The exact eigenstates of the neutrino mass matrix without CP-phase violation

ADRIAN PALCU

Department of Theoretical and Computational Physics - West University of Timișoara, V. Pârvan Ave. 4, RO - 300223 Romania

Abstract

In this paper we obtain the exact mass-eigenstates of the Majorana physical neutrinos. We start by taking into account a general $3 \times 3$ mass matrix without any CP-phase violation. It is then diagonalized by exactly solving an appropriate set of equations. The solution supplies straightforwardly the mass eigenvalues depending on the diagonal entries and mixing angles. Finally, the consequences of these analytical expressions are discussed assuming various phenomenological restrictions such as conserving the global lepton number $L = L_e - L_\mu - L_\tau$ and the $\mu - \tau$ interchange symmetry. The minimal absolute mass in the neutrino sector is also obtained since the two plausible scenarios invoked above are employed.

PACS numbers: 14.60.St; 14.60.Pq.

Key words: neutrino masses, mixing angles.

1 Introduction

Detecting neutrino oscillations [1] in atmospheric and solar neutrino fluxes (and then observing the same phenomenon in reactor and accelerator experiments) stands as a milestone in particle physics of the last decade. This compelling experimental evidence proves that a suitable theory must rely - among many other ingredients - on the fact that lepton flavor can not be conserved. Therefore, one must take into account that neutrinos mix (as quarks do). In other words they can oscillate into one another. That is: when the neutrino flavor is subject of a measurement in a neutrino flux, one obtaines different results if a macroscopic distance separates the detectors interacting with that flux. On the other hand, the Quantum Field Theory suggests that the neutrinos have to be represented by fermion massive fields, let them be of Majorana or Dirac type. At the same time, the unified theory design to describe the interactions must explain why neutrino masses are so tiny when compared to the charged lepton ones. A striking feature also arises since the data favor surprisingly large atmospheric and solar mixing angles, in contrast with the quark mixing pattern.

All these features make difficult the attempts to precisely establish the structure of the neutrino mass matrix able to fit the available data supplied by global analysis [2].
Specific textures have been proposed taking into consideration certain discrete symmetries that could govern the lepton families. Some of these symmetries are compatible with the elegant seesaw mechanism [3] designed to predict the observed small order of magnitude for the masses of physical neutrinos.

A different approach consists of advancing certain gauge models that give rise to specific Yukawa sectors able to supply a concrete mass matrix. Encouraged by the success of such a strategy in the case of a particular 3-3-1 gauge model, we propose here an analytical diagonalization of a general neutrino mass matrix just by taking into account arbitrary diagonal entries instead of the particular ones considered in the 3-3-1 model previous papers of the author [4] based on the general method of exactly solving gauge models with high symmetries [5].

The paper is organized as follows. In Section 2 the theoretical framework [6] of the neutrino mixing is briefly presented with the standard notations of the field. Section 3 deals with the exact neutrino mass eigenstates and eigenvalues obtained by solving a set of equations corresponding to the diagonalization of the general mass matrix. Certain phenomenological restrictions are introduced in Section 4 where the main results of the paper are presented. Last section is devoted to conclusions and comments on the obtained results.

2 The neutrino mass matrix

We start by assuming the neutrino mixing formula: \( \nu_{\alpha L}(x) = \sum_{i=1}^{3} U_{\alpha i} \nu_{i L}(x) \), where \( \alpha = e, \mu, \nu \) label the flavor space (flavor gauge eigenstates), while \( i = 1, 2, 3 \) denote the massive physical eigenstates. We consider throughout this paper the physical neutrinos as Majorana fields, i.e. \( \nu_{\alpha L}^c(x) = \nu_{\alpha L}(x) \). The mass term in the Yukawa sector of any gauge unified theory that generate Majorana neutrinos stands:

\[
-\mathcal{L}_Y = \frac{1}{2} \bar{\nu}_L M \nu_L^c + H.c
\]

with \( \nu_L = (\nu_e \nu_\mu \nu_\tau)^T_L \) where the superscript \( T \) denotes "transposed". The complex mixing matrix \( U \) that diagonalizes the mass matrix \( M \) in the manner \( U^* M U = m_{ij} \delta_{ij} \) has in the standard parametrization the form:

\[
U = \begin{pmatrix}
c_2 c_3 & s_2 c_3 e^{i\delta} & s_3 e^{-i\delta} \\
-s_2 c_1 - c_2 s_1 s_3 e^{i\delta} & c_1 c_2 - s_2 s_3 s_1 e^{i\delta} & c_3 s_1 \\
 s_2 s_1 - c_2 c_3 s_3 e^{i\delta} & -s_1 c_2 - s_2 s_3 c_1 e^{i\delta} & c_3 c_1
\end{pmatrix}
\] (2)

with natural substitutions: \( \sin \theta_{23} = s_1, \sin \theta_{12} = s_2, \sin \theta_{13} = s_3, \cos \theta_{23} = c_1, \cos \theta_{12} = c_2, \cos \theta_{13} = c_3 \) for the mixing angles, and \( \delta \) for the CP Dirac phase.

Let us assume the most general symmetric mass matrix for the neutrino sector as:

\[
M = \begin{pmatrix}
A & D & E \\
D & B & F \\
E & F & C
\end{pmatrix}
\] (3)

and try to obtain its eigenvalues. More specifically, this reduces to solving the set of equations:
we have taken into account from the very beginning observed by inspecting the shape of Eqs. (6) - but the proposed values for the other two (7).

\[ m_i = m_i (A, B, C, \theta_{12}, \theta_{23}, \theta_{13}) \]  

with \( i = 1, 2, 3 \). In these expressions \( m_i \) are analytical functions depending only on the mixing angles and the diagonal entries in the general mass matrix. At this stage, we do not make any assumption on the specific textures that can occur in the mass matrix when particular symmetries are added or ad hoc hypothesis are enforced.

The concrete forms of \( m_i \) remain to be determined by solving the following set of equations:

\[
\begin{align*}
  m_1 &= c_2^2 A + c_1^2 B + s_1^2 s_2^2 C - 2c_1 c_2 s_2 D + 2s_1 s_2 c_2 E - 2c_1 s_1 s_2^2 F \\
  0 &= c_2 s_2 A - c_1^2 c_2 s_2 B - s_1^2 s_2 c_2 C - (1 - 2s_2^2) s_1 E + 2s_1 s_2 c_2 F \\
  0 &= -c_1 s_1 s_2 B + c_1 s_1 s_2 C + c_2 s_1 D + c_1 c_2 E - (1 - 2s_1^2) s_2 F \\
  m_2 &= s_1^2 A + c_1^2 B + s_1^2 c_2^2 C + 2c_1 c_2 s_2 D - 2s_1 s_2 c_2 E - 2c_1 s_1 c_2^2 F \\
  0 &= s_1 c_1 c_2 B - s_1 c_1 c_2 C + s_1 s_2 D + c_1 s_2 E + (1 - 2s_1^2) c_2 F \\
  m_3 &= s_1^2 B + c_1^2 C + 2c_1 s_1 F
\end{align*}
\]

Since the actual data are not sensitive to any CP-phase violation in the lepton sector, we have taken into account from the very beginning \( \sin^2 \theta_{13} \approx 0 \) - as it can be easily observed by inspecting the shape of Eqs. (6) - but the proposed values for the other two mixing angles will be embedded only in the resulting formulas for the neutrino masses (7).

Furthermore, one obtains after a few manipulations the following analytical equations:

\[
\begin{align*}
  m_1 &= \frac{C \sin^2 \theta_{12} \sin^2 \theta_{23} - B \sin^2 \theta_{12} (1 + \sin^2 \theta_{23})}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} + \frac{A (1 - \sin^2 \theta_{12})}{(1 - 2 \sin^2 \theta_{12})}, \\
  m_2 &= \frac{B(1 - \sin^2 \theta_{12} - \sin^2 \theta_{23} + 3 \sin^2 \theta_{12} \sin^2 \theta_{23}) - C \sin^2 \theta_{23} (1 - \sin^2 \theta_{12})}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} \\
    &+ \frac{A \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})},
\end{align*}
\]
\[ m_3 = \frac{C \left( 1 - \sin^2 \theta_{23} \right) - B \sin^2 \theta_{23}}{1 - 2 \sin^2 \theta_{23}}. \] (7)

Assuming the available data concerning the mixing angles \([2]\) and the mass matrix diagonal entries, one can proceed to a detailed investigation of the resulting equations. They can reveal some interesting features, not only with respect to the type of the mass hierarchy (normal, inverted or degenerate) but also regarding the minimal absolute value in the neutrino mass spectrum and the mass splitting ratio (which imposes finally a certain relation between the diagonal entries).

Note that some of the masses could come out negative (for certain combinations of angles), but this is not an impediment since for any fermion field a \(\gamma_5\psi\) transformation can be performed at any time, without altering the physical content of the theory. As a result of this manipulation the mass sign changes or, equivalently, some neutrinos have opposite CP-phases.

Let us observe that the analytical mass equations (7) strictly impose \(\sin^2 \theta_{12} \neq 0.5\) and \(\sin^2 \theta_{12} \neq 0.5\), yet this does not forbid any closer approximation to the bi-maximal neutrino mixing. However, in the case of solar mixing angle this behaviour does not seem to be disturbing, since data confirm a large but not maximal mixing. Eventually, some radiative corrections can also be employed in order to get a more precise account for these angles, but let us observe that no particular mixing case is excluded \textit{ab initio}.

These equations do not contradict the trace condition which requires indeed a finite neutrino mass sum independently of the values of the mixing angles. As a matter of fact, if one sums the three masses in Eqs. (7), then the troublesome denominators disappear and the value required by Eq. (3) is recovered.

The particular shape of the analytical neutrino masses is due to both the choice of the \(\theta_{13} = 0\) and the nonzero diagonal entries in the mixing matrix. Any other choice - as one can observe in subsequent section - definitely leads to a different set of equations to be solved and, thus, to a different form of the solution.

4 Phenomenological restrictions

We will analyze - in the following subsections - some particular cases of the analytical solution presented above and emphasize the most appealing setting. We are guided in our choice by the need to obtain plausible predictions, and even a rough estimate regarding the absolute masses in the spectrum.

4.1 Conserving the global lepton number \(L = L_e - L_\mu - L_\tau\)

One of the most invoked symmetries in the lepton sector was the total lepton number \(L = L_e + L_\mu + L_\tau\), which still holds when one deals with Dirac neutrinos, while Majorana neutrinos violate this symmetry with two units. Therefore, it had to be abandoned in scenarios with Majorana neutrinos, as here is the case.

In the particular case of conserving the global lepton number \(L = L_e - L_\mu - L_\tau\)
the shape of the mass matrix (3) becomes:

\[
M = \begin{pmatrix} 0 & D & E \\ D & 0 & 0 \\ E & 0 & 0 \end{pmatrix}
\] (8)

The concrete forms of \(m_i\)'s remain to be computed by solving the modified set of equations:

\[
\begin{align*}
 m_1 &= -2c_1c_2s_2D + 2s_1c_2s_2E \\
 0 &= -c_1s_2^2D + c_1s_2^2D - s_1c_2^2E + s_1s_2^2E \\
 0 &= c_2s_1D + c_2c_1E \\
 0 &= c_1c_2^2D - s_1c_2^2E + s_1s_2^2E - s_1c_2^2E \\
 m_2 &= 2c_1c_2s_2D - 2s_1c_2s_2E \\
 0 &= s_1s_2D + c_1s_2E \\
 0 &= s_1c_2D + c_1c_2E \\
 0 &= s_1s_2D + c_1s_2E \\
 m_3 &= 0
\end{align*}
\] (9)

obtained straightforwardly from Eq. (6) if one puts \(A = B = C = F = 0\). The lines 3, 6, 7 and 8 in Eqs. (9) express the same condition, namely: \(D = -E \cot \theta_{23}\) giving rise to a \(\mu - \tau\) interchange symmetry if \(\cot \theta_{23} = -1\). The lines 2 and 4 in the set of equations (9) are fulfilled simultaneously if and only if \(\cos^2 \theta_{12} = \sin^2 \theta_{12}\) (maximal solar mixing angle).

Under these circumstances, taking into consideration the maximal atmospheric mixing angle too, the solution reads:

\[
|m_1| = |m_2| = \sqrt{2}D
\] (10)

\[
m_3 = 0
\] (11)

If the lepton number \(L = L_e - L_\mu - L_\tau\) is rigorously conserved the mass spectrum exhibits an inverted mass hierarchy with two degenerate nonzero masses and bimaximal mixing. The minimal neutrino mass is identical zero.

### 4.2 Mass matrix with \(\mu - \tau\) interchange symmetry

Many papers [8] develop scenarios with the \(\mu - \tau\) interchange symmetry. It seems more appealing, since the mass matrix of the neutrino sector

\[
M = \begin{pmatrix} A & D & D \\ D & B & F \\ D & F & B \end{pmatrix}
\] (12)

can predict interesting results.
We have to insert in Eqs. (7) the restrictive condition $B = C$ and simply express the resulting masses. They stand:

$$m_1 = -B \frac{\sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{23})(1 - 2 \sin^2 \theta_{12})} + A \frac{1 - \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})},$$

$$m_2 = B \frac{\sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{23})(1 - 2 \sin^2 \theta_{12})} + A \frac{\sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})},$$

$$m_3 = B. \tag{13}$$

Evidently, it is required a $\gamma^5$ transformation performed on the first neutrino field in order to get the sign change for its mass ($m_1$), if we assume that $A$ and $B$ have the same order of magnitude and a suitable close-to-maximal atmospheric mixing is invoked. In case $A \gg B$ and the atmospheric angle has a reasonable value, no mass could need a chiral transformation to get positive values.

The mass spectrum in the neutrino sector becomes:

$$|m_1| = \left[ B \frac{\sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{23})(1 - 2 \sin^2 \theta_{12})} - \frac{A(1 - \sin^2 \theta_{12})}{(1 - 2 \sin^2 \theta_{12})} \right],$$

$$m_2 = \left[ B \frac{\sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{23})(1 - 2 \sin^2 \theta_{12})} + \frac{A\sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})} \right],$$

$$m_3 = B. \tag{14}$$

The physical relevant magnitudes in neutrino oscillation experiments are the mass squared differences for solar and atmospheric neutrinos, defined as: $\Delta m_{12}^2 = m_2^2 - m_1^2$ and $\Delta m_{23}^2 = m_3^2 - m_2^2$ respectively. They result from the above expressions (Eqs. (13)):

$$\Delta m_{12}^2 \cong \frac{2AB \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})^2(1 - 2 \sin^2 \theta_{23})} \tag{15}$$

$$\Delta m_{23}^2 \cong \frac{B^2 \sin^4 \theta_{12}}{(1 - 2 \sin^2 \theta_{23})^2(1 - 2 \sin^2 \theta_{23})^2} \tag{16}$$

The mass splitting ratio defined as $r_\Delta = \Delta m_{12}^2/\Delta m_{23}^2$ yields in our scenario:

$$r_\Delta = 2 \frac{A}{B} \frac{(1 - 2 \sin^2 \theta_{12})}{\sin^2 \theta_{23}} \tag{17}$$

It is natural to presume that $A$ and $B$ have the same order of magnitude and consequently $A/B \simeq 1$. Under these circumstances $\sin^2 \theta_{23} \simeq 0.497$ in order to fulfill the phenomenological requirement $r_\Delta \simeq 0.033$.

Regarding the neutrino mass sum
\[ \sum_{i=1}^{3} m_i \simeq \frac{2AB \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12}) (1 - 2 \sin^2 \theta_{23})} \]  

(18)

This is experimentally restricted to: \( \sum_{i=1}^{3} m_i \sim 1 \text{eV} \), if we take into consideration the Troitsk [9] and Mainz [10] experiments. On the other hand, combining Eqs. (13) and (17) one obtains:

\[ \sum_{i=1}^{3} m_i = \frac{2 \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12}) (1 - 2 \sin^2 \theta_{23})} m_0 \]  

(19)

with minimal neutrino mass \( m_0 = m_3 \). This leads to

\[ m_0 = \frac{(1 - 2 \sin^2 \theta_{12}) (1 - 2 \sin^2 \theta_{23})}{2 \sin^2 \theta_{12}} \sum_{i=1}^{3} m_i \]  

(20)

Assuming the phenomenological values for the sum of the neutrino masses and the solar mixing angle \( \sin^2 \theta_{12} \simeq 0.31 \) one can analyze the behaviour of the \( m_0 \) in terms of the atmospheric mixing angle by studying the function:

\[ m_0 (\sin^2 \theta_{23}) = 0.613 \left(1 - 2 \sin^2 \theta_{23}\right) \sum_{i=1}^{3} m_i \]  

(21)

A plausible value (with the above considered values for mixing angles) can now be inferred: \( m_0 \simeq 0.0035 \text{eV} \). It is very close to the value obtained by the author (second reference in [4]) in a particular 3-3-1 model where the diagonal entries of the neutrino mass matrix were obtained in a specific manner, without resorting to any additional symmetry.

5 Concluding remarks

In this paper we have proved that the exact mass-eigenstates of a general neutrino mass matrix with no CP-phase violation can be exactly computed. The results accomodate the observed solar mixing angle and exclude the exact maximal mixing for the atmospheric angle, but do not forbid any closer approximation for such a setting. Therefore, they could be in good agreement with the data and can predict the correct mass splitting ratio. Our predictions also include the inverted mass hierarchy in the neutrino sector and the minimal absolute mass - \( m_0 \simeq 0.0035 \text{eV} \) - since the \( \mu - \tau \) interchange symmetry is employed. The global lepton symmetry \( L = L_e - L_\mu - L_\tau \) supplies two degenerate nonzero masses and one identical to zero, within the inverted hierarchy as well. However, the general case can be regarded as a perturbation that softly breaks this lepton symmetry by introducing small nonzero diagonal entries. The amazing feature seems to be the unexpected similarity of these general results with the ones obtained by the author in a particular 3-3-1 model with specific diagonal entries (proportional to charged lepton masses).
References

[1] Super-Kamiokande Collaboration (Y. Fukuda et al.), Phys. Rev. Lett. 81, 1562 (1998); Super-Kamiokande Collaboration (Y. Ashie et al.), Phys. Rev. Lett. 93, 101801 (2004); Super-Kamiokande Collaboration (Y. Ashie et al.), Phys. Rev. D 71, 112005 (2005); K2K Collaboration (E. Aliu et al.), Phys. Rev. Lett. 94, 081802 (2005); SNO Collaboration (Q. R. Ahmad et al.), Phys. Rev. Lett. 89, 011301 (2002); SNO Collaboration (B. Aharmim et al.), Phys. Rev. D 72, 052010 (2005); KamLAND Collaboration (K. Eguchi et al.), Phys. Rev. Lett. 90, 021802 (2003); KamLAND Collaboration (T. Araki et al.), Phys. Rev. Lett. 94, 081801 (2005).

[2] M. Maltoni, T. Schwetz, M. A. Torola and J. F. W. Valle, New J. Phys. 6, 122 (2004); A. Strumia and F. Vissani, Neutrino masses and mixings and..., arXiv: hep-ph/0606054; Particle Data Group (S. Eidelman et. al.), Phys. Lett. B 592, 1 (2004).

[3] M. Gell-Mann, P. Ramond and R. Slansky, in Supergravity, eds. F. van Neuwenhuizen and D. Freedman (North Holland, 1979), p.315; T. Yanagida, in Proc. of the Workshop on Unified Theory and the Baryon Number of the Universe, KEK, Japan (1979); R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).

[4] A. Palcu, Mod. Phys. Lett. A 21, 1203 (2006); A. Palcu, Mod. Phys. Lett. A 21, 2027 (2006); A. Palcu, Mod. Phys. Lett. A 21, 2591 (2006), A. Palcu, to be published in Mod. Phys. Lett. A 22 (arXiv: hep-ph/0701066).

[5] I. I. Cotăescu, Int. J. Mod. Phys. A 12, 1483 (1997).

[6] S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. 59, 671 (1987).

[7] S. T. Petcov, Phys. Lett. B 110, 245 (1982); R. Barbieri, L. Hall, D. Smith, A. Strumia and N.Weiner JHEP 12, 017 (1998); A. Joshipura and S. Rindani, Eur. Phys. J. C 14, 85 (2000); R. N. Mohapatra, A. Perez-Lorenzana and C. A. de S. Pires, Phys. Lett. B 474, 355 (2000); Q. Shafi and Z. Tavartkiladze, Phys. Lett. B 482, 145 (2000); L. Lavoura, Phys. Rev. D 62, 093011 (2000); W. Grimus and L. Lavoura, Phys. Rev. D 62, 093012 (2000); T. Kitabayashi and M. Yassue, Phys. Rev. D 63, 095002 (2001); R. N. Mohapatra, Phys. Rev. D 64, 091301 (2001); S. Babu and R. N. Mohapatra, Phys. Lett. B 532, 77 (2002); H. S. Goh, R. N. Mohapatra and S. P. Ng, Phys. Lett. B 542, 116 (2002); D. A. Dicus and H.-J. He and J. N. Ng, Phys. Lett. B 536, 83 (2002); G. K. Leontaris, J. Rizos and A. Psallidas, Phys. Lett. B 597, 182 (2004); W. Grimus and L. Lavoura, J. Phys. G 31, 683 (2005); G. Altarelli and R. Franceschini, JHEP 03, 047 (2006).

[8] T. Fukuyama and H. Nishiura, hep-ph/9702253; R. N. Mohapatra and S. Nussinov, Phys. Rev. D 60, 013002 (1999); E. Ma and M. Raidal, Phys.Rev. Lett. 87, 011802 (2001); C. S. Lam, Phys. Lett. B 507, 214 (2001); W. Grimus and L. Lavoura, Phys. Lett. B 572, 189 (2003); T. Kitabayashi and M. Yasue, Phys. Rev.
\[ D \textbf{67}, \text{015006 (2003); W. Grimus and L. Lavoura, J. Phys. G} \textbf{30}, 73 (2004); Y. Koide, \textit{Phys. Rev. D} \textbf{69}, 093001 (2004); A. Ghosal, \textit{Mod. Phys. Lett. A} \textbf{19}, 2579 (2004). \]

[9] V. M. Lobashev et al., \textit{Prog. Part. Nucl. Phys.} \textbf{48}, 123 (2002).

[10] C. Weinheimer et al., \textit{Nucl. Phys. Proc. Suppl.} \textbf{118}, 279 (2003); Ch. Krauss et al., \textit{Eur. Phys. J. C} \textbf{40}, 447 (2005).