Quatum nonlocality as a useful resource is of great interest and high importance in quantum information science. It can be witnessed by violations of Bell-type inequalities and is arguably among various fundamental features of quantum mechanics. In analogy to maximally entangled mixed state, here we introduce the notion of genuine multipartite maximally nonlocal mixed state, which is defined as one that maximally violates Svetlichny’s inequality among other states for a given linear entropy. Thus this state is the most noise-resistant quantum resource. To show its power, we also construct a quantum nonlocal game with multiple players based on Svetlichny’s inequality. The probability of winning the game has an upper bound of $3/4$ when the state shared by players is biseparable, while, with the genuine multipartite maximally nonlocal mixed state, there is a quantum strategy for them that succeeds with probability up to $(2 + \sqrt{1 + \gamma^2})/4$, beating the bound with nonvanishing $\gamma$.

Initially arising from the debate on the incompleteness of quantum mechanics [1], quantum nonlocality has now played an influential role of a valuable resource in various aspects of quantum information science [2, 3]. Quantum nonlocality can be witnessed by the violation of Bell-type inequalities that hold for local-hidden-variable (LHV) models [4, 5], and is arguably among various fundamental features of quantum mechanics. So far, investigations on Bell-type inequalities for quantum systems of arbitrary parties and dimensions have been undertaken [6–10]. For multipartite systems, however, the description of quantum nonlocality is subtle. The subtlety arises from the question as to whether the nonlocality of an $N$-qubit system is attributed to the genuine $N$-qubit correlations, or just the convex combinations of nonlocal correlations between subsystems. To show this subtlety, the simplest situation one needs to consider, as Svetlichny [11] first noted in 1987, is a three-qubit system. Svetlichny derived an inequality, satisfied by all hybrid local-nonlocal-hidden-variable models, then based on this he showed that the inequality could be violated by some quantum states. Hence, these states bear the genuine three-qubit nonlocality. Note that the genuine multipartite nonlocality is distinct from the genuine multipartite entanglement (i.e., full entanglement), while the latter states the mathematical impossibility of biseparating a quantum state. The original Svetlichny’s inequality (SI) has recently been generalized to multipartite and also arbitrarily-dimensional scenarios [12–15].

Environment-induced noise is generally unavoidable in real experiments. The system one considers in practice is better described by a mixed state. It is hence important and nontrivial to investigate the states that maximize the quantum resource for a given measure of mixedness. Examples include entanglement (maximally entangled mixed states (MEMS)) and quantum discord (maximally discordant mixed states (MDMS)) [16–18]. We extend this line of investigation and seek mixed states bearing maximal nonlocality for a given amount of purity. In this work, we present a family of mixed states which have maximal degree of nonlocality (in the sense of maximally violating the SI) for a given value of linear entropy. We call them maximally nonlocal mixed states (MNMS), by analogy with MEMS and MDMS. Genuine $N$-qubit MNMS are of importance and interest in quantum information protocols, as such states can be regarded as mixed-state generalizations of maximally nonlocal pure states (for instance Bell states for two qubits).

In this work, we first present the genuine $N$-qubit MNMS which violate the SI maximally for a given linear entropy. As a case study, we show that the two-qubit MNMS do not overlap with MEMS and there is clearly a gap between them, by explicitly exploring the maximal violation of Clauser-Horne-Shimony-Holt (CHSH) inequality [5].
we investigate the genuine multipartite MNMS as a quantum resource in the context of a quantum nonlocal game, named Svetlichny game. In the Svetlichny game, the players are divided into two groups and each player is required to provide two-bit answers to the binary questions received. The players win the Svetlichny game if a certain criterion is satisfied, as we show in the Methods section. We note that when \( N = 2 \), the Svetlichny game is reduced to the CHSH game [5]. Classically, the players cannot win the Svetlichny game with probability exceeding \( 3/4 \). We show, however, there is a quantum strategy for them that succeeds with probability up to \((2 + \sqrt{2})/4 \) [19], beating the classical bound. The genuine multipartite MNMS are found to be essential to increase the winning probability. The quantum winning probability of the nonlocal game based on two-qubit MEMS is also explored for comparison.

Results
Genuine multipartite MNMS. We now introduce the notion of genuine multipartite MNMS, which refers to the state that violates SI most for a given linear entropy. This definition is analogous to that of MEMS, for which the entanglement is the greatest for a given linear entropy. The genuine \( N \)-qubit MNMS are given by (up to local unitary transformations)

\[
\rho_{N}^{\text{MNMS}} = \frac{1 + \gamma}{2} \rho_1 + \frac{1 - \gamma}{2} \rho_2, 
\]

where \( \gamma \) is a measure of admixture between the two orthogonal states \( \rho_i = |\psi_i\rangle\langle\psi_i| \), with \( |\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\cdots00\rangle + |11\cdots11\rangle) \) and \( |\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\cdots01\rangle + |11\cdots10\rangle) \). One may group the first \((N - 1)\) qubits together as a single effective qubit, then the whole state can be simply regarded as a “two-qubit” system. The normalized linear entropy can be computed through \( S_L(\rho) = \frac{1}{d-1}(1 - \Tr\rho^2) \) with \( d = 2^N \). Let Pauli operator \( \sigma_{n_i k} = \hat{\sigma} \cdot \hat{n}_{i k} \) denote the \( i_k \)-th measurement on the \( k \)-th qubit. To obtain the quantum maximum, without loss of generality, we constrain the measurement settings \( n_{i k} \) to the \( xy \)-plane. We hence have \( \sigma_{n_i k} = \begin{pmatrix} 0 & e^{i\theta_{i k}} \\ e^{-i\theta_{i k}} & 0 \end{pmatrix} \), where \( \theta_{i k} \) indicates the measurement direction for \( n_{i k} \). The measurement on the \( N \)-qubit quantum system reads \( M_J = \sigma_{n_1} \otimes \sigma_{n_2} \otimes \cdots \otimes \sigma_{n_N} \), where \( J = i_1 i_2 \cdots i_N \). It is not difficult to calculate the correlation functions \( Q_J = \Tr(\rho M_J) \) by noting that all entries in the density matrix \( \rho \) vanish except only four \( 2 \times 2 \) blocks in the matrix corners, as shown in the following,

\[
\rho = \begin{pmatrix}
\frac{1 + \gamma}{2} & 0 & \cdots & 0 & \frac{1 + \gamma}{2} \\
0 & \frac{1 - \gamma}{2} & \cdots & \frac{1 - \gamma}{2} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{1 - \gamma}{2} & \cdots & \frac{1 - \gamma}{2} & 0 \\
\frac{1 + \gamma}{2} & 0 & \cdots & 0 & \frac{1 + \gamma}{2}
\end{pmatrix}.
\]

Thus we have an effective state

\[
\rho = \begin{pmatrix}
\frac{1 + \gamma}{2} & 0 & 0 & \frac{1 + \gamma}{2} \\
0 & \frac{1 - \gamma}{2} & \frac{1 - \gamma}{2} & 0 \\
0 & \frac{1 - \gamma}{2} & \frac{1 - \gamma}{2} & 0 \\
\frac{1 + \gamma}{2} & 0 & 0 & \frac{1 + \gamma}{2}
\end{pmatrix}.
\]

This fact allows us to elegantly pick up nontrivial entries in \( M_J \) considered as effective measurements and compute the correlation functions to be

\[
Q_J = \Tr(\rho M_J) = \cos(\theta_{i_1} + \theta_{i_2} + \cdots + \theta_{i_{N-1}}) \cos(\theta_{i_N}) - \gamma \sin(\theta_{i_1} + \theta_{i_2} + \cdots + \theta_{i_{N-1}}) \sin(\theta_{i_N}).
\]

As it is well-known, the \( N \)-qubit SI reads

\[
S_N = \frac{1}{2^{N-2}} \sum_{\alpha=1}^{2^{N-2}} T_{\alpha} \leq 1,
\]

where \( T_{\alpha} \)’s are the Bell-CHSH-type operators shown as below [15]

\[
T_{\alpha} = \frac{1}{2} \left( s_{\{k_1 k_2\}} Q_{k_1 k_2} + s_{\{k_1 k_2\}} Q_{k_1 k_2} + s_{\{k'_1 k'_2\}} Q_{k'_1 k'_2} + s_{\{k'_1 k'_2\}} Q_{k'_1 k'_2} \right),
\]
with $\mathcal{J} = K_1 K_2$, $K_1 = i_1 i_2 \cdots i_k$, $K_2 = i_{k+1} \cdots i_N$. Here $\alpha$ is related to different settings denoted by indices (i.e., setting arrangement for all qubits) $\mathcal{J} = K_1 K_2$. Different $\mathcal{J}$ correspond to different $\alpha$, which ranges from 1 to $2^{N-2}$. Due to the correlation functions in Eq. (2), one can obtain the maximal quantum violation of SI as (see Methods)

$$S_{\alpha}^{\text{max}}(\rho_{\alpha}^{\text{MNMS}}) = \frac{1}{2^{N-2}} \sum_{\alpha=1}^{2^{N-2}} I_{\alpha}^{\text{max}} = \sqrt{1 + \gamma^2}. \quad (5)$$

Eq. (5) implies that the state (1) maximally violates SI for a given linear entropy. The simple reason is that each Bell-CHSH-type inequality $I_{\alpha}$ has already reached its maximum for the state with the fixed linear entropy, so no other states can violate SI more than MNMS. Consequently the MNMS given by Eq.(1) are indeed genuine multipartite MNMS based on maximal violation of $N$-qubit SI.

**MNMS and MEMS for two-qubit:** In the following we focus on two-qubit MNMS to elucidate the difference between MEMS and MNMS. For $N = 2$, the state (1) is equivalent to $\rho_{\alpha}^{\text{MNMS}} = \frac{1}{4}(1 + \gamma)|\mu\rangle\langle\mu| + \frac{1}{2}(1 - \gamma)|\nu\rangle\langle\nu|$ up to local unitary transformations. Here $|\mu\rangle$ and $|\nu\rangle$ are two of the four orthogonal Bell states. For the convenience of comparing MNMS with MEMS, however, we would like to present a somewhat different way to show $\rho_{\alpha}^{\text{MNMS}}$ are exactly MNMS. The state of any two-qubit system, up to local unitary operations, can be written as [20]

$$\rho_2 = \frac{1}{4}\left(I + \vec{a} \cdot \vec{\sigma} \otimes I + I \otimes \vec{b} \cdot \vec{\sigma} + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j\right). \quad (6)$$

The linear entropy for this state reads $\mathcal{E}_L(\rho_2) = 1 - \frac{1}{4}(1 - \text{Tr}(\rho_2^2)) = 1 - \frac{1}{4}(|\vec{a}|^2 + |\vec{b}|^2 + c^2)$, where $c^2 = \sum_{j=1}^{3} c_j^2$. Without loss of generality, we assume $|c_1| \geq |c_2| \geq |c_3|$. It then follows that the quantum nonlocality of state (6) can be described by the maximal quantum expectation value of the Bell-CHSH operator $\mathcal{B} = \frac{1}{2}(Q_{00} + Q_{01} + Q_{10} - Q_{11})$. Here $Q_{ml}$ with $m, l = 0, 1$ are two-qubit correlation functions, and we have $\mathcal{B}(\rho_2) = \sqrt{c_1^2 + c_2^2}$, as given in [21].

Consider $|\phi\rangle$ as a superposition state of arbitrary two of the Bell states, the state (6) should satisfy $\langle\phi|\rho_2|\phi\rangle \geq 0$, which leads to three conditions $c_1^2 + c_2^2 \leq 1 + c_3^2$, $-a_1^2 + b_1^2$, with $ijk \in \{123, 231, 312\}$. Under these conditions, for a given $\mathcal{B}(\rho_2) = \sqrt{c_1^2 + c_2^2}$, the maximal $\mathcal{E}_L$ occurs at $\vec{a} = \vec{b} = 0$ and $c_3 = \pm \gamma$. This maximal value is given by $\max \mathcal{E}_L = 1 - \frac{1}{4}(1 + 2\gamma^2)$. When $\vec{a} = \vec{b} = 0$, we can compute the eigenvalues of (6) to be $\{\frac{1}{4}(1 - c_1 - c_2 - c_3), \frac{1}{4}(1 + c_1 - c_2 + c_3), \frac{1}{4}(1 - c_1 + c_2 + c_3), \frac{1}{4}(1 + c_1 + c_2 - c_3)\}$. Due to the positivity of eigenvalues of the density matrix, we readily have the constraints upon the coefficients $c_j$, which elegantly belong to a tetrahedron $ABCD$ represented by the set of vertices $A : (-1, -1, -1), B : (-1, 1, 1), C : (1, -1, 1),$ and $D : (1, 1, -1)$ in the space of $(c_1, c_2, c_3)$ [22]. Under the assumption $|c_1| \geq |c_2| \geq |c_3|$, the optimal case corresponds to the point at $(1, -\gamma, \gamma)$ on the sides $AB$ and $CD$ of the tetrahedron. In fact, the states on the sides of tetrahedron $ABCD$ can be written in the form of $\rho_{\alpha}^{\text{MNMS}}$. Hence $\rho_{\alpha}^{\text{MNMS}}$ is the state giving maximal $\mathcal{B}(\rho_2)$, i.e., the two-qubit MNMS.

Also, we would like to point out that the concurrence of the MNMS is equal to $\gamma$. In particular, if $c_3 = 0$, then the linear entropy $\mathcal{E}_L$ of state (6) achieves its maximum at $\vec{a} = \vec{b} = 0$ under the condition $|c_1 \pm c_2| \leq 1$. In this case, we have $\mathcal{B}(\rho_2) \in [0, 1]$.

The $\mathcal{E}_L \cdot \mathcal{B}$ relation for MEMS [16, 17] can be computed straightforwardly by definition. In Fig. 1 we plot the variation of quantum expectation value of the Bell-CHSH operator with respect to linear entropy for different two-qubit states. For comparison, we also plot the result for the MEMS denoted in the figure. The expectation value of $\mathcal{B}$ for the MNMS and the MEMS do not overlap, emphasizing the difference between the two sets of states.

**Svetlichny game.** We explore MNMS as useful quantum resources in the context of the Svetlichny game. As far as the mixed state is concerned, genuine multipartite MNMS are the most noise-resistant in the sense that it is defined as the states which give maximal violations of SI for a given linear entropy. In the Svetlichny game, the referee chooses an $N$-bit question $\mathcal{J} = i_1 i_2 \cdots i_N$ uniformly from the complete $N$-bit set. He then sends $\mathcal{J}_1 = i_1 \cdots i_j$ to one group with $j$ players and $\mathcal{J}_2 = i_{j+1} \cdots i_N$ to another with $N - j$ players. Each player $k \in \{1, 2, \ldots, N\}$ must reply with a single bit $a_k$ as an answer to the question $i_k$. They win the game if and only if the answers $A = a_1 \cdots a_N$ satisfy the following criterion $\text{Mod}(|\frac{\mathcal{J}_1}{\mathcal{J}_2}|, 2) = \bigoplus_{k=1}^{N} a_k$, with $T = T_1 + T_2$ where $T_1$ and $T_2$ denote the times of bit “1” appeared in $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively, and $\lfloor \cdot \rfloor$ means to take the integer part. We would like to point out that in the $N = 2$ case, the criterion is equivalent to $r \cdot s = a \oplus b$ in the CHSH game.
We define the winning probability of \( N \) players as

\[
\Pr_N = \frac{1}{2^N} \sum_{\mathcal{J}} P(\text{Mod}([\frac{T}{2}], 2) = \bigoplus_{k=1}^{N} a_k).
\]

(7)

By writing \( P(a_k|i_k) = \frac{1}{2}(1 + (-1)^{a_k}A_{i_k}) \), where \( A_{i_k} \equiv \vec{\sigma} \cdot \vec{n}_{i_k} \) (with \( \vec{n}_{i_k} = \{\theta_{i_k}, \phi_{i_k}\} \)) is the observable of the \( k \)-th qubit, the equivalence between the quantum game and the \( N \)-qubit SI is straightforward. Note that for given questions \( \mathcal{J} \) and answers \( \mathcal{A} = a_1 \cdots a_N \), the joint probability \( P(a_1 \cdots a_N|i_1 \cdots i_N) \) has non-zero contributions only from the identity 1 and the full correlation \( A_{i_1} \cdots A_{i_N} \). The other correlations do not contribute to the joint probability due to the symmetry of Svetlichny game criterion under permutation of any pair of players. Thus the winning probability (7) becomes

\[
\Pr_N = \frac{1}{2^N} \sum_{\mathcal{J}, \mathcal{A}} \delta_{\mathcal{J}, \mathcal{A}} P(a_1 \cdots a_N|i_1 \cdots i_N) = \frac{1}{2^N} \sum_{\mathcal{J}} \frac{1}{2} \left( 1 + (-1)^{\frac{T}{2}} A_{i_1} \cdots A_{i_N} \right)
\]

\[
= \frac{2 + \mathcal{S}_N}{4},
\]

where \( \delta_{\mathcal{J}, \mathcal{A}} = 1 \) if the answer \( \mathcal{A} \) satisfies the Svetlichny game criterion for each question \( \mathcal{J} \), otherwise \( \delta_{\mathcal{J}, \mathcal{A}} = 0 \). The fact that the \( N \)-qubit SI \( \mathcal{S}_N \leq 1 \) is a sum of \( 2^{N-2} \) CHSH-type inequalities \( \mathcal{I}_\alpha \leq 1 \) (see Ref. [15] for details) can be understood in the sense that a group of \( j \) observers is denoted by Alice and the other group of \( N-j \) observers is denoted by Bob, and that the measuring results in Alice’s group are independent of those in Bob’s, thus observers in the same group may be nonlocally correlated. The probability \( \Pr_N \) can never exceed \( 3/4 \) in classical context. Quantum mechanically, the maximal value of \( \mathcal{S}_N \) is independent of \( N \) and can always reach \( \sqrt{1 + \gamma^2} \), if the quantum state shared by \( N \) players is the \( N \)-qubit MNMS and the quantum winning probability is \( \Pr_N^{MNMS} = (2 + \sqrt{1 + \gamma^2})/4 \) in the Svetlichny game based on genuine multipartite MNMS which accounts for the increased winning probability. The probability \( \Pr_N^{MNMS} \) ranges from 3/4 to \( (2 + \sqrt{2})/4 \approx 0.8535 \) depending on the degree of the nonlocality of the genuine multipartite MNMS.

In Fig. 2, we consider the CHSH game and analyze the performance of MNMS and MEMS. It is clear that MNMS demonstrate the maximal violation of the CHSH inequality for a given linear entropy. Therefore it is reasonable to conclude that the genuine multipartite MNMS are indeed the most noise-resistant resource for any quantum information and computation protocols involving Bell nonlocality, such as quantum nonlocal game, Bell’s-theorem-based quantum cryptography, Bell’s-theorem-based random number generator, etc.

**Discussions**

To summarize, we have introduced the notion of genuine multipartite MNMS. The genuine multipartite MNMS are defined through maximal violation of Svetlichny’s inequality. MNMS would be important in quantum information protocols where genuine \( N \)-qubit nonlocality is needed at a given value of linear entropy. Our result is of practical interest in cases where decoherence, imperfect state preparation and measurements introduce entropy to the state of the system. The presence of genuine multipartite MNMS provides the promising noise-resistant resource for quantum protocols based on Bell nonlocality. We have investigated the Svetlichny game with the multipartite MNMS to show the application of the resource in quantum protocols. In the game, \( N \) players are divided into two groups, one with \( j \) and the other one with \( N-j \) players. Each player receives binary questions and returns two-bit answers respectively. With classical resources (defined in this context as those states that do not possess genuine \( N \)-qubit correlations), the players win the game with probability not exceeding \( 3/4 \). Employing quantum resources in the form of MNMS increases this probability to \( (2 + \sqrt{1 + \gamma^2})/4 \). Specifically, we have explored the performance of different mixed states in the CHSH game as an example to explicitly show that MNMS maximally violate Svetlichny’s inequality.

**Methods**

Before working out the quantum maximum of the SI for the state (1), let us first give a sketch of it to elucidate several concerns. Our aim is to prove that the state (1) is the genuine multipartite MNMS, i.e., maximally violates the SI (see the relation (3)) for fixed linear entropy. The \( N \)-qubit SI is comprised of \( 2^N \) correlation functions. For the convenience of analysis, the SI can be decomposed into sub-inequalities \( \mathcal{I}_\alpha \), each of which has four correlation functions. Clearly, the summing over \( \alpha \) (and then divided by a constant \( 2^{N-2} \) for the sake of normalization) yields precisely the \( N \)-qubit SI. Admittedly, the manner of decomposition is quite arbitrary. However, a convenient manner is the one we use in (4) or (8) below, such that each \( \mathcal{I}_\alpha \) is CHSH-like. Now, it is adequate for us to concentrate only on \( \mathcal{I}_\alpha \): If (1) maximally violates each \( \mathcal{I}_\alpha \) simultaneously, then, it will, naturally, maximally violate the \( N \)-qubit SI.
To this end, we need the knowledge of two qubits. As we have proved in the “MNMS and MEMS for two-qubit” part, the state (1) maximally violates the CHSH inequality for two qubits, with quantum maximum $\sqrt{1 + \gamma^2}$. For an arbitrary $N$, since $I_\alpha$ we consider is CHSH-like, it will share the same quantum maximum as $N = 2$. The reason for this is that, for a “$k|(N - k)$” cut (see Eq. (4)), nonlocal correlations between the first $k$ parties do not contribute to the violation of $I_\alpha$; neither do those between the last $N - k$ parties. Hence one may group the first $k$ parties as a single effective party, and the last $N - k$ as another effective party. The scenario then goes back to the bipartite case that we have already studied. That is, were the genuine multipartite MNMS found, it would violate the SI with quantum maximum in the form of $\sqrt{1 + \gamma^2}$. Here $\gamma$ is understood as a parameter of MNMS which has been unknown to us so far. We shall see that the state (1) exactly gives $\sqrt{1 + \gamma^2}$, and is henceforth the genuine multipartite MNMS.

Now we begin to compute the quantum violation of the SI. Since the SI is a sum of CHSH-type inequalities, it suffices to concentrate on one of the CHSH-type inequalities first, namely,

$$I_\alpha = \frac{1}{2} \left( s_{(K)0} Q_{K0} + s_{(K)1} Q_{K1} + s_{(K')0} Q_{K'0} + s_{(K')1} Q_{K'1} \right) \leq 1,$$

with $J = K i N$. We explicitly write out indices $i_n \in \{0, 1\}$ for the $N$-th party, leaving the other $N - 1$ parties indicated by $K$. Note also that $s_{(K)0} = (-1)^{\lfloor t(K) \rfloor / 2}$ is a sign function for the SI, with $t(K)$ being the number of “1” in $K$ and $t(K') = t(K) + 1$. The index $\alpha$ runs from 1 to $2^{N-2}$, and the correlations in $I_\alpha$ are different from those in other $I_{\alpha'}$. We note that $K$ and $K'$ depend on $\alpha$. A simple relation between them could be taken as:

$$K = \text{Bin}(2\alpha - 2), \quad K' = \text{Bin}(2\alpha - 1),$$

so that

$$s_{(K)0} = \left\lfloor \frac{1}{2} \left( \text{Bin}(2\alpha - 2) \right) \right\rfloor,$$

where Bin$(x)$ denotes a layout of binary digits of $x$. For instance, in the tripartite case, we have $K = 00$, $K' = 01$ for $\alpha = 1$, and $K = 10$, $K' = 11$ for $\alpha = 2$. Then it follows that the three-qubit Svetlichny inequality reads $S_3 = \frac{1}{2}(I_1 + I_2) \leq 1$.

Relations (8) are the elementary constituents of the SI, in that a sum of such inequalities exactly leads to the SI. It can be shown that we have $I_\alpha \leq \sqrt{1 + \gamma^2}$ by taking the measurement settings $\theta_{i_k = 0} = 0$, $\theta_{i_k = 1} = \pi/2$ for $k = 1, 2, ..., N - 1$, and $\theta_{i_N = 1} = -\theta_{i_N = 0}$. With these specified measurement settings, we list all options for correlations and signs with respect to various $t(K)$ and $t(Ki_N)$ as follows:

$$Q_{Ki_N} = \cos \theta_{i_N}, \quad \text{if } t(K) = 4\xi$$
$$= -\gamma \sin \theta_{i_N}, \quad \text{if } t(K) = 4\xi + 1$$
$$= -\cos \theta_{i_N}, \quad \text{if } t(K) = 4\xi + 2$$
$$= \gamma \sin \theta_{i_N}, \quad \text{if } t(K) = 4\xi + 3$$

and

$$s_{(K)0} = +1, \quad \text{if } t(K) = 4\xi$$
$$= +1, \quad \text{if } t(K) = 4\xi + 1$$
$$= -1, \quad \text{if } t(K) = 4\xi + 2$$
$$= -1, \quad \text{if } t(K) = 4\xi + 3,$$

with $\xi = 0, 1, 2, ...$. Then it is found that

$$s_{(K)0} Q_{Ki_N} = \cos \theta_{i_N = 0}, \quad \text{if } t(K) = 4\xi \text{ or } 4\xi + 2$$
$$= -\gamma \sin \theta_{i_N = 0}, \quad \text{if } t(K) = 4\xi + 1 \text{ or } 4\xi + 3,$$

so that we have

$$I_\alpha = \cos \theta_{i_N = 0} - \gamma \sin \theta_{i_N = 0} \leq \sqrt{1 + \gamma^2}.$$
This inequality is strict at \( \tan \theta_{i,N=0} = -\gamma \), with \( \cos \theta_{i,N=0} = 1/\sqrt{1+\gamma^2} \) and \( \sin \theta_{i,N=0} = -\gamma/\sqrt{1+\gamma^2} \). By similar reasoning, we can always find \( I_\alpha \leq \sqrt{1+\gamma^2} \) for \( \alpha = 1, \ldots, 2^{N-2} \) with the same measurement settings as shown above. Thus one readily has Eq. (5) for the MNMS (1).

In Fig. 3, we plot genuine tripartite nonlocality versus linear entropy for three qubits by randomly choosing \( 10^6 \) states, and the red solid curve AB numerically confirms the state (6) is indeed MNMS based on maximal violations of the three-qubit SI. Suppose there exists a state that violates the SI in the range \([d/2(d-1), 1]\). Let us denote such a state by a point “F” in Fig. 3. Then a convex combination of three states, which correspond to points A, B and F, respectively, would saturate the region ABF bounded by curves AB, BF and FA. Now, although we do not work out the exact expression of the curve FA, it is enough to see that there must be many points lying above the curve AB. This clearly contradicts our proof that curve AB corresponds to the MNMS. Therefore no state above a linear entropy \( d/2(d-1) \) would violate the SI. Apparently, the analysis above is not confined within two qubits or three qubits, but also applicable to arbitrary qubits. That is, if a point above the linear entropy \( d/2(d-1) \) violates the SI, then by a convex combination some points above the MNMS would be found, contradicting our previous proof of MNMS for states below \( d/2(d-1) \).

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**Author Contributions**
J.L.C. initiated the idea. H.Y.S., F.L.Z., C.W., X.Z.P. and J.L.C. derived the main results. H.Y.S., C.W., J.L.C., M.G., S.V., and L.C.K. wrote the main manuscript text. F.L.Z., H.Y.S., and X.Z.P. prepared the figures. All authors reviewed the manuscript.

**Additional Information**

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FIG. 1: Bell nonlocality versus linear entropy for two-qubit states. The red solid curve $B(\rho_{\text{MNMS}}^2) = \sqrt{2 - 3E_L}/2$ is for the MNMS, which maximizes $B(\rho^2)$ for each fixed value of $E_L$ ($E_L \leq 2/3$ so that the CHSH inequality is violated). Pink and blue areas indicate, respectively, arbitrary two-qubit nonlocal and local states in the $E_L$-$B$ plane. The green dotted curve $B = \sqrt{3 - 3E_L}/2$ (corresponds to the state (6) with $\vec{a} = \vec{b} = 0$ and $c_3 = 0$) denotes the maximal $B$ for each $E_L > 2/3$. The blue solid curve $B = \sqrt{1 - 3E_L}/2$ (corresponds to the states $\rho = p|00\rangle\langle00| + (1-p)|11\rangle\langle11|$) denotes the minimal $B$ for each fixed value of $E_L \leq 2/3$. For comparison, we also plot numerical results for the MEMS denoted by the blue dashed curve $B(\rho_{\text{MEMS}}^2) = (2 + \sqrt{1 + \gamma^2})/4$ for $\gamma \in [0, 2/3]$ and $B(\rho_{\text{MEMS}}^2) = (2 + \sqrt{2} \gamma)/4$ for $\gamma \in (2/3, 1]$. It is clearly shown that $B_{\text{MNMS}}$ surpasses $B_{\text{MEMS}}$ except $\gamma = 1$ in which case MNMS and MEMS are Bell states resulting in the same probability $\frac{1}{4}(2 + \sqrt{2})$. We also consider the game with arbitrary nonlocal states (see pink region).

FIG. 2: The quantum winning probability versus concurrence $\gamma$ for the MNMS, MEMS and arbitrary nonlocal states in the CHSH game. For the purpose of qualitatively comparison, we plot the winning probability in CHSH game as a function of the entanglement degree $\gamma$ (measured by concurrence) for MNMS (red solid curve) and MEMS (blue dashed curve). If the quantum state shared by Alice and Bob is the MNMS, the quantum winning probability is $Pr_{\text{MNMS}}^2 = (2 + \sqrt{1 + \gamma^2})/4$. For the MEMS, the winning probability is found to be $Pr_{\text{MEMS}}^2 = (6 + \sqrt{1 + 18\gamma^2 - \min(1, 9\gamma^2)})/12$ for $\gamma \in [0, 2/3]$ and $Pr_{\text{MEMS}}^2 = (2 + \sqrt{2}\gamma)/4$ for $\gamma \in (2/3, 1]$. It is clearly shown that $Pr_{\text{MNMS}}^2$ surpasses $Pr_{\text{MEMS}}^2$ except $\gamma = 1$ in which case MNMS and MEMS are Bell states resulting in the same probability $\frac{1}{4}(2 + \sqrt{2})$. We also consider the game with arbitrary nonlocal states (see pink region).
FIG. 3: **Genuine tripartite nonlocality versus linear entropy for three-qubit states.** The red solid curve AB is for the three-qubit MNMS, based on maximal violations of the three-qubit Svetlichny inequality. The orange solid line is the classical bound for Svetlichny’s inequality. The blue points between the curve AB and the orange line are quantum states that violate the three-qubit Svetlichny inequality, showing genuine tripartite nonlocality. Point A corresponds to the state \((|000\rangle + |111\rangle)/\sqrt{2}\), with \(S_{3}^{\text{max}} = \sqrt{2}\); Point B is degenerated by some quantum states which are not locally equivalent, such as \(\rho_{3}^{\text{MNMS}} (\gamma = 0)\) and \(|0\rangle\langle 0| \otimes (|\psi\rangle\langle \psi| + |\psi^{+}\rangle\langle \psi^{+}|)/2\), with \(|\psi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle\) and \(|\psi^{+}\rangle = \sin \theta |00\rangle - \cos \theta |11\rangle\); Point C corresponds to the maximally mixed state \(1/8\), with \(S_{3}^{\text{max}} = 0\); Point D corresponds to \(\cos \vartheta |000\rangle + \sin \vartheta |111\rangle\) with \(\sin 2\vartheta = 1/\sqrt{3}\), and \(S_{3}^{\text{max}} = \sqrt{2}/3\) [23]; Point E corresponds to \((|000\rangle\langle 000| + |111\rangle\langle 111|)/2\), with \(S_{3}^{\text{max}} = 0\).