Mathematical model of two-contour tomography scanner

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Abstract. In the conditions of transformation and modernization of society, the use of mathematical modeling is of great importance. This is due to the fact that at the present stage a distinctive feature is the introduction of science and technology, complex technical systems in all areas of activity, one of which is the development of diagnostic systems, in particular, the construction of a mathematical model of tomography scanner. The article develops a mathematical model of a two-contour tomography scanner, describes the fields created by a single-point, two-point, single-circuit source, shows the construction of the Green function for the parameter \( k = 1 \). The advantage of a two-contour tomography scanner is the presence of a sphere among the equipotential surfaces created by this device, which is convenient for analytical calculations. The final results are accurate, not numerical. This plays a significant role in medical diagnostic studies, where the patient’s health depends on the accuracy of the diagnosis.

1. Introduction

A large number of research works are dedicated to description of a tomography scanner. In [1], [2], [3], [4] ad-hoc mathematical models were considered as well as problems and algorithms for thermoacoustic (TAT) and photoacoustic (PhAT) tomography. Both TAT and PhAT versions of tomography represent seemingly the most developed methods among various “hybrid” technologies for medical visualization. These new complex techniques combine various types of physical waves (electromagnetic and acoustic in cases of PhAT and TAT) in such a way that the resolution and contrast ratio of the image obtained become substantially higher than those in applying solely acoustic or purely electromagnetic measurements. In the research work [5, 6] the progress in establishing uniqueness and in development of inversion formulae and algorithms of thermo-acoustic tomography was shown. In terms of a mathematical analysis, one can talk on rather specific inverse problem for the wave equation. In case of a constant sound speed, this can also be interpreted as a problem of spherical mean transform. It should be noted, and other researchers in this area [7–9].

Therefore the aim of the present work consists in development of a mathematical model of a device which we call a “two-contour tomography scanner”.

2. Statement of Problem

By a tomography scanner (further TS) we mean a medical device for scanning patients. Such a device may irradiate or emit electromagnetic waves, ultrasounds, or create an electrostatic field.

Let there be given a contour of TS in the form of a circumference \( y^2 + z^2 = \eta^2 \), where \( \eta = \text{const} \), shifted along the axis \( Ox \) for a distance \( \xi \) from the origin. The ray \( Ox \) is its symmetry axis, namely
along this ray a patient is disposed. It is evident that equipotential surfaces created by the contour $\omega$ possess the axial symmetry. Consequently, one can limit oneself by considering a problem on the semi-plane $Oxy$, $y \geq 0$. Denote by $M_0$ a point where the circumference $\omega$ “passes through” the semi-plane $Oxy$, $y \geq 0$. In Figure 1, a cut of equipotential surfaces created by intersection of the emitter $\omega$ with the plane $Oxy$ is depicted. These closed curves by their shape are close to Cassini ovals and, in particular, to the Bernoulli lemniscate. Unfortunately, these curves are not exactly the mentioned objects. Moreover, one cannot write their equations in an explicit form. Among these closed curves, one cannot find a circumference. In other words, a one-contour TS fails to create a spherical equipotential surface. The latter leads to difficulties in recognition of images imported by TS with $\omega$ and results in lowering of information treatment speed and data exactness. Consequently, the reliability of diagnosis is decreased.

In the present paper we suggest a model of a two-contour tomography scanner. With a certain choice of parameters, such a device (among others) creates equipotential surface with spherical shape. Both recognition of images and their interpreting by such a TS increase. Moreover, it turns out that an invention of TS with a greater number of contours becomes unnecessary.

![Figure 1](image_url)

**Figure 1.** A cut of equipotential surfaces created by intersection of the emitter $\omega$ with the plane $Oxy$ is depicted.

### 3. Potential created by one-point TS

For medical purposes, TS with a fixed spot emitter for an external scanning is not of wide use. For transviewing of deep parts of body one must use large power of emitting which might deteriorate close internal organs.

As known (see [10], [11]), a mass $m$ disposed at a point $M_0(\xi, \eta, \zeta)$ creates at an arbitrary point $M(x, y, z) \neq M_0$ a potential which is equal to

$$\varphi = \gamma \frac{m}{\rho}, \quad \rho^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$$

(1), (2)

where $\gamma$ is the gravitation constant.

The function (1) is a fundamental solution of the Laplace equation (see, e.g., [12])

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

(3)

with a singularity at the point $M_0(\xi, \eta, \zeta)$.

Equipotential surfaces of a spot emitter are evidently spheres given by equations
where \( C = \text{const} > 0 \). In what follows, for generality, let us call the mass \( m \) a gravitational charge. The study of fields generated by unlike sources is of importance for us. Therefore we will operate with potentials created by electric spotted charges \( q_i \):

\[
\varphi_i = \frac{k_i q_i}{\varepsilon \rho_i}.
\]

Here \( i \) stands for charge’s number, \( \varepsilon \) is the dielectric permeability of air (for the sake of simplicity it is assumed to be equal to 1), \( k = 10^9 \frac{m^2}{kg^2} \) is the proportionality factor. In Figure 2 the cut made by the plane \( Oxy \) of equipotential surfaces created by two charges \( q \) and \( (-\lambda q) \), where \( \lambda \neq 1 \), is depicted. One can easily see that among those lines a circumference is present being a cut of a spatial sphere.

These lines are given by equations: \( \varphi_1 + \varphi_2 = C \).

\[
\varphi = C, \quad (4)
\]

\[
\varphi_i = \frac{k_i q_i}{\varepsilon \rho_i}. \quad (5)
\]

\[
\varphi_1 + \varphi_2 = C. \quad (6)
\]

\[
\text{Figure 2. The cut made by the plane } Oxy \text{ of equipotential surfaces created by two charges } q \text{ and } (-\lambda q) \text{, where } \lambda \neq 1, \text{ is depicted.}
\]

By substituting (5) into (6):

\[
\frac{k q}{\varepsilon \rho_1} + \frac{k (-\lambda q)}{\varepsilon \rho_2} = C. \quad (7)
\]

Due to its axial symmetry with respect to the strait-line connecting two charges, we may limit ourselves to a plain cut passing through this straight-line. For the sake of definiteness, let the charge \( q \) be situated at the origin \( M_0(0,0) \), while at distance \( d \) in positive direction of the axis \( Ox \) at the point \( M_0^*(d,0) \) another charge \( (-\lambda q) \) be disposed. Therefore equations (7) are transformed to the form

\[
\frac{k}{\varepsilon} \left( \frac{q}{\sqrt{x^2 + y^2}} - \frac{\lambda q}{\sqrt{(x-d)^2 + y^2}} \right) = C. \quad (8)
\]

Among an infinite quantity of equipotential surfaces there exists a sphere. The meaning of potential on the sphere is equal to zero: \( C = 0 \). From (8) at \( C = 0 \) it follows that

\[
\frac{q}{\sqrt{x^2 + y^2}} = \frac{\lambda q}{\sqrt{(x-d)^2 + y^2}}.
\]
hence \((x-d)^2 + y^2 = \lambda^2 (x^2 + y^2)\). After transformation, we get thus the following circumference \(\omega\) with the equation:

\[
\left(x + \frac{d}{\lambda^2 - 1}\right)^2 + y^2 = \left(\frac{d\lambda}{\lambda^2 - 1}\right)^2
\]

with the center at the point \(L(-\frac{d}{\lambda^2 - 1},0)\) and of the radius \(R = \frac{d\lambda}{\lambda^2 - 1}\).

Note that \(\omega\) is the circumference of inversion for points \(M_0^*\) and \(M_0\). Indeed, let \(Oy\) intersect the upper semicircle \(\omega\) at a point \(K(0, k), k > 0\). By substituting coordinates of the point \(K\) in (9), we get

\[
0 + \frac{d}{\lambda^2 - 1} + k^2 = \left(\frac{d\lambda}{\lambda^2 - 1}\right)^2.
\]

Hence \(k^2 = \frac{d^2\lambda^2}{(\lambda^2 - 1)^2} - \frac{d^2}{(\lambda^2 - 1)^2} = \frac{d^2}{\lambda^2 - 1}\); thus we have \(k = \frac{d}{\sqrt{\lambda^2 - 1}}\) and the point \(K\) possesses coordinates \(\left(0, \frac{d}{\sqrt{\lambda^2 - 1}}\right)\). Consider two vectors \(\overline{LK} = \left(\frac{d}{\lambda^2 - 1}, -\frac{d}{\sqrt{\lambda^2 - 1}}\right)\). Their scalar product is equal to zero, \(\overline{LK} \perp \overline{M_0^* K}\). \(KM_0^*\) is a tangent restored to \(\omega\) at the point \(K\) and perpendicular to the radius \(LK\). By its construction the point \(M_0^*\) is inverse to the point \(M_0\).

4. Potential created by one-contour TS

Assume that a one-contour TS has the shape of a circle \(\omega\) with the center at \(A\) and radius \(\eta\). Let us fix there points \(M_0\) and \(P\). Draw a concentric circumference of radius \(y\) and fix there a point \(Q\). By the cosine theorem one has

\[
PQ = \sqrt{y^2 + \eta^2 - 2\eta y \cos \theta},
\]

where \(\theta\) is the angle between radii \(AP\) and \(AQ\). Let us arrange a coordinate system so that its axis \(Ox\) were passing through the point \(A\) perpendicular to the plane of the contour while the testing point \(M\) were on the plane \(Oxy\). \(M_0\) is the point at which TS’ contour “hits” the semi-plane \(Oxy\). Every charged element \(dl\) of contour \(\omega\) creates at the point \(M\) potential \(d\varphi = \frac{dq}{\rho}\), where \(\rho = |PM|\). Here we took \(k = 1\) and \(\varepsilon = 1\) for the sake of convenience. Assume that linear density of charges of contour \(\omega\) is equal to a constant value \(\mu\). We then have \(\mu = \frac{dq}{dl}\) and with regard for relation between element of arc and element of polar angle

\[
dl = \eta d\theta,
\]

we get \(dq = \mu dl = \mu\eta d\theta\). For determination of potential at the point \(M\), which is created by contour \(\omega\), one must sum the elements of potentials \(d\varphi\) within the limits from 0 to \(2\pi\) for varying angle \(\theta\):

\[
\varphi = \int_{\varphi(0)}^{\varphi(2\pi)} d\varphi = \int_{\theta(0)}^{\theta(2\pi)} \frac{\mu\eta d\theta}{\rho}.
\]

Evidently: \(QM = |x - \xi|\). Then \(\rho^2 = (x - \xi)^2 + y^2 + \eta^2 - 2\eta y \cos \theta\) and for potential (12) we obtain the expression
\[ \varphi = \mu \eta \int_{0}^{2\pi} \frac{d\theta}{\sqrt{(x-\xi)^2 + y^2 + \eta^2 - 2\eta y \cos \theta}}. \]  \hspace{1cm} (13)

Parts of the semi-circle \( \omega \) situated in the semi-spaces \( z \geq 0 \) and \( z \leq 0 \) are symmetric with respect to the plane \( Oxz \). Consequently, at the point \( M \), similar potentials are created.

\[ \varphi = g(x, y; \xi, \eta) = 2\mu \eta \int_{0}^{\pi} \frac{d\theta}{\sqrt{(x-\xi)^2 + y^2 + \eta^2 - 2\eta y \cos \theta}}. \]  \hspace{1cm} (14)

Namely in this form a fundamental solution of the axially-symmetric problem was obtained first by E. Beltrami [13]. If in Laplace equation (3) one passes to cylindrical coordinates:

\[ y = R \cos \theta \]
\[ x = R \sin \theta \]
\[ x = x \]

then it can be rewritten as follows:

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial u}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \]  \hspace{1cm} (16)

For axially-symmetric fields, the function \( u \) with respect to polar angle \( \theta \) is a constant value:

\[ u(\theta) = \text{const}, \quad \forall \theta. \]  \hspace{1cm} (17)

Consequently, \( \frac{\partial u}{\partial \theta} = 0 \Rightarrow \frac{\partial^2 u}{\partial \theta^2} = 0. \) Then equation (16) turns to be simpler:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial u}{\partial R} \right) = 0. \]  \hspace{1cm} (18)

Differential equation (18) is singular. The replacement of \( R \) to \( y \) is thus justified. Obtain:

\[ \frac{\partial}{\partial x} \left( y \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( y \frac{\partial u}{\partial y} \right) = 0. \]  \hspace{1cm} (19)

Equation (19) is termed Laplace-Beltrami equation. It can be easily seen that both the closest to \( M(x, y) \) point of circle \( M_0(x_0, y_0) \) and the farthest point \( M_0(x_0, y_0) \) are positioned at the half-plane \( Oxz \).

Denote by \( r \) and \( r_1 \) the respective distances. Then

\[ r^2 = (x-\xi)^2 + (y-\eta)^2. \]  \hspace{1cm} (20)
\[ r_1^2 = (x-\xi)^2 + (y+\eta)^2. \]  \hspace{1cm} (21)

Transform the expression under radical in (14). Get

\[ (x-\xi)^2 + y^2 + \eta^2 - 2\eta y \cos \frac{\theta}{2} = (x-\xi)^2 \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) + \]
\[ +y^2 \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) + \eta^2 \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) - 2\eta y \left( \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) = \]
\[ = \left( (x-\xi)^2 + y^2 + \eta^2 - 2\eta y \right) \cos^2 \frac{\theta}{2} + \left( (x-\xi)^2 + y^2 + \eta^2 + 2\eta y \right) \sin^2 \frac{\theta}{2} = \]
\[ = r^2 \cos^2 \frac{\theta}{2} + r_1^2 \sin^2 \frac{\theta}{2}. \]

Rewrite potential (14) with regard for the last representation.
\[ \varphi = 2 \mu \eta \int_{0}^{\pi} \frac{d\theta}{\sqrt{r^2 \cos^2 \theta + r_1^2 \sin^2 \theta}} \cdot \frac{dr}{2}. \]  

(22)

As in [10], proceed in (22) with the change of variables:  
\[ \alpha = \frac{\pi - \theta}{2}. \]  

(23)

Then for new limits of integration we obtain \( \alpha = \frac{\pi}{2} \) with \( \theta = 0 \) and \( \alpha = 0 \) with \( \theta = \pi \). From (23) it follows that \( \theta = \pi - 2\alpha \); hence \( d\theta = (\pi - 2\alpha) ' d\alpha = -2d\alpha \).

The potential expression (22) takes the form:

\[ \varphi = 2 \mu \eta \int_{0}^{\pi/2} \frac{-2d\alpha}{\sqrt{r^2 \cos^2 \theta/2 + r_1^2 \sin^2 \theta/2}} = 4 \mu \eta \int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{r^2 \cos^2 \theta/2 + r_1^2 \sin^2 \theta/2}}. \]  

(24)

By comparing (24) with the definition of a complete elliptic integral written in symmetric form (see [14]),

\[ I(r, r_1) = I(r_1, r) = \int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{r_1^2 \cos^2 \theta + r^2 \sin^2 \theta}}. \]  

(25)

We conclude that the potential of circumference \( \omega \) at the point \( M \) is equal to

\[ \varphi = 4 \mu \eta I(r_1, r). \]  

(26)

The following relation takes place:

\[ I(r_1, r) = \frac{\pi}{2 \text{agm}(r_1, r)} \]  

(27)

where \( \text{agm}(r_1, r) \) stands for the arithmetic-geometric mean of values \( r_1 \) and \( r \) by Gauss [14]. Equation of every line in Fig.1 can be written in the form \( \text{agm}(r_1, r) = \text{const} \).

The function (26) is a fundamental solution of equation (19). In the semi-plane \( Oxy, y \geq 0 \) the point \( M_0 \) can be treated as a specific emitter. However, in moving the testing point \( M \) outwards farther from this “emitter” to a distance \( r \), the potential drops down not as \( \frac{1}{r} \) but as \( \frac{1}{\text{agm}(r_1, r)} \). In other words, we are insisted to consider a “phantom point” \( \bar{M}_0 \) symmetric for point \( M_0 \) with respect to \( Oxy \) from a “prohibited” zone \( y < 0 \). Moreover, the function (26) will have here a logarithmic singularity and not a power one.

5. Potential created by two-contour TS

Assume that charges’ contours are co-axial circumferences \( \omega \) and \( \omega^* \) with centers \( A \) and \( A^* \) and situated in respective parallel planes \( x = \xi \) and \( x = \xi^* \), possessing the radii \( \eta \) and \( \eta^* \), respectively. The symmetry axis \( Ox \) is assumed to coincide with the straight-line \( AA^* \) and be perpendicular to the planes of these circles. Due to the axial symmetry, we may limit ourselves by a 2D image.

For the sake of convenience we take the point of origin \( O \) on the straight-line \( M_0 M_0^* \) at the place where it intersects the axis \( Ox \). Here \( M_0 \) and \( M_0^* \) are the points at which the circumferences \( \omega \) and \( \omega^* \) are “hitting” the semi-plane \( Oxy, y > 0 \), respectively. Let the density of charge of the first contour be equal to \( (+\mu) \), while that of the second one be \( (-\lambda\mu) \). Analogously to (26), we then get that
\( \varphi^* = -4\lambda \mu \eta I(r^*, r^*) \) (28)

is the potential created by the contour \( \omega^* \) at the same point under investigation \( M \). Denote by \( R \) the distance \( OM \), so we have that

\[ R^2 = x^2 + y^2. \] (29)

Here, analogously to (20) and (21), we assume

\[ (r^*)^2 = (x - \xi^*)^2 + (y - \eta^*)^2, \quad (r^*)^2 = (x + \xi^*)^2 + (y + \eta^*)^2. \] (30), (31)

Denote \( \rho = OM_0 \) and \( \rho^* = OM_0^* \). Then

\[ \rho^2 = \xi^2 + \eta^2, \quad (\rho^*)^2 = (\xi^*)^2 + (\eta^*)^2. \] (32), (33)

The similarity of triangles \( AOM_0 \) and \( A^*OM_0^* \) implies that

\[ \frac{\eta^*}{\eta} = \frac{\rho^*}{\rho} = \frac{\xi^*}{\xi}. \] (34)

Now let \( t \) be the proportionality factor. Then

\[ \xi^* = \xi t, \quad \eta^* = \eta t. \] (35), (36)

The combined cumulative potential created by the contours \( \omega \) and \( \omega^* \) at the point \( M(x, y) \) can be found by summing the potentials (26) and (28). Thus we obtain that equipotential lines are given by the equations

\[ \varphi + \varphi^* = C. \] (37)

Let us find the geometrical place of points satisfying condition (37) with \( C = 0 \). With regard for expressions (26) and (28) for potentials \( \varphi \) and \( \varphi^* \), by eliminating by \( 4\mu \), instead of (37) with \( C = 0 \) we get the following relation

\[ \eta I(r, r) - \lambda \eta^* I(r^*, r^*) = 0. \] (38)

Write in another form: \( I(r, r) = \lambda I(r^*, r^*). \) (39)

Note that, as it follows from the definition (25), one easily gets the relation

\[ \sqrt{I} I(r^*, r^*) = I\left(\frac{r^*}{\sqrt{I}}, \frac{r^*}{\sqrt{I}}\right). \] (40)

Indeed, the proof is evident from the chain of equalities

\[ \sqrt{I} I(r^*, r^*) = \sqrt{I} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{(r^*)^2 \cos^2 \alpha + (r^*)^2 \sin^2 \alpha}} = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{2}} \frac{1}{\sqrt{I}} \sqrt{(r^*)^2 \cos^2 \alpha + (r^*)^2 \sin^2 \alpha} = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{2}} \sqrt{(r^*)^2 \cos^2 \alpha + (r^*)^2 \sin^2 \alpha} = I\left(\frac{r^*}{\sqrt{I}}, \frac{r^*}{\sqrt{I}}\right). \] (41)

Let us choose the parameter \( t \) so that the following relations be valid
\[
\begin{align*}
\frac{r^*}{r} &= r, \\
\sqrt{t} &= r.
\end{align*}
\] (42)

Hence
\[
\begin{align*}
\left(\frac{r^*}{r}\right)^2 &= r^2t, \\
\left(r_i^*/r_i\right)^2 &= r_i^2t.
\end{align*}
\] (43)

Expand (30):
\[
\begin{align*}
\left(r^*\right)^2 &= (x - \xi^*)^2 + (y - \eta^*)^2 = (x - \xi t)^2 + (y - \eta t)^2 = \\
&= x^2 + y^2 + t^2(\xi^2 + \eta^2) - 2t(\xi x + \eta y) = \\
&= R^2 + y^2 + t^2\rho^2 - 2t(\xi x + \eta y).
\end{align*}
\]

Therefore one gets (with (29), (32), (35), (36)):
\[
\left(r^*\right)^2 = R^2 + t^2\rho^2 - 2t(\xi x + \eta y). \tag{44}
\]

Let us expand as well (20)
\[
r^2 = (x - \xi)^2 + (y - \eta)^2 = (x^2 + y^2) + (\xi^2 + \eta^2) - 2(\xi x + \eta y) = \\
= R^2 + \rho^2 - 2(\xi x + \eta y).
\]

Thus,
\[
r^2 = R^2 + \rho^2 - 2(\xi x + \eta y). \tag{45}
\]

Multiplying (45) by \(t\) and equaling to (44) on the basis of the first relation in (43), we obtain
\[
R^2 + t^2\rho^2 - 2t(\xi x + \eta y) = tR^2 + t\rho^2 - 2t(\xi x + \eta y). \tag{46}
\]

The solution \(t = 1\) is obvious. However, at the same time, this case is a special one. More exactly, in accordance with (36) the latter means that the contours \(\omega\) and \(\omega^*\) possess the same radius. Among equipotential fields created by them there is a plane given by the equation
\[
x = \frac{\xi + \xi^*}{2}. \tag{47}
\]

Now let \(t \neq 1\). We get from (46): \(t = \left(\frac{R}{\rho}\right)^2\). \tag{48}

Formula (39) can be rewritten with the use of relation (40) and (42) as follows:
\[
I(r_i, r) = \lambda I\left(r_i^*, r^*\right) = \lambda \sqrt{t} I\left(r_i^*, r^*\right) = \lambda \sqrt{t} I\left(r_i^*, \frac{r^*}{\sqrt{t}}, \frac{r}{\sqrt{t}}\right) = \lambda \sqrt{t} I(r_i, r). \tag{49}
\]

The last relation, for \(r_i^*\) and \(r^*\) not equaling zero simultaneously, is possible only with
\[
\lambda \sqrt{t} = 1. \tag{50}
\]

Taking into account (48), from (50) we obtain:
\[
\lambda = \frac{\rho}{R}. \tag{51}
\]

The values \(\xi\) and \(\eta\) are fixed, whence \(\rho = \text{const}\). Introduced by formula (36) ratio of contours’ radii is constant: \(t = \text{const}\). Then from this and (48) it follows that
\[
R = C, \tag{52}
\]

where \(C = \text{const}\). In other words, the geometric place of points corresponding to zero potential is a sphere (figure 3).
From (34) – (36): \[ \frac{\rho^*}{\rho} = t. \] (53)

From (48) and (53): \[ \rho \rho^* = R^2. \] (54)

Having fixed the distance \( |\xi^* - \xi| \) between the contours’ planes, we get the unchanging point \( O \). As it is seen from (51), by changing the ration of charges \( \lambda \) for fixed contours, one can change the radius \( R \).

Among equipotential surfaces created by coaxial circles of various radii, there exists a sphere. This the inversion sphere which translates points from one circumference to points of another circumference and vice versa. The radius of the sphere is expressed via the initial data.

In the paper [15] the following Green function was constructed

\[ G(x, y; \xi, \eta) = \frac{1}{2\pi} \left( -g(x, y; \xi, \eta) + \left( \frac{R}{\rho} \right)^k g\left(x, y; \xi^*, \eta^*\right) \right), \] (55)

where

\[ g(x, y; \xi, \eta) = \int_0^{\pi} \sin^{k-1} \theta \, d\theta \frac{\sin \theta}{\left( (x - \xi)^2 + \eta^2 + y^2 - 2y\eta \cos \alpha \right)^{\frac{k}{2}}} \] (56)

is a fundamental solution of Beltrami equation

\[ B_k u \equiv \frac{\partial}{\partial x} \left( y^k \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( y^k \frac{\partial u}{\partial y} \right), \] (57)

found by Weinshtein [16]. Note that with the parameter value \( k = 1 \) the equation (57) takes the form (19). We followed the notation accepted in the paper [15], where studies were carried out with the limitation \( k > 1 \). In our article we consider the classical case of \( k = 1 \). With the value \( k = 1 \), from (56) we obtain (14). Green function (55) degenerates into the simple form

\[ G(x, y; \xi, \eta) = \frac{1}{2\pi} \left( -g(x, y; \xi, \eta) + \frac{R}{\rho} g\left(x, y; \xi^*, \eta^*\right) \right), \] (58)

where the function \( g\left(x, y; \xi^*, \eta^*\right) \) has the same shape as (14) does; however, here instead of \( \xi \) and \( \eta \) we take \( \xi^* \) and \( \eta^* \), respectively. The condition

\( G|_{x^2+y^2=R^2} = 0 \) (59)
is then written in the form
\[ R_\rho g(x, y; \xi^+, \eta^+) = g(x, y; \xi, \eta). \] (60)

It follows from formula (48) that \( \frac{R}{\rho} = \sqrt{f} \). With regard for formulae (14), (26), (35), (36), (48) and the remark after formula (58) we conclude that (60) coincides with (40).

6. Conclusion
A mathematical model for the two-contour TS is developed in the present work. The advantages of the new device in comparison with existing old one-contour models are shown.

A prospective direction for use of the developed mathematical model of two-contour TS can be found in application of designing medical diagnostic examinations. In using two-contour TS instead of one-contour device, the computations for reading computer equipment seem to be essentially reduced. The efficiency is attained due to wider application of analytical means and methods instead of numerical transform methods. The results of scanning become more exact than purely numerical.

More detailed information on notions and terms used in the article can be found in the list of references concluding the paper. Equipment lines represented in the particle content the reader may obtain on his/her PC by running the program code depicted in the respective figures.

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