A CERTAIN SYNCHRONIZING PROPERTY OF SUBSHIFTS
AND FLOW EQUIVALENCE

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ABSTRACT. We will study a certain synchronizing property of subshifts called
λ-synchronization. The λ-synchronizing subshifts form a large class of irre-
ducible subshifts containing irreducible sofic shifts. We prove that the λ-
synchronization is invariant under flow equivalence of subshifts. The λ-synchronizing
K-groups and the λ-synchronizing Bowen-Franks groups are studied and proved
to be invariant under flow equivalence of λ-synchronizing subshifts. They are
new flow equivalence invariants for λ-synchronizing subshifts.

Keywords: λ-synchronizing subshifts, flow equivalence, K-groups, Bowen-Franks
groups.

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1. INTRODUCTION

In [23], a certain synchronizing property called λ-synchronization has been in-
troduced. The λ-synchronizing property for a subshift Λ is an equivalent property
to the property D for the transpose of Λ, which has been introduced by W. Krieger
in [18]. As the λ-synchronizing property is weaker than the usual synchroniz-
ing property, synchronizing subshifts are λ-synchronizing. Hence irreducible sofic
shifts are λ-synchronizing as well as Dyck shifts, β-shifts, Morse shifts, etc. are
λ-synchronizing. Many irreducible subshifts have this property. A λ-graph system
is a labeled Bratteli diagram with an additional structure called ι-map ([26]). A
finite directed labeled graph gives rise to a λ-graph system with stationary vertices
so that a sofic shift is presented by a λ-graph system with stationary vertices. Not
only sofic shifts but also all subshifts may be presented by λ-graph systems. There
is a canonical method to construct a λ-graph system from an arbitrary subshift. If a
subshift is sofic, the canonically constructed λ-graph system is one given by the left
Krieger cover graph for the sofic shift. Hence the canonically constructed λ-graph
system from a subshift is regarded as a generalization of a left Krieger cover graph.
In [23], a construction of λ-graph systems from λ-synchronizing subshifts has been
introduced. If a λ-synchronizing subshift is sofic, the constructed λ-graph system is
one given by the left Fischer cover graph for the sofic shift. Hence the constructed
λ-graph system from a λ-synchronizing subshift is regarded as a generalization of a
left Fischer cover graph. It is called the canonical λ-synchronizing λ-graph system
for a λ-synchronizing subshift.

In this paper, we will first characterize the canonical λ-synchronizing λ-graph
system in an intrinsic way, and prove that it has a unique synchronizing property.
We will also prove that it is minimal in the sense that there exists no proper λ-graph
subsystem that presents the subshift. In [23], it has been proved that the K-groups
and the Bowen-Franks groups for the $\lambda$-synchronizing $\lambda$-graph system are invariant under topological conjugacy, so that they yield topological conjugacy invariants of $\lambda$-synchronizing subshifts. In the second part of this paper, we will prove that $\lambda$-synchronization is invariant under flow equivalence of subshifts. Furthermore we will prove that the K-groups and the Bowen-Franks groups for the canonical $\lambda$-synchronizing $\lambda$-graph systems for $\lambda$-synchronizing subshifts are invariant under flow equivalence of $\lambda$-synchronizing subshifts. They are new nontrivial flow equivalence invariants for a large class of subshifts. In [28], the author has extended the Bowen-Franks groups to general subshifts. The Bowen-Franks groups are computed as the Bowen-Franks groups for the nonnegative matrices of the left Krieger covers if the subshifts are sofic. The Bowen-Franks groups for the $\lambda$-synchronizing $\lambda$-graph system are computed as the Bowen-Franks groups for the nonnegative matrices of the left Fischer covers if the subshifts are sofic.

Throughout the paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of nonnegative integers and the set of positive integers respectively.

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2. $\lambda$-synchronizing subshifts

Let $\Sigma$ be a finite set with its discrete topology. We call it an alphabet and each member of it a symbol. Let $\Sigma^\mathbb{Z}, \Sigma^\mathbb{N}$ be the infinite product spaces $\prod_{i=-\infty}^\infty \Sigma_i, \prod_{i=1}^\infty \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation $\sigma$ on $\Sigma^\mathbb{Z}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ is called the full shift. Let $\Lambda$ be a shift invariant closed subset of $\Sigma^\mathbb{Z}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift and simply written as $\Lambda$. We denote by $X_\Lambda(\subset \Sigma^\mathbb{N})$ the set of all right one-sided sequences appearing in $\Lambda$. We denote by $|\mu|$ the length $k$ for a word $\mu = \mu_1 \cdot \mu_k, \mu_i \in \Sigma$. For a natural number $l \in \mathbb{N}$, we denote by $B_l(\Lambda)$ the set of all words appearing in some $(x_i)_{i\in\mathbb{Z}}$ of $\Lambda$ with length equal to $l$. Put $B_0(\Lambda) = \bigcup_{i=0}^\infty B_i(\Lambda)$ where $B_0(\Lambda) = \{\emptyset\}$ the empty word. For a word $\mu = \mu_1 \cdot \mu_k \in B_\ast(\Lambda)$, a right infinite sequence $x = (x_i)_{i\in\mathbb{N}} \in X_\Lambda$ and $l \in \mathbb{Z}_+$, put

$$\Gamma^{-}_l(\mu) = \{v_1 \cdot v_l \in B_l(\Lambda) \mid v_1 \cdot \cdot v_l \mu_1 \cdot \mu_k \in B_\ast(\Lambda)\},$$
$$\Gamma^{-}(x) = \{v_1 \cdot v_l \in B_l(\Lambda) \mid (v_1, \ldots, v_l, x_1, x_2, \ldots) \in X_\Lambda\},$$
$$\Gamma^{+}(\mu) = \{\omega_1 \cdot \cdot \omega_l \in B_l(\Lambda) \mid \mu_1 \cdot \cdot \mu_k \omega_1 \cdot \cdot \omega_l \in B_\ast(\Lambda)\},$$
$$\Gamma^{+}_{\infty}(\mu) = \{y \in X_\Lambda \mid \mu y \in X_\Lambda\},$$

and

$$\Gamma^{-}(\mu) = \bigcup_{l=0}^\infty \Gamma^{-}_l(\mu), \quad \Gamma^{+}_{\infty}(\mu) = \bigcup_{l=0}^\infty \Gamma^{+}_l(\mu).$$

A word $\mu = \mu_1 \cdot \cdot \mu_k \in B_\ast(\Lambda)$ for $l \in \mathbb{Z}_+$ is said to be $l$-synchronizing if the equality

$$\Gamma^{-}_l(\mu) = \Gamma^{-}(\mu_\omega)$$

holds for all $\omega \in \Gamma^{+}_l(\mu)$. Denote by $S_l(\Lambda)$ the set of all $l$-synchronizing words of $\Lambda$. It is easy to see that a word $\mu \in B_\ast(\Lambda)$ is $l$-synchronizing if and only if $\Gamma^{-}_l(\mu) = \Gamma^{-}(\mu x)$ for all $x \in \Gamma^{+}_\infty(\mu)$. Recall that an irreducible subshift $\Lambda$ is defined to be $\lambda$-synchronizing if for any $\eta \in B_l(\Lambda)$ and $k \geq \lambda$ there exists $\nu \in S_k(\Lambda)$ such that $\eta \nu \in S_{k-l}(\Lambda)$ (see [23]).
Balchinard and Hansel have introduced a class of subshifts called synchronizing shifts which contains the irreducible sofic shifts ([1]). Let \( \Lambda \) be a subshift over \( \Sigma \). A word \( \omega \in B_s(\Lambda) \) is said to be intrinsically synchronizing if \( \mu \omega, \omega \nu \in B_s(\Lambda) \) for \( \mu, \nu \in B_s(\Lambda) \) implies \( \mu \omega \nu \in B_s(\Lambda) \). An irreducible subshift \( \Lambda \) is said to be synchronizing if \( \Lambda \) has an intrinsically synchronizing word.

**Proposition 2.1.** A synchronizing shift is \( \lambda \)-synchronizing. Hence an irreducible sofic shift is \( \lambda \)-synchronizing.

**Proof.** Let \( \omega \) be an intrinsically synchronizing word. Then the words of the form \( \xi \omega \in B_s(\Lambda) \) for \( \xi \in B_s(\Lambda) \) are intrinsically synchronizing by definition. As for any intrinsically synchronizing word \( \zeta \) of \( \Lambda \) and \( l \in \mathbb{Z}_+ \), we have

\[
\Gamma_l^{-1}(\zeta) = \Gamma_l^{-1}(\zeta \mu)
\]

for \( \mu \in \Gamma_l^{+}(\zeta) \). Hence any intrinsically synchronizing word is \( l \)-synchronizing for all \( l \in \mathbb{N} \). Since \( \Lambda \) is irreducible, for \( \eta \in B_l(\Lambda) \) and \( k \geq l \), there exists \( \xi \in B_s(\Lambda) \) such that \( \eta \xi \omega \in B_s(\Lambda) \). As \( \nu = \xi \omega \) is intrinsically synchronizing and hence \( k \)-synchronizing, one has \( \eta \nu \in S_{k-1}(\Lambda) \). This means that \( \Lambda \) is \( \lambda \)-synchronizing. \( \square \)

There exists a concrete example of an irreducible subshift that is not \( \lambda \)-synchronizing (see [23]).

We note that it has been proved that property D is invariant under topological conjugacy ([18]). The following proposition is written in [23].

**Proposition 2.2** ([18],[23]). The \( \lambda \)-synchronization is invariant under topological conjugacy of subshifts.

**Proof.** Suppose that two subshifts \( \Lambda \) over \( \Sigma \) and \( \Lambda' \) over \( \Sigma' \) are bipartitely related in the sense of [34]. There exist alphabets \( C, D \), specifications \( \kappa : \Sigma \to CD \) \( \kappa' : \Sigma' \to DC \) and a bipartite subshift \( \Lambda \) over \( C \cup D \) related to a bipartite conjugacy between \( \Lambda \) and \( \Lambda' \). We may naturally extend \( \kappa \) and \( \kappa' \) to \( B_s(\Lambda) \) and \( B_s(\Lambda') \) respectively. Let us assume that \( \Lambda \) is \( \lambda \)-synchronizing. For \( \eta' \in B_{l'}(\Lambda') \) and \( k' \geq l' \), with \( \eta' = \alpha_1' \cdots \alpha_{l'}' \) and \( \kappa'(\alpha_i') = d_{i}c_{i} \), \( i = 1, \ldots, l' \). Take symbols \( c_0 \in C, d_{l'+1} \in D \) such that \( c_0d_1c_1d_2c_2 \cdots d_{l'}c_{l'}d_{l'+1} \in B_s(\Lambda) \). Set \( \eta = \kappa^{-1}(c_0d_1c_1d_2c_2 \cdots d_{l'}c_{l'}d_{l'+1}) \) that belongs to \( B_{l'+1}(\Lambda) \). Put \( l = l' + 1 \) and \( k = k' + 1 \) and then we have \( \eta \in B_l(\Lambda) \) with \( k \geq l \). Since \( \Lambda \) is \( \lambda \)-synchronizing, there exists \( \nu \in S_k(\Lambda) \) such that \( \eta \nu \in S_{k-1}(\Lambda) \). Let \( \kappa(\nu) = \hat{c}_1 \hat{d}_1 \hat{c}_2 \hat{d}_2 \cdots \hat{c}_n \hat{d}_n \). Take \( \hat{c}_{n+1} \in C \) such that the word

\[
e_0d_1c_1d_2c_2 \cdots d_{l'}c_{l'}d_{l'+1}c_1d_1c_2d_2cd_2 \cdots \hat{c}_n \hat{d}_n \hat{c}_{n+1}
\]

is admissible in \( \Lambda \). Put

\[
\nu' = \kappa^{-1}(d_{l'+1}c_1d_1c_2d_2d_2 \cdots \hat{c}_n \hat{d}_n c_{n+1})
\]

that belongs to \( B_{n+1}(\Lambda') \). As \( \kappa'(\nu') = d_{l'+1}c_{l'+1} \) \( c_{n+1} \) and \( \nu \in S_k(\Lambda) \), one has \( \nu' \in S_k(\Lambda') \) so that \( \nu' \in S_k(\Lambda') \). By the equality

\[
e_0 \kappa'(\eta') \kappa'(\nu') = \kappa(\eta) \kappa(\nu) \hat{c}_{n+1},
\]

the word \( \eta \nu' \) is admissible in \( \Lambda' \), which belongs to \( S_{l'+1}(\Lambda') \). Therefore \( \Lambda' \) is \( \lambda \)-synchronizing. \( \square \)
A \(\lambda\)-graph system is a graphical object presenting a subshift ([26]). It is a generalization of a finite labeled graph and has a close relation to a construction of a certain class of \(C^*\)-algebras ([29]). Let \(\mathcal{L} = (V, E, \lambda, \iota)\) be a \(\lambda\)-graph system over \(\Sigma\) with vertex set \(V = \bigcup_{l \in \mathbb{Z}_+} V_l\) and edge set \(E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1}\) with a labeling map \(\lambda : E \to \Sigma\), and that is supplied with surjective maps \(\iota = \iota_{l,l+1} : V_{l+1} \to V_l\) for \(l \in \mathbb{Z}_+\). Here the vertex sets \(V_l, l \in \mathbb{Z}_+\) are finite disjoint sets. An edge \(e\) in \(E_{l,l+1}\) has its source vertex \(s(e)\) in \(V_l\) and its terminal vertex \(t(e)\) in \(V_{l+1}\) respectively. Every vertex in \(V\) has a successor and every vertex in \(V_l\) for \(l \in \mathbb{N}\) has a predecessor. It is then required that there exists an edge in \(E_{l,l+1}\) with label \(\alpha\) and its terminal is \(v \in V_{l+1}\) if and only if there exists an edge in \(E_{l-1,l}\) with label \(\alpha\) and its terminal is \(\iota(v) \in V_l\). For \(u \in V_{l-1}\) and \(v \in V_{l+1}\), put

\[
E_{l,l+1}^1(u,v) = \{ e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u \},
\]
and

\[
E_{l-1,l}^{l-1}(u,v) = \{ e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v) \}.
\]

Then we require a bijective correspondence preserving their labels between \(E_{l,l+1}^1(u,v)\) and \(E_{l-1,l}^{l-1}(u,v)\) for each pair of vertices \(u, v\). We call this property the local property of \(\lambda\)-graph system. We call an edge in \(E\) a labeled edge and a finite sequence of connecting labeled edges a labeled path. If a labeled path \(\gamma\) labeled \(\nu\) starts at a vertex \(v\) in \(V_l\) and ends at a vertex \(u\) in \(V_{l+n}\), we say that \(\nu\) leaves \(v\) and write \(s(\gamma) = v, t(\gamma) = u, \lambda(\gamma) = \nu\). We henceforth assume that \(\mathcal{L}\) is left-resolving, which means that \(t(e) \neq t(f)\) whenever \(\lambda(e) = \lambda(f)\) for \(e, f \in E\). For a vertex \(v \in V_l\) denote by \(\Gamma^-_l(v)\) the predecessor set of \(v\) which is defined by the set of words of length \(l\) appearing as labeled paths from a vertex in \(V_0\) to the vertex \(v\). \(\mathcal{L}\) is said to be predecessor-separated if \(\Gamma^-_l(v) \neq \Gamma^-_l(u)\) whenever \(u, v \in V_l\) are distinct. A subshift \(\Lambda\) is said to be presented by a \(\lambda\)-graph system \(\mathcal{L}\) if the set of admissible words of \(\Lambda\) coincides with the set of labeled paths appearing somewhere in \(\mathcal{L}\).

Two \(\lambda\)-graph systems \(\mathcal{L} = (V, E, \lambda, \iota)\) over \(\Sigma\) and \(\mathcal{L}' = (V', E', \lambda', \iota')\) over \(\Sigma\) are said to be isomorphic if there exist bijections \(\Phi_V : V \to V'\) and \(\Phi_E : E \to E'\) satisfying \(\Phi_V(V_l) = V'_l\) and \(\Phi_E(E_{l,l+1}) = E'_{l',l+1}\) such that they give rise to a labeled graph isomorphism compatible to \(\iota\) and \(\iota'\). We note that any essential finite directed labeled graph \(\mathcal{G} = (V, E, \lambda, \iota)\) over \(\Sigma\) with vertex set \(V\), edge set \(E\) and labeling map \(\lambda : E \to \Sigma\) gives rise to a \(\lambda\)-graph system \(\mathcal{L}_\mathcal{G} = (V, E, \lambda, \iota)\) by setting \(V_l = V, E_{l,l+1} = \mathcal{E}, \iota = \text{id}\) for all \(l \in \mathbb{Z}_+\) ([29]).

Two points \(x, y \in X_\Lambda\) are said to be \(l\)-past equivalent, written as \(x \sim_l y\), if \(\Gamma^-_l(x) = \Gamma^-_l(y)\). For a fixed \(l \in \mathbb{Z}_+\), let \(F^l_i, i = 1, 2, \ldots, m(l)\) be the set of all \(l\)-past equivalence classes of \(X_\Lambda\) so that \(X_\Lambda\) is a disjoint union of \(F^l_i, i = 1, 2, \ldots, m(l)\). Then the canonical \(\lambda\)-graph system \(\mathcal{L}_\Lambda = (V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)\) for \(\Lambda\) is defined as follows ([26]). The vertex set \(V^\Lambda_l\) at level \(l\) consist of the sets \(F^l_i, i = 1, 2, \ldots, m(l)\). We write an edge with label \(\alpha\) from the vertex \(F^l_i \in V^\Lambda_l\) to the vertex \(F^{l+1}_{j,l+1}\) in \(V^\Lambda_{l+1}\) if \(\alpha x \in F^l_i\) for some \(x \in F^l_j\). We denote by \(E^\Lambda_{l,l+1}\) the set of all edges from \(V^\Lambda_l\) to \(V^\Lambda_{l+1}\). There exists a natural map \(\iota^\Lambda_{l,l+1}\) from \(V^\Lambda_{l+1}\) to \(V^\Lambda_{l}\) by mapping \(F^{l+1}_{j,l+1}\) to \(F^l_i\) when \(F^l_i\) contains \(F^{l+1}_{j,l+1}\). Set \(V^\Lambda_l = \bigcup_{l \in \mathbb{Z}_+} V^\Lambda_l\) and \(E^\Lambda_l = \bigcup_{l \in \mathbb{Z}_+} E^\Lambda_{l,l+1}\). The labeling of edges is denoted by \(\lambda^\Lambda : E^\Lambda \to \Sigma\). The canonical \(\lambda\)-graph system \(\mathcal{L}_\Lambda\) is left-resolving and predecessor-separated and it presents \(\Lambda\).
A λ-graph system

\[ \Omega^\lambda(\Lambda) = (V^\lambda(\Lambda), E^\lambda(\Lambda), \lambda^\lambda(\Lambda), \iota^\lambda(\Lambda)) \]

for a λ-synchronizing subshift Λ has been introduced in [23]. It is regarded as a left Fischer cover version for a λ-synchronizing subshift Λ whereas the canonical λ-graph system \( \Omega^\lambda \) is regarded as a left Krieger cover version for a subshift Λ. For \( \mu, \nu \in B_\lambda(\Lambda) \), if \( \Gamma_i^- (\mu) = \Gamma_i^- (\nu) \), we say that \( \mu \) is \( i \)-past equivalent to \( \nu \) and write it as \( \mu \simL i \nu \).

**Lemma 3.1** ([23]). Let Λ be a λ-synchronizing subshift. Then we have

(i) For \( i \in \mathbb{N} \) and \( \eta \in B_\lambda(\Lambda) \), there exists \( \mu \in S_i(\Lambda) \) such that \( \eta \in \Gamma_i^- (\mu) \).

(ii) For \( \mu \in S_i(\Lambda) \), there exists \( \mu' \in S_{i+1}(\Lambda) \) such that \( \mu \simL i \mu' \).

(iii) For \( \mu \in S_i(\Lambda) \), there exist \( \beta \in \Sigma \) and \( \nu \in S_{i+1}(\Lambda) \) such that \( \mu \simL i \beta \nu \).

**Proof.** (i) The assertion is direct from definition of λ-synchronization.

(ii) For \( \mu \in S_i(\Lambda) \) with \( |\mu| = n \), put \( k = n + i + 1 \). As Λ is λ-synchronizing, there exists \( \nu \in S_k(\Lambda) \) such that \( \nu \mu \in S_{k-n}(\Lambda) \). Put \( \mu' = \nu \in S_{i+1}(\Lambda) \) so that \( \mu \simL i \mu' \).

(iii) For \( \mu \in S_i(\Lambda) \) with \( \mu = \mu_1 \cdots \mu_n \), put \( k = n + i \). As Λ is λ-synchronizing, there exists \( \omega \in S_k(\Lambda) \) such that \( \mu \omega \in S_{k-n}(\Lambda) \). Set \( \beta = \mu_1 \) and \( \nu = \mu_2 \cdots \mu_n \omega \). Since \( \omega \in S_k(\Lambda) \), one has \( \nu \in S_{k-(n+1)}(\Lambda) \) so that \( \nu \in S_{i+1}(\Lambda) \) and \( \mu \simL i \beta \nu \). \( \square \)

Let \( V_i^\lambda(\Lambda) \) be the \( i \)-past equivalence classes of \( S_i(\Lambda) \). We denote by \( [\mu]_i \) the equivalence class of \( \mu \in S_i(\Lambda) \). For \( \nu \in S_{i+1}(\Lambda) \) and \( \alpha \in \Gamma_i^- (\nu) \), define a labeled edge from \( [\alpha \nu]_i \in V_i^\lambda(\Lambda) \) to \( [\nu]_{i+1} \in V_{i+1}^\lambda(\Lambda) \) labeled \( \alpha \). The set of such labeled edges are denoted by \( E_i^\lambda(\Lambda) \). Since \( S_{i+1}(\Lambda) \subset S_i(\Lambda) \), we have a natural map \( [\mu]_{i+1} \in V_{i+1}^\lambda(\Lambda) \rightarrow [\mu]_i \in V_i^\lambda(\Lambda) \) denoted by \( \iota_i^\lambda(\Lambda) \).

**Proposition 3.2** ([23]). \( \Omega^\lambda(\Lambda) = (V^\lambda(\Lambda), E^\lambda(\Lambda), \lambda^\lambda(\Lambda), \iota^\lambda(\Lambda)) \) defines a λ-graph system that presents Λ.

**Proof.** We will show that the local property of λ-graph system holds. For \( [\mu]_i \in V_i^\lambda(\Lambda) \) and \( [\mu]_{i+2} \in V_{i+2}^\lambda(\Lambda) \) with \( \mu \in S_i(\Lambda), \nu \in S_{i+2}(\Lambda) \), suppose that there exists a labeled edge from \( [\mu]_i \) to \( [\nu]_{i+1} \) labeled \( \alpha \in \Sigma \). Hence \( \alpha \nu \simL i \mu \). There exist an edge from \( [\alpha \nu]_{i+1} \) to \( [\nu]_{i+2} \) labeled \( \alpha \) and an \( \iota \)-map from \( [\alpha \nu]_{i+1} \) to \( [\alpha \nu]_i \). On the other hand, suppose that there exist an \( \iota \)-map from \( [\nu]_{i+1} \) to \( [\mu]_i \) and an edge from \( [\nu]_{i+1} \) to \( [\nu]_{i+2} \) labeled \( \alpha \) so that \( \omega \simL i \alpha \nu \). Since \( \iota_i^\lambda(\Lambda)([\alpha \nu]_{i+1}) = [\alpha \nu]_i \), one has \( \mu \simL i \alpha \nu \). Hence there exists an edge from \( [\mu]_i \) to \( [\nu]_{i+1} \) labeled \( \alpha \). Therefore the local property of λ-graph system holds. By definition of λ-synchronization of Λ, an admissible word in Λ appears in \( \Omega^\lambda(\Lambda) \) as a labeled edge. Hence \( \Omega^\lambda(\Lambda) \) presents Λ. \( \square \)

We call \( \Omega^\lambda(\Lambda) \) the canonical λ-synchronizing λ-graph system of Λ. It is direct to see that \( \Omega^\lambda(\Lambda) \) is left-resolving and predecessor-separated.

Let \( \mathcal{L} = (V, E, \lambda, \iota) \) be a λ-graph system over Σ that presents a subshift Λ.

**Definition.** A λ-graph system \( \mathcal{L}' = (V', E', \lambda', \iota') \) over \( \Sigma' \) is called a λ-graph subsystem of \( \mathcal{L} \) if \( \Sigma' \subset \Sigma \) and the following conditions hold for \( l \in \mathbb{Z}_+ \):

\[ V_l' \subset V_l, \quad E_{l,l+1}' \subset E_{l,l+1}, \quad \lambda_{l,l+1}' = \lambda_{l,l+1}|E_{l,l+1}, \quad \iota_{l,l+1}' = \iota_{l,l+1}|E_{l,l+1}. \]
Corollary 3.3. \( \mathcal{L}^{\lambda(\Lambda)} \) is a \( \lambda \)-graph subsystem of \( \mathcal{L}^{\Lambda} \).

Proof. Let \( \mathcal{L}^{\Lambda} = (V^{\Lambda}, E^{\Lambda}, \lambda^{\Lambda}, \iota^{\Lambda}) \) be the canonical \( \lambda \)-graph system for \( \Lambda \). An \( \iota \)-synchronizing word \( \mu \in S_i(\Lambda) \) satisfies \( \Gamma_{\overline{i}}(\mu) = \Gamma_{\overline{i}}(\mu x) \) for \( x \in \Gamma_{\overline{m}}(\mu) \). Hence \( [\mu]_i \) naturally defines a vertex of \( V_i^{\Lambda} \) so that the vertex set \( V_i^{\lambda(\Lambda)} \) is regarded as a subset of \( V_i^{\Lambda} \). Similarly the edge set \( E_{i,l+1}^{\lambda(\Lambda)} \) is regarded as a subset of \( E_{i,l+1}^{\Lambda} \). The \( \iota \) map \( \iota^{\lambda(\Lambda)} \) of \( \mathcal{L}^{\lambda(\Lambda)} \) is obtained by restricting the \( \iota \)-map \( \iota^{\Lambda} \) of \( \mathcal{L}^{\Lambda} \). Therefore \( \mathcal{L}^{\lambda(\Lambda)} \) is a \( \lambda \)-graph subsystem of \( \mathcal{L}^{\Lambda} \).

We will characterize the canonical \( \lambda \)-synchronizing \( \lambda \)-graph system in an intrinsic way. Let \( \mathcal{L} = (V, E, \lambda, \iota) \) be a left-resolving, predecessor-separated \( \lambda \)-graph system over \( \Sigma \) that presents a subshift \( \Lambda \). Denote by \( \{v_1^{l}, \ldots , v_m^{l}\} \) the vertex set \( V_l \) at level \( l \). For an admissible word \( \nu \in B_n(\Lambda) \) and a vertex \( v_i^{l} \in V_l \), we say that \( v_i^{l} \) launches \( \nu \) if the following two conditions hold:

(i) There exists a path labeled \( \nu \) in \( \mathcal{L} \) leaving \( v_i^{l} \) and ending at a vertex in \( V_{l+n} \).

(ii) The word \( \nu \) does not leave any other vertex in \( V_l \) than \( v_i^{l} \).

The vertex \( v_i^{l} \) is called the launching vertex for \( \nu \).

Definition. A \( \lambda \)-graph system \( \mathcal{L} \) is said to be \( \lambda \)-synchronizing if for any \( l \in \mathbb{Z}_+ \) and any vertex \( v_i^{l} \in V_l \), there exists a word \( \nu \in B_n(\Lambda) \) such that \( v_i^{l} \) launches \( \nu \). We set

\[
S_{v_i^{l}}(\Lambda) = \{ \nu \in B_n(\Lambda) \mid v_i^{l} \text{ launches } \nu \}.
\]

Lemma 3.4. Keep the above notations. Assume that \( \mathcal{L} = (V, E, \lambda, \iota) \) is \( \lambda \)-synchronizing. Then we have:

(i) \( \cap_{i=1}^{m(l)} S_{v_i^{l}}(\Lambda) = S_i(\Lambda) \).

(ii) The \( l \)-past equivalence classes of \( S_i(\Lambda) \) is \( S_{v_i^{l}}(\Lambda), i = 1, \ldots , m(l) \).

(iii) For any \( l \)-synchronizing word \( w \in S_i(\Lambda) \), there exists a vertex \( v_i^{l}(\omega) \in V_l \) such that \( v_i^{l}(\omega) \) launches \( \omega \) and \( \Gamma_{\overline{i}}(\omega) = \Gamma_{\overline{i}}(v_i^{l}(\omega)) \).

Proof. (i) The inclusion relation \( S_{v_i^{l}}(\Lambda) \subset S_i(\Lambda) \) is obvious. We will show that \( \cap_{i=1}^{m(l)} S_{v_i^{l}}(\Lambda) \supset S_i(\Lambda) \). For \( \mu \in S_i(\Lambda) \), suppose that \( \mu \) leaves two vertices \( v_i^{l}, v_j^{l} \in V_l \), for \( i \ne j \). Since \( \mathcal{L} \) is left-resolving, there exist two distinct terminal vertices \( v_i^{l+n}, v_j^{l+n} \in V_{l+n} \) with \( i \ne j \) of the labeled paths labeled \( \mu \). As \( \mathcal{L} \) is \( \lambda \)-synchronizing, for the vertices \( v_i^{l+n}, v_j^{l+n} \in V_{l+n} \) there exist admissible words \( \nu(i'), \nu(j') \in B_s(\Lambda) \) such that \( v_i^{l+n} \) launches \( \nu(i') \) and \( v_j^{l+n} \) launches \( \nu(j') \). As \( \nu(i') \) does not leave any other vertex than \( v_i^{l+n} \) in \( V_{l+n} \), one has \( \nu(i') \ne \nu(j') \). One knows then that

\[
\Gamma_{\overline{i}}(v_i^{l}) = \Gamma_{\overline{i}}(\mu \nu(i')), \quad \Gamma_{\overline{j}}(v_j^{l}) = \Gamma_{\overline{j}}(\mu \nu(j')).
\]

Since \( \mathcal{L} \) is predecessor-separated, one has

\[
\Gamma_{\overline{i}}(\mu \nu(i')) \ne \Gamma_{\overline{i}}(\mu \nu(j'))
\]

and a contradiction to the hypothesis that \( \mu \) is a \( l \)-synchronizing word.

(ii) For \( \mu \in S_{v_i^{l}}(\Lambda), \nu \in S_{v_j^{l}}(\Lambda) \) with \( i \ne j \), one has

\[
\Gamma_{\overline{i}}(\mu) = \Gamma_{\overline{i}}(v_i^{l}) \ne \Gamma_{\overline{j}}(v_j^{l}) = \Gamma_{\overline{j}}(\nu)
\]

and hence \( [\mu]_i \ne [\nu]_j \). Conversely for \( \omega, \zeta \in S_i(\Lambda) \) with \( [\omega]_i \ne [\zeta]_j \), as in the above discussion, there uniquely exists a vertex \( v_i^{l}(\omega) \in V_l \) such that \( \omega \) leaves \( v_i^{l}(\omega) \). This
Lemma 3.6. Conversely, we have irreducible. For \( LV \), then synchronizing, for two vertices \( \pi \) and \( \kappa \) such that \( \lambda \) is irreducible, there exists a word \( \zeta \). Since 
\[
\Gamma^- (v^i_{(o)}) = \Gamma^- (\omega), \quad \Gamma^- (v^i_{(f)}) = \Gamma^- (\zeta)
\]
and \( \Gamma^- (\omega) \neq \Gamma^- (\zeta) \), we have \( i(\omega) \neq i(\zeta) \).

(iii) The assertion is now clear from the above discussions. \( \square \)

Definition. A \( \lambda \)-graph system \( \mathcal{L} = (V, E, \lambda, \iota) \) is said to be \( \iota \)-irreducible if for any two vertices \( v, u \in V_l \) and a labeled path \( \gamma \) starting at \( u \), there exist a labeled path from \( v \) to a vertex \( u' \in V_l + n \) such that \( \nu^n(u') = u \), and a labeled path \( \gamma' \) starting at \( u' \) such that \( \nu^n(t(\gamma')) = t(\gamma) \) and \( \lambda(\gamma') = \lambda(\gamma) \), where \( t(\gamma') \), \( t(\gamma) \) denote the terminal vertices of \( \gamma' \), \( \gamma \) respectively and \( \lambda(\gamma') \), \( \lambda(\gamma) \) the words labeled by \( \gamma' \), \( \gamma \) respectively.

A finite directed labeled graph \( G \) is irreducible as a directed graph if and only if the \( \lambda \)-graph system \( \mathcal{L}_G \) is \( \iota \)-irreducible.

Lemma 3.5. Let \( \mathcal{L} = (V, E, \lambda, \iota) \) be a \( \lambda \)-graph system that presents a subshift \( \Lambda \). If \( \mathcal{L} \) is \( \iota \)-irreducible, then \( \Lambda \) is irreducible.

Proof. For \( \mu, \nu \in B_\nu(\Lambda) \), put \( k = |\mu|, l = |\nu| \). Take a labeled path labeled \( \nu \) from a vertex in \( V_0 \) to a vertex \( v \) in \( V_l \). Take a labeled path labeled \( \mu \) from a vertex \( u \) in \( V_l \) to a vertex in \( V_{l+k} \). Since \( \mathcal{L} \) is \( \iota \)-irreducible, there exists a labeled path \( \pi \) from \( v \) to a vertex \( u' \) such that the word \( \mu \) leaves \( u' \). Denote by \( \omega \) the word of the path \( \pi \). We then have a labeled path which presents the word \( \nu \omega \mu \in B_\nu(\Lambda) \). Hence \( \Lambda \) is irreducible.

Conversely, we have

Lemma 3.6. Assume that \( \mathcal{L} = (V, E, \lambda, \iota) \) is \( \lambda \)-synchronizing. If \( \Lambda \) is irreducible, then \( \mathcal{L} \) is \( \iota \)-irreducible.

Proof. For two vertices \( v, u \in V_l \) and a labeled path \( \gamma \) starting at \( u \), put \( \mu = \lambda(\gamma) \) and \( k = |\mu| \). Let \( w \in V_{l+k} \) denote the terminal vertex \( t(\gamma) \) of \( \gamma \). Since \( \mathcal{L} \) is \( \lambda \)-synchronizing, \( w \) is a launching vertex for a word \( \eta \in B_\nu(\Lambda) \). Similarly \( v \) is a launching vertex for a word \( \zeta \in B_\nu(\Lambda) \). Put \( n = |\zeta| \). By the hypothesis that \( \Lambda \) is irreducible, there exists a word \( \xi \in B_\nu(\Lambda) \) such that \( \xi \mu \eta \in B_\nu(\Lambda) \). Since the word \( \zeta \) must leave \( v \) in \( V_l \), any labeled path labeled \( \xi \mu \eta \) must leave \( v \) in \( V_l \). Put \( m = |\zeta| \). Let \( u' \in V_{l+n+m} \) be a vertex in \( V_{l+n+m} \) at which the word \( \zeta \) ends and the word \( \mu \eta \) starts. Denote by \( \gamma' \) the labeled path labeled \( \mu \) starting at \( u' \). Let \( w' \) be the terminal vertex of \( \gamma' \). By the local property of \( \lambda \)-graph system, there exists a labeled path labeled \( \mu \) which starts at \( \nu^n+m(u') \in V_l \) and ends at \( \nu^{n+m}(w') \in V_{l+k} \), and there exists a labeled path labeled \( \eta \) starting at \( \nu^{n+m}(w') \). As \( w \) is a launching vertex for \( \eta \) and \( \mathcal{L} \) is left-resolving, we have \( \nu^{n+m}(w') = w \) and \( \nu^{n+m}(w') = u' \). This implies that \( \mathcal{L} \) is \( \iota \)-irreducible.

Therefore we have

Proposition 3.7. Let \( \mathcal{L} \) be a \( \lambda \)-synchronizing \( \lambda \)-graph system that presents a subshift \( \Lambda \). Then \( \Lambda \) is irreducible if and only if \( \mathcal{L} \) is \( \iota \)-irreducible.

Proposition 3.8. A subshift \( \Lambda \) is \( \lambda \)-synchronizing if and only if there exists a left-resolving, predecessor-separated, \( \iota \)-irreducible, \( \lambda \)-synchronizing \( \lambda \)-graph system that presents \( \Lambda \).
Proof. Let $\mathcal{L}$ be a left-resolving, predecessor-separated, $\iota$-irreducible, $\lambda$-synchronizing $\lambda$-graph system that presents $\Lambda$. For any $\eta \in B_2(\Lambda)$ and $l \in \mathbb{N}$ with $k \leq l$, there exists a terminal vertex $v^l_\eta \in V_l$ of a path labeled $\eta$. $\lambda$-synchronization of $\mathcal{L}$ implies $S_{v^l_\eta}(\Lambda) \neq \emptyset$. Take a word $\mu \in S_{v^l_\eta}(\Lambda)$. As $S_{v^l_\eta}(\Lambda) \subset S_l(\Lambda)$, we have $\mu \in S_l(\Lambda)$ and $\eta \mu \in B_l(\Lambda)$. By the previous lemma, $\Lambda$ is irreducible. Hence $\Lambda$ is $\lambda$-synchronizing.

Conversely suppose that $\Lambda$ is $\lambda$-synchronizing. The canonical $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^{\lambda}(\Lambda)$ for $\Lambda$ is a left-resolving, predecessor-separated, $\lambda$-synchronizing $\lambda$-graph system that presents $\Lambda$. As $\Lambda$ is irreducible, the $\lambda$-synchronization of $\mathcal{L}$ implies that $\mathcal{L}^{\lambda}(\Lambda)$ is $\iota$-irreducible. \hfill \Box

Theorem 3.9. For a $\lambda$-synchronizing subshift $\Lambda$, there uniquely exists a left-resolving, predecessor-separated, $\iota$-irreducible, $\lambda$-synchronizing $\lambda$-graph system that presents $\Lambda$. The unique $\lambda$-synchronizing $\lambda$-graph system is the canonical $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^{\lambda}(\Lambda)$ for $\Lambda$.

Proof. We will prove that the canonical $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^{\lambda}(\Lambda)$ for $\Lambda$ is a unique left-resolving, predecessor-separated, $\iota$-irreducible, $\lambda$-synchronizing $\lambda$-graph system that presents $\Lambda$. Let $\mathcal{L} = (V, E, \lambda, \iota)$ be a $\lambda$-graph system satisfying these properties. For $u \in V_{l}^{\lambda}(\Lambda)$, take a word $\mu(u) \in S_l(\Lambda)$ such that $u = [\mu(u)]_l \in V_{l}^{\lambda}(\Lambda)$. Since $\mathcal{L}$ is $\lambda$-synchronizing, there exists a unique vertex $v^l_{\mu(u)} \in V_l$ that launches $\mu(u)$ and satisfies $\Gamma^{-}_{l}(\mu(u)) = \Gamma^{-}_{l}(v^l_{\mu(u)})$. Define $\Phi_{V} : V_{l}^{\lambda}(\Lambda) \rightarrow V_l$ by $\Phi_{V}(u) = v^l_{\mu(u)}$. We will show that $\Phi_{V}$ yields an isomorphism between $\mathcal{L}$ and $\mathcal{L}^{\lambda}(\Lambda)$ as in the following way.

1. Well-definedness of $\Phi_{V}$: Let $\mu'(u) \in S_l(\Lambda)$ be another word such as $u = [\mu'(u)]_l$. One then has $\Gamma^{-}_{l}(\mu(u)) = \Gamma^{-}_{l}(\mu'(u))$. Since $\mathcal{L}$ is predecessor-separated, one sees $v^l_{\mu(u)} = v^l_{\mu'(u)}$ in $V_l$.

2. $\iota \circ \Phi_{V} = \Phi_{V} \circ \iota^{\lambda}(\Lambda)$: For $w \in V_{l+1}^{\lambda}(\Lambda)$, put $w' = \iota_{l+1}(\Lambda)(w) \in V_{l+1}^{\lambda}(\Lambda)$. Take a word $\mu(w) \in S_{l+1}(\Lambda)$ such that $w = [\mu(w)]_{l+1} \in V_{l+1}^{\lambda}(\Lambda)$. Let $v^{l+1}_{\mu(w)} \in V_{l+1}$ be the launching vertex for $\mu(w)$ so that $\Gamma^{-}_{l+1}(\mu(w)) = \Gamma^{-}_{l+1}(v^{l+1}_{\mu(w)})$. Take a word $\mu(w') \in S_l(\Lambda)$ such that $w' = [\mu(w')]_l$ so that $[\mu(w)]_l = [\mu(w')]_l$ and $\Gamma^{-}_{l}(\mu(w)) = \Gamma^{-}_{l}(\mu(w'))$. Hence we have $\Phi_{V} \circ \iota^{\lambda}(\Lambda)(w) = \Phi_{V}(w') = v^{l+1}_{\mu(w')}$. On the other hand, one knows that $\iota \circ \Phi_{V}(w) = \iota(v^{l+1}_{\mu(w)})$. By the local property of $\lambda$-graph system, $\mu(w)$ leaves the vertex $\iota(v^{l+1}_{\mu(w)}).$ As $\mathcal{L}$ is $\lambda$-synchronizing, $\iota(v^{l+1}_{\mu(w)})$ is the unique vertex which $\mu(w)$ leaves. Hence $\iota(v^{l+1}_{\mu(w)})$ is the launching vertex in $V_l$ for $\mu(w)$. Since $\Gamma^{-}_{l}(\mu(w)) = \Gamma^{-}_{l}(\mu(w'))$, one sees that $\iota(v^{l+1}_{\mu(w)})$ is the launching vertex in $V_l$ for $\mu(w')$ so that $v^{l+1}_{\mu(w')} = \iota(v^{l+1}_{\mu(w)}).$ This means $\Phi_{V} \circ \iota^{\lambda}(\Lambda)(w) = \iota \circ \Phi_{V}(w)$.

3. Injectivity of $\Phi_{V}$: Suppose that $u, u' \in V_{l}^{\lambda}(\Lambda)$ satisfy $\Phi_{V}(u) = \Phi_{V}(u')$. Take $\mu(u), \mu(u') \in S_l(\Lambda)$ such that $u = [\mu(u)]_l, u' = [\mu(u')]_l$. Let $v^{l}_{\mu(u)}, v^{l}_{\mu(u')}$ be the launching vertices in $V_l$ for $\mu(u), \mu(u')$ respectively so that $\Gamma^{-}_{l}(v^{l}_{\mu(u)}) = \Gamma^{-}_{l}(\mu(u)), \Gamma^{-}_{l}(v^{l}_{\mu(u')}) = \Gamma^{-}_{l}(\mu(u')).$ Since $\Phi_{V}(u) = \Phi_{V}(u')$, one sees that $v^{l}_{\mu(u)} = v^{l}_{\mu(u')}$. So that $\Gamma^{-}_{l}(\mu(u)) = \Gamma^{-}_{l}(\mu(u'))$ and $[\mu(u)]_l = [\mu(u')]_l$. Hence we have $u = u'.$

4. Surjectivity of $\Phi$: For a vertex $v^l_\eta \in V_l$, take a word $\nu \in S_{v^l_\eta}(\Lambda)$ such that $v^l_\eta$ launches $\nu$. By Lemma 3.4, one sees $\nu \in S_l(\Lambda)$. Hence $\nu$ defines a vertex $[\nu]_l$ in $V_{l}^{\lambda}(\Lambda)$. By definition, one has $\Phi_{V}([\nu]_l) = v^l_\eta$ so that $\Phi_{V}$ is surjective.
5. Existence of edge map $\Phi_E$ : For $e \in E_{l+1}^{\Lambda(\alpha)}$, put $u = s(e) \in V_{l+1}^{\Lambda(\alpha)}$, $v = t(e) \in V_{l+1}^{\Lambda(\alpha)}$, $\alpha = \lambda(e) \in \Sigma$. Take a word $\nu \in S_{l+1}(\Lambda)$ such that $\nu$ starts at $v$. Put $\zeta = \alpha \nu \in S_l(\Lambda)$. As $\mathcal{E}$ is $\lambda$-synchronizing, there exist launching vertices $v^l_i \in V_i$ for $\zeta$ and $v^{l+1}_i \in V_{l+1}$ for $\nu$ such that $\Gamma_i^-(\zeta) = \Gamma_i^- (v^l_i) \in \mathcal{E}$ and $\Gamma_{l+1}^- (\nu) = \Gamma_{l+1}^- (v^{l+1}_i)$ respectively. Put $u' = \Phi_V(u) = v^l_i \in V_i$ and $v' = \Phi_V(v) = v^{l+1}_i \in V_{l+1}$. Since $\Gamma^-_{l+1}(\nu) = \Gamma^-_{l+1}(v')$ one has $\Gamma^-_{l+1}(v') = \Gamma^-_{l+1}(v)$. Hence, there exist $e' \in E_{l+1}$ in $\mathcal{E}$ such that $\lambda(e') = \alpha$ and $t(e') = v'$. We set $\Phi_E(e) = e'$. Since $\mathcal{E}$ is left-resolving, such $e'$ is unique. It then follows that
\[ \Phi_V(t(e)) = v' = t(e') = t(\Phi_E(e)), \quad \Phi_V(s(e)) = u' = s(e') = s(\Phi_E(e)). \]
Both $\mathcal{E}^{\Lambda(\alpha)}$ and $\mathcal{E}$ are left-resolving, the map $\Phi_E$ is injective. Surjectivity of $\Phi_E$ is easily shown so that $\mathcal{E}^{\Lambda(\alpha)}$ and $\mathcal{E}$ are isomorphic as $\lambda$-graph systems. \qed

We call $\mathcal{E}^{\Lambda(\alpha)}$ the $\lambda$-synchronizing $\lambda$-graph system for a $\lambda$-synchronizing subshift $\Lambda$.

**Definition.** A $\lambda$-graph system $\mathcal{E}$ is said to be *minimal* if there is no proper $\lambda$-graph subsystem of $\mathcal{E}$ that presents the same subshift presented by $\mathcal{E}$. This means that if $\mathcal{E}'$ is a $\lambda$-graph subsystem of $\mathcal{E}$ and presents the same subshift as the subshift presented by $\mathcal{E}$, then $\mathcal{E}'$ coincides with $\mathcal{E}$.

**Proposition 3.10.** For a $\lambda$-synchronizing subshift $\Lambda$, the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{E}^{\Lambda(\alpha)}$ is minimal.

**Proof.** Suppose that $\mathcal{E}' = (V', E', \lambda', \nu')$ be a $\lambda$-graph subsystem of $\mathcal{E}^{\Lambda(\alpha)}$ that presents $\Lambda$. Hence we have $V_i' \subset V_i^{\Lambda(\alpha)}$, $E_{l+1}' \subset E_{l+1}^{\Lambda(\alpha)}$ for all $l \in \mathbb{Z}_+$. Suppose that there exists a vertex $v^l_i \in V_i^{\Lambda(\alpha)}$ such that $v^l_i \notin V_i'$. By the $\lambda$-synchronizing of $\mathcal{E}^{\Lambda(\alpha)}$, there exists a synchronizing word $\mu \in S_l(\Lambda)$ such that $v^l_i$ launches $\mu$. There is no any other vertex in $V_i^{\Lambda(\alpha)}$ than $v^l_i$ which the word $\mu$ leaves. Hence the word $\mu$ does not appear in the presentation of $\mathcal{E}'$, a contradiction. Therefore we have $V_i' = V_i^{\Lambda(\alpha)}$ for all $l \in \mathbb{Z}_+$. We will next show that $E_{l+1}' = E_{l+1}^{\Lambda(\alpha)}$ for all $l \in \mathbb{Z}_+$. Take an arbitrary edge $e \in E_{l+1}^{\Lambda(\alpha)}$. Put $\alpha = \lambda(e) \in \Sigma$ and $v^l_i = s(e) \in V_i^{\Lambda(\alpha)}$, $v^{l+1}_j = t(e) \in V_{l+1}^{\Lambda(\alpha)}$. Take an $l+1$-synchronizing word $\nu \in S_{l+1}(\Lambda)$ such that $v^{l+1}_j$ launches $\nu$. Hence $\alpha \nu$ leaves $v^l_i$. Suppose that $\alpha \nu$ leaves a vertex $v^l_i$. Let $e' \in E_{l+1}^{\Lambda(\alpha)}$ be an edge labeled $\alpha$ such that $s(e') = v^l_i$. Since $v^{l+1}_j$ is the launching vertex for $\nu$, the word $\nu$ must leave $v^{l+1}_j$ so that $t(e') = t(e)$. As $\mathcal{E}^{\Lambda(\alpha)}$ is left-resolving, one has $e = e'$. Hence $v^l_i = v^l_j$. Hence $\alpha \nu$ must leave the vertex $v^l_i$. This means that $e$ is the only edge in $E_{l+1}$ whose label is the leftmost of the word $\alpha \nu$. Therefore $e \in E_{l+1}'$ and we have $E_{l+1}' = E_{l+1}^{\Lambda(\alpha)}$ for all $l \in \mathbb{Z}_+$. We thus conclude $\mathcal{E}' = \mathcal{E}^{\Lambda(\alpha)}$. \qed

4. $\lambda$-Synchronization and Flow Equivalence

As in Proposition 2.2, $\lambda$-synchronization is invariant under topological conjugacy. In this section we will prove that $\lambda$-synchronization is invariant even under flow equivalence. Parry-Sullivan showed that the flow equivalence relation on homeomorphisms of Cantor sets is generated by topological conjugacy and expansion of symbols ([36]). Let $\Lambda$ be a subshift over alphabet $\Sigma = \{1, 2, \ldots, N\}$. We define a new subshift $\bar{\Lambda}$ over the alphabet $\bar{\Sigma} = \{0, 1, 2, \ldots, N\}$ as the subshift consisting
of all biinfinite sequences of $\tilde{\Sigma}$ obtained by replacing the symbol 1 in a biinfinite sequence in the subshift $\Lambda$ by the word 01. This operation is called expansion that corresponds to the equivalence relation called Kakutani equivalence. An argument in [36, Proposition] (cf. [28, Lemma 2.1]) says:

**Lemma 4.1** ([36]). Flow equivalence relation of subshifts is generated by topological conjugacy and the expansion $\Lambda \to \tilde{\Lambda}$.

For a subshift $\Lambda$ over $\Sigma$, recall that $X_\Lambda \subset \Sigma^\mathbb{N}$ is defined by the set of all right one-sided sequences $(x_i)_{i \in \mathbb{N}} \in \Sigma^\mathbb{N}$ such that $(x_i)_{i \in \mathbb{Z}} \in \Lambda$, and $X_\Lambda \subset \Sigma^\mathbb{N}$ is similarly defined. We set for $l \in \mathbb{N}$,

$$B_{1,l}(\Lambda) = \{\mu_1 \cdots \mu_l \in B_l(\Lambda) \mid \mu_1 = 1\}, \quad B_{1,\ast}(\Lambda) = \cup_{l=1}^\infty B_{1,l}(\Lambda),$$
$$B_{1,l}(\tilde{\Lambda}) = \{\nu_1 \cdots \nu_l \in B_l(\tilde{\Lambda}) \mid \nu_1 = 1\}, \quad B_{1,\ast}(\tilde{\Lambda}) = \cup_{l=1}^\infty B_{1,l}(\tilde{\Lambda}),$$
$$B_{1,0}(\tilde{\Lambda}) = \{\nu_1 \cdots \nu_l \in B_l(\tilde{\Lambda}) \mid \nu_l = 0\}, \quad B_{\ast,0}(\tilde{\Lambda}) = \cup_{l=1}^\infty B_{l,0}(\tilde{\Lambda}),$$
$$B_{1,l,0}(\tilde{\Lambda}) = B_{1,l}(\tilde{\Lambda}) \cup B_{l,0}(\tilde{\Lambda}), \quad B_{1,\ast,0}(\tilde{\Lambda}) = \cup_{l=1}^\infty B_{1,l,0}(\tilde{\Lambda}).$$

Define

$$\xi^B : B_\ast(\Lambda) \longrightarrow B_\ast(\tilde{\Lambda}) \setminus B_{1,\ast,0}(\tilde{\Lambda})$$

by putting the word 01 in place of 1 in the words $\mu \in B_\ast(\Lambda)$ from the left in order such as

$$\xi^B(112121321) = 01012012013201.$$

The symbol 1 and the symbol 0 can never appear in the leftmost and in the rightmost of the form $\xi^B(\mu)$ for $\mu \in B_\ast(\Lambda)$ respectively so that we have

$$\xi^B(B_\ast(\Lambda)) \cap B_{1,\ast,0}(\tilde{\Lambda}) = \emptyset.$$

We write $\xi^B(\mu)$ as $\bar{\mu}$ for brevity. Define

$$\eta^B : B_\ast(\tilde{\Lambda}) \setminus B_{1,\ast,0}(\tilde{\Lambda}) \longrightarrow B_\ast(\Lambda)$$

by putting 1 in place of 01 in the words $\nu \in B_\ast(\tilde{\Lambda}) \setminus B_{1,\ast,0}(\tilde{\Lambda})$ from the left in order such as

$$\eta^B(01012012013201) = 112121321.$$

We write $\eta^B(\nu)$ as $\bar{\nu}$ for brevity. Hence we have

$$\eta^B \circ \xi^B = \text{id}_{B_\ast(\Lambda)}, \quad \xi^B \circ \eta^B = \text{id}_{B_\ast(\tilde{\Lambda}) \setminus B_{1,\ast,0}(\tilde{\Lambda})}.$$

In the set $S_l(\Lambda)$ of $l$-synchronizing words, put

$$S_{1,l}(\Lambda) = \{\mu_1 \cdots \mu_n \in S_l(\Lambda) \mid \mu_1 = 1\}.$$

We similarly use the notation $S_l(\tilde{\Lambda})$ for the subshift $\tilde{\Lambda}$ as the set of $l$-synchronizing words of $\tilde{\Lambda}$. Put

$$S_{1,l}(\tilde{\Lambda}) = \{\nu_1 \cdots \nu_n \in S_l(\tilde{\Lambda}) \mid \nu_1 = 1\},$$
$$S_{1,0}(\tilde{\Lambda}) = \{\nu_1 \cdots \nu_n \in S_l(\tilde{\Lambda}) \mid \nu_n = 0\},$$
$$S_{1,l,0}(\tilde{\Lambda}) = S_{1,l}(\tilde{\Lambda}) \cup S_{l,0}(\tilde{\Lambda}).$$

**Lemma 4.2.** Assume that $\Lambda$ is irreducible.

(i) $\xi^B : B_\ast(\Lambda) \longrightarrow B_\ast(\tilde{\Lambda}) \setminus B_{1,\ast,0}(\tilde{\Lambda})$ induces a map $\xi^S_l : S_l(\Lambda) \longrightarrow S_l(\tilde{\Lambda}) \setminus S_{1,l,0}(\tilde{\Lambda})$. 

Hence we have

\[ \xi \]

Define such as that we have

\[ \eta \]

(i) Put

\[ X_{A_1} = \{(x_1, x_2, \cdots) \in X_A | x_1 = 1 \}, \quad X_{A_1}^- = \{(y_1, y_2, \cdots) \in X_A^- | y_1 = 1 \}. \]

We will prove that if \( \eta \in B_1(\overline{\Lambda}) \) is not 01 by 1 in \( \Lambda \), then \( \xi = \xi_l(\mu) \) is an l-synchronizing word in \( \Lambda \). For \( \eta \in X_{A_1}^- \) with \( y \notin 1 \) from the left in order such as

\[ \eta(0, 1, 0, 1, 2, 0, 1, 0, 1, 3, 2, 0, 1, 0, 1, 2, 0, \cdots) = (1, 1, 2, 1, 3, 2, 1, 1, 2, \cdots). \]

Hence we have

\[ \eta \circ \xi = \text{id}_{X_A}, \quad \xi \circ \eta = \text{id}_{X_A^- \backslash X_{A_1}^-}. \]

We will prove that if \( \mu \in B_1(\Lambda) \) is an l-synchronizing word in \( \Lambda \), the word \( \tilde{\mu} = \xi_l(\mu) \) is an l-synchronizing word in \( \Lambda \). For \( \eta \in X_{A_1}^- \) with \( y \in \Gamma_\infty(\tilde{\mu}) \), we will show that

\[ \Gamma_\infty^-(\tilde{\mu}) \subset \Gamma_\infty^- (\tilde{\mu}y). \]

Take an arbitrary word \( \omega = \omega_1 \cdots \omega_l \in \Gamma_\infty^- (\tilde{\mu}). \)

Case 1: \( \omega_1 \neq 1 \).

Since the leftmost of \( \omega \tilde{\mu} \) is not 1 and the rightmost of \( \omega \tilde{\mu} \) is not 0, replace the symbol 01 by 1 in \( \omega \tilde{\mu} \), one has \( \eta^B(\omega \tilde{\mu}) \in B_1(\Lambda) \). Since the leftmost of \( \tilde{\mu} \) is not 1, the rightmost of \( \omega \tilde{\mu} \) is not 0 so that \( \omega \notin B_1(\Lambda) \). It follows that

\[ \eta^B(\omega \tilde{\mu}) = \eta^B(\omega) \eta^B(\tilde{\mu}) = \eta^B(\omega) \mu. \]

The leftmosts of both \( \tilde{\mu} \) and \( y \) are not 1. Hence one has

\[ \eta(\tilde{\mu}y) = \eta^B(\tilde{\mu}) \eta(y) = \mu \eta(y). \]

Now \( \mu \) is l-synchronizing in \( \Lambda \) and \( \eta^B(\omega) \in \Gamma_\infty^-(\mu) \) with \( |\eta^B(\omega)| \leq l \). We have \( \eta^B(\omega) \mu \eta(y) \in X_{A_1}^- \). As \( \eta^B(\omega) \mu \eta(y) = \eta^B(\omega) \eta^B(\tilde{\mu}) \eta(y) \), it follows taht

\[ \omega \tilde{\mu} y = \xi(\eta^B(\omega) \eta^B(\tilde{\mu}) \eta(y)) \in X_{A_1}^- \backslash X_{A_1}^- \]

so that \( \omega \tilde{\mu} y \in X_{A_1}^- \) and hence \( \omega \in \Gamma_\infty^- (\tilde{\mu}y) \).

Case 2: \( \omega_1 = 1 \).

For \( \omega = \omega_1 \cdots \omega_l \in \Gamma_\infty^- (\tilde{\mu}) \), consider \( \omega' = 0 \omega_1 \cdots \omega_l \in B_{l+1}(\Lambda) \). Since \( |\eta^B(\omega')| \leq l \), one may apply the above discussion for \( \omega' \) so that \( \omega' \tilde{\mu} y \in X_{A_1}^- \). Hence \( \omega \in \Gamma_\infty^- (\tilde{\mu} y) \) and

\[ \Gamma_\infty^- (\tilde{\mu}) \subset \Gamma_\infty^- (\tilde{\mu}y) \quad \text{for all} \ y \in X_{A_1}^- \text{with} \ y \in \Gamma_\infty^+(\tilde{\mu}). \]

Therefore \( \tilde{\mu} \) is l-synchronizing in \( \overline{\Lambda} \) and \( \xi^B : B_1(\Lambda) \rightarrow B_1(\overline{\Lambda}) \) induces a map

\[ \xi^B : S_1(\Lambda) \rightarrow S_1(\overline{\Lambda}) \]

(ii) For \( \nu \in B_1(\overline{\Lambda}) \), put \( \tilde{\nu} = \eta^B(\nu) \). Suppose that \( \nu \) is 2l-synchronizing in \( \Lambda \). We will show that \( \tilde{\nu} \) is l-synchronizing in \( \Lambda \). For \( \gamma \in \Gamma_\infty^- (\tilde{\nu}) \), one sees

\[ \tilde{\nu} \]
that $\xi^B(\gamma\check{\nu}) \in B_*(\tilde{\Lambda}) \setminus B_{1,*}(\tilde{\Lambda})$. Since $\nu \not\in B_{1,*}(\tilde{\Lambda})$, one has $\xi^B(\check{\nu}) = \nu$ so that $\xi^B(\gamma)\nu \in B_*(\tilde{\Lambda})$. As $|\gamma| = l$, one has $|\xi^B(\gamma)| \leq 2l$. For $x \in \Gamma^+_\nu(\check{\nu})$, one has

$$\xi(\check{\nu}x) = \nu \xi(x) \in X_{\tilde{\Lambda}} \setminus X_{\check{\Lambda}}.$$  

Since $\nu$ is $2l$-synchronizing in $\tilde{\Lambda}$ and $\xi^B(\gamma)\nu \in B_*(\tilde{\Lambda})$ with $|\xi^B(\gamma)| \leq 2l$, we have $\xi^B(\gamma)\nu \xi(x) \in X_{\tilde{\Lambda}}$. As the leftmost of $\xi^B(\gamma)$ is not 1, we have $\xi^B(\gamma)\nu \xi(x) \in X_{\tilde{\Lambda}} \setminus X_{\check{\Lambda}}$. Hence we have

$$\gamma\check{\nu}x = \eta(\xi(\gamma)\nu \xi(x)) \in X_{\tilde{\Lambda}}$$

so that $\gamma \in \Gamma^-_{\tilde{\nu}}(\check{\nu}x)$. Therefore we have

$$\Gamma^-_{\tilde{\nu}}(\check{\nu}) \subseteq \Gamma^-_{\nu}(\check{\nu}x)$$

so that $\check{\nu}$ is $l$-synchronizing in $\Lambda$. We thus have $\eta^B : B_*(\tilde{\Lambda}) \setminus B_{1,*}(\tilde{\Lambda}) \to B_*(\Lambda)$ induces a map

$$\eta_S : S_{2l}(\tilde{\Lambda}) \setminus S_{1,2l}(\tilde{\Lambda}) \to S_l(\Lambda).$$

We note that an irreducible subshift $\Lambda$ is $\lambda$-synchronizing if and only if for $\mu \in B_*(\Lambda)$ and $k \in \mathbb{N}$ there exists $\nu \in S_k(\Lambda)$ such that $\mu\nu \in B_*(\Lambda)$.

**Proposition 4.3.** An irreducible subshift $\Lambda$ is $\lambda$-synchronizing if and only if so is $\tilde{\Lambda}$.

**Proof.** It is easy to see that $\Lambda$ is irreducible if and only if so is $\tilde{\Lambda}$. Suppose that $\Lambda$ is $\lambda$-synchronizing. For $\mu = \mu_1 \cdots \mu_l \in B_*(\tilde{\Lambda})$ and $k \in \mathbb{N}$, we have three cases.

Case 1: $\mu_1 \neq 1$, $\mu_l \neq 0$.

As $\mu \in B_*(\tilde{\Lambda}) \setminus B_{1,*}(\tilde{\Lambda})$, by putting 1 in place of 01, we have $\tilde{\mu} = \eta^B(\mu) \in B_*(\Lambda)$. Since $\Lambda$ is $\lambda$-synchronizing, there exists $\nu \in S_k(\Lambda)$ such that $\tilde{\mu}\nu \in B_*(\Lambda)$. Put $\check{\nu} = \xi^B(\nu) \in S_k(\tilde{\Lambda})$ so that one sees

$$\mu\check{\nu} = \xi^B(\tilde{\mu})\xi^B(\nu) = \xi^B(\tilde{\mu}\nu) \in B_*(\tilde{\Lambda}).$$

Case 2: $\mu_1 = 1$, $\mu_l \neq 0$.

Consider the word $0\mu = 0\mu_1 \cdots \mu_l \in B_*(\tilde{\Lambda})$ so that $0\mu \in B_*(\tilde{\Lambda}) \setminus B_{1,*}(\tilde{\Lambda})$. By the above discussion of Case 1 for the word $0\mu$, there exists $\nu \in S_k(\Lambda)$ such that $0\mu\xi^B(\nu) \in B_*(\tilde{\Lambda})$. Put $\check{\nu} = \xi^B(\nu)$ so that $\mu\check{\nu} \in B_*(\tilde{\Lambda})$.

Case 3: $\mu_l = 0$.

Put $\mu_1 = \mu_1 \cdots \mu_l \in B_*(\tilde{\Lambda})$. As the rightmost of $\mu_1$ is not 0, by applying the above two cases to $\mu_1$ and $k + 1$, one finds $\tilde{\nu} \in S_k(\tilde{\Lambda})$ such that $\mu\tilde{\nu} \in B_*(\tilde{\Lambda})$. Put $\check{\nu} = 1\tilde{\nu} \in S_k(\Lambda)$ so that we have $\mu\check{\nu} \in B_*(\Lambda)$.

Therefore we conclude that $\tilde{\Lambda}$ is $\lambda$-synchronizing.

Conversely assume that $\Lambda$ is $\lambda$-synchronizing. For $\mu = \mu_1 \cdots \mu_l \in B_*(\Lambda)$ and $k \in \mathbb{N}$, put $\tilde{\mu} = \xi^B(\mu) \in B_*(\tilde{\Lambda}) \setminus B_{1,*}(\tilde{\Lambda})$. By $\lambda$-synchronization of $\Lambda$, for $\tilde{\mu}$ and $2k \in \mathbb{N}$, there exists $\nu' = \nu_1' \cdots \nu'_n \in S_{2k}(\Lambda)$ such that $\tilde{\mu}\nu' \in B_*(\tilde{\Lambda})$. If $\nu'_n = 0$, consider the word $\nu' = \nu'_1 \cdots \nu'_n \check{1}$ instead of $\nu'$ so that we may assume that $\nu'_n \neq 0$. As $\tilde{\mu}$ does not end at 0, $\nu'$ does not begin with 1 so that $\nu' \not\in B_{1,*}(\tilde{\Lambda})$. As both $\tilde{\mu}, \nu' \in B_*(\tilde{\Lambda}) \setminus B_{1,*}(\tilde{\Lambda})$, it follows that

$$\mu\eta^B(\nu') = \eta^B(\tilde{\mu}\nu') \in B_*(\Lambda).$$
By putting $\nu = \eta^B(\nu')$, one sees that $\nu \in S_k(\Lambda)$ and $\mu \nu \in B_*(\Lambda)$. This shows that $\Lambda$ is $\lambda$-synchronizing.

Therefore we conclude

**Theorem 4.4.** The $\lambda$-synchronization is invariant under flow equivalence of subshifts.

**Proof.** By Proposition 2.2 and Proposition 4.3, $\lambda$-synchronization is invariant under topological conjugacy and expansions of subshifts. Therefore by Lemma 4.1, $\lambda$-synchronization is invariant under flow equivalence of subshifts. \qed

5. **K-groups and Bowen-Franks groups**

In this section, we will prove that the $K$-groups and the Bowen-Franks groups for the $\lambda$-synchronizing $\lambda$-graph system for $\lambda$-synchronizing subshifts are invariant under expansion $\Lambda \rightarrow \tilde{\Lambda}$. The line of the proof basically follows the proof of [28, Theorem]. As a result, the groups yield invariants for flow equivalence of subshifts. Therefore by Lemma 4.1, $\lambda$-synchronizing Bowen-Franks groups are formulated by $\Lambda$. This shows that the $\lambda$-synchronizing $\lambda$-graphs are defined to be those groups for the $\lambda$-graph system $\mathfrak{A}(\lambda)$ ([23]). They are called the $\lambda$-synchronizing $K$-groups for $\Lambda$ and the $\lambda$-synchronizing Bowen-Franks groups of $\Lambda$ respectively. We will briefly describe them. Let $\lambda \beta (l)$ be the cardinal number of the vertex set $V^\lambda(\lambda)$ denoted by $\{v^1_\lambda, \ldots, v^\beta_{\lambda l}(t)\}$. Define $m_\lambda(l) \times m_\lambda(l+1)$ matrices $I_{\lambda}^\lambda(l, i, j)$ and $A_{\lambda}^\lambda(l, i, j)$ by setting

$$I_{\lambda}^\lambda(l, i, j) =\begin{cases} 1 & \text{if } I^\lambda_1(v^1_j l, i, j) = u^l_j, \\ 0 & \text{otherwise}, \end{cases}$$

$$A_{\lambda}^\lambda(l, i, j) =\text{ the number of the labeled edges from } v^l_i \text{ to } v^{l+1}_j$$

for $i = 1, \ldots, m_\lambda(l)$ and $j = 1, \ldots, m_\lambda(l+1)$. The sequence $(I_{\lambda}^\lambda(l, i, j), A_{\lambda}^\lambda(l, i, j), l \in \mathbb{Z}_+)$ of pairs of matrices becomes a nonnegative matrix system ([26]). The $\lambda$-synchronizing $K$-groups $K_\lambda(\lambda), i = 0, 1$ are defined as the $K$-groups $K_0(A^\lambda_{\lambda(l)}, I^\lambda_{\lambda(l)}), i = 0, 1$ for the nonnegative matrix system $(A^\lambda_{\lambda(l)}, I^\lambda_{\lambda(l)})$ that are formulated by

$$K^\lambda_0(\lambda) = \lim_{l \rightarrow +} \{ I^\lambda_{\lambda(l)} : \text{Coker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)) \rightarrow \text{Coker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)))\},$$

$$K^\lambda_1(\lambda) = \lim_{l \rightarrow +} \{ I^\lambda_{\lambda(l)} : \text{Ker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)) \rightarrow \text{Ker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)))\},$$

where

$$\text{Coker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)) = \mathbb{Z}^{m_\lambda(l+1)}/(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)), \mathbb{Z}^{m_\lambda(l)},$$

$$\text{Ker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)) = \text{Ker}(I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)) \in \mathbb{Z}^{m_\lambda(l)},$$

and $I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)$ is the natural homomorphism induced by $I^\lambda_{\lambda(l)}(I_{\lambda}^\lambda(l, i, j) - tA_{\lambda}^\lambda(l, i, j)) : \mathbb{Z}^{m_\lambda(l)} \rightarrow \mathbb{Z}^{m_\lambda(l+1)}$. Denote by $\mathbb{Z}^{\lambda(\lambda)}$ the projective limit $\lim_{l \rightarrow +} (I_{\lambda}^\lambda(l, i, j) : \mathbb{Z}^{m_\lambda(l+1)} \rightarrow \mathbb{Z}^{m_\lambda(l)})$ of abelian group. The sequence $A^\lambda_{\lambda(l), i, j}, l \in \mathbb{Z}_+$ of matrices acts on $\mathbb{Z}^{\lambda(\lambda)}$ as an endomorphism that we denote by $\beta^\lambda_{\lambda(\lambda)}$. The identity on $\mathbb{Z}^{\lambda(\lambda)}$ is denoted by $I$. The $\lambda$-synchronizing Bowen-Franks groups are formulated by

$$BF^0_{\lambda(\lambda)}(\lambda) = \mathbb{Z}^{\lambda(\lambda)}/(I - A^\lambda_{\lambda(\lambda)})\mathbb{Z}^{\lambda(\lambda)}, \quad BF^1_{\lambda(\lambda)}(\lambda) = \text{Ker}(I - A^\lambda_{\lambda(\lambda)}) \in \mathbb{Z}^{\lambda(\lambda)}.$$
The group $BF_0^0(\Lambda)\Lambda$ is computed from the $K$-groups $K^i(\Lambda), i = 0, 1$ by the universal coefficient type theorem ([26, Theorem 9.6]) described as

$$0 \to \text{Ext}^1_\Lambda(R_0, R_\Lambda) \overset{\delta}{\to} BF_0^0(\Lambda) \overset{\gamma}{\to} \text{Hom}_\Lambda(R_1, R_\Lambda) \to 0$$

that splits unnaturally. For the group $BF_1(\Lambda)\Lambda$, there exists an isomorphism ([26]):

$$BF_1(\Lambda)\Lambda \cong \text{Hom}_\Lambda(R_0, R_\Lambda).$$

We henceforth fix a $\lambda$-synchronizing subshift $\Lambda$ over alphabet $\Sigma = \{1, 2, \ldots, N\}$. Recall that $S_1(\Lambda)$ denotes the set of $l$-synchronizing words, and $S_{1,l}(\Lambda)$ denotes the subset $\{\mu_1 \cdots \mu_n \in S_1(\Lambda) | \mu_1 = 1\}$. Set

$$\Omega^l_1(\Lambda) = S_1(\Lambda)/\sim_l \quad \text{and} \quad \Omega^l_1(\Lambda_1) = S_{1,l}(\Lambda)/\sim_l$$

the $l$-past equivalence classes of $S_1(\Lambda)$ and $S_{1,l}(\Lambda)$ respectively. Similarly for the subshift $\tilde{\Lambda}$, set

$$\Omega^l_1(\tilde{\Lambda}) = S_1(\tilde{\Lambda})/\sim_l \quad \text{and} \quad \Omega^l_1(\tilde{\Lambda}_1) = S_{1,l}(\tilde{\Lambda})/\sim_l$$

the $l$-past equivalence classes of $S_1(\tilde{\Lambda})$ and $S_{1,l}(\tilde{\Lambda})$ respectively. Denote by $(S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}))/\sim_l$ the $l$-past equivalence classes of $S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda})$.

**Lemma 5.1.** $(S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}))/\sim_l = \Omega^l_1(\tilde{\Lambda})\setminus \Omega^l_1(\tilde{\Lambda}_1)$.

**Proof.** Since $S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}) \subset S_1(\tilde{\Lambda})$, one has

$$(S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}))/\sim_l \subset \Omega^l_1(\tilde{\Lambda}).$$

For $\mu = \mu_1 \cdots \mu_n \in S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda})$, as $\mu_1 \neq 1$ one has $\nu \not\in \Gamma_l^- (\mu)$ for all $\nu \in B_{l,0}(\tilde{\Lambda})$. Since for $\eta \in S_{1,l}(\tilde{\Lambda})$, one has $\nu \in \Gamma_l^- (\eta)$ for some $\nu \in B_{l,0}(\tilde{\Lambda})$. Hence $|\mu|_l \not\in \Omega^l_1(\Lambda_1)$ so that

$$(S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}))/\sim_l \subset \Omega^l_1(\tilde{\Lambda})\setminus \Omega^l_1(\tilde{\Lambda}_1).$$

On the other hand, for $|\mu|_l \in \Omega^l_1(\tilde{\Lambda})\setminus \Omega^l_1(\tilde{\Lambda}_1)$ with $\mu = \mu_1 \cdots \mu_n$, one has $\mu_1 \neq 1$. If $\mu_n = 0$, consider $\mu_1 = \mu_1 \cdots \mu_{n-1}$ so that $|\mu|_l = |\mu_1|_l$ and $\mu_1 \not\in S_{1,l,0}(\tilde{\Lambda})$. Hence $|\mu|_l \in (S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}))/\sim_l$. Therefore we have

$$(S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda}))/\sim_l = \Omega^l_1(\tilde{\Lambda})\setminus \Omega^l_1(\tilde{\Lambda}_1).$$

\[ \square \]

**Lemma 5.2.** Assume that $\Lambda$ is irreducible.

(i) $\xi^S_l : S_1(\Lambda) \to S_1(\tilde{\Lambda})\setminus S_{1,l,0}(\tilde{\Lambda})$ induces a map $\xi^l : \Omega^l_1(\Lambda) \to \Omega^l_1(\tilde{\Lambda})\setminus \Omega^l_1(\tilde{\Lambda}_1)$.

(ii) $\eta^S_l : S_2(\Lambda)\setminus S_{1,2,l,0}(\tilde{\Lambda}) \to S_1(\Lambda)$ induces a map $\eta^l : \Omega^l_2(\Lambda)\setminus \Omega^l_2(\Lambda_1) \to \Omega^l_1(\Lambda)$.

**Proof.** (i) We will prove that for $\mu, \nu \in B_*(\Lambda) \mu \sim_l \nu$ in $\Lambda$ implies $\bar{\mu} \sim_l \bar{\nu}$ in $\bar{\Lambda}$. Suppose that $\mu \sim_l \nu$ in $\Lambda$. Take an arbitrary word $\omega = \omega_1 \cdots \omega_l \in \Gamma_l^- (\mu)$. Case 1: $\omega_1 \neq 1$.

Since the leftmost of $\omega \bar{\mu}$ is not 1 and the rightmost of $\omega \bar{\mu}$ is not 0, replace the symbol 01 by 1 in $\omega \bar{\mu}$, one has $\eta^B(\omega \bar{\mu}) = \eta^B(\omega)\mu \in B_*(\Lambda)$. As $|\eta^B(\omega)| \leq l$ and $\Gamma_l^- (\mu) = \Gamma_l^- (\nu)$, we have $\eta^B(\omega) \in \Gamma_l^- (\nu)$. Hence $\xi^B(\eta^B(\omega)\nu) \in B_*(\Lambda)$. Since
the leftmost of $\tilde{\mu}$ is not 1, the rightmost of $\omega$ is not 0 so that $\omega \notin B_{1,*0}(\tilde{A})$ and $\xi^B(\eta^B(\omega)\nu) = \omega\nu$. Hence we have $\omega \in \Gamma^+(\tilde{\nu})$.

Case 2: $\omega_1 = 1$.

For $\omega = \omega_1 \cdots \omega_l \in \Gamma^+(\tilde{\mu})$, consider $\omega' = 0\omega_1 \cdots \omega_l \in B_{1,+1}(\tilde{A})$. Since $|\eta^B(\omega')| \leq l$, one may apply the above discussion for $\eta^B(\omega')$ so that the condition $\eta^B(\omega')\mu \in B_*(\Lambda)$ implies $\eta^B(\omega')\nu \in B_*(\Lambda)$ and $\xi^B(\eta^B(\omega')\nu) \in B_*(\tilde{A})$. Hence we have $\omega'\nu \in B_*(\tilde{A})$ so that $\omega\nu \in B_*(\tilde{A})$ and $\omega \in \Gamma^+(\tilde{\nu})$. We thus conclude $\Gamma^+(\tilde{\mu}) \subset \Gamma^+(\tilde{\nu})$ and then $\Gamma^+(\tilde{\mu}) = \Gamma^+(\tilde{\nu})$.

Therefore by the preceding lemma, the map $\xi^S : S_1(\Lambda) \rightarrow S_2(\tilde{A})\setminus S_{1,t,0}(\tilde{A})$ induces a map $\xi^1 : \Omega^1_\Lambda(\Lambda) \rightarrow \Omega^1_\Lambda(\tilde{A}_1)$ defined by $\xi^1([\mu]) = [\tilde{\mu}]$.

(ii) It is easy to see that for $\nu_1, \nu_2 \in S_2(\tilde{A})\setminus S_{1,t,0}(\tilde{A})$ the condition $\Gamma_{\nu_1} = \Gamma_{\nu_2} = \Gamma(\tilde{\nu})$ in $\Lambda$. Hence by the preceding lemma, the map $\eta^S : S_1(\Lambda) \rightarrow S_1(\tilde{A})$ induces a map $\eta^1 : \Omega^1_\Lambda(\Lambda) \rightarrow \Omega^1_\Lambda(\tilde{A}_1)$ defined by $\eta^1([\mu]) = [\tilde{\mu}]$.

The restriction of the natural surjection

$$i^\Lambda_{t,t+1} : [\mu]_{t+1} \in \Omega^{t+1}_\Lambda(\Lambda) \rightarrow [\mu]_t \in \Omega^t_\Lambda(\Lambda)$$

yields the surjection

$$\Omega^{t+1}_\Lambda(\Lambda) \rightarrow \Omega^t_\Lambda(\Lambda),$$

which we still denote by $i^\Lambda_{t,t+1}$. Similarly we have the natural surjections

$$\Omega^{t+1}_\Lambda(\tilde{A}_1) \rightarrow \Omega^t_\Lambda(\tilde{A}_1), \quad \Omega^{t+1}_\Lambda(\tilde{A}) \rightarrow \Omega^t_\Lambda(\tilde{A}).$$

by restricting the surjection

$$i^\Lambda_{t,t+1} : \Omega^{t+1}_\Lambda(\tilde{A}) \rightarrow \Omega^t_\Lambda(\tilde{A})$$

which we still denote by $i^\Lambda_{t,t+1}$. We set the projective limits of the compact Hausdorff spaces:

$$\Omega_\Lambda(\Lambda) = \lim_{\Lambda} \Omega^t_\Lambda(\Lambda), \quad \Omega_\Lambda(\tilde{A}) = \lim_{\Lambda} \Omega^t_\Lambda(\tilde{A}),$$

$$\Omega_\Lambda(\Lambda_1) = \lim_{\Lambda} \Omega^t_\Lambda(\Lambda_1), \quad \Omega_\Lambda(\tilde{A}_1) = \lim_{\Lambda} \Omega^t_\Lambda(\tilde{A}_1).$$

We note

$$\lim_{\Lambda} \Omega^t_\Lambda(\tilde{A}) \setminus \Omega^t_\Lambda(\tilde{A}_1) \cong \lim_{\Lambda} \Omega^t_\Lambda(\tilde{A}) \setminus \lim_{\Lambda} \Omega^t_\Lambda(\tilde{A}_1) = \Omega_\Lambda(\tilde{A}) \setminus \Omega_\Lambda(\tilde{A}_1).$$

We put

$$i^\Lambda_{t,t+n} = i^\Lambda_{t,t+1} \circ i^\Lambda_{t+1,t+2} \circ \cdots \circ i^\Lambda_{t+n-1,t+n} : \Omega^{t+n}_\Lambda(\Lambda) \rightarrow \Omega^t_\Lambda(\Lambda)$$

and

$$i^\Lambda_{t,t+n} : \Omega^{t+n}_\Lambda(\tilde{A}) \rightarrow \Omega^t_\Lambda(\tilde{A})$$

is similarly defined. The following lemma is straightforward.

Lemma 5.3.

(i) $\eta^1 \circ \xi^\Lambda_2 = i^\Lambda_{t+2t} : \Omega^{2t}_\Lambda(\Lambda) \rightarrow \Omega^t_\Lambda(\Lambda)$. 
Proposition 5.8. Through the homeomorphism \( \psi \)
5.4 and Corollary 5.7, we have homeomorphisms
of the projective limits as compact Hausdorff spaces.

\[ \lim_{\ell} \xi^\lambda_i : \Omega_{\lambda}(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}) \]

Corollary 5.4. The sequence \( \{ \xi^\lambda_i \}_{\ell \in \mathbb{N}} \) of the maps \( \xi^\lambda_i : \Omega_{\lambda}(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}) \), \( \ell \in \mathbb{N} \) induces a homeomorphism

\[ \lim_{\ell} \xi^\lambda_i : \Omega_{\lambda}(\Lambda) = \lim_{\ell} \Omega_{\lambda}^i(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}) = \lim_{\ell} \Omega_{\lambda}^i(\Lambda) \]
of the projective limits as compact Hausdorff spaces.

We denote by \( \Phi_d \) the homeomorphism \( \lim_{\ell} \xi^\lambda_i : \Omega_{\lambda}(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}) \).

Let \( \varphi^B : B_{1,*}(\Lambda) \to B_{1,*}(\tilde{\Lambda})/B_{*,0}(\tilde{\Lambda}) \) be the map defined by putting 01 in place of 1 in the subword \( x_2 \cdots x_n \) of a word \( 1x_2 \cdots x_n \in B_{1,*}(\Lambda) \) such as

\[ \varphi^B(1121321301) = 101013201301. \]

Let \( \psi^B : B_{1,*}(\tilde{\Lambda})/B_{*,0}(\tilde{\Lambda}) \to B_{1,*}(\Lambda) \) be the map defined by putting 1 in place of
01 in the subword \( y_2 \cdots y_k \) in a word \( 1y_2 \cdots y_k \in B_{1,*}(\tilde{\Lambda}) \) such as

\[ \psi^B(1012013201301) = 112132131. \]

The following lemmas are similarly shown to Lemma 4.2 and Lemma 5.2.

Lemma 5.5.

(i) \( \varphi^B : B_{1,*}(\Lambda) \to B_{1,*}(\tilde{\Lambda})/B_{*,0}(\tilde{\Lambda}) \) induces the maps

\[ \varphi^B_i : S_{1,i-1}(\Lambda) \to S_{1,i}(\tilde{\Lambda}) \quad \text{and} \quad \varphi^B_i : \Omega_{\lambda}^{i-1}(\Lambda_1) \to \Omega_{\lambda}^i(\tilde{\Lambda}_1). \]

(ii) \( \psi^B : B_{1,*}(\tilde{\Lambda})/B_{*,0}(\tilde{\Lambda}) \to B_{1,*}(\Lambda) \) induces the maps

\[ \psi^B_i : S_{1,2i}(\tilde{\Lambda}) \to S_{1,i}(\Lambda) \quad \text{and} \quad \psi^B_i : \Omega_{\lambda}^{2i}(\tilde{\Lambda}_1) \to \Omega_{\lambda}^i(\Lambda_1). \]

Lemma 5.6.

(i) \( \psi^B_i \circ \varphi^B_i = \lim_{\ell} \lambda^i(\Lambda_1) \to \Omega_{\lambda}^{2i-1}(\Lambda_1) \to \Omega_{\lambda}^{2i+1}(\Lambda_1) \).

(ii) \( \varphi^B_i \circ \psi^B_i = \lim_{\ell} \lambda^i(\Lambda_1) \to \Omega_{\lambda}^{2i}(\tilde{\Lambda}_1) \to \Omega_{\lambda}^{2i+1}(\tilde{\Lambda}_1) \).

Corollary 5.7. The sequence \( \{ \varphi^B_i \}_{i \in \mathbb{Z}_+} \) of the maps \( \varphi^B_i : \Omega_{\lambda}^{i-1}(\Lambda_1) \to \Omega_{\lambda}^i(\tilde{\Lambda}_1) \), \( i \in \mathbb{Z}_+ \) induces a homeomorphism

\[ \lim_{\ell} \varphi^B_i : \Omega_{\lambda}(\Lambda_1) = \lim_{\ell} \Omega_{\lambda}^i(\Lambda_1) \to \Omega_{\lambda}(\tilde{\Lambda}_1) = \lim_{\ell} \Omega_{\lambda}^i(\tilde{\Lambda}_1) \]
of the projective limits as compact Hausdorff spaces.

We denote by \( \Phi_1 \) the homeomorphism \( \lim_{\ell} \varphi^B_i : \Omega_{\lambda}(\Lambda_1) \to \Omega_{\lambda}(\tilde{\Lambda}_1) \). By Corollary 5.4 and Corollary 5.7, we have homeomorphisms

\[ \Phi_d : \Omega_{\lambda}(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}) \Omega_{\lambda}(\Lambda), \quad \Phi_1 : \Omega_{\lambda}(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}) \]

Therefore we have

**Proposition 5.8.** The disjoint union \( \Omega_{\lambda}(\Lambda_1) \cup \Omega_{\lambda}(\Lambda) \) is homeomorphic to \( \Omega_{\lambda}(\tilde{\Lambda}) \)
through the homeomorphism

\[ \Phi_1 \cup \Phi_d : \Omega_{\lambda}(\Lambda_1) \cup \Omega_{\lambda}(\Lambda) \to \Omega_{\lambda}(\tilde{\Lambda}_1) \cup (\Omega_{\lambda}(\tilde{\Lambda}) \Omega_{\lambda}(\tilde{\Lambda}_1)) = \Omega_{\lambda}(\tilde{\Lambda}). \]
Put for $\alpha \in \Sigma$
\[
S_{\alpha,l}(\Lambda) = \{ \mu_1 \cdots \mu_n \in S_l(\Lambda) \mid \mu_1 = \alpha \},
\]
\[
S_{l+1}(\alpha \Lambda) = \{ \omega_1 \cdots \omega_n \in S_{l+1}(\Lambda) \mid \omega_1 \cdots \omega_n \in S_l(\Lambda) \}.
\]
For $\omega = \omega_1 \cdots \omega_n, \gamma = \gamma_1 \cdots \gamma_m \in S_{l+1}(\alpha \Lambda)$, we write
\[
\omega \sim_{l,\alpha} \gamma
\]
if $\Gamma_1^- (\alpha \omega_1 \cdots \omega_n) = \Gamma_1^- (\alpha \gamma_1 \cdots \gamma_m)$. We denote by $\Omega_{\lambda}^l(\alpha \Lambda)$ the equivalence classes of $S_{l+1}(\alpha \Lambda)$ under the equivalence relation $\sim$ with its discrete topology, that is
\[
\Omega_{\lambda}^l(\alpha \Lambda) = S_{l+1}(\alpha \Lambda)/ \sim_{l,\alpha}.
\]
The equivalence class of $\omega \in S_{l+1}(\alpha \Lambda)$ is denoted by $[\omega]_{l,\alpha}$.

**Lemma 5.9.** Assume that $\Lambda$ is $\lambda$-synchronizing. For $\mu \in S_{\alpha,l}(\Lambda)$, there exists a word $\zeta = \zeta_1 \cdots \zeta_n \in S_{l+1}(\alpha \Lambda)$ such that $\mu \sim \alpha \zeta$, where $\alpha \zeta$ denotes $\alpha \zeta_1 \cdots \zeta_n$.

**Proof.** For $\mu = \mu_1 \cdots \mu_n \in S_{\alpha,l}(\Lambda)$ with $\mu_1 = \alpha$, put $k = n + l$. As $\Lambda$ is $\lambda$-synchronizing, there exists $\omega \in S_k(\Lambda)$ such that $\mu \omega \in S_{k-l}(\Lambda)$. Let $\zeta = \mu_2 \cdots \mu_n \omega \in S_{k-(n-1)}(\Lambda) = S_{l+1}(\Lambda)$ so that $\alpha \zeta = \mu \omega$ and $\Gamma_1^- (\mu) = \Gamma_1^- (\mu \omega) = \Gamma_1^- (\alpha \zeta)$. $\square$

Since $S_{l+2}(\alpha \Lambda) \subset S_{l+1}(\alpha \Lambda)$ and $\sim_{l+1,\alpha}$ implies $\sim_{l,\alpha}$, the map $\iota_{l,l+1}^{\lambda(\Lambda)} : \Omega_{\lambda}^{l+1}(\alpha \Lambda) \rightarrow \Omega_{\lambda}^l(\alpha \Lambda)$ induces a surjection
\[
\Omega_{\lambda}^{l+1}(\alpha \Lambda) \longrightarrow \Omega_{\lambda}^l(\alpha \Lambda) \quad \text{for } l \in \mathbb{Z}_+
\]
which gives rise to a compact Hausdorff space
\[
\varprojlim_{l} \Omega_{\lambda}^l(\alpha \Lambda)
\]
of a projective limit along the surjections. We denote it by $\Omega_{\lambda}(\alpha \Lambda)$.

Let us consider the case for $\alpha = 1$ in the above setting. By the preceding lemma, for $\mu \in S_{1,l}(\Lambda)$, one may take $\zeta \in S_{l+1}(1 \Lambda)$ such that $\mu \sim 1 \zeta$. One then sees the map
\[
[\mu]_l \in \Omega_{\lambda}^l(1 \Lambda) \longrightarrow [\zeta]_{l,1} \in \Omega_{\lambda}^1(1 \Lambda)
\]
is well-defined. We denote it by $s_{1}^\lambda$. Conversely, any element $\omega_2 \cdots \omega_n \in S_{l+1}(1 \Lambda)$ yields $1 \omega_2 \cdots \omega_n \in S_{l,1}(\Lambda)$. This correspondence yields the inverse of $s_{1}^\lambda$, so that $s_{1}^\lambda$ is bijective and hence homeomorphic. Since the sequence of the maps $\{ s_{1}^\lambda \}_{l \in \mathbb{Z}_+}$ is compatible to $\iota$-maps
\[
\Omega_{\lambda}^{l+1}(1 \Lambda) \longrightarrow \Omega_{\lambda}^{l}(1 \Lambda) \quad \text{and} \quad \Omega_{\lambda}^{l+1}(1 \Lambda) \longrightarrow \Omega_{\lambda}^{l}(1 \Lambda),
\]
they induce a homeomorphism
\[
\varprojlim_{l} s_{1}^\lambda : \Omega_{\lambda}(1 \Lambda) \longrightarrow \Omega_{\lambda}(1 \Lambda)
\]
denoted by $s_{\lambda}$. By Proposition 5.8, we thus have

**Proposition 5.10.** The disjoint union $\Omega_{\lambda}(1 \Lambda) \sqcup \Omega_{\lambda}(\Lambda)$ is homeomorphic to $\Omega_{\lambda}(\Lambda)$ through the homeomorphism $\Phi_1 \circ s_{\lambda}^{-1} \sqcup \Phi_d$. 

Denote by $\Phi_\lambda$ the homeomorphism
\[
\Phi_1 \circ s_\lambda^{-1} \cup \Phi_d : \Omega_\lambda(\Lambda_1) \sqcup \Omega_\lambda(\Lambda) \to \Omega_\lambda(\Lambda).
\]
For $\alpha \in \Sigma$, denote by $\Omega_\lambda^\alpha(\Lambda_\alpha)$ the $l$-past equivalence classes of $S_{\alpha,l}(\Lambda)$. The map
\[
\mu_1 \cdots \mu_n \in S_{l+1}(\alpha \Lambda) \to \alpha \mu_1 \cdots \mu_n \in S_{\alpha,l}(\Lambda) \subset S_l(\Lambda)
\]
duces a map
\[
\lambda_\alpha([\mu]_{l,\alpha} \in \Omega_\lambda^l(\alpha \Lambda)) \to [\alpha \mu]_{l} \in \Omega_\lambda^l(\Lambda) \subset \Omega_\lambda^l(\Lambda).
\]
The family $\{\lambda_\alpha(\mu)\}_{\mu \in \mathbb{Z}_+}$ yields a continuous map $\Omega_\lambda(\alpha \Lambda) \to \Omega_\lambda(\Lambda)$ which we write $\lambda_\alpha(\alpha)$. Let $\mathbb{Z}_{\lambda}(\Lambda)$ and $\mathbb{Z}_{\lambda}(\alpha \Lambda)$ be the abelian groups $C(\Omega_\lambda(\Lambda), \mathbb{Z})$ and $C(\Omega_\lambda(\alpha \Lambda), \mathbb{Z})$ of all $\mathbb{Z}$-valued continuous functions on the compact Hausdorff spaces $\Omega_\lambda(\Lambda)$ and $\Omega_\lambda(\alpha \Lambda)$ respectively. The map $\lambda_\alpha(\alpha)$ induces a homomorphism of abelian groups
\[
\lambda_\alpha(\alpha)^* : \mathbb{Z}_{\lambda}(\Lambda) \to \mathbb{Z}_{\lambda}(\alpha \Lambda)
\]
such that
\[
\lambda_\alpha(\alpha)^*(f)(t) = f(\lambda_\alpha(\alpha)(t)) \quad \text{for} \quad f \in \mathbb{Z}_{\lambda}(\Lambda), t \in \Omega_\lambda(\alpha \Lambda).
\]
Let $\alpha \Omega_\lambda^{l+1}(\Lambda)$ be the $l + 1$-past equivalence classes of $S_{l+1}(\alpha \Lambda)$ so that
\[
\alpha \Omega_\lambda^{l+1}(\alpha \Lambda) = \{[\mu]_{l+1} \in \Omega_\lambda^{l+1}(\Lambda) \mid \alpha \in \Gamma(\mu)\} \subset \Omega_\lambda^{l+1}(\Lambda).
\]
The surjective map
\[
n_\alpha^{\lambda}(\alpha) : \Omega_\lambda^{l+1}(\Lambda) \to \Omega_\lambda^{l}(\Lambda)
\]
works for the restriction
\[
\alpha \Omega_\lambda^{l+1}(\Lambda) \to \alpha \Omega_\lambda^{l}(\Lambda)
\]
which yields a compact Hausdorff space
\[
\alpha \Omega_\lambda(\Lambda) = \lim_{\Lambda} \alpha \Omega_\lambda^{l}(\Lambda)
\]
by taking a projective limit along the above restrictions. We may regard $\alpha \Omega_\lambda(\Lambda)$ as a clopen subset of $\Omega_\lambda(\Lambda)$.

For $\mu, \nu \in S_{l+1}(\alpha \Lambda)$, the condition $[\mu]_{l+1} = [\nu]_{l+1}$ implies $[\mu]_{l,\alpha} = [\nu]_{l,\alpha}$ so that the map
\[
e_\alpha^{l}(\alpha) : [\mu]_{l+1} \in \alpha \Omega_\lambda^{l+1}(\Lambda) \to [\mu]_{l,\alpha} \in \Omega_\lambda^{l}(\alpha \Lambda) \quad \text{for} \quad \mu \in S_{l+1}(\alpha \Lambda)
\]
is well-defined and surjective. Since the maps $e_\alpha^{l}(\alpha) : \alpha \Omega_\lambda^{l+1}(\alpha \Lambda) \to \alpha \Omega_\lambda^{l}(\alpha \Lambda), l \in \mathbb{Z}_+$ are compatible to the surjections
\[
\nu_\alpha^{l+1}(\alpha) : \alpha \Omega_\lambda^{l+1}(\alpha \Lambda) \to \alpha \Omega_\lambda^{l}(\alpha \Lambda), \quad \nu_\alpha^{l}(\alpha) : \alpha \Omega_\lambda^{l+1}(\alpha \Lambda) \to \alpha \Omega_\lambda^{l}(\alpha \Lambda),
\]
we have a continuous surjection
\[
e_\alpha^{\lambda}(\alpha) = \lim_{\Lambda} e_\alpha^{l}(\alpha) : \alpha \Omega_\lambda(\Lambda) \to \Omega_\lambda(\alpha \Lambda).
\]
We then define a homomorphism of abelian groups
\[
e_\alpha^{\lambda}(\alpha) : \mathbb{Z}_{\lambda}(\alpha \Lambda) \to \mathbb{Z}_{\lambda}(\Lambda)
\]
by setting
\[
e_\alpha^{\lambda}(\alpha)(f)(x) = \begin{cases} f(e_\alpha^{\lambda}(\alpha)(x)) & \text{if } x \in \alpha \Omega_\lambda(\Lambda), \\ 0 & \text{otherwise} \end{cases}
\]
Proposition 5.11.

Proof. Let \( \lambda \) be identified with the matrix \( \left( \lambda_{ij} \right) \). By definition of the \( \lambda \)-synchronizing \( K \)-groups, we see

\[
\lambda(\Lambda) = \sum_{\alpha=1}^{N} \epsilon^{(\alpha)}(\alpha) \circ \lambda_{\Lambda}(\alpha)^* : \mathbb{Z}_{\Lambda(\Lambda)} \longrightarrow \mathbb{Z}_{\Lambda(\Lambda)}.
\]

Let us define an endomorphism \( \lambda(\Lambda) \) of \( \mathbb{Z}_{\Lambda(\Lambda)} \) by setting:

\[
\lambda(\Lambda) = \sum_{\alpha=1}^{N} \epsilon^{(\alpha)}(\alpha) \circ \lambda_{\Lambda}(\alpha)^* : \mathbb{Z}_{\Lambda(\Lambda)} \longrightarrow \mathbb{Z}_{\Lambda(\Lambda)}.
\]

By definition of the \( \lambda \)-synchronizing \( K \)-groups, we see

Proposition 5.11.

(i) \( K_0^\lambda(\Lambda) = \mathbb{Z}_{\Lambda(\Lambda)}/(\text{id} - \lambda(\Lambda))\mathbb{Z}_{\Lambda(\Lambda)} \).

(ii) \( K_1^\lambda(\Lambda) = \text{Ker}(\text{id} - \lambda(\Lambda)) \) in \( \mathbb{Z}_{\Lambda(\Lambda)} \).

Proof. Let \( \{ v_i^{m_\lambda(l)} \}_{i=1}^{m_\lambda(l)} \) be the vertex set of \( V_\lambda^{\Lambda(\Lambda)} \) of the \( \lambda \)-synchronizing \( \lambda \)-graph system \( \Omega^{\Lambda(\Lambda)} \). As we identify \( \Omega^{\Lambda(\Lambda)} \) with \( V_\lambda^{\Lambda(\Lambda)} \), we may write \( v_i^l = [\mu_l(i)]_l \) for some \( \mu_l(i) \in S_l(\Lambda), i = 1, \ldots, m_\lambda(l) \). For \( v_i^l \in V_\lambda^{\Lambda(\Lambda)}, v_j^{l+1} \in V_\lambda^{\Lambda(\Lambda)} \) and \( \alpha \in \Sigma \), we set \( A_{i,j}^{\Lambda(\Lambda)}(\Lambda, \alpha) = 1 \) if there exists an labeled edge labeled \( \alpha \) which starts at \( v_i^l \) and ends at \( v_j^{l+1} \), and \( A_{i,j}^{\Lambda(\Lambda)}(\Lambda, \alpha) = 0 \), otherise. Hence we have \( A_{i,j}^{\Lambda(\Lambda)}(\Lambda, \alpha) = \sum_{\alpha=1}^{N} A_{i,j+1}^{\Lambda(\Lambda)}(\Lambda, \alpha) \). We then have for \( j = 1, \ldots, m_\lambda(l+1) \)

\[
\{ [\alpha \mu_j^{l+1}(j)]_l \in V_\lambda^{\Lambda(\Lambda)} \mid \alpha \in \Gamma_1^\lambda(\mu_j^{l+1}(j)) \}
\]

\[
= \{ [\mu_l(i)]_l \in V_\lambda^{\Lambda(\Lambda)} \mid A_{i,j}^{\Lambda(\Lambda)}(\Lambda, \alpha) = 1, i = 1, \ldots, m_\lambda(l), \alpha \in \Sigma \}.
\]

For \( f \in C(\Omega^{\Lambda(\Lambda)}_\lambda, \mathbb{Z}) \) and \( v_j^{l+1} \in V_\lambda^{\Lambda(\Lambda)} \), \( j = 1, \ldots, m_\lambda(l+1) \) it follows that

\[
\lambda(\Lambda)(f)(v_j^{l+1}) = \sum_{\alpha=1}^{N} \epsilon^{(\alpha)}(\alpha) \circ \lambda_{\Lambda}(\alpha)^* (f)([\mu_l(i)]_l) = \sum_{\alpha=1}^{N} \sum_{i=1}^{m_\lambda(l)} A_{i,j}^{\Lambda(\Lambda)}(\Lambda, \alpha, j) f([\mu_l(i)]_l)
\]

\[
= \sum_{i=1}^{m_\lambda(l)} \sum_{\alpha=1}^{N} A_{i,j+1}^{\Lambda(\Lambda)}(\Lambda, \alpha, j) f([\mu_l(i)]_l)
\]

\[
= \sum_{i=1}^{m_\lambda(l)} f(v_i^l) A_{i,j+1}^{\Lambda(\Lambda)}(\Lambda, \alpha, j).
\]

The group \( C(\Omega^{\Lambda(\Lambda)}_\lambda, \mathbb{Z}) \) is identified with \( \mathbb{Z}^{m_\lambda(l)} \) through the map

\[
f \in C(\Omega^{\Lambda(\Lambda)}_\lambda, \mathbb{Z}) \longrightarrow [f(v_i^l)]_{i=1}^{m_\lambda(l)} \in \mathbb{Z}^{m_\lambda(l)}.
\]

Hence the homomorphism

\[
\lambda(\Lambda)|_{C(\Omega^{\Lambda(\Lambda)}_\lambda, \mathbb{Z})} : C(\Omega^{\Lambda(\Lambda)}_\lambda, \mathbb{Z}) \longrightarrow C(\Omega^{\Lambda+1(\Lambda)}_\lambda, \mathbb{Z})
\]

is identified with the matrix

\[
t A_{i,j+1}^{\Lambda(\Lambda)} : \mathbb{Z}^{m_\lambda(l)} \longrightarrow \mathbb{Z}^{m_\lambda(l+1)}.
\]
Therefore by definition of $K^i_\lambda(\Lambda), i = 0, 1$, we have desired formulae.

**Lemma 5.12.** There exists a natural identification

$$\Omega^l_\lambda(\Lambda_1) = 0\Omega^l_\lambda(\Lambda)$$ for all $l \in \mathbb{Z}_+$ so that $\Omega_\lambda(\Lambda_1) = 0\Omega_\lambda(\Lambda)$.

**Proof.** Since for $\mu = \mu_1 \cdots \mu_n \in S_l(\Lambda)$ the condition $\mu_1 = 1$ is equivalent to the condition $0\mu_1 \cdots \mu_n \in S_{0,l-1}(\Lambda)$, one has $S_{1,l}(\Lambda) = S_{l}(0,\Lambda)$ so that $\Omega_\lambda(\Lambda_1) = 0\Omega^l_\lambda(\Lambda)$.

**Lemma 5.13.**

(i) $\Phi^{-1}_d \circ \lambda_\lambda(0) \circ e^\lambda(0) \circ \Phi_1 \circ s^{-1}_\lambda = \lambda_\lambda(1)$ on $\Omega_\lambda(\Lambda)$.

(ii) $\Phi^{-1}_\lambda \circ \lambda_\lambda(\alpha) \circ e^\lambda(\alpha) \circ \Phi_d = \begin{cases} e^{\lambda}(1) & \text{on } \Omega_\lambda(\Lambda), \\ \lambda_\lambda(\alpha) \circ e^\lambda(\alpha) & \text{on } \alpha \Omega_\lambda(\Lambda) \end{cases}$ for $\alpha = 1, 2, 3, \cdots N$.

**Proof.** (i) We will prove that the equality

$$\Phi_d \circ \lambda_\lambda(1) \circ s_{\lambda} = \lambda_\lambda(0) \circ e^\lambda(0) \circ \Phi_1$$ holds for $\mu = \mu_1 \cdots \mu_n \in S_{1,l}(\Lambda)$, take $\zeta \in S_{l+1}(\Lambda)$ such that $\mu \sim \zeta$. It then follows that

$$\left(\xi^\lambda_1 \circ \lambda_1(1) \circ s^\lambda_1(0)[\mu_1]\right) = \left(\xi^\lambda_1 \circ \lambda_1(1)\right)([\zeta]_{l,1}) = \xi^\lambda_1([\mu_1]) = [\xi^B(\mu)]_{l}.$$

On the other hand,

$$\lambda_0(0) \circ e^\lambda(0) \circ \varphi_{l+1}^\lambda([\mu])$$

$$= \lambda_0(0) \circ e^\lambda(0)\circ \varphi_{l+1}^\lambda([\varphi^B(\mu)]_{l+1}) = \lambda_0([\varphi^B(\mu)]_{l+1}) = [0\varphi^B(\mu)]_l.$$

As $\varphi^B(\mu) = 1\varphi^B(\mu_2 \cdots \mu_n)$, one has

$$0\varphi^B(\mu) = 01\xi^B(\mu_2 \cdots \mu_n) = \xi^B(\mu)$$

so that we have

$$\Phi_d \circ \lambda_\lambda(1) \circ s_{\lambda} = \lambda_\lambda(0) \circ e^\lambda(0) \circ \Phi_1.$$

(ii) For $\alpha = 1$, the map $\lambda_\lambda(1)$ is defined from $\Omega_\lambda(1,\Lambda)$ to $\Omega_\lambda(\Lambda)$. As $\Phi_\lambda|\Omega_\lambda(1,\Lambda) = \Phi_1 \circ s_{\lambda}^{-1} : \Omega_{\lambda}(1,\Lambda) \rightarrow \Omega_{\lambda}(\Lambda)$, one sees $\Phi_{\lambda}^{-1}|_{\Omega_{\lambda}(\Lambda)} = s_{\lambda} \circ \Phi_{1}^{-1}$. As $\gamma \in S_{l}(\Lambda)$, it follows that

$$(\lambda_{l-1}(1) \circ e_{l-1}(1) \circ s_{l-1}^\gamma(1))(\gamma)_{l-1} = (\lambda_{l-1}(1) \circ e_{l-1}(1))(\gamma)_{l-1} = \lambda_{l-1}(1)(\gamma)_{l-1} = [\lambda_{l-1}]_{l-1}.$$

Since

$$(\varphi^\lambda_1 \circ (s^\lambda_{l-1})^{-1} \circ e_{l-1}(1))(\gamma)_{l-1} = (\varphi^\lambda_1 \circ (s^\lambda_{l-1})^{-1})(\gamma)_{l-1} = \varphi^\lambda_1([\lambda_{l-1}]_{l-1}) = [\varphi^\lambda_1].$$

As $\lambda_{l-1}(1)(\gamma)_{l-1} = [\lambda_{l-1}]_{l-1}$, we have $\lambda_\lambda(1) \circ e^\lambda(1) \circ \Phi_1 = \lambda_\lambda(1) \circ e^\lambda(1)$.

For $\alpha = 2, 3, \cdots N$, the map $\lambda_{\lambda}(\alpha)$ is defined from $\Omega_\lambda(\alpha,\Lambda)$ to $\Omega_\lambda(\Lambda,\alpha)$. As $\Omega_\lambda(\Lambda,\alpha)$ is contained in $\Omega_\lambda(\Lambda) \setminus \Omega_\lambda(\Lambda_1)$, one has $\Phi_{\lambda}^{-1} \circ \lambda_{\lambda}(\alpha) = \Phi^{-1}_d \circ \lambda_{\lambda}(\alpha)$. For $\gamma \in \alpha \Omega_\lambda(\Lambda)$, it follows that

$$(\lambda_{l-1}(\alpha) \circ e_{l-1}(\alpha)) \circ s_{l-1}^\gamma(\alpha)(\gamma)_{l-1} = (\lambda_{l-1}(\alpha) \circ e_{l-1}(\alpha))(\gamma)_{l-1} = \lambda_{l-1}(\alpha)(\gamma)_{l-1,\alpha} = [\alpha\gamma]_{l-1}.$$
Consider an endomorphism on the group $\mathbb{Z}_{\lambda(1)} \oplus \mathbb{Z}_{\lambda(1)}$ defined by

$$(g, h) \in \mathbb{Z}_{\lambda(1)} \oplus \mathbb{Z}_{\lambda(1)} \rightarrow (\lambda_{\lambda(1)}(h), \epsilon_{(1)}(g) + \sum_{\alpha=2}^{N} (\epsilon_{(\alpha)} \circ \lambda_{\alpha}(\alpha))(h)) \in \mathbb{Z}_{\lambda(1)} \oplus \mathbb{Z}_{\lambda(1)}$$

that is represented by

$$A_{\lambda(1)} = \begin{bmatrix} 0, & \lambda_{\lambda(1)} \circ \epsilon_{(1)} \circ \lambda_{\alpha}(\alpha) \end{bmatrix} \text{ on } \mathbb{Z}_{\lambda(1)} \oplus \mathbb{Z}_{\lambda(1)}.$$ 

For the subshift $\tilde{\Lambda}$, we will similarly formulate the endomorphism $\lambda(\tilde{\Lambda})$ on $\mathbb{Z}_{\lambda(\tilde{\Lambda})}$ as in the following way. Recall that for $\alpha \in \tilde{\Sigma}$

$$S_{t+1}(\alpha \tilde{\Lambda}) = \{ \gamma_1 \cdots \gamma_m \in S_{t+1}(\tilde{\Lambda}) \mid \alpha \gamma_1 \cdots \gamma_m \in S_{t}(\tilde{\Lambda}) \}.$$ 

The sets $\alpha \Omega_{\alpha}^{t+1}(\tilde{\Lambda})$ and $\Omega_{\alpha}^{t}(\tilde{\Lambda})$ are defined by the equivalence classes $S_{t+1}(\alpha \tilde{\Lambda})/_{\tilde{\Lambda}}$ and $S_{t+1}(\alpha \tilde{\Lambda})/_{\tilde{\Lambda}}$ respectively. The compact Hausdorff spaces $\alpha \Omega_{\alpha}(\tilde{\Lambda})$ and $\Omega_{\alpha}(\tilde{\Lambda})$ are defined by the projective limits $\lim_{\tilde{\Lambda}} \alpha \Omega_{\alpha}^{t+1}(\tilde{\Lambda})$ and $\lim_{\tilde{\Lambda}} \alpha \Omega_{\alpha}^{t}(\tilde{\Lambda})$ respectively. The maps $\tilde{e}_{(\alpha)} : \alpha \Omega_{\alpha}^{t+1}(\tilde{\Lambda}) \rightarrow \Omega_{\alpha}^{t}(\tilde{\Lambda})$ and $\tilde{\lambda}_{t}(\alpha) : \Omega_{\alpha}^{t}(\tilde{\Lambda}) \rightarrow \Omega_{\alpha}^{t}(\tilde{\Lambda})$ are defined by

$$\tilde{e}_{(\alpha)}(\gamma) = [\gamma]_{t, \alpha}, \quad \text{for } \gamma \in S_{t+1}(\alpha \tilde{\Lambda}),$$

$$\tilde{\lambda}_{t}(\alpha)(\mu) = [\alpha \mu]_{t}, \quad \text{for } \mu \in S_{t+1}(\alpha \tilde{\Lambda}).$$

Set

$$\lambda_{\lambda}^{(\alpha)} = \lim_{t} \tilde{e}_{(\alpha)} : \alpha \Omega_{\alpha}(\tilde{\Lambda}) \rightarrow \Omega_{\alpha}(\tilde{\Lambda}),$$

$$\lambda_{\lambda}^{(\alpha)}(\alpha) = \lim_{t} \tilde{\lambda}_{t}(\alpha) : \Omega_{\alpha}(\tilde{\Lambda}) \rightarrow \Omega_{\alpha}(\tilde{\Lambda}).$$

The map $\epsilon_{(\alpha)} : \mathbb{Z}_{\lambda(\tilde{\Lambda})} \rightarrow \mathbb{Z}_{\lambda(\tilde{\Lambda})}$ is defined by

$$\epsilon_{(\alpha)}(f)(x) = \begin{cases} f(\epsilon_{(\alpha)}(x)) & \text{if } x \in \alpha \Omega_{\alpha}(\tilde{\Lambda}), \\ 0 & \text{otherwise,} \end{cases}$$

for $f \in \mathbb{Z}_{\lambda(\tilde{\Lambda})}$, $x \in \Omega_{\alpha}(\tilde{\Lambda})$. The endomorphism $\lambda(\tilde{\Lambda})$ on $\mathbb{Z}_{\lambda(\tilde{\Lambda})}$ is defined by

$$\lambda(\tilde{\Lambda}) = \sum_{\alpha=0}^{N} \epsilon_{(\alpha)} \circ \lambda_{\lambda}^{(\alpha)}.$$ 

The homeomorphism $\Phi_{\lambda} : \Omega_{\lambda(1)} \sqcup \Omega_{\lambda(1)} \rightarrow \Omega_{\lambda(\tilde{\Lambda})}$ naturally yields an isomorphism $\Phi_{\lambda}^{\ast}$ of abelian groups from $\mathbb{Z}_{\lambda(\tilde{\Lambda})}$ to $\mathbb{Z}_{\lambda(1)} \oplus \mathbb{Z}_{\lambda(1)}$. We then have

**Lemma 5.14.** $\lambda(\tilde{\Lambda}) = \Phi_{\lambda}^{\ast} \circ A_{\lambda(1)} \circ \Phi_{\lambda}^{\ast}$ on $\mathbb{Z}_{\lambda(\tilde{\Lambda})}$.

**Proof.** The proof is completely similar to the proof of [28, Lemma 2.8]. We will give the proof for the sake of completeness. For $(g, h) \in \mathbb{Z}_{\lambda(1)} \oplus \mathbb{Z}_{\lambda(1)}$ and $\gamma \in \Omega_{\lambda(1)} \sqcup \Omega_{\lambda(1)}$, we see

$$[(\Phi_{\lambda}^{\ast} \circ \lambda(\tilde{\Lambda}) \circ \Phi_{\lambda}^{\ast} - 1)(g, h)](\gamma) = (\lambda(\tilde{\Lambda}))(\Phi_{\lambda}^{\ast} - 1)(g, h)(\Phi_{\lambda}(\gamma)).$$

We have two cases.
Case 1: \( \gamma \in \Omega_\lambda(\Lambda) \).
Since \( \Phi_1 \circ s_\lambda^{-1} : \Omega_\lambda(\Lambda) \rightarrow \Omega_\lambda(\Lambda_1) \) and \( \Omega_\lambda(\Lambda_1) = \partial \Omega_\lambda(\Lambda) \), one has \( \Phi_\lambda(\gamma) = \Phi_1 \circ s_\lambda^{-1}(\gamma) \in \partial \Omega_\lambda(\Lambda) \). We have \( \epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_\lambda(\gamma)) = 0 \) for \( \alpha \neq 0 \).

By the preceding lemma, one sees

\[
[(\lambda(\Lambda) \circ \Phi_{\lambda}^{*^{-1}}(g, h))(\Phi_\lambda(\gamma))] = \sum_{\alpha=0}^{N} \epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_\lambda(\gamma))
\]

\[
= \epsilon_{(0)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(0) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_1 \circ s_\lambda^{-1}(\gamma))
\]

\[
= (g, h)(\Phi_{\lambda}^{-1} \circ \lambda_{\Lambda}^{-1}(0) \circ \epsilon_{(0)}^{(\lambda)} \circ \Phi_1 \circ s_\lambda^{-1}(\gamma))
\]

\[
= (g, h)(\lambda_{\Lambda}(1)(\gamma))
\]

\[
h(\lambda_{\Lambda}(1)(\gamma)) = (\lambda_{\Lambda}(1)^*(h))(\gamma).
\]

Case 2: \( \gamma \in \Omega_\lambda(\Lambda) \).
As \( \Phi_d(\gamma) \notin \Omega_\lambda(\Lambda_1) = \partial \Omega_\lambda(\Lambda) \) one sees that

\[
\epsilon_{(0)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(0) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_d(\gamma)) = 0.
\]

It follows that

\[
[(\lambda(\Lambda) \circ \Phi_{\lambda}^{*^{-1}}(g, h))(\Phi(\gamma))] = \sum_{\alpha=1}^{N} \epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_d(\gamma)).
\]

It is easy to see that \( \Phi_d(\gamma) \in \partial \Omega_\lambda(\Lambda) \) if and only if \( \gamma \in \alpha \Omega_\lambda(\Lambda) \) for \( \alpha = 1, 2, \ldots, N \).

Hence \( \epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_d(\gamma)) = 0 \) for \( \gamma \notin \alpha \Omega_\lambda(\Lambda) \). If \( \gamma \in \alpha \Omega_\lambda(\Lambda) \), we have by the preceding lemma

\[
\epsilon_{(1)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(1) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_d(\gamma)) = (g, h)(\Phi_{\lambda}^{-1} \circ \lambda_{\Lambda}^{-1}(1) \circ \epsilon_{(1)}^{(\lambda)} \circ \Phi_d(\gamma))
\]

\[
= (g, h)(\epsilon_{(1)}^{(\lambda)}(\gamma))
\]

\[
= g(\epsilon_{(1)}^{(\lambda)}(\gamma)) = \epsilon_{(1)}^{(\lambda)}(g(\gamma)).
\]

If \( \gamma \in \alpha \Omega_\lambda(\Lambda) \) for \( \alpha = 2, 3, \ldots, N \), we have by the preceding lemma,

\[
\epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \Phi_{\lambda}^{*^{-1}}(g, h)(\Phi_d(\gamma)) = (g, h)(\Phi_{\lambda}^{-1} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \epsilon_{(\alpha)}^{(\lambda)} \circ \Phi_d(\gamma))
\]

\[
= h(\Phi_{\lambda}^{-1} \circ \lambda_{\Lambda}^{-1}(\alpha) \circ \epsilon_{(\alpha)}^{(\lambda)} \circ \Phi_d(\gamma))
\]

\[
= h(\lambda_{\Lambda}(\alpha) \circ \epsilon_{(\alpha)}^{(\lambda)}(\gamma))
\]

\[
= [(\epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}(\alpha)^*)(h)](\gamma).
\]

Therefore we conclude

\[
[\lambda(\Lambda) \circ \Phi_{\lambda}^{*^{-1}}(g, h)](\Phi_\lambda(\gamma)) = \epsilon_{(1)}^{(\lambda)}(g(\gamma)) + \sum_{\alpha=2}^{N} [(\epsilon_{(\alpha)}^{(\lambda)} \circ \lambda_{\Lambda}(\alpha)^*)(h)](\gamma)
\]

so that

\[
\Phi_{\lambda}^{*} \circ \lambda(\Lambda) \circ \Phi_{\lambda}^{*^{-1}} = A_{\lambda(\Lambda)}.
\]
Lemma 5.15.

(i) $\mathbb{Z}_{\lambda(\Lambda)}/(id - \lambda(\Lambda)) \mathbb{Z}_{\lambda(\Lambda)} \cong (\mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)})/(id - A_{\lambda(\Lambda)})(\mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)})$

(ii) $\text{Ker}(id - \lambda(\Lambda))$ in $\mathbb{Z}_{\lambda(\Lambda)}$ isomorphic to $\text{Ker}(id - A_{\lambda(\Lambda)})$ in $\mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}$.

The proof given here is similar to the proof of [28, Lemma 2.9] that is basically due to the original proof of Bowen-Franks in [3].

Proof. (i) Define a homomorphism

$$\delta : (g, h) \in \mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)} \longrightarrow [\xi_{(1)}^{\lambda(\Lambda)}(g) + h] \in \mathbb{Z}_{\lambda(\Lambda)}/(id - \lambda(\Lambda))\mathbb{Z}_{\lambda(\Lambda)}.$$  

It is clear that $\delta$ is surjective. We will show that

$$\text{Ker}(\delta) = (id - A_{\lambda(\Lambda)})(\mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}).$$

For $(g, h) \in \mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}$, it follows that

$$\delta((id - A_{\lambda(\Lambda)})(g, h)) = \delta((g - \lambda(\Lambda)^*(h), h - \epsilon_{(1)}^{\lambda(\Lambda)}(g) - \sum_{\alpha=2}^{N} \epsilon_{(\alpha)}^{\lambda(\Lambda)} \circ \lambda_{\alpha}(\alpha)^*(h)))$$

$$= [\epsilon_{(1)}^{\lambda(\Lambda)}(g - \lambda(\Lambda)^*(h)) + h - \epsilon_{(1)}^{\lambda(\Lambda)}(g) - \sum_{\alpha=2}^{N} \epsilon_{(\alpha)}^{\lambda(\Lambda)} \circ \lambda_{\alpha}(\alpha)^*(h)]$$

$$= [h - \sum_{\alpha=1}^{N} \epsilon_{(\alpha)}^{\lambda(\Lambda)} \circ \lambda_{\alpha}(\alpha)^*(h)]$$

$$= [(id - \lambda(\Lambda))(h)] = 0.$$

Conversely, we see for $(g, h) \in \text{Ker}(\delta)$,

$$\epsilon_{(1)}^{\lambda(\Lambda)}(g) + h = (id - \lambda(\Lambda))(g') \quad \text{for some } g' \in \mathbb{Z}_{\lambda(\Lambda)}.$$

It follows that

$$(id - A_{\lambda(\Lambda)})(\lambda(\Lambda)^*(g') + g, g')$$

$$= (\lambda(\Lambda)^*(g') + g - \lambda(\Lambda)^*(g'), g' - \epsilon_{(1)}^{\lambda(\Lambda)}(\lambda(\Lambda)^*(g') + g) - \sum_{\alpha=2}^{N} \epsilon_{(\alpha)}^{\lambda(\Lambda)} \circ \lambda_{\alpha}(\alpha)^*(g'))$$

$$= (g, (id - \lambda(\Lambda))(g') - \epsilon_{(1)}^{\lambda(\Lambda)}(g)) = (g, h).$$

Hence we have

$$\text{Ker}(\delta) = (id - A_{\lambda(\Lambda)})(\mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}).$$

(ii) Define a homomorphism

$$\xi : h \in \mathbb{Z}_{\lambda(\Lambda)} \longrightarrow (\lambda(\Lambda)^*(h), h) \in \mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}.$$

An element $(g, h) \in \mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}$ belongs to $\text{Ker}(id - A_{\lambda(\Lambda)})$ if and only if $(g, h) = (\lambda(\Lambda)^*(h), \epsilon_{(1)}^{\lambda(\Lambda)}(g) + \sum_{\alpha=2}^{N} \epsilon_{(\alpha)}^{\lambda(\Lambda)} \circ \lambda_{\alpha}(\alpha)^*(h))$. This condition is equivalent to the equalities: $g = \lambda(\Lambda)^*(h)$ and $h = \sum_{\alpha=1}^{N} \epsilon_{(\alpha)}^{\lambda(\Lambda)} \circ \lambda_{\alpha}(\alpha)^*(h)(= \lambda(\Lambda)(h))$. Since $\xi$ is clearly injective, it gives rise to an isomorphism: $\text{Ker}(id - \lambda(\Lambda))$ in $\mathbb{Z}_{\lambda(\Lambda)}$ $\cong$ $\text{Ker}(id - A_{\lambda(\Lambda)})$ in $\mathbb{Z}_{\lambda(\Lambda)} \oplus \mathbb{Z}_{\lambda(\Lambda)}$.

Therefore we conclude:
Theorem 5.16. For a $\lambda$-synchronizing subshift $\Lambda$, the $\lambda$-synchronizing $K$-groups $K^\lambda_0(\Lambda), K^\lambda_1(\Lambda)$ and the $\lambda$-synchronizing Bowen-Franks groups $BF^\lambda_0(\Lambda), BF^\lambda_1(\Lambda)$ are all invariant under flow equivalence of $\lambda$-synchronizing subshifts.

Proof. It is a direct consequence that the $K$-groups are invariant under flow equivalence from Theorem 4.4, Proposition 5.11 Lemma 5.14 and Lemma 5.15. As the Bowen-Franks groups are determined by the $K$-groups from the universal coefficient type theorem, the groups $BF^\lambda_\ast(\Lambda)$ are also invariant under flow equivalence. \hfill \Box

6. Examples

1. Sofic shifts.
Let $\Lambda$ be an irreducible sofic shift and $G_F(\Lambda)$ a finite directed labeled graph of the minimal left-resolving presentation of $\Lambda$, that is called the left Fischer cover graph for $\Lambda$ ([7], cf. [14], [15], [37]). Let $L$ be the minimal left-resolving presentation of $\Lambda$, that is called the left Fischer cover for the sofic shift $\Lambda$. Let $N$ be the number of the vertices of the graph $G_F(\Lambda)$. Let $M_F(\Lambda)$ be the $N \times N$ symbolic matrix of the graph $G_F(\Lambda)$. Let $A_F(\Lambda)$ be the $N \times N$ nonnegative matrix defined from $M_F(\Lambda)$ by setting all symbols in each components $M_F(\Lambda)(i,j), i,j = 1, \ldots, N$ equal to 1. Then the $\lambda$-synchronizing $K$-groups and the $\lambda$-synchronizing Bowen-Franks groups are easily calculated as:

$$K^\lambda_0(\Lambda) = \mathbb{Z}^N/(I_N - tA_F(\Lambda))\mathbb{Z}^N, \quad K^\lambda_1(\Lambda) = \text{Ker}(I_N - tA_F(\Lambda)) \quad \text{in } \mathbb{Z}^N$$

and

$$BF^\lambda_0(\Lambda) = \mathbb{Z}^N/(I_N - A_F(\Lambda))\mathbb{Z}^N, \quad BF^\lambda_1(\Lambda) = \text{Ker}(I_N - A_F(\Lambda)) \quad \text{in } \mathbb{Z}^N.$$ 

Therefore we have

Proposition 6.1. Let $\Lambda$ be an irreducible sofic shift and $A_F(\Lambda)$ the $N \times N$ edge matrix of its left Fischer cover graph with entries in nonnegative integers. Then the abelian groups

$$\mathbb{Z}^N/(I_N - A_F(\Lambda))\mathbb{Z}^N, \quad \text{Ker}(I_N - A_F(\Lambda)) \quad \text{in } \mathbb{Z}^N$$

are invariant under flow equivalence of subshifts (cf. [9]).

We note that $\mathbb{Z}^N/(I_N - tA_F(\Lambda))\mathbb{Z}^N$ is isomorphic to $\mathbb{Z}^N/(I_N - A_F(\Lambda))\mathbb{Z}^N$ and $\text{Ker}(I_N - tA_F(\Lambda)) \in \mathbb{Z}^N$ is isomorphic to $\text{Ker}(I_N - A_F(\Lambda)) \in \mathbb{Z}^N$.

2. Dyck shifts.
Let $N > 1$ be a fixed positive integer. Consider the Dyck shift $D_N$ with alphabet $\Sigma = \Sigma^- \sqcup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \ldots, \alpha_N\}, \Sigma^+ = \{\beta_1, \ldots, \beta_N\}$. The symbols $\alpha_i, \beta_i$ correspond to the brackets $(i, i)$, respectively. The Dyck inverse monoid $\mathbb{D}_N$ has the relations

$$\alpha_i \beta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad (6.1)$$

for $i, j = 1, \ldots, N$ ([13], [20]). A word $\omega_1 \cdots \omega_n$ of $\Sigma$ is admissible for $D_N$ precisely if $\prod_{m=1}^{n} \omega_m \neq 0$. For a word $\omega = \omega_1 \cdots \omega_n$ of $\Sigma$, we denote by $\bar{\omega}$ its reduced form,
which is a word of \( \Sigma \cup \{0, 1\} \) obtained after the operation (6.1). Hence a word \( \omega \) of \( \Sigma \) is forbidden for \( D_N \) if and only if \( \omega = 0 \).

In [22], a \( \lambda \)-graph system \( \mathcal{L}^{Ch(D_N)} \) that presents \( D_N \) has been introduced. It is called the Cantor graph \( \lambda \)-graph system for \( D_N \) (cf. [31]). Let \( \Sigma^N \) be the full \( N \)-shift \( \{1, \ldots, N\}^\mathbb{Z} \). We denote by \( B_l(D_N) \) and \( B_l(\Sigma^N) \) the set of admissible words of length \( l \) of \( D_N \) and that of \( \Sigma^N \) respectively. The vertices \( V_l^{Ch(D_N)} \) of \( \mathcal{L}^{Ch(D_N)} \) are given by the words of length \( l \) consisting of the symbols of \( \Sigma^+ \). That is,

\[
V_l^{Ch(D_N)} = \{ \beta_{\mu_1} \cdots \beta_{\mu_l} \in B_l(D_N) \mid \mu_1 \cdots \mu_l \in B_l(\Sigma^N) \}.
\]

The cardinal number of \( V_l^{Ch(D_N)} \) is \( N^l \). The mapping \( t(= t_{l+1}) : V_l^{Ch(D_N)} \rightarrow V_{l+1}^{Ch(D_N)} \) deletes the rightmost of a word such as

\[
t(\beta_{\mu_1} \cdots \beta_{\mu_l}) = \beta_{\mu_1} \cdots \beta_{\mu_l}, \quad (\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_{l+1}^{Ch(D_N)}.
\]

There exists an edge labeled \( \alpha_j \) from the vertex \( \beta_{\mu_1} \cdots \beta_{\mu_l} \in V_l^{Ch(D_N)} \) to the vertex \( \beta_{\mu_0} \beta_{\mu_1} \cdots \beta_{\mu_l} \in V_{l+1}^{Ch(D_N)} \) precisely if \( \mu_0 = j \), and there exists an edge labeled \( \beta_j \) from \( \beta_j \beta_{\mu_1} \cdots \beta_{\mu_l+1} \in V_l^{Ch(D_N)} \) to \( \beta_{\mu_1} \cdots \beta_{\mu_l+1} \in V_{l+1}^{Ch(D_N)} \). The resulting labeled Bratteli diagram with \( t \)-map is the Cantor horizon \( \lambda \)-graph system \( \mathcal{L}^{Ch(D_N)} \) of \( D_N \).

It is easy to see that each word of \( V_l^{Ch(D_N)} \) is \( l \)-synchronizing in \( D_N \) such that \( V_l^{Ch(D_N)} \) represent the all \( l \)-past equivalence classes of \( D_N \). Hence we know that \( V_l^{Ch(D_N)} = V_l^{\lambda(D_N)} \). We then know that the canonical \( \lambda \)-synchronizing \( \lambda \)-graph system \( \mathcal{L}^{\lambda(D_N)} \) is \( \mathcal{L}^{Ch(D_N)} \).

**Proposition 6.2.** The Dyck shift \( D_N \) is \( \lambda \)-synchronizing, and the \( \lambda \)-synchronizing \( \lambda \)-graph system \( \mathcal{L}^{\lambda(D_N)} \) is the Cantor horizon \( \lambda \)-graph system \( \mathcal{L}^{Ch(D_N)} \).

The \( K \)-groups of the \( \lambda \)-graph system \( \mathcal{L}^{Ch(D_N)} \) have been computed in [22] and [31] so that we have

\[
K_0^\lambda(D_N) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\hat{\mathbb{R}}, \mathbb{Z}), \quad K_1^\lambda(D_N) \cong 0,
\]

where \( C(\hat{\mathbb{R}}, \mathbb{Z}) \) denotes the abelian group of all \( \mathbb{Z} \)-valued continuous functions on a Cantor discontinuum \( \hat{\mathbb{R}} \). By Theorem 5.16, we have

**Proposition 6.3.** For the Dyck shifts \( D_N, D_{N'} \) with \( N, N' \geq 2, D_N \) is flow equivalent to \( D_{N'} \) if and only if \( N = N' \).

3. Topological Markov Dyck shifts.

We will state a generalization of the Dyck shifts. Let \( A = [A(i,j)]_{i,j=1,\ldots,N} \) be an \( N \times N \) matrix with entries in \( \{0, 1\} \). Consider the Dyck inverse monoid for the alphabet \( \Sigma = \Sigma^- \cup \Sigma^+ \) where \( \Sigma^- = \{\alpha_1, \ldots, \alpha_N\}, \Sigma^+ = \{\beta_1, \ldots, \beta_N\} \) as in the above. Let \( O_A \) be the Cuntz-Krieger algebra for the matrix \( A \) that is the universal \( C^* \)-algebra generated by \( N \) partial isometries \( t_1, \ldots, t_N \) subject to the following relations:

\[
\sum_{j=1}^N t_j t_j^* = 1, \quad t_i^* t_i = \sum_{j=1}^N A(i,j) t_j t_j^* \quad \text{for } i = 1, \ldots, N
\]

([5]). Define a correspondence \( \varphi_A : \Sigma \rightarrow \{t_i^*, t_i \mid i = 1, \ldots, N\} \) by setting

\[
\varphi_A(\alpha_i) = t_i^*, \quad \varphi_A(\beta_i) = t_i, \quad i = 1, \ldots, N.
\]
We denote by $\Sigma^*$ the set of all words $\gamma_1 \cdots \gamma_n$ of elements of $\Sigma$. Define the set

$$\mathfrak{F}_A = \{ \gamma_1 \cdots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \cdots \varphi_A(\gamma_n) = 0 \text{ in } \mathcal{O}_A \}.$$ 

Let $D_A$ be the subshift over $\Sigma$ whose forbidden words are $\mathfrak{F}_A$. The subshift is called the topological Markov Dyck shift defined by $A$. These kinds of subshifts have first appeared in semigroup setting ([10]) and in more general setting ([22]) without using $C^*$-algebras. If all entries of $A$ are 1, the subshift is nothing but the Dyck shift $D_N$ with $2N$ bracket, because the partial isometries \{ $\varphi_i(\alpha_i)$, $\varphi_i(\beta_i)$ | $i = 1, \ldots, N$ \} yield the Dyck inverse monoid. We note the fact that $\alpha_i/\beta_j \in \mathfrak{F}_A$ if $i \neq j$, and $\alpha_i \cdots \alpha_n \in \mathfrak{F}_A$ if and only if $\beta_i \cdots \beta_n \in \mathfrak{F}_A$. Consider the following two subsystems of $D_A$

$$D_A^+ = \{ (\gamma_i)_{i \in \mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^+, i \in \mathbb{Z} \},$$
$$D_A^- = \{ (\gamma_i)_{i \in \mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^-, i \in \mathbb{Z} \}.$$ 

The subshift $D_A^+$ is identified with the topological Markov shift

$$\Lambda_A = \{ (x_i)_{i \in \mathbb{Z}} \in \{ 1, \ldots, N \}^\mathbb{Z} \mid A(x_i, x_{i+1}) = 1, i \in \mathbb{Z} \}$$

defined by the matrix $A$ and similarly $D_A^-$ is identified with the topological Markov shift $\Lambda_A$ defined by the transposed matrix $A^t$ of $A$. Hence $D_A^+$ contains the both topological Markov shifts $\Lambda_A$ and $\Lambda_A$ that do not intersect each other. If $A$ satisfies condition (I) in the sense of Cuntz-Krieger [5], the subshift $D_A$ is not sofic ([32, Proposition 2.1]). We may define a $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ called the Cantor horizon $\lambda$-graph system for $D_A$ ([32]). We denote by $B_l(D_A)$ and $B_l(\Lambda_A)$ the set of admissible words of length $l$ of $D_A$ and that of $\Lambda_A$ respectively. The vertices $V_l^{Ch(D_A)}$ of $\mathcal{L}^{Ch(D_A)}$ at level $l$ are given by the admissible words of length $l$ consisting of the symbols of $\Sigma^+$. They are $l$-synchronizing words of $D_A$ such that the $l$-past equivalence classes of them coincide with the $l$-past equivalence classes of the set of $l$-synchronizing words of $D_A$. Hence $V_l^{Ch(D_A)} = V_l^{\Lambda(D_A)}$. Since $V_l^{Ch(D_A)}$ is identified with $B_l(\Lambda_A)$, we may write $V_l^{Ch(D_A)}$ as

$$V_l^{Ch(D_A)} = \{ v_{\mu_1 \cdots \mu_l} l \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_A) \}.$$ 

The mapping $\ell(=\ell_{l+1}) : V_{l+1}^{Ch(D_A)} \rightarrow V_l^{Ch(D_A)}$ is defined by deleting the rightmost symbol of a corresponding word such as

$$\ell(v_{\mu_1 \cdots \mu_{l+1}}) = v_{\mu_1 \cdots \mu_l}, \quad v_{\mu_1 \cdots \mu_{l+1}} \in V_{l+1}^{Ch(D_A)}.$$ 

There exists an edge labeled $\alpha_j$ from $v_{\mu_1 \cdots \mu_l} \in V_l^{Ch(D_A)}$ to $v_{\mu_0 \mu_1 \cdots \mu_l} \in V_{l+1}^{Ch(D_A)}$ precisely if $\mu_0 = j$, and there exists an edge labeled $\beta_j$ from $v_{\mu_1 \cdots \mu_{l-1}} \in V_l^{Ch(D_A)}$ to $v_{\mu_1 \cdots \mu_{l+1}} \in V_{l+1}^{Ch(D_A)}$. The resulting labeled Bratteli diagram with $\ell$-map becomes a $\lambda$-graph system written $\mathcal{L}^{Ch(D_A)}$ that presents $D_A$. It is called the Cantor horizon $\lambda$-graph system for $D_A$.

**Proposition 6.4.** The subshift $D_A$ is $\lambda$-synchronizing, and the $\lambda$-synchronizing $\lambda$-graph system $\mathcal{L}^{\Lambda(D_A)}$ is the Cantor horizon $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$.

By [32, Lemma 2.5] if $A$ satisfies condition (I) in the sense of Cuntz-Krieger [5], the $\lambda$-graph system $\mathcal{L}^{Ch(D_A)}$ satisfies $\lambda$-condition (I). If $A$ is irreducible, it is $\lambda$-irreducible. In this case we have that the $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{\Lambda(D_A)}}$ associated with $\mathcal{L}^{\Lambda(D_A)}$ is simple, purely infinite (cf.[32]).
One knows that $\beta$-shifts for $1 < \beta \in \mathbb{R}$, a synchronizing counter shift named as the context free shift in [24, Example 1.2.9], Motzkin shifts and the Morse shift are all $\lambda$-synchronizing. Their proofs are essentially seen in the papers [11], [23], [25], [30] respectively.

We study $C^*$-algebras associated with $\lambda$-synchronizing $\lambda$-graph systems in [33].

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