ON BLOWUP SOLUTIONS TO THE FOCUSING $L^2$-SUPERCRITICAL NONLINEAR FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. In this paper we study dynamical properties of blowup solutions to the focusing $L^2$-supercritical nonlinear fractional Schrödinger equation

$$i\partial_t (-\Delta)^su = -|u|^\alpha u, \quad u(0) = u_0, \quad \text{on } [0, \infty) \times \mathbb{R}^d,$$

where $d \geq 2$, $\frac{d}{2s} \leq s < 1$, $\frac{d}{4} < \alpha < \frac{d}{2s}$ and $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ is radial with the critical Sobolev exponent $s_c$. To this end, we establish a compactness lemma related to the equation by means of the profile decomposition for bounded sequences in $\dot{H}^{s_c} \cap \dot{H}^s$. As a result, we obtain the $\dot{H}^{s_c}$-concentration and the limiting profile with critical $\dot{H}^{s_c}$-norm of blowup solutions with bounded $\dot{H}^{s_c}$-norm.

1. Introduction

In this paper, we consider the Cauchy problem for the focusing $L^2$-supercritical nonlinear fractional Schrödinger equation

$$\left\{ \begin{array}{l}
i\partial_t u - (-\Delta)^su = -|u|^\alpha u, \quad \text{on } [0, +\infty) \times \mathbb{R}^d, \\
u(0) = u_0, \end{array} \right.$$  \quad (1.1)

where $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{C}$, $s \in (0, 1) \setminus \{1/2\}$ and $\alpha > 0$. The operator $(-\Delta)^s$ is the fractional Laplacian which is the Fourier multiplier by $|\xi|^{2s}$. The fractional Schrödinger equation was discovered by N. Laskin [24] as a result of extending the Feynmann path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The fractional Schrödinger equation also appears in the study of water waves equations (see e.g. Refs. [21, 26]). The study of the nonlinear fractional Schrödinger equation has attracted a lot of interest in the last decade (see e.g. Refs. [2, 6, 7, 8, 13, 14, 15, 19, 21, 23, 27, 29] and references cited therein).

The equation (1.1) enjoys the scaling invariance

$$u_\lambda(t, x) := \lambda^{\frac{2s}{\alpha}} u(\lambda^{2s} t, \lambda x), \quad \lambda > 0.$$  

A calculation shows

$$\|u_\lambda(0)\|_{\dot{H}^{s_c}} = \lambda^{s_c + \frac{2s}{\alpha} - \frac{d}{2}} \|u_0\|_{\dot{H}^{s_c}}.$$  

From this, we define the critical Sobolev exponent

$$s_c := \frac{d}{2} - \frac{2s}{\alpha}, \quad (1.2)$$  

as well as the critical Lebesgue exponent

$$\alpha_c := \frac{2d}{d - 2s_c} = \frac{d\alpha}{2s}. \quad (1.3)$$  

By definition, we have the Sobolev embedding $\dot{H}^{s_c} \hookrightarrow L^{\alpha_c}$. The equation (1.1) is called $L^2$-subcritical ($L^2$-critical or $L^2$-supercritical) if $s_c < 0$ ($s_c = 0$ or $s_c > 0$) respectively.

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The local well-posedness for (1.1) in Sobolev spaces with non-radial initial data was studied in Ref. [19] (see also Ref. [10]). In the non-radial setting, the unitary group $e^{-it(-\Delta)^s}$ enjoys Strichartz estimates (see Ref. [5] or Ref. [10]):

$$
\|e^{-it(-\Delta)^s}\psi\|_{L^p(R\times R^n)} \lesssim \|\nabla|\gamma|^{\frac{p}{q}}\psi\|_{L^2},
$$

where $(p, q)$ satisfies the Schrödinger admissible condition

$$
p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2},
$$

and

$$
\gamma_{p, q} = \frac{d}{2} - \frac{d}{q} - \frac{2s}{p}.
$$

It is easy to see that the condition $\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}$ implies $\gamma_{p, q} > 0$ for all Schrödinger admissible pairs $(p, q)$ except $(p, q) = (\infty, 2)$. This means that for non-radial data, Strichartz estimates for $e^{-it(-\Delta)^s}$ have a loss of derivatives except for $(p, q) = (\infty, 2)$. This makes the study of local well-posedness in the non-radial case more difficult. The local theory for (1.1) showed in Refs. [19, 10] is much weaker than the one for classical nonlinear Schrödinger equation, i.e. $s = 1$. In particular, in the $H^s$-subcritical case (i.e. $s_c < s$) the equation (1.1) is locally well-posed in $H^s$ only for dimensions $d = 1, 2, 3$. The loss of derivatives in Strichartz estimates can be removed if one considers radial initial data. More precisely, we have for $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$ and $\psi$ radial,

$$
\|e^{-it(-\Delta)^s}\psi\|_{L^p(R\times R^n)} \lesssim \|\psi\|_{L^2},
$$

provided that $(p, q)$ satisfies the fractional admissible condition

$$
p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q) \neq \left(\frac{2d}{d-3}, \frac{4d-2}{2d-3}\right), \quad \frac{2s}{p} + \frac{d}{q} = \frac{d}{2}.
$$

These Strichartz estimates with no loss of derivatives allow us to show a better local theory for (1.1) with radial initial data. We refer the reader to Section 2 for more details.

The existence of blowup solutions to (1.1) was studied numerically in Ref. [23]. Later, Boulenger-Himmelsbach-Lenzmann [2] established blowup criteria for radial $H^s$ solutions to (1.1). Note that in Ref. [2], they considered $H^{2s}$ solutions due to the lack of a full local theory at the time of consideration. Thanks to the local theory given in Section 2, we can recover $H^s$ solutions by approximation arguments. More precisely, they proved the following:

**Theorem 1.1** (Ref. [2]). Let $d \geq 2$, $s \in (1/2, 1)$ and $\alpha > 0$. Let $u_0 \in H^s$ be radial and assume that the corresponding solution to (1.1) exists on the maximal forward time interval $[0, T)$.

- **Mass-critical case**: If $s_c = 0$ or $\alpha = \frac{d}{4}$ and $E(u_0) < 0$, then the solution $u$ either blows up in finite time, i.e. $T < +\infty$ or blows up infinite time, i.e. $T = +\infty$ and

  $$
  \|u(t)\|_{H^s} \geq Ct^s, \quad \forall t \geq t_*,
  $$

  for some $C > 0$ and $t_* > 0$ depending only on $u_0, s$ and $d$.

- **Mass-supercritical and energy-subcritical case**: If $0 < s_c < s$ or $\frac{4s}{d} < \alpha < \frac{4s}{d-2s}$ and $\alpha < 4s$ and either $E(u_0) < 0$, or if $E(u_0) \geq 0$, we assume that

  $$
  E^{s_c}(u_0)M^{s-s_c}(u_0) < E^{s_c}(Q)M^{s-s_c}(Q), \quad \|u_0\|_{H^{s_c}}\|u_0\|_{L^{2-s_c}}^{s_c} > \|Q\|_{H^{s_c}}\|Q\|_{L^{2-s_c}}^{s_c},
  $$

  where $Q$ is the unique (up to symmetries) positive radial solution to the elliptic equation

  $$
  (-\Delta)^s Q + Q - |Q|^\alpha Q = 0,
  $$

  then the solution blows up in finite time, i.e. $T < +\infty$. 

• **Energy-critical case:** If \( s_c = s \) or \( \alpha = \frac{4s}{d-2s} \) and \( \alpha < 4s \) and either \( E(u_0) < 0 \), or if \( E(u_0) \geq 0 \), we assume that

\[
E(u_0) < E(W), \quad \|u_0\|_{\dot{H}^s} > \|W\|_{\dot{H}^s},
\]

where \( W \) is the unique (up to symmetries) positive radial solution to the elliptic equation

\[
(-\Delta)^s W - |W|^{\frac{4s}{d-2s}} W = 0,
\]

then the solution blows up in finite time, i.e. \( T < +\infty \).

Here \( M(u) \) and \( E(u) \) are the conserved mass and energy respectively.

The blowup criteria of Boulenger-Himmelsbach-Lenzmann \([2]\) naturally lead to the study of dynamical properties such as blowup rate, concentration and limiting profile, of blowup solutions to \((1.1)\).

In the mass-critical case \( s_c = 0 \) or \( \alpha = \frac{4s}{d} \), the dynamics of blowup \( H^s \) solutions was recently considered in Ref. \([11]\) (see also Ref. \([13]\)). The study of blowup \( H^s \) solutions to the focusing mass-critical nonlinear fractional Schrödinger equation is connected to the notion of ground state which is the unique (up to symmetries) positive radial solution of the elliptic equation

\[
(-\Delta)^s Q + Q - |Q|^\frac{4s}{d} Q = 0.
\]

Note that the existence and uniqueness (modulo symmetries) of ground state to \((1.4)\) were shown in Refs. \([14, 15]\). Using the sharp Gagliardo-Nirenberg inequality

\[
\|f\|_{L^\frac{4s}{d-2s}}^2 \leq C_{GN} \|f\|_{L^2} \|f\|_{\dot{H}^s}^2,
\]

with

\[
C_{GN} = \frac{2s+d}{d} \|Q\|_{L^2}^{-\frac{4s}{d}},
\]

the conservation of mass and energy show that if \( u_0 \in H^s \) satisfies \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), then the corresponding solution exists globally in time. This suggests that \( \|Q\|_{L^2} \) is the critical mass for formation of singularities. To study dynamical properties of blowup \( H^s \) solutions to the mass-critical \((1.1)\), the author in Ref. \([11]\) proved a compactness lemma related to the equation by means of the profile decomposition for bounded sequences in \( H^s \).

**Proposition 1.2** (Compactness lemma \([11]\)). Let \( d \geq 1 \) and \( 0 < s < 1 \). Let \( (v_n)_{n \geq 1} \) be a bounded sequence in \( H^s \) such that

\[
\limsup_{n \to \infty} \|v_n\|_{\dot{H}^s} \leq M, \quad \limsup_{n \to \infty} \|v_n\|_{L^{\frac{4s}{d-2s}}} \geq m.
\]

Then there exists a sequence \( (x_n)_{n \geq 1} \) in \( \mathbb{R}^d \) such that up to a subsequence,

\[
v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly in } H^s,
\]

for some \( V \in H^s \) satisfying

\[
\|V\|_{L^2}^\frac{4}{d} \geq \frac{d+2s}{d} \frac{m^{\frac{4s}{d-2s}}}{M^2} \|Q\|_{L^2}^\frac{4s}{d}.
\]

Thanks to this compactness lemma, the author in Ref. \([11]\) showed that the \( L^2 \)-norm of blowup solutions must concentrate by an amount which is bounded from below by \( \|Q\|_{L^2} \) at the blowup time. He also showed the limiting profile of blowup solutions with minimal mass \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), that is, up to symmetries of the equation, the ground state \( Q \) is the profile for blowup solutions with minimal mass.

The main goal of this paper is to study dynamical properties of blowup solutions to \((1.1)\) in the mass-supercritical and energy-subcritical case with initial data in \( \dot{H}^{s_c} \cap \dot{H}^s \). To this end, we
first show the local well-posedness for (1.1) with initial data in $\dot{H}^{s_c} \cap \dot{H}^s$. For data in $H^s$, the local well-posedness in non-radial and radial cases was showed in Refs. [19, 11]. In the non-radial setting, the inhomogeneous Sobolev embedding $W^{s,q} \hookrightarrow L^r$ plays a crucial role (see e.g. Ref. [19]). Since we are considering data in $\dot{H}^{s_c} \cap \dot{H}^s$, the inhomogeneous Sobolev embedding does not help. We thus have to rely on Strichartz estimates without loss of derivatives and the homogeneous Sobolev embedding $W^{s,q} \hookrightarrow L^r$. We hence restrict ourself to radially symmetric initial data, $d \geq 2$ and $\frac{4}{d} \leq s < 1$ for which Strichartz estimates without loss of derivatives are available. After the local theory is established, we show the existence of blowup $\dot{H}^{s_c} \cap \dot{H}^s$ solutions. The existence of blowup $H^s$ solutions for (1.1) was shown in Ref. [2] (see Theorem 1.1). Note that the conservation of mass plays a crucial role in the argument of Ref. [2]. In our consideration, the lack of mass conservation laws makes the problem more difficult. We are only able to show blowup criteria for negative energy initial data in $\dot{H}^{s_c} \cap \dot{H}^s$ with an additional assumption

$$\sup_{t \in [0,T]} \|u(t)\|_{\dot{H}^{s_c}} < \infty,$$  \hspace{1cm} (1.5)

where $[0,T)$ is the maximal forward time of existence. In the mass-critical case $s_c = 0$, this assumption holds trivially by the conservation of mass. We refer to Section 2 for more details. To study blowup dynamics for data in $\dot{H}^{s_c} \cap \dot{H}^s$, we prove the profile decomposition for bounded sequences in $\dot{H}^{s_c} \cap \dot{H}^s$ which is proved by following the argument of Ref. [20] (see also Refs. [18, 12]). This profile decomposition allows us to study the variational structure of the sharp constant to the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{n+2}}^{n+2} \leq A_{GN} \|f\|_{\dot{H}^{s_c}}^{\alpha} \|f\|_{\dot{H}^s}^{2 - \alpha}.$$  \hspace{1cm} (1.6)

We will see in Proposition 3.2 that the sharp constant $A_{GN}$ is attained at a function $U \in \dot{H}^{s_c} \cap \dot{H}^s$ of the form

$$U(x) = aQ(\lambda x + x_0),$$

for some $a \in \mathbb{C}^*$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$, where $Q$ is a solution to the elliptic equation

$$(-\Delta)^s Q + (-\Delta)^s Q - |Q|^\alpha Q = 0.$$  \hspace{1cm} (1.7)

Moreover,

$$A_{GN} = \frac{\alpha + 2}{2} \|Q\|_{\dot{H}^{s_c}}^{-\alpha}.$$  \hspace{1cm} (1.8)

The sharp Gagliardo-Nirenberg inequality (1.6) together with the conservation of energy yield the global existence for solutions satisfying

$$\sup_{t \in [0,T)} \|u(t)\|_{\dot{H}^{s_c}} < \|Q\|_{\dot{H}^{s_c}}.$$  \hspace{1cm} (1.9)

Another application of the profile decomposition is the compactness lemma, that is, for any bounded sequence $(v_n)_{n \geq 1}$ in $\dot{H}^{s_c} \cap \dot{H}^s$ satisfying

$$\limsup_{n \to \infty} \|v_n\|_{\dot{H}^s} \leq M, \quad \limsup_{n \to \infty} \|v_n\|_{L^{n+2}} \geq m,$$

there exists a sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } \dot{H}^{s_c} \cap \dot{H}^s,$$

for some $V \in \dot{H}^{s_c} \cap \dot{H}^s$ satisfying

$$\|V\|_{\dot{H}^{s_c}}^\alpha \geq \frac{2}{\alpha + 2} \frac{m^{\alpha + 2}}{M^2} \|Q\|_{\dot{H}^{s_c}}^\alpha.$$  \hspace{1cm} (1.10)
As a consequence, we show that the $\dot{H}^{s_c}$-norm of blowup solutions satisfying (1.5) must concentrate by an amount which is bounded from below by $\|Q\|_{\dot{H}^{s_c}}$ at the blowup time (see Theorem 4.1). We finally show in Theorem 5.2 the limiting profile of blowup solutions with critical norm

$$\sup_{t \in [0,T]} \|u(t)\|_{\dot{H}^{s_c}} = \|Q\|_{\dot{H}^{s_c}}.$$  

(1.7)

The paper is organized as follows. In Section 2, we recall Strichartz estimates and show the local well-posedness for data in $\dot{H}^{s_c} \cap \dot{H}^{s}$. We also prove blowup criteria for negative energy data in $\dot{H}^{s_c} \cap \dot{H}^s$ as well as the profile decomposition of bounded sequences in $\dot{H}^{s_c} \cap \dot{H}^{s}$. In Section 3, we give some applications of the profile decomposition including the sharp Gagliardo-Nirenberg inequality (1.6) and the compactness lemma. In Section 4, we show the $\dot{H}^{s_c}$-concentration of blowup solutions. Finally, the limiting profile of blowup solutions with critical norm (1.7) will be given in Section 5.

2. Preliminaries

2.1. Homogeneous Sobolev spaces. We recall the definition of homogeneous Sobolev spaces needed in the sequel (see e.g. Refs. [1], [16] or [28]). Denote $S_0$ the subspace of the Schwartz space $S$ consisting of functions $\phi$ satisfying $D^\beta \phi(0) = 0$ for all $\beta \in \mathbb{N}^d$, where $:\mathcal{F}$ is the Fourier transform on $S$. Given $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, the generalized homogeneous Sobolev space $\dot{W}^{\gamma,q}$ is defined as a closure of $S_0$ under the norm

$$\|u\|_{\dot{W}^{\gamma,q}} := \|\nabla^\gamma u\|_{L^q} < \infty.$$ 

Under this setting, the spaces $\dot{W}^{\gamma,q}$ are Banach spaces. We shall use $\dot{H}^{\gamma} := \dot{W}^{\gamma,2}$. Note that the spaces $\dot{H}^{\gamma_1}$ and $\dot{H}^{\gamma_2}$ cannot be compared for the inclusion. Nevertheless, for $\gamma_1 < \gamma < \gamma_2$, the space $\dot{H}^{\gamma}$ is an interpolation space between $\dot{H}^{\gamma_1}$ and $\dot{H}^{\gamma_2}$.

2.2. Strichartz estimates. We next recall Strichartz estimates for the fractional Schrödinger equation. To do so, we define for $I \subset \mathbb{R}$ and $p, q \in [1, \infty]$ the mixed norm

$$\|u\|_{L^p(I; L^q)} := \left( \int_I \left( \int_{\mathbb{R}^d} |u(t,x)|^q \, dx \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}},$$

with a usual modification when either $p$ or $q$ are infinity. The unitary group $e^{-i t (\Delta)^s}$ enjoys several types of Strichartz estimates, for instance non-radial Strichartz estimates, radial Strichartz estimates and weighted Strichartz estimates (see e.g. Ref. [6]). We only recall here two types: non-radial and radial Strichartz estimates.

- **Non-radial Strichartz estimates** (see e.g. Refs. [5, 10]): for $d \geq 1$ and $s \in (0,1) \setminus \{1/2\}$, the following estimates hold:

$$\|e^{-i t (\Delta)^s} \psi\|_{L^p(\mathbb{R}, L^q)} \lesssim \|\nabla^{\gamma_{p,q}} \psi\|_{L^2},$$

$$\left\| \int_0^t e^{-i (t-\tau) (\Delta)^s} f(\tau) \, d\tau \right\|_{L^p(\mathbb{R}, L^q)} \lesssim \|\nabla^{\gamma_{p,q} - \gamma_{a',\mu'}} - 2s f\|_{L^{a'}(\mathbb{R}, L^{\mu'})},$$

where $(p, q)$ and $(a, b)$ are Schrödinger admissible pairs, i.e.

$$p \in [2, \infty], \quad q \in [2, \infty], \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2},$$

and

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{2s}{p},$$

and similarly for $\gamma_{a',\mu'}$ As mentioned in the introduction, these Strichartz estimates have a loss of derivatives except for $(p, q) = (a, b) = (\infty, 2)$. 



Remark 2.3. When \( t = \infty \), the equation \( e^{-it(-\Delta)^s}u \) is radially symmetric and \( (p, q, (a, b)) \) satisfies the fractional admissible condition:

\[
p \in [2, \infty], \; q \in [2, \infty), \; (p, q) \neq \left( 2, \frac{4d - 2}{2d - 3} \right), \quad \frac{2s}{p} + \frac{d}{q} = \frac{d}{2}. \tag{2.3}
\]

2.3. Local well-posedness. In this subsection, we show the local well-posedness for (1.1) with initial data in \( \dot{H}^{s_c} \cap \dot{H}^s \). Before entering some details, let us recall the local well-posedness for (1.1) with initial data in \( \dot{H}^s \).

**Proposition 2.1** (Local well-posedness in \( \dot{H}^s \) [11]). Let

\[
\begin{align*}
d = 1, \quad & \frac{1}{2} < s < \frac{1}{3}, \quad 0 < \alpha < \frac{4d}{d - 2s}, \; u_0 \in \dot{H}^s \text{ non-radial,} \\
d = 1, \quad & \frac{1}{2} < s < 1, \quad 0 < \alpha < \infty, \; u_0 \in \dot{H}^s \text{ non-radial,} \\
d = 2, \quad & \frac{1}{2} < s < 1, \quad 0 < \alpha < \frac{4s}{4s - 2}, \; u_0 \in \dot{H}^s \text{ non-radial,} \\
d = 3, \quad & \frac{3}{5} \leq s \leq \frac{3}{4}, \quad 0 < \alpha < \frac{4s}{4s - 2}, \; u_0 \in \dot{H}^s \text{ radial,} \\
d = 3, \quad & \frac{3}{5} < s < 1, \quad 0 < \alpha < \frac{4s}{4s - 2}, \; u_0 \in \dot{H}^s \text{ non-radial,} \\
d \geq 4, \quad & \frac{d}{2d - 1} < s < 1, \quad 0 < \alpha < \frac{4s}{4s - 2}, \; u_0 \in \dot{H}^s \text{ radial.}
\end{align*}
\tag{2.4}
\]

Then the equation (1.1) is locally well-posed in \( \dot{H}^s \). In addition, the maximal forward time of existence satisfies either \( T = +\infty \) or \( T < +\infty \) and \( \lim_{t \uparrow T} \|u\|_{\dot{H}^s} = \infty \). Moreover, the solution enjoys the conservation of mass and energy, i.e. \( M(u(t)) = M(u_0) \) and \( E(u(t)) = E(u_0) \) for all \( t \in [0, T) \), where

\[
\begin{align*}
M(u(t)) & = \int |u(t, x)|^2 dx, \\
E(u(t)) & = \frac{1}{2} \int |(-\Delta)^{s/2} u(t, x)|^2 dx - \frac{1}{\alpha + 2} \int |u(t, x)|^\alpha + 2 dx.
\end{align*}
\]

We now give the local well-posedness for (1.1) with initial data in \( \dot{H}^{s_c} \cap \dot{H}^s \).

**Proposition 2.2** (Local well-posedness in \( \dot{H}^{s_c} \cap \dot{H}^s \)). Let \( d \geq 2, \; \frac{d}{2d - 1} \leq s < 1 \) and \( \frac{4s}{d} \leq \alpha < \frac{4s}{d - 2s} \). Let

\[
p = \frac{4s(\alpha + 2)}{\alpha(d - 2s)}, \quad q = \frac{d(\alpha + 2)}{d + \alpha s}.
\tag{2.5}
\]

Then for any \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^s \) radial, there exist \( T > 0 \) and a unique solution \( u \) to (1.1) satisfying

\[
u \in C([0, T), \dot{H}^{s_c} \cap \dot{H}^s) \cap L^p_{\text{loc}}([0, T), \dot{W}^{s_c, q} \cap \dot{W}^{s, q}).
\]

The maximal forward time of existence satisfies either \( T = +\infty \) or \( T < +\infty \) and \( \lim_{t \uparrow T} \|u(t)\|_{\dot{H}^{s_c}} + \|u(t)\|_{\dot{H}^s} = \infty \). Moreover, the solution enjoys the conservation of energy, i.e. \( E(u(t)) = E(u_0) \) for all \( t \in [0, T) \).

**Remark 2.3.** When \( s_c = 0 \) or \( \alpha = \frac{4s}{d} \), Proposition 2.2 is a consequence of Proposition 2.1 since \( \dot{H}^0 = L^2 \) and \( L^2 \cap \dot{H}^s = \dot{H}^s \).
Proof of Proposition 2.2. It is easy to check that $(p, q)$ satisfies the fractional admissible condition (2.3). We next choose $(m, n)$ so that

$$\frac{1}{p'} = \frac{1}{p} + \frac{\alpha}{m}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{n}.$$ 

We see that

$$\theta := \frac{\alpha}{m} - \frac{\alpha}{p} = 1 - \frac{(d - 2s)\alpha}{4s} > 0, \quad q \leq n = \frac{dq}{d - sq}.$$ 

The later fact ensures the Sobolev embedding $\dot{W}^{s, q} \hookrightarrow L^q$. Consider

$$X := \left\{ u \in C(I, \dot{H}^s \cap \dot{H}^s) \cap L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q}) : \|u\|_{L^\infty(I, \dot{H}^s \cap \dot{H}^s)} + \|u\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})} \leq M \right\},$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^p(I, L^q)},$$

where $I = [0, \zeta]$ and $M, \zeta > 0$ to be determined later. Thanks to Duhamel’s formula, it suffices to show that the functional

$$\Phi(t) := e^{-it(-\Delta)^s} u_0 + i \int_0^t e^{-i(t-\tau)(-\Delta)^s} |u(\tau)|^\alpha u(\tau) d\tau$$

is a contraction on $(X, d)$. Thanks to Strichartz estimates (2.1) and (2.2),

$$\|\Phi(u)\|_{L^\infty(I, \dot{H}^s \cap \dot{H}^s)} + \|\Phi(u)\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})} \lesssim \|u_0\|_{\dot{H}^s} + \|u\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})},$$

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(I, L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I, L^q)} \lesssim \|u - v\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})}.$$ 

By the fractional derivatives (see e.g. Proposition 3.1 of Ref. [9]) and the choice of $(m, n)$, the Hölder inequality implies

$$\|u\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})} \lesssim \|u\|_{L^m(I, \dot{H}^n)} \|u\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})} \lesssim \|u\|_{L^m(I, \dot{H}^n)} \|u\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})} \lesssim \|u\|_{L^m(I, \dot{H}^n)} \|u\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})}.$$ 

Similarly,

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(I, L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I, L^q)} \lesssim \|u - v\|_{L^p(I, L^q)}.$$ 

This shows that for all $u, v \in X$, there exists $C > 0$ independent of $\zeta$ and $u_0 \in \dot{H}^s \cap \dot{H}^s$ such that

$$\|\Phi(u)\|_{L^\infty(I, \dot{H}^s \cap \dot{H}^s)} + \|\Phi(u)\|_{L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q})} \leq C \|u_0\|_{\dot{H}^s} + C\zeta^\theta M^{\alpha + 1},$$

$$d(\Phi(u), \Phi(v)) \leq C\zeta^\theta M^{\alpha} d(u, v).$$

If we set $M = 2C\|u_0\|_{\dot{H}^s}$ and choose $\zeta > 0$ so that

$$C\zeta^\theta M^{\alpha} \leq \frac{1}{2},$$

then $\Phi$ is a strict contraction on $(X, d)$. This proves the existence of solution

$$u \in C(I, \dot{H}^s \cap \dot{H}^s) \cap L^p(I, \dot{W}^{s, q} \cap \dot{W}^{s, q}).$$

Note that by radial Strichartz estimates, the solution belongs to $L^a(I, \dot{W}^{s, b} \cap \dot{W}^{s, b})$ for any fractional admissible pairs $(a, b)$. The blowup alternatives is easy since the time of existence depends
only on the \( \dot{H}^s \cap H^s \)-norm of initial data. The conservation of energy follows from the standard approximation. The proof is complete.

**Corollary 2.4** (Blowup rate). Let \( d \geq 2, \frac{d}{d-1} \leq s < 1, \frac{4s}{d} \leq \alpha < \frac{4s}{d-2s} \) and \( u_0 \in \dot{H}^s \cap H^s \) be radial. Assume that the corresponding solution \( u \) to (1.1) given in Proposition 2.2 blows up at finite time \( 0 < T < +\infty \). Then there exists \( C > 0 \) such that

\[
\|u(t)\|_{\dot{H}^s \cap H^s} > \frac{C}{(T-t)^{\frac{s-\alpha}{2s}}},
\]

for all \( 0 < t < T \).

**Proof.** Let \( 0 < t < T \). If we consider (1.1) with initial data \( u(t) \), then it follows from (2.6) and the fixed point argument that if for some \( M > 0 \),

\[
C\|u(t)\|_{\dot{H}^s \cap H^s} + C(\zeta - t)^{\theta} M^{\alpha+1} \leq M,
\]

then \( \zeta < T \). Thus,

\[
C\|u(t)\|_{\dot{H}^s \cap H^s} + C(T-t)^{\theta} M^{\alpha+1} > M,
\]

for all \( M > 0 \). Choosing \( M = 2C\|u(t)\|_{\dot{H}^s \cap H^s} \), we see that

\[
(T-t)^{\theta} \|u(t)\|_{H^s} > C.
\]

This implies

\[
\|u(t)\|_{\dot{H}^s \cap H^s} > \frac{C}{(T-t)^{\frac{s-\alpha}{2s}}},
\]

which is exactly (2.7) since \( \frac{d}{\alpha} = \frac{4s - \alpha(d-2s)}{4s} = \frac{s-\alpha}{2s} \). The proof is complete. \( \Box \)

### 2.4 Blowup criteria

In this subsection, we prove blowup criteria for \( \dot{H}^s \cap \dot{H}^s \) solutions to the mass-supercritical and energy-subcritical (1.1). For initial data in \( H^s \), Boulenger-Himmelsbach-Lenzmann proved blowup criteria for the equation (see Theorem 1.1 for more details). The main difficulty in our consideration is that the conservation of mass is no longer available. We overcome this difficulty by assuming that the solution satisfies the uniform bound (1.5). More precisely, we have the following:

**Proposition 2.5** (Blowup criteria). Let \( d \geq 2, \frac{d}{d-1} \leq s < 1, \frac{4s}{d} < \alpha < \frac{4s}{d-2s} \) and \( \alpha < 4s \). Let \( u_0 \in \dot{H}^s \cap H^s \) be radial satisfying \( E(u_0) < 0 \). Assume that the corresponding solution to (1.1) defined on a maximal forward time interval \([0,T]\) satisfies (1.5). Then the solution \( u \) blows up in finite time, i.e. \( T < +\infty \).

**Remark 2.6.** The condition \( \alpha < 4s \) comes from the radial Sobolev embedding (a analogous condition appears in Ref. [2] (see again Theorem 1.1)).

**Proof of Proposition 2.5.** Let \( \chi : [0,\infty) \to [0,\infty) \) be a smooth function such that

\[
\chi(r) = \begin{cases} 
 r^2 & \text{if } r \leq 1, \\
 0 & \text{if } r \geq 2,
\end{cases}
\]

and \( \chi''(r) \leq 2 \) for \( r \geq 0 \).

For a given \( R > 0 \), we define the radial function \( \chi_R : \mathbb{R}^d \to \mathbb{R} \) by

\[
\varphi_R(x) = \varphi_R(r) := R^2 \chi(r/R), \quad |x| = r.
\]

It is easy to see that

\[
2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi'_R(r)}{r} \geq 0, \quad 2d - \Delta \varphi_R(x) \geq 0, \quad \forall r \geq 0, \forall x \in \mathbb{R}^d.
\]

Moreover,

\[
\|\nabla^j \varphi_R\|_{L^\infty} \lesssim R^{2-j}, \quad j = 0, \ldots, 4,
\]
Using the fact \( \text{supp}(\nabla^{j} \varphi_{R}) \subset \{ |x| \leq 2R \} \) for \( j = 1, 2 \), \( \{ R \leq |x| \leq 2R \} \) for \( j = 3, 4 \).

Now let \( u \in \dot{H}^{\infty} \cap \dot{H}^{s} \) be a solution to (1.1). We define the local virial action by

\[
M_{\varphi_{R}}(t) := 2 \int \nabla \varphi_{R}(x) \cdot \text{Im}(\bar{u}(t, x) \nabla u(t, x)) dx.
\]

The virial action \( M_{\varphi_{R}}(t) \) is well-defined. Indeed, we first learn from the Hölder inequality and the Sobolev embedding \( \dot{H}^{\infty} \rightarrow L^{\alpha_{c}} \) that

\[
\|u\|_{L^{2}(|x| \leq R)} \leq R^{\frac{\alpha_{c}}{2}} \|u\|_{L^{\alpha_{c}}(|x| \leq R)} \leq R^{\frac{\alpha_{c}}{2}} \|u\|_{\dot{H}^{\infty}(|x| \leq R)}. \tag{2.8}
\]

Using the fact \( \text{supp}(\nabla \varphi_{R}) \subset \{ |x| \leq R \} \), (2.8) and the estimate given in Lemma A.1 of Ref. [2], we have

\[
|M_{\varphi_{R}}(t)| \leq C(\chi, R) \left( \| \nabla^{\frac{3}{2}} u(t) \|_{L^{2}(|x| \leq R)}^{2} + \| u(t) \|_{L^{2}(|x| \leq R)} \| \nabla^{\frac{1}{2}} u(t) \|_{L^{2}(|x| \leq R)} \right)
\leq C(\chi, R) \left( \| u(t) \|_{H^{\infty}(|x| \leq R)}^{2} \right)
\leq C(\chi, R) \left( \| u(t) \|_{H^{\infty}(|x| \leq R)}^{2} \right).
\tag{2.9}
\]

This shows that \( M_{\varphi_{R}}(t) \) is well-defined for all \( t \in [0, T) \). Note that in the case \( \chi(r) = r^{2} \) or \( \varphi_{R}(x) = |x|^{2} \), we have formally the virial identity (see Lemma 2.1 of Ref. [2]):

\[
M'_{|x|^{2}}(t) = 8s \| u(t) \|_{H^{s}}^{2} - \frac{4\alpha}{\alpha + 2} \| u(t) \|_{L^{\alpha_{c}}+2}^{\alpha_{c}+2} = 4\alpha E(u(t)) - 2(\alpha - 4s) \| u(t) \|_{H^{s}}^{2}. \tag{2.10}
\]

We also have from Lemma 2.1 of [2] that for any \( t \in [0, T) \),

\[
M'_{\varphi_{R}}(t) = - \int_{0}^{\infty} m^{s} \int \Delta^{2} \varphi_{R} |u_{m}(t)|^{2} dxdm + 4 \sum_{j,k=1}^{d} \int_{0}^{\infty} m^{s} \int \partial_{jk}^{2} \varphi_{R} \partial_{j} \bar{u}_{m}(t) \partial_{k} u_{m}(t) dxdm - \frac{2\alpha}{\alpha + 2} \int \Delta \varphi_{R} |u(t)|^{\alpha_{c}+2} dx,
\]

where

\[
\hat{u}_{m}(t) := c_{s} \frac{1}{\Delta + m} u(t) = c_{s} \frac{\hat{u}(t)}{|\xi|^{2} + m}, \quad m > 0,
\]

with

\[
c_{s} := \sqrt{\frac{\sin \pi s}{\pi}}.
\]
Since $\varphi_R(x) = |x|^2$ for $|x| \leq R$, we use (2.10) to write

\[
M'_{\varphi_R}(t) = 8s\|u(t)\|_{H^s}^2 - \frac{4d\alpha}{\alpha + 2}\|u(t)\|_{L^{\infty+2}}^{\alpha+2} - 8s\|u(t)\|_{H^s(|x| > R)}^2 + \frac{4d\alpha}{\alpha + 2}\|u(t)\|_{L^{\infty+2}(|x| > R)}^{\alpha+2}
- \int_0^\infty m^s \int_{|x| > R} \Delta^2 \varphi_R|u_m(t)|^2 dx dm
+ 4 \sum_{j,k=1}^{\infty} \int_0^\infty m^s \int_{|x| > R} \partial^2_{jk} \varphi_R \partial_j \partial_k u_m(t) dx dm - 8s\|u(t)\|_{H^s(|x| > R)}^2.
\]

Using

\[
\partial^2_{jk} = \left( \delta_{jk} - \frac{x_j x_k}{r^2} \right) \frac{\partial}{\partial r} + \frac{x_j x_k}{r^2} \partial^2_r,
\]

we write

\[
4 \sum_{j,k=1}^{\infty} \int_0^\infty m^s \int_{|x| > R} \partial^2_{jk} \varphi_R \partial_j \partial_k u_m(t) dx dm = 4 \int_0^\infty m^s \int_{|x| > R} \varphi_R |\nabla u_m(t)|^2 dx dm.
\]

Note that (see (2.12) in Ref. [2])

\[
\int_0^\infty m^s \int |\nabla f_m|^2 dx dm = \int \left( \frac{\sin \pi s}{\pi} \int_0^\infty m^s \int_0^\infty \frac{m^s}{(|\xi|^2 + m^2)^2} dx \right) |\xi|^2 |\hat{f}(\xi)|^2 d\xi = s\|f\|_{H^s}^2.
\]

We thus get

\[
4 \sum_{j,k=1}^{\infty} \int_0^\infty m^s \int_{|x| > R} \partial^2_{jk} \varphi_R \partial_j \partial_k u_m(t) dx dm
= 8s\|u(t)\|_{H^s(|x| > R)}^2 - 4 \int_0^\infty m^s \int_{|x| > R} (2 - \varphi'_R)|\nabla u_m(t)|^2 dx dm
\leq 8s\|u(t)\|_{H^s(|x| > R)}^2.
\]

Thanks to Lemma A.2 of Ref. [2], the definition of $\varphi_R$ and the uniform bound (1.5), we estimate

\[
\left| \int_0^\infty m^s \int_{|x| > R} \Delta^2 \varphi_R|u_m(t)|^2 dx dm \right| \lesssim \|\Delta^2 \varphi_R\|_{L^\infty} \|\Delta \varphi_R\|_{L^\infty}^{-1} \|u(t)\|_{H^s(|x| \leq R)}^2
\lesssim R^{-2s} R^{2s} \|u(t)\|_{H^s(|x| \leq R)}^2 \lesssim R^{2(2s-s_c)}.
\]

We thus obtain

\[
M'_{\varphi_R}(t) \leq 4d\alpha E(u(t)) - 2(d\alpha - 4s)\|u(t)\|_{H^s}^2 + CR^{-2(s-s_c)}
+ \frac{2\alpha}{\alpha + 2} \int_{|x| > R} (2d - \Delta \varphi_R)|u(t)|^{\alpha+2} dx.
\]
Since \( \|2d - \Delta \varphi_R \|_{L^\infty} \lesssim 1 \), it remains to bound \( \|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|> R)} \). To do this, we make use of the argument of Ref. [25] (see also Ref. [12]). Consider for \( A > 0 \) the annulus \( C = \{A < |x| \leq 2A\} \), we claim that for any \( \epsilon > 0 \),
\[
\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|> R)} \leq \epsilon \|u(t)\|^{2}_{H^s} + C(\epsilon)A^{-2(s-s_c)},
\]  
(2.11)
To show (2.11), we recall the radial Sobolev embedding (see e.g. Ref. [4]):
\[
\sup_{x \neq 0} |x|^{\frac{\alpha}{d} - \beta} |f(x)| \leq C(d, \beta) \|f\|_{\dot{H}^\beta},
\]
for all radial functions \( f \in \dot{H}^\beta(\mathbb{R}^d) \) with \( \frac{1}{2} < \beta < \frac{d}{2} \). Thanks to radial Sobolev embedding and (2.8), we have
\[
\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(C)} \lesssim \left( \sup_{C} |u(t, x)| \right)^\alpha \|u(t)\|^{2}_{L^2(C)}
\]
\[
\lesssim A^{-\left(\frac{d}{2} - \beta\right)\alpha} \|u(t)\|^{\alpha}_{\dot{H}^\beta(C)} \|u(t)\|^{2}_{L^2(C)}
\]
\[
\lesssim A^{-\left(\frac{d}{2} - \beta\right)\alpha} \left( \|u(t)\|^{\frac{\alpha}{d}}_{\dot{H}^\beta(C)} \|u(t)\|^{1-\frac{\beta}{d}}_{L^2(C)} \right)^\alpha \|u(t)\|^{2}_{L^2(C)}
\]
\[
\lesssim A^{-\left(\frac{d}{2} - \beta\right)\alpha} \|u(t)\|^{\frac{\alpha}{d}}_{\dot{H}^\beta(C)} \|u(t)\|^{1+\frac{\alpha}{d}}_{L^2(C)}
\]
\[
\lesssim A^{-\theta} \|u(t)\|^{\frac{\alpha}{d}}_{\dot{H}^\beta(C)},
\]  
(2.12)
where
\[
\theta := \left( \frac{d}{2} - \beta \right) \alpha - \left( 1 - \frac{\beta}{s} \right) \alpha + 2 s_c.
\]
It is easy to check that
\[
\theta = 2(s-s_c) \left( 1 - \frac{\alpha \beta}{2s} \right).
\]
By our assumption \( \alpha < 4s \), we can choose \( \frac{1}{2} < \beta < s \) so that \( \theta > 0 \). We next apply the Young inequality to have for any \( \epsilon > 0 \),
\[
A^{-\theta} \|u(t)\|^{\frac{\alpha}{d}}_{\dot{H}^\beta(C)} \lesssim \epsilon \|u(t)\|^{2}_{\dot{H}^\beta(C)} + C(\epsilon)A^{-\frac{2\alpha s}{2s}} = \epsilon \|u(t)\|^{2}_{\dot{H}^\beta(C)} + C(\epsilon)A^{-2(s-s_c)}.
\]
This combined with (2.12) prove (2.11). We now write
\[
\int_{|x|> R} |u(t)|^{\alpha+2} dx = \sum_{j=0}^{\infty} \int_{2^j R < |x| \leq 2^{j+1} R} |u(t)|^{\alpha+2} dx,
\]
and apply (2.11) with \( A = 2^j R \) to get
\[
\int_{|x|> R} |u(t)|^{\alpha+2} dx \leq \epsilon \sum_{j=0}^{\infty} \|u(t)\|^{2}_{\dot{H}^\beta(2^j R < |x| \leq 2^{j+1} R)} + C(\epsilon) \sum_{j=0}^{\infty} (2^j R)^{-2(s-s_c)}
\]
\[
\leq \epsilon \|u(t)\|^{2}_{\dot{H}^\beta(|x|> R)} + C(\epsilon)R^{-2(s-s_c)}.
\]
This shows that for any \( \epsilon > 0 \),
\[
\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|> R)} \leq \epsilon \|u(t)\|^{2}_{\dot{H}^\beta(|x|> R)} + C(\epsilon)R^{-2(s-s_c)},
\]
and hence
\[
M'_{\varphi R}(t) \leq 4d\alpha E(u(t)) - 2(d\alpha - 4s)\|u(t)\|^{2}_{\dot{H}^s} + O \left( R^{-2(s-s_c)} + \epsilon \|u(t)\|^{2}_{\dot{H}^s} + C(\epsilon)R^{-2(s-s_c)} \right).
\]
By the conservation of energy with $E(u_0) < 0$ and the fact $d\alpha > 4s$, we take $\epsilon > 0$ small enough and $R > 0$ large enough to obtain

$$M_{\varphi R}'(t) \leq 2d\alpha E(u_0) - \delta \|u(t)\|_{H^s}^2, \tag{2.13}$$

where $\delta := d\alpha - 4s > 0$. We now follow the argument of Ref. [2]. Since $E(u_0) < 0$, we learn from (2.13) that $M_{\varphi R}'(t) \leq -c$ for $c > 0$. From this, we conclude that $M_{\varphi R}(t) < 0$ for all $t > t_1$ for some sufficiently large time $t_1 \gg 1$. Taking integration over $[t_1, t]$, we have

$$M_{\varphi R}(t) \leq -\delta \int_{t_1}^t \|u(\tau)\|_{H^s}^2 d\tau \leq 0, \quad \forall t \geq t_1. \tag{2.14}$$

We have from (2.9) and the assumption (1.5) that

$$|M_{\varphi R}(t)| \leq C(\chi, R) \left( \|u(t)\|_{H^s}^\frac{1}{2} + \|u(t)\|_{H^s}^\frac{1}{2} \right). \tag{2.15}$$

We also have

$$\|u(t)\|_{H^s} \gtrsim 1, \quad \forall t \geq 0. \tag{2.16}$$

Indeed, suppose it is not true. Then there exists a sequence $(t_n)_n \subset [0, +\infty)$ such that $\|u(t_n)\|_{H^s} \to 0$ as $n \to \infty$. Thanks to the Gagliardo-Nirenberg inequality (1.6) and the assumption (1.5), we see that $\|u(t_n)\|_{L^{s+2}} \to 0$. We thus get $E(u(t_n)) \to 0$, which is a contradiction to $E(u(t)) = E(u_0) < 0$. This shows (2.16). Combining (2.15) and (2.16), we obtain

$$|M_{\varphi R}(t)| \leq C(\chi, R) \|u(t)\|_{H^s}^\frac{1}{2}. \tag{2.17}$$

Therefore, (2.14) and (2.17) yield

$$M_{\varphi R}(t) \leq C(\chi, R) \int_{t_1}^t |M_{\varphi R}(\tau)|^{2s} d\tau, \quad \forall t \geq t_1.$$

By nonlinear integral inequality, we get

$$M_{\varphi R}(t) \lesssim C(\chi, R)|t - t_*|^{1-2s},$$

for $s > 1/2$ with some $t_* < +\infty$. Therefore, $M_{\varphi R}(t) \to -\infty$ as $t \uparrow t_*$. Hence the solution cannot exist for all times $t \geq 0$. The proof is complete. \qed

2.5. **Profile decomposition.** In this subsection, we recall the profile decomposition for bounded sequences in $H^\infty \cap H^s$.

**Theorem 2.7** (Profile decomposition). Let $d \geq 1$, $0 < s < 1$ and $\frac{4s}{d} < \alpha < 2^*$, where

$$2^* := \begin{cases} \frac{4s}{d} & \text{if } d > 2s, \\ \infty & \text{if } d \leq 2s. \end{cases} \tag{2.18}$$

Let $(v_n)_{n \geq 1}$ be a bounded sequence in $H^\infty \cap H^s$. Then there exist a subsequence still denoted $(v_n)_{n \geq 1}$, a family $(x_n^j)_{j \geq 1}$ of sequences in $\mathbb{R}^d$ and a sequence $(V^j)_{j \geq 1}$ of functions in $H^\infty \cap H^s$ such that

- for every $k \neq j$,
  $$|x_n^k - x_n^j| \to \infty, \quad \text{as } n \to \infty, \tag{2.19}$$
- for every $l \geq 1$ and every $x \in \mathbb{R}^d$,
  $$v_n(x) = \sum_{j=1}^t V^j(x - x_n^j) + v_{n,l}(x),$$
with
\[ \limsup_{n \to \infty} \|v_n^l\|_{L^q} \to 0, \quad \text{as } l \to \infty, \] (2.20)
for every \( q \in (\alpha_c, 2 + 2^*) \), where \( \alpha_c \) is given in (1.3). Moreover,
\[ \|v_n\|^2_{\dot{H}^s} = \sum_{j=1}^l \|v_j\|^2_{\dot{H}^s} + \|v_n\|^2_{\dot{H}^s} + o_n(1), \] (2.21)
\[ \|v_n\|^2_{\dot{H}^s} = \sum_{j=1}^l \|v_j\|^2_{\dot{H}^s} + \|v_n\|^2_{\dot{H}^s} + o_n(1), \] (2.22)
as \( n \to \infty \).

**Remark 2.8.** In the case \( s_c = 0 \) or \( \alpha = \frac{4s}{d} \), Theorem 2.7 is exactly Theorem 3.1 in Ref. [11] due to the fact \( \dot{H}^0 = L^2 \) and \( L^2 \cap \dot{H}^s = \dot{H}^s \).

**Proof of Theorem 2.7.** The proof is based on the argument of Ref. [20] (see also Refs. [18, 12]). For reader’s convenience, we give some details. Since \( \dot{H}^s \cap \dot{H}^s \) is a Hilbert space, we denote \( \Omega(v_n) \) the set of functions obtained as weak limits of sequences of the translated \( v_n(\cdot + x_n) \) with \( (x_n)_{n \geq 1} \) a sequence in \( \mathbb{R}^d \). Set
\[ \eta(v_n) := \text{sup}\{\|v\|_{\dot{H}^s} + \|v\|_{\dot{H}^s} : v \in \Omega(v_n)\}. \]
Clearly,
\[ \eta(v_n) \leq \limsup_{n \to \infty} \|v_n\|_{\dot{H}^s} + \|v_n\|_{\dot{H}^s}. \]
We will show that there exist a sequence \((V^j)_{j \geq 1}\) of \( \Omega(v_n) \) and a family \((x^j_n)_{j \geq 1}\) of sequences in \( \mathbb{R}^d \) such that for every \( k \neq j \),
\[ |x^k_n - x^j_n| \to \infty, \]
as \( n \to \infty \) and up to a subsequence, we can write for every \( l \geq 1 \) and every \( x \in \mathbb{R}^d \),
\[ v_n(x) = \sum_{j=1}^l V^j(x - x^j_n) + v_n^l(x), \]
with \( \eta(v_n^l) \to 0 \) as \( l \to \infty \). Moreover, (2.21) and (2.22) hold as \( n \to \infty \).

Indeed, if \( \eta(v_n) = 0 \), then we take \( V^j = 0 \) for all \( j \geq 1 \) and the proof is done. Otherwise we choose \( V^1 \in \Omega(v_n) \) such that
\[ \|V^1\|_{\dot{H}^s} + \|V^1\|_{\dot{H}^s} \geq \frac{1}{2} \eta(v_n) > 0. \]
By definition, there exists a sequence \((x^1_n)_{n \geq 1}\) in \( \mathbb{R}^d \) such that up to a subsequence,
\[ v_n(\cdot + x^1_n) \rightharpoonup V^1 \text{ weakly in } \dot{H}^s \cap \dot{H}^s. \]
Set \( v_n^1(x) := v_n(x) - V^1(x - x^1_n) \). It follows that \( v_n^1(\cdot + x^1_n) \rightharpoonup 0 \) weakly in \( \dot{H}^s \cap \dot{H}^s \) and thus
\[ \|v_n\|^2_{\dot{H}^s} = \|V^1\|^2_{\dot{H}^s} + \|v_n^1\|^2_{\dot{H}^s} + o_n(1), \]
\[ \|v_n\|^2_{\dot{H}^s} = \|V^1\|^2_{\dot{H}^s} + \|v_n^1\|^2_{\dot{H}^s} + o_n(1), \]
as \( n \to \infty \). We next replace \((v_n)_{n \geq 1}\) by \((v_n^1)_{n \geq 1}\) and repeat the same argument. If \( \eta(v_n^1) = 0 \), then we take \( V^j = 0 \) for all \( j \geq 2 \) and the proof is done. Otherwise there exist \( V^2 \in \Omega(v_n^1) \) and a sequence \((x^2_n)_{n \geq 1}\) in \( \mathbb{R}^d \) such that
\[ \|V^2\|_{\dot{H}^s} + \|V^2\|_{\dot{H}^s} \geq \frac{1}{2} \eta(v_n^1) > 0. \]
and
\[ v_n^1(\cdot + x_n^2) \rightarrow V^2 \] weakly in \( \dot{H}^{s_c} \cap \dot{H}^s \).

Set \( v_n^2(x) := v_n^1(x) - V^2(x - x_n^2) \). It follows that \( v_n^2(\cdot + x_n^2) \rightarrow 0 \) weakly in \( \dot{H}^{s_c} \cap \dot{H}^s \) and
\[
\|v_n^1\|_{H^{s_c}}^2 = \|V^2\|_{H^{s_c}}^2 + \|v_n^2\|_{H^{s_c}}^2 + o_n(1),
\]
\[
\|v_n^1\|_{H^s}^2 = \|V^2\|_{H^s}^2 + \|v_n^2\|_{H^s}^2 + o_n(1),
\]
as \( n \rightarrow \infty \). We now show that
\[
|x_n^1 - x_n^2| \rightarrow \infty,
\]
as \( n \rightarrow \infty \). Indeed, if it is not true, then up to a subsequence, \( x_n^1 - x_n^2 \rightarrow x_0 \) as \( n \rightarrow \infty \) for some \( x_0 \in \mathbb{R}^d \). Rewriting
\[
v_n^1(x + x_n^2) = v_n^1(x + (x_n^2 - x_n^2) + x_n^1),
\]
and using the fact \( v_n^1(\cdot + x_n^2) \) converges weakly to 0, we see that \( V^2 = 0 \). This implies that \( \eta(v_n^1) = 0 \), which is a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family \( (x_n^j)_{j \geq 1} \) of sequences in \( \mathbb{R}^d \) and the sequence \( (V^j)^{j \geq 1} \) of functions in \( \dot{H}^{s_c} \cap \dot{H}^s \) satisfying the claim above. Moreover, the convergence of the series \( \sum_{j=1}^{\infty} \|V^j\|_{H^{s_c}}^2 + \|V^j\|_{H^s}^2 \) implies that
\[
\|V^j\|_{H^{s_c}}^2 + \|V^j\|_{H^s}^2 \rightarrow 0, \quad \text{as } j \rightarrow \infty.
\]
By construction,
\[
\eta(v_n^j) \leq 2 \left( \|V^{j+1}\|_{H^{s_c}} + \|V^{j+1}\|_{H^s} \right),
\]
which shows that \( \eta(v_n^j) \rightarrow 0 \) as \( j \rightarrow \infty \). It remains to show (2.20). To this end, we introduce for \( R > 1 \) a function \( \phi_R \in \mathcal{S} \) satisfying \( \hat{\phi}_R : \mathbb{R}^d \rightarrow [0,1] \) and
\[
\hat{\phi}_R(\xi) = \begin{cases} 
1 & \text{if } 1/R \leq |\xi| \leq R, \\
0 & \text{if } |\xi| \leq 1/2R \vee |\xi| \geq 2R.
\end{cases}
\]
We write
\[
v_n^j = \phi_R \ast v_n^j + (\delta - \phi_R) \ast v_n^j,
\]
where \( \delta \) is the Dirac function and \( * \) is the convolution operator. Let \( q \in (\alpha_c, 2 + 2\alpha) \) be fixed. By Sobolev embedding and the Plancherel formula,
\[
\| (\delta - \phi_R) \ast v_n^j \|_{L^q} \lesssim \| (\delta - \phi_R) \ast v_n^j \|_{\dot{H}^s} \lesssim \left( \int |\xi|^{2\beta} (1 - \hat{\phi}_R(\xi)) \hat{v_n^j}(\xi)^2 \, d\xi \right)^{1/2} 
\]
\[
\lesssim \left( \int_{|\xi| \leq 1/R} |\xi|^{2\beta} |\hat{v_n^j}(\xi)|^2 \, d\xi \right)^{1/2} + \left( \int_{|\xi| \geq R} |\xi|^{2\beta} |\hat{v_n^j}(\xi)|^2 \, d\xi \right)^{1/2} 
\]
\[
\lesssim R^{\alpha - \beta} \|v_n^j\|_{H^{s_c}} + R^{\beta - \alpha} \|v_n^j\|_{H^s},
\]
where \( \beta = \frac{d}{2} - \frac{d}{q} \in (s_c, s) \). Besides, the H"{o}lder interpolation inequality yields
\[
\| \phi_R \ast v_n^j \|_{L^q} \lesssim \| \phi_R \ast v_n^j \|_{L^q}^{1 - \frac{1}{\nu_c}} \| \phi_R \ast v_n^j \|_{L^{\infty}}^{\frac{1}{\nu_c}} 
\]
\[
\lesssim \|v_n^j\|_{H^{s_c}} \| \phi_R \ast v_n^j \|_{L^\infty}^{1 - \frac{1}{\nu_c}}. 
\]
Observe that
\[
\limsup_{n \rightarrow \infty} \| \phi_R \ast v_n^j \|_{L^\infty} = \sup_{x_n} \limsup_{n \rightarrow \infty} |\phi_R \ast v_n^j(x_n)|.
\]
By the definition of \( \Omega(v_n^j) \), we see that
\[
\limsup_{n \rightarrow \infty} \| \phi_R \ast v_n^j \|_{L^\infty} \leq \sup \left\{ \left| \int \phi_R(-x)v(x) \, dx \right| : v \in \Omega(v_n^j) \right\}.
\]
The Plancherel formula then implies
\[ \left| \int \hat{\phi}_R(-x) \hat{v}(x) dx \right| = \left| \int \hat{\phi}_R(\xi) \hat{\eta}(\xi) d\xi \right| \lesssim \left\| \hat{\xi}^{-s_c} \hat{\phi}_R \right\|_{L^2} \left\| \xi^{s_c} \hat{\eta} \right\|_{L^2} \lesssim R^{\frac{2}{1-s_c}} \| \hat{\phi}_R \|_{\dot{H}^{-s_c}} \| v \|_{\dot{H}^{s_c}} \lesssim R^{\frac{2}{1-s_c}} \eta(v^{l_n}). \]
Thus, for every \( l \geq 1 \),
\[ \limsup_{n \to \infty} \| v^{l_n} \|_{L^q} \lesssim \limsup_{n \to \infty} \| (\delta - \phi_R) * v^{l_n} \|_{L^q} + \limsup_{n \to \infty} \| \phi_R * v^{l_n} \|_{L^q} \]
\[ \lesssim R^{\frac{2}{1-s_c}} \| v^{l_n} \|_{\dot{H}^{s_c}} + R^{\frac{2}{1-s_c}} \| v^{l_n} \|_{\dot{H}^s} + \| v^{l_n} \|_{\dot{H}^{s_c}} \left( R^{\frac{2}{1-s_c}} \eta(v^{l_n}) \right) \left( 1 - \frac{\alpha}{\beta} \right). \]
Choosing \( R = \left[ \eta(v^{l_n})^{-1} \right]^{\frac{1}{2(1-s_c)}} \) for some \( \epsilon > 0 \) small enough, we learn that
\[ \limsup_{n \to \infty} \| v^{l_n} \|_{L^q} \lesssim \eta(v^{l_n})^{(\beta-s_c)(\frac{1}{2(1-s_c)}-\epsilon)} \| v^{l_n} \|_{\dot{H}^{s_c}} + \eta(v^{l_n})^{(s-\beta)(\frac{1}{2(1-s_c)}-\epsilon)} \| v^{l_n} \|_{\dot{H}^s} + \eta(v^{l_n})^{\frac{1}{2(1-s_c)}} \| v^{l_n} \|_{\dot{H}^{s_c}}. \]
Letting \( l \to \infty \) and using the uniform boundedness of \( (v^{l_n})_{l \geq 1} \) in \( \dot{H}^{s_c} \cap \dot{H}^s \) together with the fact that \( \eta(v^{l_n}) \to 0 \) as \( l \to \infty \), we obtain
\[ \limsup_{n \to \infty} \| v^{l_n} \|_{L^q} \to 0, \quad \text{as} \ l \to \infty. \]
This completes the proof of Theorem 2.7. \( \square \)

3. Variational analysis

Let \( d \geq 1, 0 < s < 1 \) and \( \frac{4s}{d} < \alpha < 2^* \) where \( 2^* \) is given in (2.18). We consider the variational problems
\[ A_{GN} := \max \{ H(f) : f \in \dot{H}^{s_c} \cap \dot{H}^s \}, \quad H(f) := \| f \|_{L^{s+2}}^{\alpha+2} \div \left[ \| f \|_{\dot{H}^{s_c}}^\alpha \| f \|_{\dot{H}^s}^2 \right], \]
\[ B_{GN} := \max \{ K(f) : f \in L^{s_c} \cap \dot{H}^s \}, \quad K(f) := \| f \|_{L^{s+2}}^{\alpha+2} \div \left[ \| f \|_{L^{s_c}}^\alpha \| f \|_{\dot{H}^s}^2 \right]. \]
Here \( A_{GN} \) and \( B_{GN} \) are respectively sharp constants in the following Gagliardo-Nirenberg inequalities
\[ \| f \|_{L^{s+2}}^{\alpha+2} \leq A_{GN} \| f \|_{\dot{H}^{s_c}}^\alpha \| f \|_{\dot{H}^s}^2, \]
\[ \| f \|_{L^{s+2}}^{\alpha+2} \leq B_{GN} \| f \|_{L^{s_c}}^\alpha \| f \|_{\dot{H}^s}^2. \]

**Lemma 3.1.** If \( g \) and \( h \) are maximizers of \( H(f) \) and \( K(f) \) respectively, then \( g \) and \( h \) satisfy
\[ A_{GN} \| g \|_{\dot{H}^{s_c}}^\alpha (-\Delta)^s g + \frac{\alpha}{2} A_{GN} \| g \|_{\dot{H}^{s_c}}^{\alpha-2} \| g \|_{\dot{H}^s}^2 (-\Delta)^s g - \frac{\alpha + 2}{2} |g|^{\alpha} g = 0, \] (3.1)
\[ B_{GN} \| h \|_{L^{s_c}}^\alpha (-\Delta)^s h + \frac{\alpha}{2} B_{GN} \| h \|_{L^{s_c}}^{\alpha-\alpha} \| h \|_{\dot{H}^s}^2 |h|^{\alpha-2} h - \frac{\alpha + 2}{2} |h|^{\alpha} h = 0, \] (3.2)
respectively.

**Proof.** Since \( g \) is a maximizer of \( H \) in \( \dot{H}^{s_c} \cap \dot{H}^s \), \( g \) satisfies the Euler-Lagrange equation
\[ \frac{d}{de} \bigg|_{e=0} H(g + e\phi) = 0, \]
Indeed, a simple computation shows
\[
\left. \frac{d}{de} \right|_{e=0} \| g + \epsilon \phi \|^2_{L^{\alpha+2}} = (\alpha + 2) \int \text{Re}(\| g^\alpha g \phi \|) dx,
\]
\[
\left. \frac{d}{de} \right|_{e=0} \| g + \epsilon \phi \|^\alpha_{H^{\alpha \epsilon}} = \alpha \| g \|^\alpha_{H^{\alpha \epsilon}} \int \text{Re}(\| (-\Delta)^{\epsilon} g \phi \|) dx,
\]
and
\[
\left. \frac{d}{de} \right|_{e=0} \| g + \epsilon \phi \|^2_{H^{\epsilon}} = 2 \int \text{Re}(\| (-\Delta)^{\epsilon} g \phi \|) dx.
\]

We thus get
\[
(\alpha + 2)\| g \|^\alpha_{H^{\alpha \epsilon}} \| g \|^2_{H^{\epsilon}}, \| g \|^\alpha_{L^{\alpha+2}} \| g \|^\alpha_{H^{\alpha \epsilon}}, \| (-\Delta)^{\epsilon} g - 2 \| g \|^\alpha_{L^{\alpha+2}} \| g \|^\alpha_{H^{\alpha \epsilon}}, (-\Delta)^{\epsilon} g = 0.
\]
Dividing by \(2\| g \|^\alpha_{H^{\alpha \epsilon}} \| g \|^2_{H^{\epsilon}}\), we obtain (3.1). The proof of (3.2) is similar by using
\[
\left. \frac{d}{de} \right|_{e=0} \| h + \epsilon \phi \|^\alpha_{L^{\alpha \epsilon}} = \alpha \| h \|^\alpha_{L^{\alpha \epsilon}} \int \text{Re}(\| (-\Delta)^{\epsilon} h \phi \|) dx.
\]

The proof is complete. \(\square\)

A first application of the profile decomposition given in Theorem 2.7 is the following variational structure of the sharp constants \(A_{\text{GN}}\) and \(B_{\text{GN}}\).

**Proposition 3.2** (Variational structure of sharp constants). Let \(d \geq 1, 0 < s < 1\) and \(\frac{d}{s} < \alpha < 2^s\).

- The sharp constant \(A_{\text{GN}}\) is attained at a function \(U \in H^{s} \cap H^{s}\) of the form
  \[
  U(x) = aQ(\lambda x + x_0),
  \]
  for some \(a \in \mathbb{C}, \lambda > 0\) and \(x_0 \in \mathbb{R}^d\), where \(Q\) is a solution to the elliptic equation
  \[
  (-\Delta)^s Q + (-\Delta)^s Q - |Q|^s Q = 0. \tag{3.3}
  \]
  Moreover,
  \[
  A_{\text{GN}} = \alpha + 2 \| Q \|^{-\alpha}_{H^{\alpha \epsilon}}.
  \]

- The sharp constant \(B_{\text{GN}}\) is attained at a function \(V \in L^{\alpha \epsilon} \cap H^s\) of the form
  \[
  V(x) = bR(\mu x + y_0),
  \]
  for some \(b \in \mathbb{C}, \mu > 0\) and \(y_0 \in \mathbb{R}^d\), where \(R\) is a solution to the elliptic equation
  \[
  (-\Delta)^s R + |R|^s R - |R|^s R = 0. \tag{3.4}
  \]
  Moreover,
  \[
  B_{\text{GN}} = \alpha + 2 \| R \|^{-\alpha}_{L^{\alpha \epsilon}}.
  \]

**Proof.** We only prove Item 1, the proof for Item 2 is similar using the Sobolev embedding \(H^{s} \hookrightarrow L^{\alpha \epsilon}\). Observe that \(H\) is invariant under the scaling
\[
f_{\mu, \lambda}(x) := \mu f(\lambda x), \quad \mu, \lambda > 0.
\]
Indeed, a simple computation shows
\[
\| f_{\mu, \lambda} \|^{\alpha+2}_{L^{\alpha+2}} = \mu^{\alpha+2} \lambda^{-d} \| f \|^{\alpha+2}_{L^{\alpha+2}}, \quad \| f_{\mu, \lambda} \|^{\alpha}_{H^{\alpha \epsilon}} = \mu^{\alpha} \lambda^{-2s} \| f \|^{\alpha}_{H^{\alpha \epsilon}}, \quad \| f_{\mu, \lambda} \|^{2}_{H^{s}} = \mu^{2} \lambda^{2s-d} \| f \|^{2}_{H^{s}}.
\]
Thus, $H(f_{\mu,\lambda}) = H(f)$ for any $\mu, \lambda > 0$. Moreover, if we set $g(x) = \mu f(\lambda x)$ with

$$
\mu = \left( \frac{\|f\|_{\dot{H}^{s}_{\infty}}}{\|f\|_{\dot{H}^{s}_{\ast}}} \right)^{\frac{1}{s-\varepsilon}}, \quad \lambda = \left( \frac{\|f\|_{\dot{H}^{s}_{\infty}}}{\|f\|_{\dot{H}^{s}_{\ast}}} \right)^{\frac{1}{s-\varepsilon}},
$$

then $\|g\|_{\dot{H}^{s}_{\infty}} = \|g\|_{\dot{H}^{s}_{\ast}} = 1$ and $H(g) = H(f)$. Now let $(v_n)_{n \geq 1}$ be the maximizing sequence of $H$, i.e. $H(v_n) \to A_{GN}$ as $n \to \infty$. By scaling invariance, we may assume that $\|v_n\|_{\dot{H}^{s}_{\infty}} = \|v_n\|_{\dot{H}^{s}_{\ast}} = 1$ and $H(v_n) = \|v_n\|_{L^{\alpha+2}}^{\alpha+2} \to A_{GN}$ as $n \to \infty$. It follows that $(v_n)_{n \geq 1}$ is bounded in $\dot{H}^{\infty} \cap \dot{H}^{s}$, and the profile decomposition given in Theorem 2.7 shows that there exist a sequence $(V^j)_{j \geq 1}$ of $\dot{H}^{\infty} \cap \dot{H}^{s}$ functions and a family $(x^j_n)_{j \geq 1}$ of sequences in $\mathbb{R}^d$ such that up to a subsequence,

$$
v_n(x) = \sum_{j=1}^{\infty} V^j(x - x^j_n) + v_n^{\dagger}(x),
$$

and (2.19), (2.20), (2.21) and (2.22) hold. In particular, for any $l \geq 1$,

$$
\sum_{j=1}^{l} \|V^j\|_{\dot{H}^{\alpha+2}}^2 \leq 1, \quad \sum_{j=1}^{l} \|V^j\|_{\dot{H}^{s}}^2 \leq 1,
$$

and

$$
\limsup_{n \to \infty} \|v_n^{\dagger}\|_{L^{\alpha+2}}^{\alpha+2} \to 0, \quad \text{as } l \to \infty.
$$

Thus,

$$
A_{GN} = \lim_{n \to \infty} \|v_n\|_{L^{\alpha+2}}^{\alpha+2} = \limsup_{n \to \infty} \left\| \sum_{j=1}^{l} V^j(\cdot - x^j_n) + v_n^{\dagger} \right\|_{L^{\alpha+2}}^{\alpha+2}
$$

$$
\leq \limsup_{n \to \infty} \left( \left\| \sum_{j=1}^{l} V^j(\cdot - x^j_n) \right\|_{L^{\alpha+2}} + \|v_n^{\dagger}\|_{L^{\alpha+2}} \right)^{\alpha+2}
$$

$$
\leq \limsup_{n \to \infty} \left\| \sum_{j=1}^{\infty} V^j(\cdot - x^j_n) \right\|_{L^{\alpha+2}}^{\alpha+2}. \quad (3.6)
$$

By the elementary inequality

$$
\left| \sum_{j=1}^{l} a_j \right|^{\alpha+2} - \sum_{j=1}^{l} \left| a_j \right|^{\alpha+2} \leq C \sum_{j \neq k} |a_j| |a_k|^{\alpha+1}, \quad (3.7)
$$

the pairwise orthogonality (2.19) leads the mixed terms in the sum (3.6) to vanish as $n \to \infty$. This shows that

$$
A_{GN} \leq \sum_{j=1}^{\infty} \|V^j\|_{L^{\alpha+2}}^{\alpha+2}.
$$

We also have from the definition of $A_{GN}$ that

$$
\frac{\|V^j\|_{L^{\alpha+2}}^{\alpha+2}}{A_{GN}} \leq \frac{\|V^j\|_{\dot{H}^{\alpha+2}}^{\alpha}}{\|V^j\|_{\dot{H}^{\infty}}} \|V^j\|_{\dot{H}^{s}}^2,
$$

which implies

$$
1 \leq \frac{\sum_{j=1}^{\infty} \|V^j\|_{L^{\alpha+2}}^{\alpha+2}}{A_{GN}} \leq \sup_{j \geq 1} \|V^j\|_{\dot{H}^{\infty}}^{\alpha} \sum_{j=1}^{\infty} \|V^j\|_{\dot{H}^{s}}^2.
$$
Since \( \sum_{j \geq 1} \|V^j\|_{H^s}^2 \) is convergent, there exists \( j_0 \geq 1 \) such that
\[
\|V^{j_0}\|_{H^s} = \sup_{j \geq 1} \|V^j\|_{H^s}.
\]

By (3.5), we see that
\[
1 \leq \|V^{j_0}\|_{H^s}^2 \sum_{j = 1}^{\infty} \|V^j\|_{H^s}^2 \leq \|V^{j_0}\|_{H^s}^2.
\]

It follows from (3.5) that \( \|V^{j_0}\|_{H^s} = 1 \) which shows that there is only one term \( V^{j_0} \) is non-zero. Hence,
\[
\|V^{j_0}\|_{H^s} = \|V^{j_0}\|_{H^s} = 1, \quad \|V^{j_0}\|_{L^{\alpha_2+2}} = A_{GN}.
\]

It means that \( V^{j_0} \) is the maximizer of \( H \), and Lemma 3.1 shows that
\[
A_{GN}(-\Delta)V^{j_0} + \frac{\alpha}{2}A_{GN}(-\Delta)^sV^{j_0} - \frac{\alpha + 2}{2}\|V^{j_0}\|^{\alpha}V^{j_0} = 0.
\]

Now if we set \( V^{j_0}(x) = aQ(\lambda x + x_0) \) for some \( a \in \mathbb{C}^* \), \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^d \), then \( Q \) solves (3.3) provided that
\[
|a| = \left( \frac{2\lambda^{2s}A_{GN}}{\alpha + 2} \right)^{\frac{1}{\alpha}}, \quad \lambda = \left( \frac{\alpha}{2} \right)^{\frac{1}{\alpha(\alpha - 2s)}}. \tag{3.8}
\]

This shows the existence of solutions to (3.3). We next compute the sharp constant \( A_{GN} \) in terms of \( Q \). We have
\[
1 = \|V^{j_0}\|_{H^s}^2 = |a|^\alpha|\lambda - 2s||Q||_{L^{\alpha}}^\alpha_{H^s} = \frac{2A_{GN}}{\alpha + 2}\|Q||_{H^s}^\alpha.
\]

This implies \( A_{GN} = \frac{\alpha + 2}{2}\|Q||_{H^s}^{-\alpha} \). The proof is complete. \( \square \)

**Remark 3.3.** By (3.8) and the fact
\[
1 = \|V^{j_0}\|_{H^s}^2 = |a|^\alpha|\lambda - 2s||Q||_{H^s}^\alpha,
1 = \|V^{j_0}\|_{H^s}^2 = |a|^\alpha|\lambda - 2s||Q||_{H^s}^\alpha,
\]
we have the following Pohozaev identities
\[
\|Q||_{H^s}^2 = \frac{\alpha}{\alpha + 2}\|Q||_{H^s}^{\alpha + 2}, \tag{3.9}
\]

The above identities can be showed by multiplying (3.3) with \( \overline{Q} \) and \( x \cdot \nabla \overline{Q} \) and integrating over \( \mathbb{R}^d \) and performing integration by parts. Indeed, multiplying (3.3) with \( \overline{Q} \) and integrating by parts, we get
\[
\|Q||_{H^s}^2 + \|Q||_{H^s}^2 - \|Q||_{L^{\alpha}+2}^{\alpha + 2} = 0. \tag{3.10}
\]

Multiplying (3.3) with \( x \cdot \nabla \overline{Q} \), integrating by parts and taking the real part, we have
\[
\left(s + \frac{d}{\alpha} \right)\|Q||_{H^s}^2 + \left(s - \frac{d}{\alpha} \right)\|Q||_{H^s}^2 + \frac{d}{\alpha + 2}\|Q||_{L^{\alpha}+2}^{\alpha + 2} = 0. \tag{3.11}
\]

From (3.10) and (3.11), we obtain (3.9). Here we use the fact that for \( \gamma \geq 0 \),
\[
\text{Re} \int (-\Delta)^\gamma Qx \cdot \nabla \overline{Q} dx = \left(\gamma - \frac{d}{2}\right)\|Q||_{H^s}^2.
\]

The Pohozaev identities (3.9) imply in particular that
\[
H(Q) = \|Q||_{L^{\alpha}+2}^{\alpha + 2} + \|Q||_{H^s}^2 \|Q||_{H^s}^2 = \frac{\alpha + 2}{2}\|Q||_{H^s}^\alpha = A_{GN}, \quad E(Q) = 0.
\]
Similarly, we have
\[ \|R\|_{L^{\infty}}^2 = \frac{\alpha}{2} \|R\|_{H^s}^2 = \frac{\alpha}{\alpha + 2} \|R\|_{L^{\alpha+2}}^{\alpha+2}. \]
In particular,
\[ K(R) = \|R\|_{L^{\alpha+2}}^{\alpha+2} \div \left( \|R\|_{L^{\infty}}^2 \|R\|_{H^s}^2 \right) = \frac{\alpha + 2}{2} \|R\|_{L^{\infty}}^\alpha = B_{GN}, \quad E(R) = 0. \]

**Definition 3.4** (Ground state). • We call **Sobolev ground states** the maximizers of \( H \) which are solutions to (3.3). We denote the set of Sobolev ground states by \( G \).

• We call **Lebesgue ground states** the maximizers of \( K \) which are solutions to (3.4). We denote the set of Lebesgue ground states by \( H \).

Note that by Lemma 3.1, if \( g, h \) are respectively Sobolev and Lebesgue ground states, then
\[
A_{GN} = \frac{\alpha + 2}{2} \|g\|_{H^{\infty}}^{-\alpha}, \quad B_{GN} = \frac{\alpha + 2}{2} \|h\|_{L^{\infty}}^{-\alpha}.
\]
This implies that Sobolev ground states have the same \( H^{\infty} \)-norm, and all Lebesgue ground states have the same \( L^{\infty} \)-norm. Denote
\[
S_{gs} := \|g\|_{H^{\infty}}, \quad \forall g \in G,
\]
\[
L_{gs} := \|h\|_{L^{\infty}}, \quad \forall h \in H. \tag{3.12}
\]
In particular, we have the following sharp Gagliardo-Nirenberg inequalities
\[
\|f\|_{L^{\alpha+2}}^{\alpha+2} \leq A_{GN} \|f\|_{H^{\infty}}^\alpha \|f\|_{H^s}^2, \tag{3.14}
\]
\[
\|f\|_{L^{\alpha+2}}^{\alpha+2} \leq B_{GN} \|f\|_{L^{\infty}}^\alpha \|f\|_{H^s}^2, \tag{3.15}
\]
with
\[
A_{GN} = \frac{\alpha + 2}{2} S_{gs}^{-\alpha}, \quad B_{GN} = \frac{\alpha + 2}{2} L_{gs}^{-\alpha}.
\]

Another application of the profile decomposition given in Theorem 2.7 is the following compactness lemma.

**Theorem 3.5** (Compactness lemma). Let \( d \geq 1, 0 < s < 1 \) and \( \frac{4s}{d} < \alpha < 2^* \). Let \( (v_n)_{n \geq 1} \) be a bounded sequence in \( \dot{H}^{\infty} \cap \dot{H}^s \) such that
\[
\limsup_{n \to \infty} \|v_n\|_{H^s} \leq M, \quad \limsup_{n \to \infty} \|v_n\|_{L^{\alpha+2}} \geq m.
\]
• Then there exists a sequence \( (x_n)_{n \geq 1} \) in \( \mathbb{R}^d \) such that up to a subsequence,
\[
v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } \dot{H}^{\infty} \cap \dot{H}^s,
\]
for some \( V \in \dot{H}^{\infty} \cap \dot{H}^s \) satisfying
\[
\|V\|_{H^s}^\alpha \geq \frac{2}{\alpha + 2} m^{\alpha+2} S_{gs}^\alpha. \tag{3.16}
\]
• Then there exists a sequence \( (y_n)_{n \geq 1} \) in \( \mathbb{R}^d \) such that up to a subsequence,
\[
v_n(\cdot + y_n) \rightharpoonup W \text{ weakly in } L^{\alpha} \cap \dot{H}^s,
\]
for some \( W \in L^{\alpha} \cap \dot{H}^s \) satisfying
\[
\|W\|_{L^{\alpha}}^\alpha \geq \frac{2}{\alpha + 2} m^{\alpha+2} L_{gs}^\alpha. \tag{3.17}
\]

**Remark 3.6.** The lower bounds (3.16) and (3.17) are optimal. In fact, if we take \( v_n = Q \in G \) in the first case and \( v_n = R \in H \) in the second case where \( Q \) and \( R \) are given in Proposition 3.2, then we get the equalities.
Proof of Theorem 3.5. We only consider the first case, the second case is treated similarly using the Sobolev embedding \( \dot{H}^s \hookrightarrow L^\infty \). By Theorem 2.7, there exist a sequence \((V_j)_{j \geq 1}\) of \( \dot{H}^s \cap \dot{H}^s \) functions and a family \((x_n^j)_{j \geq 1}\) of sequences in \( \mathbb{R}^d \) such that up to a subsequence, the sequence \((v_n)_n \geq 1\) can be written as

\[
v_n(x) = \sum_{j=1}^l V^j(x - x_n^j) + v_n^l(x),
\]

and (2.20), (2.21) and (2.22) hold. This implies that

\[
m^{\alpha + 2} \leq \sup_{n \to \infty} \|v_n\|_{L^{\alpha + 2}}^{\alpha + 2} = \sup_{n \to \infty} \left\| \sum_{j=1}^l V^j(\cdot - x_n^j) + v_n^l \right\|_{L^{\alpha + 2}}^{\alpha + 2}
\]

\[
\leq \sup_{n \to \infty} \left( \left\| \sum_{j=1}^l V^j(\cdot - x_n^j) \right\|_{L^{\alpha + 2}} + \|v_n^l\|_{L^{\alpha + 2}} \right)^{\alpha + 2}
\]

\[
\leq \sup_{n \to \infty} \left\| \sum_{j=1}^\infty V^j(\cdot - x_n^j) \right\|_{L^{\alpha + 2}}^{\alpha + 2}.
\]

(3.18)

By the elementary inequality (3.7) and the pairwise orthogonality (2.19), the mixed terms in the sum (3.18) vanish as \( n \to \infty \). We thus get

\[
m^{\alpha + 2} \leq \sum_{j=1}^\infty \|V_j\|_{L^{\alpha + 2}}^{\alpha + 2}.
\]

By the sharp Gagliardo-Nirenberg inequality (3.14), we bound

\[
\sum_{j=1}^\infty \|V_j\|_{L^{\alpha + 2}}^{\alpha + 2} \leq \alpha + \frac{2}{2} \sup_{j \geq 1} \|V_j\|_{\dot{H}^s}^\alpha \sum_{j=1}^\infty \|V_j\|_{\dot{H}^s}^2.
\]

By (2.22), we infer that

\[
\sum_{j=1}^\infty \|V_j\|_{\dot{H}^s}^2 \leq \sup_{n \to \infty} \|v_n\|_{\dot{H}^s}^2 \leq M^2.
\]

Therefore,

\[
\sup_{j \geq 1} \|V_j\|_{\dot{H}^s}^\alpha \geq \frac{2}{\alpha + 2} \frac{m^{\alpha + 2}}{M^2} S_{\beta}^\alpha.
\]

Since the series \( \sum_{j \geq 1} \|V_j\|_{\dot{H}^s}^2 \) is convergent, the supremum above is attained. That is, there exists \( j_0 \) such that

\[
\|V_{j_0}\|_{\dot{H}^s}^\alpha \geq \frac{2}{\alpha + 2} \frac{m^{\alpha + 2}}{M^2} S_{\beta}^\alpha.
\]

Rewriting

\[
v_n(x + x_n^{j_0}) = V_{j_0}(x) + \sum_{j \leq j_0} V^j(x + x_n^{j_0} - x_n^j) + v_n^l(x),
\]

with \( v_n^l(x) := v_n^l(x + x_n^{j_0}) \), it follows from the pairwise orthogonality of the family \((x_n^j)_{j \geq 1}\) that

\[
V^j(\cdot + x_n^j - x_n^j) \rightharpoonup 0 \text{ weakly in } \dot{H}^s \cap \dot{H}^s,
\]

as \( n \to \infty \) for every \( j \neq j_0 \). This shows that

\[
v_n(\cdot + x_n^{j_0}) \rightharpoonup V_{j_0} + v^l, \quad \text{as } n \to \infty,
\]

(3.19)
where \( \tilde{v}^l \) is the weak limit of \( (\tilde{v}^l_n)_{n \geq 1} \). On the other hand,
\[
\| \tilde{v}^l \|_{L^{\alpha+2}} \leq \limsup_{n \to \infty} \| \tilde{v}^l_n \|_{L^{\alpha+2}} = \limsup_{n \to \infty} \| v_n^l \|_{L^{\alpha+2}} \to 0, \quad \text{as } l \to \infty.
\]
By the uniqueness of the weak limit (3.19), we get \( \tilde{v}^l = 0 \) for every \( l \geq j_0 \). Therefore, we obtain
\[
v_n(\cdot + x^{j_0}_n) \to V^{j_0}.
\]

The sequence \( (x^{j_0}_n)_{n \geq 1} \) and the function \( V^{j_0} \) now fulfill the conditions of Theorem 3.5. This ends the proof. \( \square \)

We end this section by giving some applications of sharp Gagliardo-Nirenberg inequalities (3.14) and (3.15).

**Proposition 3.7** (Global existence in \( \dot{H}^{s_c} \cap \dot{H}^s \)). Let \( d \geq 2, \frac{d}{2d-1} \leq s < 1 \) and \( \frac{4s}{d} < \alpha < \frac{4s}{d-2s} \). Let \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^s \) be radial and the corresponding solution \( u \) to (1.1) defined on the maximal forward time interval \([0, T)\). Assume that
\[
\sup_{t \in [0, T]} \| u(t) \|_{\dot{H}^{s_c}} < S_{gs}.
\]
Then \( T = +\infty \), i.e. the solution exists globally in time.

**Proof.** Note that the assumption on \( d, s, \alpha \) and \( u_0 \) comes from the local theory (see Section 2). By the sharp Gagliardo-Nirenberg inequality (3.14), we bound
\[
E(u(t)) = \frac{1}{2} \| u(t) \|_{\dot{H}^s}^2 - \frac{1}{\alpha + 2} \| u(t) \|_{L^{\alpha+2}}^{\alpha+2} \geq \frac{1}{2} \left( 1 - \frac{\| u(t) \|_{\dot{H}^{s_c}}}{S_{gs}} \right) \| u(t) \|_{\dot{H}^s}^2.
\]
Thanks to the conservation of energy and the assumption (3.20), we obtain \( \sup_{t \in [0, T]} \| u(t) \|_{\dot{H}^s} < \infty \). By the blowup alternative given in Proposition 2.2 and (3.20), the solution exists globally in time. The proof is complete. \( \square \)

**Proposition 3.8.** Let \( d \geq 2, \frac{d}{2d-1} \leq s < 1 \) and \( \frac{4s}{d} < \alpha < \frac{4s}{d-2s} \). Let \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^s \) be radial and the corresponding solution \( u \) to (1.1) defined on the maximal forward time interval \([0, T)\). Assume that
\[
S_{gs} \leq \sup_{t \in [0, T]} \| u(t) \|_{\dot{H}^{s_c}} < \infty, \quad \sup_{t \in [0, T]} \| u(t) \|_{L^{\alpha_c}} < L_{gs}.
\]
Then \( T = +\infty \), i.e. the solution exists globally in time.

The proof is similar to the one of Proposition 3.7 by using the sharp Gagliardo-Nirenberg inequality (3.15).

4. **Blowup concentration**

**Theorem 4.1** (Blowup concentration). Let \( d \geq 2, \frac{d}{2d-1} \leq s < 1 \) and \( \frac{4s}{d} < \alpha < \frac{4s}{d-2s} \). Let \( u_0 \in \dot{H}^{s_c} \cap \dot{H}^s \) be radial such that the corresponding solution \( u \) to (1.1) blows up at finite time \( 0 < T < +\infty \). Assume that the solution satisfies (1.5). Let \( a(t) > 0 \) be such that
\[
a(t) \| u(t) \|_{\dot{H}^s} \to \infty, \quad (4.1)
\]
as \( t \uparrow T \). Then there exist \( x(t), y(t) \in \mathbb{R}^d \) such that
\[
\liminf_{t \uparrow T} \int_{|x-y(t)| \leq a(t)} |(-\Delta)^{\frac{\alpha}{2}} u(t, x)|^2 \, dx \geq S_{gs}^2, \quad (4.2)
\]
and
\[ \liminf_{t \uparrow T} \int_{|x-y(t)| \leq \alpha(t)} |u(t, x)|^{\alpha} \, dx \geq L_{gs}^2. \] (4.3)

**Remark 4.2.** By the blowup rate given in Corollary 2.4 and the assumption (1.5), we have
\[ \|u(t)\|_{\dot{H}^s} > \frac{C}{(T-t)^{\frac{\alpha}{2}}} \]
for \( t \uparrow T. \) Rewriting
\[ \frac{1}{a(t) \|u(t)\|_{\dot{H}^s}^{\frac{1}{\alpha}}} = \frac{2\sqrt{T-t}}{\dot{a}(t)} \frac{1}{\|u(t)\|_{\dot{H}^s}^{\frac{1}{\alpha}}} = \frac{2\sqrt{T-t}}{\dot{a}(t)} \left( \frac{1}{\frac{T-t}{\|u(t)\|_{\dot{H}^s}}} \right)^{\frac{1}{\alpha}} < C \frac{2\sqrt{T-t}}{\dot{a}(t)}, \]
we see that any function \( a(t) > 0 \) satisfying
\[ \frac{2\sqrt{T-t}}{a(t)} \to 0 \quad \text{as} \quad t \uparrow T \]
fulfills the conditions of Theorem 4.1.

**Proof of Theorem 4.1.** Let \( (t_n)_{n \geq 1} \) be a sequence such that \( t_n \uparrow T \) and \( g \in G. \) Set
\[ \lambda_n := \left( \frac{\|g\|_{\dot{H}^s}}{\|u(t_n)\|_{\dot{H}^s}} \right)^{\frac{1}{\alpha}}, \quad v_n(x) := \lambda_n^{\frac{\alpha}{2}} u(t_n, \lambda_n x). \]
By the blowup alternative and the assumption (1.5), we see that \( \lambda_n \to 0 \) as \( n \to \infty. \) Moreover, we have
\[ \|v_n\|_{\dot{H}^s} = \|u(t_n)\|_{\dot{H}^s} < \infty, \]
uniformly in \( n \) and
\[ \|v_n\|_{\dot{H}^s} = \lambda_n^{s-s_c} \|u(t_n)\|_{\dot{H}^s} = \|g\|_{\dot{H}^s}, \]
and
\[ E(v_n) = \lambda_n^{2(s-s_c)} E(u(t_n)) = \lambda_n^{2(s-s_c)} E(u_0) \to 0, \quad \text{as} \quad n \to \infty. \]
This implies in particular that
\[ \|v_n\|_{L^{\alpha+2}}^{\alpha+2} \to \frac{\alpha+2}{2} \|g\|_{\dot{H}^s}^2, \quad \text{as} \quad n \to \infty. \]
The sequence \( (v_n)_{n \geq 1} \) satisfies the conditions of Theorem 3.5 with
\[ m^{\alpha+2} = \frac{\alpha+2}{2} \|g\|_{\dot{H}^s}^2, \quad M^2 = \|g\|_{\dot{H}^s}^2. \]
Therefore, there exists a sequence \( (x_n)_{n \geq 1} \) in \( \mathbb{R}^d \) such that up to a subsequence,
\[ v_n(\cdot + x_n) = \lambda_n^{\frac{\alpha}{2}} u(t_n, \lambda_n \cdot + x_n) \to V \quad \text{weakly in} \quad \dot{H}^{s_c} \cap \dot{H}^s, \]
as \( n \to \infty \) with \( \|V\|_{\dot{H}^{s_c}} \geq S_{gs}. \) In particular,
\[ (-\Delta)^{\frac{s}{2}} v_n(\cdot + x_n) = (-\Delta)^{\frac{s}{2}} u(t_n, \lambda_n \cdot + x_n) \to (-\Delta)^{\frac{s}{2}} V \quad \text{weakly in} \quad L^2. \]
This implies for every \( R > 0, \)
\[ \liminf_{n \to \infty} \int_{|x| \leq R} \lambda_n^\alpha \left( \int_{|x'| \leq R} |(-\Delta)^{\frac{s}{2}} u(t_n, \lambda_n x + x_n)|^2 \, dx \right)^2 \, dx \geq \int_{|x| \leq R} \left| (-\Delta)^{\frac{s}{2}} V(x) \right|^2 \, dx, \]
or
\[ \liminf_{n \to \infty} \int_{|x-x_n| \leq R \lambda_n} \left| \int_{|x'| \leq R} \left| (-\Delta)^{\frac{s}{2}} u(t_n, x') \right|^2 \, dx' \right| \, dx \geq \int_{|x| \leq R} \left| (-\Delta)^{\frac{s}{2}} V(x) \right|^2 \, dx. \]
In view of the assumption $\frac{a(t)}{x_0} \to \infty$ as $n \to \infty$, we get

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t_n)} |(-\Delta)^{\frac{\alpha}{2}} u(t_n, x)|^2 \, dx \geq \int_{|x| \leq R} |(-\Delta)^{\frac{\alpha}{2}} V(x)|^2 \, dx,$$

for every $R > 0$, which means that

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t_n)} |(-\Delta)^{\frac{\alpha}{2}} u(t_n, x)|^2 \, dx \geq \int_{|x-y| \leq a(t_n)} |(-\Delta)^{\frac{\alpha}{2}} V(x)|^2 \, dx \geq S_{g^s}^2.$$

Since the sequence $(t_n)_{n \geq 1}$ is arbitrary, we infer that

$$\liminf_{t \uparrow T} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t)} |(-\Delta)^{\frac{\alpha}{2}} u(t, x)|^2 \, dx \geq S_{g^s}^2.$$

But for every $t \in (0, T)$, the function $y \mapsto \int_{|x-y| \leq a(t)} |(-\Delta)^{\frac{\alpha}{2}} u(t, x)|^2 \, dx$ is continuous and goes to zero at infinity. As a result, we get

$$\sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t)} |(-\Delta)^{\frac{\alpha}{2}} u(t, x)|^2 \, dx = \int_{|x-x(t)| \leq a(t)} |(-\Delta)^{\frac{\alpha}{2}} u(t, x)|^2 \, dx,$$

for some $x(t) \in \mathbb{R}^d$. This shows (4.2). The proof for (4.3) is similar using Item 2 of Theorem 3.5. The proof is complete. \qed

5. Limiting profile with critical norms

Let us start with the following characterization of the ground state.

**Lemma 5.1.** Let $d \geq 1$, $0 < s < 1$ and $\frac{4s}{d} < \alpha < 2^*$.

- If $u \in \dot{H}^{s_c} \cap \dot{H}^s$ is such that $\|u\|_{H^{s_c}} = S_{g^s}$ and $E(u) = 0$, then $u$ is of the form
  $$u(x) = e^{i\theta} \lambda^{\frac{\alpha}{2s}} g(\lambda x + x_0),$$
  for some $g \in \mathcal{G}$, $\theta \in \mathbb{R}, \lambda > 0$ and $x_0 \in \mathbb{R}^d$.
- If $u \in L^{s_c} \cap \dot{H}^s$ is such that $\|u\|_{L^{s_c}} = L_{g^s}$ and $E(u) = 0$, then $u$ is of the form
  $$u(x) = e^{i\theta} \mu^{\frac{\alpha}{2s}} h(\mu x + y_0),$$
  for some $h \in \mathcal{H}$, $\theta \in \mathbb{R}, \mu > 0$ and $y_0 \in \mathbb{R}^d$.

**Proof.** We only prove Item 1, Item 2 is treated similarly. Since $E(u) = 0$, we have

$$\|u\|_{\dot{H}^s}^2 = \frac{2}{\alpha + 2} \|u\|_{L^{\frac{2s}{\alpha}}}^{\alpha + 2}.$$

Thus

$$H(u) = \|u\|_{L^{\frac{2s}{\alpha}}}^{\alpha + 2} = \frac{\alpha + 2}{2} \|u\|_{\dot{H}^s}^{-\alpha} = \frac{\alpha + 2}{2} S_{g^s}^{-\alpha} = A_{GN}.$$

This shows that $u$ is the maximizer of $H$. Proposition 3.2 then implies that $u$ is of the form $u(x) = ag(\lambda x + x_0)$ for some $g \in \mathcal{G}$, $a \in \mathbb{C}^*$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$. Since $\|u\|_{H^{s_c}} = S_{g^s} = \|g\|_{H^{s_c}}$, we have $|a| = \lambda^{\frac{\alpha}{2s}}$. The proof is complete. \qed

We are now able to show the limiting profile of blowup solutions with critical norms.

**Theorem 5.2** (Limiting profile with critical norms). Let $d \geq 2$, $\frac{d}{2s-1} \leq s < 1$ and $\frac{4s}{2d-1} < \alpha < \frac{4s}{2d}$.

Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ be radial such that the corresponding solution $u$ to (1.1) blows up at finite time $0 < T < +\infty$. 

• Assume that
\[ \sup_{t \in [0,T)} \| u(t) \|_{H^s} = S_{gs}. \]  
(5.1)

Then there exist \( g \in \mathcal{G}, \theta(t) \in \mathbb{R}, \lambda(t) > 0 \) and \( x(t) \in \mathbb{R}^d \) such that
\[ e^{i\theta(t)}\lambda_n^{2s}(t)u(t, \lambda(t) \cdot + x(t)) \rightarrow g \] strongly in \( \dot{H}^{s_c} \cap \dot{H}^s \) as \( t \uparrow T \).

• Assume that
\[ \sup_{t \in [0,T)} \| u(t) \|_{H^s} < \infty, \quad \sup_{t \in [0,T)} \| u(t) \|_{L^\infty} = L_{gs}. \]  
(5.2)

Then there exist \( h \in \mathcal{H}, \vartheta(t) \in \mathbb{R}, \mu(t) > 0 \) and \( y(t) \in \mathbb{R}^d \) such that
\[ e^{i\vartheta(t)}\mu_n^{2s}(t)u(t, \mu(t) \cdot + y(t)) \rightarrow h \] strongly in \( L^\infty \cap \dot{H}^s \) as \( t \uparrow T \).

**Proof.** We only prove the first item, the second one is treated similarly. We will show that for any \( (t_n)_{n \geq 1} \) satisfying \( t_n \uparrow T \), there exists a subsequence still denoted by \( (t_n)_{n \geq 1}, g \in \mathcal{G} \), sequences of \( \theta_n \in \mathbb{R}, \lambda_n > 0 \) and \( x_n \in \mathbb{R}^d \) such that
\[ e^{i\vartheta_n}\lambda_n^{2s}(t_n, \lambda_n \cdot + x_n) \rightarrow g \] strongly in \( \dot{H}^{s_c} \cap \dot{H}^s \) as \( n \rightarrow \infty \).  
(5.3)

Let \( (t_n)_{n \geq 1} \) be a sequence such that \( t_n \uparrow T \). Set
\[ \lambda_n := \left( \frac{\| Q \|_{\dot{H}^s}}{\| u(t_n) \|_{\dot{H}^s}} \right)^{-1}, \quad v_n(x) := \lambda_n^{2s}u(t_n, \lambda_n x), \]
where \( Q \) is as in Proposition 3.2. By the blowup alternative and (5.1), we see that \( \lambda_n \rightarrow 0 \) as \( n \rightarrow \infty \). Moreover, we have
\[ \| v_n \|_{\dot{H}^{s_c}} = \| u(t_n) \|_{\dot{H}^{s_c}} \leq S_{gs} = \| Q \|_{\dot{H}^{s_c}}, \]  
(5.4)

and
\[ \| v_n \|_{\dot{H}^s} = \lambda_n^{2-s_c} \| u(t_n) \|_{\dot{H}^s} = \| Q \|_{\dot{H}^s}, \]  
(5.5)

and
\[ E(v_n) = \lambda_n^{2(s-s_c)}E(u(t_n)) = \lambda_n^{2(s-s_c)}E(u_0) \rightarrow 0, \quad \text{as} \ n \rightarrow \infty. \]

This yields in particular that
\[ \| v_n \|_{L^{\alpha+2}}^{\alpha+2} \rightarrow \frac{\alpha+2}{2} \| Q \|_{\dot{H}^s}^2, \quad \text{as} \ n \rightarrow \infty. \]  
(5.6)

The sequence \( (v_n)_{n \geq 1} \) satisfies the conditions of Theorem 3.5 with
\[ n^{\alpha+2} = \frac{\alpha+2}{2} \| Q \|_{\dot{H}^s}^2, \quad M^2 = \| Q \|_{\dot{H}^s}^2. \]

Therefore, there exists a sequence \( (x_n)_{n \geq 1} \) in \( \mathbb{R}^d \) such that up to a subsequence,
\[ v_n(\cdot + x_n) = \lambda_n^{2s}u(t_n, \lambda_n \cdot + x_n) \rightarrow V \] weakly in \( \dot{H}^{s_c} \cap \dot{H}^s \),
as \( n \rightarrow \infty \) with \( \| V \|_{\dot{H}^{s_c}} \geq S_{gs} \). Since \( v_n(\cdot + x_n) \rightarrow V \) weakly in \( \dot{H}^{s_c} \cap \dot{H}^s \) as \( n \rightarrow \infty \), the semi-continuity of weak convergence and (5.4) imply
\[ \| V \|_{\dot{H}^{s_c}} \leq \liminf_{n \rightarrow \infty} \| v_n \|_{\dot{H}^{s_c}} \leq S_{gs}. \]

This together with the fact \( \| V \|_{\dot{H}^{s_c}} \geq S_{gs} \) show that
\[ \| V \|_{\dot{H}^{s_c}} = S_{gs} = \lim_{n \rightarrow \infty} \| v_n \|_{\dot{H}^{s_c}}. \]  
(5.7)
Therefore, \( v_n(\cdot + x_n) \to V \) strongly in \( \dot{H}^s \) as \( n \to \infty \).

On the other hand, the Gagliardo-Nirenberg inequality (3.14) shows that \( v_n(\cdot + x_n) \to V \) strongly in \( L^{n+2} \) as \( n \to \infty \). Indeed, by (5.5),

\[
\|v_n(\cdot + x_n) - V\|_{L^{n+2}}^\alpha \lesssim \|v_n(\cdot + x_n) - V\|_{\dot{H}^s}^\alpha + \|v_n(\cdot + x_n) - V\|_{\dot{H}^s}^\alpha \lesssim (\|Q\|_{\dot{H}^s} + \|V\|_{\dot{H}^s})^2 \|v_n(\cdot + x_n) - V\|_{\dot{H}^s}^\alpha \to 0,
\]
as \( n \to \infty \). Moreover, using (5.6) and (5.7), the sharp Gagliardo-Nirenberg inequality (3.14) yields

\[
\|Q\|_{\dot{H}^s}^2 = \frac{2}{\alpha + 2} \lim_{n \to \infty} \|v_n\|_{L^{n+2}}^\alpha = \frac{2}{\alpha + 2} \|V\|_{L^{n+2}}^\alpha \leq \frac{\|V\|_{\dot{H}^s}^\alpha}{S_{gs}} \|V\|_{\dot{H}^s}^\alpha = \|V\|_{\dot{H}^s}^\alpha,
\]
or \( \|Q\|_{\dot{H}^s} \leq \|V\|_{\dot{H}^s} \). By the semi-continuity of weak convergence and (5.5),

\[
\|V\|_{\dot{H}^s} \leq \liminf_{n \to \infty} \|v_n\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}.
\]

Therefore,

\[
\|V\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s} = \lim_{n \to \infty} \|v_n\|_{\dot{H}^s}. \tag{5.8}
\]

Combining (5.7), (5.8) and using the fact \( v_n(\cdot + x_n) \rightharpoonup V \) weakly in \( \dot{H}^s \cap \dot{H}^s \), we conclude that

\( v_n(\cdot + x_n) \to V \) strongly in \( \dot{H}^s \cap \dot{H}^s \) as \( n \to \infty \).

In particular, we have \( E(V) = \lim_{n \to \infty} E(v_n) = 0 \). This shows that there exists \( V \in \dot{H}^s \cap \dot{H}^s \) such that

\[
\|V\|_{\dot{H}^s} = S_{gs}, \quad E(V) = 0.
\]

By Lemma 5.1, there exists \( g \in G \) such that \( V(x) = e^{i\theta} \hat{\lambda}^{\frac{2s}{n}} g(\lambda x + x_0) \) for some \( \theta \in \mathbb{R}, \lambda > 0 \) and \( x_0 \in \mathbb{R}^d \). Thus

\[
v_n(\cdot + x_n) = \hat{\lambda}^{\frac{2s}{n}} u(t_n, \lambda_n \cdot + x_n) \to V = e^{i\theta} \hat{\lambda}^{\frac{2s}{n}} g(\lambda \cdot + x_0) \text{ strongly in } \dot{H}^s \cap \dot{H}^s \text{ as } n \to \infty.
\]

Redefining variables as

\[
\hat{\lambda}_n := \lambda_n \lambda^{-1}, \quad \hat{x}_n := \lambda_n \lambda^{-1} x_0 + x_n,
\]

we get

\[
e^{-i\theta} \hat{\lambda}_n^{\frac{2s}{n}} u(t_n, \hat{\lambda}_n \cdot + \hat{x}_n) \to g \text{ strongly in } \dot{H}^s \cap \dot{H}^s \text{ as } n \to \infty.
\]

This proves (5.3) and the proof is complete.

\[\square\]

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