Interpolation properties for some scales of approximation spaces

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Abstract

We obtain a result concerning the stability under the interpolation with functional parameter method for the approximation spaces of Lorentz-Marcinkiewicz type and also for the approximation spaces generated by symmetric norming functions of a certain type.

1 Introduction and general notions

The subject of this paper is the real interpolation in approximation spaces.

Many of the approximation scales used in approximation theory are modeled by some sequence ideals such as $l_p$, Lorentz, Lorentz-Zygmund or Lorentz-Marcinkiewicz.

A natural framework which includes most of these scales and gives a unified model for their study is provided by symmetric norming functions. Unfortunately the abstract shape of a symmetric norming function is too general to allow us to extend the basic interpolation results which hold for $l_p$, Lorentz, Lorentz-Zygmund or Lorentz-Marcinkiewicz scales to this generality.

More precisely, while in the particular cases of $l_p$, Lorentz etc. scales the stability under interpolation by the real method holds (the classical version for $l_p$ and Lorentz cases, respectively the extended one with functional parameter for Lorentz-Zygmund and Lorentz-Marcinkiewicz cases), nothing is known about the behavior under the real interpolation of the scale generated by symmetric norming functions.

Results given below give a partial extension of the known classical stability properties to the general framework given by a scale of approximation spaces generated by a certain class of symmetric norming functions. In fact we prove the following result (for the notation see below)

The main observation which leads to the results below is that the Boyd functions, which are the essential ingredient of the interpolation method with functional parameter, have a shape close enough to the symmetric norming functions of $\Phi^\varepsilon$ type to allow us to obtain informations about the interpolation process on the approximation spaces of $\Phi^\varepsilon$ type (for the notation see below).

Before continuing, let us fix the notation used throughout the paper. By $E, F$ we denote Banach spaces over $\Gamma$, where $\Gamma$ is the real or the complex field. We
let $L(E, F) := \{ T : E \to F \mid T \text{ is linear and bounded} \}$. By $\hat{k}$ we denote the set of all decreasing positive sequences $x := (x_n)_n$ such that $x_n = 0$ eventually. We denote by $l_\infty$ the set of all scalar sequences, $(x_n)_{n \in \mathbb{N}}$, with the property $\|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n| < \infty$, and by $c_0$ the set of all scalar sequences, $(x_n)_{n \in \mathbb{N}}$, with the property $\lim_{n \to \infty} |x_n| = 0$. For $0 < p < \infty$ we let $l_p$ be the set of all scalar sequences, $(x_n)_{n \in \mathbb{N}}$, such that $\|x\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty$.

2 Approximation schemes

2.1 Generalities

We start with the general scheme described by a quasi-normed abelian group, i.e. an abelian group $G$ endowed with a non-negative valued function $\|\cdot\| : G \to \mathbb{R}_+$ satisfying:

1. $\|g\| > 0$, for every $g \in G$, $g \neq 0$;
2. $\|-g\| = \|g\|$, for every $g \in G$;
3. there is a constant, depending on $G$, $c \geq 1$ such that $\|g + f\| \leq c (\|g\| + \|f\|)$, for every $g, f \in G$.

Definition 1 ([3], [15]) An approximation scheme on a quasi-normed abelian group $(G, \|\cdot\|)$ consists of a pair $(G, (G_n)_{n \in \mathbb{N}})$, where $(G_n)_{n \in \mathbb{N}}$ is a sequence of subsets of $X$ satisfying the following conditions:

1. $G_0 = \{0\}$;
2. $G_n \subseteq G_{n+1}$, for every $n \in \mathbb{N}$;
3. $G_n + G_m \subseteq G_{n+m}$, for every $n, m \in \mathbb{N}$;
4. $G_n - G_m \subseteq G_{n+m}$, for every $n, m \in \mathbb{N}$.

Associated with an approximation scheme $(G, (G_n)_{n \in \mathbb{N}})$ one introduces the notion of an approximation space ([3], [15]). We will use a function $E : G \to l_\infty$, defined by

$$E(g) := (E_n(g))_n, \quad \text{for every } g \in G,$$

where, $E_n(g)$ is the best approximation of $g$ by elements of $G_{n-1}$, i.e.

$$E_n(g) := \inf \{ \|g - h\| : h \in G_{n-1} \}.$$

The's basic properties of $E$ are as follows:
1. \( \|g\| = E_1(g) \geq E_2(g) \geq ... \geq 0 \), for every \( g \in G \) ([15]);

2. \( E_n(-g) = E_n(g) \), for every \( n \in \mathbb{N} \) and \( g \in G \) ([15]);

3. \( E_{n+m}(f+g) \leq c(E_n(f) + E_m(g)) \), for every \( n, m \in \mathbb{N} \) and all \( f, g \in G \) ([15]).

### 2.2 Examples

We shall describe some important examples.

1. **(Operator ideals)** Let us take \( G := L(E, F) \) and \( G_n := F_n(E, F) \), where \( F_n(E, F) := \{ T_n \in L(E, F) : \text{dim} T_n \leq n \} \).

   Then for any \( T \in L(E, F) \), the number \( E_n(T) \) is the \( n \)-th approximation number of \( T \), denoted by \( a_n(T) \) ([11], [13], [14]).

2. **(Sequence ideals)** Let us now take \( G := l_p(0 < p \leq \infty) \) and \( G_n := f_p^{(n)} := \{ x := (x_m)_{m} \in l_p : \text{card} \{ x_m : x_m \neq 0 \} \leq n \} \).

   Then for any \( x \in l_p \), the number \( E_n(x) \) is the \( n \)-th approximation number of \( x \), denoted by \( a_n(x) \). Let us remark that, if the sequence \( x := (x_n)_{n} \in l_p \) is ordered such that \( |x_n| \geq |x_{n+1}| \), for every \( n \), then \( a_n(x) = |x_n| \) ([9], [13], [14]).

### 2.3 Boyd functions

One way to obtain approximation spaces is to use Boyd functions. We start with a brief review of the notion of Boyd functions and the related Lorentz-Marcienkiewicz scale.

**Definition 2** ([2], [10], [14]) We denote by \( B \) the class of all functions \( \varphi : (0, \infty) \to (0, \infty) \) which have the following properties:

1. \( \varphi \) is continuous;
2. \( \varphi(1) = 1 \);
3. \( \overline{\varphi}(t) := \sup_{s>0} \frac{\varphi(ts)}{\varphi(s)} < \infty \) for any \( t > 0 \).

A straightforward consequence of this definition is that \( \varphi(st) \leq \varphi(s) \overline{\varphi}(t) \), for any \( s, t \in (0, \infty) \).
Definition 3 ([2], [10], [14]) Given \( \varphi \) in \( B \), the \( \varphi \) function’s Boyd indices \( \alpha_{\varphi}, \beta_{\varphi} \) are defined by:

\[
\alpha_{\varphi} := \inf_{1 < t < \infty} \frac{\log \varphi(t)}{\log t} = \lim_{t \to \infty} \frac{\log \varphi(t)}{\log t},
\]

and

\[
\beta_{\varphi} := \sup_{0 < t < 1} \frac{\log \varphi(t)}{\log t} = \lim_{t \to 0} \frac{\log \varphi(t)}{\log t}.
\]

The Boyd indices satisfy the following relation

\[-\infty < \beta_{\varphi} \leq \alpha_{\varphi} < \infty.\]

For future reference we collect some of the basic properties of the Boyd functions in the following.

Proposition 4 If \( \varphi, \varphi_1, \varphi_2 \) are in \( B \) and \( a \) is a real number, then \( \varphi_1 \varphi_2, \varphi_1^a \varphi_2^a \) are also in \( B \) and

1. \( \varphi_1 \varphi_2 \leq \varphi_1 \varphi_2 \)
2. \( \varphi_1(t) \leq \varphi_1(t); \)
3. \( \beta_{\varphi} > 0 \) if and only if \( \lim_{t \to 0} \varphi(t) = 0 \)

Definition 5 ([14], [15]) Let \( (G, (G_n)_n) \) be an approximation scheme, \( \varphi \in B \) and \( 0 < q < \infty \). An approximation space of Lorentz-Marcinkiewicz type is defined as follows:

\[
G_{\varphi,q} := \left\{ g \in G : \sum_{n=1}^{\infty} [\varphi(n) E_n(g)]^q n^{-1} < \infty \right\}.
\]

We also define an functional \( \| \cdot \|_{\varphi,q} : G_{\varphi,q} \to \mathbb{R} \) by

\[
\|g\|_{\varphi,q} := \left( \sum_{n=1}^{\infty} [\varphi(n) E_n(g)]^q n^{-1} \right)^{1/q} \text{ for every } g \in G_{\varphi,q}.
\]

We remark that \( (G_{\varphi,q}, \| \cdot \|_{\varphi,q}) \) is a quasi-normed abelian group, and for the particular case \( \varphi(t) := t^p, 0 < p < \infty \), we obtain the definition of the classical approximation spaces \( G_{p,q} \), where

\[
G_{p,q} := \left\{ g \in G : \|g\|_{p,q} := \left( \sum_{n=1}^{\infty} [n^p E_n(g)]^q n^{-1} \right)^{1/q} < \infty \right\}.
\]

Now if we take for \( (G, (G_n)_n) \) the particular case \( (l_\infty, (f_n^{(m)})_n) \) we obtain the well known quasi-normed sequence ideal.
\[ l_{\varphi,q} := \left\{ x \in l_{\infty} : \|x\|_{\varphi,q} := \left( \sum_{n=1}^{\infty} [\varphi(n)a_n(x)]^q n^{-1} \right)^{\frac{1}{q}} < \infty \right\}. \]

Likewise, if we take for \((G, (G_n)_n)\) the particular case \((L(E, F), (F_n(E, F))_n)\) we obtain the well known quasi-normed operator ideal

\[ L_{\varphi,q} := \bigcup_{E,F \text{ Banach spaces}} L_{\varphi,q}(E, F), \]

where

\[ L_{\varphi,q}(E, F) := \left\{ T \in L(E, F) : \|T\|_{\varphi,q} := \left( \sum_{n=1}^{\infty} [\varphi(n)a_n(T)]^q n^{-1} \right)^{\frac{1}{q}} < \infty \right\}. \]

We note that equivalently we can define \(G_{\varphi,q}\) as the set of those \(g\) in \(G\) for which the sequence \((E_n \langle g \rangle)_n\) belongs to \(l_{\varphi,q}\).

### 2.4 Symmetric norming functions

Another way for constructing approximation spaces uses a symmetric norming function. We recall the definitions involved here.

**Definition 6** ([14], [15]) A function \(\Phi : \hat{k} \to \mathbb{R}\) is called a **symmetric norming function** if the following conditions are fulfilled:

1. \(\Phi(x) > 0\) whenever \(x \neq 0\);
2. \(\Phi(\alpha x) = \alpha \Phi(x)\) for every \(\alpha > 0\) and \(x\) in \(\hat{k}\);
3. \(\Phi(x + y) \leq \Phi(x) + \Phi(y)\) for every \(x\) and \(y\) in \(\hat{k}\);
4. \(\Phi\{1, 0, 0, \ldots\} = 1\);
5. If \(\sum_{n=1}^{m} x_n \leq \sum_{n=1}^{m} y_n\) for some \(x = (x_n)_n\) and \(y = (x_n)_n\) in \(\hat{k}\) and for every \(m\) in \(\mathbb{N}^\ast\), then \(\Phi(x) \leq \Phi(y)\).

**Remark 7** [14] Notice that the above definition can be extended to the space \(l_{\infty}\) of all bounded sequences in the following way. If \(\Phi : \hat{k} \to \mathbb{R}\) is a symmetric norming function and \(x := (x_n)_n \in l_{\infty}\) we define

\[ \Phi(x) := \lim_{n} \Phi\{a_1(x), \ldots, a_n(x), 0, 0, \ldots\}, \]

when \(\sup_{n} \Phi\{a_1(x), \ldots, a_n(x), 0, 0, \ldots\} < \infty\).
The most important examples of symmetric norming functions are the extremal symmetric norming functions $\Phi_1$ and $\Phi_\infty$. These are defined as follows:

$$
\Phi_1 (x) := \sum_{n=1}^{n_x} x_n \quad \text{and} \quad \Phi_\infty (x) := \max_n x_n, \quad x \in \mathcal{K}.
$$

It is easily seen that

$$\Phi_\infty (x) \leq \Phi (x) \leq \Phi_1 (x)$$

for any symmetric norming function $\Phi$ and any $x \in \mathcal{K}$.

In our future considerations a significant place is taken by the symmetric norming functions of a certain type. These are the so-called $\Phi_\varepsilon$ function. Their definition is presented in the next proposition.

**Proposition 8** ([13]) Let $\varepsilon := (\varepsilon_n)_n$ be a decreasing sequence of positive real numbers with $\varepsilon_1 = 1$. The function $\Phi_\varepsilon : \mathcal{K} \to \mathbb{R}$ defined by

$$
\Phi_\varepsilon (x) = \sum_{m=1}^{n_x} \varepsilon_m x_m, \quad \text{for every} \quad x := (x_m)_{m \in \mathbb{N}} \in \mathcal{K},
$$

is a symmetric norming function. If the sequence $\varepsilon$ has in addition the properties

$$
\lim_{m \to \infty} \varepsilon_m = 0 \quad \text{and} \quad \sum_{m=1}^{\infty} \varepsilon_m = \infty \quad \text{then} \quad \Phi_\varepsilon \sim \Phi_\infty, \quad \text{and} \quad \Phi_\varepsilon \sim \Phi_1 \quad (\Phi \sim \Psi \text{ means sup}_{x \in \mathcal{K}} \frac{\Phi (x)}{\Psi (x)} < \infty, \quad \text{and sup}_{x \in \mathcal{K}} \frac{\Psi (x)}{\Phi (x)} < \infty). \quad \text{Let} \quad \Phi_\varepsilon \text{ be a function like above and} \quad 1 \leq p < \infty. \quad \text{The function} \quad \Phi_\varepsilon^{(p)} : \mathcal{K} \to \mathbb{R} \text{ defined by}
$$

$$
\Phi_\varepsilon^{(p)} (x) = \left( \Phi_\varepsilon ((x^p)_m) \right)^{\frac{1}{p}}, \quad \text{for every} \quad x := (x_m)_{m \in \mathcal{K}},
$$

is a symmetric function.

**Definition 9** ([5]) Let $(G, (G_n)_n)$ be an approximation scheme and $\Phi$ a symmetric norming function. An approximation space of $\Phi$ type is defined as follows:

$$
G_\Phi := \{ g \in G : \Phi ((E_n(g))_n) < \infty \}.
$$

We also define an functional $\| \cdot \|_\Phi : G_\Phi \to \mathbb{R}$ by

$$
\| g \|_\Phi := \Phi ((E_n(g))_n) \quad \text{for every} \quad g \in G_\Phi.
$$

We remark that $(G_\Phi, \| \cdot \|_\Phi)$ is a quasi-normed abelian group and for the particular case $\Phi := \Phi_\alpha$, with $\alpha := \left( n^{pq-\frac{1}{q}} \right)$, we obtain, again, the definition of the classical approximation spaces $G_{p,q}$.

Now if we take for $(G, (G_n)_n)$ the particular case $\left( l_\infty, (f^{(m)}_\infty)_n \right)$ we obtain the well known quasi-normed sequence ideal.
\[ l_{\Phi} := \{ x \in l_\infty : \| x \|_\Phi := \Phi ((a_n(x))_n) < \infty \}. \]

Likewise, if we take for \((G, (G_n)_n)\) the particular case \((L(E, F), (F_n(E, F))_n)\) we obtain the well known quasi-normed operator ideal

\[ L_{\Phi} := \bigcup_{E, F \text{ Banach spaces}} L_{\Phi}(E, F), \]

where

\[ L_{\Phi}(E, F) := \{ T \in L(E, F) : \| T \|_\Phi := \Phi ((a_n(T))_n) < \infty \}. \]

Equivalently we can define \(G_{\Phi}\) as the set of those \(g \in G\) for which the sequence \((E_n(g))_n\) belongs to \(l_{\Phi}\).

### 3 Stability results

#### 3.1 Real interpolation

Before continuing let us recall some results on real interpolation.

We consider couples \((E_0, E_1)\) of quasi-normed spaces \(E_0\) and \(E_1\), which are both continuously embedded in a quasi-normed space \(E\). This means that \(E_i \subset E\) and there is a constant \(c_i\) such that \(\| x \|_E \leq c_i \| x \|_{E_i}\) for every \(x\) in \(E_i\), \(i \in \{0, 1\}\). In the sequel we let \(\hookrightarrow\) denote a continuous embedding. We say that such a couple \((E_0, E_1)\) is a quasi-normed interpolation couple.

Let \((E_0, E_1)\) be a quasi-normed interpolating couple. For every \(x\) in \(E_0 + E_1\) J. Peetre defined the functional

\[ K(t, x, E_0, E_1) = K(t, x) := \inf_{x=x_0+x_1} (\| x_0 \|_{E_0} + t \| x_1 \|_{E_1}), \]

where \(x_i \in E_i\), \(i \in \{0, 1\}\) and \(0 < t < \infty\). Let now \((E_0, E_1)\) be a quasi-normed interpolation couple, \(0 < q < \infty\) and \(\varphi \in \mathbf{B}\). We shall consider the set

\[ (E_0, E_1)_{\varphi, q} := \left\{ x \in E_0 + E_1 : \int_{0}^{\infty} \left[ \varphi(t)^{-1} K(t, x) \right]^q \frac{dt}{t} < \infty \right\}. \]

It is important to notice that for any \(\varphi \in \mathbf{B}\) and \(0 < q < \infty\), the functional \(\| \cdot \|_{(E_0, E_1)_{\varphi, q}} : (E_0, E_1)_{\varphi, q} \rightarrow \mathbb{R}_+\), defined by

\[ \| x \|_{(E_0, E_1)_{\varphi, q}} := \left( \int_{0}^{\infty} \left[ \varphi(t)^{-1} K(t, x) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, \]

for every \(x \in (E_0, E_1)_{\varphi, q}\), is a quasi-norm.
Consider now the space \( E_\Sigma := E_0 + E_1 \) equipped with the quasi-norm
\[
\|x\|_\Sigma := \inf_{x = x_0 + x_1, x_i \in E_i} (\|x_0\|_0 + \|x_1\|_1) = K(1, x)
\]
and also the space \( E_\Delta := E_0 \cap E_1 \) equipped with the quasi-norm
\[
\|x\|_\Delta := \max (\|x\|_0, \|x\|_1).
\]
We remark that \((E_0, E_1)_{\phi,q} \hookrightarrow E_\Sigma\) \(1\)
and, if \(0 < \beta_\phi \leq \alpha_\phi < 1\) then \((E_0, E_1)_{\phi,q} \hookrightarrow (E_0, E_1)_{\phi,q}\).

The above construction which starts with an interpolation couple \((E_0, E_1)\)
and give us the space \((E_0, E_1)_{\phi,q}\) (called real interpolation method with functional parameter) was introduced by T.F. Kalugina ([2], [4], [6], [10]) as an extension of the classical real method due to J. Peetre ([1], [11], [13], [14]).

More precisely, if we take \(\phi(t) = t^{-\theta}, 0 < \theta < 1\), we get the classical real interpolation space \((E_0, E_1)_{\theta,q}\).

We recall now the reiteration theorem of the real interpolation method with functional parameter which is the main ingredient of our proofs.

**Theorem 10 ([10])** Let \(\mathcal{A} = \{A_0, A_1\}\) be an interpolation couple of quasi-normed spaces. Take \(f, f_0, f_1\) in \(B\), where \(f\) in addition satisfies \(0 < \beta_{f_0} \leq \alpha_{f_1} < 1\) and let \(u = \frac{1}{f_0}, g = f_0(f \circ u)\), \(0 < p, q_0, q_1 \leq \infty\). By \(E_i\) we denote the interpolation space \((A_0, A_1)_{f_i,q_i}\), where \(i \in \{0, 1\}\). If one of the following two hypotheses is fulfilled:
1. \(\beta_{f_0} > 0\) in the case \(q_0 < \infty\), respectively \(\sup_{t \leq 1} f_0(t) < \infty\) in the case \(q_0 = \infty\), and \(\alpha_{f_1} < 1\) in the case \(q_1 < \infty\), respectively \(\sup_{t \geq 1} f_1(t) < \infty\) in the case \(q_1 = \infty\), when \(\beta_{f_0} > 0\) or
2. \(\beta_{f_1} > 0\) in the case \(q_1 < \infty\), respectively \(\sup_{t \leq 1} f_1(t) < \infty\) in the case \(q_1 = \infty\), and \(\alpha_{f_0} > 0\) in the case \(q_0 < \infty\), respectively \(\sup_{t \geq 1} f_0(t) < \infty\) in the case \(q_0 = \infty\), when \(\alpha_{f_1} < 0\), then
\(g \in B\) and \((E_0, E_1)_{f,p} = (A_0, A_1)_{g,p}\).

Let us mention that, from this abstract reiteration theorem, F. Cobos has obtained the stability under functional parameter interpolation’s process for the operator ideals introduced by him.
Theorem 11 ([2]) Take $E, F$ Banach spaces, the numbers $q_0, q_1, q$ in $(0, \infty]$ and the functions $\chi, \varphi_0, \varphi_1$ in $B$. Let now consider the functions $\varphi : (0, \infty) \to (0, \infty), \rho : (0, \infty) \to (0, \infty)$ defined by
\[
\varphi (t) := \frac{\varphi_0 (t)}{\varphi_1 (t)},
\]
respectively by
\[
\rho (t) = \frac{\varphi_0 (t)}{\chi (\varphi (t))}.
\]
If $0 < \beta_\chi \leq \alpha_\chi < 1, \beta_{\varphi_i} > 0$ ($i = 0, 1$) and $\beta_{\varphi_i} > 0$ or $\alpha_{\varphi_i} < 0$, then
\[
\rho \in B \quad \text{and} \quad (L_{\varphi_0, q_0} (E, F), L_{\varphi_1, q_1} (E, F))_{\chi, q} = L_{\rho, q} (E, F),
\]
with equivalent quasi-norms.

3.2 Stability results for approximation spaces: The Lorentz-Marcinkiewicz case

The main result of this section is as follows.

Theorem 12 Take the numbers $q_0, q_1, q$ in $(0, \infty]$ and the functions $\chi, \varphi_0, \varphi_1$ in $B$. Let now consider the functions $\varphi, \rho : (0, \infty) \to (0, \infty)$ defined by
\[
\varphi (t) := \frac{\varphi_0 (t)}{\varphi_1 (t)}, \quad \rho (t) := \frac{\varphi_0 (t)}{\chi (\varphi (t))},
\]
If the following conditions are fulfilled:
\[
0 < \beta_\chi \leq \alpha_\chi < 1, \beta_{\varphi_i} > 0 \quad \text{where} \quad i \in \{0, 1\} \quad (3)
\]
and
\[
\beta_{\varphi_i} > 0 \quad \text{or} \quad \alpha_{\varphi_i} < 0 \quad (4)
\]
then
\[
\rho \in B \quad \text{and} \quad (G_{\varphi_0, q_0}, G_{\varphi_1, q_1})_{\chi, q} = G_{\rho, q},
\]
with equivalent quasi-norms.

The proof is constructed in two steps, the first of which has independent interest. We shall use the notation
\[
G_p := \left\{ g \in G : \sum_{n=1}^{\infty} |E_n (g)|^p < \infty \right\}.
\]

Theorem 13 Take $\varphi$ in $B$ which satisfies the condition $0 < \beta_\varphi$ and $q$ in $(0, \infty]$. If $0 < p_0 < p_1 \leq \infty$ are such that
\[
\frac{1}{p_1} < \beta_\varphi < \alpha_\varphi < \frac{1}{p_0}
\]
and if $\rho : (0, \infty) \to (0, \infty)$ is defined by

$$\rho (t) = t^{\frac{p_1}{p_0 - p_1}} (\varphi (t^{p_0/p_1}))^{-1}$$

(in the case $p_1 < \infty$), respectively

$$\rho (t) = t (\varphi (t^{p_0}))^{-1}$$

(in the case $p_1 = \infty$), then $\rho \in B$ and $\mathcal{G}_{p_0, G_{p_1}}^\rho, q = \mathcal{G}_{\varphi, q}$,

with equivalent quasi-norms.

**Proof.** The fundamental observation is that the equivalence

$$K \left( t, T, L^{(a)}_p (E, F), L^{(a)}_p (E, F) \right) \simeq K (t, (a_n (T))_n, l_{p_0}, l_{p_1}),$$

proved by H. König in [7], where "\( \simeq \) " indicates equivalence with constants that do not depend on $t$ or $T$, remain valid for the abstract case of the application $E : X \to l_\infty$, the proof being the same. Hence

$$K (t, g, G_{p_0}, G_{p_1}) \simeq K (t, (E_n (g))_n, l_{p_0}, l_{p_1}).$$

Now we obtain the following equivalences

$$g \in (G_{p_0}, G_{p_1})^\rho, q \Leftrightarrow \int_0^\infty \left[ \rho (t)^{-1} K (t, f, G_{p_0}, G_{p_1}) \right]^q \frac{dt}{t} < \infty \leftrightarrow$$

$$\Leftrightarrow \int_0^\infty \left[ \rho (t)^{-1} K (t, (E_n (f))_n, l_{p_0}, l_{p_1}) \right]^q \frac{dt}{t} < \infty \Leftrightarrow (E_n (f))_n \in (l_{p_0}, l_{p_1})^\rho, q \leftrightarrow$$

$$\leftrightarrow (E_n (g))_n \in l_{\varphi, q} \Leftrightarrow \sum_n [\varphi (n) E_n (g)]^q n^{-1} < \infty \Leftrightarrow g \in G_{\varphi, q}.$$
the number \( r \) being chosen in such a way that \( \frac{1}{r} > \max \{ \alpha \varphi_0, \alpha \varphi_1, \alpha \varphi \} \), and

\[
u(t) = \frac{f_1(t)}{f_0(t)} = \varphi(t').
\]

Applying now Theorem 10 we obtain

\[
(G_{\varphi_0, q_0}, G_{\varphi_1, q_1})_{\chi, q} = \left((G_r, G)_{f_0, q_0}, (G_r, G)_{f_1, q_1}\right)_{\chi, q} = (G_r, G)_{g, q},
\]

where

\[
g(t) = f_0(t)(f \circ u)(t) = \frac{t}{\varphi_0(t')} \chi(\varphi(t')).
\]

Applying again the previous theorem, this time with \( \rho = g \), we obtain

\[
(G_r, G)_{g, q} = G_{\varphi, q},
\]

where

\[
\varphi(t') = \frac{t}{g(t)} = \frac{\varphi_0(t')}{\chi(\varphi(t'))}.
\]

Hence we have obtained

\[
\varphi(t) = \frac{\varphi_0(t)}{\chi(\varphi(t))} = \rho(t).
\]

In conclusion,

\[
(G_r, G)_{g, q} = G_{\rho, q},
\]

and moreover we can write the desired equality

\[
(G_{\varphi_0, q_0}, G_{\varphi_1, q_1})_{\chi, q} = G_{\rho, q}.
\]

\[\blacksquare\]

3.2.1 Stability results for approximation schemes: The symmetric norming functions case

Speaking now about the frame constructed with symmetric norming functions the basic idea for finding interpolation results for the approximation spaces of type \( G_\Phi \) is that the symmetric norming functions of \( \Phi^c \) can be arranged as some Boyd functions like we shall present in the following construction.

**Definition 14** Let \( \alpha = (\alpha_n)_n \) be a sequence of real numbers with the following properties:

1. \( 1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq 0 \);
2. \( \lim_{n \to \infty} \alpha_n = 0 \);
3. \( \sum_{n=1}^{\infty} \alpha_n = \infty \);
4. \( M(p) := \sup_n \frac{\alpha_n}{\alpha_p} < \infty \) for every fixed \( p > 1 \), where by \( [t] \) we denote the
greatest integer less or equal than \( t \).
For every sequence like above and every positive number \( p \) we define the function \( 
\varphi_{\alpha, p} : (0, \infty) \to (0, \infty) \) as follows

\[
\varphi_{\alpha, p} (t) := \begin{cases} 
  t^p & \text{if } t \in (0, 1) \\
  (\alpha t)^p & \text{if } t \in \mathbb{N}^* \\
  (1 - \{t\}) \varphi_{\alpha, p} (\lfloor t \rfloor) + \{t\} \varphi_{\alpha, p} (\lfloor t \rfloor + 1) & \text{if } t \in (1, \infty) \setminus \mathbb{N}
\end{cases}
\]

where \( \{t\} := t - \lfloor t \rfloor \).

To prove that Definition 14 is not void we present the following.

**Example 15** The sequence \( \left( \frac{1}{n} \right)_n, a \leq 1 \) has the properties 1-4 from Definition 14.

**Proof.** The properties 1-3 are obviously fulfilled. To verify the fourth condition
fix \( p \) in \( \mathbb{N} \), \( p > 1 \). It is known that

\[
\left\lfloor \frac{s}{p} \right\rfloor = \left\lfloor \frac{s}{p} \right\rfloor = \left\lfloor \frac{1}{p} \right\rfloor \geq \left\lfloor \frac{1}{p} \right\rfloor
\]

for every positive number \( s \). Hence we shall obtain

\[
\alpha \left( \frac{t}{p} \right) \leq \alpha \left( \frac{t}{p} \right)
\]

and furthermore

\[
\frac{\alpha \left( \frac{t}{p} \right)}{\alpha_n} \leq \frac{\alpha \left( \frac{t}{p} \right)}{\alpha_n} = \frac{1}{\left\lfloor \frac{1}{p} \right\rfloor}.
\]

In conclusion

\[
\sup_n \frac{\alpha \left( \frac{t}{p} \right)}{\alpha_n} \leq \frac{1}{\left\lfloor \frac{1}{p} \right\rfloor} < \infty.
\]

It is obvious that the function \( \Phi : l_\infty \to \mathbb{R} \) defined by

\[
\Phi (x) := \sum_{n=1}^{\infty} \alpha_n a_n (x),
\]

for all \( x \in l_\infty \) is a symmetric norming function. More interesting is the fact that the above definition leads to a Boyd function.

**Theorem 16** Let \( \alpha = (\alpha_n)_n \) be a sequence which has the properties 1-4 from Definition 14 and \( p \in (0, \infty) \). Then \( \varphi_{\alpha, p} \in \mathbf{B} \).

**Proof.** We shall verify the axioms from the definition of a \( \mathbf{B} \)-function. The construction ensures the continuity of the function \( \varphi_{\alpha, p} \) and also the equality

\[
\varphi_{\alpha, p} (1) = (\alpha_1)^\frac{1}{p} = 1.
\]
In order to verify the third condition, we consider an arbitrary number \( t \in (0, \infty) \). We shall evaluate from above \( \frac{\varphi_{\alpha,p}(st)}{\varphi_{\alpha,p}(s)} \) for \( s \in (0, \infty) \). This way in the sequel shall take account of the inequality \([xy] \geq [x][y]\) which is true for all positive \( x, y \). Without loss of generality we may assume that \( t \neq 1 \). There are two cases. The first one is to consider all \( s \in (0, \infty) \) such that \( st \neq 1 \). Then we’ll have

\[
\frac{\varphi_{\alpha,p}(st)}{\varphi_{\alpha,p}(s)} = \frac{(st)^{\frac{1}{\alpha}}}{s^{\frac{1}{\alpha}}} = t^{\frac{1}{\alpha}}.
\]

The second one is to consider all \( s \in (0, \infty) \) such that \( st > 1 \). In that case we have to analyze two situations. For the moment we fix \( t < 1 \). Then we obtain the following relations:

\[
\frac{\varphi_{\alpha,p}(st)}{\varphi_{\alpha,p}(s)} = \frac{(1 - \{st\}) \varphi_{\alpha,p}([st]) + \{st\} \varphi_{\alpha,p}([st] + 1)}{(1 - \{s\}) \varphi_{\alpha,p}([s]) + \{s\} \varphi_{\alpha,p}([s] + 1)} = \frac{(1 - \{st\}) (\alpha_{[st]}([st]))^{\frac{1}{\alpha}} + \{st\} (\alpha_{[st]+1}([st] + 1))^{\frac{1}{\alpha}}}{(1 - \{s\}) (\alpha_{[s]}([s]))^{\frac{1}{\alpha}} + \{s\} (\alpha_{[s]+1}([s] + 1))^{\frac{1}{\alpha}}} = \frac{(1 - \{st\}) (\alpha_{[st]}([st]))^{\frac{1}{\alpha}} + \{st\} (\alpha_{[st]+1}([st] + 1))^{\frac{1}{\alpha}}}{(1 - \{s\}) (\alpha_{[s]}([s]))^{\frac{1}{\alpha}} + \{s\} (\alpha_{[s]+1}([s] + 1))^{\frac{1}{\alpha}}} \leq \frac{\frac{1}{\alpha_{[s]+1}} (1 - \{st\}) [st]^{\frac{1}{\alpha}} + \{st\} ([st] + 1)^{\frac{1}{\alpha}}}{(1 - \{s\}) [s]^{\frac{1}{\alpha}} + \{s\} ([s] + 1)^{\frac{1}{\alpha}}} = \frac{\frac{1}{\alpha_{[s]+1}} (1 - \{st\}) [st]^{\frac{1}{\alpha}} + \{st\} ([st] + 1)^{\frac{1}{\alpha}}}{(1 - \{s\}) [s]^{\frac{1}{\alpha}} + \{s\} ([s] + 1)^{\frac{1}{\alpha}}} \leq \frac{1}{\left[\frac{1}{\alpha}\right]^{\frac{1}{\alpha}} \frac{1}{\alpha_{[s]+1}} (1 - \{st\}) + \{st\} \left(1 + \frac{1}{[st]}\right)^{\frac{1}{\alpha}}}{(1 - \{s\}) + \{s\} \left(1 + \frac{1}{[s]}\right)^{\frac{1}{\alpha}}}.
\]

Now we assume that \( t > 1 \). Then

\[
\frac{\varphi_{\alpha,p}(st)}{\varphi_{\alpha,p}(s)} = \frac{(1 - \{st\}) \varphi_{\alpha,p}([st]) + \{st\} \varphi_{\alpha,p}([st] + 1)}{(1 - \{s\}) \varphi_{\alpha,p}([s]) + \{s\} \varphi_{\alpha,p}([s] + 1)} = \frac{(1 - \{st\}) (\alpha_{[st]}([st]))^{\frac{1}{\alpha}} + \{st\} (\alpha_{[st]+1}([st] + 1))^{\frac{1}{\alpha}}}{(1 - \{s\}) (\alpha_{[s]}([s]))^{\frac{1}{\alpha}} + \{s\} (\alpha_{[s]+1}([s] + 1))^{\frac{1}{\alpha}}} \leq \frac{1}{\left[\frac{1}{\alpha}\right]^{\frac{1}{\alpha}} \alpha_{[s]+1} (1 - \{st\}) + \{st\} \left(1 + \frac{1}{[st]}\right)^{\frac{1}{\alpha}}}{(1 - \{s\}) + \{s\} \left(1 + \frac{1}{[s]}\right)^{\frac{1}{\alpha}}}.
\]
\[
\leq \frac{\alpha_s^{\frac{1}{p}}([st] + 1)^{\frac{1}{p}}}{\alpha_s^{\frac{1}{p} + 1} [s]^{\frac{1}{p}}} = \left( \frac{\alpha([s])}{\alpha([s] + 1)} \right)^{\frac{1}{p}} \left( \frac{([st] + 1)^{\frac{1}{p}}}{[s]} \right) \leq \left( \frac{([st] + 1)^{\frac{1}{p}}}{[s]^{\frac{1}{p}}} \right)^{\frac{1}{p}} \leq \left( \frac{st + 1}{s - 1} \right)^{\frac{1}{p}}.
\]

In conclusion
\[
\sup_{s > 0} \frac{\varphi_{\alpha,p}(st)}{\varphi_{\alpha,p}(s)} < \infty \text{ for every } t \in (0, \infty).
\]

We are prepared now for the main result of this section.

**Theorem 17** Consider an approximation scheme \((G, (G_n)_{n \in \mathbb{N}})\). Let \(\alpha := (\alpha_n)_n\) and \(\beta := (\beta_n)_n\) be sequences having the properties 1-4 from Definition 14. If \(\alpha, \beta\) in addition satisfy the conditions
1. \(\beta_n \leq \alpha_n\) for every \(n \in \mathbb{N}^*\) and
2. \(\lim_{t \to 0} M_\alpha (\frac{1}{t}) = 0\) respectively \(\lim_{t \to 0} M_\beta (\frac{1}{t}) = 0\)
and we choose the positive numbers \(p, q, l\) satisfying the following four relations

\[
1 \leq p < q < \infty, \quad l > 1, \quad p + ql > q \quad \text{and} \quad \frac{pql}{p + ql - q} > 1,
\]

then

\[
(G^{\varphi_{\alpha,p}^\alpha}, G^{\varphi_{\alpha,p}^\beta})_{f,r} = G^{\varphi_{\alpha,p}^\gamma}_{f,r}, \quad \text{with equivalent quasi-norms},
\]

where \(f : (0, \infty) \to (0, \infty)\) is given by

\[
f(t) := t^\frac{1}{p}
\]

and \(\gamma := (\gamma_n)_n\) is given by

\[
\gamma_n := \alpha_n^{\frac{1}{p} - \frac{1}{q}} \beta_n^\frac{q}{pq}.
\]

**Proof.** We start by proving that \(l_{\varphi_{\alpha,p}} = l_{\varphi_{\alpha,p}^\gamma}\). From the definitions

\[
l_{\varphi_{\alpha,p}^\alpha} = \left\{ x \in l_\infty : \left( \sum_{n=1}^\infty \alpha_n [a_n(x)]^p \right)^{\frac{1}{p}} < \infty \right\}
\]

and

\[
l_{\varphi_{\alpha,p}} = \left\{ x \in l_\infty : \left( \sum_{n=1}^\infty [\varphi_{\alpha,p}(n) a_n(x)]^p n^{-1} \right)^{\frac{1}{p}} < \infty \right\}.
\]

Consequently we obtain

\[
l_{\varphi_{\alpha,p}} = \left\{ x \in l_\infty : \left( \sum_{n=1}^\infty (\alpha_n)^{\frac{1}{p}} n^{-1} \right)^{\frac{1}{p}} < \infty \right\}.
\]
We only have to evaluate \( \|\cdot\|_{\Phi(p)}^\alpha \). From the definition of the approximation spaces \( G_{\Phi(p)}^\alpha \), we obtain the following equivalences

\[
g \in G_{\Phi(p)}^\alpha \iff (E_n(g)) \in l_{\Phi(p)}^\alpha \iff (E_n(g)) \in l_{\varphi_{\alpha,p},p} \iff g \in G_{\varphi_{\alpha,p},p}.
\]

In conclusion

\[
G_{\Phi(p)}^\alpha = G_{\varphi_{\alpha,p},p}.
\]

Similarly

\[
G_{\Phi(q)}^\beta = G_{\varphi_{\beta,q},q}, \text{ respectively } G_{\Phi(r)}^\gamma = G_{\varphi_{\gamma,r},r}.
\]

The last step is to verify the hypotheses of Theorem 10 for

\[
\varphi_0 := \varphi_{\alpha,p}, \varphi_1 := \varphi_{\beta,q}, \chi := f.
\]

From the definition of \( f \) we obtain that \( \alpha_f = \beta_f = \frac{1}{2} \) and hence

\[
0 < \alpha_f = \beta_f < 1.
\]

Because

\[
\varphi = \frac{\varphi_0}{\varphi_1} = \frac{\varphi_{\alpha,p}}{\varphi_{\beta,q}}
\]

we can write

\[
\varphi(t) = \sup_{s > 0} \frac{\varphi_{\alpha,p}(st)}{\varphi_{\beta,q}(st)} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)}.
\]

Now we are interested in computing \( \lim_{t \to 0} \varphi(t) \). To start we notice that if \( t \to 0 \) then \( st < 1 \). Consequently,

\[
\lim_{t \to 0} \varphi(t) = \lim_{t \to 0} \sup_{s > 0} \frac{\varphi_{\alpha,p}(st)}{\varphi_{\beta,q}(st)} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)} = \lim_{t \to 0} \frac{(st)^{\frac{1}{p}}}{(st)^{\frac{1}{q}}} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)} = \lim_{t \to 0} \frac{1}{(st)^{\frac{1}{q}}} \sup_{s > 0} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)}.
\]

We only have to evaluate \( \sup_{s > 0} s^{1-p} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)} \). If \( s \in (0,1] \) then

\[
\sup_{0 < s < 1} s^{\frac{1}{p}} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)} = \sup_{0 < s < 1} s^{\frac{1}{p}} \frac{1}{s^{\frac{1}{q}}} = 1.
\]
If \( s \in (1, \infty) \) then

\[
\sup_{1 < s < \infty} s^{\frac{1}{p} - \frac{1}{q}} \frac{\varphi_{\beta,q}(s)}{\varphi_{\alpha,p}(s)} = \sup_{1 < s < \infty} s^{\frac{1}{p} - \frac{1}{q}} \frac{(1 - \{s\}) \left( \beta_{\lfloor s \rfloor} [s] \right)^{\frac{1}{q}} + \{s\} \left( \beta_{\lfloor s \rfloor + 1} ([s] + 1) \right)^{\frac{1}{q}}}{(1 - \{s\}) \left( \alpha_{\lfloor s \rfloor} [s] \right)^{\frac{1}{q}} + \{s\} \left( \alpha_{\lfloor s \rfloor + 1} ([s] + 1) \right)^{\frac{1}{q}}} \leq
\]

\[
\leq \sup_{1 < s < \infty} s^{\frac{1}{p} - \frac{1}{q}} \frac{\beta_{\lfloor s \rfloor}}{\alpha_{\lfloor s \rfloor + 1}} \cdot \left( \frac{\lfloor s \rfloor + 1}{\lfloor s \rfloor} \right)^{\frac{1}{q}} \leq \sup_{1 < s < \infty} \left( \frac{\alpha_{\lfloor s \rfloor}}{\alpha_{\lfloor s \rfloor + 1}} \right)^{\frac{1}{p}} s^{\frac{1}{p} - \frac{1}{q}} \left( \frac{\lfloor s \rfloor + 1}{\lfloor s \rfloor} \right)^{\frac{1}{q}} \leq
\]

\[
\leq M \sup_{1 < s < \infty} s^{\frac{1}{p} - \frac{1}{q}} \left( 1 + \frac{1}{\lfloor s \rfloor} \right)^{\frac{1}{q}} \leq M \sup_{1 < s < \infty} \left( \lfloor s \rfloor + 1 \right)^{\frac{1}{p} - \frac{1}{q}} \lfloor s \rfloor^{\frac{1}{q}} \left( 1 + \frac{1}{\lfloor s \rfloor} \right)^{\frac{1}{q}} \leq
\]

\[
\leq M_1 \sup_{1 < s < \infty} \left( 1 + \frac{1}{\lfloor s \rfloor} \right)^{\frac{1}{q}} < 2^{\frac{1}{q}} M_1.
\]

In conclusion for the both cases we obtain \( \lim_{t \to 0} \varphi_0(t) = 0 \) and furthermore \( \beta_{\varphi_0} > 0 \).

Similarly we analyze the functions \( \varphi_0 \) and \( \varphi_1 \). We have

\[
\overline{\varphi_{\alpha,p}}(t) = \sup_{s > 0} \frac{\varphi_{\alpha,p}(st)}{\varphi_{\alpha,p}(s)}, \text{ for every } t \in (0, \infty).
\]

The following inequality is true:

\[
\overline{\varphi_{\alpha,p}}(t) \leq \left( M \left( \frac{1}{t} \right) \frac{1}{[t]} \right)^{\frac{1}{p}},
\]

hence we can write

\[
0 \leq \lim_{t \to 0} \overline{\varphi_{\alpha,p}}(t) \leq \lim_{t \to 0} \left( M \left( \frac{1}{t} \right) \frac{1}{[t]} \right)^{\frac{1}{p}}.
\]

But

\[
\lim_{t \to 0} \left( M \left( \frac{1}{t} \right) \frac{1}{[t]} \right)^{\frac{1}{p}} = 0
\]

so we conclude that \( \lim_{t \to 0} \overline{\varphi_{\alpha,p}}(t) = 0 \) and furthermore

\( \beta_{\varphi_0} > 0 \).

Similarly we proof that \( \lim_{t \to 0} \overline{\varphi_{\beta,q}}(t) = 0 \) from which we derive again

\( \beta_{\varphi_1} > 0 \).
The hypotheses of Theorem 10 being fulfilled we can apply it to obtain

\[
\left( G_{\varphi_{\alpha,p}}, G_{\varphi_{\beta,q}} \right)_{f,r} = G_{\rho,r},
\]

where the function \( \rho : (0, \infty) \to (0, \infty) \) is defined by

\[
\rho(t) := \frac{\varphi_{\alpha,p}(t)}{\varphi_{\beta,q}(t)}.
\]

We compute

\[
\rho(n) = \frac{\varphi_{\alpha,p}(n)}{\varphi_{\beta,q}(n)} = \varphi_{\alpha,p}(n)^{1 - \frac{1}{r}} \varphi_{\beta,q}(n)^{\frac{1}{r}} =
\]

\[
= \frac{1}{n^{\frac{1}{r}}} \frac{\beta_n}{\alpha_n} n^{\frac{\alpha_n - \beta_n}{\alpha_n}} = \left( \frac{r_{\alpha_n} \beta_n}{\alpha_n} n \right)^{\frac{1}{r}} n^{\frac{1}{r}} = (\gamma_n)^{\frac{1}{r}}.
\]

In conclusion \( l_{\rho,r} = l_{\varphi_{\gamma,r}} \), where \( \gamma = (\gamma_n)_n \) and hence \( G_{\rho,r} = G_{\varphi_{\gamma,r},r} \). Adding the two equalities proved before \( G_{\varphi_{\alpha,p},p} = G_{\Phi_{\alpha,p}} \) and \( G_{\varphi_{\beta,q},q} = G_{\Phi_{\beta,q}} \) we obtain the desired result.

\[ \Box \]

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