HOLME TYPE THEOREM FOR SPECIAL LINEAR GROUPS

SHULIM KALIMAN

Abstract. Let $Z$ be an affine algebraic variety and $X$ be a smooth algebraic variety isomorphic to a semi-simple linear algebraic group whose Lie algebra is a sum of special linear Lie algebras. We show that if $\dim X \geq \max(2 \dim Z + 1, \dim TZ)$, then $Z$ admits a closed embedding into $X$. We also show that for every smooth affine flexible variety $Y$ there is a closed embedding of $Z$ into $Y \times \mathbb{A}^n$ provided that $n \geq \dim Z - 1$ and $\dim Y + n \geq \max(2 \dim Z + 1, \dim TZ)$.

1. Introduction

All algebraic varieties which appear in this paper are considered over an algebraically closed field $k$ of characteristic zero. If $Z$ is an affine algebraic variety and $TZ$ is its Zariski tangent bundle then we call $ED(Z) = \max(2 \dim Z + 1, \dim TZ)$ the embedding dimension of $Z$. Holme's theorem [Hol, Theorem 7.4] (later rediscovered in [Ka91] and [Sr]) states that $Z$ admits a closed embedding into any affine space $\mathbb{A}^n$ with $n \geq ED(X)$. In the smooth case (when $ED(Z) = 2 \dim Z + 1$) this fact was proven earlier by Swan [Swan, Theorem 2.1]. The latter result is sharp - examples of smooth irreducible $d$-dimensional affine algebraic varieties with $d \geq \frac{n}{2}$ such that they do no admit closed embeddings in $\mathbb{A}^n$ were constructed in [BMS]. Recently Feller and van Santen [FS21] proved that if $X$ is an affine variety isomorphic to a simple linear algebraic group and $Z$ is smooth, then $Z$ admits a closed embedding into $X$, provided that $\dim X > ED(Z)$. They also proved that for every $n$-dimensional algebraic group $G$ there exist smooth irreducible $d$-dimensional affine algebraic varieties with $d \geq \frac{n}{2}$ such that they do not admit closed embeddings in $G$ [FS21, Corollary 4.4]. In particular, their embedding result is optimal for even dimensions of $X$. However, they did not know whether their result is sharp in the case of an odd dimension of $X$ and, in particular, a question posed in [FS21] asks whether a smooth affine algebraic variety of dimension 7 can be embedded properly into $\text{SL}_4(k)$. The main result of this paper is the following.

Date: July 15, 2021.

2020 Mathematics Subject Classification: 14E25, 14L30, 14R10.

Key words: closed embedding, injective immersion, affine algebraic variety, flexible variety, semi-simple Lie group.
**Theorem 1.1.** Let \(Z\) be an affine algebraic variety, \(Y\) be an algebraic variety of the form \(\mathbb{A}^{n_0} \times \text{SL}_{n_1}(k) \times \text{SL}_{n_2}(k) \times \ldots \times \text{SL}_{n_l}(k)\) where \(n_0 \geq 0, l \geq 1\) and \(n_i \geq 2\) for \(i \geq 1\). Suppose that \(\varphi : Y \to X\) is a finite morphism into a normal variety \(X\), \(\dim X \geq \text{ED}(Z)\) and \(\dim Z < \text{codim}_X X_{\text{sing}}\). Then \(Z\) admits a closed embedding into \(X\) with the image contained in \(X_{\text{reg}}\).

Thus, the question of Feller and van Santen has the positive answer. Recall that starting from dimension 2 affine spaces and linear algebraic groups without nontrivial characters are examples of so-called flexible varieties (a normal quasi-affine variety \(X\) is flexible if \(\text{SAut}(X)\) acts transitively on the smooth part \(X_{\text{reg}}\) of \(X\) where \(\text{SAut}(X)\) is the subgroup of the group \(\text{Aut}(X)\) of algebraic automorphisms of \(X\) generated by one-parameter unipotent subgroups). For such varieties we prove the following.

**Theorem 1.2.** Let \(Z\) be an affine algebraic variety, \(Y\) be a smooth affine algebraic variety of the form \(X_1 \times \mathbb{A}^n\) where \(X_1\) is flexible, \(n \geq \dim Z - 1\) and \(\dim Y \geq \text{ED}(Z)\). Suppose that \(\psi : Y \to W\) is a finite morphism into a normal variety \(W\) and \(\dim Z < \text{codim}_W W_{\text{sing}}\). Then \(Z\) admits a closed embedding into \(W\) with the image contained in \(W_{\text{reg}}\).

The proofs of Theorems 1.1 and 1.2 are heavily based on the theory of flexible varieties and the technique developed in [AFKKZ], [Ka20], [KaUd] and [Ka21] whose survey can be found in Section 2. In particular, we describe injective immersions of affine algebraic varieties into smooth flexible varieties. In section 3 we develop a criterion of properness for such injective immersion. Using this criterion and some simple facts about matrices we prove Theorem 1.1 in Section 4. Theorem 1.2 is proven in Section 5.

**Acknowledgement.** The author is grateful to L. Makar-Limanov for useful consultation.

2. Flexible varieties

Let us start with the main definitions in the theory of flexible varieties.

**Definition 2.1.** (1) Given an irreducible algebraic variety \(\mathcal{A}\) and a map \(\varphi : \mathcal{A} \to \text{Aut}(X)\) we say that \((\mathcal{A}, \varphi)\) is an algebraic family of automorphisms of \(X\) if the induced map \(\mathcal{A} \times X \to X, (\alpha, x) \mapsto \varphi(\alpha).x\) is a morphism (see [Ra]).

(2) If we want to emphasize additionally that \(\varphi(\mathcal{A})\) is contained in a subgroup \(G\) of \(\text{Aut}(X)\), then we say that \(\mathcal{A}\) is an algebraic \(G\)-family of automorphisms of \(X\).

(3) In the case when \(\mathcal{A}\) is a connected algebraic group and the induced map \(\mathcal{A} \times X \to X\) is not only a morphism but also an action of \(\mathcal{A}\) on \(X\) we call this family a connected algebraic subgroup of \(\text{Aut}(X)\).
(4) Following [AFKKZ, Definition 1.1] we call a subgroup $G$ of $\text{Aut}(X)$ algebraically generated if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic subgroups of $\text{Aut}(X)$.

**Definition 2.2.** (1) A nonzero derivation $\delta$ on the ring $A$ of regular functions on an affine algebraic variety $X$ is called locally nilpotent if for every $0 \neq a \in A$ there exists a natural $n$ for which $\delta^n(a) = 0$. This derivation can be viewed as a vector field on $X$ which we also call locally nilpotent. The set of all locally nilpotent vector fields on $X$ will be denoted by $\text{LND}(X)$. The flow of $\delta \in \text{LND}(X)$ is an algebraic $\mathbb{G}_a$-action on $X$, i.e., the action of the group $(\mathbb{k}, +)$ which can be viewed as a one-parameter unipotent group $U$ in the group $\text{Aut}(X)$ of all algebraic automorphisms of $X$. In fact, every $G_a$-action is a flow of a locally nilpotent vector field (e.g., see [Fr, Proposition 1.28]).

(2) If $X$ is a quasi-affine variety, then an algebraic vector field $\delta$ on $X$ is called locally nilpotent if $\delta$ extends to a locally nilpotent vector field $\tilde{\delta}$ on some affine algebraic variety $Y$ containing $X$ such that $\tilde{\delta}$ vanishes on $Y \setminus X$ where $\text{codim}_Y (Y \setminus X) \geq 2$. Note that under this assumption $\delta$ generates a $\mathbb{G}_a$-action on $X$ and we use again the notation $\text{LND}(X)$ for the set of all locally nilpotent vector fields on $X$.

**Definition 2.3.** (1) For every locally nilpotent vector fields $\delta$ and each function $f \in \text{Ker} \delta$ from its kernel the field $f \delta$ is called a replica of $\delta$. Recall that such replica is automatically locally nilpotent.

(2) Let $\mathcal{N}$ be a set of locally nilpotent vector fields on $X$ and $G_{\mathcal{N}} \subset \text{Aut}(X)$ denotes the group generated by all flows of elements of $\mathcal{N}$. We say that $G_{\mathcal{N}}$ is generated by $\mathcal{N}$.

(3) A collection of locally nilpotent vector fields $\mathcal{N}$ is called saturated if $\mathcal{N}$ is closed under conjugation by elements in $G_{\mathcal{N}}$ and for every $\delta \in \mathcal{N}$ each replica of $\delta$ is also contained in $\mathcal{N}$.

**Definition 2.4.** Let $X$ be a normal quasi-affine algebraic variety of dimension at least 2, $\mathcal{N}$ be a saturated set of locally nilpotent vector fields on $X$ and $G = G_{\mathcal{N}}$ be the group generated by $\mathcal{N}$. Then $X$ is called $G$-flexible if for any point $x$ in the smooth part $X_{\text{reg}}$ of $X$ the vector space $T_x X$ is generated by the values of locally nilpotent vector fields from $\mathcal{N}$ at $x$ (which is equivalent to the fact that $G$ acts transitively on $X_{\text{reg}}$ [FKZ, Theorem 2.12]). In the case of $G = \text{SAut}(X)$ we call $X$ flexible without referring to $\text{SAut}(X)$ (recall that $\text{SAut}(X)$ is the subgroup of $\text{Aut} X$ generated by all one-parameter unipotent subgroups).

**Notation 2.5.** Further in this paper $X$ is always a smooth quasi-affine variety and $G$ is group acting transitively on $X$. By $\mathcal{G}$ we denote a collection of subgroups of $G$ such that $G$ is algebraically generated by $\mathcal{G}$. Given a sequence $\mathcal{H} = (H_1, \ldots, H_s)$ of elements of $\mathcal{G}$ we consider
the map
\[
(1) \quad \Phi_H : H \times X \rightarrow X \times_{p} X, \quad (h_s, \ldots, h_1, x) \mapsto ((h_s \cdot \ldots \cdot h_1) \cdot x, x)
\]
where \( H = H_s \times \ldots \times H_1 \). By \( \varphi_H : H \rightarrow X \) we denote the restriction of \( \Phi_H \) to \( H \times x_0 \) where \( x_0 \) is a fixed point of \( X \).

**Proposition 2.6.** Suppose that \( \mathcal{G} \) is closed under conjugations in \( G \). Then \( \mathcal{H} \) can be chosen so that for a dense open subset \( U \) of \( H \) the morphism \( \Phi_H \) is smooth on \( U \times X \) (in particular, \( \varphi_H \) is smooth on \( U \)). Furthermore, one can suppose that the codimension of \( H \setminus U \) in \( H \) is arbitrarily large.

**Proof.** The first statement follows from [AFKKZ, Proposition 1.16] and the second statement from [AFKKZ, p. 778, footnote]. \( \square \)

We shall use the notion of a perfect \( G \)-family of automorphisms of \( X \) (see [Ka21, Definition 2.7]). Without stating the formal definition of such families we need to emphasize some of their properties.

**Proposition 2.7.** ([Ka21, Proposition 2.8(3)]) Let \( \mathcal{A} \) be a perfect \( G \)-family of automorphisms of a smooth \( G \)-flexible variety \( X \) and \( H_0 \in \mathcal{G} \). Then \( H_0 \times \mathcal{A} \) is also a perfect \( G \)-family of automorphisms of \( X \).

**Theorem 2.8.** Let \( X \) be a smooth quasi-affine \( G \)-flexible variety, \( \mathcal{A} \) be a perfect \( G \)-family of automorphisms of \( X \), \( Q \) be a normal algebraic variety and \( \varphi : X \rightarrow Q \) be a dominant morphism. Suppose that \( Q_0 \) is a smooth open dense subset of \( Q \), \( X_0 \subset \varphi^{-1}(Q_0) \) and
\[
(2) \quad X_0 \times_{Q_0} X_0 = 2 \dim X - \dim Q.
\]
Let \( Y \) be the closure of \( \bigcup_{x \in X_0} \text{Ker}\{ \varphi : T_x X_0 \rightarrow T_{\varphi(x)} Q_0 \} \) in \( TX \) and
\[
(3) \quad \dim Y = 2 \dim X - \dim Q.
\]
Let \( Z \) be a locally closed reduced subvariety of \( X \) with \( \text{ED}(Z) \leq \dim Q \) and \( \dim Z < \text{codim}_{\varphi^{-1}(Q_0)}(\varphi^{-1}(Q_0) \setminus X_0) \). Then for a general element \( \alpha \in \mathcal{A} \) the morphism \( \varphi|_{\alpha(Z) \cap X_0} : \alpha(Z) \cap X_0 \rightarrow Q_0 \) is an injective immersion.

**Proof.** In the case of \( X_0 = \varphi^{-1}(Q_0) \) the statement of the combination of [Ka21, Theorem 2.6] and [Ka21, Proposition 2.8(5)]. In the general case the proof goes without change if one observes that \( \alpha(Z) \) does not meet \( \varphi^{-1}(Q_0) \setminus X_0 \) by the transversality theorem ([AFKKZ, Theorem 1.11]). \( \square \)

**Proposition 2.9.** Let the assumptions and conclusions of Proposition 2.6 hold. Suppose that \( H \) itself is an \( F \)-flexible variety. Let \( Z \) be a locally closed reduced subvariety of \( H \) with \( \text{ED}(Z) \leq \dim X \) (and by the conclusions of Proposition 2.6 with \( \dim Z < \text{codim}_{H}(H \setminus U) \)). Then for a general element \( \beta \in \mathcal{B} \) in any perfect \( F \)-family \( \mathcal{B} \) of automorphisms of \( H \) the morphism \( \varphi_{\mathcal{H}}|_{\beta(Z)} : \beta(Z) \rightarrow X \) is an injective immersion.
Proof. Since \( \varphi_H|_U : U \to X \) is a smooth morphism, Formulas (2) and (3) hold with \( \varphi : X \to Q, Q_0 \) and \( X_0 \) replaced by \( \varphi_H : H \to X, X \) and \( U \) respectively. Hence, the desired conclusion follows from Theorem 2.8. \( \square \)

Corollary 2.10. Let the assumptions and conclusions of Proposition 2.6 hold and \( Z \) be an affine algebraic variety with \( \text{ED}(Z) \leq \dim X \) (and by the conclusions of Proposition 2.6 with \( \dim Z < \text{codim}_H(H \setminus U) \)). Suppose that each element of \( G \) is a unipotent group, i.e. \( H \simeq \mathbb{A}^t \) where \( t \geq \dim X \). Then \( Z \) can be treated as a closed subvariety of \( H \) and for a general element \( \beta \in B \) in any perfect \( F \)-family \( B \) of automorphisms of \( H \) the morphism \( \varphi_H|_{\beta(Z)} : \beta(Z) \to X \) is an injective immersion.

Proof. The first statement follows from Holme’s theorem and the second from Proposition 2.9. \( \square \)

Since every smooth flexible variety \( X \) admits a morphism \( \varphi_H : H \to X \) as Corollary 2.10, we have the following.

Theorem 2.11. ([Ka21, Theorem 3.7]) Let \( Z \) be an affine algebraic variety and \( X \) be a smooth quasi-affine flexible variety of dimension at least \( \text{ED}(Z) \). Then \( Z \) admits an injective immersion into \( X \).

Remark 2.12. It is worth mentioning that if \( \varphi : Z \to X \) is an injective immersion, then it may happen that \( Z \) is not isomorphic to \( \varphi(Z) \). As an example one can consider the morphism \( \mathbb{A}^* \to \mathbb{A}^2, t \mapsto (x, y) = (t^2+1, t(t^2+1)) \) where \( t \) is a non-vanishing coordinate on \( \mathbb{A}^* \). It maps \( \mathbb{A}^* \) onto the polynomial curve given in \( \mathbb{A}^2 \) by the equation \( y^2 = x^3(x-1) \).

We have also the following fact in our disposal.

Theorem 2.13. ([Ka21, Theorem 3.2]) Let \( \psi : X \to Y \) be a finite morphism where \( X \) is a smooth flexible variety and \( Y \) is normal. Let \( Z \) be a quasi-affine algebraic variety which admits a closed embedding in \( X \). Suppose also that \( \dim Z < \text{codim}_Y Y_{\text{sing}} \). Then \( Z \) admits a closed embedding in \( Y \) with the image contained in \( Y_{\text{reg}} \).

3. Criterion of properness

In this section we describe some conditions under which injective immersion from Theorem 2.11 are proper (implicitly these conditions appeared already in [Ka20] and [Ka21]).

Lemma 3.1. Let Notation 2.5 hold. Then \( \mathcal{H} = (H_1, \ldots, H_s) \) can be chosen so that the morphism \( \varphi_{\mathcal{H}} : H \to X \) is surjective and equidimensional. In particular, \( \dim H \times_X H = 2 \dim H - \dim X \).

Proof. Since \( G \) acts transitively on \( X \) one can choose \( \mathcal{H} \) such that \( \varphi_{\mathcal{H}} : H \to X \) is surjective [APKKZ, Proposition 1.5]. By Chevalley’s theorem [Ha, Chap. II, Exercise 3.22] we can present \( X \) as a disjoint union \( X = \bigcup_i X^i \) of locally closed irreducible subsets \( X^i \) of \( X \) such
that the morphism $\varphi_{\mathcal{H}|\tilde{\mathcal{H}}^{-1}(X^i)} : \varphi_{\tilde{\mathcal{H}}}^{-1}(X^i) \to X^i$ is equidimensional of relative dimension $k_i$ and, furthermore, $k_i > k_j$ if $X^i$ is contained in the closure $\overline{X^j}$ of $X^j$. In particular, we can suppose that $X^1$ is a dense open subset of $X$ and $k_1 = \dim H - \dim X$. Assume that $X \neq X^1$ and $k := \max_j k_j$. Let us show that increasing the number of elements in $\mathcal{H}$ we can reduce $k - k_1$ which would yield the desired conclusion.

Put $\mathcal{H} = (\mathcal{H}, \mathcal{H}') = (H_1, \ldots, H_s, H_{s+1}, \ldots, H_t)$ and $\tilde{\mathcal{H}} = H_t \times \ldots \times H_1$, i.e., we have $\varphi_{\tilde{\mathcal{H}}} : \tilde{\mathcal{H}} \to X$. Recall that by definition $\varphi_{\tilde{\mathcal{H}}}^{-1}(x) = \{h = (h_s, \ldots, h_1) | h_s \circ \ldots \circ h_1(x_0) = x\}$ and $\varphi_{\tilde{\mathcal{H}}}^{-1}((h_t \circ \ldots \circ h_{s+1})^{-1}(x)) = k_i < k$ for a general $h'$ and, hence, $\dim \varphi_{\tilde{\mathcal{H}}}^{-1}(x) < \dim H' + k$.

Now consider the case when $k_i = k$. Since the action of $G$ is transitive by [AFKKZ, Proposition 1.5] we can choose $\mathcal{H}'$ so that for every $X^i$ with $k_i = k$ and each $x \in X^i$ the orbit $H'.x$ meets $X^i$. Since $k_i < k$ and $(h_t \circ \ldots \circ h_{s+1})^{-1}(x) \in X^i$ for a general $h' \in H'$ we see again that $\dim \varphi_{\tilde{\mathcal{H}}}^{-1}(x) < \dim H' + k$.

In particular, the maximal dimension $\tilde{k}$ of fibers of $\varphi_{\tilde{\mathcal{H}}}$ is less that $\dim H' + k$, while the dimension of general fibers of $\varphi_{\tilde{\mathcal{H}}}$ is $\dim H' + k_1$. Hence, $\tilde{k} - (\dim H' + k_1) < k$ and we are done.

\begin{lemma}
Let Notation 2.5 hold. Suppose that $Y$ is the closure of $\bigcup_{h \in H} \ker\{(\varphi_{\mathcal{H}})_* : T_h H \to T_{\varphi_{\mathcal{H}}(h)} X\}$ in $TH$. Then $\mathcal{H}$ can be chosen so that $\dim Y = 2 \dim H - \dim X$.
\end{lemma}

\begin{proof}
By Lemma 3.1 we can suppose that $\varphi_{\mathcal{H}} : H \to X$ is surjective and equidimensional. By [H19, Chap. III, Corollary 10.7] we can present $H$ as a disjoint union $H = \bigcup H^i$ of smooth subset $H^i$ of $H$ such that the morphism $\varphi_{\mathcal{H}|H^i} : H^i \to X^i := \varphi_{\mathcal{H}}(H^i)$ is smooth. Note that $Y$ is the union of the closures $Y^i$ of $\bigcup_{h \in H^i} \ker\{(\varphi_{\mathcal{H}})_* : T_h H \to T_{\varphi_{\mathcal{H}}(h)} X\}$ in $TH$. Thus, it suffices to show that $\dim Y^i \leq 2 \dim H - \dim X$.

Since $\varphi_{\mathcal{H}|H^i}$ is smooth we see that $\dim (\varphi_{\mathcal{H}})_*(T_h H) \geq \dim X_i$ for every $h \in H^i$. Hence, $\dim \ker(\varphi_{\mathcal{H}})_* |_{T_h H} \leq \dim H - \dim X_i$ and, therefore, $\dim Y^i = \dim H + \dim H^i - \dim X^i$. Since $\varphi_{\mathcal{H}}$ is equidimensional its fibers have dimension $\dim H - \dim X$ and, consequently, the dimension of the fibers of $\varphi_{\mathcal{H}|H^i}$ does not exceed $\dim H - \dim X$. On the other hand, because of smoothness the latter dimension is $\dim H^i - \dim X^i$. Hence, $\dim Y^i \leq 2 \dim H - \dim X$ which yields the desired conclusion.
\end{proof}
Remark 3.3. It may happen that under the assumptions of Lemmas 3.1 and 3.2 one cannot find a sequence $\mathcal{H}$ for which the morphism $\varphi_{\mathcal{H}} : H \to X$ is smooth on $H$ (see, [AFKKZ, Remark 1.9]).

Now we can remove the assumption that $G$ is closed under conjugations in $G$ which was used in Proposition 2.6 and, consequently, in Proposition 2.9.

Proposition 3.4. Let Notation 2.5 hold and $\mathcal{H}$ be such that the conclusions of Lemmas 3.1 and 3.2 are true. Suppose that $H$ itself is an $F$-flexible variety and $Z$ is a closed subvariety of $H$ with $\text{ED}(Z) \leq \dim X$. Then for a general element $\beta \in B$ in any perfect $F$-family $B$ of automorphisms of $H$ the morphism $\varphi_{\mathcal{H}}|_{\beta(Z)} : \beta(Z) \to X$ is an injective immersion.

Proof. Note that the assumption of Theorem 2.8 are satisfied with $g : X \to Q, Q_0$ and $X_0$ replaced by $\varphi_{\mathcal{H}} : H \to X, X$ and $H$ respectively. This yields the desired conclusion. □

Now we can strengthen Corollary 2.10.

Corollary 3.5. Let Notation 2.5 hold and $\mathcal{H}$ be such that the conclusions of Lemmas 3.1 and 3.2 are true. Let $Z$ be an affine algebraic variety with $\text{ED}(Z) \leq \dim X$. Suppose that each element of $G$ is a unipotent group, i.e. $H \simeq \mathbb{A}^t$ where $t \geq \dim X$. Then $Z$ can be treated as a closed subvariety of $H$ and for a general element $\beta \in B$ in any perfect $F$-family $B$ of automorphisms of $H$ the morphism $\varphi_{\mathcal{H}}|_{\beta(Z)} : \beta(Z) \to X$ is an injective immersion.

Proof. The first statement follows from Holme’s theorem and the second one from Proposition 3.4. □

Switching from injective immersions to closed embeddings requires the following consideration. Every nonzero polynomial $f \in \mathbb{A}^{[0]}_t$ of degree $k \geq 0$ can be presented as $f = f_0 + f_1 + \ldots + f_k$ where $f_0$ is a constant term and each $f_i$, $i \geq 1$ is a homogeneous polynomial of degree $i$. We call $f_i$ the leading homogeneous part of $f$ and denote it by $\hat{f}_i$. Given a subalgebra $A$ of the polynomial ring $\mathbb{A}^{[t]}$ we denote by $\hat{A}$ the algebra generated by $\{\hat{f} | f \in A\}$.

Proposition 3.6. Let the assumptions and conclusions of Corollary 3.5 be satisfied (in particular, $\varphi_{\mathcal{H}} : H \to X$ is surjective and equidimensional and $\text{ED}(Z) \leq \dim X$) and let $X$ be affine. Suppose that $A \subset \mathbb{A}^{[t]}$ is the algebra $\varphi_{\mathcal{H}}^*({k[X]})$ (where $k[X]$ is the algebra of regular functions on $X$). Let $V$ be the zero locus of the ideal of $\hat{A}$ that consists of all polynomials in $\hat{A}$ with zero constant terms. Suppose that the codimension of $V$ in $H = \mathbb{A}^t$ is at least $\dim Z$. Then $Z$ admits a closed embedding into $X$. Furthermore, if $S$ is a closed subvariety of $X$ such that $\dim Z < \text{codim}_X S$, then image of $Z$ does not meet $S$. 

Proof. Let $\mathcal{A}$ be a perfect SAut$(H)$-family of automorphisms of $H \simeq \mathbb{A}^t$. Consider the natural embedding $\mathbb{A}^t \hookrightarrow \mathbb{P}^t$, $D = \mathbb{P}^t \setminus \mathbb{A}^t \simeq \mathbb{P}^{t-1}$ and $K = \text{SL}_t(k)$. Then we have the natural $K$-action on $\mathbb{P}^t$ such that $D$ is invariant under it and the restriction of the action to $D$ is transitive. By Proposition 2.6 $K \times \mathcal{A}$ is still a perfect SAut$(H)$-family of automorphisms of $H$. That is, for a general $\beta$ in $K$ and a general $\alpha$ in $\mathcal{A}$ the morphism $\varphi_H|_{\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X$ is still an injective immersion by Proposition 2.7 and Corollary 2.10.

By the assumption we can find generators $b_1, \ldots, b_m$ of $k[X]$ such that for the polynomials $a_i = b_i \circ \varphi_H$ the set of common zeros of $\hat{a}_1, \ldots, \hat{a}_m$ in $\mathbb{A}^t$ is $V$. Note also that $a_1, \ldots, a_m$ can be viewed as coordinate functions of $\varphi_H : \mathbb{A}^t \to X \subset \mathbb{A}^m$ and they can be extended to rational functions on $\mathbb{P}^t$. The intersection $R$ of the indeterminacy sets of these extensions is given by the common zeros of the homogeneous polynomials $\hat{a}_1, \ldots, \hat{a}_m$ in $D$. In particular, $R$ has codimension at least $\dim Z$ in $D$. As before we treat $Z$ as a subvariety of $\mathbb{A}^t$. Let $P$ be the intersection of $D$ with the closure of $\beta \circ \alpha(Z)$ in $\mathbb{P}^t$, i.e., $\dim P \leq \dim Z - 1$. Since the restriction of the $K$-action to $D$ is transitive $P$ does not meet $R$ for general $\beta \in K$ and $\alpha \in \mathcal{A}$ by [AFKKZ, Theorem 1.15]. Hence, $\varphi_H|_{\beta \circ \alpha(Z)} : \beta \circ \alpha(Z) \to X$ is a proper morphism by [Ka20, Corollary 5.4]. Consequently, it is a closed embedding which yields the first claim. Since $\varphi_H$ is equidimensional codim$_{\mathbb{A}^t} T = \text{codim}_X S$ for $T = \varphi^{-1}(S)$. Hence, by [AFKKZ, Theorem 1.15] we can suppose also that $\beta \circ \alpha(Z)$ does not meet $T$ which yields the second claim and concludes the proof. □

Remark 3.7. The problem of finding the codimension of $V$ in $H \simeq \mathbb{A}^t$ is the bottleneck of our technique. However, it is worth mentioning, perhaps, that the Krull dimension of $\hat{A}$ coincides with the Krull dimension of $A$ (in particular, polynomials $a_1, \ldots, a_m$ as in the proof of Proposition 2.6 can be chosen so that for general $c_1, \ldots, c_m \in k$ the locus of common zeros of $\hat{a}_1 - c_1, \ldots, \hat{a}_m - c_m$ has codimension $\dim X \geq \text{ED}(Z) > \dim Z$). Let us sketch the proof of this fact supplied to the author by L. Makar-Limanov (furthermore, according to him the fact is well-known but, unfortunately, the author did not find an appropriate reference). Let $B$ be a commutative finitely generated algebra and $W$ be a finite-dimensional subspace of $B$ which generates $B$ as algebra and which contains 1 (we call such $W$ a generating subset of $B$). Let $W^n$ be the span of all products of $n$ elements of $W$ and let $f(n) = \dim W^n$. Recall that $B$ has polynomial growth if there exist real positive numbers $c$ and $r$ such that for every $n$ one has $f(n) \leq cn^r$. The Gelfand-Kirillov dimension $\text{gk}(B)$ of $B$ is the infimum of the real numbers $r$ such that the latter inequality holds for some $c$ and all $n$. 
It is known that every affine domain $B$ has polynomial growth and the Gelfand-Kirillov dimension $\text{gk}(B)$ coincides with the Krull dimension of $B$ \cite{BoKr}. Let $U$ and $\hat{U}$ be generating subspaces of $A$ and $\hat{A}$ respectively. Assume that $\text{gk}(\hat{A}) < \text{gk}(A)$. Choose $p \in \mathbb{R}$ such that $\text{gk}(\hat{A}) < p < \text{gk}(A)$. Then for a sufficiently large $n$ one has $\dim \hat{U}^n < n^p$ and $\dim U^n > n^p$. However, we can always find a basis in $U^n$ with linearly independent leading homogeneous forms. That is, $\dim U^n$ cannot exceed $n^p$. This contradiction yields the desired conclusion.

**Definition 3.8.** Let $\psi : \mathbb{A}^n \to X$ be a dominant morphism, $A = \psi^*(\mathbb{K}[X])$, $\hat{A}$ be the algebra generated by the leading homogeneous parts of all polynomials from $A$ and $V \subset \mathbb{A}^n$ be the zero locus of the ideal of $\hat{A}$ that consists of all polynomials from $\hat{A}$ with zero constant terms. Then we call $V$ the associate subvariety of $\psi$.

The following trivial fact will be used for computing of the codimension of $V$.

**Lemma 3.9.** Let $\psi_i : \mathbb{A}^{n_i} \to X_i$, $i = 1, \ldots, l$ be dominant morphisms. Suppose that $\mathbb{A}^n = \mathbb{A}^{n_1} \times \ldots \times \mathbb{A}^{n_l}$ and $\psi = (\psi_1, \ldots, \psi_l) : \mathbb{A}^n \to X := X_1 \times \ldots \times X_l$. Let $V$ (resp. $V_i$) be the associate subvariety of $\psi$ (resp. $\psi_i$). Then $\text{codim}_{\mathbb{A}^n} V = \text{codim}_{\mathbb{A}^{n_1}} V_1 + \ldots + \text{codim}_{\mathbb{A}^{n_l}} V_l$.

**Proof.** A straightforward induction on $l$ implies that $V$ coincides with $(V_1 \times \mathbb{A}^{n_2} \times \ldots \times \mathbb{A}^{n_l}) \cap \ldots \cap (\mathbb{A}^{n_1} \times \ldots \times \mathbb{A}^{n_{l-1}} \times V_l)$ which yields the desired conclusion. \qed

4. **Main Theorem I**

In this section we shall prove our main result. Let us start with notations.

**Notation 4.1.** Given a polynomial $f$ in several variables $\vec{z} = (z_1, \ldots, z_m)$ we denote as before its leading homogeneous part by $\hat{f}$. Given a matrix $f = [f_{i,j}]$ over the ring $\mathbb{K}[\vec{z}]$ we denote by $\hat{f}$ the matrix $[\hat{f}_{i,j}]$.

The next fact is straightforward.

**Lemma 4.2.** Let $a^i$, $i = 1, 2$ be $(n \times n)$ matrices over $\mathbb{K}[\vec{z}]$ such that

$$a^i = \begin{bmatrix} \alpha^i & \beta^i \\ \gamma^i & \delta^i \end{bmatrix}$$

where $\delta^i \in \mathbb{K}[\vec{z}]$ and $\alpha^i$ is an $(n-1) \times (n-1)$ matrix. Let all entries of $\alpha^i$, $\beta^i$, $\gamma^i$ or $\delta^i$ be of degree $k_i$, $l_i$, $m_i$ and $\deg \delta^i$. Suppose also that every entry of $\alpha^i$, $\beta^i$ or $\gamma^i$ has only positive integer coefficients as an element of $\mathbb{K}[\vec{z}]$. Let

$$a^1 a^2 =: a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$
where \( d \in k[z] \) and \( \alpha \) is an \((n - 1) \times (n - 1)\) matrix. Then
\[
\hat{a} = \begin{pmatrix}
\hat{\alpha} & \hat{\beta} \\
\hat{\gamma} & \hat{d}
\end{pmatrix} = \begin{pmatrix}
\hat{\alpha}^1 \hat{\alpha}^2 & \hat{\alpha}^1 \hat{\beta}^2 \\
\hat{\gamma}^1 \hat{\alpha}^2 & \hat{\gamma}^1 \hat{\beta}^2
\end{pmatrix}.
\]

In particular, every entry of \( \hat{\alpha} \) is of degree \( k_1 + k_2 \), every entry of \( \hat{\beta} \) (resp. \( \hat{\gamma} \)) is of degree \( k_1 + l_2 \) (resp. \( k_2 + m_1 \)), \( \deg \hat{d} = m_1 + l_2 \) and each of these entries has positive integer coefficients.

**Lemma 4.3.** Let \( U \) (resp. \( L \)) be the unipotent subgroup of upper (resp. lower) triangular matrices in \( SL_n(k) \), i.e., \( U \times L \) is the affine space with coordinates \((a, b)\) where \( a = \{a_{i,j} | 1 \leq i < j \leq n\} \) and \( b = \{b_{i,j} | 1 \leq j < i \leq n\} \). Let \( a \in SL_n(k[a, b]) \) be of the form \( a = I + a_0 \) where \( I \) is the identity matrix and the \((i, j)\)-entry of \( a \) is \( a_{i,j} \) if \( i < j \) and 0 otherwise. Let \( b \in SL_n(k[a, b]) \) be of the form \( b = I + b_0 \) where the \((i, j)\)-entry of \( b \) is \( b_{i,j} \) if \( i > j \) and 0 otherwise. Suppose that \( c = ab \). Then
\[
\hat{c} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]
where the \((n - 1) \times (n - 1)\) matrix \( \alpha \) coincides with the matrix obtained from \( a_0b_0 \) by the removal of the \( n \)th row and the \( n \)th column, \( \gamma = (b_{n,1}, \ldots, b_{n,n-1}) \) and \( \beta \) is the transpose of \((a_{1,n}, \ldots, a_{n-1,n})\). In particular, all entries of \( \alpha \) are homogeneous quadratic polynomials with positive integer coefficients.

**Proof.** The statement follows from the fact that \( ab = I + a_0 + b_0 + a_0b_0 \) and the \( n \)th row and the \( n \)th column of \( a_0b_0 \) have all zero entries. \( \square \)

**Lemma 4.4.** Let for \( k = 1,\ldots,r \) the notations \( U^k, L^k, a^k, b^k, a^k_{i,j}, b^k_{i,j} \) play the same role as \( U, L, a, b, a_{i,j}, b_{i,j} \) in Lemma 4.3, i.e., \( \prod_{k=1}^r U^k \times L^k \) can be viewed as the affine space with coordinates \((a, b)\) where \( a = (a^k_{i,j})_{i,j,k} \) and \( b = (b^k_{i,j})_{i,j,k} \). Let \( c = a^1b^1a^2b^2\ldots a^rb^r \in SL_n(k[a, b]) \). Then
\[
\hat{c} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]
where every entry of the \((n - 1) \times (n - 1)\) matrix \( \alpha \) is a homogeneous polynomial of degree \( 2r \), \( \deg \hat{d} = 2r - 2 \), every entry of \( \beta \) and \( \gamma \) is a homogeneous polynomial of degree \( 2r - 1 \) and each of these polynomials has positive integer coefficients.

**Proof.** For \( r = 1 \) the statement follows from Lemma 4.3. Lemma 4.2 provides the induction step \( r - 1 \implies r \) and the desired conclusion. \( \square \)

**Proposition 4.5.** Let \( U^k, L^k, a^k, b^k, a^k_{i,j}, b^k_{i,j}, H \) be the same as in Lemma 4.4, i.e., the entries of the matrix \( c = a^1b^1a^2b^2\ldots a^rb^r \) are polynomials on the affine space \( H \). Let \( A \) be the subalgebra of \( k[H] \) generated by the entries of \( c \) and \( \hat{A} \) be the algebra generated by the leading homogeneous parts of elements of \( A \). Suppose that \( r \geq 2 \) and \( V \).
is the zero locus of the ideal in $\hat{\mathcal{A}}$ consisting of all polynomials with zero constant terms. Then the codimension of $V$ in $H$ is at least $n(n-2) + 1$.

Proof. Let $t = \dim H$. Recall that if a closed subvariety $Z$ of $\mathbb{P}^{n-1}$ does not meet a linear subspace of dimension $l-1$ then the codimension of $Z$ is at least $l$. Hence, since $V$ is the set of common zeros of a collection of homogeneous polynomials of positive degrees it suffices to find a linear subspace $L$ of $H$ of dimension $n(n-2) + 1$ which meets $V$ at the origin only. Denote $b_{i,n}^1$ by $v$ and consider $L$ given by

(i) $a_{i,j}^1 = b_{i,j}^k = v$ for all $k = 3, \ldots, r$ and $\{(i, j)|1 \leq i < j \leq n\}$ and $\{(i', j')|1 \leq j' < i' \leq n\}$;

(ii) $a_{i,j}^1 = b_{i,j}^2 = 0$ for $\{(i, j)|1 \leq i < j \leq n\}$ and $\{(i', j')|1 \leq j' < i' \leq n\}$;

(iii) $a_{2,n}^1 = v$ and $a_{2,n}^2 = \ldots = a_{n-1,n}^2 = 0$.

Counting the number of equations we see that $\dim L = n(n-1) - (n-1) = n(n-2) + 1$. Since $V$ is contained in the common zeros of non-constant entries of $\hat{\mathbf{c}}$ it remains to show that $\hat{\mathbf{c}}|L$ has all entries proportional to a power of $v$ with one of them being nonzero. Let $\mathbf{c}' = \mathbf{a}^1 \mathbf{b}^1 \mathbf{a}^2 \mathbf{b}^2$ and $\mathbf{c}' = \mathbf{a}^3 \mathbf{b}^3 \ldots \mathbf{a}^r \mathbf{b}^r$, i.e., $\mathbf{c} = \mathbf{c}' \mathbf{c}'^r$. By Lemma 4.3 the matrices $\mathbf{a}^r \mathbf{b}^r$ have the same form as matrices participating in Lemma 4.2. Hence, Lemma 4.2 implies that $\hat{\mathbf{c}}|L = \mathbf{a}^1 \mathbf{b}^1 |L \mathbf{a}^2 \mathbf{b}^2|L$. In combination with Lemma 4.3 and the description of $L$ this shows that $\hat{\mathbf{c}}|L$ is the matrix with all zero entries except for the one in position $(n, n)$ which is $v^2$. Note again that $\hat{\mathbf{c}}|L = \hat{\mathbf{c}}' |L \hat{\mathbf{c}}'^r|L$ by Lemma 4.2. It follows from Lemma 4.4 that up to a nonzero coefficient every entry of $\hat{\mathbf{c}}'^r|L$ is a power of $v$. Hence, up to coefficients every entry of $\hat{\mathbf{c}}$ is a power of $v$ and the one in position $(n, n)$ is nonzero which yields the desired conclusion.

\[\square\]

Theorem 4.6. Let $Z$ be an affine algebraic variety, $Y$ be a variety of the form $\mathbb{A}^{n_0} \times \text{SL}_{n_1}(k) \times \text{SL}_{n_2}(k) \times \ldots \times \text{SL}_{n_l}(k)$ where $n_0 \geq 0, l \geq 1$ and $n_i \geq 2$ for $i \geq 1$. Suppose that $\varphi : Y \to X$ is a finite morphism into a normal variety $X$, $\dim X \geq \text{ED}(Z)$ and $\dim Z < \text{codim}_X X_{\text{sing}}$. Then $Z$ admits a closed embedding in $X$ with the image contained in $X_{\text{reg}}$.

Proof. By Theorem 2.13 it suffices to consider the case when $X$ is isomorphic to $\mathbb{A}^{n_0} \times \text{SL}_{n_1}(k) \times \ldots \times \text{SL}_{n_l}(k)$. First we let $X = \text{SL}_n(k)$. The natural action of $G = \text{SL}_n(k)$ on $X$ is transitive and $G$ is generated by subgroups $U$ and $L$ as in Lemma 4.3. Consider $H = \prod_{k=1}^r U^k \times L^k$ as in Lemma 4.4 and the morphism

$\varphi : H \to X$, $\left(\mathbf{a}^1, \mathbf{b}^1, \ldots, \mathbf{a}^r, \mathbf{b}^r\right) \mapsto \mathbf{c} = \mathbf{a}^1 \mathbf{b}^1 \ldots \mathbf{a}^r \mathbf{b}^r$.

By Lemmas 3.1 and 3.2 we can suppose that $\varphi$ is surjective, $\dim H \times_X H = 2 \dim H - \dim X$ and $\dim Y = 2 \dim H - \dim X$ where $Y$ is the closure of the constructible set $\bigcup_{h \in H} \ker\{\varphi_* : T_hH \to T_{\varphi h(h)}X\}$ in $TH$. Since $H \simeq \mathbb{A}^t$ where $t > \dim X = n^2 - 1$ we can treat $Z$ as a
closed subvariety of $H$ by Holme’s theorem. By Corollary 3.5 we can suppose that $\varphi|_Z : Z \to X$ is an injective immersion. Let $V$ be as in Proposition 4.5, i.e., $V \subseteq H$ is the associate subvariety of $\varphi$ (see, Definition 3.8). Since $\text{ED}(Z) \leq \dim X$ we see that $\dim Z$ is at most $\frac{\dim X - 1}{2}$, whereas by Proposition 4.5 $\codim_H V \geq \frac{\dim X - 1}{2}$ (where the equality occurs only in the case $n = 2$). Hence, by Proposition 3.6 we can suppose that $\varphi|_Z : Z \to X$ is a closed embedding.

Returning to the case of $X = X_0 \times X_1 \times \ldots \times X_l$ where $X_0 \simeq \mathbb{A}^n_0$ and $X_i \simeq \text{SL}_{m_i}(k)$ for $i \geq 1$ we suppose that $\varphi_i : H^i \to \text{SL}_{m_i}(k)$ and $V_i \subseteq H^i$ play the same role for $X_i$ as $\varphi : H \to \text{SL}(k)$ and $V \subseteq H$ for $\text{SL}(k)$. We also let $H^0 = \mathbb{A}^n_0$ act on $X_0 \simeq \mathbb{A}^n_0$ by translations which yields a linear isomorphism $\varphi_0 : H^0 \to X_0$. Let again $V_0 \subseteq H^0$ be the associate subvariety of $\varphi_0$. Then $V_0$ is the origin of $\mathbb{A}^n_0$ since $\varphi_0$ is linear, i.e., it has codimension $n_0$ in $X_0$. Consider $\tilde{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_l) : \tilde{H} = H^0 \times H^1 \times \ldots \times H^l \to X$ and $B = \tilde{\varphi}^*(k[X])$. Let $\tilde{V} \subseteq \tilde{H}$ be the associate variety of $\tilde{\varphi}$. By Lemma 3.9 $\codim_{\tilde{H}} \tilde{V} = \codim_{H^0} V_0 + \ldots + \codim_{H^l} V_l$. Thus, by the earlier consideration $\codim_{\tilde{H}} \tilde{V} \geq \frac{\dim X - 1}{2}$ in $\tilde{H}$, whereas $\dim Z \leq \frac{\dim X - 1}{2}$. Treating $Z$ as a closed subvariety of $\tilde{H}$ we can suppose by Proposition 3.6 that $\varphi|_Z : Z \to X$ is a closed embedding which concludes the proof.

Corollary 4.7. Let $Z$ be an affine algebraic variety and $X$ be an algebraic variety isomorphic to a semi-simple linear algebraic group whose Lie algebra is a sum of special linear Lie algebras. Suppose that $\dim X \geq \text{ED}(Z)$. Then $Z$ admits a closed embedding into $X$.

Proof. By virtue of Theorem 4.6 the claim follows from the fact that $X$ is isomorphic to the quotient of a group $\text{SL}_{m_1}(k) \times \text{SL}_{m_2}(k) \times \ldots \times \text{SL}_{m_l}(k)$ with respect to a finite subgroup in its center. \hfill \Box

5. Main Theorem II

Lemma 5.1. Let $\varphi_i : H^i \simeq \mathbb{A}^n_i \to X_i$ be dominant morphisms to affine varieties $X_i$, $i = 1, 2$ and $V_i \subseteq H^i$ be the associate subvariety of $\varphi_i$. Let $V \subseteq H = H^1 \times H^2$ be the associate subvariety of $\varphi = (\varphi_1, \varphi_2) : H \to X = X_1 \times X_2$. Then $\codim_H V \geq \codim_{H^1} V_1 + 1$.

Proof. The proof follows from Lemma 3.9 and the fact that $\codim_{H^1} V_1 \geq 1$. \hfill \Box

Proposition 5.2. Let the assumptions of Lemma 5.1 hold, each $X_i$ be smooth affine and $\varphi_i$ be surjective and equidimensional. Suppose that $S$ is a closed subvariety of $X$ and $Z$ is an affine algebraic variety such that $\codim_H V_2 \geq \dim Z - 1$ and $\dim Z < \text{ED}(X)$. Then there exists a closed embedding of $Z$ into $X$ such that the image of $Z$ does not meet $S$. 
Proof. By Lemma 5.1 \( \text{dim} \ Z \leq \text{codim}_H V \). Hence, the desired conclusion follows from Proposition 3.6. \( \square \)

**Theorem 5.3.** Let \( Z \) be an affine algebraic variety, \( Y \) be a smooth affine algebraic variety of the form \( X_1 \times \mathbb{A}^n \) where \( X_1 \) is flexible, \( n \geq \text{dim} Z - 1 \) and \( \text{dim} Y \geq \text{ED}(Z) \). Suppose that \( \psi : Y \rightarrow W \) is a finite morphism onto a normal variety \( W \) and \( \text{dim} Z < \text{codim}_W W_{\text{sing}} \). Then \( Z \) admits a closed embedding into \( W \) with the image contained in \( W_{\text{reg}} \). Furthermore, if \( R \) is a closed subvariety of \( W \) such that \( \text{dim} Z < \text{codim}_W R \), then one can suppose that the image does not meet \( R \).

Proof. Since \( X_1 \) is flexible \( \text{SAut}(X) \) acts transitively on \( X \). Hence, by Lemma 3.1 there exists a surjective equidimensional morphism \( \varphi_1 : H^1 \cong \mathbb{A}^n \rightarrow X_1 \). Let \( H^2 \cong \mathbb{A}^n \) and \( \varphi_2 : H^2 \rightarrow \mathbb{A}^n \) be a linear isomorphism. As we saw in the proof of Theorem 4.6 the associate subvariety \( V_2 \) of \( \varphi_2 \) is a singleton, i.e., its codimension in \( H^2 \) is \( n \). By Proposition 5.2 there exists a closed embedding of \( Z \) into \( Y \) such that the image does not meet \( \psi^{-1}(R) \). Hence, the desired conclusion follows from Theorem 2.13. \( \square \)

**References**

[AFKKZ] I. V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, M. Zaidenberg, Flexible varieties and automorphism groups. Duke Math. J. 162 (2013), no. 4, 767–823.

[BMS] S. Bloch, M. Pavaman Murthy, L. Szpiro, Zero cycles and the number of generators of an ideal, 38, 1989, Colloque en l’honneur de Pierre Samuel (Orsay, 1987), pp. 51-74.

[BoKr] W. Borho, H. Kraft, Über die Gelfand-Kirillov-Dimension, Math. Ann. 220 (1976), no. 1, 1-24.

[FS21] P. Feller, I. van Stamphli, Existence of embedding of smooth varieties into linear algebraic groups, preprint, arXiv:2007.16164.

[FKZ] H. Flenner, S. Kaliman, and M. Zaidenberg, A Gromov-Winkelmann type theorem for flexible varieties, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 11, 2483-2510.

[Fr] G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations, Encyclopaedia of Mathematical Sciences, Springer, Berlin-Heidelberg-New York, 2006.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York-Heidelberg, 1977.

[Hol] Holme, Audun, Embedding-obstruction for singular algebraic varieties in \( \mathbb{P}^N \), Acta Math. 135 (1975), no. 3-4, 155-185.

[Ka91] S. Kaliman, Extensions of isomorphisms between affine algebraic subvarieties of \( k^n \) to automorphisms of \( k^n \), Proc. Amer. Math. Soc. 113 (1991), no. 2, 325-334.

[Ka20] S. Kaliman, Extensions of isomorphisms of subvarieties in flexible varieties, Transform. Groups 25 (2020), no. 2, 517-575.

[Ka21] S. Kaliman, Lines in affine toric varieties, Israel J. of Mathematics (to appear).
[KaUd] S. Kaliman, D. Udumyan, *On automorphisms of flexible varieties*, arXiv:2008.02221.

[Ra] C. P. Ramanujam, *A note on automorphism groups of algebraic varieties*, Math. Ann. 156 (1964), 25–33.

[Sr] V. Srinivas, *On the embedding dimension of an affine variety*, Math. Ann., 289 (1991), no.1, 25-132.

[Swan] R. G. Swan, *A cancellation theorem for projective modules in the metastable range*, Invent. Math. 27 (1974), 23-43.

University of Miami, Department of Mathematics, Coral Gables, FL 33124, USA

Email address: kaliman@math.miami.edu