On the analytical approximation of the quadratic non-linear oscillator by modified extended iteration method

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Abstract

A modified analytical solution of the quadratic non-linear oscillator has been obtained based on an extended iteration method. In this study, truncated Fourier terms have been used in each step of iterations. The frequencies obtained by this technique show good agreement with the exact frequency. The percentage of error between the exact frequency and our third approximate frequency is as low as 0.001%. There is no algebraic complexity in our calculation, which is why this technique is very easy. The results have been compared with the exact and other existing results, which are both convergent and consistent.

Keywords: extended iteration procedure, quadratic oscillator, non-linearity, non-linear oscillations.

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1 Introduction

Non-linear dynamic problems have fascinated applied mathematicians, physicists and engineers from a long time. Over the past few decades, applications in solid and structural mechanics as well as fluid mechanics have appeared, and currently, there is widespread interest in non-linear oscillators, strange attractors, as well as the chaotic and dynamical systems theories in the engineering and applied science communities.

Physical and mechanical oscillatory systems are often governed by non-linear differential equations. Unfortunately, with the exception of a few particular cases, the exact analytical solutions of such equations cannot be determined. In many cases, it is possible to replace the non-linear differential equation by a corresponding linear differential equation that approximates the original non-linear equation closely, to give useful results. Often such linearisation is not feasible or possible, and for this situation, the original differential equation itself must be directly dealt with.
However, in many cases, it is possible to compute accurate approximate analytical solutions of the equations. A large number of approximate methods, such as perturbation [1–6], harmonic balance (HB) [7–13], homotopy perturbation [14], iteration [15–28] and so on, are commonly used for solving non-linear oscillatory systems. The perturbation method is mainly used for small non-linear problems. On the other hand, HB and iteration methods are mostly used for strong and small non-linear problems.

One important class of non-linear oscillators is conservative oscillators, in which the restoring force is not dependent on time, the total energy is constant and any oscillation is stationary. Despite the great elegance and simplicity of such equations, the solutions of specific problems are significantly hard to derive. Finding innovative methods to analyse and solve these equations has become an interesting subject in the field of ordinary and partial differential equations and dynamical systems. Using non-linear equations for most real-life problems is not always possible, and sometimes, it is not even advantageous to express exact solutions of non-linear differential equations explicitly in terms of elementary functions or independent spatial and/or temporal variables; however, it is possible to find approximate solutions.

Perturbation means grossly small change; so, the method is adopted when the non-linearity is small. Thus, in case of strong non-linearities, the perturbation method is not generally adopted. It is used to construct a uniformly valid periodic solution to second-order non-linear differential equations. A critical feature of the technique is a middle step that breaks the problem into ‘solvable’ and ‘perturbation’ parts. Perturbation theory is applicable if the problem at hand cannot be solved exactly but can be formulated by adding a ‘small’ term to the mathematical description of the exactly solvable problem.

The HB method is a procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. An important advantage of the method is that it can be applied to non-linear oscillatory problems for which the non-linear terms are not ‘small’, i.e. no perturbation parameter need to exist. A disadvantage of the method is that it is difficult to predict, for a given non-linear differential equation, whether a first-order HB calculation will provide a sufficiently accurate approximation to the periodic solution.

The iterative technique is particularly useful for calculating approximate periodic solutions and the corresponding frequencies of truly non-linear oscillators for small and large amplitudes of oscillation.

The main intention of this research is to investigate the approximate analytical solutions using the modified extended iterative method to decompose the secular term, so that the solution can be obtained by the iterative procedure. This means that we can use the extended iterative method to investigate many non-linear problems. The main thrust of this technique is that the obtained solution rapidly converges to the exact solutions.

2 Methodology

An extended iterative method is used to obtain the analytical solution of the quadratic non-linear oscillator. The procedure may be briefly described as follows.

A non-linear oscillator is modelled by the following expression:

\[
\ddot{x} + f(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}
\]

where the overdots denote differentiation with respect to time \(t\).

First, we choose the natural frequency \(\Omega\) of this system. Then, adding \(\Omega^2 x\) on both sides of Eq. (1), we obtain

\[
\ddot{x} + \Omega^2 x = \Omega^2 x - f(x) \equiv G(x, \Omega). \tag{2}
\]

The extended iterative scheme is

\[
\ddot{x}_{k+1} + \Omega^2_{k+1} x_{k+1} = G(x_{k-1}, \Omega) + G_x(x_{k-1}, \Omega)(x_k - x_{k-1}); \quad k = 1, 2, \ldots \tag{3}
\]

where \(G_x = \frac{\partial G}{\partial x}\).
The right-hand side of Eq. (3) is essentially the first term in a Taylor series expansion of the function $G(x_k, \dot{x}_k)$ at the point $(x_{k-1}, \dot{x}_{k-1})$ [29].

We have the direct iteration scheme of Eq. (2), as shown in Eq. (4):

$$\ddot{x}_{k+1} + \Omega_{k}^2 x_{k+1} = G(x_k, \Omega_k); \ k = 0, 1, 2, \ldots$$

(4)

and $x_{k+1}$ satisfies the condition

$$x_{k+1}(0) = A.$$  

(5)

The initial estimate is considered to be the following [10]:

$$x_0(t) = A \cos(\Omega_0 t).$$

(6)

The above procedure gives the sequence of following solutions: $x_1(t), x_2(t), x_3(t), \ldots$. The method can be extrapolated to any order of approximation; but due to growing algebraic complexity, the solution is confined to a lower order, usually the second [15].

3 Solution procedure

Let us consider the following non-linear inverse oscillator:

$$\ddot{x} + x^2 = 0$$

(7)

Adding $\Omega^2 x$ on both sides of Eq. (7), we get

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^2 = G(x, \Omega)$$

(8)

where $G(x, \Omega) = \Omega^2 x - x^2$ and $G_x(x, \Omega) = \Omega^2 - 2x$.

According to Eq. (4), the direct iterative scheme of Eq. (8) is

$$\ddot{x}_{k+1} + \Omega_{k}^2 x_{k+1} = \Omega_{k}^2 x_k - x_k^2$$

(9)

The first approximation $x_1(t)$ and the frequency $\Omega_0$ are obtained by substituting $k = 0$ in Eq. (9), and using Eq. (6), we get

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 x_0 - x_0^2$$

(10)

where $x_0(t) = A \cos(\Omega_0 t) = A \cos \theta$ and $\theta = \Omega_0 t$.

Now, substituting and expanding the right-hand side in a Fourier cosine series, Eq. (10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A \cos \theta - (A \cos \theta)^2$$

$$- 0.024252 \cos 5\theta + 0.008084 \cos 7\theta - 0.003675 \cos 9\theta$$

(11)

To avoid secular terms in the solution, we have to remove $\cos \theta$ from the right-hand side of Eq. (11). Thus, we have

$$\Omega_0^2 A - 0.848826 A^2 = 0, \ \Omega_0^2 = \frac{0.848826 A^2}{A}, \ \Omega_0 = 0.921318 \sqrt{A}.$$
This is the first approximate frequency of the oscillator. Note that \( \Omega_{\text{exact}}(A) = 0.914681\sqrt{A} \). After simplification, Eq. (11) reduces to

\[
\ddot{x}_1 + \Omega_1^2 x_1 = A^2 (-0.169765 \cos 3\theta + 0.024252 \cos 5\theta - 0.008084 \cos 7\theta + 0.003675 \cos 9\theta - 0.001979 \cos 11\theta).
\]

The particular solution, \( x_1^{(p)}(t) \), is as follows:

\[
x_1^{(p)}(t) = \frac{-0.169765A^2}{-9\Omega_0^2 + \Omega_0^2} \cos 3\theta + \frac{0.024252A^2}{-25\Omega_0^2 + \Omega_0^2} \cos 5\theta + \frac{-0.008084A^2}{-49\Omega_0^2 + \Omega_0^2} \cos 7\theta \\
+ \frac{0.003675A^2}{-81\Omega_0^2 + \Omega_0^2} \cos 9\theta + \frac{0.001979A^2}{-121\Omega_0^2 + \Omega_0^2} \cos 11\theta \\
= \frac{A^2}{\Omega_0^2} \left( \frac{0.169765}{8} \cos 3\theta - \frac{0.024252}{24} \cos 5\theta + \frac{0.008084}{48} \cos 7\theta \\
- \frac{0.003675}{80} \cos 9\theta - \frac{0.001979}{120} \cos 11\theta \right)
\]

\[
= A(0.025 \cos 3\theta - 0.00119 \cos 5\theta + 0.000199 \cos 7\theta - 0.000054 \cos 9\theta - 0.000191 \cos 11\theta)
\]

Therefore, the complete solution is

\[
x_1(t) = B_1 \cos \theta + 0.025A \cos 3\theta - 0.00119 \cos 5\theta + 0.000199 \cos 7\theta - 0.000054 \cos 9\theta - 0.000191 \cos 11\theta.
\]

Using \( x_1(0) = A \), we have \( B_1 = 0.976066A \). Then, we obtain

\[
x_1(t) = A(0.976066 \cos \theta + 0.025 \cos 3\theta - 0.00119048 \cos 5\theta + 0.00198413 \cos 7\theta \\
- 0.0000541126 \cos 9\theta - 0.000019425 \cos 11\theta).
\]

This is the first approximate solution of the oscillator.

According to Eq. (3), the extended iterative scheme of Eq. (8) is

\[
\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = (\Omega_k^2 x_k - x_k^2) + (\Omega_k^2 - 2x_k) (x_k - x_{k-1}).
\]

The second approximation \( x_2(t) \) and the frequency \( \Omega_1 \) are obtained by substituting \( k = 1 \) in Eq. (17), and using Eq. (7), we get Eq. (18):

\[
\ddot{x}_2 + \Omega_1^2 x_2 = x_1^2 + \Omega_1^2 x_0 - 2x_1x_0
\]

where \( x_0(t) \) and \( x_1(t) \) are given by Eqs (6) and (16).

Now substituting \( x_0(t) \) and \( x_1(t) \) and expanding the right-hand side in a Fourier cosine series, Eq. (17) reduces to

\[
\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^2 - 0.976066A - 0.816744A^2) \cos \theta + (\Omega_1^2 - 0.025000A \\
- 0.193883A^2) \cos 3\theta + (-\Omega_1^2 - 0.001190A + 0.014431A^2) \cos 5\theta \\
+ (\Omega_1^2 - 0.000198A - 0.005635A^2) \cos 7\theta + (-\Omega_1^2 - 0.000054A \\
+ 0.002713A^2) \cos 9\theta + (-\Omega_1^2 - 0.000019A - 0.001441A^2) \cos 11\theta \\
+ 0.000911A^2 \cos 13\theta
\]

To avoid secular terms in the solution, we have to remove \( \cos \theta \) from the right-hand side of Eq. (19). Thus, we have

\[
\Omega_1^2 - 0.976066A - 0.816744A^2 = 0, \ \Omega_1 = 0.914752\sqrt{A}.
\]
The particular solution, \( x_2(p)(t) \), is
\[
x_2(p)(t) = \frac{1}{9\Omega_1^2 + \Omega_2^2} (\Omega_2^2 0.025000A - 0.193883A^2) \cos \theta + \frac{1}{25\Omega_1^2 + \Omega_2^2} (-\Omega_2^20.001190A + 0.014431A^2) \cos 5\theta + \frac{1}{48\Omega_1^2 + \Omega_2^2} (\Omega_2^20.000054A + 0.002712A^2) \cos 9\theta + \frac{1}{112\Omega_1^2 + \Omega_2^2} (\Omega_2^20.000001A - 0.00144A^2) \cos 11\theta + \frac{1}{168\Omega_1^2 + \Omega_2^2} (0.00091A^2) \cos 13\theta
\]
\[
= -\frac{0.025000A}{8} \cos 3\theta + \frac{0.193883A^2}{8\Omega_1^2} \cos 3\theta - \frac{0.001190A}{24} \cos 5\theta - \frac{0.005635A^2}{48\Omega_1^2} \cos 7\theta + \frac{0.000091A^2}{120\Omega_1^2} \cos 9\theta - \frac{0.002712A^2}{168\Omega_1^2} \cos 9\theta + \frac{0.000015A}{120\Omega_1^2} \cos 11\theta + \frac{0.00091A^2}{168\Omega_1^2} \cos 13\theta
\]
\[
= 0.025838\cos 3\theta - 0.000669\cos 5\theta - 0.00136A\cos 7\theta - 0.000040A\cos 9\theta + 0.000015A\cos 11\theta - 6.48356 \times 10^{-6}A\cos 13\theta
\]
Therefore, the complete solution is
\[
x_2(t) = B_2 \cos \theta + 0.025838\cos 3\theta - 0.000669\cos 5\theta + 0.00136A\cos 7\theta - 0.000040A\cos 9\theta + 0.000015A\cos 11\theta - 6.48356 \times 10^{-6}A\cos 13\theta
\]
Using \( x_2(0) = A \), we have \( B_2 = 0.974727A \). Then, we obtain
\[
x_2(t) = 0.974727A\cos \theta + 0.0258379A\cos 3\theta - 0.000669\cos 5\theta + 0.00136A\cos 7\theta - 0.000040A\cos 9\theta + 0.000015A\cos 11\theta - 6.48356 \times 10^{-6}A\cos 13\theta
\]
This is the second approximate solution of the oscillator.

Proceeding to the third level of iteration, satisfies the following equation:
\[
\ddot{x}_3 + \Omega_3^2 x_3 = x_2^2 + \Omega^2 x_1 - 2x_2 x_1
\]
where
\[
x_1(t) = A(0.976066\cos \theta + 0.025\cos 3\theta - 0.00119048\cos 5\theta + 0.000198413\cos 7\theta - 0.0000541126\cos 9\theta - 0.000019425\cos 11\theta)
\]
and
\[
x_2(t) = A(0.974727\cos \theta + 0.0258379\cos 3\theta - 0.000669\cos 5\theta + 0.00136\cos 7\theta - 0.000040\cos 9\theta + 0.000015\cos 11\theta - 6.48356 \times 10^{-6}\cos 13\theta)
\]
Now substituting \( x_1(t) \) and \( x_2(t) \) and expanding the right-hand side in a Fourier cosine series, Eq. (24) reduces to
\[
\ddot{x}_3 + \Omega_3^2 x_3 = (-\Omega_2^20.974727A - 0.815477A^2) \cos \theta + (\Omega_2^20.025838A - 0.193801A^2) \cos 3\theta + (-\Omega_2^20.000069A + 0.013352A^2) \cos 5\theta + (\Omega_2^20.000136A - 0.0058897A^2) \cos 7\theta + (-\Omega_2^20.0000399A + 0.002779A^2) \cos 9\theta + (\Omega_2^20.0000015A - 0.001523A^2) \cos 11\theta + (-\Omega_2^20.0000006A + 0.000924A^2) \cos 13\theta - 0.00597A^2 \cos 15\theta + 0.00412A^2 \cos 17\theta - 0.000296A^2 \cos 19\theta + 0.000219A^2 \cos 21\theta + 0.000167A^2 \cos 23\theta + 0.0001298A^2 \cos 25\theta - 0.000103A^2 \cos 27\theta
\]
To avoid secular terms in the solution, we have to remove from the right-hand side of Eq. (25). Thus, we have
\[-\Omega_3^2 0.974727A - 0.815477A^2 = 0, \quad \Omega_2 = 0.91467\sqrt{A}. \quad (26)\]

The particular solution, \(x_3^{(p)}(t)\), is
\[x_3^{(p)}(t) = -\frac{1}{8\Omega_2^2} (\Omega_2^2 0.025838A - 0.193801A^2) \cos 3\theta - \frac{1}{24\Omega_2^2} (-\Omega_2^2 0.000669A + 0.013352A^2) \cos 5\theta - \frac{1}{48\Omega_2^2} (\Omega_2^2 0.0000399A + 0.002779A^2) \cos 7\theta - \frac{1}{80\Omega_2^2} (-\Omega_2^2 0.000015A - 0.001523A^2) \cos 11\theta - \frac{1}{168\Omega_2^2} (-\Omega_2^2 0.000006A + 0.000924A^2) \cos 13\theta - \frac{1}{224\Omega_2^2} (-0.000597A^2) \cos 17\theta - \frac{1}{360\Omega_2^2} (0.000412A^2) \cos 19\theta - \frac{1}{536\Omega_2^2} (0.0001298A^2) \cos 21\theta - \frac{1}{728\Omega_2^2} (-0.000103A^2) \cos 23\theta
= 0.025726A \cos 3\theta - 0.000637A \cos 5\theta + 0.000144A \cos 7\theta - 0.000015A \cos 9\theta + 3.18798 \times 10^{-6}A \cos 11\theta - 6.53694 \times 10^{-6}A \cos 13\theta + 9.381306 \times 10^{-7}A \cos 15\theta - 1.71121 \times 10^{-6}A \cos 17\theta + 3.18798 \times 10^{-6}A \cos 19\theta - 5.94814 \times 10^{-7}A \cos 21\theta + 9.81306 \times 10^{-7}A \cos 23\theta - 2.4863 \times 10^{-7}A \cos 25\theta + 1.69177 \times 10^{-7}A \cos 27\theta \quad (27)\]

Therefore, the complete solution is
\[x_3(t) = 0.974798A \cos \theta + 0.025726A \cos 3\theta - 0.000637A \cos 5\theta + 0.000144A \cos 7\theta - 0.000015A \cos 9\theta + 3.18798 \times 10^{-6}A \cos 11\theta - 6.53694 \times 10^{-6}A \cos 13\theta + 9.381306 \times 10^{-7}A \cos 15\theta - 1.71121 \times 10^{-6}A \cos 17\theta + 3.18798 \times 10^{-6}A \cos 19\theta - 5.94814 \times 10^{-7}A \cos 21\theta - 3.77331 \times 10^{-7}A \cos 23\theta - 2.4863 \times 10^{-7}A \cos 25\theta + 1.69177 \times 10^{-7}A \cos 27\theta \quad (28)\]

Using \(x_3(0) = A\), we have \(B_3 = 0.974798A\). Then, we obtain
\[x_3(t) = 0.974798A \cos \theta + 0.025726A \cos 3\theta - 0.000637A \cos 5\theta + 0.000144A \cos 7\theta - 0.000015A \cos 9\theta + 3.18798 \times 10^{-6}A \cos 11\theta - 6.53694 \times 10^{-6}A \cos 13\theta + 9.381306 \times 10^{-7}A \cos 15\theta - 1.71121 \times 10^{-6}A \cos 17\theta + 3.18798 \times 10^{-6}A \cos 19\theta - 5.94814 \times 10^{-7}A \cos 21\theta - 3.77331 \times 10^{-7}A \cos 23\theta - 2.4863 \times 10^{-7}A \cos 25\theta + 1.69177 \times 10^{-7}A \cos 27\theta \quad (29)\]

This is the third approximate solution of the oscillator.

Thus, \(\Omega_1\), \(\Omega_2\), respectively obtained from Eqs (12), (20) and (26), represent the approximation of the frequencies, and \(x_1(t)\), \(x_2(t)\), \(x_3(t)\), respectively obtained from Eqs (16), (23) and (29), represent the corresponding approximate solutions of the oscillator represented by Eq. (7).
4 Another example

Let us consider the non-linear oscillator

\[ \ddot{x} + x = -x^2 \]  

(30)

with initial condition \( x(x) = A, \dot{x}(0) = 0 \).

Obviously, Eq. (30) can be written as

\[ \ddot{x} + \Omega^2 x = \Omega^2 x - x^2 - x = G(x, \dot{x}) \quad \text{Say} \]

(31)

According to Eq. (4), the direct iterative scheme of Eq. (30) is as follows:

\[ \ddot{x}_{k+1} + \Omega^2 x_{k+1} = \Omega^2 x_k - x_k^2 \]

(32)

The first approximation \( x_1(t) \) and corresponding frequency \( \Omega_0 \) are obtained from Eq. (32) by substituting \( k = 0 \):

\[ x_1 = \left( A - \frac{1}{32} (-4 + A^2) a_{1,3} \right) \cos \theta + \frac{1}{32} (-4 + A^2) a_{1,3} \cos 3\theta \]

(33)

where

\[ A_{1,3} = \frac{1}{4} A^3 \Omega^2_0 \]

(34)

and

\[ \Omega_0 = \frac{2}{\sqrt{4 - A^2}} \]

(35)

Now, the extended iteration scheme, according to Eq. (3), is as follows:

\[ \ddot{x}_{k+1} + \Omega^2 x_{k+1} = G(x_{k-1}, \dot{x}_{k-1}) + G_x(x_{k-1}, \dot{x}_{k-1})(x_k - x_{k-1}) + G_{\dot{x}}(x_{k-1}, \dot{x}_{k-1})(\dot{x}_k - \dot{x}_{k-1}); \ k = 1, 2, \ldots \]

(36)

The second approximation \( x_2(t) \) and the corresponding frequency \( \Omega_1 \) are obtained from Eq. (36) by substituting \( k = 1 \):

\[ y_2 = \left( A + \frac{1}{\Omega_1^2} \left( \frac{a_{23}}{8} + \frac{a_{25}}{24} \right) \right) \cos \theta - \frac{1}{\Omega_1^2} \left( \frac{a_{23}}{8} \cos 3\theta + \frac{a_{25}}{24} \cos 5\theta \right) \]

(37)

where

\[ a_{23} = \frac{7A^5}{2(4 - A^2)(-4 + A^2)} - \frac{9A^5}{8(4 - A^2)(-4 + A^2)} + \frac{A^7}{16(4 - A^2)(-4 + A^2)} - \frac{4A^3 \Omega_1}{\sqrt{4 - A^2}(-4 + A^2)} + \frac{7A^5 \Omega_1}{8\sqrt{4 - A^2}(4 - A^2)} - \frac{7A^5 \Omega_1}{8\sqrt{4 - A^2}(4 - A^2)} + \frac{32A^7 \Omega_1}{32\sqrt{4 - A^2}(-4 + A^2)} - \frac{A^3 \Omega_1^2}{2(4 - A^2)(-4 + A^2)} - \frac{4A^3 \Omega_1^2}{4(4 - A^2)(-4 + A^2)} + \frac{32A^7 \Omega_1^2}{32(4 - A^2)(-4 + A^2)} \]

(38)

\[ a_{25} = \frac{A^5}{8(4 - A^2)(-4 + A^2)} - \frac{A^7}{32(4 - A^2)(-4 + A^2)} + \frac{3A^5 \Omega_1}{8\sqrt{4 - A^2}(-4 + A^2)} - \frac{3A^7 \Omega_1}{32\sqrt{4 - A^2}(-4 + A^2)} \]

(39)

\[ \Omega_1 = (-128A^2 + 40A^4 - 2A^6 - 2(4096A^4 - 2560A^6 + 528A^8 - 40A^{10} + A^{12} + 16384(4 - A^2)} - 11264A^2(4 - A^2) + 1680A^4(4 - A^2) + 32A^6(4 - A^2) - A^8(4 - A^2))^{1/2}/(2(-128\sqrt{4 - A^2} + 2A^2\sqrt{4 - A^2} + A^4\sqrt{4 - A^2})) \]

(40)

In a similar way, the method can be used for higher-order approximations.
5 Results and discussion

An iterative approach to obtain the approximate solution of the ‘quadratic non-linear oscillator’ is presented. The presented technique is very simple for solving algebraic equations analytically, and the approach is different from other existing approaches for adopting truncated Fourier series. This process significantly improves the results.

Here, the first, second and third approximate frequencies $\Omega_0$, $\Omega_1$ and $\Omega_2$ have been calculated, and the results are given in Table 1.

| Amplitude $A$ | First approximate frequencies, $\Omega_0$ | Second approximate frequencies, $\Omega_1$ | Third approximate frequencies, $\Omega_2$ |
|---------------|-------------------------------------|----------------------------------------|----------------------------------------|
|               | $0.921318\sqrt{A}$ 0.73            | $0.914752\sqrt{A}$ 0.0078             | $0.91467\sqrt{A}$ 0.0012              |

To compare the approximate frequencies, we have also given the existing results determined by Mickens and Ramadhani [9], Belendez et al. [14], Hosen [13] and Haque and Hossain [24], which are shown in Table 2. Fortunately, this current method gives significantly better results than the other formulae.

| Amplitude $A$ | First approximate frequencies, $\Omega_0$ & Error (%) | Second approximate frequencies, $\Omega_1$ & Error (%) | Third approximate frequencies, $\Omega_2$ & Error (%) |
|---------------|------------------------------------------------------|------------------------------------------------------|------------------------------------------------------|
| Mickens and Ramadhani [9] | $0.921318\sqrt{A}$ 0.73 & 0.0078 | $0.914044\sqrt{A}$ 0.0078 & 0.0078 | $0.914711\sqrt{A}$ 0.0032 & 0.0032 |
| Belendez et al. [14] | $0.921318\sqrt{A}$ 0.73 & 0.0078 | $0.914274\sqrt{A}$ 0.045 & 0.045 | $0.914733\sqrt{A}$ 0.0056 & 0.0056 |
| Hosen M A [13] | $0.921318\sqrt{A}$ 0.73 & 0.0078 | $0.914427\sqrt{A}$ 0.028 & 0.028 | $0.914705\sqrt{A}$ 0.0026 & 0.0026 |
| Haque and Hossain [24] | $0.921318\sqrt{A}$ 0.73 & 0.0078 | $0.914752\sqrt{A}$ 0.0078 & 0.0078 | $0.91467\sqrt{A}$ 0.0012 & 0.0012 |

To show the accuracy, the percentage of errors is calculated by the following definition:

$$\text{Error} = \left| \frac{\Omega_e - \Omega_k}{\Omega_e} \right| \times 100\%$$

where $\Omega_k (k = 0, 1, 2, \ldots)$ represents the approximate frequencies obtained by the present method, and $\Omega_e$ represents the corresponding exact frequency of the oscillator.

Though the method has been illustrated for a quadratic non-linear oscillator, it is also valid for other oscillators. The same technique has been applied to solve the non-linear jerk oscillator. Herein, we have calculated the approximate frequencies $\Omega_0$, $\Omega_1$ and $\Omega_2$, respectively, in Section 4. The results are given in Table 3 for five particular values of $A$; to compare the approximate frequencies, we have also given the existing results determined by Gottlieb [30]. We see that the presented technique gives us emphatically better results than the Gottlieb [30] technique.
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Table 3 Comparison of the approximate periods with exact periods $T_e$ of $\ddot{x} + x = -x\dot{x}^2$.

| A  | $T_e$       | $T_e$ Er(%) | $T_G$ Er(%) |
|----|-------------|-------------|-------------|
| 0.1| 6.275334    | 6.275333 3.43 e^{-6} | 6.2753264 1.21 e^{-4} |
| 0.2| 6.251809    | 6.251809 3.61 e^{-7} | 6.251690 1.90 e^{-3} |
| 0.5| 6.088449    | 6.088449 3.01 e^{-6} | 6.083668 7.85 e^{-2} |
| 1  | 5.527200    | 5.527434 4.25 e^{-3} | 5.441398 1.55   |
| 1.5| 4.690247    | 4.709049 4.7 e^{-1}  | 4.155936 11.39  |

$T$ denotes the modified approximate period and $T_G$ denotes approximate period obtained by Gottlieb [30]. Er(%) denotes percentage error.

6 Convergence and consistency analysis

We know that the basic idea of iterative methods is to construct a sequence of solutions $x_k$ (as well as frequencies $\Omega_k$) that have the property of convergence:

$$x_e = \lim_{k \to \infty} x_k; \text{ or } \Omega_e = \lim_{k \to \infty} \Omega_k.$$

Here, $x_e$ is the exact solution of the given non-linear oscillator.

In the present method, the solution yields less error in each iterative step compared to the previous iterative step, and finally, $|\Omega_2 - \Omega_e| = 0.91467 - 0.914681 < \varepsilon$, where $\varepsilon$ is a small positive number and $A$ is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by Eq. (4), with initial estimate given in Eq. (5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0, \text{ or } \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0$$

In the present analysis, we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0, \text{ as } |\Omega_2 - \Omega_e| = 0.$$

Thus, the consistency of the method is achieved.

7 Conclusion

It is noted that Mickens and Ramadhani [9] found only the second approximate frequencies by the HB method. Belendez et al. [14] found up to the third approximate frequencies by using a modified He’s homotopy perturbation method. Again, Hosen [13] found up to the third approximate frequencies by using a modified HB method; Haque and Hossain [24] found up to the fourth approximate frequencies by the iteration method.

In our study, it is seen that the third-order approximate frequency obtained by the adopted method is almost same as the exact frequency. It is found that, in most of the cases, our solution gives significantly better results than other existing results. The advantages of this method include its simplicity and computational efficiency.

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