On Eccentricity Matrices of Wheel Graphs

I. Jeyaraman\textsuperscript{1} and T. Divyadevi\textsuperscript{2}

Department of Mathematics
National Institute of Technology Tiruchirappalli-620 015, India
(e-mail: jeyaraman@nitt.edu\textsuperscript{1} and tdivyadevi@gmail.com\textsuperscript{2})

Abstract

The eccentricity matrix $E(G)$ of a simple connected graph $G$ is obtained from the distance matrix $D(G)$ of $G$ by retaining the largest distance in each row and column, and by defining the remaining entries to be zero. This paper focuses on the eccentricity matrix $E(W_n)$ of the wheel graph $W_n$ with $n$ vertices. By establishing a formula for the determinant of $E(W_n)$, we show that $E(W_n)$ is invertible if and only if $n \not\equiv 1 \pmod{3}$.

We derive a formula for the inverse of $E(W_n)$ by finding a vector $w \in \mathbb{R}^n$ and an $n \times n$ symmetric Laplacian-like matrix $\hat{L}$ of rank $n-1$ such that

$$E(W_n)^{-1} = -\frac{1}{2} \hat{L} + \frac{6}{n-1}ww'.$$

Further, we prove an analogous result for the Moore-Penrose inverse of $E(W_n)$ for the singular case. We also determine the inertia of $E(W_n)$.

Keywords: Wheel graph; Eccentricity matrix; Inverse; Moore-Penrose inverse.

Subject Classification (2020): 05C12; 05C50; 15A09.

1 Introduction

Let $G$ be a simple connected graph on $n$ vertices with the vertex set $\{v_1, v_2, \ldots, v_n\}$. Corresponding to $G$, several (graph) matrices have been introduced and studied, namely, the incidence matrix, the adjacency matrix, the Laplacian matrix, the distance matrix, the resistance matrix etc., see [5, 10]. We recall the adjacency and distance matrices which are relevant to the discussion here. The adjacency matrix $A(G) := (a_{ij})$ of $G$ is an $n \times n$ matrix with $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and zero elsewhere. Let $d(v_i, v_j)$ be the length of a shortest path between $v_i$ and $v_j$. The distance matrix of $G$, denoted by $D(G) := (d_{ij})$, is an $n \times n$ matrix with $d_{ij} = d(v_i, v_j)$ for all $i$ and $j$. Then $A(G)$ and $D(G)$ are symmetric matrices with diagonal entries all equal to zero. These matrices have been well studied in the literature and have a wide range of applications in chemistry, physics, computer science, etc., see [5, 9, 10] and the references therein. Note that the adjacency matrix can be derived from the distance matrix by keeping the smallest non-zero distance (which is equal to one) in each row and column and by setting the remaining entries to be zero.

Randić [18] introduced a new graph matrix known as $D_{\text{MAX}}$ which is obtained from the distance matrix $D(G)$ by retaining the largest distance in each row and each column of $D(G)$,
and setting the rest of the entries to zero. Wang et al. [19] introduced the $D_{\text{MAX}}$ matrix in a slightly different form using the notion of the eccentricity of a vertex and called it the eccentricity matrix $E(G)$. The eccentricity of a vertex $v_i$, denoted by $e(v_i)$, is defined by $e(v_i) = \max \{d(v_i, v_j) : 1 \leq j \leq n\}$. Let us recall the equivalent formulation of $D_{\text{MAX}}$ given in [19]. The eccentricity matrix of $G$ is an $n \times n$ matrix with

$$(E(G))_{ij} = \begin{cases} 
 d(v_i, v_j) & \text{if } d(v_i, v_j) = \min \{e(v_i), e(v_j)\}, \\
 0 & \text{otherwise},
\end{cases}$$

where $(E(G))_{ij}$ is the $(i,j)$-th entry of $E(G)$. It is also referred to as the anti-adjacency matrix [19]. Note that $E(G)$ is a real symmetric matrix whose entries are non-negative. This new matrix $E(G)$ is applied to study the boiling point of hydrocarbons [21]. For more details on the applications of $E(G)$ in terms of molecular descriptor, we refer to [19, 21, 18]. Motivated by the concepts and results of other graph matrices, several spectral properties have been studied for the eccentricity matrix in [19, 20, 21, 16, 17, 18]. The relation between the eigenvalues of $A(G)$ and $E(G)$ has been investigated for certain graphs in [19].

Unlike the adjacency and distance matrices of a connected graph, the eccentricity matrix is not irreducible [19]. The problem of characterizing the irreducible eccentricity matrix posed by Wang et al. [19] remains open. They showed that the eccentricity matrix of a tree is irreducible. Mahato et al. [16] gave an alternative proof. In [20], this result was generalized by providing a class of graphs whose eccentricity matrices are irreducible. In this paper, we study the eccentricity matrix of the wheel graph $W_n$ with $n$ vertices. We show that $E(W_n)$ is irreducible.

An important problem in graph matrices is to find the determinants and inertias of these matrix classes. A famous result in the theory of distance matrix states that the determinant of a tree $T$ with $n$ vertices is $(-1)^{n-1}(n-1)^{2^{n-2}}$ which depends only on the number of vertices [12]. The inertia of $D(T)$ was also found in [12]. These results were extended to the distance matrix of a weighted tree by Bapat et al. [6]. The determinant and inertia of $D(W_n)$ have been computed in [23] and the inertias of the eccentricity matrices of certain graphs have been investigated in [16]. In this paper, we determine a formula to find the determinant of $E(W_n)$ and derive the inertia of $E(W_n)$. We show that the determinant of $E(W_n)$ is $2^{n-2}(1 - n)$ if $n \not\equiv 1 \mod 3$, which depends only on the number of vertices, and zero if $n \equiv 1 \mod 3$, see Theorem 3.4.

The problem of finding the inverses of different graph matrices has been extensively studied in the literature, see [3, 5, 7, 13, 11, 25]. In order to motivate our next result, we recall the Laplacian matrix $L$ of $G$. The Laplacian matrix of $G$ is given by $L := \tilde{D} - A(G)$, where $\tilde{D}$ is the diagonal matrix whose $i$-th diagonal entry is $\deg(v_i)$, the degree of the vertex $v_i$. Then $L$ is an $n \times n$ symmetric matrix whose row sums are zero and rank of $L$ is $n - 1$, see [5]. Graham and Lovász [14] gave an interesting formula for the inverse of the distance matrix $D(T)$ of a tree $T$ which is given by

$$D(T)^{-1} = \frac{1}{2}L + \frac{1}{2(n-1)}\tau \tau',$$

where $\tau' = (2 - \deg(v_1), 2 - \deg(v_2), \ldots, 2 - \deg(v_n))$. This result was extended to the distance matrix of a weighted tree in [6] and to the resistance matrix of $G$, see [3].

Let us recall that a real square matrix $\tilde{L}$ is called a Laplacian-like matrix [25] if $\tilde{L}e = 0$ and $e'\tilde{L} = 0$, where $e$ denotes the column vector whose entries are all one. Inspired by the
result of Graham and Lovász [13], similar inverse formulae have been obtained for the distance matrices of wheel graph \( W_n \) when \( n \) is even [3], helm graphs [11], cycles [6], block graphs [7] and complete graphs [25]. For all these matrices, the inverse formula is expressed as the sum of a Laplacian-like matrix and a rank one matrix. Almost all the distance matrices of the above mentioned graphs are Euclidean distance matrix (EDM), see Section 2.3 for the definition. The inverse formula has been studied for EDM in [2] where the result of Graham and Lovász [13] was generalized. It would be interesting and challenging to study the inverse formula for non-EDM. Note that \( E(W_n) \) is not an EDM (see Section 2.3). Therefore studying \( E(W_n) \) will give new ideas and results in the class of irreducible non-EDM. Motivated by the inverse formulae of the distance matrices of several graphs, we find an inverse formula for \( E(W_n) \) when \( n \not\equiv 1 \pmod{3} \) which is of the form (1.1), see Theorem 5.11.

Let us turn our attention to the Moore-Penrose (generalized) inverse of graph matrices. Let \( A \) be an \( m \times n \) real matrix. An \( n \times m \) matrix \( X \) is called the Moore-Penrose inverse of \( A \) if \( AXA = A, XAX = X, (AX)' = AX \) and \( (XA)' =XA \). It is known that Moore–Penrose inverse, denoted by \( A^\dagger \), exists and is unique. Further, if \( A \) is a non-singular matrix, then \( A^\dagger \) coincides with the usual inverse \( A^{-1} \). For more details, we refer to [8]. The Moore-Penrose inverse has been presented for the incidence matrices of complete multipartite graph, bi-block graph, distance regular graph, tree etc., and for EDM, see [1, 5, 2] and references therein. In a recent paper, Balaji et al. [4] provided a formula to compute the Moore-Penrose inverse of the distance matrix of a wheel graph with an odd number of vertices, which is given by

\[
D(W_n)^\dagger = -\frac{1}{2}\tilde{L} + \frac{6}{n-1}ww',
\]

where \( \tilde{L} \) is a real symmetric Laplacian-like matrix of order \( n \) and \( w \in \mathbb{R}^n \). It is natural to ask whether an analogous result can be studied for the eccentricity matrix of \( W_n \) for the singular case. Answering this question positively, in this paper, we obtain a formula for \( E(W_n)^\dagger \) when \( n \equiv 1 \pmod{3} \) in the form of (1.1) by introducing a symmetric Laplacian-like matrix \( \tilde{L} \) (Theorem 6.10).

The article is organized as follows. In Section 2, we fix the notations, collect some known results and study some properties of the eccentricity matrix of the wheel graph \( E(W_n) \). In Section 3, we establish a formula to compute the determinant of \( E(W_n) \). As a consequence, we show that \( E(W_n) \) is invertible if and only if \( n \not\equiv 1 \pmod{3} \). Section 4 deals with the inertia of \( E(W_n) \). As a by-product, we obtain the determinant and the inertia of the eccentricity matrix of an edge-deleted subgraph of \( W_n \) where the edge lies on the cycle. In Section 5, we obtain an explicit formula similar to (1.1) for the inverse of \( E(W_n) \) when \( n \not\equiv 1 \pmod{3} \). The inverse formula is expressed as the sum of a symmetric Laplacian-like matrix \( \tilde{L} \) and a rank one matrix which is given by \( E(W_n)^{-1} = -\frac{1}{2}\tilde{L} + \frac{6}{n-1}ww' \), where \( w =\frac{1}{6}(7-n, 1, \cdots, 1)' \in \mathbb{R}^n \) and the rank of \( \tilde{L} \) is one less than that of \( E(W_n) \). In Section 6, an analogous result is presented for the Moore-Penrose inverse of \( E(W_n) \) when \( n \equiv 1 \pmod{3} \) by constructing a symmetric Laplacian-like matrix \( \hat{L} \) of rank \( n-3 \).

2 Preliminaries and Properties of \( E(W_n) \)

In this section, we first introduce a few notations and present some preliminary results on circulant matrices that are needed in the paper. Next, we compute the eccentricity matrix
of the wheel graph and study some of its properties. We conclude this section by finding its spectral radius.

We assume that all the vectors are column vectors and are denoted by lowercase boldface letters. As usual \( I_n \) is the identity matrix of order \( n \) and \( e_1 \) is the vector in \( \mathbb{R}^n \) with the \( i \)-th coordinate 1 and 0 elsewhere. We write \( e_{n \times 1}(0_{n \times 1}) \) to represent the vector in \( \mathbb{R}^n \) whose coordinates are all one (respectively, zero). We use the notation \( J_{n \times m}(0_{n \times m}) \) to denote the matrix with all elements equal to 1 (respectively, 0) and we simply write \( J_n \) (respectively, \( 0_n \)) if \( n = m \). We omit the subscript if the orders of the vector and the matrix are clear from the context. For the matrix \( A \), we denote the transpose of \( A \), the null space of \( A \), the range of \( A \), the \( i \)-th row of \( A \), and the \( j \)-th column of \( A \) by \( A^t \), \( N(A) \), \( R(A) \), \( A_i \), and \( A_j \), respectively. The notation \( A > 0 \) (\( A \geq 0 \)) means that all the entries of \( A \) are positive (respectively, non-negative). The determinant of a square matrix \( A \) is denoted by \( \det(A) \).

For a vector \( c = (c_1, \cdots, c_n) \in \mathbb{R}^n \), the notation \( \text{Circ}(c') \) stands for the circulant matrix of order \( n \), and \( T_n(a, b, c) \) denotes the tridiagonal matrix of order \( n \) which are given by

\[
\text{Circ}(c') = \begin{bmatrix}
    c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_n \\
    c_n & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    c_{n-1} & c_n & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_3 & c_4 & c_5 & \cdots & c_1 & c_2 \\
    c_2 & c_3 & c_4 & \cdots & c_n & c_1
\end{bmatrix}
\]
and

\[
T_n(a, b, c) = \begin{bmatrix}
a & b & 0 & \cdots & 0 & 0 & 0 \\
c & a & b & 0 & 0 & 0 & 0 \\
0 & c & a & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & a & b & 0 \\
0 & \cdots & 0 & c & a & b \\
0 & \cdots & 0 & 0 & c & a
\end{bmatrix},
\]

respectively where \( a, b \), and \( c \) are real numbers. The following properties of the circulant matrix will be used several times. We refer to the books [14, 24] for more details.

Let \( x, y \in \mathbb{R}^n \). Suppose that \( A = \text{Circ}(x') \) and \( B = \text{Circ}(y') \). Then

\[
AB = BA,
\]

\[(\text{Circ}(ax' + by')) = aA + bB,
\]

and

\[
AB = \text{Circ}(x'B).
\]

Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be defined by

\[
T((f_1, f_2, \cdots, f_n)') = (f_n, f_1, \cdots, f_{n-1})'.
\]

**Lemma 2.1.** Let \( T \) be the operator defined in (2.4). Let \( n \equiv 0 \) (mod 3) and \( C := \text{Circ}(c') \). If \( g' = (\alpha, \beta, \gamma, \alpha, \beta, \gamma, \cdots, \alpha, \beta, \gamma) \in \mathbb{R}^n \), then \( g'C = (\tau_1, \tau_2, \tau_3, \tau_1, \tau_2, \tau_3, \cdots, \tau_1, \tau_2, \tau_3) \), where \( \tau_1 = g'C_{+1} \), \( \tau_2 = (T^2(g))'C_{+1} \) and \( \tau_3 = (T(g))'C_{+1} \).

_Proof._ Let \( 1 \leq k \leq n \). Note that \( C_{+k} = (c_k, c_{k-1}, \cdots, c_2, c_1, c_n, c_{n-1}, \cdots, c_{k+1})' \) and the \( k \)-th coordinate of \( g'C \) is \( g'C_{+k} \). By the usual matrix multiplication, we see that

\[
g'C_{+k} = \alpha(c_k + c_{k-3} + \cdots + c_{k+3}) + \beta(c_{k-1} + c_{k-4} + \cdots + c_{k+2}) + \gamma(c_{k-2} + c_{k-5} + \cdots + c_{k+1}).
\]

If \( k \equiv 1 \) (mod 3), then \( g'C_{+k} = \alpha(\sum_{i=1} ( mod \ 3) c_i) + \beta(\sum_{i=0} ( mod \ 3) c_i) + \gamma(\sum_{i=2} ( mod \ 3) c_i) \). That is, \( g'C_{+1} = g'C_{+k} \) for all \( k \equiv 1 \) (mod 3). The remaining two cases \( k \equiv 2 \) (mod 3) and \( k \equiv 0 \) (mod 3) can be proved similarly. \( \square \)
2.1 Eccentricity matrix of Wheel Graph

For $n \geq 4$, the wheel graph on $n$ vertices $W_n$ is a graph containing a cycle of length $n - 1$ and a vertex (called the hub) not in the cycle which is adjacent to every other vertex in the cycle. Throughout this paper, we label the vertices of $W_n$ by $v_1, v_2, \ldots, v_n$, where the hub of $W_n$ is labelled as $v_1$ and the vertices in the cycle are labelled as $v_2, \ldots, v_n$ clockwise (see figure 1 for $W_7$).

We observe that $D(W_4) = E(W_4)$. We assume that $n \geq 5$. Since $v_1$ is adjacent to all other $v_i$'s in $W_n$, we have $e(v_1) = 1$. Note that all $v_i$'s ($i \neq 1$) are adjacent to exactly three vertices. Therefore there exists a vertex $v_j$ such that $d(v_i, v_j) = 2$, where the existence of $v_j$ follows from the assumption $n \geq 5$. Hence, $e(v_i) = 2$ for $i = 2, 3, \ldots, n$. Let $d = (0, 1, 2, 2, \ldots, 2, 1)'$ and $u = (0, 0, 2, 2, \ldots, 2, 0)'$ be the vectors fixed in $\mathbb{R}^{n-1}$. Then the distance matrix $D(W_n)$ and the eccentricity matrix $E(W_n)$ of the wheel graph are $n \times n$ symmetric matrices which can be written in the block form

$$D(W_n) = \begin{bmatrix} 0 & e' \\ e & D \end{bmatrix} \quad \text{and} \quad E(W_n) = \begin{bmatrix} 0 & e' \\ e & E \end{bmatrix},$$

(2.5)

respectively, where $D = \text{Circ}(d')$ and $E = \text{Circ}(u')$ are circulant matrices of order $n - 1$.

2.2 The irreducible property of $E(W_n)$

Let us recall that a square matrix $A$ of order $n$ is said to be irreducible if there is no permutation matrix $P$ of order $n$ such that

$$P'AP = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},$$

where $A_1$ and $A_3$ are square matrices of order at least one. An equivalent condition for the irreducibility of a non-negative matrix is given below.

**Theorem 2.2** ([14]). Let $A$ be a square matrix of order $n$ whose entries are non-negative. Then $A$ is irreducible if and only if $(I_n + A)^{n-1} > 0$.

The distance matrix of a connected graph is always irreducible but the eccentricity matrix is not irreducible [19]. Characterizing the irreducible eccentricity matrices remains an open problem. It has been shown that the eccentricity matrix of a tree is irreducible [19] and this result was extended in [20].

In the following proposition, we show that the eccentricity matrix of the wheel graph is irreducible.
Proposition 2.3. The eccentricity matrix of $W_n$ is irreducible.

Proof. For $n = 4$, we have $E(W_n) = D(W_n)$ which is irreducible. If $n \geq 5$, then 

$$(I_n + E(W_n))^2 = \begin{bmatrix} n & 2(n-3)e' \\ 2(n-3)e & J_{n-1} + \widetilde{E}^2 + 2E + I_{n-1} \end{bmatrix}.$$ 

Since $\widetilde{E} \geq 0$, each entry in $J_{n-1} + \widetilde{E}^2 + 2E + I_{n-1}$ is at least one. Now, the proof follows from Theorem 2.2.

2.3 Euclidean distance matrix

We investigate the concept of the Euclidean distance matrix for $E(W_n)$. A real square matrix $M = (m_{ij})$ of order $n$ is said to be an Euclidean distance matrix (EDM) if there exist $n$ vectors $x_1, \cdots, x_n$ in $\mathbb{R}^r$ such that $m_{ij} = \|x_i - x_j\|^2_2$, where $\| . \|_2$ stands for the Euclidean norm on $\mathbb{R}^r [15]$. It was shown that $D(W_n)$ is an EDM [15]. It is natural to ask whether $E(W_n)$ is an EDM. The answer to this question is negative. We show that $E(W_n)$ is not an EDM for all $n \geq 5$. Note that $E(W_4)$ is an EDM because $E(W_4) = D(W_4)$. We use the following characterization of EDM to prove our result.

Theorem 2.4 (see [15]). Let $M$ be a real symmetric matrix of order $n$ whose diagonal entries are zero. Then $M$ is an EDM if and only if $x'Mx \leq 0$ for all $x \in \mathbb{R}^n$ such that $x'e = 0$.

Proposition 2.5. For $n \geq 5$, the matrix $E(W_n)$ is not an Euclidean distance matrix.

Proof. For each $n$, we find a vector $x \in \mathbb{R}^n$ such that $x'e = 0$ and $x'E(W_n)x > 0$. By a simple manipulation, the matrix $E(W_n)$ given in (2.5), can be written as

$$E(W_n) = \begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} - 2 \begin{bmatrix} 0 & 0' \\ 0 & F \end{bmatrix},$$

where $F = \text{Circ} ((1,1,0,\cdots,0,1) \underbrace{0}_{(n-4)\text{times}})$. (2.6)

We first assume that $n$ is odd. Consider the vector

$$y' = (0, \tilde{y}') \in \mathbb{R}^n, \text{ where } \tilde{y} = (1, -1, 1, \cdots, 1, -1)' \in \mathbb{R}^{n-1}.$$ 

Then $e'y = 0$ and $F\tilde{y} = -\tilde{y}$. This implies that

$$E(W_n)y = \begin{bmatrix} e'y \\ 2J_{n-1}\tilde{y} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ F\tilde{y} \end{bmatrix} = 2 \begin{bmatrix} 0 \\ \tilde{y} \end{bmatrix} = 2y.$$ 

Further, $y'E(W_n)y = 2y'y = 2(n - 1) > 0$. We now consider the case $n$ is even. Then $n = 2m$ where $m \geq 3$.

Case(i): Assume that $m$ is odd. Consider the vector

$$x = (0, 1, -1, 1, \cdots, 1, -1, 0, 1, -1, \cdots, 1, -1)' \in \mathbb{R}^n,$$

where zero occurs in the first and $(m+1)$-th coordinates. Let $x' = (0, \tilde{x}')$. Then $e'_{n \times 1}x = e'_{(n-1)\times 1}\tilde{x} = 0$. Therefore, $E(W_n)x = -2 \begin{bmatrix} 0 \\ (F\tilde{x})' \end{bmatrix}$. It can be verified that $E(W_n)x =$
\[-2(0, -1, 1, -1, 1, \cdots, -1, 1, -1, 0, 0, 1, -1, 1, \cdots, 1, -1, 1)^{\prime}\] where zero occurs in the first, \(m\)-th, \((m + 1)\)-th and \((m + 2)\)-th coordinates. Note that the signs of non-zero coordinates of \(E(W_n)x\) are the same as those of \(x\). Hence, \(x^\prime E(W_n)x = 2(n - 4) > 0\).

**Case(ii):** Suppose that \(m\) is even. In this case, take
\[x = (0, 1, -1, 1 - 1, \cdots, 1, -1, 1, 0, -1, 1, -1, 1, \cdots, -1, 1, -1, 1)^{\prime} \in \mathbb{R}^n,\]
where zero in the first and \((m+1)\)-th coordinates. Proceeding similar to the proof of the previous case, we get \(x^\prime E(W_n)x = 2(n - 4) > 0\). Therefore, by Theorem 2.4, the result follows. \(\square\)

**Remark 2.6.** In fact, the proof of the above proposition shows that if \(n\) is odd, then \(y\) is an eigenvector of \(E(W_n)\) corresponding to the eigenvalue 2.

### 2.4 Spectral radius of \(E(W_n)\)

The **spectral radius** of a square matrix \(A\), denoted by \(\rho(A)\), is the maximum of the moduli of the eigenvalues of \(A\). We recall the fact that the matrix \(E(W_n)\) is non-negative and irreducible. Hence, by Perron-Frobenius Theorem \([14]\), the spectral radius \(\rho(E(W_n))\) of \(E(W_n)\) is an eigenvalue of \(E(W_n)\) and there exists a non-negative eigenvector \(x\) corresponding to \(\rho(E(W_n))\). That is, \(E(W_n)x = \rho(E(W_n))x\) and \(x \geq 0\).

We end this section by computing the spectral radius \(\rho(E(W_n))\) and a non-negative eigenvector corresponding to \(\rho(E(W_n))\). Let us recall the concepts of equitable partition and the associated characteristic matrix.

Let \(A\) be a real symmetric matrix of order \(n\) whose rows and columns are indexed by \(X = \{1, 2, \cdots, n\}\). Suppose that \(\{X_1, X_2\}\) is a partition of \(X\) such that the number of elements in \(X_1\) and \(X_2\) are \(n_1\) and \(n_2\) respectively. Let \(A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\), where each \(A_{ij}\) denotes the block submatrix of \(A\) of order \(n_i \times n_j\) formed by rows in \(X_i\) and the columns in \(X_j\). If \(q_{ij}\) denotes the average row sum of \(A_{ij}\) (the sum of all entries in \(A_{ij}\) divided by the number of rows), then the matrix \(Q = (q_{ij})\) is said to be a **quotient matrix** of \(A\) with respect to the given partition \(X\). If each row sum of the block \(A_{ij}\) is constant (that is, \(A_{ij}e_{nj} = q_{ij}e_{ni}\), for every \(i\) and \(j\)), then the partition is called **equitable**. The **characteristic matrix** \(C = (c_{ij})\) with respect to the partition \(X\) is the \(n \times 2\) matrix such that \(c_{ij} = 1\), if \(i \in X_j\) and 0 otherwise.

We state a result which gives the relation between \(\rho(A)\) and \(\rho(Q)\).

**Theorem 2.7** (\([22]\)). Let \(A\) be a non-negative matrix. If \(Q\) is the quotient matrix of \(A\) with respect to an equitable partition, then the spectral radius of \(A\) and \(Q\) are equal.

Using the notion of equitable partition, we now determine the spectral radius of \(E(W_n)\).

**Theorem 2.8.** The spectral radius of \(E(W_n)\) is \((n - 4) + \sqrt{n^2 - 7n + 15}\).

**Proof.** We have \(E(W_n) = \begin{bmatrix} 0 & e^\prime \\ e & \tilde{E} \end{bmatrix}\), where \(\tilde{E} = \text{Circ}((0, 0, 2, \cdots, 2, 0))\). Since \(\tilde{E}\) is a circulant matrix, it follows that all the row sums of \(\tilde{E}\) are the same and equal to \(2(n - 4)\). Now partition the vertex set of \(W_n\) as \(X = \{\{v_1\}, \{v_2, \cdots, v_n\}\}\). Then \(X\) is an equitable partition. So, the quotient matrix of \(E(W_n)\) with respect to \(X\) is \(Q(W_n) = \begin{bmatrix} 0 & n^{-1} \\ n^{-2}(n-4) & 0 \end{bmatrix}\). As the eigenvalues of \(Q\) are \((n - 4) \pm \sqrt{n^2 - 7n + 15}\), the proof follows by Theorem 2.7. \(\square\)
From an eigenvector of the quotient matrix \( Q \) corresponding to \( \rho(Q) \), we can obtain an eigenvector of \( E(W_n) \) corresponding to \( \rho(E(W_n)) \) where the precise statement is given below.

**Lemma 2.9** ([9]). Let \( C \) be a characteristic matrix of an equitable partition \( X \) of \( A \). Let \( v \) be an eigenvector of the quotient matrix \( Q \) with respect to \( X \) for an eigenvalue \( \lambda \). Then \( Cv \) is an eigenvector of \( A \) for the same eigenvalue \( \lambda \).

**Remark 2.10.** Let \( \rho = (n-4) + \sqrt{n^2 - 7n + 15} \). The characteristic matrix of \( E(W_n) \) with respect to \( X \) is of order \( n \times 2 \), whose first row is \( e_1 \), and the remaining rows are \( e_2 \), where \( e_1, e_2 \in \mathbb{R}^2 \). Note that \( \left( \frac{n-1}{\rho+1}, 1 \right)' \) is an eigenvector of \( Q(W_n) \) corresponding to \( \rho(Q(W_n)) \). By Lemma 2.9, \( \left( \frac{1}{\rho}(n-1), 1, \ldots, 1 \right)' \) is an eigenvector of \( E(W_n) \) corresponding to \( \rho(E(W_n)) \).

### 3 Determinant of \( E(W_n) \)

A result due to Graham and Pollak [12] showed that \( \det(D(T)) = (-1)^{n-1}(n-1)2^{n-2} \), where \( T \) is a tree on \( n \) vertices. It is clear that the formula depends only on the number of vertices of \( T \). This result was generalized to the distance matrices of weighted trees in [6]. The determinant of the distance matrix of the wheel graph \( W_n \) was studied in [23]. More precisely, \( \det(D(W_n)) = 1 - n \) if \( n \) is even, and \( \det(D(W_n)) = 0 \) if \( n \) is odd.

The primary aim of this section is to establish a formula to find the determinant of \( E(W_n) \). We show that

\[
\det(E(W_n)) = \begin{cases} 
0 & \text{if } n \equiv 1 \pmod{3}, \\
2^{n-2}(1 - n) & \text{if } n \not\equiv 1 \pmod{3}, 
\end{cases}
\]

which is given in terms of the number of vertices alone. To prove the result, we obtain the recurrence relation (3.1) involving the determinant of the tridiagonal matrix \( T_n(-2, -2, -2) \) and a bordered matrix \( B_n \) (defined in Lemma 3.3).

We need the following result, which gives the determinant of a tridiagonal matrix.

**Theorem 3.1** ([24], Thm. 5.5). Let \( T_n(a, b, c) \) be a tridiagonal matrix of order \( n \), where \( a, b \) and \( c \) are real numbers. If \( a^2 \neq 4bc \), then

\[
\det(T_n(a, b, c)) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta},
\]

where \( \alpha = \frac{a + \sqrt{a^2 - 4bc}}{2} \) and \( \beta = \frac{a - \sqrt{a^2 - 4bc}}{2} \).

In the next two lemmas, we evaluate the determinants of two matrices which will be used to find the \( \det(E(W_n)) \).

**Lemma 3.2.** Let \( T_n = T_n(-2, -2, -2) \) be a tridiagonal matrix of order \( n \). Then

\[
\det(T_n) = \begin{cases} 
2^n & \text{if } n \equiv 0 \pmod{3}, \\
-2^n & \text{if } n \equiv 1 \pmod{3}, \\
0 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** We have \( \det(T_n) = (-2)^n \det(T_n(1, 1, 1)) \). Using Theorem 3.1, we get

\[
\det(T_n) = (-2)^n \left( \frac{(1 + i\sqrt{3})}{2} \right)^{n+1} - \left( \frac{(1 - i\sqrt{3})}{2} \right)^{n+1}.
\]
Since \((1 + i\sqrt{3})^{n+1} = \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^{n+1} = \cos((n+1)\frac{\pi}{3}) + i \sin((n+1)\frac{\pi}{3})\), we have

\[\det(T_n) = (-2)^n \left( \frac{2 \sin \left( \frac{(n+1)\pi}{3} \right)}{\sqrt{3}} \right). \tag{3.1}\]

Note that

\[\sin \left( \frac{(n+1)\pi}{3} \right) = \begin{cases} (-1)^k \frac{\sqrt{3}}{2} & \text{if } n = 3k \text{ or } n = 3k+1, \\ 0 & \text{if } n = 3k+2. \end{cases} \tag{3.2}\]

The result follows by substituting (3.2) in (3.1).

Let \(\alpha\) be a non-zero number and \(i \neq j\). The addition of \(\alpha\) times row \(j\) (column \(j\)) of a matrix \(A\) to row \(i\) (column \(i\)) of \(A\) is denoted by \(R_i \rightarrow R_i + \alpha R_j\) \((C_i \rightarrow C_i + \alpha C_j)\). Note that the effect of this operation does not change the determinant of \(A\).

**Lemma 3.3.** Let \(T_n\) be defined as in Lemma [3.2]. Let \(B_n = \begin{bmatrix} 0 & e' \\ e & T_{n-1} \end{bmatrix}\) be a square matrix of order \(n\). Then

\[\det(B_n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3}, \\ 2^{n-2} \left( \frac{n}{3} - 1 \right) & \text{if } n \equiv 1 \pmod{3}, \\ -2^n \left( \frac{n-1}{3} - 1 \right) & \text{if } n \equiv 2 \pmod{3}. \end{cases}\]

**Proof.** We prove the result by induction on \(n\). It is easy to see that the determinants of \(B_2, B_3\) and \(B_4\) are \(-1, 0\) and \(4\) respectively.

Let \(n \geq 5\). Suppose that the result is true for all positive integers less than \(n\). By performing the row operation \(R_3 \rightarrow R_3 - R_2\) first and then the column operation \(C_3 \rightarrow C_3 - C_2\) on \(B_n\), we get

\[
\begin{array}{c|ccc|ccc}
& 0 & 1 & 0 & 1 & 1 & \cdots & 1 \\
1 & -2 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -2 & 0 & \cdots & 0 \\
1 & 0 & -2 & T_{n-3} \\
e & 0 & 0 & T_{n-3} \\
\end{array}
\]

We first expand \(\det(B_n)\) along the second row. Thus,

\[
\det(B_n) = (-1) \begin{array}{c|ccc|ccc}
1 & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & -2 & 0 & \cdots & 0 & 0 & 0 & -2 & 0 & \cdots & 0 \\
0 & -2 & T_{n-3} & 1 & -2 & & e & 0 & T_{n-3} \\
\end{array} + (-2) \begin{array}{c|ccc|ccc}
0 & 0 & 1 & 0 & -2 & 0' \\
0 & 0 & 0 & 1 & -2 & 0' \\
0 & 0 & 0 & 0 & e & 0' \\
\end{array}.
\]

Next expanding the above determinants along the second column, and the resulting determinants along the second row, we have

\[
\det(B_n) = (-1)(2) \begin{array}{c|ccc|ccc}
1 & 1 & e' & 0 & 0 & 1 & e' \\
0 & -2 & 0' & 0 & 0 & -2 & 0' \\
0 & 0 & T_{n-4} & 1 & -2 & e & T_{n-4} \\
\end{array} + (-2)(2) \begin{array}{c|ccc|ccc}
0 & 1 & e' & 0 & 0 & 1 & e' \\
0 & -2 & 0' & 0 & 0 & -2 & 0' \\
0 & 0 & T_{n-4} & 1 & -2 & e & T_{n-4} \\
\end{array}.
\]
\[
\begin{vmatrix}
1 & e' \\
0 & T_{n-4}
\end{vmatrix} + (-4)(-2) \det(B_{n-3}).
\]

Finally, expand the first determinant along the first column, we obtain the recursive formula

\[
\det(B_n) = 4 \det(T_{n-4}) + 8 \det(B_{n-3}). \tag{3.3}
\]

**Case (i):** Let \(n \equiv 0 \pmod{3}\). Then, \(n - 3 \equiv 0 \pmod{3}\) and \(n - 4 \equiv 2 \pmod{3}\). From Lemma 3.2 and by induction hypothesis, it follows that \(\det(T_{n-4}) = \det(B_{n-3}) = 0\). Hence \(\det(B_n) = 0\).

**Case (ii):** Suppose \(n \equiv 1 \pmod{3}\). Then, \(n - 3 \equiv 1 \pmod{3}\) and \(n - 4 \equiv 0 \pmod{3}\). By Lemma 3.2, we have \(\det(T_{n-4}) = 2^{n-4}\). Using the recurrence relation (3.3) and the induction hypothesis, we get

\[
\det(B_n) = 4(2^{n-4}) + 8 \left( 2^{n-5} \left( \frac{n-4}{3} \right) \right) = 2^{n-2} \left( \frac{n-1}{3} \right).
\]

**Case (iii):** If \(n \equiv 2 \pmod{3}\), then the proof is similar to that of **Case (ii)**. \(\square\)

Later in Remark 4.2, we will show that \(\det(B_n)\) is equal to \(\det(W_n - e)\), where \(W_n - e\) is a graph obtained from \(W_n\) by deleting an edge \(e\) which lies on the cycle.

The main result of this section is the following theorem, which gives the determinant of \(E(W_n)\) in terms of the number of vertices of \(W_n\).

**Theorem 3.4.** Let \(n \geq 5\). Then

\[
\det(E(W_n)) = \begin{cases} 
2^{n-2}(1-n) & \text{if } n \not\equiv 1 \pmod{3}, \\
0 & \text{if } n \equiv 1 \pmod{3}.
\end{cases}
\]

**Proof.** From (2.5), we have \(\det(E(W_n)) = \begin{vmatrix} 
0 & e' \\
e & -E
\end{vmatrix} \).

For each \(i = 2, 3, \ldots, n\), perform the row operations \(R_i \rightarrow R_i - 2R_1\) on \(E(W_n)\), we get

\[
\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & -2 & -2 & 0 & \cdots & 0 & -2 \\
1 & 0 & \ddots & \vdots & \ddots & \vdots & \ddots \\
1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & -2 & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & -2 & \ddots & \ddots & \ddots & \ddots & \ddots
\end{vmatrix} = T_{n-2}.
\]

We first add the third, fourth, \(\cdots\), \(n\)-th rows to the second row and then do the similar operations for columns, we get

\[
\begin{vmatrix}
0 & 1 & n-1 & e' \\
1 & n-1 & -6(n-1) & -6e' \\
e & -6e & T_{n-2}
\end{vmatrix}.
\]

10
By performing the row operation $R_2 \rightarrow R_2 + 6R_1$ first and then the column operation $C_2 \rightarrow C_2 + 6C_1$, we get

$$
\det(E(W_n)) = \begin{vmatrix}
0 & n - 1 & e' \\
n - 1 & 6(n - 1) & 0' \\
e & 0 & T_{n-2}
\end{vmatrix}.
$$

Expanding the above determinant along the second row, we have

$$
\det(E(W_n)) = (-1)(n - 1) \begin{vmatrix}
n - 1 & e' \\
0 & T_{n-2}
\end{vmatrix} + 6(n - 1) \det(B_{n-1}).
$$

Then expand the first determinant along the first column, we get the recursive formula

$$
\det(E(W_n)) = (-1)(n - 1)^2 \det(T_{n-2}) + 6(n - 1) \det(B_{n-1}). \tag{3.4}
$$

Case (i): Let $n \equiv 0 \pmod{3}$. Then $n - 1 \equiv 2 \pmod{3}$ and $n - 2 \equiv 1 \pmod{3}$. Using Lemmas 3.2 and 3.3 in the above recurrence relation (3.4), we get

$$
\det(E(W_n)) = (-1)(n - 1)^2(-2^{n-2}) + 6(n - 1)
\left(-2^{n-3} \frac{n}{3}\right) = 2^{n-2}(1 - n).
$$

Case (ii): Suppose $n \equiv 1 \pmod{3}$. Then, $n - 1 \equiv 0 \pmod{3}$ and $n - 2 \equiv 2 \pmod{3}$. Again, from Lemmas 3.2 and 3.3 we have $\det(T_{n-2}) = \det(B_{n-1}) = 0$. Hence, $\det(E(W_n)) = 0$.

Case (iii): If $n \equiv 2 \pmod{3}$, then the proof is similar to that of Case (i).

As an immediate consequence of Theorem 3.4, we have the following result.

**Theorem 3.5.** Let $n \geq 5$. Then $E(W_n)$ is invertible if and only if $n \not\equiv 1 \pmod{3}$.

4 **Inertia of $E(W_n)$**

Let us recall that for a real symmetric matrix $A$ of order $n$, the **inertia** of $A$, denoted by $\text{In}(A)$, is the ordered triple $(i_+(A), i_-(A), i_0(A))$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ respectively denote the number of positive, negative, and zero eigenvalues of $A$ including the multiplicities. It is well known that $i_+(A) + i_-(A)$ is equal to the rank of $A$.

The inertias of the distance matrices of weighted trees and unicyclic graphs (i.e., connected graphs containing exactly one cycle) have been studied in [6]. It has been shown in [23] that

$$
\text{In}(D(W_n)) = \begin{cases} (1, n - 1, 0) & \text{if } n \text{ is even,} \\
(1, n - 2, 1) & \text{if } n \text{ is odd.} 
\end{cases}
$$

From this result, it is evident that $D(W_n)$ has exactly one positive eigenvalue which is the spectral radius because $D(W_n) \succeq 0$. The inertias of the eccentricity matrices of paths and lollipop graphs were investigated in [16].

In this section, we compute the inertia of the eccentricity matrix of the wheel graph using the notion of interlacing property. We also obtain the determinant and the inertia of $E(W_n-e)$, where $W_n-e$ is the subgraph obtained from the wheel graph $W_n$ by deleting an edge $e$ which lies on the cycle.
Remark 4.1. We observe that

Moreover, $i$ is a leading principal submatrix of $E$ (24) and $i$ are $M$ leading principal submatrix of $E$. Therefore, if we delete the last row and the last column of $E$, we assume that $v_2v_n$. Then,

\[
D(W_n - e) = \begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} - \begin{bmatrix} 0 & 0' \\ 0 & T_{n-1}(2,1,1) \end{bmatrix}.
\]

Note that the eccentricities of the vertices with respect to the graphs $W_n$ and $W_n - e$ are the same. So,

\[
E(W_n - e) = \begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} - \begin{bmatrix} 0 & 0' \\ 0 & T_{n-1}(2,2,2) \end{bmatrix}.
\]

As $T_{n-1}(2,2,2)$ is a leading principal submatrix of $T_n(2,2,2)$, it follows that $E(W_n - e)$ is a leading principal submatrix of $E(W_{n+1} - e)$.

**Remark 4.2.** After performing the row operations $R_i \to R_i - 2R_1$ on $E(W_n - e)$ for each $i = 2, 3, \cdots, n$, it is evident that the resultant matrix is just $B_n$ (defined in Theorem 3.3). Therefore,

\[
\det\left(E(W_n - e)\right) = \det(B_n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{3}, \\
2^{n-2} - \frac{(n-1)}{3} & \text{if } n \equiv 1 \pmod{3}, \\
-2^{n-2} + \frac{(n+1)}{3} & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

We need the following interlacing theorem to prove the inertia of $E(W_n)$.

**Theorem 4.3 (24).** Let $M = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ be a symmetric matrix of order $n$, where $A$ is a leading principal submatrix of $M$ of order $m$ ($1 \leq m \leq n$). If the eigenvalues of $M$ and $A$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$ respectively, then $\lambda_i \geq \beta_i \geq \lambda_{n-m+i}$, for all $i = 1, 2, \cdots, m$. In particular, when $m = n - 1$, we have

\[
\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \beta_{n-1} \geq \lambda_n.
\]

Moreover, $i_+(M) \geq i_+(A)$ and $i_-(M) \geq i_-(A)$.

We first obtain the inertia of $E(W_n - e)$ which will be used to study the inertia of $E(W_n)$.
Lemma 4.4. Let \( n \geq 5 \). Then

\[
\text{In} \left( E(W_n - e) \right) = \begin{cases} 
\left( \frac{2n-5}{3}, \frac{2n-3}{3}, 1 \right) & \text{if } n \equiv 0 \pmod{3}, \\
\left( \frac{n+2}{3}, \frac{2n-2}{3}, 0 \right) & \text{if } n \equiv 1 \pmod{3}, \\
\left( \frac{2n+1}{3}, \frac{2n-1}{3}, 0 \right) & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

Proof. The proof is by induction on the number of vertices \( n \). If \( n = 5 \) then, by Remark 4.2, \( \det \left( E(W_5 - e) \right) \) is negative. Therefore, \( i_0 \left( E(W_5 - e) \right) = 0 \). Consider the matrix \( B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \) and note that \( B \) is a leading principal submatrix of \( E(W_5 - e) \).

Then the eigenvalues of \( B \) are \( \beta_1 = \sqrt{2}, \beta_2 = 0 \) and \( \beta_3 = -\sqrt{2} \). Let \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \), be the eigenvalues of \( E(W_5 - e) \). Note that \( B \) is a leading principal submatrix of \( E(W_5 - e) \).

By Theorem 4.3, we have \( \lambda_1 \geq \beta_1 \), \( \lambda_2 \geq \beta_2 \geq \lambda_3 \) and \( \beta_3 \geq \lambda_5 \). As \( \lambda_i \)'s are non-zero, we have \( \lambda_2 \geq 0 \) and \( \lambda_4 < 0 \). Since \( \det \left( E(W_5 - e) \right) < 0 \), \( i_\pm \left( E(W_5 - e) \right) \) must be 3. Thus, the inertia of \( E(W_5 - e) \) is \( (2, 3, 0) \).

Assume that the result is true for \( n - 1 \). Since \( E(W_{n-1} - e) \) is a leading principal submatrix of \( E(W_n - e) \), we use interlacing theorem to the following three cases.

Case (i): Let \( n \equiv 0 \pmod{3} \). Then \( n - 1 \equiv 2 \pmod{3} \). Since \( \det \left( E(W_{n-1} - e) \right) = 0 \), we have \( i_0 \left( E(W_n - e) \right) \geq 1 \). By induction hypothesis, \( \text{In} \left( E(W_{n-1} - e) \right) = \left( \frac{n}{3}, \frac{2n-3}{3}, 0 \right) \). Then \( \text{In} \left( E(W_n - e) \right) = \left( \frac{n}{3}, \frac{2n-3}{3}, 1 \right) \).

Case (ii): If \( n \equiv 1 \pmod{3} \), then \( \det \left( E(W_n - e) \right) > 0 \). Therefore \( i_0 \left( E(W_n - e) \right) = 0 \). Notice that, in this case, \( \text{In} \left( E(W_{n-1} - e) \right) = \left( \frac{n-1}{3}, \frac{2n-5}{3}, 1 \right) \). By Theorem 4.3, together with the fact that \( i_0 \left( E(W_n - e) \right) = 0 \), we get \( i_+ \left( E(W_n - e) \right) \geq \frac{n-1}{3} + 1 \) and \( i_- \left( E(W_n - e) \right) \geq \frac{2n-5}{3} + 1 \). Hence, \( \text{In} \left( E(W_n - e) \right) = \left( \frac{n+2}{3}, \frac{2n-2}{3}, 0 \right) \).

Case (iii): Suppose \( n \equiv 2 \pmod{3} \). Then \( \det \left( E(W_n - e) \right) < 0 \). This implies that \( i_0 \left( E(W_n - e) \right) < 0 \). It follows from induction hypothesis that \( \text{In} \left( E(W_{n-1} - e) \right) = \left( \frac{n+1}{3}, \frac{2n-4}{3}, 0 \right) \). Again, by Theorem 4.3, we have \( i_+ \left( E(W_n - e) \right) \geq \frac{n+1}{3} + \frac{2n-4}{3} = n - 1 \). Since \( \det \left( E(W_n - e) \right) < 0 \), \( i_- \left( E(W_n - e) \right) \) must be odd. Thus, \( \text{In} \left( E(W_n - e) \right) = \left( \frac{2n+1}{3}, \frac{2n-1}{3}, 0 \right) \).

From Theorem 4.3, it follows that the rank of \( E(W_n) \) is \( n \) if and only if \( n \not\equiv 1 \pmod{3} \). The next result gives the rank of \( E(W_n) \) if \( n \equiv 1 \pmod{3} \).

Theorem 4.5. If \( n \equiv 1 \pmod{3} \), then the rank of \( E(W_n) \) is \( n - 2 \).

Proof. Since \( n \equiv 1 \pmod{3} \), we have \( n = 3k + 1 \) for some \( k \geq 1 \). Consider the vectors \( x \) and \( y \) in \( \mathbb{R}^n \) which are given by

\[
x = (0, 1, 0, -1, 1, 0, -1, \ldots, 1, 0, -1)^t \quad \text{(4.1)}
\]

and

\[
y = (0, 0, 1, -1, 0, 1, -1, \ldots, 0, 1, -1)^t. \quad \text{(4.2)}
\]

By simple verification, we can see that \( E(W_n)x = E(W_n)y = 0 \). Moreover, \( x \) and \( y \) are linearly independent. Therefore, the dimension of the null space of \( E(W_n) \) is at least two and hence the rank of \( E(W_n) \) is at most \( n - 2 \). To prove the result, it is enough to find an \( (n - 2) \times (n - 2) \) submatrix of \( E(W_n) \) whose determinant is non-zero. As \( n \equiv 1 \pmod{3} \), \( n - 2 \equiv 2 \pmod{3} \). From Remark 4.2, we have \( \det \left( E(W_{n-2} - e) \right) \neq 0 \). Since \( E(W_{n-2} - e) \) is a leading principal submatrix of \( E(W_n) \), the result follows.
We now proceed to compute the inertia of the eccentricity matrix of the wheel graph.

**Theorem 4.6.** Let \( n \geq 5 \). The inertia of \( E(W_n) \) is

\[
\text{In}(E(W_n)) = \begin{cases} 
\left(\frac{n+3}{3}, \frac{2n-3}{3}, 0\right) & \text{if } n \equiv 0 \pmod{3}, \\
\left(\frac{n-1}{3}, \frac{2n-5}{3}, 2\right) & \text{if } n \equiv 1 \pmod{3}, \\
\left(\frac{n+1}{3}, \frac{2n-1}{3}, 0\right) & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** If \( n = 5 \), then the proof is similar to that of Lemma 4.3. Let \( n \geq 6 \). We consider the following three cases.

**Case (i):** Let \( n \equiv 0 \pmod{3} \). Then from Theorem 3.3 \( \det(E(W_n)) < 0 \). Note that \( E(W_{n-1} - e) \) is a leading principal submatrix of \( E(W_n) \). By Theorem 4.3 and Lemma 4.4 we have \( i_+(E(W_n)) \geq i_+(E(W_{n-1} - e)) = \frac{n}{3} \), and \( i_-(E(W_n)) \geq i_-(E(W_{n-1} - e)) = \frac{2n-3}{3} \). As \( \left(\frac{2n-3}{3}\right) \) is an odd number and \( \det(E(W_n)) < 0 \), we get \( i_+(E(W_n)) = \frac{n}{3} + 1 \), the result follows in this case.

**Case (ii):** Suppose \( n \equiv 1 \pmod{3} \). By Theorem 4.3, rank of \( E(W_n) \) is \( n - 2 \) and hence \( i_0(E(W_n)) = 2 \). Again, using Theorem 4.3 and Lemma 4.4 we get \( i_+(E(W_n)) \geq i_+(E(W_{n-1} - e)) \geq \frac{n-1}{3} \), and \( i_-(E(W_n)) \geq i_-(E(W_{n-1} - e)) \geq \frac{2n-5}{3} \). Since the sum of \( i_+(E(W_n)), i_-(E(W_n)) \) and \( i_0(E(W_n)) \) is equal to \( n \), we have \( \text{In}(E(W_n)) = \left(\frac{n-1}{3}, \frac{2n-5}{3}, 2\right) \).

**Case (iii):** If \( n \equiv 2 \pmod{3} \), then the proof goes similar to that of Case (i). \( \square \)

**Remark 4.7.** It has been proved that the distance matrix of \( W_n \) has a unique positive eigenvalue [23]. But this property does not hold for \( E(W_n) \) because it has at least two positive eigenvalues for all \( n \geq 5 \).

## 5 Inverse formula for \( E(W_n) \)

Graham and Lovász [13] obtained the formula (1.1) for the inverse of the distance matrix \( D(T) \) of a tree \( T \) which is expressed as the sum of the Laplacian matrix and a rank one matrix. Motivated by this result, a similar inverse formula has been derived for the distance matrices of several graphs, see [5, 6, 7, 11, 25].

Balaji et al. [3] obtained an inverse formula similar to (1.1) for the distance matrix of \( W_n \) when \( n \) is even. The formula for \( D(W_n)^{-1} \) is given as the sum of a Laplacian-like matrix and a rank one matrix. The objective of this section is to find a similar inverse formula for \( E(W_n) \) when \( n \not\equiv 1 \pmod{3} \). We derive this formula by adapting the common technique of finding a suitable Laplacian-like matrix \( \tilde{L} \) and a vector \( w \in \mathbb{R}^n \) such that

\[
E(W_n)w = \alpha e \text{ and } \tilde{L}E(W_n) + 2I = \beta we',
\]

where \( \alpha \) and \( \beta \) are real numbers. We start with computing the inverse of the eccentricity matrix of the wheel graph on six vertices.

**Example 5.1.** Consider the eccentricity matrix of \( W_6 \) which is given by

\[
E(W_6) = \begin{bmatrix} 
0 & e' \\
e_5 & \text{Circ}((0, 0, 2, 2, 0))
\end{bmatrix}.
\]
The Laplacian-like matrix for $E(W_6)$ is

$$
\tilde{L}(W_6) = \frac{1}{3} \begin{bmatrix}
5 & -1 & -1 & -1 & -1 \\
-1 & -1 & 2 & -1 & -1 \\
-1 & 2 & -1 & 2 & -1 \\
-1 & -1 & 2 & -1 & 2 \\
-1 & 2 & -1 & -1 & 2
\end{bmatrix}.
$$

Set $c_1' = (0, 1, 0, 0, 1)$, $c_2' = (0, 0, 1, 1, 0)$ and $M = \frac{1}{3} \text{Circ}(-6e_1' + 2e_1' - c_2')$. Then $\tilde{L}(W_6)$ can be rewritten as

$$
\tilde{L}(W_6) = \frac{5}{3} I_6 - \frac{1}{3} \begin{bmatrix} 0 & e' \\ e & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}.
$$

If $w = \frac{1}{6} (1, 1, 1, 1, 1)'$, then by direct verification, we see that

$$
E(W_6)^{-1} = \frac{1}{5} \begin{bmatrix}
-4 & 1 & 1 & 1 & 1 \\
1 & \frac{-3}{2} & 1 & \frac{-3}{2} & 1 \\
1 & 1 & \frac{-3}{2} & 1 & \frac{-3}{2} \\
1 & 1 & 1 & \frac{-3}{2} & 1 \\
1 & \frac{-3}{2} & 1 & 1 & \frac{-3}{2}
\end{bmatrix} = -\frac{1}{2} \tilde{L}(W_6) + \frac{6}{5} ww'.
$$

The above example shows that the inverse of $E(W_6)$ follows some nice patterns. Particularly, the submatrix of $E(W_6)^{-1}$ obtained by deleting the first row and the first column is a circulant matrix defined by the vector $x' = \frac{1}{10} (2, -3, 2, 2, -3) \in \mathbb{R}^5$, which follows symmetry in its last four coordinates. That is, the second coordinate is equal to the last coordinate of $x$, and the third and fourth coordinates of $x$ are the same. In fact, it is observed that the same kind of pattern is retained for the matrices $E(W_n)^{-1}$ of higher orders. So, most of the entries in the inverse can be obtained by finding a vector similar to $x$, which is given below.

The following significant vectors are identified from the observations made from numerical examples which play a central role in finding a formula for $E(W_n)^{-1}$.

For $n \geq 4$, fix the vectors $c_k \in \mathbb{R}^{n-1}$, where

$$
c_k = e_{k+1} + e_{n-k} \quad \text{where} \quad k \in \begin{cases}
\{1, 2, \cdots, \frac{n-2}{2}\} & \text{if } n \text{ is even,} \\
\{1, 2, \cdots, \frac{n-3}{2}\} & \text{if } n \text{ is odd.}
\end{cases} \quad (5.2)
$$

Consider the case $n \equiv 2 \pmod{3}$ and $n \geq 8$. Define the vectors

$$
x_1 = (2 - n) e_1 + \sum_{k=1}^{n/2} (c_{3k-2} - 2c_{3k-1} + c_{3k}) \quad \text{if } n \text{ is even,} \quad (5.3)
$$

and

$$
x_2 = (2 - n) e_1 + \sum_{k=1}^{n/2} (c_{3k} - 2c_{3k-1}) + \sum_{k=1}^{n/2} c_{3k-2} - 2e_{n/4+1} \quad \text{if } n \text{ is odd.} \quad (5.4)
$$
In the case $n \equiv 0 \pmod{3}$ and $n \geq 9$, we define
\[
x_3 = (-n)e_1 + \sum_{k=1}^{n/3} (2c_{3k-2} - c_{3k-1}) - \sum_{k=1}^{n/3-1} c_{3k} \quad \text{if } n \text{ is even}, 
\]
and
\[
x_4 = (-n)e_1 + \sum_{k=1}^{n/3} (2c_{3k-2} - c_{3k-1} - c_{3k}) + 2e_{n/2+1} \quad \text{if } n \text{ is odd}. 
\]

Lemma 5.2. If $x_i, i = 1, 2, 3, 4$ are the vectors given in (5.3) – (5.6), then
\[
x_1 = x_2 = (2 - n, 1, -2, 1, \cdots, 1, -2, 1)', \tag{5.7}
\]
and
\[
x_3 = x_4 = (-n, 2, -1, -1, \cdots, 2, -1, -1, 2)'. \tag{5.8}
\]

Proof. Suppose $n \equiv 0 \pmod{3}$ and $n$ is even. Then $n - 2 = 6l$, for some positive integer $l$. For $1 \leq k \leq \frac{n-2}{6}$,
\[
c_{3k-2} - 2c_{3k-1} + c_{3k} = (e_{3k-1} - 2e_{3k} + e_{3k+1}) + (e_{n-3k+2} - 2e_{n-3k+1} + e_{n-3k}).
\]
Thus
\[
\sum_{k=1}^{n/6} (c_{3k-2} - 2c_{3k-1} + c_{3k}) = \left[ (e_2 - 2e_3 + e_4) + \cdots + (e_{n/2-2} - 2e_{n/2-1} + e_{n/2}) \right] + \left[ (e_{n-1} - 2e_{n-2} + e_{n-3}) + \cdots + (e_{n/2+3} - 2e_{n/2+2} + e_{n/2+1}) \right].
\]
This implies that $x_1 = (2 - n, 1, -2, 1, \cdots, 1, -2, 1)'$. The remaining cases are proved similarly. \qed

Remark 5.3. Balaji et al. [3] derived an inverse formula similar to (1.1) for $D(W_n)$ when $n$ is even. They had to deal only with the case when $n$ is even and introduced only one special vector. In the case of $E(W_n)^{-1}$, we have to deal with four cases as mentioned in (5.3) – (5.6). However, by Lemma 5.2 we have to consider only two cases $n \equiv 2 \pmod{3}$ and $n \equiv 0 \pmod{3}$.

Now we construct a Laplacian-like matrix $\tilde{L}$ of order $n$ for $W_n$ using the above defined vectors. Later we will show that $\tilde{L}$ is a symmetric matrix of rank $n-1$ (see Lemma 5.8 and Theorem 5.13).

Definition 5.4. Let $n \not\equiv 1 \pmod{3}$ and $n \geq 8$. Fix the symbols $\tilde{x}$ and $\tilde{y}$ to denote the vectors given in (5.7) and (5.8) respectively. Define
\[
\tilde{L} := \frac{n-1}{3} I_n - \frac{1}{3} \begin{bmatrix} 0 & e' \\ e & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0' \\ 0 & M \end{bmatrix}, \tag{5.9}
\]
where
\[
M = \frac{1}{3} \begin{cases} \text{Circ}(\tilde{x}') & \text{if } n \equiv 2 \pmod{3}, \\ \text{Circ}(\tilde{y}') & \text{if } n \equiv 0 \pmod{3}. \end{cases} \tag{5.10}
\]
Throughout this paper, we deal with a circulant matrix of order \( n - 1 \), which is defined by some vector \( \mathbf{x} \) in \( \mathbb{R}^{n-1} \). On most occasions, the vector \( \mathbf{x} \) follows symmetry in its last \( n - 2 \) coordinates. The precise definition of symmetry is given below.

**Definition 5.5** \( \text{(3)} \). Let \( n \) be a positive integer such that \( n \geq 4 \). A vector \( \mathbf{x} = (x_1, x_2, \cdots, x_{n-1})' \) in \( \mathbb{R}^{n-1} \) is said to follow symmetry in its last \( n - 2 \) coordinates if 
\[
x_i = x_{n+1-i} \quad \text{for all } i = 2, 3, \cdots, n-1.
\]

Equivalently, \( \mathbf{x} \) follows symmetry in its last \( n - 2 \) coordinates if it has the form 
\[
\mathbf{x} = \begin{cases} 
(x_1, x_2, x_3, \cdots, x_{n-2}, x_{n-1}, x_2)' & \text{if } n \text{ is even}, \\
(x_1, x_2, x_3, \cdots, x_{n-2}, x_{n-1}, x_2, x_1)' & \text{if } n \text{ is odd}.
\end{cases}
\]

**Remark 5.6.** Let \( \mathbf{x} \) and \( \mathbf{y} \) be vectors in \( \mathbb{R}^{n-1} \). If each of \( \mathbf{x} \) and \( \mathbf{y} \) follows symmetry in its last \( n - 2 \) coordinates, then \( \alpha \mathbf{x} + \beta \mathbf{y} \) also follows symmetry in its last \( n - 2 \) coordinates, where \( \alpha \) and \( \beta \) are real numbers.

In general, a circulant matrix \( \mathbf{C} \) need not be symmetric, but if the defining vector of \( \mathbf{C} \) follows symmetry in its last \( n - 2 \) coordinates, then \( \mathbf{C} \) is symmetric, which we show next.

**Lemma 5.7.** Let \( \mathbf{C} = \text{Circ}(\mathbf{c}') \), where \( \mathbf{c} \in \mathbb{R}^{n-1} \). If \( \mathbf{c} \) follows symmetry in its last \( n - 2 \) coordinates, then \( \mathbf{C} \) is a symmetric matrix.

**Proof.** Let \( \mathbf{c} = (c_1, c_2, \cdots, c_{n-1})' \) and let \( \mathbf{C} = (a_{ij}) \). Then \( a_{ii} = c_1 \) and \( i \)-th row of \( \mathbf{C} \) is \( T^{i-1}(\mathbf{c}') = (c_{n-(i-1)}, c_{n-(i-2)}, \cdots, c_{n-2}, c_{n-1}, c_1, c_2, \cdots, c_{n-i}) \), where the operator \( T \) is defined in (2.4). Assume that \( i < j \). Then \( j = i + k \) for some positive integer \( k \). Now,
\[
\begin{align*}
\quad a_{ij} &= C_{ij} e_j = T^{i-1}(\mathbf{c}') e_j = T^{i-1}(\mathbf{c}') e_{i+k} = c_{k+1}, \quad \text{and} \\
\quad a_{ji} &= C_{ji} e_i = T^{j-1}(\mathbf{c}') e_i = c_{n-(j-i)} = c_{n-k}.
\end{align*}
\]

Since \( \mathbf{c} \) follows symmetry in its last \( n - 2 \) coordinates, we have \( c_{k+1} = c_{n-k} \). Hence the proof. \( \square \)

**Lemma 5.8.** The matrices \( \overline{\mathbf{L}} \) and \( \mathbf{M} \) given in Definition \( 5.4 \) are symmetric.

**Proof.** Since each of \( \overline{\mathbf{x}} \) and \( \overline{\mathbf{y}} \) follows symmetry in its last \( n - 2 \) coordinates, we have \( \mathbf{M} \) is symmetric by the above theorem. Therefore, \( \overline{\mathbf{L}} \) is symmetric. \( \square \)

To obtain the main result of this section, we need the following two lemmas.

**Lemma 5.9.** Let \( \mathbf{M} \) be defined as in (5.10). Then \( \mathbf{M} \mathbf{e} = \frac{1}{3}(2 - n) \mathbf{e} \).

**Proof.** Since \( \mathbf{M} \) is a circulant matrix, all the row sums of \( \mathbf{M} \) are equal. Therefore,
\[
\mathbf{M} \mathbf{e} = \alpha \mathbf{e}, \quad \text{where} \quad \alpha = \begin{cases} 
\frac{1}{3} \mathbf{e}' \overline{\mathbf{x}} & \text{if } n \equiv 2 \pmod{3}, \\
\frac{1}{3} \mathbf{e}' \overline{\mathbf{y}} & \text{if } n \equiv 0 \pmod{3}.
\end{cases}
\]

As \( \overline{\mathbf{x}}' = (2 - n, 1, -2, 1, 1, -2, 1, \cdots, 1, -2, 1) \in \mathbb{R}^{n-1} \), we have \( \mathbf{e}' \overline{\mathbf{x}} = 2 - n \). Also, the sum of the coordinates of \( \overline{\mathbf{y}}' = (-n, 2, -1, -1, 2, -1, -1, \cdots, 2, -1, -1) \in \mathbb{R}^{n-1} \), except the first and the last coordinates, is zero. So, we get \( \mathbf{e}' \overline{\mathbf{y}} = 2 - n \). Thus, \( \alpha = \frac{1}{3}(2 - n) \) which gives the desired result. \( \square \)
Lemma 5.10. Let $n \not\equiv 1 \pmod{3}$ and $M$ be defined as in (5.10). Then $M\tilde{E} = \frac{1}{3} \text{Circ}(z')$, where $z = (-4, 2, 4 - 2n, 4 - 2n, \ldots, 4 - 2n, 2) \in \mathbb{R}^{n-1}$.

Proof. Since $\tilde{E} = \text{Circ}(u')$ where $u' = (0, 0, 2, \ldots, 2, 0) \in \mathbb{R}^{n-1}$, the columns of $\tilde{E}$ can be written as

$$\tilde{E}_{si} = \begin{cases} 2(e - e_1 - e_2 - e_{n-1}) & \text{if } i = 1, \\ 2(e - e_{i-1} - e_i - e_{i+1}) & \text{if } 2 \leq i \leq n - 2, \\ 2(e - e_1 - e_{n-2} - e_{n-1}) & \text{if } i = n - 1. \end{cases}$$

First, assume that $n \equiv 2 \pmod{3}$. Note that $M$ and $\tilde{E}$ are circulant. Using (2.3), we write $M\tilde{E} = \frac{1}{3} \text{Circ}(\tilde{x}'\tilde{E})$, where $\tilde{x} = (2 - n, 1, -2, 1, \ldots, 1, -2, 1) \in \mathbb{R}^{n-1}$.

Let $z' = \tilde{x}'\tilde{E} = (z_1, \ldots, z_{n-1})$. Since $\tilde{x}'e = 2 - n$, we have

$$z_1 = \tilde{x}'\tilde{E}_{s1} = 2(2 - n) - 2((2 - n) + 1 + 1) = -4.$$ 

Now, $z_2 = \tilde{x}'\tilde{E}_{s2} = 2\tilde{x}'e - 2((2 - n) + 1 - 2) = 2\tilde{x}'\tilde{E}_{s(n-1)} = z_{n-1}$. If $3 \leq i \leq n - 2$, then

$$z_i = \tilde{x}'\tilde{E}_{si} = 2\tilde{x}'e - 2(x_{i-1} + x_i + x_{i+1}),$$ 

where $x_i$ is the $i$-th coordinate of $\tilde{x}$. Note that starting from the second coordinate, the sum of any three consecutive coordinates of $\tilde{x}$ is zero. So, we get $z_i = 2\tilde{x}'e = 2(2 - n)$. Thus, $z' = (-4, 2, 4 - 2n, \ldots, 4 - 2n, 2)$. Suppose $n \equiv 0 \pmod{3}$.

Then $M\tilde{E} = \frac{1}{3} \text{Circ}(\tilde{y}'\tilde{E})$. By the same arguments as above, we have $\tilde{y}'\tilde{E} = z'$. Hence the proof.

We are now ready to give a formula for the inverse of the eccentricity matrix of $W_n$ which is similar to the form of (1.1). That is, the inverse of $E(W_n)$ is expressed as a sum of a symmetric Laplacian-like matrix of rank $n - 1$ and a rank one matrix. Hereafter, we denote $E(W_n)$ simply by $E$.

Theorem 5.11. Let $n \geq 8$ and $n \not\equiv 1 \pmod{3}$. Consider the matrix $\tilde{L}$ given in (5.9). Suppose that $w = \frac{1}{9}(7 - n, 1, \ldots, 1) \in \mathbb{R}^n$. Then

$$E^{-1} = -\frac{1}{2} \tilde{L} + \frac{6}{n - 1}ww'.$$

Proof. In order to get identities in (5.1), multiplying $\tilde{L}$ and $E$ gives

$$\tilde{LE} = \begin{bmatrix} \frac{1-n}{3} & A \\ N & S \end{bmatrix},$$

where

$$A = \frac{n - 1}{3}e' - \frac{1}{3}e'\tilde{E},$$

$$N = \frac{n - 1}{3}e + Me,$$

(5.11)
and
\[S = \frac{n-1}{3}E - \frac{1}{3}ee' + M\tilde{E}.\]

Note that \(\tilde{E}e = 2(n-4)e\). Therefore,
\[A = \frac{n-1}{3}e' - \frac{1}{3}(2n-8)e' = \frac{7-n}{3}e'.\]

Using Lemma 5.9 in (5.11), we get,
\[N = \frac{n-1}{3}e + \frac{2-n}{3}e = \frac{1}{3}e.\]

Combining Lemma 5.10 and (2.2), \(S\) can be written as
\[S = \frac{1}{3}\text{Circ}((n-1)u' - e' + z'), \text{ where } z = (-4, 2, 4 - 2n, \cdots, 4 - 2n, 2)'\]

It is easy to verify that \((n-1)u' - e' + z' = (-5, 1, \cdots, 1)'\). Therefore,
\[S = \frac{1}{3}\text{Circ}(v'), \text{ where } v = (-5, 1, \cdots, 1)' \in \mathbb{R}^{n-1}.
\]

Hence,
\[\tilde{L}E = \frac{1}{3} \begin{bmatrix} 1 - n & (7-n)e' \\ e & \text{Circ}(v') \end{bmatrix}.\]

Since \(2e_1' + \frac{1}{3}v' = \frac{1}{3}e'\), we have
\[\tilde{L}E + 2I = \frac{1}{3} \begin{bmatrix} 7-n & (7-n)e' \\ e & \text{Circ}(e') \end{bmatrix} = 2we'.\]

As \(n \not\equiv 1 \pmod{3}\), we have \(E\) is invertible which follows from Theorem 3.5. So, from the above identity, we get
\[2E^{-1} = -\tilde{L} + 2we' E^{-1}. \quad (5.12)\]

By an easy to verification, we see that \(Ew = \left(\frac{n-1}{6}\right)e\). Therefore,
\[w = \left(\frac{n-1}{6}\right)E^{-1}e. \quad (5.13)\]

Note that \(\tilde{L}\) and \(E\) are symmetric. This implies that \(E^{-1}\) is also symmetric. Taking the transpose of (5.12), we get \(2E^{-1} = -\tilde{L} + 2E^{-1}ew'\). Using (5.13), we get the desired result
\[E^{-1} = -\frac{1}{2}\tilde{L} + \frac{6}{n-1}ww'.\]
Next, we prove that the matrix $\tilde{L}$ is a Laplacian-like matrix and of rank $n - 1$. Hence $\tilde{L}$ has some of the properties of the Laplacian matrix.

**Theorem 5.12.** If $\tilde{L}$ is defined as in (5.9), then $\tilde{L}$ is a Laplacian-like matrix.

**Proof.** We have

$$\tilde{L}e = \frac{n - 1}{3}e - \frac{1}{3}\begin{bmatrix} n - 1 \\ e_{(n-1)\times 1} \end{bmatrix} + \begin{bmatrix} 0 \\ Me_{(n-1)\times 1} \end{bmatrix} = \begin{bmatrix} 0 \\ (\frac{n-2}{3})e_{(n-1)\times 1} + Me_{(n-1)\times 1} \end{bmatrix}.$$  

Then, $\tilde{L}e = 0$ which follows from Lemma 5.9. Since $\tilde{L}$ is symmetric, row sums of $\tilde{L}$ are also zero. \qed

The proof of the following theorem is similar to that of Theorem 2 in [3], which is given for the sake of completeness.

**Theorem 5.13.** Consider the $n \times n$ matrix $\tilde{L}$ given in (5.9). Then the rank of $\tilde{L}$ is $n - 1$.

**Proof.** Suppose $\tilde{L}x = 0$ where $x$ is a non-zero vector in $\mathbb{R}^n$. Then $x^t\tilde{L}E = 0$. Using the relation $\tilde{L}E + 2I = 2we'$, we get $x^t = (x^t\mathbf{w})e'$ where $\mathbf{w} = \frac{1}{6}(7 - n, 1, 1, \ldots, 1)'$. Thus, $x$ is a scalar multiple of $e$. This gives that the dimension of nullspace of $\tilde{L}$ is at most one. Since $\tilde{L}e = 0$, we have nullity of $\tilde{L}$ is one. Hence rank of $\tilde{L}$ is $n - 1$. \qed

### 6 Moore-Penrose Inverse of $E(W_n)$

The Moore-Penrose Inverses of the incidence matrices of the different graphs have been studied in the literature, see [1, 5]. A formula for $D(W_n)\dagger$ is given by Balaji et al. [4] when $n$ is odd and it is written as the sum of a Laplacian-like matrix and a rank one matrix, see (1.2). This result motivates us to find a similar Moore-Penrose inverse formula for $E(W_n)$ when $n \equiv 1 \pmod{3}$. Now, we compute $E(W_7)\dagger$ in the following example.

**Example 6.1.** Consider the eccentricity matrix of the wheel graph on seven vertices. Take $c_1 = (0,1,0,0,0,1)'$ and $c_2 = (0,0,1,0,1,0)'$, $c_4 = (0,0,0,1,0,0)'$ and

$$P = \text{Circ}\left(\frac{-35}{18}e_1 + \frac{11}{36}c_1 - \frac{7}{36}c_2 + \frac{1}{18}e_4\right) = \text{Circ}\left(\frac{-35}{18}, \frac{11}{36}, \frac{-7}{36}, \frac{1}{18}, \frac{-7}{36}, \frac{11}{36}\right).$$

The Laplacian-like matrix for $E(W_7)$ is given by

$$\hat{L}(W_7) = \frac{1}{6} \begin{bmatrix} 2 & -2e \\ -2e_{6\times 1} & \text{Circ}\left(\left(\frac{11}{3}, \frac{11}{6}, \frac{-7}{6}, \frac{11}{6}, \frac{11}{6}\right)\right) \end{bmatrix} = 2I_7 - \frac{1}{3} \begin{bmatrix} 0 & e' \\ e & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0' \end{bmatrix}. $$

Setting $\mathbf{w} = \frac{1}{6}(0,1,1,1,1,1)'$, it is easy to verify that

$$E(W_7)\dagger = \frac{1}{6} \begin{bmatrix} -1 & e' \\ e & \text{Circ}\left((0, \frac{-6}{8}, \frac{6}{8}, 0, \frac{6}{8}, \frac{-6}{8})\right) \end{bmatrix} = -\frac{1}{2} \hat{L}(W_7) + \mathbf{w}\mathbf{w}'. $$

Hereafter we assume that $n \geq 10$ and $n \equiv 1 \pmod{3}$. To establish a formula for $E(W_n)\dagger$, we fix the following vectors in $\mathbb{R}^{n-1}$, which are identified from the numerical computations.
If $n$ is even, then fix $z_1$, which is given by
\[ z_1 = (2n - n^2)e_1 + \sum_{k=1}^{\frac{n-4}{2}} \left( \frac{3n - 18k + 8}{2}c_{3k-2} - \frac{3n - 18k + 4}{2}c_{3k-1} + c_{3k} \right) + c_{n-1}. \] (6.1)

Suppose $n$ is odd, then fix $z_2$ as
\[ z_2 = (2n - n^2)e_1 + \sum_{k=1}^{\frac{n-7}{2}} \left( \frac{3n - 18k + 8}{2}c_{3k-2} - \frac{3n - 18k + 4}{2}c_{3k-1} \right) + \sum_{k=1}^{\frac{n-7}{2}} c_{3k} + e_{n+1}, \] (6.2)

where the vectors $c_k$'s are defined in (5.2).

**Remark 6.2.** If $n$ is even, then $n = 6k + 4$ where $k \geq 1$, and if $n$ is odd, then $n = 6k + 1$ where $k \geq 2$. So, the vectors (6.1) and (6.2) are well defined.

In order to obtain the desired formula for $E(W_n)^\dagger$, we introduce the following matrix $\hat{L}$ which is analogues to $\tilde{L}$ given in Definition 5.4 for the invertible case. Similar to the properties of $\tilde{L}$, we will show that $\hat{L}$ is a symmetric Laplacian-like matrix (see Lemma 6.4 and Theorem 6.11).

**Definition 6.3.** Let $n \equiv 1 \pmod{3}$ and $n \geq 10$. Define
\[ \hat{L} := \frac{n-1}{3}I - \frac{1}{3} \left[ \begin{array}{ccc} 0 & e' \\ e & 0 \\ 0 & P \end{array} \right], \] (6.3)

where
\[ P = \frac{1}{3(n-1)} \begin{cases} \text{Circ}(z_1') & \text{if } n \text{ is even}, \\ \text{Circ}(z_2') & \text{if } n \text{ is odd}. \end{cases} \] (6.4)

We now show that the matrix $\hat{L}$ is symmetric and row sums of $P$ are equal.

**Lemma 6.4.** Let $\hat{L}$ and $P$ be the matrices as defined above. Then $P$ and $\hat{L}$ are symmetric.

**Proof.** It is easy to see that each $c_k$ in (5.2) and $e_{n+1}$, if $n$ is odd, follows symmetry in its last $n-2$ coordinates. Therefore, by Remark 5.6, each of $z_1$ and $z_2$ follows symmetry in its last $n-2$ coordinates. By Lemma 5.7, the matrix $P$ and hence $\hat{L}$ are symmetric. \( \square \)

**Lemma 6.5.** Let $P$ be the matrix defined in (6.4). Then $P = \frac{1}{3}(2 - n)e$.

**Proof.** Since $P$ is a circulant matrix, we have $Pe = \alpha e$, where
\[ \alpha = \frac{1}{3(n-1)} \begin{cases} z_1'e & \text{if } n \text{ is even}, \\ z_2'e & \text{if } n \text{ is odd}. \end{cases} \]

First consider the case when $n$ is even. Let $z_1 = (z_1^{(1)}, z_2^{(1)}, \ldots, z_{n-1}^{(1)})'$ and $1 \leq k \leq \frac{n-4}{6}$. Then, $z_{3k-1}^{(1)} = \frac{3n-18k+8}{2}$, $z_{3k}^{(1)} = \frac{3n+18k-4}{2}$, $z_{3k+1}^{(1)} = 1$, and $z_k^{(1)} = 1$. Note that each $c_k$ follows
Assume that $n - 2$ coordinates and has only two non-zero entries which are one. So, we have
\[
\alpha = \frac{1}{3(n - 1)} z_1 e = \frac{1}{3(n - 1)} \left( (2n - n^2) + 2 \left( \sum_{k=1}^{n-4} \left( \frac{8 - 4k}{2} \right) + 1 \right) + 2 \right)
\]
\[
= \frac{1}{3(n - 1)} (3n - n^2 - 2) = \frac{2 - n}{3}.
\]
If $n$ is odd, then from the definition of $z_2$, we have
\[
\alpha = \frac{1}{3(n - 1)} \left( (2n - n^2) + 2 \left( \sum_{k=1}^{n-4} \frac{8 - 4k}{2} \right) + \sum_{k=1}^{n-4} 1 \right) + 1 = \frac{2 - n}{3}.
\]

In the following, we present two lemmas which will be used to compute $\tilde{LE}$. This helps in establishing identities (5.1) to achieve a formula for $E^\dagger$.

**Lemma 6.6.** Let $\tilde{V}' = (2, -1, -1, 2, -1, -1, \ldots, 2, -1, -1) \in \mathbb{R}^{n-1}$. Let $V = \text{Circ}(\tilde{V}')$ and $P$ be defined as in (6.4). Then $PV = \frac{1}{3}(1 - n)V$.

**Proof.** Consider the operator $T$ defined in (2.4). Then $T(\tilde{V}) + T^2(\tilde{V}) = -\tilde{V}$ and
\[
V_{(k+1)\ast} = T^k(\tilde{V}) = \begin{cases} 
\tilde{V} & \text{if } k \equiv 0 \pmod{3}, \\
T(\tilde{V}) & \text{if } k \equiv 1 \pmod{3}, \\
T^2(\tilde{V}) & \text{if } k \equiv 2 \pmod{3}.
\end{cases}
\] (6.5)

Assume that $n$ is even. Let $1 \leq k \leq \frac{n-4}{6}$. Since $c_k = e_{k+1} + e_{n-k}$, we have
\[
c'_{3k-2} V = e'_{3k-1} V + e'_{n-3k+2} V = V_{(3k-1)\ast} + V_{(n-3k+2)\ast} = T^{3k-2}(\tilde{V}) + T^{n-3k+1}(\tilde{V}).
\]

As $3k - 2 \equiv 1 \pmod{3}$ and $n - 3k + 1 \equiv 2 \pmod{3}$, it follows from (6.5) that
\[
c'_{3k-2} V = T(\tilde{V}) + T^2(\tilde{V}).
\] (6.6)

Also,
\[
c'_{3k-1} V = T^2(\tilde{V}) + T(\tilde{V}), \quad \text{and} \quad c'_{3k} V = 2\tilde{V}.
\] (6.7)

Since $P = \frac{1}{3(n-1)} \text{Circ}(z'_1)$ and $V$ is a circulant matrix, it follows from (2.3) that $PV = \frac{1}{3(n-1)} \text{Circ}(z'_1 V)$. From (6.1), we have
\[
z'_1 V = (2n - n^2)e'_1 V + \sum_{k=1}^{n-4} \left( \frac{3n - 18k + 8}{2} c'_{3k-2} V - \frac{3n - 18k + 4}{2} c'_{3k-1} V + c'_{3k} V \right) + c'_{n-1} V.
\]

By the assumption on $n$, we have $n = 6l + 4$, where $l \geq 1$. Therefore
\[
c'_{n-1} V = V_{(n)\ast} + V_{(n+1)\ast} = T(\tilde{V}) + T^2(\tilde{V}).
\] (6.8)
Using (6.6), (6.7) and (6.8), we get

\[
zh_1V = (2n - n^2) \bar{v} + \sum_{k=1}^{n-1} \left( \frac{8 - 4}{2} (T(\bar{v}) + T^2(\bar{v})) + 2\bar{v} \right) + (T(\bar{v}) + T^2(\bar{v})).
\]

As \(T(\bar{v}) + T^2(\bar{v}) = -\bar{v}\), we get \(zh_1V = (2n - n^2) \bar{v} - \bar{v} = -(n-1)^2 \bar{v}\). Thus, \(PV = \frac{1}{3}(1-n) \text{Circ}(\bar{v})\).

If \(n\) is odd, the computation of \(PV\) is quite similar to that of even case. \(\blacksquare\)

**Lemma 6.7.** Let \(P\) be defined as in (6.4). Let \(\bar{u}' = (1, 1, 0, 0, \ldots, 0, 1)' \in \mathbb{R}^{n-1}\) and \(U = \text{Circ}(\bar{u}')\). If \(z' = (5n - n^2 - 10, 2n - n^2 + 2, 3, -6, 3, \ldots, 3, -6, 3, 2n - n^2 + 2) \in \mathbb{R}^{n-1}\), then

\[
PU = \frac{1}{3(n-1)} \text{Circ}(z').
\]

**Proof.** Since \(P\) and \(U\) are circulant matrices, we write \(PU = \frac{1}{3(n-1)} \begin{cases} \text{Circ}(zh_1U) & \text{if } n \text{ is even}, \\ \text{Circ}(zh_2U) & \text{if } n \text{ is odd}, \end{cases}\) by (2.3). Note that the columns of \(U\) are

\[
U_{si} = \begin{cases} e_1 + e_2 + e_{n-1} & \text{if } i = 1, \\ e_{i-1} + e_i + e_{i+1} & \text{if } 2 \leq i \leq n - 2, \\ e_1 + e_{n-2} + e_{n-1} & \text{if } i = n - 1. \end{cases}
\]

Assume that \(n\) is even. We need to compute the vector \(zh_1U\). Note that the \(i\)-th coordinate of \(zh_1U\) is \(zh_1U_{si}\). As \(z_1\) follows symmetry in its last \(n - 2\) coordinates, \(zh_1e_i = zh_1e_{n+1-i}\) for \(i = 2, 3, \ldots, n - 1\). Now,

\[
zh_1U_{s1} = zh_1(e_1 + e_2 + e_{n-1}) = zh_1e_1 + 2zh_1e_2 = (2n - n^2) + 2 \left( \frac{3n - 18 + 8}{2} \right) = 5n - n^2 - 10.
\]

We now claim that \(zh_1U\) follows symmetry in its last \(n - 2\) coordinates. To see this, first show the second and \((n - 1)\)-th coordinates of \(zh_1U\) are equal. Now

\[
zh_1U_{s2} = zh_1(e_1 + e_2 + e_3) = zh_1(e_1 + e_{n-2} + e_{n-1}) = zh_1U_{s(n-1)}, \quad (6.9)
\]

Also,

\[
zh_1U_{s2} = (2n - n^2) + \frac{3n - 18 + 8}{2} - \frac{3n - 18 + 4}{2} = 2n - n^2 + 2.
\]

If we assume \(3 \leq i \leq n - 2\), then

\[
zh_1U_{si} = zh_1(e_{i-1} + e_i + e_{i+1}) = zh_1(e_{n-i+2} + e_{n-i+1} + e_{n-1}) = zh_1U_{s(n-i+1)}, \quad (6.10)
\]

From (6.9) and (6.10), the vector \(zh_1U\) follows symmetry in its last \(n - 2\) coordinates. Therefore, to find \(zh_1U\), it is enough to compute the first \(\frac{n}{2}\) coordinates of \(zh_1U\).

Let \(3 \leq i \leq \frac{n}{2} - 2\). Then we have the following three cases.

**Case (i):** Suppose \(i = 3k\) form some \(k \geq 1\). Then \(1 \leq k \leq \frac{n-4}{6}\). Now

\[
zh_1U_{si} = zh_1(e_{i-1} + e_i + e_{i+1}) = zh_1(e_{3k-1} + e_{3k} + e_{3k+1})
\]
Hence, \( z \) Lemma 6.9. Let \( z \)

\[
\left( \frac{3n - 18k + 8}{2} \right) - \left( \frac{3n - 18k + 4}{2} \right) + 1 = 3.
\]

\textbf{Case(ii):} If \( i = 3k + 1 \), then

\[ z_1' U_{si} = z_1'(e_{3k + 1} + e_{3k + 2} + e_{3k + 3}) + 1 + \left( \frac{3n - 18(k + 1) + 8}{2} \right) = -6. \]

\textbf{Case(iii):} Let \( i = 3k + 2 \). In this case,

\[ z_1' U_{si} = z_1'(e_{3k+1} + e_{3k+2} + e_{3k+3}) = 1 + \left( \frac{3n - 18(k + 1) + 8}{2} \right) = 3. \]

When \( k = \frac{n - 1}{6} \), we have \( c_{3k} = c_{\frac{n}{2} - 2} = c_{\frac{n}{2} - 1} + c_{\frac{n}{2} + 2} \). Since \( n \equiv 1 \mod 3 \) and \( n \) is even, \( \frac{n}{2} \equiv 2 \mod 3 \). If \( i = \frac{n}{2} - 1 \) then

\[ z_1' U_{si} = z_1'(e_{\frac{n}{2} - 2} + e_{\frac{n}{2} - 1} + e_{\frac{n}{2}}) = -\left( \frac{3n - 18(\frac{n - 1}{6}) + 4}{2} \right) + 1 = -6. \]

Finally, \( z_1' U \) gives a circulant matrix defined by a vector. The vector is computed coordinate-wise as in [4].

\[ z_1' U_{si} = \begin{cases} 3 & \text{if } i \not\equiv 1 \mod 3, \\ -6 & \text{if } i \equiv 1 \mod 3. \end{cases} \]

Hence, \( z_1' U = (5n - n^2 - 10, 2n - n^2 + 2, 3, -6, 3, 3, -6, 3, \ldots, 3, -6, 3, 2n - n^2 + 2) = z' \). If \( n \) is odd, we have to calculate \( z_0' U \). Computing the vector \( z_1' U \) is similar to that of \( z_1' U \) and it can be verified that \( z_2' U = z' \). \( \Box \)

\textbf{Remark 6.8.} While deriving the Moore-Penrose inverse of \( D(W_n) \), the matrix \( \hat{L}D(W_n) \) is computed in [6] for some Laplacian-like matrix \( \hat{L} \). Deleting the first row and the first column of \( \hat{L}D(W_n) \) gives a circulant matrix defined by a vector. The vector is computed coordinate-wise where the calculations are lengthy. In our case, by rewriting \( E(W_n) \) as in (2.6), the particular type of vector is found directly rather than by coordinate-wise computation as in [3].

The following lemma is essential for the proof of \textbf{Theorem 6.10}.

\textbf{Lemma 6.9.} Let \( \hat{L} \) be defined as in (6.3). Then \( \hat{L}E = \frac{1}{3} \begin{bmatrix} 1 - n & (7 - n)e' \\ e & \text{Circ}(v') \end{bmatrix} \), where

\[ v' = \frac{1}{n - 1}(17 - 5n, n - 7, n - 7, n + 11, \ldots, n - 7, n - 7, n + 11, n - 7, n - 7) \in \mathbb{R}^{n-1}. \]

\textbf{Proof.} Let \( \mathbf{u}' = (1, 1, 0, 0, \ldots, 0, 1) \in \mathbb{R}^{n-1} \). Then, from (2.6), we write

\[ E(W_n) = \begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} - 2 \begin{bmatrix} 0 & 0' \\ 0 & \text{Circ}(\mathbf{u}') \end{bmatrix}. \]

To find \( \hat{L}E \), we first compute

\[ \hat{L} \begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} = \frac{n - 1}{3} \begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} - \frac{1}{3} \begin{bmatrix} e'e & 2e'J_{n-1} \\ 0 & J_{n-1} \end{bmatrix} + \begin{bmatrix} 0 & 0' \\ Pe & 2PJ_{n-1} \end{bmatrix}. \quad (6.11) \]
Note that
\[
\frac{n-1}{3}e' - \frac{2}{3}e' J_{n-1} = \frac{1}{3}((n-1) - 2(n-1))e' = \frac{1-n}{3}e'.
\tag{6.12}
\]

From Lemma 6.5
\[
\frac{n-1}{3}e + Pe = \frac{1}{3}((n-1) + (2-n))e = \frac{1}{3}e.
\tag{6.13}
\]

As \( PJ_{n-1} = \frac{1}{3}(2-n)J_{n-1} \), we have
\[
\frac{n-1}{3}e' + Pe = \frac{1}{3}\left( (n-1) + (2-n) \right) e = \frac{1}{3}e.
\tag{6.14}
\]

Substituting (6.12), (6.13) and (6.14) in the matrix equation (6.11), we get
\[
\hat{L}\begin{bmatrix} 0 & e' \\ e & 2J_{n-1} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1-n & (1-n)e' \\ e & J_{n-1} \end{bmatrix}.
\tag{6.15}
\]

Let \( U = \text{Circ}(u') \). Now consider
\[
\hat{L}\begin{bmatrix} 0 & 0' \\ 0 & U \end{bmatrix} = \frac{n-1}{3}\begin{bmatrix} 0 & 0' \\ 0 & U \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 0 & e'U \\ 0 & 0_{n-1} \end{bmatrix} + \begin{bmatrix} 0 & 0' \\ 0 & PU \end{bmatrix} = \begin{bmatrix} 0 & -e' \\ 0 & \frac{n-1}{3}U + PU \end{bmatrix}.
\tag{6.16}
\]

Multiplying \( \hat{L} \) and \( E \), and by (6.15) and (6.16), we get
\[
\hat{L}E = \frac{1}{3}\begin{bmatrix} 1-n & (1-n)e' \\ e & J_{n-1} \end{bmatrix} - 2\begin{bmatrix} 0 & -e' \\ 0 & \frac{n-1}{3}U + PU \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1-n & (7-n)e' \\ e & J_{n-1} - 2(1-n)U - 6PU \end{bmatrix}.
\tag{6.17}
\]

Let \( z \) be defined as in Lemma 6.7. Then a direct computation yields that the vector \( e' - 2(n-1)\bar{u}' - \left( \frac{2}{n-1} \right) z' = v' \). By Lemma 6.7 we have
\[
J_{n-1} - 2(n-1)U - 6PU = \text{Circ}\left( e' - 2(n-1)\bar{u}' - \left( \frac{2}{n-1} \right) z' \right) = \text{Circ}(v').
\]

The proof follows by combining the above equation and (6.17).

Now, we are in a position to give a formula for the Moore-Penrose inverse of the eccentricity matrix of the wheel graph, which is expressed in the form of (1.1).

**Theorem 6.10.** Let \( \hat{L} \) be the matrix given in (6.3) and \( w = \frac{1}{6}(7-n, 1, \cdots, 1)' \in \mathbb{R}^n \). Then
\[
E^\dagger = -\frac{1}{2}\hat{L} + \frac{6}{n-1}ww'.
\tag{6.18}
\]

**Proof.** From Lemma 6.9
\[
\hat{L}E = \frac{1}{3}\begin{bmatrix} 1-n & (7-n)e' \\ e & \text{Circ}(v') \end{bmatrix}.
\]
By a simple manipulation, we write

$$\hat{L}E = \frac{1}{3} \begin{bmatrix} 7 - n & (7 - n)e' \\ e & \text{Circ}(\hat{v}') \end{bmatrix} - 2I_n,$$

(6.19)

where

$$\hat{v}' = \frac{1}{n-1} (n+11, n-7, n-7, \ldots, n+11, n-7, n-7).$$

Let

$$X = -\frac{1}{2} \hat{L} + \frac{6}{n-1} \text{ww}' E.$$

To prove that $X$ is the Moore-Penrose generalized inverse of $E$, we first show that $XE$ is symmetric. Now,

$$XE = -\frac{1}{2} \hat{L}E + \frac{6}{n-1} \text{ww}' E.$$

Note that $\text{w'E} = \frac{1}{6}(n-1)e'$. Applying (6.19) in the above equation, we get

$$XE = I_n - \frac{1}{6} \begin{bmatrix} 7 - n & (7 - n)e' \\ e & \text{Circ}(\hat{v}') \end{bmatrix} + \text{we'} = I_n - \frac{1}{6} \begin{bmatrix} 0 & 0' \\ 0 & \text{Circ}(\hat{v}' - e') \end{bmatrix}.$$

(6.20)

Also,

$$\hat{v}' - e' = \frac{6}{n-1} (2, -1, -1, \ldots, 2, -1, -1).$$

Therefore, (6.20) reduces to

$$XE = I_n - \frac{1}{n-1} \begin{bmatrix} 0 & 0' \\ 0 & \text{Circ}(\tilde{v}') \end{bmatrix}, \quad \text{where} \quad \tilde{v}' = (2, -1, -1, \ldots, 2, -1, -1) \in \mathbb{R}^{n-1}. \quad (6.21)$$

Let $V = \text{Circ}(\tilde{v}')$. Then $V$ is a symmetric matrix that readily follows from Lemma 5.7 together with the fact that $\tilde{v}$ follows symmetry in its last $n-2$ coordinates. This gives $XE$ is symmetric. Also, since $X$ and $E$ are symmetric, so is $EX$. To prove $X = E^\dagger$, we need to show $EXE = E$ and $XEX = X$.

From (6.21), we have

$$EXE = E - \frac{1}{n-1} \begin{bmatrix} 0 & e' \\ e & \tilde{E} \end{bmatrix} \begin{bmatrix} 0 & 0' \\ 0 & V \end{bmatrix}.$$

Since $\tilde{E} = \text{Circ}(u')$, we write $\tilde{E}V = \text{Circ}(u'V)$. Therefore,

$$EXE = E - \frac{1}{n-1} \begin{bmatrix} 0 & e'V \\ 0 & \text{Circ}(u'V) \end{bmatrix}.$$
Since $n \equiv 1 \pmod{3}$, we have $e'\mathbf{v} = 0$. This implies that $e'V = 0'$. Also, $u'\mathbf{v} = 0$ and hence $u'V = 0'$. Thus, $XE = E$.

To complete the proof, we claim that $XEX = X$. From (6.21), we get

$$XEX = X - \frac{1}{n-1} \left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] X.$$  (6.22)

So, consider

$$\left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] X = -\frac{1}{2} \left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] \hat{L} + \frac{6}{n-1} \left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] \mathbf{w}\mathbf{w}' .$$  (6.23)

Now,

$$\left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] \hat{L} = \frac{n-1}{3} \left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] - \frac{1}{3} \left[ \begin{array}{cc} 0 & 0' \\
Ve & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0' \\
0 & VP \end{array} \right].$$

Since $P$ and $V$ are circulant, we have $VP = PV$ by (2.1). Also, $\mathbf{v}'e = 0$ and $V = \text{Circ}(\mathbf{v}')$, we get $Ve = 0$. Hence,

$$\left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] \hat{L} = \left[ \begin{array}{cc} 0 & 0' \\
0 & \frac{1}{3}(n-1)V + PV \end{array} \right] = 0,$$  (6.24)

which follows from Lemma 6.6. Also,

$$\left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] \mathbf{w} = \frac{1}{6} \left[ \begin{array}{cc} 0 & 0' \\
0 & Ve \end{array} \right] \left[ \begin{array}{c} \hat{L} - n \\
e \end{array} \right] = \frac{1}{6} \left[ \begin{array}{c} 0 \\
Ve \end{array} \right] = 0. $$

Substituting the above equation and (6.24) in (6.23), it readily follows that $\left[ \begin{array}{cc} 0 & 0' \\
0 & V \end{array} \right] X = 0$. Using this in (6.22), we get $XEX = X$. Thus, $X = E^\dagger$. $\square$

Next, we present some properties of $\hat{L}$ which are similar to that of $\tilde{L}$ given in the previous section. We show that $\hat{L}$ is a Laplacian-like matrix and rank($\hat{L}$) = rank($E^\dagger$) − 1 which is analogous to the result rank($\tilde{L}$) = rank($E^{-1}$) − 1, proved in Theorem 5.13.

**Theorem 6.11.** The matrix $\hat{L}$, defined in (6.3), is a Laplacian-like matrix.

**Proof.** We have

$$\hat{L}e = \frac{1}{3}(n-1)e - \frac{1}{3} \left[ \begin{array}{c} (n-1) \\
e_{(n-1)\times 1} \end{array} \right] + \left[ \begin{array}{c} 0 \\
Pe_{(n-1)\times 1} \end{array} \right].$$

By Lemma 6.5, $Pe_{(n-1)\times 1} = \frac{1}{3}(2-n)e_{(n-1)\times 1}$. Hence, $\hat{L}e = 0$. Since $\hat{L}$ is symmetric, the result follows. $\square$

In Theorem 4.15 we proved that rank($E(W_n)$) is $n - 2$ if $n \equiv 1 \pmod{3}$. Note that rank($E$) = rank($E^\dagger$) (see [8],p.39). From (6.18), it is clear that $E^\dagger$ is expressed as the sum of two matrices, where the rank of the second matrix $\frac{6}{n-2}\mathbf{w}\mathbf{w}'$ is one.

We will prove that rank($\hat{L}$) is $n - 3$. First, in the following lemma, we show that rank($\hat{L}$) is less than or equal to $n - 3$. To prove the result, we recall a property of the Moore-Penrose inverse, which states that $N(A^\dagger) = N(A')$ (see [8],p.63).
Lemma 6.12. The rank of $\hat{L}$ given in (6.3) is at most $n - 3$.

**Proof.** Consider the vectors $x$ and $y$ given in (4.1) and (4.2) respectively. Then, $Ex = 0$ and $Ey = 0$. We now show that $x$ and $y$ belong to $N(\hat{L})$. By the result mentioned above, we have $N(E) = N(E^\dagger)$, because $E$ is symmetric. Therefore, $E^\dagger x = 0$ and $E^\dagger y = 0$. Using Theorem 6.10 we get

$$-\frac{1}{2} \hat{L} x + \frac{6}{n-1} w w' x = 0$$

and

$$-\frac{1}{2} \hat{L} y + \frac{6}{n-1} w w' y = 0.$$ 

Note that $w' x = 0$ and $w' y = 0$. So, we have $\hat{L} x = 0$ and $\hat{L} y = 0$. Moreover, $\hat{L} e = 0$ and $e, x$ and $y$ are linearly independent. Therefore, the dimension of $N(\hat{L})$ is at least three, and hence the rank of $\hat{L}$ is at most $n - 3$.

In order to show that the rank of $\hat{L}$ is equal to $n - 3$, we find matrices $X$ and $C$ of order $n \times (n - 3)$ such that $\hat{L} EX = C$ and rank($C$) = $n - 3$. To define the matrix $C$, we need the three vectors $p, q, r \in \mathbb{R}^{n-3}$ which are given by

$$p' = (-1, -3, 0, 0, -3, 0, 0, \ldots, -3, 0, 0),$$

$$q' = (-1, 0, -3, 0, 0, -3, 0, \ldots, 0, -3, 0),$$

$$r' = (-1, 0, 0, -3, 0, 0, -3, \ldots, 0, 0, -3).$$

In the following lemma, we denote the row vector of size $n$ whose entries are all one by $e_{1 \times n}$.

Lemma 6.13. Let $S = \text{Circ}(s')$, where $s' = (-2, 0, 0, -1, 0, 0, -1, 0, 0, \ldots, -1, 0, 0) \in \mathbb{R}^{n-4}$. Consider the matrix $\hat{L}$ given in (6.3). Let $p, q, r$ be the vectors defined above. If

$$X = \frac{1}{2} \begin{bmatrix} n - 10 & (n - 7) e_{1 \times (n-4)} \\ -e_{(n-4) \times 1} & 3S \\ 0_{3 \times 1} & 0_{3 \times (n-4)} \end{bmatrix} \text{ and } C = \begin{bmatrix} 3I_{n-3} \\ p' \\ q' \\ r' \end{bmatrix},$$

then $\hat{L} EX = C$.

**Proof.** Recall from (6.19),

$$\hat{L} E = \frac{1}{3} \begin{bmatrix} 7 - n & (7 - n) e' \\ e' & \text{Circ}(\tilde{v}') \end{bmatrix} - 2I_n,$$

where $\tilde{v}' = \frac{1}{n-1}(n + 11, n - 7, n - 7, \ldots, n + 11, n - 7, n - 7) \in \mathbb{R}^{n-1}$. To multiply $\hat{L} E$ with $X$, we partition $\hat{L} E$ accordingly. That is,

$$\hat{L} E = \frac{1}{3} \begin{bmatrix} 7 - n & (7 - n) e_{1 \times (n-4)} & (7 - n) e_{1 \times 3} \\ e_{(n-4) \times 1} & \text{Circ}(\tilde{v}'_1) & R \\ e_{3 \times 1} & R' & Q \end{bmatrix} - 2 \begin{bmatrix} 1 & 0' & 0' \\ 0 & I_{n-4} & 0_{(n-4) \times 3} \\ 0 & 0_{3 \times (n-4)} & I_3 \end{bmatrix},$$

where

$$\tilde{v}'_1 = \frac{1}{n-1}(n + 11, n - 7, n - 7, \ldots, n + 11, n - 7, n - 7) \in \mathbb{R}^{n-4},$$

28
\[ R = [\mathbf{v}_1 \ T(\mathbf{v}_1) \ T^2(\mathbf{v}_1)] \text{, and } Q = \frac{1}{n-1} \begin{bmatrix} n+11 & n-7 & n-7 \\ n-7 & n+11 & n-7 \\ n-7 & n-7 & n+11 \end{bmatrix}. \]

Then,
\[
\hat{\text{LEX}} = \frac{1}{6} \begin{bmatrix} -6(7-n) & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} - \begin{bmatrix} n-10 & (n-7)e_{1\times(n-4)} \\ -e_{(n-4)\times1} & 3S \\ 0_{3\times1} & 0_{3\times(n-4)} \end{bmatrix}, \tag{6.25}
\]

where
\[
A_{12} = -(7-n)^2 e_{1\times(n-4)} + 3(7-n)e_{1\times(n-4)}S,
\]
\[
A_{21} = (n-10)e_{(n-4)\times1} - \text{Circ}(\mathbf{v}_1')e_{(n-4)\times1},
\]
\[
A_{22} = (n-7)J_{n-4} + \text{Circ}(\mathbf{v}_1')S,
\]
\[
A_{31} = (n-10)e_{3\times1} - R'e_{(n-4)\times1},
\]
\[
A_{32} = (n-7)J_{3\times(n-4)} + 3R'S.
\]

As \( S \) is a circulant matrix defined by the vector \( \mathbf{s} \), we have \( e_{1\times(n-4)}S = \alpha e_{1\times(n-4)} \), where \( \alpha \) is the sum of the coordinates of \( \mathbf{s} \). Since the non-zero coordinates of \( \mathbf{s} \) are \(-2\), which occurs at exactly one place, and \(-1\), which occurs in \((6.25)\) places,
\[
\alpha = e_{1\times(n-4)}s' = -2 + (-1)\left(\frac{n-7}{3}\right) = \frac{1-n}{3}.
\]

Therefore,
\[
A_{12} = (- (7-n)^2 + (7-n)(1-n)) e_{1\times(n-4)} = 6(n-7)e_{1\times(n-4)}. \tag{6.26}
\]

Now,
\[
\mathbf{v}_1'e_{(n-4)\times1} = \frac{1}{n-1}\left(\left[(n+11) + 2(n-7)\right]\left(\frac{n-4}{3}\right)\right) = n-4,
\]
and hence \( \text{Circ}(\mathbf{v}_1')e_{(n-4)\times1} = (n-4)e_{(n-4)\times1} \). So,
\[
A_{21} = ((n-10) - (n-4))e_{(n-4)\times1} = -6e_{(n-4)\times1}. \tag{6.27}
\]

We have
\[
\mathbf{v}_1's = \frac{1}{n-1}\left((n+11)(-2) + (n+11)(-1)\left(\frac{n-7}{3}\right)\right) = \frac{-1}{3}(n+11), \tag{6.28}
\]
and
\[
(T(\mathbf{v}_1))'s = \frac{1}{n-1}\left((n-7)(-2) + (n-7)(-1)\left(\frac{n-7}{3}\right)\right) = \frac{-1}{3}(n-7) = (T^2(\mathbf{v}_1))'s. \tag{6.29}
\]
We need to compute the vector $\vec{v}_1'S$. Note that $n - 4 \equiv 0 \pmod{3}$. Using Lemma 22.1 to the matrix $S = \text{Circ}(s')$ and the vector $\vec{v}_1$, it is enough to compute the first three coordinates of $\vec{v}_1'S$. From (6.28) and (6.29), we have

$$\vec{v}_1'S = -\frac{1}{3}(n + 11, n - 7, n - 7, \ldots, n + 11, n - 7, n - 7) = \left(\frac{1-n}{3}\right)\vec{v}_1'. \quad (6.30)$$

Similarly, we have

$$(T^i(\vec{v}_1))'S = \left(\frac{1-n}{3}\right)(T^i(\vec{v}_1))', \quad \text{for } 1 \leq i \leq 2. \quad (6.31)$$

Therefore, $\text{Circ}(\vec{v}_1')S = \text{Circ}(\vec{v}_1'S) = \frac{1}{3}(1-n)\text{Circ}(\vec{v}_1')$, and hence

$$A_{22} = (n - 7)J_{n-4} + (1 - n)\text{Circ}(\vec{v}_1'). \quad (6.32)$$

Note that the sum of the entries of $\vec{v}_1, T(\vec{v}_1)$ and $T^2(\vec{v}_1)$ are the same and equal to $(n - 4)$. Therefore, we have $R'e_{(n-4)x1} = [\vec{v}_1 \quad T(\vec{v}_1) \quad T^2(\vec{v}_1)]'e_{(n-4)x1} = (n - 4)e_{3x1}$, and

$$A_{31} = (n - 10)e_{3x1} - R'e_{(n-4)x1} = -6e_{3x1}. \quad (6.33)$$

By (6.30) and (6.31), we have

$$R'S = \left[(\vec{v}_1'S)' \quad ((T(\vec{v}_1))'S)' \quad ((T^2(\vec{v}_1))'S)'ight]' = \left(\frac{1-n}{3}\right)R'.$$

Consequently,

$$A_{32} = (n - 7)J_{3x(n-4)} + 3R'S = (n - 7)J_{3x(n-4)} + (1 - n)R'. \quad (6.34)$$

Substituting (6.30), (6.31), (6.32), (6.33) and (6.34) in (6.25), we get

$$LEX = \begin{bmatrix} 3 & 0_{1x(n-4)} \\ 0_{(n-4)x1} & \frac{1}{6}(n - 7)J_{n-4} + (1 - n)\text{Circ}(\vec{v}_1') - 3S \\ -e_{3x1} & \frac{1}{6}(n - 7)J_{3x(n-4)} + (1 - n)R' \end{bmatrix}. \quad (6.35)$$

Note that

$$\frac{1}{6}(n - 7)e_{1x(n-4)} + (1 - n)\vec{v}_1' = (-3, 0, 0, -3, 0, 0, \ldots, -3, 0, 0),$$

$$\frac{1}{6}(n - 7)e_{1x(n-4)} + (1 - n)(T(\vec{v}_1))' = (0, -3, 0, 0, -3, 0, \ldots, 0, -3, 0),$$

and

$$\frac{1}{6}(n - 7)e_{1x(n-4)} + (1 - n)(T^2(\vec{v}_1))' = (0, 0, -3, 0, 0, -3, \ldots, 0, 0, -3).$$

Hence,

$$\frac{1}{6}(n - 7)e_{1x(n-4)} + (1 - n)\vec{v}_1' = 3s' = 3e_1', \quad (6.36)$$

and

$$\left[-e_{3x1} \quad \frac{1}{6}(n - 7)J_{3x(n-4)} + (1 - n)R'\right]' = [p \quad q \quad r]'. \quad (6.37)$$

Substituting (6.36) and (6.37) in (6.35), it follows that $LEX = C$. \hfill \Box
Theorem 6.14. Let $\hat{L}$ be the matrix given in (6.3). Then the rank of $\hat{L}$ is $n - 3$.

Proof. From Lemma 6.13 we have $\text{rank}(\hat{L}) \geq \text{rank}(C) = n - 3$. The reverse inequality follows from Lemma 6.12. ☐

Concluding remarks

In this article, we studied the eccentricity matrix $E(W_n)$ of the wheel graph $W_n$. We computed the determinant and the inertia of $E(W_n)$. For the non-singular case, we established a formula for the inverse of $E(W_n)$ which was expressed as the sum of a symmetric Laplacian-like matrix and a rank one matrix. A similar result was proved for the Moore-Penrose inverse of $E(W_n)$ for the singular case. It would be interesting to study similar inverse formulae for the eccentricity matrices of other graphs.

References

[1] A. Azimi and R. B. Bapat, The Moore-Penrose inverse of the incidence matrix of complete multipartite and bi-block graphs, Discrete Math. 342 (2019), 2393–2401.

[2] R. Balaji and R. B. Bapat, On Euclidean distance matrices, Linear Algebra Appl. 424 (2007), 108–117.

[3] R. Balaji, R. B. Bapat and S. Goel, An inverse formula for the distance matrix of a wheel graph with an even number of vertices, Linear Algebra Appl. 610 (2021), 274–292.

[4] Balaji R, Bapat RB, Goel S. On distance matrices of wheel graphs with an odd number of vertices. Linear Multilinear Algebra. (2020), 1–32. DOI: 10.1080/03081087.2020.1840499.

[5] R. B. Bapat, Graphs and matrices, Springer, London, 2014.

[6] R.B. Bapat, S. J. Kirkland and M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl. 401 (2005), 193–209.

[7] R. B. Bapat and S. Sivasubramanian, Inverse of the distance matrix of a block graph, Linear Multilinear Algebra 59 (2011), 1393–1397.

[8] A. Ben-Israel and T. N. E. Greville, Generalized inverses, Springer-Verlag, New York, 2003.

[9] A. E. Brouwer and W. H. Haemers, Spectra of graphs, Springer, New York, 2012.

[10] D. Cvetković, P. Rowlinson and S. Simić, An introduction to the theory of graph spectra, Cambridge University Press, Cambridge, 2010.

[11] S. Goel, On distance matrices of helm graphs obtained from wheel graphs with an even number of vertices, Linear Algebra Appl. 621 (2021), 86–104.

[12] R. L. Graham and H. O. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971), 2495–2519.

[13] R. L. Graham and L. Lovász, Distance matrix polynomials of trees, Adv. Math. 29 (1978), 60–88.
[14] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 2013.

[15] G. Jaklič and J. Modic, Euclidean graph distance matrices of generalizations of the star graph, Appl. Math. Comput. 230 (2014), 650–663.

[16] I. Mahato, R. Gurusamy, M. Rajesh Kannan and S. Arockiaraj, Spectra of eccentricity matrices of graphs, Discrete Appl. Math. 285 (2020), 252–260.

[17] I. Mahato, R. Gurusamy, M. Rajesh Kannan and S. Arockiaraj, On the spectral radius and the energy of eccentricity matrix of a graph, Linear Multilinear Algebra. (2021), 1–11. DOI: 10.1080/03081087.2021.2015274.

[18] M. Randić, DMAX—matrix of dominant distances in a graph, MATCH Commun. Math. Comput. Chem. 70 (2013), 221–238.

[19] J. Wang, M. Lu, F. Belardo and M. Randić, The anti-adjacency matrix of a graph: eccentricity matrix, Discrete Appl. Math. 251 (2018), 299–309.

[20] J. Wang, M. Lu, L. Lu and F. Belardo, Spectral properties of the eccentricity matrix of graphs, Discrete Appl. Math. 279 (2020), 168–177.

[21] J. Wang, X. Lei, W. Wei, X. Luo and S. Li, On the eccentricity matrix of graphs and its applications to the boiling point of hydrocarbons. Chemometr Intell Lab. 207 (2020), 104173.

[22] L. You, M. Yang, W. So and W. Xi, On the spectrum of an equitable quotient matrix and its application, Linear Algebra Appl. 577 (2019), 21–40.

[23] X. Zhang and C. Song, The distance matrices of some graphs related to wheel graphs, J. Appl. Math. 2013, Art. ID 707954, 5 pp.

[24] F. Zhang, *Matrix Theory*, Springer, New York, 2011.

[25] H. Zhou, The inverse of the distance matrix of a distance well-defined graph, Linear Algebra Appl. 517 (2017), 11–29.