Linear rank inequalities on five or more variables

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July 21, 2010

Abstract

Ranks of subspaces of vector spaces satisfy all linear inequalities satisfied by entropies (including the standard Shannon inequalities) and an additional inequality due to Ingleton. It is known that the Shannon and Ingleton inequalities generate all such linear rank inequalities on up to four variables, but it has been an open question whether additional inequalities hold for the case of five or more variables. Here we give a list of 24 inequalities which, together with the Shannon and Ingleton inequalities, generate all linear rank inequalities on five variables. We also give a partial list of linear rank inequalities on six variables and general results which produce such inequalities on an arbitrary number of variables; we prove that there are essentially new inequalities at each number of variables beyond four (a result also proved recently by Kinser).

∗This work was supported by the Institute for Defense Analyses, the National Science Foundation, and the UCSD Center for Wireless Communications.

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1 Introduction

It is well-known that the linear inequalities always satisfied by ranks of subspaces of a vector space (referred to here as \textit{linear rank inequalities}) are closely related to the linear inequalities satisfied by entropies of jointly distributed random variables (often referred to as \textit{information inequalities}). For background material on this relationship and other topics used here, a useful source is Hammer, Romashchenko, Shen, and Vereshchagin [9].

The present paper is about linear rank inequalities; nonetheless, the basic results from information theory will be useful enough that we choose to use the notation of information theory here. We use the following common definitions:

\[
\begin{align*}
H(A|B) &= H(A, B) - H(B) \\
I(A; B) &= H(A) + H(B) - H(A, B) \\
I(A; B|C) &= H(A, C) + H(B, C) - H(A, B, C) - H(C)
\end{align*}
\]

There are two interpretations of these equations. When \( A, B, \) and \( C \) are random variables, \( A, B \) denotes the joint random variable combining \( A \) and \( B \); \( H(A) \) is the entropy of \( A \) given \( B \); \( I(A; B) \) is the mutual information of \( A \) and \( B \); and \( I(A; B|C) \) is the mutual information of \( A \) and \( B \) given \( C \).

But when \( A, B, \) and \( C \) denote subspaces of a vector space, then \( A, B \) denotes the space spanned by \( A \) and \( B \), which is \( \langle A, B \rangle \) or, since \( A \) and \( B \) are subspaces, just \( A + B \); \( H(A) \) is the rank of \( A \); \( H(A|B) \) is the excess of the rank of \( A \) over that of \( A \cap B \); \( I(A; B) \) is the rank of \( A \cap B \); and \( I(A; B|C) \) is the excess of the rank of \( (A + C) \cap (B + C) \) over that of \( C \). In either interpretation, the equations above are valid.

The basic Shannon inequalities state that \( I(A; B|C) \) (as well as the reduced forms \( I(A; B) \), \( H(A|B) \), and \( H(A) \)) is nonnegative for any random variables \( A, B, C \). Any nonnegative linear combination of basic Shannon inequalities is called a Shannon inequality. We will use standard Shannon computations such as \( I(A; B|C) = I(A; B, C) - I(A; C) \) (one can check this by expanding into basic \( H \) terms) and \( H(A|C) \leq H(A|B, C) \) (because the difference is \( I(A; B|C) \)) throughout this paper; an excellent source for background material on this is Yeung [15].

A key well-known fact is that all information inequalities (and in particular the Shannon inequalities) are also linear rank inequalities for finite-dimensional vector spaces. To see this, first note that in the case of a \textit{finite} vector space \( V \) over a finite field \( F \), each subspace can be turned into a random variable so that the entropy of the random variable is the same (up to a constant factor) as the rank of the subspace: let \( X \) be a random variable ranging uniformly over \( V^* \) (the set of linear functions from \( V \) to \( F \)), and to each subspace \( A \) of \( V \) associate the random variable \( X \upharpoonright A \). The entropy of this random variable will be the rank of \( A \), if entropy logarithms are taken to base |\( F \)|. For the infinite case, one can use the theorem of Rado [14] that any representable matroid is representable over a finite field, and hence any configuration of finite-rank vector spaces over any field has a corresponding configuration over some finite field.

The converse is not true; there are linear rank inequalities which are not information inequalities. The first such example is the Ingleton inequality, which in terms of basic ranks or joint
entropies is
\[ H(A) + H(B) + H(C, D) + H(A, B, C) + H(A, B, D) \]
\[ \leq H(A, B) + H(A, C) + H(B, C) + H(A, D) + H(B, D), \]
but which can be written more succinctly using the \( I \) notation as
\[ I(A; B) \leq I(A; B|C) + I(A; B|D) + I(C; D). \]
Ingleton [10] proved this inequality and asked whether there are still further independent inequalities of this kind.

A key tool used by Hammer et al. [9] is the notion of common information. A random variable \( Z \) is a common information of random variables \( A \) and \( B \) if it satisfies the following conditions: \( H(Z|A) = 0 \), \( H(Z|B) = 0 \), and \( H(Z) = I(A; B) \). In other words, \( Z \) encapsulates the mutual information of \( A \) and \( B \). In general, two random variables \( A \) and \( B \) might not have a common information. But in the context of vector spaces (or the random variables coming from them), common informations always exist; if \( A \) and \( B \) are subspaces of a vector space, one can just let \( Z \) be the intersection of \( A \) and \( B \), and \( Z \) will have the desired properties.

Hammer et al. [9] showed that the Ingleton inequality (and its permuted-variable forms) and the Shannon inequalities fully characterize the cone of linearly representable entropy vectors on four random variables (i.e., there are no more linear rank inequalities to be found on four variables).

## 2 New five-variable inequalities

We will answer Ingleton’s question here. Using the existence of common informations, one can prove the following twenty-four new linear rank inequalities on five variables (this is a complete and irreducible list, as will be explained below).

\[
\begin{align*}
I(A; B) & \leq I(A; B|C) + I(A; B|D) + I(C; D|E) + I(A; E) & (1) \\
I(A; B) & \leq I(A; B|C) + I(A; C|D) + I(A; D|E) + I(B; E) & (2) \\
I(A; B) & \leq I(A; C) + I(A; B|D) + I(B; E|C) + I(A; D|C, E) & (3) \\
I(A; B) & \leq I(A; C) + I(A; B|D, E) + I(B; D|C) + I(A; E|C, D) & (4) \\
I(A; B) & \leq I(A; C) + I(B; D|C) + I(A; E|D) & \\
& \quad + I(A; B|C, E) + I(B; C|D, E) & (5) \\
I(A; B) & \leq I(A; C) + I(B; D|E) + I(D; E|C) & \\
& \quad + I(A; B|C, D) + I(A; C|D, E) & (6) \\
I(A; B) & \leq I(A; C|D) + I(A; E|C) + I(B; D) & \\
& \quad + I(B; D|C, E) + I(A; B|D, E) & (7) \\
2I(A; B) & \leq I(A; B|C) + I(A; B|D) + I(A; B|E) & \\
& \quad + I(C; D) + I(C, D; E) & (8) \\
2I(A; B) & \leq I(A; C) + I(A; B|D) + I(A; B|E) & \\
& \quad + I(D; E) + I(B; D, E|C) & (9)
\end{align*}
\]
\[2I(A; B) \leq I(A; B|C) + I(A; B|D) + I(C; D) + I(A; E)\]
\[+ I(B; D|E) + I(A; C|D, E)\]  \hspace{1cm} (10)

\[I(A; B, C) \leq I(A; C|B, D) + I(A; C, E) + I(A; B|D, E) + I(B; D|C, E)\]  \hspace{1cm} (11)

\[I(A; B, C) \leq I(A; C) + I(A; B|D) + I(A; D|E) + I(B; E|C)\]
\[+ I(A; C|B, E) + I(C; E|B, D)\]  \hspace{1cm} (12)

\[I(A; B, C) \leq I(A; B|D) + I(A; C, E) + I(B; D|C, E)\]
\[+ I(A; C|B, E) + I(C; E|B, D)\]  \hspace{1cm} (13)

\[I(A; B, C) \leq I(A; D) + I(B; E|D) + I(A; B, C|E)\]
\[+ I(A; C|B, D) + I(A; C|D, E)\]  \hspace{1cm} (14)

\[I(A; B, C) \leq I(A; D) + I(B; E|D) + I(A; C|E) + I(A; B|C, D)\]
\[+ I(A; C|B, D) + I(B; D|C, E)\]  \hspace{1cm} (15)

\[I(A; B, C) \leq I(A; B|C, D) + I(A; C|B, D) + I(B; C; D|E)\]
\[+ I(B; C|D, E) + I(A; E)\]  \hspace{1cm} (16)

\[I(A, B; C, D) \leq I(A, B; D) + I(A; D|B, C) + I(B; D|A, C) + I(A; C|B, E)\]
\[+ I(B; C|A, E) + I(A; B|D, E) + I(C; E|B, D)\]  \hspace{1cm} (17)

\[I(A; B) + I(A; C) \leq I(B; C) + I(A; B|D) + I(A; C|D) + I(B; D|E)\]
\[+ I(C; D|E) + I(A; E)\]  \hspace{1cm} (18)

\[I(A; B) + I(A; C) \leq I(B; D) + 2I(A; C|D) + I(A; B|E) + I(D; E)\]
\[+ I(B; E|C, D) + I(C; D|B, E)\]  \hspace{1cm} (19)

\[I(A; B) + I(A; C) \leq I(B; C) + I(B; D) + I(A; C|D) + I(A; B|E)\]
\[+ I(A; E|B) + I(C; D|E) + I(B; E|C, D)\]  \hspace{1cm} (20)

\[I(A; B) + I(A; C) \leq I(B; D) + I(A; C|D) + I(A; D|E) + I(C; E)\]
\[+ I(A; B|C, E) + I(B; C|D, E) + I(B; E|C, D)\]  \hspace{1cm} (21)

\[2I(A; B) + I(A; C) \leq I(A; B|C) + I(A; B|D) + I(C; D) + I(A; C|E)\]
\[+ I(A; D|E) + 2I(B; E) + I(B; C|D, E) + I(C; E|B, D)\]  \hspace{1cm} (22)

\[I(A; B) + I(A; B, C) \leq I(A; B|D) + 2I(A; C|E) + I(B; E) + I(D; E)\]
\[+ I(A; B|C, D) + 2I(B; D|C, E) + I(C; E|B, D)\]  \hspace{1cm} (23)

\[I(A; C, D) + I(B; C, D) \leq I(B; D) + I(B; C|E) + I(C; E|D) + I(A; E) + I(A; C|B, D)\]
\[+ I(A, B; D|C) + I(A; D|B, E) + I(A; B|D, E)\]  \hspace{1cm} (24)

(Note that there is much more variety of form in these inequalities than there is in the four-variable non-Shannon-type inequalities from [5].)

Each of these inequalities is provable from the Shannon inequalities if we assume that each mutual information on the left-hand side of the inequality is in fact realized by a common information. (Hence, since such common informations always exist in the linear case, the inequalities are all linear rank inequalities.) For instance, inequalities (1)–(10) all hold if we assume that there is a random variable \(Z\) such that \(H(Z|A) = H(Z|B) = 0\) and \(H(Z) = I(A; B)\); inequality (23) holds if there exist random variables \(Z\) and \(Y\) such that \(H(Z|A) = H(Z|B) = H(Y|A) = H(Y|B) = 0\).
\[ H(Y|B, C) = 0, \quad H(Z) = I(A; B), \quad \text{and} \quad H(Y) = I(A; B, C); \] and so on. These assertions can all be verified using the program ITIP \[\text{[16]}.\] In fact, all of these become Shannon inequalities if we replace the left-hand mutual information(s) with terms \(H(Z)\) or \(H(Y)\) and add to the right-hand side appropriate terms like \(kH(Z|A) + kH(Z|B)\) for a sufficiently large coefficient \(k\) \((k = 5\) suffices for all of these inequalities). For example, for inequality (1), one can show that

\[
H(Z) \leq I(A; B|C) + I(A; B|D) + I(C; D|E) + I(A; E) + 5H(Z|A) + 5H(Z|B)
\]

is a Shannon inequality; if we set \(Z\) to be a common information for \(A\) and \(B\), we get inequality (1). Again the verifications of these Shannon inequalities can be performed using ITIP, or one can work them out explicitly. In Section 3 we will present various alternate proof techniques.

These inequalities can be written in other equivalent forms.

Obvious rewrites (move the first term on the right to the left):

\[
I(A; B|C) \leq I(A; B|D) + I(A; D|E) + I(B; E|C)
\]

\[
+ I(A; C|B, E) + I(C; E|B, D)
\]

\[
I(A, B; C|D) \leq I(A; D|B, C) + I(B; D|A, C) + I(A; C|B, E)
\]

\[
+ I(B; C|A, E) + I(A; B|D, E) + I(C; E|D)
\]

\[
I(A; C, D) + I(B; C|D) \leq I(B; C|E) + I(C; E|D) + I(A; E) + I(A; C|B, D)
\]

\[
+ I(A, B; D|C) + I(A; D|B, E) + I(A; B|D, E)
\]

Obvious rewrites (enlarge terms on the left so they can be combined):

\[
2I(A; B, C) \leq I(A; C|B) + I(A; B|C) + I(B; C) + I(A; B|D) + I(A; C|D)
\]

\[
+ I(B; D|E) + I(C; D|E) + I(A; E)
\]

\[
2I(A; B, C, D) \leq I(A; C|B) + I(A; B|C) + I(B; C) + 2I(A; B|D) + 2I(A; C|D) + I(A; B|E)
\]

\[
+ I(D; E) + I(B; E|C, D) + I(C; D|B, E)
\]

\[
2I(A; B, C, D) \leq I(A; C|B) + I(A; B|C) + I(B; C) + I(D; E) + I(A; C|D)
\]

\[
+ I(A; E) + I(A; E|B) + I(C; D|E) + I(B; E|C, D)
\]

\[
2I(A; B, C, D) \leq I(A; C|B) + I(A; B|C) + I(B; C) + I(D; E) + I(C; E|D)
\]

\[
+ I(A; E) + I(A; E|B) + I(C; D|E) + I(B; E|C, D)
\]

\[
3I(A; B, C) \leq 2I(A; C|B) + 2I(A; B|C) + I(A; B|D) + I(C; D) + I(A; C|E)
\]

\[
+ I(A; D|E) + 2I(B; E) + I(B; C|D, E) + I(C; E|B, D)
\]

\[
2I(A; B, C) \leq I(A; C|B) + I(A; B|D) + 2I(A; C|E) + I(B; E) + I(D; E)
\]

\[
+ I(A; B|C, D) + 2I(B; D|C, E) + I(C; E|B, D)
\]

\[
2I(A; B, C, D) \leq I(B; C, D|A) + I(A; C, D|B) + I(B; D) + I(B; C|E)
\]

\[
+ I(C; E|D) + I(A; E) + I(A; C|B, D) + I(A, B; D|C)
\]

\[
+ I(A; D|B, E) + I(A; B|D, E)
\]
Non-obvious rewrites:

\[
I(A; C) \leq I(A; C|B) + I(A; B|D) + I(C; D|E) + I(A; E) \tag{1c}
\]

\[
I(A; B|C) \leq I(A; E|C) + I(A; C|B, D) + I(A; B|D, E) + I(B; D|C, E) \tag{11c}
\]

\[
I(A; B|C) \leq I(A; B|D) + I(A; E|C) + I(B; D|C, E)
+ I(A; C|B, E) + I(C; E|B, D) \tag{13c}
\]

\[
I(B; C|D) \leq I(B; C|A, D) + I(A; D|B, C) + I(B; E|D) \tag{15c}
\]

\[
I(B; C) \leq I(B; D) + I(A; C|D) + I(C; D|A)
+ I(B; E|A) + I(B; C|D, E) + I(D; E|B, C) \tag{19c}
\]

\[
I(C; D|E) \leq I(A; D|E) + I(C; D|A) + I(B; D|C, E)
+ I(B; C, E|A) + I(C; E|B, D) \tag{21c}
\]

\[
2I(A; C, D) \leq I(A; D|C) + I(C; D|A) + I(A; C|B)
+ I(A; D|B) + I(A; C|E) + I(A; D|E)
+ 2I(B; E) + I(B; C|D, E) + I(C; E|B, D) \tag{22c}
\]

\[
I(B; D|E) \leq I(B; D|A) + I(A; C|E) + I(C; E|A) + I(B; D|A, C)
+ I(D; E|B, C) + I(B; E|C, D) + I(B; D|C, E) \tag{23c}
\]

\[
I(A, E; D) \leq I(B; D) + I(C; E|B) + I(D; E|C) + I(A; B|C, D)
+ I(A; D|B, C) + I(A; D|B, E) + I(A; E|B, D) \tag{24c}
\]

Note that, for these variant forms, we do not make the claim that the inequality follows from the existence of common informations corresponding to the left-hand-side terms. For instance, inequality (19c) does not follow from the Shannon inequalities and the existence of a common information for B and C. It turns out that inequality (24b) is provable from existence of a common information for \((A, B)\) and \((C, D)\), and inequalities (19b), (21b), (22b), and (23b) are provable from existence of a common information for A and \((B, C)\), but inequalities (18b) and (20b) are not; in fact, no single common information (together with the Shannon inequalities) suffices to prove (18) or (20).

3 Alternate proofs and generalizations

In this section we will provide some alternate proof techniques for the inequalities. This will lead to natural generalizations.

**Lemma 1.** The inequality \(H(Z|R) + I(R; S|T) \geq I(Z; S|T)\) is a Shannon inequality.

**Proof.** Using Shannon inequalities, we see that

\[
H(Z|R) + H(S|Z, T) \geq H(Z|R, T) + H(S|Z, T)
\geq I(S; Z|R, T) + H(S|Z, T)
\geq I(S; Z|R, T) + H(S|R, Z, T)
= H(S|R, T).
\]
So $H(Z|R) - H(S|R,T) \geq -H(S|Z,T)$; add $H(S|T)$ to both sides to get the desired result. □

**Corollary 2.** If $H(Z|R) = 0$, then $I(R; S|T) \geq I(Z; S|T)$.

**Proof of the Ingleton inequality.** Let $Z$ be a common information of $A$ and $B$, so that $H(Z|A) = H(Z|B) = 0$ and $H(Z) = I(A; B)$. Then

\[
I(A; B|C) + I(A; B|D) + I(C; D) \\
\geq I(Z; B|C) + I(Z; B|D) + I(C; D) \quad \text{[from Corollary 2 using $H(Z|A) = 0$]} \\
\geq I(Z; Z|C) + I(Z; Z|D) + I(C; D) + I(C; D) \quad \text{[from Corollary 2 using $H(Z|B) = 0$]} \\
= H(Z|C) + H(Z|D) + I(C; D) + I(C; D) \\
\geq H(Z|C) + I(Z; C) \quad \text{[from Lemma 1]} \\
\geq I(Z; Z) \quad \text{[from Lemma 1]} \\
= H(Z) \\
= I(A; B).
\]

This is essentially the proof given in Hammer et al. [9].

**Proof of inequality (1).** Let $Z$ be a common information of $A$ and $B$; then

\[
I(A; B|C) + I(A; B|D) + I(C; D|E) + I(A; E) \\
\geq I(Z; Z|C) + I(Z; Z|D) + I(C; D|E) + I(C; D) \quad \text{[from Corollary 2 five times]} \\
= H(Z|C) + H(Z|D) + I(C; D|E) + I(Z; E) \\
\geq I(Z; Z|E) + I(Z; E) \quad \text{[from Lemma 1 twice]} \\
= H(Z|E) + I(Z; E) \\
= H(Z) \\
= I(A; B).
\]

**Proof of inequality (2).** Let $Z$ be a common information of $A$ and $B$; then

\[
I(A; B|C) + I(A; C|D) + I(A; D|E) + I(B; E) \\
\geq I(Z; Z|C) + I(Z; C|D) + I(Z; D|E) + I(Z; E) \quad \text{[from Corollary 2]} \\
= H(Z|C) + H(Z|D) + I(Z; C|D) + I(Z; D|E) + I(Z; E) \quad \text{[from Lemma 1]} \\
\geq I(Z; Z|D) + I(Z; D|E) + I(Z; E) \quad \text{[from Lemma 1]} \\
= H(Z|D) + I(Z; D|E) + I(Z; E) \\
\geq I(Z; Z|E) + I(Z; E) \quad \text{[from Lemma 1]} \\
= H(Z|E) + I(Z; E) \\
= H(Z) \\
= I(A; B).
\]
The same pattern allows us to prove more general inequalities: if $A_0$ and $B_0$ have a common information, then:

\[
I(A_0; B_0) \leq I(A_0; B_0|B_1) + I(A_0; B_1|B_2) + \cdots + I(A_0; B_{n-1}|B_n) + I(B_0; B_n)
\]

(25)

\[
I(A_0; B_0) \leq 2^{n-1}I(A_0; B_0|A_1) + 2^{n-2}I(A_1; B_1|A_2) + 2^{n-2}I(A_1; B_1|B_2) + \cdots + I(A_{n-1}; B_{n-1}|A_n) + I(A_{n-1}; B_{n-1}|B_n) + I(A_n; B_n)
\]

(26)

(Note that (26) is related to results in Makarychev and Makarychev [12].) These can be generalized further; for instance, in the right hand side of (25) any number of $A_0$’s may be replaced by $B_0$’s and/or vice versa.

In fact:

**Theorem 3.** Suppose we have a finite binary tree where the root is labeled with an information term $I(x; y)$ and each other node is labeled with a term $I(x; y|z)$. These terms may involve any variables. We single out two variables or combinations of variables, called $A$ and $B$. Suppose that, for each node of the tree, if its label is $I(x; y|z)$ [we allow $z$ to be empty at the root], then:

(a) $x$ is $A$ or $B$ and there is no left child, or
(b) there is a left child and it is labeled $I(r; s|x)$ for some $r$ and $s$;

and

(a’) $y$ is $A$ or $B$ and there is no right child, or
(b’) there is a right child and it is labeled $I(r’; s’|y)$ for some $r’, s’$.

Then the inequality

\[
I(A; B) \leq \text{sum of all the node labels in the tree}
\]

(27)

is a linear rank inequality (in fact, it is true whenever $A$ and $B$ have a common information).

**Proof.** Let $Z$ be a new variable. We prove by induction in the tree (from the leaves toward the root) that, for each node $n$, if $T_n$ is the subtree rooted at $n$, and the node label at $n$ is $I(r; s|t)$, then we have as a Shannon inequality

\[
H(Z|t) \leq \text{sum of node labels in } T_n + j_n H(Z|A) + k_n H(Z|B)
\]

(28)

for some $j_n, k_n \geq 0$. (The inductive step uses Lemma[1].) Applying this when $n$ is the root and $Z$ is a common information of $A$ and $B$ gives the desired result. ■
We get the Ingleton inequality and inequalities (1) and (2) by applying this to the trees:

Ingleton:

\[
\begin{array}{c}
I(C; D) \\
/ \\
I(A; B|C) \quad I(A; B|D)
\end{array}
\]

(1):

\[
\begin{array}{c}
I(A; E) \\
/ \\
I(C; D|E) \\
/ \\
I(A; B|C) \quad I(A; B|D)
\end{array}
\]

(2):

\[
\begin{array}{c}
I(B; E) \\
/ \\
I(A; D|E) \\
/ \\
I(A; C|D) \\
/ \\
I(A; B|C)
\end{array}
\]

A longer "linear" tree like the last one gives (25), while a complete binary tree of height \( n \) gives (26).

Here is another version of Theorem 3:

**Theorem 4.** Let \( I(x_1; y_1|w_1), I(x_2; y_2|w_2), \ldots, I(x_m; y_m|w_m) \) be a list of information terms, where each \( x_i, y_i, w_i \) is chosen from the list \( A, B, r_1, r_2, \ldots, r_k \) with the exception that \( w_1 \) is empty (i.e., the first information term is just \( I(x_1; y_1) \)). Suppose that each of the variables \( r_j \) is used exactly twice, once as a \( w_i \) and once as an \( x_i \) or \( y_i \); while variables \( A \) and \( B \) may be used as many times as desired as an \( x_i \) or \( y_i \), but are not used as a \( w_i \). Then the inequality

\[
I(A; B) \leq \sum_{i=1}^{m} I(x_i; y_i|w_i)
\]

is a linear rank inequality (in fact, it is true whenever \( A \) and \( B \) have a common information).

**Proof.** We build a tree for use in Theorem 3. Each node will be labeled with one of the terms \( I(x_i; y_i|w_i) \). The root is labeled \( I(x_1; y_1) \). If we have a node \( I(x_i; y_i|w_i) \) where \( x_i \) is not \( A \) or \( B \),
then create a left child for this node and label it \( I(x_j; y_j | w_j) \) for the unique \( j \) such that \( w_j = x_i \). Similarly, if \( y_i \) is not \( A \) or \( B \), then create a right child for this node and label it \( I(x_j; y_j | w_j) \) for the unique \( j \) such that \( w_j = y_i \). It is easy to show that no term \( I(x_i; y_i | w_i) \) will be used more than once in this construction (look for the counterexample nearest the root). Hence, the construction will terminate, and the sum of the labels used is less than or equal to \( \sum_{i=1}^{m} I(x_i; y_i | w_i) \) (it does not matter if some of the terms \( I(x_i; y_i | w_i) \) are not used as labels). Now Theorem 3 gives the desired result.

Theorem 4 directly gives the Ingleton inequality and inequalities (1) and (2). It also gives a number of the other listed inequalities once we write them in an equivalent form using equations such as \( I(A; B | C) = I(A; B, C | C) \):

\[
\begin{align*}
I(A; B) &\leq I(A; C) + I(A; B|D) + I(B; C, E|C) + I(A; D|C, E) \\
I(A; B) &\leq I(A; C) + I(A; B|D, E) + I(B; C, D|C) + I(A; D, E|C, D) \\
I(A; B) &\leq I(A; C) + I(B; D|C) + I(A; D|E|D) \\
&+ I(A; B|C, E) + I(B; C, E|D, E) \\
I(A; B) &\leq I(A; C|D) + I(A; C, E|C) + I(B; D) \\
&+ I(B; D, E|C, E) + I(A; B|D, E) \\
I(A; B, C) &\leq I(A; B, C|B, D) + I(A; C, E) + I(A; B, D|D, E) \\
&+ I(B, C, D|C, E) \\
I(A; B, C) &\leq I(A; C) + I(A; B, D|D) + I(A; D|E) \\
&+ I(B, C; D|B, E) + I(A; B, C, E|B, D) \\
I(A; B, C) &\leq I(A; B, D|D) + I(A; C, E) + I(B, C; D|C, E) \\
&+ I(A; B, C|B, E) + I(C; B, E|B, D) \\
I(A; B, C) &\leq I(A; D) + I(B, D, E|D) + I(A; C, E|E) \\
&+ I(A; B, C|B, D) + I(A; C, E|D, E) \\
I(A; B, C) &\leq I(A; B, C|C, D) + I(A; B, C|B, D) + I(B, C; D|E) \\
&+ I(B; D, C, D|D, E) + I(A; E) \\
I(A; B, C; D) &\leq I(A, B; D) + I(A, B; C, D|B, C) + I(A, B; C, D|A, C) \\
&+ I(A, B; C|B, E) + I(A, B; A, C|A, E) + I(A, E; B, E|D, E) \\
&+ I(C; D, D, E|D)
\end{align*}
\]

For instance, inequality (5d) is obtained from Theorem 4 using the list of random variables

\[ A, B, C, D, (C, E), (D, E). \]

Another approach is to prove the inequality

\[
I(A; B) \leq I(A; C) + I(B; D|C) + I(A; F|D) + I(A; B|E) + I(B; E|F)
\]
directly from Theorem 4 and then apply the variable substitution
\[(A, B, C, D, E, F) \rightarrow (A, B, C, D, (C, E), (D, E))\]
to get (5d). Similarly, the other inequalities listed above are substitution instances of linear-variable inequalities on five to eight variables. (Note that (3d), (4d), and (11d) are substitution instances of (1c).)

We will now generalize Theorem 3 so as to generate additional inequalities. One easy but apparently useless generalization is to replace the binary tree with a binary forest (a finite disjoint union of binary trees). Then the hypotheses of Theorem 3 can be stated just as before (with “the root” replaced by “each root”); and the conclusion is the same except that the inequality becomes
\[mI(A; B) \leq \text{sum of all the node labels in the trees}\]  \hspace{1cm} (29)
where \(m\) is the number of trees (equivalently, the number of root nodes).

This modification alone is useless because the resulting inequality is just a sum of Theorem 3 inequalities, one for each tree. But it will become useful when combined with another modification. For this we need a tightening of Lemma 1:

**Lemma 5.** The inequality\[
H(Z|R) + I(R; S|T) \geq I(Z; S|T) + H(Z|R, S, T)\]
is a Shannon inequality.

**Proof.** The proof is just as for Lemma 1 with the slack made explicit in one step. Using Shannon inequalities, we see that
\[
H(Z|R) + H(S|Z, T) \geq H(Z|R, T) + H(S|Z, T)
\]
\[
= H(Z|R, S, T) + I(S; Z|R, T) + H(S|Z, T)
\]
\[
\geq H(Z|R, S, T) + I(S; Z|R, T) + H(S|R, Z, T)
\]
\[
= H(Z|R, S, T) + H(S|R, T).
\]
So \(H(Z|R) - H(S|R, T) \geq H(Z|R, S, T) - H(S|Z, T)\); add \(H(S|T)\) to both sides to get the desired result. \(\blacksquare\)

Using this twice (and noting that \(I(Z; Z|T) = H(Z|T)\) and \(H(Z|Z, S, T) = 0\)), we get
\[H(Z|R) + H(Z|S) + I(R; S|T) \geq H(Z|T) + H(Z|R, S, T).\] \hspace{1cm} (30)

The case where \(T\) is a null variable gives
\[H(Z|R) + H(Z|S) + I(R; S) \geq H(Z) + H(Z|R, S).\] \hspace{1cm} (31)

These give us additional options in proving inequalities, as shown below.
Proof of inequality (8). Let Z be a common information of A and B; then

\[ I(A; B|C) + I(A; B|D) + I(A; B|E) + I(C; D) + I(C, D; E) \]
\[ \geq I(Z; Z|C) + I(Z; Z|D) + I(Z; Z|E) + I(C; D) + I(C, D; E) \] [from Corollary 2]
\[ = H(Z|C) + H(Z|D) + H(Z|E) + I(C; D) + I(C, D; E) \]
\[ \geq H(Z) + H(Z|C, D) + H(Z|E) + I(C, D; E) \] [from (31)]
\[ \geq H(Z) + H(Z) + H(Z|C, D, E) \] [from (31)]
\[ \geq 2H(Z) \]
\[ = 2I(A; B). \]

This proof immediately generalizes to give: If A and B have a common information, then

\[(n - 1)I(A; B) \leq I(A; B|C_1) + I(A; B|C_2) + \ldots + I(A; B|C_n) + \]
\[ + [I(C_1; C_2) + I(C_1C_2; C_3) + \ldots + I(C_1C_2\ldots C_{n-1}; C_n)]. \] (32)

The expression in brackets is actually symmetric in \(C_1, C_2, \ldots, C_n\); it is equal to

\[H(C_1) + H(C_2) + \cdots + H(C_n) - H(C_1C_2\ldots C_n).\]

One can use Lemma 5 to produce an extended form of Theorem 3 in which an additional option is available: instead of having a left child, a node can have a left pointer pointing to some other node anywhere in the tree or forest, and similarly on the right side.

**Theorem 6.** Suppose we have a finite binary forest where each node is labeled with an information term \(I(x; y|z)\), where \(z\) is empty at each root node (i.e., the root labels are of the form \(I(x; y)\)). These terms may involve any variables. We single out two variables or combinations of variables, called A and B. Suppose that, for each node of the forest, if its label is \(I(x; y|z)\) [with \(z\) possibly empty], then:

(a) \(x\) is A or B and there is no left child, or
(b) there is a left child of this node and it is labeled \(I(r; s|x)\) for some \(r, s\), or
(c) there is a left pointer at this node pointing to some other node whose label is \(I(r'; s'|t')\) where \(x = (r', s', t')\);

and

(a') \(y\) is A or B and there is no right child, or
(b') there is a right child of this node and it is labeled \(I(r'|s'|y)\) for some \(r', s'\), or
(c') there is a right pointer at this node pointing to some other node whose label is \(I(r'; s'|t')\) where \(y = (r', s', t')\).

Suppose further that no node is the destination of more than one pointer. Let \(m\) be the number of trees in the forest (equivalently, the number of root nodes). Then the inequality

\[mI(A; B) \leq \text{sum of all the node labels in the trees}\] (33)

is a linear rank inequality (in fact, it is true whenever A and B have a common information).
Proof. As with Theorem 5, let \( Z \) be a new variable. For any left or right pointer, if \( I(r; s|t) \) is the label at the destination of the pointer, we say that the term associated with the pointer is \( H(Z|r, s, t) \). We prove by induction in the forest (upward from the leaves toward the roots) that, for each node \( n \), if \( T_n \) is the subtree rooted at \( n \), and the node label at \( n \) is \( I(r; s|t) \), then we have as a Shannon inequality

\[
H(Z|t) \leq \text{sum of node labels in } T_n + \text{Out}_n - \text{In}_n + j_n H(Z|A) + k_n H(Z|B)
\]

for some \( j_n, k_n \geq 0 \), where \( \text{Out}_n \) is the sum of the terms associated with pointers from nodes in \( T_n \) and \( \text{In}_n \) is the sum of the terms associated with pointers to nodes in \( T_n \). (A pointer whose source and destination are both in \( T_n \) will contribute to both sums, but these contributions will cancel each other out.) The inductive step uses Lemma 5; the new term in that lemma is used to handle the case where there is a pointer with destination \( n \) (note that, by assumption, there is at most one such pointer). Once (34) is proved, apply it to all of the root nodes and add the resulting inequalities together to get

\[
mH(Z) \leq \text{sum of all the node labels in the trees} + jH(Z|A) + kH(Z|B)
\]

for some \( j, k \geq 0 \); the pointer sums cancel out because each pointer contributes to one Out sum and one In sum. Applying (35) when \( Z \) is a common information of \( A \) and \( B \) gives the desired result (33).

Theorem 6 can be used to prove inequalities (8) and (9) using the following diagrams (pointers are represented as dashed curves):

And by using equivalent forms of terms as was done in formulas (3d) through (17d), one can use Theorem 6 to prove formulas (6), (10), (19b), and (21b)–(24b) via the following diagrams:
One can also get a new extended version of Theorem 4 in the same way, though it is harder to state precisely. It is also slightly less flexible because it disallows reuse of the same variable or combination of variables; and the forest diagrams are easier to verify by inspection.

Here are two more explicit proofs.

Proof of inequality (18). Let $Z$ be a common information of $A$ and $B$, and let $Y$ be a common information of $A$ and $C$; note that we have $H(Y, Z|A) = 0$. Then

\[
I(B; C) + I(A; B|D) + I(A; C|D) + I(B; D|E) \\
+ I(C; D|E) + I(A; E) \\
\geq I(Z; Y) + I(Y, Z; D) + I(Y, Z; Y|D) + I(Z; D|E) \\
+ I(Y; D|E) + I(Y; Z; E) \\
= I(Z; Y) + H(Z|D) + H(Y|D) + I(Z; D|E) \\
+ I(Y; D|E) + I(Y; Z; E) \\
\geq I(Z; Y) + H(Y|D) + I(Z; Z|E) + I(Y; D|E) + I(Y, Z; E) \\
\geq I(Z; Y) + I(Z; Z|E) + I(Y; Y|E) + I(Y, Z; E) \\
\geq I(Z; Y) + H(Z|E) + H(Y|E) + I(Y, Z; E) \\
= I(Z; Y) + H(Y, Z) \\
= H(Z) + H(Y) \\
= I(A; B) + I(A; C).
\]

Proof of inequality (20). Let $Z$ be a common information of $A$ and $B$, and let $Y$ be a common information of $A$ and $C$; note that we have $H(Y, Z|A) = 0$ and $H(C, Y|C) = H(C|C, Y) = 0$. 

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

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\[\]

\[\]
Then

\[ I(B; C) + I(B; D) + I(A; C|D) \]
\[ + I(A; B|E) + I(A; E|B) + I(C; D|E) + I(B; E|C, D) \]
\[ \geq I(B; Y) + I(Z; D) + I(Y, Z; C, Y|D) \]
\[ + I(Z; E) + I(Y; E|B) + I(Y; D|E) + I(Z; E|C, D) \]
\[ = I(B, E; Y) + I(Z; D) + I(Y, Z; C, Y|D) \]
\[ + I(Z; E) + I(Y; D|E) + I(Z; E|C, Y, D) \]
\[ \geq I(E; Y) + I(Z; D) + I(Y, Z; C, Y|D) \]
\[ + I(Z; E) + I(Y; D|E) + I(Z; E|C, Y, D) \]
\[ = I(D, E; Y) + I(Z; D) + I(Y, Z; C, Y|D) \]
\[ + I(Z; E) + I(Z; E|C, Y, D) \]
\[ = I(D, E; Y) + I(Z; D) + I(Z; C, Y|D) + H(Y|D, Z) \]
\[ + H(Z|E) + I(Z; E|C, Y, D) \]
\[ = I(D, E; Y) + I(Z; D) + I(Z; C, E, Y|D) + H(Y|D, Z) + H(Z|E) \]
\[ \geq I(D, E; Y) + I(Z; D) + I(Z; E, Y|D) + H(Y|D, Z) + H(Z|E) \]
\[ = I(D, E; Y) + I(Z; D, E, Y) + H(Y|D, Z) + H(Z|E) \]
\[ = I(D, E; Y) + I(Z; D, E) + I(Z; Y|D, E) + H(Y|D, Z) + H(Z|E) \]
\[ \geq I(D, E; Y) + I(Z; D, E) + I(Z; Y|D, E) \]
\[ + H(Y|D, E, Z) + H(Z|D, E) \]
\[ = I(D, E; Y) + I(Z; D, E) + H(Y|D, E) + H(Z|D, E) \]
\[ = I(D, E; Y) + H(Z) + H(Y|D, E) \]
\[ = H(Z) + H(Y) \]
\[ = I(A; B) + I(A; C). \]

It is not yet clear how to generalize these.

4 Completeness

The complete (and verified nonredundant) list of linear-variable inequalities on five variables consists of:
• the elemental Shannon inequalities:
\[ 0 \leq I(A; B) \]
\[ 0 \leq I(A; B|C) \]
\[ 0 \leq I(A; B|C, D) \]
\[ 0 \leq I(A; B|C, D, E) \]
\[ 0 \leq H(A|B, C, D, E) \]
and the inequalities obtained from these by permuting the five variables \( A, B, C, D, E \) (see Yeung [15] for a proof that these imply all other 5-variable Shannon inequalities);

• the following instances of the Ingleton inequality:
\[ I(A; B) \leq I(A; B|C) + I(A; B|D) + I(C; D) \]  
(36)
\[ I(A; B) \leq I(A; B|C) + I(A; B|D, E) + I(C; D, E) \]  
(37)
\[ I(A; B, C) \leq I(A; B, C|D) + I(A; B, C|E) + I(D; E) \]  
(38)
\[ I(A, B; A, C) \leq I(A, B; A, C|A, D) + I(A, B; A, C|A, E) + I(A, D; A, E) \]  
(39)
and the ones obtained from these by permuting the five variables \( A, B, C, D, E \) (see Guillé, Chan, and Grant [8] for a proof that these imply all other 5-variable instances of the Ingleton inequality); and

• inequalities (1)–(24) and their permuted-variable forms.

To verify the completeness of this list, we consider the 31-dimensional real space whose coordinates are labeled by the subsets of \( \{ A, B, C, D, E \} \) in the usual binary order:
\[ \{ A \}, \{ B \}, \{ A, B \}, \{ C \}, \{ A, C \}, \{ B, C \}, \ldots, \{ A, B, C, D, E \}. \]
Each of the listed inequalities, once it is rewritten in terms of the basic entropy terms
\[ H(A), H(B), H(A, B), H(C), H(A, C), \ldots, H(A, B, C, D, E), \]  
(40)
defines a half-space of this space; the intersection of these half-spaces is a polyhedral cone which can also be described as the convex hull of its extreme rays. If one of these extreme rays contains a nonzero point \( v \) which is (linearly) representable (i.e., there exist a vector space \( U \) and subspaces \( U_A, U_B, U_C, U_D, U_E \) of \( U \) such that \( \dim(U_A) = v(A) \), \( \dim(U_B) = v(B) \), \( \dim(U_A, U_B) = v(A, B) \), and so on), then this extreme ray can never be excluded by any as-yet-unknown linear rank inequality. If we verify that all of the extreme rays contain linearly representable points, then there can be no linear rank inequality which cuts down the polyhedral cone further, so the list of inequalities must be complete.

There are 7943 extreme rays in \( \mathbb{R}^{31} \) determined by the elemental Shannon inequalities and inequalities (1)–(24) and (36)–(39) (and permutations). If one considers two such rays to be essentially the same when one can be obtained from the other by a permutation of the five variables, then there are 162 essentially different extreme rays. A full list of the vectors generating these rays is available at:
The authors have shown that each of these vectors is representable over the field of real numbers; in fact, up to a scalar multiple, this representation can be done using matrices with integer entries which actually represent the vector over any field (finite or infinite). For instance, consider the extreme ray given by the vector

\[ \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 3 & 3 & 3 \end{bmatrix} \]

(a list of 31 ranks or entropies in the order given by (40)). To this we associate the five matrices:

\[
\begin{align*}
M_A &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
M_B &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\
M_C &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\
M_D &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\
M_E &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]

The interpretation here is that we have a fixed field \( F \), and the row space of each of these matrices specifies a subspace of \( F^3 \). The specified vector gives \( H(A) = 1 \), and the row space of \( M_A \) has dimension 1; the vector gives \( H(B) = 1 \), and the row space of \( M_B \) has dimension 1; the vector gives \( H(A, B) = 2 \), and the vector sum of the row spaces of \( M_A \) and \( M_B \) (i.e., the row space of \( M_A \)-on-top-of-\( M_B \)) has dimension 2; and so on. Equivalently, if we take three random variables \( x_1, x_2, x_3 \) chosen uniformly and independently over the finite field \( F \), and let \( A = x_1 \), \( B = x_2 \), \( C = x_3 \), \( D = x_1 + x_2 + x_3 \), and \( E = (x_1 + x_2, x_3) \), then the entropies of all combinations of \( A, B, C, D, E \) (with logarithms to base \( |F| \)) are as specified by the above vector.

The dimensions of the row spaces listed above are easily computed over the real field (as ranks of the corresponding matrices). In order to verify that the same dimensions would be obtained over any field, one just has to note that, in each case where a matrix rank is computed to be \( k \), there is actually a \( k \times k \) submatrix whose determinant is \( \pm 1 \), so the selected \( k \) rows will still be independent even after being reduced modulo any prime. (Actually, it would suffice to verify that the greatest common divisor of the determinants of all \( k \times k \) submatrices is 1.)

All of the other listed vectors turn out to be representable in the same way, except that for a few of them a scalar multiplier must be applied. For instance, consider the vector

\[ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} \]

To represent this, we would normally take \( M_A \) to be a \( 0 \times 2 \) matrix and \( M_B, M_C, M_D, M_E \) to be \( 1 \times 2 \) matrices whose unique rows have the property that any two are independent but any three are dependent. (In other words, these row vectors are a linear representation for the uniform matroid.
For example, we could take

\[ M_A = [ \] \]
\[ M_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
\[ M_C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]
\[ M_D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]
\[ M_E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

over the real field, but these would not work over the field of two elements. In fact, no such choice of row vectors works over the field of two elements (the first two row vectors would be independent, but then the only choice for the third vector would be the sum of the first two, and the same would hold for the fourth vector, contradicting the independence of the third and fourth vectors). But if we instead take the vector

\[ 0 \ 2 \ 2 \ 2 \ 4 \ 4 \ 2 \ 4 \ 4 \ 4 \ 4 \ 4 \ 2 \ 2 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \]

which is twice the preceding vector and hence determines the same extreme ray, then we can get suitable representing matrices

\[ M_A = [ \] \]
\[ M_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]
\[ M_C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
\[ M_D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \]
\[ M_E = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \]

which work over any field. The same doubling is needed for 13 more of the 162 vectors; and one additional vector, the vector

\[ 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ 2 \ 1 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 1 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \]

corresponding to the uniform matroid \( U_{2,5} \), had to be tripled in order to get a matrix representation that works over all fields.

5 Methodology; testing representability of polymatroids

The list of five-variable linear rank inequalities was produced by the following iterative process. Initially, we had the Shannon and Ingleton inequalities. At each stage, we took the current list of inequalities and used Komei Fukuda’s cddlib software \(^7\) to get the corresponding list of
extreme rays. We then examined the vectors generating the extreme rays to see whether they were representable (over the reals; we did not try to get representations working over all fields until after the iterative process was complete). When such a vector provably could not be represented, the proof (in each case we ran into here) yielded a new linear rank inequality provable via common informations; when we examined a vector where we had difficulty determining whether it was representable or not, we ran exhaustive tests on all ways of specifying a single common information (toward the end, we had to try a pair of common informations) to see whether ITIP could verify that the specified vector contradicted the Shannon inequalities together with the common information specification. Again each such verification led to a new linear rank inequality. (Of course, this is a highly sanitized version of the process as it actually occurred.)

The testing of extreme rays for linear representability soon became a large task, so we gradually developed software to automatically find such representations in a number of cases (and we added more cases when we found new ways to represent vectors). This software used combinatorial rather than linear-algebra methods; for instance, the output of the program for the sample vector

$$1 1 2 1 2 2 3 1 2 2 3 2 3 3 3 2 3 3 3 2 3 3 3 2 3 3 3 2 3 3 3 2 3 3 3 2 3 3 3$$

used above was a specification of five vector spaces $A, B, C, D, E$ which could be paraphrased as: “$A$ is generated by one vector, $B$ is generated by one vector not in $A$, $C$ is generated by one vector not in $A + B$ [the space spanned by $A$ and $B$], $D$ is generated by one vector in general position in $A + B + C$, and $E$ is generated by two vectors, one in $(A + B) \cap (C + D)$ and one in $C$.” The development of the software involved recognizing as many cases as possible where one could find such a specification which could be met over the reals (or over any sufficiently large finite field) and would yield the desired rank vector.

The (attempted) construction of a representation is done one basic subspace at a time: first the representation of $A$ is constructed (this step is trivial), then the representation of $B$ given $A$, then the representation of $C$ given $A$ and $B$, and so on. And each of these subspace representations is constructed one basis vector at a time. Given the representation of $A$, $B$, $C$, and $D$, the algorithm will determine how many basis vectors are needed for subspace $E$ and successively try to choose them in suitable positions relative to the existing subspaces. At each step, a new vector will be chosen in general position in a subspace which is a sum of some of the already-handled subspaces $A, B, C, D$. (Here “general position” means in the selected subspace but not in any relevant proper subspace of it. Which subspaces are relevant depends on the current situation; we avoid having to determine this explicitly by just saying that the underlying field is sufficiently large, or infinite.) If there is a problem with specifying that the vector is in such a sum of basic subspaces, then we may have to specify that the vector is in the intersection of two sums of basic subspaces.

Once the first vector is chosen, we take quotients of all of the existing spaces by this vector to get the new situation in which the second vector needs to be chosen. This is all done by counting dimensions, not by constructing actual numerical vectors. For instance, suppose the first vector is chosen to be in general position in subspace $R$ which is a sum of basic subspaces from $A, B, C, D$ (e.g., $R = A + B$). For each other sum subspace $T$, if the new vector is in $T$, then the quotient by the chosen vector will reduce the dimension of $T$ by 1; if the chosen vector is not in $T$, then the quotient will not change the dimension of $T$. Since the vector is in general position in $R$, the vector will be in $T$ if and only if $R \subseteq T$, and to check whether $R \subseteq T$ one simply has to see
whether \( \dim(R + T) = \dim T \). The case where the vector is chosen from an intersection of two sum subspaces \( R \) and \( S \) is more complicated; more on this below.

Consider the example (41). Suppose that we have already constructed the representations for subspaces \( A, B, C, \) and \( D \), and we are now ready to construct the representation for subspace \( E \). The current situation can be summarized by the following two-row array:

\[
\begin{array}{cccccccccccccc}
0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 3 \\
2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

(42)

Here the first row is the ranks of sums from \( A, B, C, D \) in the order given by (40), but starting with the empty space. For each of these sums, the second row gives the amount by which adding the new subspace \( E \) will increase the dimension of the sum. (So the second entry in this row is \( H(A + E) - H(A) = 3 - 1 = 2 \), the fourth entry is \( H(A + B + E) - H(A + B) = 3 - 2 = 1 \), and so on.)

From this array, we can see that, since \( E \) has dimension 2 but only increases the dimension of \( A + B \) by 1, one of the nonzero vectors in \( E \) must be in \( A + B \). So let us start by assuming that one of the vectors in \( E \) is a vector chosen in general position in \( R = A + B \). We can now check for all sums from \( A, B, C, D \) whether the sum will contain this chosen vector; this information is summarized in the row

\[
\begin{array}{cccccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

(43)

where 1 means the chosen vector is in the corresponding sum. To get the result of taking a quotient by (the subspace generated by) the chosen vector, we subtract (43) from the first row of (42) (because we have used up one vector from each of the indicated subspaces) and subtract the one’s complement of (43) from the second row of (42) (because we have taken care of one of the new vectors for \( E \) beyond each of the indicated subspaces). So the situation after the first vector is chosen is given by:

\[
\begin{array}{cccccccccccccc}
0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\end{array}
\]

Of course, the negative entry in this array means that a problem has occurred: we tried to take a new vector not in \( C + D \), but the given ranks require all vectors in \( E \) to be in \( C + D \). So we will try again; instead of taking a vector in general position in \( R = A + B \), we take a vector in general position in \( R \cap S \), where \( S = C + D \).

This leaves the problem of determining, for each sum subspace \( T \), whether the chosen vector is in \( T \); as before, this is equivalent to determining whether \( R \cap S \subseteq T \). This is not as straightforward as it was to determine whether \( R \subseteq T \); in fact, there are situations where the given data on ranks of sum subspaces simply do not determine whether \( R \cap S \subseteq T \). But we have identified many situations where the given data do allow this determination to be made. Here is a list; note that (a) each such test can also be applied with \( R \) and \( S \) interchanged, and (b) reading this list is not necessary for understanding the rest of the algorithm.

- If \( R \subseteq T \), then \( R \cap S \subseteq T \).
• If the dimensions of $R \cap S$, $R \cap T$, and $R \cap (S + T)$ are all equal, then $R \cap S \subseteq T$. [If two subspaces have the same (finite) dimension and one is included in the other, then the two subspaces are equal. Hence, we get $R \cap S = R \cap (S + T) = R \cap T$, so $R \cap S = (R \cap S) \cap (R \cap T) = R \cap S \cap T$, so $R \cap S \subseteq T$. Also, recall that the dimension of $R \cap S$ can be determined from the given data; it is equal to $I(R; S) = H(R) + H(S) - H(R, S).$]

• If the dimensions of $R \cap T$, $S \cap T$, $(R + S) \cap T$, and $R \cap S$ are all equal, then $R \cap S \subseteq T$. [We have $R \cap T = (R + S) \cap T = S \cap T$, so $R \cap T = R \cap S \cap T$. But now $\dim(R \cap S) = \dim(R \cap T) = \dim(R \cap S \cap T)$, so $R \cap S = R \cap S \cap T$, so $R \cap S \subseteq T.$]

• If $\dim(R \cap T) < \dim(R \cap S)$, then $R \cap S \nsubseteq R \cap T$, so we must have $R \cap S \nsubseteq T$.

• Let $R \cap^* S$ be the “nominal intersection” of $R$ and $S$ (i.e., the sum of the basic subspaces listed both in the sum $R$ and the sum $S$). Clearly $R \cap^* S \subseteq R \cap S$, so, if $R \cap^* S \nsubseteq T$, then $R \cap S \nsubseteq T$.

• If $\dim(R \cap T) < \dim(R \cap ((R \cap^* S) + S))$, then $R \cap S \nsubseteq T$. [First note that, if $U, V, W$ are subspaces such that $V \subseteq U$, then $U \cap (V + W) = V + (U \cap W)$. (The right-to-left inclusion is easy. For the left-to-right inclusion, if $u = v + w$ where $u \in U$, $v \in V$, and $w \in W$, then $u - v = w \in U \cap W$, so $v + w \in V + (U \cap W)$. Hence, if $R \cap S \subseteq T$, then $R \cap ((R \cap^* T) + S) = (R \cap^* T) + (R \cap S) \subseteq R \cap T$, so $\dim(R \cap ((R \cap^* T) + S)) \leq \dim(R \cap T)$.

• If $T' \subseteq T$ and $R \cap S \subseteq T'$, then $R \cap S \subseteq T$. If $T \subseteq T'$ and $R \cap S \nsubseteq T'$, then $R \cap S \nsubseteq T$.

• Let $R \setminus^* S$ be the “nominal difference” of $R$ and $S$ (i.e., the sum of the basic subspaces listed in the sum $R$ but not in the sum $S$), and let $U = (R \setminus^* S) + (S \setminus^* R)$. If $\dim(U \cap (R \cap^* S)) = 0$, then $R \cap S = ((R \setminus^* S) \cap (S \setminus^* R)) + (R \cap^* S)$.

[The right-to-left inclusion is easy. For the left-to-right inclusion, note that $R = (R \setminus^* S) + (R \cap^* S)$ and $S = (S \setminus^* R) + (R \cap^* S)$. Hence, if $x \in R \cap S$, then we have $x = y_1 + z_1 = y_2 + z_2$ for some $y_1 \in R \setminus^* S$, $y_2 \in S \setminus^* R$, and $z_1, z_2 \in R \cap^* S$. Then $y_2 - y_1 = z_1 - z_2$ is in $U \cap (R \cap^* S)$, so we have $y_2 = y_1$ and $z_2 = z_1$; hence, $y_1 \in (R \setminus^* S) \cap (S \setminus^* R)$ and $x = y_1 + z_1$ is in the desired form.] Hence, if $\dim(U \cap (R \cap^* S)) = 0$, $R \cap^* S \subseteq T$, and $((R \setminus^* S) \cap (S \setminus^* R)) \subseteq T$, then $R \cap S \subseteq T$.

These tests do suffice for the example here; the resulting membership vector is

$$0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 1$$

and the new array after taking a quotient by the first chosen vector is:

$$0 1 1 1 1 2 2 2 2 1 2 2 2 1 2 2 2$$
$$1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0$$
Let us call the new quotient spaces $A', B', C', D', E'$. The new ranks indicate that the remaining vector in $E'$ must be chosen to be in $C'$. If we take the new vector in general position in $C'$, then the resulting membership vector is:

$$0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1$$

(Note that we needed the chosen vector to be in $D'$ as well as in $C'$, but this turned out to be automatic, because the given ranks implied $C' = C' + D' = D'$.) And the result of taking a quotient by the second chosen vector is:

$$0 1 1 1 0 1 1 1 1 1 1 1 0 1 1 1 0 1 1 1$$

The all-0 row means that the representation of $E$ has been successfully completed.

The current algorithm does not try many possibilities for the next vector to choose; it simply chooses one sum subspace (usually at the beginning of the list of available ones) to try to add a vector to, and, if that yields an immediate contradiction, perhaps tries one intersection of two sum subspaces. If any such step fails (either because of a contradiction, or because the algorithm cannot determine whether $R \cap S \subseteq T$ in some case), the algorithm gives up. However, the algorithm does give itself up to 120 chances by trying all permutations of the 5 basic variables.

Each time a new extreme ray was produced, the above algorithm was applied as a positive test for representability, while tests against common informations were used as negative tests. If both sides failed, the ray was examined by hand. Sometimes this examination yielded a representation because we found a new way of determining whether $R \cap S \subseteq T$; if so, this new test was added to the algorithm. At the end, the algorithm was able to verify representability of 152 of the final 162 extreme rays, leaving only 10 to be done by hand (by methods which did not fit in the framework of this algorithm).

There are other possibilities for improving the algorithm that we have not yet implemented. One is doing a backtrack search to consider more possibilities for choosing vectors to add; another is to use the information on representation of previous subspaces in the construction of the representation of the current subspace. (In the preceding example, we used only the dimension data for $A, B, C, D$ in the construction of the representation for $E$; we did not use the actual representations constructed for $A, B, C, D$.) More ambitious would be to allow more options for choosing new vectors in terms of the known relations between the current subspaces.

6 Six-variable inequalities (ongoing work)

This iterative process for finding all linear rank inequalities is likely to be infeasible to complete for six or more variables. (Each cddlib polytope computation in 31 dimensions took about 2–3 days; in 63 dimensions it would take far longer, as well as rapidly exceeding the memory available.) But we plan to continue the study, because we expect to find new phenomena at higher levels, possibly including extreme rays that are representable over some fields but not over others (hence yielding rank inequalities which hold only over those other fields), and inequalities which hold for ranks of vector spaces but are not provable via common informations. For instance, such situations could
come from the variables associated with the Fano and non-Fano networks in [4], or the network in [3].

In order to make any progress at all, we had to take some shortcuts (since, as noted above, 63-dimensional polytope computations were out of the question). One of these was to reduce the dimension of the search by assuming equality for one or more of the inequalities found so far; in effect, this is just concentrating on one face, corner, or intermediate-dimensional extreme part of the current region. Another was to work hard on trying to improve already-obtained inequalities, find additional instances of them, or strengthen them in multiple ways if they were not already faces of the region.

We will show here some of the 6-variable inequalities we have found so far; a much longer list is available at:

http://zeger.us/linrank

All of these have been verified to be faces of the linear rank region (so they cannot be improved). To do this, we used a stockpile of linearly representable 6-variable polymatroids (the representability was proved by the algorithm described in the preceding section) encountered during the polytope computations. If a 6-variable linear rank inequality is satisfied with equality by 62 linearly independent vectors from the stockpile, then it must give a face of the linear rank region. (The stockpile currently contains 3220 polymatroids, or 1846734 after one takes all instances obtained by permuting the six basic variables. It is also available at the above website.)

First, there are the 6-variable elemental Shannon inequalities; there are 6 of these if one lists just one of each form, but 246 of them if all of the permuted-variable versions are counted. Then there are 12 instances of the Ingleton inequality (1470 counting permuted forms). Again, see Yeung [15] and Guillé, Chan, and Grant [8] for the proof that these inequalities imply all of the other Shannon and Ingleton inequalities.

Next come the instances of the 5-variable inequalities (1)–(24). The initial computation found 183 of these instances that (with permuted forms) proved all of the others. However, 16 of these instances did not pass the face verification above and were later superseded by other 6-variable inequalities; this left 167 (61740 counting permuted forms) 5-variable instances which were faces of the 6-variable rank region.

Finally, there are the true 6-variable inequalities. We have found 3490 of these so far (2395095 counting permuted forms) which pass the face verification, along with several hundred more which do not pass and which we expect to be superseded later (though this is not guaranteed; perhaps our stockpile of representable polymatroids is insufficient, although the face test has been very reliable so far). We give some examples of these here; see the website mentioned above for the full list.

Some inequalities follow directly from Theorem [3] such as:

\[
I(A; B) \leq I(A; C) + I(B; D|C) + I(A; E|D) + I(B; F|E) + I(A; B|F) \tag{44}
\]
\[
I(A; B) \leq I(A; C) + I(B; D|C) + I(A; E|D) + I(A; F|E) + I(A; B|F) \tag{45}
\]
\[
I(A; B) \leq I(A; C) + I(B; D|C) + I(E; F|D) + I(A; B|E) + I(A; B|F) \tag{46}
\]
\[
I(A; B) \leq I(A; C) + I(D; E|C) + I(A; B|D) + I(B; F|E) + I(A; B|F) \tag{47}
\]
\[
I(A; B) \leq I(C; D) + I(A; B|C) + I(E; F|D) + I(A; B|E) + I(A; B|F) \tag{48}
\]
And others follow directly from Theorem 6, such as:

\[
2I(A; B) \leq I(A; C) + I(D; E; F|C) + I(A; B|D) + I(E; F) + I(A; B|E) + I(A; B|F)
\]  
(49)

\[
2I(A; B) \leq I(A; C) + I(B; D|C) + I(A; E; F|D) + I(E; F) + I(A; B|E) + I(A; B|F)
\]  
(50)

\[
2I(A; B) \leq I(C; D) + I(A; B|C) + I(B; E; F|D) + I(E; F) + I(A; B|E) + I(A; B|F)
\]  
(51)

\[
2I(A; B) \leq I(C; D; E) + I(C; D) + I(A; F|C) + I(A; B|D) + I(A; B|E) + I(A; B|F)
\]  
(52)

\[
3I(A; B) \leq I(C; D; E, F) + I(C; D) + I(E; F) + I(A; B|C) + I(A; B|D) + I(A; B|E) + I(A; B|F)
\]  
(53)

Then there are inequalities which follow from Theorem 3 or Theorem 6 using equivalent forms:

\[
I(A; B, C) \leq I(D; E) + I(C; F|D) + I(A; B|D, F) + I(A; B|C, D) + I(A; C|B, F) + I(A; B|C|E)
\]  
(54)

\[
I(A, B; C, D) \leq I(A; C, D) + I(B; E|A) + I(B; D|A, C, F) + I(D; F|A, E) + I(B; C|A, E, F) + I(B; C|D, E) + I(A; D|B, C, F) + I(A; C|B, E, F) + I(A; F|B, D, E)
\]  
(55)

\[
2I(A; B) \leq I(D; F) + I(A; C) + I(B; D|C) + I(A; B|F) + I(A; E|D) + I(A; F|C, D) + I(A; B|E)
\]  
(56)

\[
I(A; B, C) \leq I(A; C) + I(B; D|C) + I(A; F|D) + I(A; B|F) + I(C; E|B, F) + I(A; C|B, E)
\]  
(57)

\[
3I(A, B; C, D, E) \leq I(A; C, F) + I(A, B; D) + I(A, B; E) + I(C; F|D) + I(D; F|E) + I(A; E|D, F) + I(B; C|A, D, F) + I(B; D|C, F) + I(A; D; E|B, C) + I(A; D; B|C, E) + I(A; C|E, F) + I(B; D|A, E, F) + I(B; C; D|A) + I(A, B; E|C, D) + I(B; E|A, C, D) + I(B; D|C, E, F) + I(A, B; C|D, E)
\]  
(58)

All of the sharp inequalities found so far using one common information have been verified to be instances of Theorem 6. It seems quite possible that this theorem generates all one-common-information inequalities, but we have no proof of this.

There are also hundreds of inequalities that required two common informations to prove. (Inequalities requiring more than two common informations are beyond the range of our software at present.) These are of two types. One type is those like inequalities (18) and (20) which have two
information terms on the left side and use the common informations corresponding to those terms:

\[
I(A; B) + I(A; C) \leq I(B; C) + I(A; D) + I(B; E|D) + I(C; F|D) \\
+ I(A; B|E) + I(A; C|F)
\]

(59)

\[
2I(A; B, C) + I(B; C, D) \leq I(A; C, E) + I(A; F) + I(A; C|D) + 2I(A; B|C, F) \\
+ I(B; C) + I(E; F|C) + 2I(B; D|C, E) + I(C; E|F) \\
+ I(A; D|E, F) + I(D; E|A, C, F) + 2I(A; F|C, D, E)
\]

(60)

The other type has just one information term on the left side but requires a second common information in addition to the one from the left term:

\[
I(A; B) \leq I(A; C) + I(B; D|C) + I(E; F|D) + I(A; B|E) + I(A; C|F) \\
+ I(B; E|C, F)
\]

(61)

\[
2I(A; B; C, D, E) \leq I(A, B; D, E) + I(A, D, F; C) + I(A, F; D|C) + I(B; C|D, E) \\
+ I(A; C|B) + I(A; D|B, C, E) + 2I(A; C|D, E, F) + I(B; C|A, D, E) \\
+ I(A; E|B, D, F) + I(B; E|A, C, F) + I(B; E|A, D, F) + I(B; E|C, D) \\
+ I(B; D|A, E, F) + I(A; F|B, D, E) + I(A; F|B, C, D)
\]

(62)

\[
2I(A; B, C) \leq I(A; B) + I(D; E) + I(A; B|C) + I(C; E|B) + I(D; F|B, E) \\
+ I(C; F|D) + I(A; B|C, D) + I(A; B; C|F) + I(A; C|E)
\]

(63)

Inequality (61) is proved using a common information for \(A\) and \(B\) along with a common information for \(E\) and \((D, F)\); inequality (62) is proved using a common information for \((A, B)\) and \((C, D, E)\) along with a common information for \((B, F)\) and \((A, D, E)\); and inequality (63) is proved using a common information \(Z\) for \(A\) and \((B, C)\) along with a common information for \(F\) and \(Z\). (The possible need for such iteration of common informations along with joining of variables makes it conceivable that an unbounded number of common informations could be needed to prove linear rank inequalities even on a fixed number of initial variables such as 6.)

Since the inequalities in this paper have been proven using only common informations and the Shannon inequalities, they apply not only to linear ranks but also in any other situation where we have random variables which are known to have common informations. For instance, Chan notes in [1] Definition 4) that abelian group characterizable random variables always have common informations (which are still abelian group characterizable random variables); hence, the inequalities proven here hold for such variables.

7 An infinite list of linear rank inequalities

The following theorem shows that there will be essentially new inequalities for each number of variables:

**Theorem 7.** For any \(n \geq 2\), the inequality

\[
(n - 1)I(A; B) + H(C_1C_2 \cdots C_n) \leq \sum_{i=1}^{n} I(A, C_i; B, C_i)
\]

(64)
is a linear rank inequality on \( n + 2 \) variables which is not a consequence of instances of linear rank inequalities on fewer than \( n + 2 \) variables.

**Proof.** First, it is not hard to show that (64) is equivalent to (32), and we have already seen that (32) is a linear rank inequality (this can also be proved using Theorem 6), so (64) is a linear rank inequality.

In the following, if \( S = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\} \), we will write \( C_S \) for \( C_{i_1}C_{i_2}\cdots C_{i_k} \).

Define a rank vector \( v \) on the subsets of \( \{A, B, C_1, \ldots, C_n\} \) as follows: for any \( S \subseteq \{1, 2, \ldots, n\} \),

\[
\begin{align*}
v(C_S) &= 2|S|, \\
v(AC_S) &= n + |S|, \\
v(BC_S) &= \min(2n - 2 + |S|, 2n), \\
v(ABC_S) &= \min(2n - 1 + |S|, 2n).
\end{align*}
\]

One can easily check that \( v \) does not satisfy (64). We will show that \( v \) does satisfy all instances (using the variables \( A, B, C_1, C_2, \ldots, C_n \)) of all linear rank inequalities on fewer than \( n + 2 \) variables; this will imply that (64) is not a consequence of these instances, as desired.

For this purpose, we construct rank vectors \( w_A, w_B, w_1, w_2, \ldots, w_n \), each of which is the same as \( v \) except for one value. The changed values are:

\[
\begin{align*}
w_A(A) &= n - 1, \\
w_B(B) &= 2n - 3, \\
w_i(BC_i) &= 2n.
\end{align*}
\]

We will show that each of these \( w \) vectors is linearly representable over any infinite or sufficiently large finite field \( F \). In each case, the representation will use a vector space \( V \) over \( F \) of dimension \( 2n \), with a basis \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \), and the variable \( C_j \) \( (1 \leq j \leq n) \) will be represented by the two-dimensional subspace \( \langle x_j, y_j \rangle \).

For the representations of \( A \) and \( B \), instead of giving explicit formulas, it will be convenient to use the following concept. Suppose \( U \) is a nontrivial subspace of \( V \). A point \( u \in U \) is said to be in general position in \( U \), relative to a given finite set \( S \) of points (if \( S \) is not specified, then we let \( S \) be the set of all points that have previously been mentioned explicitly), if \( u \) does not lie in any subspace \( U' \) of \( V \) spanned by a subset of \( S \) unless \( U' \) includes all of \( U \). If the set \( S \) is of size bounded by \( N \), then the “in general position” condition excludes at most \( 2^N \) proper subspaces of \( U \) (including the trivial subspace), so there is no problem finding points in general position as long as the field size is greater than \( 2^N \). If we refer to multiple points being chosen in general position, then they should be considered as chosen successively, with later points being in general position relative to earlier points as well as the previous set \( S \). This concept has been referred to by various terms; for instance, in in [15] such points are referred to as “freely placed”. Points chosen in this way make it easy to compute augmented subspace dimensions: if \( u \) is in general position in \( U \) relative to \( S \) and \( U' \) is a subspace spanned by points in \( S \), then \( \dim(\langle U', u \rangle) \) is equal to \( \dim(U') + 1 \) unless \( U \subseteq U' \), in which case it is equal to \( \dim(U') \).
For each \( i \leq n \), a representation of \( w_i \) is obtained by assigning to \( A \) the space
\[
X = \langle x_1, x_2, \ldots, x_n \rangle
\]
and assigning to \( B \) the space spanned by all of the \( x \) vectors except \( x_i \), together with \( n-1 \) additional points chosen in general position in \( V \).

For the representation of \( w_B \), we again assign to \( A \) the space \( X \); \( B \) is assigned a space spanned by \( n-2 \) points in general position in \( X \) together with \( n-1 \) additional points in general position in \( V \).

To represent \( w_A \), choose points \( z_1, z_2, \ldots, z_{n-1} \) in general position in \( X \), and assign to \( A \) and \( B \) the spaces \( \langle z_1, z_2, \ldots, z_{n-1} \rangle \) and \( \langle z_1, z_2, \ldots, z_{n-2}, y_1, y_2, \ldots, y_n \rangle \), respectively.

It remains to show that, if \( C(t_1, \ldots, t_k) \geq 0 \) is a linear rank inequality on \( k \) variables with \( k < n+2 \), then no instance of this inequality fails for \( v \). An instance of this inequality which applies to \( v \) is given by a map \( f \) from \( \{t_1, \ldots, t_k\} \) to the subsets of \( \{A, B, C_1, \ldots, C_n\} \). (Then the definition of \( f \) can be immediately extended to the subsets of \( \{t_1, \ldots, t_k\} \) by the formula \( f(\{t_{j_1}, \ldots, t_{j_m}\}) = f(t_{j_1}) \cup \cdots \cup f(t_{j_m}) \).) So suppose we have an instance, given by \( C \) and \( f \) as above, which fails for \( v \). Since \( C(t_1, \ldots, t_k) \geq 0 \) is a linear rank inequality, the instance must not fail for the representable vector \( w_A \). Therefore, the instance must use the value where \( v \) disagrees with \( w_A \). This means that there is a subset of \( \{t_1, \ldots, t_k\} \) which is mapped by \( f \) to \( \{A\} \); it follows that there is some single value \( j_A \in \{1, 2, \ldots, k\} \) such that \( f(t_{j_A}) = \{A\} \). Similarly, since the instance must not fail for \( w_B \), there is a subset of \( \{t_1, \ldots, t_k\} \) which is mapped by \( f \) to \( \{B\} \), so there exists \( j_B \in \{1, 2, \ldots, k\} \) such that \( f(t_{j_B}) = \{B\} \). And, for each \( i \leq n \), the instance must not fail for \( w_i \), so there is a subset of \( \{t_1, \ldots, t_k\} \) which is mapped by \( f \) to \( \{B, C_i\} \); hence, there exists \( j_i \in \{1, 2, \ldots, k\} \) such that \( f(t_{j_i}) \) is either \( \{C_i\} \) or \( \{B, C_i\} \). It is clear from these \( f \) values that the numbers \( j_A, j_B, j_1, j_2, \ldots, j_n \) are distinct; but this is impossible because \( \{1, 2, \ldots, k\} \) has fewer than \( n+2 \) members. This contradiction completes the proof of the theorem.

\[\blacksquare\]

## 8 Concurrent work and open questions

During the preparation of this paper, the authors became aware of closely related concurrent work. Chan, Grant, and Kern [2] show nonconstructively that there exist linear rank inequalities not following from the Ingleton inequality. Kinser [11] presents a sequence of inequalities which can be written in the form
\[
I(A_2; A_3) \leq I(A_1; A_2) + I(A_3; A_n|A_1) + \sum_{i=4}^{n} I(A_2; A_{i-1}|A_i) \tag{65}
\]
for \( n \geq 4 \). (This is a variant of (25) which follows from Theorem 4, the instance for \( n = 4 \) and \( n = 5 \) are permutated-variable forms of the Ingleton inequality and inequality (1c), respectively.) Kinser shows that (65) is a linear rank inequality for each \( n \geq 4 \) and uses a method similar to the proof of Theorem 7 above to show that instance \( n \) of (65) is not a consequence of linear rank inequalities on fewer than \( n \) variables. (The authors found the proof of Theorem 7 after the initial posting date of [11], but independently.)

Here are some fundamental open questions that this research has not yet answered.
1) For each fixed $n$, are there finitely many linear rank inequalities on $n$ variables which imply all of the others?

2) Is the method of using common informations incomplete? That is, are there linear rank inequalities that cannot be proved from the basic technique of assuming the existence of common informations?

The authors would like to thank James Oxley for helpful discussions.
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