Spherically Symmetric Braneworld Solutions with \((^{(4)}R)\) term in the Bulk

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Abstract
An analysis of a spherically symmetric braneworld configuration is performed when the intrinsic curvature scalar is included in the bulk action; the vanishing of the electric part of the Weyl tensor is used as the boundary condition for the embedding of the brane in the bulk. All the solutions outside a static localized matter distribution are found; some of them are of the Schwarzschild-\((A)dS_4\) form. Two modified Oppenheimer-Volkoff interior solutions are also found; one is matched to a Schwarzschild-\((A)dS_4\) exterior, while the other does not. A non-universal gravitational constant arises, depending on the density of the considered object; however, the conventional limits of the Newton’s constant are recovered. An upper bound of the order of \(TeV\) for the energy string scale is extracted from the known solar system measurements (experiments). On the contrary, in usual brane dynamics, this string scale is calculated to be larger than \(TeV\).

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1. Introduction

The desire to explore physics beyond the Standard Model has led us to explore the ideas that spacetime is of a dimension larger than four, and that we are essentially confined to a four-dimensional hypersurface. String theories provide a framework for exploring such ideas, but nevertheless we are still far away of having a viable low-energy realization of these theories. Braneworld models consist relevant world realizations in which some underlying features are often minimized. Replacing, for example, a whole field with a constant (solitonic solution) may probably oversimplifies the reality, but at the same time makes it possible to obtain a more concrete picture, with the hope that any new behavior appearing will be still present in the more complete theory. Not only at the cosmological level, but also at a local one - concerning stars, galaxies, clusters of galaxies - has a brane solution to be consistent with the various astrophysical observations, which are often more reliable than the cosmological ones.

Attempts for obtaining braneworld solutions are cast into two categories. First, the bulk space assumes a given geometry, a coordinate system is adopted and the influence on the brane geometry is somehow extracted. It seems as a disadvantage of this approach that the bulk is prefixed and also that the brane imbedding obtained is not gauge-invariant (independent of the coordinate system chosen). Second, do not specify the exact bulk geometry, adopt a coordinate system adapted to the brane (gauss normal coordinates or some relevant one) and deduce a brane dynamics, containing imprints from the bulk. Assumptions on the brane geometry are often sufficient for obtaining an exactly closed brane dynamics. This approach allows of a dynamically interacting brane with bulk, though this situation is not necessarily considered. A disadvantage of this method is that finding a bulk geometry in which the brane consists its boundary may be a very difficult task. A probable advantage would be the extraction of common braneworld characteristics holding for a broad class of bulk backgrounds. In both approaches, if the codimension is one, Israel matching conditions are necessarily used. In the present paper we shall elaborate on the second approach.

The effective brane equations have been obtained [1] when the effective low-energy theory in the bulk is higher-dimensional gravity. However, a more fundamental description of the physics that produces the brane could include [2] higher order terms in a derivative expansion of the effective action, such as a term for the scalar curvature of the brane, and higher powers of curvature tensors on the brane. A brane action that contains powers of the brane curvature tensors has also been used in the context of the $AdS/CFT$
correspondence (e.g. [3]) to regularize the action of a bulk $AdS$ space which diverges when the radius of the $AdS$ space becomes infinite. If the dynamics is governed not only by the ordinary five-dimensional Einstein-Hilbert action, but also by the four-dimensional Ricci scalar term induced on the brane, new phenomena appear. In [4, 5] it was observed that the localized matter fields on the brane (which couple to bulk gravitons) can generate via quantum loops a localized four-dimensional worldvolume kinetic term for gravitons (see also [6, 7, 8, 9]). That is to say, four-dimensional gravity is induced from the bulk gravity to the brane worldvolume by the matter fields confined to the brane. It was also shown that an observer on the brane will see correct Newtonian gravity at distances shorter than a certain crossover scale, despite the fact that gravity propagates in extra space which was assumed there to be flat with infinite extent; at larger distances, the force becomes higher-dimensional. The first realization of the induced gravity scenario in string theory was presented in [10]. Furthermore, new closed string couplings on Dp-branes for the bosonic string were found in [11]. These couplings are quadratic in derivatives and therefore take the form of induced kinetic terms on the brane. For the graviton in particular these are the induced Einstein-Hilbert term as well as terms quadratic in the second fundamental tensor. Considering the intrinsic curvature scalar in the bulk action, the effective brane equations have been obtained in [12]. Results concerning cosmology have been discussed in [13, 14, 15, 16, 17].

The original Randall-Sundrum models [18], based on a Minkowski brane and a specific relation between the bulk cosmological constant and the brane tension, have drawn much attention because they might be realizable in supergravity and superstring compactifications [19, 20, 21, 22]. However, any Ricci-flat four-dimensional metric can be embedded (with the common warped embedding) in $(A)dS_5$ (e.g. [23, 24]). This way, a black-string solution [23, 23, 20, 27] can be easily constructed. Furthermore, it is known that any four-dimensional Einstein spaces can foliate an $(A)dS_5$ bulk [23, 23, 30, 31, 32, 33]. Thus, asymptotically non-flat black holes (Schwarzschild-$(A)dS_4$) can be obtained as slices of the above precise bulks. Almost all treatments on spherically symmetric braneworld solutions, as the previously mentioned, representing, for example, the exterior of a star, do not take care of the finite extension of the object. Till now, there is no known exact five-dimensional solution for astrophysical brane black holes. Furthermore, looking for bulks having some interior star solution as part of their boundaries is even harder. In [34, 35, 36], some interior and exterior solutions were found, without including the $(^4 R)$ term.

In the present paper, we discuss the gravitational field of an uncharged, non-rotating
spherically symmetric rigid object when in the dynamics there is a contribution from the
brane intrinsic curvature invariant. In section 2, we find all the possible exterior solutions,
containing one undetermined parameter which is the parameter of the Newtonian term.
Some of these solutions are of the Schwarzschild-($A$)d$S_4$ form. In two cases, we can solve
also the interior problem which reduces to a generalization of the Oppenheimer-Volkoff
solution, and thus determine the unknown parameter. This is found different from the
conventional value of a localized spherically symmetric distribution within the framework of four-dimensional general relativity. Hence, a non-universal Newton’s constant,
depending on the density of the object, naturally arises. In section 3, taking care of the
classical experiments of gravity in the solar system, we can set an upper bound for the
five-dimensional Planck mass being of the order of $TeV$. The revival of the conventional
results is discussed, and also, a comparison with the more standard brane dynamics is
presented. Finally, in section 4 are our conclusions.

2. Four-Dimensional Spherically Symmetric Solutions

We consider a 3-dimensional brane $\Sigma$ (with normal vector field $n^A$) embedded in
a 5-dimensional spacetime $M$. Capital Latin letters $A, B, ... = 0, 1, ..., 4$ will denote full
spacetime, lower Greek $\mu, \nu, ... = 0, 1, ..., 3$ run over brane worldvolume, while lower Latin
ones span some 3-dimensional spacelike surfaces foliating the brane, i.e. $i, j, ... = 1, ..., 3$.
For convenience, we can quite generally, choose a coordinate $y$ such that the hypersurface
$y = 0$ coincides with the brane. The total action for the system is taken to be:

$$S = \frac{1}{2\kappa_5^2} \int_M \sqrt{-g} \left( R - 2\Lambda_5 \right) d^5x + \frac{1}{2\kappa_4^2} \int_\Sigma \sqrt{-g} \left( R - 2\Lambda_4 \right) d^4x$$
$$+ \int_M \sqrt{-g} L_5^{\text{mat}} d^5x + \int_\Sigma \sqrt{-g} L_4^{\text{mat}} d^4x. \tag{1}$$

For clarity, we have separated the cosmological constants $\Lambda_5, \Lambda_4$ from the rest matter
contents $L_5^{\text{mat}}, L_4^{\text{mat}}$ of the bulk and the brane respectively. $\Lambda_4/\kappa_4^2$ can be interpreted as
the brane tension of the standard Dirac-Nambu-Goto action, or as the sum of a brane
worldvolume cosmological constant and a brane tension. We basically concern on the case
with no fields in the bulk, i.e. $(5)^2T_{AB} = 0$.

From the dimensionful constants $\kappa_5^2, \kappa_4^2$ the Planck masses $M_5, M_4$ are defined as:

$$\kappa_5^2 = 8\pi G_5 = M_5^{-3}, \quad \kappa_4^2 = 8\pi G_4 = M_4^{-2}, \tag{2}$$
with $M_5$, $M_4$ having dimensions of $(\text{length})^{-1}$. Then, a distance scale $r_c$ is defined as:

$$r_c \equiv \frac{\kappa_5^2}{\kappa_4^2} = \frac{M_4^2}{M_5^3}.$$  

(3)

Varying (1) with respect to the bulk metric $g_{AB}$, we obtain the equations

$$(5) G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 \left( (5) T_{AB} + (\text{loc}) T_{AB} \delta(y) \right),$$

(4)

where

$$(\text{loc}) T_{AB} \equiv -\frac{1}{\kappa_4^2} \sqrt{-g} \left( (4) G_{AB} - \kappa_4^2 (4) T_{AB} + \Lambda_4 h_{AB} \right)$$

(5)

is the localized energy-momentum tensor of the brane. $(5) G_{AB}$, $(4) G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics respectively. Clearly, $(4) G_{AB}$ acts as an additional source term for the brane through $(\text{loc}) T_{AB}$. The tensor $h_{AB} = g_{AB} - n_A n_B$ is the induced metric on the hypersurfaces $y = \text{constant}$, with $n^A$ the normal vector on these.

The way the $y$-coordinate has been defined, allows us to write, at least in the neighborhood of the brane, the 5-line element in the block diagonal form

$$ds^2(5) = -N^2 dt^2 + g_{ij} dx^i dx^j + dy^2,$$

(6)

where $N, g_{ij}$ are generally functions of $t, x^i, y$. The distributional character of the brane matter content makes necessary for the compatibility of the bulk equations (4) the following modified (due to $(4) G_{\mu\nu}$) Israel-Darmois-Lanczos-Sen conditions $[37, 38, 39, 40]$

$$[K_{\mu\nu}] = -\kappa_5^2 \left( (\text{loc}) T_{\mu\nu} - \frac{(\text{loc}) T}{3} \delta_{\mu\nu} \right),$$

(7)

where the bracket means discontinuity of the extrinsic curvature $K_{\mu\nu} = \partial_y g_{\mu\nu}/2$ across $y = 0$. A $\mathbb{Z}_2$ symmetry on reflection around the brane is considered throughout.

One can derive from equations (4), (7) the induced brane gravitational dynamics $[12]$, which consists of a four-dimensional Einstein gravity, coupled to a well-defined modified matter content. More explicitly, one gets

$$(4) G_{\nu}^{\mu} = \kappa_4^2 (4) T_{\nu}^{\mu} - \left( \Lambda_4 + \frac{3}{2} \alpha^2 \right) \delta_{\nu}^{\mu} + \alpha \left( L_{\nu}^{\mu} + \frac{L}{2} \delta_{\nu}^{\mu} \right),$$

(8)

where $\alpha \equiv 2/r_c$, while the quantities $L_{\nu}^{\mu}$ are related to the matter content of the theory through the equation

$$L_{\lambda}^{\mu} L_{\nu}^{\lambda} - \frac{L^2}{4} \delta_{\nu}^{\mu} = T_{\nu}^{\mu} - \frac{1}{4} (3 \alpha^2 + 2 T_{\lambda}^{\lambda}) \delta_{\nu}^{\mu},$$

(9)
and \( L \equiv L^\mu_\mu \). The quantities \( T^\mu_\nu \) are given by the expression

\[
T^\mu_\nu = \left( \Lambda_4 - \frac{1}{2} \Lambda_5 \right) \delta^\mu_\nu - \kappa_4 \frac{(4)}{2} T^\nu_\nu + \\
+ \frac{2}{3} \kappa_5 ^{2} \left( (5) T^\mu_\nu + \frac{(5) T^\nu_\nu}{4} \right) \delta^\mu_\nu - E^\mu_\nu ,
\]

(10)

with \((5) T = (5) T_A^A\), \((5) T_A^B = g^{AC} (5) T^C_B\). Bars over \((5) T^A_B\) and the electric part \( E^\mu_\nu = \omega^A_B n^A n^B \) of the 5-dimensional Weyl tensor \( C^A_{BCD} \) mean that the quantities are evaluated at \( y = 0 \). \( E^\mu_\nu \) carries the influence of non-local gravitational degrees of freedom in the bulk onto the brane \([1]\) and makes the brane equations \((8)\) not to be, in general, closed. This means that there are bulk degrees of freedom which cannot be predicted from data available on the brane. One needs to solve the field equations in the bulk in order to determine \( E^\mu_\nu \) on the brane. In the present paper, for making \((8)\) closed, we shall set \( E^\mu_\nu = 0 \) as a boundary condition of the propagation equations in the bulk space. This is somehow simplified from the viewpoint of geometric complexity, but it is the first step for investigating the characteristics carried by the brane curvature invariant on the local brane dynamics we are interested in. Treatments and solutions without this assumption, in the context of usual brane dynamics, have been given in \([26, 41, 42, 43, 34, 35, 36, 44]\).

Due the block-diagonal form of the metric \((6)\) the solution of the algebraic system \((9)\), whenever

\[
T^i_j = \tau \delta^i_j ,
\]

(11)
is

\[
L^0_0 = \pm \frac{1}{2B} \left( (7 - 4n_+ n_-) T^0_0 - (3 - 4n_+ n_-) \tau + 3 \alpha^2 \right) ,
\]

(12)

\[
L^i_j = S E^i_j ,
\]

(13)

\[
L^0_i = L^i_0 = 0 ,
\]

(14)

where

\[
S = \frac{1}{2B} \left| T^0_0 + 3 (\tau + \alpha^2) \right| ,
\]

(15)

\[
B = \left( -6(n_+ - 1)(n_- - 1) T^0_0 + 2n_+ n_- (1 - n_+ n_-) \tau + 3 (3 - 2n_+ n_-) \alpha^2 \right)^{\frac{1}{2}} ,
\]

(16)
while the matrix $E^i_j$ is either $\text{diag}(+1,+1,+1)$ (with $n_+ = 3, n_- = 0$) or $\text{diag}(+1,+1,-1)$ (with $n_+ = 2, n_- = 1$).

Inspecting equation (8), we see that the inclusion of the term $^{(4)}R$ has brought a convenient decomposition of the matter terms. First, standard energy-momentum tensor enters without having made any choice for the brane tension $\Lambda_4$ in terms of $M_4, M_5$ (in [15, 1] it has to be $\Lambda_4 = 3\alpha^2/2$). Note that if $^{(4)}R$ is not included in the action, for $\Lambda_4 = 0$, ordinary energy-momentum terms cannot arise. Furthermore, in that case, $\Lambda_4$ has to be positive in order for $\kappa_4^2$ to be positive. Second, the additional matter terms (which rather appear here as square roots instead of squares of the four-dimensional energy-momentum tensor) all contain the factor $\alpha$ of energy string scale. Thus, conventional four-dimensional General Relativity revives on some region of a 4-spacetime, whenever these extra terms remain suppressed relative to the conventional ones; the specific value of $\alpha$ determines the region validity of General Relativity.

From now on, we are interested on static (non-cosmological) local braneworld solutions arising from the action (4). More specifically we consider a spherically symmetric line element

$$ds^2_{(4)} = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

The matter content of the 3-universe is a localized spherically symmetric untilted perfect fluid (e.g. a star) $^{(4)}T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ with $\rho = p = 0$ for $r > R$, plus the cosmological constant $\Lambda_5$. These matter contents enter $T_{\mu\nu}$ in equation (10) and thus determine $L^\mu_\nu$ on the right hand side of our dynamical equations (8). The result is

$$L^0_0 = \pm \frac{1}{2B}\left(4\Lambda_4 - 2\Lambda_5 + 3\alpha^2\right) + \kappa_4^2 \left((7 - 3n_+ n_-)\rho + (n_+ + 3n_-)p\right),$$

$$S = \frac{1}{2B} \left|4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + \kappa_4^2 (\rho - 3p)\right|,$$

$$B = \left((3 - 4n_-)(4\Lambda_4 - 2\Lambda_5 + 3\alpha^2) - 4\kappa_4^2 (3(n_- - 1)\rho + (n_+ - 3)p)\right)^{1/2},$$

with the only restriction imposed by the square root appeared in $B$. Thus, necessarily $4\Lambda_4 - 2\Lambda_5 + 3\alpha^2$ is non-negative (non-positive) for $E^i_j = \delta^i_j$ (similarly the other choice of $E^i_j$).

For the metric (17), one evaluates the Ricci tensor $^{(4)}R_{\mu\nu}$ and then constructs the field equations (8). The combination $^{(4)}R_{rr}/2A + ^{(4)}R_{\theta\theta}/r^2 + ^{(4)}R_{00}/2B$ provides the following
differential equation for $A(r)$:

$$\left(\frac{r}{A}\right)' = 1 - \kappa_4^2 \rho(r) r^2 - \left(\Lambda_4 + \frac{3}{2} \alpha^2\right) r^2 + \frac{\alpha}{2} \left(3L_0^0 + (n_+ - n_-)S\right) r^2,$$  

(21)

($' \equiv \frac{d}{dr}$). Eliminating $A'$ from (21) in the $(\theta\theta)$ component of (8), we get an equation for $B'$ from which we obtain

$$\frac{(AB)'}{AB} = Ar \left(\kappa_4^2 (\rho + \rho) - \alpha \left(L_0^0 + (2 - n_+ + n_-)S\right)\right).$$  

(22)

There are various different cases (namely eight) according to the choice of $E_j^i$ and the alternative signs in $L_0^0, S$. However, in the outside region, there are only four different cases, according to $n_+, n_-$ and the ± sign in (18); in all these cases, we can integrate equation (21) in the outside region, obtain the solution $A_>(r)$ and from (22) get the solution $B_>(r)$. The result is

$$\frac{1}{A_>(r)} = 1 - \frac{\gamma}{r} - \beta r^2, \quad r \geq R,$$  

(23)

$$B_>(r) = \frac{1}{A_>(r)} F_{n_+, n_-}(r), \quad r \geq R,$$  

(24)

with

$$\beta = \frac{1}{3} \Lambda_4 + \frac{1}{2} \alpha^2 - \alpha \frac{n_+ - n_- \pm 3}{12 \sqrt{3 - 4n_-}} \sqrt{|4\Lambda_4 - 2\Lambda_5 + 3\alpha^2|},$$  

(25)

$$F_{n_+, n_-}(r) = 1 + \left(f(r)^{\alpha r_1^{n_+ - (2 + 3\sqrt{3})n_-}} |4\Lambda_4 - 2\Lambda_5 + 3\alpha^2| - 1\right) \delta_{n_+ \neq 1, 4 - 3n_-},$$  

(26)

$$f(r) = (r - r_1) \left(\frac{r}{A_>}\right)^{\frac{1}{r_1}} g(r) \sqrt{\frac{|r_1 - \gamma|}{|r_1 + 3\gamma|}},$$  

(27)

where $r_1$ is the minimum horizon distance and $g(r)$ is equal to $\frac{r + r_1/2 + \sqrt{(r_1 + 3\gamma)/(4\beta r_1)}}{r + r_1/2 - \sqrt{(r_1 + 3\gamma)/(4\beta r_1)}}$ for $\beta > 0$, or $e^{2\arctan(\sqrt{4\beta r_1/(r_1 + 3\gamma)} / (r + r_1/2))}$ for $\beta < 0$. For $\beta = 0$, $g(r) \sqrt{|r_1 - \gamma|/\beta}$ is replaced by $(r - \gamma)^{4\gamma/5} e^{2\sqrt{7}(r + 3\gamma)(r - \gamma)}$. The ±, ⊕ signs appearing in (23), (24) correspond to the ± sign of (18). A multiplicative constant of integration for $B_>$ has been absorbed to a redefinition of time and $\gamma$ is a constant of integration. Note that for $\beta \leq 0$ there is only one horizon $r_1 \leq \gamma$, while for $\beta > 0$ (and $27\beta^2 < 4$ to have well defined horizons) there are two horizons $\gamma < r_1 < 3\gamma$ and $1/\sqrt{3\beta} < r_2 < 1/\sqrt{\beta}$. 


Solutions (23), (24) are not yet completely defined unless the parameter $\gamma$ is determined, i.e. the interior solution is found. In the case $E^i_j = \delta^i_j$, we can find two situations where equation (21) does not contain $p$ and so, we can integrate it in the interior region (we give these solutions below, equation (31)).

As it is seen from (24) and (26), all the exterior solutions are either of the form where $A > B$ are inverse to each other, or of the form where the product $A B$ is equal to $f(r)$ to a power appeared in (26). The first class of these solutions is of the Schwarzschild-$(A)dS_4$ form, while the second is not. For zero $\beta$ (we can interpret $\beta$ as the effective brane cosmological constant) the first class of these solutions reduces to Schwarzschild-like, while the second does not. Non-Schwarzschild-like exterior solutions were also obtained in [34, 41, 36, 44], but this fact was rendered to the non-vanishing $E^{\mu\nu}$. Such irregular behavior also appears here, due to the intrinsic curvature invariant, without having involved non-local bulk effects onto the brane. There is one case of our non-Schwarzschild-$(A)dS_4$ solutions with $\beta > 0, \gamma/r_1 = 2/3$, where at large distances – given that the second horizon is actually at cosmological distances – $A > B$ is almost one, i.e. the solution asymptotes the Schwarzschild-$dS_4$.

If we take the covariant derivative (denoted by $|$) with respect to the induced brane metric $h_{AB} = g_{AB} - n_A n_B$ of the equations (7), and make use of the Codacci’s equations, and of the bulk equations (4), we arrive at the equations

\[ T_A^B |_A = -\left[ (5)T_{CD}^A n_C h^B_D \right]. \]

When the matter content of the bulk space is only a cosmological constant, then the common conservation law of our world is obtained. For the static case we are discussing, this law is equivalent to the equation

\[ \frac{B'}{B} = -\frac{2\rho'}{\rho + p}. \]  

Thus, for $r \leq R$ we get the equation for $p(r)$:

\( \frac{p'}{\rho + p} = \frac{1 - A}{2r} - \frac{Ar}{2} \left( \kappa_4^2 p - \left( \Lambda_4 + \frac{3}{2} \alpha^2 \right) + \frac{\alpha}{2} L_0 - \frac{3}{2} \alpha \left( \frac{4}{3} - n_+ + n_- \right) S \right). \)

We assume a uniform distribution $\rho(r) = \rho_0 = \frac{3M}{4\pi r^3}$ for $r \leq R$. Then, the immediate integration of (28) gives:

\[ B_<(r) = \frac{\left( 1 - \frac{\gamma}{r} - \beta R^2 \right) F_{n_+ n_-}(R)}{\left( 1 + \frac{4\pi R^3}{3M} \rho(R) \right)^2}, \quad r \leq R, \]

in which, continuity of $B(r)$ at $r = R$ and the condition $p(R) = 0$ have been used. The vanishing of the pressure at the surface, which is certainly physically reasonable, is a
consequence of the application of the Israel matching conditions at the stellar surface \([16, 17]\). The pressure \(p(r)\) in (30) is found from (29).

Now, we proceed, as we said before, with the two cases where we can solve the system of equations (21), (29). Both have \(E_j^i = \delta_j^i\). The first case corresponds to the upper sign of the \(\pm\) sign in (18), and the quantity inside the absolute value of (19) in the interior of the star being positive. The second case corresponds to the lower \(\pm\) sign in (18) and negative quantity in (18). In these cases, integration of (21) gives

\[
\frac{1}{A_<(r)} = 1 - \left(\beta + \frac{\gamma}{R^3}\right)r^2, \quad r \leq R,
\]

where the parameters \(\gamma\) and \(\beta\) (from (25)) are given in terms of \(M, \alpha, \Lambda_5, \Lambda_4\) by:

First solution

\[
\frac{\gamma}{R^3} = \frac{\kappa_4^2 M}{4 \pi R^3} + \frac{\alpha}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2} - \frac{\alpha}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2 + \frac{3 \kappa_4^2 M}{\pi R^3}},
\]

\[
\beta = \frac{1}{3} \Lambda_4 + \frac{1}{2} \alpha^2 - \frac{\alpha}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2}
\]

Second solution

\[
\frac{\gamma}{R^3} = \frac{\kappa_4^2 M}{4 \pi R^3} + \frac{\alpha}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2 + \frac{3 \kappa_4^2 M}{\pi R^3}},
\]

\[
\beta = \frac{1}{3} \Lambda_4 + \frac{1}{2} \alpha^2.
\]

The first solution, as it is seen from (26) and (24), is matched to a Schwarzschild-(\(A\))dS\(_4\) exterior solution, while the second solution is matched to a non-Schwarzschild-(\(A\))dS\(_4\) exterior solution. Note that no additional constant of integration enters the above solutions by requiring that the metric is non-degenerate at \(r = 0\). In the special case with \(4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2 = 0\), (34) is matched to an exterior Schwarzschild-(\(A\))dS\(_4\).

From (29), \(p(r)\) for our two solutions, is found to be

\[
p(r) = -\rho_o \frac{\sqrt{1 - \left(\beta + \frac{\gamma}{R^3}\right)r^2} \ominus \sqrt{1 - \left(\beta + \frac{\gamma}{R^3}\right) R^2}}{\sqrt{1 - \left(\beta + \frac{\gamma}{R^3}\right)r^2} \ominus \omega \sqrt{1 - \left(\beta + \frac{\gamma}{R^3}\right) R^2}},
\]

where

\[
\omega^{-1} = 1 - \frac{2}{\kappa_4^2 \rho_o} \left(\beta + \frac{\gamma}{R^3}\right) \left(1 + \frac{\sqrt{3} \alpha}{\sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2 + 4 \kappa_4^2 \rho_o}}\right)^{-1}.
\]
The symbol ⊕ means −, except from the (rather irregular) case with \( \omega < 0, p > \rho_o/|\omega| \), where it becomes +. In the limit \( \alpha, \Lambda_4 \to 0 \) both solutions for \( A_e(r), B_e(r), p(r) \) reduce to the known Oppenheimer-Volkoff solution. Also, in the limit \( \alpha \to 0 \), the exterior solutions corresponding to (32), (33) and (34), (35) reduce to the Kottler [48, 49] solution of 4-dimensional General Relativity.

It is of some importance to notice the following. Although three unrelated parameters \( \alpha, \Lambda_4, \Lambda_5 \) (which are rather supposed to be fundamental) enter our problem, the final exterior solutions contain only two combinations of them, namely the parameters \( \gamma, \beta \). Thus, from exterior experimental data only two constraints on \( \alpha, \Lambda_4, \Lambda_5 \) can be extracted. However, the interior solutions contain, furthermore, the parameter \( \omega \), which means that a third combination of \( \alpha, \Lambda_4, \Lambda_5 \) could be obtained from possible astrophysical information. Thus, \( \alpha, \Lambda_4, \Lambda_5 \) can be uniquely determined from local measurements. Of course, as it is seen from (37), if the parameters \( \alpha, \Lambda_4, \Lambda_5 \) are extremely small (as will be seen in the next section), the influence of the bulk effects onto the interior solution is also small.

It can be seen from (82) that for a given set of parameters \( \alpha, \Lambda_4, \Lambda_5 \), the relative change \( \gamma/2G_4(M) - 1 \) on the parameter of the Newtonian term is negative and it is an increasing function of \( \rho_o \). This deviation from the common situation can be interpreted as an object-dependent gravitational constant, while \( M \) remains unchanged, i.e. \( \gamma = 2G_4(\rho_o)M, \) where \( G_4(\rho_o)/G_4 = 1 + 2 \left(1 + 4\Lambda_5 - 2\Lambda_4 \right)/\alpha^2 \left(1 - s\rho_o \right) \left(1 - \sqrt{1 + s\rho_o} \right) \) and \( s = 32\pi G_4 \left(1 + 4\Lambda_5 - 2\Lambda_4 \right)/\alpha^2 \left(1 + 4\Lambda_5 - 2\Lambda_4 \right) \). Then, \( G_4(\rho_o) \) starts from the value \( G_4 \left(1 - (1 + 4\Lambda_5 - 2\Lambda_4 \right)/\alpha^2 \) \) when \( \rho_o \to 0 \), and asymptotically tends to \( G_4 \) for \( \rho_o \to \infty \). In this picture, \( G_N \), the measured Newton’s constant, is not a universal quantity, but simply corresponds to \( G_4(\rho_{o, everyday}) \), where \( \rho_{o, everyday} \) is the density of common matter \( \sim gr/cm^2 \). There is a characteristic value of energy, which can be advocated to these densities, namely \( \alpha_e = \sqrt{G_N \rho_{o, everyday}} \sim 10^{-14} cm^{-1} \). If \( 4\Lambda_4 - 2\Lambda_5 \gg 3\alpha^2 \) (plot 1a in Fig. 1), \( G_4(\rho_o) \) is always almost equal to \( G_N \approx G_4 \), and no significant deviations from Newton’s constant universality exist. Otherwise (plot 1b in Fig. 1), significant deviations from \( G_N \) can arise. Then, there exist only two situations which do not contradict with the everyday experience of no deviation from Newton’s constant universality. These are: \( \alpha \ll \alpha_e \) or \( \alpha \gg \alpha_e \).

In the first case, \( G_N \approx G_4 \) and significant deviations from \( G_N \) appear at extremely low densities \( \rho_o \ll \alpha^2/G_N \). In the second case, \( G_N \approx G_4 \left(1 - (1 + 4\Lambda_5 - 2\Lambda_4 \right)/\alpha^2 \) \) and significant deviations appear at extremely dense objects. In the next section, we will set upper bounds on \( \alpha \), similar to \( \alpha \ll \alpha_e \), from solar system experiments, and thus the second case is excluded. If this is really the situation, the possibility for the parameters having \( 4\Lambda_4 - 2\Lambda_5 < 0 \), which leads to repulsive gravity on very low density objects, is
possible. Similar behavior to the above described, but with extra attraction, appears in the solution (34) (plot 2 in Fig. 1).

Figure 1: The $\rho_0$ dependence of Newton’s constant in various models.

Solution (32), (33), being matched to an exterior Schwarzschild-(A)$dS_4$, will be used in the next section in order to bound the parameters encountered, from experimental data of our solar system (deflection of light coming from distant stars, precession of perihelia and radar echo delay). Since there are two parameters $\beta, \gamma$ in the exterior solution, connected to the three $\alpha, \Lambda_4, \Lambda_5$, it is rather necessary to drop one of these three by hand, in order to get an estimation of the other two. It is obvious that $\alpha$ cannot be this one, since this is too restrictive and actually, analyses of this case have been performed [50, 51]. Also, we do not set $\Lambda_5 = 0$, since then, $\beta$ cannot be negative ($\beta$ negative implies $\Lambda_5 < 0$). Although $\beta$ is the same quantity which in cosmology plays the role of the effective cosmological constant [10, 12] and it is rather positive, in the present work, we do not claim any connection with cosmology, so we would prefer to be able to also deal with a negative $\beta$. As will be discussed in the next section, this may be of importance to galactic scale phenomena. In the following, we choose $\Lambda_4 = 0$. Then, from (32), (33) we find that for $G_{(4)} \simeq G_N$ it is

$$\alpha^2 = \frac{1}{\gamma R^3} (2G_N M - \gamma) (2G_N M - \gamma - 2\beta R^3),$$  \hspace{1cm} (38)

$$\Lambda_5 = 6\beta \left( 1 - \frac{\beta \gamma R^3}{(2G_N M - \gamma)(2G_N M - \gamma - 2\beta R^3)} \right),$$  \hspace{1cm} (39)
while in the other limiting case

\[ \alpha^2 = 2\beta + \frac{2\gamma^2}{R^3} \left( \gamma - 2G_NM - \frac{2\gamma}{\beta R^3} G_NM \right) \]

\[ \pm \sqrt{(2G_NM - \gamma)(2G_NM - \gamma + \frac{4\gamma}{\beta R^3} G_NM)} \right)^{-1}. \]  \( (40) \)

Finally, we note that the non-Schwarzschild-(A)dS\(_4\) solution \((34), (35)\) could be also used for extracting phenomenological bounds on the string parameters from the solar-system experiments. We have chosen in this paper the simplest solution for this purpose. However, it is known \([52]\) that the agreement with the solar-system tests of some metric-based relativistic theory requires on kinematical grounds that \(AB \simeq 1\) to high accuracy in the vicinity of the sun.

3. Constraints from Classical Tests

A difficulty arising with the calculations of the measurable quantities (integrals) comes from the fact that the solution \((32), (33)\) is not asymptotically flat, but diverges at large distances, thus, an expansion in powers of \(1/r\), performed in the standard PPN (parametrized post-Newtonian) analysis, does not work here. Hence, one has to make expansions according to parameters of the problem which are sufficiently small, and fortunately, such parameters exist.

The motion of a freely falling material particle or photon in a static isotropic gravitational field \([17]\) is described \([53]\) by the equation

\[ \left( \frac{d\phi}{dr} \right)^2 = A_\phi \left( \frac{1}{J^2 B_\phi} - \frac{1}{r^2} - \frac{E}{J^2} \right)^{-1}, \]  \( (41) \)

where \(J, E\) are constants of integration \((E > 0\) for material particles and \(E = 0\) for photons). At the points of minimum or maximum distance \(r_o\), it is \(dr/d\phi = 0\) and thus

\[ J = r_o \left( \frac{1}{B_\phi (r_o)} - E \right)^{1/2}. \]  \( (42) \)

We will analyze the three classical solar scale experiments - deflection of electromagnetic waves coming from distant stars by the sun, precession of the perihelia of planets, and time delay of radio waves.

1) Deflection of light. Although the metric is not asymptotically flat, the photon, as it can be seen from equation \((11)\), has \(d\phi/dr \rightarrow 0\) as \(r \rightarrow \infty\), and thus, it moves in a “straight” line of the background geometry in that region. The deviation from this line
is measured by the total deflection angle $\Delta \phi_d = 2|\phi(r_o) - \phi(\infty)| - \pi$, where $r_o$ is the minimum distance of the orbit to the sun (when a ray grazes sun $r_o = R_\odot$). As it is expected and will be shown below, $|\beta|$ has an extremely small value, thus, for $\beta > 0$ the horizon $r_2$ is of cosmological scale and scattering of light can be practically defined even in this case. The angular momentum $J$ is related to the “impact parameter” $b$ through the relation $J = b(1 - \beta b^2)^{-1/2}$. Integrating (41) we arrive at

$$\phi(r) - \phi(\infty) = \sqrt{\frac{r_o^3}{r_o - \gamma}} \int_r^\infty \left[r(r - r_o) \left(r^2 + r_o r - \frac{\gamma r_o^2}{r_o - \gamma}\right)\right]^{-1/2} dr. \quad (43)$$

For the above expression being well-defined, it has to be $r_o \geq \frac{3}{2} \gamma$ (which is always the case for common stars such as our sun). Note that the parameter $\beta$ has disappeared in the expression (43), i.e. the deflection phenomenon is the same as if it had been occurred in a Schwarzschild field of parameter $\gamma$. Expression (43) leads to an elliptic integral. Since $\gamma$ is almost $2G_N M_\odot$ and $r_o$ is of the order of $R_\odot$, thus, $\frac{1}{r_o}$ is of the order of $10^{-6}$. Hence, we can expand the integrand of (43) to first order in this parameter before integration [54]. It is convenient, simultaneously, to set $\sin u = \frac{r_o}{r}$, and the result is

$$\Delta \phi_d = \frac{2\gamma}{r_o}. \quad (44)$$

The best measurements on the deflection of light from the sun were obtained using radio-interferometric methods [55] and found to be (for $r_o = R_\odot$) $\Delta \phi_d = 1.761 \pm 0.016$ arc sec. Then, from (44),

$$29.440 \times 10^4 \text{ cm} < \gamma < 29.979 \times 10^4 \text{ cm}, \quad (45)$$

which is around the conventional value $2G_N M_\odot = 29.539 \times 10^4 \text{ cm}$.

2) Precession of perihelia. Here, there are two values $r_+, r_-$ of maximum and minimum distance, satisfying equation (42). The two constants of motion $J, E$ are expressed in terms of $r_+, r_-$ and are plugged in (41). The expression arising is very complicated, but referring to [50], [51], we can write the precession per orbit $\Delta \phi_p = 2|\phi(r_+) - \phi(r_-)| - \pi$ as

$$\Delta \phi_p = \frac{3\pi \gamma}{L} + \frac{6\pi \beta L^3}{\gamma}, \quad (46)$$

where $L^{-1} = (r_+^{-1} + r_-^{-1})/2$ is the *semilatus rectum* of the orbit. Both [50], [51] agree on the result (46). Actually, they refer to the Gibbons-Hawking metric, but their methods are immediately applied in our case. They disagree on the next order terms, which are, however, negligible compared to the second term of (46), for stars with small Schwarzschild radius and for slightly eccentric orbits.
For Mercury, the uncertainty in the quantity $\Delta \phi_p - \frac{6\pi G N M}{L}$ is $\pm 10^{-9}$ rad/orbit. Then, taking into account the range (45) of $\gamma$ received from deflection, we obtain

$$-7.908 \times 10^{-43} \text{ cm}^{-2} < \beta < 2.465 \times 10^{-43} \text{ cm}^{-2}.$$  \hspace{1cm} (47)

The bounds (45), (47) give from equation (38):

$$\alpha < 4.379 \times 10^{-16} \text{ cm}^{-1}.$$  \hspace{1cm} (48)

Actually, as far as, the upper bound of $|\beta|$ remains many orders of magnitude smaller than $\frac{G N M}{R}$, the above result, as can be seen from (38), is insensitive to the exact value of $\beta$. Furthermore, the fact that $\alpha$ has upper, instead of lower, bound is due to the specific functional form of the expression (38) in terms of $\gamma$. This means that the crossover scale $r_c > 4.567 \times 10^{15} \text{ cm}$, i.e. the lower bound of $r_c$ is a few times the diameter of our planetary system. Thus, the five-dimensional fundamental Planck scale $M_5$ is less than 0.9 TeV.

From equation (40), one can see that for $\beta \to 0$, $\alpha \to 0$ and then, from (45), (47), an upper bound of the order $10^{-22} \text{ cm}^{-1}$ is set for $\alpha$, which is incompatible with $\alpha \gg \alpha_e$. Thus, this case is not acceptable.

From (32), an upper bound for $\Lambda_5$ can be obtained

$$\Lambda_5 < 3.804 \times 10^{-43} \text{ cm}^{-2}.$$  \hspace{1cm} (49)

Uncertainties in the measurement of the precession of perihelion are known to exist, due to the rotation of sun; thus, it is better to examine the bounds on $\beta$ from the radar echo delay independently.

3) Radar echo delay. The time required for a radar signal to go from a point $r$ to the closest to the sun point $r_o$ of its orbit is

$$t(r, r_o) = \int_{r_o}^{r} \left( \frac{A_{>}}{B_{>}} \right)^{1/2} \left( 1 - \frac{r_o^2}{r^2} \frac{B_{>}(r_o)}{B_{>}(r)} \right)^{-1/2} dr.$$  \hspace{1cm} (50)

As in the deflection of light, expanding to first order in $\frac{\gamma}{R}$ we obtain

$$t(r, R) = \frac{1}{\sqrt{|\beta|}} \arctan h \left( \frac{\sqrt{r^2 - R^2}}{\sqrt{1 - \beta R^2}} \right) + \gamma \left( \ln \frac{1 - \beta R^2 r + \sqrt{r^2 - R^2}}{R \sqrt{1 - \beta r^2}} + \frac{1}{2 \sqrt{1 - \beta R^2}} \frac{r - R}{r + R} \frac{1 + \beta r R}{1 - \beta r^2} \right).$$  \hspace{1cm} (51)

This expression holds for $\beta > 0$, while for $\beta < 0$, $\arctan h \left( \frac{\sqrt{r^2 - R^2}}{\sqrt{1 - \beta R^2}} \right)$ has to be replaced by $\pi/2 - \arctan \left( \frac{1}{\sqrt{|\beta|}} \sqrt{\frac{1 - \beta R^2}{1 - \beta r^2}} \right)$. Whenever $|\beta| r^2 \ll 1$, the above expression takes
the form
\[
t(r, R) \simeq \sqrt{r^2 - R^2} + \gamma \ln \frac{r + \sqrt{r^2 - R^2}}{R} + \frac{\gamma}{2} \sqrt{\frac{r - R}{r + R}} + \frac{\beta}{3} (r^2 - R^2)^{3/2}.
\] (52)

We will use this expression to get bounds from the solar radar echo experiments. Notice, however, that (51) may be applicable to some more general cases.

In [56], the time delay on solar system scales was measured to an accuracy of 0.1 per cent. A ray that leaves the Earth, grazes the sun, reaches Mars and comes back would have a time delay of 248±0.25 µs where the 248 µs is the exact prediction of the “Shapiro” time delay and the uncertainty ±0.25 µs can be used to constrain β. At superior conjunction, the radius of the sun to the Earth, \(r_e\), and to the Mars, \(r_m\), are much greater than the radius of the sun \(R_{\odot}\), and thus, \(\frac{\beta}{3}(r_e^3 + r_m^3) = \pm 0.25 \mu s\). This constrains β to the range
\[
|\beta| < 7.555 \times 10^{-37} \text{ cm}^{-2}.
\] (53)

It is interesting to compare the bounds on the various parameters of the brane theory with an \(R\) term, with the bounds on the parameters which are resulting from the brane dynamics without the \(R\) term. In [1], the dynamics on the brane is given, instead of (8), by the following equation
\[
(4) G_{\mu \nu} = \frac{\kappa_4^4}{6 \kappa_4^4} (4) T_{\mu}^{\nu} - \frac{1}{2} \left( \Lambda_5 + \frac{\kappa_4^4}{6 \kappa_4^4} \Lambda_4^2 \right) \delta_{\mu}^{\nu}
\]
\[
- \frac{\kappa_4^4}{24} \left( 6 \frac{(4) T_{\rho}^{(4)} T_{\nu}^{\rho}}{(4) T_{\nu}^{(4)} T_{\rho}^{\nu}} - 2 (4) T_{\nu}^{(4)} T_{\rho}^{\sigma} (4) T_{\sigma}^{\rho} + (4) T_{\nu}^{(4)} T_{\rho}^{\sigma} \delta_{\mu}^{\nu} + (4) T_{\rho}^{(4)} \delta_{\mu}^{\nu} \right) - E_{\mu}^{\nu}.
\] (54)

For \(E_{\nu}^{\mu} = 0\), following the same steps for solving (54), as before, we arrive to the unique Schwarzschild-(A)dS4 exterior solution \(B_>(r) = \frac{1}{A_>(r)}\), where \(A_>(r)\) is given by (23). The parameters of this solution, denoted by the subscripts SMS, are given by
\[
\gamma_{SMS} = \frac{\kappa_4^2 \Lambda_{4SMS}}{6 \pi \alpha_2^{SMS}} \left( 1 + \frac{3 \kappa_4^2 M}{8 \pi \Lambda_{4SMS} R^3} \right) M,
\] (55)
\[
\beta_{SMS} = \frac{1}{6} \Lambda_{5SMS} + \frac{\Lambda_{2SMS}}{9 \alpha_2^{SMS}}.
\] (56)

It is obvious that the conventional value \(2G_N M\) of the Newtonian term can dominate \(\gamma_{SMS}\) only if \(\Lambda_{4SMS} = 3 \alpha_2^{SMS}/2\). This is the same value which revives the common four-dimensional energy-momentum terms in the general equation (54). This value is substituted in (55), (56) and then, using the bounds (45), (47) from the classical tests,
we can set bounds on $\alpha_{SMS}$, $\Lambda_{5,SMS}$. More specifically, since $\Lambda_{5,SMS}$ is not contained in (55), equation (45) is enough for finding

$$\alpha_{SMS} > 2.425 \times 10^{-13} \text{ cm}^{-1},$$  \hspace{1cm} (57)$$

which means $M_5 > 7 \text{ TeV}$. Then, from (56)

$$\Lambda_{5,SMS} < -8.818 \times 10^{-26} \text{ cm}^{-2},$$  \hspace{1cm} (58)$$

and only a bulk of negative curvature is allowed in this approach. The above results are exact, since now, there are only two unknown parameters $\alpha_{SMS}$, $\Lambda_{5,SMS}$ to be determined from the two $\gamma_{SMS}, \beta_{SMS}$. It is seen from equation (54) (plot 3 in Fig. 1) that the point particle limit of infinite density cannot be obtained (contrary to the plots 1a, 1b, 2 of Fig. 1), since then, $G_{(4)} \rightarrow \infty$. Even for different boundary conditions [34] the above limit, sometimes, is not defined at all.

Finishing, we make the following comment. In our second solution (34), we have obtained $\gamma > 2G_N M$. Thus, from (14), the deflection angle $\Delta \phi_d$ is larger than the corresponding “Einstein” deflection $\frac{4G_N M}{r_0}$. This situation of increased deflection (compared to that caused from the luminous matter) has been well observed in galaxies or clusters of galaxies, and the above solution might serve as a possible way for providing an explanation. In Weyl gravity [57, 58, 54], the above increase is advocated to some parameter like our $\beta$ (with the difference of a linear instead of quadratic term), which has to be positive in order to account for this (see also [52, 59]). But, then, a $\beta > 0$ cannot account for the additional attractive force needed to explain the galactic rotation curves. In our solution, instead, there is the additional freedom for the parameter $\beta$ being negative, which can be used for the galactic rotation curve fittings. Notice also that the Gibbons-Hawking solution cannot explain this way the extra deflection in galaxies. Alternative gravity theories may probably not have succeeded their best in illuminating the missing mass problem, but this does not mean that a new gravity modification must not be tested in the arena of local phenomena; it is certain that the whole topic deserves a more thorough investigation.

4. Conclusions

In the present paper, we have investigated the influence of the brane curvature invariant included in the bulk action, on the local spherically symmetric braneworld solutions. The brane dynamics is made closed by assuming the vanishing of the electric part of the Weyl tensor as a boundary condition for the propagation equations in the bulk space. All
the exterior solutions of a compact rigid object have been obtained. Some of them are of the Schwarzschild-(A)dS$_4$ form. Furthermore, two generalized interior Oppenheimer-Volkoff solutions have been found, one of which is matched to a Schwarzschild-(A)dS$_4$ exterior, while the other does not. A remarkable consequence is that the bulk space "sees" the finite region of the body and modifies the parameter of the Newtonian term in the outside region. No contradiction with the everyday Newton’s constant universality leads to bounds on the string scale. The known classical solar system tests, which were used in the past to check the validity of General Relativity, are here used to put precise bounds on the parameters of our model. More specifically, the crossover scale is found to be beyond our planetary system diameter, which means that the upper bound for the energy string scale is of the order of TeV. The limit of the idealized infinite density point particle is obtained, and significant deviations from the known Newton’s constant might occur on extremely low density matter distributions. In usual brane dynamics, contrary to our case, the solar tests impose a lower, instead of upper, bound of the above order on the string scale. Furthermore, in that case, for obtaining exterior non-Schwarzschild-(A)dS$_4$ solutions, one has to consider non-local bulk effects.

We have followed a braneworld viewpoint for obtaining braneworld solutions, ignoring the exact bulk space. We have not provided a description of the gravitational field in the bulk space, but confined our interest to effects that can be measured by brane-observers. However, our formalism assures the existence of a 5-dimensional Einstein space as the bulk space. By making assumptions for obtaining a closed brane dynamics, there is no guarantee that the brane is embeddable in a regular bulk. This is the case for a Friedmann brane [15], whose symmetries imply that the bulk is Schwarzschild-AdS$_5$ [55, 61]. A Schwarzschild brane can be embedded in a "black string" bulk metric, but this has singularities [23, 62]. The investigation of bulk backgrounds which reduce to Schwarzschild (or Schwarzschild-(A)dS$_4$) black holes is in progress.

It is clear that a density-dependent gravitational constant, generally violates, at the weak field limit, Newton’s third law of equal action-reaction. This, furthermore, means violation of the conservation of linear momentum and incapability to define precisely the potential energy for a system of two masses. Since the point particle limit does not meet any problem in our model, and also the Newtonian limit arises in metric-based theories for point particles moving along geodesics, we think that the understanding of the motion of the extended body in General Relativity (or more generally) would shed light to the above subteties. Beyond these, in our everyday phenomena, where very low density distributions do not contribute gravitationally, no such difficulties arise. However, such situations may
be relevant to early stages of the universe, before or during structure formation.

As a motivation for further speculation, we refer that it would be quite interesting, even for the formal status of the theory, if the existence of the $\rho_o \rightarrow \infty$ asymptotic behavior of the solutions found here, keeps valid, whenever $(^4R)$ term is present. Besides this, it is known that in cosmology the $(^4R)$ term revives the desirable early universe of standard General Relativity. However, to conclude, as braneworld solutions are continuously investigated, they have to confront with the accumulated cosmological and astrophysical observations, if the underlying theories wish to be considered as viable generalizations of General Relativity.

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