Mobility edges in topological Anderson insulating phase induced by a slowly varying modulation

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Abstract. We uncover the relationship of topology and disorder in a one-dimensional Su-Schrieffer-Heeger chain subjected to an incommensurate modulation. By numerically calculating the disorder-averaged winding number and analytically studying the localization length of the zero modes, we obtain the topological phase diagram, which implies that the topological Anderson insulator (TAI) can be induced by a slowly varying incommensurate modulation. Moreover, unlike the localization properties in the TAI phase caused by random disorder, mobility edges can enter into the TAI region identified by the fractal dimension, the inverse participation ratio, and the spatial distributions of the wave functions, the boundaries of which coincide with our analytical results.

Keywords: disorder; topological Anderson insulator; mobility edge.
1. Introduction

Topological phases of matter have attracted broad interest over the past decades \[1, 2, 3, 4, 5, 6\]. A topological insulator displays the gapped bulk states and the gapless edge modes lied in the bulk gap, which can be characterized by a non-trivially topological number. The Su-Schrieffer-Heeger (SSH) model is the simplest two-band topological system initially introduced to research the polyacetylene that exhibits rich physical phenomena \[7\], such as fractional charge soliton excitation \[8, 9\] and nontrivial edge modes \[10\]. Its chiral symmetry leads to nontrivial topology confirmed by a non-zero winding number and the emergence of the zero-energy edge modes under open boundary conditions (OBCs) \[11, 12, 13\].

On the other hand, Anderson localization in a disordered medium \[14, 15\] was first proposed in 1958, which has been realized in various experimental platforms such as cold atoms \[16, 17\] and microwaves cavity \[18, 19\]. Compared with traditional Anderson models, where even an infinitesimal random potential leads to localization in one- and two-dimensional cases, the quasi-periodic systems display distinctive localization properties. The one-dimensional Aubry-Andr´e-Harper (AAH) model as one of the paradigmatic examples undergoes a metal-insulator transition characterized by the self-duality property when the strength of the quasi-periodic potential exceeds a finite critical value \[20, 21, 22, 23\]. Remarkably, the presence of the mobility edge, one of the essential concepts in disordered systems seen only in the traditional three-dimensional Anderson model \[24\], has been demonstrated in various generalized AAH models even in one dimensional systems. Some generalized AAH models displaying the mobility edges in compact analytic forms are proposed when the so-called self-dual symmetry is broken \[25, 26, 27, 28, 29, 30\], or the system is in the form of slowly varying modulations \[31, 32, 33\].

Recently, great effort has been devoted to understand the interplay of topology and disorder, which brings new perspectives. One of the hallmark characteristics of a topological insulator is the robustness of the nontrivial edge states against weak disorder in the underlying lattice \[2, 34\]. When the amplitude of disorder is large enough, the topological features eventually disappear, accompanied by collapse of the nontrivial topological number \[35\]. Conversely, a modest disorder added to a trivial system can lead to the emergence of protected edge modes and quantized topological charges. Such disorder-driven topological phase is named the topological Anderson insulator (TAI) \[36, 37, 38, 39, 40, 41, 42, 43\]. The experimental observation of the disorder-induced TAI has been reported in two-dimensional (2D) photonics \[44, 45\], one-dimensional (1D) engineering synthetic 1D chiral symmetric wires with a precisely controllable random disorder \[46\]. Recent advances of the TAI induced by a random disorder have been extended to non-Hermitian systems \[47, 48\] and spin-orbit coupled superconductors \[49, 50\]. On the other hand, the TAI phase is also shown to appear in the SSH model with a quasi-periodic modulated intercell hopping term with all the eigenstates being localized \[51\].
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Some interesting questions arise here: whether a slowly varying modulation can lead to a TAI phase, and what the localization properties in such case are. To address these, we consider the SSH model subjected to a slowly varying intracell hopping and open boundary condition, focusing on the topology and localization features in the system. The topological phase diagrams will be obtained by numerically calculating the topological invariants in real space, i.e., the disorder-averaged winding number, and the topological phase transition points can be determined by the localization length of the zero modes. Moreover, the localization properties of the system will be explored by some proven means such as the inverse participation ratio, the fractal dimension, and the spatial distributions of the eigenstates. Specifically, we try to obtain analytic expressions for the mobility edges, if they exist.

The arrangement of the paper is as follows: In Sec.2, we briefly introduce the Hamiltonian of the SSH model with a slowly varying intracell hopping term. In Sec.3, we obtain the topological phase diagram and discuss the fate of topological zero-energy modes. Furthermore, we study the localization properties of the slowly varying SSH model in Sec.4. Finally, a summary is presented in Sec.5.

2. Model and Hamiltonian

We consider the SSH model with a slowly varying intracell modulation, which can be described by the Hamiltonian

$$H = \sum_j v_j \left( c_{j,B}^\dagger c_{j,A} + h.c. \right) + w \sum_j \left( c_{j+1,A}^\dagger c_{j,B} + h.c. \right).$$

Here $c_{j,\alpha}^{\dagger}$ ($c_{j,\alpha}$) creates (annihilates) a particle on the sublattice site $A$($B$) in the $j$-th lattice cell, and $L$ is the length of the system. As shown in Fig. 1(a), the intercell hopping strength $w$ is denoted by the black dashed line, which is set as the unit energy, i.e., $w = 1$, and the $j$-dependent intracell hopping

$$v_j = v + \Delta_j,$$

with

$$\Delta_j = \Delta \cos(2\pi \beta j^u + \phi),$$

is denoted by the red solid line. Here $v$ is the site-independent hopping inside the unit cell, $\Delta$ is the strength of incommensurate modulation, $\beta$ is the modulation frequency, and $\phi \in [0, 2\pi]$ is an arbitrary phase. In the thermodynamic limit, we have

$$\lim_{j \to \infty} \left| \frac{d\Delta_j}{dj} \right| = \lim_{j \to \infty} 2\pi \Delta \beta u |\sin(2\pi \beta j^u + \phi)| j^{1-u} \to 0,$$

with the slowly varying parameter $0 < u < 1$. The result indicates the difference of $\Delta_j$ tends to 0 when $j$ is large enough as shown in Fig. 1(b), which corresponds to a slowly varying modulation $^{[31, 32]}$. For $\Delta = 0$ or $u = 0$, the system reduces to a standard SSH model $^7$ with a uniform intracell hopping amplitude. When the intercell hopping strength exceeds the intracell hopping strength, system undergoes a
topological phase transition characterized by the emergence of zero-energy edge modes and nontrivial winding number. For $\Delta \neq 0$ and $u = 1$, $\Delta_j$ corresponds to an AA-type modulation illustrated in Fig. 1(c). The emergence of the TAI in SSH model with an AA-type modulation has been reported [51]. In the following, we testify the existence of the TAI in the SSH model with a slowly varying intracell modulation and discuss the corresponding localization properties. We fix $\beta = \sqrt{5} - 1$ and $u = 0.7$ for our discussion.

3. TAI induced by the slowly varying modulation

A TAI is characterized by the emergence of the protected edge states and the quantized topological charge induced by the addition of sufficient disorder or incommensurate modulation to a trivial band structure. To detect the TAI phase, we first utilize the open-bulk winding number to characterize the topological properties for our slowly varying SSH model. For a given modulation configuration, we can diagonalize the open-chain Hamiltonian as $H|\psi^n\rangle = E_n|\psi^n\rangle$ and $H|\tilde{\psi}^n\rangle = -E_n|\tilde{\psi}^n\rangle$ to obtain a pair of chiral-symmetric partners $|\psi^n\rangle$ and $|\tilde{\psi}^n\rangle$ with the relation $|\tilde{\psi}^n\rangle = S|\psi^n\rangle$, where the entries of $S$ are $S_{ij,\alpha\gamma} = \delta_{ij} (\sigma_z)_{\alpha\gamma}$ with $i, j$ referring to the unit cell and $\alpha, \gamma$ to the sublattice. We introduce an open-boundary $Q$ matrix given by

$$Q = \sum_n \left( |\psi^n\rangle\langle\psi^n| - |\tilde{\psi}^n\rangle\langle\tilde{\psi}^n| \right),$$  

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where \( \sum_n \) is the sum over the eigenstates in the bulk spectrum without the edge modes. The open-bulk winding number in real space is defined as \[ W_c = \frac{1}{2L'} \text{Tr}'(SQ[Q,X]). \] Here, \( X \) is the coordinate operator, namely \( X_{i\alpha,j\gamma} = j \delta_{ij} \delta_{\alpha\gamma} \). The length of the system \( L \) can be divided into three intervals with length \( l, L' \) and \( l \), i.e., \( L = L' + 2l \). The symbol \( \text{Tr}' \) represents the trace over the middle interval with length \( L' \). When \( L \to \infty \), the open-bulk winding number method can well serve for the system deviated from periodicity, and \( W_c \) is quantized to an integer, while a modest size is enough in the practical calculation. We also define the disorder-averaged winding number \[ \overline{W} = \frac{1}{N_c} \sum_{c=1}^{N_c} W_c \] with the configuration number \( N_c \). Here, we take \( N_c = 100 \) disorder realizations for different \( \phi = [0, 2\pi] \) throughout the work.

\[ \text{Figure 2. (Color online) (a) Disorder-averaged winding number } \overline{W} \text{ as a function of the modulation strength } \Delta \text{ and the amplitude of the intracell hopping } v \text{ with } L = 300 \text{ and } N_c = 100 \text{ disorder realizations. The colorbar shows the value of disorder-averaged winding number } \overline{W}. \text{ Two red dot lines represent the analytic critical lines Eq. (11) for the divergence of localization length } \lambda. \text{ (b) } \lambda^{-1} \text{ as a function of } \Delta \text{ and } v \text{ for } L = 600 \text{ and } N_c = 100 \text{ disorder realizations. The colorbar shows the value of } \lambda^{-1}. \text{ (c)(d) Two disorder-averaged energies } \overline{E}_L \text{ and } \overline{E}_{L+1} \text{ in the center of the spectrum, and the disorder-averaged winding number } \overline{W} \text{ as a function of the modulation strength } \Delta \text{ under OBCs with } v = 0.5(c) \text{ and } v = 1.3(d), \text{ respectively.} \]

Figure 2(a) shows the topological phase diagram characterized by \( \overline{W} \) versus the modulation strength \( \Delta \) and the amplitude of the intracell hopping \( v \). In the absence of the modulation \( \Delta = 0 \), the standard SSH model exhibits a topological phase transition point at \( v = 1 \) corresponding to a jump of \( \overline{W} \) from 1 to 0 with the increase of \( v \). When we turn on the slowly varying modulation \( \Delta_j \) to the nontrivial band structure with \( v < 1 \), the zero-energy modes are robust against the slowly varying modulation. With
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The increase of $\Delta \lesssim 2$, the two disorder-averaged zero modes $E_L$ and $E_{L+1}$ start to break into non-zero pairs, accompanied by the jump of the winding number to zero, as shown in Fig. 2(c) for $v = 0.5$ under OBCs. Here, $E_n = 1/N_c \sum_{c=1}^{N_c} E_n^c$ with $E_n^c$ being the $n$-th eigenenergy for a given modulation configuration. As shown in Fig. 2(a), we find that the topological transition point is independent of $v$ at $\Delta = 2$. It indicates that the topologically nontrivial phase will collapse as long as the strength of the slowly varying modulation reaches a critical value. When $v \in (1, 2)$ the system remains a topologically trivial phase as in a standard SSH model for $\Delta > 0$ and the TAI phase emerges in a finite regime between $v = \Delta^2/4 + 1$ and $\Delta = 2$, which is characterized by the nontrivial winding number and the zero-energy edge modes. The deviation from the non-zero integer of the disorder-averaged winding numbers $W$ near the phase transition boundaries comes from the instability of $W$ for a given modulation configuration near the boundary, though the one-shot $W$ is always an integer in the thermodynamic limit. We take $v = 1.3$ as an example shown in Fig. 2(d), the moderate slowly varying modulation $\Delta$ leads to a change of the nontrivial winding number in the region $\Delta \in (1, 2)$, meanwhile the disorder-averaged zero modes are also detected. When $v > 2$, the TAI phase disappears, no matter how strong the modulation amplitude is, as can be seen in Fig. 2(a). Our numerical results show that the slowly varying intracell modulation will evoke the TAI phase in a finite region for $v > 1$.

According to Refs. [42, 46, 47, 51], the topological transition is accompanied by the divergence of the localization length of the zero modes. Hence, one can obtain the phase diagram by studying the localization length of the zero modes. For the zero modes, the Schrödinger equation of the SSH model (1) $H \psi = 0$ reads:

$$
\begin{align*}
wp_{j,B} + v_{j+1}p_{j+1,B} &= 0 \\
v_jp_{j,A} + wp_{j+1,A} &= 0,
\end{align*}
$$

(7)

where $p_{j,A}(p_{j,B})$ is the probability amplitude of the zero mode on the sublattice site $A(B)$ in the $j$-th lattice cell. By solving the coupled equations, one has $p_{n+1,A} = (-1)^n \prod_{j=1}^{n} (v_j/w)p_{1,A}$, leading to the localization length $\lambda$ of the zero modes given by [51]

$$
\lambda^{-1} = \lim_{L \to \infty} \frac{1}{L} \ln \left| \frac{p_{L+1,A}}{p_{1,A}} \right|
= \lim_{L \to \infty} \frac{1}{L} \sum_{j=1}^{L} \ln |v + \Delta \cos(2\pi \beta j^w)|.
$$

(8)

According to Weyl’s equidistribution theorem [52, 53], we can use the ensemble average to evaluate the last expression

$$
\lambda^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \ln |v + \Delta \cos q|.
$$

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The integration can be performed straightforwardly as

\[
\lambda^{-1} = \begin{cases} 
\ln \frac{v + \sqrt{v^2 - \Delta^2}}{2} & v > \Delta, \\
\ln \frac{\Delta}{2} & v < \Delta.
\end{cases}
\]  

(10)

The divergence of this localization length \( \lambda \), i.e. \( \lambda^{-1} \to 0 \), gives the two critical lines

\[
v = \Delta^2/4 + 1 \quad v > \Delta,
\Delta = 2 \quad v < \Delta.
\]  

(11)

The localized critical points match the topological phase transition points shown in Fig. 2(a). Numerically the value of \( \lambda^{-1} \) for the \( L \)-th eigenstate is computed by \[27\]

\[
\lambda^{-1} = \lim_{L \to \infty} \frac{1}{L} \|T\|,
\]  

(12)

where \( \| \cdot \| \) denotes the norm of the total transfer matrix \( T = \prod_{j=2}^{L} T_j T_1 \) with

\[
T_j = \begin{bmatrix}
\frac{E_j^2 - v_j^2 - 1}{v_j} & -\frac{v_{j+1}}{v_j} \\
1 & 0
\end{bmatrix}
\]  

(13)

and

\[
T_1 = \begin{bmatrix}
\frac{(E_1^2 - v_1^2)}{v_1} & -1 \\
1 & 0
\end{bmatrix}.
\]  

(14)

The numerical result of \( \lambda^{-1} \) as a function of \( v \) and \( \Delta \) by means of the transfer matrix method \[42, 54\] is shown in Fig. 2(b), and the two diverging critical lines match with the analytic solutions Eq. (11) pretty well.

4. Localization transition and mobility edges

A common opinion is that when the system is in the TAI phase, all states are localized, showing Anderson’s localization phenomenon. In this section, we study another aspect of the localized nature of the TAI phase, i.e. the access of mobility edges to the TAI phase in the SSH model under a slowly varying incommensurate modulation. We follow the method in Ref. \[55\] to derive the expression of mobility edges for our slowly varying model. Due to the chiral symmetry, the system’s spectrum is symmetric about \( E = 0 \). To simplify, we only focus on the upper band(\( E > 0 \)), and the lower one displays identical behaviors.

The localization in the slowing varying model exhibits different features for two cases of the site-independent intracell hopping strength \( v \). As following results imply, the delocalization-localization transition is not connected to a topological phase transition in both regimes.

(a) For \( v > 1 \), the mobility edges are given by (see Appendix for details):

\[
E_c = v \pm (\Delta - 1).
\]  

(15)
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Figure 3. (Color online) (a) The fractal dimension \( \Gamma \) of different eigenstates as a function of the corresponding eigenvalues and the modulation strength \( \Delta \). The colorbar indicates the magnitude of \( \Gamma \). The black and red solid lines represent the mobility edges given in Eq. (15). The shaded regime represents the TAI regime. The black and red dots represent two eigenstates correspond to \( E = 1.9695 \) and \( E = 1.1108 \), respectively. The corresponding spatial distributions are shown in (c). (b) The distribution of the IPR as a function of the eigenenergy \( E \) with different \( \Delta \). The black dashed lines correspond to the mobility edges given in Eq. (15). Here, only \( E > 0 \) states are shown for \( \beta = (\sqrt{5} - 1)/2, \phi = 0, v = 1.1 \) and \( L = 10000 \).

Here and in Eq. (17) \( (+) \) denotes the upper(lower) mobility edge marked by the black(red) solid line shown in Fig. 3(a), respectively. We find that \( \Delta_c = 1 \) is the localization transition point. When \( \Delta > \Delta_c \), all eigenstates become localized, and there exist mobility edges in the regime \( 0 < \Delta < \Delta_c \). According to the topological phase diagram Fig. 2(a), we find that the TAI phase is compatible with the regime hosting mobility edges.

To characterize the mobility edges, we define the fractal dimension of the wave function as following:

\[
\Gamma = - \lim_{L \to \infty} \frac{\ln \text{IPR}}{\ln (2L)},
\]

where the inverse participation ratio \( \text{IPR} = \sum_{j=1}^{L} \sum_{\alpha} |\psi_{j,\alpha}^{n}|^4 \). It is known that \( \Gamma \to 1 \) for extended states and \( \Gamma \to 0 \) for localized ones [27]. In Fig. 3(a), we show the fractal dimension \( \Gamma \) of the eigenstates as a function of the modulation strength \( \Delta \) for \( v = 1.1 \). The solid lines represent the mobility edges defined by Eq. (15), and the shaded regime denotes the region with TAI. One can see that there exists an overlap between the TAI regime and the extended states regime with mobility edges. To clarify the mobility edges more clearly, Fig. 3(b) plots the IPR as a function of the eigenenergy for different \( \Delta \) with \( v = 1.1 \). The magnitude of the IPR jumps from the order of
Figure 4. (Color online) (a) The fractal dimension $\Gamma$ of different eigenstates as a function of the corresponding eigenvalues and $\Delta$. The colorbar indicates the magnitude of $\Gamma$. The black and red solid lines represent the mobility edges given in Eq. (17). (b) The distribution of the IPR as a function of the eigenenergy $E$ with different $\Delta$. The black dashed lines correspond to the positions of mobility edges given in Eq. (17). Here, only $E > 0$ states are considered, $\beta = (\sqrt{5} - 1)/2$, $\phi = 0$, $v = 0.3$ and $L = 10000$.

The magnitude of $10^{-2}$ (a typical value for the localized states [33]) to $10^{-4}$ (a typical value scales as $1/L$ for the extended states [33]), when the eigenenergy crosses some critical values for $\Delta = 0.3$ and 0.6. This clearly signals the transition between localized and extended states, and these critical values correspond to the positions of the mobility edges in the spectrum marked by the black dotted lines in Fig. 3(b) defined by Eq. (17). When $\Delta = 1.2 > \Delta_c$, the IPR magnitude for all eigenstates is up to $10^{-2}$, which indicates there is no mobility edges in the spectrum. Furthermore, the mobility edges can also be confirmed intuitively by the spatial distributions of the wave functions in Fig. 3(c). The wave functions are localized or extended when their eigenvalues satisfy $|E| > v + (\Delta - 1)$ or $v + (\Delta - 1) < |E| < v - (\Delta - 1)$, corresponding to the black star and red dot shown in Fig. 3(a), respectively.

(b) For $0 < v < 1$, the mobility edges are instead of

$$E_c = 1 \pm (v - \Delta).$$

Figure 4(a) shows the fractal dimension $\Gamma$ of the eigenstates as a function of eigenenergy $E$ and modulation strengthen $\Delta$ for $v = 0.3$. In the regime $0 < \Delta < v$, as expected from the analytical results, $\Gamma$ approximately exhibits sharp jumps from 1 to 0 for energies upper or lower lines defined by Eq. (17). When $\Delta > v$, the mobility edges vanish, and all the states are localized. Figure 4(b) plots the IPR as a function of the eigenenergy for different $\Delta$ with $v = 0.3$, where the jumping points denote the positions of the mobility edges.
5. Conclusions

In summary, we study the topological phase transition and localization transition of a SSH model with a slowly incommensurate modulation. We numerically and analytically obtain the topological phase diagram, and our results imply that a slowly vary incommensurate modulation can induce the TAI. Different from the random disorder induced the TAI, the mobility edges can enter into the TAI regime in our case.

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Note added: on submission of this manuscript, we notice a very recent preprint arXiv:2201:00988 entitled ‘Topological Anderson insulators with different bulk states in quasiperiodic chains’. The authors studied a similar problem, and main results of the TAI regime with the intermediate phase were obtained. In this paper, we derive the exact expression of mobility edges in this TAI phase.
Appendix A. Derivation of the analytical expression for mobility edges

In this Appendix, we present a detailed derivation of the expression of mobility edges Eqs. (15) and (17) in details following the method proposed in Ref. [55]. To do this, we approximate the slowly varying modulation model with an ensemble of different periodic models $M_a$. And the extended state regime in the spectrum of the slowly varying modulation model corresponds to the range of energy bands of these periodic models $M_a$. Thus, the mobility edges can be determined as the smallest overlaps of these ranges.

We first assume the slowly varying incommensurate modulation $\Delta_j$ may take a series of constants $\Delta_a (\Delta_a \in [-\Delta, \Delta])$ in the thermodynamic limit, and the system becomes an ensemble of periodic models $M_a$. Under periodic boundary conditions (PBCs) with $c_{L+1,\alpha}^\dagger = c_{1,\alpha}^\dagger$, the energy spectra of $M_a$ can be obtained by the tight-binding approximation as follows:

$$ E = \pm \sqrt{(v + \Delta_a)^2 + 1 + 2(v + \Delta_a) \cos k}, \quad (A.1) $$

where $k \in (0, 2\pi]$ is the wave number. Here we only focus on the upper band ($E > 0$) for positive $v$ and $\Delta$.

The extended state regime for $M_a$ is just the range of the energy band:

$$ |v + \Delta_a - 1| < E < |v + \Delta_a + 1| \quad (A.2) $$

or

$$ |v + \Delta_a + 1| < E < |v + \Delta_a - 1|. \quad (A.3) $$
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There exists two limit situations for \( \Delta_a \). When \( \Delta_a = \Delta \), we have
\[
|E_1| < E < |E_4|,
\]  
with \( E_1 = v + \Delta - 1 \) and \( E_4 = v + \Delta + 1 \). The other limit is \( \Delta_a = -\Delta \), for that we have
\[
|E_2| < E < |E_3|
\]
or
\[
|E_3| < E < |E_2|,
\]
with \( E_2 = v - \Delta + 1 \) and \( E_3 = v - \Delta - 1 \). To determine the smallest overlaps of energy bands of \( M_a \), one shall first sort the values of \( |E_1|, |E_2|, \) and \( |E_3| \). We first determine the sign of \( E_1, E_2, \) and \( E_3 \) in the \( v-\Delta \) plane. As shown in Fig. \ref{fig:A1} the \( v-\Delta \) plane is divided into four regions by three lines corresponding to \( E_1 = 0, E_2 = 0, \) and \( E_3 = 0 \), respectively, each labeled with sign combinations of \( E_1, E_2, \) and \( E_3 \). In region I, for instance, we have \( |E_1| < |E_3| \) and \( |E_1| < |E_2| \). When \( \Delta < v, |E_1| < |E_3| < |E_2| \), and the smallest energy overlaps for the extended states is \( E \in (|E_3|, |E_2|) \). Thus, the mobility edges reads
\[
E_c = 1 \pm (v - \Delta).
\]  
When \( \Delta > v, |E_1| < |E_2| < |E_3| \). The smallest energy overlaps of the extended states is given by \( E \in (|E_2|, |E_3|) \), and the expression of mobility edges is
\[
E_c = 1 \pm (v - \Delta).
\]  
For the regions II, III and IV, the smallest energy overlaps can be obtained by the same procedure. Table \ref{tab:AI} lists the smallest energy overlaps for different \( v \) and \( \Delta \) in four regions.

|    | \( v \) | \( \Delta \) | \( (v, \Delta) \) | the smallest energy overlaps |
|----|--------|---------|----------------|--------------------------|
| I  | \( v < 1 \) | \( \Delta < 1 \) | \( \Delta < v \) | \( (|E_3|, |E_2|) \) |
|    |        |         | \( \Delta > v \) | \( (|E_2|, |E_3|) \) |
| II | \( v > 1 \) | \( \Delta < 1 \) | \( \Delta < v \) | \( (|E_1|, |E_2|) \) |
|    |         |         | \( \Delta > v \) | \( (|E_2|, |E_1|) \) |
| III| \( v < 1 \) | \( \Delta > 1 \) | \( \Delta > v \) | \( (|E_2|, |E_3|) \) |
|    |         |         | \( \Delta < v \) | \( (|E_3|, |E_2|) \) |
|    | \( v > 1 \) | \( \Delta > 1 \) | \( \Delta > v \) | \( (|E_1|, |E_3|) \) |
|    |         |         | \( \Delta < v \) | \( (|E_3|, |E_1|) \) |
| IV | \( v < 1 \) | \( \Delta > 1 \) | \( \Delta > v \) | \( (|E_1|, |E_3|) \) |
|    | \( v > 1 \) |         | \( \Delta > 1 \) | \( (|E_3|, |E_1|) \) |
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Figure A2. (Color online) The distribution of the smallest energy overlaps for different $v$ and $\Delta$ in different regions. The shaded area indicates the regime of the system with mobility edges.

According to Table A1, there are three forms for mobility edges, i.e., $E_{c1} = |E_1|$, $E_{c2} = |E_2|$, and $E_{c3} = |E_3|$. In Fig A2, we show the region of the smallest energy overlaps of the extended states in the $v$-$\Delta$ plane for $M_a$. We find that the smallest energy overlaps of the extended states change from $(E_{c3}, E_{c2})$ to $(E_{c2}, E_{c3})$, when $\Delta$ goes beyond $v$ in the region I. At $E_{c2} = E_{c3}$, the extended state regime shrinks to a point with $\Delta = v$. In the regions II and III, the localization transition occurs at $\Delta = 1$ and $\Delta = v$. The shaded area in Fig A2 indicates the regime of the system where the mobility edges actually exist. Outside this region, all states are kept localized. As a conclusion, when $v > 1$, $E_c = v \pm (\Delta - 1)$, and the localization transition occurs at $\Delta = 1$. And for $0 < v < 1$, $E_c = 1 \pm (v - \Delta)$, and the localization transition occurs at $\Delta = v$.

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