the dynamic models of an ultra-precision fly-cutting machine tool consisting of rigid body elements, beam elements, and hinge elements were developed by MSTMM. It was also found that the deformation of the cutter head and the lathe bed may influence the quality of the machined surface.\textsuperscript{26} However, the deformations of these two components have not been investigated in previous studies. Recently, at the theoretical level, a hybrid dynamic method was developed, extending MSTMM with FEM and the Craig–Bampton (CB) reduction method for dynamic modeling and analysis of multibody systems with flexible-body elements.\textsuperscript{26}

In this paper, the proposed hybrid dynamic method is used to develop a novel model of the ultra-precision fly-cutting machine tool, where the cutter head and the lathe bed are treated as general flexible elements, and the overall transfer equation is formulated. Compared with previous MSTMM studies, this study improves MSTMM computational performance in the general case of flexible elements while maintaining the above-mentioned advantages.

In Section 2, the novel model of an ultra-precision fly-cutting machine tool system is introduced. In Section 3, the mechanical elements in the model are categorized into M subsystems and F subsystems. In Section 4, M subsystems and F subsystems are analyzed using MSTMM and FEM-CB, respectively. In Section 5, all subsystems are recombined, and the overall transfer equation of the whole system is derived. In Section 6, the vibration characteristics of the machine tool system are simulated and validated using the modal test. Conclusions are presented in Section 7.

2 | MODEL OF AN ULTRA-PRECISION FLY-CUTTING MACHINE TOOL SYSTEM

In this section, a novel model of an ultra-precision fly-cutting machine tool system considering the deformations of the cutter head and lathe bed is developed.

2.1 | Ultra-precision fly-cutting machine tool

An ultra-precision fly-cutting machine tool is used to process optical precision workpieces such as KH₂PO₄ (KDP) crystals, aluminum alloys, and polycarbonate. The main components of the machine tool, shown in Figure 1, are the lathe bed, the bridge, the columns, the slider, the workpiece, the guideways, the spindle, the cutter head, the tool posts, and so forth. The lathe bed, the bridge, the columns, the slider, and the guideways are made of natural marble with very stable properties.

![The photograph of the ultra-precision fly-cutting machine tool](image-url)

**FIGURE 1** The photograph of the ultra-precision fly-cutting machine tool
2.2 Dynamic modeling

In the dynamic model, all the components are regarded as individual mechanics elements, including body and hinge elements. The body elements include rigid components, beam-like components, and general flexible components. The hinge elements are used to link the body elements, such as elastic hinges and fixed hinges.

The dynamic model and the elements of the machine tool system are depicted in Figure 2, including 19 body elements and 27 hinge elements. All the elements are numbered uniformly, with details shown in Tables 1 and 2, respectively.

The transfer matrix of hinge element $j$ is expressed as

$$
\mathbf{U}_j = \begin{bmatrix}
I_3 & 0_{3\times3} & 0_{3\times3} & K \\
0_{3\times3} & I_3 & K' & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & I_3 & 0_{3\times3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_3
\end{bmatrix},
$$

where the linear and rotational stiffnesses are

$$
K = \begin{bmatrix}
-\frac{1}{k_x} & 0 & 0 \\
0 & -\frac{1}{k_y} & 0 \\
0 & 0 & -\frac{1}{k_z}
\end{bmatrix}, \quad K' = \begin{bmatrix}
\frac{1}{k_x} & 0 & 0 \\
0 & \frac{1}{k_y} & 0 \\
0 & 0 & \frac{1}{k_z}
\end{bmatrix}.
$$

For the hydrostatic bearings, the gas and liquid hydrostatic stiffnesses are calculated using computational fluid dynamics (CFD) software according to the actual working conditions. For the bolted connections, the bolt joint stiffnesses are calculated using fractal theory modeling. For the lifting jacks, the stiffness was obtained.

### Table 1 Body elements in the dynamic model

| Components | Element type | Body number |
|------------|--------------|-------------|
| Foundation | Rigid body   | 45          |
| Lathe bed with a linear motor primary | Flexible body | 43 |
| Left side guideway | Rigid body | 11 |
| Right side guideway | Rigid body | 13 |
| Linear motor secondary with a slider, a vacuum chunk, and a workpiece fixed by vacuum power | Rigid body | 9 |
| Left upper guideway | Rigid body | 4 |
| Right upper guideway | Rigid body | 7 |
| Left column | Beam | 39 |
| Right column | Beam | 41 |
| Combined bridge, stator part of the fork-type flexible connection mechanism, and the stator part of the torque motor | Rigid body | 37 |
| Rotor of the torque motor | Rigid body | 23 |
| Bearing spindle | Rigid body | 25 |
| Dummy body without mass and volume | Dummy body | 18 |
| Front tool holder | Rigid body | 29 |
| Counterweight tool holder | Rigid body | 31 |
| Thrust spindle | Rigid body | 35 |
| Aerostatic spindle | Beam | 20 and 15 |
| Cutter head | Flexible body | 33 |

FIGURE 2 Dynamic model and body elements of the ultra-precision fly-cutting machine tool system
from the relation between the applied force and the displacement of the jacks. These calculated results of the hydrostatic bearings, the bolted connections, and the lifting jacks show that, for each hinge element, the force and the displacement (torque and angle of rotation) satisfy linear relations in the operating range of the machine tool. The stiffnesses of the different types of hinges are incorporated into Equation (2).

Figure 3 shows the topology of the dynamic model and the elements and the corresponding index more clearly. It can be observed that the dynamic model consists of 11 closed loops. At virtual cuts at points \( P_{11,1}, P_{11,3}, P_{13,2}, P_{13,6}, P_{20,17}, P_{24,45(2)}, P_{27,45(3)}, P_{28,45(3)}, P_{44,45(3)}, \) and \( P_{44,45(5)} \), the ultra-precision fly-cutting machine tool system becomes a tree system shown in Figure 4. At the “cutting points,” the following relations can be formulated:

\[
\begin{align*}
\{ P_{11,1} : Z_{11,1} &= CZ_{1,0}, & P_{11,3} : Z_{13,3} &= CZ_{3,0}, & \\
P_{13,2} : Z_{13,2} &= CZ_{2,0}, & P_{13,6} : Z_{13,6} &= CZ_{6,0}, & \\
P_{20,17} : Z_{20,17} &= CZ_{17,0}, & P_{24,45(2)} : Z_{44,45(2)} &= CZ_{45(2),0}, & \\
P_{27,45(3)} : Z_{44,45(3)} &= CZ_{45(3),0}, & P_{44,45(4)} : Z_{44,45(4)} &= CZ_{45(4),0}, & \\
P_{44,45(5)} : Z_{44,45(5)} &= CZ_{45(5),0}.
\end{align*}
\] (3)

where \( Z_{ij} \) is defined as the state vector at the point \((i, j)\). According to MSTMM,\(^2\) the state vector contains displacements and internal torques/forces in modal coordinates:

\[
Z_{ij} = [X, Y, Z, \Theta_x, \Theta_y, \Theta_z, M_x, M_y, M_z, Q_x, Q_y, Q_z]^T. \tag{4}
\]

The machine tool system is further analyzed using the hybrid dynamic method in the following sections.
3 | SUBSYSTEMS OF THE ULTRA-PRECISION FLY-CUTTING MACHINE TOOL SYSTEM

In this section, the mechanical elements in the ultra-precision fly-cutting machine tool system model are classified into M subsystems and F subsystems. Rigid-body elements, beam elements, and hinge elements are included in M subsystems. On the other hand, the general flexible-body elements are considered in F subsystems.

As shown in Figure 5, the machine tool system is divided into 14 subsystems, including 12 M subsystems and 2 F subsystems, and the members of the subsystems are shown in Table 3.

M subsystems 1, 3, and 4 are connected to F subsystem 9 at points C1, C2, and C3, namely, the connection points; M subsystems 2, 5, 6, 7, 8, 11, 12, 13, and 14 are connected to F subsystem 10 at connection points C4, C5, C6, C7, C8, C9, C10, C11, C12, and C13. The state vectors of the machine tool system are defined as

\[
\mathbf{Z} = \begin{bmatrix}
Z_{\text{sub1}}, Z_{\text{sub2}}, Z_{\text{sub3}}, Z_{\text{sub4}}, Z_{\text{sub5}}, Z_{\text{sub6}}, Z_{\text{sub7}} \\
\end{bmatrix}^T
\]  

The state vectors of the ends and the topology structure of M subsystems 1–8, 11–14 are presented in Table 4. The overall state vectors of F subsystems 9 and 10 are defined as the modal coordinates of the flexible elements after CB reduction, that is,

\[
\mathbf{Z}_{\text{sub9}} = \mathbf{p}^{33} = \begin{bmatrix} p_1^{33}, p_2^{33}, ..., p_{n_{\text{33}}}^{33} \end{bmatrix}^T, \\
\mathbf{Z}_{\text{sub10}} = \mathbf{p}^{43} = \begin{bmatrix} p_1^{43}, p_2^{43}, ..., p_{n_{\text{43}}}^{43} \end{bmatrix}^T,
\]

where \(n_{\text{33}} = 28\) and \(n_{\text{43}} = 70\) are the numbers of the modal coordinates of flexible elements 33 and 43, respectively.

In this section, the elements in the machine tool dynamic model are divided into 12 M subsystems and 2 F subsystems. The overall transfer equations of each subsystem will be obtained in the following section.

FIGURE 3 Topology of the dynamic model

FIGURE 4 Tree topology of the dynamic model
**FIGURE 5** Topology of the dynamic model (after division)

**TABLE 3** Members of the subsystems

| Subsystem | Type | Element index | Rigid-body elements | Beam elements | Hinge elements | Flexible-body element |
|-----------|------|---------------|---------------------|---------------|----------------|----------------------|
| 1         | M    |               | 18, 23, 25, 35, 37 | 15, 20, 39    | 16, 17, 19, 21, 22, 24, 26, 27, 28, 34, 36, 38, 40 | /                     |
| 2         | M    | 4, 7, 9       | /                   | /             | 1, 2, 3, 5, 6, 8, 10 | /                     |
| 3         | M    | 29            | /                   | /             | 30             | /                     |
| 4         | M    | 31            | /                   | /             | 32             | /                     |
| 5         | M    | 11            | /                   | /             | 12             | /                     |
| 6         | M    | 13            | /                   | /             | 14             | /                     |
| 7         | M    | /             | 41                   | 42            | /              | /                     |
| 8         | M    | 45            | /                   | /             | 44(1), 46      | /                     |
| 9         | F    | /             | /                   | /             | /              | 33                    |
| 10        | F    | /             | /                   | /             | /              | 43                    |
| 11        | M    | /             | /                   | /             | 44(2)          | /                     |
| 12        | M    | /             | /                   | /             | 44(3)          | /                     |
| 13        | M    | /             | /                   | /             | 44(4)          | /                     |
| 14        | M    | /             | /                   | /             | 44(5)          | /                     |
DYNAMIC ANALYSIS OF EACH SUBSYSTEM

The overall transfer equations and the transfer matrices of M subsystems are derived based on MSTMM in Section 4.1. Then, F subsystems are analyzed by FEM and further reduced using the CB reduction method in Section 4.2.

4.1 Dynamic analysis of M subsystems

According to Table 4, M subsystems 3, 4, 7, 11, 12, 13, and 14 are chain subsystems whose overall transfer equations are obtained by multiplying all their transfer matrices. On the other hand, M subsystems 1, 2, 5, 6, and 8 belong to tree subsystems, where the main transfer equation and the geometric equations are formulated in the overall transfer equation.

| Subsystem | Topology | The state vector of the output end | State vectors of input ends |
|-----------|----------|-----------------------------------|-----------------------------|
| 1         | Tree     | Z_{10,0}                          | Z_{15,8}, Z_{17,0}, Z_{19,17}, Z_{20,2}, Z_{21,0}, Z_{22,0}, Z_{23,0}, Z_{24,0}, Z_{25,0} |
| 2         | Tree     | Z_{10,0}                          | Z_{10,0}, Z_{0,0}, Z_{0,0} |
| 3         | Chain    | Z_{12,0}                          | Z_{0,0} |
| 4         | Chain    | Z_{12,0}                          | Z_{0,0} |
| 5         | Tree     | Z_{12,0}                          | Z_{11,0}, Z_{11,1} |
| 6         | Tree     | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |
| 7         | Chain    | Z_{12,0}                          | Z_{11,0} |
| 8         | Tree     | Z_{12,0}                          | Z_{11,0} |
| 9         | Chain    | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |
| 10        | Chain    | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |
| 11        | Chain    | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |
| 12        | Chain    | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |
| 13        | Chain    | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |
| 14        | Chain    | Z_{12,0}                          | Z_{13,0}, Z_{13,1} |

TABLE 4 Features of M subsystems

4.2 Overall transfer equation of chain subsystems

The overall transfer equations of M subsystems 3, 4, 7, 11, 12, 13, and 14 are given as

\[ Z_{\text{sub}3} = \begin{bmatrix} Z_{10,0} \\ Z_{12,0} \\ Z_{14,0} \\ Z_{16,0} \\ Z_{18,0} \\ Z_{20,0} \\ Z_{22,0} \\ Z_{24,0} \\ Z_{26,0} \\ Z_{28,0} \end{bmatrix}^{T} \]

and the overall transfer matrices are deduced as

\[
\begin{align*}
U_{\text{sub3}} &= \begin{bmatrix} U_{30,0} & U_{31} & -I_{2 \times 12} & 12 & 24 \end{bmatrix}, \\
U_{\text{sub4}} &= \begin{bmatrix} U_{30,0} & U_{31} & -I_{2 \times 12} & 12 & 24 \end{bmatrix}, \\
U_{\text{sub7}} &= \begin{bmatrix} U_{30,0} & U_{31} & -I_{2 \times 12} & 12 & 24 \end{bmatrix}, \\
U_{\text{sub11}} &= \begin{bmatrix} U_{44(1),0} & U_{44(2),0} & -I_{4 \times 2} & 4 & 4 & 24 & 24 & 24 \end{bmatrix}, \\
U_{\text{sub12}} &= \begin{bmatrix} U_{44(1),0} & U_{44(2),0} & -I_{4 \times 2} & 4 & 4 & 24 & 24 & 24 \end{bmatrix}, \\
U_{\text{sub13}} &= \begin{bmatrix} U_{44(1),0} & U_{44(2),0} & -I_{4 \times 2} & 4 & 4 & 24 & 24 & 24 \end{bmatrix}, \\
U_{\text{sub14}} &= \begin{bmatrix} U_{44(1),0} & U_{44(2),0} & -I_{4 \times 2} & 4 & 4 & 24 & 24 & 24 \end{bmatrix},
\end{align*}
\]

Overall transfer equation of tree subsystem 1

According to the input and output ends of subsystem 1 in Table 4 and the constraints in Equation (3), the overall state vectors of subsystem 1 are defined as

\[ Z_{\text{sub1}} = \begin{bmatrix} Z_{10,0} \\ Z_{12,0} \\ Z_{14,0} \\ Z_{16,0} \\ Z_{18,0} \\ Z_{20,0} \end{bmatrix}^{T} \]
The main transfer equations of M subsystem 1 are derived as

\[
Z_{40,C_8} = U_{40}Z_{39}U_{38}U_{37,1}Z_{37,1} + U_{37,2}Z_{37,2} + U_{37,3}Z_{37,3} + U_{37,4}Z_{37,4} + U_{37,5}Z_{37,5} + U_{37,6}Z_{37,6} + U_{37,7}Z_{37,7} + U_{37,8}Z_{37,8} + U_{37,9}Z_{37,9}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{15.28}Z_{15,28}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{17.0}Z_{17.0}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{20.17}Z_{20.17}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{22.0}Z_{22.0}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{27.0}Z_{27.0}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{30.40}Z_{30.40}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{34.34}Z_{34.34}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{37.0}Z_{37.0}
\]

+ \frac{U_{40}Z_{39} + U_{38}U_{37,1} + U_{37,2} + U_{37,3} + U_{37,4} + U_{37,5} + U_{37,6} + U_{37,7} + U_{37,8} + U_{37,9}}{40.40}Z_{40.40}
\]

\rightarrow -i_{12}Z_{40,C_8} + (T_{17.0} + T_{20.40}C)Z_{17.0} + (T_{22.40} + T_{37.40}C)Z_{22.0} + (T_{27.40} - T_{28.40} + T_{15.40}C)Z_{28.0} + T_{34.40}Z_{34.34} = 0.
\]

The geometric equation of element 18 is supplemented. For example, the geometric equation of element 25 among the first input end and the other input ends need to be

\[
G_{15.37} - (G_{18.37} + G_{19.37}) = 0.
\]

Similarly, the geometric equations of element 25 (two input ends), element 35 (two input ends), and element 37 (five input ends) are written as

\[
G_{15.37} - (G_{18.37} + G_{19.37}) + G_{20.37} = 0.
\]

The overall transfer equation of tree subsystem 2

\[
Z_{10,C_5} = U_{10}Z_{9,2} + U_{10}Z_{9,2}Z_{2,0} + U_{10}Z_{9,2}U_{5,2}Z_{5,0} + U_{10}Z_{9,2}U_{5,2}U_{5,2}Z_{5,0} + U_{10}Z_{9,2}U_{5,2}U_{5,2}Z_{5,0}
\]

+ \frac{U_{10}Z_{9,2} + U_{5,2}Z_{5,0}}{15.10}Z_{15.10}
\]

+ \frac{U_{10}Z_{9,2} + U_{5,2}Z_{5,0}}{20.10}Z_{20.10}
\]

+ \frac{U_{10}Z_{9,2} + U_{5,2}Z_{5,0}}{30.10}Z_{30.10}
\]

+ \frac{U_{10}Z_{9,2} + U_{5,2}Z_{5,0}}{40.10}Z_{40.10}
\]

\rightarrow -i_{12}Z_{10,C_5} + T_{0.10}Z_{0,0} + T_{2.10}Z_{2,0} + T_{3.10}Z_{3,0} = 0.$
The geometric equations of element 9 with four input ends are

\[\begin{align*}
G_{1,9}Z_{1,0} + G_{2,9}Z_{2,0} &= 0, \\
G_{4,9}Z_{4,0} + G_{5,9}Z_{5,0} &= 0, \\
G_{6,9}Z_{6,0} + G_{0,9}Z_{0,0} &= 0.
\end{align*}\]  

(18)

where

\[\begin{align*}
G_{1,9} &= -H_{9,6}U_1, \\
G_{2,9} &= H_{9,7}U_2, \\
G_{4,9} &= H_{9,12}U_4, \\
G_{6,9} &= H_{9,6}U_6.
\end{align*}\]  

(19)

The overall transfer equation, the overall state vector, and the overall transfer matrix of subsystem 2 are

\[U_{\text{sub}2(30\times60)}Z_{\text{sub}2(60\times1)} = 0,\]

\[Z_{\text{sub}2} = \begin{bmatrix}
  Z_{1,0}^T, Z_{2,0}^T, Z_{6,0}^T \end{bmatrix},\]

(20)

\[U_{\text{sub}2(30\times60)} = \begin{bmatrix}
  I_{12}, T_{11-10}, T_{22-10}, T_{13-10}, T_{55-10}
\end{bmatrix}.\]

(21)

4.2.3 | Overall transfer equation of tree subsystem 5

The main transfer equation of subsystem 5 is

\[Z_{12,C_5} = \frac{U_{12_{11,1}U_{11,1}}}{T_{11-12,C_1}}Z_{11,1} + \frac{U_{12_{11,1}U_{11,2}}}{T_{11-12,C_2}}Z_{11,3}\]

\[\rightarrow -h_{12}Z_{12,C_6} + T_{11-12,C_1}Z_{11,0} + T_{11-12,C_2}Z_{3,0} = 0.\]

(22)

The geometric equations of element 11 with two input ends is

\[\begin{align*}
G_{11,11,1}Z_{1,0} + G_{11,11,2}Z_{2,0} &= 0, \\
G_{11,11,3} &= -H_{11,11}, \\
G_{11,11,4} &= H_{11,11}.
\end{align*}\]  

(23)

The overall transfer equation, the overall state vector, and the overall transfer matrix of subsystem 5 are

\[U_{\text{sub}5(18\times36)}Z_{\text{sub}5(36\times1)} = 0,\]

\[Z_{\text{sub}5} = \begin{bmatrix}
  Z_{12,C_5}^T, Z_{1,0}^T, Z_{3,0}^T \end{bmatrix},\]

\[U_{\text{sub}5(18\times36)} = \begin{bmatrix}
  -h_{12}, T_{11-12,C_1}, T_{11-12,C_2} \end{bmatrix}.\]

(24)

4.2.4 | Overall transfer equation of tree subsystem 6

The main transfer equation and the geometric equation of tree subsystem 6 are

\[Z_{14,C_7} = \frac{U_{14_{13,1}U_{13,1}}}{T_{13-14,C_1}}Z_{13,2} + \frac{U_{14_{13,1}U_{13,2}}}{T_{13-14,C_2}}Z_{13,6}\]

\[\rightarrow -h_{12}Z_{14,C_7} + T_{13-14,C_1}Z_{2,0} + T_{13-14,C_2}Z_{6,0} = 0,\]

\[G_{13,13,C_1}Z_{2,0} + G_{13,13,C_2}Z_{6,0} = 0,\]

\[G_{13,13,C_1} = -H_{13,13},\]

\[G_{13,13,C_2} = H_{13,13}.
\]

(25)

The overall transfer equation, the overall state vector, and the overall transfer matrix of subsystem 6 are

\[U_{\text{sub}6(18\times36)}Z_{\text{sub}6(36\times1)} = 0,\]

\[Z_{\text{sub}6} = \begin{bmatrix}
  Z_{14,C_7}^T, Z_{1,0}^T, Z_{3,0}^T \end{bmatrix}_{(36\times1)},\]

\[U_{\text{sub}6(18\times36)} = \begin{bmatrix}
  -h_{12}, T_{13-14,C_1}, T_{13-14,C_2} \end{bmatrix}_{(36\times1)}.\]

(26)

4.2.5 | Overall transfer equation of tree subsystem 8

The main transfer equation tree of subsystem 8 is obtained as

\[Z_{46,0} = \frac{U_{46_{45,1}U_{45,1}}}{T_{45-46,C_1}}Z_{44,C_4} + \frac{U_{46_{45,2}U_{45,2}}}{T_{45-46,C_2}}Z_{45,44(2)} + \frac{U_{46_{45,3}U_{45,4}}}{T_{45-46,C_4}}Z_{45,44(3)} + \frac{U_{46_{45,4}U_{45,4}}}{T_{45-46,C_4}}Z_{45,44(4)} + \frac{U_{46_{45,5}U_{45,5}}}{T_{45-46,C_5}}Z_{45,44(5)}\]

\[\rightarrow -h_{12}Z_{46,0} + T_{46-45,C_4}Z_{44,C_4} + T_{46-45,C_5}Z_{45,44(2)} + T_{46-45,C_5}Z_{45,44(3)} + T_{46-45,C_5}Z_{45,44(4)} + T_{46-45,C_5}Z_{45,44(5)} = 0.\]

(27)

The geometric equations of element 45 with five input ends are

\[\begin{align*}
G_{44,44,44,44}&+G_{45,45,44,44}Z_{44,C_4} = 0, \\
G_{44,44,44,45}&+G_{45,45,44,45}Z_{44,C_5} = 0, \\
G_{44,44,45,44}&+G_{45,45,45,44}Z_{44,C_4} = 0, \\
G_{44,44,45,45}&+G_{45,45,45,45}Z_{44,C_5} = 0, \\
G_{44,44,45,45}&+G_{45,45,45,45}Z_{44,C_5} = 0.
\end{align*}\]  

(28)

\[\begin{align*}
G_{44,44,44,44}&+G_{45,45,44,44}Z_{44,C_4} = 0, \\
G_{44,44,44,45}&+G_{45,45,44,45}Z_{44,C_5} = 0, \\
G_{44,44,45,44}&+G_{45,45,45,44}Z_{44,C_4} = 0, \\
G_{44,44,45,45}&+G_{45,45,45,45}Z_{44,C_5} = 0, \\
G_{44,44,45,45}&+G_{45,45,45,45}Z_{44,C_5} = 0.
\end{align*}\]  

(29)
and $\Psi_-$ represent $|\Psi|$ are much smaller than $\Psi_k$ are the solutions of $KK\Psi_0$ are the numbers of nodes in the FEM models. $\Psi_f$ are to the FEM models extracted from the FEM models.

4.3 | Dynamic analysis of F subsystems

4.3.1 | FEM models

There are two F subsystems in the machine tool system. The FEM models of these two F subsystems are developed in Software Ansys APDL, as shown in Figure 6. The parameters used in the FEM model are listed in Table 5. The lathe bed (subsystem 10) is meshed into 887 Solid72 elements, and the cutter head (subsystem 9) is meshed into 623 Solid72 elements. Through Software Ansys APDL, the mass and the stiffness matrices of these two components can be extracted for the following analysis.

4.3.2 | Derivation and reduction

The dynamic equations of the two F subsystems (elements 43 and 33) are

$$
\begin{align*}
M^{43}_{(6n^3 \times 6n^3)} \ddot{\delta}^{43}_{(6n^3 \times 1)} + K^{43}_{(6n^3 \times 6n^3)} \delta^{43}_{(6n^3 \times 1)} &= f^{43}_{(6n^3 \times 1)}, \\
M^{33}_{(6n^3 \times 6n^3)} \ddot{\delta}^{33}_{(6n^3 \times 1)} + K^{33}_{(6n^3 \times 6n^3)} \delta^{33}_{(6n^3 \times 1)} &= f^{33}_{(6n^3 \times 1)},
\end{align*}
$$

(29)

where $n^{43}$ and $n^{33}$ are the numbers of nodes in the FEM models. $M^{43}$, $K^{43}$, and $K^{33}$ are, respectively, the mass and the stiffness matrices extracted from the FEM models. $\delta^{43}$, $\delta^{33}$, $f^{43}$, and $f^{33}$ are, respectively, the displacement and the force vectors.

Using the CB reduction method, the displacement vectors in Equation (29) can be reduced as

$$
\begin{align*}
\delta^{43}_{(6n^3 \times 1)} &= \Phi^{43}_{CB(6n^3 \times 6n^3)} \Phi_{CB} \Phi_{CB}^T \delta^{43}_{(6n^3 \times 1)}, \\
\delta^{33}_{(6n^3 \times 1)} &= \Phi^{33}_{CB(6n^3 \times 6n^3)} \Phi_{CB} \Phi_{CB}^T \delta^{33}_{(6n^3 \times 1)},
\end{align*}
$$

(30)

where

$$
\begin{align*}
\Phi^{43}_{CB(6n^3 \times 6n^3)} &= I_{6n^3} \Phi_{CB(6n^3 \times 6n^3)}, \\
\Phi^{33}_{CB(6n^3 \times 6n^3)} &= I_{6n^3} \Phi_{CB(6n^3 \times 6n^3)}.
\end{align*}
$$

(31)

$\Phi^{43}_{CB}$ and $\Phi^{33}_{CB}$ are the CB transformation matrixes of elements 43 and 33, respectively. In Equation (31), $\Psi^{43}$ and $\Psi^{33}$ represent the constraint modes of the elements, solved by the statics equations $\Psi^{43} = (K^{43} - \omega\omega^T)M^{43} \Psi^{43}$ and $\Psi^{33} = (K^{33} - \omega\omega^T)M^{33} \Psi^{33}$, respectively. $\Phi^{43}_{CB}$ and $\Phi^{33}_{CB}$ are the solutions of $(K^{43} - \omega\omega^T)M^{43} \Phi^{43}_{CB} = 0$ and $(K^{33} - \omega\omega^T)M^{33} \Phi^{33}_{CB} = 0$, respectively, implying the first $k$ fixed-interface modes. $\eta^{43} = [\eta_1, \eta_2, ..., \eta_{n^{43}}]^T$ and $\eta^{33} = [\eta_1, \eta_2, ..., \eta_{n^{33}}]^T$ are the modal coordinates of elements 43 and 33, respectively. $n^{43} = 28$ is the number of modal coordinates of flexible element 33, consisting of 18 DOFs at three connection points and the first 10 fixed-interface modes. For element 43, $n^{43} = 70$, including 60 DOFs at 10 connection points and the first 10 fixed-interface modes. The values of $n^{43}$ and $n^{33}$ are much smaller than those of $n^{43}$ and $n^{33}$.

Substituting Equation (30) into (29) gives

$$
\begin{align*}
M^{43}_{(6n^3 \times 6n^3)} \Phi^{43}_{CB(6n^3 \times 6n^3)} \Phi_{CB} \Phi_{CB}^T \delta^{43}_{(6n^3 \times 1)} + K^{43}_{(6n^3 \times 6n^3)} \Phi^{43}_{CB(6n^3 \times 6n^3)} \Phi_{CB} \Phi_{CB}^T \delta^{43}_{(6n^3 \times 1)} &= f^{43}_{(6n^3 \times 1)}, \\
M^{33}_{(6n^3 \times 6n^3)} \Phi^{33}_{CB(6n^3 \times 6n^3)} \Phi_{CB} \Phi_{CB}^T \delta^{33}_{(6n^3 \times 1)} + K^{33}_{(6n^3 \times 6n^3)} \Phi^{33}_{CB(6n^3 \times 6n^3)} \Phi_{CB} \Phi_{CB}^T \delta^{33}_{(6n^3 \times 1)} &= f^{33}_{(6n^3 \times 1)},
\end{align*}
$$

(32)

Pre-multiplying by $\Phi^{43}_{CB}$ and $\Phi^{33}_{CB}^T$, Equation (32) can be rewritten as

FIGURE 6 | FEM models of the lathe bed and the cutter head: (A) the lathe bed and (B) the cutter head. FEM, finite element method.
\[
\begin{align*}
(\Phi_{\text{CB}}^{32})^T M_{\text{CB}(6 \times 9)}^{32} \Phi_{\text{CB}}^{32} + (\Phi_{\text{CB}}^{32})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{32} & = (\Phi_{\text{CB}}^{32})^T f_{32}^{32} \\
(\Phi_{\text{CB}}^{33})^T M_{\text{CB}(6 \times 9)}^{33} \Phi_{\text{CB}}^{33} + (\Phi_{\text{CB}}^{33})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{33} & = (\Phi_{\text{CB}}^{33})^T f_{33}^{33}.
\end{align*}
\]

where \((\Phi_{\text{CB}}^{32})^T M_{\text{CB}(6 \times 9)}^{32} \Phi_{\text{CB}}^{32}\), \((\Phi_{\text{CB}}^{32})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{32}\), and \((\Phi_{\text{CB}}^{33})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{33}\) are not diagonal matrices and need to be orthogonalized. The initial step of orthogonalization involves solving the vector \(\psi\) from the following equations:

\[
\begin{align*}
(\Phi_{\text{CB}}^{32})^T M_{\text{CB}(6 \times 9)}^{32} \Phi_{\text{CB}}^{32} + (\Phi_{\text{CB}}^{32})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{32} \omega^2 & = \psi^{32} = 0, \\
(\Phi_{\text{CB}}^{33})^T M_{\text{CB}(6 \times 9)}^{33} \Phi_{\text{CB}}^{33} + (\Phi_{\text{CB}}^{33})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{33} \omega^2 & = \psi^{33} = 0.
\end{align*}
\]

Then, the orthogonalized CB transformation matrices are obtained by

\[
\begin{align*}
\Phi_{\text{CB}}^{32} & = \Phi_{\text{CB}}^{32} \psi^{32}, \\
\Phi_{\text{CB}}^{33} & = \Phi_{\text{CB}}^{33} \psi^{33}.
\end{align*}
\]

Meanwhile, the orthogonalized modal coordinates are

\[
\begin{align*}
P^{33}(\alpha_{33} \times 1) & = (\psi^{33})^{-1} \bar{h}^{33(\alpha_{33} \times 1)}, \\
P^{33}(\beta_{33} \times 1) & = (\psi^{33})^{-1} \bar{h}^{33(\beta_{33} \times 1)}.
\end{align*}
\]

Substituting Equation (35) and Equation (36) into Equation (33) yields

\[
\begin{align*}
M_{\text{CB}(9 \times 9)}^{43} P^{43}_{\text{CB}(9 \times 1)} + K_{\text{CB}(9 \times 9)}^{43} P^{43}_{\text{CB}(9 \times 1)} & = (\Phi_{\text{CB}}^{32})^T f^{43}_{\text{CB}(9 \times 1)}, \\
M_{\text{CB}(9 \times 9)}^{33} P^{33}_{\text{CB}(9 \times 1)} + K_{\text{CB}(9 \times 9)}^{33} P^{33}_{\text{CB}(9 \times 1)} & = (\Phi_{\text{CB}}^{33})^T f^{33}_{\text{CB}(9 \times 1)},
\end{align*}
\]

where \(M_{\text{CB}(9 \times 9)}^{43} = (\psi^{32})^{-1} (\Phi_{\text{CB}}^{32})^T M_{\text{CB}(6 \times 9)}^{32} \Phi_{\text{CB}}^{32} \psi^{32}\), \(M_{\text{CB}(9 \times 9)}^{33} = (\psi^{33})^{-1} (\Phi_{\text{CB}}^{33})^T M_{\text{CB}(6 \times 9)}^{33} \Phi_{\text{CB}}^{33} \psi^{33}\), \(K_{\text{CB}(9 \times 9)}^{43} = (\psi^{32})^{-1} (\Phi_{\text{CB}}^{32})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{32} \psi^{32}\), \(K_{\text{CB}(9 \times 9)}^{33} = (\psi^{33})^{-1} (\Phi_{\text{CB}}^{33})^T K_{\text{CB}(6 \times 9)} \Phi_{\text{CB}}^{33} \psi^{33}\) are, respectively, mass matrices and stiffness matrices of flexible elements 43 and 33 after reduction and orthogonalization.

Substituting \(p' = Pf_{\text{CB}}\) and \(f' = Ff_{\text{CB}}\) into Equation (37), one can obtain the overall transfer equations of F subsystems 9 and 10 as

\[
\begin{align*}
\begin{pmatrix} -\omega^2 M_{\text{CB}(6 \times 9)}^{33} + K_{\text{CB}(6 \times 9)}^{33} \end{pmatrix} Z_{\text{sub}9} & = \begin{pmatrix} \Phi_{\text{CB}}^{33} \end{pmatrix}^T F^{33}, \\
\begin{pmatrix} -\omega^2 M_{\text{CB}(6 \times 9)}^{43} + K_{\text{CB}(6 \times 9)}^{43} \end{pmatrix} Z_{\text{sub}10} & = \begin{pmatrix} \Phi_{\text{CB}}^{43} \end{pmatrix}^T F^{43}.
\end{align*}
\]

After obtaining the overall transfer equation of each subsystem, the overall transfer equation of the machine tool system is derived in the following section.

## 5 | RECOMBINATION OF THE ULTRA-PRECISION FLY-CUTTING MACHINE TOOL SYSTEM

### 5.1 | Connection relations

All subsystems in the ultra-precision fly-cutting machine tool system are linked together through 13 connection points. The two ends belonging to different subsystems at one connection point satisfy the continuum position equations and the force restriction equations, which are summarized into three connection conditions in Table 6.

The features of the connection points in the machine tool system are summarized and listed in Table 7. Points \(C_1\) and \(C_4\) are regarded as the F–M condition, and the other connection points belong to the M–F condition. Using Table 7 and Figure 5, the connection relations are formulated in the following two conditions.

(a) F–M condition: Points \(C_1\) and \(C_4\)

\[
\begin{align*}
C_1 & : \left[ \begin{array}{c} \Phi_{\text{CB}}^{33} \end{array} \right]_0 p^{33} = C_p Z_{34,1}, \\
C_4 & : \left[ \begin{array}{c} \Phi_{\text{CB}}^{43} \end{array} \right]_0 p^{43} = C_p Z_{44,1}.
\end{align*}
\]

(b) M–F condition: Points \(C_2, C_3,\) and \(C_5–C_{13}\)

\[
\begin{align*}
C_2 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{33} \end{array} \right]_0 p^{33} = C_p Z_{20,0}, \\
C_3 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{43} \end{array} \right]_0 p^{43} = C_p Z_{20,0}, \\
C_5 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{33} \end{array} \right]_0 p^{33} = C_p Z_{10,0}, \\
C_6 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{43} \end{array} \right]_0 p^{43} = C_p Z_{10,0}, \\
C_7 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{33} \end{array} \right]_0 p^{33} = C_p Z_{14,0}, \\
C_8 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{43} \end{array} \right]_0 p^{43} = C_p Z_{14,0}, \\
C_9 : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{33} \end{array} \right]_0 p^{33} = C_p Z_{22,0}, \\
C_{10} : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{43} \end{array} \right]_0 p^{43} = C_p Z_{22,0}, \\
C_{11} : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{33} \end{array} \right]_0 p^{33} = C_p Z_{44,0}, \\
C_{12} : & \left[ \begin{array}{c} \Phi_{\text{CB}}^{43} \end{array} \right]_0 p^{43} = C_p Z_{44,0}.
\end{align*}
\]
TABLE 6  Connection relations in three connection conditions

| Condition | Output | Input | Position | Force/torque |
|-----------|--------|-------|----------|--------------|
| F-M | F subsystem | M subsystem | \((\phi_1)_{h}^T p_i = C_i Z_i\) | \(F_0 = -Q_{ij}/M_i\) |
| M-F | M subsystem | F subsystem | \((\phi_1)_{h}^T p_i = C_i Z_i\) | \(F_i = -Q_{ij}/M_i\) |
| F-F | F subsystem | F subsystem | \((\phi_1)_{h}^T p_i = (\phi_1)_{h}^T p_i\) | \(F_0 = -F_i\) |

TABLE 7  Features of connection points

| Connection point | Output point | Subsystem | Input point | Subsystem | Condition |
|------------------|--------------|-----------|-------------|-----------|-----------|
| C_1              | N_i   | F subsystem | (34, I) | M subsystem | F-M       |
| C_2              | (30, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_3              | (32, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_4              | N_i   | F subsystem | (44(1), I) | M subsystem | F-M       |
| C_5              | (10, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_6              | (12, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_7              | (14, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_8              | (40, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_9              | (42, O) | M subsystem | N_i   | F subsystem | M-F       |
| C_10             | (44(2), O) | M subsystem | N_i   | F subsystem | M-F       |
| C_11             | (44(3), O) | M subsystem | N_i   | F subsystem | M-F       |
| C_12             | (44(4), O) | M subsystem | N_i   | F subsystem | M-F       |
| C_13             | (44(5), O) | M subsystem | N_i   | F subsystem | M-F       |

\[
(-\omega^2 M_{R}^{23} + K_{R}^{23})Z_{5,6,10} = (\phi_1)^T F^{33}
\]

where \(C_{FO} = [0_{28 \times 3} \ 0_{28 \times 3} \ \phi_1^{33 \times 1} \ - (\phi_1^{33 \times 1})\), \(C_{RI} = [0_{28 \times 3} \ 0_{28 \times 3} \ - (\phi_1^{33 \times 1})\), and \(C_{R_i} = [0_{28 \times 3} \ 0_{28 \times 3} \ - (\phi_1^{33 \times 1})\). \n
Similarly, substituting the force relations of points C_6-C_13 into the overall transfer equation of subsystem 10 in Equation (38) gives

\[
(-\omega^2 M_{R}^{23} + K_{R}^{23})Z_{5,6,10} = (\phi_1)^T F^{33}
\]

where \(C_{FO} = [0_{28 \times 3} \ 0_{28 \times 3} \ (\phi_1)^T_{\Omega, \Theta} \ - (\phi_1)^T_{\Omega, \Theta} \), and \(C_{RI} = [0_{28 \times 3} \ 0_{28 \times 3} \ - (\phi_1)^T_{\Omega, \Theta} \ - (\phi_1)^T_{\Omega, \Theta} \) \(i = 1, 2, 3, \ldots, 8, 9\). The displacement relations of the connection points are included as new rows in the overall transfer matrix of the whole system.

5.2 Overall transfer equation of the ultra-precision fly-cutting machine tool system

Based on the overall transfer equation of each subsystem in Equations (7), (15), (20), (23), (25), (28), (41), and (42) and the displacement relations of the connection points in Equations (39)–(40), the overall transfer matrix of the machine tool system follows the same sequence as the overall state vectors in Equation (5). The overall transfer equation of the whole system is written in the form

\[
U_{all}(5524 \times 542)Z_{all}(542 \times 1) = 0.
\]
is the overall transfer matrix of the machine tool system.

The boundary conditions of the machine tool system are as follows:

(a) The foundation is fixed on all six DOFs, which is expressed by the state vector as

\[ Z_{\text{all}}(0) = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \tag{44} \]

(b) The two tool posts are free, that is,

\[ Z_{\text{all}}(1) = [X, Y, Z, \Theta_x, \Theta_y, \Theta_z, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \tag{45} \]

(c) The other boundary conditions due to the virtual cut are unknown.

According to the boundary conditions in Equations (44) and (45), zeros in the overall state vectors and the relevant columns in the overall transfer matrix are deleted. Then, one can obtain

\[ \tilde{U}_{\text{all}}[524 \times 524] \tilde{Z}_{\text{all}}[524 \times 1] = 0, \tag{46} \]

where \( \tilde{Z}_{\text{all}} \) includes unknown state vectors of the whole system; \( \tilde{U}_{\text{all}} \) is the square matrix related to \( \tilde{Z}_{\text{all}} \). For the machine tool system, Equation (46) must have nontrivial solutions. Hence, the determinant of the coefficient matrix \( \tilde{U}_{\text{all}} \) in Equation (46) equals zero,

\[ \det(\tilde{U}_{\text{all}}(\omega)) = 0. \tag{47} \]

Equation (47) is the characteristic equation of the machine tool system, which can be solved by the improved determinant algorithm.

**FIGURE 7** Distribution of measurement points of the machine tool
FIGURE 8 Stabilization diagrams of the x-direction frequency obtained using the PolyMax method: (A) 0–512 Hz, (B) 512–1024 Hz, and (C) 1024–2048 Hz
that we previously proposed. Substituting the obtained natural frequencies into Equation (46) gives
\[ \vartheta_{\text{all}}(\omega_k)|Z_{\text{all}}\omega_k = 0, \] (48)

where \( Z_{\text{all}} \) can be calculated by extracting any one row from the cofactor matrix of \( \vartheta_{\text{all}} \). Based on \( Z_{\text{all}} \), the state vector of the input and output ends in the machine tool system can be obtained by the inversion of the transfer relations in Figure 4. The mode shapes of the machine tool system can be further obtained, which can be visualized using MSTMMSim Software.

After solving for the modal characteristics of the machine tool system, the results and validation are introduced in the following section.

6 | RESULTS AND VALIDATION

6.1 | Experimental modal test

The modal test of the ultra-precision fly-cutting machine tool system was designed and conducted to validate the dynamic model and the results. In the test, 364 measurement points were scattered around the whole machine tool, shown in Figure 7, among which 48 measurement points were in the cutter head and the lathe bed, respectively. The machine tool was excited with a modal hammer (PCB086C03) to generate excitation with frequencies from 3 to 1500 Hz. The single-input-single-output (SISO) testing technique was used to measure the frequency response functions (FRFs). Due to the reciprocity of FRF, the “moving hammer” test and the “moving accelerometer” test were conducted according to the physical situation. More specifically, an accelerometer (PCB355B04) weighing 11.2 g was fixed to the driving point at the machine lathe bed by petro wax, and then several measurement points were impacted in turns. For the remaining measurement points where there is not enough space for the “moving hammer” test, such as the measurement points at the cutter head, the positions of the accelerometer and the hammer were exchanged: the accelerometer was moved around, and then the machine tool was impacted at the driving point by the hammer. Afterwards, the modal parameters were identified using LMS software and the PolyMAX identification method from the average of all measured FRFs. The stabilization diagrams of the x-direction frequency obtained using the PolyMax method are shown in Figure 8, where the whole frequency domain is split into three ranges to avoid omissions.

| Mode number | Test (Hz) | Simulation (Hz) | Error (%) | Descriptions of mode shapes |
|-------------|-----------|-----------------|-----------|-----------------------------|
| 1           | 7.45      | 7.26            | -2.51     | Coherent rotational vibration of the foundation and the machine tool around the z-axis |
| 2           | 9.99      | 10.2            | 2.31      | Coherent translational vibration of the foundation and the machine tool along the z-axis |
| 3           | 42.5      | 41.0            | -3.52     | Reverse rotational vibration of the foundation and the machine tool around the y-axis |
| 4           | 51.2      | 51.6            | -0.63     | Reverse rotational vibration of the foundation and the machine tool around the x-axis |
| 5           | 74.9      | 74.1            | 0.45      | Reverse translational vibration of the foundation and the machine tool along the z-axis |
| 6           | 75.0      | 75.7            | 0.94      | Reverse translational vibration of the bridge and the slider along the y-axis |
| 7           | 100.0     | 100.4           | 0.90      | Translational vibration of the cutter head along the y-axis |
| 8           | 109.3     | 102.4           | -6.27     | Coherent rotational vibration of the cutter head and the lathe bed around the x-axis |
| 9           | 180.9     | 168.7           | -6.75     | Reverse rotational vibration of the cutter head and the lathe bed along the x-axis |
| 10          | 181.6     | 182.3           | 0.37      | Reverse translational vibration of the cutter head and the slider along the x-axis |
| 11          | 202.7     | 216.1           | 6.59      | Reverse translational vibration of the cutter head and the bridge along the y-axis |
| 12          | 225.0     | 218.7           | -2.80     | Reverse translational vibration of the bridge and the lathe bed along the z-axis |
| 13          | 242.9     | 248.2           | 2.16      | Reverse translational vibration of the cutter head and the bridge along the y-axis |
| 14          | 252.8     | 252.4           | 0.16      | Reverse translational vibration of the cutter head and the bridge along the x-axis |
| 15          | 258.6     | 270.2           | 4.49      | Translational vibration of the cutter head along the z-axis |
| 16          | 442.5     | 436.5           | 1.36      | Deformation of the lathe bed |
| 17          | 585.0     | 586.5           | 0.25      | Bending deformation on both sides of the cutter head |
| 18          | 690.6     | 682.3           | 1.20      | Reverse rotational vibration of the slider and the rest around the x-axis |
| 19          | 815.0     | 810.7           | -0.53     | Coherent translational vibration of two tool posts and the slider along the z-axis |
| 20          | 961.2     | 970.8           | 1.00      | Reverse translational vibration of two tool posts along the z-axis |
6.2 | Results

According to the overall transfer matrix and the overall transfer equation in Section 5.2, the vibration characteristics, including the natural frequencies and the mode shapes of the machine tool system, are simulated by coding in C++. The first 20 modes simulated results by the proposed method are compared with the test ones, as shown in Table 8. The maximum relative error is 6.75%, confirming the accuracy of the dynamic model and the proposed method.

As shown in Table 8, all the components in the machine tool vibrate coherently in the first five mode shapes, indicating that the stiffnesses of the components and the coupling surface were selected to be high enough. In our previous work,25 we found that the waviness on machined surfaces is affected by the mode shapes related to the vibrations of the tool tip. Herein, the relative vibration between the tool tip and the workpiece along the z-direction in the 15th, 16th, 17th, and 20th modes will significantly influence the quality of ultra-precision machining. Since the modal shapes of the 15th and 20th modes are translational vibrations, which were calculated by previous work,25 these modes are not presented and discussed in detail in this study.

By using our MSTMMSim Software,27 two typical modal shapes (16th and 17th modes) are visualized and are shown in Figures 9B and 10B. It can be seen in Figure 9 that the deformations of the lathe bed appear in the 16th mode in both experimental and simulated results. The cutter head bends on both sides in the 17th mode in Figure 10. These two modes will generate the relative vibrations between the tool tip and the workpiece, resulting in large waviness on the machined surfaces. To achieve higher machining precision, the structure and material of the lathe bed and the cutter head should be optimized with higher bending stiffness.

7 | CONCLUSIONS

This study presents a comprehensive modeling process of an ultra-precision fly-cutting machine tool using a hybrid dynamic method, which provides theoretical insights and guidance for ultra-precision machining. The vibration characteristics of the ultra-precision fly-cutting machine tool have been successfully calculated and validated experimentally by the modal test.

The proposed modeling and analysis method maintains the advantages of the transfer matrix method for multibody systems (MSTMMs): low matrix order, fast calculation speed, small memory storage, and no need for the global dynamic equation. In addition, the proposed method can also be applied to the dynamic modeling and analysis of general flexible elements such as the lathe bed and the
cutter head, which extends the computational capability of MSTMMs.

The proposed modeling process of the ultra-precision fly-cutting machine tool is efficient, highly accurate, and easy to program. Once the system’s overall transfer matrix is derived, as long as the structure of the multiterminal input elements remains unchanged, there is no need to re-derive it even if the structural parameters of the system change. The transfer matrix of the corresponding component can simply be added or deleted in the overall transfer matrix of the original system. The proposed method lays the foundation for our ongoing research efforts focused on the dynamic control and the optimization of an ultra-precision machine tool.

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CONFLICT OF INTEREST
The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID
Hanjing Lu ORCID: http://orcid.org/0000-0003-1351-0272
Xiaoting Rui ORCID: http://orcid.org/0000-0002-6114-7685

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AUTHOR BIOGRAPHIES

Hanjing Lu received her PhD degree in engineering from Nanjing University of Science and Technology, Nanjing, China, in 2022. She is currently working as a postdoctoral researcher at the College of Engineering in Peking University, Beijing, China. From 2019 to 2021, she was a joint PhD fellow in Aarhus University, Aarhus, Denmark. Her research interests include multibody system dynamics, dynamics, and control of machine tools and mechanical systems with deformation, and transfer matrix method for multibody systems.

Xiaoting Rui received his PhD degree in dynamics, vibration, and control from Nanjing University of Science and Technology. He is the director of the Academic Committee of Nanjing University of Science & Technology and President of the Key Laboratory of System Dynamics of Complex Equipment, Ministry of Industry and Information Technology of China. He was elected as a member of the Chinese Academy of Sciences. He was a guest professor at the University of Stuttgart, Cottbus Technology University, University of Hanover, Fraunhofer Institute for High-Speed Dynamics, Karlsruhe Institute of Technology, and Hamburg University of Technology in Germany. His research interests include multibody system dynamics, transfer matrix method for multibody systems, and launch dynamics.

Ziyao Ma received his PhD degree in mechanical engineering from Central South University, Changsha, China, in 2021. He is currently working as a postdoctoral researcher at the College of Engineering in Peking University, Beijing, China. From 2019 to 2021, he was a joint PhD fellow in Aarhus University, Aarhus, Denmark, sponsored by the Chinese Scholarship Committee. His research interests include crystal plasticity, metal forming and microstructure characterization, and so forth.

Yuanyuan Ding is a PhD candidate at the Institute of Launch Dynamics in Nanjing University of Science and Technology, China. She majored in the dynamics performance of the ultra-precision fly-cutting machine tool and its influences on the surface topography, and has published several papers on this topic.

Yiheng Chen is a PhD candidate at the Institute of Launch Dynamics in Nanjing University of Science and Technology, China. He is engaged in research on rotor dynamics, nonlinear vibration, and multibody system dynamics.

Yu Chang is currently a PhD candidate at the School of Science, Nanjing University of Science and Technology, China. From 2022 to 2023, he participated in the joint-supervision doctoral program at Nanyang Technological University. His research interests include contact mechanics and multibody system dynamics.

Xuping Zhang received his BEng degree in mechanical engineering from Chongqing University, Chongqing, China, in 1992, and his PhD degree in mechanical engineering from the University of Toronto, Toronto, Canada, in 2009. He is currently an associate professor with the Department of Mechanical and Production Engineering, Aarhus University, Denmark. His research interests include dynamics and control of robotic and mechanical systems with deformation, collaborative robots for SME manufacturing and production, robotic rehabilitation, robotic in-pipe inspection, and robotic single-cell handling.

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