Nonabelian Gauge Antisymmetric Tensor Fields

S.N. Solodukhin

Department of Theoretical Physics, Physics Faculty of Moscow University, Moscow 117234, Russia

Abstract

We construct the theory of non-abelian gauge antisymmetric tensor fields, which generalize the standard Yang-Mills fields and abelian gauge p-forms. The corresponding gauge group acts on the space of inhomogeneous differential forms and it is shown to be a supergroup.

We construct the theory of non-abelian gauge antisymmetric tensor fields, which generalize the standard Yang-Mills fields and abelian gauge p-forms. The corresponding gauge group acts on the space of inhomogeneous differential forms and it is shown to be a supergroup. The wide class of generalized Chern-Simons actions is constructed. The wide class of generalized Chern-Simons actions is constructed.

1 Introduction

Recently much attention has been payed to the study of theoretical models with abelian gauge antisymmetric tensor fields (AGATF). It turned out that AGATF play an important role in the string theory, supergravity and gauge theory of gravity [1]. An important circumstance is that the AGATF can be described using differential-geometric methods as differential forms (in which case the parameters of gauge transformations are also differential forms). On the other hand, the fermionic matter fields can be described with help of a set of antisymmetric tensor fields of arbitrary rank (i.e. inhomogeneous differential forms) [2,3].

During the last years the surprising relation of gauge fields to the space-time topology was discovered. In the works of Donaldson [4] and Witten [5] such a relation was demonstrated for the standard gauge fields. On the other hand it was
shown by Horowitz [6], and earlier by Schwarz [7], that AGATF give us an example of so-called "topological field theory" and lead to appearance of topological invariants (such as the Ray-Singer torsion and linking numbers) under quantization. In [6] the minimal non-abelian generalization of AGATF was also considered. However, so far there doesn’t exist a satisfactory generalization of AGATF for non-abelian case. Attempts of such a generalization [8] result in an essentially nonlocal and nonlinear theory. Hence an actual problem is to construct a theory of non-abelian gauge antisymmetric tensor fields (NGATF). These fields could find numerous applications in many theoretical models where so far one has used the AGATF. There is also a hope that NGATF will give us new tools for the study of space-time topology. It is a goal of this paper to attempt in construction of NGATF theory.

At present we know two kinds of gauge fields. First is the standard (Yang-Mills) non-abelian gauge field which is described as the Lie algebra valued one-form $A^{(1)}$ with the infinitesimal gauge transformation law:

$$\delta_{\alpha} A^{(1)} = -d\alpha(0) - A^{(1)}\alpha(0) + \alpha(0) A^{(1)}$$  \hspace{1cm} (1.1)

where $A^{(1)} = A^a_{\mu} \lambda^a dx^{\mu}$, matrices $\lambda^a$ are the generators of a gauge group $G$, $\alpha(0) = \alpha^a \lambda^a$ is an infinitesimal transformation parameter - the zero-form with values in the Lie algebra of the group $G$.

Second is the theory of AGATF which are n-forms $A^{(n)}$ with the gauge transformation law:

$$\delta_{\alpha} A^{(n)} = -d\alpha^{(n-1)}$$  \hspace{1cm} (1.2)

where $(n-1)$-form $\alpha^{(n-1)}$ is an infinitesimal transformation parameter. Our aim is to construct a theory of NGATF which would be n-forms $A^{(n)}$ with values in the Lie algebra of a group $G$:

$$A^{(n)} = \frac{1}{n!} A^a_{\mu_1...\mu_n} \lambda^a dx^{\mu_1} \wedge ... \wedge dx^{\mu_n}$$  \hspace{1cm} (1.3)

Such a theory should generalize both the Yang-Mills model and AGATF theory in that the transformation laws (1.1),(1.2) are recovered as the particular cases of a gauge transformation law for fields (1.3).

One could naively expect the following transformation law for $A^{(n)}$:

$$\delta_{\alpha} A^{(n)} \sim -d\alpha^{(n-1)} - A \wedge \alpha^{(n-1)} + \alpha^{(n-1)} \wedge A$$  \hspace{1cm} (1.4)

where

$$\alpha^{(n-1)} = \frac{1}{(n-1)!} \alpha^a_{\mu_1...\mu_n} \lambda^a dx^{\mu_1} \wedge ... \wedge dx^{\mu_{n-1}}$$  \hspace{1cm} (1.5)

is an infinitesimal parameter $(n-1)$-form with values in the algebra of group $G$.

However the formula (1.4) is evidently senseless, since the last terms in (1.4) are differential form of rank $(2n-1)$, while the left-hand side of (1.4) is the n-form.
Only for the case \( n=1 \) this formula has sense. In order to make sense of (1.4) let us consider a set of gauge forms \( A_{(n)}, A_{(1)} \) such that (1.4) is replaced by

\[
d_\alpha A_{(n)} \sim -\alpha_{(n-1)} \wedge A_{(1)} + \alpha_{(n-1)} \wedge A_{(1)}
\]  

\[\tag{1.6}\]

Natural generalization appears to take a complete set of forms \( A_{(n)}, A_{(n-1)}, ..., A_{(p)}, ..., A_{(1)} \) with a transformation law

\[
d_\alpha A_{(n)} \sim -d\alpha_{(n-1)} - \sum_k A_{(k)} \wedge \alpha_{(n-k)} + \alpha_{(n-k)} \wedge A_{(k)}
\]  

\[\tag{1.7}\]

where \( \alpha_{(0)}, \alpha_{(1)}, ..., \alpha_{(p)}, ..., \alpha_{(n-1)} \) are infinitesimal gauge parameters - the set of \( k \)-forms with values in algebra of group \( G \).

In the standard Yang-Mills theory the gauge transformations act on a multiplet of matter fields:

\[
d_\alpha \phi^i = \alpha^i_j \phi^j
\]  

\[\tag{1.8}\]

Both \( \phi \) and parameters are zero-forms.

A natural generalization is to describe the matter fields by \( n \)-forms, and parameters by \( n \)-forms. However,

\[
\phi^i_{(n)} \rightarrow \phi^i_{(n)} + \alpha^i_{(k)j} \wedge \phi^j_{(n)}
\]  

\[\tag{1.9}\]

transfers \( n \)-forms into \((n+k)\)-forms. Hence one should consider the set of forms of all possible ranks on a given manifold, i.e. the multiplet of inhomogeneous differential forms:

\[
\Psi^i = \sum_k \frac{1}{k!} \phi^i_{\mu_1...\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}
\]  

\[\tag{1.10}\]

Then the transformation generated by inhomogeneous forms

\[
\alpha^i_j = \sum_k \frac{1}{k!} \alpha^i_{j\mu_1...\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}
\]  

\[\tag{1.11}\]

acts on the field (1.10) as follows

\[
d_\alpha \Psi^i = \alpha^i_j \wedge \Psi^j
\]  

\[\tag{1.12}\]

Notice, that such a generalization was initiated by the study of fermionic matter description in terms of inhomogeneous differential forms [2,3]. In particular in [3] the example of supersymmetry transformation mixing the forms of different rank was considered.
2 Infinitesimal gauge transformations

Let us consider an action for matter fields described by differential form Ψ taking values in an linear space \( \mathcal{L}^N \) (1.10) which is invariant under the global transformation (1.12):

\[
\Psi \rightarrow \Psi + \alpha \wedge \Psi
\]  
(2.1)

where the infinitesimal transformation parameter \( \alpha \) (which is matrix-valued inhomogeneous form (1.11) ) has constant coefficients. The kinetic term of an action depends on \( d\Psi \).

Under global transformations (2.1) one has

\[
d\Psi \rightarrow d\Psi + \eta \alpha \wedge d\Psi
\]  
(2.2)

where the operator \( \eta \) acts on the forms as follows:

\[
\eta \alpha = \sum_k \frac{(-1)^k}{k!} \alpha_{\mu_1...\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}
\]  
(2.3)

This is the involution, one easily sees that \( \eta^2 = id \). Then under local transformations (2.1) (when coefficients \( \alpha_{\mu_1...\mu_k} \) depend on \( x \)) one obtains

\[
d\Psi \rightarrow d\Psi + \eta \alpha \wedge d\Psi + d\alpha \wedge \Psi
\]  
(2.4)

In order to compensate the last term one should introduce the gauge field - an matrix-valued inhomogeneous form \( A \) (1.3):

\[
A = \sum_k A(k)
\]  
(2.5)

and consider the covariant derivative:

\[
\nabla \Psi = d\Psi + A \wedge \Psi
\]  
(2.6)

which transforms under local transformations (2.1) as follows

\[
\nabla \Psi \rightarrow \nabla \Psi + \eta \alpha \wedge \nabla \Psi
\]  
(2.7)

Then the gauge invariant action is obtained substituting external derivative \( d\Psi \) by covariant derivative \( \nabla \Psi \).

From (2.7) one obtains the transformation law for the gauge field \( A \):

\[
A \rightarrow A - d\alpha + \eta \alpha \wedge A - A \wedge \alpha
\]  
(2.8)

In terms of the inhomogeneous components (2.5) this formula reads

\[
A(n) \rightarrow A(n) - d\alpha(n-1) + \sum_{k=0}^{n-1} [(-1)^k \alpha(k) \wedge A(n-k) - A(n-k) \wedge \alpha(k)]
\]  
(2.9)
This formula makes more precise our preliminary discussion in the Introduction (see (1.7)).

It is easy to see that indeed the transformation law (2.8), (2.9) contains the laws (1.1), (1.2) as particular cases.

The curvature form for AGATF
\[ F_{(n+1)} = dA_{(n)} \] (2.10)
is invariant under gauge transformation (1.2), while the curvature for the standard Yang-Mills gauge field
\[ F_{(2)} = dA_{(1)} + A_{(1)} \wedge A_{(1)} \] (2.11)
under a gauge transformations (1.1) changes as
\[ \delta_\alpha F_{(2)} = \alpha(0)F_{(2)} - F_{(2)}\alpha(0) \] (2.12)

Let us find the generalized expression for the NGATF curvature with the transformation law
\[ F \rightarrow F + \alpha \wedge F - F \wedge \alpha \] (2.13)

It is easy to see that the proper generalization reads
\[ F = dA - (\eta A) \wedge A \] (2.14)

This satisfies all necessary conditions.

For homogeneous components \( F_{(k)} : F = \sum_k F_{(k)} \) one has
\[ F_{(n+1)} = dA_{(n)} - \sum_{k=1}^{n} (-1)^k A_{(k)} \wedge A_{(n-k+1)} \] (2.15)

It is also easy to verify the Jacobi identity for the curvature \( F \),
\[ dF + A \wedge F - (\eta F) \wedge A = 0 \] (2.16)

In the standard Yang-Mills theory one usually consider the gauge one-form \( A_{(1)} \) to be antihermitean. However, there are no reasons for such a restrictions in our general construction.

3 The gauge group

Let us discuss now the group structure of transformations (1.12) acting on the space of inhomogeneous forms \( \Psi \) with values in a N-dimensional linear space (see (1.10)).
Let $G$ be an inhomogeneous form with values in a space of matrices $GL(N,C)$:

$$G = \sum_k G_{(k)}$$

$$G_{(k)} = \frac{1}{k!} G^i_{j\mu_1...\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$$  \hspace{1cm} (3.1)$$

We determine the group multiplication of such forms as follows:

$$G' \circ G'' = \xi_{ij}^{G'} \wedge G''_{ij}$$  \hspace{1cm} (3.2)$$

In general case the exterior algebra is not a group, since not for every element one can determine the inverse element with respect to the wedge product $\wedge$. The existence condition for such an inverse element is the nondegenerate scalar part $G_{(0)} \neq 0$ in abelian case (when $\{G\} = \Lambda^* M$), and

$$\det[G_{(0)ij}] \neq 0$$  \hspace{1cm} (3.3)$$

in the non-abelian case. Consequently one must consider the space of all forms (3.1) satisfying the condition (3.3).

In order to construct an action for the matter fields (1.10) it is necessary to have an invariant scalar product on the space of inhomogeneous differential forms (1.10). The standard bilinear form

$$\int * \Psi^+ \wedge \Psi$$  \hspace{1cm} (3.4)$$

is invariant under transformations

$$\Psi \rightarrow G \wedge \Psi$$  \hspace{1cm} (3.5)$$

if $G$ has only a zero rank component. Notice that (3.4) demands a metric on manifold M which enters via the Hodge operator $*$. Instead, let us consider the following scalar product for forms on a D-dimensional manifold $M^D$.

$$\langle \phi, \psi \rangle = \int_{M^D} \xi \phi^+ \wedge \psi = \sum_{k=0}^D \int_{M^D} \xi \phi^+_{(k)} \wedge \psi_{(D-k)}$$  \hspace{1cm} (3.6)$$

where the operator $\xi$ acts on differential forms as follows

$$\xi \phi = \sum_{k=0}^D \frac{1}{k!} (-1)^{k(k-1)/2} \phi_{\mu_1...\mu_k} dx^{\mu_1} \wedge ... dx^{\mu_k}$$  \hspace{1cm} (3.7)$$

Notice that $\xi$ is the anti-involution on $\Lambda^* M^D$,

$$\xi(\phi \wedge \psi) = (\xi \psi) \wedge (\xi \phi)$$  \hspace{1cm} (3.8)$$
For invariance of (3.6) under transformations (3.5) it is sufficient to set the condition

$$\xi G^+ \wedge G = 1$$  \hspace{1cm} (3.9)

Clearly, all such $G$ form a group.

Let us consider

$$G = 1 + \alpha$$  \hspace{1cm} (3.10)

where $\alpha$ is an infinitesimal differential form.

Then from (3.9) one finds

$$\xi \alpha^+ = -\alpha$$  \hspace{1cm} (3.11)

General solution of this equation reads

$$\alpha = \sum_{k=0}^{D} \frac{(k+1)}{k!} (\rho_{(k)}^a \lambda^a + \rho_{(k)} I)$$  \hspace{1cm} (3.12)

where $\rho_{(k)}^a = (\rho_{(k)}, \rho_{(k)}^a)$ are k-forms with real coefficients, and $N \times N$ matrices $\lambda^a = (I, \lambda^a)$ are the antihermitean generators of the unitary group $U(N)$ with the structure constants $f^{abc}$. It should be noted that there is no additional condition (like $\det g = 1$ for usual unitary group) for group elements $G$ (3.9) in order to take away abelian subgroup generated by $\rho_{(k)} I$ (3.12).

Thus the generators of the group (3.10) has the form

$$X^a_{(s)} = \lambda^a I_{(s)};$$

$$I_{(s)} = \frac{1}{k!} dx^i \wedge ... \wedge dx^i, k = 0, ..., D$$  \hspace{1cm} (3.13)

These generators satisfy

$$X^a_{(s)} X^b_{(s')} - (-1)^{|s||s'|} X^b_{(s')} X^a_{(s)} = f^{abc} X^c_{(s+s')}$$  \hspace{1cm} (3.14)

and so they define a superalgebra. Hence the group (3.10) is a supergroup. Roughly, it is unitary group $U(N)$ with inhomogeneous differential forms as parameters. Notice, that the supergroups appear in ref.[9] in a similar manner.

In Appendix we find the general form for a group $G$ elements satisfying equation (3.9):

$$G = g_0 (1 + H + F)$$  \hspace{1cm} (3.15)

where zero-form $g_0$ is an element of the unitary group $U(N)$:

$$g_0^+ g_0 = 1$$  \hspace{1cm} (3.16)

and $H$ and $F$ are inhomogeneous forms of rank $\geq 1$ which satisfy the conditions:

$$\xi H^+ = -H$$

$$\xi F^+ = F$$  \hspace{1cm} (3.17)
The form $\mathcal{F}$ is constructed as a polynomial of $H$ (see Appendix): $\mathcal{F} = \sum a_k H^{2k}$ relatively of wedge product.

We can also define the subgroup of group $G$ (3.15) as follows:

$$G^K_D \equiv \{ G = g_0(1 + H + \mathcal{F})| H_{(2k-1)} = 0, k = 1, \ldots, K \} \quad (3.18)$$

This subgroup will be useful in the next Section to write the Chern-Simons terms for NGATF.

At the end of this Section let us write the explicit matter field actions invariant under global transformations (3.5), (3.9):

$$S = \int_{M^D} \eta \tilde{\psi} \wedge d\phi$$

$$S = \int_{M^D} d\tilde{\psi} \wedge d\phi$$

(3.19)

where

$$\tilde{\psi} = \xi \psi^+$$

(3.20)

The gauge invariant version of these actions is

$$S = \int_{M^D} \eta \tilde{\psi} \wedge \nabla \phi$$

$$S = \int_{M^D} \nabla \tilde{\psi} \wedge \nabla \phi$$

(3.21)

where $\nabla$ is the covariant derivative (2.6).

Notice that we do not use metric on the space-time manifold $M^D$ to write the actions (3.19),(3.21). Hence one can expect to obtain the description of some topological invariants as quantum observables in the models (3.19), (3.21).

4 Generalized Chern-Simons for NGATF

In order to treat NGATF as a dynamical field one must formulate the action for its description. It should be noted that the usual action

$$\int Tr(* F \wedge F)$$

(4.1)

which is invariant in the abelian case cannot be directly generalized to the non-abelian case. This was the main problem in earlier works on NGATF (see [8]). In our formalism there is a natural possibility to overcome the difficulty with the help of above introduced invariant scalar product for inhomogeneous differential forms (3.6). One can suggest the two expressions for the NGATF action:

$$(F, F) = \int_{M^D} Tr(\tilde{F} \wedge F)$$

(4.2)
and
\[ (\tilde{F}, F) = \int_{M^D} Tr(F \wedge F) \quad (4.3) \]

It is easy to see that both these expressions are invariant with respect to the gauge transformations (2.13) if the space-time dimension D is odd. However if D is even we have for the variation of (4.2), (4.3):

\[ \Delta_\alpha(F, F) = \int Tr[(\alpha - \eta \alpha) \wedge F \wedge F] \quad (4.4) \]
\[ \Delta_\alpha(\tilde{F}, F) = \int Tr[(\alpha - \eta \alpha) \wedge \tilde{F} \wedge F] \quad (4.5) \]

Consequently, in order to (4.2), (4.3) be gauge invariant, the infinitesimal form \( \alpha \) should be even:

\[ \eta \alpha = \alpha \quad (4.6) \]

More precisely, since the curvature form \( F \) has rank \( \geq 2 \), so the homogeneous components of \( \alpha \) should satisfy condition:

\[ \alpha_{(2k+1)} = 0, k = 0, \ldots, \frac{D}{2} - 3 \quad (4.7) \]

In other words for even space-time dimension D the gauge group for (4.2-3) is \( G_{D/2-3}^D \) (see (3.18)).

Here we can restrict ourselves to the particular case in even D (see also [10]):

\[ \eta A = -A, \]
\[ \eta F = F \quad (4.8) \]

Then from action (4.2) one obtains the equations of motion:

\[ d\tilde{F} + A \wedge \tilde{F} - (\eta \tilde{F}) \wedge A = 0, \text{ (for odd D)} \]
\[ d\tilde{F} + A \wedge \tilde{F} - \tilde{F} \wedge A = 0, \text{ (for even D)} \quad (4.9) \]

It is easy to see that if the curvature \( F \) satisfy to the one of the conditions

\[ \tilde{F} = +F \]
\[ \tilde{F} = -F \quad (4.10) \]

then the equations (4.9) hold identically as a consequence of the Jacobi identity (2.16). These conditions (4.10) are similar to the self- and antiself-duality conditions for instantons in the standard Yang-Mills theory. As concerns the functional (4.3) one can show that under the same conditions on the gauge group (4.7) and on the
gauge field (4.8) in even D case, the (4.3) is the total derivative in view of closure of the form

\[ d\omega^{(2)} = 0 \]
\[ \omega^{(2)} = Tr(F \wedge F) \]  (4.11)

Hence integral over the form \( \omega^{(2)} \) (4.3) is not an action but is an analog of the Pontryagin class. It is well known, that given the closed invariant differential form

\[ d\omega = 0, \]
\[ \Delta_\omega \omega = 0 \]  (4.12)
one can determine the Chern-Simons term

\[ \omega = d\Gamma_{cs} \]  (4.13)

For the form \( \omega^{(2)} \) (4.11) the Chern-Simons term \( \Gamma_{cs} \) has the form

\[ \Gamma_{cs} = AdA - \frac{2}{3} A(\eta A)A \]  (4.14)

independently of the value of D. Notice that both \( \omega^{(2)} \) and \( \Gamma_{cs} \) have sense in arbitrary dimension.

Now one can write the gauge invariant Chern-Simons action for every closed (D-1)-dimensional manifold \( M^{D-1} \):

\[ S_{cs} = \int_{M^{D-1}} \Gamma_{cs} \]  (4.15)

This action exists in arbitrary space-time dimension starting with dimension 3. Such a generalization of the Chern-Simons term was suggested in [10]. It is also particular result in our general construction.

To remind, for the standard Yang-Mills gauge fields (one-forms) the Chern-Simons terms exist only in odd dimension (2k-1) and are determined as follows:

\[ d\tilde{\Gamma}_{cs} = \tilde{\omega}^{(2k)} \]  (4.16)

where \( \omega^{(2k)} \) is closed invariant 2k-form:

\[ \tilde{\omega}^{(2k)} = Tr[F^k] \]  (4.17)

here F is the curvature 2-form.

The form \( \omega^{(2)} \) (4.11) is similar to \( \tilde{\omega}^{(4)} \). However \( \omega^{(2)} \) comprise components of any rank \( N \geq 4 \), so the generalized Chern-Simons term \( \Gamma_{cs} \) (4.14) exists in any dimension \( D \geq 3 \) and, in particular, in even dimension D, contrary to the standard
gauge fields case (the four-dimensional Chern-Simons example will be considered in the next Section).

However, the above discussion doesn’t exhaust the rich structure of NGATF Chern-Simons terms. Indeed, in dimension \( D \geq 6 \) one can consider the D-form:

\[
\omega^{(3)} = Tr(F \wedge F \wedge F)
\]  

which is closed and invariant under the same conditions (4.7). Hence in high dimension \( D \) along with (4.14) there exists also the following Chern-Simons term \( \Gamma^{(3)}_{cs} \):

\[
d\Gamma^{(3)}_{cs} = \omega^{(3)}
\]  

Further, in higher dimension \( D \) any invariant closed D-form:

\[
\omega^{(k)} = Tr F^k
\]  

determines the corresponding Chern-Simons \( \Gamma^{(k)}_{cs} \):

\[
d\Gamma^{(k)}_{cs} = \omega^{(k)}
\]

Any such term \( \Gamma^{(k)}_{cs} \) defines the gauge invariant NGATF action

\[
S^{(k)} = \int_{M^{D-1}} \Gamma^{(k)}_{cs}
\]

For a closed \((D-1)\)-dimensional manifold \( M^{D-1} \).

As a direct generalization one can also consider the NGATF taking values in an Grassman algebra \( \Lambda \) and write the corresponding generalized Chern-Simons terms \( \Gamma^{(k)}_{cs} \). In three dimension we then obtain the field theory whose partition function is the Casson invariant [11,12]. Hence, one can also expect the appearance of relevant invariants under quantization of these super Chern-Simons actions in arbitrary dimension.

Summarizing, we obtained very rich Chern-Simons terms structure for NGATF. One can use these along with (4.2) as actions in order to determine the dynamical description of NGATF.

5 The High-Dimensional Chern-Simons

In three dimensions (4.14) gives us usual Chern-Simons term for Yang-Mills field [5]. As an next example let us consider the generalized Chern-Simons term in four dimensions. For standard Yang-Mills gauge fields such a term doesn’t exist in \( D=4 \).

From our general expression (4.14) we obtain:

\[
\Gamma_{cs} = Tr(A_{(1)} \wedge dA_{(2)} + A_{(2)} \wedge dA_{(1)} + 2A_{(1)} \wedge A_{(1)} \wedge A_{(2)})
\]  

11
It is easy to see that integral of (5.1) over a four-dimensional closed manifold $M^4$

$$S_{cs} = \int_{M^4} \Gamma_{cs}$$ (5.2)

coincides with

$$S_{cs} = \int_{M^4} 2Tr[F(2) \wedge A(2)]$$ (5.3)

and it is thus invariant under the gauge transformations (2.9):

$$\delta_\alpha A(1) = -d\alpha(0) + \alpha(0)A(1) - A(1)\alpha(0),$$
$$\delta_\alpha A(2) = -d\alpha(1) - \alpha(1)\wedge A(1) - A(1)\wedge \alpha(1) + \alpha(0)A(2) - A(2)\alpha(0)$$ (5.4)

The Chern-Simons (5.3) with respect to (5.4) is transformed by the total differential.

It is worth noting that the action (5.3) is already well known in the literature as the BF-system. This model was proposed in [6] (see also [11,12]) as an example of an exactly soluble diffeomorphism invariant theory. (A similar action was also considered by Freedman and Townsend in [8]). As shown in [6], this model when quantized describes the topological invariants such as linking numbers. The transformations (5.4) contrary to [6] are not accidental but appear as a part of more general and rich structure.

As another example, let us consider the gauge gravity. The corresponding generalization of the Lorentz group will be described by (see Appendix for more details):

$$\xi G^k_i \wedge \eta_{kl}G^{l}_{j} = \eta_{ij}$$ (5.5)

where $G^i_j = \sum_{p=0}^{D} G^i_{(p)} j$ is a matrix-valued inhomogeneous differential form; $\eta_{ij} = diag(+1, -1, ..., -1)$ is the Minkowski D-dimensional metrical tensor, $i, j = 1, 2, ..., D$.

Localizing the group (5.5) along the lines discussed in Sect.2.3, one introduces the generalized k-forms connections $\omega^i_{(k)} j$. However, not all of these are dynamically present in the Chern-Simons action. In four dimensions, for example, one has the expression similar to (5.3):

$$S_{cs} = \int_{M^4} \Gamma_{cs},$$
$$\Gamma_{cs} = Tr[d\omega(1) \wedge \omega(2) + \omega(1) \wedge \omega(1) \wedge \omega(2)]$$ (5.6)

This action was also considered in details by Horowitz [6].

It is worth noting that the group (5.5) is a super-generalization of the Lorentz group. Hence the corresponding gauge gravity theory may be treated as the new version of a supergravity theory.

Now let us write precisely the generalized Chern-Simons term (4.14) for closed manifold $M^D, D \geq 5$:

$$D=5$$

12
\[ S_{cs} = \int Tr[A_{(3)} \wedge F_{(2)}] \]  
\[ \mathbf{D=6} \]
\[ S_{cs} = \int Tr[F_{(2)} \wedge A_{(4)} - F_{(3)} \wedge A_{(3)} - \frac{1}{3} A_{(2)} \wedge A_{(2)} \wedge A_{(2)}], \]
\[ F_{(3)} = dA_{(2)} + A_{(1)} \wedge A_{(2)} - A_{(2)} \wedge A_{(1)} \]  
\[ \mathbf{D=7} \]
\[ S_{cs} = \int Tr[A_{(5)} \wedge F_{(2)} + A_{(3)} \wedge F_{(4)}], \]
\[ F_{(4)} = dA_{(3)} + A_{(3)} \wedge A_{(1)} + A_{(1)} \wedge A_{(3)} \]  
\[ \mathbf{D=8} \]
\[ S_{cs} = \int Tr[A_{(6)} \wedge F_{(2)} + A_{(5)} \wedge F_{(3)} + A_{(4)} \wedge F_{(4)} + A_{(2)} \wedge A_{(3)} \wedge A_{(3)}], \]
\[ F_{(3)} = dA_{(2)} + A_{(1)} \wedge A_{(2)} - A_{(2)} \wedge A_{(1)} \]
\[ F_{(4)} = dA_{(3)} + A_{(3)} \wedge A_{(1)} + A_{(1)} \wedge A_{(3)} - A_{(2)} \wedge A_{(2)} \]  

One can see that the generalized Chern-Simons actions in \( \mathbf{D=6,8} \) don’t have form of the B-F system in these dimensions. It should be noted that though our Chern-Simons action for closed manifold \( M^{D}, D = 4, 5, 7, \ldots \) has the form of B-F system, it differs on B-F system for manifold with boundary which is appropriate for canonical quantization of the model. For open manifold this action is similar to standard Chern-Simons term for Yang-Mills field in three dimensions [5].

### 6 Further generalizations and development

There is a wide field for further generalizations and applications of the suggested theory. Here we would like to mention some, in our opinion, most interesting.

#### 6.1 Generalization of gauge group

The gauge group in Sect.3 (see (3.9)) is the super-generalization of the unitary group \( \text{U}(N) \). In general case, one can take any group \( g \) (orthogonal, simplicitic and etc.), which satisfies the condition

\[ g^{\star} \eta g = \eta \]  

where \( \star \) and \( \eta \) are appropriate conjugation and invariant metric tensor. Then one can determine the group on differential forms as follows:

\[ \xi G^{\star} \wedge \eta G = \eta \]
which will give us the superization of group \( g \).

Moreover there is a further generalization of (6.2) [13]. Indeed, let \( P \) be an inhomogeneous differential form with values in an matrix space. Let us consider the differential form \( G \), which satisfy the condition:

\[
\xi G^* \land P \land G = P
\]  

(6.3)

The differential form \( P \) plays the role of fundamental form on a manifold \( M \) which defines the scalar product for matter fields. In general case there are two natural fundamental forms: zero-form and D-form of volume. If \( P \) is a zero-form \( P_{ij} \) one obtains the definition (6.2). On the other hand if \( P \) is a volume form then \( G \) in (6.3) has only zero rank component \( G(0) \). Hence (6.3) is equivalent to eq.(6.1) in this case.

Notice that in these two described cases (a zero-form \( P \) and a volume-form \( P \)) the form \( P \) belongs to the cohomology space \( H^{(0)}(M) \) and \( H^{(D)}(M) \) respectively. In general case one may assume that \( P \) is closed form, \( dP = 0 \), and represents a relevant cohomology classes

\[
P_{ij} = \sum_{k=0}^{D} P^{(k)}_{ij} \quad P^{(k)}_{ij} \in H^{(k)}(M)
\]  

(6.4)

For example, if \( M \) is a compact Kahler manifold then the even cohomology groups \( H^{(2k)}(M) \) are not trivial [14]. So the construction (6.3), (6.4) has a nontrivial realization.

This superization of a group \( g \) (6.1) seems to be interesting since it inherits the properties of initial classical group \( g \) (6.1) and feels the geometry of underlying manifold \( M^D \).

6.2. Quantization and new topological invariants

It is well known that quantization of diffeomorphism invariant metric independent actions reproduces topological invariants in terms of quantum expectation values [6,7]. One obtains the Ray-Singer torsion when quantizing AGATF [7] and linking numbers for BF-system described in Sect.5 [6]. All these theories are particular cases of the above considered NGATF theory. There exist two kinds of actions for NGATF:

\[
(F, F) = \int_{M^D} Tr(\tilde{F} \land F)
\]  

(6.5)

and a wide class of the Chern-Simons actions

\[
S_{cs}^{(k)} = \int_{M^D} \Gamma_{cs}^{(k)}
\]  

(6.6)
Both of these actions are metric independent, so one can expect that the topological invariants arise as a path integrals with any of these actions:

$$Z(O_1, ..., O_j) = \int D AO_1...O_j \exp(-S[A])$$  \hspace{1cm} (6.7)

where $O_1, ..., O_j$ are gauge invariant operators. In the abelian case one usually takes as $O_i$ integrals over the k-dimensional cycles $\gamma(k)$ [6]:

$$O = \int_{\gamma(k)} A(k)$$  \hspace{1cm} (6.8)

For the one-form $A_{(1)}$ one has the gauge invariant operator [5]

$$O = Tr P \exp \int_{\gamma(1)} A_{(1)}$$  \hspace{1cm} (6.9)

which is holonomy of connection $A_{(1)}$ around loop $\gamma(1)$. In our paper [15] we find the corresponding generalization of these formulas for NGATF.

Notice that actions (3.21) for matter fields are also metric independent. Hence they are also subject for quantization leading to topological invariants.

6.3 **Physical models with NGATF**

The abelian gauge antisymmetric tensor fields play an important role in the supergravity, in the gauge theory of gravity and in the string theory [1]. The NGATF theory is generalization of both AGATF and the standard gauge fields. Hence NGATF may find a wide applications.

As it was shown in Section 3, the supergroup appears naturally in our approach. In fact, it is not necessary to introduce an additional grassman variables in the theory. The superspace has a natural realization as a space of variables $x^\mu, dx^\nu$, i.e. the space $\Lambda^*(M)$. Hence this approach is appropriate for description of supergauge theory. For instance, one may attempt in construction of the super-Poincare gravity in the framework of gauge approach with NGATF. In particular, it allows to write the gauge gravity action without a help of metric using only the gauge fields. Such a theory would describe a phase in which general covariance is unbroken. The metric will appear as a Goldstone boson of a spontaneously broken (local) general covariance, as discussed in [16].

It should be noted here that the possibility to describe gravity with help of $SL(3)$-algebra valued two-form instead of metric was also considered in [17].

Another promising application of NGATF is the string field theory. There an approach was developed which gives a nonperturbative description of string with the help of infinit-dimensional analog of the Chern-Simons action [18]. One may consider the corresponding gauge field as the inhomogeneous differential form on the space of string configurations. The appropriate gauge group would be the diffeomorphism
group \( \text{Diff}(S^1) \) probably together with the Kac-Moody extension of the Poincare group [19].

One may also expect the NGATF to play a role in condensed matter systems where the standard Chern-Simons approach has proved to be very useful [20].

Acknowledgements
I am grateful to Yu.N.Obukhov for valuable discussions and careful reading the manuscript. I also thank D.Leites, V.Serganova, D.Fursaev and O.Zakharov for conversations.

Appendix.
Let us obtain the general solution for the equation

\[
\xi G^+ \wedge G = 1 \tag{A.1}
\]

which determines the gauge group element \( G \):

\[
G^i_j = \sum_k G^i_{(k)j} \tag{A.2}
\]

In general case \( G \) can be represented in the form

\[
G = g_0 h, \tag{A.3}
\]

where \( g_0 \) is a matrix-valued zero-form, and \( h \) is a matrix valued inhomogeneous differential form with a zero rank component equal to \( h_{(0)} = 1 \).

Inserting (A.3) into (A.1) we assume that the matrix \( g_0 \) is unitary:

\[
g_0^* g_0 = 1 \tag{A.4}
\]

Then we obtain the equation for \( h \):

\[
\xi h^+ \wedge h = 1, \ h_{(0)} = 1 \tag{A.5}
\]

Any form \( h \) with \( h_{(0)} = 1 \) is decomposed into a sum

\[
h = 1 + H + \mathcal{F} \tag{A.6}
\]

where the forms \( H \) and \( \mathcal{F} \) are both of rank \( \geq 1 \) and satisfy the conditions:

\[
\xi H^+ = -H, \ \xi \mathcal{F}^+ = \mathcal{F} \tag{A.7}
\]

From (A.6) and (A.7) we obtain that

\[
h^{-1} = \xi h^+ = 1 - H + \mathcal{F} \tag{A.8}
\]

Inserting this expression into (A.5) one finds

\[
\mathcal{F} \wedge \mathcal{F} + 2 \mathcal{F} - H \wedge H = H \wedge \mathcal{F} - \mathcal{F} \wedge H \tag{A.9}
\]
It is a consequence of eqs. (A.5)-(A.8) that

\[ H = \frac{1}{2}(g - g^{-1}), \]
\[ \mathcal{F} = \frac{1}{2}(g + g^{-1}) - 1 \]  
(A.10)

One concludes from this that \( H \) and \( \mathcal{F} \) commute:

\[ H \land \mathcal{F} = \mathcal{F} \land H \]  
(A.11)

The last equation and conditions (A.7) suggest representing \( \mathcal{F} \) as an even polynomial of \( H \):

\[ \mathcal{F} = \sum_k a_k H^{2k} \]  
(A.12)

Substituting (A.12) into (A.9) and taking into account (A.11) one gets:

\[ \sum_k 2a_k H^{2k} + \sum_{p,m} a_p a_m H^{2(p+m)} - H^2 = 0 \]  
(A.13)

Then one obtains the following recurrent identities for \( a_k \):

\[ a_1 = \frac{1}{2} \]
\[ a_2 = -\frac{1}{8} \]
\[ a_k = -\frac{1}{2} \sum_{p=1}^{k-1} a_p a_{k-p}, k \geq 2 \]  
(A.14)

Then the parameter space of the group \( G \) (A.1) consists of parameters of unitary matrix \( g_0 \) (A.4) and independently of differential form

\[ H = \sum_{k \geq 1} \frac{(k(k-1)}{k!} H^{\alpha}_{\mu_1...\mu_k} \lambda^\alpha dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}, \]  
(A.15)

where \( \{\lambda^\alpha\} \) is a set of antihermitean matrices.

It is easy to obtain the group multiplication law for \( H \). We should note here that this is nonabelian even in the abelian case (when \( g_0 \in U(1) \)).

Let us consider in brief the case of the Lorentz group. The gauge group is determined in this case by the condition:

\[ \xi \mathcal{G}^T \land \eta \mathcal{G} = \eta \]  
(A.16)

where \( \eta_{ij} = \text{diag}(+1,...,-1) \) is the metric tensor, and \( G = \sum G_{(k)j}^i \) is matrix-valued differential form. The element \( \mathcal{G} \) (A.16) can also be represented in the form

\[ G = g_0(1 + H + \mathcal{F}), \]  
(A.17)
where \( g_0 \) is the Lorentz group element \((g_0^T \eta g_0 = \eta)\), and \( H \) and \( F \) satisfy

\[
\xi H_{ij} = -H_{ji} \\
\xi F_{ij} = F_{ji}
\]

(A.18)

where we denote \( H_{ij} = H^k_{j} \eta_{ik} \).

As earlier the form \( F \) can be represented as polynomial of \( H \) (A.12). From (A.18) one obtains for the k-rank component \( H^{(k)} \):

\[
H^{(k)}_{ij} = (-1)^{\frac{k(k+1)}{2}} H^{(k)}_{ji}
\]

(A.19)

In four-dimensional case we have

\[
H^{(k)}_{ij} = -H^{(k)}_{ji}, \quad k = 0, 1, 4 \\
H^{(k)}_{ij} = H^{(k)}_{ji}, \quad k = 2, 3
\]

(A.20)

Hence \( H \) is not decomposed with respect to basis of the Lorentz group generators, as we had in unitary group case (see eq.(A.15)). One should consider additionally the symmetric matrices basis for \( H^{(k)}, k = 2, 3 \). This is the main difference in the definition of the group \( G \) for complex and real cases.

References

[1] E.Nambu, Phys.Repts. C23 (1976) 250; E.Sezgin, P.van Nieuwenhuizen, Phys.Rev. D22 (1980) 301; W.Siegel, Phys.Lett. B93 (1980) 170; P.K.Townsend, Phys.Lett. B90 (1980) 275; Yu.N.Obukhov, Phys.Lett. B109 (1980) 195; R.Rohm, E.Witten, Ann.Phys.(NY) V.170 (1986) 454; D. Olivier, Phys.Rev. D33 (1986) 2462; C.Teitelboim, Phys.Lett. B167 (1986) 63; J.M.Rabin, Phys.Lett. B172 (1986) 333.

[2] D.Ivanenko, L.Landau, Zeitsch.f.Phys. 48 (1928) 340; E.Kahler, Rend.Math. 21 (1962) 4255; W.Graf, Ann.Inst.H.Poncare 29 (1978) 85; P.Becher, H.Joos, Z.Phys. A15 1983) 343; J.M.Benn, R.W.Tucker, Comm.Math.Phys. 89 (1983) 341; D.Ivanenko, Yu.N.Obukhov, Ann.d.Phys. 42 (1985) 59; D.Ivanenko, Yu.N.Obukhov, S.N.Solodukhin, Preprint IC/85/2. (ICTP, 1985). J.A.Bullinaria, Ann.Phys. 168 (1986) 301; Yu.N.Obukhov, S.N.Solodukhin, Preprint IC/91/147. (ICTP, 1991).

[3] S.N.Solodukhin, Ann.d.Phys. 46 (1989) 439; Int.J.Theor.Phys. 31 (1992) 47.

[4] S.Donaldson, J.DiffGeom. 18 (1983) 269; 26 (1987), 397; Topology 29 (1990) 257.

[5] E.Witten, Comm.Math.Phys. 121 (1989) 351.
[6] G.T.Horowitz, Comm.Math.Phys. 125 (1989) 417; G.T.Horowitz, M.Srednicki, Comm.Math.Phys. 130 (1990) 83.

[7] A.S.Schwarz, Lett.Math.Phys. 2 (1978) 247.

[8] P.K.Townsend, in: Supergravity, P.van Nieuwenhuizen and D.Z.Freedman (eds). North-Holland Publ.Comp. 1979; D.Z.Freedman, P.K.Townsend, Nucl.Phys. B177 (1981) 282; R.I.Nepomnechie, Nucl.Phys. B212 (1983) 301.

[9] P.F.Picken, J.Phys.A16 (1983) 3457.

[10] R.C.Myers, V.Periwal, Phys.Lett. B225 (1989) 352.

[11] E.Witten, Nucl.Phys. B323 (1989) 113.

[12] D.Birmingham, M.Blau and G.Tompson, Int.J.M.Phys.A5 (1990) 4721; M.F.Atiyah, L.Jeffrey, J.Geom.Phys. 7 (1990) 119; C.H.Taubes, J.Diff.Geom.31 (1990) 301.

[13] I thank Yu.N.Obukhov for this remark.

[14] S.Kobayashi, K.Namizu, Foundations of differential geometry vol.2. Interscience publishers. NY. 1969.

[15] Yu.N.Obukhov, S.N.Solodukhin, Non-abelian gauge differential forms and their holonomies, to be published.

[16] E.Witten, Comm.Math.Phys. 117 (1988) 353.

[17] G.’t Hooft, Nucl.Phys.B357 (1991) 211.

[18] E.Witten, Nucl.Phys. B268 (1986) 253; J.L.Gervais, Nucl.Phys. B287 (1987) 815.

[19] M.J.Bowick, S.G.Rajeev, Nucl.Phys.B293 (1987) 348; Y.Bars, S.Yankielowich, Phys.Rev. D35 (1987) 3878; L.Castellani, Phys.Lett. B206 (1988) 47; Nucl.Phys. B317 (1989) 46.

[20] Y.N.Chen, B.I.Halperin, F.Wilczek and E.Witten, Int.J.Mod.Phys. B3 (1989) 1001; J.D.Lykken, J.Sonnenschein, N.Weiss, The theory of anyonic superconductivity: a review FERMILAB-PUB-91/41-T, Jan.1991.