A Liouville comparison principle for sub- and super-solutions of the equation
\[ w_t - \Delta_p(w) = |w|^{q-1}w. \]

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Abstract

We establish a Liouville comparison principle for entire weak sub- and super-solutions of the equation \((\ast)\) \[ w_t - \Delta_p(w) = |w|^{q-1}w \] in the half-space \( S = \mathbb{R}_+ \times \mathbb{R}^n \), where \( n \geq 1 \), \( q > 0 \) and \( \Delta_p(w) := \text{div}_x (|\nabla_x w|^{p-2} \nabla_x w) \), \( 1 < p \leq 2 \). In our study we impose neither restrictions on the behaviour of entire weak sub- and super-solutions on the hyper-plane \( t = 0 \), nor any growth conditions on their behaviour and on that of any of their partial derivatives at infinity. We prove that if \( 1 < q \leq p - 1 + \frac{2}{n} \), and \( u \) and \( v \) are, respectively, an entire weak super-solution and an entire weak sub-solution of \((\ast)\) in \( S \) which belong, only locally in \( S \), to the corresponding Sobolev space and are such that \( u \geq v \), then \( u \equiv v \). The result is sharp. As direct corollaries we obtain known Fujita-type and Liouville-type theorems.

1 Introduction and definitions.

The purpose of this work is to obtain a Liouville comparison principle of elliptic type for entire weak sub- and super-solutions of the equation
\[ w_t - \Delta_p(w) = |w|^{q-1}w \quad (1) \]
in the half-space \( S = (0, +\infty) \times \mathbb{R}^n \), where \( n \geq 1 \) is a natural number, \( q > 0 \) is a real number and \( \Delta_p(w) := \sum_{i=1}^n \frac{d}{dx_i} A_i(\nabla w) \), with \( A_i(\xi) = |\xi|^{p-2}\xi_i \) for all
\[ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \text{ and } p > 1, \text{ defines the well-known } p\text{-Laplacian operator.} \]

Under entire sub- and super-solutions of (1) we understand sub- and super-solutions of (1) defined in the whole half-space \( S \), and under Liouville results of elliptic type for sub- and super-solutions of the parabolic equation (1) in the half-space \( S \) we understand Liouville-type results which, in their formulations, have no restrictions on the behaviour of sub- and super-solutions of (1) on the hyper-plane \( t = 0 \). Also, we would like to underline that we impose neither growth conditions on the behaviour of sub- and super-solutions to (1) or on that of any of their partial derivatives at infinity.

**Definition 1** Let \( n \geq 1 \), \( p > 1 \) and \( q > 0 \). A function \( u = u(t, x) \) defined and measurable in \( S \) is called an entire weak super-solution of the equation (1) in \( S \) if it belongs to the function space \( L^{q,\text{loc}}(S) \), with \( u_t \in L^{1,\text{loc}}(S) \) and \( |\nabla_x u|^p \in L^{1,\text{loc}}(S) \), and satisfies the integral inequality

\[
\int_S \left[ u_t \phi + \sum_{i=1}^n |\nabla_x u|^{p-2} u_{x_i} \phi_{x_i} - |u|^{q-1} u \phi \right] \, dt \, dx \geq 0 \tag{2}
\]

for every non-negative function \( \phi \in C^{\infty}(S) \) with compact support in \( S \), where \( C^{\infty}(S) \) is the space of all functions defined and infinitely differentiable in \( S \).

**Definition 2** A function \( v = v(t, x) \) is an entire weak sub-solution of (1) if \( u = -v \) is an entire weak super-solution of (1).

2 **Results.**

**Theorem 1** Let \( n \geq 1 \), \( 2 \geq p > 1 \) and \( 1 < q \leq p - 1 + \frac{2}{n} \), and let \( u \) be an entire weak super-solution and \( v \) an entire weak sub-solution of (1) in \( S \) such that \( u \geq v \). Then \( u = v \) in \( S \).

The result in Theorem 1, which evidently has a comparison principle character, we term a Liouville-type comparison principle, since, in the particular cases when \( u \equiv 0 \) or \( v \equiv 0 \), it becomes a Liouville-type theorem of elliptic type, respectively, for entire weak sub-solutions or entire weak super-solutions of (1).

Since in Theorem 1 we impose no conditions on the behaviour of entire weak sub- or super-solutions of the equation (1) on the hyper-plane \( t = 0 \), we
can formulate, as a direct corollary of Theorem 1, the following comparison principle, which in turn one can term a Fujita comparison principle, for entire weak sub- and super-solutions of the Cauchy problem for the equation (1). It is clear that in the particular cases when \( u \equiv 0 \) or \( v \equiv 0 \), it becomes a Fujita-type theorem, respectively, for entire weak sub-solutions or entire weak super-solutions of the Cauchy problem for the equation (1).

**Theorem 2** Let \( n \geq 1, \ 2 \geq p > 1 \) and \( 1 < q \leq p - 1 + \frac{2}{n} \), and let \( u \) be an entire weak super-solution and \( v \) an entire weak sub-solution of the Cauchy problem, with possibly different initial data for \( u \) and \( v \), for the equation (1) in \( S \) such that \( u \geq v \). Then \( u = v \) in \( S \).

**Remark 1** The initial data for \( u \) and \( v \) in Theorem 2 may be different.

Note that the results in Theorems 1 and 2 are sharp and that the hypotheses on the parameter \( p \) in these theorems in fact force \( p \) to be greater than \( \frac{2n}{n+1} \). The sharpness of these results for \( q > p - 1 + \frac{2}{n} \geq 1 \) follows, for example, from the existence of non-negative self-similar entire solutions to (1) in \( S \), which was shown in [1]. Also, there one can find a Fujita-type theorem on the non-existence of non-negative entire solutions of the Cauchy problem for (1), which was obtained as a very interesting generalization of the famous blow-up result established in [4], [5] and [9] to quasilinear parabolic equations. For \( 0 < q \leq 1 \), it is evident that the function \( u(t, x) = e^t \) is a positive entire classical super-solution of (1) in \( S \).

We would also like to note that the results of the present work were announced in [13] and that similar results for solutions of semilinear parabolic inequalities were obtained in [7]. To prove the results we use the \( \alpha \)-monotonicity property of the \( p \)-Laplacian operator which was established in [11] and continue to develop an approach in [7] and [8], the elliptic analogue of which was proposed [11]. That approach was subsequently used and developed in the same framework by E. Mitidieri, S. Pokhozhayev and many others, almost none of which cite the original research in [11].

For a survey of the literature on the asymptotic behaviour of and blow-up results for solutions, sub- and super-solutions of the Cauchy problem for nonlinear parabolic equations we refer to [2], [3], [6], [14], [15] and [16].
3 Proofs.

In what follows, for $q > 1$ and $2 \geq p > 1$, let

$$\omega = \frac{p(q - 1)}{q - p + 1}$$

and

$$P(R) = \{(t, x) \in S: t^{2/\omega} + |x|^2 < R^{2/\omega}\}$$

for all $R > 0$. In this case it is clear that $0 < \omega \leq 2$ and that the inequality

$$\text{volume of } P(R) \leq cR^{\frac{n+\omega}{\omega}},$$

with $c$ some positive constant which depends possibly only on $n$ and $\omega$, holds for all $R > 0$.

**Proof of Theorem 1.** Let $n \geq 1$, $2 \geq p > 1$ and $1 < q \leq p - 1 + \frac{2}{n}$, and let $u$ be an entire weak super-solution and $v$ an entire weak sub-solution of (1) in $S$ such that $u \geq v$. By the well-known inequality

$$(|u|^{q-1}u - |v|^{q-1}v)(u - v) \geq 2^{1-q}|u - v|^{q+1}$$

which holds for every $q \geq 1$ and all $u, v \in \mathbb{R}^1$ we obtain from (2) the relation

$$\int_S \left[ (u - v)\varphi + \sum_{i=1}^n \varphi_{x_i}(|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) \right] dtdx \geq 2^{1-q} \int_S (u - v)^q \varphi dtdx,$$

which holds for every non-negative function $\varphi \in C^\infty(S)$ with compact support in $S$. Let $\tau > 0$ and $R > 0$ be real numbers. Let $\eta: [0, +\infty) \to [0, 1]$ be a $C^\infty$-function which has the non-negative derivative $\eta'$ and equals 0 on the interval $[0, \tau]$ and 1 on the interval $[2\tau, +\infty)$, and let $\zeta: [0, +\infty) \times \mathbb{R}^n \to [0, 1]$ be a $C^\infty$-function which equals 1 on $P(R/2)$ and 0 on $\{(0, +\infty) \times \mathbb{R}^n \} \setminus P(R)$. Let $\varphi(t, x) = (w(t, x) + \varepsilon)^{-\nu} \zeta^s(t, x) \eta^2(t)$, where $w(t, x) = u(t, x) - v(t, x)$, $\varepsilon > 0$ and the positive constants $s > 1$ and $\nu \in (0, p - 1)$ will be chosen.
below. Substituting the function $\varphi$ in (5) and then integrating by parts we arrive at

\[-s \int_{P(R)} (w + \varepsilon)^{1-\nu} \zeta^{s-1} \eta^2 dt dx - \frac{2}{1-\nu} \int_{P(R)} (w + \varepsilon)^{1-\nu} \zeta s \eta^2 dt dx \]

\[-\nu \int_{P(R)} \sum_{i=1}^{n} w_{x_i} (|\nabla u|^{p-2} u_{x_i} - |\nabla v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu} \zeta^{s-1} \eta^2 dt dx \]

\[+ s \int_{P(R)} \sum_{i=1}^{n} \zeta_{x_i} (|\nabla u|^{p-2} u_{x_i} - |\nabla v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu} \zeta^{s-1} \eta^2 dt dx \]

\[= I_1 + I_2 + I_3 + I_4 \geq 2^{1-q} \int_{P(R)} w^q (w + \varepsilon)^{-\nu} \zeta^{s-1} \eta^2 dt dx. \quad (6)\]

First, observing that $I_3$ is non-positive, we estimate $I_4$ in terms of $I_3$ using the fact, which is a key point in our proof, that for $1 < p \leq 2$ the $p$-Laplacian operator $\Delta_p$ satisfies the $\alpha$-monotonicity condition (see, e.g., [12]) with $\alpha = p$. This in our case consists mostly of the fact that there exists a positive constant $K$ such that the coefficients $A_i, i = 1, \ldots, n,$ of the $p$-Laplacian operator satisfy the inequality

\[\left( \sum_{i=1}^{n} (A_i (\xi^1) - A_i (\xi^2))^2 \right)^{\alpha/2} \leq K \left( \sum_{i=1}^{n} (\xi^1_i - \xi^2_i) (A_i (\xi^1) - A_i (\xi^2)) \right)^{\alpha-1} \]

for all pairs $\xi^1, \xi^2 \in \mathbb{R}^n$ and $\alpha = p$, provided $1 < p \leq 2$. As a result, we have the relation

\[|I_4| \leq \int_{P(R)} c_1 |\nabla_x \zeta| \left( \sum_{i=1}^{n} w_{x_i} (|\nabla u|^{p-2} u_{x_i} - |\nabla v|^{p-2} v_{x_i}) \right)^{\frac{p-1}{p}} (w + \varepsilon)^{-\nu} \zeta^{s-1} \eta^2 dt dx. \quad (8)\]

Here we use the symbols $c_i, i = 1, \ldots, 8$, to denote constants depending possibly on $n, p, q, s$ or $\nu$ but not on $R, \varepsilon$ or $\tau$. Further, estimating the integrand on the right-hand side of (8) by Young’s inequality

\[AB \leq \rho A^\frac{\beta}{\beta+1} + \rho^{1-\beta} B^\beta \]

(9)

5
with \( \rho = \frac{\eta}{2}, \beta = p \),

\[
\mathcal{A} = \left( \sum_{i=1}^{n} w_i \left( |\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i} \right) \right)^{\frac{p-1}{p}} (w + \varepsilon)^{\frac{q-\nu}{q}} \eta \frac{2(1-\nu)}{q-\nu} \]

and

\[
\mathcal{B} = c_1 |\nabla_x \zeta| \left( w + \varepsilon \right)^{\frac{p-1-\nu}{p}} \eta^{\frac{q-1}{q}} \]

we arrive at

\[
|I_4| \leq \frac{\nu}{2} \int_{P(R)} \sum_{i=1}^{n} w_i \left( |\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i} \right) (w + \varepsilon)^{-\nu-1} \zeta^2 \eta^2 dt dx
\]

\[
+ \int_{P(R)} c_2 (w + \varepsilon)^{p-1-\nu} |\nabla_x \zeta|^p \zeta^s \eta^2 dt dx. \quad (10)
\]

Now, observing that \( I_2 \) in (6) is also non-positive, we obtain from (6) and (10) the relation

\[
\int_{P(R)} c_2 (w + \varepsilon)^{1-\nu} |\zeta|^\nu \zeta^s \eta^2 dt dx + \int_{P(R)} c_2 (w + \varepsilon)^{p-1-\nu} |\nabla_x \zeta|^p \zeta^s \eta^2 dt dx
\]

\[
\geq \int_{P(R)} \omega^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dt dx
\]

\[
+ \int_{P(R)} \sum_{i=1}^{n} w_i \left( |\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i} \right) (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx. \quad (11)
\]

Estimating both integrands on the left-hand side of (11) by Young’s inequality (9) with \( \rho = \frac{1}{2}, \beta = \frac{q-\nu}{q-1} \),

\[
\mathcal{A} = (w + \varepsilon)^{1-\nu} \zeta^{\frac{q(1-\nu)}{q-\nu}} \eta \frac{2(1-\nu)}{q-\nu},
\]

\[
\mathcal{B} = c_2 |\zeta|^\nu \zeta^{\frac{2(1-\nu)}{q-\nu}} \eta \frac{2(1-\nu)}{q-\nu}
\]

and \( \rho = \frac{1}{2}, \beta = \frac{q-\nu}{q-p+1} \),

\[
\mathcal{A} = (w + \varepsilon)^{p-1-\nu} \zeta^{\frac{(p-1-\nu)}{q-p}} \eta \frac{2(p-1-\nu)}{q-p},
\]
\[ B = c_2 |\nabla_x \zeta|^p \zeta^{\frac{p(a-p+1)}{q-p}} - \eta^{\frac{p(a-p+1)}{q-p}}, \]

respectively, we have the relation
\[
\frac{1}{2} \int_{P(R)} (w + \epsilon)^{q-\nu} \zeta^{s} \eta^2 dt dx + c_3 \int_{P(R)} |\zeta_t|^{\frac{q-\nu}{q-p+1}} \zeta^{s-\frac{q-\nu}{q-p+1}} \eta^2 dt dx
\]
\[
+ \frac{1}{2} \int_{P(R)} (w + \epsilon)^{q-\nu} \zeta^{s} \eta^2 dt dx + c_3 \int_{P(R)} |\nabla_x \zeta|^{\frac{p(q-\nu)}{q-p+1}} \zeta^{s-\frac{p(q-\nu)}{q-p+1}} \eta^2 dt dx
\]
\[ \geq \int_{P(R)} w^q (w + \epsilon)^{-\nu} \zeta^{s} \eta^2 dt dx + \]
\[
\int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w + \epsilon)^{-\nu-1} \zeta^{s} \eta^2 dt dx. \quad (12)
\]

Further, we estimate the integral
\[
\int_{P(R)} w^q \zeta^{s} \eta^2 dt dx
\]
by the inequality (12). To this end, we substitute
\[ \varphi(t, x) = \zeta^s(t, x) \eta^2(t) \]
in (5) and after integration by parts there we obtain
\[
-s \int_{P(R)} w \zeta_t \zeta^{s-1} \eta^2 dt dx - 2 \int_{P(R)} w \zeta^s \eta_t \eta dt dx
\]
\[ + s \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \]
\[ \geq 2^{1-q} \int_{P(R)} w^q \zeta^{s} \eta^2 dt dx. \quad (13)\]

Since the second term on the left-hand side of (13) is non-positive we have
\[
\int_{P(R)} w |\zeta_t| \zeta^{s-1} \eta^2 dt dx
\]

7
\[ + s \int_{P(R)} \sum_{i=1}^{n} \zeta_{x_i} (|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) \zeta^{s-1} \eta^2 \, dtdx \geq 2^{1-q} \int_{P(R)} w^q \zeta^s \eta^2 \, dtdx. \]  

(14)

Now, estimating the first integral on the left-hand side of (14) by Hölder’s inequality, we arrive at

\[ s \left( \int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 \, dtdx \right)^{\frac{1}{q}} \left( \int_{P(R)} |\zeta_t|^{\frac{q}{q-1}} \zeta^{s-\frac{q}{q-1}} \eta^2 \, dtdx \right)^{\frac{q-1}{q}} \]

\[ + s \int_{P(R)} \sum_{i=1}^{n} \zeta_{x_i} (|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) \zeta^{s-1} \eta^2 \, dtdx \]

\[ \geq 2^{1-q} \int_{P(R)} w^q \zeta^s \eta^2 \, dtdx. \]  

(15)

On the other hand, by (7) we have

\[ \int_{P(R)} \sum_{i=1}^{n} \zeta_{x_i} (|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) \zeta^{s-1} \eta^2 \, dtdx \]

\[ \leq c_4 \int_{P(R)} |\nabla_x \zeta| \left( w_{x_i} (|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) \right)^{\frac{q-1}{p}} \zeta^{s-1} \eta^2 \, dtdx. \]  

(16)

Estimating the right-hand side of (16) by Hölder’s inequality we arrive at the relation

\[ \int_{P(R)} \sum_{i=1}^{n} \zeta_{x_i} (|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) \zeta^{s-1} \eta^2 \, dtdx \]

\[ \leq c_4 \left( \int_{P(R)} (w + \varepsilon)^{(1+\nu)(p-1)} (\nabla_x \zeta)^{p} \zeta^{s-p} \eta^2 \, dtdx \right)^{1/p} \]

\[ \times \left( \int_{P(R)} \sum_{i=1}^{n} w_{x_i} (|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}) (w + \varepsilon)^{-\nu-1} \zeta \eta^2 \, dtdx \right)^{\frac{p-1}{p}} \]  

(17)
which holds for every \( \varepsilon > 0 \) and \( p - 1 > \nu > 0 \). Further, for any \( d > 1 \) we have

\[
\int_{P(R)} (w + \varepsilon)^{(1+\nu)(p-1)} |\nabla_x \zeta|^p \eta^2 dt \, dx \\
\leq \left( \int_{P(R) \setminus P(R/2)} (w + \varepsilon)^{d(1+\nu)(p-1)} \zeta^s \eta^2 dt \, dx \right)^{\frac{1}{d}} \\
\times \left( \int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^s \eta^2 dt \, dx \right)^{\frac{d-1}{pd}}.
\]

(18)

Now, we choose for every \( q > 1 \) and a sufficiently small \( \nu \) from the interval \((0, p - 1)\) the parameter \( d \) such that \( d(1+\nu)(p-1) = q \). Then (17) and (18) yield

\[
\int_{P(R)} \sum_{i=1}^{n} \zeta_x \left( |\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i} \right) \zeta^s \eta^2 dt \, dx \\
\leq c_4 \left( \int_{P(R) \setminus P(R/2)} (w + \varepsilon)^q \zeta^s \eta^2 dt \, dx \right)^{\frac{1}{pd}} \\
\times \left( \int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^s \eta^2 dt \, dx \right)^{\frac{d-1}{pd}} \\
\times \left( \int_{P(R)} \sum_{i=1}^{n} \frac{w_{x_i}}{d} \left( |\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i} \right) (w + \varepsilon)^{\nu-1} \zeta^s \eta^2 dt \, dx \right)^{\frac{p-1}{p}}.
\]

(19)

Estimating the last term on the right-hand side of (19) by (12), we have

\[
\int_{P(R)} \sum_{i=1}^{n} \zeta_x \left( |\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i} \right) \zeta^s \eta^2 dt \, dx \\
\leq c_4 \left( \int_{P(R) \setminus P(R/2)} (w + \varepsilon)^q \zeta^s \eta^2 dt \, dx \right)^{\frac{1}{pd}} \\
\times \left( \int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^s \eta^2 dt \, dx \right)^{\frac{d-1}{pd}}.
\]

9
Further, (15) and (21) yield

$$
\times \left( \int_{P(R)} (w + \varepsilon)^{q - \nu} \zeta^s \eta^2 dtdx - \int_{P(R)} w^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dtdx \right)
$$

$$
+ c_3 \int_{P(R)} |\zeta|^{\frac{q - \nu}{q - 1}} \zeta^s \eta^2 dtdx + c_3 \int_{P(R)} |\nabla \zeta|^{\frac{p(q - \nu)}{q - p + 1}} \zeta^s \eta^2 dtdx \right)^{\frac{p - 1}{p}}.
$$

In (20), passing to the limit as $\varepsilon \to 0$ as justified by Lebesgue’s theorem (see, e.g., [10, p. 303]) we obtain

$$
\int_{P(R)} \sum_{i=1}^n \zeta_i (|\nabla u|^{p - 2} u_{x_i} - |\nabla^2 u|^{p - 2} u_{x_i}) \zeta^{s - 1} \eta^2 dtdx
$$

$$
\leq c_5 \left( \int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dtdx \right)^{\frac{1}{q}} \left( \int_{P(R)} |\nabla \zeta|^{\frac{p(q - \nu)}{q - p + 1}} \zeta^s \eta^2 dtdx \right)^{\frac{p - 1}{p}} \times \left( \int_{P(R)} |\zeta|^{\frac{q - \nu}{q - 1}} \zeta^s \eta^2 dtdx + \int_{P(R)} |\nabla \zeta|^{\frac{p(q - \nu)}{q - p + 1}} \zeta^s \eta^2 dtdx \right).
$$

Further, (15) and (21) yield

$$
\int_{P(R)} w^q \zeta^s \eta^2 dtdx \leq c_6 \left( \int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dtdx \right)^{\frac{1}{q}} \left( \int_{P(R)} |\zeta|^{\frac{q - \nu}{q - 1}} \zeta^s \eta^2 dtdx \right)^{\frac{2 - 1}{q}}
$$

$$
+ c_6 \left( \int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dtdx \right)^{\frac{1}{p}} \left( \int_{P(R)} |\nabla \zeta|^{\frac{p(q - \nu)}{q - p + 1}} \zeta^s \eta^2 dtdx \right)^{\frac{p - 1}{p}} \times \left( \int_{P(R)} |\zeta|^{\frac{q - \nu}{q - 1}} \zeta^s \eta^2 dtdx + \int_{P(R)} |\nabla \zeta|^{\frac{p(q - \nu)}{q - p + 1}} \zeta^s \eta^2 dtdx \right)^{\frac{2 - 1}{p}}.
$$

Now, for arbitrary $(t, x) \in S$ and $R > 0$, we choose in (22) the function
\( \zeta = \zeta(t, x) \) in the form

\[
\zeta(t, x) = \psi \left( \frac{t^{2/\omega} + |x|^2}{R^{2/\omega}} \right),
\]

(23)

where \( 0 < \omega \leq 2 \) is given by (3) and \( \psi : [0, \infty) \to [0, 1] \) is a \( C^\infty \)-function which equals 1 on \([0, 2^{-\frac{2}{\omega}}]\) and 0 on \([1, \infty)\) and is such that the inequalities

\[
|\zeta_t| \leq c_7 R^{-1} \quad \text{and} \quad |\nabla_x \zeta| \leq c_7 R^{-\frac{2}{\omega}}
\]

(24)

hold. Note that it is always possible to find such a function \( \zeta \). Indeed, this can be easily verified by direct calculation of the corresponding derivatives of the function \( \zeta \) given by (23). Also, choosing in (22) the parameter \( s \) sufficiently large, we have from (22) by (4) and (24) the relation

\[
\int_{P(R)} w_q \zeta^s \eta^2 \, dt \, dx \leq c_8 \left( R^{\frac{n+s-\frac{d}{q}+1}{q-1}} + R^{\frac{n+s-\frac{d(q-p)}{q-p+1}}{p}} \right)^{\frac{1}{q}}
\]

(25)

which in turn by (3) implies

\[
\int_{P(R)} w_q \zeta^s \eta^2 \, dt \, dx \leq c_8 \left( R^{\frac{n+s-\frac{d}{q}+1}{q-1}} \right)^{\frac{1}{q}} \left( \int_{P(R) \setminus P(R/2)} w_q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}}
\]

(25)

Making simple calculation in (25) we arrive at

\[
\int_{P(R)} w_q \zeta^s \eta^2 \, dt \, dx \leq c_8 R^{\frac{n}{pq} \left[ q-p+1-\frac{p}{n} \right]} \left( \int_{P(R) \setminus P(R/2)} w_q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}}
\]

11
Further, since for $q > 1$, $2 \geq p > 1$ and $p - 1 > \nu > 0$ the quantities
\[
\frac{n}{pq} \quad \text{and} \quad \frac{n(pq - p + 1 - \nu(p - 1))}{p^2(q - 1)}
\]
are positive, we obtain from (26) for $1 < q < p - 1 + \frac{\nu}{n}$ the relation
\[
\int_{\Sigma} u^q \eta^2 \, dt \, dx = 0. \tag{27}
\]
Also, for $q = p - 1 + \frac{\nu}{n}$ we deduce from (26) that
\[
\int_{\Sigma} u^q \eta^2 \, dt \, dx < \infty.
\]
The latter yields the relation
\[
\int_{\frac{P(R_k)}{P(R_k) \cap P(R_k/2)}} u^q \eta^2 \, dt \, dx \to 0 \tag{28}
\]
which holds for every sequence $R_k \to \infty$. On the other hand, the inequality
\[
\int_{\frac{P(R/2)}{P(R/2) \cap P(R/2)}} u^q \eta^2 \, dt \, dx \leq c_8 R^\frac{n}{pq} \left[ q - p + 1 - \frac{\nu}{n} \right] \left( \int_{\frac{P(R)}{P(R) \cap P(R/2)}} u^q \eta^2 \, dt \, dx \right)^\frac{1}{q}
\]
\[
+ c_8 R \frac{n(pq - p + 1 - \nu(p - 1))}{p^2(q - 1)} \left[ q - p + 1 - \frac{\nu}{n} \right] \left( \int_{\frac{P(R)}{P(R) \cap P(R/2)}} u^q \eta^2 \, dt \, dx \right)^\frac{1}{pq} \tag{29}
\]
follows easily from (26). In turn, (28) and (29) imply for $q = p - 1 + \frac{\nu}{n}$ that the relation
\[
\int_{\frac{P(R_k)}{P(R_k)}} u^q \eta^2 \, dt \, dx \to 0
\]
holds for every sequence $R_k \to \infty$. The latter implies that (27) holds for every $q$ satisfying

$$1 < q \leq p - 1 + \frac{p}{n}.$$  \hspace{1cm} (30)

In (27), by letting the parameter $\tau$ in the definition of the function $\eta$ tend to zero, we obtain that $u(t, x) = v(t, x)$ a.e. in $\mathcal{S}$ for every $q$ which satisfies (30). \hfill \Box

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