Oscillator approach to quantization of $AdS_5 \times S^5$ superparticle in twistor formulation

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Abstract

Using known relation between $SU(2, 2|4)$ supertwistors and $SU(2)$ bosonic and fermionic oscillators we identify the physical states of quantized massless $AdS_5 \times S^5$ superparticle in supertwistor formulation and discuss how they fit into the spectrum of fluctuations of IIB supergravity on $AdS_5 \times S^5$ superbackground.

1 Introduction

Significant progress attained over the last decade and a half in perturbative $N = 4$ super-Yang-Mills theory (see, e.g. [1] for review) was substantially triggered by the seminal work [2] that, based on the previous findings [3], [4], showed how string and twistor theories can be coupled in a synergetic way. One of the developments was the construction of (ambi)twistor action for $N = 4$ super-Yang-Mills theory in [5], [6], [7] that results in efficient rearrangement of the perturbative series expansion.

Since $N = 4$ super-Yang-Mills theory has a dual description as Type IIB string theory on $AdS_5 \times S^5$ superbackground it is interesting to apply twistor methods also to the exploration of this duality [8], [9]. This necessitates working out supertwistor formulation on the $AdS_5 \times S^5$ superbackground for the superstring [10], [11] and Type IIB supergravity [12], [13] that should start from identifying proper supertwistor variables. One of the feasible ways towards this goal is to consider supertwistor formulation on the $AdS_5 \times S^5$ superbackground of the massless superparticle model that arises in the infinite-tension limit of the superstring.

Such approach was pursued in [14] (see also [15]) based on the generalization of extended superparticle model [16] that has extra dynamical variables and gauge symmetries to linearly realize $SU(2, 2|4)$ symmetry. It was found in [14] that appropriately fixing part of the gauge symmetries yields expression for the superparticle’s Lagrangian in terms of the supertwistors arranged into two $SU(2)$ doublets, one of the doublets being the straight-forward generalization of the Ferber supertwistors [17], while components of another $SU(2)$ doublet of supertwistors have different Grassmann parity, i.e. $SU(2, 2)$ components are Grassmann odd, whereas $SU(4)$ components are Grassmann even. Besides that these supertwistors satisfy the set of $SU(2, 2|4)$-invariant constraints that are the generators of $su(2|2)$ gauge superalgebra.

In [18], [19] we have found that this supertwistor Lagrangian originates also from the supertwistor formulation of the conventional massless superparticle model on $AdS_5 \times S^5$ superbackground [20], [21], [22], [23], [24]. One of the benefits of this approach is that it yields the incidence relations connecting supertwistor components and $AdS_5 \times S^5$ superspace coordinates via the $PSU(2, 2|4)/(SO(1, 4) \times SO(5))$ supercoset representative. Let us note that unlike the superspace formulation of the superparticle, in which the Lagrangian involves highly non-linear expressions for the $AdS_5 \times S^5$ supervielbein bosonic components, kinetic term and the constraints in the supertwistor formulation are quadratic facilitating Dirac
quantification. In [19] it was obtained the set of equations for the superparticle’s wave function in the space of superambitwistors, however, the problem is to elaborate on the details of the Penrose transform in order to establish a mapping of the components of the wave function to the fields of Type IIB supergravity multiplet compactified on $AdS_5 \times S^5$ superbackground.

In this note we use close relationship of $SU(2,2|4)$ supertwistors and $SU(2)$ bosonic and fermionic oscillators [25], [26] that are used to construct positive energy unitary irreducible representations (uirs) of $SU(2,2|4)$ [27], [12], to provide representation-theoretic characterization of the quantum states of $AdS_5 \times S^5$ superparticle. After briefly recapitulating the four-supertwistor formulation of the massless superparticle, we proceed to study oscillator realization of $u(2,2|4)$ global symmetry and $su(2|2)$ gauge symmetry generators, analyze the gauge-invariant subspace of the quantum states of the superparticle and show how it fits the fluctuation spectrum of IIB supergravity over the $AdS_5 \times S^5$ superbackground.

2 Four-supertwistor formulation of massless superparticle on $AdS_5 \times S^5$ superbackground

In [18], [19] it was found the action of the massless $AdS_5 \times S^5$ superparticle based on the four-supertwistor form of the Lagrangian

$$S = \int d\tau L_{4-stwistor}$$

$$L_{4-stwistor} = \frac{i}{2} \left( \dot{Z}_A^i \dot{Z}_A^i - \hat{Z}_A^i \hat{Z}_A^i \right) + \frac{i}{2} (\bar{\Psi}_A^i \dot{\Psi}_A^i - \dot{\bar{\Psi}}_A^i \bar{\Psi}_A^i) + \Lambda^i_j (\bar{Z}_A^j \dot{Z}_A^i - \frac{1}{2} \delta^i_j \dot{\bar{Z}}_A^j \bar{Z}_A^i) + \Lambda^p_q (\bar{\Psi}_A^p \dot{\Psi}_A^q - \frac{1}{2} \delta^p_q \dot{\bar{\Psi}}_A^p \bar{\Psi}_A^q) + \Lambda (\bar{Z}_A^i \dot{\bar{Z}}_A^i + \bar{\Psi}_A^i \dot{\bar{\Psi}}_A^i) + i \Lambda_i^j \bar{\Psi}_A^j \bar{Z}_A^i + i \bar{\Lambda}_i^j \bar{\Psi}_A^j \bar{Z}_A^i,$$  

(1)

where components of the two doublets of $SU(2,2|4)$ supertwistors have different Grassmann parities, that is Ferber or $c$-type supertwistors

$$Z_A^i = \left( \begin{array}{c} Z_A^i \\ \eta_A^i \end{array} \right), \quad \bar{Z}_A^i = (Z_\alpha^i)^\dagger H_B^A = (\bar{Z}_\alpha^i, \bar{\eta}_A^i), \quad H_B^A = \left( \begin{array}{cc} H^B_{\alpha} & 0 \\ 0 & \delta^B_A \end{array} \right)$$

(2)

have Grassmann-even $SU(2,2)$ components $Z_A^i, \bar{Z}_\alpha^i$ and Grassmann-odd $SU(4)$ components $\eta_A^i, \bar{\eta}_A^i$, whereas $a$-type supertwistors

$$\Psi_A^i = \left( \begin{array}{c} \xi_A^i \\ L_A^i \end{array} \right), \quad \bar{\Psi}_A^i = (\Psi_\alpha^i)^\dagger H_B^A = (\bar{\xi}_\alpha^i, L_A^\alpha)$$

(3)

have Grassmann-odd $SU(2,2)$ components $\xi_A^i, \bar{\xi}_\alpha^i$ and Grassmann-even $SU(4)$ components $L_A^i, \bar{L}_A^\alpha$. Action [13] coincides with that obtained in Ref. [14] by partial gauge fixing $2T$ superparticle model in $2 + 10$ dimensions. The advantage of the approach pursued in [18], [19] is not only in establishing the direct connection with the superparticle’s formulation in $AdS_5 \times S^5$ superspace but also in obtaining the incidence relations for the components of supertwistors and superspace coordinates parametrizing the $PSU(2,2|4)/(SO(1,4) \times SO(5))$ representative $G$

$$Z_A^i = G_B^{AB} v_B^i, \quad v_B^i = \left( \begin{array}{c} v_B^i \\ 0 \end{array} \right), \quad \bar{Z}_A^i = v_B^i G^{-1B}_A$$

(4)

and

$$\Psi_A^i = G_B^{AB} \ell_B^q, \quad \ell_B^q = \left( \begin{array}{c} 0 \\ \ell_B^q \end{array} \right), \quad \bar{\Psi}_A^i = \ell_B^q G^{-1B}_A.$$  

(5)
Unlike the eight-supertwistor formulation [19], in the four-supertwistor formulation [18], there are only the first-class constraints introduced via the Lagrange multipliers – seven bosonic

\[ \mathcal{Z}_A^i Z_A^j - \frac{1}{2} \delta_i^j \mathcal{Z}_A^k Z_A^k \approx 0, \]  
\[ \bar{\Psi}_A^q \Psi_A^p - \frac{1}{2} \delta_q^p \bar{\Psi}_A^r \Psi_A^r \approx 0, \]  
\[ T = \mathcal{Z}_A^i Z_A^i + \bar{\Psi}_A^q \Psi_A^q \approx 0 \]  

and eight fermionic

\[ \bar{\Psi}_A^q Z_A^i \approx 0, \]  
\[ Z_A^i \Psi_A^q \approx 0 \]  

that are the generators of the classical \( su(2|2) \) gauge symmetry of the superparticle’s action. The presence of only the first-class constraints greatly facilitates Dirac quantization of the model but requires insights into the Penrose transform of the functions of ambitwistors of \( c \) and \( a \)-type as was discussed in [19], where the set of the first-order differential equations for the superparticle’s wave function was derived.

3 From \( SU(2, 2|4) \) supertwistors to \( SU(2) \) oscillators

In this work, due to close relationship between twistors and oscillators, we analyze the physical states of superparticle in the twistor formulation using the oscillator approach. Similar analysis of the twistor formulation of massive bosonic particle model on \( AdS_5 \) was performed in [26]. From the expression for the kinetic term of the Lagrangian (1) it follows that odd components of \( c \)-type supertwistors (2) satisfy Dirac bracket (D.B.) relations

\[ \{ \eta_A^i, \bar{\eta}_B^j \}_{D.B.} = i \delta_i^j \delta_A^B. \]  

At the quantum level they become anticommutators

\[ \{ \eta_A^i, \bar{\eta}_B^j \} = \delta_i^j \delta_A^B \]  

that can be viewed as anticommutation relations of the two sets of \( SU(4) \) fermionic oscillators. Decomposing \( SU(4) \) (anti)fundamental representation indices on the (anti)fundamental representation indices of its two \( SU(2) \) subgroups introduces two kinds of \( SU(2) \) fermionic oscillators [12], [30]

\[ \eta_A^i = \left( \begin{array}{c} \alpha_a^i \\ \beta_a^i \end{array} \right), \quad \bar{\eta}_A^i = \left( \begin{array}{c} \alpha_a^i \\ \beta_a^i \end{array} \right) \]  

that satisfy

\[ \{ \alpha_a^i, \alpha_b^j \} = \delta_i^j \delta_a^b, \quad \{ \beta_a^i, \beta_b^j \} = \delta_i^j \delta_a^b. \]  

Admissible choice is to treat \( \alpha_a^i \) and \( \beta_a^i \) as raising oscillators, then \( \alpha_a^i \) and \( \beta_a^i \) are lowering ones annihilating the oscillator vacuum \( |0\rangle \).

\[ \text{This twistor formulation of the bosonic particle on } AdS_5 \text{ was recently revisited and generalized to dimensions 4 and 7 in [29], for which one can benefit from using two-component spinors with complex, real and quaternionic entries respectively.} \]

\[ \text{Not to overburden the notation we do not place hats over the quantum operators. Hopefully this will not cause a confusion.} \]

\[ \text{Another option – to use } SU(4) \text{ oscillators as they stand to construct uirs was considered in [31], [32], [33].} \]
To define bosonic $SU(2)$ oscillators one decomposes Grassmann-even components of $c$-type supertwistors on the $SL(2, \mathbb{C})$ constituents

$$Z^\alpha_i = \begin{pmatrix} \mu_i^\alpha \\ \bar{\Lambda}_{\dot{\alpha}} \end{pmatrix}, \quad \bar{Z}_\alpha^i = (\Lambda_i^\alpha \bar{\mu}_{\dot{\alpha}}),$$

and takes their linear combinations

$$a^\alpha_i = \frac{1}{\sqrt{2}} (-\mu^\alpha_i + \bar{\Lambda}_{\dot{\alpha}}), \quad a^i_\alpha = \frac{1}{\sqrt{2}} (-\bar{\mu}_{\dot{\alpha}} + \Lambda_i^\alpha)$$

and

$$b^\alpha_i = \frac{1}{\sqrt{2}} (\mu^\alpha_i + \bar{\Lambda}_{\dot{\alpha}}), \quad b^i_\alpha = \frac{1}{\sqrt{2}} (\bar{\mu}_{\dot{\alpha}} + \Lambda_i^\alpha)$$

that gives two species of $SU(2)$ bosonic oscillators in accordance with the doubleton-type structure of the uirs of $SU(2, 2)$ and $SU(2, 2|4)$ [27], [12]. This unitary transformation diagonalizes matrix $H^\alpha_\beta$ that enters the definition of dual supertwistors in (2) and (3)

$$H^\alpha_\beta = \begin{pmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{pmatrix}. \quad (15)$$

D.B. relations for $Z^\alpha_i$ and $\bar{Z}^i_\beta$

$$\{Z^\alpha_i, \bar{Z}^j_\beta\}_{D.B.} = i \delta^i_j \delta^\alpha_\beta$$

that in quantum theory become commutators

$$[Z^\alpha_i, \bar{Z}^j_\beta] = \delta^i_j \delta^\alpha_\beta$$

upon above transformation turn into commutators of $a$- and $b$-oscillators

$$[a^i_\alpha, a^j_\beta] = \delta^i_j \delta^\alpha_\beta, \quad [b^i_\alpha, b^j_\beta] = \delta^i_j \delta^\alpha_\beta$$

according to which $a^\alpha_i$ and $b^i_\alpha$ are raising operators, whereas $a^i_\alpha$ and $b^\alpha_i$ are lowering operators annihilating the vacuum $|0\rangle$.

Similarly one can introduce oscillators associated with the $a$-type supertwistors [3]. Namely, taking linear combinations of the $SL(2, \mathbb{C})$ constituents of odd components of $a$-type supertwistors

$$\xi^\alpha_q = \begin{pmatrix} m^\alpha_q \\ -\bar{n}_{\dot{\alpha}q} \end{pmatrix}, \quad \bar{\xi}_\alpha^q = (n^q_\alpha \bar{m}^{\dot{\alpha}}_q)$$

defines another two species of $SU(2)$ fermionic oscillators

$$\omega^\alpha_q = \frac{1}{\sqrt{2}} (-m^\alpha_q + \bar{n}_{\dot{\alpha}q}), \quad \omega^q_\alpha = \frac{1}{\sqrt{2}} (-\bar{m}^{\dot{\alpha}}_q + n^q_\alpha),$$

$$\varphi^\alpha_q = \frac{1}{\sqrt{2}} (m^\alpha_q + \bar{n}_{\dot{\alpha}q}), \quad \varphi^q_\alpha = \frac{1}{\sqrt{2}} (\bar{m}^{\dot{\alpha}}_q + n^q_\alpha)$$

in addition to [10]. The fact that Grassmann parity of the $SU(2, 2)$ components of $a$-type supertwistors differs from that of $c$-type supertwistors has far-reaching consequences for the future analysis. In particular, from the anticommutators of $\xi^\alpha_q$ and $\bar{\xi}_\alpha^q$

$$\{\xi^\alpha_q, \bar{\xi}_\beta^p\} = \delta^\alpha_q \delta^\beta_p$$

it follows that

$$\{\omega^\alpha_q, \omega^p_\beta\} = -\delta^\alpha_q \delta^p_\beta, \quad \{\varphi^\alpha_q, \varphi^p_\beta\} = \delta^\alpha_q \delta^p_\beta,$$
i.e. $\omega$-oscillators satisfy wrong-sign anticommutation relations.

Grassmann-even components of $a$-type supertwistors in quantum theory satisfy the following commutation relations

$$[L^A_q, \bar{L}^A_B] = \delta^A_q \delta^A_B$$

(23)

and decompose on the $SU(2) \times SU(2) \subset SU(4)$ representations analogously to (10)

$$L^A_q = \left( \begin{array}{c} c^a_q \\ \bar{a}^a_q \end{array} \right), \quad \bar{L}^A_q = (c^a_q \bar{d}^a_q),$$

(24)

introducing extra two species of $SU(2)$ bosonic oscillators. Commutation relations for which can be read off from (23)

$$[c^a_q, c^b_p] = \delta^a_q \delta^b_p, \quad [d^a_q, d^b_p] = \delta^a_q \delta^b_p.$$

(25)

As will be shown below for one of the introduced types of $SU(2)$ bosonic oscillators the commutation relations are necessarily the wrong-sign ones.

4 Oscillator realization of global $u(2, 2|4)$ symmetry superalgebra

Constraints (6) and (7) generate $su(2|2)$ gauge superalgebra with two $su(2)$ bosonic subalgebras named color superalgebra in [15], [14] that generalizes bosonic color algebras emerging in oscillator construction of $su(2, 2)$ and $su(2, 2|4)$ uirs [12], [30], [35] and in twistor formulation of bosonic particle on AdS$_5$ [28], [26]. Superparticle’s action (1) is also invariant under $U(2, 2|4)$ global symmetry generated by

$$Z^A_i \bar{Z}_B^i - \Psi^A_q \bar{\Psi}^q_B = u(2, 2|4).$$

(26)

Important property is that these $u(2, 2|4)$ generators (anti)commute with the generators of the $su(2|2)$ gauge symmetry due to the color symmetry invariance. In the oscillator approach these global symmetry generators are used to build the space of states of quantized superparticle model as the certain representation space of $u(2, 2|4)$. Above relations between the supertwistor components and oscillators [11], [13], [14], [20] and [24] are used to convert the generators (26) in the oscillator basis, in which they can be classified as quadratic in creation oscillators, quadratic in annihilation oscillators and containing products of both creation and annihilation oscillators in accordance with the three-grading structure of the $(s)u(2, 2|4)$ superalgebra [30]. This three-grading structure allows to fix the ambiguity in the choice of creation and annihilation oscillators associated with the $a$-type supertwistors once such a choice is made for the oscillators related to the $c$-type supertwistors.

Consider first the $su(2, 2) \subset u(2, 2|4)$ algebra. Three-grading decomposition of its generators is as follows

$$g_{(0)} = \{E, L^\alpha_\beta, R^\alpha_\beta\}, \quad g_{(-1)} = \{T^{(-)}\alpha_\beta\}, \quad g_{(+1)} = \{T^{(+)\alpha_\beta}\} :$$

(27)

$$2E = a^a_i a^i_\alpha + b^i_i b^i_\alpha - \omega^a_i \omega^i_\alpha + \varphi^a_i \varphi^i_\alpha,$$

(28a)

$$L^\alpha_\beta = a^a_i a^i_\beta - \omega^a_i \omega^i_\beta - \frac{1}{2} \alpha_\beta (a^a_i a^i_\gamma - \omega^a_i \omega^i_\gamma),$$

(28b)

$$R^\alpha_\beta = b^i_i b^i_\beta - \varphi^i_\alpha \varphi^i_\beta - \frac{1}{2} \alpha_\beta (b^i_i b^i_\gamma - \varphi^i_\alpha \varphi^i_\gamma),$$

(28c)

$$T^{(-)}\alpha_\beta = b^i_i a^i_\beta - \varphi^a_i \omega^i_\beta,$$

(28d)

$$T^{(+)}\alpha_\beta = a^i_i b^i_\beta - \omega^a_i \varphi^i_\beta.$$
Above expressions deserve a number of comments. From (28d) and (28e) it follows that \( \omega^q_\beta \) and \( \varphi^q_\beta \) are annihilation operators, whereas \( \omega^q_\alpha \) and \( \varphi^q_\beta \) are creation operators. In (27) \( g_{(0)} \) is spanned by the generators of the maximal compact subalgebra \( u(1)_E \oplus su(2) \oplus su(2) \) of \( su(2,2) \), where \( E \) is the \( AdS_5 \) energy (conformal dimension) operator that defines grading of the \( su(2,2) \) generators:

\[
[E, g_{(\sigma)}] = \sigma g_{(\sigma)} \quad \sigma = 0, \pm 1.
\]  

(29)

As a result the \( su(2,2) \) commutation relations can be written in concise schematic form

\[
[g_{(\sigma)}, g_{(\sigma')} ] = g_{(\sigma+\sigma')} .
\]  

(30)

Expression (28a) corresponds to the creation-annihilation ordering of oscillators and was obtained from the manifestly Hermitian one by moving raising oscillators to the left. Introducing the oscillator number operators \( N_{(a)} = a^\dagger_a a^i_a, N_{(b)} = b^\dagger_b b^i_b, N_{(\omega)} = -\omega^q_\alpha \omega^q_\alpha \) and \( N_{(\varphi)} = \varphi^q_\alpha \varphi^q_\alpha \) it can be brought to the form

\[
E = \frac{1}{2} (N_{(a)} + N_{(b)} + N_{(\omega)} + N_{(\varphi)}).
\]  

(31)

\( N_{(\omega)} \) includes extra minus sign that correlates with the wrong sign in the anticommutator of the \( \omega \)-oscillators in (22). Though \( E \) is defined to be non-negative, in (31) there is no positive \( c \)-number term unlike the case of \( su(2,2) \) generators realized solely in terms of the bosonic oscillators [12], [30]. In the latter case such positive \( c \)-number contribution ensures that the described irreducible representations (irreps) lie above or saturate respective unitarity bounds. There is no ordering ambiguity in the expressions (28d) and (28e) for \( L^\alpha_\beta \) and \( R^\alpha_\beta \) that generate two \( su(2) \) subalgebras of \( su(2,2) \). Their spin eigenvalues \( s_1 \) and \( s_2 \) together with the eigenvalue of \( E \) are, as is common, used to label the \( su(2,2) \) irreps that are constructed by repeated application of \( T^{(+)} i i_\alpha \beta \) to the vacuum \( |0 \rangle \) or another lowest-weight vector (lsvv) that is annihilated by \( T^{(-)} i i_\alpha \beta \) and the \( su(2|2) \) color symmetry generators.

Generators of the \( su(4) \subset u(2,2|4) \) also admit three-grading decomposition

\[
h_{(0)} = \{ C, A^\alpha_b, B^\beta_b \}, \quad h_{(-1)} = \{ V^{(-)} i \ b \}, \quad h_{(+1)} = \{ V^{(+)} i \ b \}:
\]  

(32)

\[
2C = -\alpha^a_\alpha \alpha^i_a - 2\beta^a_\beta \beta^i_a + c^a_q \alpha^i_a - d^a_q \beta^i_a,
\]  

(33a)

\[
A^\alpha_b = \beta^a_\alpha \alpha^i_a - c^a_q \alpha^i_a - \frac{1}{2} \delta^a_b (\beta^d_\beta \beta^i_d - c^d_q \beta^i_d),
\]  

(33b)

\[
B^\beta_b = \beta^a_\beta \beta^i_a - d^a_q \beta^i_a - \frac{1}{2} \delta^a_b (\beta^d_\beta \beta^i_d - c^d_q \beta^i_d),
\]  

(33c)

\[
V^{(-)} i b = \beta^a_\beta \alpha^i_a - c^a_q \beta^i_a,
\]  

(33d)

\[
V^{(+)} i b = \alpha^a_\beta \beta^i_a - d^a_q \beta^i_a
\]  

(33e)

so that commutation relations of the \( su(4) \) algebra can be brought to the form respecting three-grading structure

\[
[h_{(\tau)}, h_{(\tau')} ] = h_{(\tau+\tau')},
\]  

(34)

where grading of a generator is read off from its commutator with \( C \)

\[
[C, h_{(\tau)}] = -\tau h_{(\tau)} \quad \tau = 0, \pm 1.
\]  

(35)

The form of the diannihilation and dicreation operators (33d) and (33e) suggests that bosonic oscillators \( c^q_b \) and \( d^q_b \) are lowering ones, while \( c^a_q \) and \( d^a_q \) are raising. In view of (25) this choice implies that \( c \)-oscillators satisfy wrong-sign commutation relations. Then the operator \( C \) can be written in the form similar to that of the \( AdS_5 \) energy operator (31)

\[
C = \frac{1}{2} (N_{(a)} + N_{(b)} + N_{(c)} + N_{(d)}),
\]  

(36)
where \( N(\alpha) = \alpha^a \alpha_a \), \( N(\beta) = \beta^a \beta_a \), \( N(c) = -\epsilon^a \epsilon_a \) and \( N(d) = d^i d_i \) are the oscillator number operators and the overall minus sign has been introduced to conform with the definition of \( C \) in [30]. It is readily seen to be non-positive definite, whereas for the \( su(4) \) nirs in [12] its eigenvalues are non-negative due to the positive c-number contribution equal the number of colors \( P \). To label \( su(4) \) irreps arising below in addition to the eigenvalue of \( C \) we choose two spin eigenvalues \( j_1 \) and \( j_2 \) for two \( su(2) \) subalgebras spanned by \( A^a_b \) and \( B^a_{-b} \). Like in the case of \( su(2,2) \) irreps, \( su(4) \) irreps in question are constructed by the multiple action of \( V^{(+)}_{ab} \) on the vacuum \( |0\) or another lwv annihilated by \( V^{(-)}_{a_b} \) and the \( su(2|2) \) color symmetry generators.

Two remaining bosonic generators of \( u(2,2|4) \) are the constraint [40] that in oscillator form equals

\[
T = -N(\alpha) + N(\beta) - N(c) + N(d) \approx 0 \tag{37}
\]

and

\[
U = -N(\alpha) + N(\beta) - N(c) + N(d). \tag{38}
\]

Since \( T \) is the constraint, for all physical states of the quantized superparticle its eigenvalue is zero but eigenvalues of \( Y = -N(\alpha) + N(\beta) + N(c) + N(d) \) can be used as an additional label together with the \( su(2,2) \) and \( su(4) \) labels.

Among odd generators of \( u(2,2|4) \) there are two constructed out of the raising oscillators

\[
Q^{(+)}_{ab} \approx a^i \beta^i_{\alpha} - \omega^\alpha d^\alpha_a, \quad Q^{(+)}_{a\alpha} = a_i^{\alpha} b^i_{\alpha} - \epsilon_\alpha \epsilon^\alpha \tag{39}
\]

that belong to the +1 eigenstate w.r.t. the grading defined by \( E-C \). Their repeated action on the \( su(2, 2) \oplus su(4) \) lwv corresponding to the ground state of the superparticle that will be identified in the next Section yields all other lwv’s of the supermultiplet of the physical states.

## 5 Physical states of quantized superparticle

As discussed above the \( u(2,2|4) \) generators [26] (anti)commute with the \( su(2|2) \) gauge symmetry generators [41] and [47], so one can consider their action directly on the lwv’s and examine the constraints imposed by \( su(2|2) \) gauge symmetry on the ground state of quantized superparticle. Bosonic constraints [43] imply that sought for lwv is a singlet of two \( su(2) \) color subalgebras of \( su(2|2) \) and is annihilated by \( T \). Such a lwv can be constructed by acting on the vacuum \( |0\) with the products of color-singlet combinations of raising oscillators that include determinants of bosonic oscillators \( \Delta(\alpha) = \frac{1}{2} \varepsilon^{ij}_{\alpha \beta} \alpha_i^a \alpha_j^{\beta} \), \( \Delta(\beta) = \frac{1}{2} \varepsilon^{ij}_{\alpha \beta} \beta_i^a \beta_j^{\beta} \), \( \Delta(c) = \frac{1}{2} \varepsilon^{pq}_{\alpha \beta} \alpha^{ap}_{\alpha} \alpha^{bp}_{\beta} \) and \( \Delta(d) = \frac{1}{2} \varepsilon^{pq}_{\alpha \beta} \beta^{ap}_{\alpha} \beta^{bp}_{\beta} \), and antisymmetrized in color indices products of the fermionic oscillators \( \alpha^{ab} = \alpha_i^a \varepsilon_{ij} \alpha_j^{b} \), \( \beta^{ab} = \beta_i^a \varepsilon_{ij} \beta_j^{b} \), \( \omega^{\alpha \beta} = \omega_{\alpha}^{\beta} \varepsilon^{pq}_{\alpha \beta} \), and \( \varphi_{\alpha \beta} = \varphi_{\alpha}^{\beta} \varepsilon^{pq}_{\alpha \beta} \) raised to the powers balanced in such a way as to be annihilated by \( T^{(-)}_{\alpha \beta}, V^{(-)}_{a_b} \) and \( T \). Among these lwv’s only the vacuum itself is annihilated by all the fermionic constraints [47] and hence is the ground state.

Consider now repeated action on the ground state of the supersymmetry generators [49]. Applying powers of \( Q^{(+)}_{ab} \) to the vacuum and decomposing the products of oscillators into irreps of respective \( su(2) \) subalgebras of \( u(2,2|4) \) yields six independent lwv’s given in the Table 1 with spin content of which is the same as that of lwv’s given in the first column of the Table 1 in [12] with units in the second and third digits, while the eigenvalues of \( E \)

\[ \text{We prefer to label lwv’s by four spins of four } su(2) \text{ subalgebras of } psu(2,2|4) \text{ rather than by Young tableau as in [12].} \]
and $Y$ correspond to another normalizations of these generators compared to those adopted there. Similarly repeated application of $Q^{(+)}^{a}_{\alpha}$ gives another five independent lwv’s with interchanged spins $s_1 \leftrightarrow s_2$ and $j_1 \leftrightarrow j_2$ presented in the Table 2. Their spin content is the same as that of the lwv’s given in the first column of the Table 1 in [12] with units in the first and fourth digits. There remains to consider simultaneous application of the powers of $Q^{(+)}^{a}_{\alpha}$ and $Q^{(+)}^{\alpha}_{a}$. To find respective lwv’s one has to decompose the products of oscillators into irreps of four $su(2)$ subalgebras of $u(2,2|4)$ and leave, according to the criterion formulated in [31], only traceless tensors of the two $su(2)$ color subalgebras. This selects 25 lwv’s, whose labels can formally be obtained by summing the labels of all pairs of the lwv’s from Tables 1 and 2, except for the vacuum, as is seen from the Table 3. Applying to these lwv’s generators (28c) and (33c) from $+1$ subspaces of $su(2,2)$ and $su(4)$ algebras yields infinite-dimensional irreps that correspond to the fields on $AdS_5 \times S^5$ listed in the utmost right column in the Tables 1-3. To make contact with the consideration of [12] we use similar notation for these fields that span the supermultiplet of fluctuations of Type IIB supergravity on the $AdS_5 \times S^5$ superbackground. Let us note that energies of some of the above $su(2,2)$ irreps with spins $s_1$ and $s_2$ lie below the mildest unitarity bound $E \geq s_1 + s_2 + 2$ for massless fields on $AdS_5$ or the bound $E \geq s_1 + s_2 + 1$ ($s_1s_2 = 0$) for massless fields on the Minkowski boundary of $AdS_5$ [36] and hence correspond to the non-unitary irreps with non-negative energy. To the best of our knowledge such irreps have not been studied in the literature though they appear to be of physical importance as our analysis suggests.

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| $su(2,2)$ labels $(E, s_1, s_2)$ | $su(4)$ labels $(C, j_1, j_2)$ | $Y$ | fields on $AdS_5 \times S^5$ |
|-------------------------------|-------------------------------|-----|--------------------------|
| $(0, 0, 0)$                  | $(0, 0, 0)$                  | 0   | $\phi^{(1)}(y, z)$     |
| $(1/2, 1/2, 0)$              | $(-1/2, 0, 1/2)$             | -1  | $\lambda_{\alphaa}^{(1)}(y, z)$ |
| $(1, 1, 0)$                  | $(-1, 0, 0)$                 | -2  | $\lambda_{\alphaa}^{(2)}(y, z)$ |
| $(1, 0, 0)$                  | $(-1, 0, 1)$                 | -2  | $\lambda_{\alphaa}^{(3)}(y, z)$ |
| $(3/2, 1/2, 0)$              | $(-3/2, 0, 1/2)$             | -3  | $\lambda_{\alphaa}^{(4)}(y, z)$ |
| $(2, 0, 0)$                  | $(-2, 0, 0)$                 | -4  | $\bar{\phi}^{(3)}(y, z)$   |

Table 1. $su(2,2)$ and $su(4)$ labels of lwv’s obtained by acting with the powers of $Q^{(+)}^{a}_{\alpha}$ on $|0\rangle$ and respective fields on $AdS_5 \times S^5$

| $su(2,2)$ labels $(E, s_1, s_2)$ | $su(4)$ labels $(C, j_1, j_2)$ | $Y$ | fields on $AdS_5 \times S^5$ |
|-------------------------------|-------------------------------|-----|--------------------------|
| $(1/2, 0, 1/2)$              | $(-1/2, 1/2, 0)$             | 1   | $\lambda_{\alphaa}^{(1)}(y, z)$ |
| $(1, 0, 1)$                  | $(-1, 0, 0)$                 | 2   | $A_{\alphaa}^{(1)}(y, z)$        |
| $(1, 0, 0)$                  | $(-1, 1, 0)$                 | 2   | $\varphi_{\alphaa}^{(2)}(y, z)$ |
| $(3/2, 0, 1/2)$              | $(-3/2, 1/2, 0)$             | 3   | $\lambda_{\alphaa}^{(2)}(y, z)$ |
| $(2, 0, 0)$                  | $(-2, 0, 0)$                 | 4   | $\bar{\varphi}^{(3)}(y, z)$   |

Table 2. $su(2,2)$ and $su(4)$ labels of lwv’s obtained by acting with the powers of $Q^{(+)}^{a}_{\alpha}$ on $|0\rangle$ and respective fields on $AdS_5 \times S^5$

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6Numbers in round brackets after an index denote a group of symmetrized indices equal to that number.
7Unitarity conditions found by G. Mack for SU(2, 2) positive energy irreps are the special cases of the unitarity bounds for the positive energy irreps of $so(2, d)$ that correspond to tensor and tensor-spinor fields on $AdS_{d+1}$ [37] and for the positive energy irreps of $su(p, q)$ [33].
| $su(2, 2)$ labels $(E, s_1, s_2)$ | $su(4)$ labels $(C, j_1, j_2)$ | $Y$ | Fields on $AdS_5 \times S^5$ |
|-----------------|-----------------|----|-------------------|
| $(1, 1/2, 1/2)$ | $(-1, 1/2, 1/2)$ | 0  | $A_{\alpha\theta\alpha\theta}^{(4)}(y, z)$ |
| $(3/2, 1, 1/2)$ | $(-3/2, 1, 2, 0)$ | -1 | $\psi_{\alpha(2)\alpha\theta}^{(1)}(y, z)$ |
| $(3/2, 0, 1/2)$ | $(-3/2, 1/2, 1)$ | -1 | $\lambda_{\alpha\theta(2)\alpha}^{(3)}(y, z)$ |
| $(3/2, 1/2, 1)$ | $(-3/2, 0, 1/2)$ | 1  | $\psi_{\alpha(2)\theta}^{(1)}(y, z)$ |
| $(3/2, 1/2, 0)$ | $(-3/2, 1, 1/2)$ | 1  | $\lambda_{\alpha\theta(2)\alpha}^{(3)}(y, z)$ |
| $(2, 1, 0)$     | $(-2, 1, 0)$    | 0  | $A_{\alpha\theta\alpha\theta}^{(2)}(y, z)$ |
| $(2, 0, 1)$     | $(-2, 0, 1)$    | 0  | $A_{\alpha(2)\alpha\theta}^{(2)}(y, z)$ |
| $(2, 1, 1)$     | $(-2, 0, 0)$    | 0  | $h_{\alpha(2)\alpha\theta}(y, z)$ |
| $(2, 0, 0)$     | $(-2, 1, 1)$    | 0  | $\varphi_{\alpha(2)\alpha\theta}^{(4)}(y, z)$ |
| $(2, 1/2, 1/2)$ | $(-2, 1/2, 1/2)$| -2 | $A_{\alpha\theta\alpha\theta}^{(2)}(y, z)$ |
| $(5/2, 1/2, 1)$ | $(-5/2, 0, 1/2)$| -1 | $\psi_{\alpha\theta(2)\alpha}^{(2)}(y, z)$ |
| $(5/2, 1/2, 0)$ | $(-5/2, 1, 1/2)$| -1 | $\lambda_{\alpha(2)\theta\alpha}^{(5)}(y, z)$ |
| $(5/2, 1/1, 1/2)$ | $(-5/2, 1/2, 0)$ | 1  | $\psi_{\alpha(2)\theta}^{(2)}(y, z)$ |
| $(5/2, 0, 1/2)$ | $(-5/2, 1/2, 1)$ | 1  | $\lambda_{\alpha\theta(2)\alpha}^{(5)}(y, z)$ |
| $(5/2, 0, 1/2)$ | $(-5/2, 1/2, 0)$ | -3 | $\lambda_{\alpha\theta}^{(4)}(y, z)$ |
| $(5/2, 1/2, 0)$ | $(-5/2, 0, 1/2)$ | 3  | $\lambda_{\alpha\theta}^{(4)}(y, z)$ |
| $(3, 1/2, 1/2)$ | $(-3, 1/2, 1/2)$ | 0  | $A_{\alpha\theta\alpha\theta}^{(3)}(y, z)$ |
| $(3, 0, 1)$     | $(-3, 0, 0)$    | -2 | $A_{\alpha(2)\alpha\theta}^{(3)}(y, z)$ |
| $(3, 0, 0)$     | $(-3, 1, 0)$    | -2 | $\varphi_{\alpha(2)\theta}^{(5)}(y, z)$ |
| $(3, 1, 0)$     | $(-3, 0, 0)$    | 2  | $A_{\alpha(2)\alpha\theta}^{(3)}(y, z)$ |
| $(3, 0, 0)$     | $(-3, 0, 1)$    | 2  | $\varphi_{\alpha(2)\theta}^{(5)}(y, z)$ |
| $(7/2, 0, 1/2)$ | $(-7/2, 1, 2, 0)$ | -1 | $\lambda_{\alpha\theta}^{(6)}(y, z)$ |
| $(7/2, 1/2, 0)$ | $(-7/2, 0, 1/2)$ | 1  | $\lambda_{\alpha\theta}^{(6)}(y, z)$ |
| $(4, 0, 0)$     | $(-4, 0, 0)$    | 0  | $\varphi^{(6)}(y, z)$ |

Table 3. $su(2, 2)$ and $su(4)$ labels of lwv’s obtained by acting with the powers of $Q^{(+)}_{\alpha\theta} = Q^{(+)}_{\alpha\theta}$ on $|0\rangle$ and respective fields on $AdS_5 \times S^5$
6 Discussion

In this note we have shown how the physical states of quantized $AdS_5 \times S^5$ massless superparticle in twistor formulation can be described in terms of the $su(2)$ bosonic and fermionic oscillators some of which were used previously to construct unitary supermultiplets of $su(2,2|4)$ [12], [30], [35]. These physical states of the superparticle are mapped to the known supermultiplet of fluctuations of IIB supergravity on the $AdS_5 \times S^5$ superbackground [12], [13]. We observe that the description of this supermultiplet in the context of superparticle model differs from the original description in [12], where it was presented in the form of an infinite sum of supermultiplets, whose lwv's are obtained by acting on the oscillator vacuum with the raising supersymmetry generators of $psu(2,2|4)$ superalgebra constructed with the aid of $P > 1$ copies of the four sets of $su(2)$ oscillators that emerge from conventional $SU(2,2|4)$ Ferber supertwistors. In these supermultiplets lwv's with the same $su(2) \oplus su(2)$ spins of $su(2,2)$ and $Y$ eigenvalues but in different tensor-(spinor) representations of $su(4)$ correspond to the Fourier modes on $S^5$ of the $AdS_5 \times S^5$ fields [13]. In contrast the dynamical variables of the $AdS_5 \times S^5$ superparticle model in the four-supertwistor formulation under consideration are two Ferber or $c$-type supertwistors and two $a$-type supertwistors, whose components have another Grassmann parity. The latter give rise to $su(2)$ oscillators some of which satisfy wrong-sign (anti)commutation relations. As a result for some states their $su(2,2)$ lwv’s have $AdS_5$ energies, though non-negative, lying below the unitarity bounds for respective spins thus corresponding to non-unitary $su(2,2)$ irreps. They also transform according to the infinite-dimensional non-unitary $su(4)$ irreps that can be thought of as resulting from summing up the $S^5$ Fourier modes giving rise to the fields on $AdS_5 \times S^5$. So our study shows importance of such non-unitary irreps of $su(2,2)$ and $su(4)$ that clearly deserve further study.

The same results could be obtained in the supertwistor approach. In [19] there were derived the first-order differential equations for the superparticle’s wave function, whose arguments are $c$- and $a$–type ambitwistors. To provide the supertwistor description of the supermultiplet of IIB supergravity fluctuations over $AdS_5 \times S^5$ as solutions of these equations it is necessary to extend the ambitwistor transform to the case of fields on $AdS_5$. This would be an important step towards the twistor formulation of the $AdS_5/CFT_4$ duality. The results reported here can also be extended to find the supertwistor description of the supermultiplets of fluctuations over highly supersymmetric backgrounds of other supergravity theories and to the twistor description of extended objects such as superstrings on $AdS_5 \times S^5$.

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