A STRENGTHENING OF THE ENERGY INEQUALITY FOR THE LERAY-HOPF SOLUTIONS OF THE 3D PERIODIC NAVIER-STOKES EQUATIONS

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Abstract. In present note we establish the following inequality for the the Leray-Hopf solutions of the 3-D Ω-periodic Navier-Stokes Equations:

\[ \phi(|u(t)|^2) - \phi(|u(t_0)|^2) \leq 2 \int_{t_0}^{t} \phi'(|u(\tau)|^2) \left[ -\nu|A^{1/2}u(\tau)|^2 + (g(\tau), u(\tau)) \right] d\tau \]

for all \( t_0 \) Leray-Hopf points, \( t \geq t_0 \), and \( \phi : \mathbb{R}_+ \to \mathbb{R} \) is an absolutely continuous non-decreasing function with bounded derivative. Here \((\cdot, \cdot)\) and \(|\cdot|\) is correspondingly the \( L^2 \) inner product and the \( L^2 \) norm on \( \Omega \), and \( A \) is the Stokes operator.

1. PRELIMINARIES

We consider three dimensional incompressible Navier-Stokes equations (NSE)

\[
\frac{\partial}{\partial t} u(t, x) - \nu \Delta u(t, x) + (u(t, x) \cdot \nabla) u(t, x) + \nabla p(t, x) = f(t, x)
\]

\[
\nabla \cdot u(t, x) = 0,
\]

where \( x \in \mathbb{R}^3, t \in \mathbb{R} \) and \( u(t, x), f(t, x) \in \mathbb{R}^3 \) and \( p(t, x) \in \mathbb{R} \) for all \( t \) and \( x \). We supplement (1.1) with periodic boundary conditions

\[
\int_{\Omega} u(t, x) \, dx = 0 \text{ and } \int_{\Omega} f(t, x) \, dx = 0 \text{ for all } t,
\]

where

\[ \Omega = [0, L]^3, \]

and the initial condition

\[
u u(0, x) = u_0(x) \text{ for all } x,
\]

(1.3)

Taking the Leray projector \( P_L \) of (1.1) we obtain

\[
\frac{d}{dt} u + \nu A u + B(u, u) = g,
\]

where

\[
A u = -P_L \Delta u, \quad B(u, u) = P_L (u \cdot \nabla) u, \quad \text{and } g = P_L f.
\]

We introduce the following functional space

\[
H = \left\{ u : u \ - \ \Omega - \text{periodic, } u \in L^2(\Omega)^3, \ \nabla \cdot u = 0, \int_{\Omega} u = 0 \right\}
\]

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For any $u, v \in H$ denote
\[(u, v) = \int_{\Omega} u \cdot v\]
and
\[|u|^2 = \int_{\Omega} u \cdot u,
\]
the $L^2$ inner product and norm on $H$. Note that $A : D(A) \subset H \to H$ is a positive self adjoint operator with a compact inverse; its domain $D(A)$ is dense in $H$. Denote by $\lambda_k$, $k \in \mathbb{N}$, its eigenvalues arranged in the increasing order and counting the multiplicities. We also introduce the functional space
\[(1.6) \quad V = \{u \in H : u \in H^1(\Omega)^3\} = \{u \in H : u \in D(A^{1/2})\}.
\]
Define the inner product and the norm on $V$ by
\[(u, v) = (A^{1/2}u, A^{1/2}v)\]
and
\[||u||^2 = |A^{1/2}u|^2\]
for all $u, v \in V$. Note that
\[\lambda_1 |u|^2 \leq ||u||^2\]
for all $u \in V$.

The bilinear operator $B$ has the following orthogonality property:
\[(B(u, v), w) = -(B(u, w), v), \quad \text{for all } u, v, w \in V,
\]
and consequently
\[(B(u, v), v) = 0, \quad \text{for all } u, v \in V.
\]

In this paper we study weak solutions of the NSE, i.e. $H$-valued functions $u(t)$ that satisfy
\[
\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + (B(u(t), u(t)), v) = (g(t), v) \quad \text{for all } v \in V.
\]

For more background on the Leray-Hopf solutions of Navier-Stokes equations and on the functional setting used here consult [1, 3, 2, 7, 6, 5, 4].

The principal result about the existence of the weak (Leray-Hopf) solutions of the NSE can be stated as follows.

**Theorem 1.1.** Let $u_0 \in H$ and $g \in L^2([0, T], H)$, where $T > 0$ given. Then there exist a weak solution $u \in L^2([0, T], V)$, which is weakly continuous as a function from $[0, T]$ to $H$.

Moreover, the following inequality holds
\[(1.7) \quad |u(t)|^2 - |u(t_0)|^2 \leq 2 \int_{t_0}^{t} [-\nu |u(\tau)|^2 + (g(\tau), u(\tau))] \, d\tau\]
for all $t_0 \geq 0$ Lebesgue points of $|u(\tau)|^2$ and all $t \in [t_0, T]$.

In fact, the set of Lebesgue points of $|u(\tau)|^2$ is dense in $[0, T]$ and includes $t_0 = 0$ and any $t_0$ such that $u(t_0) \in V$.

Under the additional assumptions on the initial data, one gets local existence of the regular solutions of the NSE.
Theorem 2.1. Let \( \phi \in L^2([0, \infty), \nu, \Omega) \) and suppose \( \phi \) is defined on \( [0, T_0] \) for some \( T_0 > 0 \). Let \( T \) be large enough so that \( \phi \) is regular on \( [T_0, T] \). Moreover, the inequality \((1.7)\) becomes equality for all \( t_0, t \in [0, T] \), \( t \geq t_0 \).

It is important to mention that any weak solution of the NSE is regular on the set \( \mathcal{G} = \bigcup_n I_n \), where \( \{I_n \}_n \) is a countable family of disjoint intervals with \( \text{cl}(\mathcal{G}) = [0, T] \) and the fractal dimension of \( [0, T] \setminus \mathcal{G} \) less or equal to 1/2.

2. Proof of the Main Result

Theorem 1.2. Suppose \( u_0 \in V \) and \( g \in L^2([0, T], H) \). Then there exists \( T_\ast = T_\ast([u_0], \nu, \Omega) > 0 \) such that the weak solution \( u \) from Theorem 1.1 is unique on \( [0, T_\ast] \) and \( u \in L^2([0, T_\ast], D(A)) \cap C([0, T_\ast], V) \) (i.e., \( u \) is regular on \( [0, T_\ast] \)). Moreover, the inequality \((1.7)\) becomes equality for all \( t_0, t \in [0, T_\ast], t \geq t_0 \).

Proof. Let \( t_0, t \in [0, T], t \geq t_0 \) - Lebesgue point for \( |u(t)|^2 \) and all \( t \in [0, T], t \geq t_0 \).

Choose an \( \epsilon > 0 \). Then there exists \( \delta = \delta(\epsilon) > 0 \) such that, for any countable family of disjoint intervals \( \{(t_n, \tau_n)\} \subset [t_0, t] \) such that \( \sum_n (\tau_n - t_n) < \delta \),

\[
M \int_{\cup_n [t_n, \tau_n]} \frac{|g(\tau)|^2}{\nu \lambda_1} d\tau < \epsilon.
\]

and

\[
\left| \int_{\cup_n [t_n, \tau_n]} \phi'(|u(\tau)|^2) \left[-\nu|u(\tau)|^2 + (g(\tau), u(\tau)) \right] d\tau \right| < \epsilon
\]

Recall that \( u \) is regular on a set \( \mathcal{G} = \bigcup_n I_n \), where \( \text{int}(I_n) = (\alpha_n, \beta_n) \) and \( \sum_n (\beta_n - \alpha_n) = t - t_0 \). Clearly, inside each of the intervals \( (\alpha_n, \beta_n) \) we have

\[
\phi(|u(s_1)|^2) - \phi(|u(s_0)|^2) = 2 \int_{s_0}^{s_1} \phi'(|u(\tau)|^2) \left[-\nu|u(\tau)|^2 + (g(\tau), u(\tau)) \right] d\tau
\]

for all \( s_0, s_1 \in (\alpha_n, \beta_n), s_0 \leq s_1 \).

Note that there exists \( N = N(\delta) \) such that

\[
\sum_{n=1}^N (\beta_n - \alpha_n) \geq t - t_0 - \delta/2.
\]
Re-arrange these first \(N\) intervals such that \(\beta_n \leq \alpha_{n+1}\) for all \(n = 1..N - 1\). Choose \(\tau_{n-1}, t_n \in (\alpha_n, \beta_n), n = 1..N\) and \(\tau_N = t\), so that \(\tau_{n-1} < t_n < \tau_n\) for \(n = 1..N\) and
\[
\sum_{n=0}^{N} (\tau_n - t_n) < \delta.
\]

Then, by the (1.4),
\[
|u(\tau_n)|^2 - |u(t_n)|^2 \leq 2 \int_{t_n}^{\tau_n} [-\nu||u(\tau)||^2 + (g(\tau), u(\tau))] \, d\tau,
\]
for all \(n = 0..N\). Then,
\[
(2.4) \quad |u(\tau_n)|^2 - |u(t_n)|^2 \leq |u(\tau_n)|^2 - |u(t_n)|^2 + \nu \int_{t_n}^{\tau_n} ||u(\tau)||^2 \, d\tau \leq \int_{t_n}^{\tau_n} \frac{|g(\tau)|^2}{\nu \lambda_1} \, d\tau,
\]
for all \(n = 0..N\). On the other hand, since \(\phi\) is increasing,
\[
\phi(|u(\tau_n)|^2) - \phi(|u(t_n)|^2) \leq \begin{cases} 0, & \text{if } |u(\tau_n)| < |u(t_n)|; \\
M(|u(\tau_n)|^2 - |u(t_n)|^2), & \text{otherwise.}
\end{cases}
\]
Use (2.4) to obtain
\[
\phi(|u(\tau_n)|^2) - \phi(|u(t_n)|^2) \leq M \int_{t_n}^{\tau_n} \frac{|g(\tau)|^2}{\nu \lambda_1} \, d\tau.
\]
Thus, taking into the account (2.3) and (2.4), we can estimate
\[
(2.5) \quad \phi(|u(t)|^2) - \phi(|u(t_0)|^2)
= \sum_{n=0}^{N} \frac{\phi(|u(\tau_n)|^2) - \phi(|u(t_n)|^2) + \sum_{n=1}^{N} (\phi(|u(t_n)|^2) - \phi(|u(\tau_{n-1})|^2))}{d\tau} + \sum_{n=1}^{N} \int_{\tau_{n-1}}^{\tau_n} \Psi(\tau) \, d\tau
\leq \epsilon + \sum_{n=1}^{N} \int_{\tau_{n-1}}^{\tau_n} \Psi(\tau) \, d\tau \leq 2\epsilon + 2 \int_{t_0}^{t} \Psi(\tau) \, d\tau
\]
where
\[
\Psi(\tau) = \phi'(|u(\tau)|^2) [-\nu||u(\tau)||^2 + (g(\tau), u(\tau))].
\]
Returning to (2.5), make \(\epsilon \to 0\) to obtain (2.4).

\[ \square \]

**Corollary 2.1.** Under the assumptions of Theorem 2.1, let \(\psi \in C^1(\mathbb{R}_+; \mathbb{R})\) be such that \(\psi'(\xi) \leq 0\) for all \(\xi \geq 0\). Then the following inequality holds
\[
(2.6) \quad \psi(|u(t)|^2) - \psi(|u(t_0)|^2) \geq 2 \int_{t_0}^{t} \psi'(|u(\tau)|^2) [-\nu||u(\tau)||^2 + (g(\tau), u(\tau))] \, d\tau
\]
for all \(t_0 \in [0, T]\) - Lebesgue point for \(|u(\tau)|^2\) and all \(t \in [0, T], t \geq t_0\).

**Proof.** This result follows by applying Theorem 2.1 to \(\phi = -\psi\). \[ \square \]

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