CONVERGENCE ANALYSIS FOR THE PINNS
YULING JIAO*, YANMING LAI †, DINGWEI LI‡, XILIANG LU§, YANG WANG ¶, AND JERRY ZHIJIAN YANG∥

Abstract. In recent years, physical informed neural networks (PINNs) have been shown to be a powerful tool for solving PDEs empirically. However, numerical analysis of PINNs is still missing. In this paper, we prove the convergence rate to PINNs for the second order elliptic equations with Dirichlet boundary condition, by establishing the upper bounds on the number of training samples, depth and width of the deep neural networks to achieve desired accuracy. The error of PINNs is decomposed into approximation error and statistical error, where the approximation error is given in $C^2$ norm with ReLU$^3$ networks, the statistical error is estimated by Rademacher complexity. We derive the bound on the Rademacher complexity of the non-Lipschitz composition of gradient norm with ReLU$^3$ network, which is of immense independent interest.

Key words. PINNs, RelU$^3$ neural network, B-splines, Rademacher complexity, Chaining method, Pseudo dimension, Covering number.

1. Introduction. Classical numerical methods successfully for solving low-dimensional PDEs such as the finite element method [6, 7, 23, 30, 14] may encounter some difficulties in both of theoretical analysis and numerical implementation for high-dimensional PDEs. Motivated by the well-known fact that deep learning method for high-dimensional data analysis has been achieved great successful applications in discriminative, generative and reinforcement learning [12, 10, 28], solving high dimensional PDEs with deep neural networks becomes an extremely potential approach and has attracted much attentions [2, 29, 18, 24, 32, 33, 5, 11]. Due to the excellent approximation power of deep neural networks, several numerical scheme have been proposed to solve PDEs with deep neural network including the deep Ritz method (DRM) [32], physics-informed neural networks (PINNs) [24], and deep Galerkin method (DGM) [33]. Both DRM and DGM are applied to PDEs with equivalent variational form. PINNs is based on residual minimization for solving PDEs [2, 29, 18, 24], which is easy to extend to general PDEs [15, 16, 22, 21].

Despite there are great empirical achievements for the above mentioned deep PDEs solver, rigorous numerical analysis for the these methods is still on its way. [19, 13, 17, 9] study the convergence rate of DRM with two layer networks and deep networks. [26, 27, 20] provide convergence of PINNs without convergence rate. Therefore, two important issues which have not been addressed are: what is the influence of the topological structure of the networks, say the depth and width, in the quantitative error analysis? How to determine these hyper-parameters to achieve a desired convergence rate? In this paper, we study the convergence rate

*School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, P.R. China. (yulingjiaomath@whu.edu.cn)
†School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. China. (laiyiming@whu.edu.cn)
‡School of Mathematics and Statistics, Wuhan University, Wuchang District, Wuhan, China (lidingv@whu.edu.cn)
§School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, Wuchang District, Wuhan, China (xllv.math@whu.edu.cn)
¶Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong (yangwang@ust.hk)
∥School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, Wuchang District, Wuhan, China (zjyang.math@whu.edu.cn)
analysis of the PINNs with ReLU$^3$ networks and give firm answers to these questions. As far as we know, we establish the first nonasymptotic convergence rate on PINNs. Our contributions are summarized as follows.

1.1. Our contributions and main results.

- Via applying the method of average Taylor expansion and quasi-interpolation, we derive the approximation rate of B-splines in general Sobolev spaces $W^{s,p}$, which is of independent interest. Furthermore, we obtain the approximation results ReLU$^3$ network in $C^2$, see Theorem 4.9, i.e., $\forall u^* \in C^3$, For any $\epsilon > 0$, there exists a ReLU$^3$ network $u$ with depth $\lceil \log_2 d \rceil + 2$ and width $C(d, \| u^* \|_{C^3(\bar{\Omega})}) \left( \frac{1}{\epsilon} \right)^d$ such that
  \[ \| u^* - u \|_{C^2(\Omega)} \leq \epsilon. \]

- We establish a bound on the statistical error in PINNs with the tools of Pseudo dimension, especially we give an bound on the Rademacher complexity of functional of derivative of ReLU$^3$ network which is non-Lipschitz composition with ReLU$^3$ network, via calculating the Pseudo dimension of networks with ReLU, ReLU$^2$ and ReLU$^3$ activation functions, see Theorem 5.13. We prove that $\forall D, W \in \mathbb{N}^+$ and $\epsilon > 0$, if the number of training samples $n$ used in PINNs satisfying $n = C_d D^6 W^2 (D + \log W) \left( \frac{1}{\epsilon} \right)^{2+\delta}$, where $\delta$ is an arbitrarily small number, then the statistical error
  \[ \mathbb{E}_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \sup_{u \in \mathcal{P}} \left| L(u) - \tilde{L}(u) \right| \leq \epsilon. \]

- Based on the above bounds on approximation error and statistical error, we establish a nonasymptotic convergence rate of PINNs for second order elliptic equations with Dirichlet boundary condition, see Theorem 6.1, i.e., $\forall \epsilon > 0$ if we set the depth
  \[ D = \mathcal{O}(\lceil \log_2 d \rceil + 2), \]
  and width
  \[ W = \mathcal{O}\left( \left( \frac{1}{\epsilon} \right)^d \right) \]
  in the ReLU$^3$ network and set the number of training samples used in PINNs as $\mathcal{O}\left( \left( \frac{1}{\epsilon} \right)^{2d+4+\delta} \right)$, where $\delta$ is an arbitrarily small number, then
  \[ \mathbb{E}_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \left[ \left\| \tilde{u}_\phi - u^* \right\|_{H^2(\Omega)} \right] \leq \epsilon. \]

1.2. Structure of this paper. The paper is organized as follows. In Section 2 we briefly introduce neural network, the parameterized function class we use. In Section 3 we present the setting of the problem and introduce the error decomposition result for the PINNs. In Section 4 we show approximation error of ReLU$^3$ networks in Sobolev spaces with the aid of approximation results of B-splines. In Section 5 we gives bound of Rademacher complexity and then provide analysis of the statistical error. In Section 6 we derive our main result concerning
convergence rate of PINNs method. We give conclusion and short discussion in Section 7.

2. Neural Network. Let $\mathcal{D} \in \mathbb{N}^+$. A function $f : \mathbb{R}^d \to \mathbb{R}^{n \times n_{\ell}}$ implemented by a neural network is defined by

$$
\begin{align*}
    f_0(x) &= x, \\
    f_{\ell}(x) &= g_{\ell} \left( A_{\ell} f_{\ell-1} + b_{\ell} \right) = \left( \ell \right) \left( \left( A_{\ell} f_{\ell-1} + b_{\ell} \right) \right) & \text{for } \ell = 1, \ldots, \mathcal{D} - 1,
    f := f_{\mathcal{D}}(x) = A_{\mathcal{D}} f_{\mathcal{D}-1} + b_{\mathcal{D}},
\end{align*}
$$

where $A_{\ell} = \left( a_{ij}^{(\ell)} \right) \in \mathbb{R}^{n_{\ell} \times n_{\ell-1}}$, $b_{\ell} = \left( b_i^{(\ell)} \right) \in \mathbb{R}^{n_{\ell}}$ and $g_{\ell} : \mathbb{R}^{n_{\ell}} \to \mathbb{R}^{n_{\ell}}$, $\mathcal{D}$ is called the depth of the network and $W := \max \{ N_{\ell}, \ell = 0, \ldots, \mathcal{D} \}$ is called the width of the network. $\sum_{\ell=1}^{\mathcal{L}} n_{\ell}$ is called the number of units of $f$ and $\phi = \{ A_{\ell}, b_{\ell} \}$ are called the weight parameters. Let $\Phi$ be a set of activation functions with $\Phi = \{ \text{ReLU}, \text{ReLU}^2, \text{ReLU}^3 \}$ and $X$ be an normed space. We define the neural network function class

$$
\mathcal{N}(\mathcal{D}, W, \{ \| \cdot \|_X, B \}, \Phi) := \{ f : f \text{ can be implemented by a neural network with depth } \mathcal{D} \text{ and width } W, \| f \|_X \leq B, \text{ and } g^{(\ell)} \in \Phi \text{ for each } i \text{ and } \ell \}. 
$$

3. The PINNS Method and Error Decomposition. Let $\Omega$ be a convex bounded open set in $\mathbb{R}^d$ and assume that $\partial \Omega \in C^\infty$. Without loss of generality we assume that $\Omega \subset [0,1]^d$. We consider linear second order elliptic equation with Dirichlet boundary condition:

$$
\begin{align*}
    \left\{ \begin{array}{ll}
    - \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu = f & \text{in } \Omega \\
    eu = g & \text{on } \partial \Omega,
    \end{array} \right.
\end{align*}
$$

(3.1)

where $a_{ij} \in C(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $e \in C(\partial \Omega)$, $g \in L^2(\partial \Omega)$. We assume that the strictly elliptic condition holds, i.e., there exists a constant $\lambda \geq 0$ such that

$$
a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^d
$$

We further assume that 3.1 has an unique strong solution $u^* \in C^3(\bar{\Omega})$. For simplicity, define $\mathfrak{A} = \max_{i,j} \{ \| a_{ij} \|_{C(\bar{\Omega})} \}$, $\mathfrak{B} = \max_i \{ \| b_i \|_{L^\infty(\Omega)} \}$, $\mathfrak{c} = \| c \|_{L^\infty(\Omega)}$, $\mathfrak{E} = \| f \|_{L^2(\Omega)}$, $\mathfrak{G} = \| g \|_{L^2(\partial \Omega)}$, $\mathfrak{F} = \{ \mathfrak{A}, \mathfrak{B}, \mathfrak{c}, \mathfrak{E}, \mathfrak{G} \}$.

Instead of directly solving problem 3.1, the PINNs method consider a minimization problem:

$$
    u^* = \arg \min_u \mathcal{L}(u)
$$

where

$$
\mathcal{L}(u) := \int_{\Omega} \left( - \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu - f \right)^2 dx + \int_{\partial \Omega} (eu - g)^2 dx.
$$
Note that $\mathcal{L}$ can be equivalently written as

$$
\mathcal{L}(u) = |\Omega| \mathbb{E}_{x \sim U(\Omega)} \left( - \sum_{i,j=1}^{d} a_{ij}(X) u_{x_i x_j}(X) + \sum_{i=1}^{d} b_{i}(X) u_{x_i}(X) + c(X) u(X) - f(X) \right)^2
$$

(3.2)

$$
+ |\partial \Omega| \mathbb{E}_{y \sim U(\partial \Omega)} (e(Y) u(Y) - g(Y))^2,
$$

where $U(\Omega)$ and $U(\partial \Omega)$ are uniform distribution on $\Omega$ and $\partial \Omega$, respectively. We then introduce a discrete version of $\mathcal{L}$:

$$
\hat{\mathcal{L}}(u) = \frac{|\Omega|}{N} \sum_{k=1}^{N} \left( - \sum_{i,j=1}^{d} a_{ij}(X_k) u_{x_i x_j}(X_k) + \sum_{i=1}^{d} b_{i}(X_k) u_{x_i}(X_k) + c(X_k) u(X_k) - f(X_k) \right)^2
$$

$$
+ \frac{|\partial \Omega|}{M} \sum_{k=1}^{M} (e(Y_k) u(Y_k) - g(Y_k))^2
$$

(3.3)

where $\{X_k\}_{k=1}^{N}$ and $\{Y_k\}_{k=1}^{M}$ are i.i.d. random variables according to the uniform distribution $U(\Omega)$ on $\Omega$ and $U(\partial \Omega)$ on $\partial \Omega$, respectively. We now consider a minimization problem with respect to $\hat{\mathcal{L}}$:

$$
\hat{u}_\phi = \arg\min_{u \in \mathcal{P}} \hat{\mathcal{L}}(u_\phi)
$$

where $\mathcal{P}$ refers to the deep neural network function class parameterized by $\phi$. Finally, we call a (random) solver $\mathcal{A}$, say SGD, to minimize $\hat{\mathcal{L}}$ with respect to $\phi$ and denote the output of $\mathcal{A}$, say $u_{\phi, \mathcal{A}}$, as the final solution.

Before we state our error decomposition, we need an $L^p$ estimate related to the elliptic equation 3.1.

**Lemma 3.1.** For $u \in H^2(\Omega) \cap L^2(\partial \Omega)$,

$$
\|u\|_{H^2(\Omega)} \leq C \left\| - \sum_{i,j=1}^{d} a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_{i} u_{x_i} + cu \right\|_{H^{-2}(\Omega)} + C \|eu\|_{L^2(\partial \Omega)}.
$$

**Proof.** See [1]. \(\blacksquare\)

The following error decomposition enables us to decouple and bound the approximation error, statistical error and optimization error separately.

**Proposition 3.2.** Assume that $\mathcal{P} \subset H^2(\Omega) \cap C(\Omega)$, then

$$
\|u_{\phi, \mathcal{A}} - u^*\|_{H^2(\Omega)}^2
$$

$$
\leq C \inf_{u \in \mathcal{P}} \left[ 3 \max \{2d^2A^2, d^2B^2, C^2 \} \|u - u^*\|_{H^2(\Omega)}^2 + |\partial \Omega| C^2 \|u - u^*\|_{C(\Omega)}^2 \right]
$$

$$
+ C \sup_{u \in \mathcal{P}} \left\| \mathcal{L}(u) - \mathcal{L}(u_\phi) \right\|_{\mathcal{E}_{\text{stat}}} + C \left[ \mathcal{L}(u_{\phi, \mathcal{A}}) - \mathcal{L}(\hat{u}_\phi) \right].
$$
Proof. For any \( \bar{u} \in \mathcal{P} \), we have

\[
\mathcal{L}(u_{\phi, \lambda}) - \mathcal{L}(u^*)
= \mathcal{L}(u_{\phi, \lambda}) - \hat{\mathcal{E}}(u_{\phi, \lambda}) + \hat{\mathcal{E}}(u_{\phi, \lambda}) - \hat{\mathcal{E}}(\bar{u}_\phi) + \hat{\mathcal{E}}(\bar{u}_\phi) - \hat{\mathcal{E}}(\bar{u})
\]

\[
+ \hat{\mathcal{E}}(\bar{u}) - \mathcal{L}(\bar{u}) + \mathcal{L}(\bar{u}) - \mathcal{L}(u^*)
\]

\[
\leq [\mathcal{L}(\bar{u}) - \mathcal{L}(u^*)] + 2 \sup_{u \in \mathcal{P}} |\mathcal{L}(u) - \hat{\mathcal{E}}(u)| + \left[\hat{\mathcal{E}}(u_{\phi, \lambda}) - \hat{\mathcal{E}}(\bar{u}_\phi)\right],
\]

where the last step is due to the fact that \( \hat{\mathcal{E}}(\bar{u}_\phi) - \hat{\mathcal{E}}(\bar{u}) \leq 0 \). Since \( \bar{u} \) can be any element in \( \mathcal{P} \), we take the infimum of \( \bar{u} \) on both side of the above display:

\[
\mathcal{L}(u_{\phi, \lambda}) - \mathcal{L}(u^*) \leq \inf_{\bar{u} \in \mathcal{P}} [\mathcal{L}(\bar{u}) - \mathcal{L}(u^*)] + 2 \sup_{u \in \mathcal{P}} |\mathcal{L}(u) - \hat{\mathcal{E}}(u)| + \left[\hat{\mathcal{E}}(u_{\phi, \lambda}) - \hat{\mathcal{E}}(\bar{u}_\phi)\right].
\]

(3.4)

By the definition of Gâteaux derivative, we have for any direction \( v \), \( w \),

\[
\mathcal{L}'(u; v) = 2 \int_\Omega \left( - \sum_{i,j=1}^d a_{ij} u_{x_i x_j} + \sum_{i=1}^d b_i u_{x_i} + cu - f \right) \left( - \sum_{i,j=1}^d a_{ij} v_{x_i x_j} + \sum_{i=1}^d b_i v_{x_i} + cv \right) \, dx
\]

\[+ 2 \int_{\partial \Omega} (eu - g) \cdot ev \, dx \]

and

\[
\mathcal{L}''(u; v, w) = 2 \int_\Omega \left( - \sum_{i,j=1}^d a_{ij} v_{x_i x_j} + \sum_{i=1}^d b_i v_{x_i} + cv \right) \left( - \sum_{i,j=1}^d a_{ij} w_{x_i x_j} + \sum_{i=1}^d b_i w_{x_i} + cw \right) \, dx
\]

\[+ 2 \int_{\partial \Omega} ev \cdot ew \, dx \]

Since \( u^* \) is the minimizer of \( \mathcal{L}(\cdot) \), \( \mathcal{L}'(u^*; v) = 0 \). Hence by Taylor formula we have for any \( u \in \mathcal{P} \subset H^2(\Omega) \cap C(\Omega) \),

\[
\mathcal{L}(u) - \mathcal{L}(u^*) = \frac{1}{2} \mathcal{L}''(u^*; u - u^*, u - u^*)
\]

\[
= \int_\Omega \left( - \sum_{i,j=1}^d a_{ij} (u - u^*)_{x_i x_j} + \sum_{i=1}^d b_i (u - u^*)_{x_i} + c(u - u^*) \right)^2 \, dx + \int_{\partial \Omega} e^2(u - u^*)^2 \, dx
\]

\[
\leq 3 \int_\Omega \left[ \left( \sum_{i,j=1}^d a_{ij} (u - u^*)_{x_i x_j} \right)^2 + \left( \sum_{i=1}^d b_i (u - u^*)_{x_i} \right)^2 + (c(u - u^*))^2 \right] \, dx
\]

(3.5)

\[
+ \int_{\partial \Omega} e^2(u - u^*)^2 \, dx
\]

\[
\leq 3 \max\{2d^2 A^2, d^2 B^2, c^2\} \|u - u^*\|^2_{H^2(\Omega)} + |\partial \Omega| c^2 \|u - u^*\|^2_{C^1(\Omega)}.
\]

On the other hand, by Lemma 3.1 and applying the inequality \( \| \cdot \|_{H^\frac{3}{2}(\Omega)} \leq C \| \cdot \|_{L^2(\Omega)} \), we
have
\[ \mathcal{L}(u) - \mathcal{L}(u^*) = \int_{\Omega} \left( -\sum_{i,j=1}^{d} a_{ij}(u - u^*) x_i x_j + \sum_{i=1}^{d} b_i (u - u^*) x_i + c(u - u^*) \right)^2 dx + \int_{\partial \Omega} e^2 (u - u^*)^2 dx \geq C \|u - u^*\|^2_{H^2(\Omega)} \]

Combining (3.4)-(3.6) yields the result. \[ \blacksquare \]

The approximation error \( E_{\text{app}} \) describes the expressive power of the parameterized function class
\[ \mathcal{P} = \mathcal{N}^{3} := \{ \text{networks with ReLU}^3 \text{ activations} \} \]
in \( H^2(\Omega) \) and \( C(\bar{\Omega}) \) norm, which corresponds to the approximation error in FEM known as the Céa’s lemma [7]. The statistical error \( E_{\text{sta}} \) is caused by the Monte Carlo discretization of \( \mathcal{L}(\cdot) \) defined in 3.2 with \( \widehat{\mathcal{L}}(\cdot) \) in 3.3. While, the optimization error \( E_{\text{opt}} \) indicates the performance of the solver \( A \) we utilized. In contrast, this error is corresponds to the error of solving the linear systems in FEM. In this paper we focus on the first two errors, i.e, considering the scenario of perfect training with \( E_{\text{opt}} = 0 \).

4. Approximation Error. We first derive approximation rate of B-splines in Sobolev spaces and then extended this result to the approximation with ReLU^3 networks.

4.1. Property of B-splines. We denote by \( \pi_\ell \) the uniform partition of \([0, 1]\):
\[ \pi_\ell : t^{(\ell)}_0 = 0 < t^{(\ell)}_1 < \cdots < t^{(\ell)}_{\ell-1} < t^{(\ell)}_\ell = 1 \]
with \( t^{(\ell)}_i = i/\ell (0 \leq i \leq \ell) \). For \( k \in \mathbb{N}_+ \), we consider an extended partition \( \widehat{\pi}_\ell \):
\[ \widehat{\pi}_\ell : t^{(\ell)}_{-k+1} = \cdots = t^{(\ell)}_0 = 0 < t^{(\ell)}_1 < \cdots < t^{(\ell)}_{\ell-1} < t^{(\ell)}_\ell = \cdots = t^{(\ell)}_{\ell+k-1} = 1 \]
with \( t^{(\ell)}_i = i/\ell (0 \leq i \leq \ell) \). Then the univariate B-spline of order \( k \) with respect to partition \( \widehat{\pi}_\ell \) is defined by
\[ N^{(k)}_{\ell,i}(x) = (-1)^k \left( t^{(\ell)}_{i+k} - t^{(\ell)}_i \right) \cdot \left[ t^{(\ell)}_{i}, \ldots, t^{(\ell)}_{i+k} \right](x - t)^{k-1}, \quad x \in [0, 1], \quad i \in I \] (4.1)
where \( I = \{-k+1, -k+2, \ldots, \ell - 1\} \). By the definition and properties of divided difference \( \left[ t^{(\ell)}_{i}, \ldots, t^{(\ell)}_{i+k} \right] \) (see, for example, [25]) and some calculations, B-splines can also be equivalently
written as:

\[
N_{\ell,i}^{(k)}(x) = \begin{cases} 
\frac{j^{k-1}}{(k-1)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \frac{x - \frac{j}{\ell}}{\ell} \right)^{k-1} + , & 0 \leq i \leq \ell - k + 1 \\
\sum_{j=0}^{k-1} a_{ij} \left( x - \frac{j}{\ell} + \sum_{n=1}^{k-2} b_{in} x^n + b_{i0}, & -k + 1 \leq i < 0 \\
\sum_{j=\ell-k+1}^{\ell} c_{ij} \left( x - \frac{j}{\ell} \right)^{k-1} + , & \ell - k + 1 < i \leq \ell - 1 
\end{cases}
\] (4.2)

where \(\{a_{ij}\}, \{b_{in}\}, \{c_{ij}\}\) are some constants. We must mention that the values of (4.1) and (4.2) at \(x = 1\) is different for \(\ell - k + 1 < i \leq \ell - 1\). In fact, for \(\ell - k + 1 < i \leq \ell - 1\), (4.1) is discontinuous at \(x = 1\) and (4.2) is its continuous modification. In the following when we mention \(N_{\ell,i}^{(k)}\), we always refer to the continuous version (4.2).

The multivariate B-spline is defined by the product of univariate B-splines:

\[
N_{\ell,i}^{(k)}(x) = \prod_{j=1}^{d} N_{\ell_{ij},i}^{(k)}(x_j), \quad i \in I^d.
\]

**Proposition 4.1** (Theorem 4.9 and Corollary 4.10, [25]).

(i) \(N_{\ell,i}^{(k)} = 0\) for \(x \notin (t_i, t_{i+k})\) and \(N_{\ell,i}^{(k)} > 0\) for \(x \in (t_i, t_{i+k})\).

(ii) \(\{N_{\ell,i}^{(k)}\}_{i \in I}\) forms a basis of space \(\{f \in C^{k-2}([0,1]): f \text{ is } k-1 \text{ degree piecewise polynomial with respect to partition } \pi_i\}\).

**Proposition 4.2** (Theorem 4.22, [25]). For \(r \in \mathbb{N}_0\),

\[
\left| D^r N_{\ell,i}^{(k)} \right| \leq 2^r \ell^r.
\]

**Proposition 4.3** (Theorem 12.5 [25]). There exists a set of linear functionals \(\{\lambda_i\}\) mapping \(L^1(\Omega)\) to \(\mathbb{R}\) such that

\[
\lambda_i(N_{\ell,i}^{(k)}) = \delta_{ij} \text{ and } |\lambda_i(f)| \leq (2k + 1)^d g^{d(k-1)} \left( \frac{k}{\ell} \right)^{-d/p} \|f\|_{L^p(\Omega)}
\]

for any \(f \in L^p(\Omega)\) with \(p \in [1, \infty]\).

For \(i \in I^d\), let \(\Omega_i = \{x: x_j \in [t_{ij}, t_{ij+k}]\}, \quad j = 1, \cdots, d\) and define the quasi-interpolation of \(f\) by

\[
Qf = \sum_{i \in I^d} \lambda_i(f) N_{\ell,i}^{(k)}.
\] (4.3)

\(Q\) is called the interpolation operator.

**Proposition 4.4.** Let \(p(x_1, \ldots, x_d)\) be a multivariate polynomial with order \(k\) with respect to each variable, then, \(Qp = p\).

**Proof.** By definition, there exists \(\{c_{\alpha}\} \subset \mathbb{R}\) such that \(p(x_1, \cdots, x_d) = \sum_{\alpha=(0, \cdots, 0)} \alpha^{(k-1)} c_{\alpha} x_1^{\alpha_1} \cdots x_d^{\alpha_d}\).
By (ii) of Proposition 4.1, there exists \( \{e_{i,j,\alpha}\} \subset \mathbb{R} \) such that \( x_j^{\alpha} = \sum_{i \in I} e_{i,j,\alpha} N_{\ell,i}^{(k)}(x_j) \). Hence
\[
p(x_1, \ldots, x_d) = \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} c_\alpha \left( \prod_{j=1}^{d} \sum_{i \in I} e_{i,j,\alpha} N_{\ell,i}^{(k)}(x_j) \right)
\]
\[
= \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} c_\alpha \left( \sum_{i \in I} e_{i,\alpha} N_{\ell,i}^{(k)}(x) \right)
\]
\[
= \sum_{i \in I} \left( \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} c_\alpha e_{i,\alpha} \right) N_{\ell,i}^{(k)}(x).
\]

Then
\[
Q p = \sum_{i \in I} \left( \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} c_\alpha e_{i,\alpha} \right) Q N_{\ell,i}^{(k)} = \sum_{i \in I} \left( \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} c_\alpha e_{i,\alpha} \right) \sum_{j \in I} \lambda_j (N_{\ell,j}^{(k)}) N_{\ell,j}^{(k)}
\]
\[
= \sum_{i \in I} \left( \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} \delta_{ij} N_{\ell,j}^{(k)} \right) = \sum_{i \in I} \left( \sum_{\alpha=(0, \ldots, 0)}^{(k-1, \ldots, k-1)} c_\alpha e_{i,\alpha} \right) N_{\ell,i}^{(k)} = p
\]
where in the third step we apply Proposition 4.3. \( \square \)

For functions in Sobolev spaces, the normal Taylor polynomials may not exist. However, the averaged Taylor polynomials defined in the following always exists. The readers are referred to [6] for more discussions.

**Definition 4.5.** Suppose \( f \in W^{s,p}(\Omega) \). \( B \) is the ball centered at \( x_0 \in \Omega \) with radius \( \rho \) and \( B \subset \subset \Omega \). The corresponding Taylor polynomial of order \( s+1 \) of \( f \) averaged over \( B \) is defined as
\[
Q_B^{s+1} f(x) = \int_B T_y^{s+1} f(x) \phi(y) dy
\]
where \( T_y^{s+1} f(x) = \sum_{|\alpha| \leq s+1} \frac{1}{\alpha!} D^\alpha f(y)(x-y)^\alpha \) and \( \phi \) is a cut-off function supported in \( B \).

**Proposition 4.6 (Bramble-Hilbert).** Let \( B \) be a ball in \( \Omega \) which is star-shaped with respect to \( B \) and such that its radius \( \rho > \frac{1}{4} \sup\{\rho : \Omega \text{ is star-shaped with respect to a ball of radius } \rho\} \).
Let \( Q_B^s f \) be the Taylor polynomial of order \( s \) of \( f \) averaged over \( B \) with \( f \in W^{s,p}(\Omega) \) and \( p \geq 1 \). Then
\[
|f - Q_B^s f|_{W^{r,p}(\Omega)} \leq C(s, d, \gamma) h^{s-r} |f|_{W^{s,p}(\Omega)}, \quad r = 0, 1, \ldots, s,
\]
where \( h = \text{diam}(\Omega) \).

**Proof.** See [6]. \( \square \)

### 4.2. Approximation of ReLU³ Networks

Now we are able to derive the approximation results of B-splines in Sobolev spaces. Approximation rate of B-splines in \( H^s \) and \( W^{s,\infty} \) has been studied [8, 25]. However, the approximation results in general Sobolev spaces \( W^{s,p} \) established in next Theorem 4.7 is new, which is of independent interest.
Theorem 4.7. Let \( f \in W^{s,p}(\Omega) \) with \( p \in [1,\infty) \) and \( Qf \) be defined by (4.3) with \( k \geq s \), then,

\[
|f - Qf|_{W^{s,p}(\Omega)} \leq C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{s-r} |f|_{W^{s,p}(\Omega)}, \quad r = 0, 1, \ldots, s.
\]

Proof. We only show the case \( p \in [1,\infty) \) and the case \( p = \infty \) can be shown similarly. Let \( f \in W^{s,p}(\Omega) \) and \( r = (r_1, \ldots, r_d) \) with \( |r| = r \). We first deal with the local integral. Denoting

\[
I_{i,k} := \prod_{j=1}^{d} \{i_j - k + 1, i_j - k + 2, \ldots, i_j + k - 1\} \quad \text{and} \quad \widetilde{\Omega}_i := \bigcup_{j \in I_{i,k}} \Omega_j.
\]

By Proposition 4.6, there exists a ball \( B_1 \subset \widetilde{\Omega}_i \) such that

\[
\int_{\Omega_i} |D^r f - T_{B_1}^s f|^p dx \leq C(s, p, d) \left( \frac{3k}{\ell} \right)^{(s-r)p} |f|^p_{W^{s,p}(\Omega_i)}, \quad (4.4)
\]

Hence

\[
\int_{\Omega_i} |D^r (f - Qf)|^p dx \leq \int_{\Omega_i} \left[ |D^r (f - T_{B_1}^s f)| + |D^r (T_{B_1}^s f - Qf)| \right]^p dx \leq 2^{p-1} \int_{\Omega_i} |D^r (f - T_{B_1}^s f)|^p dx + 2^{p-1} \int_{\Omega_i} |D^r (T_{B_1}^s f - Qf)|^p dx \leq C(s, p, d)(3kh)^{(s-r)p}|f|^p_{W^{s,p}(\widetilde{\Omega}_i)} + 2^{p-1} \int_{\Omega_i} |D^r Q(T_{B_1}^s f - f)|^p dx \quad (4.5)
\]

where in the final step we apply (4.4) and Proposition 4.4. Now we deal with the integral related to operator \( Q \). For \( x \in \Omega_i \), we have

\[
|D^r Q(T_{B_1}^s f - f)| = \left| \sum_{j \in I_{i,k}} \lambda_j(T_{B_1}^s f - f)D^r N_{1,j}^{(k)} \right| = \sum_{j \in I_{i,k}} \lambda_j(T_{B_1}^s f - f)D^r N_{1,j}^{(k)} \leq 2^{r_\ell} \sum_{j \in I_{i,k}} |\lambda_j(T_{B_1}^s f - f)| \leq 2^{r_\ell} (2k + 1)^d g^{d(k-1)-d/p+\ell d/p} \sum_{j \in I_{i,k}} \|f - T_{B_1}^s f\|_{L^p(\Omega_j)} \leq 2^{r_\ell} (2k + 1)^d g^{d(k-1)-d/p+\ell d/p} |I_{i,k}| \|f - T_{B_1}^s f\|_{L^p(\widetilde{\Omega}_i)} \leq 2^{r_\ell} (4k^2 - 1)^d g^{d(k-1)-d/p+\ell d/p} \|f - T_{B_1}^s f\|_{L^p(\widetilde{\Omega}_i)} \leq 2^{r_\ell} (4k^2 - 1)^d g^{d(k-1)-d/p} \left( \frac{1}{\ell} \right)^{s-r-d/p} (3k)^s |f|^s_{W^{s,p}(\Omega_i)}
\]
where in the second step we apply (i) of Proposition 4.1, in the fourth step we use Proposition 4.2, in the fifth step we use Proposition 4.3 and the final step uses (4.4). Hence
\[
\int_{\Omega} |D^r Q(T_{B_i} f - f)|^p d\mathbf{x} \leq C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{(s-r)p-d} |f|_{W^{s-r,p}(\bar{\Omega})}^{p} |\Omega_i| \\
= C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{(s-r)p} |f|_{W^{s-r,p}(\bar{\Omega})}^{p}.
\]

Combining (4.5) and (4.6) and summing over all the subregions, we have the global estimate:
\[
\int_{\Omega} |D^r (f - Qf)|^p d\mathbf{x} = \sum_{i \in I} \int_{\Omega_i} |D^r (f - Qf)|^p d\mathbf{x} \leq C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{(s-r)p} \sum_{i \in I} |f|_{W^{s-r,p}(\bar{\Omega})}^{p} \\
\leq C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{(s-r)p} k^d |f|_{W^{s-r,p}(\Omega)}^{p} = C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{(s-r)p} |f|_{W^{s-r,p}(\Omega)}^{p}.
\]
The above display implies,
\[
|f - Qf|_{W^{s-r,p}(\Omega)}^{p} = \sum_{|\mathbf{r}|=r} \int_{\Omega} |D^r (f - Qf)|^p d\mathbf{x} \leq C(k, s, r, p, d) \left( \frac{1}{\ell} \right)^{(s-r)p} |f|_{W^{s-r,p}(\Omega)}^{p}.
\]

When \( f \in C^r(\Omega) \), from the case \( p = \infty \) in Theorem 4.7 we immediately obtain the following corollary.

**Corollary 4.8.** Let \( f \in C^s(\Omega) \) with \( s \geq 0 \) and \( Qf \) be defined by (4.3) with \( k \geq s \), there holds
\[
|f - Qf|_{C^r(\Omega)} \leq C(k, s, r, d) \left( \frac{1}{\ell} \right)^{s-r} |f|_{C^r(\Omega)}, \quad r = 0, 1, \ldots, s.
\]

Based on the above approximation results of B-splines in Theorem 4.7 and Corollary 4.8, we obtain the following Theorem on the approximation of ReLU\(^3\) networks.

**Theorem 4.9.** For any \( \epsilon > 0 \), there exists a ReLU\(^3\) network \( u \) with depth \( \lceil \log_2 d \rceil + 2 \) and width \( C(d, \| u^* \|_{C^3(\bar{\Omega})}) \left( \frac{1}{\ell} \right)^d \) such that
\[
\| u^* - u \|_{C^2(\Omega)} \leq \epsilon.
\]

**Proof.** By Theorem 4.7 and Corollary 4.8, we know that there exists \( \{ \alpha_i \}_{i \in I^d} \subset \mathbb{R} \) such that
\[
\left\| u^* - \sum_{i \in I^d} \alpha_i N_{i,1}^{(4)} \right\|_{C^2(\Omega)} \leq C(d) \| u^* \|_{C^3(\bar{\Omega})} \frac{1}{\ell}, \tag{4.7}
\]
Next we show that \( \{ N_{i,1}^{(4)} \} \) can be implemented by ReLU\(^3\) networks \( N^3 \) without any error.
Firstly for \( x \geq 0 \), \( x \) and \( x^2 \) can be implemented by ReLU\(^3\) exactly.

\[
x^2 = -\frac{1}{6} \sigma(x + 2) - 4\sigma(x + 1) + 3\sigma(x) - 4 \\
x = -\frac{1}{12} \sigma(x + 3) - 5\sigma(x + 2) + 7\sigma(x + 1) - 3\sigma(x) + 6
\]  \hspace{1cm} (4.8)

where \( \sigma(x) \) refers to the ReLU\(^3\) activation function \( \sigma(x) = \begin{cases} x^3, & x \geq 0 \\ 0, & \text{else} \end{cases} \). Hence by Proposition 4.2 we know that each univariate B-spline \( N_{t,i}^{(4)} \) can be realized by a ReLU\(^3\) network with depth \( 2 \) and width no more than 11. Secondly multiplication on \([0, x) \times [0, y)\] can be implemented by ReLU\(^3\) exactly. In fact, for \( x, y \geq 0 \), by (4.8) we have

\[
x \cdot y = \frac{1}{2}[(x + y)^2 - x^2 - y^2] \\
\quad = -\frac{1}{12} \sigma(x + y + 2) - 4\sigma(x + y + 1) + 3\sigma(x + y) \\
\quad \quad - \sigma(x + 2) + 4\sigma(x + 1) - \sigma(y + 2) + 4\sigma(y + 1) - 3\sigma(y) + 4.
\]

Noticing \( N_{t,i}^{(4)} \geq 0 \) by (4.1), we conclude that each multivariate B-spline \( N_{t,i}^{(4)} \) can be implemented by a ReLU\(^3\) network with depth \( \log_2 d + 2 \) and width \( \max\{11d, \frac{2}{7}d\} = 11d \). Hence by (4.7) we conclude that there exists a ReLU\(^3\) network \( u \) with depth \( \log_2 d + 2 \) and width \( C(d, \| u^* \|_{C^2(\Omega)}) \left( \frac{1}{d} \right)^d \) such that

\[
\| u^* - u \|_{C^2(\Omega)} \leq \epsilon.
\]

\( \square \)

5. Statistical Error. In this section we bound the statistical error with \( \mathcal{P} = N^3 \). To this end, we need to bound the Rademacher complexity of the non-Lipschitz composition of partial derivative and ReLU\(^3\) network. We believe that the technique we used here is helpful for bounding the statistical errors for other deep PDEs solvers where the main difficulty is bounding the Rademacher complexity of non-Lipschitz composition induced by the derivative operator.

**Lemma 5.1.**

\[
\mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in A^3} \left| L(u) - \tilde{L}(u) \right| \leq \sum_{j=1}^{13} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in A^3} \left| L_j(u) - \tilde{L}_j(u) \right|
\]

where

\[
L_1(u) = \| \mathbb{E}_{X \sim \mathcal{U}(\Omega)} \left( \sum_{i,j=1}^d a_{ij}(X) u_{x_i x_j} (X) \right) \|^2, \quad L_2(u) = \| \mathbb{E}_{X \sim \mathcal{U}(\Omega)} \left( \sum_{i=1}^d b_i(X) u_{x_i} (X) \right) \|^2, \\
L_3(u) = \| \mathbb{E}_{X \sim \mathcal{U}(\Omega)} \left( c(X) u(X) \right) \|^2, \quad L_4(u) = \| \mathbb{E}_{X \sim \mathcal{U}(\Omega)} f(X) \|^2, \\
L_5(u) = -2 \| \mathbb{E}_{X \sim \mathcal{U}(\Omega)} \left( \sum_{i,j=1}^d a_{ij}(X) u_{x_i x_j} (X) \right) \left( \sum_{i=1}^d b_i(X) u_{x_i} (X) \right) ,
\]
\[ \mathcal{L}_6(u) = -2|\Omega|E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^d a_{ij}(X) u_{x_i, x_j}(X) \right) \cdot c(X)u(X), \]

\[ \mathcal{L}_7(u) = 2|\Omega|E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^d a_{ij}(X) u_{x_i, x_j}(X) \right) \cdot f(X), \]

\[ \mathcal{L}_8(u) = 2|\Omega|E_{X \sim U(\Omega)} \left( \sum_{i=1}^d b_i(X) u_{x_i}(X) \right) \cdot c(X)u(X), \]

\[ \mathcal{L}_9(u) = -2|\Omega|E_{X \sim U(\Omega)} \left( \sum_{i=1}^d b_i(X) u_{x_i}(X) \right) \cdot f(X), \]

\[ \mathcal{L}_{10}(u) = -2|\Omega|E_{X \sim U(\Omega)} c(X)u(X)f(X), \quad \mathcal{L}_{11}(u) = |\partial \Omega|E_{Y \sim U(\partial \Omega)} \left( e(Y)u(Y) \right)^2, \]

\[ \mathcal{L}_{12}(u) = |\partial \Omega|E_{Y \sim U(\partial \Omega)} \left( g(Y) \right)^2, \quad \mathcal{L}_{13}(u) = -2|\partial \Omega|E_{Y \sim U(\partial \Omega)} e(Y)u(Y)g(Y), \]

and \( \hat{\mathcal{L}}_j(u) \) is the discrete sample version of \( \mathcal{L}_j(u) \) by replacing expectation with sample average \( j = 1, \ldots, 13 \).

**Proof.** Direct consequence of triangle inequality. \( \blacksquare \)

### 5.1. Rademacher complexity, Covering Number and Pseudo-dimension

By the technique of symmetrization, we can bound the difference between continuous loss \( \mathcal{L}_i \) and empirical loss \( \hat{\mathcal{L}}_i \) via Rademacher complexity.

**Definition 5.2.** The Rademacher complexity of a set \( A \subseteq \mathbb{R}^N \) is defined as

\[ \mathcal{R}(A) = \mathbb{E}_{\{\sigma_k\}_{k=1}^N} \left[ \sup_{a \in A} \frac{1}{N} \sum_{k=1}^N \sigma_k a_k \right], \]

where, \( \{\sigma_k\}_{k=1}^N \) are \( N \) i.i.d Rademacher variables with \( \mathbb{P}(\sigma_k = 1) = \mathbb{P}(\sigma_k = -1) = \frac{1}{2} \). The Rademacher complexity of function class \( \mathcal{F} \) associate with random sample \( \{X_k\}_{k=1}^N \) is defined as

\[ \mathcal{R}(\mathcal{F}) = \mathbb{E}_{\{X_k, \sigma_k\}_{k=1}^N} \left[ \sup_{u \in \mathcal{F}} \frac{1}{N} \sum_{k=1}^N \sigma_k u(X_k) \right]. \]

For Rademacher complexity, we have following structural result.

**Lemma 5.3.** Assume that \( w : \Omega \to \mathbb{R} \) and \( |w(x)| \leq B \) for all \( x \in \Omega \), then for any function class \( \mathcal{F} \), there holds

\[ \mathcal{R}(w(x)\mathcal{F}) \leq B \mathcal{R}(\mathcal{F}), \]

where \( w(x)\mathcal{F} = \{ \bar{u} : \bar{u}(x) = w(x)u(x), u \in \mathcal{F} \} \).
Proof.

$$
\mathfrak{N}(w(x) \cdot \mathcal{F}) = \frac{1}{N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \sup_{u \in \mathcal{F}} \sum_{k=1}^N \sigma_k w(X_k) u(X_k)
$$

$$
= \frac{1}{2N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \mathbb{E}_{(\sigma_k)_{k=2}^N} \sup_{u \in \mathcal{F}} \left[ w(X_1) u(X_1) + \sum_{k=2}^N \sigma_k w(X_k) u(X_k) \right]
$$

$$
+ \frac{1}{2N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \mathbb{E}_{(\sigma_k)_{k=2}^N} \sup_{u \in \mathcal{F}} \left[ -w(X_1) u(X_1) + \sum_{k=2}^N \sigma_k w(X_k) u(X_k) \right]
$$

$$
= \frac{1}{2N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \mathbb{E}_{(\sigma_k)_{k=2}^N} \sup_{u, u' \in \mathcal{F}} \left[ w(X_1)[u(X_1) - u'(X_1)] + \sum_{k=2}^N \sigma_k w(X_k) u(X_k) + \sum_{k=2}^N \sigma_k w(X_k) u'(X_k) \right]
$$

$$
\leq \frac{1}{2N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \mathbb{E}_{(\sigma_k)_{k=2}^N} \sup_{u, u' \in \mathcal{F}} \left[ B[u(X_1) - u'(X_1)] + \sum_{k=2}^N \sigma_k w(X_k) u(X_k) + \sum_{k=2}^N \sigma_k w(X_k) u'(X_k) \right]
$$

$$
= \frac{1}{2N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \mathbb{E}_{(\sigma_k)_{k=2}^N} \sup_{u, u' \in \mathcal{F}} \left[ B[u(X_1) - u'(X_1)] + \sum_{k=2}^N \sigma_k w(X_k) u(X_k) + \sum_{k=2}^N \sigma_k w(X_k) u'(X_k) \right]
$$

$$
= \frac{1}{N} \mathbb{E}_{(x_k, \sigma_k)_{k=1}^N} \sup_{u \in \mathcal{F}} \sum_{k=1}^N \sigma_k u(X_k) = \mathfrak{M}(\mathcal{F})
$$

Lemma 5.4.

$$
\mathbb{E}_{(x_k)_{k=1}^N} \sup_{u \in \mathcal{N}^3} \left| \mathcal{L}_j(u) - \mathcal{L}_j(u) \right| \leq C_d \mathfrak{M}(\mathcal{F}_j), \quad j = 1, \ldots, 13,
$$

where

$$
\mathcal{F}_1 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \text{ and } 1 \leq i, j, j' \leq d \text{ s.t. } f(x) = u_{x_i x_j}(x) u_{x_{i'} x_{j'}}(x) \},
$$

$$
\mathcal{F}_2 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \text{ and } 1 \leq i, i' \leq d \text{ s.t. } f(x) = u_{x_{i}}(x) u_{x_{i'}}(x) \},
$$

$$
\mathcal{F}_3 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \text{ s.t. } f(x) = u(x)^2 \}, \quad \mathcal{F}_4 = \{ f : \Omega \to \mathbb{R} \mid -1, 0, 1 \},
$$

$$
\mathcal{F}_5 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \text{ and } 1 \leq i, j, j' \leq d \text{ s.t. } f(x) = u_{x_{i} x_{j}}(x) u_{x_{i'} x_{j'}}(x) \},
$$

$$
\mathcal{F}_6 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \text{ and } 1 \leq i, j \leq d \text{ s.t. } f(x) = u_{x_{i} x_{j}}(x) u(x) \},
$$

and 13
\[ \mathcal{F}_7 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \quad \text{and} \quad 1 \leq i, j \leq d \quad \text{s.t.} \quad f(x) = u_{x,i}(x) \} \]

\[ \mathcal{F}_8 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \quad \text{and} \quad 1 \leq i \leq d \quad \text{s.t.} \quad f(x) = u_{x,i}(x)u(x) \} \]

\[ \mathcal{F}_9 = \{ f : \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3 \quad \text{and} \quad 1 \leq i \leq d \quad \text{s.t.} \quad f(x) = u_{x,i}(x) \}, \]

\[ \mathcal{F}_{10} = \mathcal{N}^3, \quad \mathcal{F}_{11} = \{ f : \partial \Omega \to \mathbb{R} \mid \exists u \in \mathcal{N}^3|_{\partial \Omega} \quad \text{s.t.} \quad f(x) = u(x)^2 \}, \]

\[ \mathcal{F}_{12} = \{ f : \partial \Omega \to \mathbb{R} \mid -1, 0, 1 \}, \quad \mathcal{F}_{13} = \mathcal{N}^3|_{\partial \Omega}, \]

with \( \mathcal{N}^3|_{\partial \Omega} = \{ f : \partial \Omega \to \mathbb{R} \mid \exists f \in \mathcal{N}^3 \quad \text{s.t.} \quad f = \bar{f}|_{\partial \Omega} \}. \]

**Proof.** We only give the proof of \( E_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{N}^3} |\mathcal{L}_1(u) - \widetilde{\mathcal{L}}_1(u)| \leq 4|\Omega|^2 d^4 \mathbb{R}(\mathcal{F}_1) \) since other inequalities can be shown similarly. We take \( \{\tilde{X}_k\}_{k=1}^N \) as an independent copy of \( \{X_k\}_{k=1}^N \), then

\[
|\mathcal{L}_1(u) - \widetilde{\mathcal{L}}_1(u)| = |\Omega| \left| E_{X \sim U(\Omega)} \left( \sum_{i,j=1}^d a_{ij}(X)u_{x,i,j}(X) \right)^2 \right| - \frac{1}{N} \sum_{k=1}^N \left( \sum_{i,j=1}^d a_{ij}(X_k)u_{x,i,j}(X_k) \right)^2 \right| 
\]

\[
= |\Omega| \left| E_{\{\tilde{X}_k\}_{k=1}^N} \frac{1}{N} \sum_{k=1}^N \left( \sum_{i,j=1}^d a_{ij}(\tilde{X}_k)u_{x,i,j}(\tilde{X}_k) \right)^2 \right| - \frac{1}{N} \sum_{k=1}^N \left( \sum_{i,j=1}^d a_{ij}(X_k)u_{x,i,j}(X_k) \right)^2 \right| 
\]

\[
\leq \frac{2|\Omega|}{N} E_{\{X_k\}_{k=1}^N} \sum_{i,i',j,j'=1}^d \sum_{k=1}^N a_{ij}(X_k)a_{i'j'}(\tilde{X}_k)u_{x,i,j}(X_k)u_{x,i',j'}(\tilde{X}_k) 
\]

\[- a_{ij}(X_k)a_{i'j'}(X_k)u_{x,i,j}(X_k)u_{x,i',j'}(X_k)]. \]
Hence

\[
\mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \left| \hat{\ell}_1(u) - \tilde{\ell}_1(u) \right|
\]

\[
\leq \frac{2|\Omega|}{N} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \mathbb{E}_{\{X_k\}_{k=1}^N} \sum_{i, i', j, j'=1}^d \left| a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) - a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) \right|
\]

\[
\leq \frac{2|\Omega|}{N} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \mathbb{E}_{\{X_k\}_{k=1}^N} \sum_{i, i', j, j'=1}^d \left| a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) - a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) \right|
\]

\[
\leq \frac{2|\Omega|}{N} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \mathbb{E}_{\{X_k\}_{k=1}^N} \sum_{i, i', j, j'=1}^d \left| a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) - a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) \right|
\]

\[
\leq \frac{2|\Omega|}{N} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \mathbb{E}_{\{X_k\}_{k=1}^N} \sum_{i, i', j, j'=1}^d \left| a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) - a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) \right|
\]

\[
= \frac{4|\Omega|^2}{N} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \sum_{i, i', j, j'=1}^d \left| a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) - a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) \right|
\]

\[
\leq \frac{4|\Omega|^2}{N} \mathbb{E}_{\{X_k\}_{k=1}^N} \sup_{u \in \mathcal{A}^3} \sum_{i, i', j, j'=1}^d \left| a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) - a_{ij}(X_k) a_{i'j'}(X_k) u_{x_{i}x_{j}}(X_k) \right|
\]

where in the second and eighth step we apply Jensen’s inequality and Lemma 5.3, respectively. The third and seventh step are due to the facts that the insertion of Rademacher variables doesn’t change the distribution and that the supremum can be achieved when the summation is positive, respectively. \( \square \)

Next we give a upper bound of Rademacher complexity in terms of the covering number of the corresponding function class.

**Definition 5.5.** Suppose that \( W \subset \mathbb{R}^n \). For any \( \epsilon > 0 \), let \( V \subset \mathbb{R}^n \) be an \( \epsilon \)-cover of \( W \) with respect to the distance \( d_\infty \), that is, for any \( w \in W \), there exists a \( v \in V \) such that \( d_\infty(w, v) < \epsilon \), where \( d_\infty \) is defined by \( d_\infty(u, v) := \|u - v\|_\infty \). The covering number \( C(\epsilon, W, d_\infty) \) is defined to be the minimum cardinality among all \( \epsilon \)-cover of \( W \) with respect to the distance \( d_\infty \).

**Definition 5.6.** Suppose that \( \mathcal{F} \) is a class of functions from \( \Omega \) to \( \mathbb{R} \). Given \( n \) sample
\( \mathbf{Z}_n = (Z_1, Z_2, \cdots, Z_n) \in \Omega^n, \mathcal{F}\mid \mathbf{Z}_n \subset \mathbb{R}^n \) is defined by
\[
\mathcal{F}\mid \mathbf{Z}_n = \{(u(Z_1), u(Z_2), \cdots, u(Z_n)) : u \in \mathcal{N}^3\}.
\]
The uniform covering number \( C_{\infty}(\varepsilon, \mathcal{F}, n) \) is defined by
\[
C_{\infty}(\varepsilon, \mathcal{F}, n) = \max_{\mathbf{Z}_n \in \Omega^n} C(\varepsilon, \mathcal{F}\mid \mathbf{Z}_n, d_{\infty}).
\]

**Lemma 5.7.** Let \( \mathcal{F} \) be a class of functions from \( \Omega \) to \( \mathbb{R} \) such that \( 0 \in \mathcal{F} \) and the diameter of \( \mathcal{F} \) is less than \( B \), i.e., \( \|u\|_{L^\infty(\Omega)} \leq B, \forall u \in \mathcal{F} \). Then
\[
\mathcal{R}(\mathcal{F}) \leq \inf_{0 < \delta < B} \left( 4\delta + \frac{12}{\sqrt{N}} \int_{\mathcal{B}} \sqrt{\log(2C_{\infty}(\varepsilon, \mathcal{F}, N))} d\varepsilon \right).
\]

**Proof.** The proof is based on the chaining method, see [31]. \( \Box \)

By Lemma 5.7, we have to bound the covering number, which can be upper bounded via Pseudo-dimension [3].

**Definition 5.8.** Let \( \mathcal{F} \) be a class of functions from \( X \) to \( \mathbb{R} \). Suppose that \( S = \{x_1, x_2, \cdots, x_n\} \subset X \). We say that \( S \) is pseudo-shattered by \( \mathcal{F} \) if there exists \( y_1, y_2, \cdots, y_n \) such that for any \( b \in \{0, 1\}^n \), there exists a \( u \in \mathcal{F} \) satisfying
\[
\text{sign}(u(x_i) - y_i) = b_i, \quad i = 1, 2, \ldots, n
\]
and we say that \( \{y_i\}_{i=1}^n \) witnesses the shattering. The pseudo-dimension of \( \mathcal{F} \), denoted as \( \text{Pdim}(\mathcal{F}) \), is defined to be the maximum cardinality among all sets pseudo-shattered by \( \mathcal{F} \).

The following proposition showing a relation between uniform covering number and pseudo-dimension.

**Proposition 5.9 (Theorem 12.2 [3]).** Let \( \mathcal{F} \) be a class of real functions from a domain \( X \) to the bounded interval \([0, B]\). Let \( \varepsilon > 0 \). Then
\[
C_{\infty}(\varepsilon, \mathcal{F}, n) \leq \sum_{i=1}^{\text{Pdim}(\mathcal{F})} \binom{n}{i} \left( \frac{B}{\varepsilon} \right)^i,
\]
which is less than \( \left( \frac{eB}{\varepsilon \cdot \text{Pdim}(\mathcal{F})} \right)^{\text{Pdim}(\mathcal{F})} \) for \( n \geq \text{Pdim}(\mathcal{F}) \).

**5.2. Bound on Statistical error.** By Lemma 5.1, 5.4, 5.7 and Proposition 5.9, we can bound the statistical error via bounding the pseudo-dimension of \( \mathcal{F}_i, i = 1, \ldots, 13 \). To this end, we show that \( \{\mathcal{F}_i\} \) are subsets of some neural network classes and then bound the pseudo-dimension of associate neural network classes.

**Proposition 5.10.** Let \( u \) be a function implemented by a ReLU\(^2\)–ReLU\(^3\) (ReLU\(^3\)) network with depth \( D \) and width \( W \). Then \( D_iu(i = 1, \cdots, d) \) can be implemented by a ReLU – ReLU\(^2\) – ReLU\(^3\) (ReLU\(^2\) – ReLU\(^3\)) network with depth \( D + 2 \) and width \( (D + 2)W \). Moreover, the neural networks implementing \( \{D_iu\}_{i=1}^d \) have the same architecture.
Before we present rigorous proof, we give some intuitions of the above results by Figure 5.1 and 5.2. The former is a ReLU³ network with 5 layers and the latter is its derivative (with respect to one variable).

Fig. 5.1: An example of ReLU³ neural network

Fig. 5.2: Derivative of neural network in Figure 5.1.

Proof. For activation function \( \rho \) in each unit, we denote \( \tilde{\rho} \) as its derivative, i.e., \( \tilde{\rho}(x) = \rho'(x) \).
We then have

\[ \tilde{\rho}(x) = \begin{cases} 
2\text{ReLU}, & \rho = \text{ReLU}^2 \\
3\text{ReLU}^2, & \rho = \text{ReLU}^3 
\end{cases} \]

Let \( 1 \leq i \leq d \). We deal with the first two layers in details and apply induction for layers \( k \geq 3 \) since there are a little bit difference for the first two layer. For the first layer, we have for any \( q = 1, 2, \ldots, n_1 \)

\[ D_iu_q^{(1)} = D_i\rho_q^{(1)} \left( \sum_{j=1}^{d} a_{qj}^{(1)} x_j + b_q^{(1)} \right) = \tilde{\rho}_q^{(1)} \left( \sum_{j=1}^{d} a_{qj}^{(1)} x_j + b_q^{(1)} \right) \cdot a_{qi}^{(1)} \]

Hence \( D_iu_q^{(1)} \) can be implemented by a ReLU − ReLU\(^2\) − ReLU\(^3\) network with depth 2 and width 1. For the second layer,

\[ D_iu_q^{(2)} = D_i\rho_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) = \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \cdot \sum_{j=1}^{n_1} a_{qj}^{(2)} D_iu_j^{(1)}. \]

Since \( \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \) and \( \sum_{j=1}^{n_1} a_{qj}^{(2)} D_iu_j^{(1)} \) can be implemented by two ReLU − ReLU\(^2\) − ReLU\(^3\) subnetworks, respectively, and the multiplication can also be implemented by

\[
(x \cdot y) = \frac{1}{4} \left[ (x + y)^2 - (x - y)^2 \right] = \frac{1}{4} \left[ \text{ReLU}^2(x + y) + \text{ReLU}^2(-x - y) - \text{ReLU}^2(x - y) - \text{ReLU}^2(-x + y) \right], \tag{5.1}
\]

we conclude that \( D_iu_q^{(2)} \) can be implemented by a ReLU − ReLU\(^2\) − ReLU\(^3\) network. We have

\[ \mathcal{D} \left( \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \right) = 3W \left( \tilde{\rho}_q^{(2)} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} u_j^{(1)} + b_q^{(2)} \right) \right) \leq W \]

and

\[ \mathcal{D} \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} D_iu_j^{(1)} \right) = 2W \left( \sum_{j=1}^{n_1} a_{qj}^{(2)} D_iu_j^{(1)} \right) \leq W. \]

Thus \( \mathcal{D} \left( D_iu_q^{(2)} \right) = 4W \left( D_iu_q^{(2)} \right) \leq \max\{2W, 4\} \). Now we apply induction for layers \( k \geq 3 \). For the third layer, \( D_iu_q^{(3)} = D_i\rho_q^{(3)} \left( \sum_{j=1}^{n_2} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) = \tilde{\rho}_q^{(3)} \left( \sum_{j=1}^{n_2} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right), \)

\( \sum_{j=1}^{n_2} a_{qj}^{(3)} D_iu_j^{(2)} \). Since \( \mathcal{D} \left( \tilde{\rho}_q^{(3)} \left( \sum_{j=1}^{n_2} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) \right) = 4W \left( \tilde{\rho}_q^{(3)} \left( \sum_{j=1}^{n_2} a_{qj}^{(3)} u_j^{(2)} + b_q^{(3)} \right) \right) \leq W \) and

\[ \mathcal{D} \left( \sum_{j=1}^{n_2} a_{qj}^{(3)} D_iu_j^{(2)} \right) = 4W \left( \sum_{j=1}^{n_2} a_{qj}^{(3)} D_iu_j^{(2)} \right) \leq \max\{2W, 4W\} = 4W, \]

we conclude that \( D_iu_q^{(3)} \) can be implemented by a ReLU − ReLU\(^2\) − ReLU\(^3\) network and
\[ \mathcal{D}\left( D, u_q^{(3)} \right) = 5, \ \mathcal{W}\left( D, u_q^{(3)} \right) \leq \max\{5\mathcal{W}, 4\} = 5\mathcal{W}. \]

We assume that \( D, u_q^{(k)} (q = 1, 2, \ldots, n_k) \) can be implemented by a ReLU-ReLU^2 network and \( \mathcal{D}\left( D, u_q^{(k)} \right) = k + 2, \ \mathcal{W}\left( D, u_q^{(k)} \right) \leq (k + 2)\mathcal{W} \). For the \((k + 1)\)-th layer, \( D, u_q^{(k+1)} = D, \rho_q^{(k+1)} \left( \sum_{j=1}^{n_k} a_q^{(k+1)} u_j^{(k)} + \tilde{b}_q^{(k+1)} \right) = \rho_q^{(k+1)} \left( \sum_{j=1}^{n_k} a_q^{(k+1)} u_j^{(k)} + \tilde{b}_q^{(k+1)} \right) \cdot \sum_{j=1}^{n_k} a_q^{(k+1)} D, u_j^{(k)} . \)

Since \( \mathcal{D}\left( D, u_q^{(k+1)} \right) = k + 2, \ \mathcal{W}\left( D, u_q^{(k+1)} \right) \leq \max\{(k + 2)\mathcal{W}, 4\mathcal{W}\} = (k + 2)\mathcal{W} \), we conclude that \( D, u_q^{(k+1)} \) can be implemented by a ReLU – ReLU^2 – ReLU^3 network and \( \mathcal{D}\left( D, u_q^{(k+1)} \right) = k + 3, \ \mathcal{W}\left( D, u_q^{(k+1)} \right) \leq \max\{(k + 3)\mathcal{W}, 4\} = (k + 3)\mathcal{W} \). Hence we derive that \( D, u = D, u_1^{\mathcal{D}} \) can be implemented by a ReLU – ReLU^2 – ReLU^3 network and \( \mathcal{D}\left( D, u \right) = \mathcal{D} + 2, \ \mathcal{W}\left( D, u \right) \leq (\mathcal{D} + 2)\mathcal{W} \). And through our argument, we know that the neural networks implementing \( \{ D, u \}_{i=1}^{\mathcal{D}} \) have the same architecture.

We now present the bound for pseudo-dimension of \( \mathcal{N}(\mathcal{D}, \mathcal{W}, \{ \text{ReLU, ReLU}^2, \text{ReLU}^3 \}) \) with \( \mathcal{D}, \mathcal{W} \in \mathbb{N}^+ \). We need the following Lemma.

**Lemma 5.11.** Let \( p_1, \ldots, p_m \) be polynomials with \( n \) variables of degree at most \( d \). If \( n \leq m \), then

\[ \left| \{(\text{sign}(p_1(x)), \ldots, \text{sign}(p_m(x))) : x \in \mathbb{R}^n\} \right| \leq 2 \left( \frac{2emd}{n} \right)^n. \]

**Proof.** See Theorem 8.3 in [3]. \( \square \)

**Proposition 5.12.** For \( \mathcal{D}, \mathcal{W} \in \mathbb{N}^+ \),

\[ \text{Pdim}(\mathcal{N}(\mathcal{D}, \mathcal{W}, \{ \text{ReLU, ReLU}^2, \text{ReLU}^3 \})) = \mathcal{O}(\mathcal{D}^2\mathcal{W}^2(\mathcal{D} + \log \mathcal{W})). \]

**Proof.** The argument follows from the proof of Theorem 6 in [4]. The result stated here is somewhat stronger than Theorem 6 in [4] since VCdim(sign(\( \mathcal{F} \))) \leq \text{Pdim}(\( \mathcal{F} \)) for any function class \( \mathcal{F} \). We consider a new set of functions

\[ \tilde{\mathcal{N}} = \{ \tilde{u}(x, y) = \text{sign}(u(x) - y) : u \in \mathcal{N}(\mathcal{D}, \mathcal{W}, \{ \text{ReLU, ReLU}^2, \text{ReLU}^3 \}) \}. \]

It is clear that \( \text{Pdim}(\mathcal{N}(\mathcal{D}, \mathcal{W}, \{ \text{ReLU, ReLU}^2, \text{ReLU}^3 \})) \leq \text{VCdim}(\tilde{\mathcal{N}}) \). We now bound the VC-dimension of \( \tilde{\mathcal{N}} \). Denoting \( \mathcal{M} \) as the total number of parameters (weights and biases) in the neural network implementing functions in \( \mathcal{N} \), in our case we want to derive the uniform bound for

\[ K_{\{x_i, y_i\}}(m) := \left| \{(\text{sign}(f(x_1, a) - y_1), \ldots, \text{sign}(u(x_m, a) - y_m)) : a \in \mathbb{R}^\mathcal{M}\} \right| \]

over all \( \{x_i\}_{i=1}^{m} \subset X \) and \( \{y_i\}_{i=1}^{m} \subset \mathbb{R} \). Actually the maximum of \( K_{\{x_i, y_i\}}(m) \) over all \( \{x_i\}_{i=1}^{m} \subset X \) and \( \{y_i\}_{i=1}^{m} \subset \mathbb{R} \) is the growth function \( G_N(m) \). In order to apply Lemma 5.11, we partition the parameter space \( \mathbb{R}^\mathcal{M} \) into several subsets to ensure that in each subset \( u(x_i, a) - y_i \) is a polynomial with respect to \( a \) without any breakpoints. In fact, our partition is exactly the
same as the partition in [4]. Denote the partition as \( \{P_1, P_2, \cdots, P_N\} \) with some integer \( N \) satisfying

\[
N \leq \prod_{i=1}^{D-1} 2 \left( \frac{2emk_i(1+(i-1)3^{i-1})}{M_i} \right)^{M_i}
\]

(5.2)

where \( k_i \) and \( M_i \) denotes the number of units at the \( i \)th layer and the total number of parameters at the inputs to units in all the layers up to layer \( i \) of the neural network implementing functions in \( \mathcal{N} \), respectively. See [4] for the construction of the partition. Obviously we have

\[
K_{\{x_i\},\{y_i\}}(m) \leq \sum_{i=1}^{N} |\{(\text{sign}(u(x_1, a) - y_1), \cdots, \text{sign}(u(x_m, a) - y_m)) : a \in P_i\}|
\]

(5.3)

Note that \( u(x, a) - y \) is a polynomial with respect to \( a \) with degree the same as the degree of \( u(x, a) \), which is equal to \( 1 + (D-1)3^{D-1} \) as shown in [4]. Hence by Lemma 5.11, we have

\[
|\{(\text{sign}(u(x_1, a) - y_1), \cdots, \text{sign}(u(x_m, a) - y_m)) : a \in P_i\}|
\leq 2 \left( \frac{2em(1+(D-1)3^{D-1})}{M_D} \right)^{M_D} .
\]

(5.4)

Combining (5.2), (5.3) and (5.4) yields

\[
K_{\{x_i\},\{y_i\}}(m) \leq \prod_{i=1}^{D-1} 2 \left( \frac{2emk_i(1+(i-1)3^{i-1})}{M_i} \right)^{M_i} .
\]

We then have

\[
G_\mathcal{K}(m) \leq \prod_{i=1}^{D} 2 \left( \frac{2emk_i(1+(i-1)3^{i-1})}{M_i} \right)^{M_i} ,
\]

since the maximum of \( K_{\{x_i\},\{y_i\}}(m) \) over all \( \{x_i\}_{i=1}^{m} \subset X \) and \( \{y_i\}_{i=1}^{m} \subset \mathbb{R} \) is the growth function \( G_\mathcal{K}(m) \). Doing some algebras as that of the proof of Theorem 6 in [4], we obtain

\[
Pdim(\mathcal{N}(\mathcal{D}, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\})) \leq O \left( D^{2}W^{2} \log U + D^{3}W^{2} \right)
= O \left( D^{2}W^{2} (D + \log W) \right)
\]

where \( U \) refers to the number of units of the neural network implementing functions in \( \mathcal{N}(\mathcal{D}, W, \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\}) \).

\[ \square \]

With the above preparations, we are able to derive our result on the statistical error.

**Theorem 5.13.** Let \( \mathcal{D}, W \in \mathbb{N}^+, \mathcal{B} \in \mathbb{R}^+ \). For any \( \epsilon > 0 \), if the number of sample

\[
N, M = C(d, \Omega, 3, B)D^6W^2(D + \log W) \left( \frac{1}{\epsilon} \right)^{2+\delta}
\]

20
where \( \delta \) is an arbitrarily small number then we have

\[
\mathbb{E}(X_k)_{k=1}^N \sup_{u \in \mathcal{P}} |\mathcal{L}(u) - \hat{\mathcal{L}}(u)| \leq \epsilon, 
\]

where \( \mathcal{P} = \mathcal{N}(\mathcal{D}, \mathcal{W}, \{\| \cdot \|_{C^2(\Omega)}, \mathcal{B}\}, \{\text{ReLU}^3\}) \).

**Proof.** We need the following Lemma. Recall that \( \Phi = \{\text{ReLU}, \text{ReLU}^2, \text{ReLU}^3\} \).

**Lemma 5.14.**

\[
\begin{align*}
\mathcal{F}_1 \subset \mathcal{N}_1 &:= \mathcal{N}(\mathcal{D} + 5, 2(\mathcal{D} + 2)(\mathcal{D} + 4)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}^2\}, \Phi) \\
\mathcal{F}_2 \subset \mathcal{N}_2 &:= \mathcal{N}(\mathcal{D} + 3, 2(\mathcal{D} + 2)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}^2\}, \Phi) \\
\mathcal{F}_3 \subset \mathcal{N}_3 &:= \mathcal{N}(\mathcal{D} + 1, \mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}^2\}, \Phi) \\
\mathcal{F}_4 \subset \mathcal{N}_4 &:= \mathcal{N}(\mathcal{D} + 5, (\mathcal{D} + 2)(\mathcal{D} + 5)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}^2\}, \Phi) \\
\mathcal{F}_5 \subset \mathcal{N}_5 &:= \mathcal{N}(\mathcal{D} + 3, (\mathcal{D} + 3)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}^2\}, \Phi) \\
\mathcal{F}_6 \subset \mathcal{N}_6 &:= \mathcal{N}(\mathcal{D} + 4, 2(\mathcal{D} + 2)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}\}, \Phi) \\
\mathcal{F}_7 \subset \mathcal{N}_7 &:= \mathcal{N}(\mathcal{D} + 3, (\mathcal{D} + 3)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}\}, \Phi) \\
\mathcal{F}_8 \subset \mathcal{N}_8 &:= \mathcal{N}(\mathcal{D} + 2, (\mathcal{D} + 2)\mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}\}, \Phi) \\
\mathcal{F}_9 \subset \mathcal{N}_9 &:= \mathcal{N}(\mathcal{D} + 1, \mathcal{W}, \{\| \cdot \|_{C(\Omega)}, \mathcal{B}\}, \Phi) \\
\mathcal{F}_{10} \subset \mathcal{N}_{10} &:= \mathcal{N}(\mathcal{D}, \mathcal{W}, \{\| \cdot \|_{C^2(\Omega)}, \mathcal{B}\}, \Phi) \\
\mathcal{F}_{11} \subset \mathcal{N}_{11} &:= \mathcal{N}(\mathcal{D} + 1, \mathcal{W}, \{\| \cdot \|_{C^2(\Omega)}, \mathcal{B}^2\}, \Phi) \\
\mathcal{F}_{12} \subset \mathcal{N}_{12} &:= \mathcal{N}(\mathcal{D}, \mathcal{W}, \{\| \cdot \|_{C^2(\Omega)}, \mathcal{B}\}, \Phi).
\end{align*}
\]

**Proof.** By Proposition 5.10, we know that for \( u \in \mathcal{N}^3 \) with depth \( \mathcal{D} \) and width \( \mathcal{W} \), \( u_{x_i} \) can be implemented by a ReLU\(^2\) - ReLU\(^3\) network with depth \( \mathcal{D} + 2 \) and width \( (\mathcal{D} + 2)\mathcal{W} \). Then by Proposition 5.10 again we have that \( u_{x_i x_j} \) can be implemented by a ReLU - ReLU\(^2\) - ReLU\(^3\) network with depth \( \mathcal{D} + 4 \) and width \( (\mathcal{D} + 2)(\mathcal{D} + 4)\mathcal{W} \). These facts combining with (5.1) yields the results. \[ \square \]

By Lemma 5.7 and Proposition 5.9, we have for \( i = 1, 2, \ldots, 10 \),

\[
\mathfrak{R}(\mathcal{F}_i) \leq \inf_{0 < \delta < B_i} \left( 4\delta + \frac{12}{\sqrt{N}} \int_{(\delta)}^{B_i} \sqrt{\log(2C(\mathcal{F}_1, \mathcal{N}))} \, d\varepsilon \right) \\
\leq \inf_{0 < \delta < B_i} \left( 2\delta + \frac{12}{\sqrt{N}} \int_{(\delta)}^{B_i} \log \left( \frac{2\left( eN\mathcal{B}_i \mathcal{P}_{\dim(\mathcal{F}_1)} \right)^{\mathcal{P}_{\dim(\mathcal{F}_i)}}} {\varepsilon \mathcal{P}_{\dim(\mathcal{F}_i)}} \right) \, d\varepsilon \right) \tag{5.5} \\
\leq \inf_{0 < \delta < B_i} \left( 4\delta + \frac{12B_i}{\sqrt{N}} + \frac{12}{\sqrt{N}} \int_{(\delta)}^{B_i} \mathcal{P}_{\dim(\mathcal{F}_1)} \cdot \log \left( \frac{eN\mathcal{B}_i}{\varepsilon \mathcal{P}_{\dim(\mathcal{F}_1)}} \right) \, d\varepsilon \right).
\]

Now we calculate the integral. Let \( t = \sqrt{\log \left( \frac{eN\mathcal{B}_i}{\varepsilon \mathcal{P}_{\dim(\mathcal{F}_1)}} \right)} \), then \( \varepsilon = \frac{eN\mathcal{B}_i}{\mathcal{P}_{\dim(\mathcal{F}_1)}} e^{-t^2} \). Denoting
Obviously, \( t_1 = \sqrt{\log(\frac{eNB_i}{\delta \cdot \text{Pdim}(F_i)})} \), \( t_2 = \sqrt{\log(\frac{eNB_i}{\delta \cdot \text{Pdim}(F_i)})} \), we have
\[
\int_{\delta}^{B_i} \log\left(\frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)}\right) \, d\epsilon = \frac{2eNB_i}{\text{Pdim}(F_i)} \int_{t_1}^{t_2} t^2 e^{-t^2} \, dt = \frac{2eNB_i}{\text{Pdim}(F_i)} \int_{t_1}^{t_2} t \left(1 - \frac{e^{-t^2}}{2}\right) \, dt = \frac{eNB_i}{\text{Pdim}(F_i)} \left[ t_1 e^{-t_1^2} - t_2 e^{-t_2^2} + \int_{t_1}^{t_2} e^{-t^2} \, dt \right]
\[
\leq \frac{eNB_i}{\text{Pdim}(F_i)} \left[ t_1 e^{-t_1^2} - t_2 e^{-t_2^2} + (t_2 - t_1) e^{-t_1^2} \right]
\leq \frac{eNB_i}{\text{Pdim}(F_i)} \cdot t_2 e^{-t_1^2} = \|u\| \sqrt{\log\left(\frac{eNB_i}{\delta \cdot \text{Pdim}(F_i)}\right)}. \tag{5.6}
\]

Combining (5.5) and (5.6) and choosing \( \delta = B_i \left(\frac{\text{Pdim}(F_i)}{N}\right)^{1/2} \leq B_i \), we get for \( i = 1, 2, 3, 5, 6, 7, 8, 9, 10 \),
\[
\Re(F_i) \leq \inf_{0 < \delta < B_i} \left( 4\delta + \frac{12B_i}{\sqrt{N}} + \frac{12}{\sqrt{N}} \int_{\delta}^{B_i} \frac{1}{\text{Pdim}(F_i)} \cdot \log\left(\frac{eNB_i}{\epsilon \cdot \text{Pdim}(F_i)}\right) \, d\epsilon \right)
\]
\[
\leq \inf_{0 < \delta < B_i} \left( 4\delta + \frac{12B_i}{\sqrt{N}} + \frac{12B_i \text{Pdim}(F_i)}{\sqrt{N}} \sqrt{\log\left(\frac{eNB_i}{\delta \cdot \text{Pdim}(F_i)}\right)} \right)
\leq 28 \frac{3}{2} B_i \left(\frac{\text{Pdim}(F_i)}{N}\right)^{1/2} \sqrt{\log\left(\frac{eN}{\text{Pdim}(F_i)}\right)} \tag{5.7}
\leq 28 \frac{3}{2} B_i \left(\frac{\text{Pdim}(N_i)}{N}\right)^{1/2} \sqrt{\log\left(\frac{eN}{\text{Pdim}(N_i)}\right)}
\leq 28 \frac{3}{2} \max\{B, B^2\} \left(\frac{\mathcal{H}}{N}\right)^{1/2} \sqrt{\log\left(\frac{eN}{\mathcal{H}}\right)},
\]
with
\[
\mathcal{H} = 4C(D + 2)^2(D + 4)^2(D + 5)^2W^2(D + 5 + \log((D + 2)(D + 4)W))
\]
where in the forth step we apply Lemma 5.14 and we use Proposition 5.12 in the last step. Similarly for \( i = 11, 13 \),
\[
\Re(F_i) \leq 28 \frac{3}{2} \max\{B, B^2\} \left(\frac{\mathcal{H}}{M}\right)^{1/2} \sqrt{\log\left(\frac{eM}{\mathcal{H}}\right)} \tag{5.8}
\]
Obviously, \( \Re(F_4) \) and \( \Re(F_{12}) \) can be bounded by the right hand side of (5.7) and (5.8), respectively. Combining Lemma 5.1 and 5.4 and (5.7) and (5.8), we have
\[
\mathbb{E}_{(x_k)_{k=1}^{N}} \sup_{u \in \mathcal{A}^3} \left| \mathcal{L}(u) - \widetilde{\mathcal{L}}(u) \right| \leq 40 \mathbb{E}_{(x_k)_{k=1}^{N}} \sup_{u \in \mathcal{A}^3} \left| \mathcal{L}_j(u) - \widetilde{\mathcal{L}}_j(u) \right|
\leq 28 \frac{3}{2} \max\{B, B^2\} \left(\frac{40\Omega C_1}{\mathcal{H}}\right)^{1/2} \sqrt{\log\left(\frac{eN}{\mathcal{H}}\right)} + 12 \mathbb{E}_{(x_k)_{k=1}^{N}} \sup_{u \in \mathcal{A}^3} \left| \mathcal{L}_j(u) - \widetilde{\mathcal{L}}_j(u) \right|
\leq 28 \frac{3}{2} \max\{B, B^2\} \left(\frac{40\Omega C_1}{\mathcal{H}}\right)^{1/2} \sqrt{\log\left(\frac{eN}{\mathcal{H}}\right)} + 12 \mathbb{E}_{(x_k)_{k=1}^{N}} \sup_{u \in \mathcal{A}^3} \left| \mathcal{L}_j(u) - \widetilde{\mathcal{L}}_j(u) \right|
\]
where
\[ C_1 = \max\{A^2d^4, B^2d^6, C^2, ABd^3, ACd^2, AFd, BC, BDd, CE\}, \]
\[ C_2 = \max\{C^2, D^2, E\}. \]

Hence for any \( \epsilon > 0 \), if the number of sample
\[ N, M = C(d, \Omega, 3, B)D^6W^2(\mathcal{D} + \log W) \left( \frac{1}{\epsilon} \right)^{2+\delta} \]
with \( \delta \) being an arbitrarily small number, then we have
\[ \mathbb{E}_{\{X_k\}_{k=1}^N, \{Y_k\}_{k=1}^M} \sup_{u \in \mathcal{A}} |\mathcal{L}(u) - \hat{\mathcal{L}}(u)| \leq \epsilon. \]

\[ \Box \]

6. Convergence rate for the PINNs. With preparation in last two sections on the bounds of approximation and statistical errors, we at the place to give our main results.

**Theorem 6.1.** Assume that \( \mathcal{E}_{opt} = 0 \). For any \( \epsilon > 0 \), we set the parameterized neural network class
\[ \mathcal{P} = \mathcal{N} \left( \lceil \log_2 d \rceil + 2, C(d, \Omega, 3, \|u^*\|_{C^2(\Omega)}) \left( \frac{1}{\epsilon} \right)^d, \{\| \cdot \|_{C^2(\Omega)}, 2\|u^*\|_{C^2(\Omega)}\}, \{\text{ReLU}^3\} \right) \]
and set the number of sample
\[ N, M = C(d, \Omega, 3, \|u^*\|_{C^2(\Omega)}) \left( \frac{1}{\epsilon} \right)^{2d+4+\delta} \]
where \( \delta \) is an arbitrarily small number, then we have
\[ \mathbb{E}_{\{X_k\}_{k=1}^N, \{Y_k\}_{k=1}^M} \|u_{\phi,a} - u^*\|_{H^\mathcal{A}(\Omega)} \leq \epsilon. \]

**Proof.** For any \( \epsilon > 0 \), by Theorem 4.9, there exists an neural network function \( \bar{u} \) with depth \( \lceil \log_2 d \rceil + 2 \) and width \( C(d, \Omega, 3, \|u^*\|_{C^2(\Omega)}) \left( \frac{1}{\epsilon} \right)^d \) such that
\[ \|u^* - \bar{u}\|_{C^2(\Omega)} \leq \left( \frac{\epsilon^2}{3C(d^2 + 3d + 4)|\Omega| \max\{2d^2A^2, d^2B^2, C^2\} + 2C|\partial\Omega|\epsilon^2} \right)^{1/2}. \]
Without loss of generality we assume that \( \epsilon \) is small enough such that
\[ \|\bar{u}\|_{C^2(\hat{\Omega})} \leq \|u^* - \bar{u}\|_{C^2(\hat{\Omega})} + \|u^*\|_{C^2(\Omega)} \leq 2\|u^*\|_{C^2(\hat{\Omega})}. \]
Hence \( \bar{u} \) belongs to the function class
\[ \mathcal{P} = \mathcal{N} \left( \lceil \log_2 d \rceil + 2, C(d, \Omega, 3, \|u^*\|_{C^2(\Omega)}) \left( \frac{1}{\epsilon} \right)^d, \{\| \cdot \|_{C^2(\Omega)}, 2\|u^*\|_{C^2(\Omega)}\}, \{\text{ReLU}^3\} \right). \]
And
\[ E_{app} \leq \frac{3}{2}(d^2 + 3d^2 + 4)|\Omega| \max\{2d^2, d, 2^2\} \| \tilde{u} - u^* \|_{C^2(\Omega)}^2 + |\partial \Omega|e^2 \| \tilde{u} - u^* \|_{C^2(\Omega)}^2 \leq \frac{\epsilon^2}{2C}. \quad (6.1) \]

By Theorem 5.13, when the number of sample
\[ N, M = C(d, \Omega, 3, B)D^6W^2(D + \log W) \left( \frac{1}{\epsilon} \right)^{4+\delta} = C(d, \Omega, 3, \| u^* \|_{C^\gamma(\Omega)}) \left( \frac{1}{\epsilon} \right)^{2d+4+\delta} \]
with \( \delta \) being an arbitrarily small number, we have
\[ E_{sta} = \mathbb{E}_{(X_k)_{k=1}^N, (Y_k)_{k=1}^M} \sup_{u \in P} \left| \mathcal{L}(u) - \hat{\mathcal{L}}(u) \right| \leq \frac{\epsilon^2}{2C}. \quad (6.2) \]

Combining Proposition 3.2), (6.1) and (6.2) yields the result. \( \square \)

In [26, 27, 20], the convergence of PINNs is studied. They proved that as the number of parameters in the neural networks and number of training samples go to infinity the solution of PINNs with converges to the solution of the PDEs’. Here we establish the nonasymptotic convergence rate of PINNs. According to our results established in Theorem 6.1, the influence of depth and width in the neural networks and number of training samples are characterized quantitatively. We give answer on how to choose the hyperparameters to archive the desired accuracy, which is import but missing in [26, 27, 20].

7. Conclusion. This paper provided an analysis of convergence rate for PINNs. Specifically, our study shed light on how to set depth and width of networks to achieve the desired convergence rate in terms of number of training samples. The estimation on the approximation error of deep ReLU³ network is established in \( C^2 \) norms. The statistical error can be derived technically by the Rademacher complexity of the non-Lipschitz composition with ReLU³ network.

There are several interesting further research directions. First, it is interesting to extend the current analysis for general second order equations with Neumann or Robin boundary conditions. Second, while we have considered only the forward problems, the error estimate can be extended to using PINNs to solve optimal control problems and inverse problems.

Acknowledgments. Y. Jiao is supported in part by the National Science Foundation of China under Grant 11871474 and by the research fund of KLATASDSMOE of China. X. Lu is partially supported by the National Science Foundation of China (No. 11871385), the National Key Research and Development Program of China (No.2018YFC1314600) and the Natural Science Foundation of Hubei Province (No. 2019CFA007), Y. Wang is supported in part by the Hong Kong Research Grant Council grants 16308518 and 16317416, as well as HK Innovation Technology Fund ITS/044/18FX. J. Yang was supported by NSFC (Grant No. 12125103, 12071362), the National Key Research and Development Program of China (No. 2020YFA0714200) and the Natural Science Foundation of Hubei Province (No. 2019CFA007).
REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. i, Communications on pure and applied mathematics, 12 (1959), pp. 623–727.

[2] C. Antunes, E. Atroshchenko, N. Alajlan, and T. Rabczuk, Artificial neural network methods for the solution of second order boundary value problems, Cmc-computers Materials & Continua, 59 (2019), pp. 345–359.

[3] M. Anthony and P. L. Bartlett, Neural network learning: Theoretical foundations, cambridge university press, 2009.

[4] P. L. Bartlett, N. Harvey, C. Liaw, and A. Meir, Nearly-tight vc-dimension and pseudodimension bounds for piecewise linear neural networks., J. Mach. Learn. Res., 20 (2019), pp. 1–17.

[5] J. Berner, M. Dablander, and P. Grohs, Numerically solving parametric families of high-dimensional kolmogorov partial differential equations via deep learning, in Advances in Neural Information Processing Systems, vol. 33, Curran Associates, Inc., 2020, pp. 16615–16627.

[6] S. Brenner and R. Scott, The mathematical theory of finite element methods, vol. 15, Springer Science & Business Media, 2007.

[7] P. G. Ciarlet, The finite element method for elliptic problems, SIAM, 2002.

[8] C. De Boor and C. De Boor, A practical guide to splines, vol. 27, springer-verlag New York, 1978.

[9] C. Duan, Y. Jiao, Y. Lai, X. Lu, and Z. Yang, Convergence rate analysis for deep ritz method, arXiv preprint arXiv:2103.13330, (2021).

[10] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio, Generative adversarial networks, Advances in Neural Information Processing Systems, 3 (2014).

[11] J. Han, A. Jentzen, and E. Weinan, Solving high-dimensional partial differential equations using deep learning, Proceedings of the National Academy of Sciences, 115 (2018), pp. 8505–8510.

[12] K. He, X. Zhang, S. Ren, and J. Sun, Delving deep into rectifiers: Surpassing human-level performance on imagenet classification, in Proceedings of the IEEE international conference on computer vision, 2015, pp. 1026–1034.

[13] Q. Hong, J. W. Siegel, and J. Xu, Rademacher complexity and numerical quadrature analysis of stable neural networks with applications to numerical pdes, arXiv preprint arXiv:2104.02903, (2021).

[14] T. J. Hughes, The Finite Element Method: Linear Static and Dynamic Finite Element Analysis, Courier Corporation, 2012.

[15] A. D. Jagtap, E. Kharazmi, and G. E. Karniadakis, Conservative physics-informed neural networks on discrete domains for conservation laws: Applications to forward and inverse problems, Computer Methods in Applied Mechanics and Engineering, 365 (2020), p. 113028.

[16] I. Lagaris, A. Likas, and D. I. Fotiadis, Artificial neural networks for solving ordinary and partial differential equations, IEEE transactions on neural networks, 9 (1998), pp. 987–1000.

[17] J. Lu, Y. Lu, and M. Wang, A priori generalization analysis of the deep ritz method for solving high dimensional elliptic equations, arXiv preprint arXiv:2101.01708, (2021).

[18] L. Lu, X. Meng, Z. Mao, and G. E. Karniadakis, Deepxde: A deep learning library for solving differential equations, SIAM Review, 63 (2021), pp. 208–228.

[19] T. Luo and H. Yang, Two-layer neural networks for partial differential equations: Optimization and generalization theory, arXiv preprint arXiv:2006.15733, (2020).

[20] S. Mishra and R. Molinaro, Estimates on the generalization error of physics informed neural networks (pinns) for approximating pdes, arXiv preprint arXiv:2006.16144, (2020).

[21] G. Pang, M. D’Elia, M. Parks, and G. Karniadakis, npinns: Nonlocal physics-informed neural networks for a parametrized nonlocal universal laplacian operator. algorithms and applications, Journal of Computational Physics, 422 (2020), p. 109760.

[22] G. Pang, L. Lu, and G. E. Karniadakis, fpinns: Fractional physics-informed neural networks, SIAM Journal on Scientific Computing, 41 (2019), pp. A2603–A2626.

[23] A. Quarteroni and A. Valli, Numerical Approximation of Partial Differential Equations, vol. 23, Springer Science & Business Media, 2008.
[24] M. Raissi, P. Perdikaris, and G. E. Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations*, Journal of Computational Physics, 378 (2019), pp. 686–707.

[25] L. Schumaker, *Spline functions: basic theory*, Cambridge University Press, 2007.

[26] Y. Shin, J. Darbon, and G. E. Karniadakis, *On the convergence of physics informed neural networks for linear second-order elliptic and parabolic type pdes*, arXiv preprint arXiv:2004.01806, (2020).

[27] Y. Shin, Z. Zhang, and G. E. Karniadakis, *Error estimates of residual minimization using neural networks for linear pdes*, arXiv preprint arXiv:2010.08019, (2020).

[28] D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. Van Den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot, et al., *Mastering the game of go with deep neural networks and tree search*, nature, 529 (2016), pp. 484–489.

[29] J. A. Sirignano and K. Spiliopoulos, *Dgm: A deep learning algorithm for solving partial differential equations*, Journal of Computational Physics, 375 (2018), pp. 1339–1364.

[30] J. Thomas, *Numerical Partial Differential Equations: Finite Difference Methods*, vol. 22, Springer Science & Business Media, 2013.

[31] A. W. van der Vaart and J. A. Wellner, *Weak convergence*, in Weak convergence and empirical processes, Springer, 1996.

[32] E. Weinan and B. Yu, *The deep ritz method: A deep learning-based numerical algorithm for solving variational problems*, Communications in Mathematics and Statistics, 6 (2017), pp. 1–12.

[33] Y. Zhang, G. Bao, X. Ye, and H. Zhou, *Weak adversarial networks for high-dimensional partial differential equations*, Journal of Computational Physics, 411 (2020), p. 109409.