Lamb Shift in Muonic Hydrogen. —II.

Analysis of the Discrepancy of Theory and Experiment

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Abstract Currently, both the $g$ factor measurement of the muon as well as the Lamb shift $2S$–$2P$ measurement in muonic hydrogen are in disagreement with theory. Here, we investigate possible theoretical explanations, including proton structure effects and small modifications of the vacuum polarization potential. In particular, we investigate a conceivable small modification of the spectral function of vacuum polarization in between the electron and muon energy scales due to a virtual millicharged particle and due to an unstable vector boson originating from a hidden sector of an extended standard model. We find that a virtual millicharged particle which could explain the muonic Lamb shift discrepancy alters theoretical predictions for the muon anomalous magnetic moment by many standard deviations and therefore is in conflict with experiment. Also, we find no parameterizations of an unstable virtual vector boson which could simultaneously explain both “muonic” discrepancies without significantly altering theoretical predictions for electronic hydrogen, where theory and experiment currently are in excellent agreement. A process-dependent correction involving electron screening is evaluated to have the right sign and order-of-magnitude to explain the observed effect in muonic hydrogen. Additional experimental evidence from light muonic atoms and ions is needed in order to reach further clarification.

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1 Introduction

Recently, two experiments involving quantum electrodynamics (QED) effects have been in disagreement with theory. The muon anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ has been measured \[1\]-\[3\] as

$$a_\mu \text{exp} = 11659208.0(6.3) \times 10^{-10}$$

in 3.4\,$\sigma$ disagreement with some of the latest theoretical analyses \[3\]

$$a_\mu \text{th} = 11659180.4(5.1) \times 10^{-10}.$$  \hspace{1em} (1.2)

The original aim of the recent muonic hydrogen Lamb shift experiment \[5\] was the determination of the proton radius. When QED theory is assumed to be correct, then the value

$$r_p = \sqrt{\langle r^2 \rangle_p} = 0.84184(67) \text{ fm}$$

is inferred for the root-mean-square proton charge radius from a comparison of theory and experiment for the transition $2S_{1/2}(F = 1) \leftrightarrow 2P_{3/2}(F = 2)$ in muonic hydrogen ($\mu$H). This value of the proton radius is in disagreement with the value obtained in the same way mainly from hydrogen and deuterium spectroscopy \[6\], which is the basis of the CODATA value \[7\],

$$r_p = 0.8768(69) \text{ fm}.$$  \hspace{1em} (1.4)

The most recent and accurate measurement of the proton radius from electron scattering \[8\], yields a value of

$$r_p = 0.879(8) \text{ fm},$$  \hspace{1em} (1.5)

when the statistical and systematic uncertainties given in Ref. \[8\] are added quadratically. The two values \[1.4\] and \[1.5\] are in excellent mutual agreement but differ from the muonic hydrogen value \[1.3\] by 5.0 standard deviations. Consequently, it may be permissible to invert the argument, and to evaluate current QED theory for the muonic hydrogen transition (as summarized in the supplementary material published with Ref. \[5\]) with the CODATA value \[1.4\]. Using the theoretical expression given in Ref. \[5\],

$$E_{\text{th}} = \left( 209.9779(49) - 5.2262 \frac{r_p^2}{\text{fm}^2} + 0.0347 \frac{r_p^3}{\text{fm}^3} \right) \text{ meV},$$  \hspace{1em} (1.6)

one obtains, using the CODATA proton radius given in Eq. \[1.4\], a theoretical prediction of

$$E_{\text{th}} = 205.984(63) \text{ fm},$$  \hspace{1em} (1.7)

which is 5.0\,$\sigma$ away from the experimental value of

$$E_{\text{exp}} = 206.2949(32) \text{ fm},$$  \hspace{1em} (1.8)

reported in Ref. \[8\]. The recent theory update \[9\] shifts theoretical predictions only minimally, to

$$E_{\text{th}} = \left( 209.9974(48) - 5.2262 \frac{r_p^2}{\text{fm}^2} \right) \text{ meV}. $$  \hspace{1em} (1.9)

Using the CODATA proton radius given in Eq. \[1.4\], one then obtains the theoretical prediction of

$$E_{\text{th}} = 205.980(63) \text{ fm},$$  \hspace{1em} (1.10)

in excellent agreement with \[1.7\], but in significant disagreement with the experimental result given in Eq. \[1.8\].
Section 2: Historical Perspective on Discrepancies in Muonic Systems

Both of the most recent QED experiments involving muons \[3,5\] are in disagreement with theory. The discrepancies have the “same sign” and read

\[
\delta a = a_{\text{exp}} - a_{\text{th}} = 2.76(81) \times 10^{-10}, \tag{1.11a}
\]

\[
\delta E = E_{\text{exp}} - E_{\text{th}} = 0.316(63) \text{ meV}. \tag{1.11b}
\]

We here proceed as follows. First, in Sec. 2, we present a historical perspective on discrepancies in muonic bound systems observed in the past, and on their eventual resolution. In view of the current discrepancy, such a historical perspective may be useful.

Possible theoretical explanations for the current discrepancy can mainly be divided into two categories: proton structure effects and modifications of the vacuum polarization charge density. Hypothetical proton structure effects are discussed in Sec. 3, and modifications of the vacuum polarization charge density in Sec. 4. Section 3 is divided into two subsections, the first of which deals with a conceivable “dip” in the proton form factor slope in the momentum transfer range studied in muonic hydrogen spectroscopy, and the second deals with a conceivable, anomalously large contribution from the inelastic part of the two-photon exchange diagram. Section 4 is divided into three parts. These deal with a conceivable non-perturbative correction to the vacuum polarization potential, with the contribution of a millicharged particle that modifies the vacuum polarization loop, and with the contribution of a conceivable virtual, unstable vector boson that modifies the vacuum polarization potential. We anticipate here that none of these considerations will lead to a definitive candidate for an explanation of the discrepancy. Still, a compilation of a number of possible explanations appears to be useful in the current situation. Some of the discussed explanations are relevant only to muonic hydrogen, which is the main subject of the current article, others may be relevant for both observed discrepancies. Possible electron screening corrections are discussed in Sec. 5. Conclusions are reserved for Sec. 6. Natural units with \(\hbar = c = \epsilon_0 = 1\) are used throughout the paper.

2 Historical Perspective on Discrepancies in Muonic Systems

Let us briefly comment on the size of the disagreement in muonic hydrogen [see Eqs. (1.11) and (1.8)]. Current predictions are based on the calculations reported in Refs. [10–17] and represent the result of independent groups. Important contributions originally calculated in Refs. [10,11] have been verified in Refs. [13,14]. Higher-order vacuum polarization effects have been given special attention in Refs. [13,14]. The theory used in the evaluation of the experiment [5] has been compiled at Laboratoire Kastler–Brossel in Paris. The disagreement of theory and experiment is on the level of \(1.508 \text{ meV}\) and which was calculated first by Kallen and Sabry [18], then recalculated in Ref. [19]. A clear exposition is given in volume III of Ref. [20] (the result has later been generalized to non-Abelian gauge theories, see Refs. [21,22]).

One may point out that the discrepancy (1.11b) amounts to (roughly) 1.5 parts per thousand of the total vacuum polarization effect in muonic hydrogen. By contrast, in a previous experiment [21] involving muonic \(3d \leftrightarrow 2p\) transitions in \(^{24}\text{Mg}\) and \(^{28}\text{Si}\), the vacuum polarization effect has already been verified to 1.0 parts per thousand (a relative accuracy of \(950 \times 10^{-6}\) is quoted in Ref. [23]) and thus, to better precision than the current disagreement. If there were any fundamental reason for a deviation of theory and experiment on this level, then one might wonder if the effect (whichever it is) might have been visible in the experiment reported in Ref. [24]. However, the muonic transition reported in Ref. [24] suffers from an uncertainty due to electron screening, and also, as it involves non-\(S\) states, the overlap of the muonic wave functions with the nucleus are not as pronounced as for \(S\) states. So, the quoted experiment [24] is not sensitive to higher-order nuclear structure effects, and it also probes the vacuum polarization at a different energy scale as compared to \(S\) states which are much closer to the nucleus.

In the 1970s, experiments involving muonic transitions were found to be in disagreement with theory [25]. Part of the discrepancies were addressed after a sign error in the calculation of the two-loop vacuum polarization correction [20] was eliminated [27]. An elucidating discussion of the status reached in 1978...
Section 3: Proton Structure Effects

3 Proton Structure Effects

3.1 Form Factor

The proton mean-square charge radius is defined in terms of the slope of the Sachs $G_E$ form factor of the proton,

$$\langle r^2 \rangle = 6 \frac{\partial G_E(q^2)}{\partial q^2} \bigg|_{q^2=0} = -6 \frac{\partial G_E(Q^2)}{\partial Q^2} \bigg|_{Q^2=0},$$

(3.1)

where $Q^2 = -q^2$ is the space-like momentum transfer. Different ranges of the momentum transfer are relevant for the calculation of the slope in different experiments. The proton radius from electronic hydrogen is determined from exchanged Coulomb photons with momentum transfers in the region

$$Q^2 \sim (\alpha m_e c)^2 = \left(3.7 \times 10^{-6} \frac{\text{GeV}}{c} \right)^2.$$

(3.2)

For muonic hydrogen, the atomic momentum is in the range of

$$Q^2 \sim (\alpha m_\mu c)^2 = \left(7.7 \times 10^{-4} \frac{\text{GeV}}{c} \right)^2.$$

(3.3)

One may point out that this is just below the electron-positron pair production threshold,

$$(2m_e c)^2 = \left(1.0 \times 10^{-3} \frac{\text{GeV}}{c} \right)^2.$$

(3.4)

The momentum transfer range probed in the recent electron scattering experiment is larger, but not excessively larger,

$$\left(6.3 \times 10^{-2} \frac{\text{GeV}}{c} \right)^2 < Q^2 < \left(1.0 \text{ GeV}/c \right)^2.$$

(3.5)

The slopes of the proton form factor determined from the electron scattering data and from electronic hydrogen spectroscopy are in excellent mutual agreement. The momentum transfer range for muonic hydrogen spectroscopy lies in between these two ranges. Consequently, it would be somewhat surprising if the proton form factor slope had a “dip” in this range that would explain the discrepancy for muonic hydrogen. Still, without a direct scattering measurement in this momentum transfer range, this possibility cannot be fully excluded at present.
3.2 Proton Polarizability

Generically, the proton polarizability can be related to the resonances of the proton (its excitation spectrum) via dispersion relations [see Eqs. (29) and (30) of Ref. [11]], and related to the inclusive reaction $e + p \rightarrow e' + X$. This is an accepted procedure for all nuclei, also for heavier nuclei [31]. For the proton, one would intuitively assume that the bulk of the contribution is from the $\Delta (1232)$ resonance, which has been measured well. Yet, the data obtained in the literature for the proton polarizability contribution scatter, and in Ref. [5], the contribution is currently estimated as $+0.015(4)\text{ meV}$ based on the scatter of values obtained from different theorists [11,36,37]. The proton polarizability correction to the $2P-2S$ Lamb shift has been calculated as $0.0012\text{ meV}$ in Ref. [11], which is an order to magnitude smaller than the discrepancy $\delta E$. Other authors [30,37] confirm the magnitude of the result and give values of $0.0015\text{ meV}$ for the proton polarizability correction to the Lamb shift in muonic hydrogen. It would thus be helpful to reevaluate the effect, and to obtain more accurate estimates, even if the current uncertainty estimate of $\pm 0.004\text{ meV}$ is numerically tiny as compared to the discrepancy of $\delta E \approx 0.31\text{ meV}$.

In addition, one may point out that in the past, nuclear radii inferred from muonic transitions and from electron scattering have agreed to better than 5% (see the discussion in Sec. 2). So, if an anomalously large proton polarizability contribution were found by a reanalysis, then one might have to revisit this effect also for other bound systems and in the more general context of the validity of the nuclear charge radius determination from muonic transitions [33,38].

4 Vacuum Polarization

4.1 Nonperturbative Vacuum Polarization

Let us briefly review why a nonperturbative vacuum polarization effect might have been considered as an explanation of the discrepancy in muonic hydrogen. The spectrum of muonic hydrogen is influenced by electronic vacuum polarization effects. The muon is heavier than the electron by a factor of $m_\mu/m_e \approx 207$, and the reduced mass of muonic hydrogen is roughly equal to the muon mass. The effective Bohr radius in muonic hydrogen is $1/(\alpha m_R) = 284.748\text{ fm}$, which is smaller than the reduced Compton wavelength of the electron, $1/m_e = 386.159\text{ fm}$. The bound muon thus enters the electronic vacuum polarization charge cloud of the proton. The electronic vacuum polarization shift in muonic hydrogen is of order $\alpha^3 m_R$, where $m_R$ is the reduced mass, and thus more pronounced than the electronic vacuum polarization shift in electronic hydrogen, where the vacuum polarization contribution to the Lamb shift is of order $\alpha^5 m_R$.

The large vacuum polarization shift is also responsible for the fact that the $2S$ level in muonic hydrogen is energetically lower than the $2P_{1/2}$ state (in contrast to electronic hydrogen, where the situation is opposite).

Superficially, the vacuum polarization effects converge very well in terms of the QED loop expansion. The one-loop effect gives a contribution of $205.0074\text{ meV}$ to the $2P-2S$ Lamb shift, in first-order perturbation theory, while the second-order effect adds $0.1509\text{ meV}$. The two-loop (Kallen–Sabry) shift is $1.5081\text{ meV}$, followed by the higher-order Wichmann-Kroll term of $-0.00103\text{ meV}$. Eventually, of course, the expansion will diverge according to a famous argument put forward by Dyson [39], but the expected nonperturbative effect would be of order $\exp(-1/\alpha)$ and thus completely negligible for the muonic hydrogen experiment.

However, the superficial convergence still does not exclude the presence of a much larger nonperturbative correction to the vacuum polarization if the local convergence of the loop expansion of the higher-order vacuum polarization potentials breaks down in close vicinity of the proton, i.e., if the higher-order (Kallen–Sabry and Wichmann–Kroll) terms are more singular than the one-loop Uehling term for $r \to 0$. In that case, a nonperturbative correction to the vacuum polarization potential might have led to a highly nonlinear, nonperturbative correction and one might have had to solve the Schrödinger equation using the full nonperturbative potential near the origin. In that case, lattice methods would probably have had to be invoked in order to calculate the full vacuum polarization potential for $r \to 0$.

The question whether this more elaborate calculation is necessary, can only be answered by a concrete calculation of the leading asymptotics of the Uehling, Kallen–Sabry, and Wichmann–Kroll potentials
for \( r \to 0 \). One finds, in agreement with Ref. \[27\], for the leading asymptotics of the one-loop Uehling potential \( V_{vp}(r) \),

\[
V_{vp}(r) \sim \frac{2 \alpha^2}{3 \pi r} \ln (m_e r), \quad r \to 0,
\]

(4.1)

for the Kallen–Sabry potential,

\[
V_{KS}(r) \sim -\frac{4 \alpha^3}{9 \pi r} \ln^2 (m_e r), \quad r \to 0,
\]

(4.2)

and for the Wichmann–Kroll potential,

\[
V_{WK}(r) \sim \frac{\alpha^4}{\pi r} \left( -\frac{2}{3} \zeta(3) + \frac{1}{6} \pi^2 - \frac{7}{9} \right), \quad r \to 0.
\]

(4.3)

These potentials have to be compared to the Coulomb potential

\[
V(r) = -\frac{\alpha}{r}.
\]

(4.4)

By inspection of these formulas, we conclude that the higher-order vacuum-polarization potentials are of the same order-of-magnitude as the Coulomb potential for distances shorter than

\[
r \sim \exp \left( \frac{-1/\alpha}{m_e} \right) = 1.2 \times 10^{-57} \text{ fm},
\]

(4.5)

which is the length scale of the Landau pole. This length scale is not sufficient to induce to any conceivably large nonperturbative effects.

### 4.2 Virtual Millicharged Particles

One of the most interesting possibilities for an explanation of both “muonic QED discrepancies” observed at present would be due to the contribution of a virtual millicharged particle. A millicharged particle was invoked as a possible explanation for the observed (later retracted) optical rotation \[40–42\] of linearly polarized laser light by a magnetic field. If the photon initiates pair production of light charged fermions with masses below the electron mass and charge on the order of \( q = \epsilon e \) with \( \epsilon \ll 1 \), then the initial observation made in Ref. \[40\] could be explained (see Refs. \[43–45\]). The non-integer charge does not contradict charge quantization if the millicharged particles are generated from a “hidden” sector of the standard model via the Stueckelberg mechanism \[46, 47\]. Such millicharged particles have been searched in devoted experiments at SLAC \[48\]. Some of these experiments are sensitive only to stable millicharged particles, because they depend on obtaining a signal from particle detectors, as pointed out in Ref. \[49\].
Several bounds have been derived regarding the mass and charge of such particles, which have otherwise been quoted as a candidate for dark matter (see Refs. [50–55]).

The muon anomalous magnetic moment discrepancy and the Lamb shift discrepancy have the same sign, i.e., the experimental result is larger than the theoretical prediction. An additional virtual excitation of a quantum field (a virtual particle) would naturally be assumed to enhance both effects. The muon anomalous magnetic moment is numerically small, the correction induced by a hypothetical virtual particle is a two-loop effect (see Fig. 1), whereas for the muonic hydrogen Lamb shift, the conceivable contribution of a millicharged particle only is a one-loop correction (see Fig. 2). Therefore, it is indicated to map out possible parameter ranges for the hypothetical millicharged particle. In muonic experiments, one is very sensitive to the mass range $m_e \ll m_M \ll m_\mu$ for hypothetical virtual particles. If the virtual particle is in this mass range, then the effect on the muon anomalous magnetic moment, and on the muonic Lamb shift is enhanced because $m_M \ll m_\mu$, but suppressed for electronic systems such as ordinary hydrogen because $m_e \ll m_M$. These models are simple-minded, straightforward ansätze that “suggest themselves” because of the pertinent mass region. We thus restrict the discussion to conceivable millicharged particles and do not consider supersymmetric graphs in which, e.g, the muon might turn into a virtual smuino, emitting a charged higgsino or wino. We also do not consider hypothetical corrections from axion electrodynamics [56].

Furthermore, since $m_e \ll m_M$ by assumption, the modification of the vacuum polarization due to the millicharged particle, for electronic hydrogen, can be absorbed into a Dirac $\delta$ potential acting on the electronic hydrogen wave functions. Its functional form is therefore indistinguishable from the nuclear finite size effect for electronic hydrogen and could be “absorbed into” a modification of the proton radius inferred from electronic hydrogen spectroscopy without any further observable consequences for atomic transitions in electronic hydrogen. For muonic hydrogen, however, since $m_M \ll m_\mu$, the hypothetical millicharged particle leads to an enhanced energy correction, different from a Dirac $\delta$. The mass range $m_e \ll m_M \ll m_\mu$, therefore is the primary parameter range probed by muonic QED experiments. For $m_M \gg m_\mu \gg m_e$, the effect of the millicharged particle amounts to a Dirac $\delta$ function and is thus indistinguishable from the contribution of the nuclear size effect for both muonic as well as electronic hydrogen.

We thus proceed as follows. First, we find a convenient parameterization of the expected modification of the spectral function of vacuum polarization, as a function of the charge and mass of the millicharged particle. Then, evaluate the shift of the anomalous magnetic moment of the muon and of the Lamb shift due to the millicharged particle (these are both proportional to the square of the charge), and relate them to the observed discrepancies, as a function of the mass of the assumed millicharged particle. Again, forming the ratio of these relative shifts, we investigate if there is a parameter range for which both the anomalous magnetic moment of the muon and the discrepancy observed in muonic hydrogen could be
Figure 3: (Color online.) In the range $10 m_e < m_M < 100 m_e$, the function $G(m_M)$ is a lot larger than unity, as shown in the plot. If a hypothetical millicharged particle in the given mass range were responsible for the discrepancy observed in muonic hydrogen, then the same particle would lead to complete disagreement for the anomalous magnetic moment of the muon.

explained by the virtual particle.

First, let us find a convenient parameterization for the spectral density of vacuum polarization. As is well known, the effect of electronic vacuum polarization on the photon propagator can be described by the replacement

$$\frac{1}{q^2 + i\epsilon} \rightarrow \frac{\alpha}{3\pi} \int_{4m_e^2}^{\infty} \frac{dt}{t} \rho_e(t) \frac{1}{q^2 - t + i\epsilon},$$

$$\rho_e(t) = \sqrt{1 - \frac{4m_e^2}{t}} \left( 1 + \frac{2m_e^2}{t} \right), \quad (4.6)$$

in the photon propagator, where we refer to $\rho_e(t)$ as the spectral density. The corresponding vacuum polarization potential (Uehling potential) induced by electronic vacuum polarization is

$$V_{vp}(r) = -\frac{\alpha^2}{3\pi} \int_{4m_e^2}^{\infty} \frac{dt}{t} \frac{e^{-\sqrt{t}r}}{r} \rho_e(t). \quad (4.7)$$

The spectral density is zero at threshold $t = 4m_e^2$ and quickly approaches the asymptotic value $\rho_e(t) \rightarrow 1$ for $t \rightarrow \infty$. Although the millicharged particle has been assumed to be of spin $1/2$, we emphasize here that similar threshold behavior can be expected regardless of the spin of the millicharged particle \cite{57,58}.

A millicharged particle modifies the spectral density of vacuum polarization according to $\rho_e(t) \rightarrow \rho_e(t) + \delta\rho(t)$. For a spin-1/2 millicharged particle, of charge $q = e\epsilon$ and mass $m_M$, we can approximate this modification as

$$\delta\rho(t) \approx e^2 \Theta(t - 4m_M^2) \quad (4.8)$$

where $\Theta$ is the step function. We intend to compare changes in the anomalous magnetic moment of the muon and in the Lamb shift induced by the millicharged particle. In line with intuition, we here need a positive spectral function $\delta\rho(t)$, increasing the Lamb shift for muonic systems and increasing the muon $g$ factor. This is the right sign because the theoretical prediction for the muon $g$ factor as well as the
Section 4: Vacuum Polarization

Figure 4: (Color online.) Both the muon anomalous magnetic moment discrepancy as well as the muonic hydrogen Lamb shift discrepancy can be explained by a millicharged particle of mass \( m_M = 0.221 \, m_e \) and charge \( q = \pm 0.0179e \), as shown in the graph. Indeed, one finds \( G(0.221 \, m_e) = 1 \). However, in the indicated mass range, the correction to the electronic hydrogen Lamb shift induced by the millicharged particle becomes so large that it leads to an inconsistent, sizeable shift of the the proton radius inferred from the hydrogen Lamb shift. See text for further explanations.

Theoretical prediction for the muonic helium Lamb shift are lower than the corresponding experimental results.

The correction to the anomalous magnetic moment due to electronic vacuum polarization is

\[
\delta a_\mu = \frac{\alpha^2}{6\pi^2} \int_0^\infty \frac{dt}{4m^2} \, f_a(t) \, \rho_e(t),
\]

where for the muon

\[
f_a(t) = 2 \int_0^1 dx \frac{x^2 (1-x)}{x^2 + (1-x)t/m^2}. \quad (4.10)
\]

We have checked that if one replaces in this expression \( m_\mu \to m_e \) and integrates over \( t \), then one obtains the known contribution \[61\],

\[
\delta a_e = \frac{139}{36} - \frac{\pi^2}{3},
\]

to the electron anomalous magnetic moment \( a_e \), due to the diagram on the left in Fig.\[4\]

Although the integral representation of \( f_a(t) \) is compact, the analytic result requires us to differentiate two cases, depending on whether \( t < 4m^2_\mu \) or \( t > 4m^2_\mu \). For \( 0 < t < 4m^2_\mu \), with \( \tau = t/(4m^2_\mu) \), one finds

\[
f_a(t) = 1 - 8 \tau - 8(1-2\tau) \ln(4\tau) - 4(1 - 8 \tau + 8 \tau^2) \left( \frac{\tau}{1-\tau} \right)^{1/2} \arctan \left( \frac{\tau}{\sqrt{1-\tau}} \right), \quad (4.12a)
\]

whereas above the muon threshold, for \( t \geq 4m^2_\mu \), with \( x = \left( 1 - \sqrt{1 - 4m^2_\mu / t} \right) / \left( 1 + \sqrt{1 - 4m^2_\mu / t} \right) \), the result for \( f_a(t) \) is

\[
f_a(t) = x^2 (2 - x^2) + 2(1+x)^2 \left( 1 + x^2 \right) \ln(1+x) - x + \frac{x^2}{x^2} + 2x^2 \left( 1 + x / (1-x) \right) \ln(x).
\]
 Consequently, the contribution to the muon anomalous magnetic moment due to the hypothetical millicharged particle, divided by the observed discrepancy $\delta a$ given in Eq. (1.11a), is

$$\chi_a = \int_0^\infty \frac{dt}{t} \eta_a(t) \delta \rho(t),$$

(4.13)

where

$$\eta_a(t) = \frac{\alpha^2}{6\pi^2} \frac{f_a(t)}{\delta a}.$$  

(4.14)

Here, $\delta \rho(t)$ is the vacuum polarization spectral density given in Eq. (4.8) due to the millicharged particle.

For the muonic hydrogen Lamb shift, the situation is as follows. The one-loop electronic vacuum polarization shifts the $2S$ level downward by $-219.6\,\text{meV}$, whereas the $2P$ is shifted downward by only $-14.6\,\text{meV}$. This is because of the enhanced probability density of the $2S$ state as compared to $2P$, near the nucleus. The total effect on the Lamb shift, by both electronic vacuum polarization and also by a hypothetical millicharged particle, can thus be approximated by taking the negative of the vacuum polarization energy shift of the (energetically lower) $2S$ level. When the resultant shift is divided by the observed discrepancy $\delta E$ given in Eq. (1.11b), one obtains the ratio

$$\chi_\mu = \int_0^\infty \frac{dt}{t} \eta_\mu(t) \delta \rho(t),$$

(4.15)

where, again with $\tau = t/(4m_\mu^2)$,

$$\eta_\mu(t) = \frac{\alpha}{3\pi} \frac{1}{\delta E} \left\langle 2S \left| \frac{\alpha}{r} \right| 2S \right\rangle = \frac{\alpha^3 m_R}{12\pi} \frac{1}{\delta E} \left( 1 + \frac{8v^2 r}{\alpha^2} \right) \left( 1 + \frac{2v\sqrt{r}}{\alpha} \right)^{-4}.$$  

(4.16)

Here, $v = m_e/m_R$ is ratio of the electron mass to the reduced mass of the muonic hydrogen system. The calculation of the ratio

$$G(m_M) = \frac{\chi_a}{\chi_\mu} = \frac{\int_0^\infty \frac{dt}{t} \eta_a(t) \delta \rho(t)}{\int_0^\infty \frac{dt}{t} \eta_\mu(t) \delta \rho(t)},$$

(4.17)

then answers the following question: Suppose that the energy discrepancy $\delta E$ in muonic hydrogen were due to the millicharged particle, then how much would the muon anomalous magnetic moment be changed by that same millicharged particle, in terms of the observed discrepancy $\delta a$? Within the approximation (4.8), the ratio $G$ depends only on the mass (not on the charge) of the millicharged particle,

$$G(m_M) = \frac{\int_{4m_\mu^2}^\infty \frac{dt}{t} \eta_a(t)}{\int_{4m_\mu^2}^\infty \frac{dt}{t} \eta_\mu(t)},$$

(4.18)

because $\epsilon^2$ as given in Eq. (4.8) cancels. If we could find $m_M$ so that $G(m_M) = 1$, then a millicharged particle would be a serious candidate to explain both the muonic anomalous magnetic moment as well as the muonic Lamb shift discrepancy.

The somewhat disappointing result of a numerical study of $G(m_M)$ is given in Fig. 3. In the mass range $10 \, m_e < m_M < 100 \, m_e$, the function $G(m_M)$ is in the range $25 < G(m_M) < 300$. The value $G(m_M) = 25$ implies that, if for given mass of the millicharged particle, the muonic hydrogen Lamb shift discrepancy is resolved, then we induce a discrepancy of theory and experiment for the muon anomalous moment by roughly $25 \times 3.4 = 85$ standard deviations. Expressed differently, the correction to the muon anomalous magnetic moment, expressed in units of the observed discrepancy $\delta a_\mu$, is larger by at least a factor 25 than the modification of the muonic Lamb shift induced by the millicharged particle, expressed in units of the observed discrepancy $\delta E$. That means that a millicharged particle in the given mass range cannot explain both observed discrepancies. The observed discrepancy $\delta E$ is too large to be explained by a
A millicharged particle with \( m_e \ll m_M \ll m_\mu \) because this would induce a prohibitively large modification of \( \delta a_\mu \). Conversely, if a virtual millicharged particle provides an explanation for the observed discrepancy \( \delta a_\mu \), then it will explain at most 4\% of \( \delta E \). Thus, a millicharged particle in the given mass range might still explain the discrepancy \( \delta a_\mu \), but if that assumption is true, then the bulk of the explanation for \( \delta E \) has to come from a different effect (e.g., proton structure).

We have previously stressed that muonic QED experiments are especially sensitive to a mass range \( m_e \ll m_M \ll m_\mu \) of the hypothetical particle. In principle, one might still explore the possibility of a millicharged particle with mass \( \alpha m_e \ll m_M \ll m_e \), because in that mass range, the correction to the hydrogen Lamb shift is still expressible in terms of a Dirac \( \delta \) function and therefore can be absorbed into a modified proton radius \( \rho \). This study is indicated even if the mass range \( m_M \ll m_e \) is not the primary range tested by muonic experiments. According to Fig. 4, we find that, in principle, a particle with \( \epsilon \approx 0.0179 \) and \( m_M \approx 0.221 m_e \) could explain both observed discrepancies \( \delta E \) and \( \delta a \), simultaneously.

However, this hypothetical particle is excluded for two reasons. First, a calculation of its contribution to the electron anomaly would be \( \delta a_e = 2.3 \times 10^{-10} \) which is much larger than the experimental uncertainty of \( \pm 2.8 \times 10^{-13} \) of the recent measurement \([62]\). A shift in the electron anomaly by \( \delta a_e = 2.3 \times 10^{-10} \) would lead to a relative shift of the fine-structure constant by \( 2 \times 10^{-7} \) and thus, to a severe disagreement with other determinations of \( \alpha \) (see Ref. \([7]\)). The second reason is as follows. Because \( m_M \approx 0.221 m_e \) is below the electron mass, the effect of the virtual particle on the electronic hydrogen spectrum is no longer parametrically suppressed. A calculation shows that because the hypothetical virtual particle now is “too light,” the proton radius inferred from the hydrogen spectroscopy experiments would increase to 0.939(7) fm, because of the concomitant Dirac \( \delta \)-like vacuum polarization potential induced by the light millicharged particle. This is \textit{a priori} not a problem, because the same modification (due to the millicharged particle) would have to be applied to the proton radius inferred from scattering experiments \([8]\).

However, in that case, since the discrepancy \( \delta E \) is to be explained by the millicharged particle, the proton radius inferred from the muonic hydrogen Lamb shift would be equal to the CODATA median value of \( r_p = 0.8768 \) fm, and there would thus be a 9\% deviation from the modified value \( r_p = 0.939(7) \) fm inferred from electronic hydrogen spectroscopy. Therefore, the parameter range \( \alpha m_e \ll m_M \ll m_e \) for the millicharged particle also can be excluded.

### 4.3 Unstable Neutral Vector Bosons

As evident from Figs. 4 and Fig. 4, our attempts to explain both muonic QED discrepancies with the simple form \([1.8]\) of the modified vacuum polarization charge density have not proven successful. In principle, one may justify a more complicated \textit{ansatz} for the modification, and with enough free parameters, it will certainly be possible to find a convenient representation that “explains” both discrepancies and is not in conflict with the electronic hydrogen Lamb shift and with the electron anomalous magnetic moment. This is not our goal.

However, one further, specific form of a modified spectral density of the vacuum polarization deserves a discussion. A light, neutral vector boson has been investigated as a possible candidate to explain a prevailing discrepancy of the decay rate of orthopositronium (experiment versus theory, see Refs. \([63, 65]\)), which has eventually been resolved \([66]\). Just like the \( \rho(770) \) vector boson, such a hypothetical, virtual neutral vector boson would induce a small hump in the spectral density of vacuum polarization, corresponding to a resonance in the photon propagator. The modification would be restricted to a finite subinterval of the \( t \) parameter. This possibility is not absolutely excluded by other searches \([63, 64]\) because the vector boson might be unstable. We have performed extensive numerical experiments, for masses (and widths) of the virtual vector boson in the primary range \( m_e \ll m_V \ll m_\mu \). One example is

\[
\delta \rho(t) = \frac{1}{30} \Theta(t - 20 m_e^2) \Theta(24m_e^2 - t). \tag{4.19}
\]

For this choice, the muon anomalous magnetic moment discrepancy is reduced to 2.5\%, with the shifted theoretical prediction now lying above the experimental value. In addition, the muonic hydrogen Lamb shift discrepancy is reduced to 2.6\%, with the theoretical value still lying below the experimental one. The \textit{muonic} hydrogen value of the proton radius would thus shift to a value of \( r_p = 0.858 \) fm. The electron
anomaly $a_e$ is shifted by $4.2 \times 10^{-14}$ which is below the current experimental uncertainty [62]. However, an evaluation of the additional vacuum polarization correction due to $\delta \rho$ as given in Eq. (4.19) for electronic hydrogen shows that the proton radius inferred from hydrogen spectroscopy would also have to be modified, namely to $r_p = 0.904(7) \text{ fm}$, because of the additional Dirac $\delta$ potential induced by the virtual vector boson. This would leave a dissatisfactory $6.7\sigma$ deviation between the modified radii from the two bound systems. Despite extensive numerical experiments, we have not found a simple, satisfactory parameter combination that might explain both muonic QED discrepancies without significantly distorting the proton radius inferred from electronic hydrogen.

5 Formation–Process Dependent Screening Corrections

All hypothetical explanations for the discrepancy of theory and experiment in muonic hydrogen discussed so far in this article do not seem to lead to a satisfactory explanation. In order to understand a physical problem, null results can also be important, but are not gratifying and leave the effect unexplained. So, in order to fully understand the problem, we also analyze the experimental procedures used in Ref. [5].

In the experiment [5], devices have been installed in the beam line to extract electrons. E.g., as revealed in Fig. 2 of Ref. [5], the beam line is designed so that muons pass two stacks of thin carbon foils, and the electrons released from the foils are then extracted from the beam line via $\vec{E} \times \vec{B}$ drift and detected in scintillators. This leads to a separation of muons and electrons, and only muons continue in the beam line, to hit the gas target. The molecular hydrogen gas target is installed in the beam line behind the foils, i.e., after the electrons have been extracted from the beam (but not from the hydrogen molecules in the gas target).

In many other experiments involving the high-precision spectroscopy of muonic transitions, the electron screening correction has been the limiting factor in analyzing the experiments [24, 31]. Quite exotic processes involving muonic atoms have been studied in the literature (see, e.g., Ref. [67]). One of the most striking surprises is the role of exotic bound states like the well-known [68] molecular state composed of a $p \mu^- e^-$ “nucleus”, and another proton, bound together by two orbiting electrons. The mentioned state is known to play a role in muon-catalyzed fusion. One might thus ask to which extent the electrons in the $H_2$ gas target may influence the observed lines. Note that the formation process of muonic hydrogen is complex and it is nontrivial to exclude the contribution of resonances from other muonic bound states; for this reason, resonances of the muonic molecules $p \mu \mu$ have been studied [69]. The first excited state of the $p \mu \mu$ molecule is predicted to have a lifetime of 0.0713 ps, close to half that of a muonic hydrogen atom in the $2P$ state. As the width of the resonance observed in the experiment [5] is close to the calculated width for the muonic hydrogen $2P$ state, the authors of [5] exclude the possibility that the molecular resonance may contribute to the observed signal.

However, these considerations do not exclude contributions from $p \mu^- e^-$ atoms composed of a proton, a negative muon and an electron, which might be formed in the gas target (these would be heavy analogues of the hydrid ion $H^-$). Because the muon’s orbit is close to the proton, the proton charge in $p \mu^- e^-$ is shielded from the outer electron. However, the muon and proton in the $2S$ or $2P$ state form a neutral core that interacts with the electron via dipole interactions. It is known that an ion-atom interaction with a functional dependence of the form $-1/r^4$ can form bound states (for a recent numerical investigation, see Ref. [70]). Here, the “ion” is the electron, whereas the “atom” is the muon-proton core. The form of the interaction potential can be derived as follows.

We first slightly generalize the problem and assume that the proton is a nucleus with charge number $Z$. If we have a nucleus with charge number $Z$ and a muon and an electron bound to it, then the unperturbed Hamiltonian for the muon reads

$$H_\mu = \frac{\vec{p}_\mu^2}{2m_\mu} - \frac{Z\alpha}{r_\mu},$$

and the unperturbed Hamiltonian for the electron is

$$H_e = \frac{\vec{p}_e^2}{2m_e} - \frac{(Z-1)\alpha}{r_e}.$$
because the outer electron merely sees the screened charge of the inner core, which is the nucleus of charge number \(Z\) plus the negative muon of negative charge, effectively reducing \(Z\) by one unit. For a molecular hydrogen gas target, we can set \(Z = 1\) (approximately), and \(H_e\) is approximately equal to the free Hamiltonian but will be corrected by a \(-1/r^4\) interaction, as detailed below. The total Hamiltonian, including the muon-electron repulsion, then is

\[
H = \frac{\mathbf{r}_\mu^2}{2m_\mu} - \frac{Z\alpha}{r_\mu} + \frac{\mathbf{r}_e^2}{2m_e} - \frac{Z\alpha}{r_e} + \frac{\alpha}{|\mathbf{r}_e - \mathbf{r}_\mu|} .
\]  \hspace{1cm} (5.3)

The perturbation is

\[
H_I = H - H_e - H_\mu = -\frac{\alpha}{r_e} + \frac{\alpha}{|\mathbf{r}_e - \mathbf{r}_\mu|} .
\]  \hspace{1cm} (5.4)

We expand this expression up to the dipole term and assume that \(r_e > r_\mu\), and obtain the interaction Hamiltonian,

\[
H_I \approx -\frac{\alpha}{r_e} + \frac{\alpha}{r_e} + \frac{r_\mu}{r_e^2} \hat{x}_e \cdot \hat{x}_\mu ,
\]  \hspace{1cm} (5.5)

where \(\hat{x}_e = \mathbf{r}_e/r_e\), and \(\hat{x}_\mu = \mathbf{r}_\mu/r_\mu\). The first two terms cancel, and the third gives the dipole interaction. We start from unperturbed states with the electron in a state with quantum numbers \(|n_e \ell_e m_e\rangle\) and the muon in the \(|2S\rangle\) state. A calculation in second-order perturbation theory then gives the energy perturbation

\[
\delta E = -\frac{\alpha^2}{2} \left\langle n_e \ell_e \left| \frac{1}{r_e^4} \right| n_e \ell_e \right\rangle_e \left\{ \frac{2}{3} \sum_{i=1}^{3} \sum_{n_i} \sum_{m_i=-1}^{1} \frac{|\langle 2S | r_i^i | n_\mu P m_\mu \rangle|}{E_{n_i \mu}^{(\mu)} - E_{2S}^{(\mu)}} \right\}^2 .
\]  \hspace{1cm} (5.6)

Here, the superscripts and subscripts identify the particles (\(e\) stands for the electron, and \(\mu\) stands for the muon). The \(|2S\rangle\) state undergoes virtual transitions to muonic \(|n_\mu P \mu \rangle\) states, and the term in curly brackets in Eq. (5.6) is recognized as the static polarizability of the 2S state. The energetically closest state to the muonic 2S state is the \(|2P_{1/2}\rangle\) state, which is energetically removed from \(|2S\rangle\) only by the Lamb shift. The energy difference is of the order

\[
E_{2P_{1/2}} - E_{2S} \sim \alpha^3 m_\mu .
\]  \hspace{1cm} (5.7)

The dipole matrix elements for the virtual transitions of the muon are of the order of

\[
\langle 2S | r_i^i | n_\mu P m_\mu \rangle \sim \frac{1}{\alpha m_\mu} .
\]  \hspace{1cm} (5.8)

We assume that the electron is in a (superposition of) states with quantum numbers \(|n_e \ell_e m_e\rangle\) whose dimensions are of the order of the (electronic) Bohr radius, and so

\[
\left\langle n_e \ell_e \left| \frac{1}{r_e^4} \right| n_e \ell_e \right\rangle \sim (\alpha m_e)^4 .
\]  \hspace{1cm} (5.9)

The result of our order-of-magnitude estimate is

\[
\delta E \sim -\alpha^2 (\alpha m_e)^4 \left( \frac{1}{\alpha^3 m_\mu^3} \right) = -0.42 \text{ meV}
\]  \hspace{1cm} (5.10)

for the energy shift of the 2S state, if an additional electron is present. As this is the lower state of the 2P–2S Lamb shift transition, the negative of \(\delta E\), i.e., +0.42meV, needs to be added to the transition frequency, potentially explaining the discrepancy. The corresponding effect, evaluated for the muonic 2P_{1/2} state, has the opposite sign and therefore shifts the transition in the same direction as the effect calculated here.

The main result of the above order-of-magnitude estimate is as follows. If, for some reason, an electron is spatially displaced from the muonic hydrogen atom by only a few Bohr radii at the time of the laser-induced
Furthermore, it would be necessary to study its inner Auger rates of its ionization cross sections in collisions with other hydrogen molecules in the gas target. Furthermore, it would be necessary to study its inner Auger rates of its ionization cross sections in collisions with other hydrogen molecules in the gas target. This is beyond the scope of the current article. The occupation numbers may depend on the formation process. In order to explain the single, well defined resonance line seen in the experiment [5], one would have to assume that formation proceeds predominantly into a specific state of the $\mu^- e^-$ atom (or into a sufficiently narrow subset of resonances). These requirements may, in the end, reveal that the current hypothesis cannot explain the observed shift of the resonance line from the predictions of QED theory. Thus, we do not claim that the calculated effect necessarily explains the discrepancy of theory and experiment. However, as we were unable to discern viable theoretical explanations, our general statement is that it may be worthwhile to study possible explanations based on a process-dependent effect.

Another mechanism by which an electron screening correction could enter the analysis of the experiment [5] might be from neighboring hydrogen molecules or atoms. In general, one assumes [71] that the initial capture into highly excited states with principal quantum number $n \approx \sqrt{m_\mu/m_e} \approx 14$ takes place via the reaction $\mu^- + H_2 \rightarrow (p p \mu^- e^-) + e^-$, where the muon is captured into a molecular orbit and an electron is ejected. Auger rates induced by neighboring hydrogen atoms and molecules dominate in the deexcitation process for liquid hydrogen [72], but are suppressed for a less dense hydrogen gas target. The deexcitation and muonic hydrogen formation process has been analyzed both theoretically [71–75] as well as experimentally [76,77]. Still, there is a multitude of available reaction channels and conceivable bound states. It may thus be worthwhile to reexamine the dynamics of the formation process of muonic hydrogen in the 5 T magnetic field with a special emphasis on the hypothetical existence of neighboring, screening electrons that may perturb the observed frequency in the experiment [5].

6 Conclusions

Recently, two serious discrepancies of theory and experiments have been observed for muonic QED systems: the muon $g$ factor discrepancy of $3.4 \sigma$ appears to persist [1–4], and for the muonic hydrogen Lamb shift [5], an even larger discrepancy is observed which amounts to $5.0 \sigma$. The proton radius inferred from the muonic hydrogen Lamb shift is $5.0 \sigma$ smaller than that determined by electronic hydrogen spectroscopy [7], or alternatively, if the proton radius from hydrogen spectroscopy or from electron scattering is used in order to obtain a theoretical prediction for muonic hydrogen, we observe a $5.0 \sigma$ discrepancy of theory and experiment. The muonic Lamb shift discrepancy is statistically more significant and thus, arguably, more urgent to be resolved than the muon $g$ factor. Proton structure effects (discussed in Sec. 3) and small modifications of the vacuum polarization potential (see Sec. 4) might be discussed as hypothetical explanations for the observed discrepancy.

As shown in Sec. 3, an elaborate reevaluation of the two-photon exchange graph for muonic hydrogen, including its inelastic (proton polarizability) contribution, might contribute to an explanation of the muonic Lamb shift discrepancy. The inelastic part has been evaluated [11,30,31] to be on the order of $0.0015(4)$ meV. Unless a reevaluation reveals a somewhat surprising enhancement of the contribution by at least an order of magnitude, the discrepancy of $\delta E = 0.31$ meV will persist. An alternative explanation due to a conceivable “dip” in the slope of the proton form factor in the momentum range probed via muonic hydrogen spectroscopy (see Sec. 3.1) could only be excluded conclusively by a measurement of the proton form factor in scattering experiments probing the momentum range indicated in Eq. (3.3). While certain dip and hump structures in the proton form factor have been seen in experiments [8] and in theoretical calculations [78], one has to admit that the “dip” hypothesis seems somewhat remote.

In Sec. 4, we investigate the role of a hypothetical millicharged particle in a numerically small, but important modification of the spectral density of vacuum polarization. We focus on millicharged particles and do not treat hypothetical supersymmetric models. The conclusion is as follows: The most immediate
theoretical ansatz for the modification $\delta \rho(t)$ given in Eq. (4.8) comes from a low-energy millicharged particle; a model for a hypothetical virtual natural vector boson resonance is described by Eq. (4.19). Our numerical experiments show that neither a simple modification of the vacuum polarization function due to a millicharged particle nor due to an unstable intermediate vector boson can simultaneously explain both discrepancies observed for the muon anomalous magnetic moment and for the muonic hydrogen Lamb shift without significantly distorting the proton radius inferred from electronic hydrogen. Millicharged particles and virtual vector bosons could explain the muon anomalous magnetic discrepancy while having a negligible effect on the muonic Lamb shift, but not vice versa. If these particles are unstable, then other parametric bounds (e.g., those following from the experimental investigation reported in Ref. [48]) may not be applicable to an analysis of their virtual contributions within vacuum polarization loops.

Our investigations severely restrict the parameter space available for a modification of the vacuum polarization spectral function due to either millicharged particles or unstable virtual vector bosons in the low-energy domain. Being statistically (barely) significant, the muon anomalous magnetic moment discrepancy is numerically so small that it restricts possible modifications of the vacuum polarization spectral function due to hypothetical millicharged particles to such small coupling strengths that a simultaneous explanation of the comparatively large observed discrepancy in muonic hydrogen becomes impossible. In a small parameter range, a virtual vector boson resonance modification somewhat reduces the statistical significance of the combined discrepancies in muonic hydrogen and for the muon anomalous magnetic moment, but at the cost of inducing a prohibitively large modification of the (electronic) hydrogen Lamb shift.

Hitherto undetected virtual particles from extensions of the standard model might still explain the two discrepancies $\delta E$ and $\delta a_{\mu}$, but a more complex structure of the modification of the vacuum polarization spectral function would have to be generated, with—possibly—both attractive as well as repulsive modifications. Repulsive modifications of the vacuum polarization potential have been discussed in the literature [60,79]. A more complex modification of the vacuum polarization has more free parameters. Therefore, a conceivable determination of these parameters becomes possible, if at all, only when more experimental data on muonic systems (e.g., the muonic helium ion) become available. Our discussion in Secs. 4.2 and 4.3 does not exhaust all possible theoretical explanations from extensions of the standard model but covers particular models that “suggest themselves.” A conceivable explanation from an extension of the standard model is constrained by the comparatively large discrepancy in the muonic hydrogen Lamb shift, the comparatively small discrepancy in the muon $g$ factor, and must respect the excellent agreement of theory and experiment for the electron $g$ factor and the hydrogen Lamb shift.

Having discussed the most attractive and far-reaching theoretical consequences of the recently observed discrepancy [5] and having found negative results, we then proceed to indicate a candidate effect for a process-dependent correction to the energy levels that may be relevant to the experiment [5] (see Sec. 5). We believe that a study of exotic bound states and a reexamination of the dynamics of the formation process indicated in Sec. 5 might be interesting in its own right even if eventually, more detailed calculations might show that these effects do not offer the explanation for the observed discrepancy in muonic hydrogen.

High-precision QED experiments are excellent probes of conceivable low-energy modifications of the standard model. Our considerations highlight the need for further experimental evidence regarding muonic bound systems, before conclusive statements are possible. As discussed in Sec. 2, discrepancies of theory and experiment in muonic systems have been encountered in the past and have eventually been resolved. Even if theoretical considerations and careful considerations of additional systematic effects concerning the experiment [5] fail to resolve the proton radius discrepancy, then there is a third way to resolve it, based on spectroscopic methods. It works as follows. In general, the fundamental constants derived from atomic spectroscopy are highly intertwined (not only the nuclear radii). One example is the Rydberg constant. If the new proton radius from Ref. [5] is inserted into the evaluation of electronic hydrogen spectra, then a very precise Rydberg constant is obtained, which is given as [5]

$$\text{c } R_{\text{uc}} = 3 289 841 960 251(5) \text{ kHz \quad [1.5 } \times 10^{-12} \text{]} ,$$

where we indicate the relative uncertainty in square brackets. In an unpublished PhD thesis [80] completed at MIT (Cambridge) in 2002, based on Rydberg transitions which are manifestly independent of the
proton charge radius, the value
\[ c R_\infty = 3 289 841 960 306(69) \text{ kHz} \quad [2.1 \times 10^{-11}], \]  \hspace{1em} (6.2)
is obtained [the difference to the value in Eq. (6.1) is 0.8σ]. The CODATA value is
\[ c R_\infty = 3 289 841 960 361(22) \text{ kHz} \quad [6.6 \times 10^{-12}], \]  \hspace{1em} (6.3)
and the difference to (6.2) also is 0.8σ, but “to the other side.” The Rydberg state value (6.2) lies in
between the values given in Eqs. (6.1) and the CODATA value (6.3). Quite surprisingly, an improved
measurement of the Rydberg constant based on ionic Rydberg states, though in itself independent of
the proton radius, could thus settle the proton radius question connected with the muonic hydrogen
measurement. Such a measurement is currently being pursued at the National Institute of Standards and
Technology [81].

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