Non-Hermitian-skin-effect-enhanced dynamical robustness of topological end states in non-reciprocal Su-Schrieffer-Heeger models and the electrical circuit’s simulation

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For non-Hermitian quantum models, the dynamics is not apparently reflected by the static properties, such as the complex energy spectrum and the non-Hermitian topological invariant, because of the non-orthogonality of the right eigenvectors, non-unitary of the time evolution, breakdown of the adiabatic theorem, etc., but it is more realistic in experiments. Here, we pay attention to the dynamics of the non-reciprocal Su-Schrieffer-Heeger model under open boundary conditions, and find that the topological end state is more robust than its Hermitian counterpart, because the non-Hermitian skin effect can suppress the part leaking to the bulk sites. To observe this, we propose a classical electric circuit with only few passive inductors and capacitors, the mapping of which is established to the quantum model, and then simulate its dynamics. This work explains how the non-Hermitian skin effect enhances the robustness of the topological end state, and offers an easy way, via the classical electric circuit, of studying the non-reciprocal quantum dynamics, which may be extended to other non-Hermitian models.

I. INTRODUCTION

In our textbooks of quantum mechanics, e.g., in Ref.6, the Hamiltonians must be Hermitian, required by the reality of the system’s energy. In 1998, Bender and Boettcher2 found that complex Hamiltonians with PT symmetry can also bear the real energy spectra when the PT symmetry of the eigenstates is not broken. Then, a theory of non-Hermitian quantum mechanics was established3, even for general non-Hermitian Hamiltonian. Along with the development of the experimental techniques, non-Hermitian Hamiltonians can be engineered in many experimental labs, especially in topological photonics12, also, the effective Hamiltonian for open quantum systems is in general non-Hermitian2. Therefore, the relevant theoretical problems naturally emerge as the concerns of the theoretical physicists. Among them, the exotic non-Hermitian topological phenomena9 distinct from the Hermitian systems, e.g., PT symmetry breaking13,14, exceptional points15,16, breakdown of the bulk-boundary correspondence of the Hermitian topological systems17,18, etc., draw considerable attentions.

Many works17–37 tried to define the topology in non-Hermitian systems by proposing various definitions of topological invariants, and restore the bulk-boundary correspondence in non-Hermitian topological systems. Among them, based on the non-reciprocal Su-Schrieffer-Heeger (SSH) model, Yao et al.22 found that the breakdown comes from the so-called “non-Hermitian skin effect” under open boundary conditions (OBCs) and established a “non-Bloch” theory to restore the bulk-boundary correspondence for this PT symmetric non-Hermitian model. This effect was theoretically studied recently in different contexts38–44, proposed to realize on various platforms45–47, and observed in electrical-circuit48,49, mechanical50, photonic51,52, and cold-atom53 experiments.

However, the bulk-boundary correspondence principle and the various definitions of the topological invariants have an implicit prerequisite — adiabatic theorem54,55, which breaks down in non-Hermitian dynamics due to the non-unitarity of the time-evolution55. Therefore, many static topological properties, e.g., topological invariants and end/edge states, of non-Hermitian systems cannot be easily manipulated and observed. Compared with the static analysis, the dynamics is more straightforward for experiments48,49,68–76 to investigate the exotic properties of non-Hermitian systems, and may also help understand the dynamics of open quantum systems45,50.

Moreover, although the large energy gaps can protect the topological end/edge states of the Hermitian model, part of the initial end/edge states will also leak to the bulk in dynamics due to the superposition with the bulk states. If the non-reciprocity-induced non-Hermitian skin effect can suppress the bulk part, it would be useful to enhance the robustness of the topological end/edge states, which may be usefully exploited in amplifiers77,78, lasers79,80, and other non-Hermitian photonic devices.

In this paper, we pay attention to the dynamics of the initial end state in topologically different regimes of the non-reciprocal SSH model, and find that the non-Hermitian skin effect indeed eliminates the bulk leakage and thus renders the non-Hermitian topological end state more robust than the Hermitian counterpart.

To demonstrate these dynamical phenomena, using the similar idea in Ref.55, we also propose an LC electrical circuit to simulate the non-reciprocal SSH model. The electrical circuit platform has been proved in recent years to be an easy-manipulating, low-cost, but powerful simulators for some topological phenomena81,82,83,84, but most of them focused on the driving schemes to study the resonance of the eigenstates85,86. Here, we use the
initial state to study the time evolution, which is much closer to the dynamical simulation of the quantum systems. This work may stimulate more dynamical study of non-Hermitian systems in other experimental platforms, such as photonics, ultracold atoms, and superconducting circuits.

II. THE NON-RECIPROCAL SSH MODEL: RECAPITULATION

While the Hermitian SSH model is a stereotypical 1D topological model, the non-reciprocal variant — the non-reciprocal SSH model, is fundamental to understand the non-Hermitian skin effect. In the second-quantization form, the Hamiltonian can be written as

\[ H = \sum_n \left[ \nu (\hat{a}_n \hat{b}_n^\dagger + \hat{b}_n \hat{a}_n) + \kappa_1 \hat{a}_{n+1}^\dagger \hat{b}_{n-1} + \kappa_1 \hat{b}_{n+1}^\dagger \hat{a}_{n-1} \right], \quad (1) \]

where \( \hat{a}_n(t) \) and \( \hat{b}_n(t) \) are the annihilation (creation) operators for the A- and B-sublattice sites, respectively, in the \( n \)th unit cell. \( \nu \) is the reciprocal intra-cell hopping amplitude, and \( \kappa_1, \kappa_2 \) are the non-reciprocal inter-cell hopping amplitudes, leading to the non-Hermiticity of the system. All the parameters are real.

This model is fundamental to understand the breakdown of the conventional bulk-boundary correspondence principle of the Hermitian topological systems in the non-Hermitian ones. According to the “non-Bloch” theory, this non-reciprocal version of the SSH model also has non-Hermitian end states under OBCs with integer unit cells if parameters satisfy \( [\kappa_1 \kappa_2] > \nu^2 \), which is continual to the Hermitian SSH model with \( \kappa_1 = \kappa_2 \).

To show the properties of the non-Hermitian end states in real space, for convenience we recast the Hamiltonian Eq. (1) under OBCs, as shown in Fig. 1 in the matrix form as \( \hat{H} = \tilde{\psi}^\dagger \hat{H} \tilde{\psi} \), where \( \tilde{\psi} = (\hat{a}_1, \hat{b}_1^\dagger, \ldots, \hat{a}_N, \hat{b}_N^\dagger) \) and \( \tilde{\psi} = (\hat{\psi}^\dagger |0\rangle, \langle |0\rangle \hat{\psi}^\dagger) \) construct the vectors of basis states \( |\psi\rangle = \tilde{\psi}^\dagger |0\rangle \) and \( \langle \psi| = \langle 0| \hat{\psi} \), and the Hamiltonian matrix of the coefficients.

\[ H = \begin{pmatrix} 0 & \nu & \kappa_1 & \cdots & \cdots & \kappa_2 & \cdots & \nu \\ \nu & 0 & \kappa_2 & \cdots & \cdots & \nu & \cdots & \nu \\ \kappa_1 & \kappa_2 & 0 & \nu & \cdots & \cdots & \kappa_2 & \nu \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \kappa_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \nu \end{pmatrix} \quad (2) \]

III. QUANTUM DYNAMICS OF END STATES

We know that the end states of the Hermitian SSH model are topological, and thus are robust in dynamics. However, because of the superposition of the bulk states, a small part of the initial end state leaks into the bulk and is reflected back by the other boundary at a certain time [Fig. 2(a)], and thus, will finally spread into the whole bulk after a sufficiently long time. When we consider the non-reciprocal SSH model, the small bulk part will be suppressed in dynamics [Fig. 2(b) and 2(d)]. This shows that the non-reciprocal end states are more robust than the Hermitian counterpart.

To understand this, we just need notice the solution of the time-dependent Schrödinger equation \( i\partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle \) under OBCs (hereafter we set \( \hbar = 1 \)), i.e.,

\[ \Psi(t) \equiv |\psi|\langle \Psi(t)\rangle = e^{-iHt}\Psi(0) = S e^{-i\hat{H}t} S^{-1}\Psi(0) \quad (3) \]
for the non-Hermitian skin effect. The parameters is a transformation, the final state $\Psi(S)$ from $\Psi(0)$ equivalently in three successive steps: (a) scaling under the basis $|\psi\rangle$ in real space, where

$$
\hat{H} = S^{-1}HS = \begin{pmatrix}
0 & \nu & \kappa & 0 & \nu & \cdots & 0 \\
\nu & 0 & \kappa & 0 & \nu & \cdots & 0 \\
\kappa & 0 & \nu & 0 & \nu & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\nu & 0 & \nu & 0 & \nu & \cdots & 0
\end{pmatrix}
$$

(4)

is a $2N \times 2N$ Hermitian matrix, mathematically similar to $H$ by an invertible matrix $S$

$$
S = \text{diag}(1, 1, e^{-\alpha}, e^{-\alpha}, \ldots, e^{-(N-1)\alpha}, e^{-(N-1)\alpha}),
$$

(5)
of which the exponentially decaying form is responsible for the non-Hermitian skin effect. The parameters $\kappa = \sqrt{\kappa_1 \kappa_2}$ and $\alpha = 2^{-1}\ln(\kappa_1/\kappa_2)$, where we assume that $\kappa_{1,2} > 0$ such that $\kappa, \alpha > 0$.

Focusing on Eq. (3), with the aid of the similarity transformation, the final state $\Psi(t)$ of $H$ can be obtained from $\Psi(0)$ equivalently in three successive steps: (a) scaling by $S^{-1}$; (b) evolving under the Hermitian matrix $\hat{H}$; (c) rescaling by $S$. Given an initial state confined in the left-most unit cell, $S^{-1}\Psi(0)$ in step (a) does not change it, preserving the signal only in the left-most unit cell. Then, the time-evolution in step (b) just follows the Hermitian case as mentioned before: depending on the parameters, transporting the signal into the bulk for the topologically trivial phase or keeping most signal in the left-most unit cell with small part transporting into the bulk for the topological phase. However, different from the Hermitian case, the rescaling $S$ in step (c) eliminates exponentially the part leaking to the bulk.

IV. ELECTRICAL CIRCUIT’S FORMALISM OF THE NON-RECIPROCAL SSH MODEL

To simulate the fate of the initial end state in two topologically different phases of the non-reciprocal SSH model, we resort to LC electrical circuits, as shown in Fig. 3 where we label the voltages of the A/B nodes in the $n$th unit cell as $V_{nA/B}(t)$, the parameters $L_{nA/B}$ as the corresponding inductances, and $C_{nA/B}$ as the capacitances.

We first do the static analysis, which corresponds to the switch in Fig. 3 connecting to the inductor. Assuming $V_{nA/B}(t) = V_{nA/B} e^{i\omega t}$, according to the Kirchhoff’s current law with the admittances of inductors and capacitors, i.e., $Y_L = 1/i\omega L$ and $Y_C = i\omega C$, where $\omega$ is the angular frequency, we have the following set of equations,

$$
\left(\Lambda_n^A - \frac{1}{\omega^2}\right) V_n^A = L_n^A C_n^A V_{n+1}^A + L_n^A C_n^B V_n^B,
$$

$$
\left(\Lambda_n^B - \frac{1}{\omega^2}\right) V_n^B = L_n^B C_n^B V_n^A + L_n^B C_n^A V_{n+1}^A,
$$

(6)

where $\Lambda_n^A = L_n^A D_n^A + L_n^A C_n^A + L_n^A C_n^B$ and $\Lambda_n^B = L_n^B D_n^B + L_n^B C_n^A + L_n^B C_n^A$. We can also rewrite them in the matrix form, $\mathcal{H}V = \mathcal{E}V$, where $\mathcal{V} = (V_1^A, V_1^B, \ldots, V_N^A, V_N^B)^T$,

$$
\mathcal{H} = \begin{pmatrix}
0 & L_n^A C_n^A & L_n^A C_n^B & 0 & \cdots & 0 \\
L_n^B C_n^A & 0 & L_n^B C_n^B & 0 & \cdots & 0 \\
\Lambda_n^A & \Lambda_n^B & 0 & L_n^A C_n^A & L_n^B C_n^B
\end{pmatrix},
$$

(7)

and

$$
\mathcal{E} = \Lambda - \frac{1}{\omega^2}I = \text{diag}(\Lambda_1^A, \Lambda_1^B, \ldots, \Lambda_N^A, \Lambda_N^B) - \frac{1}{\omega^2}I,
$$

(8)

where $I$ is the identity matrix.

To map this set of equations to the non-reciprocal SSH model, we just regard $H$ as the Hamiltonian matrix $\mathcal{H}$, $\mathcal{V}$ and $\mathcal{E}$ respectively as its right eigenvector and eigenenergy. Thus, $\mathcal{H} = \mathcal{E}\mathcal{V}$ can simulate the time-independent Schrödinger equation of the non-reciprocal SSH model if $\mathcal{E}$ and thus $\Lambda$ are proportional to the identity matrix.

To establish the mapping between the parameters of two Hamiltonian systems, Eqs. (2) and (7), we must have the following relations (in dimensionless way),

$$
\frac{L_n^A C_n^A}{LC} = \frac{L_n^B C_n^B}{LC} = 1,
$$

$$
\frac{L_n^B C_n^A}{LC} = \frac{\kappa_1}{\nu}, \quad \frac{L_n^A C_n^A}{LC} = \frac{\kappa_2}{\nu},
$$

$$
\frac{\Lambda_n^A}{LC} = \frac{\Lambda_n^B}{LC} = \lambda, \quad (n = 1, \ldots, N),
$$

(9)

where $\lambda$ is a dimensionless constant taken as needed, and $L$ and $C$ are just reference parameters of inductors and capacitors, respectively, taken for practical need. The solution for the inductors and capacitors in the LC circuit
is as follows,
\[ C^A_n = C \left( \frac{k_1}{k_2} \right)^{n-1}, \quad C^B_n = C^A_n \cdot \frac{\nu}{k_2}, \]
\[ D^A_n = C^A_n \frac{\lambda \nu - \nu - k_2}{k_2}, \quad D^B_n = C^A_n \cdot \frac{\lambda \nu - \nu - k_1}{k_2}, \]
\[ L^A_n = L^B_n = L \left( \frac{k_2}{k_1} \right)^{n-1} \cdot \frac{k_2}{\nu}, \quad (n = 1, \cdots, N). \tag{10} \]

Note that the inductances and the capacitances must be non-negative, which can be realized by setting \( L, C > 0 \) and \( \lambda \geq 1 + \max(\kappa_1/\nu, \kappa_2/\nu) \), accompanied by the model parameters \( \kappa_{1,2,\nu} > 0 \) mentioned before. For convenience, we can set \( \lambda = 1 + \kappa_1/\nu \) for the case of \( \kappa_1 \geq \kappa_2 \) to remove the inductances \( D^B_n \) in the circuit, i.e., \( D^B_n = 0 \), and thus, \( D^A_n = C^A_n(\kappa_1 - \kappa_2)/\kappa_2 \). In these settings, we have the effective Hamiltonian matrix \( \mathcal{H} = (LC/\nu)H \) and the effective eigenenergy \( E = (\lambda LC - 1/\omega^2)I \).

We can also use another circuit scheme by just exchanging inductors and capacitors. The results can be easily obtained by the duality replacement: \( \omega^2 \rightarrow 1/\omega^2, \quad C^{AB}_n \leftrightarrow 1/L^{AB}_n, \quad D^{AB}_n \rightarrow 1/D^{AB}_n \) in Eqs. \([7], [10]\), where \( l^{AB}_n \) is another label of inductors.

V. ELECTRICAL CIRCUIT’S DYNAMICAL SIMULATION

For the dynamical simulation, we need consider the original differential equation, i.e.,
\[ (\mathcal{H} - \Lambda)\dot{V}(t) - V(t) = 0, \tag{11} \]
where \( \mathcal{H} \) and \( \Lambda \) are defined generally in Eqs. \([7], [8]\), and \( V(t) = [V_1^A(t), V_1^B(t), \cdots, V_N^A(t), V_N^B(t)]^\top \). The dot in \( V \) means the derivative with respect to time, i.e., \( dV/dt \). Under the substitutions, Eqs. \([9], [10]\), the general solution reads
\[ V(t) = \sum_{n=1}^{2N} (\alpha_n \cos \omega_n t + \beta_n \sin \omega_n t) \cdot V_n, \tag{12} \]
where \( V_n \) is \( n \)th right eigenvector of \( \mathcal{H} \), and \( \omega_n \) is the corresponding eigen angular frequency of the circuit. \( \{\alpha_n, \beta_n\} \) are a set of complex coefficients to be determined by the initial conditions, coming from the second order derivatives with respect to time in Eq. \(11\). The detailed derivation can be referred to in Appendix. The form of the superposition coefficients in Eq. \(12\) is different from our familiar form of the solution of the time-dependent Schrödinger equation, but it is qualitatively the same due to the same fact that the solution is the superposition of the Hamiltonian’s eigenvectors.

To simulate the previously mentioned dynamics of the two topologically different phases, we adopt the LC circuit in Fig. 3. For the following selected values of parameters \( (\kappa_1/\nu, \kappa_2/\nu, \lambda) \), we set the reference parameters in Eq. \(10\) as \( L = 1\,\text{mH} \) and \( C = 100\,\text{pF} \), and the number of unit cells \( N = 5 \), which ensures that all values of circuit elements drop into the available regime in the realistic experiment. The typical oscillating angular frequency of the circuit is \( \omega_0 = 1/\sqrt{LC} \) in the order of MHz.

For the topological phase, we choose \( (\kappa_1/\nu, \kappa_2/\nu, \lambda) = (4, 1, 5) \) to simulate the dynamics of the non-reciprocal SSH model, corresponding to \( C^A_n = C^B_n = 100\,\text{pF} \sim 25.6\,\text{nF} \), \( D^A_n = 300\,\text{pF} \sim 78.6\,\text{nF} \), \( D^B_n = 0 \), and \( L^A_n = L^B_n = 1\,\text{mH} \sim 3.9\,\mu\text{H} \); as comparison, we also choose \( (\kappa_1/\nu, \kappa_2/\nu, \lambda) = (2, 2, 5) \) for the Hermitian counterpart, corresponding to \( D^A_n = D^B_n = C^A_n = 100\,\text{pF} \), \( C^B_n = 50\,\text{pF} \), and \( L^A_n = L^B_n = 2\,\text{mH} \). Here, we use \( n = 1, \cdots, 5 \). These two cases are said to be topological and have the topological end states because \( |\kappa_1 \kappa_2/\nu|^2 = 4 > 1 \).

For the topologically trivial phase, we choose \( (\kappa_1/\nu, \kappa_2/\nu, \lambda) = (1, 0.25, 2) \), corresponding to \( C^A_n = 100\,\text{pF} \sim 25.6\,\text{nF} \), \( C^B_n = 400\,\text{pF} \sim 102.4\,\text{nF} \), \( D^A_n = 300\,\text{pF} \sim 78.6\,\text{nF} \), \( D^B_n = 0 \), and \( L^A_n = L^B_n = 250 \sim 1\,\mu\text{H} \); as comparison, we also choose \( (\kappa_1/\nu, \kappa_2/\nu, \lambda) = (0.5, 0.5, 2) \) for the Hermitian counterpart, corresponding to \( D^A_n = D^B_n = C^A_n = 100\,\text{pF} \), \( C^B_n = 200\,\text{pF} \), and \( L^A_n = L^B_n = 500\,\mu\text{H} \). They are topologically trivial and have no topological end states because \( |\kappa_1 \kappa_2/\nu|^2 = 0.25 < 1 \).

The initial state can be prepared by connecting the switch to the DC voltage source, and after a sufficient time, only capacitors \( C^A_1 \) and \( C^B_1 \) are charged, which renders only \( V_1^A(0) = 0 \) with \( V_0 \) being the value of the DC voltage source, and thus determines the values of \( \{\alpha_n\} \). By changing the switch to the inductor in 1st unit cell at time \( t = 0 \), the zero transient current renders \( \dot{V}_1^A/B = 0 \) for all \( n \), and thus vanishes \( \{\beta_n\} \). The determination of \( \{\alpha_n, \beta_n\} \) can be referred to in Appendix. Then, \( V(t) \) of any time can be simulated according to Eq. \(12\).

Figure 4 shows the simulation results. As expected, compared with their Hermitian counterparts [Fig. 4(a) and 4(b)], we can clearly see that the non-reciprocity of both topological and topologically trivial cases can suppress the bulk leakage [Fig. 4(c) and 4(d)], and thus make the topological end state more robust. Figure 4(e) just confirms that the topological end state of the non-reciprocal SSH circuit is dynamically more localized than the Hermitian counterpart. Figure 4(d) also shows dynamically the non-Hermitian skin effect in the topologically trivial circuit, where the non-reciprocal state is accumulated to the left end, in opposite to the Hermitian counterpart, which spreads uniformly in the whole circuit.

To quantitatively reflect the extent of localization, we calculate the average inverse participation ratio (aIPR), defined as
\[ \text{aIPR} = \frac{\sum_{n} |[V^A_n(t)]^4 + [V^B_n(t)]^4|}{\sum_{n} |[V^A_n(t)]^2 + [V^B_n(t)]^2|^2}, \tag{13} \]
where
\[ \dot{V}^{A/B}_n(t) = \frac{2}{L} \int_{t/2}^{t} |V^{A/B} (\tau)| d\tau. \tag{14} \]
FIG. 4: Time evolution of the relative voltages, \( V_{\alpha/B}(t)/V_0 \), in the circuit of Fig. 3 given the initial condition by the switch changing described in the main text, for the Hermitian SSH circuits with (a) \((\kappa_1/\nu, \kappa_2/\nu, \lambda) = (2, 2, 5)\) and (b) \((0.5, 0.5, 2)\), and for the non-reciprocal SSH circuits with (c) \((\kappa_1/\nu, \kappa_2/\nu, \lambda) = (4, 1, 5)\) and (d) \((1, 0.25, 2)\). (e,f) show the average voltages, \( \bar{V}_{\alpha/B}(t)/V_0 \) (defined in the main text) at \( \omega_0 t = 100 \) for the (a,c) topological and (b,d) topologically trivial cases, where the non-reciprocal ones are labelled by the triangle (red) lines and the Hermitian ones by the circle (black) lines.

AIPR = 1 means that the averaged voltage distribution in the time interval \((t/2, t)\) is totally localized at one circuit node, while AIPR = 1/2N means the uniformly extended in the whole circuit. As the quantitative verification of the previous judgment, the AIPRs are equal to 0.8243 (red triangle) and 0.4209 (black circle) for the curves in Fig. 4(e), and 0.2611 (red triangle) and 0.1031 (black circle) in Fig. 4(f).

Actully, the dynamical behaviors in Fig. 4 can also be understood using the circuit’s version of Eq. (5). To achieve this, we rewrite Eq. (11) into the form of a first order differential equation as follows,

\[
\begin{pmatrix}
\dot{V}(t) \\
\dot{W}(t)
\end{pmatrix} = \begin{pmatrix}
0 & I \\
(\tilde{H} - \Lambda)^{-1} & 0
\end{pmatrix} \begin{pmatrix}
V(t) \\
W(t)
\end{pmatrix}
\]

(15)

by introducing \( W(t) = \dot{V}(t) \) as new variables. And now, it is more like the time-dependent Schrödinger equation,

\[
\begin{pmatrix}
\dot{V}(t) \\
\dot{W}(t)
\end{pmatrix} = \begin{pmatrix}
0 & I \\
\tilde{H} & 0
\end{pmatrix} \begin{pmatrix}
V(t) \\
W(t)
\end{pmatrix}
\]

where \( \tilde{H} = S^{-1}H S \) is the Hermitian counterpart with \( S \) being defined in Eq. (6). This is qualitatively the same as Eq. (3), and thus, the same explanation also applies to the circuit SSH systems.

VI. CONCLUSION AND DISCUSSION

The non-reciprocal SSH model is fundamental to the understanding of the non-Hermitian physics. In this paper, we transfer the focus from the static properties to the direct dynamics, showing that the non-Hermitian skin effect of this model under OBCs can enhance the robustness of the dynamics of the topological end states with a succinct explanation. Then, we propose an electrical circuit with only few passive elements of linear inductors and capacitors to simulate the non-reciprocal SSH model, clearly demonstrating the advantage in dynamics over the Hermitian counterparts for the topological end states.

In this paper, we restrict the model parameters as \( \kappa_1, \kappa_2, \nu > 0 \), which can be easily extended to \( (\kappa_1/\nu)(\kappa_2/\nu) > 0 \), showing the same dynamical properties, because the similarity transformation \( S \) still works to map it back to a Hermitian counterpart. However, for \( (\kappa_1/\nu)(\kappa_2/\nu) < 0 \), the eigenenergy or eigen angular frequency can in general be complex, and thus, the amplitude will be amplifying or decaying in time; furthermore, the circuit scheme with Eq. (10) cannot simulate this case due to the positivity of the circuit elements, which is beyond our discussion.

For experiments, the inductors and capacitors don’t have to be selected with the exact values in the main text, because the non-Hermitian skin effect here just depends on the increasing or decreasing of the values, not the exact power law in Eq. (10). Therefore, the small intrinsic inductances and capacitances of the circuit cannot qualitatively affect the results either; so do the small resistances of the connecting wires in the circuit, which only quantitatively decays the amplitudes in a long time.

For the starting of the time evolution in circuit experiments, in principle the switching time must be much smaller than the discharging time of the capacitors to ensure the little change of the initial state at \( t = 0 \). Alternatively, we may consider the driving scheme to investigate the robustness of the non-reciprocal SSH end state as in Ref. [53] but it’s another story.
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Dynamical solution of the electrical circuit

Under the substitutions, Eqs. (2) or (10), where note that \( \Lambda = (\Lambda LC) I \) is proportional to the identity matrix, we can make the following eigenvalue decomposition from Eqs. (7) and (9),
\[
M^{-1}H = \mathcal{E} = \Lambda - (\Omega^2)^{-1},
\]
where the columns of \( M \) are just the \( 2N \) right eigenvectors of \( \mathcal{H} \), i.e., \{\( \mathcal{V}_n \)\}, and \( \Omega^2 = \text{diag}(\omega_1^2, \cdots, \omega_{2N}^2) \) is the set of corresponding eigen angular frequencies of the circuit, \{\( \omega_n \)\}.

For the differential equation Eq. (11), applying the similarity transformation, Eq. (17), to both sides, we have
\[
M^{-1}(\mathcal{H} - \Lambda)M[M^{-1}\dot{\mathcal{V}}(t)] - [M^{-1}\mathcal{V}(t)] = 0
\Rightarrow -(\Omega^2)^{-1}[M^{-1}\dot{\mathcal{V}}(t)] - M^{-1}\mathcal{V}(t) = 0
\Rightarrow [M^{-1}\dot{\mathcal{V}}(t)] + (\Omega^2)^{-1}[M^{-1}\mathcal{V}(t)] = 0.
\]
Because \( \Omega^2 \) is a diagonal matrix, we can consider the solution for each element independently, i.e.,
\[
[M^{-1}\mathcal{V}(t)]_n = C_n e^{i\omega_n t} + D_n e^{-i\omega_n t} = \alpha_n \cos \omega_n t + \beta_n \sin \omega_n t,
\]
where \( \{C_n, D_n\} \) and \( \{\alpha_n = C_n + D_n, \beta_n = i(C_n - D_n)\} \) are two sets of complex coefficients to be determined by the initial conditions. Then, the final solution reads
\[
\mathcal{V}(t) = MT(t),
\]
where \( T(t) \) is an \( 2N \times 1 \) coefficient vector with entries \( \{C_n e^{i\omega_n t} + D_n e^{-i\omega_n t}\} \) or \( \{\alpha_n \cos \omega_n t + \beta_n \sin \omega_n t\} \). Then we get the formula Eq. (12).

Given the initial conditions,
\[
(\alpha_1, \cdots, \alpha_{2N})^T = T(0) = M^{-1}\mathcal{V}(0),
(\beta_1\omega_1, \cdots, \beta_{2N}\omega_{2N})^T = \dot{T}(0) = M^{-1}\dot{\mathcal{V}}(0),
\]
the coefficients, say \( \{\alpha_n, \beta_n\} \), can be determined, and thus, \( \mathcal{V}(t) \) at any time is obtained.

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