A UNIVERSAL BOUND ON THE VARIATIONS OF BOUNDED
CONVEX FUNCTIONS

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Abstract. Given a convex set $C$ in a real vector space $E$ and two points $x, y \in C$, we investigate which are the possible values for the variation $f(y) - f(x)$, where $f : C \rightarrow [m, M]$ is a bounded convex function. We then rewrite the bounds in terms of the Funk weak metric, which will imply that a bounded convex function is Lipschitz-continuous with respect to the Thompson and Hilbert metrics. The bounds are also proved to be optimal. We also exhibit the maximal subdifferential of a bounded convex function at a given point $x \in C$.

1. The Variations of Bounded Convex Functions

Let $C$ be a convex set of a real vector space $E$. Given two points $x, y \in C$, we define the following auxiliary quantity:

$$\tau_C(x, y) = \sup \{ t \geq 1 \mid x + t(y - x) \in C \}.$$ 

Clearly, $\tau_C$ takes values in $[1, +\infty]$. Intuitively, it measures how far away $x$ is from the boundary in the direction of $y$, taking the “distance” $xy$ as unit. Clearly, $\tau_C(x, y) = +\infty$ if and only if $x + \mathbb{R}_+(y - x) \subset C$. Our first result is the following.

**Theorem 1.1.** Let $m \leq M$ be two real numbers. Let $C$ be a convex set of a real vector space $E$ and $f : C \rightarrow [m, M]$ a convex function. For every couple of points $(x, y) \in C^2$, $f$ satisfies:

$$-\frac{M - m}{\tau_C(x, y)} \leq f(y) - f(x) \leq \frac{M - m}{\tau_C(x, y)}.$$ 

**Proof.** It is enough to prove the result for functions with values in $[0, 1]$, since we can consider $(M - m)^{-1}(f - m)$. Let $x, y$ be two points in $C$. Let $t$ be such that $1 \leq t < \tau_C(x, y)$. By definition of $\tau_C$, and because $C$ is convex, we have $x + t(y - x) \in C$. We can write $y$ as a convex combination of $x + t(y - x)$ and $x$ with coefficients $1/t$ and $(t - 1)/t$ respectively:

$$y = \frac{x + t(y - x) + (t - 1)x}{t}.$$ 

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By convexity of \( f \), we get:

\[
    f(y) - f(x) \leq \frac{f(x + t(y - x)) + (t - 1)f(x)}{t} - f(x)
\]

\[
    \leq \frac{f(x + t(y - x)) - f(x)}{t} \leq \frac{1}{t},
\]

where the last inequality comes from the fact that \( f \) has values in \([0, 1]\). By taking the limit as \( t \to \tau_C(x, y) \), we get:

\[
    f(y) - f(x) \leq \frac{1}{\tau_C(x, y)}.
\]

The lower bound is obtained by exchanging the roles of \( x \) and \( y \). \( \square \)

2. The Funk, Thompson and Hilbert Metrics

In this section, we rewrite the result from Theorem 1.1 as a Lipschitz-like property in the framework of convex sets in normed spaces. But \( 1/\tau_C \) is far from being a distance. We thus consider the Funk, Thompson and Hilbert metrics (which were introduced in [1], [4] and [2] respectively) and establish the link with \( \tau_C \).

We restrict our framework to the case where \( C \) is an open convex subset of a normed space \((E, \| \cdot \|)\). Let \( x, y \in C \). If \( \tau_C(x, y) < +\infty \), we can define \( b(x, y) \) to be the following point:

\[
    b(x, y) = x + \tau_C(x, y)(y - x).
\]

Note that since \( C \) is open, when \( b(x, y) \) exists, it is necessarily different from \( y \). This will be necessary to state the following definitions.

**Definition 2.1.** Let \( C \) be an open convex subset of a normed space \((E, \| \cdot \|)\). We define

(i) the Funk weak metric:

\[
    F_C(x, y) = \begin{cases} 
        \log \frac{\|x - b(x, y)\|}{\|y - b(x, y)\|} & \text{if } \tau_C(x, y) < +\infty; \\
        0 & \text{otherwise}
    \end{cases}
\]

(ii) the Thompson pseudometric:

\[
    T_C(x, y) = \max (F_C(x, y), F_C(y, x));
\]

(iii) the Hilbert pseudometric:

\[
    H_C(x, y) = \frac{1}{2} (F_C(x, y) + F_C(y, x)).
\]

**Remark 2.2.** Even if we will abusively call them *metrics*, they fail to satisfy the separation axiom in general. The Thompson and the Hilbert metrics are thus *pseudometrics*. Moreover, the Funk metric not being symmetric, it actually is a *weak* metric. The Thompson and the Hilbert metrics are respectively the *max-symmetrization* and *meanvalue-symmetrisation* of the Funk metric. For a detailed presentation of these notions, see e.g. [3].

We now establish the link between \( \tau_C(x, y) \) and \( F_C(x, y) \).
Let $| \cdot |$ denote the absolute value. From (ii), by using the inequality $\frac{1}{|C(x,y)|} \cdot |b(x,y)| \leq 1$, we get:
\[
\tau_C(x,y) = \frac{||x-b(x,y)||}{||x-y||} \quad \text{and} \quad \tau_C(x,y) - 1 = \frac{||y-b(x,y)||}{||x-y||}.
\]
And thus:
\[
\frac{||x-b(x,y)||}{||y-b(x,y)||} = \left(1 - \frac{1}{\tau_C(x,y)}\right)^{-1}.
\]
Therefore,
\[
F_C(x,y) = -\log \left(1 - \frac{1}{\tau_C(x,y)}\right).
\]

By combining Theorem 1.1 and the above proposition, we get the following corollary.

**Corollary 2.4.** Let $C$ be an open convex subset of a normed space $(E, \| \cdot \|)$. For every points $x, y \in C$, the following bounds hold:

(i) $-(M-m) \left(1 - e^{-F_C(x,y)}\right) \leq f(y) - f(x) \leq (M-m) \left(1 - e^{-F_C(x,y)}\right)$.

(ii) $|f(y) - f(x)| \leq (M-m) \left(1 - e^{-T_C(x,y)}\right)$.

(iii) $|f(y) - f(x)| \leq (M-m) \left(1 - e^{-2H_C(x,y)}\right)$.

**Remark 2.5.** From (ii), by using the inequality $e^{-s} \geq 1 - s$, we get:
\[
|f(x) - f(y)| \leq (M-m) \left(1 - e^{-T_C(x,y)}\right) \leq (M-m)T_C(x,y),
\]
and similarly for (iii). Every convex function $f : C \rightarrow [m, M]$ is thus $(M-m)$-Lipschitz (resp. $2(M-m)$-Lipschitz) with respect to the Thompson metric (resp. the Hilbert metric).

### 3. Optimality of the Bounds

We show in this section that the bounds obtained in Theorem 1.1 are optimal in the following sense. For a given convex set, and for a given couple $a$ points, there is a function which attains the upper bound (resp. the lower bound). In other words, for $x, y \in C$:

\[
\begin{align*}
\max_{f: C \rightarrow [m, M]} (f(y) - f(x)) &= \frac{M-m}{\tau_C(x,y)} \\
\min_{f: C \rightarrow [m, M]} (f(y) - f(x)) &= -\frac{M-m}{\tau_C(y,x)}.
\end{align*}
\]
In the proof of the following theorem, it will be very convenient to extend the notion of convexity to functions defined on $C$ and taking values in $\mathbb{R} \cup \{-\infty\}$ (and not $\mathbb{R} \cup \{+\infty\}$). Obviously, the result according to which the upper envelope of two convex functions is also a convex function remains true.

**Theorem 3.1.** Let $m \leq M$ be two real numbers. Let $C$ be a convex set of a real vector space $E$. For every couple of points $(x, y) \in C^2$, there exists a convex function $f : C \rightarrow [m, M]$ (resp. $g : C \rightarrow [m, M]$) such that the upper bound (resp. lower bound) of Theorem 1.1 is attained; in other words:

$$f(y) - f(x) = \frac{M - m}{\tau_C(x, y)} \quad \text{(resp. } g(y) - g(x) = -\frac{M - m}{\tau_C(y, x)} \text{)}.$$  

**Proof.** Let $x$ and $y$ be two points in $C$, and let us construct a convex function $f : C \rightarrow [0, 1]$ satisfying the equality. If $\tau_C(x, y) = +\infty$, the bound is zero, and $f = 0$ is adequate. From now on, we assume that $\tau_C(x, y) < +\infty$. The idea of the construction is the following. Let us first consider the line through $x$ and $y$. We want $f$ to increase from 0 at $x$ to 1 at the boundary in the direction of $y$, in an affine way; and to be equal to zero in the other direction. Then, we will have to extend $f$ to all $C$ in a convex way. Let $\bar{u} = \tau_C(x, y)(y - x)$. For every $z \in C$, let us define $\sigma(z) = \sup \{t \geq 0 \mid z + t\bar{u} \in C\}$. $\sigma$ clearly takes values in $[0, +\infty]$. Consider the following function.

$$\phi : \quad C \rightarrow [\sigma(z)]$$

$$z \quad \mapsto \quad 1 - \sigma(z).$$

Let us prove that $\phi$ is convex. Let $z_1$ and $z_2$ be two points in $C$ and $z_3 = \lambda z_1 + (1 - \lambda)z_2$ (with $\lambda \in (0, 1)$) a convex combination. By definition of $\sigma$, if we take two real numbers $s_1$ and $s_2$ such that $0 \leq s_1 \leq \sigma(z_1)$ and $0 \leq s_2 \leq \sigma(z_2)$, we have:

$$\begin{cases} z_1 + s_1\bar{u} \in C \\ z_2 + s_2\bar{u} \in C. \end{cases}$$

And thus, the convex combination of these two points with coefficients $\lambda$ and $1 - \lambda$ also belongs to $C$:

$$\lambda(z_1 + s_1\bar{u}) + (1 - \lambda)(z_2 + s_2\bar{u}) \in C.$$

This point can be rewritten with $z_3$:

$$z_3 + (\lambda s_1 + (1 - \lambda)s_2)\bar{u} \in C.$$

By definition of $\sigma(z_3)$, we have $\lambda s_1 + (1 - \lambda)s_2 \leq \sigma(z_3)$. This inequality is true for every $s_1 \leq \sigma(z_1)$ and $s_2 \leq \sigma(z_2)$. Consequently:

$$\lambda \sigma(z_1) + (1 - \lambda)\sigma(z_2) \leq \sigma(z_3).$$

We can now prove the convexity inequality.

$$\phi(z_3) = 1 - \sigma(z_3) \leq 1 - (\lambda \sigma(z_1) + (1 - \lambda)\sigma(z_2))$$

$$= \lambda(1 - \sigma(z_1)) + (1 - \lambda)(1 - \sigma(z_2))$$

$$= \lambda \phi(z_1) + (1 - \lambda)\phi(z_2).$$
We now choose $f = \max(\phi, 0)$. Since $\phi \leq 1$, $f$ takes values in $[0, 1]$. Let us prove that $f$ satisfies the desired equality. Let us compute $f(x)$ and $f(y)$.

$$
\sigma(x) = \sup \{ t \geq 0 \mid x + t\vec{u} \in C \}
$$

$$
= \sup \{ t \geq 0 \mid x + t\tau_C(x, y)(y - x) \in C \}
$$

$$
= \frac{1}{\tau_C(x, y)} \sup \{ t' \geq 0 \mid x + t'(y - x) \in C \}
$$

$$
= \frac{1}{\tau_C(x, y)} \tau_C(x, y)
$$

$$
= 1.
$$

Thus $\phi(x) = 1 - \sigma(x) = 0$ and $f(x) = \max(0, 0) = 0$. Similarly, we can prove:

$$
\sigma(y) = \frac{\tau_C(x, y)}{\tau_C(x, y)} - 1,
$$

and thus, $\phi(y) = 1 - \sigma(y) = \tau_C(x, y)^{-1}$ and $f(y) = \max(\tau_C(x, y)^{-1}, 0) = \tau_C(x, y)^{-1}$. We finally get:

$$
f(y) - f(x) = \frac{1}{\tau_C(x, y)}.
$$

The construction of $g$ is analogous. □

4. The Maximal Subdifferential

In the case of a nonempty convex subset $C \subset \mathbb{R}^n$, and a given point $x_0 \in C$, we wonder what is the maximal subdifferential at $x_0$ (in the sense of inclusion) for a function $f : C \rightarrow [m, M]$. We will prove that there is a maximal one, and will express it in terms of the subdifferential of a translation of the Minkowski gauge.

For each $x_0 \in C$, we define $g_{C, x_0} : C \rightarrow [0, 1]$ by

$$
g_{C, x_0}(x) = \inf \{ \lambda > 0 \mid x - x_0 \in \lambda(C - x_0) \}.
$$

This function is obviously well-defined, and can be seen as a Minkowski gauge centered in $x_0$ and restricted to $C$. It is well-known fact that the Minkowski gauge is a convex function. So is this one.

**Theorem 4.1.** Let $C$ be a nonempty convex subset of $\mathbb{R}^n$ and $x \in C$. We have

$$\max_{f : C \rightarrow [m, M]} \partial f(x) = (M - m)\partial g_{C, x}(x),$$

where the maximum is understood in the sense of inclusion.

**Proof.** Let us first relate $g_{C, x_0}$ to $\tau$. Let $x_0, x \in C$. We have

$$
g_{C, x_0}(x) = \inf \{ \lambda > 0 \mid x - x_0 \in \lambda(C - x_0) \}
$$

$$
= \sup \left\{ t > 0 \mid x - x_0 \in \frac{1}{t}(C - x_0) \right\}^{-1}
$$

$$
= \sup \left\{ t > 0 \mid x_0 + t(x - x_0) \in C \right\}^{-1}
$$

$$
= \frac{1}{\tau(x_0, x)}.
$$
Let us prove the result in the case \( m = 0 \) and \( M = 1 \), from which the general case follows immediately. Let \( f : C \to [0,1] \) be a convex function and \( x_0 \in C \). Let us show that \( \partial f(x_0) \subset \partial g_{C,x_0}(x_0) \). This is true if \( \partial f(x_0) \) is empty. Otherwise, let \( \zeta \in \partial f(x_0) \). For every \( x \in C \), we have

\[
\langle \zeta | x - x_0 \rangle \leq f(x) - f(x_0) \leq \frac{1}{\tau(x_0,x)} = g_{C,x_0}(x) - g_{C,x_0}(x_0),
\]

where we used Theorem 1.1 for the second inequality. If \( x \notin C \), the equality also holds, since \( g_{C,x_0}(x) = +\infty \). We thus have \( \partial f(x_0) \subset \partial g_{C,x_0}(x_0) \). We conclude by saying that \( g_{C,x_0} \) is a convex function on \( C \) with values in \([0,1]\).

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