Numerical evidence of linear response violations in chaotic systems.

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Abstract: It has been rigorously shown in [11] that the complex susceptibility for chaotic maps of the interval can have a pole in the upper complex plane. I develop a numerical procedure allowing to exhibit this pole from time series. I then apply the same analysis to the Hénon map and conjecture that the complex susceptibility has also a pole in the upper half complex plane.

In many situations, physical systems submitted to a weak external perturbation display a response which is proportional to this perturbation. This “linear response” property is manifested in many laws of physics (Ohm’s law, Fourier’s law, Fick’s law, ...). On fundamental grounds, a linear response theory can be established via different approaches, either by considering situations close to equilibrium, perturbing the corresponding Gibbs distribution and performing a Taylor expansion for the average value of the observable of interest, or by writ-
ing phenomenological equations/models (Langevin equations, Drude model, ...) allowing to obtain linear response from an ad hoc microscopic dynamics.

The foundation of linear response theory from a realistic microscopic dynamics (e.g. Hamilton equations) is more problematic. In the context of classical (non quantum) dynamics, the properties invoked in the foundation of equilibrium statistical mechanics (molecular chaos, ergodicity, mixing) suggest that “chaotic” dynamical systems may be good candidates to establish a rigorous connexion between the microscopic dynamics and a mesoscopic linear response. However, the typical trajectories of chaotic systems exhibit initial conditions sensitivity and (local) exponential amplification of small perturbations. This “butterfly effect” is a serious obstacle toward a linear response theory in chaotic systems, as raised by Van Kampen [14]. On the other hand, statistical mechanics is not really concerned with microscopic trajectories but instead with *ensemble averages or time averages* [these averages being related by the ergodic hypothesis]. Averaging integrates all nonlinear effects of the microscopic dynamics and one may expect that “the nonlinear deviations of the microscopic motions somehow combine to produce a linear macroscopic response” (Van Kampen, [14], page 282).

In the context of “chaotic” dynamical systems one can write down an explicit [but formal] expression relating the variation of the average value of an observable to a power expansion in the perturbation of the vector field [12] [for an example see eq. (5) below]. Ruelle has shown rigorously that this expression is well defined in uniformly hyperbolic mappings [8, 9, 10] and that the variation is indeed proportional to suitable perturbations with a sufficiently small amplitude [more precisely the variation of the average value of the observable is differentiable with respect to the perturbation]. This expression reveals clearly
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the role of mixing (exponential correlation decay) in the convergence of the time series defining the linear response [for an illustration see eq. \[8\] below] and gives a striking illustration of the Van Kampen statement quoted in the end of the previous paragraph. On physical grounds one recovers in this setting the usual linear response theory close to equilibrium, the fluctuation-dissipation theorem, Kubo coefficients and Onsager theory [12]. But this expression also holds away from equilibrium and for dynamical systems having microscopic dissipation. This last result allows to use linear response out of the field of physics, for example to investigate the interplay between the network structure and dynamics in neural networks [1, 2, 3].

However, even in classical examples of one dimensional “chaotic” dynamical systems, one can exhibit situations where linear response is violated: the series defining the complex susceptibility diverges. Obviously, this can occur for systems close to a bifurcation point, and the corresponding divergence of the susceptibility is reminiscent of second order phase transitions in statistical physics. Recently, Ruelle has shown a deeper result for chaotic maps of the interval: the complex susceptibility is well defined on the real axis (real frequencies) for some class of perturbations and diverges for some others [11]. More precisely, the complex susceptibility exhibits poles in the upper half complex plane, corresponding to certain types of perturbations. This surprising result is not a violation of causality but only means that there is no linear response to this type of perturbation. This suggests that obeying linear response is not an intrinsic property of the dynamical system, but depends also on the way it is perturbed.
From this discussion two natural questions arise. Does this property exist in larger dimensional chaotic dynamical systems? Is it possible to have an experimental/numerical characterization of this effect? The first question is difficult to address from a mathematical point of view and is still unresolved even for classical examples such as the Hénon map. The second one is the main concern of the present paper. I indeed propose a numerical experimentation protocol allowing to compute the complex susceptibility and to investigate the numerically observable effects of having a pole in the upper half complex plane. This protocol is first used in the case of the logistic map where one can use the guidelines of Ruelle’s mathematical results [11] as a validation. Then I apply it to the Hénon map for the standard parameter values and conjecture that this map may also exhibit a pole in the upper half complex plane. In the present paper I shall thus regard linear response theory, not in a statistical mechanics perspective, but instead from the point of view of dynamical systems theory and ergodic theory, in the spirit of [11]. Nevertheless, the mere fact that canonical examples of “chaotic maps” exhibit responses that are not proportional to an infinitesimal perturbation, raises interesting questions in the context of non equilibrium statistical mechanics [see discussion].

The paper is organized as follows. In the first section I recall briefly Ruelle’s results and I give a qualitative explanation of the mechanism inducing a violation of linear response for certain perturbations. This is a key point to understanding the numerical effects observed. Then, I present, in a second section, the numerical method and apply it to the logistic map, with a careful discussion of the effects induced by numerics. Finally, I apply this method to the Hénon map.
1. The logistic map.

1.1. Theoretical results. In this section I briefly recall Ruelle’s results for the logistic map:

\[ f(x) = 1 - 2x^2 \]  

(Note that Ruelle’s results holds for more general maps of the interval [1].)

The corresponding dynamical system

\[ x_{t+1} = f(x_t) \]  

\([-1, 1] \to [-1, 1]\) has an absolutely continuous ergodic measure with density:

\[ \rho(x) = \frac{1}{\pi \sqrt{1 - x^2}} \]  

Denote by \( < A > = \int_{-1}^{+1} \rho(x) A(x) dx \) the average value of some observable \( A : [-1, 1] \to \mathbb{R} \) [in the sequel I shall assume that \( A \) is a continuously differentiable function on \([-1, 1]\)]. The goal of linear response theory is to compute the variation of the average value of \( A \) when perturbing the dynamical system with a suitable function \( X(x, t) \), so that the perturbed dynamical system is:

\[ x_{t+1} = f [x_t + \epsilon X(x_t, t)] \]  

By “suitable” it is meant that \( X(x_t, t) \) is such that the trajectories of the perturbed system stay inside the interval \([-1, 1]\), to avoid a trivial divergence of the trajectories.
If the perturbation is applied permanently from a distant past (in order to reach a “stationary regime”) the variation of the average value of $A$, at time $t$, is formally given by:

$$\delta \rho_t(A) = \epsilon \sum_{n=-\infty}^{t} \int_{-1}^{+1} \rho(x) A'(f^{t-n})x f'(t-n)(x) X(x, n) dx + O(\epsilon^2)$$ (5)

If the series converges it is called the linear response of the observable $A$ to the perturbation $X$. Note that the expression (5) includes all the “microscopic” nonlinearities, via the iterates $f^{t-n}x$.

I shall consider perturbations of type $X(x, t) \equiv X(x) e^{-i\omega t}$. Then:

$$\delta \rho_t(A) = \epsilon e^{-i\omega t} \chi(\omega) + O(\epsilon^2)$$ (6)

where $\chi(\omega)$ is called the complex susceptibility, and is formally given by:

$$\chi(\omega) = \Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{+1} \rho(x) A'(f^n)x f'(n)x X(x) dx$$ (7)

where $\lambda = e^{i\omega}$. As discussed in this paper, the convergence of the series depends on the class of perturbations applied.

The study of this series is easier if one uses the variable change $x = \sin(\frac{\pi}{2} y) = \omega(y)$ which maps $f$ on $g(y) = 1 - 2|y|$ [tent map]. Then the density $\rho$ is mapped onto the density $\sigma_0(y) = \frac{1}{2}$. In the new variable $y$ we have $\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} \mathcal{L}_0^n Y(s) B'(s) ds$ where $B = A \omega, Y(y) = \sigma_0(y) \frac{X(\omega(y))}{\omega(y)}$ and $\mathcal{L}_0 \Phi(y) = \Phi \left( \frac{y-1}{2} \right) - \Phi \left( \frac{1-y}{2} \right)$.

Ruelle has shown the following results [11]. One can decompose $X$ in the form $X = C_- X_- + C_+ X_+ + X_0$ where $C_-, C_+$ are some constants. The functions
X_-, X_+ will be defined below. X_0 is any square integrable holomorphic function that vanishes at \{-1, 1\}. Then \( \Psi(\lambda) = C_- \Psi(\lambda) + C_+ \Psi(\lambda) + \Psi_0(\lambda) \) has the following properties.

1. The series \( \Psi_0(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int \rho(x) A'(f^{(n)}(x)) f^{(n)}(x) X_0(x) dx \) has a meromorphic extension in \( \mathbb{C} \) and has no pole inside the unit disc. This corresponds to having no pole in the upper half complex plane when considering the “frequency” \( \omega \). Moreover, \( \Psi_0 \) is holomorphic in \( \lambda = 1 \) (resp. \( \omega = 0 \)). [This implies that the dynamical system is robust to time-constant perturbations]. Indeed, for square integrable, holomorphic perturbations, vanishing at \{-1; 1\} one has, using the variable \( y \):

\[
\Psi(\lambda) = -\sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} (L^n Y'_0)(s) B(s) ds
\]

where \( Y_0(y) = \sigma_0(y) \frac{X_0(\omega(y))}{\omega'(y)} \) and \( L \) is the Perron-Frobenius operator:

\[
L \Phi(y) = \frac{1}{2} \Phi \left( \frac{y - 1}{2} \right) + \frac{1}{2} \Phi \left( \frac{1 - y}{2} \right)
\]

The eigenvalues of the Perron-Frobenius operator have the form \( \lambda_n = \frac{1}{4^n} \), \( n = 0 \ldots \infty \) [Ruelle-Pollicott resonances \[7, 5\]]. The eigenvalue \( \lambda_0 = 1 \) corresponds to the invariant density \( \sigma_0(y) = \frac{1}{2} \). More generally the eigenmode \( \sigma_n, \ n = 0 \ldots \infty \) are polynomials with a degree \( n \). Resuming the series \[8\] for a perturbation \( Y'_0(y) = \sum_{n=1}^{\infty} C_n \sigma_n(y) \) one finds easily that the susceptibility has poles \( \lambda_n^{-1} \). Thus, all the poles of the susceptibility, related to a function \( X_0(y) = Y_0(y) \frac{\omega'(y)}{\sigma_0(y)} \), where \( Y'_0 \) decomposes on the eigenbasis \( \sigma_n \), lie outside the unit disc [resp. in the lower half of the complex plane for the frequency \( \omega \)]. Therefore, the series converge for real frequencies and the corresponding susceptibility is well defined.
Note that eq. (8) is a version of the fluctuation-dissipation theorem. Indeed, the integral corresponds to a correlation function. Therefore, the convergence of the series is “physically” ensured by exponential mixing where the Ruelle-Pollicott resonances give the decay rates. The first resonance \((\frac{1}{4})\) corresponds to the characteristic time for relaxation to equilibrium [see next section]. The corresponding eigenmode is given by:

\[
\sigma_1(y) = -\frac{1}{3} - 2y + y^2
\]  

(10)

2. The series \(\Psi_-(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int \rho(x) A'(f^{(n)}x)f^{(n)}(x)X_-(x)dx\), converges if \(|\lambda|\sqrt{|f'(-1)|} < 1\) and, in its domain of convergence, has the form \(\frac{C_-}{1-\lambda|\sqrt{|f'(-1)|}|G_A}\) where \(G_A\) is a constant depending on \(A\). The complex susceptibility has therefore a pole in \(\lambda = \frac{1}{\sqrt{|f'(-1)|}} = \frac{1}{2}\), inside the unit disk (resp. above \(\omega = i \log(2)\)). Therefore, the series \(\Psi_-\) diverges for real frequencies \(\omega\) and there is no linear response for a perturbation \(X_-\).

3. From [1] \(X_-\) writes:

\[
X_-(x) = \pi \sqrt{1 - x^2} \left[ \frac{1}{\pi \arcsin(x) + 1} + w\left(\frac{2}{\pi \arcsin(x) + 1}\right) \right]
\]  

(11)

where \(w\) is solution of:

\[
w \left(\frac{z - 1}{2}\right) - w \left(\frac{1 - z}{2}\right) - 2w(z) = \frac{2}{3 - z}
\]  

(12)

\(w\) is not an analytic function. It can be obtained via the recursion:

\[
w(z) = \begin{cases} 
  w(2z + 1) + \frac{1}{2z - 2}; & z < 0; \\
  -w(1 - 2z) - \frac{1}{2z + 2}; & z > 0;
\end{cases}
\]  

(13)
reminiscent of the Tagaki function discussed by Gaspard in the context of the multibaker map [4].

4. In the case of unimodal maps the function $X_+$ has the same form as $X_-$ and I shall not discuss it anymore.

This result shows that there exist a violation of linear response theory for the mapping (1) and certain class of observables: an infinitesimal variation of the mapping does not induce a proportional variation. Of course there is an easy way to having a divergent response in the present context. If one applies a perturbation that “pushes” the trajectories out of the interval $[-1, 1]$ one obtains divergent trajectories and a divergent response. This is however not the effect sought here [and I shall pay attention to this in the numerics]. The reason why there is a pole in the upper half plane is more subtle and can be explain with the following argument [13].

Consider the formal definition of the susceptibility [5]. $f^{(t-n)}$ has $t-n$ zeros and oscillates rapidly (for large $t-n$) with a period $\sim 4^{n-t}$. One can decompose the integral $\int_{-1}^{+1} \rho(x)A'(f^{(t-n)}x)f'(t-n)(x)X(x)dx$ over intervals delimited by the zeros of $f^{(t-n)}$. The density $\rho$ has a strong variation about $\pm1$ and small variations on the central part. The contribution of the first interval (containing $-1$) is given, for large $t-n$, by:

$$\int_{-1}^{-1+4^{n-t}} \rho(x)A'(f^{(t-n)}x)f'(t-n)(x)X(x)dx \sim \frac{4^{t-n}}{\pi} \int_0^{4^{n-t}} \frac{A'(f^{(t-n)}(-1+u))X(-1+u)}{\sqrt{u}} du$$

(14)

where $x = -1 + u$. If $A'(f^{(t-n)}(-1+u))X(-1+u)$ does not vanish about $u = 0 (x = -1)$ and has small variations on this interval, this contribution is $\sim 4^{t-n} \int_0^{4^{n-t}} \frac{1}{\sqrt{u}} du = 2^{t-n}$, which diverges when $t-n \to \infty$, with a rate exactly
given by the pole $\lambda = \frac{1}{2}$. The corresponding susceptibility is given by a series containing a contribution $\sim \epsilon \sum_{n=-\infty}^{t} e^{-\omega_n 2^{t-n}}$ which converges only if $\omega_i > \log(2)$, where $\omega = \omega_r + i\omega_i$, and diverges on the real axis. If one considers now the contribution of the 'bulk' to the integral (14) one can figure out that it does not diverge, essentially because the derivative $f'(t-n)(x)$ has an alternating sign and because the density $\rho$ has small variations in the bulk. More precisely, the contribution of the bulk is provided by a decomposition on the Ruelle-Pollicott eigenmodes $\sigma_n$ and decays exponentially due to mixing. Thus, the existence of a pole in the upper half complex plane is due to a boundary effect obtained when the perturbation weights the points $\{-1, 1\}$ (where the density diverges). It is remarkable [but easily explained with the same argument] that the rate of exponential divergence, $\log(2)$, is exactly the value of the Lyapunov exponent $\lambda$. Indeed, if one takes $A(x) = x$ and $X(x, n) = cste$ one obtains an integral $\int_{-1}^{1} \rho(x)f'(t-n)(x)dx$ which behaves like $e^{\lambda(t-n)} = 2^{t-n}$ on the first interval$^1$. Thus, the divergence of the linear part of the response is exactly due to the microscopic instability, even after performing a time average over one or several trajectories.

$^1$ In fact, it is easy to show that the integral corresponding to this situation, $\int_{-1}^{1} \rho(x)f'(t)(x)dx = \frac{(-4)^t}{\pi} \int_{-1}^{1} \frac{f'(x)f'^{-1}(x)\ldots x}{\sqrt{1-x^2}}dx$ ... vanishes. This is because $f'(x)$ is even $\forall t > 0$ thus $f'(x)f'^{-1}(x)\ldots x$ is odd. Therefore the complete integral is zero and the response is $(-4)^t \times 0$. This is due to the particular symmetry of the map. The remarkable thing is that in a numerical computation of type (19) below, one does not compute the exact integral of a strictly constant function, but instead there are round off fluctuations such that the numerically computed integral does not exactly vanish. One obtains, instead of 0, something like $2^t \times \eta$ where $\eta$ can be very small, but nevertheless finite. Thus, one can observe the instability of the response by considering a constant perturbation and the observable $A(x) = x$, as I checked.
This effect may however look rather specific since the perturbation $X_-$ is specific. Actually, this non rigorous argument suggests that one may observe an effect of the upper pole provided that one considers perturbations that do not vanish at $\pm 1$ (but vanish outside the interval $[-1, 1]$).

1.2. Numerical results for the logistic map. The goal is now to define a numerical method allowing to compute the linear response, whenever it is defined, but also to investigate the numerical effects observed when there is a pole in the upper half complex plane. I use the following algorithm introduced in [6] and later on in [1, 2, 3]. An analytical justification of this method can be found in [6]. I give here the main idea.

Let us first consider the case where the complex susceptibility is well defined on the real axis by the series (7) (perturbations of type $X_0$). Then:

$$\chi(\omega) = \frac{\delta \rho_t(A) e^{i \omega t}}{\epsilon}$$

Since this quantity does not depend on time one may write:

$$\chi(\omega) = \frac{1}{\epsilon T} \sum_{t=1}^{T} \delta \rho_t(A) e^{i \omega t}$$

The idea is to replace $\delta \rho_t(A)$ by $A(x'(t)) - A(x(t))$ where $x'(t)$ is a typical trajectory of the perturbed system, with a perturbation $X(x) e^{-i \omega t}$ and $x(t)$ is a typical trajectory of the unperturbed system. As shown in [6] this approximation holds provided that $\epsilon T$ is sufficiently large and $\omega T \gg 1$ (basically one uses $\epsilon T \omega \gg 1$).
Any typical trajectory of (1) approaches the points ±1 within a distance of order $\epsilon$ with a characteristic time of order $\frac{1}{\sqrt{\epsilon}}$. In a numerical simulation where $\epsilon$ is small but finite this arises often, especially if one respects the condition $\epsilon T \omega > 1$. However, in this case, a perturbation $X(x)e^{-i\omega t}$ can push the trajectory out of the interval $[-1, 1]$ leading to an exponential divergence. To avoid this effect I have used instead a perturbation $X(x, t) = X(x)(1 + e^{-i\omega t})$ where $X$ is positive about $x = -1$ and negative about $x = +1$, so that the perturbation is always directed inside the interval $[-1, 1]$ whenever $x = \pm 1$. For such a equation gives:

$$\delta \rho_t(A) = \epsilon(C + e^{-i\omega t}\chi(\omega))$$

where:

$$C = \sum_{n=0}^{\infty} \int \rho(x)A'(f^{(n)}(x)f^{(n)}(x))X(x)dx$$

In the case where the complex susceptibility is defined on the real axis this only adds a constant finite term $C = \chi(0)$ to $\delta \rho_t(A)$. Then:

$$\chi(\omega) \sim \frac{1}{T} \sum_{t=1}^{T} \left[ \frac{\delta \rho_t(A)}{\epsilon} - C \right] e^{i\omega t} \sim \frac{1}{T} \sum_{t=1}^{T} \frac{\delta \rho_t(A)}{\epsilon} e^{i\omega t}$$

I have used the following procedure. I iterate the dynamical system with a perturbation $X(x, t) = X(x)(1 + e^{-i\omega t})$ and I compute $S_T(\omega, x) = \frac{1}{\epsilon T} \sum_{t=1}^{T} [A(x'(t)) - A(x(t))] e^{i\omega t}$ [actually, for $\omega \neq 0$ it is sufficient to compute $\frac{1}{\epsilon T} \sum_{t=1}^{T} A(x'(t)) e^{i\omega t}$]. This quantity depends on the initial condition $x$ and I performed an average over a large number of initial conditions. This allows to compute error bars and reduces the fluctuations.
**Remark** The quantity obtained by this procedure is a priori not the linear response but the total response since one can apply it for arbitrary large $\epsilon$. Thus, one must a priori check that the susceptibility is independent of $\epsilon$ on a certain range of small $\epsilon$ values [e.g. it does not vary if one replaces $\epsilon$ by $2\epsilon$].

I have first considered the case of a perturbation, corresponding to the Ruelle-Pollicott resonance $\frac{1}{4}$, where the complex susceptibility for real $\omega$ is well defined. In the variable $x$ this perturbation writes:

$$X_0(x) = \pi \sqrt{1-x^2} \left[ 1 - \frac{2}{3\pi} \arcsin(x) - \frac{4}{\pi^2} \arcsin^2(x) + \frac{8}{3\pi^3} \arcsin^3(x) \right]$$

(20)

The observable $A$ is also $X_0$, in order to have a projection on the first Ruelle Pollicott mode and to select only the resonance $\frac{1}{4}$. In this case the complex susceptibility writes $\Psi(\lambda) = -\frac{4\langle \sigma_1 | g_x \rangle}{\lambda^4}$.

In the figure 1a, the complex susceptibility, computed with this algorithm and compared to the theoretical value, is drawn. The amplitude of the perturbation was fixed to $\epsilon = 10^{-2}, 5.10^{-3}, 10^{-3}$. Note that numerical noise is large and that the resonance curve is very flat, requiring to have small error bars. This requires an average over very long times $T$, roughly given by the condition $\epsilon T \omega >> 1$, and limits the range of $\epsilon$ values that one can reach. Note that the condition $\epsilon T \omega >> 1$ is always violated as $\omega$ approaches zero, whatever the [finite] value of $T$. This explains the discrepancy observed for small frequencies. If one excepts this, the agreement is quite good. Note also that the experimental curve has a magnitude that does not depend of $\epsilon$, in this range of $\epsilon$ values, as expected. The linear response can be easily obtained by computing the inverse Fourier series.
It is drawn in Figure 1b [in log scale for the $y$ axis]. One observes an exponential decay very close to the theoretically expected decay $4^{-t}$, corresponding to the mixing rate given by the first Ruelle-Pollicott resonance [and also to the characteristic decay rate toward equilibrium, in agreement with the fluctuation-dissipation theorem]. Note that the range of validity for the interpolation is very thin. Indeed, rapidly the perturbation becomes so weak that one measures only the numerical noise.

What happens now if one applies the same procedure to a perturbation inducing a pole in the upper half plane? As we saw the susceptibility is not defined on the real axis because the series diverges. However, when computing the expression one is not dealing with a series, but with a finite sum. Moreover, according to the previous section, the susceptibility diverges because of the term $2^t$ in (14). Numerically, the initial perturbation, of size $\epsilon$, is thus amplified by the term $2^t$. When $\epsilon 2^t$ is small one computes numerically the first term of the Taylor expansion for the susceptibility. But when $\epsilon 2^t$ becomes too large the numerical computation includes also the nonlinear terms and one computes in fact the complete susceptibility, including non linear terms. This one does not diverge because the trajectories of the perturbed system remain in the interval $[-1,1]$, but it does not depend linearly on $\epsilon$.

Therefore, one expects the following. There is a time cut-off for $t_m(\epsilon) \sim -\frac{\log(\epsilon)}{\log(2)}$ beyond which one does not compute the linear response but the complete response, including non linear effects which saturate the growth of the perturbation. Then one should observe an exponential growth $2^t$ up to $t_m(\epsilon)$ in the linear response that have roughly the form $2^t H_{[0,t_m(\epsilon)]}(t)$, where $H_I()$
is the characteristic function of the interval $I$. The corresponding susceptibility
(Fourier transform) is:

$$\sum_{n=0}^{t_m} (2\lambda)^n = \frac{1 - (2\lambda)^{t_m}}{1 - 2\lambda}$$

which writes, using the frequency $\omega$:

$$\frac{(1 - 2\cos(\omega))(1 - 2^{t_m}\cos(\omega t_m)) + 2^{t_m+1}\sin(\omega)\sin(\omega t_m)}{5 - 4\cos(\omega)} + i\frac{2\sin(\omega)(1 - 2^{t_m}\cos(\omega t_m)) - 2^{t_m}\sin(\omega t_m)(1 - 2\cos(\omega))}{5 - 4\cos(\omega)}$$

(21)

Therefore, it exhibits oscillations due to the cut off $t_m$. For longer time, mixing
should lead to an exponential decay of the response.

To check this I have first computed (19) for the perturbation (11) and for the
observable $A(x) = x - \frac{x^2}{2}$ [the derivative of $A$ gives a maximal value for the term
$A'(f^{(t-n)}(-1+u))$ in equation (14)]. The function $X_-$ has been computed with
the recursion (13) up to the order 4. The $\epsilon$ values $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$
have been considered [but only $10^{-4}, 10^{-5}, 10^{-6}$ are represented in Fig. 2a,b
for the legibility of the figure.]. I have used a fit procedure to compare the
experimental data with the form (21). I have also computed the linear response.
The results are presented in Fig. 2a,b.

One observes indeed a clear dependency with $\epsilon$. Moreover, the real and imaginary
part exhibit the expected oscillations due to the cut-off, with a perfect
agreement with the form (21). Taking the inverse Fourier series one obtains the
linear response in Fig. 2b. One sees clearly the exponential growth $2^\epsilon$ and the
$\epsilon$ dependent cut-off. Performing a fit in log scale one obtains (fig. 3) an exponential
increase with a rate 0.65 very close to the expect value $\log(2) = 0.693$.
After this there is an exponential decay with an approximate rate $-0.24$. [I have
only represented the fit for $\epsilon = 10^{-6}$ but one sees easily in Fig.3 that the decay rate is similar for $\epsilon = 10^{-4}, 10^{-5}$. This rate is slower than the first Ruelle Pollicott resonance $-\log(4) = -1.386$. It might be that we are observing a crossover regime where exponential amplification and exponential mixing are competing. [Note also that we are outside the linear regime].

Observe that the linear response vanishes for negative times: the presence of a pole in the upper half plane is not a sign of a violation of causality, but an evidence that linear response diverges as $t$ grows.

2. Hénon map.

The Hénon map:

$$
\begin{cases}
    x_{t+1} = 1 - ax_t^2 + y_t \\
    y_{t+1} = bx_t
\end{cases}
$$

(22)

with $a = 1.4, b = 0.3$ has an attractor with a positive Lyapunov exponent but it is not uniformly hyperbolic: there are neutral point where the dynamics is neither expanding nor contracting. These points may be responsible for phenomena analogous to those described in the previous section [divergence of the susceptibility], for specific perturbations weighting those points. From this observation, it is conjectured that there may also exist pole in the upper half complex plane [and possibly more complex singularities] \[13\]. However, there are no mathematical results for this and the form of the perturbations/observable leading to such a singularity is not known. Thus, a natural empirical approach consists in applying a perturbation $\epsilon e^{i\omega t}$ in the direction $i$ and investigating the effect on the variable $x_j$. Then:
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\( \delta \rho_t(x_j) = \epsilon R_{ij}(t) \)  

(23)

where the linear response is a \( 2 \times 2 \) matrix:

\[ R_{ij}(t) = \langle DF_{ij}^t \rangle \]

(24)

I have numerically computed the response of \( x \) to a constant perturbation in the direction \( y \), with amplitude \( \epsilon \). Denote by \( \chi_{xy}(\omega) \) and \( R_{xy}(t) \) the corresponding susceptibility and response. The complex susceptibility \( \chi_{xy}(\omega) \) and the linear response \( R_{xy}(t) \) are drawn in Fig. 4a,b respectively, for \( \epsilon = 10^{-3}, 10^{-4}, 2.10^{-5}, 10^{-5} \).

One observes similar effects as in the previous section. The amplitude of the complex susceptibility increases with \( \epsilon \) and this effect is not stabilized, either by increasing the time \( T \) of average well beyond the criterion \( \epsilon \omega T >> 1 \), or by increasing the number of sample trajectories. The linear response has three parts. For short times (\( t < 20 \)) there is a small bump for the modulus of the response: its height does not seem to depend on \( \epsilon \). The perturbation/observables have a nonzero projection onto the Ruelle-Pollicott modes. It might be that this bump comes from this part, having a well behaved linear response. The second part exhibits an exponential increase with a time cut off depending on \( \epsilon \). This is very similar to the observations made for the logistic map: exponential amplification of the linear response until nonlinearities induce saturation of this effect. Then, mixing leads to an exponential decay (third part of the curve).

The exponential instability is well fitted by the curve \( \alpha e^{\lambda t} \) with \( \alpha = -6.80 \pm 0.87 \) and \( \lambda = 0.455 \pm 0.038 \) (see Fig. 5). Note that the positive Lyapunov exponent of the Hénon map is 0.42(2) for these values of \( a, b \). The decay in the third part
is well interpolated by a sum of exponential terms. The dominant term is $\beta e^{\gamma t}$ with $\gamma = -0.129 \pm 0.015$. The value of $\gamma$ is in agreement with the exponential decay of the correlation function $C_{xy}(t)$ (Fig. 5). Thus the last part apparently obeys the fluctuation-dissipation theorem.

Let us also remark that there are no thin peak in the susceptibility (the frequency resolution is 0.00612) and the thickness of the peaks does not change with $\epsilon$. This suggests that the imaginary part of the poles is bounded away from zero. This leads me to conjecture that there is no pole on the real axis.

3. Conclusion.

In this paper I have presented a numerical procedure allowing to detect a subtle effect in the linear response theory. In some chaotic dynamical systems the complex susceptibility related to specific perturbations may exhibit a pole in the upper complex plane, associated to the divergence of the linear response. This effect can be observed and analysed by a correct treatment of the numerical data.

Thus, for chaotic dynamical systems and specific perturbations one can observe amplifications of the perturbation, induced by the microscopic instability [positive Lyapunov exponent] even after performing an ensemble average [see eq. 14]. Obviously, perturbations such as $X_-$ in section 1.1 given by the recursion (13), may look very particular, but I have shown that similar effects can be observed numerically for approximations of this function (even for a constant function) when the ensemble average is approximated by a sum over a [finite] number of samples.
This raises several questions. Does this phenomenon appear in other low dimensional systems? This can be easily investigated with the same numerical procedure, though some mathematical statement about the genericity of this effect would be preferable. What happens for larger dimensions (“thermodynamic” limit)? This is an interesting question, since the existence of a macroscopic violation of the linear response induced by microscopic instability is closely related to the Van Kampen discussion [14]. However, it remains to know whether this effect persists when increasing the dimension of the system. One possibility of investigation, which is somehow a direct continuation of the present work, could be to consider a lattice of coupled logistic maps, to apply an harmonic perturbation at some point and look at the induced effects, in the spirit of this work. There is a huge literature on coupled map lattices, but I don’t know if such an experiment has already been done.

On a more mathematical ground, a last question is: “what is there beyond the pole”? In some sense, the pole in the upper half complex plane may “hide” more complex singularities lying behind it. Is it possible to have any [numerical] idea of which type of singularities are there? A natural way of doing this is to remove the exponential instability by multiplying the perturbation by a damping factor [or equivalently to use a complex frequency $\omega$]. This is natural for a mathematical point of view, but tricky in the numerics, because the damping factor is either smaller than the exponential instability [and the response grows rapidly inducing non linear effects, as we saw] or it is bigger [and the perturbation becomes rapidly numerically 0, then we are measuring short time transients]. One has thus to make small variations of the damping factor around the pole and look at the changes in the susceptibility/response curve. This is under current
investigations.

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Fig. 1. (a) Susceptibility for a perturbation \(2\omega\) and an observable also given by \(2\omega\), for \(\epsilon = 10^{-2}, 5 \cdot 10^{-3}, 10^{-3}\). (b) Corresponding linear response [denoted by \(R(t)\)], in log scale and theoretically expected curve \(y = a - \log(4) t\).
Fig. 2. (a) Susceptibility for the perturbation $X_-(x)$, where $w$ was computed up to order 4, and observable $A(x) = x - \frac{x^2}{2}$. The parameter $\epsilon$ takes the value $10^{-5}, 10^{-6}$. In full line are drawn the fitting curves obtained from (21). (b) Linear response.
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Fig. 3. Fit of the linear response corresponding to Fig. 2b.)
Fig. 4. (a) Susceptibility for a constant perturbation $X = \epsilon$ in the direction $y$ for the observable $A(x, y) = x$. (b) Corresponding linear response.
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Fig. 5. (a) Fit of the linear response $R_{xy}(t)$, $\epsilon = 10^{-6}$ for the Hénon map. (b) Fit of the correlation function $C_{xy}(t)$ for the Hénon map.
