LOG HOMOGENEOUS VARIETIES

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Abstract. Given a complete nonsingular algebraic variety $X$ and a divisor $D$ with normal crossings, we say that $X$ is log homogeneous with boundary $D$ if the logarithmic tangent bundle $T_X(-\log D)$ is generated by its global sections. Then the Albanese morphism $\alpha$ turns out to be a fibration with fibers being spherical (in particular, rational) varieties. It follows that all irreducible components of $D$ are nonsingular, and any partial intersection of them is irreducible. Also, the image of $X$ under the morphism $\sigma$ associated with $-K_X - D$ is a spherical variety, and the irreducible components of all fibers of $\sigma$ are equivariant compactifications of semiabelian varieties. Generalizing the Borel–Remmert structure theorem for homogeneous varieties, we show that the product morphism $\alpha \times \sigma$ is surjective, and the irreducible components of its fibers are toric varieties. We reduce the classification of log homogeneous varieties to a problem concerning automorphism groups of spherical varieties, that we solve under an additional assumption.

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0. Introduction

Consider a complete nonsingular algebraic variety $X$ over the field of complex numbers. If $X$ is homogeneous, then its structure is well understood by a classical result of Borel and Remmert: $X$ is the product of an abelian variety (the Albanese variety) and a flag variety (the basis of the “Tits fibration”).

More generally, one would like to classify the almost homogeneous varieties $X$, i.e., those where a connected algebraic group $G$ acts with an open orbit $X_0$. However, this central problem of equivariant geometry seems to be too general, and complete results are only known under assumptions of smallness (in various senses) for the boundary $X \setminus X_0$. We refer to [10] for an exposition of such classification results, in the setting of holomorphic actions on complex analytic spaces.

In another direction, there is a well-developed structure theory for certain classes of almost homogeneous varieties, where the acting group and the open orbit are prescribed: torus embeddings, spherical embeddings and, more generally, equivariant embeddings of a homogeneous space $X_0$ under a reductive group $G$ (see the recent survey [21]). A subclass of special interest is that of $G$-regular varieties in the sense of [3]; in loose words, their orbit structure is that of a nonsingular toric variety. Another important subclass consists of semiabelic varieties; in loose words again, they are toric bundles over abelian varieties. (Semiabelic varieties are introduced in [2], where their role in degenerations of abelian varieties is investigated.)

These restrictive assumptions on $G$ and $X_0$ are convenient in the setting of algebraic transformation groups, but somewhat unnatural from a geometric viewpoint. This motivates the search for a class of almost homogeneous varieties containing all regular varieties under a reductive group and all semiabelic varieties, having geometric significance and an accessible structure. Also, it is natural to impose that the boundary be a divisor with normal crossings.

In the present paper, we introduce the class of log homogeneous varieties and show that it fulfills the above requirements. Given a normal crossings divisor $D$ in $X$, we say that $X$ is log homogeneous with boundary $D$ if the associated logarithmic tangent bundle $T_X(−\log D)$ is generated by its global sections. It follows readily that $X$ is almost homogeneous under the connected automorphism group $G := \text{Aut}^0(X, D)$, with boundary being $D$. More generally, the $G$-orbits in $X$ are exactly the strata defined by $D$ (Prop. 2.1.2); in particular, their number is finite.

All $G$-regular varieties are log homogeneous, as shown in [4]. Semiabelic varieties satisfy a stronger property, namely, the bundle $T_X(−\log D)$ is trivial; we then say that $X$ is log parallelizable with boundary $D$. Conversely, log parallelizable varieties are semiabelic by a theorem of Winkelmann [22] which has been the starting point for our investigations. The class of log homogeneous varieties turns out to be stable under several natural operations: induction (Prop. 2.2.1), equivariant blow-ups (Prop. 2.3.2), étale covers (Prop. 2.4.4) and taking invariant subvarieties or irreducible components of fibers of morphisms (Cor. 3.2.2).
Any log homogeneous variety $X$ comes with two natural morphisms which turn out to play opposite roles. The first one is the Albanese map $\alpha : X \to \mathcal{A}(X)$; in our algebraic setting, this is the universal map to an abelian variety. We show that $\alpha$ is a fibration with fibers being log homogeneous under the maximal connected affine subgroup $G_{\text{aff}}$ of $G$, and spherical under any Levi subgroup of $G_{\text{aff}}$; this yields our main structure theorem (Thm. 3.2.1).

The second morphism is the Tits map $\tau$ to a Grassmannian, defined by the global sections of the logarithmic tangent bundle. Let $\sigma : X \to S(X)$ be the Stein factorization of $\tau$, so that $\sigma$ is the morphism associated with the globally generated divisor $-K_X - D$ (the determinant of $T_X(-\log D)$). We show that $S(X)$ is also a spherical variety under any Levi subgroup of $G_{\text{aff}}$, and the irreducible components of fibers of $\sigma$ are semiabelic varieties (Prop. 3.3.5). It follows that the product morphism $\alpha \times \sigma$ is surjective with fibers being finite unions of toric varieties (Thm. 3.3.3). This generalizes the Borel–Remmert theorem; note, however, that $S(X)$ may be singular, see Remark 3.5.3(ii).

Actually, these results hold in a slightly more general setting, namely, for a faithful action of a connected algebraic group $G$ on $X$ preserving a normal crossings divisor $D$ and such that the associated global vector fields generate $T_X(-\log D)$. Then $G$ has the same orbits as the full group $\text{Aut}^0(X, D)$, but may well be strictly contained in that group. For example, let $X$ be the projective $n$-space and $D$ the union of $m$ hyperplanes in general position. If $m \leq n$, then $X$ is log homogeneous with boundary $D$ under the action of $\text{Aut}^0(X, D)$. If, in addition, $n - m$ is odd and at least 3, then $X$ is still log homogeneous with boundary $D$ for a smaller group $G$, namely, the largest subgroup of $\text{Aut}^0(X, D)$ that acts on the intersection of the $m$ hyperplanes as the projective symplectic group.

This article is organized as follows. Section 1 contains a number of useful preliminary results on algebraic groups and their homogeneous spaces, that we did not see explicitly stated in the literature. We discuss algebraic analogues of the Albanese and Tits fibrations, which are classical tools in the theory of complex homogeneous manifolds. In our algebraic setting, the Borel–Remmert theorem admits a very short proof (Theorem 1.4.3).

In Section 2, we consider nonsingular almost homogeneous varieties, possibly not complete. We obtain a criterion for log homogeneity resp. parallelizability, as well as stability properties and simple local models. We also show that the Albanese map is a fibration; this yields an important reduction to the case where the acting group is affine. Finally, we obtain an algebraic version of Winkelmann’s theorem, again with a very short proof (Theorem 2.5.1).

The final Section 3 is devoted to complete log homogeneous varieties. After some preliminary lemmas about their relations to spherical varieties, we describe their structure and analyze the Tits map and its Stein factorization $\sigma$. We also show that the group of equivariant automorphisms is an extension of the Albanese variety by a diagonalizable group, so that its connected component is a semiabelian variety, acting on the general fibers of $\sigma$ with an open orbit (Props. 3.4.1 and 3.4.2). In fact, the
relative logarithmic tangent bundle of $\sigma$ is trivial, with fiber being the Lie algebra of the equivariant automorphism group (Prop. 3.3.5). Finally, we introduce the subclass of strongly log homogeneous varieties, closely related to regular varieties under a reductive group. We obtain a simple characterization of this subclass (Prop. 3.5.1) and we apply it to the classification of complete log homogeneous surfaces (Prop. 3.5.2).

Our arguments are purely algebraic; they involve results of the theory of algebraic transformation groups, and some basic notions borrowed from logarithmic birational geometry (for which we refer to [15]) and holomorphic transformation groups (see [1, 10]). Our results represent only the first steps in the understanding of log homogeneous varieties. They raise many open problems, among which the most important ones seem to be:

A) Characterize those homogeneous spaces that admit a log homogeneous completion.
B) Classify the triples $(L, Y, N)$, where $L$ is a connected reductive group, $Y$ is a complete nonsingular spherical $L$-variety, and $N$ is a unipotent group of automorphisms of $Y$, normalized by $L$ and such that $Y$ is log homogeneous under the semidirect product $LN$.

By Theorem 3.2.1, Problem A may be reduced to homogeneous spaces $X_0$ under an affine group $G$, and then a necessary condition is that $X_0$ be spherical under a Levi subgroup of $G$. We do not know if this condition is not sufficient.

Problem B is equivalent to the classification of complete log homogeneous varieties, by Theorem 3.2.1 again. It fits into the problem of describing automorphism groups of complete nonsingular spherical varieties, which is very much open in general (see [18, Sec. 3.4] for the toric case, and [5] for the regular case).

Another natural question concerns the case where the field of complex numbers is replaced with an algebraically closed field of positive characteristics. Here a number of results carry over with only small changes, but the Borel–Remmert theorem requires strong additional assumptions of separability. Also, some important ingredients of Section 3 (the existence of a Levi decomposition, the local structure of spherical varieties, Knop’s vanishing theorem for the logarithmic tangent sheaf) are not available in this setting.

**Notation and conventions.** Throughout this article, we consider algebraic varieties and algebraic groups over an algebraically closed field $k$ of characteristic zero. By a variety, we mean an integral separated scheme of finite type over $k$, and by an algebraic group, a group scheme of finite type over $k$. Morphisms are understood to be morphisms of varieties over $k$; points are understood to be $k$-rational. As a general reference for algebraic geometry, we use the book [8], and [7] for algebraic groups.

Any algebraic group $G$ is nonsingular, since $k$ has char. 0. Thus, the neutral component $G^0$, that is, the connected component of $G$ containing the identity element, is a nonsingular variety. Also, recall that being affine or linear are equivalent properties of $G$ (or $G^0$). But we shall consider algebraic groups that are not necessarily affine, e.g., abelian varieties for which we refer to the expository article [16].
1. Homogeneous varieties

1.1. Algebraic groups. Let $G$ be a connected algebraic group. By a result of Chevalley (see [6] for a modern proof), there exists a unique closed connected normal affine subgroup $G_{\text{aff}} \subseteq G$ such that the quotient group $G/G_{\text{aff}}$ is an abelian variety. We denote by $\alpha : G \to G/G_{\text{aff}} =: \mathcal{A}(G)$ the quotient morphism.

Recall also that any morphism from a connected affine algebraic group to an abelian variety is constant, see [16, Cor. 3.9]. It follows that $G_{\text{aff}}$ contains every closed connected affine subgroup of $G$. Moreover, every morphism (of varieties) $f : G \to A$, where $A$ is an abelian variety, factors uniquely as $\varphi \circ \alpha$, where $\varphi : \mathcal{A}(G) \to A$ is a morphism. By [16, Cor. 2.2], $\varphi$ is the composition of a translation and a group homomorphism. So $\alpha$ is the Albanese morphism of the variety $G$, as defined in [19]. In particular, $\mathcal{A}(G)$ depends only on $G$ regarded as a variety.

We record the following easy result, where we denote the center of $G$ by $C(G)$, or simply by $C$ if this yields no confusion. Also, recall that an isogeny is a surjective homomorphism of algebraic groups, with finite kernel.

Lemma 1.1.1. The algebraic groups $G/C$ and $G/C^0$ are both affine, and

\[ G = G_{\text{aff}} C^0. \]

Moreover, $C^0_{\text{aff}}$ is the neutral component of both groups $G_{\text{aff}} \cap C$ and $G_{\text{aff}} \cap C^0$, and the natural map $\mathcal{A}(C^0) \to \mathcal{A}(G)$ is an isogeny.

Proof. Via the adjoint representation of $G$ in its Lie algebra $g$, the quotient group $G/C$ is isomorphic to a closed subgroup of $\text{GL}(g)$. In particular, $G/C$ is affine.

The natural map $G/C^0 \to G/C$ is the quotient by the finite group $C/C^0$, and hence is an affine morphism. Thus, $G/C^0$ is affine as well.

Note that $G_{\text{aff}} C^0$ is a closed normal subgroup of $G$, and the corresponding quotient $G/G_{\text{aff}} C^0 \cong (G/C^0)/(G_{\text{aff}} C^0/C^0)$ is affine. On the other hand, $G/G_{\text{aff}} C^0$ is a quotient of the abelian variety $\mathcal{A}(G)$, and hence is complete. This yields \((1.1.1)\).

The remaining assertions follow readily from the definitions. \qed

1.2. The Albanese fibration. Let $X$ be a $G$-variety and $X_0 \subseteq X$ a $G$-orbit. Choosing a base point $x \in X_0$, we identify the homogeneous variety $X_0 = G \cdot x$ with the homogeneous space $G/H$, where $H$ denotes the isotropy group $G_x$.

Since $G/G_{\text{aff}}$ is an abelian variety, the product

\[ I(X_0) := G_{\text{aff}} H \]

is a closed normal subgroup of $G$, independent of the choice of the base point $x$, and the quotient $G/I(X_0)$ is an abelian variety as well. Moreover, the natural map

\[ \alpha : X_0 = G/H \to G/G_{\text{aff}} H = X_0/G_{\text{aff}} =: \mathcal{A}(X_0) \]

is the Albanese morphism of $X_0$. Note that $\alpha$ is a $G$-equivariant fibration with fiber

\[ G_{\text{aff}} H/H \cong G_{\text{aff}}/(G_{\text{aff}} \cap H), \]
and this $G_{\text{aff}}$-homogeneous variety is also independent of the choice of $x$.

Further properties of $H$ and $I(X_0)$ are gathered in the following:

**Lemma 1.2.1.** Let $X$ be a variety on which $G$ acts faithfully, and $X_0 = G \cdot x \subseteq X$ a $G$-orbit. Then the isotropy group $H = G_x$ is affine; equivalently, $H^0 \subseteq G_{\text{aff}}$. Moreover, there exists a finite subgroup $F \subseteq C^0$ such that

\[ I(X_0) = G_{\text{aff}} F. \]

**Proof.** Note that $H$ acts on the local ring $\mathcal{O}_x$ and on its quotients $\mathcal{O}_x / \mathfrak{m}_x^n$ by powers of the maximal ideal; these quotients are finite-dimensional $k$-vector spaces. Let $H_n$ be the kernel of the resulting homomorphism $H \to \text{GL}(\mathcal{O}_x / \mathfrak{m}_x^n)$. Then the $H_n$ form a decreasing sequence of closed subgroups of $H$, and their intersection acts trivially on $\mathcal{O}_x$. Since the $G$-action is faithful, this intersection is trivial, and hence so is $H_n$ for $n \gg 0$. Thus, the algebraic group $H$ is affine. As a consequence, $I(X_0)^0 = G_{\text{aff}}$, and $I(X_0)$ is affine as well. Moreover,

\[ I(X_0) = G_{\text{aff}} (C^0 \cap I(X_0)) \]

by (1.1.1). Thus, $C^0 \cap I(X_0)$ is a commutative affine algebraic group: it is the direct product of a connected algebraic group (contained in $G_{\text{aff}}$) with a finite group. □

### 1.3. The Tits fibration.

Consider again a $G$-variety $X$. The tangent sheaf

\[ T_X = \text{Der}_k(\mathcal{O}_X) \]

is a $G$-linearized sheaf on $X$, equipped with a $G$-equivariant map

\[ \text{op}_X : \mathfrak{g} \to \Gamma(X, T_X). \]

This yields a morphism of $G$-linearized sheaves

\[ \text{op}_X : \mathcal{O}_X \otimes \mathfrak{g} \to T_X. \]

Clearly, $\text{op}_X$ is surjective if and only if $X$ is $G$-homogeneous. Under this assumption, the kernel of $\text{op}_X$ is a $G$-linearized locally free sheaf, and its fiber at any point $x$ is the isotropy Lie subalgebra $\mathfrak{g}_x \subseteq \mathfrak{g}$. Likewise, the fiber at $x$ of the locally free $G$-linearized sheaf $T_X$ is the quotient $\mathfrak{g} / \mathfrak{g}_x$. In particular, $\text{op}_X$ is an isomorphism if and only if $X$ is the quotient of $G$ by a finite subgroup; we then say that $X$ is $G$-parallelizable.

If $X$ is homogeneous under a faithful action of $G$, then $\mathfrak{g}_x$ is contained in the Lie algebra $\mathfrak{g}_{\text{aff}}$ of $G_{\text{aff}}$ by Lemma 1.2.1. So we obtain a morphism

\[ \tau : X \to \mathcal{L}, \quad x \mapsto \mathfrak{g}_x, \]

where $\mathcal{L}$ denotes the scheme of Lie subalgebras of $\mathfrak{g}_{\text{aff}}$ (alternatively, we may consider the Grassmann variety of subspaces of $\mathfrak{g}_{\text{aff}}$). Clearly, $\tau$ is $G$-equivariant, where $G$ acts on $\mathcal{L}$ via its adjoint action on $\mathfrak{g}_{\text{aff}}$; in particular, $\tau$ is invariant under $C$. Thus, the image of $\tau$ is a unique $G_{\text{aff}}$-orbit that we denote by $\mathcal{L}(X)$, and $\tau : X \to \mathcal{L}(X)$ is a $G$-equivariant fibration, called the *Tits fibration*.

Choosing a base point $x \in X$ with isotropy group $H$, we obtain with obvious notation

\[ \mathcal{L}(X) = G \cdot \mathfrak{h} \cong G/N_G(\mathfrak{h}) = G/N_G(H^0) \cong G_{\text{aff}} / N_{G_{\text{aff}}}(H^0) \cong G_{\text{aff}} / N_{G_{\text{aff}}}(\mathfrak{h}), \]
where the second isomorphism follows from Lemma 1.1.1. Thus, the fiber of $\tau$ at $x$ is the homogeneous space $N_G(\mathfrak{h})/H$, quotient of the (not necessarily connected) algebraic group $N_G(\mathfrak{h})/H^0$ by the finite subgroup $H/H^0$. In particular, all connected components of fibers are parallelizable varieties.

**Remarks 1.3.1.**

(i) The Tits fibration may be defined, more generally, for a homogeneous $G$-variety $X$ on which the action is not necessarily faithful: just replace $\mathfrak{g}_x$ with $\mathfrak{g}_{\text{aff},x}$ (the isotropy Lie algebra of $x$ in $\mathfrak{g}_{\text{aff}}$). But several statements take a much simpler form in the setting of faithful actions.

(ii) Unlike the Albanese fibration, the Tits fibration may depend on the $G$-action on $X$. Consider, for example, a connected affine group $G$ regarded as a homogeneous $G$-variety under its action via left multiplication; then the Tits fibration is trivial. But this fibration is nontrivial if $G$ is regarded as a homogeneous $G \times G$-variety under its action via left and right multiplication, and $G$ is noncommutative.

(iii) The Tits fibration yields an exact sequence of $G$-linearized locally free sheaves on $G/H$

$$0 \to T_\tau \to T_X \to \tau^* T_L(X) \to 0,$$

where $T_\tau$ denotes the relative tangent sheaf. By taking fibers at the base point, this corresponds to an exact sequence of $H$-modules

$$0 \to n_\mathfrak{g}(\mathfrak{h})/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/n_\mathfrak{g}(\mathfrak{h}) \to 0.$$

Note that $H$ acts on $n_\mathfrak{g}(\mathfrak{h})/\mathfrak{h}$ via its finite quotient $H/H^0$. Thus, the pull-back of $T_\tau$ under the finite cover $G/H^0 \to G/H$ is trivial.

(iv) Likewise, the Albanese fibration yields an exact sequence of $G$-linearized locally free sheaves

$$0 \to T_\alpha \to T_X \to \alpha^* T_A(X) \to 0$$

corresponding to the exact sequence of $H$-modules

$$0 \to \mathfrak{g}_{\text{aff}}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{g}_{\text{aff}} \to 0.$$

Note that the $H$-module $\mathfrak{g}/\mathfrak{g}_{\text{aff}}$ is trivial; this also follows from the triviality of the tangent sheaf of the abelian variety $A(X)$.

1.4. **The product fibration.** We now consider the product map

$$\alpha \times \tau : X \to A(X) \times L(X),$$

where $X \cong G/H$.

**Lemma 1.4.1.** With the above notation, $\alpha \times \tau$ is surjective; in particular, it is a $G$-equivariant fibration. Its fiber at the base point of $G/H$ is the homogeneous space $N_{G_{\text{aff}}}(\mathfrak{h})/(G_{\text{aff}} \cap H)$, quotient of the affine algebraic group $N_{G_{\text{aff}}}(\mathfrak{h})/H^0$ by the finite subgroup $(G_{\text{aff}} \cap H)/H^0$.

In particular, the morphism $\alpha \times \tau$ is affine, and the connected components of its fibers are parallelizable varieties.
Thus, the fiber of $\alpha \times \tau$ equals $N_{G_{\text{aff}}} (h) / H \cong N_{G_{\text{aff}}} (h) / (G_{\text{aff}} \cap H)$.

**Remark 1.4.2.** As in Remark 1.3.1(iii), the relative tangent sheaf of $\alpha \times \tau$ corresponds to the $H / H^0$-module $n_{G_{\text{aff}}} (h) / h$. Thus, the pull-back of this sheaf under the finite cover $G / H^0 \to G / H$ is trivial.

The above lemma yields an algebraic version of the Borel–Remmert structure theorem for compact homogeneous Kähler manifolds, see e.g. [1, §3.9]:

**Theorem 1.4.3.** For a complete homogeneous $G$-variety $X$, the map $\alpha \times \tau$ is an isomorphism.

**Proof.** We may assume that $G$ acts faithfully on $X = G / H$. Since the fiber of $\alpha$ is $G_{\text{aff}} / (G_{\text{aff}} \cap H)$, then $G_{\text{aff}} \cap H$ is a parabolic subgroup of $G_{\text{aff}}$. In particular, $G_{\text{aff}} \cap H$ is connected and equals its normalizer in $G_{\text{aff}}$. It follows that

$$H^0 = G_{\text{aff}} \cap H = N_{G_{\text{aff}}} (H^0) = N_{G_{\text{aff}}} (h)$$

so that the morphism $\alpha \times \tau$ is bijective by Lemma 1.4.1. Since $\mathcal{A}(X) \times \mathcal{L}(X)$ is nonsingular, $\alpha \times \tau$ is an isomorphism.

## 2. Log homogeneous varieties

### 2.1. Definitions and basic properties.** Consider a pair $(X, D)$, where $X$ is a nonsingular variety and $D \subset X$ is a divisor with normal crossings. That is, $D$ is an effective divisor and its local equation at an arbitrary point $x \in X$ decomposes in the completed local ring $\hat{O}_x$ into a product $t_1 \cdots t_r$, where $t_1, \ldots, t_r$ form part of a regular system of parameters $(t_1, \ldots, t_n)$ of $\hat{O}_x$. Let

$$\mathcal{T}_X (- \log D) \subseteq \mathcal{T}_X = \text{Der}_k (\mathcal{O}_X)$$

be the subsheaf consisting of those derivations that preserve the ideal sheaf $\mathcal{O}_X (-D)$. One easily checks that the logarithmic tangent sheaf $\mathcal{T}_X (- \log D)$ is a locally free sheaf of Lie subalgebras of $\mathcal{T}_X$, having the same restriction to $X \setminus D$, and hence the same rank $n = \dim (X)$. The dual of $\mathcal{T}_X (- \log D)$ is the sheaf $\Omega_X^1 (\log D)$ of logarithmic differential forms, that is, of differential 1-forms on $X \setminus D$ having at most simple poles along $D$. The top exterior power $\wedge^n \mathcal{T}_X (- \log D)$ is the invertible sheaf $\mathcal{O}_X (-K_X - D)$, where $K_X$ denotes the canonical divisor.

If $D$ is defined at $x$ by the equation $t_1 \cdots t_r = 0$ as above, then a local basis of $\mathcal{T}_X (- \log D)$ (after localization and completion at $x$) consists of

$$t_1 \partial_1, \ldots, t_r \partial_r, \partial_{r+1}, \ldots, \partial_n,$$

where $(\partial_1, \ldots, \partial_n)$ is the local basis of $\mathcal{T}_X$ dual to the local basis $(dt_1, \ldots, dt_n)$ of $\Omega_X^1$. 
Next, assume that a connected algebraic group $G$ acts on $X$ and preserves $D$; we say that $(X, D)$ is a $G$-pair. Then the sheaf $\mathcal{T}_X(-\log D)$ is $G$-linearized and the map $\text{op}_X$ of (1.3.1) factors through a map
\begin{equation}
\text{op}_{X,D} : \mathfrak{g} \to \Gamma(X, \mathcal{T}_X(-\log D)).
\end{equation}
This defines a morphism of $G$-linearized sheaves
\begin{equation}
\text{op}_{X,D} : \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X(-\log D).
\end{equation}
We say that the pair $(X, D)$ is homogeneous (resp. parallelizable) under $G$, if $\text{op}_{X,D}$ is surjective (resp. an isomorphism).

More generally, given a locally closed $G$-stable subvariety $S \subseteq X$, we say that $(X, D)$ is homogeneous (resp. parallelizable) under $G$ along $S$, if $\text{op}_{X,D}$ is surjective (resp. an isomorphism) at all points of $S$.

**Remarks 2.1.1.** (i) For any homogeneous $G$-pair $(X, D)$, the complement $X_0 := X \setminus D$ is a unique $G$-orbit, since $\text{op}_{X_0}$ is surjective. In particular, the $G$-variety $X$ is almost homogeneous, and $D$ is uniquely determined by the pair $(X, G)$. We then say that $X$ is a log homogeneous $G$-variety with boundary $D$. Log parallelizable $G$-varieties are defined similarly.

(ii) If $X$ is complete, then the connected automorphism group $\text{Aut}^0(X)$ is algebraic with Lie algebra being $\Gamma(X, \mathcal{T}_X)$, see e.g. [17]. It follows that $\text{Aut}^0(X, D)$ is algebraic as well, with Lie algebra being $\Gamma(X, \mathcal{T}_X(-\log D))$. Thus, $(X, D)$ is homogeneous (resp. parallelizable) under some group $G$ if and only if the sheaf $\mathcal{T}_X(-\log D)$ is generated by its global sections (resp. is trivial). We then say that the pair $(X, D)$ is homogeneous (resp. parallelizable) without specifying the group $G$.

(iii) The log homogeneous $G$-pairs $(X, D)$, where $X$ is a curve and $G$ acts faithfully, are easily classified. One obtains the following list of triples $(X, D, G)$:

- (parallelizable) $(G, \emptyset, G)$, where $G$ is a connected algebraic group of dimension 1, i.e., the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, or an elliptic curve.
- (homogeneous) $(\mathbb{P}^1, \emptyset, \text{PGL}_2)$ and $(\mathbb{A}^1, \emptyset, B)$, where $B$ denotes the automorphism group of the affine line, i.e., the Borel subgroup of $\text{PGL}_2$ that fixes $\infty \in \mathbb{P}^1$.
- (log parallelizable) $(\mathbb{A}^1, \{0\}, \mathbb{G}_m)$ and $(\mathbb{P}^1, \{0, \infty\}, \mathbb{G}_m)$.
- (log homogeneous) $(\mathbb{P}^1, \{\infty\}, B)$.

The complete log homogeneous surfaces will be listed in Proposition 3.5.2.

We shall obtain a criterion for an arbitrary $G$-pair $(X, D)$ to be homogeneous or parallelizable; to state it, we first recall the definition of a stratification of $X$ associated with $D$. Let $X^0 := X$, $X^1 := D$ and define inductively $X^r$ to be the singular locus of $X^{r-1}$ for $r \geq 2$. Then the strata are the connected components of the locally closed subvarieties $X^r \setminus X^{r+1}$, where $r = 0, 1, \ldots$ Every stratum is nonsingular and $G$-stable; the open stratum is $X \setminus D$.

Next, we analyze the normal space to a stratum $S$ at a point $x$. Let again $t_1 \cdots t_r = 0$ be a local equation of $D$ at $x$. Then the ideal sheaf of $S$ in $X$ is generated (after localization and completion at $x$) by $t_1, \ldots, t_r$; in other words, $S$ is locally the complete intersection of all branches of $D$. As a consequence, the normal
space $N_{S/X,x}$ admits a canonical splitting into a direct sum of lines $L_1, \ldots, L_r$; the corresponding hyperplanes $\bigoplus_{j \neq i} L_j$ are the normal spaces to these branches. The isotropy group $G_x$ preserves $D$ and permutes its branches at $x$, and hence the lines $L_1, \ldots, L_r$; the connected component $G_0^x$ stabilizes each line. Thus, the representation of $G_x$ in $N_{S/X,x}$ yields a homomorphism

$$
\rho_x : G_0^x \to \text{GL}(L_1) \times \cdots \times \text{GL}(L_r) \cong G_r^r,
$$

where $G_r^r$ denotes the product of $r$ copies of the multiplicative group.

We may now state our criterion:

**Proposition 2.1.2.** The following conditions are equivalent for a $G$-pair $(X, D)$ and a stratum $S$:

(i) $(X, D)$ is homogeneous (resp. parallelizable) under $G$ along $S$.

(ii) $S$ is a unique $G$-orbit and for any $x \in S$, the homomorphism $\rho_x$ is surjective (resp. an isogeny).

If one of these conditions holds, then the sequence

$$
(2.1.4) \quad 0 \longrightarrow \mathfrak{g}_{(x)} \longrightarrow \mathfrak{g} \overset{\text{op}_{x,X,D}}{\longrightarrow} T_xX(-\log D) \longrightarrow 0
$$

is exact for any $x \in S$, where $\mathfrak{g}_{(x)}$ denotes the kernel of the representation of the isotropy Lie algebra $\mathfrak{g}_x$ in the normal space $N_{S/X,x}$ and $T_xX(-\log D)$ denotes the fiber of $T_X(-\log D)$ at $x$.

**Proof.** Clearly, $T_X(-\log D)$ preserves the ideal sheaf of $S$ in $X$. This yields a morphism $T_X(-\log D)|_S \to T_S$ and, in turn, a linear map

$$
p : T_xX(-\log D) \to T_xS
$$

between fibers. Given a regular system of parameters $(t_1, \ldots, t_n)$ as above, the map $p$ is just the projection of the space $kt_1 \partial_1 \oplus \cdots \oplus kt_r \partial_r \oplus k \partial_{r+1} \oplus \cdots \oplus k \partial_n$ onto its subspace $k \partial_{r+1} \oplus \cdots \oplus k \partial_n$. Thus, $p$ fits into an exact sequence

$$
0 \to \bigoplus_{i=1}^r kt_i \partial_i \to T_xX(-\log D) \to T_xS \to 0.
$$

Moreover, composing the map $\text{op}_{x,X,D} : \mathfrak{g} \to T_xX(-\log D)$ with $p$ yields the map $\text{op}_{x,S} : \mathfrak{g} \to T_xS$ which factors through an injective map

$$
i : \mathfrak{g}/\mathfrak{g}_x \to T_xS.
$$

So we obtain a commutative diagram of exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathfrak{g}_x & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{g}_x & \longrightarrow & 0 \\
& & \downarrow & & \downarrow i & & \downarrow i & & \\
0 & \longrightarrow & \bigoplus_{i=1}^r kt_i \partial_i & \longrightarrow & T_xX(-\log D) & \overset{p}{\longrightarrow} & T_xS & \longrightarrow & 0,
\end{array}
$$

where the left vertical map may be identified with the differential $d\rho_x : \mathfrak{g}_x \to k^r$. As a consequence, the surjectivity of the middle vertical map is equivalent to the
surjectivity of both maps $d\rho_x$ and $\text{op}_{x,S}$. If the latter condition holds, then we obtain an exact sequence

$$0 \to \ker(d\rho_x) \to \mathfrak{g} \to T_xX(-\log D) \to 0.$$ 

This implies all assertions. \hfill \Box

A direct consequence of this criterion is the following:

**Corollary 2.1.3.** Let $(X, D)$ be a homogeneous pair under $G$ acting faithfully. Then:

(i) The $G$-orbits in $X$ are exactly the strata; in particular, their number is finite.

(ii) The Tits fibration of the open orbit $X \setminus D$ extends to a $G$-equivariant morphism

$$X \to \mathcal{L}, \quad x \mapsto \mathfrak{g}(x)$$

which is invariant under $C$.

Next, we compare our notion of log homogeneity to the (earlier) notions of pseudofreeness and regularity. Recall from [12, Sec. 2] that a nonsingular $G$-variety $X$ is pseudofree if the image of $\text{op}_X$ (i.e., the coherent subsheaf of $T_X$ generated by the image of $\mathfrak{g}$) is locally free. Clearly, any log homogeneous variety is pseudofree, but the converse does not hold in general. For example, the projective line where the additive group acts by affine translations is pseudofree but not log homogeneous.

Also, recall from [3] that our nonsingular $G$-variety $X$ is $G$-regular if it satisfies the following three conditions:

(i) $X$ contains an open $G$-orbit whose complement is a union of irreducible divisors with normal crossings (the boundary divisors).

(ii) Any $G$-orbit closure in $X$ is the transversal intersection of those boundary divisors that contain it.

(iii) For any $x \in X$, the normal space $N_{G,x/X,x}$ contains an open orbit of $G_x$.

**Corollary 2.1.4.** Every regular $G$-variety is log homogeneous.

**Proof.** By (i), the complement of the open orbit is a divisor with normal crossings. Moreover, by (ii), the strata are exactly the $G$-orbits, and each normal space $N_{G,x/X,x} \cong \mathbb{A}^r$ is a direct sum of lines $L_1, \ldots, L_r$, where each $L_i$ is preserved under $G_x$ (as $L_i$ is the intersection of normal spaces to certain boundary divisors). Thus, $G_x$ acts on $N_{G,x/X,x}$ via a homomorphism

$$\rho_x : G_x \to \mathbb{G}_m^r$$

which is surjective by (iii). So $X$ is log homogeneous under $G$ by Proposition 2.1.2. \hfill \Box

The converse of this corollary also fails in general. Indeed, all $G$-orbit closures in a regular $G$-variety are also $G$-regular, and hence nonsingular. But there exist log parallelizable varieties in which most orbit closures are singular, see Example 2.2.2.

However, any complete log homogeneous variety $X$ under a reductive group $G$ is regular by [4, Prop. 2.5]. This will be generalized to certain non-complete varieties $X$ (and reductive groups $G$) in Lemma 3.1.2, and then to arbitrary groups $G$ (and complete varieties $X$) in Corollary 3.2.2.
2.2. **Induced actions.** Recall that a $G$-variety $X$ is *induced* from a homogeneous space $G/H$ if there exists a $G$-equivariant morphism $f : X \to G/H$. Equivalently, 
$$X \cong G \times^H Y,$$
where $Y$ denotes the fiber of $f$ at the base point, and $G \times^H Y$ stands for the quotient of $G \times Y$ by the action of $H$ via $h \cdot (g, y) = (gh^{-1}, h \cdot y)$. We now show that induction preserves homogeneity or parallelizability:

**Proposition 2.2.1.** Let $X$ be a $G$-variety and $D \subset X$ a reduced $G$-stable divisor. Assume that $X = G \times^H Y$ for some closed subgroup $H \subseteq G$ and some closed $H$-stable subvariety $Y \subseteq X$, and let $E := D \cap Y$ (this is a reduced $H$-stable divisor in $Y$). Then the following conditions are equivalent:

(i) $(X, D)$ is a homogeneous (resp. parallelizable) $G$-pair.
(ii) $(Y, E)$ is a homogeneous (resp. parallelizable) $H^0$-pair.

**Proof.** Clearly, $X$ is nonsingular if and only if so is $Y$. Consider then the morphism 
$$f : G \times Y \to X, \quad (g, y) \mapsto g \cdot y.$$ 

Since $f$ is smooth, $D$ has normal crossings in $X$ if and only if so has $f^{-1}(D)$ in $G \times Y$. As $f^{-1}(D) = G \times E$, the latter condition means that $E$ has normal crossings in $Y$.

Next, consider the exact sequence of $H$-linearized sheaves on $Y$:
$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0.$$ 

The normal sheaf $N_{Y/X}$ is isomorphic as a $H$-linearized sheaf to $O_Y \otimes g/h$, where $h$ denotes the Lie algebra of $H$, and $g/h$ is regarded as a $H$-module. The sequence
$$0 \to T_Y(- \log E) \to T_X(- \log D)|_Y \to N_{Y/X} \to 0$$
is also exact, since the composition $O_Y \otimes g \to T_X(- \log D)|_Y \to O_Y \otimes g/h$ is surjective. This yields a commutative diagram of exact sequences

$$
\begin{array}{ccccccc}
0 & \to & O_Y \otimes h & \to & O_Y \otimes g & \to & O_Y \otimes g/h & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & T_Y(\log E) & \to & T_X(- \log D)|_Y & \to & N_{Y/X} & \to & 0,
\end{array}
$$

where the right vertical map is an isomorphism. Thus, the surjectivity (resp. bijectivity) of the middle vertical map is equivalent to that of the left vertical map. □

As a first application of this lemma, we construct a log parallelizable variety which is not regular:

**Example 2.2.2.** Consider the group $G := \text{GL}_n$, where $n \geq 2$, and the subgroup $H$ of matrices having exactly one non-zero coefficient on each line and each column; in other words, $H$ is the semidirect product of the maximal torus $T$ of diagonal invertible matrices with the symmetric group $S_n$ of permutation matrices, where $S_n$ acts on $T$ by permuting the diagonal entries. Let $Y$ be the affine $n$-space on which $H$ acts linearly via its standard representation. Then $Y$ is log parallelizable under $T = H^0$, and its boundary $E$ (the union of the coordinate hyperplanes $E_1, \ldots, E_n$)
is $H$-stable. The $H$-orbit closures in $E$ are exactly the subsets $H \cdot (E_1 \cap \cdots \cap E_r)$, where $r = 1, \ldots, n$. In particular, $E$ is the closure of a unique $H$-orbit.

Since the $H$-action on $Y$ extends to a $G$-action, the induced variety $X := G \times^H Y$ is just the product $G/H \times^H Y$ on which $G$ acts diagonally; since $H$ is reductive, $X$ is affine. By the above lemma, $X$ is log parallelizable under $G$; its boundary $D := G \times^H E$ is a $G$-orbit closure, and hence is irreducible. More generally, the $G$-orbit closures in $X$ are the subsets

$$X^r := G \cdot (E_1 \cap \cdots \cap E_r),$$

where $r = 1, \ldots, n$, together with $X$.

We claim that the singular locus of each $X^r$ is $X^{r+1}$. This claim is a consequence of Corollary 2.1.3 or may be seen directly as follows. Let $H_r$ be the stabilizer in $H$ of the subset $E_1 \cap \cdots \cap E_r$, so that $H_r = TS_{n-r}$. Then the natural map

$$H \times^{H_r} (E_1 \cap \cdots \cap E_r) \to H \cdot (E_1 \cap \cdots \cap E_r)$$

is finite and restricts to an isomorphism over the complement of $H \cdot (E_1 \cap \cdots \cap E_{r+1})$, but over no larger open subset. It follows that the natural map

$$G \times^{H_r} (E_1 \cap \cdots \cap E_r) \to X^r$$

is finite and birational with exceptional set $X^{r+1}$. Since $X^r \setminus X^{r+1}$ is a unique orbit, this implies our claim.

As a consequence, $X$ is not $G$-regular. This may also be seen by considering the base point $x := H \cdot (1,0)$ of the closed orbit. Then $G_x = H$ acts on the normal space $N_{G,x/X,x} \cong \mathbb{A}^n$ via its standard representation; in particular, $G_x$ permutes transitively the coordinate lines $L_1, \ldots, L_n$.

As another application of Proposition 2.2.1 we show that homogeneity and parallelizability are also preserved by taking general fibers:

**Corollary 2.2.3.** Consider a homogeneous (resp. parallelizable) $G$-pair $(X, D)$, a variety $X'$, and a proper morphism $f : X \to X'$. Let $Y$ be a connected component of the fiber of $f$ at a point of the open orbit $X_0 = X \setminus D$. Then $(Y, D \cap Y)$ is a homogeneous (resp. parallelizable) pair.

**Proof.** Using the Stein factorization, we may assume that the natural map $O_{X'} \to f_* O_X$ is an isomorphism; then all the fibers of $f$ are connected, and the general fibers are irreducible.

By a result of Blanchard (see [1 Sec. 2.4] for a proof in the setting of complex geometry, which may be adapted readily to our algebraic setting), the $G$-action on $X$ descends to an action on $X'$ such that $f$ is equivariant. Let $X'_0 \cong G/H'$ be the open $G$-orbit in $X'$. Then $f^{-1}(X'_0) = f^{-1}(X_0)$ is an open $G$-stable subset of $X$, equivariantly isomorphic to $G \times^{H'} Y$. In particular, all fibers over $X'_0$ are isomorphic to $Y$, and hence the latter is a nonsingular variety. So the assertions follow from Proposition 2.2.1.

2.3. Local models. A simple example of a log parallelizable variety is the affine space $\mathbb{A}^r$ with its standard action of $G_m^r$; one may also construct induced versions $G \times^H \mathbb{A}^r$, where $H$ is a subtorus of $G$, acting on $\mathbb{A}^r$ via an isogeny to $G_m^r$. We now show that this construction yields local models for all log homogeneous varieties:
Proposition 2.3.1. Let \((X, D)\) be a homogeneous \(G\)-pair. Choose \(x \in X\) and let \(\rho_x : G^0_x \to \mathbb{G}_m^r\) be as in (2.1.3). Then there exist a subtorus \(H \subseteq G^0_x\) and a \(H\)-stable locally closed subvariety \(Y \subseteq X\) such that:

(i) The restriction of \(\rho_x\) to \(H\) is an isogeny to \(\mathbb{G}_m^r\).

(ii) \(Y \cong \mathbb{A}^r\) on which \(H\) acts via the restriction \(\rho_x : H \to \mathbb{G}_m^r\) and the standard action of \(\mathbb{G}_m^r\) on \(\mathbb{A}^r\).

(iii) The natural map \(p : G \times^H Y \to X\) is smooth and \(p^{-1}(D) \cong G \times^H E\), where \(E \subseteq \mathbb{A}^r\) denotes the union of all coordinate hyperplanes.

Moreover, \((X, D)\) is parallelizable if and only if \(p\) is étale.

Proof. We use some of the ingredients of the Luna slice theorem [14]. Since \(\rho_x\) is surjective, we may choose a subtorus \(H \subseteq G^0_x\) satisfying (i). Then there exists a \(H\)-stable decomposition

\[T_x X = T_x(G \cdot x) \oplus N = \mathfrak{g} \cdot x \oplus N,\]

where the \(H\)-module \(N\) is isomorphic to \(N_{G \cdot x/X, x}\). Moreover, the \(H\)-fixed point \(x\) admits an affine \(H\)-stable neighborhood \(X_x \subseteq X\) together with an \(H\)-equivariant map

\[\varphi : X_x \to T_x X\]

such that the differential of \(\varphi\) at \(x\) is the identity map of \(T_x X\). In particular, \(\varphi\) is étale at \(x\).

Let \(Y := \varphi^{-1}(N)\). This is an affine \(H\)-stable locally closed subvariety of \(X\), containing \(x\) and nonsingular at that point. Moreover, \(T_x Y \cong N_{G \cdot x/X, x}\) as \(H\)-modules.

By the graded Nakayama lemma, it follows that \(Y\) satisfies (ii). For (iii), note that \(p\) is smooth at the point \(H \cdot (1, x)\) of \(G \times^H Y\), since \(Y\) is a slice to the orbit \(G \cdot x\) at \(x\). Thus, \(p\) is smooth in a \(G\)-stable neighborhood of \(H \cdot (1, x)\), which must be the whole \(G \times^H Y\).

Finally, \(p\) is étale if and only if \(H = G^0_x\); equivalently, \(\rho_x\) is an isogeny. By Proposition 2.1.2 this means that \((X, D)\) is parallelizable.

From the above proposition, we deduce that homogeneity and parallelizability are preserved under equivariant blowing up:

Proposition 2.3.2. (i) Let \((X, D)\) be a homogeneous (resp. parallelizable) \(G\)-pair, \(X'\) a nonsingular \(G\)-variety, \(f : X' \to X\) a birational \(G\)-equivariant morphism, and \(D'\) the reduced inverse image of \(D\). Then \((X', D')\) is a homogeneous (resp. parallelizable) \(G\)-pair.

(ii) As a partial converse, given two \(G\)-pairs \((X, D)\), \((X', D')\) and a surjective birational \(G\)-equivariant morphism \(f : X' \to X\) such that \(D'\) is the reduced inverse image of \(D\), if \((X', D')\) is \(G\)-parallelizable, then so is \((X, D)\).

Proof. (i) Choose \(x \in X\) and let \(H, Y\) be as in Proposition 2.3.1. Form the cartesian square

\[
\begin{array}{ccc}
W & \xrightarrow{\varphi} & G \times^H Y \\
\downarrow{\pi} & & \downarrow{p} \\
X' & \xrightarrow{f} & X
\end{array}
\]
Then $W = G \times H Y'$, where $Y' := f^{-1}(Y)$. Since $p$ is smooth, then so is $\pi$. Thus, $Y'$ is a nonsingular variety; it is also toric under $G'_{m}$. Hence $Y'$ is log parallelizable by [18] Prop. 3.1; its boundary $\partial Y'$ equals the reduced inverse image $f^{-1}(D \cap Y)_{\text{red}}$. It follows that the $G$-variety $W$ is log parallelizable with boundary \[
abla W = G \times H \partial Y' = \varphi^{-1} p^{-1}(D)_{\text{red}}.
\]
Since $\pi$ is smooth and $\partial W = \pi^{-1}(D')$, the divisor $D'$ has normal crossings. Moreover, the surjective map $d\pi : T_{W} \to \pi^{*}T_{X}$ restricts to a surjective map \[
abla_{W}(- \log \partial W) \to \pi^{*}T_{X'}(- \log D').\]
It follows that $(X', D')$ is $G$-homogeneous (resp. parallelizable if so is $(X, D)$).

(ii) The assumption implies that the map $op_{X,D}$ of (2.1.2) is an isomorphism in codimension 1. Since this is a map between locally free sheaves of the same rank, it is an isomorphism. \[\square\]

Remark 2.3.3. The above statement (ii) does not extend to homogeneous pairs. For example, let $G := \text{PGL}_{n+1}$ acting diagonally on $X := \mathbb{P}^{n} \times \mathbb{I}_{n}$, where $\mathbb{I}_{n}$ denotes the incidence variety consisting of pairs $(p, h)$ such that $p \in \mathbb{P}^{n}$ and $h \subset \mathbb{P}^{n}$ is a hyperplane through $p$. Then the nonsingular $G$-variety $X$ contains an open $G$-orbit with complement $D$ consisting of those triples $(p_{1}, p_{2}, h)$ such that $p_{1}, p_{2} \in h$. Thus, $D$ is a nonsingular prime divisor in $X$, consisting of two $G$-orbits; the closed orbit $Y$ (where $p_{1} = p_{2}$) is isomorphic to $\mathbb{I}_{n}$. Hence $X$ consists of two strata but three orbits.

In particular, $(X, D)$ is a nonhomogeneous $G$-pair. But one may check that the blowing up of $Y$ in $X$ is log homogeneous under $G$.

2.4. The Albanese fibration. For any compact Kähler almost homogenous manifold, one knows that the Albanese map is an equivariant fibration with connected fibers; see e.g. [11, Sec. 3.9]. The following result is an algebraic analogue for varieties that need not be complete:

Proposition 2.4.1. Let $X$ be a nonsingular almost homogeneous $G$-variety with open orbit $X_{0}$ and let $A(X_{0}) = G/I(X_{0})$ be the Albanese variety (as defined in Subsec. 1.2). Then:

(i) The Albanese morphism \[
\alpha : X \to A(X)
\]
is a $G$-equivariant fibration, and $A(X) = A(X_{0})$. In particular, 
\[
X \cong G \times^{I} Y,
\]
where $I := I(X_{0})$ is a closed subgroup of $G$ containing $G_{\text{aff}}$, and the fiber $Y$ of $\alpha$ is a nonsingular $I$-variety, almost homogeneous under $G_{\text{aff}}$.

(ii) If $G$ acts faithfully on $X$, then $I$ (and hence $G_{\text{aff}}$) acts faithfully on $Y$.

(iii) Given a reduced $G$-stable divisor $D$ in $X$, the pair $(X, D)$ is homogeneous (resp. parallelizable) under $G$ if and only if $(Y, D \cap Y)$ is homogeneous (resp. parallelizable) under $G_{\text{aff}}$. Then the restriction to $T_{X}(- \log D)$ of the natural map
\[
T_{X} \to \alpha^{*}T_{A(X)} \cong O_{X} \otimes g/g_{\text{aff}}
\]
fits into an exact sequence
\[ 0 \to \mathcal{T}_a(-\log D) \to \mathcal{T}_X(-\log D) \to O_X \otimes g_{\text{aff}} \to 0, \]
where \( \mathcal{T}_a(-\log D) \) is a \( G \)-linearized locally free sheaf, generated by the image of \( g_{\text{aff}} \). Moreover, \( \mathcal{T}_a(-\log D)|_Y \cong \mathcal{T}_Y(-\log (D \cap Y)) \).

Proof. (i) By [16, Thm. 3.1], the Albanese map \( X_0 \to A(X_0) \) extends to a morphism \( \alpha : X \to A(X_0) \). Clearly, \( \alpha \) is a \( G \)-equivariant fibration with fibers being almost homogeneous \( G_{\text{aff}} \)-varieties. In particular, any morphism from a fiber to an abelian variety is constant; it follows that \( \alpha \) is the Albanese morphism of \( X \).

By (1.1.1), \( X = C^0 \cdot Y \) which implies (ii).

(iii) The equivalences follow from (i) together with Proposition 2.2.1. The final assertions are consequences of the commutative diagram
\[
\begin{array}{c}
0 \to O_X \otimes g_{\text{aff}} \to O_X \otimes g \to O_X \otimes g_{\text{aff}} \to 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \cong \\
0 \to \mathcal{T}_a(\log D) \to \mathcal{T}_X(-\log D) \to \alpha^* \mathcal{T}_{A(X)} \to 0.
\end{array}
\]

Using this Albanese fibration, we reduce the description of the \( G \)-equivariant automorphism group \( \text{Aut}_G(X) \) to that of \( \text{Aut}_I(Y) \):

Lemma 2.4.2. With the notation and assumptions of the above proposition, the group \( \text{Aut}_G(X) \) is algebraic and fits into an exact sequence of such groups
\[
(2.4.1) \quad 1 \to \text{Aut}_I(Y) \to \text{Aut}_G(X) \to A(X) \to 0.
\]
Moreover, \( \text{Aut}_I(X) \) is affine.

Proof. We may identify \( \text{Aut}_G(X) \) with a subgroup of \( \text{Aut}_G(X_0) \cong N_G(H)/H \), where \( X_0 \cong G/H \). To show that \( \text{Aut}_G(X) \) is algebraic, we check that it is closed in \( N_G(H)/H \). For this, consider the “graph”
\[
Y_0 := \{ (x, g \cdot x, g) \mid x \in X_0, \ g \in N_G(H)/H \} \subseteq X_0 \times X_0 \times N_G(H)/H
\]
and its closure \( Y \) in \( X \times X \times N_G(H)/H \). Then the projection
\[
p_1 \times p_3 : Y \to X \times N_G(H)/H
\]
is an isomorphism over the dense open subset \( X_0 \times N_G(H)/H \). Let \( E \subseteq Y \) be the exceptional set of \( p_1 \times p_3 \). Then \( \text{Aut}_G(X) \) consists of those \( \gamma \in N_G(H)/H \) such that \( X \times X \times \{ \gamma \} \) does not meet \( E \). Thus, \( \text{Aut}_G(X) \) is a constructible subgroup of \( N_G(H)/H \), and hence is closed.

Next, we obtain the exact sequence (2.3.1). By the universal property of the Albanese morphism, any \( G \)-equivariant automorphism of \( X \) induces an equivariant automorphism of the abelian variety \( A(X) \), that is, a translation. This yields a group homomorphism
\[
f : \text{Aut}_G(X) \to A(X)
\]
which is the restriction of the natural homomorphism
\[
\text{Aut}_G(X_0) = N_G(H)/H \to G/G_{\text{aff}} H = A(G/H) = A(X)
\]
and hence is algebraic. The composition $C^0 \to \text{Aut}_G(X) \to \mathcal{A}(X)$ is surjective by Lemma 1.1.1 so that $f$ is surjective. Since $X \cong G \times^f Y$, the kernel of $f$ is isomorphic to $\text{Aut}_I(Y)$.

As in the first step of the proof, $\text{Aut}_I(Y)$ is a closed subgroup of $N_I(H)/H$, and hence is affine since $I$ is.

\[ \square \]

**Remarks 2.4.3.** We consider again a nonsingular variety $X$, almost homogeneous under a faithful action on $G$, and we use the notation of Proposition 2.4.1.

(i) Let $I = G_{\text{aff}}F$ be as in (1.2.1). Then $F$ acts on $Y$ via a homomorphism \( \varphi : F \to \text{Aut}_I(Y) \subseteq \text{Aut}_{G_{\text{aff}}}(Y) \).

Thus, if every $G_{\text{aff}}$-equivariant automorphism of $Y$ comes from some element of $G_{\text{aff}}$ then $I = G_{\text{aff}}$ (since $I$ acts faithfully on $Y$). Equivalently, the isogeny $\mathcal{A}(G) \to \mathcal{A}(X)$ is an isomorphism under this assumption.

(ii) In general, we may reduce to the case where $\mathcal{A}(G) = \mathcal{A}(X)$, just by replacing $X$ with the finite etale cover $G \times^{G_{\text{aff}}} Y$ and keeping $G$ unchanged.

We may reduce further to the case where the isogeny $\mathcal{A}(C^0) \to \mathcal{A}(G)$ (Lemma 1.1.1) is an isomorphism, by replacing $X$ with the finite etale cover

\[ X' := C^0 \times^{C^0_{\text{aff}}} Y \]

and $G$ with the finite cover

\[ G' := C^0 \times^{C^0_{\text{aff}}} G_{\text{aff}}. \]

Then the Albanese map

\[ \alpha' : C^0 \times^{C^0_{\text{aff}}} Y \to C^0/C^0_{\text{aff}} \]

is locally trivial for the Zariski topology, since the algebraic group $C^0_{\text{aff}}$ is commutative, connected and affine.

In the above remark, both covers of $X$ are just the pull-backs under the Albanese map of the isogenies $\mathcal{A}(G) \to \mathcal{A}(X)$, resp. $\mathcal{A}(C^0) \to \mathcal{A}(G)$. We now show that every finite etale cover of an almost homogeneous variety is obtained in that way, under the assumption of completeness:

**Proposition 2.4.4.** Let $X$ be a complete nonsingular variety, almost homogeneous under a faithful $G$-action. Let \( f : X' \to X \) be an etale morphism, where $X'$ is a complete (nonsingular) variety. Then $X'$ is almost homogeneous under a faithful action of a finite cover $G'$ of $G$. Moreover, $f$ induces an isogeny $\varphi : \mathcal{A}(X') \to \mathcal{A}(X)$, and the square

\[ \begin{array}{ccc} X' & \xrightarrow{\alpha'} & \mathcal{A}(X') \\
 f \downarrow & & \downarrow \varphi \\
 X & \xrightarrow{\alpha} & \mathcal{A}(X) \end{array} \]

is cartesian.

**Proof.** Note that any derivation of $\mathcal{O}_X$ extends to a unique derivation of $f_*\mathcal{O}_{X'}$. This yields an injective homomorphism of Lie algebras

\[ i : \Gamma(X, T_X) \to \Gamma(X', T_{X'}). \]
Moreover, the image of $i$ is an algebraic Lie subalgebra, since it consists of those derivations that preserve the field of rational functions on $X$. Thus, $i(\mathfrak{g})$ is algebraic as well; let $G' \subseteq \text{Aut}^0(X')$ be the associated algebraic group. Then $G'$ is a finite cover of $G$, and has an open orbit in $X'$.

The universal property of the Albanese morphism yields a commutative square (2.4.2), where $\varphi$ is a homomorphism. We also have a commutative square

$$
\begin{array}{ccc}
\mathcal{A}(G') & \longrightarrow & \mathcal{A}(X') \\
\downarrow & \downarrow \varphi & \\
\mathcal{A}(G) & \longrightarrow & \mathcal{A}(X),
\end{array}
$$

where the three nonlabeled arrows are isogenies. It follows that $\varphi$ is an isogeny as well. So the fibers $Y, Y'$ of $\alpha, \alpha'$, have the same dimension and satisfy $Y' \subseteq f^{-1}(Y)$. But $Y$ is unirational and hence simply connected (see [20, Exp. XI, Cor. 1.3]); thus, $f$ restricts to an isomorphism $Y' \rightarrow Y$. This implies that the square (2.4.2) is cartesian. $\square$

2.5. **The complete log parallelizable case.** We shall obtain an algebraic version of the main result of [22] describing the log parallelizable compact “weakly Kähler” manifolds. To state our version, recall that a semiabelian variety is an algebraic group obtained as an extension of an abelian variety by a torus; any semiabelian variety is connected and commutative.

**Theorem 2.5.1.** Let $X$ be a complete nonsingular variety on which a connected algebraic group $G$ acts faithfully. Then the following conditions are equivalent:

(i) $X$ is log parallelizable under $G$.

(ii) $X \cong G \times^{G_{\text{aff}}} Y$, where $G_{\text{aff}}$ is a torus and $Y$ is a complete nonsingular toric variety under $G_{\text{aff}}$.

(iii) $G$ is a semiabelian variety and has an open orbit in $X$.

Under one of these conditions, $G = \text{Aut}^0(X, D)$, where $D$ denotes the boundary.

**Proof.** (i)$\Rightarrow$(ii) We use the notation of Proposition 2.4.1. Note that $Y$ is log homogeneous under $G_{\text{aff}}$. Choose $y \in Y$ such that the orbit $G_{\text{aff}} \cdot y$ is closed. Then the isotropy group $G_{\text{aff},y}$ is a parabolic subgroup of $G_{\text{aff}}$. Hence $G_{\text{aff},y}$ is connected and contains a maximal unipotent subgroup of $G_{\text{aff}}$. On the other hand, $G_0^0$ is contained in $G_{\text{aff}}$, and is isogenous to a torus by Proposition 2.1.2. It follows that $G_{\text{aff},y}$ is a torus and, in turn, that $G_{\text{aff}}$ is a torus as well.

Hence $Y$ is a toric $G_{\text{aff}}$-variety, so that the natural map $G_{\text{aff}} \rightarrow \text{Aut}_{G_{\text{aff}}}(Y)$ is an isomorphism. This implies that $I = G_{\text{aff}}$ by Remark 2.4.3(i).

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i) Since $G$ is commutative and acts faithfully in $X$, its open orbit is isomorphic to $G$ itself. Together with Proposition 2.4.1, it follows that $X = G \times^{G_{\text{aff}}} Y$, where $Y$ contains an open orbit of the torus $G_{\text{aff}}$. Thus, $Y$ is log parallelizable (see e.g. [18 Prop. 3.1]) and hence so is $X$.

Under the condition (i), we have the equalities of Lie algebras

$$\mathfrak{g} = \Gamma(X, \mathcal{O}_X \otimes \mathfrak{g}) = \Gamma(X, \mathcal{T}_X (-\log D))$$
and hence $G = \text{Aut}^0(X, D)$. □

This displays the close relationship between log parallelizable and semiabelic varieties; recall from [2] that the latter are the normal varieties on which a semiabelian variety acts with finitely many orbits, such that all isotropy groups are tori. By the above theorem, being log parallelizable or semiabelic are equivalent for complete nonsingular varieties.

3. Complete log homogeneous varieties

3.1. Relation to spherical varieties. We consider a complete log homogeneous $G$-variety $X$ and write $X \cong G \times^Y Y$ as in Proposition [2.4.1] so that $Y$ is a complete nonsingular variety, log homogeneous under $G_{\text{aff}}$. In this subsection, we obtain several preliminary results about the $G_{\text{aff}}$-variety $Y$; together, they will imply our main structure theorem in the next subsection.

Choose a Levi subgroup $L \subseteq G_{\text{aff}}$, i.e., a maximal closed connected reductive subgroup. Recall the semidirect product decomposition $G_{\text{aff}} = R_u(G_{\text{aff}})L$, where $R_u(G_{\text{aff}})$ denotes the unipotent radical; moreover, any two Levi subgroups are conjugate by an element of $R_u(G_{\text{aff}})$.

**Lemma 3.1.1.** For a $G_{\text{aff}}$-pair $(Y, E)$ where $Y$ is complete, the following conditions are equivalent:

(i) $(Y, E)$ is homogeneous under $G_{\text{aff}}$.

(ii) $(Y, E)$ is homogeneous under $L$ along each closed $G_{\text{aff}}$-orbit.

If one of these conditions holds, then there exists a smallest open $L$-stable subset $Y_L \subseteq Y$ which contains every closed $G_{\text{aff}}$-orbit. Then the pair $(Y_L, E \cap Y_L)$ is homogeneous under $L$, and every $G_{\text{aff}}$-orbit in $Y$ meets $Y_L$ along a unique $L$-orbit. In particular, the closed $L$-orbits in $Y_L$ are exactly the closed $G_{\text{aff}}$-orbits in $Y$.

**Proof.** (i)$\Rightarrow$(ii) As in the proof of Theorem [2.5.1] choose $y \in Y$ such that the orbit $G \cdot y$ is closed, i.e., $G_y$ is a parabolic subgroup of $G_{\text{aff}}$. Then $G_{\text{aff}, y} = R_u(G_{\text{aff}})L_y = G^0_{\text{aff}, y}$ and $G_{\text{aff}} \cdot y = L \cdot y$. Clearly, the homomorphism $\rho_y : G^0_y \to \mathbb{G}_m$ of (2.1.3) has a trivial restriction to $R_u(G_{\text{aff}})$. Thus, by Proposition [2.1.2] $(Y, E)$ is homogeneous under $L$ along $G_{\text{aff}} \cdot y$.

(ii)$\Rightarrow$(i) The assumption implies that the map $\text{op}_{Y, E}$ is surjective along any closed $G_{\text{aff}}$-orbit. Thus, $\text{op}_{Y, E}$ is surjective everywhere.

If (ii) holds, then there exists an open $L$-stable subvariety $Y' \subseteq Y$ containing all the closed $G_{\text{aff}}$-orbits, such that the pair $(Y', E \cap Y')$ is homogeneous under $L$. In particular, $Y'$ contains only finitely many $L$-orbits, and every closed $G_{\text{aff}}$-orbit in $Y$ is a closed $L$-orbit in $Y'$. Let $Y_L$ be the set of those $y \in Y$ such that the orbit closure $L \cdot y$ contains a closed $G_{\text{aff}}$-orbit. Then $Y_L$ is contained in $Y'$ and is the smallest common open $L$-stable neighborhood of all the closed $G_{\text{aff}}$-orbits.

As a consequence $G_{\text{aff}} \cdot Y_L$ is an open $G$-stable neighborhood of the closed $G$-orbits, so that $G_{\text{aff}} \cdot Y_L = Y$. In other words, every $G_{\text{aff}}$-orbit $\Omega$ in $Y$ meets $Y_L$. Since the $L$-orbits in $Y_L$ are the strata of the pair $(Y_L, E \cap Y_L)$, i.e., the intersections of $Y_L$ with the strata of $(Y, E)$, it follows that $\Omega \cap Y_L$ is a unique $L$-orbit. □
Next, we obtain a partial converse of Corollary 2.1.4.

**Lemma 3.1.2.** Let $Y$ be a log homogeneous variety under a connected reductive group $L$. If every closed orbit is complete, then the $L$-variety $Y$ is spherical and regular. Moreover, its closed orbits are all isomorphic.

*Proof.* This is proved in [11, Prop. 2.2.1] under the assumption that $Y$ is complete; the argument there may be adapted as follows. Choose $y \in Y$ such that the orbit $L \cdot y$ is closed, so that $L_y$ is a parabolic subgroup of $L$. Let $P \subseteq L$ be a parabolic subgroup opposite to $L_y$, so that $M := P \cap L_y$ is a Levi subgroup of both $P$ and $L_y$. By the local structure theorem (see e.g. [11, Sec. 2.1]), there exists a locally closed affine $M$-stable subvariety $S \subseteq Y$ containing $y$, such that the map

$$R_u(P) \times S \to Y, \quad (g, s) \mapsto g \cdot s$$

is an open immersion. In particular, $S$ is nonsingular and meets transversally $L \cdot y$ at the unique point $y$. This yields a $M$-equivariant isomorphism

$$T_yS \cong N_{L \cdot y}/Y, y.$$

In particular, $M$ acts on $T_yS \cong \mathbb{A}^r$ via a surjective homomorphism to $\mathbb{G}_m^r$, and hence the derived subgroup of $M$ fixes $T_yS$ pointwise. As in the proof of Proposition 2.3.1 it follows that $S \cong \mathbb{A}^r$ as $M$-varieties, and the boundary of $Y$ meets $S$ along the union of all the coordinate hyperplanes in $\mathbb{A}^r$. Thus, the $L$-variety $Y$ is regular in a neighborhood of $L \cdot y$. Moreover, the image of $R_u(P) \times \mathbb{G}_m^r$ in $Y$ is the open orbit of a Borel subgroup $B$ of $L$, so that $Y$ is spherical as well. Finally, one checks that the stabilizer in $L$ of this open $B$-orbit equals the parabolic subgroup $P$. In particular, the conjugacy class of $P$ is independent of the closed orbit $L \cdot y$. Thus, the same holds for the conjugacy class of the opposite parabolic subgroup $L_y$. \qed

Finally, we record the following result, a consequence of [13, Thm. 5.1, Cor. 5.6].

**Lemma 3.1.3.** Let $Y$ be a spherical $L$-variety, and $Y' \subseteq Y$ an open $L$-stable subset. Then $\text{Aut}_L(Y)$ is a diagonalizable algebraic group and preserves $Y'$; in particular, $\text{Aut}_L^0(Y)$ is a torus. Moreover, the restriction map $\text{Aut}_L^0(Y') \to \text{Aut}_L^0(Y')$ is an isomorphism.

### 3.2. Structure

We now come to our main result:

**Theorem 3.2.1.** Let $G$ be a connected algebraic group and choose a Levi subgroup $L \subseteq G_{\text{aff}}$. Then any complete log homogeneous $G$-variety $X$ may be written uniquely as $G \times^X Y$, where

(i) $I \subseteq G$ is a closed subgroup containing $G_{\text{aff}}$ as a subgroup of finite index.

(ii) $Y$ is a complete nonsingular $I$-variety containing an open $L$-stable subset $Y_L$ such that

(a) the $L$-variety $Y_L$ is regular, and

(b) every $G_{\text{aff}}$-orbit in $Y$ meets $Y_L$ along a unique $L$-orbit.
In particular, the $L$-variety $Y$ is spherical, and the projection $X \to G/I$ is the Albanese morphism.

Conversely, given $I$ and $Y$ satisfying (i) and (ii), the $G$-variety $X := G \times^I Y$ is log homogeneous. Moreover, each $I$-orbit in $Y$ is a unique $G_{\text{aff}}$-orbit; in particular, each $G$-orbit in $X$ meets $Y$ (resp. $Y_L$) along a unique orbit of $G_{\text{aff}}$ (resp. $L$).

**Proof.** All the assertions follow from Proposition 2.4.1 and Lemmas 3.1.1, 3.1.2 except for the equality of $I$-orbits and $G_{\text{aff}}$-orbits. For the latter, let again $I = G_{\text{aff}}F$ be as in (1.2.1). The finite subgroup $F \subseteq C^0$ acts on $Y$ by $G_{\text{aff}}$-equivariant automorphisms. Thus, $F$ preserves all $L$-orbits by Lemma 3.1.3, and hence all $G_{\text{aff}}$-orbits.

Next, we obtain some remarkable consequences of this structure theorem.

**Corollary 3.2.2.** Any complete log homogeneous $G$-variety $X$ is $G$-regular; in particular, any $G$-stable subvariety is log homogeneous. Moreover, the irreducible components of fibers of any morphism $f : X \to X'$ are log homogeneous.

**Proof.** The first assertion follows from Theorem 3.2.1 together with the $L$-regularity of $Y_L$ (Lemma 3.1.2).

For the second assertion, let $C$ be an irreducible component of a fiber and let $X'$ be the closure of $G \cdot C$ in $X$. Then $X'$ is a $G$-stable subvariety of the regular variety $X$, and hence is regular as well. So the assertion follows by applying Corollary 2.2.3 to $f|_{X'}$. □

Also, we rephrase the most delicate condition (b) in the statement of Theorem 3.2.1, to make it easier to check:

**Lemma 3.2.3.** Let $Y$ be a complete nonsingular $G_{\text{aff}}$-variety, and $Y_L \subseteq Y$ an $L$-stable open subset. Assume that $Y_L$ is $L$-regular and contains all the closed $G_{\text{aff}}$-orbits in $Y$. Let $\partial Y_L$ be the boundary of the $L$-variety $Y_L$, and $E := G_{\text{aff}} \cdot \partial Y_L$. Then the following conditions are equivalent:

(b) Every $G_{\text{aff}}$-orbit in $Y$ meets $Y_L$ along a unique $L$-orbit.

(c) $Y_L$ is not contained in $E$.

(d) $\partial Y_L$ is stable under $g_{\text{aff}}$ (acting on $Y_L$ by vector fields).

Under one of these conditions, $(Y, E)$ is a $G_{\text{aff}}$-homogeneous pair.

**Proof.** Notice that $G_{\text{aff}} \cdot Y_L$ is a $G$-stable open subset of $Y$ containing all the closed $G$-orbits, whence $G_{\text{aff}} \cdot Y_L = Y$. In other words, every $G_{\text{aff}}$-orbit in $Y$ meets $Y_L$.

(b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(d) The assumption implies the equality $Y_L \cap E = \partial Y_L$, which in turn implies the desired statement.

(d)$\Rightarrow$(b) By our assumption, $E$ does not meet the open $G_{\text{aff}}$-orbit in $Y$. It follows that $E \cap Y_L = \partial Y_L$. On the other hand, $E$ is a divisor with normal crossings (since so is $\partial Y_L$), and its complement is the open $G_{\text{aff}}$-orbit in $Y$. Moreover, each stratum of $(Y, E)$ meets $Y_L$ along a unique stratum of $(Y_L, \partial Y_L)$, i.e., a unique $L$-orbit. Together with Proposition 2.1.2 this completes the proof. □

### 3.3. The Tits morphism.

In this subsection, we consider a complete log homogeneous variety $X$ under a faithful action of $G$. Let

$$\tau : X \to \mathcal{L}, \quad x \mapsto g(x)$$
be the $G$-morphism defined in Corollary 3.1.3. We say that $\tau$ is the **Tits morphism** of $X$, and we denote its image by $\mathcal{L}(X) \subseteq \mathcal{L}$.

Clearly, $\mathcal{L}(X)$ is a projective $G$-variety, fixed pointwise by $C$. Moreover, in the notation of Theorem 3.2.1, $\mathcal{L}(X) = \mathcal{L}(Y)$, and the restriction $\tau|_{\mathcal{L}(X)}$ is the Tits morphism of the complete log homogeneous $G_{aff}$-variety $Y$. The sheaf $\mathcal{T}_X(-\log D)$ is the pull-back under $\tau$ of the quotient sheaf on $\mathcal{L}(X)$ (regarded as a subvariety of the Grassmannian of subspaces of $\mathfrak{g}$). Further properties of $\tau$ are gathered in the following:

**Proposition 3.3.1.** (i) The $G$-variety $\mathcal{L}(X)$ contains a unique closed orbit $\mathcal{F}(X)$, and $\tau$ maps every closed $G_{aff}$-orbit in $Y$ isomorphically to $\mathcal{F}(X)$.

(ii) The irreducible components of the fibers of $\tau$ are log parallelizable varieties.

*Proof.* (i) By the above observations, we may assume that $G$ is affine, i.e., $X = Y$. Let $x \in X$ such that the orbit $G \cdot x$ is closed. Then $G \cdot \tau(x) \cong G/N_G(\mathfrak{g}_x)$. Moreover, the Lie subalgebra $\mathfrak{g}_x \subseteq \mathfrak{g}$ is the intersection of the kernels of differentials of characters of the isotropy group $G_x$, a parabolic subgroup of $G$. It follows that $N_G(\mathfrak{g}_x) = G_x$; hence $\tau$ restricts to an isomorphism

$$G \cdot x \cong G \cdot \tau(x) = G \cdot \mathfrak{g}_x.$$  

Moreover, $G_x = R_a(G)L_x$, where $L$ is a Levi subgroup of $G$ and the conjugacy class of $L_x$ in $L$ is independent of the closed $G$-orbit $G \cdot x$ by Lemma 3.1.2. Thus, the orbit $G \cdot \mathfrak{g}_x$ is also independent of the closed $G$-orbit.

(ii) By Corollary 3.2.2, the irreducible components of fibers of $\tau$ are nonsingular. To show that they are log parallelizable, it suffices to consider fibers at points $y \in Y$. Then

$$\tau^{-1}(y) = C^0(\tau^{-1}(y) \cap Y)$$

is mapped by $\alpha$ onto $A(X) = C^0/(C^0 \cap I)$. Thus,

$$(3.3.1) \quad \tau^{-1}(y) \cong C^0 \times \mathbb{C}^{\nu_d}(\tau^{-1}(y) \cap Y).$$

Since $\tau^{-1}(y) \cap Y$ is a fiber of the Tits morphism of $Y$, we are reduced to checking that the irreducible components of fibers of that morphism are toric varieties. In other words, we may assume again that $G$ is affine.

Choose a point $x \in \mathcal{F}(X)$. Applying the local structure theorem as in the proof of Lemma 3.1.2, we obtain a parabolic subgroup $P \subseteq L$ and a locally closed affine subvariety $S \subseteq \mathcal{L}(X)$ containing $x$, such that the natural map

$$R_a(P) \times S \to \mathcal{L}(X)$$

is an open immersion. Moreover, $S$ is stable under $M := P \cap L_x$, a Levi subgroup of both $P$ and $L_x$. It follows that $\tau^{-1}(S)$ is a locally closed $M$-stable subvariety of $X$, and the natural map

$$R_a(P) \times \tau^{-1}(S) \to X$$

is an open immersion as well.

Together with (i), this implies that $G \cdot \tau^{-1}(S)$ is an open subset of $X$, containing all the closed $G$-orbits. Thus, $\tau^{-1}(S)$ meets all $G$-orbits, and it suffices to check the desired assertion for the fibers of the restriction $\tau^{-1}(S) \to S$. But one shows as in
the proof of Lemma 3.1.2 that $\tau^{-1}(S)$ is fixed pointwise by the derived subgroup of $M$, and is a toric variety under a quotient of $C(M)^0$. It follows that the irreducible components of every fiber $\tau^{-1}(s)$ are toric varieties under a quotient of the isotropy group $C(M)^0$.

**Remarks 3.3.2.** (i) The Tits map is constant if and only if $X$ is log parallelizable.

(ii) Likewise, the flag variety $F(X)$ is a point if and only if every closed $G$-orbit in $X$ is an abelian variety. This is also equivalent to the solvability of the group $G$ (or of $G_{\text{aff}}$ by Lemma 1.1.1).

Indeed, if $G_{\text{aff}}$ is solvable, then every closed $G_{\text{aff}}$-orbit in $Y$ is a point. Thus, every closed $G$-orbit in $X$ is isomorphic to $A(X)$. Conversely, if $F(X)$ is a point, then $Y_L$ contains a fixed point of $L$. By Proposition 2.1.2, it follows that $L = T$ is a torus, and hence $G_{\text{aff}} = R_u(G_{\text{aff}})T$ is solvable.

Next, we consider the product morphism $\alpha \times \tau : X \to A(X) \times L(X)$ for which we obtain a generalization of the Borel–Remmert theorem:

**Theorem 3.3.3.** With the above notation, $\alpha \times \tau$ is surjective, and all irreducible components of its fibers are nonsingular toric varieties.

Moreover, $\alpha \times \tau$ maps isomorphically every closed $G$-orbit in $X$ to $A(X) \times F(X)$, the unique closed $G$-orbit in $A(X) \times L(X)$.

**Proof.** The surjectivity of $\alpha \times \tau$ follows from Lemma 1.4.1 and the assertion on its fibers has been established in the proof of Proposition 3.3.1.

By that proposition, any closed $G$-orbit in $A(X) \times L(X)$ is contained in $A(X) \times F(X)$. But the latter is a unique $G$-orbit, since $C^0$ acts transitively on $A(X)$ and fixes pointwise $L(X)$.

The closed $G$-orbits in $X$ are exactly the subsets $G \times^I Y'$, where $Y' \subseteq Y$ is a closed orbit of $I$ or, equivalently, of $G_{\text{aff}}$. Moreover, the restriction of $\alpha$ to $G \times^I Y'$ is the projection to $G/I$, and $\tau$ restricts to an isomorphism $Y' \cong F(X)$ by Proposition 3.3.1 again. This implies that the restriction of $\alpha \times \tau$ to $G \times^I Y'$ is an isomorphism to $A(X) \times F(X)$. □

In fact, the connected components of fibers of $\alpha \times \tau$ (regarded as reduced subschemes of $X$) are “stable toric varieties” in the sense of [2], as shown by the argument of Proposition 3.3.1. Likewise, the connected components of fibers of $\tau$ are “stable semiabelic varieties”.

The morphism $\alpha \times \tau$ may also be used to describe the complete log homogeneous varieties having a nonconnected boundary, analogously to a result of complex geometry (see [10, I.2.6] and its references):

**Proposition 3.3.4.** The complete log homogeneous varieties $X$ having a nonconnected boundary are exactly the projective line bundles over the product $A \times F$ of an abelian variety with a flag variety, obtained as projective completions of line bundles $L \boxtimes M$, where $L \to A$ is a line bundle of degree 0, and $M \to F$ is an arbitrary line bundle. Then the Albanese map of $X$ is the projection to $A = A(X)$, and the Tits map is the projection to $F = L(X) = F(X)$. 

□
Proof. Let \((X,D)\) be a homogeneous pair with \(X\) complete and \(D\) nonconnected. Write \(X = G \times^I Y\) and \(D = G \times^I E\), so that \(E\) is not connected as well. Let \(U \subseteq G_{\text{aff}}\) be a maximal unipotent subgroup. By [9], the fixed point set \(Y^U\) is connected. Hence the open \(G_{\text{aff}}\)-orbit \(Y \setminus E\) contains \(U\)-fixed points. In particular, \(R_u(G_{\text{aff}})\) fixes \(Y\) pointwise; we may thus assume that \(G_{\text{aff}}\) is reductive. Let \(J \subseteq G_{\text{aff}}\) be the isotropy group of \(y \in (Y \setminus E)^U\). Since \(J \supseteq U\), then \(P := N_{G_{\text{aff}}}(J)\) is a parabolic subgroup of \(G_{\text{aff}}\) and the quotient \(P/J\) is a torus (see e.g. [21, Sec. 7]). Thus, the normalization \(W\) of the closure \(\overline{P \cdot y}\) is a complete toric variety under \(P/J\). Since the natural morphism

\[
\pi : G_{\text{aff}} \times^P W \to Y
\]

is birational and \(G\)-equivariant, and the boundary of any complete toric variety of dimension \(\geq 2\) is connected, we must have \(\text{dim}(W) = 1\), i.e., \(P/J \cong \mathbb{G}_m\) and \(W \cong \mathbb{P}^1\).

And since \(Y\) is nonsingular and \(E\) is a divisor, it follows that \(\pi\) is an isomorphism. In other words, \(Y\) is the projective line bundle over \(F := G_{\text{aff}}/P\) associated with the principal \(\mathbb{G}_m\)-bundle \(G_{\text{aff}}/J \to G_{\text{aff}}/P\) or with the corresponding line bundle \(M \to F\). The Tits morphism is just the structure map \(Y \to F\), so that \(F = \mathcal{F}(X) = \mathcal{L}(X)\).

As a consequence, \(\alpha \times \tau\) is a projective line bundle over \(\mathcal{A}(X) \times F\), associated with a line bundle of the form \(L \boxtimes M\) for some line bundle \(L \to \mathcal{A}(X) =: A\). Since \(L\) is \(G\)-linearized, it has degree 0.

Conversely, any such projective bundle is a complete log homogeneous variety with boundary being the union of the zero and infinity sections. \(\square\)

Returning to an arbitrary complete log homogeneous variety \(X\) under a faithful action of \(G\), we show that the Stein factorization of the Tits morphism \(\tau\) is intrinsic (although \(\tau\) is not):

**Proposition 3.3.5.** With the above notation, let

\[
X \xrightarrow{\sigma} S(X) \xrightarrow{\mathcal{F}} \mathcal{L}(X)
\]

be the Stein factorization of the Tits morphism. Then:

(i) \(S(X)\) is a normal projective \(G\)-variety, fixed pointwise by \(C^0\), and spherical under any Levi subgroup of \(G_{\text{aff}}\).

(ii) \(G\) has a unique closed orbit in \(S(X)\); it is isomorphic to \(\mathcal{F}(X)\).

(iii) \(\sigma\) is the morphism associated with the section ring of the globally generated divisor \(-K_X - D\), where \(D\) denotes the boundary of \(X\). In particular, \(\sigma\) depends only on the pair \((X,D)\).

(iv) \(\tau\) (or, equivalently, \(\sigma\)) is finite if and only if \(-K_X - D\) is ample. Likewise, \(\alpha \times \tau\) (or \(\alpha \times \sigma\)) is finite if and only if \(-K_X - D\) is ample relatively to \(\alpha\).

**Proof.** (i) follows from the construction of \(\sigma\).

(ii) It suffices to show that \(\mathcal{F}\) restricts to an isomorphism over \(\mathcal{F}(X)\). As in the proof of the above proposition, choose a maximal unipotent subgroup \(U \subseteq G_{\text{aff}}\). Then \(B := N_{G_{\text{aff}}}(U)\) is a Borel subgroup of \(G_{\text{aff}}\). The fixed point subset \(\mathcal{L}(X)^B\) consists of a unique point \(z\), which lies in \(\mathcal{F}(X)\). Moreover, \(\mathcal{L}(X)^U\) is connected and \(B\)-stable, where \(B\) acts on this set through the torus \(B/U\). It follows that \(\mathcal{L}(X)^U\) also consists of \(z\). Since \(\mathcal{F}\) is finite, then \(S(X)^U = \mathcal{F}^{-1}(z)\) (as sets). But \(S(X)^U\) is connected as
well, so that it consists of a unique point $z'$. Since $G_{\text{aff},z'}$ is a subgroup of finite index of $G_{\text{aff},z}$ and the latter is a parabolic subgroup of $G_{\text{aff}}$, then

$$\varphi^{-1}(F(X)) = G_{\text{aff}} \cdot z' \to G_{\text{aff}} \cdot z = F(X)$$

is an isomorphism.

(iii) The map $op_{X,D}$ induces a surjective map

$$\mathcal{O}_X \otimes \wedge^n g \to \wedge^n T_X(-\log D) = \mathcal{O}_X(-K_X - D),$$

where $n = \dim(X)$. Taking global sections, we obtain a linear map

$$\wedge^n g \to \Gamma(X, \mathcal{O}_X(-K_X - D))$$

with base-point-free image. The corresponding morphism

$$X \to \mathbb{P}(\wedge^n g^*)$$

may be identified with $\tau$, regarded as a morphism to the Grassmann variety of $n$-dimensional quotients of $g$. This yields our assertion.

(iv) follows from (iii) together with Proposition 3.3.1. \qed

In view of these results, one may ask if the Stein factorization of $\alpha \times \tau$ is $\alpha \times \sigma$, i.e., if the fibers of the latter morphism are connected. The answer is generally negative, as shown by the following:

**Example 3.3.6.** Let $A$ be an abelian variety and choose a point $a \in A$ of order 2. Consider

$$X := A \times^F (\mathbb{P}^1 \times \mathbb{P}^1),$$

where $F$ denotes the group of order 2 acting on $A$ via translation by $a$, and on $\mathbb{P}^1 \times \mathbb{P}^1$ by exchanging the two copies. The group

$$G := A \times \text{PGL}_2$$

acts on $X$ via its action on $A \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$(x, g) \cdot (y, z_1, z_2) = (x + y, g \cdot z_1, g \cdot z_2)$$

(which commutes with $F$). Then $X$ is a projective nonsingular variety, log homogeneous under a faithful $G$-action; the boundary is the prime divisor

$$D := A \times^F \text{diag}(\mathbb{P}^1) \cong A/F \times \mathbb{P}^1.$$

Moreover, $A(X) = A/F$, $I = F \times \text{PGL}_2$, $G_{\text{aff}} = \text{PGL}_2$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$.

One checks that the Tits morphism $\tau$ may be identified with the natural map

$$A \times^F (\mathbb{P}^1 \times \mathbb{P}^1) \to (\mathbb{P}^1 \times \mathbb{P}^1)/F \cong \mathbb{P}^2.$$ 

As a consequence, $\tau$ has connected fibers (isomorphic to $A$ over $X \setminus D$, resp. to $A/F$ over $D$). But $\alpha \times \tau$ is identified with the natural map

$$A \times^F (\mathbb{P}^1 \times \mathbb{P}^1) \to A/F \times \mathbb{P}^2,$$

a double cover ramified along $D$.

However, $\alpha \times \sigma$ has connected fibers when $X$ is replaced with a suitable étale cover, see Lemma 3.4.5 below.
3.4. The equivariant automorphism group. We consider the group $\text{Aut}_G(X)$, where $X \cong G \times^L Y$ still denotes a complete log homogeneous $G$-variety. We first obtain a stronger version of Lemma 2.4.2.

**Proposition 3.4.1.** $\text{Aut}_{G_{\text{aff}}}(Y)$ is diagonalizable and equals $\text{Aut}_L(Y)$. Moreover, $\text{Aut}_G(X)$ is commutative and preserves every $G$-orbit in $X$.

*Proof.* By Lemma 3.1.3, the subgroup $\text{Aut}_{G_{\text{aff}}}(Y) \subseteq \text{Aut}_L(Y)$ is diagonalizable; in particular, commutative. Moreover, writing $I = G_{\text{aff}}F$ as in (2.4.1), the group $F$ acts on $Y$ via $G_{\text{aff}}$-equivariant automorphisms. Thus, $\text{Aut}_L(Y) = \text{Aut}_{G_{\text{aff}}}(Y)$.

The group $\text{Aut}_G(X)$ is generated by $C^0$ and $\text{Aut}_{G_{\text{aff}}}(Y)$, and these commutative groups commute pairwise. Thus, $\text{Aut}_G(X)$ is commutative (alternatively, this follows from the exact sequence (2.4.1) together with the commutativity of any extension of an abelian variety by a diagonalizable group). Moreover, $\text{Aut}_{G_{\text{aff}}}(Y)$ preserves all the $G_{\text{aff}}$-orbits in $Y$ by Lemma 3.1.3. Thus, $\text{Aut}_G(X)$ preserves all the $G$-orbits. \hfill \Box

Next, we describe the connected equivariant automorphism group:

**Proposition 3.4.2.** (i) $\text{Aut}_G(X)^0$ is a semiabelian variety, and its Albanese variety is $A(X)$.

(ii) Let $X_0 \cong G/H$ be the open $G$-orbit in $X$. Then

\begin{equation}
\text{Aut}_G(X)^0 \cong \text{Aut}_G(X_0)^0 \cong (N_G(H)/H)^0 = N_G(H)^0H/H.
\end{equation}

Moreover, $N_G(H) = N_G(H^0) = N_G(h)$.

(iii) $\text{Aut}_G(X)^0$ is the connected center of $\text{Aut}^0(X, D)$, and acts with an open orbit on any component of a general fiber of the Tits morphism $\tau$.

*Proof.* (i) follows from the above proposition together with (2.4.1).

(ii) We claim that the (injective) restriction map

$$\text{Aut}_G^0(X) \to \text{Aut}_G^0(G/H)$$

is surjective. For this, we may reduce to the case where $G = G_{\text{aff}}$ and $X = Y$ by using (2.4.1). Then any element of $\text{Aut}_G^0(G/H)$ extends to an automorphism of $X$, by Lemma 3.1.3. This extension must be $G$-equivariant; this proves our claim.

It remains to show that $N_G(H) = N_G(H^0)$. For this, by the inclusions

$$H^0 \subseteq H \subseteq N_G(H) \subseteq N_G(H^0),$$

it suffices to check that $N_G(H^0)/H^0$ is commutative. By Lemma 1.1.1, we may assume again that $G = G_{\text{aff}}$; then

$$N_G(H^0)/H^0 = \text{Aut}_G(G/H^0) \subseteq \text{Aut}_L(G/H^0)$$

and the latter group is commutative, since the $L$-variety $G/H^0$ (a finite covering of $G/H$) is spherical by Lemma 3.1.2.

(iii) Clearly, $\text{Aut}_G^0(X) = \text{Aut}^0_G(X, D)$ contains the connected center of $\text{Aut}^0(X, D)$. To show the equality, it suffices to check that both groups have the same dimension. But by (ii), $\dim \text{Aut}_G^0(X) = \dim N_G(h) - \dim H$ equals the dimension of the general fibers of $\tau$, and the latter dimension depends only on $(X, D)$ by Proposition 3.3.5.
Recall that the restriction $\tau|_{X_0}$ is the natural map $G/H \to G/N_{G}(h)$. Together with (3.4.1), this implies that any component of a fiber of $\tau|_{X_0}$ is a unique $\text{Aut}_{G}(X)^0$-orbit.

We now investigate the relation between the connected equivariant automorphism group and the morphism $\sigma : X \to \mathcal{S}(X)$:

**Proposition 3.4.3.** Let $X$ be a complete log homogeneous variety with boundary $D$ and put $G := \text{Aut}^0(X, D)$. Denote by $\mathcal{E}$ the subsheaf of the tangent sheaf $\mathcal{T}_{\mathcal{S}(X)}$, generated by the image of $\mathfrak{g}$. Then $\mathcal{E}$ is locally free, i.e., the $G$-variety $\mathcal{S}(X)$ is pseudo-free. Moreover, there is an exact sequence of $G$-linearized sheaves
\[
0 \to \mathcal{O}_X \otimes \mathfrak{c} \to \mathcal{T}_X(-\log D) \to \sigma^* \mathcal{E} \to 0,
\]
where $\mathfrak{c}$ denotes the Lie algebra of $C$.

**Proof.** We claim that $\mathfrak{c} \cap \mathfrak{g}(c) = 0$ for all $x \in X$. Indeed, $\mathfrak{c} \cap \mathfrak{g}(c)$ is the Lie algebra of the group $C^0_{(x)}$, and the latter fixes pointwise $G \cdot x$ and its normal space at $x$. Moreover, $C^0_{(x)}$ is a torus by Lemma 1.2.1 and Proposition 3.4.2(i). Thus, $C^0_{(x)}$ is trivial.

By that claim, the image of the Tits map (regarded as a morphism to the Grassmannian $\text{Gr}({\mathfrak{g}})$ of subspaces of $\mathfrak{g}$) consists of subspaces $V$ such that $V \cap \mathfrak{c} = 0$. Let $\text{Gr}^0(\mathfrak{g}) \subseteq \text{Gr}(\mathfrak{g})$ be the corresponding open subset, and $p : \text{Gr}^0(\mathfrak{g}) \to \text{Gr}(\mathfrak{g}/\mathfrak{c})$ the projection. Then the restriction to $\text{Gr}^0(\mathfrak{g})$ of the quotient sheaf $\mathcal{Q}_\mathfrak{g}$ on $\text{Gr}(\mathfrak{g})$ fits into an exact sequence
\[
0 \to \mathcal{O}_{\text{Gr}^0(\mathfrak{g})} \otimes \mathfrak{c} \to \mathcal{Q}_\mathfrak{g} \to p^* \mathcal{Q}_{\mathfrak{g}/\mathfrak{c}} \to 0.
\]
Pulling back to $X$ yields an exact sequence
\[
0 \to \mathcal{O}_X \otimes \mathfrak{c} \to \mathcal{T}_X(-\log D) \to \tau^* \mathcal{F} \to 0,
\]
where $\mathcal{F}$ is a $G$-linearized locally free sheaf on $\mathcal{L}(X)$ (the pull-back of the quotient sheaf). In turn, this implies the exact sequence (3.4.2), where $\mathcal{E} = \varphi^* \mathcal{F}$ for the morphism $\varphi : \mathcal{S}(X) \to \mathcal{L}(X)$. Also, $\mathcal{E}$ is generated by a subspace of global sections, the image of the composite map
\[
\mathfrak{g} \to \mathfrak{g}/\mathfrak{c} \to \Gamma(\text{Gr}(\mathfrak{g}/\mathfrak{c}), \mathcal{Q}_{\mathfrak{g}/\mathfrak{c}}) \to \Gamma(\mathcal{S}(X), \mathcal{E}).
\]
The restriction of (3.4.2) to the open orbit $X_0 = G \cdot x \subseteq X$ corresponds to the exact sequence of $H$-modules
\[
0 \to \mathfrak{c} \to \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{n}_H(h) \to 0,
\]
where $H = G_x$. It follows that the restriction of $\mathcal{E}$ to the open orbit $G \cdot \sigma(x) \subseteq \mathcal{S}(X)$ is the tangent sheaf. Moreover, the locally free sheaf $\mathcal{E}$ is a subsheaf of $i_*(i^* \mathcal{E})$, where $i : G \cdot \sigma(x) \to \mathcal{S}(X)$ denotes the inclusion. Thus, $\mathcal{E}$ may be identified with the $\mathcal{O}_{\mathcal{S}(X)}$-subsheaf of derivations of the function field of $\mathcal{S}(X)$, generated by the image of $\mathfrak{g}$.

**Remark 3.4.4.** The exact sequence (3.4.2) means that the relative logarithmic tangent sheaf $\mathcal{T}_\varphi(-\log D)$ is isomorphic to $\mathcal{O}_X \otimes \mathfrak{c}$. This may be seen as a differential version of the fact that the general fibers of $\sigma$ are $C^0$-orbits (Proposition 3.4.2(iii)).
Likewise, there is an exact sequence
\[ 0 \to \mathcal{O}_X \otimes \mathfrak{c}_{\text{aff}} \to \mathcal{T}_X(-\log D) \to (\alpha \times \sigma)^*(\mathcal{T}_{\mathcal{A}(X)} \boxtimes \mathcal{E}) \cong (\mathcal{O}_X \otimes \mathfrak{g}/\mathfrak{g}_{\text{aff}}) \oplus \sigma^*\mathcal{E} \to 0, \]
where \( \mathfrak{c}_{\text{aff}} \) denotes the Lie algebra of \( C^0_{\text{aff}} \). In other words, the relative logarithmic tangent sheaf of \( \alpha \times \sigma \) is isomorphic to \( \mathcal{O}_X \otimes \mathfrak{g}_{\text{aff}} \).

Finally, we consider again the morphism \( \sigma : X \to S(X) \), under the assumption that \( \mathcal{A}(X) = \mathcal{A}(C^0) \) (this may be achieved by replacing \( X \) with a finite étale cover, see Remark 2.4.3(ii)).

**Lemma 3.4.5.** If the isogeny \( \mathcal{A}(C^0) \to \mathcal{A}(X) \) is an isomorphism, then all fibers of \( \alpha \times \sigma \) are connected. Equivalently, the Stein factorization of \( \alpha \times \tau \) is \( \alpha \times \sigma \).

**Proof.** By assumption, both isogenies \( \mathcal{A}(C^0) \to \mathcal{A}(G) \) and \( \mathcal{A}(G) \to \mathcal{A}(X) \) are isomorphisms. Thus, \( C^0 \cap I \) is connected, and \( I = G_{\text{aff}} \). So \( H \subseteq G_{\text{aff}} \) and \( C^0 \cap H = C^0_{\text{aff}} \).

Hence the restriction \( \tau |_{X_0} \) has fiber at the base point
\[ N_G(h)/H = C^0 N_{G_{\text{aff}}}(h)/H \cong C^0 \times C^0_{\text{aff}} N_{G_{\text{aff}}}(h)/H. \]

The components of this fiber are the orbits of the group
\[ N_G(h)^0H/H \cong C^0 \times C^0_{\text{aff}} N_{G_{\text{aff}}}(h)^0H/H. \]

Thus, the restriction of \( \alpha \) to every such component has irreducible fibers, isomorphic to \( N_{G_{\text{aff}}}(h)^0H/H \). The assertion follows. \( \square \)

3.5. **Strongly log homogeneous varieties.** Consider again a complete log homogeneous \( G \)-variety \( X \cong G \times^I Y \). We say that \( X \) is strongly log homogeneous if \( Y \) is regular under some connected reductive subgroup of \( G_{\text{aff}} \).

This condition holds, for example, if \( G \) is solvable (since \( Y \) is then a toric variety under a maximal torus of \( G \)). But it fails, e.g., when \( X = Y \) is the projective completion of the tangent bundle \( T_{P^2} \), and \( D \) is the divisor at infinity. Then one checks that the pair \( (X, D) \) is homogeneous, and a Levi subgroup of \( \text{Aut}^0(X, D) = L \cong \text{PGL}_3 \times \mathbb{G}_m \), where \( \text{PGL}_3 \) acts naturally on \( X \), and \( \mathbb{G}_m \) acts by multiplication on the fibers of \( T_{P^2} \). Moreover, the \( L \)-orbits in \( X \) are the zero section \( \mathbb{P}^2 \), the projectivization \( \mathbb{P}(T_{P^2}) \) (isomorphic to the flag variety of \( \mathbb{P}^2 \)), and the complement \( T_{P^2} \setminus \mathbb{P}^2 \); each of them is a unique \( \text{PGL}_3 \)-orbit. Thus, \( X \) is not regular under \( L \) or \( \text{PGL}_3 \). Moreover, both closed \( L \)-orbits are fixed pointwise by \( \mathbb{G}_m \), and they are not simultaneously regular under any proper reductive subgroup of \( \text{PGL}_3 \). So \( X \) cannot be strongly log homogeneous with boundary \( D \).

We shall obtain a criterion for strong log homogeneity which is much simpler than our characterization of log homogeneity (Theorem 3.2.1). To motivate the following statement, observe that the boundary of the strongly log homogeneous \( G \)-variety \( X \) is \( G \times^I E \), where \( E \) is a subdivisor of the boundary of the \( L \)-variety \( Y \).

**Proposition 3.5.1.** Let \( L \) be a connected reductive group, \( Y \) a complete regular \( L \)-variety, \( \partial Y \) its boundary, and \( E \subset \partial Y \) a subdivisor. Write
\[ \partial Y = E + D_1 + \cdots + D_r, \]
where \( D_1, \ldots, D_r \) are \( L \)-stable prime divisors. Then the pair \( (Y, E) \) is (strongly) log homogeneous if and only if \( D_1, \ldots, D_r \) are all generated by their global sections.
Under one of these assumptions, we have an exact sequence

\[ 0 \to \Gamma(Y, \mathcal{T}_Y(-\log \partial Y)) \to \Gamma(Y, \mathcal{T}_Y(-\log E)) \to \bigoplus_{i=1}^r \Gamma(D_i, N_{D_i/Y}) \to 0, \]

where \( N_{D_i/Y} \) denotes the normal sheaf. Moreover, \( H^j(Y, \mathcal{T}_Y(-\log E)) = 0 \) for all \( j \geq 1 \).

**Proof.** We adapt arguments from \[4\] Sec. 4.1; we provide details for completeness. Consider the map

\[ p : \mathcal{T}_Y(-\log E) \to \bigoplus_{i=1}^r N_{D_i/Y} \]

obtained by taking the direct sum of the natural maps \( \mathcal{T}_Y \to N_{D_i/Y} \) and then restricting to \( \mathcal{T}_Y(-\log E) \). Clearly, the kernel of \( p \) is \( \mathcal{T}_Y(-\log \partial Y) \). Moreover, by using a local system of parameters, one checks that \( p \) is surjective. This yields a short exact sequence

\[ 0 \to \mathcal{T}_Y(-\log \partial Y) \to \mathcal{T}_Y(-\log E) \to \bigoplus_{i=1}^r N_{D_i/Y} \to 0. \]

Taking the associated long exact sequence of cohomology groups and using the vanishing of \( H^j(Y, \mathcal{T}_Y(-\log \partial Y)) \) for \( j \geq 1 \) (a consequence of \[12\] Thm. 4.1), we obtain the short exact sequence (3.5.1) and isomorphisms

\[ H^j(Y, \mathcal{T}_Y(-\log E)) \cong \bigoplus_{i=1}^r H^j(D_i, N_{D_i/Y}) \]

for all \( j \geq 1 \).

From the exact sequences (3.5.1), (3.5.2) and the global generation of \( \mathcal{T}_Y(-\log \partial Y) \) (Corollary 2.1.4), it follows that \( \mathcal{T}_Y(-\log E) \) is globally generated if and only if so is each \( N_{D_i/Y} \). Moreover, from the standard exact sequence

\[ 0 \to \mathcal{O}_Y \to \mathcal{O}_Y(D_i) \to N_{D_i/Y} \to 0 \]

and the vanishing of \( H^j(Y, \mathcal{O}_Y) \) for \( j \geq 1 \) (which follows e.g. from \[21\] Cor. 31.1], we obtain an exact sequence

\[ 0 \to k \to \Gamma(Y, \mathcal{O}_Y(D_i)) \to \Gamma(D_i, N_{D_i/Y}) \to 0 \]

and isomorphisms

\[ H^j(Y, \mathcal{O}_Y(D_i)) \cong H^j(D_i, N_{D_i/Y}) \]

for all \( j \geq 1 \). Hence the global generation of \( N_{D_i/Y} \) is equivalent to that of \( \mathcal{O}_Y(D_i) \). The latter implies the vanishing of \( H^j(Y, \mathcal{O}_Y(D_i)) \) for all \( j \geq 1 \), since any globally generated invertible sheaf on a complete spherical variety has vanishing higher cohomology (see \[21\] Cor. 31.1] again). By (3.5.3) and (3.5.4), this implies in turn the vanishing of the \( H^j(Y, \mathcal{T}_Y(-\log E)) \). Also, we have shown that the global generation of \( \mathcal{T}_Y(-\log E) \) is equivalent to that of \( D_1, \ldots, D_r \). \( \square \)
This criterion may be formulated in combinatorial terms by using the characterization of globally generated divisors on spherical varieties, see e.g. [21, Sec. 17]. For example, if $L$ is a torus, i.e., $Y$ is a toric variety, let $\Delta$ be the corresponding fan, and $\rho_1, \ldots, \rho_r \in \Delta$ the rays associated with $D_1, \ldots, D_r$. Then the global generation of $D_i$ is equivalent to the convexity of the union of all the cones of $\Delta$ that do not contain $\rho_i$ (in other words, the complement of the star of $\rho_i$).

Proposition 3.5.1 is the main ingredient in the classification of complete logarithmic homogeneous surfaces (see Remark [2.1.1(iii)] for the much easier classification of log homogeneous curves):

**Proposition 3.5.2.** Up to isomorphism, the pairs $(X, D)$, where $X$ is a complete log homogeneous surface with boundary $D$, are those in the following list:

- (parallelizable) $(A, \emptyset)$, where $A$ is an abelian surface.
- (homogeneous) $(E \times \mathbb{P}^1, \emptyset)$, where $E$ is an elliptic curve, $(\mathbb{P}^1 \times \mathbb{P}^1, \emptyset)$, and $(\mathbb{P}^2, \emptyset)$.
- (log parallelizable) $X$ is the projective completion of a line bundle of degree 0 on an elliptic curve, and $D$ is the union of the zero and infinity sections.
- (log homogeneous)
  - (a) $(E \times \mathbb{P}^1, E \times \{\infty\})$, where $E$ is an elliptic curve.
  - (b) $(\mathbb{P}^1 \times \mathbb{P}^1, \text{diag}(\mathbb{P}^1))$.
  - (c) $(\mathbb{P}^2, C)$, where $C$ is a conic.
  - (d) $(\mathbb{P}^2, D)$, where $D$ is a subdivisor of the boundary of the toric variety $\mathbb{P}^2$, i.e., a union of coordinate lines.
  - (e) $(\mathbb{P}^1 \times \mathbb{P}^1, D)$, where $D$ is a subdivisor of the boundary of the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$.
  - (f) $(\mathbb{F}_n, D)$, where $\mathbb{F}_n$ is the rational ruled surface of index $n \geq 1$ and $D$ is a subdivisor of the boundary of the toric variety $\mathbb{F}_n$ that contains the unique curve $C_{-n}$ of self-intersection $-n$.
  - (g) All pairs obtained from those in (d), (e), (f) by successively blowing up intersection points of boundary curves.

We refer to [18, Sec. 1.7] for a description of $\mathbb{F}_n$ regarded as a toric surface. We outline the proof of Proposition 3.5.2, leaving the easy but rather long details to the reader. The parallelizable resp. homogeneous cases are straightforward, and the log parallelizable case follows either from Theorem 2.5.1 or from Proposition 3.3.4. So we may assume that $D$ is nonempty and connected. If $D$ is irreducible, then by looking at the minimal model of $X$, one obtains Cases (a), (b), (c), (d) and (e) (where $D$ is a line), as well as (f) (where $D = C_{-n}$). Otherwise, $X$ contains a fixed point of $G$, namely, the intersection of two boundary curves. By Theorem 3.3.3 and Remark 3.3.2(ii), it follows that $G$ is affine and solvable. In particular, $X$ is a toric surface and $D$ is a subdivisor of the toric boundary. Then the statement follows from Proposition 3.5.1 by elementary arguments of two-dimensional toric geometry.

**Remarks 3.5.3.** (i) In Cases (a)-(e), the only possibility for the acting group $G$ is the full automorphism group $\text{Aut}^0(X, D)$. But this does not extend to Case (f): consider, for example, the pair

$$ 3.5.5 \quad (X, D) = (\mathbb{F}_n, C_{-n} + F_0 + F_\infty), $$
where $F_0$ and $F_\infty$ are the fibers of the ruling
$$F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$$

at the corresponding points. Then $\text{Aut}^0(X, D)$ is the semidirect product of the group $U := H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ (acting by translations) with a torus $T$ of dimension 2 (acting on $U$ via the image of a maximal torus of $\text{GL}_2$). Moreover, we may take for $G$ the semidirect product of $V$ with $T$, where $V \subseteq U$ is any $T$-stable subspace without base points; there are $2^n-1$ such subspaces.

(ii) For $(X, D)$ as in (3.5.5) and $G = \text{Aut}^0(X, D)$, the image of $\sigma$ is the normal surface obtained from $F_n$ by contracting $C_{-n}$. Indeed, the group $\text{Aut}_G(X)^0$ is trivial, so that $\sigma$ is birational. Moreover, $\sigma$ maps the two $G$-fixed points $F_0 \cap C_{-n}$ and $F_\infty \cap C_{-n}$ to the same point.

In particular, the variety $\mathcal{S}(X)$ is singular if $n \geq 2$. One may also check that $\varphi : \mathcal{S}(X) \to \mathcal{L}(X)$ is bijective for all $n$, so that $\mathcal{L}(X)$ is also singular if $n \geq 2$.

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