A REVISION OF "ON ASYMPTOTIC STABILITY IN ENERGY SPACE OF GROUND STATES OF NLS IN 1D"

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Abstract. This is a revision of the author's paper "On asymptotic stability in energy space of ground states of NLS in 1D" [C3]. We correct an error in Lemma 5.4 [C3] and we simplify the smoothing argument.

§1 Introduction

We consider even solutions of a NLS

\[ iu_t + u_{xx} + \beta(|u|^2)u = 0 = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \]

We assume \( \beta(t) \) smooth, with

(H1) \( \beta(0) = \beta'(0) = 0, \beta \in C^\infty(\mathbb{R}, \mathbb{R}); \)

(H2) there exists a \( p \in (1, \infty) \) such that for every \( k = 0, 1, \)

\[ \left| \frac{d^k}{dv^k} \beta(v^2) \right| \lesssim |v|^{p-k-1} \quad \text{if} \quad |v| \geq 1; \]

(H3) there exists an open interval \( \mathcal{O} \) such that \( u_{xx} - \omega u + \beta(u^2)u = 0 \) admits a \( C^1 \)-family of ground states \( \phi_\omega(x) \) for \( \omega \in \mathcal{O}; \)

(H4) \( \frac{d^k}{d\omega^k} \| \phi_\omega \|_{L^2(\mathbb{R})}^2 > 0 \) for \( \omega \in \mathcal{O}. \)

By [ShS] the \( \omega \rightarrow \phi_\omega \in H^1(\mathbb{R}) \) is \( C^2 \) and by [We1,GSS1-2] (H4) yields orbital stability of the ground state \( e^{i\omega t} \phi_\omega(x) \). Here we investigate asymptotic stability.

We need some additional hypotheses.

(H5) For any \( x \in \mathbb{R}, u_0(x) = u_0(-x). \) That is, the initial data \( u_0 \) of (1.1) are even.

(H6) Let \( H_\omega \) be the linearized operator around \( e^{i\omega t} \phi_\omega \), see (1.3). \( H_\omega \) has a positive simple eigenvalue \( \lambda(\omega) \) for \( \omega \in \mathcal{O}. \) There exists an \( N \in \mathbb{N} \) such that \( N\lambda(\omega) < \omega < (N+1)\lambda(\omega). \)

(H7) The Fermi Golden Rule (FGR) holds (see Hypothesis 4.2 in Section 4).

(H8) The point spectrum of \( H_\omega \) consists of 0 and \( \pm \lambda(\omega). \) The points \( \pm \omega \) are not resonances.

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Theorem 1.1. Let $\omega_0 \in \mathcal{O}$ and $\phi_{\omega_0}(x)$ be a ground state. Let $u(t,x)$ be a solution of (1.1). Assume (H1)–(H8). Then, there exist an $\epsilon_0 > 0$ and a $C > 0$ such that if $\inf_{\gamma \in [0,2\pi]} \|u_0 - e^{i\gamma}\phi_\omega\|_{H^1} < \epsilon < \epsilon_0$, then there exist $\omega_+ \in \mathcal{O}$, $\theta \in C^1(\mathbb{R};\mathbb{R})$ and $h_+ \in H^1$ with $\|h_+\|_{H^1} \leq C\epsilon$ such that

$$\lim_{t \to \infty} \|u(t,\cdot) - e^{i\theta(t)}\phi_{\omega_+} - e^{it\partial^2}h_+\|_{H^1} = 0.$$ 

Theorem 1.1 is the one dimensional version of Theorem 1.1 [CM], which is valid for dimensions $D \geq 3$. In [CM] there is also a version of the theorem with (H8) replaced by a more general hypothesis, with more than one positive eigenvalue allowed (but then a more restrictive (FGR) hypothesis (H7) is required). A similar result could be proved here, but we prefer to skip the proof. We recall that results of the sort discussed here were pioneered by Soffer & Weinstein [SW1], see also [PW], followed by Buslaev & Perelman [BP1-2], about 15 years ago. In this decade these early works were followed by a number of results [BS,C1-2,GNT,M1-2,P,RSS,SW2,TY1-3,Wd1]. It was heuristically understood that the rate of the leaking of energy from the so called “internal modes” into radiation, is small and decreasing when $N$ increases, producing technical difficulties in the closure of the nonlinear estimates. For this reason prior to Gang Zhou & Sigal [GS1], the literature treated only the case when $N = 1$ in (H6). [GS1] sheds light for $N > 1$, with the eigenvalue $\lambda(\omega)$ possibly very close to 0. Here we strengthen the result in [GS1] for $D = 1$, in analogy to the way [CM] strengthens [GS1] for dimensions $D \geq 3$. For a detailed introduction to the problem of asymptotic stability we refer to [CM]. There are three hypotheses in [GS1] which we relax here. First of all, the (FGR) hypothesis in [GS1] is more restrictive than (H7). Specifically, [GS1] require a sign assumption on a coefficient of a certain equation obtained during a normal forms expansion. In [CM] and later in this paper, it is shown that it is enough to assume that the coefficient be nonzero, a generic condition, and then it is proved that it has the right sign. Second, [GS1] deals with solutions whose initial datum $u_0(x)$ satisfies more stringent conditions than being of finite energy. Finally, in the 1D case, [GS1] requires that $\beta(t)$ be very small near 0, specifically $|\beta(t)| \lesssim |t|^{3N+2}$ for $|t| \leq 1$, which we ease considerably here, since we only need $|\beta(t)| \lesssim |t|^2$. Notice that the symmetry restriction (H5) is only required to avoid moving ground states, and that if we add to (1.1) some spacial inhomogeneity, thus eliminating translation invariance, then (H5) is unnecessary. So in particular our result, dropping (H5), will apply to equations like in [GS1] of the form $iu_t + u_{xx} + V(x)u + \beta(|u|^2)u = 0$ with $V(x)$, a short range real valued potential. As remarked in [CM], our result is relevant also to equations of the form $iu_t + u_{xx} + V(x)|u|^4u = 0$ in the cases treated Fibich and Wang [FW] where ground states are proved to be orbitally stable.

The proof of Theorem 1.1 is inspired by Mizumachi [M1] and its use of Kato smoothing for the linearization which, given $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 2 \end{bmatrix}$.
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix},
\]
is defined by

(1.2) \[ H_\omega = \sigma_3 \left[-d^2/dx^2 + \omega - \beta(\phi_\omega^2) - \beta'(\phi_\omega^2)\phi_\omega^2\right] + i\beta'(\phi_\omega^2)\phi_\omega^2. \]

We exploit plane waves expansions for \( H_\omega \) and dispersive estimates for the group \( e^{-itH_\omega}P_{\ell}(\omega) \) proved in \([KS,GS1]\). We also improve the Strichartz estimates proved in \([KS]\) by means of a \( TT^* \) argument similar to the flat case.

We end with some notation. We set \( \langle x \rangle = \sqrt{1 + x^2} \). We set \( \| \langle x \rangle^\tau u \|_{H^k} \). We set \( \langle f, g \rangle = \int f(x)\overline{g(x)}dx \), with \( f(x) \) and \( g(x) \) column vectors, \( ^tA \) the transpose and \( \overline{g} \) the complex conjugate of \( g \). Given \( x \in \mathbb{R} \) set \( x^+ = x \lor 0 \) and \( x^- = (-x) \land 0 \). \( R_H(z) = (H - z)^{-1} \). \( W^{k,p}(\mathbb{R}) \) is the space of tempered distributions \( f(x) \) such that \((1 - \partial_x^2)^{k/2} f \in L^p(\mathbb{R})\).

\section{2 Linearization, modulation and set up}

We will use the following classical result, \([We1,GSS1-2]\):

**Theorem 2.1.** Suppose that \( e^{it\omega t}\phi_\omega(x) \) satisfies (H4). Then \( \exists \epsilon > 0 \) and a \( A_0(\omega) > 0 \) such that for any \( \| u(0,x) - \phi_\omega \|_{H^1} < \epsilon \) we have for the corresponding solution \( \inf\{\| u(t,x) - e^{it\phi_\omega(x - x_0)} \|_{H^1(x \in \mathbb{R})} : \gamma \in \mathbb{R} \land x_0 \in \mathbb{R} \} < A_0(\omega)\epsilon \).

This statement is stronger than the one in \([We1,GSS1-2]\) since we state a more precise estimate for the \( \delta(\epsilon) \) than in these papers. We sketch the proof in \( \S 9 \). Now we review some well known facts about the linearization at a ground state. We can write the ansatz \( u(t,x) = e^{i\Theta(t)}(\phi_\omega(t)(x) + r(t,x)) \), \( \Theta(t) = \int_0^t \omega(s)ds + \gamma(t). \) Inserting the ansatz into the equation we get

\[
ir_t = -r_{xx} + \omega(t)r - \beta(\phi_\omega^2(t))r - \beta'(\phi_\omega^2(t))\phi_\omega^2(t)r
- \beta'(\phi_\omega^2(t))\phi_\omega^2(t)r^\gamma + \gamma(t)\phi_\omega(t) - i\omega(t)\partial_\omega\phi_\omega(t) + \gamma(t)r + O(r^2).
\]

We set \( ^tR = (r, \bar{r}) \), \( ^t\Phi = (\phi_\omega, \phi_\omega) \) and we rewrite the above equation as

(2.1) \[ iR_t = H_\omega R + \sigma_3\gamma R + \sigma_3^2\Phi - i\omega\partial_\omega\Phi + O(R^2). \]

Set \( H_0(\omega) = \sigma_3(-d^2/dx^2 + \omega) \) and \( V(\omega) = H_\omega - H_0(\omega) \). The essential spectrum is

\( \sigma_\epsilon = \sigma_\epsilon(H_\omega) = \sigma_\epsilon(H_0(\omega)) = (-\infty, -\omega] \cup [\omega, +\infty) \).

0 is an isolated eigenvalue. Given an operator \( L \) we set \( N_g(L) = \bigcup_{j \geq 1} N(L^j) \) and \( N(L) = \ker L \). \([We2]\) implies that, if \( \cdot \) means span, \( N_g(H_\omega^\omega) = \{ \Phi, \sigma_3\partial_\omega\Phi \} \). \( \lambda(\omega) \) has corresponding real eigenvector \( \xi(\omega) \), which can be normalized so that \( \langle \xi, \sigma_3\xi \rangle = 1 \). \( \sigma_1\xi(\omega) \) generates \( N(H_\omega + \lambda(\omega)) \). The function \( (\omega, x) \in O \times \mathbb{R} \rightarrow \xi(\omega, x) \) is \( C^2 \);
\[ |\xi(\omega, x)| < ce^{-a|x|} \] for fixed \( c > 0 \) and \( a > 0 \) if \( \omega \in K \subset \mathcal{O}, K \) compact. \( \xi(\omega, x) \) is even in \( x \) since by assumption we are restricting ourselves in the category of such functions. We have the \( H_\omega \) invariant Jordan block decomposition

\[ L^2 = N_g(H_\omega) \oplus (\oplus_{j, \pm} N(H_\omega \mp \lambda(\omega))) \oplus L_c^2(H_\omega) = N_g(H_\omega) \oplus N^\perp_g(H_\omega^*) \]

where we set \( L_c^2(H_\omega) = \{ N_g(H_\omega^*) \oplus \oplus_{ \pm} N(H_\omega \mp \lambda(\omega)) \}^\perp \). We can impose

\[ (2.2) \quad R(t) = (z\xi + \bar{z}\sigma_1\xi) + f(t) \in \left[ \sum_{\pm} N(H_\omega(t) \mp \lambda(\omega(t))) \right] \oplus L^2_c(H_\omega(t)). \]

The following claim admits an elementary proof which we skip:

**Lemma 2.2.** There is a Taylor expansion at \( R = 0 \) of the nonlinearity \( O(R^2) \) in (2.1) with \( R_{m,n}(\omega, x) \) and \( A_{m,n}(\omega, x) \) real vectors and matrices rapidly decreasing in \( x \): \( O(R^2) = \)

\[ \sum_{2 \leq m+n \leq 2N+1} R_{m,n}(\omega)z^m\bar{z}^n + \sum_{1 \leq m+n \leq N} z^m\bar{z}^n A_{m,n}(\omega)f + O(f^2 + |z|^{2N+2}). \]

In terms of the frame in (2.2) and the expansion in Lemma 2.2, (2.1) becomes

\[ if_t = (H_\omega(t) + \sigma_3\gamma) f + \sigma_3\dot{\gamma}\Phi(\omega) - i\dot{\omega}\partial_\omega\Phi(t) + (z\lambda(\omega) - iz)m(\omega) \]

\[ - (\bar{z}\lambda(\omega) + iz)\sigma_1\xi(\omega) + \sigma_3\dot{\gamma}(z\xi + \bar{z}\sigma_1\xi) - i\dot{\omega}(z\partial_\omega\xi + \bar{z}\sigma_1\partial_\omega\xi) \]

\[ + \sum_{2 \leq m+n \leq 2N+1} z^m\bar{z}^n R_{m,n}(\omega) + \sum_{1 \leq m+n \leq N} z^m\bar{z}^n A_{m,n}(\omega)f + \]

\[ + O(f^2) + O_{\text{loc}}(|z|^{2N+2}) \]

where by \( O_{\text{loc}} \) we mean that the there is a factor \( \chi(x) \) rapidly decaying to 0 as \( |x| \to \infty \). By taking inner product of the equation with generators of \( N_g(H_\omega^*) \) and \( N(H_\omega^* - \lambda) \) we obtain modulation and discrete modes equations:

\[ i\dot{\omega} \frac{d}{d\omega} \|\phi_\omega\|^2 = \langle \sigma_3\dot{\gamma}(z\xi + \bar{z}\sigma_1\xi) - i\dot{\omega}(z\partial_\omega\xi + \bar{z}\sigma_1\partial_\omega\xi) + \sum_{m+n=2}^{2N+1} z^m\bar{z}^n R_{m,n}(\omega) \]

\[ + \sum_{m+n=1}^N z^m\bar{z}^n A_{m,n}(\omega) \rangle f + O(f^2) + O_{\text{loc}}(|z|^{2N+2}), \Phi \]

\[ \dot{\gamma} \frac{d}{d\omega} \|\phi_\omega\|^2 = \langle \text{same as above} , \sigma_3\partial_\omega\Phi \rangle \]

\[ i\dot{z} - \lambda(\omega)z = \langle \text{same as above} , \sigma_3\xi \rangle. \]
§3 Spacetime estimates for $H_\omega$

We collect some linear estimates needed for the proof of Theorem 1.1 in §4. First of all we prove that the group $e^{-itH_\omega} P_c(\omega)$ satisfies the same Strichartz estimates of the flat case. The proof is almost the same of the flat case. In particular we are able to implement a $TT^*$ argument. For a different proof without the $L_t^4 L_x^\infty$ estimate, see Corollary 7.3 [KS].

**Lemma 3.1 (Strichartz estimate).** There exists a positive number $C = C(\omega)$ upper semicontinuous in $\omega$ such that for any $k \in [0, 2]$:
(a) for any $f \in L_c^2(\omega)$,
$$
\|e^{-itH_\omega} f\|_{L_t^4 W_x^{k, \infty} \cap L_x^\infty H_x^k} \leq C\|f\|_{H^k}.
$$
(b) for any $g(t, x) \in S(\mathbb{R}^2)$,
$$
\|\int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot) ds\|_{L_t^4 W_x^{k, \infty} \cap L_x^\infty H_x^k} \leq C\|g\|_{L_t^{4/3} W_x^{k,1} + L_t^4 H_x^k}.
$$

**Proof.** First of all, the case $0 < k \leq 2$ follows by the case $k = 0$ by a simple argument in Corollary 7.3 [KS]. Now we focus on the $k = 0$ case. For any $2 \leq p \leq \infty$ by [BP1,KS,GS2] $\exists C = C(\omega)$ upper semicontinuous in $\omega$ such that

(1)
$$
\|e^{-itH_\omega} P_c(\omega)f\|_{L_p^p} \leq Ct^{-\frac{1}{4} + \frac{1}{p}}\|f\|_{L_p^{p-1}}.
$$

(2) is a consequence of (1) and of Hardy Littlewood theorem. The $L_t^\infty L_x^2$ estimate in (a) is an immediate consequence of (1) for $p = 2$. The quadratic form $\langle f, \sigma_3 g \rangle$ defined in $L_t^1(\omega) \times L_x^\infty(\omega)$ establishes an isomorphism $(L_t^1(\omega))^* \simeq L_x^\infty(\omega)$. Based on $\langle e^{-itH_\omega} f, \sigma_3 g \rangle = \langle f, \sigma_3 e^{-itH_\omega} g \rangle$ the following operators are formally adjoints

$$
g(t, x) \in L_t^{4/3} L_x^1(\omega) \rightarrow Tg = \int_{\mathbb{R}} e^{itH_\omega} g(t)(x) dt \in L_c^2(\omega)
$$

and $f \in L_c^2(\omega) \rightarrow T^* f = e^{-itH_\omega} f \in L_t^4 L_x^\infty(\omega)$.

Then we can perform a slight modification of the standard $TT^*$ argument. Preliminarily, we split $P_c(\omega) = P_+(\omega) + P_-(\omega)$ the projections in the positive and negative part of $\sigma_c(H_\omega)$, see Appendix B and [BP2,BS,C2]. We bound separately $P_\pm(\omega) \circ Tf$. The operator $T^* \circ P_\pm(\omega) \circ T$ is bounded thanks to (1) and Hardy Littlewood theorem. We write, for $L_p^p = L_c^p(\omega)$,

$$
|\langle P_\pm(\omega) \circ Tf, \sigma_3 P_\pm(\omega) \circ Tf \rangle_{\mathcal{L}}| = |\langle T^* \circ P_\pm(\omega) \circ Tf, \sigma_3 f \rangle_{\mathcal{L}}| \leq \|T^* \circ P_\pm(\omega) \circ T : L_t^{4/3} L_c^1 \rightarrow L_t^4 L_c^\infty\| \|f\|_{L_t^{4/3} L_c^1}^2.
$$
Assuming

\[ \langle P_\pm(\omega)h, \sigma_3 P_\pm(\omega)h \rangle_x \approx \pm \| P_\pm(\omega)h \|_{L_x^2(\omega)} \]

we conclude \( \| P_\pm(\omega) \circ T f \|_{L_x^2(\omega)} \lesssim \| f \|_{L_x^{4/3}L_x^1(\omega)} \). Adding up we get \( \| T f \|_{L_x^2(\omega)} \lesssim \| f \|_{L_x^{4/3}L_x^1(\omega)} \). For \( \psi \in C_0([0, \infty) \times \mathbb{R}) \) we get the following which yields (a):

\[ \langle T^* f, \sigma_3 \psi \rangle_{tx} = \langle f, \sigma_3 T \psi \rangle_{tx} \leq C \| f \|_{L_x^2(\omega)} \| \psi \|_{L_x^{4/3}L_x^1}. \]

To obtain (2) we observe that there exists a wave operator \( W : L^2(\mathbb{R}) \to L^2_\omega(\omega) \) which is an isomorphism with inverse \( Z \) such that for \( h = WH \) and \( \tilde{h} = (\tilde{h}_1, \tilde{h}_2) \) we have

\[ \langle P_+(\omega)h, \sigma_3 P_+(\omega)h \rangle = \| \tilde{h}_1 \|^2_2 \approx \| P_+(\omega)h \|_{L_x^2(\omega)} \] \( \) and
\[ \langle P_-(\omega)h, \sigma_3 P_-(\omega)h \rangle = -\| \tilde{h}_2 \|^2_2 \approx -\| P_-(\omega)h \|_{L_x^2(\omega)}. \]

\( W \) and \( Z \) above can be defined in a standard way, \( Z \) thanks to (1) and Proposition 8.1 [KS], as strong limits \( W(\omega) = \lim_{t \to +\infty} e^{-itH_\omega} e^{it\sigma_3(-\Delta + \omega)} \), \( Z(\omega) = \lim_{t \to +\infty} e^{it\sigma_3(\Delta - \omega)} e^{itH_\omega} \) and by standard theory they are inverses of each other.

**Lemma 3.2.** Fix \( \tau > 3/2 \).

1. There exists \( C = C(\tau, \omega) \), upper semicontinuous in \( \omega \) such that for any \( \varepsilon \neq 0 \)

\[ \| R_{H_\omega}(\lambda + i\varepsilon) P_\varepsilon(H_\omega)u \|_{L_x^{2,\tau}} \leq C \| u \|_{L^2}. \]

2. For any \( u \in L_x^{2,\tau} \) the following limits:

\[ \lim_{\varepsilon \searrow 0} R_{H_\omega}(\lambda \pm i\varepsilon)u = R_{H_\omega}^\pm(\lambda)u \text{ in } C_0^0(\sigma_e(H_\omega), L_x^{2,\tau}). \]

3. We have

\[ \| R_{H_\omega}^\pm(\lambda) P_\varepsilon(H_\omega) \|_{B(L_x^{2,\tau}, L_x^{2,\tau})} < C(\lambda)^{-\frac{1}{2}}. \]

4. Given any \( u \in L_x^{2,\tau} \) we have

\[ P_\varepsilon(H_\omega)u = \frac{1}{2\pi i} \int_{\sigma_e(H_\omega)} (R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda))u d\lambda. \]

These are consequences of the fact that \( \sigma_e(H_\omega) \) does not contain eigenvalues and that \( \pm \omega \) are not resonances, and of the theory in [KS].
Lemma 3.3. For any $k$ and $\tau > 3/2 \exists C = C(\tau, k, \omega)$ upper semicontinuous in $\omega$ such that:

(a) for any $f \in S(\mathbb{R})$,
$$
\| e^{-it\omega} P_c(H_\omega) f \|_{L^2_t H^k_x} \leq C \| f \|_{H^k}.
$$

(b) for any $g(t, x) \in S(\mathbb{R}^2)$
$$
\left\| \int_{\mathbb{R}} e^{it\omega} P_c(H_\omega) g(t, \cdot) dt \right\|_{H^k_x} \leq C \| g \|_{L^2_t H^k_x}.
$$

Proof. It is enough to prove Lemma 3.3, as well as Lemmas 3.4 below, for $k = 0$.

(a) implies (b) by duality:
$$
| \langle f, \sigma_3 \int e^{it\omega} P_c(\omega) g(t) dt \rangle_x | = | \langle \langle x \rangle^{-\tau} e^{-it\omega} P_c(\omega) f, \sigma_3 \langle x \rangle^\tau g \rangle_x | \\
\leq \| e^{-it\omega} P_c(\omega) f \|_{L^2_t L^2_x} \| g \|_{L^2_t L^2_x} \leq \| f \|_{L^2_t} \| g \|_{L^2_t L^2_x}.
$$

We now prove (a) for $k = 0$. Let $g(t, x) \in S(\mathbb{R}^2)$ with $g(t) = P_c(H_\omega) g(t)$. Then

$$
\langle e^{-it\omega} f, \sigma_3 g \rangle_{t,x} = \frac{1}{\sqrt{2\pi i}} \int e^{-i\lambda t} \left\langle \left( R^+_H(\lambda) - R^-_H(\lambda) \right) f, \sigma_3 \overline{g}(\lambda) \right\rangle_x d\lambda
$$

$$
= \frac{1}{\sqrt{2\pi i}} \int_{\sigma_c(H_\omega)} e^{-i\lambda t} \left\langle \left( R^+_H(\lambda) - R^-_H(\lambda) \right) f, \sigma_3 \overline{g}(\lambda) \right\rangle_x d\lambda.
$$

Then from Fubini and Plancherel and by (1) Lemma 3.3 we have

$$
\| \langle e^{-it\omega} f, \sigma_3 g \rangle_{t,x} \| \leq (2\pi)^{-1/2} \| (R^+_H(\lambda) - R^-_H(\lambda)) f \|_{L^2_t L^2_x} \times \| \overline{g}(\lambda, \cdot) \|_{L^2_t L^2_x} \lesssim \| f \|_{L^2_t} \| g \|_{L^2_t L^2_x}.
$$

Lemma 3.4. For any $k$ and $\tau > 3/2 \exists C = C(\tau, k, \omega)$ as above such that $\forall g(t, x) \in S(\mathbb{R}^2)$

$$
\left\| \int_0^t e^{-i(t-s)\omega} P_c(H_\omega) g(s, \cdot) ds \right\|_{L^2_t H^k_x} \leq C \| g \|_{L^2_t H^k_x}.
$$

Proof. By Plancherel and Hölder inequalities and by (3) Lemma 3.2 we have

$$
\| \int_0^t e^{-i(t-s)\omega} P_c(H_\omega) g(s, \cdot) ds \|_{L^2_t L^2_x} \leq
$$

$$
\leq \| R^+_H(\lambda) P_c(H_\omega) \chi_{[0, +\infty)} * \lambda \overline{g}(\lambda, x) \|_{L^2_t L^2_x} \leq
$$

$$
\leq \| R^+_H(\lambda) P_c(H_\omega) \|_{B(L^2_t L^2_x)} \| \chi_{[0, +\infty)} * \lambda \overline{g}(\lambda, x) \|_{L^2_t L^2_x} \leq
$$

$$
\leq \| R^+_H(\lambda) P_c(H_\omega) \|_{L^\infty(\mathbb{R}, B(L^2_t L^2_x))} \| g \|_{L^2_t L^2_x} \leq C \| g \|_{L^2_t L^2_x}.
$$
Lemma 3.5. \( k \) and \( \tau > 3/2 \) \( \exists C = C(\tau, k, \omega) \) as above such that \( \forall g(t, x) \in S(\mathbb{R}^2) \)

\[
\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(H_\omega) g(s, \cdot) ds \right\|_{L_t^\infty L_x^2 \cap L_t^4 W_x^{1,10} \cap L_t^4 L_x^\infty} \leq C \|g\|_{L_t^2 H_x^\tau}.
\]

Proof. For \( g(t, x) \in S(\mathbb{R}^2) \) set

\[
Tg(t) = \int_0^{+\infty} e^{-i(t-s)H_\omega} P_c(H_\omega) g(s) ds.
\]

Lemma 3.3 (b) implies \( f := \int_0^{+\infty} e^{isH_\omega} P_c(\omega) g(s) ds \in L^2(\mathbb{R}) \). Then Lemma 3.5 is a direct consequence of [CK].

§4 Proof of Theorem 1.1

We restate Theorem 1.1 in a more precise form:

Theorem 4.1. Under the assumptions of Theorem 1.1 we can express

\[
u(t, x) = e^{i\Theta(t)} \left( \phi_{\omega(t)}(x) + \sum_{j=1}^{2N} p_j(z, \bar{z}) A_j(x, \omega(t)) + h(t, x) \right)
\]

with \( p_j(z, \bar{z}) = O(z) \) near 0, with \( \lim_{t \to +\infty} \omega(t) \) convergent, with \( |A_j(x, \omega(t))| \leq C e^{-a|x|} \) for fixed \( C > 0 \) and \( a > 0 \), \( \lim_{t \to +\infty} z(t) = 0 \), and for fixed \( C > 0 \)

\[
\|z(t)\|_{L_t^{N+1} L_x^{N+2}} + \|h(t, x)\|_{L_t^\infty H_x^1 \cap L_t^4 W_x^{1,10} \cap L_t^4 L_x^\infty} < C \epsilon.
\]

Furthermore, there exists \( h_\infty \in H^1(\mathbb{R}, \mathbb{C}) \) such that

\[
\lim_{t \to -\infty} \left\| e^{i \int_0^t \omega(s) ds + i \gamma(t)} h(t) - e^{i t \frac{\partial^2}{\partial x^2}} h_\infty \right\|_{H^1} = 0.
\]

The proof of Theorem 4.1 consists in a normal forms expansion and in the closure of some nonlinear estimates. The normal forms expansion is exactly the same of [CM], in turn an adaptation of [GS1].

§4.1 Normal Form Expansion

We repeat [CM]. We pick \( k = 1, 2, \ldots N \) and set \( f = f_k \) for \( k = 1 \). The other \( f_k \) are defined below. In the ODE’s there will be error terms of the form

\[
E_{ODE}(k) = O(|z|^{2N+2}) + O(z^{N+1} f_k) + O(f_k^2) + O(\beta (|f_k|^2) f_k).
\]
In the PDE’s there will be error terms of the form

$$E_{PDE}(k) = O_{loc}(|z|^{N+2}) + O_{loc}(zf_k) + O_{loc}(f_k^2) + O(\beta(|f_k|^2)f_k).$$

In the right hand sides of the equations (2.3-4) we substitute $\dot{\gamma}$ and $\dot{\omega}$ using the modulation equations. We repeat the procedure a sufficient number of times until we can write for $k = 1$ and $f_1 = f$

$$i\omega \frac{d}{d\omega} \|\phi_\omega\|^2_2 = \left( \sum_{m+n=2}^{2N+1} z^m \bar{z}^n A_{m,n}^{(k)}(\omega) + \sum_{m+n=1}^{N} z^m \bar{z}^n A_{m,n}^{(k)}(\omega) f_k + E_{ODE}(k), \Phi(\omega) \right)$$

$$i\dot{z} - \lambda z = \langle \text{same as above} , \sigma_3 \xi(\omega) \rangle$$

$$i\partial_t f_k = (H_\omega + \sigma_3 \dot{\gamma}) f_k + E_{PDE}(k) + \sum_{k+1 \leq m+n \leq N+1} z^m \bar{z}^n R_{m,n}^{(k)}(\omega),$$

with $A_{m,n}^{(k)}$, $R_{m,n}^{(k)}$ and $A_{m,n}^{(k)}(\omega, x)$ real exponentially decreasing to 0 for $|x| \to \infty$ and continuous in $(\omega, x)$. Exploiting $|(m-n)\lambda(\omega)| < \omega$ for $m+n \leq N$, $m \geq 0$, $n \geq 0$, we define inductively $f_k$ with $k \leq N$ by

$$f_{k-1} = - \sum_{m+n=k} z^m \bar{z}^n R_{H_\omega}((m-n)\lambda(\omega)) R_{m,n}^{(k-1)}(\omega) + f_k.$$

Notice that if $R_{m,n}^{(k-1)}(\omega, x)$ is real exponentially decreasing to 0 for $|x| \to \infty$, the same is true for $R_{H_\omega}((m-n)\lambda(\omega)) R_{m,n}^{(k-1)}(\omega)$ by $|(m-n)\lambda(\omega)| < \omega$. By induction $f_k$ solves the above equation with the above notifications. Now we manipulate the equation for $f_N$. We fix $\omega_1 = \omega(0)$. We write

$$i\partial_t P_c(\omega_1)f_N = \{H_{\omega_1} + (\dot{\gamma} + \omega - \omega_1)(P_+(\omega_1) - P_-(\omega_1))\} P_c(\omega_1)f_N +$$

$$+ P_c(\omega_1)\tilde{E}_{PDE}(N) + \sum_{m+n=N+1} z^m \bar{z}^n P_c(\omega_1) R_{m,n}^{(N)}(\omega_1)$$

where we split $P_c(\omega_1) = P_+(\omega_1) + P_-(\omega_1)$ with $P_\pm(\omega_1)$ the projections in $\sigma_c(H_{\omega_1}) \cap \{\lambda : \pm \lambda \leq \omega_1\}$, see [BP2,BS,C2] and Appendix B, and with

$$\tilde{E}_{PDE}(N) = E_{PDE}(N) + \sum_{m+n=N+1} z^m \bar{z}^n \left( R_{m,n}^{(N)}(\omega) - R_{m,n}^{(N)}(\omega_1) \right) + \varphi(t, x)f_N$$

$$\varphi(t, x) := (\dot{\gamma} + \omega - \omega_1)(P_c(\omega_1) \sigma_3 - (P_+(\omega_1) - P_-(\omega_1))) f_N + (V(\omega) - V(\omega_1)) f_N$$

$$+ (\dot{\gamma} + \omega - \omega_1)(P_c(\omega) - P_c(\omega_1)) \sigma_3 f_N.$$

By Appendix B for $C_N(\omega_1)$ upper semicontinuous in $\omega_1$, $\forall N$ we have

$$\|\langle x \rangle^N (P_+(\omega_1) - P_-(\omega_1) - P_c(\omega_1) \sigma_3) f \|_{L^2_x}^2 \leq C_N(\omega_1) \|\langle x \rangle^{-N} f \|_{L^2_x}^2,$$
see also [BP2,BS]. Then \( \varphi(t,x) \) can be treated as a small cutoff function. We write

\[
(4.4) \quad f_N = - \sum_{m+n=N+1} z^m \bar{z}^n R_{H_{-1}} ((m-n)\lambda(\omega_1) + i0) P_c(\omega_1) R_{m,n}^{(N)}(\omega_1) + f_{N+1}.
\]

Then

\[
(4.5) \quad i\partial_t P_c(\omega_1) f_{N+1} = (H_{\omega_1} + (\hat{\gamma} + \omega - \omega_1)(P_+(\omega_1) - P_-(\omega_1))) P_c(\omega_1) f_{N+1} +
\]

\[+ \sum_{\pm} O(\epsilon |z|^{N+1}) R_{H_{-1}} ((\pm(N+1)\lambda(\omega_1) + i0) R_{\pm}(\omega_1) + P_c(\omega_1) \tilde{E}_{PDE}(N))
\]

with \( R_+ = R_{N+1,0}^{(N)} \) and \( R_- = R_{0,N+1}^{(N)} \) and \( \tilde{E}_{PDE}(N) = \tilde{E}_{PDE}(N) + O_{loc}(\epsilon |z|^{N+1}) \),

where we have used that \( (\omega - \omega_1) = O(\epsilon) \) by Theorem 2.1. Notice that \( R_{H_{-1}} ((\pm(N+1)\lambda(\omega_1) + i0) R_{\pm}(\omega_1) \in L^\infty \) do not decay spatially. In the ODE’s with \( k = N \), by the standard theory of normal forms and following the idea in Proposition 4.1 [BS], see [CM] for details, it is possible to introduce new unknowns

\[
(4.6) \quad \tilde{\omega} = \omega + q(\omega, z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \langle f_N, \alpha_{mn}(\omega) \rangle,
\]

\[
\tilde{z} = z + p(\omega, z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \langle f_N, \beta_{mn}(\omega) \rangle,
\]

with \( p(\omega, z, \bar{z}) = \sum p_{m,n}(\omega) z^m \bar{z}^n \) and \( q(z, \bar{z}) = \sum q_{m,n}(\omega) z^m \bar{z}^n \) polynomials in \((z, \bar{z})\) with real coefficients and \( O(|z|^2) \) near 0, such that we get

\[
(4.7) \quad \dot{\tilde{\omega}} = \langle E_{PDE}(N), \Phi \rangle
\]

\[
\dot{\tilde{z}} - \lambda(\omega) \tilde{z} = \sum_{1 \leq m \leq N} a_m(\omega) |\tilde{z}|^m \tilde{z} + \langle E_{ODE}(N), \sigma_3 \xi \rangle +
\]

\[
+ \tilde{z}^N \langle A_{0,N}(\omega) f_N, \sigma_3 \xi \rangle.
\]

with \( a_m(\omega) \) real. Next step is to substitute \( f_N \) using (4.4). After eliminating by a new change of variables \( \tilde{z} = \tilde{z} + p(\omega, \tilde{z}, \bar{z}) \) the resonant terms, with \( p(\omega, \tilde{z}, \bar{z}) = \sum \tilde{p}_{m,n}(\omega) z^m \bar{z}^n \) a polynomial in \((z, \bar{z})\) with real coefficients \( O(|z|^2) \) near 0, we get

\[
(4.8) \quad \dot{\tilde{\omega}} = \langle E_{PDE}(N), \Phi \rangle
\]

\[
\dot{\tilde{z}} - \lambda(\omega) \tilde{z} = \sum_{1 \leq m \leq N} \tilde{a}_m(\omega) |\tilde{z}|^m \tilde{z} + \langle E_{ODE}(N), \sigma_3 \xi \rangle -
\]

\[
- |\tilde{z}|^2 \tilde{z} \langle \tilde{A}_{0,N}(\omega) R_{H_{-1}} ((N+1)\lambda(\omega_1) + i0) P_c(\omega_0) R_{N+1,0}^{(N)}(\omega_1), \sigma_3 \xi \rangle
\]

\[
+ \tilde{z}^N \langle \tilde{A}_{0,N}(\omega) f_{N+1}, \sigma_3 \xi \rangle
\]

10
with \( \tilde{\alpha}_m, \tilde{A}^{(N)}_{0, N} \) and \( R^{(N)}_{N+1, 0} \) real. By \( \frac{1}{x - i0} = PV \frac{1}{x} + i\pi \delta_0(x) \) and by [BP2, BS] we can denote by \( \Gamma(\omega, \omega_0) \) the quantity

\[
\Gamma(\omega, \omega_1) = \mathbb{E} \left< A^{(N)}_{0, N}(\omega) R_{H, \omega_1} \left( (N + 1)\lambda(\omega_1) + i0 \right) P_c(\omega_1) R^{(N)}_{N+1, 0}(\omega_1 \sigma_3 \xi(\omega)) \right>
\]

\[
= \pi \langle \tilde{A}^{(N)}_{0, N}(\omega) \delta(H_{\omega_1} - (N + 1)\lambda(\omega)) P_c(\omega_1) R^{(N)}_{N+1, 0}(\omega_1 \sigma_3 \xi(\omega)) \rangle.
\]

Now we assume the following:

**Hypothesis 4.2.** There is a fixed constant \( \Gamma > 0 \) such that \( |\Gamma(\omega, \omega)| > \Gamma \).

Notice that the FGR hypothesis in [GS1] asks \( \Gamma(\omega, \omega) > 0 \). We will prove in Corollary 4.7 that in fact \( \Gamma(\omega, \omega) > \Gamma \). By continuity and by Hypothesis 4.2 we can assume \( |\Gamma(\omega, \omega_1)| > \Gamma/2 \). Then we write

\[
\frac{d}{dt} \frac{|z|^2}{2} = -\Gamma(\omega, \omega_1)|z|^{2N+2} + \mathbb{E} \left< \langle \tilde{A}^{(N)}_{0, N}(\omega)f_{N+1, \sigma_3 \xi(\omega)} \rangle \mathbb{E} \right>
\]

\[
+ \mathbb{E} \left< \langle EODE(N), \sigma_3 \xi(\omega) \rangle \right>.
\]

**§4.2 Nonlinear estimates**

By an elementary continuation argument, the following a priori estimates imply inequality (1) in Theorem 4.1, so to prove (1) we focus on:

**Lemma 4.3.** There are fixed constants \( C_0 \) and \( C_1 \) and \( \epsilon_0 > 0 \) such that for any \( 0 < \epsilon \leq \epsilon_0 \) if we have

\[
\| \tilde{z} \|_{L^2_{2N+2}}^{N+1} \leq 2C_0 \epsilon \quad \& \quad \| f_N \|_{L^\infty_t H^1_x \cap L^5_{t} W^{4,10}_x \cap L^6_{t} L^{\infty}_x \cap L^2_t H^{1,-2}_x} \leq 2C_1 \epsilon
\]

then we obtain the improved inequalities

\[
\| f_N \|_{L^\infty_t H^1_x \cap L^5_{t} W^{4,10}_x \cap L^6_{t} L^{\infty}_x \cap L^2_t H^{1,-2}_x} \leq C_1 \epsilon,
\]

\[
\| \tilde{z} \|_{L^2_{2N+2}}^{N+1} \leq C_0 \epsilon.
\]

**Proof.** Set \( \ell(t) := \gamma + \omega - \omega_0 \). First of all, we have:

**Lemma 4.4.** Let \( g(0, x) \in H^1_x \cap L^2_t(\omega_1) \) and let \( \omega(t) \) be a continuous function. Consider \( ig_\ell = \{ H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1)) \} g + P_c(\omega_1)F \). Then for a fixed \( C = C(\omega_1) \) upper semicontinuous in \( \omega_1 \) we have

\[
\| g \|_{L^\infty_t H^1_x \cap L^5_{t} W^{4,10}_x \cap L^6_{t} L^{\infty}_x} \leq C \| g(0, x) \|_{H^1} + C \| F \|_{L^1_t H^1_x + L^4_t W^{1,1}_x + L^2_t H^{1,2}_x}.
\]

Lemma 4.4 follows from Lemmas 3.1 and 3.5 and \( P_\pm(\omega_1)g(t) = e^{-iH_{\omega_1}t} e^{+i\int_0^t \ell(t)\, dt} P_\pm(\omega_1)g(0) - i \int_0^t e^{-i(s-t)H_{\omega_1}} e^{+i\int_s^t \ell(t)\, dt} P_\pm(\omega_1)F(s)\, ds \).
Lemma 4.5. Consider equation (4.1) for \( f_N \) and assume (4.10). Then we can split \( \tilde{E}_{PDE}(N) = X + O(f_N^5) \) such that \( \|X\|_{H^1_x L^2_t} \lesssim \epsilon^2 \) for any fixed \( M \) and \( \|O(f_N^5)\|_{L^1_t H^1_x} \lesssim \epsilon^5 \).

Proof of Lemma 4.5. Schematically we have for a cutoff \( \psi(x) \)
\[
\tilde{E}_{PDE}(N) = O(\epsilon)\psi(x)f_N + O_{loc}(|z|^{N+2}) + O_{loc}(zf_N) + O_{loc}(f_N^2) + O(\beta(|f_N|^2)f_N).
\]
By (4.10) for all the terms in \( \tilde{E}_{PDE}(N) \) except the last one and whose sum we call \( X \), we have:

\begin{enumerate}
  \item \( \|\langle x \rangle^M O(\epsilon)\psi(x)f_N\|_{H^1_x L^2_t} \lesssim \epsilon \langle x \rangle^{-5} f_N \|_{H^1_x L^2_t} \lesssim \epsilon^2; \)
  \item \( \|\langle x \rangle^M O_{loc}(zf_N)\|_{H^1_x L^2_t} \lesssim \|z\|_{\infty} \langle x \rangle^{-5} f_N \|_{H^1_x L^2_t} \lesssim \epsilon^2; \)
  \item \( \|\langle x \rangle^M O_{loc}(f_N^2)\|_{H^1_x L^2_t} \lesssim \|\langle x \rangle^{-5} f_N \|_{H^1_x L^2_t}^2 \lesssim \epsilon^2. \)
  \item \( \|\langle x \rangle^M O_{loc}(|z|^{N+2})\|_{H^1_x L^2_t} \lesssim \epsilon |z|^{N+1} \|_{L^2_t} \lesssim \epsilon^2. \)
\end{enumerate}

This yields \( \|\langle x \rangle^M X\|_{H^1_x L^2_t} \lesssim \epsilon^2. \) Observe that schematically \( \|\beta(|f_N|^2)f_N\|_{W^{1,r}_x} \lesssim \|f_N^5\|_{W^{1,r}_x} \) for all \( r \in (1, \infty) \), if on the right hand side we mean all the fifth powers of the components of \( f_N \). Then we have

\[
\|f_N^5\|_{L^1_t H^1_x} \lesssim \|f_N\|_{W^{1,10}_x} \|f_N\|_{L^{10}_x}^4 \|f_N\|_{L^1_x} \lesssim \|f_N^5\|_{L^1_t W^{1,10}_x} \lesssim \epsilon^5.
\]

Proof of (4.11). Recall that \( f_N \) satisfies equation (4.1) whose right hand side is \( P_\epsilon(\omega_1)\tilde{E}_{PDE}(N) + O_{loc}(z^{N+1}) \). In addition to Lemma 4.5 we have the estimate \( \|O_{loc}(z^{N+1})\|_{L^2_t H^1_x} \lesssim \|z\|_{L^{N+1}} \lesssim 2C_0 \epsilon \). So by Lemmas 3.1-4, for some fixed \( c_2 \) we get schematically

\[
\|f_N\|_{L^2_t H^1_x \cap L^2_t W^{1,10}_x \cap L^4_x L^\infty_x} \lesssim 2c_2 C_0 \epsilon + \|f_N(0)\|_{H^1_x} + O(\epsilon^2)
\]

where \( \|f_N(0)\|_{H^1_x} \leq c_2 \epsilon \) for fixed \( c_2 \geq 1 \), \( O(\epsilon^2) \) comes from all the terms on the right of (4.1) save for the \( R^{(N)}_{m,n}(\omega_0)z^m z^n \) terms which contribute the \( 2c_2 C_0 \epsilon \). Let now \( f_N = g + h \) with
\[
ig_t = \{H_{\omega_1} + \ell(t)(P_+ - P_-)\} g + X, \quad g(0) = f_N(0)
\]
\[
 ih_t = \{H_{\omega_1} + \ell(t)(P_+ - P_-)\} h + O(f_N^5), \quad h(0) = 0
\]
in the notation of Lemma 4.5. Then \( \|g\|_{H^1_x \cap L^2_t L^\infty_x} \lesssim 2C_0 \epsilon + O(\epsilon^2) + c_0 \epsilon \) by Lemmas 3.3-4 for a fixed \( c_0 \). Finally by Lemma 3.3

\[
\int_0^\infty \|e^{-i(t-s)H_{\omega_1}} e^{\pm i f_N}(s)\|_{H^1_x \cap L^2_t} \lesssim \int_0^\infty \|O(f_N^5)(s)\|_{H^1_x \cap L^2_t} \lesssim \epsilon^5.
\]

So if we set \( C_1 \approx 2C_0 + 1 \) we obtain (4.11). We need to bound \( C_0 \).

Proof of (4.12). We first need:
**Lemma 4.6.** We can decompose $f_{N+1} = h_1 + h_2 + h_3 + h_4$ with for a fixed large $M > 0$:

1. $\|\langle x \rangle^{-M} h_1 \|_{L^2_t x} \leq O(\epsilon^2)$;
2. $\|\langle x \rangle^{-M} h_2 \|_{L^2_t x} \leq O(\epsilon^2)$;
3. $\|\langle x \rangle^{-M} h_3 \|_{L^2_t x} \leq O(\epsilon^2)$;
4. $\|\langle x \rangle^{-M} h_4 \|_{L^2_t x} \leq c(\omega_1)\epsilon$ for a fixed $c(\omega_1)$ upper semicontinuous in $\omega_1$.

**Proof of Lemma 4.6.** We set

$$i\partial_t h_1 = (H_{\omega_1} + \ell(t)(P_+ - P_-)) h_1$$
$$h_1(0) = \sum_{m+n = N+1} R_{H_{\omega_1}}((m-n)\lambda(\omega_1) + i0) R^{(N)}_{m,n}(\omega_1) z^m(0) \bar{z}^n(0).$$

We get $\|\langle x \rangle^{-M} h_1 \|_{L^2_t x} \leq c(\omega_1)|z(0)|^2 \sum \|\langle x \rangle^\gamma R^{(N)}_{m,n}(\omega_1)\|_{L^2_x} = O(\epsilon^2)$ by the inequality (4.13) below, see [BP1,BS], which says that for any $\gamma > \gamma_0$ for some given $\gamma_0$,

$$(4.13) \quad \|\langle x \rangle^{-\gamma} e^{-iH_{\omega} t} R_{H_{\omega}} (\Lambda + i0) P_c(\omega) g \|_2 < C(\Lambda, \omega) \langle t \rangle^{-\frac{3}{2}} \|\langle x \rangle^\gamma g \|_2, \quad \Lambda > \omega,$$

with $C(\Lambda, \omega)$ upper semicontinuous in $\omega$ and in $\Lambda$. Next, we set $h_2(0) = 0$ and

$$i\partial_t h_2 = (H_{\omega_1} + \ell(t)(P_+ - P_-)) h_2 + O(\epsilon z^{N+1}) R_{H_{\omega_1}}((N + 1)\lambda(\omega_1) + i0) R_{N+1,0}^{(N)}(\omega_0)$$
$$+ O(\epsilon z^{N+1}) R_{H_{\omega_1}}(-(N + 1)\lambda(\omega_1) + i0) R_{0,N+1}^{(N)}(\omega_1).$$

Then we have $h_2 = h_{21} + h_{22}$ with $h_{2j} = \sum_{\pm} h_{2j\pm}$ with $h_{21\pm}(t) =$

$$\int_0^t e^{-iH_{\omega_1}(t-s)} e^{\pm i \int_0^s \ell(\tau) d\tau} P_{\pm} z^{N+2}(s) R_{H_{\omega_1}}((N + 1)\lambda(\omega_1) + i0) R_{N+1,0}^{(N)}(\omega_1) ds$$

and $h_{22\pm}$ defined similarly but with $R_{H_{\omega_1}}(-(N + 1)\lambda(\omega_1) + i0) R_{0,N+1}^{(N)}$. Now by (4.13) we get

$$\|\langle x \rangle^{-M} h_{21\pm}(t) \|_{L^2_x} \leq C\epsilon \int_0^t \langle t - s \rangle^{-\frac{3}{2}} |z(s)|^{N+1} ds$$

and so $\|\langle x \rangle^{-M} h_2 \|_{L^2_t x} \leq \epsilon \|z\|_{L^{N+2}}^{N+2} = O(\epsilon^3)$. Let $h_3(0) = 0$ and

$$i\partial_t P_c(\omega_1) h_3 = (H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1))) P_c(\omega_1) h_3 + P_c(\omega_1) \bar{E}_{PDE}(N).$$
Then by the argument in the proof of (4.11) we get claim (3). Finally let \( h_4(0) = f_N(0) \) and

\[
i\partial_t P_c(\omega_1)h_4 = (H_{\omega_1} + \ell(t)(P_+(\omega_1) - P_-(\omega_1))) P_c(\omega_1)h_4.
\]

Then by Lemma 3.3 \( \| \langle x \rangle^M h_4 \|_{L^2_x} \lesssim \| f_N(0) \|_{L^2} \leq c(\omega_1)\epsilon \) we get (4).

**Continuation of proof of Lemma 4.3.** We integrate (4.9) in time. Then by Theorem 2.1 and by Lemma 4.4 we get, for \( A_0(\omega) \) of Theorem 2.1,

\[
\| \tilde{z} \|_{L^2_t}^{2N+2} \leq A_0\epsilon^2 + 2c(\omega_1)\epsilon \| \tilde{z} \|_{L^2_t}^{N+1} + o(\epsilon^2).
\]

Then we can pick \( C_0 = (A_0 + 2c(\omega_1 + 1) \) and this proves that (4.10) implies (4.12). Furthermore \( \tilde{z}(t) \to 0 \) by \( \frac{d}{dt} \tilde{z}(t) = O(\epsilon) \).

As in [CM] in the above argument we did not use the sign of \( \Gamma(\omega, \omega_1) \). As in [CM] it is nonnegative.

**Corollary 4.7.** If Hypothesis 4.2 holds, then \( \Gamma(\omega, \omega) > \Gamma \).

Suppose we have \( \Gamma(\omega, \omega_1) < -\Gamma \). We can pick initial datum so that \( f_{N+1}(0) = 0 \) and \( z(0) \approx \epsilon \). Then following the proof of Lemma 4.6, by integrating (4.9) and using \( h_4 = 0 \), we get

\[
|\tilde{z}(t)|^2 - |\tilde{z}(0)|^2 \geq \Gamma \int_0^t |\tilde{z}|^{2N+2} + o(\epsilon) \left( \int_0^t |\tilde{z}|^{2N+2} \right)^{\frac{1}{2}} + o(\epsilon^2).
\]

For large \( t \) we have \( |\tilde{z}(t)| < |\tilde{z}(0)| \) since \( z(t) \to 0 \), so for large \( t \) we get \( \int_0^t |\tilde{z}|^{2N+2} = o(\epsilon^2) \). In particular for \( t \to \infty \) we get \( \epsilon^2 \leq o(\epsilon^2) \) which is absurd for \( \epsilon \to 0 \).

The proof that, for \( t f_N(t) = (h(t), \overline{h}(t)) \), \( h(t) \) is asymptotically free for \( t \to \infty \), is similar to the analogous one in [CM] and we skip it.

§APPENDIX A. ORBITAL STABILITY: SKETCH OF PROOF OF THEOREM 2.1

We sketch the proof of Theorem 2.1.

**Lemma A.1.** Suppose that \( e^{i\omega t} \phi_\omega(x) \) satisfies (H6). In dimension \( n > 1 \) assume also that

\[
(A.1) \quad L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - 2\beta'(\phi_\omega^2)\phi_\omega
\]

has exactly one negative eigenvalue. Then \( \exists \epsilon > 0 \) and a \( A_0(\omega) > 0 \) such that for any \( \| u(0, x) - \phi_\omega \|_{H^1(\mathbb{R}^n)} < \epsilon \) we have for the corresponding solution

\[
\inf\{\| u(t,x) - e^{i\gamma} \phi_\omega(x-x_0)\|_{H^1(x\in\mathbb{R}^n)} : \gamma \in \mathbb{R} & x_0 \in \mathbb{R}^n \} < A_0(\omega)\epsilon.
\]
The proof consists in the argument in [We1] with a minor change due to D. Stuart [S]. We have invariants:

\[
Q(f) = \frac{1}{2} \int_{\mathbb{R}^n} |f(x)|^2 dx, \quad M(f) = \frac{1}{2} \int_{\mathbb{R}^n} f(x) \nabla f(x) dx,
\]

\[
E(f) = \int_{\mathbb{R}^n} \left( \frac{\|\nabla f(x)\|^2}{2} + F(|f|) \right) dx.
\]

For \( \Theta(t, x) = \frac{v \cdot x}{2} + \partial(t) \) we have

\[
M(e^{i\Theta} f) = \frac{1}{2} \int_{\mathbb{R}^n} e^{-i\Theta} f(x) e^{i\Theta} (\nabla f(x) + i \frac{v}{2} f(x)) dx = M(f) + \frac{v}{2} Q(f)
\]

\[
E(e^{i\Theta} f) = \int_{\mathbb{R}^n} \left( \frac{\|\nabla f(x)\|^2}{2} + F(|f|) \right) dx = E(f) + \frac{v^2}{4} Q(f) + v \cdot M(f).
\]

We define now from the invariants of motion

\[
H(u) = E(u) + \omega(t) Q(u) - v(0) \cdot M(u) = E(u_0) + \omega(t) Q(u_0) - v(0) \cdot M(u_0)
\]

with \( v(0) \) initial velocity, \( \omega(t) \) a function defined later, \( u_0(x) = u(0, x) \). The idea of choosing \( v(0) \) is in [S]. For \( y \) the coordinate in the moving frame, we consider the ansatz \( u = e^{i\Theta}(\phi_\mu(y) + r(t, y)) \) satisfying the usual modulation equations

\[
\langle Q'(\varphi_\mu), r(t) \rangle = \langle M'(\varphi_\mu), r(t) \rangle = 0.
\]

After the above preparation we start the usual expansion

\[
H(e^{i\Theta}(\phi_\mu + r)) = E(e^{i\Theta}(\phi_\mu + r)) + \omega Q(\phi_\mu + r) - v(0) \cdot M(e^{i\Theta}(\phi_\mu + r)) =
\]

\[
= E(\phi_\mu + r) + \left( \omega + \frac{v^2 - 2v(0) \cdot v}{4} \right) Q(\phi_\mu + r) + (v - v(0)) \cdot M(\phi_\mu + r) =
\]

\[
= E(\phi_\mu + r) + \left( \omega - \frac{v^2(0)}{4} + \frac{(v - v(0))^2}{4} \right) Q(\phi_\mu + r) + (v - v(0)) \cdot M(\phi_\mu + r).
\]

Define \( \omega = \frac{v^2(0)}{4} + \mu. \) Then, setting \( d(\mu) = E(\phi_\mu) + \mu Q(\phi_\mu) \) and \( q(\mu) = Q(\phi_\mu) \)

\[
H(u) = d(\mu) + \frac{(v - v(0))^2}{4} q(\mu) + \langle E'(\phi_\mu) + \mu Q'(\phi_\mu) + (v - v(0)) M'(\phi_\mu), r \rangle +
\]

\[
+ \frac{1}{2} \langle [E''(\phi_\mu) + \mu Q''(\phi_\mu)] r, r \rangle + \frac{(v - v(0))^2}{4} \langle Q'(\phi_\mu) + \frac{Q''}{2}(\phi_\mu), r \rangle +
\]

\[
+ \frac{1}{2} (v - v(0)) \cdot \langle M''(\phi_\mu) r, r \rangle + o(||r||^2_{H^1}).
\]
From modulation and from $E'(\phi_\mu) + \mu Q'(\phi_\mu) = 0$ we get

$$\langle E'(\phi_\mu) + \mu Q'(\phi_\mu) + (v-v(0)) \cdot M'(\phi_\mu), r \rangle = 0.$$ 

So

$$H(u) = d(\mu) + \frac{(v-v(0))^2}{4}q(\mu) + \frac{1}{2}\langle [E''(\phi_\mu) + \mu Q''(\phi_\mu)] r, r \rangle + o(\|r\|_{H^1}^2).$$

Proceeding similarly

$$E(u_0) + \omega(t)Q(u_0) - v(0) \cdot M(u_0) = d(\mu(0)) +$$

$$\frac{1}{2}\langle [E''(\phi_{\mu(0)}) + \mu(0)Q''(\phi_{\mu(0)})] r(0), r(0) \rangle + (\mu - \mu(0))q(\mu(0)) + o(\|r(0)\|_{H^1}^2).$$

Recall now that $d'(\mu(0)) = q(\mu(0))$ so by equating the last two displayed formulas and after Taylor expansion of $d(\mu)$ we get the following result:

$$\frac{d''(\mu(0))}{2}(\mu - \mu(0))^2 + \frac{(v-v(0))^2}{4}q(\mu) + \frac{1}{2}\langle [E''(\phi_\mu) + \mu Q''(\phi_\mu)] r, r \rangle \leq$$

$$\leq \frac{1}{2}\langle [E''(\phi_{\mu(0)}) + \mu(0)Q''(\phi_{\mu(0)})] r(0), r(0) \rangle + o(\|r\|_{H^1}^2) + o(\|r(0)\|_{H^1}^2).$$

This implies $(\mu - \mu(0))^2 + (v-v(0))^2 + \|r\|_{H^1}^2 \leq C\|r(0)\|_{H^1}^2$, because of the fact that

$$\langle [E''(\phi_\mu) + \mu Q''(\phi_\mu)] r, r \rangle \approx \|r\|_{H^1}^2.$$ 

§ Appendix B. Proof of estimate (4.3)

**Lemma B.1.** The following operators $P_\pm(\omega)$ are well defined:

$$P_+(\omega)u = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \lim_{M \to +\infty} \int_\omega^M [R(\lambda + i\epsilon) - R(\lambda - i\epsilon)] u d\lambda$$

$$P_-(\omega)u = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \lim_{M \to +\infty} \int_{-M}^{-\omega} [R(\lambda + i\epsilon) - R(\lambda - i\epsilon)] u d\lambda.$$ 

For any $M > 0$ and $N > 0$ and for $C = C(N,M,\omega)$ upper semicontinuous in $\omega$, we have

$$\|\langle x \rangle^M (P_+(\omega) - P_-(\omega) - P_c(\omega)\sigma_3) f\|_{L^2} \leq C\|\langle x \rangle^{-N} f\|_{L^2}. \quad (2)$$

(2) for $M = 2$ is stated in [BP2] with the proof sketched in [BS].

**Proof.** The first part is a consequence of [KS]. We prove (2) following the argument in §7 [C2]. For this proof we set $L^2_s = \langle x \rangle^{-s}L^2$, $H = H_\omega$, $H_0 = \sigma_3(-\Delta + \omega)$,
\( R_0(z) = (H_0 - z)^{-1} \) and \( R(z) = (H - z)^{-1} \). To prove (2) it is enough to write \( P_z = P_+ + P_- \) and to prove \( \| [P_{\pm} \sigma_3 \mp P_{\pm}] g \|_{L^2_M} \leq c \| g \|_{L^2_N} \). It is not restrictive to consider only \( P_+ \). Setting \( H = H_0 + V \), we write

\[
\sum_{\pm} \pm R(\lambda \pm i\epsilon) = \sum_{\pm} \pm (1 + R_0(\lambda \pm i\epsilon)V)^{-1} R_0(\lambda \pm i\epsilon).
\]

By elementary computation

\[
R_0(\lambda \pm i\epsilon)\sigma_3 = R_0(\lambda \pm i\epsilon) - 2(-\Delta + \omega + \lambda \pm i\epsilon)^{-1} \text{diag}(0, 1).
\]

Therefore

\[
\text{rhs (3)}\sigma_3 = \text{rhs (3)} + 2 \sum_{\pm} \pm (1 + R_0(\lambda \pm i\epsilon)V)^{-1} \text{diag}(0, 1)(-\Delta + \omega + \lambda \pm i\epsilon)^{-1}.
\]

Hence we are reduced to show that

\[
Ku = \lim_{\epsilon \to 0^+} \lim_{M \to +\infty} \sum_{\pm} \pm \int_{\omega}^M (1 + R_0(\lambda \pm i\epsilon)V)^{-1} \text{diag}(0, 1)(-\Delta + \omega + \lambda \pm i\epsilon)^{-1} u d\lambda
\]

defines an operator such that for some fixed \( c \)

\[
\|Ku\|_{L^2_M} \leq c\|u\|_{L^2_N}.
\]

For \( m \geq 1 \) we expand \( (1 + R_0 V)^{-1} = \sum_{j=0}^{m+1} [-R_0 V]^j + R_0 V R V (-R_0 V)^N \) and we consider the corresponding decomposition

\[
K = \sum_{j=0}^{m+1} K_0^j + \mathcal{K}.
\]

We have \( K_0^0 = 0 \) since for any \( u \in L^2 \) we have

\[
\lim_{\epsilon \to 0^+} \lim_{M \to +\infty} \int_{\omega}^M \sum_{\pm} \pm (-\Delta + \omega + \lambda \pm i\epsilon)^{-1} \text{diag}(0, 1) u d\lambda = 0.
\]

We next consider \( K_1^0 \) and prove

\[
\|K_1^0 u\|_{L^2_M} \leq c\|u\|_{L^2_N}.
\]

The operator \( (-\Delta + \omega + z)^{-1} \) has symbol satisfying, for \( \Re z \geq 0 \):

\[
\left| \partial_\beta \partial_\xi^\alpha (|\xi|^2 + \omega + z)^{-1} \right| \leq c_{\alpha, \beta} (|\xi| + 1)^{-|\alpha|} \langle z \rangle^{-1-\beta}.
\]
Therefore we have

\[
\| \langle x \rangle^M (-\Delta + \omega + z)^{-1} u \|_{L^2} = \| \langle \sqrt{-\Delta} \rangle^M (\xi^2 + \omega + z)^{-1} \hat{u} \|_{L^2} \leq C \langle z \rangle^{-1} \| u \|_{L^2_M}
\]

and so for any \( M \in \mathbb{R} \)

\[
(8) \quad \| (-\Delta + \omega + z)^{-1} : L^2_M \to L^2_M \| \leq C \langle z \rangle^{-1}.
\]

We can assume \( u \) smooth and rapidly decreasing. Since for \( s > 1 \) we have \( \| R_0(\lambda \pm i\varepsilon) : L^2_s \to L^2_{-s} \| \leq C \langle \lambda \rangle^{-\frac{s}{2}} \), the following limit is well defined

\[
K^0_1 u = \lim_{\varepsilon \to 0^+} \lim_{M \to +\infty} \int_{\omega}^M \sum_{\pm} \sum_{\pm} [R_0(\lambda \pm i\varepsilon)V(-\Delta + \omega + \lambda \pm i\varepsilon)^{-1}] \text{diag}(0,1) ud\lambda
\]

\[
= \int_{\omega}^{+\infty} [R_0(\lambda + i0) - R_0(\lambda - i0)] V(-\Delta + \omega + \lambda)^{-1} \text{diag}(0,1) ud\lambda.
\]

By \( R_0(\lambda + i0) - R_0(\lambda - i0) = 2i\pi \delta(\Delta - \omega + \lambda) \text{diag}(1,0) \) and for \( ^t u = (u_1, u_2) \)

\[
K^0_1 u = \int_{\omega}^{+\infty} \delta(\Delta - \omega + \lambda) \text{diag}(1,0) V(-\Delta + \omega + \lambda)^{-1} u_2 \bar{e}_2 d\lambda.
\]

Up to a constant factor, this is schematically

\[
\int_{\mathbb{R}^2} e^{ix \cdot \xi} \frac{\hat{V}(\xi - \eta) \hat{u}(\eta)}{\xi^2 + \eta^2 + 2\omega} d\eta d\xi.
\]

By the correspondence \( \partial_x \leftrightarrow i\xi \) and by Parseval equality, (6) will follow by

\[
\left\| \int_{\mathbb{R}} d\eta \hat{u}(\eta) \hat{V}(\xi_1) (\xi - \eta) \partial_\xi^2 (\xi^2 + \eta^2 + 2\omega)^{-1} \right\|_{L^2_\xi} \leq C(\ell_1, \ell_2) \| \hat{u} \|_{L^2_\xi}
\]

which is a consequence of Young inequality. We consider now

\[
K^0_j u = (-)^j \lim_{\varepsilon \to 0^+} \sum_{\pm} \sum_{\pm} \int_{\omega}^{+\infty} [R_0(\lambda \pm i\varepsilon)V]^j \text{diag}(0,1)(-\Delta + \omega + \lambda \pm i\varepsilon)^{-1} ud\lambda.
\]

For some \( \delta > 0 \) small but fixed we can deform the path of integration and write

\[
K^0_j u = (-)^j \int_{\omega - \delta - i\infty}^{\omega - \delta + i\infty} [R_0(\zeta)V]^j \text{diag}(0,1)(-\Delta + \omega + \zeta)^{-1} ud\zeta.
\]
By (8) we conclude

$$\|K^0_j u\|_{L^2_M} \leq c\|u\|_{L^2_N}.$$  (9)

Next we consider also the reminder term in (5). Arguing as above

$$(-)^{m+2}Ku = \lim_{\epsilon \to 0^+} \sum \pm \int_{\omega}^{+\infty} R_0(\lambda \pm i\epsilon)V R(\lambda \pm i\epsilon)V [R_0(\lambda \pm i\epsilon)V]^m \text{diag}(0,1)(-\Delta + \omega + \lambda \pm i\epsilon)^{-1} u d\lambda$$

$$= \int_{\omega-\delta-i\infty}^{\omega+\delta+i\infty} R_0(\zeta)V R(\zeta)V [R_0(\zeta)V]^m \text{diag}(0,1)(-\Delta + \omega + \zeta)^{-1} u d\zeta.$$  

For $\Re \zeta = \omega - \delta$, (7) implies $(1 + |\zeta|)^{-1} \gtrsim \|R_0(\zeta)V: L^2_M \to L^2_N\| +

+ \|V [R_0(\zeta)V]^m : L^2_{-N} \to L^2_{-N}\| + \|(-\Delta + \omega + \zeta)^{-1} : L^2_{-N} \to L^2_{-N}\|.$

So $\|K: L^2_M \to L^2_{-N}\| < \infty$ and this with (6) and (9) yields (4) and proves $\| [P_+ \sigma_3 - P_+] u \|_{L^2_M} \leq c\|u\|_{L^2_N}.$

REFERENCES

[BP1] V.S. Buslaev, G.S.Perelman, Scattering for the nonlinear Schrödinger equation: states close to a soliton, St. Petersburg Math.J. 4 (1993), 1111–1142.

[BP2] , On the stability of solitary waves for nonlinear Schrödinger equations, Nonlinear evolution equations (N.N. Uraltseva, eds.), Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995, pp. 75–98.

[BS] V.S.Buslaev, C.Sulem, On the asymptotic stability of solitary waves of Nonlinear Schrödinger equations, Ann. Inst. H. Poincaré. An. Nonlin. 20 (2003), 419–475.

[C1] S.Cuccagna, Stability of standing waves for NLS with perturbed Lamé potential, J. Differential Equations 223 (2006), 112–160.

[C2] , On asymptotic stability of ground states of NLS, Rev. Math. Phys. 15 (2003), 877–903.

[C3] , On asymptotic stability in energy space of ground states of NLS in 1D, J. Differential Equations 245 (2008), 653-691.

[CM] S.Cuccagna, T.Mizumachi, On asymptotic stability in energy space of ground states for Nonlinear Schrödinger equations, http://www.dismi.unimo.it/Members/scuccagna/pubblicazioni.

[CPV] S.Cuccagna, D.Pelinovsky, V.Vougalter, Spectra of positive and negative energies in the linearization of the NLS problem, Comm. Pure Appl. Math. 58 (2005), 1–29.

[CK] M.Christ, A.Kieslev, Maximal functions associated with filtrations, J. Funct. Anal. 179 (2001), 409–425.

[DT] P.Deift, E.Traubowitz, Inverse scattering on the line, Comm. Pure Appl. Math. 32 (1979), 121–251.
[FW] G.Fibich, X.P.Wang, Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, Physica D 175 (2003), 96-108.

[GNT] S.Gustafson, K.Nakanishi, T.P.Tsai, Asymptotic Stability and Completeness in the Energy Space for Nonlinear Schrödinger Equations with Small Solitary Waves, Int. Math. Res. Notices 66 (2004), 3559–3584.

[GS1] Zhou Gang, I.M.Sigal, Asymptotic stability of nonlinear Schrödinger equations with potential, Rev. Math. Phys. 17 (2005), 1143–1207.

[GS2] , Relaxation of Solitons in Nonlinear Schrödinger Equations with Potential, [http://arxiv.org/abs/math-ph/0603060].

[GSS1] M.Grillakis, J.Shatah, W.Strauss, Stability of solitary waves in the presence of symmetries, I, Jour. Funct. An. 74 (1987), 160–197.

[GSS2] , Stability of solitary waves in the presence of symmetries, II, Jour. Funct. An. 94 (1990), 308–348.

[KS] J.Krieger, W.Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, J. Amer. Math. Soc. 19 (2006), 815–920.

[M1] T.Mizumachi, Asymptotic stability of small solitons to 1D NLS with potential, [http://arxiv.org/abs/math.AP/0605031].

[M2] , Asymptotic stability of small solitons for 2D Nonlinear Schrödinger equations with potential, [http://arxiv.org/abs/math.AP/0609323].

[P] G.S.Perelman, On the formation of singularities in solutions of the critical nonlinear Schrödinger equation, Ann. Henri Poincaré 2 (2001), 605–673.

[PW] C.A.Pillet, C.E.Wayne, Invariant manifolds for a class of dispersive, Hamiltonian partial differential equations, J. Diff. Eq. 141 (1997), 310–326.

[RSS] I.Rodnianski, W.Schlag, A.Soffer, Asymptotic stability of N-soliton states of NLS, preprint, 2003, [http://arxiv.org/abs/math.AP/0309114].

[S] D.M.A.Stuart, Modulation approach to stability for non topological solitons in semilinear wave equations, J. Math. Pures Appl. 80 (2001), 51–83.

[ShS] J.Shatah, W.Strauss, Instability of nonlinear bound states, Comm. Math. Phys. 100 (1985), 173–190.

[SmS] H.F.Smith, C.D.Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian, Comm. Partial Differential Equations 25 (2000), 2171–2183.

[SW1] A.Soffer, M.Weinstein, Multichannel nonlinear scattering II. The case of anisotropic potentials and data, J. Diff. Eq. 98 (1992), 376–390.

[SW2] , Selection of the ground state for nonlinear Schrödinger equations, Rev. Math. Phys. 16 (2004), 977–1071.

[TY1] T.P.Tsai, H.T.Yau, Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and radiation dominated solutions, Comm. Pure Appl. Math. 55 (2002), 153–216.

[TY2] , Relaxation of excited states in nonlinear Schrödinger equations, Int. Math. Res. Not. 31 (2002), 1629–1673.

[TY3] , Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data, Adv. Theor. Math. Phys. 6 (2002), 107–139.

[Wd1] R. Weder, Center manifold for nonintegrable nonlinear Schrödinger equations on the line, Comm. Math. Phys. 170 (2000), 343–356.

[Wd2] , $L^p \to L^p$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, J. Funct. Anal. 170 (2000), 37–68.

[We1] M.Weinstein, Lyapunov stability of ground states of nonlinear dispersive equations, Comm. Pure Appl. Math. 39 (1986), 51–68.
[We2] ———, Modulation stability of ground states of nonlinear Schrödinger equations, Siam J. Math. Anal. 16 (1985), 472–491.

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