RESONANT PROBLEMS FOR FRACTIONAL LAPLACIAN

YUTONG CHEN AND JIABAO SU

School of Mathematical Sciences, Capital Normal University
Beijing 100048, People’s Republic of China

(Communicated by Zhi-qiang Wang)

ABSTRACT. In this paper we consider the following fractional Laplacian equation
\[
\begin{cases}
(-\Delta)^s u = g(x, u) & x \in \Omega, \\
u = 0 & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \(s \in (0, 1)\) is fixed, \(\Omega\) is an open bounded set of \(\mathbb{R}^N\), \(N > 2s\), with smooth boundary, \((-\Delta)^s\) is the fractional Laplace operator. By Morse theory we obtain the existence of nontrivial weak solutions when the problem is resonant at both infinity and zero.

1. Introduction. The nonlocal equations have been experiencing impressive applications in different subjects, such as the thin obstacle problem, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasigeostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, elliptic problems with measure data, optimization, finance, etc. See [24] and the references therein. In the recent years many mathematicians have made efforts to apply the minimax methods ([25]) such as the mountain pass theorem [1], the saddle point theorem [25] or other linking type of critical point theorems in the study of the non-local fractional Laplacian equations with different nonlinearities having subcritical or critical growth, see [3, 4, 7, 8, 14, 15, 16, 23, 24, 26, 27, 29, 30, 32, 33] and references therein.

In this paper we apply the Morse theory to study the non-local equations with the nonlinearity being asymptotically linear at both infinity and zero. Precisely, we deal with the problem
\[
\begin{cases}
(-\Delta)^s u = g(x, u) & x \in \Omega, \\
u = 0 & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \(s \in (0, 1)\) is fixed, \(\Omega\) is an open bounded subset of \(\mathbb{R}^N\) with Lipschitz boundary, \(N > 2s\), and \((-\Delta)^s\) is the fractional Laplace operator, which (up to normalization factors) is defined as
\[
-(\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.
\]
The nonlinear function $g \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies $g(x,0) \equiv 0$ and the following growth condition

\[(g)\] there is $C > 0$ and $p \in (2, \frac{2N}{N-2})$ such that

$$|g_t'(x,t)| \leq C(1 + |t|^{p-2})$$

for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$. (1.3)

We are concerned with the existence of weak solutions of (1.1). A weak solution for (1.1) is a function $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dxdy = \int_{\Omega} g(x,u(x))\varphi(x)dx \forall \varphi \in H^s_0(\Omega),$$

$$u \in H^s_0(\Omega).$$

(1.4)

We refer to Proposition 9 and Appendix A in [30] for the existence and basic properties of the eigenvalue of (1.7) that will be collected in the next section.

We will consider the problem (1.1) under the situation that the following asymptotically limits exist:

$$\lim_{|t| \to \infty} \frac{g(x,t)}{t} = \lambda_\ell \text{ uniformly in } x \in \bar{\Omega},$$

$$\lim_{t \to 0} \frac{g(x,t)}{t} = \lambda_m \text{ uniformly in } x \in \bar{\Omega},$$

where $\lambda_\ell$ and $\lambda_m$ are two eigenvalues of the eigenvalue problem

$$\left\{ \begin{array}{ll}
(-\Delta)^s u = \lambda u & x \in \Omega, \\
u = 0 & x \in \mathbb{R}^N \setminus \Omega.
\end{array} \right.$$  (1.7)

We refer to Proposition 9 and Appendix A in [30] for the existence and basic properties of the eigenvalue of (1.7) that will be collected in the next section.

We note here that (1.5) characterizes the problem (1.1) as asymptotically linear resonance at infinity while (1.6) characterizes the problem (1.1) is rasonant at zero, namely, the trivial solution of (1.1) is degenerate.

As we will deal with (1.1) at resonance, we need to make on the eigenvalues of (1.7) the following assumption:

$$\left\{ \begin{array}{ll}
\lambda \text{ is an eigenvalue of } (-\Delta)^s \text{ such that all the eigenfunctions corresponding to } \lambda \text{ have nodal set with zero Lebesgue measure.} \\
\end{array} \right.$$  (1.8)

It is well known that the assumption (1.8) is always valid for the classical Laplacian $-\Delta$ in a bounded domain $\Omega([13, 18])$. For the fractional Laplacian $(-\Delta)^s$, it makes sense at least when $\lambda$ is the first eigenvalue of $(-\Delta)^s$(see [30, 31]).

As ones have pointed out in [16] it is a quite interesting problem that whether or not (1.8) is valid for all the eigenvalues of $(-\Delta)^s$.

The problem (1.1) admits a trivial solution $u = 0$ due to $g(x,0) \equiv 0$. The aim of this paper is to find nontrivial solutions of (1.1) under (1.5) and (1.6). The existence of nontrivial solutions for (1.1) depends on the interplay of the behaviors of $g$ near zero and near infinity with the eigenvalues of $(-\Delta)^s$, therefore further conditions are needed.
Set \( f(x,t) = g(x,t) - \lambda_t t \) and \( f_0(x,t) := g(x,t) - \lambda_m t \). We make the following assumptions on \( f \) and \( f_0 \).

\( (f^+) \) There are \( R > 1, r \in (0,1) \) and \( c_1, c_2 > 0 \) such that
\[
\pm f(x,t)t \geq 0, \quad c_1|t|^r \leq |f(x,t)| \leq c_2|t|^r
\]
for all \( x \in \bar{\Omega} \) and \( t \in \mathbb{R} \) with \( |t| \geq R \).

\( (f_0^+) \) There are \( \delta \in (0,1), \zeta \in \left(1, \frac{N+2s}{N-2s}\right) \) and \( c_3, c_4 > 0 \) such that
\[
\pm f_0(x,t)t \geq 0, \quad c_3|t|^\zeta \leq |f_0(x,t)| \leq c_4|t|^\zeta
\]
for all \( x \in \Omega \) and \( t \in \mathbb{R} \) with \( |t| \leq \delta \).

Notice that \((f^+)\) implies \((1.5)\) while \((f_0^+)\) implies \((1.6)\). There are a large class of functions satisfying the above assumptions. We give an example as follows.

\[
g(t) = \begin{cases} 
\lambda m t \pm |t|^{\zeta}, & |t| \leq \delta < 1, \quad \zeta \in \left(1, \frac{N+2s}{N-2s}\right); \\
\text{smooth}, & \delta \leq |t| \leq R; \\
\lambda et \pm |t|^r, & |t| \geq R > 1, \quad r \in (0,1).
\end{cases}
\]

The main results of the present paper are stated as follows.

**Theorem 1.1.** Assume \((g), (1.6)\) and \((1.8)\). Then the problem \((1.1)\) admits at least one nontrivial weak solution in each of the following cases:

(i) \((f^+)\) and there is an eigenvalue \( \lambda \) of \((1.7)\) such that \( \lambda_t < \lambda < \lambda_m \).

(ii) \((f^-)\) and there is an eigenvalue \( \hat{\lambda} \) of \((1.7)\) such that \( \lambda_m < \hat{\lambda} < \lambda_t \).

**Theorem 1.2.** Assume \((g)\) and \((1.8)\). Then the problem \((1.1)\) admits at least one nontrivial weak solution in each of the following cases:

(i) \((f^-), (f_0^+)\), \( \lambda_{t-1} < \lambda_t \) and \( \lambda_{t-1} \neq \lambda_m \);

(ii) \((f^+), (f_0^-)\), \( \lambda_{m-1} < \lambda_m \) and \( \lambda_t \neq \lambda_{m-1} \);

(iii) \((f^+), (f_0^+)\) and \( \lambda_t \neq \lambda_m \);

(iv) \((f^-), (f_0^-)\) and \( \lambda_t \neq \lambda_m \).

In particular, the conclusion is valid in (i) and (ii) for the case that \( \lambda_t = \lambda_m \).

**Theorem 1.3.** Assume \((g), (f^-)\) and \((f_0^+)\). If \( \lambda_t = \lambda_m = \lambda_1 \), then the problem \((1.1)\) admits at least two nontrivial weak solutions.

Now we give some remarks and comments. Resonance problems have received much attention in the literature since the appearance of a well-known work [19] by Landesman and Lazer. A pioneering condition is the famous Landesman-Lazer resonance condition introduced in [19] in studying semilinear elliptic problem via topological degree. A variational formulation for this condition was given by Rabinowitz (see [25]) in verifying the saddle point theorem.

The nonlocal fractional Laplacian problem with resonance may have its own meanings and has been considered in [14, 15, 16] by the saddle point theorem([25]). In [14, 16], some existence results related to \((1.1)\) had been obtained for the Landesman-Lazer resonance setting([19]). One version of the Landesman-Lazer resonance condition ([19]) can be formulated as

\[
\left\{ \begin{array}{l}
\text{\( f(x,t) \) is bounded for all \( (x,t) \in \bar{\Omega} \times \mathbb{R}; \)} \\
\text{\( F(x,t) = \int_0^t f(x,s)ds \to -\infty \) as \( |t| \to \infty. \)}
\end{array} \right.
\]

The main feature of \((1.9)\) is the boundedness of nonlinear term \( f \).

In the present paper we focus on applying Morse theory to \((1.1)\) with resonance at both infinity and zero. The nonlinear term \( f \) is an unbounded function when it
satisfies \((f^\pm)\). The type of conditions \((f^\pm)\) and \((f_0^\pm)\) were first introduced in the works [21, 20, 34] for semilinear problems at resonance and was first used in [35] to find multiple solutions for the semilinear elliptic equation

\[
\begin{aligned}
-\Delta u &= g(x, u) & x &\in \Omega, \\
 0 &= & x &\in \partial\Omega.
\end{aligned}
\] (1.10)

The problem (1.10) may be seen as the one reduced to \(s = 1\) from (1.1). In this sense, Theorems 1.1 and 1.2 may be regarded as the fractional versions of the existence results in [35]. Theorem 1.3 is new.

We prove the main results employing the Morse theory [12, 22] and critical groups computations. Precisely, we will work under the abstract framework built in [6] and modified in [34, 36]. The main difficulties in working such a framework are obviously related to the nonlocal nature of the problem.

The paper is organized as follows. In Section 2 we collect some preliminary in building the variational formulas related to (1.1) and then recall some abstract tools about Morse theory and critical groups. In Section 3 we give some technical lemmas for verifying conditions required by the abstract tools. In Section 4 the proofs of main results are given including further comments, comparisons and remarks.

2. Preliminary. In this section we will give the preliminaries for the variational structure of (1.1) and preliminary results in Morse theory.

2.1. Variational formulations for (1.1). We first recall some basic results on the functional spaces \(H^s(\mathbb{R}^N)\) and \(H^s_0(\Omega)\). See [24, 26, 27, 29, 30, 31, 32] for details.

The functional space \(H^s(\mathbb{R}^N)\) denotes the fractional Sobolev space of the functions \(u \in L^2(\mathbb{R}^N)\) such that the map \((x, y) \mapsto u(x) - u(y)\) is in \(L^2(\mathbb{R}^{2N}, dxdy)\).

The space \(H^s(\mathbb{R}^N)\) is endowed with the so-called Gagliardo norm

\[
\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy\right)^{\frac{1}{2}}.
\]

Let \(H^s_0(\Omega)\) be the function space defined as

\[
H^s_0(\Omega) := \{u \in H^s(\mathbb{R}^N) : u(x) = 0, \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega\}.
\]

Then \(H^s_0(\Omega)\) is a Hilbert space ([27]) with the norm

\[
\|u\|_{H^s_0(\Omega)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy\right)^{\frac{1}{2}},
\]

and the scalar product

\[
(u, v)_{H^s_0(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} dxdy.
\]

By Lemma 8 in [27] and Lemma 9 in [32], we have following embedding results.

**Proposition 2.1.** For each \(q \in \left[1, \frac{2N}{N - 2s}\right]\), the embedding \(H^s_0(\Omega) \hookrightarrow L^q(\mathbb{R}^N)\) is continuous and there is \(C_q > 0\) such that

\[
\|u\|_{L^q(\mathbb{R}^N)} \leq C_q \|u\|_{H^s_0(\Omega)}, \quad \forall u \in H^s_0(\Omega).
\]

This embedding is compact whenever \(q \in \left[1, \frac{2N}{N - 2s}\right]\).
We have the following variational formulations for (1.1) due to the above embedding.

**Proposition 2.2.** Assume (g) and let \( G(x,t) = \int_0^t g(x,\varsigma)\,d\varsigma \). Define the functional \( \mathcal{I} : H^s_0(\Omega) \to \mathbb{R} \) as

\[
\mathcal{I}(u) = \int_{\Omega} G(x,u)\,dx, \quad u \in H^s_0(\Omega).
\]

Then \( \mathcal{I} \) is well defined on \( H^s_0(\Omega) \) and is belonging to \( C^2(H^s_0(\Omega),\mathbb{R}) \) with derivatives given by

\[
\langle \mathcal{I}'(u),v \rangle = \int_{\Omega} g(x,u)v\,dx, \quad \forall u,v \in H^s_0(\Omega),
\]

\[
\langle \mathcal{I}''(u)v,w \rangle = \int_{\Omega} g'(x,u)vwdx, \quad \forall u,v,w \in H^s_0(\Omega).
\]

**Proof.** We follow the arguments in [2]. From (g) we deduce that

\[
|g(x,t)| \leq C(1 + |t|^{p-1}) \quad (\forall (x,t) \in \Omega \times \mathbb{R}); \quad (2.1)
\]

\[
|G(x,t)| \leq C(1 + |t|^p) \quad (\forall (x,t) \in \bar{\Omega} \times \mathbb{R}). \quad (2.2)
\]

By (2.2) and Proposition 2.1 we see that \( \mathcal{I} \) is well defined on \( H^s_0(\Omega) \).

(i) We first check that \( \mathcal{I} \) is Gâteaux differentiable on \( H^s_0(\Omega) \). For each \( u \in H^s_0(\Omega) \), we have to prove that for all \( v \in H^s_0(\Omega) \), it holds that

\[
\lim_{\varepsilon \to 0} \frac{\mathcal{I}(u + \varepsilon v) - \mathcal{I}(u)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{G(x,u + \varepsilon v) - G(x,u)}{\varepsilon} \,dx = \int_{\Omega} g(x,u)v\,dx.
\]

It is obvious that, for almost every \( x \in \Omega \),

\[
\lim_{\varepsilon \to 0} \frac{G(x,u(x) + \varepsilon v(x)) - G(x,u(x))}{\varepsilon} = g(x,u(x))v(x).
\]

By the Lagrange Theorem there exists a real number \( \theta \) such that \(|\theta| \leq |g|\) and

\[
\left| \frac{G(x,u(x) + \varepsilon v(x)) - G(x,u(x))}{\varepsilon} \right| = |g(x,u(x) + \varepsilon \theta v(x))v(x)| \leq C(1 + |u(x) + \varepsilon \theta v(x)|^{p-1})|v(x)| \leq C(|v(x)| + |u(x)|^{p-1}|v(x)| + |v(x)|^p).
\]

By the continuous embedding \( H^s_0(\Omega) \hookrightarrow L^q(\Omega) \) for all \( q \in [1, \frac{2N}{N-2s}] \), we get easily that

\[
|v| + |u|^{p-2}|v| + |v|^p \in L^1(\Omega),
\]

by the Dominated Convergence Theorem we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \frac{G(x,u + \varepsilon v) - G(x,u)}{\varepsilon} \,dx = \int_{\Omega} g(x,u)v\,dx.
\]

By (2.1) and Proposition 2.1, we deduce that

\[
\int_{\Omega} |g(x,u)v|\,dx \leq C \int_{\Omega} (1 + |u|^{p-1})|v|\,dx \leq C(C_1 + C_p^p\|u\|_{H^s_0(\Omega)})\|v\|_{H^s_0(\Omega)}.
\]

Thus, as a function on \( v \), the linear functional \( v \mapsto \int_{\Omega} g(x,u)v\,dx \) is a continuous on \( H^s_0(\Omega) \), it is the Gâteaux differential of \( \mathcal{I} \) at \( u \) on \( H^s_0(\Omega) \), we denote it by \( \mathcal{I}'_u \), that is

\[
\mathcal{I}'_u(u)v = \int_{\Omega} g(x,u)v\,dx. \quad (2.3)
\]
2.1, up to a subsequence, we may assume that
\[ u_n \to u \quad \text{in } L^p(\mathbb{R}^N) \]
\[ u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N \] as \( n \to \infty \) and there exists \( \psi \in L^p(\mathbb{R}^N) \) such that
\[ |u_n(x)| \leq \psi(x) \quad \text{a.e. in } \mathbb{R}^N \quad \text{for any } n \in \mathbb{N}. \]

We have, by the H"older inequality,
\[ \left| \int g(x, u_n) - g(x, u) \right| \leq \int \left| g(x, u_n) - g(x, u) \right| dx \]
\[ \leq \left( \int \left| g(x, u_n) - g(x, u) \right|^\frac{p}{p-1} dx \right)^{\frac{p-1}{p}} \left( \int |g|^p dx \right)^{\frac{1}{p}} \]
\[ \leq C_p \left( \int \left| g(x, u_n) - g(x, u) \right|^\frac{p}{p-1} dx \right)^{\frac{p-1}{p}} \|g\|_{L^p(\Omega)}. \] (2.5)

By (2.4) and the fact that the map \( t \to g(\cdot, t) \) is continuous in \( t \in \mathbb{R} \) we get
\[ \lim_{n \to \infty} \int \left| g(x, u_n(x)) - g(x, u(x)) \right| = 0, \quad \text{a.e. in } \mathbb{R}^N, \] (2.6)
and moreover,
\[ |g(x, u_n) - g(x, u)|^\frac{p}{p-1} \leq C(1 + |u_n|^{p-1} + |u|^{p-1})^\frac{p}{p-1} \leq C(1 + |\psi|^p + |u|^p) \in L^1(\Omega), \]
here we have used the elementary inequality \(|a + b|^q \leq c_q(|a|^q + |b|^q)\) for all \( a, b \in \mathbb{R} \).

Then by the Dominated Convergence Theorem,
\[ \lim_{n \to \infty} \int \left| g(x, u_n) - g(x, u) \right|^\frac{p}{p-1} dx = 0. \] (2.7)

By (2.5), (2.7) and Proposition 2.1 we get
\[ \|I_G(u_n) - I_G(u)\| = \sup \{ |I_G(u_n) - I_G(u)| : v \in H_0^s(\Omega), \|v\|_{H_0^s(\Omega)} = 1 \} \]
\[ \leq C \left( \int \left| g(x, u_n) - g(x, u) \right|^\frac{p}{p-1} dx \right)^{\frac{p-1}{p}} \to 0 \quad \text{as } n \to \infty. \]

Therefore \( I_G \) is continuous on \( H_0^s(\Omega) \) and then we deduce that \( I \) is Fréchet differentiable on \( H_0^s(\Omega) \) with derivative
\[ \langle I'(u), v \rangle = \int g(x, u)v dx. \]

(ii) We prove that \( I \in C^2(H_0^s(\Omega), \mathbb{R}) \). First, we prove that \( I' \) is Gâteaux differentiable. Since \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is of class \( C^1 \) in \( t \), for almost every \( x \in \Omega \), we have that
\[ \lim_{\theta \to 0} \frac{g(x, u(x) + \theta v(x)) - g(x, u(x))}{\theta} = g'(x, u(x))v(x). \]

By the Lagrange Theorem there exists a real number \( \theta \) such that \( |\theta| \leq |g| \) and
\[ \left| \frac{g(x, u(x) + \theta v(x)) - g(x, u(x))}{\theta} w(x) \right| = \left| g'(x, u(x) + \theta v(x))v(x)w(x) \right| \]
\[ \leq C(1 + |u(x) + \theta v(x)|^{p-2})|v(x)||w(x)| \]
\[ \leq C(1 + |u(x)|^{p-2} + |v(x)|^{p-2})|v(x)||w(x)| \]
By Proposition 2.1 we see that
\[(1 + |u|^{p-2}|v| |w| + |v|^{p-1}|w|) \in L^1(\Omega),\]
then it follows from the Dominated Convergence Theorem we have for all \(v, w \in H^s_0(\Omega),\)
\[
\lim_{\varepsilon \to 0} \frac{\langle \mathcal{I}'(u + \varepsilon v) - \mathcal{I}'(u), w \rangle}{\varepsilon} = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{g(x, u + \varepsilon v) - g(x, u)}{\varepsilon} \, w \, dx
\]
\[
= \int_{\Omega} \frac{g(x, u + \varepsilon v) - g(x, u)}{\varepsilon} \, w \, dx
\]
\[
= \int_{\Omega} g'_\varepsilon(x, u) w \, dx.
\]
Since
\[(v, w) \mapsto \int_{\Omega} g'_\varepsilon(x, u) w \, dx
\]
defines a conjugate bilinear function on \(H^s_0(\Omega),\) and by the Hölder inequality and
the continuous embedding \(H^s_0(\Omega) \hookrightarrow L^q(\Omega)\) for \(q \in [1, \frac{2N}{N-2s}],\) we have for any fixed
\(u \in H^s_0(\Omega)\) that
\[
\left| \int_{\Omega} g'_\varepsilon(x, u) w \, dx \right| \leq C \int_{\Omega} (1 + |u|^{p-2}) |v| |w| \, dx
\]
\[
\leq C \int_{\Omega} |v| |w| \, dx + C \int_{\Omega} |u|^{p-2} |v| |w| \, dx
\]
\[
\leq C |v|_{L^2(\Omega)} |w|_{L^2(\Omega)} + C |u|_{L^p(\Omega)} |v|_{L^p(\Omega)} |w|_{L^p(\Omega)}
\]
\[
\leq C(1 + \|u\|_{H^s_0(\Omega)}) ||v||_{H^s_0(\Omega)} ||w||_{H^s_0(\Omega)}.
\]
Hence, by applying Riesz’s representation theorem, \(\mathcal{I}' \) is Gâteaux differentiable at \(u\) with derivative
\[
\mathcal{I}'_g(u)(v, w) = \int_{\Omega} g'_\varepsilon(x, u) w \, dx
\]
and
\[
\|\mathcal{I}'_g(u)\| = \sup \{|\mathcal{I}'_g(u)(v, w)| : v, w \in H^s_0(\Omega), |v||v|_{H^s_0(\Omega)} = 1, ||w||_{H^s_0(\Omega)} = 1\}.
\]
Now we show that the map \(u \mapsto \mathcal{I}'_g(u)\) is continuous on \(H^s_0(\Omega).\) We only need to show that \(\|u_n - u\|_{H^s_0(\Omega)} \to 0\) as \(n \to \infty\) implies that
\[
\lim_{n \to \infty} \|\mathcal{I}'_g(u_n) - \mathcal{I}'_g(u)\| = 0.
\]
Up to a subsequence, we may assume that
\[
u_n \to u \quad \text{in } L^p(\mathbb{R}^N)
\]
\[
u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N
\]
as \(n \to \infty\) and there exists \(\psi \in L^p(\mathbb{R}^N)\) such that
\[
|\nu_n(x)| \leq \psi(x) \quad \text{a.e. in } \mathbb{R}^N \quad \text{for any } n \in \mathbb{N}.
\]
By the fact that the map \(t \mapsto g'_\varepsilon(\cdot, t)\) is continuous in \(t \in \mathbb{R}\) we get
\[
\lim_{n \to \infty} |g'_\varepsilon(x, \nu_n(x)) - g'_\varepsilon(x, u(x))| = 0 \quad \text{a.e. in } \mathbb{R}^N,
\]
moreover, by (1.3) and Proposition 2.1 we have
\[
|g'_\varepsilon(x, \nu_n) - g'_\varepsilon(x, u)|^{\frac{p}{p-2}} \leq C(1 + |\nu_n|^{p-2} + |u|^{p-2})^{\frac{p}{p-2}} \leq C(1 + |\psi|^p + |u|^p) \in L^1(\Omega),
\]
we then obtain by the Dominated Convergence Theorem that
\[ \lim_{n \to \infty} \int_{\Omega} |g'_i(x, u_n) - g'_i(x, u)|^{\frac{p+2}{p}} \, dx = 0. \]

Since
\[ \left| \int_{\Omega} (g'_i(x, u_n) - g'_i(x, u))v \, dx \right| \leq \int_{\Omega} |g'_i(x, u_n) - g'_i(x, u)||v| \, dx \]
\[ \leq \left( \int_{\Omega} |g'_i(x, u_n) - g'_i(x, u)|^{\frac{p+2}{p}} \, dx \right)^{\frac{p}{p+2}} \|v\|_{L^p(\Omega)} \]
\[ \leq C \left( \int_{\Omega} |g'_i(x, u_n) - g'_i(x, u)|^{\frac{p+2}{p}} \, dx \right)^{\frac{p}{p+2}} \|v\|_{H^s(\Omega)} \|w\|_{H^s(\Omega)}, \]
we get
\[ \lim_{n \to \infty} \|I'_G(u_n) - I'_G(u)\| = \lim_{n \to \infty} \sup_{v \in H^s_0(\Omega), \|v\|_{H^s(\Omega)} = 1} \left| \langle I'_G(u_n) - I'_G(u), v \rangle \right| \]
\[ \leq C \lim_{n \to \infty} \left( \int_{\Omega} |g'_i(x, u_n) - g'_i(x, u)|^{\frac{p+2}{p}} \, dx \right)^{\frac{p}{p+2}} = 0. \]

Now we conclude that \( I \in C^2(H^s_0(\Omega), \mathbb{R}) \) and
\[ \langle I''(u)v, w \rangle = \int_{\Omega} g'_i(x, u)vwdx, \quad \forall u, v, w \in H^s_0(\Omega). \]

The proof is complete. \( \Box \)

Now we define the functional \( J : H^s_0(\Omega) \to \mathbb{R} \) as
\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy - \int_{\Omega} G(x, u) \, dx, \quad u \in H^s_0(\Omega). \]  \( \text{(2.9)} \)

Since the functional \( u \to \|u\|_{H^s(\Omega)}^2 \) is in \( C^2(H^s_0(\Omega), \mathbb{R}) \), and \( J \) can be written as
\[ J(u) = \frac{1}{2} \|u\|_{H^s(\Omega)}^2 - I(u), \quad u \in H^s_0(\Omega), \]
it follows from Proposition 2.2 that under the assumption \((g)\), the functional \( J \in C^2(H^s_0(\Omega), \mathbb{R}) \) with derivatives given by
\[ \langle J'(u), v \rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy - \int_{\Omega} g(x, u)v \, dx, \quad \forall u, v \in H^s_0(\Omega); \]  \( \text{(2.10)} \)
\[ \langle J''(u)v, w \rangle = \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} \, dxdy - \int_{\Omega} g'_i(x, u)vwdx, \quad \forall u, v, w \in H^s_0(\Omega). \]  \( \text{(2.11)} \)

Therefore by (1.4) and (2.10), weak solutions of (1.1) are exactly critical points of the functional \( J \) on \( H^s_0(\Omega) \).
2.2. **The eigenvalue problem (1.7).** In this subsection we consider the eigenvalue problem mentioned in Section 1 and rewrite (1.7) as follows:

\[
\begin{aligned}
(-\Delta)^s u &= \lambda u \quad x \in \Omega, \\
0 &= u \quad x \in \mathbb{R}^N \setminus \Omega.
\end{aligned}
\] (2.12)

The number \( \lambda \in \mathbb{R} \) is an eigenvalue of (2.12) if there is a nontrivial function \( u \in H^s_0(\Omega) \) such that

\[
\int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \lambda \int_{\Omega} u(x) \varphi(x) \, dx, \quad \forall \varphi \in H^s_0(\Omega).
\]

From Proposition 9 in [30] we have the following conclusions.

**Proposition 2.3.** Let \( s \in (0, 1) \) with \( N > 2s \) and let \( \Omega \) be an open bounded subset of \( \mathbb{R}^N \). Then

(i) (2.12) admits an eigenvalue \( \lambda_1 \) which is positive and that can be characterized as follows

\[
\lambda_1 = \min_{u \in H^s_0(\Omega), \|u\|_{L^2(\Omega)} = 1} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
\]

\( \lambda_1 \) is simple, and there is a non-negative function \( \phi_1 \in H^s_0(\Omega) \) corresponding to \( \lambda_1 \), such that \( \|\phi_1\|_{L^2(\Omega)} = 1 \) and

\[
\lambda_1 = \int_{\mathbb{R}^N} \frac{|\phi_1(x) - \phi_1(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
\]

(ii) the set of the eigenvalue of (2.12) consists of a sequence \( \{\lambda_k\}_{k \in \mathbb{N}} \) with

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty.
\]

For any \( k \in \mathbb{N} \) the eigenvalues can be characterized as follows:

\[
\lambda_k = \min_{u \in \mathbb{P}_k, \|u\|_{L^2(\Omega)} = 1} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy,
\]

and moreover, \( \lambda_k \) is attained at some \( \phi_k \in \mathbb{P}_k \), that is

\[
\|\phi_k\|_{L^2(\Omega)} = 1, \quad \lambda_k = \int_{\mathbb{R}^N} \frac{\phi_k(x) - \phi_k(y))^2}{|x - y|^{N+2s}} \, dx \, dy.
\]

where

\[
\mathbb{P}_k := \{ u \in H^s_0(\Omega) : \langle u, \phi_j \rangle_{H^s_0(\Omega)} = 0 \ \text{for all} \ j = 1, 2, \cdots, k - 1 \}.
\]

(iii) Each eigenvalue \( \lambda_k \) has finite multiplicity, more precisely, if \( \lambda_k \) is such that

\[
\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+\nu_k-1} < \lambda_{k+\nu_k}
\]

for some \( \nu_k \geq 1 \), then the set of all the eigenfunctions corresponding to \( \lambda_k \) agrees with

\[
E(\lambda_k) := \text{span}\{\phi_k, \phi_{k+1}, \cdots, \phi_{k+\nu_k-1}\}, \quad \dim E(\lambda_k) = \nu_k.
\]

(iv) The sequence \( \{\phi_k\}_{k \in \mathbb{N}} \) of eigenfunctions corresponding to \( \lambda_k \) is an orthonormal basis of \( L^2(\Omega) \) and an orthogonal basis of \( H^s_0(\Omega) \).
For each eigenvalue \( \lambda_k \), we define a linear operator \( \mathcal{A}_k : H^s_0(\Omega) \to (H^s_0(\Omega))^* \) as

\[
\langle \mathcal{A}_k u, v \rangle = \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda_k \int_{\Omega} uv \, dx.
\]

(2.13)

Then \( \mathcal{A}_k \) is a bounded self-adjoint linear operator, and corresponding to the eigenvalue \( \lambda_k \) with multiplicity \( \nu_k \), \( H^s_0(\Omega) \) can be split as

\[
H^s_0(\Omega) = W^-_k \oplus V_k \oplus W^+_k
\]

(2.14)

so that \( \langle \mathcal{A}_k \phi, \phi \rangle = 0 \) for all \( \phi \in V_k := E(\lambda_k) \) and the following variational inequalities hold true:

\[
\langle \mathcal{A}_k u, u \rangle \leq \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \|u\|_{H^s_0(\Omega)}^2, \quad \forall \, u \in W^-_k,
\]

(2.15)

\[
\langle \mathcal{A}_k v, v \rangle \geq \left( 1 - \frac{\lambda_k}{\lambda_{k+\nu_k}} \right) \|v\|_{H^s_0(\Omega)}^2, \quad \forall \, v \in W^+_k.
\]

(2.16)

We conclude this subsection by citing an \( L^\infty \)-regularity result for the eigenfunctions of the fractional Laplacian \( (-\Delta)^s \).

**Proposition 2.4** ([29]). Let \( \phi \in H^s_0(\Omega) \) and \( \lambda > 0 \) be such that

\[
\int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} \, dx \, dy = \lambda \int_{\Omega} \phi(x)\psi(x) \, dx, \quad \forall \, \psi \in H^s_0(\Omega).
\]

(2.17)

Then \( \phi \in L^\infty(\Omega) \) and there exists \( C > 0 \), possible depending on \( N, s \) and \( \lambda \), such that

\[
\|\phi\|_{L^\infty(\Omega)} \leq C \|\phi\|_{L^2(\Omega)}.
\]

(2.18)

### 2.3. Preliminaries about Morse theory.

In this subsection we collect some results on Morse theory (see [12, 22]) for a \( C^2 \) functional \( \mathcal{J} \) defined on a Hilbert space \( E \).

Let \( \mathcal{J} \in C^2(E, \mathbb{R}) \). Denote for \( c \in \mathbb{R} \)

\[
\mathcal{J}^c = \{ u \in E : \mathcal{J}(u) \leq c \}, \quad \mathcal{K}_c = \{ u \in E : \mathcal{J}'(u) = 0, \mathcal{J}(u) = c \}.
\]

We say that \( \mathcal{J} \) possesses the deformation property at the level \( c \in \mathbb{R} \) if for any \( \bar{c} > 0 \) and any neighborhood \( \mathcal{N} \) of \( \mathcal{K}_c \), there are \( \epsilon > 0 \) and a continuous deformation \( \eta : E \times [0, 1] \to E \) such that

(1) \( \eta(u, t) = u \) for either \( t = 0 \) or \( z \notin \mathcal{J}^{-1}[c - \bar{c}, c + \bar{c}] \);

(2) \( \mathcal{J}(\eta(u, t)) \) is non-increasing in \( t \) for any \( u \in E \);

(3) \( \eta(\mathcal{J}^c \setminus \mathcal{N}) \subset \mathcal{J}^{c-\epsilon} \).

We say that \( \mathcal{J} \) possesses the deformation property if \( \mathcal{J} \) possesses the deformation property at each level \( c \in \mathbb{R} \). We say that \( \mathcal{J} \) satisfies the Palais-Smale condition at the level \( c \in \mathbb{R} \) if any sequence \( \{u_n\} \subset E \) satisfying \( \mathcal{J}(u_n) \to c \) and \( \mathcal{J}'(u_n) \to 0 \) as \( n \to \infty \) has a convergent subsequence. \( \mathcal{J} \) satisfies the Palais-Smale condition if \( \mathcal{J} \) satisfies the Palais-Smale condition at each \( c \in \mathbb{R} \). If \( \mathcal{J} \) satisfies the Palais-Smale condition, then \( \mathcal{J} \) possesses the deformation property ([12]).

Let \( u_0 \) be an isolated critical point of \( \mathcal{J} \) with \( \mathcal{J}(u_0) = c \in \mathbb{R} \), and \( U \) be a neighborhood of \( u_0 \). The group \( C_q(\mathcal{J}, u_0) := H_q(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U \setminus \{u_0\}), q \in \mathbb{Z} \), is called the \( q \)-th critical group of \( \mathcal{J} \) at \( u_0 \), where \( H_*(A, B) \) denotes a singular relative homology group of the pair \((A, B)\) with integer coefficients.
Let $K = \{u \in E : J'(u) = 0\}$. Assume that $J(K)$ is bounded from below by $a \in \mathbb{R}$ and $J$ possesses the deformation property at all $c \leq a$. The group $C_q(J, \infty) := H_q(E, J^\alpha), q \in \mathbb{Z}$, is called the $q$-th critical group of $J$ at infinity ([6]).

Assume that $J$ satisfies the deformation property and $K$ is a finite set. The Morse type numbers of the pair $(E, J^\alpha)$ are defined by $M_q := \sum_{u \in K} \dim C_q(J, u)$, and the Betti numbers of the pair $(E, J^\alpha)$ are defined by $\beta_q := \dim C_q(J, \infty)$. If all $M_q$ and $\beta_q$ are finite and only finitely many of them are nonzero, then the relationship between the Morse type numbers $M_q$ and the Betti numbers $\beta_q$ is expressed by the following Morse inequalities ([12, 22])

$$\sum_{j=0}^{q} (-1)^{q-j} M_j \geq \sum_{j=0}^{q} (-1)^{q-j} \beta_j \quad \text{(Morse inequality),}$$

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q \quad \text{(Morse equality).}$$

From (2.19) one can deduce that $M_q \geq \beta_q$ for all $q \in \mathbb{Z}$. If $K = \emptyset$ then $\beta_q = 0$ for all $q \in \mathbb{Z}$. Thus if $\beta_q \neq 0$ for some $q_0 \in \mathbb{Z}$, then $J$ must have a critical point $u^*$ with $C_q(J, u^*) \neq 0$. If $K = \{u^*\}$ then $C_q(J, \infty) \cong C_q(J, u^*)$ for all $q \in \mathbb{Z}$. Thus if $C_q(J, \infty) \neq C_q(J, u^*)$ for some $q \in \mathbb{Z}$ then $J$ must have a new critical point. Therefore the basic idea in applying Morse theory to find critical points of $J$ is to compute critical groups both at infinity and at known critical points clearly and then to find unknown critical points by applying formulas (2.19) and (2.20).

Now we state an abstract result for the critical groups at infinity.

**Proposition 2.5.** Let the functional $J : E \to \mathbb{R}$ take the form

$$J(u) = \frac{1}{2} \langle Au, u \rangle + F(u),$$

where $A : E \to E$ is a self-adjoint linear operator such that 0 is isolated in $\sigma(A)$, the spectrum of $A$, and $F \in C^1(E, \mathbb{R})$ satisfies

$$\|F'(u)\| = o(\|u\|) \quad \text{as} \quad \|u\| \to \infty.$$  

Denote $V := \ker A$, $W := V^{\perp} = W^+ \oplus W^-$ where $W^+$ and $W^-$ are positive definite and negative definite invariant subspaces of $A$, respectively. Assume that $\mu = \dim W^-$ and $\nu = \dim V$ are finite, and $J$ possesses the deformation property.

(i) If $\nu = 0$, then $C_q(J, \infty) \cong \delta_{q,\mu} \mathbb{Z}$ for all $q \in \mathbb{Z}$.

(ii) If $\nu > 0$, then $C_q(J, \infty) \cong \delta_{q,\mu} \mathbb{Z}$, where $\mu^+ = \mu$ and $\mu^- = \mu + \nu$, provided $J$ satisfies the angle conditions with respect to $E = V \oplus W$:

$$(AC_{\infty}^\perp) \text{ there exist } M > 0 \text{ and } \epsilon \in (0, 1) \text{ such that}$$

$$\pm \langle J'(u), v \rangle \geq 0 \quad \text{for} \quad u = v + w, \quad \|u\| \geq M, \quad \|w\| \leq \epsilon \|u\|, \quad v \in V, w \in W.$$

Proposition 2.5(i) was obtained in [37](see Remark 5.2 in [12]) and Proposition 2.5(ii) is a revision of Proposition 3.10 in [6]. The revision was made first in [34] and was remade in [36].

Let the origin 0 be an isolated critical point of $J \in C^2(E, \mathbb{R})$ such that $J''(0)$ is a Fredholm operator, it is always the case in many applications. Let $V_0 = \ker J''(0)$ and $W_0 = V_0^{\perp} = W_0^+ \oplus W_0^-$ where $W_0^+$ and $W_0^-$ are positive definite and negative definite invariant subspaces of $J''(0)$, respectively. Then the nullity $\nu_0 := \dim V_0$ of 0 is finite(see [22]). If the Morse index $\mu_0$ of $J$ at 0 is finite, then we have the following basic facts(see [12, 22]).
Proposition 2.6 \([6]\). Let the eigenvalue \(\lambda\) of \((-\Delta)^{s}\) verify \((1.8)\). Then for any given \(\tau > 0\) small, there is \(b_{\tau} > 0\) such that
\[
\text{meas}\{x \in \Omega : |v(x)| \geq b_{\tau} \|v\|_{H^{s}_{0}(\Omega)}\} > |\Omega| - \tau, \quad \forall \ v \in E(\lambda) \setminus \{0\}.
\]
By \((3.5)\), we get the conditions required by Propositions 2.5 and 2.6 for computing the critical groups.

**Proof.** We follow the arguments in [5]. Since \(E(\lambda)\) is the eigenfunction space of \((-\Delta)^s\) corresponding to the eigenvalue \(\lambda\), we have by Proposition 2.3 that \(E(\lambda)\) is finite dimensional and by Proposition 2.4 that \(E(\lambda) \subset L^\infty(\Omega)\).

Denote \(S = \{v \in E(\lambda) : \|v\|_{L^\infty(\Omega)} = 1\}\) where \(\|u\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |u(x)|\). Since \(E(\lambda)\) is finite dimensional, all norms on \(E(\lambda)\) are equivalent, to prove (3.2), it is equivalent to prove

\[
\text{meas}\{x \in \Omega : |v(x)| < b_\tau\} < \tau, \quad \forall \ v \in S.
\] (3.3)

We first show for each \(v \in S\), there is \(b_\tau(v) > 0\) such that

\[
w \in E(\lambda), \|w - v\|_{L^\infty(\Omega)} < b_\tau(v) \Rightarrow \text{meas}\{x \in \Omega : |w(x)| < b_\tau(v)\} < \tau.
\] (3.4)

For this, we argue once more by contradiction and we suppose that there were \(v_0 \in S\) such that for each \(n \in \mathbb{N}\), there exists \(w_n \in E(\lambda)\) such that

\[
\|w_n - v_0\|_{L^\infty(\Omega)} < \frac{1}{2n}, \quad \text{meas}\{x \in \Omega : |w_n(x)| < \frac{1}{2n}\} \geq \tau.
\] (3.5)

Set

\[
\omega_n = \left\{ x \in \Omega : |w_n(x)| < \frac{1}{2n} \right\}, \quad \forall \ n \in \mathbb{N},
\]

then

\[
\omega_n \subset \left\{ x \in \Omega : |v_0(x)| < \frac{1}{n} \right\}, \quad \forall \ n \in \mathbb{N}.
\]

By (3.5), we get

\[
\text{meas}\{x \in \Omega : |v_0(x)| < \frac{1}{n}\} \geq \tau, \quad \forall \ n \in \mathbb{N}.
\]

By letting \(n \to \infty\), we get

\[
\text{meas}\{x \in \Omega : v_0(x) = 0\} \geq \tau > 0.
\]

This contradicts the assumption (1.8), thus (3.4) is proved.

We next consider the covering of \(\{B_{b_\tau(v_0)}(v)\}_{v \in S}\) of \(S\), where \(B_{b_\tau(v_0)}(v)\) is the open \(L^\infty\) ball in \(E(\lambda)\) centered at \(v\) with radial \(b_\tau(v)\). As \(E(\lambda)\) is finite dimensional, by the compactness of \(S\), there is a finite refinement \(\{B_{b_\tau(v_i)}(v_i)\}_{i=1}^j\) of \(\{B_{b_\tau(v_0)}(v)\}_{v \in S}\) such that \(S \subset \{B_{b_\tau(v_i)}(v_i)\}_{i=1}^j\). Take \(b_\tau = \min\{b_\tau(v_i) : i = 1, \cdots, j\}\). Then for any \(v \in S \subset E(\lambda)\), it holds that \(v \in B_{b_\tau(v_i)}(v_i)\) for some \(i\). Hence, as a consequence of this and (3.4), we get

\[
\text{meas}\{x \in \Omega : |v(x)| < b_\tau\} \leq \text{meas}\{x \in \Omega : |v(x)| < b_\tau(v_i)\} < \tau.
\]

This proves (3.3) and the proof is complete. \(\square\)

The type of result of Lemma 3.2 was a modification of a result in [5, Lemma 3.2].

From now on we prove two technical lemmas that will be used to verify the angle conditions required by Propositions 2.5 and 2.6 for computing the critical groups.

Let (1.5) hold. Then corresponding to the eigenvalue \(\lambda_\ell\) of \((-\Delta)^s\) with multiplicity \(\nu_\ell\), \(H^s_0(\Omega)\) can be split as

\[
H^s_0(\Omega) = W^-_\infty \oplus V_\infty \oplus W^+_\infty = V_\infty \oplus W_\infty,
\]

where

\[
W^-_\infty = \bigoplus_{\lambda_k < \lambda_\ell} \ker((-\Delta)^s - \lambda_k), \quad V_\infty = \ker((-\Delta)^s - \lambda_\ell)
\]
and
\[ W^+_\infty = (W^-_\infty \oplus V_\infty)^\perp = \bigoplus_{\lambda_k > \lambda_c} \ker((-\Delta)^s - \lambda_k), \quad W^-_\infty = W^-_\infty \oplus W^+_\infty. \]

**Lemma 3.3.** Assume \((f^\pm)\). Then there exists \(M > 0\), \(\beta > 0\) and \(\epsilon \in (0,1)\) such that
\[ \pm \int_\Omega f(x,u)vdx \geq \beta \|v\|^{1+r}_{H^s_0(\Omega)}, \tag{3.6} \]
for any \(u = v + w \in H^s_0(\Omega) = V_\infty \oplus W_\infty\) with \(\|u\|_{H^s_0(\Omega)} \geq M\) and \(\|w\|_{H^s_0(\Omega)} \leq \epsilon \|u\|_{H^s_0(\Omega)}\).

**Proof.** We give the proof for the case that \((f^+)\) holds. For \(u = v + w \in H^s_0(\Omega) = V_\infty \oplus W_\infty\), we set
\[ C(M, \epsilon) = \{u = v + w : \|u\|_{H^s_0(\Omega)} \geq M, \|w\|_{H^s_0(\Omega)} \leq \epsilon \|u\|_{H^s_0(\Omega)}\}, \]
where \(M > 0\) and \(\epsilon \in (0,1)\) will be chosen below.

For \(u \in C(M, \epsilon)\), we have
\[ \|w\|_{H^s_0(\Omega)} \leq \epsilon \|u\|_{H^s_0(\Omega)}, \quad \|v\|_{H^s_0(\Omega)} \geq \sqrt{1-\epsilon^2} \|u\|_{H^s_0(\Omega)}. \tag{3.7} \]

Denote
\[ \Omega_1 = \{x \in \Omega : |u(x)| \geq R\}, \quad \Omega_2 = \Omega \setminus \Omega_1, \tag{3.8} \]
where \(R > 1\) was given in the assumptions \((f^\pm)\). For \(\sigma > 0\), we set by Lemma 3.1 that
\[ \Omega_\sigma = \{x \in \Omega : |u(x)| \leq a_\sigma \|u\|_{H^s_0(\Omega)}, \quad \forall \, u \in H^s_0(\Omega). \tag{3.9} \]

For \(\tau > 0\), we set by Lemma 3.2 that
\[ \Omega_\tau = \{x \in \Omega : |v(x)| \geq b_\tau \|v\|_{H^s_0(\Omega)}\}, \quad \forall \, v \in V_\infty \setminus \{0\}. \tag{3.10} \]

Take \(u \in C(M, \epsilon)\). Then \(u = v + w\) verifies (3.7).

1. Set \(\omega_1 := \Omega_\sigma \cap \Omega_\tau\). Then by (3.9) and (3.10), we have
\[ |v(x)| \geq b_\tau \|v\|_{H^s_0(\Omega)}, \quad |w(x)| \leq a_\sigma \|w\|_{H^s_0(\Omega)}, \quad \forall \, x \in \omega_1. \]

Thus by (3.7) we deduce that
\[ |u(x)| \geq |v(x)| - |w(x)| \geq (\sqrt{1-\epsilon^2} b_\tau - \epsilon a_\sigma) \|u\|_{H^s_0(\Omega)}. \tag{3.11} \]

For \(\epsilon \in (0,1)\) small such that \(\sqrt{1-\epsilon^2} b_\tau - \epsilon a_\sigma > 0\) and \(M > 0\) large such that \((\sqrt{1-\epsilon^2} b_\tau - \epsilon a_\sigma)M \geq R\), we have
\[ |u(x)| \geq R, \quad \forall \, x \in \omega_1. \]

It follows from \((f^+)\), (3.9) and (3.11) that
\[ \int_{\omega_1} f(x,u)vdx = \int_{\omega_1} f(x,u)udx - \int_{\omega_1} f(x,u)wdx \geq \int_{\omega_1} f(x,u)udx - \int_{\omega_1} |f(x,u)||w|dx \geq c_1 \int_{\omega_1} |u|^{1+r}dx - c_2 \int_{\omega_1} |u|^r|w|dx \geq (c_1 \sqrt{1-\epsilon^2} b_\tau - \epsilon a_\sigma)^{r+1} - c_2 \epsilon a_\sigma^{1+r})|\omega_1| \|u\|^{1+r}_{H^s_0(\Omega)}. \tag{3.12} \]
(2) Set $\omega_2 := \Omega_{\sigma} \setminus \Omega_r$. By Lemma 3.2 we see that $|\omega_2| \leq |\Omega \setminus \Omega_r| \leq \tau$. For $x \in \omega_2 \cap \Omega_1$, we have
\[ |u(x)| \geq R, \quad |u(x)| \leq a_\sigma \|u\|_{H^s_0(\Omega)}, \quad |v(x)| < b_\tau \|v\|_{H^s_0(\Omega)}, \]

it follows from $(f^+)$ that
\[ \int_{\omega_2 \cap \Omega_1} |f(x, u)v| \, dx \leq c_2 \int_{\omega_2 \cap \Omega_1} |u|^r |v| \, dx \leq c_2 \tau a_\sigma^r b_\tau \|u\|^{1+r}_{H^s_0(\Omega)}. \quad (3.13) \]

Denote $\hat{c} = \sup_{x \in \Omega, |t| \leq R} |f(x, t)|$. For $x \in \omega_2 \cap \Omega_2$, we have $|u(x)| \leq R$. Thus
\[ \int_{\omega_2 \cap \Omega_2} |f(x, u)v| \, dx \leq \hat{c} \int_{\omega_2 \cap \Omega_2} |v| \, dx \leq \hat{c}|\omega_2 \cap \Omega_2| \|b_\tau\|_{H^s_0(\Omega)} \leq \hat{c} \tau b_\tau \|u\|_{H^s_0(\Omega)}. \quad (3.14) \]

Therefore
\[ \int_{\omega_2} f(x, u)v \, dx \leq \int_{\omega_2} |f(x, u)v| \, dx \]
\[ = \int_{\omega_2 \cap \Omega_1} |f(x, u)v| \, dx + \int_{\omega_2 \cap \Omega_2} |f(x, u)v| \, dx \]
\[ \leq c_2 \tau a_\sigma^r b_\tau \|u\|^{r+1}_{H^s_0(\Omega)} + \hat{c} \tau b_\tau \|u\|_{H^s_0(\Omega)}. \quad (3.15) \]

(3) Set $\omega_3 := \Omega_{\tau} \setminus \Omega_\sigma$. Then for $u \in C(M, \epsilon)$ with $M \geq \frac{R}{a_\sigma}$,
\[ |u(x)| > a_\sigma \|u\|_{H^s_0(\Omega)} \geq a_\sigma M \geq R, \quad \forall x \in \omega_3. \quad (3.16) \]

By $(f^+)$ we have
\[ \int_{\omega_3} f(x, u)u \, dx \geq c_1 \int_{\omega_3} |u|^{r+1} \, dx \geq c_1 |\omega_3| a_\sigma^{r+1} \|u\|^{r+1}_{H^s_0(\Omega)}. \quad (3.17) \]

By $(f^+)$, $(3.7)$, Proposition 2.1 and the Hölder inequality we have
\[ \int_{\omega_3} |f(x, u)|w| \, dx \leq \int_{\omega_3} |f(x, u)| |w| \, dx \]
\[ \leq c_2 \int_{\omega_3} |u|^r |w| \, dx \]
\[ \leq c_2 \left( \int_{\Omega} |u|^{r+1} \, dx \right)^{\frac{r}{r+\tau}} \left( \int_{\Omega} |w|^{r+1} \, dx \right)^{\frac{1}{r+\tau}} \]
\[ = c_2 \|u\|_{L^{r+\tau}(\Omega)} \|w\|_{L^{1+r}(\Omega)} \]
\[ \leq c_2 C_{1+r}^{1+r} \|u\|^{1+r}_{H^s_0(\Omega)}, \quad (3.18) \]

where $C_{1+r}$ is the constant of the embedding $H^s_0(\Omega) \hookrightarrow L^{1+r}(\Omega)$. Therefore
\[ \int_{\omega_3} f(x, u)v \, dx = \int_{\omega_3} f(x, u)u \, dx - \int_{\omega_3} f(x, u)w \, dx \]
\[ \geq \int_{\omega_3} f(x, u)u \, dx - \int_{\omega_3} |f(x, u)|w| \, dx \]
\[ \geq c_1 |\omega_3| a_\sigma^{r+1} \|u\|^{r+1}_{H^s_0(\Omega)} - c_2 C_{1+r}^{r+1} \epsilon \|u\|^{1+r}_{H^s_0(\Omega)} \quad (3.19) \]
(4) Set \( \omega_4 = \Omega \setminus (\Omega_\sigma \cup \Omega_\tau) \). For \( u \in C(M, \epsilon) \) and \( x \in \omega_4 \), we have (3.16) and then have the estimate as (3):

\[
\int_{\omega_4} f(x, u)vdx = \int_{\omega_4} f(x, u)u dx - \int_{\omega_4} f(x, u)w dx \\
\geq \int_{\omega_4} f(x, u)u dx - \int_{\omega_4} |f(x, u)| w dx \\
\geq c_1 |\omega_4| A_{\sigma}^{1+r} \| u \|_{H_0^s(\Omega)}^{1+r} - c_2 C_{1+r} \epsilon \| u \|_{H_0^s(\Omega)}^{1+r} \tag{3.20}
\]

Now it follows from (3.12), (3.15), (3.19) and (3.20) that

\[
\int_{\Omega} f(x, u)vdx = \int_{\omega_4} f(x, u)vdx + \int_{\omega_2} f(x, u)vdx + \int_{\omega_3} f(x, u)vdx + \int_{\omega_4} f(x, u)vdx \\
\geq c_1 (|\omega_4| (\sqrt{1-\epsilon^2 b_\sigma} - c a_\sigma)^{r+1} + (|\omega_3| + |\omega_4|) A_{\sigma}^{1+r}) \| u \|_{H_0^s(\Omega)}^{1+r} \\
- c_2 (c a_\sigma^{1+r} |\omega_4| + a_\sigma \tau b_\sigma + 2C_{1+r} \epsilon \| u \|_{H_0^s(\Omega)}^{1+r} - \epsilon \tau b_\sigma \| u \|_{H_0^s(\Omega)}^{1+r} \\
:= (\eta_1 - \eta_2) \| u \|_{H_0^s(\Omega)}^{1+r} - \epsilon \tau b_\sigma \| u \|_{H_0^s(\Omega)}^{1+r}. \tag{3.21}
\]

where

\[
\eta_1 = c_1 (|\omega_4| (\sqrt{1-\epsilon^2 b_\sigma} - c a_\sigma)^{r+1} + (|\omega_3| + |\omega_4|) A_{\sigma}^{1+r}), \\
\eta_2 = c_2 (c a_\sigma^{1+r} |\omega_4| + a_\sigma \tau b_\sigma + 2C_{1+r} \epsilon).
\tag{3.22}
\]

In the above arguments, we first choose \( \tau > 0 \) small and fixed so that the constant \( a_\sigma \) can be fixed. From (3.22), on sees that if \( \tau > 0 \) was chosen small enough and then \( \epsilon \in (0, 1) \) was chosen small enough, then

\[
\eta := \eta_1 - \eta_2 > 0. \tag{3.23}
\]

Therefore for \( u \in C(M, \epsilon) \),

\[
\int_{\Omega} f(x, u)vdx \geq (\eta - \epsilon \tau b_\sigma M^{-r}) \| u \|_{H_0^s(\Omega)}^{1+r} =: \beta \| u \|_{H_0^s(\Omega)}^{1+r}. \tag{3.24}
\]

Now we take \( M > 0 \) large enough once more then \( \beta > 0 \) and hence

\[
\int_{\Omega} f(x, u)vdx \geq \beta \| u \|_{H_0^s(\Omega)}^{1+r} \geq \beta \| v \|_{H_0^s(\Omega)}^{1+r} \text{ for } u \in C(M, \epsilon). \tag{3.25}
\]

The proof is complete. \( \square \)

Let (1.6) hold. Then corresponding to \( \lambda_m \) with multiplicity \( \nu_m \), \( H_0^s(\Omega) \) can be split as

\[
H_0^s(\Omega) = W_0^- \oplus V_0 \oplus W_0^+ = V_0 \oplus W_0
\]

where

\[
W_0^- = \bigoplus_{\lambda_k < \lambda_m} \ker((-\Delta)^s - \lambda_k), \quad V_0 = \ker((-\Delta)^s - \lambda_m)
\]

and

\[
W_0^+ = (W_0^- \oplus V_0)^\perp = \bigoplus_{\lambda_k > \lambda_m} \ker((-\Delta)^s - \lambda_k), \quad W_0 = W_0^- \oplus W_0^+.
\]

**Lemma 3.4.** Assume (1.5) and \((f_0^\pm)\). Then there exists \( \rho > 0 \), \( \alpha > 0 \) and \( \epsilon \in (0, 1) \) such that

\[
\pm \int_{\Omega} f_0(x, u)vdx \geq \alpha \| v \|_{H_0^s(\Omega)}^{1+r}, \tag{3.26}
\]

for any \( v = v + w \in H_0^s(\Omega) = V_0 \oplus W_0 \) with \( \| u \|_{H_0^s(\Omega)} \leq \rho \) and \( \| w \|_{H_0^s(\Omega)} \leq \epsilon \| u \|_{H_0^s(\Omega)}. \)
Proof. Let \((f^+_0)\) hold. Then for each given \(\gamma \in (\zeta, \frac{N+2\zeta}{N-2\zeta}]\), there is \(\tilde{c} = \tilde{c}(\gamma, \delta) > 0\) such that
\[
|f_0(x, t)| \leq \tilde{c}|t|^\gamma, \quad x \in \bar{\Omega}, \quad |t| \geq \delta. \tag{3.27}
\]
We fix such a \(\gamma \in (\zeta, \frac{N+2\zeta}{N-2\zeta}]\) for later use. Denote
\[
\Omega_1 = \{x \in \Omega : |u(x)| \leq \delta\}, \quad \Omega_2 = \Omega \setminus \Omega_1.
\]
For \(u = v + w \in H_0^s(\Omega) = V_0 \oplus W_0\), we set
\[
C(\rho, \epsilon) = \{u = v + w : \|u\|_{H_0^s(\Omega)} \leq \rho, \|w\|_{H_0^s(\Omega)} \leq \epsilon\|u\|_{H_0^s(\Omega)}\},
\]
where \(\rho > 0, \epsilon \in (0, 1)\) will be chosen below.
For \(u \in C(\rho, \epsilon)\), we have
\[
\|w\|_{H_0^s(\Omega)} \leq \epsilon\|u\|_{H_0^s(\Omega)}, \quad \|v\|_{H_0^s(\Omega)} \geq \sqrt{1 - \epsilon^2}\|u\|_{H_0^s(\Omega)}. \tag{3.28}
\]
For \(\sigma > 0\), we set by Lemma 3.1 that
\[
\Omega_\sigma = \{x \in \Omega : |u(x)| \leq a_\sigma\|u\|_{H_0^s(\Omega)}\}, \quad \forall u \in H_0^s(\Omega). \tag{3.29}
\]
Let \(\sigma > 0\) be fixed, then from (3.29) one sees that for \(\rho > 0\) small enough,
\[
\Omega_\sigma \subset \Omega_1, \quad \forall u \in C(\rho, \epsilon). \tag{3.30}
\]
For \(\tau > 0\), we set by Lemma 3.2 that
\[
\Omega_\tau = \{x \in \Omega : |v(x)| \geq b_\tau\|v\|_{H_0^s(\Omega)}\}, \quad \forall v \in V_0 \setminus \{0\}. \tag{3.31}
\]
Take \(u \in C(\rho, \epsilon)\). Then \(u = v + w\) verifies (3.28).
1. Set \(\omega_1 := \Omega_\sigma \cap \Omega_\tau\). For \(x \in \omega_1\), we have
\[
|u(x)| \leq \delta, \quad |v(x)| \geq b_\tau\|v\|_{H_0^s(\Omega)}.
\]
Thus by (3.28),
\[
|u(x)| \geq |v(x)| - |w(x)| \geq b_\tau\|v\|_{H_0^s(\Omega)} - a_\sigma\|w\|_{H_0^s(\Omega)} \geq (\sqrt{1 - \epsilon^2}b_\tau - \epsilon a_\sigma)\|u\|_{H_0^s(\Omega)}. \tag{3.32}
\]
For \(\epsilon \in (0, 1)\) small such that \(\sqrt{1 - \epsilon^2}b_\tau - \epsilon a_\sigma > 0\), we have by \((f^+_0)\), that
\[
\int_{\omega_1} f_0(x, u)vdx = \int_{\omega_1} f_0(x, u)udx - \int_{\omega_1} f_0(x, u)wdx \geq c_3 \int_{\omega_1} |u|^{1+\zeta}dx - c_4 \int_{\omega_1} |u|^\zeta|w|dx \
\geq (c_3(\sqrt{1 - \epsilon^2}b_\tau - \epsilon a_\sigma)^{1+\zeta} - c_4\epsilon a_\sigma^{1+\zeta})\|w\|_{H_0^s(\Omega)} \|u\|_{H_0^s(\Omega)}^{1+\zeta}. \tag{3.32}
\]
2. Set \(\omega_2 := \Omega_\sigma \setminus \Omega_\tau\). By Lemma 3.2 it follows that \(|\omega_2| \leq \tau\). For \(x \in \omega_2\),
\[
|u(x)| \leq \delta \text{ and } |v(x)| < b_\tau\|v\|_{H_0^s(\Omega)} \text{.}
\]
Thus
\[
\int_{\omega_2} f_0(x, u)vdx \leq \int_{\omega_2} |f_0(x, u)||v|dx \leq c_4\tau a_\sigma^{1+\zeta}b_\tau\|u\|_{H_0^s(\Omega)}^{1+\zeta}. \tag{3.33}
\]
3. Set \(\omega_3 := \Omega_\tau \setminus \Omega_\sigma\). We have
\[
|u(x)| > a_\sigma\|u\|_{H_0^s(\Omega)}, \quad |v(x)| \geq b_\tau\|v\|_{H_0^s(\Omega)}, \quad \forall x \in \omega_3.
\]
Therefore
\[
\int_{\omega_3 \cap \Omega_1} f_0(x, u) v dx \geq \int_{\omega_3 \cap \Omega_1} f_0(x, u) v dx - \int_{\omega_3 \cap \Omega_1} |f_0(x, u)| u dx
\]
\[
\geq c_3 \int_{\omega_3 \cap \Omega_1} |u|^{1+\zeta} dx - c_4 \int_{\omega_3 \cap \Omega_1} |u|^{1+\zeta} dx
\]
\[
\geq c_3 |\omega_3 \cap \Omega_1| a_1^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta} - c_4 e C_{1+\zeta}^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta},
\]
(3.34)
where \(C_{1+\zeta}\) is the embedding constant of \(H^{1+\zeta}_0(\Omega) \hookrightarrow L^{1+\zeta}(\Omega)\) as \(1 + \zeta \in (2, \frac{2N}{N-2})\).
By (3.27) we have
\[
\int_{\omega_3 \cap \Omega_2} f_0(x, u) v dx \leq \int_{\omega_3 \cap \Omega_2} |f_0(x, u)| v dx \leq \bar{c} \int_{\omega_3 \cap \Omega_2} |u|^{\gamma} v dx \leq C_{1+\gamma}^{1+\gamma} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma}.
\]
(3.35)
It follows from (3.34) and (3.35) that
\[
\int_{\omega_3} f_0(x, u) v dx = \int_{\omega_3 \cap \Omega_1} f_0(x, u) v dx + \int_{\omega_3 \cap \Omega_2} f_0(x, u) v dx
\]
\[
\geq c_3 |\omega_3 \cap \Omega_1| a_1^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta} - c_4 e C_{1+\zeta}^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta} - c_{1+\gamma}^{1+\gamma} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma},
\]
(3.36)
(4) Set \(\omega_4 := \Omega \setminus (\Omega_2 \cup \Omega_\sigma)\). We have
\[
|u(x)| > a_\sigma \|u\|_{H^{1}_0(\Omega)}, \quad |v(x)| < b_\sigma \|v\|_{H^{1}_0(\Omega)}, \quad \forall x \in \omega_4.
\]
In the same way as in (3) we get
\[
\int_{\omega_4} f_0(x, u) v dx = \int_{\omega_4 \cap \Omega_1} f_0(x, u) v dx + \int_{\omega_4 \cap \Omega_2} f_0(x, u) v dx
\]
\[
\geq c_3 |\omega_4 \cap \Omega_1| a_1^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta} - c_4 e C_{1+\zeta}^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta} - c_{1+\gamma}^{1+\gamma} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma},
\]
(3.37)
Now it follows from (3.32), (3.33), (3.36) and (3.37) that
\[
\int_{\Omega} f_0(x, u) v dx = \int_{\omega_1} f_0(x, u) v dx + \int_{\omega_2} f_0(x, u) v dx + \int_{\omega_3} f_0(x, u) v dx + \int_{\omega_4} f_0(x, u) v dx
\]
\[
\geq c_3 (|\omega_1| \sqrt{1 - \bar{c} b_\sigma} - c a_\sigma)^{1+\zeta} + |\omega_3 \cap \Omega_1| a_1^{1+\zeta} \|u\|_{H^{1+\zeta}_0(\Omega)}^{1+\zeta}
\]
\[
- c_4 (c_a^{1+\zeta} |\omega_1| + a_\sigma^{\gamma} b_\sigma + 2C_{1+\zeta}^{1+\zeta} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma}) - 2c C_{1+\gamma}^{1+\gamma} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma},
\]
(3.38)
where
\[
\xi_1 = c_3 (|\omega_1| \sqrt{1 - \bar{c} b_\sigma} - c a_\sigma)^{1+\zeta} + |\omega_3 \cap \Omega_1| a_1^{1+\zeta} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma},
\]
\[
\xi_2 = c_4 (c_a^{1+\zeta} |\omega_1| + a_\sigma^{\gamma} b_\sigma + 2C_{1+\zeta}^{1+\zeta} \|u\|_{H^{1+\gamma}_0(\Omega)}^{1+\gamma}).
\]
(3.39)
Since \(\sigma > 0\) was small and fixed so that \(a_\sigma\) was fixed, from (3.39) on sees that if \(\tau > 0\) was chosen small enough and then \(\epsilon \in (0, 1)\) was chosen small enough, then
\[
\xi := \xi_1 - \xi_2 > 0.
\]
(3.40)
Since \(\gamma > \zeta\), if we take \(\rho > 0\) small enough more once, then
\[
\alpha := \xi - 2c C_{1+\gamma}^{1+\gamma} \rho^{\gamma-\zeta} > 0.
\]
Now for \( u \in C(\rho, \epsilon) \), we have
\[
\int_{\Omega} f_0(x, u)vdx \geq \xi \|u\|_{H_0^s(\Omega)}^{1+\zeta} - 2\epsilon C^{1+\gamma}\|u\|_{H_0^s(\Omega)}^{1+\gamma}
= \|u\|_{H_0^s(\Omega)}^{1+\zeta}(\xi - 2\epsilon C^{1+\gamma}\|u\|_{H_0^s(\Omega)}^{\gamma-\zeta})
\geq (\xi - 2\epsilon C^{1+\gamma}\rho^{\gamma-\zeta})\|u\|_{H_0^s(\Omega)}^{1+\zeta}
= \alpha \|u\|_{H_0^s(\Omega)}^{1+\zeta}.
\]
(3.41)
\[
\int_{\Omega} f_0(x, u)vdx \geq \alpha \|u\|_{H_0^s(\Omega)}^{1+\zeta} \geq \alpha \|v\|_{H_0^s(\Omega)}^{1+\zeta} \text{ for all } u \in C(\rho, \epsilon).
\]
(3.42)
The proof is complete. \( \square \)

4. Proofs of main results. This section is devoted to the proofs of Theorems 1.1–1.3 in this paper by applying Morse theory. The main abstract tools we will use have been listed in Subsection 2.3. From now on we make the convention that all assumptions made in Theorems 1.1–1.3 hold.

We rewrite the functional \( J \) defined by (2.19) as follows:
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \frac{1}{2} \lambda \int_{\Omega} |u|^2dx - \int_{\Omega} F(x, u)dx, \ u \in H_0^s(\Omega),
\]
(4.1)
where \( F(x, t) = \int_{0}^{t} f(x, \varsigma) d\varsigma. \) Define the functional \( F : H_0^s(\Omega) \rightarrow \mathbb{R} \) by
\[
F(u) = -\int_{\Omega} F(x, u(x))dx, \ u \in H_0^s(\Omega).
\]
(4.2)
Using \((f^\pm)\) we can deduce that
\[
\|F'(u)\| = o(\|u\|_{H_0^s(\Omega)}) \text{ as } \|u\|_{H_0^s(\Omega)} \rightarrow \infty.
\]
(4.3)
Indeed, by \((f^\pm)\) we deduce that
\[
|f(x, t)| \leq c(1 + |t|^r), \ x \in \Omega, t \in \mathbb{R}.
\]
Thus for \( u, \phi \in H_0^s(\Omega), \)
\[
|\langle F'(u), \phi \rangle| \leq \int_{\Omega} |f(x, u)\phi|dx \leq c \int_{\Omega} (|\phi| + |u|^r|\phi|)dx \leq c(C_1 + C_1^{1+r} \|u\|_{H_0^s(\Omega)}^{1+r})\|\phi\|_{H_0^s(\Omega)}.
\]
As \( r \in (0, 1) \), (4.3) is verified.

According to (2.13), the functional \( J \) can be written as
\[
J(u) = \frac{1}{2} \langle A_{\rho}u, u \rangle + F(u), \ u \in H_0^s(\Omega).
\]
(4.4)
Therefore \( J \) fits the the abstract framework required by Proposition 2.5.

Now we verify that \( J \) satisfies the (PS) condition.

Lemma 4.1. Assume \((g)\). Let the sequence \( \{u_n\} \subset H_0^s(\Omega) \) be bounded in \( H_0^s(\Omega) \) and satisfy
\[
J'(u_n) \rightarrow 0, \ n \rightarrow \infty.
\]
(4.5)
Then there exists \( u^* \in H_0^s(\Omega) \) such that, up to a subsequence,
\[
\|u_n - u^*\|_{H_0^s(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
(4.6)
Proof. Since $H^s_0(\Omega)$ is a reflexive Banach space and $\{u_n\}$ is bounded in $H^s_0(\Omega)$, there is a subsequence of $\{u_n\}$, still denote by $\{u_n\}$, and there exists $u^* \in H^s_0(\Omega)$, such that

$$u_n \rightharpoonup u^* \text{ weakly in } H^s_0(\Omega) \text{ as } n \to \infty. \quad (4.7)$$

By Proposition 2.1, up to a subsequence, it holds that

$$u_n \to u^* \quad \text{in } L^p(\mathbb{R}^N),$$

$$u_n(x) \to u^*(x) \quad \text{a.e. in } \mathbb{R}^N. \quad (4.8)$$

as $n \to \infty$ and there exists $\psi \in L^p(\mathbb{R}^N)$ such that

$$|u_n(x)| \leq \psi(x) \quad \text{a.e. in } \mathbb{R}^N \text{ for any } n \in \mathbb{N}, \quad (4.9)$$

where $p \in (2, \frac{2N}{N+2s})$ was given in the assumption (g) which implies that $g$ satisfies the growth condition (2.1).

By (2.1), (4.7)–(4.9), the fact that the map $t \mapsto g(\cdot, t)$ is continuous in $t \in \mathbb{R}$ and the Dominated Convergence Theorem we get

$$\lim_{n \to \infty} \int_\Omega g(x, u_n(x))u_n(x)dx = \int_\Omega g(x, u^*(x))u^*(x)dx, \quad (4.10)$$

and

$$\lim_{n \to \infty} \int_\Omega g(x, u_n(x))u^*(x)dx = \int_\Omega g(x, u^*(x))u^*(x)dx. \quad (4.11)$$

From (4.7) one sees that for all $\phi \in H^s_0(\Omega)$, it holds

$$\int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}}dxdy$$

$$\to \int_{\mathbb{R}^{2N}} \frac{(u^*(x) - u^*(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}}dxdy, \quad \text{as } n \to \infty. \quad (4.12)$$

Since $\{u_n\}$ is bounded, by (4.5) we have that

$$\langle J'(u_n), u_n \rangle = \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}}dxdy - \int_\Omega g(x, u_n(x))u_n(x)dx$$

$$\to 0, \quad \text{as } n \to \infty. \quad (4.13)$$

Consequently, recalling also (4.10), we deduce that

$$\lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}}dxdy = \int_\Omega g(x, u^*(x))u^*(x)dx \quad (4.14)$$

Furthermore, using (4.5) again, we have

$$\langle J'(u_n), u^* \rangle = \int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(u^*(x) - u^*(y))}{|x - y|^{N+2s}}dxdy$$

$$- \int_\Omega g(x, u_n(x))u^*(x)dx \to 0, \quad \text{as } n \to \infty. \quad (4.15)$$

By (4.11), (4.12), (4.14) and (4.15) we obtain

$$\int_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^2}{|x - y|^{N+2s}}dxdy = \int_\Omega g(x, u^*(x))u^*(x)dx. \quad (4.16)$$

Thus, (4.14) and (4.16) give that

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}}dxdy \to \int_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^2}{|x - y|^{N+2s}}dxdy \quad \text{as } n \to \infty.$$
That is
\[ \lim_{n \to \infty} \|u_n\|_{H^s_0(\Omega)} = \|u^*\|_{H^s_0(\Omega)}. \quad (4.17) \]

Finally we have that
\[ \|u_n - u^*\|_{H^s_0(\Omega)} \]
\[ = \|u_n\|_{H^s_0(\Omega)} + \|u^*\|_{H^s_0(\Omega)} - 2 \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(u^*(x) - u^*(y))}{|x - y|^{N+2s}} \, dx \, dy \]
\[ \to 2\|u^*\|_{H^s_0(\Omega)} - 2 \int_{\mathbb{R}^N} \frac{|u^*(x) - u^*(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0 \]
as \( n \to \infty \), thanks to (4.12) and (4.17). This proves (4.6) and the proof is complete. \( \square \)

**Lemma 4.2.** Assume \((f^\pm)\). Then the functional \(J\) defined by (4.1) satisfies the Palais-Smale condition.

**Proof.** Let the sequence \(\{u_n\} \subset H^s_0(\Omega)\) be such that
\[ J'(u_n) \to 0, \quad n \to \infty. \quad (4.18) \]
By Lemma 4.1, we only need to prove that \(\{u_n\}\) is bounded in \(H^s_0(\Omega)\). Suppose, by the way of contradiction, that
\[ \|u_n\|_{H^s_0(\Omega)} \to \infty \text{ as } n \to \infty. \quad (4.19) \]
Write \(u_n = v_n + w_n\), where \(v_n \in V_\infty\) and \(w_n \in W_\infty\). By (2.15) and (2.16), we have
\[ |\langle A_\ell w_n, w_n \rangle| \geq \vartheta \|w_n\|^2_{H^s_0(\Omega)}, \quad \forall n \in \mathbb{N}, \quad (4.20) \]
where
\[ \vartheta = \min \left\{ 1 - \frac{\lambda_{\ell}}{\lambda_{\ell+\nu}}, \frac{\lambda_{\ell}}{\lambda_{\ell-1}} - 1 \right\}. \]
By (4.18), there is \(N_1 \in \mathbb{N}\) such that
\[ |\langle J'(u_n), w_n \rangle| \leq \|w_n\|_{H^s_0(\Omega)}, \quad \forall n \geq N_1. \quad (4.21) \]
By (4.4), we have
\[ \langle J'(u_n), w_n \rangle = \langle A_\ell w_n, w_n \rangle + \langle F'(u_n), w_n \rangle, \quad (4.22) \]
By (4.3) and (4.19) we have
\[ \|F'(u_n)\| = o(\|u_n\|_{H^s_0(\Omega)}), \quad n \to \infty. \quad (4.23) \]
It follows that for any given \(\varepsilon > 0\) and all \(n\) sufficiently large,
\[ \vartheta \|w_n\|^2_{H^s_0(\Omega)} \leq |\langle A_\ell w_n, w_n \rangle| \]
\[ \leq |\langle J'(u_n), w_n \rangle| + |\langle F'(u_n), w_n \rangle| \]
\[ \leq \|w_n\|_{H^s_0(\Omega)} \|w_n\|_{H^s_0(\Omega)} + \varepsilon \|u_n\|_{H^s_0(\Omega)}^2 \|w_n\|_{H^s_0(\Omega)}. \quad (4.24) \]
Since \(\varepsilon > 0\) was chosen arbitrarily, from (4.19) and (4.24) we deduce that
\[ \frac{\|w_n\|_{H^s_0(\Omega)}}{\|u_n\|_{H^s_0(\Omega)}} \to 0 \text{ as } n \to \infty. \quad (4.25) \]
It follows that there is \(N_2 \in \mathbb{N}\) with \(N_2 \geq N_1\) such that
\[ \|u_n\|_{H^s_0(\Omega)} \geq M \quad \text{and} \quad \|w_n\|_{H^s_0(\Omega)} \leq \varepsilon \|u_n\|_{H^s_0(\Omega)} \quad \text{for } n \geq N_2, \quad (4.26) \]
where $M > 0$ and $\epsilon \in (0, 1)$ was given in Lemma 3.3. Therefore by Lemma 3.3 we have that

$$\pm \int_{\Omega} f(x, u_n)v_n dx \geq \beta \|v_n\|_{H_0^1(\Omega)}^{1+\tau} \quad \text{for} \quad n \geq N_2. \quad (4.27)$$

On the other hand, by (4.18), we get

$$\lim_{n \to \infty} \left| \int_{\Omega} f(x, u_n) \frac{v_n}{\|v_n\|_{H_0^1(\Omega)}} dx \right| = \lim_{n \to \infty} \left| \langle J'(u_n), \frac{v_n}{\|v_n\|_{H_0^1(\Omega)}} \rangle \right| = 0. \quad (4.28)$$

This contradicts (4.27). Hence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. The proof is complete. \(\square\)

Now we are ready to give the proofs of Theorems 1.1–1.3. We denote

$$\mu_t := \dim W_{-t}, \quad \nu_t := \dim V_{-t}, \quad \mu_m := \dim W_0^+, \quad \nu_m := \dim V_0. \quad (4.29)$$

**Proof of Theorem 1.1.** We give the proof for the case (i). By Lemma 4.2, the $C^2$ functional $J$ satisfies the (PS) condition.

At infinity, $J$ takes the form (4.4) and fits the framework of Proposition 2.5. Since

$$\langle J'(u), v \rangle = -\int_{\Omega} f(x, u)v dx, \quad \forall \, v \in V_\infty,$$

it follows from $(f^+)$ and Lemma 3.3 that $J$ satisfies the angle condition $(AC)$ in Proposition 2.5 at infinity with respect to $H_0^1(\Omega) = V_{\infty} \oplus W_{-t}$. Thus by Proposition 2.5(ii) we have

$$C_q(J, \infty) \cong \delta_{q, \mu_t + \nu_m} Z, \quad q \in \mathbb{Z}, \quad (4.30)$$

Therefore $J$ has a critical point $u_*$ satisfying

$$C_{\mu_t + \nu_m}(J, u_*) \neq 0. \quad (4.31)$$

By (1.6) and (2.11) we see that

$$\langle J''(0)v, w \rangle = \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy - \lambda_m \int_{\Omega} vw dx, \quad (4.32)$$

Therefore by (2.13), near zero $J$ takes the form

$$J(u) = \frac{1}{2} \langle A_m u, u \rangle - \int_{\Omega} F_0(x, u) dx, \quad u \in H_0^1(\Omega) \quad (4.33)$$

where $F_0(x, t) = \int_0^t f_0(x, \xi) d\xi$. Moreover, by (1.6) we see that $0$ is a degenerate critical point of $J$ with Morse index $\mu_m$ and the nullity $\nu_m$. By Gromoll-Meyer result we have that

$$C_q(J, 0) \cong 0, \quad \text{for} \quad q \notin [\mu_m, \mu_m + \nu_m]. \quad (4.34)$$

By assumption, there is an eigenvalue $\hat{\lambda}$ of (1.7) such that $\lambda_t < \hat{\lambda} < \lambda_m$, it follows that $\mu_t + \nu_t < \mu_m$, we see from (4.31) and (4.34) that $u_* \neq 0$. The proof is complete. \(\square\)

**Proof of Theorem 1.2.** We prove the case (i). With the same arguments, we deduce from $(f^-)$ that $J$ satisfies the angle condition $(AC)$ in Proposition 2.5 at infinity with respect to $H_0^1(\Omega) = V_{\infty} \oplus W_\infty$. Thus by Proposition 2.5(ii) we have

$$C_q(J, \infty) \cong \delta_{q, \mu_t} Z, \quad q \in \mathbb{Z}. \quad (4.35)$$

Therefore $J$ has a critical point $u_*$ satisfying

$$C_{\mu_t}(J, u_*) \neq 0. \quad (4.36)$$
Since \((f_0^+)\) implies (1.6), \(\mathcal{J}\) has \(0\) as a degenerate critical point of with Morse index \(\mu_m\) and the nullity \(\nu_m\). Furthermore, we have

\[
(\mathcal{J}'(u), v) = -\int_{\Omega} f_0(x,u)vdx, \quad v \in V_0.
\]

It follows from \((f_0^+)\) and Lemma 3.4 that \(\mathcal{J}\) satisfies the angle condition \((\mathcal{AC}_0^-)\) in Proposition 2.6 at zero with respect to \(H^s_0(\Omega) = V_0 \oplus W_0\). Hence

\[
C_q(\mathcal{J}, 0) \cong \delta_{q, \mu_m + \nu_m}Z, \quad q \in \mathbb{Z}.
\]  \hspace{1cm} (4.37)

Keeping in mind the definitions of those indices \(\mu_*\) and \(\nu_*\) and in particular, if \(\lambda_\ell = \lambda_m\) then \(\mu_\ell = \mu_m\) and \(\nu_\ell = \nu_m\). From \(\lambda_\ell - 1 < \lambda_\ell\) and \(\lambda_\ell - 1 \neq \lambda_m\) we deduce that \(\mu_\ell \neq \mu_m + \nu_m\). We obtain \(u_* \neq 0\) from (4.36) and (4.37). The proof is complete. \(\square\)

**Proof of Theorem 1.3.** In the case \(\lambda_\ell = \lambda_m = \lambda_1\), we have that \(\mu_\ell = \mu_m = 0\) and \(\nu_\ell = \nu_m = 1\). With the same arguments, we deduce from \((f^-)\) and Proposition 2.5(ii) that

\[
C_q(\mathcal{J}, \infty) \cong \delta_{q,0}Z, \quad q \in \mathbb{Z}.
\]  \hspace{1cm} (4.39)

Therefore \(\mathcal{J}\) has a critical point \(u_*\) satisfying

\[
C_0(\mathcal{J}, u_*) \neq 0.
\]  \hspace{1cm} (4.40)

It follows that ([12])

\[
C_q(\mathcal{J}, u_*) \cong \delta_{q,0}Z.
\]  \hspace{1cm} (4.41)

From \((f_0^+)\) and Proposition 2.6 we deduce that

\[
C_q(\mathcal{J}, 0) \cong \delta_{q,1}Z, \quad q \in \mathbb{Z}.
\]  \hspace{1cm} (4.42)

The second nontrivial critical point is obtained by (4.39), (4.41), (4.42) and the Morse equality (2.20). The proof is complete. \(\square\)

We conclude the paper with some comments, comparisons and remarks.

**Remark 4.3.** (1) The result for one nontrivial solution in this paper is valid without further conditions on \(f\) near infinity when the limit in (1.5) is not an eigenvalue of (1.7). In Theorem 1.3, if \((f^-)\) and \((f_0^+)\) are replaced with \((f^+)\) and \((f_0^-)\), then one nontrivial critical point \(u^*\) can be obtained with

\[
C_1(\mathcal{J}, u^*) \neq 0.
\]  \hspace{1cm} (4.43)

This means that \(u^*\) is a mountain pass point of \(\mathcal{J}\). In the classical semilinear Laplacian (1.10), from (4.43) one can deduce that \(C_q(\mathcal{J}, u^*) \cong \delta_{q,1}Z\). However, in the nonlocal fractional Laplacian setting, this is left open, to the best of our knowledge. Due to the nonlocal nature of the problem, many multiplicity results for (1.10) with resonance may not be easily extended to the nonlocal case. We will focus on the multiplicity results for the problem (1.1) in near future.

(2) The assumption (1.8) is necessary in this paper and can not be cancelled in the present setting. We note that (1.8) is always valid in the classical Laplacian \((-\Delta)\) which may be regarded as deducing from \((-\Delta)^s\) to \(s = 1\). The problem whether or not (1.8) is valid for each eigenvalue of \((-\Delta)^s\) is still open and is very interesting. One refers to [16] for some explanations.

(3) In this paper we have been working on the functional space \(H^s_0(\Omega)\). We would like to point out that all the results and the arguments above are valid if one
wants to work on a more general functional space in which the function \(|x|^{-(N+2s)}\) is replaced by a function \(K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)\) with properties
\[
\begin{cases}
    mK \in L^1(\mathbb{R}^N) \quad \text{with} \\
    \quad m(x) = \min\{|x|^2, 1\}, \quad \text{and there exists} \quad \vartheta > 0
\end{cases}
\]
\[
\text{such that} \quad K(x) \geq \vartheta |x|^{-(N+2s)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\},
\]
as what have done in [7, 8, 14, 15, 16, 23, 26, 27, 29, 30, 32] and some references therein.

REFERENCES

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal., 14 (1973), 349–381.
[2] M. Badiale and E. Serra, Semilinear Elliptic Equations for Beginners, Springer, Berlin, 2011.
[3] B. Barrios, E. Colorado, A. De Pablo and U. Sanchez, On some critical problems for the fractional Laplacian operator, J. Differential Equations, 252 (2012), 6133–6162.
[4] B. Barrios, E. Colorado, R. Servadei and F. Soria, A critical fractional equation with concave-convex power nonlinearities, Ann. Inst. H. Poincaré Anal. Non Linéaire, 32 (2015), 875–900.
[5] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to nonlinear problems with “strong” resonance at infinity, Nonlinear Anal., 7 (1983), 981–1012.
[6] T. Bartsch and S. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, Nonlinear Anal., 28 (1997), 419–441.
[7] G. M. Bisci and R. Servadei, A Brezis-Nirenberg splitting approach for nonlocal fractional equations, Nonlinear Anal., 119 (2015), 341–353.
[8] G. M. Bisci, D. Mugnai and R. Servadei, On multiple solutions for nonlocal fractional problems via \(\nabla\)-theorems, arXiv:1510.08701.
[9] H. Brezis, Analyse fonctionelle, Theorie et applications, Masson, Paris, 1983.
[10] X. Cabré and J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Adv. Math., 224 (2010), 2052–2093.
[11] A. Capella, Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains, Commun. Pure Appl. Anal., 10 (2011), 1645–1662.
[12] K. C. Chang, Infinite Dimensional Morse Theory and Multiple Solutions Problems, Birkhauser, Boston, 1993.
[13] D. G. De Figueiredo and J. P. Gossez, Strict monotonicity of eigenvalues and unique continuation, Comm. Partial Differential Equations, 17 (1992), 339–346.
[14] A. Fiscella, R. Servadei and E. Valdinoci, A resonance problem for non-Local elliptic operators, Zeitschrift für Analysis und ihre Anwendungen, 32 (2013), 411–431.
[15] A. Fiscella, Saddle point solutions for nonlocal elliptic operators, arxiv:1210.8401.
[16] A. Fiscella, R. Servadei and E. Valdinoci, Asymptotically linear problems driven by fractional operators, Math. Methods Appl. Sci., to appear.
[17] D. Gromoll and M. Meyer, On differential functions with isolated point, Topology, 8 (1969), 361–369.
[18] D. Jerison and C. E. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math., 121 (1985), 463–494.
[19] E. M. Landesman and A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech., 19 (1970), 609–623.
[20] S. Li and J. Su, Existence of multiple solutions of a two-point boundary value problem at resonance, Topo. Meth. Nonl. Anal., 10 (1997), 123–135.
[21] S. Li and J. Su, Existence of multiple critical points for asymptotically quadratic functional with applications, Abst. Appl. Anal., 1 (1996), 283–305.
[22] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, 1989.
[23] D. Mugnai and D. Pagliardini, Existence and multiplicity results for the fractional Laplacian in bounded domains, Adv. Calc. Var., to appear.
[24] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), 521–573.
[25] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., 65, American Mathematical Society, Providence, RI 1986.
[26] R. Servadei, A critical fractional Laplace equation in the resonant case, *Topol. Methods Nonlinear Anal.*, 43 (2014), 251–267.

[27] R. Servadei and E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, 389 (2012), 887–898.

[28] R. Servadei and E. Valdinoci, Lewy-Stampacchia type estimates for variational inequalities driven by (non) local operators, *Rev. Mat. Iberoam.*, 29 (2013), 1091–1126.

[29] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.*, 12 (2013), 2445–2464.

[30] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.*, 33 (2013), 2105–2137.

[31] R. Servadei and E. Valdinoci, Weak and viscosity solutions of the fractional Laplace equation, *Publ. Mat.*, 58 (2014), 133–154.

[32] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, 367 (2015), 67–102.

[33] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, 60 (2007), 67–112.

[34] J. Su, Nontrivial periodic solutions for the asymptotically linear Hamiltonian systems with resonance at infinity, *J. Differential Equations*, 145 (1998), 252–273.

[35] J. Su and C. Tang, Multiple results for semilinear elliptic equations with resonance at higher eigenvalues, *Nonlinear Anal.*, 44 (2001), 311–321.

[36] J. Su and L. Zhao, An elliptic resonance problem with multiple solutions, *J. Math. Appl. Appl.*, 319 (2006), 604–616.

[37] Z.-Q. Wang, Multiple solutions for indefinite functionals and applications to asymptotically linear problems, *Acta Math. Sinica(N.S).*., 5 (1989), 101–113.

Received February 2016; revised August 2016.

E-mail address: chenyutong@cnu.edu.cn
E-mail address: sujb@cnu.edu.cn