Bootstrapping PT symmetric Hamiltonians

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Abstract: Bootstrapping in Quantum Mechanics uses positivity condition to derive the Eigen spectrum. For non-hermitian systems usual positivity condition does not work. In this paper we define positivity condition for special class of non-hermitian Hamiltonian, the PT symmetric Hamiltonian. We illustrate this modified positivity condition with several examples and obtain eigen spectrum.
1 Introduction

One of the main tools that one uses to solve quantum mechanical systems is perturbation theory. However, perturbation theory cannot be used to strongly coupled systems. For this case, one uses numerical techniques to solve Schrödinger equations. Alternatively, one can use Bootstrap techniques, proposed recently in [1] to solve for Eigen values. Important aspect of this technique is that one does not need to solve for Schrödinger equation, rather one imposes some fundamental constraints such as positivity of the norm consistent with symmetries of the system. In principle since this procedure requires no further information than symmetries and positivity of the norm, traditionally it has been used to search consistent theories as well as its spectrum [2]. However, in the context of quantum mechanics, since there is no restriction on kind of potential that one can write down, the main focus has been to obtain the spectrum of a given quantum mechanical system [3–13].

Our aim in this paper is to generalise the bootstrapping technique to non-hermitian systems where the positivity of norm can not be demanded. We focus on PT symmetric potentials where the spectrum can be shown to be real [14, 15]. PT symmetric Hamiltonian naturally appears in various physically relevant situations such as in studies of the Lee-Yang edge singularity [16, 17]. In this paper we show that with some modified definition of norm for the PT symmetric system, one can impose positivity constraint and obtain the spectrum. This article is organised as follows. In section 2 we review the bootstrapping technique for hermitian Hamiltonians and solve for exactly solvable model Poschel-Teller potential and discuss number of interesting observations. In section 3 we set up bootstrapping technique for PT symmetric Hamiltonian and discuss several examples in section 4 to illustrate the point. In section 5 we present discussions and future directions. Section A discusses some of the details that is required in the main text.

Note added: Just before posting this paper, [13] appeared in ArXiv which also deals with non-hermitian systems. However the approach is different than one employed in this paper.

2 Quantum Bootstrapping

In a recent paper [1] presented an elegant way of solving the energy Eigen spectrum of a hermitian Hamiltonian. They showed that using the following constraints we can obtain
the Eigen spectrum numerically of any hermitian Hamiltonian,

\[ \langle R_n | [H, O] | R_m \rangle = 0 \]
\[ \langle R_n | H O | R_m \rangle = E_n \langle R_n | O | R_m \rangle \]
\[ \langle R_n | O^\dagger O | R_n \rangle \geq 0 \] (2.1)

where O is an arbitrary operator, \(| R_n \rangle\) is the n’th energy eigenstate and \(\langle R_n | \) is the complex conjugate of \(| R_n \rangle\). The last constraint of the above equation is the positivity condition of state i.e. the norm of any state must be positive. There are many examples that has been solved in the literature [1, 3–13] using this technique. As an illustration we use this technique to solve exactly solvable model, the Poschel-Teller potential.

**Poschl-Teller potential**

This is an exactly solvable potential, \(V(x) = -\frac{\lambda(\lambda+1)}{2} \text{sech}^2 x\). Putting this into the Schrodinger equation and substituting \(u = \tanh x\), we get the following differential equation

\[ ((1 - u^2)\psi'(u))^t + \lambda(\lambda + 1)\psi(u) + \frac{2E}{1 - u^2}\psi(u) = 0 \] (2.2)

The solution to this differential equation is given by the Legendre functions

\[ \psi(u) = P_\mu^\lambda(\tanh x). \] (2.3)

The energy eigenvalues are given by \(E = -\frac{\mu^2}{2}\) where \(\lambda = 1, 2, 3, \ldots\), and \(\mu = 1, 2, 3, \ldots, \lambda - 1, \lambda\). We will use bootstrap technique to find out the spectrum for this potential. To do this, we choose \(O = \text{sech}^n x \tanh x\) to compute

\[ \langle [\mathcal{H}, O] \rangle, \langle \mathcal{H} O \rangle = E \langle O \rangle. \] (2.4)

Using these two equations in (2.4), we are able to write down a recursion relation as follows

\[ (2tE + \frac{t^3}{2})\langle \text{sech}^t x \rangle + (\frac{t^3}{2} + 3t^2 + 4t + 2 + \frac{3t + 2}{2} - 2\lambda(\lambda + 1)(t + 2))\langle \text{sech}^{t+2} x \rangle \]
\[ + (\frac{2}{2} - 2(t + 1)E + 2\lambda(\lambda + 1)(t + 1) - t^3 - 3t^2 - 4t - 2)\langle \text{sech}^{t+4} x \rangle = 0 \] (2.5)

This recursion relation is used to obtain \(\langle \text{sech}^{2t} x \rangle\) in terms of \(\langle \text{sech}^{2} x \rangle\) and energy E. This allows us to form a matrix \(\mathcal{M}\) given by \(\mathcal{M}_{ij} = \text{sech}^{2(i+j)} x\). We impose the condition that \(\mathcal{M}\) be positive semi-definite to get the energy spectrum. The Eigen spectrum is plotted in Fig1.
Figure 1. Here we plot the results of bootstrap for $H=p^2 - \frac{\lambda(\lambda+1)}{2} \sech^2 x$.

For the ease of understanding, we have redefined the energy eigenvalues to be $E=-\mu^2$. As expected, we find that for $\lambda = 3$, there are 3 peaks at $E=-1,-4$ and -9 corresponding to $\mu = 1,2,3$ respectively. $K$ here refers to the size of the matrix $M$ on which positivity constraint is applied. When we compute expectation values of higher powers of $\sech^2 x$ using the recursion relation, the size of the matrix $M$ becomes larger, hence $K$ also increases. As expected, with increase in $K$ (from $K=5$ to $K=10$), the peaks become sharper. We also notice that even if the energy values become very localised and occur at their expected positions, there is a very wide spread in the allowed values of $\langle \sech^2 x \rangle$ and this feature does not vanish even when we increase $K$. We wish to address this issue in future.

One would expect that it is possible to, at least write down a closed form of recursion relation for hermitian potentials that are exactly solvable like in the case of (2.5). However, it turns out that this is not the case. For Poschl Teller potential, we were able to write down the recursion relation in terms of expectation values of even powers of $\sech x$. If we consider the potential of the form $V = \sech^2 x + \sech x \tanh x$, which is also exactly solvable, we find that its not possible to write down a closed form of recursion relation \footnote{In this case we get odd powers of hyperbolic trigonometric functions which cannot be written back in terms of $\sech^2 x$. In general this will lead to much larger class of search space and hence complicating the problem. To tackle this kind of problems, we may need to impose more general positivity condition. We discuss one such generalisation for two dimensional systems where also obtaining a closed form recursion relation is problematic.}.

However, this method is limited to the hermitian Hamiltonian case only. In [15], it was shown that PT-symmetric non-hermitian Hamiltonian can have real eigenvalues, so an
important question is then whether it is possible to generalize the "Bootstrapping" method for PT-symmetric case or not. We discuss this issue in the next section.

3 Bootstrapping for PT symmetric non-hermitian Hamiltonian

PT symmetric hamiltonians are important class of Hamiltonians which appears in variety of physical situations. They are interesting because eventhough the Hamiltonian is not hermitian, its eigen values are real. One can solve them either analytically or numerically using Schrodinger equations. In this section we develop bootstrapping technique for solving PT symmetric systems. The first challenge in the PT-symmetric Hamiltonian case is to find the positivity condition. In this section, we are going to obtain a suitable positivity condition.

3.1 Bootstrapping condition for PT symmetric systems

For hermitian system the the left and right eigenvectors are the same and one can define orthonormality condition using them. However, in non-hermitian case it is little non-trivial to define orthonormality condition because in this case we have to define a Hamiltonian dependent inner product. Let $|R_n\rangle$ is the n’th eigen state i.e.

$$H |R_n\rangle = E_n |R_n\rangle \quad (3.1)$$

If we define $\langle R_n |$ as the complex conjugate of $|R_n\rangle$, then unlike in the case of hemitian systems, for PT symmetric cases $\langle R_m | R_n \rangle \neq \delta_{mn}$ that is they are not orthonormal. In [18] introduced a new kind of norm "V norm" which is both positive and orthonormal,

$$\langle R_m | V | R_n \rangle = \delta_{n,m} \quad (3.2)$$

where V operator satisfies the following properties,

1. $V H V^{-1} = H^\dagger$
2. $V^\dagger = V$
3. V is a positive operator \[(3.3)\]

Using equation (3.2) we can show that the positivity condition in PT symmetric case becomes,

$$\langle \bar{R}_n | (O^\dagger V O) | R_n \rangle \geq 0 \quad (3.4)$$
where, $\langle \bar{R}_m | = \langle R_m | V$ and $(O^{\dagger})^V = V^{-1}O^V$. Using the following constraints we can obtain the eigenspectrum of any PT symmetric non-hermitian Hamiltonian,

$$
\begin{align*}
\langle \bar{R}_n | [H, O] | R_n \rangle &= 0 \\
\langle \bar{R}_n | HO | R_n \rangle &= E_n \langle \bar{R}_n | O | R_n \rangle \\
\langle \bar{R}_n | (O^{\dagger})^V O | R_n \rangle &\geq 0
\end{align*}
$$

(3.5)

In the next section we illustrate with few examples how to use these conditions to obtain Eigen spectrum of PT symmetric hamiltonians.

4 Example of PT symmetric potentials and implementation of bootstrap

In this section we are going to find the eigenvalues of some PT symmetric Hamiltonian using the above formalism.

4.1 (2 x 2) PT-symmetric Hamiltonian:

Let’s consider the following (2 x 2) PT symmetric non-hermitian Hamiltonian,

$$
H = \begin{pmatrix}
   i & \sqrt{2} \\
   \sqrt{2} & -i
\end{pmatrix}
$$

(4.1)

For this Hamiltonian we find the $V$ operator and it is given by,

$$
V = \begin{pmatrix}
   \sqrt{2} & -i \\
   i & \sqrt{2}
\end{pmatrix}
$$

(4.2)

Now to get the eigenvalues of $H$, let’s consider the following operator,

$$
O = a_0 E_{2x2} + a_1 \sigma_x + a_3 \sigma_z
$$

(4.3)

where, $a_0, a_1$ and $a_3$ are all real numbers. Using the equation (3.5) we get,

$$
a_0^2 + 3a_1^2 + a_3^2 + \sqrt{2}a_0a_1(E + 1/E) + a_1a_2(E + 1/E) \geq 0
$$

(4.4)

where, $E$ is the eigenvalue of $H$. The above inequality must hold for every values of $a_0, a_1$ and $a_3$. We numerically find that the inequality holds for only $E = 1.00$ and $E = -1.00$, so the possible eigenvalues of $H$ are 1.00 and -1.00. The exact eigen values for this Hamiltonian are 1 and -1.
4.1.1 General (2 × 2) PT-symmetric Hamiltonian:

Let’s take the following general (2 × 2) PT symmetric non-hermitian Hamiltonian,

\[
H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}
\]  

(4.5)

where, \(r, s\) and \(\theta\) are all real number. The energy eigen values of this Hamiltonian are real as long as the following inequality holds, \(s^2 - r^2 \sin^2 \theta \geq 0\). The Hamiltonian defined in equation (4.1) is a special case of the above Hamiltonian. For this Hamiltonian we find the \(V\) operator and it is given by,

\[
V = \frac{1}{\cos \alpha} \begin{pmatrix} 1 & -i \sin \alpha \\ i \sin \alpha & 1 \end{pmatrix}
\]  

(4.6)

where, \(\sin \alpha = \frac{r}{s} \sin \theta\). Now to get the eigenvalues of \(H\), let’s consider the following operator,

\[
O = a_0 \sigma_{2×2} + a_1 \sigma_x + a_3 \sigma_z
\]  

(4.7)

where, \(a_0, a_1\) and \(a_3\) are all real numbers. Using the equation (3.5) we get,

\[
a_0^2 + 3a_1^2(1 + \frac{2r^2 \sin^2 \theta}{s^2 - r^2 \sin^2 \theta}) + a_2^2 + a_0a_1 e + a_1a_2 \frac{r \sin \theta}{s} e \geq 0
\]  

(4.8)

where, \(e = \frac{s^2 - r^2 \sin^2 \theta + (E - r \cos \theta)^2}{(s^2 - r^2 \sin^2 \theta)(E - r \cos \theta)}\) and here \(E\) is the eigenvalue of \(H\). The above inequality must hold for every values of \(a_0, a_1\) and \(a_3\). We can now numerically find the eigenvalues of \(H\) for arbitrary values of \(r, s\) and \(\theta\). Here, we have given one example: Let’s consider the following case where, \(r = s = 1\) and \(\theta = \frac{\pi}{4}\). We substitute these values in (4.8) and numerically find that the possible energy eigen values are 0.00 and 1.41. The exact eigen values for this Hamiltonian are 0 and \(\sqrt{2}\).

4.2 Shifted Simple Harmonic Oscillator:

Let’s consider the following PT symmetric non-hermitian Hamiltonian,

\[
H = p^2 + x^2 + 2i\epsilon x
\]  

(4.9)

For this Hamiltonian the \(V\) operator is [19],

\[
V = e^{2\epsilon p}
\]  

(4.10)
Now under this operator $V$, $x$ and $p$ changes in the following way,

$$
x^V = V^{-1}xV = (x + 2i\epsilon)
$$

$$
p^V = V^{-1}pV = p
\tag{4.11}
$$

By choosing $O = \sum_k a_k p^k$ and using the positivity condition i.e. $\langle \bar{R}_n | (O^\dagger) V O | R_n \rangle \geq 0$, we got the following inequality,

$$
\sum_{j,k} a_j^* a_k \langle \bar{R}_n | p^{j+k} | R_n \rangle \geq 0
\tag{4.12}
$$

The above inequality implies the matrix, $\bar{M}_{jk} = \langle \bar{R}_n | p^{j+k} | R_n \rangle$, should be positive semi-definite. Using equation (3.5) we find the following recursion relation,

$$
4t(E - \epsilon^2) \langle \bar{R}_n | p^{t-1} | R_n \rangle + t(t - 1)(t - 2) \langle \bar{R}_n | p^{t-3} | R_n \rangle - 4(t + 1) \langle \bar{R}_n | p^{t+1} | R_n \rangle = 0
\tag{4.13}
$$

Now using the positivity condition or $\bar{M}_{jk}$ is a positive semi definite matrix, we numerically find the eigenvalues of $H$. In this problem there is only one independent variable i.e. $E$, so the search space is $1D$. We have shown the ground and first excited state in two different plots because we have to take different matrix size of $\bar{M}_{jk}$ for the ground and excited state.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{We have plotted the ground state and the first excited state energy eigen values of the shifted harmonic oscillator at $\epsilon = 0.5$. The exact ground state and the first excited state energy values of the shifted harmonic oscillator at $\epsilon = 0.5$ are, $E = 1.25$ and $E = 3.25$ respectively.}
\end{figure}
We have plotted the ground state and the first excited state energy eigen values of the shifted harmonic oscillator at $\epsilon = 1$. The exact ground state and the first excited state energy values of the shifted harmonic oscillator at $\epsilon = 1$ are, $E = 2$ and $E = 4$ respectively.

4.3 Swanson Hamiltonian:

Here we consider the non-Hermitian Swanson Hamiltonian [19] which has the following form,

$$H = p^2 + x^2 + ic(xp + px) \quad (4.14)$$

The $V$ operator for this Hamiltonian is [19],

$$V = e^{-cx^2} \quad (4.15)$$

Now under this operator $V$, $x$ and $p$ changes in the following way,

$$x^V = V^{-1}xV = x$$

$$p^V = V^{-1}pV = (p + 2icx) \quad (4.16)$$

By choosing $O = \sum_k b_k x^k$ and using the positivity condition i.e. $\langle \tilde{R}_n | (O^t)^V O | R_n \rangle \geq 0$, we got the following inequality,

$$\sum_{j,k} b_j^* b_k \langle \tilde{R}_n | x^{j+k} | R_n \rangle \geq 0 \quad (4.17)$$
The above inequality implies the matrix, $\bar{M}_{jk} = \langle \bar{R}_n | x^{j+k} | R_n \rangle$, should be positive semi-definite. Using equation (3.5) we find the following recursion relation,

$$4tE \langle \bar{R}_n | x^{t-1} | R_n \rangle + t(t-1)(t-2) \langle \bar{R}_n | x^{t-3} | R_n \rangle - 4(1 + c^2)(t + 1) \langle \bar{R}_n | x^{t+1} | R_n \rangle = 0$$

(4.18)

Now using the positivity condition or $\bar{M}_{jk}$ is a positive semi definite matrix, we numerically find the eigenvalues of $H$. For this Hamiltonian also the search space is 1D.

Figure 4. We have plotted the ground state and the first excited state energy eigen values of the Swanson Hamiltonian oscillator at $\epsilon = 0.5$. The exact ground state and the first excited state energy values of the Swanson Hamiltonian at $\epsilon = 0.5$ are, $E = 1.12$ and $E = 3.35$ respectively.

Figure 5. We have plotted the ground state and the first excited state energy eigen values of the Swanson Hamiltonian at $\epsilon = 1$. The exact ground state and the first excited state energy values of the Swanson Hamiltonian at $\epsilon = 1$ are, $E = 1.41$ and $E = 4.24$ respectively.
4.4 PT-symmetric Pöschl–Teller potential:

The Hamiltonian for PT-symmetric Pöschl–Teller potential is given by,

\[ H = \frac{p^2}{2} - \frac{\lambda(\lambda + 1)}{2} \text{sech}^2 (x + i\epsilon) \]  

(4.19)

For this Hamiltonian the \( V \) operator is,

\[ V = e^{2\epsilon p} \]  

(4.20)

Now under this operator \( V \), \( x \) and \( p \) changes in the following way,

\[ xV = V^{-1}xV = (x + 2i\epsilon) \]  

\[ pV = V^{-1}pV = p \]  

(4.21)

Now let’s define, \( z = x + i\epsilon \) and substitute this in equation (4.19) then the equation (4.19) becomes,

\[ H = \frac{p^2}{2} - \frac{\lambda(\lambda + 1)}{2} \text{sech}^2 (z) \]  

(4.22)

We can easily show that, \( z \) has the following property,

\[ (z^\dagger)^V = V^{-1}(x - i\epsilon)V = z \]  

(4.23)

By choosing \( O = \sum_k b_k \text{sech}^{2k} (z) \) and using the positivity condition i.e. \( \langle \bar{R}_n | (O^\dagger)^V O | R_n \rangle \geq 0 \), we got the following inequality,

\[ \sum_{j,k} b_j^* b_k \langle \bar{R}_n | \text{sech}^{2j+k} (z) | R_n \rangle \geq 0 \]  

(4.24)

The above inequality implies the matrix, \( \tilde{M}_{jk} = \langle \bar{R}_n | \text{sech}^{2j+k} | R_n \rangle \), should be positive semi-definite. Using equation (3.5) we find the following recursion relation,

\[ (2tE + \frac{t^3}{2}) \langle \bar{R}_n | \text{sech}^t (z) | R_n \rangle + (\frac{t^3}{2} + 3t^2 + 4t + 2 + \frac{3t+2}{2} - 2\lambda(\lambda + 1)(t+2)) \langle \bar{R}_n | \text{sech}^{t+2} (z) | R_n \rangle + ( -2(t + 1)E + 2\lambda(\lambda + 1)(t + 1) - t^3 - 3t^2 - 4t - 2) \langle \bar{R}_n | \text{sech}^{t+4} (z) | R_n \rangle = 0 \]  

(4.25)

Now using the positivity condition or \( \tilde{M}_{jk} \) is a positive semi definite matrix, we numerically find the eigenvalues of \( H \). This potential is exactly solvable whose eigenvalues The plots obtained after bootstrap are shown in Fig 1
4.5 \( p^2 - x^4 \) potential:

Let’s consider the following Hamiltonian,

\[
H = p^2 - x^4 \tag{4.26}
\]

The potential \( V = -x^4 \) is unbounded below on the real line but if we take \( x \) on a contour in the lower-half complex plane then it can give rise to a well-posed bound state problem. However, the potential becomes PT-symmetric rather than Hermitian \([14, 20]\) and the equivalent PT-symmetric Hamiltonian is given by,

\[
H = \frac{1}{2} \{(1 + ix), p^2\} - \frac{1}{2} p - \alpha (1 + ix)^2 \tag{4.27}
\]

where, \( \alpha = 16 \). The \( V \) operator for this Hamiltonian is \([20]\),

\[
V = e^{\left(\frac{x^2}{2} - 2p\right)} \tag{4.28}
\]

Now under this operator \( V \), \( x \) and \( p \) changes in the following way,

\[
x^V = V^{-1}xV = x + i\left(\frac{p^2}{\alpha} - 2\right)
\]

\[
p^V = V^{-1}pV = p
\]

By choosing \( O = \sum_k a_k p^k \) and using the positivity condition i.e. \( \langle \bar{R}_n | (O^\dagger) V O | R_n \rangle \geq 0 \), we got the following inequality,

\[
\sum_{j,k} a_j^* a_k \langle \bar{R}_n | p^{j+k} | R_n \rangle \geq 0 \tag{4.30}
\]

The above inequality implies the matrix, \( \bar{M}_{jk} = \langle \bar{R}_n | p^{j+k} | R_n \rangle \), should be positive semi-definite. Using equation (3.5) we find the following recursion relation,

\[
4\alpha t E \langle \bar{R}_n | p^{t-1} | R_n \rangle + (2t + 1) \alpha \langle \bar{R}_n | p^t | R_n \rangle + \alpha^2 t(t - 1)(t - 2) \langle \bar{R}_n | p^{t-3} | R_n \rangle - (t + 2) \langle \bar{R}_n | p^{t+3} | R_n \rangle = 0 \tag{4.31}
\]

Now using the positivity condition or \( \bar{M}_{jk} \) is a positive semi definite matrix, we numerically find the eigenvalues of \( H \). A similar analysis can be done for Hamiltonian of the form \( H = p^2 + m^2 x^2 - x^4 \) which can be mapped to \( \frac{1}{2} \{(1 + ix), p^2\} - \frac{1}{2} p - m^2(1 + ix) - \alpha(1 + ix)^2 \).
Figure 6. We have plotted the ground state and the first excited state energy eigen values of eq 4.26

Figure 7. We have plotted the ground state and the first excited state energy eigen values of eq 4.26
4.6 Simple Harmonic Oscillator Coupled to a Shifted Simple Harmonic Oscillator

The Hamiltonian of the two coupled PT-symmetric harmonic oscillator is given by,

\[ H = p^2 + x^2 + y^2 + q^2 + 2iy + 2\epsilon xy \]  (4.32)

The \( V \) operator for this Hamiltonian is

\[ V = e^{-2(\alpha p + \beta q)} \]  (4.33)

where,

\[ \alpha = \frac{\epsilon}{1-\epsilon^2}, \quad \beta = -\frac{1}{1-\epsilon^2} \]  (4.34)

Now under this operator \( V \), \( p \) and \( q \) does not change but \( x \) and \( y \) changes in the following way,

\[ x^V = V^{-1}xV = (x - 2i\alpha) \]
\[ y^V = V^{-1}yV = (y - 2i\beta) \]  (4.35)

Now let’s define, \( x_1 = x - i\alpha, \ y_1 = y - i\beta \) and substitute this in equation (4.32) then the equation (4.32) becomes,

\[ H = p^2 + x_1^2 + q^2 + y_1^2 + 2\epsilon x_1 y_1 + \frac{1}{1-\epsilon^2} \]  (4.36)

We can easily show that, \( x_1 \) and \( y_1 \) has the following property,

\[ (x_1^\dagger)^V = V^{-1}(x + i\alpha)V = x_1 \]
\[ (y_1^\dagger)^V = V^{-1}(y + i\beta)V = y_1 \]  (4.37)

For this two-dimensional problem it is difficult to find a closed form recursion relation like (4.13) or (4.18). However, in the Ref. [1], Xizhi Han, Sean A. Hartnoll, etc. explained how to deal with this type of problem. First consider the following trial operators \( I, x_1, y_1, p, \) and \( q \). From the positivity condition defined in equation (3.5), the following bootstrap matrix should be positive semi definite and this condition is used to get Fig8:
| $I$ | $x_1$ | $y_1$ | $p$ | $q$ |
|-----|-------|-------|-----|-----|
| $I$ | 1     | 0     | 0   | 0   |
| $x_1$ | 0 | $\langle x_1^2 \rangle$ | $\frac{\langle p^2 \rangle - \langle x_1^2 \rangle}{\epsilon}$ | $\frac{i}{2}$ | 0 |
| $y_1$ | 0 | $\frac{\langle p^2 \rangle - \langle x_1^2 \rangle}{\epsilon}$ | $\langle x_1^2 \rangle$ | 0 | $\frac{i}{2}$ |
| $p$ | 0 | $\frac{-i}{2}$ | 0 | $\frac{\langle p^2 \rangle}{\epsilon}$ | $\langle (\epsilon^2 - 1)x_1^2 \rangle$ |
| $q$ | 0 | 0 | $\frac{-i}{2}$ | $\frac{\langle p^2 \rangle - (\epsilon^2 - 1)x_1^2}{\epsilon}$ | $\langle p^2 \rangle$ |

**Table 1.** Matrix for Bootstrapping $H = p^2 + x_1^2 + q^2 + y_1^2 + 2\epsilon x_1 y_1 + \frac{1}{1-\epsilon^2}$

**Figure 8.** Here we plot the results of bootstrap for $H = p^2 + x_1^2 + q^2 + y_1^2 + 2\epsilon x_1 y_1 + \frac{1}{1-\epsilon^2}$ and compare it against the exact spectrum. The exact energy eigenvalue solution which is plotted in red line in the above figure is given by $E(\text{ground state}) = \sqrt{1 + \epsilon} + \sqrt{1 - \epsilon} + \frac{1}{1-\epsilon^2}$

### 4.7 Simple Harmonic Oscillator Coupled to Swanson Hamiltonian

Let’s consider the following PT-symmetric non hermitian two-dimensional potential,

$$H = p^2 + x^2 + q^2 + y^2 + i\epsilon \{q, y\} + 2\epsilon xy$$  \hspace{1cm} (4.38)

The $V$ operator for this Hamiltonian is

$$V = e^{-c y^2}$$  \hspace{1cm} (4.39)
Now under this $V$ operator, $x$, $y$ and $p$ does not change but $q$ changes in the following way,

$$q^V = V^{-1}qV = (q + 2icy) \quad (4.40)$$

Now let’s define, $q_1 = q + icy$ and substitute this in equation (4.38) then the equation (4.38) becomes,

$$H = p^2 + x^2 + q_1^2 + (1 + c^2)y^2 + 2\epsilon xy \quad (4.41)$$

We can easily show that, $q_1$ has the following property,

$$(q_1^\dagger)^V = V^{-1}(q - icy)V = q_1 \quad (4.42)$$

For this two-dimensional problem it is difficult to find a closed form recursion relation like (4.13) or (4.18). However, in the Ref. [1], Xizhi Han, Sean A. Hartnoll, etc. explained how to deal with this type of problem. First consider the following trial operators $I, x, y, p,$ and $q_1$. From the positivity condition defined in equation (3.5), the following bootstrap matrix should be positive semidefinite and this condition is used to get Fig9:

| $i$ | $f$ | $x$ | $y$ | $p$ | $q_1$ |
|-----|-----|-----|-----|-----|-----|
| $x$ | 0   | $(x^2)$ | 0   | 0   | 0   |
| $y$ | 0   | $(x^2)$ | $(x^2)$ $(\frac{1}{2\alpha - 1}) + (x^2) (\frac{1}{2a - 1})$ | 0   | $\frac{1}{2}$ |
| $p$ | $\frac{1}{2}$ | 0   | $(p^2)$ | 0   | $(p^2) \frac{1}{2} + (x^2) (\alpha - \frac{1}{2})$ |
| $q_1$ | 0   | 0   | $\frac{1}{2}$ | $(p^2) \frac{1}{2} + (x^2) (\alpha - \frac{1}{2})$ | $(p^2) (1 - \frac{\alpha (\alpha - 1)}{2}) + (x^2) (1 - \frac{\alpha (\alpha - 1)}{2})$ |

Table 2. Matrix for Bootstrapping $H = p^2 + x^2 + q_1^2 + (1 + c^2)y^2 + 2\epsilon xy$
Figure 9. Here we plot the results of bootstrap for $H = p^2 + x^2 + q^2 + (1 + c^2)y^2 + 2\epsilon xy$ and compare it against the exact spectrum. The exact ground state energy for this system is given by $1 + \alpha$ where $\alpha = 1 + c^2$; hence it is a constant with respect to $\epsilon$. We have plotted for 3 different values of $\alpha$. The scatter plot in black is the actual bootstrap plot and the lines in red blue and green are the constant energy eigenvalues.

5 Discussion

Bootstrapping in quantum mechanics is the use of positivity condition to obtain spectrum. However for non-hermitian systems, the usual positivity condition does not work. In this paper we deal with particular class of non-hermitian hamiltonian, the PT symmetric hamil-tonian for which case the eigen spectrum is real. In this paper we point out how to generalise bootstrap technique for PT symmetric case. We show that one can define a modified positivity condition and use it to solve for the Eigen spectrum. We illustrate this with several examples.

One of the important application of the PT symmetric Hamiltonian is to describe the gain-loss system. The gain-loss system consists of two subsystem and it is not an isolated system because it is in contact with external environment. When this system is in dynamical equilibrium i.e. loss and gain are equal, it exhibits PT symmetry i.e. the
Hamiltonian of the composite system is PT symmetric. There are also many situations where PT symmetry\(^2\) comes out naturally like in studies of the Lee-Yang edge singularity [16, 17]. Our study of PT symmetric potentials motivates us to develop bootstrapping for more general class of non-hermitian hamiltonians. This might have potential application on quantum open systems [22]. One important goal would be to see if one can obtain the spectrum of the Lindbladian operator. This development will have far reaching applications such as in transport phenomenon and bootstrapping may be used to derive various bounds. We shall report on these exciting possibilities in future.

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A Some details

In this appendix, we describe the method used to bootstrap potentials in 4.6 and 4.7 First, a matrix is formed as mentioned in Table A. Then \(\langle [\mathcal{H}, \mathcal{O}] \rangle = 0\) identity is used to establish constraints on the expectation values of observables, \(\langle \mathcal{O} = \{I, X, Y, P, Q, XP, PX, \ldots \} \rangle\). These relations are used to find out the independent variables and rewrite the matrix in Table A in terms of the independent variables. For example, for the potential in 4.6, all other expectation values were written in terms of \(\langle x^2 \rangle\) and \(\langle p^2 \rangle\); this is shown in Table 4.6. Next, a simple program searches for different values of \(\langle x^2 \rangle\) and \(\langle p^2 \rangle\) for which the matrix in Table 2 is positive semi-definite and minimises the expression for ground state energy of the Hamiltonian. A similar computation was done for Sec 4.7 as well.

\(^2\)The ground state of a Bose system of hard spheres is described by a non-Hermitian Hamiltonian [21]
Table 3. General Matrix for Bootstrapping. This matrix will be used in Sec 4.6 and Sec 4.7 where a good recursion relation cannot be obtained.

| I | x | y | p | q |
|---|---|---|---|---|
| I | 1 | ⟨x⟩ | ⟨y⟩ | ⟨p⟩ | ⟨q⟩ |
| x | ⟨x⟩ | ⟨x²⟩ | ⟨xy⟩ | ⟨xp⟩ | ⟨xq⟩ |
| y | ⟨y⟩ | ⟨yx⟩ | ⟨y²⟩ | ⟨yp⟩ | ⟨yq⟩ |
| p | ⟨p⟩ | ⟨px⟩ | ⟨py⟩ | ⟨p²⟩ | ⟨pq⟩ |
| q | ⟨q⟩ | ⟨qx⟩ | ⟨qy⟩ | ⟨qp⟩ | ⟨q²⟩ |

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