Application of $\Psi$-transformation to the search for continuous function’s global extremum on simplex

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Abstract. A nonconvex problem of mathematical programming, the acceptance region of which is simplex. A two-stage algorithm is suggested to solve the problem. At the first stage, the global optimum region is determined; at the second stage, local clean-up of the solution is carried out. The first stage is realized by the $\Psi$-transformation method, which is an alternative to direct random search techniques. The method is to build and use $\Psi$-function. $\Psi$-function is built empirically based on statistic tests. To perform the tests, the generator of random points evenly distributed in simplex is used. Even distribution in simplex is achieved through affine and linear transformations of points evenly distributed in a unit hypercube. For refinement of the approximate solution obtained at the first stage, the method of regular simplex reflection is used. Examples are discussed. The example of algorithm usage to optimize the hydrocarbon mixture make-up is presented.

1. Introduction
In the wide area of applications, a non-linear and in a general way a nonconvex problem has to be solved:

$$f(\mathbf{x}) \rightarrow \max,$$
$$\sum_{i=1}^{n} x_i = 1,$$
$$x_i \geq 0, \ i = 1, 2, \ldots, n,$$

where $f(\mathbf{x})$ is a continuous and probably a non-smooth function; $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

To solve multietremial optimization problems, various randomized heuristics are mostly used. They all use global and local random search in various combinations and realizations [1, 2]. For example, in the simulated annealing method, transition from the global search to the local one is done gradually [3, 4]. At the initial stage, a random search across the whole acceptance region is carried out. Thereafter, still being random, it becomes more and more local. In the cross-entropy method [5, 6], gradual search localization is implemented by changing search sample distribution in such a way that promising options would occur more likely. There are methods utilizing multi-start: a start of local searches across multiple points of the acceptance region [7, 8]. The whole region of the search space is scanned. It reduces likelihood that the search would localize in a region that is far from the global extremum. Recently, genetic algorithm simulating the natural evolution process are used more and more frequently [9, 10]. In such algorithms, the global component is realized through generation of
populations (sets of legitimate options), and localization – by simulation of the inheritance mechanism and natural selection in the course of their gradual updates.

The drawback of randomized heuristics as applied to problem (1) is that as the problem size grows, the likelihood of finding the global extremum within acceptable time noticeably decreases [11]. Indeed, let us assume that to find the global extremum it is necessary to enter into one of regions produced as a result of breaking up the acceptance set of problem (1) by each coordinate into \( m \) equal intervals. The likelihood of getting into such region as a result of one random sample generated by the transducer of random points evenly distributed across the search space is determined by formula \( p = m^{-n} \). Already at comparatively small \( m \) and \( n \), the likelihood decreases almost to zero. If the likelihood of an event is remote, then, with a sufficiently high number of tests \( K \), the likelihood of its occurrence would be equal to \( k \) and determined according to Poisson formula

\[
P_k(k) = \frac{(Kp)^k e^{-Kp}}{k!}.
\]

Therefore, the likelihood that there will be no hits into the global extremum region would be \( P_k(0) = e^{-Kp} \). The likelihood that there will be at least one hit into the global extremum region would be \( P_k(k \geq 1) = 1 - P_k(0) = 1 - e^{-Kp} \). If we specify the hit likelihood equal to 0.95, we will obtain: 0.95 = 1 - \( e^{-Kp} \). It follows that in order to hit the region sought at least once with practical likelihood, it is necessary to do \( K \approx 3n^m \) assays.

2. Materials and methods

As an alternative to direct random search methods, the method of \( \Psi \)-transformation is suggested [12]. For problem \( \max \{f(x) \mid x \in E\} \), where \( E \) is some subset of \( n \)-dimensional Euclidean space, function \( \Psi(\zeta) \) is introduced as a measure of set \( E'(\zeta) = \{x \mid x \in E, f(x) \geq \zeta\} \). Evidently, \( \Psi(\zeta) \) steadily decreases and reaches zero in point \( \zeta = f_{\max} \), where \( f_{\max} \) is the target function value in the global maximum point.

The method is realized not using function \( \Psi(\zeta) \) itself, but using its empirical analogue built based on statistic tests. The set of values \( f(x) \) is broken down into levels in increments of \( \Delta \), starting from \( \zeta_0 : \zeta_l = \zeta_{l-1} + \Delta \), \( l = 1, 2, \ldots \). For each level, a series of random assays is performed and characteristics of set \( E'(\zeta_l) \) are determined: the statistic evaluation of measure \( \Psi(\zeta_l) \) and coordinates of the center of gravity \( \bar{x}(\zeta_l) \). As evaluation of \( \Psi(\zeta_l) \), \( \psi_j \) - the relative number of points that entered \( E'(\zeta_l) \) - is taken. Variable \( \ell = (\zeta - \zeta_0) \Delta^{-1} \) is introduced, and on its basis, trends \( \psi(\ell) \) and \( \bar{x}(\ell) \) are built. The presumable global extremum coordinates are found through extrapolation along trends \( \bar{x}(\ell) \) into point \( \ell' \), which is the solution of equation \( \psi(\ell) = 0 \).

For the evaluation of characteristics of sets \( E'(\zeta_l), \ l = 1, 2, \ldots \) to be adequate, statistic tests should be done using random points evenly distributed in \( E \). In our case, \( E \) is a regular \((n-1)\)-dimensional simplex. It can be placed into a respective \((n-1)\)-dimensional cube. Points that are evenly distributed in the cube would be evenly distributed in the simplex, too. However, such method of obtaining even distribution in the simplex is non-productive. The volume of a regular \((n-1)\)-dimensional simplex with a unit length side is equal to

\[
\frac{1}{(n-1)!} \left( \frac{n}{2^{n-1}} \right)^{\frac{1}{2}}.
\]
and as \( n \) increases it fast tends to zero. Correspondingly, the likelihood that a random point generated in the hypercube would be in the simplices tends to zero as well.

2.1. Generation of points evenly distributed in simplex

Even distribution in the simplex is achieved through affine and linear transformations of points evenly distributed in the unit hypercube.

\((n-1)\) of random numbers evenly distributed in a unit segment are generated. Numbers are numbered in the ascending order: \( \xi_1 \leq \xi_2 \leq \ldots \leq \xi_{n-1} \). Vector \( \xi' = (\xi'_1, \xi'_2, \ldots, \xi'_{n-1}) \) is formed, where \( \xi'_i = \xi_i - \xi_{i-1}, \, \xi_0 = 0, \, i = 1, 2, \ldots, n-1 \). This vector, as it has been shown in [13], is evenly distributed over set \( M = \{ x \in \mathbb{R}^{n-1} | \sum_{i=1}^{n-1} x_i \leq 1, x_i \geq 0, i = 1, 2, \ldots, n \} \).

All angular points of polyhedron \( M \), except for zero, are spaced at \( \sqrt{2} \). Polyhedron \( M \) is magnified to regular \( \bar{M} \). The vertex at the origin of coordinates is translated so that the distance from it all other vertexes would also be equal to \( \sqrt{2} \). This point will have coordinates \( x'_0 = r = (1 - \sqrt{n})/(n-1) \), \( i = 1, 2, \ldots, n-1 \). Thus, we have obtained a polyhedron congruent to \((n-1)\)-dimensional simplices in the \( n \)-dimensional space. Let us denote the point of set \( \bar{M} \), which is correspondent to point \( \xi' \) of set \( M \), by \( \bar{\xi} \). It is determined by reflection of \( \xi' \) relative to hyperplane \( \sum_{i=1}^{n-1} x_i = 1 \). We make projection \( \bar{\xi}' \) onto this hyperplane: \( \bar{\xi}'_i = \xi'_i + \delta \), \( i = 1, 2, \ldots, n-1 \), where \( \delta \) is deduced from \( \sum_{i=1}^{n-1} \bar{\xi}'_i = 1 \) according to formula \( \delta = (1 - \sum_{i=1}^{n-1} \xi'_i)(n-1)^{-1} \). The magnification ratio is equal to \( \sqrt{n} \), hence, \( \bar{\xi} = \xi' + \sqrt{n}(\xi' - \xi') \). Point \( \bar{\xi} \) is translated into the \((n-1)\)-dimensional simplex of \( n \)-dimensional space of the variables of problem (1). To this end, it is represented as a convex linear combination of angular points of \( \bar{M} \) \( n \)-dimensional space. Coefficients of this combination \( a_1, a_2, \ldots, a_n \) are nothing but new coordinates of a point. They are determined as follows:

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{bmatrix} = 
\begin{pmatrix}
1 & 0 & \ldots & 0 & r \\
0 & 1 & \ldots & 0 & r \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & r \\
1 & 1 & \ldots & 1 & 1
\end{pmatrix}^{-1}
\begin{bmatrix}
\bar{\xi}_1 \\
\bar{\xi}_2 \\
\vdots \\
\bar{\xi}_{n-1} \\
1
\end{bmatrix},
\]

2.2. Finding an approximate solution by the \( \Psi \)-transformation method

The first series of random assays is carried out. For each assay \( k = 1, 2, \ldots, K \) the admissible value of argument \( x^{(k)} \) is generated and \( f(x^{(k)}) \) is deduced. Based on the results of all assays, arithmetic mean \( f_0 \) and maximal \( f_m \) of the target function’s observed values are determined. Levels’ \( L \) is formed:

\( \zeta_l = f_0 + \Delta(l-1), \, l = 1, 2, \ldots, L, \) where \( \Delta = (f_m - f_0)L^{-1} \).

The second series of random assays is carried out. For levels \( l = 1, 2, \ldots, L, \) \( \psi_l = k_i K^{-1} \), is determined, where \( k_i \) is the number of assays in which \( f(x^{(k)}) \geq \zeta_l \), as well as mean values of coordinates: \( \bar{x}_l = k_i^{-1} \sum_k \delta_j x^{(k)}_i, \, i = 1, 2, \ldots, n \), where \( \delta = 1 \) if \( f(x^{(k)}) \geq \zeta \), otherwise \( \delta = 0 \).

The obtained sequences \( \Psi = (\psi_l, l = 1, 2, \ldots, L) \), \( \bar{x}_l = (\bar{x}_{il}, l = 1, 2, \ldots, L) \), \( i = 1, 2, \ldots, n \), are used to draw quadratic trends. Parameters \( a = (a_0, a_1, a_2)^T \) of trend \( \psi(l) = a_0 + a_1 l + a_2 l^2 \) are calculated according to formula \( a = (A^T A)^{-1} A^T \Psi \), where
\[
\Lambda = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 4 \\
\vdots & \vdots & \vdots \\
1 & L & L^2
\end{pmatrix}
\]

Similarly, parameters \(b_i = (b_{i0}, b_{i1}, b_{i2})^T \) of trend \(\tilde{x}_i(l) = b_{i0} + b_{i1}l + b_{i2}l^2\) are calculated according to formula \(b_i = (A^T\Lambda)^{-1}A^T\tilde{x}_i\).

\(\ell^*\) is determined as the least positive root of quadratic equation \(\psi(\ell) = 0\). It usually exists because \(\psi(\ell)\) in segment [1, L] is the trend of sequence \(\psi\) steadily decreasing to a value close to zero. However, if equation \(\psi(\ell) = 0\) does not have a solution, then \(\ell^* = L - (a_0 + a_1L + a_2L^2)(a_1 + 2a_2L)^{-1}\).

The approximate solution of the problem is: \(x_i^* = \tilde{x}_i(\ell^*), i = 1, 2, ..., n\).

2.3. Solution refinement, local extremum search

Point \(x^*\) is in the best case within the global extremum region, but it is not the extremum itself. That is why it should be regarded as a start point of further search, namely, a search for local extremum of the hit region.

Methods implying use of derivatives of the first or second order are inapplicable in this instance due to possible non-smoothness of function \(f(x)\). A method has to be used, for which realization values of function \(f(x)\) itself would be sufficient. Taking into account the problem specifics, the method of consecutive reflection of simplexes is proposed [14]. In this case, we are talking about a search for the extremum of function \(f(x)\) in \((n)\)-dimensional simplex using \((n)\)-dimensional simplexes themselves. The algorithm consists in realization of a sequence of series. Within one series, the process continues until it becomes cyclic; thereafter, the ‘work’ simplex’s rib length is reduced, and the next series starts.

In \(n\)-dimensional Euclidean space, we construct a regular \(n\)-dimensional simplex, one of the vertexes of which (let us call it zero) lies at the origin of coordinates. Let the rib length be equal to \(\Delta x\), then vertexes’ coordinates would correspond to the columns of matrix:

\[
\begin{pmatrix}
\begin{array}{cccc}
x_i^{(0)} & x_i^{(1)} & x_i^{(2)} & \ldots & x_i^{(n)} \\
x_2^{(0)} & x_2^{(1)} & x_2^{(2)} & \ldots & x_2^{(n)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_n^{(0)} & x_n^{(1)} & x_n^{(2)} & \ldots & x_n^{(n)}
\end{array}
\end{pmatrix} = \begin{pmatrix}
0 & \hat{q} & q & \ldots & q \\
0 & q & \hat{q} & q & \ldots \\
0 & q & q & \ldots & \hat{q}
\end{pmatrix},
\]

where \(\hat{q} = \frac{\Delta x}{n\sqrt{2}}(\sqrt{n+1}+n-1)\), \(q = \frac{\Delta x}{n\sqrt{2}}(\sqrt{n+1}-1)\).

We translate this simplex from the origin so that all of its vertexes except for \(x^{(0)}\) would be on hyperplane \(\sum_{i=1}^{n} x_i = 1\) and \((n)\)-dimensional simplex with vertexes \(x^{(1)}, x^{(2)}, ..., x^{(n)}\) would be in the search start region.

Upon parallel translation of the zero vertex to point \(\tilde{x}^{(0)}\), the coordinates of all others become equal to \(\tilde{x}^{(k)} = \tilde{x}^{(0)} + x^{(k)}, k = 1, 2, ..., n\). We deduce \(\tilde{x}^{(0)}\) from condition \(\sum_{i=1}^{n} \tilde{x}^{(k)} = 1\). It would be sufficient to take any one vertex, for example, the first one:

\[
\sum_{i=1}^{n} \tilde{x}^{(1)} = \sum_{i=1}^{n} x^{(0)} + \frac{\Delta x}{n\sqrt{2}}(\sqrt{n+1}+n-1) + \sum_{i=2}^{n} \frac{\Delta x}{n\sqrt{2}}(\sqrt{n+1}-1) = \sum_{i=1}^{n} x^{(0)} + \frac{\Delta x}{n\sqrt{2}}(\sqrt{n+1}+1) = 1.
\]

From here, we obtain:
We determine the initial location of the \( n \)-dimensional simplex by referring it to point \( x^* \), obtained by the \( \Psi \)-transformation method. In this case, the coordinates of the zero vertex of simplex would be determined as follows:

\[
\tilde{x}_i^{(0)} = x_i^* \left( 1 - \frac{\Delta x}{\sqrt{2}} \sqrt{n+1} \right), \quad i = 1,2,...,n.
\]

The coordinates of all other vertexes, which form the initial ‘work’ simplex, are represented as columns of matrix \( \tilde{x}^{(k)} \) with element:

\[
\tilde{x}_i^{(k)} = x_i^* \left( 1 - \frac{\Delta x}{\sqrt{2}} \sqrt{n+1} \right) + \left\{ \begin{array}{ll}
\frac{\Delta x}{n\sqrt{2}} \left( \sqrt{n+1} + n - 1 \right) & i = k, \\
\frac{\Delta x}{n\sqrt{2}} \left( \sqrt{n+1} - 1 \right) & i \neq k.
\end{array} \right.
\]

The search process consists in realization of a sequence of steps, at each one of which a new work simplex is constructed by reflecting the vertex having the minimal value of target function \( f(x) \) from the opposite face of the initial simplex. Let \( v \in [1, n] \) be the number of such vertex. Its coordinates are determined according to formula:

\[
\tilde{x}^{(v)} = \frac{2}{n-1} \left( \sum_{i=1}^{n} x_i^{(k)} - x^{(v)} \right) - x^{(v)}.
\]

Since the search is carried out in a limited space, \( \tilde{x}^{(v)} \) might turn out beyond the acceptance region. Let us assume that at the current step of the algorithm, it turned out that \( f(x^{(v)}) < f(x^{(2)}) < ... < f(x^{(n)}) \). Then it is necessary to go through vertexes consequently until we obtain an acceptable reflection. Reflection of the best point will signify the end of the current series. The next series starts from \( \Delta x \) rib length reduction and translation of a new ‘work’ simplex into the region of the best point of the previous series.

3. Results and discussion

A series of test computations was performed using the \( \Psi \)-transformation method and Excel/ Solver/ Evolutionary algorithm. In all cases, it took much longer time for Evolutionary to solve problems. In some cases, a wrong result was obtained. Below is an example of solution of one of the problems.

3.1. Test problem solution

A multietremal problem with a known solution was taken:

\[
\sum_{i=1}^{10} i x_i^2 \rightarrow \text{max},
\]

\[
\sum_{i=1}^{10} x_i = 1,
\]

\[
x_i \geq 0, i = 1,2,...,10.
\]

Evolutionary failed to solve the problem. \( \Psi \)-transformation method produced the following results. 9 levels with increment \( \Delta = 1 \) were setup. For each level, 1000 random assays were performed. The obtained rows of \( \Psi \) and \( \tilde{x}_i \), \( i = 1,2,...,9 \) are given in table 1.

For \( \Psi \) and \( \tilde{x}_i \), \( i = 1,2,...,9 \), trends were calculated. Trends for \( \Psi \) and variables \( x_1, x_5, x_{10} \) are shown on figure 1.
was found on extremum on simplex has applied importance. Many problems 
\[ \Delta x_i = 0.1s^{-1}, \]
where \( s \) is the series number. The solution took a few seconds.

3.2. Mixing problem solution
The problem of finding a function extremum on simplex has applied importance. Many problems arising in oil refinery, petrochemical industry, construction are reduced thereto. The problem of optimization of a hydrocarbon mixture makeup is an example. Some parameters of hydrocarbon

| \( i \) | \( \Psi \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) | \( x_{10} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.982 | 0.087 | 0.100 | 0.102 | 0.102 | 0.103 | 0.106 | 0.099 | 0.104 | 0.096 | 0.102 |
| 2 | 0.855 | 0.055 | 0.061 | 0.103 | 0.108 | 0.111 | 0.117 | 0.109 | 0.115 | 0.107 | 0.114 |
| 3 | 0.643 | 0.039 | 0.041 | 0.046 | 0.097 | 0.116 | 0.132 | 0.127 | 0.136 | 0.128 | 0.138 |
| 4 | 0.429 | 0.029 | 0.031 | 0.031 | 0.034 | 0.095 | 0.139 | 0.144 | 0.166 | 0.156 | 0.176 |
| 5 | 0.259 | 0.022 | 0.023 | 0.023 | 0.026 | 0.026 | 0.026 | 0.108 | 0.151 | 0.195 | 0.233 |
| 6 | 0.145 | 0.017 | 0.017 | 0.016 | 0.020 | 0.021 | 0.022 | 0.121 | 0.219 | 0.235 | 0.311 |
| 7 | 0.078 | 0.014 | 0.014 | 0.013 | 0.016 | 0.016 | 0.016 | 0.013 | 0.202 | 0.279 | 0.418 |
| 8 | 0.034 | 0.009 | 0.011 | 0.011 | 0.011 | 0.012 | 0.010 | 0.009 | 0.010 | 0.287 | 0.630 |
| 9 | 0.008 | 0.004 | 0.007 | 0.008 | 0.004 | 0.008 | 0.007 | 0.008 | 0.005 | 0.007 | 0.942 |

We see that the result obtained is quite close to optimal. The solution was improved following the above method of reflection of regular simplexes. To obtain the optimal solution accurate to a tenth, it was enough to run 20 series reducing the simplex rib length according to formula 

\[ 2 \psi = 27.33 \psi + 0.0144 \ell^2 \]

intersects x-axis at \( \ell^* = 8.915 \). By substitution of \( \ell^* \) into trends of variables \( x_i(\ell) = b_{i0} + b_{i1} \ell + b_{i2} \ell^2 \), \( i = 1, 2, \ldots, 10 \), the approximate solution for \( x^* \) was obtained. The parameters of trends and solution coordinates are given in table 2.

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) | \( x_{10} \) |
|---|---|---|---|---|---|---|---|---|---|
| \( b_0 \) | 0.102 | 0.118 | 0.143 | 0.147 | 0.122 | 0.122 | 0.067 | -0.012 | -0.043 | 0.212 |
| \( b_1 \) | -0.024 | -0.029 | -0.036 | -0.030 | 0.004 | 0.004 | 0.037 | 0.091 | 0.097 | -0.091 |
| \( b_2 \) | 0.001 | 0.002 | 0.002 | 0.002 | -0.002 | -0.002 | -0.005 | -0.010 | -0.009 | 0.018 |
| \( x^* \) | 0.009 | 0.012 | 0.012 | 0.004 | -0.016 | -0.016 | -0.018 | 0.007 | 0.126 | 0.866 |

The function \( \psi(\ell) = 1.2923 - 0.2733 \ell + 0.0144 \ell^2 \) intersects x-axis at \( \ell^* = 8.915 \). By substitution of \( \ell^* \) into trends of variables \( x_i(\ell) = b_{i0} + b_{i1} \ell + b_{i2} \ell^2 \), \( i = 1, 2, \ldots, 10 \), the approximate solution for \( x^* \) was obtained. The parameters of trends and solution coordinates are given in table 2.

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\[ \Delta x_i = 0.1s^{-1}, \]
where \( s \) is the series number. The solution took a few seconds.

3.2. Mixing problem solution
The problem of finding a function extremum on simplex has applied importance. Many problems arising in oil refinery, petrochemical industry, construction are reduced thereto. The problem of optimization of a hydrocarbon mixture makeup is an example. Some parameters of hydrocarbon
mixtures are calculated according to special, sometimes, rather elaborate empirical formulas [15], therefore, applicability of linear programming and conventional non-linear methods as regards such problems is limited.

The mixing problem can be cast as follows:

\[
C = \sum_{i=1}^{n} c_i x_i \to \min, \\
\left\{ \phi_q(x_1, x_2, \ldots, x_n) \in [p_q^-, p_q^+] \cdot q \in Q, \\
\sum_{i=1}^{n} x_i = 1, \\
x_i \geq 0, \ i = 1, 2, \ldots, n, \right. \]

where \(x_i\) is the weight fraction of \(i\)-th component in the mixture; \(c_i\) is the price of \(i\)-th component; \(Q\) is the set of controlled parameters of the mixture; \(\phi_q(x_1, x_2, \ldots, x_n)\) is the dependence of \(q\)-the parameter on the mixture makeup; \(p_q^-, p_q^+\) are the boundary values for \(q\)-th parameter.

The mixture quality requirements can be taken into account as a penalty component of the target function:

\[
f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i + \sum_{q \in Q} \delta_q^2(x_1, x_2, \ldots, x_n) \to \min, \\
\left\{ \sum_{i=1}^{n} x_i = 1, \\
x_i \geq 0, \ i = 1, 2, \ldots, n, \right. \]

where \(\delta_q(x_1, x_2, \ldots, x_n)\) is the percentage deviation of the computed value of \(q\)-th parameter from the specified interval boundary, which is determined by formula:

\[
\delta_q(x_1, x_2, \ldots, x_n) = \begin{cases} 
100 \left( p_q^- - \phi_q(x_1, x_2, \ldots, x_n) \right) / p_q^- & p_q^- - \phi_q(x_1, x_2, \ldots, x_n) \geq 0, \\
100 \left( \phi_q(x_1, x_2, \ldots, x_n) - p_q^+ \right) / p_q^+ & p_q^+ - \phi_q(x_1, x_2, \ldots, x_n) \leq 0, \\
0 & \phi_q(x_1, x_2, \ldots, x_n) \in [p_q^-, p_q^+]. 
\end{cases}
\]

As an example, a three-component mixture was taken. Its price is: \(C = 1.5x_1 + 1.3x_2 + 1.0x_3\). Limitations by parameters (octane number, density, sulfur content) are:

\[
\begin{align*}
\phi_1(x) &= 90.2x_1 + 88.5x_2 + 73.6x_3 - 14.3x_1x_2 - 19.8x_1x_3 + 28.4x_2x_3 \geq 85, \\
\phi_2(x) &= 0.82x_1 + 0.59x_2 + 0.67x_3 \leq 0.68, \\
\phi_3(x) &= 0.001x_1 + 0.003x_2 + 0.002x_3 \leq 0.002.
\end{align*}
\]

According to the results of 1000 random assays, the mean observed value of the target function was equal to 23.42, the minimal - 11.85. 10 levels with increment \(\Delta = -1.16\) were formed. For each level, 1000 random assays were performed. Based on statistic test results, trends were calculated (figure 2):

\[
\begin{align*}
\psi(\ell) &= 0.68346 - 0.00745\ell + 0.00611\ell^2, \\
\bar{x}_1(\ell) &= 0.34104 - 0.01304\ell + 0.00199\ell^2, \\
\bar{x}_2(\ell) &= 0.36006 - 0.00095\ell + 0.00089\ell^2, \\
\bar{x}_3(\ell) &= 0.29959 - 0.01323\ell + 0.00283\ell^2.
\end{align*}
\]
Figure 2. Empirical $\psi$-function and trends of variables.

Function $\psi(\xi) = 0.68346 - 0.00745\xi + 0.0061\xi^2$ intersects x-axis at $\xi^* = 10.07$. By substituting $\xi^*$ into trends of variables, we obtain an approximate solution. The results of the first approximation $x^*$ and improved solution $\bar{x}$ obtained by reflection of simplexes are given in table 3. Evolutionary algorithm produced practically the same result for 3 minutes.

Table 3. Results of the approximate and improved solutions.

|    | $x_1$ | $x_2$ | $x_3$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $f$   |
|----|-------|-------|-------|------------|------------|------------|------|
| $x^*$ | 0.410 | 0.439 | 0.149 | -1.256     | 2.396      | 1.498      | 11.152|
| $\bar{x}$ | 0.426 | 0.445 | 0.129 | -0.939     | 2.699      | 0.928      | 10.637|

One can see from the table 3, that a rather good approximation was obtained already at the first stage. As a result of algorithm realization, incompatibility of the system of limitations was detected, and a solution concerning the criterion which main component is the sum of squares of percentage deviations of parameters from required values was obtained.

4. Conclusion

The $\Psi$-transformation method is an alternative to evolution algorithms when we are talking about multie xtremal continuous problems. The efficiency of evolution algorithms noticeably decreases as the number of variables grows. The scope of computations realized within the $\Psi$-transformation method depends on the problem dimension to a lesser degree than in evolution algorithms because statistic tests are carried out in this case to obtain statistic evaluations and construct trends used to find the solution rather than directly for the search.

The $\Psi$-transformation method is an exactly global search method. Combined with local search methods, it highly likely gives either the optimal result or a result that differs little from optimal in terms of the target function value.

It can be considered a disadvantage of the $\Psi$-transformation method that during statistic tests, for evaluations to be adequate, it is necessary to use random points evenly distributed in the search region. But the same applies to evolution algorithms, too. There, it is also desirable to use exactly even distribution, at least at the initial stage.

Obtaining even distribution in the search region is a separate problem solved only in special cases. This limits applicability of the $\Psi$-transformation method. This paper offers a solution for a special, but having high applied relevance case.
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