A MODULAR INTERPRETATION OF BBGS TOWERS

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Abstract. In 2000, based on his procedure for constructing explicit towers of modular curves, Elkies deduced explicit equations of rank-2 Drinfeld modular curves which coincide with the asymptotically optimal towers of curves constructed by Garcia and Stichtenoth. In 2015, Bassa, Beelen, Garcia, and Stichtenoth constructed a celebrated (recursive and good) tower (BBGS-tower for short) of curves and outlined a modular interpretation of the defining equations. Soon after that, Gekeler studied in depth the modular curves coming from sparse Drinfeld modules. In this paper, to establish a link between these existing results, we propose and prove a generalized Elkies’ Theorem which tells in detail how to directly describe a modular interpretation of the equations of rank-\(m\) Drinfeld modular curves with \(m \geq 2\).

Key words: Drinfeld module; Drinfeld modular curve; Ihara’s quantity; BBGS tower.

INTRODUCTION

From Ihara’s Quantity to Recursive Towers. Estimation of the number of rational points on an algebraic curve over the finite field \(\mathbb{F}_q\) is an important subject in number theory and algebraic geometry. Let \(C\) be a geometrically irreducible and smooth curve over \(\mathbb{F}_q\) and \(g = g(C)\) its genus. The number of \(\mathbb{F}_q\)-rational points on \(C\) has a well-known upper bound due to Hasse-Weil [49]:

\[
\#(C(\mathbb{F}_q)) \leq q + 1 + 2g\sqrt{q}.
\]

An improved bound is obtained by Serre [41]:

\[
\#(C(\mathbb{F}_q)) \leq q + 1 + g[2\sqrt{q}].
\]

A curve that attains the Hasse-Weil bound is called maximal. The interested reader is referred to [10, 11, 16, 23, 24, 44] for standard examples of Hermitian, Garcia-Güner-Stichtenoth, Giulietti-Korchmáros, Suzuki, and Ree curves, and to [6, 43] for recent progress on maximal curves.

Ihara [35] noted that the Hasse-Weil bound becomes weak when the genus \(g\) is relatively large with respect to the size \(q\) of the base field \(\mathbb{F}_q\). Also in [35], Ihara introduced an asymptotic bound of the number of rational points, now known as the Ihara’s quantity:

\[
A(q) := \limsup_{g \to \infty} \frac{N_q(g)}{g},
\]

where

\[
N_q(g) = \max \{ \#(C(\mathbb{F}_q)) | C \text{ is a curve over } \mathbb{F}_q \text{ with genus } g \}.
\]

The upper bound of Ihara’s quantity

\[
A(q) \leq \sqrt{q} - 1
\]

was discovered by Drinfeld-Vlăduţ [47]. Meanwhile, in search of lower bounds of \(A(q)\), people have invented various constructions of towers of curves over \(\mathbb{F}_q\). Roughly speaking, a tower \(T\) of curves over \(\mathbb{F}_q\) consists of a family of curves \(C_n\) together with a sequence of successive surjective maps:

\[
C_1 \xrightarrow{p_1} C_2 \xrightarrow{p_2} \cdots \xleftarrow{p_n-1} C_{n-1} \xrightarrow{p_n} C_n \xleftarrow{p_n} \cdots
\]

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such that all $C_n$ and $p_n$ are defined over $\mathbb{F}_q$ and $g(C_n) \to \infty$ as $n \to \infty$. The limit
\[ \lambda(T) := \lim_{n \to \infty} \frac{\#(C_n(\mathbb{F}_q))}{g(C_n)}, \]
which always exists (see [44, Lemma 7.2.3]), certainly gives a lower bound of $\lambda(T)$. A tower $T$ is called (asymptotically) good if $\lambda(T) > 0$. Though it is not normally easy to construct good towers, there are two approaches to construct them, either by class fields or by modular curves (classical, Shimura, and Drinfeld). In this paper, we will discuss three good towers arising from Drinfeld modules (elaborated in the subsequent Main Theorem).

Let us list some remarkable lower bounds of Ihara’s quantity achieved by good towers.

1. Serre [42] obtained the result
\[ A(q) \geq c \cdot \log q \]
for some constant $c > 0$. A particular value $c = \frac{1}{56}$ appeared in [39].

2. For $q$ being small prime numbers, some known results are found by Angles and Maire [3] ($A(5) \geq \frac{8}{9}$), Hajir and Maire [27] ($A(3) \geq \frac{12}{11}$), Li and Maharaj [36] ($A(7) \geq \frac{9}{8}$), Niederreiter and Xing [38] ($A(2) \geq \frac{81}{50}$), Xing and Yeo [50] ($A(2) \geq \frac{97}{50}$), and Hall-Seelig [28] ($A(7) \geq \frac{9}{8}$ and $A(11) \geq \frac{9}{7}$). This list is not complete.

3. For square numbers $q$, a sharp bound is discovered: $A(q) \geq \sqrt{q} - 1$ (hence $A(q) = \sqrt{q} - 1$), independently, by Ihara [35] and Tsfasman, Vladuț, and Zink [45], one using families of Shimura modular curves, the other using families of classical modular curves. By Gekeler [21], certain families of Drinfeld modular curves also attain this lower bound.

4. When $q$ is a cubic number, say $q = p^3$, Zink [52] got the result $A(q) \geq \frac{2(p^3 - 1)}{p^2 + 2}$ under the assumption that $p$ is a prime. Bezerra, Garcia, and Stichtenoth [7] proved that this inequality holds for arbitrary cubic numbers $q$.

5. When $q = p^{2m+1}$ where $m \geq 1$, Bassa, Beelen, Garcia, and Stichtenoth [5] proved that
\[ A(q) \geq \frac{2(p^{m+1} - 1)}{p + 1 + (p - 1)/(p^m - 1)}, \]
which is a source of inspiration of the present paper.

By Goppa’s construction [25], good towers yield good linear error-correcting codes. A celebrated discovery by Tsfasman et al. [45] — the existence of long linear codes with the relative parameters above the well-known Gilbert-Varshamov bound [44, Proposition 8.4.4], provided a vital link between Ihara’s quantity and the realm of coding theory.

Good towers that are recursive play important roles in the studies of Ihara’s quantity, coding theory, and cryptography [1, 9, 33, 34, 48, 51]. A tower $T$ is called recursive by an absolutely irreducible polynomial $f(x, y) \in \mathbb{F}_q[x, y]$ (see [44, Sections 3.6 and 7.2]) if

1. The initial curve $C_1$ is the projective line with coordinate $x_1$;
2. For $n \geq 2$, $C_n$ is the nonsingular projective model of an affine curve defined by
\[ f(x_1, x_2) = f(x_2, x_3) = \cdots = f(x_{n-1}, x_n) = 0. \]

A first concrete example of good tower which is recursive over $\mathbb{F}_{q^2}$ is given in 1995 by Garcia and Stichtenoth [17] with the recursive polynomial
\[ f(x, y) = x^{q-1}y^q + y - x^q. \]

Soon after that, they gave another tower with the recursive function [18]
\[ f(x, y) = y^q + y - \frac{x^q}{x^{q-1} + 1}. \]
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which turns out to be a subtower of the previous one. An excellent fact is that each of the two towers fulfills \( \lambda(T) = q - 1 \), the lower bound. We call such kind of towers **optimal**.

The subject for investigation in this paper is concerned with what Bassa, Beelen, Garcia, and Stichtenoth had presented in \([4,5]\) — a general construction of recursive towers over non-prime fields. For short, we call them **BBGS towers**. Below is a brief account of such towers.

Suppose that \( m = j + k \geq 2 \) is a positive integer, where \( j \) and \( k \) are coprime positive integers. Let \( a \) and \( b \) be non-negative integers such that \( ak - bj = 1 \). Consider the tower \( F \) (over \( \mathbb{F}_{q^m} \)), respectively \( H \), arising from the recursive polynomial

\[
F(x, y) = \text{Tr}_j \left( \frac{y}{x^q} \right) + \text{Tr}_k \left( \frac{y^{q^j}}{x} \right) - 1,
\]

respectively

\[
H(x, y) = \frac{\text{Tr}_j(y) - a}{\text{Tr}_j(x)^q - a} - \frac{\text{Tr}_k(y^{q^j}) - b}{\text{Tr}_k(x) - b},
\]

where \( \text{Tr}_l(x) := \sum_{i=0}^{l-1} x^q^i \). A key result in \([4]\) is the inequality

\[
\lambda(H) \geq \lambda(F) \geq 2 \left( \frac{1}{q^x - 1} + \frac{1}{q^y - 1} \right)^{-1}
\]

from which one obtains the lower bound in Equation (1). Note that the \( m = 2 \) case of the tower \( F \) coincides with the one constructed by Equation (2).

**The Motivation and Main Result.** Part of the motivation behind this work is to better understand the modular interpretation of the BBGS towers presented in \([5]\). We are also inspired by the other two works — one is \([14]\) by Elkies, where it is shown that the towers in (2) and (3) both arise from Drinfeld modular curves, the other is a recent work \([2]\) by Anbar, Bassa, and Beelen, where a particular tower \( H \) in (5) with \((m, j, k) = (3, 2, 1)\) is investigated and proved to be modular.

It is natural to ask whether one can work out explicit modular explanations of the BBGS towers \( F \) and \( H \) in, respectively, (4) and (5) with general \((m, j, k)\), and if so, what information can be derived from such explanations. For this purpose, the present paper will follow a framework described by Gekeler who proposed an abstract construction of Drinfeld modular curves \([22]\), and our answer is an explicit description of the relevant curves.

**The Main Theorem** [Generalized Elkies’ Theorem] Let \( F \) and \( H \) be functions defined as earlier by Equations (4) and (5). Let \( X_{m,j}(T^n), \hat{X}_{m,j}(T^n), \) and \( \hat{X}_{m,j}(T^n) \) be Drinfeld modular curves defined as in Definition 2.2. Assume that \( k \) is not divided by the characteristic \( p \). Then,

A. The function field \( \tilde{F}_{m,j}^{(a)} \) of the Drinfeld modular curve \( \hat{X}_{m,j}(T^n) \) over \( \mathbb{F}_{q^m} \) is generated by variables \( x_1, x_2, \ldots, x_n \) that are subject to the following recursive equations

\[
F(x_{i-1}, x_i) = 0, \quad i = 2, 3, \ldots, n.
\]

B. The function field \( \tilde{F}_{m,j}^{(a)} \) of the Drinfeld modular curve \( \hat{X}_{m,j}(T^n) \) over \( \mathbb{F}_{q^m} \) is generated by variables \( X_1, X_2, \ldots, X_n \) that are subject to the following recursive equations

\[
G(X_{i-1}, X_i) = 0, \quad i = 2, 3, \ldots, n,
\]

where

\[
G(x, y) = y \left( \sum_{i=0}^{j-1} \frac{y^{N_i}}{x^{N_{i+1}}} + \sum_{i=j}^{m-1} \frac{y^{N_i}}{x^{N_{i+j}}} \right)^{q-1} - x \quad \text{and} \quad N_i = \frac{q^i - 1}{q - 1}.
\]
C. The function field $F_{m,j}^{(1)}$ of $X_{m,j}(T)$ over $\mathbb{F}_{q^n}$ equals the rational function field $\mathbb{F}_{q^n}(z)$ with variable $z$. If $n \geq 2$, then the function field $F_{m,j}^{(n)}$ of $X_{m,j}(T^n)$ over $\mathbb{F}_{q^n}$ is generated by variables $u_2, \ldots, u_n$ satisfying the recursion

$$H(u_{i-1}, u_i) = 0, \quad i = 3, 4, \ldots, n. \quad (8)$$

We remark that

1. Parts A and B of the Main Theorem can be adapted to arguments over the base field $\mathbb{F}_q$;
2. The original Elkies’ Theorem in [14, Section 4] corresponds to the $(m, j, k) = (2, 1, 1)$ case;
3. It is tempting to mimic Elkies’ approach to handle the general $(m, j, k)$ cases $(m \geq 2)$. However, it does not simply yield what the theorem desired. Instead, we find another but equivalent description of Drinfeld modular curves, and thereby obtaining recursive formulas of the corresponding curves.

From the Main Theorem described as above, we are able to find supersingular points (which are necessarily rational) on the three modular curves (see Remark 3.3). We hope our result will shed light on finding explicit equations of certain towers, and in particular, provide valuable geometric insight into the nature of the BBGS-towers.

To this day, little is known about the general description of recursive towers from modular curves. Li, Maharaj, Stichtenoth, and Elkies [37] exhibited four optimal towers over $\mathbb{F}_{q^2}$ ($p = 2, 3, 5, 7$); Garcia, Stichtenoth, and Rück [19] computed an optimal tower over $\mathbb{F}_{p^2}$; Hasegawa, Inuzuka, and Suzuki [30–32] provided a number of classical and Shimura modular curves by using Elkies’ procedure; Hallouin and Perret [29] proposed a systematic method to produce potentially good recursive towers over finite fields. Our result and approach should be useful in the studies of Drinfeld modular curves in a wider range. In fact, based on the current work, we have come up with a sequence of Drinfeld modular curves which are organized in an elegant manner.

We also would like to point out works of others that are related to the present paper. In the work of Hu and Zhao [33,34], varies bases of certain Riemann-Roch spaces associated to the BBGS tower $\mathcal{F}$ are investigated. The interlink between explicit towers and modular curves emerges in Elkies’ works [13–15], leading to the Elkies’ modularity conjecture — All asymptotically optimal recursive towers defined over $\mathbb{F}_{q^2}$ arise from reductions of elliptic, Shimura, or Drinfeld modular curves.

This paper is organized as follows. Section 1 gives a succinct account of standard facts about Drinfeld modules, whose purpose is to fix the notation. The Drinfeld modular curves $X_{m,j}(T^n)$, $\tilde{X}_{m,j}(T^n)$, $\tilde{X}_{m,j}(T^n)$, and $\tilde{M}_{m,j}(T^n)$ ($n \geq 1$) are defined in Section 2 and a relation between $\tilde{X}_{m,j}(T^n)$ and $\tilde{M}_{m,j}(T^n)$ is then proved. Sections 1 and 2 also establish a list of important facts and identities that are subsequently used in Section 3 to prove the statements of our Main Theorem.

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1. Preliminaries

1.1. Drinfeld Modules. In this part we give a brief introduction to the notion of Drinfeld module, which was introduced by Drinfeld in his celebrated work [12]. For more in-depth studies, please see [8,12,20,26,46]. Some notations are in order. Let $\mathbb{F}_q$ be a finite field of cardinality $q$. Denote by $A := \mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$. Let $L$ be a field containing $\mathbb{F}_q$ together with a fixed $\mathbb{F}_q$-algebra homomorphism $\iota : A \rightarrow L$. Denote by $L\{\tau\}$ the non-commutative $L$-algebra which is generated...
by the \( q \)-Frobenius endomorphism \( \tau \) such that \( \tau \cdot a = a^q \) for all \( a \in L \). We refer to the \( L \)-algebra \( L\{\tau\} \) as a \textit{twisted polynomial ring} (also known as an Ore ring [40]). Denote by \( G_a \) the additive group scheme over \( L \). It is standard that the ring of \( F_q \)-linear endomorphisms \( \text{End}_{F_q}(G_a) \) of \( G_a \) is isomorphic to \( L\{\tau\} \).

Let \( \bar{L} \) be an algebraic closure of \( L \). By restricting \( \text{End}_{F_q}(G_a) \) to the \( \bar{L} \)-geometric points of \( G_a \), we obtain an induced action of \( L\{\tau\} \) on \( \bar{L} \). Explicitly, the action of a twisted polynomial \( f = \sum_{i=0}^{m} g_i \tau^i \in L\{\tau\} \) on \( \bar{L} \) is given by

\[
f : \bar{L} \to \bar{L}, \quad \mu \mapsto f(\mu) := \sum_{i=0}^{m} g_i \mu^q^i.
\]

The kernel of \( f \) is defined and denoted by

\[
\text{Ker}(f) := \{ \mu \in \bar{L} | f(\mu) = 0 \},
\]

which is a finite dimensional \( F_q \)-linear subspace of \( \bar{L} \). It is more suitable to consider \( \text{Ker}(f) \) as a group subscheme of \( G_a \), rather than a subgroup of \( \bar{L} \). However, we shall not need this refinement in this work.

The \textit{point derivation} \( \partial_0 \) of a twisted polynomial \( f \) at 0 is standard:

\[
\partial_0 : \ L\{\tau\} \to L, \quad f = \sum_{i=0}^{m} g_i \tau^i \mapsto g_0.
\]

Note that \( \partial_0 \) is a homomorphism of \( F_q \)-algebras.

A \textit{Drinfeld module} over \( L \) is an \( F_q \)-algebra homomorphism

\[
\phi : A \to L\{\tau\}, \quad a \mapsto \phi_a,
\]

satisfying the conditions

1. there exists \( a \in A \) such that \( \phi_a \neq \iota(a) \); and
2. \( \partial_0 \circ \phi = \iota \), i.e., the following diagram of \( F_q \)-algebra homomorphisms

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & L\{\tau\} \\
\downarrow{\iota} & & \downarrow{\partial_0} \\
L & & 
\end{array}
\]

is commutative.

For a Drinfeld module \( \phi \) as above, the kernel of \( \iota \), which is a prime ideal in \( A \), is called the \textit{characteristic} of \( \phi \). As \( A \) is the polynomial ring \( F_q[T] \), a Drinfeld module \( \phi \) is uniquely determined by a twisted polynomial \( \phi_T \) over \( L \). We suppose that

\[
\phi_T = g_m \tau^m + \cdots + g_2 \tau^2 + g_1 \tau + g_0,
\]

where \( g_m \) is not 0 for some integer \( m > 0 \). The number \( m \) is called the \textit{rank} of \( \phi \). If \( g_i = 0 \) for \( 1 \leq i \leq m-1 \), then \( \phi \) is said to be \textit{supersingular}.

For a polynomial \( a \in A \), the kernel of \( \phi_a \) is an \( A \)-submodule of \( \bar{L} \) thanks to the commutativity \( \phi_a \phi_b = \phi_b \phi_a \), for all \( b \in A \). In a special situation described below, the \( A \)-module structure of \( \text{Ker}(\phi_a) \) is explicit.

\textbf{Lemma 1.1} ([20, Proposition I.1.6]). \textit{If} \( a \in A \) \textit{is coprime to the characteristic of} \( \phi \), \textit{then}

\[
\text{Ker}(\phi_a) \cong (A/(aA))^{\otimes m}
\]

\textit{as} \( A \)-modules.
One recognizes this fact in parallel with a well-known result of elliptic curves:
\[ E[n] \cong \mathbb{Z}/(n\mathbb{Z}) \oplus \mathbb{Z}/(n\mathbb{Z}), \]
where \( E[n] \) is the group of \( n \)-torsion points on an elliptic curve \( E \).

### 1.2. Isomorphisms of Drinfeld Modules

We make some important conventions in subsequent analysis:

- We assume that \( \mathbb{F}_{q^n} \subseteq L \);
- We only consider Drinfeld modules of the form
  \[ \phi_T = g_m \tau^m + g_j \tau^j + 1. \]  
  \hspace{1cm} (9)

Here \( m = j + k \geq 2 \), and \( j \) and \( k \) are mutually coprime positive integers.

By the second assumption, the characteristic of a Drinfeld module is the ideal \((T - 1)\). In other words, \( \iota \) maps \( T \in A \) to \( 1 \in L \). Indeed, Bassa et al. studied this type of Drinfeld modules in \([4,5]\). Clearly, \( \phi \) is of rank \( m \). We call \( \phi \) normalized if \( g_m = -1 \).

**Notation 1.2.** We denote by \( \mathcal{D}_{m,j} \) the set of normalized Drinfeld modules that are of the form
\[ \phi_T = -\tau^m + g_j \tau^j + 1. \]  
\hspace{1cm} (10)

**Definition 1.3.** Two Drinfeld modules \( \phi \) and \( \psi \) over \( L \) are said to be isomorphic over \( \bar{L} \), if there exists an element \( \lambda \in \bar{L}^* \) such that for all \( a \in A \), the equation
\[ \lambda \phi_a = \psi_a \lambda \]  
\hspace{1cm} (11)
holds in \( \bar{L}\{\tau\} \).

Certainly, Equation (11) amounts to the condition \( \lambda \phi_T = \psi_T \lambda \). A Drinfeld module of the form (9) is isomorphic to a normalized one over \( \bar{L} \) provided that \((-g_m)\) is a \((q^m - 1)\)-st power. In fact, one takes \( \lambda \in \bar{L} \) which is a root of \((-g_m)\) of order \((q^m - 1)\). Then the scalar multiplication by \( \lambda \) gives an isomorphism from \( \phi \) to the normalized Drinfeld module \( \psi \) by setting
\[ \psi_T = -\tau^m + \lambda^{1-q^j} g_j \tau^j + 1, \]
because \( \lambda \phi_T = \psi_T \lambda \).

Recall that \( N_m = \frac{q^m - 1}{q-1} \). We call \( J(\phi) := g_j^{N_m} \in L \) the \( J\)-invariant of the normalized Drinfeld module \( \phi \in \mathcal{D}_{m,j} \). A well-known fact is that the isomorphism class of an elliptic curve is completely determined by its \( j \)-invariant. A similar fact for Drinfeld modules is the following

**Lemma 1.4** ([5, Section 4]). For two normalized Drinfeld modules \( \phi \) and \( \phi' \) represented, respectively, by
\[ \phi_T = -\tau^m + g_j \tau^j + 1 \text{ and } \phi'_T = -\tau^m + g'_j \tau^j + 1, \]  
\hspace{1cm} (12)
the following statements are equivalent:

1. The Drinfeld modules \( \phi \) and \( \phi' \) are isomorphic over \( \bar{L} \);
2. There exists some \( \lambda \in \mathbb{F}_{q^n}^* \) such that \( g_j = g'_j \lambda^{q^d-1} \);
3. The \( J \)-invariants of \( \phi \) and \( \phi' \) coincide: \( J(\phi) = J(\phi') \).

We sketch a proof for completeness.

**Proof.** The implication (1) \( \iff \) (2) is easy. If \( \lambda \phi_T = \phi'_T \lambda \) holds for some \( \lambda \in \bar{L} \), then
\[ -\lambda \tau^m + g_j \lambda \tau^j + \lambda = -\lambda^{q^m} \tau^m + g'_j \lambda^{q^d} \tau^j + \lambda. \]
Hence, \( \lambda^{q^m-1} = 1 \) and \( g_j = g'_j \lambda^{q^d-1} \). The converse is also obvious.
It is straightforward to see the implication $(2) \Rightarrow (3)$. If $g_j = g'_j \lambda^{q^j-1}$ holds for some $\lambda \in \mathbb{F}_q^*$, then
\[
g_j^{N_m} = \left( g'_j \lambda^{q^j-1} \right)^{N_m} = (g'_j)^{N_m}.
\]

We finally show the implication $(3) \Rightarrow (2)$. Note that if $g_j = g'_j = 0$ (i.e. supersingular Drinfeld modules), the proof is trivial. Below we assume that $g'_j \neq 0$.

If $J(\phi) = J(\phi')$ holds, then
\[
\left( \frac{g_j}{g'_j} \right)^{N_m} = 1,
\]
and hence $g_j/g'_j \in (\mathbb{F}_q^*)^{q^j-1}$. Since $m$ and $j$ are coprime, the image of the map
\[
\mathbb{F}_{q^m} \to \mathbb{F}_{q^m}, \quad \mu \mapsto \mu^{q^j-1}
\]
is identically $(\mathbb{F}_q^*)^{q^j-1}$. Therefore, there exists $\lambda \in \mathbb{F}_q^*$ such that $g_j^{-1} = \lambda^{q^j-1}$, as required. \hfill \Box

### 1.3. Isogenies of Drinfeld Modules.

**Definition 1.5.** An isogeny of Drinfeld modules from $\phi$ to $\psi$ is a twisted polynomial $\lambda \in \bar{L}\{\tau\}$ such that for all $a \in A$, the equation
\[
\lambda\phi_a = \psi_a\lambda
\]
holds in $\bar{L}\{\tau\}$.

Apparently, Equation (13) amounts to the condition $\lambda\phi_T = \psi_T\lambda$. In this case, $\text{Ker}(\lambda) \subseteq \bar{L}$ admits an $A$-module structure which is defined by
\[
a \cdot \mu := \phi_a(\mu), \quad \text{for } a \in A, \mu \in \text{Ker}(\lambda).
\]

Here $\phi_a(\mu)$ on the right hand side belongs to $\text{Ker}(\lambda)$, by Equation (13).

**Notation 1.6.** Let us set up some constantly used notations. For $0 \neq x \in \bar{L}$, define three types of twisted polynomials over $\bar{L}$:

1. $\eta_x := 1 + x^{1-q}T + x^{1-q^2}T^2 + \cdots + x^{1-q^{k-1}}T^{k-1};$
2. $\lambda_x := x^{q^k-1} - x^k = (x^{q^k-1} - x^{q^k-q}T)\eta_x; \text{ and}$
3. $Q_x := x^{1-q^k} + x^{1-q^{k+1}}T + \cdots + x^{1-q^{m-1}}T^{j-1} + T^j + x^{1-q^j+1} + \cdots + x^{1-q^{k-1}}T^{m-1}.$

Now we can reformulate the function $F$ defined by Equation (4):
\[
F(x, y) = \frac{Q_x(y)}{x} - 1.
\]

**Notation 1.7.** For $0 \neq x \in \bar{L}$, let $\phi_T^x$ be the Drinfeld module in $\mathcal{D}_{m,j}$ such that $\phi_T^x(x) = 0$. In other words, $\phi_T^x$ is represented by
\[
\phi_T^x := -x^m + g(x)T^j + 1,
\]
where $g(x) = x^{q^m-q^j} - x^{1-q^j}$.

We need a lemma which is generalized from [14, Equation (11)] (for the $(m, k) = (2, 1)$ case) and [2, Section 3.1] (for the $(m, k) = (3, 1)$ case).

**Lemma 1.8.** Let $\eta_x$, $Q_x$, $\phi_T^x$, and $\lambda_x$ be defined as earlier. We have
\[
\eta_x\phi_T^x = Q_x\lambda_x.
\]
Proof. The proof is by direct calculations. Let us first assume that \( k > j \). On the one hand, we have

\[
\eta_\ast \phi_T^\ast = \left( 1 + x^1 q^1 + x^1 q^1 \tau^1 + \cdots + x^1 q^{k-1} \tau^{k-1} \right) (-\tau^m + g(x) \tau^j + 1)
\]

\[
= \sum_{s=0}^{j-1} x^{1-q^s} \tau^s + \sum_{s=j}^{k-1} (x^{1-q^s} + x^{1-q^s} g(x) \tau^{s-j}) \tau^s
\]

\[
+ \sum_{s=k}^{m-1} \left( x^{1-q^s} g(x) \tau^{s-j} \right) \tau^s - \sum_{s=m}^{2m-j-1} x^{1-q^{s-m}} \tau^s.
\]

On the other hand, we have

\[
Q_x \lambda_x = \left( x^{1-q^s} + x^{1-q^{s+1}} + \cdots + x^{1-q^{k+j-1}} \tau^{j-1} + \tau^j + x^{1-q} \tau^j + \cdots + x^{1-q^{k+j-1}} \tau^{j-1} \right) \left( x \tau^s - \tau^k \right)
\]

\[
= \sum_{s=0}^{j-1} x^{1-q^s} \tau^s + \sum_{s=j}^{k-1} x^{1-q^s} \tau^s + \sum_{s=k}^{m-1} \left( -x^{1-q^s} + x^{1-q^{s-j}} \tau^{j-1} \right) \tau^s - \sum_{s=m}^{2m-j-1} x^{1-q^{s-m}} \tau^s.
\]

By carefully examining coefficients of the relevant \( \tau^s \)-terms, we see that \( \eta_\ast \phi_T^\ast \) and \( Q_x \lambda_x \) are identical.

If \( k \leq j \), the statement is proved in a similar manner. \( \square \)

The following theorem is a minor modification of a result obtained in [5, Proposition 4.2].

**Theorem 1.9.** Let \( x, y \in L \) be nonzero elements satisfying \( Q_x(y) = y \). Then the twisted polynomial \( \lambda_x \) is an isogeny from \( \phi^x \) to \( \phi^y \).

**Proof.** We wish to show that \( (\lambda_x \phi_T^\ast - \phi_T^\ast \lambda_x) \) is identically zero. Note that

\[
\lambda_x \phi_T^\ast = \left( x \tau^s - \tau^k \right) (-\tau^m + g(x) \tau^j + 1)
\]

\[
= x^{m+k} - \left( g(x) x^k + x^k \tau^1 \right) \tau^m - \tau^k + x^{k-1} g(x) \tau^j + 1,
\]

and

\[
\phi_T^\ast \lambda_x = \left( -\tau^m + g(y) \tau^j + 1 \right) \left( x \tau^s - \tau^k \right)
\]

\[
= x^{m+k} - \left( g(y) + x^{k+m-q^m} \right) \tau^m - \tau^k + x^{k-m} g(y) \tau^j + 1.
\]

Hence we have

\[
\lambda_x \phi_T^\ast - \phi_T^\ast \lambda_x = g_m \tau^m - g_j \tau^j,
\]

where

\[
g_m := -\left( g(x) x^k + x^k \tau^1 \right) + \left( g(y) + x^{k+m-q^m} \right),
\]

and

\[
g_j := x^{k-1} g(x) - x^{k-m-q^m} g(y).
\]

We will show that \( g_m \) and \( g_j \in \bar{L} \) are both identically zero. For this purpose, let us set up a polynomial \( G(X) := g_m X^{q-1} - g_j \in \mathbb{L}[X] \) and it suffices to show that \( G(X) = 0 \).

Let \( H_{x,y} := \{ h^q \mid h \in \bar{L}, \lambda_x(h) = y \} \) be a subset in \( L \). Obviously, it has exactly \( q^k \) elements.
Take an element $h^{q^d} \in H_{x,y}$. We observe that

$$h^{q^d}G(h^{q^d}) = (\lambda_x \phi_T^x - \phi_T^y \lambda_y)(h) \quad \text{(by Equation (14))}$$

$$= \lambda_x \phi_T^x(h) - \phi_T^y \lambda_y(h)$$

$$= (x^{q^d-1} - x^{q^d-q_T}) \eta \phi_T^x(h) - \phi_T^y(y)$$

$$= (x^{q^d-1} - x^{q^d-q_T}) Q_x \lambda_x(h) \quad \text{(by Lemma 1.8)}$$

$$= (x^{q^d-1} - x^{q^d-q_T}) Q_x(y)$$

$$= (x^{q^d-1} - x^{q^d-q_T})(x) = x^{q^d-1}x - x^{q^d-q} = 0.$$

It follows that $G(h^{q^d}) = 0$ and thus $G$ has at least $q^k$ zeros. However, by construction, $G$ is of degree $(q^d - 1)$. Hence $G$ must be trivial. \hfill \Box

**Corollary 1.10.** Let $x_1, \ldots, x_n$ be nonzero elements in $L$ such that $Q_{x_i}(x_{i+1}) = x_i$ for $i = 1, \ldots, n-1$. Then the twisted polynomial $\lambda_{x_{n-1}} \cdots \lambda_{x_1}$ is an isogeny from $\phi^{x_1}$ to $\phi^{x_n}$.

**Proof.** We use Theorem 1.9 repeatedly:

$$\lambda_{x_{n-1}} \cdots \lambda_{x_1} \phi_{T}^{x_1} = \lambda_{x_{n-1}} \cdots \lambda_{x_2} \phi_{T}^{x_2} \lambda_{x_1}$$

$$= \lambda_{x_{n-1}} \cdots \lambda_{x_i} \phi_{T}^{x_i} \lambda_{x_{i-1}} \cdots \lambda_{x_1}$$

$$= \phi_{T}^{x_n} \lambda_{x_{n-1}} \cdots \lambda_{x_1},$$

as claimed. \hfill \Box

## 2. Drinfeld Modular Curves and their Generalizations

### 2.1. Drinfeld Modular Curves

By Lemma 1.1, for a polynomial $N \in A$ satisfying $(T - 1, N) = 1$, the kernel $\text{Ker}(\phi_N)$ is isomorphic to $(A/(NA))^{\pm m}$ as an $A$-module.

**Notation 2.1.** Denote by $\mathcal{G}(N; \phi)$ the set of all rank 1 $N$-torsion submodules $G \subseteq \text{Ker}(\phi_N)$ (i.e., $G \cong A/(NA)$) such that $\xi(G) = G$ for all $\xi \in \text{Aut}(L/L)$.

Let $N$ and $\mathcal{G}(N; \phi)$ be as above. Two pairs $(\phi, G)$ and $(\phi', G')$, where $\phi$ and $\phi'$ are Drinfeld modules of the form (9), $G \in \mathcal{G}(N; \phi)$, and $G' \in \mathcal{G}(N; \phi')$, are said to be equivalent, if there exists an isomorphism $\lambda$ from $\phi$ to $\phi'$ such that $\lambda G = G'$.

The following three types of Drinfeld modular curves, all adapted from those of Elkies [14], can be seen as analogues to the classical modular curves which parameterize elliptic curves associated with certain level structures.

**Definition 2.2.** Suppose that $N \in A$ is a polynomial satisfying $(T - 1, N) = 1$ and $T|N$.

1. The Drinfeld modular curve $X_{m,j}(N)$ with respect to the polynomial $N$ is the algebraic curve that parameterizes equivalent classes of pairs $(\phi, G)$, where $\phi$ is a Drinfeld module which is isomorphic to a normalized one and $G \in \mathcal{G}(N; \phi)$.

2. The Drinfeld modular curve $\tilde{X}_{m,j}(N)$ is the algebraic curve which parameterizes pairs $(\phi, G)$, where $\phi \in \mathcal{D}_{m,j}$ is a normalized Drinfeld module and $G \in \mathcal{G}(N; \phi)$.

3. The Drinfeld modular curve $\bar{X}_{m,j}(N)$ is the algebraic curve which parameterizes triples $(\phi, G, x)$, where $\phi \in \mathcal{D}_{m,j}$ is a normalized Drinfeld module, $G \in \mathcal{G}(N; \phi)$, and $x$ (called marked point) is a nonzero element of $G \cap \text{Ker}(\phi_T) \cap L$ (which is isomorphic to $\mathbb{F}_q$).

**Remark 2.3.** (1) In the particular case that $N = 1$, the curve $X_{m,j}(1)$ coincides with the $J$-line, $J$ being the coordinate that tells the $J$-invariant of Drinfeld modules (see Lemma 1.4).
2.2. Towers and Galois Coverings. Let us consider the particularly interesting polynomials $N = T^n$, for $n = 1, 2, \ldots$. There associates three natural towers of modular curves. The one formed by $\tilde{X}_{m,j}(T^n)$ is drawn below:

$$\tilde{X}_{m,j}(T) \xleftarrow{p_1} \tilde{X}_{m,j}(T^2) \xleftarrow{p_2} \tilde{X}_{m,j}(T^3) \xleftarrow{p_3} \cdots,$$

where $\{p_n\}_{n \geq 1}$ is defined by

$$p_n : \tilde{X}_{m,j}(T^{n+1}) \to \tilde{X}_{m,j}(T^n) \to (\phi, G_{n+1}, x_1) \mapsto (\phi, \phi_T G_{n+1}, x_1).$$

The second tower of Drinfeld modular curves $X_{m,j}(T^n)$ and the third one for $X_{m,j}(T^n)$ are built similarly. Moreover, the three towers of curves are organized in the following diagram:

$$\begin{array}{ccccccc}
\tilde{X}_{m,j}(T) & \xleftarrow{\pi_1} & \tilde{X}_{m,j}(T^2) & \xleftarrow{\pi_1} & \tilde{X}_{m,j}(T^3) & \xleftarrow{\pi_1} & \cdots & \tilde{X}_{m,j}(T^n) & \xleftarrow{\pi_1} & \cdots \\
\downarrow{\pi_1} & & \downarrow{\pi_1} & & \downarrow{\pi_1} & & & \downarrow{\pi_1} & & \\
\tilde{X}_{m,j}(T) & \xleftarrow{\pi_1} & \tilde{X}_{m,j}(T^2) & \xleftarrow{\pi_1} & \tilde{X}_{m,j}(T^3) & \xleftarrow{\pi_1} & \cdots & \tilde{X}_{m,j}(T^n) & \xleftarrow{\pi_1} & \cdots \\
\downarrow{\pi_2} & & \downarrow{\pi_2} & & \downarrow{\pi_2} & & & \downarrow{\pi_2} & & \\
X_{m,j}(T) & \xleftarrow{\pi_2} & X_{m,j}(T^2) & \xleftarrow{\pi_2} & X_{m,j}(T^3) & \xleftarrow{\pi_2} & \cdots & X_{m,j}(T^n) & \xleftarrow{\pi_2} & \cdots \\
\end{array}$$

(15)

The vertical morphisms $\pi_1$ and $\pi_2$ are defined using their $L$-points as specified below:

$$\pi_1 : \tilde{X}_{m,j}(T^n) \to X_{m,j}(T^n), \quad (\phi, G_n, x_1) \mapsto (\phi, G_n),$$

and

$$\pi_2 : \tilde{X}_{m,j}(T^n) \to X_{m,j}(T^n), \quad (\phi, G_n) \mapsto [(\phi, G_n)].$$

Let us denote the composition of $\pi_1$ and $\pi_2$ by

$$\pi_3 : \tilde{X}_{m,j}(T^n) \to X_{m,j}(T^n), \quad (\phi, G_n, x_1) \mapsto [(\phi, G_n)].$$

Our Main Theorem in the introduction claims that the three towers above are all recursive. Note that the associated recursive polynomials coincide with those investigated by Bassa et al. in [4, 5].

Let $\hat{F}^{(n)}_{m,j}$ (resp. $F^{(n)}_{m,j}$, $\check{F}^{(n)}_{m,j}$) be the function field of $\tilde{X}_{m,j}(T^n)$ (resp. $X_{m,j}(T^n)$, $X_{m,j}(T^n)$). In parallel with (15), we are able to draw a diagram of function fields:

$$\begin{array}{ccccccc}
\hat{F}^{(1)}_{m,j} & \rightarrow & \hat{F}^{(2)}_{m,j} & \rightarrow & \hat{F}^{(3)}_{m,j} & \rightarrow & \cdots & \hat{F}^{(n)}_{m,j} & \rightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & & \uparrow & & \\
F^{(1)}_{m,j} & \rightarrow & F^{(2)}_{m,j} & \rightarrow & F^{(3)}_{m,j} & \rightarrow & \cdots & F^{(n)}_{m,j} & \rightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & & \uparrow & & \\
\check{F}^{(1)}_{m,j} & \rightarrow & \check{F}^{(2)}_{m,j} & \rightarrow & \check{F}^{(3)}_{m,j} & \rightarrow & \cdots & \check{F}^{(n)}_{m,j} & \rightarrow & \cdots \\
\end{array}$$
In the rest of this section, we establish some facts about relative degrees of morphisms appeared in Diagram (15). For convenience, we write $\mathcal{G}_n(\phi) := G(T^n; \phi)$.

**Lemma 2.4.** The relative degree of the morphism $p_n : \tilde{X}_{m,j}(T^{n+1}) \to \tilde{X}_{m,j}(T^n)$ is $q^{n-1}$.

**Proof.** Without loss of generality, we may assume that $L = \bar{L}$. Let $G_n \in \mathcal{G}_n(\phi)$ be fixed. The lemma is proved if we can show that there are exactly $q^{n-1}$ elements $G_{n+1} \in \mathcal{G}_{n+1}(\phi)$ such that $\phi_T(G_{n+1}) = G_n$.

We will list such $G_{n+1}$ explicitly. First, one can find some $\mu \in \bar{L}$ such that $G_n = A \cdot \mu$, because $G_n \cong A/(T^nA)$. Second, consider the set

$$\mathcal{S} := \left\{ \nu \in \bar{L}|\phi_T(\nu) = \mu \right\}.$$ 

Each $\nu \in \mathcal{S}$ gives rise to an $A$-module $G_{n+1} = A \cdot \nu \in \mathcal{G}_{n+1}(\phi)$ which certainly satisfies $\phi_T(G_{n+1}) = G_n$. It is also easy to see that all solutions $G_{n+1}$ to $\phi_T(G_{n+1}) = G_n$ must be of this form.

Finally, $\nu$ and $\nu' \in \mathcal{S}$ give rise to the same $G_{n+1}$ if and only if $\nu - \nu' \in G_{n+1} \cap \text{Ker}(\phi_T) \cong A/(TA)$. Therefore, the number of such $A$-modules $G_{n+1}$ is computed by:

$$\frac{\#(A/(TA))}{\#(\mathcal{S})} = \frac{q^n}{q} = q^{n-1}.$$ 

This completes the proof. \qed

For $\mu \in \mathbb{F}_q^*$, there associates an automorphism on $\tilde{X}_{m,j}(T^n)$ defined by

$$(\phi, G_n, x_1) \mapsto (\phi^{\mu x_1}, \mu G_n, \mu x_1).$$

(16)

This automorphism is also denoted by $\mu$.

**Lemma 2.5.**

1. The automorphism $\mu$ is compatible with the covering $\pi_3$, i.e., the following diagram

$$\begin{array}{ccc}
\tilde{X}_{m,j}(T^n) & \xrightarrow{\pi_3} & \tilde{X}_{m,j}(T^n) \\
\mu & \mapsto & \mu \\
\pi_3 & \xrightarrow{\pi_3} & \pi_3 \\
X_{m,j}(T^n) & \xrightarrow{\pi_3} & X_{m,j}(T^n)
\end{array}$$

of algebraic curves is commutative. Moreover, if $\mu \in \mathbb{F}_q^*$, then $\mu$ is compatible with $\pi_1$.

2. The morphism $\pi_1 : \tilde{X}_{m,j}(T^n) \to X_{m,j}(T^n)$ is a Galois covering whose Galois group is isomorphic to the multiplicative group $\mathbb{F}_q^*$. 

3. The morphism $\pi_3 : \tilde{X}_{m,j}(T^n) \to X_{m,j}(T^n)$ is a Galois covering whose Galois group is isomorphic to the multiplicative group $\mathbb{F}_q^*$. 

**Proof.** Note that the statement in part (2) follows immediately from that of part (3). So we only need to prove part (1) and (3).

1. Let $\phi$ be the Drinfeld module with

$$\phi_T = -\tau^m + g_j \tau^j + 1,$$

and hence we have

$$\phi_T^{\mu x_1} = -\tau^m + (\mu^{1-q^j}) g_j \tau^j + 1 \quad \text{and} \quad \mu \phi_T = \phi_T^{\mu x_1} \mu,$$

by direct calculation. It implies that $[(\phi, G_n)] = [(\phi^{\mu x_1}, \mu G_n)]$, i.e. the first statement of part (1). The second statement follows by observing that $\phi = \phi^{\mu x_1}$ and $G_n = \mu G_n$, if $\mu \in \mathbb{F}_q^*$. 


(3) According to part (1), we only need to show that the degree of \( \pi_3 \) equals \( (q^m - 1) \).

First, consider the situation that \( g_j \neq 0 \) in the expression of \( \phi_T \). Since \( j \) and \( m \) are coprime, we have \( \mathbb{F}_{q^m} = \mathbb{F}_q^{q^m} \). Hence the number of Drinfeld modules of the form \( \phi^\mu \), for \( \mu \in \mathbb{F}_q^* \), is equal to \( \frac{q^m - 1}{q - 1} \). In the meantime, the number of nonzero elements in \( \phi_{\tau^{n-1}}G_n \) is \( (q - 1) \). Thus the number of preimages of \( [(\phi, G_n)] \) under \( \pi_3 \) is obtained: 

\[
\frac{q^m - 1}{q - 1} (q - 1) = q^m - 1.
\]

Second, if \( g_j = 0 \), i.e., \( \phi_T = -\tau^m + 1 \), then \( \mu \phi = \phi \mu \), for all \( \mu \in \mathbb{F}_q^* \), and hence \( [(\phi, G_n)] = [(\phi, \mu G_n)] \). Moreover, \( (\phi, G_n, x_1) = (\phi, \mu G_n, x_1) \) if and only if \( \mu \in \mathbb{F}_q^* \). Again, by the fact that the number of nonzero elements in \( \phi_{\tau^{n-1}}G_n \) is \( (q - 1) \), we get the the number of preimages of \( [(\phi, G_n)] \) under \( \pi_3 \), which is \( \frac{q^m - 1}{q - 1} (q - 1) = q^m - 1 \).

This shows that the degree of \( \pi_3 \) is \( q^m - 1 \) and the assertion is thus confirmed.

\[
\square
\]

**Corollary 2.6.** All horizontal morphisms

\[
\tilde{X}_{m,j}(T^{n+1}) \to \tilde{X}_{m,j}(T^n), \quad \tilde{X}_{m,j}(T^{n+1}) \to \tilde{X}_{m,j}(T^n), \quad \text{and} \quad X_{m,j}(T^{n+1}) \to X_{m,j}(T^n)
\]

in Diagram (15), have the same relative degree \( q^m - 1 \).

**Proof.** The conclusion follows directly by Lemmas 2.4, 2.5, and the commutativity of Diagram (15).

\[
\square
\]

2.3. Modular Curves \( \tilde{M}_{m,j}(T^n) \). In subsequent analysis, we assume that \( k \) (\( = m - j \)) is not divided by the characteristic \( p \) of \( \mathbb{F}_q \). Let \( A_k := \mathbb{F}_q[T] \) be the obvious extension of the ring \( A = \mathbb{F}_q[T] \), and moreover, we treat \( L \) as an \( A_k \)-field in an obvious way. Set

\[
F(T) := 1 - (1 - T)^k = T \cdot f(T) \in A,
\]

where

\[
f(T) := \sum_{i=1}^{k} \binom{k}{i} (-T)^{i-1}.
\]

Evidently, \( f(0) = k \neq 0 \) and \( (T, f(T)) = 1 \).

For a normalized Drinfeld module \( \phi \) as in (12), there associates another Drinfeld module

\[
\Phi : A \to L\{\tau\}, \quad T \mapsto \phi_{F(T)}
\]

over \( L \). In other words,

\[
\Phi_T = \phi_{F(T)} = \phi_{1-(1-T)^k} = \phi_{(1-T)^k} = 1 - (\tau^m - g_j \tau^j)^k
\]

\[= 1 - \sum_{i=0}^{k} \binom{k}{i} \tau^{mi}, \quad \tau^{j(k-i)}
\]

\[= 1 - \sum_{i=0}^{k} \binom{k}{i} \tau^{(i+j)k},
\]

where the coefficients are defined by

\[
\binom{k}{i} = 1; \quad \binom{k}{0} = (-1)^{k_j} \tau^{\text{Tr}(q^j)}; \quad \text{and iteratively},
\]

\[
\binom{k}{i} = \binom{k-1}{i-1} + g_j \binom{k-1}{i} \tau^j, \quad \text{for } 1 \leq i \leq k - 1.
\]

The \( \tau \)-twisted polynomial \( \Phi_T \) can be regarded as a \( \tau^k \)-twisted polynomial. Therefore, one can alternatively treat \( \Phi \) as a Drinfeld module over the \( A_k \)-field \( L \) of characteristic \( (T-1) \). Recall
that \( \phi \) gives rise to an \( A \)-module structure on \( \tilde{L} \). Similarly, \( \Phi \) gives rise to an \( A_k \)-module structure on \( \tilde{L} \). According to Lemma 1.1, we have

\[
\ker(\Phi_{T^n}) \cong (A_k/(T^n A_k))^\oplus m, \quad \text{as an } A_k\text{-submodule of } \tilde{L},
\]

and

\[
\ker(\Phi_{T^n}) \cong (A/(T^n f(T)^n A))^\oplus m, \quad \text{as an } A\text{-submodule of } \tilde{L}.
\]

**Notation 2.7.** Let \( \Phi \) be the Drinfeld module arising from \( \phi \) explained as above. Denote by \( \mathcal{E}_n(\phi) \) the set consisting of \( \mathbb{F}_q^* \)-vector spaces \( E_n \subseteq \ker(\Phi_{T^n}) \), such that

1) \( E_n \) is stable under \( \text{Aut}(\tilde{L}/L) \), i.e., \( \xi(E_n) = E_n \) for all \( \xi \in \text{Aut}(\tilde{L}/L) \);

2) \( E_n \cong A/(F(T)^n A) \) as an \( A \)-submodule of \( \tilde{L} \); and

3) \( E_n \cong A_k/(T^n A_k) \) as an \( A_k \)-submodule of \( \tilde{L} \).

**Definition 2.8.** The **twisted Drinfeld modular curve** \( \tilde{M}_{m,j}(T^n) \) is the algebraic curve that parameterizes triples \( (\phi, E_n, x) \), where \( \phi \in \mathcal{D}_{m,j} \), \( E_n \in \mathcal{E}_n(\phi) \), and \( x \in \phi T^{-1}f(T)^n E_n \cap L \) is a nonzero marked point.

**Remark 2.9.** If \( j = m-1 \) (or \( k = 1 \)), then \( \Phi = \phi \) and the twisted Drinfeld modular curve \( \tilde{M}_{m,m-1}(T^n) \) coincides with the Drinfeld modular curve \( \tilde{X}_{m,m-1}(T^n) \) (see Definition 2.2).

The following key theorem is needed.

**Theorem 2.10.** The curves \( \tilde{M}_{m,j}(T^n) \) and \( \tilde{X}_{m,j}(T^n) \) are isomorphic over \( \mathbb{F}_q^* \).

A direct consequence of this theorem is that the function field of twisted Drinfeld modular curves and that of normalized Drinfeld modular curves are one and the same. Before we come to the proof of this theorem, let us establish some useful lemmas.

**Lemma 2.11.** Let \( x \in \tilde{L} \) be nonzero. Endow \( \tilde{L} \) with the \( A \)-module structure induced by the Drinfeld module \( \phi = \phi^* \). We have

1) The \( \mathbb{F}_q^* \)-vector space \( \mathbb{F}_q^* \cdot x \) is an \( A \)-submodule of \( \tilde{L} \);

2) The annihilator ideal of \( \mathbb{F}_q^* \cdot x \) is generated by \( F(T) \);

3) \( \mathbb{A} \)-modules \( \mathbb{F}_q^* \cdot x \) and \( A/(F(T)A) \) are isomorphic.

**Proof.** (1) The \( A \)-module structure of \( \mathbb{F}_q^* \cdot x \) is presented by

\[
(1 - T) \cdot (\mu x) := \phi_{1 - T}(\mu x) = (\tau m - g(x)\tau) (\mu x) = \mu \phi^m x^m - g(x)\mu^q x^q = -\mu \phi^q x + \mu^q x,
\]

for all \( \mu \in \mathbb{F}_q^* \).

(2) Let \( \{\rho, \rho^2, \ldots, \rho^{k-1}\} \) be a normal basis of \( \mathbb{F}_q^*/\mathbb{F}_q \). Since \( k \) and \( j \) are coprime, the action by \( (1 - T) \) on \( \mathbb{F}_q^* \cdot x \) is a circulant permutation to this basis. Thus the minimal polynomial of \( (1 - T) \) equals \( (\lambda^k - 1) \). This means that the annihilator ideal of \( \mathbb{F}_q^* \cdot x \) is generated by \( (1 - T)^k - 1 = -F(T) \).

(3) By construction, \( \mathbb{F}_q^* \cdot x \) and \( A \cdot \rho x \) are identical. It follows that \( \mathbb{F}_q^* \cdot x \cong A/(F(T)A) \). \( \square \)

Recall the notation \( \mathcal{E}_n(\phi) \) that we introduced earlier in Notation 2.7.

**Lemma 2.12.** For \( G_n \in \mathcal{G}_n(\phi) \), the \( \mathbb{F}_q^* \)-subspace of \( \tilde{L} \) spanned by \( G_n \), denoted by \( \mathbb{F}_q^* \langle G_n \rangle \), belongs to \( \mathcal{E}_n(\phi) \).

**Proof.** According to the definition of \( \mathcal{E}_n(\phi) \), we divide the proof into three parts.
This completes the proof.

We are now ready to give the

(1) For any \(\xi \in \text{Aut}(\overline{L}/L)\), we have \(\xi(G_n) = G_n\), by definition of \(\mathcal{G}_n(\phi)\). For \(\sum_i \mu_ig_i \in \mathbb{F}_{q^k} \langle G_n \rangle\), we see that
\[
\xi(\sum_i \mu_ig_i) = \sum_i \xi(\mu_i)g_i \in \mathbb{F}_{q^k} \langle G_n \rangle,
\]
which means that \(\text{Aut}(\overline{L}/L)\) preserves \(\mathbb{F}_{q^k} \langle G_n \rangle\).

(2) Regarding the \(A_k\)-module structure, we first examine that \(\mathbb{F}_{q^k} \langle G_n \rangle\) is closed under the \(\Phi_T\)-action. For \(\sum_i \mu_ig_i \in \mathbb{F}_{q^k} \langle G_n \rangle\), we have
\[
\Phi_T(\sum_i \mu_ig_i) = \sum_i \mu_i\Phi_T(g_i) \in \mathbb{F}_{q^k} \langle G_n \rangle.
\]

We next show that \(\mathbb{F}_{q^k} \langle G_n \rangle \cong A_k/(T^n A_k)\). This fact is due to the following observations:

(a) The subspace \(\mathbb{F}_{q^k} \langle G_n \rangle\) is contained in \(\text{Ker}(\Phi_{T^n})\). This is easy as
\[
\Phi_{T^n}(\sum_i \mu_ig_i) = \sum_i \mu_i\Phi_{T^n}(g_i) = \sum_i \mu_i\phi_{T^n}(g_i) = 0.
\]

(b) We can find \(u \in G_n\) such that \(\Phi_{T^{n-1}}(u) \neq 0\). In fact, we have
\[
\Phi_{T^{n-1}}(G_n) = \Phi_{T^{n-1}}(\phi_{T^n}(T^{n-1})(G_n)) = \Phi_{T^{n-1}}(G_n) \neq \{0\},
\]
where we used the fact that \(G_n \cong A/(T^n A)\) and \((T, f(T)) = 1\).

(c) We show that \(A_k/(T^n A_k) \cong A_k \cdot u\). In fact, by (a) and the isomorphism in (17),
\[
u \in \text{Ker}(\Phi_{T^n}) \cong (A_k/(T^n A_k))^{\oplus m}.
\]

By (b), one has \(\text{Ann}(u) = (T^n)\), and thus \(A_k \cdot u = A_k/\text{Ann}(u) = A_k/(T^n A_k)\).

(d) As \(G_n \cong A/(T^n A)\), we have \(\dim_{\mathbb{F}_q}(G_n) = n\). Comparing the \(F_{q^k}\)-dimensions of \(A_k \cdot u \subseteq \mathbb{F}_{q^k} \langle G_n \rangle\), we see that they must be equal.

(3) Regarding the \(A\)-module structure of \(G_n\), we examine that \(\mathbb{F}_{q^k} \langle G_n \rangle\) admits a \(\phi_T\)-action. In fact, for \(\sum_i \mu_ig_i \in \mathbb{F}_{q^k} \langle G_n \rangle\), we have
\[
\phi_T(\sum_i \mu_ig_i) = \sum_i \mu_i^q T \left( \phi_T(g_i) + (u - u^q)g_i \right) \in \mathbb{F}_{q^k} \langle G_n \rangle.
\]

We now show that \(\mathbb{F}_{q^k} \langle G_n \rangle \cong A/(F(T)^n A)\) as \(A\)-modules. This is accomplished following three steps.

(a) By what we concluded in the second part, we have \(A_k/(TA_k) \cong \Phi_{T^{n-1}}(\mathbb{F}_{q^k} \langle G_n \rangle)\) as \(A_k\)-modules. This implies that \(\Phi_{T^{n-1}}(\mathbb{F}_{q^k} \langle G_n \rangle) = \mathbb{F}_{q^k} \cdot x\), for some \(x \in L\).

(b) According to Lemma 2.11,
\[
\phi_{F(T)^n}(\mathbb{F}_{q^k} \langle G_n \rangle) = \Phi_{T^{n-1}}(\mathbb{F}_{q^k} \langle G_n \rangle) = A \cdot px \cong A/(F(T) A),
\]
as \(A\)-modules. And we are able to find some \(v \in \mathbb{F}_{q^k} \langle G_n \rangle\) such that
\[
\phi_{F(T)^n}(v) = \Phi_{T^{n-1}}(v) = px.
\]

Because \(\text{Ann}(px) = (F(T))\), we have \(\text{Ann}(v) = (F(T)^n)\) and
\[
A \cdot v \cong A/(\text{Ann}(v)) = A/(F(T)^n A).
\]

(c) It follows from the previous argument that the dimension of the \(F_{q^k}\)-vector space \(A \cdot v\) is \(q^{nk}\), the same as that of \(\mathbb{F}_{q^k} \langle G_n \rangle\). So one must have \(A \cdot v = \mathbb{F}_{q^k} \langle G_n \rangle\).

This completes the proof. \(\Box\)

We are now ready to give the
Proof of Theorem 2.10. We construct a map

$$\alpha : \tilde{M}_{m,j}(T^n) \to \tilde{X}_{m,j}(T^n),$$

$$(\phi, E_n, x) \mapsto (\phi, \phi f(T)^n E_n, x).$$

Here we have used the fact that

$$A/(T^n A) \cong \phi f(T)^n E_n \in G_n(\phi),$$

which is due to the definition of $E_n$. In the mean time, we construct

$$\beta : \tilde{X}_{m,j}(T^n) \to \tilde{M}_{m,j}(T^n),$$

$$(\phi, G_n, x) \mapsto (\phi, \mathbb{F}_{q^k} \langle G_n \rangle, x).$$

By Lemma 2.12, the map $\beta$ is well-defined. We now prove that $\alpha$ and $\beta$ are mutually inverse maps.

(1) First, we show that $\alpha \circ \beta = \text{Id}$, which amounts to the following identity

$$\phi f(T)^n \mathbb{F}_{q^k} \langle G_n \rangle = G_n, \quad \text{for all } G_n \in G_n(\phi). \quad (18)$$

In fact, by Lemma 2.12, we know that

$$\mathbb{F}_{q^k} \langle G_n \rangle \cong A/(F(T)^n A),$$

and hence

$$\phi f(T)^n \mathbb{F}_{q^k} \langle G_n \rangle \cong A/(T^n A).$$

It implies that

$$G_n = \phi f(T)^n G_n \subseteq \phi f(T)^n \mathbb{F}_{q^k} \langle G_n \rangle.$$ By the fact that

$$\#G_n = \# \left( A/(T^n A) \right) = \# \left( \phi f(T)^n \mathbb{F}_{q^k} \langle G_n \rangle \right),$$

we proved the desired equality (18).

(2) Second, we show that $\beta \circ \alpha = \text{Id}$, or

$$E_n = \mathbb{F}_{q^k} \langle \phi f(T)^n E_n \rangle, \quad \text{for all } E_n \in E_n(\phi).$$

In fact, $\phi f(T)^n E_n$ is isomorphic to $A/(T^n A)$ (as an $A$-submodule of $\tilde{L}$) by its definition. Using Lemma 2.12, we get

$$\mathbb{F}_{q^k} \langle \phi f(T)^n E_n \rangle \cong A_k/(T^n A_k).$$

Counting cardinalities of the two sides of

$$\mathbb{F}_{q^k} \langle \phi f(T)^n E_n \rangle \subseteq E_n,$$

we see that they must be equal (to the same number $q^{kn}$). The proof is thus completed. \[\square\]

3. Proof of Generalized Elkies’ Theorem

This section is denoted to proving the Main Theorem declared in the introduction part.
3.1. **Part A.** Recall that in Notation 1.7, we introduced the Drinfeld module $\phi^x$ such that $\phi^x_f(x) = 0$, where $0 \neq x \in \bar{L}$. Recall also that in Notation 2.7, we defined the set

$$E_n(\phi) := \left\{ E_n \subseteq \text{Ker} \left( \phi_{f(T)} \right) | E_n \cong A/(F(T)^nA) \text{ as } A\text{-modules}, \quad \text{and} \quad E_n \cong A_k/(T^nA_k) \text{ as } A_k\text{-modules} \right\}.$$ 

For $0 \neq x_1 \in \bar{L}$, let us define the set

$$E_n^*(\phi^{x_1}) := \left\{ E_n \in E_n(\phi^{x_1}) | x_1 \in \Phi_{T(T)}^{-1} \phi_{f(T)}^{x_1} E_n \right\}.$$

Next, we consider another set

$$X_n := \{ (x_2, \ldots, x_n) \in (\bar{L})^{n-1} | Q_{x_1}(x_2) = x_1, Q_{x_1}(x_3) = x_2, \ldots, Q_{x_1}(x_n) = x_{n-1} \}.$$ 

Apparently, $(x_2, \ldots, x_n) \in X_n$ is subject to Equation (6).

We need a preparatory theorem.

**Theorem 3.1.** Let $x_1 \in \bar{L}$ be nonzero, the sets $E_n^*(\phi^{x_1})$ and $X_n$ as above. For $n \geq 2$, there is an one-to-one correspondence between $E_n^*(\phi^{x_1})$ and $X_n$ as sets.

**Proof.** We will establish the following fact — For each step $n \geq 2$, there is a bijective map

$$\Theta_n : E_n^*(\phi^{x_1}) \to X_n.$$ 

Moreover, the correspondence $E_n \in E_n^*(\phi^{x_1}) \mapsto (x_2, \ldots, x_n) \in X_n$ is characterized by the following conditions:

1. As a submodule in $\bar{L}$, we have

$$E_n = \text{Ker} (\lambda_{x_n} \cdots \lambda_{x_2} \lambda_{x_1});$$

2. The set $H_n := E_n \cap (\Phi_{T^n})^{-1}(x_1)$ is non-empty, and for each $h \in H_n$, one has

$$x_n = \lambda_{x_{n-1}} \cdots \lambda_{x_1}(h).$$ 

We prove by induction on $n$, and start from $n = 2$.

1. The existence of $h \in H_2$ is due to the definition of

$$E_2^*(\phi^{x_1}) := \left\{ E_2 \in E_2(\phi^{x_1}) | x_1 \in \Phi_{T}^{x_2} \phi_{f(T)}^{x_1} E_2 \right\}.$$

2. The value $\lambda_{x_1}(h)$ does not depend on the choice of $h$. In fact, if $h' \in E_2$ satisfies $\Phi_{T}^{x_2}(h') = x_1$, then $h - h' \in \text{Ker} (\Phi_{T}) \cap E_2$. So we only need to show that $\text{Ker} (\Phi_{T}) \cap E_2 = \text{Ker} (\lambda_{x_1})$.

The fact that $\text{Ker} (\lambda_{x_1}) \subseteq \text{Ker} (\Phi_{T}) \cap E_2$ is obvious. Also note that $\#(\text{Ker} (\lambda_{x_1})) = q^k$. Let us show that $\#(\text{Ker} (\Phi_{T}) \cap E_2)$ is also $q^k$. This can be examined by observing that $E_2 \cong A_k/(T^2 A_k)$, and $\text{Ker} (\Phi_{T}) \cap E_2 \cong A_k/(TA_k)$. Hence $\text{Ker} (\lambda_{x_1})$ and $\text{Ker} (\Phi_{T}) \cap E_2$ must be identical.

3. The value $x_2 := \lambda_{x_1}(h)$ belongs to $X_2$, i.e.,

$$Q_{x_1}(x_2) = x_1.$$ (20)


We will use the equality $\phi_{f(T)}^{x_2}(x_2) = f(x_2) = kx_2$ to prove this equation. Let us do some computation:

LHS of Equation (20) $= Q_{x_1} \left( \frac{1}{k} \phi_{f(T)}^{x_2}(x_2) \right)$

$= \frac{1}{k} Q_{x_1} \left( \phi_{f(T)}^{x_2} \lambda_{x_1}(h) \right)$

$= \frac{1}{k} Q_{x_1} \left( \lambda_{x_1} \phi_{f(T)}^{x_2}(h) \right)$ (by Theorem 1.9)

$= \frac{1}{k} \eta_{x_1} \phi_{f(T)}^{x_2} \phi_{x_1}(h)$ (by Lemma 1.8)

$= \frac{1}{k} \eta_{x_1}(\Phi_T^{x_2}(h)) = \frac{1}{k} \eta_{x_1}(x_1) = \text{RHS of Equation (20)}$.

The last step is due to the definition of $\eta_x$.

(4) By the above facts, it is eligible to build a map $\Theta_2 : E_2^*(\phi^{x_1}) \rightarrow X_2$ by setting

$$\Theta_2(E_2) := \lambda_{x_1}(h), \quad \text{where } h \in H_2.$$ 

We will show that $\Theta_2 : E_2^*(\phi^{x_1}) \rightarrow X_2$ is a bijection.

(5) Let us show that

$$E_2 = \text{Ker}(\lambda_{x_2} \lambda_{x_1}), \quad \text{if } x_2 = \Theta_2(E_2). \quad (21)$$

Indeed, if we take the afore mentioned $h \in E_2$, such that $x_2 = \Theta_2(E_2) = \lambda_{x_1}(h)$, then

$$E_{q^k}(x_1, h) \subseteq E_2.$$ 

Moreover, by the choice of $h$, one has $E_{q^k}(x_1, h) \subseteq \text{Ker}(\lambda_{x_2} \lambda_{x_1})$. Consequently, we have

$$E_{q^k}(x_1, h) \subseteq E_2 \cap \text{Ker}(\lambda_{x_2} \lambda_{x_1}).$$

As $F_{q^k}$-vector spaces, $E_{q^k}(x_1, h)$, $E_2$, and $\text{Ker}(\lambda_{x_2} \lambda_{x_1})$ are all 2-dimensional, and hence they must be one and the same. This proves Relation (21).

(6) Next, we proceed to show that $\Theta_2 : E_2^*(\phi^{x_1}) \rightarrow X_2$ is a bijection. The fact that $\Theta_2$ is injective is implied by Relation (21). So we only need to show that the two sets $E_2^*(\phi^{x_1})$ and $X_2$ have the same cardinality:

(a) The number $\#(X_2) = q^{m-1}$, because the twisted degree of $Q_{x_1}$ is $(m - 1)$.

(b) Elements in $E_2^*(\phi^{x_1})$ are solutions to the relation

$$\Phi_T(E_2) = F_{q^k} \cdot x_1, \quad E_2 \in E_2^*(\phi^{x_1}).$$

In other words, the triple $(\phi, E_2, x_1)$ is the preimage of $(\phi, F_{q^k} \cdot x_1, x_1)$ under $p'_1$, the right arrow in the following commutative diagram of algebraic curves:

$$\begin{array}{ccc}
\tilde{X}_{m,j}(T^2) & \xrightarrow{\cong} & \tilde{M}_{m,j}(T^2) \\
\downarrow p_1 & & \downarrow p'_1 \\
\tilde{X}_{m,j}(T) & \cong & \tilde{M}_{m,j}(T).
\end{array}$$

The two horizontal isomorphisms $\cong$ are due to Theorem 2.10. By Lemma 2.4, the degree of the left $p_1$ is $q^{m-1}$, and so is the right $p'_1$. This confirms the fact that the number of such subspaces $E_2$ is $q^{m-1}$.

This completes proof of the $n = 2$ case.

Suppose that the statement is true for the $(n-1)$ case — We have a bijective map $\Theta_{n-1} : E_{n-1}^*(\phi^{x_1}) \rightarrow X_{n-1}$ satisfying conditions (I) and (II) with all $n$ replaced by $(n - 1)$.
(7) Given $E_n \in \mathcal{E}_n^*(\phi^{\mathbf{x}_1})$, we set $E_{n-1} := \Phi_T(E_n) \in \mathcal{E}_{n-1}^*(\phi^{\mathbf{x}_1})$. By induction assumption, we get the image

$$\Theta_{n-1}(E_{n-1}) := (x_2, \ldots, x_{n-1}) \in X_{n-1},$$

such that $E_{n-1} = \text{Ker}(\lambda_{x_{n-1}} \cdots \lambda_{x_1})$ and $x_{n-1} = \lambda_{x_{n-2}} \cdots \lambda_{x_1}(h)$, for all $h \in H_{n-1} := E_{n-1} \cap (\Phi_{T_{n-2}})^{-1}(x_1)$.

(8) The fact that $H_n$ is non-empty is due to the definition of $E_n$. Take any $h \in H_n$. We define

$$x_n := \lambda_{x_{n-1}} \cdots \lambda_{x_1}(h)$$

and

$$\Theta_n(E_n) := (x_2, \ldots, x_{n-1}, x_n).$$

Of course, we need to verify that $x_n$ does not depend on the choice of $h$. This is completely analogous to what we did in the previous Step (2) and thus omitted.

We also need to show that the data $(x_2, \ldots, x_n)$ belongs to $X_n$, which amounts to the verification of $Q_{x_{n-1}}(x_n) = x_{n-1}$. Indeed, this is a routine job done as below.

$$Q_{x_{n-1}}(x_n) = Q_{x_{n-1}} \frac{1}{k} \phi_T^{x_n}(x_n)$$

$$= \frac{1}{k} Q_{x_{n-1}} \left( \phi_T^{x_n} \lambda_{x_{n-1}} \cdots \lambda_{x_1}(h) \right)$$

$$= \frac{1}{k} Q_{x_{n-1}} \lambda_{x_{n-1}} \cdots \lambda_{x_1} \phi_T(h) \quad \text{(by Corollary 1.10)}$$

$$= \frac{1}{k} q_{x_{n-1}}^n \phi_T \lambda_{x_{n-2}} \cdots \lambda_{x_1} \phi_T(h) \quad \text{(by Lemma 1.8)}$$

$$= \frac{1}{k} q_{x_{n-1}}^n \lambda_{x_{n-2}} \cdots \lambda_{x_1} \phi_T(h) \quad \text{(by Corollary 1.10)}$$

$$= \frac{1}{k} q_{x_{n-1}}^n \lambda_{x_{n-2}} \cdots \lambda_{x_1} \Phi_T(h)$$

$$= \frac{1}{k} q_{x_{n-1}}^n (x_{n-1}) \quad \text{(by $\Phi_T(h) \in E_{n-1}$ and $\Phi_{T_{n-2}}(\Phi_T(h)) = x_1$)}$$

$$= x_{n-1}.$$

(9) So far we have constructed the map $\Theta_n$. Moreover, Relation (19) can be similarly approached as that of Step (5), and thus we omit the details of verification. Note that (19) implies that $\Theta_n$ is injective.

(10) Finally, from Lemma 2.4 and Theorem 2.10, we are able to derive the fact that

$$\#(\mathcal{E}_n^*(\phi^{\mathbf{x}_1})) = \#(X_n) = q^{(m-1)(n-1)},$$

which forces $\Theta_n$ to be bijective.

The proof is thus completed. □

Now we are ready to prove part A of the Main Theorem. Based on Theorem 2.10, it suffices to determine the function field of $\tilde{M}_{m,j}(T^n)$.

Proof of part A. We consider $L$-points of curves $\tilde{M}_{m,j}(T^n)$.

1. Case $n = 1$.

Take a geometric point $(\phi, E_1, x_1)$ of $\tilde{M}_{m,j}(T)$. It follows from $\phi_T(x_1) = 0$ that $\phi = \phi^{x_1}$. Since $0 \neq x_1 \in E_1$ and $E_1$ is a 1-dimensional $\mathbb{F}_q^k$-vector space, we get

$$E_1 = \text{Ker}(\lambda_{x_1}).$$
It means that \((\phi, E_1, x_1)\) is determined by \(x_1\). Hence the function field \(\tilde{F}^{(1)}_{m,j}\) of \(\tilde{M}_{m,j}(T)\) is equal to \(F_{q^n}(x_1)\).

2. Case \(n \geq 2\).

Let \(0 \neq x_1 \in \bar{L}\) be fixed. By definition, \(L\)-points \((\phi, E_n, x_1)\) of \(\tilde{M}_{m,j}(T^n)\) are in one-to-one correspondence to \(E_n\) in \(E_n^\phi(\phi^{x_1})\). Then by the previous Theorem 3.1, the function field \(\tilde{F}^{(n)}_{m,j}/\tilde{F}^{(1)}_{m,j}\) is generated by variables \(x_2, \ldots, x_n\) satisfying a sequence of equations

\[Q_{x_i}(x_2) = x_1, \ldots, Q_{x_i}(x_{i+1}) = x_i, \ldots, Q_{x_{n-1}}(x_n) = x_{n-1}.\]

One obtains relations in Equation (6) immediately.

\[
\square
\]

3.2. Part B. For \(\mu \in F_q^*\), recall the automorphism \(\mu\) on \(\tilde{X}_{m,j}(T^n)\) over \(\tilde{X}_{m,j}(T^n)\) given by Equation (16):

\[\mu(\phi, G_n, x_1) := (\phi^\mu x_1, \mu G_n, \mu x_1) = (\phi, G_n, \mu x_1).\]

Let \(x_1, \ldots, x_n\) be the coordinates of \(\tilde{X}_{m,j}(T^n)\) that are subject to Equation (6). Then \(\mu\) sends \((x_1, \ldots, x_n)\) to \((\mu x_1, \mu x_2, \ldots, \mu x_n)\), because the point \((\mu x_1, \mu x_2, \ldots, \mu x_n)\) satisfies Equation (6). Relying on this fact, we now prove part B of the Main Theorem.

Proof of part B. We prove the claim by induction on \(n\).

1. Case \(n = 1\).

It follows from part (2) of Lemma 2.5 that the function field \(\tilde{F}^{(1)}_{m,j}\) of \(\tilde{X}_{m,j}(T)\) is generated by \(X_1 = x_1^{q-1}\).

2. Case \(n \geq 2\). Suppose that the function field \(\tilde{F}^{(n-1)}_{m,j}\) of \(\tilde{X}_{m,j}(T^{n-1})\) is generated by \(X_1, X_2, \ldots, X_{n-1}\) with \(X_i = x_i^{q-1}\) (for \(i = 1, \ldots, n - 1\)) satisfying Equation (7) (up to \((n - 1)\)). Set \(X_n = x_n^{q-1}\). Then \(X_n\) is fixed by the action of the multiplicative group \(F_q^*\), and therefore \(F^{(n-1)}_{m,j}(X_n) \subseteq F^{n}_{m,j}\). In the meantime, it follows from Equation (6) for \(i = n\) that

\[\left(\frac{1}{X_{n-1}} + \frac{X_n}{X_{n-1}^{N_{j-1}}} + \cdots + \frac{X_n^{N_{j-1}}}{X_{n-1}^{N_{j-1}}} + \frac{X_n^{N_{j+1}}}{X_{n-1}} + \cdots + \frac{X_n^{N_{m-1}}}{X_{n-1}^{N_{k-1}}} \right)^{q-1} = \frac{X_{n-1}}{X_n},\]

i.e.,

\[G(X_{n-1}, X_n) = 0.\]

Therefore, \(\tilde{F}^{(n-1)}_{m,j}(x_n)\) is a degree \(q^{n-1}\) extension over \(\tilde{F}^{(n-1)}_{m,j}\). Combining this fact with Corollary 2.6, the function field of \(\tilde{X}_{m,j}(T^n)\) over \(\tilde{X}_{m,j}(T^{n-1})\) is generated by \(X_1, \ldots, X_n\) satisfying Equation (7). This completes the proof of part B.

\[
\square
\]

3.3. Part C. We wish to find the function field of \(X_{m,j}(T^n)\) over \(F_{q^n}\). First we recall some results in [4] and [5]. Let \(x\) and \(y\) be in \(\bar{L}\), where \(x \neq 0\), and suppose that they are related by the equation

\[\mathcal{F}(x, y) = 0.\]  

(22)

We adopt the following bivariant fractional functions:

\[R(x, y) := \frac{y}{x^{q^n}},\]
and
\[
S(x, y) = \frac{y^q}{x^q},
\]
and
\[
u(x, y) = \sum_{r=0}^{a-1} R^{r,k} + \left( \sum_{s=0}^{k-1} S^{r,s} \right)^q.
\]
(Recall that \(a\) and \(b\) are two non-negative integers satisfying \(ak - bj = 1\).

**Lemma 3.2** ([4, Proposition 3] and [5, Proposition 2.2, 2.3]). Let \(R\), \(S\), and \(u\) be as above. We have the following facts.

1. The functions \(R\), \(S\), and \(u\) are related by
\[
R = \text{Tr}_k(u) - b, \quad S = -\text{Tr}_j(u) + a,
\]
and, therefore, the function field \(\mathbb{F}_q(R, S)\) is identically \(\mathbb{F}_q(u)\) and rational;
2. The function fields \(\mathbb{F}_q(x, y)\), \(\mathbb{F}_q(u, x)\), and \(\mathbb{F}_q(u, y)\) are one and the same;
3. The field extension \(\mathbb{F}_q(x, y)/\mathbb{F}_q(u)\) is Galois, and its Galois group is isomorphic to \(\mathbb{F}_q^*\).

Now we are ready to give the

**Proof of part C.** We observe a basic fact. Let \(x_1, \ldots, x_n\) be the coordinates of \(X_{m,j}(T^n)\) that are subject to Equation (6). Then for each \(\mu \in \mathbb{F}_q^*\), the automorphism \(\mu\) defined by (16) sends \((x_1, \ldots, x_n)\) to \((\mu x_1, \mu^q x_2, \ldots, \mu^{q(n-1)} x_n)\). Indeed, one can easily check that the data \((\mu x_1, \mu^q x_2, \ldots, \mu^{q(n-1)} x_n)\) is subject to Equation (6).

We proceed to accomplish the proof.

1. Case \(n = 1\).
   Since \(X_{m,j}(T)\) is a Galois covering over \(X_{m,j}(T)\) of degree \(q^n - 1\), the function field \(F_{m,j}^{(1)}\) of \(X_{m,j}(T)\) is generated by \(z = x_1^{q^n-1}\).
2. Case \(n = 2\), In this situation, \((x_1, x_2) = (x, y)\) is subject to Equation (22). We adopt the notations \(R_2 = R(x_1, x_2)\), \(S_2 = S(x_1, x_2)\), and \(u_2 = u(x_1, x_2)\). By direct calculations, we see that \(R_2\) and \(S_2\) are stable under the action by \(\mu\). It follows from Lemmas 2.5 (part (3)) and 3.2 (parts (1) and (3)) that the function field \(F_{m,j}^{(2)}\) of \(X_{m,j}(T^2)\) is equal to \(\mathbb{F}_q(R_2, S_2) = \mathbb{F}_q(u_2)\).
3. Case \(n \geq 3\), We induct on \(n\) from the preceding \(n = 2\) case. Assume that in the \((n-1)\)-step, the function field \(F_{m,j}^{(n-1)}\) of \(X_{m,j}(T^{n-1})\) over \(\mathbb{F}_q^*\), is generated by variables \(u_2, \ldots, u_{n-1}\), with \(u_i = u(x_{i-1}, x_i)\).

Consider variables
\[
R_n := R(x_{n-1}, x_n), \quad S_n := S(x_{n-1}, x_n), \quad \text{and} \quad u_n := u(x_{n-1}, x_n).
\]
By part (2) of Lemma 3.2, we have \(F_{m,j}^{(n)} = F_{m,j}^{(n-1)}(x_{n-1}, x_n)\). Then along the same lines as in the \(n = 2\) case, we use Lemmas 2.5 and 3.2, and see that the function field \(F_{m,j}^{(n)}\) of \(X_{m,j}(T^n)\) must be identical to \(F_{m,j}^{(n-1)}(u_n)\).

According to the definitions of \(S\) and \(R\), we have
\[
q^{m-1} s_{n-1} = \frac{S_{n-1} q^{m-1}}{R_{n-1}} = \frac{S_n q^m}{R_n q^m},
\]
Finally, we take advantage of part (1) of Lemma 3.2, and get a relation between \(u_{n-1}\) and \(u_n\):
\[
\frac{\text{Tr}_j(u_{n-1}) q^k - a}{\text{Tr}_k(u_{n-1}) - b} = \frac{\text{Tr}_j(u_{n}) - a}{\text{Tr}_k(u_{n}) q^k - b}.
\]
Remark 3.3. We conclude this paper by an easy application — to interpret the lower bound of $\mathbb{F}_q$-rational points appeared in the BBGS towers.

Let $x_1, \ldots, x_n$ be coordinates on $\tilde{X}_{m,j}(T^n)$ that are subject to Equation (6). From the standpoint of Drinfeld modular curves, the condition “$x_1$ belongs to $\mathbb{F}_q^*$” is equivalent to “$(\phi, G_n, x_1)$ is a supersingular point,” (i.e., $\phi$ is supersingular). The covering map $\tilde{X}_{m,j}(T^n) \to \tilde{X}_{m,j}(T)$ is exactly the projection $(x_1, \ldots, x_n)$ to $x_1$. Thus there are as many as $(q^m-1)q^{(m-1)(n-1)}$ points on $\tilde{X}_{m,j}(T^n)$ which are supersingular. This recovers the result obtained by Bassa et al. [5, Corollary 3.2].

For $n \geq 2$, let $u_2, \ldots, u_n$ be the coordinates of $X_{m,j}(T^n)$ that are subject to Equation (7). From our description of supersingular points on $\tilde{X}_{m,j}(T^n)$, we see that these $q^{(m-1)(n-1)}$ supersingular points on $X_{m,j}(T^n)$ comprise the set

$$\left\{(u_2, \ldots, u_n) \in (\mathbb{F}_q^*)^{n-1} \mid \text{Tr}_m(u_i) = a + b, \quad i = 2, \ldots, n\right\}.$$ 

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