A CONSTRUCTION OF HOPF ALGEBRA COCYCLES FOR THE DOUBLE YANGIAN $DY(\mathfrak{sl}_2)$

B. ENRIQUEZ AND G. FELDER

Abstract. We construct a Hopf algebra cocycle in the Yangian double $DY(\mathfrak{sl}_2)$, conjugating Drinfeld’s coproduct to the usual one. To do that, we factorize the twist between two “opposite” versions of Drinfeld’s coproduct, introduced in earlier work by V. Rubtsov and the first author, using the decomposition of the algebra in its negative and non-negative modes subalgebras.

Introduction. The purpose of this paper is to show that Drinfeld’s coproduct of the Yangian double $DY(\mathfrak{sl}_2)$ ([3]) is conjugated to the usual one. For that, we construct a Hopf algebra cocycle in the Yangian double $DY(\mathfrak{sl}_2)$.

Actually, we note that $DY(\mathfrak{sl}_2)$ is endowed with two variants of Drinfeld’s coproduct. These coproducts are associated with two decompositions of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathbb{C}((z^{-1}))$, the first one being $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, with $\mathfrak{g}_+ = (\mathfrak{h} \otimes \mathbb{C}[z]) \oplus (\mathfrak{n}_+ \otimes \mathbb{C}((z^{-1})))$, $\mathfrak{g}_- = (\mathfrak{h} \otimes z^{-1}\mathbb{C}[[z]]) \oplus (\mathfrak{n}_- \otimes \mathbb{C}((z^{-1})))$, and the second one being its transform by the nontrivial Weyl group element of $\mathfrak{sl}_2$. Here $\mathfrak{h}$ and $\mathfrak{n}_\pm$ are the standard Cartan and opposite nilpotent subalgebras of $\mathfrak{sl}_2$. In [3], we considered Hopf algebras $U_{h\mathfrak{g}}$ quantizing more general Lie bialgebra structures associated with curves in higher genus, and showed that they were conjugated by a twist $F$.

The next step of [3] was the construction of a deformation $U_{h\mathfrak{g}_R}$ of the enveloping algebra of an algebra of regular functions with values in $\mathfrak{sl}_2$; in our “rational” situation, this Lie algebra corresponds to $\mathfrak{sl}_2 \otimes \mathbb{C}[z]$ and $U_{h\mathfrak{g}_R}$ to the Yangian $Y(\mathfrak{sl}_2)$. This subalgebra also had the property that

$$\Delta(U_{h\mathfrak{g}_R}) \subset U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}_R}, \quad \hat{\Delta}(U_{h\mathfrak{g}_R}) \subset U_{h\mathfrak{g}_R} \otimes U_{h\mathfrak{g}}.$$ 

The last step of that paper was to decompose $F$ as a product

$$F_2 F_1, \quad \text{with} \quad F_1 \in U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}_R}, \quad F_2 \in U_{h\mathfrak{g}_R} \otimes U_{h\mathfrak{g}},$$

and then to construct a quasi-Hopf algebra structure on $U_{h\mathfrak{g}_R}$ by twisting the coproduct $\Delta$ by $F_1$. $F_1$ and $F_2$ are constructed by applying to

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one factor of $F$ a projection of $U\mathfrak{g}$ on $U\mathfrak{g}_R$, which is a right $U\mathfrak{g}_R$-module map. In this construction, the choice of the projection is not unique. Changing the projection has the effect of changing $(F_1, F_2)$ into $(uF_1, F_2 - 1)$, for some $u \in U\mathfrak{g}_R$. This changes the coproduct $\Ad(F_1) \circ \Delta$ on $U\mathfrak{g}_R$ by some twist.

The question naturally arises whether the same technique can be applied in Hopf algebra situations. In this paper, we treat the case of the rational Manin triple $\mathfrak{g} = \mathfrak{g}^{\geq 0} \oplus \mathfrak{g}^{< 0}$, where $\mathfrak{g}^{\geq 0} = \mathfrak{sl}_2 \otimes \mathbb{C}[z]$ and $\mathfrak{g}^{< 0} = \mathfrak{sl}_2 \otimes z^{-1}\mathbb{C}[[z^{-1}]]$. In this situation, both $\mathfrak{g}^{\geq 0}$ and $\mathfrak{g}^{< 0}$ are Lie subbialgebras of $\mathfrak{g}$, and there are also deformations of their enveloping algebras in $DY(\mathfrak{sl}_2)$, $A^{\geq 0} = Y(\mathfrak{sl}_2)$ and $A^{< 0}$. Therefore we require that the projection $\Pi_{< 0, r}$ be at the same time a right $A^{< 0}$-module map. Then it is uniquely determined. We show that the first part $F_1$ of the decomposition of $F$ constructed in this way satisfies the Hopf algebra cocycle condition. This is the main result of this text.

The proof of this fact relies on the following results. We first prove that the second part $F_2$ of the decomposition of $F$ is obtained by applying to $F$ a projection $\Pi_{\geq 0, l}$ similar to $\Pi_{< 0, r}$ (eqs. (28), (29)). We give two proofs (sections 2.3, 2.4) of this result, both of them relying on some study of the duality theory within $DY(\mathfrak{sl}_2)$ (section 2.2); the first proof directly applies results from [6]. This enables to show that the defect of the cocycle identity for $F_1$ belongs to two spaces with intersection $1 \otimes DY(\mathfrak{sl}_2) \otimes 1$. The fact that the pentagon identity is automatically satisfied by such defects (4) then shows that it is indeed equal to 1.

After we twist by $F_1$ the universal $R$-matrix of $DY(\mathfrak{sl}_2)$ associated to Drinfeld's coproduct, we obtain a new solution of the Yang-Baxter equation. Applying to it 2-dimensional representations of $DY(\mathfrak{sl}_2)$, we construct $L$-operators satisfying the Yangian exchange (or $RLL$) relations of [1, 4] (section 4). This connection between Yangian $RLL$ relations and quantum current relations had earlier been obtained in [2] (see [2] in the trigonometric case). After this connection is clarified we are in position to show (section 5) that $F_1$ conjugates $\Delta$ to the Yangian coproduct on $DY(\mathfrak{sl}_2)$.

We will consider an elliptic version of the construction of this paper in a separate article ([5]). There we will construct “twisted cocycles” providing solutions to the dynamical Yang-Baxter equation; this will lead us to the construction of quantum currents of elliptic quantum groups.

The first step towards generalizing our results to the case of a general Lie algebra is to generalize the twist $F$. This has been done by N. Reshetikhin (see Rem. 2). In the general case, $F$ is then a product of
factors corresponding to each simple root; one might expect that these factors satisfy braid relations. We would therefore obtain a “quantum currents” version of braid group representations. The next step of that generalization would be the study of the duality theory within general double Yangians.

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1. The double Yangian \(DY(\mathfrak{sl}_2)\) and its coproducts

The double Yangian is a Hopf algebra that was introduced in [8] (see also [11] in the non centrally extended case), and is associated to any semisimple Lie algebra \(g\). In the case where \(g = \mathfrak{sl}_2\), this algebra is denoted by \(DY(\mathfrak{sl}_2)\). We will also denote it by \(A\). It is an algebra over the ring of formal power series in the variable \(\hbar\), with generators \(x_n, n \in \mathbb{Z}\) (\(x = e, f, h\)), \(D\) and \(K\), and the following relations:

\[
k^+(z)e(w)k^+(z)^{-1} = \frac{z - w + \hbar}{z - w} e(w),
\]

\[
k^+(z)f(w)k^+(z)^{-1} = \frac{z - w}{z - w + \hbar} f(w),
\]

\[
k^-(z)e(w)k^-(z)^{-1} = \frac{w - z + \hbar K}{w - z} e(w),
\]

\[
k^-(z)f(w)k^-(z)^{-1} = \frac{z - w + \hbar}{z - w} f(w),
\]

\[
(z - w - \hbar)e(z)e(w) = (z - w + \hbar)e(w)e(z),
\]

\[
(z - w + \hbar)f(z)f(w) = (z - w - \hbar)f(w)f(z),
\]

\[
[e(z), f(w)] = \frac{1}{\hbar} \left( \delta(z, w)K^+(z) - \delta(z, w - \hbar K)K^-(w) \right),
\]

\[
[K, \text{anything}] = 0, \quad [k^\pm(z), k^\pm(w)] = 0, \quad [D, x(z)] = \frac{dx}{dz}(z), x = e, f, k^\pm,
\]

\[
(z - w - \hbar)(z - w + \hbar - \hbar K)K^+(z)K^-(w)
\]

\[
= (z - w + \hbar)(z - w - \hbar + \hbar K)K^-(w)K^+(z),
\]
where for $x = e, f, h$, we set

$$x_{\geq 0}(z) = \sum_{n \geq 0} x_n z^{-n-1}, \quad x^{< 0}(z) = \sum_{n < 0} x_n z^{-n-1}, \quad x(z) = x_{\geq 0}(z) + x^{< 0}(z);$$

we also set

$$k^+(z) = \exp \left( h_0 \ln \left( \frac{z + \hbar}{z} \right) + \sum_{n > 0} h_n \frac{z^{-n} - (z + \hbar)^{-n}}{n} \right),$$

$$K^-(z) = \exp \left( \hbar \sum_{n < 0} h_n z^{-n-1} \right),$$

and $K^+(z) = k^+(z)k^+(z - \hbar)$, $K^-(z) = k^-(z)k^-(z - \hbar)$. In (10), (11), the arguments of the exponentials are viewed as formal power series in $\hbar$, with coefficients in $A \otimes z^{-1} \mathbb{C}((z^{-1}))$ in the first case, and in $A \otimes \mathbb{C}[z]$ in the second one. Finally, $\delta(z, w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$.

**Remark 1.** The $x_n$ correspond in the notation of [6], to $x_n[z^n]$, for $x = e, f, h$ and $n \in \mathbb{Z}$.

The double Yangian Hopf structure $\Delta_{Yg}$ is defined as follows (see [3]). Set

$$L_{\geq 0}(z) = \begin{pmatrix} 1 & \hbar f_{\geq 0}(z) & 0 \\ 0 & 1 & k^+(z) \\ 0 & k^+(z)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & h e_{\geq 0}(z) \\ 0 & 1 \end{pmatrix},$$

and

$$L^{< 0}(z) = \begin{pmatrix} 1 \\ \hbar e^{< 0}(z - hK) \\ 1 \end{pmatrix} \begin{pmatrix} k^-(z - \hbar) & 0 \\ 0 & k^-(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $L_{\geq 0, < 0}(z)$ are formal series in $z$ with values in $A \otimes \text{End}(\mathbb{C}^2)$, and we set

$$\Delta_{Yg}(K) = K \otimes 1 + 1 \otimes K, \quad \Delta_{Yg}(D) = D \otimes 1 + 1 \otimes D,$$

$$\left( \Delta_{Yg} \otimes 1 \right) L_{\geq 0}(z) = L_{\geq 0}(z)^{(13)} L_{\geq 0}(z)^{(23)},$$

$$\left( \Delta_{Yg} \otimes 1 \right) L^{< 0}(z) = L^{< 0}(z - \hbar K_1)^{(23)} L^{< 0}(z)^{(13)},$$

with $K_1 = K \otimes 1$.

The algebra $A$ can also be endowed with Drinfeld’s Hopf structures $(\Delta, \varepsilon, S)$ and $(\check{\Delta}, \check{\varepsilon}, \check{S})$. They are given, on the one hand, by the co-product $\Delta$ defined by

$$\Delta(k^+(z)) = k^+(z) \otimes k^+(z), \quad \Delta(K^-(z)) = K^-(z) \otimes K^-(z + hK_1),$$

(14)
\begin{align*}
\Delta(e(z)) &= e(z) \otimes K^+(z) + 1 \otimes e(z), \\
\Delta(f(z)) &= f(z) \otimes 1 + K^-(z)^{-1} \otimes f(z + \hbar K_1), \\
\Delta(D) &= D \otimes 1 + 1 \otimes D, \quad \Delta(K) = K \otimes 1 + 1 \otimes K,
\end{align*}

the counit \( \varepsilon \), and the antipode \( S \) defined by them; and on the other hand, by the coproduct \( \bar{\Delta} \) defined by

\begin{align*}
\bar{\Delta}(k^+(z)) &= k^+(z) \otimes k^+(z), \quad \bar{\Delta}(K^-)(z)) = K^-(-z) \otimes K^-(-z + \hbar K_1), \\
\bar{\Delta}(e(z)) &= e(z - \hbar K_2) \otimes K^-(z - \hbar K_2)^{-1} + 1 \otimes e(z), \\
\bar{\Delta}(f(z)) &= f(z) \otimes 1 + K^+(z) \otimes f(z), \\
\bar{\Delta}(D) &= D \otimes 1 + 1 \otimes D, \quad \bar{\Delta}(K) = K \otimes 1 + 1 \otimes K,
\end{align*}

the counit \( \varepsilon \), and the antipode \( \bar{S} \) defined by them.

As we remarked in [6], \( \Delta \) and \( \bar{\Delta} \) are linked by a twist operation. Let us set

\begin{equation}
F = \exp \left( \hbar \sum_{n \in \mathbb{Z}} e_n \otimes f_{-n-1} \right),
\end{equation}

then we have

\begin{equation}
\bar{\Delta} = \text{Ad}(F) \circ \Delta.
\end{equation}

(Here and later, we use the notation \( \text{Ad}(u)(x) = uxu^{-1} \), for \( x \) and \( u \) elements of some algebra, with \( u \) invertible.)

\( F \) satisfies the cocycle condition

\begin{equation}
(F \otimes 1)(\Delta \otimes 1)(F) = (1 \otimes F)(1 \otimes \Delta)(F).
\end{equation}

(see [6]).

\textbf{Remark 2.} N. Reshetikhin informed us that he obtained the conjugation equation (22) in the general case (that is, with \( \mathfrak{sl}_2 \) replaced by a semisimple Lie algebra \( \mathfrak{g} \)). Then \( F \) is equal to the product \( \prod_{k=1}^\nu F_{i_k} \), where \( w_0 = s_{i_1} \ldots s_{i_\nu} \) is a decomposition of the longest Weyl group element as a product of simple reflections, and \( F_{i_k} = q_{n \in \mathbb{Z}} e_{i_k n} \otimes f_{i_k - n - 1} \), with \((e_{i_k n})_{n \in \mathbb{Z}}, (f_{i_k n})_{n \in \mathbb{Z}}\) the components of the fields corresponding to the \( i_k \)th simple root. The crossed vertex relations seem to imply that all elements \( F_{i_k} \)'s commute together. However, this is not quite true: the relations

\begin{equation}
(z - w + \hbar a_{ij})(z - w - \hbar a_{ij}) \left[ e_{i}(z) \otimes f_{i}(z), e_{j}(w) \otimes f_{j}(w) \right] = 0
\end{equation}
It would be interesting to check using these fields, whether the existence of fields \( A_{ij}^\pm(z) \) such that

\[
[e_i(z) \otimes f_i(z), e_j(w) \otimes f_j(w)] = \delta(z, w+ha_{ij})A_{ij}^+(z) + \delta(z, w-ha_{ij})A_{ij}^-(z).
\]

It would be interesting to check using these fields, whether the \( F_i \)’s satisfy the braid relations. In the same spirit, one is led to construct fields corresponding to non-simple roots using the relation \((z - w - ha_{ij})[e_i(z), (kje_j)(w)] = 0\).

2. Decomposition of \( F \)

2.1. Subalgebras of \( A \). We will call \( A^{\geq 0} \) and \( A^{< 0} \) the subalgebras of \( A \) generated by \( D \) and the \( x_n, n \geq 0 \), resp. by \( K \) and the \( x_n, n < 0 \) (with \( x = e, f, h \)). The multiplication induces isomorphisms from \( A^{\geq 0} \otimes A^{< 0} \) and \( A^{< 0} \otimes A^{\geq 0} \) to \( A \); moreover, the intersection of \( A^{\geq 0} \) with \( A^{< 0} \) is reduced to \( C1 \).

Let \( U_h n_+ \) and \( U_h n_- \) be the subalgebras of \( A \) generated by the \( e_n, n \in \mathbb{Z} \), resp. the \( f_n, n \in \mathbb{Z} \).

Let \( U_h n_{\geq 0}^\epsilon \) and \( U_h n_{< 0}^\epsilon \) be the subalgebras generated by the \( x_n, n \geq 0 \), resp. by the \( x_n, n < 0 \), with \( x = e \) for \( \epsilon = + \) and \( x = f \) for \( \epsilon = - \).

The linear maps \( U_h n_{\geq 0}^\epsilon \otimes U_h n_{< 0}^\epsilon \to U_h n_\epsilon \) and \( U_h n_{< 0}^\epsilon \otimes U_h n_{\geq 0}^\epsilon \to U_h n_\epsilon \), defined by the composition of the inclusion with the multiplication, are linear isomorphisms; moreover, the inclusions of algebras \( U_h n_{\geq 0}^\epsilon \subset U_h n_\epsilon \) and \( U_h n_{< 0}^\epsilon \subset U_h n_\epsilon \) are flat deformations of the inclusions of commutative algebras \( \mathbb{C}[x_n, n \geq 0] \subset \mathbb{C}[x_n, n \in \mathbb{Z}] \) and \( \mathbb{C}[x_n, n < 0] \subset \mathbb{C}[x_n, n \in \mathbb{Z}] \) (see e.g. [3]).

Remark 3. Relations for generating currents \( x_{\geq 0}^\epsilon(z) = \sum_{n \geq 0} x_n z^{-n-1} \) and \( x_{< 0}^\epsilon(z) = \sum_{n < 0} x_n z^{-n-1} \) of \( U_h n_{\geq 0}^\epsilon \) and \( U_h n_{< 0}^\epsilon \) are

\[
(z - w - h)e^\eta(z)e^\eta(w) - (z - w + h)e^\eta(w)e^\eta(z) = -h \left( e^\eta(z)^2 + e^\eta(w)^2 \right)
\]

and

\[
(z - w + h)f^\eta(z)f^\eta(w) - (z - w - h)f^\eta(w)f^\eta(z) = h \left( f^\eta(z)^2 + f^\eta(w)^2 \right),
\]

\( \eta \in \{ \geq 0, < 0 \} \).

On the other hand, the relations between these currents are

\[
(z - w - h)e^{\eta}(z)e^{\eta'}(w) - (z - w + h)e^{\eta'}(w)e^{\eta}(z) - h[e^{\eta}(z)^2 + e^{\eta'}(w)^2] = 0,
\]

\[
(z - w + h)f^{\eta}(z)f^{\eta'}(w) - (z - w - h)f^{\eta'}(w)f^{\eta}(z) + h[f^{\eta}(z)^2 + f^{\eta'}(w)^2] = 0.
\]

if \( \{ \eta, \eta' \} = \{ \geq 0, < 0 \} \).
2.2. Hopf algebra pairings. Let $U_\hbar h_+$ be the subalgebra of $A$ generated by $D$ and the $h_n, n \geq 0$, and $U_\hbar h_-$ be the subalgebra of $A$ generated by $K$ and the $h_n, n < 0$.

Let $U_\hbar g_\pm$ be the subalgebras of $A$ generated by $U_\hbar h_\pm$ and $U_\hbar n_\pm$, and $U_\hbar h_\pm$ the subalgebras of $A$ generated by $U_\hbar h_\pm$ and $U_\hbar n_\pm$.

$(U_\hbar g_+, \Delta)$ are Hopf subalgebras of $(A, \Delta)$; $(U_\hbar g_+, \Delta)$ and $(U_\hbar g_-, \Delta')$ are dual to each other, and the duality $(\cdot)$ is expressed by the rules

$$\langle e_n, f_m \rangle = \frac{1}{\hbar} \delta_{n+m+1,0}, \quad \langle h_a, h_b \rangle = \frac{2}{\hbar} \delta_{a+b+1,0}, \quad \langle D, K \rangle = \frac{1}{\hbar},$$

$n, m \in \mathbb{Z}, a \geq 0, b < 0$, the other pairings between generators being trivial.

In a similar way, $(U_\hbar g_+, \Delta)$ are Hopf subalgebras of $(A, \Delta)$; $(U_\hbar g_+, \Delta')$ and $(U_\hbar g_-, \Delta)$ are dual to each other, and the duality $(\cdot)'$ is expressed by the rules

$$\langle e_n, f_m \rangle' = \frac{1}{\hbar} \delta_{n+m+1,0}, \quad \langle h_a, h_b \rangle' = \frac{2}{\hbar} \delta_{a+b+1,0}, \quad \langle K, D \rangle' = \frac{1}{\hbar},$$

$n, m \in \mathbb{Z}, a \geq 0, b < 0$, the other pairings between generators being trivial.

The restrictions of $(\cdot)$ and $(\cdot)'$ to $U_\hbar n_+ \times U_\hbar n_-$ coincide and are denoted by $(\cdot)_{U_\hbar n_\pm}$.

Moreover, we have

**Lemma 2.1.** (see [3]) 1) The annihilator of $U_\hbar n_+^{\geq 0}$ for $(\cdot)_{U_\hbar n_\pm}$ is $\sum_{n \geq 0} e_n$.

2) The annihilator of $U_\hbar n_-^{< 0}$ for $(\cdot)_{U_\hbar n_\pm}$ is $\sum_{n < 0} f_n \cdot U_\hbar n_-$.  

3) The annihilator of $U_\hbar n_0^{\geq 0}$ is $\sum_{n \geq 0} U_\hbar n_- \cdot f_n$.

4) The annihilator of $U_\hbar n_-^{< 0}$ for $(\cdot)_{U_\hbar n_\pm}$ is $\sum_{n < 0} U_\hbar n_+ \cdot e_n$.

**Proof.** 1) and 3) are consequences of [3], Prop. 6.2, and 2) and 4) are shown in a similar way.

Finally, the link between $F$ the pairing $(\cdot)_{U_\hbar n_\pm}$ can be described as follows. Let us first introduce the notation

$$\langle a, id \otimes b \rangle_{V,W} = \sum_i a_i \langle a_i', b \rangle_{V,W}, \quad \langle a, b \otimes id \rangle_{V,W} = \sum_i \langle a_i, b \rangle_{V,W} a_i',$$

for $a \in V \otimes W$ and $b \in W$, for $V, W$ some vector spaces and $(\cdot)_{V,W}$ some pairing between them, $a$ being decomposed as $\sum_i a_i \otimes a_i'$.

**Lemma 2.2.** (see [3], (66) and (68)) 1) For any $x \in U_\hbar n_+$, we have

$$\langle F, id \otimes x \rangle_{U_\hbar n_\pm} = x.$$

2) For any $y \in U_\hbar n_-$, we have

$$\langle F, y \otimes id \rangle_{U_\hbar n_\pm} = y.$$
2.3. Decomposition of $F$.

**Proposition 2.1.** There exists a decomposition $F = F_2 F_1$, with $F_1 \in U_h n_{+}^{\geq 0} \otimes U_h n_{-}^{\geq 0}$ and $F_2 \in U_h n_{+}^{> 0} \otimes U_h n_{-}^{\leq 0}$. It is unique up to changes of $(F_1, F_2)$ into $(\lambda F_1, \lambda^{-1} F_2)$, with $\lambda \in \mathbb{C}^\times$.

**Proof.** Let us denote by $\Pi_{\geq 0, r}$, $\Pi_{\leq 0, r}$, and by $\Pi_{< 0, r}$, $\Pi_{< 0, r}$ the linear maps from $U_h n_+$ to $U_h n_{+}^{\geq 0}$ and $U_h n_{+}^{\geq 0}$ defined by

$$\Pi_{\eta, l}(a_\eta a_\eta') = a_\eta \varepsilon(a_\eta'), \quad \Pi_{\eta, r}(a_\eta a_\eta') = \varepsilon(a_\eta') a_\eta,$$

for $\{\eta, \eta'\} = \{\geq 0, < 0\}$ and $a_\eta \in U_h n_\eta$.

**Lemma 2.3.**

1) $(\Pi_{< 0, r} \otimes 1)(F)$ belongs to $U_h n_{+}^{\leq 0} \otimes U_h n_{-}^{\geq 0}$.

2) $(1 \otimes \Pi_{\geq 0, r})(F)$ belongs to $U_h n_{+}^{< 0} \otimes U_h n_{-}^{\geq 0}$.

**Proof.**

1) $(\Pi_{< 0, r} \otimes 1)(F)$ clearly belongs to $U_h n_{+}^{\leq 0} \otimes U_h n_{-}$. On the other hand, we have for any $a \in U_h n_+$ and $n \geq 0$,

$$\langle (\Pi_{< 0, r} \otimes 1)(F), id \otimes e_n a \rangle = \Pi_{< 0, r}(e_n a) = 0;$$

the first equality follows from Lemma 2.2, 1, and the second from the fact that $\Pi_{< 0, r}$ is a left $U_h n_{+}^{\geq 0}$-module map. From Lemma 2.2, 1, now follows that $(\Pi_{< 0, r} \otimes 1)(F)$ also belongs to $U_h n_{+} \otimes U_h n_{-}^{\geq 0}$.

2) is proved in the same way, using Lemma 2.2, 2, and Lemma 2.1.

**Lemma 2.4.** $(\Pi_{< 0, r} \otimes 1)(F)$ is equal to $(1 \otimes \Pi_{\geq 0, r})(F)$.

**Proof.** Let $a_+$ belong to $U_h n_{-}^{\geq 0}$ and let $a_-$ belong to $U_h n_{+}^{> 0}$. Let us compute

$$\langle (\Pi_{< 0, r} \otimes 1)(F) - (1 \otimes \Pi_{\geq 0, r})(F), a_+ \otimes a_- \rangle_{U_h n_{+}^{\geq 2}}. \quad (23)$$

Due to Lemma 2.2, this is equal to $(\Pi_{< 0, r}(a_-), a_+)_{U_h n_{\pm}} - (a_-, \Pi_{\geq 0, r}(a_+))_{U_h n_{\pm}}$. Since $\Pi_{< 0, r}(a_-) = a_-$, $\Pi_{\geq 0, r}(a_+) = a_+$, (23) is equal to zero.

The pairing $\langle , \rangle_U h n_{\pm}$ is a flat deformation of the symmetric power of the pairing between $\mathbb{C}[z]$ and $z^{-1} \mathbb{C}[z^{-1}]$, defined by $\langle f, g \rangle = \text{res}_\infty(f g dz)$. Therefore, it defines an injection of $U_h n_{+}^{\geq 0}$ in the dual of $U_h n_{+}^{\leq 0}$ and of $U_h n_{+}^{> 0}$ in the dual of $U_h n_{+}^{\leq 0}$. That (23) is equal to zero then implies that $(\Pi_{< 0, r} \otimes 1)(F) = (1 \otimes \Pi_{\geq 0, r})(F)$. \qed

As $\Pi_{\geq 0, r}$ is a right $U_h g_R^{> 0}$-module map, we can apply [6], (74), second statement, with $U_h g_R = A^{> 0}$, and obtain

$$(1 \otimes \Pi_{\geq 0, r})(F) F^{-1} \in U_h n_{+}^{\geq 0} \otimes U_h n_{-}. \quad (24)$$
We can now apply the arguments of [6], Prop. 7.2, to the Hopf algebra
\((A, \Delta')\). The role of \(U_\hbar g_R\) is now played by \(A^{<0}\), \(F\) is replaced by \(F^{(21)}\).
The analogue of the second statement of [6], (74) is then
\[
((1 \otimes \Pi_{<0,r})(F^{(21)}))(F^{(21)})^{-1} \in U_\hbar n_{>0} \otimes U_\hbar n_+.
\] (25)
We can show in a similar way that
\[
((\Pi_{\geq 0, l} \otimes 1)(F))^{-1} F'(\Pi_{<0,r} \otimes 1)(F)^{-1} (\Pi_{\geq 0, l} \otimes 1)(F).
\] (27)
Since \((\Pi_{<0,r} \otimes 1)(F) \in U_\hbar n_{>0} \otimes U_\hbar n_{>0}\), and by (26), this product belongs
to \(U_\hbar n_{>0} \otimes U_\hbar n_{>0}\). On the other hand, since \((\Pi_{\geq 0, l} \otimes 1)(F) \in U_\hbar n_{\geq 0} \otimes U_\hbar n_{\geq 0}\), and by (24), it belongs to \(U_\hbar n_{\geq 0} \otimes U_\hbar n_{\geq 0}\). It follows that this
product is scalar. Since the constant term in its expansion is equal t o one, (27) is equal to one.
Therefore we can set
\[
F_1 = (\Pi_{<0,r} \otimes 1)(F) = (1 \otimes \Pi_{\geq 0,r})(F) \quad (28)
\]
and
\[
F_2 = (\Pi_{\geq 0, l} \otimes 1)(F) = (1 \otimes \Pi_{<0,l})(F). \quad (29)
\]

2.4. Another proof of Prop. 2.1. Let us define \(F_1\) and \(F_2\) by
\[
F_1 = (\Pi_{<0,r} \otimes 1)(F), \quad F_2 = (\Pi_{\geq 0,l} \otimes 1)(F), \quad (30)
\]
and show directly that \(F = F_2 F_1\). For this, we will consider the linear
endomorphism \(\ell\) of \(U_\hbar n_+\) defined by
\[
\ell(x) = \langle F_2 F_1, id \otimes x \rangle_{U_\hbar n_+}. \quad (31)
\]
Let us denote by \(\pi\) the linear map from \(U_\hbar g_+\) to \(U_\hbar n_+\), defined by \(\pi(t x) = \varepsilon(t)x, \) for \(x \in U_\hbar n_+, t \in U_\hbar h_+.\) Let us also denote by \(\pi'\) the linear map from \(U_\hbar g_+\) to \(U_\hbar n_+\), defined by \(\pi'(x't') = x' \varepsilon(t'), \) for \(x' \in U_\hbar n_+, t' \in U_\hbar h_-\).

Lemma 2.5. 1) For \(y \in U_\hbar g_+\), we have
\[
\langle F, id \otimes y \rangle = \pi(y). \quad (32)
\]
2) For \(z \in U_\hbar g_+\), we have
\[
\langle F, id \otimes z \rangle' = \pi'(z). \quad (33)
\]
Proof. Let us prove 1). Let us first show that for any \( y' \in U_h n_- \), we have
\[
\langle y', y \rangle = \langle y', \pi(y) \rangle .
\] (34)
To prove this, consider the case where \( y = t_0 y_0, y_0 \in U_h n_+, t_0 \in U_h b_+ \).
Then \( \langle y', y \rangle = \langle \Delta'(y'), t_0 \otimes y_0 \rangle (2) \); but \( \Delta'(y') \) belongs to \( U_h n_- \otimes U_h g_- \), and for \( a \in U_h b_+, b \in U_h n_- \), \( \langle a, b \rangle = \varepsilon(a) \varepsilon(b) \). It follows that \( \langle y', y \rangle = \langle (\varepsilon \otimes 1) \circ \Delta'(y') \varepsilon(t_0), y_0 \rangle = \langle y', \pi(y) \rangle \), so that (34) holds. (32) then follows from (34) and Lemma 2.2.

2) is proved in a similar way. \( \square \)

We then compute \( \ell(x) \) as follows, for \( x \in U_h n_+ \). Set \( \Delta(x) = \sum_i x_i^l \otimes x_i^r \), with \( x_i^l \in U_h n_+, x_i^r \in U_h g_+ \). Then
\[
\ell(x) = \sum_i \langle F_2, id \otimes x_i^r \rangle \langle F_1, id \otimes x_i^l \rangle = \sum_i \Pi_{\geq 0,l}(x_i^r) \Pi_{< 0,r}(\langle F, id \otimes x_i^l \rangle)
= \sum_i \Pi_{\geq 0,l}(x_i^r)(\Pi_{< 0,r} \circ \pi)(x_i^l)
\] (35)
We deduce from this expression the following property of \( \ell \).

Lemma 2.6. \( \ell \) is a left \( U_h n^\geq_+ \)-module map.

Proof. \( \Pi_{< 0,r} \circ \pi \) is defined as follows. Recall that the product operation defines a linear isomorphism from the tensor product \( U_h b_+ \otimes U_h n^\geq_+ \otimes U_h n^<_+ \) onto \( U_h g_+ \). \( \Pi_{< 0,r} \circ \pi \) is then defined by \( (\Pi_{< 0,r} \circ \pi)(x) = \varepsilon(tx_{\geq 0})x_{< 0} \), for \( x \) decomposed as \( tx_{\geq 0}x_{< 0}, t \in U_h b_+, x_{\geq 0} \in U_h n^\geq_+, x_{< 0} \in U_h n^<_+ \). On the other hand, denote by \( U_h b_+ \) the subspace of \( U_h g_+ \) corresponding to \( U_h b_+ \otimes U_h n^\geq_+ \otimes 1 \). We can check that this is a subalgebra of \( U_h g_+ \). It follows that \( \Pi_{< 0,r} \circ \pi \) satisfies
\[
(\Pi_{< 0,r} \circ \pi)(bx) = \varepsilon(b)(\Pi_{< 0,r} \circ \pi)(x),
\] (36)
for \( b \in U_h b_+, x \in U_h g_+ \).

Finally, (33) implies that for any \( n \in \mathbb{Z} \), \( \Delta(e_n) = 1 \otimes e_n + \sum_{p \geq 0} e_{n-p} \otimes K^+_p \) (we set \( K^+(z) = \sum_{p \geq 0} K^+_p z^{-p} \)), so that for \( n \geq 0 \) this belongs to \( U_h n_+ \otimes U_h b_+ \); it follows that \( \Delta(U_h n^<_+) \subset U_h n_+ \otimes U_h b_+ \).

Let us fix then \( b \) in \( U_h n^\geq_+ \) and \( x \) in \( U_h n_+ \). Set \( \Delta(x) = \sum_i x_i^l \otimes x_i^r \), \( x_i^l \in U_h n_+, x_i^r \in U_h g_+ \), and \( \Delta(b) = \sum_j b_j^l \otimes b'_j \), \( b_j^l \in U_h n_+, b'_j \in U_h b_+ \). Then
\[
\ell(bx) = \sum_{i,j} \Pi_{\geq 0,l}(b'_j x_i^l)(\Pi_{< 0,r} \circ \pi)(b_i^r x_i^l) = \sum_{i,j} \Pi_{\geq 0,l}(b'_j x_i^l) \varepsilon(b_i^r)(\Pi_{< 0,r} \circ \pi)(x_i^l)
= \sum_i \Pi_{\geq 0,l}(bx_i^l)(\Pi_{< 0,r} \circ \pi)(x_i^l) = b \sum_i \Pi_{\geq 0,l}(x_i^l)(\Pi_{< 0,r} \circ \pi)(x_i^l)
= b \ell(x);
\]
the second equality follows from (36), the third from the properties of the counit, and the fourth from the fact that \( \Pi_{\geq 0,r} \) is a left \( U^+_n \)-module map.

We now deduce from this expression:

**Lemma 2.7.** \( \ell \) is a right \( U^+_n \)-module map.

*Proof.* As above, the product operation defines an isomorphism of vector spaces from \( U^+_n \otimes U^+_n \otimes U^-_n \otimes U^+_n \) to \( U^-_n \). The image by this map of \( 1 \otimes U^+_n \otimes U^-_n \otimes U^+_n \) is a subalgebra of \( U^-_n \), that we denote by \( U^+_n \). \( \Pi_{\geq 0,l} \circ \pi' \) is then defined by \( (\Pi_{\geq 0,l} \circ \pi')(x) = \sum \alpha x_{>0} \varepsilon(b\alpha) \), if \( x \) is decomposed as \( \sum \alpha x_{>0} b\alpha \), \( x_{>0}, b\alpha \in U^+_n \). Therefore, we have

\[
(\Pi_{\geq 0,l} \circ \pi')(x) = (\Pi_{\geq 0,l} \circ \pi') (x) \varepsilon(b)
\]

for \( x \in U^+_n, b \in U^+_n \).

Finally, (13) implies that for \( n > 0 \), \( \tilde{\Delta}(e_{-n}) = \sum_{p \geq 0} e_{-n+p} \otimes ((K^-)^{-1})_{-p} + 1 \otimes e_{-n} \) (we set \( (K^-)^{-1}(z) = \sum_{p \leq 0} ((K^-)^{-1})_{p} z^{-p} \) ), and so belongs to \( U^+_n \). \( \Pi_{\geq 0,l} \circ \pi' \) it follows that \( \Delta(U^+_n) \subset U^+_n \otimes U^+_n \).

Fix then \( x \in U^+_n, b \in U^+_n \), with \( \bar{\Delta}(x) = \sum_i x_i' \otimes x_i'' \), \( x_i' \in U^+_n, x_i'' \in U^+_n \), \( b = \sum_j b_j' \otimes b_j'' \). Then

\[
\ell(xb) = \sum_{i,j} (\Pi_{\geq 0,l} \circ \pi')(x_i'' b_j') \Pi_{\leq 0,r}(x_i' b_j') = \sum_{i,j} (\Pi_{\geq 0,l} \circ \pi')(x_i'' \varepsilon(b_j') \Pi_{\leq 0,r}(x_i' b_j')
\]

the second equality follows from (38), the third one from the properties of \( \varepsilon \), and the fourth one from the fact that \( \Pi_{\leq 0,r} \) is a right \( U^+_n \)-module map.

Let us now prove Prop. 2.7. We have \( \ell(1) = 1 \). Since any element of \( U^+_n \) can be expressed as a sum of products \( \sum x_i^0 x_i^0 \), with \( x_i^0 \in U^+_n \), \( x_i^0 \in U^+_n \), and by Lemmas 2.4 and 2.7, \( \ell \) coincides with the identity.

☐
3. Cocycle properties

**Theorem 3.1.** $F_1$ satisfies the cocycle equation

$$(F_1 \otimes 1)(\Delta \otimes 1)(F_1) = (1 \otimes F_1)(1 \otimes \Delta)(F_1).$$

**Proof.** First note that

$$\Delta(A^{\geq 0}) \subset A \otimes A^{\geq 0}, \quad \Delta(A^{< 0}) \subset A^{< 0} \otimes A,$$

$$\bar{\Delta}(A^{\geq 0}) \subset A^{\geq 0} \otimes A, \quad \bar{\Delta}(A^{< 0}) \subset A \otimes A^{< 0}.$$  

Let us set

$$\Phi = \bar{F}_1^{(12)}(\Delta \otimes 1)(F_1)(\bar{F}_1^{(23)}(1 \otimes \Delta)(F_1))^{-1},$$

we have clearly $\Phi \in A^{< 0} \otimes A \otimes A^{\geq 0}$. Since we also have

$$\Phi = (\bar{\Delta} \otimes 1)(F_2)\bar{F}_2^{(12)}(1 \otimes \bar{\Delta})(F_2)\bar{F}_2^{(23)},$$

we also see that $\Phi \in A^{\geq 0} \otimes A \otimes A^{< 0}$.

Therefore $\Phi = 1 \otimes a \otimes 1$, for a certain $a \in A$. On the other hand, as $\Phi$ is obtained by twisting a quasi-Hopf structure, it should satisfy the compatibility condition (see [4])

$$(\Delta_1 \otimes id \otimes id)(\Phi)(id \otimes id \otimes \Delta_1)(\Phi) = (\Phi \otimes 1)(id \otimes \Delta_1 \otimes id)(\Phi)(1 \otimes \Phi),$$

where $\Delta_1 = \text{Ad}(F_1) \circ \Delta$. This implies that

$$1 \otimes a \otimes a \otimes 1 = (1 \otimes a \otimes 1 \otimes 1)(1 \otimes \Delta_1(a) \otimes 1)(1 \otimes 1 \otimes a \otimes 1),$$

and so $\Delta_1(a) = 1$; applying the counit to one of the factors of the tensor product where this equality takes place, we obtain $a = 1$. \hfill \Box

**Remark 4.** An other way to show that $\Phi$ is scalar is the following. We can use the third expression of $\Phi$ in [4] Prop. 7.4 to show that $\Phi$ belongs to $A \otimes A^{\geq 0} \otimes A$. By writing a similar expression for $\Phi$, we get that $\Phi \in A \otimes A^{< 0} \otimes A$. Together with the fact that $\Phi$ belongs to $1 \otimes A \otimes 1$, this shows that $\Phi$ is scalar.

**Remark 5.** First order computations lead us to believe that $F_1$, resp. $F_2$ can be expressed polynomially in terms of the res$_0(e^{<0}(z) \otimes f^{>0}(z))^n dz$, resp. of the res$_0(e^{>0}(z) \otimes f^{<0}(z))^n dz$. Product formulas for $F_{1,2}$ can be found in [5].
4. Yangian RLL relations

Its follows from Thm. 3.1 that we can twist the Hopf algebra structure \((A, \Delta)\) by \(F_1\), and get another Hopf algebra structure. The twisted coproduct is \(\Delta_1 = \text{Ad}(F_1) \circ \Delta\).

Let

\[
R = q^{D \otimes K} q^\frac{1}{2} \sum_{i \geq 0} h_i \otimes h_{-i-1} q^\sum_{i \in \mathbb{Z}} e_i \otimes f_{-i-1};
\]

this is the universal \(R\)-matrix for \((A, \Delta)\) (see [6]). The universal \(R\)-matrix for the twisted Hopf algebra \((A, \Delta_1)\) is then

\[
R_1 = F_1^{(21)} R F_1^{-1}.
\]

We then have the Yang-Baxter equation

\[
R_1^{(12)} R_1^{(13)} R_1^{(23)} = R_1^{(23)} R_1^{(13)} R_1^{(12)}.
\]

Recall now the formulas for 2-dimensional representations of \(A\) (see e.g. [1]). Let \(\zeta\) be a formal variable, \(k_\zeta\) the field of formal Laurent power series \(\mathbb{C}(\langle \zeta \rangle)\), \(\partial_\zeta\) the derivation of \(k_\zeta\) defined as \(d/d\zeta\), and \(k_\zeta[\partial_\zeta]\) the associated ring of differential operators.

**Lemma 4.1.** There is a morphism of algebras \(\pi_\zeta\) from \(A\) to \(\text{End}(\mathbb{C}^2) \otimes k_\zeta[\partial_\zeta][[\hbar]]\), defined by

\[
\pi_\zeta(K) = 0, \quad \pi_\zeta(D) = \text{Id}_{\mathbb{C}^2} \otimes \partial_\zeta,
\]

\[
\pi_\zeta(h_n) = \begin{pmatrix} \frac{2}{1+q^{n+1}} \zeta^n & 0 \\ 0 & -\frac{2}{1+q^{-n-1}} \zeta^{-n-1} \end{pmatrix} (\zeta), \quad n \geq 0,
\]

\[
\pi_\zeta(h_n) = \begin{pmatrix} 1 - q^{-\partial_\zeta} \zeta^n & 0 \\ 0 & -\frac{q^{\partial_\zeta}-1}{\hbar \partial_\zeta} \zeta^n \end{pmatrix} (\zeta), \quad n < 0,
\]

\[
\pi_\zeta(e_n) = \begin{pmatrix} 0 & \zeta^n \\ 0 & 0 \end{pmatrix}, \quad \pi_\zeta(f_n) = \begin{pmatrix} 0 & 0 \\ \zeta^n & 0 \end{pmatrix}, \quad n \in \mathbb{Z}.
\]

**Lemma 4.2.** We have

\[
(1 \otimes \pi_\zeta)(\mathcal{R}_1) = L^{\geq 0}(\zeta), \quad (1 \otimes \pi_\zeta)(\mathcal{R}_1^{(21)}) = q^{K \partial_\zeta} L^{< 0}(\zeta).
\]

**Proof.** Let us denote by \(U_{h\mathfrak{n}_+^{\geq i}}\) the linear spans in \(U_{h\mathfrak{n}_+}\) of products of more than \(i\) factors \(e_k\), resp. \(f_k\). Then the various \(\Pi_{\pm,}^{\pm,}\) preserve the \(U_{h\mathfrak{n}_+^{\geq i}}\). The formulas (28) for \(F_1\) then imply that \(F_1\) belongs to \(1 + \hbar \sum_{i \geq 0} e_{-i-1} \otimes f_i + U_{h\mathfrak{n}_+^{\geq 2}} \otimes U_{h\mathfrak{n}_+^{\geq 2}}\). The lemma now follows from the decomposition (39), and from the fact that the \(U_{h\mathfrak{n}_+^{\geq 2}}\) are contained in the kernel of \(\pi_\zeta\). \(\square\)
Lemma 4.3. The image of \( \mathcal{R}_1 \) by \( \pi_\zeta \otimes \pi_{\zeta'} \) is

\[
(\pi_\zeta \otimes \pi_{\zeta'})(\mathcal{R}_1) = A(\zeta, \zeta') R^{<0}(\zeta - \zeta'),
\]
where

\[
R^{<0}(z) = \frac{1}{z - \hbar} (z \text{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2} - hP),
\]
where \( P \) is the permutation operator of the two factors of \((\mathbb{C}^2)^{\otimes 2}\), and \( A(\zeta, \zeta') \) is the formal series \( \exp \left( \sum_{i \geq 0} \left( \frac{1}{\hbar^i} - \zeta^{i+1} \right) \zeta'^{i-1} \right) \).

Proof. Since the image by \( \pi_\zeta \) and \( \pi_{\zeta'} \) of \( U_h \mathfrak{n}_+^2 \) is equal to zero, and using again the fact that \( F_1 \) belongs to \( 1 + h \sum_{i \geq 0} e_{-i-1} \otimes f_i + U_h \mathfrak{n}_+^2 \otimes U_h \mathfrak{n}_-^2 \), we find that this image is the same as that of

\[
\left( 1 + h \sum_{i \geq 0} f_i \otimes e_{-i-1} \right) q^\frac{1}{2} \sum_{i \geq 0} h_i \otimes h_{-i-1} \left( 1 - h \sum_{i \geq 0} e_{-i-1} \otimes f_i \right).
\]

Let us denote by \( E_{ij} \) the endomorphism of \( \mathbb{C}^2 \) such that \( E_{ij} v_\alpha = \delta_{ij} v_\alpha \), where \((v_1, v_{-1})\) is the standard basis of \( \mathbb{C}^2 \). We find that

\[
(\pi_\zeta \otimes \pi_{\zeta'})(\mathcal{R}_1) = A(\zeta, \zeta') \left( 1 + \frac{\hbar}{\zeta' - \zeta} E_{-1,1} \otimes E_{1,-1} \right) \left( E_{1,1} \otimes E_{1,1} + E_{-1,-1} \otimes E_{-1,-1} \right) + \frac{\zeta' - \zeta}{\zeta' + \hbar} E_{1,1} \otimes E_{1,-1} + \frac{\zeta' - \zeta - \hbar}{\zeta' + \zeta} E_{-1,-1} \otimes E_{1,1} \left( 1 - \frac{\hbar}{\zeta' - \zeta} E_{-1,1} \otimes E_{1,1} \right);
\]
the lemma follows. \( \square \)

Define \( R^{>0}(z) \) as the inverse of \( R^{<0}(z) \). We have

\[
R^{>0}(z) = \frac{1}{z + \hbar} (z \text{Id}_{\mathbb{C}^2 \otimes \mathbb{C}^2} + hP).
\]

Let us now apply to \( \mathcal{R}_1 \) \( 1 \otimes \pi_\zeta \otimes \pi_{\zeta'} \), \( \pi_\zeta \otimes \pi_{\zeta'} \otimes 1 \) and \( \pi_\zeta \otimes \pi_{\zeta'} \otimes 1 \). We find the following relations between matrices \( L^\pm(\zeta) \):

**Proposition 4.1.** We have

\[
R^0(\zeta - \zeta') L^{(1)}(\zeta) L^{(2)}(\zeta') = L^{(2)}(\zeta') L^{(1)}(\zeta) R^0(\zeta - \zeta')
\]

\[
L^{<0}(\zeta) R^{<0}(\zeta - \zeta') L^{>0}(\zeta') = L^{>0}(\zeta') R^{<0}(\zeta - \zeta' - hK) L^{<0}(\zeta) \frac{A(\zeta, \zeta - hK)}{A(\zeta, \zeta')}, \tag{42}
\]

\[
\eta \in \{ \geq 0, < 0 \}.
\]
Remark 6. After analytic continuation in the variables $\zeta, \zeta'$, we see that $A(\zeta, \zeta')$ only depends on $\zeta - \zeta'$. If we set $A(\zeta, \zeta') = A(\zeta - \zeta')$, we then have

$$A(z)A(z + \hbar) = \frac{z}{z + \hbar},$$

so that $A$ is equal to

$$A(z) = \frac{\Gamma\left(\frac{z}{2\hbar} + \frac{1}{2}\right)^2}{\Gamma\left(\frac{z}{2\hbar} + 1\right) \Gamma\left(\frac{z}{2\hbar}\right)}.$$

5. $F_1$ and the Yangian coproduct

Since $\Delta(A^{\geq 0}) \subset A \otimes A^{\geq 0}$ and $F_1$ and $F_1^{-1}$ belong to $A \otimes A^{\geq 0}$, $\Delta_1(A^{\geq 0}) \subset A \otimes A^{\geq 0}$. On the other hand, $\Delta_1 = \text{Ad}(F_1^{-1}) \circ \bar{\Delta}$; since $\bar{\Delta}(A^{\geq 0}) \subset A^{\geq 0} \otimes A$, and $F_2$ and $F_2^{-1}$ belong to $A^{\geq 0} \otimes A$, $\Delta_1(A^{\geq 0}) \subset A^{\geq 0} \otimes A$. This shows that $A^{\geq 0}$ is a Hopf subalgebra of $(A, \Delta)$.

We can show in the same way that $A^{< 0}$ is a Hopf subalgebra of $(A, \Delta_1)$.

Therefore it is natural to expect that $\Delta_1$ coincides with the Yangian coproduct $\Delta_{Yg}$. In this section, we show that this is indeed the case.

Since $(A, \Delta_1, R_1)$ is a quasi-triangular Hopf algebra, we have

$$\Delta_1 \otimes 1)(R_1) = R_1^{(13)}R_1^{(23)}, \quad (1 \otimes \Delta_1)(R_1) = R_1^{(13)}R_1^{(12)}.$$

(43)

Apply now $id \otimes id \otimes \pi_\zeta$ to the first equation of (43) and $\pi_\zeta \otimes id \otimes id$ to the second one. We find

$$(\Delta_1 \otimes 1)L^{\geq 0}(\zeta)^{(12)} = L^{\geq 0}(\zeta)^{(13)}L^{\geq 0}(\zeta)^{(23)}, \quad (\Delta_1 \otimes 1)L^{< 0}(\zeta)^{(12)} = L^{< 0}(\zeta)^{(13)}L^{< 0}(\zeta)^{(23)},$$

where $L^{< 0}(\zeta) = q^{K}\zeta L^{< 0}(\zeta)$; the last equation implies, since $\Delta(K) = K \otimes 1 + 1 \otimes K$, that

$$(\Delta_1 \otimes 1)L^{< 0}(\zeta)^{(12)} = L^{< 0}(\zeta - \hbar K_1)^{(13)}L^{< 0}(\zeta)^{(23)},$$

Since we also have $[D \otimes 1 + 1 \otimes D, F_1] = 0$ (the algebras $A_{\pm}$ being $\text{ad}(D)$-invariant), and $[K \otimes 1 + 1 \otimes K, F_1] = 0$, and comparing the above formulas with (12), (13), we conclude:

**Proposition 5.1.** $\Delta_1 = \text{Ad}(F_1) \circ \Delta$ coincides with $\Delta_{Yg}$.

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CENTRE DE MATHEMATIQUES, URA 169 DU CNRS, ECOLE POLYTECHNIQUE, 91128 PALAIS-SÉAM, FRANCE

D-MATH, ETH-ZENTRUM, HG G46, CH-8092 ZURICH, SUISSE