Iterants and the Dirac Equation

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Abstract. This paper introduces the concept of iterant algebra and applies it to the formation of basic Clifford algebras including a reconstruction of the complex numbers in terms of a formalization of temporal process. Iterant algebra is shown to include all of matrix algebra and applications are given to a representation of the $su(3)$ Lie algebra for the Standard Model and to the construction of the Dirac Equation. This construction of the Dirac Equation makes it clear how solutions arise from nilpotent elements in the Clifford algebra and how Fermion algebra and the algebra of Majorana Fermions emerges in this context. The paper ends with a formulation of the original Majorana Dirac Equation in terms of Clifford algebra in the context of iterants.

1. Introduction

The simplest discrete system corresponds directly to the square root of minus one, when the square root of minus one is seen as an oscillation between plus and minus one. This way thinking about the square root of minus one as an iterant is explained below. More generally, by starting with a discrete time series of positions, one has immediately a non-commutativity of observations and this non-commutativity can be encapsulated in an iterant algebra as defined in Section 2 of the present paper. Iterant algebra generalizes matrix algebra and we shall see how it can be used to formulate the Lie algebra $su(3)$ for the Standard Model for particle physics and the Clifford algebra for Majorana Fermions. This paper is a sequel to and exposition for [9] and [6, 7, 8, 9, 26, 27] and it uses material from these papers. This paper incorporates new results of the author that first appear in the joint paper of the author and Rukhsan Ul- Haq [10].

Distinction and processes arising from distinction are at the base of the described world. Distinctions are elemental bits of awareness. The world is composed not of things but processes and observations. We will discuss how basic Clifford algebra comes from very elementary processes such as an alternation of $+ - + - + - \cdots$ and the fact that one can think of $\sqrt{-1}$ itself as a temporal iterant, a product of an $\epsilon$ and an $\eta$ where the $\epsilon$ is the $- + - + - + \cdots$ and the $\eta$ is a time shift operator. Clifford algebra is at the base of this mathematical world, and the fermions are composed of these things.
Sections 2 and 3 are an introduction to the process algebra of iterants and how the square root of minus one arises from an alternating process. Section 3 shows how iterants give an alternative way to do $2 \times 2$ matrix algebra. The section ends with the construction of the split quaternions. Section 4 considers iterants of arbitrary period (not just two). We then generalize this construction to arbitrary non-commutative finite groups $G$. Such a group has a multiplication table ($n \times n$ where $n$ is the order of the group $G$). We show that by rearranging the multiplication table so the identity element appears on the diagonal, we get a set of permutation matrices that represent the group faithfully as $n \times n$ matrices. In Section 5 we give an iterant interpretation of the $su(3)$ Lie algebra for the Standard Model using [32]. Section 6 discusses the structure of the Dirac equation and how the nilpotent and the Majorana operators arise naturally in this context. This section provides a link between our work and the work on nilpotent structures and the Dirac equation of Peter Rowlands [35]. We end this section with an expression in split quaternions for the the Majorana Dirac equation in one dimension of time and three dimensions of space. The Majorana Dirac equation can be written as follows:

\[
(\partial/\partial t + \hat{\eta}\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\partial/\partial z - \hat{\epsilon}\hat{\eta}\psi)\psi = 0
\]

where $\eta$ and $\epsilon$ are the simplest generators of iterant algebra with $\eta^2 = \epsilon^2 = 1$ and $\eta\epsilon + \epsilon\eta = 0$, and $\hat{\epsilon}, \hat{\eta}$ form a copy of this algebra that commutes with it. This combination of the simplest Clifford algebra with itself is the underlying structure of Majorana Fermions, forming indeed the underlying structure of all Fermions.

This paper is part of a larger story of mathematics and physics that we are in the process of telling and exploring. To begin the story, we conclude this introduction with a formulation of the Schroedinger equation that can motivate the iterants.

### 1.1. Iterants and the Schroedinger Equation

We begin with the Diffusion Equation

\[
\partial\psi/\partial t = \tau \partial^2 \psi/\partial x^2.
\]

We reformulate this equation as a difference equation in space and time. In writing it as a difference equation, I shall use $dt$ for a finite increment in time and $dx$ for a finite increment in space.

\[
(\psi_{t+dt} - \psi_t)/dt = \tau(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx))/(dx)^2.
\]

This is equivalent to

\[
(\psi_{t+dt} - \psi_t) = \frac{dt}{(dx)^2}\tau(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx)),
\]

or to

\[
(\psi_{t+dt} - \psi_t) = \kappa(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx)),
\]

where $\kappa = \frac{dt}{(dx)^2}\tau$, since for the continuum limit to exist we need to assume that $\frac{dt}{(dx)^2}$ is constant as $dt$ and $dx$ go to zero. We shall use $dt = 1$ for convenience.

Then the above equation becomes

\[
\psi_{t+1} - \psi_t = \kappa(\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx)).
\]
Consider the possibility of putting a “plus or minus” ambiguity into this equation, like so:

$$\frac{\partial \psi}{\partial t} = \pm \kappa \frac{\partial^2 \psi}{\partial x^2}.$$ 

The ± coefficient should be lawful not random, for then we can follow an algebraic formulation of the process behind the equation. We shall take ± to mean the alternating sequence

$$\pm = \cdots + - + - + - \cdots$$

and time will be discrete. Then the new equation will become a difference equation in space and time

$$\psi_{t+1} - \psi_t = (-1)^t \kappa (\psi_t(x - dx) - 2\psi_t(x) + \psi_t(x + dx)).$$

But we wish to consider the continuum limit. However, there is no meaning to

$$(-1)^t$$

in the realm of continuous time. What do do? In the discrete world the wave function ψ divides into ψ_e and ψ_o where the (discrete) time is either even or odd. So we can write (thinking of these as the corresponding discrete equations or as the continuum limits).

$$\frac{\partial_t \psi_e}{\partial t} = \kappa \frac{\partial^2 \psi_o}{\partial x^2},$$

$$\frac{\partial_t \psi_o}{\partial t} = -\kappa \frac{\partial^2 \psi_e}{\partial x^2}.$$ 

We take the continuum limit of ψ_e and ψ_o separately.

In fact we can interpret the {±} as the complex number i. Recall that the complex number i has the property that $i^2 = -1$ so that

$$i(A + iB) = iA - B$$

when A and B are real numbers,

$$i = -1/i,$$

and so if $i = 1$ then $i = -1$, and if $i = -1$ then $i = 1$. So i can be interpreted as oscillating between +1 and -1, and so we shall regard i as a definition of ±1.

$$i = \pm 1.$$ 

In fact, when we multiply $ii = (\pm 1)(\pm 1)$, we get -1 because (in this temporal interpretation) the i takes a little time to oscillate and so by the time this second term multiplies the first term, they are just out of phase and so we get either $(+1)(-1) = -1$ or $(-1)(+1) = -1$. We will formalize this point of view later in the paper.

Now $i = \pm 1$ behaves quite lawfully and we can write

$$\psi = \psi_e + i\psi_o$$

so that

$$i\partial_t \psi = i\partial_t (\psi_e + i\psi_o) = i\partial_t \psi_e - \partial_t \psi_o$$

$$= i\kappa \frac{\partial^2 \psi_o}{\partial x^2} + \kappa \frac{\partial^2 \psi_e}{\partial x^2} = \kappa \frac{\partial^2 \psi_e + i\psi_o}{\partial x^2}$$

$$= \kappa \frac{\partial^2 \psi}{\partial x^2}.$$
Thus
\[ i\frac{\partial \psi}{\partial t} = \kappa \frac{\partial^2 \psi}{\partial x^2}. \]
This the Schroedinger equation. Instead of the simple diffusion equation, we have a mutual de-
pendency where the temporal variation of \( \psi_e \) is mediated by the spatial variation of \( \psi_o \) and the
temporal variation of \( \psi_o \) is mediated by the spatial variation of \( \psi_e \). We arrive at the Schroedinger
equation in the context of \( i = \pm \) as an iterant.

\[ \partial_t \psi_e = \kappa \partial_x^2 \psi_o \]
\[ \partial_t \psi_o = -\kappa \partial_x^2 \psi_e. \]
\[ \psi = \psi_e + i\psi_o \]
\[ i\frac{\partial \psi}{\partial t} = \kappa \frac{\partial^2 \psi}{\partial x^2}. \]

**Remark.** The discrete recursion, just discussed, can actually be implemented to approximate
solutions to the Schroedinger equation. A further study of this recursion is intended. For now
the point is that this way of thinking about the Schroedinger equation shows that it is intimately
connected with a generalization of the discrete diffusion process with a temporal parity oscillation
that becomes \( i \) in the continuum limit. The temporal interpretation of \( i \) indicated here will be
given an algebraic context in the body of this paper.

**Acknowledgement.** It gives the author pleasure to thank James Flagg, Peter Rowlands, Sam
Lomonaco, Bernd Schmeikal and Rukhsan Ul Haq for conversations related to the considerations
in this paper. Kauffman’s work was supported by the Laboratory of Topology and Dynamics,
Novosibirsk State University (contract no. 14.Y26.31.0025 with the Ministry of Education and
Science of the Russian Federation).

### 2. Iterants and Idempotents

An iterant is a sum of elements of the form
\[ [a_1, a_2, ..., a_n]\sigma \]
where \([a_1, a_2, ..., a_n]\) is a vector of elements that are scalars (usually real or complex numbers)
and \(\sigma\) is a permutation on \(n\) letters. Such elements are themselves sums of elements of the form
\[ [0, 0, ..., 0, 1, 0, ..., 0]\sigma = e_i\sigma \]
where the 1 is in the \(i\)-th place. The elements \(e_i\) are the basic idempotents that generate the
iterants with the help of the permutations.

Note that if \(a = [a_1, a_2, ..., a_n]\), then we let \(a^\sigma\) denote the vector with its elements permuted
by the action of \(\sigma\). If \(a\) and \(b\) are vectors then \(ab\) denotes the vector where \((ab)_i = a_ib_i\), and \(a + b\)
denotes the vector where \((a + b)_i = a_i + b_i\). Then
\[ (a\sigma)(b\tau) = (ab^\tau)\sigma\tau, \]
\[ (ka)\sigma = k(a\sigma) \]
for a scalar \(k\), and
\[ (a + b)\sigma = a\sigma + b\sigma \]
where vectors are multiplied as above and we take the usual product of the permutations. All of matrix algebra and more is naturally represented in the iterant framework, as we shall see in the next sections.

Iterant algebra is generated by the elements

\[ e_i \sigma \]

where \( e_i \) is a vector with a 1 in the \( i \)-th place and zeros elsewhere, and \( \sigma \) is an arbitrary element of the symmetric group \( S_n \). We have that

\[ e_i \sigma = \sigma e_{\sigma^{-1}(i)} \]

so that the multiplication of iterants is defined in terms of the action of the symmetric group. We have

\[ e_i \sigma e_j \tau = e_i e_{\sigma(j)} \sigma \tau = \delta(i, \sigma(j)) e_i \sigma \tau. \]

By themselves, the elements \( e_i \) are idempotent and we have

\[ 1 = e_1 + \cdots + e_n. \]

The iterant algebra is generated by these combinations of idempotents and permutations.

For example, if \( \eta \) is the order two permutation of two elements, then \( [a, b] \eta = [b, a] \). The appearance of a square root of minus one unfolds naturally from iterant considerations. Define the “shift” operator \( \eta \) on iterants by the equation

\[ \eta[a, b] = [b, a] \eta \]

with \( \eta^2 = 1 \). Sometimes it is convenient to think of \( \eta \) as a delay operator, since it shifts the waveform \( \ldots ababab\ldots \) by one internal time step. We can define

\[ i = [-1, 1] \eta \]

and then

\[ i^2 = [-1, 1] \eta[-1, 1] \eta = [-1, 1][-1, 1]^3 \eta^2 = [-1, 1][1, -1] = [-1, -1] = -1. \]

In this way the complex numbers arise naturally from iterants. One can interpret \([ -1, 1 ]\) as an oscillation between \(-1\) and \(+1\) and \( \eta \) as denoting a temporal shift operator. The \( i = [-1, 1] \eta \) is a time sensitive element and its self-interaction has square minus one. In this way iterants can be interpreted as a formalization of elementary discrete processes. Let \( \epsilon = [-1, 1] \). By writing \( i = \epsilon \eta \) we recognize an active version of the waveform that shifts temporally when it is observed. This theme of including the result of time in observations of a discrete system occurs at the foundation of our construction.

Note that we can write \( a = [1, 0], b = [0, 1] \) and \( A = a \eta, B = b \eta \) where \( \eta \) denotes the transposition so that \([x, y] \eta = \eta[y, x] \) and \( \eta^2 = 1. \) Then we have

\[ a^2 = a, b^2 = b, ab = 0, a + b = 1, A^2 = 0 = B^2, AB = a, BA = b. \]

This is the mixed idempotent and permutation algebra for \( n = 2. \) Then we have

\[ i = B - A \]
as we can see by

\[ ii = (B - A)(B - A) = AA - AB - BA + BB = -a - b = -1. \]

This is the beginning of the relationships between idempotents, iterants and Clifford algebras.

Note that we construct an elementary Clifford algebra via

\[ \epsilon = [-1, 1] = b - a \]

and

\[ \eta. \]

Then we have

\[ \epsilon^2 = \eta^2 = 1 \]

and

\[ \epsilon \eta + \eta \epsilon = 0. \]

Note also that the non-commuting of \( \epsilon \) and \( \eta \) is directly related to the interaction of the idempotents and the permutations.

\[ \epsilon \eta = [-1, 1] \eta = \eta [1, -1] = -\eta [-1, 1] = -\eta \epsilon. \]

3. Iterants, Discrete Processes and Matrix Algebra

In this section we consider examples of iterants and related their algebra to matrix algebra. The primitive idea behind an iterant is a periodic time series or “waveform”

\[ \cdots ababababab \cdots. \]

The elements of the waveform can be any mathematically or empirically well-defined objects. We can regard the ordered pairs \([a, b]\) and \([b, a]\) as abbreviations for the waveform or as two points of view about the waveform (\(a\) first or \(b\) first). Call \([a, b]\) an iterant. One has the collection of transformations of the form \(T[a, b] = [ka, k^{-1}b]\) leaving the product \(ab\) invariant. This tiny model contains the seeds of special relativity, and the iterants contain the seeds of general matrix algebra! For related discussion see [2, 3, 4, 5, 6, 7, 10, 13, 11, 14, 1].

Define products and sums of iterants as follows

\[ [a, b][c, d] = [ac, bd] \]

and

\[ [a, b] + [c, d] = [a + c, b + d]. \]

The operation of juxtaposition of waveforms is multiplication while + denotes ordinary addition of ordered pairs. These operations are natural with respect to the structural juxtaposition of iterants:

\[ \cdots ababababab \cdots \]

\[ \cdots cdcdcdcdcd \cdots \]

Structures combine at the points where they correspond. Waveforms combine at the times where they correspond. Iterants combine in juxtaposition.
If \( \bullet \) denotes any form of binary composition for the ingredients \((a,b,...)\) of iterants, then we can extend \( \bullet \) to the iterants themselves by the definition \([a, b] \bullet [c, d] = [a \bullet c, b \bullet d]\).

We now show how the iterant algebra is related to matrix algebra. In order to keep track of this patterning, let's write

\[
[a, b] + [c, d] \eta = \begin{pmatrix} a & c \\ d & b \end{pmatrix}.
\]

where

\[
[x, y] = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},
\]

and

\[
\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Recall the definition of matrix multiplication.

\[
\begin{pmatrix} a & c \\ d & b \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + ch & ag + cf \\ de + bh & dg + bf \end{pmatrix}.
\]

Compare this with the iterant multiplication.

\[
([a, b] + [c, d] \eta)([e, f] + [g, h] \eta) =
\]

\[
[a, b][e, f] + [c, d] \eta[g, h] \eta + [a, b][g, h] \eta + [c, d] \eta[e, f] =
\]

\[
[ae, bf] + [c, d][h, g] + ([ag, bh] + [c, d][f, e]) \eta =
\]

\[
[ae, bf] + [ch, dg] + ([ag, bh] + [cf, de]) \eta =
\]

\[
[ae + ch, dg + bf] + [ag + cf, de + bh] \eta.
\]

Thus matrix multiplication is identical with iterant multiplication. The concept of the iterant can be used to motivate matrix multiplication.

**Notation.** We have the shift operation \( \eta[x, y] \eta = [y, x] \) which we shall denote by an overbar as shown below

\[
\overline{x, y} = [y, x].
\]

Ordinary matrix multiplication can be written in a concise form using the following rules:

\[
\eta \eta = 1
\]

\[
\eta Q = Q \eta
\]

where Q is any two element iterant. Note the correspondence

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [a, d]1 + [b, c] \eta.
\]

This means that \([a, d]\) corresponds to a diagonal matrix.

\[
[a, d] = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.
\]
η corresponds to the anti-diagonal permutation matrix.

\[ \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

and \([b, c] \eta\) corresponds to the product of a diagonal matrix and the permutation matrix.

\[ [b, c] \eta = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}. \]

Note also that

\[ \eta[c, b] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}. \]

This is the matrix interpretation of the equation

\[ [b, c] \eta = \eta[c, b]. \]

The fact that the iterant expression \([a, d]1 + [b, c] \eta\) captures the whole of \(2 \times 2\) matrix algebra corresponds to the fact that a two by two matrix is combinatorially the union of the identity pattern (the diagonal) and the interchange pattern (the antidiagonal) that correspond to the operators 1 and \(\eta\).

\[ \left( \begin{array}{cc} * & \@ \\ \@ & * \end{array} \right) \]

In the formal diagram for a matrix shown above, we indicate the diagonal by \(\ast\) and the anti-diagonal by \(\oplus\).

In the case of complex numbers we represent

\[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = [a, a] + [-b, b] \eta = a1 + b[-1, 1] \eta = a + bi. \]

In this way, we see that all of \(2 \times 2\) matrix algebra is a hypercomplex number system based on the symmetric group \(S_2\). In the next section we generalize this point of view to arbitrary finite groups.

We have reconstructed the square root of minus one in the form of the matrix

\[ i = \epsilon \eta = [-1, 1] \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

In this way, we arrive at this well-known representation of the complex numbers in terms of matrices. Note that if we identify the ordered pair \((a, b)\) with \(a + ib\), then this means taking the identification

\[ (a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \]

Thus the geometric interpretation of multiplication by \(i\) as a ninety degree rotation in the Cartesian plane,

\[ i(a, b) = (-b, a), \]

takes the place of the matrix equation

\[ i(a, b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix} = b + ia = (-b, a). \]
In iterant terms we have

\[ i[a, b] = \epsilon \eta[a, b] = [-1, 1][b, a]\eta = [-b, a]\eta, \]

and this corresponds to the matrix equation

\[ i[a, b] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} = [-b, a]\eta. \]

All of this points out how the complex numbers, as we have previously examined them, live naturally in the context of the non-commutative algebras of iterants and matrices. The factorization of \( i \) into a product \( \epsilon \eta \) of non-commuting iterant operators is closer both to the temporal nature of \( i \) and to its algebraic roots.

More generally, we see that

\[ (A + B\eta)(C + D\eta) = (AC + B\overline{D}) + (AD + B\overline{C})\eta \]

writing the 2 \( \times \) 2 matrix algebra as a system of hypercomplex numbers. Note that

\[ (A + B\eta)(\overline{A} - B\eta) = A\overline{A} - B\overline{B} \]

The formula on the right equals the determinant of the matrix. Thus we define the conjugate of \( Z = A + B\eta \) by the formula

\[ Z = A + B\eta = \overline{A} - B\eta, \]

and we have the formula

\[ D(Z) = ZZ \]

for the determinant \( D(Z) \) where

\[ Z = A + B\eta = \begin{pmatrix} a & c \\ d & b \end{pmatrix} \]

where \( A = [a, b] \) and \( B = [c, d] \). Note that

\[ A\overline{A} = [ab, ba] = ab1 = ab, \]

so that

\[ D(Z) = ab - cd. \]

Note also that we assume that \( a, b, c, d \) are in a commutative base ring.

Note also that for \( Z \) as above,

\[ \overline{Z} = \overline{A} - B\eta = \begin{pmatrix} b & -c \\ -d & a \end{pmatrix}. \]

This is the classical adjoint of the matrix \( Z \).
We leave it to the reader to check that for matrix iterants $Z$ and $W$,

$$ZZ = ZZ$$

and that

$$ZW = WZ$$

and

$$Z + W = Z + W.$$

Note also that

$$\eta = -\eta,$$

whence

$$B\eta = -B\eta = -\eta B = \eta B.$$

We can prove that

$$D(ZW) = D(Z)D(W)$$

as follows

$$D(ZW) = ZWZ = ZW\overline{W}Z = Z\overline{Z}W\overline{W} = D(Z)D(W).$$

Here the fact that $W\overline{W}$ is in the base ring which is commutative allows us to remove it from in between the appearance of $Z$ and $\overline{Z}$. Thus we see that iterants as $2 \times 2$ matrices form a direct non-commutative generalization of the complex numbers.

It is worth pointing out the first precursor to the quaternions (the so-called split quaternions):

This precursor is the system

$$\{\pm 1, \pm \epsilon, \pm \eta, \pm i\}.$$  

Here $\epsilon = [-1, 1]$ corresponds to a diagonal matrix with entries $-1$ and $1$. We have $\epsilon \epsilon = 1 = \eta \eta$ while $i = \epsilon \eta$ so that $ii = -1$. The basic operations in this algebra are those of $\epsilon$ and $\eta$. $\eta$ is the delay shift operator that reverses the components of the iterant. $\epsilon$ negates one of the components, and leaves the order unchanged. The quaternions arise directly from these two operations once we construct an extra square root of minus one that commutes with them. Call this extra root of minus one $\sqrt{-1}$. Then the quaternions are generated by

$$I = \sqrt{-1}\epsilon, J = \epsilon\eta, K = \sqrt{-1}\eta$$

with

$$I^2 = J^2 = K^2 = IJK = -1.$$  

One way to generate the quaternions is to start at the bottom iterant level with boolean values of 0 and 1 and the operation EXOR (exclusive or). Build iterants on this, and matrix algebra from these iterants. This gives the square root of negation. Now take pairs of values from this new algebra and build $2 \times 2$ matrices again. The coefficients include square roots of negation that commute with constructions at the next level and so quaternions appear in the third level of this hierarchy.
4. Iterants of Arbitrarily High Period
As a next example, consider a waveform of period three.

\[ \cdots abcabcabcabcabcabc \cdots \]

Here we see three natural iterant views (depending upon whether one starts at \(a\), \(b\) or \(c\)).

\[ [a, b, c], [b, c, a], [c, a, b]. \]

The appropriate shift operator is given by the formula

\[ [x, y, z]S = S[z, x, y]. \]

Thus, with \( T = S^2 \),

\[ [x, y, z]T = T[y, z, x] \]

and \( S^3 = 1 \). With this we obtain a closed algebra of iterants whose general element is of the form

\[ [a, b, c] + [d, e, f]S + [g, h, k]S^2 \]

where \( a, b, c, d, e, f, g, h, k \) are real or complex numbers. Call this algebra \( Vect_3R \) when the scalars are in a commutative ring with unit \( F \). Let \( M_3(F) \) denote the \( 3 \times 3 \) matrix algebra over \( F \). We have the

**Lemma.** The iterant algebra \( Vect_3(F) \) is isomorphic to the full \( 3 \times 3 \) matrix algebra \( M_3(F) \).

**Proof.** Map 1 to the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

Map \( S \) to the matrix

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]

and map \( S^2 \) to the matrix

\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\]

Map \([x, y, z]\) to the diagonal matrix

\[
\begin{pmatrix}
x & 0 & 0 \\
0 & y & 0 \\
0 & 0 & z
\end{pmatrix},
\]

Then it follows that

\[ [a, b, c] + [d, e, f]S + [g, h, k]S^2 \]

maps to the matrix

\[
\begin{pmatrix}
a & d & g \\
h & b & e \\
f & k & c
\end{pmatrix},
\]

preserving the algebra structure. Since any \( 3 \times 3 \) matrix can be written uniquely in this form, it follows that \( Vect_3(F) \) is isomorphic to the full \( 3 \times 3 \) matrix algebra \( M_3(F) \).
We can summarize the pattern behind this expression of $3 \times 3$ matrices by the following symbolic matrix.

$$
\begin{pmatrix}
1 & S & T \\
T & 1 & S \\
S & T & 1 \\
\end{pmatrix}
$$

Here the letter $T$ occupies the positions in the matrix that correspond to the permutation matrix that represents it, and the letter $T = S^2$ occupies the positions corresponding to its permutation matrix. The 1’s occupy the diagonal for the corresponding identity matrix. The iterant representation corresponds to writing the $3 \times 3$ matrix as a disjoint sum of these permutation matrices such that the matrices themselves are closed under multiplication. In this case the matrices form a permutation representation of the cyclic group of order 3, $C_3 = \{1, S, S^2\}$.

It should be clear to the reader that this construction generalizes directly for iterants of any period and hence for a set of operators forming a cyclic group of any order. In fact we can generalize further to any finite group $G$. See [ ] for more information about these generalizations.

(i) In this example we consider the group $G = C_2 \times C_2$, often called the “Klein 4-Group.”

We take $G = \{1, A, B, C\}$ where $A^2 = B^2 = C^2 = 1, AB = BA = C$. Thus $G$ has the multiplication table, which is also its $G$-Table for $\text{Vect}_4(G, F)$.

$$
\begin{pmatrix}
1 & A & B & C \\
A & 1 & C & B \\
B & C & 1 & A \\
C & B & A & 1 \\
\end{pmatrix}
$$

Thus we have the following permutation matrices that I shall call $E, A, B, C$:

$$
E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix},
C = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}.
$$

The reader will have no difficulty verifying that $A^2 = B^2 = C^2 = 1, AB = BA = C$. Recall that $[x, y, z, w]$ is iterant notation for the diagonal matrix

$$
[x, y, z, w] = \begin{pmatrix}
x & 0 & 0 & 1 \\
0 & y & 1 & 0 \\
0 & 1 & z & 0 \\
1 & 0 & 0 & w \\
\end{pmatrix}.
$$
Let \( \alpha = [1, -1, -1, 1], \beta = [1, 1, -1, -1], \gamma = [1, -1, 1, -1]. \)

And let \( I = \alpha A, J = \beta B, K = \gamma C. \)

Then the reader will have no trouble verifying that

\[
I^2 = J^2 = K^2 = IJK = -1, IJ = K, KI = -K.
\]

Thus we have constructed the quaternions as iterants in relation to the Klein Four Group. In Figure 1 we illustrate these quaternion generators with string diagrams for the permutations. The reader can check that the permutations correspond to the permutation matrices constructed for the Klein Four Group. For example, the permutation for \( I \) is (12)(34) in cycle notation, the permutation for \( J \) is (13)(24) and the permutation for \( K \) is (14)(23). In the Figure we attach signs to each string of the permutation. These “signed permutations” act exactly as the products of vectors and permutations that we use for the iterants. One can see that the quaternions arise naturally from the Klein Four Group by attaching signs to the generating permutations as we have done in this Figure.

(ii) One can use the quaternions as a linear basis for \( 4 \times 4 \) matrices just as our theorem would use the permutation matrices \( 1, A, B, C \). If we restrict to real scalars \( a, b, c, d \) such that \( a^2 + b^2 + c^2 + d^2 = 1 \), then the set of matrices of the form \( aI + bJ + cK + dK \) is isomorphic to the group \( SU(2) \). To see this, note that \( SU(2) \) is the set of matrices with complex entries \( z \) and \( w \) with determinant 1 so that \( z\bar{z} + w\bar{w} = 1 \).

Letting \( z = a + bi \) and \( w = c + di \), we have

\[
M = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.
\]

Letting \( z = a + bi \) and \( w = c + di \), we have

\[
M = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]
If we regard \( i = \sqrt{-1} \) as a commuting scalar, then we can write the generating matrices in terms of size two iterants and obtain

\[ I = \sqrt{-1} \epsilon, \quad J = \epsilon \eta, \quad K = \sqrt{-1} \eta \]

as described in the previous section. IF we regard these matrices with complex entries as shorthand for \( 4 \times 4 \) matrices with \( i \) interpreted as a \( 2 \times 2 \) matrix as we have done above, then these \( 4 \times 4 \) matrices representing the quaternions are exactly the ones we have constructed in relation to the Klein Four Group.

Since complex numbers commute with one another, we could consider iterants whose values are in the complex numbers. This is just like considering matrices whose entries are complex numbers. For this purpose we shall allow given a version of \( i \) that commutes with the iterant shift operator \( \eta \). Let this commuting \( i \) be denoted by \( \iota \). Then we are assuming that

\[ \iota^2 = -1, \quad \eta \iota = \iota \eta, \quad \eta^2 = +1. \]

We then consider iterant views of the form \([a + bi, c + di] + [a + bi, c + di] \eta = \eta [c + di, a + bi] \]

In particular, we have \( \epsilon = [1, -1] \), and \( i = \epsilon \eta \) is quite distinct from \( \iota \). Note, as before, that \( \epsilon \eta = -\eta \epsilon \) and that \( \epsilon^2 = 1 \). Now let

\[ I = \iota \epsilon, \quad J = \epsilon \eta, \quad K = \iota \eta. \]

We have used the commuting version of the square root of minus one in these definitions, and indeed we find the quaternions once more.

\[ I^2 = \iota \epsilon \iota \epsilon = \iota \epsilon \epsilon = (-1)(+1) = -1, \]
\[ J^2 = \epsilon \eta \epsilon \eta = \epsilon (-\epsilon) \eta \eta = -1, \]
\[ K^2 = \iota \eta \iota \eta = \iota \eta \eta = -1, \]
\[ IJK = \iota \epsilon \epsilon \eta \iota \eta = \iota \iota \eta \eta = \iota = -1. \]

Thus

\[ I^2 = J^2 = K^2 = IJK = -1. \]

This construction shows how the structure of the quaternions comes directly from the non-commutative structure of period two iterants. In other, words, quaternions can be represented by \( 2 \times 2 \) matrices. This is the way it has been presented in standard language. The group \( SU(2) \) of \( 2 \times 2 \) unitary matrices of determinant one is isomorphic to the quaternions of length one.

(iii) Similarly, \( H = [a, b] + [c + di, c - di] \eta = \begin{pmatrix} a & c + di \\ c - di & b \end{pmatrix} \)

represents a Hermitian \( 2 \times 2 \) matrix and hence an observable for quantum processes mediated by \( SU(2) \). Hermitian matrices have real eigenvalues.
If in the above Hermitian matrix form we take \( a = T + X, b = T - X, c = Y, d = Z \), then we obtain an iterant and/or matrix representation for a point in Minkowski spacetime.

\[
H = [T + X, T - X] + [Y + Zt, Y - Zt] = \begin{pmatrix} T + X & Y + Zt \\ Y - Zt & T - X \end{pmatrix}.
\]

Note that we have the formula

\[
Det(H) = T^2 - X^2 - Y^2 - Z^2.
\]

It is not hard to see that the eigenvalues of \( H \) are \( T \pm \sqrt{X^2 + Y^2 + Z^2} \). Thus, viewed as an observable, \( H \) can observe the time and the invariant spatial distance from the origin of the event \((T, X, Y, Z)\). At least at this very elementary juncture, quantum mechanics and special relativity are reconciled.

(iv) Hamilton's Quaternions are generated by iterants, as discussed above, and we can express them purely algebraically by writing the corresponding permutations as shown below.

\[
I = [+1, -1, -1, +1]s \\
J = [+1, +1, -1, -1]l \\
K = [+1, -1, +1, -1]t
\]

where

\[
s = (12)(34) \\
l = (13)(24) \\
t = (14)(23).
\]

Here we represent the permutations as products of transpositions \((ij)\). The transposition \((ij)\) interchanges \(i\) and \(j\), leaving all other elements of \(\{1, 2, ..., n\}\) fixed.

One can verify that

\[
I^2 = J^2 = K^2 = IJK = -1.
\]

For example,

\[
I^2 = [+1, -1, -1, +1]s[+1, -1, -1, +1]s \\
= [+1, -1, -1, +1][-1, +1, +1, -1]ss \\
= [-1, -1, -1, -1] \\
= -1.
\]

and

\[
IJ = [+1, -1, -1, +1]s[+1, +1, -1, -1]l \\
= [+1, -1, -1, +1][+1, +1, -1, -1]st \\
= [+1, -1, +1, -1](12)(34)(13)(24) \\
= [+1, -1, +1, -1](14)(23) \\
= [+1, -1, +1, -1]t.
\]
The reader will also note that we have moved into a different conceptual domain from an original emphasis in this paper on eigenform in relation to recursion. That is, we take an eigenform to mean a fixed point for a transformation. Thus $i$ is an eigenform for $R(x) = -1/x$. Indeed, each generating quaternion is an eigenform for the transformation $R(x) = -1/x$. The richness of the quaternions arises from the closed algebra that arises with its infinity of eigenforms that satisfy the equation $U^2 = -1$:

$$U = aI + bJ + cK$$

where $a^2 + b^2 + c^2 = 1$.

(v) In all these examples, we have the opportunity to interpret the iterants as short hand for matrix algebra based on permutation matrices, or as indicators of discrete processes. The discrete processes become more complex in proportion to the complexity of the groups used in the construction. We began with processes of order two, then considered cyclic groups of arbitrary order, then the symmetric group $S_3$ in relation to $6 \times 6$ matrices, and the Klein Four Group in relation to the quaternions. In the case of the quaternions, we know that this structure is intimately related to rotations of three and four dimensional space and many other geometric themes. It is worth reflecting on the possible significance of the underlying discrete dynamics for this geometry, topology and related physics.

5. Iterants and the Standard Model

In this section we shall give an iterant interpretation for the Lie algebra of the special unitary group $SU(3)$. The Lie algebra in question is denoted as $su(3)$ and is often described by a matrix basis. The Lie algebra $su(3)$ is generated by the following eight Gell Man Matrices [31].

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
We now give an iterant representation for these matrices that is based on the pattern
\[
\begin{pmatrix}
1 & A & B \\
B & 1 & A \\
A & B & 1
\end{pmatrix}
\]
as described in the previous section. That is, we use the cyclic group of order three to represent all \(3 \times 3\) matrices at iterants based on the permutation matrices
\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]
Recalling that \([a, b, c]\) as an iterant, denotes a diagonal matrix
\[
[a, b, c] = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix},
\]
the reader will have no difficulty verifying the following formulas for the Gell-Mann Matrices in the iterant format:
\[
\begin{align*}
\lambda_1 &= [1, 0, 0]A + [0, 1, 0]B \\
\lambda_2 &= [-i, 0, 0]A + [0, i, 0]B \\
\lambda_3 &= [1, -1, 0] \\
\lambda_4 &= [1, 0, 0]B + [0, 0, 1]A \\
\lambda_5 &= [i, 0, 0]B + [0, 0, -i]A \\
\lambda_6 &= [0, 1, 0]A + [0, 0, 1]B \\
\lambda_7 &= [0, -i, 0]A + [0, 0, i]B \\
\lambda_8 &= \frac{1}{\sqrt{3}}[1, 1, -2].
\end{align*}
\]
Letting \(F_a = \lambda_a/2\), we can now rewrite the Lie algebra into simple iterants of the form \([a, b, c]G\) where \(G\) is a cyclic group element. Compare with [32]. Let
\[
\begin{align*}
T_+ &= F_1 \pm iF_2, \\
U_+ &= F_6 \pm iF_7, \\
V_+ &= F_4 \pm iF_5, \\
T_3 &= F_3, \\
Y &= \frac{2}{\sqrt{3}}F_8.
\end{align*}
\]
Then we have the specific iterant formulas
\[
\begin{align*}
T_+ &= [1, 0, 0]A, \\
T_- &= [0, 1, 0]B.
\end{align*}
\]
\[ U_+ = [0, 1, 0]A, \]
\[ U_- = [0, 0, 1]B, \]
\[ V_+ = [0, 0, 1]A, \]
\[ V_- = [1, 0, 0]B, \]
\[ T_3 = [1/2, -1/2, 0], \]
\[ Y = \frac{1}{\sqrt{3}}[1, 1, -2]. \]

We have that \( A[x, y, z] = [y, z, x]A \) and \( B = A^2 = A^{-1} \) so that \( B[x, y, z] = [z, y, x]B \). Thus we have reduced the basic \( su(3) \) Lie algebra to a very elementary patterning of order three cyclic operations. In a subsequent paper, we will use this point to view to examine the irreducible representations of this algebra and to illuminate the Standard Model’s Eightfold Way.

6. The Dirac Equation and Majorana Fermions

We now construct the Dirac equation. This may sound circular, in that the fermions arise from solving the Dirac equation, but in fact the algebra underlying this equation has the same properties as the creation and annihilation algebra for fermions, so it is by way of this algebra that we will come to the Dirac equation. If the speed of light is equal to 1 (by convention), then energy \( E \), momentum \( p \) and mass \( m \) are related by the (Einstein) equation

\[ E^2 = p^2 + m^2. \]

Dirac constructed his equation by looking for an algebraic square root of \( p^2 + m^2 \) so that he could have a linear operator for \( E \) that would take the same role as the Hamiltonian in the Schroedinger equation. We will get to this operator by first taking the case where \( p \) is a scalar (we use one dimension of space and one dimension of time.). Let \( E = \alpha p + \beta m \) where \( \alpha \) and \( \beta \) are elements of a a possibly non-commutative, associative algebra. Then

\[ E^2 = \alpha^2 p^2 + \beta^2 m^2 + pm(\alpha \beta + \beta \alpha). \]

Hence we will satisfy \( E^2 = p^2 + m^2 \) if \( \alpha^2 = \beta^2 = 1 \) and \( \alpha \beta + \beta \alpha = 0 \). This is our familiar Clifford algebra pattern and we can use the iterant algebra generated by \( e \) and \( \eta \) if we wish. Then, because the quantum operator for momentum is \(-i\partial/\partial x\) and the operator for energy is \( i\partial/\partial t\), we have the Dirac equation

\[ i\partial \psi/\partial t = -i\alpha \partial \psi/\partial x + \beta m \psi. \]

Let

\[ \mathcal{O} = i\partial/\partial t + i\alpha \partial/\partial x - \beta m \]

so that the Dirac equation takes the form

\[ \mathcal{O} \psi(x, t) = 0. \]

Now note that

\[ \mathcal{O} e^{i(px-Et)} = (E - \alpha p - \beta m)e^{i(px-Et)}. \]

We let

\[ \Delta = (E - \alpha p - \beta m) \]
and let

\[ U = \Delta \beta \alpha = (E - \alpha p - \beta m) \beta \alpha = \beta \alpha E + \beta p - \alpha m, \]

then

\[ U^2 = -E^2 + p^2 + m^2 = 0. \]

This nilpotent element leads to a (plane wave) solution to the Dirac equation as follows: We have shown that

\[ \mathcal{O} \psi = \Delta \psi \]

for \( \psi = e^{i(px-Et)} \). It then follows that

\[ \mathcal{O}(\beta \alpha \Delta \beta \alpha \psi) = \Delta \beta \alpha \Delta \beta \alpha \psi = U^2 \psi = 0, \]

from which it follows that

\[ \psi = \beta \alpha U e^{i(px-Et)} \]

is a (plane wave) solution to the Dirac equation.

In fact, this calculation suggests that we should multiply the operator \( \mathcal{O} \) by \( \beta \alpha \) on the right, obtaining the operator

\[ \mathcal{D} = \mathcal{O} \beta \alpha = i \beta \alpha \partial / \partial t + i \beta \partial / \partial x - \alpha m, \]

and the equivalent Dirac equation

\[ \mathcal{D} \psi = 0. \]

In fact for the specific \( \psi \) above we will now have \( \mathcal{D}(U e^{i(px-Et)}) = U^2 e^{i(px-Et)} = 0 \). This idea of reconfiguring the Dirac equation in relation to nilpotent algebra elements \( U \) is due to Peter Rowlands [35]. Rowlands does this in the context of quaternion algebra. Note that the solution to the Dirac equation that we have found is expressed in Clifford algebra or iterant algebra form. It can be articulated into specific vector solutions by using an iterant or matrix representation of the algebra.

We see that \( U = \beta \alpha E + \beta p - \alpha m \) with \( U^2 = 0 \) is really the essence of this plane wave solution to the Dirac equation. This means that a natural non-commutative algebra arises directly and can be regarded as the essential information in a Fermion. It is natural to compare this algebra structure with algebra of creation and annihilation operators that occur in quantum field theory. To this end we recapitulate and start again in the next subsection.

6.1. \( U \) and \( U^\dagger \)

We start with \( \psi = e^{i(px-Et)} \) and the operators

\[ \hat{E} = i \partial / \partial t \]

and

\[ \hat{p} = -i \partial / \partial x \]

so that

\[ \hat{E} \psi = E \psi \]

and

\[ \hat{p} \psi = p \psi. \]

The Dirac operator is

\[ \mathcal{O} = \hat{E} - \alpha \hat{p} - \beta m. \]
and the modified Dirac operator is

\[ D = \mathcal{O} \beta \alpha = \beta \alpha \hat{E} + \beta \hat{p} - \alpha m, \]

so that

\[ D \psi = (\beta \alpha E + \beta p - \alpha m) \psi = U \psi. \]

If we let

\[ \tilde{\psi} = e^{i(px + Et)} \]

(reversing time), then we have

\[ D \tilde{\psi} = (-\beta \alpha E + \beta p - \alpha m) \psi = U^\dagger \tilde{\psi}, \]

giving a definition of \( U^\dagger \) corresponding to the anti-particle for \( U \psi \).

We have

\[ U = \beta \alpha E + \beta p - \alpha m \]

and

\[ U^\dagger = -\beta \alpha E + \beta p - \alpha m \]

Note that here we have

\[(U + U^\dagger)^2 = (2\beta p + \alpha m)^2 = 4(p^2 + m^2) = 4E^2,\]

and

\[(U - U^\dagger)^2 = -(2\beta \alpha E)^2 = -4E^2.\]

We have that

\[ U^2 = (U^\dagger)^2 = 0 \]

and

\[ UU^\dagger + U^\dagger U = 4E^2. \]

Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation. Furthermore, the decomposition of \( U \) and \( U^\dagger \) into the corresponding Majorana Fermion operators corresponds to \( E^2 = p^2 + m^2 \). Normalizing by dividing by \( 2E \) we have

\[ A = (\beta p + \alpha m)/E \]

and

\[ B = i\beta \alpha, \]

so that

\[ A^2 = B^2 = 1 \]

and

\[ AB + BA = 0. \]

then

\[ U = (A + Bi)E \]

and

\[ U^\dagger = (A - Bi)E, \]

showing how the Fermion operators are expressed in terms of the simpler Clifford algebra of Majorana operators (split quaternions once again).
6.2. Writing in the Full Dirac Algebra
We have written the Dirac equation so far in one dimension of space and one dimension of time. We give here a way to boost the formalism directly to three dimensions of space. We take an independent Clifford algebra generated by $\sigma_1, \sigma_2, \sigma_3$ with $\sigma_i^2 = 1$ for $i = 1, 2, 3$ and $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$. Now assume that $\alpha$ and $\beta$ as we have used them above generate an independent Clifford algebra that commutes with the algebra of the $\sigma_i$. Replace the scalar momentum $p$ by a 3-vector momentum $p = (p_1, p_2, p_3)$ and let $p \cdot \sigma = p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3$. We replace $\partial / \partial x$ with $\nabla = (\partial / \partial x_1, \partial / \partial x_2, \partial / \partial x_3)$ and $\partial p / \partial x$ with $\nabla \cdot p$.

We then have the following form of the Dirac equation.

$$i \frac{\partial \psi}{\partial t} = -i \alpha \nabla \cdot \sigma \psi + \beta m \psi.$$ 

Let

$$\mathcal{O} = i \frac{\partial}{\partial t} + i \alpha \nabla \cdot \sigma - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O} \psi (x, t) = 0.$$ 

In analogy to our previous discussion we let

$$\psi (x, t) = e^{i(p \cdot x - Et)}$$

and construct solutions by first applying the Dirac operator to this $\psi$. The two Clifford algebras interact to generalize directly the nilpotent solutions and Fermion algebra that we have detailed for one spatial dimension to this three dimensional case. To this purpose the modified Dirac operator is

$$\mathcal{D} = i \beta \alpha \partial / \partial t + \beta \nabla \cdot \sigma - \alpha m.$$ 

And we have that

$$\mathcal{D} \psi = U \psi$$

where

$$U = \beta \alpha E + \beta p \cdot \sigma - \alpha m.$$ 

We have that $U^2 = 0$ and $U \psi$ is a solution to the modified Dirac Equation, just as before. And just as before, we can articulate the structure of the Fermion operators and locate the corresponding Majorana Fermion operators. We leave these details to the reader.

6.3. Majorana Fermions
There is more to do. We will end with a brief discussion making Dirac algebra distinct from the one generated by $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3$ to obtain an equation that can have real solutions. This was the strategy that Majorana [28] followed to construct his Majorana Fermions. A real equation can have solutions that are invariant under complex conjugation and so can correspond to particles that are their own anti-particles. We will describe this Majorana algebra in terms of the split quaternions $\epsilon$ and $\eta$. For convenience we use the matrix representation given below. The reader of this paper can substitute the corresponding iterants.

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Let $\hat{\epsilon}$ and $\hat{\eta}$ generate another, independent algebra of split quaternions, commuting with the first algebra generated by $\epsilon$ and $\eta$. Then a totally real Majorana Dirac equation can be written as follows:

$$(\partial / \partial t + \hat{\eta} \partial / \partial x + \epsilon \partial / \partial y + \hat{\epsilon} \eta \partial / \partial z - \hat{\epsilon} \hat{\eta} m) \psi = 0.$$
To see that this is a correct Dirac equation, note that

\[ \hat{E} = \alpha_x \hat{p}_x + \alpha_y \hat{p}_y + \alpha_z \hat{p}_z + \beta m \]

(Here the “hats” denote the quantum differential operators corresponding to the energy and momentum.) will satisfy

\[ \hat{E}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + m^2 \]

if the algebra generated by \( \alpha_x, \alpha_y, \alpha_z, \beta \) has each generator of square one and each distinct pair of generators anti-commuting. From there we obtain the general Dirac equation by replacing \( \hat{E} \) by \( i \partial / \partial t \), and \( \hat{p}_x \) with \( -i \partial / \partial x \) (and same for \( y, z \)).

\[(i \partial / \partial t + i \alpha_x \partial / \partial x + i \alpha_y \partial / \partial y + i \alpha_z \partial / \partial y - \beta m) \psi = 0.\]

This is equivalent to

\[(\partial / \partial t + \alpha_x \partial / \partial x + \alpha_y \partial / \partial y + \alpha_z \partial / \partial y + i \beta m) \psi = 0.\]

Thus, here we take

\[ \alpha_x = \hat{\eta} \eta, \alpha_y = \epsilon, \alpha_z = \hat{\epsilon} \eta, \beta = i \hat{\epsilon} \hat{\eta}, \]

and observe that these elements satisfy the requirements for the Dirac algebra. Note how we have a significant interaction between the commuting square root of minus one \( i \) and the element \( \hat{\epsilon} \hat{\eta} \) of square minus one in the split quaternions. This brings us back to our original considerations about the source of the square root of minus one. Both viewpoints combine in the element \( \beta = i \hat{\epsilon} \hat{\eta} \) that makes this Majorana algebra work. Since the algebra appearing in the Majorana Dirac operator is constructed entirely from two commuting copies of the split quaternions, there is no appearance of the complex numbers, and when written out in \( 2 \times 2 \) matrices we obtain coupled real differential equations to be solved. Clearly this ending is actually a beginning of a new study of Majorana Fermions. That will begin in a sequel to the present paper.

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