When is Network Lasso Accurate: The Vector Case

Nguyen Tran, Saeed Basirian, Alexander Jung
Department of Computer Science
Aalto University
firstname.lastname(at)aalto.fi

Abstract

A recently proposed learning algorithm for massive network-structured data sets (big data over networks) is the network Lasso (nLasso), which extends the well-known Lasso estimator from sparse models to network-structured datasets. Efficient implementations of the nLasso have been presented using modern convex optimization methods. In this paper we provide sufficient conditions on the network structure and available label information such that nLasso accurately learns a vector-valued graph signal (representing label information) from the information provided by the labels of a few data points.

1 Introduction

We consider datasets which are represented by a “data graph”. The nodes of the data graph represent individual data points (e.g., one image out of a image collection) which are are connected by edges according to some notion of similarity. This similarity might be induced naturally by the application at hand (e.g., in social networks) or obtained from statistical models (probabilistic graphical models) [Tran and Jung, 2017b,a]. Beside graph structure, the datasets typically carry label information which we represent by a graph signal [Sandryhaila and Moura, 2013].

The acquisition of graph signal values (labels) is often expensive, and therefore we are interested in methods for learning the entire graph signal from a (small) subset of nodes (sampling set), which is a crucial task in many machine learning problems. The learning of the graph signals from a small number of signal samples, which are obtained by manually labelling few data points, is enabled by exploiting the tendency of natural graph signals to be smooth. More precisely, the smoothness hypothesis, which underlies most (semi-) supervised machine learning methods [Bishop, 2006, Chapelle et al., 2006], requires the graph signal to be nearly constant over well connected subset of nodes (clusters).

In this paper, by extending the program initiated in [Jung et al., 2017] for scalar graph signals, we present sufficient conditions on the network topology and available label information such that the nLasso can recover an underlying vector-valued graph signal. In particular, we extend the network compatibility condition (NCC) introduced in [Jung et al., 2017] to vector-valued graph signals. The NCC ensures accurate recovery of a smooth vector-valued graph signal from only few signal values (initial labels) using nLasso. We then relate the NCC to the existence of certain network flows.

2 Problem Formulation

We consider a graph signal defined over an undirected graph $G = (V, E)$, with nodes $V$ representing individual data points and undirected edges $E$ encoding domain-specific notions of similarity between data points. The strength of the connections $(i, j) \in E$ is quantified by non-negative edge weights $W_{ij}$, which we collect into a weighted adjacency matrix $W \in \mathbb{R}^{N \times N}$ (which is also known as the graph shift matrix [Chen et al., 2013]).
In addition to the graph structure $\mathcal{G}$, datasets typically convey additional information, e.g., features, labels or model parameters associated with individual data points $i \in \mathcal{V}$. We represent this additional information as a graph signal $x[:i] : \mathcal{V} \to \mathbb{R}^p$, which maps the node $i \in \mathcal{V}$ to the signal vector $x[i] := (x_1[i], \ldots, x_p[i])^T \in \mathbb{R}^p$. The graph signal vector $x[i]$ might represent, e.g., the weight vector for a local pricing model in a house price prediction application (cf. [Hallac et al., 2015]).

The clustered graph signal is different from the model of band-limited graph signals which is championed in graph signal processing [Chen et al., 2015]. Indeed, while band-limited graph signals have sparse graph Fourier transform (GFT) coefficients, clustered graph signals (1) have dense GFT with only if $\|i\|_1 \ll |\mathcal{V}|$. In particular, we observe $y[i] = x[i] + \varepsilon[i]$ for a sampled node $i \in \mathcal{M}$. The error component $\varepsilon[i]$ in (2) covers any data curation or labelling errors.

In order to be able to learn the entire graph signal $x[:i]$ from partial noisy measurements $\{y[i]\}_{i \in \mathcal{M}}$ we exploit that the true graph signal is smooth, i.e., have small TV $\|x[:i]\|_\mathcal{E}$. Moreover, any reasonable learning algorithm should deliver a graph signal with a small empirical error

\[
\hat{E}(\hat{x}[:i]) := \sum_{i \in \mathcal{M}} \|x[i] - y[i]\|_1.
\]

Note that we use the $\ell_1$-norm for the empirical error $\hat{E}(\hat{x}[:i])$ (3), which is different from the original lasso, where the squared $\ell_2$-norm was used [Bühlmann and van de Geer, 2011].

A straightforward recovery method aiming to a small TV $\|\hat{x}[:i]\|_\mathcal{E}$ and small empirical error $\hat{E}(\hat{x}[:i])$ can be formulated as a regularized optimization problem

\[
\hat{x}[:i] \in \arg\min_{\hat{x}[:i] \in \mathbb{R}^p} \hat{E}(\hat{x}[:i]) + \lambda \|\hat{x}[:i]\|_\mathcal{E}.
\]

The tuning parameter $\lambda$ in (4) trades off a small empirical error $\hat{E}(\hat{x}[:i])$ against signal smoothness $\|\hat{x}[:i]\|_\mathcal{E}$ of the learned signal $\hat{x}[:i]$. A small value of $\lambda$ enforces the solutions of (4) to obtain a small empirical error, whereas, a large value of $\lambda$ enforces the solutions of (4) to obtain a small TV, i.e., to be smooth. The recovery problem (4) is a convex problem and can be approached by modern convex optimization methods [Jung et al., 2016, Hannak et al., 2016, Jung et al., 2017].
3 When is Network Lasso Accurate?

We now introduce the network compatibility condition (NCC), which generalizes the compatibility conditions for Lasso type estimators [Bühlmann and van de Geer, 2011] of ordinary sparse signals. Our main contribution is to show that the NCC guarantees any solutions of (4) allows to accurately learn the true underlying graph signal.

**Definition 1.** Consider a data graph \( G = (V, E) \) with a particular partition \( F \) of its nodes \( V \). A sampling set \( M \subseteq V \) is said to satisfy NCC with constants \( K, L > 0 \), if

\[
K \sum_{i \in M} \Vert z[i] \Vert_2 + \Vert z[-] \Vert_{\partial F} \geq (L/\sqrt{p}) \Vert x[-] \Vert_{\partial F}
\]

for any graph signal \( z[-] \).

It turns out that, if the sampling set satisfies the NCC, any solution of (4) provides an accurate estimate of the true underlying graph signal (1).

**Theorem 2.** Consider a data set represented by data graph \( G \) and a graph signal \( x[-] \) of the form (1). If the sampling set \( M \) satisfies NCC with parameters \( L > \sqrt{p} \) and \( K > 0 \), then any solution \( \hat{x}[-] \) of (4) with \( \lambda := 1/K \) satisfies

\[
\Vert \hat{x}[-] - x[-] \Vert_\varepsilon \leq K (1 + 4 \sqrt{p}/(L - \sqrt{p})) \sum_{i \in M} \Vert \varepsilon[i] \Vert_1.
\]

**Proof.** Consider an arbitrary solution \( \hat{x}[-] \) of (4) and denote the difference between \( \hat{x}[-] \) and the true underlying clustered signal \( x[-] \) as \( \hat{x}[-] - x[-] \). By (4),

\[
\sum_{i \in M} \Vert \hat{x}[i] - y[i] \Vert_1 + \lambda \Vert \hat{x}[-] \Vert_\varepsilon \leq \sum_{i \in M} \Vert \varepsilon[i] \Vert_1 + \lambda \Vert x[-] \Vert_\varepsilon.
\]

Since the true graph signal \( x[-] \) satisfies (1), we have \( \Vert x[-] \Vert_{\partial F} = 0 \) and \( \Vert \hat{x}[-] \Vert_{\partial F} = \Vert \hat{x}[-] \Vert_{\partial F} \). Combining the decomposition property and triangle inequality for the semi-norm \( \Vert \cdot \Vert_\varepsilon \) with (7),

\[
\sum_{i \in M} \Vert \hat{x}[i] - y[i] \Vert_1 + \lambda \Vert \hat{x}[-] \Vert_{\partial F} \leq \sum_{i \in M} \Vert \varepsilon[i] \Vert_1 + \lambda \Vert \hat{x}[-] \Vert_{\partial F} \leq \sum_{i \in M} \Vert \varepsilon[i] \Vert_1 + \lambda \Vert \hat{x}[-] \Vert_{\partial F}.
\]

By triangle inequality,

\[
\sum_{i \in M} \Vert \hat{x}[i] - y[i] \Vert_1 \leq \sum_{i \in M} \Vert \hat{x}[i] - x[i] \Vert_1 + \sum_{i \in M} \Vert x[i] - y[i] \Vert_1 \leq \sum_{i \in M} \Vert x[i] \Vert_2 + \sum_{i \in M} \Vert \varepsilon[i] \Vert_1,
\]

where we have used \( \Vert \hat{x}[i] \Vert_1 \geq \Vert \hat{x}[i] \Vert_2 \). Therefore,

\[
\max (0, \sum_{i \in M} \Vert x[i] \Vert_2 - \sum_{i \in M} \Vert \varepsilon[i] \Vert_1) \leq \sum_{i \in M} \Vert x[i] - y[i] \Vert_1.
\]

Applying (9) into (8) yields

\[
\lambda \Vert \hat{x}[-] \Vert_{\partial F} \leq \sum_{i \in M} \Vert \varepsilon[i] \Vert_1 + \lambda \Vert \hat{x}[-] \Vert_{\partial F},
\]

and

\[
\sum_{i \in M} \Vert \hat{x}[i] \Vert_2 + \lambda \Vert \hat{x}[-] \Vert_{\partial F} \leq \sum_{i \in M} \Vert \varepsilon[i] \Vert_1 + \lambda \Vert \hat{x}[-] \Vert_{\partial F}.
\]

Since we assume NNC holds for \( M \), inequality (13) applies to \( \hat{x}[-] \), i.e.,

\[
(1/K)(L/\sqrt{p}) \Vert \hat{x}[-] \Vert_{\partial F} \leq \sum_{i \in M} \Vert \hat{x}[i] \Vert_2 + (1/K) \Vert \hat{x}[-] \Vert_{\partial F}.
\]

Inserting (12) into (11) and using \( \lambda := 1/K \), yields

\[
\lambda(L/\sqrt{p} - 1) \Vert \hat{x}[-] \Vert_{\partial F} \leq \sum_{i \in M} \Vert \varepsilon[i] \Vert_1.
\]

Combining (11) with (12) yields

\[
\Vert \hat{x}[-] \Vert_\varepsilon = \Vert \hat{x}[-] \Vert_{\partial F} + \Vert \hat{x}[-] \Vert_{\partial F} \leq \frac{1}{\lambda} \sum_{i \in M} \Vert \varepsilon[i] \Vert_1 + 2 \Vert \hat{x}[-] \Vert_{\partial F} \leq \frac{1}{\lambda} + \frac{4 \sqrt{p} \lambda}{(L - \sqrt{p})} \sum_{i \in M} \Vert \varepsilon[i] \Vert_1.
\]
We highlight that the nLasso (4) does not require the partition $\mathcal{F}$ used for our signal model (1). This partition is only used for the analysis of nLasso (4). Moreover, if the true underlying graph signal is of the form (1) and nLasso accurately learns this signal (Theorem 2), we can obtain the partition $\mathcal{F}$ by thresholding the graph signal differences $|x[i] - y[i]|$ for $\{i, j\} \in \mathcal{E}$ [Wang et al., 2016].

The bound (6) characterizes the recovery error in terms of the semi-norm $\|x[\cdot] - x[\cdot]|_E$, and in general does not imply a small mean squared error. However, if $\|x[\cdot] - x[\cdot]|_E$ is small, we can identify the edges $\{i, j\}$ having large $\|x[i] - x[j]|_2$ to obtain underlying clusters $C_l$ (cf. (1)).

Our second main contribution, beside Theorem 2, is to relate the NCC (cf. Definition 1) to the network structure of the data graph $G$ via the existence of certain network flows [Kleinberg and Tardos, 2006].

Let us denote the neighborhood of node $i$ by $\mathcal{N}(i) := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\}$ and $[p]:=\{1, 2, \ldots, p\}$.

**Definition 3.** Consider a graph $G = (\mathcal{V}, \mathcal{E})$ with capacity matrix $C \in \mathbb{R}^{N \times N}$. A flow with demands $d[i] \in \mathbb{R}^p$, for $i \in \mathcal{V}$, is a mapping $h[\cdot] : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}^p$ satisfying, for any $k \in [p]$, 
\[
\sum_{j \in \mathcal{N}(i)} h_k(i, j) = d_k[i], \text{ for any } i \in \mathcal{V}, \text{ and } |h_k(i, j)| \leq C_{i,j} \text{ for any edge } \{i, j\} \in \mathcal{E}.
\]

We can characterize the network topology by verifying the existence of certain network flows. In particular, the next results relates the existence of certain network flows with the NCC.

**Lemma 4.** Consider a dataset with data graph $G = (\mathcal{V}, \mathcal{E})$, whose nodes are partitioned into clusters $\mathcal{F}$, capacity matrix $C \in \mathbb{R}^{N \times N}$ with $C_{i,j} = W_{i,j}$ for all edges $\{i, j\} \in \partial \mathcal{F}$ and $C_{i,j} = LW_{i,j}$ for $\{i, j\} \in \partial \mathcal{F}$, and a sampling set $\mathcal{M}$. If there exists, for any graph signal $z[\cdot]$ and any $k \in [p]$, a flow $h_k[\cdot]$ on $G$ with $h_k(i,j) = \text{sign}(z_k[i] - z_k[j])L \cdot W_{i,j}$ for $\{i, j\} \in \partial \mathcal{F}$ and demands $|d_k[i]| \leq K$ for every node $i \in \mathcal{M}$ and $d_k[i] = 0$ for every node $i \in \mathcal{V} \setminus \mathcal{M}$, then $\mathcal{M}$ satisfies the network compatibility condition with parameters $K, L > 0$.

**Proof.** For a graph signal $z[\cdot]$ we denote, for each edge $\{i, j\} \in \partial \mathcal{F}$,
\[
b_k(i,j) = -b_k(j,i) = \text{sign}(z_k[i] - z_k[j]) \quad \text{for each } k \in [p].
\]

Consider flows $h_k(i,j)$ on the data graph $G$ satisfying
\[
\sum_{j \in \mathcal{N}(i)} h_k(i,j) = 0 \quad \text{for all } i \notin \mathcal{M}, \quad \left| \sum_{j \in \mathcal{N}(i)} h_k(i,j) \right| \leq K \quad \text{for all } i \in \mathcal{M} \tag{15}
\]
\[
|h_k(i,j)| \leq W_{i,j} \quad \text{for all } \{i,j\} \in \partial \mathcal{F}, \quad h_k(i,j) = b_k(i,j)LW_{i,j} \quad \text{for all } \{i,j\} \in \partial \mathcal{F}. \tag{16}
\]

This yields, in turn,
\[
L\|z[\cdot]\|_{\partial \mathcal{F}} = \sum_{\{i,j\} \in \partial \mathcal{F}} \|z[i] - z[j]\|_2LW_{i,j} \leq \sum_{\{i,j\} \in \partial \mathcal{F}} \|z[i] - z[j]\|_1LW_{i,j} \tag{13,10}
\]
\[
= \sum_{k \in [p]} \sum_{\{i,j\} \in \partial \mathcal{F}} (z_k[i] - z_k[j])h_k(i,j). \tag{17}
\]

Since $\partial \mathcal{F} = \mathcal{E} \setminus \partial \mathcal{F}$, developing (17) yields
\[
L\|z[\cdot]\|_{\partial \mathcal{F}} \leq \sum_{k \in [p]} \left( \sum_{\{i,j\} \in \mathcal{E}} (z_k[i] - z_k[j])h_k(i,j) - \sum_{\{i,j\} \in \partial \mathcal{F}} (z_k[i] - z_k[j])h_k(i,j) \right)
\]
\[
= \sum_{i \in \mathcal{V}} \sum_{k \in [p]} z_k[i] \sum_{j \in \mathcal{N}(i)} h_k(i,j) - \sum_{\{i,j\} \in \partial \mathcal{F}} \sum_{k \in [p]} (z_k[i] - z_k[j])h_k(i,j). \tag{18}
\]

Applying (15), (16) into (18) yields further
\[
L\|z[\cdot]\|_{\partial \mathcal{F}} \leq K \sum_{i \in \mathcal{M}} \|z[i]\|_1 + \sum_{\{i,j\} \in \partial \mathcal{F}} W_{i,j} \|z[i] - z[j]\|_1 \leq \sqrt{p}(K \sum_{i \in \mathcal{M}} \|z[i]\|_2 + \|z[\cdot]\|_{\partial \mathcal{F}}).
\]

Thus, the condition (5) is verified. \qed
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