Instantons, Monopoles and Toric HyperKähler Manifolds

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Abstract:
In this paper, the metric on the moduli space of the \( k = 1 \) \( SU(n) \) periodic instanton -or caloron- with arbitrary gauge holonomy at spatial infinity is explicitly constructed. The metric is toric hyperKähler and of the form conjectured by Lee and Yi. The torus coordinates describe the residual \( U(1)^{n-1} \) gauge invariance and the temporal position of the caloron and can also be viewed as the phases of \( n \) monopoles that constitute the caloron. The \((1,1,\ldots,1)\) monopole is obtained as a limit of the caloron. The calculation is performed on the space of Nahm data, which is justified by proving the isometric property of the Nahm construction for the cases considered. An alternative construction using the hyperKähler quotient is also presented. The effect of massless monopoles is briefly discussed.

1 Introduction

Moduli spaces of instantons \[2\] and Bogomol’nyi-Prasad-Sommerfield (BPS) monopoles \[3\] have been subject to long-time investigation. The moduli space, quotient of the set of self-dual gauge connections by the group of gauge transformations, is a subset of the configuration space and its geometry therefore reflects physical properties of the system.

In this paper periodic instantons \[17\] on \( \mathbb{R}^3 \times S^1 \), or calorons, are studied for gauge group \( SU(n) \). Calorons are composed out of elementary BPS monopoles \[29\], as is seen from the action density \[24\]. This becomes clear for small compactification lengths when the constituents are far apart. In particular, removing one of the monopoles to spatial infinity turns the \( k = 1 \) caloron into a BPS \( SU(n) \) monopole. In contrast, the situation of all monopoles nearly coalescing -in appropriate units corresponding to an infinite compactification length- gives back the ordinary instanton on \( \mathbb{R}^4 \). These various aspects are respected by the corresponding limits in the metric. The form of the metric was conjectured by Lee and Yi \[29\], using considerations of D-brane constructions and asymptotic monopole interactions. This paper addresses the explicit calculation of the metric for the caloron moduli space and its limits.

Metric properties of moduli spaces of self-dual connections play an important role in the study of non-perturbative effects of gauge theories. For instantons the metric appears
through the bosonic zero modes in the background of the charge one $SU(2)$ instanton in a calculation to study its physical effects [13]. The scattering of monopoles can be described as the geodesic motion on the moduli space [33], relating the metric to the Lagrangian of the interacting monopole system [34].

The metrics on these moduli spaces are hyperKähler [18]. This property derives formally from the nature of the selfduality equations themselves [1, 10]. It also appears in the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instantons of higher charge, as well as in the Nahm construction for monopoles as a hyperKähler structure on the space of data [8, 4]. The Nahm formalism first appeared as a generalisation of the ADHM construction to construct the BPS monopole [34]. In its extension to selfdual monopoles for arbitrary group and charge [37, 21], the Nahm data in terms of which the monopole is obtained can be constructed in terms of the Weyl zero modes in the background of the monopole. A similar scheme was set up for the caloron [38, 12], which up to very recently [22, 23, 26] had not resulted in explicit solutions. This reciprocity idea could be applied to instantons on $\mathbb{R}^4$ as well [7]. Extended to the four-torus $T^4$, the involutive property of the Nahm transformation preserves the metric and hyperKähler structure [4]. These ideas fit in a programme of studying the Nahm transformation on generalised tori $M = \mathbb{R}^4/H$, where $H$ is the isometry group of the selfdual connection. The calorons correspond to $M = \mathbb{R}^3 \times S^1, H = \mathbb{Z}$. This compactification provides a smooth interpolation between instantons and monopoles, adding to the understanding of both objects and the formalism to study them.

The incorporation of both instanton and monopole-like aspects by calorons is read off from the topological characteristics of selfdual gauge connections $A_\mu dx_\mu$ on $\mathbb{R}^3 \times S^1$ [16]. These are related to the properties of the vacuum which the solution necessarily approaches at spatial infinity in order for the action to be finite. The homotopy class of the gauge transformation connecting the vacuum at infinity with the connection near the origin gives the instanton number $k \in \pi_3(SU(n)) = \mathbb{Z}$. The vacuum itself can be nontrivial, due to the non-trivial topology of the asymptotic boundary of the base manifold $S^2 \times S^1$. This leads to extra labels for the solution which are studied in terms of the gauge holonomy $\mathcal{P}(\vec{x})$ along $S^1$. In the periodic gauge $(A_\mu(\vec{x}, x_0 + T) = A_\mu(\vec{x}, x_0))$, $\mathcal{P}(\vec{x})$ is defined as

$$\mathcal{P}(\vec{x}) = P \exp\left(\int_0^T A_0(\vec{x}, x_0) dx_0\right),$$

where $P$ denotes path ordering and $T$ the circumference of $S^1$, which we set 1. In a zero curvature background, continuous deformations of the loop do not affect $\mathcal{P}(\vec{x})$. Its eigenvalues at spatial infinity are topological invariants. Therefore, the gauge holonomy at infinity is diagonal up to an $\hat{x}$ dependent gauge transformation $V$

$$\lim_{|\vec{x}| \to \infty} \mathcal{P}(\vec{x}) = \mathcal{P}_\infty = V \mathcal{P}_\infty^0 V^{-1}, \quad \mathcal{P}_\infty^0 = \exp[2\pi i \text{diag}(\mu_1, \ldots, \mu_n)].$$

The eigenvalues can be ordered such that

$$\mu_1 < \ldots < \mu_n < \mu_{n+1} \equiv \mu_1 + 1, \quad \sum_{m=1}^n \mu_m = 0,$$
using the gauge symmetry and assuming maximal symmetry breaking for the moment. For later use, we define $\nu_m = \mu_{m+1} - \mu_m$, related to the mass of the $m^{th}$ constituent monopole. Asymptotically,

$$A_0 = 2\pi i \text{diag}(\mu_1, \cdots, \mu_n) - i \text{diag}(k_1, \cdots, k_n)/(2r) + O(r^{-2}), \sum_i k_i = 0, \quad (4)$$

up to the gauge transformation $V(\hat{x})$ that induces a map from $S^2$ to $SU(n)/H_\infty$, with $H_\infty$ the isotropy group of $\exp[2\pi i \text{diag}(\mu_1, \cdots, \mu_n)]$. The maps $V(\hat{x}) \to SU(n)/H_\infty$ are classified according to the fundamental group of $H_\infty$. Generically, $H_\infty$ consists of several $U(1)$ and $SU(N)$, $N > 1$ subgroups. Each $U(1)$ gives rise to a monopole winding number, related to the integers $k_i$. The enhanced residual gauge symmetry described by the $SU(N)$ subgroups arises when there is non-maximal symmetry breaking, $\nu_m = \mu_{m+1} - \mu_m = 0$ for some value(s) of $m$, giving rise to massless constituent monopoles. A non-trivial value of $P_\infty$ breaks the gauge symmetry. This makes calorons very similar to BPS monopoles, [36, 37, 20] which fit in the above classification as $S^1$ invariant selfdual connections, classified according to the magnetic charges $(m_1, \ldots, m_{n-1})$, where $m_i = k_1 + \cdots + k_i$. The $k = 1$ $SU(n)$ coralon studied in this paper has no magnetic charges, and its only nontrivial topological labels are the instanton number $k = 1$ and the eigenvalues $\mu_m$ of the holonomy.

The explicit computation of the metrics in this paper is based on the isometric property of the Nahm transformation, known to hold for instantons on $\mathbb{R}^4$ and $T^4$, as well as for certain types of BPS monopoles [39]. It is believed to hold generally. For most situations considered in this paper, an explicit proof seems not to be present in the literature, and will be given here. This allows for a determination of the metric on the moduli space of Nahm data. For monopoles, such a calculation was first done in [3] showing that the metric of the $(1,1)$ data is a Taub-NUT space with positive mass parameter. Considerations based on asymptotic monopole interactions [14] reproduced this result [11]. For the $(1,1,\ldots,1)$ monopole a similar equivalence was found [27, 35]. All these metrics are of so-called toric hyperKähler type [12, 13], and can be efficiently obtained as metrics on hyperKähler quotients [15]. An explicit calculation of the $k = 1$ $SU(2)$ coralon is extended here to $SU(n)$, generalising the techniques in [22, 23]. An alternative derivation using the hyper-Kähler quotient will also be given. There we will greatly benefit from the formalism in [35, 15], due to the similarity between the coralon and monopole Nahm data.

The outline of this paper is as follows. In section 2, some aspects of hyperKähler manifolds are presented, mostly to fix notation and to give some identities used throughout. Crucial in the ability to handle the coralon is that the infinite matrices of the ADHM construction are converted by Fourier transformation to functions on $S^4$. This translates ADHM to the Nahm formulation and allows one to keep track of crucial delta-function singularities. In section 3, to define notation, we summarise the ADHMN formalism for calorons as developed in refs. [22, 23, 30] based on the ADHM construction for instantons, rather than following [38, 12]. The coralon metric is calculated in section 4. The instanton and monopole limits of the coralon are discussed in section 5. A unified description of instantons, calorons and monopoles is thus achieved. Other aspects of the coralon, among which the effect of massless constituents, are commented on in the discussion. The appendix contains some technicalities on the $(1,1,\ldots,1)$ monopole.
2 Preliminaries

Manifolds with metric $g$ are hyperKähler if they have three independent complex structures $I, J, K$ that satisfy the quaternion algebra, $IJ = -JI = K$ and cyclic, whose associated Kähler forms $\omega^I(\cdot, \cdot) = g(\cdot, I\cdot), \omega^J(\cdot, \cdot) = g(\cdot, J\cdot), \omega^K(\cdot, \cdot) = g(\cdot, K\cdot)$ are closed. As will be outlined in section [4.1], the moduli spaces of selfdual connections inherit their hyper-Kähler property from the hyperKähler structure of the base space manifold $M = \mathbb{R}^4/H$, where $H = \emptyset, \mathbb{Z}, \mathbb{R}$ for instantons, caloron and monopoles respectively. The position coordinate on $\mathbb{R}^4$ will be denoted as a quaternion, $x = x_\mu \sigma_\mu$. Here the unit quaternions are defined as $\sigma_\mu = (1, -i\tau)$ and $\bar{\sigma}_\mu = (1, i\tau)$, with $ij = -ji = k$ and $\tau$ the Pauli matrices. We introduce the selfdual, resp. anti-selfdual quaternionic tensors $[19]$ $\eta_{\mu\nu} \equiv \eta^\mu_\nu \sigma_i \equiv \frac{1}{2} (\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu)$ and $\bar{\eta}_{\mu\nu} \equiv \bar{\eta}^\mu_\nu \sigma_i \equiv \frac{1}{2} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)$, and $\epsilon_{0123} = 1$.

Identifying the tangent space to $\mathbb{H} = \mathbb{R}^4$ with the vector space itself, the complex structures act on $x$ as right multiplication with $-i, -j, -k$ such that $(I, J, K)_\mu = \bar{\eta}^{1,2,3}_{\mu\nu}$. It is convenient to combine the metric and Kähler forms into one quaternion,

$$(g, \vec{\omega}) = g\sigma_0 + \vec{\omega} \cdot \bar{\sigma}.$$  

This implies for $\mathbb{R}^4$,

$$(g, \vec{\omega}) = d\vec{x} \otimes dx,$$

$$g = ds^2 = (dx_\mu)^2,$$

$$\vec{\omega} \cdot \bar{\sigma} = \frac{1}{4} d\vec{x} \wedge dx = \frac{1}{4} \bar{\eta}_{\mu\nu} dx_\mu \wedge dx_\nu = dx_0 \wedge d\vec{x} - \frac{1}{4} d\vec{x} \wedge d\vec{x}.$$  

(6)

Here, $(d\vec{a} \wedge d\vec{b})^i = \epsilon_{ijk} da^j \wedge db^k$. One extends to $\mathbb{H}^N$ by replacing $d\vec{x}$ in eq. (6) by $dx^\dagger = d\vec{x}^\dagger$.

Many examples of hyperKähler manifolds emerge as hyperKähler quotients $[18]$. Consider a hyperKähler manifold $\mathcal{M}$ acted upon freely by a group $G$ (with algebra $g$) of isometries, $L_X g = 0, L$ denoting the Lie derivative and $X \in g$. When $G$ preserves the complex structures, $L_X \vec{\omega} = 0$, the isometries are called triholomorphic and the moment map $\vec{\mu} : \mathcal{M} \rightarrow g^* \otimes \mathbb{R}^3$ can be defined as $X_\mu \vec{\omega}_{\mu\nu} = \partial_\nu \vec{\mu}^X$. The manifold $\vec{\mu}^{-1}(\vec{c})/G$, with $\vec{c} \in \mathbb{R}^3 \otimes g^*$ ($\mathbb{R}^3$ the center of $g^*$) obtained by taking the quotient of the level set $\vec{\mu}^{-1}(\vec{c})$ by $G$ is then hyperKähler itself. Isometries commuting with $G$ descend to the quotient. When they are also triholomorphic, this property is preserved.

The relevant example is provided by the moduli space of ADHM data in the construction of charge $k$ instantons on $\mathbb{R}^4$ for gauge group $SU(n)$ $[2, 7]$. The caloron will be constructed using an infinite-dimensional version of the ADHM construction which we therefore review here, to establish conventions. One considers the set $\hat{A}$ of matrices

$$\Delta = \begin{pmatrix} \lambda & \nu \\ 0 & B \end{pmatrix},$$  

(7)

with $\lambda \in \mathbb{C}^{n,2k}$ and the $2k \times 2k$ dimensional matrix $B = B_\mu \otimes \sigma_\mu$, where $B_\mu$ are $k \times k$ dimensional hermitean matrices. With metric and Kähler forms on $\hat{A}$ defined as

$$(g, \vec{\omega}) = \text{Tr} \left( dB^\dagger \otimes dB + 2d\lambda^\dagger \otimes d\lambda \right),$$  

(8)

$\hat{A}$ is hyperKähler. The $U(k)$ transformations

$$\lambda \rightarrow \lambda T^\dagger, \quad B_\mu \rightarrow T B_\mu T^\dagger, \quad T \in U(k),$$  

(9)
leave \((g, \vec{\omega})\) in eq. (8) invariant and therefore form a group of triholomorphic isometries of \(\hat{A}\). The associated moment map reads (\(\text{tr}_2\) denoting the trace associated with quaternions)

\[
\vec{\mu} = \frac{1}{2} \text{tr}_2 \left[ (B^\dagger B + \lambda^\dagger \lambda) \vec{\sigma} \right].
\]

Its zero set \(\vec{\mu}^{-1}(0)\) is formed by the solutions to the ADHM constraint

\[
\bar{\eta}_{\mu\nu} B_\mu B_\nu + \frac{1}{4} \tau_a \text{tr}_2 (\tau_a \lambda^\dagger \lambda) = 0.
\]

The instanton gauge connection corresponding to a solution to \(\Delta \in \vec{\mu}^{-1}(0)\) is obtained as

\[
A_\mu(x) = v^\dagger(x) \partial_\mu v(x),
\]

in terms of the \((2k + n) \times n\) dimensional complex matrix \(v(x)\) containing the normalised zero modes of \(\Delta^\dagger(x) = \Delta^\dagger - x^\dagger b^\dagger\), where \(b^\dagger = (0, 1_k)\). For \(A_\mu\) to be an \(SU(n)\) gauge potential, \(B^\dagger B + \lambda^\dagger \lambda\) should be invertible, implying the existence of a \(k \times k\) dimensional hermitean matrix \(f_x\) commuting with the quaternions,

\[
\Delta^\dagger(x) \Delta(x) = \sigma_0 \otimes f_x^{-1}.
\]

This matrix features in the expression for the curvature,

\[
F_{\mu\nu} = 2v^\dagger(x) b \eta_{\mu\nu} f_x b^\dagger v(x),
\]

showing it to be self-dual. It also appears in the formula for the action density [10],

\[
\text{Tr} F_{\mu\nu}^2(x) = -\partial_\mu^2 \partial_\nu^2 \log \det f_x
\]

from which it follows that the topological charge is \(k\), because of the asymptotic behaviour

\[
f_x = 1_k / x^2, \quad x^2 \to \infty.
\]

Thus it is shown that an element \(\Delta \in \vec{\mu}^{-1}(0)\) corresponds to a charge \(k\) instanton solution. The gauge connection (12) is not affected by the \(U(k)\) transformations (9), which therefore have to be divided out to obtain the instanton moduli space \(\vec{\mu}^{-1}(0)/U(k)\) (its isometry with the moduli space of instantons is discussed later). This reduces the dimension of the instanton moduli space to \(4kn\). As it is a hyperKähler quotient, this space is hyperKähler [8, 10]. Global gauge transformations of the instanton which are included as moduli, are realised by the action

\[
\lambda \to g \lambda, \quad g \in SU(n),
\]

which is a triholomorphic isometry, as follows from eq. (8). As \(SU(n)\) acts on the left, it commutes with \(U(k)\) acting on the right. Therefore, \(SU(n)\) descends as a group of triholomorphic isometries to the moduli space of ADHM data, the hyperKähler quotient \(\vec{\mu}^{-1}(0)/U(k)\), reflecting the gauge symmetry of the instanton solution.

At this place we recall a frequently used \(U(1)\) fibration over \(\mathbb{R}^3\), physically interpreted as a monopole phase and position. It is presented in terms of complex row 2-vectors that feature in the ADHM matrix \(\lambda\). Specifically, for a 2-dimensional complex row vector \(\varsigma = (\varsigma_1, \varsigma_2)\), describing \(\mathbb{R}^4\), the metric and Kähler forms read

\[
(g, \vec{\omega}) = d\varsigma^\dagger \otimes d\varsigma, \quad g = \frac{1}{4} \text{tr}_2 d\varsigma^\dagger d\varsigma, \quad \vec{\omega} \cdot \vec{\sigma} = \frac{1}{2} d\varsigma^\dagger \wedge d\varsigma.
\]
The complex structures act on \( \zeta \) by right multiplication with \(-\sigma_i\). There is a triholomorphic \( U(1) \) isometry with associated moment map
\[
\zeta \rightarrow e^{it}\zeta, \quad \bar{\mu} = \frac{1}{2} \text{tr}_2 (-i\zeta^\dagger \bar{\zeta}) = \frac{1}{4|\bar{r}|}.
\]
(19)
The level sets are \( U(1) \) fibres due to the phase ambiguity in defining \( \zeta \) from \( \bar{r} \), which becomes more manifest upon introducing new coordinates,
\[
\zeta = \zeta^0 e^{i\frac{\psi}{2}}, \quad \psi \in \mathbb{R}/(4\pi \mathbb{Z})
\]
with for example \( \zeta^0(\bar{r}) \) chosen real. A useful identity is
\[
\frac{1}{2} \text{tr}_2 (\delta \zeta^\dagger_0 \zeta_0 - \zeta^\dagger_0 \delta \zeta_0) = -i|\bar{r}| \bar{w}(\bar{r}) \cdot d\bar{r},
\]
(21)
where \( \bar{w}(\bar{r}) \) is the vector potential of the abelian Dirac monopole,
\[
\bar{\nabla}_{\bar{r}} \times \bar{w}(\bar{r}) = \frac{1}{4|\bar{r}|}.
\]
(22)
In the present form, the Dirac string lies along the positive \( z \) axis, other gauges are obtained by allowing for \( \bar{r} \) dependent phase ambiguities. In terms of \((\bar{r}, \psi)\), the metric and Kähler forms on \( \mathbb{R}^4 \) read
\[
ds^2 = \frac{1}{4} \left( \frac{1}{|\bar{r}|} d\bar{r}^2 + |\bar{r}|(d\psi + \bar{w}(\bar{r}) \cdot d\bar{r})^2 \right), \quad \bar{\omega} = \frac{1}{4} (d\psi + \bar{w}(\bar{r}) \cdot d\bar{r}) \wedge d\bar{r} - \frac{1}{4|\bar{r}|} d\bar{r} \wedge d\bar{r}.
\]
(23)
The \( U(1) \) isometry is equivalent to a linear action
\[
\psi \rightarrow \psi + 2t, \quad t \in \mathbb{R}/(2\pi \mathbb{Z}).
\]
(24)

The moduli spaces we will encounter are all so-called toric hyperKähler manifolds [42]. These manifolds have coordinates consisting of \( N \) three vectors \( \bar{x}_a \in \mathbb{R}^3, a = 1, \ldots, N, \) and \( N \) torus variables \( \phi_a \), generalising the \( U(1) \) in the previous example. Metric and Kähler forms read
\[
g = d\bar{x}_a \Phi_{ab} \cdot d\bar{x}_b + \left( \frac{d\phi_a}{4\pi} + \bar{\Omega}_{ac} \cdot d\bar{x}_c \right) (\Phi^{-1})_{ab} \left( \frac{d\phi_b}{4\pi} + \bar{\Omega}_{bd} \cdot d\bar{x}_d \right),
\]
\[
\bar{\omega} = \left( \frac{d\phi_a}{4\pi} + \bar{\Omega}_{ab} \cdot d\bar{x}_b \right) \wedge d\bar{x}_a - \frac{1}{4} \Phi_{ab} d\bar{x}_b \wedge d\bar{x}_a.
\]
(25)
The potentials \( \Phi \) and \( \bar{\Omega} \) are \( \phi_a \) independent, giving rise to \( N \) commuting triholomorphic isometries \( \partial/\partial \phi_a \), corresponding to shifts on the torus. Closure of the Kähler forms is equivalent to
\[
\frac{\partial}{\partial x_a^c} \Omega_{bc}^j - \frac{\partial}{\partial x_c^b} \Omega_{ac}^i = \epsilon_{ijk} \frac{\partial}{\partial x_c} \Phi_{ab}, \quad \forall a, b, c, i, j.
\]
(26)
These equations are therefore called hyperKähler conditions [12, 13], and generalise eq. (22). The metric in eq. (25) has an \( SO(3) \) isometry, acting on the vectors \( \bar{x}_a \), that rotates the complex structures. Toric hyperKähler manifolds are torus bundles over \( (\mathbb{R}^3)^N \) [14]. Physically, the \( \mathbb{R}^3 \) vectors \( \bar{x}_a \) are (relative) constituent monopole positions, whereas the torus describes the phases of the monopoles. In the Lagrangian interpretation of the metric, \( \Phi \) and \( \bar{\Omega} \) denote retarded interaction potentials for the constituents [34, 14] and it was considerations of this kind that led to the conjectures for the metric in [29, 27].
3 The ADHM-Nahm formalism

We will construct the caloron in the so-called algebraic gauge, related to the periodic gauge by the non-periodic gauge transformation $g(\vec{x}, x_0) = V \exp[2\pi i x_0 \text{diag}(\mu_1, \ldots, \mu_n)] V^{-1}$. In this gauge, the background field $2\pi i \text{diag}(\mu_1, \ldots, \mu_n)$ in eq. (4) is removed and we have the alternative boundary condition,

$$A_\mu(\vec{x}, x_0 + T) = P_\infty A_\mu(\vec{x}, x_0) P_\infty^{-1}. \tag{27}$$

Since in the absence of magnetic windings, $P_\infty$ can always be gauged to a constant diagonal form, we assume henceforth $P_\infty = P_\infty^0$ without loss of generality. The periodic instanton of charge one is obtained in the algebraic gauge (27) by taking an infinite array of elementary instantons, relatively gauge-rotated by $P_\infty$.

To implement this in the ADHM formalism we take a specific solution for the zero mode vector $v(x)$ in the ADHM construction,

$$v(x) = \begin{pmatrix} -1_n \\ u(x) \end{pmatrix} \varphi^{-\frac{1}{2}}(x), \quad u(x) = (B^\dagger - x^\dagger 1_k)^{-1} \lambda^\dagger, \quad \varphi(x) = 1_n + u^\dagger(x) u(x), \tag{28}$$

where $\varphi$ is an $n \times n$ positive hermitean matrix. In terms of these, one obtains

$$A_\mu(x) = \varphi^{-\frac{1}{2}}(x) (u^\dagger(x) \partial_\mu u(x)) \varphi^{-\frac{1}{2}}(x) + \varphi^{\frac{1}{2}}(x) \partial_\mu \varphi^{-\frac{1}{2}}(x). \tag{29}$$

For eq. (27) to hold, it is then required that

$$u_{p+1}(x + 1) = u_p(x) P_\infty^{-1}, \quad p \in \mathbb{Z}. \tag{30}$$

This imposes periodicity constraints on the data

$$\lambda_{p+1} = P_\infty \lambda_p, \quad B(x + 1)_{p,p'} = B(x)_{p-1,p'-1}, \tag{31}$$

which imply

$$\lambda_p = P_\infty^0 \zeta, \quad B_{p,p'} = \sigma_0 \delta_{p,p'} + \hat{A}_{p-p'}, \quad p, p' \in \mathbb{Z}. \tag{32}$$

The off-diagonal part $\hat{A}$ is still to be determined. Fourier transformation translates the ADHM formalism to the Nahm language. $B$ is cast into a Weyl operator,$$
\sum_{p,p' \in \mathbb{Z}} B_{p,p'}(x) e^{2\pi i (pz - p'z')} = \frac{\delta(z - z')}{2\pi i} \hat{D}_x(z'), \quad \hat{D}_x(z) = \sigma_\mu \hat{D}_x^\mu(z) = \frac{d}{dz} + \hat{A}(z) - 2\pi i x,$$and $\lambda^\dagger \lambda$ into a singularity structure describing the matching conditions for $\hat{A}(z)$,

$$\sum_{p \in \mathbb{Z}} e^{-2\pi ipz} \lambda_p = \sum_{p \in \mathbb{Z}} e^{2\pi ip(\mu_m - z)} P_m \zeta = \hat{\lambda}(z), \quad \hat{\lambda}(z) = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) P_m \zeta, \tag{34}$$

$$\sum_{p,p' \in \mathbb{Z}} \lambda^\dagger_p e^{2\pi i (pz - p'z')} \lambda_{p'} = \delta(z - z') \hat{\Lambda}(z), \quad \hat{\Lambda}(z) = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) \zeta^\dagger P_m \zeta = \zeta^\dagger \hat{\lambda}(z).$$
Here we introduced the projection operators $P_m = e_m e_m^t$, where $e_m$ is the $m^{th}$ unit vector, in terms of which $\mathcal{P}_\infty = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \exp(2\pi i \mu_m) P_m$ and $\lambda_p = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \exp(2\pi i p_m) P_m \zeta$. The group index $m \in \mathbb{Z}/n\mathbb{Z}$ is a cyclic variable. We also used that for any two objects $a, b$ of type $a_p = \mathcal{P}_\infty^p \alpha$, $p \in \mathbb{Z}$, the Fourier transforms defined as $\hat{a}(z) = \sum_{p \in \mathbb{Z}} \exp(-2\pi i p z) a_p$, have the property

$$\hat{a}^t(z) \hat{b}(z') = \delta(z-z') \hat{a}(z)^t < \hat{b}>= \delta(z-z') \hat{a}^t(z) = \delta(z-z') \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) a^t P_m \beta,$$

where $<H> \equiv \int_{S^1} H(z) dz$. The quadratic ADHM constraint translates into

$$\frac{1}{2} [\hat{D}_\mu(z), \hat{D}_\nu(z)] \hat{\eta}_{\mu\nu} = 4\pi^2 \Im \hat{\Lambda}(z),$$

where $\Im$ is introduced to act on a $2 \times 2$ matrix as $\Im W = \frac{1}{2} [W - \tau_2 W^\tau \tau_2]$ ($\Re W \equiv \frac{1}{2} \text{tr}_2 W$).

We use the $U(1)$ fibration over $\mathbb{R}^3$ (eq. (19)) to write

$$\zeta^t P_m \zeta = \zeta^t_{(m)} \zeta_{(m)} = \frac{1}{2\pi} (\rho_m + \bar{\rho}_m \cdot \vec{\tau}), \quad \rho_m = |\bar{\rho}_m|.$$

This leads to the caloron Nahm equation

$$\frac{d}{dz} \hat{A}_j(z) = 2\pi i \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) \rho_j^m,$$

which is abelian in the $k = 1$ situation at hand, see [24, 38]. The phase ambiguity in defining $\zeta_{(m)}$ from $\bar{\rho}_m$ is resolved later. As integration of eq. (38) over $S^1$ gives a constraint on $\zeta$,

$$\sum_{m \in \mathbb{Z}/n\mathbb{Z}} \bar{\rho}_m = \pi \text{tr}_2 (\vec{\tau} \zeta^t \zeta) = 0,$$

we can introduce vectors $\vec{y}_m, m \in \mathbb{Z}/n\mathbb{Z}$, such that $\bar{\rho}_m = \vec{y}_m - \vec{y}_{m-1}$. The vectors $\vec{y}_m$ are to be interpreted as the constituent monopole positions. We now find for the spacelike components of $\hat{A}(z)$,

$$\hat{A}_j(z) = 2\pi i \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m, \mu_{m+1}]}(z) y_j^m,$$

where $\chi_{[\mu_m, \mu_{m+1}]}(z) = 1$ for $z \in [\mu_m, \mu_{m+1}]$ and 0 elsewhere, extended periodically. Note that the Nahm equations determine $\vec{y}_m$ up to the global $\mathbb{R}^3 \times S^1$ position variable

$$\xi = \frac{1}{2\pi i} \int_{S^1} \hat{A}(z) dz, \quad \bar{\xi} = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \nu_m \vec{y}_m.$$

The $T$ symmetry eq. (1) in the ADHM construction is mapped to a $U(1)$ gauge symmetry, with gauge group $\mathcal{G} = \{g(z)|g : z \rightarrow e^{-ih(z)} \in U(1)\}$, acting as

$$\hat{A}(z) \rightarrow \hat{A}(z) + i \frac{d}{dz} h(z), \quad \zeta_m \rightarrow \zeta_m e^{ih(\mu_m)}.$$

For calorons, $g(z)$ is periodic and can be used to set $\hat{A}_0(z)$ to a constant. A piecewise linear $U(1)$ gauge function $h(z)$ shifts the $U(1)$ phase ambiguities in $\zeta_{(m)}$ to $\hat{A}_0(z)$, which
thus becomes piecewise constant. Therefore, all $4n$ moduli are included in the following solution to the Nahm equations

$$
\dot{A}(z) = 2\pi i \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m, \mu_{m+1}]}(z) (\frac{\tau_m}{4\pi \nu_m} \sigma_0 + \vec{y}_m \cdot \vec{\sigma}),
$$

where $\tau = (\tau_1, \ldots, \tau_n)'$ takes values in $\mathbb{R}^n$. Using the gauge function

$$
g(z) = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m, \mu_{m+1}]}(z) \exp(2\pi i (z - \mu_m) k_m / \nu_m), \quad k_m \in \mathbb{Z},
$$

which leave the $U(1)$ phases of $\zeta$ unaffected, $\tau$ can be restricted to the torus $\mathbb{R}^n/(4\pi \mathbb{Z})^n$. In this gauge, the moduli describing the general caloron are the position vectors $\vec{y}_m$, comprised in $\vec{y} = (\vec{y}_1, \ldots, \vec{y}_n)$ and the torus coordinate $\tau$ describing the $U(1)^{n-1}$ residual gauge symmetry and the temporal position of the caloron. Strictly speaking, these variables are coordinates on the cover of the moduli space of framed calorons. The true moduli space is obtained by dividing out the center of the gauge group. This leads to orbifold singularities.

Under Fourier transformation, the Green’s function $f_x$ (eq. (13)) for calorons becomes $\hat{f}_x(z, z') = \sum_{p, p' \in \mathbb{Z}} f_{x, p, p'} e^{2\pi i (pz - p'z')}$ and is a solution of the differential equation

$$
\left\{ \left( \frac{1}{2\pi i} \frac{d}{dz} x_0 \right)^2 + \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m, \mu_{m+1}]}(z) r_m^2 \frac{1}{2\pi} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) \vec{y}_m \cdot \vec{\sigma}_m \right\} \hat{f}_x(z, z') = \delta(z - z'),
$$

in the gauge with $\dot{A}_0(z)$ constant. Here $r_m = |\vec{x} - \vec{r}_m|$ is the center of mass radius of the $m$th constituent. Expressions for $\hat{f}_x$ in other gauges are obtained by using that under the action of $\hat{G}$, $\hat{f}_x$ transforms as

$$
\hat{f}_x(z, z') \rightarrow g(z) \hat{f}_x(z, z') g(z')^*, \quad g(z) \in \hat{G}.
$$

The Nahm construction of the $(1, 1, \ldots, 1)$ monopole, later obtained by as a special limit of the caloron, is discussed in the appendix.

## 4 The caloron metric

### 4.1 Moduli spaces of selfdual connections

The metric on the moduli space $\mathcal{M}$ of selfdual connections on the manifold $M = \mathbb{R}^4/H$ is computed as the $L_2$ norm of its tangent vectors. These are gauge orthogonal variations of the connections with respect to their moduli. Specifically, $Z_\mu$ is tangent to the moduli space when it is a solution of the deformation equation and the gauge orthogonality condition requiring it to be a zero mode of the covariant derivative $D_{\mu}^{ad} = \partial_\mu + [A_\mu, \cdot]$, 

$$
D_{[\mu}^{ad} Z_{\nu]} = \tilde{\epsilon}_{\mu\nu\alpha\beta} D^{ad}_{[\alpha} Z_{\beta]}, \quad D^{ad}_{[\mu}(A) Z_{\mu]} = 0.
$$

Written in terms of quaternions, these equations are concisely expressed as $D^{ad} Z = 0$, from which one reads off the tangent space to admit three almost complex structures
I, J, K acting as \(-i, -j, -k\) on the right. Metric and Kähler forms read

\[
(g, \bar{\omega})_M(Z, Z') = \frac{1}{4\pi^2} \int_M d_4 x \text{Tr} \left( Z^i(x) Z'^i(x) \right),
\]

where \(Z, Z'\) are any two tangent vectors. Gauge orthogonality of a general variation \(\delta A_\mu\) of the selfdual connection can be achieved by applying an infinitesimal gauge transformation \(\Phi\),

\[
Z_\mu = \delta A_\mu + D^{\text{ad}}_\mu \Phi, \quad (D^{\text{ad}}_\nu)^2 \Phi = -D^{\text{ad}}_\mu \delta A_\mu.
\]

implying for the metric

\[
g = \frac{-1}{4\pi^2} \int_M d_4 x \text{Tr} (\delta A_\mu - D^{\text{ad}}_\mu (D^{\text{ad}}_\nu)^{-2} D^{\text{ad}}_\rho \delta A_\rho)^2.
\]

The hyperKähler property of the moduli space follows formally from considering it as the infinite dimensional hyperKähler quotient of the space of general connections \(\mathcal{A}\) by the triholomorphic action of the group of gauge transformations \(G\) \([1, 10]\). The moment map is \(\bar{\mu}_G = \bar{\eta}_{\mu\nu} F_{\mu\nu}/8\pi^2\), so that the zero set is formed by the space of self-dual solutions, which quotiented by \(G\) gives the moduli space. That this quotient is well defined follows from the invariance of the Kähler forms

\[
\bar{\omega}_{rs} \cdot \bar{\sigma} = \frac{1}{4\pi^2} \int_M d_4 x \bar{\eta}_{\mu\nu} \text{Tr} (\delta_r A_\mu \delta_s A_\nu),
\]

under infinitesimal gauge transformations, which is seen by adding arbitrary \(D^{\text{ad}}_\mu \Phi\) to the deformations. For the caloron the boundary condition eq. (27) is consistent with complex structures acting as \(\bar{\eta}_{\mu\nu}\), i.e. the non-trivial holonomy is compatible with the hyperKähler structure. One therefore expects caloron moduli spaces to be hyperKähler.

For practical purposes the formal reasoning above is of little use. Computing metrics on moduli spaces with the techniques presented depends crucially on the construction of the Green’s function of the covariant Laplacian and in the present situation, we do not even have an expression for \(A_\mu\) readily available. We take a different route which uses multi-instanton calculus, suitably adapted to the caloron situation. This allows for calculating the metric in terms of the ADHMN data and makes it thus feasible to find a compensating gauge transformation or to perform the hyperKähler quotient.

Moduli spaces of selfdual connections can usually be written as a product of the base space \(M\), describing the center of mass and the non-trivial relative moduli space \(M_{\text{rel}},\)

\[
\mathcal{M} = M \times M_{\text{rel}}.
\]

In the metric this corresponds to a part describing the flat metric on the base space \(M\) and one for the relative or centered metric on \(M_{\text{rel}},\) containing the nontrivial part. However, in the case at hand, where we want to take particular limits, it will be preferable to work with the full metric on \(\mathcal{M}\).
4.2 Isometric properties of the ADHM-Nahm construction.

We first recall the computation of the metric on the moduli space of instantons on \( \mathbb{R}^4 \) which can be entirely performed using ADHM techniques. Adapted to the caloron situation, this will translate into the formalism to calculate metrics in terms of Nahm data.

A tangent vector to the instanton moduli space is given by

\[
Z_{\mu}(C) = v^\dagger(x)C\sigma_\mu f_x u(x)\varphi^{-\frac{1}{2}}(x) - \varphi^{-\frac{1}{2}}(x)u^\dagger(x)f_x\sigma_\mu C^\dagger v(x),
\]

where \( C \) is a tangent vector to the moduli space of ADHM data,

\[
C = \begin{pmatrix} c \\ Y \end{pmatrix},
\]

which satisfies

\[
(\Delta^\dagger(x)C) = (\Delta^\dagger(x)C)^t.
\]

Here the \( \Re \) part is the deformation of the ADHM constraint and the \( \Im \) part guarantees gauge orthogonality. Using an infinitesimal \( U(k) \) transformation eq. (9) \( T = \exp(-i\delta X) \), where \( \delta X = \delta X^1 \), the tangent vectors can be constructed as

\[
C = \delta\Delta + \delta X\Delta = \begin{pmatrix} \delta\lambda + i\lambda\delta X \\ \delta B + i[B,\delta X] \end{pmatrix},
\]

which automatically satisfy the deformation equation. Gauge orthogonality imposes

\[
\text{tr} \left( B^\dagger[B, i\delta X] - [B^\dagger, i\delta X]B + 2i\delta X\Lambda + \lambda^\dagger\delta\lambda - \delta\lambda^\dagger\lambda + B^\dagger\delta B - \delta B^\dagger B \right) = 0.
\]

The integral to compute the \( L_2 \) norm in eq. (48) is reduced to a boundary term corresponding with \( x^2 \to \infty \), where \( f_x \) is known, compare eq. (16). Using that \( Z(C)\bar{\sigma}^i = Z(C\bar{\sigma}^i) \), and identifying the tangent space to the ADHM data with the vector space itself, the well-known (see also [32]) hyperKähler isometric property of the ADHM construction is proven

\[
(g,\bar{\omega})_M(Z, Z') = \text{Tr} \left( Y^\dagger Y' + 2c^i\bar{c}^i \right).
\]

The right hand side of eq. (59) explains why eq. (8) gives the natural metric and Kähler forms on the space \( \hat{A} \) of ADHM matrices \( \Delta \). As the ADHM construction is an isometry and the moduli space of ADHM data \( \hat{\mu}^{-1}(0)/U(k) \) is hyperKähler the same holds for the moduli space of instantons on \( \mathbb{R}^4 \).

In employing the metric properties of the ADHM construction in the caloron case, one has -in addition to the deformation equation and gauge orthogonality- the algebraic gauge condition eq. (27) to be satisfied

\[
Z_{\mu}(x + 1) = P_\infty Z_{\mu}(x)P_\infty^{-1}.
\]
This requires
\[ Y_{p,p'} = Y_{p-1,p'-1}, \quad c_{p+1} = \mathcal{P}_\infty c_p, \quad \delta X_{p,p'} = \delta X_{p-p'}. \quad (61) \]
The compatibility of periodicity and nontrivial holonomy with the hyperKähler structure on the level of the ADHM-Nahm construction can be seen from the complex structures acting on \( Y \) and \( c \) as multiplication by \(-i, -j, -k\) on the right.

We define the Fourier transforms of the tangent vector
\[
\hat{\mathcal{C}}(z) = \sum_{p \in \mathbb{Z}} \exp(-2\pi i pz) c_p = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) \hat{c}_m, \quad \delta(z - z') \hat{Y}(z) = \sum_{p, p' \in \mathbb{Z}} e^{2\pi i (pz - p'z')} Y_{p,p'},
\]
and find after Fourier transformation of eqs. (55, 56) the analogues of eq. (47) as the deformation of the Nahm equation and a gauge orthogonality condition
\[
\frac{d}{dz} \hat{Y}_i(z) = -i\pi \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \text{tr}_2 \delta^i (\zeta_m^\dagger \hat{c}_m + \hat{c}_m^\dagger \zeta_m) \delta(z - \mu_m),
\]
\[
\frac{d}{dz} \hat{Y}_0(z) = -i\pi \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \text{tr}_2 (\zeta_m^\dagger \hat{c}_m - \hat{c}_m^\dagger \zeta_m) \delta(z - \mu_m).
\] (63)

To evaluate the caloron metric we use eq. (58) and closely follow the reasoning in [23]. By Fourier transformation, Corrigan’s formula is cast into
\[
\text{Tr} Z^i(x) Z'(x) = -\partial^2 \int_{S^1} dz \left( |\hat{Y}^\dagger(z) \hat{Y}'(z) + \hat{c}^\dagger(z) < \hat{c}' > |\hat{f}_x(z, z) \right)
\]
\[ + \frac{i}{2\pi} \partial^2 \int_{S^1} dz dz' \left( [\hat{C}(z) + \hat{Y}(z)] \hat{f}_x(z, z') [\hat{Y}'^\dagger(z') + \hat{c}^\dagger(z')] \hat{f}_x(z', z) \right), \quad (64) \]
where we introduced the shorthand notation
\[
\hat{C}(z) = \hat{c}^\dagger(z) < \hat{\lambda} >, \quad \hat{Y}_x(z) = (2\pi i)^{-1} \hat{Y}^\dagger(z) \hat{D}_x(z).
\] (65)

In evaluating the integral over \( \mathbb{R} \times S^1 \), the \( \partial^2_0 \) term gives no contribution because of periodicity. The term involving \( \partial^2_0 \) is evaluated by partial integration as a boundary term at spatial infinity, for which the asymptotic behaviour of the Green’s function \( \hat{f}_x(z, z') \) is needed. Since the asymptotic expression for the Green’s function is independent of \( n \),
\[
\hat{f}_x(z, z') = \frac{\pi}{|\vec{x}|} e^{-2\pi |\vec{x}| |z - z'| + 2\pi i x_0 (z - z')} + \mathcal{O}(|\vec{x}|^{-2}),
\] (66)
we can use the analysis for \( SU(2) \) in [23]. Combining the first line in eq. (58) with the only surviving term of the second, we find the following gauge independent expression
\[
(g, \omega)_M(Z, Z') = \left( < \hat{Y}^\dagger Y' > + 2 < \hat{c}^\dagger > < \hat{c}' > \right).
\] (67)

This proves that the metric and Kähler forms on the caloron moduli space can be computed as the metric on the Nahm data. In other words, for \( k = 1 \) \( SU(n) \) calorons, the Nahm construction is a hyperKähler isometry. A slightly modified proof shows this for monopoles of type \((1, 1, \ldots, 1)\) and can be found in the appendix.
The isometric property is essential for what follows. The metric on the caloron moduli spaces can now be calculated in terms of tangent vectors to the space of solutions to the Nahm equations, with infinitesimal gauge transformations performed where needed. This method, used in section 4.3, is called direct as it concentrates on the gauge orthogonal tangent vectors to the moduli space. An alternative method, given in section 4.4, uses the fact that the moduli space of data is an infinite dimensional hyperKähler quotient. It proceeds by using part of the \( U(k) \) gauge symmetry to embed the moduli in a finite dimensional hyperKähler space. The metric on the moduli space is then found as the metric on a finite dimensional hyperKähler quotient, with the remaining gauge action to be divided out.

### 4.3 Direct computation

In the direct approach a compensating gauge function \( \hat{\delta} \hat{X}(z) = \sum_{p \in \mathbb{Z}} X_p \exp(2\pi i p z) \) has to be found to account for the tangent vectors

\[
\hat{c}(z) = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) \left( \delta \zeta_m + i \zeta_m \hat{\delta} \hat{X}(\mu_m) \right), \quad \hat{Y}(z) = \frac{1}{2\pi i} \left( \delta \hat{A}(z) + i \frac{d}{dz} \delta \hat{X}(z) \right)
\]

(68)

to be gauge orthogonal, eq. (63). The gauge orthogonality of \( \hat{Y}(z) \) implies for the compensating gauge function \( \delta \hat{X}(z) \)

\[
-\frac{1}{2\pi} \frac{d^2 \delta \hat{X}(z)}{dz^2} + 2 \frac{\delta \hat{X}(z)}{z} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) |\bar{\rho}_m| = \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m) \left[ \frac{d\tau_m}{4\pi \nu_m} - \frac{d\tau_{m-1}}{4\pi \nu_{m-1}} - |\bar{\rho}_m| \bar{\omega}_m(\bar{\rho}_m) \cdot d\bar{\rho}_m \right],
\]

(69)

where we used eq. (21). This differential equation implies that \( \delta \hat{X}(z) \) is continuous and piecewise linear. Therefore, \( \delta \hat{X}(z) \) is fully determined by the values \( \delta \hat{X}_m \) it takes at \( z = \mu_m \), which are comprised in the vector \( \delta \hat{X} = (\delta \hat{X}_1, \ldots, \delta \hat{X}_n) \in \mathbb{R}^n \). In the gauge chosen, all functions are either constants on the subintervals \( (\mu_m, \mu_{m+1}) \), or fixed by values at \( z = \mu_m \). Therefore, the entire computation can conveniently be performed in terms of \( n \) dimensional vectors and \( n \times n \) matrix operators acting thereon, at the cost of introducing some extra notation. For taking derivatives, we will use the \( n \times n \) matrix

\[
S = \begin{pmatrix}
1 & -1 & -1 \\
1 & -1 & \\
& \ddots & \\
& & 1 & -1 \\
& & & 1 & -1 \\
-1 & & & & 1
\end{pmatrix},
\]

(70)

with unspecified entries generally put to zero. In addition we introduce the vector \( \bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_n) \in \mathbb{R}^{3n} \) and diagonal matrices

\[
N = \text{diag}(\nu_1, \ldots, \nu_n), \quad \bar{\omega} = \frac{1}{4\pi} \text{diag}(\bar{\omega}_1(\bar{\rho}_1), \ldots, \bar{\omega}_n(\bar{\rho}_n)), \quad V^{-1} = 4\pi \text{diag}(\rho_1, \ldots, \rho_n).
\]

(71)
Introducing the symbol $V$ anticipates its later interpretation as potential. In the sequel, all matrix multiplications between $n$-dimensional objects are implicitly assumed. The transpose $^t$ acts only on the indices running from 1 to $n$.

The Nahm connection is now represented by the $n$ dimensional vector

$$\hat{A} = (\hat{A}_1, \ldots, \hat{A}_n)^t = 2\pi(N^{-1} \frac{\tau}{4\pi} + \hat{g} \cdot \hat{\sigma}), \tag{72}$$

where $i\hat{A}_m$ is the value of $\hat{A}(z)$ on $(\mu_m, \mu_{m+1})$. The Nahm equation reduces to $\tilde{\rho} = S^t \hat{g}$. Similarly $\hat{c}(z) = \sum_{m\in \mathbb{Z}/n\mathbb{Z}} \delta(z - \mu_m)\hat{c}_m$ and $\hat{Y}(z) = i \sum_{m\in \mathbb{Z}/n\mathbb{Z}} \chi_{[\mu_m, \mu_{m+1}]}(z)\hat{Y}_m$ are fixed by

$$\hat{c}_m = \delta \zeta_m + i \zeta_m \delta \hat{X}_m, \quad \hat{Y} = \frac{1}{2\pi} \delta \hat{A} - \frac{1}{2\pi} N^{-1} S \delta \hat{X}. \tag{73}$$

Integrating the differential equation (69) for $\delta \hat{X}(z)$ over small intervals $[\mu_m - \epsilon, \mu_m + \epsilon]$, $\epsilon \downarrow 0$, gives conditions on the values $\delta \hat{X}_m$. This yields

$$\frac{1}{2\pi} (S^t N^{-1} S + V^{-1}) \delta \hat{X} = (S^t N^{-1} \frac{d\tau}{4\pi} - V^{-1} \tilde{W} S^t \cdot d\hat{g}), \tag{74}$$

where we used that $\int -d^2 \delta \hat{X}(z)/dz^2 dz$ contributes

$$- \left( \delta \hat{X}'(\mu_m+) - \delta \hat{X}'(\mu_m-) \right) = - \left( \frac{1}{\nu_m} (\delta \hat{X}_{m+1} - \delta \hat{X}_m) - \frac{1}{\nu_{m-1}} (\delta \hat{X}_m - \delta \hat{X}_{m-1}) \right). \tag{75}$$

Eq. (74) is solved by

$$\frac{\delta \hat{X}}{2\pi} = VS^t G^{-1} \frac{d\tau}{4\pi} - (1 - VS^t G^{-1} S) \tilde{W} S^t \cdot d\hat{g}, \tag{76}$$

such that

$$\hat{Y} = d\hat{g} \cdot \hat{\sigma} + N^{-1} \frac{d\tau}{4\pi} - \frac{1}{2\pi} N^{-1} S \delta \hat{X} = d\hat{g} \cdot \hat{\sigma} + G^{-1} (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g}), \tag{77}$$

where we defined $G = N + SVS^t$. The integration over $S^1$ to evaluate the metric on the Nahm data in eq. (67) is carried out as $< \hat{Y}^t \otimes \hat{Y} > = \hat{Y}^t N \otimes \hat{Y}$ using that each subinterval has length $\nu_m = \mu_{m+1} - \mu_m$. Thus we obtain

$$\frac{1}{2} \text{tr}_2 < \hat{Y}^t \hat{Y} > = d\hat{g}^t \cdot ND\hat{g} + (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g})^t G^{-1} N G^{-1} (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g}),$$

$$\frac{1}{2} < \hat{Y}^t \wedge \hat{Y} > = - \frac{1}{2} d\hat{g}^t N \wedge d\hat{g} \cdot \hat{\sigma} + \left( NG^{-1} (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g}) \right)^t \wedge d\hat{g} \cdot \hat{\sigma}. \tag{78}$$

Using the properties (21,23) of $\zeta_m$, the contribution to the metric of $\hat{c}_m$ defined in eq. (73) is found. One obtains

$$\text{tr}_2 < \hat{c}^l > < \hat{c} > = d\hat{g}^t \cdot SVS^t d\hat{g} + (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g})^t G^{-1} SVS^t G^{-1} (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g}),$$

$$< \hat{c}^l > \wedge < \hat{c} > = - \frac{1}{2} (SVS^t d\hat{g})^t \wedge d\hat{g} \cdot \hat{\sigma} + \left( SVS^t G^{-1} (\frac{d\tau}{4\pi} + \tilde{W} S^t \cdot d\hat{g}) \right)^t \wedge d\hat{g} \cdot \hat{\sigma},$$
where it is used that in the gauge chosen the phases of $\zeta$ are fixed. The metric and Kähler forms on moduli space of the uncentered caloron are now readily obtained

$$ds^2 = dy^t G \cdot dy + (d\tau + W \cdot dy)^t G^{-1} (d\tau + W \cdot dy),$$ (78)

$$\tilde{\omega} = (d\tau + W \cdot dy)^t \wedge dy - \frac{1}{4} (G dy)^t \wedge dy,$$ (79)

$$G = N + SVS^t, \quad \tilde{W} = S\tilde{W}S^t.$$ (80)

Equivalently writing

$$G_{m,m'} = \nu_m \delta_{mm'} - \frac{\delta_{m-1,m'}}{4\pi \rho_m} + \delta_{m,m'} \left( \frac{1}{4\pi \rho_m} + \frac{1}{4\pi \rho_{m+1}} \right) - \frac{\delta_{m+1,m'}}{4\pi \rho_{m+1}}, \quad m, m' \in \mathbb{Z}/n\mathbb{Z},$$ (80)

reveals the form of $G$ as given in [29]; thus we confirm the conjectured form for the metric in [29]. As is readily checked, from eqs. (22, 71) it follows that $G$ and $\tilde{W}$ satisfy the hyperKähler conditions (26)

$$\tilde{\nabla}_y G = \tilde{\nabla}_y \times \tilde{W}, \quad \partial_i G_{m',m''} = \epsilon_{ijk} \partial_j^{m'} \left( \tilde{W} \right)_{m'',m''}^k,$$ (81)

$$(\partial_i^m = \partial/\partial y^i_m),$$ which implies the Kähler forms in (79) to be closed and the caloron metric to be hyperKähler.

The metric has $n$ commuting triholomorphic isometries,

$$\frac{\partial}{\partial \tau_m}, \quad m = 1, \ldots, n,$$ (82)

as $G$ and $\tilde{W}$ are $\tau$ independent. The isometries correspond to shifts on the $n$-torus $\mathbb{R}^n/(4\pi \mathbb{Z})^n$ which describe the residual $U(1)^{n-1}$ gauge invariance and the temporal position

$$\xi_0 = \frac{1}{4\pi} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \tau_m \in S^1,$$ (83)

of the caloron. Therefore, the caloron moduli space is a toric hyperKähler manifold, with dimension $4n$. $3n$ coordinates describe the monopole positions and $n$ phase angles parameterise the temporal position and residual $U(1)^{n-1}$ gauge invariance in the case of maximal symmetry breaking. From the uncentered caloron metric in eq. (78), all other metrics discussed in this paper can be obtained by taking suitable limits. In the next subsection the caloron metric will be obtained using the hyperKähler quotient.

The non-trivial part of the metric is obtained by splitting off the center of mass coordinate $\xi$ in eq. (11). To this aim, we express the metric in terms of $\xi$ and $n-1$ relative monopole position vectors $\tilde{r}_m$, using that $\tilde{r}_n = - \sum_{n=1}^{n-1} \tilde{r}_m$ because of eq. (39). The two sets of coordinates are related by the $n \times n$ dimensional "centering matrix" $F_c$,

$$F_c = (S_c, Ne), \quad \left( \begin{array}{c} \tilde{r} \\ \xi \end{array} \right) = F_c^\dagger \vec{y}.$$ (84)

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Here, the \( n \times (n-1) \) dimensional matrix \( S_c \) is obtained from \( S \) by omitting its last column, and we defined \( e = (1, \ldots, 1)^t \in \mathbb{R}^n \). A tilde denotes from now on the restriction to the first \( n-1 \) coordinates, e.g. \( \tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_{n-1})^t \). New torus coordinates \( \tilde{\nu} = (\nu_1, \ldots, \nu_{n-1})^t \) are introduced as well

\[
\tau = F_c \left( \frac{\tilde{\nu}}{4\pi \xi_0} \right). \tag{85}
\]

The centered metric will be again hyperKähler, as splitting of the center off mass metric amounts to taking the hyperKähler quotient under the \( U(1) \) action

\[
\tau_m \to \tau_m + \nu_m t_c, \quad m = 1, \ldots, n, \quad t_c \in \mathbb{R}. \tag{86}
\]

From eqs. (78, 79) it is seen that this action is a triholomorphic isometry whose moment map gives the center of mass of the caloron

\[
\bar{\mu} = \frac{1}{4\pi} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \nu_m \bar{y}_m - \frac{\bar{\xi}}{4\pi}. \tag{87}
\]

Indeed, the phase variables \( \tilde{\nu} \) are invariant under the \( U(1) \) action and can serve as coordinates on the quotient whereas the fibre coordinate \( \xi_0 \) changes as \( \xi_0 \to \xi_0 + t_c \).

In the new basis the relative metric is expressed in terms of a relative mass matrix and relative interaction potentials

\[
F_c^{-1}G(F_c^{-1})^t = \begin{pmatrix} \tilde{G}_{\text{rel}} & 1 \\ 1 & 1 \end{pmatrix}, \quad \tilde{G}_{\text{rel}} = \tilde{M} + \tilde{V}_{\text{rel}},
\]

\[
(\tilde{V}_{\text{rel}})_{mm'} = \tilde{V}_{mm'} + \frac{1}{4\pi |\tilde{\rho}_n|}, \quad (\tilde{W}_{\text{rel}})_{mm'} = \tilde{W}_{mm'} + \frac{\tilde{W}_n(\tilde{\rho}_n)}{4\pi}. \tag{88}
\]

where \( m, m' = 1, \ldots, n-1 \), \( \tilde{\rho}_n = \sum_{m=1}^{n-1} \tilde{\rho}_m \). The relative mass matrix \( \tilde{M} \) is defined as

\[
F_c^t N^{-1} F_c = \begin{pmatrix} \tilde{M}^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{M}^{-1} = \begin{pmatrix} \frac{1}{\nu_1} + \frac{1}{\nu_2} & \frac{-1}{\nu_2} & \frac{-1}{\nu_3} & \cdots & \frac{-1}{\nu_{n-2}} & \frac{-1}{\nu_{n-1}} \\ \frac{-1}{\nu_2} & \frac{1}{\nu_1} + \frac{1}{\nu_2} & \frac{1}{\nu_3} & \cdots & \frac{1}{\nu_{n-2}} & \frac{1}{\nu_{n-1}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{-1}{\nu_{n-2}} & \frac{1}{\nu_{n-3}} + \frac{1}{\nu_{n-2}} & \frac{1}{\nu_{n-2}} & \cdots & \frac{1}{\nu_{n-2}} & \frac{1}{\nu_{n-1}} \\ \frac{-1}{\nu_{n-1}} & \frac{1}{\nu_{n-2}} + \frac{1}{\nu_{n-1}} & \frac{1}{\nu_{n-2}} & \cdots & \frac{1}{\nu_{n-2}} & \frac{1}{\nu_{n-1}} \end{pmatrix}. \tag{89}
\]

its explicit form allowing one to take limits that correspond to massless monopoles

\[
\tilde{M} = \tilde{M}^t, \quad \tilde{M}_{mm'} = (\nu_m + \cdots + \nu_{n-1})(1 - \nu_{m'} \cdots - \nu_{n-1}) \quad \text{for} \ m \geq m', \ m, m' = 1, \ldots, n-1. \tag{90}
\]

The centered metric and Kähler forms now read

\[
g = d\xi_\mu d\xi_\mu + d\tilde{\rho}^t \tilde{G}_{\text{rel}} \cdot d\tilde{\rho} + \frac{d\tilde{\nu}}{4\pi} + \tilde{W}_{\text{rel}} \cdot d\tilde{\rho}^t \tilde{G}_{\text{rel}}^{-1} \left( \frac{d\tilde{\nu}}{4\pi} + \tilde{W}_{\text{rel}} \cdot d\tilde{\rho} \right),
\]

\[
\bar{\omega} = d\xi_0 \wedge d\tilde{\xi} - \frac{1}{2} d\tilde{\xi} \wedge d\tilde{\xi} + \frac{d\tilde{\nu}}{4\pi} + \tilde{W}_{\text{rel}} \cdot d\tilde{\rho}^t \wedge d\tilde{\rho} - \frac{1}{2} (\tilde{G}_{\text{rel}} d\tilde{\rho})^t \wedge d\tilde{\rho}. \tag{91}
\]

The first terms give the center of mass metric on \( \mathbb{R}^3 \times S^1 \), the other terms represent the non-trivial part of the metric. Both are toric hyperKähler, and have an \( SO(3) \) invariance corresponding to spatial rotations.
4.4 HyperKähler quotient construction

We follow the approach in [35] for BPS monopoles of type (1, 1, ..., 1) and consider the right hand side of eq. (67) as the natural metric on the space of caloron Nahm data $\hat{A}$

$$(g, \bar{\omega})_A = \left( <d\hat{A}^\dagger \otimes d\hat{A}> + 2 <d\hat{\lambda}^\dagger \otimes d\hat{\lambda}> \right).$$  \hfill (92)

One then notes that the group $\hat{G}$ of $U(1)$ gauge transformations on $\hat{S}^1$ acts triholomorphically on $\hat{A}$. The zero set of the associated moment map is formed by the set $\mathcal{N}$ of solutions to the Nahm equations, which after quotienting by the $U(1)$ gauge action $\hat{G}$ on the dual $S^1$ gives the moduli space of Nahm data. By virtue of eq. (67) this quotient is isometric to the caloron moduli space, $\mathcal{M} = \mathcal{N}/\hat{G}$. \hfill (93)

As both $\mathcal{N}$ and $\hat{G}$ are infinite dimensional, it is not obvious that this procedure is well-defined. However, using the gauge action we can restrict to those solutions $\mathcal{N}_0$ to the Nahm equations which have constant $\hat{A}_0(z)$ on the subintervals $(\mu_m, \mu_{m+1})$. As the Nahm equations force $\hat{A}_i(z)$ to be piecewise constant, there are $n$ quaternions specifying the Nahm connection, denoted by $y \in \mathbb{H}^n$. The singularities (or matching data) are described by $n$ complex two-component vectors $\zeta_m$, denoted by $\zeta \in \mathbb{C}^{n,2}$. Hence, $\mathcal{N}_0$ is a subset of the space $\hat{A}_0 = \mathbb{H}^n \times \mathbb{C}^{n,2}$ of possible piecewise constant data, which has metric and Kähler forms

$$(g, \bar{\omega}) = dy^\dagger \otimes Ndy + 2d\zeta^\dagger \otimes d\zeta,$$ \hfill (94)

as is natural from eq. (94). On $\hat{A}_0$, the gauge action $\hat{G}$ is restricted to the set $\hat{G}_0$ of gauge functions with piecewise linear and continuous log. These are determined by the values $h$ assumes at $z = \mu_m$. Under these gauge transformations, $\hat{A}$ and $\zeta$ change according to

$$\zeta_m \to e^{it_m} \zeta_m, \quad \psi \to \psi + 2t, \quad y \to y - \frac{1}{2\pi} N^{-1} St,$$ \hfill (95)

where $t = (h(\mu_1), \ldots, h(\mu_n)) \in \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and $\psi = (\psi_1, \ldots, \psi_n)/(4\pi\mathbb{Z})^n$ denotes the phases of $\zeta$. The lattices correspond to gauge transformations of type (14). Therefore the action of the restriction $\hat{G}_0$ of $\hat{G}$ on $\hat{A}_0$ is equivalent to an $\mathbb{R}^n$ action on $\mathbb{H}^n \times \mathbb{C}^{n,2}$. Thus we reduced the infinite dimensional hyperKähler quotient to a finite dimensional.

This technique was also used for the $(1, 1, \ldots, 1)$ monopole metric [35]. The metric on the moduli space of Nahm data can now be computed as a metric on a hyperKähler quotient of a finite dimensional euclidean space by a toric group action. To do this we follow [15]. From the metric and Kähler forms on $\hat{A}_0$, determined by inserting eqs. (6, 23) in eq. (94),

$$ds^2 = dy^\dagger Ndy + d\bar{\rho}^t V \cdot d\bar{\rho} + \left( \frac{d\psi}{4\pi} + \bar{W} \cdot d\bar{\rho} \right)^t V^{-1} \left( \frac{d\psi}{4\pi} + \bar{W} \cdot d\bar{\rho} \right),$$ \hfill (96)

the action (95) is seen to be triholomorphic. The moment map for this $\mathbb{R}^n$ action reads

$$\vec{\mu} \cdot \vec{\sigma} = -\frac{1}{4\pi} S^t(y - \bar{y}) - 3i \zeta^\dagger P \zeta,$$ \hfill (97)
where $P = (P_1, \ldots, P_n)^t$, and has a zero set $\bar{\mu}^{-1}(0)$ given by the solutions $\hat{A}$ corresponding to $\bar{\mu} = \tilde{S}^t \tilde{y}$. Therefore, the space of piecewise constant solutions to the Nahm data is $(\hat{A}, \zeta) \in \mathcal{N}_0 = \bar{\mu}^{-1}(0) \subset \hat{A}_0$. The moduli space of Nahm data is this set quotiented by the reduction of the gauge action in eq. (42), or equivalently $\mathbb{R}^n$. Hence

$$\mathcal{M} = \mathcal{N}/\hat{\mathcal{G}} = \mathcal{N}_0/\hat{\mathcal{G}}_0 = \bar{\mu}^{-1}(0)/\mathbb{R}^n.$$  \hfill (98)

The metric on $\bar{\mu}^{-1}(0)$ reads

$$ds^2 = d\bar{y}(SVS^t + N) d\bar{y} + \left( \frac{d\psi}{4\pi} + \tilde{W} \cdot S^t d\bar{y} \right) V^{-1} \left( \frac{d\psi}{4\pi} + \tilde{W} \cdot S^t d\bar{y} \right) + dy_0^2 N dy_0 \tag{99}$$

$$\bar{\omega} = \left( \frac{d\tau}{4\pi} + \tilde{W} \cdot d\bar{y} \right) \wedge d\bar{y} - \frac{1}{2} (Gd\bar{y})^t \wedge d\bar{y}. \tag{100}$$

The $n$ vector

$$\tau = \frac{S \psi}{4\pi} + N y_0, \tag{101}$$

is invariant under the $\mathbb{R}^n/(2\pi\mathbb{Z})^n$ action (92) and can therefore be used as coordinate on the quotient $\bar{\mu}^{-1}(0)/\mathbb{R}^n = \mathcal{M}$, together with $\bar{y}$. Cotangent vectors involving $d\psi$ have a vertical component, i.e. lie along the $\mathbb{R}^n$ fibre. The horizontal and vertical part of the metric are separated by inserting $y_0 = \frac{1}{4\pi} N^{-1} (\tau - S \psi)$ and completing the squares to obtain

$$ds^2 = d\bar{y}^t G d\bar{y} + \frac{1}{4\pi} N^{-1} \frac{d\tau}{4\pi} + \frac{S^t \tilde{W} V^{-1} \tilde{W} \cdot S^t d\bar{y}}{1 + S^t \tilde{W} V^{-1} S}$$

$$- \left( \frac{1}{4\pi} N^{-1} \frac{d\tau}{4\pi} - V^{-1} \tilde{W} S^t \cdot d\bar{y} \right)^t \frac{1}{1 + S^t \tilde{W} V^{-1} S} \left( \frac{1}{4\pi} N^{-1} \frac{d\tau}{4\pi} - V^{-1} \tilde{W} S^t \cdot d\bar{y} \right) + \varphi^t (V^{-1} + S^t N^{-1} S) \varphi, \tag{102}$$

where the one form $\varphi$ denotes the component along the $\mathbb{R}^n$ fibre

$$\varphi = \frac{d\psi}{4\pi} + \frac{1}{V^{-1} + S^t \tilde{W} V^{-1} S} \left( V^{-1} \tilde{W} S^t \cdot d\bar{y} \right)^t - \frac{1}{V^{-1} + S^t \tilde{W} V^{-1} S} \left( \frac{1}{4\pi} N^{-1} \frac{d\tau}{4\pi} \right). \tag{103}$$

Horizontal projecting to the metric on $\bar{\mu}^{-1}(0)/\mathbb{R}^n$ amounts to discarding the last term in eq. (102) and one obtains (after reorganising) the metric on the caloron moduli space $\mathcal{M}$ given in eq. (78). For the Kähler forms, this projection is generally not necessary: eq. (100) is precisely the Kähler form in eq. (79). This is a manifestation of the degeneracy of the Kähler forms along the gauge orbit, needed for the hyperKähler quotient to be well defined.

## 5 Instanton and monopole limits of the caloron

From the caloron metric, other toric hyperKähler manifolds can be obtained by taking suitable limits. For large $\mathcal{T}$ or equivalently all $\rho_m$ small, one expects the metric to approach the moduli space for $k = 1$ $SU(n)$ instantons on $\mathbb{R}^4$. To study this limit, we consider the centered metric eq. (94). For small $\rho_m$, the elements of the relative mass matrix $\tilde{M}$ in eq. (88) are dominated by the $\rho_m^{-1}$ terms in $\tilde{V}_{rel}$,

$$F_c^{-1} G (F_c^{-1})^t = \begin{pmatrix} \hat{G}_{rel} & \hat{V}_{rel} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{V}_{rel} \end{pmatrix}, \quad \rho_m \rightarrow 0, \quad m = 1, \ldots, n - 1, \quad \tag{104}$$
resulting in the asymptotic form for the non-trivial part of the metric and Kähler forms

\[
g_{\text{limit}} = d\tilde{\rho}^\dagger \tilde{V}_\text{rel} \cdot d\tilde{\rho} + \left( \frac{d\tilde{\nu}}{4\pi} + \tilde{\mathcal{V}}_\text{rel} \cdot d\tilde{\rho} \right)^\dagger \tilde{V}_\text{rel}^{-1} \left( \frac{d\tilde{\nu}}{4\pi} + \tilde{\mathcal{V}}_\text{rel} \cdot d\tilde{\rho} \right),
\]

\[
\tilde{\omega}_{\text{limit}} = \left( \frac{d\tilde{\nu}}{4\pi} + \tilde{\mathcal{V}}_\text{rel} \cdot d\tilde{\rho} \right)^\dagger \cdot d\tilde{\rho} - \frac{i}{2} (\tilde{V}_\text{rel} d\tilde{\rho})^\dagger \wedge d\tilde{\rho}.
\]

The caloron with trivial gauge holonomy has the same limiting metric, as follows directly from taking the limit \( \nu_1, \ldots, \nu_{n-1} \to 0, \nu_n \to 1 \) of the caloron relative mass matrix in eq. (90). The phase variables are now given by \( \nu_m = \tau_m + \ldots + \tau_{n-1} \in \mathbb{R}/(4\pi \mathbb{Z}) \), cf. eq. (85). The Kähler forms \( \tilde{\omega}_{\text{limit}} \) are closed, since the hyperKähler conditions (26) are satisfied

\[
\tilde{\nabla}_\rho \tilde{G}_\text{rel} = \tilde{\nabla}_\rho \times \tilde{\mathcal{V}}_\text{rel},
\]

hence the limiting metric for large \( T \) is hyperKähler. It is known as the Calabi metric.

This limit was discussed in [23] using indirect arguments. With the techniques presented in this paper, it is easy to prove explicitly that the limiting metric is indeed the metric for both the ordinary \( k = 1 \) \( SU(n) \) instantons on \( \mathbb{R}^4 \) and the calorons with trivial holonomy. It follows immediately when realising that the \( 4(n-1) \) dimensional Calabi space can be obtained as the hyperKähler quotient of \( \mathbb{H}^n \) by a \( U(1) \) action [13]. This quotient emerges naturally from both the construction of the charge one \( SU(n) \) instanton and the trivial holonomy caloron. First note that there is a one to one correspondence between the ADHM data of the \( k = 1 \) \( SU(n) \) instanton and the Nahm data of the trivial holonomy caloron in the \( \tilde{G} \) gauge with constant \( A_0(z) \). The latter are given in terms of \( (\xi, \zeta) \in \mathbb{H} \times \mathbb{C}^{n,2} \) as \( \tilde{A}(z) = 2\pi i \xi, \quad \lambda(z) = \delta(z) \zeta \) and directly translate into ADHM data \( \lambda = \zeta, \quad B = \xi \) for the instanton. With only one subinterval, the metric on the Nahm data now reduces to the expression for the instanton (8). Having restricted to constant \( A_0(z) \), the remaining transformations in \( \tilde{G}_0 \) leave \( \xi \) invariant, apart from confining \( \xi_0 \) to the circle through \( g(z) = \exp(2\pi ipz), p \in \mathbb{Z} \). For their action on the matching data only the \( U(1) \) formed by the values \( g(0) \) is relevant. Therefore, in both cases the nontrivial part of the moduli space is the quotient of \( \mathbb{C}^{n,2} \) (with \( (g, \bar{\omega}) = 2d\zeta^\dagger \otimes d\zeta \)) by the \( U(1) \) action

\[
\zeta_m \to e^{it} \zeta_m, \quad \psi_m \to \psi_m + 2t, \quad m = 1, \ldots, n, \quad t \in \mathbb{R}/(2\pi \mathbb{Z}).
\]

(Identifying \( \mathbb{C}^2 \) and \( \mathbb{H} \), this quotient is readily seen to be equivalent to that discussed in eq. (36) of [13]). The corresponding moment map, zero set and invariants are given by

\[
\tilde{\mu} = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \tilde{\rho}_m, \quad \sum_{m \in \mathbb{Z}/n\mathbb{Z}} \tilde{\rho} = 0, \quad \tilde{v}_m = \psi_m - \psi_n, \quad m = 1, \ldots, n-1.
\]

Expressing the metric on the zero set in terms of invariants and the terms involving \( d\psi_n \) describing the fibre part, one obtains [13] the Calabi metric in eq. (105).

The Calabi metric has an \( SU(n) \) triholomorphic isometry, reflecting the \( SU(n) \) gauge symmetry of the \( k = 1 \) instanton and trivial holonomy caloron. As explained in section 2 for the instanton, it emerges as the \( SU(n) \) acting on \( \zeta \) in eq. (107) on the left, commuting with \( U(1) \), and descending to the quotient. A direct calculation using a compensating gauge transformation gives the same result.
In [23, 25], it was explicitly shown from the action density that removing one of the constituent monopoles of the caloron to spatial infinity, $|\vec{y}_n| \to \infty$ turns it into a static selfdual $SU(n)$ solution, i.e. a monopole in the BPS limit. Indeed, this limit corresponds to the compactification length going to zero. The Nahm data suggest that the remnant is the $(1, 1, \ldots, 1)$ monopole. We will show indeed that the metric in this limit has the required form.

Removing a constituent is described by a hyperKähler quotient. Consider the $U(1)$ action that changes the phase of the $m$th monopole in the uncentered caloron
\[ \tau_m \to \tau_m + t, \quad t \in \mathbb{R}/(4\pi \mathbb{Z}). \] (109)
It is a triholomorphic isometry as follows from eqs. (78, 79). Its moment map $\vec{\mu}_{\text{fix}}$ is exactly the position of the $m$th monopole, $\vec{\mu}_{\text{fix}} = \vec{y}_m/(4\pi)$. Therefore, the metric on the quotient, the caloron moduli space with the $m$th constituent fixed, is hyperKähler irrespective of its position. For finite $|\vec{y}_m|$, the resulting metric on the quotient $\vec{\mu}_{\text{fix}}^{-1}(\vec{y}_m)/\mathbb{R}$ is complicated, and no longer $SO(3)$ symmetric. Removing the constituent, $|\vec{y}_m| \to \infty$, i.e. fixing it at spatial infinity, gives the hyperKähler metric of the remnant BPS monopole, with a simple form and $SO(3)$ symmetry restored.

The metric with the $n$th monopole far away, in which case $\rho^{-1}_1, \rho^{-1}_n \to 0$, reads
\[ (g, \vec{\omega}) = (g_n, \vec{\omega}_n) + (g_m, \vec{\omega}_m). \] (110)
Here the removed monopole is described by $g_n = \nu_n d\vec{y}_n^2 + \nu_n^{-1} d\tau_n^2$, and the remnant by
\[ g_m = d\vec{y}_m^t G_m d\vec{y}_m + (d\tau_m/4\pi + \vec{W}_m \cdot d\vec{y}_m) G_m^{-1} (d\tau_m/4\pi + \vec{W}_m \cdot d\vec{y}_m), \]
\[ \vec{\omega}_m = -\frac{1}{4}(G_m d\vec{y}_m)^t \wedge d\vec{y}_m + (d\tau_m/4\pi + \vec{W}_m \cdot d\vec{y}_m)^t \wedge d\vec{y}_m, \] (111)
where
\[ G_m = N_m + S_m V_m S_m^t, \quad \vec{W}_m = S_m \vec{W}_m S_m^t, \]
\[ V_m^{-1} = 4\pi \text{diag}(\rho_2, \ldots, \rho_{n-1}), \quad \vec{W}_m = \text{diag}(\vec{w}_2(\rho_2), \ldots, \vec{w}_{n-1}(\rho_{n-1}))/4\pi, \]
\[ N_m = \text{diag}(\nu_1, \ldots, \nu_{n-1}), \quad y_m = (y_1, \ldots, y_{n-1})^t, \quad \vec{\rho}_m = (\vec{\rho}_2, \ldots, \vec{\rho}_{n-1})^t, \quad \tau_m = (\tau_1, \ldots, \tau_{n-1})^t. \] (112)

More explicitly, the potential term in eq. (111) reads
\[ 4\pi S_m V_m S_m^t = \begin{pmatrix}
\frac{1}{\rho_2} & -\frac{1}{\rho_2} & \cdots & -\frac{1}{\rho_2} \\
-\frac{1}{\rho_2} & \frac{1}{\rho_2} + \frac{1}{\rho_3} & \cdots & -\frac{1}{\rho_3} \\
\cdots & \cdots & \cdots & \cdots \\
-\frac{1}{\rho_{n-2}} & \frac{1}{\rho_{n-2}} + \frac{1}{\rho_{n-1}} & \cdots & -\frac{1}{\rho_{n-1}} \\
\end{pmatrix} \in \mathbb{R}^{n-1,n-1}. \] (114)
The vector potential \( \vec{W}_m \) has a similar structure. The metric in eq. (111) is that of the uncentered \( SU(n) \) monopole of type \((1,1,\ldots,1)\). The calculation of the metric on its space of Nahm data was performed in \[35, 15\]. Details on the Nahm construction of the \((1,1,\ldots,1)\) monopole and a proof of its isometric property as well as an outline of the calculation of the metric can be found in the appendix.

To connect with \[27\], we have to center the monopole. We introduce

\[
F_m = \left( S_m, \frac{1}{\nu} N_m e_m \right) \in \mathbb{R}^{n-1,n-1}
\]  

where \( e_m = (1,\ldots,1) \in \mathbb{R}^{n-1} \) and \( \nu = \sum_{m=1}^{n-1} \nu_m \) denotes the mass of the monopole. The relative position variables \( \tilde{\rho}_m \) are reinstated and the center of mass \( \mathbb{R}^3 \) position is separated off using

\[
\tilde{y}_m = (F^t_m)^{-1} \left( \begin{array}{c}
\tilde{\rho}_m \\
\xi_m
\end{array} \right), \quad \xi_m = \frac{1}{\nu} \sum_{m=1}^{n-1} \nu_m \tilde{y}_m.
\]

The mass matrix in this basis is given by

\[
F^t_m N_m^{-1} F_m = \left( \begin{array}{c}
M_m^{-1} \\
\nu^{-1}
\end{array} \right), \quad M_m^{-1} = \left( \begin{array}{cccc}
\frac{1}{\nu_1} + \frac{1}{\nu_2} & -\frac{1}{\nu_2} & \frac{1}{\nu_2} & -\frac{1}{\nu_3} \\
-\frac{1}{\nu_2} & \frac{1}{\nu_1} + \frac{1}{\nu_3} & \frac{1}{\nu_2} & -\frac{1}{\nu_3} \\
\frac{1}{\nu_2} & \frac{1}{\nu_2} & \frac{1}{\nu_n-1} & \frac{1}{\nu_n-2} \\
-\frac{1}{\nu_3} & -\frac{1}{\nu_3} & \frac{1}{\nu_n-2} & \frac{1}{\nu_n-1}
\end{array} \right),
\]

\[
M_m = M^t_m, \quad (M_m)_{m,m'} = \nu^{-1} (\nu_1 + \cdots + \nu_m)(\nu_{m'+1} + \cdots + \nu_{n-1}), \quad \text{for } m' \geq m.
\]

Furthermore, alternative torus coordinates \( \chi_m = (\chi_1,\ldots,\chi_{n-2}) \) are introduced, as well as a global \( U(1) \) phase \( \xi_{0,m} \)

\[
\tau_m = F_m \left( \begin{array}{c}
\chi_m \\
\xi_0
\end{array} \right), \quad \xi_{0,m} = \sum_{m=1}^{n-1} \tau_m.
\]

In the new coordinates, the uncentered metric is the sum of the center of mass and relative metric

\[
g_m = \nu d\tilde{\xi}_m \cdot d\tilde{\xi}_m + \nu^{-1} d\xi_0^2 + g^c_m.
\]

where the nontrivial part

\[
g^c_m = d\tilde{\rho}_m \cdot (M_m + V_m) \cdot d\tilde{\rho}_m + \frac{d\chi_m}{4\pi} + \vec{W}_m \cdot d\tilde{\rho}_m)(M_m + V_m)^{-1}(\frac{d\chi_m}{4\pi} + \vec{W}_m \cdot d\tilde{\rho}_m)
\]

is the Lee-Weinberg-Yi metric \[27\]. It is of toric hyperKähler form. Thus we proved that the \((1,1,\ldots,1)\) monopole is a limit of the caloron, identifying the static remnant in \[24, 25\].

Finally, we note that the \((1,1,\ldots,1)\) monopole has only one magnetic winding, as explained in the introduction. It is opposite to the winding of the removed monopole, and hence, we can apply the reasoning in \[13\] explaining how the instanton charge arises also for \( SU(n) \) from braiding two monopoles \[23\].
6 Discussion

Since the metric describes the Lagrangian for adiabatic motion on the moduli space, it reflects the interactions of the monopole constituents. The constituent nature of the caloron solution, easily extracted from the action density, should therefore also be reflected in the metric. The action density of the \( k = 1 \) \( SU(n) \) caloron \[24\] is derived from eq. (15) employing Green’s function techniques and reads

\[
-\frac{i}{4} \text{Tr} F_{\mu \nu}^2 = -\frac{i}{4} \partial_{\mu}^2 \partial_{\nu}^2 \log \Psi.
\]  

(121)

Here the positive scalar potential \( \Psi \) is defined as

\[
\Psi(x) = \frac{i}{4} \text{tr} \prod_{m=1}^{n} \{ A_m \} - \cos(2\pi x_0),
\]  

(122)

where

\[
A_m = \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} c_m & s_m \\ s_m & c_m \end{pmatrix} \frac{1}{r_m}
\]  

given in terms of the center of mass radii \( r_m = |\vec{x} - \vec{y}_m| \) of the \( m \)th constituent monopole, \( c_m = \cosh(2\pi \nu_m r_m) \), \( s_m = \sinh(2\pi \nu_m r_m) \) and \( \prod_{m=1}^{n} A_m = A_n \cdots A_1 \). The energy density for the \((1,1,\ldots,1)\) monopole is obtained from it by sending the \( n \)th constituent to infinity, which gives \[25\]

\[
\tilde{\Psi}_m(\vec{x}) = \frac{1}{4} \text{tr} \left\{ \frac{1}{r_{n-1}} \begin{pmatrix} s_{n-1} & c_{n-1} \\ c_{n-1} & s_{n-1} \end{pmatrix} \prod_{m=1}^{n-2} A_m \right\} .
\]  

(125)

(see \[31\] for some special cases). These densities allow for an unambiguous identification of elementary BPS monopoles as constituents of coronons, and \((1,1,\ldots,1)\) monopoles, as in the limit where \( r_m \ll r_l \) for all \( l \neq m \) the action density approaches that of the single BPS monopole \[24\]. The corresponding limit in the uncentered metrics reveals

\[
ds_m^2 = \nu_m d\vec{y}_m \cdot d\vec{y}_m + \frac{1}{\nu_m} dr_m^2
\]  

(126)

for the part describing the \( m \)th constituent, as all interaction potentials approach zero with the other constituents far away. Eq. (126) is the flat metric on \( \mathbb{R}^3 \times S^1 \), the twofold cover of the moduli space for the elementary BPS monopole. Therefore the limit of the moduli space corresponding to all monopoles well separated- of the (cover of the) caloron moduli space can be seen as a product of elementary BPS monopole moduli spaces.

We obtained the metric for the \( k = 1 \) \( SU(n) \) caloron assuming symmetry breaking to the maximal torus \( U(1)^{n-1} \) with arbitrarily chosen holonomy eigenvalues \( \mu_m \). In the situation of non-maximal breaking, some of the eigenvalues of the holonomy become equal, resulting in some monopoles acquiring zero mass. The form of the relative mass matrices defined as inverses suggests that dramatic things happen when one or more of the constituents acquire zero mass. However, as is clear from the explicit forms of \( M, M_m \) in equations (71, 90, 112, 117), all limits can be taken smoothly. This assertion was explicitly checked for
the trivial holonomy caloron, with all but one monopoles having zero mass. Therefore one can study most efficiently all symmetry breaking patterns, both for \( k = 1 \) calorons and for monopoles of type \((1,1,\ldots,1)\), just by inserting the proper values for \( \mu_m \), rather than having to calculate the metric for each case separately. Consider, both for the caloron and for the \((1,1,\ldots,1)\) monopole, the situation of \( N - 1 \) monopoles turning massless

\[
\nu_K, \ldots, \nu_{K+N-2} = 0, \quad \mu_K = \ldots = \mu_{K+N-1},
\]

resulting in an enhanced residual symmetry to \( SU(N) \times U(1)^{n-N} \). The corresponding center of mass radii no longer appear in the expression for the action and energy densities \[24\], as follows from

\[
\prod_{m=1}^{n} A_m \rightarrow \left\{ \prod_{m' = K+N-1}^{n} A_{m'} \right\} \left( \begin{array}{cc} r_{K-1} & R_c \\ 0 & r_{K+N-1} \end{array} \right) \left( \begin{array}{cc} c_{K-1} & s_{K-1} \\ s_{K-1} & c_{K-1} \end{array} \right) \frac{1}{r_{K-1}} \prod_{m=1}^{K-2} A_m \right\}
\]

Here

\[
R_c = |\vec{\rho}_K| + \ldots + |\vec{\rho}_{K+N-1}| = \pi tr_2 \sum_{m=K}^{K+N-1} \zeta_{(m)}^\dagger \zeta_{(m)}
\]

(128)

denotes what is known in the monopole literature as the "non-abelian cloud" parameter \[28\]. It is seen from the right hand side of eq. (129) that it is \( SU(N) \) invariant. From the ADHM-Nahm construction \[28,29\], this \( SU(N) \) symmetry is seen to leave the holonomy invariant. It will descend to the quotient in the hyperKähler quotient construction of the metric, and therefore, the metric will be \( SU(N) \) invariant as well, much like in the case of the trivial holonomy caloron. As the explicit form of the metric can readily be found by inserting eq. (127) in the mass matrices (112, 71), it will not be given here. The \( SU(N) \) transformations mixes the positions of the massless monopoles, which therefore do not exist as individual particles. A way of seeing this physically is that the intrinsic length scales of the monopoles, proportional to their inverse masses, become infinitely large as their masses become small, so that they overlap and lose their indentities. This appearance of massless particles and infinite length scales illustrates a very general feature of systems near a transition to a more symmetric phase.

The fact that the \( SU(n+1) \) \((1,1,\ldots,1)\) monopole and the \( SU(n) \) \( k = 1 \) caloron both consist out of \( n \) constituent BPS monopoles in combination with the fact that the former can be obtained out of an \( SU(n+1) \) caloron, suggests a great similarity between their metrics. We consider the relevant situation for quantum chromodynamics, the \( SU(3) \) caloron. Removing one monopole to infinity gives the \( SU(3) \) monopole of type \((1,1)\). There remain two constituents, of masses proportional to \( \nu_1, \nu_2 \). The relative metric of the \((1,1)\) monopole is Taub-NUT with positive mass parameter.

\[
g_{TN} = U(\vec{\rho}) d\vec{\rho}^2 + U(\vec{\rho})^{-1} \left( \frac{d\psi}{4\pi} + \frac{\vec{w}(\vec{\rho})}{4\pi} \cdot d\vec{\rho} \right)^2, \quad U(\vec{\rho}) = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} + \frac{Q}{4\pi |\vec{\rho}|},
\]

(130)

\( \vec{\rho} \) denoting the separation of the constituents, \( Q = 1 \). The relative metric for the \( SU(2) \) caloron is also a Taub-NUT \[22, 23\]. (The metric obtained there checks with eq. (130) apart from the normalisation \( 4\pi^2 \), as \( \pi \rho^2, \Psi \) in \[23\] corresponds to \( |\vec{\tau}|, \nu \) in eq. (130)). However, the interaction strength, depending on the distance between the monopoles, for
the caloron is \( Q = 2 \), twice that of the \( SU(3) \) monopole. Both solitons can be considered as built out of two interacting constituent BPS monopoles, and have a four dimensional relative moduli space. Each matching point in the Nahm construction gives rise to an interaction between monopoles of distinct type, this is to be expected. The \( SU(3) \) \((1,1)\) monopole has one matching point, at \( z = \mu_2 \) whereas the \( SU(2) \) caloron has one additional at \( z = \mu_1 + 1 \) to close the circle, in the situation of two constituents equal to the other. In \cite{26} this was attributed to the fact that the constituent monopoles in the \( SU(3) \) \((1,1)\) case are charged with respect to different \( U(1) \), whereas for the caloron, they are oppositely charged with respect to the same \( U(1) \), generated by \( \vec{\omega} \cdot \vec{\tau} \).

In conclusion, we have presented results for the metric on moduli spaces in a unified description that incorporates instantons, calorons and monopoles.

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Appendix

The \((1,1,\ldots,1)\) monopole

The Nahm construction of the \((1,1,\ldots,1)\) monopole is similar to that of the \( k = 1 \) \( SU(n) \) caloron. The main difference is that the circle is replaced by the interval \([\mu_1, \mu_{n-1}]\). For the \((1,1,\ldots,1)\) monopole, the singularities reside at \( z = \mu_2, \ldots, \mu_n - 1 \) \cite{37,21,44}. Like for the caloron we introduce \( \Delta^\dagger = (\lambda^\dagger(z), \frac{1}{2\pi i} \hat{D}^\dagger_x(z)) \),

\[
\lambda(z) = \sum_{m=2}^{n-1} \delta(z - \mu_m)\zeta_m, \quad \hat{D}_x(z) = \sigma_\mu \hat{D}_x^\mu(z) = \frac{d}{dz} + \hat{A}(z) - 2\pi ix, \quad (131)
\]

where \( \hat{A}(z) \) is now defined on \([\mu_1, \mu_n]\). The Nahm construction is performed in terms of the normalised zero modes \( v(x) \) of \( \Delta(x) \)

\[
v(x) = \left( \begin{array}{c} s_x^\dagger \hat{\psi}_x(z) \end{array} \right), \quad \frac{1}{2\pi i} \hat{D}_x^\dagger(z) \hat{\psi}_x^m(z) + \sum_{m'=2}^{n-1} \delta(z - \mu_{m'})\zeta_{m'}^\dagger s_{x m'} = 0, \quad (132)
\]

\[
v^\dagger(x)v(x) = s_x s_x + \int_{\mu_1}^{\mu_n} dz \hat{\psi}_x^\dagger(z) \hat{\psi}_x(z) = 1_n, \quad (133)
\]

where \( \hat{\psi}_x(z) = (\hat{\psi}_x^1(z), \ldots, \hat{\psi}_x^n(z)) \) contains the \( n \) two-spinors defined on the interval \([\mu_1, \mu_n]\), and \( s \in \mathbb{C}^{n-1,n} \). (The equation for \( \hat{\psi}_x^m(z) \) is readily seen to have \( n \) solutions for fixed \( s_x \) \cite{21}). Though the monopole is a static solution, it is preferable to have \( x_0 \) included as a dummy variable, the \( x_0 \) dependence trivially being implemented by \( v(x) = e^{2\pi i x_0} v(\bar{x}) \), so as to write concisely

\[
A_\mu(x) = (\Phi(x), \bar{A}(x)) = v^\dagger(x) \partial_\mu v(x) \quad (134)
\]
with the inner product defined as in eq. (133). Performing all monopole calculations in terms of $\Delta(x)$ and $v$, the caloron formalism can be copied. In particular, it follows that for eq. (134) to be selfdual, $\Delta^\dagger(x)\Delta(x)$ should commute with the quaternions. This is equivalent to the monopole Nahm equation

$$\frac{d}{dz} \hat{A}_j(z) = 2\pi i \sum_{m=2}^{n-1} \delta(z - \mu_m) \rho^j_m. \quad (135)$$

Its solution $\hat{A}_j(z)$ can be written in terms of $n - 1$ position vectors $\vec{y}_m$, $\vec{\rho}_m = \vec{y}_m - \vec{y}_{m-1}$, comprised in $\vec{y}_m = (\vec{y}_1, \ldots, \vec{y}_n)^t$,

$$\vec{\rho}_m = S^t_m \vec{y}_m \quad (136)$$

implying

$$\hat{A}_j = 2\pi i \sum_{m=1}^{n-1} \chi_{[\mu_m, \mu_{m+1}]}(z) \vec{y}_m. \quad (137)$$

Like for the caloron, there is a gauge action on the Nahm data

$$\hat{A}(z) \to \hat{A}(z) + i \frac{d}{dz} h(z), \quad \zeta_m \to \zeta_m e^{i \theta(\mu_m)}, \quad m = 2, \ldots, n - 1 \quad (138)$$

with gauge group $\hat{G}_m = \{ g(z) | g : z \to e^{-i\theta(z)} \in U(1), g(\mu_1) = g(\mu_n) = 1 \}$. The condition at the endpoints is required for $id/dz$ to be hermitean on the space of gauge functions. Hence, for the monopole $\hat{G}_m = \{ g(z) | g : z \to e^{-i\theta(z)} \in U(1), g(\mu_1) = g(\mu_n) = 1 \}$. The $G_m$ action can be used to set $\hat{A}_0(z)$ constant, and to undo the $U(1)$ phase ambiguities in relating $\zeta_m$ to $\vec{\rho}_m, m = 2, \ldots, n - 1$, hence $\zeta_m$ can be considered to have fixed phase. The monopole Nahm data can then be expressed in terms of $n - 1$ quaternions

$$\hat{A}_m = (\hat{A}_1, \ldots, \hat{A}_{n-1})^t = 2\pi (N_m^{-1} \vec{r}_m + \vec{y}_m \cdot \vec{\sigma}), \quad (139)$$

$i \hat{A}_{m,m}$ denoting the value $\hat{A}(z)$ takes on $(\mu_m, \mu_{m+1})$.

In the gauge with constant $\hat{A}_0(z)$, the Green’s function $f_x$ in the monopole Nahm construction is the solution to the differential equation

$$\left\{ \left( \frac{1}{2\pi i} \frac{d}{dz} - x_0 \right)^2 + \sum_{m=2}^{n-1} \chi_{[\mu_m, \mu_{m+1}]}(z) r^2_m + \frac{1}{2\pi} \sum_{m=2}^{n-1} \delta(z - \mu_m) |\vec{y}_m - \vec{y}_{m-1}| \right\} \hat{f}_x(z, z') = \delta(z - z'). \quad (140)$$

whereas transformations to other gauges are realised by

$$\hat{f}_x(z, z') \to g(z) \hat{f}_x(z, z') g(z)^*, \quad g(z) \in \hat{G}_m. \quad (141)$$

The boundary condition for the monopole Green’s function is determined by the requirement that $i \frac{d}{dz}$ be a hermitean operator, therefore the eigenfunctions of the left hand side of eq. (140) vanish in the endpoints. This imposes by standard Sturm-Liouville theory

$$\hat{f}(\mu_1, z') = \hat{f}(\mu_n, z') = 0 \quad (142)$$
for the Green’s function. This boundary condition is automatically satisfied when obtaining the monopole Green’s function from the caloron Green’s function, taking the limit $|\vec{y}_n| \to \infty$. The $x_0$ dependence of the monopole Green’s function is trivial
\[
\hat{f}_x(z, z') = e^{2\pi i x_0 (z - z')} \hat{f}_x(z, z').
\]

The metric on the monopole moduli space is determined in terms of the $L_2$ norm of gauge orthogonal solutions $Z_m$ to the linearised Bogomol’nyi equations. With $A_0$ identified as the Higgs field, and assuming all fields and zero modes being static, the conditions for a tangent vector to the monopole moduli space are identical to those for on the tangent vector to an instanton moduli space, hence $Z_m$ satisfies
\[
D^{ad\dagger}(A)Z_m = 0,
\]
where $\partial_0$ acts trivially, but is kept to make later derivations more transparent. Metric and Kähler forms read
\[
(g, \bar{\omega})(Z_m, Z_m') = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} d^3x \text{Tr} Z_m^\dagger(\vec{x}) Z_m'(\vec{x}).
\]

The formalism to compute the metric is copied from the caloron case. A tangent vector to the monopole moduli space is given by
\[
Z_{m\mu}(\vec{x}) = \int_{[\mu_1, \mu_2]^2} dz dz' \left( \sum_{m'=2}^{n-1} s_x^\dagger \hat{c}_{m'} \delta(z - \mu_{m'}) + \hat{\bar{\psi}}_x(z) \right) \hat{f}_x(z, z') \sigma_{\mu}^\dagger \bar{\psi}_x(z') - h.c.
\]
in terms of a tangent vector to the moduli space of monopole Nahm data
\[
C = \left( \frac{\hat{c}(z)}{\hat{Y}(z)} \right), \quad \hat{c}(z) = \sum_{m=2}^{n-1} \hat{c}_m \delta(z - \mu_m),
\]
satisfying the deformation and gauge orthogonality equations
\[
\frac{d}{dz} \hat{Y}_i(z) = -i\pi \text{tr}_2 \sum_{m=2}^{n-1} \sigma_{m}^\dagger \hat{c}_m + \hat{c}_m^\dagger \sigma_{m} \delta(z - \mu_m),
\]
\[
\frac{d}{dz} \hat{Y}_0(z) = -i\pi \sum_{m=2}^{n-1} \text{tr}_2(\hat{c}_m^\dagger \hat{c}_m - \hat{c}_m^\dagger \hat{c}_m) \delta(z - \mu_m).
\]

To derive the analogue for monopoles of Corrigan’s formula we trade each matrix multiplication in eq. (58) for an integration over $[\mu_1, \mu_2]$ or an inner product of type (133) and use the trivial $x_0$ dependence of $v(x)$ and $f_x(z, z')$ for the monopole to obtain
\[
\text{Tr} Z_m^\dagger(x) Z_m(x) = -\nabla^2 \int_{[\mu_1, \mu_2]^2} dz \left( [\hat{Y}^\dagger(z) \hat{Y}(z) + \hat{c}^\dagger(z) < \hat{c} | \hat{f}_x(z, z)] \right)
\]
\[
+ \frac{i}{4} \nabla^2 \int_{[\mu_1, \mu_2]^2} dz dz' \left( [\hat{C}(z) + \hat{Y}(z)] \hat{f}_x(z, z') [\hat{Y}^\dagger(z') + \hat{C}(z')] \hat{f}_x(z', z) \right),
\]
with $\hat{C}(z) = \sum_{m=2}^{n-1} \hat{c}_m^\dagger \delta(z - \mu_m)$, $\hat{Y}_x(z) = (2\pi i)^{-1} \hat{Y}^\dagger(z) \hat{D}_x(z)$. The monopole metric is evaluated from eqs. (143, 149) by partial integration, along the lines of the derivation in
The monopole Green’s function \( f_x(z, z') \) behaves as in eq. (66). Thus we arrive at the isometric property of the Nahm construction for \((1, 1, \ldots, 1)\) monopoles,

\[
(g, \vec{\omega})_M(Z_m, Z_m') = \text{Tr} \left( < \hat{\gamma}^\dagger \gamma' > + 2 < \hat{c}^\dagger > < \hat{c}' > \right), \quad < H > = \int_{[\mu_1, \mu_n]} H(z) dz. \quad (150)
\]

An infinitesimal gauge transformation \( \delta \hat{X}(z) \) is applied to obtain gauge orthogonality of the tangent vector \( C \)

\[
\dot{c}(z) = \sum_{m=2}^{n-1} \delta(z - \mu_m) \dot{c} = \sum_{m=2}^{n-1} \delta(z - \mu_m) \left( \delta \zeta_m + i \zeta_m \delta \hat{X}(\mu_m) \right), \quad (151)
\]

\[
\dot{\gamma}(z) = i \sum_{m=1}^{n-1} \chi[\mu_m, \mu_{m+1}] \dot{\gamma}_m = \frac{1}{2 \pi i} \left( \delta \hat{A}(z) + i \frac{d}{dz} \delta \hat{X}(z) \right).
\]

It vanishes in the endpoints \( z = \mu_1, z = \mu_n \) and satisfies

\[
\frac{1}{2 \pi} \frac{d^2 \delta \hat{X}(z)}{dz^2} + 2 \delta \hat{X}(z) \sum_{m=2}^{n-1} \delta(z - \mu_m) |\vec{\rho}_m| = \sum_{m=2}^{n-1} \delta(z - \mu_m) \left[ \frac{d \tau_m}{4 \pi \nu_m} - \frac{d \tau_{m-1}}{4 \pi \nu_{m-1}} - |\vec{\rho}_m| \vec{w}_m(\vec{\rho}_m) \cdot d\vec{\rho}_m \right]. \quad (152)
\]

Therefore, it is piecewise linear and fixed by \( \delta \hat{X} = (\delta \hat{X}_2, \ldots, \delta \hat{X}_{n-1})^t \), \( \dot{\hat{X}}_m = \delta \hat{X}(\mu_m), m = 2, \ldots, n-1 \) where

\[
\frac{1}{2 \pi} (S_m^t N_{m-1} S_m + V_{m-1}) \delta \hat{X} = (S_m^t N_{m-1} \frac{d \tau_m}{4 \pi} - V_{m-1} \vec{W}_m S_m^t \cdot d \vec{\gamma}_m), \quad (153)
\]

(see eqs. (112, 113) for definitions). With the compensating gauge function found, the remaining manipulations to retrieve the uncentered monopole metric in eq. (111) from eqs. (112, 113) differ only in the \( m \) label and the dimensions of the matrices from those in section 4.4 and are therefore not repeated here.

To compute the metric using the hyperKähler quotient construction we follow and summarise the reasoning in [15, 16] and section 4.4. We have to find the metric on \( N_m / \hat{G}_m \), where \( N_m \) is the subset of the space \( \hat{A}_m \) of monopole Nahm data containing the solutions to the Nahm equations. Making use of the \( U(1) \) gauge symmetry for monopole in eq. (138), we can restrict ourselves to piecewise constant \( \hat{A}(z) \), characterised by \( n-1 \) quaternions corresponding to its values on the subintervals. Together with the \( n-2 \) complex two vectors giving the matching data, form the space \( \hat{A}_{0m} = \mathbb{H}^{n-1} \times \mathbb{C}^{n-2,2} \ni (y_m, \zeta_m) \). This space has natural metric and Kähler forms

\[
(g, \vec{\omega}) = dy_m^t N_m \otimes dy_m + 2 d\zeta_m^t \otimes d\zeta_m \quad (154)
\]

The set of piecewise constant solutions to the Nahm equations form \( N_{0,m} \), which is a subset of \( \hat{A}_{0m} \). The vector part of a piecewise constant solution to the monopole Nahm equation (i.e.\( N_{m,0} \)) is fixed by eq. (136). We introduce the phases of \( \zeta_m \) as \( \psi_m = (\psi_2, \ldots, \psi_{n-1})^t \).

Having gauge fixed to constant \( \hat{A}(z) \), the residual \( U(1) \) gauge symmetry consists of gauge functions having piecewise linear and continuous logarithms, which vanish in the endpoints \( z = \mu_1 \) and \( z = \mu_n \). This results in an \( \mathbb{R}^{n-2} \) action on \( \hat{A}_{0m} \), characterised by

\[
y_m \rightarrow y_m - \frac{1}{2 \pi} N_m^{-1} S_m t_m, \quad \psi_m \rightarrow \psi_m + 2 t_m, \quad t_m \in \mathbb{R}^{n-2}, \quad (155)
\]
with moment map, zero set and invariants given by

\[ \tilde{\mu}_m = -\frac{1}{2\pi} S^t_m \tilde{y}_m + \frac{\tilde{p}_m}{2\pi}, \quad \tilde{p}_m = S^t_m \tilde{y}_m, \quad \tau_m = 4\pi N_m y_{0m} + S_m \psi_m. \tag{156} \]

Having established a suitable notation, the algebra to obtain the metric and Kähler forms for the uncentered monopole in eq. (111) is now nearly identical to the hyperKähler quotient construction of the uncentered caloron metric, and one readily retrieves eq. (111). Actually, one only has to insert the \( m \) labels at appropriate places, just realising that the dimensionalities of the objects are slightly different.

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