THE STEINBERG CURVE

HÉLÈNE ESNAULT AND MARC LEVINE

Abstract. Let $E$ and $E'$ be elliptic curves over $\mathbb{C}$, with Tate parametrizations $p : \mathbb{C}^* \to E$, $p' : \mathbb{C}^* \to E'$. We have the map $p*p' : \mathbb{C}^* \otimes \mathbb{C}^* \to F^2\text{CH}^2(E \times E')$ sending $u \otimes v$ to the class of the zero cycle $(x, y) - (x, 0) - (0, y) + (0, 0)$, where $x = p(u)$, $y = p'(v)$.

We show that, for general $u \in \mathbb{C}^*$, $p*p'(u \otimes (1 - u))$ is not zero in $\text{CH}^2$. We also show that the cycle $p*p'(u \otimes (1 - u))$ is not detectable by a certain class of cohomology theories, including the cohomology of the analytic motivic complex involving the dilogarithm function defined by S. Bloch in [3]. This is in contrast to its étale version defined by S. Lichtenbaum [8], which contains the Chow group.

1. Tate curves and line bundles

For a scheme $X$ over $\mathbb{C}$, we let $X_{\text{an}}$ denote the set of $\mathbb{C}$-points with the classical topology. We let $\mathcal{O}_{X_{\text{an}}}$ denote the sheaf of holomorphic functions on $X_{\text{an}}$.

We begin by describing a construction of the universal analytic Tate curve over $\mathbb{C}$. We first form the analytic manifold $\hat{\mathbb{C}}^*$ as the quotient of the disjoint union $\bigcup_{i=-\infty}^{\infty} U_i$, with each $U_i = \mathbb{C}^2$, by the equivalence relation $(x, y) \in U_i \setminus \{Y = 0\} \sim (\frac{1}{y}, xy^2) \in U_{i+1} \setminus \{X = 0\}$.

The function $\tilde{\pi}(x, y) = xy$ on $\hat{\mathbb{C}}^*$ is globally defined. Letting $D \subset \mathbb{C}$ be the disk $\{|z| < 1\}$, we define $\mathbb{C}^* = \tilde{\pi}^{-1}(D)$, so $\tilde{\pi}$ restricts to the analytic map $\pi : \mathbb{C}^* \to D$. We let $0 : D \to \mathbb{C}^*$ be the section $z \mapsto (z, 1) \in U_0$.

Let $D^* \subset D$ be the punctured disk $z \neq 0$. Since the map $(x, y) \mapsto (\frac{1}{y}, xy^2)$ is an automorphism of $(\mathbb{C}^*)^2$, the open submanifold $\pi^{-1}(D^*)$ of $\mathbb{C}^*$ is isomorphic to $(\mathbb{C}^*)^2$, and the restriction of the map $\pi$ is just

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the map \((x, y) \mapsto xy\). Thus, the projection \(p_2 : (\mathbb{C}^*)^2 \to \mathbb{C}^*\) gives an isomorphism of the fiber \(C^*_t := \pi^{-1}(t)\) with \(\mathbb{C}^*\), for \(t \in D^*\).

The fiber \(\pi^{-1}(0)\), on the other hand, is an infinite union of projective lines. Indeed, define the map \(f_i : \mathbb{CP}^1 \to C_0^*\) by sending \((a : 1) \in \mathbb{CP}^1 \setminus \infty\) to \((a, 1) \in U_i\), and \(\infty = (1 : 0)\) to \((0, 0) \in U_{i+1}\), and let \(C_i = f_i(\mathbb{CP}^1)\). Then \(\pi^{-1}(0) = \bigcup_{i=-\infty}^{\infty} C_i\), with \(\infty \in C_i\) joined with \(0 \in C_{i+1}\). Note in particular that the value \(\tilde{0}(0)\) of the zero section avoids the singularities of \(\pi^{-1}(0)\).

Define the automorphism \(\phi\) of \(\mathcal{C}\) over \(D\) by sending \((x, y) \in U_i\) to \((x, y) \in U_{i-1}\). This gives the action of \(\mathbb{Z}\) on \(\mathbb{C}^*\), with \(n\) acting by \(\phi^n\). It is easy to see that this action is free and proper, so the quotient space \(\mathcal{E} := \mathcal{C}^*/\mathbb{Z}\) exists as a bundle \(\pi : \mathcal{E} \to D\). The section \(0 : D \to \mathcal{C}^*\) induces the section \(\tilde{0}(0) : \mathcal{E}_0 \setminus \ast\).

Take \(t \in D^*\). Identifying \(C^*_t\) with \(\mathbb{C}^*\) as above, we see that \(\phi\) restricts to the automorphism \(z \mapsto tz\). Thus, the fiber \(\mathcal{E}_t := \pi^{-1}(t)\) for \(t \in D^*\) is the Tate elliptic curve \(\mathbb{C}^*/t\mathbb{Z}\), with identity \(0(t)\). On \(C_0^*\), however, \(\phi\) is the union of the “identity” isomorphisms \(C_i \to C_{i-1}\). Thus \(\phi(\infty \in C_i) = 0 \in C_i\), so the restriction of \(\mathcal{C}_0^* \to \mathcal{E}_0\) to \(C_0\) identifies \(\mathcal{E}_0\) with the nodal curve \(\mathbb{CP}^1/0nds \sim \infty\). We let \(* \in \mathcal{E}_0\) denote the singular point. Then \(\tilde{0}(0) \in \mathcal{E}_0 \setminus \ast\).

The map \((t, w) \in D \times \mathbb{C}^* \mapsto (t, w, w) \in U_0 \) gives an isomorphism \(\psi : D \times \mathbb{C}^* \to U_0 \setminus \{Y = 0\}\) over \(D\). The composition
\[
D \times \mathbb{C}^* \to \mathbb{C}^* \to \mathcal{E}
\]
defines the map \(p : D \times \mathbb{C}^* \to \mathcal{E}\) over \(D\), with image \(\mathcal{E} \setminus \ast\).

Take \(u \in \mathbb{C}^*\). We have the local system on \(\mathcal{E}\)
\[
\mathcal{L}_u := \mathcal{C}^* \times \mathbb{C}/(z, \lambda) \sim (\phi(z), u\lambda) \to \mathcal{E},
\]
and the associated holomorphic line bundle \(\mathcal{L}^\text{an}_u\) on \(\mathcal{E}\).

Let \(E_t\) be the algebraic elliptic curve associated to the analytic variety \(\mathcal{E}_t\), let \(L_u(t)\) and \(L^\text{an}_u(t)\) denote the restriction of \(L_u\) and \(\mathcal{L}^\text{an}_u\) to \(\mathcal{E}_t\), and let \(L^\text{alg}_u(t)\) be the algebraic line bundle on \(E_t\) corresponding to \(L^\text{an}_u(t)\) via [13]. The restriction of \(p\) to \(t \times \mathbb{C}^*\) defines the map \(p_t : \mathbb{C}^* \to E_{t\text{an}}\). For \(t \neq 0\), \(p_t\) is a covering space of \(E_{t\text{an}}\). The map \(p_0 : \mathbb{C}^* \to E_{0\text{an}}\) is the analytic map associated to the algebraic open immersion
\[
\mathbb{P}^1 \setminus \{0, \infty\} \xrightarrow{j} \mathbb{P}^1 \to \mathbb{P}^1/0nds \sim \infty = E_0.
\]

If \(E\) be an elliptic curve over \(\mathbb{C}\), then \(E_{\text{an}} \cong \mathbb{C}/\Lambda\), where \(\Lambda \subset \mathbb{C}\) is a lattice spanned by 1 and some \(\tau\) in the upper half plane. Taking \(t = e^{2\pi i\tau}\) gives the isomorphism \(E_{\text{an}} \cong \mathcal{E}_t\), so each elliptic curve over \(\mathbb{C}\) occurs as an \(E_t\) for some (in fact for infinitely many) \(t \in D^*\).
Sending $u \in \mathbb{C}^*$ to the isomorphism class of $L_u^{\text{alg}}(t)$ defines a homomorphism $\tilde{p}_t : \mathbb{C}^* \to \text{Pic}(E_t)$. We denote the identity $0(t) \in E_t$ simply by $0$ if $t$ is given.

**Lemma 1.1.** For all $t \in D$, $c_1(L_u^{\text{alg}}(t)) = (p_t(u)) - (0)$.

**Proof.** We first handle the case $t \neq 0$. Let $q : \mathbb{C} \to E := E_t$ be the map $q(z) = p_t(e^{2\pi iz})$, let $\tau \in \mathbb{C}$ be an element with $e^{2\pi i\tau} = t$, and let $\Lambda \subset \mathbb{C}$ be the lattice generated by $1$ and $\tau$. The map $q$ identifies $E$ with $\mathbb{C}/\Lambda$, and $L_u(t)$ with the local system defined by the homomorphism $\rho : \Lambda \to \mathbb{C}^*$, $\rho(a + b\tau) = u(b)$.

There is a unique cocycle $\theta$ in $Z^1(\Lambda, H^0(\mathbb{C}, \mathcal{O}_E^{\text{can}}))$ with $\theta(1) = 1$, $\theta(\tau) = e^{-2\pi iz}$; let $L$ be the corresponding holomorphic line bundle on $E$. Computing $c_1^{\text{top}}(L) \in H^2(E, \mathbb{Z})$ by using the exponential sequence, we find that $\deg(L) = 1$. By Riemann-Roch, we have $H^0(E, L) = \mathbb{C}$; let $\Theta(z)$ be the corresponding global holomorphic function on $\mathbb{C}$, i.e.,

$$\Theta(z + 1) = \Theta(z), \quad \Theta(z + \tau) = e^{-2\pi iz}\Theta(z),$$

and the divisor of $\Theta$ on $E$ is $(x)$, with $L \cong \mathcal{O}_E(x)$.

Take $v, w \in \mathbb{C}$ with $u = e^{2\pi iv}$ and $q(w) = x$. Let $f(z) = \frac{\Theta(z+w-v)}{\Theta(z+w)}$. Then

$$f(z + 1) = f(z), \quad f(z + \tau) = uf(z),$$

and $\text{Div}(f) = (p(u)) - (0)$. Thus, multiplication by $f$ defines an isomorphism

$$\times f : \mathcal{O}_{E_0}(\mathbb{C}) \to \mathcal{O}_{E_0}(\mathbb{C}).$$

The proof for $E_0 = \mathbb{P}^1/0 \sim \infty$ is essentially the same, where we replace $\frac{\Theta(z+w-v)}{\Theta(z+w)}$ with the rational function $\frac{X}{X-1}$. \hfill \Box

Thus, the image of $\tilde{p}_t$ in $\text{Pic}(E_t)$ is $\text{Pic}^0(E)$. After identifying the smooth locus of $E_t^0$ of $E_t$ with $\text{Pic}^0(E_t)$ by sending $x \in E_t^0$ to the class of the invertible sheaf $\mathcal{O}_{E_t}(x) - (0))$, we have $\tilde{p}_t = p_t$.

## 2. The Albanese kernel and the Steinberg relation

Let $X$ be a smooth projective variety. We let $\text{CH}_0(X)$ denote the group of zero cycles on $X$, modulo rational equivalence, $F^1\text{CH}_0(X)$ the subgroup of cycles of degree zero, and $F^2\text{CH}_0(X)$ the kernel of the Albanese map $\alpha_X : F^1\text{CH}_0(X) \to \text{Alb}(X)$. The choice of a point $0 \in X$ gives a splitting to the inclusion $F^1\text{CH}_0(X) \to \text{CH}_0(X)$.

Let $E, E'$ be smooth elliptic curves. As $\text{Alb}(E \times E') = E \times E'$, the inclusion $F^1\text{CH}_0(E \times E') \to F^1\text{CH}_0(E \times E')$ is split by sending $(x, y) - (0, 0)$ to $(x, y) - (x, 0) - (0, y) + (0, 0)$. Thus $F^2\text{CH}_0(E \times E')$
is generated by zero-cycles of the form $(x, y) - (x, 0) - (0, y) + (0, 0)$. Choosing an isomorphism $E \cong E_1$, $E' \cong E_1'$, we have the covering spaces $p : \mathbb{C}^* \to E_{\text{an}}$, $p' : \mathbb{C}^* \to E'_{\text{an}}$, and the map

$$(2.1) \quad p * p' : \mathbb{C}^* \otimes \mathbb{C}^* \to F^2 \text{CH}_0(E \times E')$$

$$u \otimes v \mapsto p(u) * p'(v) := (p(u), p'(v)) - (p(u), 0) - (0, p'(v)) + (0, 0).$$

By the theorem of the cube [10], the map $p * p'$ is a group homomorphism, and thus is surjective.

In case one or both of $E$, $E'$ is the singular curve $E_0$, we will need to use the theory of zero-cycles mod rational equivalence defined in [6]. If $X$ is a reduced, quasi-projective variety over a field $k$ with singular locus $X_{\text{sing}}$, the group $\text{CH}_0(X)$ (denoted $\text{CH}_0(X, X_{\text{sing}})$ in [7]) is defined as the quotient of the free abelian group on the regular closed points of $X$, modulo the subgroup generated by zero-cycles of the form $\text{Div} f$, where $f$ is a rational function on a dimension one closed subscheme $D$ of $X$ such that

1. No irreducible component of $D$ is contained in $X_{\text{sing}}$.
2. In a neighborhood of each point of $D \cap X_{\text{sing}}$, the subscheme $D$ is a complete intersection.
3. $f$ is in the subgroup $\mathcal{O}_{D,D\cap X_{\text{sing}}}^* k(D)^*$. 

It follows in particular from these conditions that $\text{Div} f$ is a sum of regular points of $X$.

For $X$ a reduced curve, sending a regular closed point $x \in X$ to the invertible sheaf $\mathcal{O}_X(x)$ extends to give an isomorphism $\text{CH}_0(X) \cong \text{Pic}(X)$.

We extend the definition of $F^i \text{CH}_0$ to $E \times E'$ with either $E = E_0$ or $E' = E_0$ or $E = E' = E_0$, by defining $F^1 \text{CH}_0(E \times E')$ as the subgroup of $\text{CH}_0(E \times E')$ generated by the differences $[x] - [y]$, and $F^2 \text{CH}_0(E \times E')$ the subgroup generated by expressions $[(x, y)] - [(x, 0)] - [(0, y)] + [(0, 0)]$, where $x$ is a smooth point of $E$ and $y$ a smooth point of $E'$. The surjection $p * p' : \mathbb{C}^* \otimes \mathbb{C}^* \to F^2 \text{CH}_0(E \times E')$ is then defined by the same formula as (2.1).

**Proposition 2.1** (The Steinberg relation). Take $E = E' = E_0$. Then $p(u) * p(1 - u) = 0$ in $\text{CH}_0(E_0 \times E_0)$ for all $u \in \mathbb{C} \setminus \{0, 1\}$.

**Proof.** Let $X$ be a quasi-projective surface over a field $k$. By [7], there is an isomorphism $\phi : H^2(X, \mathcal{K}_2) \to \text{CH}_0(X)$. The product $\mathcal{O}_X^* \otimes \mathcal{O}_X^* \to \mathcal{K}_2$ gives the cup product

$$H^1(X, \mathcal{O}_X^*) \otimes H^1(X, \mathcal{O}_X^*) \cup H^2(X, \mathcal{K}_2).$$
In addition, let $D, D'$ be Cartier divisors which intersect properly on $X$, and suppose that $\text{supp } D \cap \text{supp } D' \cap X_{\text{sing}} = \emptyset$. Then
\begin{equation}
(2.2) \quad \phi(\mathcal{O}_X(D) \cup \mathcal{O}_X(D')) = [D \cdot D'],
\end{equation}
where $\cdot$ is the intersection product and $[-]$ denotes the class in $\text{CH}_0$.

Since $L_{u}^\text{alg} = \mathcal{O}_{E_0}(p(u) - 0)$, (2.2) implies
\begin{equation}
\phi(u) \ast \phi(1 - u) = \rho(p_1^* L_{u}^\text{alg} \cup p_2^* L_{1-u}^\text{alg}),
\end{equation}
so it suffices to show that $p_1^* L_{u}^\text{alg} \cup p_2^* L_{1-u}^\text{alg} = 0$ in $H^2(E_0 \times E_0, \mathcal{K}_2)$.

Write $X$ for $E_0 \times E_0$. Let $\bar{\mathcal{K}}_2$ be the image of $\mathcal{K}_2$ in the constant sheaf $\mathcal{K}_2(\mathbb{C}(X))$. By Gersten’s conjecture, the surjection $\pi: \mathcal{K}_2 \to \bar{\mathcal{K}}_2$ is an isomorphism at each regular point of $X$, hence $\pi$ induces an isomorphism on $H^2$.

Let $q: \mathbb{P}^1 \to E_0$ be the normalization, giving the normalization $q \times q: \mathbb{P}^1 \times \mathbb{P}^1 \to X$. Let $i: * \to E_0$ be the inclusion of the singular point. We have the exact sequence of sheaves on $E_0$
\begin{equation}
(2.3) \quad q_* \mathcal{K}_1 \xrightarrow{\beta} i_* \mathcal{K}_1(\mathbb{C}) \to 0
\end{equation}
and the exact sequence of sheaves on $X$:
\begin{equation}
(2.4) \quad (q \times q)_* \mathcal{K}_2 \xrightarrow{\alpha} (i \times q)_* \mathcal{K}_2 \oplus (q \times i)_* \mathcal{K}_2 \to (i \times i)_* \mathcal{K}_2(\mathbb{C}) \to 0,
\end{equation}
with augmentations $\epsilon_1: \mathcal{K}_1 \to (2.3)$, $\epsilon_2: \bar{\mathcal{K}}_1 \to (2.4)$. The various cup products in $K$-theory give the map of complexes
\begin{equation}
(2.5) \quad p_1^* (2.3) \otimes p_2^* (2.3) \to (2.4)
\end{equation}
over the cup product
\begin{equation}
(2.6) \quad p_1^* \mathcal{K}_1 \otimes p_2^* \mathcal{K}_1 \to \bar{\mathcal{K}}_2.
\end{equation}

The augmentation $\epsilon_1: \mathcal{K}_1 \to \ker \beta$ is an isomorphism. The augmentation $\epsilon_2: \bar{\mathcal{K}}_1 \to \ker \alpha$ is an injection, and the cokernel is supported on $\ast \times \ast$, so $\epsilon_2: \bar{\mathcal{K}}_2 \to \ker \alpha$ induces an isomorphism on $H^2$. Thus, the complexes (2.3) and (2.4) give rise to maps
\begin{align*}
\delta_2 &: K_2(\mathbb{C}) \to H^2(X, \ker \alpha) = H^2(X, \bar{\mathcal{K}}_2) = H^2(X, \mathcal{K}_2) \\
\delta_1 &: \mathbb{C}^* = K_1(\mathbb{C}) \to H^1(E_0, \mathcal{K}_1).
\end{align*}

The compatibility of (2.5) with (2.6) yields the commutativity of the diagram
\begin{equation}
\begin{array}{ccc}
\mathbb{C}^* \otimes \mathbb{C}^* & \xrightarrow{\cup} & K_2(\mathbb{C}) \\
\downarrow{\delta_1 \otimes \delta_1} & & \downarrow{\delta_2} \\
H^1(E_0, \mathcal{K}_1) \otimes H^1(E_0, \mathcal{K}_1) & \xrightarrow{p_1^* \cup p_2^*} & H^2(X, \mathcal{K}_2).
\end{array}
\end{equation}
Since $L_v^{\text{alg}} = \delta_1(v)$ for each $v \in \mathbb{C}^*$, we have
\[
p_1^* L_u^{\text{alg}} \cup p_2^* L_{1-u}^{\text{alg}} = \delta_2(\{u, 1-u\}) = 0.
\]

The main point of this section is that the Steinberg relation is not satisfied in $\text{CH}_0(E \times E')$ except in the case $E = E' = E_0$. We first require the following lemma:

**Lemma 2.2.** Let $s : \mathbb{C} \setminus \{0, 1\} \to E \times E'$ be the analytic map $s(u) = (p(u), p'(1-u))$. Then $s(\mathbb{C} \setminus \{0, 1\})$ is not contained in any algebraic curve on $E \times E'$, except in case $E = E' = E_0$.

**Proof.** We first consider the case in which both $E$ and $E'$ are smooth elliptic curves, $E = E_t$, $E' = E_{t'}$, where $t$ and $t'$ are in $\mathbb{C}^*$ and $|t| < 1$, $|t'| < 1$. We have the maps
\[p : \mathbb{C}^* \to E, \; p' : \mathbb{C}^* \to E',\]
which are group homomorphisms with $\ker p = t^\mathbb{Z}$, $\ker p' = t'^\mathbb{Z}$.

Suppose that $s(\mathbb{C}^*)$ is contained in an algebraic curve $D \subset E \times E'$. For each $x \in E$, $(x \times E') \cap D$ is a finite set (possibly empty), hence, for each $u \in \mathbb{C} \setminus \{0, 1\}$, the set of points of $\mathbb{C}^* \times \mathbb{C}^*$ of the form $(t^n u, 1 - t^n u)$ has finite image in $E \times E'$. Thus, for each $u$, there are integers $n$, $m$ and $p$, depending on $u$, such that $n \neq m$ and
\[
1 - t^m u = t^p (1 - t^n u). \tag{2.7}
\]

Since there are uncountably many $u$, there is a single choice of $n$, $m$ and $p$ for which (2.7) holds for uncountably many $u$. But then
\[
(t^p t^n - t^m) u = 1 - t^p. \tag{2.8}
\]

If $t^p t^n - t^m = 0$, then $|t'| = 1$, contradicting the condition $|t'| < 1$. If $t^p t^n - t^m \neq 0$, then we can solve (2.8) for $u$, so (2.7) only holds for this single $u$, a contradiction.

If say $E' = E_0$, then $p' : \mathbb{C}^* \to E'$ is injective, and we have the infinite set of points $p'(1 - t^p u)$ in the image of $s$, all lying over the single point $p(u)$.

**Theorem 2.3.** Let $E = E_t$, $E' = E_{t'}$, with at least one of $E$, $E'$ non-singular. Then, for all $u$ outside a countable subset of $\mathbb{C} \setminus \{0, 1\}$, $p(u) * p'(1-u)$ is not a torsion element in $F^2 \text{CH}_0(E \times E')$. 

\[\square\]
Proof. We first give the proof in case $E$ and $E'$ are both non-singular. For a quasi-projective $\mathbb{C}$-scheme $X$, we let $S^n X$ denote the $n$th symmetric power of $X$. For $X$ smooth, we have the map

$$\rho_n : S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) \to \text{CH}_0(X)$$

$$(\sum_{i=1}^n x_i, \sum_{j=1}^n y_j) \mapsto [\sum_{i=1}^n x_i - \sum_{j=1}^n y_j].$$

For each integer $n \geq 1$, we have the morphism

$$\phi_n : E \times E' \to S^{2n}(E \times E') \times S^{2n}(E \times E')$$

$$(x, y) \mapsto (n(x, y) + n(0, 0), n(x, 0) + n(0, y)),$$

By [12, Theorem 1], $(\rho_n \circ \phi_n)^{-1}(0)$ is a countable union of Zariski closed subsets of $E \times E'$.

On the other hand, since $p_g(E \times E') = 1$, the Albanese kernel $F^2 \text{CH}_0(E \times E')$ is “infinite dimensional” [9]; in particular, $F^2 \text{CH}_0(E \times E') \otimes \mathbb{Q} \neq 0$. Since $F^2 \text{CH}_0(E \times E')$ is generated by cycles of the form $p(u) \ast p(v)$, it follows that $(\rho_n \circ \phi_n)^{-1}(0)$ is a countable union of proper closed subsets of $E \times E'$. If $D$ is a proper algebraic subset of $E \times E'$, then, by Lemma 2.2, $s^{-1}(D)$ is a proper analytic subset of $\mathbb{C} \setminus \{0, 1\}$, hence $s^{-1}(D)$ is countable. Thus, the set of $u \in \mathbb{C} \setminus \{0, 1\}$ such that $p(u) \ast p'(1 - u)$ is torsion is countable, which completes the proof in case both $E$ and $E'$ are non-singular.

If say $E' = E_0$, we use essentially the same proof. We let $X$ be the open subscheme $E \times (E_0 \setminus \{\ast\})$ of $E \times E_0$. We have the map $\rho_n : S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) \to \text{CH}_0(E \times E_0)$ defined as above. By [6, Theorem 4.3], $(\rho_n \circ \phi_n)^{-1}(0)$ is a countable union of closed subsets $D_i$ of $X$. By [14], we have the similar infinite dimensionality result for $\text{CH}_0(E \times E_0)$ as in the smooth case, from which it follows that each $D_i$ is a proper closed subset of $X$. Thus, the closure of each $D_i$ in $E \times E_0$ is a proper algebraic subset of $E \times E_0$. The same argument as in the smooth case finishes the proof.

3. Indetectability

The zero-cycle $p(u) \ast p(1 - u)$ is indetectable by cohomology theories built on the sheaf $\mathcal{O}_{E_{an} \times E_{an}}^*$. We first consider the following abstract situation.

Let $\Gamma_0(2)$ be the complex:

$$\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \to \mathbb{C}^* \otimes \mathbb{C}^*$$

$$u \mapsto u \otimes (1 - u),$$

with $\mathbb{C}^* \otimes \mathbb{C}^*$ in degree two.
Let $X = E \times E'$, and let $\Gamma(2)_{\text{an}}$ be a complex of sheaves on $X_{\text{an}}$ with the following properties:

\[(3.1)\]

1. There is a group homomorphism $\text{cl} : \text{CH}_0(X) \to \mathbb{H}^4(X_{\text{an}}, \Gamma(2)_{\text{an}})$.
2. There is a map in the derived category of sheaves $D^b(\text{Sh}_{X_{\text{an}}})$, $\rho : \mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^*[-2] \to \Gamma(2)_{\text{an}}$.
3. The composition

\[\mathbb{C}^* \otimes \mathbb{C}^*[-2] \to \mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^*[-2] \to \Gamma(2)_{\text{an}}\]

extends to a map in $D^b(\text{Sh}_{X_{\text{an}}})$, $\Gamma_0(2) \to \Gamma(2)_{\text{an}}$.
4. The composition

\[\text{Pic}(X) \otimes \text{Pic}(X) \cong H^1(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}^*) \otimes H^1(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}^*)\]

\[\cup \to H^2(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^*) \xrightarrow{\rho} \mathbb{H}^4(X_{\text{an}}, \Gamma(2)_{\text{an}})\]

agrees with the composition

\[\text{Pic}(X) \otimes \text{Pic}(X) \xrightarrow{\cup} \text{CH}_0(X) \xrightarrow{\text{cl}} \mathbb{H}^4(X_{\text{an}}, \Gamma(2)_{\text{an}}).\]

**Theorem 3.1.** Let $E = E_t$ and $E' = E_{t'}$, and let $\Gamma(2)_{\text{an}}$ be a complex of sheaves on $E_{\text{an}} \times E'_{\text{an}}$ satisfying the conditions (3.1). Then $\text{cl}(p(u) * p(1 - u)) = 0$ for all $u \in \mathbb{C} \setminus \{0, 1\}$.

**Proof.** We give the proof in case both $E$ and $E'$ are non-singular; the singular case is similar, but easier, and is left to the reader.

Since

\[p(u) * p(1 - u) = [p_1^* c_1(L_{u})] \cap [p_2^* c_1(L_{1-u})],\]

it follows from (3.1)(4) that we need to show that $\rho([L_{an}^u] \cup [L_{an}^{1-u}]) = 0$. The class $[L_{an}^u] \in H^1(E_{\text{an}}, \mathcal{O}_{E_{\text{an}}}^*)$ is the image of $[L_u] \in H^1(E_{\text{an}}, \mathbb{C}^*)$ under the map of sheaves $\mathbb{C}^* \to \mathcal{O}_{E_{\text{an}}}^*$, and similarly for $L_{1-u}$ and $L_{an}^{1-u}$. Thus, by (3.1)(3), it suffices to see that $p_1^*[L_u] \cup p_2^*[L_{1-u}] \in H^2(E \times E', \mathbb{C}^* \otimes \mathbb{C}^*)$ vanishes in $\mathbb{H}^4(E \times E', \Gamma_0(2))$.

The $\mathbb{Z}$-covers $p : \mathbb{C}^* \to E = E_t$, $p' : \mathbb{C}^* \to E' = E_{t'}$ give natural maps

\[\alpha : H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) \to H^*(E_{\text{an}}, \mathbb{C}^*),\]
\[\beta : H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) \to H^*(E'_{\text{an}}, \mathbb{C}^*).\]

Similarly, the $\mathbb{Z}^2$-cover $p \times p' : \mathbb{C}^* \times \mathbb{C}^* \to E \times E'$ gives the natural map

\[\gamma : \mathbb{H}^*(\mathbb{Z}^2, H^0(\mathbb{C}^* \times \mathbb{C}^*, \Gamma_0(2))) \to \mathbb{H}^*(E_{\text{an}} \times E'_{\text{an}}, \Gamma_0(2)).\]
Letting $\iota : \mathbb{C}^* \otimes \mathbb{C}^* \rightarrow \Gamma_0(2)$ denote the natural inclusion, the maps above are compatible with the respective cup products:
\[
\iota \circ (\alpha(a) \cup \beta(b)) = \gamma \circ \iota(a \cup b).
\]

Each $v \in \mathbb{C}^*$ gives the corresponding homomorphism $v : \mathbb{Z} \rightarrow \mathbb{C}^*$, $v(n) = v^n$. Since $[L_u] \in H^1(E_{\text{an}}, \mathbb{C}^*)$ is $\alpha(u : \mathbb{Z} \rightarrow \mathbb{C}^*)$ and $[L_{1-u}] \in H^1(E_{\text{an}}', \mathbb{C}^*)$ is $\beta(1-u : \mathbb{Z} \rightarrow \mathbb{C}^*)$, it suffices to show that $\iota(p_1^*u \cup p_2^*(1-u)) = 0$ in $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2))$, where $p_1^*u, p_2^*(1-u) : \mathbb{Z}^2 \rightarrow \mathbb{C}^*$ are the respective homomorphisms $(a, b) \mapsto u^a$, and $(a, b) \mapsto (1-u)^b$

We have the spectral sequence
\[
E_2^{p,q} = H^p(\mathbb{Z}^2, H^q(\Gamma_0(2))) \Rightarrow \mathbb{H}^{p+q}(\mathbb{Z}^2, \Gamma_0(2)).
\]

Since $\mathbb{Z}^2$ has cohomological dimension two, and since $H^q(\Gamma_0(2)) = 0$ for $q \neq 1, 2$, it follows that the natural map $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2)) \rightarrow H^2(\mathbb{Z}^2, H^2(\Gamma_0(2)))$ is an isomorphism. Since $H^2(\Gamma_0(2)) = K_2(\mathbb{C})$, we need to show that the image of $p_1^*u \cup p_2^*(1-u)$ in $H^2(\mathbb{Z}^2, K_2(\mathbb{C}))$ is zero.

By definition of the cup product in group cohomology, we have
\[
[p_1^*u \cup p_2^*(1-u)]((a, b), (c, d)) = p_1^*u(a, b) \otimes p_2^*(1-u)(c-a, d-b) = u^a \otimes (1-u)^{d-b},
\]

which clearly vanishes in $K_2(\mathbb{C})$.

\[\square\]

**Example 3.2.** In [B], S. Bloch defines a quotient complex $\mathcal{B}(2)$ of the analytic complex $\mathcal{O}_{X_{\text{an}}}^*(1) \xrightarrow{2\pi i \otimes 1} \mathcal{O}_{X_{\text{an}}} \otimes \mathcal{O}_{X_{\text{an}}}^*$ fulfilling $\mathcal{H}^i(\mathcal{B}(2)) = 0$ for $i \neq 1, 2$,

\[
\mathcal{H}^1(\mathcal{B}(2)) = \operatorname{Im}(r : K_{3, \text{ind}}(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Z}(2)) =: \Delta^*(1),
\]

where $r$ is the regulator map, and $\mathcal{H}^2(\mathcal{B}(2)) = K_{2, \text{an}}$. He shows in the same article that $r(K_{3, \text{ind}}(\mathbb{C})) = r(K_{3, \text{ind}}(\overline{\mathbb{Q}}))$, thus $\Delta^*(1)$ is a countable subgroup of $\mathbb{C}/\mathbb{Z}(2)$, and also that $\mathcal{B}(2)$ maps to the complex $\mathbb{Z}(2) \rightarrow \mathcal{O}_{X_{\text{an}}} \rightarrow \Omega^1_{X_{\text{an}}}$ which computes the Deligne cohomology $H^4_D(X, \mathbb{Q})$ when $X$ is projective smooth over $\mathbb{C}$. In fact, the cycle map $\text{CH}^2(X) \rightarrow H^4_D(X, 2)$ is shown to factor through $H^4_D(X_{\text{an}}, \mathcal{B}(2))$ (B). S. Bloch ([B]) asked whether the cycle map $\text{CH}^2(X) \rightarrow H^4_D(X_{\text{an}}, \mathcal{B}(2))$ could possibly be injective. The computations of this article show that it is not. Indeed, by Lemma (1.3) of [B], the complex $\Gamma_0(2)$ maps to the complex
\[
e (\mathbb{Z}/\mathbb{C} \setminus \{0, 1\}) \rightarrow \mathbb{C} \otimes \mathbb{C}^*,
\]
where $\epsilon$ is defined via the dilogarithm function

$$
\epsilon(a) = [\log(1 - a) \otimes a] + [2\pi i \otimes \exp\left(\frac{-1}{2\pi i} \int_0^a \log(1 - t) \frac{dt}{t}\right)],
$$

and the latter complex maps to

$$
\mathcal{B}(2)_X : \mathcal{O}^*_{X_{an}}(1) \xrightarrow{2\pi i \otimes 1} \mathcal{O}_{X_{an}} \otimes \mathcal{O}^*_{X_{an}}/\left(\epsilon(\mathbb{Z}\setminus \{0, 1\})\right)
$$

for $X = \text{Spec} \mathbb{C}$. Let us take $\Gamma(2)_{an} = \mathcal{B}(2)$. We now verify the conditions 3.1. The condition 1 is given by [5]. Indeed, one computes the Leray spectral sequence associated to $\alpha : X_{an} \to X_{\text{Zar}}$ and the first term entering $H^4(\mathcal{B}(2))$ is

$$
E_{2}^{2,2} = H^2_{\text{Zar}}(R\alpha_* \mathcal{B}(2)) = H^2(\mathcal{K}_{2,\mathbb{Z}}),
$$

where $\mathcal{K}_{2,\mathbb{Z}} := \text{Ker} \left(\alpha_* \mathcal{K}_{2,an} \xrightarrow{d\log \wedge d\log} H^2(\mathbb{C}/\mathbb{Z}(2))\right)$. Then the cycle map $\text{cl}$ is induced by $\mathcal{K}_2 \to \mathcal{K}_{2,\mathbb{Z}}$ on $X_{\text{Zar}}$, which is obviously compatible with the product in Pic. Thus we have 4. We have already discussed 2 and 3. Hence we can apply Theorem 2.3 to take a 0-cycle $p(u) * p(1 - u)$ on $E \times E'$ where both $E$ and $E'$ are smooth elliptic curves which does not die in the Chow group $\text{CH}_0(E \times E')$, whereas it dies by Theorem 3.1 in $H^4(\mathcal{B}(2))$.

In [8], S. Lichtenbaum constructs an étale version $\Gamma(2)$ of S. Bloch’s analytic complex $\mathcal{B}(2)$, the cohomology of which contains $\text{CH}_2(X)$. This contrasts with the examples discussed above.

Over a $p$-adic field, W. Raskind and M. Spieß ([11]) show that the Albanese kernel modulo $n$ of a product of two Tate elliptic curves is dominated by $K_2(k)/n$. This result is not immediately comparable to ours, but is obviously related.

### 4. The Relative Situation

In this section, we study the cycles constructed in section 2 on $X = E \times E_0$, where as there, $E$ is smooth, and $E_0$ is a nodal curve. Let $\nu = 1 \times q : E \times \mathbb{P}^1 \to X$ be the normalization. We define

$$
\mathcal{K}_2 = \text{Ker} \left(\nu_* \mathcal{K}_2 \xrightarrow{|E \times 0| \wedge |E \times \infty|} \mathcal{K}_2|_E\right)
$$

**Lemma 4.1.** One has

$$
\text{CH}_2(X) = H^2(X, \mathcal{K}_2),
$$

and the Chow group $\text{CH}_0(X)$ fits into an exact sequence

$$
0 \to H^1(E, \mathcal{K}_2) \xrightarrow{\gamma} \text{CH}_0(X) \xrightarrow{\nu^*} \text{CH}_0(E \times \mathbb{P}^1) = \text{Pic}(E) \otimes \text{Pic}(\mathbb{P}^1) \to 0.
$$
Moreover, the map $\gamma$ is defined by

$$\gamma(\sum_{x \in E(1)} x \otimes \lambda_x) = \sum_{x \in E(1)} (x, p_0(\lambda_x)) - (x, 0).$$

**Proof.** The map $\nu^* : \mathcal{K}_2 \to \mathcal{K}_2$ is obviously surjective, and by the Gersten resolution on the smooth points of $X$, the kernel is supported in codimension 1. Thus $\nu^*$ induces an isomorphism on $H^2$.

On the other hand,

$$H^1(E \times \mathbb{P}^1, \mathcal{K}_2) = H^1(E, \mathcal{K}_2) \oplus H^0(E, \mathcal{K}_1) \cup c_1(\mathcal{O}(1)).$$

The term $H^1(E, \mathcal{K}_2)$ maps to 0 via the difference of the restrictions to $E \times 0$ and $E \times \infty$, while $c_1(\mathcal{O}(1))$ restricts to 0 to either $E \times 0$ or $E \times \infty$. This shows the long exact sequence associated to the short one defining $\mathcal{K}_2$.

Finally, the value $\gamma(x \otimes \lambda_x)$ of the map is given by the boundary morphism $C^* \to H^1(X, \mathcal{O}^*_X)$ induced by the normalization sequence

$$0 \to \mathcal{O}^*_X \to q_*\mathcal{O}^*_{\mathbb{P}^1} \xrightarrow{[0]-\infty} \mathbb{C}^* \to 0$$

on the right argument $\lambda_x$. The formula for $\gamma$ thus follows from Lemma 1.1. \hfill \Box

Let $Nm : H^1(E, \mathcal{K}_2) \to \mathbb{C}^*$ be the norm map defined by

$$Nm\left(\sum_{x \in E(1)} x \otimes \lambda_x\right) = \prod_{x \in E(1)} \lambda_x. \tag{4.2}$$

We set

$$V(E) = \text{Ker}Nm. \tag{4.3}$$

One has

**Lemma 4.2.** $F^2\text{CH}_0(X) = \gamma\left(V(E)\right)$.\hfill \Box

**Proof.** By the definition given in §2, $F^2\text{CH}_0(X)$ is generated by the expressions $[(x, y)] - [(x, 0)] - [(0, y)] + [(0, 0)]$, with $x \in E(\mathbb{C})$ and $y \in E_0(\mathbb{C}) \setminus \{\ast\}$. By the formula for $\gamma$ given in Lemma 1.1, this expression is $\gamma(x \otimes y - 0 \otimes y)$, after identifying $y \in \mathbb{C}^*$ with $p_0(y) \in E_0(\mathbb{C})$. Clearly $V(E)$ is generated by the elements of $H^1(E, \mathcal{K}_2)$ of the form $x \otimes y - 0 \otimes y$, whence the lemma. \hfill \Box

Next we want to map $\text{CH}_0(X)$ to a relative version of S. Bloch’s analytic motivic cohomology. So we define

$$\mathcal{B}(2) = \text{Ker}\left(\nu^*\mathcal{B}(2) \xrightarrow{|E \times 0| - |E \times \infty|} \mathcal{B}(2)|_E\right) \tag{4.4}$$
In particular, $\bar{B}(2)$ is an extension of

$$\bar{K}_{2,an} = \text{Ker}\left(\nu_*, \bar{K}_{2,an} \xrightarrow{|E \times 0 - |E \times \infty} \bar{K}_{2,an}|_E\right)$$

placed in degree 2, by $\Delta^*(1)$, placed in degree 1. In other words, $\bar{B}(2)$ is the pull-back of $\bar{B}(2)$ via the map $\nu^* : \bar{K}_{2,an} \to \bar{K}_{2,an}$, and in particular, it receives the complex $\Gamma_0(2)$ as explained in the example 3.2.

Considering again the Leray spectral sequence attached to the identity $\alpha : X_{an} \to X_{zar}$, we see that

$$(\bar{K}_{2,\mathbb{Z}} := \text{Ker}\left(\alpha_* \bar{K}_{2,an} \to \mathcal{H}^2(\mathbb{C}/\mathbb{Z}(2))\right))$$

receives $\bar{K}_2$ and that the first map of the spectral sequence is then

$$(4.6)\quad H^2(X, \bar{K}_{2,\mathbb{Z}}) \to \mathbb{H}^4(X_{an}, \bar{B}(2)).$$

In conclusion, we have shown

**Lemma 4.3.** One has a cycle map

$$\psi_X : \text{CH}_0(X) \to \mathbb{H}^4(X_{an}, \bar{B}(2))$$

compatible with the cycle map

$$\psi_{E \times \mathbb{P}^1} : \text{CH}_0(E \times \mathbb{P}^1) \to \mathbb{H}^4((E \times \mathbb{P}^1)_{an}, \mathcal{B}(2))$$

on the normalization. Moreover, $\psi_X$ fulfills the conditions described in 3.1.

**Proof.** We just have to verify the condition 4 of 3.1. From the normalization sequence

$$0 \to \mathcal{O}_X^* \xrightarrow{\nu_* \mathcal{O}_{E \times \mathbb{P}^1}^*} \xrightarrow{|E \times 0 - |E \times \infty} \mathcal{O}_E^* \to 0,$$

one has a natural map

$$\mathcal{O}_{X_{an}}^* \otimes \mathcal{O}_{X_{an}}^* \to \bar{K}_{2,an}$$

which obviously fulfills [3.1] 4. $\Box$

Now we can apply Theorem 3.1 to conclude

**Theorem 4.4.** The 0-cycles defined by the Steinberg curve on $E \times E_0$ die in the analytic motivic cohomology $\mathbb{H}^4(X_{an}, \bar{B}(2))$.

Let $K$ be a subfield of $\mathbb{C}$. We next consider for any algebraic variety $Z$ defined over $K$, the cycle map with values in the absolute Hodge cohomology

$$(4.7)\quad H^m(Z, \mathcal{K}_2) \xrightarrow{d \log \wedge d \log} H^m(Z, \Omega^2_{Z/\mathbb{Q}}).$$
induced by the absolute \( d \log \) map

\[
\mathcal{O}_Z \xrightarrow{d \log} \Omega^1_{Z/Q}.
\]

This cycle map is obviously compatible with the map \( \gamma \), and with extension of scalars.

Let \( E \to \text{Spec} \, K \) be an elliptic curve over a subfield \( K \) of \( \mathbb{C} \). We have the exact sheaf sequence

\[
0 \to \mathcal{O}_E \otimes \Omega^1_{K/Q} \to \Omega^1_{E/Q} \to \Omega^1_{E/K} \to 0,
\]

which induces a two-term filtration \( F^i \Omega^2_{E/Q} \) of \( \Omega^2_{E/Q} \) with \( F^2 \Omega^2_{E/Q} = \mathcal{O}_E \otimes \Omega^2_{K/Q} \). This gives us the natural maps

\[
\begin{align*}
\gamma_1 & : H^*(E, \mathcal{O}_E) \otimes \Omega^1_{K/Q} \to H^*(E, \Omega^1_{E/Q}) \\
\gamma_2 & : H^*(E, \mathcal{O}_E) \otimes \Omega^2_{K/Q} \to H^*(E, \Omega^2_{E/Q}).
\end{align*}
\]

We have the norm map \( \text{Nm} : H^1(E, \mathcal{K}_2) \to H^0(K, \mathcal{K}_1) = K^* \) as in \( \text{[1.2]} \), but over \( K \); we let \( V(E) \subset H^1(E, \mathcal{K}_2) \) be the kernel of \( \text{Nm} \) (see \( \text{[4.3]} \)).

**Lemma 4.5.** Let \( K \) be an algebraically closed subfield of \( \mathbb{C} \), \( E \to \text{Spec} \, K \) an elliptic curve over \( K \). Then the cycle map with values in absolute Hodge cohomology maps \( V(X) \) to the subgroup \( \gamma_2[H^1(E, \mathcal{O}_E) \otimes \Omega^2_{E/Q}] \) of \( H^1(E, \Omega^2_{E/Q}) \).

**Proof.** The kernel of the composition

\[
\text{Pic}(E) = H^1(E, \mathcal{K}_1) \xrightarrow{d \log} H^1(E, \Omega^1_{E/Q}) \to H^1(E, \Omega^1_{E/K}) \cong K
\]

is the composition

\[
\text{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \subset K,
\]

hence the \( d \log \) map sends \( \text{Pic}^0(E) \) to the subgroup \( \gamma_1[H^1(E, \mathcal{O}_E) \otimes \Omega^1_{K/Q}] \) of \( H^1(E, \Omega^1_{E/Q}) \).

Take \( \tau \in \text{Pic}^0(E) \), \( u \in H^0(E, \mathcal{K}_1) = K^* \), and let \( \xi = \tau \cup u \in H^1(E, \mathcal{K}_2) \). Then

\[
d \log(\xi) = d \log(\tau) \cup d \log(u).
\]

Since \( d \log : K^* \to \Omega^1_{K/Q} \) is just the absolute \( d \log \) map, we see that \( d \log(\xi) \) lands in the image of the cup product map

\[
[H^1(E, \mathcal{O}_E) \otimes \Omega^1_{K/Q}] \otimes \Omega^1_{K/Q} \to H^1(E, \Omega^2_{E/Q}),
\]

which is \( \gamma_2[H^1(E, \mathcal{O}_E) \otimes \Omega^2_{K/Q}] \).

Since \( K \) is algebraically closed, the cup product \( \text{Pic}(E) \otimes K^* \to H^1(E, \mathcal{K}_2) \) is surjective, from which one sees that the cup product maps
Pic\(^0\)(E) \(\otimes\) K\(^*\) onto V(E). Combining this with the computation above completes the proof.

From the surjectivity of the cup product Pic\(^0\)(E) \(\otimes\) K\(^*\) \(\rightarrow\) V(E) for K algebraically closed, we see that the injection \(H^1(E, \mathcal{K}_2) \rightarrow \text{CH}_0(X)\) sends V(E) isomorphically onto \(F^2\text{CH}_0(X)\).

Let K be a subfield of \(\mathbb{C}\). We say that an element \(\xi\) of \(\text{CH}_0(X)\) is defined over K if there is a K-scheme \(X^0\), an element \(\xi^0\) of \(\text{CH}_0(X^0)\) and an isomorphism \(\alpha : X^0_C \rightarrow X\) such that \(\xi = \alpha_* (\xi^0_C)\). From Lemma 4.5 and the compatibility of \(d\log\) with extension of scalars, we have

**Lemma 4.6.** Take \(K = \mathbb{C}\), and let \(\xi\) be an element of \(F^2\text{CH}_0(X) = V(E)\). If \(\xi\) is defined over a field of transcendence degree one over \(\mathbb{Q}\), then \(\xi\) vanishes under the cycle map to absolute Hodge cohomology.

**Corollary 4.7.** If \(E\) is an elliptic curve with complex multiplication, then there are non-torsion cycles \(\xi \in F^2\text{CH}_0(X)\) dying in the analytic motivic cohomology as well as in absolute Hodge cohomology.

**Proof.** By the remark above, we may replace \(F^2\text{CH}_0(X)\) with \(V(E)\). Let \(\bar{E}\) be a model for \(E\), with equation \(y^2 = 4x^3 - ax - b\) defined over a number field \(K \subset \mathbb{C}\). Let \(\omega = \frac{dx}{y}\) be the standard global one-form on \(\bar{E}\).

Choosing an isomorphism \(\bar{E}_C \cong E_C\) defines the period lattice \(L_\omega \subset \mathbb{C}\) for \(\omega\). Choose a basis for \(L_\omega\) of the form \(\{\Omega, \tau\Omega\}\), and let \(t = e^{2\pi i \tau}\). Let

\[\mathcal{P} : \mathbb{C} \rightarrow \mathbb{CP}^1\]

be the Weierstraß \(P\)-function for the lattice \(L_\omega\).

The map \(\times \Omega^{-1} : \mathbb{C} \rightarrow \mathbb{C}\) gives rise to the isomorphism of Riemann surfaces \(\alpha_{an} : \bar{E}_{an} \rightarrow \bar{E}_t\) making the diagram

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\times \Omega^{-1}} & \mathbb{C} \\
\downarrow & & \downarrow \text{exp} \\
\mathbb{C}^* & \xrightarrow{p} & \bar{E}_{an} \rightarrow \bar{E}_t^{an} \\
\downarrow & & \downarrow \alpha_{an} \\
\end{array}
\]

commute, i.e.,

\[p(u) = \alpha_{an}(\mathcal{P}(\frac{\Omega}{2\pi i} \log u), \mathcal{P}'(\frac{\Omega}{2\pi i} \log u)).\]

We let

\[\alpha : \bar{E}_C \rightarrow E_t\]
be the corresponding isomorphism of algebraic elliptic curves over \( \mathbb{C} \).

By [1], th\' eor\' eme 1, \( \mathcal{P}(\frac{\Omega}{2\pi i}\log u) \) has transcendence degree 1 over \( \bar{\mathbb{Q}} \) for all \( u \in \mathbb{N}, u \geq 2 \). (We thank Y. Andr\'e for giving us this reference). Fix a \( u \geq 2 \), let \( K \) be the algebraic closure of the field \( \mathbb{Q}(\mathcal{P}(\frac{\Omega}{2\pi i}\log u)) \), and let \( x \in \bar{E}(K) \) be the point \( (\mathcal{P}(\frac{\Omega}{2\pi i}\log u), \mathcal{P}'(\frac{\Omega}{2\pi i}\log u)) \). Then \( x \) is a generic point of \( \bar{E} \) over \( \bar{\mathbb{Q}} \).

We take

\[ \xi := p(u) * p(1 - u). \]

By construction, \( \xi = \alpha(\xi_K \times_K \mathbb{C}) \), where \( \xi_K \in H^1(\bar{E}, \mathcal{K}_2) \) is the element \( [(x) - (0)] \cup [1 - u] \). Here \( [(x) - (0)] \) denotes the class in \( \text{Pic}(\bar{E}) = H^1(\bar{E}, \mathcal{K}_1) \), and \( [1 - u] \) denotes the class in \( H^0(\bar{E}, \mathcal{K}_1) = K^\ast \). Since \( K \) has transcendence degree one over \( \bar{\mathbb{Q}} \), the class of \( \xi \) in the absolute Hodge cohomology of \( \bar{E} \) vanishes, by Lemma 4.6. By Theorem 4.4, \( \xi \) dies in the analytic motivic cohomology of \( \bar{E} \) as well. It remains to show that \( \xi \) is a non-torsion element of \( H^1(E_K, \mathcal{K}_2) \).

We give an analytic proof of this using the regulator map with values in Deligne-Beilinson cohomology.

Let \( Y \) be a smooth projective surface over \( \mathbb{C} \), and let \( \text{NS}(Y) \) denote the Néron-Severi group of divisors modulo homological equivalence. Then Hodge theory implies that

\[ \text{NS}(Y) = \{(z, \varphi) \in (H^2(Y_{\text{an}}, \mathbb{Z}(1)) \times F^1H^2(Y_{\text{an}}, \mathbb{C})), z \otimes \mathbb{C} = \varphi\}, \]

and that

\[ \text{NS}(Y) \cap F^2H^2_{DR}(Y) = \emptyset. \]

We note that the map \( \text{Pic}(Y) \otimes \mathbb{C}^\ast \to H^2_{DR}(Y, \mathbb{Z}(2)) \) induced by the cup product in Deligne cohomology factors through \( \text{NS}(Y) \otimes \mathbb{C}^\ast \), and that the induced map \( \iota : \text{NS}(Y) \otimes \mathbb{C}^\ast \to H^2_{DR}(Y, \mathbb{Z}(2)) \) is injective. Indeed,

\[ H^2_{DR}(Y, \mathbb{Z}(2)) = H^2(Y_{\text{an}}, \mathbb{C}/\mathbb{Z}(2))/F^2. \]

Now take \( Y = E \times E \), and let \( U \subset E \) be the complement of a non-empty finite set \( \Sigma \) of points of \( E \). Let \( [E \times 0] \) be the class of \( E \times 0 \) in \( \text{NS}(Y) \), and let \( \gamma : \mathbb{C}^\ast \to \text{NS}(Y) \otimes \mathbb{C}^* \) be the map \( \gamma(v) = [E \times 0] \otimes v \).

Let

\[ \iota_U : \text{NS}(Y) \otimes \mathbb{C}^\ast \to H^3_{DR}(E \times U, \mathbb{Z}(2)) \]

be the composition of \( \iota \) with the restriction map \( H^3_{DR}(Y, \mathbb{Z}(2)) \to H^3_{DR}(E \times U, \mathbb{Z}(2)) \). We claim that the sequence

\[ \mathbb{C}^\ast \xrightarrow{\gamma} \text{NS}(Y) \otimes \mathbb{C}^\ast \xrightarrow{\iota_U} H^3_{DR}(E \times U, \mathbb{Z}(2)) \]

is exact. Indeed, we have the localization sequence

\[ \bigoplus_{s \in \Sigma} H^1_{DR}(E \times s, \mathbb{Z}(1)) \xrightarrow{\oplus \iota_{ts}} H^3_{DR}(Y, \mathbb{Z}(2)) \to H^3_{DR}(E \times U, \mathbb{Z}(2)) \to, \]
the isomorphism $H^2_b(E \times s, \mathbb{Z}(1)) \cong \mathbb{C}^*$ and the identity
\[ \iota_s(v) = \gamma(v), \quad v \in \mathbb{C}^*, \]
which proves our claim.

In particular, let $[\Xi] = [\Delta - \{0\} \times E] \otimes v$, where $\Delta$ is the diagonal, $v$ is an element of $\mathbb{C}^*$ which is not a root of unity, and $[\Delta - \{0\} \times E]$ is the class in $\text{NS}(Y)$. Since $[\Delta - \{0\} \times E]$ is not torsion in $\text{NS}(Y)/[E \times \{0\}]$, we see that $[\Xi]$ has non-torsion image $[\Xi]$ in
\[ H^3_D(E \times \mathbb{C} \mathbb{C}(E), \mathbb{Z}(2)) := \lim_{\varnothing \neq U \subset E} H^3_D(E \times U, \mathbb{Z}(2)), \]
where the limit is over non-empty Zariski open subsets $U$ of $E$.

Let $\Xi$ be the image of $(\Delta - 0 \times E) \otimes v$ in $H^1(Y, \mathcal{K}_2)$. Then $[\Xi]$ is the image of $\Xi$ under the regulator map $H^1(Y, \mathcal{K}_2) \to H^3_D(Y, \mathbb{Z}(2))$. Similarly, letting $\Xi_{\mathbb{C}(E)}$ be the pull-back of $\Xi$ to $E \times \mathbb{C} \mathbb{C}(E)$, $[\Xi_{\mathbb{C}(E)}]$ is the image of $\Xi_{\mathbb{C}(E)}$ under the regulator map $H^1(E \times \mathbb{C} \mathbb{C}(E), \mathcal{K}_2) \to H^3_{\mathbb{D}}(E \times \mathbb{C} \mathbb{C}(E), \mathbb{Z}(2))$. Thus, $\Xi_{\mathbb{C}(E)}$ is a non-torsion element of $H^1(E \times \mathbb{C} \mathbb{C}(E), \mathcal{K}_2)$ for each non-torsion element $v \in \mathbb{C}^*$.

Let $\Delta$ be the diagonal in $\bar{E} \times \bar{E}$, let $\xi$ be the image of $(\Delta - 0 \times \bar{E}) \otimes (1 - u)$ in $H^1(E, \mathcal{K}_2)$, and let $\xi_{\mathbb{Q}(E)}$ be the image of $\xi$ in $H^1(\bar{E} \times \mathbb{Q} \mathbb{Q}(\bar{E}), \mathcal{K}_2)$. Clearly, after choosing a complex embedding $\mathbb{Q} \subset \mathbb{C}$, $\Xi_{\mathbb{C}(E)}$ (for $v = 1 - u$) is the image of $\xi_{\mathbb{Q}(E)}$ under the extension of scalars $\mathbb{Q}(\bar{E}) \to \mathbb{C}(\bar{E}) \cong \mathbb{C}(E)$, hence $\xi_{\mathbb{Q}(E)}$ is a non-torsion element of $H^1(\bar{E} \times \mathbb{Q} \mathbb{Q}(\bar{E}), \mathcal{K}_2)$.

Since $x$ is a geometric generic point of $\bar{E}$ over $\bar{Q}$, there is an embedding $\sigma : \bar{Q}(E) \to \mathbb{C}$ such that $x : \text{Spec} \mathbb{C} \to \bar{E}$ is the composition $\text{Spec} \mathbb{C} \to \text{Spec} \mathbb{Q}(\bar{E}) \to \bar{E}$. Thus, $\xi$ is the image of $\xi_1$ under $(\text{id} \times x)^* : H^1(E \times \mathbb{Q} \mathbb{Q}(\bar{E}), \mathcal{K}_2) \to H^1(E, \mathcal{K}_2)$, and hence $\xi$ is the image of $\xi_{\mathbb{Q}(E)}$ under the map $\text{id} \times \sigma_* : H^1(\bar{E} \times \mathbb{Q} \mathbb{Q}(\bar{E}), \mathcal{K}_2) \to H^1(E, \mathcal{K}_2)$ induced by the extension of scalars $\sigma$.

Since the kernel of $\text{id} \times \sigma_*$ is torsion, it follows that $\xi$ is a non-torsion element of $H^1(E, \mathcal{K}_2)$, as desired. \[\square\]

Remark 4.8. Going back to $X = E \times E'$, where both elliptic curves are smooth, we are lacking the transcendence theorem which would force the existence of a cycle $0 \neq \xi = p(u) * p(1 - u) \in F^2\text{CH}_{0}(X)$ dying both in $\mathbb{H}^4(X, \mathcal{B}(2))$ and in absolute Hodge cohomology.

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Universität GH Essen, FB6 Mathematik und Informatik, 45117 Essen, Germany, and Department of Mathematics, Northeastern University, Boston, MA 02115, USA

E-mail address: esnault@uni-essen.de marc@neu.edu