RANKS OF TENSOR NETWORKS FOR EIGENSPACE PROJECTIONS AND
THE CURSE OF DIMENSIONALITY

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Abstract. The hierarchical (multi-linear) rank of an order-

d

rank of a tensor is key in determining the cost of

representing a tensor as a (tree) Tensor Network (TN). In general, it is known that, for a fixed accuracy,
a tensor with random entries cannot be expected to be efficiently approximable without the curse of
dimensionalit,
e.g., a complexity growing exponentially with d. In this work, we show that the ground
state projection (GSP) of a class of unbounded Hamiltonians can be approximately represented as an
operator of low effective dimensionality that is independent of the (high) dimension d of the GSP. This
allows to approximate the GSP without the curse of dimensionality.

1. Introduction

Tensor product methods naturally arise in high-dimensional problems, e.g., when one is interested in
representing d-dimensional arrays \( \psi \in \mathbb{R}^{n \times \ldots \times n} \) or approximating d-dimensional functions \( \psi : \mathbb{R}^d \to \mathbb{R} \),
and their application is ubiquitous across many scientific fields. We refer to [21,30,32,36] for overviews
and examples.

In the case of an array \( \psi \in \mathbb{R}^{n^d} \), the number of entries \( n^d \) scales exponentially with \( d \) – an expression
of a phenomenon commonly referred to as the curse of dimensionality. In order to recover tractable computational methods, a common technique is tensor product approximation

\[
\psi \approx \psi_r = \sum_{k=1}^{r} v_1^k \otimes \ldots \otimes v_d^k,
\]

(1.1)

where \( r \) is referred to as the tensor rank of \( \psi_r \). Here, the representation complexity of \( \psi_r \) is \( ndr \), i.e.,
linear in \( d \) and \( r \). The utility of such an approximation, of course, depends on how \( r \) scales w.r.t. \( d \) for
a given approximation accuracy \( \varepsilon > 0 \)

\[
(1.2)
\]

\[
r_{\varepsilon} := \min \{ r \in \mathbb{N}_0 : \| \psi - \psi_r \| \leq \varepsilon \} = O(?),
\]

for some norm \( \| \cdot \| \) on \( \mathbb{R}^{n^d} \).

With regard to answering eq. (1.2), it is well-known that the following dichotomy holds: while for
“most” tensors the rank \( r_{\varepsilon} \) will scale exponentially with \( d \), in many practical applications \( r_{\varepsilon} \) will scale
at most polynomially with \( d \). For instance, in [22,23], the authors show that if the tensor in question
is an analytic function of a sum of the independent variables or it is a power of a discrete Laplacian
operator, then, such tensors can be very efficiently approximated – without the curse of dimensionality.
In other words, the curse of dimensionality can be circumvented provided one assumes enough structure.

What is then enough structure? One can assume a highly structured expression for a tensor \( \psi \in \mathbb{R}^{n^d} \)
or a linear operator \( A : \mathbb{R}^{n^d} \to \mathbb{R}^{n^d} \), and an approximation method tailored to that expression, similarly

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4 Parts of this work were completed while the author was at the Institute for Numerical Mathematics, Ulm University, Germany.
5 Coined by R. Bellman [9].
6 For a definition of tractability see [45].
7 E.g., the set of singular matrices, as a subset of \( \mathbb{R}^{n \times n} \), has Lebesgue measure zero. By extension, any \( \nu \)-unfolding of a tensor is a matrix in \( \mathbb{R}^{\nu \times n \times \ldots \times n} \), and thus the set of tensors with exact low-rank representations has Lebesgue measure zero. Similar statements apply for \( \varepsilon \)-approximations in an appropriate sense, to be explained shortly.
as in [22, 23, 24]. One can assume \( \psi \) is a solution to the Poisson equation with low-rank data – as in, e.g., [14]. These assumptions are certainly sufficient. If, however, one only assumes \( \psi \) belongs to a regularity class – such as a Sobolev space – the curse of dimensionality cannot be avoided (see, e.g., [41]).

The main contribution of this work is along those lines. Our setting is that of a system described by a Hamiltonian \( H \) – a self-adjoint densely defined differential operator; and we consider the scaling of \( r_\varepsilon \) of the ground state projection \( \mathcal{P} \) (GSP) – the orthogonal projection onto the eigenspace of the smallest eigenvalue of \( H \). This setting is frequently encountered in quantum mechanics. Our assumptions involve \( H \) and its spectrum, i.e., in particular, we do not assume any specific expression for the eigenfunctions of \( H \).

Naturally, one can make stronger statements about approximability if one has explicit representations of \( H \), and devise efficient computational methods tailored to \( H \). However, as we will argue in Remark 3.2, many of the assumptions made are sufficient and necessary. The assumptions have a physical interpretation and are thus not merely technicalities of the mathematical formalism. Moreover, we indicate shortly (see Section 2.1) that our approach offers the flexibility to choose more general structures of \( H \). The framework of our results thus comes close to walking the fine line of just enough assumptions to guarantee a target function or operator does not suffer from the curse of dimensionality.

1.1. Tensor Networks and Hierarchical Ranks. At this point we have to specify that our results do not concern approximation as in eq. (1.1) but rather approximation with Tensor Networks (TNs). Approximations as in eq. (1.1) are frequently referred to as \( r \)-term approximation or Canonical Polyadic (CP) decomposition. Though CP certainly has its uses – most notably in chemometrics [26] – this format is not flexible enough for efficiently approximating more complex tensor product structures. The type of format we need for this work is a Tensor Train (TT) [37], a.k.a. a Matrix Product State (MPS) [33] – a particular type of a TN. The TT of an array \( \psi \in \mathbb{R}^{n_1 \times \ldots \times n_d} \) has the form (for \( d = 2, 3, 4 \))

\[
\psi(i_1, i_2) = \sum_{k_1=1}^{r_1} U^1(i_1, k_1) U^2(k_1, i_2),
\]

\[
\psi(i_1, i_2, i_3) = \sum_{k_1=1}^{r_1} \sum_{k_2=1}^{r_2} U^1(i_1, k_1) U^2(k_1, i_2, k_2) U^3(k_2, i_3),
\]

\[
\psi(i_1, i_2, i_3, i_4) = \sum_{k_0=1}^{r_0} \sum_{k_1=1}^{r_1} \sum_{k_2=1}^{r_2} \sum_{k_3=1}^{r_3} U^1(k_0, i_1, k_1) U^2(k_1, i_2, k_2) U^3(k_2, i_3, k_3) U^4(k_3, i_4, k_0),
\]

with order-2 or order-3 tensors \( U^j \in \mathbb{R}^{r_j \times n_\alpha \times n_\beta} \). It is convenient to visualize and manipulate tensor networks via tensor diagrams [8], see Figure 1.1. As we will indicate in Section 2.1 our work can be in principle extended to more general networks.

Our main result concerns hierarchical or multi-linear ranks. These are ranks of bi-partite cuts, i.e., for any \( \alpha \subset \{1, \ldots, d\} \) and \( \alpha^c := \{1, \ldots, d\} \setminus \alpha \), the hierarchical rank \( r_\alpha = r_{\alpha^c} \) is the smallest \( r_\alpha \in \mathbb{N}_0 \) such that

\[
\psi = \sum_{k_{\alpha} = 1}^{r_\alpha} v^\alpha_{k_{\alpha}} \otimes v^{\alpha^c}_{k_{\alpha}}, \quad v^\alpha_{k_{\alpha}} \in \mathbb{R}^{\otimes \#_\alpha}, \quad w^{\alpha^c}_{k_{\alpha}} \in \mathbb{R}^{\otimes \#^c_{\alpha^c}}.
\]

These ranks essentially determine the representation and computational complexity of a TN. The case of TTs corresponds to bi-partite cuts of the form \( \alpha = \{1, \ldots, j\} \) for \( j = 1, \ldots, d - 1 \). In this case we abbreviate \( r_j := r_{\{1, \ldots, j\}} \). From Figure 1.1 it is not difficult to see that the representation complexity of tree tensor networks scales roughly linearly (or log-linearly) with \( d \) and polynomially in ranks \( \mathcal{O}(r_\alpha^p) \), where \( p \) is the number of edges connecting a vertex.

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4This is not surprising as Kolmogorov showed that, for a fixed precision, Sobolev balls require an amount of information exponential in the dimension \( d \) to encode [43], i.e., these sets are too large and do not have enough structure to avoid the curse of dimensionality.

5A.k.a. Ground state density matrix/operator.

6In the sense that, if one assumption is not satisfied, there are examples of physical systems for which the ranks will scale exponentially.

7Apart from also being numerically unstable [42].

8A.k.a. Penrose graphical notation.
Notice the one-way relationship between the hierarchical ranks $r_\alpha$ and the tensor rank $r$ from eq. (1.1): $r_\alpha \leq r$ for any $\alpha$, while the reverse inequality is not true in general. Loosely speaking, eq. (1.1) demands simultaneous separability w.r.t. all coordinate directions, while eq. (1.6) demands separability only w.r.t. bi-partite cuts. And thus one can have TNs with small ranks $r_\alpha$ but exponentially growing tensor rank $r = \mathcal{O}(2^d)$. On the other hand, once again assuming, e.g., only classical Sobolev regularity, $r_\alpha$ will scale exponentially with $d$ as well [41].

1.2. Contribution. To put this work into context, we summarize what is known about low-rank approximation with TNs and how our work contributes to that body of knowledge. We do not intend to give a complete overview of all the research on TN-approximation – a lot has been done in the past decades. Instead, we highlight a few works that will serve as representatives of a general flavor of results relevant to us. We apologize in advance if some contributors feel left out by this summary.

Previous Work. There are various results that construct efficient approximations to functions and operators with explicit or highly structured expressions, mostly in the context of numerical approximation
and finite-dimensional Hilbert spaces as in, e.g., [19, 22, 23]. Techniques such as exponential sum approximation or sinc quadrature are key for these methods.

Some results bound the ranks of an iterative method on finite-dimensional Hilbert spaces, see, e.g., [25,31]. There are a few results for infinite-dimensional Hilbert spaces as in [26,78,14]. In [26,78], the authors consider adaptive numerical methods for PDEs and show that, if the exact solution is low-rank approximable, then the same holds for the numerical approximation. In [14], the authors show for the Poisson equation that the exact solution has similar low-rank approximability as the right-hand-side data.

In [41], the authors compute the optimal rank scaling for periodic functions in Sobolev and mixed Sobolev spaces, with the curse of dimensionality present in both cases, though much milder for mixed Sobolev. In [20], the authors used weighted Sobolev spaces where, loosely speaking, the weighting ensures that for $d \to \infty$ the additional dimensions have smaller and smaller impact on the derivative of the function. For these spaces the optimal hierarchical ranks scale without the curse of dimensionality.

Note that classical regularity is certainly important for a complete approximation scheme, as even if one has a rank-one function/operator, in general, one still has to approximate the low-dimensional components. If the latter is done via, e.g., piece-wise polynomials, then its efficiency depends on the degree of classical Sobolev/Besov/analytic regularity of said low-dimensional components.

However, if one is interested solely in the question of rank scaling – as we are in this work – then this property should be independent of any notion of smoothness. An analytic function of the sum of $d$ variables can be certainly approximated with ranks scaling without the curse of dimensionality – for any TN. But a tensor product of delta distributions has rank one in any representation – despite the absence of any notion of classical regularity. Hence, we emphasize that the techniques used in this work do not rely on any classical notion of regularity.

1.2.1. This Work.

(i) We provide a rigorous, discretization- and (classical) regularity-independent proof of how a TN that is adapted to the Hamiltonian structure avoids the curse of dimensionality.

(ii) Many of the required assumptions are necessary, have a physical interpretation and establish a direct link to the presence or absence of the curse of dimensionality. They provide further insight into when and why said curse appears, which goes beyond the mere intuition “Hamiltonian structure $\approx$ TN structure”.

(iii) The utilized technique is at least as valuable as the result itself. It demonstrates how similar results can be derived for more complex TNs, evolution equations and PDEs. E.g., the approach can be extended to Hamiltonians with $k$-neighbor or long-range interactions that decay sufficiently fast, see also [10]. It can be extended to more general networks using Lieb-Robinson bounds from, e.g., [35]. It provides an alternative to exponential sum approximations in the case of a unitary evolution via $\exp[-iHt]$, without the usual decay of eigenvalues. Finally, an additional benefit of this technique is that it provides valuable physical insight.

(iv) Our goal is to identify the right framework for describing low-rank approximability or, more generally, approximation without the curse of dimensionality. We assume just enough to guarantee approximability, while still leaving flexibility in the setting where possible. This contribution is an important step towards that goal.

Outline. In Section 2, we introduce the concept of operator support, our main result and discuss implications. Section 3 is dedicated to the proof of the main result in Theorem 3.11. The technical proofs are postponed to Appendix A.

2. MAIN RESULT AND DISCUSSION

Our main statement requires the concept of operator support. Though our result concerns general (possibly infinite-dimensional) Hilbert spaces, for the purpose of introduction we mainly stick with finite vectors and matrices.

Let $A \in \mathcal{L}(\mathbb{R}^n \otimes^d)$ be a bounded linear operator from $(\mathbb{R}^n)^\otimes^d$ onto itself (an $n^d \times n^d$ matrix). If for some $\alpha \subset \{1, \ldots, d\}$ and some $A_\alpha \in \mathcal{L}(\mathbb{R}^n \otimes^{\#\alpha})$ we can write $A$ as

$$A = A_\alpha \otimes I_{\alpha^c},$$

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where \( I_{\alpha} : (\mathbb{R}^n)^{\# \alpha} \rightarrow (\mathbb{R}^n)^{\# \alpha} \) is the identity, then we say \( A \) is supported on the Hilbert space \( \mathcal{H}_\alpha := (\mathbb{R}^n)^{\# \alpha} \), or simply supported on \( \alpha \), or acts on \( \mathcal{H}_\alpha \). We write in this case \( \text{supp}(A) := \mathcal{H}_\alpha \) to denote the support of \( A \).

For operators \( A, B \), we write \( \text{dist}(A, B) \) to denote the distance between supports. E.g., suppose \( A, B \in \mathcal{L}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_d) \), \( \text{supp}(A) = \mathcal{H}_2 \otimes \mathcal{H}_3 \) and \( \text{supp}(B) = \mathcal{H}_3 \otimes \mathcal{H}_4 \). Then, \( \text{dist}(A, B) = 0 \). If \( \text{supp}(A) = \mathcal{H}_1 \otimes \mathcal{H}_2 \) and \( \text{supp}(B) = \mathcal{H}_4 \otimes \mathcal{H}_5 \), then \( \text{dist}(A, B) = 2 \), and so on.

The relevance of operator support is as follows: if \( A \) is supported on \( \alpha \), then, since the identity \( I_{\alpha} \) is a tensor product rank-one operator, we know that a low-rank approximation of \( A \), w.r.t. any cut \( \gamma \subset \alpha \) grows at most exponentially in \( \# \alpha / 2 \) and not \( \# \alpha / d \). Thus, if \( \# \alpha \) can be guaranteed to be small independently of \( d \), we have decoupled the global (large) dimension \( d \) from the effective dimension \( \# \alpha \) of \( A \), i.e., the dimension of the space on which \( A \) acts non-trivially.

The core technique of our proof is the quantification of support growth due to Lieb-Robinson \([34]\): one starts with a rank-one operator, which is then smoothly evolved/continued into the target state. In the process of said continuation\(^9\), the support of the operator grows at a speed determined by the properties of the system Hamiltonian \( H \).

The main ideas behind this approach are based on the physical principle of holography: the informational content/entropy of many physical systems is proportional to its area rather than volume. Specifically, we build on the quantum entanglement area laws research of \([27, 28]\) and Lieb-Robinson bounds from \([5, 35, 38]\). A more detailed exposition of our work including connections to entanglement entropy area laws can be found in \([1]\).

### 2.1 Main Result

Now that we have introduced all the necessary concepts for this work, we state our main result. We consider separable complex-valued Hilbert spaces \( \mathcal{H}_j, j = 1, \ldots, d \), and their tensor product space \( \mathcal{H} := \bigotimes_{j=1}^d \mathcal{H}_j \). Here, we consider general topological tensor spaces \( \mathcal{H} \) equipped and completed with the canonical Hilbert space inner product. If at least \( d - 1 \) of the Hilbert spaces are finite-dimensional, then the distinction between algebraic and topological tensor product spaces is irrelevant. We also introduce the short-hand notation

\[
\mathcal{H}_{i,j} := \bigotimes_{\nu=i}^j \mathcal{H}_\nu,
\]

for the partial tensor product, once again equipped with the canonical inner product.

Our Hamiltonian is a densely defined self-adjoint operator \( H : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{H} \). Our first primary assumption is on the locality structure of \( H \)

\[
H = \sum_{j=1}^{d-1} H_{j,j+1},
\]

where each \( H_{j,j+1} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{H} \) is a Hamiltonian in its own right, supported on \( \mathcal{H}_{j,j+1} \).

This is (loosely) to be interpreted as follows: for a multi-partite (e.g., many-body) system with \( d \) constituents, \( H_{j,j+1} \) models the joint energy of subsystems \( j \) and \( j+1 \), while the total energy of the entire system is described by \( H \) and it is the sum of the subsystem energies of pairs \((j, j+1)\). Such a description of a physical system is a reasonable approximation if the many-body system has only pairwise interactions – as is the case for some systems of condensed matter, see \([13]\).

The particular local structure of \( H \) in eq. (2.1) fits the TT format. However, since Lieb-Robinson bounds hold for more general graphs (see, e.g., \([35]\)), one could extend our results to more general TNs, using similar ideas but a more cumbersome proof.

The format of writing \( H \) as in eq. (2.1) might seem unorthodox if one is more accustomed to

\[
H := K + V = -\frac{1}{2} \Delta + V,
\]

where \( \Delta \) is the Laplacian modeling the kinetic part and \( V \) is some potential. Note, however, that if we can write the potential as a superposition of local potential terms \( V = \sum_{j=1}^{d-1} V_{j,j+1} \), where each local potential \( V_{j,j+1} \) acts only on \( \mathcal{H}_{j,j+1} \), then this provides a representation as in eq. (2.1), since \( -\Delta = -\sum_{j=1}^{d-1} \Delta_j \), where \( \Delta_j \) is a Laplacian acting on \( \mathcal{H}_j \).

\(^9\)Which is often referred to as adiabatic continuation in physics.
Our second primary assumption concerns the spectrum of $H$: we suppose $H$ has a smallest eigenvalue $\lambda_0$, which we set w.l.o.g. to $\lambda_0 = 0$, and that the spectral (energy) gap $\Delta E := \lambda_1 - \lambda_0 > 0$ is non-vanishing for $d \to \infty$, i.e., $\Delta E \not\to 0$ for $d \to \infty$. This assumption separates critical systems with topological degeneracy ($\Delta E \to 0$) from non-critical systems without degeneracy ($\Delta E \not\to 0$): for the former it is known, in general, that $r_j$ will scale exponentially with $d$. Quantum criticality and the related exponential rank growth are a physical property of the underlying system described by $H$ that are in fact necessary to explain certain phenomena, such as, e.g., superconductivity (see [13]).

Finally, let the eigenspace of $H$ corresponding to $\lambda_0$ be finite-dimensional and $\{\psi_0^k\}_{k=1}^N$ be an orthonormal basis of said eigenspace. Then, we call the (normalized trace-class) projection

$$P_0 = \frac{1}{N} \sum_{k=1}^N \langle \cdot, \psi_0^k \rangle_{\mathcal{H}} \psi_0^k \in \mathcal{L}(\mathcal{H}),$$

a ground state density operator or a ground state projection (GSP).

Then, we have

**Main Result** (Theorem 3.11). For any $j = 1, \ldots, d-1$ and any $l = 1, \ldots, \min\{j-1, d-j\}$, there exist bounded, positive, self-adjoint operators $B = B(j,l)$, $L = L(j,l)$ and $R = R(j,l)$ such that

(i) $L$ and $R$ are supported on $\mathcal{H}_{1,j}$ and $\mathcal{H}_{j+1,d}$, respectively;

(ii) $B$ is supported on $\mathcal{H}_{j-l,j+l}$;

(iii) the operator norms are bounded by unity $\|B\|_L, \|L\|_L, \|R\|_L \leq 1$;

such that we have the error estimate

$$\|P_0 - BLR\|_L \leq C \exp[-cd],$$

where the constants $C, c > 0$ depend only on $\Delta E$ and not $d$, $j$ or $l$.

2.2. **Implications.** The result of the above theorem provides a rigorous and discretization-independent justification of the well-accepted intuition that TNs that fit the (interaction) structure of the problem – in this case Hamiltonian – do not incur the curse of dimensionality.

2.2.1. **Hierarchical Ranks.** If for any $j = 1, \ldots, d-1$, $P_0 \approx B(j,l) L(j,l) R(j,l)$, then the scaling of TT-ranks $r_j$ corresponding to bi-partite cuts $\mathcal{H} := \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d}$ can be inferred from $BLR$. Note that the product $LR$ has rank-one w.r.t. this bi-partite cut. Thus, a rank-$r_j$ approximation to $B$ is also a rank-$r_j$ approximation to $P_0$.

If the Hilbert spaces $\mathcal{H}_j$ are finite-dimensional with $\dim(\mathcal{H}_j) = n$, then $B$ can be represented exactly with at most $r_j \leq n^j$, i.e., independently of $d$. More generally, one would have to approximate $B$ as well. Note that – whatever the scaling of the ranks of an approximation to $B$ is – it will always be independent of $d$ as long as the constants $C, c > 0$ are independent of $d$. We elaborate with an example.

Suppose $\mathcal{H}_j = L^2([0,1]^2)$ or $\mathcal{H}_j = L^2(\mathbb{R}^3)$. As we will see in the proof of Theorem 3.11 $B$ is constructed from parts of the spectrum of $H$ and, in many cases (see, e.g., [46]), the ground state has some Sobolev regularity $s > 0$. Recall that

$$B = \hat{B}_l \otimes I, \quad \hat{B}_l \in \mathcal{L}(\mathcal{H}_{j-l,j+l}), \quad l \in \mathcal{L}(\mathcal{H}_{1,j-l-1} \otimes \mathcal{H}_{j-l+1,d}),$$

and given Sobolev regularity of the kernel of $\hat{B}_l$,

$$\left\| \hat{B}_l - \sum_{k=1}^{r_j} U_k \otimes V_k \right\|_L = O(r_j^{-s/l}), \quad U_k \in \mathcal{L}(\mathcal{H}_{j-l,j}), \quad V_k \in \mathcal{L}(\mathcal{H}_{j-l+1,j+l}).$$

Thus, for a fixed approximation accuracy $\varepsilon > 0$, from eq. (2.2) we first choose $l := \lceil 1/c \ln(C/\varepsilon) \rceil$, and hence

$$r_j^{-s/l} \leq \varepsilon \iff r_j \geq \varepsilon^{-l/s} = \varepsilon^{(1/sc)\ln(\varepsilon/C)}.$$

That is, exponential dependence on $d$ can only be re-introduced through $c$ or $C$. E.g., if $c \sim 1/d$ or $C \sim 2^d$. This in turn depends on $\Delta E$, i.e., the curse of dimensionality is now determined by the spectral gap $\Delta E$ and, if the latter is bounded from below independently of $d$, then the curse of dimensionality can be (at least asymptotically) avoided.

The bound eq. (2.2) decouples the dimension $d$ from the scaling of hierarchical ranks or, alternatively, shifts this dependence onto $C, c$. The physical relationship between ranks $r_j$ and constants $C, c$ is that

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10In principle, one can consider any weighted average of the one-dimensional projections $\langle \cdot, \psi_0^k \rangle_{\mathcal{H}} \psi_0^k$. 

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both reflect the entanglement properties of the ground state, which in turn depend on the spectral gap, local interaction length (in our case equal to two) and interaction strength (see Assumption 3.1). For gapped Hamiltonians, with fixed interaction length and strength, the entanglement of the different subsystems saturates for \( d \to \infty \): there exist finite \( C, c > 0 \), independent of \( d \). Consequently, ranks saturate as well.

### 2.2.2. Evolution Equations

A similar technique as in the proof of eq. (2.2) can be used to analyze evolution equations, such as the Schrödinger equation (in Planck units)

\[
\frac{d}{dt} \psi(t) = H \psi(t), \quad \psi(0) = \psi_0.
\]

The density operator \( P(t) := \langle \cdot, \psi(t) \rangle \mathcal{H} \psi(t) \) evolves as

\[
P(t) = \exp[iHt]P(0)\exp[-iHt].
\]

Starting with, e.g., a product state \( \psi_0 = \psi_1^0 \otimes \ldots \otimes \psi_d^0 \), similarly to eq. (2.2), one could show for any \( j = 1, \ldots, d - 1 \)

\[
P(t) \approx B(t,j,l)L(t,j,l)R(t,j,l).
\]

The support of \( B(t,j,l) \) grows with \( t \to \infty \) such that, for small times \( t > 0 \), \( P(t) \) has small hierarchical ranks independent of \( d \).

Note the similarity to approximation techniques as in [22, 23] or [19], exploiting matrix exponentials with subsequent exponential sum approximations. Unlike in the case of exponentials as \( \exp[-At] \) with a positive definite \( A \) – as for, e.g., parabolic evolution equations – the unitary evolution via \( \exp[-iHt] \) has no damping effect, i.e., the eigenvalues do not decay for \( t \to \infty \).

For the case of unitary evolution, in [19] the authors employ a quantized TT approximation based on a series expansion of \( \exp[-iHt] \) via they Cayley transform. This guarantees highly efficient approximations with low QTT-ranks of \( \psi(t) \) for the semi-discrete problem (discrete in space), provided the initial value is itself of low rank, analytic and the application of the Cayley transform remains of low rank.

In contrast, the technique for proving eq. (2.2) using Lieb-Robinson bounds relies solely on operator support. Hence, the bound does not require finite-dimensional Hilbert spaces or any notion of classical regularity, but solely those assumptions that have direct influence on the TN-ranks:

1. A TN that fits the Hamiltonian structure,
2. A Hamiltonian that does not allow interactions to spread arbitrarily fast (local/short-range and gapped),
3. An initial state with small TN-ranks,

### 2.2.3. General Equations

It is a common technique in functional calculus to express functions of an operator as an integral, e.g.,

\[
A^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^t t^{\sigma-1} \exp[-tA] \, dt, \quad \sigma > 0,
\]

\[
F(A) = \frac{1}{2\pi i} \int_\Gamma F(z) (z - A)^{-1} \, dz,
\]

see again [19, 22, 23]. This can be used, e.g., to solve or pre-condition partial differential equations (PDEs). Since the proof of eq. (2.2) relies on integral representations of \( P_0 \) as well, combined with subsequent Lieb-Robinson bounds, we believe there is potential to extend such results to PDEs.

### 2.2.4. Pure Ground States

In many cases of interest the GS of \( H \) is unique or pure, i.e., \( P_0 = \langle \cdot, \psi_0 \rangle \mathcal{H} \psi_0 \). In this case eq. (2.2) does not immediately guarantee low-rank approximability of \( \psi_0 \). To infer approximability of \( \psi_0 \) one would have to show eq. (2.2) with the operator norm replaced by the trace norm. Or, alternatively, one could extend the technique of [34] to infinite-dimensional Hilbert spaces, though this is not trivial. In [34], the authors use an approximate GSP – similar to BLR from eq. (2.2) but with stronger properties – to conclude that the pure GS \( \psi_0 \) has non-exponentially growing TT/MPS-ranks.
3. Proof of \((2.2)\)

We assume the Hamiltonian operator satisfies the following properties.

**Assumption 3.1.** Let \(H : \mathcal{D}(H) \rightarrow \mathcal{H}\) be a densely defined self-adjoint (possibly unbounded) operator.

1. **(Locality).** We assume \(H\) can be decomposed as
   \[
   H = \sum_{j=1}^{d-1} H_{j,j+1},
   \]
   where each \(H_{j,j+1}\) is supported on \(\mathcal{H}_{j,j+1}\).

2. **(Gap).** We assume the spectrum is bounded from below with a non-vanishing spectral gap
   \[
   \Delta E := \lambda_1 - \lambda_0 > 0,
   \]
   the GS eigenspace is finite-dimensional with dimension \(N\) and the GSP is defined as
   \[
   P_0 := \frac{1}{N} \sum_{k=1}^{\infty} \langle \cdot, \psi_k^0 \rangle \mathcal{H} \psi_k^0.
   \]

3. **(Finite Interaction Strength).** We assume for all \(1 \leq j \leq d-1\), \(H_{j,j+1} = H_j + H_{j+1} + \Phi_{j,j+1}\), where \(H_j\) and \(H_{j+1}\) are possibly unbounded operators supported on \(\mathcal{H}_j\) and \(\mathcal{H}_{j+1}\), respectively, and \(\Phi_{j,j+1}\) is a densely defined uniformly bounded operator supported on \(\mathcal{H}_{j,j+1}\). I.e., there exists a constant \(J\) such that
   \[
   \|\Phi_{j,j+1}\|_\mathcal{L} \leq J,
   \]
   for all \(1 \leq j \leq d-1\).

4. **(Bounded Commutators).** The commutators of the neighboring interaction and single site operators are densely defined and uniformly bounded, i.e.,
   \[
   \|[\Phi_{j,j+1}, H_{j+1}]\|_\mathcal{L} \leq J \quad \text{and} \quad \|[H_j, \Phi_{j,j+1}]\|_\mathcal{L} \leq J,
   \]
   for all \(1 \leq j \leq d-1\), where \([A, B] := AB - BA\).

5. **(Self-Adjoint).** The interaction and single site operators \(\Phi_{j,j+1}\) and \(H_j\) are self-adjoint.

**Remark 3.2.** Assumption 1 means we only consider local 2-site interactions. Our results would remain unchanged for \(k\)-site interactions, for a fixed \(k\). The point is that the complexity of approximating an eigenfunction scales exponentially with \(k\) and not \(d\). Moreover, we expect similar results could be obtained for long range interactions that decay sufficiently fast.

We require Assumption 2 since the proof heavily relies on the spectral decomposition. One could possibly extend the proofs presented here to sectorial operators. We are not certain to what extent approximability actually depends on the form of the resolvent/spectrum of the operator in \(\mathbb{C}\).

Assumption 2 is necessary for an area law to hold. Systems with degenerate ground states are at a quantum critical point and have been observed to exhibit divergent entanglement entropies (see [12, 16, 44]). Finite-dimensionality of the GS eigenspace is required to ensure the GSP is trace-class-normalizable, otherwise the GSP makes no sense.

Assumptions 3 and 4 are required for the application of Lieb-Robinson bounds, i.e., finite speed information propagation. There are essentially two difficulties when considering information propagation for dynamics prescribed by an unbounded operator.

First, unlike with classic Lieb-Robinson bounds (see [32]), bounded local operators do not have to remain local when evolved via the unitary operator \(\exp[iHt]\) (see [17]). This can be remedied as in [32] by, e.g., assuming the interactions in \(H\) are of a certain type: bounded, as in this work, or specific types of unbounded operators that we do not consider here. Hence, we require Assumption 3.

Second, when applying time dynamics to an unbounded local operator, it is not clear in which sense the operator remains approximately local. Thus, Assumption 4 ensures that the non-local part is bounded.

However, we essentially require only an application of Lieb-Robinson. Although Assumptions 3 and 4 are certainly sufficient, they are perhaps not necessary.

\[\footnote{11}{That models interactions between particles \(j\) and \(j+1\).}\]

\[\footnote{12}{E.g., think of the canonical commutation relation for position and momentum operators.}\]
Example 3.3 (Nearest Neighbor Interaction (NNI)). We provide an example of how the general structure of an NNI Hamiltonian might look like. Perhaps the most famous example of an NNI Hamiltonian is the Ising model (see [11]).

In this work we consider infinite-dimensional Hilbert spaces and unbounded Hamiltonians. A typical example to keep in mind is \( H = \bigotimes_{j=1}^{d} H_j = \bigotimes_{j=1}^{d} L^2(\mathbb{R}^n, \mathbb{C}) \), where \( n \in \{1, 2, 3\} \) if \( H \) is to model a physical phenomenon.

Let the Hamiltonian operator be given as
\[
H = -\Delta + V.
\]
The Laplacian \( \Delta \) is the one-site unbounded operator where \( H_j = -\frac{\partial^2}{\partial x_j^2} \). The potential \( V \) contains the bounded interaction operators. E.g., \( V = \sum_{j=1}^{d-1} \Phi_{j,j+1} \), where \( \Phi_{j,j+1} : \mathcal{H} \to \mathcal{H} \) is a bounded operator such as
\[
(\Phi_{j,j+1}\psi)(x) = c(x_j, x_{j+1})\psi(x), \quad \text{or} \quad (\Phi_{j,j+1}\psi)(x) = \int_{\mathbb{R}^{2n}} \kappa(x_j, x_{j+1}, y_j, y_{j+1})\psi(x_1, \ldots, y_j, y_{j+1}, \ldots, x_d) \, d(y_j, y_{j+1}),
\]
where \( c(\cdot) \) is a bounded coefficient function and \( \kappa(\cdot) \) is an integral kernel.

One would have to check the gap property and the degeneracy of the GS. Spectral properties and degeneracy of GSs have been extensively studied before and we refer to, e.g., [40, Chapter XIII] for more details.

We begin with a lemma that shows how we can approximately express the GSP through the Hamiltonian operator. This will provide the necessary link between the local operator structure and the local structure of the GSP.

Lemma 3.4. Let Assumption 3.1 (2) hold. Then, for any \( q > 0 \) and
\[
(3.3) \quad \rho^q := \frac{1}{N\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp[\mathrm{i}Ht] \exp\left[-\frac{t^2}{2q}\right] \, dt,
\]
we have
\[
\|\rho^q - P_0\|_{\mathcal{L}} \leq \exp\left[-\frac{1}{2}(\Delta E)^2 q\right],
\]
with \( \Delta E \) from (3.1).

See proof on page 13.

Next, we want to approximate \( H \) by a sum of three local operators, where each operator approximately annihilates the ground state. To this end, we apply Hasting’s quasi-adiabatic continuation technique (see [27,28]), which was also studied in [5] in the infinite-dimensional setting.\[13\]

Lemma 3.5. Suppose Assumption 3.1 (1)-(3) holds.

For a fixed \( l \in \mathbb{N} \) and a fixed \( 1 + l \leq j \leq d - 2 - l \),
\[
H_L := \sum_{k \leq j - l - 2} H_{k,k+1}, \quad H_B := \sum_{j-l-1 \leq k \leq j + l + 1} H_{k,k+1}, \quad H_R := \sum_{k \geq j + l + 2} H_{k,k+1}.
\]

Let \( \psi_0 \) be an arbitrary GS and w.l.o.g.\[14\] let \( \langle \psi_0, H_L \psi_0 \rangle_{\mathcal{H}} = \langle \psi_0, H_B \psi_0 \rangle_{\mathcal{H}} = \langle \psi_0, H_R \psi_0 \rangle_{\mathcal{H}} = 0 \). Then, for any \( q > 0 \) and
\[
\tilde{H}_L := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_L(t) \exp\left[-\frac{t^2}{2q}\right] \, dt,
\]
\[
\tilde{H}_B := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_B(t) \exp\left[-\frac{t^2}{2q}\right] \, dt,
\]
\[
\tilde{H}_R := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_R(t) \exp\left[-\frac{t^2}{2q}\right] \, dt.
\]

\[13\]There the authors considered this technique in order to describe states that belong to the same phase.

\[14\]Otherwise the integrals can be re-centered around the expectation value.
where $H_{\ldots}(t)$ is the Heisenberg evolution of $H_{\ldots}$ defined as
\[ H_L(t) := \exp[iHt]H_L \exp[-iHt], \]
we have
\[ \|\hat{H}_L\psi_0\|_{\mathcal{H}} \leq 3J^2(\Delta E)^{-1} \exp\left[\frac{-1}{2}(\Delta E)^2q\right], \]
\[ \|\hat{H}_B\psi_0\|_{\mathcal{H}} \leq 3J^2(\Delta E)^{-1} \exp\left[\frac{-1}{2}(\Delta E)^2q\right], \]
\[ \|\hat{H}_R\psi_0\|_{\mathcal{H}} \leq 3J^2(\Delta E)^{-1} \exp\left[\frac{-1}{2}(\Delta E)^2q\right]. \]
The constant $J$ is the interaction strength from (3.2).

See proof on page 13.

Remark 3.6. Note that the above bound depends explicitly only on $q$, $\Delta E$ and $J$. In fact, more precisely, the bound depends on an estimate for
\[ \|\|H, H_L\|\|_{\mathcal{L}}, \]
and the latter was assumed in (3.2) to be uniformly bounded, i.e., in particular it is independent of $j$ or $l$.

However, the subsequent lemmas will employ Lieb-Robinson bounds that depend explicitly on the parameter $l$. Thus, we will eventually use the above lemma and choose the constant $q$ depending on the spectral gap $\Delta E$ and the parameter $l$.

Next, we show that $\hat{H}_L$, $\hat{H}_B$ and $\hat{H}_R$ are approximately local. This is mainly due to Lieb-Robinson type estimates.

Lemma 3.7. Under Assumption (3.1), there exist local bounded operators $\Theta_L$, $\Theta_B$ and $\Theta_R$ supported on $\mathcal{H}_{j-2l-2,j}$, $\mathcal{H}_{j-2l-2,j+2l+3}$ and $\mathcal{H}_{j+1,l+1}$, respectively, such that for
\[ M_L := H_L + \Theta_L, \quad M_B := H_B + \Theta_B, \quad M_R := H_R + \Theta_R, \]
there exist constants $c_1 > 0$, $C_1 > 0$ such that
\[ \|\hat{H}_L - M_L\|_{\mathcal{L}} \leq C_1J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1l], \]
\[ \|\hat{H}_B - M_B\|_{\mathcal{L}} \leq C_1J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1l], \]
\[ \|\hat{H}_R - M_R\|_{\mathcal{L}} \leq C_1J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1l], \]
where $q$, $l$ are the parameters from Lemma 3.5 and $J$ is the interaction strength from (3.2).

See proof on page 14.

We can conclude the existence of the first two operators that we will need to approximate $P_0$.

Lemma 3.8. Under Assumption (3.1), there exist local, bounded and self-adjoint (projection) operators $L = L(j,l)$, $R = R(j,l)$ with the property
\[ \|(L - I)\psi_0\|_{\mathcal{H}} \leq \exp[-c_1l/2], \]
\[ \|(R - I)\psi_0\|_{\mathcal{H}} \leq \exp[-c_1l/2], \]
for any GS $\psi_0$. The operators $L$ and $R$ have the same support as $M_L$ and $M_R$, respectively, and $\|L\|_{\mathcal{L}} = \|R\|_{\mathcal{L}} = 1$.

See proof on page 15.

What remains is a step by step approximation of $\rho^B$ as a product of three local operators.

Lemma 3.9. Under Assumption (3.1), we can further approximate $P_0$ as
\[ \tilde{\rho}^B := \frac{1}{N\sqrt{2\pi q}} \int_{-\infty}^{\infty} T \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* \exp \left[ -\frac{t^2}{2q} \right] LR \, dt, \]
\[ A(t) := \exp[i(M_L + M_R)t]iM_B \exp[-i(M_L + M_R)t], \]
\[ \text{in the sense specified by the following lemma.} \]
where \( \mathcal{T} \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* \) is the negative time-ordered exponential and \( q = c_1 \frac{2l}{\Delta E} \). We have
\[
\| \tilde{\rho}^q - P_0 \|_\mathcal{L} \leq C_2 (1/N) J^2 \max \{ q, q^2 \} \left( \exp \left[ -c_1 l / 2 \right] + \exp \left[ -c_1 l \right] \right),
\]
for some constant \( C_2 > 0 \).

See proof on page 16.

It remains to show how we can obtain a local operator \( B \), maintaining the same approximation order. This follows once more from a Lieb-Robinson bound.

**Lemma 3.10.** Consider the operator
\[
\tilde{B} := \frac{1}{N \sqrt{2\pi q}} \int_{-\infty}^{\infty} \mathcal{T} \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* \exp \left[ -\frac{t^2}{2q} \right] \, dt,
\]
with \( A(t) \) as above
\[
A(t) := \exp[i(M_L + M_R)t]iM_B \exp[-i(M_L + M_R)t].
\]
Then, there exists a local bounded operator \( B \) supported on \( \mathcal{H}_{j-3l-2,j+3l+3} \), with \( \| B \|_\mathcal{L} \leq 1 \) such that
\[
\| \tilde{B} - B \|_\mathcal{L} \leq C_3 J^2 \max \{ q^{1/2}, q \} \exp[-c_3 l],
\]
for some constants \( C_3 > 0, c_3 > 0 \).

See proof on page 18.

We now have all the ingredients for the main result.

**Theorem 3.11.** Under Assumption 3.7, there exist local, bounded and self-adjoint operators \( L = L(j,l) \), \( B = B(j,l) \), \( R = R(j,l) \) with norms bounded by 1, such that for some constants \( C_4 > 0, c_4 > 0 \) independent of \( d, j \) or \( l \)
\[
\| P_0 - BLR \|_\mathcal{L} \leq C_4 J^2 \exp[-c_4 l].
\]
The respective supports are \( \mathcal{H}_{1,j} \), \( \mathcal{H}_{j-3l-2,j+3l+3} \) and \( \mathcal{H}_{j+1,d} \). The operator \( B \) can be chosen w.l.o.g. to be positive.

**Proof.** The operators \( L \) and \( R \) were defined in Lemma 3.8 and their properties follow therefrom. The operator \( B \) was defined in Lemma 3.10. W.l.o.g. we can assume it is positive, otherwise the same arguments as in [27] Lemma 4] apply.

By Lemmas 3.9 and 3.11 and since \( \| LR \|_\mathcal{L} \leq 1 \), we obtain an error bound with asymptotic dependence on \( l \) of the form \( I^2 \exp[-c l] \). Hence, we can pick constants \( C_4 > 0, c_4 > 0 \) to satisfy (3.1). This completes the proof.

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Proof of Lemma 3.4. The operator $U(t) := \exp[iHt] \exp\left[-\frac{t^2}{2q}\right]$ is strongly continuous for all $t \in \mathbb{R}$. Thus, a finite integral of $U(t)$ is well defined. For any $\psi \in \mathcal{H}$

$$
\lim_{c \to \infty} \left\| \frac{1}{\sqrt{2\pi q}} \int_{-c}^{c} U(t) \psi \ dt \right\|_{\mathcal{H}} \leq \lim_{c \to \infty} \left\| \psi \right\|_{\mathcal{H}} \frac{1}{\sqrt{2\pi q}} \int_{-c}^{c} \exp\left[-\frac{t^2}{2q}\right] \ dt = \left\| \psi \right\|_{\mathcal{H}}.
$$

Thus, the integral \(3.3\) is well defined.

Since $H$ is self-adjoint, we have the spectral decomposition $H = \int_{\sigma(H)} \lambda \ dP(\lambda)$, where $P : \sigma(H) \to \mathcal{L}(\mathcal{H})$ is a projection valued measure. Due to the gap assumption, we get that $NP_0 = P(\lambda_0)$.

Applying functional calculus for self-adjoint operators

$$
\exp[iHt] = \int_{\sigma(H)} \exp[i\lambda t] \ dP(\lambda).
$$

Equation (3.3) is to be interpreted as the unique operator such that for any $\psi \in \mathcal{H}$

$$
\left\langle \psi, \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} U(t) \psi \ dt \right\rangle_{\mathcal{H}} = \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp\left[-\frac{t^2}{2q}\right] \langle \psi, U(t)\psi \rangle_{\mathcal{H}} \ dt = \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp\left[-\frac{t^2}{2q}\right] \int_{\sigma(H)} \exp[i\lambda t] \ dP(\lambda) \ dt,
$$

where $P_{\psi}(\cdot) = \langle \psi, P(\cdot)\psi \rangle_{\mathcal{H}}$ and the equality follows from the linearity and continuity of the $\mathcal{H}$-inner product. For the last integral we can apply Fubini’s Theorem for general product measures. This allows us to write

$$
\rho^Q = \frac{1}{N} \sqrt{2\pi q} \int_{-\infty}^{\infty} \int_{\sigma(H)} \exp[i\lambda t] \exp\left[-\frac{t^2}{2q}\right] \ dP(\lambda) \ dt
$$

$$
\text{gap} = \frac{1}{N} \sqrt{2\pi q} \int_{-\infty}^{\infty} P_0 \exp\left[-\frac{t^2}{2q}\right] + \int_{\sigma(H)\setminus\{\lambda_0\}} \exp[i\lambda t] \exp\left[-\frac{t^2}{2q}\right] \ dP(\lambda) \ dt
$$

$$
\text{Fubini} = P_0 + \frac{1}{N} \sqrt{2\pi q} \int_{\sigma(H)\setminus\{\lambda_0\}} \int_{-\infty}^{\infty} \exp[i\lambda t] \exp\left[-\frac{t^2}{2q}\right] \ dt \ dP(\lambda).
$$

The last term is the Fourier transform of the density of the normal distribution. Thus,

$$
\|\rho^Q - P_0\|_{L^2} = \frac{1}{N} \left\| \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] \ dP(\lambda) \right\|_{L^2} \leq \frac{1}{N} \exp\left[-\frac{1}{2} (\Delta E)^2 q\right],
$$

which completes the proof. \(\square\)

Proof of Lemma 3.5. By the same arguments as in Lemma 3.4 the integrals are well defined. Next, application to the ground state yields

$$
H_L(t)\psi_0 = \exp[iHt]H_L \int_{\sigma(H)} \exp[-i\lambda t] \ dP(\lambda)\psi_0 \lambda_0 = \exp[iHt]H_L\psi_0 = \int_{\sigma(H)} \exp[i\lambda t] \ dP(\lambda)H_L\psi_0
$$

$$
= \langle \psi_0, H_L\psi_0 \rangle_{\mathcal{H}} + \int_{\sigma(H)\setminus\{\lambda_0\}} \exp[i\lambda t] \ dP(\lambda)H_L\psi_0.
$$

Thus,

$$
\tilde{H}_L\psi_0 = \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_L(t)\psi_0 \exp\left[-\frac{t^2}{2q}\right] \ dt
$$

$$
= \int_{\sigma(H)\setminus\{\lambda_0\}} \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp[i\lambda t] \exp\left[-\frac{t^2}{2q}\right] \ dt \ dP(\lambda)H_L\psi_0
$$

$$
= \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] dP(\lambda)H_L\psi_0.
$$
On the other hand,
\[ H \tilde{H}_L \psi_0 = \int_{\sigma(H) \setminus \{\lambda_0\}} \lambda \exp \left[ -\frac{1}{2} \lambda^2 q \right] \, dP(\lambda) \tilde{H}_L \psi_0. \]

Hence,
\[
(A.1) \quad \| H \tilde{H}_L \psi_0 \|_\mathcal{H} \geq \Delta E \left\| \int_{\sigma(H) \setminus \{\lambda_0\}} \exp \left[ -\frac{1}{2} \lambda^2 q \right] \, dP(\lambda) H \tilde{H}_L \psi_0 \right\|_\mathcal{H} = \Delta E \left\| \tilde{H}_L \psi_0 \right\|_\mathcal{H}.
\]

Next, since \( H \psi_0 = 0 \),
\[
H \tilde{H}_L \psi_0 = (H \tilde{H}_L - \tilde{H}_L H) \psi_0 = \int_{\sigma(H) \setminus \{\lambda_0\}} \exp \left[ -\frac{1}{2} \lambda^2 q \right] \, dP(\lambda) [H, H \tilde{H}_L] \psi_0
\]
\[
= \int_{\sigma(H) \setminus \{\lambda_0\}} \exp \left[ -\frac{1}{2} \lambda^2 q \right] \, dP(\lambda) \left\{ [H_j - l, j - l], H_j - l, j - l \right\} \psi_0
\]
\[
= \left\{ [H_j - l, j - l], (\Phi_j - l, j - l - 1) \right\} \psi_0,
\]
where the last equality follows from Assumption [3]. And thus
\[
\| H \tilde{H}_L \psi_0 \|_\mathcal{H} \leq 3J^2 \exp \left[ -\frac{1}{2} (\Delta E)^2 q \right].
\]
Together with \( A.1 \)
\[
\| \tilde{H}_L \psi_0 \|_\mathcal{H} \leq 3J^2 (\Delta E)^{-1} \exp \left[ -\frac{1}{2} (\Delta E)^2 q \right],
\]
and analogously for \( \tilde{H}_B, \tilde{H}_R \). This completes the proof. \( \square \)

**Proof of Lemma 3.7.** First, note that we can differentiate \( H_L(t) \) to obtain
\[
\frac{d}{dt} H_L(t) = \frac{d}{dt} \exp[iHt] H_L \exp[-iHt]
\]
\[
= \exp[iHt] [iH_L] \exp[-iHt] - \exp[iHt] [iH_L] H_L \exp[-iHt]
\]
\[
= \exp[iHt][iH, H_L] \exp[-iHt] = [i[H, H_L](t),
\]
and \( H_L(0) = H_L \). Thus, we can write
\[
(A.2) \quad H_L(t) = H_L + \int_0^t i[H, H_L](\tau) \, d\tau.
\]
By Assumption, \( [H, H_L] \) is bounded and supported on \( \mathcal{H}_{j-l-2,l-1} \). Consequently, we can write
\[
\tilde{H}_L = H_L + \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{t} \int_0^t i[H, H_L](\tau) \, d\tau \exp \left[ -\frac{t^2}{2q} \right] \, dt.
\]
Since the commutator is bounded and local, and the interactions in \( H \) are bounded, by [35] Corollary 2.2), we know a Lieb-Robinson bound applies to \( [H, H_L] \). I.e., there exists a constant (velocity) \( v \geq 0 \) and constants \( C, a > 0 \), such that
\[
(A.3) \quad \|[H, H_L](\tau), B]\|_\mathcal{L} \leq C \|[H, H_L]\|_\mathcal{L} \|[B]\|_\mathcal{L} \exp[-a \{ \text{dist}([H, H_L], B) - \tau \}],
\]
for all bounded and local \( B \). Thus, by [35] Lemma 3.2, there exists a map \( \Pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) such that \( \Pi(A) \) is supported on \( \mathcal{H}_{j-2l-2,l-j} \) and, for any \( A \in \mathcal{L}(\mathcal{H}) \) satisfying \( A.3 \) with \( \text{dist}(A, B) \geq l \), we have
\[
\|[A - \Pi(A)]\|_\mathcal{L} \leq 2C \|[A]\|_\mathcal{L} \exp[-a \{ l - \tau \}],
\]
Then, using [35] Theorem 3.4, we integrate over time and further estimate
\[
\left\| \int_0^t [H, H_L](\tau) \, d\tau - \int_0^t \Pi([H, H_L](\tau)) \, d\tau \right\|_\mathcal{L} \leq |t| CJ^2 \exp[-a \{ l - \tau \}].
\]
We define the operator
\[ \Theta_L := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \int_0^t \Pi (i[H, H_L](\tau)) \, d\tau \exp \left[ -\frac{t^2}{2q} \right] \, dt. \]

By all of the above, this operator is bounded and supported on \( \mathcal{H}_{j-2l-2,j} \). Analogously, we define \( \Theta_R \) and \( \Theta_R \) with supports in \( \mathcal{H}_{j-2l-2,j+2l+3} \) and \( \mathcal{H}_{j+1,j+2l+3} \), respectively.

What remains is to truncate the tails of the integral to obtain an overall error of the same order as the Lieb-Robinson bound. Let \( T = \frac{1}{q} \). Then,
\[
\| \hat{H}_L - M_L \|_{\mathcal{L}} =
\leq C J^2 \exp[-al] \left( \exp[avT] \frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} |t| \exp \left[ -\frac{t^2}{2q} \right] \, dt + \frac{1}{\sqrt{2\pi q}} \int_{|t| > T} |t| \exp \left[ avT - \frac{t^2}{2q} \right] \, dt \right).
\]

For the first term
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} |t| \exp \left[ -\frac{t^2}{2q} \right] \, dt \leq \sqrt{\frac{q}{2\pi}}.
\]

For the second
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| > T} |t| \exp \left[ avT - \frac{t^2}{2q} \right] \, dt = \sqrt{\frac{2}{\pi q}} \int_T^\infty t \exp \left[ avT - \frac{t^2}{2q} \right] \, dt
\]
\[= q \exp \left[ avT - \frac{T^2}{2q} \right] + avq \int_T^\infty \exp \left[ avT - \frac{t^2}{2q} \right] \, dt.
\]

For the latter term
\[
\int_T^\infty \exp \left[ avT - \frac{t^2}{2q} \right] \, dt = \int_T^\infty \exp \left[ (av)^2 q - \left( \sqrt{\frac{1}{2q}} - av \sqrt{\frac{q}{2}} \right)^2 \right] \, dt
\]
\[= \sqrt{2q} \int_T^\infty \exp \left[ -\tau^2 \right] \, d\tau \leq \sqrt{\frac{q}{2\pi}} \exp \left[ avT - \frac{T^2}{2q} \right].
\]

And thus
\[
\| \hat{H}_L - M_L \|_{\mathcal{L}} \leq C J^2 \exp[-al] \times
\left( \exp[al/2] \sqrt{\frac{q}{2\pi}} + q \exp[al/2] \exp \left[ -\frac{T^2}{2q} \right] + \frac{1}{2q} \exp[al/2] \exp \left[ -\frac{T^2}{2q} \right] \right)
\leq C J^2 \exp \left[ -\frac{a l}{2} \right] \max \left\{ \sqrt{\frac{q}{2\pi}}, q, \sqrt{\frac{q}{2}} \right\} \left( 1 + 2 \exp \left[ -\frac{T^2}{8avq} \right] \right)
\leq C_1 J^2 \max \left\{ q^{1/2}, q^{3/2} \right\} \exp[-c_1 l],
\]

with \( C_1 \) and \( c_1 \) defined in an obvious way as above. This completes the proof. \( \square \)

**Proof of Lemma 3.8**  Recall from Lemma 3.7 that we applied Lieb-Robinson to operators such as \( i[H, H_L](\tau) \). By Assumption 3.1 since \( H \) and \( H_L \) are self-adjoint, the commutators are bounded and one has \( i[H, H_L] = -i(H_L H - HH_L) = i[H, H_L] \Rightarrow \) we can conclude that the commutator is self-adjoint. Applying Lieb-Robinson as in [5] Lemma 3.2], one can construct the local approximation to preserve self-adjointness. We briefly elaborate.
Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. For the construction of $\Pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ in [5, Lemma 3.2], the authors use an arbitrary state $\rho : \mathcal{H}_2 \to \mathcal{H}_2$, though the resulting bound does not depend on the choice of $\rho$. Then, for this state, applying the spectral decomposition, we have
\[
\rho = \sum_{k=1}^{\infty} \lambda_k \langle \cdot, \psi_k \rangle_{\mathcal{H}_2} \psi_k,
\]
for $\{\psi_k : k \in \mathbb{N}\}$ orthonormal in $\mathcal{H}_2$. Next, for each $\psi_k$, the authors define the map $A_k$ as
\[
\langle v, A_k w \rangle_{\mathcal{H}_1} = \langle v \otimes \psi_k, A w \otimes \psi_k \rangle_{\mathcal{H}_1}, \quad \forall v, w \in \mathcal{H}_1.
\]
Finally, the map $\Pi(A)$ is defined as
\[
\Pi(A) := \left( \sum_{k=1}^{\infty} \lambda_k A_k \right) \otimes I.
\]
Note that each $A_k$ is self-adjoint if $A$ is self-adjoint. Therefore, $\Pi(A)$ is self-adjoint.

Thus, $M_L$, $M_B$ and $M_R$ can be chosen self-adjoint. By Lemmas 3.7 and 3.5 picking $q = c_1 \frac{2l}{\Delta E}$, we get $\|M_L \psi_0\|_{\mathcal{H}} \leq C_1 J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1 l]$. Moreover, since $M_L$ is self-adjoint, there exists a projection valued measure $P(\cdot)$ such that
\[
\|M_L \psi_0\|_{\mathcal{H}}^2 = \langle M_L \psi_0, M_L \psi_0 \rangle_{\mathcal{H}} = \langle \psi_0, M_L^2 \psi_0 \rangle_{\mathcal{H}} = \int_{\sigma(M_L)} \lambda^2 \, dP_0(\lambda).
\]
We split the spectrum of $M_L$ as
\[
\sigma_1(M_L) := \{ \lambda \in \sigma(M_L) : |\lambda| \leq C_1 J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1 l/2] \},
\]
\[
\sigma_2(M_L) := \sigma(M_L) \setminus \sigma_1(M_L).
\]
Define $L$ as
\[
L := \int_{\sigma_1(M_L)} dP(\lambda).
\]
Clearly, $L$ is a bounded self-adjoint operator with $\|L\|_{\mathcal{L}} = 1$ and the same support as $M_L$. Moreover, by orthogonality of the spectral subspaces
\[
C_1^2 J^4 \max\{q, q^3\} \exp[-c_1 l] \geq \|M_L \psi_0\|_{\mathcal{H}}^2 = \int_{\sigma_1(M_L)} \lambda^2 \, dP_0(\lambda) \psi_0 + \int_{\sigma_2(M_L)} \lambda^2 \, dP_0(\lambda) \psi_0
\]
\[
\geq C_1^2 J^4 \max\{q, q^3\} \exp[-c_1 l] \int_{\sigma_2(M_L)} dP_0(\lambda).
\]
Thus, $\int_{\sigma_2(M_L)} dP_0(\lambda) \leq \exp[-c_1 l]$. Finally, this gives
\[
\| (L - I) \psi_0 \|_{\mathcal{H}} = \left( \int_{\sigma_2(M_L)} dP(\lambda) \psi_0 \right)_{\mathcal{H}} = \left( \int_{\sigma_2(M_L)} dP_0(\lambda) \right)^{1/2} \leq \exp[-c_1 l/2].
\]
Analogously for $R$. This completes the proof.

Proof of Lemma 3.9 Note that $H = \hat{H}_L + \hat{H}_B + \hat{H}_R$. By utilizing the estimates from Lemma 3.7, we have
\[
\left\| \frac{1}{N \sqrt{2\pi q}} \int_{-\infty}^{\infty} \left\{ \exp\left[i(\hat{H}_L + \hat{H}_B + \hat{H}_R)t\right] - \exp[i(M_L + M_B + M_R)t] \right\} \exp\left[-\frac{t^2}{2q}\right] \, dt \right\|_{\mathcal{L}}
\]
\[
\leq 3C_1 (1/N) J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1 l] \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} |t| \exp\left[-\frac{t^2}{2q}\right] \, dt
\]
\[
= 3(2\pi)^{-1/2} C_1 (1/N) J^2 \max\{q, q^2\} \exp[-c_1 l],
\]
where the inequality can be shown using [24, Chapter 9, Theorem 2.12, Equation (2.22)].

\footnote{Positive operator with unit trace norm.}
Next, for the exponential term we can write
\[
\exp[i(M_L + M_B + M_R)t] = \exp[i(M_L + M_B + M_R)t] \exp[-i(M_L + M_R)t] \exp[i(M_L + M_R)t],
\]
and we define \( U(t) := \exp[i(M_L + M_B + M_R)t] \exp[-i(M_L + M_R)t] \). In \( U(t) \) the term in the exponent commutes for different \( t \), since the time-dependence is a simple multiplication by \( t \). We thus compute
\[
\frac{d}{dt} U(t) = \exp[i(M_L + M_B + M_R)t]i(M_L + M_B + M_R)\exp[-i(M_L + M_R)t]
- \exp[i(M_L + M_B + M_R)t]i(M_L + M_R)\exp[-i(M_L + M_R)t]
= \exp[i(M_L + M_B + M_R)t]iM_B\exp[-i(M_L + M_R)t]
= \exp[i(M_L + M_B + M_R)t] \exp[-i(M_L + M_R)t]iM_B\exp[-i(M_L + M_R)t]
= U(t) \exp[i(M_L + M_R)t]iM_B\exp[-i(M_L + M_R)t]
\]
and \( U(0) = I \). We abbreviate
\[
M_B(t) := \exp[i(M_L + M_R)t]M_B\exp[-i(M_L + M_R)t].
\]
Due to the simple form of \( iM_B(t) \), the solution to this initial value problem exists, is unique and is given by the (negative) time-ordered exponential of \( iM_B(t) \), see \[39\] Chapter X.12 (interaction representation) or \[18\] for ordered exponentials of more general unbounded time-dependent Hamiltonians.

Thus, our approximation so far is
\[
\hat{\rho}^t = \frac{1}{N\sqrt{2\pi q}} \int_{-\infty}^{\infty} T \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* \exp[i(M_L + M_R)t] \exp \left[ -\frac{t^2}{2q} \right] dt.
\]
By multiplying \( L, R \) from the right we obtain
\[
\|\hat{\rho}^tLR - P_0\|_\mathcal{L} = \|(\hat{\rho}^t - P_0 + P_0)LR - P_0\|_\mathcal{L} = \|(\hat{\rho}^t - P_0)LR + P_0LR - P_0\|_\mathcal{L}
\leq \|\hat{\rho}^t - P_0\|_\mathcal{L} + \|P_0LR - P_0\|_\mathcal{L}
\]
for the latter term we use Lemma \[33\] set \( q = c_1 \frac{2l}{(2\pi)^2} \) and obtain
\[
\|P_0LR - P_0\|_\mathcal{L} = \|P_0\{ (L-I) + (R-I) + I \}\|_\mathcal{L}
= \|P_0(L-I)(R-I) + P_0(L-I) + P_0(R-I)\|_\mathcal{L}
\leq 3 \|(L-I)P_0\|_\mathcal{L} + \|(R-I)P_0\|_\mathcal{L} \leq 4 \exp[-c_1l/2].
\]
since \( \|P_0\|_\mathcal{L} = \|L\|_\mathcal{L} = 1 \) and \( P_0, L, R \) are self-adjoint. Thus, overall
\[
\|\hat{\rho}^tLR - P_0\|_\mathcal{L} \leq 3(2\pi)^{-1/2}(1/N)C_1J^2 \max\{q, q^2\} \exp[-c_1l] + 4 \exp[-c_1l/2].
\]
Finally, by definition, \( L \) and \( R \) project onto the spectral subspaces of \( M_L \) and \( M_R \) corresponding to small eigenvalues. I.e.,
\[
\| \exp[i(M_L + M_R)t] - I \|_\mathcal{L} \leq 2 |t|C_1J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1l/2],
\]
hence
\[
\|\hat{\rho}^tLR - \hat{\rho}^t\|_\mathcal{L} \leq \left\| \frac{1}{N\sqrt{2\pi q}} \int_{-\infty}^{\infty} T \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* \exp[i(M_L + M_R)t]LR - LR \right\|_\mathcal{L}
\leq 2C_1(1/N)J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1l/2] \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} |t| \exp \left[ -\frac{t^2}{2q} \right] dt
\]
\[
= C_1(1/N)J^2(2\pi)^{-1/2} \max\{q, q^2\} \exp[-c_1l/2].
\]
Overall we obtain
\[
\|\hat{\rho}^t - P_0\|_\mathcal{L} \leq \|\hat{\rho}^tLR - P_0\|_\mathcal{L} + \|\hat{\rho}^tLR - \hat{\rho}^t\|_\mathcal{L}
\leq C_2J^2 \max\{q, q^2\} \left( \exp[-c_1l/2] + \exp[-c_1l] \right),
\]
for an appropriately chosen constant \( C_2 > 0 \). This completes the proof. \( \square \)
Thus, as in Lemma 3.7, we can approximate $M_B(t)$ by a local operator $\tilde{M}_B(t)$ supported on $\mathcal{H}_{j-3l-2j+3l+3}$ such that

$$ \left\| M_B(t) - \tilde{M}_B(t) \right\|_L \leq C|t| \exp[-a \{l - v|t|\}]. $$

We utilize this estimate in a similar way as in (A.5). To this end, we write the time-ordered exponential as a product integral and use a step function approximation to the integral (see [15] Chapter 3.6), namely

$$ T \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^{\tau} = \prod_0^t \exp[A(\tau)] \, d\tau = \lim_{K \to \infty} \left( \exp[A(t_K)\Delta t] \cdots \exp[A(t_0)\Delta t] \right), $$

where $t_i = i\Delta t$, $\Delta t = t/K$ and the convergence is meant in the strong sense. This is possible due to the simple form of $A(t)$, i.e., $\exp[A(t)]$ is bounded with norm 1.

For $K = 1$, we obtain as in (A.5)

$$ \left\| \exp[iM_B(t)\Delta t] - \exp[i\tilde{M}_B(t)\Delta t] \right\|_L \leq |\Delta t| \left\| M_B(t) - \tilde{M}_B(t) \right\|_L, $$

with $|\Delta t| = |t|$. For $K - 1 \to K$, by induction

$$ \left\| \exp[iM_B(t_K)\Delta t] \cdots \exp[iM_B(t_0)\Delta t] - \exp[i\tilde{M}_B(t_K)\Delta t] \cdots \exp[i\tilde{M}_B(t_0)\Delta t] \right\|_L $$

$$ \leq \left\| \exp[iM_B(t_K)\Delta t] - \exp[i\tilde{M}_B(t_K)\Delta t] \right\| \left\| \exp[iM_B(t_{K-1})\Delta t] \cdots \exp[iM_B(t_0)\Delta t] \right\|_L $$

$$ + \left\| \exp[i\tilde{M}_B(t_K)\Delta t] \times \left[ \exp[iM_B(t_{K-1})\Delta t] \cdots \exp[iM_B(t_0)\Delta t] - \exp[i\tilde{M}_B(t_{K-1})\Delta t] \cdots \exp[i\tilde{M}_B(t_0)\Delta t] \right] \right\|_L $$

$$ \leq |\Delta t| K \left\| M_B(t) - \tilde{M}_B(t) \right\|_L, $$

where by definition $|\Delta t| K = |t|$. Thus, we can estimate for the ordered exponential

$$ \left\| T \left( \exp \left[ \int_0^t iM_B(\tau) \, d\tau \right] \right) \right\|_L $$

$$ \leq |t| \left\| M_B(t) - \tilde{M}_B(t) \right\|_L \leq C|t|^2 \exp[-a \{l/3 - v|t|\}]. $$

We define the local operator $B$ as

$$ B := \frac{1}{N \sqrt{2\pi q}} \int_0^\infty T \left( \exp \left[ \int_0^t i\tilde{M}_B(\tau) \, d\tau \right] \right)^* \exp \left[ -\frac{t^2}{2q} \right] \, dt. $$

Estimating as in (A.4) for $T = \frac{1}{\sqrt{2\pi q}}$, we obtain

$$ \left\| \hat{B} - B \right\|_L \leq \left\| \int_{|t| \leq T} + \int_{|t| > T} \cdots \right\|_L \leq C(1/N)J^2 \exp[-a l/3] $$

$$ \times \left( \exp[a vT] \frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} t^2 \exp \left[ -\frac{t^2}{2q} \right] \, dt + \frac{1}{\sqrt{2\pi q}} \int_{|t| > T} t^2 \exp \left[ av|t| - \frac{t^2}{2q} \right] \, dt \right), $$

with
In analogy to (A.4), for the first term we obtain
\[ \frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} t^2 \exp \left[ -\frac{t^2}{2q} \right] dt \leq \frac{q}{2}. \]

For the second term, applying integration by parts, we obtain
\[ \frac{1}{\sqrt{2\pi q}} \int_{|t| > T} t^2 \exp \left[ av|t| - \frac{t^2}{2q} \right] dt = \sqrt{\frac{2}{\pi q}} \int_{T}^{\infty} t^2 \exp \left[ avt - \frac{t^2}{2q} \right] dt. \]

And hence
\[ \frac{1}{\sqrt{2\pi q}} \int_{|t| > T} t^2 \exp \left[ av|t| - \frac{t^2}{2q} \right] dt \leq \exp \left[ avT - \frac{T^2}{2q} \right] \frac{\sqrt{2}}{\sqrt{\pi |1 - avq|}} \sqrt{q} \left( \frac{q\pi}{2} + T \right). \]

The final estimate is thus
\[ \| \tilde{B} - B \|_{L} \leq C_3 J^2 \max \{ q^{1/2}, q \} \exp[-3c_3l], \]
for appropriate constants $C_3 > 0$, $c_3 > 0$. This completes the proof. \qed