SOLITON SOLUTIONS, LIOUVILLE INTEGRABILITY AND GAUGE EQUIVALENCE OF SASA SATSUMA EQUATION

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Abstract

Exact integrability of the Sasa Satsuma equation (SSE) in the Liouville sense is established by showing the existence of an infinite set of conservation laws. The explicit form of the conserved quantities in terms of the fields are obtained by solving the Riccati equation for the associated $3 \times 3$ Lax operator. The soliton solutions, in particular, one and two soliton solutions, are constructed by the Hirota’s bilinear method. The one soliton solution is also compared with that found through the inverse scattering method. The gauge equivalence of the SSE with a generalized Landau Lifshitz equation is established with the explicit construction of the new equivalent Lax pair.
I. Introduction

Nonlinear Schrödinger equation and its various generalized versions (higher order nonlinear Schrödinger equation) is well known in describing various physical phenomena [1]. A common property in all these physical systems is the appearance of solitons, as a result of a balance between the nonlinear and dispersive terms of the wave equations. With the advancement of experimental accuracy, solitons having more complicated dynamics can also be detected and observed now [2]. The Sasa Satsuma equation [3]

$$iQ_T + \frac{1}{2}Q_{XX} + |Q|^2Q + \frac{i}{6\epsilon}(Q_{XXX} + 6|Q|^2Q_X + 3|Q|^2Q) = 0 \quad (1)$$

describing the evolution of a complex scalar field, is an example of such a system, whose soliton solutions have been obtained through inverse scattering method (ISM) in [3].

A limited class of soliton bearing equations exhibits further interesting properties and belongs to the exclusive club of integrable systems. The most prominent definition of integrability is the integrability in the Liouville sense, i.e., the existence of a set of infinite numbers of conserved quantities in involution [4], which can be considered as the action variables. This criterion of integrability is extendable also to the quantum case. The Lax pair associated with the model is usually a sign of such integrability, while the Painlevé singularity analysis [5] is supposed to be a direct test of integrability for the given equation.

It should be mentioned that the Lax pair for the SSE as well as its one soliton through ISM were found in [3], while the Painleve analysis for the equation was
carried out in [6]. However, extraction of the higher conserved quantities for the Sasa Satsuma system and thus establishing the integrability of the whole hierarchy in the Liouville sense remained unexplored. A possible reason of this may be the difficulty involved due to the unusual $3 \times 3$ matrix form of the Lax operator, associated with the SSE.

Our objective is, therefore, to find the Riccati equation for the $3 \times 3$ Lax operator, associated with the SSE and consequently, to obtain the whole hierarchy of conserved charges in a systematic way. This construction will be somewhat involved due to the extended form of the Lax operator.

For investigating SSE from a different viewpoint, we further find the explicit soliton solutions of the equation through Hirota’s bilinear method. This is a direct and much more effective method compared to ISM for obtaining the soliton solutions, since it does not require the knowledge of the Lax pair. Moreover, the construction of the $\tau$ function becomes straightforward in this method.

One should recall in this context another interesting fact about the NLS equation that it is gauge related to the well-known Landau Lifshitz equation (LLE) [7]. This equivalence can also be established through the space curve method [8]. It is, therefore, natural to ask what is the gauge equivalent equation to the SSE. Though such an equivalent system has been discovered by the space curve method in [6], we complete the investigation for SSE by showing the equivalence of it through an explicit gauge transformation, which not only reproduces the generalized LLE (GLLE), but also constructs the associated Lax operator.
pair for the GLLE.

The organization of this paper is as follows. In section II we study the soliton solutions of SSE by Hirota’s bilinear method. We compute explicitly the one and two soliton solutions and compare our result of one soliton solution with the known one. In section III, we construct the related Riccati equation using the $3 \times 3$ Lax operator of the SSE and subsequently find the infinite number of conserved quantities through the recursion relation. The time invariance of the conserved quantities is checked directly by using the evolution equation. This proves the Liouville integrability of the SSE. The section IV, provides the gauge equivalent generalized LLE and gives the associated new Lax operators in the explicit form. Section V is the concluding one.

II. Soliton Solutions Through Hirota’s Method

Let us begin with the SSE, which through a change of variable and a Galelian transformation:

$$Q(X, T) = u(x, t) \exp \{ i \epsilon (x + \frac{\epsilon t}{6}) \}$$

$$T = t$$

$$X = x + \frac{\epsilon t}{2}$$

may be simplified to the form

$$u_t + \frac{1}{6 \epsilon} (u_{xxx} + 6 |u|^2 u_x + 3(|u|^2)_x u) = 0$$

This is an example of a complex modified KdV type equation and goes to mKdV
for the real valued field. The associated spectral problem can be studied through the pair of linear equations

\[ \Psi_x = U(x, t, \lambda)\Psi \quad (4a) \]
\[ \Psi_t = V(x, t, \lambda)\Psi \quad (4b) \]

where \( U(x, t, \lambda) \) and \( V(x, t, \lambda) \) are \( 3 \times 3 \) matrices and \( \lambda \) is the spectral parameter. The explicit form of \( U \) and \( V \) may be given using the result of \( \mathbb{3} \) as

\[ U = -i\lambda \Sigma + A \quad (5a) \]
\[ V = -i4\epsilon \lambda^2 \Sigma + 4\epsilon(\lambda^2 - |u|^2)A \\
-2\epsilon \lambda \Sigma(A^2 - A_x) - \epsilon A_{xx} + \epsilon(A_x A - AA_x) \quad (5b) \]

with

\[ \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]
\[ A = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix} \]

Compatibility of (4a) and (4b) leads to SSE (3), which can be shown easily by using the following properties of \( \Sigma \) and \( A \) matrices

\[ \{\Sigma, A\} = 0 \]
\[ \Sigma^2 = 1 \]
\[ A^3 + 2|u|^2 A = 0 \]

Note that SSE in the form (3), though suitable for studying inverse scattering technique, is not convenient for casting it into Hirota’s bilinear form. On the
other hand, the higher order nonlinear Schrödinger equation form (1) for SSE is more suitable for this purpose. Now, in order to write (1) in the bilinear form, we make the transformation

$$Q(X, T) = G(X, T)/F(X, T)$$

(6)

where, $G$ is complex and $F$ is real. Consequently, in these new variables, we have the following set of equations

$$iD_T + \frac{1}{2}D_X^2 + \frac{i}{6\epsilon}D_X^3)G.F = 0$$

(7a)

$$D_X^2 F.F = 4G^*G$$

(7b)

$$(1 - \frac{2i}{\epsilon}D_X)G^*G = 0$$

(7c)

which follow from (1). $D_T, D_X, D_{XX}$ etc. in (7) are Hirota derivatives [9]. (7) belongs to a new class of bilinear equations, whose general form would be of the type

$$B(D_X, D_T, \cdots)G.F = 0$$

(8a)

$$A(D_X, D_T, \cdots)F.F = C(D_X, D_T, \cdots)G^*G$$

(8b)

$$E(1 - D_X, D_T, \cdots)G^*G = 0$$

(8c)

The additional bilinear equation (7c) involving $G^*G$ imposes one further condition on the complex parameter $P$ (shown below), which is absent in other examples of higher order nonlinear Schrödinger equations [10].

For obtaining one soliton solution of SSE (1), we choose $G$ and $F$ in the
following form

\[ G = L \exp(\eta) \]  \hspace{1cm} (9a)

\[ F = 1 + K \exp(\eta + \eta^*) \]  \hspace{1cm} (9b)

where, \( L \) is a complex c-number parameter and

\[ \eta = PX + \Omega T + \ldots \]  \hspace{1cm} (10)

with \( P, \Omega \) are in general complex parameters. Substituting the expressions (9) for \( G \) and \( F \) in (7), we see that \( G \) and \( F \) are the solutions of (7) provided the following relations hold

\[ i\Omega + \frac{1}{2}p^2 + i\frac{p^3}{6\epsilon} = 0 \]  \hspace{1cm} (11a)

\[ K = \frac{LL^*}{2\mu'^2} \]  \hspace{1cm} (11b)

\[ P - P^* = 2i\epsilon \]  \hspace{1cm} (11c)

Where the complex parameter, \( P \) is of the form

\[ P = \mu + i\epsilon \]  \hspace{1cm} (12a)

(11a) is nothing but the dispersion relation and (11b) determines \( K \). In the above solution, so far, \( L \) is an arbitrary complex parameter. We will see shortly that in order to compare our result with the one obtained through the ISM \[ 3 \], the parameter \( L \) is to be chosen in a specific form. It follows from the dispersion relation (11a) and the expression of \( P \) (12a) that \( \Omega \) should be of the form

\[ \Omega = -\mu \left( \frac{\mu^2}{6\epsilon} + \frac{\epsilon}{2} \right) - i\frac{\epsilon^2}{3} \]  \hspace{1cm} (12b)
Substituting (12a) and (12b) in (10) and using (11b) the one soliton solution in the explicit form becomes

\[ Q(X, T) = L \exp \eta \left( \frac{1}{1 + K \exp(\eta + \eta^*)} \right) \]
\[ = \frac{L \exp \{ (\mu + i\epsilon)X - (\frac{\mu^3}{3\epsilon} + \frac{\mu^2}{2} + i\frac{\epsilon}{3})T \}}{1 + \frac{L^2}{2\mu^2} \exp \{ 2\mu X - (\frac{\mu^2}{3\epsilon} + \mu \epsilon)T \}} \]  \hspace{1cm} (13)

To compare (13) with that of ISM, we choose \( L \) as

\[ L = 2\mu \exp (-i\epsilon X^{(1)} - \mu X^{(0)}) \]  \hspace{1cm} (14)

which reduces (13) to the form

\[ Q(X, T) = \frac{\frac{\mu}{\sqrt{2}} \exp \{ i\epsilon (X - \frac{\epsilon}{3}T - X^{(1)}) \}}{\cosh (\mu X - \mu (\frac{\mu^2}{6\epsilon} + \frac{\epsilon}{2})T - \mu X^{(0)}) + \log \sqrt{2})} \]  \hspace{1cm} (15)

and this is in agreement with the ISM result of [3].

Two soliton solution of the Sasa Satsuma equation may be obtained, following [10], by choosing \( G \) and \( F \) of the form

\[ G = L(\exp \eta_1 + M \exp (\eta_1 + \eta_A)) \]  \hspace{1cm} (16a)

\[ F = 1 + k \exp (\eta_1 + \eta_1^*) + \exp (\eta_A) + kMM^* \exp (\eta_1 + \eta_1^* + \eta_A) \]  \hspace{1cm} (16b)

where, \( \eta_1 \) is complex as in the case of one soliton, but \( \eta_A \) is chosen to be real.

Substituting \( F \) and \( G \) (16) in the bilinear forms (7), we observe that \( \eta_1 \) soliton satisfies the same dispersion relation as (11a), whereas the dispersion relation for \( \eta_A \) becomes \( P_A^2 = 0 \). The degree 2 term in (7a) determines the parameter \( M \) to be unity. Once again following [10], we define the degree of a term by the number of \( \eta \)'s present in the exponent. Now the degree 2 and 4 terms in (7b),
yield the value of $k$ as in (11b). However, because of the additional bilinear form (7c) in our case, we obtain one more relation, $\epsilon(P_1^* - P_1) = 2i$. Note that, in general for complex bosonic systems having complex parameters, the two soliton solutions may have some relation, which is analogous to the three soliton conditions [10]. But, the choice of the real parameter $\eta_A$, makes the three soliton condition trivial in our case. A more general choice of $F$ and $G$ for two soliton solutions will give such a nontrivial condition.

III. Riccati Equation and The Conserved Quantities

To show the Liouville integrability, i.e. the existence of an infinite number of conserved quantities related to SSE (3), we first write the associated Riccati equation. Since the Lax operators in this case are $3 \times 3$ matrices, the Riccati equation becomes more complicated, though tactable. Let us write the auxiliary field $\Psi$ in the component form as

$$\Psi = \begin{pmatrix} \chi_{1x} \\ \chi_{2x} \\ \chi_{3x} \end{pmatrix}$$

Substituting (17) in (4a), we get a set of three coupled equations

$$\begin{align*}
\chi_{1x} &= -i\lambda\chi_1 + u\chi_3 \\
\chi_{2x} &= -i\lambda\chi_2 + u^2\chi_3 \\
\chi_{3x} &= i\lambda\chi_3 - u^*\chi_1 - u\chi_2
\end{align*}$$

Now expressing (18) in terms of $\Gamma_1 = (\chi_1/\chi_3)$ and $\Gamma_2 = (\chi_2/\chi_3)$, and eli-
nating $\chi_1$, $\chi_2$ and $\chi_3$, one obtains

\begin{align}
\Gamma_{1x} &= u - 2i\lambda \Gamma_1 + u^* \Gamma_1^2 + u \Gamma_1 \Gamma_2 \quad (19a) \\
\Gamma_{2x} &= u^* - 2i\lambda \Gamma_2 + u \Gamma_2^2 + u^* \Gamma_1 \Gamma_2 \quad (19b)
\end{align}

The first order nonlinear coupled equations (19) for $\Gamma_1$ and $\Gamma_2$ are the Riccati equations in our case. Notice that neither the integral of $\Gamma_1$ nor of $\Gamma_2$, plays the role of generating functions for conserved quantities, but a suitable combination of them does. The infinite number of conserved quantities (Hamiltonians), $H_{2n+1}$; $n = 0, 1, 2, \ldots$ can be obtained from (19) by identifying

\begin{align}
a(\lambda) &= \exp(-i\lambda x) \mid_{x \to \infty} \Psi_3(\infty, \lambda) = \exp\left\{ -\int_{-\infty}^{\infty} (u^* \Gamma_1 + u \Gamma_2) dx \right\} \quad (20a)
\end{align}

where $H_{2n+1}$ are related to $a(\lambda)$ as

\begin{align}
\ln a(\lambda) &= -2 \sum_{n=0}^{\infty} (2i)^{-2n-1} H_{2n+1} \lambda^{-2n-1} \quad (20b)
\end{align}

We will see that Hamiltonians with odd indices only survive, while the terms with even indices become trivial. This property is similar to that of the real KdV or the modified KdV equation.

We may look for series solutions of (19) by assuming $\Gamma_1$ and $\Gamma_2$ in the form

\begin{align}
\Gamma_1 = \sum_{n=0}^{\infty} C_n^1 \lambda^{-n} \quad (21a) \\
\Gamma_2 = \sum_{n=0}^{\infty} C_n^2 \lambda^{-n} \quad (21b)
\end{align}

which yield the following recursion relations from (19a) and (21):
\[ C_0^1 = 0 \quad C_1^1 = \frac{u}{2i} \]

\[ 2i C_{n+2}^1 = -\left( C_{n+1}^1 \right)_x + \sum_{m=0}^{n+1} \left( u^* C_m^1 C_{n-m+1}^1 + u C_m^1 C_{n-m+1}^2 \right). \]  \hspace{1cm} (22)

Similarly (19b) and (21) determine another, though quite similar, set of recursion relations given as

\[ C_0^2 = 0 \quad C_1^2 = \frac{u^*}{2i} \]

\[ 2i C_{n+2}^2 = -\left( C_{n+1}^2 \right)_x + \sum_{m=0}^{n+1} \left( u^* C_m^2 C_{n-m+1}^2 + u C_m^2 C_{n-m+1}^1 \right). \]  \hspace{1cm} (23)

Inserting the expressions of \( C_n^1 \) and \( C_n^2 \), thus obtained through the recursion relations (22), (23) in (21), we get from (20a,b) the explicit form of all conserved quantities, \( H_{2n+1} \). These expressions are the integrals taken over the functions of the fields \( u \) and \( u^* \) and their derivatives. The first few conserved quantities of the infinite set of the SSE hierarchies are given by

\[ H_1 = \int u^* u \, dx \]  \hspace{1cm} (24)

\[ H_3 = \int (-u^* u_x + 2 | u |^4) \, dx \]  \hspace{1cm} (25)

\[ H_5 = \int (u^* u_{xx} u_{xxx} - 4 | u |^2 u^* u_x + (| u |^2)_x^2 + 8 | u |^6) \, dx \]  \hspace{1cm} (26)

We have checked explicitly by using equations of motion (3) that \( H_1, H_3 \) and \( H_5 \) are, indeed, the constants of motion.
IV. Generalised Landau Lifshitz Type Equation as The Gauge Equivalent System

We now show an interesting connection between the SSE in the form (3) and the generalized Landau Lifshitz type equation, by exploiting the gauge equivalence of the Lax pairs of these two dynamical systems. The procedure is similar to that between the NLS and the standard Landau Lifshitz equation [7].

Under a local gauge transformation, the Jost function, $\Psi(x, t, \lambda)$ changes as

$$\tilde{\Psi}(x, t, \lambda) = g^{-1}(x, t)\Psi(x, t, \lambda)$$  \hspace{1cm} (27)

where $g(x, t) = \Psi(x, t, \lambda)|_{\lambda=0}$, may be taken as an element of the gauge group. Consequently, the Lax equations (4) under this gauge transformation (27) become

$$\tilde{\Psi}_x = \tilde{U}(x, t, \lambda)\tilde{\Psi}$$ \hspace{1cm} (28a)

$$\tilde{\Psi}_t = \tilde{V}(x, t, \lambda)\tilde{\Psi}$$ \hspace{1cm} (28b)

where $\tilde{U}$ and $\tilde{V}$ are the new gauge transformed Lax pair, given by

$$\tilde{U}(x, t, \lambda) = g^{-1}(U - U_0)g$$ \hspace{1cm} (29a)

$$\tilde{V}(x, t, \lambda) = g^{-1}(V - V_0)g$$ \hspace{1cm} (29b)

with $U_0 = U|_{\lambda=0} = g_x(x, t)g^{-1}(x, t)$ and $V_0 = V|_{\lambda=0} = g_t(x, t)g^{-1}(x, t)$.

We may identify the spin field of the Landau Lifshitz type equation as

$$S = g(x, t)^{-1} \Sigma g(x, t) \hspace{1cm} S^2 = 1.$$ \hspace{1cm} (30)
With this identification, the gauge transformed Lax pair (29) can be expressed in terms of the spin field $S$ (30) and its derivatives only, yielding

\begin{align}
\dot{U} &= -i\lambda S \\
\dot{V} &= -4i\epsilon\lambda^3 S + 2\epsilon\lambda^2 SS_x + i\epsilon\lambda(S_{xx} + \frac{3}{2}SS_x^2)
\end{align}

(31a) (31b)

In deriving (31) one has to use the following important identities

$SS_x = 2g^{-1}A g$, $SS_x^2 = -4g^{-1}\Sigma A^2 g$ and $S_{xx} + SS_x^2 = 2g^{-1}\Sigma A_x g$.

The zero curvature condition of (31);

\[ \ddot{U} - \dot{V}_x + [\dot{U}, \dot{V}] = 0 \]

(32)

leads to the generalised Landau Lifshitz type equation

\[ S_t + \epsilon S_{xxx} + \frac{3}{2}\epsilon(S^3 + SS_x^2) = 0 \]

(33)

with $S \in SU(3)/U(1)$.

V. Conclusion

In this paper, we have bilinearised higher order nonlinear Schrödinger equation, vis a vis the SSE following Hirota’s method. The Hirota’s method is an effective and important method to obtain multi soliton solutions. We have found explicitly one and two soliton solutions and recovered the ISM result of one soliton solution from that of Hirota’s method after a specific choice of the parameter involved. The result related to higher soliton solutions are more complicated and will be given elsewhere. It is found that the SSE falls under a new class of bilinear forms.
The linear problem of SSE is a nontrivial one in the sense that the Lax operator corresponding to this dynamical equation is a $3 \times 3$ matrix, which makes the related Riccati equation more involved. However, by solving such coupled Riccati equations we are able to compute explicitly the infinite number of conserved quantities through the recursion relations and to show that terms with odd indices only contribute to the conserved charges like KdV and mKdV systems. This result establishes explicitly the integrability of the SSE in the Liouville sense. Finding out the Poisson bracket structures among the dynamical fields and consequently revealing the explicit form of the hierarchy of SSE from the conserved charges obtained here will be an interesting future problem.

The gauge equivalence of the GLLE with the SSE has been established here. This equivalence not only gives a direct relationship between the fields, which would help to find the soliton solutions of the GLLE using those of the SSE, but also yields explicit Lax operators for GLLE from which one would be able to extract the related higher conserved quantities in the similar way following the present results of SSE.

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