Ergodic Convergence Rates of Markov Processes—Eigenvalues, Inequalities and Ergodic Theory

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Abstract

This paper consists of four parts. In the first part, we explain what eigenvalues we are interested in and show the difficulties of the study on the first (non-trivial) eigenvalue through examples. In the second part, we present some (dual) variational formulas and explicit bounds for the first eigenvalue of Laplacian on Riemannian manifolds or Jacobi matrices (Markov chains). Here, a probabilistic approach—the coupling methods is adopted. In the third part, we introduce recent lower bounds of several basic inequalities; these are based on a generalization of Cheeger’s approach which comes from Riemannian geometry. In the last part, a diagram of nine different types of ergodicity and a table of explicit criteria for them are presented. These criteria are motivated by the weighted Hardy inequality which comes from Harmonic analysis.

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I. Introduction

We will start by explaining what eigenvalues we are interested in.

1.1 Definition. Consider a birth-death process with a state space $E = \{0, 1, 2, \ldots, n\}$ ($n \leq \infty$) and an intensity matrix $Q = (q_{ij})$: $q_{k,k-1} = a_k > 0$ ($1 \leq k \leq n$), $q_{k,k+1} = b_k > 0$ ($0 \leq k \leq n-1$), $q_{k,k} = -(a_k + b_k)$, and $q_{ij} = 0$ for other $i \neq j$.

Since the sum of each row equals 0, we have $Q1 = 0 = 0 \cdot 1$. This means that the $Q$-matrix has an eigenvalue 0 with an eigenvector 1. Next, consider the finite case of $n < \infty$. Then, the eigenvalues of $-Q$ are discrete: $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n$. We are interested in the first (non-trivial) eigenvalue $\lambda_1 = \lambda_1 - \lambda_0$ (also called spectral gap of $Q$). In the infinite case ($n = \infty$), $\lambda_1$ can be 0. Certainly, one can

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consider a self-adjoint elliptic operator in \( \mathbb{R}^d \), the Laplacian \( \Delta \) on manifolds, or an infinite-dimensional operator as in the study of interacting particle systems.

1.2 Difficulties. To get a concrete feeling about the difficulties of this topic, let us first look at the following examples with a finite state space. When \( E = \{0, 1\} \), it is trivial that \( \lambda_1 = a_1 + b_0 \). The result is nice because when either \( a_1 \) or \( b_0 \) increases, so does \( \lambda_1 \). When \( E = \{0, 1, 2\} \), we have four parameters \( b_0, b_1, a_1, a_2 \) and
\[
\lambda_1 = 2^{-1} \left[ a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1b_1} \right].
\]
When \( E = \{0, 1, 2, 3\} \), we have six parameters: \( b_0, b_1, b_2, a_1, a_2, a_3 \). In this case, the expression for \( \lambda_1 \) is too lengthy to write. The roles of the parameters are inter-related in a complicated manner. Clearly, it is impossible to compute \( \lambda_1 \) explicitly when the size of the matrix is greater than five.

Next, consider the infinite state space \( E = \{0, 1, 2, \cdots\} \). Denote the eigenfunction of \( \lambda_1 \) by \( g \) and the degree of \( g \) by \( D(g) \) when \( g \) is polynomial. Three examples of the perturbation of \( \lambda_1 \) and \( D(g) \) are listed in Table 1.1.

| \( b_i (i \geq 0) \) | \( a_i (i \geq 1) \) | \( \lambda_1 \) | \( D(g) \) |
|----------------------|----------------------|-------------|-------------|
| \( i + c (c > 0) \) | 2i | 1 | 1 |
| \( i + 1 \) | 2i + 3 | 2 | 2 |
| \( i + 1 \) | \( 2i + (4 + \sqrt{2}) \) | 3 | 3 |

Table 1.1 Three examples of the perturbation of \( \lambda_1 \) and \( D(g) \)

The first line is the well known linear model for which \( \lambda_1 = 1 \), independent of the constant \( c > 0 \), and \( g \) is linear. Keeping the same birth rate, \( b_i = i + 1 \), changes the death rate \( a_i \) from \( 2i \) to \( 2i + 3 \) (resp. \( 2i + 4 + \sqrt{2} \)), which leads to the change of \( \lambda_1 \) from one to two (resp. three). More surprisingly, the eigenfunction \( g \) is changed from linear to quadratic (resp. triple). For the other values of \( a_i \) between \( 2i, 2i + 3 \) and \( 2i + 4 + \sqrt{2} \), \( \lambda_1 \) is unknown since \( g \) is non-polynomial. As seen from these examples, the first eigenvalue is very sensitive. Hence, in general, it is very hard to estimate \( \lambda_1 \).

In the next section, we find that this topic is studied extensively in Riemannian geometry.

II. New variational formula for the first eigenvalue

2.1 Story of estimating \( \lambda_1 \) in geometry. At first, we recall the study of \( \lambda_1 \) in geometry.

Consider Laplacian \( \Delta \) on a compact Riemannian manifold \( (M, g) \), where \( g \) is the Riemannian metric. The spectrum of \( \Delta \) is discrete: \( \cdots \leq -\lambda_2 \leq -\lambda_1 < -\lambda_0 = 0 \) (may be repeated). Estimating these eigenvalues \( \lambda_k \) (especially \( \lambda_1 \)) is very important in modern geometry. As far as we know, five books, excluding those books on general spectral theory, have been devoted to this topic: Chavel (1984), Bérard (1986), Schoen and Yau (1988), Li (1993) and Ma (1993). For a manifold \( M \), denote its dimension, diameter and the lower bound of Ricci curvature by \( d, D \),
and $K$ (Ricci$_M \geq Kg$), respectively. We are interested in estimating $\lambda_1$ in terms of these three geometric quantities. It is relatively easy to obtain an upper bound by applying a test function $f \in C^1(M)$ to the classical variational formula:

$$\lambda_1 = \inf \left\{ \int_M \|\nabla f\|^2 dx : f \in C^1(M), \int f dx = 0, \int f^2 dx = 1 \right\}, \quad (2.0)$$

where “$dx$” is the Riemannian volume element. To obtain the lower bound, however, is much harder. In Table 2.1, we list eight of the strongest lower bounds that have been derived in the past, using various sophisticated methods.

| Author(s) | Lower bound |
|-----------|-------------|
| A. Lichnerowicz (1958) | $\frac{d}{d-1} K, \ K \geq 0$. (2.1) |
| P. H. Béard, G. Besson & S. Gallot (1985) | $d \left( \int_0^{\pi^2/d} \cos (d-1) t^2 dt \right)^{2/d}, \ K = d - 1 > 0$. (2.2) |
| P. Li & S. T. Yau (1980) | $\pi^2 \left[ \frac{\alpha}{D^2} \right], \ K \geq 0$. (2.3) |
| J. Q. Zhong & H. C. Yang (1984) | $\pi^2 D^2, \ K \geq 0$. (2.4) |
| P. Li & S. T. Yau (1980) | $\frac{1}{D^2(d-1) \exp \left[ \frac{1 + \sqrt{1 + 16\alpha^2}}{2} \right]}, \ K \leq 0$. (2.5) |
| K. R. Cai (1991) | $\frac{\pi^2}{D^2} \exp \left[ \frac{\alpha}{D^2} \right], \ K \leq 0$. (2.6) |
| H. C. Yang (1989) & F. Jia (1991) | $\frac{\pi^2}{2 D^2} e^{-\alpha'}, \ K \leq 0$. (2.7) |
| H. C. Yang (1989) & F. Jia (1991) | $\frac{\pi^2}{2 D^2} e^{-\alpha'}$, if $2 \leq d \leq 4, \ K \leq 0$. (2.8) |

Table 2.1 Eight lower bounds of $\lambda_1$

In Table 2.1, the two parameters $\alpha$ and $\alpha'$ are defined as $\alpha = D \sqrt{\frac{1}{K}[(d-1)/2]}$ and $\alpha' = D \sqrt{\frac{1}{K}[(d-1) \lor 2)/2]}$. Among these estimates, five ((2.1), (2.2), (2.4), (2.6) and (2.7)) are sharp. The first two are sharp for the unit sphere in two or higher dimensions but fail for the unit circle; the fourth, the sixth, and the seventh are all sharp for the unit circle. As seen from this table, the picture is now very complete, due to the efforts of many geometers in the past 40 years. Our original starting point is to learn from the geometers and to study their methods, especially the recent new developments. In the next section, we will show that one can go in the opposite direction, i.e., studying the first eigenvalue by using probabilistic methods. Exceeding our expectations, we find a general formula for the lower bound.

2.2 New variational formula. Before stating our new variational formula, we introduce two notations:

$$C(r) = \cosh^{d-1} \left( \frac{r}{2} \sqrt{\frac{-K}{d-1}} \right), \ r \in (0, D). \quad \mathcal{F} = \{ f \in C[0, D] : f > 0 \text{ on } (0, D) \}.$$
Here, we have used all the three quantities: the dimension \(d\), the diameter \(D\), and the lower bound \(K\) of Ricci curvature.

**Theorem 2.1 [General formula] (Chen & Wang (1997a)).**

\[
\lambda_1 \geq \sup_{f \in \mathcal{F}} \inf_{r \in (0, D)} \frac{4f(r)}{\int_0^r C(s)^{-1}ds \int_0^D C(u)f(u)du} =: \xi_1. \tag{2.9}
\]

The new variational formula has its essential value in estimating the lower bound. It is a dual of the classical variational formula in the sense that “inf” in (2.0) is replaced by “sup” in (2.9). The classical formula can be traced to Lord S. J. W. Rayleigh (1877) and E. Fischer (1905). Noticing that these two formulas (2.0) and (2.9) look very different, which explains that why such a formula (2.9) has never appeared before. This formula can produce many new lower bounds. For instance, the one corresponding to the trivial function \(f \equiv 1\) is non-trivial in geometry. Applying the general formula to the test functions \(\sin(\alpha r)\) and \(\cosh^{d-1}(\alpha r)\sin(\beta r)\) with \(\alpha = D\sqrt{|K|/(d-1)/2}\) and \(\beta = \pi/(2D)\), we obtain the following:

**Corollary 2.2 (Chen & Wang (1997a)).**

\[
\begin{align*}
\lambda_1 &\geq \frac{dK}{d-1} \left(1 - \cos^d \left[\frac{D}{2} \sqrt{\frac{K}{d-1}}\right]\right)^{-1}, \quad d > 1, \quad K \geq 0, \quad (2.10) \\
\lambda_1 &\geq \frac{\pi^2}{D^2} \sqrt{1 - \frac{2D^2K}{\pi^4} \cosh^{1-d} \left[\frac{D}{2} \sqrt{\frac{-K}{d-1}}\right]}, \quad d > 1, \quad K \leq 0. \quad (2.11)
\end{align*}
\]

Applying this formula to some very complicated test functions, we can prove the following result:

**Corollary 2.3 (Chen, Scacciatelli and Yao (2002)).**

\[
\lambda_1 \geq \frac{\pi^2}{D^2} + K/2, \quad K \in \mathbb{R}. \tag{2.12}
\]

The corollaries improve all the estimates (2.1)—(2.8). Especially, (2.10) improves (2.1) and (2.2), (2.11) improves (2.7) and (2.8), and (2.12) improves (2.3) and (2.6). Moreover, the linear approximation in (2.12) is optimal in the sense that the coefficient 1/2 of \(K\) is exact.

A test function is indeed a mimic of the eigenfunction, so it should be chosen appropriately in order to obtain good estimates. A question arises naturally: does there exist a single representative test function such that we can avoid the task of choosing a different test function each time? The answer is seemingly negative since we have already seen that the eigenvalue and the eigenfunction are both very sensitive. Surprisingly, the answer is affirmative. The representative test function, though very tricky to find, has a rather simple form: \(f(r) = \sqrt{\int_0^r C(s)^{-1}ds}\). This is motivated from the study of the weighted Hardy inequality, a powerful tool in harmonic analysis (cf. Muckenhoupt (1972), Opic and Kufner (1990)).
Corollary 2.4 (Chen (2000)). For the lower bound $\xi_1$ of $\lambda_1$ given in Theorem 2.1, we have

\[ 4\delta^{-1} \geq \xi_1 \geq \delta^{-1}, \quad \text{where} \quad \delta = \sup_{r \in (0, D)} \left( \int_0^r C(s)^{-1} ds \right) \left( \int_r^D C(s) ds \right), \quad C(s) = \cosh^{d-1} \left[ \frac{s \sqrt{-K}}{2 d - 1} \right]. \tag{2.13} \]

Theorem 2.1 and its corollaries are also valid for manifolds with a convex boundary endowed with the Neumann boundary condition. In this case, the estimates (2.1)–(2.8) are conjectured by the geometers to be correct. However, only the Lichnerowicz’s estimate (2.1) was proven by J. F. Escobar in 1990. The others in (2.2)–(2.8) and furthermore in (2.10)–(2.13) are all new in geometry.

On the one hand, the proof of this theorem is quite straightforward, based on the coupling introduced by Kendall (1986) and Cranston (1991). On the other hand, the derivation of this general formula requires much effort. The key point is to find a way to mimic the eigenfunctions. For more details, refer to Chen (1997).

Applying similar proof techniques to general Markov processes, we also obtain variational formulas for non-compact manifolds, elliptic operators in boundary endowed with the Neumann boundary condition. In this case, the estimates (2.1)–(2.8) are conjectured by the geometers to be correct. However, only the Lichnerowicz’s estimate (2.1) was proven by J. F. Escobar in 1990. The others in (2.2)–(2.8) and furthermore in (2.10)–(2.13) are all new in geometry.

To conclude this part, we return to the matrix case introduced at the beginning of the paper.

2.3 Birth-death processes. Let $b_i > 0 (i \geq 0)$ and $a_i > 0 (i \geq 1)$ be the birth and death rates, respectively. Define $\mu_0 = 1$, $\mu_i = b_0 \cdots b_i/a_1 \cdots a_i (i \geq 1)$. Assume that the process is non-explosive:

\[ \sum_{k=0}^{\infty} (b_k \mu_k)^{-1} \sum_{i=0}^{k} \mu_i = \infty \quad \text{and moreover} \quad \mu = \sum \mu_i < \infty. \tag{2.14} \]

The corresponding Dirichlet form is $D(f) = \sum_i \pi_i b_i (f_i+1 - f_i)^2$, $D(D) = \{ f \in L^2(\pi) : D(f) < \infty \}$. Here and in what follows, only the diagonal elements $D(f)$ are written, but the non-diagonal elements can be computed from the diagonal ones by using the quadrilateral role. We then have the classical formula $\lambda_1 = \{ D(f) : \pi(f) = 0, \pi(f^2) = 1 \}$. Define $T' = \{ f : f_0 = 0, \text{there exists } k : 1 \leq k \leq \infty \text{ so that } f_i = f_{i,k} \text{ and } f \text{ is strictly increasing in } [0,k] \}$, $T'' = \{ f : f_0 = 0, f \text{ is strictly increasing} \}$, and $I_i(f) = [\mu_i b_i (f_i+1 - f_i)]^{-1} \sum_{j \geq i+1} \mu_j f_j$. Let $\bar{f} = f - \pi(f)$. Then we have the following results:

Theorem 2.5 (Chen (1996, 2000, 2001))$^1$. Under (2.14), we have

1. Dual variational formula. $\inf_{f \in T'} \sup_{i \geq 1} I_i(\bar{f})^{-1} = \lambda_1 = \sup_{f \in T''} \inf_{i \geq 0} I_i(\bar{f})^{-1}.$

$^1$Due to the limitation of the space, the most of the author’s papers during 1993–2001 are not listed in References, the readers are urged to refer to [11].
(2) Explicit estimate. \( \mu \delta^{-1} \geq \lambda_1 \geq (4\delta)^{-1} \), where \( \delta = \sup_{i \geq 1} \sum_{j \leq i-1} (\mu_j b_j)^{-1} \sum_{j \geq i} \mu_j \).

(3) Approximation procedure. There exist explicit sequences \( \eta'_n \) and \( \eta''_n \) such that

\[
\eta'_n - 1 \geq \lambda_1 \geq \eta''_n - 1 \geq (4\delta)^{-1}.
\]

Here the word “dual” means that the upper and lower bounds are interchangeable if one exchanges “sup” and “inf”. With slight modifications, this result is also valid for finite matrices, refer to Chen (1999).

### III. Basic inequalities and new forms of Cheeger’s constants

#### 3.1 Basic inequalities.

We now go to a more general setup. Let \((E, \mathcal{E}, \pi)\) be a probability space satisfying \( \{(x, x) : x \in E\} \in \mathcal{E} \times \mathcal{E} \). Denote by \( L^p(\pi) \) the usual real \( L^p \)-space with norm \( \| \cdot \|_p \). Write \( \| \cdot \| = \| \cdot \|_2 \).

For a given Dirichlet form \((D, D(D))\), the classical variational formula for the first eigenvalue \( \lambda_1 \) can be rewritten in the form of (3.1) below with an optimal constant \( C = \lambda_1^{-1} \). From this point of view, it is natural to study other inequalities. Two additional basic inequalities appear in (3.2) and (3.3) below:

- **Poincaré inequality:** \( \text{Var}(f) \leq CD(f), \quad f \in L^2(\pi), \) \hspace{1cm} (3.1)

- **Logarithmic Sobolev inequality:** \( \int f^2 \log \frac{f^2}{\|f\|^2} d\pi \leq CD(f), \quad f \in L^2(\pi), \) \hspace{1cm} (3.2)

- **Nash inequality:** \( \text{Var}(f) \leq CD(f)^{1/p} \|f\|_1^{2/q}, \quad f \in L^2(\pi), \) \hspace{1cm} (3.3)

where \( \text{Var}(f) = \pi(f^2) - \pi(f)^2, \; \pi(f) = \int f d\pi, \; p \in (1, \infty) \) and \( 1/p + 1/q = 1 \). The last two inequalities are due to Gross (1976) and Nath (1958), respectively.

Our main object is a symmetric (not necessarily Dirichlet) form \((D, D(D))\) on \( L^2(\pi) \), corresponding to an integral operator (or symmetric kernel) on \((E, \mathcal{E})\):

\[
D(f) = \frac{1}{2} \int_{E \times E} J(dx, dy)(f(y) - f(x))^2, \quad D(D) = \{ f \in L^2(\pi) : D(f) < \infty \}, \hspace{1cm} (3.4)
\]

where \( J \) is a non-negative, symmetric measure having no charge on the diagonal set \( \{(x, x) : x \in E\} \). A typical example is the reversible jump process with a \( q \)-pair \((q(x), q(x, dy))\) and a reversible measure \( \pi \). Then \( J(dx, dy) = \pi(dx)q(x, dy) \).

For the remainder of this part, we restrict our discussions to the symmetric form of (3.4).

#### 3.2 Status of the research. An important topic in this research area is to study under what conditions on the symmetric measure \( J \) do the above inequalities hold. In contrast with the probabilistic method used in Part (II), here we adopt a generalization of Cheeger’s method (1970), which comes from Riemannian geometry. Naturally, we define \( \lambda_1 := \inf \{ D(f) : \pi(f) = 0, \| f \| = 1 \} \). For bounded jump processes, the fundamental known result is the following:
Theorem 3.1 (Lawler & Sokal (1988)). \( \lambda_1 \geq \frac{k^2}{2M} \), where

\[
k = \inf_{\pi(A) \in (0,1)} \frac{\int_A \pi(dx)q(x, A^c)}{\pi(A) \wedge \pi(A^c)} \quad \text{and} \quad M = \sup_{x \in \mathcal{E}} q(x).
\]

In the past years, the theorem has been collected into six books: Chen (1992), Sinclair (1993), Chung (1997), Saloff-Coste (1997), Colin de Verdière (1998), Aldous, D. G. & Fill, J. A. (1994–). From the titles of the books, one can see a wide range of the applications. However, this result fails for the unbounded operator. Thus, it has been a challenging open problem in the past ten years to handle the unbounded case.

As for the logarithmic Sobolev inequality, there have been a large number of publications in the past twenty years for differential operators. (For a survey, see Bakry (1992) or Gross (1993)). Still, there are very limited results for integral operators.

3.3 New results. Since the symmetric measure can be unbounded, we choose a symmetric, non-negative function \( r(x, y) \) such that

\[
J^{(\alpha)}(dx, dy) := \mathbb{1}_{r(x, y) > 0} \frac{J(dx, dy)}{r(x, y)^{\alpha}} \quad (\alpha > 0)
\]

satisfies \( J^{(1)}(dx, E) = \frac{J^{(1)}(dx, E)}{\pi(dx)} \leq 1, \pi\text{-a.s.} \).

For convenience, we use the convention \( J^{(0)} = J \). Corresponding to the three inequalities above, we introduce the following new forms of Cheeger’s constants.

| Inequality | Constant \( k^{(\alpha)} \) |
|------------|-----------------------------|
| Poincaré   | \( \inf_{\pi(A) \in (0,1)} \frac{J^{(\alpha)}(A \times A^c)}{\pi(A) \wedge \pi(A^c)} \) | (Chen & Wang(1998)) |
| Log. Sobolev | \( \lim_{r \to 0} \frac{\int_{\pi(A) \in (0,r)} J^{(\alpha)}(A \times A^c)}{\pi(A) \sqrt{\log e + \pi(A)^{-1}}} \) | (Wang (2001a)) |
| Log. Sobolev | \( \lim_{\delta \to \infty} \inf_{\pi(A) > 0} \frac{J^{(\alpha)}(A \times A^c) + \delta \pi(A)}{\pi(A) \sqrt{1 - \log \pi(A)}} \) | (Chen (2000)) |
| Nash       | \( \inf_{\pi(A) \in (0,1)} \frac{J^{(\alpha)}(A \times A^c)}{\pi(A) \wedge \pi(A^c)^{(2q-3)/(2q-2)}} \) | (Chen (1999)) |

Table 3.1 New forms of Cheeger’s constants

Our main result can be easily stated as follows.

Theorem 3.2. \( k^{(1/2)} > 0 \implies \text{the corresponding inequality holds.} \)

In other words, we use \( J^{(1/2)} \) and \( J^{(1)} \) to handle the unbounded \( J \). The first two kernels come from the use of Schwarz inequality. This result is proven in four papers quoted in Table (3.1). In these papers, some estimates which are sharp or qualitatively sharp for the upper or lower bounds are also presented.

IV. New picture of ergodic theory and explicit criteria

4.1 Importance of the inequalities. Let \( (P_t)_{t \geq 0} \) be the semigroup determined
by a Dirichlet form \((D,D(D))\). Then, various applications of the inequalities are based on the following results:

**Theorem 4.1** (Liggett (1989), Gross (1976) and Chen (1999)).

1. **Poincaré inequality** \(\iff \|P_t f - \pi(f)\|^2 = \text{Var}(P_t f) \leq \text{Var}(f) \exp[-2\lambda_1 t]\).
2. **Logarithmic Sobolev inequality** \(\implies\) exponential convergence in entropy:
   \[\text{Ent}(P_t f) \leq \text{Ent}(f) \exp[-2\sigma t],\]  
   where \(\text{Ent}(f) = \pi(f \log f) - \pi(f) \log \|f\|_1\).
3. **Nash inequality** \(\iff\) \(\text{Var}(P_t f) \leq C\|f\|_1/t^{1-q}\).

In the context of diffusions, one can replace “\(\iff\)” by “\(\Rightarrow\)” in part (2). Therefore, the above inequalities describe some type of \(L^2\)-ergodicity for the semigroup \((P_t)_{t \geq 0}\). These inequalities have become powerful tools in the study on infinite-dimensional mathematics (phase transitions, for instance) and the effectiveness of random algorithms.

### 4.2 Three traditional types of ergodicity

The following three types of ergodicity are well known for Markov processes.

- **Ordinary ergodicity**: \(\lim_{t \to \infty} \|p_t(x,\cdot) - \pi\|_{\text{Var}} = 0\)
- **Exponential ergodicity**: \(\|p_t(x,\cdot) - \pi\|_{\text{Var}} \leq C(x) e^{-\alpha t}\) for some \(\alpha > 0\)
- **Strong ergodicity**: \(\lim_{t \to \infty} \sup_x \|p_t(x,\cdot) - \pi\|_{\text{Var}} = 0\) \(\iff\) \(\lim_{t \to \infty} e^{\beta t} \sup_x \|p_t(x,\cdot) - \pi\|_{\text{Var}} = 0\) for some \(\beta > 0\)

where \(p_t(x,dy)\) is the transition function of the Markov process and \(\|\cdot\|_{\text{Var}}\) is the total variation norm. They obey the following implications:

Strong ergodicity \(\implies\) Exponential ergodicity \(\implies\) Ordinary ergodicity.

It is natural to ask the following question. does there exist any relation between the above inequalities and the three traditional types of ergodicity?

### 4.3 New picture of ergodic theory

**Theorem 4.2** (Chen (1999), ...). For reversible Markov processes with densities, we have the diagram shown in Figure 4.1.

```
Nash inequality
/\                          /\
Logarithmic Sobolev inequality \downarrow \downarrow \downarrow
\downarrow Exponential convergence in entropy
\downarrow Poincaré inequality
\downarrow \iff \downarrow \iff
L^1\text{-exponential convergence} \iff \iff \iff \iff \iff
\downarrow L^2\text{-algebraic ergodicity}
\downarrow \iff \iff
\downarrow
Ordinary ergodicity
```
In Figure 4.1, $L^2$-algebraic ergodicity means that $\text{Var}(P_t f) \leq CV(f)t^{1-q} (t > 0)$ holds for some $V$ having the properties (cf. Liggett (1991)): $V$ is homogeneous of degree two (in the sense that $V(cf + d) = c^2V(f)$ for any constants $c$ and $d$) and $V(f) < \infty$ for all functions $f$ with finite support.

The diagram is complete in the following sense: each single-side implication cannot be replaced by double-sides one. Moreover, strong ergodicity and logarithmic Sobolev inequality (resp. exponential convergence in entropy) are not comparable. With exception of the equivalences, all the implications in the diagram are suitable for more general Markov processes. Clearly, the diagram extends the ergodic theory of Markov processes.

The diagram was presented in Chen (1999), originally for Markov chains only. Recently, the equivalence of $L^1$-exponential convergence and strong ergodicity was mainly proven by Y. H. Mao. A counter-example of diffusion was constructed by Wang (2001b) to show that strong ergodicity does not imply exponential convergence in entropy. For other references and a detailed proof of the diagram, refer to Chen (1999).

4.4 Explicit criteria for several types of ergodicity. As an application of the diagram in Figure 4.1, we obtain a criterion for the exponential ergodicity of birth-death processes, as listed in Table 4.2. To achieve this, we use the equivalence of exponential ergodicity and Poincaré inequality, as well as the explicit criterion for Poincaré inequality given in part (3) of Theorem 2.5. This solves a long standing open problem in the study of Markov chains (cf. Anderson (1991), §6.6 and Chen (1992), §4.4).

Next, it is natural to look for some criteria for other types of ergodicity. To do so, we consider only the one-dimensional case. Here we focus on the birth-death processes since the one-dimensional diffusion processes are in parallel. The criterion for strong ergodicity was obtained recently by Zhang, Lin and Hou (2000), and extended by Zhang (2001), using a different approach, to a larger class of Markov chains. The criteria for logarithmic Sobolev, Nash inequalities, and the discrete spectrum (no continuous spectrum and all eigenvalues have finite multiplicity) were obtained by Bobkov and Götze (1999) and Mao (2000, 2002a,b), respectively, based on the weighted Hardy inequality (see also Miclo (1999), Wang (2000), Gong and Wang (2002)). It is understood now the results can also be deduced from generalizations of the variational formulas discussed in this paper (cf. Chen (2001b)). Finally, we summarize these results in Theorem 4.3 and Table 4.2. The table is arranged in such an order that the property in the latter line is stronger than the property in the former line. The only exception is that even though the strong ergodicity is often stronger than the logarithmic Sobolev inequality, they are not comparable in general, as mentioned in Part III.

Theorem 4.3 (Chen (2001a)). For birth-death processes with birth rates $b_i (i \geq 0)$ and death rates $a_i (i \geq 1)$, ten criteria are listed in Table 4.2. Recall the sequence $(\mu_i)$
defined in Part II and set $\mu[i, k] = \sum_{i \leq j \leq k} \mu_j$. The notion “$(\ast)$ & $\cdots$” appeared in Table 4.2 means that one requires the uniqueness condition in the first line plus the condition “$\cdots$”. The notion “$(\varepsilon)$” in the last line means that there is still a small room $(1 < q \leq 2)$ left from completeness.

| Property                      | Criterion                                                                 |
|-------------------------------|---------------------------------------------------------------------------|
| Uniqueness                    | $\sum_{n \geq 0} \frac{1}{\mu_n b_n} \mu[0, n] = \infty \ (\ast)$       |
| Recurrence                    | $\sum_{n \geq 0} \frac{1}{\mu_n b_n} = \infty$                          |
| Ergodicity                    | $(\ast)$ & $\mu[0, \infty) < \infty$                                    |
| Exponential ergodicity        | $(\ast)$ & $\sup_{n \geq 1} \mu[n, \infty) \sum_{j \leq n-1} \frac{1}{\mu_j b_j} < \infty$ |
| $L^2$-exp. convergence        | $(\ast)$ & $\lim_{n \to \infty} \sup_{k \geq n} \mu[k, \infty) \sum_{j \leq k-1} \frac{1}{\mu_j b_j} = 0$ |
| Discrete spectrum             | $(\ast)$ & $\sup_{n \geq 1} \mu[n, \infty) \log[\mu[n, \infty) \sum_{j \leq n-1} \frac{1}{\mu_j b_j} < \infty$ |
| Log. Sobolev inequality       | $(\ast)$ & $\sup_{n \geq 1} \mu[n, \infty) \log[\mu[n, \infty) \sum_{j \leq n-1} \frac{1}{\mu_j b_j} < \infty$ |
| Strong ergodicity             | $(\ast)$ & $\sum_{n \geq 1} \mu[n+1, \infty) = \sum_{n \geq 1} \frac{1}{\mu_n} \sum_{j \leq n-1} \mu_j b_j < \infty$ |
| $L^1$-exp. convergence        | $(\ast)$ & $\sup_{n \geq 1} \mu[n, \infty) \left[q-2\right]/\left[q-1\right] \sum_{j \leq n-1} \frac{1}{\mu_j b_j} < \infty \ (\varepsilon)$ |

Table 4.2  Ten criteria for birth-death processes

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