Improved finite volume method for solving 1-D advection equation

Siyuan Zhao¹, Junjie Zhou¹, Chongbo Jing¹ and Lingquan Li²,*

¹School of Mechanical Engineering, Beijing Institute of Technology, Beijing, China
²School of Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, USA

*Corresponding author e-mail: lli32@ncsu.edu

Abstract. In the framework of the second-order finite volume method, a new improved finite volume method (FVM) for solving one-dimensional advection equations is proposed based on its conservation form. The new method first applies the scalar conservation law to the cells in the FVM, ensuring that it is conserved in time and space, and that the flat flow (i.e., the transport physical quantity) is conserved. Secondly, the time integral values of adjacent grids boundary are equalized. Finally, by establishing an equation, numerical solution values are obtained. A strong discontinuity function was used in the paper to test the new method described in this paper and compare it to the central difference method (CDM) and traditional FVM. Without the limiter, the results show that the new method described in this paper has less dissipation and better stability than CDM and traditional FVM. In addition, after adjusting the convergence condition criterion number CFL to 2, the accuracy of the numerical solution can still be guaranteed.

Keywords: finite volume method, conservation form, advection equation, discontinuity, computational fluid dynamics.

1. Introduction
The advection process is one of the important processes in atmospheric motion. The equations of motion, heat flux equations and water vapor equations in the equations of atmospheric motion all contain advection terms. Therefore, the study of advection equations is of great significance in the study of atmospheric motion [1]. In numerical weather prediction, the advection equation is mostly solved by differential numerical method [2]. However, this method does not satisfy the stability condition. With the forward integration of time, the amplitude becomes larger and larger, which eventually leads to the calculation failure [2]. Therefore, many people expect to use FVM with conservation characteristics to solve the advection equation.

FVM is widely used to solve fluid flow and heat transfer problems [3]. Compared with other numerical calculation methods, the basic idea of FVM is easy to understand and can give a direct physical explanation. Moreover, the discrete equations derived from FVM require that the integral conservation of the dependent variable be satisfied for any set of control volumes, and the entire calculation area is naturally satisfied. By using the conservation of the control volume, the instability of the numerical simulation function over time can be eliminated to some extent. However, so far, despite
many improvements to FVM, FVM still does not have high simulation accuracy in the face of strong discontinuities in functions. The application object of FVM described in the article, the distribution of water vapor in the atmosphere, is also discontinuous. Besides, the gradient of the distribution is usually large, which directly affects the accuracy of precipitation prediction. In fact, most of the current numerical calculation methods for discontinuous distribution of water vapor are not very accurate [4].

In order to better deal with numerical calculations of discontinuities, the researchers have proposed some new numerical methods [5]. Thuburn [6] developed a conserved atmospheric model based on the FVM, however this method has lower computational accuracy. Harten [7] constructed a second-order Total Variation Diminishing (TVD) format by flux correction method. Song [8] et al. combined TVD and FVM to process two-dimensional unstructured meshes, and obtained better calculation accuracy. Cockburn et al. [9] obtained a high-precision Discontinuous Galerkin (DG) method by increasing the local degree of freedom, which has great advantages in terms of computational accuracy and convergence. However, the DG method requires more computational complexity than the conventional FVM, which affects the computational economy. The multi-moment finite volume method proposed by Feng Xiao [10] not only has strict conservation, but also can construct high-precision interpolation function based on a single numerical grid. Li [4] et al. applied it to semi-Lagrangian advection transmission. The results show that it has great advantages for the numerical calculation of discontinuities. On the basis of this paper, by making the time integrals of adjacent grid boundaries equal, the fluxes in the adjacent grid interface are conserved (equal), thus improving the accuracy and instability of numerical simulation over time when meeting strong discontinuities.

2. Numerical calculation process

2.1. 1-D advection equation

First consider the 1-D advection equation,

\[ \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \]

Where \( t \) is time, \( x \) is the space coordinate, \( a \) is the characteristic velocity, and \( u \) is the flat flow (ie, the physical quantity of the transmission), which may be water vapor.

2.2. Limitations of traditional FVM

For the sake of explanation, first divide the cell:

Suppose that the space is divided into a number of cells \([x_{i-1/2}, x_{i+1/2}]\), where \( x_{i-1/2} \) is called the interface of the cell. At the same time, it is assumed that time is also divided into a number of intervals \([t^n, t^{n+1}]\), where \( t = t^n \) is called the time layer. Here, it is considered that the amount in the \( t = t^n \) time layer is a known amount, and it is considered that the amount in the \( t=t^{n+1} \) time layer, that is, the amount in the next time layer is an unknown amount. As shown in Figure 1.

![Figure 1. Time and space division of one cell](image)

For the cell \( i \), if only integrate the time of equation (1), we can get equation (2):

\[ \Delta t \times a \times \left( \frac{\partial u}{\partial x} \right)_i + (u_{i+1}^{n+1} - u_i^n) = 0 \]
Wherein, \( i \) represents the \( i \)th cell and \( n \) represents the \( t = t^n \) time layer, that is, the \( n \)th moment, and \( n+1 \) represents the next moment.

The key to solving the numerical simulation problem of the 1-D advection equation is to construct and solve the unknown flat flow (ie, the forecast term), thus realizing the numerical forecast. However, although the above calculations can be obtained with the forecast term \( u^{n+1} \), for 2-D (time and space) cells, only the time integral does not necessarily guarantee that the flat flow \( u \) is conserved. Therefore, the scalar conservation law can be applied to equation (1) and FVM is used to ensure that the flat flow \( u \) is conserved.

2.3. FVM using scalar conservation law

The content of the scalar conservation law [4] is: according to the cell division in 2.2, the conserved quantity is within the time range \([t^n, t^{n+1}]\), and the change in the space area \([x_{i-1/2}, x_{i+1/2}]\) is equal to the net flux passing through the boundary \([x_{i-1/2}, x_{i+1/2}]\) in the \([t^n, t^{n+1}]\) time, which can be expressed in the mathematical language as follows,

\[
\int_{t^n}^{t^{n+1}} \nabla \cdot \bar{F} \, dt = 0 \quad \text{(3)}
\]

or

\[
\int_{x_{i-1/2}}^{x_{i+1/2}} \left[ u(x, t^{n+1}) - u(x, t^n) \right] \, dx = -\int_{t^n}^{t^{n+1}} \left[ f(x_{i+1/2}, t) - f(x_{i-1/2}, t) \right] \, dt \quad \text{(4)}
\]

Where, \( u \) represents the conserved quantity and \( f \) represents the scalar flux.

However, the boundary value of function is not given in the section 2.2. So, an interpolation function needs to be constructed to represent the boundary value. The specific construction method is as follows.

Define the midpoint of the \( i \)th cell on the \( n \)th time layer as the function value \( u_i^n \).

By using first-order Interpolation, the boundary node of the cell on the \( n \)th time layer can be expressed as

\( u_i^n - \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^n \) and \( u_i^n + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^n \). Similarly, the boundary nodes of the cell at the time layer \( n+1 \) can be represented as

\( u_i^{n+1} - \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^{n+1} \) and \( u_i^{n+1} + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^{n+1} \).

As shown in Figure 2.

\[ u_i^n - \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^n \]
\[ u_i^{n+1} - \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^{n+1} \]
\[ u_i^n + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^n \]
\[ u_i^{n+1} + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^{n+1} \]

**Figure 2.** The value of the cell boundary node and the midpoint of the boundary

However, although the value of the cell boundary node is indicated, the relationship between the values on the spatial boundary layer is still unknown, and the right side of the equation (4) cannot be integrated.
Here, let the boundary value of each cell have a linear relationship on each spatial boundary layer such as \( x_{i-1/2} \), and then it is possible to take the average value of the two boundary nodes on the left side in Figure 2 as the midpoint value of the \( i \)th cell on the space layer \( x_{i-1/2} \). Similarly, take the average value of the two boundary nodes on the right side in Figure 2 as the midpoint value of the \( i \)th cell on the space layer \( x_{i+1/2} \).

Thus, the function value of the midpoint of each boundary on the \( i \)th cell is obtained. As shown in Figure 3.

![Figure 3. Function value of the midpoint of each boundary on the ith cell](image)

Now, there is a linear relationship between adjacent boundary nodes value of each cell and the midpoint of each boundary is the average of the sum of adjacent boundary nodes value. Therefore, the equation (4) can be integrated into:

\[
\Delta x \left[ u_{mid} \left( x, t^{n+1} \right) - u_{mid} \left( x, t^n \right) \right] = -\Delta t \left[ f_{mid} \left( x_{i+1/2}, t \right) - f_{mid} \left( x_{i-1/2}, t \right) \right] \tag{5}
\]

By combining the equation (1) and (5), equation (6) is obtained:

\[
\frac{\Delta t \times a}{2} \left[ u_i^{n+1} + u_i^n + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)^{n+1}_i + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)^n_i - u_i^{n+1} - u_i^n + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)^{n+1}_i + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)^n_i \right] + \Delta x \left( u_i^{n+1} - u_i^n \right) = 0 \tag{6}
\]

Equation (6) can be simplified as:

\[
\frac{\Delta t \times a}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^{n+1}_i + \left( \frac{\partial u}{\partial x} \right)^n_i \right] + \left( u_i^{n+1} - u_i^n \right) = 0 \tag{7}
\]

In this section, by using the scalar conservation law in FVM and applying it to 1-D advection equation, the cell can be conserved under two-dimensional conditions, thus ensuring the flat flow \( u \) conserved. At the same time, equation (7) containing the prediction term \( u^{n+1} \) is obtained.

However, equation (7) also contains the unknown quantity \( \left( \frac{\partial u}{\partial x} \right)^{n+1} \), which cannot solve \( u^{n+1} \) and at the same time, so a new equation must be constructed to solve the unknown.

### 2.4. Conserved flux in and out of adjacent grid interface

After ensuring the conservation of the flat flow, it is necessary to solve the strong discontinuity problem in the 1-D advection equation. The method proposed here is to make the flux of the adjacent interface in and out conserved by making the adjacent grid boundary time integrals equal. In addition to this, this method can also create new equations for solving unknowns \( \left( \frac{\partial u}{\partial x} \right)^{n+1} \) and \( u^{n+1} \).

As shown in Figure 4, the right boundary time integral value of the cell \( i-1 \) is expressed as
\[
\int_{t_n}^{t_{n+1}} f_t(x_{i-1/2}, t) dt \quad \text{and the left boundary time integral value of the cell } i \text{ is expressed as}
\int_{t_n}^{t_{n+1}} f_r(x_{i-1/2}, t) dt
\]

Herein, define
\[
u_{l,n} = u_{i-1} + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_{i-1}^{n+1}, \quad u_{l,n+1} = u_{i-1} + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_{i-1}^{n+1}, \quad u_{l,n} = u_{i} - \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_{i}^{n},
\]
\[
u_{r,n} = u_{i} - \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_{i}^{n+1}, \quad u_{r,n+1} = u_{i} + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_{i}^{n+1}
\]

separately. Since there is a linear relationship between adjacent boundary nodes value of each cell and the midpoint of each boundary is the average of the sum of adjacent boundary nodes value, \( \int_{t_n}^{t_{n+1}} f_t(x_{i-1/2}, t) dt \) can be simplified as \( a \left( u_{l,n} + u_{l,n+1} \right) / 2 \) and \( \int_{t_n}^{t_{n+1}} f_r(x_{i-1/2}, t) dt \) can be simplified as \( a \left( u_{r,n} + u_{r,n+1} \right) / 2 \).

![Figure 4. Boundary value of adjacent cell](image)

Finally, make the time integral values of adjacent cell boundary equal and we will get equation (8)

\[
\Delta t \times a \left( u_{l,n+1} + u_{l,n} \right)/2 = \Delta t \times a \left( u_{r,n+1} + u_{r,n} \right)/2
\]

Equation (8) can be simplified as:

\[
u_{l,n+1} - u_{l,n} - \Delta x \left( \frac{\partial u}{\partial x} \right)_{i-1}^{n+1} - \Delta x \left( \frac{\partial u}{\partial x} \right)_{i-1}^{n+1} = -u_{l,n} + u_{l,n} + \Delta x \left( \frac{\partial u}{\partial x} \right)_{i-1}^{n+1} + \Delta x \left( \frac{\partial u}{\partial x} \right)_{i-1}^{n+1}
\]

In this section, by making the adjacent grid boundary time integrals equal, we make the adjacent interface to be conserved in flux and obtain the new equation (9) described at the end of section 2.4.

2.5. **Calculate the forecast term by combining the boundary conditions**

Assume the subscript \( i \) is a natural number from 1 to \( m \). Then, the boundary condition of the advection equation is \( u_0 = u_{m} \). By combining equation (7), equation (9) and the boundary condition, giving a test function with an initial value and a value to \( m \), then writing a Fortran program for solving equations, finally we will obtain the value of forecast term and numerical simulation function.

3. **Results and analysis**

3.1. **Test function**

Use the piecewise function (10) with strong discontinuity as the test function:

\[
\begin{cases}
  e^{(x-0.3)^2} & 0 \leq x < 0.6 \\
  1 & 0.6 \leq x \leq 0.8 \\
  0 & x > 0.8
\end{cases}
\]

(10)
3.2. Results

By comparing the simulation function image (as shown in Figures 5, 6, 7, and 8) obtained by the traditional FVM, CDM, and the new method described in this paper, it can be found that when the time step is 1, the CDF has a similar simulation effect with the new method, and the image obtained by the conventional FVM does not match the original function image well. When the time step is gradually increased to 2 and 5, the traditional FVM dissipates too much, the CDM loses stability, and the two methods completely lose the numerical simulation function. However, the new method can still complete the function simulation well. In addition, as shown in Figure 9, changing the convergence condition criterion number CFL to 2, the accuracy of the numerical solution can still be ensured.

Figure 5, 6 and 7. Comparison of three method function simulation abilities with time step changing

Figure 8. New method function simulation ability with time step changing

Figure 9. New method function simulation ability with time step changing under the condition that CFL is 2

3.3. Analysis

The difference between the improved FVM and the traditional FVM proposed in this paper is that the traditional FVM can only satisfy the time-integral conservation of the dependent variable as shown in equation (2). The result is that the dissipation is large and it loses the accuracy of numerical simulation.
However, after applying the scalar conservation law, as shown in equation (4), the dependent variable can satisfy the integral conservation condition under 2-D (time and space) conditions, which ensures the stability of the numerical simulation. Secondly, by making the time integral values of adjacent cell boundaries equal, the flux of the adjacent cell interface is conserved, ensuring the continuity and accuracy of the flat flow transmission in the time dimension. Combined with the superiority of the above methods, numerical simulation can be simultaneously accurate and stable.

4. Conclusion
On the basis of the traditional FVM, by using the scalar conservation method to conserve the dependent variable and conserving the flux in the adjacent interface, a new method for dealing with the strong discontinuous in function simulation is obtained. The verification shows that the function image simulated by this method is in good agreement with the original function image. The simulation accuracy and stability are significantly improved compared with the traditional FVM and the CDM. After the convergence condition criterion CFL is adjusted to 2, the accuracy and stability of the numerical solution can still be guaranteed. So, the feasibility of the method in solving the 1-D advection equation is proved, which has broad application prospects. The next step will be to use this method to further explore the applicability of the second-order Burgers equation.

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References
[1] HUANG Ronghui, CHEN Wen, WEI Ke. Atmospheric Dynamics in the Stratosphere and Its Interaction with Tropospheric Processes: Progress and Problems [J]. Chinese Journal of Atmospheric Sciences (in Chinese), 2018, 42(3): 463-487
[2] Shi Ning. An example of nonlinear computational instability in Numerical Weather Prediction [J]. Education and Teaching Forum, 2013, (42): 160-160, 161.
[3] Wang Jiwen, Liu Ruxun. Finite Volume Method for Intermittent Solution Problem [J]. Chinese Journal of Computational Physics, 2001, 18(2): 97-105
[4] Li Xingliang. Study on the application of spherical Yin and Yang grid and new multi-moment finite volume method [D]. Nanjing: Nanjing University of Information Science and Technology, 2007: 1-169.
[5] Cheng J., Shu C.W. High order schemes for CFD: A review [J]. Jisuan Wuli/Chinese Journal of Computational Physics, 2009, 26(5).
[6] Thuburn J. A PV-Based Shallow-Water Model on a Hexagonal – Icosahedral Grid [J]. Monthly Weather Review, 1997, 125(9): 2328-2347.
[7] Harten A. High Resolution Schemes for Hyperbolic Conservation Laws [J]. Journal of Computational Physics, 1983, 49.
[8] Song Songhe, Chen Maozhang. A TVD-type finite volume method for two-dimensional unstructured grids [J]. Acta Aeronautica Sinica, 2001, 22(3).
[9] Cockburn B., Karniadakis G.E., Shu C.W. The Development of Discontinuous Galerkin Methods. Computational Science and Engineering [J], 2000, vol 11. Springer, Berlin, Heidelberg.
[10] Xiao F. Unified formulation for compressible and incompressible flows by using multi-integrated moments I: one-dimensional inviscid compressible flow [J]. Journal of Computational Physics, 2004, 195(2): 629-654.