An $O(n^2)$ algorithm for Many-To-Many Matching of Points with Demands in One Dimension

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Abstract Given two point sets $S$ and $T$, we study the many-to-many matching with demands problem (MMD problem) which is a generalization of the many-to-many matching problem (MM problem). In an MMD, each point of one set must be matched to a given number of the points of the other set (each point has a demand). In this paper we consider a special case of MMD problem, the one-dimensional MMD (OMMD), where the input point sets $S$ and $T$ lie on the line. That is, the cost of matching a pair of points is equal to the distance between the two points. we present the first $O(n^2)$ time algorithm for computing an OMMD between $S$ and $T$, where $|S| + |T| = n$.

Keywords Many-to-many point matching · One dimensional point-matching · points with demands

1 Introduction

Suppose we are given two point sets $S$ and $T$, a many-to-many matching (MM) between $S$ and $T$ assigns each point of one set to one or more points of the other set [4]. Eiter and Mannila [7] solved the MM problem using the Hungarian method in $O(n^3)$ time. Finally, Colanitto et al. [4] presented an $O(n \log n)$-time dynamic programming solution for finding an MM between two sets on the real line. The matching has different applications such as computational biology [1], operations research [2], pattern recognition [3], computer vision [8].

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A general case of MM problem is the limited capacity many-to-many matching problem (LCMM) where each point has a capacity. Schrijver [11] proved that a minimum-cost LCMM can be found in strongly polynomial time. A special case of the LCMM problem is that in which both $S$ and $T$ lie on the real line. Rajabi-Alni and Bagheri [10] proposed an $O(n^2)$ time algorithm for the one dimensional minimum-cost LCMM.

In this paper we consider another generalization of the MM problem, where each point has a demand, that is each point of one set must be matched to a given number of the other set. Let $S = \{s_1, s_2, \ldots, s_y\}$ and $T = \{t_1, t_2, \ldots, t_z\}$. We denote the demand sets of $S$ and $T$ by $D_s = \{\alpha_1, \alpha_2, \ldots, \alpha_y\}$ and $D_T = \{\beta_1, \beta_2, \ldots, \beta_z\}$, respectively. In a many-to-many matching with demand (MMD), each point $s_i \in S$ must be matched to $\alpha_i$ points in $T$ and each point $t_i \in T$ must be matched to $\beta_i$ points in $S$. We denote the demand of each point $a \in S \cup T$ by $\text{demand}(a)$. We study the one dimensional MMD (OMMD), where $S$ and $T$ lie on the line and propose an $O(n^3)$ algorithm for finding a minimum cost OMMD.

2 Preliminaries

In this section, we proceed with some useful definitions and assumptions. Fig. 1 provides an illustration of them. Let $S = \{s_i\}_i \text{for } 1 \leq i \leq y$ and $T = \{t_i\}_i \text{for } 1 \leq i \leq z$. We denote the elements in $S$ in increasing order by $(s_1, s_2, \ldots, s_y)$, and the elements in $T$ in increasing order by $(t_1, t_2, \ldots, t_z)$. Let $s_1$ be the smallest point in $S \cup T$. Let $S \cup T$ be partitioned into maximal subsets $A_0, A_1, A_2, \ldots$ alternating between subsets in $S$ and $T$ such that all points in $A_i$ are smaller than all points in $A_{i+1}$ for all $i$: the point of highest coordinate in $A_i$ lies to the left of the point of lowest coordinate in $A_{i+1}$ (Fig. 1).

Let $A_w = \{a_1, a_2, \ldots, a_s\}$ with $a_1 < a_2 < \ldots < a_s$ and $A_{w+1} = \{b_1, b_2, \ldots, b_t\}$ with $b_1 < b_2 < \ldots < b_t$. We denote $|b_1 - a_i|$ by $e_i$, $|b_i - b_1|$ by $f_i$. Obviously $f_1 = 0$. Note that $a_0$ represents the largest point of $A_{w-1}$ for $w > 0$. In an OMMD, a point matching to its demand number of points is a satisfied point.

First, we briefly describe the algorithm in [4]. The running time of their algorithm is $O(n \log n)$ and $O(n)$ for the unsorted and sorted point sets $S$ and $T$, respectively. We denote by $C(q)$ the cost of a minimum MM for the set.
Fig. 2 Suboptimal matchings. (a) \((a, c)\) and \((b, d)\) do not both belong to an optimal matching. (b) \((a, d)\) does not belong to an optimal matching.

of the points \(\{p \in S \cup T \mid p \leq q\}\). The algorithm in \cite{4} computes \(C(q)\) for all points \(q\) in \(S \cup T\). Let \(m\) be the largest point in \(S \cup T\), then the cost of the minimum MM between \(S\) and \(T\) is equal to \(C(m)\).

Lemma 1 Let \(b < c\) be two points in \(S\), and \(a < d\) be two points in \(T\) such that 
\[a \leq b < c \leq d\]. Then a minimum cost many-to-many matching that contains \((a, c)\) does not contain \((b, d)\), and vice versa (Fig. 2(a)) \cite{4}.

Lemma 2 Let \(b, d \in T\) and \(a, c \in S\) with \(a < b < c < d\). Then, a minimum cost many-to-many matching contains no pairs \((a, d)\) (Fig. 2(b)) \cite{4}.

Towards a contradiction, suppose that \(M\) is a minimum cost MM that contains such a pair \((a, d)\) (Fig. 2(b)). We can construct a new MM, denoted by \(M'\), by removing the pair \((a, d)\) from \(M\) and adding the pairs \((a, b)\) and \((c, d)\). The cost of \(M'\) is smaller than the cost of \(M\). This is a contradiction to our assumptions that \(M\) is a minimum-cost MM.

Corollary 1 Let \(M\) be a minimum-cost MM. For any matching \((a, d)\) \(\in M\) with \(a < d\), we have \(a \in A_i\) and \(d \in A_{i+1}\) for some \(i \geq 0\).

Lemma 3 In a minimum cost MM, each \(A_i\) for all \(i > 0\) contains a point \(q_i\), such that all points \(a \in A_i\) with \(a < q_i\) are matched with the points in \(A_{i-1}\) and all points \(a' \in A_i\) with \(q_i < a'\) are matched with the points in \(A_{i+1}\) \cite{4}.

Proof By Corollary 1 each point \(a \in A_i\) must be matched with a point \(b\) such that either \(b \in A_{i-1}\) or \(b \in A_{i+1}\). Let \(b \in A_{i-1}\), \(b' \in A_{i+1}\), and \(a, a' \in A_i\) with \(b < a < a' < b'\). By way of contradiction, suppose that \(M\) is a minimum cost MM containing both \((b, a')\) and \((a, b')\). Contradiction with Lemma 1.

The point \(q_i\) defined in Lemma 3 is called the separating point. In fact, the aim of their algorithm is exploring the separating points of each partition \(A_i\) for all \(i\). They assumed that \(C(p) = \infty\) for all points \(p \in A_0\). Let \(A_w = \{a_1, a_2, \ldots, a_s\}\) and \(A_{w+1} = \{b_1, b_2, \ldots, b_t\}\). Their dynamic programming algorithm computes \(C(b_i)\) for each \(b_i \in A_{w+1}\), assuming that \(C(p)\) has been computed for all points \(p \leq b_i\) in \(S \cup T\). Depending on the values of \(w\), \(s\), and \(t\) there are five possible cases.
Case 0: \( w = 0 \). In this case there are two possible situations.
- \( i \leq s \). We compute the optimal matching by assigning the first \( s - i \) elements of \( A_0 \) to \( b_1 \) and the remaining \( i \) elements pairwise (Fig. 3(a)). So we have

\[
C(b_i) = \sum_{j=1}^{s} e_j + \sum_{j=1}^{i} f_j.
\]

- \( i > s \). The cost is minimized by matching the first \( s \) points in \( A_1 \) pairwise with the points in \( A_0 \), and the remaining \( i - s \) points in \( A_1 \) with \( a_s \) (Fig. 3(b)). So we have

\[
C(b_i) = (i - s) e_s + \sum_{j=1}^{s} e_j + \sum_{j=1}^{i} f_j.
\]

Case 1: \( w > 0, s = t = 1 \). Fig. 4(a) provides an illustration of this case. By Lemma 3, \( b_1 \) must be matched with the point \( a_1 \). Therefore, we can omit the point \( a_1 \), unless it reduces the cost of \( C(b_1) \).

Case 2: \( w > 0, s = 1, t > 1 \). By Lemma 3, we can minimize the cost of the many-to-many matching by matching all points in \( A_{w+1} \) with \( a_1 \) as presented in Fig. 4(b). As case 1, \( C(b_i) \) includes \( C(a_1) \) if \( a_1 \) covers other points in \( A_{w-1} \); otherwise, \( C(b_i) \) includes \( C(a_0) \).

Case 3: \( w > 0, s > 1, t = 1 \). By Lemma 3, we should find the point \( a_i \in A_w \) such that all points \( a_j \in A_w \) with \( a_j < a_i \) are matched to points in \( A_{w-1} \) and all points \( a_k \in A_w \) with \( a_i < a_k \) are matched to points in \( A_{w+1} \) (Fig. 4(c)).
Case 4: $w > 0, s > 1, t > 1$. In this case, we should find the point $q$ that splits $A_w$ to the left and right. Let $X(b_i)$ be the cost of connecting $b_1, b_2, \ldots, b_i$ to at least $i+1$ points in $A_w$ (Fig. 5(a)). Let $Y(b_i)$ be the cost of connecting $b_1, b_2, \ldots, b_i$ to exactly $i$ points in $A_w$ (Fig. 5(b)). Finally, let $Z(b_i)$ denote the cost of connecting $b_1, b_2, \ldots, b_i$ to fewer than $i$ points in $A_w$, as depicted in Fig. 5(c).

Then, we have:

$$C(b_i) = \begin{cases} 
\min(X(b_i), Y(b_i), Z(b_i)) & 1 \leq i < s \\
\min(Y(b_i), Z(b_i)) & i = s \\
C(b_{i-1}) + e_s + f_i & s < i \leq t
\end{cases}$$

Lemma 4 Give $A_w = \{a_1, a_2, \ldots, a_s\}$ with $a_1 < a_2 < \ldots < a_s$ and $A_{w+1} = \{b_1, b_2, \ldots, b_t\}$ with $b_1 < b_2 < \ldots < b_t$. In a minimum cost MM, for $i = 2$ we have:

$$C(b_2) = \min(C(b_1) + f_2 - f_1 and \deg(b_1) = \deg(b_1) - 1 \text{ if } \deg(b_1) > 1, C(b_1) + e_s + f_2, C(b_1) + e_s + f_i + C(a_{s-1}) - C(a_{s-1}) \text{ if } \deg(b_1) = 1.$$

For $i \geq 3$ we have:

$$C(b_i) = \min(C(b_{i-1}) + f_i - f_1 and \deg(b_1) = \deg(b_1) - 1 \text{ if } \deg(b_1) > 1, C(b_{i-1}) + e_s + f_i, C(b_{i-1}) + e_s + f_i + C(a_{s-1}) - C(a_{s-1}) \text{ if } \deg(b_1) = 1 \text{ and } \deg(a_s) = 1, C(b_{i-1}) + f_i + e_s \text{ if } \deg(b_1) = 1 \text{ and } \deg(a_s) > 1.$$
For $i = b_3$ see Figure 6. We have two above cases and the third case

\[\text{deg}(b_1) = 1 \text{ and } \text{deg}(a_s) > 1\]

In this case we observe that

\[C(b_3) = C(b_2) + f_3 + e_s\]

**Lemma 5** If $b_i$ selects $a_s$ as the separating point (the best point to be matched), then $a_s$ is also the separating point (the optimal point) for $b_{i+1} \ldots b_t$ for $i \geq 2$.

**Proof** It can be simply proved according to:

\[X(b_i) = \frac{1}{j=a_s-2} \left( \sum_{p=j}^{s} e_p + C(a_{p-2}) \right) + \sum_{p=1}^{i} f_p\]
An Algorithm for OMMD problem

In this section, we present an $O(n^2)$ algorithms for finding an OMMD between two sets $S$ and $T$ lying on the line. Our recursive dynamic programming algorithm is based on the algorithm of Colannino et al. [4] and Rajabi and Bagheri [10]. We begin with some useful lemmas.

**Lemma 6** Let $a \in S, b \in T$ and $c \in S, d \in T$ such that $a \leq b < c \leq d$. Let $M$ be an minimum-cost OMMD. If $(a, d) \in M$, then either $(a, b) \in M$ or $(c, d) \in M$ or both.

**Proof** The proof of this lemma is essentially the same as the proof of Lemma 2.

**Corollary 2** Let $a \in A_i$ and $d \in A_j$ for some $i \geq 0$. For any matching $(a, d)$ in an OMMD if $j > i + 1$, then either $(a, b) \in M$ for all $b \in A_{i+1} \cup A_{i+3} \cup \ldots \cup A_{i+2}$ or $(a, b) \in M$ for all $a \in A_{i+2} \cup A_{i+4} \cup \ldots \cup A_{j-1}$.
Note that we use this corollary for satisfying the demands of \( a \in A_i \) by the points of sets \( A_j \) or for satisfying the demands of \( b \in A_j \) by the points of sets \( A_i \).

**Lemma 7** Let \( b < c \) be two points in \( S \), and \( a < d \) be two points in \( T \) such that \( a \leq b < c \leq d \). If an OMMD contains both of \((a,c)\) and \((b,d)\), then \((a,b) \in M\) or \((c,d) \in M\).

**Proof** Suppose that the lemma is false. Let \( M \) be an OMMD that contains both \((a,c)\) and \((b,d)\), and neither \((a,b) \in M\) nor \((c,d) \in M\) (Fig. 2a)). Then we can remove the pairs \((a,c)\) and \((b,d)\) from \( M \) and add the pairs \((a,b)\) and \((c,d)\); the result \( M' \) is still an OMMD which has a smaller cost, a contradiction.

**Lemma 8** Let \( a < a' \leq b < b' \) such that \( a, a' \in S \) and \( b, b' \in T \). Assume that we must match the points \( a, a' \) to the points \( b, b' \). Then, in an OMMD it does not matter that we use the pairs \((a,b), (a',b')\) or the pairs \((a,b'), (a',b)\).

**Proof** The cost of the two pairs \((a,b), (a',b')\) is equal to the cost of the two pairs \((a,b'), (a',b)\). Since we have

\[(b-a) + (b'-a') = (b'-a) + (b-a').\]

**Lemma 9** Let \( A = \{a_1, a_2, \ldots, a_s\} \) and \( B = \{b_1, b_2, \ldots, b_t\} \) be two distinct sorted sets points with demands lying on the real line. Let \( D_A = \{\alpha_1, \alpha_2, \ldots, \alpha_s\} \) and \( D_B = \{\beta_1, \beta_2, \ldots, \beta_t\} \) be the demands of the points of \( A \) and \( B \), respectively, such that \( s \geq \max_{i=1}^s \beta_j \) and \( t \geq \max_{i=1}^t \alpha_i \). Then, we can compute an OMMD between \( A \) and \( B \) in \( O(s + t) \) time.

**Proof** Obviously, for reaching the minimum cost we must match each point in \( B \) with the first unsatisfied point in \( A \) (if exists). Notice by Lemma 8 the order of matching is arbitrary. Without loss of generality we assume \( a_s \leq b_1 \) and match two sets as follows.

First, we match each point \( a_i \in A \) with the first unsatisfied point in \( B \). If all points in \( B \) are satisfied, then we match \( a_i \in A \) with the closest point of \( B \) that is not matched with it. Then, if there exists any unsatisfied point in \( B \), starting from \( b_1 \), we match each point of \( B \) with the closest point of \( A \) not matched with it previously.

**Theorem 1** Let \( S \) and \( T \) be two sets of points on the real line with \(|S| + |T| = n\). Then, an OMMD between \( S \) and \( T \) can be determined in \( O(n^2) \) time.

Let \( \text{Demand}(q) \) denote the demand of the point \( q \), i.e., the number of the points that must be matched to \( q \). For any point \( q \), let \( C(q,j) \) be the cost of an OMMD for the set of points \( \{p \in S \cup T, \text{ with } p \leq q \text{ and } 1 \leq j \leq \text{Demand}(q)\} \). Initially \( C(q,j) = \infty \) for all \( q \in S \cup T \) and \( 1 \leq j \leq \text{Demand}(q) \). If \( m \) and \( m' \) are the largest point and largest demand in \( S \cup T \), respectively then \( C(m, m') \) is the cost of an OMMD.

In this algorithm, we use this idea that in an OMMD there exists an intersection since the points are previously used for satisfying the demands of the
Observation 1 Let \( b \leq a < a' \leq b' \), in a minimum cost OMMD \( M \), if \((b, a), (a', b), (a, b') \subseteq M \), then: (i) \((b, a)\) and \((b', a)\) are used for satisfying the demands of \(a\), (ii) either \(b\) is closer to \(a'\) or \(a'\) satisfies the demand of \(b\) (Figure 7).

Observation 2 Let \( b \leq a < a' \leq b' \), in a minimum cost OMMD \( M \), if \((a', b'), (a', b), (a, b') \subseteq M \), then: (i) \((b, a')\) and \((b', a')\) are used for satisfying the demands of \(a'\), (ii) either \(b'\) is closer to \(a\) or \(a\) satisfies the demand of \(b'\) (Figure 8).

Observation 3 Let \( b \leq a < a' \leq b' \), in a minimum cost OMMD \( M \), if \((b, a), (a', b), (a', b') \subseteq M \), then: (i) these pairs are used for satisfying the demands of at least one of the sets \(\{a, b\}, \{a', b'\}, \{a, a'\}\) or \(\{b, b'\}\) (Figure 9).

Lemma 10 In a minimum cost OMMD, each \( A_i \) for all \( i > 0 \) and \( 1 \leq k \leq m' \) contains a point \( q(i, k) \), such that all points \( a \in A_i \) with \( a < q(i, k) \) and \( \text{demand}(a) \geq k \) are matched with the points in \( A_{i-1} \) and all points \( a' \in A_i \) with \( q(i, k) < a' \) and \( \text{demand}(a') \geq k \) are matched with the points in \( A_{i+1} \) [9].
Given $b \leq a < a' \leq b'$, when there exists an intersection, that is the pairs $(b, a')$ and $(a', b)$ are in $M$, we have $(b, a) \in M$ or $(a', b') \in M$ or both. Without loss of generality, we assume that the pairs $(b, a')$ and $(a', b)$ are satisfying the $k$th and $k'$th demands of the points with $k \neq k'$. We assume that $(b, a')$ and $(a', b)$ are not satisfying the same demands. Then, by Corollary 1 and Observations 1, observation 2, and observation 3 we prove the lemma.

Main step. Our algorithm is that we first add the points $a \in T \cup S$ one by one such that the first demand of all points with $\text{demand}(a) \geq 1$ are satisfied (the first repeat $k = 1$), then we add the points $a \in T \cup S$ one by one such that the second demand of all points with $\text{demand}(a) \geq 2$ are satisfied (the second repeat $k = 2$), then we add the points $a \in T \cup S$ one by one such that the
third demand of all points with demand$(a) \geq 3$ are satisfied (the third repeat $k = 3$), and so on. In fact, our purpose is to find the optimal point $q(i, k)$ for each partition $A_{w+1}$.

Consider $A_w = \{a_1, a_2, \ldots, a_s\}$ and $A_{w+1} = \{b_1, b_2, \ldots, b_t\}$. Let $D_w = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ and $D_{w+1} = \{\beta_1, \beta_2, \ldots, \beta_t\}$ be the demand sets of the points in $A_w$ and $A_{w+1}$, respectively. Assume that we have computed $C(p, k)$ for all points $\{p \in S \cup T \text{ with } p \leq a_s \text{ and } 1 \leq k \leq \text{Cap}(p)\}$ and $C(p, k-1)$ for all points $\{p \in S \cup T\}$, and now we want to compute $C(b_i, k)$ for each $b_i \in A_{w+1}$ and $1 \leq k \leq \beta_i$.

**Lemma 11** Let $A = \{a_1, a_2, \ldots, a_s\}$ and $B = \{b_1, b_2, \ldots, b_t\}$ be two distinct sorted sets of points with demands lying on the real line that are matched with each other such that $m$ demands of each point $q \in A \cup B$ are satisfied, where $m = \min(\text{demand}(q), k)$. Let $D_A = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ and $D_B = \{\beta_1, \beta_2, \ldots, \beta_t\}$ be the demands of the points of $A$ and $B$, respectively, such that $s \geq k + 1$ and $t \geq k + 1$. Then, we can compute an OMMD between $A$ and $B$ in $O(s + t)$ time, such that $k + 1$th demand of each point $q \in A \cup B$ with demand$(q) \geq k + 1$ is satisfied.

**Proof** Without loss of generality, we assume that $a_s \leq b_1$. We use three steps. Starting form $b_1$, in the first step, we match each point $b \in B$ with demand$(b) \geq k + 1$ to the closest unmatched point $a$ of $A$ with demand$(a) \geq k + 1$. Then, if there exists still unmatched point $b_i$ in $B$, we find the first $b' \in B$ with $\text{deg}(b') > k \geq \text{demand}(b')$ that is matched with $a'$, then we remove $(b', a')$ and add $(b_i, a')$. Finally, if still $k + 1$th demand of any point $b_i \in B$ is not satisfied, we select the closest unmatched point $a''$, and add $(b_i, a'')$.

In the second phase, if there exists any point $a \in A$ with unsatisfied $k + 1$th demand, we repeat above step 2 and step 3 for $A$.

In this step, we have two general cases: Case A, and Case B. In the first case, Case A, we have $s \geq \min(k + 1, \max_{j=1}^{i-1} \beta_j)$. So for each matching $(a, d)$ we have $a \in A_i$ and $d \in A_{i+1}$. But in the second case, Case B, we have $s < \min(k + 1, \max_{j=1}^{i-1} \beta_j)$. Therefore, by Corollary[2] we must investigate the partitions $A_{w-2}, A_{w-4}, A_{w-6}, \ldots$ to find the points for satisfying the $k + 1$th demands of the points.

Case A: $s \geq \min(k + 1, \max_{j=1}^{i-1} \beta_j)$. In this case, the $k + 1$th demands of the points in $A_{w+1}$ can be satisfied with the points in $A_w$.

Case A.0: $w = 0$. In this case we can get the minimum cost according to Lemma[11].

Case A.1: $w \geq 0$, $s = 1$, $i = 1$. This case is as Case 1 of [4].

Case A.1: $w \geq 0$, $s = 1$, $i \geq 1$. Recall that in Case A we have $s \geq \min(k + 1, \max_{j=1}^{i-1} \beta_j)$. So, $\beta_j = 1$ for all $1 \leq j \leq t$. This case is similar to Case 2 of [4].

Case A.2: $w \geq 0$, $s \geq 1$, $i = 1$. Two cases arise:

- $\text{deg}(b_1) < k + 1$. In this case $b_1$ has no demand but can be used for satisfying $k + 1$th demands of other points in $A_w$. For all points
in $A_w$ with demand $k + 1$ that are not matched to $b_1$ we test if matching to $b_1$ decreases the cost of matching or not. We find the separating point for all points $p_i$ in $A_w$ with $\text{demand}(p_i) \geq k + 1$. Assume that $m$ is the number of unmatched points with $b_1$ which have $k + 1$th demand, then we have

$$C(b_1, k + 1) = \min_{i=1}^{m} \left( \sum_{p=1}^{m} e_{p_i} + \min(C(b_1, k), C(b_1, k - (m - i + 1)) \right)$$

where $e_{p_i}$ is the $i$th closest point to $b_1$ with $\text{demand}(p_i) \geq k + 1$ and $\text{deg}(b_1) \geq k + 1$. In this case we have:

- There is no point in $A_{w+1}$ with demands larger than $k + 1$. We select the closest point $q \in A_w$ not matched to $b_1$. Then, if $\text{deg}(q) = \text{demand}(q)$ and $q$ is matched to a point $q'$ with $\text{deg}(q') \geq \text{demand}(q')$, we test if we remove the pair $(q, q')$ and add the pair $(q, b_1)$ the cost decreases or not and select the optimal point.
- There are $h$ points in $A_{w+1}$ with demands larger than $k + 1$, denoted by $p_i$ with $1 \leq i \leq h$. We find the separating points of the points $p_i$.

Case A.3: $w \geq 0$, $s \geq 1$, $i \geq 1$. This case is as Case 4 of [4]. The difference is that when we want to find the separating points of $A_w$ we use Lemma [11] for matching the points of two sets. Another difference is that in the $k$th repeat of our algorithm, by Lemma [14] we find the separating points of the subsets of $A_w$ ($A_{w1}, \ldots, A_{wk}$) independent form each other. Since, the points that effects them are different and distinct.

**Lemma 12** Let $b_2$ be the first point in $A_{w+1}$ with $\text{demand}(b_2) \geq k + 1$, then for $b'_2$ with $\text{demand}(b'_2) \geq k + 1$:

In the $k + 1$th phase of our algorithm, each point $p$ with $\text{demand}(p) \geq k + 1$ only can effect a single point $q'$ with either $\text{demand}(q') \geq k + 1$ or $\text{demand}(q) < k + 1$. unless $q$ is $k + 1$th optimal point for more than one point.

**Proof** Assume that $q$ is matched with a point $q'$, then other points select the $k + 1$th closest point. Note that if $q$ can be $k + 1$th optimal point of many points, but we consider the general case.

**Lemma 13** Give $A_w = \{a_1, a_2, \ldots, a_s\}$ with $a_1 < a_2 < \ldots < a_s$ and $A_{w+1} = \{b_1, b_2, \ldots, b_t\}$ with $b_1 < b_2 < \ldots < b_t$. In an OMMD if $p_i$ denotes the $i$th point in $A_{w+1}$ with $\text{demand}(p_i) \geq k + 1$ that are affected by the same subsets of $A_w$, for $i \geq 2$ we have:

let $h$ be the number of the point in $A_w$ with demand larger than $k + 1$ that are not matched with the points $p_i$, previously. If there exists $b \in A_{w+1}$ with $\text{deg}(b) > \text{demand}(b)$

$$C(p_i) = C(p_{i-1}) + f_{p_i} - f_b.$$
Else if \( p_{i-1} \) is not matched to a point \( a \in A_{wk} \) with demands larger than \( k+1 \); \( p_i \) is matched to the closest unmatched point in \( A_w \).

But if \( p_{i-1} \) is matched to a point \( a \in A_{wk} \) with demands larger than \( k+1 \); \( p_i \) tests whether be matched to the closest unmatched point in \( A_w \) or the closest unmatched point of \( A_{wk} \) with demand larger than \( k+1 \).

Proof This Lemma is proved similar to Lemma 4.

**Lemma 14** In \( k \)th repeat of our algorithm, each \( A_i \) for all \( i > 0 \) and \( 1 \leq k \leq m' \) consists of \( k \) partitions \( A_{i1}, \ldots, A_{ik} \), each contains a point \( q(i, k) \), such that all points \( a \in A_i \) with \( a < q(i, k) \) and \( \text{demand}(a) \geq k \) are matched with the points in \( A_{i-1} \) and all points \( a' \in A_i \) with \( q(i, k) < a' \) and \( \text{demand}(a') \geq k \) are matched with the points in \( A_{i+1} \).

Proof Given \( b \leq a < a' \leq b' \), when there exists an intersection, that is the pairs \((b, a')\) and \((a', b)\) are in \( M \), we have \((b, a) \in M\) or \((a', b') \in M\) or both. Consider the first repeat, \( A_w \) is partitioned to \( A_{w1} \) and \( A_{w2} \).

Without loss of generality, consider the point \( b_1 \), it can be used for satisfying the second demand of the points that are not matched with it for their first demand. We can prove it for the \( k \)th demands of points as the same. So, we can find the separating points of each subset independent from other subsets. This lemma does not contradict Lemma 10. In fact, we use another algorithm in this lemma for finding the separating point.

Case B: \( w > 0 \), \( \max_{j=1}^i \beta_j > s \). In this case, the number of the points in \( A_w \) is less than the maximum demand number of the points in \( A_{w+1} \), so by Corollary 2 we should seek the previous sets to satisfy demands of the points \( b_1, b_2, \ldots, b_i \).

This backward process is followed until finding the first set, called \( A_{w'} \), that can satisfy the demands of the points in the set \( A_{w+1} \). Then, we must find OMMD for the sets \( A_{w-1} \ldots A_{w'} \) again. It is possible that we do not reach such a set \( A_{w'} \), in this situation \( C(b_i, k) = \infty \) for all \( 1 \leq k \leq \beta_i \).

**Final step.** In this step, we consider the situation where there are not enough points in \( A_{w+1} \) for demands of the points \( a \in A_w \) for \( w > 0 \), and so by Corollary 2 we must seek the partitions \( A_{w+3}, A_{w+5}, \ldots \) for finding new points.

This forward process is followed until finding the first partition, called \( A_{j'} \), which can satisfy the demands of the points in \( A_w \). Then, we must find OMMD for the sets \( A_{w+2} \ldots A_{j'} \) again. It is possible that there does not exist a set such \( A_{j'} \). If \( A_{j'} \) exists, we should match the unmatched points in \( A_{j'} \) with the points in \( A_{j'} \) as follows.

**Lemma 15** Given \( A_w = a_1, a_2, \ldots, a_s \), let \( C(a_i, k) \) be the cost of satisfying \( k \) number of demands of the first \( i \) points in \( A_w \) with the points in \( A_j \) with \( j < i \). That is, assume that \( C(a_i, k) \) denotes the cost of matching \( k \) demands of \( a_1, a_2, \ldots, a_i \) with the points \( p \) where \( p \leq a_1 \). Also let \( \text{Cost}(a_1, k - j) \) be
the cost of matching \( k - j \) demands of \( a_i \) with the points \( p \) where \( p \leq a_1 \). Then we have:

\[
C(a_i, k) = \min_{j=1}^{k} (C(a_{i-1}, j) + \text{Cost}(a_i, k - j)).
\]

Proof Given \( A_i = a_1, a_2, \ldots, a_q \), for \( a_i \) there exists \( k \) cases, depending on satisfying its demands with \( p < a_i \) or \( p' > a_i \). Assume that \( j \) demands of \( a_i \) is satisfied with the points \( p < a_i \). Then, by Lemma 7 the points \( a_1, a_2, \ldots, a_{i-1} \) have only \( k - j \) options for satisfying their demands (see Figure 10), since \( j \) demands of them must be satisfied with the \( j \) points \( a_i \) is satisfied with them.

**Lemma 16** Given the partition \( A_w = a_1, a_2, \ldots, a_s \) of points with maximum demand \( k \), let \( T(s, k) \) be the number of options for satisfying \( k \) demands of \( a_1, a_2, \ldots, a_i \), then:

\[
T(s, k) = \sum_{j=1}^{k} T(s-1, j)
\]

Proof For demands of \( a_s \), it can satisfy \( j \) number of its demands by previous partitions. As Lemma 15 this lemma is proved.

So, by Lemma 15 using dynamic programming we have \( n*k \) states, that one of them is the optimal state, denoted by \( \text{optimal}(b_t, k) \). That is, the state that minimizes the cost of matching.

We can compute an OMMD by finding the optimal state \( \text{optimal}(b_t, k) \) of each set \( A_i \).

**4 Concluding Remarks**

Many-to-many point matching with demands MMD is a many-to-many matching where each point has a demand. We studied the one dimensional MMD, called OMMD, where we match two point sets on the line. We presented an algorithm for getting an OMMD between two point sets with total cardinality \( n \) in \( O(n^2) \) time.
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