EXPLICIT SOLVING OF THE SYSTEM OF NATURAL PDE’S OF MINIMAL SURFACES IN THE FOUR-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT. The fact that minimal surfaces in the four-dimensional Euclidean space admit natural parameters implies that any minimal surface is determined uniquely up to a motion by two curvature functions, satisfying a system of two PDE’s (the system of natural PDE’s). In fact this solves the problem of Lund-Regge for minimal surfaces. Using the corresponding result for minimal surfaces in the three-dimensional Euclidean space, we solve explicitly the system of natural PDE’s, expressing any solution by virtue of two holomorphic functions in the Gauss plane. We find the relation between two pairs of holomorphic functions (i.e. the class of pairs of holomorphic functions) generating one and the same solution of the system of natural PDE’s.

1. INTRODUCTION

Studying minimal surfaces in the four-dimensional Euclidean space $\mathbb{R}^4$, Itoh proved in [4] that any minimal non-superconformal surface $M^2$ admits locally special isothermal parameters. On the base of this result, de Azevero Tribuzy and Guadalupe proved in [1] the following theorem:

The Gauss curvature $K$ and the curvature of the normal connection $\varkappa$ (the normal curvature) of a minimal non-superconformal surface, parameterized by special isothermal parameters, satisfy the following system of partial differential equations:

$$
(K^2 - \varkappa^2)^{1/4} \Delta \ln |\varkappa - K| = 2(2K - \varkappa); \\
(K^2 - \varkappa^2)^{1/4} \Delta \ln |\varkappa + K| = 2(2K + \varkappa).
$$

Conversely, any solution $(K, \varkappa)$ to system (1) determines uniquely (up to a motion in $\mathbb{R}^4$) a minimal non-superconformal surface with Gauss curvature $K$ and normal curvature $\varkappa$.

Further we call system (1) the system of natural PDE’s of minimal surfaces in $\mathbb{R}^4$ and our aim is to solve explicitly this system.

All considerations in the paper are local.

Introducing natural parameters on any minimal surface in $\mathbb{R}^4$ reduces the number of the invariants determining the surface to two: $K$ and $\varkappa$. Further, these two

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invariants satisfy the system of two natural PDE’s and determine the minimal surface uniquely up to a motion. It is clear that the number of the invariants and the number of the PDE’s can not be reduced further. Therefore this solves the problem of Lund-Regge \cite{7} for minimal surfaces in $\mathbb{R}^4$.

The basic theorem in this paper is the following statement

**Theorem 1. (Explicit solving of the system of natural PDE’s of minimal surfaces)** Let $K$ and $\kappa$ be solutions of the system (6). Then we have locally

\[
K = \frac{-8|w'_1 w'_2|}{(|w_1|^2 + 1)(|w_2|^2 + 1)} \left( \frac{|w'_1|^2}{(|w_1|^2 + 1)^2} + \frac{|w'_2|^2}{(|w_2|^2 + 1)^2} \right),
\]

\[
\kappa = \frac{8|w'_1 w'_2|}{(|w_1|^2 + 1)(|w_2|^2 + 1)} \left( \frac{|w'_1|^2}{(|w_1|^2 + 1)^2} - \frac{|w'_2|^2}{(|w_2|^2 + 1)^2} \right),
\]

for some holomorphic functions $w_k$ ($k = 1, 2$) in $\mathbb{C}$.

Conversely, any two functions $K$ and $\kappa$, given by (2), satisfy the system (6).

We shall say that the pair $(w_1, w_2)$ generates the solution (2).

There arises the following natural question: **When two pairs of holomorphic functions generate one and the same solution of (6)?**

The answer is given by the following statement

**Theorem 2.** Let $(w_1, w_2)$ and $(\tilde{w}_1, \tilde{w}_2)$ be two pairs of holomorphic functions generating one and the same solution $(K, \kappa)$ of the system (6).

Then

\[
\tilde{w}_k = \frac{-\bar{b}_k + \bar{a}_k w_k}{\bar{a}_k + b_k w_k}, \quad a_k = \text{const}, \quad b_k = \text{const}, \quad |a_k|^2 + |b_k|^2 = 1; \quad (k = 1, 2).
\]

Conversely, any two pairs of holomorphic functions related by (3) generate one and the same solution of (6).

The basic idea we use in solving the system of natural PDE’s of minimal surfaces in $\mathbb{R}^4$ is to reduce it to the solving of the natural equation of minimal surfaces in $\mathbb{R}^3$.

2. **Explicit solving of the system of natural PDE’s**

In this section we find an explicit form of the solutions of the system (6).

In \cite{2} the first author proved the following result:

**Theorem A.** (Explicit solving of the natural PDE of minimal surfaces in $\mathbb{R}^3$) Any solution $\nu > 0$ of the natural partial differential equation of minimal surfaces

\[
\Delta \ln \nu + 2\nu = 0
\]

locally is given by the formula
where \( w \) is a holomorphic function in \( \mathbb{C} \).

Conversely, any function \( \nu(x, y) \) of the type (5) is a solution to (4).

**Proof of Theorem 1**

System (1) can be rewritten in the following form:

\[
\begin{align*}
(K^2 - \kappa^2)^{\frac{1}{4}} \Delta \ln (K^2 - \kappa^2)^{\frac{1}{4}} &= 4K, \\
(K^2 - \kappa^2)^{\frac{1}{4}} \Delta \frac{K - \kappa}{K + \kappa} &= -4\kappa;
\end{align*}
\]

Let \( K \) and \( \kappa \) be solutions of the system (6).

We introduce the functions \( \alpha \) and \( \beta \) by the formulas

\[
K = -2(\alpha^2 + \beta^2), \quad \kappa = 2(\alpha^2 - \beta^2);
\]

\( \alpha > 0, \beta > 0 \)

and system (6) takes the following form:

\[
\begin{align*}
2\sqrt{\alpha\beta} \Delta \ln \alpha \beta + 8(\alpha^2 + \beta^2) &= 0, \\
2\sqrt{\alpha\beta} \Delta \ln \frac{\alpha}{\beta} + 4(\alpha^2 - \beta^2) &= 0,
\end{align*}
\]

or

\[
\begin{align*}
\Delta \ln \frac{4\alpha^3}{\beta} + 4\sqrt{\frac{4\alpha^3}{\beta}} &= 0, \\
\Delta \ln \frac{4\beta^3}{\alpha} + 4\sqrt{\frac{4\beta^3}{\alpha}} &= 0.
\end{align*}
\]

Next we put

\[
p^2 = \frac{4\alpha^3}{\beta}, \quad q^2 = \frac{4\beta^3}{\alpha}; \quad (p > 0, q > 0).
\]

Then

\[
K = -\frac{1}{2}\sqrt{pq}(p + q), \quad \kappa = \frac{1}{2}\sqrt{pq}(p - q)
\]

and system (6) becomes

\[
\begin{align*}
\Delta \ln p + 2p &= 0, \\
\Delta \ln q + 2q &= 0.
\end{align*}
\]

Now we apply Theorem A to any of the equations (11) and obtain
\[ p = \frac{4|w_1'|^2}{(|w_1|^2 + 1)^2}, \quad w_1' \neq 0, \]
\[ q = \frac{4|w_2'|^2}{(|w_2|^2 + 1)^2}, \quad w_2' \neq 0, \]

where \( w_k (k = 1, 2) \) are holomorphic functions in \( \mathbb{C} \). Substituting (12) in (10), we obtain (2).

For the inverse, let \( p \) and \( q \) be two functions, given by (12). Then equations (11) are satisfied and the functions (2) are solutions to the system (6). \( \square \)

The basic question for the system of natural PDE’s (6) concerns the integrability of this system. Theorem 1 gives explicitly the solutions of this system and therefore (6) occurs to be the first example of an integrable system, describing a geometric class of surfaces in \( \mathbb{R}^4 \), determined by curvature conditions.

3. When two pairs of holomorphic functions generate one and the same solution to the system of natural PDE’s?

In this section we find the relation between two pairs \((w_1, w_2)\) and \((\hat{w}_1, \hat{w}_2)\) of holomorphic functions, which generate one and the same solution \((K, \kappa)\) of the system (6).

Proof of Theorem 2

First we shall clear up when two holomorphic functions generate one and the same solution to the equation (4).

Let \( \mathcal{M} : z = z(u, v), (u, v) \in \mathcal{D} \) be a minimal surface in \( \mathbb{R}^3 \) parameterized by canonical principal parameters \((u, v)\) [3].

In [2] the first author proved that \( \mathcal{M} \) has the following canonical principal representation:

\[ z'_1 = \frac{1}{2} \frac{w^2 - 1}{w'}, \]
\[ z'_2 = \frac{i}{2} \frac{w^2 + 1}{w'}, \]
\[ z'_3 = -\frac{w}{w'}, \]

where \( w \) is a holomorphic function in \( \mathbb{C} \), which generates the solution (5) to (4).

Putting
\[ F = \frac{1}{\sqrt{-2w'}}, \quad G = \frac{w}{\sqrt{-2w'}}, \quad \left(2FG = -\frac{w}{w'}\right)\]

we obtain the standard Weierstrass representation.
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\[ z_1' = F^2 - G^2, \]
\[ z_2' = i(F^2 + G^2), \]
\[ z_3' = 2FG. \]  

Further, let \( \hat{\mathcal{M}} \) be another minimal surface with Weierstrass representation

\[ \hat{z}_1' = \hat{F}^2 - \hat{G}^2, \]
\[ \hat{z}_2' = i(\hat{F}^2 + \hat{G}^2), \]
\[ \hat{z}_3' = 2\hat{F}\hat{G}. \]

If \( \mathcal{M} \) and \( \hat{\mathcal{M}} \) are congruent, i.e. \( \hat{\mathcal{M}} \) is obtained from \( \mathcal{M} \) by a motion, then \( (F, G) \) and \( (\hat{F}, \hat{G}) \) are related by the special unitary transformation (See e.g. Chapter II of \[5\])

\[ \hat{F} = aF + bG, \]
\[ a = \text{const}, \ b = \text{const}, \ |a|^2 + |b|^2 = 1 \]

and vice versa.

The correspondence between the rotation and the unitary transformation is the standard spin representation of \( SO(3) \).

Replacing the expressions (16) in the formula \( \hat{w} = \frac{\hat{G}}{\hat{F}} \) we obtain

\[ \hat{w} = -\frac{\bar{b} + \bar{a}w}{a + bw}. \]

Conversely, let \( \hat{w} \) and \( w \) be two analytic functions related by (17).

Then

\[ \hat{F} = \frac{1}{\sqrt{-2w'}} = a \frac{1}{\sqrt{-2w'}} + b \frac{w}{\sqrt{-2w'}} = aF + bG, \]
\[ \hat{G} = \frac{w}{\sqrt{-2w'}} = -\bar{b} \frac{1}{\sqrt{-2w'}} + \bar{a} \frac{w}{\sqrt{-2w'}} = -\bar{b}F + \bar{a}G. \]

Hence \( (\hat{F}, \hat{G}) \) and \( (F, G) \) are related by a special unitary transformation and therefore \( \mathcal{M} \) and \( \hat{\mathcal{M}} \) are related by a motion.

Another approach to the above question has been used in \[6\].

Thus we obtained the following

**Theorem 3.1.** Let \( w \) and \( \hat{w} \) be two holomorphic functions generating one and the same solution \( \nu \) of the natural equation \[4\].
Then

\[ \hat{w} = \frac{-\bar{b} + \bar{a} w}{a + b w}, \quad a = \text{const}, \ b = \text{const}, \ |a|^2 + |b|^2 = 1. \]

Conversely, any two holomorphic functions related by (17) generate one and the same solution of (4).

Applying two times formula (17), we obtain the proof of Theorem 2. □

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