BOUNDS ON EMBEDDINGS OF RATIONAL HOMOLOGY BALLS IN SYMPLECTIC 4-MANIFOLDS

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Abstract. The rational homology balls $B_n$ appeared in Fintushel and Stern’s rational blow-down construction [FS2]. Later, Symington [Sy1], defined this operation in the symplectic category. In [Kh2], the author defined the inverse procedure, the symplectic rational blow-up. In this paper, we study the obstructions to symplectically rationally blowing up a symplectic 4-manifold, i.e. the obstructions to symplectically embedding the rational homology balls $B_n$ into a symplectic 4-manifold. We prove a theorem and give additional examples which suggest that in order to symplectically embed the rational homology balls $B_n$, for high $n$, a symplectic 4-manifold must at least have a high enough $c_1^2$ as well.

1. Introduction

In 1997, Fintushel and Stern [FS2] defined the rational blow-down operation for smooth 4-manifolds, a generalization of the standard blow-down operation. For smooth 4-manifolds, the standard blow-down is performed by removing a neighborhood of a sphere with self-intersection $(-1)$ and replacing it with a standard 4-ball $B^4$. The rational blow-down involves replacing a negative definite plumbing 4-manifold with a rational homology ball. In order to define it, we first begin with a description of the negative definite plumbing 4-manifold $C_n, n \geq 2$, as seen in Figure 1, where each dot represents a sphere, $S_i$, in the plumbing configuration. The integers above the dots are the self-intersection numbers of the plumbed spheres: $[S_1]^2 = -(n + 2)$ and $[S_i]^2 = -2$ for $2 \leq i \leq n - 1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{plumbing_diagram.png}
\caption{Plumbing diagram of $C_n, n \geq 2$}
\end{figure}

The boundary of $C_n$ is the lens space $L(n^2, n - 1)$, thus $\pi_1(\partial C_n) \cong H_1(\partial C_n; \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$. (Note, when we write the lens space $L(p, q)$, we
mean it is the 3-manifold obtained by performing \(-\frac{p}{q}\) surgery on the unknot.) This follows from the fact that \([-n-2, -2, \ldots, -2]\), with \((n-2)\) many \((-2)\)'s is the continued fraction expansion of \(\frac{n^2}{1-n}\). (Note, we will often abuse notation and write \(C_n\) both for the actual plumbing 4-manifold and the plumbing configuration of spheres in that 4-manifold.)

Let \(B_n\) be the 4-manifold as defined by the Kirby diagram in Figure 2 (for a more extensive description of \(B_n\), see section 2.1). The manifold \(B_n\) is a rational homology ball, i.e. \(H_*(B_n; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q})\). The boundary of \(B_n\) is also the lens space \(L(n^2, n-1)\) [CH]. Moreover, any self-diffeomorphism of \(\partial B_n\) extends to \(B_n\) [FS2]. Now, we can define the rational blow-down of a 4-manifold \(X\):

**Definition 1.1.** ([FS2], also see [GoSt]) Let \(X\) be a smooth 4-manifold. Assume that \(C_n\) embeds in \(X\), so that \(X = C_n \cup_{L(n^2, n-1)} X_0\). The 4-manifold \(X_{(n)} = B_n \cup_{L(n^2, n-1)} X_0\) is by definition the rational blow-down of \(X\) along the given copy of \(C_n\).

Fintushel and Stern [FS2] also showed how to compute Seiberg-Witten and Donaldson invariants of \(X_{(n)}\) from the respective invariants of \(X\). In 1998, Symington [Sy1] proved that the rational blow-down operation can be performed in the symplectic category. More precisely, she showed that if in a symplectic 4-manifold \((M, \omega)\) there is a symplectic embedding of a configuration \(C_n\) of symplectic spheres, then there exists a symplectic model for \(B_n\) such that the rational blow-down of \((M, \omega)\), along \(C_n\) is also a symplectic 4-manifold. In [Kh2], the author defined the symplectic rational blow-up operation, where the symplectic structure of \(B_n\) is presented as an entirely standard symplectic neighborhood of a certain Lagrangian 2-cell complex, enabling one to replace the \(B_n\) with \(C_n\) and obtain a new symplectic 4-manifold.

The main goal of this paper is to investigate the following question: **what are the obstructions to symplectically embedding the rational homology balls \(B_n\) into a symplectic 4-manifold?** Note, in [Kh1], the author showed that in the smooth category there is little obstruction to embedding a rational homology ball \(B_n\):
Theorem 1.2. \cite{Kh1} Let $V_{-4}$ be a neighborhood of a sphere with self-intersection number $(-4)$. For all $n \geq 3$ odd, there exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-4}$. For all $n \geq 2$ even, there exists an embedding of the rational homology balls $B_n \hookrightarrow B_2\#\mathbb{CP}^2$.

Theorem 1.2 above implies that if a smooth 4-manifold $X$ contains a sphere with self-intersection $(-4)$, then one can smoothly embed the rational homology balls $B_n \hookrightarrow X$ for all odd $n \geq 3$. One of the implications of this is that for a given smooth 4-manifold $X$, there does not exist an $N$, such that for all $n \geq N$ one cannot find a smooth embedding $B_n \hookrightarrow X$. In the setting of this sort in algebraic geometry, for rational homology ball smoothings of certain surface singularities, such a bound on $n$ does exist, in terms of $(c_1^2, \chi_h)$ invariants of an algebraic surface \cite{KSB, Wa}. Therefore, for the case of symplectic embeddings of the rational homology balls $B_n$, if we model our symplectic manifold such that it resembles a surface of general type, we can make the following conjecture:

Conjecture 1.3. Let $(X, \omega)$ be a symplectic 4-manifold, such that:

- $b_2^+(X) > 1$
- $[c_1(X, \omega)] = -[\omega]$ as cohomology classes,

then there exists an $N$, such that for all $n \geq N$ there does not exist a symplectic embedding $B_n \hookrightarrow (X, \omega)$.

The condition $[c_1(X, \omega)] = -[\omega]$, implies that $(X, \omega)$ does not contain any spheres of self-intersection $(-1)$ or $(-2)$ and $c_1^2(X, \omega) \geq 1$, resembling a surface of general type with an ample canonical divisor.

We prove a result (Theorem 1.4) that is a first step in proving the above conjecture. We observe that if we impose the condition $n \geq c_1^2(X, \omega) + 2$ on $(X, \omega)$, then if we symplectically rationally blow up a $B_n \hookrightarrow (X, \omega)$, we would obtain a symplectic manifold $(X', \omega')$ for which $c_1^2(X', \omega') \leq -1$. As a consequence of a theorem of Taubes \cite{Ta2, Ta4, Ta3}, we would then obtain, for a generic $\omega$-compatible almost-complex structure $J_\epsilon$, a $J_\epsilon$-holomorphic embedded sphere $\Sigma_{-1}$ with self-intersection $(-1)$. The consequences of the existence of such a sphere in the symplectic rational blow-up $(X', \omega')$ leads to various contradictions of adjunction formulas and results on Seiberg-Witten invariants.

We show that if $(X, \omega)$ is such that $n \geq c_1^2(X, \omega) + 2$ (in addition to the two conditions on $(X, \omega)$ in Conjecture 1.3), then a symplectic embedding $B_n \hookrightarrow (X, \omega)$ will fall into two types: $\mathcal{A}$ and $\mathcal{E}_k$, $2 \leq k \leq n - 1$, (see Definitions 3.2, 3.3, 3.4). The types $\mathcal{A}$ and $\mathcal{E}_k$ are determined by the intersection patterns of a sphere $\Sigma_{-1}$, with self-intersection $(-1)$ (obtained as consequence of the sphere $\Sigma_{-1}$), with the spheres of $C_n \subset (X', \omega')$. We then prove the following theorem:

Theorem 1.4. If $B_n \hookrightarrow (X, \omega)$ is a symplectic embedding, where $(X, \omega)$ is a symplectic 4-manifold, such that:

- $b_2^+(X) > 1$, 

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• \([c_1(X,\omega)] = -[\omega]\) as cohomology classes,
• \(n \geq c_1^2(X,\omega) + 2\) and
• \(\text{Bas}_X = \{\pm c_1(X,\omega)\}\), (\(\text{Bas}_X\) denotes the set of Seiberg-Witten basic classes of \(X\))

then it cannot be of type \(A\) or of type \(E_k\), \(k \geq c_1^2(X,\omega) + 2\).

Note, in Theorem 1.4 above, the condition \(\text{Bas}_X = \{\pm c_1(X,\omega)\}\) on \((X,\omega)\) is also true for surfaces of general type.

We also describe a family of symplectic manifolds, \(X\), constructed from the elliptic surfaces \(E(m)\), which contain an embedded \(B_n\) of type \(E_2\) (not covered by Theorem 1.4), in such a way that

\[
n < 3 + \frac{4}{3} c_1^2(X,\omega),
\]

for all \((X,\omega) \subset X\). Thus, also providing evidence for Conjecture 1.3 that every symplectic manifold has a bound on \(n\), above which one can no longer embed a rational homology ball \(B_n\). Both Theorem 1.4 and this family of examples suggest that in order for there to exist a symplectic embedding \(B_n \hookrightarrow (X,\omega)\) for high \(n\), the manifold \((X,\omega)\) needs to at least have a high enough \(c_1^2(X,\omega)\).

It is worthwhile to note, that obstructions to symplectically embedding the rational homology balls \(B_n\), was the subject of some recent research [LM]. Their obstructions arose from nonvanishing symplectic cohomology, however, their results do not depend on \(n\).

This paper is organized as follows. In section 2, we give some brief reviews: the structure of the rational homology balls \(B_n\) and the symplectic rational blow-up construction appearing in [Kh2]; Seiberg-Witten invariants and basic classes; and toric and almost-toric fibrations of symplectic 4-manifolds, which is used the proof of the main theorem.

In section 3, after separating the symplectic embeddings of \(B_n \hookrightarrow (X,\omega)\) into types \(A\) and \(E_k\), we prove the main theorem (Theorem 1.4). We prove Theorem 1.4 in four steps, by assuming that there exists a symplectic embedding \(B_n \hookrightarrow (X,\omega)\) and obtaining a contradiction. In Step 1, section 3.1, we show that symplectic embeddings of \(B_n\) will indeed be of type \(A\) or \(E_k\). In Step 2, section 3.2, we construct a cycle \(\gamma\) and compute \(c_1(X,\omega) \cdot \gamma\). In Step 3, section 3.3, we show that if \(c_1(X,\omega) \cdot \gamma > 0\) then \(\omega \cdot \gamma > 0\), contradicting the \([c_1(X,\omega)] = -[\omega]\) assumption. In Step 4, section 3.4, we show that if \(c_1(X,\omega) \cdot \gamma \leq 0\), then the condition \(\text{Bas}_X = \{\pm c_1(X,\omega)\}\) or the adjunction formula will be violated. Additionally, in section 4 we provide explicit examples of symplectic embeddings of \(B_n \hookrightarrow (X,\omega)\) of type \(E_2\), which adhere to Conjecture 1.3.

2. Background

2.1. Description of the rational homology balls \(B_n\). There are several ways to give a description of the rational homology balls \(B_n\). One of them
is a Kirby calculus diagram seen in Figure 2. This represents the following handle decomposition: Start with a 0-handle, a standard 4-disk \( D^4 \), attach to it a 1-handle \( D^1 \times D^3 \). Call the resultant space \( X_1 \), it is diffeomorphic to \( S^1 \times D^3 \) and has boundary \( \partial X_1 = S^1 \times S^2 \). Finally, we attach a 2-handle \( D^2 \times D^2 \). The boundary of the core disk of the 2-handle gets attached to the closed curve, \( K \), in \( \partial X_1 \) which wraps \( n \) times around the \( S^1 \times \ast \) in \( S^1 \times S^2 \). We can also represent \( B_n \) by a slightly different Kirby diagram, which is more cumbersome to manipulate but is more visually informative, as seen in Figure 3 where the 1-handle is represented by a pair of balls.

![Figure 3. Another Kirby diagram of \( B_n \)](image)

The rational homology ball \( B_2 \) can also be described as an unoriented disk bundle over \( \mathbb{R}P^2 \). Since \( \mathbb{R}P^2 \) is the union of a Mobius band \( M \) and a disk \( D \), we can visualize \( \mathbb{R}P^2 \) sitting inside \( B_2 \), with the Mobius band and its boundary \( (M, \partial M) \) embedded in \( (X_1 \cong S^1 \times D^3, \partial X_1 \cong S^1 \times S^2) \) (Figure 4 with the ends of the cylinder identified), and the disk \( D \) as the core disk of the attaching 2-handle. We will construct something similar for \( n \geq 3 \). Instead of the Mobius band sitting inside \( X_1 \), as for \( n = 2 \), we have a “\( n \)-Mobius band” (a Moore space), \( L'_n \), sitting inside \( X_1 \). The case of \( n = 3 \) is illustrated in Figure 5 again with the ends of the cylinder identified. In other words, \( L'_n \) is a singular surface, homotopic to a circle, in \( X_1 \cong S^1 \times D^3 \).
whose boundary is the closed curve $K$ in $\partial X_1 \cong S^1 \times S^2$, and it includes the circle, $S = S^1 \times 0$ in $S^1 \times D^3$. Let $L_n = L'_n \cup_K D$, where $D$ is the core disk of the attached 2-handle (along $K$). We will call $L_n$ the core of the rational homology ball $B_n$; observe, that $L_2 \cong \mathbb{R}P^2$.

These cores $L_n$ were used as geometrical motivation in the construction of a symplectic structure on the rational homology balls $B_n$, in the definition of the symplectic rational blow-up operation $[Kh2]$. For $n = 2$, if we have an embedded $\mathbb{R}P^2$ in $(X, \omega)$, such that $\omega|_{\mathbb{R}P^2} = 0$, (i.e. a Lagrangian $\mathbb{R}P^2$) then the $\mathbb{R}P^2$ will have a totally standard neighborhood, which will be symplectomorphic to the rational homology ball $B_2$. In this vein, for $n \geq 3$, we can define $\mathcal{L}_n$ (labeled $\mathcal{L}_{n,1}$ in $[Kh2]$) as a cell complex consisting of an embedded $S^1$ and a 2-cell $D^2$, whose boundary “wraps” $n$ times (winding number) around the embedded $S^1$ (the interior of the 2-cell $D^2$ is an embedding). Furthermore, the cell complex $\mathcal{L}_n$ is embedded in such a way that the 2-cell $D^2$ is Lagrangian. It is shown in $[Kh2]$, by mirroring the Weinstein Lagrangian embedding theorem, that a symplectic neighborhood of such an $\mathcal{L}_n$ is entirely standard, and is a symplectic model for $B_n$. Therefore, given the existence of such an $\mathcal{L}_n$, we can replace the $B_n$ with $C_n$ and obtain a new symplectic 4-manifold $(X', \omega')$, the symplectic rational blow-up of $(X, \omega)$.

2.2. Review of Seiberg-Witten invariants and basic classes. Here we give a brief overview of Seiberg-Witten invariants and basic classes, and state some relevant results. For a full description of Seiberg-Witten invariants see $[Mo]$, and for a short overview see $[GoSt]$, section 2.4 (which this summary is based on). The Seiberg-Witten invariant is a powerful invariant of smooth manifolds. More precisely, these are invariants of a smooth 4-manifold together with a $\text{spin}^c$ structure.

We let $X$ be a smooth, closed, oriented 4-manifold, with $b^+_2(X) > 1$ odd. Given a $\text{spin}^c$ structure $s$, we can associate to it a determinant line bundle $L$. If $H^2(X; \mathbb{Z})$ has no 2-torsion, then the set of $\text{spin}^c$ structures of $X$, $\mathcal{S}^c(X)$, is in 1-1 correspondence (via $c_1(L)$) with the set of characteristic elements of $X$, $\mathcal{C}_X$:

**Definition 2.1.** The set of characteristic elements of $X$ (as above) is:

$$\mathcal{C}_X = \{ K \in H^2(X; \mathbb{Z}) | K \equiv w_2(X) (\text{mod} 2) \}.$$

We will assume for simplicity of the exposition that $H^2(X; \mathbb{Z})$ has no 2-torsion. Let $\mathcal{M}_{X}^{s, g}(K)$ be the moduli space of solutions to certain perturbed monopole equations, where $K \in \mathcal{C}_X$, $g$ is a given metric on $X$ and $\delta \in \Omega^+(X)$ is a perturbation. The moduli space $\mathcal{M}_{X}^{s, g}(K)$ is itself a closed and orientable manifold (for a generic metric $g$) of dimension $\frac{1}{2}(K^2 - (3\sigma(X) + 2\chi(X)))$. In addition, $\mathcal{M}_{X}^{s, g}(K)$ is a subspace of an infinite-dimensional manifold $\mathcal{B}_K$, which is homotopy equivalent to $\mathbb{C}P^\infty$, in particular, implying that $H^*\big(\mathcal{B}_K^s; \mathbb{Z}\big) \cong \mathbb{Z}[\mu]$ and $[\mathcal{M}_{X}^{s, g}(K)] \in H_{2n}(\mathcal{B}_K^s; \mathbb{Z})$ is a homology class.
Definition 2.2. For $X$ as above, the Seiberg-Witten invariant is $SW_X : \mathcal{C}_X \to \mathbb{Z}$ is defined by $SW_X(K) = \langle \mu^m, [\mathcal{M}_X^{k,g}(K)] \rangle$, where $\dim \mathcal{M}_X^{k,g}(K) = 2m$ and if $\dim \mathcal{M}_X^{k,g}(K) < 0$ then $SW_X(K) = 0$. (If $\dim \mathcal{M}_X^{k,g}(K)$ is odd then $b_2^+(X)$ is even, and we are assuming $b_2^+(X)$ is odd.)

The Seiberg-Witten invariant is $SW_X$ is indeed a diffeomorphism invariant: it does not depend on the choices made in its construction.

Definition 2.3. A cohomology class $K \in \mathcal{C}_X \subset H^2(X; \mathbb{Z})$ is a Seiberg-Witten basic class if $SW_X(K) \neq 0$, and the set of basic classes denoted by $\text{Bas}_X$.

Definition 2.4. A simply connected 4-manifold is said to be of simple type if for each $K \in \text{Bas}_X$ we have $K^2 = c_1^2(X) = 3\sigma(X) + 2\chi(X)$ (implying that $\dim \mathcal{M}_X^{k,g}(K) = 0$).

Now we will state some useful results of Seiberg-Witten invariants:

The Seiberg-Witten invariants behave very well under blow-ups ([FS1] for general case):

Theorem 2.5. The blow-up formula [GoSi]. Let $X$ be a simply connected 4-manifold of simple type with $\text{Bas}_X = \{K_i|i = 1, \ldots, s\}$. If $X' = X \# CP^2$ is the blow-up of $X$ and $E \in H^2(X'; \mathbb{Z})$ denotes the Poincaré dual of the homology class $e \in H_2(X'; \mathbb{Z})$ of the exceptional sphere, then the set of basic classes of $X'$ equals $\{K_i \pm E|i = 1, \ldots, s\}$.

For Seiberg-Witten behavior under rational blow-downs, we have the following results, [FS2], also see [GoSi]:

Proposition 2.6. Let the sphere configuration $C_n \subset X$, and $X_{(n)} = X^o \cup B_n$ (where $X^o = X - C_n$) be the rational blow-down of $X$ along $C_n$. Then for every characteristic element $K \in \mathcal{C}_{X_{(n)}}$ there is an element $K \in \mathcal{C}_X$ such that $K|_{X^o} = K|_{X^o}$ and $K^2 - \overline{K}^2 = -(n - 1)$. The class $K$ is called a lift of $\overline{K}$.

Theorem 2.7. Suppose that $X$ and $X_{(n)}$ (as above) are simply connected 4-manifolds. Choose $K \in \mathcal{C}_{X_{(n)}}$, and fix a lift $\overline{K} \in \mathcal{C}_X$ for it. If $K^2 \geq 3\sigma(X) + 2\chi(X)$, then $SW_{X_{(n)}}(\overline{K}) = SW_X(K)$. Consequently, the Seiberg-Witten invariants of $X$, $SW_X$, determine the Seiberg-Witten invariants of the rational blow-down of $X$, $SW_{X_{(n)}}$.

Remark 2.8. The theorem above expresses the SW basic classes of $X_{(n)}$ in terms of the SW basic classes of $X$. Consequently, it tells us which SW basic classes $X$ “pass down” to $X_{(n)}$. It does not, however, provide us a way to reconstruct the SW basic classes of $X$ from those of $X_{(n)}$. In fact, the only basic classes that can “pass down” from $X$ to $X_{(n)}$, are those which when restricted to $\partial X^o \cong L(n^2, n - 1)$, correspond to an element of order $n$ in $H^2(L(n^2, n - 1), \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$. 
For complex surfaces $S$, and a smooth, nonsingular, connected, complex curve $C \subset S$, the standard adjunction formula says that:

$$2g(C) - 2 = |C|^2 - \langle c_1(S), C \rangle,$$

where $g(C)$ is the genus of $C$. The Seiberg-Witten invariants give us the following adjunction formula result for smooth manifolds $X$:

**Theorem 2.9. Generalized adjunction formula** [KM, OzSz], also see [GoSt]. Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ (and $|\Sigma| \neq 0$). Then for every Seiberg-Witten basic class $K \in \text{Bas}_X$ we have $2g(\Sigma) - 2 \geq |\Sigma|^2 + |K(\Sigma)|$. If $X$ is of simple type and $g(\Sigma) > 0$, the same inequality holds for $\Sigma \subset X$ with arbitrary square $|\Sigma|^2$.

There is also further generalization of this result for immersed spheres. We state here a simplified version, where $\dim M^{g,0}(K) = 0$:

**Theorem 2.10. [FS1]. Generalized adjunction formula for immersed spheres.** Suppose that $X$ is an arbitrary smooth 4-manifold with $b^+_2(X) > 1$ and that $K \in C_X$ with $SW_X(K) \neq 0$ and $\dim M_X(K) = 0$. If $x \neq 0 \in H_2(X; \mathbb{Z})$ is represented by an immersed sphere with $p$ positive double points, then either

$$2p - 2 \geq x^2 + |x \cdot L|$$

or

$$SW_X(K) = \begin{cases} SW_X(K + 2x), & \text{if } x \cdot K \geq 0 \\ SW_X(K - 2x), & \text{if } x \cdot K \leq 0. \end{cases}$$

The Seiberg-Witten invariants also have interesting behavior if the 4-manifold $X$ is equipped with a symplectic form $\omega$. For example, if a 4-manifold has a symplectic structure then it must be of simple type. Additionally, we have the following important results of Taubes:

**Theorem 2.11. [Ta1] If $(X, \omega)$ is a simply connected symplectic manifold with $b^+_2(X) > 1$, then $SW_X(\pm c_1(X, \omega)) = \pm 1$.**

**Theorem 2.12. [Ta2, Ta4], also see [Ko] (and [GoSt], chapter 10, for this simpler statement). Suppose that $(X, \omega)$ is a symplectic 4-manifold with $b^+_2(X) > 1$ and $SW_X(K) \neq 0$ for a given $K \in C_X$. Assume furthermore that the class $c = \frac{1}{2}(K - c_1(X, \omega))$ is nonzero in $H^2(X; \mathbb{Z})$. Then for a generic compatible almost-complex structure $J$ on $X$, the class $PD(c) \in H_2(X; \mathbb{Z})$ can be represented by a pseudo-holomorphic submanifold (not necessarily connected).

From the above result, one can also conclude the following:

**Theorem 2.13. [Ta4, Ta3, Ko], also see [GoSt]. If $X$ is a minimal symplectic 4-manifold (i.e. does not contain symplectic spheres with self-intersection $(-1)$) with $b^+_2(X) > 1$, then $c_2^+(X, \omega) \geq 0$.**
From the above two results and the generalized adjunction formula, we can further conclude the following:

**Corollary 2.14.** [Ta3], also see [GoSt]. If \((X, \omega)\) is a symplectic 4-manifold with \(c_1^2(X, \omega) \leq -1\), then for a generic compatible almost-complex structure \(J\) on \(X\), there exists a \(J\)-holomorphic sphere of self-intersection \((-1)\).

**Proof.** From Theorem 2.13 it follows that if \(c_1^2(X, \omega) \leq -1\), then there exists a symplectic sphere, \(\Sigma \in X\), with \([\Sigma]^2 = -1\). However, since for the homology class \(\pm [\Sigma]\) we have \(c_1(X, \omega) \cdot PD([\Sigma]) = 1\) and \(SW_X(c_1(X, \omega) + 2PD([\Sigma])) \neq 0\), meaning that \(c_1(X, \omega) + 2PD([\Sigma]) \in Bas_X\), then from Theorem 2.12 we have that the homology class \([\Sigma]\) can be represented by a pseudo-holomorphic submanifold. Finally, the generalized adjunction formula forces the pseudo-holomorphic submanifold to be a sphere. □

### 2.3. Toric and almost-toric fibrations of symplectic 4-manifolds.

In this section we introduce toric and almost-toric models of symplectic 4-manifolds, which will be used in Step 3 (section 3.3) of the proof of the main theorem. The goal is to introduce enough terminology, so that we can present the almost-toric models of manifolds \(C_n\) and \(B_n\), as well as illustrate how to see the “core” \(L_n\) of \(B_n\) (see section 2.1) in these models.

In [Sy1], Symington showed that the rational blow-down construction can be performed in the symplectic category. She did this by describing the symplectic structure of \(C_n\) and a collar neighborhood of \(\partial B_n\) with the help of toric fibrations. In [Sy2], she generalized this construction to show that the generalized rational blow-down can also be performed in the symplectic category. In [Sy3], she presented a way of describing symplectic 4-manifolds through almost-toric fibrations and used this to prove the existence of the symplectic rational blow-down in a less cumbersome manner than using just toric fibrations.

The goal of toric and almost-toric fibrations of symplectic 4-manifolds is to be able to depict various topological and symplectic properties of these manifolds with polytopes and curves in plane. The basis for doing this comes from a theorem of Delzant:

**Theorem 2.15.** [De] If a closed symplectic manifold \((M^{2n}, \omega)\) is equipped with an effective Hamiltonian \(n\)-torus action, then the image of the moment map \(\Delta\) determines the manifold \(M\), its symplectic structure \(\omega\) and the torus action.

Additionally, we have the following key result on Hamiltonian torus actions:

**Theorem 2.16.** [At, GuSt] The moment map image \(\Delta\) for a Hamiltonian \(k\)-torus action on a closed symplectic manifold \((M, \omega)\) is a convex polytope.

When \(k = n\), the manifold \((M^{2n}, \omega)\) is called toric. For our purposes we will only be dealing with the case \(n = 2\), and while several of the following results hold in any even dimension, we will only state them for \(n = 2\). The main goal of [Sy3], with the almost-toric fibrations is to extend the above two
theorems to work for a larger class of symplectic 4-manifolds, and generalize the class of moment-map images.

Since the symplectic form vanishes on the fibers of a moment map, implying that the regular fibers are Lagrangian submanifolds, the moment map actually provides us with a Lagrangian fibration:

**Definition 2.17.** [Sy3] A projection \( \pi : (M^4, \omega) \to B^2 \) is a Lagrangian fibration if it restricts to a regular Lagrangian fibration (locally trivial fibration where the fibers are Lagrangian) over an open dense set \( B_0 \subset B \).

The most basic example is \( \pi : (\mathbb{R}^2 \times T^2, \omega_0) \to \mathbb{R}^2 \), with \( \omega_0 \) the standard symplectic structure, which serves as a model for all other examples. The goal is to make use of the standard lattice \( \Lambda_0 \) on the tangent bundle \( T\mathbb{R}^2 \), spanned by \( \left\{ \frac{\partial}{\partial p_i} \right\} \) and \( \left\{ \frac{\partial}{\partial q_i} \right\} \), where \((p,q)\) are the standard coordinates on \( \mathbb{R}^2 \times T^2 \). In relation to this, Symington shows the following:

**Theorem 2.18.** [Sy3] If \( \pi : (M, \omega) \to B \) is a regular Lagrangian fibration then there are lattices \( \Lambda \subset T\mathbb{B} \), \( \Lambda^* \subset T^*B \) and \( \Lambda_{\text{vert}} \) in the vertical bundle of TM (induced by \( \pi \)) that, with respect to standard local coordinates, are the standard lattice, its dual, and the standard vertical lattice.

This induced lattice \( \Lambda \) on the tangent bundle of the base \( B \), as above, gives \( B \) an integral affine structure \( A \).

**Proposition 2.19.** [Sy3] An n-manifold \( B \) admits an integral affine structure if and only if it can be covered by coordinate charts \( \{ U_i, h_i \} \), \( h_i : U_i \to \mathbb{R}^n \) such that the map \( h_j \circ h_i^{-1} \), wherever defined, is an element of \( \text{AGL}(n, \mathbb{Z}) \), i.e. a map of the form \( \Phi(x) = Ax + b \) where \( A \in \text{GL}(n, \mathbb{Z}) \) and \( b \in \mathbb{R}^n \).

Symington denotes the toric (and almost-toric) bases with \((B, A, S)\), where \( B \) is the polytope base in \( \mathbb{R}^n \) (see Theorem 2.16), \( A \) is an integral affine structure, and \( S \) is a natural stratification of the base \( B \): the l-stratum is the set of points \( b \in B \) such that \( \pi^{-1}(b) \) is a torus of dimension \( l \). Additionally, \( \partial_R B \) denotes the collection of all the k-strata, with \( k < n \), which is the reduced boundary of the base \((B, A, S)\). Symington gives the following definition of the toric fibration and base:

**Definition 2.20.** [Sy3] A Lagrangian fibration \( \pi : (M^4, \omega) \to (B, A, S) \) is a toric fibration if there is a Hamiltonian 2-torus action and an immersion \( \Phi : (B, A) \to (\mathbb{R}^2, A_0) \) such that \( \Phi \circ \pi \) is the corresponding moment map and \( S \) is the induced stratification. In this case we call \((B, A, S)\) a toric base.

Since we are looking to represent symplectic 4-manifolds, we will be working with bases of dimension 2, and with 2, 1 and 0-strata. In other words, the 1-stratum are the edges of our polytope \( B \) in the plane, and the 0-stratum are its vertices. Consequently, Symington’s goal was to put the appropriate conditions on the base \((B, A, S)\) to ensure that it determines a unique symplectic 4-manifold. To reconstruct a symplectic 4-manifold from a toric base
(B, A, S), one can start with a regular Lagrangian fibration over (B, A) and collapse certain fibers to get the desired stratification S. Symington does this with the help of symplectic boundary reduction, introduced in [Sy1], which is defined in the proposition below:

**Proposition 2.21.** Let (M, ω) be a symplectic manifold with boundary such that a smooth component Y of ∂M is a circle bundle over a manifold Σ. Suppose also that the tangent vectors to the circle fibers lie in the kernel of ω|Y. Then there is a projection ρ : (M, ω) → (M′, ω′) and an embedding φ : Σ → M′ such that ρ(Y) = φ(Σ), ρ|M−Y is a symplectomorphism onto M′ − φ(Σ) and φ(Σ) is a symplectic submanifold. The manifold (M′, ω′) = ρ(M, ω) is the **symplectic boundary reduction** of (M, ω) along Y.

Connecting the above proposition to the toric bases (B, A, S), Symington gives the following definition:

**Definition 2.22.** Given a toric fibration π : (M4, ω) → (B, A, S), the **boundary recovery** is the unique Lagrangian fibered manifold (B × T2, ω0) that yields (M, ω) via boundary reduction.

**Example 2.23.** A basic example is the toric base for a symplectic 4-manifold diffeomorphic to CP2, which is a simple triangle with vertices on (0, 0), (1, 0) and (0, 1), as depicted in Figure 6, with the bold edges representing the 1-stratum. This base represents CP2 as the boundary reduction of (B4, ω0), where the circles of the Hopf fibration are collapsed.

In reading such diagrams, it is important to remember that the pre-image of each interior point in the diagram, is a torus, S1 × S1, the pre-image of each point on the thick edges of the diagram (the 1-stratum) is a circle S1, and the pre-image of each vertex in the diagram is just a point. Before describing the toric model of Cn, we first introduce the following important element of toric bases:

**Definition 2.24.** [Sy3] Let π : (M, ω) → (B, A, S) be a toric fibration and γ a compact embedded curve with one endpoint b1 in the 1-stratum of ∂RB (both the 1 and 0-stratum in this case) and such that γ − {b1} ⊂ B0 = B −
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\[ \partial_R B \]. Let \( b_0 \) be the other endpoint of \( \gamma \). The \textit{collapsing class}, with respect to \( \gamma \), for the smooth component of \( \partial_R B \) containing \( b_1 \) is the primitive class \( a \in H_1(F_{b_0}; \mathbb{Z}) \) that spans the kernel of \( \iota_* : H_1(F_{b_0}; \mathbb{Z}) \to H_1(\pi^{-1}(\gamma); \mathbb{Z}) \), where \( \iota \) is the inclusion map. Corresponding to the \textit{collapsing class} is the \textit{collapsing covector}, with respect to \( \gamma \), which is the primitive covector \( v^* \in T^*_{b_0} B \) that determines vectors \( v(x) \in T^*_{\text{vert}} M \) for each \( x \in \pi^{-1}b_0 \) such that the integral curves of this vector field represent \( a \).

\[ L(n^2, n-1) \]

Figure 7. Toric model for \( C_n \)

**Example 2.25.** We can construct a toric fibration of the \( C_n \) configuration of spheres, (see Figure 7). In this diagram, the slopes of the edges are \( 0, \frac{1}{n+2}, \frac{2}{2n+3}, \frac{3}{3n+4}, \ldots, \frac{n}{n^2} \), thus the corresponding collapsing covectors are:

\[
v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ n+2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 2n+3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -3 \\ 3n+4 \end{bmatrix}, \ldots, \quad v_n = \begin{bmatrix} 1-n \\ n^2 \end{bmatrix}.
\]

Consequently, we have \( v_{i+1} \times v_{i-1} = [S_i]^2 \), giving us the desired self-intersection numbers of the spheres \( S_i \).

The pre-image of the (thin) curve on the top of the diagram is the boundary \( \partial C_n = L(n^2, n-1) \), since the collapsing covectors on both endpoints of the curve are \( v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( v_n = \begin{bmatrix} 1-n \\ n^2 \end{bmatrix} \).

Symington proves (Theorem 3.19, [Sy3]) that such diagrams of toric bases \((B, A, S)\), determine unique toric manifolds, presented as the boundary reduction of \((B \times T^2, \omega_0)\) (assuming certain technical conditions, see [Sy3] for details). The uniqueness of the toric manifold fibering over the base \((B, A, S)\), was shown earlier by [BM].

One can push these diagrams further, to depict Lagrangian fibrations with (nodal) singularities:
Definition 2.26. \[\text{Sy3}\] A nondegenerate Lagrangian fibration \(\pi : (M, \omega) \to B\) of a symplectic 4-manifold is an \textit{almost-toric fibration} if it is a nondegenerate topologically stable fibration with no hyperbolic singularities (e.g. a fibration with a nodal singularity). A triple \((B, \mathcal{A}, S)\) is an \textit{almost-toric base} if it is the base of such a fibration. A symplectic 4-manifold equipped with such a fibration is an \textit{almost-toric manifold}.

Thus if \(\{s_i\} \subset B\) are the images of such singularities, then \(\mathcal{A}\) is the affine structure on \(B - \{s_i\}\). Generally, a Lagrangian fibration can be arranged such that nodal singularities occur in distinct fibers. Also, a nodal fiber is the singular fiber of a Lefschetz fibration, and its neighborhood is diffeomorphic to \(T^2 \times D^2\) with a \((-1)\)-framed two-handle attached along a simple closed curve in \(T^2 \times \{x\}\). One can also compute the topological monodromy around the nodal fiber with respect to the basis \(\{[\gamma_1], [\gamma_2]\} \in H_1(F_b; \mathbb{Z})\):

\[
\Psi(\gamma) = A_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

If we choose a different basis for \(H_1(F_b; \mathbb{Z})\), then we conjugate the matrix \(A_{(1,0)}\), giving us the following monodromy matrix with eigenvector \((a, c)\):

\[
A_{(a,c)} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}.
\]

This leads us to the following definition and lemma:

Definition 2.27. \[\text{Sy3}\] Let \(\pi : (M, \omega) \to B\) be an almost-toric fibration with a node at \(s\). Let \(\eta\) be an embedded curve with endpoints at \(s\) and a point \(b \in B_0 = B - \partial_R B\) such that \(\eta - \{s\} \subset B_0\) contains no other nodes. A \textit{vanishing class} in \(H_1(F_b; \mathbb{Z})\), associated to \(s\) and \(\eta\), is the class whose representatives bound a disk in \(\pi^{-1}(\eta)\). The \textit{vanishing covector} \(w^* \in T^*_b B\) is the primitive covector that determines vectors \(w(x) \in T^{\text{vert}}_x M\) for each \(x \in \pi^{-1}b\) such that the integral curves of this vector field represent the vanishing class.

Lemma 2.28. \[\text{Sy3}\] Suppose \(\gamma\) is a positively oriented loop based at \(b\) that is the boundary of a closed neighborhood of \(s\) containing \(\eta\). Then the vanishing class is the unique class (up to scale) that is preserved by the monodromy along \(\gamma\). With respect to the basis for \(H_1(F_b; \mathbb{Z})\) for which the monodromy matrix is \(A_{(a,c)}\), the vanishing class is the class \((a, c)\).

Notice, that given such an almost-toric fibration \(\pi : (M, \omega) \to B\) with a node at \(s\), the fibration over \(B - \{s\}\) is regular and has an induced affine structure \(\mathcal{A}\). However, there is non-trivial monodromy around the node \(s\), thus there is no affine immersion of \((B - \{s\}, \mathcal{A})\) into \((\mathbb{R}^2, \mathcal{A}_0)\). To salvage this, we can remove a ray \(R\), based at the node \(s\), from the base \(B\), giving us an immersion of \((B - R, \mathcal{A})\) into \((\mathbb{R}^2, \mathcal{A}_0)\). An example of such a base \(B\) with a removed ray \(R\) is seen in Figure 8. This ray (or eigenray) is an eigenvector of the monodromy matrix of the node. Symington shows, that
any integral affine punctured plane \((V, \mathcal{A})\) will be isomorphic to the one depicted in Figure 8, where the ray \(R\) is the eigenray \((1, 0)\). Consequently, we can model an almost-toric manifold with bases \(B\) containing nodes \(s_i\): 

**Definition 2.29.** [Sy3] An integral affine manifold with nodes \((B, \mathcal{A})\) is a two-manifold \(B\) equipped with an integral affine structure on \(B - \{s_i\}\) such that each \(s_i\) has a neighborhood \(U_i\) such that \((U_i - s_i, \mathcal{A})\) is affine isomorphic to a neighborhood of the puncture in \((V^k, \mathcal{A}^k)\) (if the node has multiplicity \(k\)).

**Theorem 2.30.** [Sy3] Consider a triple \((B, \mathcal{A}, S)\) such that \((B, \mathcal{A})\) is an integral affine manifold with nodes \(\{s_i\}_{i=1}^N\). Then \((B, \mathcal{A}, S)\) is an almost-toric base if and only if every point in \(B - \{s_i\}_{i=1}^N\) has a neighborhood that is a toric base.

Symington also defined various operations, like the *nodal slide* and the *nodal trade* to get from an almost-toric base \((B, \mathcal{A}, S)\) to another \((B', \mathcal{A}', S')\), with both representing the same manifold with isotopic symplectic structures. Now we are ready to describe the almost-toric base for the rational homology balls \(B_n\):

**Example 2.31.** Figure 9 depicts an almost-toric base for the rational homology balls \(B_n\), in this diagram, the ray \(R\) has a slope of \(\frac{1}{n}\), corresponding to the eigenvector \((n, 1)\) of the monodromy

\[
A_{(n,1)} = \begin{pmatrix}
1 - n & n^2 \\
-1 & 1 + n
\end{pmatrix}.
\]

thus making \(\begin{pmatrix}
-1 \\
n
\end{pmatrix}\) be the vanishing covector of the node \(s\). The slope of the line on the right is \(\frac{n+1}{n^2}\), therefore the preimage of the thin line on the top of the diagram is \(L(n^2, n - 1)\) as was the case for the toric diagram for \(C_n\).
Symington then proves that the symplectic rational blow-down can be performed in the symplectic category, by simply removing the images of the neighborhoods of the symplectic spheres from the toric model of $C_n$ (Figure 7) and gluing below it, the almost-toric model for $B_n$ (Figure 9). They match up, since the slopes of the right-most edge is $\frac{n-1}{n^2}$, as illustrated in Figure 10.

![Figure 9. Almost toric base for $B_n$](image1)

![Figure 10. Rational blow-down in almost-toric diagrams](image2)

It is useful for our purposes to illustrate where on this almost-toric model of $B_n$ can we “see” the image of the “Lagrangian cores” $L_n$ of the rational homology $B_n$ (see section 2.1 and [Kh2]). Before we do this, we must first introduce the concept of visible surfaces in these almost-toric fibrations, as was done in [Sy3].

If one draws a curve $\nu$ in an (almost)-toric base $B$, then the pre-image of every point $b \subset B_0$ in the curve will be a torus $F_b \cong S^1 \times S^1$. However, if for every point $b$ in the curve we choose a closed curve in $F_b \cong S^1 \times S^1$, then the
entire collection of those closed curves over all points in the curve $\nu$ could potentially be a surface in the original 4-manifold. This is precisely what visible surfaces are, they are a coherent collection of such closed curves, in the pre-images of the points in a toric (almost-toric) base. Here is a more precise definition that Symington gives:

**Definition 2.32.** [SY3] A visible surface $\Sigma_\nu$ in an almost-toric fibered manifold $\pi : (M, \omega) \to (B, A, S)$ is an immersed surface whose image is an immersed (connected) curve $\nu$ with transverse self-intersections such that $\pi|_{\Sigma_\nu \cap \pi^{-1}(B_0)}$ is a submersions onto $\nu \cap B_0$, any non-empty intersection of $\Sigma_\nu$ with a regular fiber is a union of affine circles, and no component of $\partial \Sigma_\nu$ projects to a node.

The following are the conditions on curves in the base to represent a visible surface and for a curve $\nu$ to represent a unique surface $\Sigma_\nu$:

**Definition 2.33.** [SY3] Given an immersed curve $\nu : I \to (B, A, S)$, let $\{\nu_i\}_{i=1}^k$ be the continuous (and connected) components of $\nu|_{\nu^{-1}(B-B_0)}$. A primitive class $a_i$ in $H_1(\pi^{-1}(\nu_i); \mathbb{Z})$ (such that $\pi_\ast a_i = 0$ if $\nu_i$ is a loop) is compatible with $\nu$ if all of the following are satisfied:

1. $a_i$ is the vanishing class of every node in $\nu$,
2. $|a_i \cdot c| \in \{0, 1\}$ for each $c$ that is the collapsing class, with respect to $\nu_i$ for a component of the 1-stratum of $\partial R B$ that intersects $\nu_i$,
3. $|a_i \cdot c| = 1$ if $\nu_i$ intersects the 1-stratum non-transversally,
4. $|a_i \cdot d| = 1$ for each $d$ that is one of the two collapsing classes at a vertex contained in the closure of $\nu_i$. (Here, $\cdot$ is the intersection pairing in $H_1(\nu_i^{-1}(\nu_i); \mathbb{Z})$ and $\nu_i$ is the closure of $\nu_i$.)

**Theorem 2.34.** [SY3] Suppose $(B, A, S)$ is an almost-toric base such that each node has multiplicity one. An immersed curve $\nu : I \to (B, A, S)$ with transverse self-intersections and a set of compatible classes $\{a_i\}_{i=1}^k$ together determine a visible surface $\Sigma_\nu$ such that for each $b \in \nu_i$,

$$(2.2) \quad \iota_\ast [\Sigma_\nu \cap F_b] = a_i$$

where $\iota : F_b \to \pi^{-1}(\nu_i)$ is the inclusion map. (Note, we will not define the “multiplicity” of a node here; all of the nodes that we will work with have “multiplicity” one, for details see [SY3].) The surface $\Sigma_\nu$ is unique up to isotopy among visible surfaces in the preimage of $\nu$ that satisfy equation (2.2).

Furthermore, no such surface exists if the classes $a_i$ are not compatible with $\nu$.

To each primitive class $a_i \in H_1(\pi^{-1}(\nu_i); \mathbb{Z})$ there is corresponding compatible vector $v_i \in \mathbb{R}^2$ such that the integral curves of the vector field $v_i \frac{\partial}{\partial t} \subset \Lambda^{\text{vert}}$ represent $a_i$. If $v$ and $w$ are compatible vectors for primitive classes $a$ and $b$ respectively, then $|a \cdot b| = |v \times w| = |\det(vw)|$. Symington also shows that if curves $\nu_1$ and $\nu_2$ intersect transversally at a point $b \in B_0$ and $\Sigma_{\nu_1}$ and $\Sigma_{\nu_2}$ intersect transversally in $F_b$, then $\Sigma_{\nu_1}$ intersects $\Sigma_{\nu_2}$ in
$|v_1 \times v_2|$ points where the signs of all intersections is $\det(u_1 u_2) \det(v_1 v_2)$. Here, $v_i$ are the compatible vectors of $\nu_i$ and the $u_i$ are the tangent vectors of $\nu_i$ at the point $b$.

In [Sy3], it is proved that one can compute the symplectic area of the visible surfaces as follows:

**Proposition 2.35.** Let $\nu : I \to (B, A, S)$ be a parameterized immersed curve and $\{v_i\}_{i=1}^N$ a set of co-oriented compatible vectors in a base diagram that define an oriented surface $\Sigma_{\nu}$. The (signed) area of $\Sigma_{\nu}$ is:

\[
\text{Area}(\Sigma_{\nu}) = \int_{\Sigma_{\nu}} \omega = 2\pi \int_0^1 \nu'(t) \cdot v(t) \, dt
\]

where $v(t) = v_i$, if $\nu(t) \in \nu_i$ and for other values of $t$ (when $\nu \subset \partial_R B$) $v(t)$ is an integral vector such that $u(t) \times v(t) = 1$ for some integral vector $u(t) = \lambda \nu'(t)$, $\lambda > 0$.

**Remark 2.36.** Note, that given such conditions for a visible surface, $\Sigma_{\nu}$ must be a sphere, disk, cylinder or torus. Therefore, if we want to “see” a Lagrangian core $\mathcal{L}_n$ in the almost-toric base for $B_n$, we can only really “see” where $\mathcal{L}_n$ is an embedding, in other words, a Lagrangian disk in $\mathcal{L}_n$, right before the edge of the disk hits the singular part of $\mathcal{L}_n$. Proposition 2.35 implies that in order for a visible surface $\Sigma_{\nu}$ to be Lagrangian, we must have that $\nu$ is a straight line. The line $\nu$ in Figure 11, extending from the node and (almost) hitting the left edge of the 1-stratum, represents a Lagrangian visible surface $\Sigma_{\nu}$. Since the line $\nu$ hits a node, its compatible covector must correspond to the vanishing covector of the node, which is $\nu = \left[ \begin{array}{c} -1 \\ n \end{array} \right]$. Notice, if $\nu$ were to actually hit the left edge of the 1-stratum, then this would violate condition (2) of Definition 2.33 since $|v \times c| = n$.
where \( c \) is the collapsing covector of the left edge of the 1-stratum. As a result, \( \Sigma_n \) can represent the Lagrangian core \( \mathcal{L}_n \) of \( B_n \), as introduced in section 2.1, since the boundary of the 2-cell \( D^2 \) in \( \mathcal{L}_n \) wraps around \( n \) times the \( S^1 \) in \( \mathcal{L}_n \).

3. Proof of Main Theorem

We now again present the statement of the main theorem on symplectic embeddings of \( B_n \), which appears in the introduction, preceded by the following crucial proposition and some definitions and terminology.

**Proposition 3.1.** Let \((X, \omega)\) be a symplectic 4-manifold, such that:

- \( b_2^s(X) > 1 \),
- \( [c_1(X, \omega)] = -[\omega] \) as cohomology classes and
- \( n \geq c_1^2(X, \omega) + 2 \).

If there exists a symplectic embedding \( B_n \hookrightarrow (X, \omega) \) and \((X', \omega')\) is the symplectic rational blow-up of \((X, \omega)\), then there exists an embedded symplectic sphere \( \Sigma_{-1} \subset (X', \omega') \), and a linear plumbing configuration \( C_n \subset (X', \omega') \) of symplectic spheres \( S_j, 1 \leq j \leq n - 1 \), such that:

- \([\Sigma_{-1}]^2 = -1\),
- \([S_j]^2 = -n - 2\) and \([S_j]^2 = -2\) for \( 2 \leq j \leq n - 1 \) (see Figure 7) and
- \( \Sigma_{-1} \) intersects the spheres \( S_j, 1 \leq j \leq n - 1 \) positively and transversally.

**Definition 3.2.** We call a symplectic embedding of \( B_n \hookrightarrow (X, \omega) \) to be of type \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \), where \( \alpha_j \in \mathbb{Z}_{\geq 0} \), if there exists an embedded symplectic sphere, \( \Sigma \subset X' \), with \([\Sigma]^2 = -1\), such that it intersects positively and transversally with the spheres \( S_j, 1 \leq j \leq n - 1 \), of the \( C_n \) configuration in \( X' \) and \( \alpha_j \) is the number of those positive transverse intersections.

**Definition 3.3.** Let \( \mathcal{A} \) be the set of \((n - 1)\)-tuples \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \) such that:

1. \( \alpha_j \neq 0 \) for at least one \( j \), where \( 2 \leq j \leq n - 1 \), or
2. \( \alpha_1 \geq n \), or
3. \( \alpha_1 = 1 \) and \( \alpha_j = 0 \) for \( 2 \leq j \leq n - 1 \).

We will call a symplectic embedding \( B_n \hookrightarrow X \) to be of type \( \mathcal{A} \) if it is of type \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \subset \mathcal{A} \).

**Definition 3.4.** Let \( \mathcal{E}_k \) denote the \((n - 1)\)-tuple \( \langle k, 0, 0, \ldots, 0 \rangle \) for \( 2 \leq k \leq n - 1 \).

We note that Proposition 3.1 implies that a symplectic embedding \( B_n \hookrightarrow (X, \omega) \) (for \( b_2^s(X) > 1 \), \([c_1(X, \omega)] = -[\omega] \) and \( n \geq c_1^2(X, \omega) + 2 \)) will always be of type \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \), for some \((n - 1)\)-tuple \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \) with \( \alpha_j \in \mathbb{Z}_{\geq 0} \). Moreover, any \((n - 1)\)-tuple \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \) with \( \alpha_j \in \mathbb{Z}_{\geq 0} \) will be in at least one of the sets \( \mathcal{A}, \mathcal{E}_k, 2 \leq k \leq n - 1 \).
Theorem 3.5. If $B_n \hookrightarrow (X, \omega)$ is a symplectic embedding, where $(X, \omega)$ is a symplectic 4-manifold, such that:

- $b_2^+(X) > 1$,
- $[c_1(X, \omega)] = -[\omega]$ as cohomology classes,
- $n \geq c_1^2(X, \omega) + 2$ and
- $\text{Bas}_X = \{\pm c_1(X, \omega)\}$, ($\text{Bas}_X$ denotes the set of Seiberg-Witten basic classes of $X$),

then it cannot be of type $A$ or of type $E_k$, $k \geq c_1^2(X, \omega) + 2$.

Remark 3.6. The condition $[c_1(X, \omega)] = -[\omega]$ holds for surfaces of general type $X$, with the canonical class $K_X$ ample. The ampleness implies that for all curves $C$ in $X$, we have that $c_1(X, \omega) \cdot [C] < 0$, implying that there are no $(-1)$ or $(-2)$ curves in $X$. For a symplectic 4-manifold $X$, the condition $[c_1(X, \omega)] = -[\omega]$ implies that there are no symplectic spheres $S$ with self-intersection $(-1)$ or $(-2)$: since for a symplectic sphere $S$, we have $f_S^* \omega > 0$ which implies $c_1(X) \cdot [S] < 0 \Rightarrow [S]^2 < -2$ by the adjunction inequality. Additionally, the condition of $(X, \omega)$ having only one Seiberg-Witten basic class (up to sign), is also true of all surfaces of general type. Consequently, these symplectic 4-manifolds are meant to mimic surfaces of general type as much as they can.

We will prove this theorem in four steps. In Step 1, section 3.1 we will prove Proposition 3.1. In Step 2, section 3.2 using the existence of the sphere $\Sigma_{-1}$ from Proposition 3.1 we construct a specific homology cycle $\gamma$, and compute $c_1(X, \omega) \cdot \gamma$ in terms of the intersection pattern of $\Sigma_{-1}$ with the spheres of the $C_n$ configuration. In Step 3, section 3.3 we show that if $c_1(X, \omega) \cdot \gamma > 0$, then $\omega \cdot \gamma > 0$, thus contradicting the $[c_1(X, \omega)] = -[\omega]$ assumption on $(X, \omega)$. As a result, we will show that $B_n \hookrightarrow (X, \omega)$ cannot be of type $A_1 \subset A$, where $A_1$ is the set of $(n-1)$-tuples corresponding to $c_1(X, \omega) \cdot \gamma > 0$. In Step 4, section 3.4 we show that if $c_1(X, \omega) \cdot \gamma \leq 0$, then this violates certain adjunction inequalities or forces $X$ to have additional Seiberg-Witten basic classes, thus preventing $B_n \hookrightarrow (X, \omega)$ to be of type $(A - A_1)$ and $E_k$, $k \geq c_1^2(X, \omega) + 2$.

In section 4 we give explicit examples of symplectic embeddings of $B_n \hookrightarrow (X, \omega)$ of type $E_2$ for $n$ odd. In these examples, we always have $n < 3 + \frac{4}{3} c_1^2(X, \omega)$.

3.1. Step 1. As a first step in proving Theorem 3.5 we will prove Proposition 3.1, that is, we will show that there exists a sphere, $\Sigma_{-1}$ of self-intersection $(-1)$ which intersects the spheres of the $C_n$ configuration, positively and transversally, in the rational blow-up of $X$.

We begin by assuming that for a given symplectic 4-manifold $(X, \omega)$, with conditions as stated in the Proposition 3.1 there is a symplectic embedding $B_n \hookrightarrow (X, \omega)$. This embedding is in the sense of the symplectic rational blow-up theorem (Theorem 3.2 in [Kh2]), meaning there is a Lagrangian core $L_n$ in $(X, \omega)$, whose neighborhood is the rational homology ball $B_n$. It
follows, according to this theorem that we can perform the symplectic rational blow-up procedure, replacing $B_n$ with $C_n$, and obtain a new symplectic manifold $(X', \omega')$ which contains a symplectic copy of a $C_n$ configuration of symplectic spheres. Since we assumed that $n \geq c_1^2(X, \omega) + 2$, and since $c_1^2(X', \omega') = c_1^2(X, \omega) - (n - 1)$, we have $c_1^2(X', \omega') \leq -1$. As a consequence of Corollary 2.14 for a generic compatible almost-complex structure $J$, on $X'$, there exists a $J$-holomorphic sphere, $\Sigma_{-1}$ with self-intersection number $(-1)$.

In order to force only positive intersections between the spheres of the $C_n$ configuration and a sphere of self-intersection $(-1)$, $\Sigma_{-1}$ (derived from $\Sigma_{-1}$ as a consequence of Proposition 3.17), we need to make the spheres of the $C_n$ configuration pseudo-holomorphic:

Lemma 3.7. With $X'$ as above, there exists an $\omega$-compatible almost-complex structure $J$ on $X'$ such that all of the spheres in the $C_n$ configuration are $J$-holomorphic.

Proof. First, we label the spheres of $C_n$ with $S_1, S_2, S_3, \ldots, S_{n-1}$, as before in Figure 1. Let the points $a_i = S_i \cap S_{i+1}$ be the points in the intersection of the spheres of $C_n$. Let $N_{a_i}$ be small Darboux neighborhoods around those points, such that

$$E = \bigcup_{i=1}^{n-1} S_i - \bigcup_{i=1}^{n-2} (N_{a_i} \cap (S_i \cup S_{i+1}))$$

is a symplectic submanifold consisting of $(n - 1)$ connected components. Then, we can choose an $\omega$-compatible almost-complex structure $J$ on $X'$ such that all the connected components of the submanifold $E$ are $J$-holomorphic submanifolds.

We can extend this almost-complex structure $J$ across the neighborhoods of the intersection points $N_{a_i}$ as follows: First, the results of [McPo] imply that for the symplectic spheres in $C_n$ configuration, which intersect transversally and positively, can always be isotoped in such a way that they intersect orthogonally (with respect to the symplectic structure) while remaining symplectic. Second, we use the following technical local result, which is a version of McDuff’s result in [Mc]:

Lemma 3.8. Let $\pi_1$ and $\pi_2$ be two orthogonal planes through $\{0\}$ in $\mathbb{R}^4$ which intersect with positive orientation and are symplectic with respect to the standard linear symplectic form $\omega_0$. Then there is a linear $\omega_0$-compatible $J$ which preserves these planes.

Proof. We can choose a basis $(e_1, e_2)$ for $\pi_1 \subset \mathbb{R}^4$ and a basis $(e_3, e_4)$ for $\pi_2 \subset \mathbb{R}^4$, such that $\omega_0(e_1, e_2) = 1$ and $\omega_0(e_3, e_4) = 1$ and $\pi_1^\perp = \pi_2$ (with respect to $\omega_0$). Then we simply choose $J$ to be such that $J(e_1) = e_2$ and $J(e_3) = (e_4).$ 

Since after (possibly) isotoping the symplectic spheres of $C_n$, the intersections of the spheres are orthogonal, in a local Darboux neighborhood,
\( N_{a_i} \), they can be modeled by two orthogonal planes through \( \{0\} \) in \( \mathbb{R}^4 \). Therefore, Lemma 3.8 implies that we can choose an \( \omega \)-compatible almost-complex structure \( J \) on \( X' \) such that the symplectic spheres of \( C_n \) are also \( J \)-holomorphic spheres.

**Remark 3.9.** McDuff’s result [Mc], says that if the planes \( \pi_1 \) and \( \pi_2 \) intersect positively and transversally then there exists an \( \omega \)-tame almost-complex structure \( J \) preserving the planes. This is not enough for our purposes, since in the next step, using Gromov compactness we will consider a sequence of almost-complex structures from \( J_\epsilon \rightarrow J \), and since \( J_\epsilon \) is required to be \( \omega \)-compatible by Taubes’ theorem, we need \( J \) to be \( \omega \)-compatible as well.

**Proposition 3.10.** Let \( X' \) be the rational blow-up of \( X \), as above, then there exists a \( J \)-holomorphic sphere of self-intersection \( (-1) \) in \( X' \), \( \Sigma_{-1} \), with \( J \) the almost-complex structure from Lemma 3.7.

**Proof.** To show the existence of this \( J \)-holomorphic sphere, \( \Sigma_{-1} \) we will use Gromov compactness to find a sequence of almost-complex structures, under which the \( J_\epsilon \)-holomorphic sphere \( \Sigma_{-1} \) will converge to a multicurve, (or a cusp-curve) with (potentially) some “bubbles”. One of the components of the multicurve will be a \( J \)-holomorphic sphere of self-intersection \( (-1) \), \( \Sigma_{-1} \).

First, we state the definition and properties of a multicurve, convergence of almost-complex structures and Gromov compactness. Let \((M, \omega)\) be a compact symplectic manifold:

**Definition 3.11.** [MS1] A **multicurve** (or **cusp-curve**) \( C \) is a connected union
\[
C = C^1 \cup C^2 \cup \cdots \cup C^N
\]
of \( J \)-holomorphic spheres \( C^j \), which are called components. Each component is parameterized by a smooth nonconstant \( J \)-holomorphic map \( u^j : \mathbb{CP}^1 \rightarrow M \), which is not required to be simple. The multicurve is denoted by \( u = (u^1, \ldots, u^N) \).

**Definition 3.12.** [MS1] A sequence of \( J \)-holomorphic curves \( u_\nu : \mathbb{CP}^1 \) is said to **converge weakly** to a multicurve \( u = (u^1, \ldots, u^N) \) if the following holds:

1. For every \( j \leq N \), there exists a sequence \( \phi^j_\nu : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) of fractional linear transformations and a finite set \( X^j \subset \mathbb{CP}^1 \) such that \( u_\nu \circ \phi^j_\nu \) converges to \( u^j \) uniformly with all derivatives on compact subsets of \( \mathbb{CP}^1 - X^j \).

2. There exists a sequence of orientation preserving (but not holomorphic) diffeomorphisms \( f_\nu : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) such that \( u_\nu \circ f_\nu \) converges in the \( C^0 \)-topology to a parametrization \( v : \mathbb{CP}^1 \rightarrow M \) of the multicurve \( u = (u^1, \ldots, u^N) \).

It follows [MS1], that for \( \nu \) sufficiently large, that the map \( u_\nu : \mathbb{CP}^1 \rightarrow M \) is homotopic to:
\[
u_1 \# u^2 \# \cdots \# u^N : \mathbb{CP}^1 \rightarrow M.
\]
In particular if $A_\nu, A^j \in H_2(M, \mathbb{Z})$ are the homology classes of $u_\nu$ and $u^j$ respectively, then we have:

\begin{equation}
(3.4) \quad c_1(M) \cdot A_\nu = \sum_{j=1}^{N} c_1(M) \cdot A^j.
\end{equation}

Finally, we can state Gromov’s compactness theorem \[Gr\], as it appears in \[MS1\]:

**Theorem 3.13. (Gromov’s compactness)** Assume $M$ is compact, and let $J_\nu \in \mathcal{J}_r(M, \omega)$ be a sequence of $\omega$-tame almost complex structures which converge to $J$ in the $C^\infty$-topology. Then any sequence $u_\nu : \mathbb{C}P^1 \to M$ of $J_\nu$-holomorphic spheres with $\sup E(u_\nu) < \infty$ has a subsequence which converges weakly to a (possible reducible) $J$-holomorphic multicurve $u = (u^1, \ldots, u^N)$.

Additionally, specifically for symplectic manifolds of dimension 4, we have the following adjunction formula:

**Theorem 3.14. Adjunction Formula (MS2 App. E).** Let $(M, J)$ be an almost-complex 4-manifold, $(\Sigma, J)$ be a closed Riemann surface, not necessarily connected, and $u : \Sigma \to M$ be a simple $J$-holomorphic curve. Denote $A \in H_2(M; \mathbb{Z})$ the homology represented by $u$. Then

\begin{equation}
(3.5) \quad 2\delta(u) \leq A \cdot A - c_1(M) \cdot A + \chi(\Sigma)
\end{equation}

with equality if and only if $u$ is an immersion and all self-intersections are transverse.

In the above, “simple” means not multiply covered and $\delta(u)$ is the number of self-intersections of $u$:

\begin{equation}
(3.6) \quad \delta(u) := \frac{1}{2} \# \{(z_0, z_1) \in \Sigma \times \Sigma | u(z_0) = u(z_1), z_0 \neq z_1\}
\end{equation}

Additionally, McDuff also proved the following corollary to the theorem above:

**Corollary 3.15. (MS2 App. E).** Let $M, \Sigma, u$ and $A$ be as in Theorem 3.14. Then

\begin{equation}
(3.7) \quad A \cdot A - c_1(M) \cdot A + \chi(\Sigma) \geq 0
\end{equation}

with equality if and only if $u$ is an embedding.

In \[MS2\], McDuff proves Theorem 3.14 by showing that in dimension 4, a homology class $A \in H_2(M, \mathbb{Z})$ which is represented by a simple $J$-holomorphic curve, $u : \Sigma \to M$, can always be represented by an immersed $J'$-holomorphic curve $v : \Sigma \to M$, with transverse self-intersections. The curves $u$ and $v$ are $C^1$-close, and the almost-complex structures $J$ and $J'$ are $C^1$-close as well. This is shown using a strong theorem of Micallef-White \[MW\] which states that a singularity of a $J$-holomorphic curve is equivalent to a singularity of a holomorphic curve, up to a $C^1$-diffeomorphism. As a result, Li in \[Li\] made the following observation:
Lemma 3.16. If a homology class \( A \in H_2(M; \mathbb{Z}) \), for a 4-dimensional symplectic manifold \( M \), is represented by a simple \( J \)-holomorphic curve \( u : \Sigma \to M \) for some \( \omega \)-tamed almost-complex structure \( J \), then \( A \) is represented by an embedded symplectic surface.

In our case, the spheres of the \( C_n \) configuration are \( J \)-holomorphi, whereas the sphere with self-intersection \((-1)\), \( \Sigma^{-1}_\epsilon \), is \( J_\epsilon \)-holomorphic. So by Gromov’s compactness theorem, we can take a sequence of almost-complex structures \( J_\epsilon \to J \), such that there will exist a subsequence under which the \( J_\epsilon \)-holomorphic sphere \( \Sigma^{-1}_\epsilon \) will converge to some multicurve \( u = (u^1, \ldots, u^N) \). Since the \( u^i \)'s can be multiply covered (multiplicity \( m_i \)), we will write \( v^i \) for the underlying simple \( J \)-holomorphic curve, giving us \([u^i] = m_i [v^i]\) as homology classes in \( H_2(X'; \mathbb{Z}) \). Also, we have:

\[
\tag{3.8} [\Sigma^{-1}_\epsilon] = m_1 [v^1] + m_2 [v^2] + \cdots + m_N [v^N]
\]

in \( H_2(X'; \mathbb{Z}) \). Next, in Proposition \[3.17\] our goal is to show that one of the \( v^i \)'s is indeed an embedded \( J \)-holomorphic sphere of self-intersection \((-1)\) in \( X' \).

**Proposition 3.17.** Let \( \Sigma^{-1}_\epsilon \) and \( v^i \), \( i \in \{1, \ldots, N\} \), as in the above paragraph. Then for at least one \( i \), the simple \( J \)-holomorphic curve \( v^i \) is an embedded sphere with self-intersection \((-1)\).

**Proof.** If \( N = 1 \), then \( m_1 = 1 \) and \( c_1(X') \cdot [v^1] = 1 \), applying the inequality \[3.7\] for \( v^1 \), we have that \([v^1]^2 \geq -1\). If \([v^1]^2 = -1\), then by Corollary \[3.15\] it must be an embedding. If \([v^1]^2 = k \geq 0\), then by Lemma \[3.16\] there exists an embedded symplectic surface \( v^1_S \), with \([u^1] = [v^1_S]\), for which we have \(-\chi(v^1_S) = [v^1_S]^2 - c_1(X') \cdot [v^1_S] = k - 1\). However, if this is the case then this violates the generalized adjunction formula, since we would then have \( k - 1 \geq k + |c_1(X') \cdot [v^1_S]| \), which cannot occur.

We will prove this proposition for general \( N \) with an inductive combinatorial argument using Corollary \[3.15\], Lemma \[3.16\] the adjunction formula for embedded symplectic surfaces, as well as the generalized adjunction formula (Theorem \[2.9\]). First, (although not strictly necessary for the proof), we will prove the proposition for \( N = 2 \), and make a slightly stronger assumption for the initial inductive case, in order to go to the general inductive step in a less cumbersome manner. If \( N = 2 \), then we have:

\[
\tag{3.9} [\Sigma^{-1}_\epsilon] = m_1 [v^1] + m_2 [v^2] + m_1 c_1(X') \cdot [v^1] + m_2 c_1(X') \cdot [v^2] = 1
\]
Case 1: Assume $[v^1]^2 = 2k \geq 0$, then by inequality (3.7), we have: $c_1(X') \cdot [v^1] \leq 2 + 2k$, therefore:

If $c_1(X') \cdot [v^1] = 2 + 2k \Rightarrow v^1$ must be embedded

If $c_1(X') \cdot [v^1] = 2k \Rightarrow \exists v^1_S$ s.t. $-\chi(v^1_S) = 0$

If $c_1(X') \cdot [v^1] = 2k - 2 \Rightarrow \exists v^1_S$ s.t. $-\chi(v^1_S) = 2$

\[ \vdots \Rightarrow \vdots \]

If $c_1(X') \cdot [v^1] = 2 \Rightarrow \exists v^1_S$ s.t. $-\chi(v^1_S) = 2k - 2$

where $v^1_S$ is an embedded symplectic surface such that $[v^1] = [v^1_S]$. This forces $c_1(X') \cdot [v^1] \leq 0$, since if $2 \leq c_1(X') \cdot [v^1] \leq 2 + 2k$, then the embedded surface $v^1_S$ fails to satisfy the generalized adjunction formula (Theorem 2.9).

Also, note that $c_1(X') \cdot [v^1]$ must be an even integer. Next, (3.9) together with $c_1(X') \cdot [v^1] \leq 0$ imply that $c_1(X') \cdot [v^2] \geq 1$. If we apply (3.7) to $v^2$, we get: $[v^2]^2 \geq -1$. Thus, if $[v^2]^2 = -1$, then $c_1(X') \cdot [v^2] = 1$ and by Corollary 3.15 $v^2$ is an embedding, if not then $[v^2]^2 = l \geq 0$ and by (3.7) we get $1 \leq c_1(X') \cdot [v^2] \leq l + 2$, so:

If $c_1(X') \cdot [v^2] = l + 2 \Rightarrow \exists v^2_S$ s.t. $-\chi(v^2_S) = -2$

If $c_1(X') \cdot [v^2] = l \Rightarrow \exists v^2_S$ s.t. $-\chi(v^2_S) = 0$

If $c_1(X') \cdot [v^2] = l - 2 \Rightarrow \exists v^2_S$ s.t. $-\chi(v^2_S) = 2$

\[ \vdots \Rightarrow \vdots \]

If $c_1(X') \cdot [v^2] = 1 \Rightarrow \exists v^2_S$ s.t. $-\chi(v^2_S) = l - 1$ (if $l$ is even)

where $v^2_S$ is an embedded symplectic surface such that $[v^2] = [v^2_S]$. Here, we must have $[v^2]^2 = -1$, since all the cases where $[v^2]^2 = l \geq 0$ and $1 \leq c_1(X') \cdot [v^2] \leq l + 2$, cannot occur because applying the generalized adjunction formula (Theorem 2.9) would result in a contradiction. Consequently, if $[v^1]^2 = 2k \geq 0$, then we must have $[v^2]^2 = -1$ and $v^2$ must be an embedded sphere.

Case 2: Assume $[v^1]^2 = 2k + 1 \geq 0$. If we apply the inequality (3.7) to $v^1$, then we have $c_1(X') \cdot [v^1] \leq 3 + 2k$. However, just as in Case 2, if $1 \leq c_1(X') \cdot [v^1] \leq 3 + 2k$, then there would exist an embedded symplectic surface $v^1_S$, with $[v^1] = [v^1_S]$, such that applying the generalized adjunction formula (Theorem 2.9) for $v^1_S$ would result in a contradiction. Also, as before, we again observe that the integer $[v^1]^2 - c_1(X') \cdot [v^1]$ must be even, thus we have $c_1(X') \cdot [v^1] \leq -1$.

We proceed as before in Case 1, and $c_1(X') \cdot [v^1] \leq -1$ together with equation (3.9), imply that $1 \leq c_1(X') \cdot [v^2]$. Therefore, by the same steps as in Case 1, if $[v^1]^2 = 2k + 1$, then we must have $[v^2]^2 = -1$, and $v^2$ must be an embedded sphere.

We can switch the roles of $v^1$ and $v^2$, in the above cases, which implies that if $[v^2]^2 = k \geq 0$ then $[v^1]^2 = -1$ and $v^1$ must be an embedded sphere. Therefore, we are left with case:
Case 3: Assume both \([v^1]^2 \leq -1\) and \([v^2]^2 \leq -1\). We can again apply inequalities (3.7) to \(v^1\) and \(v^2\), multiplying the first by \(m_1\) and the second one by \(m_2\), adding them together, and using (3.9), we get:

\[
1 - 2m_1 - 2m_2 \leq m_1[v^1]^2 + m_2[v^2]^2,
\]

implying that both \([v^1]^2\) and \([v^2]^2\) can’t be \(\leq 2\). Therefore, we are left with a finite number of possibilities: Either \([v^1]^2 = -1\) and \([v^2]^2 = -k \leq -1\) (satisfying inequality (3.1)) or the same with roles of \(v^1\) and \(v^2\) switched. In this case we have the following:

\[
\begin{align*}
[v^1]^2 &= -1 \quad \text{and} \\
[v^2]^2 &= -k \leq -1 \\
\end{align*}
\]

implies that at least one of \(c_1(X') \cdot [v^1]\) or \(c_1(X') \cdot [v^2]\) must be 1, in turn implying that either \(v^1\) or \(v^2\) is an embedded sphere with self-intersection \((-1)\). If \(k = 1\), then \(c_1(X') \cdot [v^1] = 1\) implying that \(v^1\) is an embedded sphere with self-intersection \((-1)\). If roles of \(v^1\) and \(v^2\) are switched, with \([v^1]^2 = -k \leq -2\) and \([v^2]^2 = -1\), we would have \(v^2\) be an embedded sphere with self-intersection \((-1)\).

This covers all the possibilities of the values for \([v^1]^2\) and \([v^2]^2\) with \(N = 2\), and in each case at least one of \(v^1\) or \(v^2\) is an embedded sphere with self-intersection \((-1)\). We observe that if we replace the heavily used equation (3.9), by:

\[
m_1c_1(X') \cdot [v^1] + m_2c_1(X') \cdot [v^2] = m \geq 1
\]

then everything in the Cases 1-3 would proceed in the same way. In Case 1, whenever we have \(c_1(X') \cdot [v^1] \leq 0\), we can still use equation (3.10) to conclude that \(c_1(X') \cdot [v^1] \geq 1\), and everything would proceed in the same way. In Case 2, whenever we have \(c_1(X') \cdot [v^2] \leq -1\), again we can still use equation (3.10) to conclude that \(c_1(X') \cdot [v^2] \geq 1\). Likewise in Case 3, \(m \geq 1\) in (3.10) is all that is needed to reach the desired conclusion.

Consequently, for a configuration of \(J\)-holomorphic curves \(m_1[v^1] + m_2[v^2]\), with the condition (3.10), at least one of the curves \(v^1\) and \(v^2\) must be an embedded sphere with self-intersection \((-1)\). We make an induction assumption, that if we have a configuration of \(J\)-holomorphic curves \(m_1[v^1] + m_2[v^2] + \cdots + m_{N-1}[v^{N-1}]\), with the condition:

\[
m_1c_1(X') \cdot [v^1] + m_2c_1(X') \cdot [v^2] + \cdots + m_{N-1}c_1(X') \cdot [v^{N-1}] = m \geq 1
\]

then one of the \(v^i\), \(1 \leq i \leq N-1\), is an embedded sphere with self-intersection \((-1)\). We will show that if we have a configuration of \(J\)-holomorphic curves \(m_1[v^1] + m_2[v^2] + \cdots + m_N[v^N]\), with the condition:

\[
m_1c_1(X') \cdot [v^1] + m_2c_1(X') \cdot [v^2] + \cdots + m_Nc_1(X') \cdot [v^N] = m' \geq 1
\]

then one of the \(v^i\), \(1 \leq i \leq N\), is an embedded sphere with self-intersection \((-1)\).
Case 1': Assume $[v^N]^2 = 2k \geq 0$. Then by (3.7), we have $c_1(X') \cdot [v^N] \leq 2k + 2$. However, as in Case 1 from $N = 2$, by Lemma 3.16 the existence of a smooth symplectic surface $v^N_S$, with $[v^N] = [v^N_S]$, together with the generalized adjunction formula (Theorem 2.9) imply that in fact $c_1(X') \cdot [v^N] \leq 0$. Combining this with (3.12), we get:

$$m' - m_1c_1(X') \cdot [v^1] - \cdots - m_{N-1}c_1(X') \cdot [v^{N-1}] = m_Nc_1(X') \cdot [v^N] \leq 0$$

which according to the induction hypothesis implies that at least one $v^i$s, $1 \leq i \leq N - 1$, is an embedded sphere with self-intersection $(-1)$.

Case 2': Assume $[v^N]^2 = 2k + 1 \geq 1$. Again, by (3.7), we have $c_1(X') \cdot [v^N] \leq 2k + 3$. However, as in Case 2 from $N = 2$, we have $c_1(X') \cdot [v^N] \leq -1$, and combining this with (3.12), we get:

$$m' - m_1c_1(X') \cdot [v^1] - \cdots - m_{N-1}c_1(X') \cdot [v^{N-1}] = m_Nc_1(X') \cdot [v^N] \leq m' - m_N \geq 1$$

which again according to the induction hypothesis implies that at least one $v^i$s, $1 \leq i \leq N - 1$, is an embedded sphere with self-intersection $(-1)$.

Case 3': Assume $[v^N]^2 = -1$. Applying (3.7) to $v^N$, we get $c_1(X') \cdot v_N \leq 1$. If $c_1(X') \cdot v_N = 1$, then $v_N$ is an embedded sphere. Otherwise, $c_1(X') \cdot v_N \leq -1$, and as in Case 2', we have:

$$(3.13) \quad m_1c_1(X') \cdot [v^1] + \cdots + m_{N-1}c_1(X') \cdot [v^{N-1}] \geq m' + m_N \geq 1$$

which by the induction hypothesis would imply that at least one of the $v^i$s, for $1 \leq i \leq N - 1$, is an embedded sphere with self-intersection $(-1)$.

Case 4': Assume $[v^N]^2 = -2$. Again, applying (3.7) to $v^N$, we get $c_1(X') \cdot v_N \leq 0$, meaning we have:

$$(3.14) \quad m_1c_1(X') \cdot [v^1] + \cdots + m_{N-1}c_1(X') \cdot [v^{N-1}] \geq m' \geq 1$$

which by the induction hypothesis again would imply that at least one of the $v^i$s, for $1 \leq i \leq N - 1$, is an embedded sphere with self-intersection $(-1)$.

Applying Cases 1'-4' to every $v^i$, $1 \leq i \leq N - 1$, and applying the induction hypothesis each time, gives us that for all the instances where $[v^i]^2 \geq -2$, for any $1 \leq i \leq N$, we will have a $v^j$, for at least one $1 \leq j \leq N$, that is an embedded sphere with self-intersection $(-1)$. Therefore, the only remaining cases is when $[v^i]^2 = -k_i \leq -3$ for all $1 \leq i \leq N$. In which case, we would have $c_1(X') \cdot [v^i] \leq 2 - k_i \leq -1$ for all $1 \leq i \leq N$, which would violate the assumption (3.12). This concludes the induction argument. As a result, when $m' = 1$, this is the case of the Proposition 3.17. \qed
As a result of Proposition 3.17 we now have a $J$-holomorphic embedded sphere of self-intersection $(-1)$ in $(X',\omega')$, which we will name $\Sigma_{-1}$, along with a $C_n$ configuration of $J$-holomorphic spheres. This proves Proposition 3.10.

An important feature of $J$-holomorphic curves, proven by McDuff [MS2], is that their intersections are always positive. In fact, we can always perturb a set of $J$-holomorphic curves and obtain embedded symplectic surfaces intersecting positively and transversally. Li-Usher in [LU], develop McDuff’s techniques further, in order to perturb several $J$-holomorphic curves at once, and obtain the following result:

**Lemma 3.18.** [LU] Any set of distinct $J$-holomorphic curves $C_0,\ldots,C_m$ can be perturbed to symplectic surfaces $C_0',\ldots,C_m'$ whose intersections are all transverse and positive, with $C_i' \cap C_j' \cap C_k' = \emptyset$ when $i,j,k$ are all distinct. Furthermore, there is an almost-complex structure $J'$ arbitrarily $C^1$-close to $J$ such that the $C_i'$ are $J'$-holomorphic.

This is shown by modeling a neighborhood around each intersection point or singularity with holomorphic coordinates, and then slightly perturbing each branch.

Proposition 3.1 is now a direct consequence of Lemma 3.18, Lemma 3.7 and Proposition 3.10.

### 3.2. Step 2.

In this next step of our proof of Theorem 3.5, we use $\Sigma_{-1}$ to construct a homology class $\gamma$ and compute $c_1(X) \cdot \gamma$ in terms of the intersection numbers of $\Sigma_{-1}$ with the spheres of the $C_n$ configuration.

We begin by rationally blowing down the $C_n$ configuration in $(X',\omega')$ symplectically. We can do so by the definition of the symplectic rational blow-down of Symington in [Sy3]. We choose a neighborhood $(N(C_n),\omega'|_{N(C_n)})$ of the spheres in $C_n$, such that $\partial(N(C_n)) \cap \Sigma_{-1} \cong S^1$, $N(C_n) \cap \Sigma_{-1} \cong D^2$ and $(X'\setminus N(C_n)) \cap \Sigma_{-1} \cong D^2$. We denote this rational blow-down of $(X',\omega')$ as $(\tilde{X}',\tilde{\omega}')$. We observe that the symplectic manifolds $(X,\omega)$ and $(\tilde{X}',\tilde{\omega}')$ differ only by the volume of the rational homology ball $B_n$. This is due to the non-uniqueness of the symplectic rational blow-up operation, in terms of the symplectic volume of the $B_n$s. This also implies that the symplectic rational blow-down and the symplectic rational blow-up are not strictly inverse operations. However, $(\tilde{X}',\tilde{\omega}')$ still has the properties that $(X,\omega)$ does: $[c_1(\tilde{X}',\tilde{\omega}')] = -[\omega']$, $b_2^+(\tilde{X}') > 1$, $\text{Bas}_X = \{ \pm c_1(\tilde{X}',\tilde{\omega}') \}$ and $n \geq c_1^2(\tilde{X}',\tilde{\omega}') + 2$. Therefore, for the remainder of the proof, we will abuse notation and write $(X,\omega)$ for $(\tilde{X}',\tilde{\omega}')$.

Back up in $X'$, we can split up rational homology classes as follows:

\[
\begin{align*}
H_2(X';\mathbb{Q}) &= H_2(C_n;\mathbb{Q}) \oplus H_2(X'\setminus C_n;\mathbb{Q}) \\
\Sigma_{-1} &= a + b \\
PD(c_1(X';\omega')) &= c + d
\end{align*}
\]
Since we have $c_1(X', \omega') \cdot [\Sigma_{-1}] = 1$, then we have $1 = a \cdot c + b \cdot d$.

Let $D$ be a 2-disk defined by:

\begin{equation}
D = (X' \setminus N(C_n)) \cap \Sigma_{-1} \subset X.
\end{equation}

Observe that $D \subset X$, since by definition $X \cong (X' \setminus N(C_n)) \cup B_n$. Also, since $\partial D \subset \partial B_n$ and $H_1(B_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, then $n\partial D \cong 0 \in H_1(B_n; \mathbb{Q})$. Back down in $X$, we can now define the class $\gamma \in H_2(X; \mathbb{Q})$ by:

\begin{equation}
\gamma = nD + e^2
\end{equation}

where $e^2$ is just a 2-cell in $B_n \subset X$ for which $\partial(e^2) = n\partial D$. Since, $c_1(X', \omega') \cdot [\Sigma_{-1}] = a \cdot c + b \cdot d$ and $H_2(B_n; \mathbb{Q})$ is trivial, we have:

\begin{equation}
c_1(X, \omega) \cdot \gamma = nb \cdot d.
\end{equation}

Our goal is to compute $c_1(X, \omega) \cdot \gamma$ explicitly in terms of the intersections of the sphere $\Sigma_{-1}$ with the spheres of $C_n$. Next, in Step 3 we will show that whenever $c_1(X, \omega) \cdot \gamma > 0$ then we also have $\omega \cdot \gamma > 0$, thus contradicting the condition $[c_1(X, \omega)] = -[\omega]$. In Step 4 we will show that the intersection configurations of $\Sigma_{-1}$ with $C_n$ yielding $c_1(X, \omega) \cdot \gamma \leq 0$ will also produce a contradiction.

In order to compute $c_1(X, \omega) \cdot \gamma$, all we need to compute is $a \cdot c$, since $nb \cdot d = n(1 - a \cdot c)$, which is fairly standard. Recall, we denote the spheres of the $C_n$ configuration by $S_1, S_2, S_3, \ldots, S_{n-1}$, with $[S_1]^2 = -n - 2$ and $[S_i]^2 = -2$ for $2 \leq i \leq n - 1$. Thus, we may denote the basis of $H_2(C_n; \mathbb{Q})$ by $[S_1], [S_2], [S_3], \ldots, [S_{n-1}]$. As a result “$a$”, the homology class of $\Sigma_{-1}$ lying in $H_2(C_n; \mathbb{Q})$, may be expressed as:

\begin{equation}
a = a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]
\end{equation}

where $a_i \in \mathbb{Q}$. Next, let $I_j$ be the intersection numbers of $[\Sigma_{-1}]$ and $[S_j]$:

\[
\begin{align*}
[\Sigma_{-1}] \cdot [S_1] &= I_1 \\
[\Sigma_{-1}] \cdot [S_2] &= I_2 \\
[\Sigma_{-1}] \cdot [S_3] &= I_3 \\
& \vdots \\
[\Sigma_{-1}] \cdot [S_{n-1}] &= I_{n-1}.
\end{align*}
\]

(Note, we have $\alpha_j = I_j$ (see Definition 3.2), since the intersections of the sphere $\Sigma_{-1}$ with the spheres $S_j$ are positive and transverse.) In order to express the $a_i$ in terms of the intersection numbers $I_j$, we need to solve the following linear system:

\[
\begin{align*}
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_1] &= I_1 \\
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_2] &= I_2 \\
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_3] &= I_3 \\
& \vdots \\
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_{n-1}] &= I_{n-1}.
\end{align*}
\]
Next, we can express “c”, the homology class of \(PD(c_1(X',\omega'))\) lying in \(H_2(C_n;\mathbb{Q})\), in terms of the basis \([S_1],[S_2],[S_3],\ldots,[S_{n-1}]\):

\[
(3.19) \quad c = c_1[S_1] + c_2[S_2] + c_3[S_3] + \cdots + c_{n-1}[S_{n-1}]
\]

where the \(c_i \in \mathbb{Q}\). Since the \(S_i\) are symplectic spheres, we have the following:

\[
\begin{align*}
    c_1(X') \cdot [S_1] & = -n \\
    c_1(X') \cdot [S_2] & = 0 \\
    c_1(X') \cdot [S_3] & = 0 \\
    \vdots & \vdots \\
    c_1(X') \cdot [S_{n-1}] & = 0.
\end{align*}
\]

As a result, the quantity \(a \cdot c\) is the dot product of the following two vectors in \(H_2(C_n;\mathbb{Q})\):

\[
(3.20) \quad [a_1, a_2, a_3, \ldots, a_{n-1}]
\]

and

\[
(3.21) \quad [-n, 0, 0, \ldots, 0].
\]

Consequently, we only have to compute \(a_1\) in terms of the intersection numbers \(I_j\), which corresponds to the first row of the inverse of the \(H_2(C_n;\mathbb{Z})\) intersection matrix, giving us:

\[
(3.22) \quad a_1 = \frac{-n+1}{n^2} I_1 + \frac{-n+2}{n^2} I_2 + \cdots + \frac{-2}{n^2} I_{n-2} + \frac{-1}{n^2} I_{n-1}.
\]

Since \(a \cdot c = a_1 \cdot n \) and \(c_1(X,\omega) \cdot \gamma = n(1 - a \cdot c)\), we finally get:

\[
(3.23) \quad c_1(X,\omega) \cdot \gamma = n - I_{n-1} - 2I_{n-2} - 3I_{n-3} - \cdots - (n-2)I_2 - (n-1)I_1.
\]

Note, that since \(\alpha_j = I_j\), then we have shown that if the symplectic embedding of \(B_n \hookrightarrow X\) is of type \(\langle \alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_{n-1} \rangle\), then there is a class \(\gamma\), such that \(c_1(X) \cdot \gamma\) is given by \((3.23)\).

### 3.3. Step 3.

In this step we will show that if \(c_1(X,\omega) \cdot \gamma > 0\), then we must also have \(\omega \cdot \gamma > 0\), thus violating the \([c_1(X,\omega)] = -[\omega]\) condition of \((X,\omega)\).

This will eliminate the possibility of embeddings \(B_n \hookrightarrow X\) of type \(\mathcal{A}_1 \subset \mathcal{A}\), where \(\mathcal{A}_1\) is the set of \((n-1)\)-tuples \(\langle \alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_{n-1} \rangle\) satisfying the inequality \((3.24)\) (with \(\alpha_j = I_j\)).

If \(c_1(X) \cdot \gamma > 0\), we have:

\[
(3.24) \quad n - I_{n-1} - 2I_{n-2} - 3I_{n-3} - \cdots - (n-2)I_2 - (n-1)I_1 > 0.
\]

First, we will use the following lemma to rule out some cases.

**Lemma 3.19.** Let \(\Sigma\) and \(S\) be embedded spheres in a smooth 4-manifold \(M\) with \(b_2^+(M) > 1\), such that \([\Sigma]^2 = -1\) and \([S]^2 = -2\). Assume \([\Sigma] \cdot [S] = k \geq 1\), then we must have \(k = 1\).

**Proof.** The proof will follow from the following proposition:
Proposition 3.20. \[\text{[FM]}\] Let \( M \) be an oriented 4-manifold and \( S^2 \subset M \) be an embedded sphere with \( \alpha \in H^2(M; \mathbb{Z}) \) the cohomology class dual to \( S^2 \). If \( \alpha^2 = -1 \) or \( -2 \), there is an orientation preserving self-diffeomorphism \( \varphi \) of \( M \) such that \( \varphi^* = R_\alpha \), where:

\[(3.25)\quad R_\alpha(x) = x + 2(x \cdot \alpha)\alpha \]

if \( \alpha^2 = -1 \) and

\[(3.26)\quad R_\alpha(x) = x + (x \cdot \alpha)\alpha \]

if \( \alpha^2 = -2 \). (Note, in both cases \( R_\alpha = -\alpha \) and \( R_\alpha^2 = \text{Id} \), hence often referred to as the reflection automorphism.)

As a result of this proposition, the spheres \( \Sigma \) and \( S \) will induce orientation preserving diffeomorphisms on \( M \), corresponding to the following reflection automorphisms on \( H_2(M; \mathbb{Z}) \):

\[(3.27)\quad R_\Sigma(x) = x + 2(x \cdot [\Sigma])[\Sigma] \]

\[(3.28)\quad R_S(x) = x + (x \cdot [S])[S] \]

We begin with applying \( R_S \) to \( x = [\Sigma] \):

\[(3.29)\quad R_S([\Sigma]) = [\Sigma] + k[S] \]

Next, we apply \( R_\Sigma \) to \( x = [\Sigma] + k[S] \):

\[(3.30)\quad R_\Sigma([\Sigma] + k[S]) = (2k^2 - 1)[\Sigma] + k[S] \]

In this manner, we can continue to alternately apply \( R_S \) and \( R_\Sigma \), and get:

\[ R_S((2k^2 - 1)[\Sigma] + k[S]) = (2k^2 - 1)[\Sigma] + (2k^3 - 2k)[S] := [A_S] \]

\[ R_\Sigma([A_S]) = (4k^4 - 6k + 1)[\Sigma] + (2k^3 - 2k)[S] := [A_\Sigma S] \]

\[ R_S([A_\Sigma S]) = (4k^4 - 6k + 1)[\Sigma] + (4k^5 - 8k^3 + 3k)[S] \]

... = ...

We observe that as long as \( k \geq 2 \), the polynomials above keep growing, thus implying that there is an infinite number of spheres with homology classes of the form \( x = s_1[\Sigma] + s_2[S] \) with \( x^2 = -1 \). This cannot occur, since if it did, it would imply that there is an infinite number of Seiberg-Witten basic classes of the manifold \( M \), which cannot happen if \( b_2^+(M) > 1 \). \( \square \)

Lemma 3.19 immediately implies the following Corollary:

Corollary 3.21. With the intersection numbers \( I_j = [\Sigma_{j-1}] \cdot [S_j] \), as in section 3.3, we must have \( I_2 + I_3 + I_4 + \cdots + I_{n-1} \leq 1 \).

Proof. The spheres \( S_j \) with \( 2 \leq j \leq n - 1 \) intersect transversally with the neighboring spheres in the plumbing configuration \( C_n \). Therefore, we can construct the sphere \( S_2^{n-1} \), which is the union of the spheres \( S_j, 2 \leq j \leq n-1 \), with all the transverse intersection points smoothed out. The sphere \( S_2^{n-1} \) has self-intersection \((-2)\), since its homology class is:

\[(3.31)\quad [S_2^{n-1}] = [S_2] + [S_3] + [S_4] + \cdots + [S_{n-1}] \].
Now we can apply Lemma 3.19 with \( \Sigma = \Sigma_{-1} \) and \( S = S^2_{n-1} \), and conclude that \([\Sigma_{-1}] \cdot [S^2_{n-1}] \) is at most 1, implying:

\[
(3.32) \quad [\Sigma_{-1}] \cdot [S^2_{n-1}] = I_2 + I_3 + I_4 + \cdots + I_{n-1} \leq 1.
\]

As a direct consequence of Corollary 3.21 and (3.23), we have the following:

**Corollary 3.22.** If \( c_1(X, \omega) \cdot \gamma > 0 \), with \( \gamma = nD + e^2 \) as defined in section 3.2, then there is only one \( j, 1 \leq j \leq n-1 \), for which \( I_j = 1 \) and \( I_k = 0 \) if \( j \neq k \).

Next, we will use toric and almost-toric fibrations, introduced in section 2.3, to show that for those cases where \( c_1(X, \omega) \cdot \gamma > 0 \), we have \( \omega \cdot \gamma > 0 \).

**Proposition 3.23.** If there is only one \( j, 1 \leq j \leq n-1 \), for which \( I_j = 1 \) and \( I_k = 0 \) if \( j \neq k \), then \( \omega \cdot \gamma > 0 \).

**Figure 12.** Visible surfaces represented by curves \( \mu^1_j \) in a toric model for \( C_n \)

*Proof.* If there is only one \( j, 1 \leq j \leq n-1 \), for which \( I_j = 1 \) and \( I_k = 0 \) if \( j \neq k \), then by definition, the sphere \( \Sigma_{-1} \) only intersects the sphere \( S_j \) of the \( C_n \) configuration once at a point \( a_j \). We can present part of \( \Sigma_{-1} \) as it intersects \( S_j \), by a visible surface (see Definition 2.32), with the curve \( \mu^1_j \) and a compatible covector \( u_j = \begin{bmatrix} -1 \\ n+1 \end{bmatrix} \), for all \( 1 \leq j \leq n-1 \), (see Figure 12). We have \([\Sigma_{-1}] \cdot [S_j] = 1 \), since \(|u_j \times v_j| = 1\) for all \( j \), where the \( v_j = \begin{bmatrix} 1-j \\ (j-1)n+j \end{bmatrix} \) are the collapsing covectors corresponding to the part of the 1-stratum that represents the spheres \( S_j \), thus satisfying item (2) in the definition of visible surfaces (Definition 2.32).

After we perform the rational blow-down, as we do in the beginning of Step 2, we obtain the almost-toric base, as seen in Figure 13 (also see Figure 10). We recall here that the class \( \gamma = nD + e^2 \), where \( D \) is the “remains” of \( \Sigma_{-1} \) in \( X; D = (X'/N(C_n)) \cap \Sigma_{-1} \). Since \( \Sigma_{-1} \) is a symplectic sphere, we have \( \omega \cdot nD > 0 \). In order to show that \( \omega \cdot \gamma > 0 \), we need to show that \( \omega \) is
positive on the 2-cell, $e^2$, which “closes up” $nD$ i.e. $\partial e^2 = \partial nD$. We will do this by exhibiting the disk $e^2 \subset B_n$ as a visible surface in the almost-toric fibration of $B_n$.

$$L(n^2, n-1) \quad L(n^2, n-1) \quad L(n^2, n-1)$$

**Figure 13.** Visible surfaces represented by curves $\mu_j^2$ in almost-toric model for $B_n$

In order to represent $e^2 \subset \gamma$ as a visible surface in the almost-toric base in Figure 13, we need to choose a curve $\mu_j^2$, such that it extends the curve $\mu_j^1$ and whose collection of compatible classes in $H_2(F_b, \mathbb{Z})$, for all $b \in \mu_j^2$, forms a disk. We can arrange the visible surface represented by $\mu_j^2$ to be symplectic, because of Proposition 2.35. Also, we can arrange $e^2$ such that it hits the Lagrangian core $L_n$, represented by the straight line $\nu$, thus the curve $\mu_j^2$ hits $\nu$ and then the node $s$.

On one hand, the compatible class of $\mu_j^2$ must be the same as the vanishing class of the node $s$, in order for $\mu_j^2$ to represent a visible surface (and be a disk). On the other hand, the curve $\mu_j^2$ is a continuation of the curve $\mu_j^1$. However, the curve $\mu_j^1$ represents the visible surface for $D \in C_n$. At the point $b_j$ in the toric fibration, which lies on the curve representing the boundary $\partial C_n = L(n^2, n-1)$, the compatible covector is $u_j = \left[\frac{-1}{n+1}\right]$, corresponding to the class $\partial D \in L(n^2, n-1)$. When we begin the curve $\mu_j^2$, at the point $b_j$, the compatible class should correspond to $n\partial D \in L(n^2, n-1)$, making the compatible covector $n\left[\frac{-1}{n+1}\right] = \left[\frac{-n}{n^2 + n}\right]$. In $\partial C_n = \partial B_n = L(n^2, n-1)$, we have the compatible class with the covector $\left[\frac{-n}{n^2 + n}\right]$ homologous to the compatible class with the covector $\left[\frac{-1}{n}\right]$. As a result, the curve $\mu_j^2$ will have a compatible covector $u_j' = \left[\frac{-1}{n}\right]$ for all $1 \leq j \leq n-1$, exactly the same as the vanishing covector of the node $s$. Consequently, the curves $\mu_j^2$ do indeed represent visible surfaces, the 2-cells $e^2$ in the construction of $\gamma$. 
As a result, we have explicitly exhibited that the class \( \gamma = nD + e^2 \) is such that \( \omega \cdot \gamma > 0 \), by representing \( D \) and \( e^2 \) as visible surfaces in the almost-toric fibrations of \( C_n \) and \( B_n \), with positive symplectic area. \( \square \)

**Corollary 3.24.** Let \( \gamma \) be as above. If \( c_1(X, \omega) \cdot \gamma > 0 \), then \( \omega \cdot \gamma > 0 \).

**Proof.** This is a direct consequence of Corollary 3.22 and Proposition 3.23. \( \square \)

As a result of Corollary 3.24, we have proved that embeddings of \( B_n \hookrightarrow X \) of type \( A_1 \subset A \) cannot occur, where \( A_1 \) is the set of \( (n-1) \)-tuples \( \langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \), such that \( c_1(X) \cdot \gamma > 0 \) in terms of the intersection numbers \( I_j = \alpha_j \).

### 3.4. Step 4.

In this final step, we will show that symplectic embeddings of \( B_n \hookrightarrow X \) of type \( (A - A_1) \) (those of type \( A \) and not \( A_1 \)) and type \( \mathcal{E}_k, k \geq c_2^2(X, \omega) + 2 \) cannot occur. These sets of \( (n-1) \)-tuples precisely correspond with \( c_1(X, \omega) \cdot \gamma \leq 0 \), i.e. the cases where:

\[
(3.33) \quad n - I_{n-1} - 2I_{n-2} - 3I_{n-3} - \cdots - (n-2)I_2 - (n-1)I_1 \leq 0.
\]

**Lemma 3.25.** Let \( I_1 = [\Sigma_{-1}] \cdot [S_1] \) be as above, then \( I_1 \leq n \).

**Proof.** First, we apply the generalized adjunction formula (Theorem 2.10) to the sphere \( \Sigma_{-1} \), which gives us that \( c_1(X', \omega') + 2[\Sigma_{-1}] \) is a SW basic class of \( X' \). Second, we apply Theorem 2.10 to the sphere \( S_1 \), since \( [S_1]^2 = -n - 2 \) for a SW basic class \( L \) we have:

\[
(3.34) \quad |L \cdot [S_1]| \leq n
\]

if we let \( L = c_1(X', \omega') + 2[\Sigma_{-1}] \), then we have:

\[
(3.35) \quad |(c_1(X', \omega') + 2[\Sigma_{-1}]) \cdot [S_1]| = |c_1(X', \omega') \cdot [S_1] + 2[\Sigma_{-1}] \cdot [S_1]| \leq n.
\]

Since \( S_1 \) is a symplectic sphere, we have \( c_1(X', \omega') \cdot [S_1] = -n \), therefore, we must have: \( I_1 = [\Sigma_{-1}] \cdot [S_1] \leq n \). \( \square \)

**Corollary 3.26.** Let \( I_i = [\Sigma_{-1}] \cdot [S_i], 1 \leq i \leq n - 1 \), be as above, then \( I_1 + I_2 + I_3 + \cdots + I_{n-1} \leq n \).

**Proof.** The spheres \( S_i \) intersect each other transversally in the \( C_n \) configuration. We can (as done in Corollary 3.21) construct the sphere \( S_i^{n-1} \), which is the union of the spheres \( S_i, 1 \leq i \leq n - 1 \), with all the intersection points smoothed out (this can be done symplectically). The self-intersection number of the sphere \( S_i^{n-1} \) is \( (-n - 2) \), since its homology class is:

\[
(3.36) \quad [S_i^{n-1}] = [S_1] + [S_2] + [S_3] + \cdots + [S_{n-1}].
\]

Consequently, by applying Lemma 3.25 since \( I_1 \leq n \) then so is \( I_1 + I_2 + I_3 + \cdots + I_{n-1} \leq n \). \( \square \)
In light of Corollaries 3.21, 3.26 and Lemma 3.25, the intersection patterns of $\Sigma_{-1}$ with the spheres of the $C_n$ configuration, giving us $c_1(X,\omega) \cdot \gamma \leq 0$, which we still have to rule out are:

1. $2 \leq I_1 \leq n$ and $I_j = 0$ for all $2 \leq j \leq n-1$
2. $1 \leq I_1 \leq n-1$ and $I_j = 1$ for one $2 \leq j \leq n-1$.

**Lemma 3.27.** The following intersection configurations:

(a) $I_1 = n$ and $I_j = 0$ for all $2 \leq j \leq n-1$
(b) $I_1 = 1$ and $I_{n-1} = 1$ ($I_j = 0$ for all $2 \leq j \leq n-2$)

will force the 4-manifold $(X,\omega)$ to have basic classes in addition to $\pm K = \mp c_1(X,\omega)$, thus contradicting the hypothesis in Theorem 3.5.

**Proof.** We begin with looking at the piece of the relative homology long exact sequence for the pair $(C_n, \partial C_n)$:

\[(3.37) \quad 0 \rightarrow H_2(C_n;\mathbb{Z}) \xrightarrow{i} H_2(C_n, \partial C_n;\mathbb{Z}) \xrightarrow{\partial} H_1(\partial C_n;\mathbb{Z}) \rightarrow 0\]

Let $\delta \in H_2(C_n, \partial C_n;\mathbb{Z})$ be a relative class that is a union of the disks $\Sigma_{-1} \cap N(C_n)$, where $N(C_n)$ is a neighborhood of the spheres of the $C_n$. For all $1 \leq j \leq n-1$, we have:

\[(3.38) \quad \delta \cdot [S_j] = \Sigma_{-1} \cdot [S_j] = I_j .\]

In case (a), for the class $-n\delta$ we have:

\[-n\delta \cdot [S_1] = -n^2\]
\[-n\delta \cdot [S_2] = 0\]
\[\vdots = \vdots\]
\[-n\delta \cdot [S_{n-1}] = 0 .\]

The relative class $-n\delta$ can be supported in the interior by the following homology class:

\[(3.39) \quad -n\delta = (n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}]\]

since,

\[\begin{align*}
((n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}]) \cdot [S_1] &= -n^2 \\
((n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}]) \cdot [S_2] &= 0 \\
\vdots &= \vdots \\
((n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}]) \cdot [S_{n-1}] &= 0 .
\end{align*}\]

In case (b), for the class $-n\delta$ we have:

\[-n\delta \cdot [S_1] = -n\]
\[-n\delta \cdot [S_2] = 0\]
\[\vdots = \vdots\]
\[-n\delta \cdot [S_{n-2}] = 0\]
\[-n\delta \cdot [S_{n-1}] = -n .\]
In this case, the relative class \(-n\delta\) can be supported in the interior by the following homology class:

\[
- n\delta = [S_1] + 2[S_2] + \cdots + (n-1)[S_{n-1}]
\]

since,

\[
\begin{align*}
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_1] &= -n \\
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_2] &= 0 \\
\vdots &= \vdots \\
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_{n-2}] &= 0 \\
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_{n-1}] &= -n.
\end{align*}
\]

In both cases (a) and (b) we have the relative class \(-n\delta \in \text{im}(i) = \ker(\partial)\), implying that \(\partial(\delta) \in H_1(\partial C_n; \mathbb{Z}) \cong H_1(L(n^2, n - 1); \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}\) is an element of order \(n\). According to Theorem 2.7 and \[Pa1\], this implies that the basic classes \(\pm(c_1(X', \omega') + 2[\Sigma_{-1}])\) of \((X', \omega')\) extend to basic classes of the rational blow-down \((X, \omega)\). Note, the classes \(\pm c_1(X', \omega')\) must extend to the basic classes of \((X, \omega)\) since they are the \pm canonical class. Moreover, \(c_1(X', \omega') + 2[\Sigma_{-1}]\) and \(c_1(X', \omega')\) must extend to different basic classes on \((X, \omega)\), otherwise we would have \([\Sigma_{-1}] = 0\). Therefore, the 4-manifold \(X\) will have at least four basic classes, which is a contradiction. \(\square\)

**Lemma 3.28.** The following intersection configurations:

i) \(I_1 = 1\) and \(I_j = 1\) for one \(j\) such that \(2 \leq j \leq n - 2\)

ii) \(I_1 = k\) with \(2 \leq k \leq n-1\) and \(I_j = 1\) for one \(j\) such that \(2 \leq j \leq n-1\)

cannot occur in \((X', \omega')\), since it is a symplectic 4-manifold with \(b_2^+ (X') > 1\).

**Proof.** Assume \(I_1 = k\) for \(1 \leq k \leq n - 1\) and let \(K = c_1(X', \omega')\) be the (negative of the) canonical class of \((X', \omega')\). Since \(S_1\) is a symplectic sphere with self-intersection \((-n - 2)\), we have that \(K \cdot [S_1] = -n\). Let \(L\) be any SW basic class of \(X'\), then according to the generalized adjunction formula for immersed spheres, Theorem 2.10, we have that

\[
|L \cdot [S_1]| \leq n
\]

or \(SW_{X'}(L + 2[S_1]) = SW_{X'}(L)\) if \(L \cdot S_1 \geq 0\), \((SW_{X'}(L - 2[S_1]) = SW_{X'}(L)\) if \(L \cdot [S_1] \leq 0\). We will produce a specific SW basic class \(L\) which will fail to satisfy (3.41) and for which \(L \pm 2S_1\) cannot be a SW basic class since the symplectic 4-manifold \(X'\) is of simple type.

First, we observe, that by smoothing out the transverse intersections of the spheres in the \(C_n\) configuration and the sphere \(\Sigma_{-1}\), we have the following
spheres in $X'$, each with self-intersection $(-1)$:
\[
\begin{align*}
\Sigma_1 \\
\Sigma_1 + S_j \\
\Sigma_1 + S_j + S_{j-1} \\
\vdots \\
\Sigma_1 + S_j + S_{j-1} + \cdots + S_2 \\
\Sigma_1 + S_j + S_{j-1} + \cdots + S_2 + S_{j+1} \\
\Sigma_1 + S_j + S_{j-1} + \cdots + S_2 + S_{j+1} + S_{j+2} \\
\vdots \\
(3.42) \quad \Sigma_1 + S_j + S_{j-1} + \cdots + S_2 + S_{j+1} + S_{j+2} + \cdots + S_{n-1}.
\end{align*}
\]

Second, using these spheres, we can construct several SW basic classes using Theorem 2.10 as follows: We start off by letting $L = K$ and $x = \Sigma_1$ as in Theorem 2.10, since $|K : [\Sigma_1]| \leq -1$ cannot happen, $K + 2[\Sigma_1]$ must be a SW basic class. Note, that $K^2 = (K + 2[\Sigma_1])^2$, as required for 4-manifolds of simple type. Next, we let $L = K + 2[\Sigma_1]$ and $x = \Sigma_1 + S_j$, and after applying Theorem 2.10 again, we get that since $|(K + 2[\Sigma_1]) : ([\Sigma_1] + [S_j])| \leq -1$ cannot happen, then $(K + 2[\Sigma_1]) + 2([\Sigma_1] + [S_j])$ is a SW basic class. Proceeding in this manner, with all the spheres of (3.42), we get that $K'$ is a SW basic class of $X'$, where $K'$ is:
\[
(3.43) \quad K' = K + 2(n - 1)[\Sigma_1] + 2(n - 2)[S_j] + \cdots + 2(n - j)[S_2] + 2(n - (j + 1))[S_{j+1}] + \cdots + 2[S_{n-1}].
\]

Next, we again apply Theorem 2.10 with $L = K'$ and $x = S_1$, and as in (3.41), we get:
\[
(3.44) \quad |K' \cdot [S_1]| = |K \cdot [S_1] + 2(n - 1)[\Sigma_1] \cdot [S_1] + 2(n - j)[S_2] \cdot [S_1]|
\]

If $k = 1$, then (3.44) becomes:
\[
(3.45) \quad |2n - 2 - 2j| \leq n,
\]

which for $n \geq 4$ and $2 \leq j \leq n - 2$ cannot occur. Therefore, $K' + 2[S_1]$ is forced to be a SW basic class, however, this is impossible since $X'$ is of simple type and $(K' + 2[S_1])^2 \neq (K')^2$. Consequently, the configurations with intersection numbers $I_1 = 1$ and $I_j = 1$ for one $j$ for which $2 \leq j \leq n - 2$ cannot occur.

If $2 \leq k \leq n - 1$, then the inequality (3.44) cannot hold if $2 \leq j \leq n - 1$. Therefore, again $K' + 2[S_1]$ must be a SW basic class, but this cannot happen either since $X'$ is of simple type. Consequently, the configurations with the intersection numbers $I_1 = k$ with $2 \leq k \leq n - 1$ and $I_j = 1$ for one $j$ for which $2 \leq j \leq n - 1$ cannot occur. \qed
The results in section 3.2 as well as Lemmas 3.27 and 3.28 imply that if \( n \geq c_1^2(X, \omega) + 2 \), then there cannot be symplectic embeddings of \( B_n \hookrightarrow (X, \omega) \) of type \( A \). The only configurations which remain are those with \( I_1 = k \) where \( 2 \leq k \leq n - 1 \) and \( I_j = 0 \) for \( j \) with \( 2 \leq j \leq n - 1 \), which correspond to symplectic embeddings of \( B_n \hookrightarrow X \) of type \( \mathcal{E}_k \) for \( 2 \leq k \leq n - 1 \). Next, we will show that symplectic embeddings \( B_n \hookrightarrow (X, \omega) \) of type \( \mathcal{E}_k, k \geq c_1^2(X, \omega) + 2 \), cannot occur.

**Remark 3.29.** The key difference between symplectic embeddings of \( B_n \hookrightarrow X \) of type \( A \) and \( \mathcal{E}_k \) is that in the embeddings of type \( \mathcal{E}_k \), the sphere \( \Sigma_{-1} \) does not intersect any sphere with self-intersection \((-2)\), which as seen in Lemma 3.28 creates quite a few Seiberg-Witten basic classes leading to contradictions because of adjunction formulas. Therefore, in order to prevent embeddings of type \( \mathcal{E}_k, k \geq c_1^2(X, \omega) + 2 \), we need \( c_1^2(X, \omega) \) to be low enough to guarantee the existence of several spheres with self-intersection \((-1)\), in addition to \( \Sigma_{-1} \).

The next Lemma will be instrumental in showing this last part of Theorem 3.5.

**Lemma 3.30.** Let \( S^d_r \subset (M, \omega) \) be an immersed symplectic sphere with self-intersection \( r \) and \( d \) double points, where \((M, \omega)\) is a symplectic 4-manifold with \( c_1^2(M, \omega) \leq -1 \) and \( b_2^+(M) > 1 \). Let \( C_n^r \subset (M, \omega) \) be the linear plumbing of symplectic spheres \( S_1^r, S_2, S_3, \ldots, S_{n-1} \), where the \( S_j \) are embedded symplectic spheres with \( [S_j]^2 = -2 \) for \( 2 \leq j \leq n - 1 \). Then there exists an embedded symplectic sphere \( \hat{\Sigma}_{-1} \subset (M, \omega) \) with \( [\hat{\Sigma}_{-1}]^2 = -1 \), and \( C_n^r \subset (M, \omega) \), a linear plumbing configuration of symplectic spheres \( S^d_r, S_2^r, S_3^r, \ldots, S_{n-1}^r \) (each \( S_i^r \) is a perturbation of \( S_i \)), such that if \( \hat{\Sigma}_{-1} \) intersects any spheres in the \( C_n^r \) configuration it must do so positively and transversally.

**Proof.** The proof of this lemma mirrors the proof of Proposition 3.1 in section 3.1. As in the proof of Proposition 3.1, we start by putting an \( \omega \)-compatible almost-complex structure \( J \) on the spheres of the \( C_n^r \) configuration. We can do so in the same manner as was done for the \( C_n \) configuration in Lemma 3.7. The only difference is that we apply Lemma 3.8 to the small Darboux neighborhoods of the double points of the immersed sphere \( S^d_r \), as well as to the small Darboux neighborhoods of the intersections between adjacent spheres in the plumbing.

As before, since \( c_1^2(M, \omega) \leq -1 \), by Corollary 2.14 of the theorems of Taubes (Theorems 2.12 and 2.13), there must exist a \( J_r \)-holomorphic sphere \( \hat{\Sigma}_{-1} \) in \((M, \omega)\), with \( [\hat{\Sigma}_{-1}]^2 = -1 \) for a generic \( \omega \)-compatible almost-complex structure \( J_r \). As in section 3.1, the spheres of the \( C_n^r \) configuration are \( J_r \)-holomorphic curves, and the sphere \( \hat{\Sigma}_{-1} \) is a \( J_r \) holomorphic curve. Therefore, we use Gromov Compactness (Theorem 3.13), and take a sequence of almost-complex structures \( J_r \to J \) of which there exists a subsequence such
that $\hat{\Sigma}_{r-1}$ converges to a multicurve $\hat{u} = (\hat{u}^1, \ldots, \hat{u}^N)$. We can then apply Proposition 3.17 and conclude that there exists at least one $i$, such that $\hat{u}^i$ is an embedded $J$-holomorphic sphere, which we will label by $\hat{\Sigma}_{-1}$.

Again, as before, we apply Lemma 3.18 to the $J$-holomorphic curves $S_r^d$, $S_2$, $S_3$, $\ldots$, $S_{n-1}$, $\hat{\Sigma}_{-1}$, and perturb these into symplectic surfaces $\hat{S}_r^d$, $\hat{S}_2$, $\hat{S}_3$, $\ldots$, $\hat{S}_{n-1}$, which will intersect each other positively and transversally. The symplectic surface $\hat{S}_r^d$ has genus $g(\hat{S}_r^d) = d$, since it was obtained from the immersed sphere $S_r^d$ by smoothing out the double points, see [LU]. However, we can replace $\hat{S}_r^d$ back with $S_r^d$, and consider the linear plumbing configuration of spheres $S_r^d$, $S_2$, $S_3$, $\ldots$, $S_{n-1}$. We can still conclude that the sphere $\hat{\Sigma}_{r-1}$ (after a possible perturbation) intersects positively and transversally with that configuration, since $S_r^d$ differs from $\hat{S}_r^d$ only in small neighborhoods around its double points. 

\begin{proposition}
Let $B_n \hookrightarrow (W, \omega)$, where $(W, \omega)$ is a symplectic 4-manifold with $b_2^+ (W) > 1$, be an embedding of type $\mathcal{E}_k$, i.e. $I_1 = k$ and $I_j = 0$ for $2 \leq j \leq n - 1$, for $k \geq c_2^2 (W, \omega) + 2$, then $(W, \omega)$ must have SW basic classes in addition to $\pm c_1 (W, \omega)$.
\end{proposition}

\begin{proof}
Assume $B_n \hookrightarrow (W, \omega)$ is an embedding of type $\mathcal{E}_k$. This implies that after symplectically rationally blowing up $(W, \omega)$, we obtain $(W', \omega')$ which contains a $C_n$ configuration of symplectic spheres, and a symplectic sphere $\Sigma_{-1}$ which intersects the sphere $S_1$ ($[S_1]^2 = -n - 2$) $k$ times positively and transversally.

We blow down the sphere $\Sigma_{-1}$, and obtain a manifold $(W^{(2)}, \omega^{(2)})$, such that $c_1^2 (W^{(2)}, \omega^{(2)}) = c_1^2 (W', \omega') + 1$. The sphere $S_1 \subset W$ descends to an immersed sphere $S_{-n-2+k}^{k-tuple}$ which has self-intersection $(-n - 2 + k^2)$ and a $k$-tuple intersection point. Since the sphere $S_1$ was in fact pseudo-holomorphic, and the blow-down map is holomorphic, the immersed sphere $S_{-n-2+k}^{k-tuple}$ is pseudo-holomorphic as well. Therefore, $S_{-n-2+k}^{k-tuple}$ can be perturbed to a pseudo-holomorphic sphere with only double point intersections (see [Mc]), of which there will be $\frac{k(k-1)}{2}$ such double points. Consequently, the manifold $(W^{(2)}, \omega^{(2)})$ will contain a linear configuration $C_{-n-2+k}^{k(k-1)/2}$ of spheres $S_{-n-2+k}^{k(k-1)/2}$, $S_2$, $S_3$, $\ldots$, $S_{n-1}$, where $S_{-n-2+k}^{k(k-1)/2}$ is an immersed symplectic sphere with self-intersection $r = -n - 2 + k^2$ and $d = \frac{k(k-1)}{2}$ double points.

Next, since $k \geq c_2^2 (W, \omega) + 2$, we have that $c_1^2 (W^{(2)}, \omega^{(2)}) \leq -1$, therefore, we can apply Lemma 3.30 and obtain an embedded symplectic sphere of self-intersection $(-1)$: $\Sigma_{-1}^{(2)} \subset W^{(2)}$. This sphere $\Sigma_{-1}^{(2)}$ must intersect the configuration $C_{-n-2+k}^{k(k-1)/2}$, since if it did not, we could blow up $(W^{(2)}, \omega^{(2)})$, obtain $(W', \omega')$ again, rationally blow down and get $(W, \omega)$, which would contain the sphere $\Sigma_{-1}^{(2)}$, a contradiction since $c_1^2 (W, \omega) \geq 1$. By Lemma 3.30...
\(\Sigma^{(2)}_{-1}\) must then intersect the spheres of the \(c_{-n-2+k^2}^{k(k-1)/2}\) configuration positively and transversally.

By Lemma 3.19, if \(\Sigma^{(2)}_{-1}\) intersects with the spheres \(S_j, 2 \leq j \leq n-1\) and \([S_j]^2 = -2\), then we must have \([\Sigma^{(2)}_{-1}] \cdot [S_j] = 1\). However, if this is the case, then we would be able to blow down repeatedly \((n-2)\) times and end up with a manifold that has a sphere of self-intersection \((-1)\) and \(c_1^2 \geq 1\), which is a contradiction. Therefore, \(\Sigma^{(2)}_{-1}\) must only intersect with the immersed sphere \(S^{k(k-1)/2}_{-n-2+k^2}\).

Since \(\Sigma^{(2)}_{-1}\) is a sphere of self-intersection \((-1)\), then \(c_1(W^{(2)}, \omega^{(2)}) + 2[\Sigma^{(2)}_{-1}]\) is a SW basic class of \(W^{(2)}\), by Theorem 2.10. If we apply Theorem 2.10 to \(x = S^{k(k-1)/2}_{-n-2+k^2}\), we obtain:

\[
|c_1(W^{(2)}, \omega^{(2)}) + 2[\Sigma^{(2)}_{-1}]| \cdot S^{k(k-1)/2}_{-n-2+k^2} \leq n - k,
\]

which implies that

\[
[S^{k(k-1)/2}_{-n-2+k^2}] \cdot [\Sigma^{(2)}_{-1}] = j_2, \quad 1 \leq j_2 \leq n - k.
\]

If \(j_2 = n - k\), then we could blow up \((W^{(2)}, \omega^{(2)})\) and obtain \((W', \omega')\), which would now contain 2 spheres with self-intersection \((-1)\): \(\Sigma_{-1}\) and \(\Sigma^{(2)}_{-1}\), where:

\[
[\Sigma_{-1}] \cdot [S_1] = k, \\
[S^{(2)}_{-1}] \cdot [S_1] = n - k.
\]

Since \([\Sigma_{-1}] + [\Sigma^{(2)}_{-1}]\) \(\cdot [S_1] = n\), as in Lemma 3.27, we can construct a relative class \(\delta \in H_2(C_n, \partial C_n; \mathbb{Z})\) that is a union of the disks \((\Sigma_{-1} \cup \Sigma^{(2)}_{-1}) \cap N(C_n)\), such that the relative class \(-n\delta\) can be supported in the interior by the following homology class:

\[
-n\delta = (n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}].
\]

As a result, as in the proof of Lemma 3.27, the SW basic class \(\pm(c_1(W', \omega') + 2[\Sigma_{-1}] + 2[\Sigma^{(2)}_{-1}]\) will extend to a SW basic class on \((W, \omega)\) after rationally blowing down, forcing \((W, \omega)\) to have basic classes in addition to \(\pm c_1(W, \omega)\).

If \(j_2 \neq n-k\), then we blow down the sphere \(\Sigma^{(2)}_{-1}\) in \((W^{(2)}, \omega^{(2)})\), and obtain the manifold \((W^{(3)}, \omega^{(3)})\). The sphere \(S^{k(k-1)/2}_{-n-2+k^2} \subset (W^{(2)}, \omega^{(2)})\) descends to the sphere \(S^{k(k-1)/2}_{-n-2+k^2+j_2}\) \(\subset (W^{(3)}, \omega^{(3)})\), (after perturbing the \(j_2\)-tuple intersection, as done before). Next, since \(k \geq c_1^2(W, \omega) + 2\), we have that \(c_1^2(W^{(3)}, \omega^{(3)}) \leq -1\), therefore, we can apply Lemma 3.30 and obtain an embedded symplectic sphere of self-intersection \((-1)\): \(\Sigma^{(3)}_{-1} \subset W^{(3)}\). Again, we have that \(c_1(W^{(3)}, \omega^{(3)}) + 2[\Sigma^{(3)}_{-1}]\) is a SW basic class of \((W^{(3)}, \omega^{(3)})\), thus
by Theorem 2.10 we have that:
\[ |c_1(W^{(2)}, \omega^{(2)})| + 2[\Sigma^{(2)}_{-1}] \cdot S^{k(k-1)/2}_{-n-2+k^2} \leq n - k, \]
which implies that
\[ [\Sigma^{(k(k-1)+j_2(j_2-1)/2)}_{-n-2+k^2+j_2^2}] \cdot [\Sigma^{(3)}_{-1}] = j_3, \quad 1 \leq j_2 \leq n - k - j_2. \]

If \( j_3 = n - k - j_2 \), then we could blow up \((W^{(3)}, \omega^{(3)})\) twice and obtain \((W', \omega')\), which would now contain 3 spheres with self-intersection \((-1)\):
\( \Sigma_{-1}, \Sigma^{(2)}_{-1} \) and \( \Sigma^{(3)}_{-1} \), where:
\[
\begin{align*}
[\Sigma_{-1}] \cdot [S_1] &= k \\
[\Sigma^{(2)}_{-1}] \cdot [S_1] &= j_2 \\
[\Sigma^{(3)}_{-1}] \cdot [S_1] &= n - k - j_2.
\end{align*}
\]
Since \( ([\Sigma_{-1}] + [\Sigma^{(2)}_{-1}] + [\Sigma^{(3)}_{-1}]) \cdot [S_1] = n \), again as in Lemma 3.27, we can construct a relative class \( \delta \in H_2(C_n, \partial C_n; \mathbb{Z}) \) that is a union of the disks \((\Sigma_{-1} \cup \Sigma^{(2)}_{-1} \cup \Sigma^{(3)}_{-1}) \cap N(C_n)\), such that the relative class \(-n\delta\) can be supported in the interior by the same class as before in (3.48). As a result, just as in the proof of Lemma 3.27 the SW basic class \( \pm(c_1(W', \omega') + 2[\Sigma_{-1}] + 2[\Sigma^{(2)}_{-1}] + 2[\Sigma^{(3)}_{-1}]) \) will extend to a SW basic class on \((W', \omega')\) after rationally blowing down, again forcing \((W', \omega')\) to have basic classes in addition to \( \pm c_1(W', \omega') \).

If \( j_3 \neq n - k - j_2 \), we can repeat the same procedure again, which will again force \((W', \omega')\) to have basic classes in addition to \( \pm c_1(W', \omega') \). We can continue this process until it terminates for some \( \ell \leq n - k \), where we will have a \( j_\ell \) so that \( j_2 + j_3 + j_4 + \cdots + j_\ell = n - k \). As a result, we will obtain the manifold \((W^{(\ell)}, \omega^{(\ell)})\), which will have a sphere \( S^{k(k-1)+j_2(j_2-1)+\cdots+j_{\ell-1}(j_{\ell-1}-1)/2}_{-n-2+k^2+j_2^2+\cdots+j_{\ell-1}^2} \) that intersects the sphere \( \Sigma^{(\ell)}_{-1} \), \( (n - k - j_2 - j_3 - \cdots - j_{\ell-1}) \) times. We can then blow up \((W^{(\ell)}, \omega^{(\ell)})\) \( (\ell - 1) \) times, and obtain the manifold \((W', \omega')\) which will have \( \ell \) spheres of self-intersection \((-1)\), such that:
\[
\begin{align*}
[\Sigma_{-1}] \cdot [S_1] &= k \\
[\Sigma^{(2)}_{-1}] \cdot [S_1] &= j_2 \\
[\Sigma^{(3)}_{-1}] \cdot [S_1] &= j_3 \\
&\vdots \\
[\Sigma^{(\ell-1)}_{-1}] \cdot [S_1] &= j_{\ell-1} \\
[\Sigma^{(\ell)}_{-1}] \cdot [S_1] &= n - k - j_2 - j_3 - \cdots - j_{\ell-1} = j_\ell.
\end{align*}
\]
Again, in this case, we will have the SW basic class \( \pm(c_1(W', \omega') + 2[\Sigma_{-1}] + 2[\Sigma^{(2)}_{-1}] + 2[\Sigma^{(3)}_{-1}] + \cdots + 2[\Sigma^{(\ell)}_{-1}]) \) which will extend to a SW basic class on
$(W, \omega)$ after rationally blowing down, again forcing $(W, \omega)$ to have basic classes in addition to $\pm c_1(W, \omega)$.

Notice, that we will have to do the greatest number of blow downs if $j_2 = j_3 = \cdots = j_\ell = 1$, in which case, $\ell = n - k$. Therefore, we require $k \geq c_1^2(W, \omega) + 2$, in order for all the manifolds $(W^{(i)}, \omega^{(i)})$ with $1 \leq i \leq \ell$ to have $c_1^2(W^{(i)}, \omega^{(i)}) \leq -1$, so that we can apply Lemma 3.30 repeatedly. □

From Proposition 3.31, we can see that if $B_n \hookrightarrow (X, \omega)$ is of type $E_k$, $k \geq c_1(X, \omega) + 2$, then $(X, \omega)$ must have SW basic classes in addition $\pm c_1(X, \omega)$, which is a contradiction.

4. SYMPLECTIC EMBDDBEDDINGS OF TYPE $E_2$

In this section we will show how to explicitly construct symplectic 4-manifolds $(X, \omega)$, such that the symplectic embeddings $B_n \hookrightarrow (X, \omega)$ are of type $E_2$, for $n$ odd. In these constructions $(X, \omega)$ will have $b_2^+ (X) > 1$, $n \geq c_1^2(X, \omega) + 2$ and $\text{Bas}_X \{ \pm (c_1(X, \omega)) \}$. It is not clear however, whether such a construction actually yields a surface of general type or just a symplectic 4-manifold with said properties. Note, these constructions appear in [Ak], however, we reinterpret them here for our purposes. First, we introduce the Fintushel and Stern knot surgery construction for 4-manifolds [FS3, FS4].

**Definition 4.1.** Let $T \subset X$ be a homologically non-trivial torus, with self-intersection 0, in a 4-manifold $X$ with $b_2^+ (X) > 1$. Let $T \times D^2$ be a tubular neighborhood of $T$ in $X$. Also, let $K \subset S^3$ be a knot, and $N(K)$ be its tubular neighborhood. Then,

$$X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)))$$

is defined to be the **knot surgery manifold**.

Note, the two pieces are attached in such a manner that the homology class $[\ast \times \partial D^2]$ is identified with $[\ast \times \lambda]$, where $\lambda$ is the longitude of the knot $K$. In other words, $X_K$ is obtained from $X$ by removing a neighborhood of the torus $T$ and replacing it with $(S^1 \times (S^3 \setminus N(K)))$. The manifold $X_K$ is homotopy equivalent to $X$ (assuming $X$ is simply-connected).

In [FS3], Fintushel and Stern proved that the Seiberg-Witten invariants of $X_K$ are determined by the Seiberg-Witten invariants of $X$ and the Alexander polynomial of the knot $K$, as long as $T$ has a cusp neighborhood. For the statement of this result, it is convenient to arrange all of the Seiberg-Witten basic classes into a Laurent polynomial as follows:

**Definition 4.2.** Let $\text{Bas}_X = \{ \pm \beta_1, \ldots, \pm \beta_m \}$ and $t_{\beta_i} = exp(\beta_i)$ be variables satisfying $t_{\beta_i + \beta_j} = t_{\beta_i} t_{\beta_j}$, then

$$SW_X = b_0 + \sum_{i=1}^{m} b_i (t_{\beta_i} + (-1)^{(\chi(X) + \sigma(X))/4} t_{\beta_i}^{-1})$$

where $b_0 = SW_X(0)$ and $b_i = SW_X(\beta_i)$. 


Example 4.3. Let \( X = E(m) \) be the elliptic surface, and \( t = \exp(T) \), where \( T \) is Poincare dual of the fiber class, then:

\[ \text{SW}_{E(m)} = (t - t^{-1})^{m-2}. \]

Theorem 4.4. Let \( T \subset X \) be as above in Definition 4.1. Assume that \( T \) lies in a cusp neighborhood in \( X \), then:

\[ \text{SW}_{X_K} = \text{SW}_{X} \cdot \Delta_K(t) \]

where \( \Delta_K(t) \) is the Alexander polynomial of the knot \( K \).

Remark 4.5. If \( \Delta_K(t) \) is not monic then \( X_K \) cannot admit a symplectic structure, since if \( X_K \) is symplectic then we must have \( \text{SW}_{X_K}(\pm c_1(X_K, \omega)) = \pm 1 \). However, if the knot \( K \) is fibered, then the knot surgery manifold \( X_K \) has a symplectic structure \([\text{FS3}]\), since it can be constructed as a symplectic fiber sum \([\text{Go}]\).

We will exhibit symplectic 4-manifolds which have symplectic embeddings \( B_n \hookrightarrow X \) of type \( E_2 \), by obtaining them from the elliptic surfaces \( E(m) \) by knot surgery, blow-ups, and rational blow-down, (these constructions appeared in \([\text{Ak}]\)). We will utilize the following Lefschetz fibration of the elliptic surfaces \( E(m) \):

Lemma 4.6. \([\text{Ak}]\) There exists an elliptic Lefschetz fibration on the surface \( E(m) \) with a section, a singular fiber \( F \) of type \( I_{8m} \), \((2m - 1)\) singular fibers of type \( I_2 \) and two additional fishtail fibers.

\[ \text{fishtail fiber} \quad I_2 \text{ fiber} \quad I_l \text{ fiber} \]

![Figure 14. Fibers in an elliptic fibration](image)

Recall, that a singular fiber of type \( I_l \) is a plumbing of \( l \) spheres of self-intersection \((-2)\) in a circle, and a fishtail fiber is an immersed sphere with one positive double point and self-intersection 0 (for more on elliptic surfaces and their singular fibers, see \([\text{HKK}]\), \([\text{KM}]\), also see Figure 1).

In \([\text{FS4}]\), Fintushel and Stern investigated the consequences of performing the knot surgery construction in certain neighborhoods in an elliptic fibration:
Definition 4.7. [FS4] A double node neighborhood $D$ is a fibered neighborhood of an elliptic fibration which contains exactly two nodal fibers with the same monodromy.

One can perform knot surgery along a regular fiber in such a double node neighborhood, $D$, for example, in a neighborhood of the $I_2$ fiber (see Figure 14). The elliptic surface $E(m)$ will have a section $R$, which is a sphere with self-intersection $(-m)$. Fintushel and Stern observed that for a family of knots, the twists knots $T(r)$, if we perform knot surgery in the neighborhood of the $I_2$ singular fiber, then a disk in the section $R$ gets replaced by a Seifert surface of the knot $T(r)$. As a result, the manifold $E(m)_{T(r)}$, will have a “pseudo-section” $R_s$, which we can think of as an immersed sphere with one double point (since $g(T(r)) = 1$), still having self-intersection $(-m)$ (see [FS4], [AK]). Note, we will use this construction only for the knot $T(1)$, which is the trefoil knot, since we are interested in our 4-manifolds retaining their symplectic structures.

Next, we will describe the general construction of a family of such manifolds, similar to the examples above, (again, see [AK]).

Proposition 4.8. There exists a family of symplectic 4-manifolds $X$, with each $(X,\omega) \in X$ having $b_2^+ (X) > 1$, $Bas_X = \{ \pm c_1 (X,\omega) \}$ and a symplectic embedding $B_n \hookrightarrow (X,\omega)$ of type $E_2$, for $n$ odd. Moreover, for all $(X,\omega) \in X$, the embeddings of $B_n \hookrightarrow (X,\omega)$ are such that $n < 3 + \frac{4}{3} c_1^2 (X,\omega)$.

**Proof.** First, we take the elliptic surface $E(m)$, $m > 2$, which has a section $R$, a sphere of self-intersection $(-m)$, and perform knot surgery in the double node neighborhoods of $s$ of the $I_2$ fibers, obtaining the manifold $E(m)_{K_1,\ldots,K_s}$, where $1 \leq s \leq 2m - 1$ and $K_i$ are copies of the trefoil knot. We now obtain a “pseudo-section” $R_s$ (see [FS4], [AK]) of $E(m)_{K_1,\ldots,K_s}$, which is an immersed sphere with self-intersection $(-m)$ and $s$ double points (see Figure 15).

We can blow up $s$ times, so that $R_s$ becomes the embedded sphere $S_{-m-4s}$ (self-intersection $(-m-4s)$) in $E(m)_{K_1,\ldots,K_s} \# s \overline{\mathbb{C}P}^2$. Additionally, in $E(m)_{K_1,\ldots,K_s} \# s \overline{\mathbb{C}P}^2$ we will have $s$ exceptional spheres $E_1,\ldots,E_s$, with $|E_i|^2 = -1$, each of which intersects the sphere $S_{-m-4s}$ twice (see Figure 16). In the fibration of $E(m)$, we also have two additional fishtail fibers, $F_1$ and $F_2$ (Lemma 4.6), which intersect the “pseudo-section” $R_s$.
obtain manifolds respectively. We can then rationally blow down these configurations and
once. Therefore, we can blow up $E(m)_{K_1,...,K_s} \# s\mathbb{CP}^2$, and after smoothing out the transverse intersection, obtain a sphere $S_{m-4s-2}$ in $E(m)_{K_1,...,K_s} \# (s+1)\mathbb{CP}^2$, such that $[S_{m-4s-2}] = [S_{F_1}^m] + [S_{m-4s}]$ (see Figure 17). Likewise, we can blow up $E(m)_{K_1,...,K_s} \# (s+2)$ times (at the double points of $R_s$ and the fishtail fibers $F_1$ and $F_2$), and after smoothing out the transverse intersections, obtain a sphere $S_{m-4s-4}$ in $E(m)_{K_1,...,K_s} \# (s+2)\mathbb{CP}^2$, such that $[S_{m-4s-4}] = [S_{F_1}^{m+2}] + [S_{m-4s}] + [S_{F_2}^m]$. (The spheres $S_{F_1}^m$ and $S_{F_2}^m$ are spheres of self-intersection $(-4)$ obtained from blowing up fibers $F_1$ and $F_2$.)

In these three cases, we obtain configurations of $C_{m+4s-2}$, $C_{m+4s}$ and $C_{m+4s+2}$ in

$$E(m)_{K_1,...,K_s} \# s\mathbb{CP}^2$$
$$E(m)_{K_1,...,K_s} \# (s+1)\mathbb{CP}^2$$
$$E(m)_{K_1,...,K_s} \# (s+2)\mathbb{CP}^2,$$

respectively, by taking the spheres $S_{m-4s}$, $S_{m-4s-2}$ and $S_{m-4s-4}$, also respectively, with the spheres of the $I_{8m}$ fiber. Note, this can be done as long we have enough spheres of self-intersection $(-2)$ in the $I_{8m}$ fiber to complete the $C_{m+4s-2}$, $C_{m+4s}$ and $C_{m+4s+2}$ configurations, so we must have $(8m - 1) \geq (m + 4s)$, $(8m - 1) \geq (m + 4s - 2)$ or $(8m - 1) \geq (m + 4s - 4)$, respectively. We can then rationally blow down these configurations and obtain manifolds $X_{m+4s-2}$, $X_{m+4s}$ and $X_{m+4s+2}$, such that:

$$B_{m+4s-2} \hookrightarrow X_{m+4s-2} \cong RBD(E(m)_{K_1,...,K_s} \# s\mathbb{CP}^2)$$
$$B_{m+4s} \hookrightarrow X_{m+4s} \cong RBD(E(m)_{K_1,...,K_s} \# (s+1)\mathbb{CP}^2)$$
$$B_{m+4s+2} \hookrightarrow X_{m+4s+2} \cong RBD(E(m)_{K_1,...,K_s} \# (s+2)\mathbb{CP}^2).$$
In all of these cases, the embeddings of $B_n$ will be symplectic (since we used the trefoil knot in the knot surgery construction) and will be of type $E_2$ (due to the exceptional spheres $E_i$). Again, if $m$ is odd, then only the top basic classes

$$
\pm(m+2s-2)T+E_1+E_2+\cdots+E_r
$$

of $E(m)_{K_1,\ldots,K_s}#r\mathbb{CP}^2$ can extend to the rational blow-down, where $r \in \{s,s+1,s+2\}$, (this follows from results in [Pa1], also see [Ak]). As a result, the manifolds $X_{(m+4s-2)}$, $X_{(m+4s)}$ and $X_{(m+4s+2)}$ will each only have one SW basic class, up to sign.

It is clear from these embeddings of the rational homology balls $B_n$, that if we want higher values of $n$, we are going to have to take higher values of $m$, thus, we need to increase the $b^+_2$. In these constructions, the number $n$ is mainly restricted by the number of spheres of self-intersection $(-2)$ in the $I_{8m}$ fiber which we use to construct the $C_n$ configuration of spheres. Consequently, even if we use all of the $(2m-1)$ of the $I_2$ for our knot surgery construction along with both of the fishtail fibers $F_1$ and $F_2$, and get a sphere $S_{m-4s-4}$, we may not be able not construct a $C_n$ configuration of spheres with $n = m + 4s + 2$ if we have $(8m-1) < (n-1)$. For this reason, for each $m$, in order to get the highest possible value for $n$, we may have to use less than the $(2m-1)$ of the $I_2$ fibers in our knot surgery construction. Consequently, the highest $n$ which will work for these constructions is when $n = 8m + 1$, where we use all the $(8m - 1)$ available spheres of the $I_{8m}$ fiber.

If $m = 4k + 1$, for $k \geq 1$, then we have:

$$
B_{8m+1} \hookrightarrow X_{(8m+1)} \cong \text{RBD}(E(m)_{K_1,\ldots,K_{7k+2}}\#(7k+3)\mathbb{CP}^2),
$$

where $b^+_2(X_{(8m+1)}) = 2m - 1$ and $c^2_2(X_{(8m+1)}) = 25k + 5$.

If $m = 4k + 3$, for $k \geq 1$, then we have:

$$
B_{8m+1} \hookrightarrow X_{(8m+1)} \cong \text{RBD}(E(m)_{K_1,\ldots,K_{7k+6}}\#(7k+6)\mathbb{CP}^2),
$$

where $b^+_2(X_{(8m+1)}) = 2m - 1$ and $c^2_2(X_{(8m+1)}) = 25k + 2$.

As a result, we can see that as $(\chi_h,c^2_2) \to \infty$ then $n \to \infty$ as well. Moreover, in all these examples we have $n < 3 + \frac{4}{9}c^2_2$. If we take $m \geq 5$, we can refine this bound to $n < 3 + \frac{32}{25}c^2_2$. $\square$

It is important to note that it is not clear whether the examples in Proposition 4.8 yield surfaces of general type or just symplectic 4-manifolds. Additionally, one could probably construct embeddings of type $E_k$ for $k \geq 3$ having the same properties as those of type $E_2$ in Proposition 4.8. This might be done by defining the knot surgery construction in double node neighborhoods for fibered knots with higher genus than the trefoil knot.

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