Abstract Recent work on non-proper gauge degrees of freedom in the context of the Casimir effect is reviewed. In his original paper, Casimir starts by pointing out that, when the electromagnetic field is confined between two perfectly conducting parallel plates, there is an additional physical polarization of the electromagnetic field at zero value for the discretized longitudinal momentum besides the standard two transverse polarizations at non-zero values. In this review, the dynamics of these additional modes is obtained from first principles. At finite temperature, their contribution to the entropy is proportional to the area of the plates and corresponds to the contribution of a massless scalar field in 2+1 dimensions. When the plates are charged, there is a further contribution to the partition function from the zero mode of this additional scalar that scales with the area but does not contribute to the entropy. It reproduces the result obtained when the Gibbons–Hawking method is applied to the vacuum capacitor. For completeness, a brief discussion of the classical thermodynamics of such a capacitor is included.

Keywords Edge modes · Casimir effect · Gibbons–Hawking entropy · Black hole micro-states

1 Introduction

That seemingly unphysical polarizations of the electromagnetic field have an important role to play in the presence of charged particles is known since the work by Dirac [1], and Fock and Podolski [2], where the Coulomb force between two non-relativistic electrons is constructed in terms of creation and destruction operators associated with the scalar potential $A_0$. 

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When all polarizations are quantized, it is important to understand how equivalence with reduced phase space quantization is achieved. Arguably the most transparent implementation is through the quartet mechanism [3] which implies the cancellation of the contributions from unphysical polarizations and ghost variables when computing matrix elements of gauge invariant operators between gauge invariant states in the context of Hamiltonian BRST operator quantization. Furthermore, the associated path integral is simply related to the manifestly Lorentz invariant Lagrangian BRST path integral by integrating out momenta (see, e.g. [4] for a comprehensive review). More generally, as is well known in the context of topological field theories, these cancellations no longer work perfectly when there is non-trivial topology.

Whereas the quartet mechanism is relatively straightforward for the free electromagnetic field where the quartets are associated with temporal oscillators for the scalar potential and to oscillators for the longitudinal part of the vector potential on the one hand, and to oscillators for the ghost fields on the other, this is no longer the case in the presence of charged sources, where gauge invariance becomes a non-trivial issue [5]. For the simplest source representing a charged point particle at rest, it turns out that the BRST invariant vacuum state is a coherent state constructed out of unphysical null oscillators that represents the quantum Coulomb solution [6]. Some technical details and clarifications on this elementary construction are provided in Appendix “Details on Quantum Coulomb Solution”.

The ultimate aim of this research is a better understanding of the degrees of freedom responsible for black hole entropy. In this context, it is intriguing to note that in one of the earliest papers on linearized quantum gravity by Bronstein [7] (see [8] for perspective), the last part of the paper follows closely the derivation by Dirac, Fock, and Podolski on the Coulomb law in order to obtain Newton’s law between two test masses from the creation and destruction operators associated with the metric fluctuations $h_{00}$. How to extend the considerations below to the case of linearized gravity will be discussed elsewhere.

In order to avoid facing the question of the detailed interaction of the quantized electromagnetic field with charged dynamical matter, it is instructive to first consider the case where these interactions can be idealized as boundary conditions imposed on the free electromagnetic field. This naturally leads one to consider the electromagnetic field in the presence of charged conducting plates. In the absence of charge, this is precisely the set-up of the Casimir effect [9] at non-zero temperature [10] (see also [11, 12] and e.g. [13–15] for reviews).

As we will discuss in details below, that there is an additional physical polarization at zero value for the discretized longitudinal momentum, besides the two transverse ones at non-zero values, is well known in this context. We will focus on how to determine the dynamics of these “edge” modes and isolate their contribution to the partition function and the entropy, which scales with the area of the plates.

We then provide a microscopic understanding of the charged vacuum capacitor, where there is an additional contribution to the partition function that comes from the zero mode of the additional polarization and that also scales with the area but does not contribute to the entropy [16].
Before turning to these issues, we will first discuss the thermodynamics of a capacitor by standard methods. In the context of general relativity, the Euclidean approach of Gibbons and Hawking [17] consists in deriving the thermodynamics of Kerr–Newman black holes or of de Sitter space by evaluating on-shell the Euclidean action improved through suitable boundary terms. What these boundary terms are in the electromagnetic sector has been discussed, for instance, in [18, 19]. That the construction and interpretation of such boundary terms is very transparent in the first Hamiltonian formulation is discussed, for instance, in the derivation of the thermodynamics of the BTZ black hole [20]. We then review how the thermodynamics of the capacitor can easily be reproduced from the Euclidean approach [16].

2 Capacitor Thermodynamics: Textbook Approach

Consider a capacitor made of two conductors with charges $+q$ and $-q$ and area $A$. Its capacity $C = \frac{q}{d}$ in Lorentz–Heaviside units is

$$C_S = \frac{4\pi R_1 R_2}{R_2 - R_1}$$  \hspace{1cm} (1)

for two concentric spheres of radii $R_1 < R_2$ and

$$C_P = \frac{A}{d}$$  \hspace{1cm} (2)

for two parallel planes of area $A$ separated by a distance $d$ (see, e.g. [21, chapter 2]).

The capacitor begins with 0 charge, energy, and entropy. Charges $\pm dq$ are added on both side until one reaches $\pm q$.

In order to charge the capacitor, one may use a circuit without any resistance so that no heat would be produced in the process. One then would quickly arrive at the standard results (see, e.g., chapter 14.2 of [22] in the absence of the system and its electric polarization). A better understanding of the absence of entropy can however be gained by considering a set-up with a resistor [23].

If a potential difference $V$ is applied on the capacitor, there will be a current $I(t) = \frac{q}{RC} e^{-\frac{t}{RC}}$. The heat lost by the system through the resistor is

$$Q = \int_0^\infty R I^2(t)dt = \frac{1}{2} CV^2.$$  \hspace{1cm} (3)

Instead of a single step, the capacitor can be charged in $N$ steps, each increasing the voltage by $\frac{V}{N}$. At each step, the circuit relaxes until the current vanishes. The heat lost in all $N$ steps is then
\[ Q_N = \frac{1}{2} C (\Delta V)^2 \times N = \frac{1}{2} \frac{CV^2}{N}. \]  

(4)

If the ambient temperature is constant, the increase of entropy is

\[ \Delta S_N = \frac{Q_N}{T} = \frac{1}{2} \frac{CV^2}{TN}. \]  

(5)

In the limit \( N \to \infty \), the charging of the capacitor becomes a quasi-static process. Since in this case, there is no increase of entropy, \( dS = 0 = S(q) \), the process is reversible.

By the first law, it now follows that the increase of internal energy \( dU \) is due to the work done by the voltage source alone,

\[ dU = dW = V dq. \]  

(6)

Since \( V = \frac{q}{C} \), we get

\[ U(q) = \frac{1}{2} \frac{q^2}{C}, \]  

(7)

which is the well-known energy of a charged capacitor. In this case, it is also the free energy,

\[ F(T, q) = [U(q, S) - TS(q)]|_{S=S(T)} = \frac{1}{2} \frac{q^2}{C}. \]  

(8)

### 3 Capacitor Thermodynamics: Euclidean Approach

In the absence of gravity and of sources between the conductors, the starting point is the first-order action

\[ I = \int d^4x \left[ \dot{A}_i \pi^i - \mathcal{H}_0 + A_0 \partial_i \pi^i \right], \quad \mathcal{H}_0 = \frac{1}{2} \left( \pi^i \pi_i + B^i B_i \right), \]  

(9)

where magnetic and electric fields are given, respectively, by \( B^i = \epsilon^{ijk} \partial_j A_k, E^i = -\pi^i \). The variation of this action is

\[ \delta I = \int d^4x \left[ \delta \dot{A}_i \left( -\ddot{\pi}^i - \epsilon^{ijk} \partial_j B_k \right) + \delta A_0 \left( \partial_i \pi^i \right) + \delta \pi^i \left( \dot{A}_i - \pi_i - \partial_i A_0 \right) \right] \]

\[ + \left[ \int d^3x \delta A_i \pi^i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \int d\sigma_i \left( \epsilon^{ijk} B_j \delta A_k + A_0 \delta \pi^i \right). \]  

(10)
We focus in this section on time-independent solutions for which the equations of motions reduce to \( \pi_i = -\partial_i A_0 \) with \( \Delta A_0 = 0 \) and \( \Delta A_i - \partial_i \partial_j A^j = 0 \). We also assume that \( \Delta \) is invertible on \( \partial_i \pi^i, \partial_j A^j \) and that the gauge condition \( \partial_i A^i = 0 \) may be imposed. Defining the transverse part of a vector field through \( \vec{V}^T = \vec{V} - \vec{D}^L \), with the longitudinal part given by \( \vec{V}^L = \vec{D}(\Delta^{-1} \vec{D} \cdot \vec{D}) \), the gauge condition is equivalent to \( \vec{A} = \vec{A}^T \), while the equations of motion determine the longitudinal part \( \vec{\pi}^L \) in terms of the harmonic potential \( A_0 \) and imply \( \vec{\pi}^T = 0 = \Delta \vec{A}^T \). We assume here that this implies \( \vec{A}^T = \vec{v} \), with \( \vec{v} \) constant.

Consider a spherical capacitor with conducting spheres at radii \( R_1 < R_2 \) and charges \( +q \) and \( -q \), respectively. Under the above assumptions, the general solution to the equations of motion is

\[
A_0 = -\frac{q}{4\pi r} + c, \quad \pi^i = -\frac{q x^i}{4\pi r^3}, \quad R_1 < r < R_2, \quad (11)
\]

with \( c \) a constant and 0 outside of the shell. The classical observable that captures electric charge is

\[
Q = -\int_S d\sigma_i \pi^i, \quad (12)
\]

with \( S \) a closed surface inside the shell, for instance \( r = R, R_1 < R < R_2 \) so that \( Q = q \) on-shell.

In the case of planar conductors at \( z = 0 \) and \( z = d \) with charge densities \( \frac{q}{A} \) and \( -\frac{q}{A} \), we have instead

\[
A_0 = -\frac{q}{A} z + c, \quad \pi^i = -\delta^i_3 \frac{q}{A}, \quad 0 < z < d, \quad (13)
\]

and 0 outside of the capacitor. In this case, the electric charge observable is \( Q \) in (12) with \( S \) a plane at \( z = L, 0 < L < d \).

For later purposes, note that both solutions (11) and (13) can be transformed into solutions with \( A_0 = 0 \) by a time dependent gauge transformation. The associated vector potential satisfies \( \nabla \cdot \vec{A} = 0 \) between the conductors and is longitudinal.

When working at fixed charge, all surface terms in the second line of (10) vanish on the solutions under consideration. In the Euclidean approach, there is a contribution to the partition function from the Euclidean action evaluated at these classical solutions. It is given by

\[
-\beta F(\beta, Q) = -\left. \frac{1}{\hbar} I^E(\beta, Q) \right|_{\text{on-shell}}, \quad (14)
\]

where

\[
I_E = \int_0^{\hbar \beta} d\tau \int d^3 x \left[ -i \dot{\vec{A}}_i \pi^i + \mathcal{H}_0 - A_0 \partial_i \pi^i \right]. \quad (15)
\]
On-shell, only the longitudinal electric field in the Hamiltonian contributes and gives

\[
F(\beta, q) = \frac{q^2}{2C},
\]

(16)

where \(C\) is the capacity given by (1) and (2) in the spherical and the flat case, respectively, in agreement with (8).

When working at fixed electric potential \(A_0 = -\phi\), with \(A_0|_{S_1} = -\phi_1\), \(A_0|_{S_2} = -\phi_2\) constants and \(\mu = \phi_1 - \phi_2\), the general solution is instead

\[
A_0 = -\frac{1}{R_2 - R_1} \left( R_2 \phi_2 - R_1 \phi_1 + \frac{\mu R_1 R_2}{r} \right), \quad \pi^i = -\frac{\mu R_1 R_2 x^i}{(R_2 - R_1) r^2},
\]

\[
Q = C_S \mu,
\]

(17)
in the spherical and

\[
A_0 = -\phi_1 + \frac{\mu}{d} z, \quad \pi^i = -\delta_i^1 \frac{\mu}{d}, \quad Q = C_P \mu,
\]

(18)
in the planar case. At fixed potential, the last surface term in (10) does no longer vanish on-shell. Instead, the improved action

\[
I' = I - \int dt \int d\sigma_i A_0 \pi^i,
\]

(19)

has a true extremum on-shell. In the Euclidean action, we have instead

\[
I'_E = I_E + \int_{0}^{\beta} d\tau \int d\sigma_i A_0 \pi^i = I_E + (\phi_2 - \phi_1) Q.
\]

(20)

When evaluated on-shell, this now leads to

\[
F(\beta, \mu) = -\frac{1}{2} C \mu^2,
\]

(21)

which is related to (16) through a standard Legendre transformation.

### 4 Boundary Conditions

In the case of the capacitor, the boundary conditions for perfect conductors are \(\vec{n} \times \vec{E} = 0 = \vec{n} \cdot \vec{B}\) on the boundary defined by the conductors, with \(\vec{n}\) the normal to the boundary. In the planar case, to which we limit ourselves in the following, one thus considers free electromagnetism on \(\mathbb{R}^2 \times [0, d]\), with boundary conditions \(E^x = 0 = E^y\) at \(z = 0\) and \(z = d\). It thus follows that \(\pi^a, a = 1, 2\), satisfy Dirichlet
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conditions. We then take Dirichlet conditions for $A_d$ as well since this guarantees well-defined Poisson brackets and a standard quantization in terms of the Fourier coefficients of sine functions. The requirement that $B^3 = \partial_1 A_2 - \partial_2 A_1$ should also satisfy Dirichlet conditions then holds automatically.

There remains the boundary conditions on $(A_3, \pi^3)$ and, in the case of BRST quantization, on $(A_0, \pi^0)$ as well as the ghost variables $(\eta, P)$, $(\bar{C}, \rho)$. A natural choice is Neumann conditions for $(A_3, \pi^3)$, and Dirichlet for all others in the case of BRST quantization. This choice implies that the divergence $\vec{\nabla} \cdot \vec{\pi}$ satisfies Dirichlet conditions. The constraint $\vec{\nabla} \cdot \vec{\pi} = 0$ in the space between the conductors can then be implemented by variation in the action principle (9) through a field $A_0$ that satisfies Dirichlet conditions as well. Proper gauge transformations are defined by gauge parameters that satisfy Dirichlet conditions, which implies the same conditions for the ghost variables. In the context of BRST quantization, this choice guarantees that the quartet mechanism for $(A_0, \pi_0)$, $(\vec{A}^L, \vec{\pi}^L)$ and ghost pairs will be effective.

If $k_3 = \frac{d}{d} n_3$, fields with Dirichlet conditions on $\mathbb{R}^2 \times [0, d]$ are expanded as

$$\phi(x^i) = \sum_{n_3 > 0} \phi^S_{k_3}(x^a) \sin k_3 z, \quad \phi^S_{k_3}(x^a) = \frac{1}{d} \int_{-d}^{d} dz \phi(x^i) \sin k_3 z,$$

while $A_3, \pi^3$ with Neumann conditions are expanded as

$$\phi(x^i) = \sum_{n_3 \geq 0} \phi^C_{k_3}(x^a) \cos k_3 z, \quad \begin{cases} c_{k_3}(x^a) = \frac{1}{d} \int_{-d}^{d} dz \phi(x^i) \cos k_3 z \\ \phi^C_{0}(x^a) = \frac{1}{2d} \int_{-d}^{d} dz \phi(x^i) \end{cases}.$$  

## 5 Physical Degrees of Freedom

In the Hamiltonian approach, the reduced physical phase space or rather functions thereon can be characterized through BRST cohomology in ghost number 0. This can be done independently of a choice of gauge fixation, which enters in the specification of the Hamiltonian.

In the case of free electromagnetism in Euclidean space $\mathbb{R}^3$, the Helmholtz decomposition of vector fields alluded to above allows one to show that this cohomology consists of functions of transverse vector potentials and their momenta. Alternatively, in terms of Fourier transforms, it consists of functions of transverse oscillator variables.

The analysis of the BRST cohomology in momentum space in the case of the capacitor [16] then shows that, at $k_3 \neq 0$, there are the standard two transverse polarizations, while there is in addition the mode at $n_3 = 0$ contained in $(A^3, \pi_3)$. This is the additional physical polarization of the Casimir effect.

In the original paper, this additional polarization was not discussed in the context of BRST quantization. Even though not explicitly stated in [9], it is clear from the...
context and from \cite{24}, that the analysis is done in radiation gauge, $A_0 = 0 = \nabla \cdot A$. Imposed together with the constraint equations $\pi^0 = 0$, $\nabla \cdot \pi = 0$. When translated to momentum space with the above boundary conditions, it follows directly that the $k_3 \neq 0$ modes of $(A, \pi)$ give rise to the two transverse polarizations, while the $k_3 = 0$ mode of $(A, \pi)$ is also divergence-free.

The divergence-free vector fields in position space associated with the $k_3 = 0$ mode are given by

$$A_{\text{NPG}}^i = \delta_3^i A_{3,0}^C(x^a), \quad \pi_{\text{NPG}}^i = \delta_3^i \pi_0^3C(x^a). \quad (24)$$

The argument why they are non-trivial from a position space viewpoint in equation (4.10) of \cite{16} is incorrect. Let us focus on $\pi_{\text{NPG}}$, which has a direct interpretation in electrostatics, the argument for $A_{\text{NPG}}$ being the same. The vector field $\pi_{\text{NPG}}$ has a non-trivial longitudinal piece. The associated 1 form is co-closed without being co-exact. This follows from the Helmholtz decomposition in the presence of boundaries. Indeed, under suitable fall-off assumptions at infinity, there is a unique decomposition

$$\pi = \nabla \varphi + \nabla \times \alpha, \quad (25)$$

$$\varphi(x) = -\int d^3x' \frac{(\nabla \cdot \pi)(x')}{4\pi |x - x'|} + \oint_S \frac{(\hat{n} \cdot \pi) d\sigma(x')}{4\pi |x - x'|}, \quad (26)$$

$$\alpha(x) = \int d^3x' \frac{\nabla \times \pi(x')}{4\pi |x - x'|} - \oint_S \frac{(\hat{n} \times \pi) d\sigma(x')}{4\pi |x - x'|}. \quad (27)$$

When this decomposition is applied to $\pi_{\text{NPG}}$ for the capacitor, the potential for the longitudinal part comes entirely from the boundary contribution and is explicitly given by

$$\varphi_{\text{NPG}}(x) = \frac{1}{4\pi} \int d^3y' \pi_0^3C(x'^a) \left( \left[ (\rho^2 + (z - d)^2)^{-\frac{1}{2}} - \left[ \rho^2 + z^2 \right]^{-\frac{1}{2}} \right) \right), \quad (28)$$

while the potential for the transverse part comes entirely from the bulk contribution and is explicitly given by

$$\alpha_{\text{NPG}}^i(x) = \frac{\delta_a^i}{4\pi} \int d^3y' \epsilon^{abc} \pi_0^3C(x'^c) \ln \frac{\sqrt{\rho^2 + (d - z)^2 + d - z}}{\sqrt{\rho^2 + z^2 - z}}, \quad (29)$$

where $\rho^2 = (x - x')^2 + (y - y')^2$.

One then has to decide how to deal with the transverse space $\mathbb{R}^2$. As usual, we will put the system in a finite two-dimensional box in an intermediate stage. In this box, we can adopt either perfectly conducting conditions as in \cite{9, 24}, or use periodic conditions, which is what was done in \cite{16}. In the large area limit, where sums go to integrals, both approaches yield the same results for finite temperature.
partition functions (without zero modes). In the latter, we thus consider expansions as in equation (4.4) of [25], with \( d = 3 \) and \( p = 1 \), but we explicitly keep the zero mode because we need it for the microscopic understanding of the Gibbons–Hawking contribution. This is reminiscent of the expansion of the complex scalar field in [26].

6 Dynamics and Charge

For the transverse degrees of freedom at \( k_3 \neq 0 \), the usual discussion applies in terms of two transverse polarizations applies. In addition, the canonical Hamiltonian \( H_0 = \int d^3x \mathcal{H}_0 \) induces a Hamiltonian for the non-proper gauge degrees of freedom given by

\[
H_{NPG} = d \int d^2x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} \partial_a \phi \partial^a \phi \right],
\]

where \( \phi = A_{30}^C, \pi = \pi_0^C \). When taking into account the kinetic term in the associated first-order action and after eliminating the momentum by its own equation of motions, the associated Lagrangian action is that of a massless scalar in \( 2 + 1 \) dimensions with prefactor \( d \),

\[
S_{NPG} = d \int dt d^2x \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \partial_a \phi \partial^a \phi \right].
\]

The electric charge observable can be written as a function on the phase space that includes the non-proper gauge degrees of freedom as,

\[
Q = - \int d^2x \pi.
\]

From this expression, it follows that charge is related to the momentum of the zero mode of the scalar field, which is a particle. For canonical commutation relations, the appropriate normalization (see e.g. [16] Appendix “Details on Quantum Coulomb Solution” for details) is

\[
q = \sqrt{\frac{d}{A}} \int d^2x \phi, \quad p = \sqrt{\frac{d}{A}} \int d^2x \pi,
\]

and the associated Hamiltonian and charge observable are given by

\[
H_{NPG}^0 = \frac{1}{2} p^2, \quad Q = -\sqrt{\frac{A}{d}} p.
\]
7 Extra Contributions to Partition Function

When naively decomposing the additional massless scalar field into its zero mode, the particle, and the remaining bulk modes in two dimensions, their contributions to the partition function is straightforward. In the charged case, the former is given by

\[ Z_{NPG}^0(\beta, \mu) = \text{Tr} e^{-\beta (H_{NPG}^0 - \mu Q)}. \]  

(35)

This can be related to the well-known partition function of a free particle of unit mass by completing the square. The result is

\[ \ln Z_{NPG}^0(\beta, \mu) = \ln \Delta q - \frac{1}{2} \ln \left( 2\pi \hbar^2 \beta \right) + \frac{A}{2d} \beta \mu^2. \]  

(36)

Here \( \Delta q \) denotes the divergent interval of integration of \( q \), while the last term reproduces the Gibbons–Hawking contribution \( -\beta F(\beta, \mu) \) to the partition function as discussed in (21).

The partition function of a massless scalar in two dimensions can be obtained as usual after putting the field in a box with periodic boundary conditions and by neglecting the zero mode. The standard result in the limit of large volume in two dimensions, which is the area of the plates in the current context, is

\[ \ln Z_{NPG}' = \frac{A}{2\pi} \zeta(3)(\hbar \beta)^{-2}. \]  

(37)

A different discussion along the lines of [27, 28] gives instead

\[ F_{NPG}^0(\beta, 0) = \beta^{-1} \{ \text{div} + \ln \beta + \text{cte} \}, \]  

(38)

which differs by a factor of 2 in the \( \ln \beta \) term from (36). Note however that this difference will not matter for the considerations below as long as the \( \mu \) dependent part will still be given by \( -\frac{A}{2d} \mu^2 \).

8 Charged Black Body Partition Function

In order to discuss the full, finite result, one may follow and adapt the discussion of the finite temperature Casimir effect [10, 11] (see e.g. [13–15] for reviews).

One considers segments on the \( z \)-axis given by

\[ I = [0, d], \quad II = [d, L_z], \quad III = [0, L_z/\eta], \quad IV = [L_z/\eta, L_z]. \]  

(39)

The analog \( F_C(\beta, \mu) \) of the Casimir free energy is defined as
\( \mathcal{F}_C(\beta, \mu) = \mathcal{F}_I(\beta, \mu) + \mathcal{F}_{II}(\beta, 0) - \mathcal{F}_{III}(\beta, 0) - \mathcal{F}_{IV}(\beta, 0). \) \( (40) \)

The zero mode will then only give the Gibbons–Hawking contribution

\[ \mathcal{F}^0_C(\beta, \mu) = -\frac{A}{2d} \mu^2. \] \( (41) \)

Non-zero modes, both those at \( k_3 \neq 0 \) and those of the additional scalar, will not contribute to the \( \mu \) dependent part. As usual, one separates the zero temperature contribution from the thermal one,

\[ \mathcal{F}'_C(\beta) = \mathcal{F}'_C(\infty) + \mathcal{F}^{IT}_C(\beta). \] \( (42) \)

In the limit of large plate area \( A \) and large \( L_z \), the former is the standard zero temperature Casimir energy that may be computed from the zero point energies. Between the plates, one finds

\[ \mathcal{F}'_I(\infty) = \frac{\hbar}{2} \frac{A}{(2\pi)^2} \int d^2 k \left[ \sqrt{k_a k^a} + 2 \sum_{n=1}^{\infty} \sqrt{k_a k^a + \frac{\pi^2 n^2}{d}} \right], \] \( (43) \)

while

\[ \mathcal{F}'_{II}(\infty) - \mathcal{F}'_{III}(\infty) - \mathcal{F}'_{IV}(\infty) = -\frac{d}{2} \frac{\hbar}{(2\pi)^2} \frac{A}{(2\pi)^2} \int d^2 k \left[ \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sqrt{k_a k^a + k_z^2} \right]. \] \( (44) \)

After a suitable cut-off regularization and with the help of the Euler–Maclaurin formula, one then finds

\[ \mathcal{F}'_C(\infty) = -\frac{A \pi^2 \hbar}{720d^3}. \] \( (45) \)

Note that the first term in the square brackets of \( (43) \) is due to the additional massless scalar and gives the first term at discrete value 0 with the correct 1/2 in the Euler–Maclaurin formula. When using \( \xi \) function regularization, this divergent term is usually omitted because it does not depend on the separation distance and thus does not contribute to the Casimir force.

In the same way, the temperature dependent contribution, which needs no regularization, is given by
\[ F_{C}^{T} (\beta) = \frac{2A}{\beta} \int \frac{d^2 k}{(2\pi)^2} \left[ \sum_{n=0}^{\infty} \ln \left( 1 - e^{-\hbar \beta \sqrt{k_\alpha k^\alpha + \left( \frac{\pi}{\alpha} \right)^2}} \right) \right] \]

\[ -d \int_{-\infty}^{+\infty} \frac{dkz}{2\pi} \ln \left( 1 - e^{-\hbar \beta \sqrt{k_\alpha k^\alpha + k^2}} \right), \]

where the prime on the sum means that the term at \( n = 0 \) comes with a factor 1/2. This term is due to the non-zero modes of the additional scalar. More explicitly, if

\[ b(d, \beta, n) = \frac{1}{2\beta} \int_{d^2} d^4 s \ln \left( 1 - e^{-\frac{\pi \hbar \beta}{d} \sqrt{s}} \right) \]

one finds

\[ F_{C}^{T} (\beta) = -\frac{A}{2\pi \hbar^2 \beta^3} \zeta(3) + \frac{A\pi}{d^2} \sum_{n=1}^{\infty} b(d, \beta, n) + \frac{V\pi^2}{45\hbar^3 \beta^4}. \]

The first term from the additional scalar coincides with the contribution from (37) to the free energy. It does not contribute to the Casimir force but does contribute to the entropy. The last term corresponds to the subtraction of the black body result, that is to say the contribution of the two transverse polarization in empty space. The middle term corresponds to the contribution of the two transverse polarizations at discretized non-zero values of \( k_3 \). Low and high temperature expansions are discussed in the cited literature. The full result is then

\[ F_C (\beta, \mu) = F_C^0 (\beta, \mu) + F_C' (\infty) + F_C^{T} (\beta). \]

**Appendix: Details on Quantum Coulomb Solution**

Consider the electromagnetic field interacting with a static point particle sitting at the origin,

\[ S \left[ A_\mu; j^\mu \right] = \int d^4 x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu \right], \quad m^\mu = \delta_0^\mu \delta^3 (\vec{x}). \]

The modified vacuum state that is annihilated by the BRST charge in the presence of the source is given by

\[ |0\rangle_Q = e^{\int d^3 k q(\vec{k}) \hat{b}^\dagger (\vec{k})} |0\rangle, \quad q(\vec{k}) = \frac{Q}{(2\pi)^{3/2} \sqrt{2\omega(\vec{k})^{3/2}}}, \]

if
\[ A_0(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega(k)}} \left[ a_0(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} + \text{c.c.} \right], \]  
(52)

\[ A_i(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega(k)}} \left[ a_m(\vec{k}, t) e^m(\vec{k}) e^{i\vec{k} \cdot \vec{x}} + \text{c.c.} \right], \]  
(53)

where \( \omega(\vec{k}) = |\vec{k}| = k \), the polarization vectors are \( e^3_i = k_i/\omega(\vec{k}) \), and \( k^i e^a_i = 0, a = 1, 2 \), while the unphysical null oscillators are defined by

\[ a(\vec{k}) = a_3(\vec{k}) + a_0(\vec{k}), \quad b(\vec{k}) = \frac{1}{2} \left[ a_3(\vec{k}) - a_0(\vec{k}) \right], \]  
(54)

(see [4] for detailed conventions including the adapted mode expansions for the momenta, up to the correction pointed out in [6]). This state is constructed so as to be annihilated by the BRST charge in the presence of the source,

\[ \hat{\Omega}^Q |0\rangle^Q = 0, \]  
(55)

where

\[ \Omega^Q = -\int d^3x \left[ i\rho \pi^0 + \eta \left( \partial^i \pi_i - \xi^0 \right) \right], \]  
(56)

\[ \hat{\Omega}^Q = \int d^3k \left[ \hat{c}^\dagger(\vec{k}) \hat{a}^Q(\vec{k}) + \hat{a}^Q(\vec{k}) \hat{c}(\vec{k}) \right], \quad \hat{a}^Q(\vec{k}) = \hat{a}(\vec{k}) - q(\vec{k}). \]  
(57)

Note that it is not the only state with this property, for instance

\[ |0\rangle'^Q = e^{-\int d^3k q(\vec{k}) \hat{a}^\dagger_0(\vec{k})} |0\rangle = e^{-\int d^3k \frac{q(\vec{k})}{\sqrt{2}} \hat{a}^\dagger(\vec{k})} |0\rangle^Q, \]  
(58)

is also annihilated by \( \hat{\Omega}^Q \), and as in [1, 2], it is constructed out of temporal oscillators alone.\(^1\)

The gauge fixed Hamiltonian

\[ H_\xi = H_0 + \left\{ \Omega^Q, K_\xi \right\}, \quad H_0 = \int d^3x \frac{1}{2} \left[ \pi^i \pi_i + B^i B_i \right], \quad B^i = \epsilon^{ijk} \partial_j A_k, \]  
(59)

is constructed by using the gauge fixing fermion

\[ K_\xi = -\int d^3x \left[ i\tilde{C} \partial_k A^k + \mathcal{P} A_0 - \frac{i}{2} \tilde{C} \pi^0 \right]. \]  
(60)

\(^1\)G.B. is grateful to M. Schmidt and S. Theisen for pointing this out and for prompting the considerations below.
Since

\[
\{\Omega^Q, K_{\xi}\} = \int d^3x \left[ \partial_k A^k \pi^0 + A_0 \left( -\partial_i \pi^i + j^0 \right) + iP \rho + i \partial^i \tilde{c} \partial_i \eta - \frac{1}{2} \xi \pi^0 \pi^0 \right],
\]

(61)

the gauge fixed Hamiltonian contains in particular the correct source term. When using the decomposition \(\pi^i = \pi^i_T + \frac{1}{\Delta} \partial^j \partial^i \pi^j_i\), it follows that

\[
\frac{1}{2} \int d^3x \pi^i \pi_i = \frac{1}{2} \int d^3x \pi^i_T \pi_i^T - \frac{1}{2} \int d^3x \partial_j \pi^j \frac{1}{\Delta} \partial_k \pi^k.
\]

(62)

The last term can be written as

\[
-\frac{1}{2} \int d^3x \partial_j \pi^j \frac{1}{\Delta} \partial_k \pi^k = \{\Omega^Q, \tilde{K}^Q_{\xi}\}.
\]

(63)

so that

\[
H_{\xi} = H^{\text{ph}} + \{\Omega^Q, \tilde{K}^Q_{\xi}\}.
\]

(64)

\[
H^{\text{ph}} = \frac{1}{2} \int d^3x \left[ \pi^i_T \pi^i_T - A_j^T \Delta A_j^T - j^0 \frac{1}{\Delta} j^0 \right],
\]

(65)

\[
\tilde{K}^Q_{\xi} = K_{\xi} - \frac{1}{2} \int d^3x \frac{1}{\Delta} \left( \partial_i \pi^i + j^0 \right).
\]

(66)

In Feynman gauge \(\xi = 1\), when expressed in terms of modes, we have

\[
\hat{H}_{\xi=1} = \hat{H}^{\text{phys}} + \left[ \hat{\Omega}^Q, \hat{\tilde{K}}^Q_{\xi=1} \right],
\]

(67)

\[
\hat{H}^{\text{phys}} = \int d^3k \omega(k) \left[ \hat{a}^+_a(k) \hat{a}^a(k) + \frac{\omega(k)}{2} \right],
\]

(68)

\[
\hat{\tilde{K}}^Q_{\xi=1} = \hat{K}_{\xi=1} - \frac{1}{2} \int d^3k \frac{1}{\Delta} \left( \partial_i \pi^i + j^0 \right).
\]

(69)

and

\[
\left[ \hat{\Omega}^Q, \hat{\tilde{K}}^Q_{\xi=1} \right] = \int d^3k \omega(k) \left[ \hat{a}^+_a(k) b(k) + \hat{b}^+_a(k) \hat{a}(k) + \hat{\tilde{c}}^+_a(k) \hat{c}(k) + \hat{\tilde{c}}^+_a(k) \hat{\tilde{c}}(k) \right]
\]

(70)
where the first line is proportional to the number operator for unphysical oscillators, while the second line contains the correct source term. Since \( \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} + q \hat{a}_0 + q \hat{a}_0^\dagger = \hat{a}_3^\dagger \hat{a}_3 - (\hat{a}_0^\dagger - q)(\hat{a}_0 - q) + q^2 \), it follows that \(|0\rangle'^Q \) is an eigenstate of this gauge fixed Hamiltonian

\[
\hat{H}_{\xi=1}|0\rangle'^Q = \int d^3k \omega(\vec{k})q(\vec{k})^2|0\rangle'^Q. \tag{71}
\]

In the context of BRST quantization, one may modify the gauge fixing fermion and remove the source dependent term therein, that is to say, one may replace \( \tilde{K}_\xi^Q \) by

\[
\tilde{K}_\xi = K_\xi - \frac{1}{2} \int d^3x \mathcal{P} \frac{1}{\Delta} \partial_i \pi^i, \tag{72}
\]

\[
\hat{\tilde{K}}_{\xi=1} = \int d^3k \omega(\vec{k}) \left[ \hat{\tilde{c}}^\dagger(\vec{k}) \hat{\tilde{b}}(\vec{k}) + \hat{\tilde{b}}^\dagger(\vec{k}) \hat{\tilde{c}}(\vec{k}) \right], \tag{73}
\]

\[
[\hat{\Omega}^Q, \hat{\tilde{K}}_{\xi=1}] = \int d^3k \omega(\vec{k}) \left[ \hat{\Omega}^Q(\vec{k}) \hat{\tilde{b}}(\vec{k}) + \hat{\tilde{b}}^\dagger(\vec{k}) \hat{\Omega}^Q(\vec{k}) \right]
+ \hat{\tilde{c}}^\dagger(\vec{k}) \hat{\tilde{c}}(\vec{k}) + \hat{\tilde{c}}^\dagger(\vec{k}) \hat{\tilde{c}}(\vec{k}) \right], \tag{74}
\]

since this modifies the ghost number 0 part of the Hamiltonian by terms that are proportional to the constraints. It now follows that \(|0\rangle'^Q \) is an eigenstate of the new gauge fixed Hamiltonian \( \hat{H}'_{\xi=1} \).

\[
\hat{H}'_{\xi=1} = \hat{H}_{\text{ph}} + [\hat{\Omega}^Q, \hat{\tilde{K}}_{\xi=1}], \quad \hat{H}'_{\xi=1}|0\rangle'^Q = \int d^3k \omega(\vec{k})q(\vec{k})^2|0\rangle'^Q, \tag{75}
\]

with the same eigenvalue than \(|0\rangle'^Q \) is of \( \hat{H}_{\xi=1} \).

Note also that the difference between \( e^{\int d^3k \frac{q^2(\vec{k})}{2}}|0\rangle'^Q \) and \(|0\rangle^Q \) is BRST exact. Indeed,

\[
e^{\int d^3k \frac{q^2(\vec{k})}{2}}|0\rangle'^Q - |0\rangle^Q = \left( e^{-\int d^3k \frac{q^2(\vec{k})}{2}} \hat{\Omega}^Q(\vec{k}) - \hat{1} \right)|0\rangle^Q. \tag{76}
\]

The result follows from the fact that \( -\int d^3k \frac{q(\vec{k})}{2} \hat{\tilde{a}}^\dagger Q^Q(\vec{k}) = [\hat{K}, \hat{\Omega}^Q], \quad \hat{K} = -\int d^3k \frac{q(\vec{k})}{2} \hat{\tilde{c}}(\vec{k}) \),

\[
-\int d^3k \frac{q(\vec{k})}{2} \hat{\tilde{a}}^\dagger Q^Q(\vec{k}) = [\hat{K}, \hat{\Omega}^Q], \quad \hat{K} = -\int d^3k \frac{q(\vec{k})}{2} \hat{\tilde{c}}(\vec{k}), \tag{77}
\]
and that the difference of the exponential of a BRST exact operator minus the unit operator is a BRST exact operator,

$$e^{i \hat{K} \cdot \hat{\Omega}^O} - 1 = \left[ \hat{L}, \hat{\Omega}^O \right],$$

(78)

for some operator $\hat{L}$ (see, e.g. [4], exercise 14.3 for the proof), so that

$$e^{\int d^3k \frac{Q(\vec{k})^2}{2} |0\rangle^Q} = |0\rangle^Q - \hat{\Omega}^O \hat{L} |0\rangle^O,$$

(79)

since $|0\rangle^Q$ is BRST closed.

Some additional comments on [6] are in order.

(i) In the computation (2.8), an obvious infrared regularization is understood since the Fourier transform of $k^{-2}$ is $\frac{1}{4\pi r}$ only when using such a regulator,

$$\frac{1}{(2\pi)^3} \int d^3k \frac{1}{k^2 + \mu^2} e^{i\vec{k} \cdot \vec{x}} = \frac{1}{4\pi r} e^{-i\mu r},$$

(80)

with the desired result obtained when $\mu \to 0^+$. Indeed, the two terms cancel since both $\omega(\vec{k})$ and $q(\vec{k})$ are even under $\vec{k} \to -\vec{k}$. There is no explanation needed for the difference of a factor 2 between (2.5) and (2.6) because $A_\mu$ is not a gauge invariant quantity, as opposed to $\vec{\pi}$ and $\vec{\nabla} \times \vec{A}$ whose associated expectation values are correctly given in (2.8) and (2.9). Note however that the Hamiltonian $\hat{H}'_{\xi=1}$ gives rise to the usual oscillating behavior for all oscillators in the Heisenberg picture, except for $\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k})$ which evolve according to

$$\hat{a}^Q(\xi, \vec{k}) \equiv \hat{a}(\xi, \vec{k}) - q(\vec{k}) = e^{-i\omega(\vec{k})\xi} \hat{a}^Q(\vec{k}),$$

(82)

and its complex conjugate.

(ii) Equation (2.7) is not correct. Starting from

$$\partial_i A^i = \frac{i}{(2\pi)^{3/2}} \int d^3k \sqrt{\omega(\vec{k})} \left[ \frac{a(\vec{k}) + 2b(\vec{k})}{2} e^{i\vec{k} \cdot \vec{x}} - \text{c.c.} \right],$$

(80)

one finds instead of (2.7) that

$$Q \langle 0 | \partial_i \hat{A}^i | 0 \rangle^Q = \frac{i}{(2\pi)^{3/2}} \int d^3k \sqrt{\omega(\vec{k})} \left[ \frac{1}{2} q(\vec{k}) e^{i\vec{k} \cdot \vec{x}} - \text{c.c.} \right] = 0.$$  

(81)

Indeed, the two terms cancel since both $\omega(\vec{k})$ and $q(\vec{k})$ are even under $\vec{k} \to -\vec{k}$. There is no explanation needed for the difference of a factor 2 between (2.5) and (2.6) because $A_\mu$ is not a gauge invariant quantity, as opposed to $\vec{\pi}$ and $\vec{\nabla} \times \vec{A}$ whose associated expectation values are correctly given in (2.8) and (2.9). Note however that the Hamiltonian $\hat{H}'_{\xi=1}$ gives rise to the usual oscillating behavior for all oscillators in the Heisenberg picture, except for $\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k})$ which evolve according to

$$\hat{a}^Q(\xi, \vec{k}) \equiv \hat{a}(\xi, \vec{k}) - q(\vec{k}) = e^{-i\omega(\vec{k})\xi} \hat{a}^Q(\vec{k}),$$

(82)

and its complex conjugate.

(iii) In order to make contact with the original [5] and subsequent work, note that the new vacuum corresponds to the old one “dressed” by
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\[ e^\int d^3x \left[ i \frac{\tilde{Q}}{2} \mathbf{x} \cdot \hat{A}^{(-)}(\tilde{x}') - \frac{i}{2} j^0(\tilde{x}')(-\Delta)^{-1/2} A_0^{(-)}(\tilde{x}') \right], \tag{83} \]

where the subscript \((-\)) denotes the creation part.

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AUTHOR QUERIES

AQ1. Please check if the edit made to the sentence “In the Hamiltonian approach, the reduced ...” is fine.
AQ2. Please check the sentence “When naively decomposing the additional ...” for clarity.
AQ3. Please check the edit made in Eq. (44), and correct if necessary.