Simplex links in determinantal hypertrees

Andrew Vander Werf *

November 11, 2022

Abstract

We deduce a structurally inductive description of the determinantal probability measure associated with Kalai’s celebrated enumeration result for higher–dimensional spanning trees of the n−1–simplex. As a consequence, we derive the marginal distributions of the simplex links in such random trees. Along the way, we also characterize the higher–dimensional spanning trees of every other simplicial cone in terms of the higher–dimensional rooted forests of the underlying simplicial complex. We also apply these new results to random topology, the spectral analysis of random graphs, and the theory of high dimensional expanders. One particularly interesting corollary of these results is that the fundamental group of a union of o(log n) determinantal 2–trees has Kazhdan’s property (T) with high probability.

1 Introduction

Forty years ago, Kalai [Kal83] introduced, to spectacular effect, a generalization of the graph–theoretic notion of a tree to higher–dimensional simplicial complexes, called Q–acyclic simplicial complexes for the triviality of their rational reduced homology groups in every dimension. However, recent authors [LP19], [KN20], [Me21] appear to have settled on simply calling these hypertrees. For 0 ≤ k < n, let T n,k denote the set of k–dimensional hypertrees on the vertex set [n] := {1, 2, . . . n}. Kalai noticed, among other things, that \( \tilde{H}_{k-1}(T) \) (assume integer coefficients throughout) is a finite group for all \( T \in T_{n,k} \) and moreover that

\[
\sum_{T \in T_{n,k}} |\tilde{H}_{k-1}(T)|^2 = n^{(n-2)},
\]

which is seen to be a generalization of Caley’s formula by recalling that \( |\tilde{H}_0(T)| = 1 \) for all \( T \in T_{n,1} \), due to trees being connected. This suggests a natural probability measure [Ly03], [Ly09] \( \nu = \nu_{n,k} \), on \( T_{n,k} \) defined on atoms by \( \nu_{n,k}(T) = n^{-\binom{n-2}{k}} |\tilde{H}_{k-1}(T)|^2 \).

Seemingly unrelated to this measure, consider the k–dimensional Linial–Meshulam complex, denoted \( Y_k(n,p) \) and defined [LM06], [MW09] to be the random k–dimensional simplicial complex on [n] with full k−1–skeleton wherein each k–face is included independently and with probability \( p \). Let \( \mu_{n,k} \) denote the probability density for \( Y_k(n,(n+1)^{-1}) \).

Let \( T_{n,k} \) denote a random complex distributed according to \( \nu_{n,k} \), and let \( Y_{n,k} \) denote a random complex distributed according to \( \mu_{n,k} \). Our main result is the following structure theorem for random hypertrees distributed according to \( \nu_{n,k} \) which, for the special case of k–dimensional spanning trees of the n−1–simplex, answers in the affirmative a question posed by Lyons ([Ly03], Question 10.1) concerning the existence of natural disjoint–union couplings of certain determinantal measures:

**Theorem 1.** Assume that 1 ≤ k < n−1. There exists a coupling of \( T_{n,k}, T_{n-1,k}, T_{n-1,k-1}, Y_{n-1,k}, Y_{n-1,k-1} \) such that \( T_{n-1,k} \) and \( T_{n-1,k-1} \) are independent of \( Y_{n-1,k} \) and \( Y_{n-1,k-1} \) respectively, \( T_{n-1,k} \) and \( T_{n-1,k-1} \) are conditionally independent given \( T_{n,k} \), and

\[
T_{n,k} = \text{Cone}(n, T_{n-1,k-1} \cup Y_{n-1,k-1}) \cup T_{n-1,k} \setminus Y_{n-1,k}.
\]

*The author gratefully acknowledges partial support from NSF-DMS # 1547357.
We can easily identify this coned term as being the link of the vertex $n$ in $\mathcal{T}_{n,k}$. So, with this coupling, we have $\text{Link}(n, \mathcal{T}_{n,k}) \cong \mathcal{T}_{n-1,k-1} \cup \mathcal{Y}_{n-1,k-1}$. This can be taken to mean that a vertex link in $\mathcal{T}_{n,k}$ can be simulated by first sampling $\mathcal{T}_{n-1,k-1}$ and then adding each missing $k-1$–face independently with probability $1/n$. The remaining term in the displayed union gives a description of those $k$–faces in $\mathcal{T}_{n,k}$ which do not contain the vertex $n$. This set of faces can be simulated by sampling $\mathcal{T}_{n-1,k}$ and then deleting each of its $k$–faces independently with probability $1/n$. Note then that these two binomial processes, one of adding $k-1$–faces which are then coned with $n$ and one of deleting $k$–faces, must be correlated at least to the point of producing the same number of faces—this is because all hypertrees of a given dimension and vertex count have the same number of top–dimensional faces ([Kal83], Proposition 2).

This simple idea of decomposing a hypertree into two collections of faces—those which contain a designated vertex and those which do not—turns out to be quite powerful. However, the idea is nothing new. For example, we can see this decomposition in use by Linial and Peled [LP19] at Kalai’s suggestion to inductively construct collapsible hypertrees. It was noted there that if $Y \in \mathcal{T}_{n-1,k-1}$, $X \in \mathcal{T}_{n-1,k}$, and both are collapsible, then $X \cup \text{Cone}(n,Y) \in \mathcal{T}_{n,k}$ and is collapsible. Corollary 23 shows that this construction produces a hypertree whether or not $X$ and $Y$ are collapsible.

By iterating our formula for the law of a vertex link, we can also determine a similar expression for the law of the link of a simplex of arbitrary dimension.

**Theorem 2.** The link of the simplex $\{n-j+1, \ldots, n\}$ in $\mathcal{T}_{n,k}$ is equivalent in law to $\mathcal{T}_{n-j,k-j} \cup \mathcal{Y}_{k-j}(n-j, j/n)$ where these two $k-j$–dimensional complexes are independent.

**Proof.** We prove this by induction. The base case follows as discussed from Theorem 1. Inducting, we have

$$\text{Link}(\{n-j, \ldots, n\}, \mathcal{T}_{n,k}) \cong \text{Link}(n-j, \mathcal{T}_{n-j,k-j} \cup \mathcal{Y}_{k-j}(n-j, j/n))$$
$$= \text{Link}(n-j, \mathcal{T}_{n-j,k-j}) \cup \text{Link}(n-j, \mathcal{Y}_{k-j}(n-j, j/n))$$
$$\cong \mathcal{T}_{n-j-1,k-j-1} \cup \mathcal{Y}_{k-j-1}(n-j-1, (n-j) \cup \mathcal{Y}_{k-j-1}(n-j-1, j/n))$$
$$\cong \mathcal{T}_{n-j-1,k-j-1} \cup \mathcal{Y}_{k-j-1}\left(n-j-1, \frac{1}{n-j} + \frac{j}{n} - \frac{j}{n(n-j)}\right)$$
$$= \mathcal{T}_{n-j-1,k-j-1} \cup \mathcal{Y}_{k-j-1}\left(n-j-1, \frac{j+1}{n}\right).$$

$\square$

### 1.1 Applications to random topology

Theorem 2 has several applications to random topology. Indeed, Garland’s method [Gar73] and its refinements (see [Z96], [Z03], [Opp18], [Opp20]), Žuk’s criterion among them, have proven to be very effective tools for extracting global information about a pure $k$–dimensional simplicial complex using only information found in the $k-2$–dimensional links of the complex.

**Theorem 3** (Garland’s method). Let $X$ be a pure $k$–dimensional simplicial complex. Suppose that, for all $(k-2)$–faces $\tau \in X$, we have $\lambda^{(0)}(\text{Link}(\tau, X)) \geq 1 - \varepsilon > 0$. Then $\lambda^{(k-1)}(X) \geq 1 - k\varepsilon$.

**Theorem 4** (Žuk’s criterion). Let $X$ be a pure $2$–dimensional simplicial complex. Suppose that, for all $0$–faces $\tau \in X$, we have that $\lambda^{(0)}(\text{Link}(\tau, X)) > 1/2$ and $\text{Link}(\tau, X)$ is connected. Then the fundamental group $\pi_1(X)$ has Kazhdan’s property (T).

The $\lambda^{(k-1)}(X)$ mentioned in the above statement of Garland’s method refers to the smallest nonzero eigenvalue of the top–dimensional up–down Laplacian of $X$ under a particular weighted inner product (see [Lub18] or [GW14] for greater detail). This eigenvalue will be referred to as the *spectral gap* of $X$. In the special case where $k = 1$ and $X$ is a connected graph, $\lambda^{(0)}(X)$ corresponds to the second smallest eigenvalue of the reduced Laplacian of $X$. Fortunately, the spectral gap of a random graph, in one form or another,
shows that the number of superimposed independent as well as several other new results about Oli09 HKP19 and Corollary COL09 and a union bound and with equally high probability.

Particularly in computer science, there is a growing interest in generating families of graphs with large spectral gap, which are usually called expander graphs, and probabilistic constructions have been offered as a way to do this quickly and successfully with exceedingly high probability. For example, the authors of [HKP19] prove that the Erdős–Rényi random graph $G(n,p)$ (in particular the random infinite family $\{G(n,p(n))\}_{n \geq 1}$) achieves the same spectral gap as found in Lemma 38 and with equally high probability when $np \geq (1/2 + \delta) \log n$ for any fixed $\delta > 0$. Moreover, they show that the assumption $\delta > 0$ is necessary for this to hold. The best known expander graph families have been constructed explicitly and have asymptotically (with respect to vertex count) constant average degree. So, while the Erdős–Rényi random graph can succeed at being a reliable expander, it is only able to do so if it is allowed an expected average degree far exceeding $\frac{1}{2} \log n$.

Our characterization of the $k – 2$-dimensional link of a determinantal $(n,k)$-tree as the union of a determinantal $(n-1,k-1)$-tree with an independent $G(n-1,(k-1)/n)$ makes it, or perhaps the union of a small number of independent copies of it, a potential naturally occurring candidate for a random expander graph with constant expected average degree. Lemma 38 shows that the number of superimposed independent copies of this graph required to match the result for $G(n,p)$ is no more than $\delta \log n$ for every $\delta > 0$. In particular, the resulting graph has an expected average degree of around $(k + 1)\delta \log n$, improving upon $G(n,p)$’s necessary expected average degree by a factor of arbitrary finite size.

1.2 Outline

Section 2 is primarily devoted to establishing notation and basic homological definitions as well as providing background for the deterministic study of simplicial spanning trees. The only truly novel result of this section is Theorem 18. In section 3 the results of section 2 are applied to the spanning trees of the $n-1$-simplex to give Theorem 1 as well as several other new results about $\nu_{n,k}$. Section 4 establishes the tools required to prove Proposition 5 and Corollary 6.

2 Homological trees: simplicial and relative

A chain complex is a sequence of abelian groups, $\{C_j\}_{j \in \mathbb{Z}}$, called chain groups, linked by group homomorphisms $\partial_j : C_j \to C_{j-1}$ called boundary maps which satisfy $\partial_j \partial_{j+1} = 0$ for all $j \in \mathbb{Z}$, or equivalently $\text{Ker} \partial_j \supseteq \text{Im} \partial_{j+1}$. The $j$th homology group of a chain complex such as this is defined to be the quotient group $\text{Ker} \partial_j / \text{Im} \partial_{j+1}$. Since we will only be considering finitely–generated free $\mathbb{Z}$–modules for our chain groups, we can represent these boundary maps by integer matrices. By the structure theorem for finitely–generated abelian groups, the $j$th homology group can be expressed as the direct sum of a free abelian group, which is isomorphic to $\mathbb{Z}^{\beta_j}$ and called its free part, and a finite abelian group called its torsion subgroup which is a direct sum of finite cyclic groups. The rank $\beta_j$ of the free part is called the $j$th Betti number. We will
require the following standard fact from homological algebra which gives two equivalent formulas for what is commonly called the Euler characteristic of a chain complex.

Lemma 7. Suppose $(C_\#, \partial_\#)$ is a chain complex such that each chain group is freely and finitely generated and only finitely many of the chain groups are nontrivial. Let $f_j$ denote the rank of $C_j$ for each $j \in \mathbb{Z}$. Then

$$
\sum_{j \in \mathbb{Z}} (-1)^j f_j = \sum_{j \in \mathbb{Z}} (-1)^j \beta_j.
$$

2.1 Simplicial

Fix an integer $n \geq 1$. The set $[n] := \{1, 2, \ldots, n\}$ will be our vertex set. For $-1 \leq j \leq n-1$, a $j$–dimensional abstract simplex, or $j$–face, is a subset of $[n]$ with cardinality $j+1$. We denote the set of all $j$–faces on $[n]$ by $\binom{[n]}{j+1}$. All faces will be oriented according to the usual ordering on $[n]$. As such, we will be denoting elements of $\binom{[n]}{j+1}$ by $\{\tau_0, \tau_1, \ldots, \tau_j\}$, where it is to be implicitly understood that $1 \leq \tau_0 < \tau_1 < \cdots < \tau_j \leq n$.

Let $\partial = \partial_{k+1}$ be the matrix that’s rows and columns are indexed respectively by $\binom{[n]}{k}$ and $\binom{[n]}{k+1}$, and for which, given $\sigma \in \binom{[n]}{k}$ and $\tau \in \binom{[n]}{k+1}$, we set

$$
\partial(\sigma, \tau) := \begin{cases} 
(-1)^j, & \sigma = \tau \setminus \{\tau_j\} \\
0, & \text{otherwise}
\end{cases}
$$

(1)
to account for our choice of orientation for each face. In particular, $\partial_0^{[n]}$ is the $\binom{[n]}{0} \times \binom{[n]}{1}$ all–ones matrix.

An abstract simplicial complex with vertices in $[n]$ is a nonempty subset $X \subseteq \bigcup_{j \geq -1} \binom{[n]}{j+1}$ which, for every pair of subsets $\sigma \subseteq \tau$, satisfies $\tau \in X \implies \sigma \in X$. Let $\mathcal{A}_n$ denote the set of all abstract simplicial complexes on $[n]$. It is easily verified with this definition that $\mathcal{A}_n$ is closed under intersection and union.

We write $X_j := \binom{[n]}{j+1} \cap X$ and define the dimension of $X$ to be $\dim X := \sup\{k \geq -1 : X_k \neq \emptyset\}$. If $X$ has dimension $k$, $X$ is said to be pure if, for every $-1 \leq j \leq k$ and every $j$–face $\sigma \in X$, there exists a $\tau \in X_k$ such that $\sigma \subseteq \tau$. An important example is $K_n^k := \bigcup_{j=0}^{k+1} \binom{[n]}{j}$, the complete $k$–dimensional complex on $[n]$.

Given a matrix $M$ with entries indexed over the set $S \times T$, for $A \subseteq S$ and $B \subseteq T$, we write $M_{A,B}$ to denote the submatrix of $M$ with rows indexed by $A$ and columns indexed by $B$. We will also occasionally write $M_{A,B}$ for the case $A = S$. As a point of clarification for this notation, transposes are handled by the convention of writing $M_{A,B}^t$ to mean $(M_{A,B})^t = (M^t)_{B,A}$.

Given $X \subseteq \mathcal{A}_n$, we define its chain complex $(C_\#, \partial_\#)$ by $C_j(X) := \mathbb{Z}^X_j$, and $\partial^X := \partial_{X,j-1} : X_j \to X_{j-1}$. Note then that we have $C_j(X) = 0$ for all $j \geq n$ and all $j < -1$—because $|X_{-1}| = |\{\emptyset\}| = 1$. We denote the $j$th homology group of this chain complex by $H_j(X)$. It is equivalent to the $j$th reduced simplicial homology group, hence the notation. Note that, with this chain sequence, it makes sense to consider $H_{-1} = \ker \partial_{-1} / \text{Im} \partial_0^{[n]}$ as well, although we can see that this group is always trivial since $\partial_0^{[n]}$ is surjective.

Lemma 8. $\partial_k^{[n]} \partial_k^{[n]} + \partial_{k-1}^{[n]} \partial_{k-1}^{[n]} = n \text{Id}$.

Proof. This follows by explicit computation of matrix entries via (1) combined with the chain sequence identity $\partial_k^{[n]} \partial_{k-1}^{[n]} = 0$ which can also be seen to hold by explicit computation of matrix entries via (1). A more detailed explanation along these lines for the cases $k \geq 2$ can be found in the proof of Lemma 3 in [Kal83], and the case $k = 0$ is straightforward. For the case $k = 1$, $\partial_0^{[n]} \partial_0^{[n]}$ is the combinatorial Laplacian of the complete graph on $[n]$. Hence $\partial_0^{[n]} \partial_0^{[n]} = n \text{Id} - J$ where $J$ is the $n \times n$ all–ones matrix. Noticing that $\partial_0^{[n]} \partial_0^{[n]} = J$ then completes the proof.

Fix a $\Delta \subseteq \mathcal{A}_n$ and set $\mathcal{C}_{n,k}(\Delta) := \{X \in \mathcal{A}_n : K_n^{k-1} \cap \Delta \subseteq X \subseteq K_n^{k} \cap \Delta\}$. We will call elements of this set the $(n, k)$–complexes of $\Delta$, or just $k$–complexes when the context is clear. Note this definition makes sense even if $\dim \Delta > k$, but in this case $\mathcal{C}_{n,k}(\Delta) = \mathcal{C}_{n,k}(K_n^k \cap \Delta)$. Building upon work done by Duval, Klivans,
and Martin [DKM08], [DKM11], [DKM15], Bernardi and Klivans [BK15] define a higher–dimensional forest of \( \Delta \) to be a subset \( F \subseteq \Delta \) such that \( \partial_{\Delta_{k-1},F} \) is injective. Our definition of a forest will be equivalent to this one except that we will consider \( F \) to be an entire element of \( \mathcal{F}_{n,k}(\Delta) \) by adding to it the \( k-1 \)–skeleton of \( \Delta \), more similar to the situation in [DKM08]. That is to say, an \( F \in \mathcal{F}_{n,k}(\Delta) \) is called a \( k \)–forest of \( \Delta \) if \( \beta_k(F) = 0 \), in which case we write \( F \in \mathcal{F}_{n,k}(\Delta) \). It is easily verified that this is equivalent to Definition 3 in [BK15] by noting that \( \tilde{H}_k(F) = \text{Ker}\tilde{\partial}_k^F \) is a free group and thus is 0 if and only if its rank is 0. A \( k \)–forest \( T \in \mathcal{F}_{n,k}(\Delta) \) with \( |T_k| = \text{rank}\tilde{\partial}_k^F \) (the maximal possible value for a forest) is said to be a \( k \)–tree of \( \Delta \). Let \( \mathcal{T}_{n,k}(\Delta) \) denote the set of \( k \)–trees of \( \Delta \)—[BK15] calls these the spanning forests of \( \Delta \). In the case \( \Delta = K_{n}^{n-1} \), or just \( \Delta \supseteq K_{n}^{n} \), these are Kalai’s \( k \)–dimensional \( \mathbb{Q} \)–acyclic simplicial complexes mentioned in the introduction, and in this case we will suppress the \( (\Delta) \) in all the above notation. The following result from [DKM15] generalizes Proposition 2 from [Kal83] to general \( k \)–trees of \( \Delta \in \mathcal{A}_n^2 \):

**Lemma 9.** For \( X \in \mathcal{F}_{n,k}(\Delta) \), if any two of the following conditions hold, then so does the other condition and moreover \( X \in \mathcal{T}_{n,k}(\Delta) \):

- \( |X_k| = \text{rank}\tilde{\partial}_k^F \),
- \( \beta_{k-1}(X) = \beta_{k-1}(\Delta) \),
- \( \beta_k(X) = 0 \).

### 2.2 Relative

Given a pair \( (X, Y) \in \mathcal{A}_n^2 \) with \( X \supseteq Y \), we can define another chain complex by \( \mathcal{C}_j(X, Y) := \mathbb{Z}^X \setminus Y_j \) with boundary maps \( \partial_j^{X/Y} := \partial_j^{X \setminus Y} \). The homology groups of this chain complex are called relative homology groups and denoted \( H_j(X, Y) \). We can recover ordinary reduced homology from this by taking \( Y \) to be \( \{ \emptyset \} \) or any complex with a single 0–face and no larger faces. Moreover, since homology is homotopy invariant, we also have that \( H_j(Y, X) \cong H_j(X) \) as long as \( Y \) is contractible.

An example of a contractible complex, the simplicial cone of a complex \( \Delta \in \mathcal{A}_{n-1} \) is defined by \( \text{Cone}(n, \Delta) := \Delta \cup \{ \sigma \cup \{n\} : \sigma \in \Delta \} \in \mathcal{A}_n \). We also define for any \( X \in \mathcal{F}_{n,k}(\text{Cone}(n, \Delta)) \) the complexes \( \text{Proj}(n, X) := X \cap \Delta \in \mathcal{F}_{n-1,k}(\Delta) \) and \( \text{Link}(n, X) := \{ \sigma \in \Delta: \sigma \cap \{n\} \in X \} \in \mathcal{F}_{n-1,k-1}(\Delta) \). The next lemma defines what we will call the binomial correspondence \( \mathcal{C}_{n,k}(\text{Cone}(n, \Delta)) \xrightarrow{\sim} \mathcal{C}_{n-1,k}(\Delta) \times \mathcal{C}_{n-1,k-1}(\Delta) \) for its relationship to the binomial recurrence formula \( \binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n-1}{k} \) in the special case that \( \text{Cone}(n, \Delta) \supseteq K_n^k \).

**Lemma 10.** For \( \Delta \in \mathcal{A}_{n-1} \), the map \( \varphi(X) := \{ \text{Proj}(n, X), \text{Link}(n, X) \} \) is an injection from \( \mathcal{F}_{n,k}(\text{Cone}(n, \Delta)) \) into \( \mathcal{C}_{n-1,k}(\Delta) \times \mathcal{C}_{n-1,k-1}(\Delta) \), and its (left) inverse is given by \( \varphi^{-1}(F, R) = F \cup \text{Cone}(n, R) \) which extends to an injection of \( \mathcal{C}_{n-1,k}(\Delta) \times \mathcal{C}_{n-1,k-1}(\Delta) \) into \( \mathcal{C}_{n,k}(\text{Cone}(n, \Delta)) \).

**Proof.** For brevity, set \( \Delta_{(j)} := K_{n}^{j} \cap \Delta \). Since \( \varphi^{-1} \) is clearly also a right inverse of \( \varphi \), the only thing to show is that the images of these restrictions lie where we claim they do. Starting with \( \varphi \), we first recall that

\[
\mathcal{C}_{n,k}(\text{Cone}(n, \Delta)) = \mathcal{C}_{n,k}(\text{Cone}(n, \Delta) |_{(k)}) = \mathcal{C}_{n,k}(\Delta) \cup \{ \sigma \cup \{n\} : \sigma \in \Delta_{(k-1)} \}.
\]

So for \( X \in \mathcal{F}_{n,k}(\text{Cone}(n, \Delta)) \) we have

\[
\Delta_{(k-1)} \cup \{ \sigma \cup \{n\} : \sigma \in \Delta_{(k-2)} \} \subseteq X \subseteq \Delta_{(k)} \cup \{ \sigma \cup \{n\} : \sigma \in \Delta_{(k-1)} \}.
\]

Thus \( \Delta_{(k-1)} \subseteq \text{Proj}(n, X) \subseteq \Delta_{(k)} \) and \( \Delta_{(k-2)} \subseteq \text{Link}(n, X) \subseteq \Delta_{(k-1)} \) as desired. As for \( \varphi^{-1} \), we have

\[
\text{Cone}(n, \Delta) |_{(k-1)} = \Delta_{(k-1)} \cup \text{Cone}(n, \Delta_{(k-2)}) \subseteq F \cup \text{Cone}(n, R) \subseteq \text{Cone}(n, \Delta) |_{(k)}
\]

due to the assumption that \( (F, R) \in \mathcal{C}_{n-1,k}(\Delta) \times \mathcal{C}_{n-1,k-1}(\Delta) \).

**Theorem 11** (Excision Theorem). For any \( A, B \in \mathcal{A}_n \), we have \( H_*(A, A \cap B) \cong H_*(A \cup B, B) \).
Theorem 12 (Long exact sequence of a triple). For \( A \subseteq B \subseteq C \in \mathcal{A}_n \), arrows exist for which the following sequence is exact:

\[
\cdots \rightarrow H_k(B, A) \rightarrow H_k(C, A) \rightarrow H_k(C, B) \rightarrow H_{k-1}(B, A) \leftarrow \cdots \rightarrow H_0(C, B) \rightarrow 0.
\]

That is, the kernel of each arrow in this diagram is equal to the image of the preceding arrow.

Lemma 13. For any \( \Delta \in \mathcal{A}_n \), \( X \in \mathcal{C}_{n,k}(\Delta) \), and \( Y \in \mathcal{C}_{n,j}(X) \), any two of the following conditions imply the other:

- \(|X_k| - |Y_k| = \text{rank } \partial_k^{\Delta/Y}\),
- \(\beta_{k-1}(X,Y) = \beta_{k-1}(\Delta,Y)\),
- \(\beta_k(X,Y) = 0\).

Proof. The long exact sequence of the triple \((\Delta, X, Y)\) contains the sequence

\[
0 \rightarrow H_{k+1}(\Delta, Y) \rightarrow H_{k+1}(\Delta, X) \rightarrow H_k(\Delta, Y) \rightarrow H_k(\Delta, X) \rightarrow 0.
\]

The exactness of this sequence implies that

\[
\beta_k(X,Y) + (\beta_{k+1}(\Delta, Y) - \beta_{k+1}(\Delta, X) - \beta_k(\Delta, Y) + \beta_k(\Delta, X)) - (\beta_{k-1}(X,Y) - \beta_{k-1}(\Delta,Y)) = 0. \tag{2}
\]

It suffices to show that the first bulleted condition is equivalent to the vanishing of the middle term in \((2)\). Indeed, by the definition of relative homology, the rank–nullity theorem, dimensional considerations, and the fact that \(\Delta, X\) have the same \(k-1\)-skeleton, we have

\[
\beta_k(\Delta, X) - \beta_{k+1}(\Delta, X) = \dim \ker \partial_k^{\Delta/X} - (\text{rank } \partial_{k+1}^{\Delta/X} + \dim \ker \partial_k^{\Delta/X}) + \text{rank } \partial_{k+2}^{\Delta/X}
\]

\[
= \dim \ker \partial_k^{\Delta/X} - |(\Delta/X)_{k+1}| + \text{rank } \partial_{k+2}^{\Delta/X}
\]

\[
= |(\Delta/X)_k| - |\Delta_{k+1}| + \text{rank } \partial_{k+2}^{\Delta/X}.
\]

By most of the same considerations, \(\beta_k(\Delta,Y) - \beta_{k+1}(\Delta,Y) = \dim \ker \partial_k^{\Delta/Y} - |\Delta_{k+1}| + \text{rank } \partial_{k+2}^{\Delta/Y}\). So

\[
\beta_{k+1}(\Delta,Y) - \beta_{k+1}(\Delta,X) - \beta_k(\Delta,Y) + \beta_k(\Delta,X) = |(\Delta/Y)_k| - \text{dim } \ker \partial_k^{\Delta/Y}.
\]

\[
= |Y_k| + |(\Delta/Y)_k| - \text{dim } \ker \partial_k^{\Delta/Y} - |X_k|
\]

\[
= |Y_k| + \text{rank } \partial_k^{\Delta/Y} - |X_k|
\]

is equal to 0 if and only if the first bulleted condition holds. The result therefore follows from \((2)\). \(\square\)

Whenever two, and therefore all three, of the conditions of this lemma hold, we will call \((X, Y)\) a relative \((n, k)\)-forest of \(\Delta\). We will denote the set of such pairs by \(\mathcal{S}_{n,k}^{rel}(\Delta)\). In the special case that \(Y\) is contractible (or just that \(\dim Y < k - 2\)) we get that \(X \in \mathcal{F}_{n,k}(\Delta)\) if and only if \((X,Y) \in \mathcal{S}_{n,k}^{rel}(\Delta)\). We therefore have the following corollary which gives the original [Kal83] three equivalent necessary and sufficient pairs of conditions for a \(k\)-complex of a homologically connected \(\Delta\) (i.e. \(\beta_{k-1}(\Delta) = 0\)) to be a \(k\)-tree of \(\Delta\):
Corollary 14. For any $\Delta \in \mathcal{A}_n$ with $\beta_{k-1}(\Delta) = 0$ and $X \in \mathcal{C}_{n,k}(\Delta)$, any two of the following conditions imply the other, and in particular are equivalent to the statement that $X \in \mathcal{T}_{n,k}(\Delta)$:

- $|X_k| = \binom{n-1}{k}$,
- $\beta_k(X) = 0$,
- $\beta_{k-1}(X) = 0$.

An $R \in \mathcal{C}_{n,k-1}(\Delta)$ is called a $k-1$-root of $\Delta$ if $\partial_k^{X/R}$ is surjective and has full rank ([BK15], Definition 9). Whenever any two of the conditions in the following lemma hold for a pair $(X, Y) \in \mathcal{C}_{n,k}(\Delta) \times \mathcal{C}_{n,k-1}(\Delta)$, we will call $(X, Y)$ a rooted $(n, k)$-forest of $\Delta$ and write $(X, Y) \in \mathcal{T}_{n,k}(\Delta)$. Equivalently, a rooted $(n, k)$-forest of $\Delta$ is a pair $(X, Y) \in \mathcal{T}_{n,k}(\Delta) \times \mathcal{C}_{n,k-1}(\Delta)$ such that $Y$ is a $k-1$-root of $X$ ([BK15], Definition 12).

Lemma 15. For $X \in \mathcal{C}_{n,k}(\Delta)$ and $Y \in \mathcal{C}_{n,k-1}(\Delta)$, any two of the following conditions imply the other:

- $|X_k| = |X_{k-1} \setminus Y_{k-1}|$,
- $\beta_{k-1}(X, Y) = 0$,
- $\beta_k(X, Y) = 0$.

Proof. By Lemma 7, we have $\sum_{j=-1}^{k} (-1)^j |X_j \setminus Y_j| = \sum_{j=-1}^{k} (-1)^j \beta_j(X, Y)$, which simplifies to $|X_k| - |X_{k-1} \setminus Y_{k-1}| = \beta_k(X, Y) - \beta_{k-1}(X, Y)$.

Indeed, since $Y_j = X_j = \Delta_j$ for all $j < k - 1$, we have $C_j(X, Y) = 0$ for all $j < k - 1$. As $H_j(X, Y)$ is a quotient of a subgroup of $C_j(X, Y)$, the former vanishes with the latter.

Corollary 16. For any $\Delta \in \mathcal{A}_n$, we have

$$\mathcal{T}_{n,k}^{\text{root}}(\Delta) = \{(X, Y) \in \mathcal{T}_{n,k}(\Delta) \cap (\mathcal{C}_{n,k}(\Delta) \times \mathcal{C}_{n,k-1}(\Delta)) : \beta_{k-1}(\Delta, Y) = 0\}.$$

Proof. Clearly $\mathcal{T}_{n,k}^{\text{root}}(\Delta) \supseteq \{(X, Y) \in \mathcal{T}_{n,k}(\Delta) \cap (\mathcal{C}_{n,k}(\Delta) \times \mathcal{C}_{n,k-1}(\Delta)) : \beta_{k-1}(\Delta, Y) = 0\}$. For the other inclusion, suppose that $(X, Y) \in \mathcal{T}_{n,k}^{\text{root}}(\Delta)$. Then the long exact sequence in the proof of Lemma 13 implies that $0 = \beta_{k-1}(X, Y) \geq \beta_{k-1}(\Delta, Y) \geq 0$. Thus $(X, Y) \in \mathcal{T}_{n,k}(\Delta)$ since $\beta_{k-1}(X, Y) = 0 = \beta_{k-1}(\Delta, Y)$ and $\beta_k(X, Y) = 0$.

Lemma 17. Suppose $\Delta \in \mathcal{A}_n$ and that $F \in \mathcal{C}_{n,k}(\Delta)$, $R \in \mathcal{C}_{n,k-1}(\Delta)$ satisfy $|F_k| = |\Delta_{k-1} \setminus R_{k-1}|$, and write $\overline{R} := \Delta_{k-1} \setminus R_{k-1}$. Then $\det \partial_{F_k} \neq 0 \iff \det \partial_{\overline{R}_{k-1}} = |H_{k-1}(\overline{F}, R)| \iff (F, R) \in \mathcal{T}_{n,k}^{\text{root}}(\Delta)$.

Proof. The relative chain complex for the pair $(F, R)$ is $0 \rightarrow C_k(F) \xrightarrow{\partial_{F_k}} C_{k-1}(R) \rightarrow \cdots$ since $R_k = \emptyset$, $F_{k-1} \setminus R_{k-1} = \Delta_{k-1} \setminus R_{k-1} = \overline{R}$, and $F_{k-2} = R_{k-2}$. We therefore have $H_{k-1}(F, R) = \mathbb{Z}^{n}/\partial_{F_k} \mathbb{Z}^{F_k}$ and so, provided $\det \partial_{F_k} \neq 0$, we have $|H_{k-1}(F, R)| = |\det \partial_{F_k}|$—this is seen most easily by putting $\partial_{F_k}$ in Smith normal form. Having $|H_{k-1}(F, R)| = |\det \partial_{F_k}|$ clearly implies $|H_{k-1}(F, R)| < \infty$ which, by the previous lemma, implies that $(F, R) \in \mathcal{T}_{n,k}^{\text{root}}(\Delta)$. Finally, $(F, R) \in \mathcal{T}_{n,k}^{\text{root}}(\Delta) \implies \beta_k(F, R) = 0 \implies \ker \partial_{F_k} = H_k(F, R) = 0 \implies \det \partial_{F_k} \neq 0$.

Essentially all of the content of this last lemma is proven in greater detail in Lemmas 14 and 17 of [BK15].

Theorem 18. Set $T = F \cup \text{Cone}(n, R)$ where $F \in \mathcal{C}_{n-1,k}(\Delta)$ and $R \in \mathcal{C}_{n-1,k-1}(\Delta)$ for any $\Delta \in \mathcal{A}_{n-1}$. Then $H_n(T) \cong H_n(F, R)$, and $T \in \mathcal{T}_{n,k}(\text{Cone}(n, \Delta))$ if and only if $(F, R) \in \mathcal{T}_{n,k}^{\text{root}}(\Delta)$. Moreover the binomial correspondence restricts to a bijection $\varphi : \mathcal{T}_{n,k}(\text{Cone}(n, \Delta)) \xrightarrow{\sim} \mathcal{T}_{n,k}^{\text{root}}(\Delta)$. 

7
Proof. Since $F$ has no $k$–faces containing $n$ and $R \subseteq F$, we have $F \cap \text{Cone}(n, R) = R$. Thus, by excision, the pairs $(T, \text{Cone}(n, R))$ and $(F, R)$ have the same relative homology in every dimension. But, since $\text{Cone}(n, R)$ is contractible, $H_\ast(T) \cong H_\ast(F, R)$. Therefore $T \in \mathcal{T}_{n,k}(\text{Cone}(n, \Delta))$ if and only if $(F, R) \in \mathcal{F}_{n-1,k}^\text{root}(\Delta)$ by Lemmas 9 and 15. The bijection then follows from Lemma 10.

**Corollary 19.** Let $T = F \cup \text{Cone}(n, R)$, where $F \in \mathcal{C}_{n-1,k}$ and $R \in \mathcal{C}_{n-1,k-1}$. Then $\tilde{H}_\ast(T) \cong H_\ast(F, R)$, and $T \in \mathcal{T}_{n,k}$ if and only if $(F, R) \in \mathcal{F}_{n-1,k}^\text{root}$. Moreover the binomial correspondence restricts to a bijection $\varphi : \mathcal{T}_{n,k} \sim \mathcal{F}_{n-1,k}^\text{root}$.

**Proof.** This follows by recalling that $\mathcal{T}_{n,k}(\text{Cone}(n, \Delta)) = \mathcal{T}_{n,k}(\Delta \cap K_{n-1}^k \cup \{ \sigma \cup \{n\} : \sigma \in \Delta \cap K_{n-1}^{k-1}\})$. Thus if $\Delta \supseteq K_{n-1}^k$, then $(\Delta \cap K_{n-1}^k) \cup \{ \sigma \cup \{n\} : \sigma \in \Delta \cap K_{n-1}^{k-1}\} = K_n^k$.

## 3 Determinantal measure

For ease of reading, we will for this section abuse notation by identifying each $j$–complex of $K_{n-1}^k$ with it’s set of $j$–dimensional faces. For any $X \in \mathcal{X}_{n,j}$, we also will let $\overline{X}$ denote the $j$–complex that’s set of $j$–faces is the complement (with respect to $\binom{[n]}{j}$) of $X_j$. Define the submatrix $\tilde{\partial}$ of $\partial$ by deleting all rows of $\partial$ that correspond to elements of $\binom{[n]}{k}$ which contain the vertex $n$ (thus $\tilde{\partial}_0[n] = \partial_0[n]$). We have the following corollary of Lemma 17:

**Corollary 20.** Suppose $T \in \mathcal{C}_{n,k}$ satisfies $|T_k| = \binom{n-1}{k}$. Then $\det \tilde{\partial}_{k,T} \neq 0 \iff |\det \tilde{\partial}_{k,T} | = |\tilde{H}_{k-1}(T)| \iff T \in \mathcal{T}_{n,k}$.

**Proof.** Let $R = \text{Cone}(n, K_{n-2}^k)$ so that $|T_k| + |R_{k-1}| = \binom{n}{k}$, and $\partial T = \tilde{\partial}_{k,T}$. This now follows from Corollary 14 and Lemma 17.

This last corollary was originally proven for the cases $k \geq 1$ ([Kal83], Lemma 2) by Kalai, who combined this with the Cauchy–Binet formula and some deft linear algebra to show ([Kal83], Theorem 1) that

$$\binom{n-2}{k} = \det \tilde{\partial}^2 = \sum_{T \in \mathcal{T}_{n,k}} \det \tilde{\partial}_{k,T}^2 = \sum_{T \in \mathcal{T}_{n,k}} |\tilde{H}_{k-1}(T)|^2.$$

Understanding that $\tilde{\partial}_0[n] = \partial_0[n]$ and $\tilde{H}_{-1}(T) = 0$, the case $k = 0$ can also be seen to hold. This gives us a natural probability measure $\nu = \nu_{n,k}$ on $\mathcal{T}_{n,k}$. Namely,

$$\nu_{n,k}(T) := \frac{\det \tilde{\partial}_{k,T} |}{\det \partial} = \frac{|\tilde{H}_{k-1}(T)|^2}{\binom{n-2}{k}}.$$

This measure was originally formulated in greater generality by Lyons ([Ly003], §12) and was further expanded upon in [Ly009]. This version of the measure has also been considered again recently by authors such as Kahle and Newman [KN20] and Mészáros [Mé21].

Measures defined in the above manner are said to be determinantal [Ly003], as are the random variables associated to them. The following lemma is a special case of Theorem 5.1 from [Ly003]:

**Lemma 21.** Let $R, S$ be finite sets and let $M$ be an $R \times S$ matrix of rank $|R|$. Let $\mu$ be the determinantal measure on $S$ which is defined by $\mu(T) = \frac{\det M_{T,S}^2}{\det(\text{det}(M^2))}$ for all $T \subseteq S$ of size $|R|$. Let $P := M^2(\text{det}(M^2))^{-1}M$ (this is the matrix of the projection onto the rowspace of $M$). Then, for any $B \subseteq S$,

$$\mu(T : T \supseteq B) = \text{det } P_{B,B} \quad \text{and} \quad \mu(T : T \subseteq S \setminus B) = \text{det } (\text{Id } - P)_{B,B}.$$
Determinantal random variables enjoy the negative associations property ([Ly003], Theorem 6.5), which can be stated for random variables on $\mathcal{C}_{n,k}$ as follows: A function $f : \mathcal{C}_{n,k} \rightarrow \mathbb{R}$ is called increasing if $f(X) \leq f(X \cup Y)$ for every $X, Y \in \mathcal{C}_{n,k}$. A random variable $X \in \mathcal{C}_{n,k}$ is said to have negative associations if, for every pair of increasing functions $f_1, f_2$ and every $Y \in \mathcal{C}_{n,k}$, we have

$$\mathbb{E}[f_1(X \cap Y)f_2(X \cap Y)] \leq \mathbb{E}[f_1(X \cap Y)]\mathbb{E}[f_2(X \cap Y)].$$

We will make use of negative associations when we go to prove our applications to random topology.

As we see from Lemma 21, for any $B \in \mathcal{C}_{n,k}$ and $A \in \mathcal{C}_{n,k-1}$, we have

$$\nu_{n,k}(T : T \supseteq B) = \det(P_{n,k}B,B) \quad \text{and} \quad \nu_{n,k-1}(T : T \subseteq \overline{A}) = \det(\text{Id} - P_{n,k-1})_{A,A}$$  \hspace{1cm} (3)

where $P_{n,k} := \partial^t(\partial\partial^k)^{-1}\partial$. Our use of $\partial$ is actually a choice of basis for the rowspace of $\partial$, and an arbitrary one at that. Let $A$ be any $k-1$–root of $K_{n-1}$—this implies that $|A_{k-1}| = \binom{n-1}{k}$ ([Kal83], Lemma 1). Then, by applying a change of basis, we also have the equivalent definition(s)

$$\nu_{n,k}(T) := \frac{\det \partial^2_{A,T}}{\det(\partial\partial^k)_{A,A}}$$ \hspace{1cm} (4)

all of which correspond to the same $P_{n,k}$. Mészáros ([Mé21], Lemma 14) determined that

$$P_{n,k} := \frac{1}{n} \partial^{[n]} k \partial^{[n]} k.$$ 

By Lemma 8, we also have $\text{Id} - P_{n,k} = \frac{1}{n} \partial^{[n]} k \partial^{[n]} k := Q_{n,k}$. By Lemma 17 and Cauchy–Binet, we therefore have the following lemma:

**Lemma 22.** Suppose $B \in \mathcal{C}_{n,k}$ and $A \in \mathcal{C}_{n,k-1}$. Then

$$\nu_{n,k-1}(T : T \subseteq \overline{A}) = \det(Q_{n,k-1})_{A,A} = n^{-|A|} \sum_{B : (B,A) \in \mathcal{F}_{n,k}^{\text{root}}} \det \partial^2_{A,B},$$

and

$$\nu_{n,k}(T : T \supseteq B) = \det(P_{n,k})_{B,B} = n^{-|B|} \sum_{A : (A,B) \in \mathcal{F}_{n,k}^{\text{root}}} \det \partial^2_{A,B}.$$ 

**Corollary 23.** For any $T \in \mathcal{F}_{n,k}, T' \in \mathcal{F}_{n,k-1}$ we have $|H_{k-1}(T, T')| = |\tilde{H}_{k-1}(T)\tilde{H}_{k-2}(T')|$. Moreover

$$\left\{ (F, E) \in \mathcal{F}_{n,k}^{\text{root}} : |F| = \frac{n-1}{k} \right\} = \mathcal{F}_{n,k} \times \mathcal{F}_{n,k-1}.$$

**Proof.** Let $A := T$ so that, as in expression (4), $|A_{k-1}| = \binom{n-1}{k}$. Then by Lemma 21, we have

$$\nu_{n,k}(T) = \frac{\det \partial^2_{A,T}}{\det(\partial\partial^k)_{A,A}} = \frac{\det \partial^2_{A,T}}{n^{\binom{n-1}{k}} \nu_{n,k-1}(\{S : S \subseteq \overline{A}\})} = \frac{\det \partial^2_{A,T}}{n^{\binom{n-1}{k}} \nu_{n,k-1}(T')},$$

Thus by Lemma 17,

$$\frac{|H_{k-1}(T, T')|^2}{n^{\binom{n-1}{k}}} = \nu_{n,k}(T)\nu_{n,k-1}(T') = \frac{|\tilde{H}_{k-1}(T)|^2 |\tilde{H}_{k-2}(T')|^2}{n^{\binom{n-1}{k}} \binom{n-1}{k}}.$$ 

The set identity in the statement of the corollary now follows from Lemma 17, as the left hand side of this is finite if and only if the right hand side is finite, and $T, T'$ are also the correct sizes for the statement to hold—i.e., $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$. \hfill $\square$
Corollary 24. Suppose $T \in \mathcal{T}_{n,k}$, and let $F = \text{Proj}(n, T)$ and $E = \text{Link}(n, T)$. Then
\[
\nu_{n,k}(F \cup \text{Cone}(n, E)) = \frac{\det \frac{\partial^2 E^2}{\partial F \partial E}}{n(n-k)} = \frac{|H_{k-1}(F, E)|^2}{n(n-k)}.
\]
Proof. This follows from Corollary 19, the original definition $\nu_{n,k}(T) = \frac{|\mathcal{H}_{k-1}(T)|^2}{n(n-k)}$, and Lemma 17. \hfill \square

Corollary 25. Suppose $T \in \mathcal{T}_{n,k}$ is such that $T' : \text{Proj}(n, T) \in \mathcal{T}_{n-1,k}$ and $T'' : \text{Link}(n, T) \in \mathcal{T}_{n-1,k-1}$. Then $\nu_{n,k}(T) = \nu_{n-1,k}(T')\nu_{n-1,k-1}(T'')(1 - 1/n)^{(n-k)}$.

Proof. By the previous corollary, Lemma 17, and Corollary 23,
\[
\nu_{n,k}(T) = \frac{|H_{k-1}(T', T'')|^2}{n(n-k)} = \frac{|\mathcal{H}_{k-1}(T')|^2}{(n-1)(n-k)} \frac{|\mathcal{H}(T'')|^2}{(n-1)(n-k-1)} \frac{n-1}{n}(n-k+1) = \frac{n}{n-k} \nu_{n-1,k}(T') \nu_{n-1,k-1}(T'')(1 - 1/n)^{(n-k)}.
\]

Lemma 26. For all $F \in \mathcal{C}_{n-1,k}$ and $E \in \mathcal{C}_{n-1,k-1}$, we have
\[
\nu_{n,k}(T : \text{Proj}(n, T) = F) = \nu_{n-1,k}(T' : T' \supseteq F)(1 - 1/n)^{|F|\eta}|F|^{-\binom{n-k}{k}}
\]
and
\[
\nu_{n,k}(T : \text{Link}(n, T) = E) = \nu_{n-1,k-1}(T'' : T'' \subseteq E)(1 - 1/n)^{|E|\eta}|E|^{-\binom{n-k}{k}}.
\]

Proof. By Lemma 22, Corollary 24, and the Cauchy–Binet formula, we have
\[
(n-1)^{|F|}\nu_{n-1,k}(T : T \supseteq F) = \det(\partial^T \partial F, F) = n(n-k)\nu_{n,k}(T : \text{Proj}(n, T) = F)
\]
and
\[
(n-1)^{|E|}\nu_{n-1,k-1}(T : T \subseteq E) = \det(\partial \partial^T, E, E) = n(n-k)\nu_{n,k}(T : \text{Link}(n, T) = E).
\]

Corollary 27. Using the notation from Theorem 1, there are couplings $\pi_{n,k}$ and $\lambda_{n,k}$ of $\mathcal{T}_{n,k}$, $\mathcal{T}_{n-1,k}$, $\mathcal{Y}_{n-1,k}$ and $\mathcal{Y}_{n-1,k-1}$, $\mathcal{Y}_{n-1,k-1}$ respectively such that, marginally, $\mathcal{T}_{n-1,k}$, $\mathcal{T}_{n-1,k-1}$ are independent of $\mathcal{Y}_{n-1,k}$, $\mathcal{Y}_{n-1,k-1}$ respectively, and
\[
\pi_{n,k}((T, T', Y') : \text{Proj}(n, T) = T' \setminus Y') = 1 = \lambda_{n,k}((T, T'', Y'') : \text{Link}(n, T) = T'' \cup Y'').
\]
Namely,
\[
\pi_{n,k}(T, T', Y') := \mu_{n-1,k}(Y')\nu_{n-1,k}(T') \frac{\nu_{n,k}(T)\mathbf{1}\{\text{Proj}(n, T) = T' \setminus Y'\}}{\nu_{n,k}(S : \text{Proj}(n, S) = T' \setminus Y')}\nu_{n,k}(S : \text{Proj}(n, S) = T' \setminus Y')
\]
and
\[
\lambda_{n,k}(T, T'', Y'') := \mu_{n-1,k-1}(Y'')\nu_{n-1,k-1}(T'') \frac{\nu_{n,k}(T)\mathbf{1}\{\text{Link}(n, T) = T'' \cup Y''\}}{\nu_{n,k}(S : \text{Link}(n, S) = T'' \cup Y'')}\nu_{n,k}(S : \text{Link}(n, S) = T'' \cup Y'').
\]

Proof. It suffices to show that $\pi_{n,k}$ has the correct marginal densities, as the proof for $\lambda_{n,k}$ is basically identical. Summing over all $T$ clearly produces the independent coupling of $\mathcal{T}_{n-1,k}$, $\mathcal{Y}_{n-1,k}$. For the remaining marginal, Corollary 26 gives us that
\[
\pi_{n,k}(T, T', Y') = \nu_{n,k}(T)\nu_{n-1,k}(T')\mathbf{1}\{T' \supseteq \text{Proj}(n, T)\} \mu_{n-1,k}(Y')\mathbf{1}\{\text{Proj}(n, T) = T' \setminus Y'\} \nu_{n-1,k}(S : S \supseteq \text{Proj}(n, T)) \mu_{n-1,k}(S : T' \setminus S = \text{Proj}(n, T))
\]
which, summed over $Y'$ and then $T'$, gives the desired marginal. \hfill \square
Proof of Theorem 1. It suffices to show that

$$\pi_n, k(T', T''', Y', Y''') \mapsto \frac{\pi_n, k(T', T''', Y', Y''')}{\nu_n, k(T)}$$

is a probability density with the claimed marginal densities. Considering expression (5), summing (6) over $Y'$ and $T'$ gives $\lambda_n, k(T', T'''', Y''')$, which we know to have the desired marginal densities. One can deduce by a symmetric argument that the marginal densities with respect to $Y'$ and $T'$ are also correct. \hfill \Box

4 Spectral Estimates

For this section, we use asymptotic notation, $o()$ and $O()$, to describe the behavior of a function of $n$ as $n \to \infty$. Let $A$ be the adjacency matrix of a random graph $G = G(n, p, E, M)$ on $[n]$ satisfying the following:

1. $P[e \in G] = p \in (0, 1)$ for all $e \in \binom{[n]}{2}$.

2. There is a fixed constant $E \geq 1$ and an $M = M(n) > 0$ such that, for every $t \in [0, M]\binom{[n]}{2}$, we have

$$\mathbb{E}\exp\left(\sum_{1 \leq i < j \leq n} t_{ij} A_{ij}\right) \leq E \prod_{1 \leq i < j \leq n} (1 - p + pe^{t_{ij}}).$$

The choice to make $E$ a fixed constant is just for convenience. All of the results of this section can be easily adapted to work for $E = E(n)$ an arbitrary fixed polynomial in the variable $n$.

Proposition 28. Let $G$ be defined as above with $p = \frac{\delta \log n}{n}$ (δ an arbitrary constant), and $M \geq \frac{3}{2}(n/p)^{\frac{3}{2}}$. Then, for any fixed $s > 0$, we have with probability at least $1 - o(n^{-s})$ that

$$v^t Av = O(\sqrt{np})$$

for all unit vectors $v \perp 1$.

Proof. This will follow from Lemmas 32, 33, 34, and 35. \hfill \Box

Corollary 29. Let $L := \text{Id} - D^{-\frac{1}{2}} AD^{-\frac{1}{2}}$ where $D$ is the degree matrix of $G = G\left(n, \frac{\delta \log n}{n}, O(1), \frac{3n}{5\sqrt{3\log n}}\right)$. Suppose there are positive integers $s = O(1)$ and $m \geq \sqrt{\log n}$ such that $P[\min_{i \in [n]} \deg_G(i) \geq m] = 1 - o(n^{-s})$. Let $\lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L)$ denote the eigenvalues of $L$. Then, with probability $1 - o(n^{-s})$, we have $\lambda_1(L) = 0$, and

$$\lambda_2(L) = 1 - O\left(\frac{\sqrt{\log n}}{m}\right).$$

Proof. Since $G$ is connected with sufficiently high probability, we will treat $G$ as though it were connected almost surely. As such, we know that $L$ has minimal eigenvalue $0$ with multiplicity one, and the corresponding eigenvector is $D^{\frac{1}{2}}1$. Thus we are interested in bounding the quantity

$$\sup \left\{ \frac{y^t D^{-\frac{1}{2}} AD^{-\frac{1}{2}} y}{y^t y} : 0 \neq y \perp D^{\frac{1}{2}}1 \right\}$$

from above by some $\lambda := O(\sqrt{\log n}/m)$. Equivalently, we would like to show that

$$x^t Ax \leq \lambda x^t Dx$$

for all $x \perp D1$.

Without loss of generality, we can assume that $x$ is a unit vector. Let

$$x = \cos \theta u + \sin \theta v \quad \text{where} \quad u = \frac{1}{\sqrt{n}}1, \quad v \perp 1, \quad \text{and} \quad |v| = 1.$$
Recalling that we were required to assume that $x \perp D1$, we have
\[
x^tAx = u^tDx = 0 \quad \text{and} \quad \cos \theta \frac{\text{tr}D}{\sqrt{v^tDv}} = -\sin \theta v^tD1 \quad (\text{both of these follow from the assumption that } x \perp D1),
\]
we have
\[
x^tAx = \sin^2 \theta v^tAv - \cos^2 \theta \frac{\text{tr}D}{n} \quad \text{and} \quad x^tDx = \sin^2 \theta v^tDv - \cos^2 \theta \frac{\text{tr}D}{n}.
\]
So we have
\[
x^t(\lambda D - A)x = v^t(\lambda D - A)v \sin^2 \theta + \frac{(1 - \lambda) \text{tr}D}{n} \cos^2 \theta
\]
\[
= v^t(\lambda D - A)v \frac{(\text{tr}D)^2}{(\text{tr}D)^2 + n(v^tD1)^2} + \frac{(1 - \lambda) \text{tr}D}{n} \frac{n(v^tD1)^2}{(\text{tr}D)^2 + n(v^tD1)^2}
\]
which we would like to show is positive. Solving for $v^tAv$, it suffices to show that
\[
v^tAv \leq (1 - \lambda) \frac{(v^tD1)^2}{\text{tr}D} + \lambda v^tDv \quad \text{for all unit vectors } v \perp 1.
\]
By our minimum degree assumption, it would even suffice to show that $v^tAv \leq \lambda m$. The result now follows from Proposition 28.

In order to prove Lemma 33, we are going to need a version of Bernstein’s inequality that works on weighted sums of centered edge indicators of $G$.

**Theorem 30** (Bernstein’s inequality). Let $G$ be as above. Suppose $|c_{ij}| \leq c$ for some fixed $c$ and all $i < j$, and that $\varepsilon \geq 0$ is such that $M \geq \frac{3}{4} \sum_i c_{ij}^2 + 2\varepsilon/3$. Then, for any $H \subseteq \binom{[n]}{2}$, we have
\[
\mathbb{P} \left[ \sum_{(i,j) \in H} c_{ij}(A_{ij} - p) \geq \varepsilon \right] \leq E \exp \left( -\frac{\varepsilon^2}{2p \sum_{(i,j) \in H} c_{ij}^2 + 2\varepsilon/3} \right).
\]

**Proof.** By a Chernoff bound, the numerical bounds $1 + x \leq e^x \leq 1 + x + \frac{3x^2}{6-2x}$ for $x \leq 3$, and our assumptions about $G$, we have for $t \leq \frac{\varepsilon}{2}$ that
\[
\mathbb{P} \left[ \sum_{(i,j) \in H} c_{ij}(A_{ij} - p) \geq \varepsilon \right] \leq E \inf_{t \in (0,M]} \exp \left( -t \varepsilon + \sum_{(i,j) \in H} \frac{3t^2 c_{ij}^2 (A_{ij} - p)^2}{6 - 2ct} \right)
\]
\[
\leq E \inf_{t \in (0,M]} \exp \left( -t \varepsilon + \frac{\sum_{(i,j) \in H} 3t^2 c_{ij}^2 (A_{ij} - p)^2}{6 - 2ct} \right)
\]
\[
\leq E \inf_{t \in (0,M]} \exp \left( \frac{(\sum_{(i,j) \in H} c_{ij}^2 + 2\varepsilon/3)t^2 - 2\varepsilon t}{2 - 2ct/3} \right)
\]
\[
= E \inf_{t \in (0,M]} \exp \left( \frac{(\sigma^2 + 2\varepsilon/3)t^2 - 2\varepsilon t}{2 - 2ct/3} \right),
\]
where $\sigma^2 := p \sum_{(i,j) \in H} c_{ij}^2$. Taking $t = \frac{\varepsilon}{\sigma^2 + 2\varepsilon/3}$, we have $2 - 2ct/3 = \frac{2\sigma^2 + 2\varepsilon/3}{\sigma^2 + 2\varepsilon/3}$, and thus
\[
\frac{(\sigma^2 + 2\varepsilon/3)t^2 - 2\varepsilon t}{2 - 2ct/3} = \left( (\sigma^2 + 2\varepsilon/3)t - 2\varepsilon \right) \frac{\varepsilon}{\sigma^2 + 2\varepsilon/3} \frac{\sigma^2 + 2\varepsilon/3}{2\sigma^2 + 2\varepsilon/3} = -\frac{\varepsilon^2}{2\sigma^2 + 2\varepsilon/3}.
\]
Recalling that we were required to assume that $t \leq \frac{3}{\varepsilon}$, what we have shown holds for our choice of $t$ as long as $\varepsilon \geq -3\sigma^2 c^{-1}$. 

We will also want to be able to apply a near–optimal Chernoff bound to uniformly weighted sums of edge indicators.

**Lemma 31.** For \( G \) as above with \( M \geq \log \frac{(1-p)\varepsilon}{1-p\varepsilon} \) where \( \varepsilon \geq 3 \), we have

\[
P \left[ \sum_{(i,j) \in H} A_{ij} \geq \varepsilon p |H| \right] \leq E \exp \left( -\frac{\varepsilon \log \varepsilon}{3} p |H| \right).
\]

**Proof.** As with the previous proof,

\[
P \left[ \sum_{(i,j) \in H} A_{ij} \geq \varepsilon p |H| \right] \leq E \inf_{t \in [0,M]} e^{-t \varepsilon |H|} (1 - p + pe^t)^{|H|}
\]

\[
= E \left( \frac{(1-p)\varepsilon}{1-p\varepsilon} \right)^{-\varepsilon |H|} \left( \frac{1-p}{1-p\varepsilon} \right)^{|H|}
\]

\[
= E e^{-\varepsilon |H|} \left( 1 + \frac{(\varepsilon-1)p}{1-p\varepsilon} \right)^{|H|}
\]

\[
\leq E \exp \left( -\frac{\varepsilon \log \varepsilon - \varepsilon + 1}{3} p |H| \right).
\]

For \( \varepsilon \geq 3 \), this is bounded above by \( E \exp \left( -\frac{\varepsilon \log \varepsilon}{3} p |H| \right) \).

To prove Proposition 28, we will adapt the Kahn–Szemerédi argument. Let

\[
S := \{ v \in \mathbb{R}^n : |v| = 1 \text{ and } v \bot 1 \} \quad \text{and} \quad T := \left\{ x \in \frac{1}{\sqrt{4n}} \mathbb{Z}^n : |x| \leq 1 \text{ and } x \bot 1 \right\}.
\]

The following lemma is a special case of Claim 2.4 in \([FO05]\).

**Lemma 32.** Suppose \( |x^t A y| \leq c \) for all \( x, y \in T \). Then \( |v^t A v| \leq 4c \) for all \( v \in S \).

We now write

\[
\sum_{(i,j) \in [n]^2} |x_i A_{ij} y_j| = \sum_{(i,j) \in \mathcal{L}} |x_i A_{ij} y_j| + \sum_{(i,j) \in \mathcal{H}} |x_i A_{ij} y_j|
\]

where \( \mathcal{L} := \{ (i,j) \in [n]^2 : (x_i y_j)^2 \leq \frac{c}{n} \} \) and \( \mathcal{H} := [n]^2 \setminus \mathcal{L} \).

### 4.1 Light Couples

**Lemma 33.** Suppose \( M \geq \frac{3}{2} (n/p)^{\frac{1}{4}} \) in the definition of \( G \). For any constant \( s > 0 \), we have

\[
P \left[ \sum_{(i,j) \in \mathcal{L}} |x_i A_{ij} y_j| \geq 7 \sqrt{np} \text{ for some } x, y \in T \right] \leq E (18e^{-3})^n = o(n^{-s}).
\]

**Proof.** We can bound the contribution from the light couples as follows: It is known ([FO05], Claim 2.9) that \( |T| \leq 18^n \). So, by applying a union bound, we have

\[
P \left[ \sum_{(i,j) \in \mathcal{L}} |x_i A_{ij} y_j| \geq 7 \sqrt{np} \text{ for some } x, y \in T \right] \leq 18^n \sup_{x,y \in T} P \left[ \sum_{(i,j) \in \mathcal{L}} |x_i A_{ij} y_j| \geq 7 \sqrt{np} \right].
\]

Towards applying Bernstein’s inequality, define centered random variables

\[
B_{ij} := (|x_i y_j| 1 \{ (i,j) \in \mathcal{L} \} + |x_j y_i| 1 \{ (j,i) \in \mathcal{L} \})(A_{ij} - p)
\]
so that
\[ \sum_{(i,j) \in L} B_{ij} = \sum_{(i,j) \in L} |x_i A_{ij} y_j| - \mathbb{E} \sum_{(i,j) \in L} |x_i A_{ij} y_j|. \]

Note that each \( B_{ij}^2 \leq \frac{4p}{n} \) a.s., and \( \sum_{i<j} \mathbb{E} B_{ij}^2 \leq p \sum_{i<j} 2 \left( (x_i y_j)^2 + (x_j y_i)^2 \right) \leq 2p \) by taking advantage of the fact that \(|x|, |y| \leq 1\). Thus, by Bernstein’s inequality,
\[
\mathbb{P} \left[ \sum_{(i,j) \in L} B_{ij} \geq \frac{6 \varepsilon \sqrt{np}}{n} \right] \leq E \exp \left( -\frac{\left( \frac{6 \varepsilon^2}{np} \right)^2}{4p + 6 \frac{1}{\sqrt{n}} \sqrt{np} \varepsilon} \right) = E \exp \left( -\frac{9 \varepsilon^2}{1 + 2 \varepsilon} \right). \]

Taking \( \varepsilon = 1 \), this shows that \( \sum_{(i,j) \in L} |x_i A_{ij} y_j| \leq 6 \sqrt{np} + E \sum_{(i,j) \in L} |x_i A_{ij} y_j| \) with probability at least \( 1 - e^{-3a} \). By Lemma 2.6 of [FO05], we also have \( \mathbb{E} \sum_{(i,j) \in L} |x_i A_{ij} y_j| \leq \sqrt{np} \), thus giving us the desired bound. Our use of Bernstein’s inequality requires us to have \( M \geq \frac{6 \sqrt{np}}{2p + \frac{2}{\sqrt{n}} \sqrt{np}} = \frac{3}{4} (n/p)^{\frac{1}{2}} \).

### 4.2 Heavy Couples

For \( B, C \subseteq [n] \), let \( e(B, C) := |\{(i, j) \in \mathcal{G} : i \in B, j \in C\}| \) and \( \mu(B, C) := p |B| |C| \). The following is a weakened form of Lemma 9.1 in [HKP19].

**Lemma 34.** Suppose we have constants \( c_0, c_1, c_2 > 1 \) and a graph \( G \) on \([n]\) with \( \max_{i \in [n]} \deg_G(i) \leq c_0 np \), and, for all \( B, C \subseteq [n] \), one or more of the following hold:

- \( e(B, C) \leq c_1 \mu(B, C) \)
- \( e(B, C) \log \frac{e(B, C)}{\mu(B, C)} \leq c_2 (|B| \lor |C|) \log \frac{n}{|B| \lor |C|} \)

Then \( \sum_{i \in H} |x_i A_{ij} y_j| = O(\sqrt{np}) \).

**Lemma 35.** Suppose that \( p = \frac{\delta \log n}{n} \) for some fixed but arbitrary \( \delta > 0 \), and \( M \geq \frac{3}{4} (n/p)^{\frac{1}{2}} \). For any \( s > 0 \), there are fixed constants \( c_0, c_1, c_2 > 1 \) so that the conditions of Lemma 34 hold for \( \mathcal{G} \) with probability at least \( 1 - o(n^{-s}) \).

In terms of probabilistic bounds, the proof of this will only rely on Lemma 31. We note then that the condition \( M \geq \frac{3}{4} (n/p)^{\frac{1}{2}} \) is overkill since Lemma 31 only ever requires that we have \( M \) be greater than a fixed constant.

**Proof.** Assume without loss of generality that \( c_0 \geq 3 \). First note that, due to the monotonicity of \( x \log x \) for \( x \geq 1 \), the second bulleted condition is equivalent to the statement \( e(B, C) \leq r_1 \mu(B, C) \) where \( r_1 \geq 1 \) solves \( r_1 \log r_1 = \frac{c_2 (|B| \lor |C|) \log \frac{n}{|B| \lor |C|}}{\mu(B, C)|B| \lor |C|} \). So we can rewrite the two bulleted conditions as the single condition \( e(B, C) \leq r_1 \mu(B, C) \) where \( r := r_1 \lor c_1 \). By Lemma 31 and a union bound, we have
\[
\mathbb{P} [\max_{i \in [n]} \deg(i) > c_0 np] \leq n \mathbb{P} [\deg(n) > c_0 np] \leq En \exp \left( -\frac{\delta_0 \log c_0}{3} \log n \right)
\]
and
\[
\mathbb{P} [e(B, C) > r \mu(B, C)] \leq \mathbb{P} [e(B, C) > r_1 \mu(B, C)] \leq E \exp \left( -\frac{c_2 (|B| \lor |C|) \log \frac{n}{|B| \lor |C|}}{3} \right).
\]

Thus we can choose a constant \( c_0 \) large enough to make \( \mathbb{P} [\max_{i \in [n]} \deg_G(i) > c_0 np] = o(n^{-s}) \). Using this fact and the resulting (high probability) inequality
\[
\frac{e(B, C)}{\mu(B, C)} \leq \frac{c_0 (|B| \lor |C|) np}{p |B| |C|} = \frac{c_0 n}{|B| \lor |C|},
\]

14
we have, in the case $|B| \lor |C| \geq n/e$, that
\[ P[\exists B, C \text{ s.t. } e(B, C) > ec_0\mu(B, C) \text{ and } |B| \lor |C| \geq n/e] \leq P[\max_{i \in [n]} \deg_G(i) > c_0np] = o(n^{-s}). \]

So suppose that $|B| \lor |C| < n/e$. By a union bound over all possible pairs \( \{i, j\} \subset \lceil n/e \rceil \) \((i \leq j, \text{ without loss of generality})\) of sizes for the sets \( B, C \), it suffices to show that
\[ \left( \begin{array}{c} n \\ i \\ \end{array} \right) \left( \begin{array}{c} n \\ j \\ \end{array} \right) \exp \left( - \frac{c_2 j \log \frac{n}{j}}{3} \right) \leq n^{-s-3}. \]

Recalling that \( \binom{n}{j} \leq \left( \frac{2n}{j} \right)^j \) for all \( j \in [n] \), this can be done by showing that
\[ (s + 3) \log n + j \left( 1 + \log \frac{n}{j} \right) + i \left( 1 + \log \frac{n}{i} \right) \leq \frac{c_2}{3} j \log \frac{n}{j} \]
for all \( 1 \leq i \leq j < n/e \). Indeed, since \( j \log \frac{n}{j} \) is monotone increasing for \( 1 \leq j < n/e \) (Lemma 2.12 of [FO05]), we have
\[ (s + 3) \log n + j \left( 1 + \log \frac{n}{j} \right) + i \left( 1 + \log \frac{n}{i} \right) \leq (s + 3) \log n + 4j \log \frac{n}{j} \leq (s + 7)j \log \frac{n}{j}. \]

Taking \( c_2 \geq 3s + 21 \) therefore gives the desired bound.

\[ \square \]

4.3 Link unions

Let \( T_1, T_2, \ldots, T_m \) be jointly independent copies of \( T_{n, k-j} \) with \( k \) fixed. Then, if \( T'_1, T'_2, \ldots, T'_m \) are jointly independent copies of \( T_{n+j, k} \), we can use Theorem 2 to couple these random trees so that
\[ L_{m}^{k,j}(n) := \text{Link} \left( \{n + 1, \ldots, n + j\}, \bigcup_{i \in [m]} T'_i \right) \]
\[ = \bigcup_{i \in [m]} \text{Link} \left( \{n + 1, \ldots, n + j\}, T'_i \right) \]
\[ = \mathcal{Y}_{k-j}(n, p) \cup \bigcup_{i \in [m]} T_i, \]

where \( p := 1 - \left( 1 - \frac{1}{n+j} \right)^m \sim \frac{m}{n} \). Since each \( T_i \) is determinantal, the set of \( k \)-faces in each \( T_i \) are negatively associated (to be abbreviated NA). Moreover, the set of \( k \)-faces in \( T := \bigcup_{i \in [m]} T_i \) are also NA, as
\[ 1\{f \in T\} = 1 - \prod_{i \in [m]} (1 - 1\{f \in T_i\}) \]
is increasing in \( T \) as a function \( \mathcal{G}_{n, k} \rightarrow \mathbb{R} \), and any set of increasing functions defined on disjoint subsets of an NA set is itself an NA set [JDP83]. By the same reasoning and the fact that sets of jointly independent random variables are NA sets, we also have that the set of \( k - j \)-faces of \( L_{m}^{k,j}(n) \) is NA. We now narrow our focus to the case \( j = k - 1 \), in which
\[ G := L_{m}^{k,k-1}(n) \]
is a graph with negatively associated edges each appearing with probability \( q := 1 - (1 - p)(1 - \frac{2}{n})^{m} \sim \frac{(k+1)m}{n} \). This implies that \( G \) is of type \( \mathcal{G}(n, q, 1, \infty) \). Toward applying Corollary 29 to this \( G \), we will assume that \( m = \lceil \delta \log n \rceil \) for some arbitrary constant \( \delta > 0 \). In order to get a sufficiently strong lower bound on the minimum degree of \( G \), we need to have a very strong bound on the moment generating function of \( \deg_G(n) \) for negative inputs.
Lemma 36. Let $p_0 := 1 - \left(1 - \frac{k-1}{n+k-1}\right)^m (1 - \frac{1}{n})^m \sim \frac{km}{n}$. Then for all $t < 0$ we have

$$\mathbb{E}[\exp(t \deg_G(n))] \leq e^{mt} (1 + p_0(e^t - 1))^{n-1-m} \exp\left(\frac{m^2e^{1-t}}{2n}\right).$$

Proof. Let $\ell := \text{Link}(n, \bigcup_{i \in [m]} T_i)$, and let $B_i := 1\{i, n \in \mathcal{G}(n,p)\}$, where $\mathcal{G}(n,p)$ is as above and, in particular, independent of $\ell$. Then

$$\mathbb{E}[\exp(t \deg_G(n)) | \ell] = \prod_{i \in [n-1]} \mathbb{E}[\exp(t(1\{i \in \ell\} + 1\{i \notin \ell\} B_i)) | \ell]$$

$$= \prod_{i \in [n-1]} (1 - p) \exp(t1\{i \in \ell\}) + pe^t.$$  

Note that $\ell \sim \text{Binom}([n-1], p') \cup R$ where $p' := 1 - (1 - \frac{1}{n})^m$, and $R$ is a random subset of $[n-1]$ formed by independently and uniformly choosing (with replacement) $m$ vertices from $[n-1]$—this follows from the $k = 1$ case of Theorem 1. Thus

$$\mathbb{E}[\exp(t \deg_G(n)) | R] = \prod_{i \in [n-1]} \mathbb{E}[(1 - p) \exp(t1\{i \in \ell\}) + pe^t | R].$$

Now note that

$$\mathbb{E}[\exp(t1\{i \in \ell\}) | R] = \mathbb{E}[\exp(t1\{i \in R\} + 1\{i \notin R\} \text{Bernoulli}(p')) | R]$$

$$= \exp(t1\{i \in R\}) (1 - p' + p' \exp(t1\{i \notin R\}))$$

$$= (1 - p') \exp(t1\{i \in R\}) + p'e^t.$$  

Thus

$$\mathbb{E}[\exp(t \deg_G(n)) | R] = \prod_{i \in [n-1]} (1 - p) ((1 - p') \exp(t1\{i \in R\}) + p'e^t)$$

$$= \prod_{i \in [n-1]} ((1 - p_0) \exp(t1\{i \in R\}) + p_0e^t).$$

We therefore have

$$\mathbb{E}[\exp(t \deg_G(n))] = \sum_{j \in [m]} e^{jt} (1 - p_0 + p_0e^t)^{n-1-j} \mathbb{P}[|R| = j]$$

$$= e^{mt} (1 - p_0 + p_0e^t)^{n-1-m} \sum_{j \in [m]} e^{(j-m)t} (1 - p_0 + p_0e^t)^{m-j} \mathbb{P}[|R| = j].$$

Since $t < 0$, we have $1 - p_0 + p_0e^t \leq 1$, and thus

$$\mathbb{E}[\exp(t \deg_G(n))] \leq e^{mt} (1 - p_0 + p_0e^t)^{n-1-m} \sum_{j \in [m]} e^{-t(m-j)} \mathbb{P}[|R| = j].$$

Toward controlling the size of $R$, let $R = R_m$, and let $R_j$ be defined as $R$ after only the first $j$ independent samplings of $[n-1]$. Then, letting $v_1, v_2, ..., v_m$ be the random vertex selections, we have

$$|R_{j+1}| = |R_j| + 1\{v_{j+1} \notin R_j\}.$$  

Thus, for $s > 0$,

$$\mathbb{E}[\exp(-s|R_{j+1}|) | R_j] = \exp(-s|R_j|) \left(\frac{|R_j|}{n-1} + \frac{n-1 - |R_j|}{n-1} e^{-s}\right)$$

$$\leq \exp(-s|R_j|) \left(\frac{j}{n-1} + \frac{n-1 - j}{n-1} e^{-s}\right) = \exp(-s|R_j| - s) \left(1 + \frac{j(e^s - 1)}{n-1}\right).$$
Taking expectations and iterating then gives
\[ \mathbb{E} \exp(-s|R_m|) \leq e^{-ms} \prod_{i=1}^{m-1} \left( 1 + \frac{i(e^s - 1)}{n - 1} \right). \]
Thus, by Chernoff’s inequality, we have
\[ \mathbb{P}[|R_m| \leq m - j] \leq e^{-js} \prod_{i=1}^{m-1} \left( 1 + \frac{i(e^s - 1)}{n - 1} \right) \leq \exp\left(-js + \left( \frac{m}{2} \right) \frac{e^s - 1}{n - 1} \right) \]
for any \( s > 0 \). Taking \( s = \log \frac{2j(n-1)}{m(m-1)} \), gives
\[ \mathbb{P}[|R_m| \leq m - j] \leq \exp\left(j - \frac{m(m - 1)}{2(n - 1)} \right) \left( \frac{m(m - 1)}{2j(n - 1)} \right)^j \leq \left( \frac{em^2}{2jn} \right)^j. \]
Thus \( |R| \geq m - j + 1 \) with probability at least \( 1 - \left( \frac{em^2}{2jn} \right)^j \). Recall the summation by parts formula:
\[ \sum_{j \in [m]} f_j g_j = f_m \sum_{j \in [m]} g_j - \sum_{j \in [m]} (f_j - f_{j-1}) \sum_{i \in [j-1]} g_i. \]
Setting \( f_j = e^{-t(m-j)} \) and \( g_j = \mathbb{P}[|R| = j] \), we have
\[
\sum_{j \in [m]} e^{-t(m-j)} \mathbb{P}[|R| = j] = 1 + (e^{-t} - 1) \sum_{j \in [m]} \mathbb{P}[|R| < j] e^{-t(m-j)}
\leq 1 + (e^{-t} - 1) \sum_{j \in [m]} \left( \frac{em^2}{2(m-j+1)n} \right)^{m-j+1} e^{-t(m-j)}
= 1 + (1 - e^t) \sum_{j \in [m]} \left( \frac{e^{-1}m^2}{2jn} \right)^j
\leq 1 + (1 - e^t) \left( \exp\left( \frac{m^2e^{1-t}}{2n} \right) - 1 \right),
\]
where the last inequality uses the bound \( j^j \geq j! \). Thus, for \( t < 0 \) we have
\[ \mathbb{E} [\exp(t \deg_G(n))] \leq e^{mt} (1 - p_0 + p_0 e^t)^{n-1-m} \sum_{j \in [m]} e^{-t(m-j)} \mathbb{P}[|R| = j]
\leq e^{mt} (1 + p_0(e^t - 1))^{n-1-m} \left( 1 + (1 - e^t) \left( \exp\left( \frac{m^2e^{1-t}}{2n} \right) - 1 \right) \right)
\leq e^{mt} (1 + p_0(e^t - 1))^{n-1-m} \exp\left( \frac{m^2e^{1-t}}{2n} \right). \]

\[ \Box \]

**Lemma 37.** For \( 0 \leq j \leq m \), we have \( \mathbb{P}\left[ \min_{i \in [n]} \deg_G(i) \leq m - j \right] \leq (1 + o(1))n^{1-j}e^{-km}. \)

**Proof.** For any \( t > 0 \), we have by a union bound that
\[ \mathbb{P}\left[ \min_{i \in [n]} \deg_G(i) \leq m - j \right] \leq n \mathbb{P}[\exp(-t \deg_G(n)) \geq e^{(j-m)t}]
\leq ne^{-jt} (1 + p_0(e^{-t} - 1))^{n-1-m} \exp\left( \frac{m^2e^{1+t}}{2n} \right)
\leq \exp\left( \log n - jt + (n - 1 - m)p_0(e^{-t} - 1) + \frac{m^2e^{1+t}}{2n} \right). \]
Letting \( t = r \log n \) for any \( r \in (0, 1) \) gives
\[
P[\min_{i \in [n]} \deg_G(i) \leq m - j] \leq \exp \left( (1 - j r) \log n - (n - 1 - m) p_0 (1 - n^{-r}) + \frac{em^2}{2n^{1-r}} \right)
\leq (1 + o(1)) \exp \left( (1 - j r) \log n - km \right).
\]
Taking \( r \to 1 \) from the left then gives \( (1 + o(1)) n^{1-j} e^{-km} \).

**Lemma 38.** Let \( L \) be the reduced Laplacian of \( G \) as defined above with \( m = \lceil \delta \log n \rceil \) and \( \delta > 0 \) an arbitrary constant (so that \( P[\min_{i \in [n]} \deg_G(i) \geq m - j + 2] = 1 - o(n^{-s}) \) for any fixed \( j \)). Then for any fixed \( s \), we have \( \lambda_1(L) = 0 \) and \( \lambda_2(L) = 1 - O \left( 1/\sqrt{\log n} \right) \) with probability \( 1 - o(n^{-s}) \).

**Proof.** This follows from the previous lemma and Corollary 29.
References

[BK15] Olivier Bernardi and Caroline Klivans. “Directed Rooted Forests in Higher Dimension”. In: The Electronic Journal of Combinatorics 23 (Dec. 2015). doi: 10.37236/5819.

[COL09] Amin Coja-Oghlan and André Lanka. “The spectral gap of random graphs with given expected degrees”. In: the electronic journal of combinatorics (2009), R138–R138.

[DKM08] Art Duval, Caroline Klivans, and Jeremy Martin. “Simplicial Matrix-Tree Theorems”. In: Transactions of the American Mathematical Society 361 (Feb. 2008). doi: 10.1090/S0002-9947-09-04898-3.

[DKM11] Art M. Duval, Caroline J. Klivans, and Jeremy L. Martin. “Cellular spanning trees and Laplacians of cubical complexes”. In: Advances in Applied Mathematics 46.1 (2011). Special issue in honor of Dennis Stanton, pp. 247–274. DOI: https://doi.org/10.1016/j.aam.2010.05.005.

[DKM15] Art M. Duval, Caroline J. Klivans, and Jeremy L. Martin. “Cuts and Flows of Cell Complexes”. In: J. Algebraic Comb. 41.4 (2015), 969–999. DOI: 10.1007/s10801-014-0561-2.

[FO05] Uriel Feige and Eran Ofek. “Spectral techniques applied to sparse random graphs”. In: Random Structures & Algorithms 27.2 (2005), pp. 251–275. DOI: https://doi.org/10.1002/rsa.20089.

[Gar73] Howard Garland. “p-Adic Curvature and the Cohomology of Discrete Subgroups of p-Adic Groups”. In: Annals of Mathematics 97 (1973), p. 375.

[GW14] Anna Gundert and Uli Wagner. “On Eigenvalues of Random Complexes”. In: Israel Journal of Mathematics 216 (Nov. 2014). DOI: 10.1007/s11856-016-1419-1.

[HKP19] Christopher Hoffman, Matthew Kahle, and Elliot Paquette. “Spectral Gaps of Random Graphs and Applications”. In: International Mathematics Research Notices 2021.11 (May 2019), pp. 8353–8404. DOI: 10.1093/imrn/rnz077.

[Lub18] Alexander Lubotzky. “High dimensional expanders”. In: Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018. World Scientific. 2018, pp. 705–730.

[Lyo03] Russell Lyons. “Determinantal probability measures”. eng. In: Publications Mathématiques de l’IHÉS 98 (2003), pp. 167–212.

[Lyo09] Russell Lyons. “Random complexes and ℓ²-Betti numbers”. In: Journal of Topology and Analysis 01.02 (2009), pp. 153–175. DOI: 10.1142/S1793525309000072.

[Opp18] Izhar Oppenheim. “Local spectral expansion approach to high dimensional expanders part I: Descent of spectral gaps”. In: Discrete & Computational Geometry 59.2 (2018), pp. 293–330.
[Opp20] Izhar Oppenheim. “Local spectral expansion approach to high dimensional expanders part II: Mixing and geometrical overlapping”. In: Discrete & Computational Geometry 64.3 (2020), pp. 1023–1066.

[TY19] Konstantin Tikhomirov and Pierre Youssef. “The spectral gap of dense random regular graphs”. In: The Annals of Probability 47.1 (2019), pp. 362 –419. DOI: 10.1214/18-AOP1263.

[Ž03] Andrzej Žuk. “Property (T) and Kazhdan constants for discrete groups”. In: Geometric And Functional Analysis 13 (June 2003), pp. 643–670. DOI: 10.1007/s00039-003-0425-8.

[Ž96] Andrzej Žuk. “La propriété (T) de Kazhdan pour les groupes agissant sur les polyédres”. In: Comptes rendus de l’Académie des sciences. Série 1, Mathématique 323.5 (1996), pp. 453–458.