We consider a brane world and its gravitational linear perturbations. We present a general solution of the perturbations in the bulk and find the complete perturbed junction conditions for generic brane dynamics. We also prove that (spin 2) gravitational waves in the great majority of cases can only arise in connection with a non-vanishing anisotropic stress. This has far reaching consequences for inflation in the brane world. Moreover, contrary to the case of the radion, perturbations are stable.

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I. INTRODUCTION

The brane world scenario is a new way of looking at the Universe from the point of view of string/membrane theory, which has the advantage of offering the opportunity of comparing prediction to observational results, an aim that was beyond the scope of this area of theoretical physics for more than two decades [1, 2, 3].

String theory is naturally defined in a higher dimensional space-time. It has been shown that string theory in various dimensions display very natural behaviour under duality, which leads to the result that there is a theory in 11 dimensions from which all string theories derive in a natural way [4].

It is thus by now a widespread idea, from a general theoretical setup, that the so-called M-theory [5] is a reasonable description of our Universe: in the field theory limit, it is described by a solution of the (eventually 11-dimensional) Einstein equations with a cosmological constant, by means of a four dimensional membrane. In this picture only gravity survives in the higher dimensions, while the remaining matter and gauge interactions are typically four dimensional. This realizes the idea that the real world is a membrane imbeded in a higher dimensional space-time, such that the Standard model fields live in the membrane, while gravity can enter into the bulk of the extra dimensions [6].

Randall and Sundrum [6] proposed a model of that type, namely there exists a Universe described inside a membrane immersed in a higher dimensional space-time, such that a so-called warp in the extra dimensions prevents information to leak at large quantities outside of the real world.

A very large amount of new physics emerges. In particular, when membranes are solutions of Einstein’s equations and matter fields reside inside the brane, the gravitational fields have to obey the Israel conditions [7] at the sides of the brane. Thus, there is a possibility that gravitational fields propagating out of the brane speed up, reaching farther distances as compared to light propagating inside the brane, a scenario that for a resident of the brane (such as ourselves) implies shortcuts [8, 9, 10, 11, 12]. Moreover, the fact that the world is higher dimensional, together with general properties of quantum field theory and further input such as the holographic principle, indicate that further conclusions can be drawn [13, 14, 15, 16].

In particular, due to the presence of the extra dimension, it is very difficult to make predictions about the cosmological consequences by studying the cosmological perturbations in the brane world. As the simplest case, the cosmological perturbations were investigated by neglecting the non-trivial evolution of perturbations in the bulk [17]. In order to solve the perturbation equations including the bulk, a simplified inflation model, in which de Sitter stage of inflation is instantaneously connected to Minkowski space, was considered in the study of gravitational wave perturbations [18]. Further applications of the approach of employing conformally minkowskian coordinates, Derulle et al [20] disentangled the contributions of the bulk gravitons and of the motion of the brane, found the restrictions of the bulk gravitons when matter on the brane is taken to be scalar and solved analytically the brane perturbation equations. In this paper we are going to generalize

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Deruelle et al’s work with a full treatment of the junction conditions leading to the equations of motion on the brane, and making direct connection with the standard perturbation theory.

Our aim here is to further develop these ideas by the description of gravitational waves generated in early stages of the Universe. These are important tools for the understanding of the model, since they can actually be observed in the near future. They can also leave their imprints in the background radiation and possibly be described by the results of the Planck satellite.

II. THE SCENARIO

We shall thus consider gravitational perturbations of a five dimensional bulk where our Universe is described by a four dimensional membrane with matter fields. We consider a scenario where the unperturbed bulk is a purely Anti-de-Sitter space-time described by a metric conformally equivalent to minkowskian coordinates, that is, \[20\],

\[
ds^2 = g_{AB}dx^A dx^B = \frac{l^2}{(X^4)^2} \eta_{AB}dX^A dX^B,
\]

with \(l\) a length scale later called the Randall-Sundrum scale and \(\eta_{AB}\) the Minkowski metric. The notation is described in the appendix A.

We define the brane by a hyper-surface moving in the bulk with \(X^0 = X^0(\eta) = T(\eta), X^i = X^i = x^i, X^4 = X^4(\eta) = A(\eta)\), where \(\eta\) is defined by \(T'(\eta) = \sqrt{1 + (A'(\eta))^2}\).

Parametrized in this way, the induced metric on the brane reads

\[
d\bar{s}^2 = \left(\frac{l}{A(\eta)}\right)^2 (-d\eta^2 + \delta_{ij} dx^i dx^j).
\]

Thus, \(\eta\) is the conformal time of a FRW brane with the scale-factor given by \(a(\eta) = l/A(\eta)\).

The components of the tangent vectors are \(\bar{V}_0^A = \frac{\partial X^A}{\partial \eta} = (T'(\eta), 0, 0, 0, A'(\eta))\) and \(\bar{V}_i^A = \frac{\partial X^A}{\partial x^i} = (0, \delta^A_i, 0)\), while the normal to the brane is \(\bar{n}^A = \left(A''(\eta), 0, 0, 0, A'(\eta)'\right)\). It obeys the normalization condition \(g_{AB}[\bar{n}^A \bar{n}^B] = 1\).

The energy-momentum tensor \(\Pi_{AB}\) in the bulk is just a negative cosmological constant. Therefore \(\Pi_{AB} \bar{V}^A \bar{V}^B = 0\) on the brane. It can thus be shown that the junction conditions for a \(Z_2\) symmetric brane imply the conservation of the brane energy-momentum,

\[
\bar{T}^\mu_{\nu\mu} = 0.
\]

On the other hand, the spatial part of the junction conditions \[7\] reads

\[
\frac{\kappa}{2} \left( \bar{T}^i_j - \frac{1}{3} \delta^i_j \bar{T} \right) = \bar{K}^i_j,
\]

where \(\bar{T}^\mu_{\nu}\) is the brane energy-momentum tensor and \(\bar{K}^\mu_{\nu}\) is the second fundamental form calculated on the brane,

\[
\bar{K}_{\mu\nu} = -\bar{V}_{(\mu} \nabla_{\nu)}A \bar{n}^B |_{\bar{n}}.
\]

For reference, we list the non-zero results for the background second fundamental form on the brane, \(\bar{K}_{\mu\nu} = \left(\frac{l}{A^2} T^{\nu\sigma} A A^{\sigma} - T^{\nu\sigma}\right)\) and \(\bar{K}_{ij} = \frac{l T^{\sigma\delta} \delta_{ij}}{A^2}\).

For an isotropic and homogeneous distribution of matter in the brane with a tension \(\sigma\), we get the usual energy conservation equation \(\frac{d\rho}{d\tau} + 3H(\rho + P) = 0\), where \(d\tau = a(\eta) d\eta\) and \(H = a/a = \frac{d\omega}{a^2} = d'(\eta)/a^2 = -A'(\eta)/l\), and, from the spatial part we obtain the modified Friedmann equation

\[
H^2 = \kappa^2 (\rho + \sigma)^2 - \frac{1}{l^2} = \left(\kappa^2 \sigma^2 - \frac{1}{l^2}\right) + 6\kappa^2 \sigma \left(\frac{\rho}{3} + \frac{\rho^2}{6\sigma}\right).
\]

This sets the whole unperturbed scenario. The next step is to consider the perturbations and the subsequent wave equations.
III. PERTURBATION

Since perturbations are being treated linearly, it is possible to separate the bulk perturbation from the perturbation of the position of the brane and of the junction conditions, what we prefer to do for the sake of simplicity.

A. Bulk Perturbation

We first perturb the AdS bulk space-time. For general perturbations in conformally minkowskian coordinates the metric reads

$$ds^2 = \left(\frac{1}{X^2}\right)^2 (\eta_{AB} + h_{AB})dX^AdX^B,$$  

while on the brane we have

$$ds^2_b = \frac{l^2}{A^2} (\eta_{\mu\nu} + \gamma_{\mu\nu}^{(b)})dx^\mu dx^\nu,$$

where

$$\gamma_{\mu\nu}^{(b)} = T'^2 h_{00}\big|_b + A'^2 h_{44}\big|_b + 2 T' A' h_{04}\big|_b,$$

$$\gamma_{\mu i}^{(b)} = T' h_{0i}\big|_b + A' h_{4i}\big|_b,$$

$$\gamma_{ij}^{(b)} = h_{ij}\big|_b.$$  

Since we leave the perturbation in the position of the brane to a later stage, the tangent vectors to the brane are still the same as before, but we must find the correction to the normal to the brane. Writing the perturbed normal as an unperturbed part calculated on the brane, $\tilde{n}^A$, and a small perturbation, $\delta n^A$, we have

$$\eta_{AB}\tilde{n}^B \delta n^A + \eta_{AB}\tilde{n}^A \delta n^B + h_{AB}\big|_b \tilde{n}^B \delta n^A = 0,$$

$$\eta_{AB}\tilde{V}_\mu^B \delta n^A + h_{AB}\big|_b \tilde{V}_\mu^A \delta n^B = 0.$$  

After some computation, we find for the covariant perturbation of the normal,

$$\delta n_A = \frac{l^2}{A^2} h_{AB}\big|_b \delta n^B + \frac{l^2}{A^2} \eta_{AB} \delta n^B,$$  

or more explicitly,

$$\delta n_0 = -h_{00}\big|_b \frac{\mathcal{A}^3}{2A} + h_{04}\big|_b \frac{\mathcal{V}^4}{A}(1 - \mathcal{A}^2) - h_{44}\big|_b \frac{T'^2 A'}{2A},$$

$$\delta n_4 = h_{00}\big|_b \frac{\mathcal{A}^2 T'}{2A} + h_{04}\big|_b \frac{\mathcal{A}^3}{A} + h_{44}\big|_b \frac{T' T}{2A}(\mathcal{A}^2 + 1),$$

$$\delta n_i = 0.$$  

B. Perturbation of the Junction Conditions

With the complete normal to the brane in the perturbed background we can calculate the perturbed second fundamental form on the brane. For the spatial part we have

$$K_{ij} = \tilde{K}_{ij} - \frac{\partial \delta n_i}{\partial x^j} + \Gamma^C_{ij} \delta n_C + \delta \Gamma^C_{ij} \tilde{n}_C.$$  

Using the bulk connections presented in the appendix and the results obtained for the perturbation of the normal, and using also $\tilde{K}_{ij} = T' \delta_{ij} l/A^2$, we find for the contribution of the perturbation of the bulk to the second fundamental form on the brane

$$\delta K^i_j = \delta^i_j \left( h_{00}\big|_b \frac{\mathcal{A}^2 T'}{2l} + h_{04}\big|_b \frac{\mathcal{A}'}{l}(A^2 - 1) + h_{44}\big|_b \frac{T'}{2l}(A^2 - 1) \right)$$

$$+ \frac{\mathcal{A}' A}{l} \left[ \frac{1}{2} \frac{l^2}{A^2} \partial^j h_{0j}\big|_b + \partial_j h_{4i}\big|_b + \partial_0 h_{4j}\big|_b \right]$$

$$+ \frac{T' A}{l} \left[ \frac{1}{2} \frac{l^2}{A^2} \partial^j h_{4j}\big|_b + \partial_j h_{4i}\big|_b - \partial_0 h_{4j}\big|_b \right].$$  

(13)
1. Einstein Equations

From the framework above developed, we learned that the general perturbations in the bulk metric are translated to perturbations in the brane metric. The junction conditions, now perturbed, relate the evolution of the perturbation of the matter on the brane to the metric perturbations introduced. However, we still have to solve the Einstein equations in the bulk. Before doing that, it is useful to observe that there are five reparametrization in the bulk coordinates that could be used to eliminate five degrees of freedom in the perturbed metric. Thus, we fix the gauge in the bulk by

$$h_{A4} = 0$$  \hspace{1cm} (14)

With this choice we present the results for the perturbation in the Ricci tensor in the bulk,

$$\delta R_{AB} = \partial_C \delta \Gamma^C_{AB} - \partial_A \delta \Gamma^C_{BC} + \tilde{\Gamma}^C_{AB} \delta \Gamma^E_{CE} + \delta \Gamma^C_{AC} \tilde{\Gamma}^E_{EB} - \tilde{\Gamma}^E_{AC} \delta \Gamma^C_{EB}.$$  \hspace{1cm} (15)

The Einstein equations in the bulk read

$$\bar{R}_{AB} = \Lambda \frac{2l^2}{3(X^4)^2} \eta_{AB} - \frac{4}{(X^4)^2} h_{AB},$$  \hspace{1cm} (16)

$$\delta R_{AB} = \Lambda \frac{2l^2}{3(X^4)^2} \eta_{AB},$$  \hspace{1cm} (17)

where we used the equations for the background metric.

We realize that the Einstein equations can be solved easily for a transverse traceless-free perturbation when \( \partial_\mu h^\mu_{\nu} = 0 = h^\mu_{\nu}. \) Then components (4,0) and (4,4) are automatically satisfied, remaining, for the five independent modes \( h_{\alpha\beta}, \)

$$\frac{l^2}{(X^4)^2} \partial^4 \partial_\alpha h_{\alpha\beta} - \frac{3}{X^4} \partial_\alpha h_{\alpha\beta} = 0.$$  \hspace{1cm} (18)

Decomposing \( h_{\alpha\beta} \) in Fourier modes in the variables \( X^0 \) and \( X^i, \)

$$h_{\alpha\beta} = \int d^3k d\kappa \chi_{\alpha\beta}(\kappa, \kappa) F(X^4) e^{i\kappa_0 X^0 + i\kappa \cdot \vec{X}},$$

we find the following differential equation for the transformed function \( F(X^4), \)

$$\frac{d^2}{d(X^4)^2} F(X^4) - \frac{3}{X^4} \frac{d}{dX^4} F(X^4) + \left((k^0)^2 - |\vec{k}|^2\right) F(X^4) = 0.$$  \hspace{1cm} (19)

This is just the Bessel differential equation of order 2. We write \( F(X^4) = (mX^4)^2 Z(mX^4), \) with \( m^2 = (k^0)^2 - |\vec{k}|^2, \) in which case the general solution of the above equation is a combination of Bessel or Hankel functions of order 2,

$$Z(mX^4) = Z_2^{(j)}(mX^4) = j_m J_2(mX^4) + n_m N_2(mX^4)$$

$$= Z_2^{(h)}(mX^4) = h_m H_2^{(1)}(mX^4) + h_m H_2^{(2)}(mX^4).$$

Thus we have been able to solve exactly the bulk’s perturbation \( h_{\alpha\beta} \) in the form

$$h_{\alpha\beta} = \int d^3k dm (mX^4)^2 \chi_{\alpha\beta}(\kappa, m) Z_2(mX^4) e^{i\kappa_0 X^0 + i\kappa \cdot \vec{X}},$$  \hspace{1cm} (20)

where \( k^0 \chi_{\alpha\beta} = \epsilon^\alpha_{\alpha} = 0 \) and \( (k^0)^2 = m^2 + |\vec{k}|^2. \)

C. Perturbing the Brane Position

In the unperturbed formalism we interpret the scale factor of the Robertson-Walker brane as the position of the brane in the extra dimension. If we introduce perturbations in the matter inside the brane, there will be a perturbed Friedmann equation and, consequently, a perturbation on the position of the brane. This brane bending effect certainly may appear in the context of a reparametrization of the bulk, showing that bulk metric perturbations and brane perturbations are related.
We work here in an unperturbed bulk, but with a perturbation in the position of the brane in the extra dimension, \[ X^A(\eta, x^i)|_b = X^A(\eta) + \zeta(\eta, x^i)\tilde{n}^A. \]

Therefore, the metric induced on the brane reads
\[
 ds_b^2 = \frac{l^2}{A^2} \left[ \eta_{\mu\nu} - 2\frac{\zeta A^2}{l^2} \tilde{K}_{\mu\nu} \right] dx^\mu dx^\nu.
\] (21)

Thus, a perturbation in the brane position induces a perturbation in the brane metric, \( \gamma^{(p)}_{\mu\nu}/A^2 \), where
\[
 \gamma^{(p)}_{\eta\eta} = -\frac{2\zeta}{l\sqrt{1 + A'^2}} \left( A A'' - A'^2 - 1 \right),
\] (22)
\[
 \gamma^{(p)}_{ij} = -\frac{2\zeta}{l} \sqrt{1 + A'^2} \delta_{ij}.
\] (23)

The junction conditions thus get perturbed, and we arrive at the result
\[
 K^i_j = \tilde{K}^i_j + \partial_j \partial^i \zeta + \frac{A^2}{l^2} \frac{\partial^i}{\partial^j} \left( \frac{A'^2}{A^2} \right) \left( \zeta A^2 + \frac{A'}{A} \zeta' \right),
\] (24)

D. Gathering all together

Gathering all results together, we conclude that
\[
 ds^2 = \frac{l^2}{(X^A)^2} \left( \eta_{AB} + h_{AB} \right) dX^A dX^B,
\]
\[
 X^A = \tilde{X}^A + \zeta n^A,
\]
\[
 ds_b^2 = a^2(\tau) \left( \eta_{\mu\nu} + \gamma^{(p)}_{\mu\nu} \right) dx^\mu dx^\nu.
\] (25)

with
\[
 \gamma_{\eta\eta} = (1 + H^2l^2)h_{00}|_b + \frac{2l(\zeta)}{\sqrt{1 + H^2l^2}} \left( \frac{\tilde{a}}{a} + \frac{1}{l^2} \right),
\] (26)
\[
 \gamma_{\eta i} = \sqrt{1 + H^2l^2} h_{0i}|_b,
\] (27)
\[
 \gamma_{ij} = h_{ij}|_b - \frac{2\zeta}{l} \sqrt{1 + H^2l^2} \delta_{ij}.
\] (28)

The dependence in the bulk coordinates of the five degrees of freedom associated with the perturbation of the bulk metric, \( h_{\alpha\beta} \), is known from the Einstein equations in the bulk. What remains to be done is to impose the boundary conditions on the general solution and to relate those fields with the matter perturbation on the brane. The junction conditions provide those relations between the perturbations. Here, however, as comparing with \[20\], we do not look for the equations restricted to a specific brane evolution, but make a full treatment of the junction conditions leading to the full equations of motion on the brane for a generic dynamics.

The unperturbed junction conditions give the usual conservation of energy-momentum on the brane and the evolution of the scale factor by a Friedmann equation. On the other hand, if we allow matter perturbations to exist on the brane, as the bulk energy continues to be just a cosmological constant, the perturbed part of the junction conditions implies
\[
 \delta \left( T^{\mu}_{\nu,\mu} \right) = 0,
\]
\[
 \delta K^i_j = \frac{\kappa}{2} \left( \delta T^i_j - \frac{1}{3} \delta^i_j \delta T \right).
\] (29)
Using the previous results we find the relations
\[
\begin{align*}
\delta T^\eta_\eta &= -\delta T^\eta_\eta + \gamma_\eta \delta T^\eta_\eta + \gamma^\eta_\alpha \delta T^\eta_\alpha = -\delta T^\eta_\eta + \gamma^\eta_\eta (P + \rho) , \\
\delta \dot{T}^\eta_\eta + \frac{1}{a} \partial_t \delta T^\eta_\eta + 3H \delta T^\eta_\eta - H \delta T^\eta_k - \frac{\rho + P}{2} \gamma_k^\eta = 0 , \\
\delta \dot{T}^\eta_i &= -\frac{1}{a} \partial_j \delta T^\eta_j + 4H \delta T^\eta_i + \frac{P + \rho}{2a} \partial_i \gamma^\eta_\eta = 0 ,
\end{align*}
\]
where \( \dot{\cdot} = \partial / \partial \tau \).

In order to put the equations in a familiar form, we parametrize the perturbations by the functions \( \delta \rho, \delta P \), the velocity \( v_i \) and the anisotropic stress \( \Sigma^j_\eta \), with the definitions
\[
\begin{align*}
\delta T^\eta_\eta &= -\delta \rho , \\
\delta T^\eta_i &= -(\rho + P) v^i , \\
\delta \dot{T}^\eta_i &= (\rho + P) (v_i - \gamma^\eta_i) , \\
\delta \dot{T}^\eta_j &= \delta P \gamma^\eta_j + \Sigma^j_\eta .
\end{align*}
\]
As a consequence we find, using \( P = \omega \rho \),
\[
\begin{align*}
\delta \rho + \frac{\rho (1 + \omega)}{a} \partial_t v^i + 3H (\delta \rho + \delta P) + \frac{\rho (1 + \omega)}{2} \gamma^\eta_k &= 0 , \\
(1 + \omega) \dot{\bar{v}}_i + \frac{1}{a} \partial_j \left( \delta P \delta_j^i + \Sigma^i_\eta \right) + \left[ H (1 - 3 \omega)(1 + \omega) + \dot{\omega} \right] (v_i - \gamma^\eta_i) \\
&\quad + (1 + \omega) \left( \frac{1}{2a} \partial_v \gamma^\eta_i - \gamma^\eta_i \right) = 0 .
\end{align*}
\]

The expressions above determine the dynamics of the perturbations on the brane. Once we solve the junction conditions and express \( \zeta \) in terms of \( h_{\alpha \beta} \), we will be able to find the expressions for \( \gamma_{\mu \nu} \) and, using (37) and (38), finally study the evolution of matter perturbations on the brane.

We first decouple the equations in scalar, vector and tensorial modes, writing, in Fourier space,
\[
\begin{align*}
v_i &= -\frac{k_i}{k} V + v^i , \\
\Sigma_{ij} &= \left( -\frac{k_i k_j}{k^2} + \frac{1}{3} \delta_{ij} \right) \Sigma - \frac{i}{2k} (k_i \Sigma_j + k_j \Sigma_i) + \Sigma^T_{ij} ,
\end{align*}
\]
where \( k^i \Sigma^T_{ij} = k^i \Sigma_i = 0 \).

We now choose \( k^i = (0, 0, k) \) and find
\[
\begin{align*}
(1 + \omega) \dot{V} - \frac{k}{a \rho} \left( \delta P - \frac{2}{3} \Sigma \right) + \left[ H (1 - 3 \omega)(1 + \omega) + \dot{\omega} \right] (V - i \gamma^\eta_i) \\
&\quad + (1 + \omega) \left( -\frac{k}{2a} \gamma^\eta_i - i \gamma^\eta_j \right) = 0
\end{align*}
\]
for the scalar part and
\[
\begin{align*}
(1 + \omega) \dot{v}^i + \frac{k}{2a \rho} \Sigma_i + \left[ H (1 - 3 \omega)(1 + \omega) + \dot{\omega} \right] (v^i - \gamma^\eta_i) \\
&\quad - (1 + \omega) \gamma^\eta_i = 0
\end{align*}
\]
for the vector part, where \( i = 1, 2 \).

\textit{Junction Conditions}

Following \cite{20}, we define the quantity
\[
F^i_j = -\frac{k}{2} \Sigma^i_j + \partial_j \partial^i \zeta \\
- \frac{H}{2a} \left[ (\partial^2 h^0)_{zb} + \partial_j h^0_{zb} \right] - \frac{\sqrt{1 + H^2 Q^2}}{2a} \partial_k h^i_{zb} .
\]
We find that this quantity has no traceless part, that is $F^j_i = \frac{1}{2} \delta^j_i F$, and the trace reads
\[
F = \frac{k}{2} \delta \rho - \left[ 3 \zeta H^2 - 3 H \dot{\zeta} + 3 h_{00,0b} \frac{H^2 l}{2} \sqrt{1 + H^2} \right] . \tag{44}
\]

If we choose the spatial coordinates such that $k^i = \delta^i_j k$, the traceless-free condition implies that
\[
\epsilon_{0i} = -\frac{k}{k^0} \epsilon_{3i} = -\frac{k}{\sqrt{m^2 + k^2}} \epsilon_{3i} , \tag{45}
\]
\[
\epsilon_{00} = \frac{k^2}{(k^0)^2} \epsilon_{33} = \frac{k^2}{m^2 + k^2} \epsilon_{33} , \tag{46}
\]
\[
\epsilon_{22} = -\epsilon_{11} - \frac{m^2}{m^2 + k^2} \epsilon_{33} . \tag{47}
\]
Only five degrees of freedom remain, and they are $\epsilon_{12}$, $\epsilon_{13}$, $\epsilon_{23}$ and $\epsilon_{11}$.

A thorough discussion of massless and massive modes is given in the appendices.

### E. Connecting with the usual cosmological perturbation theory

In order to clarify what is hidden behind those equations, let us take a closer look into the usual cosmological perturbation theory. The standard approach is to perturb the RW metric in the form
\[
ds^2 = a^2(\eta) \left( -(1 + 2A) d\eta^2 - B_i d\eta d\xi^i + [(1 + 2D) \delta_{ij} + E_{ij}] dx^i dx^j \right) , \tag{48}
\]
with $E_{ij}$ traceless.

Then we define the quantities $B_i^V$, $E_i$ and $E_{ij}^T$ such that
\[
B_i = -\frac{i k^i}{k} B + B_i^V \tag{49}
\]
\[
E_{ij} = \left( -\frac{k^i k^j}{k^2} + \frac{1}{3} \delta_{ij} \right) E - \frac{i}{2k} \left( k_i E_j + k_j E_i \right) + E_{ij}^T , \tag{50}
\]
with $k^i B_i^V = k^i E_i = k^i E_{ij}^T = 0$. This general decomposition of the metric perturbations, together with and decouple the equations between what is called scalar perturbations ($\delta \rho$, $\delta P$, $V$, $\Sigma$, $A$, $B$, $D$, $E$), vectorial perturbations ($v_i^V$, $\Sigma_i$, $B_i^V$, $E_i$) and tensorial perturbations ($\Sigma_{ij}^T$, $E_{ij}^T$).

The gauge invariant tensorial perturbations in usual cosmology satisfy the wave equation
\[
(E_{ij}^T)^{\prime\prime} + 2aH(E_{ij}^T)^{\prime} + k^2 E_{ij}^T = 8\pi Ga^2 P \Sigma_{ij}^T . \tag{51}
\]

For the scalar and vector sectors, we must fix the gauge and the field equations, give the evolution of the perturbations and some constraint equations. For example, the scalar sector in the longitudinal gauge ($B = E = 0$) reads
\[
\frac{\delta \dot{\rho}}{\rho} = -(1 + \omega) \frac{k}{a} V - 3H \left( \frac{\delta \rho}{\rho} + \frac{\delta P}{\rho} \right) - 3(1 + \omega) \dot{D} , \tag{52}
\]
and
\[
\dot{V} = -H(1 - 3\omega) V - \frac{\dot{\omega}}{1 + \omega} V + \frac{k}{a(\rho + P)} \left( \delta P - \frac{2}{3} \Sigma \right) + \frac{k}{a} A . \tag{53}
\]
The constraint equations relate directly the metric perturbation to the matter perturbation. In this gauge, for the scalar sector, we find
\[
k^2 D = 4\pi Ga^2 \left( \delta \rho + \frac{3aH}{k} \rho + P \right) V \tag{54}
\]
and
\[
k^2 (A + D) = -8\pi Ga^2 \Sigma . \tag{55}
\]
Now, returning to the brane world cosmology, we can find a connection between our treatment and the usual cosmology from the perturbed brane metric. Comparing (25) with (48) and separating the trace part of \( h_{ij} \) (for \( h_{00} = h_{11} + h_{22} + h_{33} \)) and of \( E_{ij} \) (for \( E_{00} = -E_{33} \)) we note that

\[
-2A = \gamma_{\eta\eta} = (1 + H^2 t^2) h_{00} + \frac{2\zeta}{1 + H^2 t^2} \left( \frac{\dot{a}}{a} + \frac{1}{t^2} \right),
\]

\[
-B_i = 2\gamma_{\eta i} = 2\sqrt{1 + H^2 t^2} h_{0i},
\]

\[
2D = \frac{1}{3} \gamma_k^k = \frac{1}{3} h_{00} - \frac{2\zeta}{t} \sqrt{1 + H^2 t^2},
\]

\[
2E_{ij} = \gamma_{ij} - \frac{1}{3} \gamma_k^k \delta_{ij}.
\]

Explicitly the last expression reads

\[
2E_{11} = \frac{2}{3} h_{11} - \frac{1}{3} h_{22} - \frac{1}{3} h_{33},
\]

\[
2E_{22} = \frac{2}{3} h_{22} - \frac{1}{3} h_{11} - \frac{1}{3} h_{33},
\]

\[
2E_{33} = \frac{2}{3} h_{33} - \frac{1}{3} h_{11} - \frac{1}{3} h_{22},
\]

\[
2E_{12} = h_{12},
\]

\[
2E_{13} = h_{13},
\]

\[
2E_{23} = h_{23},
\]

and we have the correct number of functions to describe perturbations on the brane.

Now, for \( k^i = (0, 0, k) \), the scalar - vector - tensorial decomposition (50) states the two polarizations of tensorial perturbations as described by

\[
E_{11}^T = \frac{1}{4} (h_{11} - h_{22}),
\]

\[
E_{12}^T = \frac{1}{2} h_{12}.
\]

This leads us to a direct connection with standard perturbation theory. We shall find that the real tensorial modes are the 11 and 22 modes, but not the 13 or 23 ones.

The vectorial components are

\[
B_1^V = -2\gamma_{\eta 1} = -2 \sqrt{1 + H^2 t^2} h_{01},
\]

\[
B_2^V = -2\gamma_{\eta 2} = -2 \sqrt{1 + H^2 t^2} h_{02},
\]

\[
E_1 = i h_{13},
\]

\[
E_2 = i h_{23},
\]

and finally, the scalar ones are

\[
A = -\frac{1}{2} \gamma_{\eta\eta} = -\frac{1}{2} (1 + H^2 t^2) h_{00} - \frac{\zeta}{\sqrt{1 + H^2 t^2}} \left( \frac{\dot{a}}{a} + \frac{1}{t^2} \right),
\]

\[
D = -\frac{1}{6} \gamma_k^k = -\frac{1}{6} h_{00} + \frac{\zeta}{t} \sqrt{1 + H^2 t^2},
\]

\[
E = \frac{1}{4} h_{00} - \frac{3}{4} h_{33},
\]

\[
B = -2i \gamma_{\eta 3} = -2i \sqrt{1 + H^2 t^2} h_{03}.
\]

The general metric perturbations are described in usual cosmology by 10 functions \( A, B, D, E, B_1, B_2, E_1, E_2, E_{11}^T \) and \( E_{12}^T \) matching the correct number of degrees of freedom given by the traceless symmetric part of \( h_{\mu\nu} \) and the field \( \zeta \). However, when we require \( k^\mu h_{\mu\nu} = 0 \), we force the existence of four constraint equations that reduce this number to 6, which is the number of independent perturbations after the gauge choice in usual cosmology. Thus, from the point of view of the brane, we completely fixed the gauge when we demand \( k^\mu h_{\mu\nu} = 0 \). This is an unusual gauge
choice, it is neither a longitudinal (Newtonian) gauge nor a total-matter gauge. It relates the scalar perturbations with each other without canceling any one of them.

We can see that the brane-world treatment described so far walks closely to the usual cosmological perturbation theory. In fact, from the analysis developed in the last section, we also find that dynamical equations are quite similar to (52) and (53). In terms of $B$, $D$, and $A$, the scalar sector, (66) and (67) read

$$\frac{\delta \rho}{\rho} + (1 + \omega) \frac{k}{a} V + 3H \left( \frac{\delta \rho}{\rho} + \frac{\delta P}{\rho} \right) + (1 + \omega) 3 \dot{D} = 0 \ ,$$

and

$$\dot{V} = \frac{k}{a(\rho + P)} \left( \delta P - \frac{2}{3} \Sigma \right) - \left[ H(1 - 3\omega) + \frac{1}{1 + \omega} \right] (V + \frac{B}{2}) + \left( \frac{k}{a} A - \frac{1}{2} B \right) \ .$$

Comparing with (52) and (53) we see that the only difference comes from the unusual gauge choice we have made (in the Newtonian gauge $B = 0$ and this expressions are identical to the usual ones).

On the other hand, the quite different behaviour of the brane-world perturbations comes from the constraint equations. Contrary to the usual cosmological scenario there are constraints between tensorial modes and the anisotropic stress on the brane. Denoting the quantities under a hat by the fourier modes as in

$$h_{\alpha \beta} = \int d^3 k dm \ h_{\alpha \beta}(m, k; X^0, X^4) \ e^{i \vec{k} \cdot \vec{x}}$$

and recording the dependence of $\hat{h}$ on $X^0$ and $X^4$, (20), we follow the procedure described in the appendix for massive and massless modes and use the connection with usual cosmology theory, (66)-(75), to find for the tensorial sector

$$\frac{\kappa}{2a} \dot{\Sigma}_{12} = -E_{12} \left[ i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{Z_1(m \lambda^{-1}(\tau))}{Z_2(m \lambda^{-1}(\tau))} \right] \ ,$$

$$\frac{\kappa}{2a} \dot{\Sigma}_{11} = -E_{11} \left[ i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{Z_1(m \lambda^{-1}(\tau))}{Z_2(m \lambda^{-1}(\tau))} \right] \ ,$$

for the vectorial sector

$$\frac{\kappa}{2a} \dot{\Sigma}_1 = -E_1 \left[ i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{Z_1(m \lambda^{-1}(\tau))}{Z_2(m \lambda^{-1}(\tau))} \right] \ ,$$

$$\frac{\kappa}{2a} \dot{\Sigma}_2 = -E_2 \left[ i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{Z_1(m \lambda^{-1}(\tau))}{Z_2(m \lambda^{-1}(\tau))} \right] \ ,$$

and for the scalar sector

$$F = \frac{\dot{\hat{h}}}{2a^{\epsilon_{00}}} \left[ i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{Z_1(m \lambda^{-1}(\tau))}{Z_2(m \lambda^{-1}(\tau))} \right] - \frac{k^2}{a^2} \hat{\Sigma} \ ,$$

$$\frac{\kappa}{a^2} \dot{\Sigma} = \frac{2}{a} \dot{E} \left[ i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{Z_1(m \lambda^{-1}(\tau))}{Z_2(m \lambda^{-1}(\tau))} \right] - 4i \chi \sqrt{m^2 + k^2 + m \sqrt{1 + H^2}} \frac{\dot{\hat{h}}}{2a^{\epsilon_{00}}} + 2 \frac{k^2}{a^2} \hat{\Sigma} \ .$$

These are our main results: we completely solve the wave equations in the bulk and found the junction conditions for arbitrary brane dynamics in each mode sector. Our equations are quite complicated because of the effect of massive modes and the unusual gauge choice we made. We should indeed expect constraints in the scalar and in the vector sectors. However equations similar to (70) and (80) do not appear in usual cosmology. When we are talking about a Minkowsky brane with $a = a_0$, we can avoid such constraints choosing $m$ such that $Z_1(m \lambda/a_0) = 0$. Indeed this is exactly what is done in the normal coordinate approach [25]. However, in normal coordinates, we can not treat the full problem analytically in generic dynamical universes. From the point of view of the bulk, this seems to be possible and, after fixing a gauge in an unusual form, we are able to solve analytically the bulk modes, finding the constraints equations for general brane behaviour. As tensorial modes are gauge independent, from our analysis, we can conclude that brane-world cosmology predicts unusual constraint equations between tensorial modes and the stress energy tensor.
In the usual cosmology, gravitational waves ($E_{11}$ and $E_{12}$) couple with the tensorial part of the anisotropic stress, but there are no algebraic constraint between them. Here we see that, besides special cases discussed in the next section, in the great majority of possibilities of brane dynamics, there are no gravitational waves without anisotropic stress on the brane. Clearly any unusual behaviour as this one comes from the fact that gravitational waves should be fundamentally affected in our model. In a generic brane world scenario, we cannot talk about a gravitational wave strictly restricted to the brane. The extra constraints appear because strictly speaking, gravitational waves are not brane objects. They belong to the bulk and so they must satisfy the bulk Einstein equations. As they travel through the bulk, we can expect to find waves that follow closely the brane evolution and even go across it. For brane world residents, the amplitude of what is usually called a gravitational wave is, in fact, an average over all amplitudes of the bulk waves that, at a certain moment, cross the brane. Because the junction conditions force the matter content on the brane to behave in correspondence to the bulk geometry, all possible tensorial perturbations calculated on the brane appear to brane world residents as anisotropic stress on the energy-momentum tensor.

IV. NORMAL COORDINATES

During a de-Sitter expansion, we can explicitly transform the bulk coordinates ($X^0, X^4$) to normal coordinates ($t, y$) where the background metric is written by

\[ ds^2 = \left( \cosh(y/l) - \sqrt{1 + H^2l^2 \sinh(y/l)} \right) \left( -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \right) + dy^2 \]  

with $0 \leq y < \tanh^{-1} \left[ (1 + H^2l^2)^{-1/2} \right]$ and the brane position is taken at $y = 0$.

We note that the relevant transformation is

\[ X^4 = X^4(t, y) = \frac{l}{a(t)\cosh(y/l) - \sqrt{1 + H^2l^2 \sinh(y/l)}} \]
\[ X^0 = X^0(t, y) = \frac{1}{Ha(t)} \sqrt{1 + \frac{H^2l^2}{\cosh(y/l) - \sqrt{1 + H^2l^2 \sinh(y/l)}}} \]

Thus, we can write the solution for pure tensor modes with respect to the brane metric found in the previous sections in normal coordinates. Writing

\[ h_{ij}(t, \bar{x}, y) = \int d^3k dm \epsilon_{ij}(\vec{k}, m) \hat{h}_{km}(t, \bar{x}, y) \]

we find for a bounded zero mode

\[ \hat{h}_{km}(t, \bar{x}, y) = \exp \left[ i \vec{k} \cdot \bar{x} \right] \frac{m^2l^2}{a(t)^2\cosh(y/l) - \sqrt{1 + H^2l^2 \sinh(y/l)}} \times \]
\[ N_2 \left( \frac{m^2l^2}{a(t)\cosh(y/l) - \sqrt{1 + H^2l^2 \sinh(y/l)}} \right) \times \]
\[ \exp \left[ i \frac{m^2}{Ha(t)} \sqrt{1 + \frac{H^2l^2}{\cosh(y/l) - \sqrt{1 + H^2l^2 \sinh(y/l)}}} \right] \]

We can plot the above solutions with respect to the extra coordinate $y$ and note the decaying behaviour of the perturbations for massive modes away from the brane.

The junction conditions are reproduced in the absence of anisotropic stress by

\[ \frac{\partial h(t, \bar{x}, y)}{\partial y} \bigg|_{y=0} = 0 \]

This equation can not be solved non trivially for all times because the solution just found is non-separable in $y$ and $t$.

In a de-Sitter background, it was shown that it is possible to separate the wave equation in normal coordinates and find a separable solution in $y$ and $t$, with gravitational waves in the absence of brane anisotropic stress. However, for a non-de Sitter evolution, this is not possible and, because we started the approach generically,
our solution is intrinsically non-separable. Clearly, from this non separability in the bulk approach, the connection with the usual de-Sitter solution becomes obscure and, for this reason, we still need a method to determine the initial conditions and choose the amplitudes \( \epsilon_{ij}(\vec{k}, m) \) to finally solve the spectrum of tensorial perturbations. However, because such non separability is inherent of generic brane dynamics, our results have far reaching consequences for inflation: it gives us, for instance, a mechanism for anisotropy creation in the brane-world. If initial conditions, during a de-Sitter inflationary phase, created tensorial perturbations propagating in the bulk with an anisotropic stress free brane world, as suggested in usual inflationary models, after inflation, and also during the reheating, those waves, interacting with the brane, would necessarily produce tensorial modes of anisotropy in our universe. Those anisotropic modes would be created by the gravitational waves as described by the complete junctions conditions found in the last section.

APPENDIX A: NOTATION.

Unperturbed tangent vectors are denoted by a bar \( \bar{\vec{V}}_A^\mu \). Quantities calculated on the unperturbed brane are followed by \( \vert_b \), while \( \vert_b \) denotes the perturbed brane.

Upper case latin indices parametrize the bulk, \( A, B, C, = 0..4 \) (4 is the extra dimension), greek indices refers to the usual space-time, \( \mu, \nu, \lambda = 0..3 \), and the spatial part in the brane is parametrized by lower case latin indices, \( i, j, k = 1..3 \).

APPENDIX B: CONNECTIONS

In this section it is presented for reference the non-null connections of the unperturbed and perturbed geometry.
1. Brane Connections

For the RW metric in conformally minkowskian coordinates,
\[
ds^2 = \frac{l^2}{A^2} \eta_{\mu\nu} dx^\mu dx^\nu,
\]
the non-zero connections are
\[
\Gamma^\eta_{\eta\eta} = - A' c \eta_{\eta} \frac{A' \eta_{\eta}}{A}, \quad \Gamma^\eta_{ij} = - A' c \eta_{\eta} \delta_{ij} \frac{A' \eta_{\eta}}{A},
\]
\[
\Gamma^i_{\eta j} = - A' c \eta_{\eta} \delta_{ij} \frac{A' \eta_{\eta}}{A}, \quad \Gamma^i_{\eta j} = \frac{1}{2} \partial_i \eta_j \frac{A' \eta_{\eta}}{A}, \quad \Gamma^i_{\eta j} = \frac{1}{2} \partial_i \eta_j \frac{A' \eta_{\eta}}{A}.
\]

Linearly in the perturbation, the non-zero perturbed connections calculated with the perturbed RW metric \( ds^2 = \frac{l^2}{A^2} (\eta_{\mu\nu} + \gamma_{\mu\nu}) dx^\mu dx^\nu \) are
\[
\delta \Gamma^\eta_{\eta\eta} = \frac{1}{2} \partial_\eta \gamma^\eta_{\eta} \frac{A' \eta_{\eta}}{A}, \quad \delta \Gamma^\eta_{ij} = - \frac{1}{2} \partial_i \gamma_j + \frac{1}{2} \partial_j \gamma_i + \eta_{ij} \frac{A' \eta_{\eta}}{A}, \quad \delta \Gamma^i_{\eta j} = - \frac{1}{2} \partial_i \gamma_j + \frac{1}{2} \partial_j \gamma_i + \eta_{ij} \frac{A' \eta_{\eta}}{A},
\]
\[
\delta \Gamma^i_{\eta j} = \frac{1}{2} \partial_\eta \gamma^i_{\eta} \frac{A' \eta_{\eta}}{A}, \quad \delta \Gamma^i_{\eta j} = - \frac{1}{2} \partial_\eta \gamma^i_{\eta} \frac{A' \eta_{\eta}}{A}, \quad \delta \Gamma^i_{\eta j} = \frac{1}{2} \partial_\eta \gamma^i_{\eta} \frac{A' \eta_{\eta}}{A}.
\]

It is worthwhile reminding that the indices of the perturbation in the metric, \( \gamma_{\mu\nu} \), are raised and lowered with \( \eta_{\mu\nu} \). The same holds for the perturbation of the bulk metric described below, \( h_{AB} \), which are raised and lowered by \( \eta_{AB} \).

2. Bulk Connections

For the AdS bulk in conformally minkowskian coordinates described by the metric \( ds^2 = \frac{l^2}{X^4} \eta_{AB} dX^A dX^B \), the non-zero connections read
\[
\Gamma^0_{04} = \Gamma^4_{00} = \Gamma^4_{44} = - \frac{1}{X^4}, \quad \Gamma^4_{ij} = \delta_{ij} \frac{1}{X^4}, \quad \Gamma^i_{4j} = - \frac{\delta_{ij}}{X^4}.
\]

In the linear approximation, the non-zero perturbations in the connections computed with the perturbed AdS metric \( ds^2 = \frac{l^2}{X^4} (\eta_{AB} + h_{AB}) dX^A dX^B \), are
\[
\delta \Gamma^0_{00} = \frac{1}{2} \partial_0 h^0_0 + \frac{h^0_0}{X^4}, \quad \delta \Gamma^0_{40} = \frac{1}{2} \partial_4 h^0_0, \quad \delta \Gamma^0_{44} = - \frac{l^2}{2(X^4)^2} \partial^0 h_{44} + \partial_4 h^0_0 - \frac{h^0_0}{X^4},
\]
\[
\delta \Gamma^0_{ij} = \frac{1}{2} (\partial_i h^0_j + \partial_j h^0_i) - \frac{l^2}{(X^4)^2} \partial^0 h_{ij} - \frac{h^0_0}{X^4} \delta_{ij}, \quad \delta \Gamma^0_{i0} = \frac{1}{2} \partial_i h^0_0.
\]
\[ \delta \Gamma_{\epsilon_4}^0 = \frac{1}{2} (\partial_0 h_0^0 + \partial_4 h_4^0 - \frac{l^2}{(X^4)^2} \partial^0 h_{44}) \],
\[ \delta \Gamma_{\epsilon_0}^4 = \partial_0 h_0^4 - \frac{l^2}{2(X^4)^2} \partial^4 h_{00} + \frac{h_{44}^4}{X^4} + \frac{h_{00}^4}{X^4} \],
\[ \delta \Gamma_{\epsilon_4}^4 = \frac{1}{2} \partial_0 h_{44} + \frac{h_{44}^4}{X^4} \],
\[ \delta \Gamma_{\epsilon_4}^4 = \frac{1}{2} \partial_4 h_{44} + \frac{h_{44}^4}{X^4} \],
\[ \delta \Gamma_{\epsilon_4}^4 = \frac{1}{2} (\partial_4 h_{44} + \partial_4 h_{44} - \frac{l^2}{(X^4)^2} \partial^4 h_{ij}) + \frac{h_{ij}^4}{X^4} - \frac{h_{ij}^4}{X^4} \delta_{ij} \],
\[ \delta \Gamma_{\epsilon_{04}}^i = \frac{1}{2} (\partial_0 h_0^i + \partial_4 h_4^i - \frac{l^2}{(X^4)^2} \partial^i h_{04}) \],
\[ \delta \Gamma_{\epsilon_4}^4 = \partial_4 h_4^i - \frac{h_4^i}{X^4} \partial^4 h_{44} \],
\[ \delta \Gamma_{\epsilon_4}^i = \frac{1}{2} (\partial_4 h_4^i + \partial_4 h_4^i - \frac{l^2}{(X^4)^2} \partial^i h_{04}) \],
\[ \delta \Gamma_{\epsilon_4}^i = \frac{1}{2} (\partial_4 h_4^i + \partial_4 h_4^i - \frac{l^2}{(X^4)^2} \partial^i h_{04}) \],
\[ \delta \Gamma_{\epsilon_{ij}}^k = \frac{1}{2} (\partial_4 h_4^k + \partial_4 h_4^k - \frac{l^2}{(X^4)^2} \partial^i h_{04}) - \frac{h_4^k}{X^4} \delta_{kj} \].

**APPENDIX C: MASSIVE MODES**

Let us write
\[ h_{\alpha \beta} = \int d^3 k d m \hat{h}(m, k; X^0, X^4) \epsilon_{\alpha \beta}(m, k) e^{i \vec{k} \cdot \vec{x}} \],
\[ h_{\alpha \beta}|_{\hat{\epsilon}} = \int d^3 k d m \hat{h}(m, k; \tau) \epsilon_{\alpha \beta}(m, k) e^{i \vec{\epsilon} \cdot \vec{x}} \],
\[ \zeta = \int d^3 k d m \hat{\zeta}(m, k; \tau) e^{i \vec{\epsilon} \cdot \vec{x}} \]
and recalling (20), for modes with \( m \neq 0 \) we have
\[ \partial_0 \hat{h} = -i \sqrt{m^2 + k^2} \hat{h} \],
\[ \partial_i h_{\alpha \beta}|_{\hat{\epsilon}} = \int d^3 k d m i k \hat{h}(m, k; \tau) \epsilon_{\alpha \beta}(m, k) e^{i \vec{k} \cdot \vec{x}} \],
\[ \partial_4 \hat{h} = \hat{h} m \left( \frac{2}{x} + \frac{Z_2'(x)}{Z_2(x)} \right) |_{x = m t/a} \]

Substituting back in (18) we find
\[ \hat{F}^i_j = -\frac{k^i \epsilon_j^i - \frac{1}{2} k^j k^i \epsilon^i}{2} \]
\[ -\frac{H^4}{2a} \left[ i k^i \epsilon_{0j} + i k^j \epsilon_{0i} + i \sqrt{m^2 + k^2} \epsilon_{ij} \right] \hat{h} \]
\[ -\frac{\sqrt{1 + H^2 t^2}}{2a} \epsilon_{ji} m \hat{h} \left( \frac{2}{x} + \frac{Z_2'(x)}{Z_2(x)} \right) |_{x = m t/a} \].
Decomposing the above expression, using (40), the traceless-free equation implies that

\[
\frac{\kappa}{2} \hat{\Sigma}_2 = -\frac{\dot{h}}{2a} \epsilon_{12} \left[ iHl \sqrt{m^2 + k^2} + m \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))} \right],
\]

\[
\frac{\kappa}{2} \hat{\Sigma}_1 = -\frac{\dot{h}}{2a} \epsilon_{13} \left[ iHl m^2 \sqrt{k^2 + m^2} + m \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))} \right],
\]

\[
\frac{\kappa}{2} \hat{\Sigma}_3 = -\frac{\dot{h}}{2a} \epsilon_{23} \left[ iHl m^2 \sqrt{k^2 + m^2} + m \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))} \right],
\]

and

\[
\frac{1}{3} F = -\frac{\kappa}{2} \hat{\Sigma}_1 - \frac{\dot{h}}{2a} \epsilon_{11} \left[ iHl \sqrt{m^2 + k^2} + m \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))} \right],
\]

\[
= -\frac{\kappa}{2} \hat{\Sigma}_2 - \frac{\dot{h}}{2a} \epsilon_{22} \left[ iHl \sqrt{m^2 + k^2} + m \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))} \right],
\]

\[
= -\frac{\kappa}{2} \hat{\Sigma}_3 - \frac{\dot{h}}{2a} \epsilon_{33} \left[ iHl \sqrt{m^2 + k^2} + m \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))} \right] - \frac{k^2}{a^2} \zeta.
\]

In dynamical universes without the presence of an anisotropic stress, we claim that

- Massive modes are such that \( \epsilon_{11}(k, m) = \epsilon_{22}(k, m) \). That means, from the transverse traceless-free condition
  \[
  \epsilon_{11}(k, m) = \epsilon_{22}(k, m) = -\frac{1}{2} \frac{m^2}{k^2 + m^2} \epsilon_{33}, \tag{C8}
  \]
  \[
  \epsilon_{00}(k, m) = \frac{k^2}{k^2 + m^2} \epsilon_{33}(k, m). \tag{C9}
  \]

- Thence, we find, recalling the form of \( \dot{h} \),

\[
\zeta(m, k; \tau) = \frac{m^2 l^2}{2k^2 a(\tau)} \epsilon_{33} e^{-i \sqrt{m^2 + k^2} \tau} \left[ \frac{iHl}{\sqrt{m^2 + k^2}} \left( k^2 - \frac{3m^2}{2} \right) \right] \frac{Z_2(m l a^{-1}(\tau))}{Z_1(m l a^{-1}(\tau))} \left( \frac{3m^2}{2} + k^2 \right) \sqrt{1 + H^2 l^2} \frac{Z_1(m l a^{-1}(\tau))}{Z_2(m l a^{-1}(\tau))}. \tag{C10}
\]

- Since the Bessel’s functions are linearly independent, we can not have massive KK modes produced in the polarizations \( \epsilon_{12}, \epsilon_{23} \) and \( \epsilon_{13} \). That means, for \( m \neq 0 \),

\[
\epsilon_{12}(k, m) = \epsilon_{13}(k, m) = \epsilon_{23}(k, m) = \epsilon_{01}(k, m) = \epsilon_{02}(k, m) = 0, \tag{C11}
\]

remaining just one degree of freedom, \( \epsilon_{33}(k, m) \).

**APPENDIX D: MASSLESS MODE**

For modes with \( m = 0 \) we write

\[
h_{\alpha\beta}|_{\hat{h}} = \int d^3 k \, \hat{h}(0, k; \tau) \epsilon_{\alpha\beta}(0, k) e^{ik \cdot \hat{x}}, \tag{D1}
\]

\[
\zeta = \int d^3 k \, \hat{\zeta}(0, k; \tau) e^{ik \cdot \hat{x}}, \tag{D2}
\]

leading to the result

\[
\dot{F}_j = \frac{\kappa}{2} \hat{\Sigma}_j - \frac{1}{a^2} k^j k^i \hat{\zeta} - \frac{Hl}{2a} \left[ ik^i \epsilon_{0j} + ik^j \epsilon_{0i} + ik \epsilon_{ij} \right] \hat{h} - \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^2} \hat{h}. \tag{D3}
\]
Then, with \( k^i = \delta^i_j k \), and the traceless-free equation we have

\[
\frac{\kappa}{2} h_{12} = -\frac{\hbar}{2a} \epsilon_{12} \left( i H l l k + \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right),
\]

\[
\frac{\kappa}{2} h_{13} = -\frac{\hbar}{2a} \epsilon_{13} \left( \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right),
\]

\[
\frac{\kappa}{2} h_{23} = -\frac{\hbar}{2a} \epsilon_{23} \left( \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right),
\]

and

\[
\frac{1}{3} F = -\frac{\kappa}{2} h_{11} = -\frac{\hbar}{2a} \epsilon_{11} \left( i H l l k + \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right),
\]

\[
= -\frac{\kappa}{2} h_{12} - \frac{\hbar}{2a} \epsilon_{12} \left( i H l l k + \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right),
\]

\[
= -\frac{\kappa}{2} h_{23} - \frac{\hbar}{2a} \epsilon_{23} \left( -i H l l k + \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right) - \frac{k^2}{a^2} \zeta.
\]

From those equations, in the absence of an anisotropic stress we may claim that

- From the transverse traceless-free condition, massless modes are such that \( \epsilon_{11} = -\epsilon_{22} \), thence \( \epsilon_{11} = \epsilon_{22} = 0 \), \( \dot{F} = 0 \) and

\[
\dot{\zeta} = \frac{\hbar a}{2k^2} \epsilon_{33} \left( -i H l l k + \sqrt{1 + H^2 l^2} \frac{4b_0 l^4 a(\tau)}{c_0 a^4(\tau) + b_0 l^4} \right).
\]

(D4)

- Massless modes are such that \( \epsilon_{12} = 0 \) and we must choose either \( \epsilon_{23} = \epsilon_{13} = 0 \) or \( b_0 = 0 \). The last choice corresponds to the bounded zero mode advocated before. In this case, there still remain the degrees of freedom \( \epsilon_{33}, \epsilon_{23} \) and \( \epsilon_{13} \) with

\[
\dot{\zeta} = \frac{i H l l k}{2k^2} \epsilon_{33}.
\]

(D5)

We can gather all results under the following statements:

1. There are, at most, three independent degrees of freedom of polarization, denoted by \( \epsilon_{33}(k, m), \epsilon_{13}(k, m) \) and \( \epsilon_{23}(k, m) \), for \( k^i = k \delta^i_3 \). All the components are written as

\[
\epsilon_{11}(k, m) = \epsilon_{22}(k, m) = -\frac{1}{2} \frac{m^2}{m^2 + k^2} \epsilon_{33}(k, m),
\]

\[
\epsilon_{00}(k, m) = \frac{k^2}{m^2 + k^2} \epsilon_{33}(k, m),
\]

\[
\epsilon_{01}(k, m) = -\epsilon_{31}(k, 0) \delta(m),
\]

\[
\epsilon_{02}(k, m) = -\epsilon_{32}(k, 0) \delta(m),
\]

\[
\epsilon_{03}(k, m) = -\frac{k}{\sqrt{k^2 + m^2}} \epsilon_{33}(k, m),
\]

\[
\epsilon_{31}(k, m) = \epsilon_{31}(k, 0) \delta(m), \quad \epsilon_{32}(k, m) = \epsilon_{32}(k, 0) \delta(m),
\]

\[
\epsilon_{33}(k, m) = \epsilon_{33}(k, m), \quad \epsilon_{12}(k, m) = 0.
\]

2. The perturbations \( h_{\alpha \beta} \) can be written as

\[
h_{\alpha \beta}(t, X^i) = \int d^3 k d m (m X^4)^2 \epsilon_{\alpha \beta}(k, m) N_2(m X^4) e^{-i\sqrt{k^2 + m^2} t + ikx^3}.
\]

(D6)

On the brane, it reads

\[
h_{\alpha \beta}(\tau) = \int d^3 k d m \frac{m^2 l^2}{a^2(\tau)} \epsilon_{\alpha \beta}(k, m) N_2(m a^{-1}(\tau)) e^{-i\sqrt{k^2 + m^2} \tau + ikx^3}.
\]

(D7)
3. Finally, the perturbation on the position of the brane is

\[
\zeta(\tau) = \int d^3k \frac{m^2l^2}{2a(\tau)} \epsilon_3 \langle k, m \rangle e^{-i \sqrt{k^2 + m^2} \tau + i k x^3} \times \\
\times \left[ \frac{iHl}{\sqrt{m^2 + k^2}} \left( 1 - \frac{3m^2}{2k^2} \right) N_2(mla^{-1}(\tau)) - \right. \\
\left. - \frac{m}{(m^2 + k^2)} \left( 1 + \frac{3m^2}{2k^2} \right) \sqrt{1 + H^2l^2N_1(mla^{-1}(\tau))} \right] .
\]

(D8)

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