Theoretically Guaranteed Online Workload Dispatching for Deadline-Aware Multi-Server Jobs

Hailiang Zhao, Shuiguang Deng, Senior Member, IEEE, Jianwei Yin, Schahram Dustdar, Fellow, IEEE, and Albert Y. Zomaya, Fellow, IEEE

Abstract—Multi-server jobs are imperative in modern computing clusters. A multi-server job has multiple task components and each of the task components is responsible for processing a specific size of workloads. Efficient online workload dispatching is crucial but challenging to co-located heterogeneous multi-server jobs. The dispatching policy should decide (i) where to launch each task component instance of the arrived jobs and (ii) the size of workloads that each task component processes. Existing policies are explicit and effective when facing service locality and resource contention in both offline and online settings. However, when adding the deadline-aware constraint, the theoretical superiority of these policies could not be guaranteed. To fill the theoretical gap, in this paper, we design an $\alpha$-competitive online workload dispatching policy for deadline-aware multi-server jobs based on the spatio-temporal resource mesh model. We formulate the problem as a social welfare maximization program and solve it online with several well designed pseudo functions. The social welfare is formulated as the sum of the utilities of jobs and the resource allocation of multiple dimensions or the workload dispatching decisions should be made on-line without the knowledge of future job arrivals. The lack of the global information of the problem space could lead to the least desirable situation where all the neural network training jobs are scheduled to the only node with GPUs and many of them stay in pending state chronically.

Index Terms—Multi-server job, workload dispatching, social welfare maximization, online algorithms.

I. INTRODUCTION

Today’s computing clusters are full of multi-server jobs. A multi-server job is composed of multiple associated task components, and each task component may take different size of input workloads and require various kinds and quantities of computation resources such as CPUs, GPUs, FPGAs, etc. A typical multi-server job is the distributed training of deep neural networks. Based on the Ring All-Reduce communication pattern[8] which is well supported by the NVIDIA Collective Communication Library (NCCL) [9], each task component is responsible for processing the input data workloads (update its local gradients with the input mini-batch data samples and take the reduction operation) and commissioning the NCCL library to send the reduced gradient chunks to the other task components.

Efficient workload dispatching is crucial but challenging to co-located heterogeneous multi-server jobs. There are three key problems need to be carefully addressed. Firstly, for each multi-server job, how many task component instances should be launched? For distributed model training, the more workers we start, the faster the training speed[7]. Secondly, which node to choose to launch each task component instance? Thirdly, how many workloads should each task component processes? The major challenges are discussed as follows.

- **Service Locality.** Service locality is common in modern computing clusters. With this constraint, the task components of a multi-server job can only be processed by a subset of nodes where the resource requirements, affinity & anti-affinity [3], and other obligatory constraints are satisfied. For a resource-constrained cluster, service locality could lead to the least desirable situation where all the neural network training jobs are scheduled to the only node with GPUs and many of them stay in pending state chronically.

- **Contention of Limited Resources.** When the total resource demands of co-located task components of different jobs exceed the available resources of that node, resource contention happens. Considering that the maximum workloads a node can process mainly depends on the CPU cycle frequency and related hardware performance indexes [4], [5], how to allocate resources exhaustively for processing workloads of different job types to reduce the resource contention is of great concern.

- **Unknown Arrival Patterns of Jobs.** In real-life scenarios, the workload dispatching decisions should be made on-line without the knowledge of future job arrivals. The lack of the global information of the problem space could lead to a local optimum.

- **Jobs may be Deadline-Sensitive.** Some multi-server jobs have an explicit deadline. For instance, in AI application related companies, a trained deep neural network model is usually guaranteed to be put into service on a particular date. The workload dispatching policy should ensure the training can be finished before deadline even at the cost of performance degeneration. In this case, we can actively reduce the input workload size of data batch to meet the deadline.

A majority of workload dispatching and scheduling policies for (multi-server) jobs are proposed by formulating either continuous or combinatorial optimization problems with scenario-oriented constraints [6]–[13]. The decision variables are either the resource allocation of multiple dimensions or the workload size that each node processes for each job. Meanwhile, the optimization target is either job complete & weighted flow

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1 Another widely used distributed training architecture is the Parameter Server (PS)-Worker architecture, which will not be the focus of the proposed algorithm in this work.

2 Although more workers lead to more communication overhead, the overall process runs faster [2].
time or the utility that measures the overall system efficiency. To solve these optimization programs, algorithms are designed based on various theoretical approaches such as relaxed integer programming [8], online primal-dual approaches [9], heuristics [10], [11], deep reinforcement learning [12], [13], etc. Many of these policies are effective when facing service locality and resource contention even in online settings. However, when adding the deadline-aware constraint, the theoretical superiority of these algorithms cannot be guaranteed. Deadline-aware job schedulers are designed mainly from a system approach [14]–[16]. For instance, Hu et al. present a scheduling framework in big-data platforms, with the purpose of minimizing the average workflow turn-around time, to meet job deadlines [14]. A preemption-supported scheduler named DAPS is designed for Hadoop YARN clusters [15]. In addition, Cheng et al. propose a deadline-aware Hadoop job scheduler that takes future resource availability into consideration when minimizing job deadline misses [16]. Nevertheless, theoretically guaranteed online workload dispatching policies for deadline-aware multi-server jobs are missing in existing literature. Here the theoretical guarantee is that, could we design an $\alpha$-competitive ($\alpha \geq 1$ and the smaller, the better) online workload dispatching policy for deadline-aware multi-server jobs, such that the utilities of jobs and the cluster can be maximized simultaneously.

To fill the theoretical gap, in this paper, we study a general online workload dispatching problem for deadline-aware multi-server jobs. The jobs we consider have a specific size of input workloads, and each of them has an explicit arrival time and deadline to be finished. For any computing cluster with heterogeneous and depletable resources on each node, we propose an $\alpha$-competitive policy where $\alpha \geq 2$ to decide (i) where to launch each task component instance of the arrived jobs and (ii) the size of workloads that each task component processes. The policy is built on the so-called spatio-temporal resource mesh of nodes and resource reservation is automatically realized. We formulate the problem as a social welfare maximization program and solve it online with several well designed pseudo functions. The social welfare is defined as the sum of the utilities of jobs and the utility of the computing cluster. From the job side, the utility of each job is proportional to the workloads that processed, which is determined by the computation resources allocated to it. For instance, for large-scale distributed training, the more data samples (or training epochs) each task component processes, the better the trained model. From the cluster side, the utility can be any zero-startup non-decreasing function with diminishing return that measures the overall system efficiency or resource fairness. All the max-min, proportional fairness or $\alpha$-fairness are good choices [17]. We provide rigorous analysis to show that the proposed policy is $\alpha$-competitive for some $\alpha$ at least 2. The theoretical superiority is also validated with simulations and the results show that it distinctly outperforms baselines. Our main contributions are summarized as follows.

1) We study a general online workload dispatching problem for deadline-aware multi-server jobs from the theoretical perspective. We establish the spatio-temporal resource mesh model and solve the problem with the target of maximizing the social welfare of the system.

2) We propose an online policy which yields a competitive ratio at least 2 for general utility settings. Particularly, it has a polynomial complexity when all the utilities are linear and share the same coefficient. The theoretical superiority of the proposed policy is rigorously analyzed and verified with simulations.

The rest of this paper is organized as follows. We formally introduce the system model and formulate the online workload dispatching problem in Sec. [II]. We present the design details of the online policy with rigorous theoretical analysis in Sec. [III]. We demonstrate the numerical results in Sec. [IV] and discuss related work in Sec. [V]. Finally, we conclude this paper in Sec. [VI].

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a computing cluster of heterogeneous VM (and physical) nodes. Let us use $K$ to denote the set of nodes and index each of them by $k$. Key notations used in this paper is summarized in Table I.

| Notation | Description |
|----------|-------------|
| $K$      | The set of nodes |
| $k \in K$| The $k$-th node in $K$ |
| $N$      | The set of multi-server jobs |
| $n \in N$| The $n$-th job in $N$ |
| $a_n, \forall n \in N$ | The arrival time of job $n$ |
| $d_n, \forall n \in N$ | The strict deadline of job $n$ |
| $\rho_n, \forall n \in N$ | The target workload size of job $n$ |
| $K_n$    | The set of nodes available to job $n$ |
| $\mathcal{T}$ | The set of time slots |
| $\tau$  | The length of each time slot in $\mathcal{T}$ |
| $\mathcal{R}$ | The resource mesh |
| $r \in \mathcal{R}$ | The $r$-th resource unit in $\mathcal{R}$ |
| $C_r, \forall r \in \mathcal{R}$ | The maximum processable workloads of $r$ |
| $\mathcal{R}_n, \forall n \in N$ | The set of resource units available to $n$ |
| $x_{nr}$ | The size of workloads of $n$ dispatched to $r$ |
| $\chi_{nr}$ | The maximum processing capacity of $r$ for $n$ |
| $f_n(\cdot)$ | The utility of job $n$ |
| $g(\cdot)$ | The utility of the computing cluster $n$ |

A. Spatio-Temporal Resource Mesh

Each node is capable of processing a set of heterogeneous multi-server jobs arriving in sequence with different input workload sizes. Let us denote the set of jobs as $N$ and index each of them by $n$. Each job $n$ has a target input workload of size $q_n$ (in MB). $\forall n \in N$, we use $a_n$ and $d_n$ to represent its arrival time and strict deadline to be finished. To maximize the social welfare from a long-term vision, we consider the time horizon from $\min_{n \in N} a_n$ to $\max_{n \in N} d_n$ and evenly divide the horizon into slots of length $\tau$. Let us use $\mathcal{T}$ to denote the set of time slots and index each of them with $t$. The time slot length $\tau$ can be set as the minimum instance reserved time, for example, 1 hour for AWS spot instance[3]
To manipulate the nodes in $\mathcal{K}$ from both dimension of time and space, we introduce a spatio-temporal resource division model called resource mesh. We use $\mathcal{R} \triangleq \mathcal{K} \times \mathcal{T}$ to denote the set of resource units and index each of them by $r$. Each resource unit $r$, defined as a node in a time slot, can process at most $C_r$ workloads, limited by its hardware performance indexes such as the CPU cycle frequency and the GPU clock speed. For distributed model training, $C_r$ indicates the maximum data samples that can be processed by $r$ during the given time range. This value could be obtained by a variety of approaches from static code analysis to profiling previous runs based on hardware heterogeneity [18].

For each job $n \in \mathcal{N}$, we use

$$\mathcal{R}_n \triangleq \{ r_{kt} \in \mathcal{R} \mid \left\lceil \frac{d_n}{T} \right\rceil \leq t \leq \left\lfloor \frac{d_n}{T} \right\rfloor, k \in \mathcal{K}_n \} \quad (1)$$

to denote its set of available resource units, where $\mathcal{K}_n \subseteq \mathcal{K}$ is the set of nodes that satisfy the service locality of job $n$. Fig. [1] gives an example.

![Fig. 1. Available resource units of four example jobs in the resource mesh. Whether resource unit $r$ is available to job $n$ is decided by the service locality constraint and the available time zone of $n$.](image)

### B. Utility Functions

For each job $n \in \mathcal{N}$, we need to decide (i) how many task component instances should be initialized, (ii) which resource units to place them, and (iii) how many workloads that each task component should process. Our target is to maximize the social welfare, i.e., the sum of all jobs’ utilities and the utility of the computing cluster. Formally, we use $x_{nr}$ to denote the size of workloads dispatched to $r \in \mathcal{R}_n$ and $\chi_{nr}$ to denote the maximum workload processing capacity of $r$ for job $n$. It results to the constraint $0 \leq x_{nr} \leq \chi_{nr}$. Note that $C_r$ is designed to indicate the overall processing capacity while $\chi_{nr}$ is the workload processing capacity to the specific job $n$ of the $r$-th resource unit.

We take a zero-startup utility $f_n : [0, \chi_{nr}] \rightarrow \mathbb{R}$, where $\chi_{nr} \triangleq [\chi_{nr}]_{r \in \mathcal{R}_n}$, as the measure of satisfaction for job $n$. As a widely accepted assumption in previous works [19–23], we require $\{f_n\}_{n \in \mathcal{N}}$ to be non-decreasing, concave, and continuously differentiable on each dimension $r$. Proportional fairness and $\alpha$-fairness are good options for $\{f_n\}_{n \in \mathcal{N}}$ [24]. Note that we allow jobs to have different utilities. For each job $n \in \mathcal{N}$, its utility is defined as the sum of separate sub-utilities achieved through each available resource unit:

$$f_n(x_n) \triangleq \sum_{r \in \mathcal{R}_n} f_{nr}(x_{nr}), \forall n \in \mathcal{N}, \quad (2)$$

where $x_{nr} \triangleq [x_{nr}]_{r \in \mathcal{R}_n}$. For a given job $n$, $f_{nr}$ can also be different on different $r \in \mathcal{R}_n$. To sum up, a multi-server job can be described with the quadruple $\{q_n, \mathcal{R}_n, \chi_{nr}, f_n\}$. To simplify the problem, the utility of the computing cluster

$$g(x) : [0, x_{nr}]^{\mathcal{N} \times \mathcal{R}} \rightarrow \mathbb{R}$$

is defined as the maximized weighted resource utilization efficiency aggregated over each job:

$$g(x) \triangleq \sum_{n \in \mathcal{N}} \sum_{r \in \mathcal{R}_n} \beta_{nr} \cdot \frac{x_{nr}}{C_r}, \forall n \in \mathcal{N}, r \in \mathcal{R}_n, \quad (3)$$

where $\frac{x_{nr}}{C_r}$ is used to indicate the fractional resource consumed by $x_{nr}$, and $\beta_{nr}$ is the weight of provisioning total resource of $r$ to $n$. We further use the function $g_{nr}(x_{nr})$ to denote the weighted fractional utility $\beta_{nr} \cdot \frac{x_{nr}}{C_r}$. In (3), $x$ is the overall decision variable and $x \triangleq [x_{nr}]_{n \in \mathcal{N}}$.

### C. Online Social Welfare Maximization

Based on the above content, we formulate the social welfare maximization problem as follows:

$$\mathcal{P}_1 : \max_{\{x_n\}_{n \in \mathcal{N}}} \sum_{n \in \mathcal{N}} f_n(x_n) + \sum_{n \in \mathcal{N}} \sum_{r \in \mathcal{R}_n} g_{nr}(x_{nr})$$

s.t. $\sum_{r \in \mathcal{R}_n} x_{nr} \leq q_n, \forall n \in \mathcal{N}$,

$$x_{nr} = 0, \forall n \in \mathcal{N}, r \in \mathcal{R} \setminus \mathcal{R}_n, \quad (4)$$

$$x_{nr} \leq C_r, \forall r \in \mathcal{R}, \quad (5)$$

$$0 \leq x_{nr} \leq \chi_{nr}, \forall n \in \mathcal{N}, r \in \mathcal{R}_n. \quad (6)$$

As an offline optimization problem, although $\mathcal{P}_1$ is difficult to solve, it is built based on complete knowledge. However, in online settings, the cluster should not have the information of the $n$-th quadruple $\{q_n, \mathcal{R}_n, \chi_{nr}, f_n\}$ until job $n$ arrives. To design an efficient online workload dispatching policy with the worst-case performance guarantee, we introduce the following notations

$$t \triangleq \min_{n \in \mathcal{N}} \min_{r \in \mathcal{R}_n} \left( \frac{\partial f_n}{\partial x_{nr}} + \frac{\beta_{nr}}{C_r} \right), \quad (7)$$

The ratio between these two constants, i.e., $\frac{\beta_{nr}}{\partial f_n/\partial x_{nr}}$, demonstrates the fluctuation of the marginal social welfare, which will be

$^2$The discrete version of problem $\mathcal{P}_1$ is actually a multi-dimensional 0-1 knapsack problem, which is proved to be NP-complete [25].
introduced later. In previous theoretical papers, this ratio is viewed as a known variable and it helps construct the resource provision decisions [22], [29]-[30]. For example, in [28], the ratio is set as 36 in default.

It is worth noting that the online decision \( x_n \) made for job \( n \) when it arrives implies the idea of resource reservation. Support that at time \( t \), job \( n \) arrives. Formally, for job \( n \), by solving \( \mathcal{P}_1 \) online, the resource provision decisions

\[
\{ x_{nr_n'} \mid r_{kt'} \in \mathcal{R}_n, t' > t \}
\]

are the resource reservation results for executing job \( n \).

III. ALGORITHM DESIGN

The key challenge to solve \( \mathcal{P}_1 \) in online settings is that the dispatching of each job’s workloads to each resource unit are coupled because of (6). Nevertheless, if we could construct several feasible dual variables corresponding to \( \{ x_n \} \) in \( \mathcal{P}_1 \), and take these dual variables as the cost for using each resource unit, a near optimal solution could be obtained. To implement this, we design several pseudo-social welfare functions with estimated marginal costs. In this design, we utilize an important principle for solving online resource provision problems, i.e., estimate the cost for processing the workloads of each task component as a function of resource surplus [22], [29], [30], [28]. In the following sections, firstly, we show how the pseudo-social welfare functions are constructed. Then, based on these constructed functions, we introduce our algorithm \textup{ONSOCMAX}. It works by solving several pseudo-social welfare maximization problems polynomially online. To guarantee that \textup{ONSOCMAX} is \( \alpha \)-competitive, we analyze the requirements that the marginal cost functions should satisfy. In addition, we give the bound of the gap between the competitive ratio achieved by \textup{ONSOCMAX} and the optimal competitive ratio of a simplified case under a particular condition. In the end, we discuss how to extend \textup{ONSOCMAX} to the tasks with non-partitionable workloads and some drawbacks.

A. Pseudo-Social Welfare Function

For each arrived job \( n \), we define the pseudo-social welfare function, denoted by \( \mathcal{W}_n(x_n) \), as

\[
\begin{align*}
f_n(x_n) - \sum_{r \in \mathcal{R}_n} \int_{\omega_r(n)+x_{nr}}^{\omega_r(n)+x_{nr}} \phi_r(u) \, du + \sum_{r \in \mathcal{R}_n} g_{nr}(x_{nr}),
\end{align*}
\]

where \( \phi_r \) is a non-decreasing estimation of the marginal cost for the resource unit \( r \) in \( \mathcal{R} \) processing unit workload when the resource surplus \( u \in [0, C_r] \). We define \( \phi_r(u) = +\infty \) when \( u > C_r \). The non-decreasing property profoundly reflects an underlying economic phenomenon, i.e., a thing is valued in proportion to its rarity. The later a job arrives, the higher cost it has to pay [22]. The first component is the pseudo-utility of executing job \( n \), which is the utility of it minus the cost to pay. The second component is the utility of the computing cluster. If we organize \( \mathcal{W}_n(x_n) \)

\[
\begin{align*}
f_n(x_n) + \sum_{r \in \mathcal{R}_n} \left[ g_{nr}(x_{nr}) - \int_{\omega_r(n)+x_{nr}}^{\omega_r(n)+x_{nr}} \phi_r(u) \, du \right],
\end{align*}
\]

the second component can be regarded as the net profit of the cluster for processing the workloads of job \( n \). In this case, the later a job arrives, the harder the resource surplus to meet its requirements before deadline, which results to higher cost. The following content applies to both of these two interpretations.

To bridge connections between the optimal dual variables of \( \mathcal{P}_1 \) and the optimal solution \( x^* \), that maximizes \( \mathcal{W}_n \), we firstly introduce the dual problem of \( \mathcal{P}_1 \) as follows.

**Proposition 1.** The dual problem of \( \mathcal{P}_1 \) is:

\[
\mathcal{P}_2 : \min_{\mu, \lambda} \sum_{n \in \mathcal{N}} \sum_{r \in \mathcal{R}} \xi_{nr}(\mu_n + \lambda_r) + \sum_{n \in \mathcal{N}} \mu_n g_n + \sum_{r \in \mathcal{R}} \lambda_r C_r
\]

s.t. \( [5], [7], \mu \geq 0, \lambda \geq 0 \),

where

\[
\xi_{nr}(p) \triangleq \max_{x_{nr} \in [0, x_{nr}]} \left[ f_{nr}(x_{nr}) + \left( g_{nr}(x_{nr}) - p \cdot x_{nr} \right) \right],
\]

\( \lambda_r \triangleq [\mu_n]_{n \in \mathcal{N}} \) and \( \lambda_r \triangleq [\lambda_r]_{r \in \mathcal{R}} \) are the dual variables corresponding to \( [4] \) and \( [6] \), respectively.

**Proof.** The result is immediate with Lagrangian. \( \square \)

Essentially, \( \xi_{nr}(\cdot) \) is the convex conjugate of the fractional social welfare \( f_{nr} + g_{nr} \). Taking a closer look at the conjugate \( \xi_{nr}(p) \) and the pseudo social welfare \( \mathcal{W}_n(x_n) \), if we could find appropriate \( p^* \) and \( x^*_n \), we can bridge their connection through

\[
\mathcal{W}_n(x^*_n) \approx \sum_{r \in \mathcal{R}_n} \xi_{nr}(p^*).
\]

Based on this, we can interpret \( p \) as the marginal cost for processing unit workload [22]. We bridge the subtle connection between \( \xi_{nr} \) and \( \mathcal{W}_n \) in the following proposition, which is crucial for the design of \textup{ONSOCMAX}.

**Proposition 2.** \( \forall n \in \mathcal{N}, r \in \mathcal{R} \), when \( \phi_r(C_r) \geq u \), if (i) \( x^*_{nr} \triangleq x_{nr}^* r \in \mathcal{R} \) and \( \mu_{n r} \) are respectively the optimal primal solution and the optimal dual solution to \( [4] \) of the following problem \( \mathcal{P}_3 \):

\[
\mathcal{P}_3 : \max_{x_n} \mathcal{W}_n(x_n)
\]

s.t. \( [4], [7] \),

and (ii) the resource usage level \( \omega_r \) is updated with

\[
\begin{align*}
\omega_r^{(n+1)} &= \omega_r^{(n)} + x^*_{nr},
\omega_r^{(1)} &= 0,
\end{align*}
\]

then, \( x^*_n \) is also the optimal solution that maximizes \( \xi_{nr}(p) \) given \( p = \phi_r(\omega_r^{(n+1)}) + \mu_{nr}^* \).

**Proof.** By the definition of the non-decreasing marginal cost function \( \phi_r(\cdot) \), we can find that it is discontinuous at \( C_r \). Thus, when \( \phi_r(C_r) \geq u \), there must exist a resource usage level \( \omega_r \leq C_r \) such that \( \phi_r(\omega_r) = u \). Note that the function \( f_n + \sum_{r \in \mathcal{R}_n} g_{nr} \) is non-decreasing and its derivative on \( r \) is not more than \( \frac{g_{nr}(x_{nr})}{x_{nr}} \). Therefore, when the input of \( \phi_r \) is \( \omega_r^{(n)} + x_{nr}^{*}, \)

\footnote{This conclusion can be obtained with \( [8] \).}
suppose \( \omega_r^{(n)} + x_{nr} \leq \varpi_r \). Consequently, the derivative of the integral function

\[
\Phi_r(x_{nr}) = \int_{\omega_r^{(n)}}^{\omega_r^{(n)}+x_{nr}} \phi_r(u) du
\]

is continuous, non-decreasing, and convex when \( x_{nr} \leq \varpi_r - \omega_r^{(n)} \). The convexity is because \( \Phi_r \), i.e., \( \phi_r \), is non-decreasing. Thus, \( \mathcal{P}_3 \) is a convex optimization program and its optimal solution can be obtained through KKT conditions. Let us use \( x_{nr}^*, \mu_n^*, \gamma_n^*, \) and \( \zeta_n^* \) to denote the optimal primal and dual solutions of \( \mathcal{P}_3 \) (\( \mu_n^* \) to (14) while \( \gamma_n^* \) and \( \zeta_n^* \) to the right part and left part of (12), respectively). The KKT conditions of \( \mathcal{P}_3 \) are listed below.

\[
\begin{align*}
& \begin{cases} 
  f_n'(r(x_{nr}^*) + \frac{\delta r}{C_r} - \phi_r(\omega_r^{(n+1)}) - \mu_n^* = \gamma_n^* - \zeta_n^* \\
  \gamma_n^*(x_{nr}^* - \chi_{nr}) = 0 \\
  \zeta_n^* \cdot x_{nr} = 0 \\
  \mu_n^*(\sum_{r \in \mathcal{R}_n} x_{nr}^* - \phi_n) = 0.
\end{cases} \\
\end{align*}
\]

(13)

With KKT conditions (13), we show that the optimal solution \( x_{nr}^* \) of \( \mathcal{P}_3 \) simultaneously optimizes the conjugate \( \xi_{nr}(\gamma) \) given \( \gamma = \phi_r(\omega_r^{(n+1)}) + \mu_n^* \), i.e.,

\[
\xi_{nr}\left(\phi_r(\omega_r^{(n+1)}) + \mu_n^*\right) = f_{nr}(x_{nr}^*) + g_{nr}(x_{nr}^*) - \left(\phi_r(\omega_r^{(n+1)}) + \mu_n^*\right) \cdot x_{nr}.
\]

(14)

Case I: When \( f_{nr}(x_{nr}^*) + \frac{\delta r}{C_r} > \phi_r(\omega_r^{(n+1)}) + \mu_n^* \), \( \bar{W}_n \) is an increasing function on dimension \( r \) under (11). Thus, we have \( x_{nr} = \bar{\chi}_{nr} \), which leads to

\[
f_{nr}(x_{nr}) + \frac{\delta r}{C_r} > \phi_r(\omega_r^{(n+1)}) + \mu_n^*.
\]

(15)

which also leads to (14).

Case III: When \( f_{nr}(x_{nr}^*) + \frac{\delta r}{C_r} = \phi_r(\omega_r^{(n+1)}) + \mu_n^* \), \( x_{nr}^* \) is a maximum of

\[
f_{nr}(x_{nr}) + \left[ g_{nr}(x_{nr}) - px_{nr} \right]
\]

given \( p = \phi_r(\omega_r^{(n+1)}) + \mu_n^* \) since \( \gamma_n^* = \zeta_n^* = 0 \). Thus,

\[
x_{nr}^* \in \arg\max_{0 \leq x_{nr} \leq \tau_{nr}} \left[ f_{nr}(x_{nr}) + g_{nr}(x_{nr}) - p \cdot x_{nr} \right],
\]

which means (14) holds.

All the three conditions are visualized in Fig. 2.

\[
\begin{align*}
\text{(a)} & \quad \begin{array}{c}
\bar{\chi}_{nr} \\
\chi_{nr}
\end{array} \\
\text{(b)} & \quad \begin{array}{c}
x_{nr}^* \\
\chi_{nr}
\end{array} \\
\text{(c)} & \quad \begin{array}{c}
\bar{x}_{nr} \\
x_{nr}
\end{array}
\end{align*}
\]

Fig. 2. A visualization on how the three conditions affect the optimal \( x_{nr}^* \) of \( f_{nr}(x_{nr}) + g_{nr}(x_{nr}) - (\phi_r(\omega_r^{(n+1)}) + \mu_n^*) \cdot x_{nr} \), respectively.

So far we have analyzed the properties of the pseudo-social welfare functions and the conjugates. In the following sections, we will firstly give the design details of the online algorithm ONSOCMAX. Then, we will illustrate that, to make ONSOCMAX \( \alpha \)-competitive for some underlying \( \alpha \), what requirements the marginal cost functions \( \{\phi_r\}_{r \in \mathcal{R}} \) should satisfy.

B. ONSOCMAX Design

ONSOCMAX is built on solving \( \mathcal{P}_3 \) for each newly arrived job \( n \) in sequence. The procedure is captured in Algorithm 1. We place a hat on top of variables that denote the variables involved in ONSOCMAX.

Although \( \mathcal{P}_3 \) is a convex program, we cannot obtain its analytic solution with (13) directly. To solve it iteratively, we transform it into the following problem \( \mathcal{P}_3' \).

**Algorithm 1: ONSOCMAX**

**Input:** \( \{C_r\}_{r \in \mathcal{R}} \) and \( \{g_{nr}\}_{n \in \mathcal{N}, r \in \mathcal{R}} \)

**Output:** Online solution to \( \mathcal{P}_1 \) and final utilizations of the resource mesh

\[ \forall r \in \mathcal{R} : \hat{\omega}_r^{(1)} \leftarrow 0 \]

1. while a new multi-server job \( n \) arrives do

2. Receive the quadruple \( \{\phi_n, R_n, \bar{\chi}_{nr}, f_n\} \)

3. Get the (near) optimal solution \( x_{nr} \) of \( \mathcal{P}_3 \)

4. for \( r \in \mathcal{R}_n \) do in parallel

5. \[ \hat{\omega}_r^{(n+1)} \leftarrow \hat{\omega}_r^{(n)} + \hat{x}_{nr} \]

6. end for

7. \( n \leftarrow n + 1 \)

8. end while

9. return \( \{x_{nr}\}_{n \in \mathcal{N}} \) and \( \{\hat{\omega}_r^{(\mathcal{N}+1)}\}_{r \in \mathcal{R}} \)
Taking (23) into (21), we have

\[
\begin{align*}
\mathcal{P}_3' : \min_{x_n} \sum_{r \in \mathcal{R}_n} \left[ \Phi_r(x_{nr}) - f_{nr}(x_{nr}) - g_{nr}(x_{nr}) \right] \\
\text{s.t.} \quad & \sum_{r \in \mathcal{R}_n} x_{nr} - q_n + s = 0, \\
& x_{nr} - \chi_{nr} + l_r = 0, \forall r \in \mathcal{R}_n, \\
& q_r - x_{nr} = 0, \forall r \in \mathcal{R}_n, \\
& s, l_r, q_r \geq 0, \forall r \in \mathcal{R}_n,
\end{align*}
\]

(17)

(18)

(19)

(20)

where \( l \triangleq [l_r]_{r \in \mathcal{R}_n}, q \triangleq [q_r]_{r \in \mathcal{R}_n}, \) and \( s \) are introduced slack variables. In addition, we define the dual variables respectively to (17), (18), and (19) as \( \mu, y \triangleq [y_r]_{r \in \mathcal{R}_n}, \) and \( z \triangleq [z_r]_{r \in \mathcal{R}_n}. \)

The augmented Lagrangian of \( \mathcal{P}_3' \) is

\[
L_\sigma(x_n, s, l, q, \mu, y, z) = \sum_{r \in \mathcal{R}_n} \left[ \Phi_r(x_{nr}) - f_{nr}(x_{nr}) - g_{nr}(x_{nr}) \right] + \mu \left( \sum_{r \in \mathcal{R}_n} x_{nr} - q_n + s \right) + \sum_{r \in \mathcal{R}_n} y_r \left( x_{nr} - \chi_{nr} + l_r \right) + \sum_{r \in \mathcal{R}_n} z_r (q_r - x_{nr}) + \frac{\sigma}{2} \rho(x_n, s, l, q),
\]

(21)

where the penalty function \( \rho(\cdot) \) is defined as

\[
\rho(x_n, s, l, q) = \sum_{r \in \mathcal{R}_n} \left( q_r - x_{nr} \right)^2 + \left( \sum_{r \in \mathcal{R}_n} x_{nr} - q_n + s \right)^2 + \sum_{r \in \mathcal{R}_n} \left( x_{nr} - \chi_{nr} + l_r \right)^2,
\]

(22)

and \( \sigma > 0 \) is the penalty coefficient. If we consider the augmented Lagrangian \( L_\sigma(x_n, s, l, q, \mu, y, z) \) as the function of the slack variables \( s, l, q \), to minimize it, we can get their optimal values:

\[
\begin{align*}
\left\{ \begin{array}{l}
\phi^* = \max \left\{ -\frac{\mu}{\sigma} + q_n - \sum_{r \in \mathcal{R}_n} x_{nr}, 0 \right\}, \\
\lambda^* = \max \left\{ -\frac{y_r}{\sigma} + \chi_{nr} - x_{nr}, 0 \right\}, \forall r \in \mathcal{R}_n, \\
q^*_r = \max \left\{ -\frac{z_r}{\sigma} + x_{nr}, 0 \right\}, \forall r \in \mathcal{R}_n.
\end{array} \right.
\]

(23)

Taking (23) into (21), we have

\[
\begin{align*}
L_\sigma(x_n, \mu, y, z) &= \sum_{r \in \mathcal{R}_n} \left[ \Phi_r(x_{nr}) - f_{nr}(x_{nr}) - g_{nr}(x_{nr}) \right] + \frac{\sigma}{2} \left[ \max \left\{ \frac{\mu}{\sigma} + \sum_{r \in \mathcal{R}_n} x_{nr} - q_n, 0 \right\}^2 - \frac{\mu^2}{\sigma^2} \right] \\
& \quad + \frac{\sigma}{2} \sum_{r \in \mathcal{R}_n} \left[ \max \left\{ \frac{y_r}{\sigma} + x_{nr} - \chi_{nr}, 0 \right\}^2 - \frac{y_r^2}{\sigma^2} \right] \\
& \quad + \frac{\sigma}{2} \sum_{r \in \mathcal{R}_n} \left[ \max \left\{ \frac{z_r}{\sigma} - x_{nr}, 0 \right\}^2 - \frac{z_r^2}{\sigma^2} \right].
\end{align*}
\]

(24)

Besides, we define the constraint violation degree \( v(\cdot) \) of a solution \( x_n \) given \( \mu, y, z \) by

\[
v(x_n | \mu, y, z) \triangleq \max \left\{ \sum_{r \in \mathcal{R}_n} x_{nr} - q_n - \frac{\mu}{\sigma}, r \right\} + \sum_{r \in \mathcal{R}_n} \max \left\{ x_{nr} - \chi_{nr} - \frac{y_r}{\sigma}, r \right\} + \sum_{r \in \mathcal{R}_n} \max \left\{ -x_{nr} - \frac{z_r}{\sigma}, r \right\}.
\]

(25)

Based on (24), we can solve \( \mathcal{P}_3' \) approximatively with the augmented Lagrangian method. The procedure is summarized in Algorithm 2, which is used to substitute step 4 of ONSOC-MAX.

C. Competitive Analysis

ONSOC-MAX is at most polynomial because \( \mathcal{P}_3 \) is convex and can be solved efficiently in polynomial time with augmented Lagrangian method. Obviously, \( \{x_n\}_{n \in \mathcal{N}} \) is feasible to \( \mathcal{P}_1 \). To quantify how “good” ONSOC-MAX is, we adopt the standard competitive analysis framework [32].

Definition 1. For any arrival instance \( A \) of all the multi-server jobs, the competitive ratio for an online algorithm is defined as

\[
\alpha \triangleq \max_{\forall A} \frac{\Theta_\alpha(A)}{\Theta_{mn}(A)},
\]

(26)

where \( \Theta_\alpha(A) \) is the maximum objective value of \( \mathcal{P}_1, \Theta_{mn}(A) \) is the objective function value of \( \mathcal{P}_1 \) obtained by this online algorithm.

The competitive ratio quantifies the worst-case ratio between the optimum and the objective obtained by the online algorithm. The smaller \( \alpha \) is, the better the online algorithm. An online algorithm is called \( \alpha \)-competitive if its ratio is upper bounded by \( \alpha \).

Now we give the requirements the marginal cost functions \( \{\hat{\phi}_r\}_{r \in \mathcal{R}} \) should satisfy to guarantee that ONSOC-MAX is \( \alpha \)-competitive for some \( \alpha \).

Theorem 1. ONSOC-MAX is \( \alpha \)-competitive for some \( \alpha \geq 1 \) if \( \forall r \in \mathcal{R}, \) the marginal cost function \( \hat{\phi}_r \) is in the form of

\[
\hat{\phi}_r(\omega) = \begin{cases} \\ t & \omega \in [0, \hat{\omega}_r), \\ \hat{\varphi}_r(\omega) & \omega \in [\hat{\omega}_r, C_r] \\ +\infty & \omega \in (C_r, +\infty), \end{cases}
\]

(27)

where \( \hat{\omega}_r \) is a resource utilization threshold, and \( \hat{\varphi}_r \) is a non-decreasing function that satisfies

\[
\begin{cases} \\ \hat{\varphi}_r(\omega)C_r \leq \alpha \int_{0}^{\omega} \hat{\varphi}_r(u)du - t \cdot \omega, & \omega \in [\hat{\omega}_r, C_r] \\ \hat{\varphi}_r(\omega) = t \cdot \hat{\varphi}_r(C_r) \geq v. \end{cases}
\]

(28)

Proof. To prove this result, we refer to the technique named instance-dependent online primal-dual approach, proposed in [28]. The key idea is to construct a dual solution to \( \mathcal{P}_2 \) based on the solution \( \{x_n\}_{n \in \mathcal{N}} \) produced by ONSOC-MAX. Then, it uses this dual objective to build the upper bound of the optimum of \( \mathcal{P}_1 \) with weak duality. When building the upper
Algorithm 2: The augmented Lagrangian method

Input: \( \{C_r\}_{r \in \mathbb{R}}, \{g_{nr}\}_{r \in \mathbb{R}}, \{\theta_n, \Re_n, \chi_n, f_n\}, \) and \( \{\omega^{(n)}_r\}_{r \in \mathbb{R}} \)

Output: The (near) optimal solution \( \hat{x}_n \) of \( P_3 \)

1. Initialize the primal and dual variables \( x^{(0)}_n, \mu^{(0)}, y^{(0)}, z^{0} \)
2. Initialize the penalty coefficient \( \sigma_0 > 0 \) and
   \[ 0 < \theta_1 \leq \theta_2 \leq 1, p > 1 \]
3. Initialize the constraint violation coefficient \( \eta_0 = \frac{1}{\sigma_0} \), the precision coefficient \( \varepsilon_0 = \frac{1}{\sigma_0} \), and their final value \( \eta, \varepsilon \)
4. \( \kappa \leftarrow 0 \)
5. while true do
6.   Get a solution \( x^{\kappa+1} \) of \( \min_{x_n} L_\sigma(x_n, \mu^{\kappa}, y^{\kappa}, z^{\kappa}) \)
   by gradient descent method which satisfies the following precision:
   \[ \|\nabla x_{\kappa} L_\sigma(x_n, \mu^{\kappa}, y^{\kappa}, z^{\kappa})\| \leq \eta^{\kappa} \]
7.   Calculate the constraint violation degree
   \[ v(x^{\kappa+1} | \mu^{\kappa}, y^{\kappa}, z^{\kappa}) \] by (25)
8.   if \( v(x^{\kappa+1} | \mu^{\kappa}, y^{\kappa}, z^{\kappa}) \leq \varepsilon^{\kappa} \) then
9.     if \( \|\nabla x_{\kappa} L_\sigma(x_n, \mu^{\kappa}, y^{\kappa}, z^{\kappa})\| \leq \eta \) and
10.    \( v(x^{\kappa+1} | \mu^{\kappa}, y^{\kappa}, z^{\kappa}) \leq \varepsilon \) then
11.       return \( x^{\kappa+1} \)
12.    end if
13.   /* Update the dual variables */
14.   for \( r \in \mathbb{R} \) do in parallel
15.     \( y^{\kappa+1} \leftarrow \max\{y^{\kappa} + \sigma_n \sum_{r \in \mathbb{R}} - \theta_n, 0\} \)
16.     \( z^{\kappa+1} \leftarrow \max\{z^{\kappa} - \sigma_n x^{\kappa+1} \} \)
17.   end for
18.   \( \sigma_{\kappa+1} \leftarrow \sigma_{\kappa} \)
19.   \( \eta_{\kappa+1} \leftarrow \eta_{\kappa}/\sigma_{\kappa+1}, \varepsilon_{\kappa+1} \leftarrow \varepsilon_{\kappa}/\sigma_{\kappa+1} \)
20. else
21.   /* Keep the dual variables unchanged */
22.   \( \mu^{\kappa+1} \leftarrow \mu^{\kappa}, y^{\kappa+1} \leftarrow y^{\kappa}, z^{\kappa+1} \leftarrow z^{\kappa} \)
23.   \( \sigma_{\kappa+1} \leftarrow \sigma_{\kappa} \)
24.   \( \eta_{\kappa+1} \leftarrow 1/\sigma_{\kappa+1}, \varepsilon_{\kappa+1} \leftarrow 1/\sigma_{\kappa+1} \)
25. end if
26. \( \kappa \leftarrow \kappa + 1 \)
27. return the solution of the final iteration \( x^{\kappa} \)

\( B_1 \) and \( B_2 \) contain the instances whose final utilizations of all resource units in the mesh are below and above the threshold \( \hat{\omega}_r \), respectively. Our goal is to prove that, under the conditions (27) and (28), \( \forall A \in B_1, B_2, B_3 \) respectively, the following relations hold:
\[ \alpha \cdot \Theta_{on}(A) \geq \Theta_{P_2}(A) \geq \Theta_{P_1}(A). \]

In the following analysis, we just drop the parentheses and \( A \) for simplification.

Case I: \( \forall A \in B_1 \), from (27) we can find that the marginal costs experienced by all jobs are the same, i.e., \( t \). In this case, each job \( n \) is processed with maximum permitted workloads \( \chi_{nr} \) on \( r \in \mathbb{R} \). Thus, \( \Theta_{P_1}/\Theta_{on} = 1 \leq \alpha \).

Case II: \( \forall A \in B_2 \), we construct a feasible dual solution to \( P_2 \) as
\[ \{ \mu_n = \mu_r^*, \quad \forall n \in \mathcal{N} \} \]
\[ \lambda_r = \hat{\phi}_r(\hat{w}_r^N), \quad \forall r \in \mathbb{R}, \]
where \( \mu_r^* \) is the optimal dual solution to \( P_3 \) introduced by (13). Let \( p \geq p' \geq 0 \) and denote the optimal solution that maximizes the conjugate \( \xi_{nr}(p) \) by \( \tilde{x}_{nr} \) given \( p \). Then,
\[ \xi_{nr}(p) = f_{nr}(\tilde{x}_{nr}) + (g_{nr}(\tilde{x}_{nr}) - p \cdot \tilde{x}_{nr}) \]
\[ \leq f_{nr}(x_{nr}) + (g_{nr}(x_{nr}) - p' \cdot x_{nr}) \]
\[ \leq \max_{x_{nr}}(f_{nr}(x_{nr}) + (g_{nr}(x_{nr}) - p' \cdot x_{nr})) = \xi_{nr}(p'), \]
which indicates that the conjugate \( \xi_{nr}(p) \) is non-increasing with \( p \). The above derivation uses the fact that \( f_{nr} + g_{nr} \) is non-decreasing. Based on weak duality and the non-increasing property of the conjugate, we have
\[ \Theta_{P_1} \leq \sum_{n \in \mathcal{N}} \sum_{r \in \mathbb{R}} \xi_{nr} \mu_{nr} + \hat{\phi}_r(\hat{w}_r^N) + \sum_{n \in \mathcal{N}} \mu_{nr} \theta_n \]
\[ + \sum_{r \in \mathbb{R}} \hat{\phi}_r(\hat{w}_r^N) C_r \quad \text{the right-side is } \Theta_{P_2} \]
\[ \leq \sum_{n \in \mathcal{N}} \sum_{r \in \mathbb{R}} \xi_{nr} \mu_{nr} + \hat{\phi}_r(\hat{w}_r^{(n+1)}) + \sum_{n \in \mathcal{N}} \mu_{nr} \theta_n \]
\[ + \sum_{r \in \mathbb{R}} \hat{\phi}_r(\hat{w}_r^N) C_r \quad \text{(32)} \]
\[ = \sum_{r \in \mathbb{R}} \hat{\phi}_r(\hat{w}_r^N) C_r - \sum_{n \in \mathcal{N}} \hat{\phi}_r(\hat{w}_r^{(n+1)}) \tilde{x}_{nr} \]
\[ + \sum_{n \in \mathcal{N}} \sum_{r \in \mathbb{R}} (f_{nr}(\tilde{x}_{nr}) + g_{nr}(\tilde{x}_{nr})) \leq \Theta_{imp} \quad \text{(13)} \]

The last equality holds because \( \tilde{x}_{nr} \) simultaneously maximizes \( P_3 \) and the conjugate \( \xi_{nr}(\mu_{nr} + \hat{\phi}_r(\hat{w}_r^N)) \) (result of Proposition 3). Since \( \hat{\phi}_r \) are non-decreasing, \( \forall n \in \mathcal{N}, r \in \mathbb{R}, \)
\[ \hat{\phi}_r(\hat{w}_r^{(n+1)}) \tilde{x}_{nr} \geq \int_{\hat{w}_r^{(n)}}^{\hat{w}_r^{(n+1)}} \hat{\phi}_r(u) du. \]
(33) is illustrated in Fig. 3. Further, we have
\[ \sum_{n \in \mathcal{N}} \hat{\phi}_r(\hat{w}_r^{(n+1)}) \tilde{x}_{nr} \geq \int_{\hat{w}_r^{(1)}}^{\hat{w}_r^N} \hat{\phi}_r(u) du, \]
where $\hat{\phi}_r^{(1)} = 0$ because of (11). Besides, from Fig. 2 we can find that $\xi_{nr}(\mu_n^* + \hat{\phi}_r(\hat{\varphi}_r^{(n+1)})) \geq 0$ holds for all the serverless functions. Thus, based on (34), we have

$$\sum_{n \in N} \sum_{r \in R} \left( f_{nr}(\bar{x}_{nr}) + g_{nr}(\bar{x}_{nr}) \right) \geq \int_0^{\hat{\varphi}_r^N} \hat{\phi}_r(u) du. \quad (35)$$

Based on the above results (34) and (35), we have

$$\Theta_{mp} \leq \sum_{r \in R} \left[ \hat{\phi}_r(\hat{\varphi}_r^N) C - \int_0^{\hat{\varphi}_r^N} \hat{\phi}_r(u) du \right]$$

$$+ \sum_{n \in N} \sum_{r \in R} \left( f_{nr}(\bar{x}_{nr}) + g_{nr}(\bar{x}_{nr}) \right) \quad \triangleright (34)$$

$$\leq \sum_{r \in R} (\alpha - 1) \int_0^{\hat{\varphi}_r^N} \hat{\phi}_r(u) du \quad \triangleright (28) \quad \text{& drop } \iota \cdot \hat{\varphi}_r^N$$

$$+ \sum_{n \in N} \sum_{r \in R} \left( f_{nr}(\bar{x}_{nr}) + g_{nr}(\bar{x}_{nr}) \right) \alpha. \quad \triangleright (35)$$

The final expression is exactly $\alpha \cdot \Theta_{mp}$. Thus, $\Theta_{p_1}/\Theta_{on} < \alpha$.

**Theorem 1** extends the Two-Point Boundary Value ODEs for designing the marginal cost functions from standard 0-1 knapsack problem to multi-dimensional fractional problems. Based on Theorem 1 and Gronwall’s Inequality [33], we have detailed the design of $\{\hat{\phi}_r\}_{r \in R}$, which is irrelevant with the utilities $\{f_n\}_{n \in N}$ and $\{g_{nr}\}_{n \in N, r \in R}$, as follows.

**Theorem 2.** For any resource unit $r \in R$, if the marginal cost function $\hat{\phi}_r$ introduced in the pseudo-social welfare function is designed as

$$\hat{\phi}_r(\omega) = \begin{cases} \frac{\iota}{\exp(\alpha) - \exp(\frac{\iota}{\alpha})} e^{\frac{\iota}{\alpha} \omega} + \frac{\iota}{\alpha} & \omega \in [0, \hat{\varphi}_r) \\ \frac{\iota}{\alpha} & \omega \in (\hat{\varphi}_r, C_r) \\ \omega \in (C_r, \infty) \end{cases}$$

where $\hat{\varphi}_r = \frac{C_r}{\alpha - 1}$, then (i) ONSocMAX is $\hat{\alpha}$-competitive, where $\hat{\alpha}$ is the solution of

$$\hat{\alpha} - 1 = \frac{1}{\hat{\alpha} - 1} + \ln \frac{\hat{\varphi}_r - 1}{\hat{\alpha} - 1} \quad (40)$$

and (ii) when $\hat{\alpha} \geq \frac{1}{\iota} + 1$, the gap between $\hat{\alpha}$ and the optimal competitive ratio when $|R| = 1$ is at least $\frac{2}{\iota \sqrt{\iota + 1}} - \ln \frac{\iota}{\iota + 1} \approx 0.1368$.

**Proof.** We firstly introduce the Gronwall’s inequality [33] as follows. $\forall \alpha \in [\underline{x}, \overline{x}]$, if $f(x) \leq a(x) + b(x) f(x) du$, then

$$f(x) \leq a(x) + b(x) \int_{\underline{x}}^{x} a(u) \int_{\underline{u}}^{x} b(w) dw du, \quad (41)$$

where $f(x)$ is continuous, $a(x)$ and $b(x)$ are integrable and $\forall x \in [\underline{x}, \overline{x}], b(x) \geq 0$. The result remains valid if all the ‘$<$’ are replaced by ‘$\leq$’. Applying (41) to (28) leads to

$$\iota \leq \hat{\phi}_r(C_r) \leq \frac{\iota}{\alpha - 1} + \frac{\iota}{\alpha} - 1 e^{\frac{\iota}{\alpha} \hat{\varphi}_r}, \quad (42)$$

Thus, the minimum $\hat{\alpha}$ is achieved when all inequalities in (28) and (42) are binding, which leads to the design of $\{\hat{\phi}_r\}_{r \in R}$ and the competitive ratio achieved by (40).

In the following, we prove the results of (ii). When $R = 1$, $P_1$ degenerates to the general one-way trading (GOT) problem [34]. The optimal competitive ratio is proved to be $1 + \ln(\frac{\iota}{\alpha})$ [22], [27], [28], [60], [34]. With $\hat{\alpha} \geq 1$, $\frac{\iota}{\alpha} \geq 1$, let us take $y \geq 1$ as a substitute for $\hat{\alpha} - 1$. Then

$$\hat{\alpha} - 1 - \ln \left( \frac{\iota}{\alpha} \right) = y - \ln \left( \frac{\iota}{\alpha} \right) \quad \triangleright \text{with (40)}$$

$$\leq \ln \left( y + 1 - \frac{\iota}{\alpha} \right) + 1 - \ln y \quad \Delta \text{GAP}(y).$$

Fig. 3. A visualization of (33). The area of the blue rectangle is not less than the area of the shaded region because of the non-decreasing property of $\hat{\phi}_r$. **Case III:** $\forall A \in B_3$, we define two disjoint sets to split the resource mesh $R$: \[
\left\{ \begin{array}{l}
R_1 = \{ r \in R | 0 \leq \hat{\varphi}_r^N < \hat{\varphi}_r, \forall r \in R \} \\
R_2 = \{ r \in R | \hat{\varphi}_r \leq \hat{\varphi}_r^N \leq C_r, \forall r \in R \}. 
\end{array} \right. \quad (36)\]

For resource unit $r$ in different sets, the corresponding dual variables are constructed in different ways. We extend $P_1$ to $P_1'$ by adding the following constraint:

$$\sum_{n \in N} \sum_{r \in R} x_{nr} \leq \sum_{r \in R} \hat{\varphi}_r^N. \quad (37)$$

Apparently, $P_1'$ is the same as $P_1$ for ONSocMAX since (37) is not violated by $\{\bar{x}_{nr}\}_{n \in N}$. The dual problem $P_2'$ to $P_1'$ is

$$P_2': \min_{\mu, \lambda, \delta} \sum_{n \in N} \left[ \sum_{r \in R_1} \xi_{nr}(\mu_n + \lambda_r + \delta) + \sum_{r \in R_2} \xi_{nr}(\mu_n + \lambda_r) \right]$$

$$+ \sum_{n \in N} \mu_n \lambda_n + \sum_{r \in R} \lambda_r C_r + \delta \sum_{r \in R} \hat{\varphi}_r^N$$

s.t. $\{5\}, \{7\}, \mu \geq 0, \lambda \geq 0, \delta \geq 0,$

where $\delta$ is the dual variable corresponding to the newly added constraint (37). Then, we construct the dual solution to $P_2'$ as

$$\begin{cases}
\hat{\lambda}_r = \{ 0 \} \quad \forall r \in R_1 \\
\hat{\phi}_r(\omega), \quad \forall r \in R_2 \\
\hat{\mu}_n = \mu_n^* \quad \forall n \in N.
\end{cases} \quad (39)$$

Based on (39), we can follow a similar approach as show in Case II to obtain that $\Theta_{P_1}/\Theta_{on} \leq \alpha$. A slight difference is that, in Case III, when applying (28) to $\Theta_{mp}'$, the result is tightly bounded. \hfill \Box
Applying \( \ln(x) \leq x - 1, \forall x \geq 1 \) to the logarithm in \( \text{Gap}(y) \), we have \( \text{Gap}(y) \leq y + \frac{1}{y} - \ln y - \frac{1}{y} \) given \( y \geq \frac{1}{y} \). By analyzing the upper bound of \( \text{Gap}(y) \), we can easily find that when \( y^* = \frac{1}{y^*} \), its upper bound is at least \( \frac{1}{y^*} - \ln y^* \), which directly leads to the result in (ii).

By the design of \( \hat{\phi}_r(\cdot) \), we observe that \( \alpha \geq 2 \) holds because \( \frac{y}{x} \geq 1 \). When \( \{f_{nr}\}_{n \in \mathcal{N}, r \in \mathcal{R}_n} \) are linear and share the same coefficient, \( \hat{\alpha} = 2 \).

D. Extending to Non-Partitionable Workloads

ONSoCMax can be easily applied to general jobs whose workloads are not allowed to be partitioned. Specifically, in this case, (7) is replaced by

\[
x_{nr} \in \{0, \overline{x}_{nr}\}, \forall n \in \mathcal{N}, r \in \mathcal{R}_n, \tag{43}
\]

and \( \overline{x}_{nr} = \theta_{nr} \). To solve the new problem in online settings, we can approximate the marginal cost defined in (12) with \( \hat{\phi}_r(\omega_r^{(n)} + \mu_{nr} x_{nr}) \). With this substitution, the Case III in Fig. 2 is merged into Case I or Case II, and ONSoCMax achieves the same competitive ratio as shown in Theorem 2. This approach is exactly the implementation of (10).

E. Discussion on Drawbacks

So far we have demonstrated the design details of ONSoCMax. It works by solving several well-designed pseudo social welfare functions. The problem we formulated is essentially an NP-hard online multi-dimensional knapsack problem. Although ONSoCMax is \( \alpha \)-competitive and easy to implement in real-life systems, there are some defects that cannot be ignored and will be in-depth studied in future.

- **Communication costs are not analytically counted.** For each multi-server job, the more task component instances we launch, the more communication costs. It can lead to the diminishing returns on the job’s utility. Although the utility we considered is allowed to be concave, the elaborate relation between the communication cost and the user satisfaction is not analytically analyzed.

- **\( C_r \) and \( \overline{x}_{nr} \) might be hard to determined.** In the established model, \( C_r \) is the maximum processable workloads and \( \overline{x}_{nr} \) is the workload processing capability of job \( n \) on the resource unit \( r \). Testing tools built on statistical code analysis and profiling techniques might be required for determining these constants.

IV. Simulation Results

In this section, we conduct several simulations to validate the theoretical superiority of ONSoCMax. The experiments are not meant to be exhaustive, rather they are used to illustrate the potential of ONSoCMax.

A. Experimental Setup

Jobs and Nodes. We consider a cluster with 10 nodes in the time horizon of 24 time slots. The processing capacity of nodes are generated from an i.i.d. Gaussian \( \mathcal{N}(\mu = 20, \sigma = 2) \). By setting \( \tau \) as 60 minutes, the time horizon represents one day.

We set the number of multi-server jobs as 20. The number of job arrivals in each time slot follows a Poisson distribution with a mean of 2.03 requests, which is independent of other time slots in this day. The deadline of each job is calculated based on the available time and the maximum service duration of it, where the latter is generated by an Exponential distribution with a mean of 4 time slots (2 hours). Each job has an input workload whose size is generated from a Normal distribution \( \mathcal{N}(\mu = 18, \sigma = 3) \). The workload processing capability of each job on each node is generated from a Normal distribution \( \mathcal{N}(\mu = 7, \sigma = 1) \).

Utilities and Parameters. \( \forall n \in \mathcal{N} \), the utility of job \( n \) is set as a zero-startup, non-decreasing concave function. We study \( f_{nr} \) in three cases: linear, logarithmic, and polynomial. Specifically, for each \( n \in \mathcal{N}, r \in \mathcal{R}_n \),

\[
f_{nr}(x) = \begin{cases} 
  ax & \text{linear} \\
  a \log(x + 1) & \log \\
  a \sqrt{x} & \text{poly}, 
\end{cases}
\]

where the coefficient \( a \) is generated from a uniform distribution in \([1, 3]\). Similarly, the parameter \( \beta_{nr} \) in \( g_{nr}(\cdot) \) is generated from the uniform distribution in \([0.1, 0.5]\).

Hyper-Parameters of ONSoCMax. There are many algorithmic hyper-parameters involved in Algorithm 2. For example, the initial penalty coefficient \( \sigma_0 \), the initial constraint violation coefficient \( \eta_0 \), the initial precision coefficient \( \epsilon_0 \), etc. Their default settings are listed in Table II. In the last line, learning rate and decay are parameters involved in step 6 of Algorithm 2.

| PARAMETER   | VALUE | PARAMETER   | VALUE |
|-------------|-------|-------------|-------|
| \( \sigma_0 \) | 0.97  | \( \rho \)  | 0.002 |
| \( \theta_1 \) | 0.99  | \( \theta_2 \) | 0.999 |
| \( \eta \) | 0.1   | \( \epsilon \) | 10    |
| learning rate | 2     | decay       | 0.95  |

Algorithms Compared. We compare ONSoCMax with two handcrafted online algorithms.

- **Max-First.** In Max-First, each node always serves the job with the highest myopic social welfare, i.e., the sum of the utility of the chosen job and the utility for serving it in each time slot is maximized. Max-First is myopic because it always maximizes the partial social welfare that it sees.

- **Equal-Share.** In Equal-Share, a node serves every newly arrived job with equal opportunity within its capacity limit.

B. Simulation Results

In the following content, firstly, we verify the performance of ONSoCMax against the handcrafted policies on the social
welfare achieved. Then, we analyze the robustness of ONSOCMAX under different parameter settings.

Theoretical Superiority. Fig. 4 and Fig. 5 show the social welfare achieved under different service duration and input workloads settings of jobs. We can observe that all the algorithms achieve higher social welfare when the two variables increase. The reason is that, when the capacity of nodes are sufficient, increasing the service duration and the maximum input workloads of jobs can increase the opportunities of being fully served. Nevertheless, ONSOCMAX always performs the best among these online algorithms. Besides, ONSOCMAX performs the best for linear job utilities. This is because the logarithmic and polynomial utilities have diminishing returns, which could increase the fluctuation ratio $\nu_i$. This will cause more jobs be served with their marginal costs fall into the second segment of $\hat{\phi}_r(\omega)$, which further leads to the decrease of social welfare.

Robustness. We verify the robustness of ONSOCMAX under different settings of maximum processable workload capacity of nodes and input workload size of jobs. These two variables actually tune the congestion level, i.e., the coverage rate of service demands, from different angles. We can conclude that ONSOCMAX is robust to the changes of the congestion level from Fig. 6.

V. RELATED WORK

Workload dispatching and job scheduling are fully investigated under classic settings, where multiple identical nodes with exponentially distributed service rates process continuous arrived jobs. With Continuous Time Markov Chains (CTMC) and Lyapunov Stability theories, policies such as JSQ [35], JIQ [36], POD [37], and JFIQ [38] are proposed and analyzed on the average response time and cross-server communication overhead. In a most recent work [38], Weng et al. proposed the JFSQ and JFIQ policies under the constraints of heterogenous service rates and service locality. They prove that, under a well-connected bipartite graph condition, these two policies achieve the minimum mean response time in both the many-server regime and the sub Halpin-Whitt regime.

Another line of works study the energy-efficient and multi-resource sharing workload dispatching from a different theoretical basis [19], [20], [39]–[44]. Generally, the objective is to improve the energy efficiency with on-demand resource allocation. Thereinto, online workload dispatching of deadline-aware (multi-server) jobs have been studied in [19], [20], [42]. In a similar work [19], the authors design online algorithms for both fractional and non-fractional workload model under concave utility settings. The optimality of designed algorithms hold when all the jobs have the same deadline and share a single type of resource. Compared with it, our work is more general with the model of resource mesh and the technique of marginal cost estimation, which makes it more applicable to heterogeneous multi-server jobs with different deadlines. Online workload dispatching under the objective of minimizing
the Nash Social Welfare is revisited in a recent paper [45]. This work provides tight bounds on the price of anarchy (PoA) of pure Nash equilibria and on the competitive ratio of the general greedy algorithm under very general latency functions.

VI. CONCLUSION

In this paper, we design an online workload dispatching policy for general multi-server jobs with the target of maximizing the social welfare. The multi-server jobs we considered have multiple task components and strict deadlines. To take both the spatio and temporal resource of the computing cluster into consideration, we establish a model of resource mesh. Each task component of jobs can only be dispatched to the nodes available to it based on the service locality constraints. With the marginal cost estimation technique, we design an online policy ONSOCMax by solving several convex pseudo-social welfare maximization problems. The algorithm is proved to be $\alpha$-competitive for an $\alpha \geq 2$. Online workload dispatching of workloads with complex communication patterns will be studied in future.

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Shuiguang Deng is currently a full professor at the College of Computer Science and Technology in Zhejiang University, China, where he received a BS and PhD degree both in Computer Science in 2002 and 2007, respectively. He previously worked at the Massachusetts Institute of Technology in 2014 and Stanford University in 2015 as a visiting scholar. His research interests include Edge Computing, Service Computing, Cloud Computing, and Business Process Management. He serves for the journal IEEE Trans. on Services Computing, Knowledge and Information Systems, Computing, and IET Cyber-Physical Systems: Theory & Application as an Associate Editor. Up to now, he has published more than 100 papers in journals and refereed conferences. In 2018, he was granted the Rising Star Award by IEEE TCSVC. He is a fellow of IET and a senior member of IEEE.

Hailiang Zhao received the B.S. degree in 2019 from the School of Computer Science and Technology, Wuhan University of Technology, Wuhan, China. He is currently pursuing the Ph.D. degree with the College of Computer Science and Technology, Zhejiang University, Hangzhou, China. His research interests include cloud and edge computing, distributed systems and optimization algorithms. He has published several papers in flagship conferences and journals such as IEEE ICWS 2019, IEEE TPDS, IEEE TMC, etc. He was a recipient of the Best Student Paper Award of IEEE ICWS 2019. He is a reviewer for IEEE TSC and Internet of Things Journal.

Jianwei Yin received the Ph.D. degree in computer science from Zhejiang University (ZJU) in 2001. He was a visiting scholar with the Georgia Institute of Technology. He is currently a full professor with the College of Computer Science, ZJU. Up to now, he has published more than 100 papers in top international journals and conferences. His current research interests include service computing and business process management. He is an associate editor of the IEEE Transactions on Services Computing.

Schahram Dustdar is a full professor of Computer Science (Informatics) with a focus on Internet Technologies heading the Distributed Systems Group at the TU Wien. He is founding co-editor-in-chief of ACM Transactions on Internet of Things (ACM TIoT) as well as Editor-in-Chief of Computing (Springer). He is an associate editor of IEEE Transactions on Services Computing, IEEE Transactions on Cloud Computing, ACM Computing Surveys, ACM Transactions on the Web, and ACM Transactions on Internet Technology, as well as on the editorial board of IEEE Internet Computing and IEEE Computer. Dustdar is recipient of multiple awards: TCI Distinguished Service Award (2021), IEEE TCSC Outstanding Leadership Award (2018), IEEE TCSC Award for Excellence in Scalable Computing (2019), ACM Distinguished Scientist (2009), ACM Distinguished Speaker (2021), IBM Faculty Award (2012). He is an elected member of the Academia Europaea: The Academy of Europe, where he is chairman of the Informatics Section, as well as an IEEE Fellow (2016), an Asia-Pacific Artificial Intelligence Association (AAIA) President (2021) and Fellow (2021). He is an EAI Fellow (2021) and an IC2ICC Fellow (2021). He is a member of the 2022 IEEE Computer Society fellow evaluating committee.

Albert Y. Zomaya is the Peter Nicol Russell Chair Professor of Computer Science and Director of the Centre for Distributed and High-Performance Computing at the University of Sydney. To date, he has published > 600 scientific papers and articles and is (co-)author/editor of > 30 books. A sought-after speaker, he has delivered > 250 keynote addresses, invited seminars, and media briefings. His research interests span several areas in parallel and distributed computing and complex systems. He is currently the Editor in Chief of the ACM Computing Surveys and served in the past as Editor in Chief of the IEEE Transactions on Computers (2010-2014) and the IEEE Transactions on Sustainable Computing (2016-2020).

Professor Zomaya is a decorated scholar with numerous accolades including Fellowship of the IEEE, the American Association for the Advancement of Science, and the Institution of Engineering and Technology (UK). Also, he is an elected Fellow of the Royal Society of New South Wales and an elected foreign member of Academia Europaea. He is the recipient of the 1997 Edgeworth David Medal from the Royal Society of New South Wales for outstanding contributions to Australian Science, the IEEE Technical Committee on Parallel Processing Outstanding Service Award (2011), IEEE Technical Committee on Scalable Computing Medal for Excellence in Scalable Computing (2011), IEEE Computer Society Technical Achievement Award (2014), ACM MSWIM Reginald A. Fessenden Award (2017), the New South Wales Premier’s Prize of Excellence in Engineering and Information and Communications Technology (2019), and the Research Innovation Award, IEEE Technical Committee on Cloud Computing (2021).