Bounds for randomly shared risk of heavy-tailed loss factors

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Abstract

For a risk vector $V$, whose components are shared among agents by some random mechanism, we obtain asymptotic lower and upper bounds for the agents’ exposure risk and the systemic risk in the market. Risk is measured by Value-at-Risk or Conditional Tail Expectation. We assume Pareto tails for the components of $V$ and arbitrary dependence structure in a multivariate regular variation setting. Upper and lower bounds are given by asymptotic independent and fully dependent components of $V$ in dependence of the tail index $\alpha$ being smaller or larger than 1. Counterexamples complete the picture.

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1 Introduction

Let $V_j$ for $j = 1, \ldots, d$ be risk variables having Pareto-tails, so that, for possibly different $K_j > 0$ and tail index $\alpha > 0$,

$$P(V_j > t) \sim K_j t^{-\alpha}, \quad t \to \infty. \tag{1.1}$$

(For two functions $f$ and $g$ we write $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t \to \infty} f(t)/g(t) = 1$.) We summarize all risk variables in a vector $V = (V_1, \ldots, V_d)^\top$.

The $d$ risks in $V$ are shared among $q$ agents by some random mechanism. Let $F_i$ denote the exposure of agent $i$ and $F = (F_1, \ldots, F_q)^\top$ the exposure vector. The risk sharing is governed by a random $q \times d$ matrix $A = (A_{ij})^{q,d}_{i,j=1}$ (independent of $V$) in such a way that $F_i = \sum_{j=1}^d A_{ij} V_j$ for $i = 1, \ldots, q$ or, equivalently, in matrix notation

$$F = AV. \tag{1.2}$$

This note has been motivated by [9], where the risk variables $V_j$ model large insurance claims and agents represent reinsurance companies. The claims are randomly shared with a mechanism given by a bipartite graph structure, resulting in

$$A_{ij} = \frac{1(i \sim j)}{\deg(j)}, \tag{1.3}$$
where \(1(i \sim j)\) indicates, whether agent \(i\) takes a (proportional) share of risk \(j\) or not. Further examples include operational risk, modelling event types (risk variables) and business lines (agents), where Pareto tails are natural (cf. [4]), and also overlapping portfolios (common asset holding) as described in [3].

In all these applications it is of interest to quantify not only the risk of single agents, but also the market risk which—as a systemic risk—is of high relevance to the regulator. In accordance with [6] we assess the systemic risk by a risk measure on the \(r\)-norm \(\|F\|\) for \(r \geq 1\) of the exposure vector \(F\).

We investigate risk based on the Value-at-Risk (VaR) and Conditional Tail Expectation (CoTE), which we assess by asymptotic approximations.

Let \(V_{\text{ind}}, V, V_{\text{dep}}\) be risk vectors as above with different dependence structures among the risk variables. Here \(V_{\text{ind}}\) corresponds to asymptotically independent variables and \(V_{\text{dep}}\) to asymptotically fully dependent variables in the framework of multivariate regular variation as in [9].

As in the copula world (see [3, 7]) it is possible to assess the two extreme dependence structures, i.e. \(V_{\text{ind}}, V_{\text{dep}}\) and it is of high relevance to understand, if or under which conditions these extreme dependences lead to upper and lower bounds of risk for arbitrary dependence structures.

This note is organised as follows. In Section 2 we present \(V\) as a regularly varying vector with different dependence structures. Here we also define the risk measures VaR and CoTE for arbitrary random variables, and summarize their asymptotic behaviour in our framework. In Section 3 we derive bounds for single and systemic risk based on asymptotically independent and fully dependent random variables. We also give counter examples to present the limitations of the bounds.

2 Preliminaries

2.1 Multivariate regular variation

We recall from [12], Ch. 6 that the positive random vector \(V \in \mathbb{R}_+^d\) is *multivariate regularly varying* if there is a Radon measure \(\nu \not\equiv 0\) on the Borel \(\sigma\)-algebra \(\mathcal{B} = \mathcal{B}(\mathbb{R}_+^d \setminus \{0\})\), where 0 denotes the zero vector in \(\mathbb{R}_+^d\), such that

\[
    n\mathbb{P}\left[n^{-1/\alpha} V \in \cdot \right] \overset{\nu}{\to} \nu(\cdot), \quad n \to \infty.
\]

The symbol \(\overset{\nu}{\to}\) stands for vague convergence. Moreover, the measure \(\nu\) is homogeneous of some order \(-\alpha\) with \(\alpha > 0\) and is called the *exponent measure of* \(V\).

Denoting by \(S_+^{d-1} = \{x \in \mathbb{R}_+^d : \|x\| = 1\}\) the positive sphere in \(\mathbb{R}^d\), the existence of the exponent measure \(\nu\) is equivalent to the existence of a Radon measure \(\rho \not\equiv 0\) on the Borel \(\sigma\)-algebra \(\mathcal{B}(S_+^{d-1})\) in such a way that for all \(u > 0\)

\[
    \frac{\mathbb{P}\left[\|V\| > ut, V\|V\|^{-1} \in \cdot \right]}{\mathbb{P}\left[\|V\| > t\right]} \overset{\nu}{\to} u^{-\alpha} \rho(\cdot), \quad t \to \infty,
\]

holds. The measure \(\rho\) is called the *spectral measure of* \(V\). The precise relation between \(\nu\) and \(\rho\) can be found in [12], Ch. 6.

Finally, we note that convergence in (2.1) also implies

\[
    \frac{\mathbb{P}\left[t^{-1} V \in \cdot \right]}{\mathbb{P}\left[\|V\| > t\right]} \overset{\nu}{\to} \nu(\{x : \|x\| > 1\}), \quad t \to \infty.
\]
The tail index \( \alpha > 0 \) is also called the index of regular variation of \( V \), and we write \( V \in \mathcal{R}(-\alpha) \).

We shall often work with the so-called canonical exponent measure \( \nu^* \) of \( V \), which is defined as the image measure \( \nu^* = \nu \circ T \) under the transformation mapping \( T : \mathbb{R}_d^+ \to \mathbb{R}_d^+ \), given by

\[
T(x) = (\nu(\{x_1 > 1\})^{1/\alpha}, \ldots, \nu(\{x_d > 1\})^{1/\alpha})^\top.
\]

Then \( \nu^* \) has standardized margins and a tail index 1, corresponding to \( P(V_j > x) \sim x^{-1} \) as \( x \to \infty \).

The corresponding spectral measure \( \rho^* \) is called the canonical spectral measure and is characterized by

\[
\int_{\mathbb{S}^{d-1}} s_j \rho^*(ds) = 1, \quad j = 1, \ldots, d,
\]

see [2], p. 259.

For the matrix \( A \) and a given norm \( \| \cdot \| \), which gives rise to an operator norm

\[
\|A\|_{op} = \sup_{\|x\|=1} \|Ax\|,
\]

we require throughout the following:

- \( A \) satisfies the moment condition \( \mathbb{E} \|A\|_{op}^{\alpha+\delta} < \infty \) for some \( \delta > 0 \) and \( \alpha \) as in (1.1);
- the vector \( V \) is independent of the random matrix \( A \), while \( V_1, \ldots, V_d \) may not be independent of each other.

If both conditions hold, then the vector \( F = AV \) is again regularly varying with exponent measure \( \mathbb{E} \nu \circ A^{-1} \) (cf. [1], Proposition A.1).

### 2.2 Risk measures

We also recall the following risk measures.

**Definition 2.1.** The Value-at-Risk (VaR) of a random variable \( X \) at confidence level \( 1 - \gamma \) is defined as

\[
\text{VaR}_{1-\gamma}(X) := \inf \{ t \geq 0 : \mathbb{P}[X > t] \leq \gamma \}, \quad \gamma \in (0, 1),
\]

and the Conditional Tail Expectation (CoTE) at confidence level \( 1 - \gamma \), based on the corresponding VaR, as

\[
\text{CoTE}_{1-\gamma}(X) := \mathbb{E}[X \mid X > \text{VaR}_{1-\gamma}(X)], \quad \gamma \in (0, 1).
\]

Throughout the following constants will be relevant

\[
C_{\text{ind}}^i = \sum_{j=1}^d \mathbb{E} K_j A_{ij}^\alpha, \quad i = 1, \ldots, q, \quad \text{and} \quad C_{\text{ind}}^S = \sum_{j=1}^d K_j \mathbb{E} \|Ae_j\|^{\alpha},
\]

\[
C_{\text{dep}}^i = \mathbb{E}(AK^{-1/\alpha}1)_i^\alpha, \quad i = 1, \ldots, q, \quad \text{and} \quad C_{\text{dep}}^S = \mathbb{E}\|A1\|^{\alpha}.
\]
Lemma 2.2 ([9], Lemmas 3.7 and 3.8). Let $F = AV = (F_1, \ldots, F_q)^{\top}$.

(a) Individual risk measures:
For $\alpha > 0$ the individual Value–at–Risk of agent $i \in \{1, \ldots, q\}$ satisfies
\[ \text{VaR}_{1-\gamma}(F_i) \sim C_1^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0. \] (2.7)

For $\alpha > 1$ the individual Conditional Tail Expectation of agent $i \in \{1, \ldots, n\}$ satisfies
\[ \text{CoTE}_{1-\gamma}(F_i) \sim \frac{\alpha}{\alpha - 1} \text{VaR}_{1-\gamma}(F_i) \sim \frac{\alpha}{\alpha - 1} C_1^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0. \]

The individual constants are either $C = C_{\text{ind}}^i$ or $C = C_{\text{dep}}^i$ for $V_1, \ldots, V_d$ asymptotically independent or asymptotically fully dependent, respectively.

(b) Systemic risk measures:
The market Value–at–Risk of the aggregated vector $\|F\|$ satisfies
\[ \text{VaR}_{1-\gamma}(\|F\|) \sim C_1^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0. \] (2.8)

The market Conditional Tail Expectation of the aggregated vector $\|F\|$ satisfies
\[ \text{CoTE}_{1-\gamma}(\|F\|) \sim \frac{\alpha}{\alpha - 1} \text{VaR}_{1-\gamma}(\|F\|) \sim \frac{\alpha}{\alpha - 1} C_1^{1/\alpha} \gamma^{-1/\alpha}, \quad \gamma \to 0. \]

The systemic constants are either $C = C_{\text{ind}}^S$ or $C = C_{\text{dep}}^S$ for $V_1, \ldots, V_d$ asymptotically independent or asymptotically fully dependent, respectively.

3 Bounds for general dependence structure

Recall from (3.7) and (3.9) of [9] that the constants (2.5) can be expressed in terms of the exponent measure via
\[ C_{\text{ind}}^i = \mathbb{E} \nu_{\text{ind}} \circ A^{-1}(\{x : x_i > 1\}), \quad i = 1, \ldots, q, \quad \text{and} \quad C_{\text{ind}}^S = \mathbb{E} \nu_{\text{ind}} \circ A^{-1}(\{x : \|x\| > 1\}) \] (3.1)
\[ C_{\text{dep}}^i = \mathbb{E} \nu_{\text{dep}} \circ A^{-1}(\{x : x_i > 1\}), \quad i = 1, \ldots, q, \quad \text{and} \quad C_{\text{dep}}^S = \mathbb{E} \nu_{\text{dep}} \circ A^{-1}(\{x : \|x\| > 1\}) \] (3.2)

with (cf. Lemma 2.2 of [9])
\[ \nu_{\text{ind}}([0,x]^c) = \sum_{j=1}^{d} K_j x_j^{-\alpha} \quad \text{and} \quad \nu_{\text{dep}}([0,x]^c) = \max_{j=1,\ldots,d} \{K_j x_j^{-\alpha}\}. \] (3.3)

The analogues of the constants $C_{\text{ind}}^i, C_{\text{dep}}^i$ as well as $C_{\text{ind}}^S$ and $C_{\text{dep}}^S$ in the case of an arbitrary extremal dependence structure of the vector $V$, represented by some exponent measure $\nu$ with $\nu_{\text{ind}} \neq \nu \neq \nu_{\text{dep}}$, are then
\[ C_{\nu}^i = \mathbb{E} \nu \circ A^{-1}(\{x : x_i > t\}) \quad \text{and} \quad C_{\nu}^S = \mathbb{E} \nu \circ A^{-1}(\{x : \|x\| > t\}). \] (3.4)

We summarize the constants $K_j$, $j = 1, \ldots, d$ from (1.11) in a diagonal matrix
\[ K^{1/\alpha} := \text{diag}(K_1^{1/\alpha}, \ldots, K_d^{1/\alpha}). \]

Then for the exponent measure $\nu$ of the vector $V$ with any dependence structure,
\[ C_{\nu}^S = \mathbb{E} \nu \circ K^{1/\alpha} \circ (AK^{1/\alpha})^{-1}(\{\|x\| > 1\}) \quad \text{and} \quad C_{\nu}^i = \mathbb{E} \nu \circ K^{1/\alpha} \circ (AK^{1/\alpha})^{-1}(\{x_i > 1\}). \]
Note that the measure $\nu \circ K^{1/\alpha}$ has balanced tails; i.e., $\nu \circ K^{1/\alpha}((x_j > 1)) = 1, j = 1, \ldots, d$. Since all marginal random variables are as in (1.1), regardless of the dependence structure of the vector $V$, for the proofs of all theorems below we can and do assume that margins are standardized; e.g. $K_j = 1$ for $j = 1, \ldots, d$. Moreover, for establishing inequalities between $C^i_{\text{ind}}, C^i_{\text{dep}}$ and $C^i_{\nu}$ or $C^S_{\text{ind}}, C^S_{\text{dep}}$ and $C^S_{\nu}$, respectively, it is sufficient to prove the corresponding inequalities for all realizations of the random matrix $A$. We obtain the following bounds for the constants defining the individual risk measures.

**Theorem 3.1.** Let the three $d$-dimensional vectors $V_{\text{ind}}, V$ and $V_{\text{dep}}$ be given with equal margins $V_1, \ldots, V_d$ with $P[V_j > t] \sim K_j t^{-\alpha}$, but different exponent measures $\nu_{\text{ind}}, \nu, \nu_{\text{dep}}$. Then for the constants $C^i$ referring to agent $i$ the following inequalities hold:

$$C^i_{\text{ind}} \leq C^i_{\nu} \leq C^i_{\text{dep}} \quad \text{for } \alpha \geq 1,$$

$$C^i_{\text{dep}} \leq C^i_{\nu} \leq C^i_{\text{ind}} \quad \text{for } \alpha < 1. \quad (3.5)$$

**Proof.** Let $a_i := A_i$ be the $i$-th row of the matrix $A$ and $V_{\text{ind}}, V, V_{\text{dep}}$ be as above the risk vectors with different dependence structures. Corollary 3.8 in [11] provides for $\alpha \geq 1$ the inequalities

$$\limsup_{t \to \infty} \frac{P[a_i V_{\text{ind}} > t]}{P[a_i V]} \leq 1 \quad \text{and} \quad \limsup_{t \to \infty} \frac{P[a_i V > t]}{P[a_i V_{\text{dep}} > t]} \leq 1 \quad (3.7)$$

and for $0 < \alpha < 1$ the inequalities

$$\limsup_{t \to \infty} \frac{P[a_i V_{\text{dep}} > t]}{P[a_i V > t]} \leq 1 \quad \text{and} \quad \limsup_{t \to \infty} \frac{P[a_i V > t]}{P[a_i V_{\text{ind}} > t]} \leq 1. \quad (3.8)$$

Regarding the left inequality in (3.7), we have

$$\limsup_{t \to \infty} \frac{P[a_i V_{\text{ind}} > t]}{P[a_i V]} = \limsup_{t \to \infty} \frac{P[a_i V_{\text{ind}} > t]}{P[\|V\| > t]} \frac{P[\|V\| > t]}{P[V_j > t]} = \frac{\nu_{\text{ind}} \circ A^{-1}(\{x_i > 1\}) \nu(\{x_i > 1\})}{\nu(\{x_i > 1\})} \leq 1,$$  

since w.l.o.g all marginals are the same. The other inequalities in (3.5) as well as in (3.6) are treated analogously. \qed

For bounds on the systemic risk measures we invoke ideas from [11]. Below we sometimes write $C^S_{\nu}(A)$ and $C^S_{\nu}(A)$ instead of $C^i_{\nu}$ and $C^S_{\nu}$, if we want to emphasize that the constants depend on a particular matrix $A$.

**Theorem 3.2.** Let the three $d$-dimensional vectors $V_{\text{ind}}, V$ and $V_{\text{dep}}$ be given with equal margins $V_1, \ldots, V_d$ with $P[V_j > t] \sim K_j t^{-\alpha}$, but different exponent measures $\nu_{\text{ind}}, \nu, \nu_{\text{dep}}$. Denote the aggregated vector $\|F\|$ for some $r$-norm for $r > 1$, representing systemic risk.

(a) For the constants $C^S$ referring to systemic risk the following inequalities hold:

$$C^S_\nu \geq C^S_{\text{ind}} \quad \text{for } \alpha \geq r,$$

$$C^S_\nu \leq C^S_{\text{ind}} \quad \text{for } 0 < \alpha < 1. \quad (3.10)$$

(b) However, there are matrices $A_1, A_2$ and an exponent measure $\nu_0$ such that

$$C^S_{\text{ind}}(A_1) > C^S_{\nu_0}(A_1) \quad \text{for } 1 < \alpha < r,$$

$$C^S_{\nu_0}(A_2) > C^S_{\text{ind}}(A_2) \quad \text{for } 1 < \alpha < r. \quad (3.12)$$
Proof. (a) In analogy to [11] we define for \( s^{1/\alpha} := (s_1^{1/\alpha}, \ldots, s_d^{1/\alpha}) \)
\[
g_{A,\alpha}(s) := \| A s^{1/\alpha} \|^\alpha \quad \text{and} \quad \rho^* g_{A,\alpha} := \int_{\mathbb{R}^d} g_{A,\alpha}(s) \rho^*(ds) \quad \text{for some canonical spectral measure} \ \rho^*. \]
Analogously to (3.9), we note that
\[
\nu_{\text{ind}} \circ A^{-1}(\{\|x\| > 1\}) = \lim_{t \to \infty} \frac{\mathbb{P}[\|A V_{\text{ind}}\| > t]}{\mathbb{P}[\|A V\| > t]} \quad \text{for} \quad \mathbb{P}[\|A V\| > t] > 0
\]
Furthermore, we get from Propositions 3.2 and 3.3 in [11] that
\[
\lim_{t \to \infty} \frac{\mathbb{P}[\|A V_{\text{ind}}\| > t]}{\mathbb{P}[\|A V\| > t]} = \frac{\rho_{\text{ind}} g_{A,\alpha}}{\rho^* g_{A,\alpha}}
\]
holds. Hence, in order to prove (3.11) and (3.11) it is sufficient to show that \( \rho^*_{\text{ind}} (g_{A,\alpha}) \leq \rho^* (g_{A,\alpha}) \)
and \( \rho^*_{\text{ind}} (g_{A,\alpha}) \geq \rho^* (g_{A,\alpha}) \), respectively.
We first show (3.11). Note that for nonnegative real numbers \( a_1, \ldots, a_n \) and \( \beta \geq 1 \) the inequality
\[
a_1^\beta + \cdots + a_n^\beta \leq (a_1 + \cdots + a_n)^\beta
\]
is valid. Since \( \rho^*_{\text{ind}} g_{A,\alpha} = \sum_{j=1}^d \| A e_j \|^\alpha \), and using (3.24), we write as in the proof of Theorem 3.7 of [11]
\[
\rho^*_{\text{ind}} g_{A,\alpha} = \int_{\mathbb{R}^d} \sum_{j=1}^d \| A e_j \|^\alpha s_{j\rho^*}(ds) = \int_{\mathbb{R}^d} \frac{\sum_{j=1}^d \| A s_{j\rho^*} \|^\alpha}{\| \sum_{j=1}^d A s_{j\rho^*} \|^{\alpha}} \frac{\sum_{j=1}^d A s_{j\rho^*}}{\| A s_{j\rho^*} \|^{\alpha}} \rho^*(ds).
\]
In order to establish \( \rho^*_{\text{ind}} g_{A,\alpha} \leq \rho^* g_{A,\alpha} \) it is sufficient to bound the fraction under the right hand integral by one. For this, we recall that all the entries in \( A \) are nonnegative and that \( \frac{\alpha}{\beta} \geq 1 \). We compute
\[
\sum_{j=1}^d \| A s_{j\rho^*} \|^\alpha \leq \sum_{j=1}^d \left( \sum_{i=1}^d (a_{ij} s_{j\rho^*})^\alpha \right) = \sum_{j=1}^d \left( \sum_{i=1}^d (a_{ij} s_{j\rho^*}) \right)^\alpha \leq \left( \sum_{i=1}^d \sum_{j=1}^d (a_{ij} s_{j\rho^*}) \right)^\alpha = \left( \sum_{j=1}^d \| A e_j \|^\alpha \right) = \| A e_j \|^\alpha
\]
where we have applied inequality (3.17) twice.
For the bound (3.11) we use the \( c_r \)-inequality, see e.g. [10], p. 157, leading to
\[
\| \sum_{i=1}^n x_i \|^\alpha \leq \left( \sum_{i=1}^n \| x_i \| \right)^\alpha \leq \sum_{i=1}^n \| x_i \|^\alpha
\]
for \( x_1, \ldots, x_n \in \mathbb{R}^d \). In particular,
\[
\rho^*_{\text{ind}} g_{A,\alpha} = \sum_{j=1}^d \| A e_j \|^\alpha = \int_{\mathbb{R}^d} \sum_{j=1}^d \| A e_j \|^\alpha s_{j\rho^*}(ds)
\]
\[
= \int_{\mathbb{R}^d} g_{A,\alpha}(s) \frac{\sum_{j=1}^d \| A s_{j\rho^*} \|^\alpha}{\| A s_{j\rho^*} \|^{\alpha}} \rho^*(ds) \geq \rho^* g_{A,\alpha}
\]
leading to
\[
C^S_{\nu} = \nu \circ A^{-1}(\{\|x\| > 1\}) \leq \nu_{\text{ind}} \circ A^{-1}(\{\|x\| > 1\}) = C^S_{\text{ind}}
\]
as expressed in (3.11).
(b) Concerning examples for (3.12) and (3.13), we choose \( \nu_0 \) to be the image measure \( \nu_0 := \nu_{\text{ind}} \circ B^{-1} \) with standard exponent measure \( \nu_{\text{ind}} \) on \( \mathbb{R}_+^3 \) given as usual by \( \nu_{\text{ind}}([0,x)^c) = \sum_{j=1}^{3} x_j^{-\alpha} \) and a matrix

\[
B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\]

Furthermore, we define the function \( T : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) as

\[
T(x) = ((\nu_0(\{y \in \mathbb{R}_+^2 : |y_1| > 1\}) x_1)^{1/\alpha}, (\nu_0(\{y \in \mathbb{R}_+^2 : |y_2| > 1\}) x_2)^{1/\alpha})^T.
\]

The measure \( \nu_0^* = \nu_0 \circ T \) is then canonical; i.e., it is homogeneous of order \(-1\) and \( \nu_0^*(\{y \in \mathbb{R}_+^2 : |y_i| > 1\}) = 1 \) for \( i = 1, 2 \). To get the canonical spectral measure, we conduct the transformation to polar coordinates by setting \( \tau(x) = (|x|, \sum_{i=1}^2 x_i^2) \). Denoting by \( \rho_0^* \) the spectral measure and defining the measure \( \pi \) by \( d\pi(x) = x^{-2}dx \), the relation \( \nu_0^* = \pi \otimes \rho_1^* \) holds. We can now calculate \( \rho_0^* \) as follows. We first note that by construction \( \nu_0 \) and hence \( \nu_0^* \) only have positive mass on the axes as well as on the diagonal \( \{t \mathbf{1} : t > 0\} \). Therefore, the canonical spectral measure, living on the sphere \( S_0^2 \), only attains mass at the points \((1,0)^T, (0,1)^T, \mathbf{1}/\|\mathbf{1}\| \). We first observe that \( \nu_0 \circ B^{-1}(\{x : |x_i| > 1\}) = 2 \) for \( i = 1, 2 \). This yields

\[
\rho_0^*((\{1,0\})^T) = \nu_0 \circ T((\{te_1 : t > 1\})
= \nu_{\text{ind}} \circ B^{-1}(\{2^{1/\alpha}te_1 : t > 1\})
= \nu_{\text{ind}}(x \in \mathbb{R}_+^2 \mid Bx \in \{2^{1/\alpha}te_1 \in \mathbb{R}_+^2 : t > 1\})
= \nu_{\text{ind}}(se_1 \in \mathbb{R}_+^2 \mid sBe_2 \in \{2^{1/\alpha}te_1 \in \mathbb{R}_+^2 : t > 1\})
= \nu_{\text{ind}}(se_1 \in \mathbb{R}_+^2 \mid s \in [2^{1/\alpha}, \infty)) = \frac{1}{2} = \rho_0^*((\{0,1\})^T)
\]

by symmetry. For the third atom we calculate

\[
\rho_0^*(\{\mathbf{1}/\|\mathbf{1}\|\}) = \nu_0 \circ T((\{t\mathbf{1}/\|\mathbf{1}\| : t > 1\})
= \nu_{\text{ind}} \circ B^{-1}(\{2^{1/\alpha}t\mathbf{1}/\|\mathbf{1}\| : t > 1\})
= \nu_{\text{ind}}(x \in \mathbb{R}_+^2 \mid Bx \in \{2^{1/\alpha}t\mathbf{1}/\|\mathbf{1}\| : t > 1\})
= \nu_{\text{ind}}(se_1 \in \mathbb{R}_+^2 \mid sBe_1 \in \{(2/\|\mathbf{1}\|)^{1/\alpha}t\mathbf{1} \in \mathbb{R}_+^2 : t > 1\})
= \nu_{\text{ind}}(se_2 \mid s \in [(2/\|\mathbf{1}\|)^{1/\alpha}, \infty)) = \frac{\|\mathbf{1}\|}{2}.
\]

Consequently, we have

\[
\rho_0^* = \frac{1}{2}\delta_{(1,0)^T} + \frac{1}{2}\delta_{(0,1)^T} + \frac{\|\mathbf{1}\|}{2}\delta_{\mathbf{1}/\|\mathbf{1}\|}.
\]

Furthermore, the canonical spectral measures for the case of asymptotical independence and full dependence are

\[
\rho_{\text{ind}}^* = \delta_{(1,0)^T} + \delta_{(0,1)^T} \quad \text{and} \quad \rho_{\text{dep}}^* = \|\mathbf{1}\|\delta_{\mathbf{1}/\|\mathbf{1}\|}
\]

In order to construct counterexamples we choose \( d = q = 2 \) and the function \( g_{A_1,\alpha} \) with \( A_1 = I_2 \) the identity matrix. Then

\[
\rho_0^*g_{A_1,\alpha} = \int_{\mathbb{S}_1^2} \|Ax\|^{\alpha} \, d\rho_0^*
= \|A(1,0)^T\|^{\alpha}\rho_1^*((\{1,0\})^T) + \|A(0,1)^T\|^{\alpha}\rho_1^*((\{0,1\})^T) + \|I_2(\mathbf{1}/\|\mathbf{1}\|)^{1/\alpha}\|^{\alpha}\rho_1^*((\mathbf{1}/\|\mathbf{1}\|))
\]
\[ = 2^{-1} + 2^{-1} + \|1\|^{-1}(1,1)^\alpha \frac{\|1\|}{2} \]
\[ = 1 + 2^{\frac{\alpha}{2}} , \]

while \( \rho_{\text{ind}}^* g_{A_1,\alpha} = 2 \). This leads to the equivalences
\[
\rho_{\text{ind}}^* g_{A_1,\alpha} < \rho_{\text{ind}}^* g_{A_1,\alpha} \iff 2 > 1 + 2^{\frac{\alpha}{2}} - 1 \iff 2^{\frac{\alpha}{2}} - 1 \iff r > \alpha . \quad (3.21)
\]

In particular, we have for \( 1 < \alpha < r \),
\[
C_{\nu_0}^S(A_1) < C_{\text{ind}}^S(A_1) .
\]

Next, we choose \( A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and calculate
\[
\rho_{\text{ind}}^* g_{A_2,\alpha} = \|1\|^\alpha + \|1\|^\alpha = 2^{\frac{\alpha}{2} + 1}
\]
as well as
\[
\rho_{\text{ind}}^* g_{A_2,\alpha} = \frac{1}{2} \|1\|^\alpha + \frac{1}{2} \|1\|^\alpha + \frac{1}{2} \|1\|^\alpha = 2^{\frac{\alpha}{2}} + 1 .
\]
Consequently,
\[
\rho_{\text{ind}}^* g_{A_2,\alpha} < \rho_{\text{ind}}^* g_{A_2,\alpha} \iff 2 < 2^\alpha .
\]

Therefore, for \( \alpha > 1 \), \( C_{\text{ind}}^S(A_2) < C_{\nu_0}^S(A_2) \).

\[ \square \]

**Theorem 3.3.** Let the three \( d \)-dimensional vectors \( V_{\text{ind}}, V \) and \( V_{\text{dep}} \) be given with equal margins \( V_1, \ldots, V_d \) with \( \mathbb{P}[V_j > t] \sim K_j t^{-\alpha} \), but different exponent measures \( \nu_{\text{ind}}, \nu, \nu_{\text{dep}} \). Denote the aggregated vector \( \|F\| \) for some \( r \)-norm for \( r > 1 \), representing systemic risk.

(a) For the constants \( C^S \) referring to systemic risk the following inequalities hold:
\[
\begin{align*}
C_{\nu}^S &\leq C_{\text{dep}}^S \quad \text{for } \alpha \geq r \quad (3.22) \\
C_{\nu}^S &\geq C_{\text{dep}}^S \quad \text{for } 0 < \alpha < 1 \quad (3.23)
\end{align*}
\]

(b) However, there are matrices \( A_1, A_2 \) and an exponent measure \( \nu_0 \) such that
\[
\begin{align*}
C_{\nu_0}^S(A_1) &> C_{\text{dep}}^S(A_1) \quad \text{for } 1 < \alpha < r , \quad (3.24) \\
C_{\text{dep}}^S(A_2) &> C_{\nu_0}^S(A_2) \quad \text{for } 1 < \alpha < r . \quad (3.25)
\end{align*}
\]

**Proof.** We need the following inequalities, which are generalizations of Theorem 202 in [8], where such inequalities are proved for integrals with respect to Lebesgue measures. The general versions below are natural extensions using Fubini's theorem and the Hölder inequality for \( \sigma \)-finite measures. Suppose \( (S_1, \mu_1), (S_2, \mu_2) \) are two \( \sigma \)-finite measure spaces and \( F : S_1 \times S_2 \rightarrow \mathbb{R} \) is a product-measurable mapping. Then for \( p > 1 \) the inequality
\[
\int_{S_2} \left( \int_{S_1} F(x,y) \, d\mu_1(x) \right)^p \, d\mu_2(y) \leq \left( \int_{S_1} \left( \int_{S_2} |F(x,y)|^p \, d\mu_2(x) \right)^{\frac{1}{p}} \, d\mu_1 \right)^p \quad (3.26)
\]
and for $0 < p < 1$ the inequality
\[
\int S_2 \left| \int S_1 F(x, y) d\mu_1(x) \right|^p d\mu_2(y) \geq \left( \int S_1 \left( \int S_2 |F(x, y)|^p d\mu_2(x) \right)^{\frac{1}{p}} d\mu_1 \right)^p
\]
(3.27)
hold true.

(a) In the case $1 < r < \alpha$ we want to show (3.22); more precisely,
\[
\int_{S_{d-1}} \|A_s^{1/\alpha}\|^{\alpha} d\rho^*(s) \leq \int_{S_{d-1}} \|A_s^{1/\alpha}\|^{\alpha} d\rho_{\text{dep}}^*(s) = \|A_1\|^{\alpha}.
\]
(3.28)
To this end, we will apply (3.26) twice. In a first step, take $S_2 = S_{d-1}^{i+1}$ with $\mu_2 = \rho$ and $S_1 = \{1, \ldots, q\}$ with $\mu_1$ the counting measure, as well as $F(i, s) = \left( \sum_{j=1}^{d} A_{ij} s_j^{1/\alpha} \right)^{r}$ and $p = \frac{\alpha}{p}$. Then
\[
\int_{S_{d-1}} \|A_s^{1/\alpha}\|^{\alpha} d\rho^*(s) = \int_{S_2} \left( \int_{S_1} F(x, y) d\mu_1(x) \right)^p d\mu_2(y)
\leq \left( \int S_1 \left( \int S_2 |F(x, y)|^p d\mu_2(x) \right)^{\frac{1}{p}} d\mu_1 \right)^p
= \left( \sum_{i=1}^{q} \left( \sum_{j=1}^{d} A_{ij} s_j^{1/\alpha} \right)^{r} d\rho^*(s) \right)^{\frac{\alpha}{p}}
\]
(3.29)
In the second step, take $S_2 = S_{d-1}$ with $\mu_2 = \rho^*$ and $S_1 = \{1, \ldots, d\}$ with the weighted counting measure $\mu_1 = \sum_{j=1}^{d} A_{ij} \delta_j$ for $i = 1, \ldots, q$. Further, let $F(j, s) = s_j^{1/\alpha}$ and $p = \alpha$. Then
\[
\int_{S_{d-1}} \left( \sum_{j=1}^{d} A_{ij} s_j^{1/\alpha} \right)^{\alpha} d\rho^*(s) \leq \left( \sum_{j=1}^{d} A_{ij} \left( \int_{S_{d-1}} \left( s_j^{1/\alpha} \right)^{\alpha} d\rho^*(s) \right)^{\alpha} \right)^{\alpha} = \left( \sum_{j=1}^{d} A_{ij}^{\alpha} \right), i = 1, \ldots, q.
\]
We continue with (3.29) and find that
\[
\left( \sum_{i=1}^{q} \left( \sum_{j=1}^{d} A_{ij} s_j^{1/\alpha} \right)^{r} d\rho^*(s) \right)^{\frac{\alpha}{r}} \leq \left( \sum_{i=1}^{q} \left( \sum_{j=1}^{d} A_{ij}^{\alpha} \right)^{\frac{\alpha}{r}} \right)^{r} = \|A_1\|^{\alpha}.
\]
Relation (3.28) is shown analogously using (3.27).

Finally, we can use $\rho_0^*$ in order to show (3.24) and (3.25). Taking again $A_1 = I_2$, we obtain
\[
\rho_0^* g_{A_1, \alpha} = 1 + 2^2 = 2^\alpha
\]
and, consequently,
\[
\rho_0^* g_{A_1, \alpha} > \rho_{\text{dep}}^* g_{A_1, \alpha} \iff 1 + 2^2 > 2^2 \iff 2 > 2 \iff \alpha < r.
\]
(3.30)
Therefore, we have $C_{\rho_0}^S(A_1) > C_{\text{dep}}^S(A_1)$ for $1 < \alpha < r$.

Next, we choose $A_2 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and compute
\[
\rho_1^* g_{A_2, \alpha} = 2^\alpha + 2^{-1} 2^{\alpha(1+\frac{1}{p})} \quad \text{and} \quad \rho_{\text{dep}}^* g_{A_2, \alpha} = 2^{\alpha(1+\frac{1}{p})}.
\]
As a matter of fact,
\[
\rho_{\text{dep}}^* g_{A_2, \alpha} > \rho_1^* g_{A_2, \alpha} \iff 2^\alpha > 2 \iff \alpha > 1;
\]
i.e., we have $C_{\rho_0}^S(A_2) < C_{\text{dep}}^S$ for $1 < \alpha < r$.
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