Accelerating Federated Learning via Sampling
Anchor Clients with Large Batches

Feijie Wu¹, Song Guo¹,²*, Zhihao Qu², Shiqi He³, Ziming Liu¹
¹ Department of Computing, The Hong Kong Polytechnic University
² School of Computer and Information, Hohai University
³ Department of Computer Science, The University of British Columbia
{harli.me, ziming.liu}@connect.polyu.hk, song.guo@polyu.edu.hk
quzhihao@hhu.edu.cn, shiqihe@cs.ubc.ca

Abstract

Using large batches in recent federated learning studies has improved convergence rates, but it requires additional computation overhead compared to using small batches. To overcome this limitation, we propose a unified framework FedAMD, which disjoints the participants into anchor and miner groups based on time-varying probabilities. Each client in the anchor group computes the gradient using a large batch, which is regarded as its bullseye. Clients in the miner group perform multiple local updates using serial mini-batches, and each local update is also indirectly regulated by the global target derived from the average of clients’ bullseyes. As a result, the miner group follows a near-optimal update towards the global minimizer, adapted to update the global model. Measured by $\epsilon$-approximation, FedAMD achieves a convergence rate of $O(1/\epsilon)$ under non-convex objectives by sampling an anchor with a constant probability. The theoretical result considerably surpasses the state-of-the-art algorithm BVR-L-SGD at $O(1/\epsilon^{3/2})$, while FedAMD reduces at least $O(1/\epsilon)$ communication overhead. Empirical studies on real-world datasets validate the effectiveness of FedAMD and demonstrate the superiority of our proposed algorithm.

1 Introduction

Federated learning (FL) [1-3] has witnessed an increasing interest over the past few years. As a distributed training paradigm, it enables a group of clients to collaboratively train a global model from decentralized data under the orchestration of a central server. By this means, the sensitive privacy is basically protected because the raw data are not shared across the clients. Due to the unreliable network connection and the rapid proliferation of FL clients, it is infeasible to require all clients to be simultaneously involved in the training. To address the issue, recent works [4-14] introduce a practical setting where merely a portion of clients participates in the training. The partial-client scenario effectively avoids the network congestion at the FL server and significantly shortens the waiting time as compared to traditional large-scale machine learning [15-18].

One intrinsic challenge in FL is data heterogeneity, indicating that the data are not independent or identical distributed (non-i.i.d.) among clients. The issue severely deteriorates the speed and the stability of the convergence when we adopt FedAvg [3-4, 19-21], a classic algorithm where participants perform multiple local SGD steps and synchronize with the server afterward. With data heterogeneity, the optimal model is subject to the local data distribution, and therefore, the local updates on the clients’ models greatly deviate from the update towards optimal global parameters [7, 22-29]. In particular, under the setting of partial client participation, Yang et al. [8] draws a

*Corresponding author
As a family of the practical solutions to data heterogeneity, variance reduced techniques \cite{6,7,29,31}, Table 1: Number of communication rounds and cumulative gradient complexity that achieve the theoretical statement that non-i.i.d. features significantly deteriorate the convergence property as much as linear to the number of local updates.

Table 1: Number of communication rounds and cumulative gradient complexity that achieve \( \mathbb{E}[\|\nabla F(\hat{x}_{\text{out}})\|^2] \leq \epsilon \) for non-convex objectives (or \( \mathbb{E}[F(\hat{x}_{\text{out}})] - F_* \leq \epsilon \) for PL condition or strongly-convex with the parameter of \( \mu \)). By default, we assume the small batch size is set to 1, and the size of the local dataset \( n \) is infinity. The symbol \( \checkmark \) or \( \times \) for "Partial Clients" is determined by the following footnote 1, while "." indicates the method is a centralized approach. "Cumulative Gradient Complexity" means the number of total samples called by all clients during the entire training.

As a family of the practical solutions to data heterogeneity, variance reduced techniques \cite{6,7,29,31,33,38} achieve an improved convergence rate when compared to FedAvg. With multiple local updates, each client corrects the SGD steps with reference to an estimated global target, which is synchronized at the beginning of every round. Although, in each transmission round, variance-reduced algorithms require the communication overhead twice as more as FedAvg, their improved performances are likely to eliminate the cost increments. Recent studies \cite{6,31,32,39} have demonstrated great potential using large batches under full client participation.\footnote{In this paper, partial client participation refers to the case where only a portion of clients take part at every round during the entire training.} Measured by \( \epsilon \)-approximation, MARINA \cite{6}, for instance, realizes \( O(1/\epsilon^1/2) \) faster while using large batches, where \( M \) indicates the number of the clients.

However, none of the prior studies address the drawbacks of using large batches. Typically, a large batch update involves several gradient computations compared to a small batch update. This increases the burden of FL clients, especially on IoT devices like smartphones, because their hardware hardly accommodates all samples in a large batch simultaneously. Instead, they must partition the large batch into several small batches to obtain the final gradient. Furthermore, regarding the critical convergent differences between various participation modes, the effect of using large batches under partial client participation cannot be affirmative. BVR-L-SGD \cite{31} and FedPAGE \cite{32} claim that they can work under partial client participant, but they require all clients’ participation when the algorithms come to the synchronization using a large batch.

Motivated by the observation above, we propose a unified framework named FedAMD under federated learning that disjoints the participants into two groups, i.e., anchor and miner group, according to time-varying probabilities. In the anchor group, clients compute the gradient using a large batch cached in the server to estimate the global orientation. In the miner group, clients perform multiple updates corrected according to the previous and the current local parameters and the last local update volume. The objective for the latter group is twofold. First, multiple local updates without serious deviation can effectively accelerate the training process. Second, we update the global model using the local models from the latter group only. Since the probability alternation mechanism can be time-varying, we separately consider constant and sequential probability settings.

| Convexity          | Method    | Partial Clients | Communication Rounds | Cumulative Gradient Complexity |
|--------------------|-----------|-----------------|----------------------|-------------------------------|
| Non-convex         | SGD (large batch) - | \( \frac{1}{\mu} - \frac{1}{\epsilon} \) | \( \frac{\mu}{\epsilon} \) |
|                    | PAGE \cite{30} \times | \( \frac{1}{\mu^{2/3}} - \frac{1}{\epsilon^{2/3}} \) | \( \frac{\mu^{2/3}}{\epsilon^{2/3}} + \frac{\mu^{2/3}}{\epsilon^{2/3}} \) |
|                    | SCAFFOLD \cite{7} \checkmark | \( \frac{1}{\mu^{2/3}} + \frac{1}{\epsilon^{2/3}} \) | \( \frac{\mu^{2/3}}{\epsilon^{2/3}} + \frac{\mu^{2/3}}{\epsilon^{2/3}} \) |
| PL condition       | BVR-L-SGD \cite{31} \times | \( \frac{1}{\mu^{2/3}} + \frac{1}{\epsilon^{2/3}} \) | \( \frac{\mu^{2/3}}{\epsilon^{2/3}} + \frac{\mu^{2/3}}{\epsilon^{2/3}} \) |
|                    | VR-MARINA \cite{6} \times | \( \frac{1}{\mu^{2/3}} + \frac{1}{\epsilon^{2/3}} \) | \( \frac{\mu^{2/3}}{\epsilon^{2/3}} + \frac{\mu^{2/3}}{\epsilon^{2/3}} \) |
|                    | FedPAGE \cite{32} \times | \( \frac{1}{\mu^{2/3}} + \frac{1}{\epsilon^{2/3}} \) | \( \frac{\mu^{2/3}}{\epsilon^{2/3}} + \frac{\mu^{2/3}}{\epsilon^{2/3}} \) |
|                    | FedAMD \checkmark | \( \frac{1}{\mu^{2/3}} + \frac{1}{\epsilon^{2/3}} \) | \( \frac{\mu^{2/3}}{\epsilon^{2/3}} + \frac{\mu^{2/3}}{\epsilon^{2/3}} \) |
Contributions. We summarize our contributions as follows:

- **Algorithmically**, we propose a unified federated learning framework FedAMD that identifies a participant as an anchor or a miner. Clients in the anchor group aim to obtain the bullseyes of their local data with a large batch, while the miners target to accelerate the training with multiple local updates using small batches.

- **Theoretically**, we establish the convergence rate for FedAMD under non-convex objectives under both constant probability settings and sequential probability settings. To the best of our knowledge, this is the first work to analyze the effectiveness of large batches under partial client participation. Our theoretical results indicate that, with the proper setting for the probability, FedAMD can achieve a convergence rate of $O(1/\epsilon)$ under non-convex objective, and linear speedup under Polyak-Łojasiewicz (PL) condition [40, 41]. Comprehensive comparisons with previous works are presented in Table 1.

- **Empirically**, we conduct extensive experiments to compare FedAMD with the most representative approaches. The numerical results provide evidence of the superiority of our proposed algorithm. Achieving the same test accuracy, FedAMD utilizes the less computational power metered by the cumulative gradient complexity.

2 Related Work

**Mini-batch SGD vs. Local SGD.** Distributed optimization is required to train large-scale deep learning systems. In mini-batch SGD, each client in parallel utilizes a mini-batch to calculate the corresponding gradient to improve the training efficiency such that mini-batch training arouses serious thinking [15, 42, 43]. However, mini-batch SGD also has the problem of low computational efficiency. Some distributed deep learning frameworks training with large-batch only [44–46] often meets generalization issues, which increases training errors [47]. Therefore, local SGD (also known as FedAvg) [21, 48–50] has become a more practical way to perform multiple local updates on each device before exchanging between devices. Bijral et al. [51] analyzed the spectral norm of different datasets and constructed a graph of different clients to study local SGD. While Yun et al. [52] focused on shuffling-based variants, that is, the practical gradient can be obtained without replacing sampling.

**Federated Learning.** FL was proposed to ensure data privacy and security [53], and now it has become a hot field in the distributed system [54–60]. The FL training methods in the past few years usually require all trainers to participate in each training session [53], but this is obviously impractical when facing the increase in FL clients. To enhance the systems’ feasibility, this work assumes that a fixed number of clients are sampled at each round, which is widely adopted in [4–10, 12]. Therefore, the server collects the data from these participation every synchronization to update the model parameters [4–14].

**Variance Reduction in Finite-sum Problems.** Variance reduction techniques [61, 62] was once proposed for traditional centralized machine learning to optimize finite-sum problems [78–80] by mitigating the estimation gap between small-batch [81, 83] and large-batch [84, 86]. SGD randomly samples a small-batch and computes the gradient in order to approach the optimal solution. Since the data are generally noisy, an insufficiently large batch results in convergence rate degradation. By utilizing all data in every update, GD can remove the noise affecting the training process. However, it is time-consuming because the period for a single GD step can implement multiple SGD updates. Based on the trade-off, variance-reduced methods periodically perform GD steps while correcting SGD updates with reference to the most recent GD steps.

**Variance Reduction in FL.** The variance-reduced techniques have critically driven the advent of FL algorithms [6, 7, 29, 31, 33–38] by correcting each local computed gradient with respect to the global orientation. However, a concern is addressed on how to attain an accurate global orientation to mitigate the update drift from the global model, especially under the communication-efficient settings where clients perform numerous local updates. SCAFFOLD [7] computes the global orientation by the caching SGD gradients. Regarding that the gradients on the small-batches cannot truly represent the expected ones, the estimated global orientation greatly differs from the real one. Therefore, it does not possess a remarkable performance in most cases [87]. Utilizing large-batches to approximate the global orientation, BVR-L-SGD [31] is able to perform model training with a single client for a better convergence rate. Nevertheless, its shortage is apparent: Prior to each model training, every
client is required to compute the gradient using large-batch. Not only is the step-by-step operation time-consuming, but also it is impractical to wait for all clients to push the gradients.

3 FedAMD

In this section, we comprehensively describe the technical details of FedAMD, a federated learning framework that probabilistically disjoints the participants into the anchor group and the miner groups, and its pseudo-code is illustrated in Algorithm 1.

Problem Formulation. In an FL system with a total of $M$ clients, the objective function is formalized as

$$
\min_{\mathbf{x} \in \mathbb{R}^d} F(\mathbf{x}) = \frac{1}{M} \sum_{m \in [M]} F_m(\mathbf{x})
$$

(1)

where we define $[M]$ for a set of $M$ clients. $F_m(\cdot)$ indicates the local expected loss function for client $m$, which is unbiased estimated by empirical loss $f_m(\cdot)$ using a random realization $B_m$ from the local training data $D_m$, i.e., $\mathbb{E}_{B_m \sim D_m} f_m(\mathbf{x}, B_m) = F_m(\mathbf{x})$. We denote $n$ by the size of a client’s local dataset, i.e., $|D_m| = n$ for all $m \in [M]$. Besides, $F_*$ and $\mathbf{x}_*$ represent the minimum loss and the optimal parameters for Equation (1), respectively.

Algorithm Description. In FedAMD, a global model is initialized with arbitrary parameters $\mathbf{x}_0 \in \mathbb{R}^d$. With such a model distributed to all clients (Line 1), client $m \in [M]$ is required to generate a large batch $B_{m,0}$ for the size of $b$ and compute the gradient $v_0^{(m)}$ (Line 3). Then, clients send $v_0^{(m)}$ to the server (Line 4), and server caches these gradients and span them as a matrix $v_0 = \{v_0^{(m)}\}_{m \in [M]}$ (Line 6).

After the initialization steps above, the algorithm comes to the model training (Line 7–27). At the beginning of each round $t$, the server randomly picks a subset $A$ with the size of $A$ from $M$ clients (Line 8). Since each client is independently selected without replacement, under the setting of Equation (1), clients have an equal chance to be selected with the probability of $\frac{A}{M}$. Subsequently, the server distributes the global model $\hat{\mathbf{x}}_t$ to the clients in the set $A$, accompanying the averaged caching gradient $\hat{\mathbf{g}}_t$, i.e., $\hat{\mathbf{g}}_t = \frac{1}{M} \sum_{m \in [M]} v_0^{(m)}$ (Line 9). With the probability of $p_t$, client $i \in A$ is classified for the anchor group (Line 13–14) or the miner group (Line 16–23), and both have different objectives and focus on various tasks.

Anchor group (Line 13–14). Clients in this group target to discover the bullseyes of their local data. According to Line 12, client $i \in A$ has the probability of $p_t$ to become a member of this group. Then, the client utilizes a large batch $B_{i,t}$ with $b$ samples to obtain the gradient $v_{t+1}^{(i)}$ (Line 13). Therefore, following the gradient $v_{t+1}^{(i)}$ can find an optimal or near-optimal solution for client $i$. Next, the client pushes the gradient to the server and updates the caching gradient (Line 14). In view that some clients do not participate in the anchor group for obtaining the bullseyes at round $t$, the server spontaneously inherits their previous calculation from $v_t$ (Line 10). As a result, $\hat{\mathbf{g}}_t$ in Line 9 indicates an approximate orientation towards global optimal parameters, which directs the local update in the miner group and affects the final global update. Referring to Line 9, $v_{t+1}^{(i)}$ influences the training from round $t + 1$ up to the next time when client $i$ is a member of anchor group.

Miner group (Line 16–23). A traditional approach to calculate the gradients is based on the empirical results, where client $i$ randomly samples a small batch $B_{i,t}^\prime$ with the size of $b'$ and computes on the model $\mathbf{x}_{t,k}^{(i)}$ using the loss function such as cross entropy loss. In this case, the empirical gradient should be $\nabla f_i \left( \mathbf{x}_{t,k}^{(i)}, B_{i,t}^\prime \right)$, which is unbiased estimated to $\nabla F_i \left( \mathbf{x}_{t,k}^{(i)} \right)$ in expectation. In reality, $\nabla f_i \left( \mathbf{x}_{t,k}^{(i)}, B_{i,t}^\prime \right)$ cannot fully recover $\nabla F_i \left( \mathbf{x}_{t,k}^{(i)} \right)$ and therefore, following $\mathbf{x}_{t,k+1}^{(i)} = \mathbf{x}_{t,k}^{(i)} - \eta_t \nabla f_i \left( \mathbf{x}_{t,k}^{(i)}, B_{i,t}^\prime \right)$ to update the local model undoubtedly deviates to the local optimizer, which seriously affects the convergence property. To avoid the significant deviation happens during the local training, clients in the miner group update the local model following a clear direction such that the training process can be accelerated.
Algorithm 1 FedAMD

Input: local learning rate $η_i$, global learning rate $η_g$, minibatch size $b'$, $b' < b$, local updates $K$, probability $p_t \in [0, 1]$ for $t \geq 0$, initial model $\tilde{x}_0$.

1: Communicate the initial model $\tilde{x}_0$ with all clients $m \in [M]$ 
2: for $m \in [M]$ in parallel do 
3: \begin{align*}
    v_0^{(m)} &= \nabla f_m(\tilde{x}_0, B_{m,0}) \text{ using } B_{m,0} \sim D_m \text{ with the size of } b
\end{align*}
4: Communicate $v_0^{(m)}$ with the server 
5: end for 
6: Initialize caching gradient $v_0 = \left\{ v_0^{(m)} \right\}_{m \in [M]}$
7: for $t = 0, 1, 2, \ldots$ do 
8: Sample clients $A \subseteq [M]$ 
9: Communicate the model $\tilde{x}_t$ and the caching gradient $\tilde{g}_t = avg(v_t)$ with clients $i \in A$
10: Initialize subsequent caching gradient $v_{t+1} = v_t$
11: for $i \in A$ in parallel do 
12: if $\text{Bernoulli}(p_t) == 1$ then 
13: \begin{align*}
    v_{t+1}^{(i)} &= \nabla f_i(\tilde{x}_t, B_{i,t}) \text{ using } B_{i,t} \sim D_i \text{ with the size of } b
\end{align*}
14: Communicate $v_{t+1}^{(i)}$ with the server and indicate the update of caching gradient 
15: else 
16: Initialize $x_{t,0}^{(i)} = x_0^{(i)} = \tilde{x}_t, g_{t,0}^{(i)} = \tilde{g}_t$
17: for $k = 0, \ldots, K - 1$ do 
18: Generate random realization $B_{i,k} \sim D_i$ with the size of $b'$ 
19: \begin{align*}
    g_{t,k+1}^{(i)} &= g_{t,k}^{(i)} - \nabla f_i \left( x_{t,k}^{(i)}, B_{i,k}' \right)
\end{align*}
20: \begin{align*}
    x_{t,k+1}^{(i)} &= x_{t,k}^{(i)} + \eta_t \cdot g_{t,k+1}^{(i)}
\end{align*}
21: end for 
22: $\Delta x_t^{(i)} = \tilde{x}_t - x_{t,K}^{(i)}$
23: Communicate $\Delta x_t^{(i)}$ with the server and indicate the update of the model 
24: end if 
25: end for 
26: $\tilde{x}_{t+1} = \tilde{x}_t - η_g \cdot avg(\Delta x_t)$ where $\Delta x_t$ aggregates $\Delta x_t^{(i)}$ where client $i$ updates model 
27: end for 

First, client $i$ initializes the model with $\tilde{x}_t$ and the target direction with $\tilde{g}_t$ (Line 16). Then, in the subsequent $K$ local updates (Line 17), the client generates a $b'$-sample realization $B_{i,k}'$ (Line 18) and calculates the update $g_{t,k+1}^{(i)}$ via the variance reduced technique, i.e., $g_{t,k+1}^{(i)} = g_{t,k}^{(i)} - \nabla f_i \left( x_{t,k}^{(i)}, B_{i,k}' \right) + \nabla f_i \left( x_{t,k-1}^{(i)}, B_{i,k}' \right)$ (Line 19). Apparently, $g_{t,0}^{(i)}$ contains the information of the global target. The update $g_{t,k+1}^{(i)}$ can be treated as the estimation of the global orientation at the model $x_{t,k}^{(i)}$ because the term $\left( \nabla f_i \left( x_{t,k}^{(i)}, B_{i,k}' \right) - \nabla f_i \left( x_{t,k-1}^{(i)}, B_{i,k}' \right) \right)$ compensates the update difference between $x_{t,k}^{(i)}$ and $x_{t,k-1}^{(i)}$ for all iterations $k \in \{0, \ldots, K - 1\}$. Compared to SCAFFOLD [7] and FedLin [20] using a constant correction term, i.e., $\tilde{g}_t - v_t^{(i)}$, this approach can dynamically adjust the calibration volume subject to the model parameters $x_{t,k}^{(i)}$. Therefore, the local model update follows $x_{t,k+1}^{(i)} = x_{t,k}^{(i)} - \eta_t \cdot g_{t,k+1}^{(i)}$ (Line 20). After $K$ local updates, the model changes on client $i$ is $\Delta x_t^{(i)} = \tilde{x}_t - x_{t,K}^{(i)}$. Then, the client transmits $\Delta x_t^{(i)}$ to the server for the purpose of global model update.

Therefore, after the separate local training on the participants, the server merges the model changes from the miner group into $\Delta x_t$ (Line 26) and updates the caching gradients from the anchor group (Line 14). It is noted that the size of $\Delta x_t$ can be within the range between 0 and $A$. When the size is 0, $\tilde{x}_{t+1} = \tilde{x}_t$, or otherwise, $\tilde{x}_{t+1} = \tilde{x}_t - \eta_g \sum |\Delta x_t|/|\Delta x_t|$ (Line 26). The reason why we solely use the changes from the miner group is that clients perform multiple local updates regulated by the
global target such that the model changes walk towards the global optimal solution. While directly incorporating the new gradients from the anchor group, the global model has a degraded performance because they perform a single update that aims to find out the local bullseye deviated from the global target. Implicitly, clients in the miner group take in the update of caching gradients at iteration \( k = 0 \) to update the local model, which will affect the next global parameters.

**Previous Algorithms as Special Cases.** The probabilities can vary among the rounds that disjoint the participants into the anchor group and the miner group. By setting \( A = M \), and the probability \( \{p_t\} \) following the sequence of \( \{1, 0, 1, 0, \ldots\} \), FedAMD reduces to distributed minibatch SGD \((K = 1)\) or BVR-L-SGD \((K > 1)\). Therefore, FedAMD subsumes the existing algorithms and takes partial client participation into consideration. To obtain the best performance, we should tune the settings of \( \{p_t\} \) and \( K \). However, accounting for the generality of FedAMD, it faces substantial additional hurdles in its convergence analysis, which is one of our main contributions, as detailed in the following section.

### 4 Theoretical Analysis

In this section, we analyze the convergence rate of FedAMD under non-convex objectives with respect to \( \epsilon \)-approximation, i.e., \( \min_{t \in [T]} \| \nabla F(\tilde{x}_t) \|^2_2 \leq \epsilon \). Specifically, when it comes to PL condition, \( \epsilon \)-approximation refers to \( F(\tilde{x}_F) - F(x_*) \leq \epsilon \). In the following discussion, we particularly highlight the setting of \( \{p_t\} \) to obtain the best performance. Before stating the convergence result, we make the following assumptions, where the first two assumptions have been widely used in machine learning studies \([7,9]\), while the last one has been adopted in some recent works \([6,31,39]\).

**Assumption 1** (L-smooth). The local objective functions are Lipschitz smooth: For all \( v, \bar{v} \in \mathbb{R}^d \),

\[
\| \nabla F_i(v) - \nabla F_i(\bar{v}) \|_2 \leq L \| v - \bar{v} \|_2, \quad \forall i \in [M].
\]

**Assumption 2** (Bounded Noise). For all \( v \in \mathbb{R}^d \), there exists a scalar \( \sigma \geq 0 \) such that

\[
\mathbb{E}_{B \sim \mathcal{D}_t} \| \nabla f_i(v, B) - \nabla F_i(v) \|_2^2 \leq \frac{\sigma^2}{|B|}, \quad \forall i \in [M].
\]

**Assumption 3** (Average L-smooth). For all \( v, \bar{v} \in \mathbb{R}^d \), there exists a scalar \( L_{\sigma} \geq 0 \) such that

\[
\mathbb{E}_{B \sim \mathcal{D}_t} \left( \| \nabla f_i(v, B) - \nabla f_i(\bar{v}, B) \|_2^2 \right) \leq \frac{L^2_{\sigma}}{|B|} \| v - \bar{v} \|_2^2, \quad \forall i \in [M].
\]

**Remark.** Assumption 3 definitely provides a tighter bound for the patterns of variance reduction. In fact, solely with Assumption 2, the term \( \mathbb{E}_{B \sim \mathcal{D}_t} \left( \| \nabla f_i(v, B) - \nabla f_i(\bar{v}, B) \|_2^2 \right) \) can be bounded by a constant. Therefore, we can easily obtain the coefficient for \( \| v - \bar{v} \|_2^2 \) which could be with the same structure as the constant in RHS of Assumption 2. Furthermore, if the loss function is also Lipschitz continuous, e.g., cross entropy loss, we can derive a similar structure as presented in Assumption 3.

#### 4.1 Sequential Probability Settings

As mentioned in Section 3, a recursive pattern appeared in the probability sequence \( \{p_t \in \{0, 1\}\} \) can reduce FedAMD to the existing works. We assume that the length of pattern is \( \tau \geq 2 \), such that

\[
p_t = \begin{cases} 
1, & t \mod \tau = 0 \\
0, & \text{Otherwise}
\end{cases}
\]

We derive the following results under sequential probability settings with full client participation.

**Theorem 1.** Suppose that Assumption 1 and 2 hold, and all clients participate in the training, i.e., \( A = M \). Let the local updates \( K \geq 1 \), the minibatch size \( b = \min \left( \frac{16\sigma^2}{3T}, n \right) \) and \( b' < b \). Additionally, the settings for the local learning rate \( \eta_l \) and the global learning rate \( \eta_g \) satisfy the following two constraints: (1) \( \eta_l \eta_g = \frac{1}{KL(1+2\tau)} \); and (2) \( \eta_l \leq \frac{1}{2\sqrt{6KL}} \). Then, to
find an $\epsilon$-approximation of non-convex objectives, i.e., \( \min_{t \in [T]} \| \nabla F(\hat{x}_t) \|_2^2 \leq \epsilon \), the number of communication rounds $T$ performed by FedAMD should be

$$T = O \left( \frac{\tau(1 + 2\tau)}{\tau - 1} \cdot \frac{1}{\epsilon} \right)$$

where we treat $\nabla F(\hat{x}_0) - F_*$ and $L$ as constants.

**Comparison with BVR-L-SGD.** Theorem [31] presents that we can use the less communication rounds (i.e., $T = O(1/\epsilon)$) to obtain the expected result at $\tau = 2$ where FedAMD reduces to BVR-L-SGD [31]. This result coincides with the complexity of BVR-L-SGD in Table [31] by the setting that (1) $nM \leq \frac{1}{\tau}$, and (2) $K \geq \sqrt{n/M}$. In other words, we theoretically prove that BVR-L-SGD still achieves $\min_{t \in [T]} \| \nabla F(\hat{x}_t) \|_2^2 \leq \epsilon$ in a looser constraint.

When we consider the online cases where the local dataset is infinity large, i.e., $n \to \infty$, a major concern for $\tau = 2$ is the cumulative gradient complexity, i.e., the number of total samples used by all clients during the training. The following corollary discusses the cumulative gradient complexity with respect to $\tau$.

**Corollary 1.** Suppose that Assumptions [7] and [3] hold, and all clients participate in the training, i.e., $A = M$, and the size of local dataset tends to be infinite, i.e., $n \to \infty$. Let the local learning rate $\eta_l$ and the global learning rate $\eta_s$ satisfy the following two constraints: (1) $\eta_l \eta_s = \frac{K L}{K L (1 + 2\tau)}$; and (2) $\eta_l \leq \min \left( \frac{1}{2\sqrt{6KL}}, \frac{\sqrt{2N/K}}{\sqrt{2sL\epsilon}} \right)$. By setting the minibatch size $b = \frac{16s^2}{M \epsilon}$ and a constant $\xi$ such that $KLb' = b/\xi$, the number of total samples called by all clients (i.e., cumulative gradient complexity) is $O \left( \frac{(1 + 2\tau)(\xi + \tau - 1)}{(\tau - 1)\xi} \cdot \frac{n^2}{s\tau} \right)$. Then, the optimal cumulative gradient complexity is $O \left( \frac{n^2}{s\tau} \right)$ at $\tau = \left[ 1 + \sqrt{\frac{2N}{s}} \right]$, where $\lceil \cdot \rceil$ indicates the closest integer.

**Remark.** This corollary reveals the relation between $\tau$ and cumulative gradient complexity. As we know, the smaller the cumulative gradient complexity is, the less time the local training requires. When the constant $\xi$ is sufficiently large, $\tau = 2$ cannot be the best setting because it requires a large amount of time on model training. However, when we use a larger $\tau$, more communication rounds are required, indicating that the total communication time increases as well. Therefore, $\tau$ should be set based on the trade-off between the local model training and the communication period.

### 4.2 Constant Probability Settings

Apparently, when we set the constant probability as 1, all participants are in the anchor group such that the model cannot be updated. Likewise, when the constant probability is 0, all participants are in the miner group such that the global target cannot be updated, leading to degraded performance. Therefore, we manually define a constant $p \in (0, 1)$ such that $\{ p_t = p \}_{t \geq 0}$. In this section, we derive the following results with partial client participation.

**Theorem 2.** Suppose that Assumptions [7] and [3] hold. Let the local updates $K \geq \max \left( 1, \frac{2L^2}{(6 - 1)\epsilon L^2} \right)$, the minibatch size $b = \min \left( \frac{6\sigma^2(1-p^2)}{M \epsilon}, n \right)$ and $b' = \max \left( 1, \frac{2L^2}{K L^2} \right)$, the local learning rate $\eta_l = \frac{1}{2\sqrt{6KL}}$, and the global learning rate $\eta_s = \frac{2\sqrt{6}}{L \sqrt{1 - p^2}}$. Then, to find an $\epsilon$-approximation of non-convex objectives, i.e., $\min_{t \in [T]} \| \nabla F(\hat{x}_t) \|_2^2 \leq \epsilon$, the number of communication rounds $T$ performed by FedAMD should be

$$T = O \left( \frac{1}{\epsilon} \left( 1 + \frac{M \sqrt{1 - p^2}}{Ap} \right) \right)$$

where we treat $\nabla F(\hat{x}_0) - F_*$ and $L$ as constants.

**Remark.** Theorem [32] conveys a conclusion that a greater $p$ can realize $\epsilon$-approximation with less communication rounds. This means when the number of the anchor nodes is sufficiently large, only one miner node still achieves a better convergence rate. However, when $n$ is super large and even infinite, such a setting is infeasible because the cumulative gradient complexity could be very large if...
\( \epsilon \) is a value extremely close to 0. Therefore, the best setting for \( p \) should be a root of the equation
\[
\frac{\sqrt{1-p^2}}{p} = \frac{A}{M} \text{, where } c \text{ is a constant greater than 1.}
\]

Next, under Theorem 4, we provide the settings for the constant probability \( p \) that leads to the optimal convergence result. Based on the value of \( p \), we further refine the settings for other parameters. The following corollary takes a setting of \( b' = 1 \) into consideration, i.e., the small batch size is 1.

**Corollary 2.** Suppose that Assumption 1, 2, 3 and 4 hold. Let the constant probability \( p = \frac{M}{Ac} < 1 \), where \( c \) is a constant greater than 1, the local updates \( K \geq \max \left( 1, \frac{2L^2}{b}, 1 \right) \), the minibatch size \( b = \min \left( \frac{A^2}{M}, n \right) \) and \( b' = 1 \), the local learning rate \( \eta_l = \frac{1}{2\sqrt{6KL}} \), and the global learning rate \( \eta_s = \frac{2\sqrt{\sigma}}{1+2\sqrt{6}} \). Then, after the communication rounds of \( T = O(1/\epsilon) \), we have \( \min_{t \in [T]} \| \nabla F(\tilde{x}_t) \|^2_2 \leq \epsilon \). Therefore, during the training process, the number of total samples called by all clients (i.e., cumulative gradient complexity) is \( O \left( \frac{\sigma^2}{\epsilon^2} + AK \right) \).

In addition to the generalized non-convex objectives, we investigate the convergence rate of PL condition, a special case under non-convex objectives. The following assumption describes this case:

**Assumption 4 (PL Condition [88]).** The objective function \( F \) satisfies the PL condition when there exists a scalar \( \mu > 0 \) such that
\[
\| \nabla F(v) \|^2_2 \geq 2\mu (F(v) - F_s), \quad \forall v \in \mathbb{R}^d.
\]

It is noted that the convergence rate under PL condition can achieve linear speedup (i.e., \( O \left( \log \frac{1}{\epsilon} \right) \)) in Corollary 3 as compared to the sublinear convergence under non-convex objectives (i.e., \( O \left( \frac{1}{\epsilon} \right) \)) in Corollary 2. The following derives the theoretical results under PL condition with partial client participation.

**Theorem 3.** Suppose that Assumption 1, 2, 3 and 4 hold. Let the local updates \( K \geq \max \left( 1, \frac{2L^2}{b-1} \right) \), the minibatch size \( b = \min \left( \frac{A^2}{M}, n \right) \) and \( b' = \max \left( 1, \frac{2L^2}{KL^2} \right) \), the local learning rate \( \eta_l = \frac{1}{2\sqrt{6KL}} \), and the global learning rate \( \eta_s = \min \left( \frac{2\sqrt{6LAp}}{M\mu}, \frac{2\sqrt{\sigma}}{1+2\sqrt{6}L^2} \right) \). Then, to find an \( \epsilon \)-approximation of PL condition, i.e., \( F(\tilde{x}_T) - F(x_*) \leq \epsilon \), the number of communication rounds \( T \) performed by FedAMD should be
\[
T = O \left( \frac{M}{Ap} + \frac{1}{\mu} \left( 1 + \frac{M}{Ap} \sqrt{1-p^A} \right) \right) \log \frac{1}{\epsilon}
\]

where we treat \( \nabla F(\tilde{x}_0) - F_s \) and \( L \) as constants.

By setting an appropriate constant \( p \), we can obtain the best convergence rate of Theorem 3. Then, we can further adjust the value of the hyper-parameters such that we can set the small batch size to be 1.

**Corollary 3.** Suppose that Assumption 1, 2, 3 and 4 hold. Let the constant probability \( p = \frac{M}{Ac} < 1 \), where \( c \) is a constant greater than 1, the local updates \( K \geq \max \left( 1, \frac{2L^2}{b}, 1 \right) \), the minibatch size \( b = \min \left( \frac{A^2}{M}, n \right) \) and \( b' = 1 \), the local learning rate \( \eta_l = \frac{1}{2\sqrt{6KL}} \), and the global learning rate \( \eta_s = \min \left( \frac{2\sqrt{6LAp}}{M\mu}, \frac{2\sqrt{\sigma}}{1+2\sqrt{6}L^2} \right) \). Then, after the communication rounds of \( T = O \left( \left( 1 + \frac{1}{p} \right) \log \frac{1}{\epsilon} \right) \), we have \( F(\tilde{x}_T) - F(x_*) \leq \epsilon \). Therefore, during the training process, the number of total samples called by all clients (i.e., cumulative gradient complexity) is \( O \left( \frac{\sigma^2}{\epsilon^2} + AK \right) \left( 1 + \frac{1}{p} \right) \log \frac{1}{\epsilon} \).

## 5 Experiments

This section presents the experiments of our proposed approach and other existing baselines that are most relative to this work. We also investigate the effectiveness of probability \( \{p_t\}_{t \geq 0} \). Account for the limited space, numerical analysis on other factors like the number of local updates is presented in the supplementary materials.
Effectiveness of probability \{p_t\}_{t \geq 0}. Figure 1 demonstrates the performance of various probability settings under the scenarios of different participants. In Figure 1a and 1b with 20 clients, the sequential probability settings are more stable than the constant probability settings. With sequential probability settings, there is no distinct difference between the patterns of \{0, 1\} and \{0, 0, 1\}. In Figure 1c and 1d with 40 clients, some different phenomena can be observed. First, with sequential probability settings, the pattern of \{0, 0, 1\} has much worse performance than the pattern of \{0, 1\}. This empirically validates Theorem 1 for the best setting \(\tau = 2\) in terms of communication complexity. Furthermore, with the constant probability settings, \(p = 0.3\) dominates \(p = 0.6\) in terms of the convergence speed, although they eventually have similar test accuracy and training loss. The result depicts that a small probability does not always negatively affect convergence performance. Last but not least, it is noted that \(p = 0.6\) in 20 clients has a similar effect to \(p = 0.3\) in 40 clients, meaning that with the same number of anchor nodes, the convergence could be quite similar. This provides empirical evidence of the validness of Corollary 2.

Comparison with the state-of-the-art works. Figure 2 compares FedAMD with the existing works under partial client participation. In Figure 2a and 2c with 20 clients, FedAMD displays its superior performance under the constant and the sequential probability settings. First, under the same gradient complexity, FedAMD always dominates BVR-L-SGD and FedPAGE in terms of test accuracy and training loss. Although FedAvg surpasses FedAMD at the small cumulative gradient complexity, our proposed method (at the accuracy of 77.16\%) eventually has a better result than FedAvg (at the accuracy of 73.82\%). In Figure 2b and 2d with 40 clients, it is noticeable that all benchmarks have great improvement, especially FedPAGE. However, such improvements still cannot make them outperform FedAMD. Knowing that each marker in these four figures represents a communication round, we infer the computation complexity per round following the order that FedAMD < FedAvg < FedPAGE < BVR-L-SGD. Therefore, this set of empirical results verifies Table 1 that FedAMD can utilize the smallest cumulative gradient complexity to achieve the best convergence performance.
6 Conclusion

In this work, we investigate a unified federated learning framework FedAMD that disjoints the clients into anchor and miner group based on time-varying probabilities. We provide the convergence analysis of our proposed algorithm for constant and sequential probability settings. Under the partial-client scenario, FedAMD achieves sublinear speedup under non-convex objectives and linear speedup under PL condition. To the best of our knowledge, this is the first work to analyze the effectiveness of large batches under partial client participation. Experimental results demonstrate that FedAMD is superior to the state-of-the-art works. A promising future study is to remove the central control from the FL server and address large batch on the dynamic device unavailability.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [No] But we point out a promising direction in Section 6, which we cannot solve at the current stage. Also, our theoretical analysis is built on the assumptions, although they are widely used in the previous works.
   (c) Did you discuss any potential negative societal impacts of your work? [No] We believe our work has the potential to be implemented in practical scenarios.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] The assumptions and the theorems are rigorously and comprehensively presented in Section 4.
   (b) Did you include complete proofs of all theoretical results? [Yes] All proof are deferred to the appendix.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] The code is included in the supplementary materials.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 5 and the appendix.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No] We run each experiment for at least three times. To clearly present the results, we remove the error bars. In our source code file, we will give the detailed settings, including the seeds we use for the training.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See the supplementary materials for more details.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] See Section 5.
   (b) Did you mention the license of the assets? [N/A] The source codes are completely produced by ourselves.
   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] We provide our source code for the experiments in the supplementary materials. Also, our code includes how to download the public dataset.
(d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [N/A] We utilize the public dataset named Fashion MNIST [91].

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]
A Useful Lemmas

Prior to giving the detailed proofs of the theorems, we cover some technical lemmas in this section, and all of them are valid to the general cases.

**Lemma 1.** Let \( \varepsilon = \{\varepsilon_1, \ldots, \varepsilon_a\} \) be the set of random variables in \( \mathbb{R}^{a \times d} \). Every element in \( \varepsilon \) is independent with others. For \( i \in \{1, \ldots, a\} \), the value for \( \varepsilon_i \) follows the setting below:

\[
\varepsilon_i = \begin{cases} e_i, & \text{probability } q \\ 0, & \text{otherwise} \end{cases}
\]

where \( q \) is a constant real number between 0 and 1, i.e., \( q \in [0, 1] \). Let \(| \cdot |\) indicate the length of a set, \( \varepsilon \setminus \{0\} \) represent a set in which an element is in \( \varepsilon \) but not 0, and \( \text{avg}(\varepsilon) \) be the averaged result with the exception of zero vectors, i.e.,

\[
\text{avg}(\varepsilon) = \begin{cases} \frac{1}{|\varepsilon \setminus \{0\}|} \sum_{i=1}^{a} \varepsilon_i, & |\varepsilon \setminus \{0\}| \neq 0 \\ 0, & |\varepsilon \setminus \{0\}| = 0 \end{cases}
\]

Then, the following formulas hold for \( \mathbb{E}(\text{avg}(\varepsilon)) \) and its second norm \( \mathbb{E}\|\text{avg}(\varepsilon)\|_2^2 \):

\[
\mathbb{E}(\text{avg}(\varepsilon)) = (1 - (1 - q)^a) \cdot \frac{1}{a} \sum_{i=1}^{a} e_i;
\]

\[
\mathbb{E}\|\text{avg}(\varepsilon)\|_2^2 \leq (1 - (1 - q)^a) \cdot \frac{1}{a} \sum_{i=1}^{a} \|e_i\|_2^2
\]

**Proof.** When \( q = 0 \), the formulas in Equation 4 obviously hold because \( \mathbb{E}(\text{avg}(\varepsilon)) = 0 \) and \( \mathbb{E}\|\text{avg}(\varepsilon)\|_2^2 = 0 \). As for \( q = 1 \), since \( \text{avg}(\varepsilon) = \frac{1}{a} \sum_{i=1}^{a} e_i \), we leverage Cauchy–Schwarz inequality and get \( \mathbb{E}\|\text{avg}(\varepsilon)\|_2^2 = \frac{1}{a^2} \sum_{i=1}^{a} \|e_i\|_2^2 \leq \frac{1}{a} \sum_{i=1}^{a} \|e_i\|_2^2 \), which is consistent with the formulas in Equation 4. In addition to the preceding cases, we consider some general cases for the probability \( q \) within 0 and 1, i.e., \( q \in (0, 1) \).

Firstly, we show the proof details for \( \mathbb{E}(\text{avg}(\varepsilon)) \). For all \( i \in \{1, \ldots, a\} \), given that \( \varepsilon_i \) is not a zero vector, the coefficient of \( e_i \) is based on the binomial distribution on how many non-zero elements in the set \( \{\varepsilon_1, \ldots, \varepsilon_{i-1}\} \cup \{\varepsilon_{i+1}, \ldots, \varepsilon_a\} \). Therefore, with the probability \( q \) that \( \varepsilon_i \) is equal to \( e_i \), the coefficient of \( e_i \) in the expectation form is

\[
q \left( \frac{1}{a} \cdot \frac{(a-1)^{a-1}}{(a-1)} \cdot \frac{1}{a} \cdot \frac{(a-1)^{a-1}}{0} \right).
\]

Then, the coefficient of \( \frac{1}{a} e_i \) can be expressed and simplified for

\[
q \left( \frac{1}{a} \cdot \frac{(a-1)^{a-1}}{(a-1)} q^{a-1} + \cdots + \frac{1}{1} \cdot \frac{(a-1)^{a-1}}{0} (1-q)^{a-1} \right).
\]

Equation 5 follows

\[
\binom{\alpha}{\beta} = \frac{\alpha}{\beta} \cdot \frac{(\alpha-1) \times \cdots \times (\alpha-\beta+1)}{1 \times \cdots \times (\beta-1)} = \frac{\alpha}{\beta} \binom{\alpha-1}{\beta-1}, \quad \forall \alpha \geq \beta > 0
\]

and Equation 8 follows

\[
(q + (1-q))^a = \frac{(a)}{0} q^a + \cdots + \frac{(a)}{0} (1-q)^a.
\]

Thus, the equation \( \mathbb{E}(\text{avg}(\varepsilon)) = (1 - (1 - q)^a) \cdot \frac{1}{a} \sum_{i=1}^{a} e_i \) holds.
Secondly, we provide the analysis for $\mathbb{E} \|\text{avg}(\varepsilon)\|^2_2$. Based on the definition for $\text{avg}(\varepsilon)$ in Equation (3), we discuss the case $|\varepsilon \setminus \{0\}| \neq 0$. By means of Cauchy-Schwarz inequality, we can obtain the following inequality:

$$\left\| \frac{1}{|\varepsilon \setminus \{0\}|} \sum_{i=1}^{a} \varepsilon_i \right\|_2^2 = \left\| \frac{1}{|\varepsilon \setminus \{0\}|} \sum_{i, i \neq 0} \varepsilon_i \right\|_2^2 \leq \frac{1}{|\varepsilon \setminus \{0\}|} \sum_{i, i \neq 0} \|\varepsilon_i\|_2^2 = \frac{1}{|\varepsilon \setminus \{0\}|} \sum_{i=1}^{a} \|\varepsilon_i\|_2^2 (9)$$

Therefore,

$$\|\text{avg}(\varepsilon)\|^2_2 \leq \begin{cases} \frac{1}{|\varepsilon \setminus \{0\}|} \sum_{i=1}^{a} \|\varepsilon_i\|_2^2, & |\varepsilon \setminus \{0\}| \neq 0 \\ 0, & |\varepsilon \setminus \{0\}| = 0 \end{cases} (10)$$

Apparently, Equation (10) is very similar to Equation (3) in terms of the expression. As a result, we can adopt the same proof framework in the analysis of $\mathbb{E}(\text{avg}(\varepsilon))$. Then, we can directly draw a conclusion $\mathbb{E} \|\text{avg}(\varepsilon)\|^2_2 \leq (1 - (1 - q)^a) \cdot \frac{1}{a} \sum_{i=1}^{a} \|e_i\|^2_2$.

**Lemma 2.** Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_a\}$ be the set of random variables in $\mathbb{R}^d$ with the number of $a$. These random variables are not necessarily independent. We can suppose that $\mathbb{E} [\varepsilon_i] = e_i$, and the variance is bounded as $\mathbb{E} \|\varepsilon_i - e_i\|^2 \leq \sigma^2$. After that we can get

$$\mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i \right\|_2^2 \right] \leq \left\| \sum_{i=1}^{a} e_i \right\|_2^2 + a^2 \sigma^2 (11)$$

If we make another suppose that the conditional mean of these random variables is $\mathbb{E} [\varepsilon_i | \varepsilon_{i-1}, \ldots, \varepsilon_1] = e_i$, and the variables $\{\varepsilon_i - e_i\}$ form a martingale difference sequence, and the bound of the variance is $\mathbb{E} \|\varepsilon_i - e_i\|^2 \leq \sigma^2$. So we can make a much tighter bound

$$\mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i \right\|_2^2 \right] \leq 2 \left\| \sum_{i=1}^{a} e_i \right\|_2^2 + 2a\sigma^2 (12)$$

**Proof.** For any random variable $X$, $\mathbb{E} [X^2] = (\mathbb{E} [X - \mathbb{E} [X]])^2 + (\mathbb{E} [X])^2$ implying

$$\mathbb{E} \left[ \left\| \sum_{i=1}^{a} e_i \right\|_2^2 \right] = \left\| \sum_{i=1}^{a} e_i \right\|_2^2 + \mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i - e_i \right\|_2^2 \right] (13)$$

Expanding above expression using relaxed triangle inequality:

$$\mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i - e_i \right\|_2^2 \right] \leq a \sum_{i=1}^{a} \mathbb{E} \left[ \|\varepsilon_i - e_i\|_2^2 \right] \leq a^2 \sigma^2 (14)$$

For the second statement, $e_i$ depends on $[\varepsilon_{i-1}, \ldots, \varepsilon_1]$. Thus we choose to use a relaxed triangle inequality

$$\mathbb{E} \left[ \left\| \sum_{i=1}^{a} e_i \right\|_2^2 \right] \leq 2 \left\| \sum_{i=1}^{a} e_i \right\|_2^2 + 2a \mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i - e_i \right\|_2^2 \right] (15)$$

then we use a much tighter expansion and we can get:

$$\mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i - e_i \right\|_2^2 \right] = \sum_{i,j} \mathbb{E} \left[ (\varepsilon_i - e_i)^T (\varepsilon_j - e_j) \right] = \sum_i \mathbb{E} \left[ \left\| \sum_{i=1}^{a} \varepsilon_i - e_i \right\|_2^2 \right] \leq a \sigma^2 (16)$$

When $\{\varepsilon_i - e_i\}$ form a martingale difference sequence, the cross terms will have zero means. \qed
Lemma 3. Suppose there is a sequence \( \{y_t \in \mathbb{R}^d\}_{t \geq 0} \) satisfying a recursive function \( y_{t+1} = y_t - \eta \Delta y_t \), where \( \eta > 0 \) is a constant and \( \Delta y_0 \in \mathbb{R}^d \) is a vector. Given a \( L \)-smooth function \( G \), the following inequality holds for any \( \eta \) and \( \Delta y_t \):

\[
G(y_{t+1}) \leq G(y_t) - \frac{\eta \eta'}{2} \| \nabla G(y_t) \|_2^2 - \left( \frac{1}{2\eta \eta'} - \frac{L}{2} \right) \| y_{t+1} - y_t \|_2^2 + \frac{\eta}{2\eta'} \| \Delta y_t - \eta' \nabla G(y_t) \|_2^2
\]

(17)

where \( \eta' > 0 \) can be any constant.

Proof. Since \( G \) is a \( \mathbb{L} \)-smooth function, for any \( v, \bar{v} \in \mathbb{R}^d \), the following inequality holds:

\[
G(\bar{v}) = G(v) + \int_0^1 \frac{\partial G(v + t(\bar{v} - v))}{\partial t} dt
\]

(18)

\[
= G(v) + \int_0^1 \nabla G(v + t(\bar{v} - v)) \cdot (\bar{v} - v) dt
\]

(19)

\[
= G(v) + \nabla G(v)(\bar{v} - v) + \int_0^1 (\nabla G(v + t(\bar{v} - v)) - G(v)) \cdot (\bar{v} - v) dt
\]

(20)

\[
\leq G(v) + \nabla G(v)(\bar{v} - v) + \int_0^1 L\|t(\bar{v} - v)\|_2\|\bar{v} - v\|_2 dt
\]

(21)

\[
\leq G(v) + \nabla G(v)(\bar{v} - v) + \frac{L}{2} \| \bar{v} - v \|_2^2.
\]

(22)

Based on the conclusion on \( \mathbb{L} \)-smooth drawn from Equation (22), we derive Equation (17) step by step:

\[
G(y_{t+1}) \leq G(y_t) + \langle \nabla G(y_t), y_{t+1} - y_t \rangle + \frac{L}{2} \| y_{t+1} - y_t \|_2^2
\]

(23)

\[
= G(y_t) + \langle \nabla G(y_t), - \eta \Delta y_t \rangle + \frac{L}{2} \| y_{t+1} - y_t \|_2^2
\]

(24)

\[
= G(y_t) - \frac{\eta}{\eta'} \| \nabla G(y_t) \|_2^2 - \| \Delta y_t \|_2^2 - \| \Delta y_t - \eta' \nabla G(y_t) \|_2^2 + \frac{L}{2} \| y_{t+1} - y_t \|_2^2
\]

(25)

\[
= G(y_t) - \frac{\eta}{2\eta'} \left( \eta^2 \| \nabla G(y_t) \|_2^2 + \| \Delta y_t \|_2^2 - \| \Delta y_t - \eta' \nabla G(y_t) \|_2^2 \right) + \frac{L}{2} \| y_{t+1} - y_t \|_2^2
\]

(26)

\[
= G(y_t) - \frac{\eta \eta'}{2} \| \nabla G(y_t) \|_2^2 - \left( \frac{1}{2\eta \eta'} - \frac{L}{2} \right) \| y_{t+1} - y_t \|_2^2 + \frac{\eta}{2\eta'} \| \Delta y_t - \eta' \nabla G(y_t) \|_2^2
\]

(27)

where Equation (26) is in accordance with \( \langle \alpha, \beta \rangle = \frac{1}{2} \left( \alpha^2 + \beta^2 - (\alpha - \beta)^2 \right) \), and Equation (27) follows \( \| \Delta y_t \|_2^2 = \frac{1}{\eta^2} \| y_{t+1} - y_t \|_2^2 \).

\[\Box\]
B Preliminary for FedAMD

Algorithm \[\text{Algorithm 1}\] describes FedAMD in details. The objective in this part is to find the recursive function for the sequence of models, i.e., \\{\tilde{x}_t\}_{t \geq 0}. As mentioned in Line 26 in Algorithm \[\text{Algorithm 1}\] let \(\Delta x_i^{(t)}\) where client \(i\) updates model, then the difference between \(x_{t+1}\) and \(\tilde{x}_t\) follows the recursive function written as

\[
x_{t+1} = \tilde{x}_t - \eta_s \cdot \text{avg}(\Delta x_i)
\]

where \(\text{avg}()\) is same as defined in Lemma \[\text{Lemma 1}\]. As we know, the length of \(\Delta x_i\) changes over rounds but does not exceed the number of participants, i.e., \(|\Delta x_i| \leq A\). Then, suppose that \(\Delta x_i^{(m)}\) is in \(\Delta x_t\), \(\Delta x_i^{(m)}\) can be expressed as

\[
\Delta x_t^{(m)} = - (\tilde{x}_i, K - \tilde{x}_i) = - \sum_{k=0}^{K-1} (x_{t,k}^{(m)} - x_{t,k+1}^{(m)}) = \sum_{k=0}^{K-1} \eta_l g_{t,k}^{(m)}
\]

where the last equal sign is according to Line 20 in Algorithm \[\text{Algorithm 1}\]. Next, with the recursive formula in Line 19, we have

\[
g_{t,k}^{(m)} = g_{t,k}^{(m)} - \nabla f_m (x_{t,k-1}^{(m)}, B_{m,k}) + \nabla f_m (x_{t,k}^{(m)}, B_{m,k})
\]

Then, Equation (29) can be rewritten as

\[
\Delta x_t^{(m)} = \eta_l K \tilde{g}_t - \eta_l \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \nabla f_m (x_{t,k}^{(m)}, B_{m,k}) + \eta_l \sum_{k=0}^{K-1} \sum_{\kappa=2}^{k} \nabla f_m (x_{t,k}^{(m)}, B_{m,k})
\]

C Proofs under Sequential Probabilistic Settings

C.1 Preliminary

Lemma 4. Suppose that Assumption \[\text{A}\] and \[\text{B}\] hold. Let the local learning rate satisfy \(\eta_l \leq \min \left(\frac{1}{2 \sqrt{4KL}}, \frac{1}{2 \sqrt{4L} \epsilon} \sqrt{\frac{\mu}{\epsilon}}\right)\). With FedAMD, \(\sum_{k=0}^{K-1} \left\| x_{t,k}^{(m)} - x_{t,k-1}^{(m)} \right\|_2^2 \) represents the sum of the second norm of every iteration’s difference. Therefore, the bound for such a summation in the expected form should be

\[
\sum_{k=0}^{K-1} \mathbb{E} \left\| x_{t,k}^{(m)} - x_{t,k-1}^{(m)} \right\|_2^2 \leq 6\eta_l^2 K \left\| \tilde{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2 + 6\eta_l^2 K \left\| \nabla F (\tilde{x}_t) \right\|_2^2
\]

Proof. According to Equation (31), the update at \((k-1)\)-th iteration is

\[
x_{t,k}^{(m)} - x_{t,k-1}^{(m)} = -\eta_l \tilde{g}_t = -\eta_l \left( \tilde{g}_t - \sum_{\kappa=0}^{k-1} \nabla f_m (x_{t,k-1}^{(m)}, B_{m,k}) + \sum_{\kappa=2}^{k-1} \nabla f_m (x_{t,k}^{(m)}, B_{m,k}) \right).
\]

To find the bound for the expected value of its second norm, the analysis is presented as follows:

\[
\mathbb{E} \left\| x_{t,k}^{(m)} - x_{t,k-1}^{(m)} \right\|_2^2
\]

\[
= \eta_l^2 \mathbb{E} \left\| \tilde{g}_t - \sum_{\kappa=0}^{k-1} \nabla f_m (x_{t,k-1}^{(m)}, B_{m,k}) + \sum_{\kappa=2}^{k-1} \nabla f_m (x_{t,k}^{(m)}, B_{m,k}) \right\|_2^2
\]

\[
\leq 3\eta_l^2 \left\| \tilde{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2 + 3\eta_l^2 \left\| \nabla F (\tilde{x}_t) \right\|_2^2
\]

\[
+ 3\eta_l^2 \mathbb{E} \left\| \sum_{\kappa=0}^{k-1} \nabla f_m (x_{t,k-1}^{(m)}, B_{m,k}) - \sum_{\kappa=2}^{k-1} \nabla f_m (x_{t,k}^{(m)}, B_{m,k}) \right\|_2^2
\]
\[ = 3\eta_2^2 \| \tilde{g}_t - \nabla F(\bar{x}_t) \|_2^2 + 3\eta_2^2 \| \nabla F(\bar{x}_t) \|_2^2 + 3\eta_2^2 \mathbb{E} \left\| \sum_{k=0}^{K-1} \left( \nabla F_m \left( x^{(m)}_{t,\kappa-1} \right) - \nabla F_m \left( x^{(m)}_{t,\kappa} \right) \right) \right\|_2^2 + 3\eta_2^2 \mathbb{E} \left\| \sum_{k=0}^{K-1} \left( \nabla f_m \left( x^{(m)}_{t,\kappa-1}, B_{m,\kappa} \right) - \nabla f_m \left( x^{(m)}_{t,\kappa}, B_{m,\kappa} \right) \right) \right\|_2^2 \]

where Equation (37) is based on the Cauchy-Schwarz inequality; Equation (38) is based on the variance Theorem 4. Therefore, by summing Equation (40) for \( k = 1, \ldots, K \), we have

\[ \sum_{k=0}^{K-1} \left\| x^{(m)}_{t,k} - x^{(m)}_{t,k-1} \right\|_2^2 \leq \sum_{k=0}^{K-1} \left\| x^{(m)}_{t,k} - x^{(m)}_{t,k-1} \right\|_2^2 \]

\[ \leq 3\eta_2^2 K \| \tilde{g}_t - \nabla F(\bar{x}_t) \|_2^2 + 3\eta_2^2 K \| \nabla F(\bar{x}_t) \|_2^2 + 3\eta_2^2 K \left( KL^2 + \frac{L^2}{b} \right) \sum_{k=0}^{K-1} \mathbb{E} \left\| x^{(m)}_{t,\kappa-1} - x^{(m)}_{t,\kappa} \right\|_2^2 \]

(41)

(42)

Obviously, according to the setting of the local learning rate in the description above, the inequality

\[ 3\eta_2^2 K \left( KL^2 + \frac{L^2}{b} \right) \leq \frac{1}{2} \]

holds. Therefore, we can easily obtain the bound for the sum of the second norm of every iteration’s difference, which is consistent with Equation (33).

\[ \square \]

C.2 Full Client Participation

**Theorem 4.** Suppose that Assumption 1, 2, and 4 hold, and all clients participate in the training, i.e., \( A = M \). Let the local updates \( K \geq 1 \), and the local learning rate \( \eta_k \) and the global learning rate \( \eta \) be \( \eta_k \eta = \frac{1}{KL(1+2\tau)} \), where \( \eta \leq \min \left( \frac{1}{2\sqrt{4\epsilon KL}}, \frac{\sqrt{b'/KL}}{4\sqrt{4\epsilon L_c}} \right) \). Therefore, the convergence rate of FedAM for non-convex objectives should be

\[ \min_{t \in [T]} \mathbb{E} \left\| \nabla F(\bar{x}_t) \right\|_2^2 \leq O \left( \frac{1+2\tau}{T - \lceil T/\tau \rceil} \right) + O \left( \left( 1_{b<n} \right) \sigma^2 \frac{M_b}{L} \right) \]

(43)

where we treat \( F(\bar{x}_0) - F_* \) and \( L \) as constants.

**Proof.** When \( p_t = 1 \), according to Algorithm 1 there is no model update between two consecutive rounds, i.e., \( \tilde{x}_{t+1} = \tilde{x}_t \).

Next, we consider the case when \( p_t = 0 \). Based on Lemma 3 we have

\[ \mathbb{E} F(\tilde{x}_{t+1}) - F(\tilde{x}_t) \leq - \frac{\eta_k \eta K}{2} \mathbb{E} \left\| \nabla F(\bar{x}_t) \right\|_2^2 - \frac{1}{2\eta_k \eta K} \mathbb{E} \left\| \tilde{x}_{t+1} - \tilde{x}_t \right\|_2^2 \]
Knowing that when $p_t = 0$ and all clients involve in the training,

$$\text{avg}(\Delta x_t) = \eta_t K \hat{g}_t - \frac{\eta_t}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \nabla f_m \left( x^{(m)}_{t,k-1}, B'_{m,k} \right)$$

$$+ \frac{\eta_t}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \nabla f_m \left( x^{(m)}_{t,k}, B'_{m,k} \right) ;$$

we have the bound for $\mathbb{E} \left\| \text{avg}(\Delta x_t) - \eta_t K \nabla F (\tilde{x}_t) \right\|_2^2$ according to the following derivation:

$$\mathbb{E} \left\| \text{avg}(\Delta x_t) - \eta_t K \nabla F (\tilde{x}_t) \right\|_2^2$$

$$\leq 2 \eta_t^2 K^2 \left\| \hat{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2$$

$$+ 2 \mathbb{E} \left\| \frac{\eta_t}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \left( \nabla f_m \left( x^{(m)}_{t,k-1}, B'_{m,k} \right) - \nabla f_m \left( x^{(m)}_{t,k}, B'_{m,k} \right) \right) \right\|_2^2$$

$$= 2 \eta_t^2 K^2 \left\| \hat{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2 + 2 \mathbb{E} \left\| \frac{\eta_t}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \left( \nabla f_m \left( x^{(m)}_{t,k-1}, B'_{m,k} \right) - \nabla f_m \left( x^{(m)}_{t,k} \right) + \nabla f_m \left( x^{(m)}_{t,k-1} \right) \right) \right\|_2^2$$

$$\leq 2 \eta_t^2 K^2 \left\| \hat{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2$$

$$+ 2 \mathbb{E} \left\| \frac{\eta_t}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \left( \nabla f_m \left( x^{(m)}_{t,k-1}, B'_{m,k} \right) - \nabla f_m \left( x^{(m)}_{t,k} \right) + \nabla f_m \left( x^{(m)}_{t,k-1} \right) \right) \right\|_2^2$$

$$\leq 2 \eta_t^2 K^2 \left\| \hat{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2 + 2 \mathbb{E} \left\| \frac{\eta_t}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \left( \nabla f_m \left( x^{(m)}_{t,k-1}, B'_{m,k} \right) - \nabla f_m \left( x^{(m)}_{t,k} \right) + \nabla f_m \left( x^{(m)}_{t,k-1} \right) \right) \right\|_2^2$$

$$\leq \frac{2 \eta_t^2 K^2 L^2}{M} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \left\| x^{(m)}_{t,k-1} - x^{(m)}_{t,k} \right\|_2^2$$

$$+ \frac{2 \eta_t^2 K^3 L^2}{M^2 b'} \sum_{m \in [M]} \sum_{k=0}^{K-1} \sum_{\kappa=0}^{k} \left\| x^{(m)}_{t,k} - x^{(m)}_{t,k-1} \right\|_2^2$$

$$\leq 2 \eta_t^2 K^2 \left\| \hat{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2 + 12 \eta_t^2 K^2 \left( \frac{L^2}{M b'} + K^2 L^2 \right) \left( \left\| \hat{g}_t - \nabla F (\tilde{x}_t) \right\|_2^2 + \left\| \nabla F (\tilde{x}_t) \right\|_2^2 \right)$$
where Equation (48) follows \((\alpha + \beta)^2 \leq 2\alpha^2 + 2\beta^2\); Equation (49) is based on variance expansion; Equation (50) is based on Cauchy-Schwarz inequality and Assumption 1; Equation (51) is based on Lemma 2 and Assumption 3; Equation (53) is based on Lemma 4. According to the constraints on the local learning rate, we can further simplify Equation (53) as:

\[
E \|\text{avg}(\Delta \mathbf{x}_t) - \eta_t \nabla F(\bar{\mathbf{x}}_t)\|^2 \leq 4\eta_t^2 K^2 \|\bar{\mathbf{g}}_t - \nabla F(\bar{\mathbf{x}}_t)\|^2 + \frac{\eta_t^2 K^2}{2} \|\nabla F(\bar{\mathbf{x}}_t)\|^2. \tag{54}
\]

Plugging Equation (54) into Equation (44), we have

\[
EF(\bar{\mathbf{x}}_{t+1}) - F(\bar{\mathbf{x}}_t) \leq -\frac{\eta_t \eta K}{4} \|\nabla F(\bar{\mathbf{x}}_t)\|^2 - \left(\frac{1}{2\eta_t \eta K} - \frac{L}{2}\right) \sum_{\theta = \Lambda(t) + 1}^{t-1} E \|\bar{\mathbf{x}}_{\theta + 1} - \bar{\mathbf{x}}_{\theta}\|^2 + 2\eta_t \eta K \|\bar{\mathbf{g}}_t - \nabla F(\bar{\mathbf{x}}_t)\|^2. \tag{55}
\]

Let \(\Lambda(t)\) indicate the most recent round where \(p(\Lambda(t)) = 1\) and \(\Lambda(t) \neq t\). It is noted that recursively using \(\Lambda(\cdot)\) can achieve the value of 0, i.e., \(\Lambda(\Lambda(\ldots \Lambda(t))) = 0\). By summing Equation (55) from \(\Lambda(t)\) to \(t - 1\), we have

\[
E F(\bar{\mathbf{x}}_t) - F(\bar{\mathbf{x}}_{\Lambda(t)}) = \sum_{\theta = \Lambda(t)}^{t-1} (E F(\bar{\mathbf{x}}_{\theta + 1}) - F(\bar{\mathbf{x}}_{\theta})) \tag{56}
\]

The bound for the last term of Equation (57) is

\[
E \|\bar{\mathbf{g}}_{\theta} - \nabla F(\bar{\mathbf{x}}_{\theta})\|^2 = E \|\bar{\mathbf{g}}_{\Lambda(\theta)} - \nabla F(\bar{\mathbf{x}}_{\theta})\|^2 \tag{58}
\]

\[
= E \|\nabla F(\bar{\mathbf{x}}_{\Lambda(\theta)})\|^2 + L^2E \|\bar{\mathbf{x}}_{\theta} - \bar{\mathbf{x}}_{\Lambda(\theta)}\|^2 \tag{59}
\]

\[
\leq L^2 \sum_{\Xi = \Lambda(\theta)}^{\theta-1} E \|\bar{\mathbf{x}}_{\theta + 1} - \bar{\mathbf{x}}_{\Xi}\|^2 \tag{60}
\]

where Equation (59) is based on the variance expansion; Equation (60) is based on Assumption 1; Equation (61) is according to Cauchy-Schwarz inequality and \(\theta - \Lambda(\theta) \leq \tau\); Equation (62) follows Assumption 3. Based on the definition of \(\Lambda(\cdot)\), for all \(\theta \in \{\Lambda(t) + 1, \ldots, t - 1\}\), \(\Lambda(\theta) = \Lambda(t)\). Therefore, with Equation (62), Equation (57) can be further simplified as:

\[
E F(\bar{\mathbf{x}}_t) - F(\bar{\mathbf{x}}_{\Lambda(t)}) \leq -\frac{\eta_t \eta K}{4} \sum_{\theta = \Lambda(t) + 1}^{t-1} \|\nabla F(\bar{\mathbf{x}}_{\theta})\|^2 - \left(\frac{1}{2\eta_t \eta K} - \frac{L}{2}\right) \sum_{\theta = \Lambda(t) + 1}^{t-1} E \|\bar{\mathbf{x}}_{\theta + 1} - \bar{\mathbf{x}}_{\theta}\|^2 + 2\eta_t \eta K \|\bar{\mathbf{g}}_t - \nabla F(\bar{\mathbf{x}}_t)\|^2 \tag{63}
\]

\[
\leq -\frac{\eta_t \eta K}{4} \sum_{\theta = \Lambda(t) + 1}^{t-1} \|\nabla F(\bar{\mathbf{x}}_{\theta})\|^2 - \left(\frac{1}{2\eta_t \eta K} - \frac{L}{2} - 2\eta_t \eta K \tau^2\right) \sum_{\theta = \Lambda(t) + 1}^{t-1} E \|\bar{\mathbf{x}}_{\theta + 1} - \bar{\mathbf{x}}_{\theta}\|^2 \tag{64}
\]
we have the bound for $E$

Theorem 5. Therefore, based on the equation above, by summing up all rounds, i.e., $t$

Proof. When $p_t = 1$, according to Algorithm 1, there is no model update between two consecutive rounds, i.e., $\tilde{x}_{t+1} = \tilde{x}_t$.

Next, we consider the case when $p_t = 0$. Based on Lemma 5, we have

Knowing that when $p_t = 0$ and a set of clients $A$ involve in the training,

we have the bound for $E \|\text{avg}(\Delta x_t) - \eta_t K \nabla F(\tilde{x}_t)\|_2^2$ according to the following derivation:

\begin{equation}
E \|\text{avg}(\Delta x_t) - \eta_t K \nabla F(\tilde{x}_t)\|_2^2
\end{equation}

\begin{equation}
+ 2\eta_s \eta_f K (t - \Lambda(t) - 1) \cdot 1_{\{b<n\}} \frac{\sigma^2}{Mb}
\end{equation}

Since $\eta_s \eta_f = \frac{1}{KL(1+2\tau)}$, Equation \((65)\) can be further simplified as

$$
EF(\tilde{x}_t) - F(\tilde{x}_{\Lambda(t)}) \leq -\frac{\eta_s \eta_f K}{4} \sum_{\theta = \Lambda(t) + 1}^{t-1} \|\nabla F(\tilde{x}_\theta)\|_2^2 + 2\eta_s \eta_f K (t - \Lambda(t) - 1) \cdot 1_{\{b<n\}} \frac{\sigma^2}{Mb}
$$

Therefore, based on the equation above, by summing up all $t \in \{T + 1, \Lambda(T + 1), \ldots, \tau\}$, we can obtain the following inequality:

$$
F_s - F(\tilde{x}_0) \leq \mathbb{E} F(\tilde{x}_{T+1}) - F(\tilde{x}_0)
$$

Thus, we have

$$
\frac{1}{T - \lceil T/\tau \rceil} \sum_{t = 0; t \mod \tau = 0}^{T} \|\nabla F(\tilde{x}_t)\|_2^2 \leq \frac{4(F(\tilde{x}_0) - F_s)}{\eta_s \eta_f K (T - \lceil T/\tau \rceil)} + 8 \cdot 1_{\{b<n\}} \frac{\sigma^2}{Mb}
$$

By using the settings of the local learning rate and the global learning rate in the description, we can obtain the desired result.

C.3 Partial Client Participation

**Theorem 5.** Suppose that Assumption 1 and 2 hold. Let the local updates $K \geq 1$, and the local learning rate $\eta$ and the global learning rate $\eta_s$ be $\eta_s \eta_f = \frac{1}{KL(1+2\tau)}$, where $\eta_s \leq \min \left( \frac{1}{2\sqrt{2}KL}, \frac{\sqrt{b/K}}{4\sqrt{2L_N}} \right)$. Therefore, the convergence rate of FedAMD for non-convex objectives should be

$$
\min_{t \in [T]} \|\nabla F(\tilde{x}_t)\|_2^2 \leq O \left( \frac{1}{T - \lceil T/\tau \rceil} \left( 1 + \frac{2M\tau}{A} \right) \right) + O \left( 1_{\{b<n\}} \frac{\sigma^2}{Mb} \right)
$$

where we treat $F(\tilde{x}_0) - F_s$ and $L$ as constants.

**Proof.** When $p_t = 1$, according to Algorithm 1, there is no model update between two consecutive rounds, i.e., $\tilde{x}_{t+1} = \tilde{x}_t$.

Next, we consider the case when $p_t = 0$. Based on Lemma 5, we have

$$
\mathbb{E} F(\tilde{x}_{t+1}) - F(\tilde{x}_t) \leq -\frac{\eta_s \eta_f K}{4} \|\nabla F(\tilde{x}_t)\|_2^2 - \left( \frac{1}{2\eta_s \eta_f K} - \frac{L}{2} \right) \mathbb{E} \|\tilde{x}_{t+1} - \tilde{x}_t\|_2^2
$$

Knowing that when $p_t = 0$ and a set of clients $A$ involve in the training,

$$
\text{avg}(\Delta x_t) = \eta_t K \tilde{y}_t - \frac{\eta_s \eta_f}{A} \sum_{i \in A} \sum_{k = 0}^{K-1} \sum_{\kappa = 0}^{k} \nabla f_m \left( x_{t,k-1}^{(m)}, B_{m,k}^t \right)
$$

we have the bound for $E \|\text{avg}(\Delta x_t) - \eta_t K \nabla F(\tilde{x}_t)\|_2^2$ according to the following derivation:

$$
\mathbb{E} \|\text{avg}(\Delta x_t) - \eta_t K \nabla F(\tilde{x}_t)\|_2^2
$$
\[
\begin{align*}
\text{where Equation (75) follows (} \alpha + \beta \text{)}^2 \leq 2 \alpha^2 + 2 \beta^2; \text{ Equation (76) is based on variance expansion; Equation (77) is based on Cauchy-Schwarz inequality and Assumption 1; Equation (78) is based on Lemma 2 and Assumption 3. Equation (79) is based on the setting of client selection, where each client is selected with a probability of } A/M; \text{ Equation (80) is based on Lemma 4. According to the constraints on the local learning rate, we can further simplify Equation (81) as}
\end{align*}
\]

\[
\begin{align*}
E \left\| \text{avg}(\Delta \mathbf{x}_t) - \eta_t K \nabla F (\bar{\mathbf{x}}_t) \right\|_2^2 &\leq 4\eta_t K^2 \left\| \bar{g}_t - \nabla F (\bar{\mathbf{x}}_t) \right\|_2^2 + \frac{\eta_t K^2}{2} \left\| \nabla F (\bar{\mathbf{x}}_t) \right\|_2^2.
\end{align*}
\]

Plugging Equation (82) into Equation (71), we have

\[
\begin{align*}
EF(\bar{\mathbf{x}}_{t+1}) - F(\bar{\mathbf{x}}_t)
\end{align*}
\]
\[
\begin{align*}
&\leq -\frac{\eta_s \eta_t K}{4} \|\nabla F (\bar{x}_t)\|_2^2 - \left( \frac{1}{2\eta_s \eta_t K} - \frac{L}{2} \right) \mathbb{E} \|\bar{x}_{t+1} - \bar{x}_t\|_2^2 + 2\eta_s \eta_t K \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 \\
&= -\frac{\eta_s \eta_t K}{4} \|\nabla F (\bar{x}_t)\|_2^2 - \left( \frac{1}{2\eta_s \eta_t K} - \frac{L}{2} \right) \mathbb{E} \|\bar{x}_{t+1} - \bar{x}_t\|_2^2 \\
&\quad + 2\eta_s \eta_t K \left( \mathbb{E} \|\bar{g}_t - \mathbb{E} \bar{g}_t\|_2^2 + \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 \right) \\
&\leq -\frac{\eta_s \eta_t K}{4} \|\nabla F (\bar{x}_t)\|_2^2 - \left( \frac{1}{2\eta_s \eta_t K} - \frac{L}{2} \right) \mathbb{E} \|\bar{x}_{t+1} - \bar{x}_t\|_2^2 \\
&\quad + 2\eta_s \eta_t K \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 \\
\end{align*}
\]

where Equation (85) is based on variance expansion, and Equation (86) is based on Assumption G.

By summing Equation (86) for all \( t \in \{0, \ldots, T\} \), we have

\[
F_s - F(\bar{x}_0) \leq \mathbb{E} F(\bar{x}_{T+1}) - F(\bar{x}_0) = \sum_{t=0}^{T} \left( \mathbb{E} F(\bar{x}_{t+1}) - F(\bar{x}_t) \right)
\]

\[
\leq -\frac{\eta_s \eta_t K}{4} \sum_{t=0; t \mod \tau = 0}^{T} \|\nabla F (\bar{x}_t)\|_2^2 - \left( \frac{1}{2\eta_s \eta_t K} - \frac{L}{2} \right) \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \|\bar{x}_{t+1} - \bar{x}_t\|_2^2 \\
&\quad + 2\eta_s \eta_t K \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 + 2\eta_s \eta_t K \mathbb{E} \|\bar{g}_t - \mathbb{E} \bar{g}_t\|_2^2 \sum_{t=0; t \mod \tau = 0}^{T} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2
\]

\[
\leq -\frac{\eta_s \eta_t K}{4} \sum_{t=0; t \mod \tau = 0}^{T} \|\nabla F (\bar{x}_t)\|_2^2 - \left( \frac{1}{2\eta_s \eta_t K} - \frac{L}{2} \right) \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \|\bar{x}_{t+1} - \bar{x}_t\|_2^2 \\
&\quad + 2\eta_s \eta_t K \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 + 2\eta_s \eta_t K \mathbb{E} \|\bar{g}_t - \mathbb{E} \bar{g}_t\|_2^2 \sum_{t=0; t \mod \tau = 0}^{T} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2
\]

Let \( \Lambda(t) \) indicate the most recent round where \( p_{\Lambda(t)} = 1 \) and \( \Lambda(t) \neq t \). It is noted that recursively using \( \Lambda(\cdot) \) can achieve the value of 0, i.e., \( \Lambda(\Lambda(\ldots \Lambda(t))) = 0 \).

To find the bound for \( \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 \), the first step is to provide the bound for \( \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 \). When \( p_t = 1 \), a client updates the caching gradient with a probability of \( A/M \), and therefore, \( \bar{g}_t = (1 - A/M) \bar{g}_{\Lambda(t)} + A/M \nabla F (\bar{x}_t) \). Based on this fact, the bound for \( \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 \) can be derived as follows:

\[
\mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2 = \mathbb{E} \|\bar{g}_{A(t)} - \nabla F (\bar{x}_{A(t)})\|_2^2
\]

\[
= \mathbb{E} \left( 1 - \frac{A}{M} \right) \mathbb{E} \|\bar{g}_{\Lambda(t)} - \nabla F (\bar{x}_{\Lambda(t)})\|_2^2 + \mathbb{E} \|\nabla F (\bar{x}_{\Lambda(t)}) - \nabla F (\bar{x}_t)\|_2^2
\]

\[
\leq \left( 1 - \frac{A}{M} \right) \mathbb{E} \|\bar{g}_{\Lambda(t)} - \nabla F (\bar{x}_{\Lambda(t)})\|_2^2 + \frac{M}{A} \mathbb{E} \|\nabla F (\bar{x}_{\Lambda(t)}) - \nabla F (\bar{x}_t)\|_2^2
\]

\[
\leq \sum_{t=0}^{[t/\tau] - 1} \left( 1 - \frac{A}{M} \right)^{[t/\tau] - \theta} \cdot \frac{M}{A} L^2 \mathbb{E} \|\bar{x}_{\theta t} - \bar{x}_{\theta (t+1) \tau}\|_2^2 + \frac{M}{A} L^2 \mathbb{E} \|\bar{x}_{\Lambda(t)} - \bar{x}_t\|_2^2
\]

where Equation (91) follows \((\alpha + \beta)^2 \leq \left( 1 + \frac{1}{\gamma} \right) \alpha^2 + (1 + \gamma) \beta^2 - \left( \frac{1}{\gamma} \alpha + \sqrt{\gamma} \beta \right)^2 \leq \left( 1 + \frac{1}{\gamma} \right) \alpha^2 + (1 + \gamma) \beta^2 \) and \( \gamma = \frac{M - A}{A} \). With Equation (92), we sum up all \( t \in \{1, \ldots, T\} \) and obtain the following result:

\[
\sum_{t=0}^{T} \mathbb{E} \|\bar{g}_t - \nabla F (\bar{x}_t)\|_2^2
\]

\[
\leq \sum_{t=0}^{[t/\tau] - 1} \left( \sum_{\theta=0}^{[t/\tau] - 1} \left( 1 - \frac{A}{M} \right)^{[t/\tau] - \theta} \right) \cdot \frac{M}{A} L^2 \mathbb{E} \|\bar{x}_{\theta t} - \bar{x}_{\theta (t+1) \tau}\|_2^2 + \frac{M}{A} L^2 \mathbb{E} \|\bar{x}_{\Lambda(t)} - \bar{x}_t\|_2^2
\]
where Equation (95) follows that, for all \( \theta \in \{0, \ldots, [T/\tau] - 1\} \), the coefficient for \( M/L^2 \) includes \((1 - \frac{A}{M}) \), \ldots, \((1 - \frac{A}{M})^{[T/\tau] - \theta}\), and each of them has a maximum of \( \tau \) ts, meaning that the upper bound of the coefficient should be

\[
\tau \left( \left( 1 - \frac{A}{M} \right) + \cdots + \left( 1 - \frac{A}{M} \right)^{[T/\tau] - \theta} \right) \leq \tau \cdot \frac{M}{2A} \left( 1 - \frac{A}{M} \right) ;
\]

Equation (96) follows Cauchy-Schwarz inequality.

Plugging Equation (97) back to Equation (88), we have:

\[
F_s - F(\tilde{x}_0) \leq \frac{-\eta_s \eta_l K}{4} \sum_{t=0; t \mod \tau = 0}^{T} \left\| \nabla F(\tilde{x}_t) \right\|_2^2
- \left( \frac{1}{2\eta_s \eta_l K} - \frac{L}{2} - \eta_s \eta_l K L^2 \tau^2 \frac{M^2}{A^2} \right) \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \left\| \tilde{x}_{t+1} - \tilde{x}_t \right\|_2^2
+ 2\eta_s \eta_l K (T - [T/\tau]) \cdot 1_{\{b<n\}} \frac{\sigma^2}{Mb} \]

(99)

Since \( \eta_s \eta_l = \frac{1}{KL} \left( 1 + \frac{2M^2}{A} \right)^{-1} \), \( \frac{1}{2\eta_s \eta_l K} - \frac{L}{2} - \eta_s \eta_l K L^2 \tau^2 \frac{M^2}{A^2} \geq 0 \) such that the term \( \sum_{t=0; t \mod \tau = 0}^{T} \mathbb{E} \left\| \tilde{x}_{t+1} - \tilde{x}_t \right\|_2^2 \) can be omitted in Equation (96). Hence, we can easily obtain the following inequality:

\[
\frac{1}{T - [T/\tau]} \sum_{t=0; t \mod \tau = 0}^{T} \left\| \nabla F(\tilde{x}_t) \right\|_2^2 \leq \frac{A (F(\tilde{x}_0) - F_s)}{\eta_s \eta_l K (T - [T/\tau])} + 8 \cdot 1_{\{b<n\}} \frac{\sigma^2}{Mb} \]

(100)

By using the settings of the local learning rate and the global learning rate in the description, we can obtain the desired result. \( \square \)
### D Proofs under Constant Probabilistic Settings

#### D.1 Preliminary

**Lemma 5.** Suppose that Assumption 7 holds, and \( p_t \in (0, 1) \) is a constant. Let \( \tilde{g}_t \) be the definition of Line 9 of Algorithm 1, i.e., the average of the caching gradients. Therefore, the recursive expression for \( \{\tilde{g}_t\}_{t \geq 0} \) in the expected form is

\[
\mathbb{E}\tilde{g}_t = \begin{cases} 
(1 - \frac{A}{M} p_t - 1) \cdot \mathbb{E}\tilde{g}_{t-1} + \frac{A}{M} p_t \cdot \nabla F(\bar{x}_{t-1}), & t > 0 \\
\n\n\n\n\n(101)
\]

Furthermore, when \( t > 0 \) we can obtain the following inequality:

\[
\mathbb{E}\|\tilde{g}_t - \nabla F(\bar{x}_t)\|_2^2 \leq \left( 1 - \frac{A}{M} p_t \right) \mathbb{E}\|\tilde{g}_{t-1} - \nabla F(\bar{x}_{t-1})\|_2^2 + \frac{M}{A p_t} \cdot L^2 \mathbb{E}\|\bar{x}_t - \bar{x}_{t-1}\|_2^2.
\]

As for \( t = 0 \), we have \( \mathbb{E}\|\tilde{g}_t - \nabla F(\bar{x}_t)\|_2^2 = 0. \)

**Proof.** According to the definition of Line 9 of Algorithm 1, \( \tilde{g}_{t+1} = \text{avg}(v_{t+1}) = \frac{1}{M} \sum_{m \in [M]} v_{t+1}^{(m)}. \)

Hence, for each element in \( v_{t+1} \), i.e., \( v_{t+1}^{(m)} \), where \( m \in [M] \), they have a probability of \( (1 - \frac{A}{M} p_t) \) to retain the previous value, or otherwise update as anchor clients using large batches. Thus, the expected value for \( \mathbb{E}v_{t+1}^{(m)} \) is:

\[
\mathbb{E}v_{t+1}^{(m)} = \frac{A}{M} p_t \cdot \mathbb{E}\nabla f_m(\bar{x}_{t}, E_{m,t}) + \left( 1 - \frac{A}{M} p_t \right) \cdot \mathbb{E}v_{t}^{(m)}
\]

\[
= \frac{A}{M} p_t \cdot \nabla F_m(\bar{x}_t) + \left( 1 - \frac{A}{M} p_t \right) \cdot \mathbb{E}v_{t}^{(m)} \quad (103)
\]

Therefore, we have

\[
\mathbb{E}\tilde{g}_{t+1} = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}v_{t+1}^{(m)} = \frac{A}{M} p_t \cdot \nabla F(\bar{x}_t) + \left( 1 - \frac{A}{M} p_t \right) \cdot \mathbb{E}\tilde{g}_t \quad (105)
\]

It is worth noting that \( \mathbb{E}\tilde{g}_0 = \nabla F(\bar{x}_0) \) as it is initialized at the beginning of the training, i.e., Line 2 – 4 in Algorithm 1. Therefore, \( \mathbb{E}\|\tilde{g}_t - \nabla F(\bar{x}_t)\|_2^2 = 0. \)

Next, we find the recursive bound for \( \mathbb{E}\|\tilde{g}_{t+1} - \nabla F(\bar{x}_{t+1})\|_2^2 \):

\[
\mathbb{E}\|\tilde{g}_{t+1} - \nabla F(\bar{x}_{t+1})\|_2^2 = \mathbb{E}\left\| \left( 1 - \frac{A}{M} p_t \right) \cdot (\mathbb{E}\tilde{g}_t - \nabla F(\bar{x}_t)) + \nabla F(\bar{x}_t) - \nabla F(\bar{x}_{t+1}) \right\|_2^2
\]

\[
\leq \left( 1 + \frac{A p_t}{M - A p_t} \right) \left( 1 - \frac{A}{M} p_t \right) \mathbb{E}\|\mathbb{E}\tilde{g}_t - \nabla F(\bar{x}_t)\|_2^2
\]

\[
+ \left( 1 + \frac{M - A p_t}{A p_t} \right) \mathbb{E}\|\nabla F(\bar{x}_t) - \nabla F(\bar{x}_{t+1})\|_2^2
\]

\[
\leq \left( 1 - \frac{A}{M} p_t \right) \mathbb{E}\|\mathbb{E}\tilde{g}_t - \nabla F(\bar{x}_t)\|_2^2 + \frac{M}{A p_t} L^2 \mathbb{E}\|\bar{x}_{t+1} - \bar{x}_t\|_2^2 \quad (108)
\]

where Equation (108) follows \( (\alpha + \beta)^2 \leq \left( 1 + \frac{1}{\gamma} \right) \alpha^2 + (1 + \gamma) \beta^2 - \left( \frac{1}{\sqrt{\gamma}} \alpha + \sqrt{\gamma} \beta \right)^2 \leq \left( 1 + \frac{1}{\gamma} \right) \alpha^2 + (1 + \gamma) \beta^2 \), and Equation (109) follows Assumption 1.

**Lemma 6.** Suppose that Assumption 7 and 2 hold. Let the local learning rate satisfy \( \eta_t \leq \min \left( \frac{1}{2\sqrt{3KL}}, \frac{1}{2\sqrt{3M^2 \gamma}} \sqrt{\frac{p_t}{\|\bar{z}_t\|}} \right) \). With FedAAMD, \( \sum_{k=0}^{K-1} \|x_{i,k}^{(m)} - x_{i,k-1}^{(m)}\|_2^2 \) represents the sum of the
second norm of every iteration’s difference. Therefore, the bound for such a summation in the expected form should be

\[
\sum_{k=0}^{K-1} \mathbb{E} \left\| \mathbf{x}_{t,k} - \mathbf{x}_{t,k-1} \right\|_2^2 \leq 2\eta_t^2 (K+1) \frac{\sigma^2}{Mb} + 6\eta_t^2 (K+1) \| \mathbb{E} \tilde{g}_t - \nabla F (\tilde{x}_t) \|_2^2 + 6\eta_t^2 (K+1) \| \nabla F (\tilde{x}_t) \|_2^2
\]  

(110)

**Proof.** According to Equation (31), the update at \((k-1)\)-th iteration is

\[
\mathbf{x}_{t,k} - \mathbf{x}_{t,k-1} = -\eta_t g_{t,k} = -\eta_t \left( \mathbf{g}_{t,0} - \sum_{\kappa=0}^{k-1} \nabla f_m \left( \mathbf{x}_{t,\kappa-1}^{(m)}, B_{m,\kappa}' \right) + \sum_{\kappa=0}^{k-1} \nabla f_m \left( \mathbf{x}_{t,\kappa}^{(m)}, B_{m,\kappa}' \right) \right).
\]  

(111)

To find the bound for the expected value of its second norm, the analysis is presented as follows:

\[
\mathbb{E} \left\| \mathbf{x}_{t,k}^{(m)} - \mathbf{x}_{t,k-1}^{(m)} \right\|_2^2
\]  

(112)

\[
= \eta_t^2 \mathbb{E} \left\| \mathbf{g}_t - \sum_{\kappa=0}^{k-1} \nabla f_m \left( \mathbf{x}_{t,\kappa-1}^{(m)}, B_{m,\kappa}' \right) + \sum_{\kappa=0}^{k-1} \nabla f_m \left( \mathbf{x}_{t,\kappa}^{(m)}, B_{m,\kappa}' \right) \right\|_2^2
\]  

(113)

\[
= \eta_t^2 \mathbb{E} \left\| \mathbf{g}_t - \mathbb{E} \mathbf{g}_t - \sum_{\kappa=0}^{k-1} \left( \nabla f_m \left( \mathbf{x}_{t,\kappa-1}^{(m)}, B_{m,\kappa}' \right) - \nabla f_m \left( \mathbf{x}_{t,\kappa}^{(m)}, B_{m,\kappa}' \right) \right) \right\|_2^2
\]  

(114)

\[
= \eta_t^2 \left( \mathbb{1}_{\{b<n\}} \frac{\sigma^2}{Mb} + \sum_{\kappa=0}^{k-1} \frac{L^2 \sigma}{b'} \mathbb{E} \left\| \mathbf{x}_{t,\kappa}^{(m)} - \mathbf{x}_{t,\kappa-1}^{(m)} \right\|_2^2 \right)
\]  

(115)

\[
= \eta_t^2 \mathbb{E} \left\| \mathbf{g}_t - \mathbb{E} \mathbf{g}_t \right\|_2^2
\]  

(116)

\[
= \eta_t^2 \left( \mathbb{1}_{\{b<n\}} \frac{\sigma^2}{Mb} + \sum_{\kappa=0}^{k-1} \frac{L^2 \sigma}{b'} \mathbb{E} \left\| \mathbf{x}_{t,\kappa}^{(m)} - \mathbf{x}_{t,\kappa-1}^{(m)} \right\|_2^2 \right)
\]  

(117)

where Equation (114) is based on the variance expansion on the first term of Equation (113); Equation (117) is based on Cauchy-Schwarz inequality.

Therefore, by summing Equation (118) for \(k = 1, \ldots, K\), we have

\[
\sum_{k=0}^{K-1} \left\| \mathbf{x}_{t,k}^{(m)} - \mathbf{x}_{t,k-1}^{(m)} \right\|_2^2 \leq \sum_{k=0}^{K-1} \left\| \mathbf{x}_{t,k}^{(m)} - \mathbf{x}_{t,k-1}^{(m)} \right\|_2^2
\]  

(119)
With the model training using FedAMD, the recursive function between $E_3$, Next, we find the bound for the third term of Equation (123), i.e.,

According to Lemma 4, we have:

Lemma 7.

Obviously, according to the setting of the local learning rate in the description above, the inequality $3\eta_i^2 K \left( KL^2 + \frac{L^2}{\beta'} \right) \leq \frac{1}{2}$ holds. Therefore, we can easily obtain the bound for the sum of the second norm of every iteration’s difference, which is consistent with Equation (110).

D.2 Proofs for Non-convex Objectives

The following lemma provides a recursive expression on $EF(\bar{x}_{t+1}) - F(\bar{x}_t)$ for time-varying probability settings.

**Lemma 7.** Suppose that Assumption 2 holds, and the time-varying probability sequence $\{p_t \in (0, 1)\}_{t \geq 0}$. Let the local updates $K \geq 1$, and the local learning rate $\eta_i \leq \min \left( \frac{1}{\sqrt{6KL}}, \frac{\sqrt{\beta/K}}{2\sqrt{M_0L_x}} \right)$. With the model training using FedAMD, the recursive function between $F(\bar{x}_{t+1})$ and $F(\bar{x}_t)$ in expected form is

$$
EF(\bar{x}_{t+1}) - F(\bar{x}_t) \leq - \frac{\eta_i \eta_i K}{4} \| \nabla F(\bar{x}_t) \|^2_2 - \left( \frac{1}{2\eta_i \eta_i K} - \frac{L}{2} \right) \| \bar{x}_{t+1} - \bar{x}_t \|^2_2
$$

$$
+ 4\eta_i \eta_i K \left( 1 - (p_t)^A \right) \| \nabla \tilde{g}_t - \nabla F(\bar{x}_t) \|^2_2
$$

$$
+ 3\eta_i \eta_i K \left( 1 - (p_t)^A \right) 1_{\{b<n\}} \frac{\sigma^2}{Mb}
$$

(121)

**Proof.** According to Lemma 4, we have:

$$
EF(\bar{x}_{t+1}) - F(\bar{x}_t)
$$

(122)

$$
\leq - \frac{\eta_i \eta_i K}{2} \| \nabla F(\bar{x}_t) \|^2_2 - \left( \frac{1}{2\eta_i \eta_i K} - \frac{L}{2} \right) \| \bar{x}_{t+1} - \bar{x}_t \|^2_2 + \frac{\eta_i}{2\eta_i K} \| \Delta x_t - \eta_i K \nabla F(\bar{x}_t) \|^2_2
$$

(123)

Next, we find the bound for the third term of Equation (123), i.e., $\mathbb{E} \| \Delta x_t - \eta_i K \nabla F(\bar{x}_t) \|^2_2$, according to the following derivation:

$$
\mathbb{E} \| \Delta x_t - \eta_i K \nabla F(\bar{x}_t) \|^2_2 = \mathbb{E} \| \text{avg} \| \Delta x_t - \eta_i K \nabla F(\bar{x}_t) \|^2_2
$$

(124)

$$
\leq (1 - (p_t)^A) \frac{1}{M} \sum_{m=1}^{M} \mathbb{E} \| \Delta x_t^{(m)} - \eta_i K \nabla F(\bar{x}_t) \|^2_2
$$

(125)

$$
= (1 - (p_t)^A) \frac{1}{M} \sum_{m=1}^{M} \left( \mathbb{E} \| \Delta x_t^{(m)} - \Delta x_t^{(m)} \|^2_2 + \mathbb{E} \| \Delta x_t^{(m)} - \eta_i K \nabla F(\bar{x}_t) \|^2_2 \right)
$$

(126)

where Equation (125) follows Lemma 1 and Equation (126) follows variance equation. To find the bound for Equation (126), we first analyze its first term, i.e., $\mathbb{E} \| \Delta x_t^{(m)} - \Delta x_t^{(m)} \|^2_2$. According to Section 8, we have:

$$
\mathbb{E} \| \Delta x_t^{(m)} - \Delta x_t^{(m)} \|^2_2
$$

(127)

$$
\leq 2\eta_i^2 K^2 \mathbb{E} \| \tilde{g}_t - \mathbb{E} \tilde{g}_t \|^2_2 + 2\eta_i^2 \sum_{k=0}^{K-1} (K-1) \frac{L^2 \sigma}{\beta'} \| x_{t+k} - x_{t,k} \|^2_2
$$

(128)

$$
\leq 2\eta_i^2 K^2 \left( 1 + 2\eta_i \frac{L^2 \sigma}{\beta'} \right) 1_{\{b<n\}} \frac{\sigma^2}{Mb} + 12\eta_i^2 K^2 \frac{L^2 \sigma}{\beta'} \left( \| \mathbb{E} \tilde{g}_t - \nabla F(\bar{x}_t) \|^2_2 + \| \nabla F(\bar{x}_t) \|^2_2 \right)
$$

(129)
where Equation (129) follows Lemma 6. According to the local learning rate setting in the description, we have

$$E\left\| \Delta x_t^{(m)} - \mathbb{E}\Delta x_t^{(m)} \right\|_2^2 \leq 4\eta_t^2 K^2 \cdot \mathbf{1}_{\{b<n\}} \frac{\sigma^2}{Mb} + 12\eta_t^4 K^2 \frac{L^2 \sigma}{\mu'} \left( \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2 + \left\| \nabla F(\tilde{x}_t) \right\|_2 \right)$$  

(130)

After finding the bound for the first term of Equation (126), we now give the bound for its second term, i.e.,

$$E\left\| \mathbb{E}\Delta x_t^{(m)} - \eta_t K \nabla F(\tilde{x}_t) \right\|_2^2$$  

(132)

$$= E\left\| \eta_t \left( \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right) + \eta_t \sum_{k=0}^{K-1} \sum_{\kappa=0}^{K-1} \left( \nabla F_m \left( x_{t,k}^{(m)} \right) - \nabla F_m \left( x_{t,k-1}^{(m)} \right) \right) \right\|_2^2$$  

(133)

$$\leq 2\eta_t^2 K^2 \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2 + 2\eta_t^2 L^2 \sum_{k=0}^{K-1} \sum_{\kappa=0}^{K-1} \left\| \nabla F_m \left( x_{t,k}^{(m)} \right) - \nabla F_m \left( x_{t,k-1}^{(m)} \right) \right\|_2^2$$  

(134)

$$\leq 2\eta_t^2 K^2 \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2 + 2\eta_t^2 K^2 \sum_{k=0}^{K-1} \sum_{\kappa=0}^{K-1} \left\| x_{t,k}^{(m)} - x_{t,k-1}^{(m)} \right\|_2^2$$  

(135)

$$\leq 2\eta_t^2 K^2 \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2 + 2\eta_t^2 K^2 \sum_{k=0}^{K-1} \sum_{\kappa=0}^{K-1} \left\| x_{t,k}^{(m)} - x_{t,k-1}^{(m)} \right\|_2^2$$  

(136)

$$\leq 2\eta_t^2 K^2 \cdot \mathbf{1}_{\{b<n\}} \frac{\sigma^2}{Mb} + 2\eta_t^4 K^2 \left( 1 + 3\eta_t^2 K^2 L^2 \right) \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2 + 6\eta_t^4 K^4 L^2 \left\| \nabla F(\tilde{x}_t) \right\|_2^2$$  

(137)

where Equation (134) follows $(\alpha + \beta)^2 \leq 2\alpha^2 + 2\beta^2$; Equation (135) follows Cauchy–Schwarz inequality and Assumption [1]; Equation (137) is based on Lemma 6. Then, according to the setting for the local learning rate in the description above, we can further simplify Equation (137)

$$E\left\| \mathbb{E}\Delta x_t^{(m)} - \eta_t K \nabla F(\tilde{x}_t) \right\|_2^2 \leq 2\eta_t^4 K^4 L^2 \cdot \mathbf{1}_{\{b<n\}} \frac{\sigma^2}{Mb} + 4\eta_t^2 K^2 \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2$$  

(138)

Plugging Equation (131) and Equation (138) back to Equation (126), we can primarily obtain the inequality below:

$$E\left\| \Delta x_t - \eta_t K \nabla F(\tilde{x}_t) \right\|_2^2 \leq 2\eta_t^2 K^2 \left( 1 - (\eta_t)^A \right) \left( 2 + 3\eta_t^2 K^2 L^2 \right) \mathbf{1}_{\{b<n\}} \frac{\sigma^2}{Mb}$$

$$+ 4\eta_t^2 K^2 \left( 1 - (\eta_t)^A \right) \left( 1 + 3\eta_t^2 \frac{L^2}{\mu'} \right) \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2$$

$$+ 6\eta_t^2 K^2 \left( 1 - (\eta_t)^A \right) \left( 2L^2 \frac{L^2}{\mu'} + K^2 L^2 \right) \left\| \nabla F(\tilde{x}_t) \right\|_2^2$$  

(139)

With the setting in the description, we have:

$$E\left\| \Delta x_t - \eta_t K \nabla F(\tilde{x}_t) \right\|_2^2 \leq 6\eta_t^2 K^2 \left( 1 - (\eta_t)^A \right) \mathbf{1}_{\{b<n\}} \frac{\sigma^2}{Mb}$$

$$+ 8\eta_t^2 K^2 \left( 1 - (\eta_t)^A \right) \left\| \mathbb{E}\tilde{g}_t - \nabla F(\tilde{x}_t) \right\|_2^2$$

$$+ 6\eta_t^2 K^2 \left( 1 - (\eta_t)^A \right) \left( 2L^2 \frac{L^2}{\mu'} + K^2 L^2 \right) \left\| \nabla F(\tilde{x}_t) \right\|_2^2$$  

(140)
Therefore, according to the upper bound analyzed in the previous inequalities, Equation (123) can be reformulated as
\[ \mathbb{E} F(\tilde{x}_{t+1}) - F(\tilde{x}_t) \leq \frac{4\eta_s K}{M} \mathbb{E} ||\tilde{g}_{t+1} - \nabla F(\tilde{x}_{t+1})||^2 \]
(141)

By using the setting of the local learning rate and the global learning rate in the description, we can obtain the desired result.

**Theorem 6.** Suppose that Assumption 1, 2, and 3 hold. Let the local updates \( K \geq 1 \), and the local learning rate \( \eta_l \) and the global learning rate \( \eta_g \) be \( \eta = \frac{1}{K} \left( 1 + \frac{2M}{2pA} \sqrt{1 - p^A} \right)^{-1} \), where

\[ \eta_l \leq \min \left( \frac{1}{2\sqrt{6K}}, \frac{\sqrt{\nu/K}}{4\sqrt{3L_o}} \right). \]

Therefore, the convergence rate of FedAMD for non-convex objectives should be

\[ \min_{t \in [T]} ||\nabla F(\tilde{x}_t)||^2 \leq O \left( \frac{1}{T} \left( 1 + \frac{2M}{2pA} \sqrt{1 - p^A} \right) \right) + O \left( (1 - p^A) 1_{\{b<n\}} \frac{\sigma^2}{Mb} \right) \]
(143)

where we treat \( F(\tilde{x}_0) - F_* \) and \( L \) as constants.

**Proof.** With Lemma 5 and Lemma 7, we can find the following recursive function under the constant probability settings:

\[ \mathbb{E} F(\tilde{x}_{t+1}) + \frac{4\eta_s K}{M} \mathbb{E} ||\tilde{g}_{t+1} - \nabla F(\tilde{x}_{t+1})||^2 \]
(144)

\[ \leq \mathbb{E} F(\tilde{x}_t) + \frac{4\eta_s K}{M} \mathbb{E} ||\tilde{g}_t - \nabla F(\tilde{x}_t)||^2 - \frac{4\eta_s K}{4} \mathbb{E} ||\tilde{x}_{t+1} - \tilde{x}_t||^2 \]
(145)

Since \( \eta_s \eta_l = \frac{1}{K} \left( 1 + \frac{2M}{2pA} \sqrt{1 - p^A} \right)^{-1} \), we have:

\[ F_* \leq \mathbb{E} F(\tilde{x}_T) \leq \mathbb{E} F(\tilde{x}_T) + \frac{4\eta_s K}{M} \mathbb{E} ||\tilde{g}_T - \nabla F(\tilde{x}_T)||^2 \]
(146)

\[ \leq \mathbb{E} F(\tilde{x}_{T-1}) + \frac{4\eta_s K}{M} \mathbb{E} ||\tilde{g}_{T-1} - \nabla F(\tilde{x}_{T-1})||^2 - \frac{4\eta_s K}{4} \mathbb{E} ||\tilde{x}_T - \tilde{x}_{T-1}||^2 \]
(147)

\[ \leq F(\tilde{x}_0) + \frac{4\eta_s K}{M} \mathbb{E} ||\tilde{g}_0 - \nabla F(\tilde{x}_0)||^2 - \frac{4\eta_s K}{4} \sum_{t=0}^{T-1} ||\nabla F(\tilde{x}_t)||^2 \]
(148)

According to Lemma 5, \( ||\tilde{g}_0 - \nabla F(\tilde{x}_0)||^2 = 0 \). Therefore, based on the derivation above, we can attain the following inequality:

\[ \frac{1}{T} \sum_{t=0}^{T-1} ||\nabla F(\tilde{x}_t)||^2 \leq \frac{4(F(\tilde{x}_0) - F_*)}{\eta_s 

By using the settings of the local learning rate and the global learning rate in the description, we can obtain the desired result.
D.3 Proofs for PL Condition

**Theorem 7.** Suppose that Assumption 7 and 4 hold. Let the local updates $K \geq 1$, and the local learning rate $\eta_l$ and the global learning rate $\eta_g$ be $\eta_l, \eta_g = \min \left( \frac{A_p}{M K_p}, \frac{1}{KL} \left( 1 + \frac{2M}{A_p} \sqrt{1 - p^A} \right)^{-1} \right)$, where $\eta_l \leq \min \left( \frac{1}{2\sqrt{\eta K L}}, \frac{\sqrt{\eta}}{4M_\sigma} \right)$. Therefore, the convergence rate of FedAMD for PL condition should be

$$
\mathbb{E} F(\bar{x}_T) - F^* \leq \left( 1 - \frac{1}{2} \min \left( \frac{A_p}{M L}, \frac{\mu}{L} \left( 1 + \frac{2M}{A_p} \sqrt{1 - p^A} \right)^{-1} \right) \right)^T (F(\tilde{x}_0) - F^*) + O \left( \frac{1 - p^A}{\mu} \cdot 1_{\{b < n\}} \frac{\sigma^2}{Mb} \right) 
$$

(150)

**Proof.** With Lemma 7 we have the recursive function on the time-varying probability settings under PL condition:

$$
\mathbb{E} F(\bar{x}_{t+1}) - F(\bar{x}_t) \leq -\frac{\eta_l \eta_g K}{4} \| \nabla F(\tilde{x}_t) \|^2 - \left( \frac{1}{2\eta_l \eta_g K} - \frac{L}{2} \right) \| \bar{x}_{t+1} - \bar{x}_t \|^2 + 4\eta_l \eta_g K \left( 1 - (p_t)^A \right) \| \mathbb{E} \tilde{g}_t - \nabla F(\tilde{x}_t) \|^2 + 3\eta_l \eta_g K \left( 1 - (p_t)^A \right) 1_{\{b < n\}} \frac{\sigma^2}{Mb} 
$$

(151)

According to the description, we consider the probability $p_t = p$ and have:

$$
\mathbb{E} F(\bar{x}_{t+1}) - F^* \leq \left( 1 - \frac{\mu \eta_l \eta_g K}{2} \right) (F(\bar{x}_t) - F^*) - \left( \frac{1}{2\eta_l \eta_g K} - \frac{L}{2} \right) \| \bar{x}_{t+1} - \bar{x}_t \|^2 + 4\eta_l \eta_g K \left( 1 - p^A \right) \mathbb{E} \| \tilde{g}_t - \nabla F(\tilde{x}_t) \|^2 + 3\eta_l \eta_g K \left( 1 - p^A \right) 1_{\{b < n\}} \frac{\sigma^2}{Mb} 
$$

(152)

Since $\eta_l \eta_g \leq \frac{A_p}{MK_p}$, we have:

$$
\mathbb{E} F(\bar{x}_{t+1}) - F^* + \frac{8\eta_l \eta_g K \left( 1 - p^A \right) M}{Ap} \mathbb{E} \| \tilde{g}_{t+1} - \nabla F(\tilde{x}_{t+1}) \|^2 \leq \left( 1 - \frac{\mu \eta_l \eta_g K}{2} \right) \left( F(\bar{x}_t) - F^* + \frac{8\eta_l \eta_g K \left( 1 - p^A \right) M}{Ap} \mathbb{E} \| \tilde{g}_t - \nabla F(\tilde{x}_t) \|^2 \right) 
$$

(153)

$$
- \left( \frac{1}{2\eta_l \eta_g K} - \frac{L}{2} \right) \frac{8\eta_l \eta_g K \left( 1 - p^A \right) M M}{Ap L^2} \| \tilde{x}_{t+1} - \tilde{x}_t \|^2 + 3\eta_l \eta_g K \left( 1 - p^A \right) 1_{\{b < n\}} \frac{\sigma^2}{Mb} 
$$

(154)

According to the description $\eta_l \eta_g \leq \frac{1}{LK \left( 1 + \frac{2M}{Ap} \sqrt{1 - p^A} \right)}$, we have:

$$
\mathbb{E} F(\bar{x}_{t+1}) - F^* \leq \mathbb{E} F(\bar{x}_{t+1}) - F^* + \frac{8\eta_l \eta_g K \left( 1 - p^A \right) M}{Ap} \mathbb{E} \| \tilde{g}_{t+1} - \nabla F(\tilde{x}_{t+1}) \|^2 \leq \left( 1 - \frac{\mu \eta_l \eta_g K}{2} \right) \left( F(\bar{x}_t) - F^* + \frac{8\eta_l \eta_g K \left( 1 - p^A \right) M}{Ap} \mathbb{E} \| \tilde{g}_t - \nabla F(\tilde{x}_t) \|^2 \right) 
$$

(155)

$$
+ 3\eta_l \eta_g K \left( 1 - p^A \right) 1_{\{b < n\}} \frac{\sigma^2}{Mb} 
$$

(156)
\[
\leq \left(1 - \frac{\mu \eta \eta K}{2}\right)^{t+1} (F(\tilde{x}_0) - F_*)
+ \left(1 + \cdots + \left(1 - \frac{\mu \eta \eta K}{2}\right)^t\right) 3\eta \eta K (1 - p^A) 1_{\{b < n\}} \frac{\sigma^2}{Mb}
\]
\[
= \left(1 - \frac{\mu \eta \eta K}{2}\right)^{t+1} (F(\tilde{x}_0) - F_*) + \frac{6}{\mu} (1 - p^A) 1_{\{b < n\}} \frac{\sigma^2}{Mb}
\]

By using the settings of the local learning rate and the global learning rate in the description, we can obtain the desired result.
E Additional Experiments

In the main text, we have analyzed some experimental results in Section 5. In this part, we conduct more thorough experiments by setting different numbers of local updates and different secondary mini-batch sizes.

E.1 Detailed Experimental Setup

Our experiment conducts on Fashion MNIST [91], an image classification task to categorize a 28×28 greyscale image into 10 labels (including T-shirt/top, Trouser, Pullover, Dress, Coat, Sandal, Shirt, Sneaker, Bag, Ankle boot). In the training dataset, each class owns 6K samples. To obtain a recognizable model on the images in the test dataset, we utilize a convolutional neural network structure LeNet-5 [89, 90]. Below comprehensively presents the structure of LeNet-5 on Fashion MNIST:

| Layer | Output Shape | Trainable Parameters | Activation | Hyperparameters |
|-------|--------------|----------------------|------------|-----------------|
| Input | (1, 28, 28)  | 0                    |            |                 |
| Conv2d | (6, 24, 24) | 156                  | ReLU       | kernel size=5   |
| MaxPool2d | (6, 12, 12) | 0                    |            | kernel size=2   |
| Conv2d | (16, 8, 8)  | 2416                 | ReLU       | kernel size=5   |
| MaxPool2d | (16, 4, 4)  | 0                    |            | kernel size=2   |
| Flatten |             | 256                  |            |                 |
| Dense  |              | 120                  | ReLU       |                 |
| Dense  |              | 84                   | ReLU       |                 |
| Dense  |              | 10                   | softmax    |                 |

Our code is implemented using PyTorch v.1.4.0. Clients are picked randomly and uniformly, without replacement in one round but with replacement in subsequent rounds. For each baseline, the learning rate picks the best one from the set \{0.02, 0.005, 0.002, 0.0005, 0.0002\}.

Throughout the training, we measure the performance on the test dataset of Fashion MNIST. The training loss is calculated by the miner clients on the average loss of all iterations. As for the test accuracy, the server utilizes the entire test dataset with a total of 10K images after the global model updates. The gradient complexity is calculated at each round according to the number of anchor clients (using the entire local dataset) and the number of miner clients (randomly selecting mini-batch with the size of \(b'\) for multiple iterations).

E.2 More Numerical Results
Figure 3: Comparison of test accuracy and training loss on different probability settings for FedAMD with $Kb' = 640$. Top: $b' = 64$; bottom: $b' = 16$. (Zoom in for the best view)

Figure 4: Comparison of test accuracy and training loss on different probability settings for FedAMD with $Kb' = 640$. Top: $b' = 64$; bottom: $b' = 16$. (Zoom in for the best view)
Figure 5: Comparison of test accuracy and training loss on different baselines with $Kb' = 320$. Top: $b' = 64$; bottom: $b' = 16$. (Zoom in for the best view)

Figure 6: Comparison of test accuracy and training loss on different probability settings for FedAMD with $Kb' = 320$. Top: $b' = 64$; bottom: $b' = 16$. (Zoom in for the best view)
Figure 7: Comparison of test accuracy and training loss on different probability settings for FedAMD with $Kb' = 64$. Top: $b' = 64$; bottom: $b' = 16$. (Zoom in for the best view)

Figure 8: Comparison of test accuracy and training loss on different probability settings for FedAMD with $Kb' = 64$. Top: $b' = 64$; bottom: $b' = 16$. (Zoom in for the best view)