LONG-TIME ASYMPTOTICS OF SOLUTIONS AND THE MODIFIED
PSEUDO-CONFORMAL CONSERVATION LAW FOR SUPER-CRITICAL
NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this paper, we discuss a class of nonlinear Schrödinger equations with the power-type
nonlinearity: \((i\frac{\partial}{\partial t} + \Delta)\psi = \lambda |\psi|^{2\eta} \psi\) in \( \mathbb{R}^N \times \mathbb{R}^+ \). Based on the Gagliardo-Nirenberg interpolation
inequality, we prove the local existence and long-time behavior (continuation, finite-time blow-up
or global existence, continuous dependence) of the solutions to the \((H^q)\) super-critical Schrödinger
equation. The corresponding scaling invariant space is homogeneous Sobolev \( \dot{H}^{q_{crit}} \) with \( q_{crit} > q \).

Based on the estimates of the quadratic terms containing the phase derivatives used in the paper by
Killip, Murphy and Visan [24, SIAM J. Math. Anal. 50(3) (2018), 2681–2739] we shall study the
stability with a stronger bound on the solutions to our problem. Moreover, from the arguments on
virial-types presented in the paper by Killip and Visan [22, Amer. J. Math. 132(2) (2010), 361–424],
a modified pseudo-conformal conservation law is proposed. The Morawetz estimate for the solutions
to the problem are also presented.

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Morawetz action.

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1. Introduction and statement of the main results

The nonlinear Schrödinger (NLS for short) equation is used to describe a wavefield that is more complex than the normal form because its oscillation has a frequency proportional to the difference of the function value and its mean, see [29] and references therein. A solution of the Schrödinger equation that can be physically visualized is an amplitude function of a quantum mechanical particle that moves freely in $\mathbb{R}^N$ (see [1, 21, 29]). In this study, we consider the following NLS equation with smooth power-type nonlinearity:

$$
\begin{cases}
\left(i \frac{\partial}{\partial t} + \Delta\right) \psi = \lambda |\psi|^{2\eta} \psi, & \text{for } x \in \mathbb{R}^N, \ t \in \mathbb{R}^+, \\
\psi = \psi_0(x), & \text{for } t = 0, \ x \in \mathbb{R}^N,
\end{cases}
$$

(NLS)

where $N \geq 1$ and $\psi(x,t)$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ is a complex-valued function; $i = \sqrt{-1}$; $|\psi(\cdot,t)|^2$ is the spatial probability density function; $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2$. The power-type nonlinearity $\lambda|\psi|^{2\eta}$, for $\lambda \in \mathbb{R}$ (or, possibly, $\lambda \in \mathbb{C}$) serves as a scaling factor, when $\lambda < 0$ (or $\lambda > 0$) the equation be called focusing (or defocusing); $1 \leq \eta \in \mathbb{N}$; the initial data $\psi_0$ belongs to $H^q(\mathbb{R}^N)$ for $N \geq q > \frac{N}{2}$ and $q \geq \eta$.

With the power-type nonlinearity case $|\psi|^{2\eta}$ ($\eta \geq 1$), equation (NLS) becomes scale invariant, that is, if $\psi(x,t)$ satisfies (NLS), then we have $\psi_\theta(x,t) = \theta^{\frac{1}{\eta}} \psi(\theta x, \theta^2 t)$, with $\theta > 0$. For the critical exponent $q_{\text{crit}} = \frac{N+2}{2\eta}$, and $\|\psi\|_{H^q(\mathbb{R}^N)} = \|\theta^{\frac{1}{\eta}} \psi(\theta^2 t)\|_{H^q(\mathbb{R}^N)}$, if we consider the case of $q_{\text{crit}} > q \implies \eta > \frac{N-2}{N-2q}$ and $\psi_0 \in H^q(\mathbb{R}^N)$ then the Problem (NLS) is called the $(H^q)$ super-critical problem. Moreover, if $q_{\text{crit}} > 0$ then the problem is called $(L^2)/\text{mass}$ super-critical case, if $q_{\text{crit}} < 1$ then the problem is called $(H^1)/\text{energy}$ sub-critical case (see [9, 26]). Problem (NLS) with condition $q > \frac{N}{2}$ called the $(H^0)$ super-critical NLS equation.

It is well known that smooth solutions to the NLS equation (NLS) satisfy the following conservation laws for all $t > 0$ (see [11, 26])

$$
L^2 \text{ - norm : } \quad \mathbb{T}[\psi](t) = \int_{\mathbb{R}^N} |\psi(x,t)|^2 \, dx = \int_{\mathbb{R}^N} |\psi_0(x)|^2 \, dx = \mathbb{T}[\psi_0]; \quad \text{(Mass)}
$$

$$
\mathbb{E}[\psi](t) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi(x,t)|^2 \, dx + \frac{\lambda}{2\eta + 2} \int_{\mathbb{R}^N} |\psi(x,t)|^{2\eta + 2} \, dx = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi_0(x)|^2 \, dx + \frac{\lambda}{2\eta + 2} \int_{\mathbb{R}^N} |\psi_0(x)|^{2\eta + 2} \, dx = \mathbb{E}[\psi_0]; \quad \text{(Energy)}
$$

$$
\mathbb{M}[\psi](t) = \Im \int_{\mathbb{R}^N} \psi(x,t) \nabla \bar{\psi}(x,t) \, dx = \Im \int_{\mathbb{R}^N} \bar{\psi}_0(x) \nabla \psi_0(x) \, dx = \mathbb{M}[\psi_0]. \quad \text{(Momentum)}
$$

Here, $\Im(z)$ is the imaginary part of the complex number $z$ and $\bar{\psi}$ is the complex conjugate term of $\psi$. More precisely, $\Psi = |\psi|^2$ (mass density) and the local conservation of mass reads (see [9, page 12])

$$
\mathbb{T}_{loc}[\psi](x,t) = \frac{\partial \Psi}{\partial t}(x,t) + 2 \text{div} \left( \frac{\overline{\overline{Q^N}}}{Q^N} \psi(x,t) \right) = \frac{\partial \Psi}{\partial t}(x,t) + 2 \nabla_1 Q_1^N(\psi)(x,t) + 2 \nabla_2 Q_2^N(\psi)(x,t) + 2 \nabla_{N-1} Q_{N-1}^N(\psi)(x,t) + 2 \nabla_N Q_N^N(\psi)(x,t) = 0. \quad \text{(loc-Mass)}
$$

Here, $\Psi = |\psi|^2 = \overline{\psi} \psi$ and we note that

$$
\overline{Q^N}(\psi) = (Q_1^N(\psi), \ldots, Q_N^N(\psi)) \quad \text{and} \quad Q_m^N(\psi) = \Im \left( \bar{\psi} \nabla_m \psi \right), \ m = 1, \ldots, N. \quad \text{(1)}
$$
The \textit{local} momentum conservation is (see [9, page 12])

\[
\mathcal{M}_{\text{loc}}[\psi](x, t) = \frac{\partial Q_m^N(\psi)}{\partial t}(x, t) + \nabla_1\left(1_{m,1}L^\nabla(\psi) + S_m^N(\psi)\right)(x, t) \\
+ \nabla_2\left(1_{m,2}L^\nabla(\psi) + S_m^N(\psi)\right)(x, t) \\
+ \nabla_{N-1}\left(1_{m,N-1}L^\nabla(\psi) + S_m^N(\psi)\right)(x, t) \\
+ \nabla_N\left(1_{m,N}L^\nabla(\psi) + S_m^N(\psi)\right)(x, t) = 0, \quad (\text{loc-Momentum})
\]

where \(m = 1, \ldots, N\) and \(1_{m,n}\) is the Kronecker delta function

\[
\begin{cases}
1_{m,n} = 1, & \text{if } m = n; \\
1_{m,n} = 0, & \text{if } m \neq n,
\end{cases}
\]

and the Lagrangian density is

\[
L^\nabla(\psi) = -\frac{1}{2} \text{div} \left(\nabla |\psi|^2\right) + \frac{\lambda \eta}{\eta + 1} |\psi|^{2\eta + 2},
\]

where \(\nabla = (\nabla_1, \ldots, \nabla_N)\). The symmetric tensor is

\[
S_m^N(\psi) = 2\Re \left(\nabla_m \psi \nabla_n \bar{\psi}\right), \quad \text{for } m = 1, N, n = 1, N,
\]

with \(\Re(z)\) be the real part of the complex number \(z\).

The pseudo-conformal conservation (PCC) law was first discovered by Ginibre and Velo in the series of papers [17–19], considered as an essential tool to study the asymptotic behavior of solutions for NLS equations. J. Bourgain [4] considered the 3D defocusing NLS with \(\eta = 2\)

\[
\begin{cases}
\left(\frac{\partial}{\partial t} + \Delta\right) \psi - |\psi|^4 \psi = 0, & x \in \mathbb{R}^3, \quad t \geq t_0, \\
\psi|_{t=t_0} = \zeta(\psi(t_0)), & x \in \mathbb{R}^3,
\end{cases}
\]

where \(0 < \zeta < 1\) is a radial bump function chosen and the author proved that the solutions of problem satisfy the following PCC law

\[
||(|(\cdot) + 2i(t-t_0)\nabla|\psi(\cdot, t)|^2||_{L^2(\mathbb{R}^3)}^2 + \frac{4}{3}(t-t_0)^2 ||\psi(\cdot, t)||_{L^6(\mathbb{R}^3)}^6 \\
= ||(|(\cdot)\psi(\cdot, t_0)||_{L^2(\mathbb{R}^N)}^2 - \frac{16}{3} \int_{t_0}^t s \int_{\mathbb{R}^3} |\psi(x, s)|^6 dx ds \\
\leq ||(|(\cdot)\psi(\cdot, t_0)||_{L^2(\mathbb{R}^3)}^2
\]

By considering Problem (5), J. Colliander et al. also pointed out PCC (6) in paper [12, Eq. (8.15)]. Obviously, Problem \((\text{NLS})\) is more general than Problem (5), then PCC law for this problem is much more difficult to establish. As far as we know, in general, the PCC law for Problem \((\text{NLS})\) has not been studied extensively. There are two formal ways to set up the PCC for Problem \((\text{NLS})\).

One is using the pseudo-conformal symmetry, indeed, for \(\eta = \frac{2}{N}\) and if \(\psi\) is a solution of \((\text{NLS})\) then \(\Theta(x, t) = \frac{1}{|t|^{\frac{2}{N}}}(\frac{x}{t})^\frac{2}{N} e^{-\frac{4\pi|x|^2}{t}}\) is also a solution to \((\text{NLS})\) for \(t \neq 0\). The function \(\Theta\) satisfies the equation

\[
\frac{i}{t} \frac{\partial \Theta}{\partial t} + \Delta \Theta = \lambda t^{\eta-2} |\Theta|^{2\eta} \Theta,
\]

and by direct arguments on this equation, we obtain the PCC for Problem \((\text{NLS})\), see [7, Subsection 7.5, page 224]. Another way is to use the virial identities and this is also our aim to prove the following quantity

\[
\forall t \geq 0 : \mathcal{P}[\psi](t) = ||(|(\cdot) + 2i\nabla|\psi(\cdot, t)|^2||_{L^2(\mathbb{R}^N)}^2
\]
\[
- \frac{4t^2 \lambda N}{N + 2} \int_{\mathbb{R}^N} |\psi(x, t)|^{\frac{4}{N} + 2} dx - 2N \int_{\mathbb{R}^N} \text{div} \left( \nabla H(\psi)(x, t) \right) dx,
\]

is conserved. Here, the function $H$ satisfies some necessary conditions:

\[
\begin{align*}
H(\psi)(x, 0) &= 0; \\
\frac{\partial H(\psi)}{\partial t}(x, t) &= t|\psi|^2,
\end{align*}
\]

for a.e. on $\mathbb{R}^N$. Based on the properties of the virial identity which is called $(1-1)$ Morawetz action (see Remark 1.2 for its definition), we prove a the modified by considering the new version of the PCC law to Problem (NLS) as shown in (7) in $\mathbb{R}^N$, i.e.,

\[
\mathbb{P}[\psi](t) \text{ is conserved } \iff \mathbb{P}[\psi](t) = \mathbb{P}[\psi_0], \quad \forall t > 0.
\]

(\text{PC})

**Remark 1.1.** Since Problem (NLS) is more general than (5), it can be compared directly to see that (PC) includes (6). The conditions in (8) of $H(\psi)$ are satisfied for a class of functions containing the solution of the problem under some time-smooth conditions. This is an extension of the previous result to show that there are more nonlinear functions containing particles that still guarantee conservation. However, these functions must satisfy the constraint conditions as (8).

The main equation in (NLS) is a particular case of the general NLS equation

\[
\frac{\partial \psi}{\partial t} = -\Delta \psi + F(\psi).
\]

Equation (9) has been studied by V. Banica [2] for $F(\psi) = -|\psi|^{q-1}\psi$, the author was concerned with the focusing NLS equation posed on a regular domain $\Omega$ of $\mathbb{R}^N$, with Dirichlet boundary conditions. The blow-up rates are bounded. Moreover, they proved that if the blow-up occurs on the boundary, the blow-up rate grows faster than $(T-t)^{-1}$. For dimension $N \geq 3$, $F(\psi) = -|\psi|^{q-1}\psi$ for $1 < q < q_s$, $q_s \leq \frac{N}{N-2}$. F. Merle and P. Raphaël [26] proved the blow up of the critical norm for some radial $L^2$ super-critical NLS equations.

In dimension 2, Y. Wang [43] discussed the NLS equation containing the general potential $F(\psi) = V(x)|\psi|^{q-1}\psi$ with some assumptions of potential $V(x)$. The paper obtained some sharp conditions for global existence and blow-up of solutions in $H^1(\mathbb{R}^2)$. For the focusing cubic NLS, C. Sulem et al. [10] proved the intermediate case of blow-up solutions from initial data $\psi_0 \in H^3(\mathbb{R}^N)$, with a very strict index satisfying $1 > q > q_s$, for $q_s \leq \frac{N}{N+2}$, it called below $H^1$ cases. We refer the readers to the excellent works of C. Sulem et al. in [30–32].

L. Vega et al. [34,37,38] studied the unique continuation of solutions to non-local non-linear dispersive equations and related to NLS. In particular, Banica and Vega [35,36] demonstrated the existence of a unique solution of the binormal flow with datum a polygonal line. Some other outstanding works of L. Vega and his group on dispersive equations can be suggested to the reader as [39–41] and some related references therein. Many researchers have studied the asymptotic behavior of solutions to NLS equations: G.M. Bisci and V.D. Rădulescu [3], N. Burq et al. [6], B.J. Campos and P.I. Naumkin [8], T. Oh et al. [27], H. Xu [42], Q. Ding et al. [13], A.J. Fernández et al. [16], L. Jeanjean et al. [20], R. Killip et al. [23], T. Caraballo et al. [44], K. Pravda-Starov [25], D.V. Duong [14,15] and the references therein. To our knowledge, there are many works on the \((L^2)_{/\text{mass}}\) super-critical [2,10,26] and \((H^1)_{/\text{energy}}\) sub-critical [10,22,28] problems, but the results on long-time asymptotics for super-critical NLS problems are still limited. Our results on \((H^q)_{/\text{super-critical}}\) include the results on energy super-critical problems.

In this paper, to solve Problem (NLS), we have tried to overcome the following challenges:

- First, without the help of Strichartz estimates, but using the Gagliardo-Nirenberg (G-N) interpolation inequality (see Lemma 2.5 below) we obtain the local well-posedness results of the solutions: local existence, continuation, global existence (or finite time blow-up) and continuous dependence on initial
data. In [26], the author used Strichartz estimates to obtain the finite-time blow-up and lower bound for the mass super-critical case in Sobolev norm. This is the reason that motivates us to study the long-term behavior for the \((H^q)\) super-critical of \((NLS)\). However, we do not rely on the usefulness estimates of Strichartz used in [26] and many other works. Thus, we face the difficulty of establishing the Lipschitz property of the nonlinear source function \(\lambda|u|^{2\eta}u\) to be able to use a complex tool as the G-N interpolation inequality. It is worth mentioning here that we need the conditions of the power \(\eta\) satisfy \(N \geq \eta \in [1, q]\). It is a necessary condition to establish techniques for the application of G-N interpolation inequality. Let us cover a little bit of the technique that is closely related to use source functions of the form \(F_\eta(\psi) = \lambda|\psi|^{2\eta}\psi\). We need to estimate the following term \(\|F_\eta(\psi)\|_{H^q(\mathbb{R}^N)}\) and

\[
\|F_\eta(\psi)\|_{H^q(\mathbb{R}^N)} = \left(\sum_{|\alpha| \leq q} \sum_{\beta \leq q} \left\| F^{(\eta)}_\eta(\psi) \prod_{j=1}^\beta D^{\alpha_j}(\psi) \right\|_{L^2(\mathbb{R}^N)}^2 \right)^{1/2}.
\]

From the above formula, we have the following two important problems. First, the estimation of the norms of the term \(F^{(\eta)}_\eta(\psi)\) explains why we need to consider the term \(F_\eta(\psi) = \lambda|\psi|^{2\eta}\psi\) (i.e., related to the exponent \(2\eta\) of \(|\psi|\)) instead of other nonlinear functions. It is a clear advantage that we can use the hypothesis \((H2)\). Second, estimating the norms of the term \(\prod_{j=1}^\beta D^{\alpha_j}(\psi)\) requires the help of G-N interpolation inequality, and it is interesting to choose the complex coefficients of this inequality related to \(\eta\) and \(q\). It also has to satisfy the conditions of Hölder inequality and generalized Hölder inequality. Through Sobolev embeddings, we establish the properties of the solution on a Sobolev space \(H^q(\mathbb{R}^N)\) with the conditions \(\frac{N}{2} < q \in \mathbb{N}\) and \(q \geq \eta\). These results are presented in Theorem 1.1 and Theorem 1.2, below.

- The second challenge is to estimate a stronger bound space-time for solutions. In [24], the authors estimate the “phase derivative” terms (see [24, in Strategy of the proof]), such terms are significantly more difficult to estimate. These estimates contain higher order gradients with \(\nabla^n f(\xi) = |\xi|^n \hat{f}(\xi)\). Also, based on ideas in the paper [24], however, we do not use the estimations for phase derivative terms, but instead, we consider the expansions of generalized virial quantity with two particles, i.e. contains both \(\psi(x, t)\) and \(\psi(y, t)\) (see [9, Eq. (1.8)])

\[
\mathcal{V}_\delta(t) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |\psi(y, t)|^2 \nabla_x \delta(x - y) \cdot \overline{Q^2}(\psi)(x, t) dxdy,
\]

where \(\delta\) is a weighted function with two variables \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\). By choosing \(\delta(x - y) = |x - y|\), \(x, y \in \mathbb{R}^N\), the estimations of this virial \((10)\) are closely related to the conservation of mass (Mass) and the local conservation laws (loc-Mass), (loc-Momentum). This result is presented in Theorem 1.3 and Proposition 1.1.

Remark 1.2. Some notes on the virial identity are as follows.

(i) The generalized virial quantity \((10)\) can be called the two-particles Morawetz action with two-variables weighted function. We rewrite \((10)\) as follows

\[
\mathcal{V}_\delta(t) = \int_{\mathbb{R}^N} |\psi(y, t)|^2 \mathcal{M}_\delta(y, t) dy, \quad \text{with} \quad \mathcal{M}_\delta(y, t) = \int_{\mathbb{R}^N} \nabla_x \delta(x - y) \cdot \overline{Q^2}(\psi)(x, t) dx.
\]

By using a reduced form of \((10)\) for the one-variable weighted function \(\delta := \delta(x), x \in \mathbb{R}^N\), the restrict virial

\[
\mathcal{M}_\delta(t) \equiv \mathcal{M}_\delta(0, t) = \int_{\mathbb{R}^N} \nabla_x \delta(x) \cdot \overline{Q^2}(\psi)(x, t) dx,
\]

can be called the one-particle Morawetz action with one-variable weighted function (see [11, Eq. (1.6)]).
(ii) From now on, we shall say the generalized virial $V_\delta(t)$ in (10) is $\{2 - 2\}$ Morawetz action (i.e., two-particles and two-variables weighted function) and the restrict virial $M_\delta(t)$ in (11) is $(1 - 1)$ Morawetz action (i.e., one-particle and one-variable weighted function).

(iii) In some situations, another form of virial quantity was also considered in [22, page 29] with

$$
\int_{\mathbb{R}^N} \delta(x)|\psi(x,t)|^2dx, \quad \text{with the weighted function } \delta(x) = \phi\left(\frac{|x|}{R}\right), \quad \phi(r) = \begin{cases} 1, & \text{for } r \leq 1, \\ 0, & \text{for } r \geq 2. \end{cases}
$$

(iv) The minimal condition to impose on the weighted function $\delta := \delta(x - y)$ in (10) is that the matrices of second partial derivatives

$$
\begin{pmatrix}
\nabla^2_{x,1}\delta & \nabla_{x,1}\nabla_{x,2}\delta & \cdots & \nabla_{x,1}\nabla_{x,N}\delta \\
\nabla_{x,2}\nabla_{x,1}\delta & \nabla^2_{x,2}\delta & \cdots & \nabla_{x,2}\nabla_{x,N}\delta \\
\vdots & \vdots & \ddots & \vdots \\
\nabla_{x,N}\nabla_{x,1}\delta & \nabla_{x,N}\nabla_{x,2}\delta & \cdots & \nabla^2_{x,N}\delta
\end{pmatrix}_{N \times N}
$$

and

$$
\begin{pmatrix}
\nabla^2_{y,1}\delta & \nabla_{y,1}\nabla_{y,2}\delta & \cdots & \nabla_{y,1}\nabla_{y,N}\delta \\
\nabla_{y,2}\nabla_{y,1}\delta & \nabla^2_{y,2}\delta & \cdots & \nabla_{y,2}\nabla_{y,N}\delta \\
\vdots & \vdots & \ddots & \vdots \\
\nabla_{y,N}\nabla_{y,1}\delta & \nabla_{y,N}\nabla_{y,2}\delta & \cdots & \nabla^2_{y,N}\delta
\end{pmatrix}_{N \times N}
$$

are positive definite. With $\delta = |x - y|$, see Appendix 5.3 for the proof that the matrices

$$\{\nabla_{x,m}\nabla_{x,n}|x - y|\}^N_{m,n=1} \text{ and } \{\nabla_{y,m}\nabla_{y,n}|x - y|\}^N_{m,n=1}$$

are positive definite.

- The next challenges are to establish a new version (modified) of the PCC law related to Problem (NLS). By choosing $\delta(x) = |x|^2$ in (1-1) Morawetz action and combining with the conservation laws (Energy), (loc-Mass) and (loc-Momentum), we prove the solutions of NLS (NLS) satisfy the modified PCC law (PC). To do this, we have combined the complex analytical properties of $M_\delta(|x|^2)(t)$ to obtain a consistent result in a conservative case that depends on the selection of power $\eta$ of the nonlinear source function related to $N$-dimensional real space (here, we pick $\eta N = 2$ with $N = 1$ or $N = 2$). Moreover, in one dimension of (NLS), as a consequence of the (1-1) Morawetz action for $\eta \in [1, 2]$, the Morawetz estimate and decay estimate on time are proposed. These results are provided in Theorem 1.4, Theorem 1.5 and Proposition 1.2, below.

Now, we shall indicate the main results of this paper. The first theorem we want to mention is the result on the existence of a local solution, from which we prove that this problem can have solutions over longer time intervals (continuation of the solutions, see Definition 2.2 below). More specifically, we consider the following theorem:

**Theorem 1.1** ($H^q$-local and large-time existence). For $N \geq 1$, $N \geq \eta \geq 1$, there exists an integer number $q$ satisfying $q > \frac{N}{2}$ and $q \geq \eta$. Let $\psi_0 \in H^q(\mathbb{R}^N)$ and the nonlinearity $\lambda|\psi|^{2\eta}\psi$ satisfies hypotheses (H1)-(H2) below. Then:

(I) There exists a positive constant $T$ (depending only on $\psi_0$) such that Problem (NLS) admits a (unique) mild solution in $L^\infty_t H^q_r (\mathbb{B}^N)$ for $\mathbb{B}^N := (0, T) \times \mathbb{R}^N$.

(II) The unique solution of Problem (NLS) on $[0, T]$ can be stretched to the interval $[0, T_+]$ for some constants $T_+ > T$, so the extension function is also the (unique) mild solution on $[0, T_+]$ of Problem (NLS).

**Remark 1.3.** A more natural and interesting way to consider NLS is to avoid the dependence on Strichatrz estimates. For dimension $\geq 1$, we have tried to set up solution properties of Problem (NLS) on $H^q(\mathbb{R}^N)$ by using the Gagliardo-Nirenberg interpolation inequality. Although there is still
a limit to the index of \( q \) (namely, has to be an integer and \( q > \frac{N}{2} \)), we have achieved encouraging results.

**Remark 1.4.** For Theorem 1.1 and Theorem 1.2, our results are limited to the cases \( 2q \leq N \) or \( q \) is not an integer number, because of the following two difficulties:

i) For \( 2q \leq N \), there does not exist the Sobolev embedding \( H^q(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \) (see Lemma 2.6).

ii) For \( q \notin \mathbb{N} \), we have more difficulties to estimate the Lipschitz property in \( H^q(\mathbb{R}^N) \) of the reaction terms.

As an immediate consequence of (II) in Theorem 1.1, we guarantee the existence of a maximal time (see Definition 2.3). Next, we prove the results on non-continuation due to a blow-up (see Definition 2.4) or global existence. Furthermore, the continuous dependence on the initial data is also presented.

**Theorem 1.2** (Blow-up alternative, continuous dependence in \( H^q \)). Assume that the conditions in Theorem 1.1 hold. For every \( \psi_0 \in H^q(\mathbb{R}^N) \) there exists a maximal time \( T^* > 0 \) and a unique solution \( \psi \in L^\infty_t H^q_x(\mathbb{R}^N) \) with \( B_N := (0, T^*) \times \mathbb{R}^N \) which satisfies (16) and, in addition, satisfies the following conclusions:

( I) If \( T_* < \infty \), then \( \lim_{t \uparrow T_*} \| \psi(\cdot, t) \|_{H^q(\mathbb{R}^N)} = \infty \), or global existence if \( T_* = \infty \).

(II) If \( \psi_{0j} \rightarrow \psi_0 \) in \( H^q(\mathbb{R}^N) \) and if \( \psi_j \) is the maximal solution corresponding to the initial data \( \psi_{0j} \), then \( \psi_j \rightarrow \psi \) in \( L^\infty_t H^q_x(\mathbb{R}^N) \) for every interval \( [0, T] \subset [0, T_*] \).

We state the following theorem to estimate the stability of solutions.

**Theorem 1.3** (Stability). For \( N \geq 1 \), assume that Problem (NLS) has a solution \( \psi \in C(\mathbb{R}^+, C^\infty_0(\mathbb{R}^N)) \) with \( \psi(x, 0) = \psi_0 \in C^\infty_0(\mathbb{R}^N) \). Then the following estimate holds

\[
\frac{C_N(\eta - 1)}{2} \| (\Delta)^{\eta/2} \psi \|^2_{L^2_t L^2_x(\mathbb{R}^\times \mathbb{R}^N)} + \frac{\lambda N(\eta - 1)}{\eta + 1} \iint_{\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(y, t)|^2 |\psi(x, t)|^{2q + 2}}{|x - y|} \, dx \, dy \, dt \\
\leq \| \psi_0 \|^2_{L^2(\mathbb{R}^N)} \sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} \vec{\nabla} \cdot |x - y| \cdot \vec{Q}^2(\psi)(x, t) \, dx.
\]

The following proposition is a sharpening of the results in Theorem 1.3.

**Proposition 1.1.** For dimension \( \geq 1 \), the solutions of the defocusing case \( (\lambda > 0) \) of Problem (NLS) satisfy the following estimate

\[
\| (\Delta)^{\eta/2} \psi \|^2_{L^2_t L^2_x(\mathbb{R}^\times \mathbb{R}^N)} \lesssim \sqrt{T^2 \| \psi_0 \|} \sqrt{\mathbb{E}[\psi_0]}.
\]

A special case of the conservation law for Problem (NLS) involving the exponent of the reaction term with \( \eta = 2/N \) is considered. From the condition \( \eta \geq 1 \) in Theorem 1.1 implies that we shall deal Problem (NLS) with \( N = 1 \) or \( N = 2 \). We present the following theorem.

**Theorem 1.4** (Modified pseudo-conformal conservation law). For \( N = 1, 2 \), suppose that the conditions in Theorem 1.1 hold. Then Problem (NLS) possesses a unique solution \( \psi \in L^\infty_t H^2_x(\mathbb{R}^N) \). For \( \eta N = 2 \), we have the pseudo-conformal conservation law

\[
\mathbb{P}[\psi](t) = \mathbb{P}[\psi](0) = \| (\cdot) \psi_0 \|_{L^2(\mathbb{R}^N)}, \quad t > 0,
\]

where \( \mathbb{P}[\psi](t) \) be defined as in (7).

When \( \eta N < 2 \) and combined with the condition \( \eta \geq 1 \) in Theorem 1.1 then we infer that the Problem (NLS) is considered for \( N = 1 \) and \( \eta \in [1, 2] \), we have space-time estimation of the solutions, which is shown in the following theorem.
Theorem 1.5 (Morawetz estimate). In one dimension of Problem \((NLS)\), assume that conditions in Theorem 1.1 are fulfilled. Then for \(1 \leq \eta < 2\), we have

\[
\int_{[0,T] \times \mathbb{R}} t|\psi(x,t)|^{2\eta+2}dxdt \lesssim \sup_{0 \leq t \leq T} \mathbb{P}[\psi](t). \tag{14}
\]

From Theorem 1.5, we investigate the following proposition.

Proposition 1.2 (Time decay estimate). For \(N = 1, \eta \in [1,2)\), suppose that

\[
\sup_{0 < t \leq T} \left( \frac{d}{dt} \mathbb{P}[\psi](t) \right) \leq C,
\]

for some \(C > 0\). Then, the decay estimate of \(\psi\) in \(L^{2\eta+2}(\mathbb{R})\) holds for all \(t \in (0,T]\), with

\[
\|\psi(\cdot,t)\|_{L^{2\eta+2}(\mathbb{R})} \lesssim t^{-\frac{1}{2\eta+2}}, \quad t > 0.
\]

2. Mathematical background and some related tools

First, we set up some basic notations and tools that are useful for our work. For the quantities \(f,g > 0\), we shall write \(f \lesssim g\) if there exists a constant \(C > 0\) such that \(f \leq Cg\). The notation \(\nabla\) denotes the gradient of a scalar-valued differentiable function at the point \(x \in \mathbb{R}^N\) and \(\nabla x = (\nabla_1 x, \ldots, \nabla_N x) = (\frac{\partial x}{\partial x_1}, \ldots, \frac{\partial x}{\partial x_N})\). In some cases, to discriminate the gradient of two spatial variables \(x = (x_1, \ldots, x_N)\) and \(y\), we also write \(\nabla x\) and \(\nabla y\). In addition, \(\pm = \pm \cdots \pm\) and \(\pm\) be the calculation \pm vertical expansion. Let us set \([u_n]_{n=1}^N = (u_1, \ldots, u_N)\) and \([f \cdots f][u_n]_{n=1}^N = f u_1 + \cdots + f u_N\).

The notation \(\| \cdot \|_X\) stands for the norm in the Banach space \(X\). For \(1 \leq m \leq \infty\), \(T > 0\), the Banach space of real-valued measurable functions \(f : (0,T) \equiv I \rightarrow X\) can be well defined with norms

\[
\|f\|_{L^m(I;X)} = \left\{ \begin{array}{ll}
\left( \int_I \|f(t)\|^m_X dt \right)^{\frac{1}{m}} & \text{for } 1 \leq m < \infty, \\
\text{ess sup}_{t \in I} \|f(t)\|_X & \text{for } m = \infty,
\end{array} \right.
\]

From now on, assume that the spatial space \(X\) has \(N\)-dimensions, we shall write \(L^m(I;X)\) with \(B^N = I \times \mathbb{R}^N\) instead of the notation \(L^m(I;X(\mathbb{R}^N))\). The norm of the function space \(C^k(I;X)\), for \(0 \leq k \leq \infty\) is denoted by

\[
\|f\|_{C^k(I;X)} = \sum_{j=0}^k \sup_{t \in I} \left\| f^{(j)}(t) \right\|_X < \infty, \tag{15}
\]

where \(f^{(j)}\) is called the \(j\)-th derivative of the function \(f\). If \(k = 0\) and \(I\) is bounded interval then the space \(C(I;X)\) is a Banach space with the norm of \(L^\infty(I;X)\). For \(k = \infty\), the space \(C^\infty(I;X)\) consists of continuous strong derivatives of all orders.

For a positive integer \(q\) and \(p \in [1,\infty)\), the Sobolev space \(W^{q,p}(\mathbb{R}^N)\) consists of all locally integrable functions \(f : \mathbb{R}^N \rightarrow \mathbb{R}\) such that the weak derivatives of the order less than or equal to \(q\) belong to \(L^p(\mathbb{R}^N)\), i.e. \(\mathcal{D}^\beta(f) \in L^p(\mathbb{R}^N)\), for \(0 \leq |\beta| \leq q\). The Sobolev space \(W^{q,p}(\mathbb{R}^N)\) is a Banach space when associated with the norm

\[
\|f\|_{W^{q,p}} = \left\{ \begin{array}{ll}
\left( \sum_{|\beta| \leq q} \int_{\mathbb{R}^N} |\mathcal{D}^\beta(f)|^p dx \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\
\max \sup_{x \in \mathbb{R}^N} |\mathcal{D}^\beta(f)| & \text{for } p = \infty.
\end{array} \right.
\]
Note that, for $p = 2$, the space $H^q(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$(f, g)_{H^q(\mathbb{R}^N)} = \sum_{|\beta| \leq q} \int_{\mathbb{R}^N} \partial^\beta f \overline{\partial^\beta g} \, dx.$$ 

If $q \in \mathbb{R}$, the inner product and norm of $f, g \in H^q(\mathbb{R}^N)$ are defined by

$$(f, g)_{H^q(\mathbb{R}^N)} = (2\pi)^N \int_{\mathbb{R}^N} \langle \xi \rangle^{2q} \, \hat{f}(\xi) \overline{\hat{g}(\xi)} \, d\xi, \quad \|f\|_{H^q(\mathbb{R}^N)} = (2\pi)^N \left( \int_{\mathbb{R}^N} \langle \xi \rangle^{2q} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$ 

where $\hat{f}$ is the Fourier transform of $f$ and $\overline{\hat{g}}$ is the conjugate term of $g$. Its corresponding norm is

$$
\|f\|_{H^q(\mathbb{R}^N)} = (2\pi)^N \left( \int_{\mathbb{R}^N} \langle \xi \rangle^{2q} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.
$$

For the above functional space definitions, we refer the interested readers to [7, Chapter 1, page 11-22].

**Definition 2.1** (Mild solution, see e.g. [9, 28]). If $X$ is a Banach space, we say that $\psi \in C([0, T]; X)$ is a mild $X$-value solution of (NLS) if it satisfies the Duhamel-type integral equation

$$\psi(t) = \mathcal{Q}(t)\psi_0 - i\lambda \int_0^t \mathcal{Q}(t - \nu) \left\{ |\psi|^{2\gamma} (\nu) \psi(\nu) \right\} \, d\nu, \quad \text{for } t \in [0, T], \quad (16)$$

where $\mathcal{Q}(t) = e^{i\nu \Delta}$.

Next, we introduce the time behavior of the solution in the three definitions below (see also [33]).

**Definition 2.2** (Continuation). Given a mild solution $\psi \in L^\infty_t H^q_x(\mathbb{B}^N)$ of (NLS) with $\mathbb{B}^N = (0, T) \times \mathbb{R}^N$, we say that $\psi_{\text{cont}}$ is a continuation of $\psi$ if $\psi_{\text{cont}} \in L^\infty_t H^q_x(\mathbb{B}^N)$ with $\mathbb{B}^N = (0, T_+ \times \mathbb{R}^N$ is also a mild solution of (NLS) for some $T_+ > T$, and $\psi_{\text{cont}}(t) = \psi(t)$ whenever $t \in [0, T]$.

**Definition 2.3** (Maximal time for existence). Let $\psi(t)$ be the (unique) mild solution of (NLS). We define a maximal time $T_*$ for existence of the solution $\psi(t)$ to Problem (NLS) by the following setting:

(i) If $\psi(t)$ exists for all $t \in [0, \infty)$, then $T_* = \infty$.

(ii) If there exists a positive constant $T < \infty$ such that the solution $\psi(t)$ to Problem (NLS) exists for $t \in [0, T)$, but this solution does not exist at $t = T$, then $T_* \equiv T$.

**Definition 2.4** (Finite-time blow-up). Let $\psi(t)$ be a mild solution of (NLS). We say $\psi(t)$ blows up in finite-time if the maximal time $T_*$ (for existence of Problem (NLS)) is finite and satisfies

$$\lim_{t \uparrow T_* < \infty} \|\psi(\cdot, t)\|_{H^q(\mathbb{R}^N)} = \infty.$$ 

The main tool for our later proofs are the following lemmas. First, we recall the Gagliardo–Nirenberg interpolation inequality.

**Lemma 2.5** (Gagliardo–Nirenberg interpolation inequality). Let $s \geq 1, r \leq \infty$ and $\gamma \in \mathbb{N}^*$. Suppose also that a real number $\nu$ and $\beta \in \mathbb{N}^*$ are such that

$$\begin{cases}
\frac{1}{\beta} = \frac{\beta}{N} + \left( \frac{1}{r} - \frac{\gamma}{N} \right) \nu + \frac{1 - \nu}{s}, \\
\frac{\beta}{2} \leq \nu \leq 1.
\end{cases} \quad (17)$$

Then, for every function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ that lies in $L^s(\mathbb{R}^N)$ with $\gamma$-th derivative in $L^r(\mathbb{R}^N)$ also has $\beta$-th derivative in $L^p(\mathbb{R}^N)$ and there exists a positive constant $C_{GN}$ which depends on $\beta, \gamma, N, s, r$ and $\nu$, such that

$$\|\partial^\beta f\|_{L^p(\mathbb{R}^N)} \leq C_{GN} \|\partial^\gamma f\|_{L^r(\mathbb{R}^N)} \|f\|_{L^s(\mathbb{R}^N)}^{1 - \nu}, \quad (18)$$

**Proof.** See [5], Theorem 1, page 2.

Next, we state a version of the Sobolev embedding by the following lemma.
Lemma 2.6 (Sobolev embedding). If $N < 2q < \infty$, then $H^q(\mathbb{R}^N) \hookrightarrow C_0(\mathbb{R}^N)$ and there exists a constant positive $C^{SE} = C(N, q)$ such that

$$\|f\|_{L^\infty(\mathbb{R}^N)} \leq C^{SE} \|f\|_{H^q(\mathbb{R}^N)}.$$  

Proof. For $N < 2q$, suppose that $f \in S(\mathbb{R}^N)$ (Schwartz space) and we have

$$\|f\|_{L^\infty(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \hat{f}(\xi) e^{i\xi x} d\xi \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{(\xi) q} \hat{f}(\xi) d\xi = \int_{\mathbb{R}^N} (\xi) q \hat{f}(\xi) d\xi, \quad \text{for} \quad (\xi) = (1 + |\xi|^2)^{\frac{1}{2}}.$$  

Using the Hölder inequality we conclude that

$$\|f\|_{L^\infty(\mathbb{R}^N)} \leq \left( \int_{\mathbb{R}^N} \frac{1}{(\xi) 2q} d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} (\xi) 2q \hat{f}(\xi) d\xi \right)^{\frac{1}{2}}.$$  

(19)

Since $S(\mathbb{R}^N) \subseteq H^q(\mathbb{R}^N)$, this implies that $f \in H^q(\mathbb{R}^N)$ and $f$ is a Schwartz function, then $f \in C_0(\mathbb{R}^N)$. Moreover, for $N < 2q$, the first integral in (19) is convergent. Hence, we deduce that

$$\|f\|_{L^\infty(\mathbb{R}^N)} \leq C^{SE} \left( \int_{\mathbb{R}^N} (\xi) 2q \hat{f}(\xi) d\xi \right)^{\frac{1}{2}} \leq \|f\|_{H^q(\mathbb{R}^N)},$$

where $C = C^{SE} = \left( \int_{\mathbb{R}^N} \frac{1}{(\xi) 2q} d\xi \right)^{\frac{1}{2}} > 0$, for $q > \frac{N}{2}$. This concludes the proof.

From now on, we write $\mathcal{F}_\eta(\psi) = \lambda |\psi|^{2q} \psi$ and assume that $\mathcal{F}_\eta$ is a smooth function and satisfies the hypotheses:

- For $\eta > 0$ and $f, g \in \mathbb{R}$, there is a Lipschitz constant $C^L = C(\lambda, \eta)$ such that
  $$|\mathcal{F}_\eta(f) - \mathcal{F}_\eta(g)| \leq C^L |f - g| \left( |f|^{2q} + |g|^{2q} \right).$$  

(H1)

- There exists an integer number $\sigma$ and $\mathcal{F}_\eta^{(\sigma)}$ is called the $\sigma-$th derivative of the function $\mathcal{F}_\eta$ such that for $f, g \in \mathbb{R}$, we have
  $$|\mathcal{F}_\eta^{(\sigma)}(f) - \mathcal{F}_\eta^{(\sigma)}(g)| \leq C^L |f - g| \left( |f|^{2q-\sigma} + |g|^{2q-\sigma} \right).$$  

(H2)

We can now prove our main results.

3. PROOF OF THE MAIN RESULTS

We shall begin to prove the results in Theorem 1.1 - Theorem 1.5, Proposition 1.1 and Proposition 1.2.

3.1. Proof of Theorem 1.1. We prove the existence and continuation of the solutions to Problem (NLS) by a fixed point argument. We split the proof into two parts.

**Part I: Local existence.** Let $N \geq 1$, $\frac{N}{2} < q \in \mathbb{N}$ and $[1, q] \ni \eta \in \mathbb{N}$. For positive constants $K, T$ to be chosen later, we define a closed set with $\mathbb{B}^N = (0, T) \times \mathbb{R}^N$ and

$$\mathcal{E}_K = \left\{ \psi \in L_t^\infty H_x^q(\mathbb{B}^N) : \psi(\cdot, 0) = \psi_0 \text{ and } \|\psi - \psi_0\|_{L_t^\infty H_x^q(\mathbb{B}^N)} \leq K \right\},$$  

(20)

associated with the distance $\text{dist}(f_1, f_2) = \|f_1 - f_2\|_{L_t^\infty H_x^q(\mathbb{B}^N)}$ for $f_j \in \mathcal{E}_K$, $j = 1, 2$. We can easily prove that $(\mathcal{E}_K, \text{dist})$ is a complete metric space. Let us define the mapping $\mathcal{J}$ as follows:

$$\mathcal{J}\psi(t) = Q(t)\psi_0 - i\lambda \int_0^t Q(t - \nu) \{ |\psi|^{2q} \psi(\nu) \} d\nu.$$  

(21)

Our goal is to prove that $\mathcal{J}$ is invariant and contraction mapping on $\mathcal{E}_K$. We write (21) as a fixed-point equation (recall that $\mathcal{F}_\eta(\psi) = \lambda |\psi|^{2q} \psi$)

$$\psi = \psi_0,$$

$\mathcal{J} : L_t^\infty H_x^q(\mathbb{B}^N) \to L_t^\infty H_x^q(\mathbb{B}^N)$,
\[
\mathcal{J} \psi(t) = Q(t) \psi_0 - i \int_0^t Q(t - \nu) \mathcal{F}_\eta(\psi)(\nu) d\nu.
\] (22)

We shall write \( \mathcal{J} \) in (22) by

\[
\mathcal{J} \psi(t) = Q(t) \psi_0 + \mathcal{G}(\psi)(t), \quad \mathcal{G}(\psi)(t) = -i \int_0^t Q(t - \nu) \mathcal{F}_\eta(\psi)(\nu) d\nu.
\] (23)

We need to prove that \( \mathcal{J} \) is invariant and a contraction mapping. Indeed, for \( \psi_0 \in H^q(\mathbb{R}^N) \), with \( N \geq q > \frac{N}{2} \) and \( q \geq \eta \geq 1 \) and since \( Q(t) \) is a unitary group, we have

\[
\begin{align*}
\| (Q(t) - I) \psi_0 \|_{H^q(\mathbb{R}^N)}^2 &\leq (2\pi)^N \sup_{\xi \in \mathbb{R}^N} \left| e^{-it|\xi|^2} - 1 \right|^2 \int_{\mathbb{R}^N} |\xi|^{2q} \left| \hat{\psi}_0(\xi) \right|^2 d\xi \\
&\leq C \| \psi_0 \|_{H^q(\mathbb{R}^N)}^2 \lesssim \| \psi_0 \|_{H^q(\mathbb{R}^N)}^2, \quad C = (2\pi)^N > 0,
\end{align*}
\] (24)

where, noting that \( \left| e^{-it|\xi|^2} - 1 \right| < 1 \) for all \( \xi \in \mathbb{R}^N \). Let \( \psi_1, \psi_2 \in \mathcal{S}_K \), arguing as in (24), for every \( t \in [0, T] \), we deduce

\[
\| \mathcal{G}(\psi_1)(t) - \mathcal{G}(\psi_2)(t) \|_{H^q(\mathbb{R}^N)} \lesssim \int_0^t \| \mathcal{F}_\eta(\psi_1)(\nu) - \mathcal{F}_\eta(\psi_2)(\nu) \|_{H^q(\mathbb{R}^N)} d\nu.
\] (25)

For a multi-index \( \beta \) with \( |\beta| = q \in \mathbb{N} \), we have that

\[
\mathcal{D}^\beta (\mathcal{F}_\eta(\psi_1) - \mathcal{F}_\eta(\psi_2)) = \sum_{\eta \leq q = |\beta|} \left( \mathcal{F}_\eta^{(\eta)}(\psi_1) \prod_{j=1}^\eta \mathcal{D}^{\alpha_j}(\psi_1) - \mathcal{F}_\eta^{(\eta)}(\psi_2) \prod_{j=1}^\eta \mathcal{D}^{\alpha_j}(\psi_2) \right),
\]

where \( \eta \) is an integer number such that

\[
\begin{cases}
1 \leq \eta \leq q, \\
|\alpha_j| \geq 1, \quad j = 1, \eta, \\
q = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_\eta| = \sum_{j=1}^\eta |\alpha_j| = |\beta|.
\end{cases}
\]

Thanks to the triangle inequality,

\[
\left\| \mathcal{F}_\eta^{(\eta)}(\psi_1) \prod_{j=1}^\eta \mathcal{D}^{\alpha_j}(\psi_1) - \mathcal{F}_\eta^{(\eta)}(\psi_2) \prod_{j=1}^\eta \mathcal{D}^{\alpha_j}(\psi_2) \right\|_{L^2(\mathbb{R}^N)} \leq (\text{Term}_1) + (\text{Term}_2),
\] (26)

where we define

\[
(\text{Term}_1) := \left\| \left( \mathcal{F}_\eta^{(\eta)}(\psi_1) - \mathcal{F}_\eta^{(\eta)}(\psi_2) \right) \prod_{j=1}^\eta \mathcal{D}^{\alpha_j}(\psi_1) \right\|_{L^2(\mathbb{R}^N)}, \quad (\text{Term}_2) := \left\| \mathcal{F}_\eta^{(\eta)}(\psi_2) \prod_{j=1}^\eta \mathcal{D}^{\alpha_j}(\psi_1 - \psi_2) \right\|_{L^2(\mathbb{R}^N)}.
\]

We estimate the first in (26). Now let

\[
p_j = \frac{2q}{|\alpha_j|}, \quad j = 1, \eta \quad \text{such that} \quad \frac{2}{p_1} + \frac{2}{p_2} + \cdots + \frac{2}{p_\eta} = \sum_{j=1}^\eta \frac{2}{p_j} = 1,
\] (27)

we continue to use the generalized Hölder inequality and we have

\[
(\text{Term}_1) \leq \left\| \mathcal{F}_\eta^{(\eta)}(\psi_1) - \mathcal{F}_\eta^{(\eta)}(\psi_2) \right\|_{L^\infty(\mathbb{R}^N)} \prod_{j=1}^\eta \| \mathcal{D}^{\alpha_j}(\psi_1) \|_{L^{p_j}(\mathbb{R}^N)}. \] (28)
Using hypothesis (H2) and using the Sobolev embedding (Lemma 2.6) \( H^q(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \), we have
\[
\| \mathcal{F}^{(q)}_\eta(\psi_1) - \mathcal{F}^{(q)}_\eta(\psi_2) \|_{L^\infty(\mathbb{R}^N)} \leq C \| |q| \mathcal{S}^\eta(\mathbb{R}^N) \|_2 \| \psi_1 \|_{H^q(\mathbb{R}^N)} + \| \psi_2 \|_{H^q(\mathbb{R}^N)} \| \psi_1 - \psi_2 \|_{H^q(\mathbb{R}^N)} . \tag{29}
\]
Next, we estimate the term \( \| \mathcal{G}^{\alpha_j}(\psi_1) \|_{L^{p_j}(\mathbb{R}^N)} \) in (28) by using the Gagliardo-Nirenberg interpolation inequality (Lemma 2.5). Indeed, from the condition (17), one gets
\[
\frac{1}{p_j} = \frac{\alpha_j}{N} + \left( \frac{1}{2} - \frac{\gamma_j}{N} \right) \nu + \frac{1 - \nu}{\infty},
\]
where, we identify \((\alpha_j, p_j, \gamma_j, \infty) \equiv (\beta, p, \gamma, s)\) in Equation 17.

From this condition, we choose \( \nu = \frac{|\alpha_j|}{\eta} \), from (27) we find that \( \gamma_j = q \). Thanks to the Gagliardo-Nirenberg interpolation inequality (Lemma 2.5) we conclude that
\[
\| \mathcal{G}^{\alpha_j}(\psi_1) \|_{L^{p_j}(\mathbb{R}^N)} \leq C^G \| \mathcal{G}^{\gamma}(\psi_1) \|_{L^2(\mathbb{R}^N)} \| \psi_1 \|_{L^\infty(\mathbb{R}^N)} .
\]
From \( q = \gamma_j \) yields that \( \| \mathcal{G}^{\gamma}(\psi_1) \|_{L^2(\mathbb{R}^N)} \leq \left( \sum_{j=1}^{\eta} |\gamma_j| \| \mathcal{G}^{\gamma_j}(\psi_1) \|_{L^2(\mathbb{R}^N)} \right)^\frac{2}{2} = \| \psi_1 \|_{H^q(\mathbb{R}^N)} \), we obtain
\[
\| \mathcal{G}^{\alpha_j}(\psi_1) \|_{L^{p_j}(\mathbb{R}^N)} \leq C^G \| \psi_1 \|_{H^q(\mathbb{R}^N)} \| \psi_1 \|_{L^\infty(\mathbb{R}^N)} \| \psi_1 - \psi_2 \|_{H^q(\mathbb{R}^N)} . \tag{30}
\]
From (28) - (30), we deduce that for \( C = C^G C^S C^L > 0 \)
\[
(Term_1) \leq C^G C^S C^L \prod_{j=1}^\eta \| \psi_1 \|_{H^q(\mathbb{R}^N)} \| \psi_1 \|_{L^\infty(\mathbb{R}^N)} \left( \| \psi_1 \|_{H^q(\mathbb{R}^N)} + \| \psi_2 \|_{H^q(\mathbb{R}^N)} \right) \| \psi_1 - \psi_2 \|_{H^q(\mathbb{R}^N)}
\]
\[
\leq \prod_{j=1}^\eta \| \psi_1 \|_{H^q(\mathbb{R}^N)} \| \psi_1 \|_{L^\infty(\mathbb{R}^N)} \left( \| \psi_1 \|_{H^q(\mathbb{R}^N)} + \| \psi_2 \|_{H^q(\mathbb{R}^N)} \right) \| \psi_1 - \psi_2 \|_{H^q(\mathbb{R}^N)} .
\]
Thanks to Lemma 2.6, we have the Sobolev embedding \( H^q(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \) and noting that from (27) we derive
\[
\frac{|\alpha_1|}{q} + \frac{|\alpha_2|}{q} + \cdots + \frac{|\alpha_\eta|}{q} = \sum_{j=1}^\eta \frac{|\alpha_j|}{q} = 1,
\]
and for \( \eta > 1 \)
\[
0 \leq \eta - 1 = \eta - \frac{|\alpha_1|}{q} - \frac{|\alpha_2|}{q} - \cdots - \frac{|\alpha_\eta|}{q} = \eta - \frac{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_\eta|}{q} = \sum_{j=1}^\eta \left( 1 - \frac{|\alpha_j|}{q} \right),
\]
thus we deduce the estimation for \( C = C^G (C^S)^2 C^L > 0 \)
\[
(Term_1) \leq \left( \| \psi_1 \|_{H^q(\mathbb{R}^N)} + \| \psi_2 \|_{H^q(\mathbb{R}^N)} \| \psi_2 \|_{H^q(\mathbb{R}^N)} \right) \| \psi_1 - \psi_2 \|_{H^q(\mathbb{R}^N)} . \tag{31}
\]
Next, we estimate Term2. Also thanks to the setting for \( j = 1, \eta \)
\[
p_j = \frac{2q}{|\alpha_j|} \quad \text{and} \quad \sum_{j=1}^\eta \frac{2}{p_j} = 1,
\]
we continue to use the generalized Hölder inequality and we obtain
\[
(Term_2) \leq \left\| \mathcal{F}^{(q)}_\eta(\psi_2) \right\|_{L^\infty(\mathbb{R}^N)} \prod_{j=1}^\eta \| \mathcal{G}^{\alpha_j}(\psi_1 - \psi_2) \|_{L^{p_j}(\mathbb{R}^N)} . \tag{32}
\]
Using hypothesis (H2) and thanks to Lemma 2.6, the Sobolev embedding $H^q(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we deduce that
\[ \left\| F^{(\eta)}_\eta(\psi_2) \right\|_{L^\infty(\mathbb{R}^N)} \leq CLC^{SE} \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{q+1}. \] (33)

From the conditions
\[ \frac{1}{p_j} = \frac{\alpha_j}{N} + \left( \frac{1}{2} - \frac{\gamma_j}{N} \right), \quad \nu = \frac{\alpha_j}{q}, \quad \gamma_j = q, \]
and the Gagliardo-Nirenberg interpolation inequality (Lemma 2.5), we infer that
\[ \left\| \mathcal{D}^{\alpha_j}(\psi_1 - \psi_2) \right\|_{L^{p_j}(\mathbb{R}^N)} \leq C^{GN} \left\| \psi_1 - \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{\frac{\alpha_j}{q}} \left\| \psi_1 - \psi_2 \right\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{\alpha_j}{q}}. \] (34)

From (32) - (34), using the Sobolev embedding $H^q(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and noting that
\[ \sum_{j=1}^\eta |\alpha_j| q = 1, \quad \text{and} \quad 0 \leq \eta - 1 = \sum_{j=1}^\eta \left( 1 - \frac{|\alpha_j|}{q} \right), \]
we have that, for $C = C^{GN}(C^{SE})^2CL > 0$,
\begin{align*}
\text{(Term2)} &\leq C^{GN}C^{SE}C^L \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{\eta+1} \prod_{j=1}^\eta \left\| \psi_1 - \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{\frac{|\alpha_j|}{q}} \left\| \psi_1 - \psi_2 \right\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{|\alpha_j|}{q}} \\
&\leq C^{GN}(C^{SE})^2C^L \left( \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{2\eta} + \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{q+1} \left\| \psi_1 \right\|_{H^q(\mathbb{R}^N)}^{1-\eta} + \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{q+1} \left\| \psi_1 \right\|_{H^q(\mathbb{R}^N)}^{1-\eta} \right) \left\| \psi_1 - \psi_2 \right\|_{H^q(\mathbb{R}^N)} \\
&\lesssim \left( \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{2\eta} + \left\| \psi_2 \right\|_{H^q(\mathbb{R}^N)}^{q+1} \left\| \psi_1 \right\|_{H^q(\mathbb{R}^N)}^{1-\eta} \right) \left\| \psi_1 - \psi_2 \right\|_{H^q(\mathbb{R}^N)}. \quad (35)
\end{align*}

For $\eta \geq 1$, combining (26), (31) and (35), we have
\begin{align*}
\left\| F_\eta(\psi_1) - F_\eta(\psi_2) \right\|_{H^q(\mathbb{R}^N)} &= \left( \sum_{|\beta| \leq q} \left\| D^\beta \left( F_\eta(\psi_1) - F_\eta(\psi_2) \right) \right\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}} \\
&\lesssim \left( \sum_{|\beta| \leq q} \sum_{q \leq \eta} \left\| F^{(\eta)}_\eta(\psi_1) \prod_{j=1}^\eta D^{\alpha_j}(\psi_1) - F^{(\eta)}_\eta(\psi_2) \prod_{j=1}^\eta D^{\alpha_j}(\psi_2) \right\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}} \\
&\lesssim L(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) \left\| \psi_1 - \psi_2 \right\|_{H^q(\mathbb{R}^N)},
\end{align*}
where $L$ is dependent on $K$ and $\left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}$. From this and (25), we derive
\[ \left\| J\psi_1(t) - J\psi_2(t) \right\|_{H^q(\mathbb{R}^N)} \lesssim TL(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) \left\| \psi_1 - \psi_2 \right\|_{L^\infty H^q(\mathbb{B}^N)}. \quad (36)\]

From (25) and (36), if we take $\psi \equiv \psi_1 \in \mathcal{E}_K$ and $\psi_2 = 0$, it follows
\begin{align*}
\left\| \mathcal{G}(\psi) \right\|_{L^p H^q(\mathbb{B}^N)} &\lesssim TL(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) \left\| \psi \right\|_{L^p H^q(\mathbb{B}^N)} \\
&\lesssim TL(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)})(K + \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) = TL_1(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}), \quad (37)
\end{align*}
where $L_1(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) = L(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)})(K + \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)})$. Hence, if we choose $K$ such that $K = 2C \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}$ and $2CTL_1(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) < K$, together with (23) and (24), to obtain
\[ \left\| J\psi \right\|_{L^p H^q(\mathbb{B}^N)} \leq \left( \left\| Q - I \right\| \psi_0 \right\|_{L^p H^q(\mathbb{B}^N)} + \left\| \mathcal{G}(\psi) \right\|_{L^p H^q(\mathbb{B}^N)} \right. \\
&\lesssim \left( \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)} + TL_1(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) \right) \leq K,
\]
then $J$ is an invariant mapping. We next prove that $J$ is a contraction. From (36), the right-hand side is independent of $t$, we deduce that dist $\left( J\psi_1, J\psi_2 \right) \lesssim \rho \text{dist} (\psi_1, \psi_2)$ for $\rho := CTL(K, \left\| \psi_0 \right\|_{H^q(\mathbb{R}^N)}) > 0$. Choosing $T$ and $K$ small enough such that $\rho < 1$, then the mapping $J$ is a contraction on $\mathcal{E}_K$. 

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Hence, by using the contraction mapping principle, we conclude that the mapping $\mathcal{J}$ admits a unique fixed point $\psi$ in $\mathcal{E}_K$.

**Part II: Continuation of solutions.** Let $\psi : [0, T] \to H^q(\mathbb{R}^N)$, $(N \geq q > \frac{N}{2}$, $[1, q] \ni \eta \in \mathbb{N})$ be a unique mild solution of Problem (NLS), and $T$ be the positive time for existence (defined as in Part I). Fix $K > 0$, and for a larger time $T_+ > T$ ($T_+$ depends on $K$), we shall prove that $\psi_{\text{cont}} : [0, T_+] \to H^q(\mathbb{R}^N)$ is also the unique mild solution of Problem (NLS). This should be well dealt with for $\max \{R_j\} \leq K$, $j = 1, 3$ where the definitions for $C > 0$ are

$$
\mathcal{R}_1 := C \|\psi_0\|_{H^q(\mathbb{R}^N)}; \\
\mathcal{R}_2 := C T_+ L_1(K, \|\psi(\cdot, T)\|_{H^q(\mathbb{R}^N)}); \\
\mathcal{R}_3 := CL(K, \|\psi(\cdot, T)\|_{H^q(\mathbb{R}^N)}) T_+.
$$

(38a) (38b) (38c)

Here, $L, L_1$ are the Lipschitz constants as in (36) and (37). For $T_+ \geq T > 0$ (noting that one can choose $T_+ > T$ and close enough to $T$) and $K > 0$, let us define for $\mathbb{B}_+^N = (0, T_+) \times \mathbb{R}^N$

$$
\mathcal{E}_K = \left\{ \psi_{\text{cont}} \in L^\infty_t H^q_x(\mathbb{B}_+^N) \left| \psi_{\text{cont}}(t) = \psi(t), \|\psi_{\text{cont}} - \psi(T)\|_{L^\infty_t H^q_x(\mathbb{B}_+^N)} \leq K, \forall t \in [0, T], \psi_{\text{cont}}(t) \in \mathcal{E}_K \right. \right\}
$$

(39)

equipped with the distance $\text{dist}^+(f_1, f_2) = \|f_1 - f_2\|_{L^\infty_t H^q_x(\mathbb{B}_+^N)}$ for $f_i \in \mathcal{E}_K, i = 1, 2$, and we note that $(\mathcal{E}_K, \text{dist}^+)$ is a complete metric space. Next, we need to clarify the following two claims.

- **Claim 1:** We show that $\mathcal{J}$ defined as in (21) is an operator on $\mathcal{E}_K$. Indeed, let $\psi_{\text{cont}} \in \mathcal{E}_K$, we consider the following two cases.

* For $t \in [0, T]$, then by virtue of Part I, we know that Problem (NLS) admits a unique solution which satisfies $\mathcal{J}\psi_{\text{cont}} = \psi_{\text{cont}}$ and $\psi_{\text{cont}}(\cdot, t) = \psi(\cdot, t)$, thus $\|\mathcal{J}\psi_{\text{cont}} - \psi\|_{L^\infty_t H^q_x(\mathbb{B}_+^N)} = 0$ in $\mathcal{E}_K$ for all $t \in [0, T]$.

* For $t \in [T, T_+]$, from (16) and (21), we have

$$
\mathcal{J}\psi_{\text{cont}}(t) - \psi(T) = (Q(t) - Q(T)) \psi_0 - i \int_0^t (Q(t - \nu) - Q(T - \nu)) F_\eta(\psi_{\text{cont}}(\nu)) d\nu \\
- i \int_0^t (Q(t - \nu) - Q(T - \nu)) F_\eta(\psi_{\text{cont}}(\nu)) d\nu = \sum_{j=3}^5 \text{(Term}_j),
$$

(40)

where we define

$$(\text{Term}_3) := (Q(t) - Q(T)) \psi_0,$$

$$(\text{Term}_4) := -i \int_0^t (Q(t - \nu) - Q(T - \nu)) F_\eta(\psi_{\text{cont}}(\nu)) d\nu,$$

$$(\text{Term}_5) := -i \int_0^t (Q(t - \nu) - Q(T - \nu)) F_\eta(\psi_{\text{cont}}(\nu)) d\nu.$$

To estimate the first term in the RHS of (40), for all $t \in [T, T_+]$, we have

$$
\|\text{(Term}_3\|_{H^q(\mathbb{R}^N)} \lesssim \|\psi_0\|_{H^q(\mathbb{R}^N)}.
$$

By (38a), the following estimate holds

$$
\|\text{(Term}_3\|_{L^\infty_t H^q_x(\mathbb{B}_+^N)} \leq \mathcal{R}_1 \leq \frac{K}{4}, \forall t \in [0, T_+].
$$

(41)

From (39), we deduce for all $t \in [T, T_+]$ that

$$
\|\psi_{\text{cont}}(\cdot, t)\|_{H^q(\mathbb{R}^N)} \leq K + \|\psi(\cdot, T)\|_{H^q(\mathbb{R}^N)},
$$

and similar to (37) for all $t \in [T, T_+]$, we have the following estimate

$$
\|\text{(Term}_4\|_{H^q(\mathbb{R}^N)} \lesssim T_+ L_1(K, \|\psi(\cdot, T)\|_{H^q(\mathbb{R}^N)}.
$$

Using (38b), we infer that

$$
\| (\text{Term}_4) \|_{L^\infty_t H^q_x(B^\infty_N)} \leq R_2 \leq \frac{K}{4},
$$

(42)

We continue the procedure with estimates for the third term in the RHS of (40), similar to (37) for all $t \in [T, T_+]$, we obtain that

$$
\| (\text{Term}_5) \|_{H^q(R^N)} \lesssim T_1(K, \| \psi(\cdot, T) \|_{H^q(R^N)}).
$$

From (38b), we have that

$$
\| (\text{Term}_5) \|_{L^\infty_t H^q_x(B^\infty_N)} \leq R_2 \leq \frac{K}{4},
$$

(43)

It follows from (41)-(43) for all $t \in [0, T_+]$ that

$$
\| J \psi_{\text{cont}} - \psi(T) \|_{L^\infty_t H^q_x(B^\infty_N)} \leq \frac{K}{4} + \frac{K}{4} + \frac{3K}{4} \leq K.
$$

Hence, we have shown that $J$ is a map from $\tilde{\mathcal{K}}_t$ to $\tilde{\mathcal{K}}_t$.

- **Claim 2:** We show that $J$ is a contraction on $\tilde{\mathcal{K}}_t$. Let $\psi_1, \psi_2 \in \tilde{\mathcal{K}}_t$, and we have that, for $t \in [0, T_+]$,

$$
J \psi_1(t) - J \psi_2(t) = i \int_0^t Q(t - \nu) \left( F_\eta(\psi_2)(\nu) - F_\eta(\psi_1)(\nu) \right) d\nu.
$$

For all $t \in [0, T]$, we conclude that $J \psi_1(t) - J \psi_2(t)$ vanishes on $[0, T]$. For $t \in [T, T_+]$, proceeding as in (36), we obtain

$$
\| J \psi_1(t) - J \psi_2(t) \|_{H^q(R^N)} \lesssim L(K, \| \psi(\cdot, T) \|_{H^q(R^N)} T_+ \| \psi_1 - \psi_2 \|_{L^\infty_t H^q_x(B^\infty_N)}),
$$

where $L$ is the Lipschitz constant as in (36). Hence, from (38c), we deduce that

$$
\| J \psi_1 - J \psi_2 \|_{L^\infty_t H^q_x(B^\infty_N)} \leq L(K, \| \psi(\cdot, T) \|_{H^q(R^N)} T_+ \| \psi_1 - \psi_2 \|_{L^\infty_t H^q_x(B^\infty_N)}
$$

$$
\leq R_3 \leq \frac{K}{4} \| \psi_1 - \psi_2 \|_{L^\infty_t H^q_x(B^\infty_N)}.
$$

Thus, for any $T_+ > 0$ and without loss of generality, we may choose the radius $K$ such that $0 \leq K < 4$, and one obtains that

$$
\text{dist}^+(J \psi_1, J \psi_2) \leq \frac{K}{4} \text{dist}^+(\psi_1, \psi_2).
$$

(44)

This implies that $J$ is a $\frac{K}{4}$-contraction. By applying the Banach contraction principle, it follows that $J$ has a unique fixed point $\psi_{\text{cont}}$ in the ball $\tilde{\mathcal{K}}_t$, and we can say that $\psi_{\text{cont}}$ is a continuation in the larger time of $\psi$. \hfill \qed

### 3.2. Proof of Theorem 1.2

We shall begin to prove two results in Theorem 1.2.

**Part I:** Finite time blow-up or global existence. For $N \geq 1$, let $\psi_0 \in H^q(R^N)$ with $N \geq q > \frac{N}{2}$, $N \geq q \in [1, q]$ and define

$$
T_* := \sup \{ T > 0 : \text{there exits a solution on } [0, T] \}.
$$

Suppose that $T_* < \infty$, and $\| \psi(\cdot, t) \|_{H^q(R^N)} \leq K_0$, for some $K_0 > 0$, and for all $t \in (0, T_*)$. Now assume that there exists a sequence $\{ t_n \} \subset (0, T_*)$ such that $t_n \uparrow T_*$. We will show that $\{ \psi(t_n) \}_{n \in \mathbb{N}^*}$ is a Cauchy sequence in $H^q(R^N)$. In fact, given $\epsilon > 0$, fix $N_\epsilon = N(\epsilon) \in \mathbb{N}^*$ such that for all $n, j > N_\epsilon$,

$$
\psi(t_j) - \psi(t_n) = (\mathcal{Q}(t_j - t_n) - I) \mathcal{Q}(t_n) \psi_0 - i \int_{t_n}^{t_j} \mathcal{Q}(t_j - \nu) F_\eta(\psi)(\nu) d\nu.
$$
Thus, we have
\[ -i \left( Q(t_j - t_n) - I \right) \int_{t_0}^{t_n} Q(t_n - \nu) F_\eta(\psi)(\nu) d\nu = (Q(t_j - t_n) - I) \psi(t_n) - i \int_{t_n}^{t_j} Q(t_j - \nu) F_\eta(\psi)(\nu) d\nu. \]

Thus, we have
\[
\| \psi(\cdot, t_j) - \psi(\cdot, t_n) \|_{H^q(R^N)} \leq \left\| \left( \frac{\partial}{\partial t} - Q(t_n - \nu) F_\eta(\psi)(\nu) \right) \right\|_{L^1(R^N)} + \left\| \int_{t_n}^{t_j} Q(t_j - \nu) F_\eta(\psi)(\nu) d\nu \right\|_{L^1(R^N)}.
\]

First, we estimate the first term in the inequality above, and we have that
\[
\text{(Term}_6) \leq \| Q(t_j - t_n) - I \|_{L^1(H^q(R^N) \to H^q(R^N))} \| \psi(\cdot, t_n) \|_{H^q(R^N)}.
\]

By an argument analogous to that used in (37), we obtain
\[
\text{(Term}_7) \leq L(K_0, \| \psi_0 \|_{H^q(R^N)}) K_0 |t_j - t_n|.
\]

Since \( \{ t_n \}_{n \in \mathbb{N}} \) is a convergent sequence, then we can take \( N_\epsilon \in \mathbb{N}^* \) with \( j \geq n \geq N_\epsilon \) such that \( |t_j - t_n| \) is as small as we want. Since the semigroup \( \{ Q(t) \}_{t \geq 0} \) is strongly continuous in \( H^q(R^N) \), given \( \epsilon > 0 \), we have
\[
K_0 \| (Q(t_j - t_n) - I) \|_{L^1(H^q(R^N) \to H^q(R^N))} < \frac{\epsilon}{2},
\]
and
\[
L(K_0, \| \psi_0 \|_{H^q(R^N)}) K_0 |t_j - t_n| < \frac{\epsilon}{2}.
\]

From this, given \( \epsilon > 0 \), there exists \( N_\epsilon \in \mathbb{N}^* \) such that
\[
\| \psi(\cdot, t_j) - \psi(\cdot, t_n) \|_{H^q(R^N)} < \epsilon, \quad \text{for } j, n \geq N_\epsilon.
\]

It follows that \( \{ \psi(\cdot, t_n) \}_{n \in \mathbb{N}} \subset H^q(R^N) \) is a Cauchy sequence. Hence, for \( \{ t_n \}_{n \in \mathbb{N}} \), an arbitrary sequence of \( (0, T_*) \), the existence of the following limit has been proved \( \lim_{t \uparrow T_* \land t \leq \infty} \| \psi(\cdot, t) \|_{H^q(R^N)} < \infty \). From the results of Theorem 1.1, we have argued that the existence of the solution can be expanded to some larger time interval (i.e., \( \psi \) may be continued beyond the maximal time \( T_* \)). This contradicts the definition of \( T_* \). Therefore, either \( T_* = \infty \) or if \( T_* < \infty \) then \( \lim_{t \uparrow T_*-} \| \psi(\cdot, t) \|_{H^q(R^N)} = \infty \).

Part II: Continuous dependence. For \( \frac{q}{2} < q \in \mathbb{N} \), \( [1, q] \ni \eta \in \mathbb{N} \), let \( \psi_0 \in H^q(R^N) \) and consider \( \psi_{0j} \in H^q(R^N) \) such that \( \psi_{0j} \to \psi_0 \) as \( j \to \infty \). For \( j \) sufficiently large we have that
\[
\| \psi(\cdot, t) - \psi_j(\cdot, t) \|_{H^q(R^N)} \leq \| Q(t) (\psi_0 - \psi_{0j}) \|_{H^q(R^N)} + \int_0^t \| Q(t - \nu) (F_\eta(\psi)(\nu) - F_\eta(\psi_j)(\nu)) \|_{H^q(R^N)} d\nu.
\]

Similar to the computations above, we obtain
\[
\| Q(t) (\psi_0 - \psi_{0j}) \|_{H^q(R^N)} \lesssim \| \psi_0 - \psi_{0j} \|_{H^q(R^N)},
\]
and we claim that
\[
\int_0^t \| Q(t - \nu) (F_\eta(\psi)(\nu) - F_\eta(\psi_j)(\nu)) \|_{H^q(R^N)} d\nu \lesssim L(K_0, \| \psi_0 \|_{H^q(R^N)}) \int_0^t \| \psi(\nu) - \psi_j(\nu) \|_{H^q(R^N)} d\nu.
\]

Combining (45)-(47), we deduce that
\[
\| \psi(t) - \psi_j(t) \|_{H^q(R^N)} \lesssim \| \psi_0 - \psi_{0j} \|_{H^q(R^N)} + L(K_0, \| \psi_0 \|_{H^q(R^N)}) \int_0^t \| \psi(\nu) - \psi_j(\nu) \|_{H^q(R^N)} d\nu.
\]
By using the Grönwall inequality, we see that
\[
\|\psi(\cdot, t) - \psi_j(\cdot, t)\|_{H^q(\mathbb{R}^N)} \lesssim \|\psi_0 - \psi_j\|_{H^q(\mathbb{R}^N)} \exp\left(CL(K_0, \|\psi_0\|_{H^q(\mathbb{R}^N)})t\right), \quad t \in [0, T_*].
\] (48)

As \( j \uparrow \infty \) we have that \( \psi_0 \to \psi \) then \( \psi_j \to \psi \) in \( H^q(\mathbb{R}^N) \). Iterating this property to cover any compact subset of \([0, T_*]\), i.e., starting from \( \psi_0 \) and \( \psi(t_1) \) over the intervals \([0, t_1]\) and \([t_1, t_2]\), respectively, the proof of Theorem 1.2 is finished.

3.3. **Proof of Theorem 1.3.** As shown above, the solutions of Problem \((NLS)\) exist and satisfy the conservation laws \((\text{Mass})\), \((\text{loc-Mass})\), and \((\text{loc-Momentum})\). We consider the \((2-2)\) Morawetz action
\[
V_\delta(t) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \psi(y, t) \right|^2 \vec{\nabla}_x \delta(x - y) \cdot \overrightarrow{Q}_N^\psi(x, t) dxdy.
\] (49)

From now on, when no confusion can arise, we denote \( \int f dx = \int_{\mathbb{R}^N} f dx \). For the mass density \( \Psi = |\psi|^2 \), the derivative of both sides with respect to the variable \( t \) of the quantity \((49)\)
\[
\frac{d}{dt} V_\delta(t) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\partial \Psi}{\partial t}(y, t) \vec{\nabla}_x \delta(x - y) \cdot \overrightarrow{Q}_N^\psi(x, t) dxdy + \int_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \vec{\nabla}_x \delta(x - y) \cdot \frac{\partial \overrightarrow{Q}_N^\psi(x, t)}{\partial t} dxdy.
\]
\[
= :(\text{Integral}_1)(t) + :(\text{Integral}_2)(t)
\] (50)

First, from \((\text{loc-Mass})\) one has \( \frac{\partial \Psi}{\partial t}(y, t) = -2\text{div} \left( \overrightarrow{Q}_N^\psi(y, t) \right) \), thus we have
\[
(\text{Integral}_1)(t) = -2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \left( \vec{\nabla}_y \cdot \overrightarrow{Q}_N^\psi(y, t) \right) \left( \vec{\nabla}_x \delta(x - y) \cdot \overrightarrow{Q}_N^\psi(x, t) \right) dxdy.
\] (51)

Second, for \( \vec{\nabla}_x \delta = (\vec{\nabla}_{x,1} \delta, ..., \vec{\nabla}_{x,N} \delta) \) and \( \overrightarrow{Q}_N^\psi = (Q_1^\psi, ..., Q_N^\psi) \), we have
\[
(\text{Integral}_2)(t) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \vec{\nabla}_{x,1} \delta(x - y) \frac{\partial Q_1^\psi}{\partial t}(x, t) dxdy
+ \int_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \vec{\nabla}_{x,2} \delta(x - y) \frac{\partial Q_2^\psi}{\partial t}(x, t) dxdy
+ \int_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \vec{\nabla}_{x,N-1} \delta(x - y) \frac{\partial Q_{N-1}^\psi}{\partial t}(x, t) dxdy
+ \int_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \vec{\nabla}_{x,N} \delta(x - y) \frac{\partial Q_N^\psi}{\partial t}(x, t) dxdy
\]
From the local momentum conservation \((\text{loc-Momentum})\), we have for \( m = 1, ..., N \)
\[
\frac{\partial Q_m^\psi(x, t)}{\partial t} = -\vec{\nabla}_{x,1} \left[ (1_{m,1} L^\psi + S_{m,1}^\psi) \right] (x, t) - \vec{\nabla}_{x,2} \left( 1_{m,2} L^\psi + S_{m,2}^\psi \right) (x, t)
- \vec{\nabla}_{x,N-1} \left( 1_{m,N-1} L^\psi + S_{m,N-1}^\psi \right) (x, t) - \vec{\nabla}_{x,N} \left( 1_{m,N} L^\psi + S_{m,N}^\psi \right) (x, t),
\]
and from the properties of Kronecker delta function \((2)\), this leads to the following conclusion
\[
(\text{Integral}_2)(t) = -\left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \psi(y, t) \vec{\nabla}_{x,m} \delta(x - y) \vec{\nabla}_{x,m} L^\psi(x, t) \right|_{m=1}^N dxdy
- \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \psi(y, t) \vec{\nabla}_{x,1} \delta(x - y) \vec{\nabla}_{x,n} S_{m,n}^\psi(x, t) \right|_{n=1}^N dxdy
- \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \psi(y, t) \vec{\nabla}_{x,2} \delta(x - y) \vec{\nabla}_{x,n} S_{m,n}^\psi(x, t) \right|_{n=1}^N dxdy
- \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \psi(y, t) \vec{\nabla}_{x,N-1} \delta(x - y) \vec{\nabla}_{x,n} S_{m,n}^\psi(x, t) \right|_{n=1}^N dxdy
- \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \psi(y, t) \vec{\nabla}_{x,N} \delta(x - y) \vec{\nabla}_{x,n} S_{m,n}^\psi(x, t) \right|_{n=1}^N dxdy
\]
Using (A combination (3) and summing up the above expressions we arrive at the following conclusion

\[ \text{Integral}_2(t) = \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \delta(x-y) \nabla \psi(x,t) \right]_{n=1}^{N} dx dy \]

Next, integrating by parts, one has

\[ \text{Integral}_2(t) = \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \left( \nabla^2 \psi(x,y) \right) L^\nabla \psi(x,t) \right]_{n=1}^{N} dx dy \]

\[ + \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \left( \nabla \psi(x,y) \right) S_{1,1}^\nabla \psi(x,t) \right]_{n=1}^{N} dx dy \]

\[ + \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \left( \nabla \psi(x,y) \right) S_{2,2}^\nabla \psi(x,t) \right]_{n=1}^{N} dx dy \]

\[ \vdots \]

\[ + \left[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \left( \nabla \psi(x,y) \right) S_{N,N}^\nabla \psi(x,t) \right]_{n=1}^{N} dx dy \]

On account of \( \nabla_{x,1} + \nabla_{x,N} = \sum_{n=1}^{N} \nabla_{x,n}, \) we obtain

\[ \text{Integral}_2(t) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \sum_{n=1}^{N} \left( \nabla^2 \psi(x,y) \right) L^\nabla \psi(x,t) dx dy \]

\[ + \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \sum_{n=1}^{N} \left( \nabla \psi(x,y) \right) S_{1,1}^\nabla \psi(x,t) dx dy \]

\[ + \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \sum_{n=1}^{N} \left( \nabla \psi(x,y) \right) S_{2,2}^\nabla \psi(x,t) dx dy \]

\[ \vdots \]

\[ + \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \sum_{n=1}^{N} \left( \nabla \psi(x,y) \right) S_{N,N}^\nabla \psi(x,t) dx dy \]

Using (3) and summing up the above expressions we arrive at the following conclusion

\[ \text{Integral}_2(t) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \Delta \delta(x-y) \left( -\frac{1}{2} \text{div} \left( \nabla \psi(x,t) \right)^2 + \frac{\lambda \eta}{\eta + 1} |\psi(x,t)|^{2\eta + 2} \right) dx dy \]

\[ + \int_{\mathbb{R}^N \times \mathbb{R}^N} \psi(y, t) \sum_{m=1}^{N} \sum_{n=1}^{N} \left( \nabla \psi(x,y) \right) S_{m,n}^\nabla \psi(x,t) dx dy. \]  \( (52) \)

A combination (50), (51) and (52) conduce us to the following estimate

\[ \frac{d}{dt} \psi_\delta(t) = \text{Integral}_1(t) + \text{Integral}_2(t) \]
\[ \begin{align*}
&= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \Delta_x \delta(x - y) \left( -\frac{1}{2} \text{div} \left( \nabla_x |\psi(x, t)|^2 \right) \right) \, dxdy \\
&+ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \Delta_x \delta(x - y) \left( \frac{\lambda \eta}{\eta + 1} |\psi(x, t)|^{2\eta + 2} \right) \, dxdy \\
&+ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_{x,m} \nabla_{x,n} \delta(x - y) \right) S_{m,n}^{\nabla}(\psi)(x, t) \, dxdy \\
&- 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \nabla_y \cdot Q^\pm(\psi)(y, t) \right) \left( \nabla_x \delta(x - y) \cdot \nabla_y \psi(x, t) \right) \, dxdy = \sum_{k=3}^6 (\text{Integral}_k)(t),
\end{align*} \]

where
\[ (\text{Integral}_3)(t) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \Delta_x \delta(x - y) \left( -\frac{1}{2} \text{div} \left( \nabla_x |\psi(x, t)|^2 \right) \right) \, dxdy, \]
\[ (\text{Integral}_4)(t) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \Delta_x \delta(x - y) \left( \frac{\lambda \eta}{\eta + 1} |\psi(x, t)|^{2\eta + 2} \right) \, dxdy, \]
\[ (\text{Integral}_5)(t) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_{x,m} \nabla_{x,n} \delta(x - y) \right) S_{m,n}^{\nabla}(\psi)(x, t) \, dxdy, \]
\[ (\text{Integral}_6)(t) = -2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \nabla_y \cdot Q^\pm(\psi)(y, t) \right) \left( \nabla_x \delta(x - y) \cdot \nabla_y \psi(x, t) \right) \, dxdy. \]

Note that the weight function \( w \) here is a function of two variables \( x, y \) and let us put \( \delta(x - y) = |x - y| \), \( x, y \in \mathbb{R}^N \), from the Appendix 5.2, we have
\[ \Delta_x |x - y| = \nabla_x \cdot \nabla_x |x - y| = \frac{N - 1}{|x - y|}. \]

Thus, we have the estimate for the first integral in (53), as follows
\[ (\text{Integral}_3)(t) = \frac{N - 1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \frac{\delta(x - y)}{|x - y|} \left( -\Delta_x \psi(x, t) \right) \, dxdy = \frac{N - 1}{2} \iint_{\mathbb{R}^N} \frac{\Psi(y, t)}{|x - y|} \left( -\Delta_x \psi(x, t) \right) \, dx. \]

From the Appendix 5.4, we derive that, for dimension \( N \geq 1 \)
\[ (\text{Integral}_4)(t) = \frac{N - 1}{2} \iint_{\mathbb{R}^N} \frac{\Psi(y, t)}{|x - y|} \left( -\Delta_x \psi(x, t) \right) \, dx \]
\[ = \frac{N - 1}{2} \iint_{\mathbb{R}^N} \left( (-\Delta_x)^{\frac{N - 1}{2}} \psi(x, t) \right) \left( -\Delta_x \psi(x, t) \right) \, dx \]
\[ = \frac{N - 1}{2} \iint_{\mathbb{R}^N} \left( (-\Delta_x)^{\frac{N - 3}{2}} \psi^2(x, t) \right) \, dx \]
\[ = \frac{C_{N, \eta}(N - 1)}{2} \left\| (-\Delta_x)^{-\frac{N - 3}{2}} \psi(\cdot, t) \right\|_{L^2(\mathbb{R}^N)}^2. \]

Now, we show that the sum of \( (\text{Integral}_5)(t) \) and \( (\text{Integral}_6)(t) \) is not negative. Indeed, by integrating by parts, we have
\[ (\text{Integral}_5)(t) + (\text{Integral}_6)(t) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \Psi(y, t) \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_{x,m} \nabla_{x,n} \delta(x - y) \right) S_{m,n}^{\nabla}(\psi)(x, t) \, dxdy \]
Thanks to Appendix 5.2, one has $\nabla_{y,m} \nabla_{x,n} |x - y| = -\nabla_{x,m} \nabla_{y,n} |x - y|$, for $x, y \in \mathbb{R}^N$, $m, n = 1, ..., N,$ and we deduce that

\[
(\text{Integral}_5)(t) + (\text{Integral}_6)(t) = 2 \int_{\mathbb{R}_t^N} \nabla_{x,n} |x - y| \int_{\mathbb{R}_t^N} \nabla_{x,m} \nabla_{x,n} |x - y| \\
\times \left[ \Psi(y, t) \mathfrak{R} \left( \nabla_{x,m} \psi(x, t) \nabla_{x,n} \bar{\psi}(x, t) - Q^\nabla_m(\psi)(y, t) Q^\nabla_n(\psi)(x, t) \right) \right] dxdy
\]

By Appendix 5.2, we have $\nabla_{x,m} \nabla_{x,n} |x - y| = \nabla_{y,m} \nabla_{y,n} |x - y|$, and because of the symmetry of the above expression, changing the roles of $x$ and $y$ does not change its value, so we deduce that

\[
(\text{Integral}_5)(t) + (\text{Integral}_6)(t) = \int_{\mathbb{R}_t^N} \nabla_{x,m} \nabla_{x,n} |x - y| \\
\times \left[ \frac{\Psi(y, t)}{\Psi(x, t)} \mathfrak{R} \left( \tilde{\psi}(x, t) \nabla_{x,m} \psi(x, t) \psi(x, t) \nabla_{x,n} \bar{\psi}(x, t) \right) - Q^\nabla_m(\psi)(y, t) Q^\nabla_n(\psi)(x, t) \right] dxdy
\]

Putting $U = \tilde{\psi}(x, t) \nabla_{x,m} \psi(x, t), V = \bar{\psi}(y, t) \nabla_{y,n} \bar{\psi}(y, t)$, we have the following assertion

\[
\Re(UV) = \Re(U)\Re(V) + \Im(U)\Im(V),
\]

and for complex number $z$, we also have $\Re(z \nabla_{x,m} \bar{z}) = \Re(\bar{z} \nabla_{x,m} z) = \frac{1}{2} \nabla_{x,m} |z|^2$, yields

\[
\Re \left( \tilde{\psi}(x, t) \nabla_{x,m} \psi(x, t) \psi(x, t) \nabla_{x,n} \bar{\psi}(x, t) \right) = \Re \left( \tilde{\psi}(x, t) \nabla_{x,m} \psi(x, t) \right) \Re \left( \bar{\psi}(x, t) \nabla_{x,n} \bar{\psi}(x, t) \right)
\]

\[
+ \Im \left( \tilde{\psi}(x, t) \nabla_{x,m} \psi(x, t) \right) \Im \left( \bar{\psi}(x, t) \nabla_{x,n} \bar{\psi}(x, t) \right)
\]

\[
= \frac{\nabla_{x,m} |\psi|^2(x, t) \nabla_{x,n} |\psi|^2(x, t)}{4} + Q^\nabla_m(\psi)(x, t) Q^\nabla_n(\psi)(x, t),
\]
and the similarity to have
\[
\mathcal{R}(\tilde{\psi}(y,t)\nabla_y,m\psi(y,t)\psi(y,t)\nabla_y,n\tilde{\psi}(y,t)) = \frac{\nabla_y,m|\psi|^2(y,t)\nabla_y,n|\psi|^2(y,t)}{4} + Q_m^\nabla(\psi)(y,t)Q_n^\nabla(\psi)(y,t).
\]

(57)

From (55), (56) and (57), one obtains that

(\text{Integral}_5)(t) + (\text{Integral}_6)(t) = (\text{Integral}_7)(t) + (\text{Integral}_8)(t) + (\text{Integral}_9)(t)

\[
\begin{align*}
&\quad = \int_{\mathbb{R}^N \times \mathbb{R}^N} \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_y,m \nabla_y,n |x - y| \right) \frac{\psi(y,t)}{\psi(x,t)} \frac{\nabla_y,m|\psi|^2(x,t)\nabla_y,n|\psi|^2(x,t)}{4} dxdy \\
&\quad + \int_{\mathbb{R}^N \times \mathbb{R}^N} \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_y,m \nabla_y,n |x - y| \right) \frac{\psi(x,t)}{\psi(y,t)} \frac{\nabla_y,m|\psi|^2(y,t)\nabla_y,n|\psi|^2(y,t)}{4} dxdy \\
&\quad + \int_{\mathbb{R}^N \times \mathbb{R}^N} \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_y,m \nabla_y,n |x - y| \right) \left[ \frac{\psi(y,t)}{\psi(x,t)} Q_m^\nabla(\psi)(x,t)Q_n^\nabla(\psi)(x,t) \\
&\qquad + \frac{\psi(x,t)}{\psi(y,t)} Q_m^\nabla(\psi)(y,t)Q_n^\nabla(\psi)(y,t) - Q_m^\nabla(\psi)(x,t)Q_n^\nabla(\psi)(y,t) - Q_m^\nabla(\psi)(y,t)Q_n^\nabla(\psi)(x,t) \right] dxdy.
\end{align*}
\]

Thanks to the Appendix 5.3 imply that the matrices \( \nabla_y,m \nabla_y,n |x - y| \) and \( \nabla_y,m \nabla_y,n |x - y| \) are positive definite, it follows that the integrals (\text{Integral}_7)(t) and (\text{Integral}_8)(t) are positive. Hence,

(\text{Integral}_5)(t) + (\text{Integral}_6)(t) \geq \int_{\mathbb{R}^N \times \mathbb{R}^N} \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_y,m \nabla_y,n |x - y| \right) \left[ \frac{\psi(y,t)}{\psi(x,t)} Q_m^\nabla(\psi)(x,t)Q_n^\nabla(\psi)(x,t) \\
&\quad + \frac{\psi(x,t)}{\psi(y,t)} Q_m^\nabla(\psi)(y,t)Q_n^\nabla(\psi)(y,t) - Q_m^\nabla(\psi)(x,t)Q_n^\nabla(\psi)(y,t) - Q_m^\nabla(\psi)(y,t)Q_n^\nabla(\psi)(x,t) \right] dxdy.
\]

Defining the following vector function

\[
\tilde{J}(x,y,t) = \sqrt{\psi(y,t)}\psi^{-1}(x,t)Q^\nabla(\psi)(x,t) - \sqrt{\psi(x,t)}\psi^{-1}(y,t)Q^\nabla(\psi)(y,t),
\]

we deduce that

(\text{Integral}_5)(t) + (\text{Integral}_6)(t) \geq \int_{\mathbb{R}^N \times \mathbb{R}^N} \sum_{m=1}^N \sum_{n=1}^N \left( \nabla_y,m \nabla_y,n |x - y| \right) J_m(x,y,t)J_n(x,y,t) dxdy \geq 0.
\]

(58)

From (53), (54) and (58) it follows that

\[
\frac{d}{dt} \mathcal{V}_\theta(t) \geq \frac{C_{N,\pi}(N - 1)}{2} \left\| (-\Delta_x)^{-\frac{N-3}{4}} \psi(\cdot,t) \right\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda \eta (N - 1)}{\eta + 1} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(y,t)|^2|\psi(x,t)|^{2\eta+2}}{|x - y|} dxdy.
\]

Integrating from 0 to \( t \) and applying conservation of mass (Mass), we obtain

\[
\frac{C_{N,\pi}(N - 1)}{2} \int_0^t \left\| (-\Delta_x)^{-\frac{N-3}{4}} \psi(\cdot,z) \right\|^2_{L^2(\mathbb{R}^N)} dz + \frac{\lambda \eta (N - 1)}{\eta + 1} \int_{[0,t] \times \mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(y,z)|^2|\psi(x,z)|^{2\eta+2}}{|x - y|} dxdydz
\]

\[
\leq \mathcal{V}_{|x - y|}(t) \leq \int_{\mathbb{R}^N} |\psi(y,t)|^2 dy \sup_{t \in \mathbb{R}^+ \atop y \in \mathbb{R}^N} \left| \int_{\mathbb{R}^N} \tilde{\nabla}_x |x - y| \cdot \overrightarrow{Q^\nabla}(\psi)(x,t) dx \right|
\]

\[
= ||\psi(\cdot,t)||^2_{L^2(\mathbb{R}^N)} \sup_{t \in \mathbb{R}^+ \atop y \in \mathbb{R}^N} \left| \int_{\mathbb{R}^N} \tilde{\nabla}_x |x - y| \cdot \overrightarrow{Q^\nabla}(\psi)(x,t) dx \right|
\]
The proof is complete.

3.4. **Proof of Proposition 1.1.** For \( N \geq 1, \lambda > 0 \) and from (12), we deduce that

\[
\left\| (-\Delta_x)^{\frac{3N}{4}} |\psi|^2 \right\|_{L^2_tL^2_x(\mathbb{R}^+ \times \mathbb{R}^N)} \lesssim \left\| \psi_0 \right\|_{L^2(\mathbb{R}^N)} \sup_{t \in \mathbb{R}^+} \left\| \nabla_x |x-y| \cdot \overrightarrow{Q}^\psi(\psi)(x,t) \right\|_{L^2(\mathbb{R}^N)}.
\]

From the Appendix 5.2, we have that \( \nabla_x |x-y| = \frac{x-y}{|x-y|} \) and the fact that \( \frac{|x-y|}{|x-y|} \leq 1 \) for all \( x, y \in \mathbb{R}^N \), yields

\[
\left| \int_{\mathbb{R}^N} \nabla_x |x-y| \cdot \overrightarrow{Q}^\psi(\psi)(x,t) dx \right| = \int_{\mathbb{R}^N} \left| \frac{x-y}{|x-y|} \cdot \overrightarrow{Q}^\psi(\psi)(x,t) dx \right| \leq \int_{\mathbb{R}^N} \left| \overrightarrow{Q}^\psi(\psi)(x,t) \right| dx.
\]

By using the Cauchy-Schwartz inequality and from (1), we get that

\[
\int_{\mathbb{R}^N} \left| \overrightarrow{Q}^\psi(\psi)(x,t) \right| dx \leq \sqrt{\int_{\mathbb{R}^N} |\psi(x,t)|^2 dx \int_{\mathbb{R}^N} \left| \nabla_x \psi(\psi)(x,t) \right|^2 dx} = \left\| \psi(\cdot, t) \right\|_{L^2(\mathbb{R}^N)} \left\| \nabla_x \psi(\cdot, t) \right\|_{L^2(\mathbb{R}^N)}. \tag{60}
\]

From (59), (60) and conservation of mass (Mass), we obtain that

\[
\left\| (-\Delta_x)^{\frac{3N}{4}} |\psi|^2 \right\|_{L^2_tL^2_x(\mathbb{R}^+ \times \mathbb{R}^N)} \lesssim \left\| \psi_0 \right\|_{L^2(\mathbb{R}^N)} \sup_{t \in \mathbb{R}^+} \left\| \nabla_x \psi \right\|_{L^2(\mathbb{R}^N)}
\]

\[
\lesssim \left\| \psi_0 \right\|_{L^2(\mathbb{R}^N)} \sup_{t \in \mathbb{R}^+} \left\| \nabla_x \psi \right\|_{L^2(\mathbb{R}^N)} \lesssim \left\| \psi_0 \right\|_{L^2(\mathbb{R}^N)} \frac{3}{2} \sup_{t \in \mathbb{R}^+} \sqrt{E[\psi]}(t).
\]

From the conservation of energy (Energy) one gets

\[
\left\| (-\Delta_x)^{\frac{3N}{4}} |\psi|^2 \right\|_{L^2_tL^2_x(\mathbb{R}^+ \times \mathbb{R}^N)} \lesssim \left\| \psi_0 \right\|_{L^2(\mathbb{R}^N)} \frac{3}{2} \sqrt{E[\psi]}.
\]

The proof is done.

3.5. **Proof of Theorem 1.4.** As argued above, Problem (NLS) possesses the solutions that satisfy the conservation laws (Energy), (loc-Mass) and (loc-Momentum). The \((1-1)\) Morawetz action is the main tool to prove this theorem and we shall prove for \( N = 2 \), the case \( N = 1 \) is simpler and admitted. In fact, we recall that

\[
\mathcal{M}_\delta(t) = \int_{\mathbb{R}^2} \nabla \delta(x) \cdot \overrightarrow{Q}^\psi(\psi)(x,t) dx,
\]

where the weight function

\[
\delta(\cdot) : \mathbb{R}^2 \to \mathbb{R} \quad \text{is the moment arbitrary.}
\]

By taking the derivative of both sides of (61) with respect to the time variable, we deduce

\[
\frac{d}{dt} \mathcal{M}_\delta(t) = \int_{\mathbb{R}^2} \nabla \delta(x) \cdot \frac{\partial \overrightarrow{Q}^\psi(\psi)}{\partial t}(x,t) dx.
\]
From the local momentum conservation (loc-Momentum), we have for $m = 1,2$

$$\frac{\partial Q_m^\nabla(\psi)}{\partial t}(x,t) = -\nabla_1 \left(1_{m,1}L^\nabla(\psi) + S_{m,1}^\nabla(\psi)\right)(x,t) - \nabla_2 \left(1_{m,2}L^\nabla(\psi) + S_{m,2}^\nabla(\psi)\right)(x,t). \quad (63)$$

From the definition of Kronecker delta function (2) and combining (62) and (63), we obtain that

$$\frac{d}{dt}M_\delta(t) = -\int_{\mathbb{R}^2} \nabla_1 \delta(x) \nabla_1 \left(L^\nabla(\psi) + S_{1,1}^\nabla(\psi)\right)(x,t) dx - \int_{\mathbb{R}^2} \nabla_1 \delta(x) \nabla_2 S_{1,2}^\nabla(\psi)(x,t) dx$$

$$- \int_{\mathbb{R}^2} \nabla_2 \delta(x) \nabla_1 S_{2,1}^\nabla(\psi)(x,t) dx - \int_{\mathbb{R}^2} \nabla_2 \delta(x) \nabla_2 \left(L^\nabla(\psi) + S_{2,2}^\nabla(\psi)\right)(x,t) dx.$$  

By applying the definition of the Kronecker delta function (2), we conclude that

$$\frac{d}{dt}M_\delta(t) = \int_{\mathbb{R}^2} \nabla_1 \delta(x) \nabla_1 \left(L^\nabla(\psi)\right)(x,t) dx + \int_{\mathbb{R}^2} \nabla_1 \delta(x) \nabla_2 \left(L^\nabla(\psi)\right)(x,t) dx$$

$$+ \int_{\mathbb{R}^2} \nabla_2 \delta(x) \nabla_1 S_{1,1}^\nabla(\psi)(x,t) dx + \int_{\mathbb{R}^2} \nabla_2 \delta(x) \nabla_2 S_{1,2}^\nabla(\psi)(x,t) dx$$

$$+ \int_{\mathbb{R}^2} \nabla_2 \nabla_1 \delta(x) S_{2,1}^\nabla(\psi)(x,t) dx + \int_{\mathbb{R}^2} \nabla_2 \nabla_2 \delta(x) S_{2,2}^\nabla(\psi)(x,t) dx.$$  

Consequently,

$$\frac{d}{dt}M_\delta(t) = \int_{\mathbb{R}^2} \text{div} \left(\nabla_\delta(\psi)\right) \nabla^\nabla(\psi)(x,t) dx + \int_{\mathbb{R}^2} \sum_{m=1,2} \sum_{n=1,2} \nabla_m \nabla_\delta(\psi) S_{m,n}^\nabla(\psi)(x,t) dx. \quad (64)$$

Introducing (3) and (4) into (64), we obtain that

$$\frac{d}{dt}M_\delta(t) = \int_{\mathbb{R}^2} \text{div} \left(\nabla_\delta(\psi)\right) \left(-\frac{1}{2} \text{div} \left(\nabla |\psi(x,t)|^2\right) + \frac{\lambda \eta}{\eta + 1} |\psi(x,t)|^{2\eta+2}\right) dx$$

$$+ \int_{\mathbb{R}^2} \sum_{m=1,2} \sum_{n=1,2} \left(\nabla_m \nabla_\delta(\psi)\right) \left(2 \Re \left(\nabla_m \psi \nabla_n \bar{\psi}\right)(x,t)\right) dx. \quad (65)$$

Now, we pick the weight function $\delta(x) = |x|^2$, from the Appendix 5.1 we have that

$$\text{div} \left(\nabla_\delta\right) = \Delta |x|^2 = 4, \quad \text{and} \quad \nabla_m \nabla_n |x|^2 = 2 \times 1_{m,n}, \quad m = 1, 2, n = 1, 2.$$ 

Combining this inequality with (65) and the fact that

$$\sum_{m=1,2} \sum_{n=1,2} \left(\nabla_m \nabla_n |x|^2\right) \left(2 \Re \left(\nabla_m \psi \nabla_n \bar{\psi}\right)(x,t)\right) = \sum_{m=1,2} \sum_{n=1,2} 2(1_{m,n}) \left(2 \Re \left(\nabla_m \psi \nabla_n \bar{\psi}\right)(x,t)\right) = 4|\nabla \psi(x,t)|^2,$$

we deduce

$$\frac{d}{dt}M_{\delta=|x|^2}(t) = \int_{\mathbb{R}^2} \frac{4\lambda \eta}{\eta + 1} |\psi(x,t)|^{2\eta+2} dx - 2 \int_{\mathbb{R}^2} \text{div} \left(\nabla |\psi(x,t)|^2\right) dx + 4 \int_{\mathbb{R}^2} |\nabla \psi(x,t)|^2 dx.$$ 

Equivalent to

$$\frac{d}{dt}M_{\delta=|x|^2}(t) = 8 \left(\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi(x,t)|^2 dx + \frac{\lambda}{2\eta + 2} \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} dx\right)$$
\[-\frac{\lambda(4-4\eta)}{\eta+1} \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} \, dx \leq 2 \int_{\mathbb{R}^2} \text{div} \left( \nabla |\psi(x,t)|^2 \right) \, dx.\]

By the total energy $\mathbb{E}[\psi](t)$ in (Energy), one obtains
\[
\frac{d}{dt} \mathcal{M}_{\delta=|x|^2}(t) = 8\mathbb{E}[\psi](t) - \frac{\lambda(4-4\eta)}{\eta+1} \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} \, dx \leq 2 \int_{\mathbb{R}^2} \text{div} \left( \nabla |\psi(x,t)|^2 \right) \, dx.
\]

Let us define the $C^2$ function by the quantity
\[
\mathcal{S}(t) = \int_{\mathbb{R}^2} |x|^2 \psi(x,t) \, dx, \quad \text{with } \psi = |\psi|^2.
\]

From (loc-Mass), we have that
\[
\frac{d}{dt} \mathcal{S}(t) = \int_{\mathbb{R}^2} |x|^2 \frac{\partial \psi}{\partial t}(x,t) \, dx = -2 \int_{\mathbb{R}^2} |x|^2 \left( \nabla \cdot \overrightarrow{Q}(\psi)(x,t) \right) \, dx = 2 \mathcal{M}_{\delta=|x|^2}(t),
\]

where we have used the integration by parts. Thus we conclude that
\[
\frac{d^2}{dt^2} \mathcal{S}(t) = 16\mathbb{E}[\psi](t) - \frac{8\lambda(1-\eta)}{\eta+1} \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} \, dx - 4 \int_{\mathbb{R}^2} \text{div} \left( \nabla |\psi(x,t)|^2 \right) \, dx.
\]

From (7), we recall that
\[
\mathbb{P}[\psi](t) = \| ( \cdot + 2it \nabla ) \psi ( \cdot , t ) \|_{L^2(\mathbb{R}^2)}^2 + \frac{4t^2 \lambda}{\eta+1} \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} \, dx - 4 \int_{\mathbb{R}^2} \text{div} \left( \nabla \mathcal{H}(\psi)(x,t) \right) \, dx.
\]

The above expression is equivalent to
\[
\mathbb{P}[\psi](t) = \left\| ( \cdot ) + 2it \right\|_{L^2(\mathbb{R}^2)}^2 + 4t^2 \left\| \nabla \psi(\cdot , t) \right\|_{L^2(\mathbb{R}^2)}^2 - 4t \int_{\mathbb{R}^2} x \cdot \overrightarrow{Q}(\psi)(x,t) \, dx
\]
\[+ \frac{4t^2 \lambda}{\eta+1} \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} \, dx - 4 \int_{\mathbb{R}^2} \text{div} \left( \nabla \mathcal{H}(\psi)(x,t) \right) \, dx.
\]

If we understand that $|x|^2 = \delta(x)$ and using the conservation of energy (Energy) then the equation above is
\[
\mathbb{P}[\psi](t) \overset{\text{Energy}}{=} \int_{\mathbb{R}^2} \delta(x)|\psi(x,t)|^2 \, dx + 8t^2 \mathbb{E}[\psi_0] - 2t \int_{\mathbb{R}^2} \nabla \delta(x) \cdot \overrightarrow{Q}(\psi)(x,t) \, dx - 4 \int_{\mathbb{R}^2} \text{div} \left( \nabla \mathcal{H}(\psi)(x,t) \right) \, dx.
\]

The derivative of the above equation according to the variable $t$ becomes
\[
\frac{d}{dt} \mathbb{P}[\psi](t) = \frac{d}{dt} \int_{\mathbb{R}^2} \delta(x)|\psi(x,t)|^2 \, dx + 16t \mathbb{E}[\psi_0]
\]
\[\quad - 2 \int_{\mathbb{R}^2} \nabla \delta(x) \cdot \overrightarrow{Q}(\psi)(x,t) \, dx - 4 \int_{\mathbb{R}^2} \text{div} \left( \nabla \left( \frac{\partial}{\partial t} \mathcal{H}(\psi)(x,t) \right) \right) \, dx, \quad \text{for } \psi = |\psi|^2.
\]

Moreover, from (loc-Mass) and using integration by parts, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \delta \psi \, dx = \int_{\mathbb{R}^2} \delta \frac{\partial \psi}{\partial t} \, dx = 2 \int_{\mathbb{R}^2} \nabla \delta \cdot \overrightarrow{Q}(\psi) \, dx,
\]

and
\[
\frac{d}{dt} \left( t \int_{\mathbb{R}^2} \nabla \delta \cdot \overrightarrow{Q}(\psi) \, dx \right) = t \int_{\mathbb{R}^2} \nabla \delta \cdot \frac{\partial}{\partial t} \overrightarrow{Q}(\psi) \, dx + \int_{\mathbb{R}^2} \nabla \delta \cdot \overrightarrow{Q}(\psi) \, dx.
\]

Consequently,
\[
\frac{d}{dt} \mathbb{P}[\psi](t) = -2t \int_{\mathbb{R}^2} \nabla \delta(x) \cdot \frac{\partial}{\partial t} \overrightarrow{Q}(\psi)(x,t) \, dx + 16t \mathbb{E}[\psi_0] - 4 \int_{\mathbb{R}^2} \text{div} \left( \nabla \left( \frac{\partial}{\partial t} \mathcal{H}(\psi)(x,t) \right) \right) \, dx.
\]
From the above observation and combined with (61), (66), recalling that \( \delta(x) = |x|^2 \), one obtains
\[
\frac{d}{dt} \mathbb{P}[\psi](t) = -2t \frac{d}{dt} \mathcal{M}_{\delta=|x|^2}(t) + 16t E[\psi_0] - 4t \int_{\mathbb{R}^2} \text{div} \left( \nabla |\psi(x,t)|^2 \right) dx
\]
\[
= \frac{2\lambda t}{\eta + 1} (4 - 4\eta) \int_{\mathbb{R}^2} |\psi(x,t)|^{2\eta+2} dx,
\]
where we have used the conditions (8) that \( \frac{\partial \mathcal{H}(\psi)}{\partial t}(x,t) = t|\psi|^2(x,t) \). Let us pick \( \eta = 1 \) in (68) and integrating this equation in time on \([0,t]\) and we obtain that \( \mathbb{P}[\psi](t) \) is conserved. The proof of Theorem 1.4 is finished.

**Remark 3.1.** Obviously, we can choose the weight function \( \delta(\cdot) : \mathbb{R}^N \to \mathbb{R} \) in other forms. However, in this proof, we choose \( \delta(x) = |x|^2 = \sum_{k=1}^{N} x_k^2 \), because of the properties for our convenience such as:
\[
\text{div} (\nabla |x|^2) = \frac{\partial^2 |x|^2}{\partial x_1^2} + \frac{\partial^2 |x|^2}{\partial x_2^2} + \cdots + \frac{\partial^2 |x|^2}{\partial x_N^2} = 2N,
\]
and
\[
\frac{\partial}{\partial x_m} \frac{\partial}{\partial x_n} |x|^2 = 2 \times 1_{mn}, \quad \text{for} \ 1 \leq m, n \leq N.
\]

### 3.6. Proof of Theorem 1.5.

The proof of this theorem can be seen as a consequence of Theorem 1.4. Indeed, from (68), we have that
\[
\frac{2\lambda t}{\eta + 1} (4 - 4\eta) \int_{-\infty}^{\infty} |\psi(x,t)|^{2\eta+2} dx = \frac{d}{dt} \mathbb{P}[\psi](t).
\]
For \( \eta < 2 \), for all \( t \in [0,T] \), we have
\[
\int_{-\infty}^{\infty} t|\psi(x,t)|^{2\eta+2} dx \lesssim \frac{d}{dt} \mathbb{P}[\psi](t).
\]
Integrating this equation in time on \([0,T] \) we obtain
\[
\int_{0}^{T} \int_{-\infty}^{\infty} t|\psi(x,t)|^{2\eta+2} dx dt \lesssim \mathbb{P}[\psi](t) - \mathbb{P}[\psi](0)
\]
\[
\lesssim \mathbb{P}[\psi](t) - \|\psi(\cdot,t)\|_{L^2(\mathbb{R})}^2.
\]
Applying the fundamental theorem of calculus we deduce (14). The proof of Theorem 1.5 is complete.

### 3.7. Proof of Proposition 1.2.

This proof is also derived from the proof of Theorem 1.4. In fact, for \( \eta \in [1,2) \), for all \( t \in (0,T] \), from (69) we have
\[
\int_{-\infty}^{\infty} t|\psi(x,t)|^{2\eta+2} dx \lesssim \sup_{0 < t \leq T} \left( \frac{d}{dt} \mathbb{P}[\psi](t) \right).
\]
Assume that \( \sup_{0 < t \leq T} \left( \frac{d}{dt} \mathbb{P}[\psi](t) \right) \) has an upper bound \( C \) for some \( C > 0 \), then
\[
\int_{-\infty}^{\infty} t|\psi(x,t)|^{2\eta+2} dx \leq C.
\]
Consequently,
\[
\|\psi(\cdot,t)\|_{L^{2\eta+2}(\mathbb{R})} \leq C.
\]
This finishes the proof of Proposition 1.2.

4. Conclusions

This study has achieved a major extension over the much more investigated nonlinear Schrödinger equation. Using the Gagliardo-Nirenberg interpolation inequality, we obtained the local well-posedness results of the solutions: local existence, continuation, blow-up alternative and continuous dependence on initial data. Moreover, by using the two types of Morawetz actions, we established the stronger upper bound of the solutions, and moreover the modified pseudo-conformal conservation law for Problem (NLS). The Morawetz and decay estimates are also investigated.

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5. Appendix

5.1. A1. On the $N$-dimensional Euclidean space $\mathbb{R}^N$, let $x = (x_1, ..., x_N) \in \mathbb{R}^N$, we define the Euclidean norm of $x$ on the $N$-dimensional vector spaces with $|x| = \sqrt{\sum_{k=1}^{N} x_k^2}$. Then we have $\nabla_j |x|^2 = (\nabla_1 |x|^2, ..., \nabla_N |x|^2) = (2x_1, ..., 2x_N)$ and $\Delta |x|^2 = (\nabla \cdot \nabla)|x|^2 = 2N$, and

$$\nabla_m \nabla_n |x|^2 = \begin{cases} 2, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} = 2 \times 1_{m=n}, \quad 1 \leq m, n \leq N.$$

5.2. A2. For $x = (x_1, ..., x_N), y = (y_1, ..., y_N) \in \mathbb{R}^N$, we have the Euclidean norm of $x - y$ as

$$|x - y| = \sqrt{\sum_{k=1}^{N} (x_k - y_k)^2},$$
and \( \mathbf{\nabla}_x |x - y| = (\mathbf{\nabla}_{x,1}|x - y|, \ldots, \mathbf{\nabla}_{x,N}|x - y|) \), then we deduce
\[
\mathbf{\nabla}_x |x - y| = \left( \frac{x_1 - y_1}{\sqrt{\sum_{k=1}^{N}(x_k - y_k)^2}}, \ldots, \frac{x_N - y_N}{\sqrt{\sum_{k=1}^{N}(x_k - y_k)^2}} \right),
\]
and we have
\[
\Delta_x |x - y| = (\mathbf{\nabla}_x \cdot \mathbf{\nabla}_x)|x - y| = \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{(x_j - y_j)^2 - (x_k - y_k)^2}{(\sqrt{\sum_{j=1}^{N}(x_j - y_j)^2})^3} = \frac{N - 1}{\sqrt{\sum_{k=1}^{N}(x_k - y_k)^2}}
\]
We can also check that
\[
\mathbf{\nabla}_{y,m} \mathbf{\nabla}_{x,n}|x - y| = -\mathbf{\nabla}_{x,m} \mathbf{\nabla}_{x,n}|x - y|,
\]
and
\[
\mathbf{\nabla}_{x,m} \mathbf{\nabla}_{x,n}|x - y| = \mathbf{\nabla}_{y,m} \mathbf{\nabla}_{y,n}|x - y|.
\]

5.3. A3. We shall prove that the following matrix is positive definite
\[
\left( \begin{array}{cccc}
\mathbf{\nabla}_{x,1}^2 |x - y| & \mathbf{\nabla}_{x,1} \mathbf{\nabla}_{x,2} |x - y| & \cdots & \mathbf{\nabla}_{x,1} \mathbf{\nabla}_{x,N} |x - y| \\
\mathbf{\nabla}_{x,2} \mathbf{\nabla}_{x,1} |x - y| & \mathbf{\nabla}_{x,2}^2 |x - y| & \cdots & \mathbf{\nabla}_{x,2} \mathbf{\nabla}_{x,N} |x - y| \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{\nabla}_{x,N} \mathbf{\nabla}_{x,1} |x - y| & \mathbf{\nabla}_{x,N} \mathbf{\nabla}_{x,2} |x - y| & \cdots & \mathbf{\nabla}_{x,N}^2 |x - y| \\
\end{array} \right)_{N \times N}
\]
Picking the vector function \( \{\vartheta_k\}_{k=1}^{N} \), with \( \vartheta_k(\cdot): \mathbb{R}^N \to \mathbb{C} \), we need to prove that
\[
\int_{\mathbb{R}^N} \sum_{m=1}^{N} \sum_{n=1}^{N} (\mathbf{\nabla}_{x,m} \mathbf{\nabla}_{x,n}|x - y|) \vartheta_m(x)\vartheta_n(x)dx \geq 0. \tag{70}
\]
We have that
\[
\mathbf{\nabla}_{x,m} \mathbf{\nabla}_{x,n}|x - y| = \frac{1}{|x - y|} \left( \frac{x_m - y_m(x_n - y_n)}{|x - y|^2} \right).
\]
Now, we can write \( x, y \) as the vector functions in \( \mathbb{R}^N \) by \( \bar{x} = (x_1, \ldots, x_N), \bar{y} = (y_1, \ldots, y_N) \) and \( \bar{\vartheta} = (\vartheta_1, \ldots, \vartheta_N) \) and one obtains
\[
\sum_{m=1}^{N} \sum_{n=1}^{N} (\mathbf{\nabla}_{x,m} \mathbf{\nabla}_{x,n}|\bar{x} - \bar{y}|) \vartheta_m(x)\vartheta_n(x)
= \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{1}{|\bar{x} - \bar{y}|} \left( \vartheta_m(x)\vartheta_n(x) - \frac{(x_m - y_m)(x_n - y_n)}{|\bar{x} - \bar{y}|^2} \vartheta_m(x)\vartheta_n(x) \right)
= \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{1}{|\bar{x} - \bar{y}|} \left( |\bar{\vartheta}(x)|^2 - \frac{(\bar{x} - \bar{y}) \cdot \bar{\vartheta}^2}{|\bar{x} - \bar{y}|^2} \right). \tag{71}
\]
From the Cauchy-Schwartz inequality, we have
\[
\frac{(\bar{x} - \bar{y}) \cdot \bar{\vartheta}}{|\bar{x} - \bar{y}|^2} \leq \frac{(x_1 - y_1)x_1 + \cdots + (x_N - y_N)x_N}{|\bar{x} - \bar{y}|^2}
\leq \frac{(x_1 - y_1)^2 + \cdots + (x_N - y_N)^2}{|\bar{x} - \bar{y}|^2}(\vartheta_1^2 + \cdots + \vartheta_N^2) \leq |\bar{\vartheta}|^2. \tag{72}
\]
Thus (70) follows from (71) and (72). Similarly, we infer that the matrix \( \{ \nabla_{y,m} \nabla_{y,n} |x - y| \}_{m,n=1}^N \) is also positive definite.

5.4. A4. Let \( f \in L^1(\mathbb{R}^N) \), we have the Fourier transform of \( f \) as
\[
\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \sum_{k=1}^N x_k \xi_k} dx.
\]
By integrating by parts, we have
\[
(-\Delta_x f)(\xi) = -\int_{\mathbb{R}^N} \Delta_x f(x) e^{-2\pi i \xi \cdot x} dx = C_\pi |\xi|^2 \hat{f}(\xi).
\]
We also have that \( |x|^{-1} = C_{N,\pi} |\xi|^{-(N-1)} \), \( N > 1 \), and with \( C_{N,\pi} = 4\pi^{N/2} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2}+1)} \). Then we may define the identity
\[
(-\Delta_x)^{-\frac{N-1}{2}} f(x) = C_{N,\pi} \int_{\mathbb{R}^N} \frac{f(y)}{|x - y|} dy.
\]
And if \( f \in C_c^\infty(\mathbb{R}^N) \) then \( f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) and
\[
\| \hat{f} \|_{L^2(\mathbb{R}^N)} = C_{N,\pi} \| f \|_{L^2(\mathbb{R}^N)}, \quad C_{N,\pi} = (2\pi)^N.
\]