Gravitational Collapse and Calogero Model

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Abstract

We study the analytic structure of the S-matrix which is obtained from the reduced Wheeler-DeWitt wave function describing spherically symmetric gravitational collapse of massless scalar fields. The simple poles in the S-matrix occur in the Euclidean spacetime, and the Euclidean Wheeler-DeWitt equation is a variant of the Calogero models, which is discussed in connection with conformal mechanics and a quantum instanton.

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In the previous work [1] we studied quantum mechanically the self-similar black hole formation by collapsing scalar fields and found the wave functions that give the correct semi-classical limit. In this brief report we consider the pole structure of the S-matrix which is obtained from the wave function. The pole corresponds to a solution in an unphysical region, namely, in the Euclidean spacetime with a quantized parameter ($c_0$), which seems like a quantum version of instanton.

The spherically symmetric geometry minimally coupled to a massless scalar field is described by the reduced action in (1+1)-dimensional spacetime of which the Hilbert-Einstein action is

$$S = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left[ R - 2 (\nabla \phi)^2 \right] + \frac{1}{8\pi} \int_{\partial M} d^3x K \sqrt{h}. \quad (1)$$

The reduced action is

$$S_{sph} = \frac{1}{4} \int d^2x \sqrt{-\gamma} r^2 \left\{ (^{(2)}R(\gamma)) + \frac{2}{r^2} \left( (\nabla r)^2 + 1 \right) - 2 (\nabla \phi)^2 \right\}, \quad (2)$$

where $\gamma_{ab}$ is the (1+1)-dimensional metric. The spherical spacetime metric is

$$ds^2 = -2 du dv + r^2 d\Omega^2, \quad (3)$$

where $d\Omega^2$ is the usual spherical part of the metric, and $u$ and $v$ are null coordinates. The self-similarity condition is imposed such that

$$r = \sqrt{-uv} y(z), \quad \phi = \phi(z), \quad (4)$$

where $z = \frac{u+v}{2} = e^{-2\tau}$, $y$ and $\phi$ depend only on $z$. We introduce another coordinates $(\omega, \tau)$ as

$$u = -\omega e^{-\tau}, \quad v = \omega e^\tau, \quad (5)$$

$$ds^2 = -2\omega^2 d\tau^2 + 2d\omega^2 + \omega^2 y^2 d\Omega^2. \quad (6)$$

The classical solutions of the field equations were obtained by Roberts [2], and studied in connection with gravitational collapse by others [3]. Classically black hole formation was only allowed in the supercritical cases ($c_0 > 1$), but even in the subcritical situation there are quantum mechanical tunneling processes to form a black hole of which the probability is semiclassically calculated [4].

In our previous work [1] we quantized the system canonically with the ADM formulation to obtain the Wheeler-DeWitt equation for the quantum black hole formation

$$\left[ \frac{1}{2K} \frac{\partial^2}{\partial y^2} - \frac{1}{2K y^2} \frac{\partial^2}{\partial \phi^2} - K \left( 1 - \frac{y^2}{2} \right) \right] \Psi(y, \phi) = 0, \quad (7)$$

where $K \equiv m_e^2 \omega_e^2 \frac{\hbar}{2}$ plays the role of a cut-off parameter of the model, and we use a unit system $\hbar = 1$, $m_p = 1$, and $c = 1$. The wave function can be factorized to the scalar and gravitational parts.
\[ \Psi(y, \phi) = \exp(\pm iKc_0 \phi) \psi(y). \] (8)

Here the scalar field part is chosen to yield the classical momentum \( \pi_\phi = \pm Kc_0 \), where \( c_0 \) is the dimensionless parameter to determining the supercritical \((c_0 > 1)\), the critical \((c_0 = 1)\), and the subcritical \((1 > c_0 > 0)\) collapse.

Now the Wheeler-DeWitt equation becomes an ordinary differential equation

\[
-\frac{1}{2} \frac{d^2}{dy^2} + \frac{K}{2} \left(2 - y^2 - \frac{c_0^2}{y^2}\right) \psi(y) = 0.
\] (9)

The solution describing the black hole formation was obtained in the Ref. \[1\]:

\[
\psi_{BH}(y) = \left(\exp\left(-\frac{i}{2}Ky^2\right)\right) \left(Ky^2\right)^{\mu_-} M(a_-, b_-, iKy^2),
\] (10)

of which the asymptotic form \[3\] at the spatial infinity is

\[
\psi_{BH}(y) \simeq \Gamma(b_-) \left[\frac{e^{i\pi a_-}}{\Gamma(a_+)} (iKy^2)^{\mu_- - a_-} e^{-\frac{i}{2}Ky^2} + \frac{(iKy^2)^{\mu_- - a_+}}{\Gamma(a_-)} e^{\frac{i}{2}Ky^2}\right].
\] (11)

Here \( M \) is the confluent hypergeometric function and

\[
a_\pm = \frac{1}{2} \pm \frac{i}{2}(Q \mp K), \quad b_- = 1 - iQ, \quad \mu_- = \frac{1}{4} - \frac{i}{2}Q
\] (12)

with

\[
Q = \left(K^2c_0^2 - \frac{1}{4}\right)^{1/2}.
\] (13)

The S-matrix component describing the reflection rate is

\[
S = \frac{\Gamma(a_+)}{\Gamma(a_-)} \frac{(iK)^{a_- - a_+}}{e^{i\pi a_-}},
\] (14)

from which we obtain the transmission rate for black hole formation

\[
\frac{j_{trans}}{j_{in}} = 1 - |S|^2 = 1 - \frac{\cosh \frac{\pi}{2}(Q + K)}{\cosh \frac{\pi}{2}(Q - K)} e^{-\pi Q},
\] (15)

where \(|\Gamma\left(\frac{1}{2} + ix\right)|^2 = \pi / \cosh(\pi x)\) is used. Eq. \(13\) gives the probability of black hole formation for the supercritical, critical, and subcritical \(c_0\)-values.

In this brief report we consider the analytic structure of the S-matrix: It is an analytic function of \(Q\) and \(K\) with simple poles which can be explicitly shown as

\[
S = \sum_{N=0}^{\infty} \frac{1}{Q - K + i(2N+1)} \left(\frac{2ie^{-\frac{i}{2}K - iK\ln K}}{N!\Gamma(-N - iK)}\right).
\] (16)
The poles reside in the unphysical region of the parameter space of \( Q \) and \( K \):

\[
Q = \sqrt{K^2 c_0^2 - \frac{1}{4}} = K - i(2N + 1), \quad N = 0, 1, 2, \ldots.
\]

(17)

For physical processes of gravitational collapse there can not be poles because \( K \) and \( c_0 \) are real valued. In ordinary quantum mechanics, the poles of S-matrix occur at the bound state [6], and in relativistic scattering at the resonances or the Regge poles [7]. In our case we can not identify the poles with such bound states or resonances.

For physical interpretation of the poles we note that the replacement

\[
K = iK_E
\]

is, in effect, the change from the Lorentzian metric to the Euclidean one, and the Wheeler-DeWitt equation in the Euclidean sector becomes

\[
\left[ -\frac{1}{2K_E} \frac{d^2}{dy^2} + \frac{K_E}{2} \left( y^2 + \frac{c_0^2}{y^2} - 2 \right) \right] \psi_E = 0.
\]

(19)

Notice that this is a variant of Calogero models with the Calogero-Moser hamiltonian [8], but the energy eigenvalue is fixed, and only a quantized \( c_0 \) is allowed. The solution to the equation is of polynomial type

\[
\psi_E = e^{-\frac{1}{2}K_E y^2} \left( y \sqrt{2K_E} \right)^{K_E - 2N - \frac{1}{2}} \left( \sum_{m=0}^{N} a_m \left( \sqrt{2K_E} y \right)^m \right),
\]

(20)

\[
a_{m+2} = \frac{m - 2N}{(m + 2)(m + 2K_E - 4N)} a_m.
\]

(21)

Here we use the normalizability condition of the wave function \( \psi_E \), which gives the quantization of \( c_0 \) as

\[
K_E^2 c_0^2 + \frac{1}{4} = (K_E - 2N - 1)^2,
\]

(22)

or

\[
c_0^2 = \left( 1 - \frac{2N - 1}{K_E} \right)^2 - \frac{1}{4K_E^2}.
\]

(23)

The condition (22) is identical to the pole position of the S-matrix with \( K = iK_E \) given in (17).

The quantum solution (20) is analogous to an instanton in the sense it is a solution in the Euclidean sector. However, it is not an instanton which is strictly a classical solution. With instantons one can semiclassically evaluate the probability of tunneling process, while our solutions (20) provide the quantum probability through the pole contribution to the matrix as in (16).

The correspondence between the poles and the Euclidean polynomial solutions breaks down for large \( N \). While the poles contribute for all \( N \) without limit, the normalizable
Euclidean solutions exist only for $N < \frac{K_f E}{2}$. The polynomial solutions for large $N$ are well defined, but are not normalizable. We have not yet understood these nonnormalizable solutions.

One notable point of the Euclidean Wheeler-DeWitt equation is that it is a Calogero type system [8] which has been extensively discussed in connection with the motion of a particle near the Reissner-Nordström black hole horizon [9]. A surprising connection between black holes and conformal mechanics of De Alfaro, Fubini and Furlan [10] (DFF) has been established [11]. By studying the conformal quantum mechanics DFF suggested to solve the problem of motion to use a compact operator $L_0$ given as

$$L_0 = \frac{a}{4} \left( \frac{x^2}{a^2} + p^2 + \frac{g}{x^2} \right),$$

which is essentially identical to the Wheeler-DeWitt hamiltonian (19). The presence of conformal mechanics both in the Reissner-Nordström black hole and in the gravitational collapse forming black holes might suggest some deep nature of gravity in connection with conformal theory.

As a final remark we consider the classical field equations corresponding to the poles of the S-matrix. The relevant equations are

$$\frac{d\phi}{d\tau} = \frac{c_0}{y^2},$$

$$\left( \frac{dy}{d\tau} \right)^2 = K^2 \left( -2 + y^2 + \frac{c_0^2}{y^2} \right),$$

where in the Lorentz metric spacetime $c_0 \simeq 1 - i(2N + 1)/K$, for large $K$. The complex $c_0$ implies complex $\frac{d\phi}{d\tau}$, which may be imagined as a bound state like complex momentum in quantum mechanics. The complex $\frac{dy}{d\tau}$ is difficult to understand unless one considers complex spacetime metric. In the Euclidean case ($K = iK_E$) these classical equations are same as those equations with quantized $c_0$ in the tunneling region in Ref. [4]. It is beyond the scope of our present work to investigate the role of complex spacetime in gravitational collapse.

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