Abstract
We study quantum processes, as one parameter families of differentiable completely positive and trace preserving (CPTP) maps. Using different representations of the generator, and the Sylvester criterion for positive semi-definite matrices, we obtain conditions for the divisibility of the process into completely positive (CP-divisibility) and positive (P-divisibility) infinitesimal maps. Both concepts are directly related to the definition of quantum non-Markovianity. For the single qubit case we show that CP- and P-divisibility only depend on the dissipation matrix in the master equation form of the generator. We then discuss three classes of processes where the criteria for the different types of divisibility result in simple geometric inequalities, among these the class of non-unital anisotropic Pauli channels.

Keywords: Quantum process, Divisibility, Quantum non-Markovianity

1. Introduction
Non-Markovianity of quantum processes has been a topic of increasing interest during approximately the last ten years [1, 2, 3]. Starting with the papers by Breuer et al. [4] and Rivas et al. [5], a definition of quantum Markovianity has been reduced to the question whether all intermediate quantum maps are physically realizable; this induces a characterization that is more closely related to the Chapman-Kolmogorov condition than to the full definition of classical Markovianity [6]. For differentiable quantum processes, the question of divisibility into physically realizable quantum maps can be further reduced to the analysis of the time dependent generator of the process. This is the approach taken for this work.

The concept of divisibility has been introduced in Refs. [7, 8]. In its original form, it refers to the condition that all intermediate maps are completely positive (CP-divisibility). However, one may as well consider P-divisibility, where it is sufficient that the intermediate maps are positive [9, 10]. If an intermediate map is positive but not completely positive, one may observe information backflow for entangled states between system and some ancillary system, but not in the system alone [5, 11].

In this work, we derive general criteria for positivity and complete positivity. In particular, for single qubit processes we show that both, CP-divisibility and P-divisibility conditions, only depend on the dissipation matrix of the master equation. We identify three different classes of single qubit processes, where the criteria for CP- and P-divisibility are reduced to simple explicit geometric inequalities. One of these classes consists of processes where the Choi-matrix has the shape of an X (it means that all non-zero elements are located on the diagonal or the anti-diagonal). Many examples considered in the literature of quantum non-Markovianity are of this type. A second class consists of those processes, where the Choi-matrix has the shape of an O. The third class is that of the non-unital anisotropic Pauli channels. While criteria applicable to the generators have been studied in the context of CP-divisibility, see for instance Ref. [12], this has rarely been done for P-divisibility.

Explicit analytical criteria are valuable for the construction of Markovian approximations to a non-Markovian process as proposed in Ref. [8] and more specifically in Ref. [5]. Another area of applications is that of quantum process tomography [13, 14, 15], where it is important to identify the independent parameters which are to be determined. Finally, it may be of interest to identify quantum channels, which are P-divisible but not CP-divisible as processes where non-Markovianity may be interpreted as a genuine quantum effect [16].

Our work relies on a few general results which have been derived previously. The most important ones are (i) the Kossakowski theorem, which establishes the equivalence between positivity and contractivity (for the domain of Helstrom matrices) [11] (and references therein); (ii) necessary and sufficient criteria which can be applied directly to the time dependent generator of the quantum process, Ref. [9] for positivity and Ref. [8] for complete positivity; and finally (iii) Sylvester’s criterion for definite and semi-definite positivity [17, 18].

The paper is organized as follows: In Sec. [2] we discuss the description of quantum processes in terms of their generators and the general conditions for CP- and P-divisibility in terms of the generator. In Sec. [3] we analyze these con-
2. Differentiable quantum processes

In this section we introduce differentiable quantum processes and the definitions of P-divisibility or CP-divisibility. For both types of divisibility, we present criteria which can be applied directly to the generator of the quantum process in question.

2.1. Processes and generators

Let us denote a quantum process $\Lambda_t$ (for $t \in \mathbb{R}_0^+$), as a one-parameter family of differentiable (with respect to $t$) completely positive and trace preserving linear maps (CPTP-maps), with $\Lambda_0 = \mathbb{I}$, the identity. For simplicity, we assume that the corresponding Hilbert space is of finite dimension, $\dim(H) = d < \infty$. The quantum process $\Lambda_t$ can be defined equivalently by the generator $L_t$, such that

$$\frac{d}{dt} \Lambda_t = L_t \Lambda_t, \quad \Lambda_0 = \mathbb{I}. \quad (1)$$

One natural question to ask would be the following: What are the properties to be fulfilled by $L_t$ in order to produce a valid quantum process of CPTP maps (very recently this question has been addressed in Ref. [20]). In the present work, we have a different objective. Assuming that $L_t$ generates a valid quantum process, we ask whether that process is CP-divisible and/or P-divisible.

Note that for a given quantum process $\Lambda_t$, we can compute its generator as

$$L_t = \frac{d\Lambda_t}{dt} \Lambda_t^{-1}. \quad (2)$$

In what follows we will assume that $\Lambda_t$ is invertible. It is common that in a given quantum process, $\Lambda_t$ is non-invertible at isolated points in time. If this is the case, one has to proceed with care [20]. In order to derive P-divisibility and CP-divisibility criteria in terms of the generator, we need to relate it to the intermediate quantum map,

$$\Lambda_{t+\delta,t} = \Lambda_{t+\delta} \Lambda_t^{-1}. \quad (3)$$

This can be achieved by considering an infinitesimal intermediate time step. In that case, it holds that

$$L_t = \lim_{\delta \to 0} \delta^{-1} \left( \Lambda_{t+\delta,t} - \mathbb{I} \right). \quad (4)$$

**Choi-matrix representation.** A direct method to represent linear quantum maps (this includes generators such as $L_t$) consists in embedding the state space into the vector space $\mathcal{M}^{d \times d}$ of complex quadratic matrices of dimension $d$. In such case, the elements $\{|i\rangle \langle j|\}_{1 \leq i,j \leq d}$ form a convenient orthonormal basis with respect to the Hilbert-Schmidt scalar product $(A,B) = \text{tr}(A^\dagger B)$. Then, we define the Choi-matrix representation [21] of any linear map $\Lambda$ in $\mathcal{M}^{d \times d}$ as

$$C_\Lambda = \sum_{i,j} |i\rangle \langle j| \otimes \Lambda |i\rangle \langle j|. \quad (5)$$

In practice this matrix is a $d \times d$ matrix of block-matrices from $\mathcal{M}^{d \times d}$ which are the images of the basis elements $|i\rangle \langle j|$ under the map $\Lambda$. The remarkable properties of this representation are the following: $C_\Lambda = C_\Lambda^\dagger$ if $\Lambda[\Delta^\dagger] = \Lambda[\Delta]$ for every bounded operator $\Delta$, $C_\Lambda \geq 0$ if $\Lambda$ is complete positive and $\text{tr}(C_\Lambda) = d$ if $\Lambda$ preserves the trace [21, 22].

**Master equation.** The generator obtained in Eq. (4) preserves Hermiticity by construction, thus we can bring it to the following standard form [8, 23] (see Appendix E for a detailed derivation):

$$\frac{d}{dt} \rho = L_t[\rho], \quad (6)$$

$$L_t[\rho] = -i[H,\rho] + \sum_{i,j=1}^{d^2-1} D_{ij} \left( F_i \rho F_j^\dagger - \frac{1}{2} \{F_j^\dagger F_i, \rho\} \right).$$

In this expression, Planck’s constant $\hbar$ has been absorbed into the Hamiltonian $H$. The matrix $D$ is Hermitian, and the set $\{F_i\}_{1 \leq i \leq d^2}$ forms an orthonormal basis in the space of operators, such that $\text{tr}(F_i^\dagger F_j) = \delta_{ij}$. In addition, the operators are chosen such that $\text{tr}(F_1) = 0$, except for the last element, which is given by $F_{d^2} = \mathbb{I}/\sqrt{d}$.

In this work, we will use the expression of Eq. (6) as one possible representation of the generator $L_t$, at some arbitrary but fixed time $t$. We call this representation the “master equation representation” of the generator $L_t$ and $D$ the “dissipation matrix”. Note that an intermediate quantum process $\Lambda_{t+\delta,t} \approx \mathbb{I} + \delta L_t$ (with $\delta > 0$), as defined in Eq. (3), is CPTP if and only if $D$ is positive semidefinite [22, 24, 25]. Moreover, it defines a one-parameter semigroup in the space of CPTP maps if the generator is time independent [21, 23, 26].

2.2. Markovianity: P-divisibility vs. CP-divisibility

In this subsection we present the definitions for the P-divisibility and the CP-divisibility of quantum processes. We use the term “Markovianity” in cases, where we want to refer to both types of divisibility, indistinctively.

**CP-divisibility.** A process $\Lambda_t$ is called CP-divisible if and only if the intermediate map $\Lambda_{t+\delta,t}$ as defined in Eq. (3) is CPTP for all $t, \delta \in \mathbb{R}_0^+$. generators which depend on $\delta > 0$, as in that case, it has been shown in [2] that a process constructed from Eq. (6) is CP-divisible if and only if $D \geq 0$ for all times.

Complete positivity of a quantum map $\Lambda$ is conveniently verified using the Choi matrix representation, introduced in Eq. (5). Provided that $\Lambda$ preserves Hermiticity
and the trace, it is CPTP if and only if the Choi matrix is positive-semidefinite \cite{21, 22}, i.e. if it has only non-negative eigenvalues.

**P-divisibility.** A process \( \Lambda_t \) is called P-divisible if and only if the intermediate map \( \Lambda_{t+\delta,t} \) as defined in Eq. (3) is PTP (positivity and trace preserving) for all \( t, \delta \in \mathbb{R}_0^+ \).

Positivity of a Hermiticity and trace preserving quantum map \( \Lambda \) is more complicated to verify. In that case, one has to show that \( \Lambda[\rho] \geq 0 \) for all density matrices \( \rho \). In practice, it is sufficient to check the condition for all density matrices representing pure states.

**Local complete positivity.** Follow Refs. \cite{8}, and \cite{21}, let \( C \) be a matrix representation of \( C_L \) in the subspace orthogonal to the Bell state

\[
|\Phi_B\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle ,
\]

where \( d \) is the dimension of the Hilbert space. Then, a quantum process \( \Lambda_t \) is locally CP at time \( t \), if and only if

\[
C_\perp \geq 0 . \tag{8}
\]

Therefore the process \( \Lambda_t \) is CP-divisible, if and only if it is locally CP for all \( t \in \mathbb{R}_0^+ \). Note that in Ref. \cite{21}, it has been shown that \( C_\perp \) is unitarily equivalent to the dissipation matrix \( D \) (see Appendix E for a detailed derivation).

**Local positivity.** A quantum process is locally positive at time \( t \), if and only if for all orthogonal states \( |\psi\rangle, |\phi\rangle \in \mathcal{H} \) it holds that

\[
\langle \psi | \mathcal{L}_t | \phi \rangle \langle \phi | \psi \rangle \geq 0 . \tag{9}
\]

Similar to the CP case, it holds that a quantum process \( \Lambda_t \) is P-divisible if and only if it is locally positive for all \( t \in \mathbb{R}_0^+ \). The equivalence between local positivity and P-divisibility follows from Eq. (6):

\[
\langle \psi | \Lambda_{t+\delta,t} | \phi \rangle \langle \phi | \psi \rangle \geq 0 \Leftrightarrow \delta \langle \psi | \mathcal{L}_t | \phi \rangle \langle \phi | \psi \rangle + \langle \psi | \phi \rangle \langle \phi | \psi \rangle \geq 0 . \tag{10}
\]

In the limit \( \delta \to 0 \), this can only happen if \( \psi \) and \( \phi \) are orthogonal, \( \langle \psi | \phi \rangle = 0 \). In fact, if \( |\langle \psi | \phi \rangle|^2 > 0 \), it might very well be that \( \langle \psi | \mathcal{L}_t | \phi \rangle \langle \phi | \psi \rangle < 0 \) even if the process is P-divisible in the neighborhood of that point.

To summarize, we may express both properties CP-divisibility and P-divisibility in terms of local conditions which have to be fulfilled by the generator \( \mathcal{L}_t \) for all times \( t \in \mathbb{R}_0^+ \). In what follows, we analyze these in more detail. To avoid overly cumbersome terminology, we denote generators which fulfill Eq. (9) and/or Eq. (10) simply as “positive” and/or “completely positive generators”.

### 3. Single qubit processes

In the case of single qubit processes, the Bloch vector representation is yet another method to represent quantum channels and their generators. In the following Sec. 3.1 we discuss the following three representations: (i) the master equation, (ii) the Choi-matrix, and (iii) the Bloch vector representation and how they are related one-to-another. In Sec. 3.2 we derive explicit criteria for local positivity and local complete positivity in terms of the dissipation matrix \( D \).

#### 3.1. Equivalent representations

**Choi matrix representation.** For our purposes, the Choi matrix representation will be the most useful. A CPTP-map \( \Lambda \), which belongs to a quantum process, may be parametrized as

\[
C_\Lambda = \begin{pmatrix} \Lambda[|0\rangle\langle0|] & \Lambda[|0\rangle\langle1|] \\ \Lambda[|1\rangle\langle0|] & \Lambda[|1\rangle\langle1|] \end{pmatrix} = \begin{pmatrix} 1 - r_1 & y_1^* & x^* & 1 - z_1^* \\ y_1 & r_1 & z_2 & -x^* \\ x & z_2^* & r_2 & y_2^* \\ 1 - z_1 & -x & y_2 & 1 - r_2 \end{pmatrix} . \tag{11}
\]

The structure of \( C_\Lambda \) is due to the fact that \( \Lambda \) must preserve Hermiticity and the trace. We have chosen the parametrization in such a way that the parameters \( r_1, r_2, y_1, y_2, x, z_1, z_2 \) as functions of time are all zero at \( t = 0 \).

Note that any intermediate map \( \Lambda_{t+\delta,t} \) is at least Hermiticity and trace preserving. Therefore, Eq. (6) implies that the Choi-matrix representation of the generator \( \mathcal{L}_t \) must be Hermitian, and in all blocks, the partial trace must be equal to zero. That leaves us with the following parametrization:

\[
C_{\mathcal{L}} = \begin{pmatrix} -q_1 & Y_1^* & X^* & -Z_1^* \\ Y_1 & q_1 & Z_2 & -X^* \\ X & Z_2^* & q_2 & Y_2^* \\ -Z_1 & -X & Y_2 & -q_2 \end{pmatrix} . \tag{12}
\]

In general, there is no simple relation between the parametrization used here, and that of Eq. (11). This is because the expression for the generator \( \mathcal{L}_t \) includes the inverse of \( \Lambda_t \).

**Master equation representation.** Note that every generator \( \mathcal{L}_t \) of a Hermiticity and trace preserving quantum process, can be written in the form of Eq. (6), with Hermitian matrices \( H \) and \( D \). Therefore, we may calculate the Choi-representation of the generator, by inserting \( g = |i\rangle \langle j| \) into the RHS of Eq. (6), and compare the result to the general form in Eq. (12). For the calculation, we choose the following orthonormal operator basis \( \{|F_i\}_{1 \leq i \leq 4} \):

\[
F_1 = \frac{1}{\sqrt{2}}(|0\rangle\langle0| - |1\rangle\langle1|) , \quad F_2 = |0\rangle\langle1| , \\
F_3 = |1\rangle\langle0| , \quad F_4 = 1/\sqrt{2} . \tag{13}
\]
As a result, we obtain a linear one-to-one correspondence between the parameters used in the master equation representation and those used in the Choi representation:

\[
\begin{pmatrix}
q_1 \\
q_2 \\
\text{Re} Z_1
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1/2 & 1/2
\end{pmatrix}
\begin{pmatrix}
D_{11} \\
D_{22} \\
D_{33}
\end{pmatrix},
\]

\[
\text{Im } Z_1 = H_{22} - H_{11},
\]

\[
Z_2 = D_{32},
\]

\[
\begin{pmatrix}
Y_1 \\
Y_2 \\
X^*
\end{pmatrix} =
\begin{pmatrix}
-\sqrt{2}/4 & \sqrt{18}/4 & -i \\
-\sqrt{18}/4 & \sqrt{2}/4 & i \\
\sqrt{2}/4 & \sqrt{2}/4 & i
\end{pmatrix}
\begin{pmatrix}
D_{12} \\
D_{31} \\
H_{21}
\end{pmatrix}.
\]

As one might have expected, the quantity \(H_{11} + H_{22}\) is irrelevant for the representation of the generator, and may be set equal to zero without loss of generality. Then, Eq. (14) is clearly an invertible linear system of equations.

**Bloch vector representation.** Any qubit density matrix can be written in terms of the Pauli matrices and the identity matrix \(I\) as follows:

\[
\varrho = \frac{1}{2} \left( v_0 I + \sum_{j=1}^3 v_j \sigma_j \right),
\]

where \(v_0 = 1\) and \(\vec{v} = (v_1, v_2, v_3)\) is a vector in \(\mathbb{R}^3\) of norm \(\|\vec{v}\| \leq 1\). Any Hermiticity and trace preserving quantum map \(\Lambda\) can then be written as an affine transformation \(28\)

\[
A : \vec{v} \rightarrow \vec{v'} = R \vec{v} + \vec{t},
\]

where \(R\) is a real not necessarily symmetric square matrix and \(\vec{t}\) is a real three-dimensional vector. The coefficients of \(R\) and \(\vec{t}\) are given by

\[
t_j = \frac{1}{2} \text{tr}(\sigma_j \mathcal{L}_t[I]), \quad R_{jk} = \frac{1}{2} \text{tr}(\sigma_j \mathcal{L}_t[\sigma_k]).
\]

For the generator with the Choi-matrix representation given in Eq. (12), we find

\[
R = \begin{pmatrix}
\text{Re}(Z_2 - Z_1) & \text{Im}(Z_1 + Z_2) & \text{Re}(Y_1 - Y_2) \\
\text{Im}(Z_2 - Z_1) & -\text{Re}(Z_1 + Z_2) & \text{Im}(Y_1 - Y_2) \\
2\text{Re}(X) & -2\text{Im}(X) & -q_1 - q_2
\end{pmatrix},
\]

\[
\vec{t} = \begin{pmatrix}
\text{Re}(Y_1 + Y_2) \\
\text{Im}(Y_1 + Y_2) \\
q_2 - q_1
\end{pmatrix}.
\]

Again, it is easy to verify that the relation between this Bloch vector representation and the Choi representation is invertible.

### 3.2. Criteria for positivity and complete positivity

**Local complete positivity.** In order to verify if the Choi-matrix (as a linear transformation) projected onto the orthogonal subspace of \(|\phi_3\rangle\langle\phi_3|\), is positive, we choose the orthonormal states

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

(19)
to span that subspace. Then we obtain for the matrix representation of the Choi matrix of \(\mathcal{L}_t\), projected on that subspace:

\[
C_{\perp} = \begin{pmatrix}
\text{Re}(Z_1) - \frac{q_1 + q_2}{2} & \frac{X^* - Y_2}{\sqrt{2}} & \frac{X + Y_1^*}{\sqrt{2}} \\
\frac{X - Y_2^*}{\sqrt{2}} & q_2 & Z_2^* \\
\frac{X^* - Y_1}{\sqrt{2}} & Z_2 & q_1
\end{pmatrix}.
\]

(20)
The second equality is obtained by solving Eq. (14) for the matrix elements \(D_{ij}\). It simply means that \(C_{\perp} = D\).

We may now use the Sylvester criterion to check whether \(D \geq 0\) or not. A general discussion of that criterion can be found in the text book \(17\); the present positive semidefinite case has been treated in Ref. \(18\). In that case, the statement of the following: A Hermitian matrix is positive semidefinite if and only if all principal minors are larger or equal to zero. Hence, for \(D \geq 0\), it must hold:

\[
D_{11}, D_{22}, D_{33} \geq 0, \quad D_{11}D_{33} - |D_{31}|^2 \geq 0,
\]

\[
D_{11}D_{22} - |D_{21}|^2 \geq 0, \quad D_{33}D_{22} - |D_{32}|^2 \geq 0,
\]

\[
D_{11}D_{22}D_{33} + 2\text{Re}(D_{12}D_{23}D_{31}) \geq
\]

\[
D_{11}|D_{32}|^2 + D_{22}|D_{31}|^2 + D_{33}|D_{21}|^2.
\]

(21)

**Local positivity.** According to the criterion in Eq. (9), we need to verify that \(|\psi\rangle \mathcal{L}[|\phi\rangle\langle\phi|] |\psi\rangle \geq 0\) for all \(|\psi\rangle \perp |\phi\rangle\). Such general orthonormal states may be written as the column vectors of a unitary matrix, taken from the group \(SU(2)\). Removing an ineffective global phase we find:

\[
|\psi\rangle = \left( \frac{\cos(\theta/2)}{e^{i\beta} \sin(\theta/2)} \right), \quad |\phi\rangle = \left( -\sin(\theta/2) e^{i\beta} \cos(\theta/2) \right).
\]

Hence, we consider \(p(\theta, \beta) = \langle \psi | \mathcal{L} [|\phi\rangle\langle\phi|] |\psi\rangle\) as a function of \(\theta\) and \(\beta\). Therefore, we may say that the \(\mathcal{L}_t\) is positive at time \(t\), if and only if \(p(\theta, \beta) \geq 0\) for all \(\theta\) and \(\beta\). Using the parametrization of Eq. (12), \(p(\theta, \beta)\) may be written as

\[
p(\theta, \beta) = \frac{q_1 + q_2}{2} \cos^2 \theta + \frac{q_2 - q_1}{2} \cos \theta + \frac{A}{2} \sin^2 \theta
\]

\[
+ \frac{\text{Re}(Y_1 + Y_2) e^{-i\beta}}{2} \sin \theta
\]

\[
+ \frac{\text{Re}(Y_2 - Y_1) e^{-i\beta} - 2X e^{i\beta}}{2} \sin \theta \cos \theta,
\]

(22)
where $A = \text{Re}[Z_1 - Z_2 e^{-2i\beta}]$. In terms of the master equation parameters, we find

$$R = D_{22} + D_{33}, \quad Y_1 + Y_2 = \sqrt{2} (D_{21} - D_{13}),$$

$$S = D_{33} - D_{22}, \quad A_1 = D_{11} - \frac{D_{33} + D_{22}}{2} - \text{Re} D_{23},$$

$$Y_2 - Y_1 - 2 X^* = -\sqrt{2} (D_{21} + D_{13}), \quad (23)$$

such that

$$2p(\theta, \beta) = R + S \cos \theta + \left(D_{11} - \frac{R}{2}\right) \sin^2 \theta$$

$$+ \text{Re} \left[-D_{23} e^{-2i\beta} \sin \theta + \sqrt{2} (D_{21} - D_{13}) e^{-i\beta} - \sqrt{2} (D_{21} + D_{13}) e^{i\beta} \cos \theta \right] \sin \theta. \quad (24)$$

This shows that positivity, just as complete positivity, only depends on the dissipation matrix $D$.

In general, one should try to find all minima of this function and verify that those are non-negative. Since the domain of $p(\theta, \beta)$ is a torus without boundaries, it is sufficient to find the critical points where the partial derivatives $\partial p/\partial \theta$ and $\partial p/\partial \beta$ are both equal to zero. The corresponding equations may be reduced to a root-finding problem for 4th order polynomials. Thus analytical expressions may be obtained in principle, even so they are probably not very useful. Still, numerical evaluations are pretty straight forward to implement. In Sec. 4 we will discuss different classes of generators, where particularly simple analytical solutions can be found.

4. Examples

In this section, we consider three different classes of generators. For each class, the set of positive (completely positive) generators is interpreted as a region in a certain parameter space (a subspace of the 9-dimensional vector space of dissipation matrices). In general, these regions must be convex, since the respective criteria involve expectation values of some linear matrix which represents the generator. Hence, if we consider the expectation value of any convex combination of two generators, it immediately decomposes into the corresponding convex combination of expectation values. Unless stated otherwise, we analyze the criteria for positivity and complete positivity in terms of the dissipation matrix $D$.

4.1. X-shaped quantum channels and generators

The term “X-shape” refers to the case, where the non-zero elements in the Choi matrix appear to form the letter “X”, that means that $Y_1 = Y_2 = X = 0$ in Eq. (22).

Hence,

$$C_L = \begin{pmatrix} -q_1 & 0 & 0 & -Z_2^* \\ 0 & q_1 & Z_2 & 0 \\ 0 & Z_2^* & q_2 & 0 \\ -Z_1 & 0 & 0 & -q_2 \end{pmatrix}. \quad (25)$$

In this case, the X-shape of the generator implies the X-shape of the quantum channel, and vice versa. Many important models lead to quantum channels of that type [3][2]. In terms of the Bloch vector representation, the X-shape implies that the dynamics along the z-axis is independent from that in the (x, y)-plane [12].

According to Eq. (14) the X-shape of the Choi matrix $C_L$ implies for $H$ and $D$ from the master equation representation in Eq. (6): $H_{12} = 0$, $D_{13} = D_{12} = 0$ as well as

$$q_1 = D_{22}, \quad q_2 = D_{33}, \quad Z_2 = D_{23}$$

and

$$Z_1 = i (H_{22} - H_{11}) + D_{11} + \frac{D_{33} + D_{22}}{2}. \quad (26)$$

For the matrix $C_L$ we thus obtain:

$$C_L = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & D_{23} \\ 0 & D_{32} & D_{33} \end{pmatrix}. \quad (27)$$

Complete positivity. Considering all principal minors of the dissipation matrix in Eq. (27), we find

$$D_{11}, D_{22}, D_{33} \geq 0, \quad D_{22} D_{33} - |D_{23}|^2 \geq 0, \quad D_{11} D_{22} \geq 0, \quad D_{11} \left[ D_{22} D_{33} - |D_{23}|^2 \right] \geq 0. \quad (28)$$

Removing redundant inequalities, we are left with

$$D_{11}, D_{22}, D_{33} \geq 0, \quad D_{22} D_{33} \geq |D_{23}|^2. \quad (29)$$

Positivity. From Eq. (24) we find:

$$2p(\theta, \beta) = R + S \cos \theta + A \sin^2 \theta \geq 0, \quad (30)$$

where $A = D_{11} - \frac{R}{2} - \text{Re} [D_{23} e^{-2i\beta}]$, $R = D_{22} + D_{33}$, and $S = D_{33} - D_{22}$. This inequality must hold for all values of $\theta$ and $\beta$, parametrizing the quantum state to which the generator is applied. Thus, we only need to verify if the minimum of this expression is larger than zero. As far as $\beta$ is concerned, this means that we may replace $A$ by its minimum (as a function of $\beta$), which is given by $A_{\text{min}} = D_{11} - R/2 - |D_{23}|$. We are then left with the condition

$$\forall \theta : R + S \cos \theta + A_{\text{min}} \sin^2 \theta \geq 0. \quad (31)$$

This condition is further evaluated in Appendix A. As a result, we find that the conditions for positivity become

$$D_{22}, D_{33} \geq 0, \quad (32)$$

and if $D_{11} < |D_{23}|$, in addition

$$\left| |D_{23}| - D_{11} \right| \leq \sqrt{D_{22} D_{33}}. \quad (33)$$

In Fig. 1, we show the parameter space $D_{22}, D_{33} \geq 0$ for visualizing the regions of positivity and complete positivity for the X-shaped generator. For complete positivity, the inequalities to fulfill are given in Eq. (29), which
According to Eq. (14), this implies for the matrix elements of $H$ and $D$ from the master equation (3):

$$q = D_{33}, \quad Z_1 = i (H_{22} - H_{11}) + D_{11} + D_{33},$$

$$Y = -i H_{21} + \frac{D_{13}}{\sqrt{2}}, \quad X^* = i H_{21} + \frac{D_{13}}{\sqrt{2}}. \quad (35)$$

states independent conditions on $D_{11}$ on the one hand and $D_{22}, D_{33}, |D_{23}|^2$ on the other. For positivity, by contrast, the conditions on $D_{22}$ and $D_{33}$ depend on $D_{11}$. Here, we observe an interesting behavior: As $D_{11}$ approaches zero from above, the region of positivity becomes more and more similar to the region of complete positivity, until they coincide for $D_{11} = 0$. When $D_{11}$ becomes negative, complete positivity is violated while positivity is still maintained sufficiently far away from the black dashed line.

### 4.2. O-shaped quantum channels

Here, we consider another subset of single qubit generators, which also allow for an analytic solution. These are in some sense complementary to the $X$-shaped channels, These are obtained from the general case by setting $q_1 = q_2 = q, Y_1 = -Y_2 = Y$, and $Z_2 = 0$. The Choi matrix, representing the generator resembles an $O$, instead of an $X$, that is why we call them $O$-shaped channels.

$$C_L = \begin{bmatrix}
-q & X^* & -Z_1^* \\
Y & q & -X^* \\
-Z_1 & -X & -Y & -q
\end{bmatrix}. \quad (34)$$

This can be reduced to

$$C_L = \begin{pmatrix}
D_{11} & D_{31} & D_{13} \\
D_{13} & D_{22} & 0 \\
0 & D_{22} & D_{12}
\end{pmatrix}, \quad (36)$$

Complete positivity. Expressed in terms of the dissipation matrix, considering all principal minors.

$$D_{11}, D_{22} \geq 0, \quad D_{11}D_{22} - |D_{13}|^2 \geq 0,$$

$$D_{11}D_{22} - D_{13}D_{31}D_{22} - D_{31}D_{22}D_{13} \geq 0 \quad (37)$$

1. **Positivity.** Under the conditions mentioned above, the function $2p(\theta, \beta)$ from Eq. (24) becomes (in terms of the master equation parameters)

$$2p(\theta, \beta) = 2D_{22} + (D_{11} - D_{22}) \sin^2 \theta$$

$$-2\sqrt{2} \text{Re}[D_{13} e^{-i\beta}] \sin \theta \cos \theta \geq 0. \quad (39)$$

Again, it is possible to derive the conditions for positivity, which do no longer involve the angles $\theta$ and $\beta$. The respective calculation is outlined in Appendix B with the result [see Eq. (B.4)]

$$3D_{22} + D_{11} \geq 0, \quad D_{22} (D_{11} + D_{22}) \geq |D_{13}|^2. \quad (40)$$

In Fig. 2 we show the different regions of positivity and complete positivity in the parameter space of $D_{22}, D_{11}$. We distinguish two qualitatively different cases, $|D_{13}| = 0$ and $|D_{13}| = 1$. In both cases, the region of positivity is considerably larger than the region for complete positivity.
4.3. Non-unital anisotropic Pauli channels

Here, $\Lambda_t$ is given as an affine transformation of state vectors in the Bloch sphere [28] (see the corresponding paragraph in Sec. 3.1):

$$\Lambda_t : \vec{v} \rightarrow \vec{v}' = R \vec{v} + \vec{s},$$

(41)

where $R$ is a real diagonal matrix and $\vec{s}$ a real vector.

$$R = \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}. \quad (42)$$

Using the general formula, Eq. (2), for constructing the generator, we find

$$\mathcal{L}_P : \vec{v} \rightarrow \vec{v}' = \frac{dR}{dt} \left[ R^{-1} (\vec{v} - \vec{s}) \right] + \frac{d\vec{s}}{dt}. \quad (43)$$

Hence, the generator for this Pauli channel is given by the affine transformation $\vec{v} \rightarrow \vec{v}' = R_{\mathcal{L}P} \vec{v} + t_{\mathcal{L}P}$, with

$$R_{\mathcal{L}P} = \begin{pmatrix} -\gamma_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & -\gamma_3 \end{pmatrix}, \quad t_{\mathcal{L}P} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix},$$

where $\gamma_j = \frac{-1}{R_{jj}} \frac{dR_{jj}}{dt}$, $\tau_j = \frac{dt_j}{dt} + \gamma_j t_j$. \quad (44)

The Choi matrix representation of $\mathcal{L}_P$ is obtained by inverting Eq. (18), with the result

$$C_{\mathcal{L}P} = \frac{1}{2} \begin{pmatrix} -\gamma_3 + \gamma_3 & \tau_1 - i\tau_2 & 0 & -\gamma_1 - \gamma_2 \\ \tau_1 + i\tau_2 & \gamma_3 - \gamma_3 & 0 & -\gamma_1 - \gamma_2 \\ 0 & 0 & -\gamma_2 - \gamma_3 & \tau_1 - i\tau_2 \\ -\gamma_1 - \gamma_2 & 0 & \tau_1 + i\tau_2 & -\gamma_2 - \gamma_3 \end{pmatrix}. \quad (45)$$

In what follows, we compute the positivity and the complete positivity condition in terms of the parameters $\gamma_j$ and $\tau_j$, since this allows for relatively simple geometric interpretations. For the parametrization in terms of the master equation [6], we obtain from Eq. (12) and (14):

$$D_{22} = \frac{\gamma_3 - \tau_3}{2}, \quad D_{33} = \frac{\gamma_3 + \tau_3}{2}, \quad H_{22} = H_{11}, \quad D_{11} = \frac{\gamma_1 + \gamma_2 - \gamma_3}{2}, \quad D_{21} = \frac{\gamma_2 - \gamma_1}{2}, \quad H_{12} = 0,$$$$

D_{21} = \frac{\tau_1 + i\tau_2}{2\sqrt{2}} = -D_{13}. \quad (46)$$

This yields

$$C_\perp = \frac{1}{2} \begin{pmatrix} \gamma_1 + \gamma_2 - \gamma_3 & w^* & -w \\ w & \gamma_3 - \gamma_1 & \gamma_2 - \gamma_1 \\ -w^* & \gamma_2 - \gamma_1 & \gamma_3 + \gamma_1 \end{pmatrix},$$

(47)

with $w = (\tau_1 + i\tau_2)/\sqrt{2}$.

4.4. Complete positivity

Figure 3: Parameter space of $\gamma_1$, $\gamma_2$ and $\gamma_3$. For the generator $\mathcal{L}_P$ in Eq. (43) to be positive, all elements $\gamma_j$ must be positive (blue transparent color). For $\mathcal{L}_P$ to be completely positive, the elements $\gamma_j$ must fulfill the conditions in Eq. (48). The corresponding region is colored in orange. Note that we show a cut through the regions of positivity and complete positivity which really extend towards arbitrary large positive values.

**Complete positivity.** The complete derivation can be found in Appendix C. It yields separate conditions for the diagonal elements $\gamma_j$ and the vector $\vec{\tau}$. For the diagonal elements $\gamma_j$ we find:

$$\forall i \neq j \neq k \neq i : |\gamma_i - \gamma_j| \leq \gamma_k \leq \gamma_i + \gamma_j. \quad (48)$$

Assuming these conditions are fulfilled, the vector $\vec{\tau}$ must lie inside the following ellipsoid:

$$\frac{\tau_i^2}{a_i^2} + \frac{\tau_j^2}{a_j^2} + \frac{\tau_k^2}{a_k^2} \leq 1,$$

$$a_1 = \gamma_2 - (\gamma_2 - \gamma_3)^2, \quad a_2 = \gamma_2 - (\gamma_1 - \gamma_3)^2, \quad a_3 = \gamma_2 - (\gamma_1 - \gamma_2)^2. \quad (49)$$

The regions of $\vec{\tau}$ where the generator $\mathcal{L}_P$ fulfills the conditions of complete positivity are shown in Fig. 4 in orange. Note that in this figure, we consider two particular cases, where $\gamma_1 = \gamma_2$ such that the resulting ellipsoid as defined above is symmetric with respect to the $\tau_3$ axis.

**Positivity.** In the general expression for $p(\theta, \beta)$ in Eq. (24), we replace the parameters with those from the Pauli channel, given in Eq. (46). This yields

$$2p(\theta, \beta) = \gamma_3 \cos^2 \theta + \left[ \gamma_1 \cos^2 \beta + \gamma_2 \sin^2 \beta \right] \sin^2 \theta + \gamma_3 \cos \theta + \left( \gamma_1 \cos \beta + \gamma_2 \sin \beta \right) \sin \theta. \quad (50)$$

We can express the general inequality $2p(\theta, \beta) \geq 0$ in a geometric form:

$$\vec{e}_r = \begin{pmatrix} \sin \theta \cos \beta \\ \sin \theta \sin \beta \\ \cos \theta \end{pmatrix} : \vec{e}_r \cdot (\vec{e}_r + \vec{\tau}) \geq 0, \quad (51)$$

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where $\gamma$ is the diagonal matrix with elements $\gamma_j$.

The interpretation of this result is easy: $-\gamma \vec{e}_r + \vec{f}$ is the image of $\vec{e}_r$ under the generator $L_P$. Thus an infinitesimal intermediate map would yield

$$\Lambda_{t,t+\delta} : \vec{e}_r \rightarrow \vec{e}_r' = \vec{e}_r + \delta \ L_P[\vec{e}_r] \ .$$

In order to have $\|\vec{e}_r'\| \leq 1$, the image under the generator must be pointing towards the center of the Bloch sphere, i.e. the scalar product between $-\gamma \vec{e}_r + \vec{f}$ and $\vec{e}_r$ must be negative. Multiplying the resulting inequality by minus one, we find

$$\forall \vec{e}_r : \vec{e}_r \cdot (\gamma \vec{e}_r - \vec{f}) \geq 0$$

This relation is equivalent to the inequality in Eq. (51), as can be seen by replacing $\vec{e}_r$ by $-\vec{e}_r$.

As shown in Appendix D the set of $\vec{f}$ for which the Pauli generator $L_P$ is positive, i.e. the inequality in Eq. (51) holds, is the convex region, which contains the origin and is limited by the surface [see Eq. (D.2)]

$$\mathcal{T} = \{ \vec{f}(\theta,\beta) = (\vec{e}_r \gamma \vec{e}_r) \vec{e}_r - 2 \gamma \vec{e}_r \} \ .$$

In Fig. 4 we show the region in $\vec{f}$-space which corresponds to positivity and complete positivity of the Pauli channel generator $L_P$. We consider two cases: $\gamma_1 = \gamma_2 = 0.255$, $\gamma_3 = 0.49$ in panel (a), and $\gamma_1 = \gamma_2 = 0.495$, $\gamma_3 = 0.01$ in panel (b). In the yellow triangle shown in Fig. 3 these points are located near the upper horizontal line (a) and near the lower corner (b), respectively. Choosing $\gamma_1 = \gamma_2$ leads to regions of (complete) positivity, which are symmetric with respect to the $\tau_3$-axis, which allows us to show two-dimensional projections. We find that the regions of positivity and complete positivity are always contained in ellipsoid with the parametrization $\vec{f}(\theta,\beta) = \gamma \vec{e}_r$.

As required, the region of complete positivity (orange) is fully contained in the region of positivity (olive green). In panel (a), we show a case where the ellipsoid $\gamma \vec{e}_r$ resembles roughly a rugby ball. In that case, the are only rather thin stripes near the border of the ellipsoid, where the generator is not positive any more. In panel (b), the ellipsoid has the shape of a flat pancake, and the region of positivity in the center is much smaller.

5. Conclusions

In order to determine whether a given differentiable quantum process is CP-divisible and/or P-divisible, we derive criteria which can be applied to the generator of the process. For the single qubit case, we discuss three common representations of the generator and work out the one-to-one mappings between them. We find criteria for CP- and P-divisibility, which can be expressed as inequalities in terms of the elements of the dissipation matrix. In the CP case, we avoid solving an eigenvalue problem by using the principal minor test for semidefinite matrices. In the P case, the corresponding inequality must be fulfilled for a whole two-parameter family of functions, which leads to an optimization problem without explicit general solution.

We then discuss three different classes of generators, where our criteria do yield explicit results for CP- and P-divisibility: the familiar X-shaped channels where the elements of the Choi matrix are non-zero in the diagonal and the anti-diagonal, only; the so called O-shaped channels, where $C_{23} = 0$, $C_{11} = C_{44}$ and $C_{12} = -C_{43}$; and most importantly the non-unital Pauli channels.

Besides its general value, as for instance the positivity criteria for the Pauli channel, we expect our results to prove useful in the area of quantum process tomography and the construction of optimal P-divisible or CP-divisible approximations to non-Markovian quantum processes. In particular there, the renouncement on the calculation of higher order roots may help to find analytical or semi-analytical solutions.

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Appendix A. Positivity of X-shaped generators

The condition in Eq. (51) can be expressed equivalently in terms of the variable $x = \cos \theta$ as follows:

$$f(x) = (R - A_{\min}) \ x^2 + S \ x + A_{\min} \geq 0 \ .$$

(A.1)
First note the following obviously necessary conditions
\[ \begin{align*}
  f(0) & : A_{\text{min}} \geq 0 \quad \text{and} \\
  f(\pm 1) & : R \pm S \geq 0 \quad \Leftrightarrow \quad 0 \leq |S| \leq R .
\end{align*} \tag{A.2} \]

To find the necessary and sufficient conditions, we will divide the problem in two cases: (i) \( \Delta = R - A_{\text{min}} \leq 0 \) and (ii) \( \Delta > 0 \).

In case (i) the conditions in Eq. \( \tag{A.2} \) are also sufficient as can be seen as follows: \( f(x) \) is convex, such that for any \( x_1, x_2 \) and \( 0 < \lambda < 1 \):
\[ f(\lambda x_1 + (1 - \lambda) x_2) \geq \lambda f(x_1) + (1 - \lambda) f(x_2) . \]
Choosing \( x_1 = -1 \) and \( x_2 = 1 \), we find
\[ f(1 - 2\lambda) \geq \lambda f(-1) + (1 - \lambda) f(1) , \]
which implies that \( f(x) \geq 0 \) in the interval \((-1, 1)\). This can be expressed as
\[ |S| \leq 2 \sqrt{A_{\text{min}} \Delta} \quad \text{or} \quad |S| \geq 2\Delta + \sqrt{S^2 - 4A_{\text{min}} \Delta} . \]
The inequality to the right is equivalent to
\[ |S| \geq 2\Delta \quad \text{and} \quad ( |S| - 2\Delta )^2 \geq S^2 - 4A_{\text{min}} \Delta , \]
which is equivalent to
\[ |S| \geq 2\Delta \quad \text{and} \quad |S| \leq R , \]
where \( |S| \leq R \) had already been identified as a necessary condition, previously. Therefore, in case (ii) the necessary and sufficient conditions for positivity are \( A_{\text{min}} \geq 0 \), \( 0 \leq |S| \leq R \) and
\[ |S| \leq 2 \sqrt{A_{\text{min}} \Delta} \quad \text{or} \quad |S| \geq 2\Delta . \]

It turns out that for \( \Delta \leq A_{\text{min}} \) it holds that \( 2 \sqrt{A_{\text{min}} \Delta} < 2\Delta \) such that the two conditions cancel each other, i.e. one of the two conditions is always fulfilled. For \( \Delta > A_{\text{min}} \) which is equivalent to \( 2\Delta > R \), by contrast, implies that \( |S| \geq 2\Delta \) cannot hold, such that \( |S| \leq 2 \sqrt{A_{\text{min}} \Delta} \) must be fulfilled. To summarize, the necessary and sufficient conditions for positivity are as follows:

- \( A_{\text{min}} = \text{Re}Z_1 - |Z_2| \geq 0 \), \( 0 \leq |S| \leq R \).
- In addition, if \( R > 2A_{\text{min}} \):
  \[ |S| \leq 2 \sqrt{A_{\text{min}} (R - A_{\text{min}})} . \]

For the parametrization in terms of the master equation, we find that positivity only depends on the dissipation matrix \( D \). Since
\[ R = D_{33} + D_{22} , \quad S = D_{22} - D_{33} , \tag{A.3} \]
the condition \( 0 \leq |S| \leq R \) implies that both \( D_{22} \) and \( D_{33} \) must be larger than or equal to zero. Furthermore, with
\[ A_{\text{min}} = D_{11} + D_{33} + D_{22} - \frac{|D_{32}|}{2} , \tag{A.4} \]
the condition \( R > 2A_{\text{min}} \) implies that \( |D_{32}| > D_{11} \). Finally,
\[ A_{\text{min}}(R - A_{\text{min}}) = \left[ \frac{D_{33} + D_{22}}{2} - (|D_{32}| - D_{11}) \right] \times \left[ \frac{D_{33} + D_{22}}{2} + (|D_{32}| - D_{11}) \right] \]
\[ = \frac{(D_{33} + D_{22})^2}{4} - (|D_{32}| - D_{11})^2 , \tag{A.5} \]
such that \( |S| \leq 2 \sqrt{A_{\text{min}} (R - A_{\text{min}})} \) is equivalent to
\[ \begin{align*}
  |D_{22} - D_{33}|^2 & \leq (D_{33} + D_{22})^2 - 4 (|D_{32}| - D_{11})^2 \\
  \Leftrightarrow \quad (|D_{32}| - D_{11})^2 & \leq D_{33} D_{22} . \tag{A.6}
\end{align*} \]

To summarize, in this parametrization, the conditions for positivity read
\[ D_{22}, D_{33} \geq 0 , \quad D_{11} - |D_{32}| + \frac{D_{33} + D_{22}}{2} \geq 0 , \tag{A.7} \]
and if \( D_{11} < |D_{32}| \), in addition
\[ |D_{32}| - D_{11} \leq \sqrt{D_{33} D_{22}} . \tag{A.8} \]

Note that this last inequality implies the second inequality of Eq. \( \tag{A.7} \), which can therefore be ignored.

**Appendix B. Positivity of O-shaped generators**

We start from the condition for positivity in Eq. \( \tag{39} \). Using the trigonometric identities \( 2\sin^2 \theta = 1 - \cos 2\theta \) and \( \sin 2\theta = 2\sin \theta \cos \theta \), Eq. \( \tag{39} \) becomes
\[ 2p(\theta, \beta) = \frac{3D_{22} + D_{11}}{2} + \frac{D_{22} - D_{11}}{2} \cos 2\theta \]
\[ - \sqrt{2} \text{Re}(D_{12} e^{-i\beta}) \sin 2\theta \geq 0 . \tag{B.1} \]

This expression is minimized with respect to \( \beta \), simply by making sure that \( \text{Re}(D_{12} e^{-i\beta}) = \pm |D_{12}| \). In other words: \( 2p(\theta, \beta) \geq 0 \) for all \( \beta \) and \( \theta \) is equivalent to
\[ 3D_{22} + D_{11} + \frac{D_{22} - D_{11}}{2} \cos 2\theta \pm \sqrt{2} |D_{12}| \sin 2\theta \geq 0 . \tag{B.2} \]
This condition is equivalent to
\[
\frac{3D_{22} + D_{11}}{2} \geq 0 \quad \text{and} \quad \frac{(3D_{22} + D_{11})^2}{4} \geq \frac{(D_{22} - D_{11})^2}{4} + 2|D_{12}|^2 \quad (B.3)
\]
These two inequalities are equivalent to
\[
3D_{22} + D_{11} \geq 0 \quad \text{and} \quad D_{22} (D_{22} + D_{11}) \geq |D_{12}|^2 \quad (B.4)
\]

**Appendix C. Complete positivity of the non-unital anisotropic Pauli channel**

For the generator $L_p$ to be completely positive, the matrix $C_1$ given in Eq. (47) must fulfill the inequalities in Eq. (21). In the present case, this yields three sets of inequalities:

\[
\begin{align*}
\gamma_1 + \gamma_2 - \gamma_3 & \geq 0 , \quad \gamma_3 - \gamma_3 \geq 0 , \quad \gamma_3 + \gamma_3 \geq 0 , \\
(\gamma_3 - \gamma_3)(\gamma_3 + \gamma_3) - (\gamma_2 - \gamma_1)^2 & \geq 0 , \quad (C.1) \\
(\gamma_1 + \gamma_2 - \gamma_3)(\gamma_3 - \gamma_3) - |w|^2 & \geq 0 , \\
(\gamma_1 + \gamma_2 - \gamma_3)(\gamma_3 - \gamma_3) - |w|^2 & \geq 0 , \quad (C.2)
\end{align*}
\]
and
\[
\begin{align*}
(\gamma_1 + \gamma_2 - \gamma_3) [(\gamma_3 - \gamma_3)(\gamma_3 + \gamma_3) - (\gamma_2 - \gamma_1)^2] - w [w^* (\gamma_3 + \gamma_3) + w (\gamma_2 - \gamma_1)] - w^* [w^* (\gamma_2 - \gamma_1) + w (\gamma_3 - \gamma_3)] & \geq 0 , \quad (C.3)
\end{align*}
\]
where $w = (\gamma_1 + i \gamma_2)/\sqrt{2}$. From Eq. (C.1), we find

\[
\gamma_3 \geq |\gamma_3| \geq 0 , \quad \gamma_1 + \gamma_2 \geq \gamma_3 , \quad (\gamma_2 - \gamma_1)^2 \leq \gamma_3^2 - \gamma_1^2 ,
\]
which yields the following conditions as necessary conditions (since we set $\gamma_3 = 0$ to arrive there):

\[
\gamma_1, \gamma_2, \gamma_3 \geq 0 , \quad |\gamma_2 - \gamma_1| \leq \gamma_3 \leq \gamma_1 + \gamma_2 . \quad (C.4)
\]

It is easy to verify that these inequalities are invariant under any permutation of indices; see Fig. 3. The remaining conditions, may be interpreted as conditions for the vector $\vec{\tau}$. These consist of the inequalities in Eq. (C.2) together with

\[
|\gamma_1| \leq \sqrt{\gamma_2^2 - (\gamma_2 - \gamma_1)^2} , \quad \text{and} \quad (C.5)
\]

\[
(\gamma_1 + \gamma_2 - \gamma_3) (\gamma_2 + \gamma_3 - \gamma_1) (\gamma_3 + \gamma_1 - \gamma_2) \geq \\
(\gamma_1 + \gamma_2 - \gamma_3) \tau_3^2 + (\gamma_2 + \gamma_3 - \gamma_1) \tau_1^2 + (\gamma_3 + \gamma_1 - \gamma_2) \tau_2^2 . \quad (C.6)
\]

In **Appendix C.1** we demonstrate that condition (C.6) implies all other conditions for the vector $\tau$, which can therefore be omitted. Reorganizing the terms in Eq. (C.6), we arrive at

\[
\begin{align*}
\frac{\tau_1^2}{a_1^2} + \frac{\tau_2^2}{a_2^2} + \frac{\tau_3^2}{a_3^2} & \leq 1 , \quad a_1 = \gamma_1^2 - (\gamma_2 - \gamma_3)^2 , \\
a_2 = \gamma_2^2 - (\gamma_1 - \gamma_3)^2 , \quad a_3 = \gamma_3^2 - (\gamma_1 - \gamma_2)^2 . \quad (C.7)
\end{align*}
\]

**Appendix C.1. Omissible inequalities for $\vec{\tau}$**

In what follows, we demonstrate that Eq. (C.5) as well as Eq. (C.2) follow from Eq. (C.7), such that we may consider Eq. (C.7) as the only condition on $\vec{\tau}$. To that end note first that setting $\tau_1 = \tau_2 = 0$ we can make the LHS of Eq. (C.7) only smaller which hence implies

\[
\tau_2^2 \leq a_2^2 = \gamma_3^2 - (\gamma_1 - \gamma_2)^2 ,
\]
which is exactly Eq. (C.5). To show that Eq. (C.7) also implies Eq. (C.2), it is convenient to express $\vec{\tau}$ in elliptical coordinates,

\[
\vec{\tau} = \lambda \begin{pmatrix} a_1 \sin \theta \cos \varphi \\ a_2 \sin \theta \sin \varphi \\ a_3 \cos \theta \end{pmatrix} ,
\]
such that Eq. (C.7) allows arbitrary values for the angles $\theta, \varphi$, and limits $\lambda$ to the range $0 \leq \lambda \leq 1$.

The two inequalities in Eq. (C.2) may be combined, and then read

\[
\gamma_3 \pm \lambda a_3 \cos \theta \geq \frac{\lambda^2 \sin^2 \theta}{2} \left[ (\gamma_3 + \gamma_1 - \gamma_2) \cos^2 \varphi + (\gamma_3 - \gamma_1 + \gamma_2) \sin^2 \varphi \right] \\
= \frac{\lambda^2 \sin^2 \theta}{2} [\gamma_3 + (\gamma_1 - \gamma_2) \cos(2\varphi)] . \quad (C.8)
\]

Due to the conditions in Eq. (C.7), we may assume that $\gamma_3 \geq a_3$ and $\gamma_3 \geq |\gamma_1 - \gamma_2|$. Therefore, in order to show that Eq. (C.2) holds, it is sufficient to prove that

\[
\gamma_3 \pm \lambda a_3 \cos \theta \geq \frac{\lambda^2 \sin^2 \theta}{2} [\gamma_3 + |\gamma_1 - \gamma_2|] .
\]

For that purpose, we substitute $x = \cos \theta$ to obtain a quadratic expression:

\[
A x^2 + \lambda a_3 x + \gamma_3 - A \geq 0 , \quad A = \frac{\lambda^2}{2} [\gamma_3 + |\gamma_1 - \gamma_2|] .
\]

The LHS describes a parabola. Therefore, the inequality holds, if we can prove that the equation $A x^2 + \lambda a_3 x + \gamma_3 - A = 0$ has no solution or at most one solution. For that purpose we consider the discriminant and show that it is less or equal to zero. For later convenience, we define $g_\pm = \lambda_3 \pm |\gamma_1 - \gamma_2|$. Then we may write:

\[
\begin{align*}
\lambda^2 a_3^2 - 4A (\gamma_3 - A) & \leq 0 \\
\pm a_3^2 - g_+ (2\gamma_3 - \lambda^2 g_+) & \leq 0 , \quad A = \frac{\lambda^2 g_+}{2} \\
g_+ g_- - 2g_+ \gamma_3 + \lambda^2 g_+^2 & \leq 0 , \quad a_3^2 = g_+ g_- \\
\pm g_+ - 2\gamma_3 + \lambda^2 g_+ & \leq 0 \\
- g_+ (1 - \lambda^2) & \leq 0 , \quad g_- - 2\lambda_3 = -g_+ .
\end{align*}
\]

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This completes the proof. The discriminant is negative semidefinite. Therefore the two inequalities in Eq. (C.2) are always fulfilled and can be omitted.

Appendix D. Positivity of the non-unital anisotropic Pauli channel

We start from the condition, given in Eq. (51),
\[
\vec{e}_r \cdot (\gamma \vec{e}_r + \vec{\tau}) \geq 0 ,
\]
where \( \vec{e}_r \) is a unit vector in spherical coordinates, parametrized by the angles \( \theta, \beta \). We aim at constructing the surface \( \mathcal{T} \) which forms the outer boundary of the region of points \( \vec{\tau} \), where the above inequality holds (note that this contains the origin \( \vec{\tau} = \vec{0} \), and that it must be convex.

The condition for \( \vec{\tau} \in \mathcal{T} \) can be cast into the following set of equations:
\[
\vec{e}_r \cdot (\gamma \vec{e}_r + \vec{\tau}) = 0 \quad \frac{\partial}{\partial \theta} \vec{e}_r \cdot (\gamma \vec{e}_r + \vec{\tau}) = 0 \quad \frac{\partial}{\partial \beta} \vec{e}_r \cdot (\gamma \vec{e}_r + \vec{\tau}) = 0 .
\]

The argument is as follows: Consider the LHS of the first equation as a function \( f(\vec{\tau}, \theta, \beta) \), then we may compute
\[
\max f(\vec{\tau}) = \max_{\theta, \beta} f(\vec{\tau}, \theta, \beta) ,
\]
by finding the critical points (there may be more than one) \( (\theta_i, \beta_i) \), where the last two equalities of Eq. (D.1) hold. Typically, for some fixed but arbitrary point \( \vec{\tau} \), some of the values of \( \{ f(\vec{\tau}, \theta_i, \beta_i) \} \) may be positive and others negative; some may correspond to local maxima, others to local minima, and still others may correspond to neither to one nor to the other group. However, the global maximum will always be among these points.

The calculation of the partial derivatives is simplified by the fact that
\[
\frac{\partial \vec{e}_r}{\partial \theta} = \begin{pmatrix} \cos \theta \cos \beta \\ \cos \theta \sin \beta \\ -\sin \theta \end{pmatrix} = \vec{e}_\theta ,
\]
\[
\frac{\partial \vec{e}_r}{\partial \beta} = \begin{pmatrix} -\sin \theta \cos \beta \\ \sin \theta \sin \beta \\ \cos \theta \end{pmatrix} = \sin \theta \vec{e}_\beta ,
\]
such that \( \{ \vec{e}_r, \vec{e}_\theta, \vec{e}_\beta \} \) form a system of orthonormal vectors. Therefore the system of equations in Eq. (D.1) becomes
\[
\vec{e}_r \cdot (\gamma \vec{e}_r + \vec{\tau}) = 0 \\
\vec{e}_\theta \cdot (\gamma \vec{e}_r + \vec{\tau}) + \vec{e}_r \cdot \gamma \vec{e}_\theta = 0 \\
\vec{e}_\beta \cdot (\gamma \vec{e}_r + \vec{\tau}) + \vec{e}_r \cdot \gamma \vec{e}_\beta = 0 ,
\]
which is equivalent to
\[
\vec{e}_r \cdot (\gamma \vec{e}_r + \vec{\tau}) = 0 \\
\vec{e}_\theta \cdot (2 \gamma \vec{e}_r + \vec{\tau}) = 0 \\
\sin \theta \vec{e}_\beta \cdot (2 \gamma \vec{e}_r + \vec{\tau}) = 0 .
\]
We started by asking for which points \( \vec{\tau} \), there exist a critical point \( (\theta, \beta) \) corresponding to a global maximum such that this set of equations is fulfilled. That point would then fore sure belong to the desired surface \( \mathcal{T} \). However, starting from this relation, we may say that it assigns to any pair of angles \( (\theta, \beta) \), a unique \( \vec{\tau} \), such that that pair of angles is a critical point (of any nature), while \( f(\vec{\tau}, \theta, \beta) = 0 \). That means that for \( \vec{\tau} \in \mathcal{T} \), it is a necessary but not sufficient condition that it satisfies this equation for some pair of angles \( (\theta, \beta) \). Therefore, the surface \( \mathcal{T} \) must be a subset of the set of solutions \( \vec{\tau} \) to this equation.

The last to equalities imply that \( 2\gamma \vec{e}_r + \vec{\tau} = \alpha \vec{e}_r \) for some unknown real parameter \( \alpha \). Inserting this into the first equality, we obtain
\[
\vec{e}_r \cdot (\alpha \vec{e}_r - \gamma \vec{e}_r) = 0 \quad \Rightarrow \quad \alpha = \vec{e}_r \cdot \gamma \vec{e}_r ,
\]
and finally
\[
\vec{\tau} = (\vec{e}_r \cdot \gamma \vec{e}_r) \vec{e}_r - 2 \gamma \vec{e}_r .
\]

Appendix E. Canonical form of quantum process generators

Here, we prove that the dissipation matrix \( D \), introduced in Eq. (6) is unitarily equivalent to \( C_\perp \), defined in Sec. 2.2. For that purpose, we start from the master equation representation of the generator \( \mathcal{L} \) of a quantum process, compute its Choi-matrix representation \( C_\perp \). Finally, we compute \( C_\perp \), the projection of \( C_\perp \) onto the subspace orthogonal to the Bell state, given in Eq. (7).

We start by rewriting Eq. (6) as
\[
\mathcal{L} [\rho] = \phi [\rho] - \kappa \rho - \rho \kappa ^\dagger , \quad \text{where} \quad (E.1)
\]
\[
[\phi [\rho] = \sum_i D_{ij} F_i \rho F_j ^\dagger , \quad \kappa = i H + \frac{1}{2} \sum_{i,j=1}^{d^2} D_{ij} F_j ^\dagger F_i .
\]

To shorten the notation, we introduce the projector on the Bell state, as \( \omega = \ket{\Phi_B} \bra{\Phi_B} \) and the projector on the complementary subspace as \( \omega_\perp = \mathbb{1} - \omega \). We find,
\[
d (\text{id} \otimes \mathcal{L}) [w] = d (\text{id} \otimes \phi) [w] - \text{id} \otimes \kappa w - w \text{id} \otimes \kappa ^\dagger ,
\]
and therefore
\[
\omega_\perp C_\perp \omega_\perp = d \omega_\perp (\text{id} \otimes \phi) [w] \omega_\perp = C_\phi , \quad (E.2)
\]
the Choi-matrix representation of the map \( \phi [\rho] \). The latter equality means that \( C_\phi \) already is orthogonal to \( w \). This
can be seen from

\[
\omega (\text{id} \otimes \phi)[\omega] = \sum_{i,j=1}^{d^2-1} D_{ij} \omega (\mathbb{1} \otimes F_i) \omega (\mathbb{1} \otimes F_j) \\
= \sum_{i,j=1}^{d^2-1} D_{ij} \text{tr}(F_i) \omega (\mathbb{1} \otimes F_j) = 0 ,
\]

and similarly \((\text{id} \otimes \phi)[\omega] \omega = 0\), also.

Finally, we consider the matrix representation \(C_\perp\) of \(C_\phi\), with respect to a basis \(\{|\phi_i\rangle\}_{i=1}^{d^2-1}\) orthogonal to \(|\Phi_B\rangle\). Then, we prove that \(C_\perp\) is related to \(D\) by an unitary transformation. For that purpose, consider the matrix elements of \(C_\perp\), given by

\[
(C_\perp)_{nm} = \langle \phi_n | C_\perp | \phi_m \rangle = d \langle \phi_n | (\text{id} \otimes \phi[\omega]) | \phi_m \rangle
\]

\[
= d \sum_{i,j=1}^{d^2-1} D_{ij} a_n a_j (\phi_n | \Phi^{(i)} \rangle \langle \Phi^{(j)} | \phi_m ) \tag{E.4}
\]

with \(\sqrt{a_i} |\Phi^{(i)}\rangle = \mathbb{1} \otimes F_i | \Phi_B \rangle\) and \(a_i = ||(\mathbb{1} \otimes F_i) \Phi_B ||^2\).

Now observe that

\[
\langle \Phi^{(i)} | \Phi^{(j)} \rangle = (\Phi_B | \mathbb{1} \otimes F_i^\dagger F_j^\dagger | \Phi_B ) = \frac{1}{d} \text{tr} \left( F_i^\dagger F_j \right) = \frac{1}{d} \delta_{ij},
\]

thus \(a_i = \frac{1}{d}\), independent of \(i\). Defining \(V_{ni} = \langle \phi_n | \Phi^{(i)} \rangle\) and substituting the value of \(a_i\) in Eq. \(E.4\) we end up with

\[
C_\perp = VDV^\dagger,
\]

with \(V\) unitary, given that \(|\phi_n\rangle\) and \(|\Phi^{(i)}\rangle\) are properly normalized quantum states.

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