Constructing weak solutions by *tyger purging* in the Burgers equation

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Finite-dimensional, inviscid equations of hydrodynamics, such as the zero-viscosity, one-dimensional Burgers equation or the three-dimensional incompressible Euler equation, obtained through a Fourier-Galerkin projection, thermalise—mediated through structures known as *tygers* [Ray *et al.*, Phys. Rev. E 84, 016301 (2011)]—with an energy equipartition. Therefore, numerical solutions of inviscid partial differential equations, which typically *have* to be Galerkin-truncated, show a behaviour at odds with the parent equation. We now propose, by using the one-dimensional Burgers equation as a testing ground, a novel numerical recipe, named *tyger purging*, to arrest the onset of thermalisation and hence recover the *true dissipative solution*.

Non-linear, partial differential equations of hydrodynamics, such as the inviscid the one-dimensional Burgers or the three-dimensional Euler equations, are often studied, numerically and theoretically, by projecting them on to a Fourier subspace with a finite number of modes bounded by a (large) wavenumber $K_G$. This projection (defined precisely later), known as a Galerkin-projection, ensures that unlike the parent partial differential equation (PDE) which has an infinite number of degrees of freedom, the Galerkin-truncated equation is constrained to have only finitely many Fourier modes. Consequently, the resulting finite-dimensional, inviscid equations of hydrodynamics, such as the three-dimensional (3D) incompressible Euler equations or the one-dimensional (1D) Burgers equation, conserve both energy and phase-space, leading to solutions which thermalise in a finite-time. These solutions are thus completely different from the solutions of the actual partial differential equation, from which they derive, with infinite degrees of freedom. Indeed the early attempts by Hopf [10] and Lee [15], working with finite-dimensional, Galerkin-truncated, inviscid equations of hydrodynamics as a model of ideal fluids, resulted in understandable contradictions with what is known through experiments and observations. The reasons for such contradictions can of course be traced back to the apparent futility of constructing a (microscopic) Hamiltonian formulation, with invariant Gibbs measures, which is at odds with any self-consistent macroscopic theory, characterised by irreversible energy losses, and hence captured by a dissipative hydrodynamic formulation. Hence, the understanding of out-of-equilibrium, dissipative turbulent flows by using tools of statistical physics remains a particularly difficult problem in theoretical physics.

In recent years, this area has received renewed interest [21]—spanning studies in turbulence [8]—beginning with the work of Majda and Timofeyev [10], who showed that the Galerkin-truncated, 1D inviscid Burgers indeed thermalises. Subsequently, Cichowlas, *et al.* [5], through state-of-the-art direct numerical simulations (DNSs) showed the existence of similar thermalised states in the Galerkin-truncated 3D incompressible Euler equation (see, also, Ref. [11]). However the precise mechanism by which solutions thermalise was discovered later by Ray, *et al.* [24] who showed that thermalisation was triggered through a resonant-wave-like interaction leading to localised structures christened *tygers*. Indeed, this phenomenon of *tygers*, and eventual thermalisation, has now been studied [5] quite extensively with a complete characterisation of the structure of these solutions as well as an estimate of the time-scales for the onset of thermalisation [25].

Understanding Galerkin-truncated systems assumes a special importance when numerically studying inviscid equations for the problem of finite-time blow-up of the incompressible Euler equation (under suitable conditions). A way to conjecture for or against a finite-time singularity is to numerically solve the Euler equation and measure the width of the analyticity strip $\delta$ [24], i.e., the distance to the real domain of the nearest complex singularity. By assuming analyticity, at least up to a hypothetical time of blow-up $t_\star$, this procedure reduces to measuring the Fourier modes of the velocity field $u_k \sim \exp [-\delta(t)k]$ (ignoring vectors for convenience), for large wavenumbers $k$, and thence, $\delta$ as a function of time $t$. Therefore, a numerically compelling proof for a finite-time blow-up is to show $\delta(t) \rightarrow 0$ in a finite time.

Simple as it sounds, such an approach unfortunately runs into a severe problem in its implementation. To solve
such equations on the computer, one has to make them finite-dimensional through a Galerkin truncation. Solutions to such truncated equations thermalise, beginning at small scales (or large wavenumbers $k$) in a finite time. Hence, asymptotically at large wavenumbers the Fourier modes of the velocity field grow as a power law $\hat{u}_k \sim k^{d-1}$ (energy equipartition), where $d$ is the spatial dimension, and not fall-off exponentially from which the width of the analyticity strip can be extracted. Hence, the measurement of $\delta(t)$ becomes unreliable soon enough to prevent us from making a reasonable conjecture of if and when $\delta(t)$ might vanish [2]. Therefore, in order to have a more reliable measurement of $\delta(t)$ for times long enough to conjecture on whether there is a finite-time blow-up of, e.g., the 3D, incompressible, Euler equation, it is vital to have a (numerical) prescription—without resorting to viscous damping—which prevents the solutions from thermalising.

We now propose such a recipe and show, by using the more tractable one-dimensional Burgers equation, how the Galerkin-truncated equation can be modified mildly to obtain solutions which do not thermalise. Furthermore, our algorithm results in obtaining weak or dissipative solutions of the inviscid partial differential equations without resorting to viscous regularisations.

Let us begin with the 1D inviscid Burgers equation on a $2\pi$ periodic line

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0$$

(1)

augmented by the initial condition $u_0(x)$ which is typically a combination of trigonometric functions containing a few Fourier modes. Since we work in the space of $2\pi$ periodic solutions, we can expand the solution of Eq. (1) in a Fourier series

$$u(x) = \sum_{k=0,\pm 1,\pm 2...} e^{i k x} \hat{u}_k.$$  

(2)

This allows us to define the Galerkin projector $P_{K_G}$ as a low-pass filter which sets all modes with wavenumbers $|k| > K_G$, where $K_G$ is a positive (large) integer, to zero via

$$v(x) = P_{K_G} u(x) = \sum_{|k| \leq K_G} e^{i k x} \hat{u}_k.$$  

(3)

These definitions allow us to write the Galerkin-truncated inviscid Burgers equation

$$\frac{\partial v}{\partial t} + P_{K_G} \frac{1}{2} \frac{\partial v^2}{\partial x} = 0;$$

(4)

the initial conditions $v_0 = P_{K_G} u_0$ are similarly projected onto the subspace spanned by $K_G$. (For the 3D Euler equations, the same definition follows mutatis mutandis [1].)

The solution of the inviscid Burgers equation [1] show one or more shocks (determined by $u_0(x)$), after an initial-condition dependent finite-time $t_*$, through which energy is dissipated for $t > t_*$. Theoretically, the solution to [1], for $t > t_*$, is obtained by adding a viscous dissipation term $\nu \frac{\partial^2 v}{\partial x^2}$ with $\nu \to 0$ (the inviscid limit), which preserves the finitely many shocks of the true solution. This generalised solution, in the limit of vanishing viscosity, converges weakly to the inviscid Burgers equation and is characterised by a dissipative anomaly: energy dissipation $\epsilon$ remains finite (with an associated non-conservation of the total energy) as $\nu \to 0$.

In contrast, the Galerkin-truncated equation [4] conserves energy for all times. For initial conditions with a finite number of non-vanishing Fourier harmonic, the solution $v$ mimics rather well that of the inviscid PDE up to time $t \lesssim t_*$. Indeed, for $t < t_*$, the two solutions are essentially indistinguishable. However, when the distance of the nearest (complex) singularity of the un-truncated equation [1] is within one Galerkin wavelength ($\approx 2\pi/K_G$) of the real domain (at time $t \approx t_* - K_G^{-1/3}$), the effect of truncation becomes important. Indeed, the harbinger of eventual thermalisation shows up as spatially localised—around points with a positive velocity gradient and with a velocity identical to the shock(s)—bulge(s) (christened tygers) and concentrated, in Fourier space, in a narrow boundary layer close to and up to $K_G$; we refer the reader to Ref. [23] for a detailed analysis of the effects of truncation at such early times and the birth of tygers.

For $t > t_*$, the solutions of the truncated-equation and the PDE are dramatically different: Whereas the former stays smooth, conserves energy, and start thermalising (beginning at small scales) with an (equipartition) energy spectrum $\langle |\hat{u}_k|^2 \rangle \sim k^0$ [12], the latter shows a monotonic decrease in its kinetic energy (dissipated through the shock(s)) and an associated scaling form $\langle |\hat{u}_k|^2 \rangle \sim k^{-2}$, commonly known as the Burgers spectrum. (The angular brackets used in calculating the energy spectrum denotes suitable ensemble averages.) Thus thermalised solutions, inevitable in numerical solutions of the Galerkin-truncated inviscid equations, are fundamentally different from—and hence do not converge to—the un-truncated parent PDE.
FIG. 1: Representative plots, for $K_G = 1000$, of the Galerkin-truncated $v$ (blue) and entropy $u$ (black) solutions of the Burgers equation at (a) $t = 0.24 \gg t_*$ and (b) $t = 5.0 \gg t_*$. For the Galerkin-truncated solution, panel (a) shows signatures of impending thermalisation through the birth of *tygers* while panel (b) shows the fully thermalised solutions. (We refer the reader to the youtube link [https://www.youtube.com/watch?v=QiioybbVi6M](https://www.youtube.com/watch?v=QiioybbVi6M) for a movie of the time evolution of the Galerkin-truncated equation (and the entropy solution) with a single-mode initial condition for clarity.)

We illustrate this phenomenon in Fig. 1 by showing the solutions to the Galerkin-truncated equation $v^{(4)}$ (in blue), with $K_G = 1000$, and the entropy solution $u^{(1)}$ (in black) for (a) an early time $t = 0.24 (\gtrsim t_*)$ and (b) at a later time $t = 5.0 (t \gg t_*)$. We choose a multimode initial condition $u_0 = v_0 = \sin(x) + \sin(2x + 0.9) + \sin(3x)$ which corresponds to $t_* = 0.2218$. (The details of how the numerical simulations were performed is given below.)

As discussed above, even at times very close to $t_*$ (Fig. 1a), the two solutions show a marked difference—*tygers*—at points which have the same velocity as the shock (and a positive fluid velocity gradient). At even later times, (Fig. 1b) we see clear signatures of thermalisation in the truncated solution which has absolutely no resemblance to the entropy solution which, as a consequence of shocks merging in time, has a saw-tooth structure with a single shock. (We refer the reader to Refs. [6, 16, 19, 21, 23, 25] for more details and illustrations of this process of thermalisation.)

All of this leads us to ask if we can, without resorting to viscous dissipation, actually suppress thermalisation kicking in in such truncated equations and obtain a solution similar to the entropy one shown in Fig. 1(b)? As we have noted before, there are at least two important reasons why this question is worth answering. Firstly, as a numerical tool to obtain a more reliable estimate of the widths of the analyticity strip and secondly as a possibility of obtaining weak solutions on a computer. We now report a novel approach based on the elimination of the boundary layer (discussed above) that we call *tyger purging*: The selective removal of the Fourier space boundary layer at discrete time-intervals *purging* resulting in the suppression of thermalisation.

The equation of motion for the purged solution $w$ is, of course, the same as that of the Galerkin-truncated equation (with the truncation wavenumber $K_G$)

$$\frac{\partial w}{\partial t} + K_G \frac{1}{2} \frac{\partial w^2}{\partial x} = 0$$

augmented by an additional constraint imposed at discrete times $t_p = t_* + n\tau (n = 0,1,2,3 \ldots)$:

$$\hat{w}_k = 0 \quad \forall \quad K_p \leq k \leq K_G.$$  

We call this truncated equation, along with the additional *purging* constraint, as simply the purged equation. We note that without the additional constraint, by definition, the solution $w$ is the same as $v$ obtained from the truncated
FIG. 2: Representative plots, for \( K_G = 1000 \), of the Galerkin-truncated \( v \) (blue), the entropy \( u \) (black) and the purged \( w \) (red) solutions of the Burgers equation at \( t = 5.0 \) for (a) \( \alpha = 0.6, \beta = 0.4 \) and (b) \( \alpha = \beta = 0.8 \). In panel (b), the purged and entropy solutions are quite close to being identical. (We refer the reader to the youtube link https://www.youtube.com/watch?v=utjyfQUuClc for a movie of the full evolution in time of the solutions shown in panel (b).)

equation and hence if purging is done continuously, and not discretely, in time, we would end up solving the Galerkin-truncated equation \( (4) \) but with a truncation wavenumber \( K_p. \)

We now make the following ansatz about the choice of the inter-purging time \( \tau \) and the purging wavenumber \( K_p: \)

\[
\tau = K_G^{-\alpha}; \quad K_p = K_G - K_G^\beta; \tag{7}
\]

with real, positive exponents \( \alpha \) and \( \beta \) and the immediate constraint that \( \beta < 1 \) for \( K_p \) to be finite and less than \( K_G. \)

Before, we engage in detailed numerical analysis of our purged model, let us try to have an estimate of what could be optimal choices of \( \alpha \) and \( \beta \) in a heuristic way. Such estimates are found keeping in mind that the eventual goal of purging is to obtain solutions \( w \) which converges (weakly), as \( K_G \to \infty \), to the entropy solution \( u. \) We know that the entropy solution for \( t > t_* \) is dissipative, \( \frac{dE}{dt} < 0 \), for \( t > t_* \), where the total energy \( E = \frac{1}{2} \sum_{k=1}^{\infty} |\hat{u}_k|^2 \), whereas Galerkin-truncation conserves energy for all times. By construction, the purging strategy also allows for a finite energy loss \( \Delta E^P \equiv \frac{1}{2} \sum_{k=K_G}^{K_G} |\hat{w}_k|^2 \) at intervals of \( \tau \) resulting in a rate of loss of energy \( \frac{\Delta E^P}{\tau} \) which, in the limit \( K_G \to \infty, \) should be \( K_G \)-independent, finite, negative and comparable to \( \frac{dE}{dt}. \)

It is hard to estimate \( E^P \) theoretically without making suitable assumptions. Since in between two purges, Eq. 5 is identical to the Galerkin-truncated equation, we assume that just at the time of purging the solution \( |\hat{w}_k| \) to be a combination of the one coming from the entropy solution and a contribution from the nascent tyger. The latter, which is the deviation of the truncated from the entropy solution, was shown in Ref. \[23\] to be confined to a narrow Fourier-space boundary layer close to and up to \( K_G \) with a form (ignoring \( O(1) \) constants) \( \frac{1}{k_G^3} \exp \left[ -\frac{K_G - k}{K_G^{1/3}} \right]. \) Keeping these factors in mind, it is trivial to show that as \( K_G \to \infty, \) \( \frac{\Delta E^P}{\tau} \sim -K_G^{\alpha + \beta - 2}. \) If we now demand that this rate of energy loss be independent of \( K_G, \) we obtain the constraint \( \alpha + \beta = 2. \)

The constraint derived above is useful but it still allows considerable freedom in choosing \( \alpha \) and \( \beta. \) However, since purging itself is an elimination of the Fourier-space boundary layer—critical in the birth of tygers and hence eventual thermalisation—near \( K_G, \) it provides an independent way to obtain a bound on \( \beta. \) Indeed the simplest approach is to measure, at each purging, the loss of energy \( \Delta E \equiv \sum_{k=K_F}^{K_G} |\hat{v}_k|^2 - \sum_{k=K_p}^{K_G} |\hat{u}_k|^2. \) This excess energy, which we lose through purging, is central to the discrepancy between the conservative dynamics of the Galerkin-truncated equation and the dissipative dynamics of the 1D Burgers PDE.
FIG. 3: (a) A plot of the total energy $E^K$ versus time, from our purged solutions, for different combinations of $\alpha$ and $\beta$ and $K_G = 1000$. We also show, in black, the energy versus time plot for the entropy solution for comparison. The dashed vertical lines correspond to the times at which the shocks, three in all because of the three-mode initial conditions, form. In the inset, we plot the relative percentage error $\varepsilon$ (see text) between the purged and entropy solution, for $\alpha = \beta = 0.8$, at $t = 5.0$, as a function of $K_G$. (b) A plot of the $L_2$ norm of the percentage relative error $\phi$ (see text) for $\alpha = \beta = 0.8$ as a function of $K_G$; the dashed line shows a power-law $K_G^{-1}$ scaling consistent with the measured error.

By using the functional form for the boundary layer for incipient *tygers*, as used above, it is easy to derive:

$$\Delta E \sim \begin{cases} K_G^{\beta-2} & \text{for } \beta < 1/3 \\ K_G^{-5/3} & \text{for } \beta > 1/3. \end{cases} \tag{8}$$

Equation (8) leads to two ranges of $\beta$, with very different asymptotics in terms of the energy lost. The question remains as to what should be the optimal choice for $\beta$. Ray, *et al.* [23] (and subsequently in Ref. [25]) had shown that Galerkin-truncation leads to a transfer of energy $\sim K_G^{-5/3}$ from the shock to the *tygers* resulting in an overall conservation of kinetic energy in the truncated problem. Hence for our purging strategy to have some chance of success, it should result in an energy loss $\Delta E \approx K_G^{-5/3}$.

Remarkably, our calculation suggests that for $\beta > 1/3$, the energy loss $\Delta E$ is identical with that derived, from completely different consideration in Ref. [23], to prove why *tygers* lead to energy conservation. Purging allows us to eliminate precisely these *tygers* and hence should make the solutions dissipative. Our derivation shows that an optimal choice of the purging wavenumber is one where $\delta t$ and for each value of $\beta$, the inter-purging time was obtained with $\alpha = 0.4, 0.6, 0.8, 0.9, \text{and } 1.2$.

The choice of time-steps in such simulations require some delicacy. For the truncated problem, since the maximum principle is violated, individual realisations of the velocity field can have excursions which are large (see Fig. 1b). Hence for the truncated simulations, as well as those where purging is ineffective in preventing thermalisation, the time-step $\delta t$ has to be kept very small. However, for the cases of *successful* purging (see below), the maximum principle is no longer violated. Hence for these cases we are able to choose $\delta t \approx 10^{-5}$ ($N = 16384$) and $\delta t \approx 10^{-6}$ ($N = 65536$); for the analogous truncated problem (and the ones where the $\alpha$-$\beta$ combination fail to prevent thermalisation), $\delta t$ was taken to be at least two orders of magnitude smaller.

In numerical simulations, $\delta t$ is typically set by the resolution $K_G$ such that $\delta t \sim \mathcal{O}(K_G^{-1})$. As we have noted before, purging if done too frequently would be akin to solving the Galerkin-truncated Burgers equation with $K_G = K_P$. This implies that $\tau/\delta t \gg 1$ which, trivially, leads to $\alpha < 1$. Hence, we assume the constraint $\alpha + \beta = 2$, derived we recall from an assumption that in-between purges the solution of $\hat{\phi}$ behaves exactly like that of the truncated Burgers...
equation at $t = t_*$, should be softened to merely $\alpha + \beta \sim \mathcal{O}(1)$. (We confirm these conjectures through numerical simulations below.)

Finally, we have studied the problem for several different initial conditions (all of which consist of linear combinations of trigonometric polynomials including the simplest single-mode case $\sin(x)$); we have checked that our results and conclusions are consistent for all such initial conditions. In this paper, for brevity, we present results only for the case $w_0 = v_0 = u_0 = \sin(x) + \sin(2x + 0.9) + \sin(3x)$.

So how effective is purging in obtaining solutions $w$ which resemble the entropy solution $u$? In Fig. 2 we show representative plots, at $t = 5.0$, of the Galerkin-truncated $v$ (in blue), the entropy $u$ (in black) and the purged solutions $w$ (in red) for (a) $\alpha = 0.6, \beta = 0.4$ and (b) $\alpha = 0.8, \beta = 0.8$; we set the truncation wavenumber $K_\alpha = 1000$. As we have seen before (Fig. 1), the Galerkin-truncated solution is thermalised and different from the entropy solution. However, on purging with $\alpha = 0.6$ and $\beta = 0.4$ (Fig. 2b), the solution $w$ approximates the entropy solution much better—in so far as picking out the ramp structure and a jump near the shock—though far from perfectly.

Remarkably, if we choose $\alpha = \beta = 0.8$ (Fig. 2b)—and hence much closer to satisfying the heuristic estimate $\alpha + \beta = 2$—the agreement between the purged and entropy solutions are near-perfect. Indeed the main point of departure between the two solutions seems to be close to the shock because of the ubiquitous Gibbs-type oscillations [20] associated with Fourier transforms of functions near discontinuities.

We have checked that for $\alpha \gtrsim 0.9$, since $\tau / \delta t \sim \mathcal{O}(1)$, the purged solutions thermalise once again as was conjectured earlier. Hence, empirically, our extensive numerical simulations show that within the range of $\alpha$ that we study, the optimal choice is $\alpha = 0.8$. Furthermore, we have checked, as already conjectured through Eqs. (5), that our results are largely insensitive to the choice of $\beta$ as long as its greater than $1/3$.

The fact that the purged and entropy solutions seem to be in agreement, visually, suggests that the purged solution is dissipative as was anticipated, by construction, earlier. However, for this solution to actually converge (weakly) to the entropy solution, the rate of dissipation should be arbitrarily close to the dissipation rate $dE / dt$ of the entropy solution. The most direct way to see this is to plot, in Fig. 3(a), the total energies of the entropy $E$ and the purged $E^\beta$ solutions, as a function of time, for different values of $\alpha$ and $\beta$; we show results for $K_\alpha = 1000$. We find, as was indicated by Fig. 2 that for the optimal choice (amongst all the parameters that we have studied) of $\alpha = \beta = 0.8$, the behaviour of the total energy versus time for the purged solution is identical to the one obtained from the entropy solution. The purged solutions for other $\alpha - \beta$ combinations are dissipative as well; however they dissipate energy at rates much slower than the entropy solution. It is worth noting, that the shock-mergers, as indicated by the vertical lines in the plot, and which lead to tiny kinks in the energy versus time profile, are faithfully reproduced by purged solutions for $\alpha = \beta = 0.8$.

A measure of how accurately the purged solution mimics the dissipation of the entropy one, we calculate the percentage relative error $e = \frac{E^\beta - E}{E} \times 100$ at $t = 5.0$. In the inset of Fig. 3(a), we plot $e$ as a function of $K_\alpha$ for the most optimal purging choice ($\alpha = \beta = 0.8$). Remarkably, this error $e$ decreases rapidly with $K_\alpha$ and for a reasonably large $K_\alpha = 5000$ the relative error is close to 0.01%.

All of this leads us inevitably to the important question: For $\alpha = \beta = 0.8$, does the purged solution indeed converge to the entropy one as $K_\alpha \to \infty$? The most precise way to answer this is to measure the percentage relative error (or the $L_2$ norm) $\phi = \sqrt{\frac{1}{N} \sum_{i=1}^{N} |u(x_i) - w(x_i)|^2} \times 100$ of the discrepancies between the solutions $u$ and $w$. (In order to avoid the contamination from Gibbsian oscillations, we avoid the narrow region close to the shock in the summation.)

Given that this is a point-wise measure, unlike the global energy measurements shown in Fig. 2(a), a sharp decrease in $\phi$ with $K_\alpha$ should be clinching evidence of the efficacy of our scheme. In Fig. 3(b), we show a log-log plot of $\phi$ as a function of $K_\alpha$ and find a steep decrease ($\phi \sim K_\alpha^{-1}$ indicated by the dashed line) in the relative error as a function of $K_\alpha$. For the large values of $K_\alpha$, the relative error $\phi < 1\%$, reaching a value of $\phi \approx 0.5\%$ for our largest truncation wavenumber $K_\alpha = 10000$. Figure 3(b) essentially summarises the main message of this work: The purged Galerkin-truncated Burgers equation, for suitable choices of $\alpha$ and $\beta$, converges weakly to the dissipative (Fig. 3b), (weak) entropy solution as $K_\alpha \to \infty$.

These results lead to the conclusion that purging, if done with the right choices of $\alpha$ and $\beta$, lead to dissipative (weak) solutions which converge to the entropy solution of the parent PDE as $K_\alpha \to \infty$. Importantly, the discrepancy between the two solutions is already minute for values of $K_\alpha$ which are easily accessible. From the point of view of numerical simulations, the $\beta \geq 1/3$ condition is extremely helpful because it allows us to choose values of $\beta$ small enough such that for a given $K_\alpha$, the loss in resolution $K_\alpha - K_P$ through purging, is insignificantly small. As an example, for $K_\alpha = 10000$ and $\beta = 0.4$, fraction of resolution lost is about 0.3%.

It is also not entirely out of place to mention that purging attempts in physical space—which consists in smoothening
out the tygers in physical space through local averaging—does not result in any significant suppression of thermalisation. Furthermore, such procedures lack the easy adaptability to different initial conditions, or even different equations of hydrodynamics, because it relies on the knowledge of the locations of the tygers in physical space to implement a smoothening procedure.

Our results, if seen in isolation for the Burgers equation, are admittedly academic. This is because for the 1D Burgers equation, we already have a way, the Fast-Legendre method (which we use in this paper) to obtain the weak, dissipative solutions. Furthermore, a measurement of the widths of the analyticity strip $\delta$ is easily achieved both analytically and numerically, through the Fast-Legendre method, for the one-dimensional equation. Indeed, given that for the Burgers equation, the effects of truncation are felt at times very close to $t_*$, we have checked that the $\delta$ obtained for the Burgers equation with and without purging, are not too dissimilar from the theoretical estimate up to times very close to $t_*$. This is of course pathological to the Burgers equation and it is reasonably to conjecture that purging in the 3D Euler equation will yield more dividends. Furthermore, if we consider the 3D Euler equations, there is no analogue of the Fast-Legendre method.

It is in the light of the 3D Euler equations that this approach assumes special importance. To the best of our knowledge, till date there is no algorithm which allows, numerically, to obtain dissipative or weak solutions of the 3D Euler equation. This algorithm allows us to do exactly that. Numerically, our algorithm is trivial to implement in codes which solve the 3D Galerkin-truncated Euler equation. From earlier studies we know that the onset of thermalisation in the 3D Galerkin-truncated Euler equation is formally similar to that in the Burgers equation. (In fact, for the 2D Euler equation, this was rigorously shown to be true.) Hence, the approach outlined in this paper, should allow us to implement it for the 3D Euler equations and study, numerically, dissipate solutions as well as, and possibly most importantly, take advantage of the suppression of thermalisation to finally have a firm, albeit numerical, answer for the celebrated blow-up problem that we had described.

Before we conclude, it is important to ask if thermalisation can be suppressed by other means (without resorting to viscosity). One possibility is of course the use of the hyperviscous term. This however has the drawback that we would end up solving not the inviscid equation but its viscous form. Moreover, as shown in Refs. [1, 7], for higher-orders of the hyperviscosity—which is similar in spirit to the idea of purging—the solutions asymptote to the Galerkin-truncated equation and thermalises. Another approach is due to Pereira, et al. [19] who showed that a wavelet-based filtering technique also leads to a suppression of the resonances leading to tygers. However, such an approach has the limitation, as mentioned by the authors themselves, that the dual operations of filtering and truncation at every time step do not commute. Hence the weak dissipation introduced in this approach is somewhat uncontrolled. To this extent we feel that the our prescription is probably more suited for generating weak solutions and, importantly, more easily adaptable to higher-dimensional systems such as the 3D Euler equations.

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