Reality conditions inducing transforms for quantum gauge field theory and quantum gravity

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Abstract

For various theories, in particular gauge field theories, the algebraic form of the Hamiltonian simplifies considerably if one writes it in terms of certain complex variables. Also general relativity when written in the new canonical variables introduced by Ashtekar belongs to that category, the Hamiltonian being replaced by the so-called scalar (or Wheeler-DeWitt) constraint. In order to ensure that one is dealing with the correct physical theory one has to impose certain reality conditions on the classical phase space which generally are algebraically quite complicated and render the task of finding an appropriate inner product into a difficult one. This article shows, for a general theory, that if we prescribe first a canonical complexification and second a representation of the canonical commutation relations in which the real connection is diagonal, then there is only one choice of a holomorphic representation which incorporates the correct reality conditions and keeps the Hamiltonian (constraint) algebraically simple! We derive a canonical algorithm to obtain this holomorphic representation and in particular explicitly compute it for quantum gravity in terms of a Wick rotation transform.

1 Introduction

The motivation for this article comes from the attempt to solve the following problem in canonical quantum gravity:
The new canonical variables introduced by Ashtekar cast the constraints of general relativity into polynomial form. This is a major achievement since the constraints of general relativity when written in the usual metric (or ADM) variables are highly non-polynomial, not even analytic in these variables and this has been one of the major roadblocks to quantizing canonical gravity non-perturbatively so far.

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However, in the Lorentzian signature, the Ashtekar connection is complex while most of the mathematically rigorous contributions to quantizing this theory are valid only for real connections. A general direction to incorporate complex connections was introduced in [13] via the notion of a "coherent state transform". The specific techniques of [13] are, however, insufficient in the case of general relativity. What is needed is a transform which incorporates the correct reality conditions of Lorentzian general relativity and retains the simplicity of the constraints. In this article, we will solve this problem for a general theory and show that the solution is essentially unique.

In applications to gauge theory we will be dealing with three different situations:

Type 1): The phase space is coordinatized by a real pair \((A_i^a, P_i^a)\) where \(A\) is the connection for a compact gauge group and \(P\) is a vector density of weight one. The complexification is of the form \(A \to A^c := A + i \delta F[P]/\delta P\), \(P \to P^c := P\) and corresponds to an imaginary point transformation induced by the functional \(F[P]\). Such complex variables have some advantages in certain gauge field theories.

Type 2): Now \(F\) is a functional of \(A\) alone and the complexification is given by \(A \to A^c = A\), \(P \to P^c := P + i \delta F/\delta A\). Such kind of transform can be seen to simplify the algebraic form of physical Yang-Mills theory!

Type 3): The phase space is coordinatized by a real pair \((K_i^a, P_i^a)\) where \(K\) is now not a connection but just a 1-form that transforms homogenously under the compact gauge group and \(P\) is as before. The complexification is of the form \(K \to A^c := \delta F[P]/\delta P - i K\), \(P \to P^c := i P\) where \(F\) is the generating functional of the spin-connection associated with \(P\). This type of transform is the required one for quantum gravity and it also corresponds to a point transformation induced by \(F\), followed by a phase space Wick rotation!

The article is organized as follows:

In section 2 we propose an algorithm for the construction of a transform for any given theory in such a way that the reality conditions are guaranteed to be incorporated. The scheme is as follows:

a) Find complex canonical variables in which the algebraic form of the Hamiltonian (constraint) simplifies, b) restrict to the real canonical variables that we started with which defines an unphysical but algebraically simple Hamiltonian (constraint), c) find a suitable map from the unphysical theory to the physical one for one and the same representation in which a real connection is diagonal, d) analytically continue the result of applying the inverse of the map found in c) and e) find an inner product such that the map consisting of steps c) and d) (known as the generalized coherent state transform) is unitary.

Another idea is to follow the strategy suggested by Ashtekar [21] in the restricted context of quantum gravity: repeat steps a), b) and c) but stay in the same real representation, i.e. forget about d) and e)!

Because this approach is technically simpler whenever it can be made to work it is natural to proceed this way and it seems that it actually does work in the context of general relativity. However, in more general contexts, certain fundamental difficulties force one to adopt the generalized Segal-Bargman representation [22] which in turn requires the development of the steps d) and e). We will comment on the
nature of these difficulties and, in the appendix, suggest an approach to circumvent them in favorable cases.

In section 3 we deal with the transform of type 1 and 2 and give interesting model theories for which this kind of transform simplifies the quantization.

In section 4 we discuss the transform of type 3 and apply it then to quantum gravity when formulated in the the Ashtekar variables. It turns out that the real unphysical theory mentioned above is related to Euclidean gravity and therefore the Wick rotation alluded to is to be expected.

Suffice it to say that with this transform at our disposal we can just forget about the complex representation and do all computations in the real representation for which powerful techniques have already been developed [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] (this is true for both approaches mentioned above, at least after having solved the constraints).

If all the steps sketched in this paper could be completed in a rigorous fashion then, together with the recent proof of existence of an anomaly-free regularization of the Hamiltonian constraint [17] of Euclidean quantum gravity, we would have shown existence of non-perturbative canonical quantum gravity as a rigorously defined theory.

2 Generalized Coherent State Transform

Before beginning to construct the algorithm of incorporating the correct reality conditions into the quantum theory for a general theory, let us sketch the main ideas. Suppose we are given some real phase space $\Gamma$ (finite or infinite dimensional) coordinatized by a canonical pair $(A, P)$ (we suppress all labels like indices or coordinates) where we would like to think of $A$ as the configuration variable and $P$ as its conjugate momentum. Let the Hamiltonian (constraint) on $\Gamma$ be given by a function $H(A, P)$ which has a quite complicated algebraic form and suppose that it turns out that it can be written in polynomial form if we write it in terms of a certain complex canonical pair $(A^c, P^c) := W^{-1}(A, P)$, that is, the function $H^c := H \circ W$ is polynomial in $(A^c, P^c)$ (the reason why we begin with the inverse of the invertible map $W$ will become clear in a moment). We will not be talking about kinematical constraints like Gauss and diffeomorphism constraint etc. which take simple algebraic forms in any kind of variables.

The requirement that the complex pair $A^c, P^c$ is still canonical is fundamental to our approach and should be stressed at this point. It should also be stressed from the outset that we are not complexifying the phase space, we just happen to find it convenient to coordinatize it by a complex valued set of functions. The reality conditions on these functions are encoded in the map $W$.

Of course, the theory will be much easier to solve (for instance computing the spectrum (kernel) of the Hamiltonian (constraint) operator) in a holomorphic representation $\mathcal{H}_c$ in which the operator corresponding to $A^c$ is diagonal rather than in the real representation $\mathcal{H}$ in which the operator corresponding to $A$ is diagonal. According to the canonical commutation relations and in order to keep the Hamiltonian (constraint) as simple as possible, we are naturally led to represent the operators
on $H$ corresponding to the canonical pair $(A, P)$ by $(\hat{A}\psi)(A) = A\psi(A), (\hat{P}\psi)(A) = -i\hbar\delta\psi(A)/\delta A$. Note that then in order to meet the adjointness condition that $(\hat{A}, \hat{P})$ be self-adjoint on $H$, we are forced to choose $H = L_2(\mathcal{C}, d\mu_0)$ where $\mathcal{C}$ is the quantum configuration space of the underlying theory and $d\mu_0$ is the unique (up to a positive constant) uniform measure on $\mathcal{C}$, that is, the Haar measure (in the special case of gauge theory $d\mu_0$ coincides precisely with the induced Haar measure on $\overline{\mathcal{A}/\mathcal{G}}$ [4]). In order to avoid confusion we introduce the following notation throughout this article:

Denote by $\hat{K} : H \rightarrow \mathcal{H}_0$; $\psi(A) \rightarrow \psi(A^c)$ and $\hat{K}^{-1}$ the operators of analytic continuation and restriction to real arguments respectively. The operators corresponding to $A^c, P_0$ can be represented on the two distinct Hilbert spaces $H$ and $\mathcal{H}_0$. On $H$ we just define them by some ordering of the function $W^{-1}$, namely $(\hat{A}^c, \hat{P}_0) := W^{-1}(\hat{A}, \hat{P})$. On $\mathcal{H}_0$, the fact that $A^c, P_0$ enjoy canonical brackets allows us to define their operator versions simply by (and this is why it is important to have a canonical complex pair) $(\hat{A}', \hat{P}') = (\hat{K}\hat{A}\hat{K}^{-1}, \hat{K}\hat{P}\hat{K}^{-1})$, i.e. they are just the analytic extension of $A, P$, that is, $(\hat{A}\psi)(A^c) = A^c\psi(A^c), (\hat{P}\psi)(A^c) = -i\hbar\delta\psi(A^c)/\delta A^c$.

But how do we know that the operators $\hat{A}^c, \hat{P}_0$ on $H$ and $\hat{A}', \hat{P}'$ on $\mathcal{H}_0$ are the quantum analogues of the same classical functions $A^c, P_0$ on $\Gamma$? To show that there is essentially only one answer to this question is the first main result of the present article.

Namely, when can two operators defined on different Hilbert spaces be identified as different representations of the same abstract operator? They can be identified iff their matrix elements coincide. Due to the canonical commutation relations we have to identify in particular also the matrix elements of the identity operator, that is, scalar products between elements of the Hilbert spaces. The only way that this is possible is to achieve that the Hilbert spaces are related by a unitary transformation and that the two operators under question are just images of each other under this transformation.

In order to find such a unitary transformation we have to relate the two sets of operators $\hat{A}^c, \hat{P}_0$ and $\hat{A}', \hat{P}'$ via their common origin of definition, namely through the set $\hat{A}, \hat{P}$.

The first hint of how to do this comes from the observation that both pairs $(A, P)$ and $(A^c, P_0)$ enjoy the same canonical commutation relations, i.e. they are related by a canonical complexification. Therefore the map $W$ must be a complex symplectomorphism, that is, an automorphism of the Poisson algebra over $\Gamma$ which preserves the algebra structure but, of course, not the reality structure. Let $iC$ be the infinitesimal generator of this automorphism. As is well-known (we repeat the argument below) it follows from these assumptions that if the Complexifier $\hat{C}$ is the operator corresponding to $C$ on $H$ then we may define the quantum analogue of $(\hat{A}^c, \hat{P}_0)$ on $H$ by $(\hat{A}'^c, \hat{P}_0) := (W^{-1}\hat{A}W, W^{-1}\hat{P}W)$ where $\hat{W} := \exp(-1/\hbar\hat{C})$ is called the Wick rotator (due to its role in quantum gravity). That is, the generator $\hat{C}$ provides for a natural ordering of the function $W^{-1}(A, P)$.

So let us write the operators on $\mathcal{H}_0$ in terms of the operators on $H$. We have

\[(\hat{A}', \hat{P}') = (\hat{K}\hat{A}\hat{K}^{-1}, \hat{K}\hat{P}\hat{K}^{-1}) = (\hat{U}\hat{A}'\hat{U}^{-1}, \hat{U}\hat{P}_0\hat{U}^{-1})\] (2.1)

where we have defined

\[\hat{U} := \hat{K}\hat{W}\] (2.2)
So if we could achieve that $\hat{U}$ is a unitary operator from $\mathcal{H}$ to $\mathcal{H}_{\mathbb{C}}$ then our identification would be complete! The operator $\hat{U}$ coincides in known examples with the so-called coherent state transform (\textsubscript{13}, there it is called $C_t$) so that we call it the \textit{generalized coherent state transform}. In other words, the generalized coherent state transform can be viewed as the \textit{unique} answer to our question. Any other unitary transformation $\hat{u}$ between the Hilbert spaces necessarily corresponds to a different complexification $\hat{w} = \hat{K}^{-1} \hat{u}$ of the classical phase space in which the Hamiltonian (constraint) takes a more complicated appearance. Note that any real canonical transformation corresponds to a unitary transformation in quantum theory, so the coherent state transform can also be characterized as the “unitarization” of the complex canonical transformation that we are dealing with.

Another characterization of the coherent state transform $\hat{U}$ follows from the simple formula $\hat{U}$ = $\hat{K}\hat{W}$: it is the unique solution to the problem of how to identify analytic extension with the particular choice of complex coordinates $A_{\mathbb{C}}, P_{\mathbb{C}}$) on the real phase space $\Gamma$ as defined by $\hat{W}$.

As a bonus, our adjointness relations are trivially incorporated! Namely, because any operator $\hat{O}_q$ on $\mathcal{H}$ written in terms of $\hat{A}_{\mathbb{C}}, \hat{P}_{\mathbb{C}}$ is defined by $\hat{W}^{-1}\hat{O}\hat{W}$ where $\hat{O}$ is written in terms of $\hat{A}, \hat{P}$ and because $\hat{O}_q$ is identified on $\mathcal{H}_{\mathbb{C}}$ with $\hat{O}' = \hat{K}\hat{O}\hat{K}^{-1} = \hat{U}\hat{O}_q\hat{U}^{-1}$ we find due to unitarity that $(\hat{O}')^\dagger = \hat{U}\hat{O}_q^\dagger\hat{U}^{-1}$ where the adjoints involved on the left and right hand side of this equation are taken on $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{H}$ respectively. Note that the adjoint of $\hat{O}_q$ follows unambiguously from the known adjoints of $\hat{A}, \hat{P}$ and coincides to zeroth order in $\hbar$ with the complex conjugate of its classical analogue. Therefore, $(\hat{O}')^\dagger$ is identified with $\hat{O}_q^\dagger$ as required.

Finally we see that in extending the algebraic programme \textsubscript{14} from a real representation to the holomorphic representation of the Weyl relations we only have one additional input, everything else follows from the machinery explained below and can be summarized as follows:

- **Input A**: define an automorphism $W$ (preferencebly such that the constraints simplify).
- **Task A**: determine the infinitesimal generator $C$ of $W$.
- **Input B**: define a real $^*$ representation $\mathcal{H}$.
- **Task B**: determine a holomorphic representation $\mathcal{H}_{\mathbb{C}}$ so that $\hat{U} = \hat{K}\hat{W}$ is unitary.

Note that input B is also part of the programme if one were dealing only with the real representation so that input A is the only additional one. Task A is necessary if we want to express a given $W$ in terms of the phase space variables which is unavoidable in order to define $W$.

In the sequel we will explain the details of the considerations made above. In particular we display a standard solution to Task B so that Task A is the only non-trivial problem left (every two solutions are related necessarily by a unitary transformation so that they are physically indistinguishable)!

In fact, the second main result of this paper is that we have found the infinitesimal generator of the Wick rotation required for canonical quantum gravity.

To be concrete we will work in the context of gauge field theory but it should be clear that everything we say can actually applied to a general theory (compare \textsubscript{17}).
2.1 The complexifier and the Wick rotation

In this section, C will be a (not necessarily positive, not necessarily real) functional, called the Complexifier, on the real phase space coordinatized by \((A^i, P^a)\) where \(A\) is a real-valued connection for a compact gauge group and \(P\) is its real-valued conjugate momentum, that is, a vector density of weight one transforming homogenously under gauge transformations.

Assume that the Hamiltonian (constraint) \(H(A, P)\) has a complicated algebraic form in terms of the real variables \((A, P)\) but that it simplifies considerably if one writes it in terms of certain complex combinations \((A^C, P^a) := W^{-1}(A, P)\), that is, the function \(H_C := H \circ W\) is a low order polynomial. Due to the fact that in quantum gravity the map \(W\) is a phase space Wick rotation we will refer to it in the sequel as the Wick rotation transformation.

The important role of \(C\) is to be the infinitesimal generator of this map, that is

\[
A^C_a (x) = W^{-1} \cdot A^i_a (x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \{ A^i_a (x), C \} (n)
\]

\[
P^a_C (x) = W^{-1} \cdot P^a_j (x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \{ P^a_j (x), C \} (n)
\] (2.3)

where, as usually, the multiple Poisson bracket is iteratively defined by \(\{ f, C \} (0) = f, \{ f, C \} (n+1) = \{ \{ f, C \} (n), C \}.\) This equation can be inverted to give

\[
A^i_a (x) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{ A^C_a (x), C \} (n)
\]

\[
P^a_j (x) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{ P^a_C (x), C \} (n)
\] (2.4)

Because of this, \(W\) is an automorphism of the Poisson algebra over the real phase space \(W \cdot f = f \circ W\), in particular it preserves the symplectic structure, but it fails to preserve the reality structure. In fact it follows immediately from (2.3) and (2.4) that the reality conditions are given by (the bar denotes complex conjugation)

\[
\bar{A}_a^C (x) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{ A^C_a (x), C + \bar{C} \} (n)
\]

\[
\bar{P}^a_C (x) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{ P^a_C (x), C + \bar{C} \} (n).
\] (2.5)

Note that the existence of \(W\) does not imply that classical solutions are mapped into solutions! That is, assume that we have found a physical solution \(H(A_0, P_0) = E = \text{const.}\), then in general \(H_C(A_0, P_0) = H(W(A_0, P_0)) \neq \text{const.}\). (this has nothing to do with the fact that \(W\) does not preserve the reality conditions, rather it follows from the fact that \(\{H, C\} \neq 0\) by construction since \(W\) is supposed to turn the complicated algebraic form of \(H\) into a simpler one). However, it will turn out that the quantum analogue of \(W\) maps generalized eigenfunctions into generalized eigenfunctions!
2.2 The generalized coherent state transform

We are now going to assume that the functional $C$ has a well-defined quantum analogue, that is, $\hat{C}$ will be a (not necessarily bounded, not necessarily positive, not necessarily self-adjoint) operator on $\mathcal{H} := L_2(\mathcal{A}/G, d\mu_0)$ (for a definition and an overview over all the constructions that have to do with $\mathcal{A}/G$ we refer the reader to [14]. A reader unfamiliar with these developments can proceed by just taking $C = \mathcal{A}/G$ as the quantum configuration space of connections modulo gauge transformations and $\mu_0$ as the unique uniform (Haar) measure thereon).

Further, we would like to take equations (2.3)-(2.5) over to quantum theory, that is, we replace Poisson brackets by commutators times $1/i\bar{\hbar}$ and we replace complex conjugation by the adjoint operation with respect to measures $\mu_0, \nu$ for the real and holomorphic representations respectively of which the latter, $\nu$, is yet to be constructed.

So let $\hat{O} = O(\hat{A}, \hat{P})$ be an operator on $\mathcal{H}$, where $O$ is some analytical function (in particular loop and strip operators [14]). Using the operator identity $e^{-A}Be^A = \sum_{n=0}^{\infty} \frac{1}{n!} [B, A]_n$ and defining on $\mathcal{H}$ $\hat{W}_t := \exp(-\bar{\hbar}\hat{C})$ (2.6)
we find that on $\mathcal{H}$ the translation of (2.3) becomes

$$\hat{O}_{\psi} := O(\hat{A}^{\Psi}, \hat{P}_{\psi}) = \hat{W}_t^{-1}\hat{O}\hat{W}_t$$

(2.7)

with $t = 1/\bar{\hbar}$. It follows from these remarks that the adjoint of $\hat{O}_{\psi}$ on $\mathcal{H}$ is given by

$$\hat{O}_{\psi}^\dagger = [\hat{W}_t^\dagger\hat{W}_t]\hat{O}_{\psi}[\hat{W}_t^\dagger\hat{W}_t]^{-1}$$

(2.8)

which can be seen to be one particular operator-ordered version of the adjointness relations that follow from the requirement that the classical reality conditions (2.5) should be promoted to adjointness-relations in the quantum theory.

So we have solved the reality conditions on $\mathcal{H}$ (actually we have done this already by turning $\hat{A}, \hat{P}$ into self-adjoint operators). Therefore, as said before, they are automatically also incorporated on $\mathcal{H}_{\psi} := L_2(\mathcal{A}^{\Psi}/G^{\Psi}, d\nu_t) \cap \text{Hol}(\mathcal{A}^{\Psi}/G^{\Psi})$ (i.e square integrable functions of complexified connections which are holomorphic; here $\mathcal{A}^{\Psi}/G^{\Psi}$ is the quantum configuration space of complexified connections modulo gauge transformations [13]) upon constructing the measure $\nu_t$ in such a way that the operator

$$\hat{U}_t := \hat{K}\hat{W}_t,$$ 

(2.9)

is unitary (as before, $\hat{K}$ means analytic extension). Namely, on $\mathcal{H}_{\psi}$ the operator corresponding to the classical function $O(A^{\Psi}, P_{\psi})$ is just given by $\hat{O}' = \hat{K}\hat{O}\hat{K}^{-1} = \hat{U}_t\hat{O}_{\psi}\hat{U}_t^{-1}$ so that on $\mathcal{H}_{\psi}$

$$(\hat{O})^\dagger = [\hat{U}_t(\hat{W}_t^\dagger\hat{W}_t)\hat{U}_t^{-1}]\hat{O}'[\hat{U}_t(\hat{W}_t^\dagger\hat{W}_t)\hat{U}_t^{-1}]^{-1}$$

$$(\hat{K}(\hat{W}_t^\dagger\hat{W}_t)\hat{K}^{-1}]\hat{O}'[\hat{K}(\hat{W}_t^\dagger\hat{W}_t)\hat{K}^{-1}]^{-1}$$

(2.10)

which is just the image of (2.8). Note that $\hat{W}_t^{\Psi} = \hat{W}_t$ on $\mathcal{H}$ corresponding to the fact that classically the complexifier is unchanged if we replace $A, P$ by $A^{\Psi}, P_{\psi}$. 

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It should be stressed at this point that if $\hat{C}$ is not a positive self-adjoint operator, we will assume that $\hat{U}_t$ for positive $t$ can still be densely defined (note that $\hat{U}_{-t} \neq \hat{U}_t^{-1}$ due to the analytic continuation involved and that in general $\hat{W}_{-t} \neq (\hat{W}_t)^{-1}$ unless $\hat{C}$ is bounded), that is, there is a dense subset $\Phi$ of $\mathcal{H}$ so that the analytic continuation of the elements of its image $\hat{W}_t \Phi$ under $\hat{W}_t$ are elements of a dense subset of $\mathcal{H}_\Phi$. We do not assume that $\hat{W}_t$ itself can be densely defined on $\Phi$ as an operator on $\mathcal{H}$! Intuitively what happens here is that while $\hat{W}_t \phi$ may not be normalizable with respect to $\mu_0$ for any $\phi \in \Phi$, its analytic continuation will be normalizable with respect to $\nu_t$ by construction since $\hat{U}_t$ is unitary and thus bounded, just because the measure $\nu_t$ falls off much stronger at infinity than $\mu_0$. So we see that going to the complex representation could be forced on us. This is a second characterization of the coherent state transform: not only is it a unique way to identify a particular complexification with analytic continuation, it also provides us with the necessary flexibility to choose a better behaved measure $\nu_t$ which enables us to work in a representation in which $\hat{W}_t$ or, rather, $\hat{W}'_t$ is well-defined which is important because only then do we quantize the original theory defined by $(\mathcal{H}, \hat{H})$.

The form of the operator $\hat{W}_t$ already suggests the parallel to the developments in $\mathbb{R}$. Namely, in complete analogy we define now the “coherent state transform” (there called $C_t$) associated with the “heat kernel” $\hat{W}_t$ for the operator $\hat{C}$ to be the following map

$$\hat{U}_t : \mathcal{H} \to \mathcal{H}_\Phi, \quad \hat{U}_t[f](A\Phi) := < A\Phi, \hat{W}_t f > := < A, \hat{W}_t f >_{|A= A\Phi} \quad (2.11)$$

which on functions cylindrical with respect to a graph consisting of $n$ edges $e_I$ reduces to (provided that $\hat{C}$ leaves that subspace invariant)

$$\hat{U}_{t,\gamma}[f_\gamma](g_1^\Phi,...,g_n^\Phi) := \int_{G^n} d\mu_{0,\gamma}(g_1,...,g_n) \rho_{t,\gamma}(g_1^\Phi,...,g_n^\Phi ; g_1,...,g_n) f_\gamma(g_1,...,g_n) \quad (2.12)$$

where $g_I := h_{e_I}(A), g_I^\Phi := h_{e_I}(A^\Phi)$ are the holonomies along the edge $e_I$. Here $\rho_{t,\gamma}(g_1^I ; h_I) := < g_1,...,g_n, \exp(-t\hat{C}_\gamma)h_1,...,h_n >$ is the kernel of $\hat{W}_t$ and $\hat{C}_\gamma$ is the projection of $\hat{C}$ to the given cylindrical subspace of $L_2(A\Phi/G, d\mu_0)$. So the map $\hat{U}_t$ is nothing else than kernel convolution followed by analytic continuation (the kernel, if it exists, is real analytic on $G^n$ and therefore has a unique analytic extension).

Note that the transform is consistently defined on cylindrical subspaces of the Hilbert space because its generator $\hat{C}$ acts primarily on the connection and does not care how we write a given cylindrical function on graphs that are related to each other by inclusion.

### 2.3 Isometry

The next step is to show that there indeed exists a cylindrical measure on $A^\Phi/G^\Phi$ such that the transform $\hat{U}_t$ is an isometry. The following developments differ considerably from the techniques applied in $\mathbb{R}^3$ because those methods turn out to be sufficient only if the operator $\hat{C}$ has three additional, quite restrictive properties:

1. it is a positive self-adjoint operator,
2. it is not only gauge invariant but also left invariant and,
3. the image of a function cylindrical with respect to some graph $\gamma$ under $\hat{C}$ is again
cylindrical with respect to the same graph and it does not change the irreducible representations of the function when written in a spin-network basis \cite{14,16}. Requirement 0) ensures existence of $\hat{W}_t$ as an operator on $\mathcal{H}$, requirement 1) implies that the kernel of $\hat{W}_t$ only depends on $\hat{g}^{-1}$ and requirement 2) implies that the measure on $C^g$ can be chosen in the $G$ averaged form \cite{22}. Requirements 1), 2) are satisfied if and only if $\hat{C}$ is a gauge-invariant operator that is constructed purely from left-invariant vector fields on $G$. This particular form of the transform may be sufficient for applications in certain quantum gauge field theories but not in quantum gravity. Although the associated operator $\hat{C}$ in quantum gravity does leave every subspace cylindrical with respect to any graph invariant, it violates the rest of the requirements mentioned in 0), 1), 2), in particular it is not manifestly positive albeit self-adjoint.

The subsequent sketch of a construction of an isometry inducing measure applies to a general theory, in particular, there are no additional requirements for $\hat{C}$. Isometry means that for any $\psi, \xi$ in the domain of $\hat{U}_t$ we have

$$\int_{A/G} d\mu_0(A) \bar{\psi}(A) \xi(A) = \int_{A^q/G^q} d\nu_t(A^q, \bar{A}^q) \bar{\psi}(A^q, \bar{A}^q) \hat{U}_t \xi(A^q). \quad (2.13)$$

Denote by $\mu_0^g(A^q)$ the holomorphic extension of $\mu_0$ and by $\bar{\mu}_0^q(A^q)$ its anti-holomorphic extension which due to the positivity of $\mu_0$ are just complex conjugates of each other.

We now make the ansatz

$$d\nu_t(A^q, \bar{A}^q) = d\mu_0^g(A^q) \otimes d\bar{\mu}_0^q(\bar{A}^q) \nu_t(A^q, \bar{A}^q), \quad (2.14)$$

where $\nu_t$ is a distribution, the virtue of which is that if $\bar{C}^t$ is the complex conjugate of the adjoint of $\hat{C}$ with respect to $\mu_0$ then its analytic extension $\overline{(\bar{C}^t)^\prime}$ is the result of moving the operator $\hat{C}^t$ from the wavefunction $\xi$ to $\nu_t$ with respect to $d\mu_0^q \otimes d\bar{\mu}_0^q$, i.e. we are able to invoke our knowledge about the adjoint of $\hat{W}_t$ on $\mathcal{H}$. We will see this in a moment.

Proceeding formally, we compute the right hand side of (2.13) using the definition (2.6) (obviously $\overline{(\bar{C}^t)^\prime}$ and $\overline{(\bar{C}^t)}^\prime$ commute, the overline denotes complex conjugation (in particular $A^q \to A^q$, $\delta/\delta A^q \to \delta/\delta \bar{A}^q$) of the operator and not anti-holomorphic extension)

$$\int d\mu_0^q(\bar{A}^q) (\hat{W}_t \psi)(\bar{A}^q) \int d\mu_0^g(A^q) \nu_t(A^q, \bar{A}^q) (\hat{W}_t \xi)(A^q)$$

$$\int d\bar{\mu}_0^q(A^q) \overline{\hat{W}_t \psi}(A^q) \bar{K} \int d\mu_0(A) \nu_t(A^q) (\hat{W}_t \xi)(A)$$

$$\int d\mu_0^q(A^q) \overline{\hat{W}_t \psi}(A^q) \bar{K} \int d\mu_0(A) \bar{W}_t \nu_t(A^q) (\hat{W}_t \xi)(A)$$

$$\int d\mu_0^q(A^q) \xi(A^q) \int d\mu_0^g(B^q) \overline{\mu_t(A^q)} (\hat{W}_t \xi)(B^q)$$

$$\int d\mu_0^q(A^q) \xi(A^q) \bar{K} \int d\mu_0(A) \mu_t(A^q) (\hat{W}_t \xi)(A)$$

$$\int d\mu_0^q(A^q) d\mu_0^q(A^q) \nu_t(A^q, \bar{A}^q) (\hat{W}_t \xi)(A^q) (\bar{W}_t^\dagger) \nu_t(A^q, \bar{A}^q). \quad (2.15)$$
where the prime means, as usual, analytic extension \( A \to A^\prime, \delta/\delta A \to \delta/\delta A^\prime \)
and the adjoint involved is taken with respect to \( \mu_0 \). Here we have abbreviated
\( \nu_{t,A^\prime}(A^\prime) = \nu_t(A^\prime, A^\prime) \) and \( \mu_{A^\prime,t}(B^\prime) := \left( \hat W_t^\dagger \right)_\nu \nu_t(A^\prime, B^\prime) \) and the bar means
complex conjugation.

Equation (2.13) can now be solved by requiring
\[
\nu_t(A^\prime, A^\prime) := \left( \left( \hat W_t^\dagger \right)_t \right)^{-1} \left( \left( \hat W_t^\dagger \right)_t \right)^{-1} \delta(A^\prime, A^\prime)
\]  
(2.16)

where the distribution involved in (2.16) is defined by
\[
\int_{A^\prime} d\mu_0(A^\prime) d\nu_t(A^\prime) f(A^\prime) \delta(A^\prime, A^\prime) = \int_{A^\prime} d\mu_0(A) f(A, A) .
\]  
(2.17)

Whenever (2.16) exists and the steps to obtain this formula can be justified we have
proved existence of an isometricity inducing positive measure on \( A^\prime / G \) by explicit
construction. The rigorous proof for this \([18]\) is by proving existence of (2.16) on
cylindrical subspaces, so strictly speaking \( dv_t \) is only a cylindrical measure. The
measure is self-consistently defined because the operator \( \hat C \) is.

Note that the proof is immediate in the case in which \( \hat C \) is a positive and self-adjoint
and therefore can be viewed by itself as an interesting extension of \([14]\). In particular
it coincides with the method introduced by Hall \([22]\) in those cases when \( \hat C \) is the
Laplacian on \( G \) but our technique allows for a more straightforward computation of
\( \nu_t \).

### 2.4 Quantization

We are now equipped with two Hilbert spaces \( \mathcal{H} := L_2(\mathbb{R}^2, d\mu_0) \) and \( \mathcal{H}_q := L_2(\mathbb{R}^2, dv_t) \cap \mathcal{H}(\mathbb{R}^2 / G) \) which are isometric and faithfully implement the
adjointness relations among the basic variables. \( \mathcal{H} \) will be called the real representation
and \( \mathcal{H}_q \) the holomorphic or complex representation.

The last step in the algebraic quantization programme is to solve the theory, that is,
to find the spectrum of the Hamiltonian (or the kernel of the Hamiltonian constraint)
and observables, that is, operators that commute with the physical constraint operators.
In more concrete terms it means the following \([14]\) :

Let \( \hat H_q := \mathcal{H}_q(\hat A, \hat P) \) be a convenient ordering of \( \mathcal{H}_q(A, P) \) such that its adjoint
on \( \mathcal{H} \) corresponds to the complex conjugate of its classical analogue (that is, write
\( H_q = a + ib \) where \( a, b \) are real and order \( a, b \) symmetrically) and let \( \hat H' = H_q(\hat A', \hat P') \)
be its analytic extension. Choose a topological vector space \( \Phi(\Phi_q) \) and denote by
\( \Phi'(\Phi_q) \) its topological dual. These two spaces are paired by means of the measure
\( \mu_0(\nu_t) \), for instance \( f[\phi] := \int_{\mathbb{R}^2} d\mu_0(A) f[A]|[\phi]. \) \( \Phi(\Phi_q) \) is by construction dense
in its Hilbert space completion \( \mathcal{H}(\mathcal{H}_q) \). We will be looking for generalized eigenvectors
\( f_\lambda(f_q) \) \([23]\), that is, elements of \( \Phi'(\Phi_q) \) with the property that there exists a complex
number \( \lambda \) such that \( f_\lambda[H_q^\dagger] = \lambda f_\lambda[\phi] f_q[(\hat H')^\dagger ] = \lambda f_q[\phi] \) for any \( \phi \in \Phi(\Phi_q) \).
Here we have assumed that \( \hat H = \hat W_t^\dagger H_q \hat W_t(H') \) are self-adjoint on \( \mathcal{H}(\mathcal{H}_q) \).

Given this general setting we have at least three strategies at our disposal :

Strategy I) :
We start working on $\mathcal{H}_Q$. This means that we would try to find a convenient ordering of the operator $\hat{H}' := \mathcal{H}_Q(\hat{A}', \hat{P}')$. The Hamiltonian (constraint) on $\mathcal{H}$ now is defined to be the image under the inverse coherent state transform $\hat{H} := \hat{U}_t^{-1}\hat{H}'\hat{U}_t$ which to zeroth order in $\hbar$ coincides with one ordering of $\mathcal{H}(A, P)$ but in general will involve an infinite power series in $\hbar$. That is, we have made use of the freedom that we always have in defining the quantum analog of a classical function, namely to add arbitrary terms which are of higher order in $\hbar$.

Of course, since the constraint is simple on $\mathcal{H}_Q$ we solve it in this representation as well as the problem of finding observables. After that we can go back to $\mathcal{H}$ which is technically easier to handle and compute spectra of the observables found and so on, thus making use of the powerful calculus on $\mathcal{A}/\mathcal{G}$ that has already been developed in \textcite{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. This calculus can in particular be used to find a regularization in which $\mathcal{C}'$, $\hat{H}_\mathcal{C}$ are self-adjoint on $\mathcal{H}$ if they classically correspond to real functions because it then follows that also their images under $\hat{W}_t^{-1}$, that is, $\hat{C}, \hat{H}$, are self-adjoint on $\mathcal{H}$ and then, by unitarity, the same holds for $\hat{C}', \hat{H}'$ on $\mathcal{H}_Q$.

In this way, $\mathcal{H}_Q$ mainly arises as an intermediate step to solve the spectral problem. Strategy II):

The following strategy is suggested by Ashtekar \textcite{20} in the restricted context of quantum general relativity: Stick with the real representation all the time. The general idea of working with real connections goes back to \textcite{1} and was revived by Barbero in \textcite{19}. The strategy now seems feasible because we now have a key new ingredient at our disposal—the Wick transform—which enables the real representation to simplify both, the reality conditions and the constraints.

Here we consider this idea in the general case. That means, we look for a convenient ordering of $\hat{H}_Q := \mathcal{H}_Q(\hat{A}, \hat{P})$, then the physical Hamiltonian (constraint) is defined by $\hat{W}_t^{-1}\hat{H}_Q\hat{W}_t$ and agrees to zeroth order in $\hbar$ with some ordering of $\mathcal{H}(A, P)$. The advantage of this approach is obvious: we never need to construct the measure $\nu_t$ which is only cylindrical so far while the measure $\mu_0$ is known to be $\sigma-$additive. Although we continue to work with a connection-dynamics formulation, the complex connection drops out of the game altogether! All the results in \textcite{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} are immediately available while in strategy II) one could do so only after having solved the spectral problem (constraint).

Why then, would we ever try to quantize along the lines of strategy I)? This is because there can be in general obstructions to find the physical spectrum or kernel directly from $\hat{H}_Q$. This is also the reason why we extended the programme as to construct the coherent state transform. Such obstructions can arise as follows: $\mathcal{H}_Q(A, P)$ will be in general neither positive nor real valued even if $\mathcal{H}(A, P)$ is, in which case it is questionable whether one is in principle able to find the correct spectrum from the former Hamiltonian (constraint). (We will see an example of such a potential problem in the next section in the context of Yang-Mills theories and also in the appendix). This is so because both $\hat{W}_t$ and $\hat{U}_t$ preserve the spectrum wherever they are defined, meaning that if we fail to have coinciding spectra for $\hat{H}, \hat{H}_Q$ then $\hat{W}_t$ is ill-defined as a map on $\mathcal{H}$ or on the dense subset $\Phi$ while $\hat{U}_t$ is always well-defined on $\Phi$ (by construction) as an operator between $\mathcal{H}$ and $\mathcal{H}_Q$ since it is unitary.

A fortunate case is when the topological vector space $\Phi$ is preserved by $\hat{W}_t$: then
generalized eigenvectors \( f_\lambda \) of \( \hat{H}_\Phi \) are mapped (as elements of the topological dual space \( \Phi' \)) into generalized eigenvectors \( \hat{W}_t^\dagger f_\lambda \) of \( H \) with the same eigenvalue. The proof is easy: we have for any \( \phi \in \Phi \):
\[
\hat{W}_t^\dagger f_\lambda [H\phi] = f_\lambda [\hat{W}_t H\phi] = f_\lambda [\hat{H}_\Phi \hat{W}_t \phi] = \lambda \hat{W}_t^\dagger f_\lambda [\phi]
\]
as claimed. Note that it was crucial in this argument that \( \hat{W}_t \phi \in \Phi \).

There are indications [18] that we are lucky in the case of quantum gravity. Even if we are not lucky we might fix the situation as follows: just decrease the size of \( \Phi \) until \( \hat{W}_t \) becomes well-defined. This increases the size of \( \Phi' \) and therefore turns more and more generalized eigenvectors into well-defined distributions on \( \Phi \).

We will see an example of this in the appendix.

A minor, disadvantage of strategy II) is as follows: while we can find physical observables \( \hat{O} \) by looking for operators \( \hat{O}_\Phi \) that commute with \( \hat{H}_\Phi \) and then map them according to \( \hat{O} := \hat{W}_t^{-1} \hat{O}_\Phi \hat{W}_t \), since \( \hat{W}_t \) is not unitary on \( H \) we need to compute the expectation values \( \langle \psi, (\hat{W}_t^{-1})^\dagger \hat{W}_t^{-1} \hat{O}_\Psi \xi \rangle \) rather than just \( \langle \psi, \hat{O}_\Psi \xi \rangle \).

Via strategy I) we could compute everything either on \( \Phi \) or \( \hat{H}_\Phi \), whatever is more convenient, because \( \hat{U}_t \) is unitary and so does not change expectation values.

Strategy III)

The following strategy is a slight modification of strategy II): the viewpoint is that the eigenvalue problem of the Hamiltonian (constraint) \( \hat{H}_\Phi \) is just an intermediate step. So we first look for all formal eigenvectors \( f_\lambda \) of \( \hat{H}_\Phi \); that is, we do not even care whether they are generalized eigenvectors. Then we map these solutions with \( \hat{W}_t^{-1} \) into formal eigenvectors of \( \hat{H}_\Phi \) with the same eigenvalue and select those as the physical ones which define well-defined distributions on \( \Phi \). In this way the requirement that \( \hat{W}_t \) is a well-defined operator on \( H \) drops out. The disadvantage is that we cannot apply the machinery of algebraic quantization [14] immediately to \( \hat{H}_\Phi \) and so have to map first with \( \hat{W}_t^{-1} \). The complications associated with this step can be seen to be similar with the construction of \( \nu_t \): so let us assume that \( \hat{H}_\Phi f_\lambda = \lambda f_\lambda \) formally then \( \hat{H} \hat{W}_t^{-1} f_\lambda = \lambda \hat{W}_t^{-1} f_\lambda \) rigorously provided that \( \hat{W}_t^{-1} f_\lambda \in \Phi' \) which defines \( \Phi \). Likewise, if formally (note that \( (\hat{H}')^\dagger = \hat{H}' = \hat{K} \hat{H}_\Phi \hat{K}^{-1} = \hat{U}_t \hat{H}_\Phi \hat{U}_t^{-1} \)) \( \hat{H}' f_\lambda = \lambda f_\lambda \) then rigorously \( \hat{H} \hat{W}_t^{-1} f_\lambda = \lambda \hat{W}_t^{-1} f_\lambda \) provided that \( f_\lambda' = \hat{K} f_\lambda \in \Phi' \) where \( \Phi' = \hat{U}_t \Phi \), namely for any \( \phi \in \Phi \) we have \( <\hat{W}_t^{-1} f_\lambda, \phi> = <\hat{U}_t \hat{W}_t^{-1} f_\lambda, \hat{U}_t \phi> = <f_\lambda', \hat{U}_t \phi> \).

So we see that the decision of whether \( \hat{W}_t^{-1} f \in \Phi' \) or \( f' \in (\hat{U}_t \Phi)' = \Phi'_\Phi \) are isomorphic. In the first case we have to map a formal eigenvector \( f \) of \( \hat{H}_\Phi \) with \( \hat{W}_t^{-1} \) to test on \( \Phi' \), in the second case we have to analytically continue this eigenvector and test on \( \Phi'_\Phi \) (we take then \( \Phi'_\Phi \) as given through \( \nu_t \) and define \( \Phi := \hat{U}_t^{-1} \Phi'_\Phi \)) whether we should accept \( f \).

We should stress here that the difference between strategy II),III) consists in the issue that in II) we have the requirement that generalized eigenvectors are well-defined elements on both spaces \( \Phi \) and \( \hat{W}_t \Phi \) whereas this unnecessary in strategy III).

To summarize:

If we proceed along strategy I) then we quantize the same physical Hamiltonian (constraint) \( \hat{H}' := H(t', \hat{P}') \) and \( \hat{U}_t^{-1} \hat{H}_\Phi \hat{U}_t \) in two different but unitarily equivalent representations \( \hat{H}_\Phi \) and \( H \). The more convenient representation is \( \hat{H}_\Phi \) because the Hamiltonian (constraint) adopts a simple form. This procedure is guaranteed to lead to the correct physical spectrum of observables while it is technically more difficult to carry out since we are asked to construct the measure \( \nu_t \).

If we proceed along strategy II) then we quantize the two distinct Hamiltonians
$\hat{H}\psi := H\psi(\hat{A}, \hat{P})$ and $\hat{H} := \hat{W}_t^{-1} \hat{H}\hat{W}_t$ (of which the latter is the physical one) in the same representation $\mathcal{H}$. While this procedure is technically easier to perform, as explained above, its validity depends on the strong condition that $\hat{W}_t$ can be densely defined on $\mathcal{H}$ which is often not the case (see the appendix)!

Finally, strategy III) can be seen to be isomorphic with strategy I).

There is a conceptual similarity between strategy III) and the theory of integrable models: we map an unsolvable problem (the quantization of $\hat{C}$) into a solvable one (the quantization of $\hat{H}$) via the map $\hat{W}_t$. In the language of integrable models one tries to find a solution $\psi$ to an unsolvable partial differential equation $\hat{H}\psi = 0$ and we map a function $\phi$ into $\psi = \hat{W}_t \phi$ where $\phi$ is supposed to satisfy certain integrability conditions $\hat{H}\phi = 0$ which are usually easier to solve.

The key ingredient in both approaches is, of course, the map $\hat{W}_t$ and the key question is how to regularize $\hat{C}$. Promising preliminary results have already been obtained \[\text{RS}\] in the context of quantum gravity.

One final comment is in order: as far as obtaining eigenvectors or solutions to the constraints is concerned, it is equally difficult in all three approaches. Namely suppose that $\psi(A)$ is a generalized eigenvector of $\hat{H}\psi = H\psi(\hat{A}, \hat{P})$ then the analytic continuation $\psi(A')$ of $\psi(A)$ is a generalized eigenvector of $\hat{H}' = H\psi(\hat{A}', \hat{P}')$ (in the same ordering) with the same eigenvalue. In a sense, only the algebraic form of the constraint operator is important, not the adjointness relations for its basic variables which satisfy the same commutation relations.

3 Transforms for quantum gauge field theory

Type 1)

In the sequel $F = F[P^a]$ will denote a manifestly positive functional of $P$ alone ($P^a$ is the usual electric field in canonical gauge field theory). We consider the complexification $(A, P) \to (A + i\delta F/\delta P, P)$. Then it is easy to see that the choice $C = F$ does the job.

An interesting model in $3 + 1$ dimensions is given by a phase space coordinatized by $(A, P)$, $A$ being an $SU(2)$ connection and $P$ transforms according to the adjoint representation of $SU(2)$. The system is subject to gauge and diffeomorphism constraints and an additional scalar constraint given by:

$$H = \text{tr}\{([F_{ab}, [P^a, P^b]])^2\} \quad (3.1)$$

where $F$ is the curvature of $A$. One can see that all the constraints remain (weakly) unchanged if we replace $A$ by $A + i\lambda e$ where $e^i_a$ is the co-triad associated with $P$.

The co-triad is easily seen to be the functional derivative of the total volume of $\Sigma$:

$$V(\Sigma) := \int_\Sigma d^3x \sqrt{|\text{det}(P^a)|} \quad (3.2)$$

which makes only sense if $\Sigma$ is compact. This model equips us with a heat kernel that is manifestly diffeomorphism invariant and does not make use of the Baez measures \[\text{RS}\]. Moreover, the heat kernel measures are by construction consistently defined and uniformly bounded by 1. They therefore admit a $\sigma$ additive extension to $\mathcal{A}/\mathcal{G}$ as.
shown in \[1\].

Consider now Yang-Mills theory in 3+1 dimensions for any compact gauge group and let \(S[A] := k \int_\Sigma \text{tr}(A \wedge [d \wedge A - 1/3 A \wedge A])\) be the Chern-Simon functional where the constant is chosen such that \(\delta S/\delta A^i_a = B^i_a\) is the magnetic field of \(A^i_a\). The complexifier now induces \(A \to A = A^\mathbf{r}, P \to P + iB = P_\mathbf{c}\). Due to the Bianchi identity the Gauss constraint remains invariant under the substitution \(P \to P_\mathbf{c}\) but the physical Hamiltonian reads now \(H(A, P) = 1/2\text{tr}[P^a_\mathbf{c}(P^b_\mathbf{c} - 2iB^b)\delta_{ab}]\) which gives rise to the unphysical Hamiltonian \(H_{\mathbf{c}}(A, P) = 1/2\text{tr}[P^a_\mathbf{r}P^b - i(P^aB^b + B^aP^b)\delta_{ab}]\). Note that the Hamiltonian has simplified: \(H\) is a polynomial of order four while \(H_{\mathbf{c}}\) is a polynomial of order three only. But while in the complex representation we are still looking for the spectrum of the positive and self-adjoint operator \(\hat{H}' := H_{\mathbf{c}}(\hat{A}', \hat{P}')\) the operator \(\hat{H}_{\mathbf{c}} := H_{\mathbf{c}}(\hat{A}, \hat{P})\) of the real representation is neither positive nor self-adjoint, not even normal. Therefore, by usual spectral theory, it is far from clear how in this example the quantization of \(\hat{H}_{\mathbf{c}}\) can possibly lead to the same spectrum as \(\hat{H}\) in which we are interested. This is one example in which the fundamental spectral problem mentioned at the end of the previous section shows up.

4 A transform for quantum gravity

Denote by \(q_{ab}, K_{ab}\) the induced metric and extrinsic curvature of a spacelike hypersurface \(\Sigma\), introduce a triad \(e^i_a\) which is an \(SU(2)\) valued one-form by \(q_{ab} = e^i_a e^b_j \delta_{ij}\) and denote by \(e^a_i\) its inverse. Then we introduce the canonical pair of Palatini gravity by \((K^i_a := K_{ab} e^b_i, P^a_i := 1/\kappa \sqrt{\det(q)} e^a_i)\) where \(\kappa\) is Newton’s constant.

We will now employ our algorithm to find the coherent state transform for quantum gravity.

The important observation due to Ashtekar \[1\] is that if we write the theory in terms of the complex canonical pair \(A^\mathbf{r}_a^j := \Gamma^j_a - iK^j_a, P^\mathbf{r}_a^j := iP^a_j\) where \(\Gamma\) is the spin-connection associated with \(P\), then the Hamiltonian constraint adopts the very simple polynomial form \(H_{\mathbf{c}}(A^\mathbf{r}, P^\mathbf{r}) = -\text{tr}(F^\mathbf{r}_a^b [P^a_\mathbf{r}, P^b_\mathbf{r}] )\) where \(F^\mathbf{r}\) is the curvature of \(A^\mathbf{r}\). The importance of this observation is that \((A^\mathbf{r}, P^\mathbf{r})\) is a canonical pair which relies on the discovery that the spin connection is integrable with generating functional \(F = \int_\Sigma d^3x \Gamma^i_a P^a_i\). Ashtekar and later Barbero \[19\] also considered the real canonical pair \((A^\mathbf{r}_a^j = \Gamma^j_a + K^j_a, P^a_j)\) in which, however, the Hamiltonian takes a much more complicated non-polynomial, algebraic form. Namely, after neglecting a term proportional to the Gauss constraint we obtain \(H_{\mathbf{r}}(A^\mathbf{r}, P^\mathbf{r}) \equiv H(A, P) = \text{tr}([F_\mathbf{r} - 2R_\mathbf{r}] [P^a_\mathbf{r}, P^b_\mathbf{r}] )\) in which \(F, R\) are respectively the curvatures of \(A, P\). Although this expression can be made polynomial after multiplying by a power of \(\det(P^a_i)\) which changes the allowed degeneracies of the metric, it clearly is still unmanagable.

The real and complex canonical pairs are related by a chain of three canonical transformations \((A = \Gamma + K, P) \to (K, P) \to (-iK, iP) \to (A^\mathbf{r} = \Gamma - iK, P^\mathbf{r} = iP)\) of which the first and the last have as its infinitesimal generator the functional \(-F\) and \(iF\) respectively. The new contribution of the present article is to derive the infinitesimal generator of the middle symplectomorphism \((K, P) \to (-iK, iP)\) which
is a phase space Wick rotation!

This generator can be found as follows:

We wish to find a functional $C$ such that we can write for instance

$$-iK = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \{K, C\}_{(n)}.$$ 

Now we write $-i = e^{-i\pi/2} = \sum_{n=0}^{\infty} \frac{(-i\pi/2)^n}{n!}$ and see that we just need to find $\tilde{C}$ with the property that $K = \{K, C\}, P = -\{P, C\}$ to reproduce the multiple Poisson bracket involved and then set $C = \pi/2\tilde{C}$. Trial and error reveals the unique solution

$$\tilde{C} := \int_{\Sigma} d^3x K_a^i P_i^a$$

(4.1)

which is easily recognized as $(1/\kappa$ times) the integral over the densitized trace of the extrinsic curvature of $\Sigma$.

So it seems that we need to apply three canonical transformations: the first one is a translation by $-\Gamma$ to get rid of $\Gamma$ generated by $F$, then a Wick rotation generated by $C$ in order to install the $i$ factors and last a translation by $\Gamma$ generated by $F$ again. That $F$ is the correct choice follows from $\{K, F\} = \Gamma, \{P, F\} = 0 = \{K, F\}_{(n)}$ for $n > 1$.

But surprisingly this is not necessary (however, we would need to do it in order to transform to the $K, P$ representation [25]): the interesting fact is that the Poisson bracket of $\tilde{C}$ with the spin-connection $\Gamma^i_a$ indeed vanishes. The elegant way of seeing it is by noticing that $\tilde{C}$ generates constant scale transformations and remembering that $\Gamma^i_a$ is a homogenous rational function in $P^i_a$ and its spatial derivatives of degree zero. The pedestrian way goes as follows: Since $\Gamma$ has a generating functional $F$ we find upon employing the Jacobi identity

$$\{\Gamma^i_a(x), \tilde{C}\} = \{\{K^i_a(x), F\}, \tilde{C}\} = -\{(\{F, \tilde{C}\}, K^i_a(x)) + \{\tilde{C}, K^i_a(x)\}, F\}$$

$$= \{F, K^i_a(x)\} \{K^i_a(x), F\} = 0.$$ (4.2)

As a side result we see that the spin-connection is not a good coordinate on the phase space (constantly scaled $P$'s give rise to the same $\Gamma$ so that there is at least a one parameter degeneracy in inverting $\Gamma(P)$).

Summarizing, the complex Ashtekar variables $(A^i_a = \Gamma^i_a - iK^i_a, P^a_{q,j} = iP^a_j)$ are the result of a Wick rotation generated by $C$ in the sense of [23], namely

$$A^i_a(x) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \{A^i_a(x), C\}_{(n)} = \Gamma^i_a(x) + \sum_{n=0}^{\infty} \frac{(-i\pi/2)^n}{2} K^i_a(x)$$

$$= \Gamma^i_a(x) + e^{-i\pi/2}K^i_a(x)$$

(4.3)

and similarly for $P^a_i$.

The unphysical Hamiltonian constraint $H_{\Psi}(A, P) = -\text{tr}(F_{ab}[P^a, P^b])$ is up to the negative sign just the Hamiltonian constraint of the formal Hamiltonian formulation of Euclidean gravity (it is easy to see that our Wick rotated Lorentzian action equals that of Euclidean gravity if we replace the lapse by its negative and the shift and Lagrange multiplier of the Gauss constraint by $-i$ times themselves), however it should be stressed that what we are doing here is not the quantization of Euclidean gravity: there is no analytic continuation of the time coordinate involved for which there is no natural choice anyway.
At this point it is worthwhile to report an interesting speculation: it is often criticized among field theorists that due to the non-renormalizability of perturbative quantum gravity the Einstein Hilbert action should be supplemented by an infinite tower of counterterms. Since we define \( \hat{H} := \hat{W} - \frac{1}{t} \hat{H}_C \hat{W} t \) we obtain due to the complicated and non-polynomial expression of \( \hat{C} \) an infinite tower of terms times powers of \( \hbar \) each of which has a classical limit (after regularization) and which could be related to those anticipated counterterms if we manage to get a finite theory within our non-perturbative canonical approach (this remains to be true after continuing to imaginary time as to obtain the Euclidean version of the theory)! However, there is no concrete evidence for this to be true at the moment.

What is more important is that \( \hat{C}, \hat{H}_C \) on \( \mathcal{H} \) can be chosen to be self-adjoint operators and regularized with the techniques already available in the literature because it is a classically real expression. It is a peculiarity of the gravitational Hamiltonian that \( H \) and \( H_C \) are both real.

The task left is to define the operator version of \( C \). This seems to be a hopeless thing to do because when written in terms of \( A, P \) it involves the spin-connection which is a non-polynomial expression. There is however a remarkable connection with Chern-Simon theory which may hint at how to make this operator well-defined:

First we write \( K^a_i = A^i_a - \Gamma^a_i \) and write the spin-connection in terms of \( P^a_i \) by recalling its definition as the unique connection that annihilates the triad \( e^i_a \). Contracting with \( E^a_i = \kappa P^a_i \) one arrives after tedious computations at

\[
K^i_a E^a_i = \epsilon^{abc} e^i_e D^b e^i_c
\]

where \( D^b \) is the covariant derivative with respect to \( A^i_a \). Now we use the fact that the triad is integrable, the generating functional being the total volume \( V = \int_{\Sigma} d^3x \sqrt{|\det(q)|} \) of the manifold \( \Sigma \). Therefore we can rewrite (4.4) in terms of Poisson-brackets as follows:

\[
K^i_a E^a_i = \epsilon^{abc} \{ A^i_a, V \} D^b \{ A^i_c, V \} = \{ A^i_a, V \} \{ B^a_i, V \}
\]

where we have introduced the magnetic field \( B^a_i = \epsilon^{abc} F_{bc}/2 \). But the magnetic field also has a generator, namely the Chern-Simon functional \( S = k \int \mathrm{tr}[A \wedge F - 1/3 A \wedge A \wedge A] \). Therefore we arrive at the following remarkable identity

\[
C = \{ S, V \}_{(2)} = \{ \{ S, V \}, V \}.
\]

We know already that the Volume operator \( \hat{V} \) can be consistently defined and has a nice action on cylindrical functions.

The operator version of \( S \) is much harder to define because it actually throws us out of the space of cylindrical functions since it contains the connection at every point of \( \Sigma \). The fact that \( S \) is not even classically a gauge-invariant function will not cause any problem because we only take the commutator of \( S \).

A speculation is now to make use of the fact that \( S \) defines a topological quantum field theory. Therefore we expect that if we approximate \( S \) by a countably generated lattice, then the approximated expression will actually be independent of the triangulation chosen so that the continuum limit is already taken in that sense. We
then use this approximated $S$ to define the operator $S$ which will leave the space of cylindrical functions invariant.

However, the steps necessary to implement this programme have not been completed yet so that we proceed along a more traditional approach [18] which we only sketch here:

The idea is to take the expression (4.4) and to write

$$e^a = \varepsilon_{abc} E^b_k E^c_i \varepsilon^{ijk} \sqrt{|\det(E)|} / (2 \det(E))$$.

Next we approximate the covariant derivative by a path-ordered exponential along some line segment between two points $x, y$ which introduces a natural point splitting. We order all the $E$ dependent terms to the right and apply it to a cylindrical function in a suitable regularization. Noting that for $\sqrt{|\det(E)|}$ already a well-defined regularization exists [21] we manage to produce an operator (symmetrically ordered) which is well-defined on diffeomorphism-invariant states (in the same sense as in [17]) which leaves every cylindrical subspace separately invariant. This operator is not positive so the latter property is quite important if one wants to exponentiate it. This property is not shared for instance by the recent regularization of the Hamiltonian constraint [17] so that a possible approach using the fact that $-h := \{S, V\} = \int d^3 x \text{tr}(F_{ab}[E^a, E^b]) / \sqrt{|\det(E)|}$ is the integrated Hamiltonian constraint (modulo the determinant which unfortunately has a large kernel [24]) will not work, unless we restrict the Hilbert space to a subspace on which $\hat{C}$ is positive. This might be, after all, an attractive thing to do because $C = \{V, h\}$ is the time derivative of the total volume with respect to the integrated Hamiltonian constraint so that we are restricting ourselves to expanding universes.

The analysis of these speculations is the subject of future research.

One final comment is in order: in [13] the coherent state transform is based on a complexifier which on functions cylindrical with respect to a graph $\gamma$ consisting of edges $e_1, \ldots, e_n$ is just the Laplacian $\hat{C}_\gamma = l(e_1)\Delta_1 + \ldots + l(e_n)\Delta_n$ on the group $G^n$ where $n$ is the number of edges of that graph and $\Delta_i$ is the standard Laplacian on $G$ corresponding to the $i$-th copy of the group coordinatized by the holonomy along the edge $e_i$, $l$ is an edge function. Since this complexifier is not shown to come from an operator $\hat{C}$ on $\mathcal{H}$ with projections $\hat{C}_\gamma$ we have to check its consistency [3]. This leads us to the requirement $l(\epsilon \circ \epsilon') = l(\epsilon) + l(\epsilon')$, $l(\epsilon^{-1}) = l(\epsilon)$ which seems to imply that this $\hat{C}$ if it exists, is not diffeomorphism invariant. This is the first hint that this $\hat{C}_\gamma$ cannot be the correct choice since the transform should be gauge-invariant and diffeomorphism invariant (it maps between objects with identical transformation properties under these gauge groups). Next, as we have seen, the correct Wick rotator has nothing to do with this $\hat{C}$ which easily follows from the fact that the Wick rotator depends on $A$ while the Laplacian does not (at best it can come from a function built from $P$ alone). The image $\hat{H}'$ under $\hat{U}_t$ of the physical Hamiltonian $\hat{H}$ therefore will be complicated and therefore useless in obtaining the kernel. On the other hand, if were working on $\mathcal{H}_F$ and were quantizing $\hat{H}'$ in the holomorphic representation with $\hat{A}', 1/\hbar\hat{P}'$ acting by multiplication and functional differentiation, then we were quantizing the wrong theory because $\hat{U}_t^{-1}\hat{H}'\hat{U}_t$ is not the correct Hamiltonian. Therefore, the results of that paper, although mathematically interesting in their own right, do not serve to quantize gravity.
A.1 The harmonic oscillator

As an application of the type 1) and 2) transform, we study the quantization of \( H(x, p) = p^2 + x^2 \). We introduce \( x_q := x - ip, p_q = p \) and have \( \hat{H}(x, p) = x_q(x_q + 2ip) \) so that \( H_q(x, p) = x(x + 2ip) \) is the unphysical Hamiltonian. Note that while \( H \) is positive, \( H_q \) is neither positive nor real so that it is doubtful already at this stage whether strategy II) will lead to a solution.

The complexifier is given by \( C := 1/2p^2 \) which in this case becomes a strictly positive self-adjoint operator on \( \mathcal{H} := L_2(\mathbb{R}, dx) \) (the role of \( d\mu_0 \) is played by the Lebesgue measure \( dx \)). Indeed we have \( \{x, C\} = p, \{x, C\}_{(n)} = 0, n > 1 \). Therefore the Hille-Yosida theorem guarantees that the heat kernel \( \hat{W}_t := \exp(-t\hat{C}) \) is well-defined on \( \mathcal{H} \). In fact \( \rho_t(x, y) = \rho_t(x - y) = <x, \hat{W}_t y> \) is the standard heat kernel \( \rho_t(x) = \exp(-x^2/2t)/\sqrt{2\pi t} \) and we have \( \hat{W}_t = \hat{W}_t^\dagger = \hat{W}_t \).

Next we compute the measure \( \nu_t \). According to our general programme we write \( d\nu_t(x_q, \bar{x}_q) = \frac{1}{2} dx_q \wedge d\bar{x}_q \nu_t(x_q, \bar{x}_q) \) and find that \( \hat{W}_t \hat{W}_t^\dagger \nu_t(x_q, \bar{x}_q) = \int dx \delta([x_q + \bar{x}_q]/(2i), 0) = \delta([x_q - \bar{x}_q]/(2i), 0) \) where \( \hat{C}_q = -1/2\partial^2_{x_q} \). We write \( x_q = x - iy \) so that \( \hat{C}_q + \tilde{C}_q = -1/4(\partial^2_x - \partial_y^2) \) and find indeed \( \nu_t(x, y) = \exp(t/4\partial^2_y)\delta(y, 0) = \rho_{t/2}(y) \). We can also prove by elementary means that this leads to the correct isometry property : \( \int dx dy \rho_{t/2}(y)\rho_t(x_q - u)\rho_t(x_q - v) = \delta(u, v) \). So we have shown that we arrive at the usual Segal-Bargman transform ([13] and references therein).

In order to see that \( \hat{\rho}' \) is still a self-adjoint operator on \( \mathcal{H}_q = L_2(\mathbb{C}, d\nu_t) \) as it should since \( \hat{C} \) commutes with \( \hat{p} \) we write \( \hat{p}_q := -i\partial_{x_q} \) and just note that \( \nu_t(x_q, \bar{x}_q) = \nu_t(x_q - x_q) \) is antisymmetric in \( x_q, \bar{x}_q \). Also the identity on \( \mathcal{H} \) given by \( \hat{\chi}_q = \hat{W}_2\hat{x}_q\hat{W}_{-2} = \hat{x} + i\hat{p} = \hat{x}_q + 2i\hat{p}_q \) is quickly verified on \( \mathcal{H}_q \), i.e. \( (\hat{x}_q)^+ = \hat{x} + 2i\hat{p}' \). Finally \( \hat{U}_t\hat{H}\hat{\psi} = \hat{K}\hat{W}_t(p^2 + \hat{x}^2)\hat{W}_t^{-1}\hat{W}_t\hat{\psi} = \hat{K}(p^2 + (\hat{x} + i\hat{p})^2)\hat{K}^{-1}\hat{U}_t\hat{\psi} = [(\hat{x}')^2 + i(\hat{x}'\hat{p}' + \hat{p}'\hat{x}')]\hat{U}_t\hat{\psi} \) as required.

As it is well-known, the spectrum of \( \hat{H} \) is \( 2n + 1, n = 0, 1, 2, \ldots \) and \( \hat{\rho} := \hat{x}^2 + i(\hat{x}\hat{p} + \hat{p}\hat{x}) \) has the property that on \( \mathcal{H} \) \( \hat{W}_t^{-1}\hat{H}\hat{W}_t = \hat{H} \). A short computation reveals that this operator is not even
normal. Let $\lambda$ be any complex number then we look for solutions to $\hat{H}_\mathcal{Q}\psi_\lambda = \lambda\psi_\lambda$ which gives $\psi_\lambda = \text{const.} \ x^{(\lambda-1)/2}e^{-x^2/4}$. These solutions are normalizable provided that $\Re(\lambda) - 1 > -1$. On the other hand, consider the spectrum of $\hat{H}'$ in the holomorphic representation. The solutions are just the analytic continuation of the $\psi_\lambda$ namely $\psi_\lambda = x^{(\lambda-1)/2}e^{-x^2/4}$ but the requirements of single-valuedness and normalizability restrict $\lambda$ to be an odd integer and to be greater than or equal to one. The Gaussian decay of the solutions and of the measure $dv_t$ ensure that the eigenvectors are normalizable. Note that in this example the spectrum of $\hat{H}$ is properly contained in that of $\hat{H}_\mathcal{Q}$ so that we get a spurious spectrum. So not all the generalized eigenvectors of $\hat{H}_\mathcal{Q}$ are physical! Note also that strategy II) works in this case because the complexifier is a positive self-adjoint operator so that $\hat{W}_t$ exists as an operator on $\mathcal{H}$.

Strategy III) on the other hand gives correctly $(\hat{W}_t^{-1}\psi_\lambda)(x) = p_{[\lambda-1]/2}(x)e^{-x^2/2}$ where $p_n$ are up to a constant just the Hermite polynomials.

The major advantage is the same in all three approaches: we have simplified the problem by going from a differential equation of 2nd order to a differential equation of first order and modulo analytic continuation both sets of physical solutions coincide!

Finally, we observe that we found the physical generalized eigenvectors by first computing all the formal solutions without caring about topological issues and second we selected them by testing them on $\mathcal{H}$ and $\mathcal{H}_\mathcal{Q}$.

A.2 Free relativistic particle

As an application of the type 3) transform we consider the quantization of the massive, free relativistic particle (in one dimension or alternatively with spatial momentum dependent, positive, nowhere vanishing mass).

The Hamiltonian is now given by $H(x,p) = p^2 - m^2$. We wish to consider the Wick rotated coordinates $x_\mathcal{Q} := -ix, p_\mathcal{Q} := ip$. The corresponding complexifier is given by $\hat{C} := \pi/2(\hat{p}\hat{x} + \hat{x}\hat{p})/2$, namely if $\hat{W}_t = \exp(-t\hat{C})$ then $\hat{x}_\mathcal{Q} = \hat{W}_t^{-1}\hat{x}\hat{W}_t, \hat{p}_\mathcal{Q} = \hat{W}_t^{-1}\hat{p}\hat{W}_t$.

We have $H_\mathcal{Q}(x,p) = -(p^2 + m^2)$ so that $\hat{H} := \hat{W}_t^{-1}\hat{H}\hat{W}_t$ according to our programme. The model displays an example of a non-positive but real complexifier which we also encounter in quantum gravity.

Let us compute the action of $\hat{W}_t$ on $\mathcal{H} = L_2(\mathbb{R}, dx)$. Let $\psi \in \mathcal{H}$ be real analytic in $x$ (for instance Hermite functions) and denote by $1(x) = 1$ the constant function with value one. Then $(\hat{W}_t\psi)(x) = (\hat{W}_\mathcal{Q}\psi(\hat{x})\hat{W}_t^{-1}\hat{W}_t1)(x) = (\psi(i\hat{x}) \sum_{n=0}^\infty \frac{(\pi/2)^n}{n!} (\hat{x}\hat{p} - i/2)^n 1)(x) = e^{-ix^2/4}\psi(ix)$ which up to a constant phase is just Wick rotation of the argument. Therefore $(\hat{U}_t\psi)(x_\mathcal{Q}) = e^{-ix^2/4}\psi(ix_\mathcal{Q})$. The isometry inducing measure is readily computed: setting $x_\mathcal{Q} = x + iy$ we have with $dv_t = dxdy\nu_t(x,y)$ the requirement $\int_x dx_\mathcal{Q} \psi(x_\mathcal{Q}) = \int_{x_\mathcal{Q}} dxdy\nu_t(x,y)\psi(ix - y)\xi(ix - y)$ and this is solved by $\nu_t(x,y) = \delta(x,0)$. The same result is obtained upon employing the general formula (2.16) to compute $\nu_t$: we have $\overline{C^\dagger} = -\hat{C}$ so $\hat{J} := \overline{C^\dagger} + \overline{C^\dagger} = -[\hat{C}, \overline{C^\dagger}] = -\pi/2(y\partial_x - x\partial_y) = \pi/2\partial_\phi$ (we have introduced usual polar coordinates) and find $\nu_t(x,y) = e^{it\phi}\delta(y,0) = (e^{\pi/2\partial_0}\delta(\phi,0)e^{-\pi/2\partial_0} \cdot 1)(r,\phi) = \delta(r\sin(\phi + \pi/2),0) = \delta(x,0) = \delta((x_\mathcal{Q} + x_\mathcal{Q}^\dagger)/2,0)$. In order to see that both $\hat{p^\dagger}$ and $\hat{x^\dagger}$ are realized as anti-selfadjoint operators on $\mathcal{H}_\mathcal{Q} = L_2(\mathcal{Q}, dv_t) \cap \mathcal{H}(\mathcal{Q})$.
it is enough to realize that \( \nu_t = \nu_t(x_q + \bar{x}_q) \) is symmetric in \( x_q, \bar{x}_q \) and that \( \bar{x}_q \delta([x_q + \bar{x}_q]/2, 0) = -x_q \delta([x_q + \bar{x}_q]/2, 0) \).

We now come to look at the spectra of the Hamiltonians.

\( \hat{H}_q = -(\hat{p}^2 + m^2) \) is a negative definite, unbounded operator on \( \mathcal{H} \) because \( \hat{p} \) is self-adjoint. Its spectrum is purely continuous so its eigenvectors are of the generalized type (compare section 2.4), meaning that they take values not in \( \mathcal{H} \) but in the topological dual \( \Phi' \) of a certain vector space \( \Phi \) which is dense in \( \mathcal{H} \). An appropriate choice is \( \Phi = S \), the space of test functions of rapid decrease. The generalized eigenvectors are given by \( \exp(i\lambda x), \lambda \in \mathbb{R} \) with eigenvalue \(- (\lambda^2 + m^2) < 0, \lambda \in \mathbb{R}\). If we were interested in the kernel of \( \hat{H}_q \) we only find the vector 0 as a solution!

Now let us look at \( \hat{H} = -(\hat{p}'^2 + m^2) \). Since \( \hat{p}' \) is an anti-self-adjoint operator on \( \mathcal{H}_q \) the generalized eigenfunctions are \( \exp(\lambda x_q), \lambda \in \mathbb{R} \) with eigenvalue \( \lambda^2 - m^2 \). The appropriate space \( \Phi_q \) in this case could be chosen as the holomorphic functions whose restriction to the imaginary axis is of rapid decrease.

Except for the point \(-m^2\) the spectra are totally disjoint! In particular it seems as if strategy II) already fails in this simple example in obtaining the kernel of the constraint.

On the other hand, if we proceed along strategy III), remembering that the viewpoint should be that the quantization of \( \hat{H}_q \) is only an intermediate step, we see that the formal eigenvectors \( e^{i\lambda x} \) are mapped by \( \hat{W}_t^{-1} \) precisely into the eigenvectors of the required form, \( e^{i\lambda x} \) (modulo a phase). So this works.

What goes wrong here with strategy II) is of course that \( \hat{W}_t \) is ill defined on \( \mathcal{H} \) or \( \Phi = \mathcal{D}_\omega \) for instance the wave packet \( e^{-x^2} \) gets mapped into \( e^{x^2}/\sqrt{i} \).

This is easy to fix: just decrease the size of \( \Phi \)!

For example we could choose \( \Phi = \mathcal{D}_\omega \) the space of smooth test functions of compact support such that they remain to be compactly supported after Wick rotation (due care is needed here: for instance the functions of the type \( e^{-a^2/x^2}, a \neq 0 \) usually used to construct smooth partitions of unity or to construct smooth functions which are positive and constant, say, in the interval \([-1, 1]\) but identically zero outside \([-2, 2]\) are inappropriate and should be replaced, for instance, by \( e^{-a^2/x^4} \). This space is still dense in \( \mathcal{H} \). Now the Wick rotated generalized eigenvectors \( f_\lambda(x) = e^{i\lambda x} \) which give the correct physical spectrum can be evaluated on elements of \( \Phi \). The spectra now coincide because the analytic continuation of the \( f_\lambda \) are just the generalized eigenvectors of the physical Hamiltonian. But there is now a new problem: The operator \( \hat{p} \) is self-adjoint on \( \mathcal{H} \) but \( f_\lambda \) is a generalized eigenvector with imaginary eigenvalue which seems to contradict the spectral theorem! More precisely we have the following:

The spectral theorem for self-adjoint operators on a rigged Hilbert space \( \Phi \subset \mathcal{H} \subset \Phi' \) says that the set of generalized eigenvectors corresponding to real eigenvalues is complete.

In our case such a complete set \( S \) is given by the plane waves \( e^{i\lambda x} = (\hat{W}_t f_\lambda)(x), \lambda \in \mathbb{R} \). So how can it be possible that \( f_\lambda \) is a generalized eigenvector of \( \hat{p} \) since clearly the Fourier coefficients of \( f_\lambda \) do not exist so that we cannot write \( f_\lambda \) as a converging linear combination (as an element of \( \mathcal{D}_\omega^* \)) of elements in \( S \)? The resolution of the contradiction consists in the reexamination of what “complete” means in this context: let \( f \in \Phi \) and let \( f_\lambda \in \Phi' \) be a generalized eigenvector of some operator \( \hat{O} \) with eigenvalue \( \lambda \), that is, \( f_\lambda[\hat{O}\phi] = \lambda f_\lambda[\phi] \). Consider the eigenspace \( \Phi_\lambda \) of generalized eigenvectors let \( X \subset \text{spec}(\hat{O}) \) be a subset of the spectrum and assign
to each $\phi \in \Phi$ a function $\tilde{\phi} : \text{spec}(\hat{O}) \rightarrow (\Phi')'$ defined by $\tilde{\phi}(\lambda)[f_{\lambda}] = f_{\lambda}[\phi]$ for all $\lambda$ and for all $f_{\lambda} \in \Phi'_{\lambda}$. Then the set of all generalized eigenvectors for the set $X$ is said to be complete whenever $\tilde{\phi} = 0$ implies $\phi = 0$ the meaning of which is that the assignment defined by $\sim$ is injective for $X$ (that is, there are enough generalized eigenvectors corresponding to $X$ to probe whether two elements of $\Phi$ are different) which is very different from the notion of completeness of a basis in the usual sense, in particular it does not mean that every generalized eigenvector can be expressed as a converging (in $\Phi'$) linear combination of elements of a complete set of generalized eigenvectors (as would be the case if we were dealing with ordinary eigenvectors, that is, elements of $\mathcal{H}$).

Assume now that we are given a complete set $S$ of generalized eigenvectors corresponding to $\hat{O}$ (exists, for instance in the case that $\hat{O}$ is self-adjoint) but that we happen to find a generalized eigenvector $f$ which is not given by a linear combination of elements in $S$. This is exactly what happens in the present case!

The question is: What we are supposed to with $f$? The suggestion is to keep these elements!

We therefore conclude this appendix with a proposal:

Shrink $\Phi$, if possible, until the spectrum of the unphysical Hamiltonian includes the spectrum of the physical Hamiltonian, i.e. until $W_t$ is well-defined on $\mathcal{H}$ which is the completion of $\Phi$.

The reader may feel feel awkward when assigning negative eigenvalues to the square of a self-adjoint operator, however, we are forced to adopt this viewpoint if we want to solve the physical theory via employing strategy II).

Strategies I),III) seem, however, more appropriate.

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