Modelling of gears for nonlinear dynamics analysis

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Abstract. The model of gear pair for three dimension nonlinear transient analysis is proposed in the article. The model of gear pair is capable of representing a wide spectra of gear types used in industry. Variable gear mesh stiffness, gear mesh damping, transmission errors and backlash are introduced in the model.

1. Introduction
Gears are used in almost every machine for transmission of rotational motion. As well as another machine parts, e.g. shafts, bearings, frames and etc., gears determine the dynamics behaviour of the rotatory machine. There is a lot of literature about gear dynamics and gear systems dynamics modelling [1]. The most popular and interesting gear dynamics models were presented by Cardona[2], Spitas[3], Kahraman[4], Ozguven[5], Kubo[6], Zhang[7], Qiu[8], Bahk[9]. However, most of them do not consider three dimensional gearing motion, the time-dependent properties of the gear mesh, the mutual effects of gears and parts of the machine and do not provide the ability of transient analysis.

In the present work the model of the gear pair for three dimensional nonlinear dynamics analysis of rotary machines has been developed. The model of the gear pair is aimed to analyse a wide spectra of gear types: spur gears, helical gears, conical gears and internal gears, and takes into account such effects as variable gear mesh stiffness, gear mesh damping, transmission errors and backlash.

2. Basic definitions and state vector
The element formed by two geared wheels with centers A and B is considered at Fig.1. Initial position of each wheel center is given by \( \mathbf{r}_{A0} \) and \( \mathbf{r}_{B0} \). Two triad of unit orthogonal vectors are defined at each wheel center: \( \mathbf{e}_{A10} \), \( \mathbf{e}_{A20} \), \( \mathbf{e}_{A30} \) is attached to the first wheel, with origin at wheel centre “A”, and \( \mathbf{e}_{B10} \), \( \mathbf{e}_{B20} \), \( \mathbf{e}_{B30} \) is attached to the second wheel, with origin at wheel centre “B”. Both triads have their first vector oriented towards the contact point between wheels, their third vector perpendicularly oriented to the wheel plane (figure 1) and their second vector define each triad as right-handed coordinate system.
Figure 1. Gear pair kinematics.

The position of each wheel center at current configuration is
\[ r_A = r_{A0} + u_A \]
\[ r_B = r_{B0} + u_B \]

\( u_A \) and \( u_B \) are the translation vectors of wheel centers.

The orientation of both triads at the current configuration is obtained applying the rotation tensor to each wheel to the vectors at the initial configuration:

\[ e^A_i = L(\mathbf{S}_A) \cdot e^A_{i0}, \quad (i = 1, 2, 3) \]
\[ e^B_i = L(\mathbf{S}_B) \cdot e^B_{i0}, \quad (i = 1, 2, 3) \]

\( L(\mathbf{S}_A) \) and \( L(\mathbf{S}_B) \) the rotation matrices at nodes \( A, B \), which are related to the rotation vectors \( \mathbf{S}_A \) and \( \mathbf{S}_B \) [2, 3].

Gear pair state vector have 12 degree of freedom and consist from the translation vectors and the rotation vectors of each wheela
\[ \mathbf{q} = \{u_A, \mathbf{S}_A, u_B, \mathbf{S}_B\}. \]

3. Normal contact offset
The first triad vectors do not oriented towards the contact point after rotation. The position of vector form the wheel A center to the contact point is determined as

\[ e^A_{1\text{ref}} = \mathbf{E}_0 \cdot \left( r_B - r_A \right) \]
\[ \mathbf{E}_0 = \mathbf{E} - e^B_3 e^A_3 \]

\( \mathbf{E}_0 \)is the planar tensor, \( \mathbf{E} \) is the unit tensor, \( e^B_3 e^A_3 \) is the dyadic product of unit vector \( e^B_3 \).

The value of angle \( \psi_A \) between vectors \( e^A_i \) and \( e^A_{1\text{ref}} \) can be calculated from equations
The value of angle \( \psi_B \) and the radius vector \( \mathbf{e}_{1\text{ref}}^B \) are determined as

\[
\psi_B = \frac{\rho_A}{\rho_B} \psi_A
\]

\[
\mathbf{e}_{1\text{ref}}^B = \mathbf{e}_1^B \cos \psi_B + \mathbf{e}_2^B \sin \psi_B
\]

Basically, the vectors \( \mathbf{e}_{1\text{ref}}^A \) and \( \mathbf{e}_{1\text{ref}}^B \) indicate at different contact points. The difference between contact points determine the vector of elastic mismatch

\[
\Delta \mathbf{u} = (\mathbf{r}_0 + \mathbf{u}_B + \rho_B \mathbf{e}_{1\text{ref}}^B) - (\mathbf{r}_0 + \mathbf{u}_A + \rho_A \mathbf{e}_{1\text{ref}}^A)
\]

Also a normal vector to a teeth surface at contact point is defined as

\[
\mathbf{n} = f_1 \mathbf{e}_{1\text{ref}}^A + f_2 \mathbf{e}_{2\text{ref}}^A + f_3 \mathbf{e}_3^A
\]

\[
f_1 = \cos \gamma \cdot \sin \alpha - \cos \alpha \cdot \sin \beta \cdot \sin \gamma
\]

\[
f_2 = \cos \alpha \cdot \cos \beta
\]

\[
f_3 = \cos \alpha \cdot \cos \gamma \cdot \sin \beta + \sin \alpha \cdot \sin \gamma
\]

\( \alpha, \beta, \gamma \) the pressure, cone and helix angels respectively (Fig.2).

**Figure 2.** Normal vector to a teeth surface at contact point.

Scalar product of \( \Delta \mathbf{u} \) and \( \mathbf{n} \) determine the normal contact offset

\[
\Delta s = \mathbf{n} \cdot \Delta \mathbf{u}.
\]
4. Elastic energy, stiffness matrix and vector of inner force of gear pair

Finally, elastic energy is determined as

\[ U = \begin{cases} 
kh \frac{(\Delta s + x_{\text{error}} - b/2)^2}{2}, & \Delta s + x_{\text{error}} > b/2 \\
0, & b/2 > x > -b/2 \\
kh \frac{(\Delta s + x_{\text{error}} + b/2)^2}{2}, & \Delta s + x_{\text{error}} < -b/2 
\end{cases} \]

\( h \) is the teeth width, \( k \) is the mesh stiffness, \( x_{\text{error}} \) the loaded transmission error and \( b \) is the hoop backlash. With respect to [2], the following expression for the loaded transmission error is used

\[ x_{\text{error}} = X \left(1 - \cos \left( z_A \psi_A \right) \right) \]

\( X \) the transmission error amplitude.

Tangent stiffness matrix and vector of inner forces are defined as follows

\[ \mathbf{F} = \frac{\partial \mathbf{U}}{\partial \mathbf{q}} \]

\[ \mathbf{K} = \frac{\partial^2 \mathbf{U}}{\partial \mathbf{q} \partial \mathbf{q}} \]

Also normal contact force can be calculated from equation

\[ F_x = \begin{cases} 
kh \frac{(\Delta s + x_{\text{error}} - b/2)^2}{2}, & \Delta s + x_{\text{error}} > b/2 \\
0, & b/2 > x > -b/2 \\
kh \frac{(\Delta s + x_{\text{error}} + b/2)^2}{2}, & \Delta s + x_{\text{error}} < -b/2 
\end{cases} \]

Mesh damping can be taken into account as

\[ c_m = 2\xi \sqrt{\frac{kh I_A I_B}{I_A \beta_A^2 + I_B \beta_B^2}} \]

\( I_A \) and \( I_B \) gear inertias of wheel “A” and “B” respectively, \( \xi \) the mesh damping ratio.

5. Test case

This test [2] concerns the analysis of a transmission composed by two axes linked through a pair formed by two equal spur gears (figure 3). The system is driven at a constant speed of 12000 RPM at wheel 1, and transmits a torque \( T_2 = 945.8 \) applied at wheel 2. Gear properties are: normal modulus \( m_n = 0.15748 \), pitch diameter \( d = 3.937 \), pressure angle \( \alpha = 20^0 \) and teeth number \( z_A = z_B = 25 \) (cone and helix angles are trivially \( \beta = \gamma = 0 \)). Their mass and rotary inertia are: \( I_A = m_B = 5.36 \cdot 10^{-3} \) and \( I_A = I_B = 0.0102 \). Mesh stiffness is \( k = 1.477 \cdot 10^7 \), gear width \( h = 1 \), mesh damping \( \xi = 0.1 \), and the amplitude of the loaded transmission error is \( X = 0.0003937 \). We have considered only the torsional stiffness of shafts in the model, with values: \( k_{11} = 1.7 \cdot 10^4 \), \( k_{12} = 3 \cdot 10^4 \), and damping: \( c_{11} = 0.076026 \), \( c_{12} = 0.12369 \). Disks at extremes have rotary inertias \( I_1 = 0.051 \) and \( I_2 = 0.0102 \).
Both axes are rigidly mounted at their supports, and backlash between gears is zero. This system is integrated by using the Newmark algorithm with an adaptive time step strategy. Figure 4 shows a graphs of time evolution of the mesh force from [2] and is determined by present model. Both graphs are similar by amplitude and phase.

![Figure 3](image)

**Figure 3.** Spur gears transmission [2].

**Figure 4.** Time evolution of mesh force is determined by [2] and present model.

**Conclusion**

This paper presents the model of gear pair for three dimension nonlinear transient analysis. Variable gear mesh stiffness, gear mesh damping, transmission errors and backlash are introduced in the model. The model can be combined with another machine parts models, such as shafts, bearing, frames and etc. Test case showed good representation with another model results.

**References**

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