Abstract

A graph is edge-distance-regular when it is distance-regular around each of its edges and it has the same intersection numbers for any edge taken as a root. In this paper we give some (combinatorial and algebraic) proofs of the fact that every edge-distance-regular graph $\Gamma$ is distance-regular and homogeneous. More precisely, $\Gamma$ is edge-distance-regular if and only if it is bipartite distance-regular or a generalized odd graph. Also, we obtain the relationships between some of their corresponding parameters, mainly, the distance polynomials and the intersection numbers.

1 Introduction

In this paper, we use standard concepts and results about distance-regular graphs (see, for example, Biggs [1], or Brouwer, Cohen, and Neumaier [2]), spectral graph theory (see Cvetković, Doob, and Sachs [4], or Godsil [15]), and spectral and algebraic characterizations of distance-regular graphs (see, for instance, Fiol [10]).

Edge-distance-regular graphs, introduced by Fiol and Garriga [13], are analogous to distance-regular graphs but considering the distance partitions induced by every edge instead
of each vertex. Thus, many known results for distance-regular graphs have their counter-
part for edge-distance-regular graphs such as, for instance, the so-called spectral excess theorem. This theorem characterizes (vertex- or edge-)distance-regular graphs by their spectra and the (average) number of vertices at extremal distance (from every vertex or edge). See Fiol and Garriga [12] and Cámara, Dalfó, Fàbrega, Fiol, and Garriga [3] for the cases of distance-regular and edge-distance-regular graphs, respectively. Also, for short proofs, see Van Dam [7] and Fiol, Gago, and Garriga [11].

A distance-regular graph Γ with diameter d and odd-girth (that is, the shortest cycle of odd length) 2d + 1 is called a generalized odd graph, also known as an almost-bipartite distance-regular graph or a regular thin near (2d + 1)-gon. The first name is due to the fact that odd graphs O_k (see Biggs [1]) are distance-regular and have such an odd-girth. Notice that, in this case, the intersection parameters of Γ satisfy a_0 = \cdots = a_{d-1} = 0 and a_d \neq 0. Recently, Van Dam and Haemers [9] showed that any connected regular graph with d + 1 distinct eigenvalues and odd-girth 2d + 1 is a generalized odd graph. Moreover, Lee and Weng [18] used a variation of the spectral excess theorem for nonregular graphs to show that, in fact, the regularity condition is not necessary, and Van Dam and FiOL [8] gave a more direct short proof of the same result.

Here, we provide some (combinatorial and algebraic) proofs that, in fact, any edge-
distance-regular graph Γ is also distance-regular. Moreover, if this is the case, Γ is either bipartite or a generalized odd graph, and the relationship between the intersection num-
bers of the corresponding distance partitions, induced by a vertex and by an edge, is made explicit. Thus, a distance-regular graph Γ is edge-distance-regular if and only if Γ is either bipartite or a generalized odd graph. In fact, the ‘only if’ part is also a consequence of a result by Martin [19], who proved that if a pair of vertices at distance h is a completely regular code in a distance-regular graph Γ with diameter d, then either h = 1 and Γ has intersection numbers a_1 = \cdots = a_{d-1} = 0, or h = d and Γ is antipodal.

In the rest of this section, we recall some concepts, terminology, and results involved. Throughout this paper, Γ = (V,E) denotes a connected graph on n = |V| vertices and m = |E| edges, having adjacency matrix A, and spectrum sp Γ = \{\lambda_0^m, \ldots, \lambda_d^m\}, where \lambda_0 > \cdots > \lambda_d, and the superscripts m_i stand for the multiplicities. The distance between two vertices u,v is denoted by dist(u,v), so that the diameter of Γ is D = \max\{dist(u,v) : u,v \in V\}. Moreover, given C \subset V, the set C_i = Γ_i(C) = \{u \in V : dist(u,C) = i\} is called the i-th subconstituent with respect to C, where dist(u,C) = \min\{dist(u,v) : v \in C\} and C_0 = C. In particular, when C is a singleton, C = \{u\}, we write Γ_i(u) for Γ_i(\{u\}) and set Γ(u) for Γ_1(u). The eccentricity or covering radius of C is ecc(C) = ε_C = \max\{i : C_i \neq \emptyset\}, so that we have the partition V = C_0 \cup C_1 \cup \cdots \cup C_{ε_C}.

Given any two vertices w,u at distance dist(w,u) = i \geq 0 of a graph Γ, we consider the numbers of neighbors of w at distance i - 1, i, i + 1 from u, that is,
c_i(w,u) = |\Gamma(w) \cap \Gamma_{i-1}(u)|, \quad a_i(w,u) = |\Gamma(w) \cap \Gamma_i(u)|, \quad b_i(w,u) = |\Gamma(w) \cap \Gamma_{i+1}(u)|,

and Γ is distance-regular if these numbers only depend on i. In this case we write c_i(w,u) = c_i, a_i(w,u) = a_i and b_i(w,u) = b_i and say that these numbers are well defined. A useful
characterization of distance-regularity is the existence of the so-called *distance-polynomials* $p_0, \ldots, p_d$ of $\Gamma$ satisfying
\begin{equation}
p_i(A) = A^i, \quad i = 0, \ldots, d,
\end{equation}
where $A_i$ is the $i$-th distance matrix of $\Gamma$, with entries $(A_i)_{uv} = 1$ if $\text{dist}(u,v) = i$, and $(A_i)_{uv} = 0$ otherwise. Recall also that, if $\Gamma$ is distance-regular, the intersection parameters $p_{ij}^k = |\Gamma_i(u) \cap \Gamma_j(v)|$, with $\text{dist}(u,v) = k$, for $i, j, k = 0, \ldots, d$, are the Fourier coefficients of the polynomial $p_ip_j$ in terms of the basis constituted by the distance-polynomials of $\Gamma$ with respect to the scalar product
\[ \langle f, g \rangle = \frac{1}{n} \text{tr}(f(A)g(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\lambda_i)g(\lambda_i). \]
Thus, with $n_i = p_i(\lambda_0) = \|p_i\|^2$, we have the well-known relations
\begin{equation}
n_k p_{ij}^k = \langle p_i p_j, p_k \rangle = \langle p_j, p_i p_k \rangle = n_j p_{ik}^d, \quad i, j, k = 0, \ldots, d.
\end{equation}
In particular, when $k = 1$ and $i = j$, we have $n_1 = \lambda_0 = \delta$ (the degree of $\Gamma$) and $p_{ii}^1 = a_i$. Thus, $\delta p_{ii}^1 = n_i a_i$ and, hence, $p_{ii}^1 = 0$ if and only if $a_i = 0$. In fact notice that, for a general graph, the condition $V_i(u,v) = |\Gamma_i(u) \cap \Gamma_i(v)| = 0$ for any two adjacent vertices $u, v$ is equivalent to say that $a_i$ is well defined and null.

In an edge-distance-regular graph $\Gamma = (V,E)$ with diameter $d$, every pair of adjacent vertices $u,v \in V$ is a completely regular code. More precisely, the distance partition $\tilde{V}_0, \ldots, \tilde{V}_d$ of $V$ induced by an edge $uv \in E$, where $\tilde{V}_i = \tilde{V}_i(uv) = \Gamma_i(uv)$ is the set of vertices at distance $i$ from $\{u,v\}$ and $\tilde{d} \in \{d-1, d\}$, is regular and with the same edge-intersection numbers for any edge. That is, the numbers
\[ \tilde{a}_i(uv) = |\Gamma(w) \cap \tilde{V}_i|, \quad \tilde{b}_i(uv) = |\Gamma(w) \cap \tilde{V}_{i+1}|, \quad \tilde{c}_i(uv) = |\Gamma(w) \cap \tilde{V}_{i-1}|, \]
do not depend neither on the edge $uv$ nor on the vertex $w \in \tilde{V}_i$, but only on the distance $i$, in which case we write them as $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ for $i = 0, \ldots, \tilde{d}$ and say that they are well defined (see Cámara, Dalfó, Fàbrega, Fiol and Garriga [3] for more details).

# The characterization

In the next result we show that every edge-distance-regular graph is either bipartite distance-regular or a generalized odd graph.

**Theorem 2.1.** Let $\Gamma$ be a graph with diameter $d$. Then, the following statements are equivalent:

(a) $\Gamma$ is edge-distance-regular;

(b) $\Gamma$ is distance-regular, either bipartite or a generalized odd graph.
Moreover, if this is the case and $\Gamma$ has intersection array $\iota(\Gamma) = \{b_0, b_1, \ldots, b_d; c_1, c_2, \ldots, c_d\}$ and it is bipartite, then its edge-intersection array is

$$i(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; \tilde{c}_1, \ldots, \tilde{c}_{d-1}\} = \{b_1, b_2, \ldots, b_{d-1}; c_1, \ldots, c_{d-2}, c_{d-1}\}, \quad (3)$$

whereas, if $\Gamma$ is a generalized odd graph, its edge-intersection array is

$$i(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; \tilde{c}_1, \ldots, \tilde{c}_{d-1}, \tilde{c}_d\} = \{b_1, b_2, \ldots, a_d; c_1, \ldots, c_{d-1}, 2c_d\}. \quad (4)$$

**Proof.** As the complete graphs clearly satisfy the result, we can assume that $d > 1$. Given two adjacent vertices $u$ and $v$ of $\Gamma$, let us consider the intersection numbers $p_j^k(u, v) = |\Gamma_i(u) \cap \Gamma_j(v)|$, so that the vertex partition induced by the distances from $u$ and $v$ is shown in Fig. 1, where $V_{i,j} = V_{i,j}(u, v) = \Gamma_i(u) \cap \Gamma_j(v)$. (Notice that $V_{i,j} = \emptyset$ when $|i - j| > 1$, as dist$(u, v) = 1$.) Let $V_0, \ldots, \tilde{V}_d$ be the distance partition induced by the edge $uv$, and define $\tilde{a}_i(uv), \tilde{b}_i(uv)$ and $\tilde{c}_i(uv)$ as above. Clearly, $\tilde{a}_0(uv) = \tilde{a}_0 = c_1 = 1$.

![Figure 1: The distance partition induced by two adjacent vertices. (Every set $V_{i,j}$ is represented by its subindices.)](image)

We have the following facts:

(i) For $i = 1, \ldots, d - 1$, all vertices adjacent to $w \in V_{i,j-1} \subset \tilde{V}_{i-1}$ and at distance $i + 1$ from $u$ are in $V_{i+1,i} \subset \tilde{V}_i$. Hence,

$$|\Gamma(w) \cap \tilde{V}_i| = |\Gamma(w) \cap \Gamma_{i+1}(u)| + |\Gamma(w) \cap V_{i,i}| = b_i(w, u) + |\Gamma(w) \cap V_{i,i}|, \quad i = 1, \ldots, d - 1. \quad (5)$$

(ii) For $i = 1, \ldots, d - 1$, all vertices adjacent to $w \in V_{i,i+1} \subset \tilde{V}_i$ and at distance $i - 1$ from $u$ are in $V_{i-1,i} \subset \tilde{V}_{i-1}$. Thus,

$$|\Gamma(w) \cap \tilde{V}_{i-1}| = |\Gamma(w) \cap \Gamma_{i-1}(u)| = c_i(w, u), \quad i = 1, \ldots, d - 1. \quad (6)$$

Moreover, assuming that $\alpha_i$ is well defined with value $\alpha_i = 0$, the vertices adjacent to vertex $w$ but at distance $i$ from $v$ are in $V_{i-1,i} \cup V_{i+1,i}$ since $V_{i,i} = \emptyset$, whereas
\[ |\Gamma(w) \cap V_{i,i+1}| = 0 \text{ since } w \in V_{i,i+1} \subset \Gamma_i(u). \] Consequently,
\[
|\Gamma(w) \cap \tilde{V}_{i-1}| + |\Gamma(w) \cap \tilde{V}_{i}| = |\Gamma(w) \cap (V_{i-1,i} \cup V_{i+1,i})| = |\Gamma(w) \cap \Gamma_i(v)| = c_{i+1}(w,v), \quad i = 1,\ldots,d-1. \tag{7}
\]

(iii) For \( i = 1,\ldots,d \), all vertices adjacent to \( w \in V_{i,i} \subset \tilde{V}_i \) and at distance \( i-1 \) from \( u \) are either in \( V_{i-1,i} \) or \( V_{i-1,i-1} \). Thus,
\[
|\Gamma(w) \cap V_{i-1,i}| + |\Gamma(w) \cap V_{i-1,i-1}| = |\Gamma(w) \cap \Gamma_{i-1}(u)| = c_i(w,u),
\]
and, analogously,
\[
|\Gamma(w) \cap V_{i,i-1}| + |\Gamma(w) \cap V_{i-1,i-1}| = |\Gamma(w) \cap \Gamma_{i-1}(u)| = c_i(w,v).
\]

Therefore,
\[
|\Gamma(w) \cap \tilde{V}_{i-1}| = c_i(w,u) + c_i(w,v) - |\Gamma(w) \cap V_{i-1,i-1}|, \quad i = 1,\ldots,d. \tag{8}
\]

(b) \( \Rightarrow \) (a):
Assume that \( \Gamma \) is distance-regular with intersection parameters \( a_i, b_i, c_i \), for \( i = 0,\ldots,d \) (the parameters \( b_i \) and \( c_i \) are indicated in Fig. \( \text{[F]} \)). If \( \Gamma \) is either bipartite or a generalized odd graph, then \( a_i = 0 \) and \( V_{i,i} = \emptyset \) for every \( i = 1,\ldots,d-1 \) (in particular, as \( a_{d-1} = 0 \), the parameters \( a_d \) and \( c_d \) are those indicated in the same figure for the nonbipartite case; otherwise, we also have \( V_{d,d} = \emptyset \) and, hence, \( a_d = c_d = 0 \)). Thus, from the above reasonings we obtain:

(i) By Eq. \( \text{[G]} \) and for every vertex \( v \in \tilde{V}_{i-1} \), we have
\[
\tilde{b}_{i-1}(uv) = |\Gamma(w) \cap \tilde{V}_i| = b_i.
\]
Hence, \( \tilde{b}_i \) is well defined for \( i = 0,\ldots,d-2 \). Moreover, if \( \Gamma \) is a generalized odd graph, \( b_{d-1} = a_d \).

(ii) By Eq. \( \text{[H]} \) and for every vertex \( v \in \tilde{V}_i \),
\[
\tilde{c}_i(uv) = |\Gamma(w) \cap \tilde{V}_{i-1}| = c_i.
\]
Consequently, \( \tilde{c}_i \) is well defined for \( i = 1,\ldots,d-1 \).

(iii) Moreover, if \( \Gamma \) is a generalized odd graph, \( \tilde{V}_{d,d} \neq \emptyset \), and Eq. \( \text{[I]} \) yields that, for every vertex \( v \in \tilde{V}_{d,d} \),
\[
\tilde{c}_d(uv) = |\Gamma(w) \cap \tilde{V}_{d-1}| = 2c_d.
\]

Summarizing, all intersection numbers \( \tilde{b}_i, i = 0,\ldots,d-1 \), and \( \tilde{c}_i, i = 1,\ldots,d \) are well defined, and \( \Gamma \) is edge-distance-regular. (Of course, the other intersection numbers are just \( \tilde{a}_i = \delta - \tilde{b}_i - \tilde{c}_i \), for \( i = 0,\ldots,d \).)
(a) $\Rightarrow$ (b):
Assume that $\Gamma$ is edge-distance-regular with edge-intersection numbers $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$, for $i = 0, \ldots, d$. Then, to show that the numbers $c_i(w, u), a_i(w, u)$ and $b_i(w, u)$ depend only on the distance $i = \text{dist}(w, u)$, we proceed by induction on $i$. To begin with, observe that $b_0$, $a_0$, and $c_1$ are well defined, with values $b_0 = b_0(u, u) = \tilde{b}_0 + \tilde{a}_0 = \tilde{b}_0 + 1$, $a_0 = a_0(u, u) = 0$, and $c_1 = c_1(w, u) = 1$.

Now, assume that, for some $i \leq d - 1$, $c_i$ and $a_{i-1}$ are well defined with $a_{i-1} = 0$ or, equivalently, $V_{i-1,i-1} = \emptyset$. Thus, in order to show that $c_{i+1}$ and $a_{i}$ exist, consider a shortest path $v, u, \ldots, w$ of length $i+1(\leq d)$, so that $w \in V_{i,i}(u, v)$. Then, Eq. (6) gives $c_i = |\Gamma(w) \cap \tilde{V}_{i-1}| = c_i$.

Now, suppose that $u'v'$ is an arbitrary edge. If we assume that $V_{i,i}(u', v') \neq \emptyset$, there exists a vertex $w' \in V_{i,i}(u', v')$ and Eq. (8) gives $c_{i+1}(w, v) = |\Gamma(w) \cap \tilde{V}_{i-1}| = 2c_i$, which would be a contradiction. Thus $V_{i,i}(u', v') = \emptyset$ and $a_i$ exists, being $a_i = 0$. Moreover, coming back to the edge $uv$, Eqs. (6) and (7) yield

$$c_{i+1}(w, v) = |\Gamma(w) \cap \tilde{V}_{i-1}| = |\Gamma(w) \cap \tilde{V}_{i-1}| = c_i + \tilde{a}_i,$$

and, since $w$ and $v$ are arbitrary vertices at distance $i+1$, $c_{i+1}$ is well defined.

Thus, the numbers $b_i = b_0 - c_i - a_i$, for $i = 1, \ldots, d - 1$, are also well defined; and the same holds for $a_d = b_0 - c_d$ when $\Gamma$ is nonbipartite. Consequently, $\Gamma$ is as claimed.

2.1 Homogeneous graphs

As a consequence of the previous results, we next prove that the edge-distance-regular graphs are, in fact, a particular case of the so-called (1-)homogeneous graphs introduced by Nomura in [20]. A graph $\Gamma$ is called homogeneous if for all nonnegative integers $r, s, i, j$ and pairs of edges $uv, u'v'$,

$$x \in V_{s,r}(u, v), x' \in V_{s,r}(u', v') \implies |\Gamma(x) \cap V_{i,j}(u, v)| = |\Gamma(x') \cap V_{i,j}(u', v')|,$$

where $V_{s,r}(u, v)$ is defined as before. In other words, $\Gamma$ is homogeneous if the partition of Fig. 1 is a regular (or equitable) partition with the same intersection numbers for every pair of adjacent vertices $u, v$. For a detailed study of regular partitions, see Godsil [15] and Godsil and McKay [16].

In [6 Lemma 5.3], Curtin and Nomura showed that every bipartite or almost bipartite distance-regular graph is homogeneous. Then, from Theorem 2.1, we have:

**Corollary 2.2.** Every edge-distance-regular graph is homogeneous.  \(\square\)
However, the converse is not true. A counterexample is, for instance, the Wells graph \( W \), which is the unique distance-regular graph on \( n = 32 \) vertices and intersection array \( \iota(W) = \{5, 4, 1; 1, 1, 4, 5\} \) (see Brouwer, Cohen and Neumaier [2, Theorem 9.2.9]). Thus, since \( a_2 = 3 \) and \( a_i = 0 \) for every \( i \neq 2 \), \( W \) is homogeneous (see Nomura [20, Lemma 2]) but not edge-distance-regular. The intersection diagrams induced by a vertex and two adjacent vertices, showing that \( W \) is homogeneous, are illustrated in Figure 2.

\[
\begin{array}{c}
1 & 5 & 1 & 5 & 4 & 1 & 20 & 1 & 4 & 1
\end{array}
\]

\[
\begin{array}{c}
1 & 4 & 1 & 1 & 1 & 4 & 1
\end{array}
\]

Figure 2: The intersection diagrams of the Wells graph as a distance-regular and homogeneous graph.

3 An algebraic approach

The matrix approach to the result of Theorem 2.1 is based on the relationship of orthogonal polynomials with (edge-)distance-regularity of graphs. Recall that every system of orthogonal polynomials of a discrete variable \( \{r_i\}_{0 \leq i \leq d} \) satisfies a three term recurrence of the form

\[
x r_i = \beta_{i-1} r_{i-1} + \alpha_i r_i + \gamma_{i+1} r_{i+1}, \quad 0 \leq i \leq d,
\]

where the terms \( b_{-1} \) and \( c_{d+1} \) are zero.

In [14] Fiol, Garriga and Yebra gave the following characterization of distance-regularity.

**Theorem 3.1.** A regular graph with \( d + 1 \) distinct eigenvalues is distance-regular if its distance matrix \( A_d \) is a polynomial of degree \( d \) in \( A \), that is,

\[
p_d(A) = A_d.
\]

Moreover, such a polynomial is the highest degree member of the predistance polynomials \( \{p_i\}_{0 \leq i \leq d} \) of \( \Gamma \)—which are orthogonal with respect to the scalar product \( \langle p, q \rangle = \)
\[ \frac{1}{n} \text{tr}(p(A)q(A)), \] and are normalized in such a way that \( \|p_i\|^2 = p_i(\lambda_0) \). When the graph is distance-regular, these polynomials satisfy a three term recurrence as in Eq. (9) with \( \{\gamma_i, \alpha_i, \beta_i\} = \{c_i, a_i, b_i\} \).

The first ingredient for the algebraic approach to our results is the following proposition from Cámara, Dalfó, Fàbrega, Fiol, and Garriga [3], which describes the possibilities for the edge-diameter and the local spectra of edges in an edge-distance-regular graph. In a regular graph, the \( e \)-local spectrum of an edge \( e = uv \) is constituted by those eigenvalues \( \lambda_i \in ev_e \Gamma \) such that the orthogonal projection of the characteristic vector of \( \{u, v\} \) on their corresponding \( \lambda_i \)-eigenspace is not null; see [3] for more details. The set of these \( e \)-local eigenvalues is denoted by \( ev_e \Gamma \).

**Proposition 3.2.** Let \( \Gamma \) be an edge-distance-regular graph with diameter \( D \) and \( d + 1 \) distinct eigenvalues. Then, the following statements hold:

(a) \( \Gamma \) is regular.

(b) \( \Gamma \) has spectrally maximum diameter \( (D = d) \) and its edge-diameter satisfies:

(b1) If \( \Gamma \) is nonbipartite, then \( \tilde{D} = d \);

(b2) If \( \Gamma \) is bipartite, then \( \tilde{D} = d - 1 \).

(c) \( \Gamma \) is edge-spectrum regular and, for every \( e \in E \), the \( e \)-spectrum satisfies:

(c1) If \( \Gamma \) is nonbipartite, then \( ev_e \Gamma = ev \Gamma \);

(c2) If \( \Gamma \) is bipartite, then \( ev_e \Gamma = ev \Gamma \ \{\lambda_0\} \).

Let \( ev_E \Gamma = \bigcup_{e \in E} ev_e \Gamma \), \( ev_E^* \Gamma = ev_E \Gamma \ \{\lambda_0\} \) and \( \tilde{d} = |ev_E^* \Gamma| \). Then, by using this notation, Proposition 3.2 establishes that, if \( \Gamma \) is edge-distance-regular and nonbipartite, then \( ev_E \Gamma = ev_e \Gamma = ev \Gamma \) for every edge \( e \in E \), and \( ev_E \Gamma = ev_e \Gamma = ev \Gamma \ \{\lambda_0\} \) otherwise. In both cases all edges have the same local spectrum (\( \Gamma \) is said to be edge-spectrum regular), with \( d + 1 = |ev_E \Gamma| \) distinct \( e \)-local eigenvalues.

The role of the distance matrices in the study of edge distance-regularity is played by the distance incidence matrices. More precisely, for a given graph \( \Gamma = (V, E) \), let \( \tilde{D} = \max_{uv \in E} \{\text{ecc}(\{u, v\})\} \) be its edge-diameter. Then, for every \( i = 0, 1, \ldots, \tilde{D} \), the \( i \)-incidence matrix of \( \Gamma \) is the \( (|V| \times |E|) \)-matrix \( B_i = (b_{ue}) \) with entries \( b_{ue} = 1 \) if \( \text{dist}(u, e) = i \), and \( b_{ue} = 0 \) otherwise. The following result corresponds to Theorems 4.4 and 4.9 of [3].

**Theorem 3.3.** A regular graph \( \Gamma = (V, E) \) with edge-diameter \( \tilde{D} \) and \( d + 1 = |ev_E \Gamma| \) is edge-distance-regular if and only if any of the following conditions hold:

(a) There exists a polynomial \( \tilde{p}_i \) of degree \( i \) such that

\[ \tilde{p}_i(A)B_0 = B_i, \quad i = 0, 1, \ldots, \tilde{D}. \]

(b) \( |\Gamma_{\tilde{D}}(e)| = 2\tilde{p}_d(\lambda_0) \) for every edge \( e \in E \).
Moreover, if this is the case, these polynomials are the edge-predistance-polynomials, \( \{ \tilde{p}_i \}_{0 \leq i \leq \tilde{d}} \) with \( \tilde{d} = \tilde{D} \), satisfying a three term recurrence as in Eq. 9 with \( \{ \gamma_i, \alpha_i, \beta_i \} = \{ c_i, \bar{a}_i, \bar{b}_i \} \). Here we use the following consequence, which can be seen as the analogue of Theorem 3.1 for edge-distance-regularity.

**Corollary 3.4.** A regular graph \( \Gamma = (V,E) \) with \( \tilde{d} + 1 = |ev_E \Gamma| \) is edge-distance-regular if and only if

\[
\tilde{p}_{\tilde{d}}(A)B_0 = B_{\tilde{d}}. \tag{12}
\]

**Proof.** We only need to prove sufficiency, which is straightforward if we multiply both sides of Eq. (12) by the all-1 vector \( j \) and apply Theorem 3.3(b). \( \square \)

### 3.1 A matrix approach to Theorem 2.1

Now, we can rewrite Theorem 2.1 and prove it by showing that the distance polynomials exist if and only if the edge-distance-polynomials do.

**Theorem 3.5.** Let \( \Gamma \) be a graph with diameter \( d \) and edge-diameter \( \tilde{d} \in \{ d - 1, d \} \). Then, the following statements are equivalent:

(a) \( \Gamma \) is distance-regular with \( a_0 = \cdots = a_{d-1} = 0 \);

(b) \( \Gamma \) is edge-distance-regular.

Moreover, if this is the case, the relationships between the corresponding edge-distance-polynomials and distance polynomials are:

\[
\tilde{p}_i = p_i - p_{i-1} + p_{i-2} - \cdots + (-1)^i p_0, \tag{13}
\]
\[
p_i = \tilde{p}_i + \tilde{p}_{i-1}, \tag{14}
\]

for \( i = 0, \ldots, d - 1 \), and, if \( \Gamma \) is nonbipartite,

\[
\tilde{p}_d = \frac{1}{2}(p_d - p_{d-1} + p_{d-2} - \cdots + (-1)^d p_0); \tag{15}
\]
\[
p_d = 2\tilde{p}_d + \tilde{p}_{d-1}, \tag{16}
\]

whereas, if \( \Gamma \) is bipartite,

\[
p_d = H - \tilde{q}_{d-1} - \tilde{q}_{d-2}, \tag{17}
\]

where \( H = p_0 + \cdots + p_d \) is the Hoffman polynomial satisfying \( H(\lambda_i) = \delta_{ij} n \), and \( \tilde{q}_i = \tilde{p}_0 + \cdots + \tilde{p}_i \), for \( i = d - 2, d - 1 \).

**Proof.** (a) \( \Rightarrow \) (b):

Since \( a_i = 0 \) we have that

\[
A_i B_0 = B_i + B_{i-1}, \quad i = 0, \ldots, d - 1. \tag{18}
\]
Moreover, when $\Gamma$ is nonbipartite, $a_d \neq 0$ and

$$A_d B_0 = 2B_d + B_{d-1}.$$  

Then, multiplying both sides of Eq. (10) by $B_0$ (on the right) and using all the above equations, we get

$$p_d(A)B_0 = 2B_d + B_{d-1}$$

$$= 2B_d + A_{d-1}B_0 - B_{d-2}$$

$$= 2B_d + A_{d-1}B_0 - A_{d-2}B_0 + B_{d-3}$$

$$\vdots$$

$$= 2B_d + (A_{d-1} - A_{d-2} + A_{d-3} - \cdots - (-1)^{d+1}A_0)B_0.$$  

Thus, using Eq. (1), we get that $\tilde{p}_d(A)B_0 = B_d$ with

$$\tilde{p}_d = \frac{1}{2}(p_d - p_{d-1} + p_{d-2} - \cdots + (-1)^d p_0),$$

as claimed in Eq. (15), and $\Gamma$ is edge-distance-regular by Corollary 3.4.

Similarly, by using again Eq. (18), we get

$$p_i(A)B_0 = B_i + B_{i-1}$$

$$= B_i + A_{i-1}B_0 - B_{i-2}$$

$$\vdots$$

$$= B_i + (A_{i-1} - A_{i-2} + \cdots + (-1)^i A_0)B_0,$$

so that

$$\tilde{p}_i = p_i - p_{i-1} + \cdots + (-1)^i p_0, \quad i = 0, \ldots, d - 1,$$

as claimed in Eq. (13). Moreover, if $\Gamma$ is bipartite, then $a_d = 0$, $d = d - 1$, and the existence of the edge-distance-polynomial $\tilde{p}_{d-1}$ imply that $\Gamma$ is edge-distance-regular, again by Corollary 3.4.

Eqs. (14) and (16) are immediate consequences from Eqs. (13) and (15). Finally, when $\Gamma$ is bipartite, we can obtain $p_d$ by adding all the equalities in Eq. (14) to get $q_{d-1} = p_0 + \cdots + p_{d-1} = \tilde{q}_{d-1} + \tilde{q}_{d-2}$, so that $p_d = H - q_{d-1}$, where $H$ is the Hoffman polynomial (see Hoffman [17]), and we have Eq. (17).

(b) $\Rightarrow$ (a):

Note first that $B_0B_0^\top = A + \delta I$, where $\delta$ is the degree of $\Gamma$. Now, we show that all distance-polynomials exist, in particular $p_d(A) = A_d$, and apply Theorem 3.1.

Suppose that $\Gamma$ is nonbipartite. By Proposition 3.2 we know that $ev_\epsilon \Gamma = ev \Gamma$ and, consequently, $d = d$. As in Theorem 2.1, the proof is by induction on $i$. First, $a_1 = 0$ (as $d > 1$), $c_1 = 1$, and the first two distance polynomials exist $p_0 = 1$ and $p_1 = x$. Now, assume that, for some $i < d - 1$, $a_i = 0$, $c_i$ is well defined, and there exist the polynomials $p_{i-1}$ and $p_i$. 

To compute the product $B_iB_0^\top$, let us consider two vertices $u, w$ at distance $\text{dist}(u, w) = i < d - 1$ and take $v \in \Gamma(u) \cap \Gamma_{i-1}(w)$. Then, $(B_iB_0^\top)_{uw} = \sum_{e \in E(B_i)_{uw}} (B_0^\top)_{ew}$. Each term of the sum equals 1 for every vertex $u' \in \Gamma(v)$ such that the edge $e = uw'$ is at distance $i$ from $u$ (since $(B_i)_{uu'} = (B_0^\top)_{ww'} = 1$); that is, for every vertex

$$u' \in \Gamma(v) \cap V_{i-1}(u, v) \cup V_{i,i}(u, v) \cup V_{i+1,i}(u, v) = \Gamma(w) \cap V_{i+1,i}(u, v),$$

where we used that, since $a_i = 0$, $\Gamma(w) \cap V_{i,i}(u, v) = \emptyset$ (notice that if $w' \in V_{i,i-1}(u, v)$, then we would have $a_i(w, u) \neq 0$) and $V_{i,i}(u, v) = \emptyset$. Hence, $(B_iB_0^\top)_{uw} = b_{i-1}$. Reasoning similarly, for a vertex $w$ such that $\text{dist}(u, w) = i + 1$, and $v \in \Gamma(u) \cap \Gamma_i(w)$ (that is, $w \in V_{i+1,i}(u, v) \subset V_i$), we have $(B_iB_0^\top)_{uw} = \tilde{a}_i + c_i$. Otherwise, if $\text{dist}(u, w) \neq i, i + 1$, then $(B_iB_0^\top)_{uw} = 0$. Consequently,

$$B_iB_0^\top = \tilde{b}_{i-1}A_i + (\tilde{a}_i + c_i)A_{i+1},$$

and, multiplying Eq. (11) by $B_0^\top$ on the right, we get

$$\tilde{p}_i(A)(A + \delta I) = \tilde{b}_{i-1}p_i(A) + (\tilde{a}_i + c_i)A_{i+1}.$$ 

Thus, the distance polynomial of degree $i + 1$ is

$$p_{i+1} = \frac{1}{\tilde{a}_i + c_i}((x + \delta)\tilde{p}_i - \tilde{b}_{i-1}p_i), \quad (19)$$

graph $\Gamma$ is $(i + 1)$-partially distance-regular (that is, $p_j(A) = A_j$ for $j = 0, \ldots, i + 1$), and $c_{i+1}$ is well defined; see Dalfo, Van Dam, Fiol, Garriga, and Gorissen [5]. (Notice that this could also be derived from Eqs. (6) and (7) yielding $c_{i+1}(w, u) = |\Gamma(w) \cap \Gamma_{i-1}(v)| + |\Gamma(w) \cap V_i| = c_i + \tilde{a}_i$, as in the proof of Theorem 2.1) Now, let $u, v$ be two arbitrary adjacent vertices. If $w \in V_{i+1,i+1} \neq \emptyset$, Eq. (8) yields $\tilde{c}_{i+1} = |\Gamma(w) \cap V_i| = 2c_{i+1} > c_{i+1}$. It follows that if a vertex is at distance at most $i + 1$ from an end of an edge, it is at distance at most $i$ from the other end. Thus, the diameter of $\Gamma$ is $i + 1 < d$, which is a contradiction. Hence, $V_{i+1,i+1}(u, v) = \emptyset$ and, since $u, v$ are generic vertices, $a_{i+1} = 0$ is well defined.

The induction above proves that there exist all the distance-polynomials $p_0, p_1, \ldots, p_{d-1}$ and, also, we have $p_d = H - p_0 - p_1 - \cdots - p_{d-1}$. Then, $\Gamma$ is distance-regular with $a_0 = \cdots = a_{d-1} = 0$.

Suppose now that $\Gamma$ is bipartite. Then $\tilde{d} = d - 1$ and $ev_{\Gamma} = ev_{\Gamma} \setminus \{-\lambda_0\}$. Now, reasoning similarly as above from a shortest path $u, v, \ldots, w$ of length $d$, we have

$$B_{d-1}B_0^\top = \tilde{b}_{d-2}A_{d-1} + (\tilde{a}_{d-1} + \tilde{c}_{d-1})A_d.$$ 

Thus,

$$A_d\tilde{p}_{d-1}(A) + \delta \tilde{p}_{d-1}(A) = \tilde{b}_{d-2}A_{d-1} + \delta A_d.$$ 

Now, the key point is that the distance matrix $A_d$ of a bipartite graph can be thought as a $2 \times 2$ block matrix such that when $i$ is odd (respectively, even) the diagonal (respectively, off-diagonal) entries are the zero matrices. Indeed, the same happens with the powers of the adjacency matrix, $A^\ell$. Assume that $d$ is odd (respectively, even), the odd part
Thus, with other for the nonbipartite one. With respect to the former, consider the cube $Q_3$.

To illustrate the above result, let us give two examples: one for the bipartite case and the other for the nonbipartite one. With respect to the former, consider the cube $Q_3$, which satisfies the conditions of the theorem and has the following parameters:

- $\iota(Q_3) = \{3, 2, 1; 1, 2, 3\}$, $p_0 = 1$, $p_1 = x$, $p_2 = \frac{1}{2}(x^2 - 3)$, $p_3 = \frac{1}{2}(x^3 - 7x)$;
- $\iota(Q_3) = \{2, 1; 1, 2\}$, $\tilde{p}_0 = 1$, $\tilde{p}_1 = x - 1$, $\tilde{p}_2 = \frac{1}{2}(x^2 - 2x - 1)$.

Then, as $d = \delta = 3$ and $\tilde{b}_1 = 1$, we have

$$x\tilde{p}_2 + \delta \tilde{p}_1 = \frac{1}{2}(x^3 - 7x) + \frac{1}{2}(x^2 - 3) = \frac{1}{2}p_3 + p_2,$$

as it should be.

Now, for the nonbipartite case we consider the odd graph $O_4$, which also satisfies the conditions of the theorem, and has parameters:

- $\iota(O_4) = \{4, 3, 3; 1, 1, 2\}$, $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 4$, $p_3 = \frac{1}{2}(x^3 - 7x)$;
- $\iota(O_4) = \{3, 3, 2; 1, 1, 4\}$, $\tilde{p}_0 = 1$, $\tilde{p}_1 = x - 1$, $\tilde{p}_2 = x^3 - x - 3$,

$$\tilde{p}_3 = \frac{1}{4}(x^3 - 2x^2 - 5x + 6).$$

Thus, with $i = 2$ and since $d = \delta = 4$, Eq. (19) gives

$$p_3 = \frac{1}{\sigma_2^2 + \sigma_3^2}((x + \delta)\tilde{p}_2 - \tilde{b}_1 p_2) = \frac{1}{2}((x + 4)(x^2 - 3) - 3(x^2 - 4)) = \frac{1}{4}(x^3 - 7x),$$

as claimed.

Using the algebraic approach, we can also derive some properties of the edge-intersection numbers, apart from the trivial one $\tilde{a}_i + \tilde{b}_i + \tilde{c}_i = \delta$. As an example, we have the following:

**Lemma 3.6.** Let $\Gamma$ be a nonbipartite edge-distance-regular graph with diameter $d$. Then,

$$\tilde{a}_i = \tilde{b}_{i-1} - \tilde{b}_1, \quad i = 1, \ldots, d - 1.$$  \hfill (20)

**Proof.** In the proof of Theorem 3.5, we have already shown that, if $w \in V_{i,i-1}$ then, $(B_iB_0^i)_{ww} = \tilde{b}_{i-1}$. Similarly, one can prove that, if $w \in V_{i,i+1}$, then $(B_iB_0^i)_{ww} = \tilde{a}_i + \tilde{b}_i$. Since, in both cases, $\text{dist}(u,w) = i$, the two values must be equal and the result follows. \qed

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