Induced scalar potentials for hypermultiplets

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Abstract

Charged BPS hypermultiplets can develop a non-trivial self-interaction in the Coulomb branch of the $N = 2$ supersymmetric gauge theory, whereas neutral BPS hypermultiplets in the Higgs branch may also have a non-trivial self-interaction in the presence of Fayet-Iliopoulos terms. The exact hypermultiplet low-energy effective action (LEEA) takes the form of the non-linear sigma-model (NLSM) with a hyper-Kähler metric. A non-trivial scalar potential is also quantum-mechanically generated at non-vanishing central charges, either perturbatively (Coulomb branch), or non-perturbatively (Higgs branch). We calculate the effective scalar potentials for (i) a single charged hypermultiplet in the Coulomb branch and (ii) a single neutral hypermultiplet in the Higgs branch. The first case corresponds to the NLSM with the Taub-NUT (or KK-monopole) metric for the kinetic LEEA, whereas the second one is attached to the NLSM having the Eguchi-Hanson instanton metric.

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1 Introduction

The $N = 1$ supersymmetric matter in four spacetime dimensions is described in terms of chiral $N = 1$ multiplets and linear $N = 1$ multiplets, that are (field-theory) dual to each other. The $N = 1$ chiral superfields $\Phi$ may have a chiral scalar superpotential described by a holomorphic function $W(\Phi)$. As regards the fundamental quantum field theory actions, the function $W(\Phi)$ should be restricted to a cubic polynomial by renormalizability, while there is no such restriction if it appears in the low-energy effective action (LEEA) of a quantum $N = 1$ supersymmetric field theory. In the exact LEEA, the full quantum-generated scalar potential $W(\Phi)$ is supposed to include all perturbative as well as all non-perturbative corrections, if any.

The $N = 2$ supersymmetric matter is described by hypermultiplets. The off-shell hypermultiplets come in two fundamental versions that are (field-theory) dual to each other in the $N = 2$ harmonic superspace. The first version is called a Fayet-Sohnius-type (FS) hypermultiplet, it is described by an unconstrained complex analytic superfield $q^+$ of $U(1)$-charge (+1), and it is on-shell equivalent to the standard Fayet-Sohnius hypermultiplet comprising two $N = 1$ chiral supermultiplets. The second version is called a Howe-Stelle-Townsend-type (HST) hypermultiplet, it is described by a real unconstrained analytic superfield $\omega$ of vanishing $U(1)$-charge, and it is on-shell equivalent to the standard $N = 2$ tensor (or $N = 2$ linear) multiplet comprising an $N = 1$ chiral multiplet and an $N = 1$ linear multiplet. Unlike that in $N = 1$ supersymmetry, there is apparently no $N = 2$ supersymmetric invariant that would generate the scalar potential for the scalar components of a hypermultiplet. It is often assumed in the literature that the hypermultiplet scalar potential (beyond the BPS mass term generated by central charges) simply does not exist, both in a fundamental $N = 2$ supersymmetric field theory action and in the corresponding LEEA provided that $N = 2$ supersymmetry is not broken.

At the fundamental level, any non-trivial hypermultiplet potential is indeed forbidden by renormalizability and unitarity. However, contrary to the naive expectations, a non-trivial scalar potential does appear in the hypermultiplet LEEA provided that the central charges do not vanish. It was recently demonstrated in the case of a single charged (FS-type) hypermultiplet minimally coupled to an abelian $N = 2$ vector multiplet (i.e. in the Coulomb branch). In this Letter we give some more details of this calculation (sect. 2), and then extend it to yet another interesting case of a single (HST-type) hypermultiplet with a Fayet-Iliopoulos (FI) term (sect. 3).

See ref. [1] for a recent review and an introduction to the harmonic superspace.
2 Taub-NUT action with central charges

A FS-type hypermultiplet is most naturally described in the \( N = 2 \) harmonic superspace, in terms of an unconstrained complex analytic superfield \( q^+ \) of \( U(1) \)-charge (+1). As was shown in ref. [2], a single FS-type charged hypermultiple \( t \) with a non-vanishing central charge (or BPS mass) gets the one-loop induced self-interaction in the Coulomb branch of an \( N = 2 \) supersymmetric gauge theory. The hypermultiplet LEEA is given by the NLSM whose target space metric is Taub-NUT.

The corresponding \( N = 2 \) harmonic superspace action in the analytic subspace \( \zeta^M = (x^m_A, \theta^+, \bar{\theta}^\alpha) \) reads as follows:

\[
S_T[q] = -\int d\zeta^{(-4)} du \left\{ \frac{\ast q^+}{q^+} D^{++} q^+ + \frac{\lambda}{2} (q^+)^2 (q^+)^2 \right\} \tag{1}
\]

where the covariant derivative \( D^{++} \), in the analytic basis at non-vanishing central charges \( Z \) and \( \bar{Z} \), has been introduced,

\[
D^{++} = \partial^{++} - 2i\theta^+ \sigma^m \bar{\theta}^\alpha \partial_m + i\theta^+ \bar{\theta}^\alpha \bar{Z} + i\bar{\theta}^+ \bar{\theta}^\alpha Z \tag{2}
\]

and \( \lambda \) is the induced (Taub-NUT) NLSM coupling constant. Eq. (2) can be most easily obtained by (Scherk-Schwarz) dimensional reduction from six dimensions [2]. For simplicity, we ignore here possible couplings to an abelian \( N = 2 \) vector superfield. The explicit expression for \( \lambda \) in terms of the fundamental gauge coupling and the hypermultiplet BPS mass can be found in ref. [2]. Our \( q^+ \) superfields are of dimension minus one (in units of length), while the coupling constant \( \lambda \) is of dimension two.

Our aim now is to find the component form of the action (1). Without central charges it was done in ref. [3]. The non-vanishing central charges were incorporated in ref. [3]. In this section, we are going to concentrate on the component scalar potential originating from the action (1). The corresponding equations of motion are

\[
D^{++} q^+ + \lambda (q^+ \ast q^+) q^+ = 0 \quad \text{and} \quad D^{++} \ast q^+ - \lambda (q^+ \ast q^+) \ast q^+ = 0
\tag{3}
\]

Since we are only interested in the purely bosonic part of the action (1), we drop all fermionic fields in the \( \theta^+, \bar{\theta}^\alpha \) expansion of \( q^+ \),

\[
q^+(\zeta, u) = F^+(x_A, u) + i \theta^+ \sigma^m \bar{\theta}^\alpha A_m^-(x_A, u) \nonumber
\]
\[
+ \theta^+ \bar{\theta}^\alpha M^-(x_A, u) + \bar{\theta}^+ \bar{\theta}^\alpha N^-(x_A, u) \nonumber
\]
\[
+ \theta^+ \bar{\theta}^\alpha \bar{\theta}^\beta \bar{\theta}^\gamma P^{(-3)}(x_A, u) \tag{4}
\]

\(^4\)We use the standard notation for the \( N = 2 \) harmonic superspace, see e.g., ref. [1].

\(^5\)We assume that \( c = \hbar = 1 \) as usual.
Being substituted into eq. (3), eq. (4) yields

\[ \partial^{++} F^+ + \lambda (F^+ \ast F^+) F^+ = 0 \]  (5)

\[ \partial^{++} A_m - 2 \partial_m F^+ + 2 \lambda A_m \ast F^+ F^+ \ast \lambda (F^+) = 0 \]  (6)

\[ \partial^{++} M^- + 2 \lambda M^- \ast \ast F^+ + \lambda \ast \ast (F^+)^2 + i \ast Z F^+ = 0 \]  (7)

\[ \partial^{++} N^- + 2 \lambda N^- \ast \ast F^+ + \lambda \ast \ast (F^+)^2 + i \ast Z F^+ = 0 \]  (8)

\[ \partial^{++} P^{(-3)} + \partial^m A_m - 2 \lambda F^+ \ast \ast F^+ \ast P^{(-3)} + \lambda (F^+)^2 \ast \ast F^+ \ast (P^{(-3)}) \]

\[ - \lambda A^{-m} A^{-m} \ast F^+ - \lambda A^{-m} \ast \ast F^+ - 2 \lambda F^+(M^- \ast \ast M^- - N^- \ast \ast N^-) \]

\[ + 2 \lambda \ast \ast F^+ M^- N^- + i \ast Z N^- + i \ast Z M^- = 0 \]  (9)

as well as their conjugates.

The integration over \( \theta^+, \bar{\theta}^+ \) in eq. (1) results in the bosonic action

\[ S_T = - \int d\zeta (-4) du \left\{ * q^+ \ast D^{++} q^+ + \frac{\lambda}{2} (q^+)^2 (\ast q^+)^2 \right\} \]

\[ \to S = - \frac{1}{2} \int d^4 x du \left\{ (A_m - \partial^m F^+ - A_m \partial^m F^+) \right\} \]

\[ + i F^+ (\ast Z \ast M^- + \ast Z \ast N^-) + i \ast \ast F^+ (Z M^- + Z N^-) \]  (10)

Since the action (11) has the global \( U(1) \) invariance

\[ q^{+'} = e^{i \alpha} q^+, \quad \ast q^{+'} = e^{-i \alpha} \ast q^+ \]  (11)

there exists the conserved Noether current \( j^{++} \)

\[ D^{++} j^{++} = 0, \quad j^{++} = i q^+ \ast q^+ \]  (12)

It implies, in particular, that \( \partial^{++} (F^+ \ast \ast F^+) = 0 \) and, hence,

\[ F^+(x, u) \ast \ast F^+(x, u) = C^{(ij)} u_i^+ u_j^+ \]  (13)

\[ (F^+ \ast \ast F^+)^\ast = - F^+ \ast \ast F^+ \to C^{(ij)} = - \epsilon_{ij} \epsilon_{mn} C^{(ln)} \]  (14)

where the new function \( C^{(ij)}(x) \) has been introduced. Changing the variables as

\[ F^+(x, u) = f^+(x, u) e^{\lambda \varphi}, \quad \varphi(x, u) = - C^{(ij)}(x) u_i^+ u_j^- = - \varphi(x, u) \]  (15)

now reduces eq. (11) to the linear equation

\[ \partial^{++} f^+(x, u) = 0 \to f^+(x, u) = f^i(x) u_i^+ \]  (16)
After taking into account that

\[ F^+ \dot{F}^+ = f^+ \dot{f}^+ \rightarrow C^{(ij)}(x) = -f^{(i}(x) \bar{f}^{j)}(x), \]  

(17)

where \( \bar{f}^i = \epsilon^{ij} \bar{f}_j \) and \( \bar{f}_j \equiv (f^j) \), we obtain a general solution in the form

\[ F^+(x, u) = f^i u^+_i e^{\lambda \varphi} = f^i(x) u^+_i \exp\{\lambda f^{(i} \bar{f}^{k)} u^+_j u^-_k\} \]  

(18)

The same conclusion also appears when using the Ansatz

\[ F^+ = e^C[f^i u^+_i + B^{ijk} u^+_i u^+_j u^-_k] \]  

(19)

in terms of functions \( C \) and \( B^{ijk} \) at our disposal. After substituting eq. (19) into the equation of motion (5), we find

\[ B^{ijk} = 0 \quad \text{and} \quad C = \lambda f^{(i} \bar{f}^{j)} u^+_i u^-_j \]  

(20)

so that

\[ F^+ = f^i u^+_i \exp\{\lambda f^{(i} \bar{f}^{k)} u^+_j u^-_k\} \]  

(21)

again. To get a similar equation for \( A^-_m \), we use eq. (3) and the Ansatz

\[ A^-_m = e^{\lambda \varphi} \left\{ a \lambda f^i u^+_i \partial_m (f^{(k} \bar{f}^{j)} u^-_k u^-_j) \\
+ b \partial_m f^i u^-_i + c \frac{\lambda f^i u^-_i}{1 + \lambda f f} (f^{j} \partial_m \bar{f}_j - \bar{f}_j \partial_m f^j) \right\} \]  

(22)

with some coefficients \( (a, b, c) \) to be determined. After substituting eq. (22) into eq. (3), we find the relations

\[ b = 2 , \quad 2a = 2 + b , \quad \text{and} \quad c = b - a + 1 \]  

(23)

so that \( a = b = 2 \) and \( c = 1 \). Therefore, we have

\[ A^-_m = e^{\lambda \varphi} \left\{ 2\lambda f^i u^+_i \partial_m (f^{(k} \bar{f}^{j)} u^-_k u^-_j) \\
+ 2 \partial_m f^i u^-_i + \frac{\lambda f^i u^-_i}{1 + \lambda f f} (f^{j} \partial_m \bar{f}_j - \bar{f}_j \partial_m f^j) \right\} \]  

(24)

To solve the remaining equations of motion (7) and (8) for the auxiliary fields \( M^- \) and \( N^- \) (the rest of equations of motion in eq. (9) is irrelevant for our purposes), we introduce the Ansatz

\[ M^- = e^{\lambda \varphi} R^- \equiv Re^{\lambda \varphi} f^i u^-_i \]  

(25)

\[ N^- = e^{\lambda \varphi} S^- \equiv Se^{\lambda \varphi} f^i u^-_i \]  

(26)
with some coefficient functions $R$ and $S$ to be determined. After substituting eq. (25) into eq. (7) we get

$$\partial^+ R - \lambda f^j \bar{f}^k u^+_{j} u^+_k R^- + \lambda f^j \bar{f}^j u^+_{j} \bar{S}^- + i\bar{Z} f^i u^+_i = 0$$

$$R f^i u^+_i - R\bar{f}^m \bar{f}^n u^+_m u^-_n + i\bar{Z} f^i u^+_i - \lambda S f^m \bar{f}^n u^+_m u^-_n = 0$$

$$R f^i u^+_i + R\bar{f}^m \bar{f}^n (u^- m u^- n + \delta^m n) u^-_m + i\bar{Z} f^i u^+_i - \lambda S f^m \bar{f}^n u^+_m u^-_n = 0$$

$$f^i u^+_i [R(1 + \lambda f^j \bar{f}^j) + i\bar{Z}] - \lambda (R + \bar{S}) f^m \bar{f}^n u^+_m u^-_n u^-_n = 0$$

It follows

$$R = -\frac{i\bar{Z}}{1 + \lambda f \bar{f}} \quad \text{and, hence,} \quad M^- = -\frac{iZ}{1 + \lambda f \bar{f}} e^{\lambda \varphi} f^i u^-_i \quad (27)$$

Similarly, we find from eqs. (8) and (26) that

$$N^- = -\frac{iZ}{1 + \lambda f \bar{f}} e^{\lambda \varphi} f^i u^-_i \quad (28)$$

Substituting now the obtained solutions for the auxiliary fields $F^+, A^-, M^-$ and $N^-$ into the action (10) yields the bosonic NLSM action

$$S = \frac{1}{2} \int d^4x \left\{ g_{ij} \partial_m f^i \partial^n f^j + g^{ij} \partial_m \bar{f}^i \partial^n \bar{f}^j + 2h^i_j \partial_m f^j \partial^n \bar{f}^i - V(f) \right\} \quad (29)$$

whose metric takes the form

$$g_{ij} = \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} \bar{f}^i \bar{f}^j$$

$$g^{ij} = \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} f^i f^j$$

$$h^i_j = \delta^i_j (1 + \lambda f \bar{f}) - \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} f^i \bar{f}^j$$

(30)

This metric is known to be equivalent to the standard Taub-NUT metric up to a field redefinition [3]. The scalar potential in eq. (31) takes the form [4]

$$V(f) = \frac{Z \bar{Z}}{1 + \lambda f \bar{f}} \bar{f} f$$

(31)

By construction, the effective scalar potential (31) for a single charged hypermultiplet is generated in the one-loop perturbation theory [2], and it is exact in the Coulomb branch. The vacuum expectation values for the scalar hypermultiplet components, which are to be calculated from this effective potential, all vanish. Notably, the BPS mass $m_{\text{BPS}}^2 = |Z|^2$ is not renormalized, as it should. Nevertheless, the whole effective scalar potential (31) is not merely the quadratic (BPS mass) contribution.
3 Eguchi-Hanson action with central charges

As was argued in refs. [1, 2], a non-trivial hypermultiplet self-interaction can be non-perturbatively generated in the Higgs branch, in the presence of non-vanishing constant FI-term \( \xi^{(ij)} = \frac{1}{2}(\vec{\tau} \cdot \vec{\xi})^{ij} \), where \( \vec{\tau} \) are Pauli matrices. The FI-term is nothing but the vacuum expectation value of the \( N = 2 \) vector multiplet auxiliary components (in a WZ-like gauge). The FI-term has a nice geometrical interpretation in the underlying ten-dimensional type-IIA superstring brane picture made out of two solitonic 5-branes located at particular values of \( \vec{w} = (x^7, x^8, x^9) \) and some Dirichlet 4- and 6-branes, all having the four-dimensional spacetime \( (x^0, x^1, x^2, x^3) \) as the common macroscopic world-volume [4]. The values of \( \vec{\xi} \) can then be identified with the angles at which the two 5-branes intersect, \( \vec{\xi} = \vec{w}_1 - \vec{w}_2 \), in the type-IIA picture [1].

The three hidden dimensions \( (\vec{w}) \) are identified by the requirements that they do not include the two hidden dimensions \( (x^4, x^5) \) already used to generate central charges in the effective four-dimensional field theory, and that they are to be orthogonal to the direction \( (x^0) \) in which the Dirichlet 4-branes are finite and terminate on 5-branes.

The simplest non-trivial LEEA for a single dimensionless \( \omega \)-hypermultiplet in the Higgs branch reads [1, 2]:

\[
S_{EH}[\omega] = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ (D^{++}\omega)^2 - \frac{(\xi^{++})^2}{\omega^2} \right\}
\] (32)

where \( \xi^{++} = u^+_i u^+_j \xi^{(ij)} \) is the FI-term, and \( \kappa \) is the coupling constant of dimension one (in units of length). When changing the variables to \( q^{+}_a = u^+_a \omega + u^-_a f^{++} \) and eliminating the Lagrange multiplier \( f^{++} \) via its algebraic equation of motion, one can rewrite eq. (32) to an equivalent form in terms of a \( q^+ \)-hypermultiplet as follows [3]:

\[
S_{EH}[q] = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ q^{a+} D^{++} q^{a+} - \frac{(\xi^{++})^2}{(q^{a+} u^-_a)^2} \right\}
\] (33)

where we have used the notation \( q^{+}_a = (q^{+}_a, q^{+}_A) \) and \( q^{a+} = \varepsilon^{ab} q^{+}_b \). In its turn, eq. (33) is classically equivalent to the following gauge-invariant action in terms of two FS hypermultiplets \( q^{+}_A (A = 1, 2) \) and the auxiliary real analytic \( N = 2 \) vector superfield \( V^{++} \) [3]:

\[
S_{EH}[q, V] = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ q^{a+} D^{++} q^{a+} + V^{++} \left( \frac{1}{2} \varepsilon^{AB} q^{a+}_A q^{a+}_B + \xi^{++} \right) \right\}
\] (34)

We now calculate the component form of this hypermultiplet self-interaction by using eq. (34) as our starting point. In a bit more explicit form, it reads

\[
S = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ \gamma_1^{+} D^{++} q^{+}_1 + \gamma_2^{+} D^{++} q^{+}_2 + V^{++} (\gamma_1^{+} + \gamma_2^{+} - \gamma_1^{+} + \gamma_2^{+} + \xi^{++}) \right\}
\] (35)
The equations of motion are given by

\[ D^{++}q_1^+ + V^{++}q_2^+ = 0 \]  
(36)
\[ D^{++}q_2^+ - V^{++}q_1^+ = 0 \]  
(37)
\[ \dot{q}_1^+q_1^+ - \dot{q}_2^+q_2^+ + \xi^{++} = 0 \]  
(38)

while the last equation is clearly the algebraic constraint on the two FS hypermultiplets. In what follows, we ignore fermionic contributions and use a WZ-gauge for the \( N = 2 \) vector superfield \( V^{++} \), so that \( D^{++} \) and \( q^+ \) are still given by eqs. (2) and (4), whereas

\[
V^{++} = -2i\theta^+ \sigma^m \bar{\theta}^+ V_m(x_A) + \theta^+\theta^+\bar{a}(x_A) + \bar{\theta}^+\theta^+ a(x_A) + \theta^+\theta^+\bar{a}(x_A)
\]  
(39)
\[
+\theta^+\theta^+\bar{a}(x_A)u^-_i u^-_j
\]  
(40)

The equation of motion (36) in components reads

\[
\partial^{++}F_1^+ = 0
\]  
(41)
\[
-2\partial_m F_1^+ + \partial^{++}A_{1m}^- - 2V_m F_2^+ = 0
\]  
(42)
\[
i\bar{Z}F_1^+ + \partial^{++}M_1^- + \bar{a}F_2^+ = 0
\]  
(43)
\[
iZF_1^+ + \partial^{++}N_1^- + aF_2^+ = 0
\]  
(44)
\[
\partial^{++}F_1^{(-3)} + \partial^m A_{1m}^- + i\bar{Z}N_1^- + iZM_1^- + V_m A_{2m}^- + \bar{a}N_2^- + aM_2^- + D^{(ij)}u^-_i u^-_j F_2^+ = 0
\]  
(45)

whereas eq. (37) gives

\[
\partial^{++}F_2^+ = 0
\]  
(46)
\[
-2\partial_m F_2^+ + \partial^{++}A_{2m}^- + 2V_m F_1^+ = 0
\]  
(47)
\[
i\bar{Z}F_2^+ + \partial^{++}M_2^- - \bar{a}F_1^+ = 0
\]  
(48)
\[
iZF_2^+ + \partial^{++}N_2^- - aF_1^+ = 0
\]  
(49)
\[
\partial^{++}F_2^{(-3)} + \partial^m A_{2m}^- + i\bar{Z}N_2^- + iZM_2^- - V_m A_{1m}^- - \bar{a}N_1^- - aM_1^- - D^{(ij)}u^-_i u^-_j F_1^+ = 0
\]  
(50)

The constraint (38) in components is given by

\[
\star F_1^+ + F_2^+ - \star F_2^- + F_1^- + \xi^{++} = 0
\]  
(51)
\[
\star A_{1a}^- - F_2^+ + \star F_1^- + A_{2a}^- - \star A_{2a}^- - F_1^- + F_2^- + \star A_{1a}^- = 0
\]  
(52)
\[
\star F_1^+ + M_2^- - \star F_2^- + M_1^- + \star N_1^- - F_2^- - \star N_2^- + F_1^- = 0
\]  
(53)
\[
\star F_1^+ + N_2^- - \star F_2^- + N_1^- + \star M_1^- - F_2^- - \star M_2^- + F_1^- = 0
\]  
(54)
Substituting the component expressions for \( q^+ \) and \( V^+ \) into the action (35) results in the following bosonic action:

\[
S = -\frac{1}{2\kappa^2} \int d^4x du \left\{ \bar{F}^1_1 + \bar{\varphi}^m A^+_1 + \bar{F}^2_2 + \bar{\varphi}^m A^+_2 + V^m \bar{F}^1_1 + A^+_2m - \bar{F}^2_2 + A^+_1m \right. \\
\left. + \bar{a} \bar{F}^1_1 + N^+_2 - \bar{F}^2_2 + N^+_1 \right) + a(\bar{F}^1_1 + M^+_2 - \bar{F}^2_2 + M^+_1) \\
+ iD^{ij}u^-_iu^-_j(\xi^{++} + \bar{F}^+_1 + F^+_2 - \bar{F}^+_2 + F^+_1) \\
+ \bar{F}^1_1 + (iZ N^-_1 + iZ M^-_1) + \bar{F}^2_2 + (iZ N^-_2 + iZ M^-_2) \right\}
\]

The next step in our calculation is to fix the harmonic dependence of the fields \( F^+_i, A^-_{ia}, M^-_i \) and \( N^-_i \). Eqs. (41) and (46) imply

\[
F^+_1 = f^+_1 u^+_i \quad \text{and} \quad F^+_2 = f^+_2 u^+_i
\]

whereas eq. (42) yields

\[
-2\partial_m F^+_1 + \partial^{++} A^+_1m - 2V_m F^+_2 = 0
\]

After introducing the Ansatz

\[
A^+_1m = A^+_1m u^-_i + B^+_1m u^+_i u^-_j u^-_k
\]

we find that

\[
A^+_1m = (2\partial_m f^+_1 + 2V_m f^+_2) u^-_i
\]

Similarly, it follows from eq. (47) that

\[
A^+_2m = (2\partial_m f^+_2 - 2V_m f^+_1) u^-_i
\]

Eqs. (43), (44), (48) and (49) now imply

\[
M^-_1 = -(\bar{a} f^+_2 + iZ f^+_1) u^-_i
\]
\[
N^-_1 = -(a f^+_2 + iZ f^+_1) u^-_i
\]
\[
M^-_2 = (\bar{a} f^+_1 - iZ f^+_2) u^-_i
\]
\[
N^-_2 = (a f^+_1 - iZ f^+_2) u^-_i
\]
After substituting all the component solutions into the action (56), we find the (abelian) gauged NLSM action

\[ S = \frac{1}{2\kappa^2} \int d^4x \left\{ (\partial_m f_1^i + V_m f_2^i)(\partial^m \bar{f}_{1i} + V^m \bar{f}_{2i}) + \left( \partial_m f_2^i - V_m f_1^i \right)(\partial^m \bar{f}_{2i} - V^m \bar{f}_{1i}) \right\} \]

(66)

where the scalar hypermultiplet components \( f_{1,2}^{i} \) are still subject to the constraint

\[ \xi^{(ij)} = \bar{f}^{(i} f_{2}^{j)} - \bar{f}^{(i} \bar{f}^{j)} \]

(67)

In calculating the action (66) we have also used the equation of motion for the \( N = 2 \) vector multiplet auxiliary field \( a \), whose solution reads

\[ a = -i Z \frac{f_1^i f_{2i} - \bar{f}_2^i \bar{f}_{1i}}{f_1^i f_{1i} + f_2^i f_{2i}} \]

(68)

A solution to the equation of motion for the vector gauge field \( V_m \) is given by

\[ 2V_m = \frac{\partial_m \bar{f}_{1j} f_2^j - \bar{f}_{1j} \partial_m f_2^j - \partial_m \bar{f}_{2j} f_1^j + \bar{f}_{2j} \partial_m f_1^j}{f_1^i f_{1i} + f_2^i f_{2i}} \]

(69)

In terms of the two complex scalar \( SU(2) \) doublets \( f_{1,2}^{i} \) subject to the three real constraints (71) and one abelian gauge invariance, we have \( 2 \times 2 \times 2 - 3 - 1 = 4 \) independent degrees of freedom, as it should. After eliminating the auxiliary vector potential \( V_m \) via eq. (63) and solving the constraint (67), one finds the NLSM with a non-trivial hyper-Kähler metric (by construction, as the consequence of \( N = 2 \) supersymmetry) and a non-trivial scalar potential

\[ V = \frac{ZZ}{(f_1^1 f_{1i} + f_2^1 f_{2i})} \left[ (f_1^1 \bar{f}_{2i} - \bar{f}_2^i \bar{f}_{1i})^2 + (f_1^i \bar{f}_{1i} + f_2^i \bar{f}_{2i})^2 \right] \]

(70)

The kinetic terms in the NLSM (66) are known to represent the Eguchi-Hanson instanton metric up to a field redefinition [5], so that to this end we concentrate on the scalar potential (70) only.

Let’s introduce the following notation

\[ \bar{f}_{1(2)}^{1} = \bar{f}_{1(2)}^{2}, \quad \bar{f}_{1(2)}^{2} = - \bar{f}_{1(2)}^{1} \]

(71)

and keep the positions of indices as above. The operator * denotes the usual complex conjugation. The constraints (71) now take the form

\[ \xi^{11} = \bar{f}_{1}^{1} f_{2}^{1} - f_{1}^{1} \bar{f}_{2}^{1} = \bar{f}_{1}^{2} f_{2}^{1} - f_{1}^{1} \bar{f}_{2}^{2} \]

\[ \xi^{12} = \frac{1}{2} (\bar{f}_{1}^{1} f_{2}^{2} + f_{1}^{2} \bar{f}_{2}^{1}) - \frac{1}{2} (\bar{f}_{1}^{1} \bar{f}_{2}^{2} + f_{1}^{2} \bar{f}_{2}^{1}) \]

\[ \xi^{22} = f_{1}^{2} f_{2}^{2} - f_{1}^{2} \bar{f}_{2}^{2} \]
When multiplying these constraints with Pauli matrices $(\tau_1, 1, \tau_3)_{ij}$, we get

\[ \xi^1 = \bar{f}_1^1 f_2^2 + \bar{f}_1^2 f_2^1 - (f_1^1 \bar{f}_2^2 + f_1^2 \bar{f}_2^1) \]
\[ \xi^2 = \bar{f}_1^1 f_2^1 - f_1^1 \bar{f}_2^1 + \bar{f}_1^2 f_2^2 - f_1^2 \bar{f}_2^2 \]
\[ \xi^3 = \bar{f}_1^1 f_2^1 - f_1^1 \bar{f}_2^1 - \bar{f}_1^2 f_2^2 + f_1^2 \bar{f}_2^2 \]

while we have $\bar{\xi}^2 \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \neq 0$. We now choose the direction $\xi^2 = \xi^3 = 0$ and $\xi^1 = 2i$, so that our constraints now take the form

\[ \bar{f}_1^1 f_2^2 + \bar{f}_1^2 f_2^1 - (f_1^1 \bar{f}_2^2 + f_1^2 \bar{f}_2^1) = 2i \]
\[ (-f_1^1)^* f_2^1 + f_1^1 (f_2^1)^*) + ((f_1^2)^* f_2^2 - f_1^2 (f_2^2)^*) = 2i \]

and

\[ f_2^1 (f_2^1)^* = f_1^1 (f_2^2)^* , \quad f_2^2 (f_1^1)^* = f_1^2 (f_2^2)^* \]

We thus end up with only two+one real constraints and one gauge invariance

\[ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}' = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \]

In the parametrization

\[ f_i^j = m_i^j \exp(i\varphi_i^j) \]

the constraints (74) and (77) read

\[ m_1^1 m_2^2 = m_1^2 m_2^1 e^{-i\varphi_1^1-i\varphi_2^2+i\varphi_1^1+i\varphi_2^2} \]
\[ m_1^1 m_2^1 \sin(\varphi_1^1 - \varphi_2^1) + m_2^2 m_1^2 \sin(\varphi_1^1 - \varphi_2^2) = 1 \]

We now want to fix the local $U(1)$ symmetry by imposing the gauge condition

\[ \varphi_2^1 + \varphi_2^2 = \varphi_1^1 + \varphi_1^2 . \]

When using

\[ |f_1^1| = m, \quad |f_2^1| = n, \quad \varphi_1^1 = \theta , \quad \varphi_2^1 = \phi , \]

as the independent fields, our constraints above can be easily solved:

\[ -\varphi_2^1 = \varphi_2^2 = \phi , \quad \varphi_1^1 = -\varphi_1^2 = \theta \]

and

\[ m_1^1 = \frac{m}{(m^2 + n^2) \sin(\theta + \phi)} , \quad m_1^2 = \frac{n}{(m^2 + n^2) \sin(\theta + \phi)} , \quad m_2^1 = m , \quad m_2^2 = n \]
It is straightforward to deduce the other fields $F^+_i, A^{-m}_i, \tilde{M}^{-i}$ and $N^{-i}$ in terms of the independent components (83). The scalar potential (84) in terms of these independent field variables takes the form (no indices and constraints any more!)

$$V = \frac{|Z|^2 \sin^2(\theta + \phi)}{m^2 + n^2} \left[ \frac{4(m^2 - n^2)^2}{1 + (m^2 + n^2)^2 \sin^2(\theta + \phi)} + \frac{1 + (m^2 + n^2)^2 \sin^2(\theta + \phi)}{\sin^2(\theta + \phi)} \right]$$

(86)

It is clear from this equation that the potential $V$ is positively definite, and it is only non-vanishing due to the non-vanishing central charge $|Z|$. It signals the spontaneous breaking of $N = 2$ supersymmetry in our model.

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