ABSTRACT. Starting from a Riemannian conformal structure on a manifold $M$, we provide a method to construct a family of Lorentzian manifolds. The construction relies on the choice of a metric in the conformal class and a smooth 1-parameter family of self-adjoint tensor fields. Then, every metric in the conformal class corresponds to the induced metric on $M$ seen as a codimension two spacelike submanifold into these Lorentzian manifolds. Under suitable choices of the 1-parameter family of tensor fields, there exists a lightlike normal vector field along such spacelike submanifolds whose Weingarten endomorphism provide a Möbius structure on the Riemannian conformal structure. Conversely, every Möbius structure on a Riemannian conformal structure arises in this way. Flat Möbius structures are characterized in terms of the extrinsic geometry of the corresponding spacelike surfaces.

1. INTRODUCTION

A Riemannian conformal structure on a manifold $M$ is an equivalence class of Riemannian metrics on $M$ where two metrics are equivalent if they differ by a factor that is a smooth positive function on the manifold $M$. Conformal structures (in Lorentzian signature) was introduced by Hermann Weyl in order to formulate a unified fields theory. Weyl wrote “To derive the values of the quantities $g_{ik}$ from directly observed phenomena, we use light-signals .... By observing the arrival of light at the points neighbouring to $O$ we can thus determine the ratios of the values of the $g_{ik}$’s ..... It is impossible, however, to derive any further results from the phenomenon of the propagation of light...” [19, Chap. 4, Sec. 27].

From a mathematical perspective, the problem of the equivalence for conformal structures on $(n \geq 3)$-dimensional manifolds was solved by E. Cartan by means of the now called canonical normal Cartan connection [5]. For dimension $n \geq 3$, Riemannian conformal structures $(M, c)$ correspond bijectively (up to isomorphism) with normal Cartan geometries of type $(G, P)$ where $G = O(1, n + 1)/\{\pm Id\}$ is the Möbius group and $P$ is the Poincaré conformal group defined to be the isotropy group of the line through an isotropic (lightlike) vector (see details in [8, Theor. 1.6.7]). That is, conformal structures on an $(n \geq 3)$-dimensional manifold $M$ gives rise to a principal $P$-bundle $P \to M$ and a unique Cartan connection $\omega \in \Omega^1(P, g)$ where $g$ is the Lie algebra of the Möbius group $G$ such that $\omega$ satisfies certain normalization conditions and conversely.

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These results have been extended to dimensions one and two by means of the notion of Möbius structure [3]. Section 2. A Möbius structure on a manifold $M$ is essentially equivalent to defining a conformal class of metrics $c$ on $M$ and a “Schouten type-tensor” for $c$, that is, a map $D : c \to \mathcal{T}_{(0,2)}M$ such that for every $g \in c$ the tensor $D(g)$ is symmetric with $\text{trace}_g D(g) = \frac{\text{scal}^g}{2(n-1)}$ and $D$ follows the same conformal transformation law that the Schouten tensor, Definition 2.1. Thus, the notion of Möbius structure provides a uniform description of Cartan geometries of type $(G, P)$ for all dimensions. More explicitly, if we start with a conformal structure $(M, c)$ on an $(n \geq 2)$-dimensional manifold $M$, the set of Cartan geometries of type $(G, P)$ is in one-to-one correspondence with the set of “Schouten type-tensor” for $c$. Hence, this notion is specially relevant for conformal structures on surfaces.

The planning of this paper is as follows. Starting from a Riemannian conformal structure $(M, c)$, by setting a metric $g \in c$ and an admissible 1-parameter family $\alpha : \mathbb{R} \to \mathcal{T}_{(1,1)}M$, see Definition 3.1, we construct a $(n + 2)$-dimensional Lorentzian manifold $(\tilde{M}, \tilde{g})$, see Proposition 3.3, such that

1. there is a distinguished lightlike hypersurface $Q \subset \tilde{M}$ (see Definition 2.6) and
2. every metric in the conformal class $e^{2u}g \in c$ is the induced metric of an immersion from $M$ to $\tilde{M}$ through $Q$. Such immersions are defined in (20) and are denoted by $\Psi^u$.

This construction is inspired by the Fefferman and Graham ambient metric for conformal structures in the 1980s, [9] (see also [10]). Roughly speaking, starting with a Riemannian conformal structure $(M, c)$, the space of scales $\mathcal{Q}$ consists of the rays of metrics $y := t^2g_x$ on $T_xM$ where $x \in M$, $t \in \mathbb{R}^+$ and $g \in c$. The ambient metric $\tilde{g}$ is defined so that $(\tilde{M}, \tilde{g})$ is a Lorentzian manifold that admits $Q$ as an embedded lightlike hypersurface. The original Fefferman-Graham metric requires certain normalisation condition (see Remark 2.5). In this paper, we will adopt the weaker notion of pre-ambient space given in [7], Definition 2.4. The pre-ambient metric $\tilde{g}$ that we define in (9) is not a warped product metric in general, Remark 3.4.

Now, every spacelike immersion $\Psi^u$ has codimension two in $(\tilde{M}, \tilde{g})$ and its normal bundle is spanned by the lightlike vector fields vector fields $\xi^u$ and $\eta^u$ given in (22). The main aim of this paper is to show Theorem 4.5 which states that:

Assume the admissible 1-parameter family $\alpha$ satisfies $\text{trace}(\alpha(0)) = \frac{\text{scal}^u}{n-1}$ where $\text{scal}^u$ is the scalar curvature of the fixed metric $g$. Then, the assignment $D : c \to \mathcal{T}_{(0,2)}M$, $e^{2u}g \mapsto e^{2u}g (A_{\eta^u}(-), -)$, defines a Möbius structure for the Riemannian conformal structure $(M, c)$, where $A_{\eta^u}$ denotes the Weingarten endomorphism of $\eta^u$. Moreover, every Möbius structure for a Riemannian conformal structure $(M, c)$ arises in this way.

The content of this paper is distributed as follows. In Section 2, taking into account ideas from [3] and [4], we recall the notion of Möbius structure on Riemannian conformal structures $(M, c)$ as was introduced in [15], extended to arbitrary dimension in an obvious way. We also include several basic facts on spacelike submanifolds in Lorentzian geometry. The spacelike submanifolds have been studied for a long time, both from the physical and mathematical points
of view (see for instance [16] and references therein). Then, we show some properties from the Lorentzian geometry perspective of the notion of pre-ambient space. Section 3 provides an explicit method to construct examples of pre-ambient spaces and includes several curvature properties of these pre-ambient spaces. In particular, we give conditions which permit to assure that the Ricci tensor of these pre-ambient spaces vanishes along $Q$, Corollary 3.9.

The main results are in Section 4 where it is essentially shown that Möbius structures agree with certain Weingarten endomorphisms of codimension two spacelike submanifolds in these pre-ambient spaces, Theorem 4.3. This result is remarkable for conformal structures in surfaces. As was mentioned, there is no preferred Möbius structure on a 2-dimensional Riemannian conformal structure. Theorem 4.5 provides an explicit method to construct such structure. We hope that our viewpoint sheds some light on the interplay between the theory of spacelike submanifolds and Möbius structures on Riemannian conformal structures. Section 4 also includes several properties on the family of spacelike immersions we need to construct the Möbius structure. In fact, Corollary 4.3 shows that the normal curvature tensor of such immersions always vanishes. Also, as a consequence of Remark 4.4, the mean curvature vector field of the isometric immersion $\Psi^u$ with induced metric $e^{2u}g$ satisfies

$$\|H^u\|^2 = \frac{\mathrm{scal} e^{2u}g}{n(n-1)},$$

see details in Remark 4.10. Particular cases of this formula have been previously obtained in [13, Cor. 4.5] and [14, Cor. 3.7]. Note that the causal character of $H^u$ in the Lorentzian manifold $\tilde{M}$ is determined by the sign of the scalar curvature of the metric $e^{2u}g$. Remark 4.10 also includes that $\nabla H^u = 0$ if and only if $\mathrm{scal} e^{2u}g$ is constant (compare with [14, Cor. 3.10]). In particular, when $M$ is compact, the positive answer to the Yamabe problem implies that there exists an immersion $\Psi^u$ with parallel mean curvature vector field.

Section 5 focuses in the two dimensional case, we write down the Codazzi equation in terms of the Cotton-York tensor, Lemma 5.1. Then, Proposition 5.3 shows that tangent spaces of $M$ along these immersions are invariant under the curvature tensor of $(\tilde{M}, \tilde{g})$ if and only if the Cotton-York tensor of $c$ vanishes. In the terminology of [4], [15], this means that the Möbius structure $D$ on $(M, c)$ is flat.

2. Preliminaries

All the manifolds are assumed to be smooth, Hausdorff, satisfying the second axiom of countability and without boundary. Let $M$ be a manifold with $\dim M = n \geq 2$. A Riemannian conformal structure on $M$ is an equivalence class $c = [g]$ of Riemannian metrics where two metrics $g$ and $g'$ are said to be equivalent when $g' = e^{2u}g$. A Möbius structure on a manifold $M$ is essentially equivalent to defining a conformal class of metrics $c$ on $M$ and a “Schouten type-tensor” for $c$. This problem was addressed in [4] and [5, Sec. 5]. For our purposes, we adopt the following definition.

Definition 2.1. ([4], [15]) A Möbius structure on an $(n \geq 2)$-dimensional manifold $M$ is a triple $(M, c, D)$ where $c$ is a Riemannian conformal structure on $M$ and
(1) $D$ is a map $D : c \to T_{(0,2)}M$ such that for every $g \in c$, the tensor $D(g)$ is symmetric with
\[
\text{trace}_g D(g) = \frac{\text{scal}_g}{2(n - 1)},
\]
where $\text{scal}_g$ is the scalar curvature of the metric $g \in c$ and $\text{trace}_g D(g)$ denotes the $g$-metric trace of the corresponding tensor $D(g)$.

(2) $D$ satisfies the following conformal transformation law
\[
D(e^{2u}g) = D(g) - \frac{\|\nabla^g u\|_g^2}{2}g - \text{Hess}^g(u) + du \otimes du,
\]
where $\nabla^g u$ and $\text{Hess}^g(u)$ are the gradient and the Hessian of the function $u \in C^\infty(M)$ for the metric $g$, respectively.

We mean the map $D$ as a Möbius structure for the conformal structure $c$. The conformal transformation law implies that a Möbius structure $D$ for a conformal class $c$ is completely determined by the value at a single $g \in c$. In fact, the relationship between the scalar curvatures of two conformally related metrics and the conformal transformation law imply that
\[
\text{trace}_{e^{2u}g} D(e^{2u}g) = \frac{\text{scal}_{e^{2u}g}}{2(n - 1)}.
\]

For $(n \geq 3)$-dimensional Riemannian conformal structures $(M, c)$, there is a preferred Möbius structure. In fact, let us recall that Schouten tensor is defined by
\[
\text{P}_g(X, Y) = \frac{1}{n - 2} \left( \text{Ric}_g(X, Y) - \frac{\text{scal}_g}{2(n - 1)}g \right),
\]
where $\text{Ric}_g$ denotes the Ricci tensor of the Riemannian metric $g \in c$. The well-known conformal transformation law for the Schouten tensor implies that $D(g) = \text{P}_g$ provides a Möbius structure for the conformal class $c$. Therefore, for conformal structures on $(n \geq 3)$-dimensional manifolds, the Schouten tensor gives a canonical Möbius structure. For the two dimensional case, there is something new. Namely, on a 2-dimensional conformal Riemannian manifold $(M, c)$, a Möbius structure is equivalent to specifying a “Schouten type-tensor”, $[3], [4]$.

**Remark 2.2.** The Uniformization Theorem states that a 2-dimensional Riemannian manifold $(M, g)$ admits a metric $g'$ conformal to $g$ with constant Gauss curvature $k$. This fact leads to a choice of the Möbius structure determined by $D(g') = (k/2)g'$ and the conformal transformation law. On the other hand, recall that for a connected oriented 2-dimensional manifold $M$, there is a well-known one-to-one correspondence between conformal classes and complex structures. A Riemann surface is a such 2-dimensional manifold endowed with a particular choice of conformal or complex structure. Thus, a Möbius structure on a connected oriented 2-dimensional manifold $M$ is equivalent to specifying a complex structure and a “Schouten type-tensor” on $M$.

**Remark 2.3.** For $n \geq 3$ and taking into account $2 \div \text{Ric}_g = d \text{scal}_g$ (see for instance $[12]$ Cor. 3.54), one gets that $\div \text{P}_g = \frac{1}{2(n - 1)} d \text{scal}_g$. This property is not satisfied for Möbius structures, in general.

In this section we also fix some terminology and notations for spacelike immersions in Lorentzian manifolds. Let $(\tilde{M}, \tilde{g})$ be an $(m \geq 2)$-dimensional Lorentzian manifold. That is, $(\tilde{M}, \tilde{g})$ is
a semi-Riemannian manifold endowed with a metric tensor $\tilde{g}$ of signature $(1, m - 1)$. A smooth immersion $\Psi : M \to (\tilde{M}, \tilde{g})$ of a (connected) $n$-dimensional manifold $M$ is said to be spacelike when the induced metric $g := \Psi^* (\tilde{g})$ is Riemannian.

Let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields along the spacelike immersion $\Psi$. Every vector field $X \in \mathfrak{X}(M)$ provides, in a natural way, the vector field $X|_\Psi = X \circ \Psi \in \mathfrak{X}(M)$. As usual, for $V \in \mathfrak{X}(M)$, we have the decomposition $V = V^\top + V^\bot$, where $V_x^\top \in T_x \Psi \cdot T_x M$ and $V_x^\bot = (T_x \Psi \cdot T_x M)^\bot$ for all $x \in M$. We have agreed to denote by $T \Psi$ the differential map of $\Psi$. We call $V^\top$ the tangent part of $V$ and $V^\bot$ the normal part of $V$. The $C^\infty(M)$-submodule of $\mathfrak{X}(M)$ of all normal vector fields along $\Psi$ is denoted by $\mathfrak{X}^\bot(M)$, that is, $\mathfrak{X}^\bot(M) = \{ V \in \mathfrak{X}(M) : V^\top = 0 \}$. The set of vector fields $\mathfrak{X}(M)$ may be seen as a $C^\infty(M)$-submodule of $\mathfrak{X}(M)$ by meaning of

$$\mathfrak{X}(M) \to \mathfrak{X}(M), \quad V \mapsto T \Psi \cdot V,$$

where $(T \Psi \cdot V)(x) := T_x \Psi \cdot V_x$ for all $x \in M$. In order to avoid ambiguities, we explicitly write the differential map $T \Psi$ when necessary.

We write $\nabla^g$ and $\nabla$ for the Levi-Civita connections of $(M, g)$ and $(\tilde{M}, \tilde{g})$, respectively. As usual, we also denote by $\nabla$ the induced connection and by $\nabla^\bot$ the normal connection on $M$. The decomposition of the induced connection $\nabla$, into tangent and normal parts, leads to the Gauss and Weingarten formulas of $\Psi$ as follows

$$(2) \quad \nabla_V W = T \Psi \cdot \nabla_V^g W + \Pi(V, W) \quad \text{and} \quad \nabla_V \xi = -T \Psi \cdot A_\xi V + \nabla_V^\bot \xi,$$

for every tangent vector fields $V, W \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^\bot(M)$. Here $\Pi$ denotes the second fundamental form and $A_\xi$ the Weingarten endomorphism (or shape operator) associated to $\xi$. For vector fields $U, V, W \in \mathfrak{X}(M)$, we let

$$(3) \quad (\nabla_U \Pi)(V, W) = \nabla_U^\bot (\Pi(V, W)) - \Pi(\nabla_U V, W) - \Pi(V, \nabla_U W).$$

Then the Codazzi equation reads as follows (see for instance [12] Prop. 4.33), taking into account that our convention on the sign of the Riemannian curvature tensor is the opposite to [12])

$$(4) \quad (\nabla_U \Pi)(V, W) - (\nabla_V \Pi)(U, W) = \left( \hat{R}(T \Psi \cdot U, T \Psi \cdot V) T \Psi \cdot W \right),$$

where $\hat{R}$ is the curvature tensor of $\nabla$. Every Weingarten endomorphism $A_\xi$ is self-adjoint and the second fundamental form is symmetric. They are also related by the following formula

$$(5) \quad g (A_\xi V, W) = \tilde{g} (\Pi(V, W), \xi).$$

The normal curvature tensor $R^\bot$ is given by

$$R^\bot (V, W) \xi = \nabla_V^\bot \nabla_W ^\bot \xi - \nabla_W^\bot \nabla_V^\bot \xi - \nabla_{[V, W]}^\bot \xi$$

and the mean curvature vector field by $H = \frac{1}{m} \text{tr}_{\tilde{g}} \Pi$.

A particular case occurs when, working with a codimension two immersion $\Psi$, we are able to find a global lightlike normal frame $\{ \xi, \eta \}$ along $\Psi$. That is, $\xi$ and $\eta$ are two globally defined
normal vector fields along $\Psi$ which are lightlike (i.e., $\tilde{g}(\xi, \xi) = \tilde{g}(\eta, \eta) = 0$) with the normalization condition $\tilde{g}(\xi, \eta) = -1$. Let $A_\xi$ and $A_\eta$ be the associated Weingarten endomorphisms. Then, for every $V, W \in \mathfrak{X}(M)$, the second fundamental form can be written as

$$II(V, W) = -g(A_\eta V, W)\xi - g(A_\xi V, W)\eta.$$ 

Taking traces in this expression, we obtain for the mean curvature vector field

$$H = -\frac{1}{n} (\text{trace}(A_\eta)\xi + \text{trace}(A_\xi)\eta).$$

Let $(M, c)$ be a Riemannian conformal structure on an $(n \geq 2)$-dimensional manifold $M$. Let us consider the $\mathbb{R}^+$-principal fiber bundle $\pi : Q \to M$ defined as the ray fiber subbundle in the fiber bundle of Riemannian metrics given by metrics in the conformal class $c$. Thus, the fiber over $x \in M$ is formed by the values of $g_x$ for all metrics $g \in c$. Every section of $\pi$ provides a Riemannian metric in the conformal class $c$ and the principal $\mathbb{R}^+$-action on $Q$ is given by $\varphi(\tau, g_x) = \tau^2 g_x, x \in M$. Let us denote by $Z_Q$ the fundamental vector field for the action $\varphi$, that is,

$$Z_Q(g_x) = \frac{d}{dt} \bigg|_{t=0} \varphi(e^t, g_x) = \frac{d}{dt} \bigg|_{t=0} (e^{2t} g_x).$$

The principal bundle $\pi : Q \to M$ is called the scale bundle of $(M, c)$.

**Definition 2.4.** ([10]) A pre-ambient space for a Riemannian conformal structure $(M, c)$ is an $(n+2)$-dimensional Lorentzian manifold $(\tilde{M}, \tilde{g})$ such that

1. There is a free $\mathbb{R}^+$-action $\tilde{\varphi}$ on $\tilde{M}$ and an embedding $\iota : Q \to \tilde{M}$ such that the following diagram commutes

   $$\begin{array}{ccc}
   \mathbb{R}^+ \times Q & \xrightarrow{\text{id}_{\mathbb{R}^+} \times \iota} & \mathbb{R}^+ \times \tilde{M} \\
   \varphi \downarrow & & \downarrow \tilde{\varphi} \\
   Q & \underset{\iota}{\rightarrow} & \tilde{M}
   \end{array}$$

   Hence, the fundamental vector field $Z \in \mathfrak{X}(\tilde{M})$ for the action $\tilde{\varphi}$ and the vector field $Z_Q \in \mathfrak{X}(Q)$ are $t$-related, i.e., $T_{g_x} \cdot Z_Q(g_x) = Z(\iota(g_x))$ for all $g_x \in Q$.

2. For $Z \in \mathfrak{X}(\tilde{M})$, we have $\mathcal{L}_Z \tilde{g} = 2\tilde{g}$, where $\mathcal{L}$ is the Lie derivative.

3. For any $g_x \in Q$ and $\xi, \eta \in T_{g_x}Q$, the following equality holds

   $$\iota^*(\tilde{g})_{g_x}(\xi, \eta) = g_x(T_{g_x} \pi \cdot \xi, T_{g_x} \pi \cdot \eta).$$

   In particular, we have $\iota^*(\tilde{g})(Z_Q, -) = 0$.

For a pre-ambient space $(\tilde{M}, \tilde{g})$ the metric $\tilde{g}$ is called a pre-ambient metric. The condition $\mathcal{L}_Z \tilde{g} = 2\tilde{g}$ tells us that the vector field $Z$ is homothetic with respect to the pre-ambient metric $\tilde{g}$. 
Remark 2.5. The notion of ambient metric in [10] satisfies a normalisation condition. In fact, in order to obtain the uniqueness of the ambient Lorentzian metric $\tilde{g}$, the ambient metric by Fefferman and Graham imposes that the Ricci tensor of the metric $\tilde{g}$ vanishes to a certain order (depending on the dimension) on $Q$, see [10] for details. The pre-ambient space has been used by Čap and Gover in order to get the relationships to the standard tractors, see [7].

We end this section with several comments from the point of view of Lorentzian geometry of the notion of pre-ambient space.

Definition 2.6. A lightlike manifold is a pair $(N, h)$ where $N$ is an $(n + 1)$-dimensional smooth manifold with $n \geq 2$ and furnished with a lightlike metric $h$. That is, $h$ is a symmetric $(0, 2)$-tensor field on $N$ such that

1. $h(\xi, \xi) \geq 0$ for all $\xi \in \mathcal{X}(N)$.
2. For every $y \in N$, the radical $\text{Rad}(h)(y) = \{\xi \in T_yN : h(\xi, -) = 0\}$ defines a 1-dimensional distribution on $N$.

A smooth immersion $\Psi: N^{n+1} \to (\tilde{M}^{n+2}, \tilde{g})$ in an arbitrary Lorentzian manifold is said to be a lightlike hypersurface when the induced tensor $\Psi^{*}(\tilde{g})$ is a lightlike metric.

Now, let $(M, c)$ be a Riemannian conformal structure on an $(n \geq 2)$-dimensional manifold $M$ and $(\tilde{M}, \tilde{g})$ a pre-ambient space for $(M, c)$. Then, condition (3) in Definition 2.4 implies that $\nu: Q \to \tilde{M}$ is a lightlike hypersurface. Moreover, the induced lightlike metric $h := \nu^{*}(\tilde{g})$ does not depend on the particular pre-ambient metric $\tilde{g}$. In the terminology of [10], the lightlike metric $h$ is called the tautological tensor. The radical distribution $\text{Rad}(h)$ is globally generated by the vector field $Z_{Q}$.

Recall that every choice of a metric $g \in c$ provides a section of $\pi: Q \to M$ and conversely. The following result is well-known. We include here a proof for the sake of completeness.

Lemma 2.7. Let $(M, c)$ be a Riemannian conformal structure and $(\tilde{M}, \tilde{g})$ a pre-ambient space for $(M, c)$. For every $g \in c$, the map

$$\Psi^{g} := \nu \circ g: M \to (\tilde{M}, \tilde{g})$$

is a codimension two spacelike immersion with induced metric $(\Psi^{g})^{*}(\tilde{g}) = g$. Moreover, the vector field $\xi := Z \mid_{\Psi^{g}}$ is normal and lightlike along $\Psi^{g}$ with $A_{\xi} = -\text{Id}$.

Proof. For every $x \in M$ a direct computation gives

$$(\Psi^{g})^{*}(\tilde{g})_{x} = g^{*}(\nu^{*}(\tilde{g}))_{g(x)} = g^{*}(\pi^{*}(g))_{\nu(x)} = (\pi \circ g)^{*}(g)_{x} = g_{x}.$$

Taking into account that $\xi_{x} = Z(\Psi^{g}(x)) = T_{g(x)}\nu \cdot Z_{Q}(g_{x})$, we get $\xi \in \mathcal{X}^{\perp}(M)$ (for the immersion $\Psi^{g}$) and $\tilde{g}(\xi, \xi) = 0$. In order to see that $A_{\xi} = -\text{Id}$, recall that the condition $L_{Z}g = 2g$ is equivalent to

$$\tilde{g}(\bar{\nabla}_{X}Z, Y) + \tilde{g}(X, \bar{\nabla}_{Y}Z) = 2\tilde{g}(X, Y), \quad X, Y \in \mathcal{X}(\tilde{M}).$$

In particular, for vector fields $V, W \in \mathcal{X}(M)$ we have

$$\tilde{g}(\bar{\nabla}_{T_{\Psi^{g}}V}Z, T_{\Psi^{g}}W) + \tilde{g}(T_{\Psi^{g}}V, \bar{\nabla}_{T_{\Psi^{g}}W}Z) = 2\tilde{g}(T_{\Psi^{g}}V, T_{\Psi^{g}}W).$$
and from the polarization identity we arrive to
\[
\tilde{g}(\nabla_{T\Psi^g} V Z, T\Psi^g \cdot W) = \tilde{g}(T\Psi^g \cdot V, T\Psi^g \cdot W).
\]

We are in position to compute \(\tilde{\nabla}_V \xi\) as follows
\[
\tilde{\nabla}_V \xi = \tilde{\nabla}_{T\Psi^g} (\nabla_{T\Psi^g} (V Z)) = (\tilde{\nabla}_{T\Psi^g} (V Z))^\top + (\tilde{\nabla}_{T\Psi^g} (V Z))^\bot = T\Psi^g \cdot V + \nabla_V \xi
\]
and now the assertion \(A_\xi = -\text{Id}\) is clear. □

3. A METHOD TO CONSTRUCT PRE-AMBIENT SPACES

Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\).

\textbf{Definition 3.1.} A smooth 1-parameter family \(\alpha : \mathbb{R} \to \mathcal{T}_{(1,1)} M\) is called admissible when

1. \(\alpha(\rho)\) is a self-adjoint tensor field with respect to any representative \(g \in c\),
2. \(\alpha(0) = \text{Id}\),
3. there is \(\delta > 0\) such that \(\alpha(\rho)\) is not singular for \(|\rho| < \delta\).

Here, the smoothness of \(\alpha\) means that for every \(V \in \mathfrak{X}(M)\) and \(x \in M\), there exists
\[
\dot{\alpha}(\rho)(V_x) = \lim_{\varepsilon \to 0} \frac{\alpha(\rho + \varepsilon)(V_x) - \alpha(\rho)(V_x)}{\varepsilon} \in T_x M.
\]
In particular, we have \(\dot{\alpha}(0) \in \mathcal{T}_{(1,1)} M\).

\textbf{Remark 3.2.} The condition (3) in the above definition can be deleted when \(M\) is compact and, at least locally, \(\delta\) always exists, in the general case.

Let us fix a metric \(g \in c\) and an admissible smooth 1-parameter family \(\alpha : \mathbb{R} \to \mathcal{T}_{(1,1)} M\). For every \(\rho \in \mathbb{R}\), we define the following symmetric tensor on \(M\),
\[
\langle V, W \rangle^g_\rho = g(\alpha(\rho)(V), W).
\]
Clearly, \(\langle , \rangle^g_0 = g\) and so \(\langle , \rangle^g_\rho\) can be seen as a 1-parameter deformation of the metric \(g\). Moreover, \(\langle , \rangle^g_\rho\) is positive definite on \(M\) for \(|\rho| < \delta\). Henceforth, let us consider the manifold \(\tilde{M} := B \times M\), where \(B := \mathbb{R}^+ \times (-\delta, +\delta)\) with coordinates \((t, \rho)\). This manifold \(\tilde{M}\) can be endowed with the Lorentzian metric
\[
\tilde{g} = dt \otimes dt + d\rho \otimes d\rho + t^2 (-\rho, -) = d(\rho t) \otimes dt + dt \otimes d(\rho t) + t^2 (-, -)\rho^g
\]
and with the free \(\mathbb{R}^+\)-action \(\tilde{\varphi}(\tau, (t, \rho, x)) = (\tau t, \rho, x)\). The choice of the metric \(g \in c\) provides the global trivialization of \(\pi : Q \to M\) given by
\[
t^2 g_x \in Q \mapsto (t, x) \in \mathbb{R}^+ \times M
\]
and the following embedding of \(Q\) in \(\tilde{M}\) at \(\rho = 0\),
\[
t_g : Q \to \tilde{M}, \quad t^2 g_x \mapsto (t, 0, x).
\]
A direct computation shows that \( t_g \circ \varphi(\tau, t^2 g_x) = \tilde{\varphi} \circ (\text{id}_{\mathbb{R}^+} \times t_g)(\tau, t^2 g_x) = (\tau t, 0, x) \). On the other hand, the fundamental vector field \( Z \in \mathcal{X}(M) \) corresponding to the action \( \tilde{\varphi} \) is \( Z = t \frac{\partial}{\partial t} \) and one directly checks that \( \mathcal{L}_Z g = 2g \). Finally, for \( t^2 g_x \in \mathcal{Q} \) and \( \xi, \eta \in T_t \tilde{\varphi}_g \mathcal{Q} \), we have

\[
(t^g \tilde{g})_{\tilde{g}_x} (\xi, \eta) = \tilde{g}((t, 0, x) (T_t \tilde{\varphi}_g \cdot t_g \cdot \xi, T_t \tilde{\varphi}_g \cdot t_g \cdot \eta)) = t^2 g_x (T_t \tilde{\varphi}_g, \pi \cdot \xi, T_t \tilde{\varphi}_g, \pi \cdot \eta).
\]

Hence, \((\tilde{M} = B \times M, \tilde{g})\) where the metric \( \tilde{g} \) is given in (9) is a pre-ambient space for \((M, c)\).

We have thus led to the following result.

**Proposition 3.3.** Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\). For every choice of a metric \(g \in \mathcal{C}\) and an admissible smooth 1-parameter family \(\alpha: \mathbb{R} \to \mathcal{T}_{(1,1)}M\), the manifold \(\tilde{M} = B \times M\) is a pre-ambient space for \((M, c)\).

**Remark 3.4.** In the particular case that \(\alpha(\rho) = f^2(\rho) \text{Id}\) with \(f: (-\delta, +\delta) \to \mathbb{R}, f(0) = 1\) and \(f > 0\), the pre-ambient space \((\tilde{M}, \tilde{g})\) with metric \(\tilde{g} = d(\rho t) \otimes dt + dt \otimes d(\rho t) + (t f(\rho))^2 g\) is a warped product in the terminology of [12 Chap. 7].

**Remark 3.5.** The one-form \(\omega\) metrically equivalent to the vector field \(Z\) is

\[
\omega = t^2 d\rho + 2t \rho dt,
\]

thus, we have \(d\omega = 0\).

As a Lorentzian manifold, the pre-ambient space \((\tilde{M}, \tilde{g})\) is timelike orientable, that is, there exists a globally defined timelike vector field, namely,

\[
T := \frac{1}{t} \partial_t - (1 + \frac{\rho}{t^2}) \partial_\rho \in \mathcal{X}(\tilde{M}),
\]

which satisfies \(\tilde{g}(T, T) = -2\). To be used later, we also introduce the spacelike vector field

\[
E := \frac{1}{t} \partial_t + (1 - \frac{\rho}{t^2}) \partial_\rho \in \mathcal{X}(\tilde{M}),
\]

with \(\tilde{g}(E, E) = 2\) and \(\tilde{g}(T, E) = 0\). The set of all natural lifts of vector fields \(V \in \mathcal{X}(M)\) to \(\mathcal{X}(\tilde{M})\) is denoted by \(\mathcal{L}(M)\). For a vector field \(V \in \mathcal{X}(M)\), its lift to \(\mathcal{L}(M) \subset \mathcal{X}(\tilde{M})\) is also denoted by \(V\).

As was mentioned in Remark 3.4, the metrics \(\tilde{g}\) in (9) are not warped product metrics, in general. Hence, the formulas for the Levi-Civita connection of warped products metrics in [12 Prop. 7.36] do not work.

**Proposition 3.6.** The Levi-Civita connection \(\tilde{\nabla}\) of \((\tilde{M}, \tilde{g})\) satisfies

\[
\tilde{\nabla}_{\partial_t} \partial_t = \tilde{\nabla}_{\partial_\rho} \partial_\rho = 0, \quad \tilde{\nabla}_{\partial_\rho} \partial_t = \tilde{\nabla}_{\partial_t} \partial_\rho = \frac{1}{t} \partial_\rho,
\]

\[
\tilde{\nabla}_{\partial_t} V = \frac{1}{t} V, \quad \tilde{\nabla}_{\partial_\rho} V = \frac{1}{2} \alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)),
\]

\[
\tilde{\nabla}_V W |_{\eta_g(\mathcal{Q})} = \frac{1}{2t} \tilde{g}(\dot{\alpha}(0)(V), W) \partial_t - \frac{1}{t^2} \tilde{g}(V, W) \partial_\rho + \nabla^\eta_t W,
\]
where $V, W \in \mathcal{L}(M)$.

**Proof.** A direct consequence of Koszul formula for the Levi-Civita connection of $(\tilde{M}, \tilde{g})$ shows
\[
\tilde{\nabla}_t \partial_t = \tilde{\nabla}_{\partial_t} \partial_t = 0 \quad \text{and} \quad \nabla_{\partial_t} \partial_t = \frac{1}{t} \partial_t.
\]
On the other hand, the Koszul formula also implies
\[
\tilde{g}(\tilde{\nabla}_t \partial_t, \partial_t) = \tilde{g}(\tilde{\nabla}_t \partial_t, \partial_\rho) = 0 \quad \text{and} \quad 2\tilde{g}(\tilde{\nabla}_t \partial_t, W) = \partial_t \tilde{g}(V, W).
\]
By definition of the metric $\tilde{g}$,
\[
\partial_t \tilde{g}(V, W) = 2tg(\alpha(\rho)(V), W) = \frac{2}{t} \tilde{g}(V, W),
\]
and then we get $\tilde{\nabla}_t \partial_t = \frac{1}{t} \partial V$. In the same manner, we compute
\[
2\tilde{g}(\tilde{\nabla}_{\partial_t} V, W) = \partial_\rho \left( t^2 g(\alpha(\rho)(V), W) \right) = t^2 g(\dot{\alpha}(\rho)(V), W) = \tilde{g} \left( \alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)), W \right).
\]
From (14), it follows that
\[
\tilde{g}(\tilde{\nabla}_V W, \partial_t) = \tilde{g}(\tilde{\nabla}_V \partial_t, W) = -\frac{1}{t^2} \tilde{g}(V, W), \quad \tilde{g}(\tilde{\nabla}_V W, \partial_\rho) = -\frac{1}{2} \tilde{g} \left( \alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)), W \right).
\]
In order to compute $\tilde{g}(\tilde{\nabla}_{U}, W)$ for $U \in \mathcal{L}(M)$ and $p = (t, 0, x) \in \iota_{\partial}(\mathcal{Q})$, we can assume $U, V, W \in \mathcal{L}(M)$ so that all their brackets are zero at the point $p$. Then, the Koszul formula yields
\[
2\tilde{g}(\tilde{\nabla}_{U} W, U_p) = V_p \tilde{g}(W, U) + W_p \tilde{g}(V, U) = U_p \tilde{g}(V, W) = t^2 \left( V_x g(W, U) + W_x g(V, U) - U_x g(V, W) \right)
\]
\[
= 2t^2 g(\tilde{\nabla}_x V, U_x) = 2\tilde{g} \left( (\tilde{\nabla}_x V)_p, U_p \right).
\]
Therefore, we conclude that
\[
\tilde{\nabla}_{U} W \big|_{\iota_{\partial}(\mathcal{Q})} = -\frac{1}{2t} \tilde{g}(\tilde{\nabla} V T + \frac{1}{2t} \tilde{g}(\tilde{\nabla} W, E) E + \tilde{\nabla}^2 W)
\]
\[
= -\frac{1}{2t} \tilde{g}(\dot{\alpha}(0)(V), W) \partial_t - \frac{1}{t^2} \tilde{g}(V, W) \partial_\rho + \tilde{\nabla}^2 W.
\]
\[
\square
\]

**Remark 3.7.** Let us fix $(t, \rho) \in B$ and consider the spacelike submanifold
\[
\mathcal{F} := \{(t, \rho)\} \times M \subset \tilde{M}.
\]
The vector fields $T|_{\mathcal{F}}$ and $E|_{\mathcal{F}}$ span the normal bundle of $\mathcal{F}$ and Proposition 3.6 implies
\[
\tilde{\nabla}_V T|_{\mathcal{F}} = \frac{1}{t^2} V - \frac{1}{2} \left( 1 + \frac{\rho}{t^2} \right) \alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)) \quad \text{and}
\]
\[
\tilde{\nabla}_V E|_{\mathcal{F}} = \frac{1}{t^2} V - \frac{1}{2} \left( 1 - \frac{\rho}{t^2} \right) \alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V))
\]
for every $V \in \mathcal{L}(M)$. Therefore, the second fundamental form $\Pi_{\mathcal{F}}$ is given by
\[
\Pi_{\mathcal{F}}(V, W) = -\frac{1}{2t^2} \tilde{g}(\alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)), W) \partial_t - \frac{1}{t^2} \left( \tilde{g}(\partial V, W) - \rho \tilde{g}(\alpha(\rho)^{-1}(\dot{\alpha}(\rho)(V)), W) \right) \partial_\rho.
\]
where \( V, W \in \mathfrak{X}(M) \). Thus, on the contrary to the warped products metrics, the fibers \( \mathcal{F} \) are not totally umbilical, in general. It is not difficult to show that for a fixed \((t, \rho)\), the corresponding fiber \( \mathcal{F} \) is totally umbilical if and only if the endomorphism field \( \alpha(\rho)^{-1} \circ \dot{\alpha}(\rho) = f \text{Id} \) for some \( f \in C^\infty(M) \).

**Remark 3.8.** For \( \alpha(\rho) = f^2(\rho) \text{Id} \) with \( f > 0 \), the metric \( \tilde{g} \) is a warped metric with warping function \( h(t, \rho) = tf(\rho) \). In this case (16) reduces to

\[
\Pi_\mathcal{F}(V, W) = -\frac{\tilde{g}(V, W)}{tf(\rho)} \left( f'(\rho) \partial_t + f(\rho) \frac{2\rho f'(\rho)}{t} \partial_\rho \right).
\]

A direct computation shows that the above formula agrees with [12, Prop. 7.35 (3)].

From [7], the Ricci tensor \( \widetilde{\text{Ric}} \) of any pre-ambient space \((\widetilde{M}, \widetilde{g})\) restricted to \( \iota_g(\mathcal{Q}) \) satisfies

\[
(17) \quad \widetilde{\text{Ric}}|_{\iota_g(\mathcal{Q})}(\partial_t, \partial_t) = \widetilde{\text{Ric}}|_{\iota_g(\mathcal{Q})}(\partial_t, V) = 0, \quad V \in \mathcal{L}(M)
\]

if and only if \( d\omega|_{\iota_g(\mathcal{Q})} = 0 \). As consequence of Remark 3.5, this formula (17) holds for the metric \( \tilde{g} \) in (9). The following result provides the other component of \( \widetilde{\text{Ric}} \) on \( \iota_g(\mathcal{Q}) \).

**Corollary 3.9.** The Ricci tensor \( \widetilde{\text{Ric}} \) of \((\widetilde{M}, \widetilde{g})\) satisfies

\[
(18) \quad \widetilde{\text{Ric}}|_{\iota_g(\mathcal{Q})}(V, W) = \text{Ric}^g(V, W) - \frac{\text{trace}(\dot{\alpha}(0))}{2} g(V, W) - \left( \frac{n-2}{2} \right) g(\dot{\alpha}(0)(V), W),
\]

where \( V, W \in \mathcal{L}(M) \). For \( \xi, \eta \in \mathfrak{X}(\mathcal{Q}) \), we have

- If \( n = 2 \),
  \[
  \widetilde{\text{Ric}}|_{\iota_g(\mathcal{Q})}(T_{tg} \cdot \xi, T_{tg} \cdot \eta) = 0
  \]
  if and only if \( \text{trace}(\dot{\alpha}(0)) = 2K^g \), where \( K^g \) is the Gaussian curvature of \( g \).

- If \( n \geq 3 \),
  \[
  \widetilde{\text{Ric}}|_{\iota_g(\mathcal{Q})}(T_{tg} \cdot \xi, T_{tg} \cdot \eta) = 0
  \]
  if and only if \( g(\dot{\alpha}(0)(-), -) = 2P^g \), where \( P^g \) is the Schouten tensor of \( g \).

**Proof.** Let \((e_1, \ldots, e_n)\) be an orthonormal local frame on \((M, g)\) and consider the orthonormal local frame for \((\widetilde{M}, \widetilde{g})\) on \( \rho = 0 \) given by

\[
\left( \frac{1}{\sqrt{2}}T, \frac{1}{\sqrt{2}}E, E_1, \ldots, E_n \right),
\]

where \( E_i = \frac{1}{t}e_i \) and the vector fields \( T, E \) are given in (11) and (12), respectively. Then, we get

\[
\begin{align*}
\widetilde{\text{Ric}}|_{\iota_g(\mathcal{Q})}(V, W) &= \sum_{i=1}^{n} \tilde{g} \left( \widetilde{\text{R}}(E_i, V)W, E_i \right) + \frac{1}{2} \tilde{g} \left( \widetilde{\text{R}}(E, V)W, E \right) - \frac{1}{2} \tilde{g} \left( \widetilde{\text{R}}(T, V)W, T \right) \\
&= \sum_{i=1}^{n} \tilde{g} \left( \widetilde{\text{R}}(E_i, V)W, E_i \right) + \frac{1}{t} \left( \tilde{g} \left( \widetilde{\text{R}}(\partial_t, V)W, \partial_t \right) + \tilde{g} \left( \widetilde{\text{R}}(\partial_\rho, V)W, \partial_\rho \right) \right).
\end{align*}
\]
For every vector field $X \in \mathfrak{X}(\tilde{M})$, we have the following decomposition

\[ X = \sum_{i=1}^{n} f_i E_i + \frac{1}{2} \tilde{g}(X, E) E - \frac{1}{2} \tilde{g}(X, T) T \]

where $f_i \in C^\infty(\tilde{M})$. Let us note that $f_i|_{\rho=0} = \tilde{g}(X, E_i)$. Now, a straightforward computation from Proposition 3.6 gives

\[ \tilde{g} \left( \tilde{R}(\partial_t, V) W, \partial_\rho \right) + \tilde{g} \left( \tilde{R}(\partial_\rho, V) W, \partial_t \right) = 0. \]

Finally, it is a standard computation, from Proposition 3.6 and (19), to check that

\[ \tilde{\text{Ric}}|_{\iota_g(Q)}(V, W) = \sum_{i=1}^{n} g \left( \nabla_{e_i}^g \nabla_e^g W, e_i \right) - \sum_{i=1}^{n} g \left( \nabla_{e_i}^g \nabla_{e_i}^g W, e_i \right) - \frac{n}{2} g(\partial_t(0)(V), W) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} g(\partial_t(0)(V), e_i) + \frac{1}{2} \sum_{i=1}^{n} g(V, e_i) g(\partial_t(0)(e_i), W) \]

\[ = \text{Ric}^g(V, W) - \frac{\text{trace}(\partial_t(0))}{2} g(V, W) - \left( \frac{n-2}{2} \right) g(\partial_t(0)(V), W). \]

The vanishing properties of the Ricci tensor on $\iota_g(Q)$ are direct consequences of (17) and (18).

4. Constructing Möbius structures from spacelike immersions

From now on, we assume $(M, c)$ is a Riemannian conformal structure on an $(n \geq 2)$-dimensional manifold $M$ and we have fixed

1. a metric $g \in c$
2. an admissible smooth 1-parameter family $\alpha : \mathbb{R} \to T(1,1)M$.

Thus, we have the pre-ambient space $(\tilde{M}, \tilde{g})$ as in Proposition 3.3.

For every $u \in C^\infty(M)$, the spacelike immersion $\Psi^{2u g}$ in Lemma 2.7 satisfies

\[ \Psi^{2u g} : M \to (\tilde{M}, \tilde{g}), \quad x \mapsto (e^{u(x)} 0, x) \]

and $(\Psi^{2u g})^*(\tilde{g}) = e^{2u} g$. For simplicity of notation, from now on, we write $\Psi^u$ instead of $\Psi^{2u g}$.

The differential map of $\Psi^u$ is

\[ T\Psi^u \cdot V = V(u)e^u \partial_t|_{\Psi^u} + V|_{\Psi^u}, \]
where $V \in \mathfrak{X}(M)$. A direct computation from (21) shows that the vector fields

$$\xi^u = e^u \partial_t |_{\Psi^u} \text{ and } \eta^u = e^{-u} \| \nabla^g u \|_g^2 \partial_t |_{\Psi^u} - e^{-2u} \partial_r |_{\Psi^u} + e^{-2u} \nabla^g u |_{\Psi^u}$$

span the normal bundle of $\Psi^u$ and one easy checks that $\{\xi^u, \eta^u\}$ is a global lightlike normal frame. The lightlike normal vector field $\xi^u$ agrees with $Z |_{\Psi^u}$ where $Z \in \mathfrak{X}(\bar{M})$ is the fundamental vector field corresponding to the action $\bar{\varphi}$.

**Lemma 4.1.** Let $\Psi^u : M \to (\bar{M}, \bar{g})$ be the immersion given in (20). For every $V \in \mathfrak{X}(M) \subset \mathfrak{X}(\bar{M})$, the following formulas hold

$$(V |_{\Psi^u})^\top = T \Psi^u \cdot V, \quad (\partial_r |_{\Psi^u})^\top = T \Psi^u \cdot \nabla^g u.$$  

**Proof.** From (21) and (22), it is easy to check that

$$(V |_{\Psi^u})^\top = V |_{\Psi^u} + \bar{g}\left(V |_{\Psi^u}, \xi^u\right)\eta^u + \bar{g}\left(V |_{\Psi^u}, \eta^u\right)\xi^u = V |_{\Psi^u} + V(u)e^u \partial_t |_{\Psi^u} = T \Psi^u \cdot V.$$  

The same proof works for $(\partial_r |_{\Psi^u})^\top$. \hfill \Box

**Proposition 4.2.** Let $A_{\xi^u}, A_{\eta^u}$ be the Weingarten endomorphisms associated to the lightlike normal vector fields $\xi^u, \eta^u$ given in (22), then $A_{\xi^u} = -\Id$ and

$$A_{\eta^u} = e^{-2u} \left[ \dot{\alpha}(0) - \| \nabla^g u \|_g^2 \Id + g(\nabla^g u, \Id) \nabla^g u - \nabla^g \nabla^g u \right],$$

where $\nabla^g \nabla^g u(V) := \nabla^g \nabla^g u$ for all $V \in \mathfrak{X}(M)$.

**Proof.** The first assertion is a direct consequence of Lemma 2.7. On the other hand, according again to (21) and Proposition 3.6 we have for $V \in \mathfrak{X}(M)$,

$$\nabla_V \left( e^{-u} \frac{\| \nabla^g u \|_g^2}{2} \partial_t \right) = V \left( e^{-u} \frac{\| \nabla^g u \|_g^2}{2} \right) \partial_t |_{\Psi^u} + e^{-2u} \frac{\| \nabla^g u \|_g^2}{2} V |_{\Psi^u},$$

$$\nabla_V (e^{-2u} \partial_r) = -2e^{-2u} V(u) \partial_r |_{\Psi^u} + e^{-2u} \left( \dot{\alpha}(0)(V) \right) |_{\Psi^u}$$

and

$$\nabla_V (e^{-2u} \nabla^g u) = -2e^{-2u} V(u) \nabla^g u |_{\Psi^u} + e^{-2u} \left( \nabla_V \nabla^g u \right) |_{\Psi^u}.$$  

Taking into account that $(\partial_t |_{\Psi^u})^\top = 0$, from Lemma 4.1 we also get

$$\left( \partial_r |_{\Psi^u} - \nabla^g u |_{\Psi^u} \right)^\top = 0.$$  

Then, from (24), (25) and (26), we arrive to

$$\left( \nabla_V \eta^u \right)^\top = e^{-2u} \left[ \frac{\| \nabla^g u \|_g^2}{2} (V |_{\Psi^u})^\top - \frac{1}{2} \left( \left( \dot{\alpha}(0)(V) \right) |_{\Psi^u} \right)^\top + \left( \nabla_V \nabla^g u \right) |_{\Psi^u} \right]^\top.$$  

Now, the proof ends by means of a straightforward computation from (15) and Lemma 4.1. \hfill \Box
Corollary 4.3. Let $\Psi^u : M \to (\tilde{M}, \tilde{g})$ be the immersion given in \eqref{eq:2}. The normal vector fields $\xi^u$ and $\eta^u$ are parallel with respect to the normal connection. In particular, the normal curvature tensor vanishes, that is, $R^\perp(V, W) = 0$ for every $V, W \in \mathfrak{X}(M)$.

Proof. From Proposition 4.2, we know that $A_{\xi^u} = -Id$. Then, the Weingarten formula reads as follows
\[
\tilde{\nabla}_V \xi^u = T\Psi^u \cdot V + \nabla^\perp_V \xi^u = V(u)e^u\partial_t|_{\Psi^u} + V|_{\Psi^u} + \nabla^\perp_V \xi^u.
\]
On the other hand, from \eqref{eq:14}, we get
\[
\tilde{\nabla}_V \xi^u = \tilde{\nabla}_V (e^u\partial_t) = V(u)e^u\partial_t|_{\Psi^u} + e^u e^{-u} V|_{\Psi^u} = V(u)e^u\partial_t|_{\Psi^u} + V|_{\Psi^u},
\]
and therefore $\nabla^\perp_V \xi^u = 0$. Now, taking into account that $\{\xi^u, \eta^u\}$ is a global lightlike normal frame, we have $V\tilde{g}(\xi^u, \eta^u) = \tilde{g}(\xi^u, \nabla^\perp_V \eta^u) = 0$ for every $V \in \mathfrak{X}(M)$. Thus, since $\Psi^u$ is a codimension two spacelike submanifold, there is a smooth function $f \in C^\infty(M)$ such that $\nabla^\perp_V \eta^u = f \xi^u$ and then $0 = \tilde{g}(\eta^u, \nabla^\perp_V \eta^u) = -f$ and so $\nabla^\perp_V \eta^u = 0$. □

Remark 4.4. From Proposition 4.2 and formula \eqref{eq:6}, one obtains the second fundamental form $\Pi^u$ of $\Psi^u$ as follows
\[
\Pi^u(V, W) = -g\left(\frac{\hat{\alpha}(0)}{2} \frac{\|\nabla^g u\|^2}{2} V + V(u)\nabla^g u - \nabla_V \nabla^g u, W\right) \xi^u + e^{2u} g(V, W) \eta^u,
\]
for every $V, W \in \mathfrak{X}(M)$. In particular, the corresponding mean curvature vector field is
\[
H^u = \frac{e^{-2u}}{n} \left(\triangle^g u - \frac{\text{trace}(\hat{\alpha}(0) - (n-2)||\nabla^g u||^2)}{2} \right) \xi^u + \eta^u,
\]
where $\triangle^g$ denotes the Laplace operator of the metric $g$.

Now, we are in position to state the main result of this paper. Assume $(M, c)$ is a Riemannian conformal structure on an $(n \geq 2)$-dimensional manifold $M$ and $\alpha : \mathbb{R} \to \mathcal{T}_{(1,1)} M$ is an admissible smooth 1-parameter family. By means of Proposition 4.2, we have $A_{\rho'} = \frac{\hat{\alpha}(0)}{2}$ and then for every $u \in C^\infty(M)$,
\[
A_{\eta^u} = e^{-2u} \left[A_{\rho^u} - \frac{1}{2} ||\nabla^g u||^2 g \text{Id} + g(\nabla^g u, \text{Id})\nabla^g u - \nabla^g \nabla^g u\right].
\]
Hence, for every $V, W \in \mathfrak{X}(M)$ we get
\[
e^{2u} g(A_{\rho^u}(V), W) = g(A_{\rho^u}(V), W) - \frac{||\nabla^g u||^2}{2} g - \text{Hess}^g(u) + du \otimes du.
\]
In other words, the assignment
\[
D : c \to \mathcal{T}_{(0,2)} M, \quad e^{2u} g \mapsto e^{2u} g(A_{\eta^u}(-), -),
\]
satisfies the conformal transformation law \eqref{eq:2} in \eqref{eq:2}. In addition, if we assume $\text{trace}_g(A_{\rho^u}) = \frac{\text{scal}^g}{2(n-1)}$, the map $D$ defines a Möbius structure for the Riemannian conformal structure $(M, c)$. Therefore, we have obtained the following result.
Theorem 4.5. Let \((M, c)\) be a Riemannian conformal structure on an \((n \geq 2)\)-dimensional manifold \(M\). Assume the admissible smooth 1-parameter family \(\alpha : \mathbb{R} \to T_{(1,1)} M\) satisfies \(\text{trace}(\dot{\alpha}(0)) = \frac{\text{scal}}{n-1}\). Then, the assignment \(D\) in (28) defines a Möbius structure for the Riemannian conformal structure \((M, c)\).

Conversely, every Möbius structure \((M, c, D)\) can be constructed (at least locally) from the above Theorem. In fact, fix \(g \in c\) and consider \(\alpha(\rho) = \text{Id} + 2\rho \hat{D}(g)\), where \(D(g)(V, W) = g(\hat{D}(g)(V), W)\) for \(V, W \in \mathfrak{X}(M)\). For any \(x \in M\), there is an open subset \(x \in O \subset M\) such that \(\alpha\) is an admissible smooth 1-parameter family on \(T_{(1,1)} O\). It is easily checked \((O, c, D)\) is obtained from \(\alpha\) by means of Theorem 4.5. Note that \(\alpha(\rho) = \text{Id} + 2\rho \hat{D}(g)\) can be replaced for any curve with \(\alpha(0) = \text{Id}\) and \(\dot{\alpha}(0) = 2 \hat{D}(g)\).

Remark 4.6. When \(M\) is compact, every Möbius structure \((M, c, D)\) is globally recovered from suitable Weingarten endomorphisms as in Theorem 4.5.

Corollary 4.7. Let \((M, g)\) be a Riemannian manifold with \(\dim M \geq 3\). Then the Schouten tensor \(P^g\) is given by \(P^g = g(A(-), -)\) (at least locally) where \(A\) is the Weingarten endomorphism of a suitable isometric codimension two immersion of \((M, g)\) in a Lorentzian manifold \((\tilde{M}, \tilde{g})\).

Remark 4.8. This result could be compared with the classical Brinkmann result \([2]\) in the 1920s which stated that an \((n \geq 3)\)-dimensional simply connected Riemannian manifold is (locally) conformally flat if and only if it can be isometrically immersed in the future lightlike cone \(N^{n+1} \subset \mathbb{L}^{n+2}\). This classical result is presented in a modern form in \([1]\).

Remark 4.9. Since, there is no preferred Möbius structure on a 2-dimensional Riemannian conformal structure, Theorem 4.5 provides an explicit method to construct such structures. Moreover, by means of Corollary 3.9, the condition \(\text{trace}(\dot{\alpha}(0)) = 2K^g\), where \(K^g\) is the Gauss curvature of fixed metric \(g\) implies that the Ricci tensor of \(\tilde{g}\) satisfies \(\text{Ric}_{\tilde{g}}(\xi, \eta) = 0\) for all \(\xi, \eta \in \mathfrak{X}(Q)\).

Remark 4.10. Under the assumption \(\text{trace}(\dot{\alpha}(0)) = \frac{\text{scal}}{n-1}\) and by means of the relationship between the scalar curvature of conformally related metrics, formula (27) reduces to

\[
\|H^u\|^2 = \frac{1}{2n(n-1)} \text{scal} e^{2u} \xi^u + \eta^u,
\]

and therefore, \(\|H^u\|^2 = \frac{\text{scal} e^{2u} g}{n(n-1)}\). This formula widely generalizes \([13, \text{Cor. 4.5}]\) and \([14, \text{Cor. 3.7}]\). Therefore, the causality of \(H^u\) is determined by the sign of \(\text{scal} e^{2u} g\). For conformal Riemannian structures on compact 2-dimensional manifolds \((M, c)\) and, as direct consequence of the Gauss-Bonnet theorem, we get

\[
\int_M e^{2u} \|H^u\|^2 d\mu_g = 2\pi \chi(M),
\]
where $\chi(M)$ is the Euler characteristic of the manifold $M$ and $d\mu_g$ is the canonical measure associated to $g$. Also, from Corollary 4.3, the condition $\nabla^2 H^u = 0$ is equivalent to $\text{scal}^{2u}g$ being constant (compare with [14, Cor. 3.10]). The positive solution to the Yamabe problem states that on every conformal Riemannian structure $(M, c)$ on a compact manifold $M$ there is a metric $g \in c$ with constant scalar curvature. Therefore, in the compact case, there exists an immersion $\Psi^u$ as in (20) with parallel mean curvature vector field.

5. AN APPLICATION

For a Möbius structure $(M, c, D)$ on a 2-dimensional manifold $M$, the Cotton-York tensor for $g \in c$ has been introduced in [4] and [15] as follows

\begin{equation}
C(g)(U, V, W) = g \left( \left( \nabla^g_U \hat{D}(g) \right)(V) - \left( \nabla^g_V \hat{D}(g) \right)(U), W \right), \quad U, V, W \in \mathcal{X}(M).
\end{equation}

This definition formally agrees with the usual Cotton-York tensor defined from the Schouten tensor of an $(n \geq 3)$-dimensional Riemannian manifold $(M, g)$. The Cotton-York tensor given in (29) for $n = 2$ satisfies $C(g) = C(e^{2u}g)$ (e.g., [15]).

In this Section, we assume $(M, c, D)$ is a Möbius structure on a 2-dimensional manifold $M$ which is achieved by means of Theorem 4.5.

**Lemma 5.1.** Let $(M, c, D)$ be a Möbius structure on a 2-dimensional manifold $M$. Then, the Cotton-York tensor satisfies

\begin{equation}
C(g)(V, U, W)\xi^u = \left( \nabla^g_U \Pi^u \right)(V, W) - \left( \nabla^g_V \Pi^u \right)(U, W),
\end{equation}

for $\Psi^u$ as in (20). Hence, the Codazzi equation (1) reduces to

\begin{equation}
\left( \tilde{R}(T\Psi^u \cdot U, T\Psi^u \cdot V)T\Psi^u \cdot W \right) = C(g)(V, U, W)\xi^u.
\end{equation}

**Proof.** According to Remark 4.4, the second fundamental form of $\Psi^u$ is

\begin{equation}
\Pi^u(V, W) = -P(e^{2u}g)(V, W)\xi^u + e^{2u}g(V, W)\eta^u.
\end{equation}

From Corollary 4.3 we have $\nabla^g_U \xi^u = \nabla^g_U \eta^u = 0$ and then, a direct computation gives

\begin{equation}
\nabla^g_U(\Pi^u(V, W)) = -e^{2u}g \left( \nabla^g_U(e^{2u}g \hat{P}(e^{2u}g)) \right)(V, W)\xi^u.
\end{equation}

Now, the covariant derivative of the second fundamental form in (3) is easily computed. The proof ends by means of (29) and $C(g) = C(e^{2u}g)$ for $n = 2$.

**Definition 5.2.** ([4], [15]) A Möbius structure $(M, c, D)$ on a 2-dimensional manifold $M$ is called flat when $C(g) = 0$ for every $g \in c$.

As a direct consequence of Lemma 5.1 we have.

**Proposition 5.3.** A Möbius structure $(M, c, D)$ on a 2-dimensional manifold $M$ is flat if and only if for every immersion $\Psi^u : M \rightarrow (\tilde{M}, \tilde{g})$ as in (20), the curvature tensor $\tilde{R}$ of the pre-ambient manifold $(\tilde{M}, \tilde{g})$ satisfies

\begin{equation}
\tilde{R}(T\Psi^u \cdot U, T\Psi^u \cdot V)T\Psi^u \cdot W \in \mathcal{X}(M) \subset \mathcal{X}(M),
\end{equation}

for all $U, V, W \in \mathcal{X}(M)$.
Remark 5.4. For a flat Möbius structure \((M, c, D)\), Proposition 5.3 states that tangent spaces of \(M\) along \(\Psi^u\) are invariant under the curvature tensor of \((\tilde{M}, \tilde{g})\). As far as we know, the theory of immersions satisfying this condition appeared for the first time in [11]. K. Ogiue called these immersions as invariant immersions. This condition generalizes properties of the immersions into manifolds of constant sectional curvature. The existence of curvature invariant tangent subspaces in a general Riemannian manifold is related with the existence of totally geodesic submanifolds (see [18] for more details).

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