A UNIVERSAL $A_{\infty}$ STRUCTURE ON BV ALGEBRAS
WITH MULTIPLE ZETA VALUE COEFFICIENTS

JOHAN ALM

Abstract. We explicitly construct a universal $A$-infinity deformation of Batalin-Vilkovisky algebras, with all coefficients expressed as rational sums of multiple zeta values. If the Batalin-Vilkovisky algebra that we start with is cyclic, then so is the $A$-infinity deformation. Moreover, the adjoint action of the odd Poisson bracket acts by derivations of the $A$-infinity structure. The construction conjecturally defines a new presentation of the Grothendieck-Teichmüller Lie algebra.

Introduction

A Batalin-Vilkovisky algebra is a differential graded commutative algebra $A$ together with a so-called BV operator $\Delta : A \to A$ of degree $-1$, such that the operation

$$\{f, g\} := \Delta(fg) - \Delta(f)g - (-1)^{|f|}f\Delta(g),$$

is a degree $-1$ graded Poisson bracket on $A$, and such that the BV operator acts as derivations of this bracket. Batalin-Vilkovisky algebras were invented by physicists as a tool in the quantization of gauge theories, but there are several natural examples also in pure mathematics, such as Hochschild cohomology groups of Frobenius algebras, symplectic homology of symplectic manifolds, homology groups of free loop spaces and polyvector fields on manifolds with a specified volume form. Every Batalin-Vilkovisky algebra is by definition a differential graded commutative algebra. In this paper we give a universal construction of an $A_{\infty}$ deformation of this graded commutative product. Since the graded commutative product is already associative the higher homotopies of the $A_{\infty}$ structure are analogous to Massey products.

The deformation that we construct is weakly canonical, in the following sense. Batalin-Vilkovisky algebras are algebras for the homology operad $H(fM_0)$ associated to the topological operad $fM_0$ of Riemann spheres with marked punctures and phase parameters (angles) at each puncture, see [8, 11]. Analogously, associative algebras are governed by the homology operad $H(X)$ of a topological operad $X$ that parametrizes configurations of points on a line. Embedding the line as the real axis on a Riemann sphere (and fixing the phase parameters to point along that axis) defines a canonical morphism of topological operads $X \to fM_0$. The induced morphism

$$m : H(X) \to H(fM_0)$$

is the morphism that trivially interprets the commutative product of a Batalin-Vilkovisky algebra as an associative product, and completely forgets the BV operator. Both operads $X$ and $fM$ are formal, meaning that there exists quasi-isomorphisms of differential graded operads

$$C(X) \xrightarrow{\simeq} P \xrightarrow{\simeq} H(X) \quad \text{and} \quad C(fM_0) \xrightarrow{\simeq} Q \xrightarrow{\simeq} H(fM_0)$$

connecting the respective differential graded operad of (singular) chains to the respective homology operad. It is natural to ask if the natural morphism $X \to fM_0$ is formal too, in the sense that the zig-zags of quasi-isomorphisms can be chosen in such a way that we obtain a commutative diagram

$$
\begin{array}{ccc}
C(X) & \xrightarrow{\simeq} & C(fM_0) \\
\downarrow & & \downarrow \simeq \\
Q & \xrightarrow{\simeq} & P \\
\downarrow & & \downarrow \simeq \\
H(X) & \xrightarrow{m} & H(fM_0).
\end{array}
$$
It follows from recent results of \cite{19} that we can not. One way of interpreting this is that the map \( m : H(X) \to H(fM_0) \) is much less canonical than we would like to think it is. Rather, it is a mere shadow or first order approximation of the “truly canonical” topological morphism \( X \to fM_0 \). The universal \( A_\infty \) structure that we construct adds the necessary higher order correction terms to the map \( m \), and is in this sense canonical. However, note that the zig-zags of quasi-isomorphisms (as above) are themselves very much not canonical, so the canonical nature of our structure is rather weak. There is essentially a unique zig-zag witnessing the formality of the operad \( X \). Formalities of the operad \( fM_0 \), on the other hand, are by \cite{20} essentially parametrized by the set of Drinfeld associators. We conjecture that the universal \( A_\infty \) structure that we construct corresponds to choosing the Knizhnik-Zamolodchikov associator, which in a sense is the canonical choice of associator once we strictify the setup from purely topological to one of algebraic geometry, since then the Knizhnik-Zamolodchikov arises canonically from the comparison isomorphism between Betti and de Rham realizations.

Assume \( k_1, \ldots, k_r \) is a sequence of strictly positive integers, with \( k_r \geq 2 \). The multiple zeta values (for short, MZVs) are the real numbers of the form

\[
\zeta(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.
\]

It is relatively easy, using the above displayed series representation, to see that the multiplication of two MZVs is a rational linear combination of MZVs, so they span a subalgebra \( \zeta \) of the real numbers. The Knizhnik-Zamolodchikov associator is known to be a kind of generating function for multiple zeta values.

Briefly, our construction proceeds as follows. Let \( M_{0,n}^{\delta} \) be Francis Brown’s partial compactification of the moduli space of genus zero curves with \( n \) marked points \cite{4}. It sits as an intermediary \( M_{0,n} \subset M_{0,n}^{\delta} \subset \overline{M}_{0,n} \) between the open moduli space and the Deligne-Mumford compactification, as the space obtained by adding only those boundary divisors of the Deligne-Mumford compactification that bound the connected component of the set of real points \( M_{0,n}(\mathbb{R}) \) which corresponds to having the marked points in the canonical order \( z_1 < \cdots < z_n \). The closure of this connected component inside \( M_{0,n}(\mathbb{C}) \), call it \( X_n \), is an associahedron of dimension \( n - 3 \). Brown’s moduli spaces constitute a planar operad (we may cyclically permute the marked points, but general permutations are not allowed). The associahedra \( X_n \) likewise form a planar operad, and using results from Brown’s thesis \cite{4} there are natural maps

\[
H(M_{0,n}^{\delta}) \to \Omega(X_n),
\]

by which the rational cohomology cooperad of the moduli spaces can be represented as certain logarithmic differential forms on associahedra. The main result of \cite{4} is that every top-dimensional form in its image has an integral over the associahedron that converges to a rational sum of MZVs. Thus, there exists a canonical morphism of planar operads

\[
\zeta \otimes_{\mathbb{Q}} A_\infty \to \zeta \otimes_{\mathbb{Q}} H(M_0^{\delta}),
\]

defined by integration pairing. Let \( \delta_n \) be Brown’s Lie algebra of dihedral braids \cite{4}. It has a generator \( \delta_{ij} = \delta_{ji} \) for each pair of indices \( 1 \leq i \neq j \leq n \) with \( i \neq j + 1 \) modulo \( n \), and relations

\[
[\delta_{i-1,j} + \delta_{j-1,i} - \delta_{i-1,j-1} - \delta_{i-1,j} + \delta_{k-1,j} - \delta_{k-1,j-1} - \delta_{k,i}] = 0
\]

for all quadruples of distinct indices. We note that this family of Lie algebras form a planar operad. Brown gave a novel treatment of the Knizhnik-Zamolodchikov connection in terms of this Lie algebra,
and based on his results we show how to write down a morphism of differential graded cooperads

\[ \alpha_{\text{reg}} : C(\mathfrak{d}) \to H(M^0_b) \].

This map can be regarded as a “regularized” version of the Knizhnik-Zamolodchikov connection, but note that it is not a morphism of cooperads of differential graded commutative algebras, only of cooperads of differential graded vector spaces, hence it is not a connection in the true sense of being defined by a connection form with values in the Lie algebra of dihedral braids.

Let \( \mathfrak{tb}_s \) be the Lie algebra of spherical ribbon braids. We use a slightly nonstandard presentation of this Lie algebra, with generators \( b_{ij} \) (\( 1 \leq i \neq j \leq n \)) and \( s_k \) (\( 1 \leq k \leq n \)), and relations saying that \( b_{ji} = b_{ij} \), that \( [b_{ij}, b_{kl}] = 0 \) whenever all four indices are distinct, that \( 2s_k + \sum_{i=1}^n b_{ik} = 0 \) for all \( k \), and that all the \( s_k \) are central. These Lie algebras assemble, for varying \( n \), to an operad. Its associated dg operad \( H(\mathfrak{tb}) \) of Chevalley-Eilenberg homologies may be identified with the operad of Batalin-Vilkovisky algebras, which we previously in the introduction denoted \( H(fM_0) \). We construct a morphism

\[ \gamma : C(\mathfrak{d}) \to C(\mathfrak{tb}) \]

of differential graded planar operads of Chevalley-Eilenberg cochain complexes. By suitably dualizing, taking homology, and composing with our regularized Knizhnik-Zamolodchikov connection, we obtain a morphism

\[ \gamma \circ \alpha^*_{\text{reg}} : H(M^0_b) \to H(\mathfrak{tb}) \].

Composing with the morphism defined by Brown’s integration pairing then defines

\[ \nu : \zeta \otimes_{\mathbb{Q}} A_{\infty} \to \zeta \otimes_{\mathbb{Q}} H(\mathfrak{tb}) \]

of planar dg operads.

In the last section of the paper we prove that the adjoint action of the odd Poisson bracket of a Batalin-Vilkovisky algebra acts by strict derivations of our \( A_\infty \) structure. To prove this we define a family of manifolds with corners \( \mathcal{X}_{p,q} \subset fM^0_{p+q} \) that encode the two-colored operadic combinatorics of the homotopies of an \( L_\infty \) action by \( A_\infty \) derivations of an \( A_\infty \) algebra. Our construction, based on integration over the associahedra \( \mathcal{X}_n \), extends to a representation by integration over also the spaces \( \mathcal{X}_{p,q} \), and inspection of the involved integrals shows that the data added by this is just the statement that the adjoint action of the odd Poisson bracket is an action by derivations of the \( A_\infty \) structure. In formulas, the adjoint action is a morphism of dg Lie algebras

\[ \text{ad} : (A[1], \{ , \}) \to \text{Der}(A, \nu). \]

It follows that if \( \kappa \) is a Maurer-Cartan element of \( A \), then \( \nu \) will be an \( A_\infty \) structure also on \( A \) with the differential twisted by addition of the term \( \text{ad}_\kappa \).

Many thanks to Dan Petersen and Sergei Merkulov for valuable discussion and to Malin Göteman for providing me with the proof of appendix A.

1. Preliminary definitions

The set \( \{1, \ldots, n\} \) is denoted \([n]\). We usually treat it as (cyclically) ordered in the obvious sense. The cardinality of a set \( S \) is denoted \#S.

Let \( \mathbb{K} \) be a field of characteristic zero. The terms differential graded vector space, henceforth abbreviated to \( dg \) vector space, and cochain complex (over \( \mathbb{K} \)) are used as synonyms. In particular, we always use cohomological grading so that differentials increase degree. A \( dg \) vector space is of finite type if it is finite-dimensional in each degree. We always apply the Koszul sign rules for tensor products of \( dg \) vector spaces and tensor products of maps of \( dg \) vector spaces. Briefly, these sign rules are as follows.

The space of maps from \( V \) to \( W \) is the \( dg \) vector space \( \text{Map}(V, W) \) with

\[ \text{Map}(V, W)^n := \prod_p \text{Hom}_{\mathbb{K}}(V^{p-n}, W^p), \]

where \( \text{Hom}_{\mathbb{K}}(V^{p-n}, W^p) \) denotes the vector space of all linear maps from \( V^{p-n} \) to \( W^p \), and differential given on \( \phi \in \text{Map}(V, W)^n \) by \( d^n_{\text{Map}(V, W)} \phi := d_W \circ \phi - (-1)^n d_V \circ \phi \). A vector \( \phi \) of \( \text{Map}(V, W)^n \) is called a map of \( dg \) vector spaces of degree \( n \). Note that a morphism from \( V \) to \( W \) is the same thing as a cocycle of degree 0 of \( \text{Map}(V, W) \). We apply the Koszul sign rules to maps, which says that for homogeneous maps \( f, g \) and homogeneous vectors \( u, v \) in their respective domains, \( f \otimes g \) is defined by
\[(f \circ g)(u \odot v) = (-1)^{|g| |u|} f(u) \odot g(v).\] Given dg vector spaces \(V\) and \(W\) their tensor product is the dg vector space \(V \otimes W\) with \((V \otimes W)^n := \bigoplus_{p+q=n} V^p \otimes_k W^q\), differential defined by \(d_{V \otimes W} := d_V \otimes \text{id}_W + \text{id}_V \otimes d_W\) (using the Koszul sign rule for maps). The Koszul symmetry for \(V \otimes W\) is the morphism

\[\sigma_{V \otimes W} : V \otimes W \to W \otimes V\]
given on vectors of homogeneous degree by \(\sigma_{V \otimes W}(v \otimes w) := (-1)^{|v||w|} w \otimes v\). The tensor product, the Koszul symmetry and the tensor unit \(k\) give the category of dg vector spaces the structure of a symmetric monoidal category. Using the space of maps and the Koszul sign rules for maps we can (and implicitly usually will do) consider the category of dg vector spaces as a category enriched in itself, because the space of maps and the tensor product satisfy the usual adjunction.

1.1. Planar operads. All dg (co)operads are assumed to be (co)augmented, and we will accordingly dispense with the distinction between dg (co)operads and dg pseudo-(co)operads. We otherwise follow the conventions concerning operads adopted in \([14]\). A notable exception is the terminology of planar (co)operads, which to our knowledge has only one precedent in the literature, see \([15]\). The idea for the concept is simple enough: just like cyclic operads are based on trees, operads on rooted trees, and nonsymmetric operads on planar rooted trees, our notion of planar operads is based on planar (non-rooted) trees.

A graph \(G\) is a finite set of flags \(F_G\) with an involution \(\tau : F_G \to F_G\), a finite set of vertices \(V_G\) and a function \(h : F_G \to V_G\). The fixed points of \(\tau\) are called legs and the orbits of length two are called edges. Let \(E_G\) denote the set of edges. Let \(v\) and \(v'\) be two vertices. They are said to share an edge if there exists a flag \(f\) such that \(h(f) = v\) and \(h(\tau(f)) = v'\), and they are said to be connected if there exists a sequence of vertices \(v = v_0, v_1, \ldots, v_k = v'\) such that \(v_i\) and \(v_{i+1}\) share an edge. A graph is called connected if any two of its vertices are connected. The valency of a vertex is the cardinality \#\(h^{-1}(v)\).

A morphism of graphs \(\phi : G \to G'\) is a function \(\phi^* : F_G \to F_{G'}\), which is required to be bijective on legs and injective on edges, together with a function \(\phi_* : V_G \to V_{G'}\), such that \(\phi_*\) is a coequalizer of the two functions \(h, h \circ \tau : F_G \setminus \phi^*(F_{G'}) \to V_{G'}\).

A graph is called a tree if it is connected and \#\(V_G\) - \#\(E_G\) = 1. A tree is planar if for each vertex \(v\) there is a specified cyclic ordering on the set \(F_v := h^{-1}(v)\) of flags attached to \(v\). A tree is stable if every vertex has at least three legs attached to it. A tree (with \(n\) legs) is said to be labeled if we are given a bijection between the set of legs and the cyclically ordered set \([n]\).

A morphism of stable, planar and labeled trees is a morphism of the underlying graphs that respects all cyclic orderings on flags and legs. With these conventions stable, planar labeled trees form a category \(\text{PT}\).

**Definition 1.1.1.** Fix a cocomplete monoidal category \(V\). A planar collection in \(V\) is an indexed family \(\{K_n : n \geq 3\}\) of objects in \(V\), such that \(K_n\) is a representation of the cyclic group \(\mathbb{Z}/n\mathbb{Z}\). Such collections form a category \(\text{PT}(V)\). Moreover, every planar collection \(K\) defines a functor

\[K[] : \text{Iso} \text{PT} \to V\]
on the category of stable, planar, labeled trees and their isomorphisms, via

\[K[T] := \bigotimes_{v \in V_T} K_{#F_v}.\]

This can be used to define an endofunctor \(\mathcal{PT} : \text{PT}(V) \to \text{PT}(V)\) by

\[\mathcal{PT}(K)_n := \colim (\text{Iso}(\text{PT} \downarrow t_n) \xrightarrow{K[]} V).\]

In the above \(t_n\) is a labeled planar tree with a single vertex and \(n\) legs, and \((\text{PT} \downarrow t_n)\) denotes the comma category of trees over \(t_n\). We call \(\mathcal{PT}\) the free planar operad functor.

Assume that \(T\) is a stable, planar labeled tree and that for every vertex \(u \in V_T\) of \(T\) we are given a stable planar labeled tree \(T_u\). Then we can build a tree \(T'\) that contains each \(T_u\) as a subtree and has the property that contracting all the \(T_u\) subtrees of \(T'\) produces the original tree \(T\). In particular, \(V_{T'} = \bigsqcup_{u \in V_T} V_{T_u}\), giving a canonical morphism

\[\bigotimes_{u \in V_T} K_{#F_u} \to \bigotimes_{u \in V_{T'}} K_{#F_u}.\]
These maps assemble to a natural transformation $\mathcal{PT} \circ \mathcal{PT} \to \mathcal{PT}$. The definition as a colimit gives a natural transformation $id \to \mathcal{PT}$. Together these two natural transformations give the free planar operad functor the structure of a monad.

**Definition 1.1.2.** A planar (pseudo-)operad in $V$ is an algebra for the free planar operad monad. A morphism of planar (pseudo-)operads is a morphism of algebras for the free planar operad monad.

Given the established terminology among operadchiks, planar operads should perhaps more properly be called “nonsymmetric cyclic” operads, but to the author’s ears that sounds a bit forced.

**Remark 1.1.3.** Just as for ordinary cyclic operads, a planar operad is determined by a planar collection $O$ and a family of composition morphisms

$$c^{\iota}_i : O_n \otimes O_k \to O_{n+k-2},$$

parametrized by $1 \leq i \leq n$, $1 \leq j \leq k$, satisfying certain associativity and equivariance conditions. These morphisms arise as follows. Graft the $i$th vertex of the tree $t_n$ to the $j$th vertex of $t_k$, to obtain a tree $t_n \circ_i t_k$: the composition of $O$ is the morphism

$$O[t_n \circ_i t_k] \to O[t_{n+k-2}]$$

defined by the algebra structure $\mathcal{PT}(O) \to O$. In fact, only the operations

$$c^{\iota}_i : O_{m+1} \otimes O_{n+1} \to O_{m+n}, \quad 1 \leq i \leq n,$$

suffice. We could have defined a planar (pseudo-)operad as a stable collection $O$ such that the collection $\{O(n) := O_{n+1}\}_{n \geq 2}$ together with the operations $c_i := c^{\iota}_{i+1}$ is a nonsymmetric (pseudo-)operad and, moreover, if $\tau : O(n) \to O(n)$ is the right action of the cycle $(n + 1 \ldots n)$, then

$$(\phi \circ_1 \psi) \tau = \psi \tau \circ_n \phi \tau, \quad \forall \phi \in O(m), \psi \in O(n), m, n \geq 2,$$

while

$$(\phi \circ_i \psi) \tau = \phi \tau \circ_{i-1} \psi \tau, \quad \forall \phi \in O(m), \psi \in O(n), m, n \geq 2, 2 \leq i \leq m.$$ 

These are the axioms we typically verify, but the freedom to graft arbitrary planar trees simplifies many abstract arguments.

Let us specialize now to the case when $V$ is the category of dg vector spaces, with Koszul sign rules. We can then define a slight variation of the free planar operad functor, as follows. Define

$$\det \otimes K[] : T \mapsto \det(V_T) \otimes_R K[T],$$

where the determinant $\det(S)$ of a finite set $S$ is defined to be the top exterior power $\wedge^{|S|} S$, placed in degree zero. The formula

$$\mathcal{PT}^- (K)_n := \text{colim}(\text{Iso}(\mathcal{PT} \downarrow t_n) \xrightarrow{\det \otimes K[]} V)$$

again defines a monad.

**Definition 1.1.4.** The free antiplanar monad is the functor $\mathcal{PT}^-$. Its algebras are called antiplanar dg (pseudo-)operads.

**Remark 1.1.5.** By suitably dualizing the definitions we obtain notions of (anti)planar (pseudo-)cooperads. The linear dual of an (anti)planar dg (pseudo)-cooperad is an (anti)planar dg (pseudo)-operad.

**Remark 1.1.6.** Every cyclic operad defines a planar operad, by simply forgetting the information that we are allowed to arbitrarily permute the inputs, retaining only the information that we are allowed to cyclically permute them. Accordingly, the operad Lie of Lie algebras, for example, is a planar operad. The operad of associative algebras $\text{Ass}$ is likewise cyclic. Moreover, as a symmetric operad it is freely generated by a nonsymmetric operad $\text{Ass}$. The same is true once we add the freedom to cyclically permute inputs: the cyclic operad $\text{Ass}_n = \text{Ind}_{\mathbb{Z}/n\mathbb{Z}}^\Sigma_n \mathbb{Q}$ is generated by the planar (pseudo)-operad $\text{Ass}_n = \mathbb{Q}$.

**Convention 1.1.7.** We will in the main body of the paper drop the qualifying prefix “pseudo” in front of operads and cooperads.
1.2. Deformation theory of planar operads. Consider the category of (reduced and augmented, symmetric) dg operads. This category has a model structure induced by that on dg vector spaces. A morphism \( f : P \to P' \) is
- a weak equivalence, also referred to as a quasi-isomorphism, if each \( f_n : P(n) \to P'(n) \) is a quasi-isomorphism of dg vector spaces.
- a fibration if each \( f_n \) is a fibration of dg vector spaces.
- a cofibration if it satisfies the lifting property.

For a proof, see [12].

**Claim 1.2.1.** The definition of a model structure repeats mutatis mutandum to define a model structure on the category of planar dg (pseudo-)operads.

The free operad functor defines a bar-cobar duality and cofibrant replacements. A consequence of the above claim is that the free planar operad functor defines a bar-cobar duality and cofibrant replacements for planar dg operads. In other words, all the theory concerning deformation complexes and Koszul duality of [14, 16, 10] applies to planar dg (co)operads, too.

1.3. Analytical manifolds with corners. We here record some facts from Brown’s [4], and [1] by Alekseev, Löffler, Rossi and Torossian.

Set \( U_{p,q} \coloneqq \mathbb{R}_{\geq 0}^d \times \mathbb{R}^p, d := p + q \). Define a non-permuting analytic isomorphism \( \varphi : U_{p,q} \to U_{p,q} \) to be an analytic diffeomorphism \( \varphi = (\varphi_1, \ldots, \varphi_d) \) of \( \mathbb{R}^d \) with positive Jacobian, restricting to a diffeomorphism of \( U_{p,q} \), where \( \varphi_i \) satisfies \( \varphi_i|_{x_i=0} = 0 \) and \( (\partial \varphi_i/\partial x_i)|_{x_i=0} = 1 \) for each \( i = 1, \ldots, q \). There is also a natural action of the permutation group \( \Sigma_p \times \Sigma_q \) on \( U_{p,q} \), and we define an analytic isomorphism \( U_{p,q} \to U_{p,q} \) to be a map as above composed with such a permutation. An analytic manifold with corners is a manifold with an atlas of that has analytic isomorphisms \( U_{p,q} \to U_{p,q} \) as transitions. Denote the algebra of complex-valued real-analytic functions on \( U_{p,q} \) by \( \mathcal{C}^r(U_{p,q}) \). Set

\[
\Omega(U_{p,q}) := \mathcal{C}^r(U_{p,q})[dx_1, \ldots, dx_{p+q}],
\Omega_1(U_{p,q}) := \Omega(U_{p,q})[d\log x_1, \ldots, d\log x_q],
\Omega_{p,\log}(U_{p,q}) := \Omega(U_{p,q})[\log x_1, \ldots, \log x_q, d\log x_1, \ldots, d\log x_q].
\]

One may check that these graded vector spaces are invariantly defined under analytic isomorphisms of \( U_{p,q} \); hence can be defined for any analytic manifold with corners. Moreover, the de Rham differential extends to give \( \Omega_{p,\log}(U_{p,q}) \) the structure of a dg vector space.

Let \( l \in [q] \) and let \( D := x_l = 0 \subset U_{p,q} \). The regularized restriction along \( D \) is the linear map

\[
\text{Reg}_D : \Omega_{p,\log}(U_{p,q}) \to \Omega_{p,\log}(D)
\]

given by

\[
\text{Reg}_D(\alpha) := \alpha(x_l = \log x_l = dx_l = d\log x_l = 0).
\]

This is compatible with analytic isomorphisms and the de Rham differential; hence defines regularized restrictions along codimension one boundary strata of any analytic manifold with corners.

2. Brown’s moduli spaces

This section recollects facts concerning Brown’s moduli spaces \( M_{0,n}^6 \), mostly borrowing from [4].

Define the open moduli space of \( n \)-pointed genus zero curves as the quotient manifold

\[
M_{0,n} := (\mathbb{CP}^1)^n \setminus \text{diagonals}/\text{PSL}_2(\mathbb{C}).
\]

It is an algebraic variety and the ring of functions has the following presentation. Define \( \chi_n \) to be the set of unordered pairs \( \{i,j\} \) of indices \( i,j \in [n] \) that are not consecutive modulo \( n \). We shall follow Brown and refer to \( \chi_n \) as the set of chords on \( [n] \). Given a chord \( \{i,j\} \in \chi_n \), let \( u_{ij} \) denote the cross-ratio

\[
u_{ij} := |i + 1 | j + 1 j := \frac{(z_i - z_{i+1})(z_{i+1} - z_j)}{(z_i - z_j)(z_{i+1} - z_{j+1})}.
\]

It is well-defined as a function on \( M_{0,n} \). Considering \( [n] \) as cyclically ordered in the natural way, any chord \( \{i,j\} \) will partition \( [n] \setminus \{i,j\} \) into two connected components. Say that two chords \( \{i,j\} \) and \( \{k,l\} \) cross if \( k \) and \( l \) belong to different connected components in the partition defined by \( \{i,j\} \). This is obviously a symmetric condition in the sense that this is true if and only if \( i \) and \( j \) lie in different
connected components of the partition defined by \{k,l\}. Given a subset \( A \subset \chi_n \), let \( A^\perp \) denote the set of chords that cross every chord in \( A \), and say that two subsets \( A, B \subset \chi_n \) cross completely if \( A^\perp = B \) and \( B^\perp = A \). The collection of arbitrary cross-ratios, \( \{[i:j|k:l]\} \), is well-known to generate the ring of functions on the moduli space. Following Brown, one can argue based on the various symmetries satisfied by cross-ratios, that the functions \( u_{ij} \) are in fact sufficient to generate the whole ring of functions:

\[
\mathcal{O}(M_{0,n}) = \mathbb{Q}[u_{ij}, u_{ij}^{-1} | \{i,j\} \in \chi_n]/(R),
\]

where \( R \) is spanned by all elements

\[
1 - \prod_{\{i,j\} \in A} u_{ij} - \prod_{\{k,l\} \in B} u_{kl},
\]
labeled by pairs of completely crossing subsets \( A, B \subset \chi_n \). We shall follow Brown and refer to the \( u_{ij} \)'s as the dihedral coordinates. The image below shows crossing chords on a pentagon and the corresponding relation between coordinate functions on \( M_{0,5} \).

```
1 5
(1) 2
(2) 3
(3) 4
u_{24} = 1 + u_{13}u_{35}.
```

**Definition 2.0.1.** Define \( A_n \) to be the de Rham complex of logarithmic algebraic forms on \( M_{0,n} \) with possible singularities on \( \overline{M}_{0,n} \setminus M_{0,n} \).

It follows from Deligne’s [6] that \( A_n \) has trivial differential and that \( A_n \to H(M_{0,n}) \) is injective. The cohomology of the spaces \( M_{0,n} \) have a pure Hodge structure and that implies that the map into the cohomology is also surjective.

**Lemma 2.0.2.** [6] The cohomology algebra \( \mathbb{A}_n \cong H(M_{0,n}) \) is the graded commutative algebra generated by the degree 1 elements

\[
\alpha_{ij} := d \log u_{ij}, \{i,j\} \in \chi_n,
\]

modulo the relations that

\[
\sum_{\{i,j\} \in A} \alpha_{ij} \left( \sum_{\{k,l\} \in B} \alpha_{kl} \right) = 0
\]

for all pairs of completely crossing subsets \( A, B \subset \chi_n \).

Note that the logarithmic forms \( \alpha_{ij} \) satisfy the relations given above as forms, not just as cohomology classes.

**Definition 2.0.3.** Brown’s moduli space \( M^\Delta_{0,n} \) is the variety

\[
M^\Delta_{0,n} := \text{Spec} \mathbb{Q}[u_{ij} | \{i,j\} \in \chi_n]/\langle R \rangle,
\]

where \( R \) is the same set of relations as that defining the open moduli space.

2.1. **Operadic structure.** Choose a chord on \([n]\). Without loss of generality we may write the chord in the form \( \{i, i+k\} \), with \( 1 \leq i \leq n \) and \( i+k \leq n \). Imagine \([n]\) as numerating the vertices of an \( n \)-gon. The chord \( \{i, i+k\} \) divides the \( n \)-gon into two: a \((k+1)\)-gon with vertices \( \{1, \ldots, i, i+k, \ldots, n\} \) and an \((n-k+1)\)-gon with vertices \( \{i, i+1, \ldots, i+k\} \). This defines a partition

\[
\chi_n = \chi_{\{1, \ldots, i+k, \ldots, n\}} \sqcup \chi_{\{i, i+1, \ldots, i+k\}} \sqcup \{\{i,j\}\} \sqcup \{\{p,q\} \in \chi_n | \{p,q\} \text{ crosses } \{i,j\}\}.
\]

Define a morphism

\[
\Delta_{\{i, i+k\}} : \mathbb{Q}[u_{ij} | \{i,j\} \in \chi_n] \to \mathbb{Q}[u_{ij} | \{i,j\} \in \chi_{\{1, \ldots, i, i+k, \ldots, n\}}] \otimes \mathbb{Q}[u_{ij} | \{i,j\} \in \chi_{\{i, i+1, \ldots, i+k\}}]
\]

by sending \( u_{rs} \) to

- \( u_{rs} \otimes 1 \), if \( \{r,s\} \in \chi_{\{1, \ldots, i, i+k, \ldots, n\}} \),
- \( 1 \otimes u_{rs} \), if \( \{r,s\} \in \chi_{\{i, i+1, \ldots, i+k\}} \),
- \( 0 \), if \( \{r,s\} = \{i, i+k\} \),
- \( 1 \otimes 1 \), if \( \{r,s\} \) crosses \( \{i, i+k\} \).
Reindexing according to the unique order-preserving bijections
\[ \{1, \ldots, i, i + k, \ldots, n\} \cong \{1, \ldots, n - k + 1\}, \quad \{i, i + 1, \ldots, i + k\} \cong \{1, \ldots, k + 1\}, \]
we obtain a morphism
\[ \Delta_{\{i, i+k\}} : \mathcal{O}(\mathcal{M}_{\delta}^\rho) \rightarrow \mathcal{O}(\mathcal{M}_{\delta}^{\rho, n-k+1}) \otimes \mathcal{O}(\mathcal{M}_{\delta}^{\rho, k+1}), \]
and hence, dually, a morphism
\[ \circ^{k+1}_i : \mathcal{M}_{\delta}^{\rho,k+1} \times \mathcal{M}_{\delta}^{\rho,n-k+1} \rightarrow \mathcal{M}_{\delta}^{\rho,n}. \]

\textbf{Lemma 2.1.1.} The morphism displayed above is the inclusion of the boundary strata of \( \mathcal{M}_{\delta}^{\rho} \) which is defined by the equation \( u_{i,i+k} = 0 \).

\textbf{Lemma 2.1.2.} The inclusions of boundary strata define a structure of planar operad (in the category of affine varieties) on the collection of Brown’s moduli spaces.

\textit{Proof.} The Deligne-Mumford compactifications \( \overline{\mathcal{M}}_{0,n} \) are well-known to assemble to a cyclic operad, with composition maps defined by inclusions of boundary strata. The necessary associativity and equivariance relations for the maps on Brown’s moduli spaces follow immediately from the corresponding ones for the Deligne-Mumford compactifications. \( \square \)

2.2. The associahedra. The condition that the \( n \) marked points appear in the cyclic order \( z_1 < \cdots < z_n \) defines a connected component \( X_n \subset \mathcal{M}_{0,n}(\mathbb{R}) \) of the set of real points of the open moduli space. Its compactification inside \( \mathcal{M}_{\delta}^{\rho,n} \), call it \( X_n \), is isomorphic as an analytical manifold with corners to the \((n−1)\)-dimensional associahedron. It is defined by the equations
\[ X_n = \{0 \leq u_{ij} \leq 1 \text{ for all } \{i, j\} \in \chi_n\}, \]
and its interior is isomorphic to the open simplex \( \{0 < z_2 < \cdots < z_{n−2} < 1\} \). The planar operad structure of Brown’s moduli spaces restricts to a planar operad structure on the \( X_n \)’s (in the category of oriented analytical manifolds with corners). The homology operad \( H(X) \) is canonically isomorphic to the planar operad \( A_\infty \) governing cyclic associative algebras. It is generated by a degree zero \( \nu \in A_3 \) satisfying the relations \( \nu \cdot (312) = \nu \) and \( \nu \circ_1 \nu = \nu \circ_3 \nu \). The boundary of \( X_n \) is stratified by products of lower-dimensional \( X_i \)’s. It follows that the face complexes \( C(X_n) \) constitute a planar dg suboperad of the operad of chains (currents) on \( X \). One easily sees that \( C(X) \) is free as a planar graded operad on the collection \( \{[X_n] \mid n \geq 3\} \) of fundamental chains.

\textbf{Lemma 2.2.1.} The identification \( \nu_n = [X_n] \) is an isomorphism between the planar dg operad \( C(X) \) of fundamental chains on Brown’s associahedra and the nonsymmetric dg operad \( A_\infty \) of \( A_\infty \) algebras, considered as a planar dg operad.

\textbf{Remark 2.2.2.} We leave the lemma without proof. The isomorphism between the nonsymmetric \( A_\infty \) operad and the cell complex operad of associahedra, and the cyclic compatibility of it, is well-known. (The statement is old enough to predate the language of operads, and has historically been one of the main reasons for inventing the language of operads!)

2.3. Cooperads of cohomology algebras. One can regard the cohomology algebras \( A_n \) of the open moduli spaces as (possibly) singular analytical forms on Brown’s moduli spaces. One may accordingly apply regularized restrictions of these forms to boundary strata.

\textbf{Lemma 2.3.1.} The collection \( A \) of cohomology algebras of the open moduli spaces is a planar dg cooperad under the cocompositions defined by regularized restriction to boundary strata. If \( \{i, j\} \in \chi_n \) equals \( \{i, j\} \) or crosses it, then the cocomposition
\[ \Delta_{\{i,j\}}^A : A_n \rightarrow A_{n-n'+1} \otimes A_{n'+1} \]
corresponding to regularized restriction of forms to the strata \( u_{ij} = 0 \) sends the form \( \alpha_{ij} \) to zero; otherwise it just sends it to the form associated to the corresponding chord on \( [n-n'+1] \) or on \([n'+1]\).

\textit{Proof.} This is just the definition of regularized restriction, given the explicit form of the cocomposition \( \mathcal{O}(\mathcal{M}_{\delta}^{\rho}) \rightarrow \mathcal{O}(\mathcal{M}_{\delta}^{\rho,n-n'+1}) \otimes \mathcal{O}(\mathcal{M}_{\delta}^{\rho,n'}) \) on functions, since \( u_{ij} = 0 \) is a global equation defining the strata. \( \square \)
In [9], Ezra Getzler defined a different cooperad structure on the cohomology algebras of the open moduli space. We shall only be concerned with the nonsymmetric data arising in the construction, and then we can paraphrase Getzler’s construction in the following way.

Define
\[ \text{Res}_{\{i,j\}} : A_n \to (A_{n-n'+1} \otimes A_{n'+1})[-1] \]
to be the Poincaré residue of logarithmic forms along the divisor \( u_{ij} = 0 \). Shift degrees to obtain a degree zero mapping
\[ \Delta_{\{i,j\}}^A : A_n[-1] \to A_{n-n'+1}[-1] \otimes A_{n'+1}[-1]. \]

Lemma 2.3.2. The maps \( \Delta_{\{i,j\}}^A \) equip the collection \( A[-1] \) with the structure of an antiplanar dg cooperad.

Lemma 2.3.3. The Poincaré residue has the explicit expression \( \text{Res}_{\{i,j\}} = \Delta_{\{i,j\}}^A \circ \partial/\partial \alpha_{ij} \).

Proof. Let \( A, B \subset \chi_n \) be a pair of completely crossing subsets and define
\[ R_{A,B} := \left( \sum_{\{i,j\} \in A} \alpha_{ij} \right) \left( \sum_{\{k,l\} \in B} \alpha_{kl} \right) \in \mathbb{Q}[\alpha_{ij} | \{i,j\} \in \chi_n]. \]
Recall that \( A_n \) is freely generated by the \( \alpha_{rs} \) modulo the relations that \( R_{A,B} = 0 \) for all pairs of completely crossing subsets. Assume first that \( \{i,j\} \notin A \cup B \). Then
\[ \frac{\partial}{\partial \alpha_{ij}} R_{A,B} = 0. \]
Assume conversely, without loss of generality, that \( \{i,j\} \in A \). Then
\[ \frac{\partial}{\partial \alpha_{ij}} R_{A,B} = \sum_{\{k,l\} \in B} \alpha_{kl}. \]
However, we note that all \( \{k,l\} \in B \) must then cross \( \{i,j\} \); and since \( \Delta_{\{i,j\}}^A \alpha_{rs} = 0 \) if \( \{r,s\} \) crosses \( \{i,j\} \), we can conclude that, in all cases,
\[ \Delta_{\{i,j\}}^A \frac{\partial}{\partial \alpha_{ij}} R_{A,B} = 0. \]
This proves that the expression is well-defined as a map on \( A_n \). That it equals the Poincaré residue is then clear since \( u_{ij} = 0 \) is the equation defining the strata and \( \alpha_{ij} = du_{ij}/u_{ij} \). \( \square \)

Definition 2.3.4. Define \( A^\delta_n \) to be the de Rham complex of algebraic logarithmic forms on \( M_{0,n} \) with possible singularities on \( \overline{M}_{0,n} \setminus M_{0,n} \).

Remark 2.3.5. The canonical \( A^\delta_n \to H(M_{0,n}^\delta) \) is an isomorphism, just as the corresponding map for the open moduli space. Moreover, \( A^\delta_n \subset A_n \), and restriction to boundary strata defines a nonsymmetric dg (sub)cooperad structure on the collection \( A^\delta \subset A \).

The algebras \( A^\delta_n \) are quite involved to define in terms of generators and relations. For our purposes the following lemma provides the best description.

Lemma 2.3.6. The cohomology algebra \( A^\delta_n \) of Brown’s moduli space is the joint kernel
\[ A^\delta_n = \bigcap_{\{i,j\} \in \chi_n} \ker(\text{Res}_{\{i,j\}}) \subset A_n \]
of all the Poincaré residue maps.

Proof. A form is regular if and only if it has no residue anywhere. \( \square \)
2.4. Brown’s theorem on periods. Let $\zeta$ denote the algebra of multiple zeta values (for short, MZVs), i.e., the subalgebra of the real numbers generated by all

$$\zeta(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

for $k_1, \ldots, k_r$ a sequence of strictly positive integers, with $k_r \geq 2$. The weight of a multiple zeta value $\zeta(k_1, \ldots, k_r)$ is the number $k_1 + \cdots + k_r$.

Recall the embedded associahedra $X_n \subset M^\delta_{0,n}$. Brown has shown the following very remarkable fact:

**Lemma 2.4.1.** [4] Let $\beta \in \mathbb{A}^d$. Its integral $\int_{X_n} \beta$ is a rational linear combination of multiple zeta values of weight at most $n - 3$. Moreover, every multiple zeta value arises as such an integral.

2.5. The dihedral KZ connection. In [4], Brown gave a novel treatment of the Knizhnik-Zamolodchikov connection, which we recall here.

**Definition 2.5.1.** Define the dihedral Lie algebra on $[n]$, to be denoted $\mathfrak{d}_n$, to be the Lie algebra generated by variables $\delta_{ij}$, $i, j \in [n]$, modulo the relations that $\delta_{ji} = \delta_{ij}$ for all indices, $\delta_{ij} = 0$ unless $\{i, j\} \in \chi_n$ is a chord, and

$$[\delta_{i-1,j} + \delta_{j-1,i} - \delta_{i-1,j-1} - \delta_{j-1,i-1}, \delta_{k-1,l} + \delta_{l-1,k} - \delta_{k-1,l-1} - \delta_{l-1,k-1} - \delta_{kl}] = 0$$

for all quadruples of indices such that $\#\{i, j, k, l\} = 4$.

**Lemma 2.5.2.** The Lie algebras $\mathfrak{d}_n$ naturally form a planar operad in the category of Lie algebras with direct sum as monoidal product.

**Proof.** Instead of mimicking gluing of trees, we interpret operadic composition graphically as gluing of polygons. Gluing together a polygon with $n - k + 1$ sides and a polygon with $k + 1$ sides, along specified edges, produces an $n$-gon. From the gluing arises a function

$$f \sqcup g : \chi_{n-k+1} \sqcup \chi_{k+1} \rightarrow \chi_n.$$

We used this previously, when we noted that a chord defines a partition of $\chi_n$ and a corresponding cooperadic cocomposition on the algebra of functions on $M^\delta_{0,n}$. Define

$$\mathfrak{d}_{n-k+1} \oplus \mathfrak{d}_{k+1} \rightarrow \mathfrak{d}_n$$

by $\delta_{ij} \oplus \delta_{kl} \mapsto \delta_{f(i)f(j)} + \delta_{g(k)g(l)}$. These compositions are obviously associative and equivariant under cyclic permutations (because gluing polygons is), and defining relations are mapped to defining relations (only the indices change). $\Box$

It follows that the Chevalley-Eilenberg cochain complexes (with rational coefficients) $C(\mathfrak{d})$ form a planar cooperad of dg commutative algebras.

**Lemma 2.5.3.** [4] The form $\alpha_n := \sum_{\{i,j\} \in \chi_n} \alpha_{ij} \delta_{ij} \in \mathbb{A}_n \otimes \mathfrak{d}_n$ is a (singular) flat connection on $M^\delta_{0,n}$, compatible with regularized restriction to boundary strata, i.e., it defines (for varying $n$) a morphism

$$\alpha : C(\mathfrak{d}) \rightarrow \mathbb{A}$$

of planar cooperads of dg commutative algebras.

We shall refer to $\alpha$ as the dihedral Knizhnik-Zamolodchikov connection.

3. Operads of graphs and chord diagrams

3.1. Chord diagrams.

**Definition 3.1.1.** Define $\mathcal{C}_n$, the cooperad of dihedral chord diagrams, to be the planar cooperad $C(\mathfrak{d} / [\mathfrak{d}, \mathfrak{d}])$ of Chevalley-Eilenberg cochain complexes on the Abelianizations of the dihedral Lie algebras.

Its linear dual, $\mathcal{G} := C(\mathfrak{d} / [\mathfrak{d}, \mathfrak{d}])$, is the planar operad of dihedral chord diagrams.

It follows that $\mathcal{C}_n = \mathbb{Q}[\delta_{ij}^r \mid \{i, j\} \in \chi_n]$. We call elements in this algebra chord diagrams, because their monomials can be represented diagrammatically by an $n$-gon with a set of chords drawn on it.
Ordering the chords in the diagram recovers the monomial. A diagrammatic example of the operadic composition is shown below.

Let us spend a few words diagrammatically describing the cocomposition. Say that a chord \( \{i,j\} \in \chi_n \) is dividing for a diagram \( \delta \in C_n \) if \( \{i,j\} \) is not contained in \( \delta \) and if, additionally, it does not cross any chord in \( \delta \). For example, the chord \( \{3,6\} \) is dividing for the heptagon displayed to the right, above. Splitting the \( n \)-gon along the chord \( \{i,j\} \) must accordingly split the diagram \( \delta \) into two diagrams, \( \delta' \) and \( \delta'' \), drawn on the respective smaller polygons that arise. The cocomposition sends \( \delta \) to \( \delta' \otimes \delta'' \), or, if \( \{i,j\} \) is not dividing for the diagram, then it sends it to zero.

We can combinatorially mimic the formula 2.3.3 for the Poincaré residue, and set

\[
\text{Res}_{\{i,j\}} := \Delta^{c}_{\{i,j\}} \circ \frac{\partial}{\partial \delta_{ij}}.
\]

Diagrammatically, this amounts to first removing the chord \( \{i,j\} \) from the diagram and then dividing the resulting diagram by cutting the \( n \)-gon along \( \{i,j\} \). If the chord is not present to begin with, or after its deletion it is not dividing, then the residue is zero. Say that a chord \( \{i,j\} \) in a diagram \( \delta \) is residual if \( \text{Res}_{\{i,j\}} \delta \neq 0 \). Below is a diagrammatic example.

Thus, in this example the chord \( \{3,5\} \) is residual for the displayed dihedral chord diagram on the heptagon. It is the only residual chord in the diagram.

**Definition 3.1.2.** A monomial in \( C_n \) is convergent if it lies in the joint kernel \( \bigcap_{\{i,j\} \in \chi_n} \text{Res}_{\{i,j\}} \subset C_n \) of all diagrammatical residue maps, i.e., if it has no residual chords. The collection of convergent dihedral chord diagrams is the collection \( C^\delta \subset C \) spanned by all convergent monomials.

**Lemma 3.1.3.** The collection of convergent dihedral chord diagrams is a planar dg cooperad, with a split inclusion

\[
id : C^\delta \hookrightarrow C \xrightarrow{\text{res}} C^\delta
\]

into the cooperad of all dihedral chord diagrams.

**Definition 3.1.4.** The linearly dual planar dg operad \( G^\delta := (C^\delta)^* \) is the operad of convergent dihedral chord diagrams.

**Proof.** Assume that the monomial \( \delta \in C_n \) is convergent. Then for every chord \( \{i,j\} \in \chi_n \), \( \Delta^{c}_{\{i,j\}} \delta \) is either zero or a tensor product \( \delta' \otimes \delta'' \) of monomials. Assume, to get a contradiction, that the cocomposition is not zero and that \( \delta' \) is not convergent, i.e., that it has a residual chord \( \{r',s'\} \). This chord must be the image under the cocomposition of some residual chord \( \{r,s\} \) of \( \delta \) because up to reindexing, if \( \text{Res}_{\{r',s'\}} \delta' = \delta_1 \otimes \delta_2 \), then \( \text{Res}_{\{r,s\}} \delta = \delta_1 \otimes \delta_2 \otimes \delta'' \), where \( \delta_2 \otimes \delta'' \) is a diagram formed by gluing together \( \delta_2 \) and \( \delta'' \) along \( \{i,j\} \). We thus have a contradiction and can conclude that the subspace spanned by convergent monomials is closed under cooperadic cocomposition.

Monomials constitute a basis of the dihedral chord algebra \( C_n \). Projecting only onto the subspace spanned by convergent monomials defines a map \( \text{reg} : C \rightarrow C^\delta \) of collections of dg vector spaces. Almost exactly the same argument by contradiction as above shows that it respects cocompositions. Namely, take a monomial \( \delta \in C_n \) and write \( \Delta^{c}_{\{i,j\}} \delta = \delta' \otimes \delta'' \). Either \( \text{reg}(\delta') = \delta \) or \( \text{reg}(\delta') = 0 \), depending on whether \( \delta \) is convergent or not. In the first case, \( \text{reg}(\delta') \otimes \text{reg}(\delta'') = \delta' \otimes \delta'' \), because a residual chord in
either factor would have to come from a residual chord in \( \delta \), and there are none. In the second case, i.e.,
when \( \delta \) has a residual chord \( \{r, s\} \) we either have \( \delta' \otimes \delta'' = 0 \) (if \( \{r, s\} = \{i, j\} \) or if \( \{r, s\} \) crosses \( \{i, j\} \)) or the image of \( \{r, s\} \) ends up as a residual chord in \( \delta' \) or in \( \delta'' \), and then \( \text{reg}(\delta') \otimes \text{reg}(\delta'') = 0 \).

**Lemma 3.1.5.** The morphism

\[
\Delta_C = C \otimes (\delta \to \overline{\delta} / \overline{[\delta]} \to A),
\]

defined by the Abelianization \( \delta \to \overline{\delta} / \overline{[\delta]} \) and the dihedral Knizhnik-Zamolodchikov connection \( \partial/\partial \alpha \) restricts to a morphism \( \Delta_C \to A^\delta \) of planar dg cooperads.

**Proof.** The mapping \( \delta \to A \) is a morphism of planar cooperads of dg commutative algebras by naturality of Abelianization. As a consequence it must commute with the respective Poincaré residue maps that we have defined, \( \Delta_C \to (\partial/\partial \delta \alpha) \) and \( \Delta_A \to (\partial/\partial \alpha \alpha) \), respectively. The result then follows from 2.3.6.

**Lemma 3.1.6.** The projection \( \text{reg} : C \to C^\delta \) onto convergent dihedral chord diagrams induces an analogous morphism \( \text{reg} : A \to A^\delta \) of planar operads, such that

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & A \\
\text{reg} \downarrow & & \downarrow \text{reg} \\
C^\delta & \xrightarrow{\text{reg}} & A^\delta
\end{array}
\]

commutes.

**Proof.** The kernel of \( \alpha : C_n \to A_n \) is the ideal generated by all

\[
\rho_{A,B} := \left( \sum_{\{i,j\} \in A} \delta_{ij} \right) \left( \sum_{\{k,l\} \in B} \delta_{kl}^* \right),
\]

def for pairs of completely crossing subsets \( A, B \subset \chi_n \). We must show that \( \alpha(\text{reg}(\rho_{A,B})) = 0 \) for all monomials \( \delta \in C_n \). This will be true if \( \text{reg}(\rho_{A,B}) \) is either zero or \( \rho_{A,B} \). By linearity,

\[
\text{reg}(\rho_{A,B}) = \sum_{\{i,j\} \in A} \sum_{\{k,l\} \in B} \text{reg}(\delta_{ij} \delta_{kl}^*).
\]

Assume that \( \text{reg}(\delta_{ij} \delta_{kl}^*) = 0 \) for some \( \{i, j\} \in A \) and \( \{k, l\} \in B \). Neither \( i, j \) nor \( k, l \) can be residual chords of \( \delta_{ij} \delta_{kl}^* \), because \( i, j \) is not dividing for \( \{k, l\} \) and vice versa. Hence \( \delta \) must have a residual chord, but then \( \text{reg}(\delta_{ij} \delta_{kl}^*) = 0 \) for all \( \{i, j\} \in A \) and \( \{k, l\} \in B \), implying \( \text{reg}(\rho_{A,B}) = 0 \). Conversely, assume that \( \text{reg}(\delta_{ij} \delta_{kl}^*) = \delta_{ij} \delta_{kl}^* \) for some \( \{i, j\} \in A \) and \( \{k, l\} \in B \). Then \( \delta \) can have no residual chord, and we conclude that \( \text{reg}(\rho_{A,B}) = 0 \).

That \( \text{reg} : A \to A^\delta \) is a morphism of planar cooperads is automatic:

\[
\Delta_A \text{reg}(\alpha(\delta)) = \Delta_A \alpha(\text{reg}(\delta)) = (\alpha \otimes \alpha) \Delta^C \text{reg}(\delta)
\]

\[
= (\alpha \otimes \alpha)(\text{reg} \otimes \text{reg}) \Delta^C \delta = (\text{reg} \otimes \text{reg})(\alpha \otimes \alpha) \Delta^C \delta
\]

\[
= (\text{reg} \otimes \text{reg}) \Delta_A \alpha(\delta).
\]

**Definition 3.1.7.** We refer to both \( \text{reg} : A \to A^\delta \) and \( \text{reg} : C \to C^\delta \) as the regularization.

**Definition 3.1.8.** The composite morphism

\[
\alpha_{\text{reg}} : C \to A \to A^\delta
\]

defined by elements \( b_{ij} (1 \leq i, j \leq n) \) and \( s_i (1 \leq i \leq n) \), modulo the relations that \( b_{ij} = b_{ji} \), \( b_{ii} = 0 \), \( 2s_k + \sum_{i=1}^{n} b_{ik} = 0 \),

\[
[b_{ij}, b_{kl}] = 0 \quad \text{if} \quad \#\{i, j, k, l\} = 4,
\]

and \( [s_k, \text{any}] = 0 \) for all \( k = 1, \ldots, n \).
Take $1 < i < n$. We define a map
\[ c_i^{k+1} : \mathfrak{tb}_{n-k+1} \oplus \mathfrak{tb}_{k+1} \to \mathfrak{tb}_n \]
by
\[
\begin{align*}
\quad b_{pq} \oplus 0 & \mapsto b_{pq} \quad \text{if } p, q < i, \\
\sum_{r=i}^{i+k-1} b_{pr} & \mapsto b_{p+i-1,0} + b_{i+1+q-1} \quad \text{if } p < i, q = i, \\
\sum_{r=i}^{i+k-1} b_{r+i-k-1} & \mapsto b_{i+p-1+q-1} \quad \text{if } p = i, i < q, \\
0 \oplus b_{pq} & \mapsto b_{i+p-1+q-1} \quad \text{if } p, q \leq k,
\end{align*}
\]

**Lemma 3.2.2.** The maps $c_i^{k+1}$ defined above make the Lie algebras of spherical ribbon braids a planar operad of Lie algebras.

We shall not prove the above statement. There are similar statements proved in the literature, e.g., the paper [18] proves that the Lie algebras $t_n$ of braids (in the plane) constitute an operad, with analogous formulas for the composition. The paper [20] states that the Lie algebras of ribbon braids in the plane form an operad, again by analogous formulas. The operadic composition which we have defined can be given a graphical interpretation, and then the necessary associativity constraints become more transparent.

Define
\[ \text{Gra}_n^{\mathcal{O}} := \mathbb{Q}[e_{ij}, e_k \mid 1 < i < j < n - 1, 1 \leq k \leq n - 1], \]
where we give all generators degree minus one. Note that because of the relations $s_n = -(1/2) \sum_{i=1}^{n-1} b_{in}$ and $b_{in} = -2s_l - \sum_{j=1}^{n-1} b_{it}$, the mapping
\[ \text{Gra}_n^{\mathcal{O}} \to C.(\mathfrak{tb}_n/[\mathfrak{tb}_n, \mathfrak{tb}_n]) \]
that sends $e_{ij}$ to $b_{ij}$ and $e_k$ to $s_k$, is an isomorphism. We use this isomorphism to transfer the canonical planar operad structure on $C.(\mathfrak{tb}/[\mathfrak{tb}, \mathfrak{tb}])$ to $\text{Gra}_n^{\mathcal{O}}$.

**Definition 3.2.3.** The operad of tadpole graphs is the planar dg operad $\text{Gra}_n^{\mathcal{O}}$.

A monomial in $\text{Gra}_n^{\mathcal{O}}$ gives rise to a graph with vertex set $[n-1]$, an edge between vertices $i$ and $j$ for every $e_{ij}$ in the monomial, and a tadpole (an edge attached at both ends to the same vertex) at the vertex $k$ for every $e_k$ in the monomial, and no legs. By ordering the edges and tadpoles one recovers the monomial from the graph. The operad composition
\[ c_i^{k+1} : \text{Gra}_{n-k+1}^{\mathcal{O}} \oplus \text{Gra}_{k+1}^{\mathcal{O}} \to \text{Gra}_n^{\mathcal{O}} \]
\[ \Gamma \otimes \Gamma' \to \Gamma \circ_i^{k+1} \Gamma' \]
can be described graphically as follows. Remove the vertex $i$ of $\Gamma$ and consider the edges previously attached to $i$ as legs, producing a graph with legs $\Gamma' \{ \{i\} \}$. The composition $\Gamma \circ_i^{k+1} \Gamma'$ is the sum $\sum \pm \Gamma''$ over all graphs $\Gamma''$ that can be obtained from $\Gamma' \{ \{i\} \}$ and $\Gamma'$ by attaching the legs of the former to the vertices of the latter. The signs in the sum arise from reordering the edge sets and can be recovered by
reverting to the purely algebraic description of the composition. In the example below the vertices are numbered left to right and the order of the edges has been omitted.

\[
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
  \node (4) at (3,0) {$4$};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \node (5) at (4,0) {$5$};
  \draw (4) -- (5);
\end{tikzpicture}
\]

\[
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
  \node (4) at (3,0) {$4$};
  \node (5) at (4,0) {$5$};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
\end{tikzpicture}
\]

The number of terms grows rapidly with the size of the graphs we compose. This is in stark contrast to the composition of dihedral chord diagrams, where the composition of two diagrams is a single new diagram. This difference notwithstanding, we claim the following result:

**Lemma 3.2.4.** The formula

\[
\gamma : \mathcal{D}_n \to \mathcal{B}_n, \quad \delta_{ij} \mapsto \sum_{i \leq r < s < j} b_{rs} + \sum_{1 \leq k < j} s_k
\]

is a morphism of planar operads of Lie algebras.

**Remark 3.2.5.** The following identity holds:

\[
\gamma(\delta_{pq}) = s_p \circ_{\mathcal{B}}^{q-p+1} 0,
\]

where the composition is \( \circ_{\mathcal{B}}^{q-p+1} : \mathcal{B}_{n+p-q+1} \otimes \mathcal{B}_{q-p+1} \to \mathcal{B}_n \).

**Proof.** To see that \( \gamma \) is well-defined as a morphism of collections of Lie algebras, we note that

\[
\gamma(\delta_{i-1} + \delta_{j-1} - \delta_{i-j-1} - \delta_{ij}) = b_{i-1} - b_{j-1} - b_{j-2}.
\]

Thus

\[
\gamma(\delta_{i-1} + \delta_{j-1} - \delta_{i-j-1} - \delta_{ij}) = b_{i-1} - s_{j-1} - b_{k-1} - b_{k-2}.
\]

The elements \( s_r \) are central, which reduces the expression to \( [b_{i-1} - s_{j-1} - b_{k-1} - b_{k-2}] \), which is zero whenever \# \{i, j, k, l\} = 4, as required. That \( \gamma \) is compatible with the cyclic group actions is a much more involved computation and we have referred the calculation to the appendix \( \Box \)

To prove that \( \gamma \) is a morphism of operads of Lie algebras, one essentially only has to note that

\[
\gamma(\delta_{pq}) = \circ_{\mathcal{B}}^{q-p+1}(s_p \otimes 0) = s_p \circ_{\mathcal{B}}^{q-p+1} 0.
\]

That \( \gamma \) respects compositions then follows from the associativity relations for the operad of spherical ribbon braids. For example, if \( p < i < q \), then

\[
\gamma(\delta_{pq} \circ_{\mathcal{B}}^{k+1} 0) = \gamma(\delta_{p,q+k-1}) = s_p \circ_{\mathcal{B}}^{q-p+k-1} 0.
\]

On the other hand,

\[
\gamma(\delta_{pq}) \circ_{\mathcal{B}}^{k+1} 0 = (s_p \circ_{\mathcal{B}}^{q-p+1} 0) \circ_{\mathcal{B}}^{k+1} 0 = s_p \circ_{\mathcal{B}}^{q-p+k-1} 0 \circ_{\mathcal{B}}^{k+1} 0 = s_p \circ_{\mathcal{B}}^{p-q+k-1} 0.
\]

\[\Box\]

**Definition 3.2.6.** We abuse notation and denote the induced morphism

\[
\gamma : G = C(\mathcal{D}/[\mathcal{D}, \mathcal{D}]) \to C(\mathcal{B}/[\mathcal{B}, \mathcal{B}]) \cong \text{Gra}^{\circ}
\]

of planar dg operads by the same symbol \( \gamma \). Note that \( s_n \) or \( b_{rs} \) never occur in the the formula for \( \gamma \), so above morphism is formally identical to the map on Lie algebras:

\[
\gamma(\delta_{ij}) = \sum_{i \leq r < s < j} e_{rs} + \sum_{1 \leq k < j} e_k = e_i \circ_{\mathcal{B}}^{j-i+1} 1.
\]

4. The main theorem

**Definition 4.0.7.** The Batalin-Vilkovisky operad \( \text{BV} \) is the planar dg operad \( H(\mathcal{B}) \) of Chevalley-Eilenberg homologies of the spherical ribbon braids.

**Remark 4.0.8.** The Batalin-Vilkovisky operad is not just planar but, in fact, cyclic. As such it is generated by a bracket operation \( \{ \cdot, \cdot \} = [b_{ij}] \in H_{-1}(\mathcal{B}_3) \), a product operation \( m = [1] \in H_0(\mathcal{B}_3) \) and a so-called BV operator \( \Delta = [s_1] \in H_{-1}(\mathcal{B}_3) \), under the following relations:

- The bracket is a dg Lie bracket of degree \(-1\).
- The product operation is a dg commutative associative multiplication.
- The BV operator \( \Delta \) squares to zero.
- The bracket is a degree $-1$ derivation of the product:
\[ \{ , \} \circ_2^1 m = m \circ_1^1 \{ , \} + (m \circ_3^1 \{ , \}) \]
- The BV operator acts as a degree $-1$ derivation of the bracket:
\[ \Delta \circ_1^1 \{ , \} + \{ , \} \circ_2^1 \Delta + \{ , \} \circ_3^1 \Delta = 0. \]

- The cyclic structure is defined by the relations
\[ s_n = -(1/2) \sum_{i=1}^{n-1} b_{in} \quad \text{and} \quad b_n = -2s_l - \sum_{j=1}^{n-1} b_{jl}. \]
In particular, if $\tau$ is the generator $i \mapsto i + 1$ of the cyclic action, then $m\tau = m$ while $\{ , \} \tau = -2m \circ_2^1 \Delta - \{ , \}$ and $(m \circ_2^1 \Delta)\tau = m \circ_1^1 \Delta + m \circ_3^1 \Delta + \{ , \}$. That $H(\mathfrak{tb})$ indeed is given by above generators and relations is proved in [20].

**Theorem 4.0.9.** The mapping $\gamma$, the regularization, and the dihedral Knizhnik-Zamolodchikov connection together define, via integration over the associahedra in Brown’s moduli spaces, a morphism
\[ \nu : \zeta \otimes \mathcal{A}_\infty \to \zeta \otimes \text{BV} \]
of planar dg operads. The coefficients of the operation $\nu_n \in \zeta \otimes \text{BV}_n$ are rational sums of multiple zeta values of weight at most $n - 3$.

**Proof.** It follows from [2.2.1] and [2.4.1] that integration of forms on Brown’s moduli spaces is a canonical morphism
\[ \zeta \otimes \mathcal{A}_\infty \to \zeta \otimes (A^1)^*, \]
whose coefficients of the $n$-ary operation are multiple zeta values of weight at most $n - 3$. The regularized dihedral Knizhnik-Zamolodchikov $\alpha_{\text{reg}} : C(\mathfrak{o}) \to A^1$ of [3.1.8] connection is a morphism of planar dg cooperads. The mapping $\gamma : \mathfrak{o} \to \mathfrak{tb}$ of [3.2.4] defines a morphism of planar dg cooperads $C(\mathfrak{o}) \to C(\mathfrak{tb})$. By taking cohomology and dualizing we get $\gamma \circ \alpha_{\text{reg}}^* : (A^1)^* \to H(\mathfrak{o}) \to H(\mathfrak{tb}) = \text{BV}$. □

**Definition 4.0.10.** We christen the cyclic $A_\infty$ structure $\nu$ the exotic structure.

4.1. **Obtaining an explicit formula.** To write a thoroughly explicit formula for the exotic structure we have to choose suitable basis elements of $A^5$ and $H(\mathfrak{tb})$.

4.1.1. **Basis of the BV cooperad.** Because of the relations
\[ s_n = -(1/2) \sum_{i=1}^{n-1} b_{iin} \quad \text{and} \quad b_n = -2s_l - \sum_{j=1}^{n-1} b_{jl}, \]
the cohomology algebra of the Lie algebra of spherical ribbon braids can be written entirely in terms of $b^*_{ij}$ (1 ≤ $i$, $j$ ≤ $n - 1$) and $s^*_k$ (1 ≤ $k$ ≤ $n - 1$).

**Definition 4.1.1.** Let $\Lambda_{n-1}$ be the free graded commutative algebra $\mathbb{Q}[b^*_{ij} | 1 \leq i, j \leq n - 1]$ on the degree 1 generators $b^*_{ij}$, modulo the Arnold relations
\[ b^*_{ij} b^*_{jk} + b^*_{ik} b^*_{kj} + b^*_{ik} b^*_{jk} = 0, \]
for all triples of distinct indices $i, j, k$. We call $\Lambda_{n-1}$ the Arnold algebra.

Define $S_{n-1}$ to be the free graded commutative algebra $\mathbb{Q}[s^*_1, \ldots, s^*_{n-1}]$ on degree 1 generators $s_{k}$.

**Lemma 4.1.2.** The cohomology algebra $H(\mathfrak{tb}_n)$ is isomorphic to the algebra $\Lambda_{n-1} \otimes S_{n-1}$.

**Proof.** This is rather well-known, see for example [20] or [11]. The proof is a mild generalization of Arnold’s computation in [2], from 1969, of the cohomology of the configuration space of points in the plane in terms of the cohomology of the corresponding braid group. □

Let $\beta^k_{n-1}$ denote the set of sets $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ of $k$ pairs $(i, j)$ of elements in $[n - 1]$ satisfying $i_r < j_r$, and such that no two pairs are equal. We refer to $\beta^k_{n-1}$ as the set of length $k$ index monomials. Define $\beta^*_{n-1}$ to be the subset consisting of those index monomials $M \in \beta^k_{n-1}$ with the property that there are no two $(i, j), (j, k) \in M$. Let $\beta^k_{n-2} := \bigcup_{k=1}^{n-2} \beta^k_{n-1}$, and define $\beta_{n-1}$ analogously.

**Lemma 4.1.3.** The function $\beta^k_{n-1} \to \Lambda^k_{n-1}$ that sends $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ to the monomial $b^*_{i_1j_1} \cdots b^*_{i_kj_k}$ identifies $\beta^*_{n-1}$ as a basis of $\Lambda^k_{n-1}$. 

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Proof. The proof is based on repeated use of the Arnold relation. First of all, it is clear that arbitrary index monomials in $\mathfrak{b}^k_{n-1}$ span the degree $k$ summand of the Arnold algebra. Say that two indices $i$ and $j$ are connected of length $q+1$ in an index monomial $M$ if there are $(i, r_1), (r_1, r_2), \ldots, (r_q, j) \in M$. Further, say that a pair $(i, j)$ has index $p$ in an index monomial $M \in \mathfrak{b}^p_{n-1}$ if the two indices are connected in $M$ and there are exactly $p+1$ other pairs $(r, s) \in M$ such that $r \leq i$ and $j \leq s$. For example, if $(1, n) \in M$, then it must necessarily have index 1. This allows us to put a decreasing filtration on the Arnold algebra, by letting $F^p \Lambda_{n-1}$ be spanned by monomials $b^*(M)$ with the property that for all connected $(i, j)$ in $M$ of index $p$, the restriction of $M$ to $\mathfrak{b}_{(i, \ldots, j)}$ lies in $\mathfrak{b}_{1 \ldots 3}$. Then $F^p \Lambda_{n-1} = \Lambda_{n-1}$. We claim that the Arnold relation implies $F^1 \Lambda_{n-1} = \Lambda_{n-1}$. To see this, note that

$$b^*_{r_1} b^*_{r_2} \ldots b^*_{s_{aq}} = (b^*_{r_1} b^*_{r_2} - b^*_{r_1 r_2} b^*_{r_2 r_3}) b^*_{r_3 r_4} \ldots b^*_{s_{aq}}.$$  

After this $(i, j)$ is path-connected of length $q - 1$, and the restriction to $\{i, \ldots, r_2\}$ will lie in $\mathfrak{b}_{(i, \ldots, r_2)}$. Iterating we can reduce to a monomial in $\mathfrak{b}_{(i, \ldots, j)}$. Arguing inductively on the filtration degree $p$, we conclude that the monomials in $\mathfrak{b}_{n-1}$ span the Arnold algebra. \qed

Let $s^q_{n-1}$ be the set of strictly increasing sequences of length $q$ in the ordered set $\{s^1_n < \cdots < s^n_{n-1}\}$.

Corollary 4.1.4. The set $\bigsqcup_{q+p=d} \beta^p_{n-1} \cup s^q_{n-1}$ is a basis of the degree $d$ component $H^d(\mathfrak{b}_n)$.

Note that the basis gives a linear map $H(\mathfrak{b}_n) \to C(\mathfrak{fr}_n)$, i.e., the basis vectors are representative cocycles.

Definition 4.1.5. Set $\beta s_{n-1} := \bigsqcup_{q+p=n-3} \beta^p_{n-1} \cup s^q_{n-1}$; so $\beta s_{n-1}$ spans $H^{n-3}(\mathfrak{fr}_n)$.

4.1.2. Basis of the top cohomology of Brown’s moduli spaces.

Definition 4.1.6. Define $L_{n-1}$ to be the set of iterated binary bracketings of the indices $1, \ldots, n-1$, subject to the following conditions:

- Each index appears exactly once. (Thus the word must be an iteration of $n-2$ binary brackets.)
- The smallest index in a bracket stands to the left and the largest to the right.

For example, $[1,[2,3]]$ and $[[1,2],3]$ both lie in $L_3$, but neither $[2,[1,3]]$ nor $[[1,3],2]$ does. For each $L \in L_{n-1}$, define an ordering on the bracketings in $L$ by reading them outside in and left to right. For example, the first bracket in $[[1,3],[2,4]]$ is that between $[1,3]$ and $[2,4]$, the second that between 1 and 3 and the third that between 2 and 4.

Definition 4.1.7. Say that a binary bracket $(i, j)$ of bracketings in an $L \in L_{n-1}$ is connected if the set of indices appearing inside $b$ is a connected subset of $[n-1]$. Define the set of prime bracketings, to be denoted $p_{n-1}$, to be the subset of $L_{n-1}$ consisting of all those $P$ with the property that only the outermost bracket is connected.

To each bracket in an iterated binary bracket, associate the pair $(i, j)$, where $i$ is the smallest index appearing in the bracket and $j$ is the largest. In this way, we associate to each iterated binary bracketing $L$ an index monomial $M_L = \{(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})\}$. For example, if $L = [[1,3],[2,4]]$, then $M_L = \{(1,4), (1,3), (2,4)\}$. Every index monomial $M_L$ obtained in this way must contain the pair $(1,n-1)$ (because the outermost bracket of $L$ encloses 1 and $n-1$). Define $A_L$ to be the index monomial obtained by removing $(1,n-1)$.

Remark 4.1.8. The index monomial $A_L$ lies in $\beta^{n-3}_{n-1}$ and is dual to the Batalin-Vilkovisky operation obtained from the Lie word $L$ by replacing the outermost bracket with a product. For example, $A_{[[1,3],[2,4]]}$ is dual to the operation $\{1,3\}[2,4]$.

Definition 4.1.9. For each prime bracketing $P \in p_{n-1}$, define a form $\alpha_P \in A^n_{n-1}$ by the following recipe. First write out the index monomial $A_P$. Then, for each $(i, j)$ in $A_P$, take a copy of the form $\alpha_{ij}$. Multiply together the factors in the order in which the corresponding brackets appear in $P$.

For example, $P = [[1,3],[2,4]]$ is a prime bracketing, and $\alpha_P = \alpha_{14} \alpha_{25}$.

Lemma 4.1.10. The set $\{\alpha_P \ | \ P \in p_{n-1}\}$ is a basis for the top degree cohomology $(A^n_{n-1})_{n-3}$ of Brown’s moduli spaces.
Proof. It is fairly evident that the map $P \mapsto \alpha_P$ is injective. That is is surjective follows from the results of [3], which computes the graded dimensions of $A_n^\delta$, and [17], which computes the cardinality of $p_{n-1}$. A more conceptual proof is the following. Getzler proved that the top degree part of the cooperad $A[-1]$, with residue as cocomposition, is a copy of the Lie cooperad. Realizing $A^\delta$ as the joint kernel of the residue maps says that it is isomorphic to the space of nonsymmetrically indecomposable Lie words, which by [17] is exactly the subspace of the Lie operad spanned by all prime bracketings. □

4.1.3. The explicit formula. The following result follows from the contents of the preceeding two subsections.

Theorem 4.1.11. The $n$-ary operation of the exotic structure has the following explicit formula:

$$\nu_n = \sum_{P \in p_{n-1}} \int_{X_n} \alpha_P \otimes g_P \in \zeta \otimes \BV_n.$$  

The sum is over the set $p_{n-1}$ of prime bracketings, defined in 4.1.9 and the corresponding forms $\alpha_P$ are defined in 4.1.9. The operation

$$g_P := \sum_{w \in \beta_{n-1}} \langle \alpha_{reg} \circ \gamma^* w, \alpha_P \rangle \phi_w \in \BV_n$$

is a sum over the basis of $H^{n-3}(\mathfrak{t} \mathfrak{b}_n)$ mentioned in 4.1.5, with $\phi_w$ denoting the conjugate dual basis vectors of $H_{3-n}(\mathfrak{t} \mathfrak{b}_n)$.

Remark 4.1.12. The operation $g_P$ contains as one of its terms the BV operation defined by replacing the outermost bracket in $P$ with a product. For example, $g_{[[1,3],[2,4]]}$ contains the operation $\{1,3\} \{2,4\}$.

4.2. Computation of the first nontrivial term. We shall here follow the algorithmic formula of 4.1.11 to compute the first few terms of the exotic structure. The term $\nu_3$ equals the binary commutative multiplication $m$. Thus the exotic structure is a deformation of the commutative one. There is no term $\nu_4$ because $A_4^\delta = 0$. In the case $n = 5$, $A_5^\delta$ is one-dimensional in the top degree, spanned by the form

$$\alpha_P = \alpha_{14}\alpha_{25}$$

corresponding to the unique prime bracketing $[[1,3],[2,4]] \in p_4$. To calculate the corresponding integral we fix coordinates $s$ and $t$ on the associahedron $X_5$ by using the symmetry of the moduli space to gauge the labeled points to

$$z_1, z_2, z_3, z_4, z_5 = 0 < s < t < 1 < \infty.$$  

The two relevant dihedral coordinates are then

$$u_{14} = 1 - s, \text{ and } u_{25} = \frac{s}{t}.$$  

Using this, the integral

$$\int_{X_5} \alpha_P = \int_{0<s<t<1} \frac{ds \, dt}{(1-s)t} = \frac{\pi^2}{6} = \zeta(2)$$

equals the second zeta value.

The next step is to calculate the Batalin-Vilkovisky operation $g_P$. We first note that there are exactly five convergent dihedral chord diagrams of the right degree:

$$\delta_{14}^* \delta_{25}^*, \delta_{13}^* \delta_{25}^*, \delta_{13}^* \delta_{24}^*, \delta_{24}^* \delta_{35}^* \text{ and } \delta_{14}^* \delta_{35}^*.$$  

Using the defining relations of $A_5$, cf. [2.0.2] we deduce that

$$\alpha_{reg}(\delta_{14}^* \delta_{25}^*) = \alpha_P, \quad \alpha_{reg}(\delta_{13}^* \delta_{25}^*) = -\alpha_P, \quad \alpha_{reg}(\delta_{13}^* \delta_{24}^*) = \alpha_P,$$

$$\alpha_{reg}(\delta_{24}^* \delta_{35}^*) = \alpha_P \text{ and } \alpha_{reg}(\delta_{14}^* \delta_{35}^*) = -\alpha_P.$$
Thus
\[
g_P = \sum_{w \in B_{p_6}} \langle \alpha_{reg} \circ \gamma^* w, \alpha_P \rangle \phi_w
= \sum_{w \in B_{p_6}} \langle \gamma^* w, \delta_{14} \delta_{25} - \delta_{13} \delta_{25} + \delta_{13} \delta_{24} + \delta_{24} \delta_{35} - \delta_{14} \delta_{35} \rangle \phi_w.
\]

By definition,
\[
\gamma(\delta_{14} \delta_{25}) = (b_{12} + b_{13} + b_{23} + s_1 + s_2 + s_3)(b_{23} + b_{24} + b_{34} + s_2 + s_3 + s_4),
\]
\[
\gamma(\delta_{13} \delta_{25}) = (b_{12} + s_1 + s_2)(b_{23} + b_{24} + b_{34} + s_2 + s_3 + s_4),
\]
\[
\gamma(\delta_{13} \delta_{24}) = (b_{12} + s_1 + s_2)(b_{23} + s_2 + s_3),
\]
\[
\gamma(\delta_{24} \delta_{35}) = (b_{23} + s_2 + s_3)(b_{34} + s_3 + s_4),
\]
\[
\gamma(\delta_{14} \delta_{35}) = (b_{12} + b_{13} + b_{23} + s_1 + s_2 + s_3)(b_{34} + s_3 + s_4).
\]

After a little calculation we find that
\[
\gamma(\delta_{14} \delta_{25}) - \gamma(\delta_{13} \delta_{25}) + \gamma(\delta_{13} \delta_{24}) + \gamma(\delta_{24} \delta_{35}) - \gamma(\delta_{14} \delta_{35})
= b_{13}b_{24} + b_{13}b_{23} + b_{12}b_{23} + b_{12}b_{24} + b_{23}b_{34} - b_{12}b_{34}
+ b_{13}s_2 - b_{24}s_3 - b_{34}s_3 + b_{12}s_2 + b_{23}s_4 - b_{34}s_1 - b_{12}s_4
+ s_1s_2 + s_3s_4 - s_1s_4.
\]

Of all these terms, only $b_{12}b_{23}$ and $b_{23}b_{34}$ have zero pairing with all representative cocycles in the preferred basis of $H^2(\text{rb}_5)$. The conjugate operation is
\[
g_P = \{1, 3\} \{2, 4\} + \{1, 3\} \{2, 3\} \{4\} - 1\{2, 3\} \{4\} - 1\{1, 2\} \{3, 4\}
+ \{1, 3\} \Delta(2)4 - 1\{2, 4\} \Delta(3) - 12\{\Delta(3), 4\} + 1\{1, \Delta(2)\} \{3, 4\}
+ 1\{2, 3\} \Delta(4) - 1\{1, 2\} \{3, 4\} - 1\{2, 3\} \{4\} - 1\{1, 2\} \Delta(4)
+ 1\{1, 2\} \Delta(4) - 1\{1, 2\} \Delta(3) \Delta(4) - 1\{1, 2\} \Delta(3) \Delta(4)
+ \Delta(1) \Delta(2) \{3, 4\} + 12\{\Delta(3), \Delta(4)\} - 1\{1, 2\} \Delta(3) \Delta(4).
\]

The notation we have used here is hopefully self-explanatory (it is, admittedly, not ideal in how it handles signs). The end result of our computations is that
\[
\nu_6 = \zeta(2)\{1, 3\} \{2, 4\} + \{1, 3\} \{2, 3\} \{4\} - 1\{2, 3\} \{4\} - 1\{1, 2\} \{3, 4\}
+ \{1, 3\} \Delta(2)4 - 1\{2, 4\} \Delta(3) - 12\{\Delta(3), 4\} + 1\{1, \Delta(2)\} \{3, 4\}
+ 1\{2, 3\} \Delta(4) - 1\{1, 2\} \{3, 4\} - 1\{2, 3\} \{4\} - 1\{1, 2\} \Delta(4)
+ \Delta(1) \Delta(2) \{3, 4\} + 12\{\Delta(3), \Delta(4)\} - 1\{1, 2\} \Delta(3) \Delta(4).
\]

4.3. The homotopy in arity 6. We shall here discuss the explicit form of
\[
\nu_6 = \sum_{p \in p_5} x_6 \alpha_P \otimes g_P.
\]

The set of prime bracketings under consideration is fairly small, $p_5$ consists of the four elements
\[
P_1 := [[[1, 3], 4], [2, 5]], \quad P_2 := [[[1, [3], [2, 4], 5]],
\]
\[
P_3 := [[1, [2, 4], [3, 5]], \quad \text{and} \quad P_4 := [[1, [4], [2, 3, 5]],
\]
so we only have to calculate four integrals to find the coefficients. However, the number of terms contributing to the respective operations $g_P$ is huge. We shall accordingly satisfy ourselves with writing out the obvious contributions.

The dihedral chord diagrams corresponding to the four prime bracketings are

![Dihedral chord diagrams for prime bracketings](image-url)
It is transparent from this that the associated differential forms are related via the cyclic action of \( \tau = (12 \ldots 6) \) and, also, via the dihedral flip \( \sigma \) that puts the 6 indices in the order \( 5 < 4 < 3 < 2 < 1 < 6 \). We have

\[
\alpha P_1 \cdot \tau = -\alpha P_2, \quad \alpha P_3 \cdot \tau = -\alpha P_4, \\
\alpha P_1 \cdot \sigma = \alpha P_3, \quad \alpha P_2 \cdot \sigma = \alpha P_4.
\]

The cyclic action \( \tau \) acts orientation reversing while the the flip \( \sigma \) acts orientation-preserving on the associahedron \( X_6 \). It follows that

\[
\int_{X_6} \alpha P_1 = \int_{X_6} \alpha P_2 = \int_{X_6} \alpha P_3 = \int_{X_6} \alpha P_4,
\]

so we only have to calculate one of the integrals, say that \( \alpha P_1 \). Fix coordinates

\[
0 < x < y < z < 1 < \infty = z_1, z_2, z_3, z_4, z_5, z_6
\]
on \( X_6 \). Then

\[
u_{15} = 1 - x, \quad u_{14} = \frac{z - x}{(1 - x)z}, \quad u_{26} = \frac{x}{y},
\]

and

\[
\alpha P_1 = d \log u_{15}d \log u_{14}d \log u_{26}\]

\[= \left( \frac{1}{(1 - x)yz} - \frac{1}{(1 - x)y(z - x)} \right) dx dy dz.
\]

The integral over \( X_6 \) equals

\[
\int_{X_6} \alpha P_1 = \zeta(3) - (-2\zeta(3)) = 3\zeta(3),
\]

cf. the computations at the end of [5]. It follows that

\[
\nu_6 = 3\zeta(3) \left( \{1, 3\}, \{2, 5\} + \{1, 3\}\{2, 4\}, \{1, 2, 4\}\{3, 5\} + \{1, 4\}\{2, 3, 5\} + \ldots \right).
\]

Not that the terms indicated by dots above include not just operations involving the BV operator, but also operations like \( \{1, 2, 3\}, 4\}5 \).

4.4. **Homotopical nontriviality of the exotic structure.** We can argue that \( \nu \) is homotopy nontrivial based on the explicit expression for \( \nu_5 \).

**Proposition 4.4.1.** The exotic structure is a homotopy nontrivial as a deformation of the strict multiplication \( m : A_S \to BV \).

**Proof.** The first higher homotopy of the exotic structure is the operation \( \nu_5 \), which contains the term \( \{1, 3\}\{2, 4\} \). This term can be identified with the degree 1 cocycle \( b_{13}b_{24} \) in the deformation complex

\[
\operatorname{Def}(A_{\infty} \xrightarrow{\partial_m} \operatorname{C}(\mathfrak{t}b)).
\]

The deformation complex has two differentials, a differential \( \partial_m = [m, ] \) defined by the binary product, and the internal Chevalley-Eilenberg differential \( \partial_C \). We have

\[
b_{13}b_{24} = \frac{1}{2}\partial_m(b_{13}b_{12} - b_{13}b_{23}) = 0,
\]

so by the relation \([b_{ij}, b_{ik} + b_{jk}] = 0,\]

\[
b_{13}b_{24} = (\partial_m + \partial_C)\frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) - \partial_C\frac{1}{2}(b_{13}b_{12} - b_{13}b_{23})
\]

\[= (\partial_m + \partial_C)\frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) - \frac{1}{2}(b_{13}b_{12} + b_{13}b_{23})
\]

\[= (\partial_m + \partial_C)\frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) + [b_{13}, b_{23}].
\]
In other words, $b_{13}b_{24}$ is cohomologous to $[b_{13}, b_{24}]$. This latter term can visibly not be $\partial_m$-exact. □

4.5. Relationship to Grothendieck-Teichmüller theory. Note that we could just as well have normalized our forms to $\alpha_{ij} = du_{ij}/(cu_{ij})$ for $c$ an arbitrary nonzero constant. Then the coefficient in front of $\nu_5$ would have been $\zeta(2)/c^2$, so the value $\zeta(2)$ is in this elementary sense a “gauge freedom” in our construction. Write $\zeta^+$ for the augmentation ideal of the algebra of multiple zeta values. It is conjectured that

$$\zeta^+/(\zeta^+ \cdot \zeta^+) \cong \text{grt}_1 \oplus \mathbb{Q}(2).$$

It follows from this that the Grothendieck-Teichmüller group $\text{GRT} = \text{GRT}_1 \times G_m$ should act freely as an automorphism group of the algebra of multiple zeta values. The factor $G_m$ acts by arbitrarily fixing a nonzero value for $\zeta(2)$. A recent theorem, with independent proofs by Furusho and Willwacher [21, 7], says that

$$H^1(\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{Ger})) \cong \text{grt}_1 \oplus \mathbb{Q}\{1, 3\}\{2, 4\}.$$  

Here Ger is the operad of Gerstenhaber algebras. Just by comparing factors one sees that the class of the operation $\{1, 3\}\{2, 3\}$ should correspond to $\zeta(2)$. The exotic structure is an explicit bridge between above result by Willwacher and Furusho, and the conjectured form of $\zeta^+/(\zeta^+ \cdot \zeta^+)$. Let us be slightly more detailed. The exotic structure is a Maurer-Cartan element

$$\nu' := \nu - m \in \zeta \otimes \text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV}).$$

In fact, $\nu'$ has coefficients in $\zeta^+$. The right hand side in the Maurer-Cartan equation

$$\partial_m \nu' = -\frac{1}{2}[\nu', \nu']$$

consequently has coefficients in the ideal $\zeta^+ \cdot \zeta^+$, so the reduction modulo this ideal is a degree 1 cocycle

$$\varpi' \in (\zeta^+/\zeta^+ \cdot \zeta^+) \otimes \text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV}).$$

It can equivalently be regarded as a morphism of complexes

$$\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV})^*[-1] \to \zeta^+/(\zeta^+ \cdot \zeta^+)$$

from the graded dual of the deformation complex. Taking cohomology gives a morphism

$$\varphi_\nu : H^1(\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV}))^* \to \zeta^+/(\zeta^+ \cdot \zeta^+).$$

One can argue that the (graded dual of the) Grothendieck-Teichmüller Lie algebra sits inside the domain of this morphism.

Kontsevich introduced in [13] a dg operad ($\text{Graphs}, \partial$) of graphs with white and black vertices, equipped with a quasi-isomorphism

$$\text{Ger} \to \text{Graphs}$$

from the operad of Gerstenhaber algebras. It follows from the results by Willwacher and Furusho that

$$H^1(\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{Graphs})) \cong \text{grt}_1 \oplus \mathbb{Q}\{1, 3\}\{2, 4\}\{2, 3, 5\}\{2, 3, 5\}.$$  

The correspondence $\varphi_\nu$, discussed above, relates the class $\{1, 3\}\{2, 4\}\{2, 3, 5\}\{2, 3, 5\}$ to the coefficient $\zeta(2)$ of

$$\nu_5 = \zeta(2)\{1, 3\}\{2, 4\} + \ldots.$$  

The so-called tetrahedal element of $\text{grt}_1$ has a representative cocycle

$$\Gamma_5 := \begin{array}{c|c|c|c}
\{1, 3\} & \{2, 4\} & \{1, 3\} & \{2, 4\}\{3, 5\}
\end{array}$$

in the deformation complex. This sum of four graphs corresponds to the displayed terms in our formula

$$\nu_6 = 3\zeta(3) \left(\{1, 3\}, 4\{2, 5\} + \{1, 3\}\{2, 4\}, 5 + \{1, 2, 4\}\{3, 5\} + \{1, 4\}\{2, 3, 5\} + \ldots\right).$$

It follows that the map $\varphi_\nu$ relates the tetrahedal element to $3\zeta(3)$. 

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5. Odd symplectic manifolds

One of the most important classes of Batalin-Vilkovisky algebras are the algebras of functions on odd symplectic manifolds. We show in this section that the exotic deformation can in this case be given an alternative but equivalent formula, more in the spirit of a perturbative quantum field theory and resembling a sum over Feynman diagrams.

Throughout this section we liberally apply the Einstein sum convention of summing over repeated tensor-indices.

Fix a finite number $d > 1$. Assume given $d$ graded variables $q^\mu$ ($1 \leq \mu \leq d$) and, additionally, also $d$ graded variables $p_\nu$ of degrees $|p_\nu| = 1 - |q^\mu|$. Define the free graded commutative algebra
\[ \mathcal{O} := \mathbb{Q}[q^\mu, p_\nu \mid 1 \leq \mu, \nu \leq d]. \]

We will be considering this as an algebra of functions on a space. Accordingly we will use geometric terminology and, e.g., refer to elements of the module of derivations
\[ X := \text{Der}(\mathcal{O}, \mathcal{O}) = \mathcal{O} \otimes \text{span}\{\partial_{p_\nu}, \partial_{q^\mu}\} \]
as vector fields. The module of vector fields is a graded Lie algebra. Elements of the dual $\mathcal{O}$-module,
\[ \Omega^1 = \mathcal{O} \otimes \text{span}\{dq^\mu, dp_\nu\}, \]
will be called differential one-forms, and the complex
\[ \Omega := S_\mathcal{O}(\Omega^1[-1]), \]
with de Rham-type differential $d$, will be referred to as the de Rham complex and its elements as differential forms. As in differential geometry, we may talk about the contraction $X_{,\beta}$ of a vector field $X$ and a differential form $\beta$, and define a Lie derivative by $L_X \beta := X_{,\beta} + d(X_{,\beta})$. Our space has a distinguished 2-form, $\omega := dp^\mu dq_\nu$, called the symplectic form. Its $d$th power $\omega^d$ will be called the volume form.

The odd symplectic structure equips $\mathcal{O}$ with a degree $-1$ Poisson bracket $\{\cdot, \cdot\}$ (so the suspension $\mathcal{O}[1]$ is a graded Lie algebra, and the adjoint action of bracket acts by graded derivations of the graded commutative product) and a Batalin-Vilkovisky operator
\[ \Delta = \frac{\partial^2}{\partial p_\nu q^\mu} : \mathcal{O} \rightarrow \mathcal{O}. \]
The bracket and the Batalin-Vilkovisky operator are related by the formula
\[ \{f, g\} = \Delta(fg) - \Delta(f)g - (-1)^{|f|}f\Delta(g). \]

Define
\[ D^n(\mathcal{O}, \mathcal{O}) := \text{Map}(\mathcal{O}^\otimes n, \mathcal{O}). \]

Then set
\[ CD^n(\mathcal{O}, \mathcal{O}) := (D^{n+1}(\mathcal{O}, \mathcal{O}) \otimes \mathcal{O} \Omega^2d)_{X}. \]
The coinvariants are taken with respect to the action
\[ D \otimes \mu \mapsto X \circ D \otimes \mu + (-1)^{|X||D|}D \otimes L_X \mu \]
of the Lie algebra of vector fields. Observe that $CD^n(\mathcal{O}, \mathcal{O})$ has an action of the permutation group $\Sigma_{n+1}$, in particular, it has an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$, induced by the natural action on $D^{n+1}(\mathcal{O}, \mathcal{O})$.

Given $D \in D^n(\mathcal{O}, \mathcal{O})$, let $id \cup D$ be the element of $D^{n+1}(\mathcal{O}, \mathcal{O})$ define by $D \cup id = m \circ (D \otimes id)$, for $m : \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}$ the multiplication.

**Lemma 5.0.1.** The map
\[ D^n(\mathcal{O}, \mathcal{O}) \xrightarrow{\cup id} D^{n+1}(\mathcal{O}, \mathcal{O}) \xrightarrow{\omega^d} D^{n+1}(\mathcal{O}, \mathcal{O}) \otimes \mathcal{O} \Omega^2d \rightarrow CD^n(\mathcal{O}, \mathcal{O}) \]
is a linear isomorphism for all $n$. 21
Define $\phi_f$ for all functions $O$ of Berezin algebras of functions on $V_{\text{Vilkovisky}}$ in the present case, mean that Berezin integration is a trace functional on the exotic structure on $O$. The cyclic compatibility of our construction will thus, transparent, but also much more technically involved to define rigorously, since it involves the subtleties of Berezin integration, questions of continuity, etc. The cyclic compatibility of our construction will thus, transparent, but also much more technically involved to define rigorously, since it involves the subtleties of Berezin integration, questions of continuity, etc. The cyclic compatibility of our construction will thus, transparent, but also much more technically involved to define rigorously, since it involves the subtleties of Berezin integration, questions of continuity, etc.

5.1. A cyclic representation. Given $1 \leq i < j \leq n-1$, define $D_{ij} := \frac{\partial}{\partial p^i} \frac{\partial}{\partial q^j} + \frac{\partial}{\partial q^i} \frac{\partial}{\partial p^j} : O^{\otimes n-1} \to O^{\otimes n-1}$. Here $\partial^i / \partial x^i := i d^{\otimes i-1} \otimes (\partial / \partial x^i) \otimes i d^{\otimes n-1-i}$ acts as derivation on the $i$th factor and leaves the remaining factors untouched. Define also for $1 \leq k \leq n-1$, $D_k := \frac{\partial^2}{\partial q^k \partial q_k} : O^{\otimes n-1} \to O^{\otimes n-1}$. (No sum over $k$.)
Take a monomial $\Gamma \in \text{Gra}_n^\otimes$. For every edge $e$ in the graph, write $D_e := D_{ij}$ if $e = e_{ij}$, or $D_e := D_k$ if the edge is a tadpole $e_k$. Finally define

$$D_\Gamma := m \circ \bigcirc_{\alpha \in \Gamma} D_e : \mathcal{O} \circ^\otimes n-1 \to \mathcal{O}.$$ 

Here $m : \mathcal{O} \circ^\otimes n-1 \to \mathcal{O}$ is the commutative product. This is unambiguous since the monomial $\Gamma$ defines an ordering of operators $D_e$ up to an even permutation.

**Lemma 5.1.1.** The association $\Gamma \mapsto D_\Gamma$ is a cyclic representation of $\text{Gra}_n^\otimes$ in the endomorphism operad of $\mathcal{O}$. Moreover, the representation is given by operations that are invariant under affine symplectomorphisms.

**Proof.** Invariance under translations is clear since the operators $D_\Gamma$ all have constant coefficients. Invariance under linear symplectomorphisms follows from noting that whenever $p_\mu$ appears in a formula it does so in a pair with its conjugate $q^\mu$, with a sum over $\mu$.

That it respects operadic composition repeats the proofs of the various related statements in [13, 22].

Remark 5.1.2. Instead of using a graphical language, we could have equivalently phrased the preceding lemma as a cyclic representation $C(\text{tb}/[\text{tb}, \text{tb}]) \to \text{End}(\mathcal{O})$.

Remark 5.1.3. The cyclic Batalin-Vilkovisky algebra-structure on the function ring $\mathcal{O}$ is given by the composite

$$\text{BV} = H(\text{tb}) \to C(\text{tb}/[\text{tb}, \text{tb}]) \cong \text{Gra}_n^\otimes \xrightarrow{D} \text{End}(\mathcal{O}).$$

5.2. **The exotic structure in Darboux coordinates.**

**Theorem 5.2.1.** The operations of the exotic $A_\infty$ structure on $\zeta \otimes \mathcal{O}$ are given by the explicit formula

$$\nu_n := \sum_{e} \int_{X_n} \alpha^e_{\text{reg}} D_{\gamma(e)}, \quad n \geq 3,$$

where the sum is taken over all monomials $e \in C_n$ of degree $n - 3$.

**Proof.** Write $\text{coGra}_n^\otimes$ for the cooperad linearly dual to $\text{Gra}_n^\otimes$, and $\text{coBV} := H(\text{tb})$ for the Batalin-Vilkovisky cooperad. By [5.1.3] and [5.1.6] we have a commutative diagram

$$
\begin{array}{ccc}
\text{coGra}_n^\otimes & \xrightarrow{\gamma^\ast} & \zeta \\
\downarrow & \circlearrowright & \downarrow \\
\text{coBV} & \xrightarrow{\alpha_{\text{reg}}} & A
\end{array}
$$

of morphisms of planar dg cooperads. The exotic structure on $\zeta \otimes \mathcal{O}$, given canonically via the composite $\zeta \otimes A_{\infty} \to \zeta \otimes \text{BV} \to \zeta \otimes \text{coGra}_n^\otimes \to \zeta \otimes \text{End}(\mathcal{O})$ corresponds to the composite $\text{coGra}_n^\otimes \to A^\otimes$ via the bottom path in the commutative diagram. The explicit formula in the theorem is the composite $\text{coGra}_n^\otimes \to A^\otimes$ obtained by following the upper path. \qed
The sheaf of functions on an arbitrary odd symplectic 2\(d\)-dimensional manifold \(M\) is locally of the form \(C^\infty(T^*[−1]U) = Γ(U, ∧TU)\), where \(U\) is an open coordinate chart in \(\mathbb{R}^d\) with coordinates \(q^\mu\) and \(p_\nu\) are conjugate sections of the tangent bundle \(TU\), such that the symplectic form has the expression \(ω|_U = dp_\nu dq^\mu\). Such coordinates \((q,p)\) are called local Darboux coordinates on the odd symplectic manifold. All of the results of this section generalize from the polynomial algebra \(Ω\) to any Darboux coordinate chart \(C^\infty(T^*[−1]U)\). Thus the preceding theorem can be read as a recipe for how to write the exotic structure on an arbitrary (finite-dimensional) odd symplectic manifold in a Darboux coordinate chart.

5.3. Alternative computation of the first nontrivial term. Let us compute the term \(ν_5\) on \(Ω\) by using the formula of theorem 5.2.1 of the preceding subsection. Recall from [12] that there are five convergent dihedral chord diagrams of the right degree:

\[
c_i := δ_{14}δ_{25}, \quad c_{ii} := δ_{13}δ_{25}, \quad c_{iii} := δ_{13}δ_{24},
\]

\[
c_iv := δ_{24}δ_{35} \quad \text{and} \quad c_v := δ_{14}δ_{35}.
\]

The five two-forms corresponding to the five convergent chord diagrams are

\[
α^i = \frac{ds}{(1 − s)t}, \quad α^{ii} = \frac{ds}{(1 − s)t}, \quad α^{iii} = \frac{ds}{(1 − s)t},
\]

\[
α^{iv} := \frac{ds}{(1 − s)t} \quad \text{and} \quad α^v := −\frac{ds}{(1 − s)t},
\]

and all of them integrate to either plus or minus

\[
\int_{0<s<t<1} \frac{ds}{(1 − s)t} = ζ(2) = \frac{π^2}{6}.
\]

In the next step we find the five graphs in \(\text{Gra}^\ddagger\) associated to the chord diagrams:

\[
γ_1 = (e_{12} + e_{14} + e_{23} + e_1 + e_2 + e_3)(e_{23} + e_{24} + e_{34} + e_2 + e_3 + e_4),
\]

\[
γ_{ii} = (e_{12} + e_1 + e_2)(e_{23} + e_{24} + e_{34} + e_2 + e_3 + e_4),
\]

\[
γ_{iii} = (e_{12} + e_1 + e_2)(e_{23} + e_{24} + e_3),
\]

\[
γ_{iv} = (e_{23} + e_2 + e_3)(e_{34} + e_3 + e_4),
\]

\[
γ_v = (e_{12} + e_{14} + e_{23} + e_1 + e_2 + e_3)(e_{34} + e_3 + e_4).
\]

Define \(γ_5 := γ_1 − γ_{ii} + γ_{iii} + γ_{iv} − γ_v\). After some algebra we find that

\[
γ_5 = e_{13}e_{24} + e_{13}e_{23} + e_{12}e_{23} + e_{23}e_{24} + e_{23}e_{34} − e_{12}e_{34} + e_{13}e_2 − e_{24}e_3 − e_{34}e_3 + e_{12}e_2 + e_{23}e_4 − e_{23}e_1 + e_{34}s_1 − e_{12}e_4 + e_1e_2 + e_3e_4 − e_1e_4.
\]

The corresponding operator \(D_5\) in \(D^4(Ω, Ω)\) is

\[
D_5 = \{1, 3\}[2, 4] + \{1, [2, 3]\}4 − 1\{[2, 3], 4\} − \{1, 2\}[3, 4]
\]

\[
+ \{1, 3\}Δ(2)4 − 1\{2, 4\}Δ(3) − 12\{Δ(3), 4\} + \{1, Δ(2)\}34
\]

\[
+ 1\{2, 3\}Δ(4) − Δ(1)\{2, 3\}4 + Δ(1)2\{3, 4\} − \{1, 2\}3Δ(4)
\]

\[
+ Δ(1)Δ(2)34 + 12Δ(3)Δ(4) − Δ(1)23Δ(4).
\]

To arrive at the above formula we have for example used that

\[
D(e_{13}e_{23} + e_{12}e_{23}) = D(e_{12}⊙ e_{12}) = \{1, 2\}⊙ \{1, 2\}.
\]

We can sum up our discussion by

\[
ν_5 = ζ(2)\{1, 3\}[2, 4] + \{1, [2, 3]\}4 − 1\{[2, 3], 4\} − \{1, 2\}[3, 4]
\]

\[
+ \{1, 3\}Δ(2)4 − 1\{2, 4\}Δ(3) − 12\{Δ(3), 4\} + \{1, Δ(2)\}34
\]

\[
+ 1\{2, 3\}Δ(4) − Δ(1)\{2, 3\}4 + Δ(1)2\{3, 4\} − \{1, 2\}3Δ(4)
\]

\[
+ Δ(1)Δ(2)34 + 12Δ(3)Δ(4) − Δ(1)23Δ(4))
\]

This agrees with the earlier computation in [12].
6. Compatibility with the Poisson bracket

This section is devoted to proving that the adjoint action of the odd Poisson bracket on a Batalin-Vilkovisky algebra acts by (strict) derivations of the exotic $\mathcal{A}_\infty$ structure.

6.1. The moduli space of Riemann spheres with marked points and phase parameters. Let $\text{Conf}_n(Z)$ denote the manifold of all injections of the set $\{n\}$ into a manifold $Z$, that is,

$$\text{Conf}_n(Z) = Z^n \setminus \text{diagonals}.$$

Define $\pi : ST(\mathbb{C}P^1) \to \mathbb{C}P^1$ to be the circle bundle of tangent directions on the Riemann sphere and let $V_n$ be the manifold

$$V_n := \{ \xi \in \text{Conf}_n(ST(\mathbb{C}P^1)) \mid \pi \circ \xi \in \text{Conf}_n(\mathbb{C}P^1) \}.$$

**Definition 6.1.1.** The open moduli space of Riemann spheres with marked points and phase parameters is the quotient manifold

$$fM_{0,n} := V_n / \text{PGL}_n(\mathbb{C}).$$

The manifold $fM_{0,n}$ has a natural compactification into a manifold with corners, $\overline{fM}_{0,n}$, which can be described as follows, cf. [11]. Let $M_{0,n}$ be the real oriented blow-up of the boundary locus of the Deligne-Mumford compactification $\overline{M}_{0,n}$. Let $L_k \to M_{0,n}$ be the line bundle whose fiber above a point $[\Sigma]$ is the tangent space $T_{\Sigma} \Sigma$ at the $k$th marked point. It extends to a line bundle on $\overline{M}_{0,n}$ and then the product

$$fM_{0,n} := SL_1 \times M_{0,n} \cdots \times M_{0,n} SL_n$$

of the associated circle bundles of directions can be taken as a compactification of $fM_{0,n}$.

**Lemma 6.1.2.** [11] The collection $fM_{0,n} := \{ fM_{0,n} \mid n \geq 3 \}$ is a cyclic (pseudo-)operad of manifolds with corners. The composition is by gluing surfaces at marked points, tensoring the tangential directions (to produce a nodal surface with a tangential direction at the node).

Let $\pi : fM_{0,n} \to M_{0,n}$ be the projection that forgets the tangent directions.

**Lemma 6.1.3.** The cohomology algebra of $fM_{0,n}$ is the algebra $f\mathcal{A}_n$ generated by the forms $\beta_{ij} := \pi^* \alpha_{ij}$ (for $\{i,j\} \in \chi_n$) and the normalized volume forms $\theta_k$ ($1 \leq k \leq n$) on the circle factors of the fibers of $\pi : fM_{0,n} \to M_{0,n}$, modulo the relations that

$$\left( \sum_{\{i,j\} \in A} \beta_{ij} \right) \left( \sum_{\{k,l\} \in B} \beta_{kl} \right) = 0$$

for all pairs of completely crossing subsets $A, B \subset \chi_n$. In short, we have an algebra isomorphism $f\mathcal{A}_n \cong \mathcal{A}_n \otimes \mathbb{Q}[\theta_k \mid 1 \leq k \leq n]$.

**Proof.** The compactified moduli space $fM_{0,n}$ is homotopy equivalent to its interior, $M_{0,n}$. The latter space is in turn homotopy equivalent to the total space of the $(\mathbb{C}^\times)^n$-bundle

$$U_{0,n} := L_1^\infty \times M_{0,n} \cdots \times M_{0,n} L_n^\infty,$$

parametrizing Riemann spheres with nonzero tangent vectors at marked points. Picking a trivialization of the bundle allows one to embed $U_{0,n}$ as the complement of a divisor in $\overline{M}_{0,n} \times \mathbb{C}^n$, and one can deduce that $H(U_{0,n})$ is isomorphic to the algebra of logarithmic forms generated by the $\alpha_{ij}$’s and forms $d \log \tau_k$ ($1 \leq k \leq n$), where $\tau_k$ is the coordinate on the $k$th copy of $\mathbb{C}$.

The forms $\theta_k$ are defined without reference to a trivialization (as they may be considered as fibrewise Haar measures on the principal bundles $SL_k \to M_{0,n}$) and represent the same cohomology classes as the forms $d \log \tau_k$. \qed

**Definition 6.1.4.** Define $f\mathcal{A}_n^\delta$ to be the image of $\mathcal{A}_n^\delta \otimes \mathbb{Q}[\theta_k \mid 1 \leq k \leq n]$ under the isomorphism $\mathcal{A}_n \otimes \mathbb{Q}[\theta_k \mid 1 \leq k \leq n] \cong f\mathcal{A}_n$. 

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6.2. The Knizhnik-Zamolodchikov with phases. Consider the 1-form
\[
\beta := \sum_{\{i,j\} \in \mathcal{X}_n} \beta_{ij} \gamma(\delta_{ij}) + \sum_{k=1}^n \theta_k s_k
\]
with values in the Lie algebra of spherical ribbon braids. Here \( \gamma : \mathfrak{h}_n \rightarrow \mathfrak{rb}_n \) is the morphism of the preceding sections. Since \( \pi^* : A_n \rightarrow fA_n \) is a morphism of algebras, \( \beta_{ij} = \pi^* \alpha_{ij}, \gamma \) is a morphism of Lie algebras, and the \( s_k \)'s are central, we deduce that \( \beta \) is a flat (singular) connection on \( fM_{\mathfrak{h}_n} \), i.e., a morphism of dg algebras
\[
\beta : C(\mathfrak{rb}_n) \rightarrow fA_n.
\]
Note that the regularization map \( \text{reg} : A_n \rightarrow A^\mathfrak{d}_n \) induces a map \( \text{reg} : fA_n \rightarrow fA^\mathfrak{d}_n \) which we abusively denote by the same symbol, and define
\[
\beta_{\text{reg}} := \text{reg} \circ \beta : C(\mathfrak{rb}_n) \rightarrow fA^\mathfrak{d}_n.
\]
We call \( \beta_{\text{reg}} \) the regularized Knizhnik-Zamolodchikov connection with phases.

6.3. Chains parametrizing homotopy derivations. Recall that we defined \( fM_{0,n} = V_n/PGL_2(\mathbb{C}) \), with
\[
V_n = \{ \xi \in \text{Conf}_n(ST(CP^1)) \mid \pi \circ \xi \in \text{Conf}_n(CP^1) \}.
\]
Let us first make a remark on an elementary geometric construction. Assume given a point \( (z, \sigma) \in ST(CP^1) \). Flow in the direction \( \sigma \) determines an oriented great circle \( C \) through \( z \) and its antipodal point \( \ell(z) \) on the Riemann sphere. Identify \( z \) as the north pole and \( \ell(z) \) as the south pole, and let \( p \) denote the stereographic projection of \( CP^1 \setminus \{z\} \) onto the plane \( \mathbb{C} = T_{\ell(z)}(CP^1) \) tangent at the antipode. The image \( p(C) \) of the great circle is an oriented Euclidean line \( L \) in the plane. Let \( \tilde{\eta} \in \Gamma(\mathbb{C}, ST(C)) \) be the constant directional vector field defined by the direction of the line \( L \). It lifts to a directional vector field \( \eta \in \Gamma(CP^1 \setminus \{z\}, ST(CP^1)) \) on the Riemann sphere with a zero of index 2 at \( z \).

**Definition 6.3.1.** Assume \( p \geq 0, q \geq 2, p + q \geq 3 \). Define \( Y_{p,q} \subset V_{p+q} \) to be the subspace consisting of those \( \xi = (z_i, \sigma_i)_{i=1}^{p+q} \) satisfying the following conditions:

- The direction \( \sigma_i \) at \( z_i \) is for all \( 1 \leq i \leq p+q-1 \) equal to the value of the directional vector field \( \eta \in \Gamma(CP^1 \setminus \{z_{p+q}\}, ST(CP^1)) \) defined by the direction \( \sigma_{p+q} \) at \( z_{p+q} \), i.e., \( \sigma_i = \eta|_{z_i} \).

- If \( q \geq 3 \), then we require that the images of the points \( z_{p+1}, \ldots, z_{p+q-1} \) under the stereographic projection \( CP^1 \setminus \{z\} \rightarrow \mathbb{C} \) are collinear on a line parallel to the line \( L \), and ordered compatibly with the orientation of \( L \).

Define \( X^\circ_{p,q} \) to be the image of \( Y_{p,q} \) in \( fM_{0,p+q} \), and define \( X_{p,q} \) to be the closure of \( X^\circ_{p,q} \) in \( fM_{0,p+q} \).

It is clear that \( X_{p,q} \) is a compact analytic (semialgebraic, in fact) real manifold with corners. Moreover, \( X_{0,q} \), for \( q \geq 3 \), is an associahedron of dimension \( q - 3 \). To see this, take \( \xi = (z_i) \in Y_{0,q} \). We can use the gauge freedom to assume \( z_q = \infty \). We are then left with \( q - 1 \) collinear points \( (z_i)_{i=1}^{q-1} \) in the plane, modulo the gauge action \( \mathbb{C} \times \mathbb{C}^\times \). (The direction \( \sigma_q \) determines the direction of the line of collinearity.) Use the translation freedom to put \( z_1 \) at the origin and the freedom of complex dilation \( \mathbb{C}^\times \) to put \( z_{q-1} \) at 1. The compatibility between numbering of the points and the orientation of the line ensures we are left with a configuration \( 0 < z_{p+2} < \cdots < z_{p+q-2} < 1 \). The compactification \( X_{0,q} \) of the open simplex of such configurations is an associahedron: we have an isomorphism \( X_{0,q} \cong X_q \) with Brown’s associahedron.

The space \( X_{1,2} \) is isomorphic to a circle, but not canonically so. We can for example either fix the three points to \( z_1, z_2, z_3 = 1, 0, \infty \) and parametrize \( X_{1,2} \) by the direction \( \sigma_3 \), or we can fix the direction \( \sigma_2 \) and parametrize \( X_{1,2} \) by the angle \( t \) such that \( z_1, z_2, z_3 = e^{it}, 0, \infty \).

In general, we can identify \( X_{p,q} \) with the following manifold. Define
\[
\text{Conf}_{p,q-1}(\mathbb{C}) := \{ (z_i) \in \text{Conf}_{p+q-1}(\mathbb{C}) \mid \text{Im}(z_{p+1}) = \cdots = \text{Im}(z_{p+q-1}), \text{Re}(z_{p+1}) < \cdots < \text{Re}(z_{p+q-1}) \}
\]
to be the subspace consisting of configurations where the last \( q - 1 \) points are collinear on a line parallel to the real axis and ordered compatibly with the canonical orientation of the line. Putting \( z_{p+q} = \infty \) and using the rotation freedom to fix the direction of the line of collinearity defines an isomorphism
\[
X^\circ_{p,q} \cong \text{Conf}_{p,q-1}(\mathbb{C})/\mathbb{C} \times \mathbb{R}_{>0}.
\]
The configuration space \( \text{Conf}_{p+q-1}(\mathbb{C}) \) has a well-known Axelrod-Singer (also called the real Fulton-McPherson) compactification \( \bar{\text{Conf}}_{p,q-1}(\mathbb{C}) \). Let \( \text{Conf}_{p,q-1}(\mathbb{C}) \) be the closure inside \( \bar{\text{Conf}}_{p+q-1}(\mathbb{C}) \). Then above isomorphism extends to
\[
X_{p,q} \cong \text{Conf}_{p,q-1}(\mathbb{C})/\mathbb{C} \times \mathbb{R}_{>0}.
\]
We can use the translation and dilation freedom to identify \( X_{p,q} \) with tuples of points \( x_j + iy_j, 1 \leq j \leq p \) and points \( 0 < t_1 < \cdots < t_{q-3} < 1 \). The form
\[
dx_1dy_1 \cdots dx_pdy_p dt_1 \cdots dt_{q-3}
\]
defines an orientation on \( X_{p,q} \). This formula is vacuous if \( q = 2 \), but the only case with \( q = 2 \) that will play a role in what follows is \( X_{1,2} \), which, as discussed above, is isomorphic to a circle. We orient it by \( \theta_3 \).

6.4. The compatibility equations. We first note that the forms in the image of \( \beta_{\text{reg}} : C(\mathfrak{rb}) \to f\mathbb{A}^4 \) are integrable over the chains \( [X_{p,q}] \). Since the points in \( X_{0,q} \) correspond to collinear configurations the angle forms \( \theta_k \) do not contribute to the integrals and
\[

will reproduce the exotic structure. Moreover, an integral over \( X_{p,q} \) with \( p \geq 1 \) and \( p + q \geq 4 \) will vanish because the forms \( \beta_{ij} \) depend holomorphically on the \( z_1, \ldots, z_p \). In more detail, we can represent a point in \( X_{p,q}^0 \) by \( q - 2 \) points between 0 and 1 on the real axis, and \( p \) points that are free to be anywhere in the plane. A top-dimensional form on \( X_{p,q} \) must accordingly contain terms \( dz_1d\bar{z}_i \), but the Knizhnik-Zamolodchikov connection contains no such anti-holomorphic dependence \( d\bar{z}_i \). It follows that the only nonzero integration with \( p \geq 1 \) is
\[
\nu_{1,2} = \int_{X_{1,2}} \beta_{\text{reg}}.
\]
Since \( A^4_3 = \mathbb{Q} \), the integrand is
\[
\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_3 = -\frac{\theta_1}{2}(b_{12} + b_{13}) - \frac{\theta_2}{2}(b_{12} + b_{23}) - \frac{\theta_3}{2}(b_{13} + b_{23})
\]
\[
= -\frac{\theta_1}{2} b_{12} - \frac{\theta_2}{2} b_{23} - \frac{\theta_3}{2} b_{13}.
\]
The defining condition \( \sigma_i = \eta|_{\bar{z}_i} \), for \( i = 1,2 \) and \( \eta \) the directional vector field determined by \( \sigma_3 \), implies that if \( \sigma_3 \) rotates with positive orientation, then \( \sigma_1 \) and \( \sigma_2 \) will rotate the same amount in the negative orientation. Since we took \( \theta_3 \) to define the orientation, we accordingly have
\[
\nu_{1,2} = \int_{St}( -\frac{\theta_3 - \theta_1}{2} b_{12} - \frac{\theta_3 + \theta_1}{2} b_{23} - \frac{\theta_3 + \theta_1}{2} b_{13} ) = b_{12}.
\]
The Batalin-Vilkovisky operation given by \( b_{12} \) is the odd Poisson bracket. In the present context however we prefer to regard it as a the adjoint action of “\( z_1 \)” on “\( z_2 \)”, with output “\( z_3 \)”.

Proposition 6.4.1. The adjoint action of the odd Poisson bracket of a Batalin-Vilkovisky algebra acts by strict derivations of the exotic \( A_\infty \) structure.

Proof. We will argue that the claim is a consequence of the Stokes’ relation
\[
0 = \int_{X_{1,n}} d\beta_{\text{reg}} = \int_{\partial X_{1,n}} \beta_{\text{reg}}.
\]
Let us put \( X_q := X_{0,q} \) and reserve the notation \( X_{p,q} \) for cases when \( p \geq 1 \). Every codimension one strata of the manifold \( X_{1,n} \) has the following form. Assume \( S_1 \sqcup S_2 = \{2, \ldots, n+1\} \) is a partition, where both subsets are cyclically consecutive, \( \#S_1 \geq 1 \) and \( \#S_2 \geq 2 \). Then there is a strata
\[
X_{1,S_1 \sqcup \{e_1\}} \times X_{S_2 \sqcup \{e_2\}},
\]
corresponding to degenerations where a nodal point develops and the points labeled by \( \{1\} \sqcup S_1 \) fall on one side of the node and the points labeled by \( S_2 \) on the other, and every codimension one strata arises
in this way. We can classify the strata further, saying that a strata is of type I if \( n + 1 \in S_1 \), and of type II if \( n + 1 \in S_2 \). Using this, we can write

\[
\partial X_{1,n} = \bigcup_{\text{type I}} X_{1,n-q+1} \times X_{q+1} \cup \bigcup_{\text{type II}} X_{n-m+1} \times X_{1,m+1}.
\]

Since \( \beta_{\text{reg}} \) can produce a top-dimensional form only on spaces \( X_r \) and \( X_{1,2} \), and since the strata correspond to operad compositions, we find

\[
\int_{\partial X_{1,n}} \beta_{\text{reg}} = \pm \nu_{1,2} \circ_2 \nu_n + \sum_{i=2}^{n+1} \pm \nu_{(2, \ldots, n+1)} \circ_i \nu_{1,2},
\]

or, more suggestively,

\[
\pm \text{ad}_1(\nu_n(2, \ldots, n)) + \sum_i \pm \nu_n(2, \ldots, \text{ad}_1(i), \ldots, n).
\]

Up to verification of the signs this proves that the adjoint action \( \text{ad}_1 \) of the input labeled 1 is a derivation of the exotic product \( \nu_n(2, \ldots, n) \) of the remaining inputs. \( \square \)

A. CYCLIC COMPATIBILITY OF \( \gamma \)

This appendix completes the proof of \( \ref{thm:main} \) by giving a direct proof of the claim that

\[
\gamma : G_n \to \text{Gr}_n^{(\gamma)}, \delta_{ij} \mapsto \sum_{i \leq r < s < j} e_{rs} + \sum_{1 \leq k < j} e_k
\]

respects the cyclic group actions. The image of \( \gamma(\delta_{1n}) \) under the cyclic action \( \tau : i \mapsto i + 1 \) is the sum

\[
S := e_n + \sum_{r+1 \leq i < j \leq n-1} e_{ij} + \sum_{r+1 \leq i \leq j \leq n-1} e_i + \sum_{r+1 \leq i \leq n-1} e_{in},
\]

under the provisio that all terms involving an index \( n \) are rewritten according to the relations

\[
e_an = -2e_a - \sum_{1 \leq i \leq n-1} e_{ia},
\]

\[
e_n = -\frac{1}{2} \sum_{1 \leq j \leq n-1} e_{jn}.
\]

We have to show that \( \tau(\gamma(\delta_{1n})) = \gamma(\delta_{1r+1}) \).

**It is a computation.** Expand the terms containing \( n \) and cancel out terms with \( e_i \),

\[
S = -\frac{1}{2} \sum_{1 \leq i \leq n-1} \left[ -2e_i - \sum_{1 \leq j \leq n-1} e_{ji} \right] + \sum_{r+1 \leq i < j \leq n-1} e_{ij}
\]

\[
+ \sum_{r+1 \leq i \leq n-1} e_i + \sum_{r+1 \leq i < j \leq n-1} \left[ -2e_i - \sum_{1 \leq j \leq n-1} e_{ji} \right]
\]

\[
= \sum_{1 \leq i \leq n-1} e_i + \frac{1}{2} \sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} - \sum_{r+1 \leq i \leq n-1} e_i - \sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji}
\]

\[
= \sum_{1 \leq i \leq r} e_i + \frac{1}{2} \sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} - \sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji}.
\]

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Now expand the terms $A$ and $B$. In sums where $j < i$, we switch summation index so that $i < j$.

$$A = \sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ij}$$

$$= \sum_{1 \leq i \leq r} \left( \sum_{1 \leq j < i} e_{ij} + \sum_{i < j \leq n-1} e_{ij} \right) + \sum_{r+1 \leq i \leq n-1} \left( \sum_{1 \leq j < i} e_{ij} + \sum_{i < j \leq n-1} e_{ij} \right)$$

$$= \sum_{1 \leq i < j \leq r} e_{ij} + \sum_{1 \leq i \leq r} \left( \sum_{1 \leq j < i} e_{ij} + \sum_{r+1 \leq j \leq n-1} e_{ij} \right) + \sum_{r+1 \leq j \leq n-1 \leq i} \sum_{1 \leq j < i} e_{ij}$$

$$= 2 \sum_{1 \leq i < j \leq r} e_{ij} + \sum_{1 \leq i \leq r} \left( \sum_{1 \leq j < i} e_{ij} + 2 \sum_{r+1 \leq j \leq n-1 \leq i} e_{ij} \right)$$

$$B = \sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j \leq i} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j \leq i} e_{ij}$$

$$= \sum_{1 \leq i \leq r} \sum_{1 \leq j < i} e_{ij} + \sum_{r+1 \leq j \leq n-1} \sum_{1 \leq i < j} e_{ij}$$

$$= \sum_{1 \leq i \leq r} \sum_{1 \leq j < i} e_{ij} + \sum_{r+1 \leq j \leq n-1} \sum_{1 \leq i < j} e_{ij}$$

$$= 2 \sum_{1 \leq i < j \leq r} e_{ij} + \sum_{1 \leq i \leq r} \left( \sum_{1 \leq j < i} e_{ij} + \sum_{r+1 \leq j \leq n-1 \leq i} e_{ij} \right)$$

When the expressions for $A$ and $B$ are inserted into the sum $S$, most terms cancel out.

$$S = \sum_{1 \leq i \leq r} e_{i} + \left[ \sum_{1 \leq i \leq j \leq r} e_{ij} + \sum_{1 \leq i \leq j \leq r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq j \leq n-1} \sum_{1 \leq i < j} e_{ij} \right]$$

$$+ \sum_{r+1 \leq i \leq n-1} e_{ij} - \left[ \sum_{1 \leq j < i \leq r} e_{ij} + \sum_{r+1 \leq i \leq n-1 \leq j} e_{ij} + \sum_{r+1 \leq i \leq n-1 \leq j} e_{ij} \right]$$

$$= \sum_{1 \leq i \leq r} e_{i} + \sum_{1 \leq i < j \leq r} e_{ij}.$$
[10] E. Getzler and M. M. Kapranov. Cyclic operads and cyclic homology, pages 167–201. International Press, Cambridge, 1995. url: math.northwestern.edu/ getzler/Papers/cyclic.pdf.
[11] J. Giansiracusa and P. Salvatore. Cyclic formality of the framed 2-discs operad and genus zero stable curves. Transactions of the American Mathematical Society, 364(11):5881–5911, 2012. url: arxiv:0911.4430.
[12] V. Hinich. Homological algebra of homotopy algebras. Communications in Algebra, 25(10):3291–3323, 1997.
[13] M. Kontsevich. Operads and Motives in Deformation Quantization. Letters in Mathematical Physics, 48(1):35–72, 1999. url: http://arxiv.org/abs/math/9904055.
[14] J.-L. Loday and B. Vallette. Algebraic Operads. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 2012. url: math.unice.fr/ brunov/Operads.pdf.
[15] L. Menichi. Batalin-Vilkovisky algebras and cyclic cohomology of Hopf algebras. K-Theory, 32(2):231–251, 2004. url: arxiv.org/abs/math/0311276.
[16] S.A. Merkulov and B. Vallette. Deformation theory of representations of prop(erad)s. Journal für die reine und angewandte Mathematik (Crelles Journal), 636:125–174, 2009.
[17] P. Salvatore and R. Tauraso. The operad Lie is free. Journal of Pure and Applied Algebra, 213:224–230, 2009. url: http://arxiv.org/pdf/0802.3010.pdf.
[18] D. Tamarkin. Action of the Grothendieck-Teichmüller Lie algebra on the operad of Gerstenhaber algebras, 2002. url: arXiv:math/0202039.
[19] V. Tourtchine and T. Willwacher. Relative (non-)formality of the little cubes operads and the algebraic Schoenflies theorem, 2014. url: arxiv.org/abs/1409.0163.
[20] P. Severa. Equivalence of formalities of the little discs operad. Letters in Mathematical Physics, 93:29–35, 2010. url: arxiv.org/abs/0902.3576.
[21] T. Willwacher. M. Kontsevich’s graph complex and the Grothendieck-Teichmuller Lie algebra, 2010. url: arxiv.org/abs/1009.1654.
[22] T. Willwacher and D. Calaque. Formality of Cyclic Cochains, 2008. url: arxiv.org/pdf/0806.4095v2.pdf.

Stockholm University, Department of Mathematics
E-mail address: alm@math.su.se