The Size of Compact Extra Dimensions from Blackbody Radiation Laws

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In this work we generalize the Stefan-Boltzmann and Wien’s displacement laws for a D-dimensional spacetime composed by 4 non-compact dimensions and $D - 4$ compact dimensions. The electromagnetic field is assumed to pervade all compact and non-compact dimensions. In particular, the total radiated power becomes $R(T) = \sigma_B T^4 + \sigma_D(a) T^D$, where $a$ is the size of compact extra dimensions. For $D = 10$, predicted from String Theory, and $D = 11$, from M-Theory, the outcomes agree with available experimental data for $a$ as high as $2 \cdot 10^{-7}$m.

Interesting proposals to solve open problems in particle physics, like the Hierarchy problem, and in cosmology, like the dark matter and dark energy, involve spacetimes with dimensions different from the usually accepted four. The original proposal of a model with one extra dimension was done by Kaluza and Klein in the 1920’s to unify gravity, described by the general theory of relativity, with Maxwell’s electromagnetism. In their model, the fifth dimension is compactified on a circle of a given radius, while the other four non-compact dimensions are identified with our usual space-time.[1]

String theory is presently the most viable candidate for unifying gravity with the other fundamental interactions described by quantum field theories and requires a 9+1 dimensional spacetime. String theory is also related to M-theory defined in 10+1 dimensional spacetime.[2]. Usually, the extra dimensions are supposed to be compact, with a compactification parameter that might be related to the Planck scale.

About a decade ago, different models have been proposed to deal with the size of extra dimensions, keeping the extra dimensions compact[3][4] or non-compact[5]. Many predictions of these and other models are being tested in the ongoing LHC experiments, with hadronic beams colliding now at 7TeV and in the forthcoming years at 14TeV.

Another way to test the predictions of string/M-theory is to look at the physics of low energy processes. It is expected that, in some way, string/M-theory shall reproduce the physics that one can observe in scales much lower than the Planck scale.

Blackbody radiation is very well described by Planck’s law which implies, and explains, the Stefan-Boltzmann and Wien laws. In particular, the Stefan-Boltzmann law predicts that the electromagnetic radiated power of a blackbody is $R(T) = \sigma_B T^4$, where $\sigma_B$ is the Stefan-Boltzmann constant.

Recently, it was pointed out that the blackbody radiation laws should be corrected if the radiation is confined into a cavity.[6]. On the other side, it has been noticed that the blackbody radiation laws should depend on the (flat and non-compact) spacetime dimensions $D$[7][8] such that the Stefan-Boltzmann law would be modified to $R(T) = \sigma_D T^D$ with $\sigma_D = \text{constant}$, for any temperature in constrast with observed data.

A blackbody is defined as a body whose surface completely absorbs all radiation falling upon it. As consequence, all blackbodies emit thermal radiation with the same spectrum. In order to study its properties one takes a small bidimensional orifice connecting an isothermal enclosure to its outside as a blackbody surface. Here we consider that the blackbody is immersed in a $D$-dimensional spacetime with $D - 4$ compact extra dimensions.

The electromagnetic radiation inside the enclosure is assumed to be composed of standing waves. Choosing a system of orthogonal coordinates with origin at one of the enclosure’s vertices, we take for the sake of simplicity[9] $\ell$ as the length of the edges parallel to the non-compact axes $x_i$ ($i = 1, 2, 3$) and $a$ as the length of the edges related to the compact axes $x_j$ ($j = 4, ..., D - 1$), so the $i$-th and $j$-th components of the electric field ($l = i, j$) are given by

$$E_l(x_1, t) = E_{0_l} \sin(k_1 x_1) e^{-i\omega t}.$$  \hfill (1)

The $i$-th electric field components satisfy Dirichlet boundary conditions $E_i(0, t) = E_i(\ell, t) = 0$, for which there is no surface currents on the isothermal cavity walls, while the $j$-th components satisfy periodic boundary conditions $E_j(x_j, t) = E_j(x_j + a, t)$. Thus

$$k_i = \frac{\pi}{\ell} n_i \ , \quad k_j = \frac{2\pi}{a} n_j \ ,$$  \hfill (2)

where $n_i, n_j = 0, 1, 2, 3, ...$ represent the possible modes of vibration.

Considering the standing waves like components of a plain electromagnetic one with wave-vector $k$ and since the frequency $\nu = |k| c/2\pi$, we get for each mode $(n_i, n_j)$

$$\nu_{ij} \equiv \nu(n_i, n_j) = \frac{c}{a} \sqrt{\frac{a^2}{4\ell^2} \sum_{i=1}^3 n_i^2 + \sum_{j=4}^{D-1} n_j^2} \ .$$  \hfill (3)
Making use of Bose-Einstein statistical prescription and accounting two helicity states for the photons associated with the standing waves, the energy density inside the isothermal enclosure maintained at temperature $T$ is

$$
\rho(T) = \frac{2}{V} \sum_{n_i, n_j} \frac{h \nu_{ij}}{e^{h \nu_{ij}/kT} - 1},
$$

with $V = \ell^3$, the 3-dimensional cavity volume.

The energy density is proportional to the radiancy $R(T)$, the energy rate per unit area of the orifice. Considering geometric factors for which the emanated power propagates only through the 3 non-compact dimensions, since the other ones are compact, one gets

$$
R(T) = \frac{c}{4} \rho(T).
$$

The compact dimensions have the same length of their corresponding edges $a$, which is originally unknown, while the orthodox edges are taken to have length $\ell$ at the cm scale or higher. Thus, although $n_i$ and $n_j$ cover the same numerical range, their respective contributions to $\nu_{ij}$ may not be on the same foot. With this in mind [4] will be worked out for two separate cases, $\sum n_j^2 = 0$ and $\sum n_j^2 \neq 0$. Namely

$$
R(T) = \frac{c}{2V} \left[ \sum_{n_j=0}^{\infty} + \sum_{n_j \neq 0}^{\infty} \right] \frac{h \nu_{ij}}{e^{h \nu_{ij}/kT} - 1}.
$$

The first sum is a single one in $n_i$. Considering $\nu_0 \equiv \nu(n_i, 0)$ and taking this sum by an integral, one gets

$$
R_4(T) = \frac{c}{2V} \sum_{n_j=0}^{\infty} \frac{h \nu_0}{e^{h \nu_0/kT} - 1} = \int_0^\infty R_4(T, \nu_0) d\nu_0
$$

where the spectral radiancy $R_4(T, \nu_0)$ for this range of modes is

$$
R_4(T, \nu_0) = \frac{2\pi h}{c^2} \frac{\nu_0^3}{e^{h \nu_0/kT} - 1}.
$$

Making the variable change $z_0 = h \nu_0/kT$ one integrates [8] arriving at

$$
R_4(T) = \frac{2\pi c}{(hc)^3} \left( \frac{k}{hc} \right)^3 \int_0^\infty \frac{z_0^3}{e^{z_0} - 1} dz_0,
$$

with the integral being expressed by the mathematical identity

$$
\int_0^\infty \frac{z^d}{e^z - 1} dz = \Gamma(d + 1) \zeta(d + 1).
$$

Taking $d = 3$ in the above identity, we obtain for the radiancy contribution due to the radiation confined within the 3 non-compact dimensions

$$
R_4(T) = \sigma_\beta T^4,
$$

which is the well-known Stefan-Boltzmann law with

$$
\sigma_\beta = \frac{2\pi^5 k}{15} \left( \frac{k}{hc} \right)^3 = 5.67 \cdot 10^{-8} \text{ W m}^{-2}\text{K}^{-4}.
$$

Now we consider the case $\sum n_j^2 \neq 0$ where both $n_i$ and $n_j$ contribute to the spectral radiancy. Taking the sum by an integral, with $\nu \equiv \nu(n_i, n_j)$, one gets

$$
R_D(T) = \frac{c}{2V} \sum_{n_j \neq 0} \frac{h \nu}{e^{h \nu/kT} - 1} = \int_0^\infty R_D(T, \nu) d\nu
$$

and the corresponding spectral radiancy is given by

$$
R_D(T, \nu) = \frac{\Omega(d-1) a^{D-4}}{2^{D-3} \pi_{D-2}^2} \frac{h \nu^{D-1}}{e^{h \nu/kT} - 1},
$$

where $\Omega(d) = 2\pi^{d/2}/\Gamma(d/2)$ is the solid angle in a $d$-dimensional space.

Making the variable change $z = h \nu/kT$ one gets

$$
R_D(T) = \frac{\Omega(d-1) h a^{D-4}}{2^{D-3} \pi_{D-2}^2} \frac{k^D}{h} \int_0^\infty \frac{z^{D-1}}{e^{z} - 1} dz,
$$

and using [10] the radiancy contribution $R_D(T)$ due to these modes is

$$
R_D(T) = \sigma_D(a) T^D,
$$

with

$$
\sigma_D(a) = 4 \Omega(d-1) k c \left( \frac{k}{2hc} \right)^{D-1} \Gamma(D) \zeta(D) a^{D-4}.
$$

Grouping the computed radiancy contributions one gets for the total blackbody energy rate per unit area,

$$
R(T) = \sigma_\beta T^4 + \sigma_D(a) T^D,
$$

which can be understood as a generalized Stefan-Boltzmann law for spacetimes with $D-4$ compact extra dimensions.

Defining $\delta_T(T, a) = R_D(T)/R_4(T)$ as the radiancy relative deviation due to the compact dimensions with respect to the non-compact contributions,

$$
\delta_T(T, a) = \frac{15 \Omega(D-1) }{4\pi^3} \Gamma(D) \zeta(D) \left( \frac{k}{2hc} \right)^{D-4} a^D \frac{1}{\Gamma(D+1)}
$$

one notes that for sufficiently low temperatures, compared with the inverse of the compact dimensions size, $T \ll 2hc/ka$, the obtained generalization for the Stefan-Boltzmann law [18] reduces to its well known form [10].

On the other side, for sufficiently high temperatures, $T \gg 2hc/ka$, the generalized Stefan-Boltzmann law assumes its higher dimensional character [10].

Also through [10] one observes that, the greater the number of extra compact dimensions, the narrower will
be the transition temperature gap between the $\sigma_0 \ T^4$ and $\sigma_0 (a) \ T^D$ domains.

To obtain the respective Wien’s law we write the total wavelength radiancy as

$$ R(T, \lambda) = R_a(T, \lambda) + R_D(T, \lambda), \quad (20) $$

where

$$ R_a(T, \lambda) = \frac{2\pihc^2}{\lambda^5} \left( e^{\frac{hc}{k\lambda T}} - 1 \right)^{-1}, \quad (21) $$

and

$$ R_D(T, \lambda) = \frac{\Omega(D-1)hc^2}{2\lambda^{D+1}} \left( \frac{a}{\lambda} \right)^{D-4} \left( e^{\frac{hc}{k\lambda T}} - 1 \right)^{-1}. \quad (22) $$

Since for a given $T$ there is a $\lambda_m$ for which $R(T, \lambda_m)$ is maximum, through $dR(T, \lambda)/d\lambda = 0$ one obtains

$$ 1 - e^{-hc/kT\lambda_m} = \frac{hc}{kT\lambda_m} \left( 1 + \epsilon_{D}(a, \lambda_m) \right), \quad (23) $$

where $\epsilon_{D}(a, \lambda) = R_D(T, \lambda)/R_a(T, \lambda)$ is defined as the wavelength radiancy relative deviation,

$$ \epsilon_{D}(a, \lambda) = \frac{\Omega(D-1)}{4\pi} \left( \frac{a}{2\lambda} \right)^{D-4}. \quad (24) $$

The Eq. (23) can be regarded as the generalized Wien’s law for spacetimes with $D - 4$ compact extra dimensions, once it relates the blackbody temperature $T$ with the wavelength $\lambda_m$ corresponding to the maximum value of the total wavelength radiancy $R(T, \lambda)$.

Note that for wavelengths much larger than the size of compact dimensions, $\lambda_m \gg a/2$, eq. (23) reduces to

$$ 1 - e^{-x_0} = \frac{x_D}{D+1}; \quad x_D \equiv \frac{hc}{k\lambda_m T}, \quad (25) $$

where $x_D = W(-\alpha - (D+1)e^{-\alpha(D+1)} + D + 1$. This yields $\lambda_m T = x_D hc/k$ and the generalized Wien’s displacement law assumes its D-dimensional behavior.

Since every blackbody radiation measurement is performed within certain wavelength ranges, the upper bound for the size of compact dimensions can be calculated by fitting $\epsilon_{D}(a, \lambda)$, for a given $D$, into the experimental uncertainty. Precise measurements performed at $\lambda \approx 250 \text{ nm}$ and $T \approx 3000 \text{ K}$ restrain the size of compact extra dimensions within our model to be not higher than $2 \cdot 10^{-7} \text{ m}$ for $D = 10$ and $D = 11$. In Fig. 1, we plot the wavelength radiancy relative deviation $\epsilon_{D}(a, \lambda)$, in these two situations.

The generalized Stefan-Boltzmann law (18) for $D = 10$ and $D = 11$ are given respectively by (in S.I. units)

$$ R(T) = 5.67 \cdot 10^{-8} T^4 + 1.32 \cdot 10^3 a^6 T^{10}, \quad (26) $$

$$ R(T) = 5.67 \cdot 10^{-8} T^4 + 3.94 \cdot 10^9 a^7 T^{11}. \quad (27) $$

These expressions are plotted in Fig. 2 and compared with the standard results. For the considered size of the compact extra dimensions the discrepancy between these functions is just noticeable for $T > 10^4 \text{ K}$.

Regarding the generalization of Wien’s displacement law eq. (23) for $D = 10$ and $D = 11$ one finds respectively ($\lambda = \lambda_m$),

$$ 1 - \exp \left( -\frac{\alpha}{T\lambda} \right) = \frac{\alpha}{T\lambda} \frac{1 + \beta \left( \frac{\alpha}{T\lambda} \right)^6}{5 + 11\beta \left( \frac{\alpha}{T\lambda} \right)^6}; \quad (28) $$

$$ 1 - \exp \left( -\frac{\alpha}{T\lambda} \right) = \frac{\alpha}{T\lambda} \frac{1 + \gamma \left( \frac{\alpha}{T\lambda} \right)^7}{5 + 12\gamma \left( \frac{\alpha}{T\lambda} \right)^7}, \quad (29) $$

FIG. 1: Wavelength radiancy relative deviation for $D = 10$ and $D = 11$ with respect to $D = 4$, for $a = 2 \cdot 10^{-7} \text{ m}$.

FIG. 2: Generalized Stefan-Boltzmann law for $D = 10$ and $D = 11$ compared with $D = 4$, for $a = 2 \cdot 10^{-7} \text{ m}$.
where
\[
\alpha = \frac{\hbar c}{k} , \quad \beta = 2.36238 , \quad \gamma = 2.02935 . \tag{30}
\]

The behavior of these generalized Wein’s laws are plotted in Fig. 3 and compared with the standard results. This graph also shows a clear signature of the extra dimensions, as can be seen in Eq. (19), while the wavelength for which deviations in the blackbody radiation becomes important is directly proportional to the size of compact dimensions, as can be noted in Eq. (24). Thus one can estimate the size of the compact extra dimensions through possible deviations measured with blackbody radiation experiments performed at high temperatures and short wavelengths.

We conclude remarking that the temperature for which deviations in the blackbody radiation becomes relevant is inversely proportional to the size of compact dimensions, as can be seen in Eq. (19), while the wavelength for which deviations in the blackbody radiation becomes important is directly proportional to the size of compact dimensions, as can be noted in Eq. (24). Thus one can estimate the size of the compact extra dimensions through possible deviations measured with blackbody radiation experiments performed at high temperatures and short wavelengths.

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[1] For a review see, e. g., M. Dine, “Supersymmetry and string theory: Beyond the standard model,” Cambridge, UK: Cambridge Univ. Pr. (2007) 515 p.
[2] K. Becker, M. Becker, J. H. Schwarz, “String theory and M-theory: A modern introduction,” Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p.
[3] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429, 263 (1998) [arXiv:hep-ph/9803315].
[4] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 436, 257 (1998) [arXiv:hep-ph/9804398].
[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); ibid. 4690 (1999).
[6] A M Garcia-Garcia, Phys. Rev. A 78 023806 (2008).
[7] H. Alnes, F. Ravndal and I. K. Wehus, J. Phys. A 40, 14309 (2007) [arXiv:quant-ph/0506131].
[8] T. R. Cardoso and A. S. de Castro, Rev. Bras. Ens. Fis. 27, 559 (2005) [arXiv:quant-ph/0510002].
[9] One could actually have taken \( t_i \approx t \) and \( a_j \approx a \) as the edges length, and the outcomes would be quite the same.
[10] P. J. Mohr et. al., CODATA 2010, NIST http://physics.nist.gov/cuu/Constants/index.html
[11] V I Sapritsky, Metrologia, 32 (1995) 411-417.
[12] H W Yoon and C E Gibson, Metrologia, 37 (2000) 429-432.
[13] R Friedrich and J Fischer, Metrologia, 37 (2000) 539-542.
[14] H W Yoon, C E Gibson and P Y Barnes, Metrologia, 40 (2003) S172-S176