Stroboscopic Symmetry-Protected Topological Phases

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Symmetry-protected topological (SPT) phases of matter have been the focus of many recent theoretical investigations, but controlled mechanisms for engineering them have so far been elusive. In this work, we demonstrate that by driving interacting spin systems periodically in time and tuning the available parameters, one can realize lattice models for bosonic SPT phases in the limit where the driving frequency is large. We provide concrete examples of this construction in one and two dimensions, and discuss signatures of these phases in stroboscopic measurements of local observables.

Since the discovery of the quantum Hall effect (QHE) [1], topological phenomena in quantum many-body systems have dramatically changed our understanding of phases of matter. In particular, the study of the fractional QHE brought about the notion of topological order [2–4], which characterizes phases of matter with emergent fractional excitations and topological ground-state degeneracy, which cannot be described within the standard Landau-Ginzburg framework.

In recent years, the prediction and discovery of topological band insulators [5, 6] has awakened a great deal of interest in gapped symmetry-protected topological (SPT) phases of matter. These phases of matter lack fractionalized degrees of freedom, but display topological properties that manifest themselves in non-trivial boundary states that are protected by global symmetries. While they do not display the long-range entanglement of topologically-ordered systems, SPT phases of matter are characterized primarily by a nontrivial short-range entanglement structure in the low-energy states [7].

Following the classification of weakly-interacting fermionic SPT states [8–10], there has been a vast amount of recent effort to classify strongly-interacting SPT phases [7, 11–15] as well as to construct models supporting them [7, 16–25]. In light of this effort, it is highly desirable to identify controlled mechanisms capable of bringing SPT states into realization.

In this Letter, we put forward a proposal to realize bosonic SPT phases as out-of-equilibrium states of quantum spin systems with periodically-driven multispin interactions. The systems we study are described by time-dependent Hamiltonians of the form

\[ H(t) = H_0 + \Theta(t) f(t) H_{\text{int}}, \]  

where \( H_0 \) is a local Hamiltonian describing a trivial paramagnet (i.e., one whose ground state is a trivial product state) and \( H_{\text{int}} \) is a local interaction with a time-periodic coupling constant \( f(t) = f(t + T) \) with zero mean and a characteristic frequency \( \omega = 2\pi/T \). \( \Theta(t) \) is the Heaviside function denoting a protocol where the drive is switched on at \( t = 0 \).

When \( H_{\text{int}} = 0 \), \( H(t) = H_0 \) can be mapped from a trivial paramagnetic Hamiltonian to an SPT Hamiltonian by a product of local unitary transformations that entangles the local degrees of freedom in a nontrivial way [7, 21]. Such transformations arise naturally in the study of many-body systems with periodically-driven interactions. In particular, we will show that, in the limit of large \( \omega \), the time-dependent unitary transformation to the “rotating frame,” (we set \( \hbar = 1 \))

\[ U_R(t) = e^{i \int_0^t dt' f(t')} H_{\text{int}} \equiv e^{i \Theta(t) H_{\text{int}}}, \]  

generates the desired entanglement if \( H_{\text{int}} \) is chosen appropriately. The transformation \( U_R(t) \) maps a state \( |\psi(t)\rangle \), whose time evolution is governed by the Hamiltonian (1), into a state \( |\psi_R(t)\rangle = U_R(t) |\psi(t)\rangle \) whose time evolution is generated by

\[ H_R(t) = U_R(t) H(t) U_R^\dagger(t) - i U_R(t) \partial_t U_R^\dagger(t), \]  

\[ = U_R(t) H_0 U_R^\dagger(t). \]  

The stroboscopic evolution of the initial state in the rotating frame, \( |\psi_R(nT)\rangle = e^{-i \mathcal{H}_F nT} |\psi_R(0)\rangle \) \( (n \in \mathbb{Z}) \), is governed by the Floquet Hamiltonian \( \mathcal{H}_F \), which can be systematically determined via a Magnus expansion [26, 27]. In the infinite-frequency limit, the Floquet Hamiltonian is nothing but the time-average of the Hamiltonian (3),

\[ \mathcal{H}_F^{(0)} = \frac{1}{T} \int_0^T dt \mathcal{H}_R(t), \]  

while the \( n \)-th order term in the Magnus expansion is of order \( 1/\omega^n \). We will refer to \( \mathcal{H}_F^{(0)} \) as the stroboscopic Hamiltonian, because in the infinite-frequency limit, where only \( \mathcal{H}_F^{(0)} \) survives, the stroboscopic evolution and the true unitary evolution of the time-dependent system coincide.
If the amplitude of the drive is small compared to the frequency, the stroboscopic Hamiltonian (4) simply reduces to $H_0$. On the other hand, when the amplitude of the drive is chosen to scale with the frequency $\omega$, the stroboscopic Hamiltonian can acquire a nontrivial form that is different from $H_0$ [27]. In this work, we show that the stroboscopic Hamiltonian (4) describes microscopic models of SPT states with $\mathbb{Z}_2 \times \mathbb{Z}_2$ [7, 20, 21] and $\mathbb{Z}_2$ [7, 17] symmetries, respectively, for one- and two-dimensional driven systems. We refer to the phases generated in this way as stroboscopic SPT (SSPT) phases. Remarkably, we find that, while the SSPT Hamiltonian (4) is not invariant under the global symmetry, the original time-dependent Hamiltonian (1) is not. Hence the global symmetry of the SSPT phase is found to be an emergent property of the high-frequency limit of $H_F$. These results can be generalized to other symmetry classes.

We also demonstrate that dynamical properties of the driven system at stroboscopic times capture physical properties of the ground state of the corresponding equilibrium SPT system, which are encoded in its entanglement structure. For the one-dimensional system studied in Eq. (5a), where the SSPT Hamiltonian possesses an emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, we find a regime in which the time-dependent expectation value of the edge spins remains constant at stroboscopic times, which, in this dynamical setting, is analogous to the protected edge modes in the ground state of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT Hamiltonian.

We start by studying an open 1D chain with $N$ sites described by the time-dependent Hamiltonian (1) with

$$H_{1D}(t) = \hbar \sum_{i=1}^{N} \sigma^x_i + \Theta(t) f(t) \sum_{i=1}^{N-1} \sigma^z_i \sigma^z_{i+1}, \quad (5a)$$

and

$$f(t) = \lambda \omega \cos(\omega t + \varphi), \quad \lambda > 0. \quad (5b)$$

Note that the driving amplitude is taken to scale linearly with the frequency, so that $\lambda$ is dimensionless. The Pauli operators $\sigma^a_i$ ($a = x, y, z$) satisfy the onsite algebra $[\sigma^a_i, \sigma^b_j] = 2i \delta_{ij} \epsilon_{abc} \sigma^c_i$ and the anticommutation relation $\{\sigma^a_i, \sigma^b_j\} = 2 \delta_{ab}$. Furthermore, notice that the Hamiltonian (5a) has an onsite $\mathbb{Z}_2$ spin flip symmetry generated by $S = \prod_{i=1}^{N} \sigma^z_i$. Henceforth, we set the energy scale $\hbar = 1$, with the understanding that the limit $\omega \to \infty$ corresponds to taking $\omega \gg \hbar$.

Upon making the transformation to the rotating frame, we find that, for $\varphi = 0$, the stroboscopic Hamiltonian (4) is given by

$$H_{\mathbb{Z}_2 \times \mathbb{Z}_2} = J_0(2\lambda) (\sigma^x_1 + \sigma^x_N) + \sum_{i=2}^{N-1} [a(\lambda) \sigma^x_i - b(\lambda) \sigma^z_{i-1} \sigma^z_{i+1}], \quad (6)$$

where $a(\lambda) = \frac{1}{2} [1 + J_0(4\lambda)]$, $b(\lambda) = 1 - a(\lambda)$ and $J_0(x)$ is the Bessel function of the first kind.

Observe that the Hamiltonian (6) possesses a global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $S_{\text{even}} = \prod_{i \text{ even}} \sigma^x_i$ and $S_{\text{odd}} = \prod_{i \text{ odd}} \sigma^x_i$, corresponding to independent spin flips on the even and odd sublattices. However, the time-dependent Hamiltonian (5a) has a $\mathbb{Z}_2$ symmetry, rather than a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry – in other words, this enlarged symmetry group is an emergent property of the high-frequency limit $\omega \to \infty$, as it appears only upon taking the time average Eq. (4).

We plot the couplings $a(\lambda)$, $b(\lambda)$, and $J_0(2\lambda)$ in Fig. 1. Observe that by varying $\lambda$, one can tune the couplings such that $a(\lambda) > b(\lambda)$ or vice versa. We will argue below that the values of $\lambda$ for which $a(\lambda) = b(\lambda)$ are critical points of the effective Hamiltonian that separate a trivial insulating phase from an SPT phase.

To continue our analysis of this $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric Hamiltonian, it is instructive to rewrite it in terms of Majorana operators [28, 29]

$$\alpha_i = \left( \prod_{j<i} \sigma^z_j \right) \sigma^x_i, \quad (7a)$$

$$\beta_i = i \alpha_i \sigma^x_i = i \left( \prod_{j<i} \sigma^z_j \right) \sigma^z_i \sigma^x_i, \quad (7b)$$

which are Hermitian and satisfy the usual fermionic algebra. In terms of these operators, the Hamiltonian (6)
FIG. 2: (Color online) Competing dimerization patterns in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT chain. Dotted and solid lines account, respectively, for the dominant dimerization patterns in the trivial and SPT phases. When only the 3-spin term is present in the model, a dangling spin is localized on the edges.

The Hamiltonian above contains two types of terms that can be thought of as projectors onto two distinct dimerization patterns that encode the entanglement structure of the ground-state wavefunction (see Fig. 2). The pattern encoded by the $\alpha_i \beta_i$ terms involves Majorana dimers on each site. It is “trivial” in the sense that, for a finite chain, the pattern pairs all Majorana operators. On the other hand, the $\beta_{i-1} \alpha_{i+1}$ terms encode dimerization between next-neighbor Majoranas of opposite types. This pattern is “nontrivial” in the sense that it leaves two unpaired Majoranas at each end of a finite chain, yielding a fourfold ground-state degeneracy that it leaves two unpaired Majoranas at each end of a finite chain, as we have verified by exact diagonalization.

The preceding discussion illustrates that the stroboscopic Hamiltonian (6) is $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric and contains one free parameter, $\lambda$, that tunes the system across the transition between the $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSPT phase and the trivial paramagnetic phase. However, it is evident from Fig. 1 that the coupling $a(\lambda) \neq 0$ for any $\lambda$. Naively, then, it seems that one cannot access the “ideal” case where the fourfold ground-state degeneracy predicted by Eqs. (6) and (8) is exact. Nevertheless, this is not the case, as the local field $J_0(2\lambda)$ at the ends of the chain vanishes identically if $2\lambda$ is equal to a zero of the Bessel function $J_0$ (see Fig. 1). In this case, the operators $\sigma_z^i$ and $\sigma_x^i$ identically commute with the Hamiltonian (6), and the system has an exact fourfold ground-state degeneracy despite the presence of a transverse field in the bulk. Deviations from these special values of $\lambda$ lift the degeneracy, but the system remains in the SSPT phase so long as the bulk gap remains open.

So far, we have demonstrated that, in the limit $\omega \to \infty$, the stroboscopic evolution of the periodically-driven spin chain of Eq. (5a) is generated by the effective Hamiltonian (6), for an appropriately-chosen driving protocol. However, it remains to be shown that the stroboscopic evolution generated by Eq. (6) has telltale signatures in local measurements. To address this point, we consider the local spin expectation value

$$\langle \sigma_z^i(t) \rangle = \langle \Psi_0 | e^{+i\mathcal{H}_{Z_2 \times Z_2}t} \sigma_z^i e^{-i\mathcal{H}_{Z_2 \times Z_2}t} | \Psi_0 \rangle,$$  

(10)

where $| \Psi_0 \rangle$ is some initial state. This quantity coincides with the true time evolution of the operator $\sigma_z^i$ in the limit $\omega \to \infty$, when the period $T$ is infinitesimally small and $t = nT$ ($n \in \mathbb{Z}$) is approximately a continuous variable. For simplicity, we choose the initial state $| \Psi_0 \rangle$ to be a tensor product of eigenstates of $\sigma_j^z$ on each site $j$, so that $\langle \sigma_z^j(t) \rangle$ is invariant under the unitary transformation $U_R(t)$ that maps the problem into the rotating frame.

The observable defined in Eq. (10) provides a clear signature of the degenerate states in the SSPT phase. When $J_0(2\lambda) = 0$, $\langle \sigma_z^i(t) \rangle$ and $\langle \sigma_x^i(t) \rangle$ are independent of time, since $\sigma_z^i$ and $\sigma_x^i$ commute with the effective Hamiltonian $\mathcal{H}_{Z_2 \times Z_2}$. For $i \neq j$, however, $\langle \sigma_z^j(t) \rangle$ evolves quasi-periodically in time, with oscillations occurring on a timescale $\tau$, on the order of the inverse bulk energy gap of $\mathcal{H}_{Z_2 \times Z_2}$ (see Fig. 3). If $\lambda$ is tuned away from one of these special values but remains within the phase boundary, then $\langle \sigma_z^i(t) \rangle$ and $\langle \sigma_x^i(t) \rangle$ acquire os-
FIG. 3: (Color online) Stroboscopic evolution of $\langle \sigma_i^z(t) \rangle$ for $i = 1, 2, \text{ and } 5$, where the initial state is chosen to be the product state $|\Psi_0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$ and $2\lambda$ equals the first zero of the Bessel function $J_0(x)$ in Fig. 1.

cilliatory dynamics on timescales much longer than $\tau_c$, so that the bulk and boundary behavior can be distinguished.

We now turn to a discussion of finite-frequency corrections to the SSPT phase. As the driving frequency $\omega$ is decreased, two effects occur that lead to deviations from the infinite-frequency case discussed here. First, the stroboscopic evolution of observables, as in Eq. (10), no longer coincides with the true time evolution of the system. In particular, expectation values of observables become dressed by intra-period effects that become significant if the system is not observed at stroboscopic times $t_n = nT$ for $n \in \mathbb{Z}$ [27]. However, the expectation values of observables at stroboscopic times is still predicted by the unitary evolution generated by the Floquet Hamiltonian $\mathcal{H}_F$. This brings us to the second effect, namely the fact that $\mathcal{H}_F$ acquires finite-frequency corrections appearing at orders $1/\omega$ and higher in the Magnus expansion. These corrections generically break the symmetry that protects the SSPT phase. Indeed, in the Supplemental Material we present the finite-frequency corrections to the infinite-frequency Hamiltonian $\mathcal{H}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ that break the emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry down to the $\mathbb{Z}_2$ symmetry of the original time-dependent Hamiltonian (5a). As a result, when finite-frequency corrections to $\mathcal{H}_F$ become significant, the critical point separating the SPT phase from the trivial paramagnetic phase can be avoided by choosing appropriate paths in parameter space. However, as long as $\omega$ is much larger than any other natural energy scale in the problem, the SSPT phase is distinct from the trivial phase.

The rationale behind the ability to engineer the SSPT Hamiltonian (6) can be stated as follows. First, recall that, when the driving vanishes, the mapping from the trivial phase to the SPT one can be achieved via a product of local unitary transformations [7]. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case, the generator of this transformation is proportional to the Ising interaction in Eq. (5a) [21]. On the other hand, in the driven system, the unitary transformation $U_R(t)$ to the rotating frame is also generated by the Ising interaction. Consequently, at infinite frequency, we found parameter regimes in which this transformation effectively mapped a trivial paramagnet to an SPT one, and gave rise to an emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry that is not shared by the time-dependent Hamiltonian (5a).

The above discussion gives us a principle for obtaining an SSPT phase from a driven system: the interaction term in Eq. (1) should be chosen to be the generator of the unitary transformation connecting a trivial to an SPT system. In order to demonstrate that this stroboscopic approach to SPT phases holds beyond the 1D case discussed above, we consider a 2D system on the triangular lattice with a driven three-spin interaction,

$$H_{2D}(t) = -\hbar \sum_j \sigma_j^y + \Theta(t) f(t) \sum_{\langle ijk \rangle} \sigma_i^z \sigma_j^x \sigma_k^y,$$

(11)

where the summation in the second term runs over all the triangles of the lattice and we assume the same form Eq. (5b) for $f(t)$. Interestingly, for $\lambda \approx 0.51$ and $\varphi \approx \pm 0.27\pi$, we find that (see Supplemental Material)

$$\mathcal{H}_F^{(0)} \approx h_{\text{eff}} \sum_j \sigma_j^z e^{i \frac{\pi}{2} \sum_{\ell \in \langle j \rangle} \left(1 - \sigma_\ell^z \sigma_\ell^{\prime z}\right)}.$$

(12)

Remarkably, the Hamiltonian (12), which involves up to seven-spin interactions, is the exactly-solvable model of a $\mathbb{Z}_2$ SPT paramagnet studied by Levin and Gu in Ref. [17]. The model (12) can be obtained from a trivial paramagnet by a product of local unitary transformations that each depend on three $\sigma^z$ spins (see Supplemental Material), which then justifies the need for a three-spin interaction in (11).

In conclusion, we have shown in this Letter that by adding appropriately chosen periodically-driven multi-spin interactions to a trivial paramagnetic Hamiltonian, it is possible to realize SPT phases in the high-frequency limit. We further illustrated via a 1D example that the SPT phase can be probed with stroboscopic measurements of local observables, and pointed out that this construction can be extended to higher dimensions and different symmetry classes, such as the above example of the $\mathbb{Z}_2$ SPT phase in 2D. Recent developments in quantum simulation with trapped ions [30–32] and superconducting quantum circuits [33, 34] have shown that it is possible to engineer tunable multi-spin interactions and transverse fields in a laboratory setting. These developments suggest the possibility that the SSPT phase could be realized in an experiment, if the appropriate sinusoidal drive can be implemented.
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Supplemental Material: Stroboscopic Symmetry-Protected Topological Phases

Appendix A: One-Dimensional SSPT Hamiltonian

Stroboscopic Hamiltonian

We derive an effective Hamiltonian that encapsulates the stroboscopic dynamics generated by

\[ H_{1D}(t) = \hbar \sum_{i=1}^{N} \sigma_i^x + \Theta(t) f(t) \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^z, \]  

(A1a)

with

\[ f(t) = \lambda \omega \cos(\omega t + \varphi), \quad \lambda > 0. \]  

(A1b)
To do this, we employ the time-dependent unitary transformation

\[ U_R(t) = \exp \left[ i \int_{-\infty}^{t} dt' \Theta(t) f(t) \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^z \right] = \exp \left[ i g(t) \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^z \right], \tag{A2} \]

where \( g(t) = \lambda [\sin(\omega t + \varphi) - \sin \varphi] \), which transforms the Hamiltonian to the rotating frame as follows:

\[ H_R(t) = U_R(t) H_{1D}(t) U_R^\dagger(t) - i U_R(t) \partial_t U_R(t) = \hbar U_R(t) \left( \sum_{i=1}^{N} \sigma_i^z \right) U_R^\dagger(t). \tag{A3} \]

Explicitly we find

\[ H_R(t) = \sum_{i=2}^{N-1} \left\{ \cos^2(2g(t)) \sigma_i^x - \sin^2(2g(t)) \sigma_{i-1}^x \sigma_{i+1}^x - \cos(2g(t)) \sin(2g(t)) \left( \sigma_i^y \sigma_{i+1}^x + \sigma_{i-1}^y \right) \right\} \]

\[ + \cos(2g(t)) \left( \sigma_1^x + \sigma_N^x \right) - \sin(2g(t)) \left( \sigma_1^y \sigma_2^x + \sigma_{N-1}^y \sigma_N^x \right). \tag{A4} \]

The time average of Eq. (A4) yields

\[ H^{(0)}_\varphi = \sum_{i=2}^{N-1} \left\{ a(\lambda, \varphi) \sigma_i^x - b(\lambda, \varphi) \sigma_{i-1}^x \sigma_i^x \right\} \]

\[ + d(\lambda, \varphi) \left( \sigma_1^x + \sigma_N^x \right) - e(\lambda, \varphi) \left( \sigma_1^y \sigma_2^x + \sigma_{N-1}^y \sigma_N^x \right), \tag{A5} \]

where the coefficients \( a(\lambda, \varphi), ..., e(\lambda, \varphi) \) are given by

\[ a(\lambda, \varphi) = 1 - b(\lambda, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} d\tau \cos^2 \left[ 2\lambda G_\varphi(\tau) \right], \tag{A6a} \]

\[ c(\lambda, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} d\tau \frac{1}{2} \sin \left[ 4\lambda G_\varphi(\tau) \right], \tag{A6b} \]

\[ d(\lambda, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} d\tau \cos \left[ 2\lambda G_\varphi(\tau) \right], \tag{A6c} \]

\[ e(\lambda, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} d\tau \sin \left[ 2\lambda G_\varphi(\tau) \right], \tag{A6d} \]

where \( G_\varphi(\tau) = \sin(\tau + \varphi) - \sin \varphi \). Observe that for the choice \( \varphi = 0 \), the two-body terms that break the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry vanish and we recover the stroboscopic Hamiltonian in Eq. (6) of the main text.

**Leading Finite-Frequency Correction to the SSPT Hamiltonian**

We now present the order-1/\( \omega \) Magnus correction to the SSPT Hamiltonian, namely

\[ H_F^{(1)} = -\frac{1}{2T} \int_0^T dt_1 \int_0^{t_1} dt_2 \{ H_R(t_1), H_R(t_2) \}, \tag{A7} \]

where \( H_R(t) \) is given by Eq. (3). We will work exclusively with an infinite chain in this section, as our aim is only to show that the bulk \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry is broken by this correction. After calculating the necessary commutators, we find that

\[ H_F^{(1)} = \hbar_1 \sum_i (\sigma_i^x \sigma_{i+1}^z - \sigma_i^y \sigma_{i+1}^y) + \hbar_2 \sum_i (\sigma_{i-1}^x \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^z - \sigma_{i-1}^y \sigma_i^z), \tag{A8a} \]
where the coefficients are given by

\begin{align}
    h_1 &= -\frac{1}{\pi \omega} \int_0^{2\pi} \int_0^{\tau_1} d\tau_1 d\tau_2 \cos(2\lambda \sin \tau_1) \cos(2\lambda \sin \tau_2) \sin[2\lambda (\sin \tau_2 - \sin \tau_1)] \\
    h_2 &= \frac{1}{\pi \omega} \int_0^{2\pi} \int_0^{\tau_1} d\tau_1 d\tau_2 \sin(2\lambda \sin \tau_1) \sin(2\lambda \sin \tau_2) \sin[2\lambda (\sin \tau_2 - \sin \tau_1)].
\end{align}

Observe that each term above breaks the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the zeroth-order Hamiltonian is therefore an emergent symmetry that appears only at high frequencies.

**Appendix B: Two-Dimensional SSPT Hamiltonian**

**Exactly Solvable $\mathbb{Z}_2$ SPT Model**

In this section, we review the 2D $\mathbb{Z}_2$ SPT model introduced by Levin and Gu in Ref. [17]. We start with the trivial paramagnetic Hamiltonian on the triangular lattice (see Fig. 4),

\( H_0 = -\sum_j \sigma_j^x, \)

and the 2D $\mathbb{Z}_2$ SPT Hamiltonian [17]

\[ H_{2D,SPT} = -\sum_j \mathcal{O}_j = \sum_j \sigma_j^x e^{i \frac{\pi}{4} \sum_{(\ell,\ell')} \langle \sigma^z_{\ell} \sigma^z_{\ell'} \rangle} \sum_{(\ell,\ell')} \langle \sigma^x_{\ell} \sigma^x_{\ell'} \rangle, \]

where the sum over $\ell, \ell'$ in Eq. (B2) extends over pairs of nearest neighbor spins around the spin at site $j$ as depicted in Fig. 4. The Hamiltonian Eq. (B2) is invariant under spin flips generated by $S_{\mathbb{Z}_2} = \prod_j \sigma_j^z$.

The Hamiltonians Eq. (B1) and (B2) are related by the unitary transformation

\[ \mathbb{W} = \prod_j e^{-i \frac{\pi}{4} \sum_{(\ell,\ell')} \langle \sigma^z_{\ell} \sigma^z_{\ell'} \rangle} \]

that implements

\[ \mathcal{O}_j = \mathbb{W} \sigma_j^x \mathbb{W}^{-1} = -\sigma_j^x e^{i \frac{\pi}{4} \sum_{(\ell,\ell')} \langle \sigma^x_{\ell} \sigma^x_{\ell'} \rangle}. \]
Now for every site \(j\) we expand the exponent:

\[
e^{\frac{i\pi}{4} \sum_{(\ell \ell')j} (1 - \sigma^z_\ell \sigma^z_{\ell'})} = e^{\frac{3\pi}{4} \prod_{(\ell \ell')j} [\cos(\pi/4) - i \sin(\pi/4) \sigma^x_\ell \sigma^x_{\ell'}]}
\]

\[
= \frac{1}{4} \left\{ \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 \right\} \tag{B5}
\]

where \(\sigma^x_1, ..., \sigma^x_6\) denote the six spin operators around the site \(j\), as in Fig. 4.

**Driven Three-Spin Interaction**

Motivated by the unitary transformation Eq. (B3), we are led to consider a time dependent three-spin interaction

\[
H_{2D}(t) = -\hbar \sum_j \sigma^y_j + \Theta(t) f(t) \sum_{(ijk)} \sigma^z_i \sigma^z_j \sigma^z_k, \tag{B6a}
\]

where the summation in the second term runs over every triangle of the lattice and

\[
f(t) = \lambda \omega \cos(\omega t + \varphi), \quad \lambda > 0. \tag{B6b}
\]

The unitary transformation to the rotating frame

\[
U_R(t) = \exp \left[ ig(t) \sum_{(ijk)} \sigma^z_i \sigma^z_j \sigma^z_k \right], \tag{B7}
\]

where \(g(t) = \lambda [\sin(\omega t + \varphi) - \sin \varphi]\), yields the rotating-frame Hamiltonian

\[
H_R(t) = U_R(t) \left( -\hbar \sum_j \sigma^y_j \right) U_R^\dagger(t). \tag{B8}
\]

The relevant object to compute is then

\[
U_R(t) \sigma^y_j U_R^\dagger(t) = \sigma^y_j \exp \left[ -i2g(t) \sigma^z_j \sum_{(\ell \ell')j} \sigma^z_\ell \sigma^z_{\ell'} \right]
\]

\[
= \sigma^y_j \prod_{(\ell \ell')j} [\cos(2g(t)) - i \sin(2g(t)) \sigma^z_\ell \sigma^z_{\ell'}]
\]

\[
\equiv \sigma^y_j \mathcal{A}_j(t). \tag{B9}
\]

Explicitly, we have

\[
\mathcal{A}_j(t) = \mathcal{A}^I_j(t) + \mathcal{A}^{II}_j(t), \tag{B10a}
\]

where

\[
\mathcal{A}^I_j(t) = i \sigma^z_j \left[ \beta_1(t) \left( \sigma^x_1 \sigma^x_2 + \sigma^x_2 \sigma^x_4 + \sigma^x_5 \sigma^x_6 \right. \right.
\]

\[
+ \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6
\]

\[
+ \sigma^x_2 \sigma^x_3 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_2 \sigma^x_3 \sigma^x_4 \sigma^x_5 \sigma^x_6 + \sigma^x_2 \sigma^x_3 \sigma^x_4 \sigma^x_5 \sigma^x_6
\]

\[
+ \sigma^x_1 \sigma^x_2 \sigma^x_4 \sigma^x_5 \sigma^x_6 \], \tag{B10b}
\]

\[
- \beta_2(t) \left( \sigma^x_1 \sigma^x_2 + \sigma^x_2 \sigma^x_4 + \sigma^x_3 \sigma^x_2 + \sigma^x_4 \sigma^x_6 + \sigma^x_5 \sigma^x_6 + \sigma^x_1 \sigma^x_6 \right) \right],
\]
\[ A_j^{(1)}(t) = \beta_1(t) + \beta_4(t) \left( \sigma^x_1 \sigma^x_2 \sigma^x_3 + \sigma^y_1 \sigma^y_2 \sigma^y_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 \right) \\
+ \sigma^z_1 \sigma^z_2 \sigma^z_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 + \sigma^z_1 \sigma^z_2 \sigma^z_3 , \tag{B10c} \]

where

\[ \begin{align*}
\beta_1(t) &= 2c^3(t) s^3(t) , \\
\beta_2(t) &= c(t) s^5(t) + c^5(t) s(t) , \\
\beta_3(t) &= c^6(t) - s^6(t) , \\
\beta_4(t) &= c^2(t) s^4(t) - c^4(t) s^2(t) ,
\end{align*} \tag{B10d} \]

and we use the shorthand notation \( c(t) \equiv \cos(2g(t)) \text{ and } s(t) \equiv \sin(2g(t)) \).

The Floquet Hamiltonian at infinite frequency, obtained from the time average of the Hamiltonian Eq. \( (B8) \),

\[ H_F^{(0)} = -\hbar \sum_j \sigma_j^y \left( \frac{1}{T} \int_0^T dt A_j(t) \right) , \tag{B11} \]

upon using Eq. \( (B10) \), depends on the following parameters

\[ \begin{align*}
\beta_1(\lambda, \varphi) &= \frac{1}{2} \int_0^{2\pi} d\tau 2 \cos^3 \left[ 2\lambda G_\varphi(\tau) \right] \sin^3 \left[ 2\lambda G_\varphi(\tau) \right] , \\
\beta_2(\lambda, \varphi) &= \frac{1}{2} \int_0^{2\pi} d\tau \left\{ \cos \left[ 2\lambda G_\varphi(\tau) \right] \sin^5 \left[ 2\lambda G_\varphi(\tau) \right] + \cos^5 \left[ 2\lambda G_\varphi(\tau) \right] \sin \left[ 2\lambda G_\varphi(\tau) \right] \right\} , \\
\beta_3(\lambda, \varphi) &= \frac{1}{2} \int_0^{2\pi} d\tau \left\{ \cos^6 \left[ 2\lambda G_\varphi(\tau) \right] - \sin^6 \left[ 2\lambda G_\varphi(\tau) \right] \right\} , \\
\beta_4(\lambda, \varphi) &= \frac{1}{2} \int_0^{2\pi} d\tau \left\{ \cos^2 \left[ 2\lambda G_\varphi(\tau) \right] \sin^4 \left[ 2\lambda G_\varphi(\tau) \right] - \cos^4 \left[ 2\lambda G_\varphi(\tau) \right] \sin^2 \left[ 2\lambda G_\varphi(\tau) \right] \right\} ,
\end{align*} \tag{B12} \]

where \( G_\varphi(\tau) = \sin(\tau + \varphi) - \sin \varphi \). Whenever

\[ \beta_1(\lambda^*, \varphi^*) = \beta_2(\lambda^*, \varphi^*) \equiv \beta^* \neq 0 , \tag{B13a} \]

\[ \beta_3(\lambda^*, \varphi^*) = \beta_4(\lambda^*, \varphi^*) = 0 , \tag{B13b} \]

the Hamiltonian Eq. \( (B11) \) acquires the form

\[ H_F^{(0)} = 4 \beta^* h \sum_j \sigma_j^y e^{i \frac{\pi}{4} \sum_{\langle \alpha' \rangle} (1 - \sigma_0^z \sigma_{\alpha'}^z) } , \tag{B14} \]

which is the same model Eq. \( (B2) \) shown in Ref. \((17)\) to describe the 2D SPT paramagnet with \( \mathbb{Z}_2 \) symmetry. We have found numerically that condition Eq. \( (B13) \) is satisfied, for example, for \( \lambda^* \approx 0.51 \text{ and } \varphi \approx \pm 0.27 \pi \). It is fundamental to stress that even though the driven Hamiltonian Eq. \( (B6a) \) does not have the \( \mathbb{Z}_2 \) symmetry, this symmetry emerges in the \( \omega \to \infty \) limit.