Cocompact imbedding theorem for functions of bounded variation into Lorentz spaces

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Abstract

We show that the imbedding $\dot{BV}(\mathbb{R}^N) \hookrightarrow L^{1^*,q}(\mathbb{R}^N)$, $q > 1$ is cocompact with respect to group and the profile decomposition for $BV(\mathbb{R}^N)$. This paper extends the cocompactness and profile decomposition for the critical space $L^{1^*}(\mathbb{R}^N)$ to Lorentz spaces $L^{1^*,q}(\mathbb{R}^N)$, $q > 1$. A counterexample for $BV(\mathbb{R}^N) \hookrightarrow L^{1^*,1}(\mathbb{R}^N)$ not cocompact is given in the last section.

Keywords: Concentration analysis; BV spaces; Lorentz spaces; Profile decomposition

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1 Introduction and Preliminaries

It is well known that the classical Sobolev imbedding may be improved within the framework of Lorentz spaces $L^{p,q}$, see [3, 10, 12, 16]. Solimini [12] shows that the space $W^{1,p}(\mathbb{R}^N)$, $1 < p < N$, imbeds into $L^{p',q}(\mathbb{R}^N)$, $p \leq q \leq +\infty$ and give the profile decomposition for Sobolev space $W^{1,p}(\mathbb{R}^N)$. Adimurthi and Tintarev [2] extends the profile decomposition proved by Solimini [12] for Sobolev spaces $W^{1,p}(\mathbb{R}^N)$ with $1 < p < N$ to the non-reflexive case $p = 1$. They replace $W^{1,1}(\mathbb{R}^N)$ by $BV(\mathbb{R}^N)$ and show the imbedding $BV(\mathbb{R}^N) \hookrightarrow L^{1^*,1}(\mathbb{R}^N)$ is cocompact. In this paper we extend the result of [2] to the framework of Lorentz spaces and give the cocompact imbedding and the profile decomposition for Lorentz spaces. In broad sense defect of compactness is known as concentration compactness by Lions [9]. It is expressed as a sum of elementary concentrations called profile decomposition, which is introduced by Struwe [15], also see Lieb [8], Gérard [5] and Jaffard [7] study the profile decomposition for the fractional Sobolev spaces and give a weaker form of remainder. Schihdler and Tintarev [11] prove the profile decomposition for general Hilbert spaces. Profile decomposition for uniformly convex Banach space is obtained by Solimini and Tintarev [13, 14]. A profile decomposition in the general non-reflexive Banach space remain an open problem, see [2, 18]. The main results of this paper are Theorem 2.1 and 2.4. The idea for the proof of the theorem is followed by [2, 18], but uses different argument for the estimate of Lorentz seminorms over lattices.

We need the following definitions and lemmas of Lorentz spaces and $BV$ spaces, see [1, 4, 6, 16, 17, 19]. Lorentz spaces are known as real interpolation spaces between Lebesgue spaces

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and can be defined via the notion of Schwarz symmetrization. Let \(u\) be a measurable function on \(\mathbb{R}^N, N \geq 2\), whose level sets have finite measure for every level. Then the function

\[
\mu(\lambda) = |\{x \in \mathbb{R}^N \mid |u(x)| > \lambda\}|, \quad \lambda \geq 0, \tag{1.1}
\]

is the distribution function of \(u\) and

\[
u^*(r) = \inf\{\lambda > 0 \mid \mu(\lambda) \leq |B_r|\} \tag{1.2}
\]
is radially symmetric and non-increasing rearrangement of \(u\), where \(B_r\) is ball centered at the origin with radius \(r\) and \(|A|\) is the \(N\) dimensional Lebesgue measure of \(A \subset \mathbb{R}^N\). We call the function \(u^*(|x|), \ x \in \mathbb{R}^N\) as Schwarz symmetrization of \(u\). Then, we define Lorentz spaces \(L^{p,q}(\mathbb{R}^N)\) as

\[
L^{p,q}(\mathbb{R}^N) = \{u : \text{measurable in } \mathbb{R}^N \mid \|u\|_{L^{p,q}} = B_1^{\frac{q-p}{pq}} \left( \int_{\mathbb{R}^N} \left(\frac{|x|^N}{|x|^N} u^*(|x|)\right)^q \frac{dx}{|x|^N} \right)^{\frac{1}{q}} < \infty \}. \tag{1.3}
\]
The definition of Lorentz spaces is also given by decreasing rearrangement \(u^*\) as the distribution function \(\mu\) of \(u\), namely

\[
u^*(t) = \sup\{\lambda > 0 \mid \mu(\lambda) > t\}. \tag{1.4}
\]
Then, we define the Lorentz spaces

\[
L^{p,q}(\mathbb{R}^N) = \{u : \text{measurable in } \mathbb{R}^N \mid \|u\|_{L^{p,q}} = \left( \int_0^\infty (u^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty \}, \tag{1.5}
\]
where (1.5) is a quasinorm and equivalent to (1.3). Lorentz spaces satisfy the following imbedding lemma.

**Lemma 1.1** ([4]). Suppose that \((\Omega, \Sigma, m)\) is non-atomic and that \(m(\Omega) < \infty\). Then if \(0 < p_1 < p_2 \leq \infty\), \(L^{p_2,q_2}(\Omega) \subset L^{p_1,q_1}(\Omega)\) for any \(q_1\) and \(q_2\), with continuous inclusion, where \(m(A)\) is the \(N\) dimensional Lebesgue measure of \(A \subset \mathbb{R}^N\).

**Definition 1.2** ([19]). Let \(\Omega\) be an open subset of \(\mathbb{R}^N\). A function \(u \in L^1(\Omega)\) is said to be of bounded variations, if the total variation of \(u\) on \(\Omega\) is

\[
\|Du\|_{\Omega} = \sup\{\int_\Omega udvdx \mid v \in C_0^\infty(\Omega; \mathbb{R}^N), \ ||\varphi||_{\infty} \leq 1\} < +\infty, \tag{1.6}
\]
where \(v(x) = (v_1(x), v_2(x), \ldots v_N(x))\) and \(||v||_{\infty} = \sup_{x \in \Omega}(\sum_{k=1}^N (v_k(x))^2)^{\frac{1}{2}}\). On \(BV(\Omega) = \{u \in L^1(\Omega) \mid \|Du\|_{\Omega} < \infty\}\), we define the norm

\[
\|u\|_{BV(\Omega)} = \|Du\|_{\Omega} + \|u\|_{L^1(\Omega)}. \tag{1.7}
\]

**Definition 1.3.** The sequence \(\{u_n\}\) converges weakly to \(u\) in \(BV(\Omega)\), written \(u_n \rightharpoonup u\) if

\[
u_n \rightharpoonup u \text{ in } L^1_{loc}(\Omega), \tag{1.9}
\]
and

\[
\partial_k u_n \rightharpoonup \partial_k u \text{ in } (C_0(\Omega))^*, \tag{1.10}
\]
for $1 \leq k \leq N$, as $n \to \infty$, where $(C_0(\Omega))^*$ denotes the space of finite measures on $\Omega$.

**Lemma 1.4.** Assume $\Omega \subset \mathbb{R}^N$ is open and bounded with $\partial \Omega$ Lipschitz. The imbedding

$$BV(\Omega) \hookrightarrow L^{1',1}(\Omega) \subset L^{1',q}(\Omega), \quad 1 < q \leq +\infty,$$

is continuous. More precisely, there exists a constant $C$ with depends only on $\Omega, q$ and $N$, such that for $\forall \ u \in BV(\Omega)$

$$\|u\|_{L^{1',q}(\Omega)} \leq \|u\|_{L^{1',1}(\Omega)} \leq C\|u\|_{BV(\Omega)}.$$  \hfill (1.11)

**Definition 1.5.** The space of functions of bounded variation $BV(\mathbb{R}^N)$ is the space of all measurable functions $u$ vanishing at infinity (i.e. $\forall M > 0, |\{x \in \mathbb{R}^N \mid |u(x)| > M\}| < \infty$) such that

$$\|u\|_{BV} = \|Du\| = \sup \left\{ \int_{\mathbb{R}^N} udivvdx \mid v \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^N), \|v\|_{L^\infty} \leq 1 \right\} < \infty,$$  \hfill (1.13)

where $v(x) = (v_1(x), v_2(x), \ldots v_N(x))$ and $\|v\|_{\infty} = \sup_{x \in \Omega}(\sum_{k=1}^N(v_k(x))^2)^{\frac{1}{2}}$. The $BV(\mathbb{R}^N)$ norm $\|u\|_{BV}$ can be interpreted as the total variation $\|Du\|$ of the measure associated with to derivative $Du$ in the sense of distributions on $\mathbb{R}^N$.

**Definition 1.6.** The sequence $\{u_n\}$ converges weakly to $u$ in $BV(\mathbb{R}^N)$, written $u_n \rightharpoonup u$ if

$$u_n \to u \quad \text{in} \quad L^{1}_{loc}(\mathbb{R}^N),$$  \hfill (1.14)

and

$$\partial_k u_n \to \partial_k u, \quad 1 \leq k \leq N$$  \hfill (1.15)

as finite measures on $\mathbb{R}^N$ and $n \to \infty$.

**Lemma 1.7** ([17]). $BV(\mathbb{R}^N)$ is continuously embedding in $L^{1',1}(\mathbb{R}^N)$ for $N \geq 2$, and in $L^\infty(\mathbb{R})$ for $N = 1$.

We need the following properties of $BV(\mathbb{R}^N)$, also see [2].

1. Invariance. The group of operators on $BV(\mathbb{R}^N)$,

$$\mathcal{D} = \{g[j, y] : u \to 2^{(N-1)j}u(2^j(-y)) \}_{j \in \mathbb{Z}},$$  \hfill (1.16)

consists for linear isometries of $BV(\mathbb{R}^N)$, which are also linear isometries on $L^{1',q}(\mathbb{R}^N), 1 \leq q \leq \infty$.

2. Chain rule. Let $\varphi \in C^1(\mathbb{R})$. Then for $\forall u \in BV(\mathbb{R}^N)$

$$\|D\varphi(u)\| \leq \|\varphi\|_{L^\infty} \|Du\|.$$  \hfill (1.17)

**Definition 1.8** ([2], [18]). Let $X$ be a Banach space and let $\mathcal{D}$ be a group of linear isometries of $X$. One says that a sequence $\{u_k\} \subset X$ is $\mathcal{D}$-vanishing (to be written $u_k \xrightarrow{\mathcal{D}} 0$) if for any sequence $\{g_k\} \subset \mathcal{D}$ one has $g_ku_k \to 0$ in $X$. A continuous imbedding of $X$ into a topological space $Y$ is called cocompact with respect to $\mathcal{D}$ if $u_k \xrightarrow{\mathcal{D}} 0$ implies $u_k \to 0$ in $Y$. 
2 Main results

Theorem 2.1. The imbedding $BV(\mathbb{R}^N) \hookrightarrow L^{1,q}(\mathbb{R}^N)$, $1 < q \leq 1^* = \frac{N}{N-1}$, $N \geq 2$, is cocompact with respect to the group $D$, i.e. if, for any sequence $\{j_k, y_k\} \subset \mathbb{Z} \times \mathbb{R}^N$, $g[j_k, y_k]u_k \to 0$ in $BV(\mathbb{R}^N)$ then $u_k \to 0$ in $L^{1,q}(\mathbb{R}^N)$.

Remark 2.2. The embedding $BV(\mathbb{R}^N) \hookrightarrow L^{1,q}(\mathbb{R}^N)$, $1^* < q \leq \infty$ is cocompact with respect to $D$, by the cocompact imbedding $BV(\mathbb{R}^N) \hookrightarrow L^1(\mathbb{R}^N)$ and $L^1(\mathbb{R}^N) \hookrightarrow L^{1,q}(\mathbb{R}^N)$, $q > 1^*$. Theorem 2.1 may hold as follows by the interpolate method, since the sequence is bounded in $L^{1^*,1}(\mathbb{R}^N)$, and we will use the estimate of Lorentz norms over lattices to prove Theorem 2.1. However, the cocompact embedding $BV(\mathbb{R}^N) \hookrightarrow L^{1^*,1}(\mathbb{R}^N)$ does not hold for $q = 1$ and we will give a counterexample in the Section 3 for this case.

Proof. Let $\{u_k\} \subset BV(\mathbb{R}^N)$ be such that for any $\{j_k, y_k\} \subset \mathbb{Z} \times \mathbb{R}^N$,

$$g_ku_k = g[j_k, y_k]u_k \to 0 \text{ in } BV(\mathbb{R}^N).$$

(2.1)

Step 1. Assume first that $\sup_{k \in \mathbb{N}} \|u_k\|_{L^\infty} < \infty$ and $\sup_{k \in \mathbb{N}} \|u_k\|_{L^1} < \infty$. Then using the $L^\infty$-boundedness of $\{u_k\}$ and $BV(\Omega_0) \hookrightarrow L^{1,q}(\Omega_0)$, $\Omega_0 = (0,1)^N$, $1 < q < 1^* = \frac{N}{N-1}$, we have

$$\|u_k\|_{L^{1,q}(\Omega_0)} = |B(0,1)|^{\frac{q-1}{q}} \left( \int_{\Omega_0} (u_k^q(|x|))q|x|^{N(q-1)-q}dx \right)^{\frac{1}{q}} \leq C\|u_k\|_{BV(\Omega_0)} = C(\|Du_k\|_{\Omega_0} + \|u_k\|_{L^1(\Omega_0)}),$$

(2.2)

which implies

$$|B(0,1)|^{\frac{q-1}{q}} \left( \int_{\Omega_0} (u_k^q(|x|))q|x|^{N(q-1)-q}dx \right)^{\frac{1}{q}} \leq C(\|Du_k\|_{\Omega_0} + \|u_k\|_{L^1(\Omega_0)})(\int_{\Omega_0} (u_k^q(|x|))q|x|^{N(q-1)-q}dx)^{\frac{1}{q}}.$$

(2.3)

Considering Lemma 1.1 and setting $p_1 = 1^* = \frac{N}{N-1} < p_2 = 3$, $q_1 = q$, $q_2 = 3$ and $m(\Omega_0) = m((0,1)^N) < \infty$, we have

$$L^3(\Omega_0) = L^3^3(\Omega_0) \subset L^{1,q}(\Omega_0),$$

(2.4)

and

$$\|u_k\|_{L^{1,q}(\Omega_0)} \leq \|u_k\|_{L^3(\Omega_0)}.$$  

(2.5)

By (2.3) and (2.5), we get

$$|B(0,1)|^{\frac{q-1}{q}} \int_{\Omega_0} (u_k^q(|x|))q|x|^{N(q-1)-q}dx \leq C(\|Du_k\|_{\Omega_0} + \|u_k\|_{L^1(\Omega_0)})(\int_{\Omega_0} (u_k^q(|x|))q|x|^{N(q-1)-q}dx)^{\frac{2}{q}} \leq C(\|Du_k\|_{\Omega_0} + \|u_k\|_{L^1(\Omega_0)})(\int_{\Omega_0} |u_k|^3dx)^{\frac{2}{q}} \leq C'(\|Du_k\|_{\Omega_0} + \|u_k\|_{L^1(\Omega_0)})(\int_{\Omega_0} |u_k|^dx)^{\frac{2}{q}},$$

(2.6)
where the last inequality is given by $L^\infty$-boundedness of $\{u_k\}$. Repeating this inequality for the domain of integration $(0, 1)^N + y$, $y \in \mathbb{Z}^N$, and adding the resulting inequalities over all $y \in \mathbb{Z}^N$, we have

$$
|B(0, 1)|^{\frac{q-1}{r}} \int_{\mathbb{R}^N} (u_k^N(|x|))^{q}|x|^{N(q-1)-q} dx
$$

$$
= |B(0, 1)|^{\frac{q-1}{r}} \int_{\mathbb{R}^N} (u_k^N(|x|))^{q}|x|^{N(q-1)-q} dx
$$

$$
\leq C' \left( ||Du_k||_{\mathbb{R}^N} + ||u_k||_{L^1(\mathbb{R}^N)} \right) \left( \sup_{y \in \mathbb{Z}^N} \int_{\Omega_0} |u_k(x - y)| dx \right)^{\frac{q-1}{q}}. \tag{2.7}
$$

Here we use the fact that the sum $\sum_{y \in \mathbb{Z}^N} ||Du_k||_{(0, 1)^N + y}$ can be split into $3^N$ sums of variations over unions of cubes with disjoint closures, each of them, as follows from Definition 1.5 bound by $||Du_k||_{\mathbb{R}^N}$, which implies $\sum_{y \in \mathbb{Z}^N} ||Du_k||_{(0, 1)^N + y} \leq 3^N ||Du_k||_{\mathbb{R}^N}$. By the assumption $g[j_k, y_k]u_k \to 0$ in $BV(\mathbb{R}^N)$, we have $u_k(\cdot - y_k) \to 0$ in $L^1((0, 1)^N)$ for $\forall \{y_k\} \subset \mathbb{Z}^N$. This implies $u_k \to 0$ in $L^1(\mathbb{R}^N)$.

**Step 2.** Now consider a general sequence $\{u_k\} \subset BV(\mathbb{R}^N)$ satisfying $g[j_k, y_k]u_k \to 0$ in $BV(\mathbb{R}^N)$ for any $\{j_k, y_k\} \subset \mathbb{Z} \times \mathbb{R}^N$. Let $\chi \in C_0^\infty((\frac{1}{2N}, 4N^{-1}))$ be such that $\chi(t) = t$ if $t \in [1, 2^{-N-1}]$. Let $\chi_j(t) = 2^{(N-1)j}\chi(2^{-(N-1)j}|t|), \ j \in \mathbb{Z}, \ t \in \mathbb{R}$ and obviously $\|\chi_j\|_{L^\infty} = \|\chi_j\|_{L^\infty}$. By Lemma 1.7 the embedding $BV(\mathbb{R}^N) \hookrightarrow L^{1,q}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ and $\chi_j(u_k) \in BV(\mathbb{R}^N)$, we have

$$
\|\chi_j(u_k)\|_{L^{1,q}(\mathbb{R}^N)} \leq C \|\chi_j(u_k)\|_{BV} = C \|D\chi_j(u_k)\| = C \|D\chi_j(u_k)\|_{B_{k_j}}. \tag{2.8}
$$

From the definition of $L^{1,q}(\mathbb{R}^N)$, we get

$$
\|\chi_j(u_k)\|_{L^{1,q}(\mathbb{R}^N)} = |B(0, 1)|^{\frac{q-1}{r}} \left( \int_{\mathbb{R}^N} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)^{\frac{1}{q}}
$$

$$
= |B(0, 1)|^{\frac{q-1}{r}} \left( \int_{\{x \in \mathbb{R}^N | 0 \leq |u_k(x)| < \infty \}} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)^{\frac{1}{q}}
$$

$$
\geq |B(0, 1)|^{\frac{q-1}{r}} \left( \int_{\{x \in \mathbb{R}^N | 2^{(N-1)j} \leq |u_k(x)| < 2^{(N-1)(j+1)} \}} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)^{\frac{1}{q}}
$$

$$
= |B(0, 1)|^{\frac{q-1}{r}} \left( \int_{A_{k_j}} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)^{\frac{1}{q}} \tag{2.9}
$$

where $A_{k_j} = \{x \in \mathbb{R}^N | 2^{(N-1)j} \leq |u_k(x)| < 2^{(N-1)(j+1)} \}, \ j \in \mathbb{Z}^N$. Considering the inequalities (2.8) and (2.9), we have

$$
|B(0, 1)|^{\frac{q-1}{r}} \left( \int_{A_{k_j}} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)^{\frac{1}{q}} \leq C \|D\chi_j(u_k)\|_{B_{k_j}}. \tag{2.10}
$$

and

$$
|B(0, 1)|^{\frac{q-1}{r}} \left( \int_{A_{k_j}} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)
$$

$$
\leq C \|D\chi_j(u_k)\|_{B_{k_j}} \left( \int_{A_{k_j}} ((\chi_j(u_k))^\| |x|^{N(q-1)-q} dx \right)^{\frac{q-1}{q}}, \tag{2.11}
$$

where $B_{k_j} = \{x \in \mathbb{R}^N | 2^{(n-1)(j-1)} \leq |u_k(x)| < 2^{(N-1)(j+2)} \} \supset A_{k_j}$ and $\chi_{k_j}(u_k) = u_k, \ x \in A_{k_j}$ by definition of $\chi_j(t)$. Let us sum up the above inequalities (2.11) over $j \in \mathbb{Z}$. Note that by
Proof. By Theorem 1.3 of [2] and Theorem 2.1, the claim easily follows.

(1.17) \( \|D\chi_j(u_k)\|_{B_{k,j}} \leq \|\chi'\|_{L^\infty} \|Du_k\|_{B_{k,j}} \). Furthermore, one can break all the integers \( j \) into four disjoint sets \( J_1, J_2, J_3, J_4 \),

\[
\begin{align*}
J_1 &= \{..., -4, 0, 4, 8...\} \\
J_2 &= \{..., -3, 1, 5, 9...\} \\
J_3 &= \{..., -2, 6, 10...\} \\
J_4 &= \{..., -1, 3, 7, 11...\}
\end{align*}
\]

such that for any \( m \in \{1, 2, 3, 4\} \), all functions \( \chi_j(u_k), j \in J_m \), have pairwise disjoint supports. Consequently, \( \sum \|Du_k\|_{B_{k,j}} \leq 4\|Du_k\| \), we have therefore

\[
\begin{align*}
|B(0,1)|^{\frac{q-1}{q'}} \left( \int_{\mathbb{R}^N} (u^q(|x|))q|x|^{N(q-1)-q} \, dx \right) \\
&\leq C \|Du_k\| \sup_j \left( \int_{A_{kj}} ((\chi_j(u_k))^q(|x|))q|x|^{N(q-1)-q} \, dx \right)^{\frac{1}{q'}} \\
&\leq C \|Du_k\| \sup_j \|\chi_j(u_k)\|_{{L^1}^q(A_{kj})}.
\end{align*}
\]

It suffices now to show that for any sequence \( \{j_k\} \subset \mathbb{Z} \), \( \chi_{jk}(u_k) \to 0 \) in \( {L^1}^q(A_{jk}) \). Taking into account the invariance of the \( \mathcal{L}^q \)-norm under the operators \( g[j_k, y_k] \), it suffices to show that \( \chi(2^{-j_k(N-1)}|u_k(2^{j_k})|) \to 0 \) in \( {L^1}^q \), which is immediate by the assumption \( g[j_k, y_k]u_k \to 0 \), \( \chi_{jk}(u_k) = u_k, x \in A_{jk} \) and the argument of the Step 1, once we take into account for sequences \( \chi(2^{-j_k(N-1)}|u_k(2^{j_k})|) \) uniformly bounded in \( \infty \). By \( \chi \in C_0(\mathbb{R}_+\times(\frac{1}{2^{N-1}}, \infty)) \) and \( \chi(t) = t \) as \( t \in (1, 2^{N-1}) \), we get \( \chi(t) \leq t + 1 \) for all \( t \in (\frac{1}{2^{N-1}}, 4^{N-1}) \). Considering this fact and

\[
|u_k(x)| \leq 2^{(1+j_k)(N-1)}, \ x \in A_{jk},
\]

we have

\[
\begin{align*}
\chi(2^{-j_k(N-1)}|u_k(2^{j_k}x)|) &\leq 2^{-j_k(N-1)}|u_k(2^{j_k}x)| + 1 \\
&\leq 2^{-j_k(N-1)}\chi(2^{j_k+1})(N-1) + 1 \\
&= 2^{(N-1)} + 1,
\end{align*}
\]

which implies \( \chi(2^{-j_k(N-1)}|u_k(2^{j_k}x)|) \) bounded in \( \infty \). The proof is completed.

Theorem 2.3. The imbedding \( \tilde{W}^{1,1}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^1 \), \( 1 < q \leq \infty, N \geq 2 \), is cocompact with respect to the group \( \mathcal{D} \).

Theorem 2.4 (profile decomposition). Let \( \{u_k\} \subset \tilde{BV}(\mathbb{R}^N) \) be a bounded sequence. For each \( n \in \mathbb{N} \) there exist \( w^{(n)} \in \tilde{BV}(\mathbb{R}^N) \) and sequences \( \{j_k^n, y_k^n\} \subset \mathbb{Z} \times \mathbb{R}^N \) with \( j_k^{(1)} = 0 \) and \( y_k^{(1)} = 0 \) satisfying

\[
|j_k^{(n)} - j_k^{(m)}| + |y_k^{(n)} - y_k^{(m)}| \to \infty
\]

whenever \( m \neq n \), such that for a renumbered subsequence, \( g[-j_k^{(n)}, -y_k^{(n)}]u_k \to w^{(n)} \) as \( k \to \infty \),

\[
r_k = u_k - \sum_n g[j_k^{(n)}, y_k^{(n)}]w^{(n)} \to 0 \text{ in } \mathcal{L}^1 \, \mathcal{Q}(\mathbb{R}^N), \ q > 1
\]

where the series \( \sum_n g[j_k^{(n)}, y_k^{(n)}]w^{(n)} \) converges in \( \tilde{BV}(\mathbb{R}^N) \) uniformly in \( k \), and

\[
\sum_n \|Dw^{(n)}\| + o(1) \leq \|Du_k\| \leq \sum_n \|Dw^{(n)}\| + \|Dr_k\| + o(1).
\]

Proof. By Theorem 1.3 of [2] and Theorem 2.1, the claim easily follows.
3 Counterexample

In this section, we will show an example which indicates the imbedding $BV(\mathbb{R}^N) \hookrightarrow L^{1*,1}(\mathbb{R}^N)$ not cocompact. In fact we shall define a bounded sequence of $BV(\mathbb{R}^N)$, which does not converge to zero in $L^{1*,1}(\mathbb{R}^N)$. To this aim, fix a function

$$\phi(x) = \begin{cases} 1, & 1 < |x| \leq 2, \\ 0, & |x| \leq 1 \text{ or } |x| > 2, \end{cases}$$

$x \in \mathbb{R}^N$, and obviously $\phi \in BV(\mathbb{R}^N)$. We take the function $v_i$ given by the rescaling of $\phi$,

$$v_i(x) = 2^{i(N-1)}\phi(2^i x), \quad 2^{-i} < |x| \leq 2^{-(i-1)}. \quad (3.2)$$

For every positive integer number $n$, we choose $u_n$ as the following function:

$$u_n(x) = \frac{1}{n} \sum_{i=1}^n v_i(x). \quad (3.3)$$

Since the functions $v_i$ satisfies $\|v_i\|_{BV} = \|\phi\|_{BV}$ and (3.3), we have

$$\|u_n\|_{BV} = \left\| \frac{1}{n} \sum_{i=1}^n v_i \right\|_{BV} = \frac{1}{n} \sum_{i=1}^n \|v_i\|_{BV} = \|\phi\|_{BV}. \quad (3.4)$$

and

$$\|u_n\|_{L^{1*,1}}^* = \frac{1}{n^{1-1}}\|\phi\|_{L^{1*,1}}^* \to 0, \quad n \to \infty. \quad (3.5)$$

Then $u_n \to 0$ strongly in $L^{1*}(\mathbb{R}^N)$. It must be clearly $u_n \to 0$ also in $L^{1*,q}(\mathbb{R}^N), 1 < q < 1^*$, as follows by interpolation since the sequence is bounded in $L^{1*,1}(\mathbb{R}^N)$. We claim that $\{u_n\}$ does not converge to 0 in $L^{1*,1}(\mathbb{R}^N)$ and therefore Theorem 2.1 does not hold with $q = 1$. In fact, if $u_n \to 0$ strongly in $L^{1*,1}(\mathbb{R}^N)$, then we have $u_n \to 0$ in $L^{1*,1}(\mathbb{R}^N) = L^{N-1,1}(\mathbb{R}^N)$. This implies for any linear functional $f \in (L^{1*,1}(\mathbb{R}^N))^* = (L^{N-1,1}(\mathbb{R}^N))^* = L^{N,\infty}(\mathbb{R}^N)$, we have

$$\lim_{n \to \infty} f(u_n) = 0. \quad (3.6)$$

However, we can find a functional $f_0 \in L^{N,\infty}(\mathbb{R}^N)$,

$$f_0(u) = \int_{\mathbb{R}^N} u(x) \frac{1}{|x|} dx. \quad (3.7)$$
Setting $u = u_n$ in (3.7), we get

$$f_0(u_n) = \int_{\mathbb{R}^N} u_n(x) \frac{1}{|x|} dx$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^N} v_i(x) \frac{1}{|x|} dx$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^N} 2^{i(N-1)} \phi(2^i x) \frac{1}{|x|} dx$$

$$= \frac{1}{n} \sum_{i=1}^{n} 2^{i(N-1)+i-N} \int_{\mathbb{R}^N} \phi(y) \frac{1}{|y|} dy$$

$$= \int_{\mathbb{R}^N} \phi(y) \frac{1}{|y|} dy$$

$$= \int_{\{y \in \mathbb{R}^N | 1 < |y| \leq 2\}} \frac{1}{|y|} dy$$

$$\geq \frac{1}{2} |\{y \in \mathbb{R}^N | 1 < |y| \leq 2\}| \neq 0,$$

(3.8)

where $|A|$ is the $N$ dimensional Lebesgue measure of $A \subset \mathbb{R}^N$ in the last inequality. Taking $n \to \infty$ in the above inequality (3.8), we have

$$\lim_{n \to \infty} f_0(u_n) \geq \frac{1}{2} |\{y \in \mathbb{R}^N | 1 < |y| \leq 2\}| \neq 0.$$  \hspace{1cm} (3.9)

This conclusion is a contradiction.

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