Modeling the spectrum of gravitational waves in the primordial Universe

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Abstract

Recent observations from type Ia Supernovae and from cosmic microwave background (CMB) anisotropies have revealed that most of the matter of the Universe interacts in a repulsive manner, composing the so-called dark energy constituent of the Universe. The analysis of cosmic gravitational waves (GW) represents, besides the CMB temperature and polarization anisotropies, an additional approach in the determination of parameters that may constrain the dark energy models and their consistence. In recent work, a generalized Chaplygin gas model was considered in a flat universe and the corresponding spectrum of gravitational waves was obtained. The present work adds a massless gas component to that model and the new spectrum is compared to the previous one. The Chaplygin gas is also used to simulate a Λ-CDM model by means of a particular combination of parameters so that the Chaplygin gas and the Λ-CDM models can be easily distinguished in the theoretical scenarios here established. The lack of direct observational data is partially solved when the signature of the GW on the CMB spectra is determined.

KEYWORDS: dark energy, gravitational waves
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1 Introduction

A large number of dark energy candidate models have been proposed since that the SNeIa [Riess et al., 1998, Perlmutter et al. 1998, Tonry et al. 2003] and CMB [Tegmark et al., 2004] experiments revealed the accelerating expansion of the universe. In order to explain the observational data all these models consider an equation of state with negative pressure. The cosmological constant models consider a simple equation of state, \( p = -\rho \), which however results a huge discrepancy with the data [Sahni, 2004]. Another class of models considers a scalar field, called quintessence [Caldwell, 2004], a third class assumes a perfect fluid with a negative pressure which is proportional to the inverse of the energy density, the Chaplygin gas [Gorini et al., 2004], and finally a forth class, considers an X-fluid with an equation of state \( p = -\omega \rho \), where \( \omega \) is a positive constant, known as phantom energy when \( \omega > 1 \) [Sahni, 2004].

The distinction among these models must be done by means of combined observations which up to the present have been unable to define what description is more appropriate. Gravitational waves represent one of the potential tools that may in near future offer an additional way to constrain the cosmological parameters and distinguish among the models, since gravitons decouple in a very early time in the universe history. In this paper, we study the spectrum of gravitational waves due to the X-fluid model and compare the results with a
With equations (3) and (4), we write the Einstein’s equations in the form

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_{de}),
\]

\[
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 = -8\pi G (p_m + p_{de}),
\]

where \( a \) is the scale factor of the universe, while \( \rho_m \) and \( \rho_{de} \) are the pressureless fluid and the dark energy densities, respectively. The pressures \( p_m \) and \( p_{de} \) of the fluids are related with their densities by the equations of state \( p_m = 0 \), and \( p_x = \omega \rho_x \), with \( \omega < 0 \) (in case of an X-fluid) or \( p_c = -\bar{A}/\rho_c^\alpha \), with \( \bar{A}, \alpha > 0 \) (in case of Chaplygin gas), respectively. We take the scale factor today as the unity, \( a_0 = 1 \) (The subscripts 0, according to the current notation, indicate the present values of the quantities.) and rewrite the dark energy density for both cases as:

\[
\rho_x = \rho_{x0} a^{-3(\omega + 1)},
\]

\[
\rho_c = \rho_{c0} \left[ \frac{\bar{A}}{\alpha^{3(\alpha + 1)}} + \frac{1}{\alpha^{3(\alpha + 1)}} \right]^{-\frac{3}{\alpha + 1}}, \quad \bar{A} = \frac{\bar{A}}{\rho_{c0}^{\alpha + 1}}.
\]

With equations (3) and (4), we write the Einstein’s equations in the form

\[
\frac{\dot{a}}{a} = H_0 \left( \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{x0}}{a^{3(\omega + 1)}} \right)^{1/2},
\]

\[
\frac{\ddot{a}}{a} = -\frac{1}{2} H_0^2 \left( \frac{\Omega_{m0}}{a^3} + \Omega_{x0} \frac{1 + 3\omega}{a^{3(\omega + 1)}} \right)^{1/2},
\]

for the X-fluid case, and

\[
\frac{\dot{a}}{a} = H_0 \left[ \frac{\Omega_{m0}}{a^3} + \Omega_{x0} \left( \frac{\bar{A}}{\alpha^{3(\alpha + 1)}} + \frac{1}{\alpha^{3(\alpha + 1)}} \right) \right]^{1/2},
\]

\[
\frac{\ddot{a}}{a} = H_0^2 \left[ \frac{\Omega_{m0}}{2a^3} + \Omega_{x0} \left( \bar{A} + \frac{1}{\alpha^{3(\alpha + 1)}} \right) \right]^{1/2} \left[ 1 - \frac{3(1 - \bar{A})}{2\alpha^{3(\alpha + 1)}} \left( \bar{A} + \frac{1}{\alpha^{3(\alpha + 1)}} \right) \right],
\]

for the Chaplygin gas. The Hubble constant \( H_0 \) is defined as \( H_0 = a_0/\bar{A} \) and, once we are restricted to a flat universe, the fractions of pressureless matter and dark energy gas today, \( \Omega_{m0} \) and \( \Omega_{de0} \), respectively, obey the relation \( \Omega_{m0} + \Omega_{de0} = 1 \).

With these two last equations we are able to write the GW amplitude differential equation as a function of the observable parameters \( H_0, a, \Omega_{m0} \) and \( \Omega_{de0} \), and of the dark energy fluid parameters: \( \bar{A}, \alpha \) or \( \omega \). It is also important to remark that

(i) if \( \bar{A} = 0 \), then the Chaplygin gas behaves like the pressureless fluid and the situation is the same as if we had set \( \Omega_{m0} = 1 \);

(ii) on the other hand, for \( \bar{A} = 1 \) it behaves like the Cosmological Constant and therefore, we can simulate the \( \Lambda \) CDM scenario;

(iii) the X-fluid is equal to the Cosmological Constant for \( \omega = -1 \).

Among the many possible cases produced by the combinations of the parameters we have chosen a few important ones which are indicated in table 1. These choices will be sufficient for the aim of the present analysis, where we compare the spectra due to each model.

2 Outline of the model

We consider a flat, homogeneous and isotropic universe described by the Friedman-Robertson-Walker metric, which makes the Einstein’s equations to assume the form

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_{de}),
\]

\[
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 = -8\pi G (p_m + p_{de}),
\]

where \( a \) is the scale factor of the universe, while \( \rho_m \) and \( \rho_{de} \) are the pressureless fluid and the dark energy densities, respectively. The pressures \( p_m \) and \( p_{de} \) of the fluids are related with their densities by the equations of state \( p_m = 0 \), and \( p_x = \omega \rho_x \), with \( \omega < 0 \) (in case of an X-fluid) or \( p_c = -\bar{A}/\rho_c^\alpha \), with \( \bar{A}, \alpha > 0 \) (in case of Chaplygin gas), respectively. We take the scale factor today as the unity, \( a_0 = 1 \) (The subscripts 0, according to the current notation, indicate the present values of the quantities.) and rewrite the dark energy density for both cases as:

\[
\rho_x = \rho_{x0} a^{-3(\omega + 1)},
\]

\[
\rho_c = \rho_{c0} \left[ \frac{\bar{A}}{\alpha^{3(\alpha + 1)}} + \frac{1}{\alpha^{3(\alpha + 1)}} \right]^{-\frac{3}{\alpha + 1}}, \quad \bar{A} = \frac{\bar{A}}{\rho_{c0}^{\alpha + 1}}.
\]

With equations (3) and (4), we write the Einstein’s equations in the form

\[
\frac{\dot{a}}{a} = H_0 \left( \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{x0}}{a^{3(\omega + 1)}} \right)^{1/2},
\]

\[
\frac{\ddot{a}}{a} = -\frac{1}{2} H_0^2 \left( \frac{\Omega_{m0}}{a^3} + \Omega_{x0} \frac{1 + 3\omega}{a^{3(\omega + 1)}} \right)^{1/2},
\]

for the X-fluid case, and

\[
\frac{\dot{a}}{a} = H_0 \left[ \frac{\Omega_{m0}}{a^3} + \Omega_{x0} \left( \frac{\bar{A}}{\alpha^{3(\alpha + 1)}} + \frac{1}{\alpha^{3(\alpha + 1)}} \right) \right]^{1/2},
\]

\[
\frac{\ddot{a}}{a} = H_0^2 \left[ \frac{\Omega_{m0}}{2a^3} + \Omega_{x0} \left( \bar{A} + \frac{1}{\alpha^{3(\alpha + 1)}} \right) \right]^{1/2} \left[ 1 - \frac{3(1 - \bar{A})}{2\alpha^{3(\alpha + 1)}} \left( \bar{A} + \frac{1}{\alpha^{3(\alpha + 1)}} \right) \right],
\]

for the Chaplygin gas. The Hubble constant \( H_0 \) is defined as \( H_0 = a_0/\bar{A} \) and, once we are restricted to a flat universe, the fractions of pressureless matter and dark energy gas today, \( \Omega_{m0} \) and \( \Omega_{de0} \), respectively, obey the relation \( \Omega_{m0} + \Omega_{de0} = 1 \).

With these two last equations we are able to write the GW amplitude differential equation as a function of the observable parameters \( H_0, a, \Omega_{m0} \) and \( \Omega_{de0} \), and of the dark energy fluid parameters: \( \bar{A}, \alpha \) or \( \omega \). It is also important to remark that

(i) if \( \bar{A} = 0 \), then the Chaplygin gas behaves like the pressureless fluid and the situation is the same as if we had set \( \Omega_{m0} = 1 \);

(ii) on the other hand, for \( \bar{A} = 1 \) it behaves like the Cosmological Constant and therefore, we can simulate the \( \Lambda \) CDM scenario;

(iii) the X-fluid is equal to the Cosmological Constant for \( \omega = -1 \).

Among the many possible cases produced by the combinations of the parameters we have chosen a few important ones which are indicated in table 1. These choices will be sufficient for the aim of the present analysis, where we compare the spectra due to each model.
\[\begin{array}{cccccc}
\omega & A & \alpha & \Omega_{de0} & \Omega_{m0} \\
\text{Pressureless Fluid} & - & - & - & 0 & 1 \\
\text{Cosmological Constant} & -1 & 1 & - & 0.96 & 0.04 \\
\text{Generalized Chaplygin Gas} & -0.5 & 1 & 0.96 & 0.04 \\
\text{X-fluid} & -10 & - & - & 0.7 & 0.3 \\
\text{X-fluid (Phantom Energy)} & -10 & - & - & 0.7 & 0.3 \\
\end{array}\]

Table 1: Models chosen for the analysis.

3 GW equations

Cosmological gravitational waves are obtained through little perturbations \(h_{ij}\) on the metric. Hence, the tensor \(g^{(0)}_{ij}\), related to the unperturbed metric, is replaced by \(g_{ij} = g^{(0)}_{ij} + h_{ij}\) and the resulting expression is [Weinberg, 1972]:

\[
\ddot{h} - \frac{\dot{a}}{a} \dot{h} + \left( \frac{k^2}{a^2} - 2 \frac{\ddot{a}}{a} \right) h = 0, \tag{9}
\]

where \(k\) is the wave number times the velocity of light (\(k = 2\pi c/\lambda\)), the dots indicate time derivatives and \(h(t)\) is defined as: \(h_{ij}(\vec{x}, t) = h(t)Q_{ij}\), where \(Q_{ij}\) are the eigenmodes of the Lagrangian operator, such that \(Q_{ii} = \partial_a Q_{ki} = 0\).

Proceeding with a variable transformation from time to the scale factor \(a\), and representing the derivatives with respect to \(a\) by primes, equation (9) assumes the form

\[
h'' + \left( \frac{\dot{a}}{a^2} - \frac{1}{a} \right) h' + \frac{1}{a^2} \left( \frac{k^2}{a^2} - 2 \frac{\ddot{a}}{a} \right) h = 0 \tag{10}
\]

and, by the application of the background equations \(4\) and \(5\), or \(8\) and \(9\), which concern to the dark energy fluid, into (10), one can easily express \(h\) in terms of the parameters of the model.

3.1 Generalized Chaplygin gas

Let us perform the last operation mentioned above and, in order to find \(h\) as a function of the redshift \(z\), let us (i) use the known relations \(1 + z = \frac{a_0}{a}, a_0 = 1\); (ii) perform a second change of variables (from \(a\) to \(z\)); and (iii) take back the dots to indicate, from now on, the new integration variable. These operations result in

\[
h + \left[ \frac{2}{1 + z} + \frac{3}{2}(1 + z)^2 \frac{f_1}{f_2} \right] h + \left[ \frac{k^2}{f_2} - \frac{2}{(1 + z)^2} + 3(1 + z) \frac{f_1}{f_2} \right] h = 0, \tag{11}
\]

\[f_1 = \Omega_{m0} + \Omega_{c0} (1 - \bar{A})(1 + z)^{3\alpha} \left[ \bar{A} + (1 - \bar{A})(1 + z)^{3(\alpha + 1)} \right]^{-1}, \tag{12}\]

\[f_2 = \Omega_{m0} (1 + z)^3 + \Omega_{c0} \left[ \bar{A} + (1 - \bar{A})(1 + z)^{3(\alpha + 1)} \right]^{1/\alpha}, \tag{13}\]

where \(k\) is redefined to absorb the Hubble constant, i.e., \(k = 2\pi c/H_0 \lambda\) and, therefore, \(h\) is a dimensionless quantity.
3.2 X-fluid

To obtain the equivalent equations for the X-fluid model, we use the same procedure described above. The result may be written in the same form as in (11), with \( f_1 \) and \( f_2 \) redefined as:

\[
\begin{align*}
    f_1 &= \Omega_m + \Omega_x(1 + \omega)(1 + z)^{3\omega}, \\
    f_2 &= \Omega_m(1 + z)^3 + \Omega_x(1 + z)^{3(\omega+1)}.
\end{align*}
\]

One can easily notice that, if \( \omega = -1 \) and \( \bar{A} = 1 \), both cases converge to the same equation and then reproduce the \( \Lambda \)-CDM scenario.

4 GW spectra analysis

The power spectrum of gravitational waves, defined as

\[
\frac{d\Omega_{GW}}{d \ln \nu} = |h_0(\nu)|^{5/2}
\]

[where \( h_0(\nu) = h(0) \) and \( \nu = H_0 k/2\pi \)], is generally obtained from the solutions of (11) which are found by means of a numerical algorithm specifically created for this problem. In the particular case of the \( \Lambda \)-CDM model, an analytical result is also possible and this fact is used to verify the accuracy of the calculation.

For each of the cases of interest, we have assigned some common parameters, namely the initial conditions \( h(z_i) = \nu^{-3}10^{-5} \), \( \dot{h}(z_i) = \nu^{-2}10^{-5} \), \( z_i = 4000 \); the range of frequencies \( 10^{-18} \text{Hz} \leq \nu \leq 10^{-15} \text{Hz} \), which is a small part of the range of observational interest for GW that goes up to up to \( 10^{10} \text{Hz} \); and the normalization constant imposed by the CMB [Riazuelo & Uzan, 2000]

\[
\left| \frac{d\Omega_{GW}}{d \ln \nu} \right|_{10^{-18}} \leq 10^{-10}.
\]

The resulting spectra are presented in figures (1-11) and we can see that they all have a similar behavior: a strong oscillatory shape; a peak at \( \nu \approx 0 \) (due to the initial condition, which is proportional to \( k^{-3} \)); after a very rapid decrease, the spectra increase slowly. Besides these features, it is important to remark that:

![Figure 1](image_url)

**Figure 1.** GW spectrum for \( \Omega_c = 0 \) and \( \Omega_m = 1 \).
(i) The plot (Fig. 1) is for a pressureless pure model and we can see that the spectrum corresponding to a Cosmological Constant dominated universe (Fig. 2) grows faster with the frequency reaching a value three times greater. Still referring to a $\Lambda$-CDM model, figure (Fig. 3) shows a behavior similar to the pressureless fluid, but the growing amplitude is just a little larger.
(ii) The graphics (4-9), however, are not so easily distinguishable. They seem to have no dependence on the $\Omega_{c0}$ parameter and this may be interpreted as a consequence of the Chaplygin gas property of interpolation between dark-matter and dark-energy.

(iii) Figures (4) and (5) refer to the Chaplygin gas in its traditional form – with the equation of state $p = -A/\rho$, since $\alpha = 1$ – while (6-9) correspond to the generalized Chaplygin gas and the six plots together show that the spectrum is not affected by $\alpha$.

(iv) The X-fluid is represented in plots (10) and (11). The later is the case of phantom energy, with $\omega = -10$, while the former is an intermediate situation between the pressureless and the $\Lambda$ cases. We notice that they also do not differ from each other.

5 Conclusions

In this work we have compared three important dark energy models in the context of the gravitational wave spectra that they are able to induce and we observe that the models are strongly degenerated in the range of frequencies studied – from $10^{-18}$ up to $10^{-15}$ Hz. This degeneracy is in part expected, since the models must converge to each other when some particular combinations of parameters are considered. In addition, for the Chaplygin gas, the negligible dependence on the density parameter $\Omega_{c0}$ is consistent with the idea of a unified dark component [Fabris et al. 2004, Gorini et al. 2004]. Since the gas interpolates both dark energy and matter, the suppression of the dark matter density parameter (see, plots (4-9)) should not affect the spectrum, as observed. However, it is interesting how it seems to be insensitive to $\alpha$ as well. This result indicates that this
model is more sensitive to $\dot{A}$ than to $\alpha$. The X-fluid component contribution seems to be completely degenerated and must be better studied in the near future, specially at larger frequency ranges.

Unfortunately, no direct observational data from cosmological gravitational waves is available up to now, but still it is important to investigate the relation between these gravitational waves produced or amplified by the presence of the dark energy component and the CMB temperature and polarization anisotropies which are already observationally determined [Bennett et al., 2003]. This correlation between CMB and gravitational waves will be properly studied once we have a larger range of frequencies in the GW spectra.

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References

C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, M. Kogut, A. and Limon, S. S. Meyer, L. Page, D. N. Spergel, G. S. Tucker, E. Wollack, E. L. Wright, C. Barnes, M. R. Greason, R. S. Hill, M. R. Komatsu, E. and Nolta, N. Odegard, H. V. Peiris, L. Verde, & J. L. Weiland. First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results. *ApJ Supplement*, 148:1–27, May 2003.

M. Caldwell, R. R. and Doran. Cosmic microwave background and supernova constrains on quintessence: concordance regions and target models. *Phys. Rev.D*, 69:103517, May 2004.

J. C. Fabris, S. V. B. Gonçalves, & M. S. Santos. Gravitational waves in the generalized Chaplygin gas model. *Gen. Rel. Grav.*, 36:2559–2572, November 2004.

V. Gorini, A. Kamenshchik, U. Moschella, & V. Pasquier. The Chaplygin gas as a model for dark energy. *gr-qc/0403062*, 2004.

S. Perlmutter, G. Aldering, M. della Valle, S. Deustua, R. S. Ellis, S. Fabbro, A. Fruchter, G. Goldhaber, I. M. Groom, D. E. and Hook, A. G. Kim, M. Y. Kim, R. A. Knop, R. G. Lidman, C. and McMahon, P. Nugent, R. Pain, N. Panagia, C. R. Pennypacker, P. Ruiz-Lapuente, B. Schaefer, & N. Walton. Discovery of a supernova explosion at half the age of the universe. *Nature*, 391:51, January 1998.

A. Riazuelo & J. P. Uzan. Quintessence and Gravitational Waves. *Phys. Rev. D*, 62:083506, October 2000.

A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, Peter M. Garnavich, Craig J. Gilliland, Ron L. and Hogan, Saurabh Jha, Robert P. Kirshner, B. Leibundgut, M. M. Phillips, David Reiss, Brian P. Schmidt, Robert A. Schommer, R. Chris Smith, J. Spyromilio, Christopher Stubbs, Nicholas B. Suntzeff, & John Tonry. Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. *AJ*, 116:1009–1038, September 1998.

V. Sahni. Dark matter and dark energy. *astro-ph/0403324*, 2004.

Max Tegmark, Michael A. Strauss, Michael R. Blanton, Kevork Abazajian, Scott Dodelson, Havard Sandvik, Xiaomin Wang, David H. Weinberg, Idit Zehavi, Neta A. Bahcall, Fiona Hoyle, David Schlegel, Roman Scoccimarro, Michael S. Vogeley, Andreas Berlind, T. Budavari, Andrew Connolly, Daniel J. Eisenstein, Douglas Finkbeiner, Joshua A. Frieman, James E. Gunn, Lam Hui, Bhuvnesh Jain, David Johnston, Stephen Kent, Huan Lin, Reiko Nakajima, Robert C. Nichol, Jeremiah P. Ostriker, Adrian Pope, Ryan Scranton, Uros Seljak, Ravi K. Sheth, Albert Stebbins, Alexander S. Szalay, I. Szapudi, Yongzhong Xu, James Annis, J. Brinkmann, Scott Burles, Istvan Castander, Francisco J. and Csabai, Jon Loveday, Mamoru Doi, Masataka Fukugita, Bruce Gillespie, Greg Hennessy, David W. and I. Hogg, Z. Ivezic, Gillian R. Knapp, Don Q. Lamb, Brian C. Lee, Robert H. Lupton, Timothy A. McKay, Peter Kunszt, Jeffrey A. Munn, Liam O’Connell, John Peoples, Jeffrey R. Pier, Michael Richardson, Constance Rockosi, Donald P. Schneider, Christopher Stoughton, Douglas L. Tucker, Daniel E. vanden Berk, Brian Yanny, & Donald G. York. Cosmological parameters from SDSS and WMAP. *Phys. Rev. D*, 69:103501, May 2004.
John L. Tonry, Brian P. Schmidt, Brian Barris, Pablo Candia, Peter Challis, Alejandro Clocchiatti, Alison L. Coil, Alexei V. Filippenko, Peter Garnavich, Craig Hogan, Stephen T. Holland, Saurabh Jha, Robert P. Kirshner, Kevin Krisciunas, Bruno Leibundgut, Weidong Li, Thomas Matheson, Mark M. Phillips, Adam G. Riess, Robert Schommer, R. Chris Smith, Jesper Sollerman, Jason Spyromilio, Christopher W. Stubbs, & Nicholas B. Suntzeff. Cosmological Results from High-z Supernovae. *ApJ*, 594:1–24, September 2003.

Steven Weinberg. *Gravitation and Cosmology*. John Wiley, New York, 1972.