A dilute atomic Fermi system with a large positive scattering length

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We show that a dilute atomic Fermi system at sufficiently low temperatures, can display fermionic superfluidity, even in the case of a repulsive atom-atom interaction, when the scattering length is positive. The attraction leading to the formation of Cooper pairs is provided by the exchange of Bogoliubov phonons if a fraction of the atoms form a BEC of weakly bound molecules.

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Ever since the pioneering experiment of DeMarco and Jin [1] there has been an ongoing search for the super-fluidity in a dilute fermionic atomic system [2]. A recent experimental result from Jin’s group [3] seem to point to the possible creation of an even more intriguing system, a dilute atomic Bose system, subsequently shown to be in a BEC state [4], coexisting with a dilute atomic fermionic fluid, perhaps even a superfluid itself as we will show here, both superfluids composed from one single atomic species. Taking advantage of a Feshbach resonance between given hyperfine states all these authors [3, 4] have manipulated the scattering length and made it positive. Starting with an almost equal mixture of these hyperfine spin sub–states and a relatively small and negative scattering length, by ramping the magnetic field at various rates, these authors have created a mixture of weakly bound fermionic atoms and unbound fermions, all of them interacting with positive scattering lengths, as we shall show here. The weakly bound state formed is bosonic in character, with a binding energy typically much larger than the temperature of the atomic cloud [4]. This binding energy is approximately given by $\hbar^2 / m a^2$, when the scattering length is large. We shall often refer to these weakly bound molecule as dimers. The fraction of such molecules depends on the rate at which the magnetic field is ramped across the Feshbach resonance, ranging from essentially zero in the sudden limit and up to about 80% in the adiabatic limit. At $T = 0$ and in the adiabatic limit one would expect a full conversion of all fermions into such weakly bound molecules, the so called BCS $\rightarrow$ BEC crossover [2, 4, 6, 8, 10]. Unless the magnetic field is very close to the Feshbach resonance, the size of these molecules is smaller than the average inter particle separation. If the weakly bound molecules form a BEC state, then the induced fermion–fermion interaction, due to the exchange of Bogoliubov bosonic sound waves, is attractive. Very close to the Feshbach resonance when the atomic scattering length is much larger than the effective range, the full fermion–fermion interaction can then become weakly attractive. Under these circumstances the BEC may coexist with a weakly coupled BCS superfluid.

Over the last two decades many authors, starting with Leggett [2, 4, 5, 6, 7, 8, 9, 11], studied the BCS–BEC crossover in Fermi systems, when the pairing coupling constant is varied from very small to relatively large values. If the pairing coupling constant is so weak, that in vacuum two fermions do not have a bound state, one can expect a relatively small pairing gap at a finite fermion density, due to the Cooper instability [12]. When $k_F |a| < 1$ and $a < 0$ the fermion kinetic energy dominates over the interaction energy and the Fermi system has a positive internal pressure. In the dilute limit (when $k_F |a| < 1$ and where $k_F$ is the Fermi wave vector and $a < 0$ the scattering length), the pairing gap is given by

$$\Delta \approx \left( \frac{2}{e} \right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp \left( \frac{\pi}{2 k_F a} \right).$$

Even in the dilute limit the determination of the pre–exponential factor requires the evaluation of many–body diagrams beyond the naive expected mean–field effects alone [13, 14]. At least theoretically, one can contemplate the following question: “How do the properties of a Fermi system evolve if one were to increase the pairing coupling constant until two fermions in vacuum could form a bound state?” The arguments presented in Refs. [2, 4, 5, 6, 8, 10, 11] lead us to believe that the many fermion system will most likely undergo a smooth crossover from a weak coupling BCS state to a BEC state, made of relatively tightly bound fermion pairs. The prospects of observing such a crossover were quite dim until the recent series of experiments [2, 4]. While for both positive and negative values of the scattering length when $k_F |a| < 1$ the properties of a many fermion system can be described perturbatively, in the intermediate region,
when the scattering length is comparable or larger than the average inter particle separation the entire system is in non–perturbative regime. The basic properties of a Fermi system with $|a| = \infty$ have been refined recently in a quantum Monte Carlo variational calculation [11].

In a Fermi–Bose mixture the interaction responsible for the formation of Cooper pairs could be an induced interaction [14, 15, 16, 17, 18]. A moving fermion can excite a sound wave in the Bose gas, which will perturb at a relatively large distance another fermion. This situation is similar to the electron–phonon model of BCS superconductivity. The boson induced interaction between fermions is in momentum–frequency representation given by

$$U_{bf}(q, \omega) = U_{bf}^2 \frac{2n_b \varepsilon_q}{\hbar^2 \omega^2 - \varepsilon_q(\varepsilon_q + 2n_b U_{bb})},$$

where $U_{bf}$ is the fermion–boson coupling constant, $q$ and $\omega$ are the wave vector and the frequency exchanged between the two interacting fermions, $\varepsilon_q = \hbar^2 q^2 / 2m_b$, $m_b$ is the boson mass, $n_b$ the boson number density, $U_{bb} = 4\pi \hbar^2 a_{bb} / m_b$ and $a_{bb}$ the boson–boson scattering length. In the weak coupling limit the two paired fermions exchange a small energy $\hbar \omega = O(\varepsilon_F)$ and a large linear momentum $q = O(k_F)$ and then the induced interaction is predominantly static and attractive. It is straightforward to show that in coordinate representation the induced interaction is of Yukawa type [14, 15, 16, 17, 18].

$$U_{bf}(r) = -\frac{U_{bf}^2}{U_{bb}} \frac{1}{4\pi \xi_b^2} \exp\left(-\frac{r}{\xi_b}\right),$$

where

$$\xi_b = \frac{\hbar}{2m_b s_b} = \sqrt{\frac{a_{bb}}{16\pi m_b s_b}}, \quad s_b = \frac{n_b U_{bb}}{m_b},$$

$\xi_b$ is the healing or coherence length and $s_b$ is the speed of the Bogoliubov sound waves in the Bose gas. It is notable that the strength of the induced interaction in momentum representation at $q = 0$ is independent of the boson density. The radius $\xi_b$ of this interaction could exceed significantly $a_{bb}$ in a very dilute Bose gas, where $n_b a_{bb}^3 \ll 1$, see Eq. (4).

Using a very simple semiclassical criterion suggested by Calogero [14], one can show that the strength of this induced interaction is typically weak. Neglecting for the moment the role of $U_{bf}$, Calogero’s condition that the potential $U_{bf}(r)$ can sustain at least one two–particle bound state is

$$2 \pi \int_0^\infty dr \sqrt{-m_f U_{bf}(r) / \hbar^2} = \sqrt{\frac{8}{\pi}} \frac{U_{bf}^2 m_f}{U_{bb} 4\pi \hbar^2 \xi_b^2} \geq 1.$$  

This criterion is saturated in the case of a simple square–well potential and thus cannot be improved. If one assumes that all scattering lengths and masses are comparable in magnitude, it is easy to see that this condition can hardly be satisfied in the dilute limit, when the healing/coherence length is large ($\xi_b \gg a_{bb}$) and the right hand side of Eq. (5) is $\propto \sqrt{a_{bb}/\xi_b} \ll 1$. We have thus naturally arrived at the conclusion that the net effect of a repulsive fermion–fermion interaction $U_{ff}$ and of the attractive boson induced fermion–fermion interaction $U_{bf}$ could lead to at most a weak coupling pairing gap in the fermion sector. Even in the absence of $U_{ff}$ the induced interaction can lead to a weak coupling BCS gap only.

An extremely interesting system to consider consists of fermionic atoms mixed with diatomic molecules of the same atom species, as in the recent experiment of Jin’s group [19]. One can change the fermion–fermion scattering length to a large and positive value $a_{ff} := a \gg r_0$ ($r_0$ is the radius of the interaction, which is also typically of the order of the effective range) by means of a Feshbach resonance. For the sake of simplicity of the argument we shall assume that there are equal amounts of two identical fermionic spin subspecies, which we shall formally identify with the two spin states of a spin 1/2. In such a system one can observe the BCS–BEC crossover predicted two decades ago [1, 2, 3, 4, 5, 6, 7, 8]. What was not appreciated until recently [20] is that because not only $a_{ff} = a > 0$, but also $a_{bb} = 1.179 a > 0$ and $a_{bb} > 0$, one can manufacture a metastable mixed Fermi–Bose system composed of such fermion atoms and weakly bound fermion pairs. The size of the fermion pair is $O(a) \gg r_0$ and a dilute system of such pairs alone is metastable, since $a_{bb} = 0.60 a > 0$ see below and Ref. [22].

This is due to the act that the transitions between atoms and molecules is a relatively slow process, see Refs. [3, 4] and in particular the discussion of the latest experiment in Jin’s group [4]. In fact, at temperatures smaller than the binding energy of a pair $T \ll \hbar^2 / m_f a^2$, the pair–pair collision is mostly elastic. Inelastic collisions with the formation a tightly bound state and one or two fast atoms with kinetic energies of the order $\hbar^2 / m_f a^2 \gg \hbar^2 / m_f a^2 \gg T$ is possible. The branching ratio for such a channel is small ($\propto (r_0/a)^{3/2}$, $\alpha \approx 3$) [22]. Moreover, since the final products will carry a large kinetic energy, they will interact with the rest of the slow constituents with cross sections of the order $r_0^2 \ll a^2 \ll n^{-2/3}$, where $n$ is either the fermion or the pair number density. A fast atom will not interact with a pair with a cross section of the order of $a^2$, as one might naively expect, for similar reasons why cosmic neutrinos do not have a cross section of the order of the Earth radius squared when they impinge on our planet. Therefore, such process could not lead to a noticeable heating of the system. Three–body recombination rates can be suppressed by choosing appropriate low densities. One should recognize however that due to the different dependence on the scattering length $a$ of the three-body recombination and of the the rate of formation of tightly bound molecules, the optimal conditions for the formation of an atomic-molecular system correspond to such values of the scattering length
for which these rates are approximately equal. The lifetimes of such a mixed atomic-molecular system could be estimated as ranging from \(10^{-2}\) s to several seconds, depending on the atom species, peak number density and specific value of the scattering length.

The total energy density of such a system of spin 1/2 fermions is given by the following expression \(\langle m_f := m, m_b := 2m n a^3 \ll 1, n = n_f + 2n_b \rangle [21]\)

\[
\mathcal{E} = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} n_f + \frac{\pi \hbar^2 a}{m} n_f + \frac{3.537 \pi \hbar^2 a}{m} n_f n_b + \frac{0.6 \pi \hbar^2 a}{m} n_f^2 + \varepsilon_2 n_b,
\]

where \(n_f = \frac{k_F^3}{3\pi^2}, k_F\) is the Fermi wave vector and \(\varepsilon_2 = -\hbar^2/ma^2\) is the ground state energy of the pair. The specific numerical values quoted here were obtained from a detailed calculation of the atom-dimer and dimer-dimer scattering lengths described in the next paragraph. There are well known corrections of higher order in \(n a^3 \ll 1\) to this expression, which we neglect here. The chemical potentials for the fermions and bosons in this case can be specified independently. Chemical equilibrium between fermions and pairs can be established only after a very long time, as at these small temperatures \(|\varepsilon_2| = \hbar^2/ma^2 \gg T\), see the discussion of recent results from Jin’s group [4]. The energy density in Eq. 6 has to be supplemented with a contribution arising from the weak coupling pairing correlations (condensation energy), due to induced interactions, if there is a Cooper instability. Since the Fermi subsystem is always in the weak coupling BCS limit, this correction to the energy is small.

In the extreme limit when \(a \gg r_0\), where \(r_0\) is the effective range appearing in the low energy expansion of the s-wave scattering phase shift \(k \cot \delta(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \ldots\), the fermion–pair scattering length is \(a_{fb} \approx 1.179 a\), a result first established by Skorniakov and Ter–Martirosian in 1957 [21]. Since the value of the atom–atom scattering length can be manipulated by means of the Feshbach resonance, so can the ratio \(r_0/a\), as \(r_0\) (unlike \(a\)) should have a small variation with the applied magnetic field in the narrow region of the resonance. The dependence of \(a_{fb}\) as a function of \(a = a_{ff}\) for fixed \(r_0\) is determined, up to corrections of order \((r_0/a)^3\), by a simple generalization of the Skorniakov and Ter–Martirosian equation 21. It turns out that the ratio \(a_{fb}/a\) varies by less than 1\% as \(r_0\) varies from \(r_0 = 0\) to \(r_0 = 0.3a\) and the likelihood of Cooper pair formation is not much affected by finite range effects. We have performed as well calculations of the dimer-dimer scattering length using a well established four-body scattering formalism. The four-body equations we solve involve four identical spin 1/2 fermions (two with spin up and two with spin down), interacting through a short range potential in \(^1S_0\) alone. The formalism we use is identical to the one developed in Refs. [23] to study the four-nucleon system at low energies and is based on the solution of the Alt, Grassberger and Sandhas (AGS) equations [26], in the form that was proposed by Fonseca and Shanley [27]. Since this method requires the solution of the underlying two- and three-body T-matrix for all relevant channels, it is appropriate to mention that we include in the calculation three-body sub-amplitudes with total angular momenta ranging from \(1/2^+\) to \(9/2^+\), since we find that \(P\)-wave atom-dimer scattering contributes as much as 20\% to the dimer-dimer scattering length. The results of the calculation indicate that \(a_{fb} = 0.60a\). Calculations were performed using both a finite and a zero range two-body interaction. In the case of a finite range we have varied the ratio \(a/r_0\) from \(O(1)\) to \(\approx 2000\). The calculations of Petrov et al. [28] of the atom-dimer and dimer-dimer scattering lengths, based on a novel theoretical approach, whose validity we were unable to assess though, agree with our result.

It is remarkable that the interaction properties of this mixed system of atoms and weakly bound molecules can be described in terms of a single parameter, the fermion–fermion scattering length \(a\). From the previous analysis [14, 15, 16, 17, 18] and the arguments given above we know now that the tightly bound Cooper pairs, which form a BEC, can lead to an additional attraction among the fermions. The full fermion–fermion interaction in momentum space (and zero frequency) is therefore

\[
U_{ff}(q) = \frac{2\pi \hbar^2 a}{m} \left[ 1 - \frac{5.213}{1 + q^2 \xi^2} \right],
\]

where \(\hbar q\) is the momentum exchanged by two incoming fermions with momenta \(\hbar k_{1,2}\). The \(s\)-wave pairing gap is determined by the \(l = 0\) partial wave of this interaction (if attractive), when both fermion momenta are at the Fermi surface \((k_1 = k_2 = k_F)\)

\[
\Delta = \left( \frac{2}{\rho} \right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \times \exp \left[ \frac{\pi}{2k_F a} \left( 1 - \frac{5.213}{1 + 4k_F^2 \xi^2} \right) \right],
\]

otherwise \(\Delta \equiv 0\). The value of the pre-exponential factor requires further analysis. The induced interaction has a relatively large radius and typically in an extremely dilute system \(2k_F \xi = 1.127(n_f/n_b)^{1/3}(n_0 a^3)^{-1/6} \gg 1\), if the densities of the fermion and boson subsystems are comparable. Pairing in the \(s\)-wave is not likely then, since in this case the \(s\)-wave full fermion–fermion interaction is repulsive. Still, for appropriate values of the parameters \(s\)-wave pairing is possible, see Fig. 14. For other values of the parameters \(p\)-wave pairing should be considered [28], but the size of the \(p\)-wave gap will typically be smaller.

The spectrum of sound waves in such a system has two branches. The bosonic pairs will naturally have a
Bogoliubov sound branch with the speed $s_b$, see Eq. [31], while in the fermion sector the sound waves will have the speed $s_f = v_F/\sqrt{3} = \hbar(3\pi^2 n_f)^{1/3}/\sqrt{3m}$, unlike in a normal Fermi gas, where the speed of the Landau’s zero sound waves is larger then $v_F$ [30]. The ratio of these two sound speeds could in principle have any value, depending on the specific values of the scattering length and fermion and boson number densities $s_f/s_b \propto (n_f/n_b)^{1/3}/(n_b a^3)^{1/6}$. For dilute systems with comparable amounts of bosons and fermions the slowest mode is the bosonic Bogoliubov sound wave. There are relatively small corrections to the values of $s_b$ and $s_f$, due to interactions and also due to some mixing between the two types of sound waves. We simply mention that for these systems (in both normal and superfluid fermion phases) there is no long–wave instability of the type suggested in Ref. [31]. Even though the present analysis was limited to the homogeneous case, the properties of a system in a trap can be easily inferred and as, it was shown in Ref. [32], an atom–molecule mixture of the kind considered here acquires a complex and quite unexpected spatial structure.

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