NODAL BUBBLE-TOWER SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM WITH COMPETING POWERS

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Abstract. In this paper, we consider the following semilinear elliptic problem
\begin{align*}
-\Delta u &= |u|^{p-1}u - |u|^{q-1}u \quad \text{in } \mathbb{R}^N, \\
u(x) &\to 0, \text{ as } |x| \to \infty,
\end{align*}
where \( \frac{N}{N-2} < q < p < p^* \) or \( q > p > p^* \), \( p^* = \frac{N+2}{N-2} \), \( N \geq 3 \). We show that if \( q \) is fixed and \( p \) is close enough to \( \frac{N+2}{N-2} \), the above problem has radial nodal bubble tower solutions, which behave like a superposition of bubbles with different orders and blow up at the origin.

1. Introduction and main results. In this paper, we study the following semilinear elliptic problem
\begin{align*}
-\Delta u &= |u|^{p-1}u - |u|^{q-1}u \quad \text{in } \mathbb{R}^N, \\
u(x) &\to 0, \text{ as } |x| \to \infty,
\end{align*}
where \( \frac{N}{N-2} < q < p < p^* \) or \( q > p > p^* \), \( p^* = \frac{N+2}{N-2} \), \( N \geq 3 \).

When \( 1 \leq q < p < p^* \), Kwong \[18\] and Kwong, Zhang \[20\] showed that there is a unique positive radial solution for (1.1). For the case \( q = 1 \), problem (1.1) becomes the classical Schrödinger equation. Jones and Kupper \[17\], and McLeod, Troy, and Weissler \[24\] obtained that for each integer \( k \geq 1 \) there exists at least one radial solution of (1.1) that has exactly \( k \) positive zeros. Similar results have been also obtained by Bartsch, Willem \[2\] and Cao, Zhu \[9\] by variational methods. Later on, Bartsch, Willem \[3\] and Lorca, Ubilla \[22\] proved that there are infinitely many nonradial solutions. Recently, Musso, Pacard and Wei \[27\] constructed sign-changing solutions with dihedral symmetry, which are different from the solutions obtained in \[3, 22\]. The literature on the subcritical case is very wide and is impossible to give a complete list of references. We refer to the reader \[1, 4, 5, 11, 21, 23\] and the references therein.

For the supercritical case \( p > p^* \), there are few results up to now. The Pohozaev argument shows that there are no finite energy solutions of (1.1) provided that \( p > p^* \), \( q = 1 \). When \( q > p > p^* \), Kwong, McLeod, Peletier and Troy \[19\] proved...
that there exists a unique positive radial solution with fast decay $O(|x|^{2-N})$, as $|x| \to \infty$. Troy [32] showed that for any integer $k \geq 1$, there is a radially symmetric solution with exactly $k$ positive zeros and decay to zero at the rate $O(|x|^{2-N})$.

The motivation for studying problem (1.1) comes from the study of the following Dirichlet problem
\[
\begin{cases}
-\Delta u = u^p - \varepsilon u^q & \text{in } B_1(0), \\
u > 0, & \text{in } B_1(0), \\
u = 0, & \text{on } \partial B_1(0),
\end{cases}
\] (1.2)

where $q > p > p^*$, $B_1(0)$ is the unit ball in $\mathbb{R}^N$. Merle and Peletier [25, 26] have investigated the asymptotic behavior of positive solutions of (1.2) as $\varepsilon \to 0$. However, their results show that the decay of the radial solutions must satisfy the fast decay, i.e. $u(|x|) = O(|x|^{2-N})$, as $|x| \to \infty$. Moreover, Dancer, Santra [12] and Dancer, Santra and Wei [13] considered the following singular perturbed problem
\[
\begin{cases}
-\varepsilon^2 \Delta u = u^p - u^q & \text{in } \Omega, \\
u > 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\] (1.3)

where $1 < q < p < p^*$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$. They studied the asymptotic behavior of the least energy solutions as $\varepsilon$ goes to zero, which is also closely related to problem (1.1).

From [18, 19, 20], we know that there is a unique positive radial solution for problem (1.1). It is natural to ask if there are some other sign-changing solutions of (1.1)? Our purpose in this paper is to give a positive answer to this question. Actually, we construct nodal bubble tower solutions for (1.1) which behave like a superposition of bubbles. It seems that this is the first existence results of nodal bubble tower solutions for (1.1).

More precisely, we consider the subcritical case
\[
\begin{cases}
-\Delta u = |u|^{p^* - 1} - \varepsilon u^q - |u|^{q - 1} u & \text{in } \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to \infty,
\end{cases}
\] (1.4)

where $\varepsilon > 0$, $q \in \left(\frac{N}{N-2}, p^*\right)$ is fixed.

Our main result concerning problem (1.4) is the following.

**Theorem 1.1.** Assume that $N \geq 3$, then for any integer $k \geq 1$, there exist $\varepsilon_k > 0$ such that for $\varepsilon \in (0, \varepsilon_k)$ problem (1.4) has a radial nodal bubble tower solution $u_\varepsilon$, which has the form
\[
u_\varepsilon(x) = C_N \sum_{i=1}^{k} (-1)^i \frac{M_i \varepsilon^{-(1+\frac{1}{p^*-q})}}{\left(1 + M_i \varepsilon^{\frac{4}{N-2} - \frac{1}{N-2} (1+\frac{1}{p^*-q}) |x|^2}\right)^{\frac{N+2}{2}}} (1 + o(1)),
\]

where $C_N = (N(N-2))^{\frac{N+2}{2}}$, $M_1, \ldots, M_k$ are positive constants depending only on $N$ and $k$, and $o(1) \to 0$ uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \to 0$.

For the supercritical case, we consider
\[
\begin{cases}
-\Delta u = |u|^{p^* - 1} + \varepsilon u^q - |u|^{q - 1} u & \text{in } \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to \infty,
\end{cases}
\] (1.5)

where $\varepsilon > 0$, $q > p^*$ is fixed. Then we have
Theorem 1.2. Assume that $N \geq 3$, then for any integer $k \geq 1$, there exist $\varepsilon_k > 0$ such that for $\varepsilon \in (0, \varepsilon_k)$ problem (1.5) has a radial nodal bubble tower solution $\hat{u}_\varepsilon$, which has the form

$$
\hat{u}_\varepsilon(x) = C_N \sum_{i=1}^{k} (-1)^i \frac{\hat{M}_i \varepsilon^{\left(i-1+\frac{1}{p^*}\right)}}{\left(1 + \hat{M}_i \varepsilon^{\left(i-1+\frac{1}{p^*}\right)} |x|^2\right)^{\frac{N-2}{2}} (1 + o(1))},
$$

where $C_N = (N(N - 2))^{\frac{N-2}{2}}$, $\hat{M}_1, \ldots, \hat{M}_k$ are positive constants depending only on $N$ and $k$, and $o(1) \to 0$ uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \to 0$.

Remark 1.3. It is very interesting to investigate what happens in the complementary cases, i.e. $q > p^*$ for (1.4) or $q \in \left(\frac{N}{N-2}, p^*\right)$ for (1.5). It follows from [8] that the reduced energy functional does not have any critical points. It is not possible to construct bubble tower solutions for (1.4) or (1.5). See [28] for more general relation between tower of positive or sign-changing bubble solutions and the sign of $\varepsilon$.

Remark 1.4. The basic elements to construct nodal bubble tower solutions are the radial functions $U_{\mu}(|x|)$, defined by

$$
U_{\mu}(|x|) = C_N \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{\frac{N-2}{2}}, \mu > 0,
$$

where $C_N = (N(N - 2))^{\frac{N-2}{2}}$. It’s well-known [7, 31] that $U_{\mu}(x)$ are the unique radial solutions of $-\Delta u = u^{p^*}, u > 0$ in $\mathbb{R}^N$.

The main idea of the paper is motivated by [8, 15]. More precisely, we will use the Lyapunov-Schmidt reduction argument to prove Theorem 1.1 and 1.2, which reduces the construction of the solutions to a finite-dimensional variational problem. It is worthwhile pointing out that bubble tower concentration phenomena for nonlinear elliptic problems with critical Sobolev exponent has been observed in [8, 10, 15, 14, 28, 29]. However, as far as we know, there are few results for problem (1.1).

As a final remark, our results seem to be connected with the work of Campos [8]. More precisely, Campos considered

$$
\begin{cases}
-\Delta u = u^{p^*+\varepsilon} + u^q \text{ in } \mathbb{R}^N, \\
u > 0 \text{ in } \mathbb{R}^N, \\
u(x) \to 0, \text{ as } |x| \to \infty,
\end{cases}
$$

(1.6)

where $\varepsilon > 0$, $q \in \left(\frac{N}{N-2}, p^*\right)$, or $\varepsilon < 0$, $q > p^*$. He constructed positive radial solutions, which look like a superposition of bubbles. Our results in this paper can been seen somewhat dual counterparts in [8].

This paper is organized as follows. In Section 2, we will give some preliminaries needed in the later sections. The finite dimensional reduction argument is carried out in Section 3. We will prove the main results in Sections 4, 5.

2. Some preliminaries. In this section, we will give some basic estimates, which will be used in the later sections. We are interested in finding radially symmetric solutions for problem (1.4), and then it can reduce the following problem

$$
\begin{cases}
-u'' - \frac{N-1}{r}u' = |u|^{p^*+1-\varepsilon}u - |u|^{q-1}u, \text{ in } \mathbb{R}, \\
u'(0) = 0, u(r) \to 0, \text{ as } r \to \infty.
\end{cases}
$$

(2.1)
Define the Emden-Fowler transformation
\[ v(y) = r^{\frac{N-2}{2}} u(r), \quad r = e^{-\frac{y}{p^*}-\frac{1}{2}}, \quad y \in \mathbb{R}. \] (2.2)

After these changes of variables, problem (2.1) becomes
\[ \begin{align*}
& v'' - v + \gamma e^{\gamma y}|v|^{p^*-1-\epsilon}v - \gamma e^{-(p^*-q)y}|v|^{q-1}v = 0 \text{ in } \mathbb{R}, \\
& v(y) \to 0 \text{ as } |y| \to +\infty,
\end{align*} \] (2.3)

where \[ \gamma = \frac{4}{(N-2)^2}. \]

The energy functional corresponding to problem (2.3) is
\[ I_\epsilon(v) = J_\epsilon(v) + \gamma \log \frac{e^{-y}}{(1 + e^{-y})^{\frac{N-2}{2}}}, \quad J_\epsilon(v) = \frac{1}{2} \int_\mathbb{R} |v'|^2 + |v|^2 - \frac{\gamma}{p^* + 1 - \epsilon} \int_\mathbb{R} e^{\gamma y}|v|^{p^*+1-\epsilon}. \]

Set \[ W(y) = C_N \frac{e^{-y}}{(1 + e^{-\frac{y}{p^*-q}})^{\frac{N-2}{2}}}, \quad C_N = (N(N-2))^{\frac{N-2}{4}}. \]

It is easy to see that \( W(y) \) is the unique solution of the problem
\[ \begin{align*}
& W'' - W + \gamma W^{p^*} = 0 \quad \text{in } \mathbb{R}, \\
& W'(0) = 0, \quad W(y) > 0, \\
& W(y) \to 0 \text{ as } y \to \pm \infty.
\end{align*} \] (2.4)

For given \( \Lambda_i > 0, \ i = 1, 2, \ldots, k, \) set
\[ \begin{align*}
\xi_1 &= -\frac{1}{p^* - q} \log \epsilon + \frac{1}{p^* - q} \log \Lambda_1, \\
\xi_{i+1} - \xi_i &= -\log \epsilon - \log \Lambda_{i+1}, \quad i = 1, 2, \ldots, k - 1.
\end{align*} \] (2.5)

Let us write
\[ W_i(y) = W(y - \xi_i), \quad V = \sum_{i=1}^k (-1)^i W_i. \] (2.6)

We will look for a solution of problem (2.3) of the following form
\[ v = \sum_{i=1}^k (-1)^i W_i + \phi, \]

where \( \phi \) is small.

Set
\[ \eta_1 = -\infty, \eta_\ell = \frac{\xi_{\ell-1} + \xi_\ell}{2}, \quad \ell = 2, \ldots, k, \eta_{k+1} = +\infty. \]

First, we give some asymptotic estimates of \( W_i \) and \( V \) in the following lemma.

**Lemma 2.1.** For fixed \( \delta > 0 \) and \( \delta < \Lambda_i < \delta^{-1}, i = 1, 2, \ldots, k, \) we have the following estimates:
\[
\int_{\mathbb{R}} |V|^{p^*+1} = k \int_{\mathbb{R}} W^{p^*+1} + o(1),
\]
\[
\int_{\mathbb{R}} \left( |V|^{p^*+1} - |V|^{p^*+1-\epsilon} \right) = k \varepsilon \int_{\mathbb{R}} W^{p^*+1} \log W + o(\varepsilon),
\]
\[
\int_{\Omega_{\ell}} |V|^{p^*+1} = \left( \sum_{\ell=1}^{k} \xi_{\ell} \right) \int_{\mathbb{R}} W^{p^*+1} + o(1),
\]
\[
\int_{\Omega_{\ell}} W_{p^*}^{\ell} W_j = o(\varepsilon), \quad i \neq \ell,
\]
\[
\int_{\Omega_{\ell}} W_{p^*}^{\ell} W_j = \alpha e^{-|\xi_{\ell} - \xi_j|} + o(\varepsilon), \quad j \neq \ell,
\]
\[
\int_{\Omega_{\ell}} \left( |V|^{p^*+1} - W_{p^*}^{\ell}^{+1} - (p^* + 1) \sum_{j \neq \ell} (-1)^{j+1} W_{p^*}^{\ell} W_j \right) = o(\varepsilon),
\]

where \( \alpha = C_N \int_{\mathbb{R}} e^{-y} W^{p^*} \), \( \Omega_{\ell} = \{ y \in \mathbb{R} : \eta_{\ell} \leq y < \eta_{\ell+1} \} \).

Proof. The proof of this lemma is similar to Lemma 4.4 in [29] and we omit the details. \( \square \)

Next, we give the asymptotic expansions of the energy functional \( I_{\varepsilon}(V) \), which will turn out to be very important in looking for critical points of \( I_{\varepsilon}(V + \phi) \).

**Proposition 2.2.** For any \( \delta > 0 \), there exists \( \varepsilon_0 > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \), we have the following asymptotic expansion

\[
I_{\varepsilon}(V) = k a_0 + k a_1 \varepsilon - k a_2 \varepsilon \log \varepsilon + \varepsilon \Psi(\Lambda) + \varepsilon \Theta_{\varepsilon}(\Lambda),
\]

where \( \Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_k) \),

\[
\Psi(\Lambda) = \frac{a_3}{\Lambda_1} + \frac{k a_4}{p^* - q} \log \Lambda_1 + \sum_{i=2}^{k} \left[ \frac{a_5 \Lambda_i - (k - i + 1) a_4 \log \Lambda_i}{\Lambda_i} \right]
\]

and \( \Theta_{\varepsilon} \to 0 \), as \( \varepsilon \to 0 \) uniformly in \( C^1 \)-norm on the set of \( \Lambda_i \)'s with \( \delta < \Lambda_i < \delta^{-1} \), \( i = 1, 2, \cdots, k \). Here \( a_i, i = 0, 0, \cdots, 5 \) are positive constants depending only on \( N \).

Proof. Recall that

\[
I_{\varepsilon}(V) = J_{\varepsilon}(V) + \frac{\gamma}{q + 1} \int_{\mathbb{R}} e^{-\left(p^* - q\right) y} |V|^{q+1}.
\]

Note that

\[
J_{\varepsilon}(V) = \frac{1}{2} \int_{\mathbb{R}} |V'|^2 + |V|^2 - \frac{\gamma}{p^* + 1 - \varepsilon} \int_{\mathbb{R}} e^{-\varepsilon y} |V|^{p^*+1-\varepsilon}.
\]

\[
= J_0(V) - \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} (e^{-\varepsilon y} - 1) |V|^{p^*+1}
\]
\[
+ \left( \frac{1}{p^* + 1} - \frac{1}{p^* + 1 - \varepsilon} \right) \gamma \int_{\mathbb{R}} e^{-\varepsilon x} |V|^{p^*+1-\varepsilon}
\]
\[
+ \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} e^{-\varepsilon y} \left( |V|^{p^*+1} - |V|^{p^*+1-\varepsilon} \right)
\]
\[
= J_0(V) - \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} (e^{-\varepsilon y} - 1) |V|^{p^*+1} + k a_1 \varepsilon + o(\varepsilon),
\]
where
\[ a_1 = \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} W^{p^*} \log W - \frac{\gamma}{(p^* + 1)^2} \int_{\mathbb{R}} W^{p^* + 1}. \]

It follows from Lemma 2.1, we find
\[ \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} (e^{-\varepsilon y} - 1) |V|^{p^* + 1} = \frac{\gamma \varepsilon}{p^* + 1} \int_{\mathbb{R}} y |V|^{p^* + 1} + o(\varepsilon) = a_4 \varepsilon \sum_{j=1}^k \xi_j + o(\varepsilon), \]

where
\[ a_4 = \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} W^{p^* + 1}. \]

It is easy to check that
\[ J_0(V) - k \sum_{i=1}^k J_0(W_i) = \frac{\gamma}{p^* + 1} \int_{\mathbb{R}} \left( \sum_{i=1}^k W_i^{p^* + 1} - |V|^{p^* + 1} \right) + \sum_{i>j} (-1)^{i+j} \gamma \int_{\mathbb{R}} W_i^{p^*} W_j. \]

Thus, from Lemma 2.1, we find
\[ J_0(V) - \sum_{i=1}^k J_0(W_i) = \frac{\gamma}{p^* + 1} \int_{\Omega} \left( \sum_{\ell=1}^k W_{\ell}^{p^* + 1} - |V|^{p^* + 1} \right) + \sum_{\ell=1}^k \sum_{j>\ell} (-1)^{i+j} \gamma \int_{\Omega} W_{\ell}^{p^*} W_j + o(\varepsilon) \]
\[ = \gamma \sum_{\ell=1}^k \int_{\Omega} W_{\ell}^{p^*} W_{\ell+1} + o(\varepsilon) \]
\[ = a_5 \sum_{\ell=1}^{k-1} e^{-|\xi_{\ell+1} - \xi_\ell|} + o(\varepsilon), \]

where
\[ a_5 = \gamma C_N \int_{\mathbb{R}} e^y W^{p^*}. \]

Thus, we find
\[ J_0(V) = k a_0 + k a_1 \varepsilon - k a_2 \varepsilon \log \varepsilon + a_4 \varepsilon \sum_{\ell=1}^k \xi_\ell + a_5 \sum_{\ell=1}^{k-1} e^{-|\xi_{\ell+1} - \xi_\ell|} + o(\varepsilon), \]

where
\[ a_0 = J_0(W), \ a_2 = \frac{a_4}{p^* - q} + \frac{(k-1)a_4}{2}. \]

Finally, it is easy to check that
\[ \frac{\gamma}{q + 1} \int_{\mathbb{R}} e^{-(p^* - q)y} |V|^{q+1} \]
\[
= \frac{\gamma}{q+1} \sum_{\ell=1}^{k} \int_{\Omega} e^{-(p^* - q)y} W_\ell^{q+1} + o(\varepsilon)
\]
\[
= a_3 e^{-(p^* - q)\xi_i} + o(\varepsilon),
\]
where
\[
a_3 = \frac{\gamma}{q+1} \int_{\mathbb{R}} e^{-(p^* - q)y} W^{q+1}.
\]
Note that
\[
\sum_{\ell=1}^{k-1} e^{-\frac{2k-1}{k}(\xi_{\ell+1} - \xi_{\ell})} = \varepsilon \sum_{\ell=1}^{k} \Lambda_{\ell}
\]
and
\[
\sum_{\ell=1}^{k} \xi_{\ell} = - \frac{k}{p^* - q} \log \varepsilon - \frac{k(k-1)}{2} \log \varepsilon + \frac{k}{p^* - q} \log A_1 - \sum_{\ell=2}^{k} (k - \ell + 1) \log \Lambda_{\ell}.
\]
Therefore, we can obtain (2.7) immediately and the proof of Proposition 2.2 is complete.

3. The finite dimensional reduction. In this section, we will perform the finite dimensional procedure, which reduces problem (2.3) to a finite-dimensional problem on \(\mathbb{R}\).

For any given points \(\xi_1, \xi_2, \cdots, \xi_k\), let
\[
\|\phi\|_* = \sup_{y \in \mathbb{R}} \left( \sum_{i=1}^{k} e^{-\sigma|y - \xi_i|} \right)^{-1} |\phi(y)|,
\]
where \(0 < \sigma < \min\{p^* - 1, 2q - p^* - 1, 1\}\). We define the Banach space
\[
\mathcal{C}_* = \{ v \in C(\mathbb{R}) : \|v\|_* < \infty \}
\]
edowed of the \(\|\cdot\|_*\)-norm defined as above.

Set
\[
Z_i(y) = W_i'(y), \ i = 1, 2, \cdots, k.
\]
Now, we consider the following linear problem
\[
\begin{align*}
\mathcal{L}_\varepsilon \phi &= h + \sum_{i=1}^{k} c_i Z_i \text{ in } \mathbb{R}, \\
\int_{\mathbb{R}} Z_i \phi &= 0, \ i = 1, 2, \cdots, k, \ \lim_{y \to \pm \infty} \phi(y) = 0,
\end{align*}
\]
(3.1)
where \(c_i, \ i = 1, 2, \cdots, k\) are some constants and
\[
\mathcal{L}_\varepsilon \phi = -\phi'' + \phi - (p^* - \varepsilon)\gamma e^{-\varepsilon y} |V|^{p^* - 1 - \varepsilon} \phi + q \gamma e^{-(p^* - q)y} |V|^{q-1} \phi.
\]

First of all, we have the following priori estimate for (3.1).

Lemma 3.1. Assume that there are sequences \(\varepsilon_n \to 0\) and points \(0 < \xi_1^n < \xi_2^n < \cdots < \xi_k^n\) depending on \(\varepsilon_n\) satisfying
\[
\xi_1^n \to \infty, \ \min_{1 \leq i \leq k} (\xi_{i+1}^n - \xi_i^n) \to +\infty, \ \xi_k^n = o(\varepsilon_n^{-1}),
\]
such that \(\phi_n\) solves (3.1) for certain scalars \(c_i^n\) and \(h_n\) with \(\|h_n\|_* \to 0\), then
\[
\lim_{n \to \infty} \|\phi_n\|_* = 0.
\]
Proof. We will first show that
\[ \lim_{n \to \infty} \| \phi_n \|_{L^\infty} = 0. \]
Arguing by contradiction, we may assume that \( \| \phi_n \|_{L^\infty} = 1 \). Multiplying (3.1) by \( Z^n_\ell \) and integrating by parts, we find
\[
\sum_{i=1}^k c_i^n \int_R Z_i^n Z^n_\ell = \int_R L_{\epsilon_n}(Z^n_\ell) \phi_n - \int_R h_n Z^n_\ell.
\]
This defines an almost diagonal system in the \( c_i^n \)'s as \( n \to \infty \). Since \( q > \frac{N}{N-2} \) and \( Z^n_\ell(y) \) is a solution of
\[-Z'' + Z - p^* \gamma W^{p^*-1} Z = 0.\]
Thus, we have
\[
\sum_{i=1}^k c_i^n \int_R Z_i^n Z^n_\ell = \int_R \left[ p^* \gamma W^{p^*-1} Z^n_\ell - (p^* - \epsilon_n) \gamma e^{-\epsilon_n y} V^{p^*-1} Z^n_\ell \right] \phi_n + q \gamma \int_R e^{-(p^*-q)y} |V|^{q-1} Z^n_\ell \phi_n - \int_R Z^n_\ell h_n.
\]
Note that \( Z^n_\ell(y) = O(e^{-|y - \xi^n_\ell|}) \), by the dominated convergence Theorem, we know that \( \lim_{n \to \infty} c_i^n = 0 \). Assume that \( y_n \in \mathbb{R} \) is such that \( |\phi_n(y_n)| = 1 \), it follows from (3.1) and the elliptic regular theory that we can assume that there is an \( \ell \) and a fixed \( M > 0 \), such that \( |\xi^n_\ell - y_n| \leq M \) for \( n \) large enough.
Set
\[ \tilde{\phi}_n(y) = \phi_n(y + \xi^n_\ell). \]
From (3.1), we see that up to a subsequence, there is \( \tilde{\phi} \) such that \( \tilde{\phi}_n \to \tilde{\phi} \) uniformly over compact sets of \( \mathbb{R} \) and \( \tilde{\phi} \) is a nontrivial bounded solution of
\[-\tilde{\phi}'' + \tilde{\phi} - p^* \gamma W^{p^*-1} \tilde{\phi} = 0.\]
Thus, by nondegeneracy in (30), \( \tilde{\phi} = cW', \ c \neq 0 \). However,
\[ 0 = \int_R Z^n_\ell \tilde{\phi}_n \to c \int_R |W'|^2, \]
which is a contradiction. Thus, \( \lim_{n \to \infty} \| \phi_n \|_{L^\infty} = 0. \)
Next we shall establish that
\[ \lim_{n \to \infty} \| \phi_n \|_* \to 0. \]
Now we see that (3.1) possesses the following form
\[-\phi''_n + \phi_n = g_n, \quad (3.2)\]
where
\[ g_n = h_n + (p^* - \epsilon) \gamma e^{-\epsilon_n y} |V|^{p^*-1-\epsilon_n} \phi_n - q \gamma e^{-(p^*-q)y} |V|^{q-1} \phi_n + \sum_{i=1}^n c_i^n Z_i^n. \]
If \( 0 < \sigma < \min\{p^* - 1, 2q - p^* - 1\} \), we find
\[ |g_n(y)| \leq \theta_n \sum_{i=1}^k e^{-\sigma|y - \xi^n_i|} \text{ with } \theta_n \to 0. \]
Choosing \( C > 0 \) large enough, we see that
\[
\varphi_n(y) = C \theta_n \sum_{i=1}^{k} e^{-\sigma|y - \xi_i^n|}
\]
is a supersolution of (3.2), and \(-\varphi_n(y)\) will be a subsolution of (3.2). Thus,
\[
|\phi_n| \leq C \theta_n \sum_{i=1}^{k} e^{-\sigma|y - \xi_i^n|}
\]
and the result follows.

The following proposition is a direct consequence of proposition 1 in [15] combining with Lemma 3.1.

**Proposition 3.2.** There exist positive numbers \( \varepsilon_0, \delta_0, R_0 \), such that if \( \xi_1, \cdots, \xi_k \) satisfy
\[
R_0 < \xi_1, \ R_0 < \min_{i=1,\cdots,k-1} (\xi_{i+1} - \xi_i), \ \xi_k < \frac{\delta_0}{\varepsilon}, \ \ (3.3)
\]
then for \( \varepsilon \in (0, \varepsilon_0) \) and \( h \in C_* \), problem (3.1) has a unique solution \( \phi = T\varepsilon(h) \).
Moreover, there exists \( C > 0 \) such that
\[
\|T\varepsilon(h)\|_* \leq C\|h\|_*, \ |c_i| \leq C\|h\|_*.
\]

For later purposes, we need to study properties of the differentiability of the operator \( T\varepsilon \) on the variables \( \xi_i \) and the space \( L(C_*) \) of the linear operator in \( C_* \). For simplicity, we will use the notation \( \xi = (\xi_1, \xi_2, \cdots, \xi_k) \). We also consider numbers \( \varepsilon_0, \delta_0 \) and \( R_0 \) given by Proposition 3.2 and let for \( 0 < \varepsilon < \varepsilon_0 \),
\[
\mathcal{M}_\varepsilon = \left\{ \xi \in \mathbb{R}^k : R_0 < \xi_1, \ R_0 < \min_{i=1,\cdots,k-1} (\xi_{i+1} - \xi_i), \ \xi_k < \frac{\delta_0}{\varepsilon} \right\}.
\]

We now define the following map
\[
S\varepsilon : \mathcal{M}_\varepsilon \times C_* \rightarrow L(C_*)
\]
\[
(\xi, h) \mapsto S\varepsilon(\xi, h) = T\varepsilon(h).
\]

We have the following results.

**Proposition 3.3.** For each \( h \in C_* \), the map \( \xi \rightarrow S\varepsilon(\xi, h) \) is of class \( C^1 \). Besides, there is a constant \( C > 0 \) such that
\[
\|D\xi T\varepsilon(h)\|_* \leq C\|h\|_*,
\]
uniformly on the vectors \( \xi \in \mathcal{M}_\varepsilon \).

**Proof.** Fix \( h \in C_* \), and let \( \phi = T\varepsilon(h) \). Recall that \( \phi \) satisfies
\[
\mathcal{L}_\varepsilon \phi = h + \sum_{i=1}^{k} c_i Z_i \text{ in } (0, +\infty)
\]
and plus orthogonality conditions for some constants \( c_i \). We define the constants \( b_j \) satisfying
\[
\sum_{i=1}^{k} b_i \int_{\mathbb{R}} Z_i Z_j = 0, \ j \neq \ell,
\]
\[
\sum_{i=1}^{k} b_i \int_{\mathbb{R}} Z_i Z_\ell = -\int_{\mathbb{R}} \phi \partial_{\xi_\ell} Z_\ell.
\]
Again this is also an almost diagonal system and \( \partial_{\xi_\ell} \phi = \chi + \sum_{j=1}^{k} Z_j \), where \( \int_{\mathbb{R}} \chi Z_j = 0 \), \( j = 1, 2, \ldots, k \). Consider differentiation with respect to \( \xi_\ell \), we have
\[
\mathcal{L}_\varepsilon \partial_{\xi_\ell} \phi = (p^* - \varepsilon) \gamma e^{-\varepsilon y}(\partial_{\xi_\ell}|V|^{p^*-1-\varepsilon})\phi - q\gamma e^{-(p^*-q)y}(\partial_{\xi_\ell}|V|^{q-1})\phi \\
+ c_\ell (\partial_{\xi_\ell} Z_\ell) + \sum_{i=1}^{k} \partial_{\xi_\ell} c_i Z_i.
\]
Define
\[
f(x) = (p^* - \varepsilon) \gamma e^{-\varepsilon y}(\partial_{\xi_\ell}|V|^{p^*-1-\varepsilon})\phi - q\gamma e^{-(p^*-q)y}(\partial_{\xi_\ell}|V|^{q-1})\phi \\
+ c_\ell (\partial_{\xi_\ell} Z_\ell) + \sum_{i=1}^{k} \partial_{\xi_\ell} c_i Z_i.
\]
Then, we find
\[
\chi = T_\varepsilon(f).
\]
Thus, \( \partial_{\xi_\ell} \phi \) satisfies
\[
\partial_{\xi_\ell} \phi = T_\varepsilon(f) + \sum_{i=1}^{k} b_i Z_i.
\]
By Proposition 3.2 we find
\[
|b_i| \leq C\|\phi\|_*, \quad |c_i| \leq C\|h\|_*, \quad \|\phi\|_* \leq C\|h\|_*.
\]
Thus, \( \|\partial_{\xi_\ell} \phi\|_* \leq C\|h\|_* \) and \( \partial_{\xi_\ell} \phi \) depends continuously on \( \xi_\ell \) and \( h \) with this norm for each \( \ell = 1, 2, \ldots, k \).

Now we consider the following intermediate problem
\[
\begin{cases}
- (V + \phi)^{\prime\prime} + (V + \phi) - \gamma e^{-\varepsilon y}(V + \phi)^{p^*-1-\varepsilon}(V + \phi) \\
+ \gamma e^{-(p^*-q)y}(V + \phi)^{q-1}(V + \phi) = \sum_{i=1}^{k} c_i Z_i \text{ in } \mathbb{R},
\end{cases}
\]
\[
\lim_{y \to \pm \infty} \phi(y) = 0, \quad \int_{\mathbb{R}} Z_i \phi = 0, \quad i = 1, 2, \ldots, k.
\]
In order to solve problem (3.4), we rewrite it as
\[
\begin{cases}
\mathcal{L}_\varepsilon \phi = N^1_\varepsilon(\phi) + N^2_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^{k} c_i Z_i \text{ in } \mathbb{R},
\end{cases}
\]
\[
\lim_{y \to \pm \infty} \phi(y) = 0, \quad \int_{\mathbb{R}} Z_i \phi = 0, \quad i = 1, 2, \ldots, k,
\]
where
\[
N^1_\varepsilon(\phi) = \gamma e^{-\varepsilon y} \left( |V + \phi|^{p^*-1-\varepsilon}(V + \phi) - |V|^{p^*-1-\varepsilon}V - (p^* - \varepsilon)|V|^{p^*-1-\varepsilon} \phi \right),
\]
\[
N^2_\varepsilon(\phi) = -\gamma e^{-(p^*-q)y} \left( |V + \phi|^{q-1}(V + \phi) - |V|^{q-1}V - q|V|^{q-1} \phi \right)
\]
and
\[
R_\varepsilon = \gamma e^{-\varepsilon y}|V|^{p^*-1-\varepsilon}V - \gamma e^{-(p^*-q)y}|V|^{q-1}V - \sum_{i=1}^{k} (-1)^i \gamma W_i^{p^*}.
\]
Let us fix a large number \( M > 0 \), \( \xi \) satisfies the following conditions
\[
\xi_1 > \frac{1}{2} \log \frac{1}{M\varepsilon}, \quad \min_{1 \leq i \leq k-1} (\xi_{i+1} - \xi_i) > \log \frac{1}{M\varepsilon}, \quad \xi_k < k \log \frac{1}{M\varepsilon}.
\]
In order to prove that (3.5) is uniquely solvable with respect to \( \|\phi\|_* \), we need to estimate \( R_\varepsilon \), \( N_\varepsilon(\phi) \) and their derivative correspondents in the \( \|\cdot\|_* \)-norm, where \( N_\varepsilon(\phi) = N^1_\varepsilon(\phi) + N^2_\varepsilon(\phi) \).

**Lemma 3.4.** There exists \( C > 0 \) such that if \( \|\phi\|_1 \leq \frac{1}{2} \|V\|_1 \), then
\[
\|N_\varepsilon(\phi)\|_* \leq C \left( \|\phi\|_*^{\min\{p^*-\varepsilon,2\}} + \|\phi\|_*^{\min\{2q-p^*,2\}} \right),
\]
\[
\|D_\phi N_\varepsilon(\phi)\|_* \leq C \left( \|\phi\|_*^{\min\{p^*-1-\varepsilon,1\}} + \|\phi\|_*^{\min\{2q-p^*+1,1\}} \right),
\]
where
\[
\|\phi\|_1 = \sup_{y \in \mathbb{R}} \left( \sum_{i=1}^{k} e^{-|y-\xi_i|} \right)^{-1} \|\phi\|.
\]

**Proof.** Since
\[
|N^1_\varepsilon(\phi)| \leq \begin{cases} C e^{-\varepsilon y} |\phi|^{p^*-\varepsilon}, & 1 < p^* \leq 2, \\ C e^{-\varepsilon y} (|V|^{p^*-2}\phi^2 + |\phi|^{p^*-\varepsilon}), & p^* > 2, \end{cases}
\]
and
\[
|N^2_\varepsilon(\phi)| \leq \begin{cases} C e^{-(p^*-q) y} |\phi|^q, & q \leq 2, \\ C e^{-(p^*-q) y} (|V|^{q-2}\phi^2 + |\phi|^q), & q > 2. \end{cases}
\]

First, we consider \( p^* > 2 \) and \( 2q - p^* > 2 \).
\[
|N^1_\varepsilon(\phi)| \leq C e^{-\varepsilon y} \|\phi\|_*^2 |V|^{p^*-2} \left( \sum_{i=1}^{k} e^{-\sigma|y-\xi_i|} \right)^2 + C e^{-\varepsilon y} \|\phi\|^{p^*-\varepsilon} \left( \sum_{i=1}^{k} e^{-\sigma|y-\xi_i|} \right)^{p^*-\varepsilon} \leq C \left( \|\phi\|_*^{p^*-\varepsilon} + \|\phi\|_*^2 \right) \left( \sum_{i=1}^{k} e^{-\sigma|y-\xi_i|} \right)
\]
and
\[
|N^2_\varepsilon(\phi)| \leq C \left( |V|^{2q-p^*-2}\phi^2 + |\phi|^{2q-p^*} \right) \leq C \left( \|\phi\|_*^2 + \|\phi\|_*^{2q-p^*} \right) \left( \sum_{i=1}^{k} e^{-\sigma|y-\xi_i|} \right).
\]

Thus,
\[
\|N_\varepsilon(\phi)\|_* \leq C \left( \|\phi\|_*^{\min\{p^*-\varepsilon,2\}} + \|\phi\|_*^{\min\{2q-p^*,2\}} \right).
\]

The case for \( p^* \leq 2 \) is similar. \( D_\phi N_\varepsilon(\phi) \) can be estimated similarly and the proof of the lemma is completed. \( \square \)

**Lemma 3.5.** Assume that (3.6) holds, then there exists \( C > 0 \) such that
\[
\|R_\varepsilon\|_* \leq C \varepsilon^{1+q/2}, \|\partial_\xi R_\varepsilon\|_* \leq C \varepsilon^{1+q/2},
\]
where \( \tau > 0 \) is a small constant.
Proof. Recall that
\[ R_\varepsilon = \gamma(e^{-\varepsilon y} - 1)|V|^{p^*-1-\varepsilon}V + \gamma(|V|^{p^*-1-\varepsilon}V - |V|^{p^*-1}V) \]
\[ + \gamma(|V|^{p^*-1-\varepsilon}V - \sum_{i=1}^k (-1)^iW_i^{p^*}) - \gamma e^{-(p^*-q)y}|V|^{q-1}V, \]
\[ =: R_{1,\varepsilon} + R_{2,\varepsilon} + R_{3,\varepsilon} + R_{4,\varepsilon}. \]

Since
\[ V = \sum_{i=1}^k (-1)^iW_i, \quad W_i = O(e^{-|y-\xi_i|}), \]
we see
\[ |R_{1,\varepsilon}| \leq C\varepsilon |y| e^{-\varepsilon \theta y} |V|^{p^*-\varepsilon} \leq C\varepsilon \sum_{i=1}^k e^{-\sigma |y-\xi_i|}, \quad \theta \in (0,1), \]
\[ |R_{2,\varepsilon}| \leq C\varepsilon |V|^{p^*-1} \log |V| \leq C\varepsilon \sum_{i=1}^k e^{-\sigma |y-\xi_i|}. \]

Next we estimate \( R_{3,\varepsilon} \) and \( R_{4,\varepsilon} \).

Define
\[ \chi_\ell = \frac{\xi_{\ell-1} + \xi_\ell}{2}, \quad \ell = 2, \ldots, k, \quad \text{where} \quad \chi_1 = -\infty, \quad \chi_{k+1} = +\infty. \]
Thus, for \( \chi_\ell < y < \chi_{\ell+1} \), we have
\[
|R_{3,\varepsilon}| \leq \left| \gamma \left( |V|^{p^*-1}V - \sum_{i=1}^k (-1)^iW_i^{p^*} \right) \right|
\leq C\sum_{i \neq \ell} e^{-(p^*-1)|y-\xi_i|} e^{-|y-\xi_i|} + \left( \sum_{j \neq \ell} e^{-|y-\xi_j|} \right)^2
\leq Ce^{-\sigma |y-\xi_\ell|} \sum_{j \neq \ell} e^{\frac{1-\varepsilon}{2}|\xi_j-\xi_{\ell-1}|} + \sum_{j \neq \ell} e^{-2|y-\xi_j|}
\leq C\varepsilon^{\frac{1-\varepsilon}{2}} \sum_{i=1}^k e^{-\sigma |y-\xi_i|}.
\]

In a similar way, using the fact that \( \frac{N}{N-2} < q < \frac{N+2}{N-2} \), we find
\[
|R_{4,\varepsilon}| \leq Ce^{-(p^*-q)y} \sum_{i=1}^k e^{-q|y-\xi_i|} \leq C\varepsilon^{\frac{1-\varepsilon}{2}} \sum_{i=1}^k e^{-\sigma |y-\xi_i|}.
\]

Therefore, \( \|R_\varepsilon\| \leq C\varepsilon^{\frac{1-\varepsilon}{2}} \). The term \( \partial_\xi R_\varepsilon \) can be estimated similarly and the results follow.

The next proposition enables us to reduce the problem of finding a solution for (2.3) to a finite dimensional problem.

**Proposition 3.6.** Suppose that conditions (3.6) hold. Then there exists a positive constant \( C \) such that, for \( \varepsilon > 0 \) small enough, problem (3.5) admits a unique solution \( \phi = \phi(\xi) \), which satisfies
\[ \|\phi\| \leq C\varepsilon^{\frac{1-\varepsilon}{2}}. \]
Moreover, \( \phi(\xi) \) is of class \( C^1 \) on \( \xi \) with the \( \| \cdot \|_* \)-norm, and
\[
\| D_\xi \phi \|_* \leq C \varepsilon^{1+\frac{s}{2}},
\]
where \( \tau > 0 \) is a small constant.

Proof. Let us consider the operator
\[
A_\varepsilon(\phi) := T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon),
\]
then we know that problem (3.5) is equivalent to the fixed point problem \( \phi = A_\varepsilon(\phi) \).
We will use the contraction mapping theorem to solve it.
Set
\[
E_\rho = \{ \phi \in C_* : \| \phi \|_* \leq \rho \varepsilon^{1+\frac{s}{2}} \},
\]
where \( \rho > 0 \) will be fixed later.

We will show that \( A_\varepsilon \) is a contraction map from \( E_\rho \) to \( E_\rho \).
In fact, for \( \varepsilon > 0 \) small enough, we find
\[
\| A_\varepsilon(\phi) \|_* \leq C \| N_\varepsilon(\phi) + R_\varepsilon \|_* \leq C \left( (\rho \varepsilon)^{\min(p^*-\varepsilon,2)} + (\rho \varepsilon)^{\min(2q-p^*,2)} + \varepsilon^{1+\frac{s}{2}} \right) \leq \rho \varepsilon^{1+\frac{s}{2}},
\]
provided \( \rho \) is chosen large enough, but independent of \( \varepsilon \).
Thus, \( A_\varepsilon \) maps \( E_\rho \) into itself. Moreover,
\[
\| A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2) \|_* \leq C \left( (\rho \varepsilon)^{\min(p^*-1-\varepsilon,1)} + (\rho \varepsilon)^{\min(2q-p^*,-1,1)} \right) \| \phi_1 - \phi_2 \|_*
\]
Hence,
\[
\| A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2) \|_* \leq C \left( (\rho \varepsilon)^{\min(p^*-1-\varepsilon,1)} + (\rho \varepsilon)^{\min(2q-p^*,-1,1)} \right) \| \phi_1 - \phi_2 \|_* \leq \frac{1}{2} \| \phi_1 - \phi_2 \|_*
\]
Thus, there is a unique \( \phi \in E_\rho \), such that \( \phi = A_\varepsilon(\phi) \).
Now consider the differentiability of \( \xi \to \phi(\xi) \).
Let
\[
B(\xi,\phi) = \phi - T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon).
\]
First, we have \( B(\xi,\phi(\xi)) = 0 \). Let us write
\[
D_\phi B(\xi,\phi)[\psi] = \psi - T_\varepsilon(\psi D_\phi N_\varepsilon(\phi)) = \psi + M(\psi),
\]
where
\[
M(\psi) = -T_\varepsilon(\psi D_\phi N_\varepsilon(\phi)).
\]
From (3.7), we find
\[
\| M(\psi) \|_* \leq C \left( \varepsilon^{1+\frac{s}{2}} \min(p^*-1-\varepsilon,1) + \varepsilon^{1+\frac{s}{2}} \min(2q-p^*,-1,1) \right) \| \psi \|_*.
\]
Thus, the linear operator \( D_\phi B(\xi,\phi) \) is invertible in \( C_* \) with uniformly bounded inverse depending continuously on its parameters. Differentiating with respect to \( \xi \), we deduce
\[
D_\xi B(\xi,\phi) = -D_\xi T_\varepsilon[N_\varepsilon(\phi) + R_\varepsilon] - T_\varepsilon[D_\xi N_\varepsilon(\xi,\phi) + D_\xi R_\varepsilon],
\]
where all these expressions depend continuously on their parameters.
By the implicit function theorem, we see that $\phi(\xi)$ is of class $C^1$ and
\[
D\xi \phi = -(D_\phi B(\xi, \phi))^{-1} [D\xi B(\xi, \phi)].
\]
Thus,
\[
\|D\xi (\phi)\|_* \leq C (\|N_\xi (\phi) + R_\xi\|_* + \|D\xi N_\xi (\xi, \phi)\|_* + \|D\xi R_\xi\|_*) \leq C \xi^{1+\tau}.
\]
The proof of Proposition 3.6 is concluded.

4. Proof of Theorem 1.1. In this section, we will prove Theorem 1.1. To do so, we will choose $\xi$ such that $V + \phi$ is a solution of (2.3), where $\phi$ is the map obtained in Proposition 3.6.

Recall that
\[
I_\varepsilon(v) = J_\varepsilon(v) + \frac{\gamma}{q + 1} \int \varepsilon^{-|p' - q|y}|v|^{q+1}, \quad (4.1)
\]

where
\[
J_\varepsilon(v) = \frac{1}{2} \int \varepsilon |v|^2 - \frac{\gamma}{p' + 1 - \varepsilon} \int \varepsilon^{-|p' + 1 - \varepsilon|}.
\]

Define
\[
K_\varepsilon(\xi) = I_\varepsilon(V + \phi).
\]

It is well-known that if $\xi$ is a critical point of $K_\varepsilon(\xi)$, then $V + \phi$ is a solution of (2.3). Next, we will prove that $K_\varepsilon(\xi)$ has a critical point. To this end, we need the next lemma, which is important in finding the critical points of $K_\varepsilon$.

Lemma 4.1. The following expansion holds
\[
K_\varepsilon(\xi) = I_\varepsilon(V) + O(\varepsilon^{1+\tau}),
\]
where $O(\varepsilon^{1+\tau})$ is uniformly in the $C^1$-sense on the vectors $\xi$ satisfying (2.5).

Proof. Note that $DI_\varepsilon(V + \phi)[\phi] = 0$, we have
\[
I_\varepsilon(V + \phi) - I_\varepsilon(V) = \frac{1}{2} D^2 I_\varepsilon(V + t\phi)[\phi, \phi]
= \int (N_\varepsilon(\phi) + R_\varepsilon) \phi + (p' - \varepsilon) \gamma \int \varepsilon^{-|p' + 1 - \varepsilon|} (|V|^{p' + 1 - \varepsilon} - |V + t\phi|^{p' + 1 - \varepsilon}) \phi^2
+ \int q\gamma \int \varepsilon^{-|p' - q|} (|V|^{q-1} - |V + t\phi|^{q-1}) \phi^2.
\]

Since $\|\phi\|_* \leq C \xi^{1+\tau}$, we find that
\[
\int \varepsilon^{-|p' + 1 - \varepsilon|} (|V|^{p' + 1 - \varepsilon} - |V + t\phi|^{p' + 1 - \varepsilon}) \phi^2 
\leq C(\|N_\varepsilon(\phi)\|_* + \|R_\varepsilon\|_*) \|\phi\|_* = O(\varepsilon^{1+\tau}),
\]
\[
\int \varepsilon^{-|p' - q|} (|V|^{q-1} - |V + t\phi|^{q-1}) \phi^2 \leq C \|\phi\|_*^2 \int \varepsilon^{-\sigma|y - \xi|} \left( \sum_{i=1}^k e^{-\sigma|y - \xi_i|} \right)^2 \leq C \|\phi\|_*^2
\]
and
\[
\int_{\mathbb{R}} e^{-(p^*-q)y}|V|^q-1 - |V + t\phi|^{q-1}|\phi|^2 \\
\leq C\|\phi\|^2 \int_{\mathbb{R}} \left( \sum_{i=1}^{k} e^{-\sigma |y-k\xi_i|} \right)^2 \leq C\|\phi\|^2.
\]
Thus,
\[
I_\varepsilon(V + \phi) = I_\varepsilon(V) + O(\varepsilon^{1+\tau}).
\]
Differentiating with respect to \(\xi_\ell\), we get that
\[
\partial_{\xi_\ell} (I_\varepsilon(V + \phi) - I_\varepsilon(V)) = \int_{\mathbb{R}} \partial_{\xi_\ell} \left[ (N_\varepsilon(\phi) + R_\varepsilon)\phi \right]
\]
\[
+ (p^* - \varepsilon) \int_{\mathbb{R}} e^{-\varepsilon y} \partial_{\xi_\ell} \left[ \left( |V|^{p^*-1-\varepsilon} - |V + t\phi|^{p^*-1-\varepsilon} \right) \phi \right]^2
\]
\[
+ q\gamma \int_{\mathbb{R}} e^{-(p^*-q)y} \partial_{\xi_\ell} \left[ \left( |V|^q - |V + t\phi|^{q-1} \right) \phi \right].
\]
Thus, the result follows.

\begin{proof}[Proof of Theorem 1.1] Recalling that
\[
\xi_1 = -\frac{1}{p^* - q} \log \varepsilon + \frac{1}{p^* - q} \log \Lambda_1,
\]
\[
\xi_{i+1} - \xi_i = -\log \varepsilon - \log \Lambda_i, \quad i = 1, 2, \ldots, k - 1.
\]
where \(\delta < \Lambda_i < \frac{1}{\varepsilon}, \delta > 0\) is a fixed constant. To simplify notation, we denote \(\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_k)\). Thus, it is sufficient to find a critical point of the function
\[
\bar{K}_\varepsilon(\Lambda) = \varepsilon^{-1} (K_\varepsilon(\xi(\Lambda)) - ka_0).
\]
From Lemma 4.1 and Proposition 2.2, we have
\[
\bar{K}_\varepsilon(\Lambda) = \Psi_k(\Lambda) + c_1 + c_2 \log \varepsilon + o(1),
\]
where the term \(o(1)\) goes to 0 uniformly as \(\varepsilon \to 0\) and \(c_1, c_2\) are constants depending only on \(k, N\).

Define
\[
F_1(\Lambda_1) = \frac{a_3}{\Lambda_1} + \frac{ka_4}{p^* - q} \log \Lambda_1
\]
and
\[
F_i(\Lambda_i) = a_5 \Lambda_i - (k - i + 1)a_4 \log \Lambda_i, \quad i = 2, \ldots, k.
\]
It is easy to see that for each \(i = 1, \ldots, k\), the function \(F_i(\Lambda_i)\) has exactly one stable minimum point \(\Lambda_i^*\) on \((0, +\infty)\), where
\[
\Lambda_i^* = \frac{(p^* - q)a_3}{ka_4}, \quad \Lambda_i^* = \frac{(k - i + 1)a_4}{a_5}, \quad i = 2, \ldots, k.
\]
Thus, the function \(\Psi_k(\Lambda)\) has a stable critical point \(\Lambda^* = (\Lambda_1^*, \ldots, \Lambda_k^*)\). Therefore, for \(\varepsilon\) small enough, there exists a critical point \(\Lambda^\varepsilon = (\Lambda_1^\varepsilon, \ldots, \Lambda_k^\varepsilon)\) of the function \(\bar{K}_\varepsilon(\Lambda)\), such that \(\Lambda_i^\varepsilon = \Lambda_i^* + o(1)\) as \(\varepsilon \to 0\) for \(i = 1, \ldots, k\).\[\Box\]
For the $\Lambda^\varepsilon$ obtained above, let
\[
\xi^\varepsilon_1 = \frac{1}{p^* - q} \log \frac{\Lambda^\varepsilon_1}{\varepsilon}, \quad \xi^\varepsilon_i = \frac{1}{p^* - q} \log \frac{\Lambda^\varepsilon_1 \cdots \Lambda^\varepsilon_i}{\varepsilon^{(i-1)(p^* - q) + 1}}, \quad i = 2, \ldots, k.
\]

Hence, $\xi^\varepsilon = (\xi^\varepsilon_1, \ldots, \xi^\varepsilon_k)$ is a critical points of $K^\varepsilon(\xi)$ and
\[
v^\varepsilon(y) = \sum_{i=1}^{k} (-1)^i W(y - \xi^\varepsilon_i) + \phi(\xi^\varepsilon) = \sum_{i=1}^{k} (-1)^i W(y - \xi^\varepsilon_i)(1 + o(1))
\]
is a solution of (2.3).

Note that
\[
e^{\xi^\varepsilon_i} = M^\varepsilon_i \varepsilon^{-(i-1) + \frac{1}{p^* - q}}, \quad i = 1, \ldots, k, \quad (4.3)
\]
where
\[
M^\varepsilon_i = (\Lambda^\varepsilon_1)^{\frac{1}{p^* - q}} (\Lambda^\varepsilon_2 \cdots \Lambda^\varepsilon_i)^{-1}.
\]

Set
\[
M_i = (\Lambda^*_1)^{\frac{1}{p^* - q}} (\Lambda^*_2 \cdots \Lambda^*_i)^{-1}, \quad i = 1, \ldots, k.
\]

Thus, using the transformation (2.2), we see
\[
u^\varepsilon(x) = C_N \sum_{i=1}^{k} (-1)^i \frac{M_i \varepsilon^{-(i-1) + \frac{1}{p^* - q}}}{1 + M_i \varepsilon^{(i-1) + \frac{1}{p^* - q}}} (1 + o(1)),
\]
and the proof of Theorem 1.1 is concluded. \qed

5. **Proof of Theorem 1.2.** In this section, we will give the proof of Theorem 1.2. Since the main idea is very similar to the proof of Theorem 1.1, we just present the needed modifications and sketch its proof. To do this, we consider
\[
\begin{cases}
-u'' - \frac{N-1}{r} u' = |u|^{p^* - 1 + \varepsilon} u - |u|^{q - 1} u, & \text{in } \mathbb{R}, \\
u'(0) = 0, \quad u(r) \to 0, & \text{as } r \to \infty,
\end{cases} \quad (5.1)
\]
where $\varepsilon > 0$, $q > p^*$ is fixed.

Define the following transformation
\[
v(y) = r^{\frac{N-2}{2}} u(r), \quad r = e^{\frac{n-1}{2} y}, \quad y \in \mathbb{R}. \quad (5.2)
\]

Then, problem (5.1) becomes
\[
\begin{cases}
v'' + v + \gamma e^{-\varepsilon y} |v|^{p^* - 1 + \varepsilon} v - \gamma e^{(p^* - q) y} |v|^{q - 1} v = 0 & \text{in } \mathbb{R}, \\
v(y) \to 0 & \text{as } |y| \to +\infty,
\end{cases} \quad (5.3)
\]
where
\[
\gamma = \frac{4}{(N-2)^2}.
\]

The energy functional associated to problem (5.3) is
\[
\hat{I}_\varepsilon(v) = \hat{J}_\varepsilon(v) + \frac{\gamma}{p + 1} \int_{\mathbb{R}} e^{(p^* - q) y} |v|^{p + 1},
\]
where
\[
\hat{J}_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}} |v'|^2 + |v|^2 - \frac{\gamma}{p + 1 + \varepsilon} \int_{\mathbb{R}} e^{-\varepsilon y} |v|^{p^* + 1 + \varepsilon}.
\]
For small $\varepsilon > 0$, we define
\[
\hat{\xi}_1 = -\frac{1}{q-p^*} \log \varepsilon + \frac{1}{q-p^*} \log \hat{\Lambda}_1,
\]
\[
\hat{\xi}_{i+1} - \hat{\xi}_i = -\log \varepsilon - \log \hat{\Lambda}_{i+1}, \quad i = 1, 2, \cdots, k-1,
\]
where $\hat{\Lambda}_i, i = 1, 2, \cdots, k$ are positive constants. We will find a solution of problem (5.1) of the following form
\[
\hat{v} = \hat{V} + \hat{\phi},
\]
where $\hat{V} = \sum_{i=1}^k (-1)^i W(y - \hat{\xi}_i)$, $\hat{\phi}$ is small.

Similar to Proposition 2.2, we have

**Proposition 5.1.** For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have the following asymptotic expansion
\[
\hat{I}_\varepsilon(\hat{V}) = k\hat{a}_1 + k\hat{a}_4 \log \varepsilon + \varepsilon \hat{\Psi}_k(\hat{\Lambda}) + \varepsilon \hat{\Theta}_\varepsilon(\hat{\Lambda}),
\]
where $\hat{\Lambda} = (\hat{\Lambda}_1, \hat{\Lambda}_2, \cdots, \hat{\Lambda}_k)$,
\[
\hat{\Psi}_k(\hat{\Lambda}) = \frac{\hat{a}_3}{\hat{\Lambda}_1} + \frac{k\hat{a}_4}{p-p^*} \log \hat{\Lambda}_1 + \sum_{i=2}^k \left[ \hat{a}_5 \hat{\Lambda}_i - (k-i+1)\hat{a}_4 \log \hat{\Lambda}_i \right]
\]
and $\hat{\Theta}_\varepsilon \to 0$, as $\varepsilon \to 0$ uniformly in $C^1$-norm on the set of $\hat{\Lambda}_i$'s with $\delta < \hat{\Lambda}_i < \delta^{-1}$, $i = 1, 2, \cdots, k$. Here $\hat{a}_i$, $i = 0, \cdots, 5$ are given by
\[
\begin{align*}
\hat{a}_0 & = \frac{1}{2} \int \frac{|W'|^2 + |W|^2}{p^*+1} \int |W|^{p^*+1}, \\
\hat{a}_1 & = \frac{1}{(p^*+1)^2} \int W^{p^*+1} \frac{\gamma}{p^*+1} \int W^{p^*+1} \log W, \\
\hat{a}_2 & = \frac{\gamma}{(q-p^*)} \frac{1}{p^*+1} \int W^{p^*+1} + (k-1) \frac{\gamma}{2(p^*+1)} \int W^{p^*+1}, \\
\hat{a}_3 & = \frac{\gamma}{p^*+1} \int e^{-(q-p^*)} W^{p^*+1}, \\
\hat{a}_4 & = \frac{\gamma}{p^*+1} \int W^{p^*+1}, \\
\hat{a}_5 & = \gamma C_N \int e^{\frac{1}{2} W^{p^*}}.
\end{align*}
\]

**Proof of Theorem 3.3.** Let $\hat{\phi}$ be the map obtained similarly as in the reduction procedure in Section 3. Define
\[
\hat{K}_\varepsilon(\xi) = \hat{I}_\varepsilon(\hat{V} + \hat{\phi}).
\]
Thus,
\[
\hat{K}_\varepsilon(\xi) = \hat{I}_\varepsilon(\hat{V}) + O(\varepsilon^{1+\tau}).
\]
It is easy to check that $\hat{\Psi}_k(\hat{\Lambda})$ has a unique stable minimum point given by
\[
\hat{\Lambda}^* = \left( \frac{\hat{a}_3}{\hat{a}_4}, \frac{\hat{a}_4}{\hat{a}_5}, \frac{\hat{a}_5}{\hat{a}_5}, \cdots, \frac{\hat{a}_5}{\hat{a}_5} \right) \quad : = (\hat{\Lambda}_1^*, \hat{\Lambda}_2^*, \cdots, \hat{\Lambda}_k^*).
\]
Set
\[
\bar{K}_\varepsilon(\hat{\Lambda}) = \varepsilon^{-1} \left( \hat{K}_\varepsilon(\hat{\xi}(\hat{\Lambda})) - k\hat{a}_0 \right).
\]
Thus, we have
\[
\bar{K}_\varepsilon(\hat{\Lambda}) = \hat{\Psi}_k(\hat{\Lambda}) + \hat{c}_1 + \hat{c}_2 \log \varepsilon + o(1),
\]
where the term $o(1)$ goes to 0 uniformly as $\varepsilon \to 0$ and $\hat{c}_1, \hat{c}_2$ are constants depending only on $k, N$. Therefore, for $\varepsilon$ small enough, there exists a critical point $\hat{\Lambda}^* = (\hat{\Lambda}_1^*, \cdots, \hat{\Lambda}_k^*)$ of the function $K_{\varepsilon}(\hat{\Lambda})$, such that $\hat{\Lambda}^* \to \hat{\Lambda}^*$ as $\varepsilon \to 0$.

For the $\hat{\Lambda}^*$ obtained above, let

$$
\hat{\xi}_i^* = \frac{1}{q - p^*} \log \frac{\hat{\Lambda}_i^{\varepsilon}}{\varepsilon}, \quad \hat{\xi}_i^* = \frac{1}{q - p^*} \log \frac{\hat{\Lambda}_i^{\varepsilon}}{(\hat{\Lambda}_2^{\varepsilon} \cdots \hat{\Lambda}_k^{\varepsilon})^{p^* - p^*}} (-1) (q - p^*) \varepsilon^{(i - 1)(q - p^*) + 1}, \quad i = 2, \cdots, k.
$$

Hence, $\hat{\xi}^* = (\hat{\xi}_1^*, \cdots, \hat{\xi}_k^*)$ is a critical points of $K_{\varepsilon}(\hat{\xi})$ and

$$
\hat{u}_\varepsilon(y) = \sum_{i=1}^{k} (-1)^i W(y - \hat{\xi}_i^*) + \hat{\phi}(\hat{\xi}^*) = \sum_{i=1}^{k} (-1)^i W(y - \hat{\xi}_i^*)(1 + o(1))
$$

is a solution of (5.3).

Note that

$$
e^{\hat{\xi}_i^*} = \hat{M}_i^{\varepsilon} (-1)^{i - 1} \varepsilon^{(q - p^*)}, \quad i = 1, \cdots, k,$
$$

where

$$
\hat{M}_i^{\varepsilon} = (\hat{\Lambda}_1^{\varepsilon} \hat{\Lambda}_2^{\varepsilon} \cdots \hat{\Lambda}_k^{\varepsilon})^{-1}.
$$

Set

$$\hat{M}_i = (\hat{\Lambda}_1^*)^{-1} \hat{\Lambda}_i^{\varepsilon} (\hat{\Lambda}_2^* \cdots \hat{\Lambda}_k^*)^{-1}, \quad i = 1, \cdots, k.
$$

Hence, we can deduce that

$$
\hat{u}_\varepsilon(x) = C_N \sum_{i=1}^{k} (-1)^i \hat{M}_i^{\varepsilon} (-1)^{i - 1} \varepsilon^{(q - p^*)} \left(1 + \hat{M}_i^{\varepsilon} (-1)^{i - 1} \varepsilon^{(q - p^*)} (\varepsilon^{(q - p^*)} |x|^2) \right)^{\frac{n - p}{2}} (1 + o(1)),
$$

and this completes the proof of Theorem 1.2.

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