A property of the interleaving distance for sheaves

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Abstract

Let $X$ be a real analytic manifold endowed with a distance satisfying suitable properties and let $k$ be a field. In [PS20], the authors construct a pseudo-distance on the derived category of sheaves of $k$-modules on $X$, generalizing a previous construction of [KS18]. We prove here that if the distance between two constructible sheaves with compact support (or more generally, constructible sheaves up to infinity) on $X$ is zero, then these two sheaves are isomorphic. This answers in particular a question of [KS18].

1 Introduction

The interleaving distance was introduced in [CCSG+] and provides a pseudo-metric on the category of persistent modules. It was generalized to multi-persistence modules by M. Lesnick in [Les12, Les15]. In his thesis [Cur14], J. Curry showed how to interpret the notion of persistent modules in the classical language of sheaves. This allows one to interpret the interleaving pseudo-distance as a pseudo-metric on the category of $\gamma$-sheaves on a finite-dimensional real vector space where $\gamma$ is a convex proper cone. In [KS18], M. Kashiwara and P. Schapira systematically treat persistent homology in the framework of derived sheaf theory. In particular, they introduce the so-called convolution pseudo-distance for derived sheaves on real normed vector spaces. There, they asked if this pseudo-distance is a distance [KS18, Rem. 2.3], that is if two sheaves with convolution distance zero are necessarily isomorphic. A similar question had been studied by Lesnick in [Les15] where he proved that the interleaving pseudo-distance restricted to finitely presented persistent modules is a distance. This result is related to the aforementioned question as it follows from [BP21] that the restriction of the convolution pseudo-distance to the subcategory of $\gamma$-sheaves is equal to the interleaving pseudo-distance. In [BG18], N. Berkouk and G. Ginot proved that the convolution pseudo-distance is in general not a distance and established, by constructing an appropriate matching distance, that it is a distance on constructible sheaves on $\mathbb{R}$. Here,
following [PS20], we consider a generalization of the convolution pseudo-distance for sheaves on “good” metric spaces (as for instance complete Riemannian manifolds with strictly positive convexity radius). We prove that on a real analytic manifold endowed with a “good” distance, the pseudo-distance on sheaves is a distance when restricted to constructible sheaves with compact support (or more generally, constructible sheaves up to infinity).

2 Review

Throughout this paper, \( k \) is a field. We shall mainly follow the notations of [KS90] for sheaf theory. For a topological space \( X \) we denote by \( \Delta \) the diagonal of \( X \times X \) and by \( a_X \) the map \( X \to \text{pt} \). We denote by \( \text{D}^b(k_X) \) the bounded derived category of sheaves of \( k \)-modules. If \( X \) is a real analytic manifold, we denote by \( \text{D}^b_{\mathbb{R}c}(k_X) \) the full triangulated subcategory of \( \text{D}^b(k_X) \) consisting of \( \mathbb{R} \)-constructible sheaves. We denote by \( \omega_X \) the dualizing complex.

2.1 Kernels

Recall that a topological space \( X \) is good if it is Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

Given topological spaces \( X_i \)’s, we set \( X_{ij} = X_i \times X_j \), \( X_{123} = X_1 \times X_2 \times X_3 \). We denote by \( q_{ij}: X_{ij} \to X_i \) and \( q_{ij}: X_{123} \to X_{ij} \) the projections.

For \( A_{ij} \subset X_{ij} \) with \( i = 1, 2, j = i + 1 \), one defines \( A_{12} \circ A_{23} \subset X_{13} \) as

\[
A_{12} \circ A_{23} := q_{13}(q_{12}^{-1}A_{12} \cap q_{23}^{-1}A_{23}).
\]

(2.1)

For good topological spaces \( X_i \)’s, one often calls an object \( K_{ij} \in \text{D}^b(k_{X_{ij}}) \) a kernel. One defines as usual the composition of kernels by

\[
K_{12} \circ K_{23} := Rq_{13}(q_{12}^{-1}K_{12} \otimes q_{23}^{-1}K_{23}).
\]

(2.2)

If there is no risk of confusion, we write \( \circ \) instead of \( \circ_2 \).

2.2 Distances

For a metric space \((X, d), x_0 \in X \) and \( a \in \mathbb{R}_{\geq 0} \), we set

\[
B_a(x_0) = \{x \in X; d(x_0, x) \leq a\}, \quad B^a(x_0) = \{x \in X; d(x_0, x) < a\}
\]

\[
\Delta_a = \{(x, y) \in X \times X; d(x, y) \leq a\}, \quad \Delta^a = \{(x, y) \in X \times X; d(x, y) < a\},
\]

\[
\Delta^+ = \{(x, y, t) \in X \times X \times \mathbb{R}; d(x, y) \leq t\}, \quad \Delta^{+,c} = \{(x, y, t) \in X \times X \times \mathbb{R}; d(x, y) < t\}.
\]

Following [PS20], we say that a metric space \((X, d)\) is good, or simply that the
distance is good, if the underlying topological space is good and moreover

\[
\begin{aligned}
&\text{there exists some } \alpha_X > 0 \text{ such that for all } 0 \leq a, b \leq \alpha_X: \\
&(i) \text{ for any } x_1, x_2 \in X, \ B_a(x_1) \cap B_b(x_2) \text{ is contractible or empty (in}
\text{ particular, for any } x \in X, \ B_a(x) \text{ is contractible}), \\
&(ii) \text{ the two projections } q_1 \text{ and } q_2 \text{ are proper on } \Delta_a, \\
&(iii) \Delta_a \circ \Delta_b = \Delta_{a+b}.
\end{aligned}
\]

(2.3)

In order to construct a distance on sheaves in this situation, the main idea of [PS20] is to use the kernel \( k_{\Delta_a} \) when \( 0 \leq a < \alpha_X \) and to replace it by a composition of kernels \( k_{\Delta_b} \) with \( 0 \leq b < \alpha_X \) otherwise.

We shall also consider the hypotheses (2.4) below which insure that the kernels \( k_{\Delta_a} \) are invertible (see [PS20, Prop. 2.2.3]).

Let \( U \) be an open subset of a real \( C^0 \)-manifold \( M \). We say that \( U \) is locally topologically convex (l.t.c. for short) in \( M \) if each \( x \in M \) admits an open neighborhood \( W \) such that there exists a topological isomorphism \( \phi: W \to V \), with \( V \) open in a real vector space, such that \( \varphi(W \cap U) \) is convex. Clearly, if \( U \) is l.t.c. then it is l.c.t.

\[
\begin{aligned}
&\text{The good metric space } X \text{ is a } C^0\text{-manifold and } \\
&(a) \text{ for } 0 < a \leq \alpha_X, \text{ the set } \Delta_a^\circ \text{ is l.t.c. in } X \times X, \\
&(b) \text{ the set } \Delta^{+, \circ} \text{ is l.t.c. in } X \times X \times -\infty, \alpha_X, \\
&(c) \text{ For } x, y \in X, \text{ setting } Z_a(x, y) = B_a(x) \cap B_a^\circ(y), \text{ one has } \\
&\text{RG}(X, k_{Z_a(x,y)}) \simeq 0 \text{ for } x \neq y \text{ and } 0 < a \leq \alpha_X.
\end{aligned}
\]

(2.4)

In loc. cit. it is shown that complete Riemannian manifolds with strictly positive convexity radius as well as normed vector spaces satisfy conditions (2.3) and (2.4). Moreover, given a good metric space \( (X, d) \), one can naturally associate a pseudo-distance on the objects of the derived category \( D^b(k_X) \), generalizing the convolution distance first introduced in [KS18].

\[ \text{2.3 Constructible sheaves} \]

Here, we work in the framework of real analytic manifolds and use the notions of being subanalytic or constructible “up to infinity”, following [Sch20].

Recall that a b-analytic manifold \( X_\infty = (X, \hat{X}) \) is the data of real analytic manifold \( \hat{X} \) and a subanalytic open relatively compact subset \( X \subset \hat{X} \). One denotes by \( j_X : X \hookrightarrow \hat{X} \) the embedding. A morphism of b-analytic manifolds is a morphism of real analytic manifolds \( f : X \to Y \) whose graph is subanalytic in \( \hat{X} \times \hat{Y} \).

One defines naturally the full triangulated subcategory \( D^b_{\text{rc}}(k_{X_\infty}) \) of \( D^b_{\text{rc}}(k_X) \) consisting of sheaves constructible up to infinity. These are the objects \( F \) of \( D^b_{\text{rc}}(k_X) \) such that \( j_{X!} F \in D^b_{\text{rc}}(k_{\hat{X}}) \). The advantage of the notion of being constructible up to infinity is that it is stable under the six operations, in particular, by direct images of morphisms of b-analytic manifolds [Sch20].
In the sequel, instead of writing “subanalytic up to infinity” or “constructible up to infinity”, we shall write “b-subanalytic” or “b-constructible”.

Let \( F \in D^b_{\mathcal{R}_c}(k_X) \). Recall that \( F \) is b-constructible if \( j_X ! F \in D^b_{\mathcal{R}_c}(k_X) \). One denotes by \( D^b_{\mathcal{R}_c}(k_X) \) the full triangulated subcategory of \( D^b_{\mathcal{R}_c}(k_X) \) consisting of sheaves b-constructible.

### 3 Main theorem

In this section, \( X_\infty \) is a b-analytic manifold endowed with a good distance.

Recall the definition of being \( a \)-isomorphic and the associated pseudo-distance \( \text{dist} \), following \([KS18]\) generalized in \([PS20]\).

**Definition 3.1.** Let \( F, G \in D^b(k_X) \) and let \( a > 0 \). One says that \( F \) and \( G \) are \( a \)-isomorphic if there exist morphisms \( u_a : F \circ k_{\Delta_a} \rightarrow G \), \( v_a : G \circ k_{\Delta_a} \rightarrow F \) such that \( F \circ k_{\Delta_2a} \rightarrow G \circ k_{\Delta_a} \rightarrow F \) and \( G \circ k_{\Delta_2a} \rightarrow F \circ k_{\Delta_a} \rightarrow G \) are the natural morphisms associated with \( k_{\Delta_2a} \rightarrow k_\Delta \).

One sets

\[
\text{dist}(F, G) = \inf\{a \in [0, +\infty]; F \text{ and } G \text{ are } a \text{-isomorphic}\}.
\]

In the sequel, when considering thickenings of the diagonal \( \Delta_a \), we shall assume \( 0 \leq a < \alpha_X \), where \( \alpha_X \) is given in (2.3). For \( 0 \leq a \leq b < \alpha_X \), the morphism

\[
k_{\Delta_b} \rightarrow k_{\Delta_a}
\]

define the morphisms

\[
(3.1) \quad \text{RHom}(F \circ k_{\Delta_a}, G) \rightarrow \text{RHom}(F \circ k_{\Delta_b}, G),
\]

\[
(3.2) \quad \text{RHom}(F, G \circ k_{\Delta_a}) \rightarrow \text{RHom}(F, G \circ k_{\Delta_b}).
\]

One of the central step of the proof is to establish that the morphisms (3.1) and (3.2) are isomorphisms. For that purpose, we will prove that these morphisms spaces can be written as the cohomology of a constructible sheaf on \( \mathbb{R} \) supported by balls of radius \( a \) and \( b \). The constructibility will imply that, for \( 0 \leq a \leq b \) sufficiently small, these cohomology groups are isomorphic which implies the result.

Denote by \( q_1 : X \times \mathbb{R} \rightarrow X \) and \( q_2 : X \times \mathbb{R} \rightarrow \mathbb{R} \) the projections. Also denote by \( I_a \) the closed interval \([-a, a]\) of \( \mathbb{R} \) and by \( I_a^o \) the open interval.

**Lemma 3.2.** Let \( F, G \in D^b_{\mathcal{R}_c}(k_{X_\infty}) \).

(i) For \( a \geq 0 \) one has,

\[
\text{R}\Gamma_{I_a} (\mathbb{R}; Rq_2_\ast \mathcal{H}\text{om}(F \circ k_{\Delta_a^+}, q_1^! G)) \sim \text{RHom}(F \circ k_{\Delta_a}, G).
\]

(ii) For \( a > 0 \) one has,

\[
\text{R}\Gamma(I_a^o; Rq_2_\ast \mathcal{H}\text{om}(F \circ k_{\Delta_a^+}, q_1^! G)) \sim \text{RHom}(F \circ k_{\Delta_2^o}, G)[1].
\]
Proof. (i) Set \( K := \mathcal{R}	ext{Hom}(F \circ k_{\Delta^+}, q_1^!G) \). Then
\[
\mathcal{R} \Gamma I_a \mathcal{R} q_{2*} K \simeq \mathcal{R} q_{2*} \mathcal{R} \Gamma q_{2*}^{-1} I_a K
\]
\[
\simeq \mathcal{R} q_{2*} \mathcal{R}	ext{Hom} \left( (F \circ k_{\Delta^+}) \otimes k_{q_{2}^{-1} I_a}, q_1^! G \right) .
\]
Therefore,
\[
\mathcal{R} \Gamma I_a (\mathbb{R}; \mathcal{R} q_{2*} K) \simeq \mathcal{R} a X_*, \mathcal{R} q_{1*} \mathcal{R} \text{Hom} \left( (F \circ k_{\Delta^+}) \otimes k_{q_{2}^{-1} I_a}, q_1^! G \right) 
\]
\[
\simeq \mathcal{R} a X_*, \mathcal{R}	ext{Hom} \left( \mathcal{R} q_{1!} \left( (F \circ k_{\Delta^+}) \otimes k_{q_{2}^{-1} I_a} \right), G \right)
\]
To conclude, let us check the isomorphism
\[
(3.3) \quad \mathcal{R} q_{1!} \left( (F \circ k_{\Delta^+}) \otimes k_{q_{2}^{-1} I_a} \right) \simeq F \circ k_{\Delta_a}.
\]
Consider the diagram
\[
\begin{array}{ccc}
X \times X \times \mathbb{R} & \xrightarrow{p_{12}} & X \times X \\
\downarrow p_{23} & & \downarrow q_1 \\
X \times \mathbb{R} & \xrightarrow{p_1} & X \\
\downarrow r_1 & & \downarrow q_2 \\
X & \xrightarrow{r_2} & X
\end{array}
\]
One has
\[
\mathcal{R} q_{1!} \left( (F \circ k_{\Delta^+}) \otimes k_{q_{2}^{-1} I_a} \right) \simeq \mathcal{R} q_{1!} \mathcal{R} p_{23!} \left( p_{1}^{-1} F \otimes k_{\Delta^+} \otimes p_{23}^{-1} k_{q_{2}^{-1} I_a} \right)
\]
\[
\simeq \mathcal{R} r_2! \mathcal{R} p_{12!} \left( p_{12}^{-1} F \otimes k_{\Delta^+ \cap \gamma_{p_{23} q_2^{-1} I_a}} \right)
\]
\[
\simeq \mathcal{R} r_2! \left( p_{12}^{-1} F \otimes \mathcal{R} p_{12!} k_{\Delta^+ \cap \gamma_{p_{23} q_2^{-1} I_a}} \right).
\]
We now remark that \( \Delta^+ \cap p_{23}^{-1} q_2^{-1} I_a \subset p_{12}^{-1}(\Delta_a) \) and \( p_{12} \) restricted to \( \Delta^+ \cap p_{23}^{-1} q_2^{-1} I_a \) is proper. Hence, there are natural morphisms (the first morphism is obtained by adjunction)
\[
(3.4) \quad k_{\Delta_a} \rightarrow \mathcal{R} p_{12*} p_{12}^{-1} k_{\Delta_a} \rightarrow \mathcal{R} p_{12*} k_{\Delta^+ \cap \gamma_{p_{23} q_2^{-1} I_a}} \xleftarrow{\sim} \mathcal{R} p_{12!} k_{\Delta^+ \cap \gamma_{p_{23} q_2^{-1} I_a}}
\]
which define
\[
(3.5) \quad k_{\Delta_a} \rightarrow \mathcal{R} p_{12!} k_{\Delta^+ \cap \gamma_{p_{23} q_2^{-1} I_a}}.
\]
It remains to prove that this last morphism is an isomorphism. One has \( p_{12}(\Delta^+ \cap p_{23}^{-1} q_2^{-1} I_a) = \Delta_a \) and the fibers of \( p_{12} \) above \((x, y) \in X \times X\) is the interval \([d(x, y), a] \) which is contractible or empty. This, together with [KS90, Prop. 2.5.2], proves that (3.5) is an isomorphism, hence proves (3.3).
(ii) Replacing \( I_a \) with \( I_a^\circ \) and \( \Delta^+ \) with \( \Delta^{+, \circ} \) in the proof of (i) it remains to show
\[
(3.6) \quad \mathcal{R} q_{1!} \left( (F \circ k_{\Delta^{+, \circ}}) \otimes k_{q_2^{-1} I_a^\circ} \right) \simeq F \circ k_{\Delta_a^\circ}[-1].
\]
By the same proof as for (3.3) we have reduced to showing:

\[(3.7) \quad Rp_{1,2}! k_{\Delta^+, \circ r_2^{-1} q_2^{-1}, F} \simeq k_{\Delta^+[-1]}.
\]

This isomorphism is deduced from (3.5) by duality. □

Lemma 3.3. Let \( F, G \in D^b_{\text{Rec}}(k_{X_\infty}) \) and let \( 0 \leq a < \alpha_X \). Then there exists \( c > a \) such that for \( a \leq b \leq c \), (3.1) and (3.2) are isomorphisms.

Proof. (i) Let us treat (3.1). Set \( H := Rq_{2*}R\mathcal{H}om(F \circ k_{\Delta^+}, q_1^* G) \). Since \( F, G \in D^b_{\text{Rec}}(k_{X_\infty}) \), then \( H \in D^b_{\text{Rec}}(k_R) \) by [Sch20, §2.] and therefore by [KS90, Lem. 8.4.7],

\[R\Gamma_{1,a}^\circ(R; H) \simeq R\Gamma_{1,b}^\circ(R; H)\]

for \( a \leq b \leq c \) for some \( c > a \). Applying Lemma 3.2 (i), we get the result.

(ii) Let us treat (3.2). First, using [PS20, Prop. 2.2.3], we obtain

\[R\text{Hom}(F, G \circ k_{\Delta_a}) \simeq R\text{Hom}(F \circ (k_{\Delta^+} \otimes r_2^{-1} \omega_X), G) \]

\[\simeq R\text{Hom}((F \otimes \omega_X) \circ k_{\Delta^+}, G)\]

Second, set \( H := Rq_{2*}R\mathcal{H}om((F \otimes \omega_X) \circ k_{\Delta^+}, q_1^* G) \). Since \( F, G \in D^b_{\text{Rec}}(k_{X_\infty}) \), then \( H \in D^b_{\text{Rec}}(k_R) \) and therefore by [KS90, Lem. 8.4.7]

\[R\Gamma(I_a^b; H) \simeq R\Gamma(I_a^b; H)\]

for \( a \leq b \leq c \) for some \( c > a \). Applying Lemma 3.2 (ii), we get the result. □

Theorem 3.4. Let \( X_\infty \) be a \( b \)-analytic manifold endowed with a good distance and satisfying (2.4). Let \( F, G \in D^b_{\text{Rec}}(k_{X_\infty}) \). If \( \text{dist}(F, G) \leq a \) with \( 0 \leq a < \alpha_X \), then \( F \) and \( G \) are \( a \)-isomorphic. In particular, if \( \text{dist}(F, G) = 0 \), then \( F \) and \( G \) are isomorphic.

Proof. The proof proceeds in four steps. Let \( 0 \leq a \leq b \) with \( b \) small enough so that Lemma 3.3 holds.

(i) Consider Diagram (3.8) below

\[
\begin{array}{ccc}
\text{Hom}(F \circ k_{\Delta_{a+b}}, G \circ k_{\Delta_a}) \times \text{Hom}(G \circ k_{\Delta_a}, F) & \overset{\circ}{\longrightarrow} & \text{Hom}(F \circ k_{\Delta_{2a}}, F) \\
\downarrow & & \downarrow \\
\text{Hom}(F \circ k_{\Delta_{a+b}}, G \circ k_{\Delta_b}) \times \text{Hom}(G \circ k_{\Delta_b}, F) & \overset{\circ}{\longrightarrow} & \text{Hom}(F \circ k_{\Delta_{2b}}, F) \\
\downarrow & & \downarrow \\
\text{Hom}(F \circ k_{\Delta_{a+b}}, G \circ k_{\Delta_b}) \times \text{Hom}(G \circ k_{\Delta_b}, F) & \overset{\circ}{\longrightarrow} & \text{Hom}(F \circ k_{\Delta_{2b}}, F)
\end{array}
\]

(3.8)
Let us show that this diagram commutes.

Diagram ① is obtained by applying the functor \( k_{\Delta_b-a} \) to the first line.

Diagram ② is obtained by composing the second line with the canonical morphism \( F \circ k_{\Delta_b-a} \rightarrow F \).

Diagram ③ is obtained by composing the third line with the canonical morphism \( F \circ k_{\Delta_b} \rightarrow F \circ k_{\Delta_a+b} \).

Hence, Diagram (3.8) commutes.

(ii) Assume \( \text{dist}(F, G) \leq a \), let \( b > a \) and let \( u_b \in \text{Hom}(F \circ k_{\Delta_b}, G) \) and \( v_b \in \text{Hom}(G \circ k_{\Delta_b}, F) \) be as in Definition 3.1. Denote by \( \varepsilon_b \) the natural morphism associated with \( k_{\Delta_b} \rightarrow k_{\Delta} \), by \( \Psi_b : D^b(k_X) \rightarrow D^b(k_X) \) the functor \( L \mapsto L \circ k_{\Delta_b} \) and by \( \Psi_F : D^b(k_{X \times X}) \rightarrow D^b(k_X), \ K \mapsto F \circ K \). Then, the \( b \)-isomorphism equations are explicitly given by

\[
(3.9) \quad v_b \circ \Psi_b(u_b) = \Psi_F(\varepsilon_{2b}).
\]

(iii) The vertical arrows in Diagram ① are isomorphisms thanks to (2.4) (see [PS20, Prop. 2.2.3]). The vertical arrows in Diagram ② and ③ are isomorphisms thanks to Lemma 3.3.

(iv) Hence, there exist \( u_a \in \text{Hom}(F \circ k_{\Delta_a}, G) \) and \( v_a \in \text{Hom}(G \circ k_{\Delta_a}, F) \) whose images are the morphisms \( u_b \) and \( v_b \). Using the vertical isomorphism, equation (3.9) translates to the same equation with \( b \) replaced by \( a \). The same result holds with \( F \) and \( G \) interchanged. This complete the proof.

Corollary 3.5. Let \( X \) be a real analytic manifold endowed with a good distance and satisfying (2.4). Let \( F, G \in D^b_{\text{bc}}(k_X) \), both with compact support. Assume that \( \text{dist}(F, G) = 0 \). Then \( F \cong G \).

Proof. Let \( Y \) be an open subanalytic subset of \( X \) containing the supports of \( F \) and \( G \). Then regard \( Y_\infty = (Y, X) \) as a \( b \)-analytic manifold and apply Theorem 3.4.

Remark 3.6. When the space \( X \) is a finite dimensional real vector space endowed with a closed proper convex subanalytic cone \( \gamma \) with nonempty interior and a vector \( v \) in the interior of \( \gamma \), then, thanks to [BP21, Cor. 5.9], Theorem 3.4 implies the same results for \( \gamma \)-sheaves endowed with the interleaving distance associated with the pair \( (\gamma, v) \) (see [BP21, Def. 4.8]).

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