A Field-theoretical Interpretation of the Holographic Renormalization Group

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Abstract

A quantum-field theoretical interpretation is given to the holographic RG equation by relating it to a field-theoretical local RG equation which determines how Weyl invariance is broken in a quantized field theory. Using this approach we determine the relation between the holographic C theorem and the C theorem in two-dimensional quantum field theory which relies on the Zamolodchikov metric. Similarly we discuss how in four dimensions the holographic C function is related to a conjectured field-theoretical C function. The scheme dependence of the holographic RG due to the possible presence of finite local counterterms is discussed in detail, as well as its implications for the holographic C function. We also discuss issues special to the situation when mass deformations are present. Furthermore we suggest that the holographic RG equation may also be obtained from a bulk diffeomorphism which reduces to a Weyl transformation on the boundary.

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1 Introduction

In recent years, significant progress has been made towards obtaining holographic renormalization group (RG) flows from deformations of AdS space, thereby generalizing the AdS/CFT correspondence to non-conformal field theories. This line of investigation began with the work of [1] and [2]. Subsequently, different approaches have been pursued in order to obtain the crucial first-order differential equations from the supergravity equations of motion, in particular using supersymmetry [3] or the Hamilton-Jacobi equation [4]. These approaches show in particular that holographic RG flows are gradient flows governed by a superpotential. Moreover, a C theorem valid in arbitrary even dimensions has been derived for deformed AdS spaces [2, 3, 5, 6]. The crucial positivity condition arises from the weak energy condition or equivalently from the Raychaudhuri equation within supergravity.

Within the AdS/CFT correspondence and its generalizations to non-conformal field theories, the supergravity fields are sources for composite operators in the dual field theory. This applies in particular to the supergravity scalars, which are sources for composite scalar operators. From the quantum field theory perspective, we may view these operators as insertions whose couplings are given by the corresponding sources originating from supergravity. Therefore the AdS/CFT correspondence and its generalizations naturally lead us to quantum field theories with space-time dependent couplings. This implies that also the holographic renormalization group is local. For instance the Callan-Symanzik equation derived in [4] is of the local form

$$-\left(2g^{\mu\nu}(x)\frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^i(x)\frac{\delta}{\delta \phi^i(x)}\right)e^{(S-S_0)}$$

$$= \text{(local anomalies involving the curvature and } \partial_\mu \phi^i\text{)}, \quad (1.1)$$

with $g^{\mu\nu}(x)$ the metric on a $d$-dimensional hypersurface at $r = r_0$. The $\beta$ functions in (1.1) are given by derivatives of a superpotential with respect to the couplings. In a given regularization scheme, the anomalies on the r.h.s. arise from divergent local counterterms $S_0$ necessary to regulate the supergravity action $S$. Note that the l.h.s. of (1.1) is local - as opposed to global - since it involves functional derivatives. Of course the l.h.s. is also non-local in the sense that it contains operator insertions, and further functional derivatives of the l.h.s. will give rise to a Callan-Symanzik equation for correlation functions. The divergent counterterms $S_0$ and the anomaly on the r.h.s. however are local also in the sense that further functional derivatives give rise to terms involving delta functions.

The aim of this paper is to give a precise field-theoretical interpretation for this local holographic flow equation. This local equation expresses how Weyl symmetry is broken in the dual field theory, as opposed to the usual ‘global’ Callan-Symanzik equation which expresses how scale invariance is broken in a quantized field theory. Within standard quantum field theory, a local RG equation for general quantum field theories expressing how Weyl symmetry is broken has already been discussed comprehensively by Osborn in...
There, the space-time dependence of the couplings was used essentially as a trick for obtaining a precise definition for insertions of finite composite operators as functional derivatives of the generating functional with respect to the couplings. Furthermore the space-time dependent couplings allow for the study of Weyl symmetry consistency conditions for the local anomalies present in the local RG equation. These have been used in [7] to give an alternative derivation of the Zamolodchikov C theorem in two dimensions, using the positivity of the Zamolodchikov metric, which is defined in terms of the two-point functions of \( d = 2 \)-dimensional operators, to give the essential positivity of the flow of the C function. Moreover in [7], the implications of the Weyl consistency conditions were also considered in four dimensions and used to relate the RG flow of a conjectured C function to a quadratic form. However in four dimensions it has so far not been possible to show that this quadratic form is positive definite.

The essential ingredient for the field-theoretical interpretation of the holographic RG is to note that the equation (1.1) is precisely of the form of the field-theoretical RG discussed in [7]. This identification between the holographic and the field-theoretical local RG equation is confirmed by calculations of the anomalies on the r.h.s. of (1.1) for specific cases within the holographic RG approach [8, 9, 10]. These results may all be expanded in the basis given for these local anomalies within field theory in [7].

The identification of the holographic RG with the field theoretical local RG expressing how Weyl symmetry is broken allows us to derive a number of new results for the holographic RG. In particular it gives a field-theoretical interpretation in both two and four dimensions to the holographic C theorem proved in [3, 5]. The holographic C function is given by

\[
C = \frac{c_0}{|\mathcal{W}(\Phi)|^{d-1}}
\]

(1.2)

with \( \mathcal{W}(\Phi) \) the superpotential generating the flow. \( c_0 \) is a constant chosen such that at the fixed points, \( C \) coincides with the coefficient of the Euler anomaly. The dimension of the bulk space is \( d + 1 \). The flow of \( C \) is positive due to the weak energy condition. Furthermore it was shown in [11] that the flow of \( C \) may also be written in the form

\[
\mu \frac{d}{d\mu} C = -\frac{2}{3} C L_{ij} \beta^i \beta^j \leq 0,
\]

(1.3)

where \( L_{ij} \) is the metric on the space of supergravity scalars. The result (1.3) is scheme dependent, and the scheme in which it holds was named ‘holographic’ scheme in [11].

Our main result is to show - at least for a particular class of flows - that (1.3) coincides with the relation

\[
\mu \frac{d}{d\mu} \tilde{a} = -\beta^i \partial_i \tilde{a} = -\chi_{ij} \beta^j \beta^i, \quad \tilde{a} \equiv a + \beta^i w_i,
\]

(1.4)

derived within field theory in [7] using the Weyl consistency condition. Here \( a \) is the coefficient of the Euler anomaly contributing to the r.h.s. of (1.4), and \( \chi_{ij}, w_i \) are the
coefficients of anomaly contributions involving derivatives of the couplings. Using the results of [7], we study in detail the scheme dependence of $a$, $\chi$, $w$ originating from the possibility of adding finite local counterterms to $S_0$ in (1.1). In two dimensions, the results of [7] allow to relate (1.3) to the field-theoretical C theorem which relies on the Zamolodchikov metric, but for $d = 4$, (1.3) remains a scheme-dependent result from the quantum field-theoretical perspective.

We calculate $a$, $\chi$, $w$ explicitly for holographic theories within a minimal subtraction scheme. The essential ingredient for this calculation is that (1.1) imposes finiteness condition on the Weyl variation of the divergent local counterterms $S_0$ since the anomalies on the r.h.s. must be finite. We solve these finiteness conditions order by order in a strong-coupling perturbative expansion, in which each order is characterized by the number of derivatives of the superpotential with respect to the supergravity scalars. Since we use a cut-off $\varepsilon$ in the radial direction as regulator, this procedure corresponds to an extension of the anomaly calculation method of [12, 13, 14] to scalars with unprotected dimension. We show that the higher order corrections to $a$, $\chi$, $w$ cancel. Since our calculation of $a$, $\chi$, $w$ is in agreement with (1.3), we conclude that the ‘holographic’ scheme corresponds in fact to minimal subtraction.

Since the holographic RG characterizes the Weyl transformation properties of holographic theories, we suggest that it may be possible to give an alternative derivation of the holographic RG by considering a bulk diffeomorphism which reduces to a Weyl transformation on the boundary. This will correspond to generalizing the so-called PBH transformation [13], a bulk diffeomorphism with this property which has been applied to holographic theories in [10], to deformed AdS spaces. As a starting point for this program we generalize the PBH transformation to scalars in the conformal case. When conformal symmetry is broken, the scalars are expected to acquire an anomalous Weyl weight related to their $\beta$ function.

Moreover we point out that for a complete field-theoretical understanding of the holographic RG, it will be necessary to show that there is agreement between the RG equation and the renormalization procedure for correlation functions, as there is in standard quantum field theory. In this context we restrict our attention again to the simple case when the $\beta$ functions vanish. For this case we show that there is consistency between the PBH transformation - or equivalently the holographic RG - at the level of the action and at the level of the two-point functions. For the general case $\beta^i \neq 0$, the field-theoretical discussion of [7] indicates a possibility for proving the agreement between RG equation and correlator renormalization in general, and we hope that a generalization of the results presented here may serve as a basis for applying this analysis in the holographic context. This would provide a link between the holographic RG and two-point functions calculated for deformed AdS spaces for instance in [17, 18, 19, 20].

Our analysis applies to smooth flows from an UV to an IR fixed point, which may be described within the supergravity approximation. Typically we consider the case where
\( \mathcal{N} = 4 \) supersymmetry at the boundary is broken down to \( \mathcal{N} = 2 \) or \( \mathcal{N} = 1 \) supersymmetry by operator deformations. The identification of the holographic RG with the field-theoretical local RG and the proposed generalization of the PBH transformation apply to theories were conformal symmetry is broken either by relevant (as in the examples given in [3, 5]) or by marginal operator deformations. However the identification of (1.3) and (1.4) for the holographic C theorem applies so far only to the case when conformal symmetry is broken only by marginal operator deformations of engineering dimension \( d \). It is essential for our analysis that these operators acquire an anomalous dimension \( \gamma^i_j = \partial^i \beta^j \) along the flow \(^1\).

In the presence of relevant operator deformations, the relation between (1.3) and (1.4) is less obvious due to the following: Scalars dual to relevant operator deformations contribute anomalies of a different structure to the r.h.s. of (1.1) as those dual to marginal operators. This is due to the fact that the l.h.s. of (1.1) - and therefore also its r.h.s. - is of dimension \( d \). Since the scalars \( \phi^i \) dual to relevant operators are dimensionful, their anomaly contribution to the r.h.s. of (1.1) will contain less derivatives than those of the scalars dual to marginal operators. This coincides with the fact that correlators of lower dimension operators are less divergent as distributions. - Of course all of the supergravity scalars \( \Phi_i \) appearing in the bulk supergravity action are dimensionless. However in an expansion in the radial cut-off \( \varepsilon \) of dimension (length)\(^2\), they have the form \(^{13}\)

\[
\Phi^i(r) = \varepsilon^{(d - \Delta^i)/2} (\phi^i_0(r_0) + \phi^i_2(r_0) \varepsilon + \ldots), \quad \varepsilon = (r - r_0)^2,
\]

such that \( \phi^i_0(r_0) \) is of mass dimension \( (d - \Delta^i) \). It is \( \phi^i_0 \) and not \( \Phi^i \) which contributes to the anomaly in (1.1), since (1.1) holds in the case when \( \varepsilon = 0 \). For flows generated by relevant operators, we leave a complete study of the field-theory interpretation of the holographic C theorem for future work. Let us emphasize however that it was shown in the field theory analysis of [4] that the relation (1.4) is valid also in the presence of relevant operators whose coupling is of dimension two, i.e. which corresponds to a supergravity scalar with \( \Delta^i = 2 \) in \( d = 4 \). Still, in the field theory analysis of [4] the anomaly coefficients \( a, \chi, w \) depend only on dimensionless couplings dual to marginal operators. Therefore when relevant operator deformations are present, the relation between (1.4) and (1.3) - which involves the \( \Phi^i \) - is less clear.

A field-theoretical interpretation of the holographic RG complementary to the one presented here was given in [21] by relating the holographic RG to the exact renormalization group. A Wilsonian interpretation of the holographic RG was given in [22].

The paper is organized as follows: In section 2 we give a brief review of the field-theory results of [7] necessary for our analysis. In section 3 we discuss the conformal case, for which we generalize the PBH transformation to scalars and show that there is agreement between the holographic RG at the level of the action and at the level of the correlation

\(^{1}\)It will be interesting to construct an explicit example for such a holographic flow. We leave this for future work.
functions. We also discuss contributions to the Weyl anomaly for scalars dual to lower-dimensional relevant operators in this context. In section 4 we suggest how to generalize the PBH transformation to deformed AdS spaces. Furthermore we show that the result (1.3) for the holographic C theorem is in agreement with the Weyl consistency condition for the holographic RG. We discuss its scheme dependence and use the results of [7] to relate (1.3) to the field-theoretical C theorem in two dimensions. In section 5 we calculate the anomaly coefficients of (1.4), which are relevant to the C theorem, from finiteness conditions for the Weyl variation of the local divergent counterterms $S_0$ in (1.1) in a strong-coupling perturbative approach. We conclude by pointing out further directions for research in section 6.

2 Local RG equations in QFT

Here we give a brief review of the results for the local RG derived in [7] which are needed below. For further details we refer to [7]. We consider a renormalizable quantum field theory in $d$ dimensions which is coupled to a curved space background. Furthermore the couplings $\lambda_i$ are considered to be space-time dependent. We define insertions of the stress tensor and of scalar composite operators by virtue of

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}; \quad O_i = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \lambda_i}, \quad (2.1)$$

where the vacuum energy functional is given by $\exp W = \int D\phi \exp(-S)$.

These definitions ensure in turn a well-defined expression for correlation functions, eg. for

$$\langle O_i(x)O_j(y) \rangle = \left. \frac{\delta^2 W}{\delta \lambda_i(x) \delta \lambda_j(y)} \right|_{\lambda_k=\text{const.}}, \quad (2.2)$$

and similarly for higher point functions and for the energy-momentum tensor. The local RG equation discussed in [7] is of the form

$$- \int d^d x \sqrt{g} \sigma(x) \left[ 2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} - \beta^i \frac{\delta}{\delta \lambda^i} \right] W = \frac{1}{4\pi} \int d^d x \sqrt{g} \left[ \sigma B + \partial_{\mu} \sigma Z^\mu \right]. \quad (2.3)$$

The right hand side is a local anomaly which involves local terms depending on the curvature tensors and on derivatives of the coupling. The exact form of these terms depends on the dimension. $\sigma(x)$ is an arbitrary function. The interpretation of (2.3) is that the terms breaking Weyl symmetry in a renormalizable quantum field theory are either non-local terms with a $\beta$ function as their coefficient or local terms. Note that we are only considering dimensionless couplings - and thus dimension $d$ operators $O_i$ - in (2.3). Mass terms may be considered separately [7].
It was shown in [23] using a perturbative approach and dimensional regularisation that examples for both scalar and gauge theories satisfy a local RG equation of the form (2.3). Using the BPHZ approach, similar local RG equations have been derived for $\phi^4$ theory in [24].

When only dimensionless couplings are present, the vacuum energy functional satisfies the relation

$$\left( \mu \frac{\partial}{\partial \mu} + 2 \int d^d x g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} \right) W = 0$$

with $\mu$ the renormalization scale.

In two dimensions the explicit form of the local anomaly terms in (2.3) is

$$- \int d^2 x \sigma \left( 2 g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} - \beta^i \frac{\delta}{\delta \lambda^i} \right) W = \frac{1}{4\pi} \int d^2 x \left[ \sigma \left( -a R + \chi_{ij} \partial^\mu \lambda^i \partial_\mu \lambda^j \right) + 2 \partial^\mu \sigma w_i \partial_\mu \lambda^i \right].$$

(2.5)

The generating functional $W$ satisfies Wess-Zumino consistency conditions for Weyl symmetry. Defining

$$\Delta[\sigma] \equiv 2 \int d^d x \sigma g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} \quad \text{and} \quad \Delta_\beta[\sigma] \equiv \int d^d x \sigma \beta^i \frac{\delta}{\delta \lambda^i},$$

(2.6)

the consistency conditions read

$$[(\Delta[\sigma] - \Delta_\beta[\sigma]), (\Delta[\sigma'] - \Delta_\beta[\sigma'])] W = 0 \quad .$$

(2.7)

These conditions imply relations for the coefficients of the local anomaly terms. In particular they imply, with $\delta g^{\mu \nu} = 2 \sigma g^{\mu \nu}, \delta R = 2 \sigma R + 2 \nabla^2 \sigma,$

$$\beta^i \partial_i \tilde{a} = \chi_{ij} \beta^j, \quad \tilde{a} \equiv a + \beta^i w_i \quad .$$

(2.8)

This relation is invariant under the addition of finite local counterterms to $W$. For instance, adding

$$\delta W = - \frac{1}{4\pi} \int d^2 x \sqrt{g} \left( b R - c_{ij} \partial_\mu \lambda^j \partial^\mu \lambda^i \right),$$

(2.9)

to $W$, we have

$$\delta a = \beta^i \partial_i b, \quad \delta \chi_{ij} = \mathcal{L}_\beta c_{ij}, \quad \delta w_i = - \partial_i b + c_{ij} \beta^j, \quad \delta \tilde{a} = c_{ij} \beta^i \beta^j, \quad$$

(2.10)
with $\mathcal{L}_\beta$ the Lie derivative
\[
\mathcal{L}_\beta c_{ij} = \beta^k \partial_k c_{ij} + \partial_i \beta^k c_{kj} + \partial_j \beta^k c_{ik}.
\] (2.11)

Differentiating (2.11) further with respect to $\lambda_i$ for $\sigma \equiv 1$ gives
\[
\mathcal{D}\langle O_i(x)O_j(0)\rangle + \partial_i \beta^k \langle O_k(x)O_j(0)\rangle + \partial_j \beta^k \langle O_i(x)O_k(0)\rangle = 8\pi \chi_{ij} \partial^2 \delta^{(2)}(x),
\] (2.12)
\[
\mathcal{D} \equiv \mu \frac{\partial}{\partial \mu} + \beta^i \frac{\partial}{\partial \lambda^i}.
\] (2.13)

The form of the scalar two-point function is determined by rotational invariance and the dimension of the operators even when conformal symmetry is broken. Furthermore it has to be ensured that the two-point function has a well-defined Fourier transform such that it does not contain any poles in particular for $x \to 0$. A form for the two-point function which meets these requirements is given by
\[
\langle O_i(x)O_j(0)\rangle = \partial^2 \partial^2 \Omega_{ij}(t), \quad t = \frac{1}{2} \ln \mu^2 x^2.
\] (2.14)

The extraction of derivatives is similar to the method of differential regularisation [25]. For the energy-momentum tensor two-point function we have similarly
\[
\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle = (\delta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu})(\delta_{\sigma\rho} \partial^2 - \partial_{\sigma} \partial_{\rho}) \Omega(t).
\] (2.15)

Inserting (2.14) into (2.12) we obtain, using $\partial_\mu (x_\mu/x^2) = 2\pi \delta^{(2)}(x)$,
\[
\mathcal{D}\Omega'_{ij} + \partial_i \beta^k \Omega'_{kj} + \partial_j \beta^k \Omega'_{ik} = 4\chi_{ij},
\] (2.16)
and in an analogous calculation a similar result for $\Omega$ defined in (2.13). The positive definite Zamolodchikov metric is defined by
\[
G_{ij}(x) \equiv \frac{1}{16} (x^2)^2 \langle O_i(x)O_j(0)\rangle.
\] (2.17)

Using (2.12) and (2.16) we find
\[
G_{ij} = \frac{1}{4} (\Omega''_{ij} - \Omega''_{ij} + \frac{1}{4} \Omega'''_{ij}) = \chi_{ij} + \mathcal{L}_\beta c_{ij},
\] (2.18)
\[
c_{ij} = \frac{1}{4} (-\Omega'_{ij} + \Omega'_{ij} - \frac{1}{4} \Omega''_{ij}).
\] (2.19)

The two-dimensional C theorem is then easily obtained by defining the C function to be
\[
C \equiv 3(\tilde{a} + c_{ij} \beta^i \beta^j) = -\frac{3}{4} (\Omega' + \Omega'' - \frac{1}{4} \Omega''' - \frac{1}{4} \Omega''')
\] (2.20)

which satisfies
\[
C'' = -\beta^i \partial_i C = -3G_{ij} \beta^i \beta^j \leq 0.
\] (2.21)
Note that the linear combination of derivatives of $\Omega$ in (2.20) corresponds exactly to the $C$ function originally considered by Zamolodchikov [26].

In four dimensions the structure of the local anomaly is more complicated due to the larger number of possible independent forms. With the definitions in (2.6) we have

$$- (\Delta[\sigma] - \Delta_\beta[\sigma]) W = \int d^4 x \sqrt{g} [\sigma B + \partial^\mu \sigma Z_\mu], \quad (2.22)$$

where

$$B = c F - a G + \frac{1}{9} \beta_c R^2$$

$$+ \frac{1}{3} \lambda_i \partial^i \lambda R + \frac{1}{6} \lambda_i \partial^i \lambda R - \frac{1}{2} \lambda_i \partial^i \lambda^i \lambda^i G_{\mu\nu} + \frac{1}{2} \lambda_i \nabla^2 \lambda \lambda^i \lambda^j + \frac{1}{2} \lambda_i \partial^i \lambda \lambda^j \lambda^k. \quad (2.23)$$

Here $R$ is the Ricci scalar, $F$ the square of the Weyl tensor $C_{\mu\sigma\rho\nu}$, $G$ the Euler density and $G_{\mu\nu}$ the Einstein tensor. For $Z^\mu$ we have

$$Z^\mu = - G_{\mu\nu} w_i \partial^\nu \lambda_i + \frac{1}{3} \partial_\mu (q R) + \frac{1}{3} R Y_i \partial_\mu \lambda^i + \partial_\mu (U_i \nabla^2 \lambda_i + \frac{1}{2} V_i \partial_\nu \lambda_i \partial^\nu \lambda_i) + S_{ij} \partial_\mu \lambda_i \lambda^i \lambda^k + \frac{1}{2} T_{ijk} \partial_\nu \lambda_i \partial^\nu \lambda^j \partial_\nu \lambda^k. \quad (2.24)$$

In this case the consistency condition (2.7) implies a number of relations for the anomaly coefficients, among which

$$\beta^i \partial_\mu \tilde{a} = \frac{1}{2} \lambda_\mu \beta^i \beta^j, \quad \tilde{a} \equiv a + \frac{1}{8} w_i \beta^i. \quad (2.25)$$

This is analogous to the relation (2.8). However in four dimensions it has not been possible so far to relate $\chi^g_\mu$ to the Zamolodchikov metric, unlike the two-dimensional result (2.18). This may be traced back to the fact that (2.23) relates $\chi^g_\mu$ to the three-point function $\langle T_{\mu\nu} O_i O_j \rangle$ rather than just to the two-point function $\langle O_i O_j \rangle$.

3 The $\beta^i = 0$ limit of the holographic RG

To show that the holographic RG is consistent with the results presented in section 2, we begin by considering the simple case in which the $\beta$ functions vanish. We extend the PBH transformation to scalars and show how it gives rise to a holographic local RG. We discuss the anomalies present in this RG, and demonstrate consistency with the local RG for correlation functions. We also discuss operators of lower dimension.

We consider $(d+1)$-dimensional AdS space with metric of Fefferman-Graham form [27]

$$ds^2 = G_{MN} dX^M dX^N = \frac{L^2}{4} \left[ \left( \frac{d\rho}{\rho} \right)^2 + \frac{1}{\rho^2} g_{\mu\nu}(x, \rho) dx^\mu dx^\nu \right]. \quad (3.1)$$
where $M, N = 1, \ldots, 5$ and $\mu, \nu = 1, \ldots, 4$. The boundary is at $\rho = 0$. $L$ is the AdS radius, and $\rho$ is of dimension (length)$^2$. Moreover we consider the bosonic part of the supergravity action

$$S = \frac{1}{16\pi G_N} \left[ \int d^{(d+1)}x \sqrt{G} \left[ R + 2\Lambda + L_{ij} \partial^{\mu} \Phi^i \partial_\mu \Phi^j \right] - \int d^d x \sqrt{\gamma} 2K \right], \quad (3.2)$$

where $L_{ij}$ is a metric on the space of supergravity scalars.

As discussed in [16], there is a $(d+1)$-dimensional diffeomorphism $v_M$ [15] which reduces to a Weyl transformation on the boundary. This is the so-called PBH transformation. With the ansatz

$$\rho = \rho' e^{2\sigma(x')} \simeq \rho'(1 + 2\sigma(x')) \quad , \quad (3.3)$$

$$x_\mu = x'_\mu + a_\mu(x', \rho') \quad , \quad v_M = (2\sigma, a_\mu), \quad (3.4)$$

where the requirement of diffeomorphism invariance of the $(d+1)$-dimensional metric constrains the form of $a_\mu$, $g^{\mu\nu}$ has the transformation property

$$\delta g^{\mu\nu} = 2\sigma (1 - \rho \partial_\rho) g^{\mu\nu}(x, \rho) + \nabla^\mu a^\nu(x, \rho) + \nabla^\nu a^\mu(x, \rho). \quad (3.5)$$

Similarly the PBH transformation for the scalars is obtained using

$$\Phi^i(x, \rho) = \rho^{(d - \Delta^i)/2} \phi^i(x, \rho). \quad (3.6)$$

This gives

$$\delta \phi^i(x, \rho) = -\sigma (\Delta^i - d) \phi^i(x, \rho) + 2\sigma \rho \partial_\rho \phi^i(x, \rho) + a^\mu \partial_\mu \phi^i(x, \rho). \quad (3.7)$$

$i$ is an index in the field space.

For $G^{MN}(x, \rho)$, $\Phi^i(x, \rho)$ which are solutions of an equation of motion, we have the expansion

$$g^{\mu\nu}(x, \rho) = \sum_{n=0}^{\infty} g_{(n)}^{\mu\nu}(x) \rho^n \quad , \quad \Phi^i(x, \rho) = \sum_{n=0}^{\infty} \phi_{(n)}^i(x) \rho^n. \quad (3.8)$$

For $\Delta^i = d/2$ we have, instead of (3.6), [28, 29, 13]

$$\Phi^i(x, \rho) = \rho \ln \rho \sum_{n=0}^{\infty} \phi_{(n)}^i(x) \rho^n. \quad (3.9)$$

$\phi_{(0)}^i$ is of mass dimension $d - \Delta^i$. At the boundary the PBH transformation reduces to a Weyl transformation under which in particular

$$\delta g_{(0)}^{\mu\nu}(x) = 2\sigma g_{(0)}^{\mu\nu}(x) \quad , \quad \delta \phi_{(0)}^i(x) = -(\Delta^i - d) \sigma \phi_{(0)}^i(x). \quad (3.10)$$

$^2$Throughout we use the conventions of [13].
As is well known, the action (3.2) is divergent near the boundary an requires regularization by subtraction of appropriate divergent counterterms $S_0$. Introducing a cut-off at $\rho = \varepsilon$, the PBH transformation of the action reduces near the boundary to

$$- \left( 2 \varepsilon \frac{\delta}{\delta \varepsilon} + 2 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}_{(0)}} - \left( \Delta^{(i)} - d \right) \phi^i_{(0)} \frac{\delta}{\delta \phi^i_{(0)}} \right) e^{S - S_0} = \mathcal{A}_4 \ .$$  \hspace{1cm} (3.11)

In (3.11) we may take the limit $\varepsilon \to 0$ such as to obtain

$$- \left( 2 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}_{(0)}} - \left( \Delta^{(i)} - d \right) \phi^i_{(0)} \frac{\delta}{\delta \phi^i_{(0)}} \right) e^W = \mathcal{A}_4 \hspace{1cm} (3.12)$$

with $e^W \equiv \lim_{\varepsilon \to 0} e^{S - S_0}$ the generating functional and $\mathcal{A}_4$ a local anomaly. This equation coincides with the field-theoretical local RG equation (2.3) for the case when $\beta^i = 0$.

We proceed by calculating the local anomaly terms on the r.h.s. of (3.12) explicitly. We begin with the contributions from supergravity scalars dual to operators of dimension $d = 4$. As discussed in [30] and later in [31], the counterterm $S_0^{(4)}$ necessary to regulate $S$ as far as fields dual to dimension $d = 4$ operators are concerned is given by the action of four-dimensional conformal supergravity,

$$S_0^{(4)} = - \frac{1}{2} \ln \varepsilon \frac{N^2}{4(4\pi^2)} S_{\text{CSG}} \ , \quad S_{\text{CSG}} = \int d^4 x \sqrt{g} L_{\text{CSG}} ,$$

$$L_{\text{CSG}} = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - \varepsilon_{\mu\rho\alpha\beta} \varepsilon_{\nu\sigma\gamma\delta} R^{\mu\nu\rho\sigma} R_{\alpha\beta\gamma\delta} + L_i^{(4)} \phi^i_{(0)} \Delta_4 \phi^i_{(0)} , \hspace{1cm} (3.13)$$

where $L_i^{(4)}$ is the metric on the space of supergravity scalars dual to dimension 4 operators and $\Delta_4$ is the Riegert operator [12].

$$\Delta_4 \equiv \nabla^2 \nabla^2 + 2 \nabla^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \nabla^\nu . \hspace{1cm} (3.14)$$

This result for $S_0^{(4)}$ agrees with the anomaly calculation of Henningson and Skenderis [12], extended to scalar fields in [13]. Inserting (3.13) into the local RG equation (3.11) with $\Delta = d = 4$, and subsequently taking the limit $\varepsilon \to 0$, we obtain for the local anomaly

$$\mathcal{A}_4 = \frac{N^2}{4(4\pi^2)} L_{\text{CSG}} . \hspace{1cm} (3.15)$$

Comparing with the field-theoretical local RG (2.22) with (2.23) we identify

$$\chi^a_{ij} = \frac{N^2}{2(4\pi^2)} L_i^{(4)} , \quad \chi^g_{ij} = \frac{N^2}{(4\pi^2)^2} L_i^{(4)} , \quad c = a = \frac{N^2}{4(4\pi^2)} . \hspace{1cm} (3.16)$$

The result for $\chi^a_{ij}$ may also be obtained from considering the implications of the local RG for the two-point function $\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle$, which provides a consistency check. With the definition

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle \equiv \delta \frac{\delta}{\delta \phi^i_{(0)}} \frac{\delta}{\delta \phi^j_{(0)}} e^{S - S_0} \bigg|_{\phi^k_{(0)} = \text{const}} , \hspace{1cm} (3.17)$$

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle \equiv \frac{\delta}{\delta \phi^i_{(0)}} \frac{\delta}{\delta \phi^j_{(0)}} e^{S - S_0} \bigg|_{\phi^k_{(0)} = \text{const}} , \hspace{1cm} (3.17)$$

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the standard AdS/CFT procedure gives the result
\[ \langle O_i(x) O_j(0) \rangle = \frac{N^2}{2(4\pi)^2} L_{ij}^{(4)} \partial^2 \partial^2 \left( \frac{1}{(x^2)^2} - \frac{1}{2} \ln \varepsilon \delta^{(4)}(x) \right) , \] (3.18)
where the local term guarantees a well-defined expression with a regular Fourier transform also when \( \bar{x} \to 0 \). In (3.18), derivatives have been extracted following the prescription of differential regularization [25] in order to obtain a less singular expression. A similar structure of the AdS/CFT correlators was discussed in [33] using dimensional regularization.

On the other hand, integrating the local RG (3.11) for \( \Delta_i = d = 4 \) using
\[ \left( \mu \frac{\partial}{\partial \mu} + 2 \int d^4x \sqrt{g} g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} \right) e^W = 0 , \varepsilon \frac{\partial}{\partial \varepsilon} = \int d^4x \sqrt{g} \varepsilon \frac{\delta}{\delta \varepsilon} , \] (3.19)
and varying the result twice with respect to \( \phi_i(0) \), we obtain
\[ \left( -2\varepsilon \frac{\partial}{\partial \varepsilon} + \mu \frac{\partial}{\partial \mu} \right) \langle O_i^{(4)}(x) O_j^{(4)}(0) \rangle = \frac{N^2}{2(4\pi)^2} L_{ij}^{(4)} \partial^2 \partial^2 \delta^4(x) . \] (3.20)
It is easy to check that the expression (3.18), which is independent of the renormalization scale \( \mu \), satisfies (3.20). From the results of [34] for general conformal field theories, we expect a similar consistency between the RG equation for the energy-momentum tensor two-point function, and the Weyl tensor squared anomaly in the RG equation at the level of the generating functional.

For scalar operators of dimension \( d - 2 = 2 \), the dual scalar \( \phi_i(0) \) as given by (3.9) is of mass dimension 2. The appropriate boundary counterterm for regulating the action \( S \) is given by
\[ S_0^{(2)} = \frac{N^2}{2\pi^2} \frac{1}{2} \ln \varepsilon \int d^4x \sqrt{g} L_{ij}^{(2)} \phi_i(0) \phi_j(0) . \] (3.21)
In the RG equation (3.11), the counterterm (3.21) gives rise to an anomaly
\[ \mathcal{A}_4^{(2)} = \frac{N^2}{2\pi^2} L_{ij}^{(2)} \phi_i(0) \phi_j(0) , \] (3.22)
and for the correlation function as defined in (3.17) we obtain
\[ \langle O_i(x) O_j(0) \rangle = \frac{N^2}{2\pi^2} L_{ij}^{(2)} \left( \frac{1}{(x^2)^2} - \frac{1}{2} \ln \varepsilon \delta^{(4)}(x) \right) , \] (3.23)
which is consistent with
\[ \left( -2\varepsilon \frac{\partial}{\partial \varepsilon} + \mu \frac{\partial}{\partial \mu} \right) \langle O_i^{(2)}(x) O_j^{(2)}(0) \rangle = \frac{N^2}{2\pi^2} L_{ij}^{(2)} \delta^4(x) . \] (3.24)
Note the different structure of the anomalies for operators with \( \Delta_i = 4 \) in (3.15) with (3.13) and with \( \Delta_i = 2 \) in (3.22).
4 Local RG equation from deformed AdS space

For the general case $\beta^i \neq 0$, we consider a $(d + 1)$-dimensional space with metric
\[
ds^2 = e^{2A(r)} g_{\mu\nu}(x) dx^\mu dx^\nu + dr^2 .
\] (4.1)

Our analysis applies to flows from an UV to an IR fixed point, i.e. the space is asymptotically AdS both for $r \to \infty$ and for $r \to -\infty$. The bosonic part of the supergravity action is given by
\[
S = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{G} \left[ R + G^{\mu\nu} L^{ij}(\Phi) \partial_\mu \Phi_i \partial_\nu \Phi_j + V(\Phi) \right] - \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} 2K
\] (4.2)

Here $G$ denotes the $(d+1)$-dimensional metric, $\gamma$ is the induced metric on the boundary and $L^{ij}$ is the metric on the space of the scalar supergravity fields which is positive definite. As may be derived using either supersymmetry [3] or the Hamilton-Jacobi approach to holographic RG flows [4], the RG flow is generated by the first-order equations
\[
\frac{dA(r)}{dr} = -\frac{g}{3} \mathcal{W}(\Phi), \quad \frac{d\Phi^i(r)}{dr} = \frac{g}{2} L^{ij} \frac{\partial \mathcal{W}(\Phi)}{\partial \Phi^j},
\] (4.3)

with the superpotential
\[
\mathcal{W}(\phi) = \frac{g^2}{8} L^{ij} \frac{\partial \mathcal{W}}{\partial \Phi^i} \frac{\partial \mathcal{W}}{\partial \Phi^j} - \frac{g^2}{3} \mathcal{W}(\Phi)^2 .
\] (4.4)

Here $g$ is the Yang-Mills coupling.

For the definition of the renormalization scale and of the $\beta$ functions we follow [35, 11, 36] and define
\[
\ln \mu \equiv A(r), \quad \beta^i \equiv \mu \frac{d}{d\mu} \Phi^i = \frac{d\Phi^i}{dA(r)} .
\] (4.5)

Using (4.3) we have
\[
\beta^i = -\frac{3}{2} \frac{1}{\mathcal{W}} L^{ij} \frac{\partial \mathcal{W}}{\partial \Phi^j} .
\] (4.6)

The choice (4.5) for the definition of the $\beta$ functions corresponds to the choice of a particular renormalization scheme. However requiring (4.6) to be covariant under a change of scheme puts some restrictions on the possible definitions for $\beta^i$ : Suppose we would define
\[
\beta^i \equiv \frac{d\Phi^i}{df(r)} .
\] (4.7)
with some arbitrary function $f(r)$. Then

$$\beta^i = \frac{d\Phi^i}{df(r)} = \frac{d\Phi^i}{dr} \cdot \frac{1}{f'} = \frac{g}{2} \frac{L^{ij}}{f'} \partial W \cdot \frac{1}{f'} = -\frac{3}{2} \frac{L^{ij}}{W} \partial \Phi^j \cdot \frac{A'}{f'}.$$  (4.8)

We see that covariance of (4.6) requires $f$ to be related to $A$ in general, though it is conceivable that for a given theory, redefinitions of $\phi_i$ and $W$ exist such as to ensure the relation (4.6) for more general $f'$s.

We assume that the PBH transformation as discussed in section 3 may be generalized to deformed AdS spaces, i.e. we assume that there is a $(d+1)$-dimensional diffeomorphism under which both the metric (4.1) - after a suitable coordinate transformation to a generalized Fefferman-Graham form - and the action $S$ are invariant, and which reduces to a Weyl transformation on the $d$-dimensional hypersurface at $r = r_0$. $A(r_0) = \ln \mu_0$ defines a renormalization scale $\mu_0$. For deformed AdS spaces, we expect the supergravity scalars to acquire an anomalous Weyl weight related to their $\beta$ function. Of course it would be very interesting to work out the exact form of the corresponding $(d+1)$-dimensional diffeomorphism. We leave this for future work. For our purposes here, we assume that the Weyl transformation induced by the PBH transformation is of the form

$$\delta \phi_0^i(x, r_0) = -\sigma(\Delta^{(i)} - d + \bar{\beta}^{(i)}) \phi_0^i(x, r_0)$$  (4.9)

for the lowest order term in the expansion of $\phi^i(r, x)$ in $\varepsilon = (r - r_0)^2$. $\beta^{(i)}$ is related to the $\beta$ function by

$$\beta^i = \beta^{(i)} \Phi^i.$$  (4.10)

In the limit $\varepsilon \to 0$, subject to suitable regularization, the equation expressing $(d+1)$-dimensional diffeomorphism invariance of the action $S$ then reduces to

$$\lim_{\varepsilon \to 0} \int d^d x \sigma(x) \left( -2\varepsilon \frac{\delta}{\delta \varepsilon} - 2g_0^{\mu\nu} \frac{\delta}{\delta g_0^{\mu\nu}} + (\Delta^{(i)} - d + \bar{\beta}^{(i)}) \phi_0^i \frac{\delta}{\delta \phi_0^i} \right) e^{S - S_0} = \int d^d x \left( \sigma(x) B + \partial^\mu \sigma Z_\mu \right),$$  (4.11)

where

$$\int d^d x \left( \sigma(x) B + \partial^\mu \sigma Z_\mu \right) = \lim_{\varepsilon \to 0} \int d^d x \sigma \left( 2\varepsilon \frac{\delta}{\delta \varepsilon} + 2g_0^{\mu\nu} \frac{\delta}{\delta g_0^{\mu\nu}} - (\Delta^{(i)} - d + \bar{\beta}^{(i)}) \phi_0^i \frac{\delta}{\delta \phi_0^i} \right) S_0.$$  (4.12)

(4.12) imposes a finiteness condition on the Weyl variation of the divergent local counterterms $S_0$ since $B$, $Z_\mu$ have to be finite. We investigate the implications of this finiteness condition in detail in section 5 below. Note also that

$$\hat{\beta}^i = \frac{d\Phi^i(r, x)}{dA(r)} = \frac{d\Phi^i(r_0, x)}{dA(r_0)} + O(\varepsilon) = \beta^i + O(\varepsilon).$$  (4.13)
Since the limit in (4.11) is finite, we obtain
\[ \int d^4x \sigma(x) \left( -2g_0^{\mu\nu} \frac{\delta}{\delta g_0^{\mu\nu}} + (\Delta^{(i)} - d + \beta^{(i)}) \phi_0^i \frac{\delta}{\delta \phi_0^i} \right) e^{S - S_0} = \int d^4x \left( \sigma(x)B + \partial^{\mu} \sigma Z_\mu \right). \]

(4.14)

Since \( \sigma(x) \) is an arbitrary function, this equation is identical to the local Callan-Symanzik equation obtained in the Hamilton-Jacobi formalism in [4], up to terms involving \( (\Delta^{(i)} - d) \phi_0^i \). These terms are due to the fact that (4.14) is a RG equation rather than a Callan-Symanzik equation. In the special case when \( \Delta^{(i)} = d \) for all \( i \), the local Callan-Symanzik equation and (4.14) coincide to give
\[ \int d^4x \sqrt{g} \sigma(x) \left( -2g_0^{\mu\nu} \frac{\delta}{\delta g_0^{\mu\nu}} + \beta^i \frac{\delta}{\delta \phi_0^i} \right) e^{S - S_0} = \int d^4x \sqrt{g} [\sigma B + \partial^{\mu} \sigma Z_\mu]. \]

(4.15)

For general \( \Delta^{(i)} \), using that
\[ \left( \mu \frac{\partial}{\partial \mu} + \beta^i \frac{\partial}{\partial \phi_0^i} \right) e^{S - S_0} = 0, \]
and in the limit when the \( x \)-dependent fluctuations of \( \phi_0^i(x, r_0) \) vanish, (4.14) gives
\[ \left( \mu \frac{\partial}{\partial \mu} + \beta^i \frac{\partial}{\partial \phi_0^i} \right) e^{S - S_0} = \int d^4x B, \]
for \( \sigma(x) = \text{const.} \) (4.17) is the well-known global form for the Callan-Symanzik equation.

The main point of our analysis is that (4.15) is exactly of the same form as the field-theoretical local RG equation (2.3) presented in section 2. As a consequence we find that the field-theoretical discussion of the implications of Weyl consistency on \( B \) and \( Z_\mu \) applies in particular to the holographic RG, such that we may use the results of the field theory discussion in order to obtain new information about the holographic flow. On the field theory side, a basis for the local anomaly contributions \( B \) and \( Z_\mu \) is given in (2.23). On the supergravity side, the local anomaly contributions were calculated for specific cases for instance in [8, 9, 10]. These results agree with the field-theoretical basis.

For analyzing the implications of the Weyl consistency conditions for the holographic RG flows, we restrict ourselves first to the case when all active supergravity scalars are dual to operators with \( \Delta^{(i)} = d \), such that (4.15) applies.

The implications of the Weyl consistency conditions enable us in particular to relate the \( C \) function of supergravity [3, 4],
\[ C(r) = \frac{c_0}{|W|^{d-1}}, \]
(4.18)

Note that in standard quantum field theory, the RG equation determines how a renormalized theory evolves under a change of the renormalization scale, while the Callan-Symanzik equation determines how a renormalized theory evolves under a scale transformation. The two equations coincide only in the case when the theory is massless in the classical limit, or more generally when all couplings are dimensionless, which corresponds to \( \Delta^{(i)} = d \).
whose flow is positive definite due to the weak energy condition, to the coefficients in the anomalies $B$ and $Z_{ \mu}$. These coefficients are functions of the scalar supergravity fields once conformal symmetry is broken. Let us first consider the case $d = 2$, for which we know within quantum field theory that the $C$ theorem holds. Weyl consistency implies the relation

$$\partial_i a = \chi_{ij} \beta^j - \beta^i \partial_j w_i - \partial_i \beta^j w_j ,$$

with the notation of (2.5). For the holographic flows this relation is satisfied in particular by

$$a = C - L_{ij} \beta^i \beta^j , \quad C = \frac{c_0}{|W|} , \quad w_i = L_{ij} \beta^j = -\frac{3}{2} \partial_i \ln W , \quad \chi_{ij} = \frac{2}{3} C L_{ij} . \quad (4.20)$$

Of course, this is just a particular solution. In section 5 below, we calculate these anomaly coefficients in a well-defined regularization scheme.

The result (4.20) corresponds just to the “holographic” scheme as defined in [11]. With (4.20) the equation

$$\beta^i \partial_i \tilde{a} = \chi_{ij} \beta^j , \quad \tilde{a} = a + \beta^i w_i , \quad (4.21)$$

for the anomaly coefficients corresponds just to the well-known holographic $C$ theorem

$$\dot{C} = -\beta^i \partial_i C = -\frac{2}{3} C L_{ij} \beta^i \beta^j \leq 0 , \quad (4.22)$$

where the inequality follows from the weak energy condition, or equivalently from the positivity of $L_{ij} \beta^i \beta^j$ and of $C$. Of course, the results (4.20) for the anomaly coefficients are scheme dependent and in particular the addition of finite local counterterms would change the result in exactly the same way as obtained in field theory, (2.9). This implies that the holographic $C$ theorem is scheme-dependent. For instance, $w_i$ may be set to zero, and correspondingly $a$ to $C$, by adding a finite local term

$$S_f^R = -\frac{3}{2} \int d^2 x \sqrt{g} \ln W \ R \quad (4.23)$$

to the vacuum energy functional. $\tilde{a} = a + \beta^i w_i$ remains invariant under this change. Moreover, when a finite local boundary term

$$S_f^\partial = \int d^2 x \sqrt{g} c_{ij} \partial^\mu \phi_0^i \partial_\mu \phi_0^j \quad (4.24)$$

is added to the vacuum energy functional such that

$$\delta \tilde{a} = c_{ij} \beta^j , \quad \delta \chi_{ij} = \mathcal{L}_\beta c_{ij} , \quad (4.25)$$

with $\mathcal{L}_\beta$ the Lie derivative as in (2.11), then although the equation (4.21) remains invariant under this change of scheme, it is not clear if the new

$$\chi'_{ij} \equiv \chi_{ij} + \delta \chi_{ij} = \frac{2}{3} C L_{ij} + \mathcal{L}_\beta c_{ij} \quad (4.26)$$
remains positive. In two dimensions however, the field theory $C$ theorem holds independently of the scheme chosen. As discussed in section 2, within the Weyl consistency approach, the key ingredient for the proof of the $C$ theorem is $G_{ij} = \chi_{ij} + \mathcal{L}_\beta c_{ij}$ with $G_{ij}$ the Zamolodchikov metric. This implies that $\chi_{ij} + \mathcal{L}_\beta c_{ij}$ is positive definite independently of the scheme, such that the flow of $C_t \equiv \tilde{a} + c_{ij} \beta^i \beta^j$ is positive for any renormalization scheme, i.e. independently of the explicit form of $c_{ij}$. For the holographic flows in $d = 2$ this implies that the $C$ theorem holds independently of the form of possible finite counterterms. We note that for the “holographic” scheme, which corresponds to $c_{ij} = 0$, i.e. to minimal subtraction, the field-theoretical and the holographic $C$ theorem coincide.

Similarly for $d = 4$ we may show that for the holographic scheme, the holographic $C$ function (4.18), which satisfies

$$- \beta^i \partial_i C = - 2 C L_{ij} \beta^i \beta^j ,$$  \hspace{1cm} (4.27)

coinsides with the field theory relation

$$\beta^i \partial_i \tilde{a} = \frac{1}{8} \chi_{ij} \beta^i \beta^j , \quad \tilde{a} = a + \frac{1}{8} \beta^i w_i , \hspace{1cm} (4.28)$$

with the anomaly coefficients defined by (2.23), if

$$a = C - L_{ij} \beta^i \beta^j , \quad w_i = 8 L_{ij} \beta^j \quad \chi_{ij}^g = 16 C L_{ij} , \quad \tilde{a} = C , \hspace{1cm} (4.29)$$

or equivalently - after adding the appropriate finite counterterm - by

$$a = C , \quad w_i = 0 \quad \chi_{ij}^g = 16 C L_{ij} . \hspace{1cm} (4.30)$$

Again the relation (4.28) is invariant under the addition of finite boundary counterterms under which $\chi_{ij}^g \rightarrow \chi_{ij}^g + \mathcal{L}_\beta c_{ij}$. However in $d = 4$ it has not yet been possible to relate $\chi_{ij}^g + \mathcal{L}_\beta c_{ij}$ to the Zamolodchikov metric, such as to show positivity. Therefore, from a field-theoretical point of view, the holographic $C$ theorem in $d = 4$ is a scheme-dependent result.

5 Anomaly coefficients within minimal subtraction

In this section we show that the particular solution for the anomaly coefficients which corresponds to the holographic $C$ theorem (4.27), with $w^i$ set to zero as in (4.30), may be obtained as a consistent solution of equation (4.12), which imposes a finiteness condition on the Weyl variation of the divergent local counterterms $S_0[^4]$. For definiteness we consider the four-dimensional case, the argument for $d = 2$ being exactly analogous. As before, we restrict the calculation to the case that all active scalars have $\Delta (i) = d$. We solve (4.12)

\footnote{Within standard perturbative quantum field theory, finiteness conditions of the form (4.12) have been considered in \cite{23}.}
order by order in a perturbative approach, expanding around $r = r_0$ and in derivatives with respect to the couplings. For instance for the superpotential we have the expansion

$$\mathcal{W}(\Phi(r)) = \mathcal{W}(\Phi(r_0)) + \partial_i \mathcal{W}(\Phi(r_0)) \partial^i \Phi + \frac{1}{2} \partial_i \partial_j \mathcal{W}(\Phi(r_0)) \partial^i \Phi \partial^j \Phi + \ldots,$$

(5.1)

$$\Phi^i(r) = \phi^i_0(r_0) + \varepsilon \phi^i_2(r_0) + \ldots,$$

(5.2)

$$\delta \Phi^i(\varepsilon) = \delta \phi^i_0(r_0) + \varepsilon \delta \phi^i_2(r_0) + \ldots, \quad \varepsilon = (r - r_0)^2.$$  

(5.3)

Here $\varepsilon$ is the regulator. According to [13], we expect logarithms of $\varepsilon$ in the expansion of $\delta \Phi^i(\varepsilon)$. $\phi_2$ and all higher order contributions may be expressed in terms of $\phi_0$, but their explicit form is not relevant here. Note that due to (4.6), the beta functions correspond to one derivative of the superpotential. Due to (4.3) we expect this expansion to be equivalent to a Taylor expansion of $A'(r)$. In the conformal case, $A''$ and all higher order derivatives vanish.

For solving (4.12) perturbatively for $d = 4$, we note that a basis for the anomaly terms $B, Z_\mu$ is given by (2.23). Here we restrict ourselves to considering only those contributions to the anomaly which are relevant for the C theorem, ie. those involving the coefficients $a, w$ and $\chi^{a_{ij}}$. The divergent local counterterm contributing to these anomaly contributions is given by

$$S_0^C = - \int d^4 x \sqrt{g} \left[ b \bar{R} R + \frac{1}{2} \eta_{ij} G^{\mu \nu} \partial_\mu \phi^i_0 \partial_\nu \phi^j_0 \right],$$

(5.4)

with divergent coefficients $b, \eta_{ij}$. Here $\bar{R} R$ is the Euler density and $G^{\mu \nu}$ is the Einstein tensor. The finiteness condition (4.12) implies

$$\begin{align*}
(2\varepsilon \partial_\varepsilon - \hat{\beta}^i \partial_i) b &= a \\
(2\varepsilon \partial_\varepsilon - L_\beta) \eta_{ij} &= \chi^{a_{ij}} \\
8 \partial_i b - \eta_{ij} \hat{\beta}^j &= w_i.
\end{align*}$$

(5.5)

$L_\beta \eta_{ij} \equiv \beta^k \partial_k \eta_{ij} + \partial_i \beta^k \eta_{kj} + \partial_j \beta^k \eta_{ik}$ is the Lie derivative. The right hand side of these equations - and therefore also the left hand side - must be finite. To lowest order, ie. up to terms involving derivatives of $\mathcal{W}(\phi)$ and $L_{ij}(\phi)$, (5.5) is solved by

$$\begin{align*}
b &= \frac{1}{2} \ln \varepsilon \frac{c_0}{|\mathcal{W}|^3} = \frac{1}{2} \ln \varepsilon C, \quad a = \frac{c_0}{|\mathcal{W}|^3} = C \\
\eta_{ij} &= 8 \ln \varepsilon C L_{ij}, \quad \chi_{ij} = 16 C L_{ij} \\
w_i &= 0.
\end{align*}$$

(5.6)

We see that there is agreement with the result $\beta^i \partial_i \tilde{a} = \frac{1}{8} \chi^{a_{ij}} \phi^i_0 \phi^j_0$ obtained from the Weyl consistency condition. To next order, ie. including terms involving two derivatives with respect to $\phi^i$, (5.3) has the solution

$$\begin{align*}
b &= \frac{1}{2} \ln \varepsilon C + \frac{1}{4} \ln^2 \varepsilon C L_{ij} \beta^i \beta^j, \quad a = C, \\
\eta_{ij} &= 8 \ln \varepsilon C L_{ij} + 2 \ln^2 \varepsilon L_{\beta}(C L_{ij}), \quad \chi_{ij} = 16 C L_{ij}, \\
w_i &= \ln \varepsilon (4 \partial_i C - 8 C L_{ij} \beta^j) = 0.
\end{align*}$$

(5.7)
The contributions to $a, \chi, w$ involving $\ln \varepsilon$ cancel and the contributions involving $\ln^2 \varepsilon$ are of higher order since they contain at least three derivatives with respect to $\phi^i$. Therefore we have obtained the finite result

$$a = C, \quad \chi_{ij} = 16 C L_{ij}, \quad w_i = 0 \quad (5.8)$$

satisfying the Weyl consistency condition to second order in the derivatives with respect to $\phi^i$. The appearance of $\ln^2 \varepsilon$ is natural in a perturbation expansion and the essential feature is that the result for the anomalies is finite to a given order in the expansion. To obtain the result (5.8) including terms involving four derivatives, we have to include terms involving $\ln^3 \varepsilon$ in order to cancel the terms involving $\ln^2 \varepsilon$:

$$b = \frac{1}{2} \ln \varepsilon C + \frac{1}{4} \ln^2 \varepsilon C L_{ij} \beta^i \beta^j + \frac{1}{24} \ln^3 \varepsilon \beta^k \partial_k (C L_{ij} \beta^i \beta^j), \quad a = C,$$

$$\eta_{ij} = 8 \ln \varepsilon C L_{ij} + 2 \ln^2 \varepsilon C L_{ij} + \frac{1}{8} \ln^3 \varepsilon C (C L_{ij}) \beta^i, \quad \chi_{ij} = 16 C L_{ij}, \quad (5.9)$$

$$w_i = \ln^2 \varepsilon (2 \partial_i (C L_{jk} \beta^j \beta^k) - 2 C L_{ij} \beta^j) = 0. \quad (5.10)$$

The calculation showing that $w_i$ in (5.10) vanishes is non-trivial and relies on the result (4.3) expressing $\beta^i$ in terms of the superpotential and its derivative. We expect that we may continue this calculation to any given order in the derivatives of $\mathcal{W}$. It seems possible to use induction to show that the result (5.8) holds to all orders in the expansion.

These results show that for operators with $\Delta^{(i)} = d$, the holographic C theorem may be obtained from a consistent perturbative solution of the finiteness condition (4.12) in a well-defined regularization scheme, together with the Weyl consistency condition originating from the local structure of the holographic RG.

### 6 Conclusion and Perspectives

For our analysis of holographic RG flows we have made use of results for local RG equations obtained within standard quantum field theory. There are further examples of field theory results relevant to holographic flows, for instance concerning the possibility of constructing a four-dimensional C function based on the energy-momentum tensor two-point function. This two-point function is generally of the form

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(0) \rangle = \frac{1}{3} S_{\mu\nu} S_{\sigma\rho} \Omega_0(x) + (S_{\mu(\sigma} S_{\rho)} - \frac{1}{3} S_{\mu\nu} S_{\sigma\rho}) \Omega_2(x), \quad (6.1)$$

$$S_{\mu\nu} \equiv \delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu.$$

It was shown in [37], based on the spectral representation approach to the C theorem [38], that in four dimensions the function $C_F(\lambda)$, which is directly related to $\Omega_2$ in (6.1) and which at the fixed points coincides with the coefficient $c$ of the Weyl tensor squared contribution to the anomaly, satisfies

$$-\beta^i \partial_i C_F = 2 C_F - G, \quad (6.2)$$
with $G$ positive definite. This result indicates that in general the flow of $C_F$ is not expected to be monotonic. In fact, counterexamples for a C theorem based on $C_F$ have been found in [31]. As discussed for instance in section 2 above, it is expected that a four-dimensional C theorem will involve a C function which at the fixed points coincides with the coefficient $a$ of the Euler density anomaly (see also [14]). However for holographic flows, whose anomaly coefficients generally satisfy $a = c$ and which therefore represent a special class of field theories, examples for which the flow of $C_F$ is indeed monotonic have been discussed in [41]. It will therefore be interesting to determine the exact form of (6.2) for holographic flows. - Moreover the field-theoretical analysis of the geometrical structure of the renormalization group in [42, 43] is also related to the holographic RG.

Furthermore it will be very interesting to investigate if the results of this paper may be applied to examples for holographic flows which go beyond the supergravity approximation. For example, the structure of the conformal anomaly in the Polchinski-Strassler model [14] has been investigated very recently in [45]. We expect that the local RG may be useful for obtaining further results about the conformal anomaly in this and in related models.

In [10], the question was raised if it is possible to find a general criterion for the existence of a holographic dual for any given quantum field theory. In [17] it was suggested for four-dimensional conformal field theories that a necessary criterion for the existence of a gravity dual is that the coefficients of the Euler and Weyl tensor squared anomalies must coincide, $a = c$. In the light of our results here, and noting that in the framework of perturbative quantum field theory, local RG equations have been shown to hold for a number of renormalizable quantum field theories, including $\phi^4$-theory and gauge theories in four dimensions, we may speculate that it may be possible to construct holographic duals also for theories with $a \neq c$, starting from their local RG. These gravity duals, if they exist, would presumably be of a very different form than those obtained from deformations of the AdS/CFT correspondence.

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5For a precise definition of $C_F$ and $G$ see [37, 8].
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