Kamlah Expansion and Gauge Theories

Oliver Schröder and Hugo Reinhardt
Eberhard-Karls Universität Tübingen
(Dated: July 9, 2002)

In Yang-Mills theories, variational calculations of the Rayleigh-Ritz type face the problem that on the one hand, calculability puts severe constraints on the space of test wave functionals; on the other hand, the test wave functionals have to be gauge invariant. The conflict between the two requirements can be resolved by introducing a projector. In this paper we present an approach to approximating the projector in a way known and successfully employed in nuclear physics: the Kamlah expansion. We discuss it both for electrodynamics and for Yang-Mills theories to leading order in a perturbative expansion, and demonstrate that the results are compatible with what one would expect from perturbation theory.

I. INTRODUCTION AND MOTIVATION

Recently, there has been a renewed interest in the application of variational techniques to Yang-Mills theories, cf. [1,2,3,4,5] and references therein. Whereas the first investigation in this line [1] did not focus explicitly on the subject of gauge invariance, subsequent investigations have put more stress upon the subject. The basic idea can already been found in the nuclear physics environment [6]: if one starts from a trial wave functional that does not share the basic symmetries of the Hamiltonian, one should not vary the expectation value of the energy but rather introduce a projection operator \( P \) that projects the wave functional onto the symmetric space and vary this new expectation value:

\[
\delta \left( \langle \psi | H P | \psi \rangle / \langle \psi | P | \psi \rangle \right) = 0.
\]  

This construction allows the usage of simple wave functionals. It turns out, unfortunately, that the projector acting on e.g. the simple class of Gaussian wave functionals cannot be evaluated exactly in the case of Yang-Mills theories. Therefore a number of approximation methods have been devised. One class of approximations treats the projection integral as an effective non-linear sigma model (with a non-local action) and uses common field theoretical approximation methods to deal with this model [2], cf. also [3]. Another approach has suggested an 'improved energy functional' [4, 5]. The idea that is followed in this approach is that the presence of the projector could be simulated by considering an improved energy functional which contains besides the energy expectation value additional terms which depend on the generator of the symmetry \( \Gamma^a(x) \):

\[
E^{eff} = \langle \psi | H | \psi \rangle - \left( \Theta^{-1} \right)^{ab}(x,y)\langle \psi | \Gamma^a(x) \Gamma^b(y) | \psi \rangle
\]  

where we have assumed that the states are properly normalized. Here \( \Theta^{ab} \) denotes the “moment of inertia” corresponding to the “rotations” generated by \( \Gamma^a(x) \). In this paper we want to investigate how one can derive such an improved energy functional and under which restrictions this can actually be done.

The structure of this paper is as follows: first, in section II we present the projector onto gauge invariant states as it is be used in this paper; we also comment shortly on how one can write down such a projector in the presence of external colour charges. In section III we introduce the general concept of the Kamlah expansion as it is used in nuclear physics [6, 7, 8], and formulate it in a way s.t. it may be used also for gauge theories. In section IV we apply the concept to the theory of a quantized electromagnetic field in the presence of classical charges. There we will see that the leading order of the expansion - which was often sufficient in the case of nuclear physics - is in fact inconsistent in quantum field theories. In section V we discuss the application to Yang-Mills theories. However, we are
only able to discuss the leading order of a subsequent perturbative expansion. Also, some additional difficulties w.r.t. the usage of external charges will become apparent. In section V I we give a summary and a critical evaluation of the Kamalhah expansion. Conventions and the canonical quantization scheme used in this paper are discussed in appendix A, some results from perturbation theory will be presented in appendix B and details of some of the calculations are given in appendix C.

II. THE PROJECTOR ONTO GAUGE INVARIANT STATES

A. Generalities

In this section we want to introduce the concept of the projector onto gauge invariant states. The basic idea is simple: if we start from a wave functional \( \psi_{\alpha_i}[A] = (A|\psi_{\alpha_i}) \) which is not gauge invariant, we can construct a gauge invariant wave functional \( \psi_{\beta_i}[A] \) by summing over all gauge transformed wave functionals with the correct Haar measure. Symbolically

\[
\psi_{\beta_i}[A] = \int D\mu[\phi] \psi_{\alpha_i}[A^U]
\]

where \( D\mu[\phi] \) symbolizes the integration over all (as we will see below: topologically trivial) gauge transformations parametrized by \( \phi \). For notation cf. also app. A. For the purposes of this paper - as outlined in app. A 4 - it is sufficient to have wave functionals that are annihilated by the Gauss law operator. This is equivalent to having wave functionals that are invariant under topologically trivial gauge transformations; therefore the domain of integration in eq. (3) can be restricted to topologically trivial gauge transformations also. In this case the gauge transformation can be written with the help of the Gauss law operator \( G \) (in absence of external charges given in eq. (6), in the presence of external charges a further discussion can be found in sec. III B):

\[
\psi_{\alpha_i}[A^U] = e^{-i \int d^3x \psi^\alpha(x)G^\alpha(x)} \psi_{\alpha_i}[A].
\]

This now allows to rewrite eq. (3) in terms of an operator \( P^\alpha \) acting upon \( \psi_{\alpha_i} \):

\[
\psi_{\beta_i}[A] = \left( \int D\mu[\phi] e^{-i \int d^3x \phi^\alpha(x)G^\alpha(x)} \right) \psi_{\alpha_i}[A] = P^\alpha \psi_{\alpha_i}[A],
\]

where \( D\mu[\phi] \) denotes the integral over all variables parametrizing the gauge transformation including the Haar measure necessary to make the integral invariant under left multiplication.

B. Projector in the presence of external charges

If we restrict ourselves to the pure gauge sector without external charges the Gauss law operator is simply (for conventions and placement of factors of \( g \) cf. appendix A, \( \Gamma^a \) is defined in eq. (A10))

\[
G^\alpha(x) = -\Gamma^a(x)
\]

Both the Gauss law operator and its pure gauge part fulfill the “angular momentum” algebra

\[
[G^\alpha(x), G^\beta(y)] = if^{abc}G^c(x)\delta_{xy}, \quad [-\Gamma^a(x), -\Gamma^b(y)] = if^{abc}(-\Gamma^c(x))\delta_{xy}.
\]

If we want also to consider external charge contributions, this can be done most economically in the following way (for definiteness, we assume the external charges to be fermions, in the following generically called ‘quarks’) 9, 10, 11: add to the Hamiltonian of the gauge sector a matter field part \( H_M \):

\[
H_M = M \int d^3x \bar{\psi}(x)\psi(x)
\]

where \( M \) denotes the quark mass, and we treat a single quark species (“flavour”) for simplicity. In concordance with the common interpretation we associate \textit{creation} operators with operators that create positive energy states, and thus write explicitly

\[
\psi^\alpha_a(x) = \begin{pmatrix} a_1^\alpha_1(x) \\ a_2^\alpha_1(x) \\ b_1^\alpha_1(x) \\ b_2^\alpha_1(x) \end{pmatrix}, \quad \psi^{\dagger}_\alpha_a(x) = \begin{pmatrix} \alpha_1^{\dagger}a_1(x) \\ \alpha_2^{\dagger}a_2(x) \\ \alpha_1^{\dagger}b_1(x) \\ \alpha_2^{\dagger}b_2(x) \end{pmatrix}.
\]
We then interpret the operators \(a, a^\dagger\) as connected to quarks, and the operators \(b, b^\dagger\) as connected to anti-quarks. We now turn to the charge operator. Let \(\lambda^a\) be the (hermitian) generators of the SU(N) representation appropriate for the quarks. The charge operator is then given by
\[
\rho^a(x) = J^a_\mu(x) = \bar{\psi}(x)\gamma^\mu\lambda^a\psi(x),
\]
where there is no integration over \(x\) just as in the remainder of this section (unless explicitly stated). In terms of the \(a, b\) operators, it can be written as
\[
\rho^a(x) = a_i^b(x)\lambda^a_{bc}a_i^c(x) - b_i^b(x)\lambda^a_{bc}b_i^c(x).
\]
The matrix \(\lambda^\dagger\) denotes simply the complex conjugate matrix of \(\lambda\). The charge operators inherit their commutation relations from the \(\lambda\) matrices:
\[
[a_i^a(x)\lambda^b\alpha(x) + b_i^a(x)\lambda^b\alpha(x)], [b_i^b(x)\lambda^c\beta(x) - a_i^b(x)\lambda^c\beta(x)] = if^{abc}(x)\delta_{\alpha\beta}.
\]
Thus, we can form the Gauss law operator in the presence of external charges
\[
G^a(x) = -\Gamma^a(x) + \rho^a(x),
\]
where \(\Gamma^a(x)\) is defined in eq. (10). We form states in the presence of external charges (called \(|\text{coupled}\rangle\)) in the following way:
\[
|\text{coupled}\rangle = \sum_{a,i} c^a_i|\text{fermion } i\rangle|\text{YM } a\rangle,
\]
where \(c^a_i\) are probability amplitudes and \(|\text{YM } a\rangle\) are purely gluonic states labelled by \(a\). The fermionic states are generated by acting with quark and anti-quark creation operators on the (gauge invariant) fermionic vacuum \(^1\):
\[
|\text{fermion } i\rangle = a_{i_1}^\dagger \cdots a_{i_n}^\dagger b_{j_1}^\dagger \cdots b_{j_m}^\dagger|0\rangle,
\]
where we have combined colour-, spinor- and spatial indices into a single super-index \(i\) (or \(j\)), and denoted the fermionic vacuum by \(|0\rangle\). We can also write suggestively
\[
\sum_{a,i_1,\ldots,i_m} c^a_{i_1,\ldots,i_m}a_{i_1}^\dagger \cdots a_{i_n}^\dagger b_{j_1}^\dagger \cdots b_{j_m}^\dagger|0\rangle|\text{YM } a\rangle = \sum_{i_1,\ldots,i_m} a_{i_1}^\dagger \cdots a_{i_n}^\dagger b_{j_1}^\dagger \cdots b_{j_m}^\dagger|\text{YM } a\rangle_i_1,\ldots,i_m
\]
with \(|\text{YM } a\rangle_i_1,\ldots,i_m = \sum_a c^a_{i_1,\ldots,i_m}|\text{YM } a\rangle\). Using
\[
e^{-i\int d^4x \phi^a(x)\rho^a(x)}\psi_i(x)\epsilon^{i\int d^4x \phi^a(x)\rho^a(x)} = (e^{i\phi^a(x)\lambda^a\alpha})_{ac}\psi_i^c(x) = q^R_{ac}[U(x)]\psi_i^c(x),
\]
\[
e^{-i\int d^4x \phi^a(x)\rho^a(x)}\psi_i(x)\epsilon^{i\int d^4x \phi^a(x)\rho^a(x)} = \psi_i^R(x)(e^{-i\phi^a(x)\lambda^a\alpha})_{ca} = p^R_{ac}[U(x)]\psi_i^{R\dagger}(x)
\]
where \(\psi\) transforms according to the representation \(q^R_{ac}[U(x)]\) and \(\psi^\dagger\) transforms according to the conjugate representation \(p^R_{ac}[U(x)] = q^{R\dagger}_{ac}[U^{-1}(x)]\), the requirement
\[
e^{-i\int d^4x \phi^a(x)\rho^a(x)\text{coupled}\rangle} = |\text{coupled}\rangle
\]
leads to (basically) the same result as known in the literature \(^2\) where one considers the Yang-Mills wave functional \(|\text{YM } a\rangle_i_1,\ldots,i_m\) with the extra colour indices:
\[
e^{i\int d^4x \phi^a(x)\Gamma^a(x)}|\text{YM } a\rangle_i_1,\ldots,i_m = q^R_{i_1l_1}[U]\cdots q^R_{i_nl_n}[U]p^R_{j_1k_1}[U]\cdots p^R_{j_mk_m}[U]|\text{YM } a\rangle_i_1,\ldots,i_m
\]
where the super-index notation for the representation matrices \(q, p\) has to be understood as follows: if we decompose the super-index \(i_1\) into the colour index \(a_1\), the spatial index \(n_1\) and the position index \(x_1\), and correspondingly \(l_1\) into \(b_1, n_1\) and \(x_1\) (the position index is in both cases identical since we do not associate an integration over position indices with gauge transformations), then
\[
(q^R, p^R)_{i_1l_1}[U] = \delta_{m_1n_1}(q^R, p^R)_{a_1b_1}[U(x_1)].
\]
The formulation chosen in this section has the advantage that one can give a closed expression for the projector even in the presence of external charges. At this point one has to note that from the fact that \(G^a|\psi\rangle = 0\) one may not conclude that one has diagonalized all generators \(\Gamma^a\) of the gauge part of the Gauss law operator, since the matter part \(\rho^a\) of \(G^a\) is also operator valued. One can only choose the Cartan subalgebra of the colour charges to be diagonalized.

\(^1\) That the fermionic vacuum is gauge invariant is a trivial consequence of the fact that it is annihilated by the annihilation operators.
C. The Projector onto Physical States

Since we now have an operator that generates gauge transformations on arbitrarily charged states, we can also write down a very simple (formal) expression for the projector onto the physical sector:

$$P^p = \int \mathcal{D}\mu[\varphi] e^{i \int d^3x \varphi^a(x)(\Gamma^a(x) - \rho^a(x))}. \tag{21}$$

The integration domain runs over the topologically trivial sector of the gauge transformations. Two points deserve attention: first, we want to consider why $P^p$ is indeed a projector onto the physical subspace, and second, why it is sufficient to restrict the domain of integration to topologically trivial gauge transformations. In fact these two points are closely related. A state in the physical subspace is characterized by the fact that it is annihilated by the Gauss law operator:

$$\int x \varphi^a(x)\rho^a(x) = 0. \tag{22}$$

(b) the left invariance of the Haar measure

$$\mathcal{D}\mu[\varphi] = \mathcal{D}\mu[\alpha(\phi, \varphi)], \tag{23}$$

and (c) the fact that a topologically trivial gauge transformation does not change the domain of integration of the projector $P^p$ to obtain

$$e^{-i \int \phi^a \mathcal{G}^a} P^p = e^{-i \int \phi^a \mathcal{G}^a} \int \mathcal{D}\mu[\varphi] e^{-i \int \varphi^b \mathcal{G}^b} = \int \mathcal{D}\mu[\varphi] e^{-i \int \phi^a \mathcal{G}^a} e^{-i \int \varphi^b \mathcal{G}^b} = \int \mathcal{D}\mu[\alpha(\varphi, \phi)] P^p. \tag{24}$$

Thus we have answered both questions: the projector projects onto the physical subspace since the projector is invariant under small gauge transformations, and since the Gauss law constraint is equivalent to topologically trivial gauge transformation only this domain of integration is required.

III. GENERAL CONCEPT OF THE KAMLAH EXPANSION

A. Abelian Case

The concept of the Kamlah expansion \cite{6, 7, 8} is most easily explained using an Abelian quantum mechanical example \cite{6, 7, 8}; assume that we have a two-dimensional system. The generator of rotations is $J$. The projector onto states having angular momentum $I$ is

$$P^I = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha(I-I)}. \tag{25}$$

That this is indeed the correct projector can again be seen most easily by

$$e^{i\beta J} P^I = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i(\alpha+\beta)J} = e^{i\beta I} P^I \tag{26}$$

where it has been used that $\int_0^{2\pi} = 1$ due to the periodicity of the integrand.

The case where the Kamlah expansion works best is the case of strong deformations. Strongly deformed states $|\phi\rangle$ are states where

$$|\langle \phi | e^{i\alpha J} |\phi\rangle| \tag{27}$$

decreases rapidly for increasing $\alpha$. This is very useful, since one can argue that

$$|\langle \phi | He^{i\alpha J} |\phi\rangle| \tag{28}$$
will vanish rapidly for increasing \( \alpha \), too, but
\[
\frac{\langle \phi | H e^{i\alpha J} | \phi \rangle}{\langle \phi | e^{i\alpha J} | \phi \rangle}
\]
is a smooth function of \( \alpha \). The argument consists basically of noting that, whereas \( \hat{J} \) is a one-particle operator (in many-body language), \( e^{i\alpha J} \) is a collective operator that affects arbitrarily large numbers of particles
\[
e^{i\alpha J} = \sum_{0 \text{-body operator}} 1 + \frac{i\alpha \hat{J}}{1 \text{-body operator}} + \frac{(i\alpha \hat{J})^2}{2 \text{-body operator}} + \frac{(i\alpha \hat{J})^3}{3 \text{-body operator}} + \ldots,
\]
(30)

\( H \) is (in nuclear physics) at most a two-body operator (A similar terminology can also be constructed for Yang-Mills theories, which is done in [13, 14]. In this terminology the Yang-Mills Hamiltonian also will be at most a two-body operator, i.e. contain up to four creation/annihilation operators.); thus the behaviour w.r.t. increasing \( \alpha \) should be qualitatively the same whether we consider \( e^{i\alpha J} \) or \( H e^{i\alpha J} \). One can now argue further that in the matrix element \( \langle \phi | H e^{i\alpha J} | \phi \rangle \) all degrees of freedom that are not affected by the collective rotation (‘internal degrees of freedom’) can be integrated out, and that this results in an effective Hamiltonian operator that is an - up to now - unknown function of \( \hat{J} \), the symmetry generator. For this function it is proposed to use a power series expansion in powers of \( \hat{J} - \langle \hat{J} \rangle \) (where \( \langle \hat{J} \rangle \) is used as an abbreviation for \( \langle \phi | \hat{J} | \phi \rangle \)), which is plausible, since one does not expect singularities for collective rotations with finite angular momenta. (The idea to expand the Hamiltonian in powers of the symmetry generators, although with a different method to obtain the coefficients, was proposed in [15, 16], cf. also [17, 18].)

However, this is only useful if we can stop the expansion after a few terms. That this is indeed possible one can see as follows:

- by assumption, \( \langle \phi | e^{i\alpha J} | \phi \rangle \) is well localized in \( \alpha \) space, since we only consider strongly deformed states; therefore, a broad range of wave functions with good \( \hat{J} \) quantum number have to be added up coherently
- \( \langle \phi | (\hat{J} - \langle \hat{J} \rangle) e^{i\alpha J} | \phi \rangle \): each wave function with good \( \hat{J} \) quantum number is now weighted by that quantum number and this begins to destroy the coherence, in other words \( \langle \phi | (\hat{J} - \langle \hat{J} \rangle) e^{i\alpha J} | \phi \rangle \) is broader in \( \alpha \) space than \( \langle \phi | e^{i\alpha J} | \phi \rangle \)
- the higher the power of \( \hat{J} - \langle \hat{J} \rangle \), the broader is the resulting matrix element in \( \alpha \) space, but we already know that \( \langle \phi | H e^{i\alpha J} | \phi \rangle \) is not very much broader in \( \alpha \) space than \( \langle \phi | e^{i\alpha J} | \phi \rangle \), hence the matrix elements containing higher powers of \( \hat{J} - \langle \hat{J} \rangle \) (besides \( e^{i\alpha J} \)) must be suppressed (We assume here that there is no ‘conspiracy’ between different orders in the expansion, but this seems to be a plausible assumption.), i.e. the corresponding expansion coefficients must be small. qed.

Thus, we can stop the expansion
\[
\langle \phi | H e^{i\alpha J} | \phi \rangle = \sum_{k=0}^{\infty} A_k \langle \phi | (\hat{J} - \langle \hat{J} \rangle)^k e^{i\alpha J} | \phi \rangle
\]
(31)
after a few terms:
\[
\langle \phi | H_{\text{approx}} e^{i\alpha J} | \phi \rangle = \sum_{k=0}^{n} A_k \langle \phi | (\hat{J} - \langle \hat{J} \rangle)^k e^{i\alpha J} | \phi \rangle.
\]
(32)

The \( (n+1) \) coefficients are determined by requiring that \( \langle \phi | H e^{i\alpha J} | \phi \rangle \) and \( \langle \phi | H_{\text{approx}} e^{i\alpha J} | \phi \rangle \) along with their first \( n \) derivatives w.r.t. \( \alpha \) are equal at the point \( \alpha = 0 \). One thus ends up with the set of equations\(^2\)
\[
\langle \phi | H (\hat{J} - \langle \hat{J} \rangle)^m | \phi \rangle = \sum_{k=0}^{n} A_k \langle \phi | (\hat{J} - \langle \hat{J} \rangle)^{k+m} | \phi \rangle,
\]
(33)

\(^2\) In order to obtain a more compact notation, we have equated \( \langle \phi | H e^{i\alpha (\hat{J} - \langle \hat{J} \rangle)} | \phi \rangle \) and \( \langle \phi | H_{\text{approx}} e^{i\alpha (\hat{J} - \langle \hat{J} \rangle)} | \phi \rangle \). In the sections in field theory, we will equate the terms equivalent to \( \langle \phi | H e^{i\alpha (\hat{J} - 1)} | \phi \rangle \) and \( \langle \phi | H_{\text{approx}} e^{i\alpha (\hat{J} - 1)} | \phi \rangle \). They differ only by a multiplicative c-number \( e^{-i\phi (\hat{J})} \) or \( e^{-i\phi I} \) respectively.
where \( m = 0, \ldots, n \). After we have solved these equations to obtain \( A_i \), we can insert \( H_{\text{approx}} = \sum_{k=0}^{n} A_k (\hat{J} - \langle \hat{J} \rangle)^k \) into the expression for the projected energy

\[
E_{\text{proj}}^{\text{approx}}(I) = \frac{\langle \phi | H_{\text{approx}} | \phi \rangle}{\langle \phi | P^I | \phi \rangle} = \sum_{k=0}^{n} A_k \frac{\langle \phi | (\hat{J} - \langle \hat{J} \rangle)^k P^I | \phi \rangle}{\langle \phi | P^I | \phi \rangle}.
\]

(34)

The advantage of the approach now becomes visible: whereas before we did not know how to evaluate the matrix elements including the projector, it has now become trivial; \( P^I \) projects onto angular momentum \( I \). Then \( J P^I = I P^I \), and numerator and denominator cancel, which means that we do not have to calculate a single matrix element containing the projector explicitly:

\[
E_{\text{proj}}^{\text{approx}}(I) = \sum_{k=0}^{n} A_k (I - \langle \hat{J} \rangle)^k.
\]

(35)

In nuclear physics, for most applications \( n \) is taken to be one or two; one then ends up with

\[
E_{\text{proj}}^{\text{approx}}(I) = \langle H \rangle + \frac{\langle H \Delta \hat{J} \rangle}{\langle \Delta \hat{J} \rangle^2} (I - \langle \hat{J} \rangle)
\]

(36)

for \( n = 1 \) and

\[
E_{\text{proj}}^{\text{approx}}(I) = \langle H \rangle - \frac{\langle \Delta \hat{J}^2 \rangle}{2 \langle \Delta \hat{J} \rangle} + \frac{\langle \hat{J} \rangle}{\mathcal{I}_{\text{sc}}} (I - \langle \hat{J} \rangle)^2 + \frac{1}{2 \mathcal{I}_{\text{sc}}} (I - \langle \hat{J} \rangle)^2,
\]

(37)

with

\[
\frac{1}{2 \mathcal{I}_{\text{sc}}} = \frac{\langle H \Delta \hat{J}^2 \rangle - \langle H \rangle \langle \Delta \hat{J}^2 \rangle}{\langle \Delta \hat{J} \rangle^2 - \langle \Delta \hat{J}^2 \rangle^2} \quad \text{and} \quad \frac{\langle \hat{J} \rangle}{\mathcal{I}_{\text{sc}}} = \frac{\langle H \Delta \hat{J} \rangle}{\langle \Delta \hat{J} \rangle}
\]

(38)

for \( n = 2 \). We have abbreviated \( \langle \phi | \ldots | \phi \rangle \) by \( \langle \ldots \rangle \) and used the notation \( \Delta \hat{J} = \hat{J} - \langle \hat{J} \rangle \). If we take simply \( n = 0 \), we end up with \( A_0 = \langle H \rangle \) which is reassuring: the lowest order term just gives the unprojected expectation value. We see that, at first order, we get no corrections to the mean field energy if \( \langle \hat{J} \rangle = I \), whereas for the expansion to second order, we even then obtain a correction. One can interpret this correction if one considers a deformed state as being a superposition of states with different angular momenta; especially, there are components in the wave function that are of higher angular momentum than the angular momentum one projects onto. Usually one would assume that higher angular momenta also contain higher kinetic energy that should not be present in the projected wave function; thus one obtains a negative correction to the ordinary energy expectation value. One notes in passing that the two moments of inertia \( \mathcal{I}_Y, \mathcal{I}_{\text{sc}} \) are quite different from one another; the latter is under certain circumstances identical to the moment of inertia of the cranking model \( \mathcal{I}_Y \), since one can show that, if the variational space considered for the cranking calculation contains not only \( |\phi\rangle \) but also \( \Delta \hat{J} |\phi\rangle \), then the solution of the cranking equations indeed solves also the variational equations derived for the \( n = 1 \) Kamlah expansion, and that the Lagrange multiplier of the cranking model has precisely the same form as the first order correction of the Kamlah expansion.

There are calculations that implicitly assume that in practice the moment of inertia \( \mathcal{I}_Y \) can be determined by a cranking type calculation. In Yang-Mills theories it turns out that to \( \mathcal{O}(g^0) \) this is indeed the case, but the general expressions given above in eq. (38), and the fact that the variational space of Gaussian wave functionals is too small to prove the equivalence of \( \mathcal{I}_{\text{sc}} \) and the cranking model moment of inertia, do not give much hope that it stays this way for higher orders in perturbation theory.

### B. Non-Abelian Case

If one goes from an Abelian type problem to a non-Abelian problem (like three dimensional angular momentum projection) one immediately has to face some problems: the first one is that a rigid body has two frames of reference (the lab frame and the intrinsic frame), two sets of angular momentum operators and (since the total

---

3 The variational space that we use in field theory, consisting of Gaussian wave functionals only, is not so large as it is required for this equivalence.
angular momentum is identical in the intrinsic and in the lab frame) three angular momentum quantum numbers (usually called $I,K,M$ belonging to $J^2 = I^2, J_3, I_3$) [8]. The basic ingredient into any projector is then

$$P_{KM}^I = |I,K⟩⟨I,M|.$$  \hspace{1cm} (39)

The quantum numbers $I,K$ are those of the state to be observed in the lab frame, whereas $M$ is only seen in the intrinsic frame. The object $P_{KM}^I$ does not have the properties usually expected of a projector, since it is neither hermitian, nor do we have a relation like $P^2 = P$. The obvious remedy cannot be used, since taking $P_{KM}^I = P_{MM}$ would avoid the mathematical problems but would violate rotational symmetry, as an 'intrinsic' state with $M = 0$ would always lead to a state to be observed in the lab frame with $K = 0$ [8]. One could also ask more physically: if $M$ is an internal quantum number that has rather little to do with what can be observed in the lab frame, why don’t we use

$$P_{K}^I = \sum_M f_M P_{KM}^I$$ \hspace{1cm} (40)

as a projector with $f_M$ as additional variational parameters (cf. e.g. [21])? In [8] it was argued that one should take all $f_M$ to be equal to one, but whatever one does, one always ends up with the problem that one has to evaluate matrix elements of operators acting upon projectors that do not carry the quantum numbers that are projected upon; to be concrete, one has e.g. to evaluate

$$\frac{\sum_{M,M'}\langle φ|\vec{J} P_{MM'}^I |φ⟩}{\sum_{M,M'}\langle φ|P_{MM'}^I |φ⟩}.$$ \hspace{1cm} (41)

One sees clearly that, unlike in the Abelian case, the projector matrix elements do not usually cancel. A number of ways have been devised in the literature to approximately evaluate these sums, but they are not of interest here, since they cannot be transcribed to the field theory case. We only want to point to the fact that the difficulties in even the formulation of the projector do not come into play if we want to project onto a state with $I = 0$, since then there is only one projector $P_{00}^0$ and no factors of $f$ can be chosen. This is the case of the Yang-Mills projector since we have combined the gluonic ($Γ$) and the matter ($ρ$) contribution into one set of operators ($G$) which now are supposed to form an ‘s-wave’. The problem of non-factorizability of projectors and other operators will nevertheless also haunt us later on, too (cf. sec. VI A).

C. General Remarks about Validity

One last comment shall be made at this point on the validity of the Kamla expansion. It is a priori not clear that it is physically sensible to perform the Kamla expansion for a local symmetry. In nuclear physics, the very deformed states usually do not come about through a very deformed nucleus in space, but are due to a very large number of participating nucleons [8]. In a gauge field theory, one has usually the situation of a very small number of excitations at every point in space. Therefore it is not clear whether it is sensible to use the ‘large-deformation’ expansion. However, it will turn out that the Kamla expansion performs very well in the cases we have considered.

IV. APPLICATION TO QUANTIZED ELECTRODYNAMICS

In this section, we want to apply the general principles outlined above to what we call electrodynamics - a quantized electromagnetic field interacting with static sources. It is illuminating since we will see that in marked contrast to nuclear physics, the first and second order expansions are even qualitatively quite different.

A. Space of Allowed Wave Functionals

In the remainder of this paper we will restrict ourselves to wave functionals of Gaussian form

$$\psi[\hat{A}] = \exp \left\{ - (\hat{A} - \bar{\hat{A}})_i \left( \frac{1}{4} G^{-1}_{ij} - i \Sigma_{ij} \right) (\hat{A} - \bar{\hat{A}})_j + i \vec{e}_i (\hat{A} - \bar{\hat{A}})_i \right\}$$ \hspace{1cm} (42)

where we use a super-index notation: when we consider electrodynamics, $i,j$ both include a spatial and a position index, which is summed or integrated over, respectively. In the case of Yang-Mills theories the super-indices furthermore include a colour index. $G^{-1}, \Sigma, \bar{\hat{A}}, \vec{e}$ are all purely real. Further conventions and a short account of canonical quantization are given in appendix A.
B. Simplified Projector

In the case of electrodynamics the expression for the projector can be simplified significantly compared to eq. (21). Since in electrodynamics there is only one charge operator, which also commutes with the Hamiltonian, the fermionic states can be chosen as eigenstates of the charge operator, and since we want to prescribe a certain external charge configuration we consider only states (cf. eq. (51) for the non-Abelian case) of the form

$$|\text{coupled}\rangle = |\text{fermion } \hat{\rho}\rangle |\text{photon}\rangle,$$

with

$$\rho(x)|\text{fermion } \hat{\rho}\rangle = \hat{\rho}(x)|\text{fermion } \hat{\rho}\rangle.$$  

Since we only want to project operators that depend solely on the gauge field part, we can use a simplified projector $P^\rho$

$$P^\rho|\text{coupled}\rangle = \int D\phi e^{i\int \phi(x) - \rho(x)} |\text{fermion } \hat{\rho}\rangle |\text{photon}\rangle = |\text{fermion } \hat{\rho}\rangle P^\rho|\text{photon}\rangle$$

which does not depend on the fermion charge operator but directly on the prescribed c-number external charge distribution. Furthermore, the (pure gauge part of the) Gauss law operator is also simpler than in Yang-Mills theories. It is given by

$$\Gamma(x) = \nabla \Pi(x).$$

C. Kamlah Expansion to First Order

We start from the expression (cf. eqs. (31, 32))

$$\langle H e^{i\int \phi(x) - \rho(x)} \rangle = A_0 \langle e^{i\int \phi(x) - \rho(x)} \rangle + A_1(y) \langle \{\Gamma(y) - \langle \Gamma(y) \rangle \} e^{i\int \phi(x) - \rho(x)} \rangle,$$

where we have again used $\int \phi(x) - \rho(x)$ as an abbreviation for $\int d^3z \phi(z)(\Gamma(z) - \rho(z))$. A useful notation that will be used throughout the remainder of this section is

$$\Delta(x) = \Gamma(x) - \langle \Gamma(x) \rangle, \quad \delta(x) = \rho(x) - \langle \Gamma(x) \rangle.$$  

Introducing the functional inverse $\Theta^{-1}(x, y)$ of $\langle \Delta(x)\Delta(y) \rangle$, s.t.

$$\Theta^{-1}(x, y)\langle \Delta(z)\Delta(y) \rangle = \delta_{xy}$$

we obtain for the projected energy functional (details of the procedure in general are outlined in app. C, where it is applied to the specific case of the second order Kamlah expansion)

$$\frac{\langle H P^\rho \rangle}{\langle P^\rho \rangle} \bigg|_{1^{\text{st}}} = \langle H \rangle + \langle H \Delta(x) \rangle \Theta^{-1}(x, y) \delta(y),$$

where we have introduced the notation $A|_{1^{\text{st}}}$ to denote the fact that $A$ is evaluated using the Kamlah expansion to first order. Note that the derivation can be carried out for any gauge-invariant few-body operator $\mathcal{O}$, since there is nothing special about the Hamiltonian:

$$\frac{\langle \mathcal{O} P^\rho \rangle}{\langle P^\rho \rangle} \bigg|_{1^{\text{st}}} = \langle \mathcal{O} \rangle + \langle \mathcal{O} \Delta(x) \rangle \Theta^{-1}(x, y) \delta(y).$$

This provides a first test of the validity of the first-order Kamlah expansion. We may set $\mathcal{O} = \Delta(z)$, and obtain

$$\frac{\langle \Delta(z) P^\rho \rangle}{\langle P^\rho \rangle} \bigg|_{1^{\text{st}}} \overset{\text{eq. (51)}}{=} \langle \Delta(z) \rangle + \langle \Delta(z)\Delta(x) \rangle \Theta^{-1}(x, y) \delta(y) = \rho(z) - \langle \Gamma(z) \rangle,$$

i.e.

$$\frac{\langle \Gamma(z) P^\rho \rangle}{\langle P^\rho \rangle} \bigg|_{1^{\text{st}}} \overset{\text{eq. (52)}}{=} \rho(z),$$

$$\left. \frac{\langle \Delta(z) P^\rho \rangle}{\langle P^\rho \rangle} \right|_{1^{\text{st}}} = \rho(z).$$

$$\left. \frac{\langle \Gamma(z) P^\rho \rangle}{\langle P^\rho \rangle} \right|_{1^{\text{st}}}$$
showing that the expression provided by the first order Kamplah expansion for $\langle \Gamma(z)P^\rho \rangle / \langle P^\rho \rangle$ is exact. It doesn’t stay that way, however, if one considers e.g. $O = \Delta(z_1)\Delta(z_2)$:

$$\frac{\langle \Delta(z_1)\Delta(z_2) \rangle}{\langle P^\rho \rangle} \bigg|_{\text{1st}} = \langle \Delta(z_1)\Delta(z_2) \rangle + \langle \Delta(z_1)\Delta(z_2) \rangle \Theta^{-1}(x,y) \delta(y) = \langle \Delta(z_1)\Delta(z_2) \rangle,$$

(53)

where we have used that $\langle \Delta(z_1)\Delta(z_2) \rangle = 0$, which is true for every Gaussian state as given by eq. (12). We see that there are no corrections due to the projector for $\Delta(z_1)\Delta(z_2)$ by the first order Kamplah formula. This gives rise to the expectation that the energy will not be correctly projected, since it contains terms of the type $\langle \Gamma(x)\Gamma(y)P^\rho \rangle$ (as we will see below, they arise from the kinetic energy).

Let us now evaluate the projected energy. For this it is useful to note that under a variety of assumptions\(^4\)

$$\langle B_i(x)B_j(y)\Gamma(z) \rangle = \langle B_i(x)B_j(y)\rangle \langle \Gamma(z) \rangle,$$

(54)

i.e. $\langle B_i(x)B_j(y)\Delta(z) \rangle = 0$. Physically it is plausible that the magnetic part of the energy should not be affected, since it contains the transversal degrees of freedom only. For the electrical field one obtains

$$\langle \Pi_i(x)\Pi_i(x)\rangle = 2\langle \Pi_i(x)\Delta(z) \rangle \langle \Pi_i(x) \rangle + \langle \Pi_i(x)\Pi_i(x) \rangle \langle \Gamma(z) \rangle,$$

(55)

which is valid even without integrating over $x$. With this, the first order energy correction is given by

$$\langle H\Delta(z) \rangle = \langle \Pi_i(x)\Delta(z) \rangle \langle \Pi_i(x) \rangle.$$

(56)

One can make further progress by assuming that both $G^{-1}$ and $\Sigma$ are translationally invariant; then one can write

$$\langle \Pi_i(x)\Pi_j(y) \rangle = \frac{1}{4}G^{-1}_{ij}(x,y) + 4\Sigma G \Sigma_{ij}(x,y) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}.(x-y)}(\partial^L \mathbf{P} \dot{L}_{ij} + \partial^P \mathbf{P} \dot{P}_{ij})$$

(57)

which defines the longitudinal and transversal components $\partial^L \mathbf{P}, \partial^P \mathbf{P}$, and where we have used the abbreviation $\mathbf{P} \dot{L}_{ij}$ for $\mathbf{p} \mathbf{p}_j / \mathbf{p}^2$; correspondingly $\dot{P}_{ij} = \delta_{ij} - \dot{P}_{ij}^P$. This allows an explicit expression for $\Theta^{-1}(x,y)$:

$$\Theta^{-1}(x,y) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}.(x-y)} \frac{1}{\mathbf{p}^2 \partial^L P}$$

(58)

and thus

$$\langle H\Delta(x_1) \rangle \Theta^{-1}(x_1,y) = \langle \Pi_i(x)\rangle \langle \Pi_i(x)\Delta(x_1) \rangle \Theta^{-1}(x_1,y) = \langle \Gamma(x)\rangle \langle \Delta(x_1) \rangle \langle \Gamma(y) \rangle - \rho(y).$$

(59)

$G_\Delta$ denotes the Green’s function of the (ordinary) Laplace operator, i.e. $-\nabla_i \nabla_i G_\Delta(x,y) = \delta_{xy}$. Then the projected energy reads

$$\frac{\langle HP^\rho \rangle}{\langle P^\rho \rangle} \bigg|_{\text{1st}} = \langle H \rangle - \langle \Gamma(x) \rangle G_\Delta(x,y) \left( \langle \Gamma(y) \rangle - \rho(y) \right).$$

(60)

We can make this expression even more transparent if we have a look at $\langle \Pi_i^L(x) \rangle \langle \Pi_i^L(x) \rangle$ where $\Pi_i^L(x)$ denotes the longitudinal component of $\Pi_i(x)$, i.e. $\Pi_i^L(x) = \int d^3y (\mathbf{P}^L)_{ij}(x,y)\Pi_j(y)$; for $\langle \mathbf{P}^L \rangle_{ij}(x,y)$ cf. eq. (A19). One can derive the identity

$$\langle \Pi_i^L(x) \rangle \langle \Pi_i^L(x) \rangle = \langle \Gamma(x) \rangle G_\Delta(x,y) \langle \Gamma(y) \rangle,$$

(61)

and therefore one can write

$$\frac{\langle HP^\rho \rangle}{\langle P^\rho \rangle} \bigg|_{\text{1st}} = \langle H \rangle^P - \frac{1}{2} \left\{ \langle \Gamma(x) \rangle G_\Delta(x,y) \langle \Gamma(y) \rangle - 2\langle \Gamma(x) \rangle G_\Delta(x,y) \rho(y) \right\}$$

(62)

$$= \langle H \rangle^P - \frac{1}{2} \langle \Gamma(x) - \rho(x) \rangle G_\Delta(x,y) \langle \Gamma(y) - \rho(y) \rangle + \frac{1}{2} \rho(x) G_\Delta(x,y) \rho(y).$$

(63)

\(^4\) Either one may assume that the kernel in the Gaussian wave functional is purely real, i.e. $\Sigma = 0$, or that both $G^{-1}$ and $\Sigma$ are translation invariant, or that the background field $B_i = e_{ijk} \nabla_j A_k$ fulfills the classical equations of motion.
\( (H)^P \) denotes here the part of the energy expectation value which is independent of \( (\Pi^L) \), but which still depends on \( (\Pi^L) \), where we have introduced the abbreviation \( (AB)_c = \langle AB \rangle - \langle A \rangle \langle B \rangle \). In formulas this is expressed simply by \( \langle H \rangle^P = \langle H \rangle - \frac{1}{2} \langle (\Pi^L(x)) (\Pi^L(x)) \rangle \). Thus \( (\Gamma(x)) \) is determined to be equal to \( \rho(x) \) at the stationary point, giving the correct charge contribution to the energy

\[
\frac{1}{2} \rho(x) G_\Delta(x, y) \rho(y). \tag{64}
\]

As can be seen by explicit computation, at the stationary point \( (\Pi^L) \) is zero, s.t. the energy depends only on the transversal degrees of freedom and on the charge density. However, the outcome is quite in contrast to what we expected from the Kamlah expansion initially: since we have computed the projected energy functional, (a) this should be completely independent of the gauge dependent degrees of freedom; thus, it should not depend on \( (\Pi^L) \), since in order to provide a non-vanishing expectation value \( (\Pi^L) \) the wave functional has to depend on \( A^L \); and (b) the state was assumed to be strongly deformed, but, if we take the first-order formalism seriously, at the stationary point our state is just not deformed, since \( (\Pi^L) = 0 \). These problems are overcome by performing the Kamlah expansion to second order. Therefore, we now turn to this.

### D. Kamlah Expansion to Second Order

Details of the derivation of the energy functional are given in app. C. It has to be noted that the factorization properties of Gaussian wave functionals, cf. eq. (12), and translational invariance of both \( G^{-1} \) and \( \Sigma \) are used heavily in this derivation. We obtain for the projected energy functional (using the notation \( \Delta H = H - \langle H \rangle \)):

\[
\frac{\langle H P^o \rangle}{\langle P^o \rangle} \bigg|_{2^{\text{nd}}} = \langle H \rangle - \frac{1}{2} \left( \Delta H \Delta(x_1) \Delta(x_2) \right) \Theta^{-1}(x_1, x_2) - \left( \Delta H \Delta(x_1) \right) \Theta^{-1}(x_1, y_1) (\Gamma(y_1) - \rho(y_1)) + \frac{1}{2} \left( \Delta H \Delta(x_1) \right) \Theta^{-1}(x_1, y_1) (\Gamma(y_1) - \rho(y_1)) (\Gamma(y_2) - \rho(y_2)), \tag{65}
\]

where we have introduced the notation \( A \big|_{2^{\text{nd}}} \) to denote the fact that \( A \) is evaluated using the Kamlah expansion to second order. As in the paragraph above, in the derivation of eq. (66) no special property of \( H \) apart from gauge invariance was used, and therefore the expression is valid for every gauge-invariant few-body operator \( O \):

\[
\frac{\langle OP^o \rangle}{\langle P^o \rangle} \bigg|_{2^{\text{nd}}} = \langle O \rangle - \frac{1}{2} \left( \Delta O \Delta(x_1) \Delta(x_2) \right) \Theta^{-1}(x_1, x_2) - \left( \Delta O \Delta(x_1) \right) \Theta^{-1}(x_1, y_1) (\Gamma(y_1) - \rho(y_1)) + \frac{1}{2} \left( \Delta O \Delta(x_1) \right) \Theta^{-1}(x_1, y_1) \Theta^{-1}(x_2, y_2) (\Gamma(y_1) - \rho(y_1)) (\Gamma(y_2) - \rho(y_2)). \tag{66}
\]

As in the first order case, we now have a look at \( \Delta \) and \( \Delta \Delta \) in (\( z = z_1 \) and \( z = z_2 \)):

\[
\frac{\langle \Delta(z) P^o \rangle}{\langle P^o \rangle} \bigg|_{2^{\text{nd}}} = eq. (66), \tag{67}
\]

which is the same result as in the first-order expansion, since all the additional terms due to the second order vanish; next

\[
\frac{\langle \Delta(z_1) \Delta(z_2) P^o \rangle}{\langle P^o \rangle} \bigg|_{2^{\text{nd}}} = eq. (66), \tag{68}
\]

Thus, in the second-order Kamlah expansion, both \( \langle \Delta(z) P^o \rangle/\langle P^o \rangle \) and \( \langle \Delta(z_1) \Delta(z_2) P^o \rangle/\langle P^o \rangle \) are reproduced exactly (cf. sec IV E). This gives rise to the hope that the energy, too, will be projected correctly, since it depends both on \( (\Gamma(x)) \) and \( (\Gamma(y)) \). The detailed derivation of the simplified expression for the projected energy is given in app. C. The main lessons are that, under the proposed assumptions, the magnetic energy does not obtain a correction from the projector (since it depends on the transversal degrees of freedom only) whereas the correction of the Kamlah expansion subtracts just the longitudinal part of the electric energy; one obtains eventually a concise formula for the projected energy:

\[
\frac{\langle H P^o \rangle}{\langle P^o \rangle} \bigg|_{2^{\text{nd}}} = \langle H \rangle - \frac{1}{2} \rho(x) G_\Delta(x, y) \rho(y). \tag{69}
\]
Note how different the outcome is from the first-order calculation: the projected energy is completely independent of the longitudinal parts of the electric field (both the expectation value and the correlator), and in turn they cannot be determined variationally. In the wave functional, the longitudinal parts are totally undetermined, which is what one wants since their value is prescribed by the projector. Thus we can conclude that, at least in electrodynamics, the second order Kamlah expansion is mandatory for a consistent formalism.

One amusing point might be noted at the end of this section: if one restricts the dimensionality of space-time to (1+1), there are no transversal components of \( A, \Pi \); thus the projection to second order as carried out by the Kamlah expansion leaves behind only the interaction of static charges. Though we have not checked explicitly (by replacing Fourier integrals by sums etc), eq. (69) seems to suggest strongly that if we consider the theory on a compact spatial manifold in (1+1) dimensions, we would also keep - in addition to the energy of the charges - the spatially constant part of \( \Pi \) (since this part is annihilated by the derivatives in the longitudinal projector, and is thus not subtracted), just as one would expect.

\[ E \]

V. APPLICATION TO ABELIANIZED YANG-MILLS THEORIES

A. Difficulties with External Charges

After the experience with the Kamlah expansion in electrodynamics, we want to try to apply it also to the Yang-Mills case. We will see, however, that we cannot incorporate charges as simply as in electrodynamics, and an explicit computation does not transcend the leading order in perturbation theory (although a possible background field is treated to all orders).

In order to discuss why we cannot include charges, the first order of the Kamlah expansion is sufficient; we don't expect the pattern found here: If we want to project correctly the product of a Gauss law operators we have to obtain a product of \( n \) charge distributions \( \rho \) from the Kamlah expression for the projected operator. We have seen that the coefficients \( A_i \) are independent of \( \rho \) so the only place where \( \rho \) does appear is in the terms \( \langle (\Gamma) - \rho \rangle \) (where for simplicity we haven't written out the product). But the highest power \( i \) that we obtain is equal to the order of the Kamlah expansion, i.e. a Kamlah expansion to second order includes at most terms \( \propto \rho^2 \). Thus the operator \( \Gamma^3 \) can never be correctly projected to give \( \rho^3 \) in the second order Kamlah expansion and a third order treatment would be called for.

\[ E \]

5 Here, a little care has to be taken w.r.t. the placement of \( g_s \). If we use 'perturbative scaling', cf. app. [A], the generator of gauge transformations in the gluonic sector is not \(-I^a\), but -\( \frac{1}{2} t^a \). Thus the projector is in principle \( \int Dg e^{-i \int g_s \phi} \frac{1}{Z} = \int Dg e^{i \int \phi^a (I^a - g_t^a)} \). However, this explicit scaling will only be used later on when we have already done away with the charges; we use the 'non-perturbative scaling' while we are developing most of the formalism, and re-install explicit factors of \( g \) later on.
obtain
\[ \langle H P^\rho \rangle_{1, \rho} = \langle H \rangle \langle P^\rho \rangle + \langle YM| H \Gamma^a(x)|YM\rangle (\Theta^{-1})^{ab}(x,y) \langle \delta^b(y) P^\rho \rangle, \]
(71)
where we have used that \( \Gamma^a(x) P^\rho = \rho^a(x) P^\rho \), and that our states \( | \rangle \) have the form \( | \text{fermion} \rangle |YM\rangle \) (This is actually a restriction - cf. eq. (4) - which is of no importance to the argument made here). At this point we see already that the results of electrodynamics do not generalize straightforwardly: there the fermionic states could be chosen as eigenstates of all charge operators \( \rho \). In Yang-Mills theories this is not possible (apart from states which are either non-charged or where the charges form locally colour-singlets), and therefore we cannot write
\[ \langle \delta^b(y) P^\rho \rangle = \langle \delta^b(y) \rangle \langle P^\rho \rangle. \]
(72)
Thus, we cannot get rid of all expectation values containing the projector simply by considering the normalized projected energy expectation value \( \langle H P^\rho \rangle / \langle P^\rho \rangle \). This, however, was the main appeal of the Kamlah expansion, and in order to rescue as much as possible of it, we will consider only uncharged states in the following: states that satisfy \( \langle f | \rho^a(x) = 0 \text{ and } \rho^a(x) | f \rangle = 0 \). In this case, the projector is reduced to its purely gluonic part, and therefore called \( \mathbf{P} \) in the remainder of this section.

### B. Kamlah Expansion to Second Order

We will now turn to the second order Kamlah expansion. While determining the coefficients \( A_0, A_1, A_2 \), we still work with general (possibly charged) states; it will turn out that the coefficients are actually independent of whether we have a charged state or not. We have set \( \Sigma \equiv 0 \) throughout the whole calculation. Then, we start from (cf. eqs. (11, 12))
\[ \langle H e^{-i f \varphi^a \varphi^b \varphi^c} \rangle = A_0 \langle e^{-i f \varphi^a \varphi^b \varphi^c} \rangle + A_1^a(y) \langle \Delta^a(y) e^{-i f \varphi^a \varphi^b \varphi^c} \rangle + A_2^a(y, z, \rho) \langle \Delta^a(y) \Delta^b(z) e^{-i f \varphi^a \varphi^b \varphi^c} \rangle. \]
(73)
It seems to be quite sensible to require that \( A_2^{ab}(y, z) \) should be symmetric under interchange of all indices, i.e. \( A_2^{ab}(y, z) = A_2^{ba}(z, y) \), since (a) most of \( \Delta^a(y) \Delta^b(z) \) is symmetric apart from \( \Gamma^a(y) \Gamma^b(z) \) \(^6\), (b) the antisymmetric part of \( \Delta^a(y) \Delta^b(z) \) is \( -\frac{1}{2} f^{abc} \Gamma^c(y) \delta_{yz} \) and should therefore rather be assigned to the term multiplying \( A_1 \) from a systematic point of view. In the course of the calculation, we have to perform two approximations: first, we have to restrict ourselves to Gaussian states (as in electrodynamics), cf. eq. (12). However, we have to make a second approximation, which is more severe: we will restrict the (gluonic part of the) Gauss law operator to its \( \mathcal{O}(g^0) \) part. In order to facilitate this perturbative treatment, we start with the ‘non-perturbative scaling’ scheme (cf. table I in app. [3]) and then replace \( \mathbf{A} \) by \( \mathbf{A} + g a \). Then, \( a \) is the new dynamical quantum field with \( \langle a \rangle = 0 \), and \( \frac{1}{g a} \) is in the following denoted by \( \mathbf{D} \). The pair \( (\mathbf{a}, \mathbf{D}) \) then follows the ‘perturbative scaling’ scheme. Note that in this scaling scheme we continue to call \( \mathbf{D}_i^{ab}(x) \Pi_i^b(x) \) the (gluonic part of the) Gauss law operator, even though the generator of gauge transformations in this scheme is \( -\frac{1}{g} \mathbf{D}_i^{ab}(x) \Pi_i^b(x) \). The Gauss law operator is now expanded w.r.t. \( g \) as follows:
\[ \Gamma^a(x) = \mathbf{D}_i^{ab}(x) \Pi_i^b(x) = \langle \mathbf{D}_i^{ab}(x) - g f^{acb} a_i^c(x) \rangle \Pi_i^b(x) = \hat{\Gamma}^a(x) - g f^{acb} a_i^c(x) \Pi_i^b(x) \]
\[ = \hat{\Gamma}^a(x) + \mathcal{O}(g), \]
(74)
i.e. \( \hat{\Gamma}^a(x) = \mathbf{D}_i^{ab}(x) \Pi_i^b(x) \), where \( \mathbf{D}_i^{ab}(x) \) is the covariant derivative in the background field \( \mathbf{A} \), explicitly \( \mathbf{D}_i^{ab}(x) = \nabla_i \delta^{ab} - f^{acb} A_i^c(x) \). This then will allow to use factorizability in a fashion identical to the case of electrodynamics. However, this approximation has consequences; the first one is that we have reduced the non-Abelian \( SU(N) \) symmetry, cf. eq. (7), to an Abelian \( U(1)^{(N^2 - 1)} \) symmetry since:
\[ \{ \hat{\Gamma}^a(x), \hat{\Gamma}^b(y) \} = 0. \]
(75)
The second one has to do with the admissible background fields \( \mathbf{A} \); it is clear that we cannot expect accuracy of the energy expectation value to orders higher than \( \mathcal{O}(g^0) \); thus we approximate the Hamiltonian by its \( \mathcal{O}(g^0) \)

\(^6\) Note that \( |\Gamma^a, \Gamma^b\rangle \neq 0 \), cf. eq. (3).
The kinetic energy remains unchanged, but the magnetic energy is simplified quite dramatically if we insert the decomposition \(A^d(x) = \hat{A}^d(x) + g \hat{a}^d(x)\) into \(\frac{1}{g^2}B^a_i(x)B^b_i(x)\) and neglect all terms of higher order than \(g^0\):

\[
\frac{1}{g^2}B^a_i(x)B^b_i(x) = \frac{1}{g^2}B^a_i(x) + \frac{2}{g}B^a_i(x)\epsilon_{i,j,k,l} (\hat{D}^{a_{c_1}}_{j_1}(x)\delta_{xx_1}) a^{c_1}_{k_1}(z_1)
+ \epsilon_{i,j,k,l}(\hat{D}^{a c_1}_{j_1}(x)\delta_{xx_1}) \epsilon_{i,j,k,l}(\hat{D}^{a_{c_2}}_{j_2}(x)\delta_{xx_2}) - B^a_i(x)f^{a c_1 c_2}\epsilon_{i,j,k,l}\delta_{xx_1}\delta_{xx_2}) a^{c_1}_{k_1}(z_1)a^{c_2}_{k_2}(z_2).
\]

(76)

For eqs. (76, 77) to be valid we do not have to integrate over \(x\). The restriction on \(\hat{A}\) comes about from the requirement that the approximated Hamiltonian should commute with the approximated Gauss law operator; otherwise the (implicit) assumption of the Kamal expansion that in the beginning the Hamiltonian commutes with the projector is incorrect. That one obtains a non-trivial condition at all results from the fact that, to \(O(g^0)\) of the commutator of the full quantities, one gets also a cross term from the \(O(g^-1)\) part of \(B^2\) with the \(O(g^1)\) part of \(\Gamma\). Thus we can express the requirement of gauge invariance of the approximated Hamiltonian in two different ways:

(a) the term of order \(g^{-1}\) in eq. (77) should vanish, since then there can be no cross term with the \(O(g^1)\) contribution to \(\Gamma\). With a partial integration, we see that this condition can be written as

\[
\epsilon_{i,j,k,l}(\hat{D}^{a c_1}_{j_1}(x)\delta_{xx_1}) a^{c_1}_{k_1}(z_1) = 0,
\]

(78)

which is nothing but the classical equation of motion for the (static) Yang-Mills field, since \(F\) is the field strength tensor.

(b) the commutator \(\frac{1}{g^2}[B^a_i(x), \Gamma]\) should vanish. We can compute the commutator and obtain

\[
\frac{1}{g^2}[B^a_i(x)B^b_i(x), \Gamma^b(y)] = 2\hat{D}^{b_{l_1}}_{l_1}(y)M^{b_{l_1}c_2}_{l_1,k_2}(x; y, z_2)a^{c_2}_{k_2}(z_2) = 0,
\]

(79)

where we have used the fact that \(M\) - as defined by eqs. (76, 77) - is symmetric under interchange of all its indices. This is of course equivalent to condition (a), and thus to the requirement of the background field fulfilling the classical equations of motion for static fields.

The computation is given in some detail in app. C2. Here we want to mention that in the course of this computation a further restriction has to be made of the class of admissible kernels \(G^{-1}\) which is stated explicitly in eq. (C20). We then obtain an overall correction to the electric energy

\[
-\frac{1}{2} \int d^3x_1 d^3x_2 (\Pi_L)_{i_2 i_1} (x_2, x_1)(\Pi^{c_1}_{i_1}(x_1)\Pi^{c_2}_{i_2}(x_2))
\]

(80)

where \(\Pi_L\) is the (generalized) longitudinal projector as defined in app. A4. Thus, the electrical energy neither contains \(\hat{D}^{\alpha b}_{i_1}e^b_i\) nor the (generalized) longitudinal component of \(G^{-1}\). Since by requirement of approximate gauge invariance the magnetic energy is transversal as well, this is true for the total energy, and to \(O(g^0)\), we can write

\[
\langle HP \rangle_P \bigg|_{2=4} = \frac{1}{2} \int d^3 x_1 d^3 x_2 (\Pi_L)_{i_2 i_1} (x_2, x_1)(\Pi^{c_1}_{i_1}(x_1)\Pi^{c_2}_{i_2}(x_2)) + \frac{1}{2} \int d^3 x \left\{ \frac{1}{g^2}\hat{B}^a_i(x)\hat{B}^a_i(x) + \text{tr}(MGxy)\big|_{y \rightarrow x} \right\}.
\]

(81)

\footnote{This approximation is also sensible from another point of view: the \(O(g^0)\) contribution of the Hamiltonian is at most quadratic in the operators \(A\) and \(\Pi\) and from the case of electrodynamics one expects that the second-order Kamal expansion treats correctly only (projected) expectation values of up to two operators, cf. sec. IV2.}

\footnote{In the electrodynamical case there is no \(O(g^1)\) contribution to the Gauss law operator and hence the whole discussion is not necessary.
with \( \text{tr}(\mathcal{M}G_{xy}) = \langle (\hat{D}(x)\hat{D}(x) - \hat{B}(x))^{ab}_{ij}f^{cde}_{ji}(x,y) \rangle \), where we have used the abbreviations \( \hat{D}_{ij}^{ab}(x) = \epsilon_{ikj}\hat{D}_{i}^{ab}(x), \hat{B}_{ij}^{ab}(x) = \hat{B}_{i}^{ab}(x)f^{acde}_{ikj} \); for \( \Pi_T \) cf. eq. (A26). Writing the kinetic energy also in terms of \( G^{-1} \) and \( \bar{e} \), the energy is given as

\[
E_{\text{approx}}^{\text{proj}} = \left\langle \langle P \rangle \right\rangle_{2\text{nd}} = \frac{1}{2} \int d^3x_1 d^3x_2 \left\{ \langle \Pi_T \rangle_{i_2i_1}(x_2,x_1)\bar{e}^{c_1}_{i_1}(x_1)\bar{e}^{c_2}_{i_2}(x_2) \right\} + \frac{1}{8} \text{Tr}(G^{-1}) + \frac{1}{2} \int d^3x \left\{ \frac{1}{g^2}\bar{B}_i^a(x)\bar{B}_i^a(x) + \text{tr}(\mathcal{M}G_{xy})|_{y=x} \right\},
\]

(82)

where \( G^{-1}_{TT} \) is defined by eq. (C26), and the trace Tr furthermore runs over colour, spatial and position indices. We see that the projected energy is in fact independent of the (generalized) longitudinal components of \( \bar{e} \) and \( G^{-1} \). This is what we would expect from the projector: we treat it to \( \mathcal{O}(g^0) \), and we obtain an energy functional that is compatible with the energy functional of a state that is annihilated to \( \mathcal{O}(g^0) \) by the Gauss law operator, cf. app. B. Perturbation theory thus seems not to invalidate the treatment of the Kamleh expansion. It is interesting to note that eq. (82) is (at least formally) identical to the improved energy functional obtained in \( B \) in the case of magnetic background fields, to the same order in perturbation theory as considered here. One difference, however, should be noted: in our treatment the condition that \( \bar{A} \) has to satisfy the classical equations of motion (and thus \( M \) has to be transversal in the generalized sense) occurred naturally due to the requirement that the approximated Hamiltonian should still commute with the approximated Gauss law operator, whereas in \( B \) it is apparently assumed that \( M \) is transversal in the generalized sense. In the application considered in \( B \), the Savvidy vacuum, the condition is of course fulfilled. In other cases, as e.g. the constant magnetic field stemming from a non-commuting gauge potential \( [22, 23] \), this need not be the case.

As a last topic in this section we want to consider the value of the energy if we consider it at the stationary point of \( \bar{e} \) and \( G \). We first consider \( \bar{e} \):

\[
\frac{\delta}{\delta \bar{e}^a_i(x)} E_{\text{approx}}^{\text{proj}} = \int d^3x_1 \left\{ (\Pi_T)^{ac_1}_{i_1i}(x,x_1)\bar{e}^{c_1}_{i_1}(x_1) \right\} = 0,
\]

(83)

thus the transversal component of \( \bar{e} \) has to be zero at the stationary point. Next we consider \( G_{TT} \) (since \( M \) is transversal, the magnetic energy also contains only \( G_{TT} \)). For the variation w.r.t. \( G_{TT} \) we need the relation

\[
\frac{\delta}{\delta (G_{TT})^{a_1a_2}_{i_1i_2}(z_1,z_2)} \text{Tr}(G^{-1}) = -(G_{TT}^{-1})(G_{TT}^{-1})^{a_1a_2}_{i_1i_2}(z_1,z_2).
\]

(84)

After a short calculation we end up with

\[
E_{\text{approx}}^{\text{proj}}(\bar{A}) = \frac{1}{2} \text{Tr}((\hat{D}\hat{D} - \hat{B})^2) + \frac{1}{2g^2} \int d^3x \bar{B}_i^a(x)\bar{B}_i^a(x).
\]

(85)

Thus we see that at the stationary point, the difference to the treatment of \( B \), where the projector was ignored, lies only in the requirement that here \( \bar{A} \) has to satisfy the classical equations of motion. The important point of the Kamleh expansion is, however, that the energy functional depends only on the transversal degrees of freedom even away from the stationary point.

**C. Treatment of Charges to Lowest Order in \( g \)**

Here we only want to insert an additional point that is inspired by the observation that, in the leading-order perturbative treatment, the gauge group \( SU(N) \) is reduced to a direct product of \( U(1) \) groups. In that case one can obviously also require the charged states to be simultaneous eigenstates of all charge operators, and can therefore factorize

\[
\langle \delta^b(y)\mathcal{P}^c \rangle = \delta^b(y)\langle \mathcal{P}^c \rangle,
\]

(86)

where the \( \delta^b(y) \) on the RHS is no longer an operator but simply a c-number function. At this point, we should make clear that in the 'perturbative scaling', cf. app. \( A \), that has been used here, \( \delta \) reads

\[
\delta^a(x) = g\rho^a(x) - (\bar{\Gamma}^a(x)),
\]

(87)
The first term can be easily calculated to give zero whereas the second term simplifies to
\[ \langle \mathcal{O} \rangle \theta^{c_{1}c_{2}}(x_{2}, x_{1}) \] (88)
\[ + \frac{1}{2} \langle \Delta H \Delta^{c_{1}}(1) \Delta^{c_{2}}(x_{2}) \rangle \Theta^{c_{1}d}(x_{1}, y_{1}) \Theta^{c_{2}d}(x_{2}, y_{2}) \times \] (\langle \mathcal{O}^{d}(y_{1}) \rangle - g \rho^{d}(y_{1})) (\langle \mathcal{O}^{d}(y_{2}) \rangle - g \rho^{d}(y_{2})).

We see that it differs from the energy projected onto the chargeless sector by
\[ -g \left\{ \left( \langle \Delta H \Delta^{c_{1}}(x_{1}) \Delta^{c_{2}}(x_{2}) \rangle \Theta^{c_{1}d}(x_{1}, y_{1}) \Theta^{c_{2}d}(x_{2}, y_{2}) \langle \mathcal{O}^{d}(y_{1}) \rangle \right) \right\} \rho^{d}(y_{2}) \]
\[ + \frac{g^{2}}{2} \langle \Delta H \Delta^{c_{1}}(x_{1}) \Delta^{c_{2}}(x_{2}) \rangle \Theta^{c_{1}d}(x_{1}, y_{1}) \Theta^{c_{2}d}(x_{2}, y_{2}) \rho^{d}(y_{1}) \rho^{d}(y_{2}).
\]
The first term can be easily calculated to give zero, thus there is no term linear in \( \rho \) that contributes to the energy, whereas the second term simplifies to
\[ \frac{g^{2}}{2} \int d^{3}y_{1} d^{3}y_{2} \rho^{d}(y_{1}) G_{\Delta}^{d}(y_{1}, y_{2}) \rho^{d}(y_{2}) \quad \text{with} \quad - \eta_{i}^{ab} \eta_{j}^{bc} \delta_{ij}^{cd}(x, y) = \delta^{ad} \delta_{xy}. \]

We see that in leading order perturbation theory, the charges interact via a potential given by the Green’s function of the background field-covariant Laplacian. Three comments are in order: first, since the Green’s function depends on the background field, the variational equations for \( \bar{\mathcal{A}} \) are not independent of the charge distribution. This is different from the electrodynamical case where radiation (transversal terms) and static charges were completely decoupled. Second, we see that obtaining the \( \beta \) function from corrections to the Coulomb potential (which would result for \( \mathcal{A} = 0 \)) is on this level of approximation not possible; it seems almost surprising that we obtained the correct leading order potential, since our projection scheme only attempts to give the correct energy to \( O(g^{0}) \). For the first quantum corrections to the interquark potential, we would need the energy up to \( O(g^{1}) \). Third, the fact that the inter-charge potential is given by the Green’s function of the covariant Laplacian strongly reminds one of the result of [24] where a rigid rotator model was considered which was also inspired by ideas from nuclear physics.

VI. SUMMARY AND CONCLUSIONS

In this paper we considered the Kamlah expansion, which is an expansion of the expectation value of the Hamiltonian between projected states in powers of the symmetry generator. This is a technique successfully employed in nuclear physics, but it is not clear a priori that the same can be done in a field theory with a local symmetry since the applicability in nuclear physics is based on the fact that the states are strongly deformed, which comes about usually from many particles participating in that deformed state. We tried the Kamlah expansion first for electrodynamics with external (static) charges, where we found that the first order of the Kamlah expansion is inconsistent with its assumptions, since at the stationary point we find a state that is not deformed. Going to second order, however, we found exactly the properties that we expect of a projected energy functional, namely that it is independent of the longitudinal parts of the \( (\mathcal{P} \mathcal{P}) \) correlation function and the longitudinal part of the expectation value of \( \mathcal{P} \), and the correct Coulomb energy appears in the projected energy.

---

9 We have already indicated above that the two apparently very different moments of inertia \( I_{x_{1}}, I_{y_{1}} \) are identical in a perturbative treatment to leading order in \( g \); this is the point where one observes this most clearly. For the evaluation of the expressions used here, we have used a different form of the longitudinal projector, \( (\mathcal{P}_{L})^{c_{1}c_{2}}(x_{2}, x_{1}) = (\hat{\mathcal{D}}^{c_{2}b_{2}}(x_{2}) \hat{\mathcal{D}}^{b_{1}c_{1}}(x_{1}) G^{b_{1}b_{2}}(x_{1}, x_{2})) \), where again we don’t integrate over \( x_{1}, x_{2} \).
This result motivated us to consider Yang-Mills theories, too, in this framework but we restricted ourselves to the leading order in a perturbative expansion in powers of the coupling constant $g$. Since we have approximated both the Gauss law operator and the Hamiltonian, we obtain consistency conditions from the requirement that the approximated Hamiltonian still commutes with the approximated Gauss law operator. Effectively, this restricts the choice of possible background fields to those which satisfy the (static) classical equations of motion. This condition makes the operator that multiplies the part of the potential energy quadratic in $\mathbf{A}$ transversal w.r.t. the covariant derivative in the background field. Since the projection subtracts off the longitudinal part (in the generalized sense) of the electrical energy, we end up with an energy functional that is independent of the longitudinal parts (in the generalized sense) of the parameters of the Gaussian wave functional. At the stationary point of all parameters save the background vector potential, the energy functional looks precisely like the one-loop functional that one obtains in the mean field considerations without taking into account the projector. Of course, this is so only formally, since on the one hand the consistency condition alluded to above ensures that the energy depends only on the transversal degrees of freedom (thereby reducing three polarization states to two), and, on the other hand, the longitudinal parts of the parameters in the Gaussian wave functional are undetermined - instead of being set to zero - just as one would expect from a projected energy functional. We then also considered how external charges can be included into the Kamalh expansion. Up to now, we have been able to allow for them only to leading order in perturbation theory, since only then the charges can be treated in a quasi-Abelian manner.

Three investigations seem worthwhile:

**First**, one has to investigate whether the Kamalh expansion is a truly non-perturbative expansion or whether at some stage an implicit expansion in powers of $g$ takes place. In this context, one should also investigate further the 'pattern' found in our discussion on electrodynamics, namely that one seems to need the Kamalh expansion to $n^{th}$ order to project correctly terms containing $n$ operators. If this would be true in general, one would need the fourth-order Kamalh expansion for a correct projection of the full Yang-Mills Hamiltonian.

**Second**, one should try to find out whether the Kamalh expansion can be carried out also to higher orders in perturbation theory since this would lift the restriction to background fields that satisfy the classical equations of motion. This should be considered also in the presence of external charges, which up to now can only be dealt with classically. If that can be done successfully, one should face the problem of computing the one-loop contribution to the interquark potential. If this reproduces the standard results, one could also try in a **third** investigation a non-perturbative evaluation of the second-order Kamalh expansion (provided the Kamalh expansion is truly non-perturbative and a second-order treatment is sufficient), although this will be a quite difficult task, since in our evaluation we relied strongly on the factorization properties; these, however, cannot be applied so easily in a non-perturbative framework, since the full Gauss law operator contains products of two operators at the same point in space.

Thus, renormalization problems, which we have ignored completely in this paper, will have to be taken into account in a non-perturbative manner before one can perform such a non-perturbative evaluation of the projected energy functional.

Of course, when considering such non-perturbative extensions one should keep in mind that the restriction to Gaussian states has tied us already to the Hartree-Fock approximation, which is basically the two-loop approximation of the total quantum energy.

**Acknowledgments**

The authors would like to thank Michael Engelhardt, Jean-Dominique Länge and Markus Quandt for continuous discussion. This work was supported by the Deutsche Forschungsgemeinschaft under grants DFG Mu 705/3, DFG III GK-GRK 683/1-01, DFG-Re 856/4-1 and DFG-Re 856/5-1.

**APPENDIX A: CONVENTIONS**

1. **Indices and Summation Conventions**

   In this paper, four different kinds of indices appear: colour indices, spatial indices, position indices and super-indices, combining several of the former three kinds as will be explicitly stated wherever they are introduced. To each kind of index a certain range of letters is attached: colour indices: $a, b, c, \ldots$, spatial indices: $i, j, k, \ldots$, position indices: $x, y, \ldots$, super-indices: $i, j, k, \ldots$.

   Unless explicitly stated otherwise, Einstein's summation convention will be used throughout. Discrete indices will be summed over, continuous indices will be integrated over. We will also use a uniform notation for $\delta$ functions, regardless of the nature of indices (discrete or continuous); thus, $\delta_{xy}$ in this paper corresponds to $\delta^{(3)}(x - y)$ in the
usual notation for Dirac’s δ, whereas e.g. \( \delta^{ab} \) is simply Kronecker’s δ.

2. Minkowski Space

We use as metric for the Minkowski space

\[
g_{\mu\nu} = \text{diag}(1, -1, -1, -1),
\]

and thus have as four-vectors \( a^\mu \) and \( a_\mu = g_{\mu\nu}a^\nu \)

\[
a^\mu = (a_0, a) , \quad a_\mu = (a_0, -a)
\]

with the scalar product

\[
a \cdot b = a_\mu b^\mu = a_0 b_0 - a \cdot b.
\]

Only for the four-derivative, signs are distributed differently:

\[
\partial^\mu = (\partial_0, -\nabla) , \quad \partial_\mu = (\partial_0, \nabla)
\]

with \( \nabla_i = \frac{\partial}{\partial x^i} \). Sometimes we need a gradient w.r.t. another variable. This we denote by attaching the variable name to \( \nabla \), e.g. \( \nabla^\gamma = \frac{\partial}{\partial y^\gamma} \).

3. Group Theory Conventions

The (hermitian) generators of the SU(N) Lie algebra satisfy the commutation relations

\[
[\lambda^a, \lambda^b] = if^{abc}\lambda^c.
\]

The generators in the fundamental representation(s) are normalized to (1/2):

\[
\text{tr}(\lambda^a_{\text{fund}}\lambda^b_{\text{fund}}) = \frac{1}{2}\delta^{ab}.
\]

The structure constants satisfy the Jacobi identity

\[
f^{abr}f^{cdr} + f^{acr}f^{dbr} + f^{adr}f^{bcr} = 0
\]

and are normalized as

\[
f^{abc}f^{dce} = N\delta^{ad}.
\]

4. Canonical Quantization

Throughout this paper we work in the Hamiltonian framework. We use the Weyl gauge \( A_0 = 0 \), but do not fix the gauge any further. The usage of the Weyl gauge implies that the physical states of the system have to satisfy the Gauss law constraint. In the absence of external charges (which we assume throughout this appendix), the condition reads

\[
\Gamma^a(x)|\psi\rangle = 0
\]

where \( \Gamma^a(x) = \hat{D}^{ab}_i(x)\Pi^b_i(x) \) with \( \hat{D}^{ab}_i(x) = \nabla_i\delta^{ab} - f^{abc}A^c_i(x) \).

\[
\hat{D}^{ab}_i(x) \quad \text{is the covariant derivative in the adjoint representation. If we consider a system with external charges, the Gauss law operator obtains additional contributions that stem from the external charges, as is outlined in sec. II.}
\]

In order to quantize the system, we do not have to resolve Gauss’ law, since even a system with redundant coordinates
may be quantized \[ \mathcal{H} \]. We work in the coordinate representation, i.e. \( \mathbf{A} \) is realized multiplicatively, whereas the canonical momenta are realized as derivative operators:

\[
\langle \mathbf{A} | \Pi_a^i(x) | \psi \rangle = \frac{1}{\delta} \frac{\partial}{\partial \mathbf{A}^a_i(x)} \langle \mathbf{A} | \psi \rangle.
\]  

(A11)

This choice ensures the satisfaction of the equal-time commutation relations automatically

\[
[A_i^a(x), \Pi_j^b(y)] = i\delta^{ab}\delta_{ij}\delta_{xy}.
\]

(A12)

Factors of \( g \) appear in several quantities at different places according to the different conventions possible as will be discussed at length in app. A.4. However, contrary to some conventions used in the literature, the meaning of \( \mathbf{A}, \Pi \) is always the same irrespective of whether we choose 'perturbative' or 'non-perturbative scaling', especially no factors of \( g \) enter the commutator eq. (A12). The wave functionals we deal with in this paper are given in the \( \mathbf{A} \)-representation

\[
\psi[\mathbf{A}] = \langle \mathbf{A} | \psi \rangle.
\]

(A13)

In the language of wave functionals the fact that the choice of Weyl gauge implies that the Gauss law operator is translated as

\[
\psi[\mathbf{A}^U] = \psi[\mathbf{A}]
\]

(A14)

for topologically trivial but otherwise arbitrary gauge transformations \( U \). In this context, \( \mathbf{A}^U \) denotes the gauge transformed field

\[
(\mathbf{A}^a_i \lambda^a)^U(x) = U(x)\mathbf{A}^a_i(x)\lambda^aU^\dagger(x) - iU(x)\nabla_i U^\dagger(x)
\]

(A15)

with \( U(x) = e^{i\phi^a(x)\lambda^a} \) and \( \lambda^a \) are the generators of SU(N) in some representation.

For further details on the canonical treatment of Yang-Mills theories, cf. [26]; more details on the Schrödinger picture can be found e.g. in [27, 28].

---

### 5. Factors of \( g \)

In Yang-Mills theories one has basically two options concerning where one wants to put the coupling constant, either in front of the action ('non-perturbative scaling') or in front of the commutator term in the field strength ('perturbative scaling'). In table I we give a short list concerning which convention leads to which placing of factors of \( g \) in other quantities of interest.

|                         | non-perturbative scaling | perturbative scaling |
|-------------------------|--------------------------|----------------------|
| covariant derivative    | \( D_\mu = \partial_\mu - iA_\mu \) | \( D_\mu = \partial_\mu - ig A_\mu \) |
| field strength          | \( F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \) | \( F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \) |
| action                  | \( S = -\frac{1}{4g^2} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) \) | \( S = -\frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) \) |
| electrical field        | \( E^a_\mu = F^a_{\mu0} \) | \( E^a_\mu = F^a_{\mu0} \) |
| magnetic field          | \( B^a_i = -\frac{1}{2} \epsilon_{ijk} F^a_{jk} \) | \( B^a_i = -\frac{1}{2} \epsilon_{ijk} F^a_{jk} \) |
| momentum               | \( \Pi^a_i = \frac{\partial}{\partial A^a_i} \) | \( \Pi^a_i = \frac{\partial}{\partial A^a_i} \) |
| Hamiltonian             | \( H = \frac{2}{g^2} \Pi^a_i(x) \Pi^a_i(x) + \frac{1}{2g^2} B^a_i(x) B^a_i(x) \) | \( H = \frac{1}{2} \Pi^a_i(x) \Pi^a_i(x) + \frac{1}{2} B^a_i(x) B^a_i(x) \) |
| wave functional of 'free' theory | \( \psi[\mathbf{A}] \sim e^{-\frac{ig}{2} g A^a_{\mu\nu} A^{a\mu\nu}} \) | \( \psi[\mathbf{A}] \sim e^{-\frac{g}{2} A^a_{\mu\nu} A^{a\mu\nu}} \) |
In other words, for expressions of the type \( e^{i \phi \delta x} \) can be given in different forms that are equivalent under the usual circumstances (i.e. all functions of \( x \) the right 'perturbative' scaling scheme. We have used the shorter form \( \Gamma^a(x) \) for \( \Gamma^a(x) \).

### 6. Generalized Projectors

In this appendix we want to give some notions on the generalized projectors as they are used in this paper. First let us note that the projection operators are thought of as bilocal objects, i.e. they depend on two spatial coordinates. Consider e.g. the (generalized) longitudinal projector \((\Pi_L)_{ij}^{ab}(x,y)\):

\[
(\Pi_L)_{ij}^{ab}(x,y).
\]

As is to be expected for a projector, \((\Pi_L)\) is idempotent (if we include an integration over the double continuous index as well):

\[
(\Pi_L)_{ij}^{ab}(x,y) = \int d^3 z \ (\Pi_L)_{ik}^{ac}(x,z)(\Pi_L)_{kj}^{cb}(z,y).
\]

The (generalized) longitudinal projector \((\Pi_L)\) is a symmetric operator in the sense that

\[
\int d^3 x \ d^3 y \ f_i^a(x)(\Pi_L)_{ij}^{ab}(x,y)g_j^b(y) = \int d^3 x \ d^3 y \ g_i^a(x)(\Pi_L)_{ij}^{ab}(x,y)f_j^b(y),
\]

where we have dropped boundary terms from two partial integrations. The latter property implies that it can be diagonalized, and from eq. \((A17)\) one can conclude that its eigenvalues are 0, 1. The generalized longitudinal projector can be given in different forms; for this it might be useful to recall that the ordinary longitudinal projector \(P^L_{ij}(x,y)\) also can be given in different forms that are equivalent under the usual circumstances (i.e. all functions of \( y \) are found to the right \(^{10}\) of \((P^L)_{ij}(x,y))

\[
(P^L)_{ij}(x,y) = -\left(\nabla_i^x G_\Delta(x,y)\right)\nabla_j^y
\]

\[
= \left(\nabla_j^y \nabla_i^x G_\Delta(x,y)\right)
\]

\[
= \int \frac{d^3 k}{(2\pi)^3} e^{ik(x-y)} \frac{k_i k_j}{k^2},
\]

where \(G_\Delta(x,y)\) denotes the Green’s function of the Laplace operator, \(-\nabla_i \nabla_j G_\Delta(x,y) = \delta_{xy}\). Under the given circumstances all three definitions are equivalent; however, only the first one really has to ‘act to the right’ in order to make sense. The (ordinary) transversal projector can also easily be defined as

\[
(P^T)_{ij}(x,y) = \delta_{ij} \delta_{xy} - (P^L)_{ij}(x,y).
\]

\(^{10}\) In other words, for expressions of the type \( f_i(x)(P^L)_{ij}(x,y)g_j(x,y) \) (with a sum or integral over all double indices) all these definitions are equivalent, otherwise they will usually be not.

---

**TABLE I:** Placement of the coupling constant \( g \) in the ‘non-perturbative’ and the ‘perturbative’ scaling scheme. We have used the shorter form \( \Gamma^a(x) \) for \( \Gamma^a(x) \).
For the generalized longitudinal projectors we can now give corresponding definitions:

\[
(\Pi_L)_{ij}^{ab}(x,y) = - (\hat{D}_i^{ab}(x)G_{\Delta}^{a_1 b_1}(x,y))\hat{D}_j^{b_1}(y) \tag{A23}
\]

\[
= (\hat{D}_j^{b_1}(y)\hat{D}_i^{ab}(x))G_{\Delta}^{a_1 b_1}(x,y) \tag{A24}
\]

\[
= \sum_n (\xi_n^a)_i(x)(\xi_n^b)_j(y) \tag{A25}
\]

where \(G_{\Delta}^{a_1 b_1}(x,y)\) is the Green’s function of the covariant Laplacian \((\hat{D}\hat{D})^{c_1 a_1}(x)\), i.e. \(-\hat{D}\hat{D})^{c_1 a_1}(x)G_{\Delta}^{a_1 b_1}(x,y) = \delta^{a_1 b_1}\delta_{xy}\). \((\xi_n^a)_i(x)\) are the normalized eigenfunctions of \((\Pi_L)_{ij}^{a_1 b_1}(x,y)\) with eigenvalue 1, and we do not integrate over \(x,y\). We assume that the covariant Laplacian has no eigenvalues zero. Here the corresponding generalized transversal projectors are given by

\[
(\Pi_T)_{ij}^{ab}(x,y) = \delta_{ij}\delta^{ab}\delta_{xy} - (\Pi_L)_{ij}^{a_1 b_1}(x,y). \tag{A26}
\]

**APPENDIX B: RESULTS FROM PERTURBATION THEORY**

In this appendix we want to address the following problem: assume that we start with a wave functional that has Gaussian form. Since this wave functional is not annihilated by the Gauss law operator \(\Gamma^a(x)\), we multiply it by a polynomial. How do we have to fix the coefficients of the polynomial s.t. Gauss’ law is satisfied to a given order in the coupling constant g? 

In order to proceed, we use the same procedure as in the main text: start out from 'non-perturbative scaling', and then replace \(A\) by \(\tilde{A} + ga\) so that the background field does not decouple in the perturbative limit. The covariant derivative can then be split into a part containing the background field (which is completely of \(O(g^0)\)) and a part that contains the fluctuation part and an explicit factor of g:

\[
\hat{D}_i^{ab}(x) = \hat{D}_i^{ab}(x) - gf^{acb}a_c^i(x) \quad \text{with} \quad \hat{D}_i^{ab}(x) = \nabla_i^{ab} - f^{acb}\tilde{A}_c^i(x). \tag{B1}
\]

Thus the Gauss law operator reads (cf. the discussion in sec. \(\sqrt{B}\))

\[
\Gamma^a(x) = \hat{D}_i^{ab}(x)\frac{\delta}{\delta a^i_b(x)} = \frac{1}{i}\hat{D}_i^{ab}(x)\frac{\delta}{\delta a^i_b(x)} - gf^{acb}a^i_c(x)\frac{1}{i}\frac{\delta}{\delta a^i_b(x)}. \tag{B2}
\]

The wave functional is of the form

\[
\Psi = P(a)e^{\frac{1}{2}\chi a + \bar{\chi}a}, \tag{B3}
\]

where we have used a (hopefully) obvious shorthand notation together with the abbreviation \(\chi = \frac{1}{2}G^{-1} - 2i\Sigma\), and \(P(a)\) indicates a power series in g:

\[
P(a) = \sum_{n=0}^{\infty} g^n P^{(n)}(a). \tag{B4}
\]

If we now require

\[
\Gamma^a(x)\Psi = 0 + O(g^{N+1}) \tag{B5}
\]

we obtain a recursion relation between the different \(P^{(n)}\)s:

\[
0 = \sum_{n=0}^{N} g^n \frac{\delta}{\delta a^i_b}((\chi a)_i^b + i\bar{\epsilon^i}_b) P^{(n)} - \sum_{n=1}^{N} g^n a^i_c f^{acb}((\chi a)_i^b + i\bar{\epsilon^i}_b) P^{(n-1)}
\]

\[+ \sum_{n=1}^{N} g^n \frac{\delta}{\delta a^i_b} P^{(n)} - \sum_{n=2}^{N} g^n a^i_c f^{acb} \frac{\delta P^{(n-1)}}{\delta a^i_b} = 0, \tag{B6}
\]
where it is sufficient to terminate the sums at \( n = N \) since we require Gauss’ law only up to \( \mathcal{O}(g^N) \). Note that the summation indices start at different lower values. Thus we can study the \( \mathcal{O}(g^0) \) and \( \mathcal{O}(g^1) \) cases separately\(^{11}\):

\[
\mathcal{O}(g^0): \quad \hat{D}^a_b((-\chi a)^b_k + i\xi^b_k) = 0. \tag{B7}
\]

Since this equation has to be valid for all \( a \), one actually obtains three conditions since \( \hat{D} \) does not mix real and imaginary parts:

\[
\hat{D}\chi = 0 \rightarrow \hat{D}^a_b(x)(G^{-1})_{ij}^{ab}(x,y) = 0 \quad \text{and} \quad \hat{D}^{ab}_i(x)\Sigma_{ij}^{ab}(x,y) = 0 \tag{B8}
\]

\[
\hat{D}\xi = 0, \quad \text{i.e.} \quad \hat{D}^a_b(x)\xi^b_k(x) = 0. \tag{B9}
\]

Note the similarity with the QED condition of having a purely transversal kernel.

Carrying this procedure further to higher orders in \( g \) will lead to certain difficulties, e.g. that \( \psi \) will be annihilated only if one takes \( a \) to stem from a subspace satisfying the background field condition \( \hat{D}a = 0 \), thereby invalidating the whole concept, since our states should be such that Gauss’ law is satisfied for all \( a \) without further gauge fixing. However, to \( \mathcal{O}(g^0) \), the concept works, and is compatible with the results from the Kalmlah expansion as seen in sec. \( \S \).

The main lesson from this section is that a pure Gaussian state can satisfy Gauss’ law only to order \( g^0 \). This is the reason why we can formulate an exactly gauge invariant Gaussian ground state for electrodynamics.

**APPENDIX C: DERIVATIONS**

This appendix contains some of the technical details of the different derivations.

1. Electrodynamics, Expansion to Second Order

   a. Generalities

   Here one starts from the expression (cf. eqs. (31, 32))

   \[
   \langle He^{i\int \phi(\Gamma-\rho)} \rangle = A_0\langle e^{i\int \phi(\Gamma-\rho)} \rangle + A_1(y)\{(\Gamma(y) - \langle \Gamma(y) \rangle)\} e^{i\int \phi(\Gamma-\rho)} + A_2(y,z)\{(\Gamma(y) - \langle \Gamma(y) \rangle)\{(\Gamma(z) - \langle \Gamma(z) \rangle)\} e^{i\int \phi(\Gamma-\rho)} \tag{C1}\n   \]

   The coefficients will be determined as follows: one needs three equations, which are obtained by

   (a) setting \( \phi = 0 \),

   (b) performing a functional derivative w.r.t. \( \phi(x_1) \) and setting \( \phi = 0 \) afterwards,

   (c) performing two functional derivatives, w.r.t. \( \phi(x_1), \phi(x_2) \) and setting \( \phi = 0 \) afterwards.

   After a bit of manipulation, one obtains

   \[
   A_0 = \langle H \rangle - A_2(y,z)\langle \Delta(y)\Delta(z) \rangle, \tag{C2}
   \]

   \[
   A_1(y) = \langle \langle H\Delta(x_1) \rangle - A_2(y_1,z_1)\langle \Delta(y_1)\Delta(z_1)\Delta(x_1) \rangle \rangle \Theta^{-1}(x_1,y) \tag{C3}
   \]

   and for \( A_2 \) one obtains an even less pleasant equation

   \[
   \langle H\Delta(x_1)\Delta(x_2) \rangle - \langle H \rangle \langle \Delta(x_1)\Delta(x_2) \rangle - \langle H\Delta(z)\Theta^{-1}(z,y)\Delta(y)\Delta(x_1)\Delta(x_2) \rangle
   = A_2(y,z)\left[ \langle \Delta(y)\Delta(z)\Delta(x_1)\Delta(x_2) \rangle - \langle \Delta(y)\Delta(z)\Delta(x_1)\Delta(x_2) \rangle \right]
   = \langle \Delta(y)\Delta(z)\Delta(x_1)\Delta(x_2) \rangle \Theta^{-1}(y_1,y_2)\langle \Delta(y_2)\Delta(x_1)\Delta(x_2) \rangle \tag{C4}
   \]

\(^{11}\) We set \( P^{(0)} = 1 \), since in the end it will be fixed by the overall normalization anyway; it has to be nonzero, s.t. the state does not vanish as \( g \rightarrow 0 \).
with $\Delta(x) = \Gamma(x) - \langle \Gamma(x) \rangle$ and $\Theta^{-1}$ is defined via $\Theta^{-1}(x,y) \langle \Delta(y) \Delta(z) \rangle = \delta_{xz}$. Eq. (C4) simplifies significantly if one exploits some of the factorization features of Gaussian states,

$$\langle \Delta(x) \Delta(y) \Delta(z) \rangle = 0$$
$$\langle \Delta(y) \Delta(z) \Delta(x_1) \Delta(x_2) \rangle = \langle \Delta(y) \Delta(z) \rangle \langle \Delta(x_1) \Delta(x_2) \rangle + \langle \Delta(x_1) \Delta(z) \rangle \langle \Delta(y) \Delta(x_2) \rangle + \langle \Delta(x_2) \Delta(z) \rangle \langle \Delta(x_1) \Delta(y) \rangle.$$  \hspace{1cm} (C5)

This allows to rewrite eq. (C4) as

$$\langle H \Delta(x_1) \Delta(x_2) \rangle - \langle H \rangle \langle \Delta(x_1) \Delta(x_2) \rangle = A_2(y, z) \langle (\Delta(y) \Delta(x_1)) \langle \Delta(z) \Delta(x_2) \rangle + \langle \Delta(y) \Delta(x_2) \rangle \langle \Delta(z) \Delta(x_1) \rangle \rangle.$$  \hspace{1cm} (C6)

In order to obtain $A_2$ explicitly, one can again use $\Theta^{-1}$:

$$\left( \langle \Delta(y) \Delta(x_1) \rangle \langle \Delta(z) \Delta(x_2) \rangle + \langle \Delta(y) \Delta(x_2) \rangle \langle \Delta(z) \Delta(x_1) \rangle \right) \frac{1}{2} \Theta^{-1}(x_1, y_1) \Theta^{-1}(x_2, y_2) = \frac{1}{2} (\delta_{yy_1} \delta_{xy_2} + \delta_{yy_2} \delta_{xy_1}),$$

(C8)

giving (since $A_2(y_1, y_2) = A_2(y_2, y_1)$)

$$A_2(y_1, y_2) = \left( \langle H \Delta(x_1) \Delta(x_2) \rangle - \langle H \rangle \langle \Delta(x_1) \Delta(x_2) \rangle \right) \frac{1}{2} \Theta^{-1}(x_1, y_1) \Theta^{-1}(x_2, y_2).$$

(C9)

This result leads directly to eq. (B5) in the main text.

\textit{b. Explicit Form of Energy Functional}

We will now turn to the evaluation of the energy. In addition to the terms we have computed in the context of the first order Kamlah expansion, we need to compute $\langle B_i(x) B_j(y) \Gamma(z_1) \Gamma(z_2) \rangle$. If we again - as in the first order calculation - take either $\Sigma = 0$ or $G^{-1}$ and $\Sigma$ translationally invariant, the matrix element factorizes,

$$\langle B_i(x) B_j(y) \Gamma(z_1) \Gamma(z_2) \rangle = \langle B_i(x) B_j(y) \rangle \langle \Gamma(z_1) \Gamma(z_2) \rangle$$

(C10)

which is again plausible, since there should be no corrections to the magnetic part of the energy as it depends on the transversal degrees of freedom only. The next ingredient we have to compute for $\langle \Delta H \Delta(x_1) \Delta(x_2) \rangle$ is the electric energy, which gives a rather simple contribution, too. We can evaluate it using the same factorization properties that underly eq. (C4), and obtain altogether:

$$\langle \Delta H \Delta(x_1) \Delta(x_2) \rangle = \langle \Gamma(x_2) \Pi_i(x) \rangle \langle \Gamma(x_1) \Pi_i(x) \rangle.$$  \hspace{1cm} (C11)

This expression enters the correction of the mean-field energy

$$\frac{1}{2} \langle \Delta H \Delta(x_1) \Delta(x_2) \rangle \Theta^{-1}(x_1, x_2) = \frac{1}{2} \langle P^L_{ji}(x_1, x_2) \Pi_j(x_2) \Pi_i(x_1) \rangle,$$

(C12)

where $P^L_{ij}(x_2, x_1)$ denotes the longitudinal projector as defined \(^{12}\) e.g. in eq. (A19); translational invariance, that has been assumed before in the first order calculation, implies that the electrical energy can be written as

$$\frac{1}{2} \langle \Pi_i(x) \Pi_i(x) \rangle = \frac{1}{2} \langle P^L_{ji}(y, x) \Pi_i(x) \Pi_j(y) \rangle + \frac{1}{2} \langle P^T_{ji}(y, x) \Pi_i(x) \Pi_j(y) \rangle$$

(C13)

and thus the second order Kamlah expansion precisely subtracts off the longitudinal (gauge variant) part of the kernel. The other term needed to compute $\langle HP^o \rangle / \langle P^p \rangle$ is even simpler,

$$\frac{1}{2} \langle \Delta H \Delta(x_1) \Delta(x_2) \rangle \Theta^{-1}(x_1, y_1) \Theta^{-1}(x_2, y_2) = G_{\Delta}(y_1, y_2).$$

(C14)

Similar to the first-order discussion, we subtract off the part $\langle \Pi^L_i \rangle \langle \Pi^L_j \rangle$ (thus completing $\langle P^L_{ji} \Pi_i \Pi_j \rangle_c$ to $\langle P^L_{ji} \Pi_i \Pi_j \rangle$) and add it back in afterwards. Thereby, we obtain eq. (B9).

\(^{12}\) In this context there is no problem with the other definitions eqs. (A20) (A21)
2. Yang-Mills Theories, Expansion to Second Order

a. Generalities

We follow the same procedure as it was outlined in the case of electrodynamics in app. C1. From eq. (73) we obtain by this procedure again three equations:

\[
\langle H \rangle = A_0 + A_2^{ab}(y,z)\langle \Delta^a(y)\Delta^b(z) \rangle,
\]

\[
\langle HG^c(x) \rangle = A_0\langle G^c(x) \rangle + A_1^a(y)\langle \Delta^a(y)G^c(x) \rangle + A_2^{ab}(y,z)\langle \Delta^a(y)\Delta^b(z)G^c(x) \rangle
\]

\[
\langle H \frac{1}{2}\{G^{c_1}(x_1),G^{c_2}(x_2)\} \rangle = A_0\left\langle \frac{1}{2}\{G^{c_1}(x_1),G^{c_2}(x_2)\}\right\rangle + A_1^a(y)\langle \Delta^a(y)\frac{1}{2}\{G^{c_1}(x_1),G^{c_2}(x_2)\}\rangle
\]

+ A_2^{ab}(y,z)\langle \Delta^a(y)\Delta^b(z)\frac{1}{2}\{G^{c_1}(x_1),G^{c_2}(x_2)\}\rangle
\]

where in the last equation, eq. (C17), we have used the abbreviation \{G^{c_1}(x_1),G^{c_2}(x_2)\} = G^{c_1}(x_1)G^{c_2}(x_2) + G^{c_2}(x_2)G^{c_1}(x_1). We then can - again - express A_0, A_1 in terms of A_2 and unprojected expectation values; the equation for A_2 is very similar to eq. (C4) in the electrodynamics calculation, but here it is even worse, since \[\Delta^a(x),\Delta^b(y)\] \neq 0. In the case of electrodynamics, significant progress could be made using the factorization features eqs. (C2, C3). In order to mimic the treatment presented there, and thus to make further progress, we have to resort to the perturbative expansion of eq. (73) as outlined in sec. 5.1.

b. Factorizations

For the quasi-Abelian form \(\hat{\Gamma}^a\) we again have the factorization properties of Gaussian states similar to eqs. (C5, C6):

\[
\langle \Delta^a(x)\Delta^b(y)\Delta^c(z) \rangle = 0,
\]

\[
\langle \Delta^a(y)\Delta^b(z)\Delta^{c_1}(x_1)\Delta^{c_2}(x_2) \rangle = \langle \Delta^a(y)\Delta^b(z)\langle \Delta^{c_1}(x_1)\Delta^{c_2}(x_2) \rangle \rangle
\]

+ \langle \Delta^{c_1}(x_1)\Delta^b(z)\langle \Delta^a(y)\Delta^{c_2}(x_2) \rangle \rangle
\]

+ \langle \Delta^{c_2}(x_2)\Delta^b(z)\langle \Delta^{c_1}(x_1)\Delta^a(y) \rangle \rangle
\]

where we have introduced the notation \(\Delta^a(x) = \hat{\Gamma}^a(x) - \langle \hat{\Gamma}^a(x) \rangle\).

c. Derivation

We now use the factorization properties to compute A_2 (or, if we hadn’t required A_2 to be symmetric from the outset, the symmetric part of A_2; but since the \(\Gamma\)’s commute anyway, the distinction does not play a role):

\[
A_2^{d_1,d_2}(y_1,y_2) = \frac{1}{2}\left[\langle H\Delta^{c_1}(x_1)\Delta^{c_2}(x_2) \rangle - \langle H \rangle\langle \Delta^{c_1}(x_1)\Delta^{c_2}(x_2) \rangle\right] \times
\]

\[
\langle \Theta^{-1}\rangle^{c_1,d_1}(x_1,y_1)\langle \Theta^{-1}\rangle^{c_2,d_2}(x_2,y_2).
\]

With this, we can give now explicitly the projected, normalized energy functional to second order in the Kamlah expansion (in absence of external charges):

\[
\frac{\langle HP \rangle}{\langle P \rangle}_{2^{\text{nd}}} = \langle H \rangle - \frac{1}{2}\langle \Delta H \Delta^{c_1}(x_1)\Delta^{c_2}(x_2) \rangle\langle \Theta^{-1}\rangle^{c_2,c_1}(x_2,x_1)
\]

\[
- \langle \Delta H \Delta^{c_1}(x_1)\langle \Theta^{-1}\rangle^{c_2,c_2}(x_1,x_2)\langle \Gamma^{c_2}(x_2) \rangle \rangle
\]

\[
+ \frac{1}{2}\langle \Delta H \Delta^{c_1}(x_1)\Delta^{c_2}(x_2) \rangle \langle \Theta^{-1}\rangle^{c_2,d_1}(x_1,y_1)\langle \Theta^{-1}\rangle^{c_2,d_2}(x_2,y_2)\langle \Gamma^{d_1}(y_1) \rangle\langle \Gamma^{d_2}(y_2) \rangle \rangle.
\]

As in the case of electrodynamics, we may here also insert \(\hat{\Gamma}\) into the Kamlah expansion expression instead of \(H\). This is due to the fact that \(\hat{\Gamma}^a\) is gauge-invariant w.r.t. gauge transformations carried out by \(e^{i\int x^a \Gamma^a}\). If we do this, the terms analogous to the second and the fourth term in eq. (C21) (with \(H\) replaced by \(\hat{\Gamma}^a\)) are zero, and the terms
with the (generalized) longitudinal projector

\[ \langle \Delta (\Pi_x^a(x)\Pi_x^b(x)) \Delta^{c_1} (x_1) \rangle = 2 \langle \Pi_x^a(x) \Delta^{c_1} (x_1) \rangle \langle \Pi_x^b(x) \rangle, \tag{C22} \]

\[ \langle \Delta (\Pi_x^a(x)\Pi_x^b(x)) \Delta^{c_2} (x_2) \rangle = 2 \langle \Pi_x^a(x) \Delta^{c_2} (x_2) \rangle \langle \Pi_x^b(x) \rangle, \tag{C23} \]

\[ \langle \Delta (\mathcal{B}_x^a(x)\mathcal{B}_x^b(x)) \Delta^{c_1} (x_1) \rangle = 0, \tag{C24} \]

\[ \langle \mathcal{B}_x^a(x)\mathcal{B}_x^b(x)\Gamma^{c_1} (x_1) \Gamma^{c_2} (x_2) \rangle = -\frac{1}{2} \int d^3x \hat{D}_{i_1i_2 d_1}(x_1) \hat{D}_{i_2i_3 d_2}(x_2) M^{d_1 d_2}_{i_1 i_2}(x_1, x_2) = 0, \tag{C25} \]

where we have set \( \Sigma \) to zero here and in the remaining computations of this appendix. Also the expression

\[ \int d^3x \hat{D}_{i_1 i_2}^{c_1 d_1}(x_1) \hat{D}_{i_2 i_3}^{c_2 d_2}(x_2) M^{d_1 d_2}_{i_1 i_2}(x_1, x_2) \]

requires a bit of special attention: there is no integration over \( x_1, x_2 \) although they appear more than once, but we have to integrate over \( x \), although it seems to appear only once (but this is an artifact of the notation introduced in eq. \( \text{(77)} \)). The last identity eq. \( \text{(C22)} \) is true since we had to restrict the allowed background fields to fulfill the classical equations as discussed above. In addition, we need a simple expression for \( \Theta^{-1} \): in the electrodynamics calculation we saw that we had to assume translational invariance of \( G, \Sigma \). Here we'll need something similar, namely we require

\[ (G^{-1})_{ij}^{ab}(x, y) \overset{!}{=} \{ \Pi_L \}^{ac}_{ii} (x, x_1) (G^{-1})^{c_1 c_2}_{ii} (x_1, x_2) (\Pi_L)^{c_2 b}_{ij} (x_2, y) \]

\[ = (G_{LL})^{ab}_{ij}(x, y) + (G_T^{ab})_{ij}(x, y) \tag{C26} \]

with the (generalized) longitudinal projector\(^{13}\)

\[ (\Pi_L)^{ab}_{ij}(x, y) = -\hat{D}_x^{ac}(x) G_x^{cd}(x, y) \hat{D}_y^{db}(y), \quad \text{where} \quad -\hat{D}_x^{ac}(x) G_x^{db}(x, y) = \delta^{ab} \delta_{xy}, \tag{C27} \]

(no integration over \( x, y \)) and the (generalized) transversal projector \( (\Pi_T)^{ij}_{ab}(x, y) = \delta^{ab} \delta_{ij} \delta_{xy} - (\Pi_L)^{ij}_{ab}(x, y) \). Thus, we don't want to allow for non-zero cross terms \( \Pi_L G^{-1} \Pi_T, \Pi_T G^{-1} \Pi_L \). For the explicit construction of \( \Theta^{-1} \) this has the advantage that one can invert the longitudinal and transversal parts individually:

\[ (G_{LL}^{-1})_{ij}^{ab}(x, y) (G_{LL})^{bc}_{ij} (y, z) = (\Pi_L)^{ac}_{ij} (x, z) \tag{C28} \]

\[ \text{and} \quad (G_T^{-1})_{ij}^{ab}(x, y) (G_T)^{bc}_{ij} (y, z) = (\Pi_T)^{ac}_{ij} (x, z), \tag{C29} \]

We now want to relate the inverse of \( \hat{D}_i \hat{D}_j (G^{-1}) \) to \( \hat{D}_i \hat{D}_j G \): we start with the observation that, since \( G_{LL}^{-1} = \Pi_L G^{-1} \Pi_L \), we can write (no integration over \( x, z \))

\[ (G_{LL}^{-1})_{ij}^{ab}(x, z) = \hat{D}_i^{ac}(x) \hat{D}^{bc}_{j}(z) (H_L)^{c_1 c_2} (x, z) \tag{C30} \]

and correspondingly (again no integration over \( x, z \))

\[ (G_{LL})_{ij}^{ab}(x, z) = \hat{D}_i^{ac}(x) \hat{D}^{bc}_{j}(z) (K_L)^{c_1 c_2} (x, z), \tag{C31} \]

where \( H_L, K_L \) are auxiliary functions that have the important characteristic that they don't carry spatial indices. From the requirement \( \int d^3y \ (G_{LL}^{-1})_{ij}^{ab}(x, y) (G_{LL})_{jk}^{bc}(y, z) = (\Pi_L)^{ac}_{ij} (x, z) \) we obtain

\[ G_{ac}^{ij}(x, y) = \int d^3z \ (\hat{D}_i \hat{D}_j)^{b_1 b_2} (z) (H_L)^{a_1 b_1} (x, z) (K_L)^{b_2 c_1} (z, y). \tag{C32} \]

\(^{13}\) Some properties of generalized projectors are given in appendix \( X \).
If on the other hand we start from the defining equation of $\Theta^{-1}$

$$
\int d^3y \left( \frac{1}{4} \hat{D}_i^{ab}(x) \hat{D}_j^{bc}(y)(G^{-1})^{b_1b_2}_{ij}(x,y) \right) (\Theta^{-1})^{bc}(y,z) = \delta^{ac}\delta_{xz}
$$

we end up with

$$
4G^4_{\Delta^{ac}}(x,y) = \int d^3z \left( (\hat{D} \hat{D})^{b_2b_1}(z)(H_L)^{a_1b_1}(x,z) \right) (\Theta^{-1})^{b_2c_1}(z,y).
$$

From the comparison of eq. (C34) with eq. (C32), we obtain

$$
(\Theta^{-1})^{b_2c_1}(z,y) = 4(K_L)^{b_2c_1}(z,y)
$$

and thus conclude

$$
\hat{D}_i^{b_2}(x) \hat{D}_j^{b_1}(y)(\Theta^{-1})^{b_1b_2}(x,y) = 4(G_{LL})^{bc}_{ij}(x,y),
$$

where again we don’t integrate over $x,y$. We can now compute the correction terms to the Kamlah expansion very easily, using the fact that e.g. $\langle \Pi_i^{(x)}(x) \Delta^{(y)}(y) \rangle = \hat{D}_j^{bc}(y)(G^{-1})^{ac}_{ij}(x,y)$:

1. $\langle (\Pi_i^{(x)}(x) \Pi_i^{(y)}(y) \Delta^{c_1}(x_1) \Delta^{c_2}(x_2) ) (\Theta^{-1})^{c_1c_2}(x_1, x_2) \rangle = 2 \langle (\Pi_i^{(x)}(x_2, x_1) \Pi_i^{(y)}(x_1) \Pi_i^{(y)}(x_2) ) \rangle$

2. $\langle (\Pi_i^{(x)}(x) \Pi_i^{(y)}(y) \Delta^{c_1}(x_1)) (\Theta^{-1})^{c_1c_2}(x_1, x_2) \rangle = 2 \langle (\Pi_i^{(x)}(x_2, x_1) \Pi_i^{(x)}(x_1) \Pi_i^{(y)}(x_2) ) \rangle$

3. $\langle (\Pi_i^{(x)}(x) \Pi_i^{(y)}(y) \Delta^{c_1}(x_1) \Delta^{c_2}(x_2) ) (\Theta^{-1})^{c_1c_2d_1}(x_1, x_2, y_1) \rangle = 2 \langle (\Pi_i^{(x)}(x_2, x_1) \Pi_i^{(x)}(x_1) \Pi_i^{(y)}(y_1) ) \rangle$

Inserting these results into eq. (C21), we obtain the results given by eqs. (80-82) in the main text.

[1] A. K. Kerman and D. Vautherin, Annals Phys. 192, 408 (1989).
[2] I. I. Kogan and A. Kovner, Phys. Rev. D 52, 3719 (1995), hep-th/9408081.
[3] D. Diakonov (1998), hep-th/9805137.
[4] C. Heinemann, C. Martin, D. Vautherin, and E. Iancu (1998), hep-th/9802036.
[5] C. Heinemann, E. Iancu, C. Martin, and D. Vautherin, Phys. Rev. D 61, 116008 (2000), hep-ph/9911515.
[6] P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer Verlag, Berlin-Heidelberg, 1980), 716 p.
[7] A. Kamlah, Z. Phys. B 216, 52 (1968).
[8] J. P. Blaizot and G. Ripka, Quantum Theory of Finite Systems (The MIT Press, Cambridge, Mass., 1986), 657 p.
[9] A. M. Polyakov, Gauge Fields And Strings (Harwood, Chur, Switzerland, 1987), (Contemporary Concepts in Physics, 3), 301 p.
[10] K. Zarembo (1998), hep-th/9808189.
[11] K. Zarembo, Mod. Phys. Lett. A 13, 2317 (1998), hep-th/9806150.
[12] P. E. Haagensen and K. Johnson (1997), hep-th/9702204.
[13] O. Schröder, Ph.D. thesis, Eberhard-Karls Universität Tübingen (2002), http://w210.ub.uni-tuebingen.de/dbt/volltexte/2002/483.
[14] O. Schröder and H. Reinhardt (2002), in preparation.
[15] F. Villars, Nucl. Phys. 74, 353 (1965).
[16] F. Villars, in Many-Body Description of Nuclear Structure and Reactions (Academic Press, 1966), vol. XXXVI of Proceedings of the International School of Physics Enrico Fermi.
[17] E. Marshak and J. Weneser, Annals Phys. 53, 569 (1969).
[18] E. Marshak and J. Weneser, Phys. Rev. C 2, 1683 (1970).
[19] R. Wolf, Phys. Rev. A 18, 325 (1978).
[20] R. Soares, Nucl. Phys. A 281, 475 (1977).
[21] A. Kerman and N. Onishi, Nucl. Phys. A 281, 373 (1977).
[22] L. S. Brown and W. I. Weisberger, Nucl. Phys. B 157, 285 (1979).
[23] S. Huang and A. R. Levi, Phys. Rev. D 49, 6849 (1994), hep-lat/9312069.
[24] S. Levit, Z. Phys. A 353, 213 (1995), hep-th/9305056.
[25] N. H. Christ and T. D. Lee, Phys. Rev. D 22, 939 (1980).
[26] R. Jackiw, Rev. Mod. Phys. 52, 661 (1980).
[27] R. Jackiw, *Functional representations for quantized fields* (1987), 1st Asia Pacific Workshop on High Energy Physics, Singapore, Jun 1987 and 6th Symp. on Theoretical Physics, Seoul, Korea, Jul 1987 and Mathematical Quantum Field Theory, Montreal, Canada, Sep 1987.

[28] R. Jackiw, *Analysis on infinite dimensional manifolds: Schroedinger representation for quantized fields* (1988), presented at Seminar on Higher Mathematics, Montreal, Canada, Jun 1988.

[29] P. Cea, Phys. Rev. **D37**, 1637 (1988).