3-connected graphs and their degree sequences

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Abstract. Necessary and sufficient conditions for a sequence of positive integers to be the degree sequence of a 3-connected simple graph are detailed. Conditions are also given under which such a sequence is necessarily 3-connected i.e. the sequence can only be realised as a 3-connected graph. Finally, a matrix is introduced whose non-empty entries partition the set of 3-connected graphs.

Mathematics Subject Classification (2010). Primary 05C40.
Keywords. 3-connected graph, degree sequence, graph construction, BG-operations.

1. Introduction

Necessary and sufficient conditions for a sequence of non-negative integers to be connected i.e. the degree sequence of some finite simple connected graph, are implicit in Hakimi [6] and have been stated explicitly by the author in [7]. This note builds upon these conditions of Hakimi and begins by describing an iterative construction of a 3-connected graph (in which all intermediate graphs are also 3-connected), due to Barnette and Grünbaum [1]. The set of all 3-connected graphs is then partitioned into equivalence classes each of which is placed separately in a unique entry of a matrix \( P_G \). The relationship between entries in \( P_G \) and the Barnette and Grünbaum construction is then described. Necessary and sufficient conditions are given for a sequence of non-negative integers to be 3-connected and/or necessarily 3-connected. Finally, the relationship between entries in \( P_G \) and (necessarily) 3-connected degree sequences is outlined.

2. Preliminaries

Let \( G = (V_G, E_G) \) be a graph where \( V_G \) denotes the vertex set of \( G \) and \( E_G \subseteq [V_G]^2 \) denotes the edge set of \( G \) (given that \([V_G]^2 \) is the set of all 2-element subsets of \( V_G \)). An edge \( \{a, b\} \) is denoted \( ab \). A graph is finite when \( |V_G| < \infty \) and \( |E_G| < \infty \), where \( |X| \) denotes the cardinality of the set \( X \). The union of
graphs \( G \) and \( H \) i.e. \((V_G \cup V_H, E_G \cup E_H)\), is denoted \( G \cup H \). By a slight abuse of notation, \( ab \cup G \) is understood to be the graph \((\{a, b\}, \{ab\}) \cup (V_G, E_G)\). A graph is simple if it contains no loops (i.e. \( aa \notin E_G \)) or parallel/multiple edges (i.e. \( \{ab, ab\} \notin E_G \)). The degree of a vertex \( v \) in a graph \( G \), denoted \( \deg(v) \), is the number of edges in \( G \) which contain \( v \). A graph where all vertices have degree \( k \) is called a \( k \)-regular graph. A path is a graph with \( n \) vertices in which two vertices, known as the endpoints, have degree 1 and \( n - 2 \) vertices have degree 2. A graph is connected if there exists at least one path between every pair of vertices in the graph. Paths \( P_1 \) and \( P_2 \), both with endpoints \( a \) and \( b \), are internally disjoint if \( P_1 \cap P_2 = (\{a, b\}, \{\} \)\). A graph \( G \) is 3-connected when there exists at least 3 internally disjoint paths in \( G \) between any two vertices in \( G \). A tree is a connected graph with \( n \) vertices and \( n - 1 \) edges. \( K_n \) denotes the complete graph on \( n \) vertices. All basic graph theory definitions can be found in standard texts such as [2], [3] or [5]. All graphs in this work are assumed to be simple, undirected and finite.

3. Constructing 3-connected graphs

Barnette and Grünbaum introduce three operations on 3-connected graphs in [1] which are used to construct a 3-connected graph from \( K_4 \) where all intermediate graphs are also 3-connected. These operations are collectively referred to as BG-operations and are described in Definition 3.1. Each operation is also individually named based on the number of additional vertices and edges, respectively, that are added to the 3-connected graph by the operation i.e. the \((2,3)\)-operation on a 3-connected graph \( H \) adds 2 vertices to \( V_H \) and 3 edges to \( E_H \). Before defining BG-operations note that an edge \( uv \in E_G \) is subdivided whenever \( uv \) is removed from \( E_G \), a vertex \( v \) is added to \( V_G \) and the edges \( uv \) and \( vw \) are added to \( E_G \).

Definition 3.1. Given a 3-connected graph \( H \) then a BG-operation on \( H \) is one of the following operations:

- (0,1)-operation: add an edge \( ab \) to \( H \) such that \( a, b \in V_H \) but \( ab \notin E_H \).
- (1,2)-operation: subdivide an edge \( xy \in E_H \) by adding a vertex \( a \notin V_H \), then adding the edge \( ab \notin E_H \) such that \( b \in V_H \) and \( b \neq x, y \).
- (2,3)-operation: subdivide edges \( xy \) and \( wz \), where \( xy \neq wz \), by adding vertices \( a \) and \( b \), respectively, then adding the edge \( ab \).

In the literature the operations described in Definition 3.1 and shown in Figure 1 are called basic BG-operations as they do not result in the introduction of parallel edges.

Theorem 3.2. A graph \( G \) is 3-connected if and only if it can be constructed from \( K_4 \) using BG-operations.

No loss of generality is incurred by using only basic BG-operations as every multigraph has a maximal simple graph which can be constructed from \( K_4 \) using basic BG-operations. Hence, it simply requires the addition of the
appropriate parallel edges and loops to complete the construction of the required multigraph. Further details on Theorem 3.2 can be found in [1] and [9] as well as in [8].

4. Listing all 3-connected graphs

Let \( G_3 \) be the set of all (unlabelled) 3-connected simple graphs. Define a graph equivalence relation \( \sim \) as \( G \sim H \) if and only if \( |V_G| = |V_H| \) and \( |E_G| = |E_H| \).

Consider a matrix \( P = (p_{i,j})_{i \in \mathbb{Z}, j \in \mathbb{N}_0} \) such that each non-empty entry \( p_{i,j} \in P \) contains a single equivalence class (with respect to \( \sim \)) with the entry \( p_{0,0} \) containing \( K_4 \).

The equivalence classes are arranged in \( P \) as follows:

- Each \( G \in p_{i,j} \) is a maximal proper subgraph of some \( G' \in p_{i,j+1} \) such that \( V_G \subset V_{G'} \).
- Each \( G \in p_{i,j} \) is a maximal proper subgraph of some \( G'' \in p_{i+1,j} \) such that \( V_G = V_{G''} \) and \( E_G \subset E_{G''} \).
- If \( p_{i,j} \) contains a maximal complete graph, then \( p_{i+1,j}, p_{i+2,j}, \ldots \) are empty.

Such an arrangement of equivalence classes within \( P \) defines the following relationships between vertex and edge set cardinalities:

- Assuming that \( p_{i,j} \) does not contain a maximal complete graph, then for any \( G \in p_{i,j} \) there exists some \( G^* \in p_{i+1,j} \) such that \( G \subset G^* \) with \( |V_{G^*}| = |V_G| \) and \( |E_{G^*}| = |E_G| + 1 \). See Figure 2.
- For any \( G \in p_{i,j} \) there exists some \( G^{**} \in p_{i,j+1} \) such that \( G \subset G^{**} \) with \( |V_{G^{**}}| = |V_G| + 1 \) and \( |E_{G^{**}}| = |E_G| + 3 \) (when adding a vertex to a 3-connected graph, so that 3-connectedness is preserved, at least 3 additional edges must also be added). See Figure 2.

For each \( G \in p_{i,j} \) it is possible to state \( |V_G| \) and \( |E_G| \) in terms of \( i \) and \( j \).

As \( p_{0,0} \) contains \( K_4 \) and all entries in the same column of \( P \) contain graphs with the same number of vertices, then column \( j \) contains graphs with \( j + 4 \) vertices, where \( j \in \mathbb{N}_0 \).
Recall that $p_{0,0}$ contains $K_4$ and that $|E_{K_4}| = \binom{4}{2}$. So that 3-connectedness is preserved then graphs in $p_{0,1}$ have 4 + 1 vertices and $\binom{4}{2} + 3$ edges. It follows that graphs in entry $p_{0,j}$ have 4 + $j$ vertices and $\binom{4}{2} + 3j$ edges. Observe that graphs in $p_{1,j}$ (if they exist) contain 1 more edge than graphs in $p_{0,j}$ and that graphs in $p_{-1,j}$ (if they exist) contain 1 edge less than graphs in $p_{0,j}$. As rows are labelled using $\mathbb{Z}$ then the entry $p_{i,j}$ contains graphs with $\binom{4}{2} + 3j + i$ edges.

It is now possible to define $P_{G_3}$, the partition matrix of $G_3$.

**Definition 4.1.** Let $G_3$ be the set of all 3-connected simple graphs, then $P_{G_3}$, the partition matrix of $G_3$, is the matrix

$$P_{G_3} := (p_{i,j})_{i \in \mathbb{Z}, j \in \mathbb{N}_0}$$

such that

$$p_{i,j} := \{G \in G_3 \mid |V_G| = j + 4 \text{ and } |E_G| = i + 3j + 6\}.$$

A portion of $P_{G_3}$ is shown in Figure 3.
5. BG-operations and $P_{G_3}$

It is now possible to see how entries in $P_{G_3}$ are related in terms of the three BG-operations.

- Performing a $(0,1)$-operation on some $G \in p_{i,j}$ results in some $G' \in p_{i+1,j}$ such that $|V_{G'}| = |V_G|$ and $|E_{G'}| = |E_G| + 1$.
- Performing a $(1,2)$-operation on some $G \in p_{i,j}$ results in some $G'' \in p_{i-1,j+1}$ such that $|V_{G''}| = |V_G| + 1$ and $|E_{G''}| = |E_G| + 2$.
- Performing a $(2,3)$-operation on some $G \in p_{i,j}$ results in some $G''' \in p_{i-3,j+2}$ such that $|V_{G'''}| = |V_G| + 2$ and $|E_{G'''}| = |E_G| + 3$.

The relationship described in the three points above is illustrated in Figure 4.

When constructing any 3-connected graph $G$ using BG-operations it is worth noting that the order in which the BG-operations are performed is not always arbitrary as is illustrated by the following example.

**Example.** Consider the entry $p_{-2,2}$ in $P_{G_3}$ as shown in Figure 5. The graph $G_1$ in Figure 5 is the result of a $(1,2)$-operation performed on $K_4$ followed by another $(1,2)$-operation. The graph $G_2$ in Figure 5 is the result of a $(2,3)$-operation performed on $K_4$ followed by a $(0,1)$-operation. Note that no graph in $p_{-2,2}$ is the result of a $(0,1)$-operation performed on $K_4$ followed by a $(2,3)$-operation. Observe that $G_1 \sim G_2$ as they are both contained in $p_{-2,2}$ but that it is not possible to construct $G_1$ using a $(2,3)$-operation followed by a $(0,1)$-operation and similarly, it is not possible to construct $G_2$ using two successive $(1,2)$-operations.
6. Degree sequences and graph partitions

A finite sequence \( s = \{s_1, ..., s_n\} \) of non-negative integers is called \textit{graphic} if there exists a finite simple graph with vertex set \( \{v_1, ..., v_n\} \) such that \( v_i \) has degree \( s_i \) for all \( i = 1, ..., n \). Necessary and sufficient conditions for a sequence of non-negative integers to be graphic were first described by Erdős and Gallai in \([4]\). Necessary and sufficient conditions for a sequence of non-negative integers to be connected are implicit in Hakimi \([6]\) and these conditions have been stated explicitly by the author in \([7]\). The maximum degree of a vertex in \( G \) is denoted \( \Delta_G \) and the minimum degree of a vertex in \( G \) is denoted \( \delta_G \). Given a graph \( G \) then the degree sequence \( d(G) \) is the monotonic non-increasing sequence of degrees of the vertices in \( V_G \). This means that every graphical sequence \( s \) is equal to the degree sequence \( d(G) \) of some graph \( G \) (subject to possible rearrangement of the terms in \( s \)).

**Definition 6.1.** A finite sequence \( s = \{s_1, ..., s_n\} \) of positive integers is called \textit{3-connected} if there exists a finite simple 3-connected graph with vertex set \( \{v_1, ..., v_n\} \) such that \( \text{deg}(v_i) = s_i \) for all \( i = 1, ..., n \).

Given a sequence of positive integers \( s = \{s_1, ..., s_n\} \) then define the associated pair of \( s \), denoted \( (\varphi(s), \epsilon(s)) \), to be the pair \( (n, \frac{1}{2} \sum_{i=1}^{n} s_i) \). Where no ambiguity can arise, \( (\varphi(s), \epsilon(s)) \) is simply denoted \( (\varphi, \epsilon) \).

**Lemma 6.2.** Given a sequence \( s \) with associated pair \( (\varphi, \epsilon) \) such that \( s = d(G) \) for some 3-connected graph \( G \in \mathcal{G}_3 \), then \( G \) is contained in \( p_{\epsilon-3\varphi+6, \varphi-4} \) in \( P_{\mathcal{G}_3} \).

**Proof.** As \( s \) is the degree sequence \( d(G) \) of some graph \( G \in \mathcal{G}_3 \) then \( |V_G| \) is the number of terms in \( s \) which is \( \varphi \) and \( |E_G| \) is half the sum of the degrees of all vertices in \( G \) which is exactly \( \epsilon \). Recall that each entry \( p_{i,j} \) in \( P_{\mathcal{G}_3} \) is defined as \( \{G \in \mathcal{G}_3 \mid |V_G| = j + 4 \text{ and } |E_G| = i + 3j + 6\} \). By rearranging \( |V_G| = \varphi = j + 4 \) and substituting into \( |E_G| = \epsilon = i + 3j + 6 \) then \( j = \varphi - 4 \) and \( i = \epsilon - 3\varphi + 6 \) hence \( s = d(G) \) for some \( G \) contained in \( p_{\epsilon-3\varphi+6, \varphi-4} \). \( \square \)
Lemma 6.3. The non-empty entries in column $j = \varphi - 4$ of $P_{G_3}$ are

- $p - \frac{3\varphi + 12}{2}, \varphi - 4$ to $p\left(\frac{\varphi - 3}{2}\right), \varphi - 4$, inclusive, when $\varphi$ is even, and
- $p - \frac{3\varphi + 13}{2}, \varphi - 4$ to $p\left(\frac{\varphi - 3}{2}\right), \varphi - 4$, inclusive, when $\varphi$ is odd.

Proof. Observe that $\delta_G \geq 3$ for every $G \in G_3$. It follows that the minimum possible $\epsilon$ for a degree sequence $s$ of any 3-connected graph is $\epsilon = \frac{3\varphi}{2}$ i.e. $s = \{3, \ldots, 3\}$. Note that $\epsilon$ must be even by definition, however, this can only occur whenever $\varphi$ is even. In other words, there cannot exist a 3-regular graph with an odd number of vertices. From Lemma 6.2, when $\varphi$ is even, the uppermost non-empty entry in column $j = \varphi - 4$ of $P_{G_3}$ is contained in row $\min\{\epsilon\} - 3\varphi + 6 = \frac{3\varphi}{2} - 3\varphi + 6 = -\frac{3\varphi + 12}{2}$.

Observe that for any $G \in G_3$ where $d(G) = s$ then $\Delta_G \leq \varphi - 1$ as $G$ is simple. It follows that the maximum possible $\epsilon$ of any 3-connected graph is $\epsilon = \frac{\varphi}{2}$ i.e. $s = \{\varphi - 1, \ldots, \varphi - 1\}$. From Lemma 6.2 the lowermost non-empty entry in column $j = \varphi - 4$ of $P_{G_3}$ is contained in row $\max\{\epsilon\} - 3\varphi + 6 = \left(\frac{\varphi}{2}\right) - 3\varphi + 6 = \frac{\varphi^2 - 7\varphi + 12}{2} = \left(\frac{\varphi - 3}{2}\right)^2$. This argument is summarised in Figure 6.

| $\varphi$ even | $\{s_1, \ldots, s_n\}$ | $\epsilon$ |
|----------------|-------------------------|------------|
| $p - \frac{3\varphi + 12}{2}, \varphi - 4$ | $\{3, \ldots, 3\}$ | $\frac{3\varphi}{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $p\left(\frac{\varphi - 3}{2}\right), \varphi - 4$ | $\{n - 1, \ldots, n - 1\}$ | $\left(\frac{\varphi}{2}\right)$ |

Figure 6. All possible 3-connected degree sequences of even length $\varphi$.

If $\varphi$ is odd, note that the minimum possible $\epsilon$ for a degree sequence $s$ of any 3-connected graphs is $\epsilon = \frac{3\varphi + 1}{2}$ i.e. $s = \{4, 3, \ldots, 3\}$. It follows from Lemma 6.2 that, when $\varphi$ is odd, the uppermost non-empty entry in column $j = \varphi - 4$ of $P_{G_3}$ is contained in row $\min\{\epsilon\} - 3\varphi + 6 = \frac{3\varphi + 1}{2} - 3\varphi + 6 = -\frac{3\varphi + 13}{2}$. As the parity of $\varphi$ is irrelevant when maximising $\epsilon$ then the lowermost non-empty entry in column $j = \varphi - 4$ of $P_{G_3}$ is contained in row $\left(\frac{\varphi - 3}{2}\right)$. This argument is summarised in Figure 7.

□

Corollary 6.4. The non-empty entries in column $j$ of $P_{G_3}$ are

- $p - \frac{3j + 1}{2}, j$ to $p\left(\frac{2j + 1}{2}\right), j$, inclusive, when $j$ is even, and
- $p - \frac{2j + 1}{2}, j$ to $p\left(\frac{2j + 1}{2}\right), j$, inclusive, when $j$ is odd.
\[
\begin{array}{c|c|c}
\varphi \text{ odd} & \{s_1, \ldots, s_n\} & \epsilon \\
\hline
p \left(\frac{3\varphi+11}{2}\right), \varphi - 4 & \{4, 3, \ldots, 3\} & \frac{3\varphi+1}{2} \\
\vdots & \vdots & \vdots \\
P \left(\frac{\varphi-3}{2}\right), \varphi - 4 & \{n-1, \ldots, n-1\} & \left(\frac{\varphi}{2}\right)
\end{array}
\]

**Figure 7.** All possible 3-connected degree sequences of odd length \(\varphi\).

**Proof.** The result follows from Lemma 6.3 by letting \(\varphi = j + 4\). \(\square\)

It follows that, when \(j\) is odd, column \(j\) in \(P_\varphi\) has \(\frac{j^2-j}{2} + \frac{3j-1}{2} + 1 = \frac{j^2+4j+1}{2}\) non-empty entries, for example, column \(j = 1\) has \(\frac{1^2+4(1)+1}{2} = 3\) non-empty entries, as shown in Figure 3. Similarly, when \(j\) is even, column \(j\) in \(P_\varphi\) has \(\frac{j^2-j}{2} + \frac{3j}{2} + 1 = \frac{j^2+4j+2}{2}\) non-empty entries, for example, column \(j = 2\) has \(\frac{2^2+4(2)+2}{2} = 7\) non-empty entries, again see Figure 3.

Note that it is possible to provide alternative justifications for Lemma 6.3 and Corollary 6.4 using the relationship between BG-operations and entries in \(P_\varphi\) which is outlined in Figure 4.

7. Results

**Theorem 7.1.** Given a sequence \(s = \{s_1, \ldots, s_n\}\) of positive integers, with the associated pair \((\varphi, \epsilon)\), such that \(s_i \geq s_{i+1}\) for \(i = 1, \ldots, n-1\) then \(s\) is 3-connected if and only if

- \(\epsilon \in \mathbb{N}\),
- \(\frac{3\varphi}{2} \leq \epsilon \leq \left(\frac{\varphi}{2}\right)\),
- \(s_1 \leq \varphi - 1\) and \(s_n \geq 3\).

**Proof.** \((\Rightarrow)\) Clearly \(\epsilon \in \mathbb{N}\) is a necessary condition for any sequence \(s\) to be realisable as half the sum of the degrees in any graph is the number of edges in that graph and this must be a natural number. The necessity of the condition \(\frac{3\varphi}{2} \leq \epsilon \leq \left(\frac{\varphi}{2}\right)\) follows from Lemma 6.3 noting that \(\epsilon > \frac{3\varphi}{2}\) whenever \(\varphi\) is odd. Finally, the necessity of the condition \(s_1 \leq \varphi - 1\) follows from the definition of a simple graph and the need for \(s_n \geq 3\) is due to the fact that every vertex in a 3-connected graph has degree at least 3. It follows from Figures 6 and 7 in Lemma 6.3 that, for a fixed \(\varphi\), all sequences with \(\frac{3\varphi}{2} < \epsilon < \left(\frac{\varphi}{2}\right)\) are also realisable.

\((\Leftarrow)\) Suppose that \(s = \{s_1, \ldots, s_n\}\) is 3-connected. This means that \(s\) is the degree sequence of a 3-connected graph \(G\), hence \(\sum_{i=1}^{n} deg(v_i) = 2|V_G|\) and
so $\epsilon \in \mathbb{N}$. As $G$ is 3-connected then $\deg(v_i) \geq 3$ for all $i = 1, \ldots, n$ hence if $G$ is a minimal 3-connected graph on $n$ vertices then $d(G) = \{3, \ldots, 3\}$ with $|E_G| = \frac{3n}{2}$ if $n$ is even or $d(G) = \{4, 3, \ldots, 3\}$ with $|E_G| = \frac{3n+1}{2}$ if $n$ is odd, hence $s_n \geq 3$ and $\epsilon \geq \frac{3\epsilon}{2}$. As $G$ is simple then $\deg(v_i) \leq n - 1$ for all $i = 1, \ldots, n$ and the maximal simple (3-connected) graph on $n$ vertices is the complete graph $K_n$ which has the degree sequence $\{n-1, \ldots, n-1\}$ and $|E_{K_n}| = \binom{n}{2}$, hence $s_1 \leq n - 1$ and $\epsilon \leq \binom{\epsilon}{2}$.

Before stating the next result the following definition is required.

**Definition 7.2.** A finite sequence $s = \{s_1, \ldots, s_n\}$ of positive integers is called **necessarily 3-connected** if $s$ can only be realisable as a 3-connected (simple) graph.

**Theorem 7.3.** Given a sequence $s = \{s_1, \ldots, s_n\}$ of positive integers, with the associated pair $(\varphi, \epsilon)$, such that $s_i \geq s_{i+1}$ for $i = 1, \ldots, n - 1$ then $s$ is necessarily 3-connected if and only if $s$ is 3-connected and $\epsilon > \left(\frac{\varphi}{2}\right)^2 + 5$.

**Proof.** ($\Rightarrow$) Clearly it is necessary for $s$ to be 3-connected if it is to be necessarily 3-connected. It is required to show that it is necessary for $\epsilon > \left(\frac{\varphi}{2}\right)^2 + 5$. Consider a sequence $s = \{s_1, \ldots, s_n\}$ such that $\epsilon = \left(\frac{\varphi}{2}\right)^2 + 5$. Observe that one such sequence is $s' = \{n-1, n-1, n-3, \ldots, n-3, 3, 3\}$ which has $\varphi(s') = \varphi(s'') = \left(n, \frac{(n-2)(n-3)+4+6}{2}\right) = \left(n, \frac{n-2}{2} + 5\right)$. Observe that $s' = d(G_1)$, see Figure 8 where $G_1 = H_1 \cup H_2$ such that $H_1 \simeq K_4$, $H_2 \simeq K_{n-2}$ and $H_1 \cap H_2 \simeq K_2$ with $V_{H_1} = \{v_1, v_2, v_{n-1}, v_n\}$, $V_{H_2} = \{v_1, \ldots, v_{n-2}\}$ and $V_{H_1 \cap H_2} = \{v_1, v_2\}$. Note that $G_1$ is 2-connected as $G_1 \setminus \{v_1, v_2\}$ is disconnected.

However, $s' = \{n-1, n-1, n-3, \ldots, n-3, 3, 3\}$ is, in fact, 3-connected as $s'$ is also the degree sequence of $G_2$, again see Figure 8 noting that $v_i v_j \in E_{G_1}$ but $v_i v_j \notin E_{G_2}$. Therefore, it is required that $\epsilon > \left(\frac{\varphi}{2}\right)^2 + 5$ if $s$ is to be necessarily 3-connected as the sequence $s' = \{n-1, n-1, n-3, \ldots, n-3, 3, 3\}$ with $\epsilon(s') = \left(\frac{\varphi}{2}\right)^2 + 5$ is realisable as a 2-connected graph.

![Figure 8](image-url) \hspace{1cm} \text{Figure 8.} d(G_1) = d(G_2) = s' = \{n-1, n-1, n-3, \ldots, n-3, 3, 3\}.

($\Leftarrow$) It is now required to show that if $s$ is 3-connected and $\epsilon > \left(\frac{\varphi}{2}\right)^2 + 5$ then $s$ is necessarily 3-connected. To show this it is required to show that the maximum number of edges in a graph with $n$ vertices which is not 3-connected is
Observe that any maximally 2-connected graph on \( n \) vertices will necessarily contain a cut set with two vertices \( \{u, v\} \) i.e. \( G \setminus \{u, v\} \) is disconnected. To maximise the number of edges in \( G \) it is clear that \( G \setminus \{u, v\} \) contains just two connected components i.e. \( G = H_1 \cup H_2 \) where \( H_1 \cap H_2 = (\{u, v\}, \{uv\}) = C \) with \( H_1 \simeq K_{a+2}, H_2 \simeq K_{b+2} \) and \( H_1 \cap H_2 \simeq K_2 \) (noting that \( a + b = n - 2 \)). So, the task of maximising \( |E_{H_1 \setminus C}| + |E_{H_2 \setminus C}| \) is equivalent to minimising the number of edges in a complete bipartite graph \( K_{a,b} \) as \( K_n \setminus (E_{H_1 \cup H_2}) \simeq K_{a,b} \).

Let \( a + b = n - 2 \), with \( a \leq b \), then \( |E_{K_{a,b}}| = ab \) where \( a, b \in \{1, \ldots, n-3\} \). Note that \( a > 0 \) as \( G \setminus \{u, v\} \) is disconnected i.e. \( K_a \neq K_0 = (\emptyset, \emptyset) \). It is straightforward to show that \( ab \) attains its maximum at \( a = b = \frac{n-2}{2} \), when \( n \) is even, and at \( a = \lfloor \frac{n-2}{2} \rfloor, b = \lceil \frac{n-2}{2} \rceil \) when \( n \) is odd. It follows that \( ab \) is minimised when \( a = 1 \) and \( b = n - 3 \). However, observe that \( a > 1 \) as \( a = 1 \) implies that \( H_1 \simeq K_3 \) which means that \( d(G) \) contains a term equal to 2, but this contradicts the \( s_n \geq 3 \) condition. Hence \( |E_{K_{a,b}}| \), with \( a + b = n - 2 \), is minimised when \( a = 2 \) and \( b = n - 4 \) and so the maximal 2-connected graph on \( n \) vertices is isomorphic to \( H_1 \cup H_2 \) where \( H_1 \simeq K_4, H_2 \simeq K_{n-2} \) and \( H_1 \cap H_2 \simeq K_2 \), see \( G_1 \) in Figure 8. Notice that the union of \( G_1 \) and any edge in \( G_1 \), the complement of \( G_1 \), results in a 3-connected graph. □

**Corollary 7.4.** All simple graphs with \( n \) vertices and at least \( \frac{n^2-5n+18}{2} \) edges are 3-connected.

**Proof.** As shown in Theorem 7.3, a maximal 2-connected graph with \( n \) vertices is isomorphic to the union of \( H_1 \simeq K_4 \) and \( H_2 \simeq K_{n-2} \) where \( H_1 \cap H_2 \simeq K_2 \) and such a graph has \( |E_{K_{n-2}}| + |E_{K_4}| - |E_{K_2}| = (\binom{n-2}{2}) + (\binom{4}{2}) - (\binom{2}{2}) \) edges. It follows that any simple graph with \( n \) vertices and at least \( (\binom{n-2}{2}) + 5 \) edges is 3-connected. □

Note that for all \( n \in \mathbb{N}, n^2 - 5n \) is even and \( n^2 - 5n + 18 > 0 \), hence \( \frac{n^2-5n+18}{2} \in \mathbb{N} \).

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