Six functor formalism for sheaves with non-presentable coefficients

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Abstract

In this paper we show that the six functor formalism for sheaves on locally compact Hausdorff topological spaces, as developed for example in [KS90], can be extended to sheaves with values in any closed symmetric monoidal ∞-category which is stable and bicomplete. Notice that, since we do not assume our coefficients to be presentable or restrict to hypercomplete sheaves, our arguments will be not obvious and substantially different from the ones contained in [KS90]. Along the way we also study locally contractible geometric morphisms and prove that, if $f : X \to Y$ is a continuous map which induces a locally contractible geometric morphism, then the exceptional pullback functor $f^!$ satisfies a formula generalizing [KS90, Proposition 3.3.2 (ii)], where it is proven only for topological submersions. At the end of our paper we also show how one can express Atiyah duality by means of the six functor formalism.

Contents

1 Introduction 2
  1.1 Linear overview ................................................. 2
  1.2 Acknowledgements .............................................. 3

2 Sheaves and tensor products 5
  2.1 Tensor product of cocomplete ∞-categories ...................... 5
  2.2 Sheaves and cosheaves ........................................... 11

3 Shape theory and shape submersions 15
  3.1 Relative shape .................................................. 15
  3.2 Locally contractible geometric morphisms ...................... 19
  3.3 Shape submersions .............................................. 23

4 Localization sequences 27

5 Verdier Duality 31
  5.1 $\K$-sheaves .................................................. 32
  5.2 Verdier duality ................................................ 34
  5.3 The pullback $f^*_e$ .......................................... 38

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1 Introduction

It is universally acknowledged that the most complete reference dealing with the *six functor formalism* for sheaves on topological spaces is Masaki Kashiwara and Pierre Schapira’s seminal book *Sheaves on Manifolds* [KS90]. However, for technical reasons related to the construction of derived functors, the authors there restrict themselves to hypersheaves with values in bounded derived categories of rings with finite global dimension. In this paper we exploit the power of the now established theory of ∞-categories (as developed for example in [Lur09] or [Cis19]) to extend the formalism to a much broader setting. More precisely, if \( f : X \to Y \) is a continuous map between locally compact Hausdorff topological spaces and \( \mathcal{C} \) is any stable bicomplete ∞-category, we use Lurie’s Verdier duality equivalence ([Lur17, Theorem 5.5.5.1])

\[
\mathcal{D}_\mathcal{C} : \text{Shv}(X; \mathcal{C}) \xrightarrow{\sim} \text{CoShv}(X; \mathcal{C})
\]

where the target of the equivalence is the ∞-category of \( \mathcal{C} \)-valued *cosheaves* on \( X \), i.e. \( \text{Shv}(X; \mathcal{C})^{\mathsf{op}} \), to construct adjunctions

\[
\begin{align*}
\text{Shv}(Y; \mathcal{C}) & \xleftarrow{\perp} \text{Shv}(X; \mathcal{C}) \\
\text{Shv}(X; \mathcal{C}) & \xleftarrow{\perp} \text{Shv}(Y; \mathcal{C})
\end{align*}
\]

with natural equivalences

\[
\mathcal{D}_\mathcal{C} f^!_\mathcal{C} \simeq (f^{\mathsf{op}})_* \mathcal{D}_\mathcal{C} \quad \mathcal{D}_\mathcal{C} f^*_\mathcal{C} \simeq (f^{\mathsf{op}})_* \mathcal{D}_\mathcal{C}.
\]

Notice that, since we do not require \( \mathcal{C} \) to be presentable, the existence of \( f^*_\mathcal{C} \) is not at all obvious (see the discussion in Remark 2.9), and so we will have to work a bit harder than one might expect. Nevertheless we would like to point out that our efforts to make the results in this paper as general as possible are not vain, and actually lead to many advantages. Namely, by working with a class of coefficients closed under the operation of passing to the opposite category, we do not break the symmetry which comes from Verdier duality, meaning that whenever we prove some result involving the functors \( f^*_\mathcal{C} \) and \( f^!_\mathcal{C} \) that is true for all \( \mathcal{C} \) stable and bicomplete, we immediately obtain a dual theorem involving the functors \( f^!_\mathcal{C} \) and \( f^*_\mathcal{C} \), and viceversa. Another way to put it is that our formalism applies with no distinctions to sheaves or cosheaves: this will be used in a follow-up paper in which we will prove a duality theorem for constructible sheaves on a conically smooth stratified space, as we will need to extend some results about constructible sheaves, such as homotopy invariance (see [Hai20]) or the exodromy equivalence (see a yet unpublished paper by Porta and Teyssier), to constructible cosheaves.

We try to outline the key ingredients in our paper that allow us to work with non-presentable coefficients. As we explained above, the main difficulty lies in showing the existence of the pullback functor \( f^*_\mathcal{C} \). The main tool we employ to carry out this purpose is
Lurie’s tensor product of cocomplete ∞-categories as defined in [Lur17, 4.8.1]: in particular, a property of this tensor product that we will use over and over is that it preserves adjunctions between cocontinuous functors (see Remark 2.1). We show in Lemma 2.1 that it restricts to a monoidal structure for Cocont_{∞}^{st} (i.e. the ∞-category of stable cocomplete ∞-categories with cocontinuous functors between them) and observe in Corollary 5.1 that the model of X-sheaves (see [Lur09, Theorem 7.3.4.9]) implies that Shv(X; Sp) is a strongly dualizable object in Cocont_{∞}^{st}. Hence, we get an equivalence

$$\text{CoShv}(X; Sp) \otimes C \simeq \text{CoShv}(X; C)$$

for any C stable and cocomplete, and so, by Verdier duality,

$$\text{Shv}(X; Sp) \otimes C \simeq \text{Shv}(X; C)$$

for any C stable and bicomplete: this last equivalence is crucial, because it will allow us to reduce a lot of arguments to the case of sheaves of spectra. At this point, we factor f as a composition of an open immersion and a proper map (see factorization (5.1)), and prove the existence of a left adjoint to $f^*_C$ in these two separate cases. The only non-trivial part is the case when f is proper: we first show in Lemma 5.1 that $f^*_C$ preserves colimits and then in Proposition 5.2 that there is a natural equivalence $f^*_{Sp} \otimes C \simeq f^*_C$, so that $f^*_C$ has a left adjoint which can be identified with $f^*_{Sp} \otimes C$. In particular, we see that the global section functor

$$\text{Shv}(X; C) \to C$$

admits a left adjoint: as a consequence, using the results in [Cis19, 6.7], we show in Theorem 5.3 that the inclusion of $\text{Shv}(X; C)$ in C-valued presheaves on X admits a left adjoint.

The discussion above involves only the four functors $f^*_C$, $f^*_C$, $f^!_C$ and $f^!_C$, but what about the other two? Our first observation is that, a priori, there is no need to require our category of coefficients to have a monoidal structure to make sense of things like projection formulas or Künneth formulas. To be more precise, one can show that there is a functor

$$\text{Shv}(X; C) \times \text{Shv}(Y; D) \to \text{Shv}(X \times Y; Sp) \otimes (C \otimes D)$$

which preserves colimits in both variables and induces an equivalence

$$\text{Shv}(X; C) \otimes \text{Shv}(Y; D) \simeq \text{Shv}(X \times Y; Sp) \otimes (C \otimes D)$$

and thus, by taking $X = Y$ and composing with $\Delta^*_{Sp} \otimes (C \otimes D)$, where $\Delta : X \to X \times X$ is the diagonal embedding, we get a variablewise colimit preserving functor denoted as

$$\text{Shv}(X; C) \times \text{Shv}(Y; D) \to \text{Shv}(X; Sp) \otimes (C \otimes D)$$

(see Remark 3.8 and Remark 5.4 for more details). For this kind of tensor product of sheaves, we prove formulas

$$f^*_{C\otimes D}(F \otimes G) \simeq f^*_{C} F \otimes f^*_{D} G$$

$$f^!_{C\otimes D}(F \otimes f^*_{D} G) \simeq f^!_{C} F \otimes G$$

$$(f \times g)^{C\otimes D}_{i}(F \boxtimes G) \simeq f^!_{C} F \boxtimes g^!_{D} G$$
respectively in Corollary \ref{cor:heine-borel}, Proposition \ref{prop:delta} and Proposition \ref{prop:delta}. In particular, when \( \mathcal{C} \) admits a monoidal structure whose tensor preserves colimits in both variables, one obtains a cocontinuous functor

\[ \text{Shv}(X; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{C}) \to \text{Shv}(X; \mathcal{C}) \]

whose composition with \ref{eq:heine-borel} induces a monoidal structure on \( \text{Shv}(X; \mathcal{C}) \): this way we can deduce all the analogous formulas in the monoidal setting as well as their dual versions involving the internal homomorphism functor when the monoidal structure in \( \mathcal{C} \) is closed (see Remark \ref{rem:monoidal}.

We describe one last advantage of our general rendition of the six functor formalism. In Definition \ref{def:shape} we define locally contractible geometric morphisms (see also \cite[Definition 3.2.1]{AC}), and later in Definition \ref{def:shape} specify a vast class of continuous maps between topological spaces called \textit{shape submersions} which induce a locally contractible geometric morphism (see Proposition \ref{prop:submersions}): this class includes topological submersions, but it’s much bigger. An easy implementation of our machinery shows that, if \( f : X \to Y \) induces a locally contractible geometric morphism and \( \mathcal{C} \) is stable and bicomplete, then \( f_! \) admits a right adjoint and we have a formula

\[ f_!^* \mathcal{F} \otimes f_!^* \mathcal{G} \simeq f_!^* ( \mathcal{F} \otimes \mathcal{G} ) \]

that highly generalizes \cite[Proposition 3.3.2]{KS}, where it is proven only in the case where \( f \) is a topological submersion.

1.1 Linear overview

We now give a linear overview of the contents of our paper.

In section 2 we recall the definition of Lurie’s tensor product of cocomplete \( \infty \)-categories and prove some of its basic properties. In particular, we will interpret the results in \cite[6.7]{Cis} in terms of this tensor product in Theorem \ref{thm:tensor}, show that it preserves the property of being stable in Lemma \ref{lem:tensor-stable} and show that compactly generated stable \( \infty \)-category is a strongly dualizable object in the symmetric monoidal \( \infty \)-category \( \text{Cocont}_{\infty}^\text{st} \) in Proposition \ref{prop:tensor-stable}. Afterwards we recall the definition of sheaves and cosheaves with values in a general \( \infty \)-category and explain how Lurie’s tensor product can be used to conveniently describe \( \text{Shv}(X; \mathcal{C}) \) at least when \( \mathcal{C} \) is presentable. Most of the results in this section are not original, but we still felt the necessity to spend some time writing them up to make our discussion as self contained and reader friendly as possible.

In section 3 we define for any geometric morphism \( Y \to X \) the relative shape \( \Pi_{X/}^\infty (Y) \) as an object of \( \text{Pro}(X) \) and describe explicitly in Proposition \ref{prop:shape} how this construction can be enhanced to a functor

\[ \Pi_{X/}^\infty : \text{Top}_{/X} \to \text{Pro}(X) \]

and, even more, to a lax natural transformation between functors \( \text{Top}^{op} \to \text{Cat}_{\infty} \) (see Remark \ref{rem:shape}): these coherent structures with which we equip the shape will be used to prove easily that the shape is homotopy invariant in Corollary \ref{cor:shape-invariant} and later in Proposition \ref{prop:shape-invariant} to show that the Thom spectrum gives a natural transformation of sheaves of \( E_{\infty} \)-spaces. Later we define locally contractible geometric morphisms and give a characterization in Proposition \ref{prop:contractible} which mimics the one in \cite[C3.3]{Joh} for the locally connected case. We also show that, when \( f \) is a geometric morphism induced by a continuous map of topological spaces, the property of being locally contractible is checked more easily. Then we define shape submersions and prove in Proposition \ref{prop:submersions} a base change formula which will imply that they induce locally contractible geometric morphisms (see Corollary \ref{cor:submersions}).

In section 4 we follow the approach of \cite{Kha} to obtain the localization sequences associated to a decomposition of a topological space into an open subset and its closed complement.
Also the results here are not so new but, after section 5, they will imply that there is a rec-
ollement of $\text{Shv}(X; \mathcal{C})$ associated to any open-closed decomposition of $X$ whenever $\mathcal{C}$ is stable
and bicomplete, while this was previously known only for $\mathcal{C}$ presentable.

Section 5 is devoted to Verdier duality, and how it can be used to show that the pushfor-
ward $f^*_\mathcal{C}$ admits a left adjoint for any $\mathcal{C}$ stable and bicomplete in the way we have sketched
at the beginning of the introduction.

In section 6 we develop the six functor formalism: as usual, we prove base change (Propo-
sition 6.1), projection (Corollary 6.2) and Küneth (Corollary 6.3) formulas for $f^*_\mathcal{C}$, and
discuss the properties of $f^*_\mathcal{C}$ when $f$ is a shape submersion in Proposition 6.4.

At last, in section 7 we show how the six functor formalism can be used to express a
relative version of Atiyah duality for any proper submersion between smooth manifolds.

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2 Sheaves and tensor products

The goal of this section will be twofold: first we are going to introduce Lurie’s tensor product
of cocomplete $\infty$-categories as defined in [Lur17], and secondly we will recall the definition
of sheaves and cosheaves with values in general $\infty$-categories. The reason why we want to
spend some time discussing this matter, aside from it being interesting on its own, is that
in the following sections this tensor product will prove to be an extremely convenient tool
to describe some categories of sheaves and functors between them: through it we will be
able to produce a vast class of essential geometric morphisms, we will extend easily some
results regarding sheaves of spaces to sheaves with values in any presentable $\infty$-category,
and later prove the existence of a sheafification functor when the $\infty$-category of coefficients is
stable and bicomplete with no presentability assumption, and construct the full six-functor
formalism in this setting. Most of the results in this section are not at all original and can
be found for example in [Cis19], in [Lur17], or are already well known.

2.1 Tensor product of cocomplete $\infty$-categories

For the whole section we will fix a universe $\mathbf{V}$ and denote by $\text{Cocont}_\infty$ the $\infty$-category of non
necessarily $\mathbf{V}$-small $\infty$-categories admitting $\mathbf{V}$-small colimits, with $\mathbf{V}$-cocontinuous functors
between them. For short, we will call an object of $\text{Cocont}_\infty$ a cocomplete $\infty$-category and,
for any two cocomplete $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$ we will denote by $\text{Fun}_!(\mathcal{C}, \mathcal{D})$ the $\infty$-category
of functors preserving $\mathbf{V}$-small colimits.

Let $\mathcal{C}$ and $\mathcal{D}$ be cocomplete. Recall that, by [Lur09, 5.3.6], there exists a cocomplete
$\infty$-category, denoted by $\mathcal{C} \otimes \mathcal{D}$, such that we have an equivalence

\[
\text{Fun}_!(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \text{Fun}_!(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})
\]  

(2.1)
functorial on $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ cocomplete, where $\text{Fun}_{\times!}$ indicates the $\infty$-category of bifunctors preserving $\mathbf{V}$-small colimits in each variable. There is a canonical functor

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D},$$

well-defined up to a contractible space of choices, which preserves colimits in both variables and corresponds to the identity of $\mathcal{C} \otimes \mathcal{D}$ through the equivalence \[2.1\] we will often denote the image through this functor of a couple $(c, d) \in \mathcal{C} \times \mathcal{D}$ by $c \otimes d$. More precisely, [Lur17, Corollary 4.8.1.4] shows that this operation provides $\text{Cocont}_\infty$ with the structure of a symmetric monoidal $\infty$-category, and the inclusion of $\text{Cocont}_\infty$ in $\text{Cat}_\infty$ is lax monoidal, where the latter if equipped with the cartesian monoidal structure. Since we obviously have a functorial equivalence

$$\text{Fun}_{\times!}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \text{Fun}_{\times!}(\mathcal{C}, \text{Fun}_{\times!}(\mathcal{D}, \mathcal{E})),$$

this monoidal structure is closed. One can also deduce by the proof of [Lur09, Proposition 5.3.6.2] that the equivalence \[2.1\] may be described explicitly by left Kan extension through $\otimes$, with an inverse given by precomposition with $\otimes$.

**Remark 2.1.** As usual, one may regard $\text{Cocont}_\infty$ as a $(\infty, 2)$-category. It follows by \[2.1\] that, for any cocomplete $\infty$-category $\mathcal{C}$, tensoring with $\mathcal{C}$ actually gives rise to a 2-functor. An important consequence of this observation is that tensoring with $\mathcal{C}$ preserves adjunctions of cocontinuous functors, since any adjunction is characterized by the classical triangular identities (see for example [Cis19, Theorem 6.1.23, (v)]).

We will now present a list of results about the tensor product of cocomplete $\infty$-categories that will turn out to be very useful later.

Let $A$ be a small $\infty$-category, $\mathcal{C}$ cocomplete. For any two objects $a \in A$ and $c \in \mathcal{C}$, denote by $a \otimes c = a[c]$ the left Kan extension of $c$ along $a$ (here we are considering $a$ and $c$ as functors $A^{op} \leftarrow \Delta^0 \to \mathcal{C}$). Thus we get a functor

$$A \times \mathcal{C} \xrightarrow{y_{A/c}} \text{Fun}(A^{op}, \mathcal{C})$$

$$(a, c) \mapsto a \otimes c$$

which preserves colimits on the $\mathcal{C}$ variable that we will call the *relative Yoneda embedding*. By definition, we have a functorial equivalence

$$\text{Hom}(a \otimes c, F) \simeq \text{Hom}(c, F(a))$$

for any $F \in \text{Fun}(A^{op}, \mathcal{C})$.

**Remark 2.2.** Recall that in a closed symmetric monoidal $\infty$-category, an object $x$ is strongly dualizable if and only if the canonical map

$$y \otimes \text{Hom}(x, 1) \to \text{Hom}(x, y)$$

(2.2)

obtained as adjoint to

$$y \otimes \text{Hom}(x, 1) \otimes x \to y \otimes 1 \xrightarrow{\simeq} y$$

is an equivalence. In the case of $\text{Cocont}_\infty$, one sees easily that the map (2.2) can be described as induced by

$$\mathcal{D} \times \text{Fun}_{\times!}(\mathcal{C}, \mathcal{S}) \simeq \text{Fun}_{\times!}(\mathcal{S}, \mathcal{D}) \times \text{Fun}_{\times!}(\mathcal{C}, \mathcal{S}) \to \text{Fun}_{\times!}(\mathcal{C}, \mathcal{D})$$

where the last functor is given by composition.
**Remark 2.3.** Consider the variablewise cocontinuous functor
\[ \text{Fun}(A^{op}, S) \times \mathcal{C} \to \text{Fun}(A^{op}, \mathcal{C}) \]  
(2.3)

which corresponds to a cocontinuous functor
\[ \text{Fun}(A^{op}, S) \otimes \mathcal{C} \to \text{Fun}(A^{op}, \mathcal{C}) \]  
(2.4)

obtained as the extension by colimits of the relative Yoneda embedding, and denote by \( F \otimes c \) the image of a couple \((F, c) \in \text{Fun}(A^{op}, S) \times \mathcal{C} \). By definition one has identifications
\[ \text{Hom}(F \otimes c, G) \simeq \text{Hom}(F, \text{Hom}_{\mathcal{C}}(c, G(\cdot))) \]
functorially on \( F \in \text{Fun}(A^{op}, S), G \in \text{Fun}(A^{op}, \mathcal{C}) \) and \( c \in \mathcal{C} \), where the hom-space on the right-hand side is taken on the \( \infty \)-category of presheaves of \( U \)-small spaces, where \( U \) is a universe such that \( C \) is \( U \)-small. In particular one sees that \( F \otimes c \) can be identified with the composition \( c \circ F \) (here, by an abuse of notation, we use the same letter to refer to the image of \( c \) through the equivalence \( C \simeq \text{Fun}(S, C) \)), and so the functor (2.4) coincides with (2.2).

For any cocomplete \( \infty \)-category \( D \), precomposition with \( y_{A/\mathcal{C}} \) induces a functor
\[ \text{Fun}(\text{Fun}(A^{op}, \mathcal{C}), D) \to \text{Fun}(A \times \mathcal{C}, D) \simeq \text{Fun}(\mathcal{C}, \text{Fun}(A, D)) \]
and since colimits are computed pointwise in functor categories, it restricts to
\[ \text{Fun}(\text{Fun}(A^{op}, \mathcal{C}), D) \to \text{Fun}(\mathcal{C}, \text{Fun}(A, D)) \]  
(2.5)

**Theorem 2.1.** The functor (2.5) is an equivalence. In particular, \( \text{Fun}(A^{op}, S) \) is strongly dualizable in the monoidal \( \infty \)-category \( \mathcal{C}ocont_{\infty} \) with dual \( \text{Fun}(A, S) \), and thus the functor (2.4) is an equivalence.

**Proof.** A complete proof of the first statement can be found in [Cis19, 6.7]. The main ingredient of the proof is that, by [Cis19, Lemma 6.7.7], any \( F \in \text{Fun}(A^{op}, \mathcal{C}) \) can be written canonically as
\[ F = \lim_{c \to F(a)} a \otimes c \]
where the colimit is indexed by the Grothendieck construction of the functor \((a, c) \to \text{Hom}_C(c, F(a)) \). Furthermore, even though this indexing category is not small a priori, [Cis19, Lemma 6.7.5] proves that it is finally small. From this one may deduce easily the theorem, in a similar spirit to how one proves that \( \text{Fun}(A^{op}, S) \) is the free cocompletion under small colimits of \( A \).

To prove the last statement, we just observe that we have canonical equivalences
\[ \text{Fun}(\text{Fun}(A^{op}, \mathcal{C}), D) \simeq \text{Fun}(\mathcal{C}, \text{Fun}(A, D)) \]
\[ \simeq \text{Fun}(\mathcal{C}, \text{Fun}(\text{Fun}(A^{op}, S), D)) \]
\[ \simeq \text{Fun}(\text{Fun}(A^{op}, S) \otimes \mathcal{C}, D) \]
whose composition is given by precomposing with (2.4), and so we may conclude by Remark 2.3. \( \square \)

**Corollary 2.1.** Let \( u : A \to B \) be a functor between small \( \infty \)-categories, and let \( \mathcal{C} \) be any cocomplete \( \infty \)-category. Then we have equivalences \( u_! \otimes \mathcal{C} \simeq u_! \) and \( u^* \otimes \mathcal{C} \simeq u^* \). Here by an abuse of notation we write \( u_! \) and \( u^* \) to indicate both left Kan extension (restriction) along \( u \) for functors with values in \( S \) and in \( \mathcal{C} \).
Proof. By Remark 2.3 and Theorem 2.1 we have an adjunction $u_! \otimes \mathcal{C} \dashv u^* \otimes \mathcal{C}$ of cocontinous functors between $\mathcal{C}$-valued presheaves. By uniqueness of adjoints, it suffices to show that $u_! \otimes \mathcal{C} \simeq u_!$. By the proof of Theorem 2.1 it suffices to show that the two functors agree on object of the type $a \otimes c$, for $a \in A$ and $c \in \mathcal{C}$. We have functorial equivalences

$$\text{Hom}((u_! \otimes \mathcal{C})(a \otimes c), F) \simeq \text{Hom}(u(a) \otimes c, F)$$

$$\simeq \text{Hom}_c(c, F(u(a)))$$

$$\simeq \text{Hom}(a \otimes c, u^* F)$$

and thus, again by uniqueness of adjoints, we may conclude. □

Denote by $\text{Cont}^p_{\infty}$ ($\text{Cont}^st_{\infty}$) the full subcategory of $\text{Ccont}_{\infty}$ spanned by pointed (stable) cocomplete $\infty$-categories.

Lemma 2.1. Let $\mathcal{C}$ be any pointed (stable) cocomplete $\infty$-category, and let $\mathcal{D}$ be cocomplete. Then $\mathcal{C} \otimes \mathcal{D}$ is pointed (stable). In particular, $\text{Cont}^p_{\infty}$ ($\text{Cont}^st_{\infty}$) inherits an obvious monoidal structure from $\text{Ccont}_{\infty}$ and the inclusion in $\text{Ccont}_{\infty}$ admits a left adjoint given by tensoring with $S_*$ ($\text{Sp}$).

Proof. First of all, notice that $\Delta^0 \otimes \mathcal{D} \simeq \Delta^0$. Since $\mathcal{C}$ is pointed, the zero object $\Delta^0 \to \mathcal{C}$ is simultaneously a right and a left adjoint of the unique functor $\mathcal{C} \to \Delta^0$, and thus one may tensor these two adjunctions with $\mathcal{D}$ and obtain by Corollary 2.1 that $\mathcal{C} \otimes \mathcal{D}$ is pointed.

Assume now that $\mathcal{C}$ is stable. Since $\mathcal{C} \otimes \mathcal{D}$ is pointed, one sees easily that the suspension functor for $\mathcal{C} \otimes \mathcal{D}$ is obtained by applying $- \otimes \mathcal{D}$ to the suspension of $\mathcal{C}$, and so it is in particular an equivalence.

To prove that last part of the statement, it suffices to show that, for any pointed (stable) and cocomplete $\infty$-category $\mathcal{C}$, the evaluation at $S^0 (\mathcal{S})$ induces an equivalence $\text{Fun}(\mathcal{S}_*, \mathcal{C}) \simeq \mathcal{C}$ ($\text{Fun}(\text{Sp}, \mathcal{C}) \simeq \mathcal{C}$), but this follows easily by noticing that $\mathcal{S}_* \simeq \text{Ind}(\mathcal{S}_*^{\text{fin}})$ ($\text{Sp} \simeq \text{Ind}(\text{Sp}^{\text{fin}})$) and that evaluation at $S^0 (\mathcal{S})$ induces an equivalence between finitely cocontinuous functors from $\mathcal{S}_* (\text{Sp}^{\text{fin}})$ to $\mathcal{C}$ and $\mathcal{C}$. □

Remark 2.4. Let $A$ be any small category and $\mathcal{C}$ any object of $\text{Ccont}^p_{\infty}$. By the previous lemma, the functor (2.3) factors as

$$\text{Fun}(A^{op}, \mathcal{S}) \times \mathcal{C} \longrightarrow \text{Fun}(A^{op}, \mathcal{C})$$

$$\begin{array}{ccc}
\downarrow^\Sigma_\mathcal{C} \\
\text{Fun}(A^{op}, \text{Sp}) \times \mathcal{C}
\end{array}$$

inducing an equivalence

$$\text{Fun}(A^{op}, \mathcal{C}) \simeq \text{Fun}(A^{op}, \text{Sp}) \otimes \mathcal{C}.$$ 

Moreover, Remark 2.3 implies that one has identifications

$$\text{Hom}(F \otimes c, G) \simeq \text{Hom}(F, \text{Hom}_c(c, G(-)))$$

operatorially on $F \in \text{Fun}(A^{op}, \text{Sp})$, $G \in \text{Fun}(A^{op}, \mathcal{C})$ and $c \in \mathcal{C}$, where $\text{Hom}_c(c, -)$ denotes the canonical enrichment of $\mathcal{C}$ in $U$-small spectra, with $U$ a universe such that $\mathcal{C}$ is $U$-small, and the hom-space on the right-hand side is taken on the $\infty$-category of presheaves of $U$-small spectra.

Proposition 2.1. Let $\mathcal{C}$ be a compactly generated stable $\infty$-category. Then $\mathcal{C}$ is a strongly dualizable object of $\text{Ccont}^p_{\infty}$. 

8
Proof. Since $\mathcal{C}$ is stable and compactly generated, it follows that there exists a small stable $\infty$-category $A$ with finite colimits such that $\mathcal{C} \simeq \text{Fun}_{\text{ex}}(A^{\text{op}}, \text{Sp})$. Thus, since $\text{Fun}(\mathcal{C}, \text{Sp}) \simeq \text{Fun}_{\text{ex}}(A, \text{Sp})$, to prove the proposition we have to show that for any $D$ stable and cocomplete there is an equivalence
\[
\text{Fun}_{\text{ex}}(A^{\text{op}}, D) \simeq \text{Fun}_{\text{ex}}(A^{\text{op}}, \text{Sp}) \otimes D.
\]
We first prove that the inclusion $i : \text{Fun}_{\text{ex}}(A^{\text{op}}, D) \hookrightarrow \text{Fun}(A^{\text{op}}, D)$ admits a left adjoint $L$.

For any $a \in A$, denote by $y^{st}(a)$ the spectrally enriched representable functor associated to $a$, obtained as usual through the equivalence
\[
\text{Fun}_{\text{ex}}(A^{\text{op}}, \text{Sp}) \simeq \text{Fun}_{\text{lex}}(A^{\text{op}}, \text{Sp}).
\]
(2.6)

We define $L$ as the unique (up to a contractible space of choices) cocontinuous functor extending
\[
A \times D \longrightarrow \text{Fun}_{\text{ex}}(A^{\text{op}}, D)
\]
\[
(a, x) \longmapsto y^{st}(a) \otimes x.
\]
Indeed, $y^{st}(a) \otimes x$ is exact as it can be modelled by the composition of two finite colimit preserving functors. When $D = \text{Sp}$, by (2.6) and the Yoneda lemma, one sees that $L$ is left adjoint to $i$.

Let $D$ be any stable cocomplete $\infty$-category, and let $U$ be a universe such that $D$ is $U$-small. To see that $L$ is the desired left adjoint, we observe that for any $F \in \text{Fun}_{\text{ex}}(A^{\text{op}}, D)$, $x \in D$, we have functorial identifications
\[
\text{Hom}(y^{st}(a) \otimes x, F) \simeq \text{Hom}(y^{st}(a), \text{Hom}_{\mathcal{D}}(x, F(-)))
\]
\[
\simeq \text{Hom}_{\mathcal{D}}(x, F(a))
\]
\[
\simeq \text{Hom}(a \otimes x, F)
\]
where the hom-space on the right-hand side is taken on the $\infty$-category of presheaves of $U$-small spectra on $A$, and the second equivalence follows by the fact that $F$, and hence $\text{Hom}_{\mathcal{D}}(x, F(-))$, is exact.

Now notice that $i : \text{Fun}_{\text{ex}}(A^{\text{op}}, \text{Sp}) \hookrightarrow \text{Fun}(A^{\text{op}}, \text{Sp})$ preserves colimits, and so, by tensoring with $D$, one obtains an adjunction between cocontinuous functors
\[
\begin{array}{ccc}
\text{Fun}(A^{\text{op}}, D) & \overset{L \otimes D}{\longrightarrow} & \text{Fun}_{\text{ex}}(A^{\text{op}}, \text{Sp}) \otimes D \\
\downarrow & \longleftarrow & \\
i \otimes D
\end{array}
\]
where $i \otimes D$ is fully faithful. Since $\text{Fun}_{\text{ex}}(A^{\text{op}}, \text{Sp}) \otimes D$ and $\text{Fun}_{\text{ex}}(A^{\text{op}}, D)$ can be respectively identified with the essential images of $(Li) \otimes D$ and $Li$, to conclude the proof it suffices to show that the two functors are naturally equivalent, but this is true because they coincide on objects of the type $a \otimes x$. \hfill $\square$

Recall that an $\infty$-category $\mathcal{C}$ is called $\mathbf{V}$-presentable (for short, when there is no possibility of confusion we will only write presentable) if there exists a $\mathbf{V}$-small $\infty$-category $A$ such that $\mathcal{C}$ is a left Bousfield localization of $\text{Fun}(A^{\text{op}}, \text{Sp})$ by a $\mathbf{V}$-small set of morphism in $\text{Fun}(A^{\text{op}}, \text{Sp})$. If we furthermore assume that the localization functor $\text{Fun}(A^{\text{op}}, \text{Sp}) \rightarrow \mathcal{C}$ is left exact, we will say that $\mathcal{C}$ is an $\infty$-topos. It follows easily by this definition that any presentable $\infty$-category is complete and cocomplete. Presentable categories are equivalently defined as follows. Recall
that, for $\mathcal{C}$ any $\infty$-category and $S$ a class of morphisms in $\mathcal{C}$, we define an object $X \in \mathcal{C}$ to be $S$-\textit{local} if, for every morphism $f : A \to B$ in $S$, the induced morphism

$$\Hom_{\mathcal{C}}(B, X) \to \Hom_{\mathcal{C}}(A, X)$$

is invertible. Then we say that an $\infty$-category $\mathcal{C}$ is $\textbf{V}$-\textit{presentable} is there exists a $\textbf{V}$-small class $S$ of morphisms in $\Fun(A^{op}, S)$ such that $\mathcal{C}$ is equivalent to the full subcategory of $\Fun(A^{op}, S)$ spanned by $S$-local objects.

We denote by $Pr_L$ the full subcategory of $\mathcal{Cocont}_\infty$ spanned by presentable $\infty$-categories and $Pr_R = Pr^p_{op}$. Notice that, by the adjoint functor theorem (see for example [Cis19, Proposition 7.11.8]), the morphisms in $Pr_L$ are functors which admit a right adjoint and, consequently, morphisms in $Pr_R$ are functors which admit a left adjoint. We also denote by $\top$ the non full subcategory of $Pr_R$ whose objects are $\infty$-topoi and morphisms are functors which admit a left exact left adjoint (such functors are called geometric morphisms).

**Proposition 2.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable $\infty$-categories. Then $\mathcal{C} \otimes \mathcal{D}$ is presentable and there is a canonical equivalence $\mathcal{C} \otimes \mathcal{D} \simeq RFun(\mathcal{C}^{op}, \mathcal{D})$. In particular, $Pr_L$ inherits a symmetric monoidal structure.

**Proof.** Let $A$ and $B$ be two small $\infty$-categories, $S$ and $S'$ two small sets of morphisms of $\Fun(A^{op}, S)$ and $\Fun(B^{op}, S)$ respectively such that $\mathcal{C}$ and $\mathcal{D}$ are equivalent the full subcategories of $S$ and $S'$-local objects. By Theorem 2.1 we have

$$\Fun(A^{op}, S) \otimes \Fun(B^{op}, S) \simeq \Fun(A^{op}, \Fun(B^{op}, S))$$

$$\simeq \Fun((A \times B)^{op}, S).$$

It then follows from the proof of [Lur17, Proposition 4.8.1.15] that $\mathcal{C} \otimes \mathcal{D}$ can be identified with the full subcategory of $\Fun((A \times B)^{op}, S)$ spanned by $S \otimes S'$-local objects, where $S \otimes S'$ is the image of $S \times S'$ through the canonical functor

$$\Fun(A^{op}, S) \times \Fun(B^{op}, S) \to \Fun((A \times B)^{op}, S).$$

The proof of the last assertion follows by [Lur17, Lemma 4.8.1.16] and [Lur17, Proposition 4.8.1.17].

**Remark 2.5.** Let $\mathcal{C}$ be a presentable $\infty$-category. One can deduce easily for Proposition 2.2 identifications $\mathcal{C} \otimes S_\ast \simeq \mathcal{C}_\ast$ and $\mathcal{C} \otimes Sp \simeq Sp(\mathcal{C})$, where $\mathcal{C}_\ast$ denotes the $\infty$-category of pointed objects of $\mathcal{C}$, and $Sp(\mathcal{C})$ denotes the $\infty$-category of spectrum objects of $\mathcal{C}$, i.e. the limit of the tower

$$\ldots \alpha \to \mathcal{C}_\ast \to \alpha \to \mathcal{C}_\ast$$

where $\alpha$ is the usual loop functor. Both these constructions come with canonical functors $\mathcal{C}_\ast \to \mathcal{C}$ and $\alpha^\infty : Sp(\mathcal{C}) \to \mathcal{C}$, and since $\mathcal{C}$ is presentable one can show that these admit left adjoints $(-)_+ : \mathcal{C} \to \mathcal{C}_\ast$ and $\Sigma_\infty : \mathcal{C} \to Sp(\mathcal{C})$. By construction, we have a factorization

$$\mathcal{C} \xrightarrow{(-)_+} \mathcal{C}_\ast \xrightarrow{\Sigma_\infty} Sp(\mathcal{C}).$$

In particular we see that, if $\mathcal{C}$ is presentable and pointed (stable), by tensoring $(-)_+ : \mathcal{C} \to \mathcal{C}_\ast$ ($\Sigma_\infty : \mathcal{C} \to Sp(\mathcal{C})$) with $\mathcal{C}$ we obtain an equivalence $\mathcal{C} \otimes S_\ast \simeq \mathcal{C}$ ($\mathcal{C} \otimes Sp \simeq Sp(\mathcal{C})$). Thus, if $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are presentable $\infty$-categories where $\mathcal{D}$ is pointed and $\mathcal{E}$ is stable, we get functorial identifications

$$\Fun_{\ast}(\mathcal{C} \otimes S_\ast, \mathcal{D}) \simeq \Fun_{\ast}(\mathcal{C}, \Fun_{\ast}(S_\ast, \mathcal{D})) \simeq \Fun_{\ast}(\mathcal{C}, \mathcal{D})$$

10
\[
\text{Fun}(C \otimes \mathbb{S}p, \mathcal{E}) \simeq \text{Fun}(C, \text{Fun}(\mathbb{S}p, \mathcal{E})) \simeq \text{Fun}(C, \mathcal{E})
\]

induced respectively by precomposing with \((-)_+^\ast\) and \(\Sigma_{+}^\infty\). Furthermore, we see that \((-)_+ : \mathbb{S} \to \mathbb{S}_\ast\) and \(\Sigma_{+}^\infty : \mathbb{S} \to \mathbb{S}p\) make \(\mathbb{S}_\ast\) and \(\mathbb{S}p\) into idempotent cocomplete \(\infty\)-categories with respect to Lurie’s tensor product: by [Lur17, Proposition 4.8.2.9] this implies that there are canonical variablewise cocontinuous symmetric monoidal structures on \(\mathbb{S}_\ast\) and \(\mathbb{S}p\) with unit objects given by \(S^0 := (*)_+^\ast\) and \(\mathbb{S} := \Sigma_{+}^\infty\), and one can show that these coincide with the usual smash products of pointed spaces and spectra. In particular, we see that the functors

\[
\mathbb{S} \xrightarrow{(-)_+} \mathbb{S}_\ast \xrightarrow{\Sigma_{+}^\infty} \mathbb{S}p
\]

are all monoidal, where \(\mathbb{S}\) is equipped with the cartesian monoidal structure.

### 2.2 Sheaves and cosheaves

We now pass to recalling the definition of sheaves with values in an \(\infty\)-category. Let \(X\) be a small \(\infty\)-category equipped with a Grothendieck topology. Recall that there is a small \(\infty\)-category \(\text{Cov}(X)\), as defined in [Lur09, Notation 6.2.2.8], which can be described informally as having for objects couples \((x, R)\), where \(x \in X\) and \(R \hookrightarrow y(x)\) is a sieve covering \(x\), and morphisms between \((x, R)\) and \((y, R')\) are just maps \(f : x \to y\) in \(X\) such that the restriction of \(y(f)\) to \(R\) factors through \(R'\). There is an obvious projection \(\rho : \text{Cov}(X) \to X\) which has a section \(s : X \to \text{Cov}(X)\) defined on objects by sending \(x\) to \((x, y(x))\).

**Definition 2.1.** Let \(\mathcal{C}\) be a complete \(\infty\)-category. With the same notations as above, we say that a functor \(F \in \text{Fun}(X^{\text{op}}, \mathcal{C})\) is a sheaf if the unit morphism

\[
\rho^s F \to s_\ast s^s \rho^s F \simeq s_\ast F
\]

is an equivalence. Dually, for a cocomplete \(\infty\)-category \(\mathcal{C}\), we say that a functor \(F \in \text{Fun}(X, \mathcal{C})\) is a sheaf if the counit morphism

\[
s_! F \simeq s_\ast s^s \rho^s F \to \rho^s F
\]

is an equivalence. We denote by \(\text{Shv}(X; \mathcal{C})\) (\(\text{CoShv}(X; \mathcal{C})\)) the full subcategory of \(\text{Fun}(X^{\text{op}}, \mathcal{C})\) (\(\text{Fun}(X, \mathcal{C})\)) spanned by (co)sheaves. When \(\mathcal{C}\) is the \(\infty\)-category of spaces \(\mathbb{S}\), we will simply write \(\text{Shv}(X)\).

**Remark 2.6.** More concretely, one can describe a sheaf as a functor \(F\) such that for any covering sieve \(R \hookrightarrow y(x)\) the canonical morphism

\[
F(x) \to \lim_{y(x') \to R} F(x')
\]

is an equivalence. Notice also that we clearly have an equivalence \(\text{CoShv}(X; \mathcal{C}) \simeq \text{Shv}(X; \mathcal{C}^{\text{op}})^{\text{op}}\).

**Remark 2.7.** It is well known that, for any \(\infty\)-site \(X\), the category \(\text{Shv}(X)\) is an \(\infty\)-topos. Unlike the case of 1-topoi, it’s still unclear whether any \(\infty\)-topos is equivalent to \(\text{Shv}(X)\) for some \(\infty\)-site \(X\) (see [Rez19]).

We now give another description of categories of sheaves and cosheaves: this can be seen as an \(\infty\)-categorical analogue of the classical equivalence between cosheaves and Lawvere distributions on \(X\) (see for example [BF06]).
Lemma 2.2. Let $X$ be an $\infty$-site, $\mathcal{C}$ be any cocomplete $\infty$-category. Then left Kan extension along the functor $X \xrightarrow{\gamma} \operatorname{Fun}(X^{\text{op}}, S) \xrightarrow{\gamma} \operatorname{Shv}(X)$ defines an equivalence

$$\mathcal{C} \mathcal{O} \operatorname{Shv}(X; \mathcal{C}) \simeq \operatorname{Fun}(\operatorname{Shv}(X), \mathcal{C}),$$

where $y$ is the Yoneda embedding and $L$ is the sheafification functor. Equivalently, a functor $X \to \mathcal{C}$ is a cosheaf if and only if its extension by colimits $\operatorname{Fun}(X^{\text{op}}, S) \to \mathcal{C}$ factors through $L$. Dually, for any complete $\infty$-category $\mathcal{C}$, we have an equivalence

$$\operatorname{Shv}(X; \mathcal{C}) \simeq \operatorname{Fun}_{s}(\operatorname{Shv}(X)^{\text{op}}, \mathcal{C}).$$

Proof. Since $L$ commutes with colimits, by the universal property of localizations composition with $L$ embeds $\operatorname{Fun}(\operatorname{Shv}(X), \mathcal{C})$ in $\operatorname{Fun}(\operatorname{Fun}(X^{\text{op}}, S), \mathcal{C})$ as the full subcategory of functors sending covering sieves $R \hookrightarrow y(x)$ to equivalences in $\mathcal{C}$. On the other hand, a functor $F : X \to \mathcal{C}$ is a cosheaf precisely if there is an equivalence

$$\lim_{y(x') \to R} F(x') \simeq F(x)$$

for any sieve $R$ on $x \in X$, thus precisely if its extension by colimits $\operatorname{Fun}(X^{\text{op}}, S) \to \mathcal{C}$ lies in $\operatorname{Fun}(\operatorname{Shv}(X), \mathcal{C})$. \hfill $\square$

Corollary 2.2. Let $X$ be an $\infty$-site, $\mathcal{C}$ be any presentable $\infty$-category. Then the inclusion $\mathcal{C} \mathcal{O} \operatorname{Shv}(X; \mathcal{C}) \hookrightarrow \operatorname{Fun}(X, \mathcal{C})$ admits a right adjoint.

Proof. By [Lur09, Proposition 5.5.3.8] and the previous lemma, the $\infty$-category $\mathcal{C} \mathcal{O} \operatorname{Shv}(X; \mathcal{C})$ is presentable. Thus, since $\mathcal{C} \mathcal{O} \operatorname{Shv}(X; \mathcal{C}) \hookrightarrow \operatorname{Fun}(X, \mathcal{C})$ obviously preserves colimits, we may conclude by the adjoint functor theorem. \hfill $\square$

Corollary 2.3. Let $X$ be an $\infty$-site, $\mathcal{C}$ be any presentable $\infty$-category. Then we have an equivalence $\operatorname{Shv}(X) \otimes \mathcal{C} \simeq \operatorname{Shv}(X; \mathcal{C})$.

Proof. It follows by the adjoint functor theorem that $\operatorname{Fun}_{s}(\operatorname{Shv}(X)^{\text{op}}, \mathcal{C}) \simeq \operatorname{RFun}(\operatorname{Shv}(X)^{\text{op}}, \mathcal{C})$. Thus, by the previous lemma and by Proposition 2.2 we get the conclusion. \hfill $\square$

Remark 2.8. It is possible to prove that, for any topological space $X$ and any $\infty$-topos $\mathcal{Y}$, $\operatorname{Shv}(X; \mathcal{Y})$ is a product of $\operatorname{Shv}(X)$ and $\mathcal{Y}$ in the $\infty$-category $\text{Top}$ (see [Lur09, Proposition 7.3.3.9]). Thus, by the previous corollary combined with [Lur09, Proposition 7.3.1.11], if $Y$ is a locally compact topological space, we have an equivalence

$$\operatorname{Shv}(X) \otimes \operatorname{Shv}(Y) \simeq \operatorname{Shv}(X \times Y).$$

Example 2.1. (i) Let $f : X \to Y$ be a continuous map between topological spaces. Recall that this induces a geometric morphism $f : \operatorname{Shv}(X) \to \operatorname{Shv}(Y)$, which amounts to an adjunction $f^{*} \dashv f_{*}$, where $f_{*} : \operatorname{Shv}(X) \to \operatorname{Shv}(Y)$ is defined by $\Gamma(U ; f_{*} F) = \Gamma(f^{-1}(U) ; F)$ for any $U \subseteq Y$. By the previous corollary, one may characterize $f^{*} : \operatorname{Shv}(Y) \to \operatorname{Shv}(X)$ as the essentially unique $\operatorname{Shv}(X)$-valued cosheaf on $Y$ with the property that $f^{*}(y(U)) = y(f^{-1}(U))$.

(ii) Let $\text{Top}$ be the 1-category of topological spaces, and $\operatorname{Kan} \hookrightarrow \text{sSet}$ be the full subcategory of all simplicial sets consisting of Kan complexes. Recall that there is a functor $\text{Top} \to \text{Kan}$ defined by assigning to each topological space $X$ its singular complex, i.e. the simplicial set defined by $n \mapsto \operatorname{Hom}_{\text{Top}}(\Delta^{n}, X)$, where $\Delta^{n}$ is the standard $n$-simplex. Recall also that, by [Cis19, Theorem 7.8.9], there is a functor $\text{sSet} \to \text{sSet}$ which identifies
as a localization of $sSet$ at the class of weak homotopy equivalences. We define $\Sing: \Top \to S$ as the composition of the two functors defined above. It is proven in [Lur17, A.3] that, for any topological space $X$, the restriction of $\Sing$ to $\mathcal{U}(X)$ is indeed a cosheaf: this may be regarded as a non-truncated version of the classical Seifert-Van Kampen theorem. Furthermore, one can also show that $\Sing$ is a hypercomplete cosheaf (see [Lur17, Lemma A.3.10]): this means that, as cocontinuous functor $\Shv(X) \to S$, $\Sing$ factors through the hypercompletion of $\Shv(X)$.

Consider now the $\infty$-category $\Shv(X; C)$, where $X$ is any $\infty$-site and $C$ is complete and cocomplete. It is natural to ask oneself whether at this level of generality one is still able to obtain a result like Corollary 2.3, at least when the inclusion $\Shv(X; C) \to \Fun(X^{op}, C)$ admits a left adjoint. In the rest of the section we will briefly outline the reason why the answer to this question doesn’t seem to be affirmative. We start with a general proposition concerning left Bousfield localizations and categories of local objects.

**Proposition 2.3.** Let $\mathcal{C}$ be an $\infty$-category and $S$ a class of morphisms in $\mathcal{C}$. Denote by $\mathcal{C}_S$ the full subcategory of $\mathcal{C}$ spanned by $S$-local objects, and assume that the inclusion $i: \mathcal{C}_S \to \mathcal{C}$ admits a left adjoint $L$. Thus, composition with $L$ gives a fully faithful functor

$$\Fun(\mathcal{C}_S, \mathcal{D}) \hookrightarrow \Fun(\mathcal{C}, \mathcal{D}) \quad (2.7)$$

whose essential image is given by left adjoints $\mathcal{C} \to \mathcal{D}$ sending all morphisms in $S$ to equivalences.

**Proof.** Let $W$ be the class of morphisms in $\mathcal{C}$ which are sent by $L$ to equivalences and denote by $\mathcal{A}$ and $\mathcal{A}'$ the full subcategories of $\Fun(\mathcal{C}, \mathcal{D})$ spanned respectively by left adjoints $\mathcal{C} \to \mathcal{D}$ sending all morphisms in $W$ to equivalences and left adjoints $\mathcal{C} \to \mathcal{D}$ sending all morphisms in $S$ to equivalences. By [Cis19, Proposition 7.1.18], we already know that (2.7) is fully faithful and that its essential image is given by $\mathcal{A}$. It follows immediately by the definition of a local object that $L$ sends all morphisms in $S$ to equivalences, thus we just need to show that $\mathcal{A}'$ is contained in $\mathcal{A}$.

Consider a functor $F: \mathcal{C} \to \mathcal{D}$ in $\mathcal{A}'$ with right adjoint $G: \mathcal{D} \to \mathcal{C}$. By definition of $\mathcal{A}'$, we have that for every morphisms in $f \in S$ and every $d \in \mathcal{D}$

\[
\begin{array}{ccc}
\Hom_{\mathcal{C}}(c, G(d)) & \xrightarrow{f} & \Hom_{\mathcal{C}}(c', G(d)) \\
\downarrow \simeq & & \downarrow \simeq \\
\Hom_{\mathcal{D}}(F(c), d) & \xrightarrow{F(f)} & \Hom_{\mathcal{D}}(F(c'), d).
\end{array}
\]

Thus $G(d)$ is $S$-local, and hence there exists a functor $G': \mathcal{D} \to \mathcal{C}_S$ such that $G = iG'$. Let now $f$ be a morphism in $W$. By definition of $W$ we have, functorially on $d \in \mathcal{D}$,

\[
\begin{array}{ccc}
\Hom_{\mathcal{C}}(c, iG'(d)) & \xrightarrow{f} & \Hom_{\mathcal{C}}(c', iG'(d)) \\
\downarrow \simeq & & \downarrow \simeq \\
\Hom_{\mathcal{C}_S}(L(c), G'(d)) & \xrightarrow{L(f)} & \Hom_{\mathcal{C}_S}(L(c'), G'(d)),
\end{array}
\]

and hence $F(f)$ is invertible, and so we may conclude. \qed
We now claim that there exists a class of morphisms $S$ of $\text{Fun}(X^{op}, \mathcal{C})$ such that $\text{Shv}(X; \mathcal{C})$ can be identified with the full subcategory of $S$-local objects of $\text{Fun}(X^{op}, \mathcal{C})$. We define $S$ as the class of morphisms

$$S = \{ R \otimes M \to y(x) \otimes M \mid R \hookrightarrow y(x) \text{ is a sieve}, M \in \mathcal{C} \}.$$ 

For any sieve $R \hookrightarrow y(x)$, $M \in \mathcal{C}$ and $F \in \text{Fun}(X^{op}, \mathcal{C})$, since $R \simeq \varprojlim_{y(x') \to R} y(x')$, we have a commutative diagram

$$\begin{array}{c}
\text{Hom}_c(-, F(x)) \\
\downarrow \simeq
\end{array} \quad \begin{array}{c}
\text{Hom}_c(-, \varprojlim_{y(x') \to R} F(x')) \\
\downarrow \simeq
\end{array} \quad \begin{array}{c}
\text{Hom}_{\text{Fun}(X^{op}, \mathcal{C})}((y(x) \otimes -, F) \\
i \text{Hom}_{\text{Fun}(X^{op}, \mathcal{C})}(R \otimes -, F)
\end{array}$$

where the upper horizontal arrow is induced by the canonical map $F(x) \to \varprojlim_{y(x') \to R} F(x')$. Thus, we see that the upper horizontal arrow is invertible if and only if the lower horizontal one is, and so $F$ is a sheaf if and only if it is $S$-local. In particular, by the previous proposition, whenever the inclusion $\text{Shv}(X; \mathcal{C}) \hookrightarrow \text{Fun}(X^{op}, \mathcal{C})$ admits a left adjoint, the $\infty$-category $\text{Shv}(X; \mathcal{C})$ is characterized by the universal property

$$\text{LFun}(\text{Shv}(X; \mathcal{C}), \mathcal{D}) \hookrightarrow \text{LFun}_S(\text{Fun}(X^{op}, \mathcal{C}), \mathcal{D})$$

where the right-hand side denotes the $\infty$-category of left adjoint functors sending all morphisms in $S$ to equivalence. On the other hand, tensoring the usual sheafification $\text{Fun}(X^{op}, S) \to \text{Shv}(X)$ with $\mathcal{C}$ gives a colimit preserving functor

$$L' : \text{Fun}(X^{op}, \mathcal{C}) \simeq \text{Fun}(X^{op}, S) \otimes \mathcal{C} \to \text{Shv}(X) \otimes \mathcal{C}$$

. Combining the universal property of the tensor product of cocomplete categories and Theorem 2.1, we see that, for any cocomplete $\infty$-category $\mathcal{D}$, precomposition with $L'$ may be factored as

$$\text{Fun}(\text{Shv}(X) \otimes \mathcal{C}, \mathcal{D}) \simeq \text{Fun}_i(\mathcal{C}, \text{Fun}_i(\text{Shv}(X), \mathcal{D}))$$

$$\hookrightarrow \text{Fun}_i(\mathcal{C}, \text{Fun}_i(\text{Fun}(X^{op}, S), \mathcal{D}))$$

$$\simeq \text{Fun}_i(\text{Fun}(X^{op}, \mathcal{C}), \mathcal{D})$$

and hence identifies $\text{Fun}_i(\text{Shv}(X) \otimes \mathcal{C}, \mathcal{D})$ with the full subcategory of $\text{Fun}_i(\text{Fun}(X^{op}, \mathcal{C}), \mathcal{D})$ spanned by those functors sending maps in $S$ to equivalences. Hence we obtain a comparison functor

$$\text{Shv}(X) \otimes \mathcal{C} \to \text{Shv}(X; \mathcal{C})$$

but unless $\mathcal{C}$ is presentable, there is no evident reason why one should expect this to be an equivalence.

**Remark 2.9.** A close inspection of the proof of [Lur09, Proposition 6.2.2.7] shows that the usual formula for sheafification provides the desired left adjoint whenever $\mathcal{C}$ is bicomplete and, for every $x \in X$ and every sieve $R \hookrightarrow y(x)$, the functor

$$\begin{array}{c}
\text{Fun}(X^{op}, \mathcal{C}) \\
\longrightarrow
\end{array} \quad \begin{array}{c}
\mathcal{C}
\end{array}$$

$$F \mapsto (s_* F)(x, R) \simeq \varprojlim_{y(x') \to R} F(x')$$
is accessible: this will be true automatically for example when $C$ is presentable, since any functor between presentable $\infty$-categories which is a right adjoint is automatically accessible. A similar observation in the case of sheaves with values in ordinary 1-categories can be found in [KS06, 17.4]. However, if we drop the presentability assumption for $C$, it is not clear a priori why the inclusion $\text{Shv}(X; C) \hookrightarrow \text{Fun}(X^{op}, C)$ should admit a left adjoint.

**Remark 2.10.** Suppose that $C$ is such that $\text{Shv}(X; C) \hookrightarrow \text{Fun}(X^{op}, C)$ admits a left adjoint $L$. Hence, for any $x \in X$ an $M \in C$, by applying $L$ to $x \otimes M$ gives an object denoted by $M_x$ with the property that, for any other sheaf $F$, we have a functorial identification

$$\text{Hom}(M_x, F) \simeq \text{Hom}(M, F(x)).$$

It follows by [Cis19, Proposition 7.1.18] and Theorem [21] that, for any cocomplete $\infty$-category $D$, we have a fully faithful functor

$$\text{Fun}(\text{Shv}(X; C), D) \hookrightarrow \text{Fun}(C, \text{Fun}(A, D))$$

and thus any cocontinuous functor with domain $\text{Shv}(X; C)$ is uniquely determined by its values on objects of the type $M_x$.

## 3 Shape theory and shape submersions

In this section we will deal with questions related to shape theory from the perspective of higher topos theory: we recommend [Lur17, Appendix A] and [Hoy18] for some good introductory accounts to this subject. We will start by defining a version of shape which is relative to a geometric morphism, and give a detailed description of its functoriality as well as a proof of its homotopy invariance. After that we will define essential and locally contractible geometric morphisms: the first notion refers to morphisms $f : X \to Y$ whose relative shape is constant (as a pro-object on $Y$) locally on $X$, while the second to essential geometric morphisms satisfying an additional push-pull formula. After that we will define shape submersions, i.e. continuous maps which are locally given by projections $X \times Y \to Y$, where $X$ is such that the unique geometric morphism $\text{Shv}(X) \to S$ is essential. These are proven to satisfy a base change formula, which will imply that they induce locally contractible geometric morphisms.

### 3.1 Relative shape

For any $\infty$-category $C$, denote by $\text{Pro}(C)$ the $\infty$-category of pro-objects in $C$, i.e. the free completion of $C$ under cofiltered limits. When $C$ is accessible and admits finite limits, one shows ([Lur09, Proposition 3.1.6]) that $\text{Pro}(C)$ is in fact equivalent to the full subcategory of $\text{Fun}(C, S)_{op}$ spanned by the left exact functors.

Let $F : C \to D$ be an accessible functor between presentable $\infty$-categories, and let $G : D \to \text{Fun}(C, S)_{op}$ be the composition of the Yoneda embedding $D \to \text{Fun}(C, S)_{op}$ with $F^* : \text{Fun}(D, S)_{op} \to \text{Fun}(C, S)_{op}$. By the adjoint functor theorem, $G$ factors through $C$ if and only if $F$ commutes with limits, and when this condition is verified $G$ is a left adjoint to $F$. If $F$ is only exact, by the characterization stated above $G$ factors through $\text{Pro}(C)$: in this case, we say that $G$ is the left pro-adjoint of $F$. Notice that, in this situation, $G$ is a genuine left adjoint of the functor $\text{Pro}(F) : \text{Pro}(C) \to \text{Pro}(D)$.

Specializing to the case of a geometric morphism between $\infty$-topoi $f : X \to Y$, we see that the pullback $f^*$ admits a pro-left adjoint, that we will denote by $f_! : X \to \text{Pro}(Y)$. More explicitly, for every object $U \in X$, $f_!(U)$ is the pro-object on $Y$ defined by the assignment

$$V \mapsto \text{Hom}_X(U, f^*(V)).$$
Definition 3.1. Let $\mathcal{X}$ be an $\infty$-topos, $f : Y \to \mathcal{X}$ a geometric morphism. We define the shape of $Y$ relative to $\mathcal{X}$ as
\[ \Pi^\infty_{\infty}(Y) := f^! 1_Y, \]
where $1_Y$ is a terminal object of $Y$. We will say that $f$ is constant shape if $\Pi^\infty_{\infty}(Y)$ belongs to $Y$. In the case where $f$ is the unique geometric morphism $a : Y \to S$, $a^! 1_Y$ will be denoted just by $\Pi_{\infty}(Y)$ and will be called the shape or fundamental pro-$\infty$-groupoid of $Y$.

Remark 3.1. Notice that, as a left exact functor $\mathcal{X} \to S$, $f^! 1_Y$ can be identified with the functor $a_* f_* f^*$, where $a : \mathcal{X} \to S$ is the unique geometric morphism.

Proposition 3.1. There exists a functor
\[ \Pi^\infty_{\infty} : \text{Top}_{/\mathcal{X}} \to \text{Pro}(\mathcal{X}) \]
whose values on objects coincides with the shape relative to $\mathcal{X}$ and whose values on morphisms $Y \to Y'$ is given by the transformation
\[ a_* f_* f^* \to a_* f_* g_* g^* f^* \simeq a_* f_* f^* \]
induced by the unit of the adjunction $g^* \dashv g_*$. We will proceed through some reduction steps.

Proof. Since we clearly have a functor
\[ \text{Fun}^{\text{lex}}(\mathcal{X}, \mathcal{X}) \to \text{Fun}^{\text{lex}}(\mathcal{X}, S) \simeq \text{Pro}(\mathcal{X})^{op} \]
given by post composition with $a_*$, it suffices to prove that there is a functor
\[ T : \Pi^\infty_{\infty} : (\text{Top}_{/\mathcal{X}})^{op} \to \text{Fun}^{\text{lex}}(\mathcal{X}, \mathcal{X}) \]
that assigns $f_* f^*$ to any $f : Y \to \mathcal{X}$ and at the level of morphisms $Y \to Y'$ is given by the transformation
\[ f'_* f'^* \to f'_* g_* g^* f'^* \simeq f_* f^* \]
induced by the unit of the adjunction $g^* \dashv g_*$. We will proceed through some reduction steps.

First of all, the Yoneda embedding induces a fully faithful functor
\[ \text{Fun}(\mathcal{X}, \mathcal{X}) \hookrightarrow \text{Fun}(\mathcal{X}, \text{Fun}(\mathcal{X}^{op}, S)) \simeq \text{Fun}(\mathcal{X}^{op} \times \mathcal{X}, S) \]
and thus it suffices to construct a functor
\[ (\text{Top}_{/\mathcal{X}})^{op} \to \text{Fun}(\mathcal{X}^{op} \times \mathcal{X}, S) \]
whose image lies in $\text{Fun}^{\text{lex}}(\mathcal{X}, \mathcal{X})$. By [Cis19, Remark 6.1.5], standard computations with adjunctions of 1-categories show that we also have, functorially on $F, G \in \mathcal{X}$, a commutative square
\[ \begin{array}{ccc}
\text{Hom}_Y(f^* F, f^* G) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{X}}(F, f_* f^* G) \\
g^* \downarrow & & \downarrow \text{unit} \\
\text{Hom}_{Y'}(g^* f^* F, g^* f^* G) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{X}}(F, f_* g_* g^* f^* G),
\end{array} \]

16
so it suffices to show that the transformation on the left hand side can be enriched to a
functor \((\text{Top}_\mathcal{X})^{\text{op}} \to \text{Fun}(\mathcal{X} \times \mathcal{X}^{\text{op}}, \mathcal{S})\). Recall also that we have a forgetful functor
\[
(\text{Top}_\mathcal{X})^{\text{op}} \simeq (\text{Top}^{\text{op}})_\mathcal{X} \quad \xrightarrow{\mathfrak{C}^\mathcal{X}/} \quad \mathfrak{Cat}^\infty_{\mathcal{X}/}
\]
\[
(f : \mathcal{Y} \to \mathcal{X}) \quad \xmapsto{\quad (f^* : \mathcal{X} \to \mathcal{Y}).}
\]
Hence, we will construct a functor
\[
\mathfrak{Cat}^\infty_{\mathcal{X}/} \to \text{Fun}(\mathcal{X}^{\text{op}} \times \mathcal{X}, \mathcal{S}).
\]
Let \(\mathfrak{Cat}^\infty_{\mathcal{X}/}\) be the full subcategory of \(\mathfrak{Set}\) spanned by \(\infty\)-categories, and let \(\text{LFib}(\mathcal{X}^{\text{op}} \times \mathcal{X})\) be the full subcategory of \(\mathfrak{Set}_{/\mathcal{X}^{\text{op}} \times \mathcal{X}}\) spanned by the left fibrations. Since \(\mathfrak{Cat}^\infty_{\mathcal{X}/}\) is the category of fibrant objects in the Joyal model structure on \(\mathfrak{Set}\), it follows by [Cis19, Theorem 7.5.18], [Cis19, Example 7.10.14] and [Cis19, Theorem 3.9.7] that one may regard \(\mathfrak{Cat}^\infty_{\mathcal{X}/}\) as a localization of \(\mathfrak{Cat}^\infty_{\mathcal{X}/}\) by the class \(\mathcal{W}\) of fully faithful and essentially surjective functors. Thus, by [Cis19, Corollary 7.6.13] and [Cis19, Proposition 7.1.7], we get an equivalence \(\mathfrak{Cat}^\infty_{\mathcal{X}/} \simeq \mathfrak{Cat}^\infty_{\mathcal{X}/}[\mathcal{W}^{-1}]\). On the other hand, \(\text{LFib}(\mathcal{X}^{\text{op}} \times \mathcal{X})\) is the category of fibrant objects in the covariant model structure on \(\mathfrak{Set}_{/\mathcal{X}^{\text{op}} \times \mathcal{X}}\), and so by [Cis19, Theorem 7.5.18], [Cis19, Theorem 7.8.9] and [Cis19, Theorem 4.4.14] we may regard \(\text{Fun}(\mathcal{X}^{\text{op}} \times \mathcal{X}, \mathcal{S})\) as the localization of \(\text{LFib}(\mathcal{X}^{\text{op}} \times \mathcal{X})\) by the class \(\mathcal{W}'\) of fibrewise equivalences. Thus, to produce \(T\) it will suffice to provide a functor \(\mathfrak{Cat}^\infty_{\mathcal{X}/} \to \text{LFib}(\mathcal{X}^{\text{op}} \times \mathcal{X})\) of 1-categories which maps \(\mathcal{W}\) into \(\mathcal{W}'\).

Recall that, for any \(\infty\)-category \(\mathcal{C}\), we have a left fibration \(\mathcal{S}(\mathcal{C}) \xrightarrow{(s,t)} \mathcal{C}^{\text{op}} \times \mathcal{C}\), called the \textit{twisted diagonal}, classifying the hom-bifunctor \(\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{S}\) (as it is defined in [Cis19, 5.6.1]). For any functor \(f : \mathcal{X} \to \mathcal{Y}\), consider the left fibration defined by the pullback
\[
\begin{array}{ccc}
T(f) & \xrightarrow{\mathcal{S}(g)} & \mathcal{S}(\mathcal{Y}) \\
\downarrow & & \downarrow \\
\mathcal{X}^{\text{op}} \times \mathcal{X} & \xrightarrow{f^{\text{op}} \times f} & \mathcal{Y}^{\text{op}} \times \mathcal{Y}
\end{array}
\]
which classifies the functor
\[
\text{Hom}_\mathcal{X}(f(-), f(-)) : \mathcal{X}^{\text{op}} \times \mathcal{X} \to \mathcal{S}.
\]
The functoriality on \(\mathcal{Y}\) of the twisted diagonal and the universal property of pullbacks imply that \(T\) defines a functor \(\mathfrak{Cat}^\infty_{\mathcal{X}/} \to \text{LFib}(\mathcal{X}^{\text{op}} \times \mathcal{X})\), as illustrated by the diagram corresponding to a morphism \(\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{\mathcal{g}} & \mathcal{Y}'
\end{array}\)

Moreover, [Cis19, Corollary 5.6.6] implies that \(T\) sends any fully faithful functor to a fibrewise equivalence, and hence we can conclude.
Recall that, for two maps \( Y \xrightarrow{f_0} Y' \) over a topological space \( X \), we say that \( f_0 \) is homotopic to \( f_1 \) over \( X \) if there exists a map \( h : Y \times I \to Y' \) over \( X \) such that \( f_t = hi_t \) \( t = 0, 1 \), where \( i_t : Y \hookrightarrow Y \times I \) is the inclusion corresponding to \( t \in I \).

**Corollary 3.1 (Homotopy invariance).** Let \( Y, Y' \) be two topological spaces over \( X \), and let \( Y \xrightarrow{f_0} Y' \) \( f_1 \) be two homotopic maps over \( X \). Then the functor \( T \) induces an equivalence \( T(f_0) \simeq T(f_1) \). In particular, \( T \) sends homotopy equivalences over \( X \) to invertible morphisms in \( \text{Fun}(X, X)^{\text{op}} \).

**Proof.** Let \( p : Y \times I \to Y \) be the canonical projection. By [Lur17, Lemma A.2.9], we know that \( p^* \) is fully faithful, and hence \( T(p) \) is invertible. Since \( pi_0 = pi_1 = \text{id}_Y \) and \( T \) is functorial, we get an equivalence \( T(i_0) \simeq T(i_1) \). Thus, since there exists a homotopy \( h \) over \( X \) such that \( f_t = hi_t \) \( t = 0, 1 \), the functoriality of \( T \) gives the desired \( T(f_0) \simeq T(f_1) \).

**Remark 3.2.** Recall that, for any \( \infty \)-topos \( \mathcal{X} \), there is a fully faithful functor
\[
\mathcal{X} \hookrightarrow \mathcal{T}_{\text{Top}/X}
\]
(3.1)

Since \( \mathcal{T}_{\text{Top}/X} \) has small cofiltered limits, the latter can be extended to a functor
\[
\beta : \text{Pro}(\mathcal{X}) \to \mathcal{T}_{\text{Top}/X}.
\]

It is possible to construct the functor
\[
\Pi^X_\infty : \mathcal{T}_{\text{Top}/X} \to \text{Pro}(\mathcal{X})
\]

directly by showing that there is an equivalence
\[
\text{Hom}_{\text{Pro}(\mathcal{X})}(\Pi^X_\infty(y), Z) \simeq \text{Hom}_{\mathcal{T}_{\text{Top}/X}}(y, \beta(Z))
\]

which is functorial on \( Z \in \text{Pro}(\mathcal{X}) \), as it is done in [Lur16, Proposition E.2.2.1]. However, we preferred to prove directly the functoriality of the relative shape, namely because the approach mentioned above leaves unclear how the functor \( \Pi^X_\infty \) would behave at the level of morphisms, which is needed to have a proof of homotopy invariance as clean and immediate as the one above.

**Remark 3.3.** As usual, since \( \text{Top} \) has pullbacks, the slice \( \mathcal{T}_{\text{Top}/X} \) can be equipped with a contravariantly functorial structure
\[
\mathcal{T}_{\text{Top}}^{\text{op}} \to \mathcal{C}_{\text{at}_\infty}
\]

that can be described for any geometric morphism \( g : \mathcal{X} \to \mathcal{Y} \) by sending an object \( (f : y' \to y) \in \mathcal{T}_{\text{Top}}/\mathcal{Y} \) to the resulting arrow over \( \mathcal{X} \) obtained by performing the pullback of \( f \) along \( g \). Thus, since we have for any \( y \in \mathcal{Y} \) a canonical pullback square
\[
\begin{array}{ccc}
\mathcal{X}_{/f \cdot y} & \longrightarrow & \mathcal{Y}/y \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]
in \(\mathcal{T}_{\text{op}}\), we see that the functor (3.1) is actually natural on \(X\), where the left hand side is functorial by the usual forgetful \(\mathcal{T}_{\text{op}}^{op} \to \mathbf{Cat}_{\infty}\) sending a geometric morphism \(f\) to \(f^*\). In particular we obtain that

\[
\beta : \text{Pro}(X) \to \mathcal{T}_{\text{op}}/X
\]

is actually natural on \(X\). Notice that, if one regards \(\mathbf{Cat}_{\infty}\) as an \((\infty, 2)\)-category, the universal property of \(\text{Pro}(X)\) and the definition of the slice imply that \(\beta\) can be seen as a natural transformation between 2-functors. Hence, by [Hau20, Theorem 3.22], by adjunction we may regard the relative shape as a lax natural transformation, where the 2-cells involved may be described as follows: any geometric morphism induces an adjunction

\[
\text{Pro}(X) \xrightarrow{g} \text{Pro}(Y)
\]

and for any commutative square of topoi

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{g'} & \mathcal{Y}' \\
\downarrow{f'} & & \downarrow{f} \\
\mathcal{X} & \xrightarrow{g} & \mathcal{Y}
\end{array}
\]

applying \(g'^*\) to the unit of the adjunction \(f'_2 \dashv f^*\) induces a natural transformation

\[
g'^* \to g'^* f'^* f_2 \simeq f'^* g^* f_2
\]

and hence by transposition

\[
f'_2 g'^* \to g^* f_2
\]

called the base change transformation, which when evaluated at \(1_{\mathcal{Y}'}\) gives the desired

\[
f'_2 g'^* 1_{\mathcal{Y}'} \simeq f'_2 1_{\mathcal{X}''} \to g^* f_2 1_{\mathcal{Y}'}.
\]

### 3.2 Locally contractible geometric morphisms

We start by recalling the definition of a locally cartesian closed \(\infty\)-category.

**Definition 3.2.** An \(\infty\)-category \(\mathcal{C}\) is cartesian closed if it admits finite products and, for any object \(c \in \mathcal{C}\), the functor \(- \times c : \mathcal{C} \to \mathcal{C}\) admits a right adjoint.

An \(\infty\)-category \(\mathcal{C}\) is locally cartesian closed if it has pullbacks and, for any object \(c \in \mathcal{C}\), the slice \(\mathcal{C}_{/c}\) is cartesian closed, or equivalently, if for any arrow \(f : c \to d\) in \(\mathcal{C}\), the functor \(\mathcal{C}_{/d} \to \mathcal{C}_{/c}\) given by pulling back along \(f\) admits a right adjoint called the dependent product along \(f\) and denoted by \(\prod_f : \mathcal{C}_{/c} \to \mathcal{C}_{/d}\). A functor \(F : \mathcal{C} \to \mathcal{D}\) between locally cartesian \(\infty\)-categories if locally cartesian closed if \(F\) commutes with pullbacks and dependent products, i.e. for any arrow \(f : c \to d\) in \(\mathcal{C}\), we have a commutative square

\[
\begin{array}{ccc}
\mathcal{C}_{/c} & \xrightarrow{\prod_f} & \mathcal{C}_{/d} \\
F \downarrow & & \downarrow{F} \\
\mathcal{D}_{/Fc} & \xrightarrow{\prod_{Ff}} & \mathcal{D}_{/Fd}
\end{array}
\]

We will denote \(\mathbf{Cat}_{\infty}^{lcc}\) the subcategory of \(\mathbf{Cat}_{\infty}\) whose objects are locally cartesian closed \(\infty\)-categories with locally cartesian closed functors between them.
Example 3.1. By universality of colimits and adjoint functor theorem, any ∞-topos is locally cartesian closed.

Let $F \to G$ be a morphism in Pro($\mathcal{Y}$) and let $H \to f^*G$ be a morphism in Pro($\mathcal{X}$). Then we have a canonical commutative square

\[
\begin{array}{ccc}
  f_\sharp(f^*F \times_{f^*G} H) & \longrightarrow & f_\sharp H \\
  \downarrow & & \downarrow \\
  f_\sharp f^* F & \longrightarrow & f_\sharp f^* G \\
  \downarrow & & \downarrow \\
  F & \longrightarrow & G
\end{array}
\]

which determines a unique morphism

\[
f_\sharp(f^*F \times_{f^*G} H) \to F \times_G f_\sharp H
\]

called the projection morphism.

Proposition 3.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism between ∞-topoi. Consider the conditions

(i) $f^*$ admits a left adjoint and, for every $F \to G$ in $\mathcal{Y}$ and $H \to f^*G$ in $\mathcal{X}$, the associated projection morphism is invertible;

(ii) $f^*$ is locally cartesian closed;

(iii) $f^*$ admits a left adjoint and, for every $F$ in $\mathcal{Y}$ and $H$ in $\mathcal{X}$, the associated projection morphism

\[
f_\sharp(f^*F \times H) \to F \times f_\sharp H
\]

is invertible.

Then (i) and (ii) are equivalent. Moreover, if $f$ is induced by a continuous map between topological spaces, then these are also equivalent to (iii).

Proof. We first show that (i) implies (ii). Since $f^*$ commutes with finite limits, it suffices to show that it commutes with dependent products. But, for any $\alpha : F \to G$ in $\mathcal{Y}$, the square

\[
\begin{array}{ccc}
  \mathcal{Y}/F & \longrightarrow & \mathcal{Y}/G \\
  f^\ast \downarrow & & \downarrow f^\ast \\
  \mathcal{X}_{/f^\ast F} & \longrightarrow & \mathcal{X}_{/f^\ast G}
\end{array}
\]

commutes if and only if the square given by the corresponding left adjoints commutes. This last assertion is equivalent to requiring the projection morphisms to be invertible, and so we are done.

We now show that (ii) implies (i). By the same argument as above, it suffices to prove that $f^*$ admits a left adjoint. Since $f^*$ is cocontinuous and preserves finite limits, we are only left to prove that $f^*$ commutes with infinite products: we will do this by exhibiting products (more generally, limits indexed by small ∞-groupoids) in any ∞-topos as a special case of dependent products, so that the result will follow by assumption (ii). Let $\mathcal{X}$ be an ∞-topos,
π : X → S the unique geometric morphism. First of all, we observe that the cocontinuos functors

\[ S \to \text{Cat}_\infty^{op} \quad \text{and} \quad A \to \text{Fun}(A, X) \]

are naturally equivalent: the functor

\[ \frac{X/\pi_* A \times A}{(F \to \pi^* A, a : \Delta^0 \to A) \to F_a} \]

where \( F_a \) is defined as the pullback

\[ \begin{array}{ccc}
F_a & \to & F \\
\downarrow & & \downarrow \\
1_X & \to & \pi^* A \\
\end{array} \]

induces the desired natural transformation, which is obviously an equivalence when A is contractible. In particular, if \( \alpha : A \to \Delta^0 \) is the unique map, we have a corresponding commutative square

\[ \frac{X/\pi^* A \times \Delta^0}{\text{Fun}(\Delta^0, X) \to \text{Fun}(A, X)} \]

where the lower horizontal arrow assigns to an object \( F \) of \( X \) the constant functor at \( F \). Thus we obtain an identification of the respective right adjoints, i.e. a commutative square

\[ \frac{X/\pi^* A \times \Delta^0}{\text{Fun}(A, X) \to \text{Fun}(\Delta^0, X)} \]

which is what we wanted.

Assume now that we have an essential geometric morphism \( f : X = \text{Shv}(X) \to Y = \text{Shv}(Y) \) induced by a continuous map \( f : X \to Y \). Clearly (iii) is a special case of (i). Assume then that \( f \) satisfies the hypothesis (iii). Let \( \alpha : F \to G \) be a morphism in \( Y \) and let \( H \to f^* G \) be a morphism in \( X \). Since the projection morphism is a natural transformation between colimit preserving functors and since for any \( H \in X/\pi^* G \) we have an equivalence \( H \simeq \lim_{y(U) \to G} (H \times f^* G y(f^{-1}U)) \), we may assume that \( H \to f^* G \) factors as \( H \to y(f^{-1}U) \to f^* G \) for some open \( U \in \text{U}(Y) \), and hence by the pasting properties of pullbacks we may also assume that \( G = y(U) \). Notice that, for any \(( -1 )\)-truncated object \( V \) in a topos \( Z \) and for any other two objects \( A, B \in Z/V \), we have an identification \( A \times_V B \simeq A \times B \) : this follows because for any other object \( C \) mapping both to \( F \) and \( G \), we have that \( \text{Hom}_Z(C, V) \) is contractible, and thus \( \text{Hom}_Z(C, A \times_V B) \simeq \text{Hom}_Z(C, A \times B) \). Thus, since both \( y(U) \) and \( y(f^{-1}U) \) are \(( -1 )\)-truncated, we are only left to prove that

\[ f_{1}(f^* F \times H) \to F \times f_{2}H \]

is invertible, which is true by assumption. □
**Definition 3.3.** Let \( f : \mathcal{X} \to \mathcal{Y} \) be a geometric morphism of \( \infty \)-topoi. We say that \( f \) is **essential** if \( f_* \) factors through \( \mathcal{Y} \), or equivalently if \( f^* \) admits a left adjoint. Furthermore, we say that an essential geometric morphism is **locally contractible** if it satisfies the equivalent conditions (i) and (ii) in Proposition 3.2. We say that a geometric morphism is of **trivial shape** if \( f_* \) is fully faithful, or equivalently if the unit transformation \( \text{id}_\mathcal{Y} \to f_* f^* \) is an equivalence. When \( f \) is the unique geometric morphism \( \mathcal{X} \to \mathcal{S} \), we will say that \( \mathcal{X} \) is locally contractible.

**Example 3.2.** (i) Recall that any object \( U \in \mathcal{X} \) of an \( \infty \)-topos determines a geometric morphism \( j : \mathcal{X}/U \to \mathcal{X} \). By [Lur09, Proposition 6.3.5.1] \( j \) is locally contractible, and \( j^* : \mathcal{X}/U \to \mathcal{X} \) can be described as the usual forgetful functor.

(ii) By [Lur17, Proposition A.1.9], an \( \infty \)-topos is locally contractible if and only if the unique geometric morphism \( \mathcal{X} \to \mathcal{S} \) is essential, since in this case the projection morphism is automatically invertible.

(iii) Let \( f : \mathcal{X} \to \mathcal{Y} \) be an essential geometric morphism. For any the \( \infty \)-topos \( \mathcal{Z} \), by Remark 2.1 we obtain a geometric morphism \( f \otimes \mathcal{Z} : \mathcal{X} \otimes \mathcal{Z} \to \mathcal{Y} \otimes \mathcal{Z} \) by applying to \( f \) the functor \( - \otimes \mathcal{Z} \). Since both \( f_* \) and \( f^* \) commute with colimits, the adjunction \( f^* \dashv f_* \) is preserved by \( - \otimes \mathcal{Z} \), and so \( f \otimes \text{id}_\mathcal{Z} \) is an essential geometric morphism.

**Remark 3.4.** Let \( f : \mathcal{X} \to \mathcal{Y} \) be a continuous map. Notice that, since the functor

\[
 f^{-1} : \mathcal{U}(\mathcal{Y}) \to \mathcal{U}(\mathcal{X})
\]

preserves open coverings, for any complete \( \infty \)-category \( \mathcal{C} \) we still have a well defined pushforward \( f_* : \text{Shv}(\mathcal{X}; \mathcal{C}) \to \text{Shv}(\mathcal{Y}; \mathcal{C}) \) given as usual by \( \Gamma(U; f_* F) = \Gamma(f^{-1}(U); F) \) for all \( U \in \mathcal{U}(\mathcal{Y}) \). Although at this level of generality there is no reason to expect \( f_* \) to have a left adjoint, if \( f \) induces an essential geometric morphism at the level of sheaves of spaces, then it actually does. Indeed, recall that there is an equivalence

\[
\text{Shv}(\mathcal{X}; \mathcal{C}) \simeq \text{Fun}_*(\mathcal{S}hv(\mathcal{X})^{op}, \mathcal{C})
\]

. Through this equivalence and Example 2.1 (i), we can identify \( f_* : \text{Fun}_*(\mathcal{Shv}(\mathcal{X})^{op}, \mathcal{C}) \to \text{Fun}_*(\mathcal{Shv}(\mathcal{Y})^{op}, \mathcal{C}) \) with precomposition with the opposite of the pullback \( f^* : \text{Shv}(\mathcal{Y}) \to \text{Shv}(\mathcal{X}) \). Thus, similarly to Remark 2.1 by applying the 2-functor \( \text{Fun}_*(-^{op}, \mathcal{C}) \) to the adjunction between cocontinuous functors \( f_* \dashv f^* \), we obtain the desired left adjoint.

It is straightforward to check that the composition of two locally contractible geometric morphism is again locally contractible (see [AC21, Corollary 3.2.5]). We observe that the properties of being essential or locally contractible can be checked locally on the source.

**Lemma 3.1.** Let \( f : \mathcal{X} \to \mathcal{Y} \) be a geometric morphism, and let \( \mathcal{B} \subseteq \mathcal{X} \) which generates \( \mathcal{X} \) under colimits. For any object \( U \in \mathcal{B} \), consider the composite geometric morphism

\[
\mathcal{X}/U \to \mathcal{X} \to \mathcal{Y}.
\]

We have the following

(i) \( f \) is essential if and only if (3.2) is of constant shape for any \( U \in \mathcal{B} \);

(ii) \( f \) is locally contractible if and only if (3.2) is locally contractible for any \( U \in \mathcal{B} \).

**Proof.** A proof can be found in [AC21, Proposition 3.1.5] and [AC21, Proposition 3.2.6]. □
Remark 3.5. The content of part (i) in Lemma 3.1 suggests that a valid alternative way to call a geometric morphism whose pullback has a left adjoint could have been *locally of constant shape*. This is actually the approach taken by Lurie in [Lur17, Appendix A]; however, we have decided to stick with the more concise nomenclature which appears also in [Joh2] and [AC21].

Corollary 3.2. Let $X$ be a locally contractible topological space, $a : X \to \ast$ the unique map, and assume that $X = \mathcal{S}hv(X)$ is hypercomplete. Then $\mathcal{S}hv(X)$ is locally contractible and $a_\sharp$ is equivalent to the extension by colimits of the cosheaf $\text{Sing}$. Consequently, for sheaves of spectra, the functor $a_\sharp : \mathcal{S}hv(X; \mathcal{S}p) \to \mathcal{S}p$ obtained by applying $- \otimes \mathcal{S}p$ is uniquely determined by the formula $a_\sharp(S_U) = \Sigma^\infty_+ U$ for any $U \in \mathcal{U}(X)$.

Proof. Let $B$ be the poset of all contractible open subsets of $X$. Notice that, even though $B$ in general is not a sieve, by the hypercompleteness assumption of $\mathcal{S}hv(X)$ we get an equivalence

$$\lim_{U \in B} y(U) \simeq y(V)$$

for any $V \in \mathcal{U}(X)$ that can be easily checked on stalks. In particular, we see that the full subcategory $\mathcal{S}hv(B) \subseteq \mathcal{S}hv(X)$ generates $\mathcal{S}hv(X)$ under colimits. Thus by Lemma 3.1 it suffices to check that $\mathcal{S}hv(X)/U \simeq \mathcal{S}hv(U)$ is of constant shape for any $U \in B$, but this is true by homotopy invariance of the shape. The last assertion follows immediately by noticing that through the equivalence $S \otimes \mathcal{S}p \simeq \mathcal{S}p$, an object $A \otimes S$ corresponds (functorially on $A$) to $\Sigma^\infty_+ A$. □

Remark 3.6. Beware that the viceversa of Corollary 3.2 is not true. For this reason, to avoid confusion, from now on we will say that a topological space $X$ is essential if $\mathcal{S}hv(X)$ is (or equivalently, if $\mathcal{S}hv(X)$ is locally contractible by Example 3.2 part (ii)).

3.3 Shape submersions

Definition 3.4. A continuous map $f : X \to Y$ between topological spaces is a *shape submersion* if for every point $x \in X$ there exist an open neighbourhood $U$ of $x$ and a space $X'$ which is essential, such that $U$ is homeomorphic to $f(U) \times X'$ and the diagram

$$f(U) \times X' \xrightarrow{\cong} U \hookrightarrow X \quad \text{and} \quad \xrightarrow{f} \downarrow \quad Y$$

commutes, where $p$ is the obvious projection.

Example 3.3. (i) By Proposition 3.2, if $X$ is locally contractible and hypercomplete, then $X \to \ast$ is a shape submersion.

(ii) Any topological submersion of fiber dimension $n$ is a shape submersion.

Remark 3.7. (i) It follows easily from the definition that shape submersions are stable under pullbacks.

(ii) If $f : X \to Y$ is a shape submersion, then the set of open subsets of $X$ of the type $X' \times V$, where $V$ is open in $Y$ and $X'$ is locally contractible, forms a basis for the topology of $X$. Although this basis is not closed under finite intersections, the set of representable sheaves corresponding to open subsets homeomorphic to the product of
an open in $Y$ and a space locally contractible generates $\text{Shv}(X)$ under colimits. To see this, consider

$$\mathcal{B} = \{ U \in \mathcal{U}(X) \mid U \cong W, \text{with } W \in \mathcal{U}(V \times S) \text{ for some } V \in \mathcal{U}(Y), S \text{ locally contractible} \}.$$  

The set $\mathcal{B}$ clearly forms a basis closed under finite intersections, and then we have $\text{Shv}(X) \simeq \text{Shv}(\mathcal{B})$. Moreover, since open immersions are locally contractible, and since $\mathcal{U}(V) \times \mathcal{U}(S)$ forms a basis of $V \times S$ which is closed under finite intersections, we get our claim.

In the particular case of a topological submersion of fiber dimension $n$, since $\mathbb{R}^n$ is hypercomplete, we have an equivalence $\text{Shv}(\mathbb{R}^n) \simeq \text{Shv}(W)$ where $W \subseteq \mathcal{U}(\mathbb{R}^n)$ is the poset of open balls inside $\mathbb{R}^n$, and thus, since $\mathbb{R}^n$ is locally compact, we have $\text{Shv}(V \times \mathbb{R}^n) \simeq \text{Shv}(V) \otimes \text{Shv}(W)$. In particular, the set of representable sheaves corresponding to open subsets homeomorphic to the product of an open in $Y$ and an open ball in $\mathbb{R}^n$ generates $\text{Shv}(X)$ under colimits.

For technical reasons that will be justified in a moment, from now on whenever we have a shape submersion $f : X \to Y$ we will assume that either $Y$ is locally compact or all spaces of locally constant shape appearing in the basis of $X$ are locally compact.

**Lemma 3.2.** Any shape submersion $f : X \to Y$ induces an essential geometric morphism. Thus, for any presentable $\infty$-category $\mathcal{C}$, we obtain an adjunction $f^* \dashv f_*$ for $\mathcal{C}$-valued sheaves.

**Proof.** Since $f^*$ is a left adjoint, it is in particular accessible. Hence, by adjoint functor theorem, it suffices to prove that $f^*$ commutes with limits. For any functor $I \to \text{Shv}(Y)$ we have a canonical map

$$f^*(\lim_{i \in I} F_i) \to \lim_{i \in I} f^* F_i$$

and it suffices to check that this is an equivalence after restricting to any open subset in the basis of $X$ associated to $f$. Hence, since the operation of restricting a sheaf to an open subset commutes with limits, we can assume that $f$ is a projection $f : X \times Y \to Y$ where either $X$ or $Y$ is locally compact and $X$ is locally contractible. Thus, by point (iii) in Example 3.2 and Remark 2.8 we get that $f$ is essential. The last assertion follows immediately by Corollary 2.3. \qed

**Lemma 3.3** (Smooth base change). Let $\mathcal{C}$ be a presentable $\infty$-category. For every given pullback square

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow g' & & \downarrow g \\
Y' & \xrightarrow{f} & Y
\end{array}$$

of topological spaces where $g$ and $g'$ are shape submersions, there is a natural equivalence

$$f^* g_! \simeq g'_! f'^*$$

and, by transposition, also

$$g^* f_* \simeq f'_* g'^*$$

for $\mathcal{C}$-valued sheaves.
Proof. First of all, the base change transformation as defined in Remark 3.3 defines a comparison natural transformation. Since all functors appearing are colimit preserving, it suffices to check that the morphism is an equivalence only on representables. Hence, applying also Remark 3.7, we see that can assume that the pullback square is of the type

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{id_X \times f} & X \times Y \\
\downarrow g' & & \downarrow g \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

where \(X\) is locally contractible, \(g\) and \(g'\) are the canonical projections. By Example 3.2 point (iii) and Remark 2.8, we have \((id_X \times f)^* \simeq (id_X)^* \otimes f^*\), \(g_* \simeq a_1 \otimes (id_Y)^*\) and \(g'_* \simeq a_2 \otimes (id_Y')^*\), where \(a : X \to *\), and so, since \(\otimes\) is a bifunctor, we have

\[
f^* g_* \simeq f^*(a_1 \otimes (id_Y)^*) \\
\simeq (a_1 \otimes (id_Y)^*)(id_X^* \otimes f^*) \\
\simeq g'_*(id_X \times f)^*.
\]

Remark 3.8. Let \(X\) and \(Y\) be two topological spaces. The functor

\[
\begin{array}{c}
\mathcal{U}(X) \times \mathcal{U}(Y) \\
\longrightarrow
\end{array}
\begin{array}{c}
\mathcal{Shv}(X \times Y) \\
(U, V) \longleftarrow
\end{array}
\begin{array}{c}
y(U \times V)
\end{array}
\]

extends by colimits to a functor

\[
\text{Fun}(\mathcal{U}(X)^{op}, S) \times \text{Fun}(\mathcal{U}(Y)^{op}, S) \to \mathcal{Shv}(X \times Y).
\]

Since it clearly sends covering sieves to equivalences in both variables, we obtain a functor

\[
\begin{array}{c}
\mathcal{Shv}(X) \times \mathcal{Shv}(Y) \\
\longrightarrow
\end{array}
\begin{array}{c}
\mathcal{Shv}(X \times Y) \\
(F, G) \longleftarrow
\end{array}
\begin{array}{c}
F \boxtimes G
\end{array}
\]

and in case \(X\) or \(Y\) is locally compact, this gives an explicit description of the equivalence in Remark 2.8. More generally, by Corollary 2.3, tensoring with two presentable \(\infty\)-categories \(\mathcal{C}\) and \(\mathcal{D}\) gives

\[
\mathcal{Shv}(X; \mathcal{C}) \times \mathcal{Shv}(Y; \mathcal{D}) \to \mathcal{Shv}(X \times Y; \mathcal{C} \otimes \mathcal{D})
\]

for which the image of a couple \((F, G)\) in the domain will still be denoted as \(F \boxtimes G\).

Let \(\Delta : X \to X \times X\) be the diagonal. By post composing with \(\Delta^*\) we get a functor denoted by

\[
\begin{array}{c}
\mathcal{Shv}(X; \mathcal{C}) \times \mathcal{Shv}(X; \mathcal{D}) \\
\longrightarrow
\end{array}
\begin{array}{c}
\mathcal{Shv}(X; \mathcal{C} \otimes \mathcal{D}) \\
(F, G) \longleftarrow
\end{array}
\begin{array}{c}
F \boxtimes G \simeq \Delta^*(F \boxtimes G).
\end{array}
\]

Suppose now that \(\mathcal{C}\) is equipped with a monoidal structure \(\otimes\) such that the functor \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) preserves colimits in each variable. Then we have a functor cocontinuous \(\mathcal{Shv}(X; \mathcal{C} \otimes \mathcal{C}) \to \mathcal{Shv}(X; \mathcal{C})\) and by composing with (3.3) we obtain an induced monoidal structure on \(\mathcal{Shv}(X; \mathcal{C})\), which will still be denoted by \(\otimes\). It is straightforward to check that the functor

\[
\otimes : \mathcal{Shv}(X; \mathcal{C}) \times \mathcal{Shv}(X; \mathcal{C}) \to \mathcal{Shv}(X; \mathcal{C})
\]

(3.4)
can be described as follows

\[(F, G) \mapsto L^\mathcal{C}(U \mapsto \Gamma(U; F) \otimes_\mathcal{C} \Gamma(U; G))\]

where \(L^\mathcal{C}\) is the sheafification for \(\mathcal{C}\)-valued presheaves. In particular, we have that any \(F \in \text{Shv}(X; \mathcal{C})\) induces a colimit preserving functor

\[- \otimes_\mathcal{C} F : \text{Shv}(X; \mathcal{C}) \to \text{Shv}(X; \mathcal{C}).\]

Since \(\text{Shv}(X; \mathcal{C})\) is presentable, this has a right adjoint denoted by

\[\text{Hom}_X(F, -) : \text{Shv}(X; \mathcal{C}) \to \text{Shv}(X; \mathcal{C}).\]

This functor supplies \(\text{Shv}(X; \mathcal{C})\) with a self-enrichment and for this reason will be called \textit{internal Hom sheaf} functor.

**Remark 3.9.** Let \(X\) and \(Y\) be two topological spaces, \(\mathcal{C}\), \(\mathcal{D}\) and \(\mathcal{E}\) be presentable \(\infty\)-categories, and let \(\alpha : \text{Shv}(X) \to \text{Shv}(Y)\) and \(\Phi : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}\) be cocontinuous functors. Notice that the functoriality in each variable of the tensor product of cocomplete \(\infty\)-categories gives a commutative diagram

\[
\begin{array}{ccc}
\text{Shv}(X \times X; \mathcal{C} \otimes \mathcal{D}) & \xrightarrow{\Delta^*} & \text{Shv}(X; \mathcal{C} \otimes \mathcal{D}) \otimes \text{Shv}(X; \mathcal{C} \otimes \mathcal{D}) \\
\downarrow{\Delta} & & \downarrow{\text{Shv}(Y) \otimes \Phi} \\
\text{Shv}(X; \mathcal{C} \otimes \mathcal{D}) & \xrightarrow{\text{Shv}(X) \otimes \Phi} & \text{Shv}(X; \mathcal{E}) \quad \xrightarrow{\alpha \otimes \varepsilon} \quad \text{Shv}(Y; \mathcal{E}).
\end{array}
\]

In particular, the diagram above shows that whenever we prove a formula involving the functor (3.3) and operations on sheaves coming from some continuous map, then we may deduce immediately a corresponding formula for the functor (3.4).

**Corollary 3.3** (Monoidality). Let \(f : X \to Y\) be a morphism of topological spaces, \(\mathcal{C}\) and \(\mathcal{D}\) two presentable \(\infty\)-categories. Then we have a canonical identification

\[f^*(F \otimes G) \simeq f^*F \otimes f^*G\]

and in particular, when \(\mathcal{C} = \mathcal{D}\) is monoidal, by transposition

\[f_* \text{Hom}_X(f^*H, K) \simeq \text{Hom}_Y(f_*H, f_*K).\]

**Proof.** The commutativity of the diagram

\[
\begin{array}{ccc}
\text{Shv}(Y; \mathcal{C}) \times \text{Shv}(Y; \mathcal{D}) & \xrightarrow{(f^*, f^*)} & \text{Shv}(X; \mathcal{C}) \times \text{Shv}(X; \mathcal{D}) \\
\otimes & & \otimes \\
\text{Shv}(Y; \mathcal{C} \otimes \mathcal{D}) & \xrightarrow{f^*} & \text{Shv}(X; \mathcal{C} \otimes \mathcal{D})
\end{array}
\]

follows from the commutativity of

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\Delta \downarrow & & \Delta \downarrow \\
X \times X & \xrightarrow{(f, f)} & Y \times Y
\end{array}
\]

that is trivially verified. The last part follows directly from the previous lemma. \(\square\)
Corollary 3.4 (Smooth projection formula). Let \( f : X \to Y \) be a shape submersion, \( \mathcal{C} \) and \( \mathcal{D} \) two presentable \( \infty \)-categories. Then for any \( F \in \text{Shv}(X; \mathcal{C}) \) and \( G, H \in \text{Shv}(Y; \mathcal{D}) \), we have a canonical equivalence

\[
f_!(F \otimes f^*G) \simeq f_!F \otimes G
\]

and hence, by transposition, when \( \mathcal{C} = \mathcal{D} \) is monoidal, equivalences

\[
f_! \text{Hom}_X(F, f^*G) \simeq \text{Hom}_Y(f_!F, G)
\]

and

\[
f^* \text{Hom}_X(G, H) \simeq \text{Hom}_Y(f^*G, f^*H).
\]

In particular, any shape submersion induces a locally contractible geometric morphism.

Proof. Let \( \Delta_f : X \to Y \) be the graph of \( f \). We have

\[
f_!(F \otimes f^*G) \simeq f_! \Delta_f^*(F \boxtimes G)
\]

\[
\simeq \Delta^*(f \times \text{id}_Y)_!(F \boxtimes G)
\]

\[
\simeq f_! F \otimes G
\]

where the second equivalence follows by applying smooth base change to the pullback square

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_f} & X \times Y \\
\downarrow f & & \downarrow f \times \text{id}_Y \\
Y & \xrightarrow{\Delta} & Y \times Y.
\end{array}
\]

The last assertion follows by specializing (3.4) to the case when \( \mathcal{C} \) is \( S \) equipped with the cartesian monoidal structure and by the commutativity of the diagram

\[
\begin{array}{ccc}
F \times f^*G & \xrightarrow{\simeq} & f^*(f_! F \times G) \\
\downarrow \simeq & & \downarrow \simeq \\
\Delta_f^*(F \boxtimes G) & \xrightarrow{\simeq} & \Delta_f^*(f \times \text{id}_Y)^*(f \times \text{id}_Y)_!(F \boxtimes G)
\end{array}
\]

where the upper horizontal arrow is the one which transposes to the projection morphism and the lower horizontal one transposes to the smooth base change transformation.

4 Localization sequences

We will now prove a version of the localization theorem for sheaves of spaces (and of spectra) on a topological space \( X \): this essentially states that, for any closed immersion \( i : Z \to X \) with open complement \( j : U \to X \), the inclusions

\[
\text{Shv}(Z) \xleftarrow{i_*} \text{Shv}(X) \xrightarrow{j_*} \text{Shv}(U)
\]

form a recollement (in the sense of [Lur17, Definition A.8.1]). Achieving this goal in our context will be slightly more complicated than in the case of [KS90, Proposition 2.3.6], namely because we don’t want to assume all our spaces to be hypercomplete. We will follow instead the strategy outlined in [Kha19]: the main ingredient will be to show that the pushforward \( i_* : \text{Shv}(Z) \to \text{Shv}(X) \) commutes with contractible colimits, i.e. colimits indexed by contractible simplicial sets. From this we will be able to reduce to checking the theorem in the case of representable sheaves, which is almost straightforward.

We start by reporting [Kha19, Definition 3.1.5] and [Kha19, Lemma 3.1.6].
**Definition 4.1.** Let $X$ and $Y$ be essentially small $\infty$-sites, and assume that $Y$ admits an initial object $\emptyset_Y$. A functor $u : X \to Y$ is topologically quasi-cocontinuous if for every covering sieve $R' \to y(u(x))$ in $Y$, the sieve $R \to y(x)$, generated by morphisms $x' \to x$ such that either $u(x')$ is initial or $y(u(x')) \to y(u(x))$ factors through $R' \to y(u(x))$, is a covering in $X$.

**Lemma 4.1.** With notation as in the previous definition, let $u : X \to Y$ be a topologically quasi-cocontinuous functor. Assume that the initial object $\emptyset_Y$ is strict in the sense that for any object $y \in Y$, any morphism $d \to \emptyset_Y$ is invertible. Assume also that, for any object $y \in Y$, the sieve $\emptyset_{\text{Fun}(Y,S)} \to y(d)$ is a covering in $Y$ if and only if $y$ is initial (where $\emptyset_{\text{Fun}(Y,S)}$ denotes the initial object of $\text{Fun}(Y,S)$). Then the functor $\text{Shv}(Y) \to \text{Shv}(X)$, given by the assignment $F \mapsto L_X(u^*(F))$, where $L_X : \emptyset_{\text{Fun}(X,S)} \to \text{Shv}(X)$ denotes the sheafification functor, commutes with contractible colimits.

**Lemma 4.2.** Let $i : Z \hookrightarrow X$ be a closed immersion. Then $i_*$ commutes with contractible colimits.

**Proof.** By the lemma above and by unraveling the definition of topologically quasi-cocontinuous functor, this amounts to check that, for any $V \in \mathcal{U}(X)$ and any open covering $\{W_i\}_{i \in I} \subseteq \mathcal{U}(Z)$ of $V \cap Z$, the family

$$T = \{ U \subseteq V \mid U \cap Z = \emptyset \text{ or } U \cap Z \subseteq W_i \text{ for some } i \in I \} \subseteq \mathcal{U}(X)$$

covers $V$. But this is clear, because $V \setminus Z \subseteq T$ and any $W_i$ can be written as $W_i \cap Z$ for some $W_i \in \mathcal{U}(V)$.

**Corollary 4.1.** Let $\mathcal{C}$ be any pointed presentable $\infty$-category. Then the pushforward $i_*^! : \text{Shv}(Z; \mathcal{C}) \to \text{Shv}(X; \mathcal{C})$ commutes with all colimits, and thus admits a right adjoint $i_*^! : \text{Shv}(X; \mathcal{C}) \to \text{Shv}(Z; \mathcal{C})$.

**Proof.** It suffices to prove the corollary for $\mathcal{C} = S_\ast$. Note that it suffices to check that $i_*$ preserves the initial object and commutes with contractible colimits: any $F : I \to \mathcal{D}$ from a simplicial set $I$ to an $\infty$-category $\mathcal{D}$ with an initial object $\emptyset_\mathcal{D}$ may be seen as $I \to \mathcal{D}_{\emptyset_\mathcal{D}/}$ and thus corresponds to a functor $\Delta^0 \ast I \to \mathcal{D}$ with the same colimit as $F$ but indexed by weakly contractible simplicial set. But $i_*$ preserves the initial object because $\text{Shv}(Z; S_\ast)$ is pointed, and thus we may conclude by the previous lemma.

Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$. For any $F \in \text{Shv}(X)$, consider the functorial commutative square

$$
\begin{array}{ccc}
j_Z j^*(F) & \longrightarrow & F \\
\downarrow & & \downarrow \\
j_Z (j^* i_* i^*(F)) & \longrightarrow & i_* i^*(F),
\end{array}
$$

where all the morphisms are given by the obvious units and counits. Notice that for any $G \in \text{Shv}(X)$ and $V \in \mathcal{U}(U)$, we have

$$
\Gamma(V; i_* G) = \Gamma(U \cap Z; G) \simeq \ast,
$$

and so we can identify $j^* i_* : \text{Shv}(Z) \to \text{Shv}(U)$ with a constant functor with value the terminal object $y(U) \in \text{Shv}(U)$. Hence the previous square may be written as

$$
\begin{array}{ccc}
j_Z j^*(F) & \longrightarrow & F \\
\downarrow & & \downarrow \\
j_Z (y(U)) & \longrightarrow & i_* i^*(F),
\end{array}
$$

(4.1)
**Theorem 4.1.** The canonical square (4.1) is a pushout.

**Proof.** Since all functors appearing in (4.1) commute with contractible colimits and any sheaf on \( X \) is canonically written as colimit indexed by the contractible category \( \mathfrak{U}(X)_{/F} = \text{Shv}(X)_{/F} \times_{\text{Shv}(X)} \mathfrak{U}(X) \) (it has an initial object), it suffices to prove the theorem when \( F = y(V) \) for some \( V \in \mathfrak{U}(X) \), and hence we just need to show that \( i_*i^*(y(V)) \simeq y(U \cup V) \). For any \( W \in \mathfrak{U}(X) \), we have

\[
\Gamma(W; i_*i^*(y(V))) \simeq \Gamma(W; i_*(y(V \cap Z))) \\
= \Gamma(W \cap Z; y(V \cap Z)) \\
= \text{Hom}_{\mathfrak{U}(Z)}(W \cap Z, V \cap Z) \\
= \text{Hom}_{\mathfrak{U}(X)}(W, V \cup U) \\
= \Gamma(W; y(V \cup U)),
\]

where the second to last identification follows by the usual exponential adjunction in the boolean algebra of all subsets of \( X \).

**Corollary 4.2.** Let \( i : Z \hookrightarrow X \) be a closed immersion with open complement \( j : U \hookrightarrow X \), and let \( i_*^C, i^*_{C}, i^!_{C}, j^*_{C} \) and \( j^!_{C} \) be the induced pushforward and pullback functors at the level of \( C \)-valued sheaves, where \( C \) is any pointed presentable \( \infty \)-category-. Then we get a canonical cofiber sequence

\[
j^!_{C}j^*_{C}F \to F \to i_*^C i^*_{C}F
\]

and dually a fiber sequence

\[
i^!_{C}i^*_{C}F \to F \to j^*_{C}j^!_{C}F.
\]

**Proof.** It suffices to treat only the case of the sequence (4.2). Furthermore, we only need to prove the case of sheaves of pointed spaces, since all functors appearing are colimit preserving and we have a canonical equivalence \( \text{Shv}(X; \mathfrak{C}) \simeq \text{Shv}(X; \mathfrak{S}_*) \otimes \mathfrak{C} \), where \( \mathfrak{C} \) is any presentable pointed \( \infty \)-category and \( X \) is any topological space. We define the canonical morphisms in (4.2) through counit and unit of the appropriate adjunctions, and we see immediately that the composition of those two morphisms is null-homotopic because \( i_*^\mathfrak{S}_* j^!_{\mathfrak{S}_*} \) is equivalent to a constant functor with value the zero object in \( \text{Shv}(Z; \mathfrak{S}_*) \).

Let \( \alpha : \text{Shv}(X; \mathfrak{S}_*) \to \text{Shv}(X) \) be the forgetful functor. A close inspection of the appropriate universal properties shows that, for any \( F \in \text{Shv}(X; \mathfrak{S}_*) \), there is a canonical pushout square

\[
j^!_{\mathfrak{S}_*}j^*_{\mathfrak{S}_*}F \to F \to i_*^\mathfrak{S}_* i^*_{\mathfrak{S}_*}F,
\]

where the left vertical map is induced by the point of \( F \). Thus, since \( \alpha \) reflects pushouts and we have an equivalence \( \alpha i_*^\mathfrak{S}_* \simeq i_*^\alpha \), it suffices to prove that the canonical square

\[
\alpha(j^!_{\mathfrak{S}_*}j^*_{\mathfrak{S}_*}F) \to \alpha(F) \\
\downarrow \quad \downarrow \\
y(X) \to i_*^\alpha \alpha(F).
\]
induced by applying \( \alpha \) to the sequence (4.2) is a pushout. For this purpose, consider the commutative diagram

\[
\begin{array}{ccc}
  j^\sharp (y(U)) & \longrightarrow & y(X) \\
  \downarrow & & \downarrow \\
  j^\sharp j^* \alpha (F) & \longrightarrow & \alpha (j^\sharp_\ast j^\ast F) \\
  \downarrow & & \downarrow \\
  j^\sharp j^* (y(X)) & \longrightarrow & y(X) \\
\end{array}
\]

The upper left square and the left vertical rectangle are both pushouts, and so also the lower left square is a pushout. But the lower horizontal rectangle is a pushout, and so we can conclude.

Corollary 4.3. Consider a pullback square

\[
\begin{array}{ccc}
  Z' & \longrightarrow & Z \\
  f' \downarrow & & \downarrow s \\
  Z'' & \longrightarrow & X \\
\end{array}
\]

where \( f \) and \( f' \) are shape submersions and \( s \) (and consequently \( s' \)) is a closed immersion. Then, for sheaves with values in a pointed presentable \( \infty \)-category, we have a canonical equivalence

\[
\begin{array}{ccc}
  s_! f'_! & \simeq & f_! s'_! \\
\end{array}
\]

or equivalently

\[
\begin{array}{ccc}
  f'^* s'_! & \simeq & s'^* f^* \\
\end{array}
\]

Proof. Let \( j \) and \( j' \) be the complement open immersions associated respectively to \( s \) and \( s' \). By the localization sequences and by smooth base change, we have a commutative diagram where all the rows are cofiber sequences and all the vertical arrows are invertible

\[
\begin{array}{ccc}
  j^\sharp j^* f_\sharp & \longrightarrow & f_\sharp \\
  \downarrow \simeq & \downarrow \simeq & \downarrow \simeq \\
  j^\sharp f'_\sharp j'^* & \longrightarrow & f_\sharp s'_\sharp s'^* \\
  \downarrow \simeq & \downarrow \simeq & \downarrow \simeq \\
  f_\sharp j'^* j^* & \longrightarrow & f_\sharp \\
\end{array}
\]

Hence we may conclude by precomposing the dotted equivalence with \( s'_! \), since \( s'_! \) is fully faithful.

Remark 4.1. It is not hard to see that one might deduce from Theorem 4.1 that \( i_* \) is fully faithful (this was already proven in [Lur09, Corollary 7.3.2.10]). From this follows immediately that, in the case of sheaves of pointed spaces or of spectra, one has the identification

\[
i_! \simeq \text{fib} (i^* i^*_\ast (\text{unit}) \to i^* j_* j^*).
\]
Remark 4.2. From Theorem 4.1 one deduces immediately that, at least when $C$ is presentable stable, the functors $i^*$ and $j^*$ are jointly conservative, i.e. a morphism $\alpha : F \to G$ in $\text{Shv}(X; \mathcal{C})$ is invertible if and only if both $i^*(\alpha)$ and $j^*(\alpha)$ are invertible. This implies in particular that the fully faithful functors $i_*$ and $j_*$ make $\text{Shv}(X; \mathcal{C})$ a recollement of $\text{Shv}(Z; \mathcal{C})$ and $\text{Shv}(U; \mathcal{C})$, in the sense of [Lur17, Definition A.8.1]. However, this is true in a much greater generality: see [Hai21] for a proof in the cases when $\mathcal{C}$ is an $\infty$-topos or compactly generated. Later on, for stable coefficients, we will also be able to relax the presentability assumption to the more general requirement for $\mathcal{C}$ to admit both limits and colimits.

5 Verdier Duality

From now on, unless otherwise specified, all the topological spaces we will deal with will be assumed to be locally compact and Hausdorff. This implies that the following are equivalent

1. $f : X \to Y$ is proper (i.e. the preimage of any compact subset of $Y$ is compact);
2. $f$ is closed with compact fibers;
3. $f$ is universally closed.

Another important consequence of the previous assumption is that any map $X \xrightarrow{f} Y$ can be factored as a composition of a closed immersion (which is in particular proper by the characterization above), an open immersion and a proper map as follows

$$
\begin{align*}
X \times Y &\xrightarrow{j \times id_Y} \overline{X} \times Y \\
X &\xrightarrow{\Gamma_f} \overline{X} \\
&\xrightarrow{p} Y
\end{align*}
$$

(5.1)

where $\Gamma_f$ is the graph of $f$, $X \xrightarrow{i} \overline{X}$ is the inclusion of $X$ into its one point compactification and $p$ is the projection to the second coordinate: this factorization will be used very often later.

This section will be devoted to the exposition of a more general version of the classical Verdier Duality, expressed as originally done in [Lur17, Theorem 5.5.5.1] through an equivalence between the categories of sheaves and cosheaves on a locally compact Hausdorff space. We try here to give a presentation of the proof of [Lur17, Theorem 5.5.5.1] that is shorter and (hopefully) clearer than the original one: we will directly produce for every stable and bicomplete $\infty$-category $\mathcal{C}$ a functor

$$
\mathbb{D}_C : \text{Shv}(X; \mathcal{C}) \longrightarrow \mathcal{C} \text{shv}(X; \mathcal{C})
$$

with an explicit inverse given by

$$
\mathbb{D}_C^{op} : \text{Shv}(X; \mathcal{C}^{op})^{op} \simeq \mathcal{C} \text{shv}(X; \mathcal{C}) \longrightarrow \mathcal{C} \text{shv}(X; \mathcal{C}^{op})^{op} \simeq \text{Shv}(X; \mathcal{C})
$$

We then exploit the full power of this equivalence (combined with the properties of the tensor product of stable cocomplete $\infty$-categories observed in section 2) to obtain some surprising results such as the equivalence $\text{Shv}(X; \mathcal{C}) \simeq \text{Shv}(X; \mathcal{S}p) \otimes \mathcal{C}$ and the existence of a sheafification functor for $\mathcal{C}$-valued presheaves whenever $\mathcal{C}$ is stable and bicomplete: these will be the building blocks that will permit us to develop the full six functors formalism in the next section.
5.1 **K-sheaves**

We start by recalling the definition of K-sheaves, that will provide a very convenient model of $\text{Shv}(X; \mathcal{C})$.

**Definition 5.1.** Let $\mathcal{K}(X)$ be the poset of compact subsets of a topological space $X$. A $\mathcal{K}$-sheaf on $X$ with values in an $\infty$-category $\mathcal{C}$ is a functor $F : \mathcal{K}(X) \to \mathcal{C}$ satisfying the following properties:

(i) $F(\emptyset)$ is a terminal object,

(ii) For every $K, K' \in \mathcal{K}(X)$ the square

$$
\begin{array}{ccc}
\Gamma(K \cup K'; F) & \to & \Gamma(K; F) \\
\downarrow & & \downarrow \\
\Gamma(K'; F) & \to & \Gamma(K \cap K'; F)
\end{array}
$$

is pullback,

(iii) For every $K \in \mathcal{K}(X)$, the canonical map

$$
\lim_{\rightarrow K \in K'} \Gamma(K'; F) \to \Gamma(K; F)
$$

is invertible, where $K \in K'$ means that $K'$ contains an open neighbourhood of $K$.

We will denote by $\text{Shv}_{\mathcal{K}}(X; \mathcal{C}) \subseteq \text{Fun}(\mathcal{K}(X)^{op}, \mathcal{C})$ the full subcategory spanned by K-sheaves.

From now to the rest of this section, $X$ will be a locally compact Hausdorff space.

**Theorem 5.1.** Let $\mathcal{C}$ be a bicomplete $\infty$-category where filtered colimits are exact. Then we have a canonical equivalence

$$
\text{Shv}(X; \mathcal{C}) \simeq \text{Shv}_{\mathcal{K}}(X; \mathcal{C}).
$$

**Proof.** We just recall here the construction of the two inverses, the full proof can be found in [Lur09, Theorem 7.3.4.9]. Let $M$ be the union $\bigcup(X) \cup \mathcal{K}(X)$ considered as a poset contained in the power set of $X$, and let $i : \bigcup(X) \hookrightarrow M$ and $j : \mathcal{K}(X) \hookrightarrow M$ be the corresponding inclusion. We thus get two adjunctions

$$
\begin{array}{ccc}
\text{Fun}(\bigcup(X)^{op}, \mathcal{C}) & \to & \text{Fun}(M^{op}, \mathcal{C}) \\
i^* & & j^* \\
\downarrow & & \downarrow \\
\text{Fun}(\mathcal{K}(X)^{op}, \mathcal{C}) & \to & \text{Fun}(\bigcup(X)^{op}, \mathcal{C}) \\
j_* & & i_*
\end{array}
$$

whose composite can be shown to restrict to an equivalence between sheaves and $\mathcal{K}$-sheaves. More explicitly, at the level of objects the functors are given by the formulas

$$
\begin{array}{ccc}
\text{Fun}(\bigcup(X)^{op}, \mathcal{C}) & \xrightarrow{\theta} & \text{Fun}(\mathcal{K}(X)^{op}, \mathcal{C}) \\
F & \mapsto & (K \mapsto \lim_{\rightarrow K \in \bigcup(U)} \Gamma(U; F)) \\
G & \mapsto & (U \mapsto \lim_{\rightarrow K \in \bigcup(U)} \Gamma(K; G)).
\end{array}
$$

\[\square\]
**Proposition 5.1.** Let $\mathcal{C}$ be a bicomplete $\infty$-category where filtered colimits are exact, and let $\mathcal{A}_X$ be the full subcategory of $\text{Fun}(\mathcal{K}(X)^{op}, \mathcal{C})$ spanned by those satisfying conditions (i) and (ii) in Definition 5.1. Then the fully faithful functor $\text{Shv}(X; \mathcal{C}) \hookrightarrow \mathcal{A}_X$ has a left inverse $\varphi : \mathcal{A}_X \rightarrow \text{Shv}(X; \mathcal{C})$ which preserves filtered colimits.

**Proof.** Let $M$ be the set whose elements are couples $(K, i)$ with $K \in \mathcal{K}(X)$ and $i = 0, 1$, where we define $(K, i) \leq (K', j)$ if $K' \subseteq K$ and $i = j$ or $K' \not\subseteq K$ and $i < j$. It is easy to see that $\leq$ actually defines a partial order on $M$. We have two functors

$$\mathcal{K}(X)^{op} \xrightarrow{i_0} M \quad \mathcal{K}(X)^{op} \xrightarrow{i_1} M$$

and consider the cocontinuous functor

$$\varphi := (i_1)^* (i_0)! : \text{Fun}(\mathcal{K}(X)^{op}, \text{Sp}) \rightarrow \text{Fun}(\mathcal{K}(X)^{op}, \text{Sp}).$$

Unravelling the definition, we see that

$$\Gamma(K; \varphi F) \simeq \lim_{(K', 0) \rightarrow (K, 1)} \Gamma(K'; F) \simeq \lim_{K \in K'} \Gamma(K'; F),$$

and so $\varphi F \simeq F$ whenever $F$ is a $\mathcal{K}$-sheaf. Hence, to conclude the proof, it suffices to show that the essential image of $\varphi|_{\mathcal{A}_X}$ is contained in $\text{Shv}_X(X; \mathcal{C})$.

Indeed, for any $F \in \mathcal{A}_X$ we have

$$\lim_{K \in K''} \Gamma(K''; \varphi F) \simeq \lim_{K \in K''} \lim_{K' \in K''} \Gamma(K'; F) \simeq \lim_{K \in K'} \Gamma(K'; F) \simeq \Gamma(K; \varphi F),$$

where the second equivalence holds by [Lur09, Remark 4.2.3.9] and [Lur09, Corollary 4.2.3.10] since for any $K$ compact, the full subposet of $\mathcal{K}(X)^{op}$ spanned by those $K'$ such that $K \subseteq K'$ is filtered, and we have

$$\bigcup_{\{K''|K \in K''\}} \{K'|K'' \subseteq K'\} = \{K'|K \subseteq K'\}.$$}

Moreover, since filtered colimits are exact in $\text{Sp}$, $\varphi F$ belongs to $\mathcal{A}_X$, and so it is a $\mathcal{K}$-sheaf.

**Corollary 5.1.** The $\infty$-category $\text{Shv}(X; \text{Sp})$ is a strongly dualizable object in $\text{Cocont}_\infty^*$. In particular, for any $\mathcal{C} \in \text{Cocont}_\infty^*$, the canonical functor

$$\text{CoShv}(X; \text{Sp}) \otimes \mathcal{C} \rightarrow \text{CoShv}(X; \mathcal{C})$$

is an equivalence.

**Proof.** By Proposition 2.1 and Theorem 5.1 it suffices to show that $\text{Shv}_X(X; \text{Sp})$ is a retract in $\text{Cocont}_\infty^*$ of a compactly generated $\infty$-category. Let $\mathcal{A}_X \subseteq \text{Fun}(\mathcal{K}(X)^{op}, \text{Sp})$ be the $\infty$-category defined in Proposition 5.1 in the case $\mathcal{C} = \text{Sp}$. $\mathcal{A}_X$ is clearly compactly generated, as it is equivalent to $\text{Fun}_{lex}(\mathcal{K}(X)^{op}, \text{Sp})$. The proof is then concluded by noticing that the inclusion $\text{Shv}_X(X; \text{Sp}) \subseteq \mathcal{A}_X$ and $\varphi : \mathcal{A}_X \rightarrow \text{Shv}_X(X; \text{Sp})$ are exact and preserve filtered colimits since filtered colimits in $\text{Sp}$ are exact, and thus preserves all colimits since $\text{Sp}$ is stable.

\[ \square \]
Let \( f : X \to Y \) be a proper continuous map between topological spaces, and let again \( \mathcal{C} \) be bicomplete \( \infty \)-category where filtered colimits are exact. We will now show how to exploit the model of \( \mathcal{K} \)-sheaves to give a very convenient description of the pushforward \( f_* \). Since for any \( K \in \mathcal{K}(Y) \) \( f^{-1}(K) \) is compact, we obtain a functor

\[
\text{Fun}(\mathcal{K}(X)^{\text{op}}, \mathcal{C}) \xrightarrow{f_*} \text{Fun}(\mathcal{K}(Y)^{\text{op}}, \mathcal{C})
\]

\[
F \xleftarrow{\theta} (K \mapsto \Gamma(f^{-1}(K); F)).
\]

Notice that the restriction of \( f_* \) to \( \text{Shv}_X(X; \mathcal{C}) \) lands in \( \text{A}_Y \), but a priori not in \( \text{Shv}_X(Y; \mathcal{C}) \), therefore we define

\[
f_*^X : \text{Shv}_X(X; \mathcal{C}) \to \text{Shv}_X(Y; \mathcal{C})
\]

as the composition of \( f_* \) restricted to \( \text{Shv}_X(X; \mathcal{C}) \) and \( \varphi : \text{A}_Y \to \text{Shv}_X(Y; \mathcal{C}) \).

**Lemma 5.1.** Let \( f : X \to Y \) be a proper continuous map between topological spaces, and let \( \mathcal{C} \) be bicomplete \( \infty \)-category where filtered colimits are exact. Then there is a commutative diagram

\[
\begin{array}{ccc}
\text{Shv}(X; \mathcal{C}) & \xrightarrow{\theta} & \text{Shv}_X(X; \mathcal{C}) \\
\downarrow f_* & & \downarrow f_*^X \\
\text{Shv}(Y; \mathcal{C}) & \xrightarrow{\theta} & \text{Shv}_X(Y; \mathcal{C})
\end{array}
\]

In particular, \( f_* \) preserves filtered colimits, and when \( \mathcal{C} \) is stable it preserves all colimits.

**Proof.** For any \( K \in \mathcal{K}(Y) \), we define

\[
T = \{ U \in \mathcal{U}(X) \mid \exists K' \ni K \text{ with } f^{-1}(K') \subseteq U \}.
\]

Notice that if \( V \in \mathcal{U}(Y) \) contains \( K \), then there exists an open neighbourhood \( W \) of \( K \) with compact closure, and thus, since \( f \) is proper, \( f^{-1}(V) \in T \). In particular we obtain a functor

\[
\alpha : \{ V \in \mathcal{U}(Y) \mid K \subseteq V \} \to T
\]

which is obviously final. We have

\[
\Gamma(K; f_*^X \theta F) \simeq \lim_{K \subseteq K'} \lim_{f^{-1}(K') \subseteq U} \Gamma(U; F)
\]

\[
\simeq \lim_{U \in T} \Gamma(U; F)
\]

\[
\simeq \lim_{K' \subseteq V} \Gamma(f^{-1}(V); F)
\]

\[
\simeq \Gamma(K; \theta f_* F)
\]

where the second equivalence follows by [Lur09, Remark 4.2.3.9] and [Lur09, Corollary 4.2.3.10], and the third one since \( \alpha \) is final.

### 5.2 Verdier duality

For the rest of the section, all \( \infty \)-categories involved will be assumed to be stable and bicomplete.
Remark 5.1. Since in any stable $\infty$-category $\mathcal{C}$ filtered colimits are exact, and since the opposite of any stable $\infty$-category is again stable, by the above theorem we get equivalences

$$\mathcal{S}hv(X; \mathcal{C}) \simeq \mathcal{S}hv_K(X; \mathcal{C})$$

and

$$\mathcal{C}o\mathcal{S}hv(X; \mathcal{C}) \simeq \mathcal{S}hv(X; \mathcal{C}^{op})^{op} \simeq \mathcal{C}o\mathcal{S}hv_K(X; \mathcal{C})$$

where we define $\mathcal{C}o\mathcal{S}hv_K(X; \mathcal{C})$ to be $\mathcal{S}hv(X; \mathcal{C}^{op})^{op}$.

Definition 5.2. Let $F \in \mathcal{S}hv(X; \mathcal{C})$, $U \in \mathcal{U}(X)$, and $K$ any closed subset of $X$. We define the sections of $F$ supported at $K$ and compactly supported sections of $F$ over $U$ respectively as

$$\Gamma_K(X; F) := \text{fib}(\Gamma(X; F) \to \Gamma(X \setminus K; F))$$

$$\Gamma_c(U; F) := \lim_{K \subseteq U} \Gamma_K(X; F),$$

where the colimit ranges over all compact subsets of $U$.

Remark 5.2. Notice that, if $K \subseteq U$ for some open $U$, we get a pullback square

$$\begin{array}{ccc}
\Gamma(X; F) & \longrightarrow & \Gamma(X \setminus K; F) \\
\downarrow & & \downarrow \\
\Gamma(U; F) & \longrightarrow & \Gamma(U \setminus K; F)
\end{array}$$

and hence the fibers of the two horizontal maps coincide: for this reason, we will often also denote it by $\Gamma_K(U; F)$. Furthermore, if $S \subseteq X$ is locally closed, we define

$$\Gamma_S(X; F) := \Gamma_Z(U; j^* F)$$

where $S = U \cap Z$, with $U$ open, $Z$ closed and $j : U \hookrightarrow X$ the inclusion, but we will also use the notation $\Gamma_S(U; F)$.

Remark 5.3. Indeed, the definition of the sections of a sheaf $F$ on a compact $K$ is functorial both in $F$ and in $K$: since we have an obvious functor

$$\mathcal{K}(X) \longrightarrow \text{Fun}(\Delta^1, \mathcal{U}(X)^{op})$$

$$K \longrightarrow (X \to X \setminus K)$$

we get

$$\begin{array}{ccc}
\mathcal{K}(X) \times \text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C}) & \longrightarrow & \text{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{\text{fib}} \mathcal{C} \\
\downarrow & & \downarrow \circ \\
\text{Fun}(\Delta^1, \mathcal{U}(X)^{op}) \times \text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C}) & \longrightarrow & \text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})
\end{array}$$

where the diagonal arrow is the composition of functors and the right horizontal arrow is given by taking the fiber of an arrow in $\mathcal{C}$, and so by adjunction we get the desired

$$\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C}) \longrightarrow \text{Fun}(\mathcal{K}(X), \mathcal{C})$$

$$F \longrightarrow (K \mapsto \Gamma_K(X; F)).$$

Finally, by further composing with the functor $\psi$ defined in Theorem 5.1 we get

$$\begin{array}{ccc}
\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C}) & \xrightarrow{\mathcal{D}_c} & \text{Fun}(\mathcal{U}(X), \mathcal{C}) \\
F & \longrightarrow & (U \mapsto \Gamma_c(U; F))
\end{array}$$

(5.2)
Theorem 5.2. The functor $\mathbb{D}_C$ restricts to an equivalence

$$\mathbb{D}_C : \text{Shv}(X; C) \longrightarrow \text{CoShv}(X; C).$$

Proof. We first prove that, if $F$ is a sheaf, then $\mathbb{D}_C(F)$ is a cosheaf. By virtue of Theorem 5.1, it suffices to prove that the functor

$$K \mapsto \Gamma_K(X; F)$$

is a $\mathcal{K}$-cosheaf.

- $\Gamma_\emptyset(X; F) \simeq 0$ since $F(X) \to F(X \setminus \emptyset)$ is an equivalence.

- Let $K, K' \in \mathcal{K}(X)$. The square

\[
\begin{array}{ccc}
\Gamma_{K \cap K'}(X; F) & \longrightarrow & \Gamma_K(X; F) \\
\downarrow & & \downarrow \\
\Gamma_{K'}(X; F) & \longrightarrow & \Gamma_{K \cup K'}(X; F)
\end{array}
\]

is the fiber of the obvious map between the pullback squares

\[
\begin{array}{ccc}
\Gamma(X; F) & \longrightarrow & \Gamma(X; F) \\
\downarrow & & \downarrow \\
\Gamma(X; F) & \longrightarrow & \Gamma(X \setminus (K \cap K'); F) \\
\downarrow & & \downarrow \\
\Gamma(X \setminus K'; F) & \longrightarrow & \Gamma(X \setminus (K \cup K'); F),
\end{array}
\]

and so it is a pullback. Thus, since $C$ is stable, it’s also a pushout.

- For any $K \in \mathcal{K}(X)$, we have a map of fiber sequences

\[
\begin{array}{ccc}
\Gamma_K(X; F) & \longrightarrow & \lim_{K' \in K'} \Gamma_{K'}(X; F) \\
\downarrow & & \downarrow \\
\Gamma(X; F) & \longrightarrow & \lim_{K' \in K'} \Gamma(X; F) \\
\downarrow & & \downarrow \\
\Gamma(X \setminus K; F) & \longrightarrow & \lim_{K' \in K'} \Gamma(X \setminus K'; F).
\end{array}
\]

To prove that $a$ is an equivalence, it suffices to prove that $b$ and $c$ are. But $b$ is an equivalence because the poset $\{K' \in \mathcal{K}(X) \mid K \subseteq K'\}$ has a contractible nerve (since it is filtered) and $c$ is an equivalence because $\{X \setminus K' \in \mathcal{U}(X) \mid K \subseteq K'\}$ is a covering $X \setminus K$.

We will now prove that $\mathbb{D}_C^{\text{op}}$ is an inverse of $\mathbb{D}_C$. By symmetry, it suffices to show that it is a left inverse. Unwrapping the definitions and using the equivalence of Theorem 5.1, this amounts to check that we have a cofiber sequence

$$\begin{array}{ccc}
\Gamma_\emptyset(X \setminus K; F) & \longrightarrow & \Gamma_\emptyset(X; F) \\
\longrightarrow & & \Gamma(K; F)
\end{array}$$

natural in $K$ and $F$. 36
First of all, we show that it suffices to prove that, for any fixed $K \in \mathcal{K}(X)$, $U \in \mathcal{U}(X)$ containing $K$ and with compact closure, and $K' \in \mathcal{K}(X)$ containing $U$, the sequence
\[
\Gamma_{K' \setminus U}(X; F) \longrightarrow \Gamma_{K'}(X; F) \longrightarrow \Gamma(U; F),
\]
where the first morphism is given by the functoriality of sections supported on a compact and the second one is given by Remark 5.2, is a cofiber sequence. To see this, we start by noticing that the sequence is natural in $K'$ and $F$, since both morphisms are canonically induced by the restrictions of $F$. Thus we can pass to the colimit ranging over all compacts $K' \supseteq U$ and get a fiber sequence
\[
\lim_{K' \supseteq U} \Gamma_{K' \setminus U}(X; F) \longrightarrow \Gamma_c(X; F) \longrightarrow \Gamma(U; F),
\]
since the poset $\{K' \in \mathcal{K}(X) \mid K' \supseteq U\}$ is filtered (it is non-empty because it contains the closure of $U$) and the inclusion $\{K' \in \mathcal{K}(X) \mid K' \supseteq U\} \subseteq \mathcal{K}(X)$ is cofinal. Since any $K' \supseteq U$ is contained in $(K' \cup \overline{U}) \setminus U$, we get an equivalence
\[
\lim_{K' \supseteq U} \Gamma_{K' \setminus U}(X; F) \simeq \lim_{\{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F),
\]
and hence, adding everything up, we obtain a fiber sequence
\[
\lim_{\{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F) \longrightarrow \Gamma_c(X; F) \longrightarrow \Gamma(U; F),
\]
which is natural in $U$, since the morphism $\Gamma_c(X; F) \to \Gamma(U; F)$ clearly is. Hence we can get the desired sequence (5.3) by passing to the colimit ranging over $P = \{U \in \mathcal{U}(X) \mid \overline{U} \in \mathcal{K}(X) \text{ and } U \supseteq K\}$ because $\Gamma(K; F) = \lim_{U \in P} \Gamma(U; F)$ (since open subsets with compact closure form a basis of $X$), and because we have equivalences
\[
\lim_{U \in P} \lim_{\{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F) \simeq \lim_{U \in P} \Gamma_{K'}(X; F)
\]
\[
\simeq \lim_{K' \subseteq X \setminus K} \Gamma_{K'}(X; F)
\]
\[
\simeq \Gamma_c(X \setminus K; F)
\]
where the first one follows by [Lur09, Remark 4.2.3.9] and [Lur09, Corollary 4.2.3.10].

We are now left to show that (5.4) is a cofiber sequence. Consider the commutative diagram
\[
\begin{array}{ccccccc}
\Gamma_{K' \setminus U}(X; F) & \longrightarrow & 0 \\
\downarrow & & \\
\Gamma_{K'}(X; F) & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\Gamma(X; F) & \longrightarrow & \Gamma(X \setminus (K' \setminus U); F) & \longrightarrow & \Gamma(X \setminus K'; F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\Gamma(U; F) & \longrightarrow & 0
\end{array}
\]
where $Z := \text{fib}(\Gamma(X \setminus (K' \setminus U); F) \rightarrow \Gamma(X \setminus K'; F))$. Since the middle big horizontal rectangle is a pullback, it follows that also the left middle square is. But since the left vertical rectangle is pullback, then also the upper left square is. Then it suffices to prove that the composition

$$Z \rightarrow \Gamma(X \setminus (K' \setminus U); F) \rightarrow \Gamma(U; F)$$

is an equivalence, but this is clear since the lower right square is pullback because $F$ is a sheaf.

**Corollary 5.2.** There is an equivalence

$$\eta : \text{Shv}(X; \mathcal{S}) \otimes \mathcal{C} \rightarrow \text{Shv}(X; \mathcal{C}).$$  \hfill (5.5)

**Proof.** The functor $\eta$ is obtained by composing

$$\text{Shv}(X; \mathcal{S}) \otimes \mathcal{C} \xrightarrow{\mathcal{D}_{\mathcal{S}} \otimes \mathcal{C}} \text{CoShv}(X; \mathcal{S}) \otimes \mathcal{C} \xrightarrow{\simeq} \text{CoShv}(X; \mathcal{C}) \xrightarrow{\mathcal{D}_{\mathcal{C}}^{-1}} \text{Shv}(X; \mathcal{C}).$$

More concretely, for any $F \in \text{Shv}(X; \mathcal{S})$ and $M \in \mathcal{C}$, we have $\eta(F \otimes M) \simeq \mathcal{D}_{\mathcal{C}}^{-1}(M \circ \mathcal{D}_{\mathcal{S}} F)$, where $M$ on the right-hand side denotes the essentially unique colimit preserving functor $\mathcal{S} \rightarrow \mathcal{C}$ corresponding to $M$.

**5.3 The pullback $f^*_C$**

**Proposition 5.2.** Let $f : X \rightarrow Y$ be a proper map, and denote by $f^*_C : \text{Shv}(X; \mathcal{C}) \rightarrow \text{Shv}(Y; \mathcal{C})$ the pushforward. Then we have a commutative square

$$\begin{array}{ccc}
\text{Shv}(X; \mathcal{S}) \otimes \mathcal{C} & \xrightarrow{\eta} & \text{Shv}(X; \mathcal{C}) \\
\downarrow f^*_S \otimes \mathcal{C} & & \downarrow f^*_C \\
\text{Shv}(Y; \mathcal{S}) \otimes \mathcal{C} & \xrightarrow{\eta} & \text{Shv}(Y; \mathcal{C}).
\end{array}$$

In particular, $f^*_C$ admits a left adjoint which is identified through $\eta$ with $f^*_S \otimes \mathcal{C}$.

**Proof.** By a slight abuse of notation, denote as $f^*_C : \text{CoShv}(X; \mathcal{C}) \rightarrow \text{CoShv}(Y; \mathcal{C})$ the pushforward for cosheaves (i.e. $(f^*_C)^{\text{op}}$). We have that the square

$$\begin{array}{ccc}
\text{CoShv}(X; \mathcal{S}) \otimes \mathcal{C} & \xrightarrow{\simeq} & \text{CoShv}(X; \mathcal{C}) \\
\downarrow f^*_{\mathcal{S}} \otimes \mathcal{C} & & \downarrow f^*_C \\
\text{CoShv}(Y; \mathcal{S}) \otimes \mathcal{C} & \xrightarrow{\simeq} & \text{CoShv}(Y; \mathcal{C})
\end{array}$$

commutes since the horizontal arrows can be modelled by a composition of functors, and thus we are only left to show that the square

$$\begin{array}{ccc}
\text{Shv}(X; \mathcal{C}) & \xrightarrow{\mathcal{D}} & \text{CoShv}(X; \mathcal{C}) \\
\downarrow f_* & & \downarrow f_* \\
\text{Shv}(Y; \mathcal{C}) & \xrightarrow{\mathcal{D}} & \text{CoShv}(Y; \mathcal{C})
\end{array}$$

commutes.

38
First of all, we show that there exists a natural transformation $f_* D \to D f_*$ even when $f$ is not proper. Fix $V \in \mathcal{U}(Y)$ and a compact $K \subseteq f^{-1}(V)$, so that $f(K)$ is a compact subset of $V$. For any $F \in \text{Shv}(X; \mathcal{C})$, the commutative triangle

$$\Gamma(X \setminus K; F) \to \Gamma(X \setminus f^{-1}(f(K)); F)$$

provides a morphism

$$\Gamma_K(X; F) \to \Gamma_{f(K)}(Y; f_* F) \to \Gamma_c(V; f_* F).$$

Since all morphisms are induced by the restrictions of $F$, the resulting map is natural on $K$ and $V$, and hence gives rise to the desired transformation. Furthermore, when $f$ is proper, each compact $K \subseteq X$ is contained in the compact $f^{-1}(f(K))$, so by cofinality we obtain an equivalence

$$\lim_{K \subseteq f^{-1}(V)} \Gamma_K(X; F) \simeq \lim_{C \subseteq \mathcal{U}} \Gamma_C(Y; f_* F)$$

and thus we may conclude.

**Lemma 5.2.** Let $j : U \hookrightarrow X$ be an open immersion, and denote by $j^* : \text{Shv}(X; \mathcal{C}) \to \text{Shv}(U; \mathcal{C})$ the restriction. Then we have a commutative square

$$\begin{array}{ccc}
\text{Shv}(X; \mathcal{S}^p) \otimes \mathcal{C} & \xrightarrow{\eta} & \text{Shv}(X; \mathcal{C}) \\
\downarrow j^*_p \otimes \mathcal{C} & & \downarrow j^*_c \\
\text{Shv}(U; \mathcal{S}^p) \otimes \mathcal{C} & \xrightarrow{\eta} & \text{Shv}(U; \mathcal{C}).
\end{array}$$

In particular, $j^*_c$ admits a left adjoint which is identified through $\eta$ with $j^*_p \otimes \mathcal{C}$.  

**Proof.** By an abuse of notation, denote by $j^* : \mathcal{C}\text{Shv}(X; \mathcal{C}) \to \mathcal{C}\text{Shv}(U; \mathcal{C})$ the restriction for cosheaves. Again we see that the square

$$\begin{array}{ccc}
\mathcal{C}\text{Shv}(X; \mathcal{S}^p) \otimes \mathcal{C} & \xrightarrow{\simeq} & \mathcal{C}\text{Shv}(X; \mathcal{C}) \\
\downarrow j^*_p \otimes \mathcal{C} & & \downarrow j^*_c \\
\mathcal{C}\text{Shv}(U; \mathcal{S}^p) \otimes \mathcal{C} & \xrightarrow{\simeq} & \mathcal{C}\text{Shv}(U; \mathcal{C})
\end{array}$$

is obviously commutative, and thus we only have to show that the square

$$\begin{array}{ccc}
\text{Shv}(X; \mathcal{C}) & \xrightarrow{D} & \mathcal{C}\text{Shv}(X; \mathcal{C}) \\
\downarrow j^* & & \downarrow j^*_c \\
\text{Shv}(U; \mathcal{C}) & \xrightarrow{D} & \mathcal{C}\text{Shv}(U; \mathcal{C})
\end{array}$$

commutes. But this follows immediately because by Remark 5.2 and by observing that $\Gamma(V; j^* F) \simeq \Gamma(V; F)$ for any $V \subseteq U$, we have that

$$\Gamma_c(V; j^* F) \simeq \Gamma_c(V; F)$$

functorially on $F$ and $V$.  

39
Corollary 5.3. Let \( f : X \to Y \) be any continuous map. Then \( \text{Shv}(X; \mathcal{C}) \to \text{Shv}(Y; \mathcal{C}) \) admits a left adjoint \( f_!^* \) such that there exists a commutative square

\[
\text{Shv}(Y; \mathcal{S}p) \otimes \mathcal{C} \xrightarrow{\eta} \text{Shv}(Y; \mathcal{C}) \\
\downarrow f_!^* \otimes \mathcal{C} \quad \downarrow f_!^*
\]

\[
\text{Shv}(X; \mathcal{S}p) \otimes \mathcal{C} \xrightarrow{\eta} \text{Shv}(X; \mathcal{C}).
\]

Proof. Using the factorization \[5.1\] this follows immediately by Proposition \[5.2\] and Lemma \[5.2\].

Theorem 5.3. Let \( i_* : \text{Shv}(X; \mathcal{C}) \hookrightarrow \text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C}) \) be the inclusion functor, and let \( L_* : \text{Fun}(\mathcal{U}(X)^{op}, \mathcal{S}p) \to \text{Shv}(X; \mathcal{S}p) \) be the left adjoint of \( i_* \). Denote by \( L_* \mathcal{C} \) the composition

\[
\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{S}p) \otimes \mathcal{C} \xrightarrow{\eta} \text{Shv}(X; \mathcal{S}p) \otimes \mathcal{C} \\
\uparrow \quad \quad \quad \uparrow \\
\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C}) \xrightarrow{L_*} \text{Shv}(X; \mathcal{C}).
\]

Then \( L_* \mathcal{C} \) is left adjoint to \( i_* \).

Proof. We have equivalences

\[
\text{Hom}_{\text{Shv}(X; \mathcal{C})}(L_*^* F, G) \simeq \lim_{M \to \Gamma(U; F)} \text{Hom}_{\text{Shv}(X, \mathcal{S}p) \otimes \mathcal{C}}(j_*^* \mathcal{S}p \otimes M, \eta^{-1} G)
\]

\[
\simeq \lim_{M \to \Gamma(U; F)} \text{Hom}_{\text{Shv}(X, \mathcal{C})}(j_*^* \mathcal{C} M, G)
\]

\[
\simeq \lim_{M \to \Gamma(U; F)} \text{Hom}_{\mathcal{C}}(M, \Gamma(U; G))
\]

\[
\simeq \lim_{M \to \Gamma(U; F)} \text{Hom}_{\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})}(U \otimes M, i^* G)
\]

\[
\simeq \text{Hom}_{\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})}(F, i^* G)
\]

where \( j : U \hookrightarrow X \) denotes an open inclusion, \( a : U \to * \) the unique map, so that the first equivalence follows since \( \mathcal{S}p \) is presentable and the second is a consequence of Lemma \[5.2\] and Corollary \[5.3\]. Since all identifications are functorial on \( F \) and \( G \), we obtain the thesis.

Remark 5.4. After the results in this section, we are now able to extend everything we have proven so far for sheaves with presentable coefficients to sheaves with values in a stable bicomplete \( \infty \)-category. The only detail we have to handle with more care is the functor \[3.3\]: since the tensor product of two stable bicomplete \( \infty \)-categories is not again complete unless one of the two is compactly generated, \[3.3\] will now take values in \( \text{Shv}(X; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{D}) \). Nevertheless, when \( \mathcal{C} \) has a monoidal structure such that its tensor \( \otimes \mathcal{C} \) preserves colimits in both variables, the composition of \[3.3\] with the obvious functor

\[
\text{Shv}(X; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{C}) \to \text{Shv}(X; \mathcal{C})
\]

still gives the usual monoidal structure on \( \text{Shv}(X; \mathcal{C}) \). As a consequence, we see that the equivalences in Corollary \[3.3\] and Corollary \[3.4\] still hold in \( \text{Shv}(X; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{D}) \) and in \( \text{Shv}(X; \mathcal{C}) \) when \[3.3\] is exchanged with the tensor product in \( \text{Shv}(X; \mathcal{C}) \) described above.
6 Six functor formalism

For the whole section we will assume that all the ∞-categories involved are stable and bi-complete. In this situation, by the results in the previous section, for any continuous map $f : X \to Y$ we have an adjunction $f^* \dashv f_*$ for $\mathcal{C}^{op}$-valued sheaves, and finally, by passing to the opposite categories and applying the inverse of Verdier duality, we get an adjunction

$$f_! \dashv f^!$$

Shv$(X; \mathcal{C}) \perp$ Shv$(Y; \mathcal{C})$.

More concretely, $f_!$ is the functor uniquely determined by the formula

$$\Gamma_c(U; f_! F) = \Gamma_c(f^{-1}(U); F)$$

for all $U \in \mathcal{U}(Y)$. The functors $f_!$ and $f^!$ are called respectively pushforward with proper support and exceptional pullback. The goal of this section will be to prove all the usual formulas that one has classically for these functors, and hence to develop the full six functors formalism in the particular case when $\mathcal{C}$ is monoidal. A first attempt towards these results can be found in the paper [BL96], even though it almost totally lacks of proofs.

6.1 The formulas for $f^!_C$

Lemma 6.1. There exists a natural transformation $f_! \to f^*$ which is an equivalence when $f$ is proper.

Proof. By Verdier duality, it suffices to construct a natural transformation between $\mathbb{D}f_! \to \mathbb{D}f^*$ and show it is an equivalence when $f$ is proper, which is the content of the second part of the proof of Proposition 5.2.

Corollary 6.1. Let $i : Z \hookrightarrow X$ be a closed immersion. Then the functor $i^! : \text{Shv}(X; \mathcal{S}p) \to \text{Shv}(Z; \mathcal{S}p)$ coincides with the one defined in Corollary 4.1.

Proof. This follows immediately by the previous lemma, since any closed immersion is proper.

Lemma 6.2. Let $X$ be a topological space, $j : U \hookrightarrow X$ an open immersion. Then we have $j_! \dashv j^*$ or equivalently $j^* \simeq j_!$.

Proof. By the proof of Lemma 5.2 we have a natural equivalence $\mathbb{D}j^* \simeq \mathbb{D}j_!$, and thus we may conclude.

Remark 6.1. Let $j : U \hookrightarrow X$ be the inclusion of any relatively compact open subset. Then a simple computation involving Lemma 6.1 and the closure of $U$ shows that, for any sheaf $F$ on $X$, one has

$$\Gamma(U; F) \simeq \Gamma_c(X; j_* j^* F).$$

Since $U$ is relatively compact, any closed subset of $U$ can be written as the intersection of $U$ with some compact subset of $X$, and thus we obtain

$$\Gamma_c(X; j_* j^* F) \simeq \lim_{\overline{K} \subseteq X} \Gamma_{U \cap K}(U; F)$$

\simeq \lim_{\overline{S} \subseteq \mathcal{U}} \Gamma_S(U; F),$$

where the last colimit ranges over all closed subsets of $U$. 41
A similar proof of the following proposition can also be found in [Jin19, Lemma 3.2].

**Proposition 6.1 (Base change).** For every given pullback square

\[
\begin{array}{ccc}
  X' & \xrightarrow{f'} & X \\
  g' \downarrow & & \downarrow g \\
  Y' & \xrightarrow{f} & Y
\end{array}
\]

of topological spaces, there is a natural equivalence

\[
f^* g_l \simeq g'_l f'^* \]

and, by transposition, also

\[
g^! f_* \simeq f'^! g'^! \]

*Proof.* By the factorization (5.1) and [Hai21, Corollary 3.2], it suffices to prove the proposition in the case where \(g\) is an open immersion.

By Lemma 6.2, we may use the unit of the adjunction \(g^! \dashv g_*\) to produce a transformation

\[
f'^* \to f'^* g^* g_l \simeq g'^* f^* g_l \]

and hence by adjunction the desired

\[
g'_l f'^* \to f^* g_l.\]

By Lemma 6.2, we see that

\[
g'_l f'^* M_U \simeq g'_l M_{f^{-1}(U)} \]
\[
\simeq M_{f^{-1}(U)} \]
\[
\simeq f^* M_U \]
\[
\simeq f^* g_l M_U.\]

Since all functors appearing are left adjoints, we thus can conclude by Remark 2.10.

**Remark 6.2.** It follows from Remark 2.5 and the definition of the tensor (3.3) that for any topological space \(X\) and any bicomplete stable \(\infty\)-category \(\mathcal{C}\), \(\text{Shv}(X; \mathcal{C})\) is tensored over \(\text{Shv}(X; S^p)\). When there is no possibility of confusion will denote by \(F \otimes G\) the image through the canonical variablewise colimit preserving functor

\[
\text{Shv}(X; \mathcal{C}) \times \text{Shv}(X; S^p) \to \text{Shv}(X; \mathcal{C})
\]

of a couple \((F, G)\), and, when \(G \in \text{Shv}(X; S^p)\) by \(\text{Hom}_\vee (G, F)\) the image of any \(F \in \text{Shv}(X; \mathcal{C})\) through the right adjoint of \(- \otimes G\).

**Proposition 6.2 (Projection formula).** Let \(f : X \to Y\) be a morphism of topological spaces, and let \(\mathcal{C}\) and \(\mathcal{D}\) be two stable and bicomplete \(\infty\)-categories. Then, for any \(F \in \text{Shv}(X; \mathcal{C})\) and \(G, H \in \text{Shv}(Y; \mathcal{D})\), we have a canonical equivalence

\[
f^! F \otimes G \simeq f_!(F \otimes f^* G)
\]

or, when \(\mathcal{C} = \mathcal{D}\) has a closed symmetric monoidal structure, by transposition

\[
f_! \text{Hom}_\vee (F, f^! G) \simeq \text{Hom}_\vee (f_! F, G)
\]

and

\[
f^! \text{Hom}_\vee (G, H) \simeq \text{Hom}_\vee (f^* G, f^! H).
\]
Proof. Exactly as for the smooth projection formula, one may deduce this result from the base change for pushforward with proper support.

**Corollary 6.2.** Let $k: Z \to X$ be the inclusion of a locally closed subset of $X$, $F \in \Shv(X; \mathcal{C})$ with $\mathcal{C}$ stable and bicomplete. Then we have a canonical equivalence

$$k_! k^* F \simeq F \otimes S_Z$$

or equivalently

$$k_* k! F \simeq \Hom_X(S_Z, F),$$

where $S_Z = k! k_* S_X$. Moreover, when $\mathcal{C}$ has a closed symmetric monoidal structure, we have

$$k_! k^* F \simeq F \otimes C_1^Z$$

or equivalently

$$k_* k! F \simeq \Hom_X(1^Z, F),$$

where $1$ and $1_X$ are respectively the monoidal units of $\mathcal{C}$ and $\Shv(X; \mathcal{C})$, and $1_Z = k_! k^* 1_X$.

**Proof.** This is an immediate consequence of Proposition (6.2) and Corollary (3.3).

Let $f: X \to X'$ and $g: Y \to Y'$ be morphisms of topological spaces, and let $f \times g: X \times Y \to X' \times Y'$ be the induced map on the products. For any two stable bicomplete $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$, the variable-wise colimit preserving functor

$$\Shv(X; \mathcal{C}) \times \Shv(Y; \mathcal{D}) \xrightarrow{f_! \times g_!} \Shv(X'; \mathcal{C}) \times \Shv(Y'; \mathcal{D}) \xrightarrow{\otimes} \Shv(X' \times Y'; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{D})$$

induces a functor

$$f_! \otimes g_! : \Shv(X \times Y; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{D}) \to \Shv(X' \times Y'; \mathcal{S}p) \otimes (\mathcal{C} \otimes \mathcal{D}).$$

**Proposition 6.3** (Künneth formula). We have a natural equivalence

$$f_! \otimes g_! \simeq (f \times g)_!.$$ 

Notice that, since $X$ and $Y$ are locally compact, $f_! \otimes g_!$ is the image of the couple $(f_!, g_!)$ through the bifunctor given by the tensor product of cocomplete $\infty$-categories.

**Proof.** Since up to Verdier Duality they correspond to usual sections of sheaves, the functors

$$a_{U!} = \Gamma_c(U; -),$$

where $a_U: U \to *$ for all $U \in U(X)$, are jointly conservative, and thus it suffices to prove the proposition in the case where $X' = Y' = *$, which amounts to check that the triangle

$$\Shv(X; \mathcal{C}) \times \Shv(Y; \mathcal{D}) \xrightarrow{\otimes} \Shv(X \times Y; \mathcal{C} \otimes \mathcal{D}) \xrightarrow{a_{X \times Y!}} \mathcal{C} \otimes \mathcal{D}$$

is commutative. This follows from a projection formula and base change applied to the pullback square

$$\begin{array}{ccc}
X \times Y & \xrightarrow{p_Y} & Y \\
\downarrow{p_X} & & \downarrow{a_Y} \\
X & \xrightarrow{a_X} & *
\end{array}$$
as below

\[ a_{X!}F \otimes a_{Y!}G \simeq a_{X!}(F \otimes a_X^*a_{Y!}G) \]
\[ \simeq a_{X!}(F \otimes p_{X!}p_Y^*G) \]
\[ \simeq a_{X\times Y!}(p_{X!}F \otimes p_Y^*G) \]
\[ \simeq a_{X\times \ast}p_{X!}a_{\ast}^*p_{Y!}^*G. \]

\[ \Box \]

**Corollary 6.3.** For every given pullback square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{g'} & & \downarrow{g} \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

of topological spaces where \( f \) and \( f' \) are shape submersions, there is a natural equivalence

\[ g_!f'_* \simeq f_!g_*' \]

or equivalently

\[ f'^* g'_! \simeq g'^* f_! \]

**Proof.** First of all we construct a natural transformation. Applying \( g'_! \) to the unit of the adjunction \( f'^* \dashv f_! \) and using base change we obtain

\[ g'_! \to g_! f'^* f_! \simeq f_! g_! f'_* \]

and hence by transposition the desired transformation. By factorization (5.1) and Corollary 4.3 we are only left to prove the case of a pullback square

\[
\begin{array}{ccc}
X \times Y' & \xrightarrow{id_X \times f} & X \times Y \\
\downarrow{a \times id_{Y'}} & & \downarrow{a \times id_Y} \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

were \( X \) is compact and \( a : X \to \ast \) is the unique map, but this follows by Künneth formula and by the functoriality of the tensor product of cocomplete \( \infty \)-categories as follows

\[ (a \times id_Y)_!(id_X \times f)_! \simeq (a_! \otimes (id_Y)_!)((id_X)_! \otimes f_!) \]
\[ \simeq a_! f_! \]
\[ \simeq ((id_a)_! (a_!)) \otimes (f_! (id_Y)_!)) \]
\[ \simeq ((id_a)_! \otimes f_!) (a_! \otimes (id_Y)_!). \]

\[ \Box \]

### 6.2 \( f'_! \) when \( f \) is a shape submersion

Let \( f : X \to Y \) be a continuous map inducing an essential geometric morphism. In particular, we have an adjunction \( f_! \dashv f^* \) for \( \mathcal{E} \)-valued sheaves, and thus, after passing to opposite categories and applying Verdier duality, we get an adjunction

\[
\begin{array}{ccc}
\mathcal{E} & \xleftarrow{f^*} & \mathcal{E} \\
\downarrow{f_!} & & \downarrow{f_!} \\
\mathbb{Sh}(Y; \mathcal{E}) & \perp & \mathbb{Sh}(X; \mathcal{E}).
\end{array}
\]
We now want to show that the functor $f_\circ$ can be expressed through other previously defined operations in the particular case when $f$ induces a locally contractible geometric morphism. In this way we will vastly generalize the classical formula relating the $f^!$ and $f^*$ when $f$ is a topological submersion.

**Remark 6.3.** Let $f : X \to Y$ be a continuous map inducing a locally contractible geometric morphism. Let $U$ be an open subset of $Y$, and consider the pullback square

\[
\begin{array}{ccc}
 f^{-1}(U) & \xrightarrow{f'} & U \\
 \downarrow{j'} & & \downarrow{j} \\
 X & \xrightarrow{f} & Y.
\end{array}
\]

Notice that the projection morphism can be written as an equivalence

\[ j_2 j^* f^* \simeq f'_2 j'_2 (j')^* \simeq j_1 f'_1 (j')^*, \]

and so, since $j_1$ is fully faithful, we obtain an equivalence

\[ j^* f_1 \simeq f'_1 (j')^*. \]

Thus, by Lemma 5.2, we also have an equivalence

\[ j^* f_0 \simeq f'_0 (j')^*. \]

**Proposition 6.4.** Let $f : X \to Y$ be a continuous map inducing a locally contractible geometric morphism, and let $\mathcal{C}$ and $\mathcal{D}$ be two stable and bicomplete $\infty$-categories. Then, for any $F \in \text{Shv}(Y; \mathcal{C})$ and $G \in \text{Shv}(Y; \mathcal{D})$, we have a canonical equivalence

\[ f^! F \otimes f^* G \simeq f^!(F \otimes G), \]

or equivalently, when $\mathcal{C} = \mathcal{D}$ has a closed symmetric monoidal structure, a canonical equivalence for any $K \in \text{Shv}(X; \mathcal{C})$

\[ \text{Hom}_Y((-), f \circ (-)) \to f_! \text{Hom}_X(f^! (-), (-)) \]

as the image through the above equivalence of $f_! f^! F \otimes G \to F \otimes G$.

Since all functors appearing are cocontinuous, it suffices to prove the proposition when $\mathcal{C} = \mathcal{D} = \mathbf{Sp}$. Thus we will prove that the transformation

\[ \text{Hom}_Y((-), f_0 (-)) \to f_! \text{Hom}_X(f^! (-), (-)) \]

is natural.
is an equivalence. Keeping the same notations as in Remark 6.3, we have that for all \( F \in \text{Shv}(Y; \text{Sp}) \) and \( G \in \text{Shv}(X; \text{Sp}) \)

\[
\Gamma(U; \text{Hom}_{\text{Shv}((-1(U)) \text{Sp})}(f^*F, (j')^*G)) \\
\simeq \text{Hom}_{\text{Shv}((-1(U)) \text{Sp})}((f')^*f^*F, (j')^*G) \\
\simeq \text{Hom}_{\text{Shv}((-1(U)) \text{Sp})}(j^*F, f'_0(j')^*G) \\
\simeq \text{Hom}_{\text{Shv}((-1(U)) \text{Sp})}(j^*F, j'^*_0 f_0 G) \\
\simeq \text{Hom}_{\text{Shv}((-1(U)) \text{Sp})}(j^*F, j'^*_0 f_0 G) \\
\simeq \Gamma(U; \text{Hom}_{\text{Sp}}(F, f_0 G))
\]

where the second equivalence follows by Lemma 5.2 and the fifth one by Remark 6.3.

**Proposition 6.5.** Let \( f : X \to Y \) be a topological submersion of fiber dimension \( n \), and let \( \mathcal{C} \) be any stable presentable \( \infty \)-category. Then:

(i) \( f^! \mathbb{S}_Y \) is a locally constant sheaf of spectra on \( X \) with stalks equivalent to \( \mathbb{S}[n] \),

(ii) for all \( F \in \text{Shv}(Y; \mathcal{C}) \) there is a canonical equivalence

\[
f^! \mathbb{S}_Y \otimes f^*F \simeq f^! F
\]

or equivalently by adjunction, for every \( G \in \text{Shv}(X; \mathcal{C}) \)

\[
f_2 G \simeq f_1(G \otimes f^! \mathbb{S}_Y).
\]

**Proof.** To prove (i) one may follow verbatim the proof of [KS90, Proposition 3.2.3], while (ii) follows by Proposition 6.4 and the fact that \( \mathbb{S} \) is an invertible object with respect to the smash product of sheaves of spectra.

# 7 Relative Atiyah Duality

Recall that, if \( X \) is a compact smooth manifold, then the **Atiyah duality** states that \( \Sigma^\infty_+ X \) is strongly dual to the **Thom spectrum** associated to the virtual vector bundle given by the inverse of the tangent bundle of \( X \), which is denoted by \( \text{Th}(-T X) \). In this section we will take advantage of the full six functors formalism to give a generalization of this important classical result in terms of sheaves. We will start by recalling the definition of the **J-homomorphism** in the context of the theory of \( \infty \)-categories (as done for example in [Cla11, Section 2]) and after we will see that this morphism defines canonically (up to a contractible space of choices) a sheaf of spectra \( \text{Th}(E) \) on a topological space \( X \) associated to any vector bundle \( E \) over \( X \). Then, for any submersion \( f : X \to Y \) between smooth manifolds, we will deduce an equivalence between \( \text{Th}(T_f) \) and \( f^! \mathbb{S}_Y \), where \( T_f \) denotes the relative tangent bundle of \( f \), from which, when \( f \) is assumed to be proper, we will be able to obtain formally a relative version of Atiyah duality (with a proof that mirrors the one of [Hoy17, Corollary 6.13]).

## 7.1 Thom spaces and the J-homomorphism

Let \( \text{Top} \) be the 1-category of all topological spaces, and let \( \text{CW} \subseteq \text{CW}^h \subseteq \text{Top} \) be respectively the full subcategories spanned by spaces homeomorphic to a CW-complex and spaces homotopy equivalent to a CW-complex. Recall that \( \text{CW} \) is canonically enriched in compactly generated weak Hausdorff spaces, and that the coherent nerve of the simplicial category associated to it is equivalent to the \( \infty \)-category of spaces \( \mathcal{S} \). Recall also that \( \text{Top} \) can be
equipped with a Grothendieck topology given by open coverings: an important remark about this topology is that, even though Top is not small, the usual formula for the sheafification functor preserves presheaves with small fibers, since for any space \( X \in \text{Top} \) the category of coverings of \( X \) is small and each covering of \( X \) is indexed essentially by a small poset. Throughout this section, we will denote by \( L \) the aforementioned sheafification functor.

Let \( \text{Vect}_R \simeq \bigcup_{n \in \mathbb{N}} \text{BGL}_n(\mathbb{R}) \) be the topological category whose objects are finite dimensional real vector spaces, where \( \text{GL}_n(\mathbb{R}) \) is equipped with the usual topology of a manifold (or equivalently the compact-open topology). The usual direct sum of vector spaces makes \( \text{Vect}_R \) into a \( \mathbb{E}_\infty \) space. Since all maps in \( \text{GL}_n(\mathbb{R}) \) are proper, the one point compactification induces a monoidal functor

\[
\text{Vect}_R \to \text{CW}_*,
\]

of topological categories, where \( \text{CW}_* \) is viewed as a monoidal topological category through the usual smash product and the one point compactification of a topological space is canonically pointed with its point at infinity. Thus by passing to coherent nerves we obtain a monoidal functor of \( \infty \)-categories

\[
J : \text{Vect}_R \to \text{S}_*
\]

that is called the \textit{J-homomorphism}, and by post composing with the infinite suspension functor

\[
J^{st} : \text{Vect}_R \to \text{Sp}
\]

called the \textit{stable J-homomorphism}. To see that it is actually possible to recover the classical \( J \)-homomorphism from \( J \), one just need to notice that it is given by a collection of functors \( \text{BGL}_n(\mathbb{R}) \to \text{S}_* \) that correspond to morphisms

\[
\text{GL}_n(\mathbb{R}) \to \text{Hom}_{\text{S}_*}(S^n, S^n) \simeq \Omega^n S^n
\]

whose restriction to \( O_n(\mathbb{R}) \) gives the desired map. Recall that, for any monoidal \( \infty \)-category \( \mathcal{C} \) we can define the Picard groupoid of \( \mathcal{C} \), denoted by \( \text{Pic}(\mathcal{C}) \), as the group-like \( \mathbb{E}_\infty \) space given by the full subcategory of the maximal subgroupoid of \( \mathcal{C} \) spanned by invertible objects. Notice that, since any monoidal functor preserves invertible objects, Pic is actually functorial on \( \mathcal{C} \), and one can show that this functor is right adjoint to the inclusion of symmetric monoidal \( \infty \)-groupoids (i.e. group-like \( \mathbb{E}_\infty \)-spaces) into symmetric monoidal \( \infty \)-categories. Hence, since any suspension of the sphere spectrum is invertible, the stable \( J \)-homomorphism factors as a functor

\[
\text{Vect}_R \to \text{Pic}(\text{Sp})
\]

and since \( \text{Pic}(\text{Sp}) \) is a group, this corresponds uniquely (up to a contractible space of choices) to

\[
J^{st} : \text{K}^{top}(\mathbb{R}) \to \text{Pic}(\text{Sp})
\]

where \( \text{K}^{top}(\mathbb{R}) \) is defined as the group completion of \( \text{Vect}_R \), and can be shown to be equivalent to the infinite loop space of the classical topological \( K \)-theory spectrum.

Let \( f : X \to Y \) be any map of topological spaces. Since the pullback functor \( f^* : \text{Shv}(Y; \text{Sp}) \to \text{Shv}(X; \text{Sp}) \) is monoidal, we obtain consequently a functor

\[
\begin{array}{ccc}
\text{Top}^{op} & \longrightarrow & \mathbb{E}_\infty \text{Grp} \\
X & \longmapsto & \text{Pic}(\text{Shv}(X; \text{Sp}))
\end{array}
\]

whose global sections are \( \text{Pic}(\text{Shv}(\ast; \text{Sp})) \simeq \text{Pic}(\text{Sp}) \). By descent for \( \infty \)-topoi, and since taking spectrum objects and Picard groupoids both commute with limits, such functor is a sheaf.
Thus, if we still indicate by $K^{\text{top}}(\mathbb{R})$ the corresponding constant presheaf on Top, $J^st$ induces a natural transformation
\[ L(K^{\text{top}}(\mathbb{R})) \to \text{Pic}(\text{Shv}(-; \mathbb{S}p)) \] (7.1)
of sheaves of $E_\infty$-groups.

**Remark 7.1.** Assume that $X$ is locally contractible and hypercomplete, and let $L(\text{Vect}_\mathbb{R})$ be the sheafification of the constant presheaf of $E_\infty$-spaces associated to $\text{Vect}_\mathbb{R}$. Then, by Proposition 3.2, we have that
\[ \Gamma(X; L(\text{Vect}_\mathbb{R})) \simeq \text{Hom}_\mathbb{S}(\text{Sing}(X), \text{Vect}_\mathbb{R}) \]
and if $X$ is also paracompact, the right-hand side is equivalent to the $\infty$-groupoid $\text{Vect}_\mathbb{R}(X)$ of real vector bundles over $X$, which is equipped with an $E_\infty$ structure through the usual Whitney sum. Thus in particular, if we restrict to presheaves on $CW^h$, $L(K^{\text{top}}(\mathbb{R}))$ can be described as the sheafification of the presheaf of $E_\infty$ groups assigning to each topological space $X$ the group completion of $\text{Vect}_\mathbb{R}(X)$. In conclusion, we see that if we restrict to $CW^h$ the morphism (7.1) corresponds uniquely to a natural transformation
\[ \text{Vect}_\mathbb{R}(-) \to \text{Pic}(\text{Shv}(-; \mathbb{S}p)). \] (7.2)

We will give in a moment a very explicit description of this transformation by means of the six functors formalism.

**Definition 7.1.** Let $X$ be any topological space. Let $p : E \to X$ be a real vector bundle over $X$, which is equipped with an $E_\infty$ structure through the usual Whitney sum. Thus in particular, if we restrict to presheaves on $CW^h$, $L(K^{\text{top}}(\mathbb{R}))$ can be described as the sheafification of the presheaf of $E_\infty$-spaces assigned to each topological space $X$ the group completion of $\text{Vect}_\mathbb{R}(X)$. In conclusion, we see that if we restrict to $CW^h$ the morphism (7.1) corresponds uniquely to a natural transformation
\[ \text{Vect}_\mathbb{R}(-) \to \text{Pic}(\text{Shv}(-; \mathbb{S}p)). \] (7.2)

We will give in a moment a very explicit description of this transformation by means of the six functors formalism.

**Remark 7.2.** Notice that $p^*_f$ exists since any vector bundle is obviously a shape submersion, so indeed our definition of the Thom spectrum makes sense. Notice also that, by localization sequences, one has $\text{Th}(E) \simeq p^*_f 8_\mathbb{S}_X$.

**Lemma 7.1.** Let $X$ be a $CW$-complex, $a : X \to *$ the unique map. Then, for any vector bundle $E$ over $X$, $a^*_f \text{Th}(E)$ is equivalent to the Thom spectrum of $E$ as classically defined.

**Proof.** Let $b : E \to *$ and $c : E^\times \to *$ be the unique maps. By Proposition 3.2, we have equivalences
\[ a^*_f \text{Th}(E) \simeq b^*_f \text{cofib}(j_{j^*S_E \to S_E}) \]
\[ \simeq \text{cofib}(c^*S_{E^\times} \to b^*_f S_E) \]
\[ \simeq \text{cofib}(\Sigma^\infty_+ E^\times \to \Sigma^\infty_+ E) \]
and the spectrum on the last line coincides with the usual Thom spectrum of $E$. \qed

**Proposition 7.1.** The Thom spectrum induces a natural transformation of sheaves of $E_\infty$-spaces
\[ \text{Vect}_\mathbb{R}(-) \to \text{Pic}(\text{Shv}(-; \mathbb{S}p)) \]
on $CW^h$ that coincides with (7.2).
Proof. First of all, we show that the Thom spectrum induces a natural transformation
\[ \text{Vect}_R(-) \to \text{Shv}(-; \text{Sp}) \] (7.3)
of presheaves of \( \infty \)-categories. Let \( p : E \to X \) be a vector bundle, \( p^\times : E^\times \to X \) be the induced map on the complement of the zero section. Since one can write \( \text{Th}(E) \) as \( \text{cofib}(p^\times_\sharp \text{Sp}_{E^\times} \to p_\sharp \text{Sp}_E) \), it will suffice to show that the associations \( E \mapsto p_\sharp \text{Sp}_E \) and \( E \mapsto p^\times_\sharp \text{Sp}_{E^\times} \) induce natural transformations. Consider a pullback square in \( \text{Top} \)
\[
\begin{array}{ccc}
 f^*E & \xrightarrow{f'} & E \\
 \downarrow{p'} & & \downarrow{p} \\
 X' & \xrightarrow{f} & X
\end{array}
\]
where \( p \) is a vector bundle. We want to show that the induced square
\[
\begin{array}{ccc}
 \text{Shv}(f^*E) & \xrightarrow{f'} & \text{Shv}(E) \\
 \downarrow{p'} & & \downarrow{p} \\
 \text{Shv}(X') & \xrightarrow{f} & \text{Shv}(X)
\end{array}
\]
is a pullback square in \( \text{Top} \). By descent, it suffices to show that the square
\[
\begin{array}{ccc}
 \text{Shv}(X' \times \mathbb{R}^n) & \xrightarrow{f \times \text{id}_{\mathbb{R}^n}} & \text{Shv}(X \times \mathbb{R}^n) \\
 \downarrow{p'} & & \downarrow{p} \\
 \text{Shv}(X') & \xrightarrow{f} & \text{Shv}(X)
\end{array}
\]
is a pullback, but this is true because by Remark 2.8 the big rectangle
\[
\begin{array}{ccc}
 \text{Shv}(X' \times \mathbb{R}^n) & \xrightarrow{f \times \text{id}_{\mathbb{R}^n}} & \text{Shv}(X \times \mathbb{R}^n) \\
 \downarrow{p'} & & \downarrow{p} \\
 \text{Shv}(X') & \xrightarrow{f} & \text{Shv}(X) \\
 \downarrow{p} & & \downarrow{p} \\
 & & \text{Sp}
\end{array}
\]
and the right square are both pullbacks. Similarly one has that the square
\[
\begin{array}{ccc}
 \text{Shv}(f^*E^\times) & \xrightarrow{f'} & \text{Shv}(E^\times) \\
 \downarrow{p^\times} & & \downarrow{p^\times} \\
 \text{Shv}(X') & \xrightarrow{f} & \text{Shv}(X)
\end{array}
\]
is a pullback. Hence we have two natural transformations
\[ \text{Vect}_R(-) \to \text{Top}/\text{Shv}(-) \]
given respectively by sending a vector bundle \( E \to X \) to \( \text{Shv}(E) \to \text{Shv}(X) \) and to \( \text{Shv}(E^\times) \to \text{Shv}(X) \), and thus, by further composing with the relative shape, by Remark 3.3 we obtain lax natural transformations
\[ \text{Vect}_R(-) \to \text{Pro}(\text{Shv}(-)) \]
which factor as
\[ \text{Vect}_R(-) \to \text{Shv}(-) \]
since any shape submersion induces a locally contractible geometric morphism. Furthermore, by smooth base change and [Hau20, Theorem 3.22], we see that these are actually natural transformations, and thus, composing with

\[ \text{Shv}(-) \to \text{Shv}(-; Sp) \]

we get the natural transformation (\ref{eq:7.3}).

We now prove that (\ref{eq:7.3}) is monoidal and that it factors through Pic(\text{Shv}(-; Sp)). Since Vect\(_R(-)\) is the constant sheaf associated to Vect\(_R\), for any sheaf \(F\) of \(E_\infty\)-spaces on CW\(^h\), we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(\text{Vect}_R(-), \text{Pic}(F)) & \xrightarrow{\cong} & \text{Hom}_{E_\infty}(\text{Vect}_R(*), \text{Pic}(F(*))) \\
\downarrow & & \downarrow \\
\text{Hom}(\text{Vect}_R(-), F) & \xrightarrow{\cong} & \text{Hom}_{E_\infty}(\text{Vect}_R(*), F(*)) \\
\downarrow & & \downarrow \\
\text{Hom}(\text{Vect}_R(-), F) & \xrightarrow{\cong} & \text{Hom}_S(\text{Vect}_R(*), F(*))
\end{array}
\]

where the horizontal arrows are induced by taking global sections, the upper vertical arrows by the natural transformation Pic(\(F\)) \to \(F\) and the lower vertical arrows by the forgetful functor from \(E_\infty\)-spaces to spaces. By Lemma \ref{lem:7.1} we know that, after taking global sections, (\ref{eq:7.3}) factors through Pic(\(S_p\)), and thus we may conclude by a simple diagram chase.

**Corollary 7.1.** Let \(X\) be any topological space in CW\(^h\). Let \(p : E \to X\) be a real vector bundle over \(X\), and denote \(s : X \hookrightarrow E\) its zero section. Then Th(\(E\)) is invertible with inverse given by \(s^! S_E\).

**Proof.** By the previous proposition, we already know that Th(\(E\)) is invertible, and thus to compute its inverse we just need to look at its dual. Then we may conclude by smooth projection formula and the projection formula for pushforward with supports since

\[
\text{Hom}_X(p_* s^! S_X, S_X) \simeq p_* \text{Hom}_E(s^! S_X, S_E) \\
\simeq p_* s_* s^! S_E \\
\simeq s^! S_E.
\]

\(\square\)

### 7.2 Relative Atiyah duality

**Theorem 7.1.** Let \(f : X \to Y\) be a submersion between smooth manifolds. Then we have an equivalence \(f^! S_Y \simeq \text{Th}(T_f)\), where \(T_f\) is the relative tangent bundle of \(f\), defined by the short exact sequence of vector bundles

\[
T_f \to TX \to f^* TY. \tag{7.4}
\]

**Proof.** First of all we prove the case when \(f\) is the unique map \(a : X \to *\) and \(X\) is a smooth manifold. Choose a closed embedding \(i : X \to \mathbb{R}^n\), and let \(a' : \mathbb{R}^n \to *\) be the unique map. By Proposition \ref{prop:6.3}, we have \(a'^! S \simeq S_{\mathbb{R}^n}[n]\) and thus, since \(T\mathbb{R}^n\) is a trivial vector bundle of fiber dimension \(n\), we have Th(\(T\mathbb{R}^n\)) \simeq \(S_{\mathbb{R}^n}[n] \simeq a'^! S\). Let \(p : N_i \to X\) be the conormal bundle of the embedding \(i\), defined by the short exact sequence

\[
TX \to i^* T\mathbb{R}^n \to N_i, \tag{7.5}
\]
s : X ↣ N, its zero section. By the ϵ-neighbourhood theorem, there exists an open immersion
\( j : U \hookrightarrow \mathbb{R}^n \) and a homomorphism \( g : N \to U \) such that we have a factorization
\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{R}^n \\
\downarrow{\iota} & & \downarrow{j} \\
N & \xrightarrow{\sim} & U.
\end{array}
\]

Thus, we obtain equivalences
\[
i^! \mathbb{S}^{n} \simeq s^! g^* f^* \mathbb{S}^{n}
\simeq s^! \mathbb{S}_{N_i}
\simeq \text{Th}(N_i)^{-1}.
\]

Then, by Proposition 6.5 and (7.5) we have
\[
a^! \mathbb{S} \simeq a^! i^! \mathbb{S}
\simeq a^! \mathbb{S} \otimes i^* \text{Th}(TR^n)
\simeq \text{Th}(N_i)^{-1} \otimes \text{Th}(i^* TR^n)
\simeq \text{Th}(TX).
\]

Suppose now that \( f : X \to Y \) is any submersion between smooth manifolds, \( a : X \to * \) and \( b : Y \to * \) be the unique maps. Then, by Proposition 6.5 and (7.4), we have
\[
\text{Th}(Tf) \simeq \text{Th}(TX) \otimes \text{Th}(f^* TY)^{-1}
\simeq a^! \mathbb{S} \otimes (f^! b^! \mathbb{S})^{-1}
\simeq a^! \mathbb{S} \otimes (f^! b^! \mathbb{S})^{-1} \otimes f^! \mathbb{S}_Y
\simeq f^! \mathbb{S}_Y.
\]

and thus we can conclude. \( \square \)

**Corollary 7.2** (Relative Atiyah Duality). Let \( f : X \to Y \) be a proper shape submersion. Then \( f^! \mathbb{S}_X \in \text{Shv}(Y; Sp) \) is strongly dualizable with dual \( f^* \mathbb{S}_X \). In particular, if \( X \) and \( Y \) are smooth manifolds and \( f \) is a proper topological submersion, then \( f^! \mathbb{S}_X \) is strongly dualizable with dual \( f^* \text{Th}(-Tf) \).

**Proof.** Since \( f \) is proper, by smooth projection formula and the projection formula for the pushforward with supports, we have, functorially on \( F \in \text{Shv}(Y; Sp) \)
\[
\text{Hom}_Y(f^! \mathbb{S}_X, F) \simeq f^* \text{Hom}_X(\mathbb{S}_X, f^* F)
\simeq f_* f^* F
\simeq f_* f^* F
\simeq f^* \text{Th}(-Tf).
\]

In particular, when \( f \) is a submersion of smooth manifolds, by Proposition 6.5 and the previous theorem, we have \( f^* \mathbb{S}_X \simeq f^* \text{Th}(-Tf) \). \( \square \)

**Remark 7.3.** Let \( X \) be a smooth manifold, \( a : X \to * \) the unique map. By specializing the previous corollary to \( a \) and Lemma 7.1 we see that we recover the classical Atiyah duality.
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