Formation of bound states and BCS-BEC crossover near a flat band: the sawtooth lattice

Giuliano Orso$^1$ and Manpreet Singh$^{1,2}$

$^1$Université de Paris, Laboratoire Matériaux et Phénomènes Quantiques, CNRS, F-75013, Paris, France
$^2$Centre for Quantum Engineering Research and Education, TCG Centres for Research and Education in Science and Technology, Sector V, Salt Lake, Kolkata 70091, India

(Dated: December 21, 2021)

We investigate pairing and superconductivity in the attractive Hubbard model on the one-dimensional sawtooth lattice. This is a periodic lattice with two sites per unit cell, which exhibits a flat band (FB) by fine-tuning the hopping rates. We first discuss the formation of pairs in vacuum, showing that there is a broad region of hopping rates values around the FB point, for which both the binding energy and the effective mass of the bound state are strongly affected even by weak interactions. Based on the DMRG method, we then address the ground-state properties of a system with equal spin populations as a function of the interaction strength and the hopping rates. We compare our results with those available for a linear chain, where the model is integrable by Bethe ansatz, and show that the multiband nature of the system substantially modifies the physics of the BCS-BEC crossover. The chemical potential of a system near the FB point remains always close to its zero-density limit predicted by the two-body physics. In contrast, the pairing gap exhibits a remarkably strong density dependence and, differently from the binding energy, it is no longer peaked at the FB point. We show that these results can be interpreted in terms of polarization screening effects, due to an anomalous attraction between pairs in the medium and single fermions. In particular, we find that three-body bound states (trimers) are allowed in the sawtooth lattice, in sharp contrast with the linear chain geometry.

I. INTRODUCTION

During the last ten years there has been a growing interest on FB lattices [1]. These are periodic systems, described by tight binding models, in which one or more dispersion relations is flat or almost flat. The corresponding eigenstates are localized on few lattice sites due to destructive quantum interference. The absence of kinetic energy together with the inherent macroscopic degeneracy of the flat-band make FB systems ideal candidates to enhance interaction effects. For instance they provide a viable route to enhance the superconducting transition temperature [2–5], generate fractional quantum Hall states at room temperature [6], and produce many other intriguing quantum effects.

Lattice models containing a flat band have been realized experimentally with optical lattices for ultracold atoms [7–9], photonic lattices [10, 11], semiconductor microcavities [12] and artificial electronic lattices [13–15]. The recent discovery [16] of unconventional superconductivity and strongly correlated phases in bilayer graphene twisted at a magic angle, causing the emergence of flat bands in the electronic structure, has further boosted the theoretical and experimental research on FB sistem.

The physics of two-body bound states in the presence of a flat band has been recently explored theoretically in different contexts, including topological matter [17–20] and the link between the inverse effective mass of the bound state and the quantum metric of the single-particle states [21–23]. This second direction is related to the more general question of understanding how transport and superconductivity can occur in system with quenched kinetic energy [24–32]. Transport in many-body bosonic flat-band systems has also been explored, see for instance [33–36].

In this work we combine the solution of the two-body problem with exact density matrix renormalization group (DMRG) calculations to study, in a unified framework, the influence of the multi-band structure and the vicinity to the flat band on pairing phenomena, going from the formation of molecules in vacuum to superconductivity in many-body fermionic systems. Our investigation is based on the one-dimensional (1D) sawtooth lattice, also known as triangular or Tasaki lattice. The sawtooth lattice is shown in Fig. 1 (a) and is described by two tunneling rates, $t$ and $t'$. For $t = 0$ the lattice reduces to the linear chain, with the usual single-band dispersion, while for $t = t'/\sqrt{2}$ (FB point) the lower energy Bloch band becomes flat. In this work we study the ground state properties of the attractive Fermi Hubbard model in the sawtooth lattice for the general multiband case, paying special attention to the FB point, and compare our results with those available for $t = 0$, where the model is integrable by Bethe ansatz. By increasing the fermion-fermion attraction, the Fermi gas progressively transforms into a bosonic gas of diatomic molecules. The evolution from a Bardeen–Cooper–Schrieffer (BCS) state to a Bose-Einstein Condensate (BEC), commonly referred to as the BCS-BEC crossover, has been investigated both theoretically and experimentally in single-band dispersive systems, going from superconductors [37] to atomic Fermi gases [38]. Recent theoretical works [39–41] have generalized the theory to two-band continuous models describing superfluid Fermi gases near an orbital Feshbach resonance, and a significant increase of the critical temperature $T_c$ has been predicted when the lower band
becomes shallow \cite{42,43}.

Superconductivity in the sawtooth lattice at the FB point has been recently explored numerically in Ref.\cite{29}, with a focus on the superfluid weight $D_s$. The authors introduced a modified multiband BCS theory with sublattice-dependent order parameters to account for the different connectivity of the A and B sites in Fig. 1 (a). The mean field theory was shown to compare well with exact DMRG calculations. Here we focus on the behavior of the (superfluid) pairing gap and of the chemical potential as a function of the interaction strength $U < 0$ and the tunneling rates. We show that, near the FB point, the chemical potential remains always closed to its zero-density limit, even in the weakly interacting regime. In contrast, the superfluid pairing gap is strongly depleted at finite density and its peak is shifted with respect to the FB point. We explain this surprising effect by studying the change in the ground state energy of the system upon adding an extra fermion. Differently from the linear chain case, the energy of the excitation falls below the bottom of the lower band, indicating that tightly-bound bound states, both at the FB point and for generic tunneling rates. We show that, near the FB point, the number of unit cells, while $t/t'_s = 1/\sqrt{2}$ (b), $t/t'_s = 1/\sqrt{2}$ (c), corresponding to the flat-band point, and $t/t'_s = 1$ (d). In this work we fix the energy scale by setting $t'_s = 1$.

The article is organized as follows. In Sec. II we review the single-particle properties of the sawtooth lattice and present the formalism used to solve the two-body problem in a multiband lattice. In Sec. III we show our results for the binding and the effective mass of the two-body bound states, both at the FB point and for generic tunneling rates. In Sec. IV we present our DMRG results for the BCS-BEC crossover and the appearance of trimer states. Finally in Sec. V we present our conclusions.

II. THEORETICAL APPROACH

A. Single-particle properties

We recall here the single-particle properties of the 1D sawtooth lattice, shown in Fig. 1 (a). Its unit cell contains two sites, called A and B. Particles can hop between two B sites with hopping rate $t$, while tunneling between A and B sites occurs at a rate $t'_s$. The tight-binding Hamiltonian is given by

$$H_{sp} = \sum_i t |iB\rangle \langle i+1B| + h.c. + t'_s |iA\rangle \langle iB| + (i+1B) + h.c.,$$

with $|iA/B\rangle$ denoting the local (site) basis. The dispersion relations of the two bands associated to the Hamiltonian (1) are given by

$$\varepsilon_{\pm}(q) = t \cos q \pm \sqrt{t^2 \cos^2 q + 2t'^2(1 + \cos q)},$$

where $q$ is the wave-vector of the Bloch state and we have set to one the distance between two adjacent B sites. In Fig. 1 (panels b-d) we show how the shapes of the two bands evolve as the tunneling ratio $t/t'_s$ is changed. While the higher band is always concave down at $q = 0$, the bottom $q = q_B$ of the lower band changes from $q_B = 0$, for $t/t'_s < 1/\sqrt{2}$ (b), to $q_B = \pi$ for $t/t'_s > 1/\sqrt{2}$ (d). Exactly at $t/t'_s = 1/\sqrt{2}$, the lower band becomes flat, $\varepsilon_{-}(q) = -\sqrt{t}$ (c). The amplitudes of the Bloch states at site $j$ associated to the energy bands $\varepsilon_{\nu}(q)$ can be conveniently written as $\Psi_{\nu,j} = \varepsilon_{\nu}/\sqrt{L} \left(\alpha_{\nu} \beta_{\nu}\right)$, where $L$ is the number of unit cells, while $\alpha_{\nu}, \beta_{\nu}$ satisfy

$$\alpha_{\nu} = t'_s(1 + e^{-iq}) / \varepsilon_{\nu}(q) \beta_{\nu} ,$$

(3)
together with the normalization condition $|\alpha_{\nu}|^2 + |\beta_{\nu}|^2 = 1$. Since $\varepsilon_{\nu}(-q) = \varepsilon_{\nu}(q)$, we are free to choose $\beta_{\nu}$ real and satisfying $\beta_{-\nu} = \beta_{\nu}$. From Eq. (3) we then find that $\alpha_{-\nu} = \alpha_{\nu}^*$, where the star indicates the complex conjugate.

B. Two-body problem

We now consider two particles hopping on the sawtooth lattice and coupled by contact interactions. The two-body Hamiltonian is given by $H = H_0 + U$, where $H_0 = H_{sp} \otimes 1 + 1 \otimes H_{sp}$ is the noninteracting Hamiltonian and $U = U(\hat{P}^A + \hat{P}^B)$ accounts for contact interactions between the two particles. Here $\hat{P}^x = \sum_i \hat{P}^x_i$ is the noninteracting Hamiltonian and $U = U(\hat{P}^A + \hat{P}^B)$ accounts for contact interactions between the two particles. Here $\hat{P}^x = \sum_i \hat{P}^x_i$
\( \sum_{m^\sigma} |m^\sigma m^\sigma\rangle \langle m^\sigma m^\sigma| \) are the pair projector operators over the doubly occupied sites of the \( \sigma = A, B \) sublattices. The properties of two-body bound states can be obtained by mapping the stationary Schrödinger equation into an effective single-particle model for the center-of-mass motion of the pair, as done in Refs. [44, 45] for continuous and lattice models, respectively. If the external potential is periodic, the momentum \( Q \) of the pair is conserved and the problem further reduces to finding the eigenvalues of a \( N_b \times N_b \) matrix, where \( N_b \) is the number of basis sites per unit cell, as we shall see below for \( N_b = 2 \). The same equation has been recently obtained by Iskin in Ref. [22] by using a different (variational) approach. Scattering states in flat bands have instead been discussed in [46], but can also be obtained by adapting the formalism below, as done in Ref.[47].

We start by writing the two-body Schrödinger equation as \( (E - \hat{H}_0)|\psi\rangle = \hat{U}|\psi\rangle \), where \( E \) is the total energy of the pair. Substituting it into the Schrodinger equation and bringing the operator \( \hat{U} \) as \( (\hat{A}) \) into the Schrodinger equation.

\[
\frac{1}{\hat{U}}|\psi\rangle = (E - \hat{H}_0)^{-1}\hat{A}|\psi\rangle + (E - \hat{H}_0)^{-1}\hat{B}|\psi\rangle. 
\tag{4}
\]

Next, by projecting the wave-function (4) on the doubly occupied states \( |n^\sigma m^\sigma\rangle \), we obtain a close equation for the corresponding amplitudes \( f(n) = \frac{\langle n^\sigma m^\sigma|\psi\rangle}{\langle n^\sigma n^\sigma|\psi\rangle} \) as

\[
f(n)\frac{1}{\hat{U}} = \sum_m K(n, m)f(m),
\tag{5}
\]

where, for given values of \( n \) and \( m \), \( K(n, m) \) is a 2 \times 2 matrix depending parametrically on the energy and whose entries are given by \( K^{\sigma\sigma}(n, m) = \langle n^\sigma n^\sigma|(E - \hat{H}_0)^{-1}|m^\sigma m^\sigma\rangle \). The latter can be conveniently expressed in terms of the components of the single-particle Bloch wave-functions \( \Psi_{q\lambda}(j) \), so that Eq.(5) takes the form

\[
f(n)\frac{1}{\hat{U}} = \frac{1}{L^2} \sum_{m, q, p, \nu, \nu'} \frac{e^{i(q+p)(n-m)}}{E - \varepsilon_\nu(q) - \varepsilon_{\nu'}(p)} M_{\nu \nu'}(q, p) f(m),
\tag{6}
\]

where

\[
M_{\nu \nu'}(q, p) = \langle \alpha_{q\nu} \beta_{q\nu'} | \alpha_{p\nu} \beta_{p\nu'} \rangle \langle \beta_{q\nu} \alpha_{p\nu'} | \alpha_{q\nu} \beta_{q\nu'} \beta_{p\nu'} \rangle. \tag{7}
\]

One can easily see that the eigenstates of Eq.(6) are plane waves \( f(n) = \frac{e^{i\alpha_n}}{\sqrt{\lambda_n}} f_Q \), with \( Q \) being the center-of-mass momentum of the pair. By substituting it into Eq.(6) and taking the continuum limit, we end up with the eigenvalue problem

\[
f_Q \frac{1}{\hat{U}} = R f_Q, \tag{8}
\]

where \( R = R(E, Q) \) is a 2 \times 2 matrix defined as

\[
R = \sum_{\nu, \nu'} \frac{d\theta}{2\pi} M_{\nu \nu'}(q, Q - q).
\tag{9}
\]

The two eigenvalues of the matrix \( R \) are given by

\[
\lambda_{\pm} = \frac{R_{11} + R_{22}}{2} \pm \frac{1}{2} \sqrt{(R_{11} - R_{22})^2 + 4|R_{12}|^2}. \tag{10}
\]

For a given interaction strength \( U \) and quasi-momentum \( Q \), the energy levels of bound states are obtained by looking for solution of \( \lambda_{\pm}(E, Q) = 1/U \), the energy \( E \) taking values outside the noninteracting two-body energy spectrum. In the following we fix the energy scale by setting \( t' = 1 \) and restrict to attractively bound states, corresponding to \( U < 0 \).

**III. TWO-BODY RESULTS**

**A. Bound states at FB point**

We present our results for the two-body bound states for the special case \( t' = 1/\sqrt{2} \), where the lower Bloch band becomes flat, see Fig. 1 (c). We will be interested in the solutions of Eq.(8) with energy \( E < E_{\text{ref}} \), where \( E_{\text{ref}} = 2\varepsilon_{-}(q_B) = -2\sqrt{2} \) is the ground state energy of the two-body system in the absence of interactions. These states are often referred to as doublons, since for large \( |U| \) the two particles sit at the same site and form a tightly bound molecule with energy \( E \sim U \).

The integration over momentum in Eq.(9) will be generally performed numerically. Analytical integration is also possible via residue techniques, although the calculation can become difficult for arbitrary combinations of the parameters \( E \) and \( Q \). As an example, we provide here the exact expression for the matrix \( R \) valid for zero center-of-mass momentum and \( E < E_{\text{ref}} \). To this end, we substitute in Eq.(7) the amplitudes of the Bloch wavefunctions obtained from Eq.(3):

\[
\alpha_{q-} = -\frac{1 + e^{-i q}}{\sqrt{2(1 + \cos q)}}, \quad \beta_{q-} = \frac{1}{\sqrt{2(1 + \cos q)}},
\tag{11}
\]

\[
\alpha_{q+} = \frac{1 + e^{-i q}}{\sqrt{2(1 + \cos q)(1 + \cos q)}}, \quad \beta_{q+} = \sqrt{\frac{1 + \cos q}{2(1 + \cos q)}},
\]

and the corresponding dispersion relations of the two bands, with \( \varepsilon_{+}(q) = \sqrt{2(1 + \cos q)} \). For \( Q = 0 \) the integration over momentum is performed by introducing the complex variable \( z = e^{i q} \), so that the integrating function takes the form of a ratio \( t(z)/y(z) \) of two analytical functions. We then calculate the integral via the Cauchy’s residue theorem of complex analysis, after identifying the poles inside the circle \(|z| = 1\). This gives
We substitute Eq. (12) in Eq. (10) and obtain the energy of the bound state by solving \( U = 1/\lambda_+ \) (the eigenvalue \( \lambda_+ \) yields the energy of the first excited bound state). We then extract the binding energy \( E_b \) from the relation \( E = -E_b + E_{\text{ref}} \), so that \( E_b > 0 \) if the state is bound. In Fig. 2 we plot our results for the binding energy as a function of the modulus of the interaction strength (blue solid curve). For weak interactions, the binding energy behaves approximately linearly in \( |U| \) and vanishes as \( U \) goes to zero, implying that \( E = -2\sqrt{2} \). Therefore, for small \( U \) the main contribution to the integral in the rhs of Eq. (9) comes from the contribution \( \nu = \nu' = - \), leading to a pole in the matrix elements of \( R \) at the same energy, while in all the other contributions \( E \) can be safely replaced by \( E_{\text{ref}} \). From Eq. (12) we then find \( U \simeq c_1(E + 2\sqrt{2}) + c_2(E + 2\sqrt{2})^2 \), where \( c_1 = 1.94189 \) and \( c_2 = 0.36479 \). Hence the weak coupling expansion of the binding energy is given by

\[
E_b \simeq -\frac{U}{c_1} + \frac{c_2}{c_1^2} U^2, \tag{13}
\]

as shown in Fig. 2 by the dot-dashed curve. Notice that the linear in \( U \) dependence of the binding energy, also reported in \[21\], is a direct consequence of the localized nature of the single particle states forming the molecule. Indeed, the same behavior was also observed \[45\] for two interacting particles in the presence of a quasi-periodic lattice, once single-particle localization sets in. The effective mass \( m_p^* \) of the bound state is defined through \( 1/m_p^* = E''(0) \), where \( E(Q) \) is the dispersion relation of the bound state. In Fig. 3 we plot the effective mass of the pair as a function of the interaction strength (solid blue line). In order to derive the corresponding weak coupling expansion for the pair effective mass, we need to calculate the matrix \( R \) for a small but finite momentum \( Q \). To do so, in Eq. (9) we perform a quadratic expansion in \( Q \) for the non singular contributions and replace therein the energy \( E \) by \( -2\sqrt{2} \). The integration can then be done analytically yielding the following approximate expressions for the matrix elements:

\[
R_{11} \simeq \frac{(\sqrt{3} - 2)(7 - \cos Q) + 4}{(7 - \cos Q)\sqrt{3}(E + 2\sqrt{2})} - \frac{1008 + 187Q^2}{1728\sqrt{6}}
\]

\[
R_{12} \simeq \frac{2e^{-iQ/2}\cos(Q/2)}{(7 - \cos Q)\sqrt{3}(E + 2\sqrt{2})} + \frac{432 - Q(185Q + i216)}{1728\sqrt{6}}
\]

\[
R_{22} \simeq \frac{4}{(7 - \cos Q)\sqrt{3}(E + 2\sqrt{2})} - \frac{5(144 + 11Q^2)}{1728\sqrt{6}}. \tag{14}
\]

We then substitute the rhs of Eqs (14) in Eq. (10) and expand \( 1/\lambda_- = U \) up to the second order in \( E + 2\sqrt{2} \). We obtain \( U \simeq f_1(Q)(E + 2\sqrt{2}) + f_2(Q)(E + 2\sqrt{2})^2 \), where \( f_i(Q) = c_i + d_iQ^2 \) are quadratic functions of the momentum. Here \( d_1 = 0.261236 \) and \( d_2 = 0.294074 \), while the constants \( c_i \) are defined as above. Solving for the energy yields \( E(Q) \simeq -2\sqrt{2} + \frac{U}{f_1(Q)} - \frac{f_2(Q)}{f_1(Q)^2} U^2 \). The effective mass of the pair is then given by

\[
\frac{1}{m_p^*} \simeq -\frac{2d_1}{c_1^2} U + \frac{6c_2d_1 - 2d_2c_1}{c_1^4} U^2, \tag{15}
\]

as displayed in Fig. 3 with the dot-dashed red line. The linear term in Eq.(15) is consistent with the numerical result in Ref. [21]. We see that the inverse effective mass takes its maximum value around \( U = -4.3 \). In this intermediate region of \( U \) values the weak coupling expansion...
is clearly seen in the numerical calculations of the superfluid weight $D_s$ calculated in [29]. Indeed both quantities scale as $|U|$ for weak interactions and as $1/|U|$ in the strongly interacting regime. It is also worth emphasizing that the linear-in-$U$ dependence of the inverse effective mass of the pair observed for weak interactions is a generic feature of FB lattices. Indeed, for $E$ sufficiently close to $2\varepsilon_b$, with $\varepsilon_b$ being the energy of the FB, the matrix $R$ in Eq. (9) takes the approximate form $R \simeq A(Q)/(E - 2\varepsilon_b)$, where $A$ is a matrix depending on the momentum $Q$ of the pair, as shown in Eq. (14).

Before continuing, we point out that the occupation of the two sublattices is asymmetric due to the different connectivity of A and B sites: in the deepest bound state the two particles reside more on the B sublattice, while in the first excited bound state the particles occupy prevelently the A sites. This point can be easily verified by calculating the normalized eigenvectors $\mathbf{v} = (v^A, v^B)$ of the matrix $R$ for $E$ large and negative. To lowest order in $1/E$ we find

$$R = \begin{pmatrix} 1/E + 4/E^3 & 4/E^3 \\ 4/E^3 & 1/E + 8/E^3 \end{pmatrix},$$

whose eigenvalues are $\lambda_{\pm} = 1/E + (6 \pm 2\sqrt{5})/E^3$. The corresponding normalized eigenvectors are $\mathbf{v}_{-} = (v_1, v_2)$ and $\mathbf{v}_{+} = (-v_2, v_1)$, where $v_1 = \sqrt{2/(5 + \sqrt{5})}$ and $v_2 = \sqrt{(5 + \sqrt{5})/10}$. Hence the probability for the pair to be in the A site is $|v_1|^2 = 0.276$ for the ground state and $|v_2|^2 = 0.724$ for the first excited state. This result is consistent with Ref. [29], also reporting an asymmetric occupation of the two sublattices for the ground state density profile of the attractive Fermi-Hubbard model on the sawtooth lattice.

**B. Bound states for generic tunneling rates**

We investigate here the properties of the lowest energy bound state in the absence of the flat band, i.e. for an arbitrary $t \neq 1/\sqrt{2}$. From Eq. (2) we find that the reference energy is given by

$$E_{ref} = \begin{cases} 2(t - \sqrt{4 + t^2}) & \text{if } t < \frac{1}{\sqrt{2}} \\ -4t & \text{if } t > \frac{1}{\sqrt{2}} \end{cases}$$

(15) and the strong coupling expansion (see Eq. (26) below) become completely inadequate, implying that higher order terms play a crucial role.

It is interesting to note that the dependence of the pair inverse effective mass on $|U|$ shown in Fig. 3 is clearly reminiscent of the behavior of the superfluid weight $D_s$ calculated in [29]. Indeed both quantities scale as $|U|$ for weak interactions and as $1/|U|$ in the strongly interacting regime. It is also worth emphasizing that the linear-in-$U$ dependence of the inverse effective mass of the pair observed for weak interactions is a generic feature of FB lattices. Indeed, for $E$ sufficiently close to $2\varepsilon_b$, with $\varepsilon_b$ being the energy of the FB, the matrix $R$ in Eq. (9) takes the approximate form $R \simeq A(Q)/(E - 2\varepsilon_b)$, where $A$ is a matrix depending on the momentum $Q$ of the pair, as shown in Eq. (14).

Before continuing, we point out that the occupation of the two sublattices is asymmetric due to the different connectivity of A and B sites: ins the deepest bound state the two particles reside more on the B sublattice, while in the first excited bound state the particles occupy prevalently the A sites. This point can be easily verified by calculating the normalized eigenvectors $\mathbf{v} = (v^A, v^B)$ of the matrix $R$ for $E$ large and negative. To lowest order in $1/E$ we find

$$R = \begin{pmatrix} 1/E + 4/E^3 & 4/E^3 \\ 4/E^3 & 1/E + 8/E^3 \end{pmatrix},$$

whose eigenvalues are $\lambda_{\pm} = 1/E + (6 \pm 2\sqrt{5})/E^3$. The corresponding normalized eigenvectors are $\mathbf{v}_{-} = (v_1, v_2)$ and $\mathbf{v}_{+} = (-v_2, v_1)$, where $v_1 = \sqrt{2/(5 + \sqrt{5})}$ and $v_2 = \sqrt{(5 + \sqrt{5})/10}$. Hence the probability for the pair to be in the A site is $|v_1|^2 = 0.276$ for the ground state and $|v_2|^2 = 0.724$ for the first excited state. This result is consistent with Ref. [29], also reporting an asymmetric occupation of the two sublattices for the ground state density profile of the attractive Fermi-Hubbard model on the sawtooth lattice.

We see that $E_b$ takes its maximum value in correspondence of the FB point (solid vertical line), for all values of the interaction strength. The discontinuous derivative of the binding energy at the FB point is inherited from the corresponding singularity of the reference energy. Notice that the origin of the peak in the weak and in the strongly interacting regimes is different. In the first case, it directly follows from the fact that at this point the binding energy scales linearly in $U$, while for any other point the growth is only quadratically in $U$. To see this, we perform a quadratic expansion around the bottom of the lower band, $\epsilon_- (q, \epsilon_F) \simeq \epsilon_- (q_b) + \epsilon''_- (q_b) (q - q_b)^2/2$. Next, we approximate the numerator in the rhs of Eq. (9) by a constant, $M_-(q, -q) \simeq M_-(q_b, -q_b)$ and integrate over momentum. From Eq. (10) we obtain

$$\frac{1}{U} \simeq \frac{|a_{q_b}|^2 + |b_{q_b}|^2}{2\sqrt{\epsilon''_- (q_b)} \sqrt{E_b}},$$

(17)

showing that for weak interactions the binding energy grows as $U^2$, as for the linear chain ($t = 0$), where $E_b = \sqrt{U^2 + 16 - 4}$. Eq. (17) breaks down for $t = 1/\sqrt{2}$ since the single-particle inverse effective mass $1/m^* = \epsilon''_- (q_b)$ associated to the lower band vanishes. For any other point $m^*$ is given by

$$\frac{1}{m^*} = \begin{cases} -t + \frac{1+t^2}{\sqrt{4+t^2}} & \text{if } t < \frac{1}{\sqrt{2}} \\ \frac{2t^2}{4t^2 - 1} & \text{if } t > \frac{1}{\sqrt{2}}. \end{cases}$$

(18)

We therefore write $E_b \simeq U^2 f(t)$, where $f$ is a function of the tunneling rate, which can be obtained by substituting in Eq. (17) the Bloch states amplitudes, Eq. (3) and the
effective mass, Eq. (18). This yields

\[ f(t) = \begin{cases} 
- \frac{(2t^2)^2 \sqrt{4t^2 - (2t^2 - t \sqrt{1+t^2})^2}}{4t^4} & \text{if } t < \frac{1}{\sqrt{2}} \\
\frac{8}{t^2} & \text{if } t > \frac{1}{\sqrt{2}} 
\end{cases} \]  

In Fig. 4a we display the result obtained from Eq. (19) for the weakest interaction considered, \( U = -0.25 \) (dashed line). There is a wide region around the FB point, where the numerical result is far apart from the weak coupling expansion. Indeed Eq. (17) relies on the assumption that \( E_b \ll w \), where \( w = \sqrt{4 + t^2 - 3t} \) is the width of the lowest energy band. This condition is necessarily violated near the FB point, where the bandwidth vanishes.

In the presence of a very strong attraction, the two particles sit at the same lattice site and form a tightly bound state. Since \( E \sim U \) is large and negative, the binding energy reduces to \( E_b \sim -U + E_{\text{ref}} \). Thus, in this regime the binding energy mirrors the reference energy \( E_{\text{ref}} \), which has a singular peak for \( t = 1/\sqrt{2} \), as displayed in Fig. 4 (b). In order to improve the strong coupling expansion for \( E_b \), we use the formula 

\[ \frac{1}{E_{\text{ref}}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{E^{n+1}} \]  

Expanding the rhs of Eq. (20) in powers of \( E \), up to second order, gives \( E \sim U + a_1/E + a_2/E^2 \), where \( a_1 = -4(1 + t^2 + \sqrt{1 + t^2}) \) and \( a_2 = -12t - (12t + 8t^3)/\sqrt{1 + t^2} \) are functions of the tunneling rate. Hence, the asymptotic behavior of the binding energy in the strongly interacting regime is given by

\[ E_b \sim -U + E_{\text{ref}} + \frac{a_1}{U} + \frac{a_2}{U^2}. \]  

In Fig. 4 (b) we compare our numerics (solid lines) with the prediction of Eq. (21), shown with dashed lines. We see that the approximation (21) works better and better as \( U \) becomes large and negative, and for \( U = -8 \) nearly coincides with the numerical data. Differently from the weak coupling expansion (17), the strong coupling expansion (21) is valid also at the FB point and it is displayed in Fig. 2 by the dashed curve. Let us now discuss the inverse effective mass of the bound state. This is shown...
in Fig. 5 as a function of the tunneling rate \( t \) and for different values of the interaction strength. The dotted line corresponds to the noninteracting limit, where the bound state breaks and the pair effective mass reduces to twice the single-particle mass, \( m_p = 2m^* \). We see that, far from the FB point, weak interactions tend to slightly increase the effective mass of the pair. In contrast, close to it, the pair mass is strongly reduced by interactions, an effect which persists until \( U \simeq -5 \). It is also interesting to note that the minimum in the inverse effective mass shifts towards smaller values of \( t \) as \( |U| \) increases.

The interaction-induced correction to \( m_p \) in the weak coupling regime can be obtained by generalizing Eq. (17) to a finite momentum of the pair. In particular, for \( Q > 0 \) the dominant contribution to the integral in Eq. (9) comes from the region centered around \( q = q_B + Q/2 \), with \( q_B \) defined as above. By replacing \( q_B \) with \( q_B + Q/2 \) in Eq. (17) and taking the square of both sides of the equation, we obtain the dispersion relation of the bound state

\[
E(Q) = 2\epsilon_{-}(q_B+Q/2) - \left( \frac{|a_{qB+Q/2}t - |b_{qB+Q/2}t|}{4\epsilon_{-}(q_B+Q/2)} \right)^2 U^2.
\]

From Eq. (22) we then find that the effective mass of the pair in the weak coupling regime reduces to

\[
\frac{1}{m_p^2} \simeq \frac{1}{2m^*} + hU^2,
\]

where \( h \) is a function of the tunneling rate, whose explicit form can be obtained by making use of Eqs (2) and (3) in Eq. (22). This gives

\[
h(t) = \begin{cases} 
\frac{4-t(8u+t(60+t(40u+t(86+t(25u+t(36+5t(t+u))))))))}{16u(t^2-1)} & \text{if } t < \frac{1}{\sqrt{2}} \\
\frac{1+t^2}{16(t^2-1)} & \text{if } t > \frac{1}{\sqrt{2}}
\end{cases}
\]

with \( u = \sqrt{4+\frac{1}{t^2}} \). Notice that \( h \) diverges for \( t = 1/\sqrt{2} \), since in Eq. (22) \( \epsilon'_-(q) = 0 \) in the entire Brillouin zone. The divergence signals that Eq. (23) does not hold at the FB point, in agreement with Eq. (15).

The strong coupling expansion for the pair effective mass can be obtained by following the same procedure used to derive Eq. (20), but this time we retain the full \( Q \) dependence of the matrix elements in the rhs of Eq. (9). The integration can still be performed analytically and from Eq. (10) we obtain

\[
\frac{1}{U} \simeq \frac{1}{E^2} \left( E(4+E^2) + 6t(1+\cos Q) + 2Et^2(1+\cos Q) - \sqrt{8(E+3t)^2(1+\cos Q) + 4t^2(1+Et + (3+Et)\cos Q)^2} \right).
\]

which provides an implicit equation for the dispersion relation of the bound state. We then expand the rhs of Eq. (25) in powers of \( 1/E^2 \), retaining up to second order terms. Finally we solve for the energy and extract the effective mass as

\[
\frac{1}{m_p} = \frac{1 + 2t^2(t^2 + \sqrt{1+t^4})}{U\sqrt{1+t^4}} \frac{1}{U^2(1+t^4)^{3/2}} - \frac{t(3+8t^2 + 6\sqrt{1+t^4} + 6t^4\sqrt{1+t^4})}{U^2(1+t^4)^{3/2}}.
\]

The strong coupling expansion (26) for the FB point is displayed in Fig. 3 by the dashed curve. Notice that the \( 1/U^2 \) correction in Eq. (26) accounts for the non-monotonic behavior of the inverse effective mass displayed in Fig. 5, including the shift of the minimum towards smaller values of \( t \) as interaction effects become stronger.

IV. BCS-BEC CROSSOVER

In this section we investigate the ground state properties of the attractive Fermi-Hubbard model in the sawtooth lattice. The Hamiltonian is given by

\[
H = \sum_{i,\sigma} \left[ t c_{i\alpha}^\dagger B_{i+1}^\sigma + c_{i\alpha}^B (c_{i+1\alpha}^\dagger c_{i\sigma} + h.c.) \right] + U \sum_i (n_{\uparrow i}^A n_{\downarrow i}^A + n_{\uparrow i}^B n_{\downarrow i}^B),
\]

where \( c_{\alpha i}^\sigma (c_{\alpha i}^\dagger) \) is the local creation (annihilation) operator for fermions with spin component \( \alpha = \uparrow, \downarrow \) in the sublattice \( \sigma = A, B \), and \( n_{\alpha i}^\sigma = c_{\alpha i}^\dagger c_{\alpha i}^\sigma \) are the corresponding local density operators. We recall that \( t' = 1 \) in our energy units. We define the density of the two spin components with respect to the total number of lattice sites, \( n_{\alpha} = N_{\alpha}/(2L) \), where \( N_{\alpha} \) is the number of fermions with spin \( \alpha \). In this work we restrict our attention to fully paired systems, corresponding to equal densities of the two spin components, \( n_\uparrow = n_\downarrow \).
Two important observables characterizing the BCS-BEC crossover in Fermi gases are the \textit{pairing gap} \( \Delta_{pg} \) and the chemical potential \( \mu \). The first, also known as the spin gap, corresponds to the energy needed to break a pair in the many-body system by reversing one spin, while the second corresponds to half the energy change upon adding a pair (one fermion with spin up and one fermion with spin down) to the system. Let \( \epsilon(n,s) \) be the ground state energy per lattice site, expressed in terms of the total fermion density \( n = n_{\uparrow} + n_{\downarrow} \) and the spin density \( s = n_{\uparrow} - n_{\downarrow} \). The pairing gap and the chemical potential are given by

\[
\mu = \left( \frac{\partial \epsilon}{\partial n} \right)_{s=0}, \quad \Delta_{pg} = 2 \left( \frac{\partial \epsilon}{\partial s} \right)_{s=0}. \tag{28}
\]

We compute the ground state energy \( E(N_{\uparrow},N_{\downarrow}) \) of the system as a function of the spin populations \( N_{\alpha} \) for a large enough system size \( L \). For our DMRG calculations, we use the open-source code of the ALPS library \cite{ALPS06}, which is based on the Matrix Product States (MPS) representation \cite{LUBOM(98)}. We consider systems sizes up to \( L = 60 \), corresponding to 120 sites, with open boundary conditions.

We evaluate the chemical potential by approximating the derivative in Eq.\,(28) by a finite difference, \( \mu \approx (E(N_{\uparrow}+1,N_{\downarrow}+1) - E(N_{\uparrow},N_{\downarrow}))/2 \). For the pairing gap, we use \( -\Delta_{pg} \approx E(N_{\uparrow}+1,N_{\downarrow}+1) - 2E(N_{\uparrow}+1,N_{\downarrow}) + E(N_{\uparrow},N_{\downarrow}) \). In the thermodynamic limit, this formula is equivalent to the finite difference \( \Delta_{pg} \approx 2\left( E(N_{\uparrow}+1,N_{\downarrow}-1) - E(N_{\uparrow},N_{\downarrow}) \right) \), but is less sensitive to finite-size effects. For the special case \( t = 0 \), the Hamiltonian \((27)\) is integrable and the pairing gap and the chemical potential can be calculated numerically by solving the integral equations of the exact Bethe-ansatz solution, as done in Ref \cite{LUBOM(96a)}. For vanishing densities, both the pairing gap and the chemical potential possess a well defined limit, which is consistent with the solution of the two-body problem. The pairing gap reduces to the binding energy, since for \( N_{\uparrow} = N_{\downarrow} = 0 \) we find from Eq. \( (28) \) that \( \Delta_{pg} = E(2,0) - E(1,1) = E_b \). Here we use the fact that \( E(2,0) = 2E(1,0) = E_{ref} \) and \( E(1,1) = E(Q) = 0 \), where \( E(Q) \) is the energy dispersion of the two-body bound state calculated in Sec.\,III. From Eq.\,(28) we instead find that \( \mu = E(1,1)/2 \), since \( E(0,0) = 0 \), implying that the chemical potential reduces to

\[
\mu = -\frac{E_b + E_{ref}}{2}. \tag{29}
\]

A peculiar feature of 1D systems is that interaction effects become stronger as the particle density decreases. As a consequence, the binding energy provides an upper bound for the pairing gap. In contrast, the two-body prediction \( (29) \) is a lower bound for the chemical potential, because the inverse compressibility \( \partial \mu / \partial n \) must be positive or null to ensure the energetic stability of the gas.

Before presenting our results, we emphasize that the pairing gap discussed here is different from the mean field superconducting order parameters \( \Delta^{A/B} \) investigated in Ref. \cite{BRO(12)}: not only those order parameters depend on the sublattice index, but they also increase as the density increase, while the pairing gap shows the opposite behavior.

### A. Results at flat band point

In Fig.\,6 we plot the chemical potential versus \( |U| \) at the FB point, together with the zero density prediction \((29)\). For weak enough interactions, finite density corrections are small, due to the infinite compressibility associated to the FB. From Eqs\,(13) and \((29) \) we obtain, to first order in \( U \), \( \mu \approx -\sqrt{2} + U/(2c_1) \), which is fully consistent with our numerics. This result differs from the BCS mean field estimate given in \cite{BRO(12)}, where the linear in \( U \) correction for the chemical potential was found to explicitly depend on the density.

In the inset of the same figure we plot the difference between the chemical potential and its zero density limit. This quantity exhibits a non-monotonic behavior, taking its maximum value around \( |U| \approx 8 \) and then decreasing steadily as the strongly interacting regime is approached. Here bound states behave as point-like hard-core bosons, hopping between neighboring sites of the sawtooth lattice and experiencing repulsive nearest-neighbor interactions as well as a uniform potential of different strength in the two sublattices. Since the coupling constants associated to these processes are all proportional to \( 1/|U| \), as demonstrated in Ref.\,[37], the leading finite-density correction to Eq.\,(29) is of the same order and therefore vanishes as \( |U| \) becomes infinite, as shown in the inset of Fig.\,6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{(Color online) Chemical potential at the flat band point \( t = 1/\sqrt{2} \) as a function of the modulus of the interaction strength for filling \( n_{\uparrow} = n_{\downarrow} = 1/3 \) (black circles). The dashed line represents to the zero density limit, see Eq.\,(29). The inset shows the difference between the solid and the dashed lines as a function of \(|U|\). Notice that the difference tends to zero for infinite attraction. The solid lines are a guide to the eye.}
\end{figure}
we show the pairing gap as a function of the modulus of the interaction strength, for filling \( n_s = n_c = 1/3 \) (black circles). The dashed line represents the zero density limit of the pairing gap, i.e., the two-body binding energy \( E_b \). The inset shows the difference between the dashed and the solid lines as a function of the interaction strength. Notice that this difference saturates to a constant value as \(|U|\) becomes infinite, while in the integrable case \( t = 0 \) the difference vanishes. The solid lines are a guide to the eye.

In Fig. 7 we show the pairing gap as a function of the modulus of the interaction strength (black circles), together with the two-body binding energy (red dashed line). For weak interactions the numerical data are well fitted by \( \Delta_{pg} = 0.44|U| \), shown by the blue dot-dashed line. For strong interactions, the difference between the binding energy and the pairing gap saturates to a constant value, as shown in the inset of Fig. 7. By comparing the insets of Fig. 6 and 7, we see that finite-density effects for the pairing gap are typically one order of magnitude larger than for the chemical potential. This result contrasts with the integrable limit \( t = 0 \), where density corrections are of order \( 1/U \) for both the chemical potential and the pairing gap. In particular, from the Bethe ansatz solution, it has been shown \([51]\) that, for \( n \ll 1 \),

\[
\mu = -\frac{\sqrt{U^2 + 16}}{2} + \frac{\pi^2 n^2}{4(U^2 + 16)}, \quad \Delta_{pg} = E_b - \frac{\pi^2 n^2}{2(U^2 + 16)},
\]

(30)

with the binding energy being equal to \( E_b = \sqrt{U^2 + 16} - 4 \). In Fig. 8 we display \( \Delta_{pg} \) at the FB point as a function of the density for \( U = -15 \). The pairing gap decreases rapidly at low densities \( (n \lesssim 0.08) \), while for higher density it diminishes at a much slower rate. Moreover our numerics suggest that the density dependence of the pairing gap could be singular, i.e., nonanalytical, at \( n = 0 \), in stark contrast with Eq. (30).

To better understand the origin of the strong finite-density effects on the pairing gap, we study the excess energy \( \mu_\tau \), corresponding to the change in the ground state energy of the system upon adding an extra spin up fermion, \( \mu_\tau = E(N^+, N^\downarrow) - E(N^+, N^\downarrow) \). From Eq. (28) we find that this quantity is related to the previous observables by the general equation

\[
\mu_\tau = \mu + \Delta_{pg}/2, \quad (31)
\]

holding for any tunneling rate \( t \) and interaction strength \( U \). We write the equation of state of the fully paired system as \( \mu(n) = \mu(0) + \Delta\mu(n) \), where \( \mu(0) \) is given by Eq. (29) and \( \Delta\mu(n) \) accounts for finite-density effects. Notice that \( \Delta\mu \geq 0 \), because the inverse compressibility \( (\partial\mu/\partial n)_{\mu=0} \) of the system must be positive or zero to ensure its mechanical stability. From Eq. (31) we then find

\[
\Delta_{pg} = E_b - E_{ref} + 2\mu_\tau - 2\Delta\mu, \quad (32)
\]

showing that the density dependence in the pairing gap comes not only from the equation of state, as in Eq. (30), but also from the excess energy.

From Eq. (31) we calculate the excess energy at the FB point as a function of the interaction strength and plot the result in Fig. 9. We see that \( \mu_\tau \) is a decreasing function of \(|U|\). For a noninteracting gas \( \mu_\tau = \mu = \epsilon_F = -\sqrt{2} \), since for \( n < 1 \) the upper dispersive band is empty. To first order in \( U \) we find \( \mu_\tau \simeq -\sqrt{2} + UF(n) \), where \( F(n) \) is a function of the density, satisfying \( F(n = 0) = 0 \). This behavior is shown in Fig. 9 by the blue dashed line for \( n = 2/3 \). From Eq. (32) we then find, to the same order, \( \Delta_{pg} \simeq U(2F(n) - 1/c_1) \) since \( \Delta\mu \simeq 0 \). For large \(|U|\) the excess energy does not scale linearly with \( U \), as the chemical potential does, because adding an extra fermion to a fully paired system does not change the number of pairs. Instead it saturates to a density-dependent value, which sits well below the energy of the flat band for the chosen density, \( \mu_\tau < E_{ref}/2 \). From Eq. (32) this implies that the pairing gap is strongly reduced by the finite density as shown in the inset of Fig. 7, while corrections from the equation of state are subleading, since \( \Delta\mu \sim 1/|U| \).

According to Eq. (32), for large \(|U|\) the pairing gap yields the ground state energy of the pair in vacuum, but
measured with respect to the many-body reference energy $2\mu_\uparrow$, instead of the bare reference energy $E_{\text{ref}}$.

The condition $\mu_\uparrow < E_{\text{ref}}/2$ indicates that the excess fermion and tightly bound pairs tend to attract each other, which can potentially lead to the formation of cluster states, like trimers [52, 53]. These states are generally forbidden in the linear chain limit $t = 0$, unless the hopping rates of the spin up and the spin down fermions are different [54–56] (a strong attractive atom-dimer interaction has indeed been observed experimentally [57] in Fermi-Fermi mixtures with unequal masses). To verify this important point, we compute by DMRG the binding energy $E_b^{\text{trim}}$ of trimers made of, say, two spin up fermions and one spin down fermion, which is defined as

\[
E_b^{\text{trim}} = -E(2, 1) + E(1, 1) + E(1, 0).
\]  

FIG. 9. (Color online) Excess energy $\mu_\uparrow$ (see text for definition) at the flat-band point as a function of $|U|$ for filling $n_\uparrow = n_\downarrow = 1/3$ (black circles). The blue dashed line represents the asymptotic behavior for weak interactions. The connecting line is a guide to the eye.

FIG. 10. (Color online) Binding energy of a three-body bound state (trimers) at the flat band as a function of the modulus of the interaction strength. The connecting line is a guide to the eye.

The obtained result for $t = 1/\sqrt{2}$ is displayed in Fig. 10 as a function of the interaction strength and indeed confirms the existence of trimers for finite $|U|$.

B. Results for generic tunneling rate

In Fig. 11 we show the chemical potential as a function of the tunneling rate $t$ plotted for different values of $U = 0, -0.25, -0.5, -1, -3, -5$ with $n_\uparrow = n_\downarrow = 1/3$. The two-body limit $\mu = (E_b + E_{\text{ref}})/2$ for $U = -5$ is shown with the dot-dashed line. The vertical solid line indicates the flat-band point $t = 1/\sqrt{2}$. The connecting lines are a guide to the eye.

FIG. 11. (Color online) Chemical potential $\mu$ as a function of the tunneling rate $t$ plotted for different values of $U = 0, -0.25, -0.5, -1, -3, -5$ with $n_\uparrow = n_\downarrow = 1/3$. The two-body limit $\mu = (E_b + E_{\text{ref}})/2$ for $U = -5$ is shown with the dot-dashed line. The vertical solid line indicates the flat-band point $t = 1/\sqrt{2}$. The connecting lines are a guide to the eye.

In Fig. 12 we plot the ratio $\Delta_{\text{pk}}/|U|$ between the spin-gap and the modulus of the interaction strength as a function of the tunneling rate $t$ for increasing values of $|U|$. The obtained results are clearly similar to their two-body counterpart, presented in Fig. 4, showing a drastic enhancement of pairing in the vicinity of the FB point for weak to moderate interactions. There are other interesting effects brought about by the finite density. First, the position of the maximum of the pairing gap drifts to smaller values of $t$ as $|U|$ increases, while the two-body binding energy always remains peaked at $t = 1/\sqrt{2}$ (see Fig. 4). Second, for strong interactions finite density corrections are more prominent near the FB point, as can be seen in Fig. 12, where the pairing gap for $U = -5$ is compared to the corresponding value of the binding
energy (dashed line). By comparing Fig. 11 with 12 we see that finite density corrections for the chemical potential can be significantly smaller than for the pairing gap, except in a neighborhood of the integrable point $t = 0$, where Eq. (30) applies.

In Fig. 13 we plot the corresponding results for the excess energy as a function of the tunneling rate. Far from the FB point, weak interactions cause a fast decrease of $\mu^\uparrow$ with respect to the Fermi energy, while near the FB point the decrease is rather modest. As a consequence, a local maximum appears, drifting towards smaller values of $t$ as $|U|$ increases and turning into a global maximum at $t \approx 0.66$. Since for finite interactions the chemical potential in Fig. 11 depends smoothly on the tunneling rate, we find from Eq. (31) that the drift of the peak in the pairing gap simply reflects the behavior of the excess energy. By further increasing $|U|$, we see from Fig. 13 that there is a larger and larger window of $t$ values around the FB point, for which $\mu^\uparrow < E_{\text{ref}}/2$, implying that the pairing gap is density-depleted for arbitrary large $|U|$, as shown in Fig. 12. Notice that for $t = 0$ the above condition is never satisfied, since Eqs (30) and (31) yield $\mu^\uparrow = -2 = E_{\text{ref}}/2$ for large $|U|$.  

V. CONCLUSION AND OUTLOOK

In this work we have investigated pairing and superconductivity in the 1D sawtooth lattice. From the solution of the two-body problem, we have extracted the binding energy and the effective mass of attractively bound states, as a function of the interaction strength and the tunneling rates. We have found that, in a broad region of tunneling rates around the FB point, both quantities are highly sensitive to weak interactions. In particular, the binding energy possesses a pronounced maximum in correspondence of the FB point, which persists for any $|U| < 0$.

Our DMRG results for the BCS-BEC crossover in fully paired spin-1/2 Fermi gases confirm that the proximity to a flat band provides a shortcut to the strong coupling regime. In particular the chemical potential remains always pinned near its two-body value, irrespective of the actual density and the value of the interaction strength. For small $|U|$, this is a consequence of the large compressibility due to the presence of the flat band, while for strong interactions it reflects the fact that the many-body energy scales associated to pair tunneling and nearest neighbor interactions between pairs decrease as $1/|U|$. In contrast, we find that the pairing gap is strongly affected by density corrections. As a consequence, it takes its maximum value not at the FB point, but at a shifted position $t^* < 1/\sqrt{2}$, which depends on the value of the density.

We have also shown that the Fermi Hubbard model with negative $U$ in the sawtooth lattice allows for three-body bound states at the FB point, due to an effective attractive interaction between an excess fermion and the surrounding pairs. This can lead to the formation of clusters and the suppression of superconductivity in systems with large spin-imbalance. The fate of trimer states for $t \neq 1/\sqrt{2}$ remains to be investigated.

It would be interesting to study by exact numerics the behavior of the superfluid weight $D_s$ in the sawtooth lattice for a generic tunneling rate and its relation with the pair inverse effective mass. Our results show that for finite $|U|$ the minimum of $1/n^\uparrow$ drifts towards smaller values of $t$. Another intriguing direction is to understand whether multiband BCS theory can correctly predict the behavior of the pairing gap observed in our numerics.
especially its density-induced depletion.

The results discussed in this work can be investigated experimentally with cold atoms in optical lattices. In particular a viable scheme to implement the sawtooth lattice has been recently proposed [33, 58]. The interaction strength can be controlled either directly, via a Feshbach resonance or indirectly, by varying the tunneling rates $t, t'$ and consequently the ratios $t/U$ and $t'/U$. While we have mainly focused on the sawtooth lattice, we expect that our results will apply also to other FB systems.

ACKNOWLEDGEMENTS

We thank S. Pilati and F. Chevy for useful comments on the manuscript. G.O. acknowledges financial support from ANR (Grant SpiFBox) and from DIM Sirteq (Grant EML 19002465 DFG). M.S. acknowledges funding from MULTIPLY fellowship under the Marie Sklodowska-Curie COFUND Action (grant agreement No. 713694).

[1] D. Leykam, A. Andreanov, and S. Flach, Artificial flat band systems: from lattice models to experiments, Advances in Physics: X 3, 1473052 (2018), https://doi.org/10.1080/23746149.2018.1473052.
[2] V. A. Khodel and V. R. Shaginyan, Superfluidity in system with fermion condensate, JETP Letters 51, 553 (1990).
[3] N. B. Kopnin, T. T. Heikkilä, and G. E. Volovik, High-temperature surface superconductivity in topological flat-band systems, Phys. Rev. B 83, 220503 (2011).
[4] T. T. Heikkilä, N. B. Kopnin, and G. E. Volovik, Flat bands in topological media, JETP Letters 94, 233 (2011).
[5] H. Aoki, Theoretical possibilities for flat band superconductivity, Journal of Superconductivity and Novel Magnetism 33, 2341 (2020).
[6] E. Tang, J.-W. Mei, and X.-G. Wen, High-temperature fractional quantum hall states, Phys. Rev. Lett. 106, 236802 (2011).
[7] G.-B. Jo, J. Guzman, C. K. Thomas, P. Hosur, A. Vishwanath, and D. M. Stamper-Kurn, Ultracold atoms in a tunable optical kagome lattice, Phys. Rev. Lett. 108, 045305 (2012).
[8] S. Taie, H. Ozawa, T. Ichinose, T. Nishio, S. Nakajima, and Y. Takahashi, Coherent driving and freezing of bosonic matter wave in an optical lieb lattice, Science Advances 1, 10.1126/sciadv.1500854 (2015).
[9] T.-H. Leung, M. N. Schwarz, S.-W. Chang, and D. M. Stamper-Kurn, Interaction-enhanced group velocity of bosonic matter wave in an optical lieb lattice, Science 348, 918–921 (2015).
[10] A. Julku, S. Peotta, T. I. Vanhala, D.-H. Kim, and P. Törmä, Interaction-induced lattices for bound states: Designing flat bands, quantized pumps, and higher-order topological insulators for doublons, Phys. Rev. Research 2, 013348 (2020).
[11] S. Mukherjee, A. Spracklen, I. Sagnes, M. Abbarchi, D. D. Solnyshkov, G. Malpuech, E. Galopin, A. Lemaitre, J. Bloch, and A. Amo, Direct observation of dirac cones and a flatband in a honeycomb lattice for polaritons, Phys. Rev. Lett. 112, 116402 (2014).
[12] R. Drost, T. Ojanen, A. Harju, and P. Liljeroth, Topological states in engineered atomic lattices, Nature Physics 13, 668 (2017).
[13] R. Drost, T. Ojanen, A. Harju, and P. Liljeroth, Topological states in engineered atomic lattices, Nature Physics 13, 668 (2017).
[14] M. R. Slot, T. S. Gardenier, P. H. Jacobse, G. C. P. van Miert, S. N. Kempkes, S. J. M. Zevenhuizen, C. M. Smith, D. Vannmaekelbergh, and I. Swart, Experimental realization and characterization of an electronic lieb lattice, Nature Physics 13, 672 (2017).
[15] M. N. Huda, S. Kezilebieke, and P. Liljeroth, Designer flat bands in quasi-one-dimensional atomic lattices, Phys. Rev. Research 2, 043426 (2020).
[16] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, Nature 556, 43 (2018).
[17] G. Palumbo, G. Palmumbo, N. Goldman, and M. Di Liberto, Interaction-induced lattices for bound states: Designing flat bands, quantized pumps, and higher-order topological insulators for doublons, Phys. Rev. Research 2, 013348 (2020).
[18] Y. Kuno, T. Mizoguchi, and Y. Hatsugai, Interaction-induced doublons and embedded topological subspace in a complete flat-band system, Phys. Rev. A 102, 063325 (2020).
[19] S. Flannigan and A. J. Daley, Enhanced repulsively bound atom pairs in topological optical lattice ladders, Quantum Science and Technology 5, 045017 (2020).
[20] G. Pelegrí, A. M. Marques, V. Ahufinger, J. Moms, and R. G. Dias, Interaction-induced topological properties of two bosons in flat-band systems, Phys. Rev. Research 2, 033267 (2020).
[21] P. Törmä, L. Liang, and S. Peotta, Quantum metric and effective mass of a two-body bound state in a flat band, Phys. Rev. B 98, 220511 (2018).
[22] M. Iskin, Two-body problem in a multiband lattice and the role of quantum geometry, Phys. Rev. A 103, 053311 (2021).
[23] M. Iskin, Exact relation between the effective-mass tensor of the two-body bound states and the quantum-metric tensor of the underlying Bloch states, arXiv e-prints, arXiv:2109.06000 (2021), arXiv:2109.06000 [cond-mat.quant-gas].
[24] S. Peotta and P. Törmä, Superfluidity in topologically nontrivial flat bands, Nature Communications 6, 8944 (2015).
[25] A. Julku, S. Peotta, T. I. Vanhala, D.-H. Kim, and P. Törmä, Geometric origin of superfluidity in the lieb-lattice flat band, Phys. Rev. Lett. 117, 045303 (2016).
[26] R. Mondaini, G. G. Batrouni, and B. Grémaud, Pairing and superconductivity in the flat band: Creutz lattice, Phys. Rev. B 98, 155142 (2018).

[27] M. Iskin, Superfluid stiffness for the attractive hubbard model on a honeycomb optical lattice, Phys. Rev. A 99, 023608 (2019).

[28] N. Verma, T. Hazra, and M. Randeria, Optical spectral weight, phase stiffness, and tc bounds for trivial and topological flat band superconductors, Proceedings of the National Academy of Sciences 118, 10.1073/pnas.2106741118 (2021), https://www.pnas.org/content/118/34/e2106741118.full.pdf.

[29] S. M. Chan, B. Grémaud, and G. G. Batrouni, Pairing and superconductivity in quasi one-dimensional flat band systems: Creutz and sawtooth lattices (2021), arXiv:2105.12761 [cond-mat.supr-con].

[30] V. A. J. Pyykkönen, S. Peotta, P. Fabritius, J. Mohan, T. Esslinger, and P. Törnä, Flat-band transport and josephson effect through a finite-size sawtooth lattice, Phys. Rev. B 103, 144519 (2021).

[31] J. S. Hofmann, E. Berg, and D. Chowdhury, Superconductivity, pseudogap, and phase separation in topological flat bands, Phys. Rev. B 102, 201112 (2020).

[32] V. Peri, Z.-D. Song, B. A. Bernevig, and S. D. Huber, Fragile topology and flat-band superconductivity in the strong-coupling regime, Phys. Rev. Lett. 126, 027002 (2021).

[33] S. D. Huber and E. Altman, Bose condensation in flat bands, Phys. Rev. B 82, 184502 (2010).

[34] Y.-Z. You, Z. Chen, X.-Q. Sun, and H. Zhai, Superfluidity of bosons in kagome lattices with frustration, Phys. Rev. Lett. 109, 265302 (2012).

[35] L. G. Phillips, G. De Chiara, P. Öhberg, and M. Valiente, Low-energy behavior of strongly interacting bosons on a flat-band lattice above the critical filling factor, Phys. Rev. B 91, 054103 (2015).

[36] F. Baboux, L. Ge, T. Jacqmin, M. Biondi, E. Gaspelin, A. Lemaître, L. Le Gratiet, I. Sagnes, S. Schmidt, H. E. Tièrce, A. Amo, and J. Bloch, Bosonic condensation and disorder-induced localization in a flat band, Phys. Rev. Lett. 116, 060402 (2016).

[37] R. Micnas, J. Ranninger, and S. Robaszkiewicz, Superconductivity in narrow-band systems with local nonretarded attractive interactions, Rev. Mod. Phys. 62, 113 (1990).

[38] J. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. 80, 885 (2008).

[39] M. Iskin, Two-band superfluidity and intrinsic josephson effect in alkaline-earth-metal fermi gases across an orbital feshbach resonance, Phys. Rev. A 94, 011604 (2016).

[40] L. He, J. Wang, S.-G. Peng, X.-J. Liu, and H. Hu, Strongly correlated fermi superfluid near an orbital feshbach resonance: Stability, equation of state, and leggett mode, Phys. Rev. A 94, 043624 (2016).

[41] J. Xu, R. Zhang, Y. Cheng, P. Zhang, R. Qi, and H. Zhai, Reaching a fermi-superfluid state near an orbital feshbach resonance, Phys. Rev. A 94, 033609 (2016).

[42] H. Tajima, Y. Yerin, A. Perali, and P. Pieri, Enhanced critical temperature, pairing fluctuation effects, and bcs-bec crossover in a two-band fermi gas, Phys. Rev. B 99, 180503 (2019).

[43] H. Tajima, A. Perali, and P. Pieri, Bcs-bec crossover and pairing fluctuations in a two band superfluid/superconductor: A t matrix approach, Condensed Matter 5, 10.3390/condmat5010010 (2020).

[44] M. Wouters and G. Orso, Two-body problem in periodic potentials, Phys. Rev. A 73, 012707 (2006).

[45] G. Dufour and G. Orso, Anderson localization of pairs in bichromatic optical lattices, Phys. Rev. Lett. 109, 155306 (2012).

[46] M. Valiente and N. T. Zinner, Quantum collision theory in flat bands, 50, 064004 (2017).

[47] G. Orso and G. V. Shlyapnikov, Superfluid fermi gas in a 1d optical lattice, Phys. Rev. Lett. 95, 260402 (2005).

[48] M. Dolfi, B. Bauer, S. Keller, A. Kosenkov, T. Ewart, A. Kiantian, T. Giamarchi, and M. Troyer, Matrix product state applications for the ALPS project, Comput. Phys. Commun. 185, 3430 (2014).

[49] U. Schollwöck, The density-matrix renormalization group in the age of matrix product states, Ann. Phys. 326, 96 (2011).

[50] F. Heidrich-Meisner, G. Orso, and A. E. Feiguin, Phase separation of trapped spin-imbalanced fermi gases in one-dimensional optical lattices, Phys. Rev. A 81, 053602 (2010).

[51] F. Woynarovich and K. Penc, Novel magnetic properties of the hubbard chain with an attractive interaction, Zeitschrift für Physik B Condensed Matter 85, 269 (1991).

[52] E. Burovski, G. Orso, and T. Jolicoeur, Multiparticle composites in density-imbalanced quantum fluids, Phys. Rev. Lett. 103, 215301 (2009).

[53] M. Dalmonte, P. Zoller, and G. Pupillo, Trimer liquids and crystals of polar molecules in coupled wires, Phys. Rev. Lett. 107, 163202 (2011).

[54] G. Orso, E. Burovski, and T. Jolicoeur, Luttinger liquid of trimers in fermi gases with unequal masses, Phys. Rev. Lett. 104, 065301 (2010).

[55] M. Dalmonte, K. Dieckmann, T. Roscilde, C. Hartl, A. E. Feiguin, U. Schollwöck, and F. Heidrich-Meisner, Dimer, trimer, and fulde-ferrell-larkin-ovchinnikov liquids in mass- and spin-imbalanced trapped binary mixtures in one dimension, Phys. Rev. A 85, 063608 (2012).

[56] G. Roux, E. Burovski, and T. Jolicoeur, Multitimer composites in density-imbalanced quantum fluids, Phys. Rev. Lett. 104, 065301 (2010).

[57] M. Dalmonte, K. Dieckmann, T. Roscilde, C. Hartl, A. E. Feiguin, U. Schollwöck, and F. Heidrich-Meisner, Dimer, trimer, and fulde-ferrell-larkin-ovchinnikov liquids in mass- and spin-imbalanced trapped binary mixtures in one dimension, Phys. Rev. A 85, 063608 (2012).