Correctness of the definition of the Laplace operator with delta-like potentials

B. E. Kanguzhin and K. S. Tulenova

Department of Mathematics, Al-Farabi Kazakh National University, Almaty, Kazakhstan; Institute of Mathematics and Mathematical Modeling, Almaty, Department of Function theory and Functional Analysis, Kazakhstan

ABSTRACT
In this paper, we give a correct definition of the Laplace operator with delta-like potentials. Correctly solvable pointwise perturbation is investigated and formulas of resolvent are described. We study some properties of the resolvent. In particular, we prove Krein’s formula for these resolvents.

1. Introduction
The main goal of this paper is to give a correct definition of formally defined operator via \(-\Delta + \delta_s\), where \(\Delta\) is the Laplacian and \(\delta_s\) is Dirac’s delta function. A number of works have been devoted to this classical problem of Mathematical Physics. There are a lot of approaches to solving such a problem. One of the main approaches, based on the theory of Neumann on extensions of symmetric operators, is a so-called expansion to a wider space [1]. Applications of Neuman’s scheme of the theory of extensions, as a rule, are limited to a circle of abstract operators in a Hilbert space. In this note, we present a method of restrictions of previously well-defined maximal operator. The method of restrictions of the maximal operator is dual to the method of extensions of the minimal operator. In the theory of extensions of the minimal operator, it is usually preferred to act in terms of the boundary form, i.e.

\[ \langle B_0^* u, v \rangle - \langle u, B_0^* v \rangle, \]

where \(B_0\) is the minimal symmetric operator. Further progress in various directions can be found in [2–7] from different points of view. A systematic application of non-standard analysis in the theory of point interactions can be found in [8].

CONTACT
K. Tulenov tulenov@math.kz

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In this work, the initial operator is the maximal operator $B_M$, which can be considered as the adjoint of the minimal operator $B_0$. After defining correctly the closed maximal operator $B_M$, its everywhere solvable restrictions will be constructed. We act in terms of the boundary form $\langle B_M u, v \rangle - \langle u, B_M v \rangle$, when constructing the restrictions of the maximal operator. Hence, at the final stage, self-adjoint operators will be selected from everywhere solvable restrictions.

An alternative approach for the one-dimensional Sturm–Liouville operator with a potential $v(x)$, which is the singular distribution of first-order, was given in [7]. In particular, operators generated by the differential expression $-\frac{d^2}{dx^2} + v(x)$ were investigated, when $v(x) = \frac{da(x)}{dx}$ and $a(x) \in L_2(0, 1)$. Authors of [7] introduced definition of the domain of the maximal operator as follows:

$$D(B) = \{ y(x), y^{[1]}(x) \in W^1_2[0, 1] : -(y^{[1]})' - ay^{[1]} - a^2 y \in L_2(0, 1) \},$$

where $y^{[1]}(x) = y'(x) - a(x)y(x)$ and $W^1_2[0, 1]$ is the Sobolev space. In case $v(x) = \delta(x - x_0)$, where $x_0 \in (0, 1)$, it is convenient to denote the domain of the operator $B$ by $W^2_{2, \gamma}(\Omega_0)$. Here, $\Omega_0 = (0, 1) \setminus \{x_0\}$,

$$\gamma_1(y) = \lim_{\delta \to 0} (y(x_0 + \delta) - y(x_0 - \delta)),
\gamma_2(y) = \lim_{\delta \to 0} (y'(x_0 + \delta) - y'(x_0 - \delta)).$$

The domain $D(B)$ is defined as follows:

$$W^2_{2, \gamma}(\Omega_0) = \{ y \in W^2_{2, \text{loc}}(\Omega_0) : \gamma_1(y) = 0, \gamma_2(y) = y(x_0),
-(y^{[1]})'(x) - \frac{1}{2} \text{sign}(x)y^{[1]}(x) - \frac{1}{4} y(x) \in L_2(0, 1) \}.$$
defined correctly both on spaces $H^{2m}(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$. In addition, in the same paper, self-adjoint extension of the operator $(-\Delta)^m + \delta_i$ is given for wider classes than $H^m(\mathbb{R}^n)$. Calculations in [5] are based on the results of [9]. In this paper, we introduce a correct definition of the operator $-\Delta + \delta_i$ in $L_2(\Omega)$, where $\Omega$ is a bounded set in $\mathbb{R}^d$ with smooth boundary. In order to reduce technical details, we set out our approach when $S$ is an inlet and $\Omega$ is an inlet ball in $\mathbb{R}^d$. Unlike [4], in this paper, the definition of the operator $-\Delta + \delta_i$ is defined directly with no use of quadratic forms. Our definition has some own advantages. More precisely, it allows us to study some properties of resolvents of the operators in more detail. As a consequence, we may obtain some results on the discrete spectrum of operators introduced in this work. The difference from [7] is that we investigate multi-dimensional case, i.e. $d > 1$. However, we consider potentials are represented by delta-function only, while in [7] authors considered potentials that can be singular distributions of the first order. Another difference of this paper from [7] is that the domain of the maximal operator contains functions which are not in $L_2(\Omega)$. Questions related to this paper are partially studied in [10–14] for the Laplace and polyharmonic operators. In particular, first regularized trace formulas for the Laplace operator were obtained in [15].

The structure of this paper is as follows. Section 2 recalls the definition of the maximal operator $B_M$. Moreover, in Theorem 2.2, we show the correct restriction of the maximal operator $B_M$. It means that the problem (5)–(7) below has a unique solution in the punctured area $\Omega_0 = \Omega \setminus S$, where $S$ is a singleton. Section 3 deals with correctly solvable pointwise perturbations. In Section 4, formulas of resolvents of correctly solvable pointwise perturbations are given. As an application, we obtain an analogue of M.G. Krein’s formula for the resolvents. In Section 5, we obtain some results when we change of the initial part of the spectrum of the Laplace operator.

2. Definition of maximal operator

In case $d = 1$ and $S = \{x_0\}$, where $0 < x_0 < 1$, the definition of the domain of maximal operator defined in the previous section is given as follows:

$$D(B) = \left\{ y \in W_{2, \text{loc}}^2(\Omega_0) : \gamma_1(y) = 0, \gamma_2(y) = y(x_0), \right.$$

$$-\frac{d^2}{dx^2}(y(x) - \gamma_2(y)G_1(x, x_0)) \in L_2(0, 1) \right\}.$$

Moreover, we have $By(x) = -\frac{d^2}{dx^2}(y(x) - \gamma_2(y)G_1(x, x_0))$, where

$$G_1(x, x_0) = \left\{ \begin{array}{ll}
    x_0(1-x), & x_0 < x, \\
    x(1-x_0), & x \leq x_0.
\end{array} \right.$$

Then a correct restriction of the operator $B$ is considered in terms of invertible operators, generated by operations $-\frac{d^2}{dx^2} + \delta(x - x_0)$. In case $d > 2$ and $S = \{x_0\}$, where $x_0 \in \Omega \subset \mathbb{R}^d$, first we need to define the domain of maximal operator, generated by operation $-\Delta + \delta_i(x)$, and then we will study its correct restriction. For this purpose, we use the following well known facts. Its proof can be found from [16, Chapter IV].
**Theorem 2.1.** The solution of the Dirichlet problem for non-homogeneous harmonic equation

\[ \Delta w(x) = f(x), \quad |x| < 1 \]  

with the boundary condition

\[ w|_{|x|=1} = 0 \]

is given by the following formula:

\[ w(x) = \int_{\Omega} G(x, \xi) f(\xi) \, d\xi, \]

where \( G(x, \xi) = C \cdot (|x - \xi|^2 - |\xi|^2 \cdot |x - \frac{\xi}{|\xi|^2}|^2 - d) \) and \( C \) is some constant.

**Remark 2.1:** Theorem 2.1 claims that the Green function of the Dirichlet problem in the ball \( \Omega \), whenever \( d > 2 \), can be written out explicitly.

Let \( \Omega \) be an open unit ball and let \( x^0 \in \Omega \). We denote by \( \Omega_0 \) an open unit ball \( \Omega \) without one fixed point \( x^0 \), i.e. \( \Omega_0 := \Omega \setminus \{x^0\} \). For \( \delta > 0 \), we denote the ball of radius \( \delta \) centered at \( x^0 \) by \( \Pi_\delta^0 = \{ x \in \Omega : |x - x^0| \leq \delta \} \). For \( s = 0, 1, 2, \ldots, d \), define the following functionals:

\[ \gamma_0(h) = - \lim_{\delta \to 0+} \int_{\partial \Pi_\delta^0} \frac{\partial}{\partial \nu_\delta} h(t) \, dS_t, \quad \gamma_s(h) = d \cdot \lim_{\delta \to 0+} \int_{\partial \Pi_\delta^0} \frac{(t_s - x_s^0)}{|t - x^0|} h(t) \, dS_t \]

for some functions \( h \), which these formulas make sense. Similar constructions were studied in [12] for the \( m \)-Laplacian (see also [10]). Throughout this paper, we denote by \( \frac{\partial}{\partial \nu_\delta} \) the normal derivative along the boundary \( \partial \Pi_\delta^0 \) at a point \( \xi \). We denote by \( W_{2,\gamma}^2(\Omega_0) \) the elements \( h \) in \( W_{2,\gamma}^2(\Omega_0) \) for which \( \omega|_{\partial \Omega} = 0 \) with the values of \( \gamma_0(h), \gamma_1(h), \ldots, \gamma_d(h) \) being finite, and satisfying

\[ \Delta_x \left( h(x) - \gamma_0(h) \cdot G(x, x^0) - \gamma_1(h) \cdot \frac{\partial G(x, x^0)}{\partial \xi_1} - \cdots - \gamma_d(h) \cdot \frac{\partial G(x, x^0)}{\partial \xi_d} \right) \in L^2(\Omega). \]

The next lemma shows that the space \( W_{2,\gamma}^2(\Omega_0) \) can be larger than \( W_2^2(\Omega) \). This result could be inferred from [12, Lemma 3.2] (see also [13, Lemma 1]); however, we present its simpler proof for the convenience of the reader.

**Lemma 2.1.** For any \( s = 1, 2, \ldots, d \), we have

\[ G(x, x^0) \in W_{2,\gamma}^2(\Omega_0), \quad \frac{\partial G(x, x^0)}{\partial \xi_s} \in W_{2,\gamma}^2(\Omega_0). \]

Moreover,

\[ \gamma_0(G) = -1, \quad \gamma_s(G) = 0, \quad \gamma_0 \left( \frac{\partial G}{\partial \xi_j} \right) = 0, \quad \gamma_s \left( \frac{\partial G}{\partial \xi_j} \right) = \delta_{sj}, \]

where \( \delta_{sj} \) is the Kronecker symbol.
Proof: Note that the Green function $G(x, x^0)$ is represented as a sum

$$G(x, x^0) = \varepsilon(x, x^0) + k(x, x^0),$$

where $\varepsilon(x, x^0)$ is the fundamental solution and $k(x, x^0)$ is a compensating function. Since the compensating function $k(x, x^0)$ is smooth in $\Omega$, it follows that

$$\gamma_j(k(\cdot, x^0)) = 0, \quad \gamma_j \left( \frac{\partial}{\partial t_s} k(\cdot, x^0) \right) = 0, \quad j = 0, 1, \ldots, d, \quad s = 1, 2, \ldots, d.$$

It remains to prove that the values of functionals

$$\gamma_0(\varepsilon), \gamma_1(\varepsilon), \ldots, \gamma_d(\varepsilon),$$

$$\gamma_0 \left( \frac{\partial \varepsilon}{\partial \xi_s} \right), \gamma_1 \left( \frac{\partial \varepsilon}{\partial \xi_s} \right), \ldots, \gamma_d \left( \frac{\partial \varepsilon}{\partial \xi_s} \right)$$

are finite. By straightforward calculation, we obtain

$$\gamma_0(\varepsilon) \overset{(4)}{=} -C_d \cdot \lim_{\delta \to 0} \int_{\partial \Omega^0} \frac{\partial}{\partial \nu_t} |t - x^0|^{2-d} \, dS_t$$

$$= -C_d \cdot \lim_{\delta \to 0} \int_{\partial \Omega^0} \sum_{k=1}^{d} \frac{\partial}{\partial t_k} |t - x^0|^{2-d} \cdot \frac{(t_k - x^0_k)}{|t - x^0|^2} \, dS_t$$

$$= -C_d \sum_{k=1}^{d} \lim_{\delta \to 0} \int_{\partial \Omega^0} \frac{\partial}{\partial t_k} (|t - x^0|^2)^{\frac{2-d}{2}} \cdot \frac{(t_k - x^0_k)}{|t - x^0|^2} \, dS_t$$

$$= -C_d \sum_{k=1}^{d} \lim_{\delta \to 0} \int_{\partial \Omega^0} (2 - d) |t - x^0|^{-d-1} (t_k - x^0_k)^2 \, dS_t$$

$$= -C_d \cdot (2 - d) \sum_{k=1}^{d} \lim_{\delta \to 0} \int_{\partial \Omega^0} |t - x^0|^{-d-1} (t_k - x^0_k)^2 \, dS_t,$$

where $C_d = -\frac{1}{(d-2)\sigma_d}$. By using the substitution $t = x^0 + \delta \cdot \frac{t - x^0}{\delta} = x^0 + \delta \eta$, where $\eta \in S^{d-1}$ and $dS_t = \delta^{d-1} \, dS_\eta$, we obtain the required formula, i.e.

$$\gamma_0(\varepsilon) = -C_d \cdot (2 - d) \sum_{k=1}^{d} \lim_{\delta \to 0} \int_{S^{d-1}} \delta^{-d+1} \eta_k^2 \delta^{d-1} \, dS_\eta$$

$$= -C_d \cdot (2 - d) \cdot \lim_{\delta \to 0} \int_{S^{d-1}} \sum_{k=1}^{d} \eta_k^2 \, dS_\eta = -\frac{1}{(2 - d)\sigma_d} (2 - d) \sigma_d = -1.$$

Similarly, again by the substitution $t = x^0 + \delta \cdot \frac{t - x^0}{\delta} = x^0 + \delta \eta$, we compute $\gamma_s(\varepsilon)$

$$\gamma_s(\varepsilon) \overset{(4)}{=} d \cdot C_d \lim_{\delta \to 0} \int_{\partial \Omega^0} \frac{(t_k - x^0_k)}{|t - x^0|^2} |t - x^0|^{2-d} \, dS_t$$
Now, we study the value of \( \gamma_0 \left( \frac{\partial G(\xi, x^0)}{\partial \xi_j} \right) \). First, we need to compute the normal derivative of the function \( \frac{\partial \varepsilon(t, x^0)}{\partial \xi_j} \), \( j = 1, 2, \ldots, d \). Indeed,

\[
\frac{\partial}{\partial v_l} \left( \frac{\partial \varepsilon(t, x^0)}{\partial \xi_j} \right) = C_d \cdot (2 - d) \sum_{k=1}^d \frac{\partial}{\partial t_k} \left( |t - x^0|^{-d} \cdot (t_j - x_j^0) \right) \cdot \left( \frac{t_k - x_k^0}{|t - x^0|} \right)
\]

\[
= C_d \cdot (2 - d) \left( \sum_{k=1}^d (t_j - x_j^0) \cdot \left( -\frac{d}{2} \right) |t - x^0|^{-d-2} \cdot 2(t_k - x_k^0) \right)
\]

\[
= C_d \cdot (d - 2)(d - 1)(t_j - x_j^0) \cdot |t - x^0|^{-d-1}.
\]

Next, we compute the values of the functional

\[
\gamma_0 \left( \frac{\partial \varepsilon(\cdot, x^0)}{\partial \xi_j} \right) \quad \text{(4)} = -C_d \cdot (d - 2)(d - 1) \cdot \lim_{\delta \to 0} \int_{\partial \Pi^0_{t\delta}} (t_j - x_j^0) \cdot |t - x^0|^{-d-1} \, dS_t.
\]

By using the substitution \( \eta = \frac{t-x^0}{\delta} \) in the last integral, we obtain the following equality:

\[
\gamma_0 \left( \frac{\partial \varepsilon(\cdot, x^0)}{\partial \xi_j} \right) = -C_d \cdot (d - 2)(d - 1) \lim_{\delta \to 0} \int_{\partial \Pi^0_{t\delta}} \delta \cdot t_j \cdot \delta^{-d-1} \delta^{d+1} \, dS_t = C_d \cdot (d - 2)(d - 1) \lim_{\delta \to 0} \left( \delta^{-1} \int_{S^{d-1}} \eta_j \, dS_\eta \right) = 0.
\]

Now, we compute the values of functionals \( \gamma_s \left( \frac{\partial \varepsilon(\cdot, x^0)}{\partial \xi_j} \right) \), \( s = 1, 2, \ldots, d \). If \( j = s \), then

\[
\gamma_s \left( \frac{\partial \varepsilon(\cdot, x^0)}{\partial \xi_j} \right) \quad \text{(4)} = C_d \cdot (2 - d) \cdot d \cdot \lim_{\delta \to 0} \int_{\partial \Pi^0_{t\delta}} \eta_s \cdot \delta^{-d} \cdot (t_s - x_s^0) \cdot |t - x^0|^{-d} \cdot (t_j - x_j^0) \, dS_t
\]

\[
= \frac{d}{\sigma_d} \cdot \lim_{\delta \to 0} \int_{S^{d-1}} \eta_s \cdot \delta^{-d} \cdot \eta_j \, dS_\eta = \frac{d}{\sigma_d} \cdot \lim_{\delta \to 0} \int_{S^{d-1}} \eta_s \cdot \eta_j \, dS_\eta.
\]

Since

\[
\int_{S^{d-1}} \eta_j^2 \, dS_\eta = \frac{\sigma_d}{d},
\]

it follows that

\[
\gamma_s \left( \frac{\partial \varepsilon(\cdot, x^0)}{\partial \xi_j} \right) = 1.
\]

In the case \( j \neq s \), it is easy to see that \( \gamma_s \left( \frac{\partial \varepsilon(\cdot, x^0)}{\partial \xi_j} \right) = 0 \). This concludes the proof. \( \blacksquare \)
**Lemma 2.2.** Every element \( h \) in \( W^{2}_{2,\gamma} (\Omega_0) \) is represented as

\[
h(x) = h_0(x) + \gamma_0(h) G(x, x^0) + \sum_{i=1}^{d} \gamma_i(h) \frac{\partial G(x, x^0)}{\partial \xi_i},
\]

where \( h_0 \in W^{2}_2(\Omega) \) and \( h_0|_{\partial \Omega} = 0 \).

Moreover, such representation is unique.

**Proof:** Since \( h \in W^{2}_{2,\gamma} (\Omega_0) \), it follows that there exist values of \( \gamma_0(h), \gamma_i(h), i = 1, 2, \ldots, d \), which are finite. Set

\[
w(x) = h(x) - \gamma_0(h) G(x, x^0) - \sum_{i=1}^{d} \gamma_i(h) \frac{\partial G(x, x^0)}{\partial \xi_i}.
\]

From the definition of \( W^{2}_{2,\gamma} (\Omega_0) \), it follows that \( w|_{\partial \Omega} = 0 \) and \( \Delta w \in L^2(\Omega) \).

Let us denote \( f(x) = \Delta w, \ x \in \Omega \). Then, the solution of the Dirichlet problem

\[
\Delta u = f(x), \quad x \in \Omega,
\]

\[
u|_{\partial \Omega} = 0
\]

exists and unique in the class \( W^{2}_2(\Omega) \). Consequently, \( u(x) \in W^{2}_2(\Omega) \). The uniqueness of the solution of the Dirichlet problem implies that \( u(x) \equiv w(x) \). Therefore, \( w(x) \) also belongs to \( W^{2}_2(\Omega) \). This completes the proof. \( \blacksquare \)

We correspond the maximal operator \( B_M \) defined by the expression

\[
B_M u = \Delta \left( u - \gamma_0(u) G(x, x^0) - \sum_{s=1}^{d} \gamma_s(u) \frac{\partial}{\partial \xi_s} G(x, x^0) \right),
\]

\[
D(B_M) = \left\{ u \in W^{2}_{2, \text{loc}} (\Omega_0) : u|_{\partial \Omega} = 0, u - \gamma_0(u) G(x, x^0) - \sum_{s=1}^{d} \gamma_s(u) \frac{\partial}{\partial \xi_s} G(x, x^0) \in W^{2}_2(\Omega) \right\}.
\]

Therefore, domain of the maximal operator \( B_M \) coincides with \( W^{2}_{2,\gamma} (\Omega_0) \).

**Remark 2.2:** The maximal operator \( B_M \) defined above is a closed operator in the following sense. Indeed, for all \( n \geq 1 \), let us consider the following sequences:

\[
w_n(x) = w_{0n}(x) + \sum_{i=0}^{d} \gamma_i(w_n) \varphi_i(x) \in W^{2}_{2,\gamma} (\Omega_0),
\]

where \( w_{0n} \in W^{2}_2(\Omega) \) and \( w_{0n}|_{\partial \Omega} = 0 \), and

\[
f_n(x) = -B_M w_n(x).
\]
Suppose that $w_{0n}(x)$ converges to $v_0(x)$ and $f_n(x)$ converges to $g(x)$ in $L_2(\Omega)$. Also, assume that there exist limits
\[
\lim_{n \to \infty} \gamma_i(w_n) = c_i, \quad i = 0, 1, \ldots, d.
\]
Then $v_0 \in W^2_2(\Omega)$, $\Delta v_0(x) = g(x)$, and
\[
\lim_{n \to \infty} w_n(x) = v_0(x) + \sum_{i=0}^{d} c_i \varphi_i(x).
\]
Moreover, it is easy to see that
\[
B_M \left( v_0(x) + \sum_{i=0}^{d} c_i \varphi_i(x) \right) = -\Delta v_0(x).
\]
This shows that the operator $B_M$ is closed in the above-specified sense.

The following result could be inferred from [12, Theorem 2.1], which shows the correct restriction of the operator $B_M$. However, we explain its proof shortly for the convenience of the reader.

**Theorem 2.2.** Boundary value problem for the Poisson equation in the punctured area $\Omega_0$

\[ B_M u(x) = f(x), \quad x \in \Omega_0 \] (5)

with the external Dirichlet condition
\[ u|_{\partial \Omega} = 0 \] (6)

and with the internal boundary condition
\[ \gamma_i(u) = \gamma_i(h), \quad i = 0, 1, 2, \ldots, d \] (7)

has a unique solution $u$ in the class $W^2_{2,\gamma}(\Omega_0)$ for any $f \in L_2(\Omega)$ and $h \in W^2_{2,\gamma}(\Omega_0)$. Moreover, it is represented by the formula
\[ u(x) = \int_{\Omega} G(x, \xi) f(\xi) \, d\xi + \gamma_0(h) G(x, x^0) + \sum_{i=1}^{d} \gamma_i(h) \frac{\partial G(x, x^0)}{\partial \xi_i}. \]

**Proof:** To prove Theorem 2.2, we need to check Equation (5), the external boundary condition (6), and the internal boundary condition (7), which are easy to verify by Lemma 2.1. If $f = 0$ and $h = 0$, then it follows from the theorem on removable singularity of harmonic function [17, Theorem III. 39, p.112] that the problem (5)–(7) has a unique solution in the punctured area $\Omega_0$. ■

**Remark 2.3:** For the solvability of the problem (5)–(6), it is necessary to add $d + 1$ boundary conditions additionally. This fact was the motivation for the operator $B_M$ to be called maximal.
Define a bilinear form \( \langle f, w \rangle \) of elements \( f \in L_2(\Omega) \) and \( w \in W^{2,\gamma}_2(\Omega_0) \). For this, first we need to find a function \( v_0 \in W^{2,\gamma}_2(\Omega) \) as the solution of the Dirichlet problem

\[
\Delta v_0(x) = f(x), \quad x \in \Omega, \\
v_0|_{\partial \Omega} = 0.
\]

Then, the bilinear form \( \langle f, w \rangle \) is computed by

\[
\langle f, w \rangle \equiv \langle -\Delta v_0, w_0 + \sum_{i=0}^d \gamma_i(w)\varphi_i(x) \rangle := \langle -\Delta v_0, w_0 \rangle_{L_2(\Omega)} + \sum_{i=0}^d \gamma_i(w)\beta_i(v).
\]

Here, we use the fact that (see Lemma 2.2) every \( w \in W^{2,\gamma}_2(\Omega_0) \) has a form

\[
w(x) = w_0(x) + \sum_{i=0}^d \gamma_i(w)\varphi_i(x),
\]

where \( \varphi_0(x) = G(x, x^0), \varphi_i(x) = \frac{\partial G(x, t)}{\partial \xi_i} \big|_{t=x^0}, i = 1, \ldots, d, \) and \( w_0 \in W^{2,\gamma}_2(\Omega) \). Also note that the numbers \( \beta_0(v), \beta_1(v), \ldots, \beta_d(v) \) are defined by formulas

\[
\beta_0(v) = v_0(x^0), \quad \beta_i(v) = \frac{\partial v_0(x)}{\partial x_i} \big|_{x=x^0}, i = 1, \ldots, d.
\]

We further let

\[
\langle w, f \rangle = \langle f, w \rangle.
\]

According to Theorem 2.2, we compute the following boundary form:

\[
\langle B_M w, v \rangle - \langle w, B_M v \rangle.
\]

Similar results to the following were studied in [13, Theorem 4] (see also [11, Theorem 3]) in two-dimensional case.

**Theorem 2.3.** For any \( w, v \in W^{2,\gamma}_2(\Omega_0) \), we have

\[
\langle B_M w, v \rangle - \langle w, B_M v \rangle = \sum_{i=0}^d \gamma_i(w)\beta_i(v) - \sum_{i=0}^d \beta_i(w)\gamma_i(v),
\]

where \( \beta_0(v) = v_0(x^0), \beta_i(v) = \frac{\partial v_0}{\partial x_i} \big|_{x=x^0}, \beta_0(w) = w_0(x^0), \beta_i(w) = \frac{\partial w_0}{\partial x_i} \big|_{x=x^0}. \)

**Proof:** By Lemma 2.2, we know that every element \( w \) in \( W^{2,\gamma}_2(\Omega_0) \) is represented uniquely as

\[
w(x) = w_0(x) + \gamma_0(w)G(x, x^0) + \sum_{i=1}^d \gamma_i(w)\frac{\partial G(x, x^0)}{\partial \xi_i},
\]

where \( w_0 \in W^{2,\gamma}_2(\Omega) \) and \( w_0|_{\partial \Omega} = 0 \). Set \( \varphi_0(x) = G(x, x^0), \varphi_i(x) = \frac{\partial G(x, x^0)}{\partial \xi_i}, i = 1, 2, \ldots, d. \)

Take two arbitrary elements \( w, v \in W^{2,\gamma}_2(\Omega_0) \). We denote by \( J(w, v) \) the boundary form
\[ \langle B_Mw, v \rangle - \langle w, B_Mv \rangle. \]

Then,

\[
J(w, v) = \left( -\Delta w_0, v_0 + \sum_{i=0}^{d} \gamma_i(v) \varphi_i \right) - \left( w_0 + \sum_{i=0}^{d} \gamma_i(w) \varphi_i, -\Delta v_0 \right)
\]

\[
= \langle -\Delta w_0, v_0 \rangle - \langle w_0, -\Delta v_0 \rangle - \sum_{i=0}^{d} \gamma_i(v) \langle w_0, \Delta \varphi_i \rangle + \sum_{i=0}^{d} \gamma_i(w) \langle \varphi_i, \Delta v_i \rangle
\]

\[
= \sum_{i=0}^{d} \gamma_i(w) \beta_i(v) - \sum_{i=0}^{d} \beta_i(w) \gamma_i(v).
\]

This completes the proof. 

It follows from Theorem 2.3 that numerical vectors

\[ \Gamma_1(w) = [\gamma_0(w), \gamma_1(w), \ldots, \gamma_d(w)] \]

and

\[ \Gamma_2(w) = [\beta_0(w), \beta_1(w), \ldots, \beta_d(w)] \]

represent boundary trace of an element \( w \in W^2_{2,\gamma}(\Omega_0) \). Since \( w \) is arbitrary element from \( W^2_{2,\gamma}(\Omega_0) \), the operators \( \Gamma_1 \) and \( \Gamma_2 \) are surjective from \( W^2_{2,\gamma}(\Omega_0) \) into \( \mathbb{C}^{d+1} \). Consequently, \( \langle \mathbb{C}^{d+1}, \Gamma_1, \Gamma_2 \rangle \) represents Boundary Triples (see [2,3,6] for more details). In [3], only boundary values \( \gamma_0(w) \) and \( \beta_0(w) \) were attended. In our case, there is more complete selection of boundary values \( \Gamma_1(w) \) and \( \Gamma_2(w) \).

### 3. Correctly solvable pointwise perturbations

Define an operator \( K \) which maps the elements from the space \( L^2(\Omega) \) into \( W^2_{2,\gamma}(\Omega_0) \), which is continuous in the following sense:

(i) If the sequence of norms \( \|h_n\|_{L^2(\Omega)} \to 0 \) as \( n \to \infty \), then for any \( j = 0, 1, 2, \ldots, d \), we have \( \gamma_j(Kh_n) \to 0 \) as \( n \to \infty \).

First, we prove the following important result.

**Theorem 3.1.** For any operator

\[ K : L^2(\Omega) \to W^2_{2,\gamma}(\Omega_0), \]

which is continuous in the sense (i), the following problem of pointwise perturbation:

\[
B_Mu(x) = f(x), \quad x \in \Omega_0,
\]

\[
u|_{\partial \Omega} = 0,
\]

\[
\gamma_j(u) = \gamma_j(KB_Mu), \quad j = 0, 1, 2, \ldots, d
\]
has a unique solution in $W^2_{2,\gamma}(\Omega_0)$ for any $f \in L^2(\Omega)$, which is represented as

$$u(x) = \int_{\Omega} G(x, \xi) f(\xi) \, d\xi + \gamma_0(Kf) G(x, x^0) + \sum_{i=1}^{d} \gamma_i(Kf) \frac{\partial G(x, x^0)}{\partial \xi_i}.$$ 

**Proof:** Let $f \in L^2(\Omega)$. Suppose $h(x) = Kf(x)$. Then, applying Theorem 2.2, where we replace boundary conditions

$$\gamma_j(u) = \gamma_j(h), \quad j = 0, 1, \ldots, d$$

with conditions

$$\gamma_j(u - KBu) = 0, \quad j = 0, 1, \ldots, d,$$

we can obtain new correctly solvable perturbations of the Dirichlet problem (1)–(2). Moreover, it holds for any $K$ in Theorem 3.1.

[Discussion of Theorem 3.1.] In Theorem 3.1, an operator, which is given by the expression $\Delta - C_0(u) \delta(x - x^0) - \sum_{s=1}^{d} C_s(u) \frac{\partial}{\partial x_s} \delta(x - x^0)$, where functionals $C_j(u) := \gamma_j(KBu), j = 0, 1, 2, \ldots, d$, is well-defined. If $C_0(u) = k(u(x^0), (C_1 = \cdots = C_d \equiv 0)$, then we obtain a correct definition of the operator $\Delta + k\delta(x - x^0)$. This fact was proved by Savchuk and Shkalikov in [7], in case $d = 1$.

Let us denote by $B_K$ an operator corresponding to the boundary value problem in Theorem 3.1. Then, operator $B_0 = \Delta$ corresponds to the Dirichlet problem (1)–(2). Note that if $\gamma_s(u) = 0, s = 0, 1, 2, \ldots, d$ for $u \in W^2_{2,\gamma}(\Omega_0)$, then $Bu = B_0u$. Similarly, if $\gamma_j(u) = \gamma_j(KBu), j = 0, 1, 2, \ldots, d$, then $Bu = B_Ku$.

**Remark 3.1:** It is possible to obtain an inverse statement of Theorem 3.1, but here we do not pursue such a goal.

**Remark 3.2:** It is easy to verify that functionals $\gamma_j(Kf), j = 0, 1, 2, \ldots, d$, are linear continuous on $L^2(\Omega)$. Therefore, by the Riesz representation theorem, we have

$$\gamma_j(Kf) = \int_{\Omega} f(t) \overline{c_j(t)} \, dt,$$

where $c_j \in L^2(\Omega), j = 0, 1, 2, \ldots, d$. Thus, the domain of $B_K$ is defined as

$$D(B_K) = \left\{ u \in D(B_M) : u|_{\partial \Omega} = 0, \gamma_j(u) = \int_{\Omega} B_Mu \cdot \overline{c_j(t)} \, dt, j = 0, 1, \ldots, d \right\},$$

where $\{c_0(\cdot), \ldots, c_d(\cdot)\}$ is a system of functions from $L^2(\Omega)$. The operator, which is continuous in the sense (i), $K : L^2(\Omega) \to W^2_{2,\gamma}(\Omega_0)$ is represented as

$$Kf(x) = h_0(x) + \gamma_0(Kf) G(x, x^0) + \sum_{j=1}^{d} \gamma_j(Kf) \frac{\partial G(x, x^0)}{\partial \xi_j}, \quad h_0 \in W^2_2(\Omega).$$
If we suppose \( \gamma_s(h_0) = 0 \) for \( s = 0, 1, \ldots, d \), and by the Theorem 3.1 the values of functionals \( \gamma_s(Kf) \) are involved in the boundary conditions, it is sufficient to consider operators \( K \) as

\[
Kf(x) = C_0(f)G(x, x^0) + \sum_{j=1}^{d} C_j(f) \frac{\partial G(x, x^0)}{\partial \xi_j},
\]

where \( C_j(\cdot), j = 0, 1, 2, \ldots, d \), are linear continuous functionals on \( L_2(\Omega) \).

We denote by \( K_{d+1} \) a set of finite rank operators as in (8) for any systems

\[
\{C_0(\cdot), C_1(\cdot), \ldots, C_d(\cdot)\},
\]

where \( C_s(\cdot) \) are linear continuous functionals on \( L_2(\Omega) \). It follows from Theorem 3.1 that for different operators \( K \in K_{d+1} \) we correspond different operators \( B_K \). Hence, a family \( \{B_K\} \) of operators \( B_K \) can be parameterized by a parameter of an operator \( K \in K_{d+1} \).

If \( c_j(\cdot) \in W^2_2(\Omega) \) and \( c_j(x^0) = 0 \), \( \frac{\partial c_j(x^0)}{\partial x_s} = 0 \), \( s = 1, \ldots, d \), then the domain of the operator \( B_K \) takes the form

\[
D(B_K) = \left\{ u \in D(B_M) : |u|_{\partial \Omega} = 0, \gamma_j(u) = \int_\Omega u(t) \cdot \Delta c_j(t) \, dt - \int_{\partial \Omega} \frac{\partial u}{\partial v_t} \cdot c_j(t) \, dt + \gamma_0(u) \int_{\partial \Omega} \frac{\partial G}{\partial v_t} \cdot c_j(t) \, dt \right\}.
\]

Moreover, if \( \Delta c_j(t) = 0 \) in \( \Omega \), then

\[
D(B_K) = \left\{ u \in D(B_M) : |u|_{\partial \Omega} = 0, \gamma_j(u) = - \int_{\partial \Omega} \frac{\partial u}{\partial v_t} \cdot c_j(t) \, dt + \gamma_0(u) \int_{\partial \Omega} \frac{\partial G}{\partial v_t} \cdot c_j(t) \, dt \right\}.
\]

If we suppose \( c_0(t) = k \cdot G(x, x^0), c_j(t) \equiv 0, j = 1, \ldots, d \), then the domain of the operator \( B_K \) has the form

\[
D(B_K) = \left\{ u \in D(B_M) : u|_{\partial \Omega} = 0, \gamma_0(u) = k \cdot \lim_{x \to x_0} (u - \gamma_0(u)G(x, x^0)), \gamma_j(u) = 0, j = 1, \ldots, d \right\}.
\]

Consequently, in this case the operator \( B_K \) generated by the differential expression \( \Delta + k\delta(x - x^0) \) is defined correctly. Moreover, by Theorem 3.1 such operator is invertible from \( L_2(\Omega) \) into \( D(B_K) \).
**Definition 3.1:** An invertible restriction $B_K$ of the maximal operator $B_M$ is called self-adjoint respect to the boundary form

$$\langle B_M w, v \rangle - \langle w, B_M v \rangle, \quad \forall \ w, v \in D(B_M), \tag{9}$$

if for all $w$ and $v$ in $D(B_K)$ the following equality holds:

$$\langle B_K w, v \rangle = \langle w, B_K v \rangle.$$

We select self-adjoint operators respect to the previous boundary form (9) from the set of invertible restrictions $B_K$ which are described in Theorem 3.1.

**Theorem 3.2.** Let $\alpha$ be a set consisting of complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_d$. An operator $B_\alpha$ corresponding to the boundary value problem

$$B_M w(x) = f(x), \quad x \in \Omega_0,$$

$$w|_{\partial \Omega} = 0,$$

$$\gamma_i(w) = \alpha_i \cdot \beta_i(w), \quad j = 0, 1, 2, \ldots, d$$

is an invertible restriction of the maximal operator $B_M$.

Moreover, the operator $B_\alpha$ is self-adjoint respect to the boundary form (9).

**Proof:** Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$ be a set of complex numbers. Define an operator $K$ in the following form:

$$Kf(x) = \sum_{i=0}^{d} \alpha_i \langle f, \phi_i \rangle \phi_i(x).$$

Then, the correct restriction $B_K$ coincides with the operator $B_\alpha$. Consequently, $B_\alpha \subset B_M$ and there exists $B_{\alpha^{-1}}$. Now, we need to show that $B_\alpha$ is a self-adjoint operator in the sense of Definition 3.1. According to Theorem 2.3, we have

$$\langle B_M w, v \rangle - \langle w, B_M v \rangle = \sum_{i=0}^{d} \gamma_i(w) \overline{\beta_i(v)} - \sum_{i=0}^{d} \beta_i(w) \overline{\gamma_i(v)}.$$

If we put the operator $B_K$ instead of $B_M$ in the previous formula, then we obtain

$$\langle B_\alpha w, v \rangle - \langle w, B_\alpha v \rangle = 0,$$

for all $w, v \in D(B_\alpha)$. This concludes the proof. ■

**Remark 3.3:** In [3,6], it was described self-adjoint extensions of the minimal operator which can be both invertible and non-invertible operators. Only invertible self-adjoint operators are selected in Theorem 3.2.
4. Resolvents of correctly solvable pointwise perturbations

In this section, we write out directly a representation of resolvents of boundary value problems from Theorem 3.1. The results similar to can be found from [12, Theorem 2.4], [13, Theorem 2], [15, Theorem 2], and [11, Theorem 3]. Since our purpose is a bit different and we need it for further investigations, we show its proof.

**Theorem 4.1.** Let us given a linear continuous operator

\[ K : L^2(\Omega) \to W^{2,\gamma}_0(\Omega_0). \]

Then, the following problem of pointwise perturbation:

\[
(B_M - \lambda)u(x) = f(x), \quad x \in \Omega_0, \\
u|_{\partial \Omega} = 0, \\
g_j(u) = g_j(KB_Mu), \quad j = 0, 1, 2, \ldots, d
\]

has a unique solution in the space \( W^{2,\gamma}_0(\Omega_0) \) for any \( f \in L^2(\Omega) \) and for any complex-valued spectral parameter \( \lambda \), save possibly for some countable set.

Moreover, for the resolvent \((B_K - \lambda I)^{-1}\), we have

\[
(B_K - \lambda I)^{-1}f(x) = (B_0 - \lambda I)^{-1}f(x) + g_0 \left( KB_0 (B_0 - \lambda I)^{-1}f \right) B_K (B_K - \lambda I)^{-1}G(x,x^0) + \sum_{j=1}^{d} g_j (KB_0 (B_0 - \lambda I)^{-1}f) B_K (B_K - \lambda I)^{-1} \frac{\partial G(x,x^0)}{\partial t_j}. \tag{10}
\]

**Proof:** We have,

\[ B_0(B_0 - \lambda I)^{-1} = I + \lambda (B_0 - \lambda I)^{-1}. \]

Similarly, we obtain

\[ B_K(B_K - \lambda I)^{-1} = I + \lambda (B_K - \lambda I)^{-1}. \]

Set

\[
u_0(x) = (B_0 - \lambda I)^{-1}f(x), \\
T_0(x) = (B_K - \lambda I)^{-1}G(x,x^0), \\
T_j(x) = (B_K - \lambda I)^{-1} \frac{\partial G(x,x^0)}{\partial t_j}.
\]

Define a function \( u \) by the formula

\[
u(x) = \nu_0(x) + g_0(K(f + \lambda \nu_0(x)))(G(x,x^0) + \lambda T_0(x)) + \sum_{j=1}^{d} g_j(K(f + \lambda \nu_0(x)))(\frac{\partial G(x,x^0)}{\partial t_j} + \lambda T_j(x)). \tag{11}
\]
Since \( u_0 \in W^2_2(\Omega) \), \( T_j \in D(B_K) \), \( j = 0, 1, \ldots, d \), it follows that \( B_Mu_0 = B_0u_0 \), \( B_MT_j = B_KT_j \), and hence,

\[
B_Mu(x) - \lambda u(x) = B_0u_0(x) - \lambda u_0(x) + \gamma_0(K(f + \lambda u_0(x)))(B_M(G(x,x^0)) - \lambda G(x,x^0) + \lambda(B_K(T_0) - \lambda T_0(x)))
\]

\[
+ \sum_{j=1}^{d} \gamma_j(K(f + \lambda u_0(x))) \left( B_M \left( \frac{\partial G(x,x^0)}{\partial t_j} \right) - \lambda \frac{\partial G(x,x^0)}{\partial t_j} \right)
\]

\[
+ \lambda(B_K(T_j(x)) - \lambda T_j(x)) \right).
\]

Since

\[
B_0u_0(x) - \lambda u_0(x) = f(x),
\]

\[
B_M(G(x,x^0)) = 0,
\]

\[
B_M \left( \frac{\partial G(x,x^0)}{\partial t_j} \right) = 0,
\]

\[
B_K(T_j(x)) - \lambda T_j(x) = \frac{\partial G(x,x^0)}{\partial t_j},
\]

\[
B_K(T_0(x)) - \lambda T_0(x) = G(x,x^0),
\]

we obtain the required formula

\[
B_Mu(x) - \lambda u(x) = f(x).
\]

Now, we compute the trace of the function \( u \) on the external boundary \( \partial \Omega \). Note that functions \( u_0(x), G(x,x^0) \) are solutions of the Dirichlet problem and their traces on the external boundary \( \partial \Omega \) equal to zero. Then, the trace of function \( \frac{\partial G(x,x^0)}{\partial t_j} \) also vanishes on the external boundary \( \partial \Omega \). Since \( u \) is a linear combination of above functions, it follows that its trace is equal to zero on \( \partial \Omega \). It remains to calculate the values of the functionals \( \gamma_j(u - KB_Mu) \), \( j = 0, 1, \ldots, d \). It is clear that \( \gamma_j(w) = \gamma_j(KBMw) \), \( j = 0, 1, \ldots, d \) when \( w(x) = T_j(x) \), since \( T_j \in D(B_K) \). Similarly, we have \( \gamma_j(u_0) = 0 \), since \( u_0 \in W^2_2(\Omega) \). Applying functionals \( \gamma_s, s = 0, 1, \ldots, d \), to the both sides of (11) and by the preceding, we obtain

\[
\gamma_s(u) = \gamma_0(K(f + \lambda u_0))\delta_{0,s} + \delta_{\mu} \sum_{j=1}^{d} \gamma_j(K(f + \lambda u_0))
\]

\[
+ \lambda(\gamma_0(K(f + \lambda u_0))\gamma_s(KBT_0) + \sum_{j=1}^{d} \gamma_j(K(f + \lambda u_0))\gamma_s(KBT_j)).
\]

On the other hand, since \( B_MT_j = B_KT_j \), and

\[
\gamma_s(BMU) = \gamma_s(K(f + \lambda u_0)) + \lambda(\gamma_0(K(f + \lambda u_0)))\gamma_s(KBT_0)
\]
thereby completing the proof.  

**Remark 4.1:** Theorem 4.1 says that to calculate the resolvent \((B_K - \lambda I)^{-1}\) for any \(f\), we need to know the values of \((B_K - \lambda I)^{-1}\) at fixed elements \(G(x, x^0), \frac{\partial G(x, x^0)}{\partial t_j}, j = 1, 2, \ldots, d\).

**Remark 4.2:** Formulas (10) represent a generalization of the second Hilbert identity for the resolvent in the case, when \(D(B_K) \neq D(B_0)\). Similar identities to (10) were studied in [18].

In particular, Theorem 4.1 implies that the difference of resolvents \((B_K - \lambda I)^{-1} - (B_0 - \lambda I)^{-1}\) is a finite rank operator, therefore, there exists finite trace

\[
Tr\left((B_K - \lambda I)^{-1} - (B_0 - \lambda I)^{-1}\right) = \sum_{i=0}^{d} \gamma_i (KB_0(B_0 - \lambda I)^{-1}B_K(B_K - \lambda I)^{-1}\varphi_i),
\]

where

\[
\varphi_0(x) = G(x, x^0), \quad \varphi_i(x) = \frac{\partial G(x, x^0)}{\partial t_i}, \quad i = 1, \ldots, d.
\]

We will define below meromorphic function \(\Delta(\lambda)\) and prove an analogue of Krein’s formula [19,20]. Set

\[
\beta_{00} = \gamma_0 (KB_0(B_0 - \lambda I)^{-1}\varphi_0), \quad \beta_{j0} = \gamma_j (KB_0(B_0 - \lambda I)^{-1}\varphi_0),
\]

\[
\beta_{0s} = \gamma_0 (KB_0(B_0 - \lambda I)^{-1}\varphi_s), \quad \beta_{js} = \gamma_j (KB_0(B_0 - \lambda I)^{-1}\varphi_s) .
\]

Then we define characteristic determinant by the following formula:

\[
\Delta(\lambda) = (-1)^{d+1} \begin{vmatrix}
\lambda_\beta_{00} - 1 & \lambda_\beta_{01} & \cdots & \lambda_\beta_{0d} \\
\lambda_\beta_{10} & \lambda_\beta_{11} - 1 & \cdots & \lambda_\beta_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_\beta_{d0} & \lambda_\beta_{d1} & \cdots & \lambda_\beta_{dd} - 1
\end{vmatrix}_{d \times d}.
\]

The determinant \(\Delta(\lambda)\) is a perturbation determinant (see [20, Chapter IV, p.156]).

**Theorem 4.2.** Let \(B_K\) and \(B_0\) be operators defined as in Theorem 4.1. We have

\[
Tr\left((B_K - \lambda I)^{-1} - (B_0 - \lambda I)^{-1}\right) = -\frac{d}{d\lambda} \ln \left(\Delta(\lambda)\right).
\]
Proof: Let us define functions

\[ \psi_s(x) = B_K (B_K - \lambda I)^{-1} \varphi_s \]

for any \( s = 0, 1, \ldots, d \). It is easy to see that for a fixed \( s \) the function \( \psi_s \) is a solution of the following problem:

\[ B_M \psi_s(x) = \lambda \psi_s(x), \quad \gamma_j(\psi_s) - \gamma_j(KB_M \psi_s) = \delta_{js}, \quad j = 0, 1, \ldots, d. \]

Let us compute \( \psi_s(x) \). For this, we define a vector-column

\[ \overrightarrow{P}(x) = (B_0(B_0 - \lambda I)^{-1} \varphi_0(x), \ldots, B_0(B_0 - \lambda I)^{-1} \varphi_d(x))^T. \]

Fix \( s \) and denote by \( M_s(x) \) the value of the determinant, which is obtained from the determinant \( \Delta(\lambda) \) by replacing \( s \)'th column with the column \( \overrightarrow{P}(x) \). It is easy to verify that \( M_s(x) \) is a solution of the equality

\[ B_M M_s(x) = \lambda M_s(x). \]

To compute the difference \( \gamma_i(M_s) - \gamma_j(KB_M M_s) \), first we consider

\[
\begin{align*}
\gamma_i(B_0(B_0 - \lambda I)^{-1} \varphi_s) - \gamma_i(KB_0(B_0 - \lambda I)^{-1} \varphi_s) \\
= \gamma_i(\varphi_s) + \lambda \gamma_i((B_0 - \lambda I)^{-1} \varphi_s) \\
- \gamma_i(KB_0 \varphi_s) - \lambda \gamma_i(KB_0(B_0 - \lambda I)^{-1} \varphi_s) \\
= \delta_{is} - \lambda \gamma_i(KB_0(B_0 - \lambda I)^{-1} \varphi_s) = \delta_{is} - \lambda \beta_{is}.
\end{align*}
\]

Obviously, \( \gamma_i(M_s) - \gamma_j(KB_M M_s) = -\Delta(\lambda) \delta_{is} \). Consequently, we have

\[ \psi_s(x) = -\frac{M_s(x)}{\Delta(\lambda)}. \quad (14) \]

After some adjustment of (12), we obtain

\[ Tr((B_K - \lambda I)^{-1} - (B_0 - \lambda I)^{-1}) = \sum_{i=0}^{d} \gamma_i(KB_0(B_0 - \lambda I)^{-1} \varphi_i). \]

It follows from (14) that

\[ Tr((B_K - \lambda I)^{-1} - (B_0 - \lambda I)^{-1}) = -\frac{1}{\Delta(\lambda)} \sum_{i=0}^{d} \gamma_i(KB_0(B_0 - \lambda I)^{-1} M_i). \quad (15) \]

Let us define a column vector

\[ \overrightarrow{V}(x) = (\varphi_0(x), \ldots, \varphi_d(x))^T. \]

For fix \( s \) again we denote by \( N_s(x) \) the value of the determinant, which is obtained from the determinant \( \Delta(\lambda) \) by replacing \( s \)'th column with the column \( \overrightarrow{V}(x) \). Since \( M_s = B_0(B_0 - \lambda I)^{-1} \varphi_s \) for any \( s = 0, 1, \ldots, d \).
\(\lambda I^{-1} N_s\), it follows from (15) that

\[
\text{Tr}((B_K - \lambda I)^{-1} - (B_0 - \lambda I)^{-1}) = -\frac{1}{\Delta(\lambda)} \sum_{i=0}^{d} \gamma_i (KB_0^2 (B_0 - \lambda I)^{-2} N_i).
\]

Finally, since

\[
\frac{d}{d\lambda} \left( \lambda B_0 (B_0 - \lambda I)^{-1} \right) = \left( B_0 (B_0 - \lambda I)^{-1} \right)^2,
\]

we have

\[
\frac{d}{d\lambda} \Delta(\lambda) = \sum_{i=0}^{d} \gamma_i (KB_0^2 (B_0 - \lambda I)^{-2} N_i).
\]

Then, the combination of (16) and (17) gives us the required formula (13) of M.G. Krein. This completes the proof. \(\blacksquare\)

Let \(f \in L_2(\Omega)\), then it is convenient to define the following expressions:

\[
\beta_0(f) = \gamma_0 (KB_0 (B_0 - \lambda I)^{-1} f), \quad \beta_j(f) = \gamma_j (KB_0 (B_0 - \lambda I)^{-1} f),
\]

\[
F(f) = \beta_0(f) G(x,x^0) + \sum_{j=1}^{d} \beta_j(f) \frac{\partial G(x,x^0)}{\partial t_j},
\]

\[
F_0 = (B_0 - \lambda I)^{-1} G(x,x^0) + \beta_{00} G(x,x^0) + \sum_{j=1}^{d} \beta_{0j} \frac{\partial G(x,x^0)}{\partial t_j},
\]

\[
F_s = (B_0 - \lambda I)^{-1} \frac{\partial G(x,x^0)}{\partial t_s} + \beta_{0s} G(x,x^0) + \sum_{j=1}^{d} \beta_{js} \frac{\partial G(x,x^0)}{\partial t_j},
\]

\[
Q(f,x,\lambda) = \begin{pmatrix}
\beta_0(f) & \beta_1(f) & \cdots & \beta_d(f) & F(f) \\
\lambda \beta_{00} - 1 & \lambda \beta_{10} & \cdots & \lambda \beta_{d0} & F_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda \beta_{0d} & \lambda \beta_{1d} & \cdots & \lambda \beta_{dd} - 1 & F_d 
\end{pmatrix}.
\]

The next theorem gives a representation of the resolvent \((B_K - \lambda I)^{-1}\).

**Theorem 4.3.** Let the assumptions of Theorem 4.1 hold. Then, the resolvent \((B_K - \lambda I)^{-1}\) is a finite dimensional perturbation of the resolvent \((B_0 - \lambda I)^{-1}\), which is represented by the following formula:

\[
(B_K - \lambda I)^{-1} f(x) = (B_0 - \lambda I)^{-1} f(x) + \frac{Q(f,x,\lambda)}{\Delta(\lambda)}.
\]

**Proof:** In Theorem 4.1, it is obtained a representation of the resolvent \((B_K - \lambda I)^{-1}\). We substitute consecutively \(G(x,x^0), \frac{\partial G(x,x^0)}{\partial t_s}, s = 1, \ldots, d\), instead of \(f(x)\) in the above
representation. As a result, we obtain a system of matrix-vector equations
\[ D \vec{P} = 0, \]
where
\[ \vec{P} = \left((B_K - \lambda I)^{-1}\varphi_0, \ldots, (B_K - \lambda I)^{-1}\varphi_d, -1\right)^T, \]
\[ D = \begin{pmatrix}
\beta_0(f) & \beta_1(f) & \cdots & \beta_d(f) & F(f) \\
\lambda\beta_{00} - 1 & \lambda\beta_{10} & \cdots & \lambda\beta_{d0} & F_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda\beta_{0d} & \lambda\beta_{1d} & \cdots & \lambda\beta_{dd} - 1 & F_d
\end{pmatrix}. \]

Since the previous system of linear algebraic equations has a non-trivial solution, the determinant of the system equals to zero, i.e. \( \det D = 0 \). This implies
\[ (B_K - \lambda I)^{-1}f(x) = (B_0 - \lambda I)^{-1}f(x) + \frac{Q(f, x, \lambda)}{\Delta_1(\lambda)}, \]
which represents a generalization of the well-known Hilbert’s identity for the resolvent. This concludes the proof. \( \blacksquare \)

\textbf{Remark 4.3:} Theorem 4.3 represents a generalization of 2nd Hilbert’s identity for the resolvent, whenever operators \( B_K \) and \( B_0 \) have different domains.

\textbf{Remark 4.4:} Since \( \beta_{sk}, F_s \) depend on a spectral parameter \( \lambda \) in a meromorphic way, it follows that \( \Delta(\lambda) \) is a meromorphic function of \( \lambda \). Consequently, the resolvent \( (B_K - \lambda I)^{-1} \) is also a meromorphic operator function and its number of poles at most countable. This fact agrees with the statement of Theorem 4.1.

Formula (10) can be generalized in the following way. For \( s = 1, \ldots, d \), we denote by \( B_s \) an operator corresponding to the boundary value problem
\[ (B_M - \lambda I)u = f, \]
\[ u|_{\partial\Omega} = 0, \]
\[ \gamma_j(u) = \gamma_j(KBMu), \quad j = 0, 1, \ldots, s - 1, \]
\[ \gamma_k(u) = 0, \quad k = s, s + 1, \ldots, d. \]

Then, we can obtain the following formula for the resolvent of the operator \( B_s \).
\[ (B_s - \lambda I)^{-1}f = (B_{s-1} - \lambda I)^{-1}f + \gamma_{s-1}(KB_{s-1}(B_{s-1} - \lambda I)^{-1}f) \frac{\theta_{s-1}}{1 - \lambda \gamma_{s-1}(K\theta_{s-1})}, \quad (18) \]
where \( \theta_s = B_s(B_s - \lambda I)^{-1}q_s, \varphi_0(x) = G(x, x^0), \varphi_s(x) = \frac{\partial G(x, x^0)}{\partial \xi}, \quad s = 1, 2, \ldots, d. \) In case \( s = d \), this implies equality \( B_K = B_d \). It follows from (18), \( s = 1, \ldots, d \), that operators \( B_1, \ldots B_{d-1} \) can be excluded and we obtain formula (10).
5. Change of initial part of spectrum of the Laplace operator

In this section, we study some spectral properties of correctly solvable pointwise perturbations. Resolvents of such operators described in Theorem 4.3. In Theorem 4.3, the perturbation determinant \( \Delta(\lambda) \) has appeared. Perturbation determinants were studied in [20, Chapter IV, p.156]. We consider operator \( B_K \) as a perturbation of the operator \( B_0 \) and the operator \( B_0 = \Delta \) corresponds to the Dirichlet problem (1) and (2). It is well known that \( B_0 \) is a self-adjoint operator with the discrete spectrum. We denote the eigenvalues of the operator \( B_0 \) and the corresponding eigenvectors by \( \{\mu_n\}_{n \geq 1} \) and \( \{\omega_n(x)\}_{n \geq 1} \), respectively. It is known that the system of eigenvectors \( \{\omega_n(x)\}_{n \geq 1} \) forms orthonormal basis in the space \( L_2(\Omega) \). Then, the elements \( \beta_{ij} \) of perturbation determinant \( \Delta(\lambda) \) are identified by formula

\[
\beta_{ij} = \gamma_1(KB_0(B_0 - \lambda I)^{-1} \varphi_j) = \sum_{n=1}^{\infty} \frac{\mu_n\langle \varphi_j, \omega_n \rangle}{\mu_n - \lambda} \cdot C_{in}, \tag{19}
\]

where \( C_{in} = \gamma_1(K\omega_n) \).

**Proposition 5.1.** For all \( n \in \mathbb{N} \), we have

\[
\langle \varphi_0, \omega_n \rangle = \frac{2\omega_n(x^0)}{\mu_n}.
\]

**Proof:** Since

\[
\lim_{\delta \to +0} \int_{\partial \Omega_0^0} \frac{\partial \varphi_0}{\partial v_\xi} \, dS_\xi = 1
\]

and

\[
\lim_{\delta \to +0} \int_{\partial \Omega_0^0} \frac{(\xi_j - x^0_j)}{|\xi - x^0|} \varphi_0 \, dS_\xi = 0,
\]

it follows that

\[
\langle \varphi_0, \omega_n \rangle = \lim_{\delta \to +0} \int_{\Omega/\Pi_0^0} \varphi_0 \omega_n(x) \, dx = \lim_{\delta \to +0} \frac{1}{\mu_n} \int_{\Omega/\Pi_0^0} \varphi_0 \frac{\partial \omega_n}{\partial v_\xi} \, dS_\xi + \lim_{\delta \to +0} \frac{1}{\mu_n} \int_{\partial \Pi_0^0} \omega_n(\xi) \frac{\partial \varphi_0}{\partial v_\xi} \, dS_\xi
\]

\[
= \frac{\omega_n(x^0)}{\mu_n} - \sum_{j=1}^{d} \frac{\partial \omega_n(x^0)}{\partial \xi_j} \frac{1}{\mu_n} \lim_{\delta \to +0} \int_{\partial \Pi_0^0} \frac{(\xi_j - x^0_j)}{|\xi - x^0|} \varphi_0 \, dS_\xi
\]

\[
= \frac{\omega_n(x^0)}{\mu_n} + \frac{\omega_n(x^0)}{\mu_n} \lim_{\delta \to +0} \int_{\partial \Omega_0^0} \frac{\partial \varphi_0}{\partial v_\xi} \, dS_\xi = \frac{2\omega_n(x^0)}{\mu_n}.
\]

This concludes the proof. ■
For all $s = 1, 2, \ldots, d$, we define operators whose resolvents are computed recurrently by

$$(B_s - \lambda I)^{-1}f = (B_{s-1} - \lambda I)^{-1}f + \gamma_{s-1}(KB_{s-1}(B_{s-1} - \lambda I)^{-1}f) \cdot \frac{\theta_{s-1}}{1 - \lambda \gamma_{s-1}(K\theta_{s-1})},$$

where $\theta_i = B_i(B_i - \lambda I)^{-1}\varphi_i$, $i = 0, 1, \ldots, d$. For $s = 1$, define a perturbation determinant

$$\Delta_{01}(\lambda) = 1 - \lambda \gamma_0(K\theta_0) = 1 - \lambda \gamma_0(KB_0(B_0 - \lambda I)^{-1}\varphi_0).$$

Since

$$\Delta_{01}(\lambda) = 1 - \lambda \sum_{n=1}^{\infty} \frac{\mu_n\langle \varphi_0, \omega_n \rangle}{\mu_n - \lambda} \cdot C_0n,$$

it follows from Proposition 5.1 that

$$\Delta_{01}(\lambda) = 1 - 2\lambda \sum_{n=1}^{\infty} \frac{\omega_n(x^0)}{\mu_n - \lambda} \cdot C_0n. \quad (20)$$

Formula (20) implies the following result. In this result, we use the approach in the proof of the main results in [21], which are used for ordinary differential operators.

**Theorem 5.1.** If one of the conditions $\omega_N(x^0) = 0$ or $\gamma_0(K\omega_N) = 0$ holds for some $N$, then we have

$$\lambda_N(B_1) = \mu_N,$$

where $\lambda_N(B_1)$ is the $N$’th eigenvalue of the operator $B_1$.

**Proof:** First, we prove the case, when the eigenvalues $\mu_N$ of the operator are simple. The proof below will require slight modification for the case of multiple eigenvalues $\mu_N$. Since

$$(B_1 - \lambda I)^{-1}f(x) = (B_0 - \lambda I)^{-1}f(x) + \gamma_0(KB_0(B_0 - \lambda I)^{-1}f) \cdot \frac{\theta_0(x)}{\Delta_{01}(\lambda)},$$

it follows from $\omega_N(x^0) = 0$ that the residue is calculated by formula

$$\text{res}_{\mu_N}(B_1 - \lambda I)^{-1}f = -\langle f, \omega_N \rangle \omega_N(x) + \frac{\theta_0(x, \lambda)}{\Delta_{01}(\lambda)} \bigg|_{\lambda = \mu_N} \cdot \mu_N(f, \omega_N) \cdot C_0N.$$

This implies that $\lambda = \mu_N$ is an eigenvalue of the operator $B_1$ corresponding to the following eigenvector:

$$\omega_N(x) + \frac{\mu_N C_0N}{\Delta_{01}(\mu_N)} \cdot \theta_0(x, \mu_N).$$

If $\gamma_0(K\omega_N) = 0$, the residue is calculated by formula

$$\text{res}_{\mu_N}(B_1 - \lambda I)^{-1}f = -\langle f, \omega_N \rangle \omega_N(x) + \frac{\gamma_0(KB_0(B_0 - \lambda I)^{-1}f)}{\Delta_{01}(\lambda)} \bigg|_{\lambda = \mu_N} \cdot \omega_N(x^0) \omega_N(x).$$
It follows from the last formula that $\lambda = \mu_N$ is again an eigenvalue of the operator $B_1$ corresponding to the eigenvector $\omega_N(x)$. This completes the proof. ■

**Remark 5.1:** Theorem 5.1 can be easily reformulated for the pair of $B_{s-1}$ and $B_s$.

**Remark 5.2:** It follows from Theorem 5.1 that the operator $B_1$ is not a self-adjoint operator as $B_0$.

Next, we give an example of a boundary problem in a punctured domain, when the spectrum of the Dirichlet problem remains unchanged.

**Example 5.1:** If the operator $K$ maps all eigenvectors $\omega_n(x)$ of the operator $B_0$ to the linear combination of derivatives $\frac{\partial}{\partial \xi_j} G(x, x^0)$, $j = 1, 2, \ldots, d$, then the assumptions of Theorem 5.1 hold. In other words, we have $C_{0n} = 0$ for all $n$. Consequently, the spectrum of the operator $B_1$ coincides with the spectrum of $B_0$.

This example can be easily modified so that only a finite number of eigenvalues of the operator $B_1$ differs from those of $B_0$. Similar examples for the Sturm–Liouville operators are usually studied by the method of Crum [22], whenever the only finite part of the spectrum is changed.

Note that the regularized trace of perturbed operator $B_1$ in Example 5.1 equals to zero. Regularized traces have been studied by many authors, in particular, [23,24]. For the perturbation in the Example 5.1, the conditions of results of [23, Theorem 1] (see also [24]), in which their results are valid, are violated.

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