OPERATOR-VALUED ANALYTIC FUNCTIONS ON THE SYMMETRIZED BIDISK

TIRTHANKAR BHATTACHARYYA AND HARIPADA SAU

Abstract. The symmetrized bidisk
\[ G = \{ (z_1 + z_2, z_1z_2) : |z_1| < 1, |z_2| < 1 \} \]
is a ripe field for complex analysis and operator theory. Operator-valued analytic functions on \( G \) are studied in this paper. We start with a realization formula for such a function of supremum norm less than or equal to 1. We apply the realization formula to prove the Pick-Nevanlinna interpolation result for Hilbert space operators. An application of the method of proof developed in the Realization Theorem proves the Toeplitz corona theorem. The statements of the three major theorems are given in the Introduction.

1. Introduction

Let \( D \) be the open unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \overline{D} \) be its closure \( \{ z \in \mathbb{C} : |z| \leq 1 \} \). The realization formula, in its simplest form, tells us that given a scalar-valued holomorphic function \( f : D \rightarrow D \), there is a Hilbert space \( H \) and a unitary \( V \) on \( \mathbb{C} \oplus H \) so that if we write \( V \) as the \( 2 \times 2 \) block operator matrix
\[ V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
we have
\[ f(\lambda) = A + \lambda B(1 - \lambda D)^{-1} C. \]

This is folklore, see Chapter 6 of [2] for a proof. Agler generalized this to the bidisk. If \( E \) is a Banach space, and \( \Omega \) is a domain in \( \mathbb{C}^n \), let \( H^\infty(\Omega, E) \) denote the Banach space of all bounded analytic \( E \)-valued functions on \( \Omega \). If \( E = \mathbb{C} \), the complex numbers, then we shorten \( H^\infty(\Omega, \mathbb{C}) \) to \( H^\infty(\Omega) \). Thus the realization formula described above is for a function in \( H^\infty(\mathbb{D}) \) with \( \|f\|_\infty := \sup_{\lambda \in \mathbb{D}} |f(\lambda)| \leq 1 \). Let us turn to the case \( \Omega = \mathbb{D}^2 \), the bidisk. Given two Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), the evaluation operator is a map from the bidisk
\[ (\lambda_1, \lambda_2) \mapsto \mathcal{E}_{(\lambda_1, \lambda_2)} \in \mathcal{B}(\mathcal{H}) \]
where \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) and \( \mathcal{E}_{(\lambda_1, \lambda_2)} = \lambda_1 I_{\mathcal{H}_1} \oplus \lambda_2 I_{\mathcal{H}_2} \). In full generality, Agler’s theorem considers two Hilbert spaces \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) and an \( f \in H^\infty(\mathbb{D}^2, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)) \) with
\[ \|f\|_{\infty, \mathbb{D}^2} := \sup_{\mathbb{D}^2} \|f(\lambda_1, \lambda_2)\| \leq 1. \]

Date: February 27, 2017.
MSC2010: 46E22, 47A13, 47A25, 47A56.

Key words and phrases: Symmetrized bidisk, Realization formula, Interpolation, Operator-valued kernel, Reproducing kernel Hilbert space, Toeplitz-corona problem.

This research is supported by University Grants Commission, India via CAS. Work of the second author was largely done at the Indian Institute of Science, Bangalore.
Agler then proved the existence of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a unitary
\[ V : L_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \to L_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \]
satisfying the following. Let us write $V$ in the operator matrix form
\[ V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
(1.3)
with $A \in \mathcal{B}(L_1, L_2), B \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, L_2), C \in \mathcal{B}(L_1, \mathcal{H}_1 \oplus \mathcal{H}_2)$ and $D \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

Then
\[ f(\lambda_1, \lambda_2) = A + B \mathcal{E}_{(\lambda_1, \lambda_2)}(I_H - D \mathcal{E}_{(\lambda_1, \lambda_2)})^{-1}C. \]
For its proof, see Chapter 11 of the book [2]. The proof is very different from the one variable case. This theorem by Agler has spawned a generation of research in multivariable operator theory. Several important results of multivariable operator theory have the bidisk realization theorem at the root of their proofs. We prove such a theorem on the symmetrized bidisk
\[ \mathbb{G} = \{(z_1 + z_2, z_1z_2) : |z_1| < 1, |z_2| < 1 \}. \]
What makes it interesting and challenging is that, instead of two co-ordinate functions, we have uncountably many for $\mathbb{G}$. These are called the parametrized co-ordinate functions, defined below.

A point of $\mathbb{G}$ is often denoted as $(s, p)$ for natural reasons. Given a point $(s, p)$ of $\mathbb{C}^2$, there are many characterizations of when it could belong to $\mathbb{G}$. Among them, none is more interesting than the following due to Agler and Young:
\[ (s, p) \in \mathbb{G} \text{ if and only if } \left| \frac{2ap - s}{2 - as} \right| < 1 \text{ for all } \alpha \in \overline{\mathbb{D}}. \]
This can be found in Theorem 2.1 in [6]. This immediately shows that if we define a function $\varphi$ by
\[ \varphi(\alpha, s, p) = \frac{2ap - s}{2 - as} \]
for all $(\alpha, s, p)$ such that $2 - as \neq 0$, then we get
\[ (s, p) \in \mathbb{G} \text{ if and only if } \sup\{|\varphi(\alpha, s, p)| : \alpha \in \overline{\mathbb{D}}\} < 1. \]
(1.4)
For every $\alpha \in \overline{\mathbb{D}}$, the function $\varphi(\alpha, \cdot)$ is in the norm unit ball of $H^\infty(\mathbb{G})$ and for every $(s, p) \in \mathbb{G}$, the function $\varphi(\cdot, s, p)$ is in the norm unit ball of $C(\overline{\mathbb{D}})$. Sometimes we shall write $\lambda$ for the pair $(s, p)$ and then we shall write $\varphi(\cdot, \lambda)$ for $\varphi(\cdot, s, p)$. We shall call $\varphi(\alpha, \cdot)$ the co-ordinate functions parametrized by $\alpha \in \overline{\mathbb{D}}$.

There are some intermediate steps in the Realization Theorem which are interesting in their own right. To describe them, we need some operator theory. The closed symmetrized bidisk
\[ \{(z_1 + z_2, z_1z_2) : |z_1| \leq 1, |z_2| \leq 1 \} \]
is denoted by $\Gamma$ and a pair $(S, P)$ of commuting bounded operators on a Hilbert space is called a $\Gamma$-contraction if $(S, P)$ has $\Gamma$ as a spectral set, i.e., if
\[ \|f(S, P)\| \leq \|f\|_{\infty, \Gamma} \]
for any polynomial $f$ in two variables, where $\|f\|_{\infty, \Gamma} = \{|f(s, p)| : (s, p) \in \Gamma\}$. Polynomial convexity of $\Gamma$, in conjunction with Oka’s Theorem takes care of the joint spectral subtleties and ensures that the inequality above holds for all functions analytic in a neighbourhood of $\Gamma$. 

A function \( k : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{C} \) that is holomorphic in the first variable and anti-holomorphic in the second, is called a kernel if for any \( n \in \mathbb{N} \), any \( n \) points \( \lambda_1, \lambda_2, \ldots, \lambda_n \) in \( \mathbb{G} \) and any \( n \) scalars \( c_1, c_2, \ldots, c_n \) not all zero, it is true that

\[
\sum_{i,j=1}^{n} c_i \overline{c_j} k(\lambda_i, \lambda_j) > 0.
\]

It is elementary that for every kernel \( k \), there is a Hilbert space \( H_k \) consisting of holomorphic functions on \( \mathbb{G} \) such that the set of functions

\[
\{ \sum_{i=1}^{n} k(\cdot, \lambda_i) : n > 0 \text{ and } \lambda_i \in \mathbb{G} \}
\]

is dense in \( H_k \) and \( f(\lambda) = \langle f, k(\cdot, \lambda) \rangle_{H_k} \) for any \( f \in H_k \) and \( \lambda \in \mathbb{G} \). The operator theory comes into play because a kernel \( k \) on \( \mathbb{G} \) is called admissible if the pair \( (M_s, M_p) \) of multiplication by the co-ordinate functions is a \( \Gamma \)-contraction on \( H_k \).

A \( \mathcal{B}(\mathcal{H}) \) valued kernel is a function \( k : \mathbb{G} \times \mathbb{G} \rightarrow \mathcal{B}(\mathcal{H}) \) which is holomorphic in the first variable, anti-holomorphic in the second and

\[
\sum_{i,j=1}^{n} \langle k(\lambda_i, \lambda_j) h_i, h_j \rangle > 0
\]

for all \( n \geq 1 \), all \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{G} \) and all \( h_1, h_2, \ldots, h_n \in \mathcal{H} \), not all zero. Again, such a kernel gives rise to a Hilbert space of holomorphic functions on \( \mathbb{G} \) in which the set

\[
\{ \sum_{i=1}^{n} k(\cdot, \lambda_i) h_i : n > 0, h_i \in \mathcal{H} \text{ and } \lambda_i \in \mathbb{G} \}
\]

is dense. If the multiplication pair \( (M_s, M_p) \) forms a \( \Gamma \)-contraction on this Hilbert space, we call \( k \) admissible. In the next section, we shall give an equivalent formation of admissibility of \( k \). If a scalar valued (or a \( \mathcal{B}(\mathcal{H}) \) valued) function \( k \) on \( \mathbb{G} \times \mathbb{G} \) does not satisfy \( \sum_{i,j=1}^{n} c_i \overline{c_j} k(\lambda_i, \lambda_j) > 0 \) (respectively \( \sum_{i,j=1}^{n} \langle k(\lambda_i, \lambda_j) h_i, h_j \rangle > 0 \)) but instead satisfies \( \sum_{i,j=1}^{n} c_i \overline{c_j} k(\lambda_i, \lambda_j) \geq 0 \) (respectively \( \sum_{i,j=1}^{n} \langle k(\lambda_i, \lambda_j) h_i, h_j \rangle \geq 0 \)) for all \( n \geq 1 \), all \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{G} \) and \( n \) scalars \( c_1, c_2, \ldots, c_n \) (respectively \( h_1, h_2, \ldots, h_n \) in \( \mathcal{H} \)), not all zero, then such a \( k \) will be called a weak kernel.

We shall need one more level of generalization of the concept of kernels. For two \( C^* \)-algebras \( \mathcal{A} \) and \( \mathcal{C} \), a function \( \Delta : \mathbb{G} \times \mathbb{G} \rightarrow \mathcal{B}(\mathcal{A}, \mathcal{C}) \) is called a completely positive function if

\[
\sum_{i,j=1}^{N} c_i^* \Delta(\lambda_i, \lambda_j)(a_i^* a_j)c_j
\]

is a non-negative element of \( \mathcal{C} \) for any positive integer \( N \), any \( n \) points \( \lambda_1, \lambda_2, \ldots, \lambda_N \) of \( \mathbb{G} \), any \( N \) elements \( a_1, a_2, \ldots, a_N \) from \( \mathcal{A} \) and any \( N \) elements \( c_1, c_2, \ldots, c_N \) from \( \mathcal{C} \). More details on these functions are found in [9].

When we use the word kernel or the phrase weak kernel, holomorphicity in the first component and anti-holomorphicity in the second component are built in whereas when we use the word function, as in a completely positive function, no such holomorphicity is implied.

Finally, given two functions \( k_1, k_2 : \mathbb{G} \times \mathbb{G} \rightarrow B(\mathcal{L}) \), the notation \( k_1 \otimes k_2 \) stands for the \( B(\mathcal{L} \otimes \mathcal{L}) \)-valued function on \( \mathbb{G} \times \mathbb{G} \) by

\[
k_1 \otimes k_2((s, p), (t, q)) = k_1((s, p), (t, q)) \otimes k_2((s, p), (t, q))
\]
for all \((s, p), (t, q)\) in \(G\).

**Operator-valued Realization Theorem.** Let \(L_1\) and \(L_2\) be two Hilbert spaces, \(Y\) be any subset of \(G\) and \(f : Y \rightarrow B(L_1, L_2)\) be any function. Then the following statements are equivalent.

- **H:** There exists a function \(F\) in \(H^\infty(G, B(L_1, L_2))\) with \(\|F\|_\infty \leq 1\) and \(F|_Y = f\);
- **M:** \((I_{L_2} - f(s, p)f(t, q)^*) \otimes k((s, p), (t, q))\) is a weak kernel for every \(B(L_2)\)-valued admissible kernel \(k\) on \(Y\);
- **D:** There is a completely positive function \(\Delta : Y \times Y \rightarrow B(C(\overline{D}), B(L_2))\) such that for every \((s, p)\) and \((t, q)\) in \(Y\),
  \[I_{L_2} - f(s, p)f(t, q)^* = \Delta((s, p), (t, q))(1 - \varphi(\cdot, s, p)\varphi(\cdot, t, q));\]
- **R:** There is a Hilbert space \(H\), a unital *-representation \(\pi : C(\overline{D}) \rightarrow B(H)\) and a unitary \(V : L_1 \oplus H \rightarrow L_2 \oplus H\) such that writing \(V\) as
  \[
  V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
  \]
  we have \(f(s, p) = A + B\pi(\varphi(\cdot, s, p))(I_H - D\pi(\varphi(\cdot, s, p)))^{-1}C\), for every \((s, p)\) in \(Y\).

As an application of the Realization Theorem, we obtain the operator-valued Pick-Nevanlinna interpolation in the space of bounded analytic functions on \(G\). The celebrated Pick-Nevanlinna interpolation, now studied for a hundred years, characterizes the data \(\lambda_1, \lambda_2, \ldots, \lambda_N\) in \(D\) and \(w_1, w_2, \ldots, w_N\) in \(\overline{D}\) in terms of existence of a function \(f \in H^\infty(D)\) interpolating the data. There is an \(f \in H^\infty(D)\) with \(\|f\|_\infty \leq 1\) and \(f(\lambda_i) = w_i\), \(i = 1, 2, \ldots, N\) if and only if
\[
\left(\frac{1 - w_i\overline{w}_j}{1 - \lambda_i\lambda_j}\right)_{i,j=1}^N
\]
is a positive semi-definite matrix. We now state the Interpolation Theorem.

**Operator-valued Interpolation Theorem.** Let \(L_1\) and \(L_2\) be Hilbert spaces and \(W_1, W_2, \ldots, W_N \in B(L_1, L_2)\). Let \((s_1, p_1), (s_2, p_2), \ldots, (s_N, p_N)\) be \(N\) distinct points in \(G\). Then there exists a function \(f\) in the closed unit ball of \(H^\infty(G, B(L_1, L_2))\) interpolating each \((s_i, p_i)\) to \(W_i\) if and only if
\[
(I_{L_2} - W_iW_j^*) \otimes k((s_i, p_i), (s_j, p_j))_{i,j=1}^N
\]
is a positive semi-definite matrix, for every \(B(L_2)\)-valued admissible kernel \(k\) on \(G\).

A judicious use of the method of proof of the Realization Theorem gives us the Toeplitz corona theorem. For the uninitiated, we briefly recall the central issues around the Corona Theorem. Obviously, \(D\) is contained in the maximal ideal space \(M_{H^\infty(D)}\) of the Banach algebra \(H^\infty(D)\) by means of identification of a \(w \in D\) with the multiplicative linear functional of evaluation, \(f \rightarrow f(w)\) for all \(f \in H^\infty(D)\). It is usually difficult to find the maximal ideal space of a Banach algebra. Kakutani asked whether the corona \(M_{H^\infty(D)} \setminus \overline{D}\) (in the weak-star topology) is empty or in other words, whether \(D\) is dense in \(M_{H^\infty(D)}\) in the natural weak-star topology. Elementary functional analysis shows that Kakutani’s question is equivalent to the following:

Given \(\varphi_1, \varphi_2, \ldots, \varphi_N\) in \(H^\infty(D)\) satisfying
\[
|\varphi_1(z)|^2 + |\varphi_2(z)|^2 + \cdots + |\varphi_N(z)|^2 \geq \delta^2 > 0 \text{ for all } z \in D,
\]
for some \( \delta > 0 \), is it true that there are functions \( \psi_1, \psi_2, \ldots, \psi_N \) in \( H^\infty(\mathbb{D}) \) such that

\[
(1.7) \quad \psi_1 \varphi_1 + \psi_2 \varphi_2 + \cdots + \psi_N \varphi_N = 1?
\]

Of course if it were true, then the ideal generated by \( \varphi_2, \ldots, \varphi_N \) would be full \( H^\infty(\mathbb{D}) \). It is easy to see that the converse implication is true, so that (1.6) is a necessary condition for (1.7). The sufficiency was proved, and hence Kakutani’s question was answered affirmatively by Carleson [13]. This triggered a rather long list of research work on issues related to the corona theorem. First, Hörmander introduced a different approach based on an appropriate inhomogeneous \( \bar{\partial} \)-equation, see [16] and references therein for a beautiful discussion and various results in this direction. Then Wolff produced a simple proof, see [16] for Wolff’s solution. Coburn and Schechter in [10] and Arveson in [8] came up with a condition similar to (1.6) but different from (1.6):

\[
(1.8) \quad M_{\varphi_1} M_{\varphi_2} + M_{\varphi_2} M_{\varphi_2} + \cdots + M_{\varphi_N} M_{\varphi_N} \geq \delta^2 > 0.
\]

Coburn and Schechter were interested in interpolation (in other words when an ideal in a Banach algebra contains the identity) and Arveson started out searching for an operator theoretic proof of the corona theorem. The notation \( M_\varphi \), for a \( \varphi \in H^\infty(\mathbb{D}) \) stands for the multiplication operator on \( H^2(\mathbb{D}) \) and is also called the Toeplitz operator with symbol \( \varphi \). Using the Szegö kernel, it is elementary to see that (1.8) implies (1.6). Both papers mentioned above proved that (1.8) implies (1.7). Arveson achieved a bound: if \( \|\psi_i\|_\infty \leq 1 \) for each \( 1 \leq i \leq N \), then \( \psi_1, \psi_2, \ldots, \psi_N \) can be so chosen that

\[
\|\psi_i\|_\infty \leq 4 Ne^{-3}.
\]

Using the corona theorem, one can show that (1.7) implies (1.8). Thus, in the disk, all three statements are equivalent. In a general domain, equivalence of (1.7) and (1.8) is called the Toeplitz Corona Theorem. Agler and McCarthy proved the Toeplitz corona theorem for the bidisk in [3]. In the last section of this paper, we shall prove the following theorem.

**Toeplitz Corona Theorem on the Symmetrized Bidisk.** Let \( \varphi_1, \varphi_2, \ldots, \varphi_N \) be functions in \( H^\infty(\mathbb{G}) \) and \( \delta > 0 \). The following statements are equivalent.

1. There are functions \( \psi_1, \psi_2, \ldots, \psi_N \) in \( H^\infty(\mathbb{G}) \) such that

\[
\psi_1 \varphi_1 + \psi_2 \varphi_2 + \cdots + \psi_N \varphi_N = 1
\]

and

\[
\sup_{(s,p) \in \mathbb{G}} \left( |\psi_1(s,p)|^2 + |\psi_2(s,p)|^2 + \cdots + |\psi_N(s,p)|^2 \right) \leq \frac{1}{\delta};
\]

2. For any admissible kernel \( k \) on \( \mathbb{G} \), the multiplication tuple \( (M_{\varphi_1}, M_{\varphi_2}, \ldots, M_{\varphi_N}) \) on the reproducing kernel Hilbert space \( H_k \) is bounded below by \( \delta \), i.e.,

\[
M_{\varphi_1} M_{\varphi_2} + M_{\varphi_2} M_{\varphi_2} + \cdots + M_{\varphi_N} M_{\varphi_N} \geq \delta.
\]

2. **Preliminaries**

In this section, we collect some preliminary concepts, notations and results that will be used in the rest of the paper. The vector-valued Hardy space of \( \mathbb{G} \) provides us a natural example of an admissible kernel. We define it in this section. Let \( \mathcal{L} \) be a Hilbert space. Let us denote by \( H^2_\mathcal{L}(\mathbb{G}) \) the class of functions holomorphic on \( \mathbb{G} \) with values in \( \mathcal{L} \) such that

\[
\sup_{0 < r < 1} \int_{\mathbb{T} \times \mathbb{T}} \| f \circ \pi(r e^{i\theta_1}, r e^{i\theta_2}) \|^2 \| J(r e^{i\theta_1}, r e^{i\theta_2}) \|^2 d\theta_1 d\theta_2 < \infty,
\]
where  \( J_{\pi}(z_1, z_2) = z_1 - z_2 \), i.e., the complex Jacobian of the map
\[
\pi(z_1, z_2) = (z_1 + z_2, z_1z_2)
\]
and \( d\theta \) is the normalized Lebesgue measure on the unit circle \( T \). The norm of a function \( f \) in \( H^2_s(\mathbb{G}) \) is defined to be
\[
\| f \| = \| J \|^{-1} \left( \sup_{0 < r < 1} \int_{T \times T} \| f \circ \pi(r e^{i\theta_1}, r e^{i\theta_2}) \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \int_{T \times T} \| J_{\pi}(r e^{i\theta_1}, r e^{i\theta_2}) \|_{L^2}^2 \right)^{\frac{1}{2}}
\]
Multiplication by the constant \( \| J \|^{-1} \) is to make sure that the constant functions \( f(s, p) = \text{constant} \) for all \((s, p) \in \mathbb{G}\) have norm \( \| l \| \) and hence we can embed \( \mathcal{L} \) into \( H^2_s(\mathbb{G}) \) isometrically.

When \( \mathcal{L} = \mathbb{C} \), the corresponding \( H^2_s(\mathbb{G}) \) is the scalar-valued Hardy space of the symmetrized bidisk, denoted by \( H^2(\mathbb{G}) \). Note that the space \( H^2_s(\mathbb{G}) \) is isometrically isomorphic to \( H^2(\mathbb{G}) \otimes \mathcal{L} \), via the unitary
\[
f(\cdot) \otimes v \mapsto f(\cdot)v
\]
for all \( f \in H^2(\mathbb{G}) \) and \( v \in \mathcal{L} \). It was observed in [13] that \( H^2(\mathbb{G}) \) is a reproducing kernel Hilbert space with the kernel
\[
k^s((s, p), (t, q)) = \frac{1}{(1 - pq)^2 - (s - tp)(t - sq)}
\]
called the Szego kernel of the symmetrized bidisk. It was also proved in [13] (Lemma 2.5) that the Szego kernel is an admissible kernel. Hence, it is clear that if we define the operator-valued function \( K^s : \mathbb{G} \times \mathbb{G} \rightarrow \mathcal{B}(\mathcal{L}) \) by
\[
K^s((s, p), (t, q)) = \frac{I_{\mathcal{L}}}{(1 - pq)^2 - (s - tp)(t - sq)},
\]
then \( H^2_s(\mathbb{G}) \) is the reproducing kernel Hilbert space with the admissible kernel \( K^s \). In other words, the pair of multiplication operator by the co-ordinate functions \((M_s, M_p)\) is a \( \Gamma \)-contraction on \( H^2_s(\mathbb{G}) \).

In the rest of this section, we give a natural example of an operator-valued kernel and a large class of completely positive functions on the symmetrized bidisk.

For two vectors \( u \) and \( v \) in a Hilbert space \( \mathcal{L} \), the following rank one operator, often called a \emph{dyad}, \([u \otimes v] : \mathcal{L} \rightarrow \mathcal{L} \) is defined by
\[
[u \otimes v](w) = \langle w, v \rangle u, \text{ for all } w \in \mathcal{L}.
\]
Elementary computations yield that
\begin{enumerate}
\item \([u \otimes v] = [v \otimes u]^*\),
\item \(A[u \otimes v] = [Au \otimes v]\) and \([u \otimes v]A = [u \otimes A^*v]\) for any \( A \) in \( \mathcal{B}(\mathcal{L}) \) and
\item \(\text{tr}[u \otimes v] = \langle u, v \rangle\).
\end{enumerate}

We shall make use of the following example of an operator-valued kernel while establishing the realization formula in the next section.

\textbf{Example 1.} Let \( u : \mathbb{G} \rightarrow \mathcal{L} \) be any function. Define the function \( d : \mathbb{G} \times \mathbb{G} \rightarrow \mathcal{B}(\mathcal{L}) \) by
\[
d((s, p), (t, q)) = [u(s, p) \otimes u(t, q)]
\]
It can be checked that \( d \) is a positive semi-definite function.
Example 2. Let $\delta : \overline{D} \times G \times G \rightarrow B(\mathcal{L})$ be any function such that for each $\alpha \in \overline{D}$, the function $\delta(\alpha, \cdot, \cdot)$ is a positive semi-definite function on $G \times G$. Also suppose $\mu$ is a positive regular Borel measure on $\overline{D}$. Define $\Delta_\mu^\delta : G \times G \rightarrow B(C(\overline{D}), B(\mathcal{L}))$ by

$$
\Delta_\mu^\delta((s, p), (t, q))(h) = \int_{\overline{D}} h(\cdot)\delta(\cdot, (s, p), (t, q))d\mu.
$$

It can be checked by a straightforward computation that $\Delta$ is a completely positive function on $G$.

Two particular examples of $\delta$ are given below. Define $b : \overline{D} \times G \times G \rightarrow B(\mathcal{L})$ by

$$
b(\alpha, (s, p), (t, q)) = \frac{I_\mathcal{L}}{1 - \varphi(\alpha, s, p)\varphi(\alpha, t, q)}.
$$

For every $\alpha \in \overline{D}$, that $b(\alpha, \cdot, \cdot)$ is a weak kernel on $G \times G$ follows from the fact that it is the composition of the Szego kernel

$$(z, w) \rightarrow \frac{I_\mathcal{L}}{1 - zw}$$

with the map $((s, p), (t, q)) \mapsto (\varphi(\alpha, s, p), \varphi(\alpha, t, q))$ from $G \times G$ into $\overline{D} \times \overline{D}$.

Define $d : \overline{D} \times G \times G \rightarrow B(\mathcal{L})$ by

$$
d(\alpha, (s, p), (t, q)) = \frac{[u(s, p) \otimes u(t, q)]}{1 - \varphi(\alpha, s, p)\varphi(\alpha, t, q)}.
$$

It follows from a simple computation that for every $\alpha \in \overline{D}$, $d(\alpha, \cdot)$ is a positive semi-definite function.

We end this section with an equivalent formation of admissibility of an operator-valued kernel on $G$.

Lemma 3. A $B(\mathcal{H})$-valued kernel $k$ on $G$ is admissible if and only if

$$(1 - \varphi(\alpha, s, p)\varphi(\alpha, t, q))k((s, p), (t, q))$$

is a $B(\mathcal{H})$-valued weak kernel for every $\alpha \in \overline{D}$.

Proof. Suppose $k$ is a $B(\mathcal{H})$-valued admissible kernel on $G$, which means by definition that the pair $(M_\alpha, M_\beta)$ is a $\Gamma$-contraction on $H_k$, which by part (v) of Theorem 1.5 of [2], is equivalent to the operator $(2\alpha M_\beta - M_\alpha)(2 - \alpha M_\beta)^{-1}$ being a contraction on $H_k$ for every $\alpha \in \overline{D}$. This is equivalent to the following $B(\mathcal{H})$-valued function

$$(1 - \varphi(\alpha, s, p)\varphi(\alpha, t, q))k((s, p), (t, q))$$

being a weak kernel, for every $\alpha \in \overline{D}$. To obtain the last equivalence, we used the fact that $M_\beta^*k(\cdot, (t, q))v = \tilde{\psi}(t, q)k(\cdot, (t, q))v$, for every $v \in \mathcal{H}$ and $\psi \in H^\infty(G)$. Now the proof follows as the function $\varphi$ is continuous on the circle. \qed

3. The realization formula - operator case

We prove the Realization Theorem here. Our strategy is to first show $(\mathcal{H}) \Rightarrow (D) \Rightarrow (R) \Rightarrow (H)$ and then $(H) \Leftrightarrow (M)$. The crux of the matter is handled with an approximation result because the full $H^\infty$ functional calculus for a $\Gamma$-contraction is not available. This result plays a major role and hence we call it a theorem.

Given a function $J : G \times G \rightarrow B(\mathcal{L})$, define $J_r$ for all $0 < r < 1$, by

$$
J_r((s, p), (t, q)) = J((rs, r^2p), (rt, r^2q))
$$
for every \((s, p)\) and \((t, q)\) in \(G\).

**Theorem 4.** Let \(J : G \times G \to \mathcal{B}(\mathcal{L})\) be a continuous self-adjoint function. If

\[
J_r \otimes k : ((s, p), (t, q)) \mapsto J((rs, r^2p), (rt, r^2q)) \otimes k((s, p), (t, q))
\]

is positive semi-definite for every \(\mathcal{B}(\mathcal{L})\)-valued admissible weak kernel \(k\) and every \(0 < r < 1\), then there exists a completely positive function \(\Delta : G \times G \to \mathcal{B}(C(\overline{D}), \mathcal{B}(\mathcal{L}))\) such that for every \((s, p), (t, q)\) in \(G\),

\[
J((s, p), (t, q)) = \Delta((s, p), (t, q)) \left(1 - \varphi(\cdot, s, p)\varphi(\cdot, t, q)\right).
\]

**Proof.** We first prove it for finite subsets of the form \(Y = \{(s_i, p_i) : 1 \leq i \leq N\}\) of \(G\) and then apply Kurosh’s theorem. Consider the following subset of \(N \times N\) self-adjoint operator matrices with entries in \(\mathcal{B}(\mathcal{L})\),

\[
\mathcal{W}_Y = \{\Delta((s_i, p_i), (s_j, p_j)) \left(1 - \varphi(\cdot, s_i, p_i)\varphi(\cdot, s_j, p_j)\right) : \Delta : G \times G \to \mathcal{B}(C(\overline{D}), \mathcal{B}(\mathcal{L}))\text{ completely positive function}\}.
\]

The subset \(\mathcal{W}_Y\) of \(\mathcal{B}(\mathcal{L}^N)\) is a wedge in the vector space of \(N \times N\) self-adjoint matrices with entries from \(\mathcal{B}(\mathcal{L})\) in the sense that it is convex and if we multiply a member of \(\mathcal{W}_Y\) by a non-negative real number, then the element remains in \(\mathcal{W}_Y\). Since \(\mathcal{B}(\mathcal{L}^N)\) is the dual of \(\mathcal{B}_1(\mathcal{L}^N)\), the ideal of trace class operators acting on \(\mathcal{L}^N\), it has its natural weak-star topology. We shall show that it is closed. This will require some work and hence we separate it out as a lemma. We shall pick up the proof of the theorem after we prove the lemma.

**Lemma 5.** \(\mathcal{W}_Y\) is closed.

**Proof.** To prove that \(\mathcal{W}_Y\) is weak-star closed, let

\[
K_\nu((s_i, p_i), (s_j, p_j)) = \Delta_\nu((s_i, p_i), (s_j, p_j)) \left(1 - \varphi(\cdot, s_i, p_i)\varphi(\cdot, s_j, p_j)\right)
\]

be a net in \(\mathcal{W}_Y\) which is indexed by \(\nu\) in some index set and which converges to an \(N \times N\) self-adjoint \(\mathcal{B}(\mathcal{L})\)-valued matrix \(K = (K_{ij})\) with respect to the weak-star topology. This means that for every \(X = (X_{ij}) \in \mathcal{B}_1(\mathcal{L}^N)\), the net of scalars \(\text{tr}(K_\nu X)\) converges to \(\text{tr}(K X)\). Let us use a special \(X\). Consider two vectors \(u\) and \(v\) in \(\mathcal{L}\) and choose \(X\) to be the that operator matrix which has \(u \otimes v\) as the \((ji)\)-th entry and zeroes elsewhere. Then we get

\[
\langle \Delta_\nu((s_i, p_i), (s_j, p_j)) \left(1 - \varphi(\cdot, s_i, p_i)\varphi(\cdot, s_j, p_j)\right) u, v \rangle \to \langle K_{ij} u, v \rangle
\]

for all \(i = 1, 2, \ldots, N\) and all \(j = 1, 2, \ldots, N\). In particular, we have

\[
\langle \Delta_\nu((s_i, p_i), (s_i, p_i)) \left(1 - |\varphi(\cdot, s_i, p_i)|^2\right) u, u \rangle \to \langle K_{ii} u, u \rangle
\]

for all \(u \in \mathcal{L}\) and \(1 \leq i \leq N\). Let us recall that for any \((s, p) \in G\), \(\sup\{|\varphi(\alpha, s, p)| : \alpha \in \overline{D}\} < 1\), so that we have an \(\epsilon > 0\) satisfying \(1 - |\varphi(\cdot, s_i, p_i)|^2 \geq \epsilon\) for each \(1 \leq i \leq N\). Hence for each \(1 \leq i \leq N\),

\[
\langle \Delta_\nu((s_i, p_i), (s_i, p_i)) \left(1 - |\varphi(\cdot, s_i, p_i)|^2\right) u, u \rangle \geq \epsilon \langle \Delta_n((s_i, p_i), (s_i, p_i))(1) u, u \rangle.
\]

Since the left hand side of the above inequality converges, we have an \(M > 0\) such that

\[
\sup_\nu \langle \Delta_\nu((s_i, p_i), (s_i, p_i))(1) u, u \rangle < M.
\]
Since we have \(\|h\|_\infty^2 - |h(\cdot)|^2 \geq 0\) for every \(h \in C(\overline{D})\), hence
\[
\langle \Delta_\nu \left((s_i, p_i), (s_j, p_j)\right)(|h|^2)u, u \rangle \leq \|h\|_\infty^2 \langle \Delta_\nu \left((s_i, p_i), (s_j, p_j)\right)(1)u, u \rangle \leq \|h\|_\infty^2 M.
\]
For a completely positive function \(\Delta\), we have for every \(h_1, h_2 \in C(\overline{D})\) and \(u, v \in \mathcal{L}\),
\[
|\langle \Delta \left((s, p), (t, q)\right)(h_1 h_2)u, v \rangle| \leq \langle \Delta \left((s, p), (s, p)\right)(|h_1|^2)u, u \rangle \langle \Delta \left((t, q), (t, q)\right)(|h_2|^2)v, v \rangle,
\]
which immediately gives a bound on off-diagonal entries
\[
|\langle \Delta_\nu \left((s_i, p_i), (s_j, p_j)\right)(|h|^2)u, v \rangle| \leq \|h\|_\infty^2 M^2,
\]
for every \(h \in C(\overline{D})\), all \(u, v \in \mathcal{L}\) and every \(\nu\). Therefore, for every \(h \in C(\overline{D})\) and \(u, v \in \mathcal{L}\),
the net \(\{\langle \Delta_\nu \left((s_i, p_i), (s_j, p_j)\right)(|h|^2)u, v \rangle\}\) is bounded, for each \(1 \leq i, j \leq N\). Since the set \(Y\) is finite, we get a subnet \(\nu_\ell\) such that \(\{\langle \Delta_\nu \left((s_i, p_i), (s_j, p_j)\right)(|h|^2)u, v \rangle\}\) converges to some complex number depending on \(h, u\) and \(v\). Now we define a completely positive function \(\Delta : Y \times Y \to \mathcal{B}(C(\overline{D}), \mathcal{B}(\mathcal{L}))\) by
\[
\langle \Delta \left((s_i, p_i), (s_j, p_j)\right)h(u, v) = \lim_{\nu} \langle \Delta_\nu \left((s_i, p_i), (s_j, p_j)\right)(h)u, v \rangle
\]
and extend it trivially to whole \(\mathbb{G} \times \mathbb{G}\). Consequently, for every \(h \in C(\overline{D})\) and \(u, v \in \mathcal{L}\),
\[
\langle \Delta \left((s_i, p_i), (s_j, p_j)\right)(h)u, v \rangle = \langle K_{ij}u, v \rangle \quad \text{for each } 1 \leq i, j \leq N
\]
proving that \(\mathcal{W}_Y\) is weak-star closed and hence operator norm closed too. \(\square\)

Continuing the proof of the theorem, let \(b \) and \(d \) be as in (2.5) and (2.6) respectively and let \(\mu\) be a probability measure on \(\overline{D}\). Then \(\Delta^\mu\) as defined in (2.4) satisfies
\[
\Delta^\mu \left((s, p), (t, q)\right)(\varphi(\alpha, s, p)\overline{\varphi(\alpha, t, q)}) = I_{\mathcal{L}} \quad \text{for every } (s, p), (t, q) \in \mathbb{G}
\]
Thus the \(N \times N\) matrix with each entry \(I_{\mathcal{L}}\) is in \(\mathcal{W}_Y\). Changing \(b\) to \(d\), we get
\[
\Delta^d \left((s, p), (t, q)\right)(1 - \varphi(\alpha, s, p)\overline{\varphi(\alpha, t, q)}) = u(s, p) \otimes u(t, q) \quad \text{for every } (s, p), (t, q) \in \mathbb{G}
\]
and hence if \(u_1, u_2, \ldots, u_N\) are any vectors in \(\mathcal{L}\), then the \(N \times N\) matrix
\[
D(i, j) = u_i \otimes u_j \quad \text{for each } 1 \leq i, j \leq N
\]
is in \(\mathcal{W}_Y\).

We now show that the restriction \(j\) of \(J\) to \(Y \times Y\) is in \(\mathcal{W}_Y\). Suppose on the contrary that \(j\) is not in \(\mathcal{W}_Y\). Then, applying part (b) of Theorem 3.4 in [22], we get a weak-star continuous linear functional \(L\) on \(\mathcal{B}(\mathcal{L}^2)\), whose real part is non-negative on \(\mathcal{W}_Y\) and strictly negative on \(j\). We replace this linear functional by its real part, i.e., \(\frac{1}{2}(L(T) + \overline{L(T)})\) and denote it by \(L\) itself. Since \(L\) is weak-star continuous, by Theorem 1.3 in Chapter V of [15] we get \(L\) to be of the form
\[
L(T) = tr(TK)
\]
for some \(N \times N\) self-adjoint operator matrix \(K\) with entries in the ideal of trace class operators. Let us define \(K^t\) by \(K^t(e_m, e_n) = K(e_n, e_m)^t\). Let \(\{e_n : n \in \mathbb{N}\}\) be an orthonormal basis for \(\mathcal{L}\). For \(u = \sum c_m e_m\) and \(v = \sum d_n e_n\) in \(\mathcal{L}\), we make a note of the following fact about \(K^t\), which will be used later in the proof.
\[
\langle K^t(e_m, e_n)u, v \rangle = \sum_{m,n} c_m \overline{d_n} \langle K(e_n, e_m)^t e_m, e_n \rangle = \sum_{m,n} c_m \overline{d_n} \langle K(e_n, e_m) e_n, e_m \rangle = \langle K(e_n, e_m) \overline{u}, \overline{v} \rangle,
\]
where $\bar{u} = \sum c_m e_m$ and $\bar{v} = \sum \tilde{d}_n e_n$.

It is simple to show that $K^t$ is a $\mathcal{B}(\mathcal{L})$-valued positive semi-definite kernel on $Y$, i.e.,

\begin{equation}
\sum_{i,j=1}^{N} \langle K^t(\lambda_i, \lambda_j) u_j, u_i \rangle \geq 0,
\end{equation}

where $u_1, u_2, \ldots, u_N$ are arbitrary vectors in $\mathcal{L}$. The following shows that (3.2) is the action of $L$ on the kernel $D(i, j) = [\bar{u}_i \otimes \bar{u}_j]$ and hence we are done.

\[0 \leq L(D) = \text{tr}(DK) = \sum_{i,j=1}^{N} \text{tr}(D_{ij}K_{ji}) = \sum_{i,j=1}^{N} \langle \bar{u}_i, K_{ji}^* \bar{u}_j \rangle = \sum_{i,j=1}^{N} K_{ji} \bar{u}_i, \bar{u}_j = \sum_{i,j=1}^{N} \langle K^t(\lambda_i, \lambda_j) u_j, u_i \rangle.\]

The next step is to show that $K^t$ is admissible. This is a matter of choosing the completely positive function judiciously. Note that for each $\alpha \in \overline{\mathbb{D}}$ and a function $u : G \rightarrow \mathcal{L}$, the function $\Delta^\alpha : \mathbb{G} \times \mathbb{G} \rightarrow \mathcal{B}(C(\mathbb{D}), \mathcal{B}(\mathcal{L}))$ defined by

\[\Delta^\alpha((s, p), (t, q))(h) = h(\alpha)[u(s, p) \otimes u(t, q)]\]

is completely positive. This implies that for each $\alpha \in \overline{\mathbb{D}}$ and vectors $u_1, u_2, \ldots, u_N$ in $\mathcal{L}$, the following $\mathcal{B}(\mathcal{L})$-valued $N \times N$ matrix

\[A(\alpha) = \left(\left(1 - \varphi(\alpha, s_i, p_i)\varphi^*(\alpha, s_j, p_j)[u_i \otimes u_j]\right)\right)_{i,j=1}^{N}\]

is in $\mathcal{W}_Y$. The fact that $L$ is non-negative on $\mathcal{W}_Y$ shows that $K^t$ is admissible.

Therefore by assumption, the $\mathcal{B}(\mathcal{L} \otimes \mathcal{L})$-valued function $j_r \otimes K^t$ on $Y \times Y$ is positive semi-definite, which means that for every choice of vectors $\{u_i\}_{i=1}^{N}$ in $\mathcal{L} \otimes \mathcal{L}$, we have

\begin{equation}
\sum_{i,j=1}^{N} \langle j_r \otimes K^t(\lambda_i, \lambda_j) u_j, u_i \rangle \geq 0.
\end{equation}

For a finite subset $F = \{1, 2, \ldots, R\}$ of $\mathbb{N}$, choose $u_i = \sum_{m=1}^{R} e_m \otimes e_m$ for each $i$. Note that for this choice of $u_i$, (3.3) is same as

\begin{equation}
\sum_{i,j=1}^{N} \sum_{m,n=1}^{R} \langle j_r(\lambda_i, \lambda_j)e_m, e_n \rangle \langle K^t(\lambda_i, \lambda_j)e_m, e_n \rangle \geq 0.
\end{equation}
On the other hand, for every $0 < r < 1$, we have

\[ L(j_r) = \sum_{i,j=1}^{N} \text{tr}(j_r(\lambda_i, \lambda_j)K(\lambda_j, \lambda_i)) \]
\[ = \sum_{i,j=1}^{N} \sum_{n=1}^{\infty} \langle j_r(\lambda_i, \lambda_j)K(\lambda_j, \lambda_i)e_n, e_n \rangle \]
\[ = \sum_{i,j=1}^{N} \sum_{m,n=1}^{\infty} \langle j_r(\lambda_i, \lambda_j)e_m, e_n \rangle \langle K(\lambda_j, \lambda_i)e_m, e_n \rangle \]
\[ = \sum_{i,j=1}^{N} \sum_{m,n=1}^{\infty} \langle j_r(\lambda_i, \lambda_j)e_m, e_n \rangle \langle K^t(\lambda_i, \lambda_j)e_m, e_n \rangle \geq 0. \]

the last inequality following from (3.4). Using continuity of $J$, we get $L(j) \geq 0$, which is contradiction to the assumption that $J$ not in $W_Y$. A standard application of Kurosh’s Theorem (Theorem 2.56 in [2]) completes the proof.

\[ \square \]

**Remark 6.** The continuity assumption in Theorem 4 is not necessary if one replaces (3.1) by the assumption that

\[ J \otimes k : ((s, p), (t, q)) \mapsto J((s, p), (t, q)) \otimes k((s, p), (t, q)) \]

is positive semi-definite for every admissible kernel $k$. Apart from getting rid of the continuity assumption, there is a different advantage with this new assumption. With this new assumption, the theorem can be extended to a function $J$ defined on any arbitrary subset of the symmetrized bidisk. We made a slightly weaker assumption, viz., (3.1), to facilitate a direct application. Why we stated Theorem 4 the way we did will be clear when we prove Lemma 7.

The following lemma shows that (H)⇒(D).

**Lemma 7.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Hilbert spaces. If $f$ is a function in $H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2))$, then there exists a completely positive function $\Delta : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(C(\overline{B}), \mathcal{B}(\mathcal{L}_2))$ such that

\[ I_{\mathcal{L}_2} - f(s, p)f(t, q)^* = \Delta((s, p), (t, q)) \left(1 - \varphi(\cdot, s, p)\overline{\varphi(\cdot, t, q)}\right). \]

**Proof.** Let $k$ be a $\mathcal{B}(\mathcal{L}_2)$-valued admissible kernel on $\mathbb{G}$. By definition, this means that the pair $(M_s, M_p)$ is a $\Gamma$-contraction on $H_k$, the reproducing kernel Hilbert space for $k$. For $f$ in $H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2))$, the function $\tilde{f}(s, p) = f(s, \tilde{p})^*$ is in $H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_2, \mathcal{L}_1))$ and has the same supremum norm as $f$ and satisfies

\[ \tilde{f}_r(s, p) = f(r\tilde{s}, r^2\tilde{p})^* \]

for $0 < r < 1$. Since $(M_s, M_p)$ is a $\Gamma$-contraction, the operator $\tilde{f}_r(M_s^*, M_p^*)$ is a contraction, by definition of a $\Gamma$-contraction. Moreover,

\[ \tilde{f}_r(M_s^*, M_p^*) \left(k(s,p)u \otimes v\right) = k(s,p)u \otimes f_r(s, p)^*v. \]
Therefore, for points \((s_1, p_1), (s_2, p_2), \ldots, (s_N, p_N)\) from \(\mathbb{G}\) and vectors \(u_{im}, m = 1, 2, \ldots\) and \(v_{in}, n = 1, 2, \ldots\) from \(\mathcal{L}_2\), we have
\[
0 \leq \langle (I - \tilde{f}_r(M^*_r, M^*_p)\tilde{f}_r(M^*_r, M^*_p)^*) \sum_{j,n} k(s_j, p_j)v_{jn} \otimes u_{jn}, \sum_{i,m} k(s_i, p_i)v_{im} \otimes u_{im} \rangle \\
= \sum_{i,j,m,n} \langle (1 - f_r(s_i, p_i)f_r(s_j, p_j)^*)u_{jn}, v_{im} \rangle k((s_i, p_i), (s_j, p_j))v_{im}, v_{jn} \rangle.
\]

In other words, \((I_{\mathcal{L}_2} - f_r(s, p)f_r(t, q)^*) \otimes k((s, p), (t, q))\) is a positive semi-definite kernel. So, by Theorem 4, the proof is complete. \(\square\)

**Lemma 8.** Let \(Y\) be a subset of \(\mathbb{G}\). If \(\Delta : Y \times Y \to \mathcal{B}(\overline{C(\mathbb{D})}, \mathcal{B}(\mathcal{L}))\) is completely positive function, then there is a Hilbert space \(\mathcal{H}\) and a function \(L : Y \to \mathcal{B}(\overline{C(\mathbb{D})}, \mathcal{B}(\mathcal{H}, \mathcal{L}))\) such that for every \(h_1, h_2 \in C(\mathbb{D})\) and \((s, p), (t, q) \in Y\),
\[
(3.5) \quad \Delta((s, p), (t, q))(h_1h_2) = L(s, p)h_1(L(t, q)h_2)^*.
\]
Moreover, there is a unital *-representation \(\pi : \overline{C(\mathbb{D})} \to \mathcal{B}(\mathcal{H})\) such that for every \(h_1, h_2 \in C(\mathbb{D})\) and \((s, p) \in Y\),
\[
(3.6) \quad L(s, p)h_2\pi(h_1) = L(s, p)(h_1h_2).
\]

**Proof.** Let \(V\) denote the vector space with basis \(\{(s, p) \in Y\}\). Define a sesquilinear form on the vector space \(V \otimes \mathcal{L} \otimes C(\mathbb{D})\) by
\[
\langle (s, p) \otimes e \otimes h, (t, q) \otimes e' \otimes h' \rangle = \langle \Delta((t, q), (s, p))\overline{h}\overline{e}'e', e'\rangle_{\mathcal{L}}
\]
and extend linearly in the first variable and anti-linearly in the second. The fact that this is a positive semi-definite sesquilinear form follows from properties of \(\Delta\) and we have
\[
\sum_{i,j=1}^N c_i^*c_j\langle \Delta((s_i, p_i), (s_j, p_j))\overline{h}\overline{e}_j, e_i \rangle \geq 0.
\]

Quotienting \(V \otimes \mathcal{L} \otimes C(\mathbb{D})\) by the null space of the sesquilinear form and then taking completion, we obtain a Hilbert space \(\mathcal{H}\). Now define \(L : Y \to \mathcal{B}(\overline{C(\mathbb{D})}, \mathcal{B}(\mathcal{H}, \mathcal{L}))\) by
\[
L(s, p)h((t, q) \otimes e' \otimes h') = \Delta((s, p), (t, q))(hh')(e')
\]
and extend linearly to whole of \(\mathcal{H}\). A straightforward inner product computation produces that
\[
(3.5)
\]
for all \(h \in C(\mathbb{D})\), \((s, p) \in Y\) and \(e \in \mathcal{L}\). Now \(3.5\) follows form the following simple computation:
\[
L(s, p)h_1(L(t, q)h_2)^* = L(s, p)h_1((t, q) \otimes e \otimes \overline{h}_2) = \Delta((s, p), (t, q))(h_1h_2)(e).
\]
The definition of the representation now nearly suggests itself. Define \(\pi : \overline{C(\mathbb{D})} \to \mathcal{B}(\mathcal{H})\) by
\[
\pi(h)((s, p) \otimes e \otimes h') = (s, p) \otimes e \otimes hh'
\]
and extend it linearly. Skipping the routine computation of checking that \(\pi\) is a unital *-representation, the following computation proves \(3.6\):
\[
L(s, p)h_2\pi(h_1)((t, q) \otimes e' \otimes h') = L(s, p)h_2((t, q) \otimes e' \otimes h_1h')
\]
\[
= \Delta((s, p), (t, q))(h_1h_2h')(e') = L(s, p)(h_1h_2)((t, q) \otimes e' \otimes h').
\]
\(\square\)
Proof of (D)⇒(R): Let \( Y \) be any subset of \( G \) and \( f : Y \to \mathcal{B}(L_1, L_2) \) be any function for which there is a completely positive function \( \Delta : Y \times Y \to \mathcal{B}(C(\overline{D}), \mathcal{B}(L_2)) \) such that for every \((s, p), (t, q)\) in \( Y \),

\[
I_{L_2} - f(s, p)f(t, q) = \Delta((s, p), (t, q))(1 - \varphi(\cdot, s, p)\bar{\varphi}(\cdot, t, q)).
\]

By Lemma \( \ref{lem:1} \) there is a Hilbert space \( \mathcal{H} \) and a function \( L : Y \to \mathcal{B}(C(\overline{D}), \mathcal{B}(\mathcal{H}, L_2)) \) such that for every \( h_1, h_2 \in C(\overline{D}) \) and \((s, p), (t, q)\) in \( Y \),

\[
\Delta((s, p), (t, q))(h_1 \otimes h_2) = L(s, p)h_1L(t, q)h_2^*,
\]

which, in particular, implies

\[
\Delta((s, p), (t, q))(\varphi(\cdot, s, p)\bar{\varphi}(\cdot, t, q)) = L(s, p)(\varphi(\cdot, s, p)L(t, q)(\varphi(\cdot, t, q))^*.
\]

Hence after a re-arrangement of terms \( \ref{eq:1} \) becomes

\[
I_{L_2} + L(s, p)(\varphi(\cdot, s, p)L(t, q)(\varphi(\cdot, t, q))^* = L(s, p)(1)L(t, q)(1)^* + f(s, p)f(t, q)^*,
\]

which implies that there exists an (lurking) isometry from

\[
\text{span}\{e \oplus L(t, q)(\varphi(\cdot, t, q))^* e : (t, q) \in Y, e \in L_2\} \subset L_2 \oplus \mathcal{H}
\]

onto

\[
\text{span}\{f(t, q)^* e \oplus L(t, q)(1)^* e : (t, q) \in Y, e \in L_2\} \subset L_1 \oplus \mathcal{H}
\]

such that for all \((t, q) \in Y \) and \( e \in L_2 \),

\[
\begin{pmatrix}
I_{L_2} \\
L(t, q)(\varphi(\cdot, t, q))^*
\end{pmatrix}
\begin{pmatrix}
e \\
f(t, q)^*
\end{pmatrix}
\begin{pmatrix}
V_1 \\
L(t, q)(1)^*
\end{pmatrix}
e.
\]

Now a standard technique of adding an infinite dimensional summand to \( \mathcal{H} \), if necessary, we can extend \( V_1 \) unitarily from whole of \( L_2 \oplus \mathcal{H} \) onto \( L_1 \oplus \mathcal{H} \). Let the following be the \( 2 \times 2 \) block operator matrix representation for \( V_1 \)

\[
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix}
\]

Lemma \( \ref{lem:1} \) also provides us with a unital \(^*\)-representation \( \pi : C(\overline{D}) \to \mathcal{B}(\mathcal{H}) \) with the property that \( L(s, p)h_2\pi(h_1) = L(s, p)(h_1h_2) \) for every \((s, p) \in Y \) and \( h_1, h_2 \in C(\overline{D}) \), which implies \( L(t, q)(\varphi(\cdot, t, q))^* = \pi(\varphi(\cdot, t, q))^*L(t, q)(1)^* \).

Hence by \( \ref{eq:1} \) we have

\[
\begin{align*}
A_1 + B_1\pi(\varphi(\cdot, t, q))^*L(t, q)(1)^* &= f(t, q)^* \\
C_1 + D_1\pi(\varphi(\cdot, t, q))^*L(t, q)(1)^* &= L(t, q)(1)^*.
\end{align*}
\]

Eliminating \( L(t, q)(1)^* \) from above two equations we get for every \((t, q) \in Y \),

\[
f(t, q)^* = A_1 + B_1\pi(\varphi(\cdot, t, q))^*(I_{\mathcal{H}} - D_1\pi(\varphi(\cdot, t, q))^*)^{-1}C_1,
\]

Taking adjoint of \( \ref{eq:2} \) we get the desired form of the function

\[
f(s, p) = A + B\pi(\varphi(\cdot, s, p))(I_{\mathcal{H}} - D\pi(\varphi(\cdot, s, p)))^{-1}C,
\]

where

\[
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}^* = V_1^*.
\]

This completes the proof of \( (D) \Rightarrow (R) \). \( \square \)
Proof of \((\textbf{R}) \Rightarrow (\textbf{H})\): Suppose \(f : Y \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\) is a function for which there is a Hilbert space \(\mathcal{H}\), a unital *-representation \(\pi : C(\mathbb{D}) \to \mathcal{B}(\mathcal{H})\) and a unitary
\[
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{L}_1 \oplus \mathcal{H} \to \mathcal{L}_2 \oplus \mathcal{H}
\]
such that \(f\) can be expressed as \(f(s, p) = A + B\pi(\varphi(\cdot, s, p))(I_H - D\pi(\varphi(\cdot, s, p)))^{-1}C\), for every \((s, p) \in Y\). Define \(F : \mathbb{G} \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)\) by
\[
F(s, p) = A + B\pi(\varphi(\cdot, s, p))(I_H - D\pi(\varphi(\cdot, s, p)))^{-1}C,
\]
for every \((s, p) \in \mathbb{G}\).

Note that to complete the proof, all we need to show is that \(F\) is in \(H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2))\).

Proof of the Interpolation Theorem: In this section, we prove the Operator Valued Pick-Nevanlinna interpolation theorem. The necessity is clear by part \((\textbf{M})\) of the Realization Theorem and hence we prove the sufficiency below.

Proof of the Interpolation Theorem: Suppose
\[
\left( (I_{\mathcal{L}_2} - W_i W_j^*) \otimes k((s_i, p_i), (s_j, p_j)) \right)_{i,j=1}^N
\]
is positive semi-definite for every \(\mathcal{B}(\mathcal{L}_2)\)-valued admissible kernel \(k\) on \(\mathbb{G}\). To find an interpolating function \(f\) in \(H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2))\) with its norm no greater than one, we
use the same technique that was used in the proof of (D)⇒(R). Let \( Y \) be the set \( \{ (s_1, p_1), (s_2, p_2), \ldots, (s_N, p_N) \} \). Define \( J \) on \( Y \times Y \) by
\[
J((s_i, p_i), (s_j, p_j)) = I_{\mathcal{L}_2} - W_i W_j^* \text{ for each } 1 \leq i, j \leq N.
\]
Then condition (1.5) says that
\[
J \otimes k : ((s, p), (t, q)) \mapsto J((s, p), (t, q)) \otimes k((s, p), (t, q))
\]
is positive semi-definite for every \( \mathcal{B}(\mathcal{L}_2) \)-valued admissible kernel \( k \). So, by Remark \( \ref{remark} \) there is a completely positive function \( \Delta : \mathbb{G} \times \mathbb{G} \to \mathcal{B}(\mathcal{L}_2) \) such that for every \( 1 \leq i, j \leq N \),
\[
J((s_i, p_i), (s_j, p_j)) = \Delta((s_i, p_i), (s_j, p_j)) \left( 1 - \varphi(\cdot, s_i, p_i) \overline{\varphi(\cdot, s_j, p_j)} \right).
\]
Now we invoke Lemma \( \ref{lemma} \) to obtain a Hilbert space \( \mathcal{H} \), a function \( L : \mathbb{G} \to \mathcal{B}(\mathcal{L}_2) \) and a unital *-representation \( \pi : C(\overline{\mathbb{D}}) \to \mathcal{B}(\mathcal{H}) \) such that (1.1) becomes
\[
I_{\mathcal{L}_2} + L(s_i, p_i)(\varphi(\cdot, s_i, p_i))L(s_j, p_j)(\varphi(\cdot, s_j, p_j))^* = L(s_i, p_i)(1)L(s_j, p_j)(1)^* + f(s_i, p_i)f(s_j, p_j)^*,
\]
which, as before, gives rise to a unitary \( V_1 \) from \( \mathcal{L}_2 \oplus \mathcal{H} \) onto \( \mathcal{L}_1 \oplus \mathcal{H} \) such that for all \( e \in \mathcal{L}_2 \) and \( 1 \leq j \leq N \),
\[
(\pi(\varphi(\cdot, s_j, p_j))^*L(s_j, p_j)(1)^*)e \overset{V_1}{\to} \left( \frac{f(s_j, p_j)^*}{L(s_j, p_j)(1)^*} \right)e.
\]
Let the following be the \( 2 \times 2 \) block operator matrix representation for \( V_1 \)
\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.
\]
Hence by (4.2) we have
\[
A_1 + B_1 \pi(\varphi(\cdot, s_j, p_j))^*L(s_j, p_j)(1)^* = f(s_j, p_j)^*
\]
(4.5)
\[
C_1 + D_1 \pi(\varphi(\cdot, s_j, p_j))^*L(s_j, p_j)(1)^* = L(s_j, p_j)(1)^*.
\]
Eliminating \( L(s_j, p_j)(1)^* \) from above two equations we get
\[
f(s_j, p_j)^* = A_1 + B_1 \pi(\varphi(\cdot, s_j, p_j))^*(I_{\mathcal{H}} - D_1 \pi(\varphi(\cdot, s_j, p_j))^*)^{-1}C_1,
\]
which is same as
\[
f(s_j, p_j) = A_1^* + C_1^* \pi(\varphi(\cdot, s_j, p_j))(I_{\mathcal{H}} - D^* \pi(\varphi(\cdot, s_j, p_j)))^{-1}B_1^*,
\]
for every \( 1 \leq j \leq N \). Therefore we define the function \( f \) on \( \mathbb{G} \) by
\[
f(s, p) = A + B \pi(\varphi(\cdot, s, p))(I_{\mathcal{H}} - D \pi(\varphi(\cdot, s, p)))^{-1}C,
\]
where
\[
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}^* = V_1^*.
\]
This function interpolates the data because of (4.2) and is in the closed unit ball of \( H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)) \). This completes the proof of the interpolation theorem. \( \square \)
5. The Toeplitz-Corona on the Symmetrized Bidisk

The aim here is to prove the Toeplitz Corona Theorem described in Section 1. However, we shall prove a more general result below.

**Theorem 9.** Let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ be Hilbert spaces and $Y$ be a subset of $\mathbb{G}$. Suppose $\Phi : Y \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ and $\Theta : Y \to \mathcal{B}(\mathcal{L}_3, \mathcal{L}_2)$ are given functions. Then the following statements are equivalent:

(i) There exists a function $\Psi$ in the closed unit ball of $H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_3, \mathcal{L}_1))$ such that $\Phi(s,p)\Psi(s,p) = \Theta(s,p)$ for all $(s,p) \in Y$;

(ii) The function $[\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^*] \otimes k((s,p), (t,q))$ is positive semi-definite on $Y$ for every $\mathcal{B}(\mathcal{L}_2)$-valued admissible kernel $k$ on $Y$;

(iii) There exists a completely positive function $\Delta : Y \times Y \to \mathcal{B}(C(\overline{D}), \mathcal{B}(\mathcal{L}_2))$ such that for all $(s,p), (t,q)$ in $Y$,

$$\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^* = \Delta((s,p), (t,q))(1 - \varphi(\cdot, s,p)\overline{\varphi(\cdot, t,q)}).$$

**Proof.** Our strategy is to show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Suppose $(i)$ holds. Since $\Psi$ is in the closed unit ball of $H^\infty(\mathbb{G}, \mathcal{B}(\mathcal{L}_3, \mathcal{L}_2))$, we apply the realization theorem with $Y = \mathbb{G}$ and $f = \Psi$ to get by part $(3)$

$$(I_{\mathcal{L}_2} - \Psi(s,p)\Psi(t,q)^*) \otimes k((s,p), (t,q))$$

is positive semi-definite for every $\mathcal{B}(\mathcal{L}_2)$-valued admissible kernel $k$ on $\mathbb{G}$. Now $(ii)$ follows from the following simple observation:

$$[\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^*] \otimes k((s,p), (t,q)) = \Phi(s,p)(I_{\mathcal{L}_1} - \Psi(s,p)\Psi(t,q)^*)\Phi(t,q)^* \otimes k((s,p), (t,q)).$$

$(ii) \Rightarrow (iii)$: Suppose $(ii)$ holds. Then $(iii)$ follows by Remark 9.

$(iii) \Rightarrow (i)$: This part of the proof uses the lurking isometry technique to construct the function $\Psi$. Suppose there exists a completely positive function $\Delta : Y \times Y \to \mathcal{B}(C(\overline{D}), \mathcal{B}(\mathcal{L}_2))$ such that for every $(s,p), (t,q)$ in $Y$,

$$\Phi(s,p)\Phi(t,q)^* - \Theta(s,p)\Theta(t,q)^* = \Delta((s,p), (t,q))(1 - \varphi(\cdot, s,p)\overline{\varphi(\cdot, t,q)}).$$

Now, as before, we re-arrange the terms in the above equation and apply Lemma 8 to obtain a Hilbert space $\mathcal{H}$, a function $L : Y \to B(C(\overline{D}), \mathcal{B}(\mathcal{H}, \mathcal{L}_2))$ and a unital $*$-representation $\pi : C(\overline{D}) \to \mathcal{B}(\mathcal{H})$ such that

$$\Phi(s,p)\Phi(t,q)^* + L(s,p)(\varphi(\cdot, s,p)L(t,q)(\varphi(\cdot, t,q))^*) = \Theta(s,p)\Theta(t,q)^* + L(s,p)(1)L(t,q)(1)^*,$$

which implies that there exists an isometry $V_1$ from

$$\overline{\text{span}}\{\Phi(t,q)^* e \odot L(t,q)(\varphi(\cdot, t,q))^* e : (t,q) \in Y e \in \mathcal{L}_2\} \subset \mathcal{L}_1 \oplus \mathcal{H}$$

onto

$$\overline{\text{span}}\{\Theta(t,q)^* e \odot L(t,q)(1)^* e : (t,q) \in Y e \in \mathcal{L}_2\} \subset \mathcal{L}_3 \oplus \mathcal{H}$$

such that for all $(t,q) \in Y$ and $e \in \mathcal{L}_2$,

$$(5.1) \quad \left(\begin{array}{c} \Phi(t,q)^* \\ \pi(\varphi(\cdot, t,q))^* L(t,q)(1)^* \end{array}\right) e \overset{V_1}{\Rightarrow} \left(\begin{array}{c} \Theta(t,q)^* \\ L(t,q)(1)^* \end{array}\right) e.$$
We add an infinite-dimensional summand to $H$, if necessary, to extend $V_1$ to a unitary from $L_1 \oplus H$ onto $L_3 \oplus H$. Decompose $V_1$ as the $2 \times 2$ block operator matrix
\[
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix}
\]
and define the function $\Psi$ on $G$ by
\[
\Psi(t, q)^* = A_1 + B_1 \pi(\varphi(\cdot, t, q))^*(I_H - D_1 \pi(\varphi(\cdot, t, q))^*)^{-1} C_1.
\]
Then $\Psi$ is a contractive multiplier and by (5.1) it satisfies $\Psi(t, q)^* \Phi(t, q)^* = \Theta(t, q)^*$ for all $(t, q) \in Y$. Hence (i) holds.

The Toeplitz Corona Theorem follows from Theorem 9 when we choose $Y = G$, $L_1 = \mathbb{C}^N$, $L_2 = \mathbb{C} = L_3$, $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_N)$ and $\Theta = \sqrt{\delta}$.

6. ACKNOWLEDGEMENT

The second author would like to thank Prof. John E. McCarthy profusely for clarifying a doubt which has been used in the proof of Theorem 9.

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(Bhattacharyya) DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560 012, INDIA.

E-mail address: tirtha@member.ams.org

(Sau) DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY, POWAI, MUMBAI 400076, INDIA.

E-mail address: haripada@math.iitb.ac.in