Effective Field Theory for Pedestrians

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Abstract

A pedagogical introduction to effective field theory is presented.

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Introduction

If in 1975 one had asked for a brief history of hadronic physics it would have undoubtedly gone something like this [1]: first there was a ‘classical age’ initiated by Yuhawa’s (1935) meson hypothesis for the nuclear force and terminated (± 1950) by invading hoards of “strange” particles and resonance. There followed a sort of ‘dark ages’ where arcane rites of dispersion relations, Regge poles and dual resonance models were practiced. Finally we are now in the ‘enlightened age’ of “quantum chromodynamics” (QCD): baryons – like the proton and neutron – are composites of three “quarks” while mesons are made of quark-antiquark pairs; these are inseparably bound by a colour force which becomes weak at short distances, and the interaction between hadrons is a colour analogue of the van der Waal’s force between neutral atoms.

Alas, some twenty five years later we still are unable to calculate many interesting quantities such as the nuclear mass or nucleon-nucleon potential directly from QCD (albeit lattice gauge enthusiasts will tell you with the next generation of computers · · ·). One is left with a variety of models (bag, Skyrme, etc.) and a sort of interpolating scheme (QCD sum rules), but nothing approaching the systematics and accuracy of quantum electrodynamics (QED). The difference is due to confinement: whereas in QED the basic entities (electrons and photons) are observable, in QCD they (quarks and gluons) are not, rather we can only observe their hadronic composites.

Still, the triumphs of QED were afforded by the realization that one did not need to be able to calculate the electron mass to determine the effects of the self energy of a bound electron – the Lamb shift [2]. That one could apply a modified version of this and work directly with hadrons in a systematic way was first suggested by Weinberg (1979) [3] and marked the birth of a new age: the age of effective field theory whose ramifications go far beyond hadronic physics alone.
There are by now a number of textbook exposition [4] and review articles for the sophisticate; in this talk I will endeavour to give the novice some feeling for what is going on using the old static model [5] as an example. Then, at the end I will return to the wider implications.

A word of warning: for simplicity (mine, not yours) I will use ‘natural units’ $h = c = 1$; mass and momenta are in units of energy, and length in units of inverse energy, a useful conversion being

$$\hbar c = 1 = 197\text{MeV} \cdot \text{fm},$$

$$(1 \text{fm} = 10^{-13} \text{cm}).$$

The Static Model

Let me begin by recalling that the impetus for pre-QCD meson theory was Yukawa’s observation that in contrast to Poisson’s equation for the electrostatic potential, the equation

$$\left( \frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \phi = gn$$

has for a static charge at the origin, $n(\vec{r}) = \delta(\vec{r})$

$$\phi(r) = \frac{g}{4\pi} \frac{e^{-mr}}{r}$$

whose range is not infinite but $1/m$. Now suppose for the moment $n = 0$; by making the Fourier expansion

$$\phi(t, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} \varphi(t, \vec{k}) e^{i\vec{k} \cdot \vec{r}}$$

one obtains for each $\vec{k}$

$$\ddot{\varphi}(\vec{k}) + \omega^2(\vec{k}) \varphi(\vec{k}) = 0 \quad , \omega^2(\vec{k}) = \vec{k}^2 + m^2. \quad (5)$$

Thus, classically one has a set of harmonic oscillators and the dispersion relation $\omega(\vec{k})$ is that for a particle of mass $m$ in relativity. Each oscillator has a “momentum”

$$\pi(\vec{k}) = \dot{\varphi}(\vec{k})$$

(6)
and the total energy is
\[ H_0 = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2} \pi^2(k) + \frac{\omega^2(k)}{2} \varphi^2(k) \right] . \]  
\(7\)

Now each oscillator can be quantized individually, but instead of \(\varphi\) and \(\pi\) it is more convenient to use \(a\) and \(a^+\)
\[ \phi = \frac{1}{\sqrt{2\omega}} (a^+ + a) \quad \pi = i \sqrt{\frac{\omega}{2}} (a^+ - a) \ . \]  
\(8\)

This gives
\[ \hat{H}_0 = \int \frac{d^3k}{(2\pi)^3} \omega(k) \hat{a}^+(\vec{k}) \hat{a}(\vec{k}) \]  
\(9\)

where we have thrown out an infinite sum at “zero point energies” which play no role here. The ‘ladder operators’ have non-vanishing commutator
\[ [\hat{a}(\vec{k}) , \hat{a}^+(\vec{k}')] = (2\pi)^3 \delta(\vec{k} - \vec{k}') \]  
\(10\)

and the lowest energy, ground or ‘vacuum’ state \(|0\rangle\) obeys
\[ \hat{a}(k)|0\rangle = 0 \]  
\(11\)

so indeed it has zero energy. The state \(|\vec{k}\rangle = \hat{a}^+(\vec{k})|0\rangle\) has the property
\[ \hat{H}_0 |\vec{k}\rangle = \omega(\vec{k}) |\vec{k}\rangle \]  
\(12\)

so describing a particle (meson) with (3-) momentum \(\vec{k}\) and energy \(\omega(\vec{k})\).

When the right hand side of (2) is nonzero, i.e. the nucleon is present, the oscillators are driven so the total energy (hamiltonian) is
\[ \hat{H} = \hat{H}_0 + \hat{H}_I . \]  
\(13\)

Now, \(H_0\) is unmodified if the nucleon is static (in practical terms this means we are neglecting recoil which is a fair approximation to reality). In writing the ‘interaction

\[ \text{Specifically : } E_{ZPE} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega(\vec{k}) \]  

2Specifically : \(E_{ZPE} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega(\vec{k}) \)
part’ $H_I$ we need to account for the fact that the light mesons (pions) are ‘pseudoscalar’\(^3\), i.e. under ‘parity’, $\vec{r} \to -\vec{r}$, $\phi \to -\phi$ whereas for a scalar $\phi$ is unchanged. Taking this together with the fact that the nucleon is spin 1/2 (occurring in two spin states, “up” and “down”), because the energy should not be changed by parity or rotations the unique choice is

$$\hat{H}_I = \int \frac{d^3k}{(2\pi)^3} \left[ -\frac{ig}{\sqrt{2\omega(k)}} K \left( \hat{a}^+(\vec{k}) + \hat{a}(\vec{k}) \right) \right]$$  \hspace{1cm} (14)$$

where $K$ is shorthand for the 2 by 2 matrix

$$K = \begin{pmatrix} k_z & k_x - iky \\ k_x + ihy & -k_z \end{pmatrix}$$  \hspace{1cm} (15)$$

and for simplicity “isospin” is neglected. Note the coupling parameter $g$ must have dimension of length to compensate that of $K$.

Finally it is also worth mentioning that if one replaces the words nucleon and meson by electron and phonon this model bears many similarities to problems in solid state physics [6].

**The Self-Energy**

Alack, unlike $\hat{H}_0$, $\hat{H}$ cannot be diagonalized exactly but can be treated by time independent perturbation theory familiar to every quantum mechanic. Taking the unperturbed state as that with no mesons and one nucleon the leading energy shift – which is to say the nucleon mass shift because it is static – is given by\(^4\)

$$\Delta E^{[1]} = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{ig}{\sqrt{2\omega}} K \right] \left[ \frac{1}{-\omega} \right] \left[ -\frac{ig}{\sqrt{2\omega}} \right]$$  \hspace{1cm} (16)$$

which can be given a diagrammatic representation

\(^3\)In QCD this follows from ‘spontaneously broken chiral symmetry’.

\(^4\)This may be compared to the usual expression $E_n^{(2)} = \sum_{s \neq n} \frac{H_{1ns} H_{1sn}}{E_n - E_s}$. Note $\langle \vec{k} | \hat{H}_I | 0 \rangle = -igK\sqrt{2\omega}$. 

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Reading from right to left: the nucleon emits a meson losing energy $\omega(\vec{k})$, remains with energy $-\omega$ for a time and then reabsorbs the meson; it can do this for any $\vec{k}$ so we add all the intermediate states. Turning this around it is easy to use these ‘Feynman rules’ to write down contributions corresponding to

(try it!). Notice these involve more “loops”.

Of course the energy shift is not a matrix but $(\vec{K})^2 = \vec{k}^2$  and after a little work (16) leads to

$$\delta M^{[1]} = -\left(\frac{g}{2\pi}\right)^2 \int_0^\infty dk \left[ k^2 - m^2 + \frac{m^4}{k^2 + m^2} \right]$$

(17)

where $k = |\vec{k}|$. It is painfully obvious that only the last integral converges to $\pi m^3/2$, the rest diverge! This is analogous to (even classical) electrodynamics where in the self-energy of a point charge is infinite. To be honest we ought to insert a convergence or ‘form’ factor all the way back in $H_I$, but then the result depends on how we choose to ‘cutoff’.
Irrespective of details we can say that the nucleon mass $M$ is of the form

$$M = \hat{M} + \kappa_1 m^2 - \frac{g^2}{8\pi} m^3 + ... \quad (18)$$

where $\hat{M}$, which is what $M$ would be were the meson massless, and $\kappa_1$ ‘renormalised’ parameters hiding the strong cutoff dependence. The ellipsis represents weakly cutoff dependent parts, higher loops, etc. The first significant thing about (18) is that as a function of $m^2$, the parameter appearing in $H$, the unknown parameters appear in the analytic part whereas the non-analytic part is calculable. It is not hard to see why: if we tried to expand (17) in powers of $m^2$ we soon encounter integrals which diverge at the lower limit only, and these do not care how we ‘regularize’. One reason why this is significant is that in QCD the pseudoscalar mass squared is proportional to the quark mass, $m^2 \propto m_q$; the first two terms in $M$ give the Gell-Mann-Okubo relation for the barren octet and the equal splitting rule for the decuplet, the last the correlation to these.

But there is something deeper: the theory we are working with is ‘non-renormalizable’, signalled by needing $\kappa_1$ as a parameter in the 1-loop calculation. At 2-loops we need more, and ultimately to hide all our ignorance would require an infinite number of parameters! Once more, with feeling this time, the bits which are cutoff sensitive are analytic in $m^2$ so

$$M = \hat{M} + \kappa_1 m^2 + \kappa_2 m^4 + \cdots + \text{calculable} \quad (19)$$

Now if we replace the upper limit in (17) by $\Lambda$ with

$$\Lambda = \frac{2\pi}{g} \quad (20)$$

our one-loop calculation says $K_1^{[1]} = \Lambda^{-1}$. Generally then

$$m = \hat{m} + \bar{\kappa}_1 m^2/\Lambda + \bar{\kappa}_2 m^4/\Lambda^3 + \cdots + \text{calculable} \quad (21)$$

with $\bar{\kappa}_i$ a pure number of order unity.
We have arrived at the crux of why field theory is effective in the usual sense of the word. The infinity of parameters do not contribute equally, and higher orders are suppressed by powers of $m/\Lambda$. If we were calculating meson-nucleon scattering the corresponding series would be in $|\vec{q}|/\Lambda$ and $m/\Lambda$, $\vec{q}$ the meson momentum, so this only works for energies low compared to $\Lambda$. For the case in hand, $m/\Lambda \approx m_\pi/m_\rho \approx 140$ MeV/770 MeV and (18) is valid up to the 20% level (the same as recoil corrections).

**More Effective Theory**

In conclusion, let me stress that our modest calculation did not require that we know anything about the underlying theory, QCD. All we needed were the low energy degrees of freedom and their interaction. Now, quantum gravity is discarded as a fundamental theory because it is nonrenormalizable, involving as it does the dimensionful newtonian coupling

$$G = \ell_{pl}^2$$

where $\ell_{pl} \approx 10^{-33}$ cm is the Planck length. As noted by Donoghue [7], however, whatever the ultimate ‘Theory of Everything’ (GOD) quantum gravity can be treated as an effective field theory and e.g. quantum corrections to the newtonian potential

$$V(r) = -\frac{G m_1 m_2}{r} \left[ 1 + \beta \left( \frac{\ell_{pl}}{r} \right)^2 + \cdots \right]$$

are calculable. $\beta$ is a computable number of order unity, and the pathetic smallness of the correction is less significant than the realization that it can be done.

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5Similarly, heavy particle contributions are suppressed by powers of $1/m_H$. They are subsumed in $\hat{M}$ and $\vec{\kappa}_i$. 
Appendix

In case the reader did try and wants to check his/her work, the expressions corresponding to figure 2 are

\[
\Delta E^{[2a]} = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \left[ \frac{ig K}{\sqrt{2\omega}} \right] \left[ \frac{1}{-\omega} \right] \left[ \frac{ig K'}{\sqrt{2\omega'}} \right] \cdot \left[ \frac{1}{-\omega - \omega'} \right] \left[ -ig K' \right] \left[ \frac{1}{-\omega} \right] \left[ -ig K \right] \frac{1}{\sqrt{2\omega'}} \frac{1}{\sqrt{2\omega}}
\]

\[
\Delta E^{[2b]} = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \left[ \frac{ig K'}{\sqrt{2\omega'}} \right] \left[ \frac{1}{-\omega'} \right] \left[ \frac{ig K}{\sqrt{2\omega}} \right] \cdot \left[ \frac{1}{-\omega - \omega'} \right] \left[ -ig K' \right] \left[ \frac{1}{-\omega} \right] \left[ -ig K \right] \frac{1}{\sqrt{2\omega'}} \frac{1}{\sqrt{2\omega}}
\]

These are most difficult to evaluate, but as noted in the text contribute only at the 20% level.
References

[1] Glimpses of pre-QCD history may be found in: R. Oppenheimer, *Physics Today*, November 1966, 51; G. Veneziano, ibid, September 1969, 31.

[2] H.A. Bethe, *Phys. Rev.* 72 (1947) 339.

[3] S. Weinberg, *Physica* 96A (1979) 327.

[4] H. Georgi, “Weak Interactions and Modern Particle Physics”, Benjamin Cumming (1984); J.F. Donoghue, E. Golowich and B.R. Holstein, “Dynamics of the Standard Model”, Cambridge University Press (1992).

[5] G.C. Wick, *Rev. Mod. Phys.* 27 (1955) 339; also references therein.

[6] H. Lipkin, “Quantum Mechanics: New Approaches to Selected Topics”, North-Holland (1973).

[7] J.F. Donoghue, *Phys. Rev. Lett.* 72 (1994) 2996.