ON GENERALIZED 3-MANIFOLDS WHICH ARE NOT HOMOLOGICALLY LOCALLY CONNECTED

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Abstract. We show that the classical example \(X\) of a 3-dimensional generalized manifold constructed by van Kampen is another example of not homologically locally connected (i.e. not HLC) space. This space \(X\) is not locally homeomorphic to any of the compact metrizable 3-dimensional manifolds constructed in our earlier paper which are not HLC spaces either.

1. Introduction

In our earlier paper [10] we constructed for every natural number \(n > 2\), examples of \(n\)-dimensional compact metrizable cohomology \(n\)-manifolds which are not homologically locally connected with respect to the singular homology (i.e. they are not HLC spaces). In the present paper we shall call them singular quotient \(n\)-manifolds.

Subsequently, we have discovered that van Kampen constructed a compact metrizable generalized 3-manifold which "is not locally connected in dimension 1 in the homotopy sense" [2, p. 573]. The description of van Kampen’s construction can be found in [2] p. 573 (see also [16, p. 245]).

The obvious modification of van Kampen’s construction gives an infinite class of examples – we shall call them van Kampen generalized 3-manifolds. The main purpose of the present paper is to prove the following theorem:

Theorem 1.1. No van Kampen generalized 3-manifold is homologically locally connected in dimension 1 with respect to the singular homology. They are neither locally homeomorphic to any singular quotient 3-manifold. Furthermore, no singular quotient 3-manifold is locally homeomorphic to any van Kampen generalized 3-manifold.

2. Preliminaries

We shall denote the singular homology (resp. Čech cohomology) groups with integer coefficients by \(H_\ast\) (resp. \(H^\ast\)). All spaces considered in this paper will be assumed to be metrizable, locally compact and finite-dimensional. Under these circumstances the classes of generalized manifolds, homology manifolds, and cohomology manifolds in the classical sense [16] coincide (cf. [3, 9, 11]). For the history and importance of generalized manifolds see [13].
Definition 2.1. (cf. [3, p. 377, Corollary 16.9]). A locally compact, cohomologically locally connected with respect to Čech cohomology \((clc)\), and cohomologically finite-dimensional space \(X\) is called a generalized \(n\)-manifold, \(n \in \mathbb{N}\), if
\[
\check{H}^p(X, X \setminus \{x\}) \cong \check{H}^p(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})
\]
for all \(x \in X\) and all \(p \in \mathbb{Z}_+\).

Definition 2.2. A closed 3-manifold is called a homology 3-sphere if its homology groups are isomorphic to the homology groups of the standard 3-sphere (cf. e.g. [8]). The complement of the interior of any 3-simplex in any triangulation of any homology 3-sphere is called a homology 3-ball.

In 1904 Poincaré constructed the first example of a homology 3-sphere with a nontrivial fundamental group [12, 14]. The following theorem is due to Brown [4]:

Theorem 2.3. (Generalized Schönflies theorem). Let \(h : S^{n-1} \times [-1, 1] \to S^n\) be an embedding. Then the closures of both complementary domains of \(h(S^{n-1} \times 0)\) in \(S^n\) are topological \(n\)-cells.

Definition 2.4. A space \(X\) is said to be locally homeomorphic to the space \(Y\) if for every point \(x \in X\) there exists an open neighborhood \(U_x \subset X\) which is homeomorphic to some open subspace of \(Y\).

Definition 2.5. (cf. e.g. [1, 5, 10]). Let \(G\) be any group and \(g \in G\) any element. The commutator length \(cl(g)\) of the element \(g\) is defined as the minimal number of the commutators of the group \(G\) the product of which is \(g\). If such a number does not exist then we set \(cl(g) = \infty\). Moreover, \(cl(e) = 0\) if and only if \(g = e\), where \(e\) denotes the neutral element.

Theorem 2.6. (cf. [6, 7]). Let \(G\) be a free product of groups \(\{G_i\}_{i=1}^k\), \(G = G_1 * G_2 * \cdots * G_k\). Let \(g \in G\), \(g_i \in G_i\) for \(i = 1, k\) and let \(g = g_1 g_2 \cdots g_k\). Then \(cl(g) = \sum_{i=1}^k cl(g_i)\).

Assertion 2.7. The commutator length function has following properties:

1. If \(\varphi : G \to H\) is a homomorphism of groups and \(g \in G\) then \(cl(\varphi(g)) \leq cl(g)\); and

2. For every element \(g \in G\), \(cl(g) = \infty \iff g \notin [G, G]\).

All undefined terms can be found in [3], [14] or [15].

3. The construction of the van Kampen example and the proof of the Main Theorem

To prove the main theorem we shall need a more detailed description of the original van Kampen construction [2, p. 573]. For nonnegative numbers \(i, p\) and \(q\) let the ball layer \(A_{i,p,q}\) be the following subspace of \(\mathbb{R}^3\):
\[
A_{i,p,q} = \{ \mathbf{a} \in \mathbb{R}^3 \mid p \leq |\mathbf{a} - (0,0,0)| \leq p + q, \ (0,0,0) \in \mathbb{R}^3 \}.
\]
In particular, \(A_{i,0,q}\) is a 3-ball with the center at the point \((0,0,0)\) in \(\mathbb{R}^3\) and of radius \(q\). For \(i \in \mathbb{N}\), let \(\{B_i\}_{i \in \mathbb{N}}\) be a sequence of a homology 3-balls with a nontrivial fundamental group. All spaces \(B_i\) are compact 3-manifolds the boundaries of which are homeomorphic to \(S^2\).
Let for \( n, k \in \mathbb{N} \)
\[
X_{n,k} = A_{0,(n-\frac{1}{2}),k} \setminus \bigcup_{j=n}^{n+k-1} A_{j,0,\frac{1}{4}} \bigcup_{i=n}^{n+k-1} B_i
\]
with the topology of identification of the boundaries of \( A_{i,0,\frac{1}{4}} \) and \( B_i \) (the bar over the space denotes the closure of that space). Let
\[
X_k = X_{1,k} \cup A_{0,0,\frac{1}{4}}.
\]
Obviously, \( X_{n,k} \) and \( X_k \) are compact 3-manifolds with boundaries. Since the homology groups of \( B_i \) are the same as the homology groups of the 3-ball \( A_{i,0,\frac{1}{4}} \) it follows from the exactness of the Mayer-Vietoris homology sequence that the homology groups of \( X_k \) are the same as the homology groups of \( A_{0,0,(k+\frac{1}{4})} \), i.e., they are trivial and \( X_k \) is a homology 3-ball.

Consider \( X = \lim\limits_{\rightarrow} X_k \subset X_{k+1} \) with the topology of the direct limit. Then
\[
H_*(X) \cong H_*(\ast) \quad \text{and} \quad \check{H}^*(X) \cong \check{H}^*(\ast), \quad \text{i.e.} \quad X \text{ is an acyclic 3-manifold.}
\]
The one-point compactification of this space, by the point \( \ast \), is a 3-dimensional compact metrizable space \( X^* \).

We have
\[
X^* \cong \lim\limits_{\leftarrow} (X_k/\partial A_{0,0,(k+\frac{1}{4})}),
\]
where \( X_k/\partial A_{0,0,(k+\frac{1}{4})} \) is the quotient of the manifold \( X_k \) via the identification of all of its boundary points to one point and where the projections
\[
X_k/\partial A_{0,0,(k+\frac{1}{4})} \rightarrow X_{k+1}/\partial A_{0,0,(k+1+\frac{1}{4})}
\]
map the subspaces \( (X_{k+1}/\partial A_{0,0,(k+1+\frac{1}{4})}) \setminus X_k \) of the spaces \( X_{k+1}/\partial A_{0,0,(k+1+\frac{1}{4})} \) to the point of identification of \( X_k/\partial A_{0,0,(k+\frac{1}{4})} \).

Obviously, the spaces \( X_k/\partial A_{0,0,(k+\frac{1}{4})} \) are homology 3-spheres and all projections in the inverse sequence generate isomorphisms of homology and cohomology groups. Therefore
\[
\check{H}^*(X^*) \cong \check{H}^*(S^3).
\]
It follows from the exact sequence
\[
\cdots \rightarrow H^{p-1}(X^* \setminus \ast) \rightarrow \check{H}^p(X^*, X^* \setminus \ast) \rightarrow \check{H}^p(X^*) \rightarrow \check{H}^p(X^* \setminus \ast) \rightarrow \cdots,
\]
from the facts that \( X^* \setminus \ast = X \), \( \check{H}^p(X^*) = 0 \) for \( p \neq 3 \), and from (1) and (3), that
\[
\check{H}^p(X^*, X^* \setminus \ast) \cong H^p(\mathbb{R}^3, \mathbb{R}^3 \setminus \{0\}).
\]
Let us show that \( X^* \) is a clc space. Since \( X \) is a 3-manifold, the space \( X^* \) is a clc space at every point of \( X^* \setminus \ast \). Since \( X = \lim\limits_{\rightarrow} X_k \subset X_{k+1} \) and \( X_k \) are compact spaces, the system of the sets \( \{X^* \setminus X_k\} \) is a basis of neighborhoods of the point \( \ast \). According to (2), we have
\[
X^* \setminus X_k = \lim\limits_{\leftarrow} (X_n/\partial A_{0,0,(n+\frac{1}{4})}) \setminus X_k, \text{ for } n > k.
\]
The spaces \( (X_n/\partial A_{0,0,(n+\frac{1}{4})}) \setminus X_k \) are homology 3-balls. It follows that \( X^* \setminus X_k \) is an acyclic space with respect to the Čech cohomology. Therefore \( X^* \) is a clc space. Hence it follows from (1) that \( X^* \) is a generalized 3-manifold (cf. Definition 2.1).
The topological type of the space $X^*$ depends on the choice of the sequence of homology 3-balls $B_i$. In the case when all $B_i$ are homeomorphic to the Poincaré homology 3-sphere, the space $X^*$ is the example of van Kampen. In general, since there exist infinitely many distinct homology 3-balls (cf. [8]), there exists an infinite class of van Kampen generalized 3-manifolds.

Let $X^*$ be any van Kampen generalized 3-manifold. We shall show that $X^*$ is not HLC in dimension 1. It suffices to prove that for every $k$ the embedding $X^* \setminus X_k \subset X^* \setminus X_1$ is not homologically trivial with respect to singular homology.

The space $X^* \setminus X_k$ is a retract of $X^* \setminus X_1$. Indeed, the 3-balls $A_{i,0,\frac{1}{4}}$ are AR spaces so there exists for every $i$, a mapping of $B_i$ on $A_{i,0,\frac{1}{4}}$ which is the identity on the boundary of $B_i$. Therefore we have a mapping of $X_{1,k}$ onto the ball layer $A_{0,\frac{1}{4},k}$.

However, the sphere $A_{0,k+\frac{1}{4},0}$ is a retract of this ball layer and $A_{0,k+\frac{1}{4},0} \subset X^* \setminus X_k$. Therefore $X^* \setminus X_k$ is a retract of $X^* \setminus X_1$. So in order to prove that $X^* \setminus X_k \subset X^* \setminus X_1$ is not homologically trivial it suffices to prove that $H_1(X^* \setminus X_k) \neq 0$.

Since for any path-connected space the 1-dimensional singular homology group is isomorphic to the abelianization of the fundamental group of this space, it suffices to show that $\pi_1(X^* \setminus X_k)$ is a nonperfect group.

Consider the union of spheres $A_{j,\frac{1}{4},0}$, for $j > k$, with the compactification point as a subspace of the space $X^* \setminus X_k$. Join them by the segments

$I_j = \{(0,y,0) \mid y \in [j + \frac{1}{4}, j + 1 - \frac{1}{4}]\}, \; j = k + 1, \cdots \infty.$

We get the compactum

$$A = \bigcup_{j=k+1}^{\infty} A_{j,\frac{1}{4},0} \cup I_j \cup \{\ast\}.$$  

The space $A$ also lies in a 3-dimensional cube with a countable number of open balls $\{\text{int} A_{j,\frac{1}{4}}\}_{j=k+1}^{\infty}$ removed and it is obviously its retract. Therefore the space

$$B = A \bigcup_{i=k+1}^{\infty} B_i$$

with the natural topology of identification is a retract of $X^* \setminus X_k$.

Consider nontrivial loops $\alpha_i \in B_i$, with their homotopy classes $[\alpha_i]$. Then $0 < l([\alpha_i])$. Since $H_1(B_i) = 0$ it follows that $l([\alpha_i]) < \infty$. Let $\alpha$ be any loop in $B$ such that for the canonical projection of this space to $B_i$, the image of $\alpha$ generates the loop $\alpha_i$ (obviously such a loop exists). We have a homomorphism

$$\pi_1(B) \to \pi_1(B_1) * \pi_1(B_2) * \cdots$$

such that the image of $[\alpha]$ has projections $[\alpha_i]$ in $\pi_1(B_i)$. Hence according to [2.30] and [2.7] the commutator length of $[\alpha] \in \pi_1(B)$ is infinite, $d([\alpha]) = \infty$, and the loop $\alpha$ generates the loop $\alpha_i$ (obviously such a loop exists). Therefore and since $B$ is a retract of $X^* \setminus X_k$ it follows that $\pi_1(X^* \setminus X_k)$ is nonperfect and hence the van Kampen generalized 3-manifold is not an HLC-space.

Let us prove that no van Kampen generalized 3-manifold is locally homeomorphic to any singular quotient 3-manifold. Consider any open neighborhood $U$ of the singular point $\ast$ of a van Kampen generalized 3-manifold. Suppose that it were
homeomorphic to some open subset of some almost 3-manifold. Now, every singular quotient 3-manifold is a quotient space of a topological 3-manifold by some continuum, which generates a singular point [10]. Therefore there should exists an index $i$ such that some neighborhood of $B_i$ in $U$ embeds into the Euclidean 3-space with some ball layer $A_{i, \frac{1}{4} + \varepsilon}$.

By Theorem 2.3, the bounded component of $\mathbb{R}^3 \setminus A_{i, \frac{1}{4} + \varepsilon}$ is a 3-cell. So the space $B_i \cup A_{i, \frac{1}{4} + \varepsilon}$ embeds in a 3-cell. This embedding cannot be an onto mapping to the 3-cell because $B_i$ is not simply connected. Therefore the sphere $A_{i, \frac{1}{4} + \varepsilon}$ must be a retract of $B_i$. However, this is impossible since the space $B_i$ is acyclic whereas the sphere $A_{i, \frac{1}{4}}$ is not acyclic. So no van Kampen generalized 3-manifold is locally homeomorphic to any almost 3-manifold.

By the argument above and since every local homeomorphism must map singular points to singular points it follows that no singular quotient 3-manifold is locally homeomorphic to any van Kampen generalized 3-manifold. □

4. Epilogue

Cohomological local connectedness does not imply local contractibility even in the category of compact metrizable generalized 3-manifolds (this follows e.g. from Theorem [11]). However, the following problem remains open:

**Problem 4.1.** Does there exist a finite-dimensional compact metrizable generalized manifold (or merely finite-dimensional compactum) which is homologically locally connected but not locally contractible?

Mardešić formulated the following interesting problem:

**Question 4.2.** Is it true that every compact generalized $n$-manifold which is an ANR can be represented as an inverse limit of compact $n$-manifolds?

As it was mentioned above (Chapter 3 of reference [2]), every van Kampen generalized 3-manifold is an inverse limit of compact 3-manifolds. In contrast with this we set forth the following conjecture:

**Conjecture 4.3.** No singular quotient 3-manifold can be represented as an inverse limit of closed 3-manifolds.

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References

[1] C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) 37 (1991), 109–150.
[2] E. G. Begle, *Locally connected spaces and generalized manifolds*, Amer. J. Math. 64 (1942), 553–574.
[3] G. E. Bredon, *Sheaf Theory*, 2nd Ed., Graduate Texts in Math. 170, Springer, Berlin, 1997.
[4] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. 646 (1960), 74–76.
[5] K. Eda, U. H. Karimov, D. Repovš, *On (co)homology locally connected spaces*, Topology Appl. 120 (2002), 397–401.
[6] R. Z. Goldstein, T. E. Turner, *A note on commutators and squares in free products*, Combinatorial Methods in Topology and Algebraic Geometry (Rochester, NY, 1982), Contemp. Math. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 69–72.
[7] H. B. Griffiths, *A note on commutators in free products, II.*, Proc. Camb. Phil. Soc. **51** (1955), 245–251.

[8] J. L. Gross, *An infinite class of irreducible homology 3-spheres*, Proc. Amer. Math. Soc. **25** (1970), 173–176.

[9] A. E. Harlap, *Local homology and cohomology, homology dimension and generalized manifolds*, Mat. Sbornik. **96** (1975), 347–373.

[10] U. H. Karimov, D. Repovš, *Examples of cohomology manifolds which are not homologically locally connected*, Topology Appl. **155** (2008), 1169–1174.

[11] W. J. R. Mitchell, *Homology manifolds, inverse system and cohomological local connectedness*, J. London. Math. Soc. (2) **19** (1979), 348–358.

[12] H. Poincaré, *Cinquième complément a l’analysis situs*, Rend. Circ. Mat. Palermo **18** (1904), 45–110.

[13] D. Repovš, *Detection of higher dimensional topological manifolds among topological spaces*, Giornate di Topologia e Geometria delle Varietà, Bologna 1990, M. Ferri, Ed., Univ. degli studi di Bologna (1992), pp. 113–143.

[14] H. Seifert, W. A. Threlfall, *Textbook of Topology*, Acad. Press, New York, 1980.

[15] E. H. Spanier, *Algebraic Topology*, Springer-Verlag, Berlin, 1966.

[16] R. L. Wilder, *Topology of Manifolds*, American Mathematical Society Colloq. Publ. **32**, New York, 1949.