The Road to Stueckelberg’s Covariant Perturbation Theory as Illustrated by Successive Treatments of Compton Scattering

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Abstract

We review the history of the road to a manifestly covariant perturbative calculus within quantum electrodynamics from the early semi-classical results of the mid-twenties to the complete formalism of Stueckelberg in 1934. We chose as our case study the calculation of the cross-section of the Compton effect. We analyse Stueckelberg’s paper extensively. This is our first contribution to a study of his fundamental contributions to the theoretical physics of twentieth century.

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One often considers the birth of quantum electrodynamics (QED) as one of the most exciting and challenging chapters of the history of modern physics. Though not a fundamentally new theory, like general relativity or quantum mechanics, QED, as an effort to obtain a consistent relativistic treatment of the interaction of matter with radiation, gave birth to most fertile concepts of modern microphysics such as quantum fields, gauge symmetries and

1 Relativistich invariante Störungstheorie des Diracschen Elektrons (Relativistic Invariant Perturbation Theory of the Dirac Electron), *Annalen der Physik*, received 10.Sept.1934.

2 Remark on the intensity of the radiative scattering of moving free electrons, Stueckelberg 1935a.
renormalization techniques. These developments have already been covered in many excellent studies and publications. There is however another aspect of the history of QED, equally symptomatic of a definite tendency that culminated with the widely used Feynman diagrams. We deal with a theory defined, maybe for the first time in such explicit terms, by its associated perturbation expansion. Indeed, most of the tools and techniques used were introduced in an attempt to give theoretical meaning and practical usefulness to the successive terms of perturbative series.

Although classical electrodynamics had existed and had been successfully applied for more than half a century, its closed-form fundamental laws (Maxwell equations) were not adequate to yield a fully successful quantum version by a straightforward process of quantization. Many difficulties had to be overcome. On the one hand, there was the need to accommodate the continuous field equations to the fundamentally discontinuous quantum processes of energy and momentum exchange between photons and matter particles. On the other hand, one faced the difficulties, already present in the classical theory, of formulating a consistent relativistic theory of several interacting particles.

The first step on the road to a full-fledged quantum theory was the quantization of the radiation field itself. The 1927 papers of Dirac opened the way. He showed how to quantize the radiation field as an assembly of non-interacting bosons and proposed a perturbation scheme enabling one to deal with its time-dependent coupling to matter. His method of quantization made possible the first quantum treatment of emission-absorption processes and provided a decisive step towards a genuine quantum theory of photon-electron interaction. Until the advent of the modern S-matrix approach within quantum field theory at the end of the forties, the spirit of this method was very influential, supplemented by Dirac’s equally famous relativistic electron equation, as the main theoretical tool of investigation in the study of matter-radiation interaction.

Following Dirac’s pioneering theory, one could distinguish several lines of research. The main one dealt with a consistent formulation of a rela-

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3For a philosophical analysis of the emergence of these concepts, see Cao 1997.

4For instance the penetrating accounts by A. Pais, *Inward Bound. Of Matter and Forces in the Physical Universe* (Pais 1986) and S. S. Schweber’s *Q.E.D. and the Men who made it*, (Schweber 1994), both invaluable sources for contemporary history of particle physics.

5For an analysis of the S-matrix approach see Cushing 1990.

6The other important source of inspiration was the general theory of quantized fields by Heisenberg and Pauli 1929 and 1930; Fermi 1932 simplified the quantization of the electromagnetical field.
tivistic theory of particles interacting with and through the radiation field. Before the advent of quantum mechanics, the problem was tackled in a "correspondence principle" way as illustrated for instance by the achievement of Kramers and Heisenberg (1925). The quantum evolution equations of matrix and wave mechanics were the foundations of semi-classical treatments, like those of Dirac (1926b and 1927a), of Gordon (1926) (for spin 0 particles), and of Klein and Nishina (1929) (spin 1/2 electrons). The fundamental insights of Dirac (Dirac 1932) enabled further progress, resulting in the interaction picture (Dirac, Fock, Podolsky 1932, see however already Dirac 1927b) and the multiple-time formalism. Of course, the problem couldn’t find a complete solution until it was realized that both radiation and matter had to be expressed as quantized fields. The point-particle singularities were then "dissolved" into field excitations and the problem of coupling consistently matter to radiation transformed into that of field-field coupling. This modern standpoint was initiated by the work of Pauli and Heisenberg (Heisenberg and Pauli 1929 and 1930) and preceded by Jordan’s insights (Jordan 1927a,b). Along this fundamental thread, one had to provide a systematic way of generating and organizing the successive terms of the perturbative series without losing the manifest relativistic invariance and other symmetries (gauge invariance) characteristic of the full equations. We shall focus our attention on these issues, leaving aside the further problem of the infinities arising in the higher-order processes, such as the self-energy or vacuum polarization, which constitute another important chapter of the history of QED.

In what follows we shall attempt to sketch an outline of the various stages in the attainment of a manifestly covariant perturbation calculus of QED processes. As an illustration for our study we shall choose the case of Compton and related effects to lowest order. Here, by Compton effect, we mean mainly the scattering of a photon on a free electron; indeed in the twenties,
the Compton effect covered also the case of the scattering of high-energy photons on bound electrons, where the latter could be considered as free. We shall discuss the early results of Dirac, Gordon, and Klein and Nishina, then shall consider the generalized Klein-Nishina type formulas of Waller (1930) and Tamm (1930), and finally shall end up with the results of Stueckelberg (1934). The contribution of Stueckelberg shall be examined in closer detail, as we believe it constitutes the first complete and easily generalizable instance of a manifest relativistically invariant perturbative calculus. It is of interest in many respects, both technical and epistemological. The "modernity" of this contribution is striking and characteristic of Stueckelberg’s original turn of mind. From this point of view, the present study can be considered as a first stage in an attempt at drawing the attention of historians to Stueckelberg’s achievements during those years.

Our choice of the landmarks on the road towards Stueckelberg’s 1934 paper is accordingly motivated by the need to illustrate both the key challenges as well as the main theoretical inputs which motivated Stueckelberg’s approach. Thus, we start with a brief reminder of the "infamous" Bohr, Kramers and Slater proposal (Bohr, Kramers, Slater 1924), which can be considered as the final expression of the fundamental impotence of the "old" quantum physics as regards the emerging problem of matter-radiation interaction followed by a brief discussion of the Heisenberg-Kramers dispersion formulas. We next present two early Dirac papers on the Compton effect first treated with the matrix formalism and then with wave mechanics, a progression characteristic of the whole of quantum studies of the time (Dirac 1926b, 1927a). Gordon’s 1926 contribution is crucial in understanding the Klein-Nishina 1929 paper, both papers sharing the same strategy and scope. Dirac’s 1927 field quantization and time-dependent perturbation theory (Dirac 1926c, 1927b) are the essential techniques used subsequently by Waller and Tamm. Stueckelberg’s 1934 paper took also advantage from Dirac’s reflection (Dirac 1932) on the backreaction problem, which lead among other thing to the interaction picture of the later joint work of Fock, Dirac and Podolsky. We end up the paper with a discussion of the little interest met by Stueckelberg’s work at his time.

Throughout the paper we have tried to adopt a unified (modern) notation with some exceptions when there was a danger of betraying the spirit of the original paper. We report in the footnotes the most important changes with respect to the original versions, and discuss our conventions in the Appendix.
2 Preliminaries: the end of the old quantum theory, the BKS crisis and the Heisenberg-Kramers dispersion formula

2.1 Introduction

By the mid-twenties, the old quantum theory of the founding fathers, Planck, Einstein, Bohr and Sommerfeld was spectacularly extended by powerful new techniques which provided a formal framework for many of the "quantum rules" used, with much brio, in the blossoming field of atomic physics. Quantum mechanics, first in its matrix form, shortly followed by the wave version, was born. It offered the promise of a unified treatment of many problems and a way of solving long-standing difficulties. Indeed, in spite of undisputed successes of the old theory, persistent difficulties were taking increasing importance. Thus, after Bohr's successful theory of the atom, there remained the problem of explaining the dynamics of quantum transitions between atomic levels and more generally that of the interaction of radiation with matter. The classical continuous wave picture of radiation, after being challenged by Einstein in 1905 with his light-quantum interpretation of the photo-electric effect, was again questioned in 1922 this time by Compton who studied X-ray scattering. It was known since the work of D. C. H. Florance in 1910 and J. A. Gray in 1913 that the secondary radiation produced by irradiating a metal with gamma rays is "softer" (less penetrating) than the primary radiation. After his experiments in 1921, Compton, who realized that there was a shift of frequency, thought that a wave-model interpretation could still hold (Compton 1921). Facing the evidence of his subsequent experiments, he progressively changed his mind and proposed an explanation (Compton 1923) in terms of collision processes between individual electrons and light-quanta. Cloud-chamber experiments by C. T. R. Wilson and W. Bothe (Wilson 1923, Bothe 1923) showed tracks of the recoil electrons which were consistent with the light-quantum hypothesis. Still, the latter was far from being accepted and several first-rank physicists kept trying to accommodate a continuous wave type explanation to the discrete reality, using more or less modified forms of classical electrodynamics.

One of the fiercest opponents to the light-quantum nature of radiation was Niels Bohr. In an attempt to push the old wave scheme to its limits, Bohr, Kramers and Slater (BKS) published a paper (Bohr Kramers and Slater 1923) in which they derived a dispersion formula for the scattered light. Debye reached the same conclusions at approximately the same time (Debye 1923).

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For a review of the theory and experiments, see Compton and Allison (1935).
Slater 1924) where they followed up on an earlier idea of Slater (Slater 1922). The latter suggested to view the atom in a stationary state as an assembly of "virtual" harmonic oscillators with eigenfrequencies determined by the transitions between stationary states. In genuine "theoretical despair", Bohr, Kramers and Slater went as far as to question the validity of the energy conservation principle at the level of individual atomic processes, and replaced it by a milder version, holding only at the statistical level, a move apparently due to Bohr and Kramers and originally rejected by Slater. Part of Bohr’s motivation was then to save the wave model of radiation, which he believed essential for the explanation of the Compton effect, since the latter implied an increase of the wave-length, the measurement of which presupposed, in his opinion, the wave picture. Secondary radiation thus resulted in his view from a continuous process of emission of coherent waves from each illuminated atom.

The empirical refutation of the BKS proposal came quickly. Indeed, in the BKS theory, the electron recoil (which paradoxically still made sense in this approach) would occur with finite probability in any direction, whereas according to the light-quantum hypothesis, it would strictly depend on that of the incident radiation. The validity of the conservation principles in the individual collisions was established beyond doubt by Bothe and Geiger (1925) on one hand, and by Compton and Simon (1925) on the other.

Be it as it may, Compton scattering vindicated the power of conservation principles and joined the ranks of other well-established processes involving light quanta and electrons, such as Bremsstrahlung, or the photo-electric effect, which were all in 1924 still lacking a fully satisfactory quantum treatment because of the absence of a genuine quantum starting point.

### 2.2 The Heisenberg-Kramers dispersion relations and the first attempts

In the quest for a quantitative approach, an important step was however taken by Kramers and Heisenberg [Kramers and Heisenberg 1924] who established a dispersion formula describing the features of the radiation reemitted by an atom after illumination by primary radiation. The authors were able to achieve a quantitative result but their method excluded the possibility to be applied to more energetic regimes, those characteristic for instance

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12 See Van der Waerden 1968 for a study and an english translation of the paper. The reader will also find in Max Dresden’s (1987) biography of Kramers an insightful account of this and other episodes of the old quantum theory.

13 See Murdoch 1987.

14 See again Van der Waerden 1968.
of light scattering off unbound electrons. Beyond its intrinsic interest, the Heisenberg-Kramers dispersion formula became *un passage obligé* in every subsequent study, a standard to be reattained and a testing ground for any new technique, only to be replaced in this role by the Klein-Nishina formula at the end of the twenties. The approach of Kramers and Heisenberg was based purely on the correspondence principle and constitutes a brilliant example of the surprisingly powerful method the latter constituted in such expert hands. Together with similar approaches by Smekal 1923, Breit 1926, Wentzel 1927, and others, it can be considered as the outmost excursion of the old theory into a territory which could be fully explored only with radically new tools which were soon to come.

The Kramers-Heisenberg paper starts with an analysis of the then current standpoint on the problem of dispersion of radiation by atoms. After recalling the usual treatment of the coherent secondary radiation, understood (classically) as the result of the harmonic motion of the electrons excited by the external radiation, the authors mention the difficulty of analyzing this effect within the quantum theoretical picture of electronic transitions between stationary states:

> We are thus faced with the problem of describing the scattering and dispersion effects of the atom in terms of the quantum theoretical picture of atomic structure. According to this picture, the appearance of a spectral line is not linked with the presence of elastically oscillating electrons, but with transitions from one stationary state to another. Bohr’s correspondence principle however gives us a significant hint on how to describe the reaction of the atom to the radiation field, using classical concepts. In a recently published paper by Bohr, Kramers and Slater, the authors sketched in rough outline how such a description can be accomplished in a relatively simple manner. What is above all typical for this theory, is the assumption that the reaction of the atom to the radiation field should primarily be understood as a reaction of an atom existing in a given stationary state; it is also assumed that transitions between two stationary states are of very short duration, and that the detailed nature of these transitions will not play any role in the description of the optical phenomena.\(^\text{15}\)

Following this path, it is assumed that, under the influence of an external radiation, given by an oscillating electric field with amplitude \(E_0\) and

\(^{15}\text{Kramers-Heisenberg (1925), p. 682. We rely here and later on the English translation in Van der Waerden (1968), pp. 223-252.}\)
frequency $\nu$

$$E(t) = \text{Re}\{E_0 e^{i2\pi \nu t}\}, \quad (1)$$

as long as the atom remains in a stationary state, say, of energy $E_k$ it reemits spherical waves, corresponding to an oscillating dipole moment

$$D(t) = \text{Re}\{d e^{i2\pi \nu t}\}, \quad (2)$$

where the vector $d$ is proportional to $E_0$. However, if one takes further into account the (spontaneous) transitions to states of lower energy $E_l$, the atom radiates, in addition to the coherent emission (2), spherical waves with frequencies given by Bohr’s conditions $\nu_q = (E_k - E_l)/\hbar$. Interferences between the coherent and the spontaneous radiation should then lead to a modification of the secondary radiation. The form of these modifications as affecting the $\nu-$ dependence had been investigated by Kramers (1924) relying on classical dispersion theory and the correspondence principle. There, spontaneous emission was analyzed in terms of classical oscillators with eigen-frequencies $\nu_q$ and the actual (quantum) behaviour obtained taking the limit of high quantum numbers. The paper of Kramers and Heisenberg further extended the analysis of Kramers:

It is the purpose of this paper to show how the correspondence idea, when pursued more closely, leads to the surprising result that the assumption (2) for the reaction of an atom to incident radiation is too restricted, and will in general have to be extended by adding a series of terms as follows:

$$D(t) = \text{Re}\left\{d e^{i2\pi \nu t} + \sum_k d_k e^{i2\pi (\nu + \nu_k) t} + \sum_l d_l e^{i2\pi (\nu - \nu_l) t}\right\} \quad (3)$$

where $\hbar \nu_k$ or $\hbar \nu_l$ denote the energy difference between two stationary states of the atom, one of which is always identical with the momentary state of the atom; the vectors $d_k$ and $d_l$ again depend on $E$ (in the form of a linear vector function) and on $\nu$. Expressed in words, the result can be stated as follows: *Under the influence of irradiation with monochromatic light, an atom not only emits coherent spherical waves of the same frequency as that of the incident light: it also emits systems of incoherent spherical waves, whose frequencies can be represented as combinations of the incident frequency with other frequencies that correspond to possible transitions to other stationary states. Such additional systems of spherical waves must evidently occur in the form of*
scattered light; but they cannot make any contribution to the dispersion and absorption of the incident light.

The analysis of Kramers and Heisenberg does not rely on the concept of light quanta, which as we have mentioned, was still considered by some as unwarranted. However, their result is related to some earlier work by Smekal (1923) who explicitly used the corpuscular hypothesis. Smekal's paper is remarkable, not only because of his use of the corpuscular hypothesis, but also because it conjectured a new effect, incoherent scattering of light by atoms. This is what Kramers and Heisenberg tell us above. This effect, nowadays called Raman scattering, was confirmed experimentally in 1928.

A more detailed discussion of the Heisenberg-Kramers would be here out of place; we wanted merely to illustrate the spirit of an approach using the correspondence principle, the latter, when coupled to the general theory of multiply periodic systems (not discussed here), being then the sole systematic theoretical tool in the obtention of quantum results before the breakthrough of quantum mechanics.

3 Semi-classical attempts to Compton scattering

3.1 Introduction

In the first attempts at studying matter-radiation interaction, as in the problem of illuminating an atom with radiation, the standpoint was to consider the re-emitted radiation as a result of some internal process involving the electrons within a well-defined regime, described as a classical multiply-periodic system to which the quantization rules could then further be applied with the guidance of the correspondence principle. The situation changed drastically with the advent of quantum mechanics and the availability of genuine quantum evolution equations. Then, the interaction of electrons with radiation could be, at least in principle, handled directly. In the first stage, where the

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16 See van der Waerden translation (1968), p. 227. The Kramers-Heisenberg result, valid in the long-wave approximation, was generalized to low energy x-rays and bound electrons by Wentzel 1927, using non-relativistic quantum mechanics. See also Waller 1929.

17 One would nowadays speak of inelastic scattering.

18 See Raman (1928), Raman and Krishnan (1928), and also Landsberg and Mandelstam 1928.

19 For an excellent survey, one can consider the book of Friedrich Hund, *Geschichte der Quantentheorie*, Hoschultaschenbücher, Bibliographisches Institut, Mannheim 1967. See also Dresden (1987).
radiation field wasn’t quantized yet, one had to use semi-classical methods deriving the features of the secondary radiation from the classical Maxwell formulas supplemented by the quantum expressions for charge and current densities.

In what follows, we shall review several contributions which we chose to group together because they all share a peculiar interplay of classical and quantum techniques. None of the authors (including Klein and Nishina) were working consistently in a purely quantum mechanical setting, until the contributions of Waller and especially that of Tamm in 1930, which were already part of the next stage, starting with the field quantization breakthrough of Dirac in 1927. This interplay illustrates how young the quantum formalism was in those years and how dominant the influence of the correspondence principle still was.

3.2 Dirac: first uses of the new quantum mechanics

We start our story with a publication of Dirac (Dirac 1926b, received 29 April 1926), who applied Heisenberg’s matrix formalism to provide a first treatment of the Compton scattering within the setting of the newly born quantum mechanics\(^{20}\). In this paper, as in the preceding ones (Dirac 1925 and 1926a), considerable effort is devoted to the development of a universal q-calculus, wherein Born, Heisenberg and Jordan’s original formalism is systematized and generalized (Born, Heisenberg and Jordan 1926). His treatment tries moreover to build relativity into the matrix formalism, a rather cumbersome task because of the necessarily privileged role of time in the canonical formalism. The key idea is to use in those terms of the matrix elements which are related to an oscillation of angular frequency \(\omega = 2\pi \nu\), instead of the non covariant expressions \(\exp(i\omega t)\), the relativistic expressions \(\exp(i\omega t')\), where

\[
t' = t - \frac{(l_1 x_1 + l_2 x_2 + l_3 x_3)}{c}
\]

Here, \((l_1, l_2, l_3)\) is the unit vector in the direction of observation and \(x_i\; i = 1, 2, 3\) the coordinates of the electron. Clearly, \((l_1, l_2, l_3)\) will eventually correspond to the direction of the scattered (secondary) photon. The 1926 paper achieves this program using only the tools of canonical commutation relations

\(^{20}\)It is interesting to mention also a paper by Schrödinger (1927) where he obtains the conservation law of the Compton scattering using the analogy between radiation ”scattering” off the square of the wave function, and radiation scattered off a pressure wave, as solved by Brillouin.
of operators\footnote{The paper was apparently not easy to understand, to witness Klein (1968) p. 79: "One day Ehrenfest brought us [Uhlenbeck and Goudsmit] a new paper by Dirac containing a quantum dynamical theory of the Compton effect which we tried to read, but without success". The same Ehrenfest wrote to Dirac (in Kragh 1990, p. 53): "Einstein is currently in Leiden (until Oct. 9th). In the few days we have left, he, Uhlenbeck and I are struggling together for hours at a time studying your work, for Einstein is eager to understand it. But we are hitting at a few difficulties, which - because the presentation is so short - we seem absolutely unable to overcome."}. Dirac will publish a second paper (\citeyear{Dirac1927}, received \text{22 Nov. 1926}) where he reaches the same goal with the help of wave mechanics. The physical treatment of both papers shares the same overall strategy. However, the use of wave mechanics makes the calculations easier because, according to Dirac:

This [matrix mechanics] method is rather artificial; particularly so since on the quantum theory there is no unique way of writing the Hamiltonian equation [...] (owing to the ambiguity in the order of the factors in $H$), and it has to be proved that all reasonable ways of doing so lead to the same results. A more natural and more easily understood method of obtaining the matrices is provided by Schrödinger’s wave mechanics\footnote{Dirac (1927\text{a}), p. 500.}.

Indeed, instead of using quantum equations for canonical operators compatible with the variable $t'$, one can in the wave formalism simply work at the level of the solutions $\psi$ of the wave equation. Assuming known a set of such solutions $\psi_{\alpha}$ parametrized by some quantum numbers (collectively denoted $\alpha$) the matrix elements of any dynamical variable $C$ are obtained through the expansion

$$ C\psi_{\alpha} = \sum_{\alpha'} C_{\alpha\alpha'} \psi_{\alpha'} \quad (5) $$

where the matrix elements $C_{\alpha\alpha'}$ can then be expressed as functions of $t'$. The off-diagonal matrix elements are further interpreted as related to the transition from state $\alpha$ to state $\alpha'$, as initiated in Heisenberg (1925), Born, Heisenberg and Jordan (1926), and in Dirac’s own papers, as for instance Dirac (1926\text{a}). This is precisely the road that Dirac will follow in his second paper as we discuss it now\footnote{The reader will find a general discussion of Dirac’s first paper in Mehra and Rechenberg 1982-87, vol. 4, p. 196-213.}

One considers, in the presence of an electromagnetic potential, the relativistic equation for a charged particle

$$ (p^\nu - \frac{e}{c} A^\nu)(p_\nu - \frac{e}{c} A_\nu) = -m^2 c^2, \quad (6) $$
where the four vector \( A = (\mathbf{A}, A_0) \), \( \mathbf{A} \) is the vector and \( A_0 \) the scalar potential, \( e \) the electron charge, \( c \) the speed of light (the summation convention over \( \nu = 0, 1, 2, 3 \) is applied throughout). This yields, after the (Schrödinger) substitutions
\[
p_i \rightarrow -i\hbar \frac{\partial}{\partial x_i} \equiv \hat{p}_i; \quad cp_0 = E \rightarrow i\hbar \frac{\partial}{\partial t} \equiv \hat{E}
\]
the wave equation
\[
\left\{ m^2c^2 - \frac{\hat{E}^2}{c^2} + \hat{p}_1^2 + (\hat{p}_2 + a' \cos k_1(ct - x_1))^2 + \hat{p}_3^2 \right\} \psi = 0
\]
where Dirac specializes to the case \( A_1 = A_3 = A_0 = 0 \), and \( A_2 = a \cos k_1(ct - x_1) \) with \( k_1 = \omega/c \), the only non zero component of the wave vector \( \mathbf{k} = (\omega/c, 0, 0) \) giving the direction of propagation of the electromagnetic wave. Moreover, \( a' = ae/c \). The solutions to (8) will be obtained under the assumption that the amplitude \( a \) is small so that higher order terms in \( a \) can be neglected. Dirac first performs the (canonical) transformations (in order not to overload the notations, we drop here and in the sequel the carot distinguishing the operators \( \hat{p} \) from the c-numbers \( p \))
\[
\tilde{x}_1 = ct - x_1; \quad p_1 = -\tilde{p}_1 + l_1 \frac{\tilde{E}}{c} \\
\tilde{x}_2 = x_2; \quad p_2 = \tilde{p}_2 + l_2 \frac{\tilde{E}}{c} \\
\tilde{x}_3 = x_3; \quad p_3 = \tilde{p}_3 + l_3 \frac{\tilde{E}}{c} \\
\tilde{t} = t - (l_1x + l_2x_2 + l_3x_3)/c; \quad E = \tilde{E} - c\tilde{p}_1
\]
These transformations enable to separate the variables because the wave equation does not involve now explicitly \( \tilde{x}_2, \tilde{x}_3 \) and \( \tilde{t}' \):
\[
(m^2c^2 + 2\frac{\tilde{E}}{c} \left\{ (1 - l_1)\tilde{p}_1 + l_2\tilde{p}_2 + l_3\tilde{p}_3 + l_2a' \cos k_1\tilde{x}_1 \right\} + \tilde{p}_2^2 + \tilde{p}_3^2 + 2a'\tilde{p}_2 \cos k_1\tilde{x}_1)\psi = 0.
\]
Consequently one can set the general solution to be
\[
\psi_{(\tilde{p}_2, \tilde{p}_3, \tilde{E})} = e^{i\tilde{p}_2\tilde{x}_2/\hbar} e^{i\tilde{p}_3\tilde{x}_3/\hbar} e^{-i\tilde{E}/\hbar} \chi(\tilde{x}_1).
\]
here, \(\tilde{p}_2, \tilde{p}_3, \tilde{E}\) are the eigenvalues of the operators \(\tilde{p}_2, \tilde{p}_3\) and \(\tilde{E}\). The substitution of this ansatz into (10) enables to determine the function \(\chi(\tilde{x}_1)\):

\[
\begin{align*}
\chi(\tilde{x}_1) &= \exp \left( -\frac{i}{(1-l_1)\hbar} \left( (l_2 \tilde{p}_2 + l_3 \tilde{p}_3 + \frac{cb}{2\tilde{E}}) \tilde{x}_1 + \frac{a'}{k_1} (l_2 + \frac{c \tilde{p}_3}{\tilde{E}}) \sin k_1 \tilde{x}_1 \right) \right) \quad (12) \\
b &= m^2 c^2 + \tilde{p}_2^2 + \tilde{p}_3^2
\end{align*}
\]

We have now a set of solutions \(\psi_{(\tilde{p}_2, \tilde{p}_3, \tilde{E})}\) each parametrized by the quantum numbers \(\tilde{p}_2, \tilde{p}_3, \tilde{E}\) corresponding to the motion of the electron in the external radiation. To study the features of the secondary radiation, one has now to determine the relevant observables. Since classically the secondary radiation is related to the oscillation of the electron, the latter are chosen to be the amplitudes of vibration along (a basis) of axes chosen perpendicular to the direction of observation \((l_1, l_2, l_3)\). Dirac sets\(^{27}\):

\[
\begin{align*}
X &= l_3 x_1 - \frac{l_2 l_3}{1-l_1} x_2 + \left( \frac{l_2^2}{1-l_1} - l_1 \right) x_3 \\
Y &= l_2 x_1 + \left( \frac{l_3^2}{1-l_1} - l_1 \right) x_2 - \frac{l_2 l_3}{1-l_1} x_3
\end{align*}
\]

One can express further \(X\) and \(Y\) (as acting on the initial state) with the help of the constants of motion which enables to make explicit their \(t'\) dependence:

\[
\begin{align*}
(1-l_1)X &= c_3 + \frac{2 \tilde{p}_2 \tilde{E}}{b} c_4 + l_3 c_1 t + \frac{2 (1-l_1) \tilde{p}_2 \tilde{E}}{b} t - \frac{2 c a' \tilde{p}_2 \tilde{p}_2}{k_1 b \tilde{E}} \sin k_1 \tilde{x}_1. \quad (13) \\
(1-l_1)Y &= c_2 + \frac{2 \tilde{p}_2 \tilde{E}}{b} c_4 + l_2 c_1 t + \frac{2 (1-l_1) \tilde{p}_2 \tilde{E}}{b} t + \left( \frac{ca'}{k_1 \tilde{E}} - \frac{2 ca' \tilde{p}_3^2}{k_1 b \tilde{E}} \right) \sin k_1 \tilde{x}_1
\end{align*}
\]

The knowledge of the constants of motion \(c_2, c_3, c_4\) above is not material here so that we refer reader to Dirac’s paper, p. 503 for their exact expression.

We know that only the oscillating terms contribute to the emitted radiation. Inspection of (13) shows that we have to concentrate on \(\sin k_1 \tilde{x}_1\). According to the scheme explained before, an off-diagonal matrix element of this term will contain the information pertaining to the secondary radiation corresponding to the transition from initial state \((\tilde{p}_2, \tilde{p}_3, \tilde{E})\) to a final one \((\tilde{p}'_2, \tilde{p}'_3, \tilde{E}')\). To obtain the general matrix element, one acts on the (initial) state \(\psi_{(\tilde{p}_2, \tilde{p}_3, \tilde{E})}\) with the operator \(\exp ik_1 \tilde{x}_1\) and observes that (compare with (11))

\[
\exp ik_1 \tilde{x}_1 \cdot \psi_{(\tilde{p}_2, \tilde{p}_3, \tilde{E})}
\]

\(^{27}\)Multiplied by the charge they represent the components of the polarisation of the system, a terminology used by Dirac.
\[
\exp \left\{ \left( \tilde{x}_2 \tilde{p}_2 + \tilde{x}_3 \tilde{p}_3 - \tilde{E} \tilde{t} \right) - \frac{l_2 \tilde{p}_2 + l_3 \tilde{p}_3 + \frac{c b}{2E} + \hbar k_1}{1 - l_1} \tilde{x}_1 \right\}
\]
\[
\equiv \exp(-i\omega' \tilde{t}) \exp \left\{ \left( \tilde{x}_2 \tilde{p}_2 + \tilde{x}_3 \tilde{p}_3 - \tilde{E}' \tilde{t} \right) - \frac{l_2 \tilde{p}_2 + l_3 \tilde{p}_3 + \frac{c b}{2E}}{1 - l_1} \tilde{x}_1 \right\}
\]
where we have set \( \omega' = 2\pi \nu' \). In this calculation, one has again neglected higher terms in \( a' \). We have then the fundamental result
\[
\exp(ik_1 \tilde{x}_1) \cdot \psi(\tilde{p}_2, \tilde{p}_3, \tilde{E}) = \exp(-i\omega' \tilde{t}) \psi(\tilde{p}_2', \tilde{p}_3', \tilde{E}')
\]
with the only non-vanishing matrix element for \((\tilde{p}_2, \tilde{p}_3, \tilde{E}) \to (\tilde{p}_2', \tilde{p}_3', \tilde{E}')\) being \( \exp(-i\omega' \tilde{t}) \). Notice that this matrix element is taken between electron states which correspond both to solutions of involving the initial radiation field. Here,
\[
\begin{align*}
\tilde{p}_2' &= \tilde{p}_2 \\
\tilde{p}_3' &= \tilde{p}_3 \\
\tilde{E}' + \hbar \omega' &= \tilde{E} \\
\frac{c b}{2E'} &= \frac{c b}{2E} + \frac{\hbar \omega}{c}
\end{align*}
\]
Concerning \( \tilde{p}_1 \), it can be obtained, relying on the analogy with the free case, as the coefficient of \( \tilde{x}_1 \) in the eq. \((12)\) at the price of neglecting \( a' \sin k_1 \tilde{x}_1 \). This yields
\[
\tilde{p}_1 = -\frac{(l_2 \tilde{p}_2 + l_3 \tilde{p}_3 + \frac{c b}{2E})}{1 - l_1}
\]
To justify this step Dirac offers the following classical "justification". The neglecting of \( a' \) enables to give meaning to the variation of \( p_1 \) because, in order to give a meaning to the recoil momentum of the electron we must neglect the oscillations of the electron due to the incident radiation.\(^{28}\) Using \((14)\), one gets now
\[
\tilde{p}_1' = \tilde{p}_1 - \frac{\hbar \omega}{c}
\]
The equations above give the relations between the quantum numbers of initial and final states. These equations correspond to energy-momentum conservation; in the original variables, recalling eq. \((9)\) and \((11)\), one finds
\[
\begin{align*}
p_1' + l_1 \frac{\hbar \omega'}{c} &= p_1 + \frac{\hbar \omega}{c}
\end{align*}
\]
\(^{28}\)Dirac (1927a), p. 506.
\[ p_2' + l_2 \frac{\hbar \omega'}{c} = p_2 \]
\[ p_3' + l_3 \frac{\hbar \omega'}{c} = p_3 \]
\[ E' + \hbar \omega' = E + \hbar \omega \]

In the rest system, (15) yield the Compton relation
\[ \frac{\omega}{\omega'} = 1 + \frac{\hbar \omega}{mc^2} (1 - \cos \theta). \] (16)
which is equivalent in terms of the wave-length \( \lambda \) to
\[ \lambda' = \lambda + \frac{\hbar}{mc} (1 - \cos \theta) \] (17)

Here, \( \theta \) is the scattering angle.

The features of the emitted radiation are then obtained in the following way. Recall that \( X \) represents a component of the polarization and its matrix elements have been determined (see (13) and (14)). Relying on Heisenberg’s prescriptions, and his own results\(^\text{29}\), Dirac identifies the product \( X_{pp'}X_{p'p} \) (here we denote collectively the quantum numbers by \( p \)) as a quarter of the square of the amplitude of vibration in the \( X \)-direction. The square of the total amplitude \( |C|^2 \) is accordingly the sum of that for \( X \) and that for the (perpendicular) \( Y \). It can be related to the intensity (energy flux) at distance \( r \) using the classical formula
\[ I = \frac{e^2 \omega^4}{8 \pi c^3 r^2} |C|^2 \] (18)

This formula is the classical dipole approximation for the radiation emitted by an oscillating density of charge and current. Since for \( \hbar = 0 \) one has in (14) \( \mathcal{E}' = \mathcal{E} \), the quantum expression for the intensity turns out to be equal to \( (\mathcal{E}/\mathcal{E}') \) times its classical value (see Dirac’s first paper p. 405-406 and the

\(^{29}\)See Dirac (1926). Dirac discusses there the quantization of multiply-periodic quantum systems through the use of uniformizing variables \( J \) and \( \omega \). For such systems, the observables \( C \) have the general expansion
\[ \sum_\alpha C_\alpha(J_1, J_2, ...) e^{i (\alpha \omega)} = \sum_\alpha e^{i (\alpha \omega)} C_\alpha'(J_1, J_2, ...) \]
where the sum is over integers. The difference of order between the coefficients \( C_\alpha \), resp. \( C_\alpha' \) and the exponentials above can be related to the two possible transitions between a given pair of stationary states. In the case of polarization, the product of the coefficients \( C \) gives the intensity. See Dirac’s paper for more details.

16
second p. 506 for the detailed expressions). In particular, for an electron initially at rest, Dirac obtains for the intensity of the secondary radiation at distance \( r \) from the emitting electron the expression

\[
I = I_0 \frac{e^4}{m^2 c^4 r^2} \left( \frac{\omega'}{\omega} \right)^3 \sin^2 \phi
\]

where \( I_0 = a^2 \omega^2 / 8 \pi c \) is the intensity of the primary radiation, and \( \phi \) is the angle between the direction of observation and the initial polarization. In terms of the (differential) scattering cross section \( d\sigma / d\Omega \), Dirac’s result is then:

\[
\frac{\omega'}{\omega} \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \left( \frac{\omega'}{\omega} \right)^3 \sin^2 \phi
\]

As noted by Dirac, the result (19) is just \( \left( \frac{\omega'}{\omega} \right)^3 \) times its value according to the classical theory. For unpolarized incident radiation, one has to take the average of \( \sin^2 \phi \) over all directions of polarization, which produces a factor of \( 1 + \cos^2 \theta \)/2, i.e.

\[
\frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \left( \frac{\omega'}{\omega} \right)^2 (1 + \cos^2 \theta)/2
\]

or, using (19)

\[
\frac{d\sigma}{d\Omega}(\theta) = \frac{e^4}{2m^2 c^4} \frac{1 + \cos^2 \theta}{(1 + \alpha(1 - \cos \theta))^2}
\]

where \( \alpha = \hbar \omega / mc^2 \). The results (19) and (21) were already obtained in Dirac 1926b. After comparing this result with the available experimental data of Compton (Compton 1923), Dirac observes:

[... ] the experimental values are all less than the values given by the present theory, in roughly the same ratio (75 per cent.), which shows that the theory gives the correct law of variation of intensity with angle, and suggests that in absolute magnitude Compton’s values are 25 per cent. too small \( \text{[\ldots]} \).

The source of this discrepancy will be clarified by Klein and Nishina 1929.

---

\(^{30}\) The differential scattering cross section \( \frac{d\sigma}{d\Omega} \) is nowadays defined as the ratio of the number of scattered particles into the unit solid angle over the number of incident particles. This is related to the energy radiated per unit time, unit solid angle and unit incident intensity, because the energy is the number of light quanta multiplied by \( \hbar \omega \). To make contact with the preceding equation for the intensity one has to multiply it by \( r^2 \), divide by the intensity of the incident (plane) wave, and multiply by \( \frac{\omega'}{\omega} \).

\(^{31}\) Dirac 1926b, p. 423.
3.3 Gordon: quantized current of a scalar electron as source of the retarded classical potential

At the end of the second paper of Dirac, a footnote added in proof mentions a paper by Walter Gordon dealing with the same subject. Indeed, before Dirac’s second paper appeared, Gordon had treated the same problem, albeit in a more detailed way (Gordon 1926, received 29 Sept. 1926). As Dirac previously, and following Schrödinger’s prescription, he derived the relativistic quantum wave equation for a spinless charged particle coupled to the classical electromagnetic potential $A_\mu$:

$$\left[ (-i\hbar \partial_\mu - \frac{e}{c} A_\mu) \left( -i\hbar \partial^\mu - \frac{e}{c} A^\mu \right) + m^2 c^2 \right] \psi = 0 \quad (22)$$

This invariant wave equation is named today after him and Oscar Klein, who had obtained the same expression in a publication that appeared a few months earlier (Klein 1926). An important feature of Gordon’s paper is his understanding of gauge transformations at the wave function level. He properly realized that to the transformation $A_\alpha \to A_\alpha + \partial_\alpha f$ there corresponds the multiplicative phase factor $\psi \to e^{i\frac{e}{\hbar c} f} \psi$, where $f$ is an arbitrary function.

In his derivation of the secondary radiation, the strategy followed by Gordon sets the standard for subsequent works. It is, together with the previous approach by Dirac, a testimony to the power of the quantum formalism which takes care of the situation way beyond what might have been expected at first sight. The electron is considered as perturbed from its free evolution by the primary radiation. Assuming its perturbed motion as known, the features of the secondary radiation are then obtained from the classical radiation formula with retardation:

$$A'_\mu (y) = \frac{1}{c} \int s_\mu (x, t - \frac{|x-y|}{c}) dx, \quad dx \equiv dx_1 dx_2 dx_3 \quad (23)$$

---

32 It appears that Klein and Uhlenbeck were also working on the wave approach to Compton scattering but didn’t reach a quantitative result before Gordon. (Klein 1968, p. 79). This seems as well to have been the case with Pauli, see Rechenberg and Mehra 1982-87, vol. 5, p. 830.

33 Gordon writes $\Phi$ instead of $A$.

34 One should at this point also mention the results of Schrödinger himself (Schrödinger 1926d) and Fock (1926). Further references can be found in Schweber (1961), p. 54. and Kragh 1984. The naming of the scalar relativistic equation after Klein and Gordon appears then quite arbitrary.

35 The same had already been realized by Fock (1926).
where the densities of charge and current of the perturbed electron enter through the quantum expressions

\[ s_\alpha = \frac{1}{i} \left( \psi^* \partial_\alpha \psi - \psi \partial_\alpha \psi^* - \frac{2i e}{\hbar c} A_\alpha \psi \psi^* \right) \]  

(24)

These densities are obtained from the wave equation in a way similar to that used by Schrödinger in his pioneering work on the non-relativistic case. Actually, there is an implicit factor of \( e \) in front of this expression which is "hidden" in the normalization of \( \psi \) as will be explained shortly, see eqs. (30) and (32).

Notice how the strategy followed by Gordon differs from that of Dirac examined earlier. Gordon uses the classical radiation formula with retardation into which he inserts the quantum expression for the current. It is through the latter that the quantum features will enter the stage, namely the proper correlation between initial and final states, and the resulting secondary radiation, as will be seen shortly. Dirac, on the other hand, inserted by hand the covariant phase \((ct - kx)\) by a change of variables, and used a final dipole approximation formula, where he injected the squared quantum amplitude for the polarization. In his case, the information about the final state and the resulting secondary radiation is obtained from the action of the polarization observable upon the initial state. In view of the fact that Gordon obtained the same final result as Dirac, the two approaches prove equivalent albeit their heuristics is different.

Gordon’s "two-step" scheme of computation (first solve the radiation-perturbed electron evolution, then compute the resulting classical radiation and interpret the latter as Compton radiation) leans upon an approximation not easily expressible within the framework of more modern (fully quantized) approaches (the same holds also for Dirac’s approach discussed earlier). Nevertheless, its success is remarkable, especially so in the Klein-Nishina computation to be discussed shortly, where, to the order considered, a correct formula is obtained in spite of using a classical electromagnetic field. The reasons of this “two-step” technique are clear. Until the advent of Dirac’s field quantization (1927), there was no way to treat the interaction of the radiation field with matter quantum mechanically. The quantum ingredient could only be introduced using the quantum evolution equations for matter subjected to (classical) radiation and radiating back classically through the associated charge and densities (quantum) expressions (this is dubbed by the various authors as a korrespondenzmässig procedure). The power of the quantum formalism is manifest in the fact that it is effectively sufficient.

\[ \text{36} \text{see for instance Klein (1926), p. 416.} \]
to use a quantum evolution equation for the electron (and keeping radiation classical) to achieve a result where the process is handled in terms of an effective matrix element between initial and final quantum states, thus introducing an element of discontinuity in a otherwise apparently continuous process. In this picture, the discrete shifts in the frequency (and wave vector) of the radiation can be understood as induced by the quantum evolution of the electron wave function itself. Notice that these features are not explicitly present in the original formulation of the problem. There, a purely classical electromagnetic radiation affects the motion of the electron which radiates according to the classical laws, all this along a purely continuous course of events: by the very nature of the approach, no such notions as initial or final states can find their way naturally in this context.

To see how the discontinuity related to a transition from initial to final states emerges out of this seemingly continuous context, let us define with Gordon the radiation field as $A_\mu = a_\mu \cos \varphi$ where $\varphi = (k \cdot x - \omega t) \equiv k \cdot x$ with $x_0 = ct$ and $k_0 = \omega/c$ The description of the perturbed electron motion is obtained by Gordon with the additional approximation that the terms $A^2$ in (22) can be neglected, in the same way as in Dirac. He thus ends up solving the truncated equation

$$\partial \cdot \partial \psi - \frac{2i}{\hbar} \left( \frac{e}{c} a \cdot \partial \psi \right) \cos \varphi - \frac{m^2 c^2}{\hbar^2} \psi = 0 \quad (25)$$

A few remarks are in order here. The neglect of the $A^2$ terms may appear surprising as far as the electrodynamics of scalar particles is considered. Indeed, one can show (using for example the modern propagator approach; see also section 7.2) that there is a natural gauge choice where the total contribution to the scattering amplitude in the laboratory frame (initial electron at rest) comes precisely from this particular term. However, In his paper, Gordon does not commit himself explicitly to any particular gauge or frame. To understand the situation, one has to remember again the peculiar nature of the semi-classical approach used here. In this picture, the current eq. (24) contains already the coupling to the primary photon as evidenced by the term proportionnal to $e$, namely

$$\frac{2i e}{\hbar c} A_\alpha \psi \psi^* \quad (26)$$

In the course of the calculation, the "coupling" to the secondary photon (to use modern expressions) will be obtained indirectly and at this point it is

---

37 Instead of $k$, Gordon uses the notation $l$.
38 See e. g. Bjorken and Drell 1964, pp. 194-195. This is further explained in section 6.2. of the present paper.
39 We thank prof. Roberto Casalbuoni for a valuable discussion concerning this point.
best to keep track of the powers of the coupling constant (proportional to $e$) as we shall see shortly.

Gordon can solve (25) choosing the ansatz

$$\psi = \exp(\frac{i}{\hbar}W),$$

(27)

with the modified phase

$$W = p \cdot x + \frac{e}{c} \frac{p \cdot a}{p \cdot k} \sin \varphi$$

(28)

provided that the momenta $p_\mu$ obey the mass-shell equation:

$$p \cdot p + m^2 c^2 = p^2 - \left(\frac{E}{c}\right)^2 + m^2 c^2 = 0.$$

(29)

Gordon considers then an arbitrary superposition of solutions

$$\psi = \int z(p) C(p) e^{\frac{i}{\hbar}W(p)} d\mathbf{p},$$

(30)

where $d\mathbf{p} = dp_1 dp_2 dp_3$. It contains two factors: $z(p)$, that Gordon identifies as a weighting of the individual solutions (27), and $C(p)$ that he considers a normalization. The reason for this splitting will appear shortly. Gordon builds then the bilinear expressions entering the current four-vector (24)

$$\psi \psi^* = \int \int e^{\frac{i}{\hbar}(W(p) - W(p'))} z(p) z(p') C(p) C(p') dp d\mathbf{p}'$$

(31)

$$\psi^* \partial_\mu \psi = \frac{i}{\hbar} \int \int (p_\mu + \frac{e}{c} \frac{p \cdot a}{p \cdot k} k_\mu \cos \varphi) e^{\frac{i}{\hbar}(W(p) - W(p'))} z(p) z(p') C(p) C(p') dp d\mathbf{p}'$$

and c.c.

As can be seen, these expressions are of the form

$$O = \sum_{ij} O_{ij}$$

where the labels $i$ and $j$ run over the set of the solutions of (25). Hence, considering superpositions (30), one can obtain the current (24) as a sum of "transition currents" related to a quantum transition from eigenstate $i$ to eigenstate $j$. This is just how the initial and final states of the electron will enter the stage.

$^{49}$This is the solution of the truncated equation (25); the solution of the full equation would involve terms quadratic in the potential.
The value of normalization factor $C(p)$ is obtained by Gordon using a comparison with the classical case and the arbitrary (but then familiar requirement) that

$$\int z^2(p)dp = 1,$$

hence the splitting[^41]. The value of $C(p)$ turns then out to be

$$C^2(p) = \frac{e c^2}{4 \pi \hbar^2 E} \tag{32}$$

where $E$ is given by (29). Notice here the presence of another $e$ factor in the numerator. This is the one alluded to in the comment after eq. (24). Combined with the one already present in the term eq. (26), it makes for the correct $e^2$ dependence of the result. This justifies *a posteriori* the neglect of the quadratic terms not only in eq. (25), but also in the next steps.

Indeed, Gordon is now in position to obtain the features of the secondary radiation using the preceding expressions to build up the current in (23): He first expands the exponentials in eq. (31) to first order in $a$, then, assuming that the charge distribution is of negligible size with respect to the distance $|y| \equiv r$ of the observation point $y$, approximates $|x - y| = r - \hat{k}' \cdot x$, where $\hat{k}'$ is the unit vector[^42] in the direction $y$. He arrives thus at the dipole expression

$$A'_\mu = \frac{ec}{2\hbar^2 r} \Re \int \frac{Z(P)Z'(P')}{\sqrt{E\Delta E'\Delta'}} T_\mu e^{i(P-P')x-i\omega'(1-\frac{E'}{E})} dP dP' dx \tag{33}$$

with

$$T_\mu = \frac{e}{c} \left[ \frac{1}{\hbar} \left( \frac{p \cdot a - p' \cdot a}{p \cdot k} + \frac{p' \cdot a}{p' \cdot k} \right) (p_\mu + p'_\mu) + \left( \frac{p \cdot a}{p \cdot k} + \frac{p' \cdot a}{p' \cdot k} \right) k_\mu - 2a_\mu \right] \tag{34}$$

$$\omega' = \frac{E - E' + \hbar \omega}{\hbar}$$

The integral above is written in terms of new integration variables

$$P = p + \hbar k - \frac{E + \hbar \omega}{c} \hat{k}'$$

$$P' = p' - \frac{E'}{c} \hat{k}'$$

[^41]: Actually, neither Gordon nor Dirac knew to normalize the initial and final states in the continuum. Therefore both had to use the classical limit in order to get the normalization in an indirect way.

[^42]: Instead of $\hat{k}$ Gordon uses $\xi$. 
which explains the jacobians
\[
\Delta = 1 - \frac{c}{E} \mathbf{p} \cdot \hat{k}' = 1 - \frac{v}{c} \cos \varphi
\]
as well as
\[
Z^2(P) \Delta(p) = z^2(p),
\]
and similarly for the corresponding primed quantities. We recognize here \(\Delta\) as the Doppler factor depending on the angle between the velocity \(v\) of the electron and the direction of observation.

The features interpretable in terms of a transition from an initial to a final state are thus ensured by the bilinearity of the above expression. Now, the final state values for the frequency and momenta are further obtained thanks to the appearance in (33) of the Dirac \(\delta\)-function hidden in its Fourier form
\[
\int e^{i \sum (P_k - P'_k) x_k} dx = (\hbar)^3 \prod_k \delta(P_k - P'_k)
\]
which ensure conservation of energy-momentum yielding the equality
\[
p + \hbar k = p' + \hbar k', \quad \text{with} \quad k' = \frac{\omega'}{c} \hat{k}', \quad k_0' = \hbar \frac{\omega'}{c}
\]
so that \(\hbar k\) and \(\hbar k'\) are the respective 4-momenta of the initial and final photon. After obtaining his final expression for the radiation
\[
A'_\mu = \frac{e c}{2r} \int \frac{Z(P)Z'(P)}{\sqrt{E \Delta E' \Delta'}} T_{\mu} \cos \omega'(t - \frac{r}{c}) dP
\]
Gordon makes the significative comment:

Since \(Z^2(P) dP\) and \(Z'^2(P) dP\) are the weights of the two state spaces and are combined together, the radiation potential corresponding to the single transition is given by
\[
\frac{e c}{2r} \frac{1}{\sqrt{E \Delta E' \Delta'}} T_{\alpha} \cos \omega'(t - \frac{r}{c}).
\]

---

\[\text{References and Notes:}\]
\[\text{Gordon 1926, p. 130.}\]
This is how in this correspondence principle way the elements of the now so-familiar picture of electron-photon process emerge albeit in a semi-classical context yet not involving Dirac’s field quantization nor perturbation theory. To make contact with modern treatments, the reader can contract the gauge invariant expression (34) with the polarization vector of the secondary photon, which in a sense provides the second external photon line supplementing the one already present in the current (24), (to be compared with for instance eq. (9.30) of Bjorken and Drell and section 6.2 of the present paper).

For the final formula giving the intensity of the secondary radiation Gordon obtains \( \frac{\omega'}{\omega} \) is the ratio of the final and the initial frequencies

\[
I = \left( \frac{\omega'}{\omega} \frac{\Delta}{(1 - \frac{v}{c} \cos \delta)} \right)^3 I_{kl}
\]

which is the Dirac’s result (19) once one substitutes for \( I_{kl} \), the Thompson formula and assumes the electron at rest, \( v = 0 \), so that the ratio of the Doppler factors \( \Delta/(1 - \frac{v}{c} \cos \delta) = 1 \) (\( \delta \) is the angle between the velocity and the wave vector of the initial radiation).

Before turning to the Klein-Nishina paper (and later to Waller and Tamm), we want to mention a contribution of Klein 1927 submitted at the very end of 1926 and entitled Elektrodynamik und Wellenmechanik vom Standpunkt des Korrespondenzprinzips. In this paper, Klein also uses the relativistic scalar equation and studies its various applications to the dispersion of light by atoms and free electrons. As indicated in the title, Klein discusses quite extensively the spirit of the correspondence principle approach. Anyone interested in the heuristics of the latter will find in Klein’s contribution a valuable discussion of this method. Another feature of Klein’s paper is an interesting perturbative scheme which the latter proposes to deal with time-dependent problems. It fits somehow between the original proposal of Schrödinger and announces that of Dirac. Klein’s scheme is however practical only in the case of equations linear (first order) in time derivative, so that, albeit in possession of the scalar relativistic equation, Klein considers only its non-relativistic limit (Schrödinger equation) when discussing the dispersion of light scattered off atoms, or uses instead an ansatz for the case of Compton scattering.

\(^{45}\)Bjorken and Drell 1964, p. 194.  
\(^{46}\)See the next section.
3.4 Klein and Nishina: spin 1/2 electrons

The Klein-Nishina paper (Klein and Nishina 1929, received 30 Oct. 1928) is the first to apply the freshly discovered spinor equation of Dirac (Dirac 1928, received 2 Jan. 1928) to Compton scattering. It reads, using the original set of \( \sigma_i \) and \( \rho_i \), \( i = 1, \ldots, 3 \) introduced by Dirac:

\[
\left\{ \frac{E + eV}{c} + \rho_1 \left( \sigma_i, p + \frac{e}{c} A \right) + \rho_3 mc \right\} \psi = 0 \quad (36)
\]

Alternatively, one can use the nowadays more familiar \( \gamma \) matrices setting

\[
\begin{align*}
\rho_1 \sigma_i &= \gamma_0 \gamma_i \quad i = 1, 2, 3 \\
\rho_2 \sigma_i &= \gamma_i \quad i = 1, 2, 3 \\
\rho_3 &= -i \gamma_0
\end{align*}
\]

in which case one can rewrite (36) as

\[
\left\{ i \left( -\gamma_0 \frac{E + eV}{c} + (\gamma, p + \frac{e}{c} A) \right) + mc \right\} \psi = 0 \quad (37)
\]

As already mentioned, the use of this relativistic equation which takes into account the electron spin constitutes the main novelty of the paper. For the rest, it closely follows the correspondence principle strategy of Gordon. Instead of the expressions for the current densities (24) pertaining to the scalar (Klein-Gordon) equation, Klein and Nishina use instead the Dirac densities

\[
\begin{align*}
\rho &= e \bar{\psi} \gamma_0 \psi' \\
J &= ec \bar{\psi} \gamma \psi'
\end{align*}
\]

Here \( \bar{\psi} \) stands for the (one-row) spinor obtained from \( \psi \) via hermitian conjugation and multiplication by the matrix \( \gamma_0 \): \( \bar{\psi} = (\psi^*) \gamma_0 \). \( \psi \) is taken as solution of the second-order equation (the scalar potential \( V \) is set to zero)

\[
\begin{align*}
\left\{ \frac{\hbar^2}{c^2} \partial_t^2 + \left( \frac{e}{c} \nabla + \frac{e}{c} A \right)^2 + m^2 c^2 \right\} \psi + \frac{e \hbar}{c} (\sigma B) \psi + \frac{i e \hbar}{c} \rho_1 (\sigma E) \psi = 0
\end{align*}
\]

47Here, the 4x4 \( \sigma \) matrices are obtained putting the 2x2 Pauli matrices on the diagonal.
48See the original paper of Dirac. The conventions used here are a departure from those of Dirac (and consequently from those of Klein-Nishina and Waller, who follow Dirac) in that he uses the euclidean metric in which \( \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_\mu\nu \), with \( \mu, \nu = 1, \ldots, 4 \), whereas we chose to work with \( \gamma_0 = i \gamma_4 \) which amounts to use the metric \( g_{\mu\nu} = (-1, 1, 1, 1) \).
obtained from (36) taking the square [9]. Klein and Nishina assume a vector potential of the form

$$A = ae^{-i(kx-ct)} + a^*e^{i(kx-ct)} = 2 \text{Re}(ae^{-ikx}); \quad k = \frac{\omega}{c}, k_0 = \frac{\omega}{c};$$

and make the following ansatz for the solution:

$$\psi = \left\{ 1 + g(p)e^{-ikx} + \tilde{g}(p)e^{ikx} \right\} \psi_0$$

(40)

where \(\psi_0\) is the free solution for \(A = 0\)

$$\psi_0 = u(p)e^{-\frac{i}{\hbar}\left(p\cdot x\right)}.$$

(41)

The ansatz (40) is the spinor version of the ansatz of Gordon (27) and (28). Indeed, the functions \(g\) and \(\tilde{g}\) are given by

$$g(p) = \frac{e}{\hbar c pk} \left\{ pa + \frac{1}{2}i\hbar \sigma \eta + \frac{1}{2}i\hbar \rho_1(\sigma \epsilon) \right\}$$

(42)

$$\tilde{g}(p) = -\frac{e}{\hbar c pk} \left\{ pa^* + \frac{1}{2}i\hbar \sigma \eta^* + \frac{1}{2}i\hbar \rho_1(\sigma \epsilon^*) \right\}$$

where \(E = e^{-ikx} + \epsilon^* e^{ikx}\), and \(B = \eta e^{-ikx} + \eta^* e^{ikx}\). In the case of linear polarization (the case studied by Gordon), one has \(\epsilon^* = \epsilon\), which implies \(a^* = -a\). Then, from (40), and (42), one sees that

$$\psi = \left\{ 1 + \frac{e}{\hbar c pk} \left\{ 2 \cos kr - \frac{e}{\hbar c pk} \left[ h(\sigma \eta) + i\hbar \rho_1(\sigma \epsilon) \right] \sin kr \right\} \right\} \psi_0$$

where we recognize the linear part of the solution (27) and (28) of Gordon supplemented by spin terms. As in the latter’s case, quadratic terms in the potential are thus further neglected. The ”initial-final” picture emerges also in a way analogous to that of Gordon, Klein and Nishina following essentially the same steps; they obtain for instance as the analog of (35) the result:

$$A(p, p') = \frac{(h)^3}{r} \frac{1}{\sqrt{\Delta \Delta'}} \left\{ e^{i\omega'(t - \frac{r}{c})} \bar{u}(p) \left( \gamma g(p') + \tilde{g}(p')^\dagger \gamma \right) u'(p') + c.c \right\}$$

49This use of the second order equations may appear puzzling. It is possible that Klein and Nishina used the latter instead of (36) to keep close to the original discussion of Gordon.

50Klein and Nishina’s notation for the potential is \(U\).

51Klein and Nishina write \(v(p)\) instead of \(u(p)\), the latter denoting in their paper the adjoint spinor \(v(p)^\dagger\). In order not to make the discussion unnecessarily cluttered with notations, we do not discuss here the conventions on the basis of spinor solutions to the free Dirac equation.
In order to obtain explicit formulas for the intensity of the scattered radiation, Klein and Nishina still have to go through the tedious steps of the evaluation of products consisting typically of initial and final spinors $\bar{u}(p)$ and $u'(p')$ sandwiching various combinations of spin matrices. The technicalities involved in these calculations don’t bring any further physical insight and we shall not comment them here\footnote{52}. Klein and Nishina finally obtain the following celebrated expression for the scattering cross section for unpolarized incident light, and \textit{in the rest frame of the initial electron}

\[
\frac{d\sigma}{d\Omega}(\theta) = \frac{e^4}{2m^2c^4} \left( \frac{1 + \cos^2 \theta}{(1 + \alpha(1 - \cos \theta))^2} \right) \left[ 1 + \alpha^2 \frac{(1 - \cos \theta)^2}{(1 + \cos^2 \theta)(1 + \alpha(1 - \cos \theta))} \right]
\]

(43)

where $\theta$ is the scattering angle, and $\alpha = \hbar\omega/mc^2$. It differs from the result obtained earlier by Dirac (and also Gordon) by terms of second order in $\alpha$ and higher (compare with (21)). We know nowadays that this expression constitutes the correct result to the lowest order. It is indeed remarkable that Klein and Nishina were able to obtain it within such a semi-classical context. The reason of this success is due to the quantum mechanism, already mentioned above, which automatically takes care of shifting the state of the photon ensuring conservation of energy-momentum, and to the fact that at the tree level, where multiple effects of emission/absorption are neglected, it is sufficient to consider the radiation field as classical. The following appreciation may serve as a closing word\footnote{53}.

\footnote{52}Let’s remark however that the technical tricks making the reduction of the spinor algebra much easier are due to Casimir 1933 and Wannier 1935.\footnote{53}Ya. S. Smorodinskii 1987, p. 823.
since they did not encounter a paradox that surprised Tamm and Waller.

4 Dirac’s perturbation theory, field-quantization and intermediate states

In this chapter, we shall examine the key role of the perturbation theory and field quantization introduced by Dirac (and used subsequently him by Waller and Tamm). The two developments are related since Dirac quantizes the radiation field within the context of his perturbation theory for Bose particles. The choice of the perturbation technique (as distinct from the choice of the approximation, or better, physical treatment of the process) is an important theme surfacing again and again in our story. The interpretation following a given perturbative scheme has a major role in the way the whole process is pictured and understood. We first start with a brief recall of the situation prior Dirac’s theory.

One of the first formulations of the time-dependent perturbation theory within quantum mechanics goes logically back to the “Drei-Männer-Arbeit” of Born, Heisenberg and Jordan (1926). It is formulated in terms of the matrix formalism and relies on the analogy with the classical canonical formalism. It enables the authors to derive as an application the Heisenberg-Kramers dispersion relations.

With the advent of wave mechanics, the community took possession of a new, mathematically more familiar tool. The perturbation theory in wave mechanics superseded then the matrix version. Schrödinger developed the perturbation theory for time independent Hamiltonians in his third paper (Schrödinger 1926c). It is essentially an application of Sturm theory. The time-dependent perturbation theory appears in the last paper of Schrödinger’s tetralogy (Schrödinger 1926d), where he introduces his celebrated time-dependent equation:

$$\frac{i}{\hbar}\partial_t \psi = \left(\Delta - \frac{1}{2\hbar^2} V(x, t)\right) \psi,$$

where $\Delta$ is the Laplacian. As an example, Schrödinger set out to solve the problem of the evolution in a time-dependent potential of the form $V = V_0(x) + v(x, t)$. For the important case of perturbation by external radiation,

The “paradox” in question is the necessity to sum over negative energy intermediate states of the electron as will be seen in the next section.
i.e. \( v(x, t) = A(x) \cos(\omega t) \), he obtained at first-order the solution

\[
\psi(x, t) = u_k(x) \exp \left( \frac{i E_k}{\hbar} t \right) 
\]

\[
+ \frac{1}{2} \sum_{n=1}^{\infty} a_{kn} u_n(x) \left[ \exp \left( \frac{i (E_k + \hbar \omega)}{\hbar} t \right) \frac{(E_k - E_n - \hbar \omega)}{E_k - E_n + \hbar \omega} \right]
\]

where the constant coefficients \( a_{kn} = [A]_{kn} \) are the matrix elements of \( A \) taken in the basis \( u_n \) of the solutions of the free \( A = 0 \) case. His technique consisted in eliminating the time dependence and use then the time-independent theory of his preceding paper. He interpreted the expression above in terms of a time-dependent potential driving the system out of its pure \( u_k \)-oscillating mode and superposing on it secondary oscillations of frequencies \( \hbar^{-1} E_k \pm \omega \). The heuristics of such a formula were clearly of particular meaning for the problem of dispersion (scattering of radiation off an atom). Indeed, computing the resulting electric moment, Schrödinger obtained a result that he declared "as formally quite identical with Kramers’ formula for secondary radiation" [55].

### 4.1 Dirac’s perturbation theory for bosons

In the paper *On the Theory of Quantum Mechanics* (Dirac 1926c)[56], where the Einstein \( B \) coefficient of stimulated emission and absorption is derived from first principles, Dirac uses for the first time his perturbation theory for Hamiltonians with explicit time dependence. It is a significant and decisive step, its importance going much beyond the purely formal content. Indeed, it offers the advantage to lend itself to quite general situations but, mainly, together with Dirac’s interpretation, it paves the way to field quantization (1927b)[57] and naturally construes elementary processes as transitions between “free states”. Dirac’s method was to be adopted in most of the following works and its spirit conserved in the modern techniques.

Dirac develops a systematic approach to deal with arbitrary (time-dependent) perturbations in the following way. Assume that we have a (complete) set of solutions \( \varphi_k \) of a problem associated with the Hamiltonian \( H_0 \), typically a non-interacting atom or particle; the general solution then reads: \( \varphi = \sum_i c_i \varphi_i \). Then consider a new Hamiltonian \( H = H_0 + H_{\text{int}} \). To solve the

---

[55] Schrödinger (1926d), p. 120.
[56] See the discussion of Bromberg 1977.
[57] See e.g. Jost 1972, Bromberg 1977 and Cao 1997.
corresponding problem, Dirac proposes to expand its solution $\psi$ on the set of the free ones:

$$\psi = \sum_k a_k(t)\varphi_k$$  \hspace{1cm} (46)

The $a_k$’s are the expansion coefficients that remain to be determined\(^{58}\). Dirac remarks further, that instead of considering the $a_k$’s to give the probabilities of being in the state $k$, one can as well think of them here as giving the number of individual (disturbed) systems occupying the state $k$. Talking of atoms, Dirac says (p. 674):

We shall consider the general solution of [the undisturbed] equation to represent an assembly of the undisturbed atoms in which $|c_n|^2$ is the number of atoms in the $n$-th state, and shall assume that $\psi$ represent in the same way an assembly of the disturbed atoms, $|a_k|^2$ being the number in the $k$-th state at any time $t$.

The exact evolution equations for the $a_k$’s are:

$$i\hbar \frac{d}{dt}a_k = \sum_n a_n[H_{\text{int}}]_{kn},$$  \hspace{1cm} (47)

where $[H_{\text{int}}]_{kn}$ are matrix elements of $H_{\text{int}}$ (in the $\varphi$ basis). All that remains to be done is to solve the above equations since the whole time-dependence of the solution is contained in the $a$’s. This can, and in most cases must be done, perturbatively. For instance, assuming that at $t = 0$, the system is entirely in the unperturbed state, say $a_0(0) = 1$, so that $a_k = 0; k \neq 0$, then one can get, inserting on the right-hand side of (47) the initial values, at first order

$$a_k = \frac{[H_{\text{int}}]_{k0}(e^{i(E_0 - E_k)t/\hbar} - 1)}{(E_0 - E_k)}$$  \hspace{1cm} (48)

If the matrix elements $[H_{\text{int}}]_{k0}$ vanish (so that there is no direct contribution to $a_k$) then the method of approximation has to be refined\(^ {59}\). If, for some

---

\(^{58}\) One should notice at this point that in his study of the relativistic scalar equation Klein (1927) developed a perturbative scheme that used already the expansion on free solutions. His technique is however less general and can be seen as a special case of Dirac’s. Klein’s approach was derived for the classical approximation of the scalar relativistic equation, involving only first order time derivative.

\(^{59}\) The formula (47) as well as (48) are not written or discussed in their general form in the 1926c paper. There Dirac considers specifically the first and second order solutions to the problem of dispersion of radiation by atoms, where the expansion (46) is for the wave
indices \( n' \), \([H_{\text{int}}]_{kn'} \neq 0 \neq [H_{\text{int}}]_{n'0} \), then one can get the following second order contribution passing through "intermediate states" \( n' \):

\[
a_k = \sum_{n'} [H_{\text{int}}]_{kn'} [H_{\text{int}}]_{n'0} \left[ \frac{e^{i(E_0 - E_k)t/\hbar} - 1}{E_0 - E_{n'}} - \frac{e^{i(E_{n'} - E_k)t/\hbar} - 1}{E_{n'} - E_k} \right] (49)
\]

The formulas above constitute the main results of perturbation theory that we shall find used by many authors in the next sections.

There is however much more to Dirac's technique. In his celebrated paper *The quantum theory of emission and absorption of radiation* (Dirac 1927b, received 2 Febr. 1927) the whole discussion is resumed and the equations (47) are cast in a Hamiltonian form. Thus, the \( a_k \) 's and the complex conjugated \( a_k^* \) 's acquire the status of the dynamical variables of some new abstract problem. The latter is classical, but Dirac, with his fascinating intuition, asks us to consider the quantum version of this new problem. He writes (p. 250):

...we can describe the effect of a perturbation on an assembly of independent systems by means of canonical variables and Hamiltonian equations of motion. The development of the theory which naturally suggests itself is to make these canonical variables q-numbers satisfying the usual quantum conditions instead of c-numbers, so that their Hamiltonian equations of motion become true quantum equations. The Hamiltonian function will now provide a Schrödinger wave equation, which must be solved and interpreted in the usual manner. The interpretation will give not merely the probable number of systems in any state, but the probability of any given distribution of the systems among the various states, this probability being, in fact, equal to the square of the modulus of the normalized solution of the wave equation that satisfies the appropriate initial conditions. We could of course calculate directly from elementary considerations the probability of any given distribution when the systems are independent, as we know the probability of each system being in any particular state. We shall find that the probability calculated directly in this way does not agree with that obtained from the wave equation except in the special case when there is only one system in the assembly. In the general case it will be shown that the wave equation leads to the correct value for the probability of any given distribution function of the atom. The case of the intermediate states will be discussed by Dirac later in his paper 1927c. Our treatment here follows Heitler 1936, p. 89.
when the systems obey Einstein-Bose statistics instead of being independent.

So, now, the \( a_k^* \)'s and the \( a_k \)'s become q-numbers, namely the now familiar creation and annihilation operators of the corresponding (undisturbed) boson states\(^{60}\). A huge step has been taken. Dirac will show next that his second-quantized procedure (now the name is much more meaningful) properly takes into account Bose-Einstein statistics. Dirac discusses this point using instead of the operators \( a_k \), \( a_k^* \) the canonical variables corresponding to the operators \( N_k = a_k^* a_k \) and their canonical conjugated \( \theta_k \). This makes the wave function of this second-quantized problem depend on the occupation numbers of the states of the bosons. Indeed, the wave function is defined on the manifold of the \( N_k \) variables, or more properly, the manifold defined by the spectrum of the \( N_k \) operators which are the positive integers\(^{61}\).

So far, Dirac’s analysis concerns an assembly of bosons the perturbation of which results from a potential expressed solely in terms of boson variables. There is as yet no “coupling” to some other system. In the same paper (1927b), Dirac will consequently generalize his technique, coupling the bosons to a given external system, typically an atom. When the bosons are light quanta, and although the interaction matrix elements are \textit{a priori} not known, Dirac is already able at this point to derive the correct dependence of the probabilities for stimulated and spontaneous emission (Einstein’s \( A \) and \( B \) coefficients) upon the intensity of the incident radiation. This is because the algebra of boson operators contains automatically the right dependence on the occupation numbers.

With Dirac’s perturbation theory and the remarkable use he will make of it in his quantization of the radiation field, we enter definitely into a new era. The basic ingredients of the machinery of QED are now set, but several important problems are not yet resolved. One feature that remains to be in-

\[^{60}\text{Actually Dirac will quantize the canonical variables}
\]

\[
\begin{align*}
b_r &= a_r e^{-\frac{i}{\hbar} W_r t} \\
b_r^* &= a_r^* e^{\frac{i}{\hbar} W_r t}
\end{align*}
\]

where \( W_r \) is the energy of state \( r \), setting

\[
[b_r, b_s^*] = \delta_{rs}
\]

This enables to factor out the trivial time dependence due to free evolution.

\[^{61}\text{For further details about the relation between a choice of state-space basis and the variables entering the wave-functions, see Dirac’s ”transformation theory” as developed in Dirac’s ”The Physical Interpretation of Quantum Dynamics”, Proc. Roy. Soc. (London) A, vol. 113 (1927), pp. 621-641, received 2 Dec. 1926.}\]
cluded is a proper way to take into account the reaction of the emitted radiation on the source itself, called Rückwirkung, which we translate here as back-reaction. This problem is related to the more fundamental problem of a relativistic and interacting theory of particles and fields. For now, we want to examine some early applications of Dirac’s perturbation/quantization method which recast Compton scattering in the now familiar formulation of a transition from initial to final states through successive absorption/emission (or emission/absorption) of an intermediate photon, but where the reaction of the radiation on the electron is still not considered.

4.2 Dirac’s quantization of the radiation field

A spectacular application of the formalism above is Dirac’s quantized theory of the interaction of matter with radiation. In the 1927b paper this is achieved in the following way: the radiation field is first quantized as an assembly of independent bosons, then formally coupled to an external system according to the scheme of the preceding section. To determine explicitly this coupling to matter, Dirac considers the ”wave” point of view, where this time he expands the vector potential $A$ into its Fourier components (which he supposes of large but finite number $A = \sum_r A_r$). In this ”wave point of view”, the coupling is explicitly given by

$$c^{-1} \sum_r \dot{X} A_r,$$

involving the time-derivative of the total ”polarization” $X$ of the (atom) system in the direction of $A_r$, where relativistic effects have been neglected \[62\]. After some manipulation, Dirac shows the equivalence of this point of view with the previous ”bosonic” one. This is satisfying, as the Fourier expansion amounts to the decomposition of the vector potential $A$ in a basis of plane waves, which are the solutions for the free radiation (see eq. 46). The payoff is that the matrix elements of the interaction are now explicitly determined. Eventually, Dirac ends up with the following formalism: One first considers the Hamiltonian of matter and free bosons $H_0 = H_{\text{matter}} + \sum_r W_r N_r$ where $H_{\text{matter}}$ is originally the Hamiltonian describing atomic states, $W_r = \hbar \omega_r$ are

\[62\] The term polarization might appear ambiguous. One may offer the following justification. In the Hamiltonian

$$\frac{1}{2m} (p + \frac{e}{c} A)^2$$

Dirac writes down the $\frac{e}{mc^2} p A$ term as $\frac{1}{c} \frac{d}{dt} (e x) A$ using $p = \frac{d}{dt} (m x)$ with $x$ the ”position” of the charge $e$. Thus $X = (e x)$. The $A^2$ term is neglected and the term $\frac{1}{2mc^2} p^2$ included in the Hamiltonian for matter.
the energies of the free photon states. $N_r$ is the occupation number operator of the $r$-th state; together with its canonically conjugated variable $\theta_r$, they are related to the algebra of creation and annihilation operators by:

$$b_r = N_r^{1/2} e^{-i \theta_r / \hbar}$$

$$b_r^* = N_r^{1/2} e^{i \theta_r / \hbar}$$

$$[b_r, b_s^*] = \delta_{r,s}$$

In the coupling term (50), the norm $A_r$ of each component $A_r$ is expressed now as

$$A_r = a_r \cos \theta_r / \hbar,$$

and finally the interaction Hamiltonian becomes

$$H_{\text{int}} = 2(\hbar^{1/2} c^{-3/2}) \sum_r \sqrt{\nu_r / \rho_r} \dot{X}_r \left\{ N_r^{1/2} e^{i \theta_r / \hbar} + (N_r + 1)^{1/2} e^{-i \theta_r / \hbar} \right\}.$$
light-quantum jumps from one state to another of the same frequency but different direction of motion (i.e. the corresponding matrix element $v_{mm'} = 0$). All the same, radiation that has apparently been scattered can appear by a double process in which a third state, $n$ say, with different proper energy from $m$ and $m'$, plays a part. [...] The scattered radiation thus appears as the result of the two processes [...] one of which must be an absorption and the other an emission, in neither of which is the total proper energy even approximately conserved.

On the other hand, the full interaction Hamiltonian which does not neglect the terms quadratic in $A$ does yield contributions to direct scattering processes.

The 1926c and 1927b,c papers of Dirac set the basic language and concepts characteristic of the modern conception of the elementary processes as sequences of absorption/emission of light quanta, passing through a sequence of intermediate states. Let us quote him again, this time with a lucid discussion of the apparent violation of conservation laws (Dirac 1927b, p. 265):

Also the theory enables one to understand how it comes about that there is no violation of the law of the conservation of energy when, say, a photo-electron is emitted from an atom under the action of extremely weak incident radiation. The energy of interaction of the atom and the radiation is a q-number that does not commute with the first integrals of the motion of the atom alone or with the intensity of the radiation Thus one cannot specify this energy by a c-number at the same time that one specifies the stationary state of the atom and the intensity of the radiation by c-numbers. In particular, one cannot say that the interaction energy tends to zero as the intensity of the incident radiation tends to zero. There is thus always an unspecifiable amount of interaction energy which can supply the energy for the photo-electron.

4.3 Waller: Compton scattering with quantized radiation

The success of the first applications of Dirac’s technique (computation of Einstein’s $A$ and $B$ coefficients in 1927b, rederivation of the Kramers’ and Heisenberg’s formulas and study of the case of resonance in 1927c) will motivate others to apply it to further problems. However almost two years pass
before the first to react, Ivar Waller (Waller 1929, received 21 July 1929 and Waller 1930, received the 12 Feb. 1930), at that time at the University of Upsala, and Igor Tamm (Tamm 1930, received the 7 Apr. 1930) in Moscow, publish their results. 

The first contribution of Waller, *Die Streuung kurzwelliger Strahlung durch Atome nach der Diracschen Strahlungstheorie*, dates from 1929 (Waller 1929). There, Waller considers again the problem of radiation scattering off an atom, neglecting relativistic effects at the level of the electron bound in the Coulomb potential. He uses on the other hand the full coupling \( \frac{1}{2m}(p+\frac{e}{c}A)^2 \) without neglecting the \( A^2 \) terms. After studying transitions at the level of the discrete spectrum, he considers also its continuous part (ionization) and ends up with a short account of the relativistic case, where he quotes some preliminary results obtained using Dirac’s equation for the electron. He explicitly mentions the need of taking into account the ”negative energy” intermediate states, but further analysis is missing. It will appear in full extent in Waller’s next paper, *Die Streuung von Strahlung durch gebundene und freie Elektronen nach der Diracschen relativistischen Mechanik* (Waller 1930, received 12 Feb. 1930). Technically, Waller’s 1930’s paper is a straightforward application of the techniques put forward by Dirac. As previously Klein and Nishina, Waller considers the interacting equation

\[ \{ i (-\gamma_0 (E + eV) + (\gamma, p + eA)) + mc^2 \} \psi = 0 \] (54)

After introducing the quantized potential \( A \) and following the steps of Dirac, Waller obtains an equation for the wave-function \( \Phi(J; \ldots, N_r; \ldots) \) depending collectively on atom (electron) variables \( J \), and light-quanta occupation numbers \( N_r \) of states \( r \)

\[
\left[ i \hbar \frac{d}{dt} - E(J') - \sum_r N_r \hbar \omega_r \right] \Phi(J'; \ldots, N_r; \ldots) = \\
+ \sum_J \sum_r \sqrt{N_r} \tilde{M}^r(J'J) \Phi(J; \ldots, N_r - 1; \ldots)
\]

(55)

Here, we have used the following notation for the matrix elements of the interaction corresponding to the \( r \)-th component:

\[ M^r(J'J) \equiv \epsilon_r \int \bar{\psi}_{J'}(x) \gamma \psi_J(x) e^{-i k \cdot x} d\mathbf{x} \] (56)

\[ \tilde{M}^r(J'J) \equiv \epsilon_r \int \bar{\psi}_{J'}(x) \gamma \psi_J(x) e^{+i k \cdot x} d\mathbf{x}, \]

\(^{64}\)One should also note here a paper by Wentzel (1929) where the latter uses as well a quantized radiation field but in a non relativistic approximation.
\( \psi_j(x) \) are eigensolutions for the motion of the electron in an external (atomic) potential \( V \) and \( E(J) \) the corresponding energies. Depending on the value of \( V \), one can deal with the scattering off a bound electron or off a free one \((V = 0)\). One has further \( e_r = \sqrt{\frac{e^2 \hbar \nu_r}{c \rho_r}} e_r \), where \( e_r \) is the polarization of the \( r \)-th (Fourier) component of the radiation, and \( k_r = \hbar \omega_r \hat{k}_r \), the momentum of the \( r \)-th state associated light quantum.

\[(55)\]

is of the form

\[
[i\hbar \partial_t - W(m)] \Phi(m) = \sum_{m'} (m|M|m') \Phi(m')
\]

where \((m|M|m')\) is the \( m,m' \) matrix element of \( M \), and this time the variables \( m \) label the states of the whole system. In particular, the emission and absorption processes correspond to the elements

\[
(J';...,N_r + 1,...|M|J;...,N_r,...) = \sqrt{(N_r + 1)M^r(J'J)}
\]

\[
(J';...,N_r - 1,...|M|J;...,N_r,...) = \sqrt{N_r \tilde{M}^r(J'J)}
\]

Assuming that at time \( t = 0 \) the system is in the state labelled by \((J;...0,0,...,N_s,...0)\) and after applying Dirac’s second-order perturbation theory formula (see [49]), Waller obtains the following expression for the transition amplitude \( a \) to the state \((J';...0,1_r,...,N_s - 1,...0)\):

\[
a(J';...0,1_r,...,N_s - 1,...0) =
\]

\[
- (e^{i\hbar \beta t} - 1) \beta^{-1} \sqrt{N_s} \sum_{J''} \frac{[M^r(J'J'')] [\tilde{M}^s(J''J)]}{E(J) - E(J'') + \hbar \omega_s}
\]

\[
- (e^{i\hbar \beta t} - 1) \beta^{-1} \sqrt{N_s} \sum_{J''} \frac{\tilde{M}^s(J'J'') [M^r(J''J)]}{E(J) - E(J'') - \hbar \omega_r}
\]

Here \( \beta = E(J') + \hbar \omega_r - E(J) - \hbar \omega_s \). The formula clearly exhibits the two contributions corresponding respectively to intermediate states \((J'';...0,...,N_s - 1)\) and \((J';...0,...,N_s,...0)\). The numerical factor in front of \( e_r \), see the discussion of Dirac’s 1927b in the previous section. Our notations differ from Waller’s, namely one has \( M \to A \), \( \tilde{M} \to B \), \( e \to \mu \), and moreover we have included the scalar product of the current \( \bar{\psi}_J(x) \gamma \psi_J(x) \) with the polarization \( e_r \) directly in the definition of the matrix elements \( M \) and \( \tilde{M} \). Other changes involve Waller’s use of primed quantities as denoting the eigenvalues of the corresponding operators, a notation introduced by Dirac (see for instance The Physical Interpretation of the Quantum Dynamics, Proc. Roy. Soc. (London) A, vol. 113 (1927), pp. 621-641. For the sake of making the formulas more transparent, we have dropped this usage.
1, ..., 0) and \((J''; 0, 1, \ldots, N_s, 0)\). The probability of scattering a photon of polarization \(e_r\) into the solid angle \(d\Omega_r\) with the simultaneous transition of the electron to state \(J'\) is then given by

\[
\rho_r d\Omega_r \int |a(J'\ldots0, 1, \ldots, N_s - 1, \ldots0)|^2 d\nu_r = \hbar^{-2} t_j \sum_{J''} |\rho_{rJ''}|^2 N_s d\Omega_r \tag{59}
\]

where the sum in the absolute value bars is the same as the one from above, and the factor \(\rho_r\) is the density of states about state \(r\). For an incident intensity \(I_0\) the power \(P_r\) radiated into the solid angle in the direction \(\hat{k}_r\), and with polarisation \(e_r\) is then given by

\[
P_r = \frac{I_0 e^4 \omega_s^2}{\omega_r^2} \left| \sum_{J''} \frac{[M^r(J'J'')][\tilde{M}^s(J''J)]}{E(J) - E(J'') + \hbar \omega_s} \right|^2 + \sum_{J''} \frac{[\tilde{M}^s(J'J'')][M^r(J''J)]}{E(J) - E(J'') - \hbar \omega_r} \]

Waller remarks that in the derivation of the formula above the back-reaction of the radiation on the scattering system has not been taken into account, therefore one can obtain the same result without quantizing the field.

One can now specialize to the free case (in the bound case, Waller neglects the relativistic and spin effects). Here, Waller takes for the \(\psi_J(x)\) the free plane-wave spinor solutions

\[
\psi^{\sigma}(p) = u^{\sigma}(p)e^{-\frac{i}{\hbar}(E^{\sigma}t - px)} \tag{60}
\]

with the conventions that for \(\sigma = 1, 2\) one deals with positive energy solutions \(E = c \sqrt{m^2c^2 + p^2}\) and \(\sigma = 3, 4\) with negative ones: the normalizations of the single-row spinors are

\[
\sum_{l=1}^{4} u^{*\sigma}(p)u^{\sigma}(p) = h^{-3}\delta(\sigma - \sigma'); \tag{61}
\]

\[
\sum_{\sigma=1}^{4} u^{*\sigma}(p)u^{\sigma}(p) = h^{-3}\delta(l - l')
\]

further

\[
\int \psi^{\sigma}(p)\psi^{\sigma'}(p')d\mathbf{x} = \delta(\sigma - \sigma')\delta(p - p')
\]
Expanding the wave-function for the electron and radiation using a 3-dimensional Fourier transform,

$$\Psi(x, N) = \int \sum_\sigma \Phi^\sigma(p, N) u^\sigma(p) e^{-\frac{i}{\hbar}(p \cdot k)} dp$$

(62)

and inserting into (54) with $V = 0$, one recovers (55), where $p$, together with the index $\sigma$, play now the role of the variable $J$. Going through the steps already discussed above, assuming that the electron initial state is a (positive energy) superposition with a momentum distribution $\alpha^\sigma(p)$, and that there are $N_s$ initial photons, the amplitude formula (58) reads, resuming our usual conventions for the labelling of in and out states

$$a^\sigma'(p_i; 1_r, \ldots, N_s - 1) \equiv a^\sigma''(p', k') = -\sqrt{N} \sum_{\sigma=1}^2 B^{\sigma'' \sigma} \alpha^\sigma(p' + k - k)(e^{\frac{i}{\hbar} \beta t} - 1)/\beta$$

with $\beta = E^{\sigma''}(p') + \hbar \omega' - E^\sigma(p) - \hbar \omega$ and where

$$B^{\sigma'' \sigma} = \hbar^6 \sum_\rho \left[ \frac{\epsilon^\rho \Pi^{\rho' \rho + k'} [\epsilon^\rho \Pi^{\rho + k}]}{E^\sigma(p) - E^\rho(p + k) + \hbar \omega} \right]$$

(63)

$$+ \hbar^6 \sum_\rho \left[ \frac{\epsilon^\rho \Pi^{\rho' \rho - k'} [\epsilon^\rho \Pi^{\rho - k}]}{E^\sigma(p) - E^\rho(p - k') - \hbar \omega'} \right]$$

with the notation for the current

$$\Pi_{\sigma \rho} = \tilde{\epsilon}^\rho(p') \gamma u^\sigma(p).$$

(64)

Again, the structure of the intermediate states is explicit; notice in particular that the index $\rho$ runs over all four values, which takes into account negative energy states as well. The probability for a transition to a final state with the electron momentum $p'$ and spin $\sigma'$, and the photon $k'$ and $e'$ is found linear in time

$$\rho' dp'd\Omega' \int \left| a^{\sigma'}(p'; k') \right|^2 dv' = \frac{\rho' N}{\hbar^2 \Delta^{\sigma'}(p', k')} \left| \sum_\sigma B^{\sigma'' \sigma} \alpha^\sigma(p) \right|^2 dp'd\Omega'$$

(65)

To obtain the right hand side above, one trades the integration over $\nu'$ for the one over $\beta$, with the Jacobian.

$$\frac{d\beta}{dv'} = \hbar \Delta^{\sigma'}(p, k') \equiv \hbar \left[ 1 - \frac{c}{E^\sigma(p)} p \cdot k' \right]; \quad p = p' + k' - k$$

(66)
and notices that the resulting integral has a sharp maximum around $\beta = 0$, which ensures energy conservation. To get an intensity formula, one has further to integrate $\int dp'\overline{\rho}(p')$ over the final electron momentum, and sum over (final) spin states. Instead of integrating over $p'$ one can as well use $p$, with the Jacobian

$$\frac{dp'}{dp} = \frac{\Delta^\sigma(p, \hat{k}')}{\Delta^{\sigma'}(p', \hat{k}')}.\quad (67)$$

The cross section formula for a positive energy final electron state is then

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{\nu^2} \frac{E(p')}{mc^2} \sum_{\sigma=1}^{2} \int \left| \sum_{\sigma'=1}^{2} B^{\sigma\sigma'}(p) \alpha^{\sigma'}(p') \right|^2 dp.$$ 

In the sequel of his calculation, Waller assumes the case of an electron initially at rest ($\alpha^{\sigma}(p)$ is vanishing except in the vicinity of $p = 0$). To obtain the total intensity (the K-N formula), it remains to work out the products of the currents (64) and add the intensities for a choice of two orthogonal polarizations $e'$. Waller devotes special care to the discussion of the role of the negative energy states in the summations over intermediate states in (63), $\rho = 3$ and 4. Thus, for the free electron case, he shows that it is crucial to take them into account in order to obtain the Klein-Nishina formula. This fact is rendered especially dramatic in the classical limit ($\hbar \nu/mc^2 \ll 1$) where only negative energy intermediate states contribute to the classical scattering formula. One has to remember that Dirac only in the same year proposed his negative-energy sea hypothesis (Dirac 1930), and that the paradoxes connected with the negative energy solutions (Klein 1929) were a major source of worry. The paradoxical need of the negative states as intermediate states of the scattering process must have certainly added to the confusion.

4.4 Tamm: matter and radiation quantized

Even if the semi-classical methods of Gordon and Klein-Nishina seem fully sufficient to derive the correct expressions for the Compton scattering at the order considered, Tamm’s 1930’s contribution goes even further than the latter in that it considers, in addition to a quantized radiation field, a quantized matter field, following the lines set up by Heisenberg and Pauli (Heisenberg and Pauli 1929). Tamm rederives thus the Klein-Nishina formula in a fully consistent quantum way, which he considers to be a proof of the validity of the semi-classical approach (p. 545):

The scattering of the radiation by free electrons has already been studied many times. However, one mainly has quantized
only the electron motion and, using the correspondence principle, determined the scattered radiation from the calculated distribution of the electron current. Not only is this procedure logically unsatisfactory, but it is ambiguous. It is therefore of interest to start the problem anew applying systematically the quantum-mechanical approach (quantization of the electromagnetic field and of the ψ wave) within the framework of Dirac’s wave equation for the electron. One rederives in this way the scattering formula of Klein and Nishina which legitimates the correspondence principle approach used by the latter.

In his paper, Tamm recognizes as well the necessity to take into account the negative energy intermediate states in the derivation of the KN scattering formula. The last part of his paper is devoted to the computation of the spontaneous transition of an electron from a positive to a negative energy state with the resulting emission of two photons. This process, resp. its inverse, is interpreted by Tamm along the lines of Dirac’s hole theory (Dirac 1930) as pair annihilation and creation respectively; the negative energy Dirac electron is interpreted further as a proton (p. 547):

This process can be treated consequently according to Dirac’s theory only when one will succeed in understanding theoretically the interaction of negative-energy electrons, the vanishing of the field of the electron and the ”hole” in the act of annihilation etc. With the present status we could calculate the spontaneous electron transition from a level of positive energy to one of negative energy only when totally neglecting these circumstances. If one interprets the unoccupied negative energy level as a proton and the transition just mentioned as an annihilation then the result

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66 Compare for instance the treatment of Smekal’s ”combination scattering” of radiation off atoms, on one hand as in E. Schrödinger (Ann. d. Phys., vol. 81 (1926), p. 109), and as in O. Klein (Zs. f. Phys., vol. 41 (1927), p. 407) on the other (note in the original text).

67 Die Streuung der Strahlung durch freie Elektronen ist schon vielfach untersucht worden. Man hat aber dabei meistenteils nur die Elektronenbewegung gequantelt und die Streustrahlung korrespondenzmässig aus der berechneten Verteilung des Elektronenstroms bestimmt. Dieses Verfahren ist aber nicht nur logisch unbefriedigend, sondern auch nicht eindeutig. Es schien deshalb von Interesse zu sein, das Problem von neuem in einer konsequenten quantenmechanischen Weise (Quantelung des elektromagnetischen Feldes und der ψ-Wellen) unter Zugrundelegung der Diracschen Wellengleichung des Elektrons zu behandeln. Man gelangt in dieser Weise wieder zu der von Klein und Nishina abgeleiteten Streuformel, wodurch die Folgerichtigkeit der von diesen Forschern durchgeführten korrespondenzmässigen Behandlung des Problems bestätigt wird.
can be summarized as follows: the annihilation occurs in a collision of the electron and the proton; the effective cross-section relevant to this process is equal to the classically computed cross-section
\[ \pi d^2 = \pi \left( \frac{e^2}{mc^2} \right)^2 \]
of the elementary charge \( e \). Since Dirac’s theory is, in the absence of interactions, symmetrical as concerns electrons and protons, one has to interpret \( m \) in the above expression as specific, but for the time unspecifiable average of the electron and proton masses.

It should be noted that contrary to the Compton effect this annihilation process does not have a classical counterpart in the context of the correspondence principle. When commenting his result, Tamm points out that the lifetime for the hydrogen atom, \( 10^{-3} \) sec., turns out to be much too small.

Tamm follows the lines set by Dirac, and his obtention of the explicit scattering formulas is a matter of technical manipulations much of the same nature as already discussed in the case of Waller. We shall therefore not dwell any further on the content of Tamm’s paper, also because the issue of second-quantization of matter waves, its most distinctive feature, is not relevant to the work of Stueckelberg.

Before closing this section, it is interesting to signal the paper of Seishi Kikuchi (1931). Using the Heisenberg-Pauli formalism, he shows that at the
same time the photon is scattered, the electron recoils, within Heisenberg’s uncertainty relations. The Heisenberg-Pauli formalism as applied to matter-radiation interaction has also been studied by Oppenheimer (1930). Indeed, at the end of his paper, Waller (1930) remarks that Oppenheimer was able to obtain the equation (55) from the Heisenberg-Pauli formalism, at the price of neglecting the self-energy of the electron.

5 The emergence of the interaction picture

As we mentioned in the previous section, the problem of properly taking into account the backreaction of the emitted radiation on the source is not solved in the above papers. Here is the problem as analyzed by Dirac (Dirac 1932):

We shall now consider in detail the question of how the information contained in classical electrodynamics can be taken over into the quantum theory. We meet at once with the difficulty that the classical theory itself is not free from ambiguity. To make the discussion precise, let us suppose we have a single electron interacting with a field of radiation and consider the radiation resolved into ingoing and outgoing waves. The classical problem is, given the ingoing radiation and suitable initial conditions for the electron, determine the motion of the electron and the outgoing radiation. The classical equations which deal with this problem are of two kinds, (i) those that determine the field produced by the electron (which field is just the difference of the ingoing and outgoing fields) in terms of the variables describing the motion of the electron, and (ii) those that determine the motion of the electron. Equations (i) are quite definite and unambiguous, but not so equations (ii). The latter express the acceleration of the electron in terms of field quantities at the point where the electron is situated and these field quantities in the complete classical picture are infinite and undefined. In the usual approximate treatment of the problem one takes for these field quantities just the contributions of the ingoing waves. This treatment is necessarily only approximate, since it does not take into account the reaction on the electron of the waves it emits. We should expect in an accurate treatment, that the field determining the acceleration of the electron would be in some way associated with both the ingoing and outgoing waves. Classical attempts have been made to improve the theory by assuming a definite structure for
the electron and calculating the effect on one part of it of the field produced by the rest, but such methods are not permissible in modern physics.\footnote{Dirac 1932, p. 457.}

In the same paper, which, albeit not well known, constitutes a major example of his intuition, Dirac conceives of a solution formulated in heuristic terms. He will give it a formal expression in a subsequent paper with Fock and Podolsky (Dirac, Fock and Podolsky 1932).

Let us make the assumption that the passage from the field of ingoing waves to the field of outgoing waves is just a quantum jump performed by one field. This assumption is permissible on account of the fact, discussed in the preceding section, that all the quantities in relativistic quantum mechanics are of the nature of probability amplitudes referring to one ingoing field and one outgoing field, so that we may associate, say, the right-hand sides of the probability amplitudes with ingoing fields and the left-hand sides with outgoing fields. In this way we automatically exclude quantities referring to two ingoing fields or to two outgoing fields and make a great simplification in the foundations of the theory.

The significance of the new assumption lies in the fact that the classical picture from which we derive our equations of motion must contain no reference to quantum jumps. This classical picture must therefore involve just one field, a field composed of waves passing undisturbed through the electron and satisfying everywhere Maxwell’s equations for empty space. With this picture the equations of motion for the electron are perfectly definite and unambiguous. There are no equations of motion for the field, as the field throughout space-time is pictured as given. Thus the interaction between electron and field is introduced into the equations in only one place.

The quantization of the equations of motion derived from this picture may conveniently be carried out in two stages. Let us first quantize only the variables describing the electron. We then get just the usual quantum theory of the motion of an electron in a given classical field, with the difference that in the present case the field must necessarily be resolvable into plane waves and must therefore contain nothing of the nature of a Coulomb force.\footnote{Dirac 1932, p. 458. The italics are his.}
So, as we see, the key observation is to recognize that because of the shortcomings of classical electrodynamics, a genuinely quantum ansatz has to be taken (the traditional conceptual path through the correspondence principle is no longer practical), namely that to attribute to the shift from the ingoing to the outgoing field a quantum jump nature. The effective classical picture which emerges is that of a free field obeying Maxwell equations for empty space.

What remains to be done is to find a formal counterpart enabling one to use free field equations without however neglecting the backreaction. This is exposed in the paper with Fock and Podolsky and named today the interaction (or Dirac) picture (Dirac, Fock and Podolsky 1932; to some extent, this was anticipated in Dirac 1927b). Assume a composite system described by the Hamiltonian $H_A + H_B + V$ where the $A$ and $B$ part interact via an interaction term $V$. A unitary transformation of the form $O \to O^* = \exp(i/\hbar H_B T)O \exp(-i/\hbar H_B T)$ yields a picture where the transformed dynamical variables of the $B$ system, say $q_B^*$, obey equations of motion for the part $B$ alone, so that the effect of the interaction term $V$ has been transformed away:

$$\partial_t q_B^* = \frac{i}{\hbar} [H_B, q_B^*]$$

The time variable of the free (B part only) equations is then to be considered as a separate time variable for $B$. If the $B$ part stands for the radiation Hamiltonian, and part $A$ for the particles, the interaction picture realizes the required trick of Dirac. The appearance of the independent time variable $t$ in addition to the collective time $T$ will directly lead, in a many-body situation, to the multi-time formalism as exposed later in the same paper. This is because the interaction transformation yields an individual (Dirac) equation for each separate particle with its individual time where only the coupling to the radiation field is present and the other particles absent (the particles interact only through the radiation field). In the Heisenberg-Pauli theory of quantized fields (Heisenberg and Pauli 1929), matter is second-quantized, the particles become an excited collective state of a field, and their "individual" character is blurred, so that there is no longer any need to use multi-particle equations and thus multiple times.

6 Stueckelberg’s 1934 paper

\[71\text{Relativistich invariante Störungstheorie des Diracschen Elektrons (Relativistic Invariant Perturbation Theory of the Dirac Electron), Annalen der Physik, received 10.Sept.1934.}\]
6.1 Introduction.

We come now to the final part of our study which concentrates on the 1934 contribution of E. C. G. Stueckelberg: *Relativistic invariant perturbation theory of the Dirac electron; Part I: radiative scattering and Bremsstrahlung*. Stueckelberg was at this time Privatdozent at the University of Zurich with professor Gregor Wentzel. In the winter of 1934 he was called by the University of Geneva to substitute for the deceased prof. A. Schildof. In 1935 he became there associate professor of theoretical physics. His whole career will be spent at the Universities of Geneva and Lausanne.

In 1934, Stueckelberg, after significative contributions to the quantum theory of molecular spectra and scattering started working on Q.E.D. At that time, it was obviously a prominent topic and many among the most renowned physicists were contributing. In a letter to the president of the Schulrat of E.T.H. in Zürich (8 March 1934), W. Pauli writes:

Dr. Stückelberg has stated his desire to get deeper involved with Q.E.D. and agrees with the nomination of Mr. Weisskopf [as assistant of Pauli, a position that Stueckelberg himself considered previously].

This quote seems to indicate that at that time Stueckelberg did not yet feel sufficiently acquainted with Q.E.D. However, only a couple of months later he was ready to submit his paper to the *Annalen*.

Stueckelberg felt very indebted to Arnold Sommerfeld whom he approached twice for extended periods. In a letter of 30 March 1935 he informed the latter of his nomination in Geneva and continued:

I owe my knowledge in the domain of theoretical physics mainly to the two years which I could spend in your Institute, once as a student and later as a National Research Fellow. Furthermore I received my appointment in Princeton as well as my habilitation in Zurich thanks to your recommendations. Thus I know that I owe it mainly to you, highly esteemed Herr Geheimrat, to have obtained this position in Geneva.

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72 *Part II* actually never appeared.
73 For a biography see Wenger (1986) and Schweber (1994).
74 Winans and Stueckelberg 1928, Morse and Stueckelberg 1929, and Stueckelberg 1932. In the first paper the authors thank E. U. Condon “for very helpful suggestions”.
75 To witness, e.g. the publications by Bethe and Fermi 1932, Fermi 1932, Bethe and Heitler 1934, Heisenberg 1934, Wentzel 1933,1934, Weisskopf 1934.
76 Enz et al. 1997, p. 57.
I regret of course to leave Zurich which like Princeton, was a permanent source of inspiration to me. I add a reprint of my work on the Dirac electron [Stueckelberg 1934]. In a note to appear shortly in Helv. Phys. Acta the Compton scattering of moving electrons is discussed [Stueckelberg 1935a]. These days I am busy generalizing the calculation method to the many body problem [Stueckelberg 1935c].

From 7 to 11 March 1937, A. Sommerfeld was hosted by Stueckelberg in his house in Geneva following a visit of son Johann Wolfgang in June 1936. Stueckelberg’s correspondence with Sommerfeld continued over the years as is shown by a letter of 31 March 1949 (two years before Sommerfeld’s death), in which Stueckelberg thanks for the reception of Sommerfeld’s textbook on Electrodynamics and discusses the substraction method of Dirac, his work with Rivier and Schwinger’s theory.

As its title shows, the aim of the 1934 paper is to provide a unified covariant perturbative treatment of Compton scattering, Bremsstrahlung as well as annihilation and creation of particle-antiparticle pairs (which is covered in the subsequent publication, Stueckelberg 1935b). In his paper, Stueckelberg thanks G. Wentzel for the suggestion to work out an invariant perturbation theory using a four-dimensional Fourier transform. S. S. Schweber makes the following comment on the relation between Wentzel and Stueckelberg, and their emphasis on manifest relativistic covariance:

When Stueckelberg came to the University of Zurich in 1933, Wentzel was working on eliminating the divergences in relativistic field theories (Wentzel 1933a,b, 1934a) and he got Stueckelberg interested in quantum field theory. Stueckelberg began working on the field-theoretical description of the interaction between
particles. It is to be noted that Dirac, Fock, and Podolsky’s (1932) article on quantum electrodynamics was the point of departure for both Wentzel’s and Stueckelberg’s research programs (Stueckelberg 1934, 1935c 1936). In 1938, again taking the Dirac, Fock Podolsky paper as his starting point, Stueckelberg gave a formulation of quantum electrodynamics and of various meson theories in what later became known as the interaction picture. He stressed its advantages, namely, the manifest covariance of the “Schrödinger” equation in that picture, and the possibility of writing covariant commutation rules for field operators at different times (Stueckelberg 1938a,b).

According to his publications, Wentzel was mainly interested in the years 1925-1930 by the problem of light scattering off free and bound electrons, besides studying atomic spectra with the methods of quantum mechanics, including the relativistic H-atom. Wentzel cared for manifest Lorentz invariance which is suggested by the fact that he used in two instances the multitime formalism of Dirac, Fock and Podolsky. First, in his tentative to eliminate the self-energy divergences in classical and quantum electrodynamics (Wentzel 1933 and 1934a). Second, in his study of the possible equivalence of a spin 1 photon and of a spin 1/2 particle and its antiparticle, following an idea of de Broglie (Wentzel 1934b).

As we show in the present paper (sections 6.2 and 6.3) Stueckelberg in 1934 used the interaction picture of Dirac, Fock and Podolsky and again in later publications (Stueckelberg 1936, 1938). He discussed the multitime formalism in 1935c and 1938.

The main innovation of the 1934 paper is the introduction of a new perturbative scheme yielding manifestly relativistic expressions for the matrix elements. This is achieved by performing a four-dimensional Fourier transformation of the wave-function, thus eliminating space and time variables. Compared to Waller 1930, Tamm 1930 or Heitler 1936, Stueckelberg’s procedure is thus the first departure from the “older (Dirac) form of the perturbation theory”. The approach proposed by Stueckelberg is far more powerful, but was not adopted by others at the time. This lack of interest appears retrospectively as very unfortunate, as was recognized by V. Weisskopf 1981,

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82Swedeber 1994 p. 577-78.
83See e.g. Wentzel 1925, 1926, 1927, 1929.
84Møller 1931 uses also the four-dimensional Fourier-transform but for the classical retarded potential in a correspondence - theoretic treatment of electron-electron scattering (see Kragh 1992 and Roqué 1992). See also Bethe-Fermi 1932.
85See the discussion in Jauch and Rohrlich 1955, p.158.
resp. 1983, in his recollections of that period. Weisskopf reminds us of the difficulties encountered in the higher order corrections to Q.E.D., and remarks:

Already in 1934 [...] it seemed that a systematic theory could be developed in which these infinities [divergent radiative corrections] are circumvented. At that time nobody attempted to formulate such a theory [...]. There was one tragic exception [...], and that was Ernst C.G. Stueckelberg. He wrote several important papers in 1934-38 putting forward a manifestly invariant formulation of field theory. This could have been a perfect basis for developing the ideas of renormalization. Later on, he actually carried out a complete renormalization procedure in papers with D. Rivier, independently of the efforts of other authors. Unfortunately, his writings and his talks were rather obscure, and it was very difficult to understand them or to make use of his methods. He came frequently to Zurich in the years 1934-6, when I was working with Pauli, but we could not follow his way of presentation. Had Pauli and I myself been capable of grasping his ideas, we might well have calculated the Lamb shift and the correction to the magnetic moment of the electron at the time.

The virtue of Stueckelberg’s approach is to include the findings of his predecessors, but making manifest the underlying general symmetry features. These trends characterize Stueckelberg’s life-long work and our paper is the first stage in exploring this broader theme. We shall return to the assessment of Stueckelberg’s contribution and its fate at the end of this paper. It is time now to get a closer look at his method.

We recall first the successive stages in the calculation of Compton scattering.

1. Dirac 1926b; Dirac 1927a

The first two papers of Dirac illustrate one of the first uses of quantum mechanics in the problem of matter-radiation interaction. The dynamical variables of the charged particle are quantized, and its wave function obeys the relativistic Schrödinger (Klein-Gordon) equation. This equation is solved with the help of canonical transformations. The electro-magnetic field is not quantized.

2 Gordon 1926, Klein-Nishina 1929

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86 Rivier-Stueckelberg 1948, Rivier 1949.
87 Weisskopf 1981, p. 78, resp. 1983, p. 74.
Their treatment is semi-classical. The electromagnetic field is not quantized, but the electron satisfies the relativistic Klein-Gordon, resp. Dirac equation. The electron current, modified by the primary field, is the source of the final retarded field, calculated using Maxwell’s equations. The treatment is “continuous” and no intermediate states appear. There is however a notion of initial and final states with a discontinuous switch from the former to the latter. The general treatment is in principle relativistic, manifest in Gordon’s case, implicit in the Klein-Nishina paper, where the main part of the calculations is done in the rest frame of the electron.

3. Waller 1930 and Tamm 1930.

The radiation field is now quantized (and the electron field as well in Tamm), and the time-dependent perturbation scheme of Dirac 1927b is applied. Intermediate states are now necessary because the interaction term linear in the field prevents a direct transition from the initial to the final state (see our discussion of Dirac’s paper section 4.1 and 4.2). At the interaction point (vertex), energy is not conserved. Although the electron obeys the relativistic Dirac equation, the formulas of the perturbation theory break manifest covariance, and this is reflected by the presence in the matrix elements of the non-covariant factors \((E - E')^{-1}\). This is because standard perturbation theory expands the full solution in a basis of unperturbed solutions with time-dependent coefficients, which, in the case of free motion (plane waves), amounts to perform a Fourier transform for space variables only. This unsymmetrical treatment of space and time makes the calculation clumsy. Furthermore, the contributions of positive and negative electron energies in the intermediate state, necessary for obtaining the correct classical limit, are considered separately and reunited only later using spinor identities. The results are obtained in the rest system of the electron.

In Stueckelberg’s 1934 paper, thanks to the use of the four-dimensional Fourier transform, manifest covariance is kept throughout. The factor \((E - E')^{-1}\) is now replaced by the covariant expression \((p^2 + M^2)^{-1}\) where \(p\) is the four-momentum of the intermediate state, \(M = \frac{\hbar}{m}\), and \(m\) is the electron mass, corresponding to a virtual particle off mass-shell \((p^2 + M^2 \neq 0)\). Energy and three-momentum are now conserved at the vertices. The contributions of positive and negative energies (corresponding to virtual electrons and positrons) are contained in the single propagation function \((p^2 + M^2)^{-1}\), which corresponds to what later was called Feynman propagator\(^{88}\) and by Rivier-Stueckelberg 1948, causal function\(^{89}\). All of Stueckelberg’s expressions

\(^{88}\)In this particular case (Born approximation) \(i\epsilon\) is not necessary.
\(^{89}\)Feynman 1948 and 1949. Rivier-Stueckelberg 1948 and Rivier 1949.
for matrix elements are identical to those obtained nowadays from Feynman diagrams.

Let us mention two other important features. First, because he doesn’t commit himself to any specific gauge, Stueckelberg’s matrix elements are manifestly gauge invariant, a feature rather unusual for those times (for instance Fermi in his review 1931, and after him Heitler 1936 worked in the radiation (or Coulomb) gauge).

Next, in evaluating his expressions, Stueckelberg uses integration over the complex energy-plane, making explicit use of the singularity structure as a device for putting external particle states on mass-shell.

6.2 Compton scattering on scalar particles following Stueckelberg’s method

As a warm-up exercise, we consider the scattering of photons on charged scalar particles, using the method developed by Stueckelberg in his 1934 paper on the Compton scattering on spin 1/2 electrons. This is the same problem which was solved by Dirac 1926b, 1927a, and Gordon 1926 (see section 3). Stueckelberg’s method is not only more direct, but also yields a matrix element for the scattering process which is manifestly Lorentz and gauge invariant. The starting point is again the relativistic equation for the scalar wave-function \( \phi(x) \) of the “electron”, interacting with the electromagnetic field \( A_{\mu} \), which is now quantized according to Dirac 1927b.

\[
(-i\partial_{\mu} + eA_{\mu}(x))(-i\partial^{\mu} + eA^{\mu}(x))\phi(x) + \frac{m^2c^2}{\hbar^2} \phi(x) = 0 \quad (68)
\]

In the sequel, we put \( M = \frac{mc}{\hbar} \). Instead of the pseudoeuclidean metric with \( x_4 = iict \) fashionable in Stueckelberg’s time, we use the real metric \( g_{\mu\nu} \) with \( g_{ii} = g_{00} = 1, \ i = 1, 2, 3 \).

In the interaction picture (see section 6)\(^91\), \( A_{\mu} \) is a free field with the Fourier expansion\(^{92}\)

\[
A^\mu = \sum_{k} \epsilon_{k}^\mu V_k \left[ a_k e^{i(k\cdot x)} + a_k^\dagger e^{-i(k\cdot x)} \right] \quad (69)
\]

\(^{90}\)Stueckelberg probably learned complex integration techniques in physics from A. Sommerfeld, whose courses he followed in Munich 1924/1925 and whom he visited, as we saw, in 1930. It is interesting to note that Wentzel used as well Sommerfeld’s technique (Sommerfeld 1916) in his 1926 paper on multiply-periodic systems within the new quantum mechanics (Wentzel 1926). See also Stueckelberg’s paper “Sur l’intégration de l’équation \( \sum_{i=1}^{4} \partial_i^2 - l^2)Q = -\rho \) en utilisant la méthode de Sommerfeld”, Stueckelberg 1939).

\(^{91}\)See Dirac 1927b and Dirac-Fock-Podolsky 1932.

\(^{92}\)Stueckelberg writes \( \Gamma_k \) instead of \( a_k \) and \( \sigma^k \) for \( e_k \)
The sum over \( k \) refers to photons in a box \( G \) with

\[
V_k^2 = \frac{2\pi}{Gk_0c\hbar} \tag{70}
\]

The 4-momentum \( k \) and 4-vector of polarization \( e_k \) satisfy the mass-shell and transversality conditions

\[
k_\mu k^\mu = k^2 = 0; k_\mu e_k^\mu = k \cdot e_k = 0; e_k^2 = 1 \tag{71}
\]

The annihilation, resp. creation operators \( a_k, a_k^\dagger \) obey the usual commutation relations

\[
[a_k, a_k^\dagger] = \delta_{kk'} \tag{72}
\]

Applied to an eigenstate \( |N_1, ..., N_k, ...\rangle \) of the photon number operator \( N \), with eigenvalues \( N_i \), they yield the well known result\(^{93}\)

\[
a_k |N_1, ..., N_k, ...\rangle = \sqrt{N_k} |N_1, ..., N_k - 1, ...\rangle \tag{73}
\]

\[
a_k^\dagger |N_1, ..., N_k, ...\rangle = \sqrt{N_k + 1} |N_1, ..., N_k + 1, ...\rangle
\]

Following Stueckelberg the interacting wave-function is expanded on the photon number eigenstates \( |N^j\rangle \) with coefficients \( \varphi(x) \), according to Dirac (1927b) (see section 4.1)

\[
\phi^j(x) = \sum_j \varphi^j(x) |N^j\rangle \tag{74}
\]

Here \( N^j \) denotes all possible photon configurations \( \{N_1, N_2, ..., N_k, ...\} \). Now comes the departure of Stueckelberg’s method from those of his predecessors, Waller, Tamm, Heitler, etc. He introduces the four-dimensional Fourier transformation for the ”electron” wave function \( \varphi(x)\)\(^{95}\)

\[
\varphi^j(x) = \int d^4p e^{i(p \cdot x)} \chi^j(p) \tag{75}
\]

The result will be the elimination in the perturbation expansion of time together with space\(^{96}\). Introducing \((69), (74)\) and \((75)\) into the Klein-Gordon

\(^{93}\)Stueckelberg writes \( T_k \) instead of \( eV_k \).

\(^{94}\)Stueckelberg writes \( u(N_1, ..., N_k, ...) \) instead of \( |N_1, ..., N_k, ...\rangle \).

\(^{95}\)In the following formula, Stueckelberg writes \( l \cdot x \) instead of \( p \cdot x \). We retain the more familiar notation \( p \), but the reader has to remember that this \( p \) includes the factor \( \hbar^{-1} \).

\(^{96}\)Notice that although the photon states are discretized (i.e. radiation is in a box), the Fourier expansion above is continuous, the states considered not being necessarily on mass-shell.
equation (68) yields

\[
\sum_j \int d^4p e^{i(p \cdot x)} \left( [p^2 + M^2] \chi^j(p) \right. \\
+ e \sum_k (2p - k) \cdot e_k V_k(a_k + \hat{a}_{-k}) \chi^j(p - k) + \\
\left. e^2 \sum_{k,k'} e_k \cdot e_{k'} V_k V_{k'}(a_k + a_{-k})(a_{k'} + a_{-k'}) \chi^j(p - k - k') \right) |N^j \rangle = 0
\]

(76)

The general approach of Stueckelberg is to use a perturbation expansion in powers of the charge \( e \). The perturbation expansion applies to the time-independent functions \( \chi^j(p) \). Thus, the zeroth order is given by the free state equation

\[
\int d^4p e^{i(p \cdot x)} \left( [p^2 + M^2] \chi^{j(0)}(p) \right) = 0
\]

(77)

To get the first order we insert \( \chi^{j(0)}(p - k) \) into the second term of (76)

\[
\sum_j \left( \int d^4p e^{i(p \cdot x)} \left( [p^2 + M^2] \chi^{j(1)}(p) \right) +
\right.
\]

\[
e \sum_k (2p - k) \cdot e_k V_k(a_k + \hat{a}_{-k}) \chi^{j(0)}(p - k) \right) |N^j \rangle = 0
\]

(78)

For the second approximation, \( \chi^{j(1)} \) is introduced into the second term and \( \chi^{j(0)} \) into the third one

\[
\sum_j \left( \int d^4p' e^{i(p' \cdot x')} \left( [p'^2 + M^2] \chi^{j(2)}(p') \right) \right)
\]

\[
\left. \langle N^j \rangle = 0
\right)
\]

(79)

This expression gives a matrix element symmetric in \( k \) and \( k' \). For Compton scattering, we take \( p, k \) for the 4-momenta of the initial "electron" and photon. \( \hat{a}_{-k} \) annihilates the photon \( k \) and creates a photon with momentum \( -k' \). The value of the matrix element of \( a_{-k} \) is one. In order to describe an outgoing photon we change the sign of \( k' \). With \( p' \) the final "electron" momentum the conservation law reads

\[
p + k = p' + k'
\]

\[^{97}\text{Stueckelberg's notation is here } l^0 \text{ for } p, l \text{ for } p', -p \text{ or } m \text{ for } k'. \text{ See the footnote preceding eq. (75).}\]
so that finally the relation between the initial wave-function $\chi^{(0)}(p)$ and the final $\chi^{(2)}(p')$ is given by

$$
\chi^{(2)}(p') = \frac{e^2}{p'^2 + M^2} V_k V_{k'} \Omega \chi^{(0)}(p)
$$

(80)

$$
\Omega = \frac{(2p' + k') \cdot e_{k'} - (2p + k) \cdot e_k}{(p + k)^2 + M^2}
$$

(81)

$$
+ \frac{(2p' - k) \cdot e_k - (2p - k') \cdot e_{k'}}{(p - k')^2 + M^2} - 2e_k \cdot e_{k'}
$$

We see that Stueckelberg’s scheme is entirely within momentum space $p$. The matrix element $\Omega(p)$ is obviously Lorentz invariant. It is also invariant under the gauge transformations $e_k \to e_k + \lambda k; e_{k'} \to e_{k'} + \lambda' k'$ if the external electrons are on mass-shell. The first two terms with denominators contain the contribution of intermediate states whereas the last term corresponds to the direct interaction. It is important to realize that Stueckelberg didn’t need to modify the overall scheme of iteration to achieve this result; such is not the case with the previous (Dirac theory) where a refinement of the perturbative method had to be made to obtain a contribution through intermediate states (see section on Dirac’s perturbation theory).

Equation (81) is a witness to the modernity of Stueckelberg’s approach, since one had to wait until 1948 to find similar expressions. $\Omega$ is actually identical (except for normalizations) to the corresponding factor given by Bjorken and Drell 1964, eq. (9.30), using Feynman techniques.

Equation (81) greatly simplifies in the rest system of the initial charged particle ($p = 0$) and in the Coulomb or radiation gauge where $e_0 = e'_0 = 0$ and hence $p \cdot e_k = p \cdot e_{k'} = 0$. In this case $\Omega_{\text{lab.}} = -2e_k \cdot e_{k'}$.

The steps above illustrate the iteration method of Stueckelberg.

We now pass to the discussion of Stueckelberg’s method for calculating the cross-section. It starts with the definition of ”on mass-shell wave-functions”. We first introduce the on mass-shell 4-momentum, denoted by

$$
\vec{p} = (p, \vec{p}_0); \vec{p}_0^2 = P^2 + M^2
$$

(82)

To solve the free Klein-Gordon equation (77), Stueckelberg makes the Ansatz

$$
\chi^{(0)}(p) = \frac{1}{i\pi} \frac{\omega^{(0)}(p)}{p^2 + M^2}
$$

(83)

See also Heitler 1936, p. 88.
where $\omega^{(0)}(p)$ is a continuous function which vanishes fast enough at infinity. Then $\chi^{(0)}(p)$ satisfies (77) if $\omega^{(0)}(\bar{p})$ obeys

$$ (p^2 + M^2)\omega^{(0)}(\bar{p}) = 0 \quad (84) $$

Notice that $p^2 + M^2 = \bar{p}_0^2 - p_0^2$ which makes explicit the singularity of (83) in the energy variable $p_0$. The reason of this special way in which Stueckelberg expresses the solution of (77) is the following. It is different from zero only for those values of $p$ where the $\bar{p}_0^2 = p^2 + M^2$. Elsewhere, it vanishes. The form of (83) with its explicit singularity makes it possible to write both cases in a single closed form, which is then suitable for being reinserted into interation steps. Each order of iteration will add a new singularity the impact of which being crucial during contour-integration. The information about the values of $p$ where the non-trivial solution is valid is then retrieved by complex contour-integration around the singularity (This feature of Stueckelberg’s method is quite original and contributes to the full automatism of his perturbative calculus99).

For instance, in the first order term there are two poles

$$ \chi^{(1)}(p) = \frac{-1}{[p^2 + M^2]} \sum_k (2p - k) \cdot \frac{1}{i\pi} \frac{\omega^{(0)}(p)}{p^2 + M^2} \chi^{(0)}(p - k) \quad (85) $$

However, to this order, it is impossible to simultaneously satisfy the mass-shell conditions $(p^2 + M^2) = 0$, $(p - k)^2 + M^2 = 0$ and $k^2 = 0$.

Accordingly, $\chi^{(0)}(p)$ is integrated over the complex $p_0$-plane, in the positive sense if $p_0 = \bar{p}_0$, negative if $p_0 = -\bar{p}_0$. The result is, using Cauchy’s formula

$$ \int_C dp_0 \chi^{(0)}(p)e^{ip_0x_0} = \int_C dp_0 e^{ip_0x_0} \frac{1}{i\pi} \frac{\omega^{(0)}(p)}{p^2 + M^2} = \frac{\omega^{(0)}(\bar{p})}{\bar{p}_0} e^{i\bar{p}_0x_0} \quad (86) $$

This should then be put into the integral

$$ \varphi^{(0)}(x) = \int d^4p e^{i(p \cdot x)} \chi^{(0)}(p) \quad (87) $$

To obtain the final ”electron” wave-function on mass-shell, we follow Stueckelberg and define, by analogy with (83)

$$ \frac{\omega^{(2)}(p', x_0)}{\bar{p}_0} = \int_{C'} dp'_0 e^{i(p'_0 - p_0)x_0} \chi^{(2)}(p') \quad (88) $$

As will be seen presently, $\omega^{(2)}$ is time dependent, while the same formula for $\omega^{(0)}$ gives a time-independent expression. The idea is to define a number

\begin{footnote}
Nowadays, instead of (83) one writes $\chi^{(0)}(p) = \delta(p^2 + M^2)\omega^{(0)}(p)$.
\end{footnote}
of final "electrons" which will turn out to be linear in \( t \) and will allow then an easy computation of the number of "electrons" scattered per unit time. This will now be sketched.

Remembering (80)

\[
\chi^{(2)}(p') = \frac{e^2}{p'^2 + M^2} V_K V_{K'} \Omega(p) \chi^{(0)}(p)
\]

we see that \( \chi^{(2)} \) has now two poles, at \( p'_0 = \bar{p}'_0 \) and \( p_0 = \bar{p}_0 \), the latter coming from \( \chi^{(0)}(p) \) (there is a third pole in \( \Omega(p) \), see (81), which does not contribute for the same reason as above) To integrate over the complex \( p'_0 \)-plane one has to substitute \( p_0 = p'_0 + k'_0 - k_0 \). Defining

\[
s_0 = \bar{p}'_0 + k'_0 - \bar{p}_0 - k_0
\]

the integration (88) gives, up to terms of order \( s^2_0 (n \geq 0) \),

\[
\frac{\omega^{(2)}(p', x_0)}{\bar{p}'_0} = -2e^2 \left(1 - e^{-i\pi x_0}\right) \frac{\Omega(\bar{p}) \omega^{(0)}(\bar{p})}{2\bar{p}'_0^2 \bar{p}_0} \tag{89}
\]

Later on, we shall integrate over the pole in \( s_0 \) justifying the neglect of higher order terms.

In the previous approach (Waller and Tamm) the transition amplitude of a process under consideration was directly related to the value of the expansion coefficients (see (49) or more specifically (58) and (59)) and thus "ready to use". Here, to obtain the scattering cross-sections, one uses the conserved currents of free particles

\[
j_\mu(x) = -i(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*) \tag{90}
\]

and defines the number of incident charged particles \( n^{(0)} \) as

\[
n^{(0)} = \int d^3x j_0(x) = -i \int d^3x (\varphi^* \partial_0 \varphi - \varphi \partial_0 \varphi^*) \tag{91}
\]

One finds consequently

\[
n^{(0)} = (2\pi)^3 \int d^3p 2\bar{p}_0 \frac{\omega^{(0)*}(\bar{p}) \omega^{(0)}(\bar{p})}{\bar{p}'_0} \frac{\omega^{(0)*}(\bar{p}) \omega^{(0)}(\bar{p})}{\bar{p}_0} \tag{92}
\]

Similarly, for the final "electrons" Stueckelberg uses the same expression but now with the perturbed solutions

\[
n^{(2)}(x_0) = (2\pi)^3 \int d^3p 2\bar{p}_0 \frac{\omega^{(2)*}(\bar{p}) \omega^{(2)}(\bar{p})}{\bar{p}'_0} \frac{\omega^{(2)*}(\bar{p}) \omega^{(2)}(\bar{p})}{\bar{p}_0} \tag{93}
\]
With the solution (89) this gives the time dependent expression

\[ n^{(2)}(x_0) = e^4 \left( 1 - \cos(s_0 x_0) \right) V_k^2 V_{k'}^2 \Omega^* \Omega \]  

(94)

where the expectation value of \( \Omega^* \Omega \) is

\[ \Omega^* \Omega = \frac{(2\pi)^3}{\pi^{(0)}} \int d^3 p 2\vec{p}_0 \frac{\omega^{(0)}(p)^\dagger}{\vec{p}_0} \Omega^*(p) \Omega(p) \frac{\omega^{(0)}(p)}{\vec{p}_0} \]  

(95)

To get the cross-section one has to integrate over the phase space of final electrons \( d^3 \vec{p}' \) and of final photons \( (2\pi)^{-3} G k_0'^2 dk_0' d\Omega \) in the solid angle \( d\Omega \) (not to be confused with the preceeding \( \Omega \)). Furthermore one assumes that the initial "electron" wave packet is sharply peaked at a initial momentum value.

After changing the variable \( k_0' \) into \( s_0 \) one meets the integral

\[ \int ds_0 \left\{ \frac{(1 - e^{-is_0 x_0})}{2 s_0} \right\}^2 = \int ds_0 \frac{(1 - \cos(s_0 x_0))}{2 s_0} = \frac{2\pi}{4\hbar} x_0 = \frac{\pi}{2\hbar} c t \]  

(96)

Now, dividing by \( t \), by the incident photon flux and the density of target particles, one finds the gauge invariant differential cross-section (all external particles on mass-shell)

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{M^2} \frac{k_0'^2}{k_0^2} \left\{ \frac{(\vec{p} \cdot e_k)(\vec{p}' \cdot e_{k'})}{\vec{p} \cdot k} - \frac{(\vec{p}' \cdot e_k)(\vec{p} \cdot e_{k'})}{\vec{p}' \cdot k'} - e_k \cdot e_{k'} \right\}^2 \]  

(97)

Here, \( \alpha = e^2/\hbar c \) is the fine structure constant, \( M^{-1} = \hbar/mc \) is the Compton wave-length and \( \alpha/M = r_0 \) is the classical radius of the electron.

### 6.3 The spinor case

We are now in a position to review Stueckelberg’s 1934 paper for spin 1/2 electrons. The essential complication with respect to the scalar case above is the necessity to take into account the spin degrees of freedom. Furthermore, Stueckelberg generalizes slightly the formalism in order to derive an expression which is then suitable to be applied to Compton scattering and also to Bremsstrahlung and pair production. The starting point is now the Dirac equation for the spinor wave-function \( \Psi(x) \)\(^{100}\)

\(^{100}\)The original notation is \( C \) instead of \( M \).
\[
\left[ \frac{1}{i} (\gamma \cdot \partial_x) + M + e V(x) \right] \Psi(x) = 0 \tag{98}
\]

where the coupling of the matter current to the electromagnetic field is

\[
V(x) = \sum_k V_k (e_k \cdot \gamma) \left[ a_k e^{i(k \cdot x)} + a_k^\dagger e^{-i(k \cdot x)} \right] \tag{99}
\]

\(V_k\) is given by (77), \((e_k \cdot \gamma)\) denotes the scalar product of the photon polarization 4-vector \(e_k\), and the Dirac matrices \(\gamma_\mu, \mu = 0, 1, 2, 3\) (we now use the following convention for \(\gamma\) matrices: \(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2g_{\mu\nu}\))[^1]. The analogue of the expansion (74) involves now spinor functions \(\varphi^j(x)\):

\[
\Psi(x) = \sum_j \varphi^j(x) |N_j >\tag{100}
\]

and as previously Stueckelberg Fourier expands[^2]

\[
\Psi(x, N) = \sum_j \int d^4 p e^{i(p \cdot x)} u^j(p) |N_j > \tag{101}
\]

with \(u^j(p)\) the spinor in momentum space[^3].

The Fourier-transform of the full Dirac equation (98) can be written

\[
\sum_j \int d^4 p e^{i(p \cdot x)} \left\{ \left[ (\gamma \cdot p) + M \right] u^j(p) + e \sum_{k,i} V_k P_{kji} (e_k \cdot \gamma) u^i(p-k) \right\} |N_j > = 0 \tag{102}
\]

Here \(P_{kji}\) is a matrix element of \(P_k\) the latter standing for one of the operators \(a_k, a_k^\dagger,\) or 1. This notation allows a unified treatment of the above mentioned processes[^4]

\[
P_k |N^i > = \sum_j |N^i > P_{kij} \tag{103}
\]

and \(V_k\) is one term in the sum of eq. (99).

[^1]: Stueckelberg works with the convention

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\delta_{\mu\nu}
\]

the minus sign of the right hand side being the opposite of that chosen by Dirac (and all the other authors discussed in our paper).

[^2]: See footnote before eq. (75). Compare also this formula to (62).

[^3]: Stueckelberg writes \(A^j(l)\) instead of \(u^j(p)\).

[^4]: \(P_k = 1\) will be relevant for Bremsstrahlung, see below section 6.6
The perturbation expansion is obtained in the same way as in the scalar case. The first approximation \( u^{(1)} \) is found to be (compare with (78))

\[
\begin{align*}
 u^{(1)} &= -\frac{1}{(\gamma \cdot p) + M} \sum_k V_k P_{k'0}(e_k \cdot \gamma) u^{(0)}(p - k) \\
\end{align*}
\]

Similarly, for the second order, the result is (compare (80) and (81)):

\[
\begin{align*}
 u^{(2)}(p') &= -\frac{1}{p'^2 + M^2} \sum_{k',k} V_{k'} V_k (P_{k'})_{ji}(P_k)_{i0} \Omega(p) u^{(0)}(p) \\
\end{align*}
\]

\[
\Omega(p) = ((\gamma \cdot p') - M) \left\{ \frac{(e_{k'} \cdot \gamma)((\gamma \cdot p' - k') - M)(e_k \cdot \gamma)}{(p' - k')^2 + M^2} + \frac{(e_k \cdot \gamma)((\gamma \cdot p' - k) - M)(e_{k'} \cdot \gamma)}{(p' - k)^2 + M^2} \right\}
\]

with \( p = p' - k - k' \) and \( (P_{k'})_{ij} \) are the matrix elements of the operators \( a_k, a_k^\dagger \) and 1, that is \( \sqrt{N_k} \) and \( \sqrt{N_k} + 1 \) for the first two. Here, one gets only the contributions corresponding to intermediate states as there is no direct transition, contrary to the scalar case.

The expressions (105) and (106) are obviously Lorentz-invariant and correspond exactly to the expressions obtained by "modern" Feynman rules.

We discuss now the mass shell solutions. The spinor analogue of the zero-order approximation, is given by the same Ansatz (see eq. (83))

\[
\begin{align*}
 u^{(0)}(p) &= \frac{1}{i\pi} \frac{\omega(p)}{p^2 + M^2} \\
\end{align*}
\]

where this time \( \omega \) satisfies the free ("mass-shell") Dirac equation

\[
(\gamma \cdot \bar{p} + M)\omega(\bar{p}) = 0
\]

\( \bar{p} \) is again given by (82). The mass-shell condition is now obtained as the condition of vanishing determinant of the four by four matrix \( (\gamma \cdot p + M) \). Equation (108) admits four linearly independent solutions. For each sign of the energy there are two spin states \( \omega^r(\bar{p}) \). In eq. (106), Stueckelberg gets automatically the projection operator

\[
\Lambda^+(\bar{p}') = \frac{\gamma \cdot \bar{p}' - M}{2M} = \sum_{r=1}^{2} \omega^r(\bar{p}') \omega^r(\bar{p}')
\]

This shows that the summation over final spin states of the electron is naturally built in. One can keep it like that, or split it into respective contributions of the two spin states.

\(^{105}\omega(p)\) is in Stueckelberg’s paper B(l^0).
The integration over the complex $p_0$ plane yields (cf. (86) and (87))

$$\phi^{(0)}(x) = \int d^4p e^{i(p \cdot x)} u^{(0)}(p)$$

$$= \int d^4p e^{i(px)} \frac{\omega(\vec{p})}{p_0} e^{-i\vec{p}_0x_0} \tag{109}$$

For the second order case one finds using eq. (105)

$$\frac{\omega^{(2)}(\vec{p}, x_0)}{\bar{p}_0} = \sum_{k', k} \frac{(1 - e^{is_0x_0})}{s_0} V_k V_{k'} (P_k)_{ji} (P_{k'})_{i0} \frac{\Omega(\vec{p}) \omega(\vec{p})}{\bar{p}_0 \bar{p}_0} \tag{110}$$

plus higher order terms in $s_0 = \vec{p}_0' - \vec{p}_0 - k_0 - k_0'$.

We now specialize to the case of Compton scattering. We choose the initial photon configuration as $N_k = (0, 0, ..., 1_k, ...)$. The problem is then to find the transition amplitude to the final configuration $N_{k'} = (0, 0, ..., 1_{k'}, ..., 0_k, ...)$, i.e. a photon has been absorbed in state $k$ and another one emitted in state $k'$. For Compton scattering, $p, p'$ are then interpreted as the initial, resp. final electron 4-momenta and $k, -k'$, the initial, resp. final photon 4-momenta. This choice of variables makes the symmetry of the result in $k$ and $k'$ explicit.

In the spinor case, Stueckelberg defines the number of electrons in state $\phi(x)$ by (instead of (91))

$$n = \int d^3x \rho = \int d^3x \phi^\dagger(x) \phi(x) \tag{111}$$

In zeroth approximation (no interaction), the number of particles (electrons) is given by the time independent and Lorentz invariant $n^{(0)}$, using Eq. (109)

$$n^{(0)} = (2\pi)^3 \int d^3p \frac{\omega(\vec{p}) \omega(\vec{p})}{\bar{p}_0 \bar{p}_0} \tag{112}$$

The expectation value of the operator $\Omega^\dagger \Omega$ of Eq. (106) is (compare with eq. (95))

$$\overline{\Omega^\dagger \Omega} = \frac{(2\pi)^3}{n^{(0)}} \int d^3p \frac{\omega(\vec{p}) \omega(\vec{p})}{\bar{p}_0 \bar{p}_0} \tag{113}$$

As before, the number of final electrons is given by the time dependent

\footnote{Here $\phi^\dagger(x)$ is the hermitian conjugate spinor, whereas for Stueckelberg $\phi^\dagger(x)$ denotes the adjoint spinor which nowadays is written $\bar{\varphi}(x) = \phi^\dagger(x) \gamma_0$.}
\[ n^{(2)} = (2\pi)^3 \int d^3p \frac{\omega^{(2)}(\bar{p}, x_0) \omega^{(2)}(\bar{p}, x_0)}{\bar{p}_0}, \]

By the same mechanism as in the scalar case (see eq. 96), the integration over \( s_0 \) yields a linear increase of \( n^{(2)} \) with time \( x_0 \). The singularity at \( s_0 = 0 \) expresses the equality of initial and final energy (the outgoing photon energy being \(-k'_0\) for Compton scattering; see the comment after eq. 100).

These are the main ingredients for obtaining the Klein-Nishina formula (after insertion of the correct expressions for the \( V \)'s, and similarly for the \( P_k \)'s) see further discussion in sections 6.5 and 6.7.

### 6.4 Lorentz and gauge invariant squared matrix element \( \Omega^\dagger \Omega \)

A systematic feature of Stueckelberg’s 1934 paper is the care he takes to make the Lorentz and gauge invariance manifest. This is particularly the case for \( \Omega^\dagger \Omega \), the square of the transition matrix element function of the polarizations \( e_k \) and \( e_{k'} \) of the electromagnetic field. In his calculation, Stueckelberg considers a final photon which is on mass-shell and transverse, i.e. \( k'^2 = k' \cdot e_{k'} = 0 \). On the other hand, he keeps terms proportional to \( k^2 \) and \( k \cdot e_k \). This allows him to use his formulas also for Bremsstrahlung (see below). For Compton scattering he will of course put also \( k^2 = k \cdot e_k = 0 \).

Stueckelberg now calculates \( \Omega^\dagger \Omega \) with the help of eq. 106. We have shown above that the sum over final spins is taken care of. What remains to be done is to apply the free Dirac equation for \( \omega(\bar{p}) \), the mass shell conditions for the initial and final electrons and the energy momentum conservation in order to obtain a simpler expression. In the end he uses the formula

\[ \omega(\bar{p})^\dagger \gamma_\mu \gamma_\nu \omega(\bar{p}) = \frac{\bar{p}_\mu}{mc}. \]

In the case of Compton scattering, Stueckelberg’s result was obtained later by Wannier 1935, taking traces of Dirac matrices.

The general formula found by Stueckelberg is the following

\[
\frac{n_0}{2p_0^0} \Omega^\dagger \Omega = \frac{(e_k' \cdot p)^2}{(k' \cdot p)^2} \left\{ 2(e_k' \cdot p)^2 + \frac{1}{2} e_k^2 k^2 - 2(e_k' \cdot p')(e_k \cdot k) \right\}
+ \frac{(e_k \cdot p')^2}{(k' \cdot p')^2} \left\{ 2(e_k \cdot p)^2 + \frac{1}{2} e_k^2 k^2 + 2(e_k \cdot p)(e_k \cdot k) \right\}
\]

(114)
\[-2\frac{(e_{k'} \cdot p)(e_{k'} \cdot p')}{(k' \cdot p)(k' \cdot p')} \left\{ 2(e_k \cdot p)(e_k \cdot p') + \frac{1}{2} e_k^2 k^2 - (e_k \cdot k')(e_k \cdot k) \right\} \\
+ 2\frac{(e_{k'} \cdot p)}{(k' \cdot p)} (e_{k'} \cdot e_k)(e_k \cdot 2p' - k) \\
- 2\frac{(e_{k'} \cdot p')}{(k' \cdot p')}(e_{k'} \cdot e_k)(e_k \cdot 2p + k) + 2(e_k \cdot e_{k'})^2 \\
+ \frac{1}{2(k' \cdot p)(k' \cdot p')} \left\{ e_k^2 (k' \cdot k)^2 - 2(e_k \cdot k')(e_k \cdot k)(k' \cdot k) + (e_k \cdot k')^2 k^2 \right\} \]

only Lorentz invariant scalar products appear. In this expression \( p' \) and \( p \) satisfy the mass-shell conditions \((p')^2 + M^2 = 0\) and \((p)^2 + M^2 = 0\). Equation (114) is again manifestly Lorentz invariant and coincides with the result which would be obtained with Feynman rules.

It is also invariant under the two independent gauge transformations\(^{108}\)

\[
e_k \rightarrow e_k + \text{const.} \cdot k \\
e_{k'} \rightarrow e_{k'} + \text{const.} \cdot k' \]

As Stueckelberg pointedly remarks, checking this invariance prevents algebraic errors. Indeed, if one specializes (114) to a transverse photon \( k \), and compares the result with Eq. 11-13 p.231 of Jauch-Rohrlich 1955, one finds an error in the latter\(^{109}\). As already mentioned, Stueckelberg’s insistence on gauge invariance was rather unusual at the time of his paper. Where the gauge is not fixed from the beginning (as Fermi’s 1932 or Heitler’s 1936 choice of the Coulomb gauge \( e_0 = 0 \) ), it is merely noticed. Again, the virtue of Eq. (114) is its generality. Since the signs of \( p_0 \) and \( p'_0 \) are not fixed, the formula can be used, besides for Compton scattering and Bremsstrahlung, to discuss pair production and annihilation\(^{110}\).

\(^{107}\)In order not to overload the notations we skipped in the above formula the "bars" on \( p \) and \( p' \).

\(^{108}\)The invariance under the first transformation holds even off mass-shell \( k^2 \neq 0 \), when the photon is coupled to a conserved current, and the electrons are on mass-shell (we thank M. Veltman for pointing out this result to us).

\(^{109}\)On p. 233 the authors however claim that their equation (11-13) is gauge invariant. Therefore the missing factor 2 in the last term of the first line must be a misprint.

\(^{110}\)Franz 1938 gives the matrix elements for arbitrary polarizations of the photon and the electron, in the electron rest-system. See also Nishina 1929a,b.
6.5 Pauli 1933: Klein-Nishina formula for moving electrons

An obvious application of Stueckelberg’s relativistic formalism was to answer a question raised by Pauli. At that time, discrepancies were found between the Klein-Nishina formula for Compton scattering and experimental data for high energy incoming photons (see Schweber 1994 p.82). As we noted before, all Compton scattering calculations for spin 1/2 electrons done before Stueckelberg were performed in the rest system of the initial electron. Pauli asked the question whether the K-N formula was still valid in the limit where the initial and final light frequencies \( \nu \) and \( \nu' \) go to infinity, their ratio being kept constant. To this end he Lorentz-transformed the \( K-N \) formula from the rest system of the initial electron to one with arbitrary velocity \( \vec{v} \). Taking then the above limit, he found that the cross-section depended explicitly on \( \vec{v} \). His very elegant calculation nevertheless took five pages, whereas Stueckelberg could very easily specialize his formula to this case.

Pauli’s formula for Compton scattering of unpolarized photons is (in modern notation, as in eq. (117))

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2M^2} \left( \frac{mc^2}{E} \right)^2 \left( \frac{\nu'}{\nu} \right)^2 \frac{1}{D^2} \left[ \frac{\nu D}{\nu' D'} + \frac{\nu' D'}{\nu D} - \sin^2 \theta \right]
\]

(116)

\( \theta \) is the scattering angle, \( \nu \) and \( \nu' \) the initial resp. final frequencies of the photon. This formula differs from the Klein-Nishina one by the necessary changes when going from the initial electron rest system to an electron moving with velocity \( v \), namely

\[
mc^2 \rightarrow E, \quad \frac{\nu'}{\nu} \rightarrow \frac{\nu' D'}{\nu D}
\]

Here \( D \) and \( D' \) are the Doppler factors \( D = 1 - \frac{\vec{v}}{c} \cos \alpha \) and \( D' = 1 - \frac{\vec{v}}{c} \cos \alpha' \), where \( \alpha \) resp. \( \alpha' \) are the angles between the initial electron and the initial, resp. final photon. This can be written in the invariant way

\[
p_0 k_0 D = p \cdot k
\]

\[
p_0 k_0' D' = p \cdot k'
\]

(117)

Stueckelberg specializes eq. (114) for \( \frac{p_0}{2p_0'} \Omega' \Omega \) to Compton scattering, i.e. \( k^2 = k \cdot e_k = 0 \) and \( e_k^2 = 1 \). He then chooses the gauge for which \( p \cdot e_k = p \cdot e_{k'} = 0 \), corresponding to transversality in the initial electron rest system. Hence he obtains

\[
W \equiv \frac{p_0}{2p_0'} \Omega' \Omega = 2(e_k \cdot e_{k'})^2 + \frac{(k \cdot k')^2}{2(k' \cdot p)(k' \cdot p')}
\]

(118)
Using energy-momentum conservation and mass-shell conditions for electrons, he recognizes that
\[
\frac{(k \cdot k')^2}{2(k' \cdot p)(k' \cdot p')} = \frac{1}{2} \left[ \frac{\nu D}{\nu' D'} + \frac{\nu' D'}{\nu D} \right] - 1
\]

(119)

Stueckelberg then averages over initial polarizations \(e_k\) and sums over final polarizations \(e_{k'}\). He finds for the first term in eq. (118)
\[
\frac{1}{2} \sum_{e_k} \sum_{e_{k'}} 2(e_k \cdot e_{k'})^2 = 1 + \cos^2 \theta
\]

(120)
\[
\cos \theta = 1 - \frac{(k \cdot k')p^2}{(k' \cdot p)(k \cdot p)}
\]

(121)

and the second term is doubled. Therefore
\[
\frac{1}{2} \sum_{e_k} \sum_{e_{k'}} W = \frac{\nu D}{\nu' D'} + \frac{\nu' D'}{\nu D} - \sin^2 \theta
\]

(122)

which is the bracket in eq. (119) naturally written in the invariant form
\[
\frac{(p \cdot k)}{(p' \cdot k')} + \frac{(p \cdot k')}{(p \cdot k)} + \left( \frac{(k \cdot k')(p \cdot p)}{(k \cdot p)(k' \cdot p)} \right)^2 - 2 \left( \frac{(k \cdot k')(p \cdot p)}{(k \cdot p)(k' \cdot p)} \right)
\]

See more details in section (6.7).

### 6.6 Bremsstrahlung

Bremsstrahlung is the emission of a free photon by an electron interacting with the Coulomb field of a nucleus. It is thus analogous to Compton scattering, where the initial photon (say \(k\)) is replaced by a classical electromagnetic field\[111\]

\[
A_k = \frac{2\hbar c}{e^2} M_k e_k \cos(k \cdot x)
\]

(123)
\[
M_k = \frac{4\pi Z e^2}{k^2 \hbar c}
\]

(124)

In a particular reference system, where \(k_0 = 0\), \(M_k\) is the Fourier transform of the Coulomb potential \(Z e^2/r\) of a nucleus. \(e_k\) is again a polarization vector.

\[111\]That’s why in the formula (103) \(P_k\) is put to 1.
To treat this problem, the strategy of Stueckelberg is to define what he calls the generalized Klein-Nishina formula. To this end he replaces \( V_k \sqrt{N_k} \) by \( M_k \) \((k^2 \neq 0, (e_k \cdot k) \neq 0)\) everywhere in the calculation for Compton scattering leading to the K-N formula. His aim is then to deduce the formula for Bremsstrahlung from this generalized K-N formula. In particular, he uses Eq. (114) with one photon off mass-shell.

Previously to Stueckelberg, the formula for Bremsstrahlung in the electrostatic field was obtained by Bethe-Heitler 1934 and by Sauter 1934 in second order of perturbation theory. On the other hand, Williams and v. Weizsäcker 1934, using qualitative arguments, found an approximate formula which for large initial energy \( E_e \) of the electron and large energy compared with \( mc^2 \) of the emitted photon, agrees with the Bethe-Heitler-Sauter formula. The idea of v. Weizsäcker-Williams, inspired by Fermi 1924, goes as follows:

In the rest-system of the nucleus the Fourier transform of the static field is given by

\[
V^L = \frac{1}{(2\pi)^3} \int d^4k \delta(k_0) M_k (e_k \cdot k) e^{i(k \cdot x)}
\] (125)

Since \( E_e \gg mc^2 \) by assumption, the electron has a large velocity \( \vec{v} \). In the rest-system of the electron, the partial waves of (125) move with velocity \( \vec{V} = -\vec{v} \). \( |\vec{V}| \) is almost equal to the light velocity \( c \), and the partial waves for small \( k^2 \) are almost transverse \((e_k \cdot k) \approx 0\). v. Weizsäcker-Williams apply to these quasi-lightwaves the Klein-Nishina formula for electrons at rest. The Bremsstrahlung, calculated in this frame, appears as an incoherent sum of the scattering amplitudes of the individual partial waves \( k \). This can be justified by the fact that one averages over wave-packets large in comparison with the nuclear field.

In his 1934 paper, Stueckelberg shows that the Bethe-Heitler formula for Bremsstrahlung, exact to second order, can be deduced from the generalized K-N formula in the same way as the approximation for large velocities was deduced by v. Weizsäcker-Williams from the ordinary K-N formula.

Here too, Stueckelberg takes advantage of the manifest relativistic and gauge invariance of his formalism, which allows him to go back and forth between the nucleus rest-system (B-H-S formula) and the electron rest-system (v. W-W formula). For example, to go from the former to the latter, a gauge transformation is used from a polarization with only a time component, to one with only space components. Finally, Stueckelberg shows that

\footnote{In Stueckelberg 1935b the same idea was used to obtain the formula for pair creation by a fast electron in the field of a nucleus}

65
the incoherent addition of the contributions of partial waves is rigorously
justified.

6.7 Invariant averaging over photon polarization

In section 6.4 we mentioned Stueckelberg’s derivation of the Pauli formula for Compton scattering by moving electrons. After the publication of Stueckelberg’s paper, Pauli pointed out to Stueckelberg that he first averaged over the photon polarization in the rest system of the electron, and only then Lorentz-transformed it back to the moving frame.

Stueckelberg replies to Pauli’s objection by remarking that, since unpolarized light is a Lorentz-invariant notion, the averaging is a Lorentz-invariant operation and can therefore be performed in any reference system.

Nevertheless, in the 1935 paper, Stueckelberg shows that one can average in a way which exhibits manifest invariance from the beginning. Instead of relating the polarization $e_k$ to the potential $A$ through

$$A^\mu = e^\mu \exp i(k \cdot x)$$

the idea is to use the anti-symmetric field-strength tensor $F$ and its conjugate (dual) $\tilde{F}$ (corresponding to the usual electric and magnetic fields).

$$F^{\mu\nu} = k^\mu e^\nu - k^\nu e^\mu$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the usual antisymmetric symbol. Consider the Lorentz force per unit charge $e^\mu$ on the electron with momentum $p$.

$$e^\mu = F^{\mu\nu} p_\nu$$

which can be normalized

$$\frac{e^2}{(k \cdot p)^2} = 1$$

and define also the dual expression

$$\tilde{e}^\mu = \tilde{F}^{\mu\nu} p_\nu$$

Because of the following orthogonality properties

$$\epsilon \cdot \tilde{e} = \epsilon \cdot k = k \cdot \tilde{e} = 0$$

113Remark on the intensity of the radiative scattering of moving free electrons, Stueckelberg 1935a.
114In particular, for $p = 0$, this is just the electric field.
the new four-vectors $\epsilon$ and $\hat{\epsilon}$ have the right properties to express the polarization, since they are mutually orthogonal and orthogonal to $k$. On the other hand, with the old polarizations $\epsilon_k$, these orthogonality properties cannot be ensured in a Lorentz invariant way.

The squared matrix element for Compton scattering (specialized to $k^2 = k \cdot e = 0$) can now simply be written

$$W = \frac{1}{2} \left\{ \frac{|\epsilon_{k'}|}{|\epsilon_k|} + \frac{|\epsilon_k|}{|\epsilon_{k'}|} \right\} - 1 + 2 \left( \frac{\epsilon_k \cdot \epsilon_{k'}}{(k \cdot p)(k' \cdot p)} \right)^2.$$ (132)

Averaging the last term of this expression over the polarizations (using now a special frame) gives

$$\frac{1}{2} \sum_{\epsilon_k} \sum_{\epsilon_{k'}} 2 \left( \frac{\epsilon_k \cdot \epsilon_{k'}}{(k \cdot p)(k' \cdot p)} \right)^2 = 1 + \left( 1 - \frac{(k \cdot k') p^2}{(k \cdot p)(k' \cdot p)} \right)^2,$$ (133)

and the same averaging doubles the other terms.

Now one finds again the result of Pauli 1933, namely

$$\frac{1}{2} \sum_{\epsilon_k} \sum_{\epsilon_{k'}} W = \frac{D k_0}{D' k'_0} + \frac{D' k'_0}{D k_0} - \sin^2 \theta.$$ (134)

7 The response to Stueckelberg’s paper

Stueckelberg’s 1934 paper, as well as others where he followed the same method, did not get much attention. The first paper which referred to it was Wannier (1935). He mentioned that his result for the trace over spinor indices has also been obtained by Stueckelberg in his eq. (114). But Stueckelberg’s calculus is for instance not mentioned by Heitler 1936. This could be an illustration of the difference of ”styles” of Heitler and Stueckelberg. While the first tries to get as simple expressions as possible, ready for applications, the second is more interested in the general structure of the theory with an emphasis on symmetries and other fundamental principles. Strangely enough, even Wentzel, who suggested the idea of Stueckelberg’s work, never quoted it.

Pauli had certainly noticed the 1934 paper, since, as discussed above, Stueckelberg 1935 wrote a sequel on a Lorentz-covariant polarization vector answering the criticism of Pauli (see preceding section). Later on, in a letter to Heisenberg (5 February 1937), Pauli drew Heisenberg’s attention to the 1934 paper.
I would also like to draw your attention to a work of Stueckelberg concerning the formalism of perturbation theory. The paper is not well written, but the basic idea (due to Wentzel) seems reasonable to me; it consists in making evident the relativistic invariance by eliminating space and time completely from the theory and examining directly the coefficients of the \textit{four}-dimensional Fourier expansion of the wave function.\footnote{On Wentzel’s influence on Stueckelberg see section 6.1.}

After the 1939-45 war, a whole new era started in quantum field theory with the renormalization program. In his report to the Solvay congress 1948, J. R. Oppenheimer insists on the necessity to preserve covariance in all steps of the calculation if one wants to eliminate the infinities. As an example of such a covariant theory he quotes Stueckelberg’s paper.

Now it is true that the fundamental equations of quantum-electrodynamics are gauge and Lorentz covariant. But they have in a strict sense no solutions expansible in powers of $e$. If one wishes to explore these solutions, bearing in mind that certain infinite terms will, in a later theory, no longer be infinite, one needs a covariant way of identifying these terms, and for that, not merely the field equations themselves, but the whole method of approximation and solution must at all stages preserve covariance. This means that the familiar Hamiltonian methods, which imply a fixed Lorentz frame $t=$constant, must be renounced; neither Lorentz frame nor gauge can be specified until after, in a given order in $e$, all terms have been identified, and those bearing on the definition of charge and mass recognized and relegated; then of course, in the actual calculation of transition probabilities and the reactive corrections to them, or in the determination of stationary states in fields which can be treated as static, and in the reactive corrections thereto, the introduction of a definite coordinate system and gauge for these no longer singular and completely well-defined terms can lead to no difficulty.

It is probable that, at least to order $e^2$, more than one covariant formalism can be developed. Thus Stueckelberg’s four-
dimensional perturbation theory would seem to offer a suitable starting point, as also do the related algorithms of Feynman\textsuperscript{117}.

In the same spirit F. J. Dyson (1949) comments in the notes added in proof to his paper "The Radiation Theories of Tomonaga, Schwinger, and Feynman": "A covariant perturbation theory similar to that of section III has previously been developed by E.C.G. Stueckelberg" referring to the 1934 paper and to Nature vol. 153 (1944), p. 143.

We have already mentioned Weisskopf’s reminiscences (section 6.1). The importance of manifest Lorentz and gauge invariance for the development of the theory in these times has been also emphasized by Pais 1986, p. 457. When commenting on Schwinger’s Lorentz invariant 1948 calculation of various terms contributing to the Lamb shift, Pais writes: "Schwinger’s direct calculation of the electric term produced a most unpleasant surprise, however: it was too small by a factor $1/3$!\textsuperscript{118}" Pais continues by quoting Schwinger:

> This difficulty is attributable to the incorrect transformation properties of the electron self-energy in the conventional Hamiltonian treatment and is completely removed in the covariant formalism now employed.\textsuperscript{119}

Pais concludes with the following discussion:

> How can a fully covariant theory yield non-covariant results? Because during the calculation one has to subtract infinity from infinity, which in general is not a well-defined step. How can one hope to avoid non-covariant answers? By computing in such a way that covariance is manifest at every stage; and likewise for gauge invariance. Take, for example, Heisenberg and Pauli’s treatment of quantum electrodynamics in the Coulomb gauge which [...] may not look covariant but is covariant nevertheless. It is not, however, manifestly covariant at every stage. Thus the Coulomb gauge does not lend itself (readily) to the evaluation of radiative corrections. Likewise the second order perturbation formula [...] and its higher-order partners, though actually covariant, are not manifestly so. One can, however, cast them in

\textsuperscript{117}see the reprint in Schwinger 1958, p. 150.
\textsuperscript{118}Pais’ source is here J. Schwinger in Brown and Hoddeson (1983), p. 329.
\textsuperscript{119}Schwinger (1949).
an equivalent manifestly covariant form, as in fact Stueckelberg
had already shown in 1934.\textsuperscript{\textcopyright{20}}

Finally this opinion of Gell-Mann 1989 (p. 702):

By about 1950 it was known that QED is renormalizable for
charged spinor particles [...] The second order renormalizability
of the charge in QED had been established in 1934 by Dirac and
Heisenberg, and that of the mass by a number of authors in 1948.
Of those, the first ones to complete correct relativistic calculations
of the Lamb shift (Willis Lamb and Norman Kroll and J. Bruce
French and Victor Weisskopf) actually used the clumsy old non-
covariant method. The place where the new covariant methods
played a crucial role, particularly those of E.C.G. Stueckelberg and
Richard P. Feynman, which are still used today, was in permitting
calculations to be done quickly, especially to fourth and higher
orders (which would have been impractical with the old methods),
and in making possible the proof of renormalizability to all orders.

So, after all, why didn’t people take advantage of Stueckelberg’s method.
? Several reasons could be proposed. The paper itself was certainly not
written in a transparent style (remember Weisskopf’s and Pauli’s comments).
His notations were clumsy (see our appendix). On the other hand, some of
the mathematical features of Stueckelberg’s theory (contour integrals instead
of Dirac’s delta function) could have made it unfamiliar to use. Also, more
generally, the paper didn’t provide a new physical result but ”only” a new
method for deriving results already obtained; no one realised (see Weisskopf’s
reminiscences) that it could have been used to treat in a more consistent way
the unsolved divergence difficulties.

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their helpful comments.

\textsuperscript{20}In his book (p. 244-45), S. S. Schweber reports on the other hand this revealing
reaction of Schwinger when asked about the disparities between his calculation of the
Lamb shift and that of French and Weisskopf: ”Well, if you do not keep the calculation
explicitly covariant, anything can happen”.

We shall address the general problem of renormalization in Quantum Field Theory and
Stueckelberg’s contribution to it in a subsequent publication.
9 Appendix

In order to make easier the comparison between the various works considered in this paper, we have adopted a set of conventions that we systematically use when discussing the technical details. The necessary changes in notations or conventions with respect to the original papers are of minor importance, but for the sake of historical accuracy, we report below the main alterations.

Four-vectors are written in normal type, with their spatial part noted in boldface: and the time-component \( a_0 \); thus \( a = (a_1, a_2, a_3, a_0) = (e, a_0) \). In particular, the space-time location of an event is given by the four-vector \((x, ct)\) where the time-component \( x_0 = ct \) and \( c \) is the speed of light in vacuum. Our relativistic metric is given by the matrix \( g_{\mu\nu} \) with \( g_{ii} = g_{00} = 1, i = 1, 2, 3 \). All the authors we consider adopt instead the euclidean metric, introducing for each time-component \( a_0 \) its euclidean counterpart \( a_4 = ia_0 \). Thus, the space-time scalar product between two four-vectors \( a \) and \( b \) is in our notation \( a \cdot b = \sum a_\mu b_\mu \equiv a_0 b_0 \equiv a_\mu b_\mu \) where the sum convention has been used in the last expression. Clearly, \( a \cdot b = ab - a_0 b_0 = ab + a_4 b_4 \).

An electromagnetic plane wave \( \psi \) of angular frequency \( \omega = 2\pi\nu \), wavelength \( \lambda \), and direction of propagation given by the unit vector \( \hat{k} \) is given by

\[
\psi = \exp(i\frac{\omega}{c}(\hat{k}x - ct)) = \exp(i\hat{k} \cdot x)
\]

where one introduced the four-vector \( k = (k, k_0) = \frac{\omega}{c}(\hat{k}, 1) \). Similarly, using the Einstein-Planck-de Broglie relations:

\[
\begin{align*}
    p &= \hbar k \\
    E &= \hbar \omega
\end{align*}
\]

we associate to a particle with four-momentum \( p = (\hat{p}, E) \) the matter-wave

\[
\exp\left(i\frac{\hbar}{c}(\hat{p}x - Et)\right).
\]

The wave vectors of the initial and final radiation are noted respectively \( \hat{k} \) and \( \hat{k}' \), and in general quantities related to the final state are primed.

The original notations of Stueckelberg appear quite unusual (which certainly doesn’t help the reader...). We present now a dictionary for going from his to our notations.

Four momenta:
- initial electron: \( l^0 \rightarrow p \)
- final electron: \( l \rightarrow p' \)
- initial photon: \( k \rightarrow k \)
final photon: $-p$, then $m \to k'$
Photon polarization: $\sigma^k \to e_k$
Photon state: $u(N) \to |N \rangle$
Photon annihilation and creation operators: $\Gamma_k, \Gamma_k^\dagger \to a_k, a_k^\dagger$
Electron spinor: $A(l^0) \to u(p)$
Electron spinor on mass-shell: $B(l^0) \to \omega(p)$
Adjoint spinor: $B^\dagger = B^\ast \gamma_4 \to \bar{\omega} = \omega^\dagger \gamma_0$

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