PURIFICATION OF ENTANGLED COHERENT STATES

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We suggest an entanglement purification scheme for mixed entangled coherent states using 50-50 beam splitters and photodetectors. This scheme is directly applicable for mixed entangled coherent states of the Werner type, and can be useful for general mixed states using additional nonlinear interactions. We apply our scheme to entangled coherent states decohered in a vacuum environment and find the decay time until which they can be purified.

Keywords: purification, entanglement, entangled coherent state, quantum information

1. Introduction

Entanglement is an important manifestation of quantum mechanics. Highly entangled states play a key role in an efficient realization of quantum information processing including quantum teleportation, cryptography and computation. When an entangled state prepared for quantum information processing is open to an environment, the pure entangled state becomes mixed one and the entanglement of the original state becomes inevitably degraded. To obtain highly entangled states from less entangled mixed ones, entanglement purification protocols have been proposed.

Recently, entangled coherent states have been studied for quantum information processing and nonlocality test. Teleportation schemes via entangled coherent states and quantum computation with coherent-state qubits using multimode entangled coherent states have been investigated. These suggestions require highly entangled coherent states for successful realization. Even though entanglement concentration for partially entangled pure states has been studied, there is a need for a purification scheme for mixed states.

In this paper, we suggest an entanglement purification scheme for mixed entangled coherent states. This scheme is based on the use of 50-50 beam splitters and photodetectors. We show that our scheme can be directly applied for entangled coherent states of the Werner form based on quasi-Bell states. The scheme can also be useful for general mixed entangled coherent states using additional nonlinear interactions.

In Sec. 2, we review entangled coherent states and their characteristics with a entanglement concentration scheme for pure states. The purification scheme for mixed states is suggested and applied to a simple example in Sec. 3. We also discuss how this scheme is related to previously suggested ones, and we show that our scheme is applicable to any Werner-type states. The required bilateral operations for general mixed states are discussed in Sec. 4. In Sec. 5, we apply our scheme to decohered entangled coherent states in a vacuum environment. Finally, we present an example of application for multi-mode entanglement purification.
2. Entangled coherent states and pure state concentration

We define entangled coherent states

\[ |\Phi_\varphi\rangle_{ab} = N_\varphi(|\alpha\rangle_a |\alpha\rangle_b + e^{i\varphi}| -\alpha\rangle_a | -\alpha\rangle_b), \]  
\[ |\Psi_\varphi\rangle_{ab} = N_\varphi(|\alpha\rangle_a | -\alpha\rangle_b + e^{i\varphi}| -\alpha\rangle_a |\alpha\rangle_b), \]  

where \( \alpha = \alpha_r + i\alpha_i \) is the complex amplitude of the coherent state \( |\alpha\rangle \), \( \varphi \) is a real local phase factor, and \( N_\varphi = \{2(1 + \cos \varphi e^{-4|\alpha|^2})\}^{-1/2} \) is the normalization factor. Entangled coherent states (1) and (2) can be generated using a nonlinear medium and lossless 50-50 beam splitter [17]. A coherent superposition state (cat state) can be generated from a coherent state \( |\pm\sqrt{2}\alpha\rangle \) by a nonlinear interaction in a Kerr medium [17]. When the coherent superposition state \( M_\varphi(|\sqrt{2}\alpha\rangle + e^{i\varphi}| -\sqrt{2}\alpha\rangle) \), where \( M_\varphi \) is the normalization factor, is input to a 50-50 beam splitter while nothing is input to the other input port, the resulting state is an entangled coherent state (1) or (2) depending on the relative phase between the reflected and transmitted fields from the beam splitter.

However, Kerr nonlinearity of currently available nonlinear media is too small to generate the required coherent superposition state. It was pointed out that one needs optical fiber of about 3,000 km to generate a coherent superposition state with currently available Kerr nonlinearity [18]. Even though it is possible to make such a long nonlinear optical fiber, the decoherence effect during the propagation is too large. Some alternative methods have been studied to generate a superposition of macroscopically distinguishable states using conditional measurements [19, 20]. One drawback of these schemes is due to the low efficiency of photon-number measurement. Cavity quantum electrodynamics has been studied to enhance nonlinearity [21], and there have been experimental demonstrations of generating cat states in a cavity and in a trap [22, 23]. Unfortunately, all the suggested schemes for quantum information processing with coherent states [8, 9, 11, 15], including the work in this paper, require propagating optical cat states. There were other suggestions to generate cat states with trapped ions [24] and with solitons [25].

Electromagnetically induced transparency (EIT) has been studied as a method to obtain giant Kerr nonlinearity [26]. There was an experimental report of an indirect measurement of a giant Kerr nonlinearity utilizing EIT [27], but this developing technology has not been exactly at hand to generate an optical cat state. In short, generating a propagating optical cat state is extremely demanding and difficult with currently available Kerr effect, so that a scheme to generate a cat state with small nonlinear effect needs to be studied. In this paper, we assume ideal nonlinear effect to generate a coherent superposition state while its generating scheme with small nonlinearity is being studied and will be demonstrated elsewhere [28].

It is possible to define a 2-dimensional Hilbert space \( H_\alpha \) with two linear independent vectors \( |\alpha\rangle \) and \( | -\alpha\rangle \). For example, an orthonormal basis for Hilbert space \( H_\alpha \) can be constructed

\[ |u\rangle = M_+ (|\alpha\rangle + | -\alpha\rangle), \]  
\[ |v\rangle = M_- (|\alpha\rangle - | -\alpha\rangle), \]  

where \( M_+ \) and \( M_- \) are normalization factors. Using the orthogonal basis, we can study the entangled coherent state in the \( 2 \times 2 \)-dimensional Hilbert space. For \( \varphi = 0 \), \( |\Phi_\varphi\rangle \) can be
represented as

\[ |\Phi_{\varphi=0}\rangle = \frac{N(\varphi = 0)}{2M^2_+} \left( |u\rangle|u\rangle + \frac{M^2_+}{M^2_-} |v\rangle|v\rangle \right). \]  

(5)

The entanglement of \( |\Phi_{\varphi}\rangle \) and \( |\Psi_{\varphi}\rangle \) can be quantified by the von Neumann entropies of their reduced density matrices. We find that the degree of entanglement \( E(|\alpha|, \varphi) \) for \( |\Phi_{\varphi}\rangle \) and \( |\Psi_{\varphi}\rangle \) are the same and

\[ E(|\alpha|, \varphi) = -\frac{N^2_\varphi}{\ln 2} \left\{ M(0) M(\varphi) \ln[N^2_\varphi M(0) M(\varphi)] + M(\pi) M(\varphi + \pi) \ln[N^2_\varphi M(\pi) M(\varphi + \pi)] \right\}, \]

(6)

where \( M(\varphi) = 1 + \cos \varphi e^{-2|\alpha|^2} \). Note that \( E(|\alpha|, \varphi) \) is the degree of entanglement defined not in continuous variables as in [29] but in the \( 2 \times 2 \) space \( H^{(1)}_\alpha \otimes H^{(2)}_\alpha \). The degree of entanglement \( E(|\alpha|, \varphi) \) varies not only by the coherent amplitude \( \alpha \) but also by the relative phase \( \varphi \). When \( \varphi = \pi \), both the entangled coherent states \( |\Phi_{\varphi}\rangle \) and \( |\Psi_{\varphi}\rangle \) are maximally entangled regardless of \( \alpha \), i.e., \( E(|\alpha|, \pi) = 1 \). When \( \varphi = 0 \), on the other hand, \( E(|\alpha|, \varphi) \) is minimized for a given coherent amplitude \( \alpha \). These characteristics of entangled coherent states have already been pointed out by some authors [16, 30].

![Fig. 1. Measure of entanglement \( E(|\alpha|, \varphi) \), quantified by the von Neumann entropy of the reduced density matrix, against the relative phase \( \varphi \) of the entangled coherent state.\( |\alpha| = 0.8 \) (solid line), \( |\alpha| = 1 \) (dashed line), \( |\alpha| = 1.2 \) (dot-dashed line), and \( 0 \leq \varphi < 2\pi \). This figure shows that when \( |\alpha| \) is large, the quasi-Bell states are good approximations to maximally entangled Bell states.](image)

Substituting \( \varphi \) by 0 and \( \pi \), we define quasi-Bell states [16]

\[ |\Phi_{\pm}\rangle_{ab} = N_{\pm}(|\alpha\rangle_a|\alpha\rangle_b \pm | -\alpha\rangle_a | -\alpha\rangle_b), \]  

(7)

\[ |\Psi_{\pm}\rangle_{ab} = N_{\pm}(|\alpha\rangle_a | -\alpha\rangle_b \pm | -\alpha\rangle_a |\alpha\rangle_b). \]  

(8)

These states are orthogonal to each other except

\[ \langle \Psi_+ | \Phi_+ \rangle = \frac{1}{\cosh 2|\alpha|^2}. \]

(9)

We immediately see that as \( |\alpha| \) grows, they rapidly become orthogonal. In Fig. 1, we also show that the entanglement \( E(|\alpha|, \varphi) \) drastically approaches to 1 as \( |\alpha| \) increases. We calculate \( E(2, 0) \approx 0.9999997 \) and \( E(3, 0) \approx 1 - 6.7 \times 10^{-16} \), which shows quasi-Bell states are good approximations to maximally entangled Bell states.
It was shown that complete Bell-state measurements on a product Hilbert space of two two-level systems are not possible using linear elements \[^{31}\]. However, a remarkable feature of quasi-Bell states is that each one of them can be unambiguously discriminated using only linear elements. Suppose that each mode of the entangled state is incident on a 50-50 beam splitter. After passing the beam splitter, the quasi-Bell states become

\[
\begin{align*}
|\Phi_+\rangle_{ab} & \rightarrow |U\rangle_f |0\rangle_g, \\
|\Phi_-\rangle_{ab} & \rightarrow |V\rangle_f |0\rangle_g, \\
|\Psi_+\rangle_{ab} & \rightarrow |0\rangle_f |U\rangle_g, \\
|\Psi_-\rangle_{ab} & \rightarrow |0\rangle_f |V\rangle_g,
\end{align*}
\]

where the even cat state \(|U\rangle = M_+ (|\sqrt{2}\alpha\rangle + | - \sqrt{2}\alpha\rangle)\) with the normalization factor \(M_+\) contains only even numbers of photons, while the odd cat state \(|V\rangle = M_- (|\sqrt{2}\alpha\rangle - | - \sqrt{2}\alpha\rangle)\) with the normalization factor \(M_-\) contains only odd numbers of photons \[^{31}\]. By setting two photodetectors for the output modes \(f\) and \(g\) respectively to perform number parity measurement, the quasi-Bell measurement can be simply achieved. For example, if an odd number of photons is detected for mode \(f\), the state \(|\Phi_-\rangle\) is measured, and if an odd number of photons is detected for mode \(g\), then \(|\Psi_-\rangle\) is measured. Even though there is non-zero probability of failure in which both of the detectors do not register a photon due to the non-zero overlap of \(|0\rangle|U\rangle^2 = e^{-2\alpha^2}/(1 + e^{-4\alpha^2})\), the failure probability is very small for an appropriate choice of \(\alpha\) and the failure is known from the result whenever it occurs. This quasi-Bell measurement scheme can be used for concentration of pure entangled coherent states \[^{3}\].

### 3. Entanglement purification for mixed states

Suppose that Alice and Bob’s ensemble to be purified is represented by

\[
\rho_{ab} = F|\Phi_-\rangle\langle\Phi_-| + (1 - F)|\Psi_-\rangle\langle\Psi_-|,
\]

where \(F\) is the fidelity defined as \(|\langle\Phi_-|\rho_{ab}|\Phi_-\rangle\) and \(0 < F < 1\). Note that \(|\Phi_-\rangle\) and \(|\Psi_-\rangle\) are maximally entangled and orthogonal to each other regardless of \(\alpha\). Alice and Bob choose two pairs from the ensemble which are represented by the following density operator

\[
\rho_{ab\alpha\alpha'} = F^2|\Phi_-\rangle\langle\Phi_-| \otimes |\Phi_-\rangle\langle\Phi_-| + F(1 - F)|\Phi_-\rangle\langle\Phi_-| \otimes |\Psi_-\rangle\langle\Psi_-| + (1 - F)^2|\Psi_-\rangle\langle\Psi_-| \otimes |\Phi_-\rangle\langle\Phi_-|.
\]

The fields of modes \(a\) and \(a'\) are in Alice’s possession while \(b\) and \(b'\) in Bob’s. In Fig. \[^{3}\](a), we show that Alice’s action to purify the mixed entangled state. The same is conducted by Bob on his fields of \(b\) and \(b'\).

There are four possibilities for the fields of \(a\) and \(a'\) incident onto the beam splitter \((BS1)\), which gives the output (In the following, only the cat part for a component of the mixed state is shown to describe the action of the apparatuses)

\[
\begin{align*}
|\alpha\rangle_a |\alpha\rangle_{a'} & \rightarrow |\sqrt{2}\alpha\rangle_f |0\rangle_{f'}, \\
|\alpha\rangle_a - |\alpha\rangle_{a'} & \rightarrow |0\rangle_f |\sqrt{2}\alpha\rangle_{f'}, \\
| - \alpha\rangle_a |\alpha\rangle_{a'} & \rightarrow |0\rangle_f | - \sqrt{2}\alpha\rangle_{f'}, \\
| - \alpha\rangle_a - |\alpha\rangle_{a'} & \rightarrow | - \sqrt{2}\alpha\rangle_f |0\rangle_{f'}.
\end{align*}
\]
Fig. 2. (a) Entanglement purification scheme for mixed entangled coherent states. P1 tests if the incident fields $a$ and $a'$ were in the same state by simultaneous clicks at $A_1$ and $A_2$. For P2, detector $B$ is set for photon parity measurement. Bob performs the same on his field of modes $b$ and $b'$ as Alice. If Alice and Bob measure the same parity, the pair is selected. By iterating this process maximally entangled pairs can be obtained from a sufficiently large ensemble of mixed states. (b) Simpler purification scheme to increase the coherent amplitude of the purified state. The success probability of this scheme is more than twice as large as the scheme with P1 and P2 shown in (a).
In the boxed apparatus P1, Alice checks if modes $a$ and $a'$ were in the same state by counting photons at the photodetectors $A1$ and $A2$. If both modes $a$ and $a'$ are in $|\alpha\rangle$ or $|\alpha\rangle$, $f'$ is in the vacuum, in which case the output field of the beam splitter $BS2$ is $|\alpha\rangle$. Otherwise, the output field is either $|2\alpha\rangle_{t1,t2}$ or $|0,2\alpha\rangle_{t1,t2}$. When both the photodetectors $A1$ and $A2$ register any photon(s), Alice and Bob are sure that the two modes $a$ and $a'$ were in the same state but when either $A1$ or $A2$ does not register a photon, $a$ and $a'$ were likely in different states. Of course, there is a probability not to register a photon even though the two modes were in the same state, which is due to the nonzero overlap of $|\langle 0\rangle \sqrt{2\alpha}|^2$.

It can be simply shown that the second and third terms of Eq. (12) are always discarded by the action of P1 and Bob’s apparatus same as P1. For example, at the output ports of $BS1$ and Bob’s beam splitter corresponding to $BS1$, $|\Phi_-\rangle_{ab}|\Psi_-\rangle_{a'b'}$ becomes

$$|\Phi_-\rangle_{ab}|\Psi_-\rangle_{a'b'} \rightarrow N_2^2 \left( |\sqrt{2\alpha},0,0,\sqrt{2\alpha}\rangle - |0,\sqrt{2\alpha},\sqrt{2\alpha},0\rangle \right)$$

$$\quad - \left| 0,-\sqrt{2\alpha},-\sqrt{2\alpha},0 \right|_g f g' f' g'$$

where $g$ and $g'$ are the output field modes from Bob’s beam splitter corresponding to $BS1$. The fields of modes $f'$ and $g'$ can never be in $|0\rangle$ at the same time; at least, one of the four detectors of Alice and Bob must not click. The third term of Eq. (12) can be shown to lead to the same result by the same analysis.

For the cases of the first and fourth terms in Eq. (12), all four detectors may register photon(s). After the beam splitter, the ket of $(|\Phi_-\rangle(|\Phi_-\rangle)_{ab} \otimes (|\Phi_-\rangle(|\Phi_-\rangle)_{a'b'})$ of Eq. (12) becomes

$$|\Phi_-\rangle_{ab}|\Phi_-\rangle_{a'b'} \rightarrow |\Phi'_{+}\rangle_{fg}|0,0\rangle_{f'g'} - |0,0\rangle_{fg}|\Phi'_{+}\rangle_{f'g'}$$

where $|\Phi'_{+}\rangle = N_{+}'(\sqrt{2\alpha},\sqrt{2\alpha} + |-\sqrt{2\alpha},-\sqrt{2\alpha}\rangle)$ with the normalization factor $N_{+}'$. The normalization factor in the right hand side of Eq. (18) is omitted. The first term is reduced to $|\Phi'_{+}\rangle_{fg}(|\Phi'_{+}\rangle_{f'g'}$ after $|0,0\rangle_{f'g'}|0,0\rangle$ is measured out by Alice and Bob’s P1’s. Similarly, the fourth term of Eq. (12) yields $|\Psi'_{+}\rangle_{fg}(|\Psi'_{+}\rangle_{f'g'}$, where $|\Psi'_{+}\rangle$ is defined in the same way as $|\Phi'_{+}\rangle$, after $|0,0\rangle_{f'g'}|0,0\rangle$ is measured. Thus the density matrix for the field of modes $f$ and $g$ conditioned on simultaneous measurement of photons at all four photodetectors is

$$\rho_{fg} = F'|\Phi'_{+}\rangle \langle \Phi'_{+} | + (1 - F')|\Psi'_{+}\rangle \langle \Psi'_{+} |$$

where

$$F' = \frac{F^2}{F^2 + (1 - F)^2}$$

and $F'$ is always larger than $F$ for any $F > 1/2$.

If the pair is selected by Alice and Bob’s P1’s, each of them performs another process (P2) for the selected pair. The pair is incident onto a 50-50 beam splitter at each site of Alice and Bob shown in Fig. 2(a). If the selected pair is $|\Phi'_{+}\rangle(|\Phi'_{+}\rangle$ of Eq. (19), then the beam splitter gives

$$|\Phi'_{+}\rangle_{fg} \rightarrow |\Phi'_{+}\rangle_{kl} \left( \frac{M_+}{M_-} |u,v\rangle_{k'\ell'} + \frac{M_+}{M_-} |v,u\rangle_{k'\ell'} \right) + |\Phi_-\rangle_{kl} \frac{N_+}{N_-} \left( |u,v\rangle_{k\ell'} + |v,u\rangle_{k\ell'} \right)$$

where $l$ and $l'$ are field modes at Bob’s site corresponding to $k$ and $k'$. The normalization factor is omitted in Eq. (21). It is known that $|u\rangle$ contains only even numbers of photons.
and \(|\nu\rangle\) contains only odd numbers of photons. The state is reduced to \(|\Phi_-\rangle\) when different parities are measured at \(k'\) and \(l'\) by Alice and Bob respectively. The same analysis shows that \(|\Psi_-\rangle\) remains by P2’s for \(|\Psi_+\rangle_{fg}\langle\Psi_+|\) of Eq. (19) which is originated from the fourth term of Eq. (12).

The total state after the full process becomes

\[
\rho_{fg} = F'|\Phi_-\rangle\langle\Phi_-| + (1 - F')|\Psi_-\rangle\langle\Psi_-|.
\]

We already saw from Eq. (20) that \(F'\) is larger than \(F\) for any \(F > 1/2\). Alice and Bob can perform as many iterations as they need for higher entanglement. The success probability \(P_s\) for one iteration is

\[
P_s = \frac{F^2 + (1 - F)^2}{4} \left(1 - \frac{2e^{-4|\alpha|^2}}{1 + e^{-8|\alpha|^2}}\right) \left(1 - \frac{e^{-4|\alpha|^2}}{1 + e^{-8|\alpha|^2}}\right) ,
\]

which approaches to \(P_s = \frac{e^2 + (1 - F)^2}{4}\) and \(1/8 \leq P_s \leq 1/4\) for \(|\alpha| \gg 1\).

By reiterating this process including P1 and P2, Alice and Bob can distill some maximally entangled states \(|\Phi_-\rangle\) asymptotically. Of course, a sufficiently large ensemble and initial fidelity \(F > 1/2\) are required for successful purification [3]. P2 may be different depending on the type of entangled coherent states to be distilled. For example, if Alice and Bob need to distill \(|\Phi_+\rangle\) instead of \(|\Phi_-\rangle\), pairs should be selected when the measurement outcomes yield the same parity.

Let us now consider the roles of P1 and P2 by comparing our scheme with refs. [3] and [3]. Pan et al. suggested a purification scheme for the entanglement of linearly polarized photons, where they use polarizing beam splitters (PBS’s) with photodetectors to test if the two photons are in the same polarization [3]. From Eqs. (13) to (16), we pointed out that P1 is to test whether the two fields \(a\) and \(a'\) are in the same state. Hence P1 plays a similar role in our scheme as PBS’s in [3]. Next, consider P2 which enables to perform orthogonal measurement based on \(|\alpha\rangle \pm | - \alpha\rangle\). This measurement is also necessary in the other schemes [3, 3]. (We will show later that this process (P2) is not always necessary in our scheme.) Pan et al. explained that a PBS in their scheme has the same effect as a controlled-NOT gate in the scheme suggested by Bennett et al. [3] except that the success probability is half as large as [3]. Both the schemes [3, 3] can be directly applied to any Werner states without additional bilateral rotations, thereby it is clear that our scheme is also applicable to any Werner-type states.

If Alice and Bob want to distill entangled coherent states \(|\Phi_+\rangle\) or \(|\Psi_+\rangle\) while increasing their coherent amplitudes, it can be simply accomplished by performing only P1 in Fig. 3(b). Suppose that Alice and Bob need to purify a type of ensemble

\[
\rho_{ab} = G_1|\Phi_+\rangle\langle\Phi_+| + G_2|\Psi_+\rangle\langle\Psi_+| ,
\]

where \(G_1 + G_2 \simeq 1\) for \(|\alpha| \gg 1\). If P1 is successful, the selected pair becomes

\[
\rho_{fg} = G'_1|\Phi'_+\rangle\langle\Phi'_+| + G'_2|\Psi'_+\rangle\langle\Psi'_+| ,
\]

where \(G'_1\) is larger than \(G_1\) for any \(G_1 > G_2\). After \(n\) iterations, they get a subensemble with higher fidelity of

\[
|\Phi'_+\rangle = \mathcal{N}_+ (|2^{n/2}\alpha\rangle|2^{n/2}\alpha\rangle + |-2^{n/2}\alpha\rangle |-2^{n/2}\alpha\rangle) ,
\]

where
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where the coherent amplitude has increased. Here, $N_+$ is a normalization factor. For example, if $G_1$ is 2/3 and coherent amplitude $\alpha$ is 2, the fidelity and the amplitude will be $\sim 0.99999$ and 8 respectively after three times of iterations.

Note that the success probability $P'_s$ of this simplified scheme is

$$P'_s = \frac{F^2 + (1 - F)^2}{2} \left( 1 - \frac{2e^{-4|\alpha|^2}}{1 + e^{-8|\alpha|^2}} \right),$$  \hspace{1cm} (27)

which is more than twice as large as that of the scheme shown in Fig. 2(a) and approaches to $1/4 < P'_s < 1/2$ for $|\alpha| \gg 1$. This is due to the fact that the process P2 is not directly for entanglement purification differently from the other two schemes \[4, 6\]. We separated P1 and P2 while the other schemes do both processes by one measurement. In our case, the process P1 purifies the mixed ensemble but the resulting state has a larger amplitude. It should be noted that even though the simplified scheme is applicable to any Werner-type states, (symmetric) entangled coherent states $|\Phi_+\rangle$ and $|\Psi_+\rangle$ can only be obtained by it.

4. Purification for general mixed states

We have shown that a mixed Werner state may be purified using beam splitters and photodetectors. A general mixed state may be transformed into a Werner state by random bilateral rotations \[5, 33\]. The Werner state can then be distilled purified. For the case of spin-1/2 systems, the required rotations are $B_x, B_y$ and $B_z$ which correspond to $\pi/2$ rotations around $x, y$ and $z$ axes.

The $B_x$ rotation can be realized using a nonlinear medium for the entangled coherent state. The anharmonic-oscillator Hamiltonian of a Kerr medium is \[17\]

$$H_{NL} = \hbar \omega a^\dagger a + \hbar \Omega (a^\dagger a)^2,$$  \hspace{1cm} (28)

where $\omega$ is the energy level splitting for the harmonic-oscillator part of the Hamiltonian and $\Omega$ is the strength of the anharmonic term. When the interaction time $t$ in the medium is $\pi/\Omega$, coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ evolve as follows:

$$|\alpha\rangle \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2}}(|\alpha\rangle + i|-\alpha\rangle),$$  \hspace{1cm} (29)

$$|-\alpha\rangle \rightarrow \frac{e^{-i\pi/4}}{\sqrt{2}}(i|\alpha\rangle + |-\alpha\rangle),$$  \hspace{1cm} (30)

which corresponds to $B_x$ up to a global phase shift.

The $B_z$ rotation can be obtained by displacement operator $D(\delta) = \exp(\delta a^\dagger - \delta^* a)$ \[1\], where $a$ and $a^\dagger$ are respectively annihilation and creation operators. We know that two displacement operators $D(\alpha)$ and $D(\delta)$ do not commute but the product $D(\alpha)D(\delta)$ is simply $D(\alpha + \delta)$ multiplied by a phase factor, $\exp[\frac{i}{2}(\alpha^*\delta - \alpha^*\delta)]$. This phase factor plays a role to rotate the logical qubit. The action of displacement operator $D(\epsilon)$, where $\epsilon (\ll 1)$ is real, on a qubit $|\phi\rangle = a|\alpha\rangle + b|-\alpha\rangle$ is the same as $z$-rotation of the qubit by $U_z(\theta/2 = 2\alpha\epsilon)$. We can easily check their similarity by calculating the fidelity:

$$|\langle \phi|U_z(2\alpha\epsilon)D(\epsilon)|\phi\rangle|^2 \simeq \exp[-\epsilon^2] \simeq 1,$$  \hspace{1cm} (31)
where $\alpha \gg 1$ is assumed and both $\alpha$ and $\epsilon$ are real. Then the rotation angle can be represented as $\theta = 4\alpha \epsilon$. Note that a small amount of $\epsilon$ suffices to make one cycle of rotation for a large $\alpha$. The displacement operation $D(i\epsilon)$ can be effectively performed using a beam splitter with the transmission coefficient $T$ close to unity and a high-intensity coherent field of amplitude $i\epsilon$, where $\epsilon$ is real. It is known that the effect of the beam splitter is described by $D(i\epsilon\sqrt{T-1})$ in the limit of $T \to 1$ and $\epsilon \gg 1$. For the $B_z$ rotation, $\epsilon$ should be taken to be $\pi/8\alpha$ so that the incident coherent field may be $|i\pi/(8\alpha\sqrt{T-1})\rangle$. The $B_y$ rotation can be realized by applying $B_x$ and $B_z$ together with $\sigma_z$ noting

$$B_y = -\sigma_z B_x B_z,$$  \hfill (32)

where $\sigma_z$ is $\pi$ rotation around $z$ axis. The coherent state $|i\pi/(4\alpha\sqrt{T-1})\rangle$ should be used to perform $\sigma_z$.

Alice and Bob can perform random bilateral rotations to transform the initial general mixed state into a Werner state. In this process, the efficiency of nonlinear interaction can affect the efficiency of the scheme.

5. Purification for decohered states in vacuum

We now apply our scheme to a physical example in a dissipative environment. When the entangled coherent channel $|\Phi_-\rangle$ is embedded in a vacuum, the channel decoheres and becomes a mixed state of its density operator $\rho_{ab}(\tau)$, where $\tau$ stands for the decoherence time. By solving the master equation \[32\]

$$\frac{\partial \rho}{\partial \tau} = \dot{J} \rho + \dot{L} \rho; \quad \dot{J} \rho = \gamma \sum_i a_i \rho a_i^\dagger, \quad \dot{L} \rho = -\frac{\gamma}{2} \sum_i (a_i^\dagger a_i \rho + \rho a_i^\dagger a_i)$$  \hfill (33)

where $\gamma$ is the energy decay rate, the mixed state $\rho_{ab}(\tau)$ can be straightforwardly obtained as

$$\rho_{ab}(\tau) = \tilde{N}(\tau) \left\{ |t\alpha, t\alpha\rangle \langle t\alpha, t\alpha| + | -t\alpha, -t\alpha\rangle \langle -t\alpha, -t\alpha| + \Gamma(|t\alpha, t\alpha\rangle \langle -t\alpha, -t\alpha| + | -t\alpha, -t\alpha\rangle \langle t\alpha, t\alpha|) \right\},$$  \hfill (34)

where $| \pm t\alpha, \pm t\alpha\rangle = | \pm t\alpha \rangle_a | \pm t\alpha \rangle_b$, $t = e^{-\gamma\tau/2}$, $\Gamma = \exp[-4(1 - t^2)|\alpha|^2]$, and $\tilde{N}(\tau)$ is the normalization factor. The decohered state $\rho_{ab}(\tau)$ may be represented by the dynamic quasi-Bell states defined as follows:

$$|\tilde{\Phi}_{\pm}\rangle_{ab} = \tilde{N}_{\pm}(|t\alpha \rangle_a |t\alpha \rangle_b \pm | -t\alpha \rangle_a | -t\alpha \rangle_b),$$  \hfill (35)

$$|\tilde{\Psi}_{\pm}\rangle_{ab} = \tilde{N}_{\pm}(|t\alpha \rangle_a | -t\alpha \rangle_b \pm | -t\alpha \rangle_a |t\alpha \rangle_b),$$  \hfill (36)

where $\tilde{N}_{\pm} = (2(1 + e^{-4t^2|\alpha|^2}))^{-1/2}$. The decohered state is then

$$\rho_{ab}(\tau) = \tilde{N}(\tau) \left\{ \frac{(1 + \Gamma)}{N_{\pm}^2} \langle \tilde{\Phi}_{-}\rangle \langle \tilde{\Phi}_{-}| + \frac{(1 - \Gamma)}{N_{\pm}^2} \langle \tilde{\Phi}_{+}\rangle \langle \tilde{\Phi}_{+}| \right\}$$

$$F(\tau)|\tilde{\Phi}_{-}\rangle \langle \tilde{\Phi}_{-}| + (1 - F(\tau))|\tilde{\Phi}_{+}\rangle \langle \tilde{\Phi}_{+}|$$  \hfill (37)

where, regardless of the decay time $\tau$, $|\tilde{\Phi}_{-}\rangle$ is maximally entangled and $|\tilde{\Phi}_{-}\rangle$ and $|\tilde{\Phi}_{+}\rangle$ are orthogonal to each other. The decohered state (37) is not in the same form as Eq. (11) so...
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Before the purification scheme is applied, we need some bilateral unitary transformations. We find that the unitary operation $B_x B_y \rho_{ab} (\tau) B^\dagger_x B^\dagger_y = F(\tau |\tilde{\Phi}_- \rangle \langle \tilde{\Phi}_- | + (1 - F(\tau)) |\tilde{\Psi}_+ \rangle \langle \tilde{\Psi}_+ |$, (38)

which is obviously distillable form using the schemes explained in Sec. 3.

A Hadamard gate $H$ for coherent states [11, 15] can also be used to transform the state (37) into a distillable form

$H_a H_b \rho_{ab} (\tau) H^\dagger_b H^\dagger_a = N_h \left\{ F(\tau |\tilde{\Psi}_+ \rangle \langle \tilde{\Psi}_+ | + (1 - F(\tau)) |\tilde{\Phi}_+ \rangle \langle \tilde{\Phi}_+ | \right\}$, (39)

where $N_h$ is the normalization factor due to the nonzero overlap between $|\tilde{\Psi}_+ \rangle$ and $|\tilde{\Phi}_+ \rangle$.

The ensemble of state (37) can be purified successfully only when $F(\tau)$ is larger than $1/2$.

Because

$F(\tau) = \frac{N^2(1 + \Gamma)}{N^2(1 + \Gamma) - N^2(1 - \Gamma)}$, (40)

it is found that purification is successful when the decoherence time $\gamma \tau < \ln 2$ regardless of $\alpha$. This result is in agreement with the decay time until which teleportation can be successfully performed via an entangled coherent state shown in ref. [9].

6. Multi-mode purification

Besides a two-mode entangled coherent state, a multi-mode entangled coherent state [34] can be used for quantum computation using coherent-state qubits [11]. There is a suggestion for multi-mode entanglement purification based on controlled-NOT operation [35]. In this section we investigate an example of application of our scheme to multi-mode entangled states.

Multi-mode entangled coherent states can be generated using a coherent superposition state and 50-50 beam splitters. The number of required beam splitters is $N - 1$, where $N$ is the number of modes for the multi-mode entangled state. For example, a four-mode entangled state can be generated as shown in Fig. 3(a). After passing the three beam splitters, the four-mode entangled state $|B_1 \rangle = N_- (|a, a, a, a \rangle + | - a, - a, - a, - a \rangle)$ is generated. Suppose Alice and Bob’s ensemble to be purified is represented by

$\rho_{ab} = F |B_1 \rangle \langle B_1 | + G |B_2 \rangle \langle B_2 |$, (41)

where $|B_2 \rangle = N_- (|a, -a, a, -a \rangle + | - a, a, - a, - a \rangle)$ and $|B_2 \rangle$ can be generated in a similar way as $|B_1 \rangle$. By extending the scheme studied above, the ensemble (41) can be purified as shown in Fig. 3(b). After one successful iteration of the purification process, the originally selected pairs become

$\rho_{ab} = F' |B_1 \rangle \langle B_1 | + G' |B_2 \rangle \langle B_2 |$, (42)

where $F' = \frac{F^2}{F^2 + (1 - F^2)}$ is always larger than $F$ for $F > G$. Alice and Bob can iterate the process as many time as required for their use. Note that this scheme can be applied to any $N$-mode entangled states of the same type and so does the simpler scheme only with P1.
7. Remarks

We have suggested an entanglement purification scheme for mixed entangled coherent states. Our scheme is based on the use of 50-50 beam splitters and photodetectors. The scheme is directly applicable for mixed entangled coherent states of the Werner type, and can be useful for general two-mode mixed states using additional nonlinear interactions. We have also suggested a simplified variation of this scheme which, however, increases the coherent amplitude of the entangled coherent state. We applied our scheme to an entangled coherent state decohered in a vacuum environment.

Finally, we would like to address possible difficulties for experimental realization of the purification scheme. We already pointed out that the nonlinear interaction required for the generation of cat states and for additional bilateral rotations to purify some non-Werner type states is extremely difficult, while the efforts to improve nonlinearity that additional noise is being continuously investigated. If the entangled coherent state is subject to a thermal environment, it is not straightforward to represent the decohered state in the simple basis of the quasi-Bell states (1) and (8). The purification of a decohered state due to thermal environment will be much more complicated and will deserve further studies. Laser phase drift can be another obstacle to realize quantum information processing with coherent states and phase stabilization methods via mixing of laser beams can be used to reduce the drifts. Apart from the phase drift, There has been a controversy on whether conventional laser sources can be used for quantum communication with coherent states [37, 38]. Most recent study [38] shows that the conventional laser can be used for quantum teleportation and for
generating continuous-variable entanglement because optical coherence is not necessary for the purpose.

For quantum information processing, an entangled coherent state is normally assumed to have a large coherent amplitude. Even though the success probability of the purification scheme is better as the coherent amplitude, $\alpha$, is larger, it does not change much. For example, the success rate is about 5% degraded for $\alpha = 1$ compared with the case for $\alpha \to \infty$. To use the first purification scheme described in Fig. 2(a), even and odd numbers of photons have to be discriminated. If the coherent amplitude is large, the efficiency to discriminate even and odd numbers of photons becomes low due to the dark current. However, when the coherent amplitude is small, a highly efficient avalanche photodiode can be used to discriminate even (0 and 2) photon numbers and odd (1) photon number because, for example, taking $\alpha = 1$ the probability of photon number being 0 and 2 for an even cat state is about 97% and the probability of photon number being 1 is about 85%. Takeuchi et al. used threefold tight shielding and viewports that worked as infrared blocking filters to eliminate the dark count. On the other hand, the second purification scheme in Fig. 2(b) is robust against detection inefficiency when $\alpha$ is large because it is enough to discern a coherent state and a vacuum in this simplified scheme. By employing a distributed photon counter or a homodyne detector, we have even a higher detection efficiency to discern a coherent state and a vacuum.

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