THE WEIGHTED PERIODIC-PARABOLIC DEGENERATE LOGISTIC EQUATION

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Abstract. The main goal of this paper is twofold. First, it characterizes the existence of positive periodic solutions for a general class of weighted periodic-parabolic logistic problems of degenerate type (see (1.1)). This result provides us with a substantial generalization of Theorem 1.1 of Daners and López-Gómez [12] even for the elliptic counterpart of (1.1), and of some previous findings of the authors in [1] and [2]. Then, it sharpens some results of [19] by enlarging the class of critical degenerate weight functions for which (1.1) admits a positive periodic solution in an unbounded interval of values of the parameter \( \lambda \). The latest findings, not having any previous elliptic counterpart, are utterly new and of great interest in Population Dynamics.

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1. Introduction

The main goal of this paper is to characterize the existence of classical positive periodic solutions, \( (\lambda, u) \), of the periodic-parabolic problem

\[
\begin{align*}
\partial_t u + \mathcal{L}u &= \lambda m(x,t)u - a(x,t)f(u)u \\
B u &= 0
\end{align*}
\]

in \( Q_T := \Omega \times (0, T) \), on \( \partial \Omega \times [0, T] \),

where:

- \((H_\Omega)\) \( \Omega \) is a bounded subdomain (open and connected subset) of \( \mathbb{R}^N \), \( N \geq 1 \), of class \( C^{2+\theta}_0 \) for some \( 0 < \theta \leq 1 \).

- \((H_\mathcal{L})\) \( \mathcal{L} \) is a non-autonomous differential operator of the form

\[
\mathcal{L} \equiv \mathcal{L}(x, t) := -\sum_{i,j=1}^{N} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_j(x,t) \frac{\partial}{\partial x_j} + c(x,t)
\]

with \( a_{ij} = a_{ji}, b_j, c \in F \) for all \( 1 \leq i, j \leq N \) and some \( T > 0 \), where

\[
F := \left\{ u \in C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) : u(\cdot, T + t) = u(\cdot, t) \text{ for all } t \in \mathbb{R} \right\}.
\]
Moreover, \( \mathcal{L} \) is assumed to be uniformly elliptic in \( \bar{Q}_T = \bar{\Omega} \times [0,T] \), i.e., there exists a constant \( \mu > 0 \) such that

\[
\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \mu |\xi|^2 \quad \text{for all } (x,t,\xi) \in \bar{Q}_T \times \mathbb{R}^N,
\]

where \(| \cdot |\) stands for the Euclidean norm of \( \mathbb{R}^N \).

\((H_B)\) \( B \) is a non-classical mixed boundary operator of the form

\[
B\xi := \begin{cases} 
\xi 
& \text{on } \Gamma_0, \\
\partial_n \xi + \beta(x)\xi 
& \text{on } \Gamma_1,
\end{cases}
\xi \in C(\Gamma_0) \oplus C^1(\Omega \cup \Gamma_1),
\]

where \( \Gamma_0 \) and \( \Gamma_1 \) are two disjoint open and closed subsets of \( \partial \Omega \) such that \( \partial \Omega := \Gamma_0 \cup \Gamma_1 \). In (1.4), \( \beta \in C^{1+\theta}(\Gamma_1) \) and \( \nu = (\nu_1, \ldots, \nu_N) \in C^{1+\theta}(\partial \Omega; \mathbb{R}^N) \) is an outward pointing nowhere tangent vector field.

\((H_a)\) \( m, a \in F \) and \( a \) satisfies \( a \geq 0 \), i.e., \( a \geq 0 \) and \( a \neq 0 \).

\((H_f)\) \( f \in C^1(\mathbb{R}; \mathbb{R}) \) satisfies \( f(0) = 0 \), \( f'(u) > 0 \) for all \( u > 0 \), and \( \lim_{u \to +\infty} f(u) = +\infty \).

Thus, we are working under the general setting of [6, 7], where \( \beta \) can change sign, which remains outside the classical framework of Hess [16] and Daners and López-Gómez [11], where \( \beta \geq 0 \). In this paper, \( B \) is the Dirichlet boundary operator on \( \Gamma_0 \), and the Neumann, or a first order regular oblique derivative boundary operator, on \( \Gamma_1 \), and either \( \Gamma_0 \), or \( \Gamma_1 \), can be empty. When \( \Gamma_1 = \emptyset \), \( B \) becomes the Dirichlet boundary operator, denoted by \( \mathcal{D} \). Thus, our results are sufficiently general as to cover the existence results of Du and Peng [14, 15], where \( m(x,t) \) was assumed to be constant, \( \Gamma_0 = \emptyset \) and \( \beta = 0 \), and of Peng and Zhao [21], where \( \beta \geq 0 \). The importance of combining temporal periodic heterogeneities in spatially heterogeneous models can be easily realized by simply having a look at the Preface of [18].

By a classical solution of (1.1) we mean a solution pair \((\lambda, u)\), or simply \( u \), with \( u \in E \), where

\[
E := \left\{ u \in C^{2+\theta,1+\theta/2}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) : u(\cdot, T + t) = u(\cdot, t) \text{ for all } t \in \mathbb{R} \right\}.
\]

The first goal of this paper is to obtain a periodic-parabolic counterpart of [12, Th. 1.1] to characterize the existence of classical positive periodic-solutions of (1.1), regardless the structure of \( a^{-1} \). Our main result generalizes, very substantially, our previous findings in [12], where we dealt with the special case \( m \equiv 1 \), and [2], where we imposed some (severe) restrictions on the structure of \( a^{-1}(0) \). All previous requirements are removed in this paper. To state our result, we need to introduce some of notation.

Throughout this paper, \( \mathcal{P} \) stands for the periodic-parabolic operator

\[
\mathcal{P} := \partial_t + \mathcal{L},
\]

and, for every tern \((\mathcal{P}, \mathcal{B}, \mathcal{Q}_T)\) satisfying the previous general assumptions, we denote by \( \sigma[\mathcal{P}, \mathcal{B}, \mathcal{Q}_T] \) the principal eigenvalue of \((\mathcal{P}, \mathcal{B}, \mathcal{Q}_T)\), i.e., the unique value of \( \tau \) for
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which the linear periodic-parabolic eigenvalue problem
\[
\begin{cases}
\mathcal{P} \varphi = \tau \varphi & \text{in } Q_T, \\
\mathcal{B} \varphi = 0 & \text{on } \partial \Omega \times [0, T],
\end{cases}
\]
admits a positive eigenfunction \( \varphi \in E \). In such case, \( \varphi \) is unique, up to a positive multiplicative constant, and \( \varphi \gg 0 \) in the sense that, for every \( t \in [0, T] \),
\[
\varphi(x,t) > 0 \text{ for all } x \in \Omega \cup \Gamma_1 \text{ and } \partial_{\nu} \varphi(x,t) < 0 \text{ if } x \in \Gamma_0.
\]
The existence of the principal eigenvalue in the special case when \( \beta \geq 0 \) is a classical result attributable to Beltramo and Hess [9]. In the general case when \( \beta \) changes of sign this result has been established, very recently, in [6, 7]. Throughout this paper, we will also consider the principal eigenvalue
\[
\Sigma(\lambda, \gamma) := \sigma[\mathcal{P} - \lambda m + \gamma a, \mathcal{B}, Q_T], \quad \lambda, \gamma \in \mathbb{R}. \tag{1.5}
\]
Since \( a \geq 0 \), by Proposition 2.1(a), \( \Sigma(\lambda, \gamma) \) is an increasing function of \( \gamma \) and hence, for every \( \lambda \in \mathbb{R} \), the limit
\[
\Sigma(\lambda, \infty) := \lim_{\gamma \to \infty} \Sigma(\lambda, \gamma) \leq \infty \tag{1.6}
\]
is well defined. The first aim of this paper is to establish the following characterization.

**Theorem 1.1.** Suppose \( (H_{\Omega}), (H_{\Sigma}), (H_{B}), (H_a) \) and \( (H_f) \). Then, (1.1) admits a positive solution if, and only if,
\[
\Sigma(\lambda, 0) < 0 < \Sigma(\lambda, \infty) \tag{1.7}
\]
Moreover, it is unique, if it exists.

As Theorem 1.1 holds regardless the structure of the compact set \( a^{-1}(0) \), it provides us with a substantial extension of [12, Th. 1.1], which was established for the elliptic counterpart of (1.1), and some of the findings of the authors in [1] and [2], where some additional (severe) restrictions on the structure of \( a^{-1}(0) \) were imposed.

From the point of view of the applications, determining whether or not \( \Sigma(\lambda, \infty) = \infty \) occurs, is imperative to ascertain the hidden structure of the set of \( \lambda \in \mathbb{R} \) for which (1.7) holds. Thanks to Theorem 1.1, this is the set of \( \lambda \)'s for which (1.1) possesses a positive periodic solution. To illustrate what we mean, suppose that \( a(x,t) \) is positive and separated away from zero in \( Q_T \), i.e.,
\[
a(x,t) \geq \omega > 0 \text{ for all } (x,t) \in Q_T.
\]
Then, by Proposition 2.1(a), it is apparent that
\[
\Sigma(\lambda, \gamma) := \sigma[\mathcal{P} - \lambda m + \gamma a, \mathcal{B}, Q_T] \geq \sigma[\mathcal{P} - \lambda m, \mathcal{B}, Q_T] + \gamma \omega
\]
and hence, for every \( \lambda \in \mathbb{R} \),
\[
\Sigma(\lambda, \infty) = \lim_{\gamma \to \infty} \Sigma(\lambda, \gamma) = \infty.
\]
Therefore, in this case, (1.1) has a positive solution if, and only if, \( \Sigma(\lambda, 0) < 0 \). Naturally, to determine the structure of the set of \( \lambda \)'s where
\[
\Sigma(\lambda, 0) = \sigma[\mathcal{P} - \lambda m, \mathcal{B}, Q_T] < 0
\]
the nodal properties of $m(x, t)$ play a pivotal role. Indeed, whenever $m \neq 0$ has constant sign, $\Sigma(\lambda, 0) < 0$ on some $\lambda$-interval which might become $\mathbb{R}$. However, when
\[
\int_0^T \min_{x \in \Omega} m(x, t) \, dt < 0 < \int_0^T \max_{x \in \Omega} m(x, t) \, dt
\]
and $\Sigma(\hat{\lambda}, 0) > 0$ for some $\hat{\lambda} \in \mathbb{R}$, then
\[
\{ \lambda \in \mathbb{R} : \Sigma(\lambda, 0) < 0 \} = (-\infty, \lambda_-) \cup (\lambda_+, \infty),
\]
where $\lambda_\pm$ stand for the principal eigenvalues of the linear weighted eigenvalue problem
\[
\begin{cases}
  P \varphi = \lambda m(x, t) \varphi & \text{in } Q_T, \\
  B \varphi = 0 & \text{on } \partial \Omega \times [0, T].
\end{cases}
\]
A complete analysis of this particular issue was carried out in [6] and [7]. So, we stop this discussion here.

Contrarily to what happens in the autonomous case, $\Sigma(\lambda, \infty)$ can be infinity if $a^{-1}(0)$ has non-empty interior in $Q_T$. The next result characterizes whether $\Sigma(\lambda, \infty) < \infty$ holds for a special family of weight functions $a(x, t)$ with a tubular vanishing set, $a^{-1}(0)$, as described on [19, Th. 6.4]. This result generalized, very substantially, some pioneering findings of Daners and Thornet [13].

**Theorem 1.2.** $\Sigma(\lambda, \infty) < \infty$ if, and only if, there is a continuous map $\tau : [0, T] \to \Omega$ such that $\tau(0) = \tau(T)$ and
\[
(\tau(t), t) \in \text{int } a^{-1}(0) \text{ for all } t \in [0, T],
\]
i.e., if we can advance, upwards in time, from time $t = 0$ up to time $t = T$ within the interior of the vanishing set of $a(x, t)$.

Actually, the fact that $\Sigma(\lambda, \infty) < \infty$ if $\tau$ exists is a general feature not depending on the particular structure of $a^{-1}(0)$. In the proof of [19, Th. 4.1] it was derived from a technical device introduced in the proof of Lemma 15.4 of Hess [16]. The converse result that $\Sigma(\lambda, \infty) = \infty$ when the map $\tau$ does not exist, has been already proven when $a^{-1}(0)$ has the structure sketched in Figure 1, thought it remained an open problem in [19] for more general situations. In Figure 1, as in all remaining figures of this paper, the $x$-variables have been plotted in abscissas, and time $t$ in ordinates. The white region represents $a^{-1}(0)$, while the dark one is the set of $(x, t) \in Q_T$ such that $a(x, t) > 0$.

In Figure 1, we are denoting $P = (x_P, t_P)$ and $Q = (x_Q, t_Q)$. The first picture shows a genuine situation where $t_P < t_Q$, while $t_P = t_Q$ in the second one, which is a limiting case with respect to the complementary case when $t_P > t_Q$, where, due to Theorem 1.2, $\Sigma(\lambda, \infty) < \infty$. All pictures throughout this paper idealize the general case when $N \geq 1$, though represent some special cases with $N = 1$.

Subsequently, as sketched by Figure 1, we are assuming
\[
a(x, t) > 0 \text{ for all } (x, t) \in \partial \Omega \times [0, T].
\]
Our second goal in this paper is to extend Theorem 1.2 to cover a more general class of critical weight functions $a(x, t)$ satisfying (1.9) for which the map $\tau$ does not exist. These cases are critical in the sense that some small perturbation of $a(x, t)$ might,
or might not, entail the existence of $\tau$. To precise what we mean, let $a_0 \in F$ be an admissible weight function such that $\text{int } a_0^{-1}(0)$ has the structure of the first plot of Figure 2 with $t_Q < t_P$. In such case, by Theorem 1.2, $\Sigma(\lambda, \infty) < \infty$ for all $\lambda \in \mathbb{R}$.

Now, let $Q$ be an open and connected subset of $\text{int } a_0^{-1}(0)$, with sufficiently smooth boundary, such that

$$M := \bar{Q} \subset \text{int } a_0^{-1}(0),$$
The right picture of Figure 2 shows an admissible \( M \) placed between the lines \( t = t_P \) and \( t = t_Q \). Lastly, we consider any function \( a_1 \in F \) such that \( a_1(x,t) > 0 \) if and only if \( (x,t) \in \mathcal{O} \).

Then, the next result holds.

**Theorem 1.3.** Suppose that \( a = a_0 + a_1 \) and there is not a continuous map \( \tau : [0,T] \to \Omega \) such that \( \tau(0) = \tau(T) \) and \( (\tau(t),t) \in \text{int } a^{-1}(0) \) for all \( t \in [0,T] \). Then, \( \Sigma(\lambda, \infty) = \infty \) for all \( \lambda \in \mathbb{R} \).

Note that the relative position of the component \( M \) with respect to the remaining components where \( a > 0 \) is crucial to determine whether or not \( \Sigma(\lambda, \infty) \) is finite. Figure 3 shows a magnification of the most significative piece of the right picture of Figure 2 in a case where the curve \( \tau \) does not exist (on the left), together with another situation where the curve \( \tau \) does exist (on the right). In the second case, according to Theorem 1.2, we have that \( \Sigma(\lambda, \infty) < \infty \) for all \( \lambda \in \mathbb{R} \).

![Figure 3. Influence of the relative position of M.](image)

Theorem 1.3 can be generalized up to cover a wider class of critical weight functions \( a(x,t) \). Indeed, suppose that the interior of \( a_0^{-1}(0) \) looks like shows the first picture of Figure 2, with \( t_Q < t_P \), and, for any given integer \( n \geq 2 \), let \( t_i \in (t_Q,t_P), \ i \in \{1,...,t_{n-1}\} \), such that

\[
t_Q < t_1 < t_2 < \cdots < t_{n-1} < t_P.
\]

Now, consider \( n \) open subsets, \( \mathcal{O}_j, 1 \leq j \leq n, \) with sufficiently smooth boundaries and mutually disjoint closures, such that

\[
M_j := \bar{\mathcal{O}}_j \subset \text{int } a_0^{-1}(0), \quad 1 \leq j \leq n,
\]
and
\[ t_Q = \min \{ t \in [0, T] : (M_1)_t \neq \emptyset \} \]
\[ t_1 = \max \{ t \in [0, T] : (M_1)_t \neq \emptyset \} = \min \{ t \in [0, T] : (M_2)_t \neq \emptyset \} \]
\[ t_2 = \max \{ t \in [0, T] : (M_2)_t \neq \emptyset \} = \min \{ t \in [0, T] : (M_3)_t \neq \emptyset \} \]
\[ t_3 = \max \{ t \in [0, T] : (M_3)_t \neq \emptyset \} = \min \{ t \in [0, T] : (M_4)_t \neq \emptyset \} \]
\[ \ldots < t_{n-1} = \max \{ t \in [0, T] : (M_{n-1})_t \neq \emptyset \} = \min \{ t \in [0, T] : (M_n)_t \neq \emptyset \} = t_P. \]
Lastly, let \( a_i \in F \) be, \( i \in \{1, \ldots, n\} \), such that \( a_i(x, t) > 0 \) if and only if \((x, t) \in \mathcal{O}_i\). Then, the next generalized version of Theorem 1.3 holds.

**Theorem 1.4.** Suppose that
\[ a = a_0 + a_1 + \cdots + a_n \]
and that there is not a continuous map \( \tau : [0, T] \to \Omega \) such that \( \tau(0) = \tau(T) \) and \((\tau(t), t) \in \text{int } a^{-1}(0) \) for all \( t \in [0, T] \). Then, \( \Sigma(\lambda, \infty) = \infty \) for all \( \lambda \in \mathbb{R} \).

The left picture of Figure 4 illustrates a typical situation with \( n = 3 \) where all the assumptions of Theorem 1.4 are fulfilled. In the case illustrated by the right picture one can construct an admissible curve \( \tau \) and hence, due to Theorem 1.2, \( \Sigma(\lambda, \infty) < \infty \) for all \( \lambda \in \mathbb{R} \). This example shows that the non-existence of \( \tau \) is necessary for the validity of Theorem 1.4.

![Figure 4](image)

**Figure 4.** Two admissible configurations with \( n = 4 \).

This paper is organized as follows: Section 2 contains some preliminaries, Section 3 analyzes some general properties of the bi-parametric curve of principal eigenvalues \( \Sigma(\lambda, \gamma) \), Section 4 analyzes the continuous dependence of \( \sigma[\mathcal{P}, \mathcal{B}, Q_T] \) with respect to the variations of \( \Omega \) along \( \Gamma_0 \), Section 5 delivers the proof of Theorem 1.1 and, finally, Section 6 delivers the proof of Theorems 1.3 and 1.4.

## 2. Preliminaries

The principal eigenvalue \( \sigma[\mathcal{P}, \mathcal{B}, Q_T] \) satisfies the next pivotal characterization. It is [7, Th. 1.2].
Theorem 2.1. The following conditions are equivalent:

(a) $\sigma[\mathcal{P}, \mathcal{B}, Q_T] > 0$.
(b) $(\mathcal{P}, \mathcal{B}, Q_T)$ possesses a positive strict supersolution $h \in E$.
(c) Any strict supersolution $u \in E$ of $(\mathcal{P}, \mathcal{B}, Q_T)$ satisfies $u \gg 0$ in the sense that, for every $t \in [0, T]$, $u(x, t) > 0$ for all $x \in \Omega \cup \Gamma_1$, and

$$\partial_\nu u(x, t) < 0 \quad \text{for all} \quad x \in u^{-1}(0) \cap \Gamma_0.$$ 

In other words, $(\mathcal{P}, \mathcal{B}, Q_T)$ satisfies the strong maximum principle.

Theorem 2.1 is the periodic-parabolic counterpart of Theorem 2.4 of Amann and López-Gómez [4], which goes back to López-Gómez and Molina-Meyer [20] for cooperative systems under Dirichlet boundary conditions, and to [5] for cooperative periodic-parabolic systems under Dirichlet boundary conditions. In [6] the most fundamental properties of the principal eigenvalue $\sigma[\mathcal{P}, \mathcal{B}, Q_T]$ were derived from Theorem 2.1. Among them, its uniqueness, algebraic simplicity and strict dominance, as well as the properties collected in the next result, which is a direct consequence of Propositions 2.1 and 2.6 of [6].

Proposition 2.1. The principal eigenvalue satisfies the following properties:

(a) Let $V_1, V_2 \in F$ be such that $V_1 \preceq V_2$. Then,

$$\sigma[\mathcal{P} + V_1, \mathcal{B}, Q_T] < \sigma[\mathcal{P} + V_2, \mathcal{B}, Q_T].$$

(b) Let $\Omega_0$ be a subdomain of $\Omega$ of class $C^{2+\theta}$ such that $\overline{\Omega}_0 \subset \Omega$. Then,

$$\sigma[\mathcal{P}, \mathcal{B}, Q_T] < \sigma[\mathcal{P}, \mathcal{D}, \Omega_0 \times (0, T)].$$

The principal eigenvalue

$$\Sigma(\lambda) := \sigma[\mathcal{P} - \lambda m, \mathcal{B}, Q_T], \quad \lambda \in \mathbb{R},$$

plays a pivotal role in characterizing the existence of positive solutions of (1.1). According to Theorems 5.1 and 6.1 of [6], $\Sigma(\lambda)$ is analytic, strictly concave and, as soon as $m(x, t)$ satisfies (1.8),

$$\lim_{\lambda \to \pm \infty} \Sigma(\lambda) = -\infty.$$ 

Therefore, if (1.8) holds and $\Sigma(\lambda_0) > 0$ for some $\lambda_0 \in \mathbb{R}$, then there exist $\lambda_- < \lambda_0 < \lambda_+$ such that

$$\Sigma(\lambda) \begin{cases} < 0 & \text{if} \ \lambda \in (-\infty, \lambda_-) \cup (\lambda_+, \infty), \\ = 0 & \text{if} \ \lambda \in \{\lambda_-, \lambda_+\}, \\ > 0 & \text{if} \ \lambda \in (\lambda_-, \lambda_+). \end{cases} \quad (2.1)$$

Actually, one can choose $\lambda_0$ such that $\Sigma'(\lambda_0) = 0$, $\Sigma'(\lambda) > 0$ if $\lambda < \lambda_0$ and $\Sigma'(\lambda) < 0$ if $\lambda > \lambda_0$. Figure 5 shows the graph of $\Sigma(\lambda)$ for this choice in such case.

The next result follows from Lemma 3.1 and Theorem 4.1 of [2]. It provides us with a necessary condition for the existence of positive periodic solutions of (1.1), as well as with its uniqueness.

Theorem 2.2. Suppose that $a \geq 0$ in $Q_T$ and that (1.1) admits a positive periodic solution, $(\lambda, u)$, with $u \in E$. Then, $\Sigma(\lambda) < 0$. Moreover, $u$ is the unique positive periodic solution of (1.1) and $u \gg 0$, in the sense that

$$u(x, t) > 0 \quad \text{for all} \quad x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \partial_\nu u(x, t) < 0 \quad \text{for all} \quad x \in \Gamma_0.$$
The next result establishes that $\Sigma(\lambda) < 0$ is not only necessary but also sufficient whenever $a^{-1}(0) = \emptyset$. It is Theorem 6.1 of [2].

**Theorem 2.3.** Suppose that $a(x,t) > 0$ for all $(x,t) \in \bar{Q}_T$. Then, (1.1) possesses a positive periodic solution if, and only if, $\Sigma(\lambda) < 0$.

### 3. A Bi-parametric Family of Principal Eigenvalues

The next result collects some important properties of the bi-parametric family of principal eigenvalues $\Sigma(\lambda, \gamma)$ introduced in (1.5).

**Theorem 3.1.** The function $\Sigma(\lambda, \gamma)$ is concave, analytic in $\lambda$ and $\gamma$, and increasing with respect to $\gamma \in \mathbb{R}$. Thus, for every $\lambda \in \mathbb{R}$, the limit $\Sigma(\lambda, \infty)$ introduced in (1.6) is well defined and, for every $\gamma > 0$, satisfies

$$\Sigma(\lambda) = \Sigma(\lambda, 0) < \Sigma(\lambda, \gamma) < \Sigma(\lambda, \infty) \quad \text{for all } \lambda \in \mathbb{R}. \quad (3.1)$$

Moreover, one of the following excluding options occurs. Either

(a) $\Sigma(\lambda, \infty) = \infty$ for all $\lambda \in \mathbb{R}$; or
(b) $\Sigma(\lambda, \infty) < \infty$ for all $\lambda \in \mathbb{R}$.

Furthermore, when (b) occurs, also the function $\Sigma(\cdot, \infty)$ is concave.

**Proof.** The fact that $\Sigma(\lambda, \gamma)$ is analytic and concave with respect to each of the parameters $\lambda$ and $\gamma$ follows from [6, Th. 5.1]. By Proposition 2.1(a), it is increasing with respect of $\gamma \in \mathbb{R}$. Thus, for every $\lambda \in \mathbb{R}$, $\Sigma(\lambda, \infty) \in \mathbb{R} \cup \{+\infty\}$ is well defined and (3.1) holds.

Now, suppose that $\Sigma(\hat{\lambda}, \infty) < \infty$ for some $\hat{\lambda} \in \mathbb{R}$. Then, according to Proposition 2.1(a), we have that, for every $\lambda, \gamma \in \mathbb{R}$,

$$\Sigma(\lambda, \gamma) = \sigma[\mathcal{P} - \lambda m + \gamma a, \mathcal{B}, Q_T]$$
$$= \sigma[\mathcal{P} - \hat{\lambda} m + \gamma a + (\hat{\lambda} - \lambda)m, \mathcal{B}, Q_T]$$
$$\leq \Sigma(\hat{\lambda}, \gamma) + |\lambda - \hat{\lambda}| \|m\|_\infty < +\infty.$$

Consequently, the option (b) occurs.

As, for every $\gamma \in \mathbb{R}$, the function $\Sigma(\lambda, \gamma)$ is concave, it is apparent that, for every $\lambda_1, \lambda_2, \gamma \in \mathbb{R}$ and $t \in [0, 1]$,

$$\Sigma(t\lambda_1 + (1-t)\lambda_2, \gamma) \geq t \Sigma(\lambda_1, \gamma) + (1-t) \Sigma(\lambda_2, \gamma).$$
Thus, by (3.1), for every $\gamma \in \mathbb{R}$,
\[
\Sigma(t\lambda_1 + (1-t)\lambda_2, \infty) > t \Sigma(\lambda_1, \gamma) + (1-t) \Sigma(\lambda_2, \gamma).
\]
Consequently, letting $\gamma \uparrow \infty$, it follows that
\[
\Sigma(t\lambda_1 + (1-t)\lambda_2, \infty) \geq t \Sigma(\lambda_1, \infty) + (1-t) \Sigma(\lambda_2, \infty),
\]
which ends the proof. \(\square\)

4. Continuous dependence of $\sigma[\mathcal{P}, \mathcal{B}, Q_T]$ with respect to $\Gamma_0$

In this section, we establish the continuous dependence of $\sigma[\mathcal{P}, \mathcal{B}, Q_T]$ with respect to a canonical perturbation of the domain $\Omega$ around its Dirichlet boundary $\Gamma_0$. Our result provides with a periodic counterpart of [17, Th. 8.5] for the perturbation
\[
O_n := \Omega \cup \left( \Gamma_0 + B_{\frac{1}{n}}(0) \right) = \Omega \cup \{ x \in \mathbb{R}^N : \text{dist}(x, \Gamma_0) < \frac{1}{n} \}. \tag{4.1}
\]
For sufficiently large $n \geq 1$, say $n \geq n_0$, we consider $a_{ij,n} = a_{ij}, b_{j,n} = b_j$ and $c_n = c$ in $Q_T$, where we are denoting
\[
F_n := \left\{ u \in C^0([0, T], \mathbb{R}) : u(\cdot, t) = u(\cdot, t) \text{ for all } t \in \mathbb{R} \right\}, \tag{4.2}
\]
as well as the associated differential operators
\[
\mathcal{L}_n = \mathcal{L}_n(x, t) := -\sum_{i,j=1}^{N} a_{ij,n}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{N} b_{j,n}(x,t) \frac{\partial}{\partial x_j} + c_n(x,t), \tag{4.3}
\]
the periodic parabolic operators $\mathcal{P}_n := \partial_t + \mathcal{L}_n$, $n \geq n_0$, and the boundary operators $\mathcal{B}_n$ defined by
\[
\mathcal{B}_n := \left\{ \begin{array}{ll}
\mathcal{D} & \text{on } \Gamma_{0,n} \equiv \partial \mathcal{O}_n \setminus \Gamma_1, \\
\mathcal{B} & \text{on } \Gamma_{1,n} \equiv \Gamma_1.
\end{array} \right. \tag{4.4}
\]
Then, the next result holds.

**Theorem 4.1.** Let $O_n$ be defined by (4.1) for all $n \geq n_0$, and set $Q_{T,n} := O_n \times (0, T)$. Then,
\[
\lim_{n \to \infty} \sigma[\mathcal{P}_n, \mathcal{B}_n, Q_{T,n}] = \sigma[\mathcal{P}, \mathcal{B}, Q_T]. \tag{4.5}
\]

**Proof.** By Proposition 2.6 of [6], we have that, for every $n \geq n_0$,
\[
\sigma_n := \sigma[\mathcal{P}_n, \mathcal{B}_n, Q_{T,n}] < \sigma[\mathcal{P}_n, \mathcal{B}_{n+1}, Q_{T_{n+1}}] < \sigma[\mathcal{P}, \mathcal{B}, Q_T].
\]
Thus, the limit in (4.5) exists and it satisfies that
\[
\sigma_\infty := \lim_{n \to \infty} \sigma_n \leq \sigma[\mathcal{P}, \mathcal{B}, Q_T]. \tag{4.6}
\]
Thus, it suffices to show that
\[
\sigma_\infty = \sigma[\mathcal{P}, \mathcal{B}, Q_T]. \tag{4.7}
\]
To prove (4.7) we can argue as follows. For every $n \in \mathbb{N}$, let $\varphi_n \gg 0$ denote the (unique) principal eigenfunction associated with $\sigma_n$ normalized so that
\[
\| \varphi_n \|_{L^{2+\theta, 1+\frac{\theta}{2}}(\Omega \times \mathbb{R})}^2 = 1, \tag{4.8}
\]
where $C^{2+\theta,1+\theta}_{T}(\tilde{\Omega} \times \mathbb{R}; \mathbb{R})$ denotes the set of functions $u \in C^{2+\theta,1+\theta}_{T}(\tilde{\Omega} \times \mathbb{R}; \mathbb{R})$ that are $T$-periodic in time. Note that, since $\tilde{\Omega} \subset \bar{\Omega}_n$, $\varphi_n \in E$ for all $n \geq n_0$. By the compactness of the imbedding $$C^{2+\theta,1+\theta}_{T}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}) \hookrightarrow C^{2,1}_{T}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}),$$ we can extract a subsequence of $\varphi_n$, relabeled by $n \geq n_0$, such that $$\lim_{n \to \infty} \varphi_n =: \varphi \quad \text{in} \quad C^{2,1}_{T}(\bar{\Omega} \times \mathbb{R}; \mathbb{R}). \quad (4.9)$$

From the definition of the $\varphi_n$'s, it follows from (4.9) that

\[
\begin{aligned}
\mathcal{P}\varphi &= \sigma_{\infty}\varphi \quad \text{in} \quad Q_T, \\
\mathcal{B}\varphi &= 0 \quad \text{on} \quad \partial Q_T.
\end{aligned}
\]

Moreover, $\varphi \geq 0$ in $Q_T$. Therefore, $\varphi$ must be a principal eigenfunction of $(\mathcal{P}, \mathcal{B}, Q_T)$.

By the uniqueness of the principal eigenvalue, this entails (4.7) and ends the proof. \hfill \Box

5. Proofs of Theorem 1.1

First, we will show that (1.7) is necessary for the existence of a positive solution of (1.1). Indeed, the fact that $\Sigma(\lambda) < 0$ has been already established by Theorem 2.2. Let $u$ be a positive solution of (1.1). Then, also by Theorem 2.2, $u \gg 0$ solves

\[
\begin{aligned}
\mathcal{P}u &= \lambda \mu(x,t) + a(x,t) f(u) \quad \text{in} \quad Q_T, \\
\mathcal{B}u &= 0 \quad \text{on} \quad \partial Q_T,
\end{aligned}
\]

and hence, by the uniqueness of the principal eigenvalue,

$$\sigma[\mathcal{P} - \lambda \mu + af(u), \mathcal{B}, Q_T] = 0.$$  

By the continuity of $f$ and $u$, there exists a constant $\gamma > 0$ such that

$$f(u(x,t)) < \gamma \quad \text{for all} \quad (x,t) \in Q_T.$$  

Consequently, thanks to Proposition 2.1(a), we can infer that

$$0 = \sigma[\mathcal{P} - \lambda \mu + af(u), \mathcal{B}, Q_T] < \sigma[\mathcal{P} - \lambda \mu + \gamma a, \mathcal{B}, Q_T] = \Sigma(\lambda, \gamma) < \Sigma(\lambda, \infty)$$

because $a \geq 0$ in $Q_T$. This shows the necessity of (1.7).

Subsequently, we will assume (1.7), i.e.,

$$\Sigma(\lambda) < 0 < \Sigma(\lambda, \infty). \quad (5.1)$$

Let $\varphi \gg 0$ denote the principal eigenfunction associated to $\Sigma(\lambda)$ normalized so that $\|\varphi\|_{\infty} = 1$. The fact that $\Sigma(\lambda) < 0$ entails that $u := \varepsilon \varphi$ is a positive subsolution of (1.1) for sufficiently small $\varepsilon > 0$. Indeed, by $(H_f)$, we find from (5.1) that

$$0 < f(\varepsilon) < -\Sigma(\lambda)/\|a\|_{\infty}$$

for sufficiently small $\varepsilon > 0$. Thus, $u := \varepsilon \varphi$ satisfies

$$(\mathcal{P} - \lambda \mu + af(u))u = (\Sigma(\lambda) + af(\varepsilon \varphi))u \leq (\Sigma(\lambda) + \|a\|_{\infty} f(\varepsilon))u < 0 \quad \text{in} \quad Q_T.$$  

Moreover,

$$\mathcal{B}u = \varepsilon \mathcal{B} \varphi = 0 \quad \text{on} \quad \partial Q_T.$$
Hence, $u$ is a positive strict subsolution of (1.1) for sufficiently small $\varepsilon > 0$.

To establish the existence of arbitrarily large supersolutions we proceed as follows. Consider, for sufficiently large $n \in \mathbb{N}$,

$$O_n := \Omega \cup \{x \in \mathbb{R}^N : \text{dist}(x, \Gamma_0) < \frac{1}{n}\}.$$  

For sufficiently large $n$, say $n \geq n_0$, $\Gamma_1$ is a common part of $\partial \Omega$ and of $\partial O_n$. Subsequently, we consider

$$\Gamma_{0,n} := \partial O_n \setminus \Gamma_1. \quad \text{(5.2)}$$

By construction,

$$\text{dist}(\Gamma_{0,n}, \Gamma_0) = \frac{1}{n} \quad \text{for all } n \geq n_0.$$  

Now, much like in Section 4, for every $n \geq n_0$, we consider $\Sigma_n$ and $B_n$ satisfying (4.3) and (4.4), as well as the parabolic operator $P_n := \partial_t + \Sigma_n$. Then, by Theorem 4.1, we have that

$$\lim_{n \to \infty} \sigma[P_n - \lambda m_n + \gamma a_n, B_n, O_n \times (0, T)] = \Sigma(\lambda, \gamma), \quad \text{(5.3)}$$

where $m_n$ and $a_n$ are continuous periodic extensions of $m$ and $a$ to $Q_{T,n}$.

Since $\Sigma(\lambda, \infty) > 0$, by Theorem 3.1, we have that $\Sigma(\lambda, \gamma) > 0$ for sufficiently large $\gamma$, say $\gamma > \gamma_0 > 0$. Subsequently, we fix one of those $\gamma$'s. According to (5.3), we can enlarge $n_0$ so that

$$\Sigma_n(\lambda, \gamma) \equiv \sigma[P_n - \lambda m_n + \gamma a_n, B_n, O_n \times (0, T)] > 0 \quad \text{for all } n \geq n_0. \quad \text{(5.4)}$$

Pick $n \geq n_0$ and let $\psi_n \gg 0$ be any positive eigenfunction associated to $\Sigma_n(\lambda, \gamma)$. We claim that $\bar{\pi} := \kappa \psi_n$ is a positive strict supersolution of (1.1) for sufficiently large $\kappa > 1$. Note that

$$\psi_n(x,t) > 0 \quad \text{for all } (x,t) \in \tilde{Q}_T. \quad \text{(5.5)}$$

Indeed, $\psi_n(x,t) > 0$ for all $(x,t) \in O_n \times [0,T]$. Thus, $\psi_n(x,t) > 0$ for all $x \in \Omega \cup \Gamma_0$ and $t \in [0,T]$. Consequently, since $\psi_n(x,t) > 0$ for all $x \in \Gamma_1$ and $t \in [0,T]$, (5.5) holds true. Thanks to (5.5), we have that, for every $\kappa > 0$,

$$B \bar{\pi}(x,t) = \bar{\pi}(x,t) > 0 \quad \text{for all } (x,t) \in \Gamma_0 \times [0,T].$$

Moreover, by construction, $B \bar{\pi} = 0$ on $\Gamma_1 \times [0,T]$. Therefore, $B \bar{\pi} \geq 0$ on $\partial \Omega \times [0,T]$.

On the other hand, by (5.4) and (5.5), we have that

$$(P_n - \lambda m_n)\psi_n = \Sigma_n(\lambda, \gamma) \psi_n - \gamma a_n \psi_n > -\gamma a_n \psi_n \quad \text{in } Q_T. \quad \text{(5.6)}$$

By construction, $P_n = P$, $a_n = a$ and $m_n = m$ in $Q_T$. Moreover, by (5.5), there exists a constant $\mu > 0$ such that $\psi_n > \mu$ in $Q_T$. Thus, owing to $(H_f)$, it follows from (5.6) that

$$(P - \lambda m + af(\bar{\pi}))\bar{\pi} > a(f(\kappa \psi_n) - \gamma)\bar{\pi} \geq a(f(\kappa \mu) - \gamma)\bar{\pi} > 0 \quad \text{in } Q_T \quad \text{(5.7)}$$

for sufficiently large $\kappa > 0$. Therefore, $\bar{\pi}$ is a positive strict supersolution of (1.1) for sufficiently large $\kappa > 0$.

Finally, either shortening $\varepsilon > 0$, or enlarging $\kappa > 0$, if necessary, one can assume that

$$0 \ll \underline{u} = \varepsilon \varphi \leq \varepsilon \leq \kappa \mu \leq \bar{\pi} = \kappa \psi_n.$$  

Consequently, by adapting the argument of Amann [3] to a periodic-parabolic context (see, e.g., Hess [16], or Dancer and Koch-Medina [10]), it becomes apparent that (1.1)
admits a positive solution. This shows the existence of a positive solution for (1.1). 

The uniqueness is already known by Theorem 2.2.

6. Proof of Theorem 1.4

6.1. Preliminaries. Throughout this section, according to [19, Def. 3.2], for any given connected open subset \( G \) of \( \mathbb{R}^N \times \mathbb{R} \), a point \((x,t)\) \(\in\partial G\) is said to belong to the flat top boundary of \( G \), denoted by \( \partial_{FT} G \), if there exists \( \varepsilon > 0 \) such that

- \( B_\varepsilon(x) \times \{t\} \subset \partial G \), where \( B_\varepsilon(x) := \{y \in \mathbb{R}^N : |y - x| < \varepsilon\} \).
- \( B_\varepsilon(x) \times (t - \varepsilon, t) \subset G \).

Then, the lateral boundary of \( G \), denoted by \( \partial_L G \), is defined by

\[ \partial_L G = \partial G \setminus \partial_{FT} G. \]

The next generalized version of the parabolic weak maximum principle holds; it is [19, Th. 3.4].

Theorem 6.1. Let \( G \) be a connected open subset of \( \mathbb{R}^N \times \mathbb{R} \) whose flat top boundary \( \partial_{FT} G \) is nonempty and consists of finitely many components with disjoint closures, and suppose \( c = 0 \), i.e., \( \Sigma \) does not have zero order terms. Let \( u \in C(\overline{G}) \cap C^2(G) \) be such that

\[ Pu = \partial_t u + \Sigma u \leq 0 \quad \text{in} \ G. \]

Then,

\[ \max_{\overline{G}} u = \max_{\partial_L G} u. \]

Subsequently, for every \( \varepsilon \geq 0 \), we set

\[ [a \geq \varepsilon] := \{(x,t) \in \overline{Q}_T : a(x,t) \geq \varepsilon\} = a^{-1}([\varepsilon, \infty)), \]

and denote by \( \varphi_\gamma \gg 0 \) the unique principal eigenfunction associated to \( \Sigma(\lambda, \gamma) \) such that \( \|\varphi_\gamma\|_{L^\infty(\overline{Q}_T)} = 1 \). Then, by [19, Th. 5.1], the next result holds.

Theorem 6.2. Assume \((H_\Omega), (H_\Sigma), (H_B), (H_a)\),

\[ a(x,t) > 0 \quad \text{for all} \ (x,t) \in \partial \Omega \times [0,T], \quad (6.1) \]

and \( \Sigma(\lambda, \infty) < \infty \) for some \( \lambda \in \mathbb{R} \). Then,

\[ \lim_{\gamma \to \infty} \varphi_\gamma = 0 \quad (6.2) \]

uniformly on compact subsets of \((\Omega \times [0,T]) \cap [a > 0]\).

6.2. Proof of Theorem 1.3. Note that, besides \((H_\Omega), (H_\Sigma), (H_B), (H_a)\), we are assuming (6.1), or, equivalently, (1.9). Our proof will proceed by contradiction. So, assume that \( \Sigma(\tilde{\lambda}, \infty) < \infty \) for some \( \tilde{\lambda} \in \mathbb{R} \). Then, by Theorem 3.1,

\[ \Sigma(\lambda, \infty) < \infty \quad \text{for all} \ \lambda \in \mathbb{R}. \quad (6.3) \]

Fix \( \lambda \in \mathbb{R} \), and assume that \( c - \lambda m \geq 0 \) in \( Q_T \). Once completed the proof in this case, we will show how to conclude the proof when \( c - \lambda m \) changes of sign.

For sufficiently small \( \varepsilon > 0 \), let \( G_\varepsilon \) be an open set such that

\[ \partial_L G_\varepsilon \subset [a > 0] \quad \text{and} \quad t = t_Q - \varepsilon \quad \text{if} \ (x,t) \in \partial_{FT} G_\varepsilon, \]
as illustrated in the first picture of Figure 6. In this figure, as in all subsequent pictures of this paper, we are representing the nodal behavior of the weight function $a(x,t)$ in the parabolic cylinder $Q_{\tau} = \Omega \times [0,\tau]$ for some $\tau \in (t_p, T)$. Since $\partial_L G_\varepsilon$ is a compact subset of the open set $[a>0]$, Theorem 6.2 implies that
\[
\lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \partial_L G_\varepsilon. \tag{6.4}
\]
Moreover, by the definition of $\varphi_\gamma$, we have that, for every $\gamma \geq 0$,
\[
\partial_t \varphi_\gamma + (\mathcal{L} - c) \varphi_\gamma = (-c + \lambda m - \gamma a + \Sigma(\lambda, \gamma)) \varphi_\gamma \leq \Sigma(\lambda, \gamma) \varphi_\gamma,
\]
because $-c + \lambda m \leq 0$ and $a \geq 0$. Thus, the auxiliary function
\[
\psi_\gamma := e^{-t \Sigma(\lambda, \gamma)} \varphi_\gamma \tag{6.5}
\]
satisfies
\[
\partial_t \psi_\gamma + (\mathcal{L} - c) \psi_\gamma = -\Sigma(\lambda, \gamma)e^{-t \Sigma(\lambda, \gamma)} \varphi_\gamma + e^{-t \Sigma(\lambda, \gamma)} \partial_t \varphi_\gamma + e^{-t \Sigma(\lambda, \gamma)} (\mathcal{L} - c) \varphi_\gamma \leq 0.
\]
Hence, it follows from Theorem 6.1 that, for every $\gamma \geq 0$,
\[
\max_{G_\varepsilon} \psi_\gamma = \max_{\partial_L G_\varepsilon} \psi_\gamma. \tag{6.6}
\]
On the other hand, by (6.3),
\[
\Sigma(\lambda, \gamma) \leq \Sigma(\lambda, \infty) < \infty. \tag{6.7}
\]
Thus, (6.4) and (6.5) imply that
\[
\lim_{\gamma \uparrow \infty} \psi_\gamma = 0 \quad \text{uniformly on } \partial_L G_\varepsilon.
\]
Consequently, according to (6.6), we actually have that
\[
\lim_{\gamma \uparrow \infty} \psi_\gamma = 0 \quad \text{uniformly in } \bar{G}_\varepsilon.
\]
Equivalently, by (6.5) and (6.7),
\[
\lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly in } \bar{G}_\varepsilon.
\]
As this holds regardless the size of $\varepsilon > 0$, we find that
\[
\lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{in } G_0 := \bigcup_{\varepsilon > 0} G_\varepsilon, \tag{6.8}
\]
uniformly in $\bar{G}_\varepsilon$ for all $\varepsilon > 0$.

To show the validity of (6.8) at the level $t = t_Q$, one needs an additional argument. Consider any closed ball, $\bar{B}$, such that
\[
\bar{B} \subset \text{int } a^{-1}(0), \quad B \cap [t > t_Q] \neq \emptyset, \quad B \cap [t < t_Q] \neq \emptyset,
\]
as illustrated in the first picture of Figure 7, where the ball has been centered at a point $R = (x_R, t_R)$ with $t_R = t_Q$. Since $\Sigma(\lambda, \infty) < \infty$, in the interior of $a^{-1}(0)$ one can combine the theorem of Eberlein–Schmulian (see, e.g., [17, Th. 3.8]) with Theorem 4 of Aronson and Serrin [8] to show the existence of a sequence $\{\gamma_n\}_{n \geq 1}$ such that
\[
\lim_{n \to \infty} \gamma_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \varphi_{\gamma_n} = \varphi_{\infty} \quad \text{uniformly in } \bar{B}.
\]
for some smooth function $\varphi_\infty \in C^{2,1}(\bar{B})$. Since
\[ \bar{B} \cap [t < t_Q] \subset G_0, \]
we can infer from (6.8) that $\varphi_\infty(x, t) = 0$ if $(x, t) \in \bar{B}$ with $t \leq t_Q$. Thus, there is a region $G$, like the one represented in the second picture of Figure 7, such that
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \partial_L G. \] (6.9)

Consequently, applying again the Theorem 6.2 to $\psi_\gamma$, we are also driven to
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \bar{G}. \] (6.10)

Next, for sufficiently small $\varepsilon > 0$, we consider any open set $H_\varepsilon$ such that
\[ t = t_P - \varepsilon \quad \text{if } (x, t) \in \partial_{FT} H_\varepsilon, \]
\[ H_\varepsilon \cap [t \leq t_Q] \subset \bar{G}, \] (6.11)
and
\[ \partial_L H_\varepsilon \cap (t_Q, t_P - \varepsilon) \subset [a > 0] \quad (6.12) \]
as illustrated by the first picture of Figure 8, where \( S = (x_S, t_S) \) stands for any point in \( M \) with \( t_S = t_Q \); \( H_\varepsilon \) exists by our structural assumptions. According to (6.10) and (6.11), we have that
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \bar{H}_\varepsilon \cap [t \leq t_S]. \quad (6.13) \]
Moreover, by Theorem 6.2, it follows from (6.12) that, for arbitrarily small \( \eta > 0 \),
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \partial_L H_\varepsilon \cap [t_S + \eta, t_P - \varepsilon]. \quad (6.14) \]
Note that, thanks to (6.13) and (6.14), we have that
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{point-wise on } \partial_L H_\varepsilon \quad (6.15) \]
and uniformly on \( \partial_L H_\varepsilon \setminus B_\delta(S) \) for sufficiently small \( \delta > 0 \), where \( B_\delta(S) \) stands for the open ball centered at \( S \) with radius \( \delta > 0 \). Naturally, by our assumptions, the function \( \psi_\gamma \) defined in (6.5) also approximates zero point-wise on \( \partial_L H_\varepsilon \) and uniformly on \( \partial_L H_\varepsilon \setminus B_\delta(S) \) for sufficiently small \( \delta > 0 \). Moreover, since
\[ \partial_t \psi_\gamma + (\mathcal{L} - c) \psi_\gamma \leq 0 \quad \text{in } Q_T, \]
\( \psi_\gamma \) provides us with a positive subsolution of the parabolic operator \( \partial_t + \mathcal{L} - c \). Thus, by the parabolic Harnack inequality, there exists a constant \( C > 0 \) such that
\[ \max_{\partial_t H_\varepsilon} \psi_\gamma \leq C \min_{\partial_t H_\varepsilon} \psi_\gamma. \quad (6.16) \]
Hence,
\[ \lim_{\gamma \uparrow \infty} \max_{\partial_t H_\varepsilon} \psi_\gamma = 0 \]
and therefore,
\[ \lim_{\gamma \uparrow \infty} \max_{\partial_t H_\varepsilon} \varphi_\gamma = \lim_{\gamma \uparrow \infty} \max_{\partial_t H_\varepsilon} \psi_\gamma = 0. \]
Consequently, once again by Theorem 6.1, one can infer that
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \bar{H}_\varepsilon \]
for sufficiently small \( \varepsilon > 0 \).

Now, consider the open set \( H_0 \) defined through
\[ H_0 := \cup_{\varepsilon > 0} H_\varepsilon, \]
as sketched in the second picture of Figure 8.

Proceeding with \( H_0 \) as we previously did with \( G_0 \), it becomes apparent that one can construct an open subset \( H \subset H_0 \), like the one sketched in the first picture of Figure 9, such that
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \partial_L H. \]
Therefore, by Theorem 6.1,
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \bar{H}. \quad (6.17) \]
Finally, considering an open subset $\mathcal{O}$ of $Q_T$ of the type described in the second picture of Figure 9, with

$$\bar{\mathcal{O}} \cap [t \leq t_P] \subset \bar{H} \quad \text{and} \quad \partial_t \mathcal{O} \cap [t_P < t \leq T] \subset [a > 0],$$

at the light of the analysis already done in this proof it is apparent that

$$\lim_{\gamma \to \infty} \varphi_\gamma = 0 \quad \text{uniformly in } \bar{\mathcal{O}}.$$

By Theorem 6.2, this implies that

$$\lim_{\gamma \to \infty} \varphi_\gamma(x, T) = 0 \quad \text{uniformly in } \bar{\Omega}.$$

Thus, since $\varphi(x, t)$ is $T$-periodic,

$$\lim_{\gamma \to \infty} \varphi_\gamma(x, 0) = 0 \quad \text{uniformly in } \bar{\Omega},$$

which implies

$$\lim_{\gamma \to \infty} \varphi_\gamma = 0 \quad \text{uniformly on } \bar{Q}_T.$$
This is impossible, because 
\[ \| \phi_\gamma \|_{L^\infty(Q_T)} = 1. \]
This contradiction shows that
\[ \Sigma(\lambda, \infty) = \infty. \]  
(6.18)
In the general case when \( c - \lambda m \) changes of sign, we can pick a sufficiently large \( \omega > 0 \) such that
\[ c - \lambda m + \omega \geq 0 \quad \text{in} \quad \bar{Q}_T. \]
Then, by the result that we have just proven,
\[ \lim_{\gamma \to \infty} \sigma[P - \lambda m + \omega + \gamma a, B, Q_T] = \Sigma(\lambda, \infty) + \omega = \infty. \]
Therefore, (6.18) also holds in this case. This ends the proof of Theorem 1.3.

6.3. **Proof of Theorem 1.4.** Assume that \( n = 2 \) and that \( \Sigma(\hat{\lambda}, \infty) < \infty \) for some \( \hat{\lambda} \in \mathbb{R} \). Then, by Theorem 3.1, (6.3) holds. Arguing as in the proof of Theorem 1.3, we fix \( \lambda \in \mathbb{R} \) and assume \( c - \lambda m \geq 0 \) in \( Q_T \). The general case follows with the final argument of the proof of Theorem 1.3.

Repeating the argument of the proof of Theorem 1.3, one can construct an open subset, \( H \), of \( Q_T \), with \( H \subset [t \leq t_{P_2}] \), as illustrated in the first picture of Figure 10, satisfying (6.17).

**Figure 10.** The construction of the open sets \( J_\varepsilon, \varepsilon > 0 \).

Arguing as in the proof of Theorem 1.3, for sufficiently small \( \varepsilon > 0 \), there exists an open set \( J_\varepsilon \) such that
\[ t = t_P - \varepsilon = t_{Q_2} - \varepsilon \quad \text{if} \quad (x, t) \in \partial_{FT} J_\varepsilon, \quad \bar{J}_\varepsilon \cap [t \leq t_{P_2}] \subset \bar{H}, \]  
(6.19)
and
\[ \partial_L J_\varepsilon \cap (t_{P_2}, t_P - \varepsilon] \subset [a > 0] \]  
(6.20)
like the one shown in the second picture of Figure 10; \( J_\varepsilon \) exists by our structural assumptions. By (6.17), (6.19) and (6.20), we find that
\[ \lim_{\gamma \uparrow \infty} \varphi_\gamma = 0 \quad \text{uniformly on} \quad \bar{J}_\varepsilon \cap [t \leq t_{P_2}]. \]  
(6.21)
Moreover, as in the proof of Theorem 1.3, it is apparent that
\[
\lim_{\gamma \to \infty} \varphi_\gamma = 0 \text{ point-wise on } \partial L J_\varepsilon
\] (6.22)
and uniformly on \(\partial L J_\varepsilon \setminus B_\delta(P_2)\) for sufficiently small \(\delta > 0\). By our assumptions, the function \(\psi_\gamma\) defined in (6.5) also approximates zero point-wise on \(\partial L J_\varepsilon\) and uniformly on \(\partial L J_\varepsilon \setminus B_\delta(P_2)\) for arbitrarily small \(\delta > 0\), and
\[
\partial_t \psi_\gamma + (\mathcal{L} - c) \psi_\gamma \leq 0 \text{ in } Q_T.
\]
Therefore, arguing as in the proof of Theorem 1.3, it follows from the parabolic Harnack inequality that
\[
\lim_{\gamma \to \infty} \varphi_\gamma = 0 \text{ uniformly on } \overline{J}_\varepsilon.
\]
Next, we consider the open set \(J_0\) defined by
\[
J_0 := \bigcup_{\varepsilon > 0} J_\varepsilon,
\]
which has been represented in the first picture of Figure 11. Arguing with \(J_0\) as we did with \(G_0\) in the proof of Theorem 1.3, it becomes apparent that there exists an open subset \(J \subset J_0\), such that
\[
\lim_{\gamma \to \infty} \varphi_\gamma = 0 \text{ uniformly on } \partial L J
\]
and \(J = \mathcal{O} \cap [t \leq t_P]\) (see the right picture of Figure 11), where \(\mathcal{O}\) is a prolongation of \(J\) up to \(t = T\) with
\[
\partial L (\mathcal{O} \setminus J) \subset [a > 0].
\]

**Figure 11.** The construction of \(J_0, J\) and \(\mathcal{O}\).
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