Compact symmetric spaces, triangular factorization, and Cayley coordinates

Derek Habermas

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Abstract

Let $U/K$ represent a connected, compact symmetric space, where $\theta$ is an involution of $U$ that fixes $K$, $\phi : U/K \to U$ is the geodesic Cartan embedding, and $G$ is the complexification of $U$. We investigate the intersection, studied by Pickrell [Pic06], of $\phi(U/K)$ with the Bruhat decomposition of $G$ corresponding to a $\theta$-stable triangular, or LDU, factorization $g = n^- \oplus h \oplus n^+$ of the Lie algebra $\mathfrak{g}$ of $G$. When $g \in \phi(U/K)$ is generic, the corresponding factorization $g = ld(g)u$ is unique, where $l \in N^-$, $d(g) \in H$, and $u \in N^+$. In this paper we present an explicit formula for $d$ in Cayley coordinates, compute it in several classes of symmetric spaces, and use it to identify representatives of the connected components of the generic part of $\phi(U/K)$.

This formula calculates a moment map for a torus action on the highest dimensional symplectic leaves of the Evens-Lu Poisson structure on $U/K$ [EL01] studied also by Foth and Otto [FO06], and Caine [Cai08].

1 Introduction

Let $U/K$ be a connected, irreducible, compact, Riemannian symmetric space on which $U$ acts isometrically on the symmetric space. Then $K$ is the fixed point set of an involution $\theta$ of $U$. Let $G$ be the complexification of $U$ and $\mathfrak{g}$ the complexification of the Lie algebra $\mathfrak{u}$ of $U$. We assume $\theta$ can be extended to a holomorphic involution of $G$ and we let $\theta$ denote this extension as well as the corresponding involutions of $\mathfrak{u}$ and $\mathfrak{g}$. In this paper we consider the intersection of the image of the Cartan embedding

$$\phi : U/K \to U \subseteq G : uK \mapsto uu^{-\theta}$$

with the Bruhat (or triangular, or LDU) decomposition

$$G = \coprod_{\mu \in W} \Sigma^G_{\mu}, \quad \Sigma^G_{\mu} = N^- wHN^+$$
relative to a \(\theta\)-stable triangular decomposition \(g = n^- \oplus h \oplus n^+\).

For a generic element \(g\) in this intersection, \(g \in \phi(U/K) \cap \Sigma^G_1\), this yields a unique triangular factorization \(g = ldu\). Our main contribution is to produce explicit formulas for the diagonal term \(d\) in classical cases when \(\theta\) is an inner automorphism, using Cayley coordinates. (This choice of coordinate is motivated by considerations beyond sheer convenience. See [Pic89].)

We choose specific representations of the various \(u\) in \(su(n)\) that are compatible with \(\theta\); that is, \(\theta\) fixes each of the subspaces \(n^-\), \(h\), and \(n^+\) which, in our setup, always consist of strictly lower triangular, diagonal, and strictly upper triangular matrices, respectively. This is made precise in section 3.

The formulas for \(d\) contain determinants such as \(\det(1 + X)\), where \(X\) is in \(i\), the \(-1\)-eigenspace of \(\theta\) acting on the Lie algebra \(u\). Due to the relatively sparse nature of these matrices, these determinants are often easily calculable, and we illustrate this with examples. The structure of the paper is as follows.

In section 2 we introduce notation and review relevant background for the intersection \(\phi(U/K) \cap \Sigma^G_1\).

In section 3 we calculate \(d\) in Cayley coordinates and use this calculation to identify representatives in each connected component of \(\phi(U/K) \cap \Sigma^G_1\).

In section 4 we apply the results of section 3 to compact symmetric spaces of type AIII, DIII, CI, CPI, and BDI.

In section 5 we identify directions for future study.

In the appendices, some non-standard representations used in the paper are more fully explained, and calculations in several low dimensional examples are presented.

2 Background

Here we review the intersection of a compact symmetric space with a compatible Bruhat decomposition; this material is presented in more detail in [Pic06]. As stated in the introduction, \(U/K\) is a connected irreducible compact symmetric space, where \(U\) is a connected Lie group acting on the symmetric space isometrically and transitively, \(G\) is the complexification of \(U\), and \(K \subseteq U\) is the connected component containing the identity of the
fixed point set of an involution $\theta$ of $U$. We then have a diagram of groups

\[ \begin{array}{ccc}
G & \rightarrow & U \\
\uparrow & & \uparrow \\
G_0 & \rightarrow & K \\
\uparrow & & \uparrow \\
K & \rightarrow & U
\end{array} \]

where the arrows represent inclusion maps, isometric embeddings of symmetric spaces. (The symmetric space structure is induced from the Killing form.) We also have a corresponding diagram of Lie algebras

\[ \begin{array}{ccc}
g & \rightarrow & u \\
\uparrow & & \uparrow \\
g_0 & \rightarrow & u \\
\uparrow & & \uparrow \\
\mathfrak{k} & \rightarrow & \mathfrak{u}
\end{array} \]

Note that $G_0/K$ is the non-compact dual symmetric space to $U/K$, though that will not be the focus of this paper.

In a slight abuse of notation, we also use $\theta$ to denote the induced involution on the Lie algebra $u$ of $U$ as well as its complex linear extension to the Lie algebra $g$ of $G$. We also assume that $\theta$ extends to a holomorphic involution on $G$ which will also be denoted $\theta$. Let $(\cdot)^{-\theta}$ denote the Cartan involution of $G$ fixing $U$, and $\sigma(\cdot) = (\cdot)^{-\star \theta}$ the involution of $G$ fixing $G_0$. Since the inversion map, $\star$, and $\theta$ commute, our notation $g^\theta := \theta(g)$ should not cause confusion.

We have Cartan embeddings of symmetric spaces

\[ \begin{array}{c}
U/K \xrightarrow{\phi} U : uK \mapsto uu^{-\theta} \\
\downarrow \\
G/G_0 \xrightarrow{\phi} G : gG_0 \mapsto gg^\theta.
\end{array} \]

These are totally geodesic embeddings of symmetric spaces. The following proposition (Theorem 1 in [Pic06]) characterizes the images of these embeddings as subsets of $G$.

**Proposition 2.1.** (a) In terms of $g \in G$,

\[ \begin{array}{c}
\phi(U/K) = \{g^{-1} = g^* = g^\theta\}_0 \rightarrow U = \{g^{-1} = g^*\} \\
\downarrow \\
\phi(G/G_0) = \{g^* = g^\theta\}_0 \rightarrow G
\end{array} \]
where \( \{ \cdot \}_0 \) denotes the connected component containing the identity.

(b) The connected components of \( \{ g^{-1} = g^* = g^\theta \} \) are determined by the map which sends \( g \) to the inner conjugacy class of the involution \( \eta = \text{Ad}(g) \circ \theta \), subject to the constraint that \( \eta \) equals \( \theta \) in \( \text{Out}(U) = \text{Ad}(U) \setminus \text{Aut}(U) \). A similar statement applies to \( \{ g^* = g^\theta \} \), with \( \sigma \) and antilinear automorphisms of \( G \) in place of \( \theta \) and involutions of \( U \).

Let \( u = k \oplus i \mathfrak{p} \) be the decomposition of \( u \) into \( +1 \) and \( -1 \) eigenspaces of \( \theta \). By Proposition 2.1 we can use the derivative of the Cartan embedding to identify the tangent space of \( U/K \) at \( eK \) with \( i \mathfrak{p} = \{ -X = X^* = X^\theta \} \subseteq \mathfrak{g} \).

The exponential map of \( g \) maps \( i \mathfrak{p} \) onto \( \phi(U/K) \). (See chapter VII of [Hel78].)

Fix a maximal abelian subalgebra \( t_0 \subseteq k \). We then obtain \( \theta \)-stable Cartan subalgebras \( \mathfrak{h}_0 = Z_{g_0}(t_0) = t_0 \oplus a_0 \), \( \mathfrak{t} = t_0 \oplus i a_0 \), and \( \mathfrak{h} = \mathfrak{h}_0 \) of \( g_0 \), \( u \), and \( g \), respectively, where \( Z_{g_0}(t_0) \) is the centralizer of \( t_0 \) in \( g_0 \) and \( a_0 \subseteq \mathfrak{p} \) (see (6.60) of [Kna02]). Let \( T_0 = \exp(t_0) \) and \( T = \exp(t) \) correspond to maximal tori in \( K \) and \( U \), respectively.

We obtain a \( \theta \)-stable triangular decomposition \( g = n^- \oplus \mathfrak{h} \oplus n^+ \) so that \( \sigma(n^\pm) = n^\mp \). (See p. 709 in [Pic06].) Let \( N^\pm = \exp(n^\pm) \) and \( H = \exp(\mathfrak{h}) \). We also let \( W = W(G,T) \) denote the Weyl group, \( W = N_U(T)/T \cong N_G(H)/H \). Corresponding to this triangular decomposition of \( g \), we have the Bruhat decomposition of the group \( G \),

\[
G = \coprod_{w \in W} \Sigma^G_w, \quad \Sigma^G_w = N^- w H N^+,
\]

where \( \Sigma^G_w \) is diffeomorphic to \( (N^- \cap wN^-w^{-1}) \times H \times N^+ \); we refer to these as the “levels” of the Bruhat decomposition. Elements in the top level, \( \Sigma^G_1 \), are called “generic.” Define

\[
d : \Sigma^G_1 \to H : g \mapsto d(g) \text{ if } g = ld(g)u \text{ for } l \in N^-, d(g) \in H \text{ and } u \in N^+.
\]

Since this factorization is unique for generic elements, the map \( d \) is well defined.

Intersecting this decomposition with \( \phi(U/K) \) we obtain a decomposition of our symmetric space whose levels are indexed by \( W \). We denote these levels by \( \Sigma^{\phi(U/K)}_w = \phi(U/K) \cap \Sigma^G_w \), keeping in mind that some levels may be the empty set.
Theorems 2 and 3 from \cite{Pic06} examine the intersections of the symmetric spaces and varieties mentioned in Proposition 2.1 with levels $\Sigma^G_w$ for arbitrary $w \in W$. The following proposition summarizes some facts from these theorems about the top level $\Sigma^G_1(U/K)$ for use in the next section.

**Proposition 2.2.** The map

$$N^- \times (T_0^{(2)} \times \exp(i\alpha_0) \times \exp(it_0)) \rightarrow \{g^* = g^\theta\} \cap \Sigma^G_1$$

$$l_1[w,m], a_0 \rightarrow g = \iota wma_\theta\iota^*$$

is a diffeomorphism, so that the connected components of $\{g^* = g^\theta\} \cap \Sigma^G_1$ are indexed by $T_0^{(2)}/\exp(i\alpha_0)^{(2)}$. Furthermore, $w \in T_0^{(2)}$ is in $\phi(U/K)$ (and, therefore, represents a connected component of $\Sigma^G_1(U/K)$) if and only if there exists $w_1 \in N_U(T_0)$ such that $\phi(w_1K) = w$.

### 3 The diagonal element in Cayley coordinates

In this section we compute, in Cayley coordinates, $d : \Sigma^G_1 \rightarrow H$ and its restriction to $\Sigma^\phi(U/K)$ for the following compact symmetric spaces.

| Type  | $U/K$                              | $\theta$          |
|-------|------------------------------------|-------------------|
| AIII  | $SU(m+n)/SU(m) \times U(n)$       | Ad($I_{m,n}$)     |
| DIII  | $SO(2n)/U(n)$                      | Ad($I_{n,n}$)     |
| CI    | $Sp(p) \times U(n)$                | Ad($I_{n,n}$)     |
| CII   | $SO(p+q)/SO(p) \times SO(q)$      | Ad($I_{p+q}$)     |
| BDI   | $SO(p+q)/SO(p) \times SO(q)$      | Ad($I_{p+q}$)     |

The matrix $I_{a,b,c,...}$ is a diagonal matrix with the first $a$ diagonal entries $-1$, the next $b$ diagonal entries $1$, the next $c$ diagonal entries $-1$, and so on. For type BDI, if $p$ and $q$ are both odd, $\theta$ is an outer automorphism of $SO(p+q)$. This special case will be addressed in section 4.5.

For the symmetric spaces $U/K$ under consideration, the complexification $G$ of $U$ is either $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, or $Sp(n, \mathbb{C})$ for some $n$. Let $N^+_SL$ (resp. $N^-_{SL}$ and $H_{SL}$) be the subgroup of $SL(n, \mathbb{C})$ consisting of upper triangular unipotent (resp. lower triangular unipotent and diagonal) matrices in $SL(n, \mathbb{C})$. Let $T$ denote the antiholomorphic involution of $SL(n, \mathbb{C})$ given by $T(g) = (g^{-1})^*$ where $*$ denotes conjugate transpose. Let $\tau$ denote anti-transpose (reflection across the antidiagonal), the holomorphic involution of $GL(n, \mathbb{C})$ given by $g^\tau = J_n g J_n^{-1}$ where $J_n \in GL(n, \mathbb{C})$ has entries equal to 1.
on the antidiagonal and 0 elsewhere. Also let \( \tau \) denote its restriction to any subgroup of \( GL(n, \mathbb{C}) \) as well as the derivatives acting on the corresponding Lie algebras.

We will embed \( SO(n, \mathbb{C}) \) into \( SL(n, \mathbb{C}) \) (resp. \( Sp(n, \mathbb{C}) \) into \( SL(2n, \mathbb{C}) \)) as the fixed point set of a holomorphic involution \( \Theta \) of \( SL(n, \mathbb{C}) \) (resp. \( SL(2n, \mathbb{C}) \)) such that \( \Theta \) preserves \( N^+_{SL}, \ N^-_{SL}, \) and \( H_{SL} \), and such that \( \mathcal{T}\Theta = \Theta \mathcal{T} \). For \( G = SO(n, \mathbb{C}) \), define \( \Theta(g) = (g^{-1})^\tau \). For \( G = Sp(n, \mathbb{C}) \), define \( \Theta(g) = I_{n,n}(g^{-1})^\tau I_{n,n}^{-1} \). Each \( \Theta \) has the specified properties; for more information see Appendix A. For each \( G \), let 

\[
N^+ = G \cap N^+_{SL}, \quad N^- = G \cap N^-_{SL} \quad \text{and} \quad H = G \cap H_{SL},
\]

and let \( n^+, n^- \) and \( \mathfrak{h} \) be their Lie algebras. Then \( g = n^+ \oplus \mathfrak{h} \oplus n^- \) is a triangular decomposition of \( g \).

For each of the five cases of \( U/K \) we choose a holomorphic involution \( \theta \) of \( SL(n, \mathbb{C}) \) such that it commutes with both \( \mathcal{T} \) and \( \Theta \), the triangular decomposition is \( \theta \)-stable, and \( G_0 = G^{T \theta} \). Note that \( T\theta = \sigma \) as defined in section 2. The restriction of \( \theta \) to \( G \), the restriction to \( U \), and the corresponding involutions of the Lie algebras \( g \) and \( u \) will still be denoted \( \theta \). The choices for \( \theta \) are given in table (3.1).

Define the Cayley map by

\[
\Phi : u(n) \to \{ g \in U(n) \mid -1 \notin \text{spec}(g) \} : X \mapsto g = \frac{1 - X}{1 + X}.
\]

Since \( 1 - X \) and \( (1 + X)^{-1} \) commute, our notation is unambiguous. Note that \( \Phi \) is invertible by \( g \mapsto \frac{1 - g}{1 + g} \).

**Lemma 3.2.** Suppose that \( \psi : GL(n, \mathbb{C}) \to GL(n, \mathbb{C}) \) is an automorphism or an anti-automorphism and that \( \psi \) can be extended to a linear operator \( \tilde{\psi} \) on \( \operatorname{End}_\mathbb{C}(\mathbb{C}^n) \). Let \( g \in GL(n, \mathbb{C}) \), and let \( X \in \operatorname{End}_\mathbb{C}(\mathbb{C}^n) \) be in the tangent space to \( GL(n, \mathbb{C}) \) at \( g \). Then the following is true.

(a) The derivative of \( \psi \) is \( \tilde{\psi} \); that is, \( d\psi\bigg|_g(X) = \tilde{\psi}(X) \).

(b) If we further suppose that \( -X = X^* = \tilde{\psi}(X) \), then \( X \) is in the domain of the Cayley map \( \Phi \), and

\[
\Phi(X) \in \{ g \in GL(n, \mathbb{C}) \mid g^{-1} = g^* = \psi(g) \}.
\]

**Proof.** (a) Let \( g \in GL(n, \mathbb{C}) \), and let \( X \in T_g(GL(n, \mathbb{C})) \cong \operatorname{End}_\mathbb{C}(\mathbb{C}^n) \). Then

\[
d\psi\bigg|_g(X) = \frac{d}{dt} \bigg|_{t=0} \tilde{\psi}(g + tX)
\]

\[
= \lim_{t \to 0} \frac{\tilde{\psi}(g + tX) - \tilde{\psi}(g)}{t} = \tilde{\psi}(X).
\]
(b) Suppose $-X = X^* = \bar{\psi}(X)$. Then $X$ is skew Hermitian, and $g := \Phi(X)$ is unitary. By part (a), we also have
\begin{equation}
\psi(g) = \bar{\psi}((1-X)(1+X)^{-1}) = \bar{\psi}(1-X)(\bar{\psi}(1+X))^{-1} = (1+X)(1-X)^{-1} = g^{-1}.
\end{equation}

Note that if $\psi$ is an anti-automorphism, (3.3) follows from the fact that $1-X$ and $(1+X)^{-1}$ commute.

Proposition 3.4. Let $U/K$ be one of the symmetric spaces in table (3.1) with corresponding involution $\theta$. Then \( \Phi(i\mathfrak{p}) \subseteq \phi(U/K) \).

Proof. We must show that, for each $U/K$,
\[ \Phi(i\mathfrak{p}) \subseteq \{ g \in U|g^{-1} = g^{\theta}\}_0 = \phi(U/K). \]

Let $X \in i\mathfrak{p}$, and $g = \Phi(X)$. Each involution $\theta$ meets the criteria of Lemma 3.2. Therefore, since $i\mathfrak{p}$ is connected, by continuity of $\Phi$ we have
\[ \Phi(i\mathfrak{p}) \subseteq \{ g \in U(n)|g^{-1} = g^{\theta}\}_0. \]

Furthermore, since the determinant is fixed under conjugation, we have $\det(g) = \det(g^{\theta}) = \det(g^{-1}) = (\det(g))^{-1}$ which implies that $\det(g) = \pm 1$. By continuity of $\Phi$, and since $0 \in i\mathfrak{p}$, we have $\det(g) = 1$. So,
\[ \Phi(i\mathfrak{p}) \subseteq \{ g \in SU(n)|g^{\theta} = g^{-1}\}_0. \]

All that remains to be shown is that $\Phi(i\mathfrak{p}) \subseteq U$. In the case where $U = SU(n)$, we are done. For $U = SO(n)$, note that $\tau$ meets the criteria of Lemma 3.2 since our representation of $\mathfrak{so}(n)$ lies in the $-1$ eigenspace of $\tau$. Therefore,
\[ \Phi(i\mathfrak{p}) \subseteq \{ g \in SU(n)|g^{\tau} = g^{-1}\} = U. \]

The case where $U = Sp(n)$ follows similarly, since our representation of $\mathfrak{sp}(n)$ lies in the $-1$ eigenspace of $Ad(I_{n,n}) \circ \tau$. This completes the proof.

Notation: Let $A$ be an $m \times n$ matrix, and let $1 \leq l \leq m, 1 \leq l' \leq n$. Let $Q_{l,m}$ denote the set of all strictly increasing sequences of $l$ integers chosen from $\{1, \ldots, m\}$. Let $\alpha = (i_1, \ldots, i_l) \in Q_{l,m}, \beta = (j_1, \ldots, j_{l'}) \in Q_{l',n}$, and let $A[\alpha, \beta]$ denote the $l \times l'$ submatrix consisting of the intersection of rows $i_1, \ldots, i_l$ and columns $j_1, \ldots, j_{l'}$ of $A$. We use the shorthand $A[k] = A[(1, \ldots, k), (1, \ldots, k)]$ for the $k \times k$ principal block of $A$. Also we use $I_k$
as shorthand for $I_{k,n-k}$ when it is understood from context to be an $n \times n$ matrix. It is useful to think of multiplication on the left (resp. right) by $I_k$ as changing the sign of the first $k$ rows (resp. columns) of the matrix.

**Lemma 3.5.** Let $X \in \mathfrak{su}(n), g = \frac{1-X}{1+X} \in SU(n)$. Then, for $1 \leq k \leq n$,

$$\det(g[k]) = \frac{\det(1 + I_k X)}{\det(1 + X)}.$$

**Proof.** Write $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ and $(1+X)^{-1} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$, where $X_1, Y_1 \in M_{k \times k}(\mathbb{C})$, and so on. Then

$$(1 + I_k X)(1+X)^{-1} = \begin{bmatrix} (1 - X_1)Y_1 - X_2 Y_2 \\ X_3 Y_1 + (1 + X_4)Y_3 \\ X_3 Y_2 + (1 + X_4)Y_4 \end{bmatrix}^*$$

$$= \begin{bmatrix} g[k] \\ 0_{(n-k) \times k} \\ 1_{(n-k) \times (n-k)} \end{bmatrix}.$$

Taking determinants, the claim follows. 

**Theorem 3.6.** Let $d : \Sigma_{\mathfrak{su}(n)}^\mathfrak{SL}(n,\mathbb{C}) \to H$ be as described above. If $X \in \mathfrak{su}(n)$ such that $g = \Phi(X)$ is generic, then $\det(1 + I_k X) \neq 0$ for $1 \leq k \leq n$, and

$$d(g) = \text{diag} \left( \frac{\det(1 + I_1 X)}{\det(1 + X)}, \frac{\det(1 + I_2 X)}{\det(1 + X)}, \frac{\det(1 + I_3 X)}{\det(1 + X)}, \ldots, \frac{\det(1 + I_n X)}{\det(1 + X)} \right).$$

**Proof.** Let $X \in \mathfrak{su}(n)$ such that $g = \Phi(X)$ is generic, and let $g = ld(g)u$ as described above. Then, by Gaussian elimination, $\det(g[k]) \neq 0$ for $1 \leq k \leq n$, and

$$d(g) = \text{diag} \left( \frac{\det(g[1])}{\det(g[1])}, \frac{\det(g[2])}{\det(g[2])}, \ldots, \frac{\det(g)}{\det(g[n-1])} \right). \quad (3.7)$$

Then the Theorem follows immediately from Lemma 3.5.

From Fredholm theory, we have the following equality.

$$\det(1 + A) = \sum_{l=0}^{\infty} \text{Tr}(\wedge^l(A))$$
This formula is for trace class operators on a separable Hilbert space, possibly infinite dimensional. In our case, this sum terminates. When \( l = 0 \), the summand is 1, and for \( 1 \leq l \leq n \) and \( \alpha = (i_1, \ldots, i_l) \in \mathcal{Q}_{l,n} \) we have

\[
\text{Tr}(\wedge^l(A)) = \sum_{1 \leq i_1 < \cdots < i_l \leq n} \langle \wedge^l(A)e_{i_1} \wedge \cdots \wedge e_{i_l}, e_{i_1} \wedge \cdots \wedge e_{i_l} \rangle = \sum_{\alpha \in \mathcal{Q}_{l,n}} \det A[\alpha, \alpha].
\]

(For consistency, we define \( \det A[\emptyset, \emptyset] = 1 \).) Thus, the individual entries of \( d(g) \) are ratios of sums of determinants of submatrices of \( X \). Specifically,

\[
[d(g)]_{kk} = \frac{\det(1 + I_kX)}{\det(1 + I_{k-1}X)} = \sum_{l=0}^{n} \sum_{\alpha \in \mathcal{Q}_{l,n}} \frac{\det(I_kX)[\alpha, \alpha]}{\det(I_{k-1}X)[\alpha, \alpha]}.
\tag{3.8}
\]

(3.8)

Note that

\[
\det(1 + X) = \sum_{l=0}^{n} \sum_{\alpha \in \mathcal{Q}_{l,n}} \det X[\alpha, \alpha].
\]

and that for each \( 1 \leq k \leq n \), \( \det(1 + I_kX) \) has the same summands as \( \det(1 + X) \) up to a sign.

Example 3.9. Let \( U/K = SU(2)/U(1) \cong \mathbb{C}P^1 \cong S^2 \). Let

\[
X = \begin{bmatrix} 0 & z \\ -\bar{z} & 0 \end{bmatrix} \in i\mathfrak{p} \subseteq \mathfrak{su}(n).
\]

and let \( g = \Phi(X) \). Applying (3.8) above, we have

\[
d(g) = \begin{bmatrix} \frac{1-|z|^2}{1+|z|^2} & 0 \\ 0 & \frac{1+|z|^2}{1-|z|^2} \end{bmatrix}.
\]

For this example, there are two connected components of \( \Sigma_{1}^{\phi(U/K)} \). By Proposition 2.2 these are indexed by

\[
T_{0}^{(2)} = \{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \}.
\]
We can use the above computation to identify these $w$. Obviously, $\Phi(0) = +1$, but one can see that $-1$ is not in the image of $\Phi$. However, letting $|z|$ tend to infinity, we see that

$$
\lim_{|z| \to \infty} \Phi\left( \begin{bmatrix} 0 & z \\ -z & 0 \end{bmatrix} \right) = \lim_{|z| \to \infty} \begin{bmatrix} \frac{1-|z|^2}{1+|z|^2} & 0 \\ 0 & \frac{1+|z|^2}{1+|z|^2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

We generalize this idea in the following theorem, which is motivated by Proposition 2.2.

**Theorem 3.10.** Let $w \in T_0^{(2)}$, and define $\alpha_w \in Q_{l,n}$ by $j \in \alpha_w$ if and only if $[w]_{jj} = -1$. Then $w \in \Sigma^{\phi(U/K)}_{1}$ if and only if $w = 1$ or there exists $X \in i\mathbb{P}$ such that the limits

$$
\lim_{t \to \infty} \frac{\det(1 + tI_k X)}{\det(1 + tX)} = \lim_{t \to \infty} \frac{\det(tI_k X)[\alpha_w, \alpha_w]}{\det(tX)[\alpha_w, \alpha_w]}
$$

exist (and are equal) for all $1 \leq k \leq n$. That is, $w$ is in $\Sigma^{\phi(U/K)}_{1}$ if the term corresponding to $\alpha_w$ in (3.8) dominates in the limit for each $1 \leq k \leq n$.

**Proof.** Suppose $w \neq 1$ and define $\alpha_w$ as above. If such an $X \in i\mathbb{P}$ satisfying (3.11) exists, then the left hand side of (3.11) is (the limit of) the numerator of the $k^{th}$ entry of the diagonal element of $\Phi(tX) \in \phi(U/K)$ as in (3.7). Since $\phi(U/K)$ is connected and complete, $\lim_{t \to \infty} \Phi(tX) \in \phi(U/K)$. The right hand side of (3.11) is $-1$ exactly when $k \in \alpha_w$, so the diagonal part of $\lim_{t \to \infty} \Phi(tX)$ is $w$. Therefore, $w \in \Sigma^{\phi(U/K)}_{1}$.

Conversely, suppose $w \in \Sigma^{\phi(U/K)}_{1}$ and $\alpha_w \in Q_{l,n}$ as above. By Proposition 2.2 there exists $w_1 \in N_U(T_0)$ such that $\phi(w_1 K) = w$. Define

$$
\psi : \{w_1 \in N_U(T_0) \mid \phi(w_1 K) \in T_0^{(2)} \} \to i\mathbb{P}
$$

$$
[\psi(w_1)]_{ij} = \begin{cases} [w_1]_{ij} & \text{if } i, j \in \alpha_w \\ 0 & \text{otherwise} \end{cases}
$$

Let $X = \psi(w_1)$. Since $[w]_{jj} = -1$ for all $j \in \alpha_w$, then $\text{proj}_k(X) = 0$ by construction, so $X \in i\mathbb{P}$.

The result follows if the determinants on the right hand side of (3.11) are non-zero. Since $w_1 \psi \in T_0^{(2)}$, and since the effect of $\theta$ is only to negate entries in certain blocks (among which are not the diagonal blocks), it follows that $w_1$ has order two. Therefore, it can be obtained by performing a sequence of pairwise disjoint row interchanges $(ij)$ on a matrix in $T_0$, where $i, j \in \alpha_w$, and the $ij^{th}$ entry of $w_1$ is in a block negated by $\theta$. Hence, $\det X[\alpha_w, \alpha_w] = \det w_1[\alpha_w, \alpha_w] \neq 0$. This completes the proof. □
4 Explicit Calculations Of $d$ In Terms Of $X$

We now apply the results of section 3 for each symmetric space in table (3.1) on a case by case basis. We let $d$ denote $d(g)$ where $g = \Phi(X)$ for $X \in i\mathfrak{p}$, and let $e_{ij}$ denote the matrix (the size shall be understood from context) with the $ij^{th}$ entry equal to 1 and the other entries equal to 0.

4.1 Type AIII

Symmetric space: $SU(m + n)/S(U(m) \times U(n)) \cong Gr(m, \mathbb{C}^{m+n})$

Involution: $\theta : X \mapsto Ad(I_m)(X)$

Block structure:

$$
\mathfrak{t} = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid \text{trace} = 0 \right\} \quad i\mathfrak{p} = \left\{ \begin{bmatrix} 0 & Z \\ -Z^* & 0 \end{bmatrix} \right\}
$$

where $A \in \mathfrak{u}(m)$, $B \in \mathfrak{u}(n)$, and $Z \in M_{m \times n}(\mathbb{C})$. Without loss of generality, we assume $m \leq n$. Note that $Z$ in (4.1) is the graph coordinate for $Gr(n, \mathbb{C}^{m+n})$.

Intrinsic formula for $d$ in terms of $X$ (easily obtained from Theorem 3.6):

$$
d = \prod_{k=1}^{m+n-1} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} = (\det(1 + X))^{-\sum h_k} \prod_{k=1}^{m+n-1} (\det(1 + I_k X))^{h_k},
$$

where $h_k = e_{kk} - e_{k+1,k+1}$ are coroots in $\mathfrak{h}_R$. In matrix form,

$$
h_k = \text{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0).
$$

Since the block form of $X$ has diagonal blocks equal to 0, we have $\det(1 + X) = \det(1 + ZZ^*)$, and

$$
\det(1 + I_k X) = \begin{cases} 
\det(1 + I_k ZZ^*) & \text{if } 1 \leq k \leq m \\
\det(1 - ZI_{k-n}Z^*) & \text{if } n \leq k \leq m + n
\end{cases}
$$

The submatrices $X[\alpha, \alpha]$, $\alpha \in Q_{l,m+n}$ also have diagonal blocks equal to 0. To see this, let $\alpha = (i_1, \ldots, i_l)$ and let $l'$ be such that $i_{l'} \leq m$, but $i_{l'+1} > m$. Then

$$
X[\alpha, \alpha] = \begin{bmatrix} 0 & Z[\alpha_m, \hat{\alpha}_m] \\ -(Z[\alpha_m, \hat{\alpha}_m])^* & 0 \end{bmatrix}.
$$

where $\alpha_m = (i_1, \ldots, i_{l'}) \in Q_{l',m}$, and $\hat{\alpha}_m = (i_{l'+1} - m, \ldots, i_l - m) \in Q_{l'-l,m}$. 

11
As a consequence of the block structure of $X$, $\det(I_k X)[\alpha, \alpha] \neq 0$ only if $|\alpha_m| = |\hat{\alpha}_m|$, that is, when $Z[\alpha_m, \hat{\alpha}_m]$ is square. So,

$$
det(1 + I_k X) = \sum_{l=0}^{m+n} \sum_{\alpha \in Q_{l,m+n}} det(I_k X)[\alpha, \alpha]
= \sum_{l=0}^{m+n} \sum_{\alpha \in Q_{l,m+n}} \pm det(Z[\alpha_m, \hat{\alpha}_m]Z[\alpha_m, \hat{\alpha}_m]^*).
$$

This sum ranges over all square submatrices of $Z$. The sign of the summand is negative if and only if $|\alpha_k| := |\{j \in \alpha | j \leq k\}|$ is odd. In particular,

$$
det(1 + X) = \sum_{l=0}^{m} \sum_{\alpha \in Q_{l,m+n}} det(Z[\alpha_m, \hat{\alpha}_m]Z[\alpha_m, \hat{\alpha}_m]^*).
$$

It follows immediately from $|\alpha_m| = |\hat{\alpha}_m|$ that for $w \in \Sigma_{1}^{(U/K)}$, it is necessary that the number of $-1$ entries in the upper diagonal $m \times m$ block is equal to the number of $-1$ entries in the lower diagonal $n \times n$ block. This is also a sufficient condition by the construction in Theorem 3.10.

Restricting our attention to complex projective space, $Gr(1, \mathbb{C}^{1+n}) \cong \mathbb{C}P^n$, for $X \in ip$ we have

$$
X = \begin{bmatrix} 0 & Z \\ -Z^* & 0 \end{bmatrix}
$$

where $Z = [z_1 \ldots z_n]$.

Applying formula (4.2), we see that

$$
det(1 + I_k X) = 1 + \sum_{i=1}^{k-1} z_i \bar{z}_i - \sum_{j=k}^{n} z_j \bar{z}_j,
$$

and so

$$
d_{kk} = \frac{1 + \sum_{i=1}^{k-1} |z_i|^2 - \sum_{j=k}^{n} |z_j|^2}{1 + \sum_{i=1}^{k-2} |z_i|^2 - \sum_{j=k-1}^{n} |z_j|^2}, \quad 1 \leq k \leq n + 1.
$$

For $\mathbb{C}P^1 \cong S^2$, we have $ip \cong \mathbb{C}$. In Cayley coordinates, the above formula yields $d_{11} = (1 - |z|^2)/(1 + |z|^2)$, which is the height function in stereographic coordinates (under projection from the south pole) or in the $z$ coordinate on the Riemann sphere.
4.2 Type DIII

Symmetric Space: $SO(2n)/U(n)$
Involution: $\theta : X \mapsto Ad(I_n)(X)$
Block structure:

$$\mathfrak{t} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \right\}$$
$$\mathfrak{i} = \left\{ \begin{bmatrix} 0 & Z \\ -Z^* & 0 \end{bmatrix} \right\}$$

where $A \in \mathfrak{u}(n)$, and $Z \in M_{n \times n}(\mathbb{C})$ such that $Z = -Z^\tau$.

Intrinsic formula for $d$ in terms of $X$:

$$d = \prod_{k=1}^{n-1} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} \left( \frac{\det(1 + I_n X)}{\det(1 + X)} \right)^{\frac{1}{2}(-h_{n-1} + h_n)}$$

where

$$h_k = \begin{cases} e_{kk} - e_{k+1,k+1} + e_{2n-k,2n-k} - e_{2n-k+1,2n-k+1}, & 1 \leq k < n \\ e_{n-1,n-1} + e_{nn} - e_{n+1,n+1} - e_{n+2,n+2}, & k = n \end{cases}$$

are coroots in $\mathfrak{h}_R$. In matrix form,

$$h_k = \text{diag}(\ldots, 1, -1, \ldots, 1, -1, \ldots), \quad h_n = \text{diag}(\ldots, 1, 1, -1, -1, \ldots)$$

(4.4)

As $SO(2n)/U(n)$ is a subspace of $Gr(n, \mathbb{C}^{2n})$, equation (4.2) still holds, but the requirement $-X = X^\tau$ restricts $\Phi(X)$ to $SO(2n)$. In concrete terms, this implies that the anti-diagonal entries of $X$ must be zero.

In applying Theorem 3.10 to $SO(2n)/U(n)$, there are 3 facts to consider as we identify the $w \in T_0^{(2)}$. First, since this is a submanifold of the complex Grassmanian, there are as many $-1$ entries in the first diagonal block as in the second. Second, since for $g \in SO(2n)$ we have $g^\tau = g^{-1}$, the second diagonal block is determined by the first, so $w^\tau = w$. Third, since $-X = X^* = X^\tau$ then $\det X[\alpha, \alpha] \neq 0$ implies that either 4 divides $|\alpha|$, or there exists $\beta \supseteq \alpha$ and 4 divides $|\beta|$. For example, if $\det X[(i, j), (i, j)] \neq 0$ such that $i < 2n - j$, then $\det X[(i, 2n - j, j, 2n - i), (i, 2n - j, j, 2n - i)] \neq 0$.

The result is that each $w \in T_0^{(2)}$ is symmetric across the anti-diagonal, and has an even number of $-1$ entries in each diagonal block.

4.3 Type CI

Symmetric Space: $Sp(n)/U(n)$
Involution: $\theta : X \mapsto Ad(I_n)(X)$
Block structure:
\[
\mathfrak{g} = \left\{ \begin{bmatrix} A & 0 & 0 & B \\ 0 & C & D & 0 \\ 0 & -D^* & -C^* & 0 \\ -B^* & 0 & 0 & -A^* \end{bmatrix} \right\}, \quad \mathfrak{i}p = \left\{ \begin{bmatrix} 0 & Z_1 & Z_2 & 0 \\ -Z_1^* & 0 & 0 & Z_2^* \\ -Z_2^* & 0 & 0 & -Z_1^* \\ 0 & -Z_2^{**} & Z_1^{**} & 0 \end{bmatrix} \right\}
\]

where \( A \in \mathfrak{u}(n) \), and \( Z \in \mathfrak{M}_{n \times n}(\mathbb{C}) \) such that \( Z = Z^\tau \).

Intrinsic formula for \( d \) in terms of \( X \):
\[
d = \prod_{k=1}^{n} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k}
\]

where
\[
h_k = \begin{cases} 
e_{kk} - \ne_{k+1,k+1} + \ne_{2n-k,2n-k} - \ne_{2n-k+1,2n-k+1}, & 1 \leq k < n \\
\ne_{nn} - \ne_{n+1,n+1}, & k = n
\end{cases}
\]

are coroots in \( \mathfrak{h}_R \). In matrix form,
\[
h_k = \text{diag}(\ldots,1,-1,\ldots,1,-1,\ldots), \quad h_n = \text{diag}(\ldots,1,-1,\ldots)
\]

As \( Sp(n)/U(n) \) is a subspace of \( Gr(n,\mathbb{C}^{2n}) \), equation (4.2) still holds, but the requirement \(-X = Ad(I_n)X^\tau \) restricts \( \Phi(X) \) to \( Sp(n) \).

In applying Theorem 3.10, we note that the symmetry in our choice of representation gives us \( g^\tau = \text{conj}(I_n)g^{-1} \), so for \( w \in T_0^{(2)} \) we have \( w^\tau = w \), just as in Type DIII. But, unlike the orthogonal case, the anti-diagonal entries of \( X \in \mathfrak{i}p \) need not be zero. Hence, \( |\alpha_m| \) need not be even for \( \det X[\alpha,\alpha] \) to be non-zero for some \( X \). Indeed, \( X \) can be chosen as in the proof of Theorem 3.10 such that \( X[\alpha,\alpha] \) has all anti-diagonal entries 1 and all others zero. Therefore, all \( w \) with an even number of \(-1\) entries that are symmetric across the anti-diagonal occur.

4.4 Type CII

Symmetric Space: \( Sp(p+q)/Sp(p) \times Sp(q) \cong Gr(p,\mathbb{H}^{p+q}) \)

Involution: \( \theta : X \mapsto Ad(I_{p,2q,p})X \)

Block Structure:
\[
\mathfrak{g} = \left\{ \begin{bmatrix} A & 0 & 0 & B \\ 0 & C & D & 0 \\ 0 & -D^* & -C^* & 0 \\ -B^* & 0 & 0 & -A^* \end{bmatrix} \right\}, \quad \mathfrak{i}p = \left\{ \begin{bmatrix} 0 & Z_1 & Z_2 & 0 \\ -Z_1^* & 0 & 0 & Z_2^* \\ -Z_2^* & 0 & 0 & -Z_1^* \\ 0 & -Z_2^{**} & Z_1^{**} & 0 \end{bmatrix} \right\}
\]
where $A = -A^*, B = B^T, C = -C^*, D = D^T, Z_1, Z_2 \in M_{p \times q}(C)$. That is, 
\[
\begin{bmatrix}
A & B \\
-B^* & -A^T
\end{bmatrix} \in \mathfrak{sp}(p), \text{ and } 
\begin{bmatrix}
C & D \\
-D^* & -C^T
\end{bmatrix} \in \mathfrak{sp}(q).
\]
So $\mathfrak{t} \cong \mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$.

Intrinsic formula for $d$ in terms of $X$:
\[
d = \prod_{k=1}^{p+q} \left( \frac{\det(1 + I_kX)}{\det(1 + X)} \right)^{h_k}
\]
where, letting $n = p + q$,
\[
h_k = \begin{cases}
  e_{kk} - e_{k+1,k+1} + e_{2n-k,2n-k} - e_{2n-k+1,2n-k+1}, & 1 \leq k < p + q = n \\
  e_{kk} - e_{k+1,k+1}, & k = p + q = n
\end{cases}
\]
are coroots in $\mathfrak{h}_R$.

To use (3.8) to understand $d$, notice that the blocks negated by $\theta$ are those where the middle $2q$ rows (resp. columns) intersect with the outer $2p$ columns (resp. rows). Therefore, if we construct $X$ as in the proof of Theorem 3.10, then $\det X|\alpha,\alpha| \neq 0$ if and only if
\[
|\{k \in \alpha \mid k \leq p \text{ or } k > p + 2q\}| = |\{k \in \alpha \mid p < k \leq p + 2q\}|.
\]
So we have $w \in T^{(2)}_0$ is in $\phi(U/K)$ if and only if $w$ is symmetric across the anti-diagonal, and has an equal (even) number of $-1$ in the center “$Sp(q)$ part” as in the outer “$Sp(p)$ part.”

### 4.5 Type BDI

Symmetric Space: $SO(p + q)/SO(p) \times SO(q) \cong Gr(p, \mathbb{R}^{p+q})$

Involution: $\theta : X \mapsto Ad(\hat{I})X$ (Inner if and only if $pq$ is even)

Case 1: $p$ and $q$ are not both odd. Without loss of generality, assume $p$ is even. Then $\hat{I} = I_{p\times q} \cdot \hat{I}$.

Block structure:
\[
\mathfrak{t} = \left\{ \begin{bmatrix} A & 0 & B \\ 0 & C & 0 \\ -B^* & 0 & -A^T \end{bmatrix} \right\}, \quad \mathfrak{ip} = \left\{ \begin{bmatrix} 0 & Z & 0 \\ -Z^* & 0 & -Z^T \\ 0 & Z^* & 0 \end{bmatrix} \right\}
\]
where $A = -A^*, B = -B^T, C = -C^* = -C^T, Z \in M_{2 \times q}(C)$. That is, $C \in \mathfrak{so}(q)$, and
\[
\begin{bmatrix}
A & B \\
-B^* & -A^T
\end{bmatrix} \in \mathfrak{so}(p).
\]
So $\mathfrak{t} \cong \mathfrak{so}(p) \oplus \mathfrak{so}(q)$.

Intrinsic formula for $d$ in terms of $X$: 

15
Subcase 1: If $q$ is even (and so $p+q$ is even) we have

\[ d = \prod_{k=1}^{\frac{p+q}{2}-1} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} \left( \frac{\det(1 + \bar{I}_{\frac{p+q}{2}} X)}{\det(1 + X)} \right)^{\frac{1}{2}h_{\frac{p+q}{2}}} \]

where coroots in $h_\mathbb{R}$ are equal to

\[ h_k = \begin{cases} 
\epsilon_{kk} - \epsilon_{k+1,k+1} + \epsilon_{p+q-k,p+q-k} - \epsilon_{p+q-k+1,p+q-k+1}, & 1 \leq k < \frac{p+q}{2} \\
\epsilon_{k-1,k-1} + \epsilon_{kk} - \epsilon_{k+1,k+1} - \epsilon_{k+2,k+2}, & k = \frac{p+q}{2},
\end{cases} \]

As matrices, the $h_k$ are as in (4.4).

Subcase 2: If $p+q$ is odd, we have

\[ d = \prod_{k=1}^{\lfloor \frac{p+q}{2} \rfloor -1} \left( \frac{\det(1 + I_k X)}{\det(1 + X)} \right)^{h_k} \left( \frac{\det(1 + I_{\lfloor \frac{p+q}{2} \rfloor} X)}{\det(1 + X)} \right)^{\frac{1}{2}h_{\lfloor \frac{p+q}{2} \rfloor}} \]

and the coroots are

\[ h_k = \begin{cases} 
\epsilon_{kk} - \epsilon_{k+1,k+1} + \epsilon_{p+q-k,p+q-k} - \epsilon_{p+q-k+1,p+q-k+1}, & 1 \leq k < \lfloor \frac{p+q}{2} \rfloor \\
2\epsilon_{kk} - 2\epsilon_{k+2,k+2}, & k = \lfloor \frac{p+q}{2} \rfloor.
\end{cases} \]

Note that, when $p+q$ is odd, the last root always has the form

\[ h_{\lfloor \frac{p+q}{2} \rfloor} = \text{diag}(\ldots, 2, 0, -2, \ldots). \]

The form of $ip$ here is similar to that of the quaternionic Grassmannian; the blocks negated by $\theta$ are those where the middle $q$ rows (resp. columns) intersect with the outer $p/2$ columns (resp. rows). Therefore, if we construct $X$ as in the proof of Theorem 3.10 then $\det X[\alpha, \alpha] \neq 0$ if and only if

\[ |\{k \in \alpha \mid k \leq p/2 \text{ or } k > p/2 + q\}| = |\{k \in \alpha \mid p/2 < k \leq p/2 + q\}|. \]

So we have $w \in T_0^{(2)}$ is in $\phi(U/K)$ if and only if $w$ is symmetric across the anti-diagonal, and has an equal (even) number of $-1$ in the center “$SO(q)$ part” as in the outer “$SO(p)$ part.” (Notice that if $q$ is odd, the middle diagonal entry must be positive 1.)

If we restrict our attention to even-dimensional real projective space, $SO(2n+1)/SO(1) \times SO(2n) \cong \mathbb{R}P^{2n}$, then by the reasoning above, there can be no $-1$ entries in any $w \in T_0^{(2)}$ in $\phi(U/K)$. Thus, the only $w$ present for $\mathbb{R}P^{2n}$ is the identity matrix, corresponding to the connectedness of $\Sigma_1^{\phi(\mathbb{R}P^{2n})}$. 

16
Case 2: $p$ and $q$ are both odd. Now $\theta = Ad(\hat{I})$ where

$$\hat{I} = \begin{bmatrix}
  1_{\frac{p-1}{2} \times \frac{p-1}{2}} & 0 & 0 & 0 & 0 & 0 \\
  0 & -1_{\frac{q-1}{2} \times \frac{q-1}{2}} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1_{\frac{q-1}{2} \times \frac{q-1}{2}} & 0 \\
  0 & 0 & 0 & 0 & 0 & 1_{\frac{p-1}{2} \times \frac{p-1}{2}} \\
\end{bmatrix}.$$ 

The automorphism $\theta$ is an outer automorphism, since $\lambda \hat{I} \not\in SO(p + q)$ for all $\lambda \in \mathbb{C}$. Still, $\theta$ meets the criteria of Lemma 3.2 and so the proof of Proposition 3.4 may be applied to this case.

Block Structure:

$$\mathfrak{t} = \begin{bmatrix}
  A & 0 & u & u & 0 & B \\
  0 & C & -v & v & D & 0 \\
  -u^* & v^* & 0 & 0 & -v^* & -u^* \\
  -u^* & -v^* & 0 & 0 & v^* & -u^* \\
  0 & -D^* & v^{\tau} & -v^{\tau} & -C^* & 0 \\
  -B^* & 0 & u^{\tau} & u^{\tau} & 0 & -A^* \\
\end{bmatrix}$$

where $A = -A^*, C = -C^*, B = -B^*, D = -D^*$. That is, $$\begin{bmatrix}
  A & u & B \\
  -u^* & 0 & -u^{\tau} \\
  -B^* & u^{\tau} & -A^* \\
\end{bmatrix} \in$$
\[ s_0(p), \text{ and} \begin{bmatrix} C & v & D \\ -v^* & 0 & -v^T \\ -D^* & v^{*T} & -C^T \end{bmatrix} \in s_0(q). \]

\[ i\mathfrak{p} = \begin{bmatrix} 0 & Z_1 & w_1 & -w_1 & Z_2 & 0 \\ -Z_1^* & 0 & w_2 & w_2 & 0 & -Z_2^T \\ -w_1^* & -w_2^* & is & 0 & -w_2^T & w_1^T \\ w_1^* & -w_2^* & 0 & -is & -w_2^T & -w_1^T \\ -Z_2^* & 0 & w_1^{*T} & w_2^{*T} & 0 & -Z_1^T \\ 0 & Z_2^{*T} & -w_1^{*T} & w_2^{*T} & Z_1^{*T} & 0 \end{bmatrix} \]

where \( Z_1, Z_2 \in M_{\frac{p+q}{2} \times \frac{q-1}{2}}(\mathbb{C}), w_1 \in \mathbb{C}^{\frac{p+q}{2}}, w_2 \in \mathbb{C}^{\frac{q-1}{2}}. \)

Intrinsic Formula for \( d \) in terms of \( X \):

\[ d = \prod_{k=1}^{\frac{p+q}{2}-1} \left( \frac{\det(1 + I_kX)}{\det(1 + X)} \right)^{h_k} \left( \frac{\det(1 + I_{\frac{p+q}{2}}X)}{\det(1 + X)} \right)^{\frac{1}{2}(-h_{\frac{p+q}{2}-1} - h_{\frac{q-1}{2}})} \]

where coroots in \( h_{\mathbb{R}} \) are equal to

\[ h_k = \begin{cases} e_{kk} - e_{k+1,k+1} + e_{p+q-k,p+q-k} - e_{p+q-k+1,p+q-k+1}, & 1 \leq k < \frac{p+q}{2} \\ e_{k-1,k-1} + e_{kk} - e_{k+1,k+1} - e_{k+2,k+2}, & k = \frac{p+q}{2}, \end{cases} \]

As matrices, the \( h_k \) are as in (4.4).

The form we have chosen for \( i\mathfrak{p} \) reveals the presence of \( ia_0 \) in the center two diagonal entries. However, by Proposition 2.2, the connected components of \( \Sigma_{1}^{\phi(U/K)} \) are indexed by \( T_{0}^{(2)}/\text{exp}(ia_0)^{(2)} \). Therefore, to use Theorem 3.10 to identify representatives of the connected components, we simply ignore all \( \alpha \) containing \( (p + q)/2 \), and \( (p + q)/2 + 1 \), the middle two rows/columns. This puts us in the same situation as in the previous case, and so the representatives of elements in \( T_{0}^{(2)}/\text{exp}(ia_0)^{(2)} \) correspond to those \( w \) that are symmetric across the anti-diagonal and have the same number of \(-1\) entries in the inner “\( O(q) \) part” (excluding the middle two entries) as in the outer “\( O(p) \) part.”

For odd-dimensional real projective space, \( SO(2n+2)/SO(1) \times SO(2n+1) \cong \mathbb{R}P^{2n+1} \), the situation is similar to that of \( \mathbb{R}P^{2n} \), there is one connected component of \( \Sigma_{1}^{\phi(\mathbb{R}P^{2n+1})} \).
5 Future Study

The two other classical compact symmetric spaces are $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$. In these cases, $\Phi(\mathfrak{ip}) \not\subseteq \phi(U/K)$. The issue here is that Proposition 3.4 does not apply since, for these spaces, we cannot say that $\det(g) = 1$ for $g \in \Phi(\mathfrak{ip})$. Nevertheless, if we relax the determinant condition, the formula could be applied to the homogeneous spaces $U(n)/SO(n)$ and $U(2n)/Sp(n)$. Perhaps our approach could be of some use in studying the large rank limit of these spaces. It would also be interesting to see if Cayley coordinates could be used to identify $w$ representing the lower levels, $\Sigma_w^{\phi(U/K)}$ where $w \neq 1$.

A $\theta$-Stable Representations

We are motivated to use the following representations of $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ in $\mathfrak{su}(n)$ because they are the fixed point sets of involutions that preserve the triangular decomposition of $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{h}$ consists of diagonal matrices, and $\mathfrak{n}^+$ ($\mathfrak{n}^-$) consists of upper (lower) triangular matrices.

Let $\tau : \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ be the map given by reflection across the anti-diagonal; that is, $X^\tau = JX^tJ^{-1}$ where $J$ is the $n \times n$ matrix whose entries are ones on the anti-diagonal and zeros elsewhere. Then $X \mapsto -X^\tau$ is an involution of $\mathfrak{sl}(n, \mathbb{C})$ that stabilizes the above triangular decomposition. The restriction to $\mathfrak{su}(n)$ is also such an involution.

Proposition A.1. $\mathfrak{so}(n) \cong \{X \in \mathfrak{su}(n) | -X^\tau = X\}$, and $\mathfrak{sp}(n) \cong \{X \in \mathfrak{su}(2n) | -X^\tau = \text{Ad}(I_n)X\}$.

Proof. Define involutions on $\mathfrak{sl}(n, \mathbb{C})$ by

$$\Theta_0(X) = -X^t, \quad \Theta_1(X) = -X^\tau = -JX^tJ,$$

for $X \in \mathfrak{sl}(n, \mathbb{C})$.

Then $\Theta_1 = \text{Ad}_J \circ \Theta_0$. Let $P = \frac{1}{\sqrt{2}}(J + i1)$. A straightforward calculation shows that $\Theta_1 = \text{Ad}_P \circ \Theta_0 \circ \text{Ad}_P^{-1}$. Thus the Lie algebra isomorphism $\text{Ad}_P : \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ maps the fixed point subalgebra of $\Theta_0$ to that of $\Theta_1$. This completes the proof for $\mathfrak{so}(n)$; the proof for $\mathfrak{sp}(n)$ follows similarly.

Note: This representation of $\mathfrak{so}(n)$ is the space of infinitesimal isometries of the following $n$ (real) dimensional subspace of $\mathbb{C}^n$:

$$\left\{ \begin{bmatrix} x_n - ix_1 \\ \vdots \\ x_1 - ix_n \end{bmatrix} \mid x_j \in \mathbb{R} \right\}.$$
B Low Dimensional Examples

To more clearly illustrate (4.2), we look at some specific examples. We have already computed (4.2) for \(\mathbb{C}P^n\) in equation (4.3).

Example B.1. \(U/K \cong SO(6)/U(3)\).

In this case, we have

\[
X = \begin{bmatrix}
0 & Z \\
-Z^* & 0
\end{bmatrix}, \quad \text{where } Z = \begin{bmatrix}
z_{11} & z_{12} & 0 \\
z_{21} & 0 & -z_{12} \\
0 & -z_{21} & -z_{11}
\end{bmatrix},
\]

and using formula (4.2), our result simplifies to

\[
d = \text{diag}\left(\frac{-z_{11}^2 + z_{21}^2 - z_{22}^2}{1 + z_{11}^2 + z_{21}^2 + z_{22}^2}, \frac{-z_{21}^2 + z_{12}^2 - z_{22}^2}{1 + z_{12}^2 + z_{21}^2 + z_{22}^2}, \frac{-z_{22}^2 + z_{11}^2 - z_{12}^2}{1 + z_{11}^2 + z_{12}^2 + z_{22}^2}\right).
\]

Example B.2. \(U/K = Sp(2)/Sp(1) \times Sp(1) \cong \mathbb{H}P^1\)

\[
X = \begin{bmatrix}
0 & Z & 0 \\
-Z^* & 0 & -Z^t \\
0 & Z^t & 0
\end{bmatrix} \quad \text{where } Z = \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix},
\]

and

\[
d = \text{diag}\left(\frac{1 - |z_1|^2 - |z_2|^2}{1 + |z_1|^2 + |z_2|^2}, \frac{1 + |z_1|^2 - |z_2|^2 + (|z_1|^2 + |z_2|^2)^2}{1 + (|z_1|^2 + |z_2|^2)^2}, \frac{1 - (|z_1|^2 + |z_2|^2)^2}{1 + (|z_1|^2 + |z_2|^2)^2}\right).
\]

Example B.3. \(U/K \cong SO(2n + 1)/SO(1) \times SO(2n) \cong \mathbb{R}P^{2n}\).

Here, \(X\) has the form

\[
X = \begin{bmatrix}
0 & Z & 0 \\
-Z^* & 0 & -Z^t \\
0 & Z^t & 0
\end{bmatrix}, \quad \text{where } Z = \begin{bmatrix}
z_n \\
\vdots \\
z_1
\end{bmatrix}.
\]
This is similar to the $\mathbb{C}P^n$ case, but each column and row is “doubled,” so each summand will appear twice, perhaps with some sign changes depending on $k$. The result is this:

$$\det(1 + I_k X) = \begin{cases} 1 + 2 \sum_{i=1}^{n-k} |z_i|^2, & 1 \leq k < n \\ 1, & k = n + 1 \\ 1 + 2 \sum_{i=1}^{k-(n+1)} |z_i|^2, & n + 1 \leq k \leq 2n + 1. \end{cases}$$

For $\mathbb{R}P^6$, we have

$$d = \text{diag} \left( \frac{1+2|z_1|^2+2|z_2|^2}{1+2|z_1|^2+2|z_2|^2+2|z_3|^2}, \frac{1+2|z_3|^2}{1+2|z_1|^2+2|z_2|^2+2|z_3|^2+4|z_1|^2}, \frac{1}{1+2|z_1|^2}, \frac{1}{1+2|z_1|^2+2|z_2|^2}, \frac{1}{1+2|z_1|^2+2|z_2|^2+2|z_3|^2+4|z_1|^2} \right).$$

**Example B.4.** $U/K \cong SO(2n + 2)/SO(1) \times SO(2n + 1) \cong \mathbb{R}P^{2n+1}$. Here, $X \in iK$ has the form

$$X = \begin{bmatrix} 0 & Z & 0 \\ -Z^* & is & 0 \\ 0 & -is & -Z^* \end{bmatrix} \text{ where } Z = \begin{bmatrix} z_n & -z_n \\ \vdots & \vdots \\ z_1 & -z_1 \end{bmatrix}.$$  

This is similar to the even-dimensional projective plane $\mathbb{R}P^{2n}$, but because of the duplication of $z_1, \ldots, z_n$, each $|z_j|^2$ has a coefficient of 4 when it appears. The presence of the phase is due to the fact that $\theta$ is an outer automorphism. The result is

$$\det(1 + I_k X) = \begin{cases} 1 + s^2 + 4 \sum_{i=1}^{n-k} z_i \bar{z}_i, & 1 \leq k < n \\ (1 + is)(1 - is) = 1 + s^2, & k = n \\ (1 - is)(1 - is) = 1 - 2is - s^2, & k = n + 1 \\ (1 - is)(1 + is) = 1 + s^2, & k = n + 2 \\ 1 + s^2 + 4 \sum_{i=1}^{k-(n+2)} z_i \bar{z}_i, & n + 2 < k \leq 2n + 2. \end{cases}$$

For $\mathbb{R}P^5$, we have

$$d = \text{diag} \left( \frac{1+s^2+4|z_1|^2}{1+s^2+4|z_1|^2+4|z_2|^2}, \frac{1+s^2}{1+s^2+4|z_1|^2}, \frac{1+is}{1+is+4|z_1|^2}, \frac{1-is}{1-is+4|z_1|^2}, \frac{1+is+4|z_1|^2+4|z_2|^2}{1+is+4|z_1|^2+4|z_2|^2} \right).$$

The lowest dimensional example of $SO(p+q)/SO(p) \times SO(q)$ where $p$ and $q$ are both odd, that is not a real projective space, is $SO(6)/SO(3) \times SO(3)$, and already $\det(1 + I_k X)$ turns out to be quite complicated. It is easy enough to compute with a program such as Maple, but a concise presentation is elusive.
References

[Cai08] Arlo Caine. Compact symmetric spaces, triangular factorization, and Poisson geometry. *Journal of Lie Theory*, 18(2):273–294, 2008.

[EL01] Sam Evens and Jiang-Hua Lu. On the variety of Lagrangian subalgebras. I. *Ann. Sci. École Norm. Sup. (4)*, 34(5):631–668, 2001.

[FO06] Philip Foth and Michael Otto. A symplectic approach to van den Ban’s convexity theorem. *Doc. Math.*, 11:407–424 (electronic), 2006.

[Hel78] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Kna02] Anthony W. Knapp. *Lie Groups: Beyond An Introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2002.

[Pic89] Doug Pickrell. Notes on harmonic analysis on infinite dimensional symmetric spaces. *Preprint*, 1989.

[Pic06] Doug Pickrell. The diagonal distribution for the invariant measure of a unitary type symmetric space. *Transform. Groups*, 11(4):705–724, 2006.