SURFACES OF GENERAL TYPE WITH $q = 2$ ARE RIGIDIFIED

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Dedicated to Professor Fabrizio Catanese on the occasion of his 65th birthday

Abstract. Let $S$ be a minimal projective surface of general type with irregularity $q = 2$. We show that, if $S$ has a nontrivial holomorphic automorphism acting trivially on the rational cohomology ring, then it is a surface isogenous to a product. As a consequence we infer that every surface of general type with $q = 2$ is rigidified, that is, such surfaces have no nontrivial holomorphic automorphism isotopic to the identity as a diffeomorphism.

Introduction

A compact complex manifold $X$ is called rigidified if there is no nontrivial (holomorphic) automorphism of $X$ that is isotopic to the identity. This is equivalent to the statement that in the identity component $\text{Diff}^0(X)$ of the diffeomorphism group only the identity preserves the given complex structure. Catanese asked in [C13] if compact complex manifolds of general type are rigidified. A positive answer would be useful in establishing a desired local homeomorphism between the Teichmüller space and the Kuranishi space at the corresponding point of the relevant complex manifold.

Obviously, any diffeomorphism from $\text{Diff}^0(X)$ preserves topological invariants such as the cohomology ring of $X$. So faithfulness of the rational cohomological representation of the automorphism group guarantees that the complex manifold at hand is rigidified. This is how we proved that surfaces of general type with irregularity $q \geq 3$ are rigidified (see [CLZ13]). If the irregularity is two or less then there indeed exist unbounded series of surfaces, all isogenous to a product of curves, whose automorphism group has a nonfaithful rational cohomological representation. We prove in this note that those examples in [CLZ13] are the only ones with irregularity two, by showing the following

Theorem 0.1. Let $S$ be a minimal surface of general type with $q(S) = 2$. If there is a nontrivial automorphism acting trivially on the rational cohomological ring $H^*(S, \mathbb{Q})$, then $S$ is a surface isogenous to a product, of unmixed type.

Recall that a surface of general type is isogenous to a product of curves if it admits a product of two smooth curves as an étale cover. For the basic properties of such surfaces we refer to the seminal paper [C00], but see also [CLZ13, Section 4] for an exposition that pertains more to the investigation here. Theorem 0.1 is parallel to a main result of [CLR], which says that minimal surfaces of general type with $q = 1$ and with a maximal possible automorphism group (of order 4) acting...
trivially on cohomology are isogenous to a product of two curves. It is not clear if such a strong geometric restriction continues to hold for regular surfaces with a “large” automorphism group acting trivially on the cohomology, though there is some evidence supporting it.

**Corollary 0.2.** Minimal surfaces of general type with \( q(S) = 2 \) are rigidified in the sense of Catanese.

**Proof.** This is a consequence of Theorem 0.1 and [CLZ13, Proposition 4.8]. □

The automorphisms of a surface of general type, as well as their possible isotopies to the identity, always descend to the (unique) minimal model of the surface. **Corollary 0.2** implies easily that all surfaces of general type with \( q = 2 \) are rigidified.

The proof of Theorem 0.1 is based on the numerical classification of our surfaces given in [CLZ13] and the following result for surfaces of maximal Albanese dimension, recently established by Lu–Zuo [LZ15], and at the same time independently by Barja–Pardini–Stoppino [BPS15].

**Theorem 0.3** (Lu–Zuo, Barja–Pardini–Stoppino). Let \( S \) be a minimal surface of general type with maximal Albanese dimension. Then \( K_S^2 = 4 \chi(S, \mathcal{O}_S) \) if and only if \( q(S) = 2 \) and the canonical model of \( S \) is a flat double cover of the Albanese surface \( \text{Alb}(S) \), branched along an ample curve with at most simple singularities.

1. **Some known information**

In the rest of the paper we will let \( S \) denote a minimal surface of general type with \( q(S) = 2 \) and with a nontrivial \( \text{Aut}_0(S) \), the group of those automorphisms acting trivially on the cohomological ring of \( S \) with rational coefficients. More notation will be introduced along the way.

We collect some useful information obtained in our previous papers and draw easy consequences thereof.

**Theorem 1.1** ([CLZ13]).

(i) The invariants of \( S \) satisfy

\[ K_S^2 = 8 \chi(S, \mathcal{O}_S). \]

(ii) The group \( \text{Aut}_0(S) \) has order two, hence is generated by an involution, say \( \sigma \). Moreover, the fixed locus \( \text{Fix}(\sigma) \) consists of exactly \( 4 \chi(S, \mathcal{O}_S) \) isolated points.

(iii) \( S \) has maximal Albanese dimension.

By **Theorem 1.1** the quotient surface \( S/\sigma \) has exactly \( 4 \chi(S, \mathcal{O}_S) \) singularities, all of which are ordinary nodes. Let \( \lambda : S \to S/\sigma \) be the quotient map. Then we have

\[ \lambda^* K_{S/\sigma} = K_S \]

which is big and nef. It follows that the minimal resolution \( X \) of singularities of \( S/\sigma \) is of general type and does not contain any \((-1)\)-curves, that is, it is already the minimal model in the birational class of \( S/\sigma \). We have

\[ (1.1) \quad K_S^2 = 2K_{S/\sigma}^2 = 2K_X^2. \]

By the Hodge decomposition \( \sigma \) acts trivially also on the Debeault cohomology groups and we have ([CL13, Section 1])

\[ H^i(S, \Omega_S^j) = H^i(X, \Omega_X^j) \quad \text{for} \quad 0 \leq i, j \leq 2. \]

It is then quite obvious that \( \chi(X, \mathcal{O}_X) = \chi(S, \mathcal{O}_S) \) and \( q(X) = q(S) = 2 \). Combining (1.1) and **Theorem 1.1** with this we obtain

\[ (1.2) \quad K_X^2 = 4 \chi(X, \mathcal{O}_X). \]
Our main concern is the surjective Albanese map \( a_S: S \to \text{Alb}(S) \) of \( S \), which factors through the quotient map \( \lambda: S \to S/\sigma \) ([CoLi13 Section 1]). Since Albanese maps contract all rational curves the Albanese map of \( X \) factors through the contraction of \((-2)\)-curves \( X \to S/\sigma \to X_{\text{can}} \), where \( X_{\text{can}} \) is the canonical model of \( X \). There is a natural identification of Albanese surfaces \( \text{Alb}(X) \) and \( \text{Alb}(S) \), so that the following diagram of morphisms is commutative:

\[
\begin{array}{ccc}
S & \xrightarrow{\lambda} & S/\sigma \\
\downarrow & & \downarrow \\
X & \xrightarrow{a} & \text{Alb}(X) \\
\end{array}
\]

where the composition of the morphisms in a row gives the corresponding Albanese morphism.

In the following we will denote by \( A \) the two identified Albanese surfaces \( \text{Alb}(S) \) and \( \text{Alb}(X) \).

One finds in (1.3) that \( X \) also has maximal Albanese dimension. With (1.2) we can apply Theorem 0.3 to get

**Proposition 1.2.** Via the Albanese map the canonical model \( X_{\text{can}} \) of \( X \) is a flat double cover of the Albanese surface \( A \), branched along an ample curve \( D \) with at most simple singularities.

It follows that the Albanese map of \( S \) is generically finite of degree 4.

## 2. The branch curve \( D \)

As is seen in Section 1 the quotient \( S/\sigma \) has many nodes. This forces the branch curve \( D \) of the Albanese map \( a_X: X \to A \) to decompose as a union of smooth elliptic curves, as we will show.

Standard computation for double covers yields \( K_X^2 = \frac{D^2}{2} \). Using the adjunction formula we calculate the arithmetic genus of \( D \):

\[
p_a(D) = 1 + \frac{D^2}{2} = 1 + K_X^2 = 1 + 4 \chi(X, \mathcal{O}_X),
\]

where the last equality follows from (1.2).

We explain now another way to compute \( p_a(D) \). Since \( D \) has only simple singularities, the contraction of \((-2)\)-curves \( \mu: X \to X_{\text{can}} \) is the canonical resolution of singularities of \( X_{\text{can}} \) as a double cover of \( A \) (cf. [BHPV III.7]): we have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mu} & X_{\text{can}} \\
\downarrow a & & \downarrow a \\
\tilde{A} & \xrightarrow{\rho} & A
\end{array}
\]

where \( \rho: \tilde{A} \to A \) is the composition of blow-ups resolving successively the singularities of the branch curve and \( \tilde{a}: X \to \tilde{A} \) is a double cover branched along the strict transform of \( D \) possibly plus some \((-2)\)-curves over the triple points of \( D \). Remark that the \((-2)\)-curves on \( X \) are exactly the inverse images of the exceptional curves of \( \rho \).
Let \( \tilde{D} \subset \tilde{A} \) be the strict transform of \( D \). Then \( \tilde{D} \) is an embedded resolution of \( D \) and there is a relation between \( p_a(D) \) and \( p_a(\tilde{D}) \) (cf. \cite{H77}, Cor. V.3.7):

\[
(2.2) \quad p_a(D) = p_a(\tilde{D}) + \sum_{p \in D_{\text{sing}}} \delta_p
\]

where \( D_{\text{sing}} \) denotes the singular locus of \( D \) and \( \delta_p \) is a positive integer for any \( p \in D_{\text{sing}} \), determined by the type of the curve singularity \( p \in D \).

**Definition 2.1.** A collection of distinguished curves \( E_1, \ldots, E_k \) on a smooth projective surface is called **even** if the sum \( \sum_{1 \leq i \leq k} E_i \) is linearly equivalent to \( 2L \) for some integral divisor \( L \).

**Lemma 2.2.** Let \( D \) be the branch curve of the Albanese map \( a_X: X \to A \) and \( p \) a singularity of \( D \). Let \( E_1, \ldots, E_k \subset X \) be an even collection of disjoint \((-2)\)-curves. Then

\[
\delta_p \geq \# \{ E_i \mid E_i \text{ is contracted to } p \},
\]

and if the equality holds then \( p \in D \) is of type \( A_{2m+1} \) from some integer \( m \geq 0 \).

**Proof.** Let \( C \) be the strict transform of \( D \) in the blow-up of \( A \) at \( p \). Then, according to type of \( p \in D \), one can determine the types of singularities of the curve \( C \) over \( p \) and in turn the value \( \delta_p \) as in the following table (cf. \cite{BHPV} Sec. II.8 and \cite{H77} Cor. V.3.7):

| \( D \) at \( p \) | \( D_{\text{sing}} \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |
|-----------------|-----------------|--------|--------|--------|
| \( C \) over \( p \) | \( A_{n-2} \) | \( A_{n-5} \) | \( A_0 \) | \( A_1 \) | \( A_2 \) |
| \( \delta_p \)   | \( \frac{n+1}{2} \) | \( 1 + \left\lfloor \frac{n}{2} \right\rfloor \) | 3     | 4      | 4      |

where points of type \( A_{-1} \) and \( A_0 \) are meant to be smooth points.

If \( p \in D \) is of type \( A_{2m} \) or \( E_n \) then there is no non-empty collection of disjoint \((-2)\)-curves over \( p \) whose sum has even intersection with each component of the exceptional locus \( a_X^{-1}(p) \). It follows that in these cases \( \# \{ E_i \mid E_i \text{ is contracted to } p \} = 0 \).

If \( p \in D \) is of type \( D_n \) then the only non-empty collection of disjoint \((-2)\)-curves over \( p \), whose sum has even intersection with each component of \( a_X^{-1}(p) \), consists of two end components. It follows that in this case \( \# \{ E_i \mid E_i \text{ is contracted to } p \} \leq 2 \).

Compared with the corresponding values of \( \delta_p \) in the above table the lemma follows. \( \square \)

Let \( \tilde{S} \to S \) be the blow-up at \( \text{Fix}(\sigma) \). Then the induced morphism \( \tilde{S} \to X \) is a double cover branched exactly along the exceptional curves \( E_1, \ldots, E_{4\chi} \) in \( X \) over the singular points of \( S/\sigma \), where \( \chi = \chi(S, O_S) \). So they form an even collection of disjoint \((-2)\)-curves, and by Lemma 2.2 we have

\[
(2.3) \quad \sum_{p \in D_{\text{sing}}} \delta_p \geq \sum_{p \in D_{\text{sing}}} \# \{ E_i \mid E_i \text{ is contracted to } p \} = 4\chi(S, O_S),
\]

with equality if and only if \( \delta_p = \# \{ E_i \mid E_i \text{ is contracted to } p \} \) for any \( p \in D_{\text{sing}} \).

Write \( \tilde{D} = \bigcup_{1 \leq i \leq k} \tilde{D}_i \) as the union of (smooth) irreducible components. Since \( \tilde{D}_i \) has a non-constant morphism to the abelian surface \( A \) we infer that \( g(\tilde{D}_i) \geq 1 \). Combining (2.2) with (2.3) we can bound from below the arithmetic genus of \( D \) as
Proposition 2.3. The morphism $S/\sigma$ has at most $A_{2m+1}$-singularities. The irreducible components $D_i = \rho(\tilde{D}_i)$ has geometric genus 1 by (2.6). Since there are no singular elliptic curves on an abelian variety, the components $D_i$ are in fact smooth. Moreover, the singularities of a union of elliptic curves on an abelian surfaces are ordinary, hence $D$ has only $A_1$-singularities.

By (2.5) and Lemma 3.1 the branch curve $D$ has at most $A_{2m+1}$-singularities. Since there are no singular elliptic curves on an abelian variety, the components $D_i$ are in fact smooth. Moreover, the singularities of a union of elliptic curves on an abelian surfaces are ordinary, hence $D$ has only $A_1$-singularities.

We summarize the above discussion by the following proposition.

Proposition 2.3. The morphism $S/\sigma \rightarrow A$ in (1.3), induced by the Albanese map of $S$ (or by that of $X$), is a flat double cover branched along a simple normal crossing ample curve $D$ whose irreducible components are elliptic curves.

We read from (1.3) that the Albanese map $\alpha_S: S \rightarrow A$ is then a finite morphism of degree 4.

3. Bidouble cover

The flat double cover $S/\sigma \rightarrow A$ induces an involution $\tau: S/\sigma \rightarrow S/\sigma$. The fixed locus Fix($\tau$) is the ramification curve of the double cover and contains all the singularities of $S/\sigma$. We want to lift $\tau$ to $S$, so that the degree 4 finite morphism $\alpha_S: S \rightarrow A$ will be recognized as a (flat) bidouble cover (see Proposition 3.2 below).

Some preparation is needed. Let $U_0$ be the smooth locus of $S/\sigma$. It is obviously invariant under the action of $\tau$.

Lemma 3.1. Let $u \in U_0$ be a $\tau$-fixed point. Then the induced automorphism of the fundamental group $\tau_*: \pi_1(U_0, u) \rightarrow \pi_1(U_0, u)$ is the identity map.

Proof. Let $p_1, \ldots, p_{4\chi} \in S$ be the fixed points of $\sigma$ where $\chi = \chi(S, \mathcal{O}_S)$. Then their images $q_i$ in $S/\sigma$ are exactly the singular points of $S/\sigma$, so we have $U_0 = (S/\sigma) \setminus \{q_1, \ldots, q_{4\chi}\}$. Furthermore, the images $r_i = \alpha_S(p_i)$ in the Albanese surface $A$ are exactly the nodes of the branch curve $D$.

We take sufficiently small and disjoint balls $B_i \subset A$ centered at $r_i$ in the Euclidean topology. Let $U_i := \alpha^{-1}(B_i)$ and $V_i = \alpha^{-1}(B_i)$ be the inverse images in $S/\sigma$ and in $S$ respectively. Then the $U_i$’s (resp. $V_i$’s) are invariant under the action of $\tau$ (resp. $\sigma$). Moreover, they are simply connected due to the conic structure at the singularities (cf. [D92, page 23]).
We view \( S/\sigma \) as the topological space obtained by patching the small neighborhoods \( U_i \)'s to \( U_0 \). More precisely, set \( X_0 = U_0 \) and define \( X_i = X_{i-1} \cup U_i \) inductively for \( 1 \leq i \leq 4\chi \). Then it is clear that \( S/\sigma = X_{4\chi} \). For each \( 1 \leq i \leq 4\chi \) we have the exact sequence of fundamental groups by van Kampen’s theorem
\[
\pi_1(U_i \setminus \{q_i\}, u_i) \to \pi_1(X_{i-1}, u_i) \to \pi_1(X_i, u_i) \to 1
\]
which is preserved by induced involution \( \bar{\tau} \) on the fundamental groups.

We prove by reversed induction on \( i \) that \( \bar{\tau}_* : \pi_1(X_{i-1}) \to \pi_1(X_{i-1}) \) is the identity map. Here we are allowed to omit the base points for the fundamental groups to simplify the notation because, for the statement to hold, the base points are irrelevant. By a result of Nori we have an isomorphism (see [LZ15, Lemma 4.5])
\[
\pi_1(X_{4\chi}) = \pi_1(S/\sigma) \cong \pi_1(A),
\]
Therefore, as the base step of induction, \( \bar{\tau}_* \) acts trivially on \( \pi_1(X_{4\chi}) \). Concerning the left end of (3.1) we have
\[
\pi_1(U_i \setminus \{q_i\}) \cong \mathbb{Z}/2,
\]
which has trivial automorphism group. In particular, \( \bar{\tau}_* \) acts as identity on it. It follows that, if \( \bar{\tau}_* \) is the identity on \( \pi_1(X_i) \), so is it on \( \pi_1(X_{i-1}) \). This finishes the induction step and we conclude that \( \bar{\tau}_* \) is the identity on \( \pi_1(U_0) \).

**Proposition 3.2.** The Albanese map \( a_S : S \to A \) is a bidouble cover.

**Proof.** We retain the notation in Lemma 3.1. Let \( S_0 \subset S \) be the inverse image of \( U_0 \) under the quotient map \( \lambda : S \to S/\sigma \) Then the map \( \lambda|_{S_0} : S_0 \to U_0 \) is an étale double cover. By Lemma 3.1 the induced automorphism \( \bar{\tau}_* \) of \( \pi_1(U_0, u) \) is the identity, where \( u \in U_0 \) is a \( \bar{\tau} \)-fixed point. In particular, the subgroup \( \langle \lambda|_{S_0} \rangle \pi_1(S_0, s) \) is invariant by \( \bar{\tau}_* \), where \( s \in S_0 \) is chosen to be over \( u \). As is known from general topology there is an automorphism \( \tau_0 \) of \( S_0 \) such that the following diagram commutes
\[
\begin{array}{ccc}
S_0 & \xrightarrow{\tau_0} & S_0 \\
\downarrow & & \downarrow \\
U_0 & \xrightarrow{\bar{\tau}_*|_{U_0}} & U_0.
\end{array}
\]
By the Riemann extension theorem \( \tau_0 \) extends to an automorphism \( \tau \) of \( S \), which is necessarily a lifting of \( \tau \in \text{Aut}(S/\sigma) \).

The group generated by \( \sigma \) and \( \tau \) sits in an extension of an order 2 group by the other:
\[
1 \to \langle \sigma \rangle \to \langle \sigma, \tau \rangle \to \langle \tau \rangle \to 1,
\]
hence is an abelian group of order 4.

Now we have a factorization of the Albanese map of \( S \):
\[
a_S : S \to S/\langle \sigma, \tau \rangle \to A.
\]
Since \( \deg(a_S) = |\langle \sigma, \tau \rangle| = 4 \), the finite morphism between normal surfaces \( S/\langle \sigma, \tau \rangle \to A \) is birational, hence is an isomorphism.

We claim \( \langle \sigma, \tau \rangle \cong (\mathbb{Z}/2)^2 \). Otherwise, \( \langle \sigma, \tau \rangle \) is isomorphic to \( \mathbb{Z}/4 \). The two automorphisms \( \tau \) and \( \sigma \) must be of order 4 and we have \( \sigma = \tau^2 = (\sigma \circ \tau)^2 \). Hence the fixed point sets \( \text{Fix}(\tau) \) and \( \text{Fix}(\sigma \circ \tau) \) are both contained in \( \text{Fix}(\sigma) \). But the later consists only of isolated points by Theorem 1.1, contradicting the fact that \( a_S : S \to A \) has a non-empty branch curve \( D \).

Therefore \( a_S : S \to A \) is a bidouble cover with Galois group \( \langle \sigma, \tau \rangle \). \( \square \)
Remark 3.3. In the proof of Lemma 3.1 we can shrink $B_i$ (if needed) and choose coordinates $y, z$ on $B_i$ such that $D \cap B_i$ is defined by $yz = 0$. Then the degree 4 map $V_i \to B_i$ is automatically a bidouble cover branched along the axes ($yz = 0$) (cf. [BHPV, page 102]). The effect of Lemma 3.1 and Proposition 3.2 is to glue these local bidouble covers to a global one.

Proof of Theorem 0.1. Let $\sigma_0 = \sigma, \sigma_1, \sigma_2$ be the three nontrivial elements of the Galois group of the bidouble cover $S \to A$. We know that $\sigma_0$ does not fix any curve. For $i = 1, 2$ let $D_i$ be the branch curve, the stabilizer over which is generated by $\sigma_i$. Since the (branched) covering space $S$ is smooth, the curves $D_1$ and $D_2$ are smooth, as is evident in the theory of bidouble covers (cf. [C84]). So they are both disjoint union of smooth elliptic curves (cf. Proposition 2.3).

Now it is easy to see that $D_i$ consists of fibres of some smooth elliptic fibration $h_i : A \to E_i$. Composing $h_1$ with the Albanese map $a_S : S \to A$ we get a fibration $f : S \to E_1$. (The fibration $h_2 \circ a_S : S \to E_2$ also works.) One sees that the singular fibres of $f$ are over $D_1$ and they are of the form $2C$ with $C$ smooth. With such a fibration structure and with the numerical equation $K_S^2 = 8\chi(S, \mathcal{O}_S)$ (see Theorem 1.1) the surface $S$ must be isogenous to a product, of unmixed type ([S95, Lemma 5]).

Remark 3.4. Surfaces isogenous to a product with $q = 2$ and with nontrivial $\text{Aut}_0(S)$ are classified in [CLZ13].

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