Relation between fundamental estimation limit and stability in linear quantum systems with imperfect measurement

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From the noncommutative nature of quantum mechanics, estimation of canonical observables \(\hat{q}\) and \(\hat{p}\) is essentially restricted in its performance by the Heisenberg uncertainty relation, 
\[
\langle \Delta \hat{q}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \hbar^2/4.
\]
This fundamental lower-bound may become bigger when taking the structure and quality of a specific measurement apparatus into account. In this paper, we consider a particle subjected to a linear dynamics that is continuously monitored with efficiency \(\eta \in (0,1]\). It is then clarified that the above Heisenberg uncertainty relation is replaced by 
\[
\langle \Delta \hat{q}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \hbar^2/4\eta
\]
if the monitored system is unstable, while there exists a stable quantum system for which the Heisenberg limit is reached.

The quantum filter \([3, 4, 5, 6, 7]\) is particularly important, because of its potential application to quantum feedback control \([8, 9, 10, 11, 12, 13, 14, 15]\). More specifically, for a continuously monitored system, the quantum filter generates an optimal estimate of a system observable, which can be fed back to control the system. The estimator is recursively computed using the Belavkin filtering equation: this completely reflects the structure of the monitored system. Hence, within the framework of quantum filtering, the estimation limit is determined by dynamical properties of the system, e.g., the stability.

In this paper, we particularly focus on a single one-dimensional particle that has a quadratic potential and a linear interaction with a vacuum electromagnetic field, the latter of which is continuously measured by a homodyne detector \([16, 17, 18, 19, 20, 21, 22, 23]\). For this system, the filtering equation is reduced to the famous Kalman filter, and eventually the estimation error can be evaluated explicitly. The goal of this paper is to show that, irrespective of parameters of the system, there exists a fundamental estimation limit determined by the dynamical stability properties of the system. In particular, we show that a new estimation limit on \(\hat{q}\) and \(\hat{p}\) appears if the system is unstable, while there exists a stable quantum system for which the Heisenberg limit is reached.

We use the following notation: for a matrix \(A = (a_{ij})\), the symbols \(A^T, A^1,\) and \(A^*\) represent its transpose, conjugate transpose, and elementwise complex conjugate of \(A\), i.e., \(A^T = (a_{ji})\), \(A^1 = (a_{ji}^*)\), and \(A^* = (a_{ji}^*) = (A^1)^T\), respectively; these rules are applied to any rectangular matrix including column and row vectors. \(\text{Re}(A)\) and \(\text{Im}(A)\) denote the real and imaginary part of \(A\), respectively, i.e., \((\text{Re}(A))_{ij} = (a_{ij} + a_{ji}^*)/2\) and \((\text{Im}(A))_{ij} = (a_{ij} - a_{ji}^*)/2i\).

We first review the quantum filtering theory with the focus on a particle interacting with a field. The interaction is given by a unitary operator subjected to the following Hudson-Parthasarathy equation \([24]\):
\[
hd\hat{U}_t = \left[\left(-i\hat{H} - \frac{1}{2}\hat{c}^\dagger \hat{c} dt + \hat{c}^\dagger \hat{B}_t^1 - \hat{c} \hat{B}_t^1 \right)\hat{U}_t, \hat{U}_0 = \hat{I}, \right.
\]
where \(\hat{c} = \hat{c}_1 \hat{q} + \hat{c}_2 \hat{p}\). The constants \(c_1, c_2 \in \mathbb{C}\) are determined according to the system-field interaction. The quantum Wiener process \(\hat{B}_t\), which is a field operator, satisfies the following quantum Ito rule:
\[
d\hat{B}_t d\hat{B}_t = 0,\quad d\hat{B}_t^1 d\hat{B}_t = 0,\quad d\hat{B}_t d\hat{B}_t^1 = h dt,\quad d\hat{B}_t^1 d\hat{B}_t^1 = 0.
\]
In addition to the interaction, the particle is trapped in a quadratic harmonic potential of the form
\[
\hat{H} = \frac{1}{2}\hat{x}^\dagger \hat{G} \hat{x} = \frac{1}{2}(g_{11} \hat{q}^2 + g_{12} \hat{q} \hat{p} + g_{13} \hat{p}^2 + g_{22} \hat{p}^2),
\]
where \(\hat{x} = (\hat{q}, \hat{p})^\dagger\), and \(G = (g_{ij})\) is a \(2 \times 2\) real symmetric matrix. In the Heisenberg picture, the time-evolved position and momentum operators \(\hat{q}_t = \hat{U}_t^\dagger \hat{q} \hat{U}_t\)

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where $\dot{\hat{p}}_t = \hat{U}_t \hat{p} \hat{U}_t^\dagger$ satisfy the following quantum stochastic differential equation:
\[
\dot{x}_t = A \dot{x}_t dt + i\Sigma[Cd\hat{B}_t + C^*d\hat{B}_t^*],
\] (2)
where $\dot{x}_t = (\dot{\hat{q}}_t, \dot{\hat{p}}_t)^T$. Here, we have defined
\[
A := \Sigma[G + \text{Im}(C^*C)], \quad C := \begin{pmatrix} c_1 & c_2 \\ c_2 & -1 & 0 \end{pmatrix}.
\]
Next, we consider to measure a field observable after the interaction. In the homodyne detection scheme, the observable to be measured is given by
\[
Y_t' = \hat{U}_t^\dagger(e^{-i\phi} \hat{B}_t + e^{i\phi} \hat{B}_t^*) \hat{U}_t + \kappa(\hat{B}_t + \hat{B}_t^*),
\]
where $\hat{B}_t$ is a noise uncorrelated from $\hat{B}_t$, and $\kappa \geq 0$ represents the strength of $\hat{B}_t$. Also, $\phi \in [0, 2\pi)$ denotes a phase-shift parameter that should be optimized. Re-defining the normalized output $Y_t$ satisfying $dY_t^2 = dt$, we have
\[
dY_t = 2\sqrt{\eta} C_t^T \dot{x}_t dt + \sqrt{\eta}(e^{-i\phi}d\hat{B}_t + e^{i\phi}d\hat{B}_t^*) + \sqrt{1-\eta}(d\hat{B}_t + d\hat{B}_t^*),
\] (3)
where $C_t := \text{Re}(e^{-i\phi} C)$ and $\eta := (1 + \kappa^2)^{-1} \in (0, 1]$. Remarkably, $Y_t$ satisfies the self-ndemolition property $[Y_s, Y_t] = 0, \forall s, t$ for a fixed $\phi$, which indicates that the observation $Y_t = \nu \{Y_s \mid 0 \leq s \leq t\}$ constructs a classical stochastic process. Furthermore, $Y_t$ satisfies the nondemolition condition $[Y_s, \dot{\hat{q}}_t] = 0, [Y_s, \dot{\hat{p}}_t] = 0, \forall s \leq t$ for a fixed $\phi$. These two properties allow us to define the quantum conditional expectations $\pi_t(\hat{q}) = \mathbb{P}(\hat{q}_t \mid Y_t)$ and $\pi_t(\hat{p}) = \mathbb{P}(\hat{p}_t \mid Y_t)$, which are the best estimates of $\hat{q}_t$ and $\hat{p}_t$ in the sense of the least mean square error. Following the quantum filtering theory, we obtain a recursive equation to calculate $\pi_t(\hat{q})$ and $\pi_t(\hat{p})$:
\[
d\pi_t(\hat{x}) = A\pi_t(\hat{x}) dt + \sqrt{\eta} \frac{2}{\hbar} V_t C_t + \Sigma^T C_t \pi_t(\hat{x}) dt, \quad (4)
\]
where $C_t := \text{Im}(e^{-i\phi} C)$ and $\pi_t(\hat{x}) := (\pi_t(\hat{q}), \pi_t(\hat{p}))^T$. Here, $V_t$ is the symmetrized covariance matrix given by
\[
V_t := \mathbb{P}(\hat{P}_t \mid Y_t)
\]
\[
\hat{P}_t := \begin{pmatrix} \Delta\hat{q}_t^2 & \frac{1}{2}(\Delta\hat{q}_t \Delta\hat{p}_t + \Delta\hat{p}_t \Delta\hat{q}_t) \\ \frac{1}{2}(\Delta\hat{q}_t \Delta\hat{p}_t + \Delta\hat{p}_t \Delta\hat{q}_t) & \Delta\hat{p}_t^2 \end{pmatrix},
\] (5)
where $\Delta\hat{q}_t := \hat{q}_t - \pi_t(\hat{q})$ and $\Delta\hat{p}_t := \hat{p}_t - \pi_t(\hat{p})$ are the estimation errors. $V_t$ satisfies the following Riccati differential equation:
\[
\dot{V}_t = AV_t + V_tA^T + D - \frac{4\eta}{\hbar} V_t C_t C_t^T V_t,
\] (6)
where
\[
A' := \Sigma[G + C_t C_t^T + (2\eta - 1)C_t C_t^*],
\]
\[
D := \hbar \Sigma^T [C_t C_t^T + (1 - \eta)C_t C_t^*] \Sigma.
\]
As Eq. (6) is deterministic, the quantum conditional expectation $V_t = \mathbb{P}((\hat{P}_t \mid Y_t)$ is replaced by the simple expectation $V_t = \langle \hat{P}_t \rangle := \text{Tr}[(\rho \otimes \Phi)\hat{P}_t]$, where $\rho$ is a system state and $\Phi$ is the field vacuum state. The set of equations (4) and (6) called the quantum Kalman filter computes the best estimate of $\dot{\hat{q}}_t$ and $\dot{\hat{p}}_t$ recursively.

We here provide an important fact: Unlike the classical case where the error covariance matrix is simply a nonnegative matrix, the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$ imposes $V_t$ to satisfy the condition
\[
V_t + \frac{i\hbar}{2} \Sigma \geq 0,
\]
that yields the Heisenberg uncertainty relation
\[
\det(V_t) \geq \frac{\hbar^2}{4} \Rightarrow \langle \Delta\hat{q}_t^2 \rangle \langle \Delta\hat{p}_t^2 \rangle \geq \frac{\hbar^2}{4}. \quad (7)
\]
This inequality does hold regardless of a measurement setup. Hence the following natural question arises. Can the Heisenberg limit $\hbar^2/4$ be reached in the linear filtering scheme discussed above? To answer this important question needs a detailed investigation of $V_\infty$, a unique steady solution of the algebraic Riccati equation $V_\infty = 0$ in Eq. (5). (If the Riccati equation does not have such a solution, it implies that the estimation fails; we do not take this bad scenario into account.) In particular, we aim to get a fundamental lower bound of $\det(V_\infty)$ that does not include $C, G$, and $\phi$, because these terms completely depend on a system under consideration. We then obtain the following result.

**Theorem.** Suppose Eq. (6) has a unique steady solution $V_\infty$. Then, the estimation error $\det(V_\infty)$ has the following achievable bounds for any $C, G$, and $\phi$:
\[
\det(V_\infty) \geq \frac{\hbar^2}{4\eta} \text{ (if } C_t^T \Sigma C_t \leq 0),
\]
\[
\det(V_\infty) \geq \frac{\hbar^2}{4} \text{ (if } C_t^T \Sigma C_t > 0).
\]

**Proof.** The proof is done by a straightforward calculation. Without loss of generality, we can assume that $C_t$ is normalized: $C_t^T C_t = 1$. Let $C_t$ be a unit real vector orthogonal to $C_t$, i.e., $C_t^T C_t = 1$ and $C_t^T C_t = 0$, and define
\[
v_1 := C_t^T V_\infty C_t, \quad v_2 := C_t^T V_\infty C_t, \quad v_3 := C_t^T V_\infty C_t.
\]
Then, as $(C_t, C_t)$ is a $2 \times 2$ orthogonal matrix, we have
\[
\det(V_\infty) = \det\left[ \begin{pmatrix} C_t^T \\ C_t^T \end{pmatrix} V_\infty C_t, C_t \right] = v_1 v_3 - v_2^2.
\]
Furthermore, let us define
\[
\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} := \begin{pmatrix} C_t^T A' C_t \\ C_t^T A' C_t \end{pmatrix},
\]
\[
\begin{pmatrix} d_1 & d_2 \\ d_2 & d_3 \end{pmatrix} := \begin{pmatrix} C_t^T D C_t \\ C_t^T D C_t \end{pmatrix}.
\]
Note that $D^T = D$. With the above notations, the algebraic Riccati equation $V_\infty = 0$ is reduced to

\begin{align}
2a_1 v_1 + 2a_2 v_2 + d_1 - \frac{4\eta}{\hbar} v_1^2 &= 0, \quad (8) \\
a_3 v_1 + (a_1 + a_4) v_2 + a_2 v_3 + d_3 - \frac{4\eta}{\hbar} v_1 v_2 &= 0, \quad (9) \\
2a_3 v_2 + 2a_4 v_3 + d_3 - \frac{4\eta}{\hbar} v_3^2 &= 0. \quad (10)
\end{align}

Then, adding $v_2^2 \times (8), -2v_1 v_2 \times (9)$, and $v_1^2 \times (10)$, we readily obtain

$$2(v_1 v_3 - v_2^2)(a_2 v_2 - a_4 v_1) = d_3 v_1^2 - 2d_2 v_1 v_2 + d_1 v_2^2.$$ 

This together with Eq. (8) leads to

$$\det(V_\infty) = \frac{\hbar}{4\eta} \cdot \frac{d_3 v_1^2 - 2d_2 v_1 v_2 + d_1 v_2^2}{v_1^2 - h(a_1 + a_4) v_1/2\eta - h a_3/4\eta}.$$ 

Note that the denominator is strictly positive from the assumption that the Riccati equation has a unique steady solution. Now, calculating $d_1$, e.g., $d_1 = h(1 - \eta)(C_i^T \Sigma C_i)^2$, the numerator of $\det(V_\infty)$ is evaluated as

$$d_3 v_1^2 - 2d_2 v_1 v_2 + d_1 v_2^2 = \hbar v_1^2 + h(1 - \eta)(|C_i^T \Sigma C_i| v_1 - (C_i^T \Sigma C_i) v_2)^2 \geq \hbar v_1^2,$$

from which we have

$$\det(V_\infty) \geq \frac{\hbar^2}{4\eta} \cdot \frac{v_1^2}{v_1^2 - h(a_1 + a_4) v_1/2\eta - h a_3/4\eta}.$$ 

The right-hand side of the above inequality is further evaluated as follows. First, if $a_1 + a_4 = 2(\eta - 1)C_i^T \Sigma C_i = 0$, which implies $d_1 = 0$, we immediately obtain $\det(V_\infty) \geq \hbar^2/4\eta$. Second, if $C_i^T \Sigma C_i < 0$, which implies $a_1 + a_4 > 0$ and $d_1 > 0$, we have $\det(V_\infty) > \hbar^2/4\eta$. Finally, let us consider the case of $C_i^T \Sigma C_i > 0$ which leads to $a_1 + a_4 < 0$ and $d_1 > 0$; a simple calculation clarifies that the function $f(v) = v_1^2/ (v_1^2 + a_1 v_1 - b)$, $(a > 0, b > 0)$ satisfies $f(v) \geq 4b/(4b+a^2)$ when $v > 0$ and $v_1^2 + a_1 v_1 - b > 0$. This lower bound becomes $\eta$ in our problem where $a = -h(a_1 + a_4)/2\eta$ and $b = h a_3/4\eta$. As a result, we obtain $\det(V_\infty) \geq \hbar^2/4 \eta$ in this case. The achievability of the above lower bounds is discussed in the example part.

We now give a physical interpretation to the sign of $C_i^T \Sigma C_i$. To do this, let us focus on the matrix $A$, which corresponds to the drift term of the quantum dynamics \( \hat{q} \) and the filter \( \hat{p} \). The characteristic polynomial of $A$ is $\lambda^2 + 2(C_i^T \Sigma C_i) \lambda + (C_i^T \Sigma C_i)^2 + \det(G) = 0$. Hence, $A$ has two stable eigenvalues if and only if the conditions

$$C_i^T \Sigma C_i > 0, \quad (C_i^T \Sigma C_i)^2 + \det(G) > 0$$

are satisfied. The latter condition is easily attained by making the coefficient of $C$ (i.e., the interaction strength) sufficiently large, if the former condition is already satisfied. Therefore, under the condition $C_i^T \Sigma C_i > 0$, both the quantum dynamics and the filter are (asymptotically) stable in the sense that, roughly speaking, those trajectories are constrained around $\hat{x} = 0$ and $\pi_t(\hat{x}) = 0$. This implies that the fundamental estimation limit $\hbar^2/4\eta$ can be violated if the dynamics we aim to track is stable. Combining the theorem with the above discussion, we deduce the following fact:

$$\langle \Delta q^2 \rangle / \langle \Delta p^2 \rangle \geq \hbar^2 / 4\eta \quad \text{(if the system is unstable)},$$

$$\langle \Delta q^2 \rangle / \langle \Delta p^2 \rangle \geq \hbar^2 / 4 \quad \text{(if the system is stable)}.$$ 

**Remark 1.** In practice we cannot construct a perfect measurement apparatus with $\eta = 1$. Thus, the condition $C_i^T \Sigma C_i > 0$ is clearly preferable from the estimation performance viewpoint. Actually, for example when $C_i^T \Sigma C_i \leq 0$ and $\eta = 1/4$, the estimation error is lower bounded by $\hbar^2$, i.e., $\langle \Delta q^2 \rangle / \langle \Delta p^2 \rangle \geq \hbar^2$, which is much bigger than the Heisenberg limit $\hbar^2/4$. However, the sign of $C_i^T \Sigma C_i$ cannot be changed by tuning the Hamiltonian matrix $G$ and the phase-shift $\phi$. (Note that $C_i^T \Sigma C_i = \text{Re}(C)^T \Sigma \text{Im}(C)$.) In other words, only the interaction term $C$ is the crucial factor that determines the estimation limit.

**Remark 2.** The Hamiltonian of the form $\hat{H} = \hat{\xi}^\dagger G \hat{\xi}/2 - \hat{\xi}^\dagger \Sigma B u_t$, where $B \in \mathbb{R}^2$, allows that the system dynamics

$$d\hat{x}_t = A\hat{x}_t dt + Bu_t + i \Sigma [C\hat{B}_t - C^* d\hat{B}_t]$$

can be controlled using a feedback input $u_t \in \mathcal{Y}$. For example, the quantum linear quadratic gaussian (LQG) controller effectively stabilizes the system. However, any control input cannot reduce the estimation limit, because the error covariance matrix $V_t$ obeys the same Riccati equation \( [9] \) without respect to $B$ and $u_t$.

We will show that the two bounds in the theorem are tight in a sense that there exists at least one example where the equality holds in each case.

**Example 1.** Doherty et al. considered in \[18\] a single particle system with the following harmonic oscillator potential and the interaction with strength $\alpha > 0$:

$$\hat{H} = \frac{m \omega^2}{2} \hat{q}^2 + \frac{1}{2m} \hat{p}^2, \quad \hat{c} = \sqrt{2\alpha} \hat{q}.$$ 

This corresponds to

$$G = \begin{pmatrix} m \omega^2 & 0 \\ 0 & 1/m \end{pmatrix}, \quad C = \sqrt{2\alpha} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$C_t = \begin{pmatrix} \sqrt{2\alpha} \cos \phi \\ 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} -\sqrt{2\alpha} \sin \phi \\ 0 \end{pmatrix}.$$ 

First, we remark that the Riccati equation \( [9] \) has a unique steady solution $V_\infty$ for all the parameters. Then,
due to $C_1^T \Sigma C_1 = 0$, the estimation error is bounded by
\[
\det(V_\infty) \geq \frac{\hbar^2}{4\eta} \Rightarrow \langle \Delta \hat{q}_\infty^2 \rangle \langle \Delta \hat{p}_\infty^2 \rangle \geq \frac{\hbar^2}{4\eta}.
\]
Actually, the drift matrix $A$ has eigenvalues $\pm i\omega$, implying that the particle is oscillating with frequency $\omega$, and thus that the system is not stable. Furthermore, in this case, we can obtain a simple explicit form of $\det(V_\infty)$:
\[
\det(V_\infty) = \frac{\hbar^2}{4\eta} \left(1 - \frac{\eta}{\cos^2 \phi} + \eta\right),
\]
which attains $\hbar^2/4\eta$ when $\phi = 0$. Therefore, the lower bound $\hbar^2/4\eta$ is indeed achievable. In particular, when $\phi = 0$ we have
\[
\langle \Delta \hat{q}_\infty^2 \rangle \langle \Delta \hat{p}_\infty^2 \rangle = \frac{\hbar^2}{4\eta} + \frac{\hbar^2/4\eta}{\sqrt{r_1^2 + r_1 + \hbar^2/4\eta}},
\]
where $r_1 = \hbar m \omega^2/8\eta \alpha$. Thus, in the limit of $r_1 \to \infty$ the estimation error satisfies the minimum uncertainty relation $\langle \Delta \hat{q}_\infty^2 \rangle \langle \Delta \hat{p}_\infty^2 \rangle = \hbar^2/4\eta$, which further attains the Heisenberg limit $\langle \Delta \hat{q}_\infty^2 \rangle \langle \Delta \hat{p}_\infty^2 \rangle = \hbar^2/4$ only when $\eta = 1$.

Example 2. Wiseman and Doherty considered in [19].

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