RECONSTRUCTING CURVES FROM THEIR HODGE CLASSES

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Abstract. Let $S$ be a smooth algebraic surface in $\mathbb{P}^3(\mathbb{C})$. A curve $C$ in $S$ has a cohomology class $\eta_C \in H^1(\Omega^1_S)$. Define $\alpha(C)$ to be the equivalence class of $\eta_C$ in the quotient of $H^1(\Omega^1_S)$ modulo the subspace generated by the class $\eta_H$ of a plane section of $S$. In the paper “Reconstructing subvarieties from their periods” the authors Movasati and Sertöz pose several interesting questions about the reconstruction of $C$ from the annihilator $I_{\alpha(C)}$ of $\alpha(C)$ in the polynomial ring $R = H^0_s(\mathcal{O}_{\mathbb{P}^3})$. It contains the homogeneous ideal of $C$, but is much larger as $R/I_{\alpha(C)}$ is artinian. We give sharp numerical conditions that guarantee $C$ is reconstructed by forms of low degree in $I_{\alpha(C)}$. We also show it is not always the case that the class $\alpha(C)$ is perfect, that is, that $I_{\alpha(C)}$ could be bigger than the sum of the Jacobian ideal of $S$ and of the homogeneous ideals of curves $D$ in $S$ for which $I_{\alpha(D)} = I_{\alpha(C)}$.

1. Introduction

The Hodge conjecture, one of the most challenging and interesting open questions in algebraic geometry, can be regarded as a reconstruction problem. Even when the Hodge conjecture is known, as for curves on surfaces, there are a series of somewhat related problems that might shed a new light on some aspects of the cycle map.

A good example is given by [GH83, Theorem 4.b.26] where it is proven that, given a smooth surface $S \subset \mathbb{P}^3$ and an integral class $\gamma$ in $H^1(\Omega^1_S)$ with the same numerical properties as the fundamental class of a curve $C \subset S$, then $\gamma$ is itself the fundamental class of an effective divisor $D \subset S$ provided $\deg(S)$ is large relative to the self-intersection of $\gamma$ and to $\deg(C)$.

In a similar vein, very interesting recent work by Movasati and Sertöz [MS19] concerns the reconstruction of subvarieties of $\mathbb{P}^N$ from their periods. Our purpose is to give an answer, in the special case of curves lying on a smooth algebraic surface $S$ in complex projective space, to two questions raised in [MS19] that we now illustrate. A curve $C$ in $S$ has a fundamental cohomology class $\eta_C \in H^1(\Omega^1_S)$. We denote by $\alpha(C)$ the equivalence class of $\eta_C$ in the quotient of $H^1(\Omega^1_S)$ modulo the subspace generated by the class $\eta_H$ of a plane section of $S$: the class $\alpha(C)$ depends on the embedding of $S$ in $\mathbb{P}^3$, and can be seen as a linear form on the primitive cohomology $H^1(\Omega^1_S)^{\perp_H}$. Following [MS19] we focus our analysis on the annihilator $I_{\alpha(C)}$ of $\alpha(C)$ in the polynomial ring $R = H^0_s(\mathcal{O}_{\mathbb{P}^3})$. Note - see Proposition 2.3 - that $I_{\alpha(C)} = I_{\alpha(D)}$ for two curves $C$ and $D$ in $S$ if and only if $mC + nD + pH$ is linearly equivalent to zero for some choice of integers $m$, $n$ and $p$ with $m$ and $n$ non zero and relatively prime. Thus the annihilator $I_{\alpha(C)}$, which
contains the homogeneous ideal \( I_C \) of \( C \) and the Jacobian ideal \( J_S \) of \( S \), in general it is much larger than \( I_C + J_S \), as it contains the ideal \( I_D \) for any curve \( D \) for which there is a relation \( mC + nD + pH \sim 0 \) as above. Still, one can ask whether \( C \) can be reconstructed from \( I_{\alpha(C)} \) when \( \deg(S) \) is large with respect to the degree or other invariants of \( C \), and Movasati and Sertöz in [MS19] investigate, in a more general context than ours, the following questions:

1. under which conditions \( I_{\alpha(C)} \) reconstructs \( C \), in the sense that forms of low degree in \( I_{\alpha(C)} \) cut out the curve \( C \) scheme-theoretically? To be precise, we will say that \( C \) is reconstructed at level \( m \) by \( I_{\alpha(C)} \) if its homogeneous ideal \( I_C \) is generated over \( R \) by \( I_{\alpha(C), \leq m} \) - that is, by forms of degree \( \leq m \) in \( I_{\alpha(C)} \).

2. as the example of complete intersection suggests, they define a class \( \alpha \in H^1(\Omega^1_S) / \mathcal{C} \eta_H \) to be perfect at level \( m \) if there exist effective divisors \( D_1, \ldots, D_q \) in \( S \) such that \( I_{\alpha(D_i)} = I_{\alpha} \) for every \( i = 1, \ldots, q \) and

\[
I_{\alpha,j} = \sum_{i=1}^q I_{D_i,j} + J_{S,j} \quad \text{for every} \ j \leq m
\]

where \( J_S \) denotes the Jacobian ideal of \( S \). The question is under which conditions this class \( \alpha(C) \) is perfect at level \( m \), and whether all classes \( \alpha(C) \) are perfect at every level \( m \).

In this paper we prove two theorems that give partial answers to these questions. Our first theorem extends known results on complete intersections [Dan17, MS19] to arithmetically Cohen-Macaulay curves (ACM curves for short). The tools we need for this are provided by a very nice paper by Ellingsrud and Peskine [EP93] which unfortunately seems to be little known. In [EP93] the authors were interested in the study of the Noether-Lefschetz locus, and the invariant \( \alpha(C) \) plays a prominent role in their work because it vanishes if and only if the curve is a complete intersection of \( S \) and another surface. Their paper connects the class \( \alpha(C) \) to the normal sequence arising from the inclusions \( C \subset S \subset \mathbb{P}^3 \) and gives an effective tool for computing its annihilator \( I_{\alpha(C)} \) - see Lemma 2.5. To state our first theorem, given a curve \( C \) in \( \mathbb{P}^3 \), we let \( s(C) \) be the minimum degree of a surface containing \( C \), and \( e(C) \) the index of speciality of \( C \), that is, the maximum \( n \) such that \( \mathcal{O}_C(n) \) is special, that is, \( h^1(\mathcal{O}_C(n)) > 0 \).

**Theorem 1.1.** Suppose \( C \) is an ACM curve on the smooth surface \( S \subseteq \mathbb{P}^3(C) \). Let \( s \) denote the degree of \( S \). Then

1. if \( s \geq 2e(C) + 8 - s(C) \), the curve \( C \) is reconstructed at level \( e(C) + 3 \) by \( I_{\alpha(C)} \);

2. the class \( \alpha(C) \) is perfect at level \( m \) for every \( m \).

For example, let \( C \) be a twisted cubic curve: \( C \) is then ACM with invariants \( s(C) = 2 \) and \( e(C) = -1 \). By Theorem 1.1, if \( S \) is a quartic surface containing \( C \), then \( C \) is cut out scheme-theoretically by the quadrics whose equations lie in \( I_{\alpha(C,S)} \). This was suggested and verified for thousands of randomly chosen quartic surfaces containing \( C \) in [MS19, Sections 2.3 and 3.2].
Our second theorem provides a first example of a non-perfect algebraic class $\alpha(C)$, giving a negative answer to Question 2.12 in [MS19].

**Theorem 1.2.** Let $C \subset \mathbb{P}^3$ be a smooth rational curve of degree 4 contained in a smooth surface $S$ of degree $s = 4$. The class $\alpha(C)$ in $S$ is not perfect at level 3.

It would be very interesting to determine conditions for a class $\alpha(C)$ to be perfect, and we don’t know whether ACM curves form the largest set of curves $C$ whose classes $\alpha(C)$, in any smooth surface $S$ containing $C$, are perfect.

Finally, we note that Movasati and Sertöz pose their questions of reconstruction and perfectness in a more general context, namely for classes in $H^n(\Omega^n_X)$ of varieties of dimension $n$ in smooth hypersurfaces $X$ in $\mathbb{P}^{2n+1}$. An interesting and challenging problem is trying to answer those questions for every $n$, generalizing as far as it is possible the results of this paper to higher dimension.

The paper is structured as follows. In section 2 we collect some well known facts we need, and, for the benefit of the reader, we recall in some detail the constructions from [EP93] we will need in the sequel of the paper. In section 3 we prove Proposition 3.1 - a numerical criterion that guarantees, when the degree of $S$ is large with respect to that of $C$, that the curve $C$ is reconstructed at a certain level $m$ by $I_{\alpha(C)}$. As an example we prove in Corollary 3.2 a reconstruction result for a general rational curve of degree $d$. In section 4 we prove Theorem 1.1, which is split in Theorems 4.1 and 4.8. In section 5 we prove Theorem 1.2.

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## 2. Preliminaries

We work in the projective space $\mathbb{P}^3$ over the field $\mathbb{C}$ of complex numbers. Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$ and $i \in \mathbb{N}$, we define

$$H^i_{\mathbb{C}}(\mathcal{F}) = \bigoplus_{n \in \mathbb{N}} H^i(\mathbb{P}^3, \mathcal{F}(n)).$$

These are graded module over the polynomial ring

$$R = H^0_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}^3}) \cong \mathbb{C}[x, y, z, w].$$

Given a subscheme $X$ of $\mathbb{P}^3$, we will denote by $\mathcal{I}_X$ its sheaf of ideals, and by $I_X = H^0(\mathcal{I}_X)$ its saturated homogeneous ideal in $R$. We will write $I_{X,n}$ to denote its $n^{th}$ graded piece $H^0(\mathbb{P}^3, \mathcal{I}_X(n))$.

If $M = \bigoplus_{n \in \mathbb{Z}} M_n$ a graded $R$-module, the graded $\mathbb{C}$-dual module $M^*$ of $M$ is defined by setting $(M^*)_m = \text{Hom}_{\mathbb{C}}(M_{-m}, \mathbb{C})$ with multiplication $R_n \times (M^*)_m \rightarrow (M^*)_m+n$ defined by

$$g\lambda(v) = \lambda(gv), \quad \forall g \in R_n, \lambda \in (M^*)_m, v \in M_{-m-n}.$$
By Serre’s duality, if $X \subseteq \mathbb{P}^N$ is an equidimensional Cohen-Macaulay sub-scheme of dimension $d$, then for any locally free sheaf $\mathcal{F}$ on $X$ there is an isomorphism of graded $R$-modules

$$(H^i(X, \mathcal{F}))^* \cong H^{d-i}_s(X, \mathcal{F}^\vee \otimes \omega_X)$$

Let $S$ be a smooth algebraic surface of degree $s$ in $\mathbb{P}^3$, and $C \subset S$ a curve, that is, an effective Cartier divisor in $S$. The curve $C$ has a cohomology class $\eta_C \in H^1(S, \Omega^1_S)$. It can be defined as follows: the curve $C$ defines a linear form $\lambda_C$ on the set of $(1,1)$ forms by integration; abstractly one can define this linear form as the image of the trace map $H^1(C, \Omega^1_C) \to \mathbb{C}$ under the transpose of the morphism $H^1(S, \Omega^1_S) \to H^1(C, \Omega^1_C)$ obtained by restricting differentials on $S$ to $C$ [Har77, Chapter III Ex. 7.4]. The cohomology class $\eta_C$ is the image of $\lambda_C$ under the Serre’s duality isomorphism $H^1(\Omega^1_S)^* \cong H^1(\Omega^1_S^*)$.

If $\mathcal{O}_S(C)$ denotes the invertible sheaf on $S$ corresponding to $C$, then $\eta_C = c(\mathcal{O}_S(C))$ where $c$ denotes the first Chern class homomorphism

$$c : \text{Pic}(S) \to H^1(S, \Omega^1_S).$$

(1)

The perfect pairing $\langle \ , \ \rangle$ of Serre’s duality is compatible with the intersection product of divisor classes [Har77, Chapter V Ex. 1.8] in the sense that for every pair of Cartier divisors $D$ and $E$ on $S$

$$\langle c(\mathcal{O}_S(D)), c(\mathcal{O}_S(E)) \rangle = D \cdot E.$$

Since $S$ is a surface in $\mathbb{P}^3$, numerically equivalent divisors on $S$ are linearly equivalent, and the first Chern class map $\text{Pic}(S) \to H^1(S, \Omega^1_S)$ is injective.

The cotangent bundles of $S$ and $\mathbb{P}^3$ are related by the exact sequence

$$0 \to \mathcal{O}_S(-s) \to \Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_S \to \Omega^1_S \to 0.$$  

(2)

It is well known (see e.g. [EGPSm85]) that $H^1(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_S) \cong \mathbb{C}$ and that its image in $H^1(\Omega^1_S)$ is the class $\eta_H$ of a plane section $H$ of $S$. We look at a portion of the long cohomology sequence arising from (2)

$$H^1(\Omega^1_S) \overset{\delta}{\to} H^2(\mathcal{O}_S(-s)) \overset{\beta}{\to} H^2(\Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_S)$$

(3)

Dualizing and using Serre’s duality we get an exact sequence

$$H^0(T_{\mathbb{P}^3}(s - 4)) \overset{\iota^*}{\to} H^0(\mathcal{O}_S(2s - 4)) \to \text{Im}(\delta)^* \to 0.$$  

(4)

We denote by $J_S$ the Jacobian ideal of $S$, that is, the ideal of $R$ generated by the partial derivatives of an equation of $S$. Then the above discussion is summarized by Griffith’s theorem: the primitive first cohomology group of $S$ is isomorphic to the $(2s - 4)$-graded piece of the Jacobian ring of $S$:

$$H^1(\Omega^1_S)^{+\mu} \cong \text{Im}(\delta)^* \cong \frac{H^0(\mathcal{O}_{\mathbb{P}^3}(2s - 4))}{J_{S,2s-4}}.$$

**Definition 2.1.** Given a curve $C$ in $S$, we will denote by $\alpha(C) = \alpha(C, S, \mathbb{P}^3)$ the image of its cohomology class $\eta_C$ under the map

$$H^1(\Omega^1_S) \overset{\delta}{\to} H^2(\mathcal{O}_S(-s)) \cong H^0(\mathcal{O}_S(2s - 4))^*.$$  

Thus $\alpha(C)$ is a linear form on $H^0(\mathcal{O}_S(2s - 4))$ that vanishes on $J_{S,2s-4}$.
Given $\alpha \in H^0(O_S(2s - 4))^*$, we denote by $I_\alpha$ the annihilator of $\alpha$ in the polynomial ring $R$: it is the homogeneous ideal in $R$ whose $n^{th}$ graded piece is

$$I_{\alpha,n} = \{ f \in R_n \mid \alpha(fg) = 0, \forall g \in H^0(O_S(2s - 4 - n)) \}.$$  

**Remark 2.2.** When writing the paper, we decided to take all ideals in the polynomial ring $R = H^0(\mathcal{O}_{\mathbb{P}^3})$: thus $J_S$ and $I_\alpha$ are for us ideals of $R$, and $J_S \subset I_{\alpha(C)}$. Our motivation is that we would like to compare $I_\alpha$ with the ideal of $C$ as a curve in $\mathbb{P}^3$. In [MS19] the author’s denote by $I_\alpha$ the annihilator of $\alpha$ in the Jacobian ring and by $I_\alpha$ its preimage in $R$.

Let $T = R/I_S = H^0_{\alpha}(O_S)$. Then $\alpha \in (T_{2s-4})^*$, and the ideal $I_\alpha$ is determined by $\text{Ker}(\alpha) \subseteq T_{2s-4}$; conversely, one can recover $\text{Ker}(\alpha)$ as the image of $I_{\alpha,2s-4}$ via the quotient map $R_{2s-4} \to T_{2s-4}$. The perfect pairing

$$R_n/I_{\alpha,n} \times (R_{2s-4-n}/I_{\alpha,2s-4-n})^* \to \mathbb{C}$$

shows $A := R/I_\alpha = \bigoplus_{n=0}^{2s-4} A_n$ is an artinian Gorenstein ring of socle $2s - 4$ [EP93, Prop 1.3].

In [EP93] the authors were interested in the study of the Noether-Lefschetz locus, and the invariant $\alpha(C)$ plays a prominent role in their work because it vanishes if and only if the curve is a complete intersection of $S$ and another surface. More generally, a Lefschetz type theorem about the Picard group of $S$ (see [SGA73, B78, Voi02],) implies the following fact:

**Proposition 2.3.** Let $C$ and $D$ be effective divisors on a smooth surface $S \in \mathbb{P}^3$, and let $H$ denote a plane section of $S$. Then $I_{\alpha(C)} = I_{\alpha(D)}$ if and only if there exist $m, n, p \in \mathbb{Z}$, $m, n \neq 0$ and relatively prime, such that $mC + nD + pH$ is linearly equivalent to zero.

**Proof.** Suppose $mC + nD + pH$ is linearly equivalent to zero and $m$ and $n$ are nonzero. The cotangent complex (2) gives rise to an exact sequence in cohomology

$$H^1(\Omega^1_{\mathbb{P}^3} \otimes O_S) \cong \mathbb{C} \xrightarrow{\gamma} H^1(\Omega^1_S) \xrightarrow{\delta} H^2_\omega(O_S(-s)) \cong H^0(O_S(2s - 4))^*$$

and one knows that $\gamma(1) = \eta_H$, so that the kernel of $\delta$ is the $\mathbb{C}$-line spanned by $\eta_H$. From $mC + nD + pH \sim 0$ we then deduce $ma(C) = -na(D)$. Since $m$ and $n$ are nonzero, the linear forms $\alpha(C)$ and $\alpha(D)$ have the same kernel, hence $I_{\alpha(C)} = I_{\alpha(D)}$.

In the other direction, suppose $I_{\alpha(C)} = I_{\alpha(D)}$, that is, $\alpha(C)$ and $\alpha(D)$ have the same kernel. Then $\alpha(C) = c\alpha(D)$ for a nonzero complex number $c$. Using (5) and the intersection pairing we deduce that are integers $m, n, p$, with $m$ and $n$ nonzero, such that $mC + nD + pH$ is linearly equivalent to zero. Finally, $m$ and $n$ can be taken relatively prime because $\text{Pic}(S)/\mathbb{Z}H$ has no torsion (see for example [B78, Theorem B]). In particular, when $D = 0$, one can take $m = 1$. $\square$

As noted in [EP93] and [MS19, Lemma 2.3], the ideal $I_{\alpha(C)}$ contains the ideal of $C$ in $S$. This follows from the remark of [EP93] that $\alpha(C) \in H^0(O_S(2s - 4))^*$ is the pull-back of a linear form $\beta(C) \in H^0(O_C(2s - 4))^*$.  


For the benefit of the reader and for later use, we give a proof of this fact. The linear form $\beta(C)$ arises from the normal bundles exact sequence:

$$0 \to N_{C/P^3} \cong \omega_C(4 - s) \to N_{C/P^3} \to N_{C/P^3} \otimes \mathcal{O}_C \cong \mathcal{O}_C(s) \to 0.$$  

Tensoring (6) with $\mathcal{O}_C(-s)$ and taking cohomology we obtain a map $H^0(\mathcal{O}_C) \to H^1(\omega_C(4 - 2s))$ and we let

$$\beta(C) \in H^0(\mathcal{O}_C(2s - 4))^* \cong H^1(\omega_C(4 - 2s))$$

denote the image of $1 \in H^0(\mathcal{O}_C)$.

**Proposition 2.4.** [EP93, Construction 1.8] The linear form $\alpha(C)$ is the pull-back of $\beta(C)$ to $S$, that is, $\alpha(C) = \rho^*(\beta(C))$ where $\rho^*$ is the transpose of the natural map $\rho : H^0(\mathcal{O}_S(2s - 4)) \to H^0(\mathcal{O}_C(2s - 4))$.

**Proof.** Observe that $\Omega^1_S$ is a rank two vector bundle with determinant $\omega_S$, hence the tangent bundle $T_S = (\Omega^1_S)^\vee$ is isomorphic to $\Omega^1_S \otimes \omega_S^{-1} = \Omega^1_S(4 - s)$. The tangent complex of $S \subseteq P^3$ and the normal bundle sequence (6) give rise to a commutative diagram

$$
\begin{array}{c}
0 \to \Omega^1_S \cong T_S(s - 4) \to T_{P^3} \otimes \mathcal{O}_S(s - 4) \to \mathcal{O}_S(s - 4) \to 0 \\
0 \to \omega_C \to N_{C/P^3}(s - 4) \to \mathcal{O}_C(s - 4) \to 0
\end{array}
$$

Taking cohomology and dualizing one sees that $\alpha(C)$ is the pull back of $\beta(C)$ to $S$. \qed

The following Lemma in [EP93] gives an effective method to compute $I_\alpha$ in many cases.

**Lemma 2.5.** [EP93, Lemma 1.10] Let $N(C)$ denote the image of the map $H^0_N(\mathcal{O}_{P^3}) \to H^0(\mathcal{O}_C)$ arising from the normal bundle sequence (6). Let $\alpha : R = H^0_N(\mathcal{O}_{P^3}) \to H^0_N(\mathcal{O}_C)$ be the natural map. Then, for every integer $n$,

$$\pi^{-1}(N(C),n) \subseteq I_{\alpha(C),n}$$

with equality if $\pi_{2s - 4 - n}$ is surjective.

**Proof.** The exact sequence

$$
\begin{array}{c}
H^0_N(\mathcal{O}_{P^3})(-s) \to H^0_N(\mathcal{O}_C) \xrightarrow{1+\beta} (H^0_N(\mathcal{O}_C)(2s - 4))^*
\end{array}
$$

shows $N(C) = \text{Ann}_{H^0_N(\mathcal{O}_C)}(\beta)$.

The map $\alpha : R \to H^0_N(\mathcal{O}_C)$ factors through $\rho : H^0_N(\mathcal{O}_S) \to H^0_N(\mathcal{O}_C)$. To simplify notation, write $T = H^0_N(\mathcal{O}_S)$ and $e = 2s - 4$. As $\alpha$ is an element of the $T$-module $T^*$, the ideal $I_\alpha$, which by definition is the annihilator of $\alpha$ in $R$, is the inverse image of $\text{Ann}_T(\alpha)$ under the surjective map $R \to T$. Hence what we have to prove is that $\rho^{-1}(N(C),n) \subseteq \text{Ann}_T(\alpha)_n$ for every integer $n$, with equality holding when $\rho_{e-n}$ is surjective. Now

$$\text{Ann}_T(\alpha = \rho^*(\beta))_n = \{ g \in T : g \rho^*(\beta)(v) = \beta(\rho(g)\rho(v)) = 0 \quad \forall v \in T_{e-n} \}$$
while the inverse image $\rho^{-1}(N(C)_n)$ of the $n^{th}$ graded piece of the annihilator of $\beta(C)$ in $H^0_*(\mathcal{O}_C)$ is equal to
\[ \{ g \in T_n : (\rho(g)\beta)(w) = \beta(\rho(g)w) = 0 \ \forall w \in H^0(\mathcal{O}_C(e-n)) \}. \]
The thesis is now evident. \qed

**Corollary 2.6.** The annihilator $I_{\alpha(C)}$ of $\alpha(C)$ contains both the homogeneous ideal of $C$ and the Jacobian ideal of the surface $S$.

To exemplify the scope of this construction, we remark that it immediately yields the following well known corollary (originally due to Griffiths and Harris, see [EGPSm85] for more details).

**Corollary 2.7.** Suppose $S$ is a smooth surface in $\mathbb{P}^3$ and $C$ is an effective divisor on $S$. Then $C$ is a complete intersection of $S$ and another surface if and only if the sequence 6 of normal bundles splits.

**Proof.** If $C$ is a complete intersection of $S$ and another surface, it is clear that the sequence splits. Conversely, if the sequence splits, then $\beta(C) = 0$. Therefore $\alpha(C) = 0$, and the thesis follows from Proposition 2.3. \qed

### 3. Reconstruction of the ideal

Motivated by [MS19], we want to compare $I_C$ and $I_{\alpha(C)}$. The following proposition gives rather sharp sufficient conditions for the curve $C$ to be reconstructed at level $p$ by $I_{\alpha(C)}$.

**Proposition 3.1.** Let $S$ be a smooth surface of degree $s$ in $\mathbb{P}^3$, and let $C$ be an effective Cartier divisor on $S$. Assume that the homogeneous ideal $I_C$ is generated by its forms of degree $\leq p$ and that the following vanishing conditions are satisfied
\begin{align*}
(1) & \ h^1(\mathcal{J}_C(2s-4-p)) = 0 \\
(2) & \ h^0(N_C/\mathbb{P}^3(p-s)) = 0
\end{align*}
then $I_{\alpha(C),p} = I_{C,p}$, therefore $C$ is reconstructed at level $p$ by $I_{\alpha(C)}$.

**Proof.** Since $h^0(N_C/\mathbb{P}^3(p-s)) = 0$, the annihilator of $\beta(C)$ in degree $p$ vanishes. Since $\pi_{2s-4-p} : R_{2s-4-p} \to H^0(\mathcal{O}_C(2s-4-p))$ is surjective, by Lemma 2.5
\[ I_{\alpha(C),p} = \pi_p^{-1}(\text{Ann}(\beta_C)_p) = I_{C,p}. \] \qed

We can now answer a question raised in [MS19, Section 2.3.1] about twisted cubics contained in quartic surfaces: if $C$ is a twisted cubic contained in a smooth quartic surface $S \subset \mathbb{P}^3$, then $C$ is cut out by quadrics in $I_{\alpha(C)}$. More generally:

**Corollary 3.2.** Suppose $C \subset \mathbb{P}^3$ is a general rational curve of degree $d \geq 3$ and let $n_0$ be the round up of $\sqrt{6d-2} - 3$, that is, the smallest positive integer $n$ such that $\binom{n+3}{3} - nd - 1 \geq 0$. If $C$ is contained in a smooth surface $S$ of degree $s \geq n_0 + 3$, then $C$ is reconstructed at level $n_0 + 1$ by $I_{\alpha(C,S)}$. 

Proof. By [HH85] a general rational curve is a curve of maximal rank, that is, \( h^0(\mathcal{I}_C(n)) = 0 \) for \( n \leq n_0 - 1 \) and \( h^1(\mathcal{I}_C(n)) = 0 \) for \( n \geq n_0 \). Hence \( C \) is \( n_0 + 1 \) regular in the sense of Castelnuovo-Mumford, and \( I_C \) is generated by its forms of degree \( \leq n_0 + 1 \). Furthermore, by [EVdV81] the normal bundle of the immersion \( \mathbb{P}^1 \to C \subset \mathbb{P}^3 \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(2d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2d-1) \). Hence \( h^0(\mathcal{N}_{C/\mathbb{P}^3}(-m)) = 0 \) for every \( m \leq -2 \). Thus we can apply Proposition 3.1 with \( p = n_0 + 1 \). \( \square \)

**Remark 3.3.** If \( C \) is a smooth irreducible curve of degree \( d \), then \( h^1(\mathcal{I}_C(n)) = 0 \) for every \( n \geq d - 3 - e \) (see [GP87] and [Han00]), where \( e := e(C) = \max\{n \mid h^1(\mathcal{O}(n)) > 0\} \) is the index of speciality of \( C \).

**Corollary 3.4.** Let \( S \) be a smooth surface of degree \( s \) in \( \mathbb{P}^3 \), and let \( C \) be an effective Cartier divisor on \( S \). Suppose \( \mathcal{I}_C \) is \( r \)-regular in the sense of Castelnuovo-Mumford. If \( s \geq 2r + 1 \), then \( C \) is reconstructed at level \( r \) by \( I_{\alpha(C)} \).

Proof. Since \( \mathcal{I}_C \) is \( r \)-regular, the ideal \( I_C \) is generated by its forms of degree \( \leq r \) and \( H^1(\mathcal{I}_C(n)) = 0 \) for every \( n \geq r - 1 \). As \( s \geq 2r + 1 \) and \( r \geq 1 \), the first condition \( h^1(\mathcal{I}_C(2s - 4 - r)) = 0 \) in Proposition 3.1 is satisfied for \( p = r \).

We are left to check that \( h^0(\mathcal{N}_{C/\mathbb{P}^3}(r - s)) = 0 \).

By [PS74, Prop 4.1], there are two surfaces \( S_1 \) and \( S_2 \) of degree \( r \) meeting properly in a complete intersection \( X = S_1 \cap S_2 = C \cup D \) so that \( C \) and \( D \) have no common component. Consider the exact sequence

\[
0 \to \mathcal{I}_X \to \mathcal{I}_C \to \mathcal{I}_{C,X} \to 0.
\]

Applying \( \text{Hom}(-, \mathcal{O}_C) \) we get

\[
0 \to \text{Hom}(\mathcal{I}_X, \mathcal{O}_C) \to \mathcal{N}_C \to \mathcal{N}_{X|C}
\]

and \( \text{Hom}(\mathcal{I}_X, \mathcal{O}_C) = 0 \) since \( C \) and \( D \) have no common component. Therefore, there is an inclusion

\[
\mathcal{N}_C \hookrightarrow (\mathcal{N}_X|_C = \mathcal{O}_C(r) \oplus \mathcal{O}_C(r)
\]

hence \( h^0(\mathcal{N}_C(m)) = 0 \) for \( m \leq -r - 1 \). In particular \( h^0(\mathcal{N}_C(r - s)) = 0 \) because \( s \geq 2r + 1 \). \( \square \)

4. **Arithmetically Cohen-Macaulay curves**

In this section we explain how Example 1.15.3 in [EP93] extends the result about the perfection of complete intersections to the much larger class of arithmetically Cohen-Macaulay curves (from now on, ACM curves). Recall that a curve \( C \subset \mathbb{P}^3 \) is called ACM if its homogeneous ring \( R_C = R/I_C \) is Cohen-Macaulay, or, equivalently, if \( C \) is locally Cohen-Macaulay of pure dimension 1 and \( H^1_1(\mathcal{I}_C) = 0 \). A smooth ACM curve is what classically was referred to as a *projectively normal curve*. We refer the reader to [HS11] for a detailed study of ACM curves on a surface in \( \mathbb{P}^3 \).

If \( C \subset \mathbb{P}^3 \) is an ACM curve, then \( I_C \) has a free graded resolution of the form...
and $I_C$ coincides with the ideal generated by the $r \times r$ minors of $\phi$ by the Hilbert-Burch theorem - cf. [MDP90, Proposition II.1.1 p. 37].

Applying the functor $\text{Hom}_R(\bullet, R/I_C)$ to (7) as in [Ell75, p. 428] one obtains a long exact sequence

$$0 \to E = \bigoplus_{j=0}^{r} R(-b_j) \xrightarrow{\phi} F = \bigoplus_{i=0}^{r+1} R(-a_i) \to I_C \to 0$$

Equation (8) 0 \to H^0(C, \mathcal{N}_C) \to \bigoplus_{i=0}^{r} R_C(a_i) \to \bigoplus_{j=0}^{r} R_C(b_j) \to H^0_*(\omega_C(4)) \to 0

The importance of this sequence for our purposes is that it allows to compute the Hilbert function $n \mapsto h^0(\mathcal{N}_{C,P^3}(n))$ of $\mathcal{N}_{C,P^3}$ as a function of the Hilbert function $n \mapsto h^0(\mathcal{O}_C(n))$ of $C$; we can then compute the dimension of $\text{Ann}(\beta_1(C))$ and of $I_{a(C)}$, in terms solely of the Hilbert function of $C$ and of the degree $s$ of $S$. To justify our assertion, one needs to observe that to compute $h^0(\mathcal{N}_{C,P^3}(n))$ out of (8) one does not need to know the numbers $a_i$’s and $b_j$’s, but only for each $n$ the difference

$$\#\{i : a_i = n\} - \#\{j : b_j = n\}$$

which depends only on the Hilbert function of $C$.

As an application of this argument, we can give for ACM curves a sharp bound for the smallest integer $n$ such that $I_{a(C),n} = I_{C,n}$. For this we will not need the full Hilbert function of $C$, but just its index of speciality $e := e(C) = \max\{n \mid h^1(\mathcal{O}(n)) = h^2(\mathcal{F}_C(n)) > 0\}$ and the minimum degree $s(C)$ of a surface containing $C$: $s(C) = \min\{n \mid h^0(\mathcal{F}_C(n)) > 0\}$. For an ACM curve $C$, the ideal $\mathcal{F}_C$ is $e+3$-regular because $H^1_*(\mathcal{F}_C) = 0$. In particular, the ideal $I_C$ is generated in degrees $\leq e+3$, and $s(C) \leq e+3$.

**Theorem 4.1.** Let $S$ be a smooth surface of degree $s$ in $\mathbb{P}^3$. Let $C \subset S$ be an ACM curve, let $s(C)$ be the minimum degree of a surface containing $C$ and let $e(C)$ be the index of speciality of $C$.

If $s \geq 2e(C) + 8 - s(C)$ then $I_{a(C),s(e+3)} = I_{C,(e+3)}$. Therefore $C$ is reconstructed at level $e + 3$ by $I_{a(C)}$.

**Proof.** The statement follows from Proposition 3.1 with $p = e + 3$ provided we can show that $h^0(\mathcal{N}_{C,P^3}(e+3-s)) = 0$. For this we use the exact sequence (8), which shows that the maximum $n$ for which $h^0\mathcal{N}_{C,P^3}(n) = 0$ is $n = s(C) - e(C) - 5$. 

**Remark 4.2.** A twisted cubic curve $C$ is ACM with invariants $s(C) = 2$ and $e(C) = -1$. Hence from Theorem 4.1 it follows once more that, if $C$ is contained in a smooth quartic surface $S$, then $C$ is cut out by quadrics in $I_{a(C),s}$.

**Remark 4.3.** Theorem 4.1 improves for ACM curves the bound of Corollary 3.4 because, since $r = e+3$, then $2e + 8 - s(C) = 2r + 2 - s(C)$.

In [MS19, Sec 2.3], motivated by the case of complete intersections, formulate the notion of a perfect class:
Definition 4.4. Let $S$ be a smooth surface of degree $s$ in $\mathbb{P}^3$. A class $\alpha \in H^1(\Omega_S)/\mathbb{C}\eta_H \subseteq H^0(\mathcal{O}_S(2s - 4))^*$ is perfect at level $m$ if there exist effective divisors $D_1, \ldots, D_q$ in $S$ such that $I_{\alpha(D_i)} = I_\alpha$ for every $i = 1, \ldots, q$ and

$$I_{\alpha,j} = \sum_{i=1}^q I_{D_i,j} + J_{S,j}$$

for every $j \leq m$.

We say the class is perfect if $I_\alpha = \sum_{i=1}^q I_{D_i} + J_S$. We make the convention that the zero class is perfect - geometrically, this amounts to consider the empty set as a (empty) curve, and is consistent with regarding the zero divisor as an effective divisor.

Example 4.5. If $C \subset S$ is the complete intersection of two surfaces meeting properly, then $\alpha(C)$ is perfect (see [MS19, Ex 2.11], [EP93, Ex 1.15.2], [Dan17, Prop. 2.14]). If one does not agree that the zero class is perfect, then one needs to add the condition that $C$ is cut out by two surfaces of degrees $< s = \deg(S)$.

We now wish to generalize the previous example to the class of ACM curves showing that, if $C$ is ACM, then the class $\alpha(C)$ is perfect. For this we need to recall more facts from [EP93]. Suppose the ACM curve $C$ is contained in a smooth surface $S$ of degree $s$ and equation $f = 0$. Then the polynomial $f$ can be written in the form

$$f = \sum_{i=1}^{r+1} g_i h_i$$

where the $h_i$’s are the images of the generators of the free module $F$ in the resolution (7) of $I_C$. Since the $h_i$’s are the signed $r \times r$ minors of $\phi$, then polynomial $f$ is the determinant of the morphism $\psi : E \oplus \mathbb{R}(-s) \to F$ obtained adding the column $[g_1, \ldots, g_{r+1}]^T$ to the matrix of $\phi$; in other words, $\psi$ coincides with $\phi$ on $E$, and sends $1 \in \mathbb{R}(-s)$ to $\sum_{i=1}^{r+1} g_i e_i$, where the $e_i$’s are the generators of $F$. We thus obtain a resolution of $I_C/I_S$:

$$0 \to E \oplus \mathbb{R}(-s) \xrightarrow{\psi} F \to I_C/I_S \to 0 \quad (9)$$

Since $S$ is smooth, the curve $C$ is Cartier on $S$ so that $I_C/I_S$ can locally be generated by one element. It follows that the ideal $I_r(\psi)$ generated by the $r \times r$ minors of $\psi$ is irrelevant, that is, its radical is the irrelevant maximal ideal $(x, y, z, w)$ of the polynomial ring $\mathbb{R}$.

Proposition 4.6. [EP93, Prop. 1.16] Let $C \subset \mathbb{P}^3$ be an ACM curve contained in the smooth surface $S$. Suppose $I_C$ has the resolution (7). Then

(1) if $\psi$ is as in exact sequence (9) the presentation of $I_{C,S}$, then

$$I_{\alpha(C)} = I_r(\psi)$$

is the ideal generated by the $r \times r$ minors of $\psi$;
(2) the $n^{th}$-graded piece $\text{Ann}(\alpha(C))_n$ of the annihilator of $\alpha(C)$ in $H^0(\mathcal{O}_S)$ is the image of the natural map
\[
\bigoplus_{m \in \mathbb{Z}} H^0(\mathcal{O}_S(C + (n + m)H) \otimes H^0(\mathcal{O}_S(-C - mH)) \rightarrow H^0(\mathcal{O}_S(n))
\]

Remark 4.7. Note that $\text{Ann}(\alpha(C)) = I_{\alpha(C)}/I_S$. The equality $I_{\alpha(C)} = I_r(\psi)$ is a non trivial fact that is not given a full proof in [EP93]; a complete proof can be found in [KMR09, Proposition 4.3 and p. 382].

We can now prove that the class $\alpha(C)$ of an ACM curve in a smooth surface $S$ is perfect:

**Theorem 4.8.** Let $C \subset S$ be an ACM curve and let $S$ be a smooth surface. Then the class $\alpha(C)$ of $C$ in $S$ is perfect.

**Proof.** Fix an integer $n$. By Proposition 4.6 $\text{Ann}(\alpha(C))_n$ is the image of the natural map
\[
\bigoplus_{m \in \mathbb{Z}} H^0(\mathcal{O}_S(C + (n + m)H) \otimes H^0(\mathcal{O}_S(-C - mH)) \rightarrow H^0(\mathcal{O}_S(n)).
\]

Note the sum on the left hand side is finite, and consists of those $m$ for which the linear systems $|C + (m + n)H|$ and $|-C - mH|$ are both non-empty. For such an $m$ we pick a basis $g_1, \ldots, g_{r_m}$ of $H^0(\mathcal{O}_S(-C - mH)$ and corresponding effective divisors $D_k = (g_k)_0 \in |-C - mH|$. The image of $H^0(\mathcal{O}_S(C + (n + m_k)H) \otimes g_k$ in $H^0(\mathcal{O}_S(n))$ is $H^0(\mathcal{I}_{D_k}/S(n))$. (If $C \sim tH$ is a complete intersection of $S$ and another surface, taking $m_k = -t$ and $n = 0$ we get $D_k$ the empty curve, and in this case $\alpha(C) = 0$ is perfect by our definition). Note that $Q\alpha(D_k) = Q\alpha(C)$ by Proposition 2.3. Now letting $k$ and $m$ vary we see that $\alpha(C)$ is perfect at level $n$, for every $n$. Since $\text{Ann}(\alpha(C))$ is finitely generated, we can let $n$ vary up to the maximum degree of a generator of $\text{Ann}(\alpha(C))$, and recover the whole $\text{Ann}(\alpha(C))$ as the sum of finitely many $I_{D_k}/S$ with $D_k \sim C + (n + m)H$ for some $m$ and $n$. Therefore $\alpha_C$ in $S$ is perfect.

\[\square\]

5. Example of a non perfect class

**Theorem 5.1.** Let $C \subset \mathbb{P}^3$ be a smooth rational curve of degree 4 contained in a smooth surface $S$ of degree $s = 4$. The class $\alpha(C)$ in $S$ is not perfect at level 3.

**Proof.** A smooth rational quartic curve $C \subset \mathbb{P}^3$ is contained in a unique quadric surface $Q$, and $Q$ is necessarily smooth (all curves on the quadric cone are arithmetically Cohen-Macaulay by [Har77, Chapter V Ex. 2.9]). We may assume $C$ is a divisor of type $(3,1)$ on $Q$. The ideal sheaf of $C$ is 3-regular, hence $I_C$ generated by quadrics and cubics.

Suppose $C$ is contained in a smooth quartic surface $S$. Then $Q \cap S$ is the union of $C$ and an effective divisor $D_0$ of type $(1,3)$ on $Q$. Note that $D_0$ is a curve of degree 4 and arithmetic genus 0; as the divisor class of $D_0$ is different from that of $C$ and $C$ is irreducible, we conclude that $C$ and $D_0$ have no common component.
The curves $C$ and $D_0$ don’t move in their linear system on the quartic surface $S$: for $C$ this follows from $C^2 = -2$, and in any case for both $D_0$ and $C_0$ one might argue that

$$h^0(\mathcal{O}_S(D_0)) = h^0(\mathcal{O}_S(2H - C)) = h^0(\mathcal{I}_C(2)) = 1.$$ 

Having established the geometric set-up, we proceed to show that $I_{\alpha(C)}$ contains too many cubics for $\alpha(C)$ to be perfect at level 3. To compute the dimension of $I_{\alpha(C),3}$, we use the fact that $R/I_{\alpha(C)}$ is a Gorenstein ring with socle in degree $2s - 4 = 4$, hence

$$\dim I_{\alpha(C),3} = \dim I_{\alpha(C),1} + \dim R_3 - \dim R_1 = \dim I_{\alpha(C),1} + 16 \geq 16.$$ 

This estimate is good enough for us to prove the theorem, but let us show anyway that $\dim I_{\alpha(C),3} = 16$: as $C$ is a divisor of type $(3,1)$ on $Q$, $h^1(\mathcal{I}_C(3)) = 0$ hence by Lemma 2.5 $I_{\alpha(C),1}$ is the pull back to $R_1$ of $N(C)_1$, the image of $H^0(N_C/\mathbb{P}^3(-3))$ in $H^0(\mathcal{O}_C(1))$: as the normal bundle of $C$ pulls-back on $\mathbb{P}^1$ to $\mathcal{O}_{\mathbb{P}^1}(7) \oplus \mathcal{O}_{\mathbb{P}^1}(7)$ by [EvDvV81, Proposition 6]), we conclude that $I_{\alpha(C),1} = 0$, hence $\dim I_{\alpha(C),3} = 16$. The same argument shows that $I_{\alpha(C),2} = I_{C;2}$ as well.

To check whether $I_{\alpha,3}$ is perfect, we need to determine curves $D$ in $S$ with $I_{\alpha(D)} = I_{\alpha(C)}$ and $h^0(\mathcal{I}_D(3)) \geq 1$ so that $D$ can contribute to $I_{\alpha,3}$. Thus suppose $D$ is such a curve. By Proposition 2.3, there exist $m, n, p \in \mathbb{Z}$, $m, n \neq 0$ and relatively prime, such that $pH + mC + nD$ is linearly equivalent to zero. By assumption $3H - D$ is effective; as $C$ is not linearly equivalent to $tH$ for any $t$, neither is $D$, hence $1 \leq \deg(D) = D \cdot H \leq 11$. Replacing $D$ with $D' = 3H - D$ we can even assume $D \cdot H \leq 6$.

Now consider the matrix

$$M = \begin{bmatrix} H^2 & C \cdot H & H \cdot D \\ C \cdot H & C^2 & C \cdot D \\ H \cdot D & C \cdot D & D^2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & x \\ 4 & -2 & y \\ x & y & z \end{bmatrix}$$

As $pH + mC + nD$ is linearly equivalent to zero, the vector $v = [p, m, n]^T$ is in the kernel of $M$. Set $x = H \cdot D$, $y = C \cdot D$ and $z = D^2$. Note that $z = D^2 = 2(p_a(D) - 1) = 2q$ is even.

The determinant of $M$ must vanish, so

$$x^2 + 4xy - 2y^2 - 24q = 0$$

From this we deduce first that $x$ and $y$ must be even, and then that $4$ divides $x$. As $1 \leq x \leq 6$, we must have $x = 4$. Thus $D$ is a curve of degree 4, and either $D = C$ or $C$ is not a component of $D$, hence $y = C \cdot D \geq 0$. Assume that $D \neq C$. Writing $y = 2t$ with $t \geq 0$, we obtain the equation

$$t^2 - 4t + 3q - 2 = 0$$

Looking at the discriminant of this quadratic equation in $t$ we deduce $6 - 3q$ is a perfect square, so that $q = 2 - 3a^2$ for an integer $a \geq 0$. Then solving for $t$ and imposing $t \geq 0$ we obtain $t = 2 + 3a$. So $H \cdot D = x = 4$, $C \cdot D = y = 4 + 6a$ and $D^2 = 4 - 6a^2$. Then solving the linear system $Mv = 0$ for $v = [p, m, n]^T$ we find $m = an$ and $p = -(a + 1)n$. Since $m$ and $n$ are relatively prime and non zero and $a \geq 0$, the only possibility is that $a = 1$. Then we can take $m = n = 1$ and conclude $C + D \sim 2H$, so that $C + D$ is the complete
intersection of the unique quadric $Q$ containing $C$ with $S$, and $D = D_0$ is the residual to $C$ in the complete intersection $Q \cap S$.

We conclude that the only curves $D$ in $S$ that are contained in a cubic surface and satisfy $I_{\alpha(D)} = I_{\alpha(C)}$ are $C$, the residual $D_0$ to $C$ in the complete intersection $Q \cap S$, and the effective divisors linearly equivalent to either $3H - C$ or $3H - D_0$. But observe that, if $D' \sim 3H - D_0 \sim C + H$ is effective, then

$$h^0 \mathcal{I}_{D'}(3) = h^0 \mathcal{I}_C(2) = 1.$$ 

Therefore there is a unique cubic containing $D'$, whose equation is contained in the ideal of $D_0$. Similarly, if $D'' \sim 3H - C$ is effective, there is a unique cubic containing $D''$, whose equation is contained in the ideal of $C$. Hence any cubic form that belongs to the ideal of a curve $D$ on $S$ satisfying $I_{\alpha(D)} = I_{\alpha(C)}$ is in the vector space spanned by $I_{C,3}$ and $I_{D_0,3}$.

To show $\alpha(C)$ is not perfect at level 3 it is now enough to show that cubics containing either $C$ or $D_0$ plus the cubics in the Jacobian ideal $J_S$ do not span $I_{\alpha(C),3}$.

To this end, note that cubic surfaces that contain both $C$ and $D_0$ are in the ideal of the complete intersection of $S$ and $Q$, and so form a vector space of dimension 4. By Grassmann’s formula

$$\dim I_{C,3} + \dim I_{D_0,3} = 7 + 7 - 4 = 10$$

There are four independent cubics in the Jacobian ideal, so

$$\dim I_{C,3} + \dim I_{D_0,3} + \dim J_{S,3} \leq 14 < 16 = \dim I_{\alpha(C),3}$$

and this shows that $\alpha(C)$ in $S$ is not perfect at level 3. \qed

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