Bi-quadratic polynomial approach for global convergent algorithm in high dimensions coefficient inverse problems

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Abstract. Sequential minimization algorithm in convexification approach established a stable approximate solution via minimizing a finite sequence of strictly convex objective function. Its application to 1D/2D dimension cases is reported in a great deal literatures. An interesting question should be asked is how about the development of the convex approach to high dimension case. That is to say, whether the approach could be applied to realistic three dimensions issue by a routing way. It becomes a much more challenging topic in present research. In this work, a newly modified global convergent algorithm for recovering the profile of coefficient inverse problems will be constructed by using so-called bi-quadratic polynomial (BQP) in 2D/3D cases. Hope the developed paradigm would be applied in a broad class application.

1. Introduction
Global convergent algorithm for coefficient inverse problems (CIP) has been used in solving a lot of problems. Our referred publications included [1],[2], [3],[4] and [8]. But for high dimension case, the approach is quite limited to be used for dealing with a variety of issues arising in real world. The motivation of this paper is to systematically develop a bi-quadratic polynomial (BQP) approach for constructing approximate solution in global convergent scheme for two/three dimensional cases. The methodology was originally proposed in [11] and relative works see [12] and [13]. Particularly, this work is to address a updated global convergent algorithm for solutions of coefficient inverse problems by using BQP approximate, see [14]. Fortunately, the later papers [5, 6, 7, 9] have used the method to give the sufficient evidences in advance.

The final goal of this paper is to apply the BQP approximate methodology for solving coefficient inverse problems in high dimension.

Let \( (x, t) \in \mathbb{R}^3 \times (0, \infty) \) for \( x = (x_1, x_2, x_3) \). In order to propose BQP approach (cf.[5]), we consider the Cauchy problem given by parabolic partial differential equation in the form of

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} - \nabla^2 u(x, t) + \rho(x) u(x, t) &= -s(x, t), \\
\frac{\partial u(x, t)}{\partial \eta} \bigg|_{t=0} + ku(x, t) \big|_{t=0} &= u_0(x),
\end{align*}
\]

where \( \rho(x) \geq 0 \) is an unknown coefficient, \( k \) is a constant and \( \frac{\partial}{\partial \eta} \) represents the unit outer
normal derivative in the Robin initial condition. Let \( R_+^3 = \{ x \in \mathbb{R}^3 \mid x_2 > 0 \text{ or } x_3 > 0 \} \) and \( R^-_+^3 = \{ x \in \mathbb{R}^3 \mid x_2 > 0 \text{ and } x_3 > 0 \} \). \( S = \{ (x_2, x_3) \mid x_1 = 0 \} \) denotes a plane. Assume that the source function \( s(x, t) = \delta(x - x_0)\delta(t) \geq 0 \) is compactly supported in \( \mathbb{R}^3 \). Let \( \Omega \subset \mathbb{R}^3_+ \) be a prism/quadrangle defined by \( \Omega = \{ x \in \mathbb{R}^3_+ : -R < x_1 < R, x_2 \in (0, L), x_3 \in (0, L) \} \). The support of \( s(x, t) \) belongs to the set \( \mathbb{R}^3_+ \setminus \Omega \). Using the convexification approach (cf. [4]) to denote the Laplace transform of \( u(x, t) \) by \( w(x, s) = \int_0^\infty e^{-st}u(x, t)dt \) for \( s > 0 \). Applying Laplace transform in Cauchy problem (1) to obtain

\[
\begin{cases}
[s + \rho(x)]w - \nabla^2w = S_0(x, s), \\
\lim_{|x| \to \infty} w = 0, \\
\frac{\partial w(x, s)}{\partial \eta} + kw(x, s) = \tilde{u}_0(x, s), \quad x \in S.
\end{cases}
\]  

(2)

Here \( \tilde{u}_0(x, s) \) and \( S_0(x, s) \) represents Laplace transforms of \( u_0 \) and source function \( s(x, t) \), respectively. For simplicity, assume that \( S_0(x, s) = \delta(x-x_0) \), where \( x_0 = (0, x_2, L-a) \in \mathbb{R}^3_+ \setminus \Omega \) is a fixed source position, and \( a \) is a small positive number. \( \frac{\partial w}{\partial \eta} \) is the outer normal derivative of \( w \) at the boundary, which corresponding to \( \frac{\partial u}{\partial \eta} \) in (1). Refer Lemma 1 and 2 in [4] for the positivity of \( w(x, s) \).

**CIP for system (1):** Given data on \( w(x, s), x \in S \) for a fixed source position and sufficiently large \( T \). The coefficient inverse problem (CIP) is to identify the unknown coefficient \( \rho(x) \) in \( \Omega \).

### 2. Bi-quadratic polynomial approach

Owing to the source function \( S_0(x, t) \) can be existed or not, so it’s convenient to neglect this term in following deduces. For \( w > 0 \) in \( \Omega \), let \( v = \ln w \) and transform (2) as

\[
\nabla^2v + (\nabla v)^2 = s + \rho(x), \quad x \in \Omega.
\]  

(3)

If introducing \( f(x, s) = v + s \cdot l(x, x_0), q(x, s) = \frac{\partial f}{\partial s}, \) then by formulation (6) in Lemma 2 of [4] to imply that \( f(x, s) = O(s^{-1}), q(x, s) = O(s^{-2}) \) as \( s \to \infty \). Hence to have approximate

\[
f(x, s) \approx -\int_\delta^\bar{s} q(x, \nu)d\nu,
\]

where \( \bar{s} \) is a sufficiently large number. For \( x \in \Omega, \Gamma = \partial \Omega \) and \( s \in (s_0, \bar{s}) \), the boundary problem (2) is converted to

\[
\begin{cases}
\int_\delta^\bar{s} \nabla^2q(x, \nu)d\nu + \left( \int_\delta^\bar{s} \nabla q(x, \nu)d\nu \right)^2 + s^2 - s = 0, \\
\frac{\partial q(x, s)}{\partial \eta} + kq(x, s) = q_0(x, s), \quad x \in \Gamma.
\end{cases}
\]  

(4)

Here \( q_0 \) correspond to \( \tilde{u}_0 \). For numerical computation, \( q_0 \) is calculated by using some regularizing scheme of numerical differential on \( s \). Obviously (4) does not contain the unknown coefficient \( \rho(x) \). Set \( H(m') = \{ q(x, s) \mid \frac{\partial q}{\partial \eta} + kq = q_0, x \in \Gamma, \quad \max_{s \in (s_0, \bar{s})} \| q(x, s) \|_{C^{\alpha}(\Omega)} \leq m' \} \), where \( m' > 0 \) is a known number.

**Existence:** For the problem (4), then there exists at most one solution \( q(x, s) \) such that \( \max_{s \in (s_0, \bar{s})} \| q(x, s) \|_{C^{\alpha}(\Omega)} < \infty \).
Let $\Omega = \Omega_1 \times [0,L] \times [0,L]$ for $L > 0$, where $\Omega_1 = \{ x_1 | - R \leq x_1 \leq R \}$. Suppose that $q_0$ is observed on $\Gamma = \Omega_1 \cap S \subset \{ x_2 = 0, x_3 = 0 \}$. Take $\{ \phi_k(x_1) \}_{k=1}^K \subset C^3(\Omega_1)$ as a set of linear (or quadratic, cubic) basis functions that approximate $q(x, s)$ and its $x$-derivatives up to the third order (cf. [10]), i.e.,

$$D_k^\alpha q(x, s) \approx D_k^\alpha \sum_{k=1}^K \eta_k(x_2, x_3, s) \phi_k(x_1), \quad (x, s) \in \bar{\Omega} \times [s_0, \bar{s}], \quad |\alpha| \leq 3. \quad (5)$$

Here, $K = n \times n$ is the total number of sub-rectangles. Then, for $q \in H(m')$, a number $n$ can be chosen corresponding to $K(\varepsilon, m')$ for sufficiently small $\varepsilon > 0$ and $\eta_k(x_2, x_3, s)$, such that $\max_{x \in (s_0, \bar{s})} \| q - \sum_{k=1}^K \eta_k(x_2, x_3, s) \phi_k(x_1) \|_{C^3(\bar{\Omega})} < \varepsilon$. Denote $p(x_2, x_3, s) = (\eta_1(x_2, x_3, s), \ldots, \eta_k(x_2, x_3, s), \ldots, \eta_K(x_2, x_3, s))^T$ and substitute (5) into (4). The result is quoted as in [4] to parabolic coefficient inverse problems. Let $q(x, s) \approx \sum_{k=1}^K \eta_k(x_2, x_3, s) \phi_k(x_1)$ for $(x, s) \in \bar{\Omega} \times [s_0, \bar{s}]$. Then we have

$$\mathcal{L}(p) = p''_1 + p''_2 + \int_s^\bar{s} F(p_1', p_2', \int_s^\nu p_1'(x_2, x_3, \nu) d\nu, \int_s^\nu p_2'(x_2, x_3, \nu) d\nu, \int_s^\nu p(x_2, x_3, \nu) d\nu, x_2, x_3).$$

From (4), it is easy to deduce that

$$\mathcal{L}(p) = 0, \quad p = p(x_2, x_3, s), x_2 \in (0, L), x_3 \in (0, L), s \in (s_0, \bar{s}),$$

$$\frac{\partial p(x_2, x_3, s)}{\partial \eta} + k p(x_2, x_3, s) = p_0(x_2, x_3, s), \quad (x_2, x_3) \in \Gamma. \quad (6)$$

Here $p_0$ corresponds to the initial value $q_0$ in (4).

Third kind boundary condition can be converted to Dirichlet boundary condition without describing the extrapolated boundary condition. In numerical approximate of finite element method (FEM) as in [10], for constructing the bi-quadratic polynomials as approximate solutions at each element, we discretize the coefficient inverse problems (6) on a rectangle $[0, L] \times [0, L]$ of two dimension domain. Denote $y = x_2, z = x_3$ with the grid

$$0 = y_0 < y_1 < \ldots < y_{n-1} < y_n = L, \quad h^i = y_i - y_{i-1},$$

$$0 = z_0 < z_1 < \ldots < z_{n-1} < z_n = L, \quad h^j = z_j - z_{j-1}.$$

The indexes $k = 1, 2, \ldots, K$ of $\eta_k(x_2, x_3, s)$ to the indexes $ij$ of $p_{ij}(y, z, s)$ have the corresponding relationship: $k = (i - 1)n + j$. See detailed array in Table 1. Using the Taylor expansion for three arguments of the function $f^T(y, z, s)$ at point $(y_0, z_0, s)$, then we obtain the polynomial in the form of

$$f^T(y, z, s) = \frac{\partial^2 f^T(y_0, z_0, s)}{\partial y^2}(y - y_0)^2 + \frac{\partial^2 f^T(y_0, z_0, s)}{\partial z^2}(z - z_0)^2 + \frac{\partial f^T(y_0, z_0, s)}{\partial y}(y - y_0)(z - z_0) + \frac{\partial f^T(y_0, z_0, s)}{\partial y}(z - z_0) + f^T(y_0, z_0, s). \quad (7)$$
Table 1. Bi-quadratic polynomials at each element

|   |   |   |   |   |
|---|---|---|---|---|
| $p_{11}$ | $p_{12}$ | $p_{1j}$ | $p_{1n}$ |   |
| $p_{21}$ | $p_{22}$ | $p_{2j}$ | $p_{2n}$ |   |
|   | $p_{i1}$ | $p_{i2}$ | $p_{ij}$ | $p_{in}$ |   |
|   | $p_{n1}$ | $p_{n2}$ | $p_{nj}$ | $p_{nn}$ |   |

Considering the Taylor expansion of $p_{ij}$ at point $(y_0, z_0, s)$, it is easy from (7) to guess the leading coefficients in the expansion of $p_{ij}$ as:

$$\frac{\partial^2 p_{ij}(y_0, z_0, s)}{\partial y^2} = a_i(s), \quad \frac{\partial^2 p_{ij}(y_0, z_0, s)}{\partial z^2} = b_j(s), \quad \frac{\partial^2 p_{ij}(y_0, z_0, s)}{\partial y \partial z} = c_{ij}(s),$$

where $a_i(s), b_j(s), c_{ij}(s)$ need to be determined by calculating at each element:

$$a_i(s) = \alpha_i s^2 + s, \quad b_j(s) = \beta_j s^2 + s, \quad c_{ij}(s) = \gamma_{ij} s^2 + s,$$

for constants $\alpha_i, \beta_j$ and $\gamma_{ij}$ in $\mathbb{R}^1$.

Therefore, we can approximate the component of the vector function $\mathbf{p}(x_2, x_3, s)$ via a quadratic polynomial in each sub-rectangle as follow:

$$\mathbf{p}(y, z, s) \approx p_{ij}(y, z, s)$$

$$= a_i(s)\frac{(y - y_{i-1})^2}{2} + b_j(s)\frac{(z - z_{j-1})^2}{2} + c_{ij}(s)(y - y_{i-1})(z - z_{j-1}) + p'_{ij}(y_{i-1}, z_{j-1}, s)(y - y_{i-1}) + p''_{ij}(y_{i-1}, z_{j-1}, s)(z - z_{j-1}) + p_i(y_{i-1}, z_{j-1}, s)$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$. For more detailed configuration, see reference [14].

By considering the minimization problem for the criteria function, we obtain the resulting functional with respect to the $(i, j)$-th leading coefficients $\alpha_i, \beta_j, \gamma_{ij}$ of the quadratic polynomial as:

$$J_{\lambda, i, j}(\mathbf{p}) = \int_{s_0}^{s} \int_{z_{j-1}}^{z_j} \int_{y_{i-1}}^{y_i} |\mathcal{L}(p_{ij}(y, z, \nu))|^2 C_{\lambda, i, j}^2(y, z) dy dz d\nu,$$

(8)

where $C_{\lambda, i, j}(y, z) = \exp[-\lambda(y - y_{i-1}) - \lambda(z - z_{j-1})]$ ($i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$) are Carleman’s weighted functions, which appear in Carleman estimates for the operator $\frac{d^2}{dy^2} + \frac{d^2}{dz^2}$.

The calculation of $J_{\lambda, i, j}$ in (8) can be executed by using Gauss Legendre quadrature. By taking norms as $\|a_i\| = \max_{s \in [a_i, b_i]} |a_i(s)|, \|b_j\| = \max_{s \in [a_i, b_i]} |b_j(s)|, \|c_{ij}\| = \max_{s \in [a_i, b_i]} |c_{ij}(s)|$, the sequential minimizers of $J_{\lambda, i, j}(p_{ij})$ will be searched on the bounded set $G_{abc}(m) = \{ (a_i(s), b_j(s), c_{ij}(s)) : \|a_i\| \leq m, \|b_j\| \leq m, \|c_{ij}\| \leq m \}$. Let the corresponding set of $p_{ij}$ be denoted by $G(m) = \{ g(s) : \|g\| \leq m \}$. By citing the result given in [4] and the convexity of $J_{\lambda, i, j}(\mathbf{p})$, the uniqueness of minimizers for these functionals and the convergence result can be proven certainly (cf. [14]).
Exact solution: Assume that there exists a vector function $p^*(y, z, s)$, such that

$$\begin{align*}
\tilde{L}(p^*) &= 0, \quad y \in (0, L), \quad z \in (0, L), \quad s \in (s_0, s), \\
\frac{\partial}{\partial \eta} p^*(y, z, s) + k p^*(y, z, s) &= p_0^*(s), \quad (y, z) \in \Gamma.
\end{align*}$$

where $\tilde{L}(p^*) \equiv p'' + p' + F^*(p''', p^*, \int_s^y p'(y, z, \nu) d\nu, \int_s^z p^*(y, z, \nu) d\nu, y, z)$. Then $p^*(y, z, s)$ is called the exact solution to (6).

Set $\Omega_0 = [0, L] \times [0, L]$, and assume that $p^* \in C^3(\Omega_0)$ satisfy $\max_{s \in (s_0, s)} \|p^*(y, z, s)\|_{C^3(\Omega_0)} \leq c_0 m$ for constant $c_0$. The function $F$ is known approximately by

$$F(p', p, \int_s^y p'(y, z, \nu) d\nu, \int_s^z p(y, z, \nu) d\nu, y, z) = F^*(p''', p^*, \int_s^y p''(y, z, \nu) d\nu, \int_s^z p^*(y, z, \nu) d\nu, y, z) + \tilde{\varepsilon}(y, z, s),$$

where $\max_{s \in [s_0, s]} \|\tilde{\varepsilon}(y, z, s)\|_{C^3(\Omega_0)} \leq \varepsilon$ and $\max_{s \in [s_0, s]} \|p_0(s) - p_0^*(s)\| \leq \varepsilon$. Denote that

$$F_{ij} = F(p'_{ij}, p_{ij}, \int_s^y p'_{ij}(y, z, \nu) d\nu, \int_s^z p_{ij}(y, z, \nu) d\nu, y, z) \bigg|_{y = y_i - 1, \quad z = z_j - 1},$$

$$F^*_{ij} = F^*(p''_{ij}', p_{ij}', \int_s^y p''_{ij} (y, z, \nu) d\nu, \int_s^z p_{ij} (y, z, \nu) d\nu, y, z) \bigg|_{y = y_i - 1, \quad z = z_j - 1}.$$

Existence of minimizer: Assume that all above assumptions are satisfied. Let $(a_0, b_0, c_{00}), (a_i, b_j, c_{ij})$ and $(a'_i, b'_j, c'_{ij})$ be the corresponding coefficients of $p_0^*, p_{ij}$ and $p_{ij}'$, respectively. Let $h_i, h_j$ be the discrete lengths of the subintervals at $y, z$ directions, respectively. Assume that $(a_0, b_0, c_{00}) \in G_{abc}(c_0 m)$. Then there exists sufficiently small positive numbers $h_{i0}(m), h_{j0}(m)$ such that for all $h_i \in (0, h_{i0}), h_j \in (0, h_{j0})$

$$(a_i, b_j, c_{ij}), (a'_i, b'_j, c'_{ij}) \in G_{abc}(m),$$

$p_{ij}(y_{i-1}, z_{j-1}, s), p'_{ij}(y_{i-1}, z_{j-1}, s) \in G(m)$,

$F_{ij} \in G(c'_0 m), \quad (i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n)$

All functionals $J_{\lambda, i, j}(p)$ are strictly convex on $G(m)$. The unique minimizer pairing $(\tilde{a}_i, \tilde{b}_j, \tilde{c}_{ij})$ of $J_{\lambda, i, j}(p)$ is an interior point of $G_{abc}(m)$. It can also be regarded as the solution of operation equation $(a_i, b_j, c_{ij}) = \Phi_{\lambda, i, j}(a_i, b_j, c_{ij})$ with the relationship: $\Phi_{\lambda, i, j} : G(m) \rightarrow G(m)$.

The proof in [14] can be obtained by citing [4], [5], [11] and [12].

Furthermore, it illustrated that the convergence of the bi-quadratic polynomials algorithm to the exact solution is guaranteed for initial guess $(a_0, b_0, c_{00})$ in the set $G_{abc}(m)$ without the lower bounded condition. It implies that the global convergence of the BQP paradigm in two-dimension.

Inversion: If the approximating field $\tilde{f}(x, \nu)$ is solved, the unknown coefficient $\rho(x)$ can be determined by (3) approximately. The inversion formula for $\tilde{\rho}(x)$ can be deduced as

$$\tilde{\rho}(x) = \nabla^2 \tilde{f} + (\nabla \tilde{f})^2 - s, \quad \forall s \in [s_0, s].$$

Hence, without needing to compute the solutions $u(x, t)$ for getting $\tilde{\rho}(x)$ (cf. [4]).

Due to the limit of page length, we omit the discussion on convergence and error estimation. We also neglect the computational procedure for simplification (cf. [14]).

Notice: For the construction of bi-quadratic polynomial in three dimensional case, we render to our further publication in details.
3. Conclusions and future work
This work solved the coefficient inverse problems of nonlinear parabolic partial differential equations (PDE) for two dimensional spaces using developed globally convergent algorithm (i.e., convexification approach). The newly proposed bi-quadratic polynomial (BQP) approach is presented for constructing approximate solution in high dimensions architecture.

Although we have worked out a great deal problems in the determination of unknown coefficient inverse problem by current level global convergent algorithm using the convexification approach as well as adaptive techniques in recovering property (i.e. profile) of materials, mechanical and medicine process. However, it still has fatal limition for realistic issues, which needs to be solved in future advantage global convergent algorithm. Some questions and answers is posed for referring.

• Q1: How about the theory and numerical evidence in 3 dimension?
  A1: In the case of unknown coefficient only dependent to spatial variable, current BQP, CIP and globally convergent algorithm would be extended and valid.

• Q2: How about the coefficient depended on the time $t$?
  A2: In that case, current methodology will not valid. It needs to establish new theory for various case arising in real world.

• Q3: Other solution for Q2?
  A3: Mathematically, one can develop a third path to convert the issue in Q2 to Q1 if any. It is also difficult obviously.

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