Interaction quench in a trapped 1D Bose gas

Paolo P Mazza¹, Mario Collura¹, Márton Kormos² and Pasquale Calabrese¹

¹ Dipartimento di Fisica dell’Università di Pisa and INFN, 56127 Pisa, Italy
² MTA-BME Momentum Statistical Field Theory Research Group, 1111 Budapest, Budafoki út 8, Hungary
E-mail: mario.collura@df.unipi.it

Received 12 July 2014
Accepted for publication 5 October 2014
Published 10 November 2014

Online at stacks.iop.org/JSTAT/2014/P11016
doi:10.1088/1742-5468/2014/11/P11016

Abstract. We studied the non-equilibrium quench dynamics from free to hard-core 1D bosons in the presence of a hard-wall confining potential. The density profile and the two-point fermionic correlation function in the stationary state as well as their full time evolution was characterised. It was found that for long times the system relaxes to a uniform density profile, but the correlation function memorises the initial state with a stationary algebraic long-distance decay, which is opposite to the exponential behaviour found for the same quench in the periodic setup. We also compute the stationary bosonic two-point correlator which was found to decay exponentially for large distances. A two-step mechanism was shown to govern the time evolution; a quick approach to an almost stationary value was followed by a slow algebraic relaxation to the true stationary state.

Keywords: correlation functions, quantum gases, quantum quenches

ArXiv ePrint: 1407.1037
1. **Introduction**

The non-equilibrium dynamics of isolated quantum systems is currently in a golden age mainly because of recent experiments on trapped ultra-cold atomic gases [1–7], which allowed for the realization and the experimental study of unitary non-equilibrium evolution over long time scales. The non-equilibrium situation that has attracted most of the theorists' attention is the so called interaction quench, in which a system evolves unitarily from an initial state, which is the ground-state of a translationally invariant Hamiltonian, differing from the one governing the evolution by an experimentally tunable interaction parameter [8].

doi:10.1088/1742-5468/2014/11/P11016

---

**Contents**

1. **Introduction** .......................................................... 2
2. **Model and quench** .......................................................... 4
   2.1. The initial setup .......................................................... 4
   2.2. The quench protocol .................................................... 5
3. **Stationary properties** .......................................................... 6
   3.1. Initial fermionic correlation function .................................................... 7
   3.2. Fermionic mode occupation .................................................... 8
   3.3. Stationary particle density .................................................... 11
   3.4. Stationary two-point fermionic correlation function .................................................... 12
   3.5. Stationary bosonic correlation function .................................................... 13
4. **Time-dependent quantities** .......................................................... 15
   4.1. The off-diagonal correlator \( \langle \hat{n}_p^{\dagger} \hat{n}_q \rangle \) .................................................... 15
   4.2. Time evolution of the density profile .................................................... 17
   4.3. Time evolution of the two-point fermionic correlation function .................................................... 18
5. **Conclusion** .......................................................... 22

**Acknowledgments** .......................................................... 23

**Appendix A. Lattice formulation** .......................................................... 24

**Appendix B. Generic confining potential** .......................................................... 26

**Appendix C. Technical details for the evaluation of the time-dependent correlation function** .......................................................... 28

**References** .......................................................... 28
One of the most interesting findings of both theoretical and experimental investigations is the different behaviours displayed by generic and integrable systems, with the latter memorising the initial state for an infinite time [2, 9–12], while the former locally relaxes to a standard Gibbs distribution in which only the initial energy determines the (local) stationary state as an effective temperature [13–15]. However, most previous studies lack a direct connection to the experiments in which the atoms are trapped by some external potential, a situation that for a truly interacting model is very difficult (if not impossible) to tackle analytically in a precise way. For this reason, one of the simplest instances of an interaction quench in the presence of a simple confining potential has been considered in the present paper. Despite this double level of simplicity, it shall be seen that the calculations are non-trivial and that some very interesting effects appear in the quench dynamics.

Consider a 1D Bose gas with Hamiltonian

$$\hat{H} = \int_0^L dx \left[ \partial_x \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) + c \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}(x) + V(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \right].$$  (1)

Here \(\hat{\phi}(x)\) is a boson field satisfying canonical commutation relations \([\hat{\phi}(x), \hat{\phi}^\dagger(y)] = \delta(x - y)\), \(c\) is the two-body coupling constant, \(V(x)\) is the confining potential and we set \(\hbar = 2m = 1\). In the absence of the external potential (i.e. for \(V(x) = 0\)) the Hamiltonian reduces to the celebrated Lieb–Liniger Hamiltonian [16], which is integrable and exactly solvable by Bethe ansatz for any value of the interaction strength \(c\). Global quenches of the coupling constant \(c\) have already been studied in several papers [17–27], as well as other interesting quench dynamics [28–35]. However, in the presence of an external potential \(V(x)\), the Hamiltonian (1) is not integrable for arbitrary values of the coupling constant \(c\).

There are only two special points in which the model is still exactly solvable for arbitrary \(V(x)\) which correspond to free bosons (\(c = 0\)) and impenetrable bosons (\(c = +\infty\)). Indeed, for periodic boundary conditions (PBC), the quench from \(c = 0\) to \(c = \infty\) has already been studied in [22] and despite the simplicity of the initial and final Hamiltonian, the non-equilibrium dynamics turned out to be extremely rich (e.g. breaking Wick’s theorem for finite time) because the initial and final modes are not linearly related.

Consequently, it is very interesting to obtain analytical results for a quench from noninteracting to strongly interacting bosons for a trapped gas. The most natural choice for the confining potential would be a harmonic one (i.e. \(V(x) \propto x^2\)), which is the most commonly used in experiments. However, while it is possible, but cumbersome, to perform analytical calculations with a harmonic trap, this does not represent the easiest choice for introducing and understanding the new effects engendered by the trap. The simplest confining potential giving rise to most of the relevant trapping effects is surely the hard-wall potential, which forces the many-body wave function to vanish outside a given interval of length \(L\) (this can be seen as a power-law confining trap \(V(x) \propto |2x/L|^\alpha\) in the limit of large exponent \(\alpha \gg 1\)). For all of these reasons, we limit ourselves here to considering quench in a hard-wall trap with the main nontrivial aspect that is the initial state, i.e. the Bose–Einstein condensate (BEC) in the trap, breaks translational invariance. As we shall see, this leads to a number of unexpected results, which we shall briefly anticipate. Firstly, although the initial state is highly inhomogeneous and the Hamiltonian governing the dynamics breaks translational invariance, in the large-time limit the density becomes homogeneous (sufficiently far from the boundaries). However, the stationary fermionic...
two-point function is very different from the periodic case. Indeed, while for PBC it decays exponentially for large distances [22], in the presence of the hard-wall trap decay is only algebraic for points deeply in the bulk of the system. This is a very unexpected result because it means physically that the system memorises the inhomogeneity of the initial state, even if the density becomes constant. Furthermore, this is different from what would have happened if the system had thermalised because, at finite temperature, the boundary conditions can only affect a small region close to the boundaries and not the bulk of the system.

The present paper is organised as follows. In section 2 we introduce the model under investigation and the quench protocol; in particular, we focus our attention on nonlinear mapping between pre- and post-quench field operators and we stress the nontrivial aspects introduced by the confining potential. In section 3 we analyse the stationary behaviour of the density and of the two-point function, which could be equivalently described in terms of a generalised Gibbs ensemble (GGE). Section 4 is devoted to the analytical evaluation of the full time-dependence of the particle density and of the two-point fermionic correlators. We also characterise exactly how the stationary values are approached for infinite time. Finally, in section 5 conclusions are drawn.

2. Model and quench

We consider a 1D Bose gas described by the Hamiltonian (1) with a hard-wall confining potential on the interval \([0, L]\), i.e. the potential forces the many-body wave function to vanish at the boundaries \(x = 0, L\). It is worth mentioning that, in the case of a hard-wall trap, the Lieb–Liniger model is integrable for arbitrary values of the coupling constant \(c\) [36]. However, in what follows, we limit our considerations to the out-of-equilibrium unitary dynamics generated by an interaction quench of the coupling constant \(c\), from noninteracting bosons (\(c = 0\)) to hard-core bosons (\(c = \infty\)) in the presence of hard-wall boundaries (HBC) at all times.

2.1. The initial setup

The many-body system is initially prepared in the \(N\)-particle ground state of the free-boson Hamiltonian, i.e. equation (1) with \(c = 0\). Since the Hamiltonian is quadratic, it can be diagonalised in terms of the modes

\[
\hat{\xi}_q = \int_0^L dx \varphi_q^*(x) \hat{\phi}(x), \quad \hat{\xi}_q^\dagger = \int_0^L dx \varphi_q(x) \hat{\phi}^\dagger(x),
\]

(2)

where the normalised one-particle eigenfunctions

\[
\varphi_q(x) = \sqrt{\frac{2}{L}} \sin(q\pi x/L), \quad q = 1, 2, \ldots,
\]

(3)

are the solutions of the one-particle eigenvalue problem

\[
\begin{cases}
\partial_x^2 \varphi_q(x) = \epsilon_q \varphi_q(x), \\
\varphi_q(0) = \varphi_q(L) = 0,
\end{cases}
\]

(4)
with $\epsilon_q = (q\pi/L)^2$. Indeed, by using the inverse of transformation (2), we can rewrite the initial Hamiltonian $\hat{H}_0$ in the diagonal form

$$
\hat{H}_0 = \sum_{q=1}^{\infty} \epsilon_q \hat{\xi}_q^{\dagger} \hat{\xi}_q.
$$

As usual for a BEC, the many-body ground state is prepared by filling the lowest energy level ($q = 1$) with $N$ particles:

$$
|\psi_0(N)\rangle = \frac{1}{\sqrt{N!}} \hat{\xi}_1^{\dagger} |0\rangle,
$$

where $|0\rangle$ is the pre-quench vacuum state characterised by $\hat{\xi}_q|0\rangle = 0$. The initial two-point bosonic correlation function

$$
\langle \psi_0(N)|\hat{\phi}^{\dagger}(x)\hat{\phi}(y)|\psi_0(N)\rangle = 2n \sin(\pi x/L) \sin(\pi y/L),
$$

In particular, the initial particle density is

$$
n_0(x) \equiv \langle \psi_0(N)|\hat{\phi}^{\dagger}(x)\hat{\phi}(x)|\psi_0(N)\rangle = 2n \sin^2(\pi x/L).
$$

The most visible effect due to the hard-wall trap is to constrain the bosonic cloud in such a way that its density distribution presents a strong inhomogeneity, which is the main physical difference when compared to the periodic setup of [22].

### 2.2. The quench protocol

At time $t = 0$ we suddenly turn on an infinitely strong interaction, i.e. we let the system evolve with the Hamiltonian (1) with $c = \infty$. In this limit, known as Tonks–Girardeau limit [37], the bosons behave as impenetrable. The Hamiltonian can be rewritten in terms of hard-core bosonic fields, $\hat{\Phi}(x)$, $\hat{\Phi}^{\dagger}(x)$, which satisfy a hybrid algebra; they commute at different space points, otherwise they obey an effective Pauli principle (induced by infinite repulsion) whenever they are evaluated at the same space point:

$$
[\hat{\Phi}(x), \hat{\Phi}^{\dagger}(y)] = 0, \quad x \neq y, \quad [\hat{\Phi}^{\dagger}(x)]^2 = [\hat{\Phi}(x)]^2 = 0.
$$

In terms of these fields the Hamiltonian is quadratic

$$
\hat{H} = \int_0^L dx \partial_x \hat{\Phi}^{\dagger}(x) \partial_x \hat{\Phi}(x),
$$

and the hybrid commutation relationships encode the infinitely strong interactions, which seem to be absent from the quadratic form (10). The relationship between the hard-core boson fields and the free bosonic ones is $\hat{\Phi}^{\dagger}(x) = P_x \hat{\phi}^{\dagger}(x) P_x$, where $P_x = |0\rangle\langle 0|_x + |1\rangle\langle 1|_x$ is the local projector on the truncated Hilbert space with at most one boson at the point $x$.

Using a Jordan–Wigner transformation, we can map the hard-core boson fields to fermion fields

$$
\hat{\Psi}(x) = \exp \left\{ i\pi \int_0^x dz \hat{\Phi}^{\dagger}(z) \hat{\Phi}(z) \right\} \hat{\Phi}(x), \quad \hat{\Psi}^{\dagger}(x) = \hat{\Phi}^{\dagger}(x) \exp \left\{ -i\pi \int_0^x dz \hat{\Phi}^{\dagger}(z) \hat{\Phi}(z) \right\},
$$

doi:10.1088/1742-5468/2014/11/P11016
which satisfy canonical anti-commutation relations \( \{ \hat{\Psi}(x), \hat{\Psi}^\dagger(y) \} = \delta(x - y) \). Jordan–Wigner mapping guarantees that the fermionic and the bosonic density operators coincide, i.e. \( \hat{\Psi}^\dagger(x) \hat{\Psi}(x) = \hat{\Phi}^\dagger(x) \hat{\Phi}(x) \).

In terms of the fermionic fields the Hamiltonian (10) is
\[
\hat{H} = \int_0^L dx \, \partial_x \hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x),
\]
and is diagonalised by the Fermi operators \( \hat{\eta}_q, \hat{\eta}_q^\dagger \), related to the fermionic fields \( \hat{\Psi}(x), \hat{\Psi}^\dagger(x) \) as
\[
\hat{\Psi}(x) = \sum_{q=1}^\infty \varphi_q(x) \hat{\eta}_q, \quad \hat{\eta}_q = \int_0^L dx \, \varphi_q^*(x) \hat{\Psi}(x),
\]
where the post-quench single-particle eigenfunctions coincide with the pre-quench single-particle ones in equation (3). The crucial difference between the two sets of modes is the different algebrae that they satisfy. In terms of fermionic modes, the Hamiltonian is diagonal
\[
\hat{H} = \sum_{q=1}^\infty \epsilon_q \hat{\eta}_q^\dagger \hat{\eta}_q = \sum_{q=1}^\infty \epsilon_q \hat{n}_q,
\]
with \( \hat{n}_q \equiv \hat{\eta}_q^\dagger \hat{\eta}_q \) being the post-quench mode occupation operators.

The main observable that we consider in the following is a two-point fermionic correlation function
\[
C(x, y; t) \equiv \langle \exp(i \hat{H} t) \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \exp(-i \hat{H} t) \rangle,
\]
where we introduced the simplified notation \( \langle \ldots \rangle \equiv \langle \Psi_0(N)| \ldots |\Psi_0(N) \rangle \) in order to indicate expected values in the initial state. The time dependence in this correlation function can be explicitly written in terms of the post-quench modes as
\[
C(x, y; t) = \sum_{p,q} \varphi_p^*(x) \varphi_q(y) e^{i(\epsilon_p - \epsilon_q)t} \langle \hat{\eta}_p^\dagger \hat{n}_q \rangle.
\]
The density profile is just given by evaluating the correlation function at coincident points
\[
n(x; t) = C(x, x; t) = \sum_{p,q} \varphi_p^*(x) \varphi_q(x) e^{i(\epsilon_p - \epsilon_q)t} \langle \hat{\eta}_p^\dagger \hat{n}_q \rangle.
\]

3. Stationary properties

In this section we report complete characterisation of the stationary properties of the system after quench. We computed the infinite time average of the density profile and of the fermionic correlation function, which are equal to their large time limit, as we will show explicitly in the following section.

Because of the integrability of the post-quench Hamiltonian, the large time limit of the reduced density matrix of any finite interval (in the sense described in [38–41]) is expected to be described by the GGE [9]
\[
\rho_{\text{GGE}} = Z^{-1} \exp \left( - \sum_i \lambda_i \hat{I}_i \right),
\]
doi:10.1088/1742-5468/2014/11/P11016
where \( \{I_i\} \) is a complete set of local integrals of motion and the Lagrange multipliers \( \lambda_i \) are fixed by the conditions \( \langle I_i \rangle = \text{Tr}[\rho_{\text{GGE}} I_i] \). However, recent results [42–47] show that in some interacting theories the stationary state differs from the GGE built with local charges [48, 49], suggesting that additional integrals of motion should be included in the GGE. Here we can ignore this issue as we are dealing with a post-quench free theory. Furthermore, we also preferred to avoid dealing with the issue of locality due to our post-quench Hamiltonian breaks translational invariance. Thus, we exploit the fact that the Tonks–Girardeau model has a simpler infinite set of conserved charges, formed by the fermionic mode occupation numbers \( \hat{n}_q \) (we recall that for PBC the local conserved charges can be expressed as linear combinations of \( \hat{n}_q \)[33,50], so GGEs built from \( \hat{n}_q \) and \( I_i \) are equivalent).

The time average for equation (16) can be worked out in a straightforward way (see also [51] for more general settings) obtaining

\[
C(x, y; t) = \sum_{q=1}^{\infty} \phi_q^*(x) \phi_q(y) \langle \hat{n}_q \rangle = \lim_{t \to \infty} C(x, y; t) \equiv C_\infty(x, y),
\]

which, as expected, only depends on the post-quench fermionic mode of occupation. We emphasise that \( \langle \hat{n}_q \rangle \) is the only ingredient needed to construct the GGE and, thanks to Wick’s theorem, it allows us to calculate any correlation function of local operators, showing that the GGE indeed captures complete stationary behaviour. It is worth mentioning that the GGE also fixes stationary two-time quantities [52], which will not be considered here.

Thus, the elementary bricks needed for the stationary (19) and time-dependent (16) fermionic correlation functions are the fermionic mode occupation \( \langle \hat{n}_q \rangle \) and the correlator \( \langle \hat{n}_q \hat{n}_q \rangle \), respectively. By analogy with the periodic case [22], these can be obtained from the initial correlator of the real-space fermionic fields \( \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle \) calculated in the next subsection.

### 3.1. Initial fermionic correlation function

Calculation of the initial fermionic correlation function is not straightforward due to the non-Gaussian nature of the initial state in terms of the post-quench fermionic operators. As a starting point, we should exploit the usual relationship between fermionic and bosonic correlations, which reads (for \( x < y \))

\[
\langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle = \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} \int_x^y dz_1 \ldots \int_x^y dz_j \langle \hat{\Phi}^\dagger(x) \hat{\Phi}^\dagger(z_1) \ldots \hat{\Phi}^\dagger(z_j) \hat{\Phi}(z_j) \ldots \hat{\Phi}(z_1) \hat{\Phi}(y) \rangle.
\]

Although the initial state does not respect the hard-core condition, we can treat the hard-core boson fields as if they were canonical bosonic fields. Indeed, following the analogous idea for PBC [22], we can assume

\[
\langle \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(z_1) \ldots \hat{\phi}^\dagger(z_j) \hat{\phi}(z_j) \ldots \hat{\phi}(z_1) \hat{\phi}(y) \rangle = \langle \hat{\Phi}^\dagger(x) \hat{\Phi}^\dagger(z_1) \ldots \hat{\Phi}^\dagger(z_j) \hat{\Phi}(z_j) \ldots \hat{\Phi}(z_1) \hat{\Phi}(y) \rangle.
\]
This equality is proved in appendix A using a rigorous lattice regularisation. The lhs of equation (21) is worked out in the initial ground state in a straightforward way: \( |\Psi_0(N)\rangle \):

\[
\langle \hat{\phi}^\dagger (x) \hat{\phi}^\dagger (z_1) \cdots \hat{\phi}^\dagger (z_j) \hat{\phi} (z_j) \cdots \hat{\phi} (z_1) \hat{\phi} (y) \rangle = \varphi_1^\dagger (x) \varphi_1 (y) \prod_{i=1}^{j} |\varphi_1 (z_i)\rangle^2 \langle \hat{\xi}_1^j \rangle^j \langle \hat{\xi}_1^j+1 \rangle^j.
\] (22)

From \( \langle \hat{\xi}_1 | \Psi_0 (N) \rangle = \sqrt{N} |\Psi_0 (N-1)\rangle \), we have \( \langle \langle \hat{\xi}_1^j \rangle^j \langle \hat{\xi}_1^j+1 \rangle^j \rangle = N!/(N-j-1)! \), which allows us to rewrite equation (20) as

\[
\langle \hat{\Psi}^\dagger (x) \hat{\Psi} (y) \rangle = \varphi_1^\dagger (x) \varphi_1 (y) \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} \frac{N!}{(N-j-1)!} \left( \int_x^y \! dz |\varphi_1 (z)\rangle \right)^j.
\] (23)

This relationship is valid in the domain \( x < y \), while in the opposite case \( x > y \), the only difference arises from the exchange of the integration limits, leading to the absolute value of the integral, which can be written as

\[
\left| \int_x^y \! dz |\varphi_1 (z)\rangle \right|^2 = \frac{|x-y|}{L} + \frac{\text{sgn}(x-y)}{2\pi} \left[ \sin \left( \frac{2\pi y}{L} \right) - \sin \left( \frac{2\pi x}{L} \right) \right]
\]

\[
= \frac{|x-y|}{L} - \frac{\text{sgn}(x-y)}{\pi} \cos \left[ \frac{\pi (x+y)}{L} \right] \sin \left[ \frac{\pi (x-y)}{L} \right],
\] (24)

which finally leads to \((\forall x, y \in [0, L])\)

\[
\langle \hat{\Psi}^\dagger (x) \hat{\Psi} (y) \rangle = 2 \frac{N}{L} \sin \left( \frac{\pi}{L} x \right) \sin \left( \frac{\pi}{L} y \right) \left[ 1 - 2 \left( \frac{|x-y|}{L} - \frac{\text{sgn}(x-y)}{\pi} \right) \cos \left( \frac{\pi (x+y)}{L} \right) \sin \left( \frac{\pi (x-y)}{L} \right) \right]^{N-1}.
\] (25)

Equation (25) is valid for any finite value of \( L \) and \( N \). Interestingly, its structure is quite general and independent of the particular shape of the confining potential (see appendix B for more details).

As we shall see, a relevant scaling regime in the quench problem is provided by taking the thermodynamic limit (TDL) with \( \tilde{\omega} = (x+y)/L \) kept fixed and \( z = x - y \) arbitrary. In this limit, equation (25) becomes

\[
\langle \hat{\Psi}^\dagger (x) \hat{\Psi} (y) \rangle = n [1 - \cos (\pi \tilde{\omega})] e^{-2n[1-\cos(\pi \tilde{\omega})]|z|}.
\] (26)

In figure 1, this asymptotic form is compared with the direct numerical evaluation of equation (25). In the following sections we will always use as a starting point equation (25), even if in some instances equation (26) would have led to the same results. We preferred to proceed in this way to retain as much information about the boundaries as possible.

### 3.2. Fermionic mode occupation

The fermionic mode occupation \( \langle \hat{n}_q \rangle \) is obtained by plugging equation (25) in to the definition (13):

\[
\langle \hat{n}_q \rangle = \int_0^L \! dx \int_0^L \! dy \varphi_q (x) \varphi_q^\dagger (y) \langle \hat{\Psi}^\dagger (x) \hat{\Psi} (y) \rangle
\]

\[
= \frac{4N}{L^2} \int_0^L \! dx \int_0^L \! dy \sin \left( \frac{q \pi x}{L} \right) \sin \left( \frac{q \pi y}{L} \right) \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi y}{L} \right)
\]

\[
\times \left[ 1 - 2 \left( \frac{|x-y|}{L} - \frac{\text{sgn}(x-y)}{\pi} \cos \left( \frac{\pi (x+y)}{L} \right) \sin \left( \frac{\pi (x-y)}{L} \right) \right) \right]^{N-1}.
\] (27)

doi:10.1088/1742-5468/2014/11/P11016

8
Interaction quench in a trapped 1D Bose gas

Figure 1. The initial fermionic correlation function equation (25) as function of $z = x - y$ for fixed $w = x + y$. The numerical data for $N = L = 200$ (symbols) are compared to the analytical scaling function given by equation (26) (full lines).

Using standard trigonometric identities and changing the integration variables to $v = \pi(x - y)/L$ and $u = \pi(x + y)/L - \pi$, we have the more compact expression

$$\langle \hat{n} q \rangle = \frac{N}{2\pi^2} \int_{-\pi}^{\pi} \left[ \cos(qv) - (-1)^q \cos(qu) \right] \left[ \cos(v) + \cos(u) \right] \left[ 1 - \frac{2}{\pi} \left\| v \right\| + \text{sgn}(v) \sin(v) \cos(u) \right]^{N-1}.$$  

(28)

In figure 2 we report the mode occupation $\langle \hat{n} q \rangle$ evaluated numerically from equation (28) for different values of $N$. All data for different $N$ collapse on a universal smooth function of the rescaled variable $q/N$, except for very small values of $q$ ($q \lesssim 10$). This implies that only modes with $q/N \sim O(1)$ are important in the TDL allowing us to simplify equation (28).

Indeed, for $q \gg 1$, the function $\cos(qu)$ is integrated over an integer multiple of its period (we recall that $q$ is an integer) and consequently its contribution is suppressed with respect to the remaining part of the integral. Thus, for large $q$ and large $N$, we can rewrite equation (28) as

$$\langle \hat{n} q \rangle = \frac{2N}{\pi^2} \int_{0}^{\pi} \left[ \cos(qv) \right] \left[ \cos(v) + \cos(u) \right] \left[ 1 - \frac{2}{\pi} \left[ v + \sin(v) \cos(u) \right] \right]^{N-1}. $$  

(29)

We can now take the large $N$ limit. Since $\left| 1 - \frac{2}{\pi} \left[ v + \sin(v) \cos(u) \right] \right| \lesssim 1$ throughout the integration domain, for large $N$, the integral in $v$ is dominated by the neighbourhood of $v = 0$ and so we can limit the integral to a region $v \in [0, \epsilon]$ with $\epsilon \ll 1$. Expanding the integrated function in $v$, we obtain

$$\langle \hat{n} q \rangle = \frac{2N}{\pi^2} \int_{0}^{\pi} \left[ \cos(qv) \right] \left[ 1 + \cos(u) \right] \left[ 1 - \frac{2v}{\pi} \left[ 1 + \cos(u) \right] \right]^{N-1}. $$  

(30)

doi:10.1088/1742-5468/2014/11/P11016
Interaction quench in a trapped 1D Bose gas

Figure 2. Fermionic mode occupation \( \langle \hat{n}_q \rangle \) as a function of the rescaled variable \( q\pi/(2N) \) for different particle numbers \( N \). The numerical data, evaluated using equation (28), collapses on the asymptotic universal function (full black line) given by equation (33). For comparison the mode occupation for PBC is also shown (dashed line) [22].

where we could not expand \( \cos(qv) \) because for large \( q \) it can oscillate many times in \([0, \epsilon]\).

For large \( N \) and small \( v \)

\[
\left[1 - \frac{2v}{\pi}[1 + \cos(u)]\right]^{N-1} \simeq e^{N\ln \left(1 - \frac{2v}{\pi}[1 + \cos(u)]\right)} \simeq e^{-N\frac{2v}{\pi}[1 + \cos(u)]},
\]

leading to

\[
\langle \hat{n}_q \rangle = \frac{2N}{\pi^2} \int_0^\pi du \int_0^\epsilon dv \cos(qv)[1 + \cos(u)]e^{-N\frac{2v}{\pi}[1 + \cos(u)]},
\]

After these simplifications, the integrated function is exponentially small in \( N \) and therefore, in the TDL, we can send the upper bounds of integration \( \epsilon \) to infinity. The \( v \) integration becomes the cosine Fourier transform of the exponential function, finally giving

\[
\langle \hat{n}_q \rangle = \frac{1}{\pi} \int_0^\pi \frac{du}{1 + \left[\frac{q\pi/(2N)}{1 + \cos(u)}\right]^2} = 1 - \sqrt{\frac{\tilde{q} (\tilde{q} + \sqrt{4 + \tilde{q}^2})}{2 (4 + \tilde{q}^2)}}, \quad \tilde{q} \equiv \frac{q\pi}{2N} = \frac{1}{2n} L,
\]

showing explicitly that the mode occupation number is indeed a function of the rescaled variable \( q/N \). This analytical result is compared with the numerical evaluation of the mode occupation in figure 2 and they match perfectly for large enough \( N \). Notice that equation (33) satisfies the normalisation condition \( \sum_q \langle \hat{n}_q \rangle = N \) in the TDL, as it should.

Before using this result to calculate the real-space properties of the system in the stationary state, it is of interest to compare this mode distribution with the same quantity for PBC \( n_{PBC}(\tilde{q}) = 1/(1 + \tilde{q}^2) \) [22] (reported for comparison as a dashed line in figure 2). Both distributions have a power law behavior for \( \tilde{q} \gg 1 \) with \( n_{PBC}(\tilde{q}) \simeq 1/\tilde{q}^2 \).
Interaction quench in a trapped 1D Bose gas

Figure 3. (Left) The stationary particle density profile near the left boundary ($x = 0$) is perfectly described by equation (35). (Right) The same data versus the rescaled variable $x/L$ showing that for large $L$, the systems become homogeneous in the interval $[0, L]$ with small corrections at the boundaries.

and $\langle \hat{n}_q \rangle \simeq 3/(2\tilde{q}^2)$. The effect of the boundaries is more apparent for small $\tilde{q}$ when $n_{\text{PBC}}(\tilde{q}) \simeq 1 - \tilde{q}^2$ and $\langle \hat{n}_q \rangle \simeq 1 - \sqrt{q}/2$, the latter being non-analytical in zero (which, as shall be seen, has strong consequences for real-space correlations).

3.3. Stationary particle density

In this section the particle density profile $n_\infty(x)$ in the stationary state is analysed. From definition (19), using the one-particle eigenfunctions (3) and the fermionic mode occupation (33), one immediately has

$$n_\infty(x) = \sum_{q=1}^{\infty} |\varphi_q(x)|^2 <\hat{n}_q> = \frac{2}{\pi L} \sum_{q=1}^{\infty} \int_0^\pi du \frac{\sin^2(q\pi x/L)}{1 + \left[\frac{q\pi/(2N)}{1+\cos(u)}\right]^2}.$$  \hspace{1cm} (34)

In figure 3, we show that the numerical evaluation of this sum approaches the uniform value $n = N/L$ when increasing the system size $L$ and particle number $N$. The HBC influences the profile only in a region $x \sim O(1)$ close to the boundaries, which shrinks to a set of zero measures when considering the scaling variable $x/L$ (right panel).

This behaviour can be easily understood analytically by replacing, in the TDL limit, the sum with an integral

$$n_\infty(x) = \frac{2}{\pi^2} \int_0^\infty dq \int_0^\pi du \frac{\sin^2(xq)}{1 + \left[\frac{q/(2n)}{1+\cos(u)}\right]^2} = n_\infty(x) = n - ne^{-4nx[I_0(4nx) - I_1(4nx)]},$$  \hspace{1cm} (35)

where $I_m(z)$ are modified Bessel functions. This shows that the thermodynamic stationary density is $n_\infty(x) = n$, i.e. the value obtained in the rescaled variable $x/L$ (see the right panel in figure 3). The correction in equation (35) is non-vanishing only in a set of measured zeroes (in $x/L$) and describes the behaviour close to the left boundary at

doi:10.1088/1742-5468/2014/11/P11016
$x = 0$ (which matches numerical results as shown in the left panel of figure 3). Notice that in equation (35) we lost the information about the right boundary at $x = L$. It is, however, obvious that close to the right boundary the density has the same profile as at the left one.

### 3.4. Stationary two-point fermionic correlation function

In this section we study the two-point fermionic correlator $C_\infty(x, y)$ in the stationary state. Let us start by noticing that as long as we are interested in the bulk properties of the system, as the stationary density is homogeneous, we expect all correlation functions to be translationally invariant. Therefore, we keep the difference $x - y \sim O(1)$ to avoid infinitely separated points in the TDL.

In terms of the mode occupation, the stationary fermionic correlation function can be written as

$$C_\infty(x, y) = \sum_{q=1}^{\infty} \varphi_q^*(x) \varphi_q(y) (\hat{n}_q) = \frac{2}{\pi L} \sum_{q} \int_{0}^{\pi} du \frac{\sin(q\pi x/L) \sin(q\pi y/L)}{1 + \left[\frac{q\pi/(2N)}{1 + \cos(u)}\right]^2},$$

(36)

that, in the TDL, becomes a double integral, which can be explicitly performed:

$$C_\infty(x, y) = \frac{1}{\pi^2} \int_{0}^{\infty} dq \int_{0}^{\pi} du \frac{\cos[q(x - y)] - \cos[q(x + y)]}{1 + \left[\frac{q\pi/(2N)}{1 + \cos(u)}\right]^2},$$

$$= ne^{-2n|x-y|[I_0(2n|x-y|) - I_1(2n|x-y|)]} - I_1(2n|x+y|).$$

(37)

This stationary correlator consists of two different parts:

$$C_\infty(x, y) = C^B_\infty(|x - y|) + C^{\text{bou}}_\infty(x, y),$$

(38)

where we have (i) a bulk correlator $C^B_\infty(|x - y|)$ depending only on the distance between the two points and that is the true thermodynamic stationary correlator and (ii) a boundary term $C^{\text{bou}}_\infty(x, y)$ depending on $x + y$ which goes to zero when $x$ and $y$ are far from the boundary $x, y \gg 1$. We stress that the information about the right boundary has been lost due to the way we performed the TDL, in analogy to the density profile.

In the two left panels of figure 4 we compare the numerically evaluated correlation function with the thermodynamic result for $y = L/2$, $L/8$ as a function of $x \in [0, L]$. The numerics perfectly agree with equation (37) as long as $x$ is far from the right boundary where, as we already stated, equation (37) does not apply.

Let us discuss the bulk stationary correlator in more detail

$$C^B_\infty(z) = ne^{-2n|z|[I_0(2n|z|) - I_1(2n|z|)]},$$

(39)

Although this result slightly resembles the real-space correlator for PBC $C_{\text{PBC}}(z) = ne^{-2n|z|[22]}$, there are qualitative differences, as evident from figure 4. The boundary conditions and the highly inhomogeneous initial profile strongly affect the two-point stationary function in the bulk. The multiplicative factor in equation (39), depending

---

doi:10.1088/1742-5468/2014/11/P11016
on the difference between two Bessel functions, modifies both small- and large-distance behaviour. For $zn \ll 1$ the correlator behaves as $C^B_{\infty}(z)/n \sim 1 - 3n|z|$ manifesting a faster short-distance decay compared to the case with PBC. For large distances the behaviour is completely different. While for PBC there is exponential decay for all distances, the trapped correlator shows an algebraic decay for $zn \gg 1$, namely $C^B_{\infty}(z)/n \sim |nz|^{-3/2}/(8\sqrt{\pi})$. This is actually a very important difference compared to the PBC case: in the TDL, the boundaries strongly affect the bulk, a phenomenon that has no direct analogue in finite temperature systems.

### 3.5. Stationary bosonic correlation function

From knowledge of the fermionic two-point function all other correlations in the stationary state can be derived with the help of Wick’s theorem. The most relevant correlation from the experimental point of view is surely the bosonic two-point correlation function whose Fourier transform is the momentum distribution function, the quantity most commonly measured in experiments on cold atoms. In the remainder of this section we are going to evaluate this bosonic correlation function in the bulk.

We consider the stationary bosonic two-point correlation function

$$C^\text{bos}_{\infty}(z) \equiv \lim_{t \to \infty} \langle \Phi^\dagger(x, t)\Phi(x + z, t) \rangle,$$

where $\Phi$ is the field operator. The algebraic decay in $z$ is simply a consequence of the fact that its Fourier transform is not analytic in zero.

---

*Figure 4.* Fermionic two-point correlation function in the stationary state. (Left) The numerically evaluated correlators (using the sum in equation (36)) are compared to the TDL result in equation (37) (full lines) for $y = L/2$ (bottom) and $y = L/8$ (top). (Right) The bulk stationary correlator in equation (39) (full black lines) is compared to the PBC one (dashed black lines). In the inset, the same correlators are shown on a log–log scale to highlight power law behaviour. The red dot–dashed line represents the asymptotic behaviour for small $z$ (i.e. $\sim 1 - 3n|z|$) and the red dot–dot–dashed line asymptotic behaviour for large $z$ (i.e. $\sim |nz|^{-3/2}/(8\sqrt{\pi})$).

---

3 From the mathematical point of view the algebraic decay in $z$ is simply a consequence of the fact that its Fourier transform is not analytic in zero.
where we explicitly used bulk translational invariance in the stationary state. We will set $x = 0$ for simplicity. The bosonic correlation function $C^\text{bos}_\infty(z)$ can be expressed in terms of the fermionic correlations using Jordan–Wigner mapping (11) and Wick’s theorem. For $z > 0$, we have:

$$C^\text{bos}_\infty(z) = \left\langle \hat{\Psi}^\dagger(0) \exp \left\{ -i\pi \int_0^z dy \hat{\Psi}^\dagger(y)\hat{\Psi}(y) \right\} \hat{\Psi}(z) \right\rangle. \tag{41}$$

Taylor expanding the exponential this becomes

$$C^\text{bos}_\infty(z) = \sum_{k=0}^{\infty} \frac{(-i\pi)^k}{k!} \int_0^z dz_1 \cdots \int_0^z dz_k \left\langle \hat{\Psi}^\dagger(0)\hat{\Psi}^\dagger(z_1)\cdots\hat{\Psi}^\dagger(z_k)\hat{\Psi}(z_1)\cdots\hat{\Psi}(z_k)\hat{\Psi}(z) \right\rangle, \tag{42}$$

which can be rearranged in normal order and, using Wick’s theorem, we finally have

$$C^\text{bos}_\infty(z) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \int_0^z dz_1 \cdots \int_0^z dz_k \det(\hat{\Psi}^\dagger(x_i)\hat{\Psi}(y_j)) \tag{43}$$

$$= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} \int_0^z dz_1 \cdots \int_0^z dz_k \det C^B_\infty(x_i - y_j),$$

where the indices $i, j$ run from 0 to $k$ and we used the convention $x_i = y_i \equiv z_i, \forall i > 0$ and $x_0 \equiv 0, y_0 \equiv z$. Equation (43) is a Fredholm’s minor of the first order [53].

It is generally very difficult to manipulate Fredholm’s minors analytically and, for this reason we decided to evaluate equation (43) numerically, which is a quite standard procedure. Indeed, this numerical evaluation can be achieved by discretising the Fredholm’s minor in equation (43) as explained in [20, 33, 54]. In order to do so we proceed as follows, (i) we discretise the space interval $[0, z]$ in $M + 1$ points, introducing lattice spacing $a = z/(M + 1)$; (ii) we define the $(M + 1) \times (M + 1)$ matrices (indices run from 1 to $M + 1$):

$$R_{nm} = \delta_{nm} - \delta_{n1}\delta_{1m}, \tag{44}$$

$$S_{nm} = C^B_\infty((n - m)a) \text{ for } n > 1, \quad S_{1m} = C^B_\infty(z - ma),$$

where $C^B_\infty(z)$ is the stationary bulk fermionic correlation (39). Therefore, the bosonic correlator is given by the limit:

$$C^\text{bos}_\infty(z) = \lim_{a \to 0} \frac{\det(2a S - R)}{2a}. \tag{45}$$

In practice, we evaluate the ratio on the rhs of the above equation for small enough spacing $a$ and check that it does not vary to the required precision by making it smaller. In this way, we calculate $C^\text{bos}_\infty(z)$ as a function of $z$ numerically and the results are illustrated in figure 5.

Let us analyse the results in figure 5 critically. From the inset, it is clear that the large-distance behaviour of the bosonic correlation function is exponential, although the behaviour of the fermionic one is algebraic. This does not come as a surprise, because in other cases [33] the algebraic decay of fermionic correlations has been found to result in an exponential in the bosonic correlation. The decay rate of the exponential (i.e. the inverse correlation length) is $\sim 2.65n$ which is larger than the decay rate for PBC $2n$ (reported for comparison in figure 5). However, while for PBC the bosonic correlator is exactly exponential for all distances [22], i.e. $C^\text{bos}_{\text{PBC}}(z) = ne^{-2n|z|}$, this is not the case for HBC; for small $z$, $C^\text{bos}_\infty(z)/n$ is well fitted by $1 - 3n|z|$ (see figure 5), which coincides with the small-distance behaviour of the fermionic correlator.

doi:10.1088/1742-5468/2014/11/P11016 14
Interaction quench in a trapped 1D Bose gas

Figure 5. Bosonic bulk two-point correlation function in the stationary state. For small distances $nz \ll 1$, $C_{\infty}^{bos}(z)/n \simeq 1 - 3n|z|$ (dot–dashed red line). The inset shows the same correlator on the log-scale to highlight exponential large-distance behaviour $C_{\infty}^{bos}(z)/n \propto e^{-2.65n|z|}$ (dot–dashed red line). In both panels dashed lines correspond with the bosonic correlator for PBC, which is shown for comparison.

4. Time-dependent quantities

In this section we analyse the time evolution of the density profile and of the two-point fermionic correlation function. We will limit considerations to both these quantities deeply in the bulk in order to have more accessible results. In the periodic case these quantities are constant in time [22] because of translational invariance, but in the confined case they present a nontrivial dynamic. The inhomogeneous initial density affects non-equilibrium dynamics for arbitrary times, leading for infinite time to the stationary bulk correlator given in equation (39). Indeed, we show that the time-averaged values calculated in the previous section are indeed approached for long times in the TDL.

4.1. The off-diagonal correlator $\langle \hat{n}_p^{\dagger} \hat{n}_q \rangle$

The elementary building block needed for the evaluation of time-dependent quantities is the initial correlator of the post-quench fermionic mode, which can be written for arbitrary $N$ and $L$ as

$$\langle \hat{n}_p^{\dagger} \hat{n}_q \rangle = \int_0^L dx \int_0^L dy \varphi_p(x)\varphi^*_q(y)\langle \hat{\Psi}^{\dagger}(x)\hat{\Psi}(y) \rangle \right.$$  

$$= \frac{4N}{L^2} \int_0^L dx \int_0^L dy \sin \left( \frac{p\pi x}{L} \right) \sin \left( \frac{q\pi y}{L} \right) \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi y}{L} \right) \times \left[ 1 - 2 \left( \frac{|x - y|}{L} - \frac{\text{sgn}(x - y)}{\pi} \cos \left( \frac{\pi(x + y)}{L} \right) \sin \left( \frac{\pi(x - y)}{L} \right) \right) \right]^{N-1}. \quad (46)$$

doi:10.1088/1742-5468/2014/11/P11016
With the change of variables $v = \pi(x - y)/L$, $u = \pi(x + y)/L$, this can be rewritten as:

$$
\langle \hat{n}_p \hat{n}_q \rangle = \frac{N}{\pi^2} \int_{-\pi}^{\pi} du \int_{-\pi}^{\pi-|u|} dv \sin\left[p(u + v)/2\right] \sin\left[q(u - v)/2\right] [\cos(v) - \cos(u)] \\
\times \left[1 - \frac{2}{\pi} |v| - \text{sgn}(v) \sin(v) \cos(u)\right]^{N-1}.
$$

(47)

We can proceed as in the case of the mode occupation, assuming that in the TDL the relevant contributions to the density and to the correlation functions only come from a large $p$ and $q$ in equation (47). Thus, we consider $p + q \gg 1$, but we make no assumptions about the difference $p - q$.

In order to make manifest the dependence on $p + q$ and $p - q$ of the above integral, let us use a few simple trigonometric identities. Let us start by expanding

$$
\sin\left[p(u + v)/2\right] \sin\left[q(u - v)/2\right] = \left[\cos(pv/2) \sin(qu/2) + \cos(qu/2) \sin(pv/2)\right] \\
\times \left[\cos(pq/2) \sin(qu/2) - \cos(qu/2) \sin(pq/2)\right].
$$

(48)

Then, let us focus attention on one of the four products (the same argument will be valid for the other terms):

$$
\cos(pv/2) \cos(qv/2) \sin(pu/2) \sin(qu/2) \\
= \frac{1}{2} \cos(pv/2) \cos(qv/2) \{\cos[u(p - q)/2] + \cos[u(p + q)/2]\}.
$$

Now, for $p + q \gg 1$, as the integration domain in the variable $u$ always contains an integer number of periods of the cosine function, we can neglect the term $\cos[u(p + q)/2]$. Collecting together the analogous results for all the four terms, one gets

$$
\sin\left[p(u + v)/2\right] \sin\left[q(u - v)/2\right] \approx \frac{1}{2} \{\sin[v(p + q)/2] \sin[u(p - q)/2] \\
+ \cos[v(p + q)/2] \cos[u(p - q)/2]\},
$$

(50)

where the approximate equality is intended to be valid only under the integration in equation (47) and in the TDL.

Therefore, making use of the fact that the integration domain in $v$ is symmetrical (thus the term proportional to $\sin[v(p + q)/2] \sin[u(p - q)/2]$ vanishes identically) and changing the integration variable $u$ to $u - \pi$, we can recast equation (47) in a straightforward way into

$$
\langle \hat{n}_p \hat{n}_q \rangle = \frac{N}{\pi^2} \int_{-\pi}^{\pi} du \int_{-\pi}^{\pi-|u|} dv \cos\left[v \frac{p + q}{2}\right] \cos\left[u \frac{(p + q)(p - q)}{2}\right] \\
\times [\cos(v) + \cos(u)] \left[1 - \frac{2}{\pi} |v| + \text{sgn}(v) \sin(v) \cos(u)\right]^{N-1}.
$$

(51)

Furthermore, whenever $p - q$ is odd, the argument of the integral is an odd function in the variable $u$, which integrated over the symmetric interval $[-\pi, \pi]$ gives zero.

At this point, following the same reasoning, which led from equation (29) to equation (33), we have

$$
\langle \hat{n}_p \hat{n}_q \rangle = \frac{1}{\pi} \int_{0}^{\pi} du \frac{\cos[(u + \pi)(p - q)/2]}{1 + \left[(p+q)\pi/(4N)\right]^2},
$$

(52)

for $p - q$ even, otherwise it is zero.

doi:10.1088/1742-5468/2014/11/P11016
In order to check the correctness of this result and of all the approximations used, we compared equation (52) with the numerical data obtained directly from the starting expression (47). We performed numerical analysis, fixing one of the modes (let us say \(q\)) and varying \(p\) in an interval centred around \(q\). We did such an analysis for several values of \(q\) and, for \(N = 50\), we obtained very good agreement for sufficiently large \(p + q\).

4.2. Time evolution of the density profile

Plugging the mode–mode correlator (52) into equation (17), the time-dependent particle density reads

\[
n(x, t) = \frac{2}{\pi L} \sum_{p, q \in D} \int_{0}^{\pi} du \sin(p\pi x/L) \sin(q\pi x/L) \frac{\cos[(u + \pi)(p - q)/2]}{1 + \left[\frac{(p+q)\pi/(4N)}{1+\cos(u)}\right]^{2}} e^{i\pi^{2}t(p+q)(p-q)/L^{2}},
\]

(53)

where the indices of the double sum run over the domain \(D\) such that their difference \(p - q\) is an even integer (i.e. \(p\) and \(q\) are either both odd or both even). Therefore, the obvious change of variable is \(p + q \equiv 2r\), \(p - q \equiv 2l\); the domain \(D\), in terms of these new variables, becomes \(1 < r < \infty, -r + 1 \leq l \leq r - 1\). Then, equation (53) can be written as

\[
n(x, t) = \frac{1}{\pi L} \sum_{r=1}^{\infty} \sum_{l=-r+1}^{r-1} \int_{0}^{\pi} du \left[\cos(2l\pi x/L) - \cos(2r\pi x/L)\right] \frac{\cos[l(u + \pi)]}{1 + \left[\frac{r\pi/(2N)}{1+\cos(u)}\right]^{2}} e^{i4\pi^{2}trl/L^{2}}.
\]

(54)

Since we are interested in the TDL, we introduce the rescaled variables \(\tilde{x} = x/L\), \(\tilde{t} = t/L\) and \(\tilde{r} = r/L\). The sum over \(r\) becomes an integral in the new variable \(\tilde{r}\) and the sum over \(l\) can be extended from \(-\infty\) to \(+\infty\), obtaining

\[
n(x, t) = \frac{1}{\pi} \int_{0}^{\infty} d\tilde{r} \sum_{l=-\infty}^{\infty} \int_{0}^{\pi} du \left[\cos(2l\tilde{r}\tilde{x}) - \cos(2\tilde{r}\pi\tilde{x})\right] \frac{\cos[l(u + \pi)]}{1 + \left[\frac{\tilde{r}\pi/(2N)}{1+\cos(u)}\right]^{2}} e^{i4\pi^{2}\tilde{r}l}\cos(4\pi^{2}\tilde{r}l)
\]

\[
\simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tilde{r} \sum_{l=-\infty}^{\infty} \int_{0}^{\pi} du \cos(2l\tilde{r}\tilde{x}) \frac{\cos[l(u + \pi)]}{1 + \left[\frac{\tilde{r}\pi/(2N)}{1+\cos(u)}\right]^{2}} e^{i4\pi^{2}\tilde{r}l},
\]

(55)

where in the last line we dropped the term \(\cos(2\tilde{r}\pi\tilde{x})\) because it rapidly oscillates for \(L \to \infty\). In performing this approximation, we lose information about the behaviour close to the boundaries, but this is exactly what was forced when we introduced the thermodynamic variable \(\tilde{x}\).

The integral in \(\tilde{r}\) in equation (55) can be explicitly performed, giving

\[
n(x, t) = n \sum_{l=-\infty}^{\infty} (-1)^{l} \cos(2l\pi\tilde{x}) \int_{0}^{\pi} \frac{du}{\pi} \cos(lu) \left[1 + \cos(u)\right] e^{-8\pi^{2}lu/\left[1+\cos(u)\right]}
\]

\[
= n + 2n \sum_{l=1}^{\infty} (-1)^{l} \cos(2l\pi\tilde{x}) \int_{0}^{\pi} \frac{du}{\pi} \cos(lu) \left[1 + \cos(u)\right] e^{-8\pi^{2}lu/\left[1+\cos(u)\right]}.
\]

(56)

This result does not depend on \(L\) and \(N\) independently, but only on their ratio \(n = N/L\), as it should do.

doi:10.1088/1742-5468/2014/11/P11016
The \(n(x, t)\) just derived correctly reproduces the two limiting cases \(t = 0\) and \(t \to \infty\). The former is obtained by setting \(\tilde{t} = 0\) in equation (56) and then the integral can be easily evaluated, giving the following for the density

\[
n(x, 0) = n \sum_{l=-\infty}^{\infty} \frac{(-1)^l \sin[\pi l]}{\pi l(l^2 - 1)} \cos(2l\pi x) = 2n \sin^2(\pi x),
\]

where we used the fact that the only non-zero contributions to the sum come from \(l = 0\) and \(|l| = 1\). The limit \(t \to \infty\) is given only by the \(l = 0\) term in the sum (since all the others are exponentially suppressed) and it agrees with the bulk stationary result \(n\).

More explicit information can be extracted from equation (56) using the integral representation of the modified Bessel functions (valid for \(m \in \mathbb{Z}\) and \(s \in \mathbb{R}\))

\[
I_m(s) = (-1)^m \int_0^\pi \frac{du}{\pi} \cos(mu)e^{-s\cos(u)},
\]

and the identity \(\partial_s I_m(s) = [I_{m-1}(s) + I_{m+1}(s)]/2\). Therefore, equation (56) can be written as

\[
n(x, t) = n - 2n \sum_{l=1}^{\infty} \cos(2l\pi x) \partial_s \left[I_l(s)e^{-s}\right] \bigg|_{s=\pi\tilde{t}l} = n + n \sum_{l=1}^{\infty} \cos(2l\pi x)e^{-8\pi l\tilde{t}} \left[2I_l(8\pi l\tilde{t}) - I_{l-1}(8\pi l\tilde{t}) - I_{l+1}(8\pi l\tilde{t})\right].
\]

This form shows clearly how time-dependent density approaches the stationary value. Indeed, even if the presence of the exponential factor could suggest a typical relaxation time, combination with the Bessel functions gives rise to an algebraic decay for large times. Indeed, relaxation of the density takes place in a two-step process. Firstly, there is a short transient for \(n\tilde{t} \ll 1\), in which the density decays very quickly to a value very close to the stationary one, see figure 6. After this transient, relaxation is slowed down dramatically to an algebraic behaviour. Indeed, the use of the asymptotic expansion \([2I_l(s) - I_{l-1}(s) - I_{l+1}(s)]\exp(-s) \sim s^{-3/2}/\sqrt{2\pi}\) leads us to the following large-time behaviour of the density (see figure 6)

\[
n(x, t) \sim n + \frac{n}{64\pi^2(n\tilde{t})^{3/2}} \left[\text{Li}_{3/2}(e^{2\pi i}) + \text{Li}_{3/2}(e^{-2\pi i})\right] \quad \text{for} \quad \tilde{t} \gg 1,
\]

in terms of the Polylogarithm function \(\text{Li}_m(s) \equiv \sum_{k=1}^{\infty} \frac{s^k}{k^m}\).

The physical interpretation of this two-step relaxation behaviour is very intuitive. Soon after the quench, the bosons experience an infinitely strong repulsion, which suddenly tends to reduce the density in the centre by moving particles close to the boundaries. However, after this rapid process the final equilibration takes place by means of a series of many bounces off the boundaries and this process needs times that are much larger than \(L/n\) (we recall that the speed of sound in the Tonks Girardeau gas and in our normalisation is \(v = 2\pi n\)).

4.3. Time evolution of the two-point fermionic correlation function

The time-dependent two-points fermionic correlation function \(C(x, y; t)\) can be evaluated along the same lines as the density. Plugging the mode–mode correlator (52) into
equation (16), we have
\[
C(x, y; t) = \frac{2}{\pi L} \sum_{p, q \in \mathbb{D}} \int_0^\pi du \sin \left[ \frac{p\pi (w + z)}{2L} \right] \sin \left[ \frac{q\pi (w - z)}{2L} \right] \frac{\cos(u + \pi)(p - q)/2}{1 + \left[ \frac{(p+q)\pi (4N)}{1+\cos(u)} \right]^2} e^{i4t\pi \frac{p+q}{2L}}.
\] (61)

where we introduced the variables \( z = x - y, \ w = x + y \). As shown in appendix C, as long as we are interested in the TDL of this correlator in the bulk, we can perform the following replacement in the integral:
\[
\sin \left[ \frac{p\pi (w + z)}{2L} \right] \sin \left[ \frac{q\pi (w - z)}{2L} \right] \rightarrow \cos \left[ (p + q)\pi z/(2L) \right] \cos \left[ (p - q)\pi w/(2L) \right]/2.
\]

Then, changing the indices of the sum as \( p + q = 2r \) and \( p - q = 2l \), the correlator can be written as
\[
C(x, y; t) = \frac{1}{4\pi} \int_{-\infty}^\infty d\tilde{r} \sum_{l = -\infty}^{\infty} \int_0^\pi du \cos(l\tilde{r}z/L) \cos(l\pi w/L) \frac{\cos[l(u + \pi)]}{1 + \left[ \frac{r\pi (2N)}{1+\cos(u)} \right]^2} e^{4i\tilde{r}tL}. \] (62)

By analogy with the density profile, we introduce the rescaled variables
\[
\tilde{r} = r/L, \quad \tilde{t} = t/L, \quad \tilde{w} = w/L,
\] (63)
but we do not rescale the distance between the two points \( z = x - y \), in such a way as to explore correlations at arbitrary distances in the bulk. We replace the sum over \( r \) with an integral over \( \tilde{r} \), let the sum over \( l \) run from \( -\infty \) to \( \infty \) and we use simple trigonometric identities to write the correlator as
\[
C(x, y; t) = \frac{1}{4\pi} \int_{-\infty}^\infty d\tilde{r} \sum_{l = -\infty}^{\infty} \int_0^\pi du \cos(l\pi \tilde{w}) \frac{\cos[l(u + \pi)]}{1 + \left[ \frac{r\pi (2N)}{1+\cos(u)} \right]^2} \left[ \cos(\tilde{r}\pi z + 4\pi^2 \tilde{r}\tilde{t}L) + \cos(\tilde{r}\pi z - 4\pi^2 \tilde{r}\tilde{t}L) \right]. \] (64)

Figure 6. (Left) The density profile \( n(x, t) \) given by equation (59) as a function of \( \tilde{x} = x/L \) for fixed rescaled times \( \tilde{t} = nt/L \). The full black line is the uniform stationary value reached for \( t \to \infty \). (Right) The time evolution of the density \( n(x, t) \) as a function of rescaled time at fixed \( \tilde{x} \). After a short transient, the density approaches the stationary value as a power-law (see inset for a comparison, of log–log scale, with the large-time behaviour given by equation (60) (red straight lines)).
Once again, the integral in $\tilde{r}$ can be easily performed, being proportional to the Fourier transformation of a Lorentzian function, obtaining

$$ C(x, y; t) = n \sum_{l=-\infty}^{\infty} (-1)^l \cos(l \pi \tilde{w}) \int_0^{\pi} \frac{du}{2\pi} \cos(lu)[1 + \cos(u)] $$

$$ \times \left\{ e^{-2n|z+4\pi\tilde{t}|[1+\cos(u)]} + e^{-2n|z-4\pi\tilde{t}|[1+\cos(u)]} \right\}, $$

$$ = -\frac{n}{2} \sum_{l=-\infty}^{\infty} \cos(l \pi \tilde{w}) \left\{ \partial_s \left[ I_l(s)e^{-s} \right] \bigg|_{s=2n|z+4\pi\tilde{t}|} + \partial_s \left[ I_l(s)e^{-s} \right] \bigg|_{s=2n|z-4\pi\tilde{t}|} \right\}. $$

(65)

We can now isolate the term $l = 0$, which corresponds to the bulk stationary result $C_{B}(z)$ in equation (39)) and we can rewrite equation (65) as

$$ C(x, y; t) = C_{B}(z) - n \sum_{l=1}^{\infty} \cos(l \pi \tilde{w}) \left\{ \partial_s \left[ I_l(s)e^{-s} \right] \bigg|_{s=2n|z+4\pi\tilde{t}|} + \partial_s \left[ I_l(s)e^{-s} \right] \bigg|_{s=2n|z-4\pi\tilde{t}|} \right\}. $$

(66)

Let us now critically analyse this time-dependent correlation function. For finite rescaled time, $\tilde{t} < \infty$, equation (66) is not a translational invariant since it depends both on $z$ and $\tilde{w}$. As $\tilde{t} \to \infty$, all terms with $l \neq 0$ in the sum vanish and only the translational invariant stationary part survives. In the opposite limit, $\tilde{t} = 0$, equation (66) should reproduce the scaling regime of the initial correlation function $\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y) \rangle$ in equation (26). This is not so apparent from the series representation of the correlation function, but can be shown by plugging the following infinite sum

$$ \sum_{l=-\infty}^{\infty} a^l I_l(s) = \exp[s(a+1/a)/2], $$

(67)

into equation (65), obtaining

$$ C(x, y; 0) = -n \partial_s \left[ e^{-s} \sum_{l=\infty}^{\infty} \text{Re} \left( e^{il\pi \tilde{w}} I_l(s) \right) \bigg|_{s=2n|z|} \right] $$

$$ = -n \partial_s e^{-\frac{1}{2}(1-\cos(\pi \tilde{w}))|z|} \bigg|_{s=2n|z|} = n \left[ 1 - \cos(\pi \tilde{w}) \right] e^{-2n(1-\cos(\pi \tilde{w}))|z|}, $$

(68)

which coincides with equation (26).

The time-evolved correlation function in equation (66) depends on both $z$ and $\tilde{w}$. These two variables work on different scales: (i) $z = x - y$ is a ‘local’ variable and is the only one that survives in the stationary state; (ii) $\tilde{w}$ is a ‘global’ variable on which $z$ is modulated. Therefore, in order to understand the physics of equation (66) for different times, it is useful to fix the value of $\tilde{w}$ and plot the time-dependent correlator as a function of the local variable $z$ as in figure 7. The resulting behaviour is reminiscent of that observed in other quench studies with initial inhomogeneous states [33, 55]. Indeed, the correlation function approaches the stationary value by expelling a series of traveling peaks from the vicinity of $z \simeq 0$ which move ballistically through the system. The velocity of the primary peaks (i.e. the highest and most visible ones in figure 7) is $v_p = 4\pi/L$. There is an infinite number of smaller secondary peaks (a second one is visible on a closer look at figure 7), which move with velocities that are integer multiples of $v_p$. This aspect is independent of
Interaction quench in a trapped 1D Bose gas

Figure 7. (Top) space-time contour plot of the time-dependent fermionic correlator $C(x, y; t)$ at fixed $\tilde{w} \equiv w/L = (x + y)/L$. Time is rescaled as $\tilde{t} = t/L$ and $z = x - y$. The plotted region corresponds to $n\tilde{t} \in [0, 1]$ and $nz \in [-20, 20]$. Notice the correlation peaks expelled from $z = 0$, which move ballistically with velocities that are integer multiples of $v_p = 4\pi/L$. (Bottom) Profiles of the fermionic correlation function as a function of $z$ for different rescaled time $n\tilde{t} = nt/L$ and $w/L = 1, 1/4$ (i.e. each curve is a horizontal cut of the contour plot on the top).

To conclude our analysis, in figure 8 we report a contour plot of $C(x, y; t)$ as a function of $nz$ and $\tilde{w}$ for different times. These plots show pictorially how the initial inhomogeneous correlation is smoothed out and made uniform by counter propagating fronts emitted from $z = 0$. Consequently, there is an effective region inside a horizon $|z| < v_p\tilde{t}$ in which the system is almost stationary and translationally invariant (and hence approximately the precise value of $\tilde{w}$, while the other details of this process (e.g. shape and amplitude of the peaks) depend on the value of $\tilde{w}$ (as should be clear from figure 7). Consequently, the characteristic time in which the correlator at distance $z$ gets close to its stationary value is almost independent of $\tilde{w}$, because it is roughly the time needed for the primary peaks to travel a distance $z$. 

"doi:10.1088/1742-5468/2014/11/P11016"
Interaction quench in a trapped 1D Bose gas

Figure 8. Color snapshots of the fermionic correlation $C(x, y, t)$ given by equation (66) for different rescaled times $n\tilde{t} = nt/L$ as a function of $n z = n(x - y) \in [-2, 2]$ and $\tilde{w} = (x + y)/L \in [0, 2]$. Notice the initially strong inhomogeneity along $\tilde{w}$, which is smoothed out during the time evolution.

We have analysed the effect of a hard-wall trapping potential in the 1D Bose gas following a quantum quench from free to hard-core bosons. Both the initial conditions and the Hamiltonian governing the time evolution break translational invariance. As a consequence, the density and the two-point correlation function exhibit a nontrivial space-time dependence. Although this is a quench between two free theories, the pre- and post-quench mode-operators are not linearly related and, therefore, the time evolution shows many non-trivial effects, such as the breaking of Wick’s theorem for finite times.

5. Conclusion

We have analysed the effect of a hard-wall trapping potential in the 1D Bose gas following a quantum quench from free to hard-core bosons. Both the initial conditions and the Hamiltonian governing the time evolution break translational invariance. As a consequence, the density and the two-point correlation function exhibit a nontrivial space-time dependence. Although this is a quench between two free theories, the pre- and post-quench mode-operators are not linearly related and, therefore, the time evolution shows many non-trivial effects, such as the breaking of Wick’s theorem for finite times.
We studied both large time behaviour and full-time evolution of the density profile and of the two-point fermionic correlation function in detail. Their large-time properties turned out to be described by the GGE constructed with the mode occupation numbers and hence Wick’s theorem is restored for large time. Although the system is not translationally invariant, the stationary density is uniform. The bulk correlation function turned out to depend only on the distance between the two points and so we conclude that translational invariance is dynamically restored (in the GGE, Wick’s theorem applies and so all correlations can be derived from the two-point one), apart from with finite size effects close to the boundaries. However, the stationary state retains memory of the initial inhomogeneous state because the asymptotic two-point fermionic correlation function decays algebraically for large (bulk) distances, while in the periodic case decay has been found to always be exponential [22]. We point out here that this very peculiar effect is mainly due to the highly inhomogeneous initial state. This is a consequence of the bosonic nature of the initial state in which all the particles are in the same one-particle state. This does not happen for quenches in ‘purely fermionic’ theories (such as the Ising chain studied in [60]), for which a hard-wall trapping potential leads to a less inhomogeneous initial state because of the effective repulsion due to the Pauli principle. To complete the analysis of the stationary state, we also computed the bosonic correlation function numerically; it turned out to decay exponentially for large distances. The decay rate (i.e. the correlation length) is however different from the one in the case of the PBC.

We also studied the full time-dependence for both particle density and the two-point fermionic correlation function. We found that relaxation takes place by a two-step mechanism; firstly, there is a rapid transient in which the density drops to an almost uniform value and then the decay to the stationary value is algebraic and it is driven by the particles bouncing off the boundaries many times. The equilibration of the two-point function takes place by the expulsion of a series of correlation peaks that move out ballistically, leaving the system almost equilibrated inside an effective horizon.

It is of considerable interest to determine whether our findings can be generalised to the case of the experimentally more relevant harmonic-trapping potential. However, immediately the algebra becomes very cumbersome because the one-particle eigenfunctions are Hermite polynomials and not simple trigonometric functions. In the light of the results that we obtained for hard-wall confinement, it seems very unlikely that an approach based on local density approximation could provide the correct answer, making an exact calculation for the harmonic trap even more desirable.

Another more difficult generalisation would be to consider the same quench in the presence of a hard-wall trap, but to a finite interaction Lieb–Liniger model (which is integrable [36]). However, the overlaps needed in the Bethe ansatz framework [11] are very difficult to calculate compared to the periodic case in which the initial many-body wave-function is constant [23,26].

Acknowledgments

PC and MC acknowledge the ERC for financial support under Starting Grant 279391 EDequations MK acknowledges financial support from the Marie Curie IIF Grant PIIF-GA-2012-330076.

doi:10.1088/1742-5468/2014/11/P11016
Appendix A. Lattice formulation

In this appendix we provide a rigorous lattice regularisation to justify the identity (21). We closely follow the analogous calculation for PBC [22].

Let us consider a system of \( N \) bosons hopping on a 1D lattice with \( M \) sites with lattice spacing \( \delta \): the length of the lattice is \( L = M \delta \). The one-particle eigenfunction associated with the lowest energy level is \( \sqrt{2/M} \sin(\pi i/M) \) and so the many-body ground state is given by

\[
|N\rangle = \sqrt{\frac{2^N}{M^N N!}} \left( \sum_{i=1}^{M} \sin\left( \frac{\pi i}{M} \right) \hat{b}_i \right)^N |0\rangle, \tag{A1}
\]

where \( \hat{b}_i \) are the canonical bosonic operators acting on the \( i \)th site and \( |0\rangle = \prod_i |0\rangle_i \) with \( |n\rangle_i \) is the \( n \)-boson state at site \( i \).

The hard-core boson operators are defined as in the continuum case

\[
\hat{a}_i = P_i \hat{b}_i P_i, \quad \hat{a}_i^\dagger = P_i \hat{b}_i^\dagger P_i, \tag{A2}
\]

with \( P_i = |0\rangle_i \langle 0|_i + |1\rangle_i \langle 1|_i \) being the on-site projector on the truncated Hilbert space. The hard-core boson operators satisfy the \textit{mixed} algebra

\[
[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j] = [\hat{a}_i, \hat{a}_j^\dagger] = 0, \quad i \neq j, \quad \hat{a}_i^2 = \hat{a}_i^{\dagger 2} = 0, \quad \{\hat{a}_i, \hat{a}_j^\dagger\} = 1. \tag{A3}
\]

Jordan-Wigner mapping from hard-core bosons to free fermions on the lattice reads

\[
\hat{a}_i = e^{-i\pi \sum_{j<i} \hat{c}_j^\dagger \hat{c}_i}, \quad \hat{c}_i = e^{i\pi \sum_{j<i} \hat{a}_j \hat{a}_i^\dagger} (1 - 2 \hat{c}_i^\dagger \hat{c}_i), \quad \hat{c}_i^\dagger = \hat{c}_i \tag{A4}
\]

with \( \{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{i,j} \). The lattice thermodynamic limit is defined as \( N, M \to \infty \), keeping the filling factor \( \nu = N/M \) constant. The continuum limit for finite systems is obtained by considering the lattice spacing \( \delta \to 0 \), the number of sites \( M \to \infty \), while the physical length \( L = M \delta \) is kept constant. Therefore, the continuum TDL can now be taken as \( N, L \to \infty \), with gas density \( n = N/L \) constant (we can equivalently think to the continuum and thermodynamic limit as the limit \( \delta, \nu \to 0 \), keeping constant \( n = \nu \delta \)). Finally, the relationships between the lattice and the continuum operators are

\[
\hat{b}_i = \sqrt{\delta} \hat{\phi}(\delta i), \quad \hat{a}_i = \sqrt{\delta} \hat{\Psi}(\delta i), \quad \hat{c}_i = \sqrt{\delta} \hat{\Phi}(\delta i). \tag{A5}
\]

The initial fermionic correlation function for \( k < l \) can be written as

\[
\langle N| \hat{c}_k^\dagger \hat{c}_l |N\rangle = \langle N|\hat{a}_k^\dagger \prod_{j=k+1}^{l-1} (1 - 2 \hat{a}_j^\dagger \hat{a}_j)\hat{a}_l |N\rangle = \sum_{r=0}^{\infty} (-2)^r \sum_{k<n_1<...<n_r<l} \langle N|\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \ldots \hat{a}_{n_r}^\dagger \hat{a}_n \hat{a}_l |N\rangle. \tag{A6}
\]

Therefore, in order to find the fermionic correlation function we have to evaluate the multipoint hard-core boson correlators

\[
\langle N|\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \ldots \hat{a}_{n_r}^\dagger \hat{a}_n \hat{a}_l |N\rangle, \tag{A7}
\]

which can be calculated by expand the multinomial in equation (A1) as

\[
|N\rangle = \sqrt{\frac{2^N}{M^N N!}} \sum_{i_1,...,i_M} \left( \begin{array}{c} N \\ i_1, \ldots, i_M \end{array} \right) (p_1 \hat{b}_1)^{i_1} \ldots (p_M \hat{b}_M)^{i_M} |0\rangle, \tag{A8}
\]

doi:10.1088/1742-5468/2014/11/P11016
where $p_i \equiv \sin(\pi i/M)$ and the sum runs over all sets of non-negative integers $\{i_1, \ldots , i_M\}$ such that $\sum_j i_j = N$. Let us start by considering the action of the hard-core boson string in equation (A7) on the many-body ground state, i.e. $\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \ldots \hat{a}_{n_1}^\dagger \hat{a}_n^\dagger \hat{a}_l^\dagger |N\rangle$. If we want a non-zero result, we must fix the value of some indices, i.e. $i_l = i_n = \ldots = i_{n_1} = 1$ and $i_k = 0$. This comes from having rewritten the hard-core boson operators $\hat{b}^{(l)}$ in terms of the canonical ones $\hat{b}^{(l)}$: the projectors $P_l$ appear and they annihilate all the multi-occupied sites, thus obtaining

$$\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \ldots \hat{a}_{n_1}^\dagger \hat{a}_n^\dagger \hat{a}_l^\dagger |N\rangle$$

$$= \sqrt{2^N/N!} \sum_{\{i_1, \ldots , i_M\}'} \left( \frac{N}{i_1, \ldots , i_k = 0, \ldots , i_{n_1} = 1, \ldots , i_{n_r} = 1, i_l = 1, \ldots , i_{M} \right) \times (p_l b_l^1)^{i_1} \ldots (p_M b_M^1)^{i_M} |0\rangle,$$

wherein $\{i_1, \ldots , i_M\}' = \{i_1, \ldots , i_M\} \setminus \{i_k, i_{n_1}, \ldots , i_{n_r}\}$. Notice that in the previous equation all $b_{n_1}^1$ and $b_l^1$ come with power one, while there is no $b_r^1$. When we consider the scalar product between the state defined in equation (A9) and the ground state $|N\rangle$ the only non-zero contributions come from those terms which perfectly match the powers of all operators. Therefore, by using $\langle 0|(p_l b_l^1)^n(p_M b_M^1)^m|0\rangle = p_l^{2n} p_M^{2m}$ we obtain

$$\langle N|\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \hat{a}_n^\dagger \hat{a}_l^\dagger |N\rangle = \frac{2^N}{M^{N} N!} p_k p_l \prod_{j=1}^r p_{n_j} \sum_{\{i_1, \ldots , i_{M}\}'} \left( \frac{N}{i_1, \ldots , i_M} \right)^2 (p_1^{2i_1} \ldots p_M^{2i_M} i_M!,$$

which, since $\sum_{j: i_j \in \{i_1, \ldots , i_{M}\}'} i_j = N - r - 1$, can be rewritten as

$$\langle N|\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \hat{a}_n^\dagger \hat{a}_l^\dagger |N\rangle = \left( \frac{2}{M} \right)^{N-r-1} p_k p_l \prod_{n_1} \ldots \prod_{n_r} N(N-1) \ldots (N-r)$$

$$\times \left( \sum_{j: i_j \in \{i_1, \ldots , i_{M}\}'} p_j^2 \right)^{N-r-1},$$

where, once again, we used the definition of the multinomial expansion. The indices $\{i_1, \ldots , i_M\}'$ are $M - r - 2$ and their distribution depends on how the other $r + 2$ indices, namely $k, l, n_1, \ldots , n_r$, have been chosen on the lattice. In the continuum limit between the site $l$ and $k$ there is an infinite number of operators, however, $r \leq N - 1$ since the string $\hat{a}_{n_1}^\dagger \hat{a}_n^\dagger \hat{a}_n^\dagger \hat{a}_n^\dagger$ acts on $(N - 1)$-particle state with $N$ finite. Moreover, in such a limit, the lattice holds an infinitely dense number of sites and, therefore, the index $j$ such that $i_j \in \{i_1, \ldots , i_{M}\}'$ runs over the whole lattice, except for the $r + 2$ positions, which represent a subset of null measures in the continuum limit. Thus the following approximation holds

$$\sum_{j: i_j \in \{i_1, \ldots , i_{M}\}'} p_j^2 \simeq \frac{1}{\delta} \int_0^L dz \sin^2 \left( \frac{\pi z}{L} \right) = \frac{M}{2},$$

which leads to

$$\langle N|\hat{a}_k^\dagger \hat{a}_{n_1}^\dagger \hat{a}_n^\dagger \hat{a}_l^\dagger |N\rangle \simeq \left( \frac{2}{M} \right)^{r+1} N(N-1) \ldots (N-r) p_k p_l \prod_{n_1} \ldots \prod_{n_r}.$$

\textbf{doi:10.1088/1742-5468/2014/11/P11016}
At this point, in order to calculate the fermionic two-point function, we have to sum terms like those in equation (A11) over the indices \( n_i \). This sum can be done using the approximation
\[
\sum_{n_1=k+1}^{l} \sum_{n_2=n_1+1}^{l} \ldots \sum_{n_r=n_{r-1}+1}^{l} p_{n_1}^2 p_{n_2}^2 \ldots p_{n_r}^2 \simeq \frac{1}{r!} \left( \sum_{m=k+1}^{l} p_m^2 \right)^r, \tag{A14}
\]
which is actually exact in the continuum limit (i.e. when sums are replaced by integrals). Therefore, inserting equation (A13) into equation (A6) and using equation (A14), we finally get
\[
\langle N| \hat{c}_k \hat{c}_l |N \rangle = 2 \frac{N}{M} \sum_{r=0}^{N-1} (-2)^r (N-1) \ldots (N-r) \left( \frac{2}{M} \sum_{m=k+1}^{l} p_m^2 \right)^r N^{-1} \tag{A15}
\]
Now using \( \langle \hat{c}_k \hat{c}_l \rangle = \delta \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle \) and \( \delta \sum_m \equiv \int dz \) with \( z = \delta m \), we can take the continuum limit of equation (A15), obtaining (for \( x < y \))
\[
\langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle = 2 \frac{N}{M \delta} \sum_{r=0}^{N-1} (-2)^r (N-1) \ldots (N-r) \left( \frac{2}{M} \sum_{m=k+1}^{l} \delta p_m^2 \right)^r N^{-1} \tag{A16}
\]
which coincides with equation (25).

**Appendix B. Generic confining potential**

The initial correlation function given in equation (25) has two special limits, which are not exclusive features of the hard-wall confining potential, but are valid for a generic potential (as long as the initial interaction is set to \( c = 0 \)), as we are going to show in this appendix.

Let us consider a generic trapping potential centred in \( x = 0 \). The potential introduces a typical length-scale \( \ell > 0 \) [61] (for example, in the presence of a harmonic confinement \( V(x) = \omega^2 x^2 / 2 \) the typical length is \( \ell \simeq 1 / \sqrt{\omega} \)), such that the eigenfunctions vanish for \( |x| \gg \ell \). The orthonormal one-particle eigenfunctions can be written as
\[
\phi_p(x) = \frac{1}{\sqrt{\ell}} \psi_p \left( \frac{x}{\ell} \right), \quad p = 0, 1, 2, \ldots, \tag{B1}
\]
where \( \psi_p(z) \) are the normalised eigenfunctions for \( \ell = 1 \).

We consider again as the initial state the BEC constructed by placing \( N \) particles in the lower energy-level with wave-function \( \phi_0(x) \). Following the same logic, which led us...
to equation (25), we find the general form for the fermionic correlation function

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y) \rangle = N\phi_0^*(x)\phi_0(y) \left[ 1 - 2 \left| \int_x^y dz \, |\phi_0(z)|^2 \right| \right]^{N-1},$$

which is valid for any finite $N$ and $\ell$. Let us introduce the integral function

$$F(x) \equiv \int_{-\infty}^x dz \, |\phi_0(z)|^2 = \frac{1}{\ell} \int_{-\infty}^{x/\ell} dz \, |\psi_0(z/\ell)|^2 = \int_0^{x/\ell} d\tilde{z} \, |\psi_0(\tilde{z})|^2 \equiv \tilde{F}(x/\ell),$$

which satisfies $|\tilde{F}(y/\ell) - \tilde{F}(x/\ell)| \leq 1$. $F(x)$ is a bounded monotonic even function, therefore $F(x) = F(y)$ implies $|x| = |y|$. We consider the thermodynamic limit $N \to \infty$, $\ell \to \infty$ with $N/\ell = n$ and with the additional constraint that the rescaled variables $x/\ell$ and $y/\ell$ are kept finite. Now, since $\psi_0(z)$ is a one-particle ground-state function with no nodes in its domain (apart from the boundaries if the domain is finite), the limit

$$\lim_{N \to \infty} N \left[ 1 - 2 \left| \tilde{F}(y/\ell) - \tilde{F}(x/\ell) \right| \right]^{N-1} = \frac{\delta(x - y)}{|\psi_0(x/\ell)|^2},$$

leads to

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y) \rangle = \delta(x - y), \quad \text{for} \quad N \to \infty.$$  \hfill (B5)

However, this result does not correspond to the correct scaling regime and applies only to the case of a very tight confining potential with an extremely localised two-point fermionic function, in which the details of the trapping are lost. The fermionic mode occupation $\langle \hat{n}_q \rangle$ corresponding to this two-point correlator is $\langle \hat{n}_q \rangle = 1$, which clearly is not physical.

In order to circumvent this problem, we could think of taking the TDL by considering the variables $x$ and $y$ finite. In this case equation (B2) can be rewritten as

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y) \rangle = n|\psi_0(x/\ell)\psi_0(y/\ell)\rangle \left[ 1 - 2 \left| \tilde{F}(y/\ell) - \tilde{F}(x/\ell) \right| \right]^{N-1},$$

which, for $\ell \gg 1$, with $x$ and $y$ fixed, can be expanded around $x, y \sim 0$ as

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y) \rangle = n|\psi_0(0)|^2 \left[ 1 - \frac{2n}{N} |\psi_0(0)|^2 |x - y| \right]^{N-1}.$$  \hfill (B7)

and finally, using $\lim_{N \to \infty} (1 + z/N)^N = e^z$, one obtains

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(y) \rangle = n|\psi_0(0)|^2 e^{-2n|\psi_0(0)|^2|x - y|},$$  \hfill (B8)

which coincides with the result for PBC [22] (when $|\psi_0(0)|^2 = 1$). This result is easily understood; for $\ell \gg 1$ and $x$ and $y$ finite, the system retains only information about the value of the initial density in the middle of the trap, i.e. equation (B8) is equivalent to considering a translationally invariant case with homogeneous initial density equals to $|\psi_0(0)|^2$ thence completely losing the effect of the trap.

Thus, we conclude that there are no shortcuts to this problem and, in order to retain properly confinement effects, the correct way to proceed is to keep $N$ and $\ell$ finite (which in our specific case corresponds to the size $L$) for the calculation of $\langle \hat{n}_q \rangle$ and $\langle \hat{n}_p^\dagger\hat{n}_q \rangle$ and only afterwards take the thermodynamic limit.
Appendix C. Technical details for the evaluation of the time-dependent correlation function

In this appendix we show the details of the calculations needed to derive equation (62) from equation (61).

From well known trigonometric identities, we can rewrite the term
\[ \sin \left( \frac{p\pi}{2L} (w + z) \right) \sin \left( \frac{q\pi}{2L} (w - z) \right), \]

as
\[ \frac{1}{2} \cos \left( \frac{\pi(p + q)z}{2L} \right) \cos \left( \frac{\pi(p - q)w}{2L} \right) - \frac{1}{2} \cos \left( \frac{\pi(p - q)z}{2L} \right) \cos \left( \frac{\pi(p + q)w}{2L} \right) + \frac{1}{2} \sin \left( \frac{\pi(p + q)z}{2L} \right) \sin \left( \frac{\pi(p - q)w}{2L} \right) - \frac{1}{2} \sin \left( \frac{\pi(p - q)z}{2L} \right) \sin \left( \frac{\pi(p + q)w}{2L} \right). \]

In the thermodynamic limit the only relevant contribution to the time-dependent correlation function comes from the first term of equation (C2). All the other terms are either identically vanishing or introduce finite-size corrections which disappear in the TDL. Indeed, since we are working in the regime \( p + q \gg 1, w = x + y \sim O(L) \) and \( z = x - y \sim O(1) \), using the rescaled variables \( \tilde{w} = w/L, \tilde{r} = (p + q)/(2L), \tilde{l} = (p - q)/2 \), we have
\[ \frac{1}{2} \cos(\pi \tilde{r} \tilde{z}) \cos(\pi \tilde{l} \tilde{w}) - \frac{1}{2} \cos \left( \frac{\pi l}{L} \right) \cos(\pi L \tilde{r} \tilde{w}) + \frac{1}{2} \sin(\pi \tilde{r} \tilde{z}) \sin(\pi \tilde{l} \tilde{w}) \]
\[ - \frac{1}{2} \sin \left( \frac{\pi l}{L} \right) \sin(\pi L \tilde{r} \tilde{w}). \]

The third term in equation (C3) does not contribute to the evaluation of the time-dependent correlation function since it is an odd function of \( l \) and the sum over \( l \) in equation (62) is symmetric around zero. Moreover, as \( L \to \infty \) the last term vanishes. Therefore, the only terms which survive are
\[ \frac{1}{2} \cos(\pi \tilde{r} \tilde{z}) \cos(\pi \tilde{l} \tilde{w}) - \frac{1}{2} \cos(\pi L \tilde{r} \tilde{w}). \]

The first term in equation (C4) is exactly the term that was considered in the main text and which leads to the correct time-dependent correlation function. The second one, instead, introduces only finite-size corrections; indeed, following the same reasoning as in section 4, it is straightforward to show that it corresponds to \( (\tilde{t} = t/L) \)
\[ - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tilde{r} \sum_{l = -\infty}^{\infty} \int_0^{\pi} du \cos(\pi L \tilde{r} \tilde{w}) \cos[l(u + \pi)] \frac{e^{i4\pi^2 \tilde{r} \tilde{u}}}{1 + \exp(u)} \sim \exp(-L), \]
thus vanishing in the thermodynamic limit.

References

[1] Greiner M, Mandel O, H"ansch T W and Bloch I 2002 Nature 419 51
[2] Kinoshita T, Wenger T and Weiss D S 2006 Nature 440 900

doi:10.1088/1742-5468/2014/11/P11016
Interaction quench in a trapped 1D Bose gas

[3] Trotzky S, Fleisch A, McCulloch I P, Schollwöck U, Eisert J and Bloch I 2012 Nat. Phys. 8 325
[4] Cheneau M, Barmettler P, Poletti D, Endres M, Schauß P, Fukuhara T, Gross C, Bloch I, Kollath C and Kuhr S 2012 Nature 481 484
[5] Gring M, Kuhnert M, Langen T, Kitagawa T, Rauer B, Schreitl M, Mazets I, Smith D A, Demler E and Schmiedmayer J 2012 Science 337 1318
[6] Schneider U et al 2012 Nat. Phys. 8 213
[7] Ronzheimer J P, Schreiber M, Braun S, Hodgman S S, Langer S, McCulloch I P, Heidrich-Meisner F, Bloch I and Schneider U 2013 Phys. Rev. Lett. 110 205301
[8] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 Rev. Mod. Phys. 83 863
[9] Rigol M, Dunjko V, Yurovsky V and Olshanii M 2007 Phys. Rev. Lett. 98 50405
[10] Rigol M and Srednicki M 2012 Phys. Rev. Lett. 108 110601
[11] Caux J-S and Essler F H L 2013 Phys. Rev. Lett. 110 257203
[12] Cazalilla M A, Iucci A and Chung M-C 2012 Phys. Rev. E 85 011133
[13] Deutsch J M 1999 Phys. Rev. A 43 2046
[14] Rigol M, Dunjko V and Olshanii M 2008 Nature 452 854
[15] Lieb E H and Liniger W 1963 Phys. Rev. A 43 2046
[16] Lieb E H and Liniger W 1963 Phys. Rev. A 43 2046
[17] Gritsev V, Rostunov T and Demler E 2010 J. Stat. Mech. P05012
[18] Muth D, Schmidt B and Fleischhauer M 2010 New J. Phys. 12 083065
[19] Calabrese P and Le Doussal P 2011 Phys. Rev. Lett. 106 050405
[20] Caux J-S and Essler F H L 2011 Phys. Rev. A 84 033640
[21] He K and Rigol M 2012 Phys. Rev. A 84 033609
[22] Brandino G P, De Luca A, Konik R M and Mussardo G 2012 Phys. Rev. B 85 214435
[23] Sirker J, Konstantinidis N P and Sedlmayr N 2014 Phys. Rev. A 89 042104
[24] Iyer D and Andrei N 2012 Phys. Rev. A 87 053607
[25] Cramer M and Eisert J 2010 Phys. Rev. Lett. 105 250401
[26] Banuls M C, Cirac J I and Hastings M B 2009 Phys. Rev. Lett. 102 050401
[27] Kormos M, Shashi A, Caux J-S and Imambekov A 2013 Phys. Rev. B 88 205131
[28] Natu S S and Mueller E J 2013 Phys. Rev. A 87 053607
[29] Caux J-S and Konik R M 2012 Phys. Rev. Lett. 109 153001
[30] Iyer D and Andrei N 2012 Phys. Rev. Lett. 109 115304
[31] Iyer D, Guan H and Andrei N 2013 Phys. Rev. A 87 053602
[32] Vicari E 2012 Phys. Rev. A 86 053628
[33] Collura M, Caux J-S and Calabrese P 2011 Phys. Rev. Lett. 107 010601
[34] Calabrese P, Essler F H L and Falati M 2011 Phys. Rev. Lett. 107 010601
[35] Minguzzi A and Gangardte D M 2005 Phys. Rev. Lett. 94 240404
[36] Buljan H, Pezer R and Gasenzer T 2006 Phys. Rev. Lett. 100 080406
[37] Caux J-S and Konik R M 2012 Phys. Rev. Lett. 109 153031
[38] Tonks L 1936 Phys. Rev. 50 955
[39] Girardeau M 1960 J. Math. Phys. 1 516
[40] Cramer M, Dawson C M, Eisert J and Osborne T J 2008 Phys. Rev. Lett. 100 030602
[41] Cramer M and Eisert J 2010 New J. Phys. 12 055020
[42] Barthel T and Schollwöck U 2008 Phys. Rev. Lett. 100 100601
[43] Calabrese P, Essler F H L and Fagotti M 2011 Phys. Rev. Lett. 106 227203
Interaction quench in a trapped 1D Bose gas

[41] Calabrese P, Essler F H L and Fagotti M 2012 J. Stat. Mech. P07022
[42] Fagotti M 2014 J. Stat. Mech. P03016
[43] Wouters B, Brockmann M, De Nardis J, Fioretto D and Caux J-S 2014 Phys. Rev. Lett. 113 117202
[44] Pozsgay B, Mestyán M, Werner M A, Kormos M, Zarand G and Takács G 2014 Phys. Rev. Lett. 113 117203
[45] Mierzejewski M, Prelovšek P and Prosen T 2014 Phys. Rev. Lett. 113 020602
[46] Goldstein G and Andrei N 2014 arXiv:1405.4224
[47] Pozsgay B 2014 J. Stat. Mech. P09026
[48] Fagotti M and Essler F H L 2013 J. Stat. Mech. P07012
[49] Fagotti M, Collura M, Essler F H L and Calabrese P 2014 Phys. Rev. B 89 125101
[50] Fagotti M and Essler F H L 2013 Phys. Rev. B 87 245107
[51] Sotiriadis S and Calabrese P 2014 J. Stat. Mech. P07024
[52] Essler F H L, Evangelisti S and Fagotti M 2012 Phys. Rev. Lett. 109 247206
[53] Jerri A 1999 Introduction to Integral Equations with Applications (New York: Wiley)
   Feinberg J 2004 J. Phys. A: Math. Gen. 37 6299
[54] Imambekov A, Mazets I E, Petrov D S, Gritsev V, Manz S, Hofferberth S, Schumm T, Demler E and
   Schmiedmayer J 2009 Phys. Rev. A 80 033604
[55] Collura M and Karevski D 2014 Phys. Rev. B 89 214308
[56] Calabrese P and Cardy J 2006 Phys. Rev. Lett. 96 136801
   Calabrese P and Cardy J 2007 J. Stat. Mech. P06008
   Calabrese P and Cardy J 2005 J. Stat. Mech. P04010
[57] Fioretto D and Mussardo G 2010 New J. Phys. 12 055015
[58] Sotiriadis S, Fioretto D and Mussardo G 2012 J. Stat. Mech. P02017
[59] Calabrese P, Essler F H L and Fagotti M 2012 J. Stat. Mech. P07016
[60] Igloi F and Rieger H 2011 Phys. Rev. Lett. 106 035701
   Rieger H and Igloi F 2011 Phys. Rev. B 84 165117
[61] Campostrini M and Vicari E 2010 Phys. Rev. A 81 023606
   Campostrini M and Vicari E 2010 Phys. Rev. A 81 063614

doi:10.1088/1742-5468/2014/11/P11016 30