Hard Thermal Loops and the Sphaleron Rate on the Lattice

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We measure the sphaleron rate (topological susceptibility) of hot SU(2) gauge theory, using a lattice implementation of the hard thermal loop (HTL) effective action. The HTL degrees of freedom are implemented by an expansion in spherical harmonics and truncation. Our results for the sphaleron rate agree with the parametric prediction of Arnold, Son and Yaffe: \(\Gamma \propto \alpha^5 T^4\).

1. MOTIVATION

Baryon number is not a conserved quantity in the Standard Model: due to the anomaly, the violation is related to the (Minkowski time) topological susceptibility of the SU(2) weak group. While at low temperatures the violation is totally negligible \(1\), at temperatures above the electroweak symmetry restoration temperature (\(\sim 100\) GeV) the rate of the baryon number violation (sphaleron rate) \(\Gamma\) is large. This can have significant repercussions for baryon number generation in the early Universe, and it opens the avenue for purely electroweak baryogenesis.

Even though the weak coupling constant is small, at high temperatures the sphaleron processes are dominated by IR momenta \(k \sim g^2 T\) and are thus inherently non-perturbative. Moreover, the IR modes behave essentially classically, which is signalled, for example, by the large occupation numbers of the Bose fields: \(n(k \sim g^2 T) = (e^{k/T} - 1)^{-1} \approx T/k \sim 1/g^2 \gg 1\). This has motivated the much utilized method of using the classical equations of motion to calculate \(\Gamma\) in hot SU(2) theories \(2\) (the Higgs and fermionic degrees of freedom essentially decouple in the hot EW phase). For recent reviews, see \(3, 4\).

The success of the classical method hinges on the efficient decoupling of the almost-classical IR modes relevant for the sphaleron processes and the strongly non-classical UV modes. However, as argued by Arnold, Son and Yaffe \(5\), this decoupling is not complete. A step beyond the classical approximation is the hard thermal loop (HTL) effective theory \(6\), which incorporates the leading order effects of the UV modes. The HTL theory can be cast in various forms; most practical for lattice computations is the one where the strongly non-classical UV modes are described by including a large number of classical massless particles with adjoint charge moving on the background of IR fields. This field + particles system can be put on a lattice as such, and it has been successfully used in simulations \(7\). In this work we use an alternative Boltzmann-Vlasov approach, where the particles are described with local density functions \(n(t, \vec{x}, k)\). For full description, see \(8\).

2. HTL THEORY ON THE LATTICE

Let us consider a system consisting of the HTL particles moving on the background of IR gauge fields. The particle density functions \(n(t, \vec{x}, k)\) obey the Vlasov equation

\[
\frac{dn}{dt} = 0 = \partial_0 \delta n + \vec{v} \cdot \vec{D} \delta n + \partial_0 \vec{k} \frac{\partial n}{\partial \vec{k}}
\]

where \(n_0 = (e^{k/T} - 1)^{-1}, n = n_0 + \delta n\), and the Lorentz-force \(\propto \vec{v} \times \vec{B}\) has been neglected. The IR gauge fields evolve according to the Yang-Mills
equations:
\[
D_\mu F^{\mu
u} = j^{\nu}_{\text{hard}} = 4g \int \frac{d^3k}{(2\pi)^3} v^\nu \delta n, \tag{2}
\]
where the 4-velocity \( v = (1, \vec{k}/k) \). These equations can be further simplified by factorizing \( \delta n^a = -gW^a(x, \vec{v})(\partial n_0/\partial k) \) and integrating over the amplitude \( |\vec{k}| \) [8]:
\[
D_\mu F^{\mu\nu} = m_D^2 \int \frac{d\Omega}{4\pi} v^\nu W(x, \vec{v})
\]
\[
v^\mu D_\mu W(x, \vec{v}) = v_i F^{0i}. \tag{3}
\]
Here \( d\Omega \) integration is over the directions of the 4-velocity \( v \). The field \( W^a(x, \vec{v}) \) is proportional to the flux of the particles at point \( x \) to direction \( \vec{v} \).

In order to perform lattice simulations the field \( W \) has to be regularized in space (standard lattice) and on the \( \vec{v} \)-sphere. We do this by expanding \( W \) in spherical harmonics: \( W^a(x, \vec{v}) = W^a_{lm}(x)Y_{lm}(\vec{v}) \), and truncating the expansion to \( l \leq l_{\text{max}} \). In terms of \( W_{lm} \), the equations finally become [8]
\[
D_i F^{0i} = (m_D^2/\sqrt{4\pi})W_{00} \tag{4}
\]
\[
D_\mu F^{\mu i} = (m_D^2/4\pi)V^{*i}_{lm}W_{1m} \tag{5}
\]
\[
D_0 W_{lm} = -c^{i\prime i^\prime}_{lm} D_i W_{i^\prime m} + \delta_{l1} V^{*i}F^{0i}. \tag{6}
\]
Here the coefficients \( c^{i\prime i^\prime}_{lm; i^\prime} = \int d\Omega Y^{*i}_{lm}Y_{i^\prime m} \) and \( V^{*i}_{lm} = \int d\Omega Y^{*i}_{lm} \). Eq. (4) is the Gauss law, and, as long as it is satisfied by the initial configuration, it is preserved by Eqs. (5) and (6).

With a finite \( l_{\text{max}} \), these equations can be readily discretized: \( SU(2) \) gauge field is defined on the links of the lattice, and the \( (l_{\text{max}} + 1)^2 \) adjoint \( W_{lm} \) fields are on lattice sites. The discretization and the properties of the theory on the lattice are discussed in detail in [8].

3. THE SPHALERON RATE
The measurement of the sphaleron rate proceeds along similar lines to the purely classical theory: first, we generate an ensemble of initial thermalized configurations (which satisfy the Gauss law), and then evolve these with Eqs. (5) and (6). We then obtain \( \Gamma \) by measuring the rate of the Chern-Simons number diffusion [8].

We have to check how \( \Gamma \) depends on (a) \( l_{\text{max}} \), (b) lattice spacing and (c) the Debye mass \( m_D \). Only the last parameter is physical (it depends on the particle content of the theory).

Let us first consider the \( l_{\text{max}} \) dependence. In Fig. [8] we show \( \Gamma \) measured from a set of lattices with \( l_{\text{max}} \leq 10 \). We note that when \( l_{\text{max}} \) is even, the rate remains remarkably constant (much better than indicated by naïve arguments [8]). The behaviour at odd \( l_{\text{max}} \) can be understood by considering the properties of the gauge field propagator [8]. Thus, we conclude that modest values of \( l_{\text{max}} \approx 4–6 \) are sufficient in order to obtain the \( l_{\text{max}} \to \infty \) behaviour within reasonable statistical errors.

Dimensionally, one would expect that \( \Gamma \propto \alpha^4 T^4 \), the non-perturbative scale to the fourth power. However, as argued by Arnold, Son and Yaffe [8], the evolution of the IR fields is Landau damped by the UV modes, and the rate is slower by one further factor of \( \alpha \), parametrically
\[
\Gamma = \kappa' \frac{g^2 T^2}{m_D^2} \alpha^5 T^4, \tag{7}
\]
where \( \kappa' \) is a constant to be determined by lattice measurements. In Fig. [8] we show the behaviour of \( \Gamma \) against \( g^2 T^2/m_D^2 \), measured using various
In leading order $\beta_0 = 4/g^2 a$, where $a$ is the lattice spacing.

lattice spacings $a \propto 1/\beta_0$. The rate is clearly not constant when the Debye mass is varied, and it goes to zero when $m_D^2 \to \infty$, as predicted by Eq. (7). We have not observed any significant dependence on the lattice spacing (provided that it is small enough). It should be noted that the physical $m_D^2$ is not only the ‘bare’ $m_D^2$ which appears in Eq. (4), but it is a sum of the bare $m_D^2$ and a contribution due to the UV lattice gauge field modes $\propto 1/a$ [10].

In Fig. 3 we compare the coefficient $\kappa'$ of the scaling law (7) as measured in this work, with the particles method [7], and with only the classical SU(2) gauge theory simulation [11]. In the last case the physical $m_D^2$ arises solely through the lattice UV modes; here the lattice spacing is up to a factor of $\sim 4$ smaller than in the two HTL approaches. The consistency of the results is remarkable, considering the very different treatments used.

To conclude with, the sphaleron rate in hot SU(2) gauge theory is now settled. Inserting the Standard Model value of $m_D^2 = 11/6 g^2 T^2$, we obtain for the rate a value $\Gamma = (25 \pm 2) \alpha^5 T^4$.

REFERENCES

1. G. t’Hooft, Phys. Rev. Lett. 37 (1976) 8.

2. J. Ambjørn, T. Askgaard, H. Porter, and M. Shaposhnikov, Nucl. Phys. B 353 (1991) 346.
3. G.D. Moore, in these proceedings [hep-lat/9907009].
4. J. Smit, Nucl. Phys. (Proc. Suppl.) 63 (1998) 89.
5. P. Arnold, D. Son and L.G. Yaffe, Phys. Rev. D55 (1997) 6264 [hep-ph/9609481].
6. E. Braaten and R. Pisarski, Nucl. Phys. B337 (1990) 569; Phys. Rev. D 45 (1992) 1827; J. Frenkel and J. Taylor, Nucl. Phys. B334 (1990) 199 (1990).
7. G.D. Moore, C. Hu and B. Müller, Phys. Rev. D58 (1998) 045001 [hep-ph/9710436].
8. D. Bödeker, G.D. Moore and K. Rummukainen, NORDITA-99/45HE [hep-ph/9907549].
9. J.P. Blaizot and E. Iancu, Phys. Rev. Lett. 70 (1993) 3376 [hep-ph/9301230]; V.P. Nair, Phys. Rev. D48 (1993) 3432 [hep-ph/9307326].
10. P. Arnold, Phys. Rev. D55 (1997) 7781 [hep-ph/9701393].
11. G.D. Moore and K. Rummukainen, McGill-99-21 [hep-ph/9906259].