Singular Regions in Black Hole Solutions in Higher Order Curvature Gravity

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Abstract

Four-dimensional black hole solutions generated by the low energy string effective action are investigated outside and inside the event horizon. A restriction for a minimal black hole size is obtained in the frame of the model discussed. Intersections, turning points and other singular points of the solution are investigated. It is shown that the position and the behavior of these particular points are defined by various kinds of zeros of the main system determinant. Some new aspects of the $r_s$ singularity are discussed.

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I. INTRODUCTION

During last years string four-dimensional dilatonic black holes attracted much attention. As this type of black holes is the solution of the string theory in its low energy limit, therefore, by studying these solutions one can hope to clarify some important unsolved problems of modern theoretical physics, for example, what is the endpoint of the black hole evaporation, the quantum coherence and black hole thermodynamics problems [1–3], so on. After the appearance of the Gibbons-Maeda-Garfinkle-Horowits-Strominger (GM-GHS) solution [4] a great interest to the investigation of the higher order curvature corrections in the Einstein-dilaton (Yang-Mills) lagrangian arised [5–11]. It is now very actual problem because in the regions where the curvature of the space-time increases the role of the $\alpha'$ expansion terms grows. The general form of this expansion is not investigated well yet [12]. So as a first step researches study the contribution of the second order curvature corrections. It was found that the black hole solutions with the $(\alpha')^1$ terms generalize the wellknown Schwarzschild solution and the GM-GHS one. The problem is to find new feathers of the black hole solutions which were introduced by the string theory.

Some researchers study only the simplified bosonic part of the low energy string action taken in the form

$$ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ m_{P}^2 (-R + 2\partial_\mu \phi \partial^\mu \phi) - e^{-2\phi} F_{\mu \nu} F^{\mu \nu} + \lambda e^{-2\phi} S_{GB} \right], \quad (1) $$

where $R$ is a Ricci scalar; $\phi$ is a dilaton field; $m_{P}$ is the Planck mass; $F_{\mu \nu}$ is the Maxwell field; $\lambda = \alpha'/4g^2$ is the string coupling parameter describing the Gauss-Bonnet (GB) term contribution ($S_{GB} = R_{ijkl}R^{ijkl} - 4R_{ij} R^{ij} + R^2$) to the action (1). It was found that the black hole solution of such action (with or without the Maxwell term) does exist and provides the non-trivial dilatonic hair [5–10]. Further, it was established that the modification of the solution by the second order curvature corrections became non-vanishing only if the black hole size (mass) was small enough (the case of the large value of the coupling constant $\lambda$).
It is necessary to emphasize that the differential equations in such class models have a very complicated structure. So, there is no possibility for the direct analytical solving them. Hence, the researchers have to use perturbative \([10,14,15]\) or numerical \([5–9]\) methods.

In our previous papers \([7,8]\) black hole type solutions of the action \((1)\) were found by the special numerical method from infinity up to a some particular point inside the event horizon \(r_h\). It was shown that in the case when the magnetic charge \(q\) was rather small (or vanished) a singular “tube” (in \(t\) direction) with the topology \(S^2 \times R^1\) has appeared inside the black hole. Asymptotically flat solution occurs from infinity up to this singular “tube” with the radius \(r_s\). It takes place in the range of the magnetic charge \(q\) to be \(0 \leq q < q_{cr}\), where \(q_{cr}\) is a new critical charge value appearing in second order curvature gravity \([8]\). In the case of \(q\) to be large enough (\(q_{cr} < q < M\sqrt{2}\) in GHS gauge, see \([4]\)) the solution occurs up to zero point where the dilatonic function \(\phi\) diverges as in the GM-GHS case. In this paper we are going to show that all the particular points (and their main feathers) of the black hole solution obtained from the set of implicit ordinary differential equations are defined by the various kinds of zeros of the main system determinant. In addition some new feathers of these solutions are discussed.

The structure of the paper is the following. Sec. II deals with the main equations, in Sec. III we briefly remind our previous results and discuss them in the light of the conclusions of \([16]\), in Sec. IV some new physical feathers of the \(r_s\) singularity are presented and Sec. V contains the main conclusions.

II. FIELD EQUATIONS

Our purpose is to study physical and mathematical feathers of the black hole solutions (static, spherically symmetric case). Therefore, the most convenient choice of metric is

\[
d s^2 = \Delta(r)dt^2 - \frac{\sigma^2(r)}{\Delta(r)}dr^2 - f^2(r)(d\theta^2 + \sin^2\theta d\varphi^2),
\]

where functions \(\Delta, \sigma\) and \(f\) depend only on a radial coordinate \(r\). Various types of the metric gauges were used in our investigations. The most convenient of them are, so-called,
“curvature gauge” \((f = r, \text{ used in } [7])\) and GHS gauge \((\sigma = 1, \text{ used in } [8])\). Corresponding field (Einstein) equations can be rewritten in the matrix form

\[
a_{i1} \Delta'' + a_{i2} f'' + a_{i3} \phi'' = b_i, \quad \text{GHS gauge}
\]

\[
\tilde{a}_{i1} \Delta'' + \tilde{a}_{i2} \sigma' + \tilde{a}_{i3} \phi'' = \tilde{b}_i, \quad \text{“curvature gauge” (2)}
\]

where \(i = 1, 2, 3\) and the coefficients \(a_{ij}\) and \(b_i\) are equal to (in the GHS gauge)

\[
a_{11} = 0,
\]

\[
a_{12} = -m_{Pl}^2 f + 4e^{-2\phi} \lambda \phi' 2\Delta f',
\]

\[
a_{13} = 4e^{-2\phi} \lambda (\Delta f'^2 - 1),
\]

\[
a_{21} = m_{Pl}^2 f + 4e^{-2\phi} \lambda \phi' 2\Delta f',
\]

\[
a_{22} = m_{Pl}^2 2\Delta + 4e^{-2\phi} \lambda \phi' 2\Delta \Delta',
\]

\[
a_{23} = 4e^{-2\phi} \lambda 2\Delta \Delta' f',
\]

\[
a_{31} = 4e^{-2\phi} \lambda (\Delta f'^2 - 1),
\]

\[
a_{32} = 4e^{-2\phi} \lambda 2\Delta \Delta' f',
\]

\[
a_{33} = (-2)m_{Pl}^2 \Delta f^2,
\]

\[
b_1 = m_{Pl}^2 f^2 (\phi')^2
\]

\[
+ 4e^{-2\phi} \lambda (\Delta f'^2 - 1) 2(\phi')^2,
\]

\[
b_2 = (-2)m_{Pl}^2 (\Delta' f' + \Delta f (\phi')^2)
\]

\[
+ 4e^{-2\phi} \lambda 2\Delta \Delta' f' 2(\phi')^2
\]

\[
+ 2e^{-2\phi} q^2 (1/f^3)
\]

\[
- 4e^{-2\phi} \lambda \phi' 2(\Delta')^2 f',
\]

\[
b_3 = 2m_{Pl}^2 \phi' f (\Delta' f + 2\Delta f')
\]

\[
+ 2e^{-2\phi} q^2 (1/f^2)
\]

\[
- 4e^{-2\phi} \lambda (\Delta')^2 f'^2.
\]
The remaining $\delta S/\delta f = 0$ additional equation fulfills automatically elsewhere on the solution trajectory.

The Arnowitt-Diezer-Misner (ADM) mass $M$ and the dilaton charge $D$ are defined by the asymptotic expansions for $\Delta$, $f$ and $\phi$

$$\Delta = 1 - \frac{2M}{r} + O\left(\frac{1}{r}\right);$$
$$f^2 = r^2 \left(1 - \frac{D}{r} + O\left(\frac{1}{r}\right)\right);$$
$$\phi = \phi_\infty + \frac{D}{r} + O\left(\frac{1}{r}\right).$$

III. INVESTIGATION OF THE PARTICULAR POINTS

For integrating outside and inside the event horizon, a method based on integrating over an additional parameter was used and described in detail in [7]. Here we briefly remind the main results and discuss their new feathers.

The results of our numerical integration are depicted in Fig.1. Shown in the Figure presents the dependence of metric function $\Delta$ against radial coordinate $r$ with the different meanings of the event horizon value $r_h$ (in the curvature gauge). It was found in [8] that the solution with the singular turning point $r_s$ exists only if $0 \leq q < q_{cr}$ where $q_{cr}$ is the critical magnitude of the magnetic charge. The appearance of $q_{cr}$ is created with the second order curvature corrections. While $q$ increases and reaches the $q_{cr}$ value, the solution behavior changes and it looks like completely to the GM-GHS one. As one can see from Fig.1, the behavior of the above-mentioned metric function $\Delta$ outside the regular horizon has the usual form and is similar to the standard Schwarzschild solution (see, for example, [5,10]). Inside the horizon the solution exists only down to the value $r = r_s$. Another solution branch (additional branch) begins from the value $r_s$ and exists only up to the “singular” horizon $r_x$.

To check the particular points of the system (2) it is necessary to consider its main determinant (here, for simplicity, we use “curvature gauge” and vanishing value of magnetic charge; in the GHS gauge with $0 \leq q < q_{cr}$ the results are completely the same but the
formulas have larger length)

\[ D_{\text{main}} = \Delta \left[ A\Delta^2 + B\Delta + C \right], \quad (4) \]

where

\[
A = (-32)e^{-4\phi}\sigma^2\lambda^2 \left[ 4\sigma^2\phi'^2 m_{pl}^2 r^2 - \sigma^2 m_{pl}^2 + 12e^{-2\phi}\Delta'\phi'/\lambda \right],
\]

\[
B = (-32)e^{-2\phi}\sigma^4 \lambda \left[ \sigma^2\phi'm_{pl}^4 r^3 + 2e^{-2\phi}\sigma^2\lambda m_{pl}^2 - 8e^{-4\phi}\Delta'\phi'\lambda^2 \right],
\]

\[
C = 32e^{-4\phi}\sigma^8\lambda^2 m_{pl}^2 - 2\sigma^8 m_{pl}^4 r^4 + 64e^{-4\phi}\sigma^6\Delta'\phi'\lambda^3.
\]

Eqs. (2) represent the system of ordinary differential equations in a non-evident form. That is why the system (2) can have the peculiarities when its main determinant \( D_{\text{main}} \) vanishes. The structure of those peculiarities depends upon the behavior of the eqs. (2) near the singular surface \( D_{\text{main}} = 0 \) in phase-space [16]. In the model with asymptotically flat solutions only three types of the \( D_{\text{main}} \) degeneracy arise. They are

(a): \( \Delta = 0, \ C \neq 0 \) \hspace{1cm} (“intersection”),

(b): \( A\Delta^2 + B\Delta + C = 0, \ \Delta \neq 0, \ C \neq 0 \) \hspace{1cm} (“turning point”),

(c): \( \Delta = 0, \ C = 0 \) \hspace{1cm} (“complicated singularity”), \quad (5)

The case (5a) is realized at the regular horizon \( r_h \) (see Fig.1) and occurs also in the Schwarzschild solution. The asymptotic form of the solutions near the point \( r_h \) has the form

\[
\Delta = d_1(r - r_h) + d_2(r - r_h)^2 + o((r - r_h)^2),
\]

\[
\sigma = s_0 + s_1(r - r_h) + o((r - r_h)),
\]

\[
e^{-2\phi} = \phi_0(1 - 2*\phi_1(r - r_h) + 2(\phi^2_1 - \phi_2)(r - r_h)^2) + o((r - r_h)^2), \quad (6)
\]
where \((r - r_h) \ll 1\). Substituting the formulas (5) into the equations (2), one obtains the following right relations between the expansion coefficients \((s_0, \phi_0\) and \(r_h\) are free independent parameters, here we do correct an unfortunate misprint in our work [7])

\[ d_1(z_1d_1^2 + z_2d_1 + z_3) = 0, \]  

(7)

where:

\[
\begin{align*}
    z_1 &= 24\lambda^2\phi_0^2, \\
    z_2 &= -m_P^4r_h^3s_0^2, \\
    z_3 &= m_P^4r_h^2s_0^4,
\end{align*}
\]

and the parameter \(\phi_1\) for \(d_1 \neq 0\) is equal to: \(\phi_1 = [(m_P^2)/((4\lambda d_1\phi_0)] \ast [r_h d_1 - s_0^2]\).

When \(d_1 = 0\), the metric function \(\Delta\) has the double (or higher order) zero. In such a situation the equation for \(d_2\) \((d_3, d_4, \ldots)\) is a linear algebraic one and there are no asymptotically flat branches.

When \(d_1 \neq 0\) the solution of the black hole type takes place only if the discriminant of the equation (7) is greater or equal to zero and, hence, \(r_h^2 \geq 2\lambda\phi_0\sqrt{14 + 8\sqrt{3}}\). One or two branches occurs and always one of them is asymptotically flat. With the supposition of \(\phi_\infty = 0\) (and, as we tested, in this case \(1.0 \leq \phi_0 < 2.0\)) the infimum value of the event horizon is

\[
    r_h^{\inf} = \sqrt{\lambda} \left( \frac{4\sqrt{6}}{m_P^2} \right)^{\frac{1}{2}},
\]

(8)

The analogous formula in the other interpretation was studied by Kanti et al. in [8].

The case (3b) is realized inside the regular horizon at \(r = r_s < r_h\) (see Fig.1) as a consequence of the intensification of the GB-term influence. The solution behavior strongly differs from the Schwarzschild one or the GM-GHS one. The \(D_{main}\) degeneracy of (3b) type reduces to the violation of the uniqueness of the solution at the point \(r_s\). Similar situations are typical for the systems of the type (2) in the neighborhood of the singular surface \(D_{main} = 0\) [10]. The asymptotic behavior of both solution branches near the position
$r_s$ can be described by the following formulas using the smooth function $\sigma$ as an independent variable [4]:

\[ \Delta = d_s + d_2(\sigma - \sigma_s)^2 + o((\sigma - \sigma_s)^2), \]
\[ r = r_s + r_2(\sigma - \sigma_s)^2 + o((\sigma - \sigma_s)^2), \]
\[ \exp(-2\phi) = \phi_s(1 - 2f_2(\sigma - \sigma_s)^2) + o((\sigma - \sigma_s)^2), \]

where $\sigma - \sigma_s \ll 1$. Free independent parameters are: $\sigma_s$, $\phi_s$, $r_s$. After the substitution of these expansions to the system [2], one can obtain that $d_2 = f_2$ and the equation for $(d_2/r_2)$ has the form

\[ z_4y^2 + z_5y + z_6 = 0, \]

where $y = (d_2/r_2)$ and the other coefficients are

\[ z_4 = m^2_{pl}\sigma_s^2d_sr_s^2 + 4\lambda\phi_s(\sigma_s^2 - 3d_s), \]
\[ z_5 = -m^2_{pl}\sigma_s^2r_s, \]
\[ z_6 = m^2_{pl}\sigma_s^2(\sigma_s^2 - d_s). \]

Eq. (10) may have either no solutions or one or two solutions dependently upon its discriminant magnitude. The case of the unique solution corresponds to the minimal value of $r_{hmin}$ (eq. (8), see Fig.1 curve (c)). The situation with the positive discriminant corresponds to the turning point of the solution. Here it is necessary to note that if one rewrites the expansions [3] against the expansion parameter $(r - r_s) \ll 1$, the result will have the following form $\Delta = d_s + y(r - r_s) + \ldots$ and so on. Hence, there are only two branches (because of two possible values of $y$) can exist near the position $r_s$. They are: the asymptotically flat one and the $(r_s r_x)$ one. No any other solution branches are in the neighborhood of $r_s$. The “curvature invariant” $R_{ijkl}R^{ijkl}$ near $r_s$ is equal to $(r - r_s \ll 1)$:

\[ R_{ijkl}R^{ijkl} = T_1/(r - r_s) + o\left(1/(r - r_s)\right) \rightarrow \infty, \text{ where } T_1 = \text{const}. \]

The components $T^0_0$ and $T^2_2$ of the stress-energy tensor $T^\mu_\nu$ also diverges (as $1/\sqrt{r - r_s}$) near the position $r_s$.

The case [5c] is realized on the singular horizon $r_x$ of the additional branch $(r_s r_x)$ (see Fig.1, curves (a) and (b)). This branch is not asymptotically flat and, hence, is non-physical.
The asymptotic form of this solution near the position \( r_x \) is shown in [7]. “Curvature invariant” \( R_{ijkl}R^{ijkl} \) also diverges near the position \( r_x \).

Here it is importantly to stress that the distance between the points \( r_x \) and \( r_h \) decreases while decreasing \( r_h \) (see Fig. 1). In the limit point defined by the restriction (8) all particular points pour together \( r_h=r_s=r_x=r_{h\text{min}} \) and the internal structure of the black hole disappears. The case (5c) is realized in this point and, therefore, the “curvature invariant” diverges. Hence, the point \( r_{h\text{min}} \) represents the event horizon and the singularity in the same point. So, such situation contradicts with the “cosmic censorship” hypothesis [17,18] but there is a question about its stability. The possibility of the realization of such situation is still open. Returning to the formula (8), one should remember that \( \lambda \) is the combination of the fundamental string constants. That is why this formula can be reinterpreted as the restriction to the minimal black hole size (mass) in the given model. This restriction appears in the second order curvature gravity and is absent in the minimal Einstein-Schwarzschild gravity. This fact can throw an additional light on the problems of the black holes in our Universe.

IV. \( R_S \) SINGULARITY

It is possible to obtain the approximate relation between \( r_s, r_h \) and \( \lambda \). Substituting the Schwarzschild values of the metric functions and the vanishing value of the dilaton charge \( D \) to the formula (4), one obtains

\[
r_s^3 = \frac{4\sqrt{3}r_h\phi_s}{m_{Pl}^2}.
\]

Fig. 2 shows the dependencies of the value \( r_s \) against the coupling parameter \( \lambda \) given by the formula (11) and by the numerical integration. From the eq.(11) it is possible to conclude that the pure Schwarzschild solution is the limit case of our one with \( r_s = 0 \). In the case with rather small value of \( \lambda \) this formula gives the good agreement with the results calculated by the numerical integration. While increasing \( \lambda \), the absolute error increases as a consequence
of ignoring the non-vanishing values \((1 - \sigma), \phi'\) and so on. It is necessary to point out that the eq. (11) represents the dependence \(r_s = \text{const} \lambda^{1/3}\) which we suppose to be right because after the appropriate selection this constant by hands the agreement between numerical data and this formula improves. Eq. (11) shows also that when the influence of the GB term (or black hole mass) increases, \(r_s\) also increases.

Further, it is possible to find the approximate relation between the \(q_{cr}, \lambda\) and \(M\). One can rewrite the \(D_{main}\) in the GHS coordinates and substitute there the GM-GHS values of the metric functions \(\Delta, f\) and the dilaton function \(\phi\) in the following form

\[
\Delta = 1 - \frac{2M}{r}, \\
f = \sqrt{r^2 - r \frac{q^2}{M}}, \\
e^{-2\phi} = 1 - \frac{q^2}{Mr}.
\]

Hence, \(D_{main}\) takes the form (we suppose \(m^2_{Pl} = 1\) for simplicity)

\[
D_{main} = \frac{T}{r^{10}M^4(rM - q^2)^2},
\]

where \(T = T(M, \lambda, q^2, r)\) is the polynomial of the sixteenth order against \(r\) (we do not write it because of its dangerous length). If one supposes the charge \(q\) to vanish in the eq. (12) (hence, the denominator of eq. (12) never vanishes) he obtains the formula (11) in the form

\[
r_s^3 = 8\sqrt{3}\lambda M.
\]

The denominator in the eq. (12) can vanish only in the case of \(r = r_k = q^2/M\). If this situation happens, \(D_{main}\) diverges. Consequently, all the senior derivatives vanishes and then in the point \(r_k\) which is also particular one has only the local minimum (maximum). So, the condition for the turning point existence is the following: \(r_s\) must be situated righter relatively to \(r_k\), i.e. \(r_s \geq r_k\). The limit condition \(r_s = r_k\) is just one for \(q_{cr}\) because when \(r_k > r_s\) one must have a local minimum (maximum) righter \(r_s\). That is why equating each other the expressions of the \(r_s\) and \(r_k\) one obtains the approximate formula for \(q_{cr}\)

\[
q_{cr} = \lambda^{1/6} \left(8\sqrt{3}M^4\right)^{1/6}.
\]
The last formula analogously to the (11) gives the good agreement with the data obtained from the numerical calculations only in the case of rather small $\lambda$. While increasing $\lambda$ the absolute error increases as well. This fact can be seen from the Figure 3 showing the dependence of the value $q_{cr}$ against the coupling constant $\lambda$. Here it is necessary to point out once more that the eq.(13) represents the right dependency $q_{cr} = \text{const} \lambda^{1/6}$.

Using the expansions (9), it is possible to study the behavior of the radial time-like and isotropic geodesics near the singularity $r_s$ (“curvature gauge” case with vanishing $q$). Based on these expansions and on the technics of the geodesical curve calculations from ref. [19], after integrating over the radial coordinate $x = r - r_s \ll 1$ one obtains the expressions for the proper time $\tau(x)$ and the coordinate time $t(x)$ for the radial time-like and radial isotropic geodesics

\[
\begin{align*}
\tau(x) &= \pm C_1 x + C_2 + \ldots, \quad \text{(time-like)} \\
t(x) &= \pm C_3 x + C_4 + \ldots, \quad \text{(time-like)} \\
\tau(x) &= \pm C_5 x + C_6 + \ldots, \quad \text{(isotropic)} \\
t(x) &= \pm C_7 x + C_8 + \ldots, \quad \text{(isotropic)}
\end{align*}
\tag{14}
\]

where $C_i$ are the constant values.

Comparing the values $t(x)$ and $\tau(x)$ with the Schwarzschild ones near the regular event horizon (see, for example, [19]), one can conclude that the behavior of the radial geodesics near the $r_s$ singularity differs from the radial geodesics behavior of the Schwarzschild solution near the regular horizon. In our model all the above-mentioned values limited, in the Schwarzschild model $t(x)$ everywhere diverges. The divergence of the $R_{ijkl}R^{ijkl}$ also differs in the models considered: $1/x^6$ in the Schwarzschild case near the origin and limited at the horizon and only $1/x$ in our case ($x = r - r_s$). Therefore, one can suppose that the $r_s$-singularity is a week one (according to the Clarke classification [20]) and it can be removed by the appropriate extension of the metric. According to the Propositions 8.2.2 and 8.2.3
from [20], this procedure is not forbidden because near the position \( v = r_s \) “criterion”

\[
K(v) = \int_0^v dv' \int_0^{v'} dv'' R_{\mu\nu}(v'')
\]

is limited. So, the question on the possibility of removing the \( r_s \) singularity by the appropriate metric extension remains open. It is necessary to emphasis that the singular turning point \( r_s \) appears in various kinds of metric parametrizations as we tested.

V. CONCLUSIONS

The black hole solutions generated by the bosonic part of the four-dimensional low energy superstring effective action with the second order curvature corrections are discussed in this paper. They are obtained by using the special numerical method described in [7]. It is demonstrated that all the particular points of the solution, namely, regular horizon \( r_h \), singular horizon \( r_x \) and \( r_s \)-singularity, are defined by the various types of zeros of the main determinant \( D_{\text{main}} \), namely, “intersection”, “complicated singularity” and “turning point” correspondingly.

The restriction for a minimal black hole size (mass) is obtained in the frame of the model with the vanishing Maxwell field contribution (Einstein-dilaton-Gauss-Bonnet model). This minimal black hole size (mass) depends upon the combination of the string fundamental constants.

The approximate formulas for the \( r_s = r_s(\lambda, r_h) \) at vanishing \( q \) and \( q_{cr} = q_{cr}(\lambda, M) \) are found. These formulas have a perturbative nature and shows that the effects of second order curvature corrections become non-vanishing at rather small sizes of black holes.

The behavior of the radial time-like and radial isotropic geodesics is studied. According to this and other criterions \( r_s \) singularity is not “strong”. We have no arguments that the \( r_s \)-singularity is coordinate one (and can be removed by the appropriate metric extension), but **nothing forbids this**. This question is still open.
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FIG. 1. The dependence of the metric function $\Delta$ versus the radial coordinate $r$ at the different values of the event horizon $r_h$. The “curvature gauge” with the vanishing contribution of the magnetic field is used during the calculations of the data shown in Figure. The curve (a) represents the case where $r_h$ is rather large and is equal to 30.0 Plank unit values (P.u.v.). The curve (b) shows the changes in the behavior of $\Delta(r)$ when $r_h$ is equal to 7.5 P.u.v. The curve (c) represents the boundary case with $r_h = r_{h_{\text{min}}}$ where all the particular points, namely, $r_h$ (regular horizon), $r_s$ (singular turning point) and $r_x$ (singular horizon), pour together and the internal structure disappears. The curve (d) shows the case where $2M \ll r_{h_{\text{min}}}$ ($2M=1.5$ P.u.v.) and any horizon is absent. The envelope curve (e) shows the position of the points $r_s$ against the different values of $r_h$. 
FIG. 2. The dependence of the position of the singular turning point $r_s$ versus $\lambda$. Squares represent the values of $r_s$ calculated from the numerical integration. The solid curve is obtained by using formula (11) with $r_h = 100.0$ P.u.v. and $\phi_s = 1.2$. 
FIG. 3. The dependence of the position of the critical value of the magnetic charge \( q_{cr} \) versus \( \lambda \). Squares represent the values of \( q_{cr} \) calculated from the numerical integration. The solid curve is obtained by using formula (13) with \( M = 5 \). P.u.v.