GEVREY WELL-POSEDNESS OF THE HYPERBOLIC PRANDTL EQUATIONS

Wei-Xi Li and Rui Xu

Abstract. We study the 2D and 3D Prandtl equations of degenerate hyperbolic type, and establish without any structural assumption the Gevrey well-posedness with Gevrey index \( \leq 2 \). Compared with the classical parabolic Prandtl equations, the loss of the derivatives, caused by the hyperbolic feature coupled with the degeneracy, can’t be overcame by virtue of the classical cancellation mechanism that developed for the parabolic counterpart. Inspired by the abstract Cauchy-Kowalewski theorem and by virtue of the hyperbolic feature, we give in this text a straightforward proof, basing on an elementary \( L^2 \) energy estimate. In particular our argument does not involve the cancellation mechanism used efficiently for the classical Prandtl equations.

1. Introduction and main result

The classical Prandtl equation was derived by L. Prandtl in 1904 as a model to describe the behavior of the flow near the fluid boundary. It is a degenerate parabolic type equation, losing one order tangential derivatives due to the absence of tangential diffusion. The mathematical study on the classical Prandtl boundary layer has a long history with various approaches developed and so far it is well-explored in various function spaces, see, e.g., [3, 7, 9, 12, 14, 16, 19, 21, 22, 23, 26, 29, 41, 43] and the references therein. Among those works, there are two main settings. One is referred to the Sobolev well/ill-posedness that usually requires Oleinik’s monotonicity assumption (cf. [3, 31, 33]), and another is referred to the Gevrey (including analyticity) well-posedness without any structural assumption. Note that Oleinik’s monotonicity condition is crucial for the Sobolev well-posedness theory. On one hand, the classical solution was established by Oleinik (see for instance [33]) and two groups Alexandre-Wang-Xu-Yang [3] and Masmoudi-Wong [31] independently proved the existence and uniqueness theory in the Sobolev spaces by using energy method. On the other hand, without Oleinik’s monotonicity, the Prandtl equation may be ill-posed in Sobolev space (Gevrey space more precisely) that was observed by Gérard-Varet and Dormy [11]. Inspired by the ill-posedness result in [11, 29], it is natural to expect the Gevrey well-posedness whenever the initial data are in Gevrey class \( \leq 2 \) without any structural assumption. This has been confirmed recently by Dietert and Gérard-Varet [9] and Li-Masmoudi-Yang [18] for the 2D and the 3D Prandtl

2020 Mathematics Subject Classification. 35Q35, 35Q30, 76D10, 76D03.

Key words and phrases. hyperbolic Prandtl boundary layer, well-posedness, Gevrey space, abstract Cauchy-Kowalewski theorem.
equations, respectively, after the earlier works of [7, 12, 22] in Gevrey class and Sammartino-Caflisch’s work [39] in the analytic setting. Recently, similar problems have been explored for MHD boundary layer equations that are of Prandtl type equations with the strong background magnetic fields; for instance the stabilizing effect of the magnetic fields has been justified recently by Liu-Xie-Yang [28] (see also [24] for the further development), and the ill-posed in Sobolev space (more precisely, in Gevrey spaces with index > 2) is established by [24, 27] and the Gevrey well-posedness is proven by [22]. The aforementioned works are mainly concerned with the local-in-time existence. We refer to Xin-Zhang’s work [41] on global weak solutions under the monotonicity condition and the recent works of Paicu-Zhang [35] and Wang-Wang-Zhang [40] on global small analytic and Gevrey solutions, respectively. The global analytic solutions to MHD boundary layer system were obtained recently by Liu-Zhang [31] and Li-Xie [17].

This work aims to study the well-posedness theory of the hyperbolic version of Prandtl equations. The hyperbolic boundary layer equations can be derived similarly as the classical Prandtl system, just following the Prandtl’s ansatz when analyzing the asymptotic expansion w.r.t. the viscosity of the hyperbolic Navier-Stokes equations which are complemented with the no-slip boundary conditions. The hyperbolic version of Navier-Stokes equation by adding a small hyperbolic perturbation to the classical one, was initiated by Cattaneo [6]. The classical Navier-Stokes equation has infinite propagation speed which is non-physical, and in order to avoid the non-physical feature Cattaneo [6] proposed the Cattaneo law instead of the classical Fourier law. Up to now, there have been extensive works on the hyperbolic Navier-Stokes equation; see for instance [2, 3, 8, 34, 37, 38] and the references therein. Similar to the parabolic Navier-Stokes equation, a boundary layer will appear when investigating the hyperbolic counterpart in a bounded domain complemented with the no-slip boundary conditions, and its governing equation can be derived as below (see Appendix A below), just following the classical Prandtl’s ansatz,

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\left( \partial_t^2 + \partial_t + u \cdot \partial_x + v \partial_y - \partial_y^2 \right) u + \partial_x p = 0, \\
\partial_x u + \partial_y v = 0,
\end{array} \right. & \quad (x, y) \in \mathbb{R}_+^n \\
u|_{y=0} = 0, \quad v|_{y=0} = 0, \quad \lim_{y \to +\infty} u(t, x, y) = U(t, x), \quad x \in \mathbb{R}^{n-1}, \\
u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \quad (x, y) \in \mathbb{R}_+^n,
\end{aligned}
\]

where the fluid domain \( \mathbb{R}_+^n = \{(x, y) \in \mathbb{R}^n; \ x \in \mathbb{R}^{n-1}, y > 0\} \) with \( n = 2 \) or \( n = 3 \), and \( u, v \) stand for the tangential and normal velocity, respectively, and \( U(t, x), p(t, x) \) are given functions from the outflow that are linked by

\[\partial_t^2 U + \partial_t U + (U \cdot \partial_x) U + \partial_x p = 0.\]

For simplicity, we may assume without loss of generality that \( U \equiv 0 \) in (1.1), and then the system (1.1) is reduced to

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\left( \partial_t^2 + \partial_t + u \cdot \partial_x + v \partial_y - \partial_y^2 \right) u = 0, \\
\partial_x u + \partial_y v = 0,
\end{array} \right. & \quad (x, y) \in \mathbb{R}_+^n \\
u|_{y=0} = 0, \quad v|_{y=0} = 0, \quad \lim_{y \to +\infty} u(t, x, y) = 0, \\
u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1.
\end{aligned}
\]
This text is motivated by the recent work of Aarach [1] and Paicu-Zhang [36], where the authors investigated the global well-posedness of hydrostatic Navier-Stokes equations of hyperbolic version. For the classical hydrostatic Navier-Stokes or Euler equations, it is far from well-explored, and there are only few works. We mention here the very recent work [13] of Gérard-Varet-Masmoudi-Vicol on the local Gevrey well-posedness of 2D parabolic hydrostatic Navier-Stokes equation under convex assumption. The convex assumption is removed for the hyperbolic version of hydrostatic Navier-Stokes equation by the recent Paicu-Zhang’s work [36] where they established the global well-posedness in Gevrey class 2, improving the earlier Aarach’s work [1] in the analytic setting. This shows the hyperbolic feature may acts as stabilizing factor for hydrostatic Navier-Stokes equation. In this work we will justify the effect of the hyperbolic feature on the Prandtl equations that admit a degeneracy structure different from the hydrostatic Navier-Stokes equation.

**Notation.** Given the domain $\Omega = \mathbb{R}^n$ with $n = 2$ or $3$, we will use $\| \cdot \|_{L^2}$ and $(\cdot, \cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(\Omega)$ and use the notation $\| \cdot \|_{L^2_{\rho}}$ and $(\cdot, \cdot)_{L^2_{\rho}}$ when the variable $x$ is specified. Similar notation will be used for $L^\infty$. Moreover, we use $L^p_{\rho}(L^q_{\sigma}) = L^p(\Omega; L^q(\mathbb{R}^n))$ for the classical Sobolev space.

**Definition 1.1.** Let $\ell > 1/2$ be a given number. With each pair $(\rho, \sigma)$, $\rho > 0$ and $\sigma \geq 1$, the Gevrey space $X_{\rho, \sigma}$ consists of all smooth functions $f(t, x)$ such that the Gevrey norm $|f|_{\rho, \sigma} < +\infty$, where $|\cdot|_{\rho, \sigma}$ is defined as below. If $f$ is a function of $x$ only but independent of $t$, then

$$|f|_{\rho, \sigma} = \sup_{|\alpha| \geq 7} \rho^{\alpha-\ell} \left( \| \langle y \rangle^\ell \partial_y \partial_x^\alpha f \|_{L^2} + |\alpha| \| \langle y \rangle^\ell \partial_x^\alpha f \|_{L^2} \right),$$

and moreover for functions $f$ of $(t, x)$ variables we define

$$|f|_{\rho, \sigma} = \sup_{|\alpha| \geq 7} \rho^{\alpha-\ell} \left( \| \langle y \rangle^\ell \partial_t \partial_x^\alpha f(t) \|_{L^2} + \| \langle y \rangle^\ell \partial_y \partial_x^\alpha f(t) \|_{L^2} + |\alpha| \| \langle y \rangle^\ell \partial_x^\alpha f(t) \|_{L^2} \right),$$

where and throughout the paper $\langle y \rangle = (1 + |y|^2)^{1/2}$. We call $\sigma$ the Gevrey index.

Note if $f \in X_{\rho, \sigma}$, then $f$ is of Gevrey class and Sobolev, respectively, in tangential variable $x$ and normal variable $y$. The main result can be stated as below.

**Theorem 1.2.** Let $1 \leq \sigma \leq 2$ and $0 < \rho_0 \leq 1$. Suppose the initial data in $X_{\rho_0, \sigma}$ satisfy that $u_0, u_1 \in X_{2\rho_0, \sigma}$, compatible with the boundary condition in $X_{\rho_0, \sigma}$. Then the $2D$ or $3D$ hyperbolic Prandtl system (1.2) admits a unique solution $u \in L^\infty([0, T]; X_{\rho, \sigma})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.

**Remark 1.3.** The above result shows that the well-posedness theory of hyperbolic Prandtl equation is not worse than the one for the parabolic counterpart. For the classical Prandtl equation it is well-known 2 is the critical Gevrey index for the well-posedness, seeing for instance [3] [11] [13]. Naturally we would ask the critical Gevrey index for the hyperbolic version of Prandtl equations, which remains unclear so far.
Remark 1.4. As to be seen below, although the cancellation mechanism is an efficient tool when investigating the well-posedness theory for the classical Prandtl equations, it can not apply to the hyperbolic version. Inspired by the abstract Cauchy-Kowalewski theorem and by virtue of the hyperbolic feature, we present in this text a straightforward proof of the main result, basing on an elementary $L^2$ energy estimate. In particular our argument does not involve the cancellation mechanism developed for the classical Prandtl equations. We believe the argument presented here can apply to other type hyperbolic equations with loss of derivatives.

2. A priori estimate and the methodology

2.1. A priori estimate. The key part for proving the main result Theorem 1.2 is to establish a priori estimate for the system (1.2). In fact the existence and uniqueness theory will follow two main strategies, one refers to the construction of approximate solutions for a regularized hyperbolic Prandtl equations which is quite standard since we can apply the classical hyperbolic theory, the other refers to the derivation of an uniform estimate for these approximate solutions, and this follows from the same argument for proving a priori estimate. For sake of simplicity, we will only prove a priori estimate for regular solutions.

Assumption 2.1. Let $X_{\rho,\sigma}$ be the Gevrey function space equipped with the norm $\| \cdot \|_{\rho,\sigma}$ given in Definition 1.1, and let $u \in L^\infty ([0, T]; X_{\rho_0,\sigma})$ solve the hyperbolic system (1.2) with initial data $u_0, u_1 \in X_{2\rho_0,\sigma}$ for some $0 < \rho_0 \leq 1$. Moreover we suppose that there exists a constant $C^*$ such that

$$\forall \, t \in [0, T], \quad \sup_{|\alpha| \leq 2} \| \partial_x^\alpha u(t) \|_{L^2} + \sup_{|\alpha| \leq 2} \| \partial_x^\alpha \partial_y u(t) \|_{L^2} \leq C^*, \tag{2.1}$$

where the constant $C^* \geq 1$ depends only on $|u_0|_{2\rho_0,\sigma}, |u_1|_{2\rho_0,\sigma},$ the Sobolev embedding constants and the numbers $\rho_0, \ell$ that are given in Definition 1.1.

Theorem 2.2. Let $1 \leq \sigma \leq 2$. Under Assumption 2.1 we can find a constant $C$, depending only on the Sobolev embedding constants and the numbers $\rho_0, \ell$ that are given in Definition 1.1, such that for any $t \in [0, T]$ the following estimate

$$|u(t)|_{\rho,\sigma}^2 \leq C(|u_0|_{2\rho_0,\sigma}^2 + |u_1|_{2\rho_0,\sigma}^2) + C \int_0^t \left( |u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^3 \right) ds + CC^* \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds$$

holds true for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq \rho_0$.

Once we have the a priori estimate in Theorem 2.2 the existence and uniqueness for the hyperbolic system (1.2) will follow, just repeating the presentation of [18, Section 6] or [22, Section 8] with slight modification.

2.2. Difficulties and Methodologies. As for the classical Prandtl equation, the main difficulty arises from the loss of tangential derivatives due the absence of diffusion in $x$ variable. So far it is well-explored to overcome the loss of derivatives when investigating the well-posedness of the classical Prandtl equation, seeing for instance the recent works [3, 9, 18, 31], where the main idea involved is the cancellation mechanism initiated by [3, 31]. Note that
these cancellation technique can not apply to the hyperbolic Prandtl system. Precisely when performing energy estimate for system (1.2) we have
\[ \frac{1}{2} \frac{d}{dt} (\| \partial_t u \|^2_{L^2} + \| \partial_y u \|^2_{L^2}) + \| \partial_t u \|^2_{L^2} = -(u \partial_x u, \partial_t u)_{L^2} - (v \partial_y u, \partial_t u)_{L^2}, \]
where the loss of one order tangential derivatives occurs in the terms on the right side with the main difficulty arising from the first one. To overcome the loss of derivatives it relies on the observation that we only lose half order rather than one order derivatives due to the hyperbolic feature. Our argument is inspired by the abstract Cauchy-Kowalewski theorem, whose statement in general Banach scales can be found in [1, 32] as well as the references therein; see [18, 22, 39] as well for its application to the Well-posedness theory of classical Prandtl equations in analytic or Gevrey spaces. Consider the Cauchy problem
\[ \partial_t h = F(t, h, \partial_x h), \quad h|_{t=0} = h_0, \]
and for given analytic data \( F \) and \( h_0 \), we may only expect, by virtue of the classical Cauchy-Kowalewski theorem, the local existence and uniqueness in analytic space for the Cauchy problem above since it loses in general one order derivative. Moreover as far as the following Cauchy problem
\[ \partial_x^2 h = F(t, h, \nabla h), \quad h|_{t=0} = h_0, \quad \partial_t h|_{t=0} = h_1, \tag{2.2} \]
is concerned, the existence theory can be extended to any Gevrey space once the Gevrey index \( \leq 2 \) by using abstract Cauchy-Kowalewski theorem in Gevrey class. We refer to [18] for the application of the abstract Cauchy-Kowalewski theorem of Gevrey version by exploring the intrinsic structure similar as in (2.2). Note the hyperbolic version of Prandtl system (1.2) may be viewed as the type of (2.2) due to the hyperbolic factor \( \partial_t^2 \) in (1.2).

3. Proof of the a priori estimate

This part is devoted to proving the a priori estimate stated in Theorem 2.2. Without loss of generality we only present in detail the proof of Theorem 2.2 for \( n = 2 \), and the three-dimensional case can be treated in the same way.

Proof of Theorem 2.2 (Two-dimensional case). We proceed through several steps to deal with the terms involved in Definition 1.1 of \( |u|_{\rho,\sigma} \). In the following argument we suppose \( t \) is fixed with \( 0 < t \leq T \). To simplify the notation we use the capital letters \( C \) to denote some generic constant that may vary from line to line, depending only on the Sobolev embedding constants and the numbers \( \rho_0, \ell \) given in Definition 1.1 but independent of the constant \( C \), in (2.1) and the order of derivatives denoted by \( m \).

Step (a). In this step, we conclude that for any \( m \geq 7 \) and any pair \((\rho, \tilde{\rho})\) with \( 0 < \rho < \tilde{\rho} \leq \rho_0 \),
\[ \frac{\rho^{2(m-7)}}{[(m-7)!]^{2\sigma}} m^2 \left\| \langle y \rangle^\ell \partial^m_x u(t) \right\|^2_{L^2} \leq C \left| u_0 \right|_{2\rho_0,\sigma}^2 + C \int_0^t \left| u(s) \right|_{\rho,\tilde{\rho}}^2 \sigma \tilde{\rho} \sigma ds, \tag{3.1} \]
In fact, it follows from Definition 1.1 of \( |u|_{r,\sigma} \) that, for any integer \( j \geq 0 \) and for any \( 0 < r \leq \rho_0 \),
\[ \left\| \langle y \rangle^\ell \partial_x \partial_t^j u \right\|_{L^2} + \left\| \langle y \rangle^\ell \partial_y \partial_x^j u \right\|_{L^2} + j \left\| \langle y \rangle^\ell \partial_x^j u \right\|_{L^2} \leq \left\{ \begin{array}{ll} \frac{[(j-7)!]^{\sigma}}{[(r-7)!]^{\sigma}} \left| u \right|_{r,\sigma}, & \text{if } j \geq 7, \\ \left| u \right|_{r,\sigma}, & \text{if } j \leq 6. \end{array} \right. \tag{3.2} \]
As a result,
\[
m^2 \int_0^t \| \langle y \rangle^{\ell} \partial_x^m u(s) \|_{L^2} \| \langle y \rangle^{\ell} \partial_t \partial_x^m u(s) \|_{L^2} ds
\leq \int_0^t m \frac{[m-7]!^2 \sigma}{\rho^2(m-\rho)} |u(s)|^2_{\rho,\sigma} ds \leq C \frac{[m-7]!^2 \sigma}{\rho^2(m-\rho)} \int_0^t \frac{|u(s)|^2_{\rho,\sigma}}{\rho - \rho} ds,
\] (3.3)
the last inequality holding because for any pair \((\rho, \tilde{\rho})\) with \(0 < \rho < \tilde{\rho} \leq \rho_0 \leq 1,
\[
\frac{m}{\rho^2(m-\rho)} \leq \frac{1}{\rho^2(m-\rho)} \frac{m}{\tilde{\rho}^2(m-\tilde{\rho})} \leq \frac{C}{\rho^2(m-\rho)} \frac{1}{\tilde{\rho}^2(m-\tilde{\rho})} \leq \frac{C}{\rho^2(m-\rho)} \frac{1}{\tilde{\rho} - \rho}
\] (3.4)
due to the fact that the inequality \(k_{r,k}^{l} \leq \frac{1}{l^2}\) holds true for any integer \(k \geq 0\) and any number \(0 < r < 1\). On the other hand, observing \(\rho \leq \rho_0\),
\[
m^2 \| \langle y \rangle^{\ell} \partial_x^m u_0 \|_{L^2}^2 \leq m^2 \left( \frac{[m-7]!^2 \sigma}{(2\rho_0)^2(m-\rho_0)} \right) |u_0|_{\rho_0,\sigma}^2
\leq m^2 \frac{\rho^2(m-\rho)}{(2\rho_0)^2(m-\rho_0)} \left( \frac{[m-7]!^2 \sigma}{\rho^2(m-\rho)} \right) |u_0|_{\rho_0,\sigma}^2 \leq C \frac{[m-7]!^2 \sigma}{\rho^2(m-\rho)} |u_0|_{\rho_0,\sigma}^2.
\]
Combining the above inequality and (3.3) with the fact that
\[
m^2 \| \langle y \rangle^{\ell} \partial_x^m u(t) \|_{L^2}^2 = m^2 \| \langle y \rangle^{\ell} \partial_x^m u(t) \|_{L^2}^2 + 2m^2 \int_0^t \| \langle y \rangle^{\ell} \partial_x^m u(s) \|_{L^2}^2 ds
\leq m^2 \| \langle y \rangle^{\ell} \partial_x^m u(t) \|_{L^2}^2 + 2m^2 \int_0^t \| \langle y \rangle^{\ell} \partial_x^m u(s) \|_{L^2}^2 \| \langle y \rangle^{\ell} \partial_t \partial_x^m u(s) \|_{L^2} ds
\leq C \frac{[m-7]!^2 \sigma}{\rho^2(m-\rho)} \left( |u_0|_{\rho_0,\sigma}^2 + \int_0^t \| u(s) \|_{\rho,\sigma}^2 \right),
\]
we obtain the assertion (3.1).

Step (b). This step is devoted to proving that, for any \(m \geq 7\) and any pair \((\rho, \tilde{\rho})\) with \(0 < \rho < \tilde{\rho} \leq \rho_0\),
\[
\frac{\rho^2(m-\rho)}{[m-7]!^2 \sigma} \left( \| \langle y \rangle^{\ell} \partial_t \partial_x^m u(t) \|_{L^2}^2 + \| \langle y \rangle^{\ell} \partial_y \partial_x^m u(t) \|_{L^2}^2 \right)
\leq C \left( |u_0|_{\rho_0,\sigma} + |u_1|_{\rho_0,\sigma} \right) + C \int_0^t \left( |u(s)|_{\rho_0,\sigma}^2 + |u(s)|_{\rho_0,\sigma}^3 \right) ds + CC_* \int_0^t \frac{|u(s)|_{\rho_0,\sigma}^2}{\rho - \rho} ds,
\] (3.5)
where \(C_*\) is the number given in (2.1). To do so, applying \(\partial_x^m\) to the first equation in (1.2) gives
\[
(\partial_t^2 + \partial_t - \partial_y^2) \partial_x^m u = - \sum_{j=0}^m \binom{m}{j} \left[ (\partial_x^j u) \partial_x^{m-j} u + (\partial_x^j v) \partial_x^{m-j} \partial_y u \right],
\] (3.6)
Furthermore, multiplying the both sides of (3.6) by $\langle y \rangle^{2\ell} \partial_t \partial_x^m u$ and then integrating over $[0, t] \times \mathbb{R}^2_+$, we obtain, observing the initial-boundary conditions in (1.2),

\[
\begin{aligned}
\frac{1}{2} \left\| \langle y \rangle^{\ell} \partial_t \partial_x^m u(t) \right\|_{L^2}^2 + \frac{1}{2} \left\| \langle y \rangle^{\ell} \partial_y \partial_x^m u(t) \right\|_{L^2}^2 + \int_0^t \left\| \langle y \rangle^{\ell} \partial_t \partial_x^m u(s) \right\|_{L^2}^2 ds
&= \frac{1}{2} \left\| \langle y \rangle^{\ell} \partial_x^m u_1 \right\|_{L^2}^2 + \frac{1}{2} \left\| \langle y \rangle^{\ell} \partial_y \partial_x^m u_0 \right\|_{L^2}^2 \\
&\quad - \sum_{j=0}^m \binom{m}{j} \int_0^t \left( \langle y \rangle^{\ell} \left[ (\partial_x^2 u) \partial_x^{m-j+1} + (\partial_x^2 v) \partial_x^{m-j} \partial_y u \right], \langle y \rangle^{\ell} \partial_t \partial_x^m u \right) ds
&\quad - \int_0^t \left( \partial_y \partial_x^m u, (\partial_y \langle y \rangle^{2\ell} \partial_t \partial_x^m u \right)_{L^2} ds.
\end{aligned}
\] (3.7)

It follows from Definition 1.1 that

\[
\frac{1}{2} \left\| \langle y \rangle^{\ell} \partial_x^m u_1 \right\|_{L^2}^2 + \frac{1}{2} \left\| \langle y \rangle^{\ell} \partial_y \partial_x^m u_0 \right\|_{L^2}^2
\leq \frac{[(m - 7)!]^{2\sigma}}{\rho^{2(m - 7)}} \left( |u_0|_{\rho, \sigma}^2 + |u_1|_{\rho, \sigma}^2 \right) \leq \frac{[(m - 7)!]^{2\sigma}}{\rho^{2(m - 7)}} \left( |u_0|_{2\rho_0, \sigma}^2 + |u_1|_{2\rho_0, \sigma}^2 \right).
\]

As for the last term on the right of (3.7) we use (3.2) to conclude

\[
\left| \int_0^t \left( \partial_y \partial_x^m u, (\partial_y \langle y \rangle^{2\ell} \partial_t \partial_x^m u \right)_{L^2} ds \right| \leq C \left\| \frac{[(m - 7)!]^{2\sigma}}{\rho^{2(m - 7)}} \right\|_{\rho, \sigma} u(s)_{\rho, \sigma}^2 ds.
\]

Then the desired estimate (3.8) will follow if we can prove that

\[
\sum_{j=0}^m \binom{m}{j} \left| \int_0^t \left( \langle y \rangle^{\ell} \left[ (\partial_x^2 u) \partial_x^{m-j+1} + (\partial_x^2 v) \partial_x^{m-j} \partial_y u \right], \langle y \rangle^{\ell} \partial_t \partial_x^m u \right)_{L^2} ds \right|
\leq C \int_0^t |u(s)|_{\rho, \sigma}^2 ds + CC_* \int_0^t \frac{|u(s)|_{\rho, \sigma}^2}{\rho - \rho} ds. \tag{3.8}
\]

To do so we write

\[
\sum_{j=0}^m \binom{m}{j} \left| \int_0^t \left( \langle y \rangle^{\ell} \left[ (\partial_x^2 u) \partial_x^{m-j+1} + (\partial_x^2 v) \partial_x^{m-j} \partial_y u \right], \langle y \rangle^{\ell} \partial_t \partial_x^m u \right)_{L^2} ds \right|
\leq \sum_{j=0}^m \binom{m}{j} \int_0^t \left\| \langle y \rangle^{\ell} \partial_x^m u \right\|_{L^2}^2 \left\| \langle y \rangle^{\ell} \partial_t \partial_x^m u \right\|_{L^2}^2 ds
\tag{3.9}
\]

\[
+ \sum_{j=0}^m \binom{m}{j} \int_0^t \left\| \langle y \rangle^{\ell} \partial_x^m \partial_y \partial_x^{m-j} u \right\|_{L^2}^2 \left\| \langle y \rangle^{\ell} \partial_t \partial_x^m u \right\|_{L^2}^2 ds.
\]
For the first term on the right side we have, denoting by \([p]\) the largest integer less than or equal to \(p\),

\[
\sum_{j=0}^{[m/2]} \binom{m}{j} \| \langle y \rangle^\ell (\partial_x^j u) \partial_x^{m-j+1} u \|_{L^2} \leq \sum_{j=0}^{[m/2]} \binom{m}{j} \| \partial_x^j u \|_{L^\infty} \| \langle y \rangle^\ell \partial_x^{m-j+1} u \|_{L^2}
\]

\[
+ \sum_{j=[m/2]+1}^{m} \binom{m}{j} \| \langle y \rangle^\ell \partial_x^j u \|_{L^2} \| \partial_x^{m-j+1} u \|_{L^\infty}.
\]

Using again (3.12) and the Sobolev embedding inequality

\[
\| F \|_{L^\infty} \leq C \left( \| F \|_{H^2_x(L^2_y)} + \| \partial_y F \|_{H^2_y(L^2_x)} \right)
\]

which holds true for \(x \in \mathbb{R}\) or \(x \in \mathbb{R}^2\), we compute, in view of (2.1),

\[
\sum_{j=0}^{[m/2]} \binom{m}{j} \| \partial_x^j u \|_{L^\infty} \| \langle y \rangle^\ell \partial_x^{m-j+1} u \|_{L^2} \leq C \sum_{j=5}^{[m/2]} \frac{m!}{j!(m-j)!} \frac{[(j-5)!]\sigma}{\rho^{j-5}} \frac{[(m-j-6)!]\sigma}{(m-j+1)\rho^{m-j-6}} |u|_{\rho,\sigma}^2
\]

\[
+ C \sum_{1 \leq j \leq 4} \frac{m!}{j!(m-j)!} \frac{[(m-j-6)!]\sigma}{(m-j+1)\rho^{m-j-6}} |u|_{\rho,\sigma}^2 + C_* \frac{[(m-6)!]\sigma}{(m+1)\rho^{m-6}} |u|_{\bar{\rho},\sigma},
\]

with \(C_*\) the constant given in (2.1). Direct verification shows, observing \(1 \leq \sigma \leq 2\),

\[
\sum_{1 \leq j \leq 4} \frac{m!}{j!(m-j)!} \frac{[(m-j-6)!]\sigma}{(m-j+1)\rho^{m-j-6}} |u|_{\rho,\sigma}^2 \leq C \frac{[(m-7)!]\sigma}{\rho^{m-7}} |u|_{\rho,\sigma}^2,
\]

\[
\frac{[(m-6)!]\sigma}{(m+1)\rho^{m-6}} |u|_{\bar{\rho},\sigma} \leq C \frac{m!}{\rho^{m-7}} \frac{[(m-7)!]\sigma}{\rho^{m-7}} |u|_{\bar{\rho},\sigma} \leq C \frac{m!}{\bar{\rho}} \frac{[(m-7)!]\sigma}{\rho^{m-7}} |u|_{\bar{\rho},\sigma}
\]

and

\[
\sum_{j=5}^{[m/2]} \frac{m!}{j!(m-j)!} \frac{[(j-5)!]\sigma}{\rho^{j-5}} \frac{[(m-j-6)!]\sigma}{(m-j+1)\rho^{m-j-6}} |u|_{\rho,\sigma}^2 \leq C \frac{|u|_{\rho,\sigma}^2}{\rho^{m-7}} \sum_{j=5}^{[m/2]} \frac{m![(j-5)!]^{\sigma-1}[(m-j-6)!]^{\sigma-1}}{j!(m-j)!^{\sigma-1}}
\]

\[
\leq C \frac{|u|_{\rho,\sigma}^2}{\rho^{m-7}} \sum_{j=5}^{[m/2]} \frac{(m-7)!m!}{j!(m-j)!^{\sigma-1}} [(m-7)!]^{\sigma-1} \leq C \frac{[(m-7)!]\sigma}{\rho^{m-7}} |u|_{\rho,\sigma}^2.
\]

Substituting the above inequalities into (3.11) gives

\[
\sum_{j=1}^{[m/2]} \binom{m}{j} \| \partial_x^j u \|_{L^\infty} \| \langle y \rangle^\ell \partial_x^{m-j+1} u \|_{L^2} \leq C \frac{[(m-7)!]\sigma}{\rho^{m-7}} |u|_{\rho,\sigma}^2 + CC_* \frac{|u|_{\bar{\rho},\sigma}}{\rho^{m-7}} |u|_{\bar{\rho},\sigma}.
\]

(3.12)
Following a similar argument for proving (3.12), we also have
\[
\sum_{j=\lfloor m/2 \rfloor + 1}^{m} \binom{m}{j} \| \langle y \rangle^\ell \partial_x^j u \|_{L^2} \| \partial_x^{m-j+1} u \|_{L^\infty} \leq C \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |u|_\rho^{2} \sigma .
\]
Substituting the above inequality and (3.12) into (3.10), we conclude
de (3.4). Similarly we follow a similar argument as above to conclude
\[
\sum_{j=\lfloor m/2 \rfloor + 1}^{m} \binom{m}{j} \| \langle y \rangle^\ell (\partial_x^j u) \partial_x^{m-j+1} u \|_{L^2} \leq C \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |u|_\rho^{2} \sigma + CC_* \frac{m}{\rho} \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |u|_\rho^{2} \sigma .
\]
This, together with (3.2), yields
\[
\int_{0}^{t} \sum_{j=0}^{m} \binom{m}{j} \| \langle y \rangle^\ell (\partial_x^j u) \partial_x^{m-j+1} u \|_{L^2} \leq C \frac{[(m-7)!]^\sigma}{\rho^{2(m-7)}} \int_{0}^{t} \| u(s) \|_{\rho, \sigma}^{3} ds + CC_* \frac{[(m-7)!]^\sigma}{\rho^{2(m-7)}} \int_{0}^{t} \| u(s) \|_{\rho, \sigma}^{2} ds,
\]
due to (3.1). Similarly we follow a similar argument as above to conclude
\[
\int_{0}^{t} \sum_{j=0}^{m} \binom{m}{j} \| \partial_x^j v \|_{L^\infty} \| \langle y \rangle^\ell \partial_y \partial_x^{m-j} u \|_{L^2} \leq C \frac{[(m-7)!]^\sigma}{\rho^{2(m-7)}} \int_{0}^{t} \| u(s) \|_{\rho, \sigma}^{3} ds + CC_* \frac{[(m-7)!]^\sigma}{\rho^{2(m-7)}} \int_{0}^{t} \| u(s) \|_{\rho, \sigma}^{2} ds,
\]
where we have used the inequality that if \( \ell > 1/2 \) and \( \partial_x u + \partial_y v = 0 \) as well as \( v|_{y=0} = 0 \),
\[
\| \partial_x^j v \|_{L^\infty_{y} L^2_x} = \int_{y=0}^{\infty} \partial_x^j u(t, x, y) dy \|_{L^\infty_{y} L^2_x} \leq \int_{y=0}^{\infty} \| \partial_x^j v(t, x, y) dy \|_{L^\infty_{y} L^2_x} \leq \int_{y=0}^{\infty} \| \partial_x^{j+1} u(t, x, y) dy \|_{L^\infty_{y} L^2_x} \leq \int_{y=0}^{\infty} \langle y \rangle^\ell \partial_x^{j+1} u(t, x, y) dy \|_{L^\infty_{y} L^2_x} \leq C \| \langle y \rangle^\ell \partial_x^{j+1} u \|_{L^2_{x,y}},
\]
for all integers \( j \geq 0 \), all \( t \in [0, T] \), and some positive constant \( C = C(\ell) \). Combining the two estimates (3.13), (3.14) with (3.9), we obtain (3.8). Thus we can get the desired estimate (3.5).
Step (c) We combine the estimates (3.1) and (3.5) to conclude, for any \( m \geq 7 \) and any pair \((\rho, \tilde{\rho})\) with \(0 < \rho < \tilde{\rho} \leq \rho_0\),
\[
\frac{\rho^{2(m-7)}}{[(m-7)!]^2} \left( \| \langle y \rangle^\ell \partial_\tau^m u(t) \|_{L^2}^2 + \| \langle y \rangle^\ell \partial_y \partial_\tau^m u(t) \|_{L^2}^2 + m^2 \| \langle y \rangle^\ell \partial_y^m u(t) \|_{L^2}^2 \right) \\
\leq CC \left( |u_0|_{2\rho_0,\sigma} + |u_1|_{2\rho_0,\sigma} \right) + C \int_0^t \left( |u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^3 \right) ds + CCs \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds.
\]
It can be checked straightforwardly that the estimate
\[
\sum_{0 \leq m \leq 6} \left( \| \langle y \rangle^\ell \partial_\tau^m u(t) \|_{L^2}^2 + \| \langle y \rangle^\ell \partial_y \partial_\tau^m u(t) \|_{L^2}^2 + m^2 \| \langle y \rangle^\ell \partial_y^m u(t) \|_{L^2}^2 \right) \\
\leq C \left( |u_0|_{2\rho_0,\sigma} + |u_1|_{2\rho_0,\sigma} \right) + C \int_0^t \left( |u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^3 \right) ds + CCs \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds
\]
holds true for any pair \((\rho, \tilde{\rho})\) with \(0 < \rho < \tilde{\rho} \leq \rho_0\). As a result, we combine the two estimates above to complete the proof of Theorem 2.2 for the two-dimensional case.

Proof of Theorem 2.2 (Three-dimensional case). It can be treated in the same way as that for two-dimensional case. In fact following the argument above gives that for any \( m \geq 7 \) we have, recalling \( x = (x_1, x_2)\),
\[
\frac{\rho^{2(m-7)}}{[(m-7)!]^2} \left( \| \langle y \rangle^\ell \partial_\tau^m u(t) \|_{L^2}^2 + \| \langle y \rangle^\ell \partial_y \partial_\tau^m u(t) \|_{L^2}^2 + m^2 \| \langle y \rangle^\ell \partial_y^m u(t) \|_{L^2}^2 \right) \\
\leq C \left( |u_0|_{2\rho_0,\sigma} + |u_1|_{2\rho_0,\sigma} \right) + C \int_0^t \left( |u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^3 \right) ds + CCs \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds,
\]
with \( j = 1 \) or \( 2 \). This along with the fact that
\[
\| \partial_\tau^\alpha F \|_{L^2} \leq C \left( \| \partial_{x_1}^{\alpha_1} F \|_{L^2} + \| \partial_{x_2}^{\alpha_2} F \|_{L^2} \right)
\]
yields
\[
\sup_{|\alpha| \geq 7} \frac{\rho^{\alpha - 7}}{[(\alpha - 7)!]^2} \left( \| \langle y \rangle^\ell \partial_\tau^\alpha f(t) \|_{L^2} + \| \langle y \rangle^\ell \partial_y \partial_\tau^\alpha f(t) \|_{L^2} + \| \langle y \rangle^\ell \partial_y^\alpha f(t) \|_{L^2} \right) \\
\leq C \left( |u_0|_{2\rho_0,\sigma} + |u_1|_{2\rho_0,\sigma} \right) + C \int_0^t \left( |u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^3 \right) ds + CCs \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds.
\]
The case when \( |\alpha| \leq 6 \) is straightforward. Then we have completed the proof of Theorem 2.2 for the three-dimensional case.

Appendix A. Derivation of the boundary layer system

In this section, we only give the derivation of two-dimensional (2D) boundary layer system in (1.1) since the 3D can be derived in the same way. We consider the following 2D hyperbolic
version of Navier-Stokes system in $\Omega := \mathbb{R} \times \mathbb{R}_+ = \{(x, y) \in \mathbb{R}^2_+; \ x \in \mathbb{R}, y > 0\}$

\[
\begin{align*}
\eta \partial^2_t u^\varepsilon + \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon^2 \Delta u^\varepsilon + \nabla p^\varepsilon &= 0, \\
\nabla \cdot u^\varepsilon &= 0, \\
\left. u^\varepsilon \right|_{t=0} &= u_0, \quad \left. \partial_t u^\varepsilon \right|_{t=0} = u_1,
\end{align*}
\]

(A.1)

where $\eta = 1$ and $u^\varepsilon = (u^\varepsilon, v^\varepsilon)$ denotes velocity fields, $p^\varepsilon = p^\varepsilon(t, x, y)$ denotes the scalar pressure. The above system is complemented with no-slip boundary conditions on the velocity fields, that is,

\[
(u^\varepsilon, v^\varepsilon)|_{y=0} = (0, 0).
\]

(A.2)

**Away from the boundary:** We construct the approximate solution by the following expansion

\[
\begin{align*}
u^\varepsilon(t, x, y) &\sim 2 \sum_{j=0}^2 \varepsilon^j u^{I,0}(t, x, y) + o(\varepsilon); \\
v^\varepsilon(t, x, y) &\sim 2 \sum_{j=0}^2 \varepsilon^j v^{I,0}(t, x, y) + o(\varepsilon); \\
p^\varepsilon(t, x, y) &\sim 2 \sum_{j=0}^2 \varepsilon^j p^{I,0}(t, x, y) + o(\varepsilon).
\end{align*}
\]

Plugging the above expansion into the hyperbolic Navier-Stokes system (A.1), and letting $\varepsilon \to 0$, then matching the $O(\varepsilon^0)$ term, we find that $(u^{I,0}(t, x, y), v^{I,0}(t, x, y), p^{I,0}(t, x, y))$ should satisfy the hyperbolic Euler equations

\[
\begin{align*}
\partial^2_t u^{I,0} + \partial_t u^{I,0} + (u^{I,0} \cdot \nabla) u^{I,0} + \nabla p^{I,0} &= 0, \\
\nabla \cdot u^{I,0} &= 0,
\end{align*}
\]

(A.3)

Naturally, we endow (A.3) with the same initial data as viscous flow in (A.1). That is

\[
\left. u^{I,0} \right|_{t=0} = (u_0, v_0)(x, y), \quad \left. \partial_t u^{I,0} \right|_{t=0} = (u_1, v_1)(x, y),
\]

For well-posedness of the system (A.3), we may impose a homogeneous Dirichlet boundary condition for the normal components of velocity field:

\[
\left. v^{I,0} \right|_{y=0} = 0.
\]

(A.4)

**Near the boundary $y = 0$:** Comparing the boundary conditions (A.2) with (A.4), there is a mismatch between the tangential component $u^\varepsilon(t, x, y)$ and $u^{I,0}(t, x, y)$ on the boundary $\{y = 0\}$. To derive the equations for boundary layers, we carry out the multi-scale analysis as Prandtl in [33]. We suppose that the solution $(u^\varepsilon, v^\varepsilon)$ of the hyperbolic Navier-Stokes system
\[ (A.1) \] accepts the following multi-scale formal expansions

\[
\begin{align*}
u^\varepsilon(t,x,y) & \sim \sum_{j=0}^{2} \varepsilon^j \left[ u^{I,j}(t,x,y) + u^{B,j}(t,x,\frac{y}{\varepsilon}) \right] + o(\varepsilon); \\
v^\varepsilon(t,x,y) & \sim \sum_{j=0}^{2} \varepsilon^j \left[ v^{I,j}(t,x,y) + v^{B,j}(t,x,\frac{y}{\varepsilon}) \right] + o(\varepsilon); \\
p^\varepsilon(t,x,y) & \sim \sum_{j=0}^{2} \varepsilon^j \left[ p^{I,j}(t,x,y) + p^{B,j}(t,x,\frac{y}{\varepsilon}) \right] + o(\varepsilon).
\end{align*}
\]  

(A.5)

where \((u^{I,j}, v^{I,j})\) are inner functions, \((u^{B,j}, v^{B,j})\) and \(p^{B,j}\) are the boundary layer functions near the boundary \(y = 0\). Moreover, the boundary layer functions \(u^{B,j}, v^{B,j}, p^{B,j}\) exponentially go to zero as \(\tilde{y} = \frac{y}{\varepsilon} \rightarrow +\infty \ (\varepsilon \rightarrow 0)\).

First, we substitute the ansatz given in \((A.5)\) into the first equation of \((A.1)\) to obtain

\[
\begin{align*}
\sum_{j \geq 0} \varepsilon^j \partial_t^2 (u^{I,j} + u^{B,j}) & + \sum_{j \geq 0} \varepsilon^j \partial_x (u^{I,j} + u^{B,j}) + \sum_{j \geq 0} \varepsilon^j \sum_{k=0}^{j} (u^{I,k} + u^{B,k}) (\partial_x u^{I,j-k} + \partial_x u^{B,j-k}) \\
& + \sum_{j \geq 0} \varepsilon^j \sum_{k=0}^{j} [(u^{I,k} + u^{B,k}) \partial_y u^{I,j-k}] + \sum_{j \geq 0} \varepsilon^{j+1} \sum_{k=0}^{j} (u^{I,k} + u^{B,k}) \partial_y u^{B,j-k+1} + \frac{1}{\varepsilon} u^{I,0} \partial_y u^{B,0} \\
& + \sum_{j \geq 0} \varepsilon^j \partial_x p^{B,j} + \sum_{j \geq 0} \varepsilon^j \partial_y p^{B,j} = \sum_{j \geq 0} \varepsilon^j \left( \varepsilon^2 \partial_x^2 u^{I,j} + \varepsilon^2 \partial_y^2 u^{I,j} + \varepsilon^2 \partial_x^2 u^{B,j} + \partial_y^2 u^{B,j} \right), \\
\sum_{j \geq 0} \varepsilon^j \partial_y^2 (u^{I,j} + u^{B,j}) & + \sum_{j \geq 0} \varepsilon^j \partial_y (u^{I,j} + u^{B,j}) + \sum_{j \geq 0} \varepsilon^j \sum_{k=0}^{j} (u^{I,k} + u^{B,k}) (\partial_x v^{I,j-k} + \partial_x v^{B,j-k}) \\
& + \sum_{j \geq 0} \varepsilon^j \sum_{k=0}^{j} (u^{I,k} + u^{B,k}) \partial_y v^{I,j-k} + \sum_{j \geq 0} \varepsilon^{j+1} \sum_{k=0}^{j} (u^{I,k} + u^{B,k}) \partial_y v^{B,j-k+1} + \frac{1}{\varepsilon} u^{I,0} \partial_y v^{B,0} \\
& + \sum_{j \geq 0} \varepsilon^j \partial_y p^{I,j} + \frac{1}{\varepsilon} \partial_y p^{B,0} + \sum_{j \geq 0} \varepsilon^j \partial_y p^{B,j+1} \\
= \sum_{j \geq 0} \varepsilon^j \left( \varepsilon^2 \partial_y^2 v^{I,j} + \varepsilon^2 \partial_y^2 v^{I,j} + \varepsilon^2 \partial_x^2 v^{B,j} + \partial_y^2 v^{B,j} \right),
\end{align*}
\]

(A.6)

Second, we substitute the ansatz given in \((A.5)\) into the div-free condition \(\nabla \cdot u^\varepsilon = 0\) of \((A.1)\) to get

\[
\begin{align*}
\partial_x u^{I,j} + \partial_y v^{I,j} = 0, & \quad j \geq 0 \\
\partial_x u^{B,j} + \partial_y v^{B,j+1} = 0, & \quad j \geq 0 \\
\partial_y v^{B,0} = 0,
\end{align*}
\]

(A.7)
Third, we substitute the ansatz given in (A.5) into the boundary condition of (A.1) to get
\[
\begin{cases}
    u^{I,j}(t, x, 0) + u^{B,j}(t, x, 0) = 0, j \geq 0 \\
v^{I,j}(t, x, 0) + v^{B,j}(t, x, 0) = 0, j \geq 0
\end{cases}
\]
Since \(v^{I,0}(t, x, 0) = 0\), we have \(v^{B,0}(t, x, 0) = 0\).

Last, by using the Taylor expansion in \(y\), we write \(u^{I,0}(t, x, y)\) as
\[
u^{I,0}(t, x, y) = u^{I,0}(t, x, 0) + y \partial_y u^{I,0}(t, x, 0) + \frac{y^2}{2} \partial_y^2 u^{I,0}(t, x, 0) + \cdots = u^{I,0} + \varepsilon \partial_y u^{I,0} + o(\varepsilon),
\]
where here and below we use the notation \(f\) to stand for the trace of a function \(f\) on the boundary \(y = 0\). Similarly, we write
\[
v^{I,0}(t, x, y) = \varepsilon \partial_y u^{I,0} + o(\varepsilon),
\]
\[
p^{I,0}(t, x, y) = p^{I,0} + \varepsilon \partial_y p^{I,0} + o(\varepsilon).
\]
Then we insert the above three terms into (A.6) and moreover consider the same order terms of \(\varepsilon\). By combining (A.7) we obtain as follows:

**At the order \(\varepsilon^{-1}\):** From the second equation in (A.6) we get
\[
\partial_y p^{B,0} = 0.
\]
this together with the assumption that \(p^{B,j}, j \geq 0\) goes to 0 as \(\tilde{y} \to +\infty\) implies
\[
p^{B,0} \equiv 0.
\]

**At the order \(\varepsilon^0\):** From the second equation in (A.6), we obtain
\[
\partial_y p^{B,1} = 0.
\]
this together with the assumption that \(p^{B,j}, j \geq 0\) goes to 0 as \(\tilde{y} \to +\infty\) implies
\[
p^{B,1} \equiv 0.
\]

On the other hand, we consider the first equation in (A.6) only with \(O(\varepsilon^0)\) terms to yield that
\[
\partial_t^2 u^{B,0} + \partial_t u^{B,0} + u^{B,0} \partial_x u^{I,0} + (u^{I,0} + u^{B,0}) \partial_x \cdot u^{B,0} + (v^{I,1} + u^{B,1} + \tilde{y} \partial_y v^{I,0}) \partial_y u^{B,0} = \partial_y^2 u^{B,0}
\]
Thus, as a result, we deduce that \((u^{B,0}, v^{B,1})\) satisfies the equations as follows:
\[
\begin{cases}
    \partial_t^2 u^{B,0} + \partial_t u^{B,0} + u^{B,0} \partial_x u^{I,0} + (u^{I,0} + u^{B,0}) \partial_x \cdot u^{B,0} \\
    + (v^{I,1} + u^{B,1} + \tilde{y} \partial_y v^{I,0}) \partial_y u^{B,0} = \partial_y^2 u^{B,0}, \\
    \partial_x \cdot u^{B,0} + \tilde{y} \partial_y v^{B,1} = 0,
\end{cases}
\]
\[
\begin{align*}
    u^{B,0}(t, x, 0) &= -u^{I,0}(t, x, 0), & v^{B,1}(t, x, 0) &= -v^{I,1}(t, x, 0), & \lim_{\tilde{y} \to +\infty} u^{B,0}(t, x, \tilde{y}) &= 0, \\
    (u^{B,0}, v^{B,1})|_{t=0} &= (0, 0), & (\partial_t u^{B,0}, \partial_t v^{B,1})|_{t=0} &= (0, 0).
\end{align*}
\]
Denoting
\[
\begin{align*}
u(t, x, \tilde{y}) &= u^{I,0} + u^{B,0}(t, x, \tilde{y}); \\
v(t, x, \tilde{y}) &= v^{I,1} + u^{B,1}(t, x, \tilde{y}) + \tilde{y} \partial_y v^{I,0}.
\end{align*}
\]
which along with the system (A.3) at $y = 0$ to yield the hyperbolic Prandtl boundary layer equations

$$
\begin{align*}
\partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u + \partial_x P &= \partial_y^2 u, \\
\partial_x u + \partial_y v &= 0, \\
u(t, x, 0) &= v(t, x, 0) = 0, \quad \lim_{\tilde{y} \to +\infty} u(t, x, \tilde{y}) = U(t, x), \\
u|_{t=0} &= u_0(0, x), \quad \partial_t u|_{t=0} = u_1(0, x),
\end{align*}
$$

(A.8)

where

$$
P(t, x) = \overline{p(t, x)}, \quad U(t, x) = \overline{u(t, x)}
$$

and the known functions $U(t, x)$ and $P(t, x)$ satisfy the following law:

$$
\partial_t^2 U + \partial_t U + (U \cdot \partial_x)U + \partial_x P = 0.
$$

For simplicity of notations, we have replaced $\tilde{y}$ by $y$ in the boundary layer system (1.1).

**Remark A.1.** Note that the boundary layer problem (A.8) can be derived by using the following scale transform:

$$
t = t, \quad x = x, \quad \tilde{y} = \frac{y}{\varepsilon},
$$

and

$$
u^\varepsilon(t, x, y) = u(t, x, \tilde{y}), \quad v^\varepsilon(t, x, y) = \varepsilon v^b(t, x, \tilde{y}),
$$

$$p^\varepsilon(t, x, y) = p(t, x, \tilde{y}).$$

**Acknowledgements.** The work was supported by NSF of China(Nos. 11961160716, 11871054, 12131017) and the Natural Science Foundation of Hubei Province No. 2019CFA007.

**References**

[1] N. Aarach. Global well-posedness of 2D Hyperbolic perturbation of the Navier-Stokes system in a thin strip. *arXiv e-prints*, page arXiv:2111.13052, Nov. 2021.

[2] B. Abdelhedi. Global existence of solutions for hyperbolic Navier-Stokes equations in three space dimensions. *Asymptot. Anal.*, 112(3-4):213–225, 2019.

[3] R. Alexandre, Y.-G. Wang, C.-J. Xu, and T. Yang. Well-posedness of the Prandtl equation in Sobolev spaces. *J. Amer. Math. Soc.*, 28(3):745–784, 2015.

[4] K. Asano. A note on the abstract Cauchy-Kowalewski theorem. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(4):102–105, 1988.

[5] Y. Brenier, R. Natalini, and M. Puel. On a relaxation approximation of the incompressible Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 132(4):1021–1028, 2004.

[6] C. Cattaneo. Sulla conduzione del calore. *Atti Sem. Mat. Fis. Univ. Modena*, 3:83–101, 1949.

[7] D. Chen, Y. Wang, and Z. Zhang. Well-posedness of the linearized Prandtl equation around a non-monotonic shear flow. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(4):1119–1142, 2018.

[8] O. Coulaud, I. Hachicha, and G. Raugel. Hyperbolic Quasilinear Navier–Stokes Equations in $\mathbb{R}^2$. *J. Dynam. Differential Equations*, 2021(Forthcoming).
[9] H. Dietert and D. Gérard-Varet. Well-posedness of the Prandtl equations without any structural assumption. *Ann. PDE*, 5(1):Paper No. 8, 51, 2019.

[10] W. E and B. Engquist. Blowup of solutions of the unsteady Prandtl’s equation. *Comm. Pure Appl. Math.*, 50(12):1287–1293, 1997.

[11] D. Gérard-Varet and E. Dormy. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.*, 23(2):591–609, 2010.

[12] D. Gérard-Varet and N. Masmoudi. Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(6):1273–1325, 2015.

[13] D. Gérard-Varet, N. Masmoudi, and V. Vicol. Well-posedness of the hydrostatic Navier-Stokes equations. *Anal. PDE*, 13(5):1417–1455, 2020.

[14] Y. Guo and T. Nguyen. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.*, 64(10):1416–1438, 2011.

[15] M. Ignatova and V. Vicol. Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.*, 220(2):809–848, 2016.

[16] I. Kukavica, N. Masmoudi, V. Vicol, and T. K. Wong. On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.*, 46(6):3865–3890, 2014.

[17] S. Li and F. Xie. Global solvability of 2D MHD boundary layer equations in analytic function spaces. *J. Differential Equations*, 299:362–401, 2021.

[18] W.-X. Li, N. Masmoudi, and T. Yang. Well-posedness in Gevrey function space for 3D Prandtl equations without Structural Assumption. *Comm. Pure Appl. Math.* doi:10.1002/cpa.21989.

[19] W.-X. Li, V.-S. Ngo, and C.-J. Xu. Boundary layer analysis for the fast horizontal rotating fluids. *Commun. Math. Sci.*, 17(2):299–338, 2019.

[20] W.-X. Li, D. Wu, and C.-J. Xu. Gevrey class smoothing effect for the Prandtl equation. *SIAM J. Math. Anal.*, 48(3):1672–1726, 2016.

[21] W.-X. Li and R. Xu. Well-posedness in Sobolev spaces of the two-dimensional MHD boundary layer equations without viscosity. *Electron. Res. Arch.*, 29(6):4243–4255, 2021.

[22] W.-X. Li and T. Yang. Well-posedness in Gevrey function spaces for the Prandtl equations with non-degenerate critical points. *J. Eur. Math. Soc. (JEMS)*, 22(3):717–775, 2020.

[23] W.-X. Li and T. Yang. Well-posedness of the MHD boundary layer system in Gevrey function space without structural assumption. *SIAM J. Math. Anal.*, 53(3):3236–3264, 2021.

[24] C.-J. Liu, D. Wang, F. Xie, and T. Yang. Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces. *J. Funct. Anal.*, 279(7):108637, 45, 2020.

[25] C.-J. Liu, Y.-G. Wang, and T. Yang. On the ill-posedness of the Prandtl equations in three-dimensional space. *Arch. Ration. Mech. Anal.*, 220(1):83–108, 2016.

[26] C.-J. Liu, Y.-G. Wang, and T. Yang. A well-posedness theory for the Prandtl equations in three space variables. *Adv. Math.*, 308:1074–1126, 2017.

[27] C.-J. Liu, F. Xie, and T. Yang. A note on the ill-posedness of shear flow for the MHD boundary layer equations. *Sci. China Math.*, 61(11):2065–2078, 2018.
[28] C.-J. Liu, F. Xie, and T. Yang. MHD boundary layers theory in Sobolev spaces without monotonicity I: Well-posedness theory. *Comm. Pure Appl. Math.*, 72(1):63–121, 2019.

[29] C.-J. Liu and T. Yang. Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. *J. Math. Pures Appl.* (9), 108(2):150–162, 2017.

[30] N. Liu and P. Zhang. Global small analytic solutions of MHD boundary layer equations. *J. Differential Equations*, 281:199–257, 2021.

[31] N. Masmoudi and T. K. Wong. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.*, 68(10):1683–1741, 2015.

[32] L. Nirenberg. An abstract form of the nonlinear Cauchy-Kowalewski theorem. *J. Differential Geometry*, 6:561–576, 1972.

[33] O. A. Oleinik and V. N. Samokhin. *Mathematical models in boundary layer theory*, volume 15 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[34] M. Paicu and G. Raugel. Une perturbation hyperbolique des équations de Navier-Stokes. In *ESAIM Proceedings*, Vol. 21 (2007) [Journées d’Analyse Fonctionnelle et Numérique en l’honneur de Michel Crouzeix], volume 21 of *ESAIM Proc.*, pages 65–87. EDP Sci., Les Ulis, 2007.

[35] M. Paicu and P. Zhang. Global existence and the decay of solutions to the Prandtl system with small analytic data. *Arch. Ration. Mech. Anal.*, 241(1):403–446, 2021.

[36] M. Paicu and P. Zhang. Global hydrostatic approximation of hyperbolic Navier-Stokes system with small Gevrey class data. *arXiv e-prints*, page *arXiv:2111.12836*, Nov. 2021.

[37] R. Racke and J. Saal. Hyperbolic Navier-Stokes equations I: Local well-posedness. *Evol. Equ. Control Theory*, 1(1):195–215, 2012.

[38] R. Racke and J. Saal. Hyperbolic Navier-Stokes equations II: Global existence of small solutions. *Evol. Equ. Control Theory*, 1(1):217–234, 2012.

[39] M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(2):433–461, 1998.

[40] C. Wang, Y. Wang, and P. Zhang. On the global small solution of 2-D Prandtl system with initial data in the optimal Gevrey class. *arXiv e-prints*, page *arXiv:2103.00681*, Feb. 2021.

[41] Z. Xin and L. Zhang. On the global existence of solutions to the Prandtl’s system. *Adv. Math.*, 181(1):88–133, 2004.

[42] C.-J. Xu and X. Zhang. Long time well-posedness of Prandtl equations in Sobolev space. *J. Differential Equations*, 263(12):8749–8803, 2017.

[43] P. Zhang and Z. Zhang. Long time well-posedness of Prandtl system with small and analytic initial data. *J. Funct. Anal.*, 270(7):2591–2615, 2016.

(W.-X. Li) School of Mathematics and Statistics, Wuhan University, 430072 Wuhan, China, & Hubei Key Laboratory of Computational Science, Wuhan University, 430072 Wuhan, China

E-mail address: wei-xi.li@whu.edu.cn

(R. Xu) School of Mathematics and Statistics, Wuhan University, 430072 Wuhan, China

E-mail address: xurui218@whu.edu.cn