THE $\ell$-MODULAR REPRESENTATION OF REDUCTIVE GROUPS OVER FINITE LOCAL RINGS OF LENGTH TWO

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Abstract. Let $\mathcal{O}_2$ and $\mathcal{O}'_2$ be two distinct finite local rings of length two with residue field of characteristic $p$. Let $G(\mathcal{O}_2)$ and $G(\mathcal{O}'_2)$ be the group of points of any reductive group scheme $G$ over $\mathbb{Z}$ such that $p$ is very good for $G \times \mathbb{F}_q$. We prove that there exists an isomorphism of group algebra $K[G(\mathcal{O}_2)] \cong K[G(\mathcal{O}'_2)]$, where $K$ is a sufficiently large field of characteristic different from $p$.

1. Introduction

Let $\mathcal{O}$ be a discrete valuation ring with a unique maximal ideal $\mathfrak{p}$ and having finite residue field $\mathbb{F}_q$, the field with $q$ elements where $q$ is a power of a prime $p$. We denote $\mathcal{O}_r$ to be the reduction of $\mathcal{O}$ modulo $\mathfrak{p}^r$. Similarly, given $\mathcal{O}'$ a second discrete valuation ring with the same residue field $\mathbb{F}_q$, we can define $\mathcal{O}'_r$. The complex representation theory of general linear groups over the rings $\mathcal{O}_r$ has been heavily studied. In particular, it has been conjectured by Onn that there is an isomorphism of group algebras $\mathbb{C}[G(\mathcal{O}_r)] \cong \mathbb{C}[G(\mathcal{O}'_r)]$. When $r = 2$, this conjecture was proven by Singla.

Moreover, assuming $p$ is odd, Singla proved a generalization of the conjecture for $r = 2$ when $G$ is either $SL_n$ with $p \not| n$ or the following classical groups $Sp_n$, $O_n$, and $U_n$. Later on, Stasinski proved it for $SL_n$ for all $p$. More generally, for a possibly non-classical reductive group, Stasinski and Vera-Gajardo compared the complex representation of the two groups in question.

Let us briefly describe the result of Stasinski and Vera-Gajardo. First recall that for any group scheme $G$ over $\mathbb{Z}$ and any commutative ring $R$, we may speak of the group $\mathbb{G}(R)$ of $R$-points. We now let $G$ be a reductive group scheme over $\mathbb{Z}$ – for details on such group schemes see e.g. [2] and [3]. Note that for any root datum – see e.g. [2] – there is a split reductive group scheme $G$ over $\mathbb{Z}$ with this root datum [2]. Exp. XXV Theorem 1.1. The group $G \times \mathbb{F}_q$ obtained by base-change with the finite field $\mathbb{F}_q$ of characteristic $p$ is a reductive algebraic group over $\mathbb{F}_q$, and we want to insist that

$$\text{(*) } p \text{ is very good for } G \times \mathbb{F}_q;$$

we observe that the condition (*) depends only on the root datum of $G$ – see [12] Section 4] for the definition of good/very good primes.

Under assumption (*), Stasinski and Vera-Gajardo proved that $\mathbb{C}[G(\mathcal{O}_2)] \cong \mathbb{C}[G(\mathcal{O}'_2)]$. When $G \times \mathbb{F}_q$ is a classical absolutely simple algebraic group not of type $A$, any $p > 2$ is very good for $G$. When $G = SL_n$ or $SU_n$, $p$ is very good for $G$ if and only if $n \not\equiv 0 (\text{mod } p)$. If $G \times \mathbb{F}_q$ is absolutely simple of exceptional type, any $p > 5$ is very good for $G$; see [12] 4.3 for the precise conditions when $p \leq 5$.

We study the $\ell$-modular representation of such group scheme over $\mathcal{O}_2$ a local ring of length two with finite residue field. One can show that $\mathcal{O}_2$ is isomorphic to one of the following rings: $\mathbb{F}_q[t]/t^2$ or $W_2(\mathbb{F}_q)$, the ring of Witt vectors of length two over $\mathbb{F}_q$ [13]. Thus, we let $G_2 = G(\mathcal{O}_2)$ and $G'_2 = G(\mathcal{O}'_2)$, where $G$ stands for a reductive group scheme over $\mathbb{Z}$ such that $p$ is very good for $G \times \mathbb{F}_q$.

In this paper, we generalize the previous results over $\mathbb{C}$ to results over a sufficiently large field $K$ of characteristic $\ell \neq p$. More precisely, we prove that there exists an isomorphism of group algebras $K G_2 \cong K G'_2$ over a sufficiently large field $K$ of characteristic $\ell$ as long as $\ell \neq p$. We take $K$ to be a
sufficiently large so that the representation theory of the groups over \( K \) is the same as the representation theory over an algebraically closed field of characteristic \( l \).

In order for us to understand those two group algebras, we study their decomposition by block algebras. Given \( A \) a \( K \)-algebra, we define a block of \( A \) to be a primitive idempotent \( b \) in the center of \( A \), which is denoted as \( Z(A) \); the algebra \( Ab \) is called a block algebra of \( A \). By an idempotent, we mean an element \( b \) such that \( b^2 = b \) and primitive if \( b = b_1 + b_2 \) is an expression of \( b \) as a sum of idempotents such that \( b_1b_2 = 0 \), either \( b_1 = 0 \) or \( b_2 = 0 \). Moreover, the block algebra \( Ab \) is an indecomposable two-sided ideal summand of \( A \). Thus for a finite group \( G \), we can write

\[
KG = B_1 \oplus B_2 \oplus \cdots \oplus B_n
\]

where the \( B_i = e_iKG \) are the unique block algebras of \( KG \) up to ordering \([1]\). Moreover \( e_i \cdot e_j = 0 \) whenever \( i \neq j \).

We investigate blocks of \( KG \) which we denote as \( Bl(G) \). To understand the block algebra of those two group algebras, we exploit the fact that those two groups are extensions of \( G(\mathbb{F}_q) \) by an abelian \( p \)-group denoted as \( N \) which is isomorphic to the Lie algebra of \( G(\mathbb{F}_q) \) denoted as \( g \) \([15] \) Lemma 2.3\]. In fact, for the rest of this section let \( G \) denote either \( G_2 \) or \( G'_2 \). Let the map

\[
\rho : G \rightarrow G(\mathbb{F}_q)
\]

be the surjective map obtained from the map \( O_2 \rightarrow \mathbb{F}_q \) with \( N = \ker(\rho) \). The two groups \( G_2 \) and \( G'_2 \) act on \( g \), via its quotient \( G(\mathbb{F}_q) \). This action is transformed by the above isomorphism into the action of \( G \) on its normal subgroup \( N \), we explore the details of this action in section 3. In section 2, we use Clifford’s theory to relate blocks of \( G \) with blocks of \( N \). Specifically, we have that any block \( b \in Bl(G) \) is equal to \( T r^H_{H} (d) = \sum_{g \in G/H} gdg^{-1} \) for \( d \) in \( Bl(H) \), where \( H \) is the stabilizer in \( G \) for some block of \( N \). In section 4, we exploit the fact that Clifford theory, in combination with the results obtained in the characteristic zero case, gives an isomorphism between certain blocks algebra of the stabilizer of those two groups. In section 5, we induce the previous isomorphism to an isomorphism between block algebras of \( G_2 \) and \( G'_2 \) by taking advantage of the fact that blocks of a group algebra are interior \( G \)-algebras. This isomorphism will then give rise to an isomorphism between those two group algebras.

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2. Background on Clifford theory of blocks

Let \( G \) be a finite group and \( N \) a normal subgroup of \( G \). We use Clifford theory to relate blocks of \( KG \) with blocks of \( KN \). Given \( b \) a block of \( KG \) and \( e \) a block of \( KN \), we say \( b \) covers \( e \) if \( be \neq 0 \). We denote the set of blocks of \( G \) that covers \( e \) by \( Bl(G|e) \). Note that there is a natural action of \( G \) on the set of blocks of \( N \) by conjugation. Moreover one can show that for a fixed \( b \), the set of blocks \( e \) of \( KN \) satisfying \( be \neq 0 \) is a \( G \)-conjugacy class of blocks of \( KN \) \([4] \) Proposition 6.8.2\].

The following Clifford theorem for blocks of \( G \) holds:

**Theorem 2.1.** Let \( G \) be a finite group and \( N \) a normal subgroup of \( G \). Given a block \( b \in Bl(G) \), we have:

1. there exists a block \( e \in Bl(N) \) such that \( be \neq 0 \) so \( b \in Bl(G|e) \)
(2) if $e$ is as in (1), then $b = \text{Tr}_H^G(d) := \sum_{g \in G/H} g dg^{-1}$ for a unique block $d \in \text{Bl}(H|e)$ where $H$ is the stabilizer of $e$ in $G$ under the conjugation action.

(3) The assignment $d \mapsto \text{Tr}_H^G(d) = b$ gives a bijection between $\text{Bl}(H|e)$ and $\text{Bl}(G|e)$.

For more information about Clifford theory of blocks consult Linckelmann [1], Nagao [6], or Craven [1]. Thus in order for us to understand the block algebra of those two groups, we start by investigating the block structure of $eKH$ for a fixed block $e \in \text{Bl}(N)$, where as above $H$ is the stabilizer of $e$ in $G$, under the conjugation action.

Let $\text{Irr}_K(N)$ be the set of irreducible characters of $KN$. Since $l$ does not divide $|N|$ (because $N$ is a $p$-group), any block of $KN$ is defined by $e = e_\chi = \frac{\chi(1)}{|N|} \sum_{g \in N} \chi(g^{-1})g$, for some $\chi$ in $\text{Irr}_K(N)$ [7]. Specifically, $KN$ is a semisimple algebra and $eKN$ is a matrix algebra, so it has defect zero. Thus, we can take advantage of the following proposition to show that $eKH \cong eKN \otimes_K K^{\alpha-1}$. For an $\alpha$ in $Z^2(G; K^*)$, we denote $K^\alpha G$ as the twisted group algebra of $G$ by $\alpha$. It has basis as $K$-algebra the elements of $G$ and given $x, y \in G$, we define $x \cdot y = \alpha(x, y)xy$ where $xy$ is the product of $x$ and $y$ in $G$. Furthermore, given a different $\beta$ in $Z^2(G; K^*)$ there is a $G$-graded algebra isomorphism $K^\alpha G \cong K^\beta G$ if and only if the classes of $\alpha$ and $\beta$ are equal in $H^2(G; K^*)$ [2]. Thus, for the purpose of this paper $\alpha$ is defined up to an element in the second cohomology group $H^2(G; K^*)$ with $G$ acting trivially on $K^*$.

**Proposition 2.2.** (Theorem 6.8.13, Linckelmann [1]) Let $H$ be a finite group, $N$ a normal subgroup of $H$ and $e$ a $H$-stable block of defect zero of $KN$. Set $S = KNe$ and suppose that $K$ is a splitting field for $S$. Let $L = H/N$. For any $x \in H$ there is $s_x \in S^*$ such that $xtx^{-1} = s_xt(s_x)^{-1}$ for all $t \in S$ and such that $s_x s_y = s_{xy}$ if at least one of $x, y$ is in $N$. Then the 2-cocycle $\alpha \in Z^2(H; K^*)$ defined by $s_x s_y = \alpha(x, y)s_{xy}$ for $x, y \in N$ depends only on the images of $x, y$ in $L$ and induces a 2-cocycle, still denoted $\alpha$, in $Z^2(L; K^*)$, and we have an isomorphism of $K$-algebras

$$KHe \cong S \otimes_K K^{\alpha-1}$$

sending $xe$ to $s_x \otimes \bar{x}$, where $x \in H$ and $\bar{x}$ is the image of $x$ in $L$.

3. The action of $G_2$ and $G_2'$ on the kernel $N$

In this section, we study the action of $G$ on the kernel $N$ so we can understand the stabilizer $H$ mod $N$ and thus we can take advantage of the Proposition 2.2. The two groups $G_2$ and $G_2'$ act on $g$, via its quotient $G(\mathbb{F}_q)$. Particularly, there is a natural adjoint action of $G(\mathbb{F}_q)$ on $g$:

$$\text{Ad} : G(\mathbb{F}_q) \rightarrow \text{GL}(g).$$

Also there is an automorphism of $\sigma : G(\mathbb{F}_q) \rightarrow G(\mathbb{F}_q)$ given by raising each matrix entry to the $p$-th power. Note that $\sigma$ composed with $\text{Ad}$ gives rise to a second action of $G(\mathbb{F}_q)$ on $g$. Thus, given $X \in g$, we note that the action of $G_2 = G(\mathbb{F}_q[t]/t^2)$ and $G_2' = G(\text{W}_2, \mathbb{F}_q)$ are as follow:

$$g \cdot_1 X = \text{Ad}(\bar{g})X \quad \text{for } g \in G_2$$

$$g \cdot_2 X = \text{Ad}(\sigma(\bar{g}))X \quad \text{for } g \in G_2'.$$

where $\bar{g} = \rho(g)$, see [15 Section 2.3] for more details. Moreover, we also have actions of both $G_2$ and $G_2'$ on $g^* = \text{Hom}_{\mathbb{F}_q}(g, \mathbb{F}_q)$ and $\text{Irr}_K(g)$. Given $F$ an element of $g^*$ or $\text{Irr}_K(g)$ and $X \in g$, we define:

$$(g \cdot_1 F)(X) = F(g^{-1} \cdot_1 X) \quad \text{for } g \in G_2$$

$$(g \cdot_2 F)(X) = F(g^{-1} \cdot_2 X) \quad \text{for } g \in G_2'.$$
Lemma 3.1. Let \( \chi \) and \( \tau \) be characters of \( g \) then \( g \chi = \tau \) for some \( g \in G' \) if and only if there is \( h \in G_2 \) such that \( h \chi = \tau \).

Proof. Let \( g \in G' \) and \( h \in \rho^{-1}(\sigma(g^{-1})) \subset G_2 \) then we have:
\[
    g \chi(X) = (g \cdot \chi)(X) = \chi(g^{-1} \cdot e X) = \chi(\rho^{-1}(\sigma(g^{-1})) \cdot e X) = h \chi(X).
\]

Given a non-trivial irreducible character \( \phi : F_q \to K^* \) and for each \( \beta \in g^* \), we define the character \( \chi_{\beta} \in \text{Irr}(g) \) by
\[
    \chi_{\beta}(X) = \phi(\beta(X)).
\]

Lemma 3.2. [15] Lemma 4.1] The map \( \beta \mapsto \chi_{\beta} \) defines an isomorphism of abelian group \( g^* \to \text{Irr}(g) \) that is \( G \)-invariant i.e. for \( g \in G \) we have
\[
    g \cdot \beta \mapsto \chi_{\beta} \quad \text{or} \quad g \cdot \beta \mapsto g \cdot \chi_{\beta}.
\]

Note, there is also an isomorphism of \( g \cong N \) which is \( G \)-invariant [15, Lemma 2.3.]. Thus, we can identify the characters of \( N \) with characters of \( g \). Given a character \( \chi \) of \( N \), we define \( H_2 \) and \( H'_2 \) to be the stabilizer of \( \chi \) in \( G_2 \) and \( G'_2 \) respectively. Thus, we have the following lemma about the stabilizer of \( e \chi \).

Lemma 3.3. The stabilizer \( H_2/N \) and \( H'_2/N \) of \( e \chi \) are isomorphic.

Proof. Since \( g \cdot e \chi = e \chi \), the stabilizer of \( e \chi \) is the same as the stabilizer of \( \chi \) [7, Section 9]. Thus, we compute the stabilizer of \( \chi \). By the discussion above, we can consider \( \chi \) to be in \( \text{Irr}(g) \), by Lemma 3.2 we have that \( \chi = \chi_{\beta} \) for some \( \beta \in g^* \). Thus:
\[
    H'_2/N = \{ g \in \text{GL}_{n}(F_q) | \rho^{-1}(\sigma(g)) \cdot \beta = \beta \} = \{ g \in \text{GL}_{n}(F_q) | \rho^{-1}(\sigma(g)) \cdot 1 = \beta \} = \sigma^{-1}(H_2/N).
\]

Since the map \( \sigma \) is automorphism, it follows that \( H_2/N \cong H'_2/N \).

4. The isomorphism of \( eKH_2 \) and \( eKH'_2 \)

In this section, we prove that there is an isomorphism of blocks of \( eKH_2 \) and \( eKH'_2 \), where \( e = e \chi \) for \( \chi \) in \( \text{Irr}_K(N) \) such that \( H_2 \) and \( H'_2 \) are the stabilizer of \( e \) in \( G_2 \) and \( G'_2 \) respectively. To prove this isomorphism, we take advantage of Proposition 2.2. In order to do so, we need to understand the cofactors \( \alpha \) associated to \( eKH_2 \) and \( eKH'_2 \). We prove that the cofactor \( \alpha \) is trivial in both cases. For this, we introduce projective \((K)\)-representation of \( G \) with factor set \( \alpha \). By which, we mean a map \( X : G \to \text{GL}_n(K) \) such that \( X(x)X(y) = \alpha(xy)X(xy) \) for all \( x, y \) in \( G \), where \( \alpha(xy) \) is in \( K^* \). In fact, one can check that \( \alpha \) is in \( Z^2(G; K^*) \), where we assume that \( G \) acts on \( K^* \) trivially. Similarly, one can define a projective representation as a \( K\alpha G \)-module [6]. We let \( H \) denote either \( H_2 \) or \( H'_2 \). The projective representations of \( H \) are closely related to Clifford theory. In fact, we can use proposition 2.2 to show that there is a projective representation \( V \) of \( H \) that extends \( \chi \). By that we mean that \( V \) viewed as \( KN \)-module is isomorphic to the \( KN \)-module associated to \( \chi \).
Proposition 4.1. The α, obtained from proposition 2.2 associated to the block e_χ of KN gives rise to a projective representation of H that extends χ.

Proof. Let Y be the representation associated to χ, i.e. Y : N → GL_n(K). Let R be a set of representative of L = H/N in H and S = KN e. For each x ∈ R, by proposition 2.2 there is a s_x ∈ S^* such that xtx^{-1} = s_x t(s_x)^{-1} for all t ∈ S. Note also since e is a H-stable block of defect zero of KN, then S = KN e ∼= M_n(K). With abuse of notation, we can think of s_x as an element of GL_n(K). Now define X to be map X : H → GL_n(K) such that for each h = xn ∈ H, we have X(h) = s_x Y(n) with x ∈ R and n ∈ N. One can check that X as defined above is a projective representation of H with factor set α, in Z^2(L; K^*) that extends Y.

In order to understand α from proposition 2.2 which is associated to a projective (K)-representation of G, we study the projective representation of H with factor set α that extends χ over a field of characteristic zero. To relate them, we need an l-modular system. By this we mean a triple (F; R; K) where F is a field of characteristic zero equipped with a discrete valuation, R is the valuation ring in F with maximal ideal (π), and K = R/(π) is the residue field of R, which is required to have characteristic l. If both F and K are splitting fields for G we say that the triple is a splitting l-modular system for G. Note, we need an l-modular system so we can relate representation over a field F of characteristic zero to representation over a field K of characteristic l. The following lemma shows that given a projective representation over F, we can obtain a projective representation over R. Notice this lemma is just a generalization of the already known fact over group algebras that can be extended to hold over twisted group algebras. In fact the prove is the same, see [9, Lemma 2.2.2].

Lemma 4.2. Let G be a finite group and M be a F^αG-module with α ∈ Z^2(G; R^*) then M contains a lattice L that is a R^αG – module.

Proof. Note by [9, Lemma 2.2.1] to show that L is a lattice of M it is enough to show that L is finitely generated as an R-module and L generates M as a F-vector space. Thus, pick a F basis e_1, ..., e_n of M then let L' := Re_1 + ... + Re_n a lattice of M. Let L := \sum_{g \in G} gL' then L is a R^αG – module. Note L is finitely generated as an R-module by \{ge_i : 1 \leq i \leq n, g \in G\} and it also generates M as vector space. Thus L is a G-invariant lattice.

Before we introduce the following theorem, recall that a block of defect zero may be defined as a matrix algebra. Moreover, by the following Proposition 4.3 such a block is a ring summand of KG which has a projective simple module.

Proposition 4.3. (Theorem 9.6.1, Webb [17]). Let (F; R; K) be a splitting p-modular system in which R is complete, and let G be a group of order p^(d+e)q where q is prime to p. Let T be an FG-module of dimension n, containing a full RG-sublattice T_0. The following are equivalent:

(1) p^d | n and T is a simple FG-module.

(2) The homomorphism RG → End_R(T_0) that gives the action of RG on T_0 identifies End_R(T_0) ∼= M_n(R) with a ring direct summand of RG.

(3) T is a simple FG-module and T_0 is a projective RG-module.

(4) The homomorphism KG → End_K(T_0/πT_0) identifies End_K(T_0/πT_0) ∼= M_n(K) with a ring direct summand of KG.

(5) As a KG-module, T_0/πT_0 is simple and projective.
Most importantly, by Proposition 4.3 a block of defect zero can have only one simple module and there is a unique ordinary simple module that reduces to it. This is used in the proof of the following theorem.

**Theorem 4.4.** Given a $H$-stable block $e_\chi$ of defect zero of $KN$ where $N \triangleleft H$. Let $V$ be the unique ordinary simple module associated to this block. According to Proposition 4.4, let $\hat{V}$ be the projective representation of $H$ that extends $V$, with cofactor $\hat{\alpha}$ in $Z^2(H/N; R^*)$. Any $H$-invariant lattice of $\hat{V}$ call it $L$, gives rises to a $K^\alpha H$-module that extends the simple projective $KN$-module associated to $e_\chi$, where $\alpha = \hat{\alpha} \mod (\pi)$.

**Proof.** Let $\hat{V}$ be as above so $\hat{V}$ is a $F^\alpha H$-module such that $\hat{V}_{\downarrow N}^H \simeq V$ as a $FN$-module. Now by Lemma 4.2 take $L$ to be an $H$-invariant lattice of $\hat{V}$ then consider its reduction to a $K^\alpha H$-module, call it $\bar{L} := K \otimes_R L$. Note that $L$ is also a $N$-invariant lattice of $V$, the unique simple ordinary module associated to the block $e_\chi$. Thus, by Proposition 4.3 the reduction of $L$ is a simple module of $KN$. Therefore, $\bar{L}_{\downarrow N}^H$ is isomorphic to the simple module associated to this block. One can conclude that $\bar{L}$ is a $K^\alpha H$-module that extends the simple projective module associated to $e_\chi$, where $\alpha = \hat{\alpha} \mod (\pi)$. □

**Proposition 4.5.** The $\alpha$ obtained from Theorem 4.4 is trivial.

**Proof.** Note by work of Stasinski and Vera-Gajardo, it was proven that any $\chi$ element of $\text{Irr}_F(N)$ extends to it is inertia group $H$ [15, Proposition 4.5]. Thus by Proposition 4.1 the cofactor associated to the extension of $\chi$ is trivial over $F$, i.e. $\hat{\alpha} = 1$. By Theorem 4.4, there is a $K^\alpha H$-module $\bar{L}$ that extends the simple projective module associated to $e_\chi$, where $\alpha = \hat{\alpha} \mod (\pi) = 1$. Thus $\alpha$ is trivial. □

We recall the following result from Lemma 3.3 that the stabilizer $H_2$ and $H'_2$ of $\chi$ are isomorphic mod $N$ i.e. $H_2/N \simeq H'_2/N$. We will denote $L := H_2/N \simeq H'_2/N$ for this quotient.

**Theorem 4.6.** The following $K$-Algebras are isomorphic $eKH_2$ and $eKH'_2$.

**Proof.** We can apply Proposition 2.2 since $e$ is $H$-stable block of defect zero of $KN$. Thus, we have an isomorphism of $K$-algebras

$$\Phi : eKH_2 \cong eKN \otimes_K KL \cong eKH'_2$$

since by Proposition 4.5 $\alpha^{-1}$ is trivial. □

5. The Isomorphism of $KG_2$ and $KG'_2$

The following results about interior $G$-algebras will be useful in order to prove the isomorphism of those two group algebras. Note first that a $G$-algebra over a field $K$ is an $K$-algebra $A$ together with an action of $G$ on $A$ by $K$-algebra automorphisms. An interior $G$-algebra is a $G$-algebra where the action of $G$ is given by inner automorphism. The example to keep in mind is that $KG$ and $bKG$ are interior $G$-algebra with $G$ acting by conjugation, where $b \in Bl(G)$.

Given $H$ a subgroup of $G$ and $B$ an interior $H$-algebra, we define $\text{Ind}_H^G(B)$ to be the $K$-module $KG \otimes_{KH} B \otimes_{KH} KG$ and one can put an interior $G$-algebra structure on $\text{Ind}_H^G(B)$. For more details on the interior $G$-algebra structure on $\text{Ind}_H^G(B)$ one can consult Thévenaz’s book on $G$-algebras and
modular representation theory \[16\]. In fact, the following lemma shows that as a $K$-algebra one can think of $\text{Ind}_H^G(B)$ as a matrix algebra over $B$.

Lemma 5.1. (Lemma 16.1, Thévenaz \[16\]) Let $H$ be a subgroup of $G$ of index $n$, and $B$ be an interior $H$-algebra. Then we have $\text{Ind}_H^G(B) \cong M_n(B)$ as $K$-algebras.

In the following proposition, we will see that given certain conditions there is a way of relating the algebra obtained by the induction from $H$ to $G$ of a certain $H$-algebra with the algebra obtained by the idempotent $\text{Tr}_H^G(i)$ where $i$ is an idempotent fixed by $H$. Given $A$ an interior $G$-algebra, we will denote the set of elements of $A$ fixed by $H$ as $A^H$ and $1_A$ as the multiplicative identity of $A$.

Proposition 5.2. (Proposition 16.6, Thévenaz \[16\]) Let $A$ be an interior $G$-algebra and let $H$ be a subgroup of $G$. Assume that there exists an idempotent $i \in A^H$ such that $1_A = \text{Tr}_H^G(i)$ and $i^g = 0$ for all $g \in G - H$. Then there is an isomorphism of interior $G$-algebras

$$F : \text{Ind}_H^G(iA) \cong A;$$

Given by $x \otimes b \otimes y \mapsto x \cdot b \cdot y$ ($x, y \in G, b \in iAi$).

We use Proposition 5.2 to prove the following:

Proposition 5.3. Given a block $b$ of $KG_2$, there is a block $\hat{\Phi}(b)$ of $KG'_2$ such that the following $K$-algebras $bKG_2$ and $\hat{\Phi}(b)KG'_2$ are isomorphic.

Proof. Fix $\chi \in \text{Irr}(N)$ up to conjugation by $G$, and let $e = e_\chi$ a primitive idempotent of $KN$, now fix a block $b \in Bl(G_2|e)$ such that by Clifford Theorem $2.1$ $b = \text{Tr}^{G_2}_{H_2}(d)$ for some primitive idempotent $d$ of $KH_2$ where $H_2$ is the stabilizer of $e$ in $G_2$. Notice by Theorem 4.6 we have that $\Phi : eKH_2 \cong eKH'_2$ as $K$-algebras. Now the map $\Phi$ gives a bijection between $Bl(H_2|e)$ and $Bl(H'_2|e)$ such that $\Phi : dKH_2 \cong \Phi(d)KH'_2$ for each $d \in Bl(H_2|e)$. Moreover, Clifford Theorem $2.1$ tells us there is a bijection between $Bl(G_2|e)$ and $Bl(H_2|e)$. Thus, we obtain a bijection between $Bl(G_2|e)$ and $Bl(G'_2|e)$ by the map $b \mapsto \hat{\Phi}(b) = \text{Tr}^{G_2}_{H'_2}(\Phi(d))$.

Let $A = bKG_2$ so $1_A = b = \text{Tr}^{G_2}_{H'_2}(d)$. Note $d \cdot b = d$ so $d \in A^H_2$ and $d^gd = 0$ for $g \in G_2 - H_2$ \[1\]. In proof of Lemma 6.8.4]. Thus, we have that

$$dAd = dbKG_2d = dKG_2d = dKH_2d = dKH_2.$$

Thus by Proposition 5.2 there is an isomorphism of interior $G_2$-algebra:

$$\text{Ind}_{H_2}^{G_2}(dKH_2) \cong bKG_2.$$

Note by Lemma 5.1 that $\text{Ind}_{H_2}^{G_2}(dKH_2) \cong M_n(dKH_2)$ as $K$-algebras, with $n = |G_2 : H_2| = |G'_2 : H'_2|$. Since, $\Phi : dKH_2 \cong \Phi(d)KH'_2$ conclude that

$$\hat{\Phi} : bKG_2 \cong \text{Ind}_{H_2}^{G_2}(dKH_2) \cong \text{Ind}_{H_2}^{G_2}(\Phi(d)KH'_2) \cong \hat{\Phi}(b)KG'_2.$$

Thus we have that $bKG_2$ and $\hat{\Phi}(b)KG'_2$ are isomorphic as $K$-algebras. \qed
With the above $K$-algebra isomorphism, we define a map $\Psi : KG_2 \to KG'_2$ such that $\Psi(a) = \sum_{b \in Bl(G_2)} \hat{\Phi}(a \cdot b)$ for $a \in KG_2$ and show that this map is an isomorphism of $K$-algebras.

**Theorem 5.4.** Let $G_2 = \mathbb{G}(O_2)$ and $G'_2 = \mathbb{G}(O'_2)$, be the group of points of any reductive group scheme $\mathbb{G}$ over $\mathbb{Z}$ such that $p$ is very good for $\mathbb{G} \times \mathbb{F}_q$. There exists an isomorphism of group algebra $K[\mathbb{G}(O_2)] \cong K[\mathbb{G}(O'_2)]$, where $K$ is a sufficiently large field of characteristic different from $p$.

**Proof.** Define the map $\Psi : KG_2 \to KG'_2$ by $\Psi(x) = \sum_{b \in Bl(G_2)} \hat{\Phi}(xb)$ for $x \in KG_2$. First, we show that $\Psi$ is an algebra homomorphism. Note that by definition it is a linear map since each of the $\hat{\Phi}$ from Proposition 5.3 are. Now given $x$ and $y$ both in $KG_2$, we want to show that $\Psi(xy) = \Psi(x)\Psi(y)$. To prove this it is sufficient to show that different blocks of $G_2$ are mapped to different blocks of $G'_2$.

Because if they are, we have that

$$\Psi(x)\Psi(y) = \sum_{b \in Bl(G_2)} \hat{\Phi}(xb) \cdot \sum_{b \in Bl(G_2)} \hat{\Phi}(yb) = \sum_{b \in Bl(G_2)} \hat{\Phi}(xb) \cdot \hat{\Phi}(yb).$$

Since the products of elements of different blocks is zero. Moreover by $\hat{\Phi}$ being a $K$-algebra homomorphism for each block $b$ we have:

$$\Psi(x)\Psi(y) = \sum_{b \in Bl(G_2)} \hat{\Phi}(xb) \cdot \hat{\Phi}(yb) = \sum_{b \in Bl(G_2)} \hat{\Phi}(xbyb) = \sum_{b \in Bl(G_2)} \hat{\Phi}(xyb) = \Psi(xy)$$

Now to show that different blocks of $G_2$ are mapped to different blocks of $G'_2$ we look at two cases. Let $b$ and $c$ be two different blocks of $G_2$:

**Case 1:** Assume $b$ and $c$ cover the same block $e$ of $KN$. Thus, by Clifford Theorem 2.1 $b = Tr_{H_2}^{G_2}(d_1)$ and $c = Tr_{H_2}^{G_2}(d_2)$ such that $d_1 \cdot d_2 \neq 0$. Recall by Theorem 4.6 that the map $\Phi : KH_2 \cong KH'_2$ is a $K$-algebra isomorphism, thus $\Phi(d_1)\Phi(d_2) = 0$. Applying Clifford Theorem 2.1 for the block of $KG'_2$ over $e$, we have that $Tr_{H_2}^{G'_2}(\Phi(d_1)) \neq Tr_{H_2}^{G'_2}(\Phi(d_2))$, since $\Phi(d_1) \neq \Phi(d_2)$. Thus $\hat{\Phi}(b) \neq \hat{\Phi}(c)$.

**Case 2:** Assume $b$ and $c$ cover different blocks $e_1$ and $e_2$ of $KN$ respectively. Note to prove that $\hat{\Phi}(b) \neq \hat{\Phi}(c)$, it is enough to show that they cover different blocks of $KN$ as blocks of $G'_2$. Assume the opposite. By definition of the map $\hat{\Phi}$, we also have that $\hat{\Phi}(b)$ and $\hat{\Phi}(c)$ cover the blocks $e_1$ and $e_2$ of $KN$ respectively. Thus by [1] Proposition 6.8.2, we have that $e_1$ and $e_2$ are $G'_2$-conjugated blocks. Since $KN$ is a semisimple algebra i.e all the blocks have defect zero, $e_1$ and $e_2$ are defined by a unique ordinary character $\chi$ and $\tau$ of $KN$. Thus $e_1$ and $e_2$ are $G'_2$-conjugated blocks if and only if $\chi$ and $\tau$ are $G'_2$-conjugated characters. By Lemma 3.1 we can conclude that $\chi$ and $\tau$ are $G'_2$-conjugated and thus so are $e_1$ and $e_2$. This is a contradiction with the fact that $b$ and $c$ cover different blocks $e_1$ and $e_2$ of $KN$.

Thus, we can conclude that different blocks of $G_2$ are mapped to different blocks of $G'_2$. Therefore, we have an algebra homomorphism from $\Psi : KG_2 \to KG'_2$ and since each $\hat{\Phi}$ is injective and maps different blocks to different blocks, we can conclude $\Psi$ is an injective map. Thus by dimension reasons, $\Psi$ is also surjective. Therefore, $KG_2 \cong KG'_2$ as $K$-algebras.
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