Extraction of Force-Chain Network Architecture in Granular Materials Using Community Detection

Danielle S. Bassett\textsuperscript{1,2,}\textsuperscript{*}, Eli T. Owens\textsuperscript{3,4}, Mason A. Porter\textsuperscript{5,6}, M. Lisa Manning\textsuperscript{7}, Karen E. Daniels\textsuperscript{3}

\textsuperscript{1} Department of Bioengineering, University of Pennsylvania, Philadelphia, PA 19104, USA; \textsuperscript{2} Department of Electrical Engineering, University of Pennsylvania, Philadelphia, PA 19104, USA; \textsuperscript{3} Department of Physics, North Carolina State University, Raleigh, NC 27695, USA; \textsuperscript{4} Department of Physics, Presbyterian College, Clinton, SC 29325, USA; \textsuperscript{5} Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK; \textsuperscript{6} CABDyN Complexity Centre, University of Oxford, Oxford, OX1 1HP, UK; \textsuperscript{7} Department of Physics, Syracuse University, Syracuse, NY 13244, USA

\textsuperscript{*} Corresponding author. E-mail address: dsb@seas.upenn.edu

(Dated: August 19, 2014)

Force chains form heterogeneous physical structures that can constrain the mechanical stability and acoustic transmission of granular media. However, despite their relevance for predicting bulk properties of materials, there is no agreement on a quantitative description of force chains. Consequently, it is difficult to compare the force-chain structures in different materials or experimental conditions. To address this challenge, we treat granular materials as spatially-embedded networks in which the nodes (particles) are connected by weighted edges that represent contact forces. We use techniques from community detection, which is a type of clustering, to find groups of closely connected particles. By using a geographical null model that is constrained by the particles’ contact network, we extract chain-like structures that are reminiscent of force chains. We propose three diagnostics to measure these chain-like structures, and we demonstrate the utility of these diagnostics for identifying and characterizing classes of force-chain network architectures in various materials. To illustrate our methods, we describe how force-chain architecture depends on pressure for both laboratory experiments and numerically-generated frictionless packings. By resolving individual force chains, we quantify statistical properties of force-chain shape and strength, which are potentially crucial diagnostics of bulk properties (including material stability). These methods facilitate quantitative comparisons between different particulate systems, regardless of whether they are measured experimentally or numerically.

PACS numbers: 89.75.Fb, 64.60.aq, 81.05.Rm

I. INTRODUCTION

Particulate matter comes in many forms: it ranges from frictionless emulsions and frictional granular materials to bonded composites and biological cells. A long-known hallmark of particulate systems is the heterogeneous distribution of inter-particle forces within them\textsuperscript{[1,4]}—a phenomenon that has come to be known as force chains. The term “force chain” arises from the appearance—particularly in two-dimensional (2D) systems—of chains of particles that transfer force from one to another along the chain. In Fig. 1, for example, the eye readily focuses on the backbone that arises from the set of bright particles. These particles experience forces that are larger than average, and one can see such forces by using photoelastic particles and placing them between crossed polarizers. In both two and three dimensions, it is increasingly common to measure inter-particle forces in a variety of systems—including frictional grains\textsuperscript{[5,7]}, frictionless grains\textsuperscript{[8]}, and frictionless emulsions\textsuperscript{[9,11]}. This opens the door for achieving a better understanding of force chains in a variety of systems.

It is known that force chains are important for resisting shear\textsuperscript{[5]} and directing sound propagation\textsuperscript{[13,16]}, but little is known about which properties of these chains are universal to particulate systems and which are sensitive to details (such as friction, adhesion, boundary conditions, and body forces like gravity). Improved methods for automatically identifying force chains and quantifying their specific properties (e.g. size and shape) would yield a deeper understanding of how they impact mechanical properties of particulate systems.

Although the presence of force chains is a generic feature of particulate systems, the term “force chain” is often used colloquially, and the field still lacks a quantitative definition of the term. Recently, several techniques have been proposed that aim to identify which subset of the particles form a “force-chain network.” Peters et al.\textsuperscript{[17]} calculated force chains in low friction by setting two requirements: particles must occur in a “quasi-linear” arrangement, and they must contain a concentrated stress. They used an algorithm that requires a choice of a threshold value for each of the two conditions, and their notion does not allow force chains to branch. Using discrete-element simulations with a flat-punch geometry, Peters et al.\textsuperscript{[17]} obtained force chains that contained approximately half of the particles in a system. Additionally, the lengths of these chains satis-
FIG. 1. (Color Online) We represent granular materials as spatially-embedded networks [12] whose nodes (particles) are connected by weighted edges that represent contact forces. (A) Image of a 2D vertical aggregate of photoelastic disks that are confined in a single layer by a pressure of $P \approx 6.7 \times 10^{-4} \text{E}$. Several particles are embedded with a piezoelectric sensor, for which the wires are visible. (B) The internal stress pattern in the photoelastic particles manifests as a network of force chains. (C) The blue line segments show the edges of a weighted graph, which we determine from image processing and overlay on the image from panel (B). An edge between two particles (i.e., nodes) exists if the two particles are in physical contact with each other; the forces between particles give the weights of the edges.

fied an approximately exponential distribution. Arevalo et al. [13] examined polygonal structures that underly a force-chain network, which they determined by selecting all inter-particle forces above some threshold. In their numerical simulations, they found that triangular structures dominate the network near a critical packing fraction. More recently, topological tools such as computational homology have been used to address the question of how to determine the structure of force chains [19-21]. For example, in numerical simulations Kondic et al. [19] were able to distinguish between frictional and frictionless packings using a topological invariant known as the 0th Betti number [22]. They demonstrated that the 0th Betti number changed with packing density, and used a force threshold to isolate strong forces from weak forces.

Each of the above techniques has been informative, but they also share a common viewpoint that the strongest inter-particle forces form the backbone of a particulate system. However, even a linear chain of strong particles would not be stable against buckling without the participation of the particles that lie alongside them [23]. Therefore, it is necessary to develop techniques that do not include a minimum threshold force to be able to consider a particle part of the network.

The purpose of the present paper is to use an approach based on network science [24] to develop an example of one appropriate method. Network science provides a powerful set of tools to represent complex systems by focusing not only on the components of such systems, but also on the interactions among those components. A network representation is particularly appropriate for particulate systems, where particles (network nodes) are connected to one another by contact forces (network edges). In Fig. 1 we show an example network, which we constructed from laboratory experiments. Network approaches in granular materials have already had several success in describing the dynamics of granular materials. For example, prior studies have investigated spatial patterns in the breaking of edges in a (binary) contact network under shear [25-28] and the influence of network topology on acoustic propagation [29]. The latter study established the importance of using weighted contact networks to take into account the strength of inter-particle forces. Several other papers have also recently contributed to this line of inquiry using a variety of different networks-based approaches [30-33].

In the present paper, we use network representations to build a set of practical methods to (1) automatically extract force chains from force networks (without the use of thresholding) and (2) define a set of three scalar quantities that one can use to characterize and classify different particulate materials. In Section II we present two commonly-used granular systems in which it is possible to measure inter-particle forces. These two case studies—one in a “wet” laboratory and the other based on a standard computational model for jammed packings—
provide a basis for describing the utility of our new technique. In Section III we describe our method to automatically extract the force-chain structure from a force network.

Our method uses community detection (a form of clustering for networks) with a geographical null model to account for the spatial embeddedness of particulate systems, and then calculate a gap factor that allows us to optimize the resolution at which the detected communities are maximally branched. This framework yields three diagnostics—size, network force, and gap factor—that are helpful for characterizing the resulting structures of particles. (We define these diagnostics in Section III B.) We then apply our methodology to force networks that we measured in two case studies. We demonstrate the utility of our diagnostics for (1) quantifying changes in force-chain structure as a function of the confining pressure applied to a granular material and (2) facilitating the comparison and classification of force-chain network architectures across different media. We conclude and discuss several implications of our work in Section IV.

II. METHODS

To better understand which features of force chains are universal and to demonstrate that our methodology for identifying force chains is insightful for particulate systems in general, we compare and contrast results from two case studies: granular experiments and frictionless simulations. In both situations, the particles are confined in two dimensions, and they each consist of packings of bidisperse disks that interact with each other in a Hertzian-like manner. Both cases are jammed under constant pressure and use approximately the same number of particles (600). The key differences between the two systems are (1) the presence of friction and gravitational pressure in the experiments and (2) the use of periodic boundary conditions in the simulations. For each of the experiments and simulations, we measure force-contact networks for (approximately) the same 7 different values of confining pressure.

A. Granular Experiments

We perform experiments on a vertical 2D granular system of bidisperse disks that are confined between two sheets of Plexiglas. The particles are 6.35 mm thick, their diameters are \( d_1 = 9 \) mm and \( d_2 = 11 \) mm (which yields a diameter ratio of approximately 1.22), and they are cut from Vishay PSM-4 photoelastic material to provide measurements of the internal forces. These particles have an elastic modulus of \( E = 4 \) MPa. We produce new configurations by rearranging the particles by hand, and we increase the pressure by placing additional brass weights on the top surface of the packing. The values of pressure, which we report in units of the elastic modulus \( E \) (recall that the configuration is two-dimensional), are \( 2.7 \times 10^{-4} \), \( 4.1 \times 10^{-4} \), \( 6.7 \times 10^{-4} \), \( 1.1 \times 10^{-3} \), \( 2.2 \times 10^{-3} \), \( 3.8 \times 10^{-3} \), and \( 5.9 \times 10^{-3} \). See Refs. [16] [29] [36] for additional details about the experiments.

For each of 21 particle configurations and the 7 values of pressure, we compute particle positions and forces using two high-resolution pictures of the system. We use one image, which we take without the polarizers, to determine the particle positions and contacts. (See [37] for a description of the technique.) We take particles to be in contact if they have a measurable-magnitude force based on our photoelastic calculations. Using a second image that we take with the polarizers, we then determine the particle contact forces by solving the inverse photoelastic problem [37].

B. Frictionless Simulations

We perform numerical simulations of a 50:50 ratio of bidisperse frictionless disks. We use a diameter ratio of 1.22, and particles interact via a Hertzian potential in a box with periodic boundary conditions in both directions and zero gravity [38][10]. This model has been well-studied and it is significantly different from our experimental system with friction, gravity, and fixed boundaries. We generate mechanically-stable packings via a standard conjugate-gradient method [38]. We then perform simulations for a fixed packing fraction and volume, and we analyze 20 mechanically-stable packings at each packing fraction \( \phi \). We choose the seven values of the packing fraction so that the mean pressure \( \bar{p} \) at that packing fraction [11] matches the ones in the experiments: \( [\phi, \bar{p}] = [0.8499, 3 \times 10^{-4}], [0.8521, 4 \times 10^{-4}], [0.8560, 7 \times 10^{-4}], [0.8621, 11 \times 10^{-4}], [0.8760, 22 \times 10^{-4}], [0.8927, 38 \times 10^{-4}], \) and \( [0.9106, 59 \times 10^{-4}] \), where the modulus \( E \) is defined as the energy scale for the Hertzian interaction \( \epsilon \) divided by the mean Voronoi area of a particle in the packing. The lowest value of \( \phi \) provides a data point for a jammed packing that is less dense than what is accessible in our experiments.

III. FORCE-CHAIN EXTRACTION AND CHARACTERIZATION

A. Force-Chain Extraction

To locate the force chains, we begin by recording which particles are in contact with (and exert force on) which other particles. In network language, we represent each particle as a node, and we represent each inter-particle
contact as an edge whose weight is given by the magnitude of the force at that contact. We thereby construct a force-weighted contact network $\mathbf{W}$ from a list of all inter-particle forces. If particle $i$ and $j$ are in contact, then $W_{ij} = f_{ij}$, where $f_{ij}$ is the normal force between them. If two particles are not in contact, then $W_{ij} = 0$. In addition, we let $W_{ii} = 0$. We also construct an un-weighted (i.e., binary) matrix $\mathbf{B}$ whose elements are

$$B_{ij} = \begin{cases} 1, & W_{ij} \neq 0, \\ 0, & W_{ij} = 0. \end{cases}$$

The matrix $\mathbf{B}$ is often called an “adjacency matrix” \[24\], and the matrix $\mathbf{W}$ is the “weight matrix.”

To obtain force chains from $\mathbf{W}$, we want to determine sets of particles for which strong inter-particle forces occur amidst densely connected sets of particles. We can obtain a solution to this problem via “community detection” \[24\] \[33\] \[42\], in which we select groups of densely connected nodes called “modules” or “communities.” A popular way to identify communities in a network is by maximizing modularity

$$Q = \sum_{i,j} [W_{ij} - \gamma P_{ij}] \delta(g_i, g_j),$$  \hspace{1cm} (1)

where node $i$ is assigned to community $g_i$, node $j$ is assigned to community $g_j$, the Kronecker delta $\delta(g_i, g_j) = 1$ if and only if $g_i = g_j$, the quantity $\gamma$ is a resolution parameter, and $P_{ij}$ is the expected weight of the edge that connects node $i$ and node $j$ under a specified null model. The most common null model for modularity optimization is the Newman-Girvan (NG) null model \[34\] \[35\] \[43\] \[44\]

$$P_{ij}^{NG} = \frac{k_i k_j}{2m},$$  \hspace{1cm} (2)

where $k_i = \sum_j W_{ij}$ is the strength (i.e., weighted degree) of node $i$ and $m = \frac{1}{2} \sum_{i,j} W_{ij}$. The NG null model is most appropriate for networks in which a connection between any pair of nodes is possible.

Importantly, many networks include (explicit or implicit) spatial constraints that exert a strong influence on which edges are present \[12\]. For particulate systems, numerous edges are simply physically impossible, so it is important to improve upon the NG null model for such applications. We will use the term \textit{geographical constraints} to describe the explicit spatial constraints in such systems. These constraints exert a significant effect on their network structure, so it is important to take them into account when choosing a null model. For granular materials (and other particulate systems), each particle can only be in contact (i.e., $W_{ij} \neq 0$) with its immediate neighbors. Therefore, we use a null model, which we call the \textit{geographical null model}, to account for this constraint. The geographical null model is

$$P_{ij} = \rho B_{ij},$$  \hspace{1cm} (3)

where $\rho$ is the mean edge weight in a network and $\mathbf{B}$ is the binary adjacency matrix of the network. (Such a null model was used previously for applications in neuroscience \[45\].) Recall that the adjacency matrix encapsulates the presence or absence of contact between each pair of particles. For a granular material, $\rho = \mathbf{f} := \langle f_{ij} \rangle$ is the mean inter-particle force.

Maximizing $Q$ yields a hard partition of a network into communities such that the total edge weight inside of modules is as large as possible relative to the chosen null model. A hard partition assigns each node to exactly one community. (An alternative is a soft partition \[46\], which allows each node in a network to be associated with multiple communities.) For the geographical null model in \[3\], maximizing $Q$ yields a partition of the particles into groups that have larger inter-group particle forces than the mean force. Such groups represent the force chains in a granular system. Because optimizing $Q$ is NP-hard \[17\], the success of the maximization is subject to the limitations of the employed computational methods. In the present paper, we use a Louvain-like locally greedy algorithm \[45\]. Additionally, given the near-degeneracy of the modularity landscape that tends to inflict networks that are constructed from empirical data \[49\] (i.e., numerous different partitions often provide comparably high $Q$ values), we report community-detection results that are ensemble averages over 20 optimizations.

\section{Properties of Force Chains}

We characterize communities using several diagnostics: size, network force, and a \textit{gap factor} (a novel notion that we introduce in the present paper). The size $s_c$ of a community $c$ is simply the number of particles in that community. The systemic size $s$ is given by the mean of $s_c$ over all communities. The modularity $Q$ is composed of sums of magnitudes of bond forces and therefore also has units of force. Therefore, we use the term \textit{network force} to indicate the contribution of a community $c$ to modularity. The network force of a community is given by the formula

$$\sigma_c = \sum_{i,j \in C} [W_{ij} - \gamma \rho B_{ij}],$$  \hspace{1cm} (4)

where $c$ is the set of nodes in community $c$. The systemic network force $\sigma$ is the mean of $\sigma_c$ over all communities. Communities that correspond to force chains composed of densely packed particles with large forces between them have large values of network force, whereas communities that correspond to force chains composed of sparsely packed particles with small forces between them have small values of network force.

To identify the presence of gaps and the degree of branching in the geometry of a force-chain network, we calculate the Pearson correlation between \"physical
distance” (which we measure using the standard Euclidean metric) and “hop distance” (which is often called the “topological distance” and counts distance measured only along network edges), and we examine how it depends on the force-chain topology. Communities that exhibit compact or linear-chain characteristics (see the main diagonal in Fig. 2) occur when there is a perfect correlation between hop distance and physical distance. This occurs because particles that are 1 contact away from each other (i.e., they are adjacent to each other in the binary contact network) are also 1 particle-distance away. In contrast, force chains with gaps, branches, and rings have a larger hop distance than physical distance (see the upper triangle in Fig. 2). Particles that are close together in space do not necessarily have strong forces binding them; the presence of more complicated shapes decreases the correlation between the physical and hop distance. (Note for particulate systems that the lower triangle is unphysical, as it would require forces between particles that are not in contact with one another to achieve a larger physical distance than hop distance.)

Therefore, to identify communities that are composed of branched structures, we measure the amount of correlation between the hop distance and the physical distance in the set of all node pairs in a given community. To compute the hop distance of a community, we define the community contact network \( B^c \). Its elements \( B^c_{ij} \) are entries of the matrix \( B \) for which the corresponding nodes have both been assigned to the same set \( C \). We then calculate the path lengths between possible pairs of nodes in a community \( c \) using the hop distance on the matrix \( B^c \). The resulting matrix of pairwise distances is \( L_t \). To compute the physical distance, we calculate the Euclidean distances between possible pairs of nodes in a community. The resulting matrix of pairwise distances is \( L_p \). In defining the community gap factor, we choose to weight each community by its size, so that larger communities are weighted more heavily than small communities in accordance with the number of particles they contain. In this case, the gap factor \( g_c \) of a community \( c \) is

\[
g_c = 1 - \frac{r_c s_c}{s_{\text{max}}},
\]

where \( r_c \) is the value of the Pearson correlation between the upper triangle of \( L_t \) and the upper triangle of \( L_p \), and \( s_{\text{max}} \) is the size of the largest community. (Note that we exclude the diagonal elements of the matrices \( L_p \) and \( L_t \).) We define the systemic gap factor as

\[
g = 1 - n \sum_c r_c \frac{s_c}{s_{\text{max}}},
\]

where the quantity \( n \) is the total number of communities (including singletons) and we again note larger communities contribute more than small communities to the averaging in (5). An alternative choice for the systemic gap factor is to weight all communities uniformly to calculate an alternative systemic gap factor

\[
g_{\text{uniform}} = 1 - n \sum_c r_c .
\]

As we discuss in more detail in the appendix, branched communities typically include more nodes than linear communities, so the weighted gap factor \( g \) tends to be larger for more-branched communities. By contrast, \( g_{\text{uniform}} \) tends to be maximized for more linear communities. For the remainder of the main manuscript, we focus on the weighted gap factor because it varies far less across packings. Further studies of the utility of \( g_{\text{uniform}} \) would also be interesting.

IV. CHARACTERIZING FORCE CHAINS

We now characterize the force-chain networks of the experiments and simulations that we described in Section II using the methodology that we described in Section III. Our community-detection procedure consists of two stages. First, we maximize modularity with the geographical null model for different values of the resolution parameter \( \gamma \): from \( \gamma = 0.1 \) to \( \gamma = 2.1 \) in increments of \( \Delta \gamma = 0.2 \). We then choose a resolution that approximately maximizes the systemic gap factor \( g \) [see Eq. (6)]. This ensures that we have extracted communities of particles that have strong and dense force-weighted contacts with one another (i.e., we have found a network partition with high modularity \( Q \)), and that they are spatially sparse (i.e., they tend to have low topological-physical distance correlation \( r_c \)). The communities that we obtain tend to take the form of chain-like structures...
FIG. 3. (Color Online) Sample community-detection results for a single granular experiment at $4.1 \times 10^{-4} E$ illustrate multisresolution structure in chain-like communities. (A) Community structure as a function of the resolution parameter $\gamma$. Color indicates the logarithm (base 10) of the network force $\sigma_c$ of community $c$. (We don’t show single-particle communities.) (B) The systemic gap factor $g$ [see Eq. (6)] as a function of the resolution parameter $\gamma$ exhibits a maximum at $\gamma = 0.7$ (for $\gamma \in \{0.1, 0.3, \ldots, 2.1\}$). Error bars indicate the standard deviation of the mean over the laboratory packings. In the inset, we show an image of the 2D vertical packing of photoelastic disks that we use for panels (A) and (B).

that are reminiscent of force chains, which provides visual support that our technique is successful. This represents a significant improvement over communities that one detects using the NG null model. In previous work using the NG null model, we observed that these latter communities always tend to be compact in form. We know, however, that communities with other qualitative features are very common. (See the schematic in Fig. 2.)

The resolution parameter $\gamma$ sets the spatial resolution of the communities [33, 34, 52] [see Eq. (1)]. By tuning $\gamma$, one can either examine large communities (using small values of $\gamma$) or small communities (using large values of $\gamma$). In Fig. 2A, we show an example computation using data from experiments. (Note that we do not show single-particle communities.) Observe that small values of $\gamma$ produce communities that are dominated by compact structures (low $g$), whereas large values of $\gamma$ produce communities that are dominated by linear structures (also low $g$). This suggests that we can identify an optimal value $\gamma_{opt}$ at which $g$ is maximized. This also corresponds to the choice for which the detected communities are most similar to the force-chain structure that we observe visually (see Fig. 3A). For this particular packing, we identify $\gamma_{opt} = 0.7$ as the optimal choice in our examination of $\gamma \in \{0.1, 0.3, \ldots, 2.1\}$. As one can see in Fig. 3A, we observe at all values of $\gamma$ that the detected communities vary in their size, network force, and gap factor. We also note that single-particle communities are common for all values of $\gamma$ near 1 (and become increasingly common as $\gamma$ increases after it exceeds 1).

Community detection via modularity maximization with a resolution parameter chosen to yield a maximal systemic gap factor provides an automated approach for detecting force chains in granular media. Once identified, we then calculate the size $s$, network force $\sigma$, and gap factor $g$ to describe key features of a force-chain network. In the sections below, we quantitatively examine how the distributions of $s$, $\sigma$, and $g$ vary as a function of pressure for both granular experiments and frictionless simulations.

A. Granular Experiments

Our experiments are at 7 different pressures, which range from $2.7 \times 10^{-4} E$ to $5.9 \times 10^{-3} E$. As illustrated in Fig. 3A, we observe that the communities become more compact (so they are closer to the diagonal line in Fig. 3) as the pressure increases. At a low value of the resolution parameter ($\gamma = 0.7$), the detected communities metamorphose from branched structures to compact domains as pressure increases. At a high value of the resolution parameter ($\gamma = 1.9$), the detected communities tend to shrink from many-particle chains to 2-particle chains as pressure increases. This provides a way to quantify our earlier observations that force-chain networks at higher pressures are more homogeneous (observations at small $\gamma$) and less chain-like (observations at large $\gamma$) than they are at lower pressures.

At all pressures (and in spite of the aforementioned differences), we find a maximum in the systemic gap factor $g$ as a function of the resolution parameter $\gamma$ (see Fig. 3B). The shape of the curve of the gap factor versus the resolution-parameter has a more pronounced peak at higher pressures than at lower pressures: larger slopes descend from and (especially) lead up to the values of $\gamma$ near the maximum. At all pressures, small values of $\gamma$ select compact communities and large values of $\gamma$ select two-particle communities. In between, we observe a value of $\gamma$ at which the most branch-like structures appear; we refer to this value that approximately maximizes the gap factor as the “optimal” value. This optimal value changes slightly as a function of pressure; for example, $\gamma_{opt} = 0.9$ at $5.9 \times 10^{-3} E$ and $\gamma_{opt} = 0.7$ for $2.7 \times 10^{-4} E$. High-pressure packings also exhibit a much smaller systemic gap factor than low-pressure packings at both low and high values of the resolution parameter. This observation...
is consistent with both the increased compactness of the communities and the increasingly homogeneous nature of the force-chain structure as one increases the pressure.

The resolution that maximizes the gap factor identifies structures in a force network that are most reminiscent of the force chains that are apparent by eye; in other words, it identifies branching communities. Therefore, to extract force chains from force networks across different packings and pressures, we determine community structure over a range of resolution parameters and identify the resolution-parameter value that approximately maximizes the gap factor. We refer to the communities detected at $\gamma_{\text{opt}}$ as the “force chains” in our calculations, and we characterize their properties in terms of their size, network force, and gap factor.

As illustrated in Fig. 5, we observe that the size and network force of the force chains is approximately exponential for all 7 pressures. This identifies that the majority of communities are relatively small and weak, but a few communities are relatively large and strong. The gap-factor distribution is skewed, with most communities having a high gap factor and only a few communities having a low gap factor. In high-pressure packings, communities tend to be less broadly distributed with respect to both size and network force; this is consistent with the homogenization of force chains as pressure increases.

B. Frictionless Simulations

To illustrate quantitative differences in force-chain structure between laboratory and numerical packings, we use our methodology to identify force chains in frictionless simulations. In addition to their lack of friction, the numerical packings also differ from our experiments in that they use periodic boundary conditions and have zero gravity. As in Section IV A, we find communities via modularity maximization at different values of the resolution parameter for different packing fractions, which we selected to match the 7 pressures from the experiments.

As shown in Fig. 6A, the simulated packings change qualitatively as a function of the resolution parameter. For relatively small values of the resolution parameter ($\gamma = 0.7$), the granular force network has collapsed into a single compact community. In contrast, for relatively large values of the resolution parameter ($\gamma = 1.3$), we

![FIG. 4. (Color Online) (A) Images of experimental 2D vertical packings of photoelastic disks. These images reveal the manifestation of the internal stress pattern in a set of photoelastic particles as a network of force chains. (B) Community structure as a function of the resolution parameter $\gamma$ for the following pressures: (top) $6.7 \times 10^{-4}$, (middle) $2.2 \times 10^{-3}$, and (bottom) $5.9 \times 10^{-3}$. Color indicates the logarithm (base 10) of the network force $\sigma_{\text{c}}$ of community $\text{c}$. (C) Gap factor as a function of both the resolution parameter $\gamma$ and the pressure. Error bars indicate a standard deviation of the mean over the laboratory packings. The arrow emphasizes increasing pressure.](image1)

![FIG. 5. (Color Online) Cumulative probability distributions of (A) community size $s_{\text{c}}$ and (B) network force $\sigma_{\text{c}}$ for all communities. (C) Histogram of the gap factor $g_{\text{c}}$ for all communities. In these calculations, we use $\gamma_{\text{opt}} = 0.9$, which we choose to maximize $g$ across all pressures (rather than for a single pressure). The arrows emphasize increasing pressure.](image2)
identify many small branch-like communities that are reminiscent of force chains. Unlike in the granular experiments, we do not observe a strong qualitative change in community structure as a function of pressure (regardless of the value of $\gamma$). Compare Figs. 4B and 5B.

We again identify a value $\gamma_{opt}$ that is associated with the maximum value of the systemic gap factor $g$ in order to select the most branched community structure. As with the experimental data, there is a clearly identifiable maximum (see Fig. 6B), which occurs at $\gamma = 0.9$ for $\gamma \in \{0.1, 0.3, \ldots, 2.1\}$. We also observe that the shape of $g(\gamma)$ is more consistent across pressures for the simulations than it was for the experiments.

As in the laboratory experiments, we refer to the communities at $\gamma_{opt}$ as our “force chains,” and we characterize their properties in terms of size, network force, and gap factor (see Fig. 7A). Across the range of pressures from $2.7 \times 10^{-4}E$ to $5.9 \times 10^{-3}E$, we observe that chain size and network force are approximately exponential, indicating that the majority of communities are relatively small and weak but a few communities are relatively large and strong. The gap factor distribution has a left skew, which indicates that most communities have a large gap factor and only a few communities have a small gap factor.

In contrast to the laboratory experiments, the shape of the distributions for size, network force, and gap factor in frictionless simulations do not change dramatically as a function of pressure. Nevertheless, there is a systematic difference in the mean size and mean network force for communities as a function of pressure even in the frictionless simulations [see Fig. 7B].

### C. Comparison Between Laboratory and Numerical Force Chains

Using our methodology of extracting force chains, it is possible to quantitatively compare and contrast two particulate systems (such as the aforementioned granular experiments and frictionless simulations). In Fig. 8 we summarize the three main diagnostics (size, network force, and gap factor), that we calculate for each of our two case studies. We anticipate that these diagnostics might provide a helpful means of characterizing how well
Finally, we observe that the size and network force of chains increases with pressure in laboratory packings, whereas the systemic gap factor decreases with pressure for such packings. These observations indicate the presence of a changing length scale, as is to be expected \cite{57,58}. As pressure increases, the force structure becomes more homogeneous, leading to extended sections of material with densely packed particles that exert strong forces on one another. However, the force structure becomes less branch-like, leading to a decrease in the systemic gap factor. The frictionless packings also exhibit increases in size and network force of chains with pressure; however, unlike in the laboratory packings, the frictionless packings do not exhibit a decrease in systemic gap factor with pressure. This quantitative difference between the two types of packings is a technique for measuring the visual differences between Figs. 4A and 6A, where it is clear that the geometry of the force chains is affected less by pressure in the simulated setting than in the laboratory setting.

Figure 8B also demonstrates that the size of force-chain-like communities decreases as pressure approaches 0, which provides some elucidation for a longstanding question in the field of disordered matter. In particulate systems at 0 temperature, the point at which pressure $P = 0$ is the jamming transition, and much recent work has been devoted to attempting to understand the nature of this transition \cite{59}. There is now strong evidence that there is a growing length scale \cite{57,58}, which corresponds to the size of regions that are mechanically unstable, near the transition. As we discuss in Section V, it is natural to postulate that our force-chain communities are negatively correlated with weak regions in a particulate packing. Therefore, we expect force chains to shrink as mechanically unstable regions grow.

V. CONCLUSIONS AND DISCUSSION

In this paper, we treated granular materials as spatially-embedded networks in which nodes (particles) are connected by edges whose weights we determine from contact forces. We developed and applied a network-based clustering method, in which we detect tightly-connected “communities” of particles via modularity maximization with a geographical null model, to extract chain-like structures that are reminiscent of force chains in both numerical and laboratory packings. From these chain-like structures, we calculated three key diagnostics (size, network force, and gap factor), and we illustrated their utility for identifying and characterizing force-chain network architectures across two types of packings (laboratory and numerical) and across 7 different pressures. To characterize force chains, we identified an “optimal” resolution-parameter value that approximately maximized the gap factor in all of these scenarios.
An alternative approach would be to examine the structure of force chains that are identified at a set of resolution parameters that one optimizes by other parameters (e.g., size, network force, etc.) separately. This is an interesting direction for future study.

After we identified an optimal value for the resolution parameter, we systematically evaluated and compared properties of force-chain communities. A common feature found in both the laboratory and numerical packings is that the distribution of force-chain sizes is consistent with an exponential distribution. In the laboratory packings of 2D granular materials, we found that high-pressure force networks exhibit compact rather than branching communities, which is consistent with the notion that increasing pressure causes a breakdown in the long-range heterogeneous structure in a material. In contrast, the geometry of the force chains in the (frictionless) numerical packings appear to be affected less by pressure. The force chains in the numerical packings also tend to be both larger and stronger than their counterparts in laboratory packings. Together, these results support the conclusion that the force-chain structure, topology, and geometry is different in the two systems.

**Methodological Insights**

An important conclusion of our paper is that choosing an appropriate null model is critical for extracting information from community detection in networks. We have developed and applied a geographically-informed null model for the detection of network communities, which are strongly reminiscent of force chains, in particular materials. The power of this null model lies in its ability to fix a network’s topology (which is given by a binary contact network) while scrambling its geometry (which is given by a force-weighted contact network). The geographical null model thereby includes more of the fundamental physics of particulate systems than the Newman-Girvan null model that is usually used in modularity maximization. An interesting future direction would be to incorporate different physical constraints and principles (e.g., force balance) and to examine the different results that one obtains by including different combinations of relevant physical ideas. A key question in community detection is what is the minimal set of physical ideas—and, more generally, the minimal amount of problem-specific information—that one can include in a null model to get answers that are more insightful than using a one-size-fits-all null model like the Newman-Girvan model.

**Practical Utility**

More broadly, we expect that our methodology will provide a framework for understanding which features of force chains are universal versus which are governed by detailed particle-particle or particle-environment interactions. For example, our methodology could be used to provide a quantification of the similarity of force chains between a simulation and a given experiment. It can also provide similar quantifications between two experiments (or two simulations) with different types of particles or boundary conditions. It could also be a tool to help predict differences in macroscopic behavior based on subtle differences in the mean or distributions of force-chain diagnostics.

Our methodology also provides additional information that is not available via traditional methods for identifying force chains. For example, we know which particles are strongly connected within communities, and we can also observe the spatial arrangement of the various communities that comprise a force network. This allows one to ask whether the arrangement of communities helps to govern the linear and nonlinear response of a disordered packing. For example, it is possible that large, strong communities indicate a region of relatively high mechanical stability or that boundaries between these communities indicate areas of weakness. In the future, these types of investigations could be used to better understand the onset of flow or failure in particulate systems.

**ACKNOWLEDGEMENTS**

DSB was supported by the Alfred P. Sloan Foundation, the Army Research Laboratory through contract number W911NF-10-2-0022 from the U.S. Army Research Office, and the National Science Foundation award #BCS-1441502. KED and ETO were supported by NSF-DMR-0647473 and NSF-DMR-1206808, and they thank James Puckett for the development of the photoelastic image-processing code. MLM was supported by the Alfred P. Sloan Foundation, NSF-BMMB-1334611 and NSF-DMR-1352184. MAP was supported by the James S. McDonnell Foundation (research award #22002177), the EPSRC (EP/J001759/1), and the FET-Proactive project PLEXMATH (FP7-ICT-2011-8; grant #317614) funded by the European Commission. The content is solely the responsibility of the authors and does not necessarily represent the official views of the funding agencies.

**APPENDIX: EFFECT OF COMMUNITY WEIGHTING ON FORCE-CHAIN EXTRACTION**

In this appendix, we describe an alternative set of measurements that we made using the uniformly-weighted...
The large variation in the maximum of $g_{\text{uniform}}$ over packings and pressures makes it difficult to choose an optimal resolution-parameter value. We choose to take $\gamma_{\text{opt}} = 1.1$ because (1) it corresponds to the maximum of $g_{\text{uniform}}$ in the numerical packings and (2) it corresponds to the mean of the maximum of $g_{\text{uniform}}$ in the laboratory packings. To facilitate the comparison of optimal $\gamma$ parameters in the two weighting schemes, we denote $\gamma_{\text{opt}}$ for $g$ as $\hat{\gamma}$ and we denote $\gamma_{\text{opt}}$ for $g_{\text{uniform}}$ as $\hat{\gamma}_{\text{uniform}}$. Note that $\hat{\gamma}_{\text{uniform}} = 1.1$ differs from (and is larger than) $\hat{\gamma} = 0.9$. 

**Force-Chain Structure at the Optimal Value of the Resolution Parameter**

The force chains that we identify for the optimal value for the uniformly-weighted gap factor (at $\hat{\gamma}_{\text{uniform}} = 1.1$) differ from those that we identified in the main text for the optimal value of the size-weighted gap factor (at $\hat{\gamma} = 0.9$). We show our comparison in Fig. 10. For both laboratory and numerical packings, the force chains that we identify at $\gamma = 0.9$ are larger and more branched than the ones that we identify at $\gamma = 1.1$ (which are smaller and more linear). Indeed, the communities identified at $\gamma = 1.1$ have more singletons than the communities identified at $\gamma = 0.9$. These results follow from the difference in the two weighting schemes for calculating a systemic gap factor. The size-weighted systemic gap factor $g$ weights larger communities more heavily than smaller ones, and the larger communities tend to be the more branched communities that we identify at smaller values of the resolution parameter (e.g., $\gamma = 0.9$). In contrast, the uniformly-weighted systemic gap factor $g_{\text{uniform}}$ gives equal weight to small and large communities, and it therefore uncovers the linear communities that are evident at larger values of the resolution parameter (e.g., $\gamma = 1.1$). We can therefore use the size-weighted gap factor to identify larger, more branched force chains and the uniformly-weighted gap factor to identify smaller, more linear force chains.

[1] P. Dantu, in *Proceedings of the fourth International Conference on Soil Mechanics and Foundation Engineering, London* (1957) pp. 144–148.
[2] C. H. Liu, S. R. Nagel, D. A. Schecter, S. N. Coppersmith, S. Majumdar, O. Narayan, and T. A. Witten, *Science* **269**, 513 (1995).
FIG. 10. (Color Online) In both (A) laboratory and (B) numerical packings, we identify larger and more branched force chains at the optimal resolution determined by (left; $\gamma = 0.9$) the size-weighted gap factor $g$, and we identify smaller and less branched force chains at the optimal resolution determined by (right; $\gamma = 1.1$) the uniformly-weighted gap factor $g_{\text{uniform}}$. These observations are consistent across all pressure values, but they are especially evident at high pressures in the laboratory packings and are least evident at low pressures in the laboratory packings. We observe little variation for different values of pressure in the numerical packings. In both panels, we highlight the network force for the identified force chains from example packings.

[3] F. Radjai, M. Jean, J. J. Moreau, and S. Roux, Physical Review Letters 77, 274 (1996).
[4] D. M. Mueth, H. M. Jaeger, and S. R. Nagel, Physical Review E 57, 3164 (1998).
[5] D. Howell, R. P. Behringer, and C. Veje, Physical Review Letters 82, 5241 (1999).
[6] T. S. Majmudar and R. P. Behringer, Nature 435, 1079 (2005).
[7] M. Saadatfar, A. P. Sheppard, T. J. Senden, and A. J. Kabla, Journal of the Mechanics and Physics of Solids 60, 55 (2012).
[8] S. Mukhopadhyay and J. Peixinho, Physical Review E 84, 2 (2011).
[9] J. Brujić, S. F. Edwards, D. V. Grinev, I. Hopkinson, D. Brujić, and H. A. Makse, Faraday Discussions 123, 207 (2003).
[10] J. Zhou, S. Long, Q. Wang, and A. D. Dinsmore, Science 312, 1631 (2006).
[11] K. W. Desmond, P. J. Young, D. Chen, and E. R. Weeks, Soft Matter 9, 3424 (2013).
[12] M. Barthelemy, Phys Rep 499, 1 (2011).
[13] H. A. Makse, N. Gland, D. L. Johnson, and L. M. Schwartz, Physical Review Letters 83, 5070 (1999).
[14] R. Cruz Hidalgo, C. U. Grosse, F. Kun, H. W. Reinhardt, and H. J. Herrmann, Physical Review Letters 89, 205501 (2002).
[15] E. Somfai, J.-N. Roux, J. H. Snoeijer, M. van Hecke, and W. van Saarloos, Phys Rev E 72, 021301 (2005).
[16] E. T. Owens and K. E. Daniels, Europhysics Letters 94, 54005 (2011).
[17] J. Peters, M. Muthuswamy, J. Wibowo, and A. Tordesillas, Physical Review E 72, 041307 (2005).
[18] R. Arévalo, I. Zuriguel, and D. Maza, Physical Review E 81, 041302 (2010).
[19] L. Kondic, A. Goulet, C. S. O’Hern, M. Kramar, K. Mischaikow, and R. P. Behringer, Europhysics Letters 97, 54001 (2012).
[20] M. Kramar, A. Goulet, L. Kondic, and K. Mischaikow, arXiv:1404.6279 (2014).
[21] S. Ardanuz-Trevijano, I. Zuriguel, R. Arévalo, and D. Maza, Phys. Rev. E 89, 052212 (2014).
[22] H. Edelsbrunner and J. Harer, Computational Topology: An Introduction (American Mathematical Society, 2010).
[23] F. Radjai, D. E. Wolf, M. Jean, and J. J. Moreau, Physical Review Letters 80, 61 (1998).
[24] M. E. J. Newman, Networks: An Introduction (Oxford University Press, 2010).
[25] M. Herrera, S. McCarthy, S. Slotterback, E. Cephas, W. Losert, and M. Girvan, Physical Review E 83, 1 (2011).
[26] A. Tordesillas, P. O’Sullivan, and D. M. Walker, Comptes Rendus Mécanique 338, 556 (2010).
[27] D. M. Walker and A. Tordesillas, International Journal Of Solids And Structures 47, 624 (2010).
[28] S. Slotterback, M. Mailman, K. Ronaszegi, M. van Hecke, M. Girvan, and W. Losert, Phys Rev E 85, 021309 (2012).
[29] D. S. Bassett, E. T. Owens, K. E. Daniels, and M. A. Porter, Phys Rev E 86, 041306 (2012).
[30] J. H. Lopez, L. Cao, and J. M. Schwarz, Phys. Rev. E 88, 062130 (2013).
[31] R. Arévalo, L. A. Pugnaloni, D. Maza, and I. Zuriguel, Philosophical Magazine 93, 4078 (2013).
[32] D. M. Walker, A. Tordesillas, M. Small, R. P. Behringer, and C. K. Tse, Chaos 24, 013132 (2014).
[33] R. Navakas, A. Diuys, and B. Peters, Physica A: Statistical Mechanics and its Applications 407, 312 (2014).
[34] M. A. Porter, J.-P. Onnela, and P. J. Mucha, Not Amer Math Soc 56, 1082 (2009).
[35] S. Fortunato, Phys Rep 486, 75 (2010).
[36] E. T. Owens and K. E. Daniels, Soft Matter 9, 1214 (2013).
[37] J. G. Puckett and K. E. Daniels, Physical Review Letters 110, 058001 (2013).
[38] C. O’Hern, S. Langer, A. Liu, and S. Nagel, Phys. Rev. Lett. 88, 075507 (2002).
[39] C. S. O’Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Physical Review E 68, 011306 (2003).
[40] M. L. Manning and A. J. Liu, Physical Review Letters 107, 108302 (2011).
[41] For finite systems, there isn’t a bijection between pressure and packing fraction. Because the packing fraction
is fixed in these simulations, there is some variance (of roughly $9 \times 10^{-5}$) in the pressure between different packings.

[42] M. E. J. Newman, Nat Phys 8, 25 (2012).
[43] M. E. J. Newman and M. Girvan, Phys Rev E 69, 026113 (2004).
[44] M. E. J. Newman, Proc Natl Acad Sci USA 103, 8577 (2006).
[45] D. S. Bassett, M. A. Porter, N. F. Wymbs, S. T. Grafton, J. M. Carlson, and P. J. Mucha, Chaos 23, 013142 (2013).
[46] D. M. Walker and A. Tordesillas, in 2014 IEEE International Symposium on Circuits and Systems (ISCAS), Vol. June (2014) pp. 1275–1278.
[47] U. Brandes, D. Delling, M. Gaertler, R. Görke, M. Hoeffer, Z. Nikoloski, and D. Wagner, IEEE Trans on Knowl Data Eng 20, 172 (2008).
[48] I. S. Jutla, L. G. S. Jeub, and P. J. Mucha, “A generalized Louvain method for community detection implemented in MATLAB,” (2011–2012).
[49] B. H. Good, Y. A. de Montjoye, and A. Clauset, Phys Rev E 81, 046106 (2010).

[50] Note that we use the term “compact” in the spirit of the mathematical sense of the word, although our exact usage is rather different.
[51] J. Reichardt and S. Bornholdt, Phys Rev E 74, 016110 (2006).
[52] J.-P. Onnela, D. J. Fenn, S. Reid, M. A. Porter, P. J. Mucha, M. D. Fricker, and N. S. Jones, Phys Rev E 86, 036104 (2012).
[53] M. Mailman, C. F. Schreck, C. S. O’Hern, and B. Chakraborty, Physical review letters 102, 255501 (2009).
[54] Z. Zeravcic, N. Xu, A. J. Liu, S. R. Nagel, and W. van Saarloos, Europhysics Letters 87, 26001 (2009).
[55] P. A. Cundall and O. D. L. Strack, Geotechnique 29, 47 (1979).
[56] S. Papanikolaou, C. S. O’Hern, and M. D. Shattuck, Physical review letters 110, 198002 (2013).
[57] M. Wyart, L. Silbert, S. Nagel, and T. Witten, Phys Rev E 72, 051306 (2005).
[58] C. P. Goodrich, W. G. Ellenbroek, and A. J. Liu, Soft Matter 9, 10993 (2013).
[59] A. J. Liu and S. R. Nagel, Annual Review of Condensed Matter Physics 1, 347 (2010).