Wave transmission through a layered piezoelectric/elastic phononic crystal with capacitors

Sergey I. Fomenko\(^1\), Mikhail V. Golub\(^1\) and Chuanzeng Zhang\(^2\)

\(^1\) Institute for Mathematics, Mechanics and Informatics, Kuban State University, 350040 Krasnodar, Russian Federation
\(^2\) Chair of Structural Mechanics, Department of Civil Engineering, University of Siegen, D-57068 Siegen, Germany
E-mail: sfom@yandex.ru; m_golub@inbox.ru; c.zhang@uni-siegen.de

Abstract. In this study, elastic wave propagation through a layered phononic crystal made of finite number of unit-cells composed of four elastic/piezoelectric layers and surface electrodes connected by an electrical circuit with a capacitor is considered. The numerical investigation based on the transfer matrix method of the influence of the condenser capacitance on the stop-bands is provided. Some critical values of capacitance providing stop-band transformation into pass-bands are revealed.

1. Introduction
Since piezoelectric materials can be controlled by using the electric field, dynamic behaviour of piezoelectric structures attracts a lot of attention \([1, 2, 3]\). At the same time, acoustic metamaterials (AMs) and phononic crystals (PnCs) are employed to manipulate wave propagation \([4, 5, 7]\). Therefore, it is natural to design structures made of AMs and PnCs incorporating piezoelectric parts and to investigate wave phenomena in them. Investigation of the active control of wave propagation through the periodical elastic structures with embedded piezoelectric layers and electrical circuits at them and energy harvesting using such systems are already started (see \([8, 9]\)). However many problems in this new field are not studied yet. For instance, wave propagation excited by inclined incidental waves, wave excitation by piezoelectric actuators on surfaces and internal interfaces of the PnCs, optimization of the structure of PnCs and electrical circuits connected to them, etc. The aim of this study is to investigate wave phenomena in layered piezoelectric PnCs and to estimate the possibility to control pass-bands and stop-bands using capacitors connected to the electrodes surrounding piezoelectric layers.

2. Mathematical model
Let us consider a three-dimensional layered PnC composed of \(N\) unit-cells between two elastic half-spaces. The unit-cell is composed of four layers: elastic layer \((A_1)\), electric isolator \((A_2)\), piezoelectric layer \((A_3)\) and the electric isolator \((A_4)\), see Figure 1. The piezoelectric layer is connected to a capacitor via two infinitely thin electrodes situated on its upper bottom surfaces. The thicknesses of each layer in the unit-cell are \(h_i\) \((i = 1, 4)\), while \(H\) is the total thickness of the unit-cell; the capacitance \(C\) \([F]\) of the capacitor is the same in each unit-cell. The wave motion in the PnC with the angular frequency \(\omega\) is excited by time-harmonic plane wave incoming from
lower half-space $x_3 < 0$). The angles $\theta_1$ and $\theta_2$ are accordingly azimuthal and radial incidence angles. The properties of the elastic materials ($A_1$, $A_2$ and $A_4$) are given by tensor of elastic constants $c_{ijmn}$ [Pa] and the mass density $\rho$ [kg/m$^3$]. The piezoelectric layer is described by $c_{ijmn}$, $\rho$, piezoelectric constants $e_{nij}$ [C/m$^2$] and dielectric constants $\epsilon_{in}$ [F/m].

![Image](image_url)

**Figure 1.** Geometry of the problem

The governing equations

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2},$$

should be supplied for piezoelectric material by the equation

$$D_{j,j} = 0.$$

Here $\sigma_{ij}$, $u_j$ and $D_j$ are the stress tensor, mechanical and electric displacement vector components respectively ($i,j,m,n \in \{1,2,3\}$). The constitutive equations are follows

$$\sigma_{ij} = c_{ijmn}s_{mn} - e_{nij}E_n, \quad D_i = e_{imn}\epsilon_{mn} + \epsilon_{in}E_n,$$

where $s_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m})$ and $E_n = -\varphi_n$ are the strain tensor and components of the electric field respectively; $\varphi$ is the electric potential. The boundary conditions at the internal interfaces $z = z_k$ are given by the continuity of the displacements and stresses

$$[u_i]_{z_k} = 0, \quad [\sigma_{3i}]_{z_k} = 0, \quad (i = 1, 2, 3).$$

Here $[f]_{z_k}$ means the jump of the function $f(z)$ at $z = z_k$.

Plane waves are studied and electroded surfaces are connected via the capacitor. Therefore, in addition to (3), the following boundary conditions for the electrical terms are to be satisfied at the interfaces between the piezoelectric layer (domain $[z_k, z_{k+1}]$) and the elastic isolators:

$$D_3(z_k) = D_3(z_{k+1}) = C\left(\varphi(z_{k+1}) - \varphi(z_k)\right).$$

The solution of (1)-(4) is based on the transfer-matrix method (see [7, 6, 9] for more details). In accordance with (4), the transfer-matrix of the piezoelectric layer is modified via keeping only mechanical terms in the generalized state vector, i.e. displacements, normal and tangential stresses. Therefore, the capacitance $C$ is considered as a control parameter in the transfer matrix. The displacements and stresses in the upper half-space can be written as

$$u_i = \sum_{n=1}^{3} u_{i,n} \lambda_n^{-N}, \quad \sigma_{3i} = \sum_{n=1}^{3} \sigma_{3i,n} \lambda_n^{-N}.$$
where \( \lambda_n \) are the eigenvalues of the unit-cell transfer matrix such as \( |\lambda_n| \geq 1 \). The semi-analytical formulae for the coefficients \( u_{i,n} \) and \( \sigma_{3i,n} \) were derived in [6], the latter provides numerically stable algorithm for computing wave-fields in considered structure. The eigenvalues \( \lambda_i \) are related also with the wavenumbers \( \zeta_i = H^{-1} \ln \lambda_i \) of Bloch waves propagating in infinite PnC [7].

In addition to the analysis of Bloch waves, the energy transmission and reflection coefficients are useful to characterize the stop-band and pass-bands for PnCs of finite number of unit cells. The energy transmission and reflection coefficients are defined as the ratio of the energy flow transferred by the incident wave: \( \kappa^\pm = \epsilon_3^+/\epsilon_3^0 \), \( \kappa^+ + \kappa^- = 1 \).

The expressions for Umov-Pointing vector can be found in [7].

3. Results and discussion

The numerical results shown in Figures 2–3 are calculated for unit-cell made of piezoelectric material PZT-5A (layer \( A_3 \)) and three elastic materials: aluminium (\( A_1 \)) and epoxy (\( A_2, A_4 \)). The half-spaces are assumed made of aluminium, \( h_1 = h_3 = 0.4 \ H \), \( h_2 = h_4 = 0.1 \ H \), while the number of unit-cells \( N = 32 \). Lame’s coefficients \( \lambda \) and \( \mu \) [GPa] of isotropic layers, mass densities \( \rho \), elastic \( C_{mn} \) [GPa], piezoelectric \( E_{ij} \) (in Voigt’s notation) and relative dielectric constants \( \varepsilon_{ij} = \varepsilon_{ij}/\varepsilon_0 \) (\( \varepsilon_0 = 8.85 \cdot 10^{-12} \) [F/m]) are the following: Aluminium (\( \rho = 2700 \), \( \lambda = 51.1, \mu = 26.3 \)); Epoxy (\( \rho = 1200 \), \( \lambda = 6.38, \mu = 1.61 \)); PZT-5A (\( \rho = 7500 \), \( C_{11} = C_{22} = 12.1 \), \( C_{33} = 11.1 \), \( C_{44} = C_{55} = 2.11 \), \( C_{66} = 2.28 \), \( C_{12} = C_{13} = C_{23} = 11.1 \), \( E_{15} = E_{24} = 12.3 \), \( E_{31} = E_{32} = -5.4 \), \( E_{33} = 15.8 \), \( \varepsilon_{11} = \varepsilon_{22} = 916 \), \( \varepsilon_{33} = 830 \). The results are presented in dimensionless form using normalization via \( \omega_0 = 2\pi v_0/H \) and \( C_0 = \varepsilon_{11}H \), where \( v_0 = (\mu/\rho)^{1/2} \) is the phase velocity of shear waves in the half-spaces.

\[ \kappa^\pm = \epsilon_3^+/\epsilon_3^0, \ \kappa^+ + \kappa^- = 1. \]

![Figure 2](https://example.com/figure2.png)

Figure 2. The energy transmission coefficient \( \kappa^+ \) for different capacitance \( C \), normal (subplot a, \( \theta_1 = \theta_2 = 0^\circ \)) and obliquely (subplot b, \( \theta_1 = 45^\circ, \theta_2 = 0^\circ \)) incident longitudinal waves

The plots of energy transmission coefficient \( \kappa^+(\omega) \) for the several discrete values of the capacitance \( C \) and normal incidence of the longitudinal plane waves are shown in Fig.2(a). The transmission for an obliquely incident longitudinal wave is depicted in Fig.2(b). The zero value of capacitance \( C \) corresponds to the absence of the capacitor or an open electric circuit. A certain discrepancy between \( \kappa^+(\omega) \) for small values of the capacitance \( (C < 2.8C_0) \)
is observed only at higher frequencies range (outside presented plots). A significant difference in the structure of stop-bands is observed in the lower frequency range at $\frac{C}{C_0} = 3$. An extra stop-band $0.25 < \omega/\omega_0 < 0.36$ occurs, but the latter also causes sufficient widening of second and third pass-bands.

Figure 3. The localization factor $\gamma(\omega, C)$ for normally incident longitudinal incident wave

More detailed analysis can be performed considering the localization factor ($\gamma = -\ln(\kappa^+/2NH)$) for various capacitances $C$ and frequencies $\omega$. Fig. 3 demonstrates surface $\gamma(\omega, C)$, where white zones show pass-bands, while coloured domains exhibit stop-bands. The critical values $C^*$ of the capacitance $C$ are visible in Fig. 3 as hyperbolic-like white trajectories. At these values, stop-bands are able to transform to pass-bands and vice versa. The effect of band transformation at $C^*$ is also observed for oblique incidence (see Fig.2b).

4. Conclusions
The method proposed in [7] based on the transfer matrix method is modified and applied to study wave propagation in a PnC with a unit-cell composed of the piezoelectric layer, three elastic layers and two electrodes connected via capacitor. The influence of the capacitors on the wave propagation and a structure of stop-bands is discussed.

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