Crossover in the log-gamma polymer from the replica coordinate Bethe Ansatz

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Abstract

The coordinate Bethe Ansatz solution of the log-gamma polymer is extended to boundary conditions with one fixed end and the other attached to one half of a one-dimensional lattice. The large-time limit is studied using a saddle-point approximation, and the cumulative distribution function of the rescaled free energy of a long polymer is expressed as a Fredholm determinant. Scaling limits of the kernel are identified, leading to a crossover from the GUE to the GOE Tracy–Widom distributions. The continuum limit reproduces the crossover from droplet to flat initial conditions of the Kardar–Parisi–Zhang equation.
1 Introduction and conclusions

The Kardar–Parisi–Zhang (KPZ) equation \[1,2\], a continuum model of one-dimensional growth of an interface in the presence of noise, can be solved exactly by mapping the height field of the interface to the free energy of a continuum directed-polymer model. The time evolution of the integer moments of the partition function of this model, is given by the Hamiltonian of the one-dimensional Lieb–Liniger model of interacting bosons \[3–5\], which is solvable by Bethe Ansatz methods \[6\]. These moments allow to reconstruct the probability density of the rescaled free energy for the most studied classes of boundary conditions for the KPZ equation. In particular, the free energy is related at large times to Tracy–Widom \[7,8\] distributions of the largest eigenvalue of large Gaussian random matrices, with classes depending on the boundary conditions, such as the Gaussian unitary ensemble (or GUE) for droplet boundary conditions and the Gaussian orthogonal ensemble (or GOE) for flat boundary conditions.

An alternative approach to the solution of the continuum model (with KPZ universality properties related to classes of boundary conditions \[11–33\]), is the study of discrete models of directed polymers (see \[34\] for a recent review, and \[35,36\] for more families of models and their classification, and \[37,38\] for rigorous solutions involving the replica approach). In particular, the homogeneous log-gamma polymer, introduced by Seppäläinen \[39\], defined by the distribution of Boltzmann weights on a lattice is a model of up-right directed paths on a square lattice with multiplicative random weights distributed according to the homogeneous log-gamma model are considered. The Boltzmann weights (at finite temperature set to unity) are independent identically-distributed variables with the the distribution:

\[
P\gamma(\omega)d\omega = \frac{1}{\Gamma(\gamma)}\omega^{-1-\gamma}e^{-1/\omega}d\omega,
\]
with fixed parameter $\gamma > 0$. The mapping from the integer coordinates $(i, j)$ of the vertices to space and time coordinates $(x, t)$ is defined by the relations $t = i + j - 2$ and $x = (i - j)/2$. The up-right constraint on the directed polymer amounts to allowing only jumps from $(x, t)$ to $(x \pm \frac{1}{2}, t + 1)$. The allowed values of the coordinate $x$ at even (resp. odd) times are therefore integer (resp. half-odd) numbers.

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The partition function of the log-gamma polymer with both ends fixed was studied in [49] by coordinate Bethe Ansatz techniques and related to the continuum directed polymer known to be mapped to the KPZ equation with droplet boundary conditions, using the replica solution of the Lieb–Liniger model (LL). It was also solved by combinatorial methods [47], and the generating function was related to a Fredholm determinant [48] (see also [50] for a rigorous approach to the model at finite time). Using a symmetry argument the rescaled free energy of the polymer with one free end was related to the GOE Tracy–Widom distribution [51]. In this paper we will extend the Bethe Ansatz approach to the limit of a long polymer with one end fixed at $(x, t)$ and the other end attached to one half of an infinite one-dimensional lattice.

We need to define a discrete analogue of the partition function of the continuum polymer model, studied (see [15,16]) with boundary conditions defined by means of a slope $w$ and denoted by

$$\mathbb{Z}_{w}^{LL}(x, t) := \sum_{y \in \mathbb{Z}} \sum_{\phi : (y, 0) \rightarrow (x, t)} \prod_{(x', t') \in \phi} e^{wy\omega_{x', t'}},$$

(3)

with space and time restricted to discrete values by the constraint of up-right directed paths as explained above. Due to the random nature of the Boltzmann weights, this partition function is a random variable. The purpose of this paper is to characterise its probability law in the large-time limit and to identify two limits, one involving large positive positions and one involving large negative positions, in which the law of a suitable rescaled free energy approaches the GOE and GUE distributions, in order to establish the existence of a crossover regime in the log-gamma polymer.

Our derivation is conjectural to the extend that we adopt the analytic-continuation prescription to complex values of the number of replicas (which is needed as the fat tail of the distribution of Boltzmann weights gives rise to divergences for replicas of order larger than $\gamma$), and the conjectural form of the norm of Bethe states proposed in [49]. Thios system of states is assumed to be complete.
We will introduce a scaling parameter $\lambda$ and by means of a saddle-point approximation in the time-evolution of replicated partition functions, relate it to time by a scaling law, and introduce a rescaled space coordinate $\tilde{x}$ and a rescaled slope $\tilde{w}$ as follows:

$$
\lambda^3 = -\psi''\left(\frac{\gamma}{2}\right) \frac{t}{8}, \quad \tilde{x} = 2\psi'\left(\frac{\gamma}{2}\right) \frac{x}{\lambda^2}, \quad \tilde{w} = \lambda w, \quad (4)
$$

where $\psi$ is the Euler digamma function. The large-time limit is therefore captured by the large-$\lambda$ limit. Let us denote by $\chi_1$ (resp. $\chi_2$) a random variable whose cumulative distribution function is given by the GOE function $F_1$ (resp. the GUE function $F_2$). In the large-$\lambda$ limit, we will identify the two identities in law:

$$
\log(Z_w(x,t)) + \psi\left(\frac{\gamma}{2}\right) t - \frac{4}{c^2} \frac{\tilde{w}^2}{\tilde{x}} \lambda \chi_1 + \lambda \left( \frac{4 \tilde{w}^2}{c^2} + \frac{\tilde{x}}{\tilde{w}} \right), \quad (5)
$$

$$
\log(Z_w(x,t)) + \psi\left(\frac{\gamma}{2}\right) t - \frac{8}{\psi''\left(\frac{\gamma}{2}\right)} \frac{x^2}{4t}, \quad (6)
$$

where $c = 4/(\gamma - 1)$.

The structure of the paper is as follows. In Section 2 we introduce notations for the model and the quantities of interest, then review the coordinate Bethe Ansatz solution and the assumptions leading to an expression the generating function, organised as a string expansion. In Section 3 we work out the one-string contribution, identify scaling laws from a saddle-point approximation of the time-evolution factor in the large-time limit. This step provides an expression for the one-string contribution as the trace of a kernel. Arguments from the solution of the Lieb–Liniger problem in the continuum with half-flat boundary condition are carried over in Section 4 to the saddle-point approximation of the discrete model, leading to an organisation of the generating function as a Fredholm determinant. By introducing a lattice spacing we show that the continuum limits of Eqs 5 and 6 (which induces a scaling of time and the limit of large $\gamma$ parameter) reproduce the GOE and GUE limits of the crossover in the KPZ equation [16].

2 Review of the model and quantities of interest

2.1 Generating function and starting formula in terms of moments

We can access the cumulative distribution function of the free energy through the large-$\lambda$ limit of the following generating function $g_\lambda$ defined by

$$
g_\lambda(s) = \exp\left(-e^{-\lambda s} Z_w(x,t)\right) = \exp\left(-e^{-\lambda(s+f)}\right), \quad (7)
$$

where the overline denotes the average over the distribution of Boltzmann weights (the disorder described by Eq. 1), and the rescaled free energy $f$ is defined as

$$
f := -\lambda^{-1} \log(\tilde{Z}_w(x,t)). \quad (8)
$$
Indeed, provided the parameter $\lambda$ can go to infinity together with time, the large-time cumulative distribution function of $f$ can be accessed through the large-$\lambda$ limit of the generating function:

$$\lim_{\lambda \to \infty} g_\lambda(s) = \theta(f + s) = \text{Prob}(f > -s).$$

(9)

On the other hand, an expansion of the exponential function in Eq. 7 at fixed $\lambda$ gives rise to a starting expression of the generating function as a formal series in the moments of the distribution of the random variable $Z_w(x,t)$:

$$g^{\text{mom}}_\lambda(s) := 1 + \sum_{n \geq 1} \frac{(-1)^ne^{-\lambda ns}}{n!} Z_w(x,t)^n$$

(10)

However, the moments of $Z_w(x,t)$ do not exist beyond a certain order depending on the value of the model parameter $\gamma$, as was appreciated in [49] in the model with two fixed ends and in [51] in the model with one fixed end, by considering the moment of order $n$ and time 0. In the present model, we observe the same divergence at large orders:

$$Z_w(x,0)^n = \sum_{y \in \mathbb{Z}} \delta_{x,y} e^{w_{y,x}} (\omega_{x,0})^n = \frac{1}{\Gamma(1 + m)} \int_0^\infty w^{-1+n-\gamma} e^{-1/w} dw = e^{wx} \frac{\Gamma(\gamma - n)}{\Gamma(\gamma)},$$

(11)

but the expression can manifestly be extended to complex values of $n$.

Let us assume that the generating function can be expressed as complex integrals using the Mellin representation of the exponential function, and use the prescription given in [49], leading from the starting formal series $g^{\text{mom}}_\lambda$ to the complex integral

$$g_\lambda(s) = -\int P(Z_w) \left( \int_C \frac{dm}{2\pi i \sin(\pi m)} \frac{1}{\Gamma(1 + m)} e^{-\lambda sm Z_w^m} \right) dZ_w$$

$$= \int_C \frac{dm}{2\pi i \sin(\pi m)} \frac{1}{\Gamma(1 + m)} e^{-\lambda sm Z_w^m},$$

(12)

where $C = -a + i\mathbb{R}$ with $a > 0$, and that the expression $Z_w^m$ can be extended to complex values of the order $m$, at all times.

### 2.2 Quantum-mechanical calculation of the moments of the partition function

By analogy with the mapping from directed polymer in the continuum to the Lieb–Liniger model, the moment of order $n$ of the polymer with two fixed ends (one being at vertex $y$ and time $t = 0$) was mapped in [49] to a wave function $\psi_t$ by the following definition:

$$\prod_{i=1}^n Z(x_i, t|y, 0) = 2^{nt} \left( \frac{\bar{c}}{4} \right)^{n(t+1)} \psi_t(x_1, \ldots, x_n), \quad \text{with} \quad \bar{c} := \frac{4}{\gamma - 1}.$$  

(13)

The multiplicativity of the Boltzmann weights, expressed between times $t$ and $t+1$ along a directed polymer path,

$$Z(x, t + 1|y, 0) = w_{x,t+1} \left( Z \left( x - \frac{1}{2}, t|y, 0 \right) + Z \left( x + \frac{1}{2}, t|y, 0 \right) \right),$$

(14)
induces a linear time-evolution equation for the wave function:

\[ \psi_{t+1} = H_n \psi_t. \]  

(15)

The expression of the Hamiltonian (or transfer matrix) \( H_n \) was worked out in Section 4 of [49]. It is the analogue for the log-gamma polymer of the Lieb–Liniger Hamiltonian in the continuum directed polymer. In the next section we will use the results of the diagonalisation problem of the Hamiltonian:

\[ H_n \Psi_\mu(x_1, \ldots, x_n) = \theta_\mu \Psi_\mu(x_1, \ldots, x_n). \]  

(16)

Given a complete system of eigenfunctions labelled by \( \mu \), with associated eigenvalues \( \theta_\mu \), and the notation \( \Psi_\mu(x_1, \ldots, x_n) = \langle x_1, \ldots, x_n | \mu \rangle \), inserting projectors onto these eigenstates into the definition of \( Z_w \) in Eq. 3 yields the expression (up to a complex conjugation that was also taken in [16] and has no effect on the l.h.s. which is a real quantity):

\[
Z_w(x,t)^n = 2^n t \left( \frac{\tilde{c}}{4} \right)^{n(t+1)} \sum_{y_1, \ldots, y_n \in \{0, \ldots, L-1\}} e^{w \sum_{p=1}^{n} y_p \langle y_1, \ldots, y_n | H_n^t | x_1, \ldots, x_n \rangle} = 2^n t \left( \frac{\tilde{c}}{4} \right)^{n(t+1)} \sum_{y_1, \ldots, y_n \in \{0, \ldots, L-1\}} e^{w \sum_{p=1}^{n} y_p \Psi_\mu(y_1, \ldots, y_n)}  
\]

(17)

2.3 Bethe Ansatz solution in the large-volume limit

Eigenfunctions of the operator \( H_n \) can be expressed by coordinate Bethe Ansatz techniques due to Brunet. They consist of a superposition of plane waves parametrised by rapidities denoted by \( (\lambda_1, \ldots, \lambda_n) \):

\[
\Psi_\mu(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} A_\sigma \prod_{i=1}^{n} e^{i \lambda_\sigma(a) x_\alpha},
\]

(18)

but weighted by symmetry factors written in terms of the tangents of the rapidities.

\[
A_\sigma = \prod_{1 \leq \alpha < \beta \leq n} \left( 1 + \frac{\tilde{c} \text{sign}(x_\beta - x_\alpha + 0^+)}{2 \left( t_{\sigma(\alpha)} - t_{\sigma(\beta)} \right)} \right), \quad t_\alpha = i \tan \left( \frac{\lambda_\alpha}{2} \right).
\]

(19)

with the notation

\[
\tilde{c} = \frac{4}{\gamma - 1}.
\]

(20)

This Ansatz implies that the plane waves can be expressed in terms of the family of parameters \( (t_\alpha)_{1 \leq \alpha \leq n} \):  

\[
z_\alpha = e^{i \lambda_\alpha} = \frac{1 + t_\alpha}{1 - t_\alpha}.
\]

(21)

In the special case \( n = 2 \), working out the eigenvalues \( \theta_\mu \) in the time-evolution problem of Eq. [13] yields \( \theta_{\mu,(n=2)} = \frac{1}{4} \sum_{\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}} z_1^\alpha z_2^\beta = \frac{1}{4} \prod_{\alpha=1}^{2} (z_\alpha^{\frac{1}{2}} + z_\alpha^{-\frac{1}{2}}) \), which generalises at higher orders,
yielding an expression of the eigenvalues in terms of the families of parameters \((t_a)_{1 \leq a \leq n}\) only:

\[
\theta_\mu = \prod_{a=1}^{n} \left( \frac{z_a^\frac{1}{2} + z_a^{-\frac{1}{2}}}{2} \right) = \left( \prod_{a=1}^{n} \frac{1}{1-t_a^2} \right)^{\frac{1}{2}}.
\]  

(22)

Moreover, imposing \(L\)-periodic boundary conditions yields the following generalisation of the Bethe equations

\[
e^{i\lambda_\alpha L} = \prod_{\beta \neq \alpha} \frac{2t_\alpha - 2t_\beta + \bar{c}}{2t_\beta - 2t_\alpha - \bar{c}}, \quad \alpha \in \{1, \ldots, n\}.
\]

(23)

In the thermodynamic limit (of a large number \(L\) of sites), when expressed in terms of the tangents of the rapidities (and not in terms of the rapidities themselves as in the Lieb–Liniger model) are arranged in strings in the complex plane, up to corrections vanishing exponentially\(^1\) at large \(L\). Indeed, if the rapidity \(\lambda_\alpha\) has a strictly positive imaginary part, the l.h.s. of Eq. (23) vanishes exponentially at \(L\), which implies that one of the factors in the numerator of the r.h.s. must be zero in the thermodynamic limit. This implies that there exists an index \(\beta\) such that \(2t_\beta = 2t_\alpha - \bar{c}\). Iterating this procedure (and repeating it using the denominator of the r.h.s in the case of a negative imaginary part of the rapidity) yields a string of parameters, specified by \(m_j\) regularly spaces values on a horizontal segment in the complex plane:

\[
t_\alpha = t_{j,a} = \frac{k_j}{2} + \frac{\bar{c}}{4}(m_j + 1 - 2a), \quad a \in \{1, \ldots, m_j\}.
\]

(24)

As such a string is invariant by complex conjugation, the eigenfunctions at identical values of the argument needed to express the moments of the partition function read, for a system of \(n_s\) strings labelled by the integer \(j\), each with \(m_j\) rapidities and an imaginary part given by \(ik_j/2\):

\[
\Psi_{\mu=\{m_j,k_j\},1\leq j\leq n_s}(x, \ldots, x) = \prod_{j=1}^{n_s} \left( \prod_{a=1}^{m_j} \frac{1 + t_{j,a}}{1 - t_{j,a}} \right)^x
\]

\[
= \prod_{j=1}^{n_s} \left( \frac{\Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{c} \right) \Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{c} \right)}{\Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{c} \right) \Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{c} \right)} \right)^x
\]

(25)

where use has been made of the identity \(\Gamma(a + m)/\Gamma(a) = \prod_{k=0}^{m-1} (a + k)\). Moreover, applying the same identity to the eigenvalues of the Hamiltonian (Eq. (22) can also be expressed as products of Gamma functions:

\[
\theta_{\mu=\{m_j,k_j\},1\leq j\leq n_s} = \prod_{j=1}^{n_s} \left( \prod_{a=1}^{m_j} \frac{1}{1-t_{j,a}^2} \right)^{\frac{1}{2}}
\]

\[
= \prod_{j=1}^{n_s} \left( \prod_{a=1}^{m_j} \frac{2}{c} \right)^{m_j} \left( \frac{\Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{c} \right) \Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{c} \right)}{\Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{c} \right) \Gamma \left( \frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{c} \right)} \right)^{\frac{1}{2}}.
\]

(26)

\(^1\)All the results of this paper are derived in the thermodynamic limit.
The norm of such a system of string states in the thermodynamic limit was conjectured in [49] to be given by:

$$||\mu \equiv \{m_j, k_j\}, 1 \leq j \leq n_s||^2 = \left(\sum_{j=1}^{n_s} m_j\right)! L^{n_s} \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + \bar{c}^2(m_i + m_j)^2}{4(k_i - k_j)^2 + \bar{c}^2(m_i - m_j)^2} \times \prod_{j=1}^{n_s} \left(\frac{m_j}{\bar{c}^m_{m_j-1}} \left(\sum_{a=1}^{m_j} \frac{1}{1 - t_{j,a}^2}\right) \prod_{b=1}^{\infty} (1 - t_{j,b}^2)\right).$$

(27)

The formula was checked in special cases and its form is inspired by the determinantal form of the Gaudin–Korepin formula [6,53], to which it reduces in the continuum limit. Let us assume it holds and use of it to express the moment of $Z_w(x, t)$ from Eq. 17.

### 2.4 Phase-space integration

The sum over Bethe eigenstates in Eq. 17 involves a sum over the Bethe states labelled by index $\mu$, which is equivalent to an integration over the space of rapidities. However, the string states are described by non-linear functions of the rapidities, Eq. 19. The contribution of a system of $n_s$ strings to the moment of a given order $n = \sum_{j=1}^{n_s} m_j$ is therefore expressed by summing over each of the numbers of rapidities $m_j$, and integrating integral over each of the continuous parameters $k_j$, up to a Jacobian factor worked out in Section 7.4 of [49]. In the thermodynamic limit, this factor compensates the sum of rational fractions from the norms of the string states (Eq. 27):

$$\sum_{\mu \equiv \{m_j, k_j\}, 1 \leq j \leq n_s} \rightarrow_{L \rightarrow \infty} n_s \prod_{j=1}^{n_s} \left(\frac{L}{2\pi} \int_{-\infty}^{\infty} dk_j \sum_{a=1}^{m_j} \frac{1}{1 - t_{j,a}^2}\right).$$

(28)

The net factor from the norm of a string state with $m_j$ rapidities after this simplification can therefore be expressed in terms of the string parameters in terms of the Gamma function:

$$\prod_{a=1}^{m_j} (1 - t_{j,a}^2)^{-1} = \frac{2}{\bar{c}^m} \frac{\Gamma\left(-\frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{\bar{c}}\right) \Gamma\left(-\frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{\bar{c}}\right)}{\Gamma\left(\frac{m_j}{2} + \frac{\gamma}{2} - i\frac{k_j}{\bar{c}}\right) \Gamma\left(\frac{m_j}{2} + \frac{\gamma}{2} + i\frac{k_j}{\bar{c}}\right)}.$$

(29)

### 3 The one-string contribution to the generating function

Reorganising the generating function as a sum over numbers of strings in the Bethe states induces a sequence of ($\lambda$-dependent) functions of as follows:

$$g_\lambda(s) =: 1 + \sum_{n_s \geq 1} \frac{1}{n_s} Z(n_s, s).$$

(30)

As in the continuum model of directed polymers [14,15], the one-string term can provide a candidate expression for a kernel in terms of which to write the higher-order terms in a determinantal form.
3.1 Saddle-point approximation of the integrand

To express the one-string contribution to the generating function, with linear momentum \( k \) (in \( t_\alpha \) parametrisation), we need to compute the moment of order \( m \), where \( m \) is the number of rapidities in the string state as expressed \([17]\). The only quantity in that moment which has not been expressed yet is the sum of the wave function over all negative values of its arguments. Let us introduce the notation

\[
\Omega_{w,c}(m, k) := \lim_{L \to \infty} \sum_{-L \leq y_1 \leq y_2 \leq \ldots \leq y_m \leq 0} e^{\sum_{p=1}^{m} y_p} \Psi_{\mu \equiv \{m,k\}}(y_1, \ldots, y_m) = \sum_{\sigma \in S_m} A_{\sigma} G_{\sigma(1) \ldots \sigma(m)}^{w,c},
\]

with

\[
G_{\sigma(1) \ldots \sigma(m)}^{w,c} = \lim_{L \to \infty} \sum_{-L \leq y_1 \leq y_2 \leq \ldots \leq y_n \leq 0} e^{\sum_{k=1}^{m} (w y_k + i \sigma_{(k)} y_k)}.
\]

The one-string term in the generating function defined by Eq. \([30]\) reads, with the continuation prescription of Eq. \([12]\) to complex values of the number of particles in the string (specialising Eqs \([25, 26, 27, 29]\) to the case of one string):

\[
Z(n_s = 1, s) = \int_C \frac{dm}{2i\pi m \sin(\pi m)} \frac{1}{\Gamma(1 + m)} e^{-\lambda s m} \int_{-\infty}^{+\infty} dk \ \Omega_{w,c}(m, k) \times \left( \frac{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} - i\frac{k}{2}\right)}{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} + i\frac{k}{2}\right)} \right)^{\frac{x}{2}} \times \left( \frac{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} - i\frac{k}{2}\right)}{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} + i\frac{k}{2}\right)} \right)^{\frac{x}{2}}.
\]

Rescaling the integration variable \( k \) by \( \tilde{c} \) yields

\[
Z(n_s = 1, s) = \tilde{c} \int_C \frac{dm}{2i\pi m \sin(\pi m)} \frac{1}{\Gamma(1 + m)} e^{-\lambda s m} \int_{-\infty}^{+\infty} dk \ \Omega_{w,c}(m, \tilde{c}k) F(x, m, k, t + 2),
\]

where the time-dependent factor reads

\[
F(x, m, k, t) = \Gamma \left(\frac{m}{2} + \frac{\gamma}{2} - ik\right)^{x+\frac{1}{2}} \Gamma \left(-\frac{m}{2} + \frac{\gamma}{2} + ik\right)^{-x+\frac{1}{2}} \times \Gamma \left(\frac{m}{2} + \frac{\gamma}{2} - ik\right)^{-x-\frac{1}{2}} \Gamma \left(\frac{m}{2} + \frac{\gamma}{2} + ik\right)^{x-\frac{1}{2}}
\]

\[
= \left( \frac{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} - ik\right)}{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} + ik\right)} \right)^{x+\frac{1}{2}} \left( \frac{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} + ik\right)}{\Gamma\left(-\frac{m}{2} + \frac{\gamma}{2} + ik\right)} \right)^{x-\frac{1}{2}},
\]

and can be studied at large time using a saddle-point approximation. Rescaling the integration variables \( m \) and \( k \) by a factor of \( \lambda \) leads to an expression with a \( \lambda \)-independent exponential factor in the integrand:

\[
Z(n_s = 1, s) = \lambda^{-1}\tilde{c} \int_C \frac{dm}{2i\pi m \sin(\pi m/\lambda)} \frac{1}{\Gamma(1 + m/\lambda)} e^{-\lambda s m} \times \int_{-\infty}^{+\infty} dk \ \Omega_{w,c}(m/\lambda, \tilde{c}k/\lambda) F(x, m/\lambda, k/\lambda, t + 1),
\]
From the expression of the time-dependent factor in Eq. 35 we can read off the following logarithm, whose behaviour close to \((m = 0, k = 0)\) is relevant in the large-\(\lambda\) limit:

\[
\Lambda(m, k) = \log \left( \frac{\Gamma \left( -\frac{m}{2} + \frac{\gamma}{2} - ik \right)}{\Gamma \left( \frac{m}{2} + \frac{\gamma}{2} - ik \right)} \right). \tag{37}
\]

as it can be used as follows to derive a saddle-point approximation of the kernel:

\[
\mathcal{F}(x, m, k, t) = \exp \left( \left( x + \frac{t}{2} \right) \Lambda(m, k) + \left( -x + \frac{t}{2} \right) \Lambda(m, -k) \right). \tag{38}
\]

The logarithm \(\Lambda(m, k)\) is odd in \(m\), and its Taylor expansion around the origin reads as follows, treating powers of \(m\) and \(k\) as being of the same order:

\[
\Lambda(m, k) = \left( -\psi \left( \frac{\gamma}{2} \right) - i\psi' \left( \frac{\gamma}{2} \right) k + \frac{1}{2} \psi'' \left( \frac{\gamma}{2} \right) k^2 \right) m - \frac{1}{24} \psi'' \left( \frac{\gamma}{2} \right) m^3 + o(m^3), \tag{39}
\]

where \(\psi = \Gamma' / \Gamma\) is the digamma function. Let us group terms that contain factors of \(x\) or \(t\) in the logarithm of the kernel:

\[
\log \mathcal{F}(x, m, k, t + 1) = -\psi \left( \frac{\gamma}{2} \right) mt - 2i\psi' \left( \frac{\gamma}{2} \right) kmx + \frac{1}{2} \psi'' \left( \frac{\gamma}{2} \right) k^2mt - \frac{1}{24} \psi'' \left( \frac{\gamma}{2} \right) m^3t + o(m^3). \tag{40}
\]

The first term, which is proportional both to time and to the number of particles in the string (or order of the moment before analytic continuation), corresponds to the additive part of the free energy. Let us discard this term from now on by shifting the energy of the configurations. This is equivalent to studying the random quantity \(\log Z_w(x, t) + \psi(\gamma/2)t\), without changing the notations. When the integration variables \(m\) and \(k\) in Eq. 12 are scaled by a factor of \(\lambda\) in a change of variables, the relevant logarithm after shift in energy levels is therefore

\[
\log \mathcal{F}(x, \lambda^{-1} \tilde{m}, \lambda^{-1} \tilde{k}, t) = -2i\psi' \left( \frac{\gamma}{2} \right) \tilde{k}\tilde{m}\lambda^2x + \frac{1}{2} \psi'' \left( \frac{\gamma}{2} \right) \tilde{k}^2\tilde{m}\lambda^{-3}t - \frac{1}{24} \psi'' \left( \frac{\gamma}{2} \right) \tilde{m}^3\lambda^{-3}t + \ldots, \tag{41}
\]

from which we deduce scaling laws that make the quadratic and cubic terms independent of the parameter \(\lambda\). The higher-order terms in the Taylor expansion can be neglected at large time using the Laplace method, as they will all carry at least a factor of \(\lambda^{-5}\) from powers or \(m\) and \(k\), provided

\[
\lambda^3 = -\frac{1}{8} \psi'' \left( \frac{\gamma}{2} \right) t, \tag{42}
\]

and

\[
x = \lambda^2 \frac{1}{2\psi' \left( \frac{\gamma}{2} \right)} \tilde{x}. \tag{43}
\]

With these prescriptions the large-time limit coincides with the large-\(\lambda\) limit, which ensures that the large-\(\lambda\) limit of the generating function can give access to the large-time probability distribution of the free energy. The Airy function \(\text{Ai}\) can be introduced to write the cubic term as a Laplace transform. Moreover the two scale-invariant terms in the argument of the exponential function can
Going back to the integral form of the one-string term in Eq. 36, we need to find an equivalent of the analytic continuation of the overlap $\Omega_{\tilde{w}, \bar{c}}(m, k)$ around $(m = 0, k = 0)$. Let us introduce the rescaled slope $\tilde{w}$, which will be kept fixed in the large-time limit:

$$w = \frac{\tilde{w}}{\lambda}. \quad (45)$$

At large $\lambda$ and small $m$ and $k$ the arguments of the exponential functions involved in the expression of the overlap are small (the tangents of the rapidities are small, as the maximum of the real part of the $t_\alpha$ parameters is proportional to $m\bar{c}$, Eqs 19, 47):

$$G_{\tilde{w}, \bar{c}}^{w, \bar{c}} = \prod_{p=1}^{m} \frac{1}{1 - e^{-(pw + i\sum_{j=1}^{p} \lambda_j)}}. \quad (46)$$

Even though the overlaps are calculated from integer values of $m$, we can trade a small value of $m$ (for which analytic continuation is needed) and a fixed value of $\bar{c}$, for a much larger (integer) value of $m$ and a much lower value of $\bar{c}$, while keeping $m\bar{c}$ fixed. In the regime of small charge the rapidities are small themselves, and are arranged as a string in the complex plane, as Eq. 19 becomes:

$$t_\alpha \simeq \lambda_\alpha \rightarrow 0 \frac{i\lambda_\alpha}{2}. \quad (47)$$

The expansion at lowest non-zero order in the rapidities yields an expression analogous to that of the overlap studied in [16] for the crossover in the continuum model:

$$G_{\tilde{w}, \bar{c}}^{\tilde{w}, \bar{c}} \sim \prod_{j=1}^{m} \frac{1}{j^{\tilde{w}/\lambda} + \frac{\bar{c}}{\lambda} \sum_{a=1}^{j} \left(k + i\bar{c} \sum_{a=1}^{m} (m + 1 - 2a)\right)} \left(1 + o(1)\right), \quad (48)$$

whose dominant contribution in the large-time limit coincides with the multiple integral of a plane wave in the Lieb–Liniger model with a string of rapidities given by $\{k/\lambda + i\bar{c}/(2\lambda)(m + 1 - 2a)\}_{1 \leq a \leq m}$. In the above expression $m$ is still manifestly an integer, however the sum of the contributions of the r.h.s. weighted by symmetry factors is known from the solution of the continuum directed polymer model to give rise to a factorised form that can be analytically continued.

### 3.2 Factorisation of the overlap factor in the Lieb–Liniger model

Let us review the factorisation of the integral over half-lines of the Bethe wave functions in the continuum directed polymer model [15]. Consider a string of rapidities with $m$ elements in the
Lieb–Liniger model, given in terms of a linear momentum $k$ and a charge $\bar{c}$ by:

$$\lambda_{\alpha}^{LL}(k, m, \bar{c}) = k + \frac{i\bar{c}}{2}(m + 1 - 2\alpha), \quad \alpha \in \{1, \ldots, m\},$$

(49)

the Bethe wave function $\Psi_{w}^{LL}$ is an eigenfunction of the Lieb–Liniger Hamiltonian, with the following expression:

$$\Psi_{m,k,\bar{c}}^{LL}(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} A_{\sigma} \exp \left( i \sum_{\alpha=1}^{m} \lambda_{\sigma(\alpha)}x_{\alpha} \right), \quad A_{\sigma}^{LL}(k, m) = \prod_{1 \leq \alpha < \beta \leq m} \left( 1 + i\bar{c}\frac{\text{sign}(x_{\beta} - x_{\alpha} + 0^+)}{\lambda_{\sigma(\alpha)} - \lambda_{\sigma(\beta)}} \right).$$

(50)

Moreover, the overlap integral of this wave function with the negative half-line can be factorised as follows in the case $\bar{c} = 1$:

$$\Omega_{w,\bar{c}=1}^{LL}(k, m) = \left( \prod_{\alpha=1}^{m} \int_{-\infty}^{\omega} dy_{\alpha} e^{wy_{\alpha}} \right) \Psi_{m,k,\bar{c}}^{LL}(y_1, \ldots, y_m) = \sum_{P \in S_m} \left( \prod_{j=1}^{m} \frac{1}{jw + i \sum_{l=1}^{j} \lambda_{\sigma(l)}(k, m, \bar{c})} \right),$$

(51)

can be factorised as follows in the case $\bar{c} = 1$ (see Section 5 in [16]):

$$\Omega_{w,\bar{c}=1}^{LL}(k, m, \bar{c} = 1, w) = \frac{m!}{\prod_{\alpha=1}^{m} (\lambda_{\alpha}(k, m, \bar{c} = 1) - iw)} \prod_{1 \leq \alpha < \beta \leq m} \frac{i + \lambda_{\alpha}(k, m, \bar{c} = 1) + \lambda_{\beta}(k, m, \bar{c} = 1) - 2iw}{\lambda_{\alpha}(k, m, \bar{c} = 1) + \lambda_{\beta}(k, m, \bar{c} = 1) - 2iw},$$

(52)

which upon substituting the string of rapidities described in Eq. 49 yields

$$\Omega_{w,\bar{c}=1}^{LL}(k, m) = \frac{(-1)^{m} \Gamma(2ik + 2w)}{\Gamma(2ik + 2w + m)}.$$  

(53)

To restore the value of $\bar{c}$, let us start from Eq. 51 by factorising one power of $\bar{c}$ per factor in each of the terms in the sum:

$$\Omega_{w,\bar{c}}^{LL}(k, m) = \bar{c}^{-m} \sum_{P \in S_m} \left( A_{P}^{LL}(k, m) \prod_{j=1}^{m} \frac{1}{jw + i \sum_{l=1}^{j} \lambda_{\sigma(l)}(k, m, \bar{c} = 1)} \right)$$

$$= \frac{m!}{(i\bar{c})^{m} \prod_{\alpha=1}^{m} (\lambda_{\alpha}(k, m, \bar{c} = 1) - i\bar{c}w)} \prod_{1 \leq \alpha < \beta \leq m} \frac{i + \lambda_{\alpha}(k, m, \bar{c} = 1) + \lambda_{\beta}(k, m, \bar{c} = 1) - 2i\bar{c}w}{\lambda_{\alpha}(k, m, \bar{c} = 1) + \lambda_{\beta}(k, m, \bar{c} = 1) - 2i\bar{c}w}.$$  

(54)

### 3.3 Large-time limit of the one-string contribution

The equivalent of the overlap factor in the large-time limit identified in [18] from the lowest non-zero order expansion in the rapidities for integer $m$ can therefore be analytically continued by substituting the above Lieb–Liniger expression:

$$\Omega_{w/\bar{c},\bar{c}}(m, k/\lambda) \simeq \lim_{m \to 0} \frac{(-1)^{m} \Gamma \left( \frac{2k+2w}{\lambda c} \right)}{\bar{c}^{m} \Gamma \left( \frac{2k+2w}{\lambda c} + m \right)}$$

(55)
It should be noted that the occurrence of this expression from the Lieb–Liniger model comes merely from the large-time limit and the continuation to complex values of \(m\), while \(\gamma\) is kept fixed. The lattice spacing is still equal to one. In the continuum limit will be taken eventually, \(\gamma\) will go to infinity as the lattice spacing will go to zero.

For a fixed value of the rescaled slope \(\bar{w}\) and a large value of time, the asymptotic form of the analytic continuation of the integral over the negative real axis can be substituted into the expression of the one-string contribution expressed in Eq. 36, together with the integral representation \(m^{-1} = \int_{0}^{\infty} e^{-mv}\), to yield:

\[
Z(n_s = 1, s) \sim_{\lambda \to \infty} \int_{0}^{\infty} dv \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy \int_{C} \frac{dm}{2i\pi \sin(\pi m/\lambda)} \frac{1}{\Gamma(1 + m/\lambda)} \\
\int_{-\infty}^{+\infty} dk \frac{\Gamma\left(\frac{2(ik + \bar{w})/\bar{c}}{\lambda}\right)}{\Gamma\left(\frac{2(ik + \bar{w})/\bar{c} + m}{\lambda}\right)} \times \text{Ai}(y + ik\bar{x} + 4k^2) \exp(-mv - ms + my),
\]

(56)

in which a shift of the integration variable \(y\) has been used to transfer the variables \(s\) to the Airy function. We can simplify the integrand in the large-time limit by using the equivalent \(\Gamma(u/\lambda) \sim_{\lambda \to \infty} \lambda/u\)

\[
Z(n_s = 1, s) \sim_{\lambda \to \infty} \int_{0}^{\infty} dv \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy \int_{C} \frac{dm}{2i\pi \sin(\pi m/\lambda)} \frac{2ik + 2\bar{w}}{2ik + 2\bar{w}/\bar{c}} \times \text{Ai}(y + v + s + ik\bar{x} + 4k^2)e^{my}
\]

(57)

where the Laplace transforms of the Dirac mass and step function \(\theta(y) = 1(y > 0)\) have been used to integrate over \(m\).

4 Large-time limit of the generating function

4.1 Determinantal form of the higher-order terms in the string expansion

As in the Lieb–Liniger case, assuming the generating function has the structure of a Fredholm determinant, we can read off the kernel from the one-string calculation. Introducing the kernel

\[
K_{\gamma,s}(v_1, v_2) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy \left(\frac{4}{\gamma - 1}(ik + \bar{w})\delta(y)\right) \times \text{Ai}(y + v_1 + v_2 + s + ik\bar{x} + 4k^2)e^{-2ik(v_1 - v_2)},
\]

(58)
the large-time limit of the one-string term reduces to its trace:

$$Z(n_s = 1, s) \sim_{\lambda \to \infty} \int_{-\infty}^{\infty} K_s(v, v) dv.$$ (59)

In order to rewrite the generating function in terms of the above kernel, we have to insert factors from the multiple-string contributions to the norm of the string states.

Going back to the formal expression of the moment of order $n$, the inter-string factors from the norms of the string states (Eq. 27) yield the starting formula for the $n_s$-string contribution (while thanks to the expression of the eigenvalues, the time-evolution factor reduces to the product of the contribution of each string):

$$Z_{\text{mom}}(n_s, s) = \sum_{m_1, \ldots, m_{n_s} = 1}^{\infty} \prod_{j=1}^{n_s} \left( \frac{2m_j}{m_j} \int_{-\infty}^{\infty} \frac{dk_j}{2\pi} F(x, m_j, k_j, t + 2) \right) \times \Omega_{w, \bar{c}}(\{m_j, \bar{c}k_j\}_{1 \leq j \leq n_s}) \prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2},$$ (60)

The inter-string factors from the norm allow to use the same crucial identity as in [15, 49]

$$\prod_{1 \leq i < j \leq n_s} \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} = \det \left( \frac{1}{2i(k_i - k_j) + m_i + m_j} \right) |_{n_s \times n_s} \prod_{j=1}^{n_s} (2m_j),$$ (61)

to give the $n_s$-string contribution the form of a determinant. Applying the integration prescription of Eq. 12, $n_s$ times leads to an expression in which each of the real integration variables $k_1, \ldots, k_{n_s}$ has been rescaled:

$$Z(n_s, s) = \left( \prod_{j=1}^{n_s} \int_{C_j} \frac{-dm_j}{2i\pi \sin(\pi m_j)} 2m_j^{m_j+1} \int_{-\infty}^{\infty} \frac{dk_j}{2\pi} F(x, m_j, k_j/\lambda, t + 2)e^{-\lambda m_j s} \right) \times \Omega_{w, \bar{c}}(\{m_j, \bar{c}k_j/\lambda\}_{1 \leq j \leq n_s}) \det \left( \frac{1}{2i(k_i - k_j) + \lambda(m_i + m_j)} \right) |_{n_s \times n_s}$$ (62)

As the maximum rapidity in a system of strings is bounded by the maximum of $m_j \bar{c}$, we use a small-rapidity expansion of the overlap integrals in each of the strings, which allows us to make use of the factorisation of the overlap of the wavefunction in the Lieb–Liniger model:

$$\Omega_{w, \bar{c}}(\{m_j, \bar{c}k_j/\lambda\}_{1 \leq j \leq n_s}) \simeq_{\lambda \to \infty} \left( \bar{c}^{-m_j} \prod_{j=1}^{n_s} \frac{(-1)^{m_j} \Gamma(z_{jj}/\lambda)}{\Gamma((z_{jj} + m_j)/\lambda)} \right) \times \prod_{1 \leq i < j \leq n_s} \frac{\Gamma(1 - (z_{ij} + (m_i + m_j)/2)/\lambda) \Gamma(1 - (z_{ij} - (m_i + m_j)/2)/\lambda) \Gamma(1 - (z_{ij} + (m_i - m_j)/2)/\lambda) \Gamma(1 - (z_{ij} - (m_i - m_j)/2)/\lambda)}{\Gamma(1 - (z_{ij} + (m_i - m_j)/2)/\lambda) \Gamma(1 - (z_{ij} - (m_i - m_j)/2)/\lambda) \Gamma(1 - (z_{ij} + (m_i + m_j)/2)/\lambda) \Gamma(1 - (z_{ij} - (m_i + m_j)/2)/\lambda)},$$ (63)

where

$$z_{ij} = i(k_i + k_j) + 2w, \quad i, j \in \{1, \ldots, n_s\}.$$ (64)
The inter-string factors coming from the overlap integral go to zero in the large-time limit once the variables are rescaled. Inserting the large-$\lambda$ expansion of the time-dependent factor with the scaling of $\lambda$ identified at the one-string level, together with the representation of the determinant as a sum over permutations, and an integral representation of each of the factors, reproduces all the algebraic steps of the derivation of the Fredholm determinant in the continuum model:

$$Z(n_s, s) \simeq_{t \to \infty} (-1)^{n_s} \left( \prod_{j=1}^{n_s} \int_0^\infty dv_j \right) \det K_{\gamma,s}^{(n_s \times n_s)} |_{n_s \times n_s} \left( 1 + o(1) \right) \tag{65}$$

Having derived the $n_s$ by $n_s$ determinantal form of the $n_s$-string in the large-time limit, we can use the identities obtained in Section 9 of [16] through variations on formulas involving integrals of products of Airy functions [54], up to a factor of $\bar{c} = 4/(\gamma - 1)$ in front of the $\delta$-function in the kernel, and obtain the Fredholm determinant:

$$g_{\infty}(s) = \text{Det} \left( 1 - K_{\gamma,s} \right) \tag{66}$$

$$K_{\gamma,s}(v_1, v_2) = \theta(v_1)\theta(v_2) \left( \int_0^\infty dy \text{Ai} \left( y + v_1 + 2^{-2/3} \left( s + \frac{\bar{x}^2}{16} \right) \right) \text{Ai} \left( y + v_2 + 2^{-2/3} \left( s + \frac{\bar{x}^2}{16} \right) \right) \right) + \frac{4\theta(v_1)\theta(v_2)}{\gamma - 1} \left( - \int_{-\infty}^0 dy \text{Ai}(v_1 + \sigma + y)\text{Ai}(v_2 + \sigma - y)e^{2yu} + 2^{-1/3} \text{Ai} \left( v_1 + v_2 + 2\sigma - 2u^2 \right) e^{-u(v_1-v_2)} \right), \tag{67}$$

with

$$u = -2^{-2/3} \left( \bar{w} + \frac{\bar{x}^2}{16} \right), \quad \text{and} \quad \sigma = 2^{-2/3} \left( s + \frac{\bar{x}^2}{16} \right). \tag{68}$$

### 4.2 GUE limit

The second part of the kernel vanishes in the limit

$$\bar{w} + \frac{\bar{x}^2}{8} \to \infty. \tag{69}$$

If we let time go to infinity while keeping $s + \bar{x}^2/16$ finite, the coordinate $x$ scales with time according to the saddle-point approximation so that it stays small in scale of time

$$\frac{x}{t} \simeq_{t \to \infty} O(t^{-1/3}). \tag{70}$$

We are therefore close to the diagonal of the square lattice (in the $(i,j)$ coordinates described in the introduction, because $x$ is proportional to the difference $i - j$), which is the sector of the model with boths fixed ends for which explicit expressions for the elastic constant were worked out in [49] in terms of the model parameter $\gamma$. Moreover, we can read off the relevant expression in the argument of the Airy kernel in terms of the coordinates $x$ and $t$ as

$$\lambda \frac{\bar{x}^2}{16} = \frac{2 \psi'' \left( \frac{\gamma}{2} \right) \bar{x}^2}{t}, \tag{71}$$
so that if the random variable $\chi_2$ has the GUE distribution $F_2$ as its cumulative distribution function, the following identity holds in law

$$-\lambda f = \lambda \chi_2 + \kappa \frac{x^2}{4t}, \quad (72)$$

with

$$\kappa = -8 \left( \frac{\psi'}{\psi''} \right)^2. \quad (73)$$

The parabolic term is Eq. (72) identical to the one worked out in the diagonal region in the model with both ends fixed (Section 10 in [49]). It is known to reduce to the droplet solution in the continuum limit, which corresponds to the GUE behaviour of the large-time limit of the model with both ends fixed in the diagonal sector.

### 4.3 GOE limit

To capture the GOE limit we need to send $\tilde{w} + \tilde{x}/8$ to $-\infty$ while keeping $\tilde{w}$ positive, in which limit only the last term in the kernel contributes:

$$K_s(v_1, v_2) \simeq_{\tilde{w} + \tilde{x}/8 \to -\infty} \theta(v_1)\theta(v_2) \text{Ai}(v_1 + v_2 + 2\sigma - 2\tilde{w}^2), \quad (74)$$

Scaling the coordinates $x$ and $t$ by a factor of $\bar{c}$ and $\bar{c}^2$ respectively restores the scale $\bar{c}$ in the statements of [49], and denoting by $\chi_1$ the random variable that has the GUE distribution $F_2$ as its cumulative distribution function we read off the identity in law:

$$-\lambda f = \lambda \chi_1 + \lambda \left( 4\frac{\tilde{w}^2}{\bar{c}^2} + \frac{\tilde{x}}{\bar{c}} \right). \quad (75)$$

Working out the last two terms in terms of the coordinates $x$, $t$ and the parameter of the model:

$$4\frac{\lambda^3}{\bar{c}^2} \tilde{w}^2 + \lambda \tilde{w} \frac{\tilde{x}}{\bar{c}} = -\frac{1}{2} \frac{\psi''}{\psi'} \left( \frac{\gamma}{2} \right) w^2 \frac{t}{\bar{c}^2} + 2 \frac{\psi'}{\psi''} \left( \frac{\gamma}{2} \right) w \frac{x}{\bar{c}}, \quad (76)$$

we can take the continuum limit by introducing a lattice spacing $a$ sent to zero with fixed Lieb–Liniger parameter $\bar{c}_{LL}$ through the prescription

$$\bar{c} = a \bar{c}_{LL}, \quad \gamma = 1 + \frac{4}{a \bar{c}_{LL}} \to \infty \quad (77)$$

where the factor of $1/8$ ensures that the factors $\theta^\mu_\mu$ reduce to the time-evolution factor of the Lieb–Liniger model in the continuum limit (see Section 5 of [49]). The asymptotic expansion of the digamma function

$$\psi(u) \simeq_{u \to \infty} \log u - \frac{1}{2u} - \frac{1}{12u^2} + O(u^{-4}) \quad (78)$$

yields

$$4\frac{\lambda^3}{\bar{c}^2} \tilde{w}^2 + \lambda \tilde{w} \frac{\tilde{x}}{\bar{c}} \simeq_{LL} -\frac{t}{8} w^2 + wx = w_{LL} x_{LL} + t_{LL} w_{LL}^2, \quad (79)$$

where the continuum coordinates $x_{LL}$ and $t_{LL}$ are scaled by powers of the lattice spacing

$$x = \frac{x_{LL}}{a}, \quad t = \frac{t_{LL}}{a^2}, \quad w = a w_{LL}, \quad (80)$$
where the factor of 8 ensures that the factors $\theta^t_\mu$ reduce to the time-evolution factor of the Lieb–Liniger model in the continuum limit (see Section 5 of [49]), and the scaling of the slope $w_{LL}$ is induced by prescribing that the shift of the free energy $wx$ should be fixed in the continuum limit. The shift in the free energy therefore coincides with the one in the GOE limit of the crossover in the KPZ equation.

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