CONVERGENCE OF NONLOCAL THRESHOLD DYNAMICS APPROXIMATIONS TO FRONT PROPAGATION

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ABSTRACT. In this note we prove that appropriately scaled threshold dynamics-type algorithms corresponding to the fractional Laplacian of order $\alpha \in (0, 2)$ converge to moving fronts. When $\alpha \geq 1$ the resulting interface moves by weighted mean curvature, while for $\alpha < 1$ the normal velocity is nonlocal of “fractional-type.” The results easily extend to general nonlocal anisotropic threshold dynamics schemes.

INTRODUCTION

We study here the convergence of a class of threshold dynamics-type approximations to moving fronts. Although the arguments extend easily to general anisotropic kernels to keep the presentation simple, here we concentrate on a particular isotropic case, namely, fractional Laplacian of order $\alpha \in (0, 2)$. The resulting interfaces move either by weighted mean curvature, if $\alpha \in [1, 2)$, or by a nonlocal fractional-type normal velocity, if $\alpha \in (0, 1)$.

Threshold dynamics is a general term used to describe approximations to motion of boundaries of open sets in $\mathbb{R}^N$ by “measuring” interactions with the environment. The general scheme we consider here is described as follows:

Let $\Omega_0$ be an open subset of $\mathbb{R}^N$ with boundary $\Gamma_0$. The goal is to come up with an explicit approximation evolution scheme with time step $h > 0$, so that, as $nh \to t$, the “approximate” front $\Gamma_{nh}^h$, which is the boundary of an open set $\Omega_{nh}$ identified as the level set of a sign-function, converges, in a suitable sense, to a moving front $\Gamma_t$, the boundary of an open set $\Omega_t$, and to identify the limiting velocity. For each $n \in \mathbb{N}$, let

$$\Gamma_{nh}^h = \partial \{ x \in \mathbb{R}^N : u_h(\cdot, nh) = 1 \} ,$$

where

$$u_h(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c} \text{ in } \mathbb{R}^N ,$$ (0.1)

and, for $n \geq 1$,

$$u_h(\cdot, (n + 1)h) = \text{sign} (J_h * u_h(\cdot, nh)) .$$ (0.2)

Here $\text{sign}(t) = 1$ if $t > 0$ and $-1$ otherwise, $1_A$ denotes the characteristic function of $A \subset \mathbb{R}^N$, and, for $\alpha \in (0, 2)$,

$$J_h(x) = p_\alpha(x, \sigma_\alpha(h)) ,$$ (0.3)
where $p_{\alpha}$ is the fundamental solution of the fractional Laplacian pde

\begin{equation}
W_{t} - L^{\alpha}W = 0 \text{ in } \mathbb{R}^{N} \times (0, \infty),
\end{equation}

with

\begin{equation}
L^{\alpha}W(x) = \int |y - x|^{-(N+\alpha)}(W(y) - W(x)) \, dy.
\end{equation}

Hence, at each time step, we solve the equation (0.4) with initial datum

\[ W(\cdot, 0) = u_{h}(\cdot, nh) \text{ in } \mathbb{R}^{N}, \]

and for time $\sigma_{\alpha}(h)$. Then we define $u_{h}(\cdot, (n + 1)h)$ by

\[ u_{h}(x, (n + 1)h) = \begin{cases} 
1 & \text{if } W(x, \sigma_{\alpha}(h)) > 0, \\
-1 & \text{otherwise}.
\end{cases} \]

The algorithm generates functions $u_{h}(\cdot, nh)$ and open sets $\Omega_{nh}^{h}$ defined by

\[ \Omega_{nh}^{h} = \{x \in \mathbb{R}^{N} : J_{h} * u_{h}(\cdot, (n - 1)h)(x) > 0\} \text{ and } u_{h}(\cdot, nh) = 1_{\Omega_{nh}^{h}} - 1_{(\Omega_{nh}^{h})^{c}} \text{ in } \mathbb{R}^{N}. \]

We prove that, when $h \to 0$, the discrete evolution $\Gamma_{0} \to \Gamma_{h}^{\Omega_{nh}^{h}} = \partial\Omega_{nh}^{h}$ converges, in a suitable sense, to the motion $\Gamma_{0} \to \Gamma_{t}$ with nonlocal fractional normal velocity, if $\alpha \in (0, 1)$, and normal velocity equal to a multiple of the mean curvature, if $\alpha \in [1, 2]$.

If the kernel $J$ is a Gaussian, it is a classical result that the algorithm generates movement by mean curvature — see below for an extensive discussion and references.

To state the main result we recall that the geometric evolution of a front $\Gamma_{t} = \partial\Omega_{t}$ with normal velocity $v(Dn, n, \Omega_{t})$, starting at $\Gamma_{0} = \partial\Omega_{0}$, is best described by “the level set” partial differential equation

\begin{equation}
\begin{cases}
 u_{t} + F(D^{2}u, Du, \{u(\cdot, t) \geq u(x, t)\}, \{u(\cdot, t) \leq u(x, t)\}) = 0 \text{ in } \mathbb{R}^{N} \times (0, \infty), \\
 u = g \text{ on } \mathbb{R}^{N} \times \{0\},
\end{cases}
\end{equation}

with $g$ such that

\begin{equation}
\Omega_{0} = \{x \in \mathbb{R}^{N} : g(x) > 0\} \text{ and } \Gamma_{0} = \{x \in \mathbb{R}^{N} : g(x) = 0\},
\end{equation}

and

\[ F(D^{2}u, Du, \{u(\cdot, t) \geq u(x, t)\}, \{u(\cdot, t) \leq u(x, t)\}) = |Du|v(-D(\frac{Du}{|Du|}), -\frac{Du}{|Du|} \{u(\cdot, t) \geq u(x, t)\}, \{u(\cdot, t) < u(x, t)\}). \]

The basic fact of the level set approach is that the sets $\Omega_{t} = \{x \in \mathbb{R}^{N} : u(x, t) > 0\}$ and $\Gamma_{t} = \{x \in \mathbb{R}^{N} : u(x, t) = 0\}$ are independent of the choice of the initial datum $g$ provided the latter is positive in $\Omega_{0}$ and zero in $\Gamma_{0}$.

The weighted mean curvature motion corresponds to the level set pde

\begin{equation}
\begin{array}{c}
u_{t} - C_{\alpha} \text{tr}(I - \hat{D}u \otimes \hat{D}u)D^{2}u = 0 \text{ in } \mathbb{R}^{N} \times (0, \infty),
\end{array}
\end{equation}
where, for \( p \in \mathbb{R}^N \setminus \{0\} \), \( \hat{p} = p/|p| \), while the equation corresponding to the nonlocal motion is
\[
(0.9) \quad \frac{\partial u}{\partial t} - C_\alpha |D u| \int_0^\infty \left( \mathbf{1}^+(u(x + y, t) - u(x, t)) - \mathbf{1}^-(u(x + y, t) - u(x, t)) \right) |y|^{-(N+\alpha)} \, dy = 0 ,
\]
where \( \mathbf{1}^+ \) and \( \mathbf{1}^- \) denote respectively the characteristic functions of \([0, \infty)\) and \((\infty, 0)\) and, in both cases, \( C_\alpha \) is an explicit constant specified later in the paper.

Although the heuristic meaning of (0.8) is well known, some discussion about (0.9) is in order. It is implicit in (0.9) that there are sufficient cancellations in the term \( \mathbf{1}^+(u(x + y, t) - u(x, t)) - \mathbf{1}^-(u(x + y, t) - u(x, t)) \) to compensate for the lack of integrability of the kernel \( y \mapsto |y|^{-N-\alpha} \) at the origin. Indeed if we write the integral in polar coordinates,
\[
\int_0^\infty r^{-(1+\alpha)} \int_{S^1} \mathbf{1}^+(u(x + r\sigma) - u(x + r\sigma)) \, d\sigma ,
\]
we see that the spherical integral measures the “deviation” of \( \partial \Omega \), where \( \Omega = \{ y \in \mathbb{R}^N : u(y + x, t) \geq u(x, t) \} \), to be perfectly balanced between positive and negative parts, a deviation that infinitesimally is given by the mean curvature of \( \partial \Omega \). Observe also that if the surface \( \{ x \in \mathbb{R}^N : u(y + x, t) = u(x, t) \} \) is smooth and \( |D u| \neq 0 \) on it, then an integration by parts leads to
\[
\frac{\partial u}{\partial t} + \frac{C_\alpha}{\alpha} |D u| \int_{\{ y \in \mathbb{R}^N : u(x+y, t) = u(x, t) \}} \frac{y}{|y|^{N+\alpha}} \cdot \frac{D u(x + y, t)}{|Du(x + y, t)|} \, d\Sigma(y) ,
\]
where \( d\Sigma \) is the \((N-1)\)-surface measure.

To state the main result of the paper we recall that, given a bounded sequence \((u_h(\cdot , nh))_{n \in \mathbb{N}}\) of bounded functions, the “half relaxed” limits \( u^* \) and \( u_* \) are defined by
\[
(0.10) \quad \begin{cases} 
  u^*(x, t) = \lim \sup_{y \to x, \, nh \to t} u_h(y, nh) , \\
  u_*(x, t) = \lim \inf_{y \to x, \, nh \to t} u_h(y, nh) .
\end{cases}
\]

It is immediate that \( u_* \leq u^* \) and, more importantly, if \( \lim \inf u_h = \lim \sup u_h \), then, as \( h \to 0 \), \( u_h \to u \) locally uniformly.

The result is:

**Theorem.** Assume that \( \Gamma_0 = \partial \Omega_0 = \partial(\mathbb{R}^N \setminus \overline{\Omega_0}) \) and consider the family \((u_h(\cdot , nh))_{n \in \mathbb{N}}\) defined by (0.1) and (0.2) with \( \sigma_\alpha(h) = h^\alpha/2 \), if \( \alpha \in (1, 2) \), \( \alpha = \sigma_1^2(h) \), if \( \alpha = 1 \), and \( \sigma_\alpha(h) = h^{1+\alpha} \), if \( \alpha \in (0, 1) \). Let \( \Omega_t = \{ x \in \mathbb{R}^N : u(x, t) > 0 \} \) and \( \Gamma_t = \{ x \in \mathbb{R}^N : u(x, t) = 0 \} \), where, for some uniformly continuous \( g \) such that \( g(\Omega_0) = \{ x \in \mathbb{R}^N : g(x) > 0 \} \) and \( \Gamma_0 = \{ x \in \mathbb{R}^N : g(x) = 0 \} \), \( u \) is the unique uniformly solution of (0.6) with \( F \) given by (0.8), if \( \alpha \geq 1 \), and (0.2), if \( \alpha < 1 \). Then \( \lim \inf u_h = 1 \) in \( \Omega_t \) and \( \lim \sup u_h = -1 \) in \((\Omega_t \cup \Gamma_t)^c \).

The Theorem asserts that the scheme characterizes the evolution of the front by assigning the values 1 inside the region \( \{u > 0\} \) and \( -1 \) outside the region \( \{u < 0\} \). Whether the regions where \( u_h \) converges to 1 and \( -1 \) are exactly the regions inside and outside the front respectively depends on whether the front develops an interior or not. Interior are regions (patches) of positive measure where \( u = 0 \). Actually the answer is yes if and only if no interior develops.

We have:
Corollary. If $\bigcup_{t > 0} \Gamma_t \times \{t\} = \partial \{(x, t) : u(x, t) > 0\} = \partial \{(x, t) : u(x, t) < 0\}$, then the set $F^n = \bigcup_{n \in \mathbb{N}} (\Gamma_{nh} \times \{nh\})$ converges, as $n \to \infty$, to $F = \bigcup_{t > 0} (\Gamma_t \times \{t\})$ in the Hausdorff distance.

The strategy of the proof of the Theorem is similar to the one in Barles and Georgelin [BG], which is based on the general scheme developed by Barles and Souganidis [BS1] for convergence of monotone, stable and consistent approximation to viscosity solutions. Once the correct scaling $\sigma_\alpha(h)$ is identified, the main step is to prove the consistency of the scheme. The key difference between previous results and the one here is that all previous works considered kernels that were either exponentials or compactly supported, while in the case at hand they only have a prescribed power decay.

Threshold dynamics schemes are used in probability and, in particular, percolation theory to find asymptotic shapes. We refer to Gravner and Griffeath [GG] and the references therein for a discussion and results from the probabilistic point of view.

In the context of moving fronts, Bence, Merriman and Osher [MBO] introduced a scheme to compute mean curvature motion by iterating the heat equation. Evans [E] and Barles and Georgelin [BG] provided proofs for the BMO-algorithm. Some extensions to other isotropic kernels were considered by Ishii [Is], while Ishii, Peres and Souganidis [IPS] studied general anisotropic schemes with compactly supported kernels, and Slepcev [S] proved the convergence of a class of nonlocal threshold dynamics. Recently, Da Lio, Forcadel, and Monneau [DFM] studied the convergence, at large scales, of a nonlocal first-order equation to an anisotropic mean curvature motion. Although related to, the results of [DFM] are different than ours. Nonlocal operators and, in particular, fractional Laplacians of order $\alpha = 1$ are often used to describe dislocation dynamics by line tension terms deriving from an energy associated to the dislocation line. We refer to Garroni and Müller [GM1], GM2 for a variational analogue of what we are doing here for $\alpha = 1$. For the case $\alpha < 1/2$, the stationary solutions of (0.9) must satisfy an integral “zero mean curvature” equation, that can be obtained as the Euler-Lagrange equation of a minimization process in the Hilbert space $H^{\alpha/2}$ of functions with “$\alpha/2$ derivatives in $L^2$. ” A regularity theory of such surfaces, similar to the classical theory of “boundaries of sets with minimal perimeter,” is being developed by Caffarelli, Roquejoffre and Savin [CRS]. Finally, Imbert and Souganidis [ImS] studied recently the onset of fronts at the asymptotic limit of fractional integral equations with reaction terms.

Nonlocal phase transition models were proposed by Chen and Fife [CF], Giacomin and Lebowitz [GL1], GL2, DeMasi, Orlandi, Presutti and Triolo [DOPT], and DeMasi, Gobron and Presutti [DGP], in the Landau-Ginzburg context of mean field theory for statistical mechanics. All the above references assume, however, fast enough decay or compact support for the diffusion kernels to guarantee an infinitesimal curvature condition for the limit. The connection between these nonlocal equations and the underlying stochastic Ising systems and moving fronts was established by Katsoulakis and Souganidis [KS1, KS2] and Barles and Souganidis [BS2].

The paper is organized as follows: In Section 1 we present some preliminaries. The proof of the Theorem begins in Section 2. The key step in the proof, i.e., the consistency of the scheme, is presented in Section 3.

1. Preliminaries

We recall some basic facts from the Crandall-Lions [CL] theory of viscosity solutions that we will be using in the paper. We begin with the definitions. Since the two equations (0.8) and (0.9) are of different nature, i.e., local versus nonlocal, we give two separate definitions.

**Definition 1.1.** An upper semicontinuous (usc) (resp. lower semicontinuous (lsc)) function $u$ is a viscosity subsolution (resp. supersolution) of (0.8) if and only if, for all $\phi \in C^2(\mathbb{R}^N \times (0, \infty))$, if
for any maximum (resp. minimum) point \((x, t) \in \mathbb{R}^N \times (0, +\infty)\) of \(u - \phi\),

\[(1.1)\]  
\[
\phi_t \leq C_\alpha \text{tr}(I - \overline{D\phi} \otimes \overline{D\phi}) D^2 \phi \text{ if } |D\phi| \neq 0 \text{ or } \phi_t \leq 0 \text{ if } |D\phi| = 0 \text{ and } D^2 \phi = 0 ,
\]

(resp.

\[(1.2)\]  
\[
\phi_t \geq C_\alpha \text{tr}(I - \overline{D\phi} \otimes \overline{D\phi}) D^2 \phi \text{ if } |D\phi| \neq 0 \text{ or } \phi_t \geq 0 \text{ if } |D\phi| = 0 \text{ and } D^2 \phi = 0 .
\]

To make precise statements for the nonlocal motion, it is necessary to introduce some additional notation. To this end, for \(v : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}\) and \(A \subset \mathbb{R}^N\), let

\[
\begin{align*}
\overline{I}[v](x, t) &= \int 1^+(v(y + x, t) - v(x, t)) - 1^- (v(y + x, t) - v(x, t)) |y|^{-(N+\alpha)} dy , \\
\overline{I}_A[v](x, t) &= \int_A 1^+(v(y + x, t) - v(x, t)) - 1^- (v(y + x, t) - v(x, t)) |y|^{-(N+\alpha)} dy ,
\end{align*}
\]

\[(1.3)\]  
\[
\begin{align*}
\overline{I}_A[v](x, t) &= \int_A 1^+(v(y + x, t) - v(x, t)) - 1^- (v(y + x, t) - v(x, t)) |y|^{-(N+\alpha)} dy , \\
\underline{I}_A[v](x, t) &= \int_A 1^- (v(y + x, t) - v(x, t)) - 1^+ (v(y + x, t) - v(x, t)) |y|^{-(N+\alpha)} dy ,
\end{align*}
\]

where \(1_+\) and \(1_-\) denote respectively the characteristic functions of \((0, \infty)\) and \((-\infty, 0]\).

We rewrite the level set pde (0.9) obtained for \(\alpha < 1\) as

\[(1.4)\]  
\[
u_t - C_\alpha |Du| \overline{I}[u] = 0 \text{ in } \mathbb{R}^N \times (0, \infty) .
\]

We have:

**Definition 1.2.** A locally bounded usc (resp. lsc) function \(u\) is a viscosity subsolution (resp. supersolution) of (1.4) if and only if, for all \(\phi \in C^2(\mathbb{R}^N \times (0, \infty))\), if for any maximum (resp. minimum) point \((x, t) \in \mathbb{R}^N \times (0, \infty)\) of \(u - \phi\), and for any ball \(B_\delta \subset \mathbb{R}^N\) centered at \((x, t)\),

\[(1.5)\]  
\[
\phi_t \leq C_\alpha |D\phi| \left[ \overline{I}_{B_\delta}[\phi] + \overline{I}_{\mathbb{R}^N \setminus B_\delta}[u] \right] ,
\]

(resp.

\[(1.6)\]  
\[
\phi_t \geq C_\alpha |D\phi| \left[ \underline{I}_{B_\delta}[\phi] + \overline{I}_{\mathbb{R}^N \setminus B_\delta}[u] \right] .
\]

Few comments are in order here. Firstly, the difference between (1.5) and (1.6) is not a typo. It is actually necessary to guarantee the well posedness and, in particular, the stability of the solution — see [S] for a discussion of a similar problem. Secondly, it turns out (see Barles and Imbert [BI]) that Definition 1.2 is actually independent of \(\delta\). Hence, we may assume in the proofs that \(\delta\) is either fixed depending on \(\phi\), or, even, that \(\delta \to 0\) in an appropriate way.

It is well known that the initial value problem (0.8) has a unique uniformly continuous solution — see, for example, Barles, Soner and Souganidis [BSS] and Ishii and Souganidis [IS] for general results. The well-posedness of uniformly continuous solutions of the initial value problem for (0.9), which follows along the lines of the analogous result for integro-differential operators, has been studied recently by Imbert [In].

It turns out, however, (see [BSS] for a general discussion) that very weak, e.g., discontinuous, viscosity solutions of (0.8) and (0.9) may not be unique. The uniqueness is very much related to the issue of the development of interior.
We say that an evolving front \((\Omega_t, \Gamma_t, \mathbb{R}^N \setminus \bar{\Omega}_t)\) does not develop interior if, for all \(t \geq 0\),
\[
\bigcup_{t>0} \{(x, t) : u(x, t) > 0\} = \partial \{(x, t) : u(x, t) < 0\}.
\]

There are several sufficient conditions on \((\Omega_0, \Gamma_0, \mathbb{R}^N \setminus \bar{\Omega}_0)\) that imply that there is no interior (see [BSS]). A general necessary and sufficient condition, which is related to the uniqueness of solutions of (0.6), is given in the next proposition. For its proof we refer to [BS2].

**Proposition 1.3.** For an open subset \(\Omega_0\) of \(\mathbb{R}^N\), let \((\Omega_t, \Gamma_t, \mathbb{R}^N \setminus \bar{\Omega}_t)\) be the level-set evolution of \((\Omega_0, \Gamma_0, \mathbb{R}^N \setminus \bar{\Omega}_0)\) with normal velocity \(-F\).

(i) The no-interior condition (1.7) holds if and only if it holds for \(t = 0\) and the initial value problem (0.6) with initial datum \(u_0 = \mathbf{1}_{\Omega_0} - \mathbf{1}_{\mathbb{R}^N \setminus \bar{\Omega}_0}\) has a unique discontinuous viscosity solution.

(ii) If (1.7) fails, then every usc subsolution (resp. lsc supersolution) \(w\) of (0.6) with \(w(\cdot, 0) \leq \mathbf{1}_{\Omega_0} - \mathbf{1}_{\bar{\Omega}_0}\) (resp. \(w(\cdot, 0) \geq \mathbf{1}_{\Omega_0} - \mathbf{1}_{\bar{\Omega}_0}\)) satisfies, in \(\mathbb{R}^N \times (0, +\infty)\),
\[
w \leq \mathbf{1}_{\Omega_t^+ \cup \Gamma_t} - \mathbf{1}_{\Omega_t^-} \quad (\text{resp. } w \geq \mathbf{1}_{\Omega_t^+} - \mathbf{1}_{\Gamma_t \cup \Omega_t^-}).
\]

As far as the kernels \(P_{\alpha}\) are concerned we recall that, for \((x, t) \in \mathbb{R}^N \times (0, \infty)\),
\[
(1.8) \quad P_{\alpha}(x, t) = t^{-N/\alpha}P_{\alpha}(xt^{-1/\alpha}),
\]
where, for some \(C_{N,\alpha} > 0\) and all \(x \in \mathbb{R}^N\), the kernel \(P_{\alpha}(x) = P_{\alpha}(x, 1)\) satisfies
\[
(1.9) \quad 0 \leq P_{\alpha}(x) \leq C_{N,\alpha}(1 + |x|^{N+\alpha})^{-1} \quad \text{and} \quad |DP_{\alpha}(x)| \leq C_{N,\alpha}|x|^{N-1+\alpha}(1 + |x|^{N+\alpha})^{-2}.
\]

We will also use here that, both locally uniformly in \(\mathbb{R}^N \setminus \{0\}\) and in \(L^1(\mathbb{R}^N)\),
\[
(1.10) \quad \lim_{t \to 0} t^{-1}P_{\alpha}(\cdot, t) = \tilde{p}_{\alpha}(\cdot),
\]
where, for some \(C_{N,\alpha} > 0\),
\[
(1.11) \quad \tilde{p}_{\alpha}(x) = C_{N,\alpha}|x|^{-(N+\alpha)}.
\]

2. The proof of the convergence

The main step of the proof of the Theorem is

**Proposition 2.1.** The functions \(\lim \sup u_h\) and \(\lim \inf u_h\) are, respectively, viscosity subsolutions and supersolutions of (0.6) for the \(F\) specified in the Theorem.

We postpone the proof of Proposition 2.1 and we proceed with the

*Proof of the Theorem.* Let \(u\) be as in the statement and denote by \(\text{sign}^*\) and \(\text{sign}_*\) the usc and the lsc envelopes respectively of the sign function in \(\mathbb{R}\).

The functions (see [BSS] for the proof)
\[
\text{sign}^*(u) = \begin{cases} 1 & \text{if } u \geq 0, \\ -1 & \text{if } u < 0, \end{cases} \quad \text{and} \quad \text{sign}_*(u) = \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{if } u \leq 0, \end{cases}
\]
are respectively the maximal usc subsolution and the minimal lsc supersolution of (0.6) with initial datum \( \mathbf{1}_{\Omega_0} - \mathbf{1}_{\Omega_0^c} \) — recall that \( \Omega_0 = \{ x : g(x) > 0 \} \). Therefore any subsolution (resp. supersolution) \( w \) of (0.6) with the same initial datum satisfies

\[
w \leq \text{sign}^* u \quad \text{(resp. } w \geq \text{sign}_+ u \text{)} \quad \text{in } \mathbb{R}^N \times (0, \infty) .
\]

It then follows from Proposition 2.1 that

\[
(2.1) \quad \limsup^* u_h \leq \text{sign}^* u \quad \text{and} \quad \liminf_* u_h \geq \text{sign}_+ u \quad \text{in } \mathbb{R}^N \times (0, \infty) .
\]

Since \( u_h \) takes only the values \( \pm 1 \), (2.1) gives

\[
\liminf_* u_h = 1 \quad \text{in } \Omega_t = \{ u > 0 \} \quad \text{and} \quad \limsup^* u_h = -1 \quad \text{in } \{ u < 0 \} .
\]

The proof of the first part of the Theorem is now complete. \( \square \)

The Corollary follows exactly as in Section 5 of [BG].

We continue with the:

**Proof of Proposition 2.1.** We only present the argument for \( \bar{u} = \limsup^* u_h \). The claim for \( \liminf_* u_h \) follows similarly.

Let \( \phi \) be a smooth test function and assume that \( (x_0, t_0) \in \mathbb{R}^N \times (0, \infty) \) is a strict global maximum point of \( \bar{u} - \phi \). To avoid any technical difficulties, we assume that \( \liminf_{(y,s) \to (0,\infty)} \phi(y,s) = +\infty \).

If either \( \bar{u}(x_0, t_0) = -1 \) or \( (x_0, t_0) \) belongs to the interior of the set \( \{ \bar{u} = 1 \} \), the facts that \( \bar{u} \) is usc and takes only the values \( \pm 1 \) yield that \( \bar{u} = -1 \) in a neighborhood of \( (x_0, t_0) \). Therefore,

\[
(2.2) \quad D\phi(x_0, t_0) = 0 , \quad D^2\phi(x_0, t_0) \geq 0 \quad \text{and} \quad \phi_t(x_0, t_0) = 0 .
\]

Next we assume that \( (x_0, t_0) \in \partial \{ \bar{u} = 1 \} \). It is standard in the theory of viscosity solutions that, since \( \phi \) grows at infinity and \( (x_0, t_0) \) is a strict maximum, there exists a subsequence \( (x_h, nh) \) such that, as \( h \to 0 \),

\[
(2.3) \quad \begin{cases} u^*_h(x_h, nh) - \phi(x_h, nh) = \max_{\mathbb{R}^N \times \mathbb{N}} (u^*_h - \phi) , \\ u^*_h(x_h, nh) \to 1 \quad \text{and} \quad (x_h, nh) \to (x_0, t_0) , \end{cases}
\]

where \( u^*_h \) is the upper semicontinuous envelop of \( u_h \).

Here we used the upper semicontinuity of \( u^*_h \) and the fact that

\[
\limsup^* u_h = \limsup^* u^*_h .
\]

Once again \( u_h \) taking only the values \( \pm 1 \) and \( u^*_h(x_h, nh) \to 1 \), as \( h \to 0 \), imply that, for \( h \) sufficiently small, \( u^*_h(x_h, nh) = 1 \). Moreover, for all \( x \in \mathbb{R}^N \) and \( n \in \mathbb{N} \),

\[
(2.4) \quad u^*_h(x, nh) \leq 1 - \phi(x_h, nh) + \phi(x, nh) .
\]

Indeed, if \( u^*_h(x, nh) = -1 \), the inequality is trivially true, while if \( u^*_h(x, nh) = 1 \), then

\[
\phi(x, nh) - \phi(x_h, nh) \geq 0 ,
\]

and, therefore,

\[
u^*_h(x, nh) \leq \text{sign}^*(\phi(x, nh) - \phi(x_h, nh)) .\]
Recall next that
\[ u_h(\cdot, n_h h) = \text{sign}^*(S(h)u_h(\cdot, (n_h - 1)h)(\cdot)) \quad \text{in} \quad \mathbb{R}^N, \]
where
\[ S(h)v = J_h * v. \]
Therefore,
\[ u_h(\cdot, n_h h) \leq \text{sign}^*(S(h)u_h(\cdot, (n_h - 1)h)) \quad \text{in} \quad \mathbb{R}^N, \]
and, hence,
\[ u_h^*(\cdot, n_h h) \leq \text{sign}^*(S(h)u_h^*(\cdot, (n_h - 1)h)) \quad \text{in} \quad \mathbb{R}^N. \]
Let \( x = x_h. \) Since
\[ 1 = u_h^*(x_h, n_h h) = \text{sign}^* S(h)(u_h^*(\cdot, (n_h - 1)h)(x_h), \]
we have
\[ 0 \leq S(h)(u_h^*(\cdot, (n_h - 1)h)(x_h) = S(h)\text{sign}^* u_h^*(\cdot, (n_h - 1)h)(x_h). \]
The definition of \( S(h) \) and \( u_h^* \) taking only the values \( \pm 1 \) yield
\[
0 \leq \int (1^+(u_h^*(y + x_h, (n_h - 1)h) - u_h^*(x_h, n_h h))
- 1^-(u_h^*(y + x_h, (n_h - 1)h) - u_h^*(x_h, n_h h))p_\alpha(y, \sigma_\alpha(h)) dy.
\]
(2.5)
The proof will then be complete if we show that (2.5) implies, that, at \((x_0, t_0)\), if \( D\phi = 0 \), then \( \phi_\alpha \leq 0 \), or, if \( |D\phi| \neq 0 \), \( \phi_\alpha \leq F(D^2\phi, D\phi, \{\bar{u}(\cdot, t) \geq \bar{u}(x_0, t_0)\}) \), with \( F \) given by (0.8) if \( \alpha \in [1, 2) \) and (0.9) if \( \alpha \in (0, 1). \)

This is exactly the consistency of the scheme, which we investigate in the next section.

\[ \square \]

3. The Consistency

Since the argument is technical, it is necessary to look at several different cases depending on the range of \( \alpha \) and whether \( D\phi \) vanishes or not. As discussed earlier we only check the subsolution property.

We begin with the case \( \alpha \in (0, 1) \). To this end let
\[
C_\alpha = \left( 2 \int_{\mathbb{R}^N} P_\alpha(0, y') dy' \right)^{-1} C_{N, \alpha},
\]
where \( C_{N, \alpha} \) is the constant in (1.11).

**Proposition 3.1.** Fix \( \alpha \in (0, 1) \), set \( \sigma_\alpha(h) = h^{\alpha/1+\alpha} \) and assume that (2.5) holds. Then, for any ball \( B_h \subset \mathbb{R}^N \) centered at \((x_0, t_0)\), (1.5) holds at \((x_0, t_0)\) with \( C_\alpha \) given by (3.1).

**Proof.** Throughout the proof, to make the notation simpler, we drop the explicit dependence of \( p_\alpha \) and \( \sigma_\alpha \) on \( \alpha \), we write \( \sigma \) instead of \( \sigma(h) \), \( t_h = nh \) and \( \phi_h(y, s) = \phi(y + x_h, s) - \phi(x_h, t_h). \)
In view (2.3), for any \( A \subset \mathbb{R}^N \), we have

\[
\begin{cases}
\{ y \in \mathbb{R}^N : u_h^a(y + x_h, t_h - h) \geq u_h^a(x_h, t_h) \} \cap A \subseteq \{ y \in \mathbb{R}^N : \phi_h(y, t_h - h) \geq 0 \} \cap A , \\
\text{and} \\
\{ y \in \mathbb{R}^N : u_h^a(y + x_h, t_h - h) < u_h^a(x_h, t_h) \} \cap A \supseteq \{ y \in \mathbb{R}^N : \phi_h(y, t_h - h) < 0 \} \cap A .
\end{cases}
\]

(3.2)

Let \( a = \phi_t(x_0, t_0) \). Since, as \( h \to 0 \), \( \phi_t(x_h, t_h) \to \phi_t(x_0, t_0) \), for any \( \gamma > 0 \) and sufficiently small \( h \), which may depend on \( \delta \), we have

\[
\phi(\cdot + x_h, t_h - h) \leq \phi(\cdot + x_h, t_h) - (a - \gamma)h \quad \text{in} \quad \tilde{B}_\delta ,
\]

(3.3)

and, hence,

\[
\phi_h(\cdot, t_h - h) \leq \phi_h(\cdot, t_h) - (a - \gamma)h \quad \text{in} \quad \tilde{B}_\delta ,
\]

where now the ball \( \tilde{B}_\delta \) is centered at the origin.

We consider next two different cases depending on whether \( |D\phi(x_0, t_0)| \) vanishes or not.

If \( |D\phi(x_0, t_0)| \neq 0 \), then choosing \( \delta \) sufficiently we may assume that the level sets of \( \phi \) in \( B_\delta \) near \( \phi(x_0, t_0) \) are graphs of smooth functions with uniformly bounded derivatives.

If

\[
I_h = \int (I^+(u_h^a(y + x_h, t_h - h) - u_h^a(x_h, t_h)) - I^-(u_h^a(y + x_h, t_h - h) - u_h^a(x_h, t_h))p(y, \sigma) \, dy ,
\]

in view of (3.2), we have

(3.4)

\[
I_h \leq I_h^\delta + I_h^\gamma ,
\]

where

\[
I_h^\delta = \int_{\tilde{B}_\delta} (I^+(u_h^a(y + x_h, t_h - h) - u_h^a(x_h, t_h)) - I^-(u_h^a(y + x_h, t_h - h) - u_h^a(x_h, t_h))p(y, \sigma) \, dy
\]

and

\[
I_h^\gamma = \int_{\tilde{B}_\delta} (I^+(\phi_h(y, t_h - h)) - I^-(\phi_h(y, t_h - h))p(y, \sigma) \, dy .
\]

Since, as \( h \to 0 \), \( u_h^a(x_h, t_h) \to u^a(x_0, t_0) \), it follows from (1.8), (1.10) and the upper semicontinuity of \( u_h^a \) that

\[
\lim_{h \to 0} \sigma^{-1} I_h^\delta \leq C_{N, \alpha}\bar{I}_{B_\delta}[u](x_0, t_0) .
\]

(3.5)

Next we concentrate on the limiting behavior, as \( h \to 0 \), of \( I_h^\delta \) for \( \delta \) sufficiently small.

Using (3.3) we find that

\[
I_h^\delta \leq I_h^\delta + IV_h^\delta + IV_h^\delta
\]

where

\[
I_h^\delta = \int_{\tilde{B}_\delta} (I^+(\phi_h(y, t_h)) - I^-(\phi_h(y, t_h)))p(y, \sigma) \, dy ,
\]

\[
IV_h^\delta = \int_{\tilde{B}_\delta} (I^+(\phi_h(y, t_h)) - I^-(\phi_h(y, t_h)))p(y, \sigma) \, dy ,
\]

\[
IV_h^\delta = \int_{\tilde{B}_\delta} (I^+(\phi_h(y, t_h)) - I^-(\phi_h(y, t_h)))p(y, \sigma) \, dy ,
\]

\[
IV_h^\delta = \int_{\tilde{B}_\delta} (I^+(\phi_h(y, t_h)) - I^-(\phi_h(y, t_h)))p(y, \sigma) \, dy .
\]
\[ i_{h}^{} = \pm \int_{B_{\delta}} (1^{+}(\phi_{h}(y, t_{h}) - (a - \gamma)h) - 1^{+}(\phi_{h}(y, t_{h}))) p(y, \sigma) \, dy. \]

We treat each one of these terms separately. As far as \( \Phi_{h}^{\pm} \) is concerned, we observe that the Dominated Convergence Theorem and (1.10) yield
\begin{equation}
\lim_{h \to 0} \sigma^{-1} \Phi_{h}^{\pm} \leq C_{N, \alpha} \int_{B_{\delta}} (1^{+}(\phi(y + x_{0}, t_{0}) - \phi(x_{0}, t_{0})) - 1^{+}(\phi(y + x_{0}, t_{0}) - \phi(x_{0}, t_{0}))) |y|^{-(N + \alpha)} \, dy
= C_{N, \alpha} \int_{B_{\delta}} [\phi](x_{0}, t_{0}),
\end{equation}
provided we can show that the integrand in \( \Phi_{h}^{\pm} \) divided by \( \sigma^{-1} \) is integrable.

To this end, using radial coordinates and (1.9), we find
\begin{equation}
|\sigma^{-1} \Phi_{h}^{\pm}| \leq C_{N, \alpha} \int_{0}^{\delta} (\sigma^{2} + r^{2})^{-\frac{N + \alpha}{2}} D_{h}^{\delta}(r) \, dr
\end{equation}
where
\[ D_{h}^{\delta}(r) = \left| \int_{\partial B(r)} (1^{+}(\phi(y + x_{0}, t_{0}) - \phi(x_{0}, t_{0})) - 1^{+}(\phi(y + x_{0}, t_{0}) - \phi(x_{0}, t_{0}))) \, d\sigma. \]

In general we know that \( D_{h}^{\delta}(r) \) is bounded for all \( r \). However, it is a calculus exercise to check that the regularity of \( \phi \) and the fact that \( |D\phi| \neq 0 \) in \( B_{\delta} \) — recall that \( \delta \) is sufficiently small and \( |D\phi(x_{0}, t_{0})| \neq 0 \), yield, for some constant \( C > 0 \) depending on \( \delta \),
\[ D_{h}^{\delta}(r) \leq C r. \]

Hence, since \( \alpha \in (0, 1) \),
\[ |\sigma^{-1} \Phi_{h}^{\pm}| \leq \int_{0}^{\delta} r(\sigma^{2} + r^{2})^{-\frac{N + \alpha}{2}} \, dr < \infty. \]

We continue now with the analysis of \( \pm \nu_{h}^{\pm} \). We show that, for \( \sigma = h^{\alpha/(1+\alpha)} \),
\begin{equation}
\lim_{h \to 0} \pm \sigma^{-1} \nu_{h}^{\pm} \leq -(a - \gamma)|D\phi(x_{0}, t_{0})|^{-1} \left( 2 \int_{\mathbb{R}^{N-1}} P(0, y') \, dy' \right).
\end{equation}

Combining (3.3) and (3.7) yields, after letting \( \gamma \to 0 \), (1.15) at least when \( |D\phi(x_{0}, t_{0})| \neq 0 \). We show (3.7) when \( a > 0 \). The case \( a \leq 0 \) follows similarly. To this end, given that \( |D\phi(x_{0}, t_{0})| \neq 0 \), by taking \( \delta \) even smaller if necessary, we may assume that \( \phi_{h}(\cdot, t_{h}) \) has the form
\begin{equation}
\phi_{h}(y, t_{h}) = \beta_{h} y + (A_{h} y, y),
\end{equation}
with \( A_{h} = 2D^{2}\phi(x_{h}, t_{h}) \) and \( \beta_{h} = |D\phi_{h}(x_{h}, t_{h})| \to |D\phi(x_{0}, t_{0})| \), as \( h \to 0 \).

We then have
\[ \{ y \in \mathbb{R}^{N} : 0 \leq \phi_{h}(\cdot, t_{h}) \} = \{ y \in \mathbb{R}^{N} : 0 \leq y_{1} + (A_{h} y, y) \} \]
and
\[ \{ y \in \mathbb{R}^N : (a - \gamma)h \leq \phi_h(\cdot, t_h) \} = \{ y \in \mathbb{R}^N : a_h h \leq y_1 + (\tilde{A}_h y, y) \}, \]
where
\[ \tilde{A}_h = \beta_h^{-1} A_h \quad \text{and} \quad a_h = \beta_h^{-1}(a - \gamma). \]

Consider the integrals
\[ J_h^{\delta, \pm} = \pm \int_{B_\delta} 1^\pm(\phi_h(y, t_h) - (a - \gamma)h) - 1^\pm(\phi_h(y, t_h))p(y, \sigma) dy. \]

In view of (1.8) and (1.9) we have, for some \( C_{N, \alpha} > 0 \)
\[ 0 \leq p(y, \sigma) \leq \sigma C_{N, \alpha} |y|^{-(N+\alpha)} \text{ in } \bar{B}_\delta. \]

Moreover, as \( h \to 0 \) and almost everywhere in \( y, \)
\[ |1^\pm(\phi_h(y, t_h) - (a - \gamma)h) - 1^\pm(\phi_h(y, t_h))| \to 0. \]

Therefore
\[ \lim_{h \to 0} \sigma^{-1} J_h^{\delta, \pm} = 0. \]

Adding and subtracting \( J_h^{\delta, \pm} \) to (3.3) we see that to conclude the proof we need to show that
\[ \lim_{h \to 0} \sigma^{-1} J_h^{\delta, \pm} \leq -(a - \gamma)|D\phi(x_0, t_0)|^{-1} \int_{\mathbb{R}^{N-1}} P(0, y') dy'. \]  

(3.9)

Let
\[ \psi(\sigma) = \int_{\mathbb{R}^N} (1^+(\phi_h(y, t_h) - (a - \gamma)h) - 1^+(\phi_h(y, t_h))p(y, \sigma) dy \\
= \int_{\mathbb{R}^N} (1^+(\phi_h(\sigma^{1/\alpha} y, t_h) - (a - \gamma)h) - 1^+(\phi_h(\sigma^{1/\alpha} y, t_h)))P(y) dy. \]

and
\[ \Psi_h(y) = (\tilde{A}_h y, y). \]

Since \( \sigma = h\sigma^{-1/\alpha} \) and
\[ \phi_h(\sigma^{1/\alpha} y, t_h) = \sigma^{1/\alpha} \beta_h y_1 + \sigma^{2/\alpha} (A_h y, y), \]
we have
\[ 1^+(\phi_h(\sigma^{1/\alpha} y, t_h) - (a - \gamma)h) = 1^+(y_1 + \sigma^{1/\alpha}(\tilde{A}_h y, y) - a_h h\sigma^{-1/\alpha}) = 1^+(y_1 + \sigma^{1/\alpha}\Psi_h(y) - a_h \sigma) \]

and
\[ 1^+(\phi_h(\sigma^{1/\alpha} y, t_h)) = 1^+(y_1 + \sigma^{1/\alpha}\Psi_h(y)). \]

It is therefore immediate that \( \psi(0) = 0. \) Hence to prove (3.9) we need to find \( \psi'(0). \)
In view of the above simplifications we rewrite $\psi$ as

$$
\psi(\sigma) = \int_{\mathbb{R}^N} (\mathbf{1}^+(y_1 + \sigma^{1/\alpha} \Psi_h(y) - a_h \sigma) - \mathbf{1}^+(y_1 + \sigma^{1/\alpha}(\Psi_h(y))) P(y) \, dy .
$$

We state below the main step of the proof of (3.9) as a separate lemma. To this end, let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$
(3.10) \quad f(\sigma) = \int_{\mathbb{R}^N} \mathbf{1}^+(y_1 + F(y, \sigma)) P(y) \, dy .
$$

We have:

**Lemma 3.2.** Let $f$ be given by (3.10) and assume that $F \in C^1(\mathbb{R}^N \times [0, \infty))$ and $\partial_{y_1}^2 F \in C(\mathbb{R}^N \times [0, \infty))$. Then

$$
(3.11) \quad f(\sigma) - f(0) = - \int_0^\sigma \int_{\mathbb{R}^N} \mathbf{1}^+(y_1 + F(y, \sigma)) \partial_{y_1}(\partial_\rho F(y, \rho)) P(y) \, dy \, d\sigma
$$

$$
+ \int_0^\sigma \int_{\mathbb{R}^N} \mathbf{1}^+(y_1 + F(y, \rho)) \partial_\rho(\partial_{y_1} F(y, \rho)) P(y) \, dy \, d\sigma .
$$

We continue the ongoing proof and return to the proof of the lemma later. We use Lemma 3.2 with

$$
F(y, \sigma) = \sigma^{1/\alpha}(\tilde{A}_h y, \sigma) - a_h \sigma \quad \text{and} \quad F(y, \sigma) = \sigma^{1/\alpha}(\tilde{A}_h y, \sigma),
$$

both of which satisfy the assumption of Lemma 3.2. In either case we find

$$
\partial_{y_1} F(y, \sigma) = 2\sigma^{1/\alpha}(\tilde{A}_{h1, 1} y_1 + \sum_{j=2}^N \tilde{A}_{h, j} y_j) .
$$

Therefore,

$$
\lim_{\sigma \to 0} \sigma^{-1} \left[ \int_0^\sigma \int_{\mathbb{R}^N} \mathbf{1}^+(y_1 + F(y, \rho)) \partial_\rho(\partial_{y_1} F(y, \rho)) P(y) \, dy \right] = 0 .
$$

Since $\alpha \in (0, 1)$, it is hence immediate that

$$
\lim_{\sigma \to 0} \sigma^{-1} \left[ \int_0^\sigma \int_{\mathbb{R}^N} \mathbf{1}^+(y_1 + F(y, \rho)) \partial_\rho(\partial_{y_1} F(y, \rho)) P(y) \, dy \right] = 0 .
$$

Next assume that $|D\varphi(x_h, t_h)| = 0$. Since, as $h \to 0$, $\beta_h = |D\phi(x_h, t_h)| \to 0$, the previous argument does not work and it is necessary to look at several cases. It is clear from the definition of the solution that we only have to show that

$$
a = \phi_t(x_h, t_h) \leq 0 .
$$

First we assume that, along some sequence $h \to 0$, $\beta_h \neq 0$ and $\sigma \beta_h^{-1} \to 0$. In this case it is possible to repeat the previous argument. The main difference is that, instead of a fixed $\delta$, here we use $\delta_h = \beta_h C$ with $C = 2||D^2\phi||$. 


We look at each of the estimates/limits of the first part. We begin with $I_{h}^0$. Using that
$$|I_{h}^0| \leq C \delta^{-\alpha}$$
we find
$$\beta_h |I_{h}^0| \leq C \beta_h \delta^{-\alpha} = C \beta_h^{1-\alpha}.$$ 
For the limit involving $I_{h}^\delta$ we observe that the integration takes place over a set $D_h$ such that
$$D_h \subseteq \{ y \in B_{\delta h} : |\beta_h y| \leq C(\|y\|^2 + |y'|^2) \} \subseteq \{ y \in B_{\delta h} : |y| \leq C \delta_h |y_1| + C|y'|^2 \}$$
$$\subseteq \{ y \in B_{\delta h} : |y| \leq \beta_h (2C)^{-1}|y'|^2 \} ,$$
Similarly, it follows that
$$\limsup_{h \to 0} \sigma^{-1} \beta_h |I_{h}^\delta| = 0 .$$
The only thing left to check now is that
$$\lim_{h \to 0} (\pm) \sigma^{-1} \beta_h \int_{B_{\delta h}} (1^+(y_1 + \sigma^{1/\alpha} \beta_h^{-1}(A_h y, y) - (a-\gamma)\sigma \beta_h^{-1}) - 1^+(y_1 + \sigma^{1/\alpha} \beta_h^{-1}(A_h y, y))p(y, \sigma) \, dy$$
$$\leq -(a-\gamma) \int_{\mathbb{R}^{N-1}} P(0, y') \, dy'.$$
This, however, follows exactly as before. Notice that, since $\alpha < 1$, $\sigma^{1/\alpha} \beta_h^{-1} = \sigma^{\alpha^{-1}} \beta_h^{-1} \to 0$, as $h \to 0$.
The next case is that, along a subsequence $h \to 0$, either $\beta_h = 0$ or $\sigma \beta_h^{-1} \to \infty$. Here we argue by contradiction and assume that $a > 0$, and, hence, $a-\gamma > 0$ for $\gamma$ sufficiently small.
Arguing as in the beginning of the proof of the $|D\phi(x_0, \ell_0)| \neq 0$ case, we find
$$\begin{cases}
0 \leq C \sigma \delta^{-\alpha} + \int_{B_{\delta_h}} (1^+(\beta_h y_1 + (A_h y, y) - (a-\gamma)) - 1^-(\beta_h y_1 + (A_h y, y) - (a-\gamma)))p(y, \sigma) \, dy \\
= C \sigma \delta^{-\alpha} + \int_{B_{\delta_h}} (1^+(\sigma^{-1} \beta_h y_1 + \sigma^{\alpha^{-1}}(A_h y, y) - (a-\gamma))) \\
- 1^-(\sigma^{-1} \beta_h y_1 + \sigma^{\alpha^{-1}}(A_h y, y) - (a-\gamma)))P(y) \, dy .
\end{cases}$$
(3.12)
It is clear that, as $h \to 0$ and a.e. in $y_1$,
$$1^+(\sigma^{-1} \beta_h y_1 + \sigma^{\alpha^{-1}}(A_h y, y) - (a-\gamma)) \to 0 \quad \text{and} \quad 1^-(\sigma^{-1} \beta_h y_1 + \sigma^{\alpha^{-1}}(A_h y, y) - (a-\gamma)) \to 1 .$$
Hence dividing (3.12) by $\sigma$ and letting $h \to 0$ we get a contradiction to $a > 0$.
The last case to consider is that, along a sequence $h \to 0$, $\beta_h \not= 0$ and $\sigma \beta_h^{-1} \to \ell$ with $\ell > 0$.
Again we rewrite (3.12) as
$$0 \leq C \sigma \delta^{-\alpha} + \int_{\mathbb{R}^{N}} (1^+(y_1 + \sigma^{1/\alpha} \beta_h^{-1}(A_h y, y) - (a-\gamma)\sigma \beta_h^{-1})$$
$$- 1^-(y_1 + \sigma^{1/\alpha} \beta_h^{-1}(A_h y, y) - (a-\gamma)\sigma \beta_h^{-1}))P(y) \, dy .$$
(3.13)
Since $\sigma^{1/\alpha} \beta^{-1} = \sigma^{1-1/\alpha} \beta^{-1} \to 0$, as $h \to 0$, letting $h \to 0$ and $\gamma \to 0$ in (3.13) gives

$$0 \leq \int_{\mathbb{R}^N} (1^+(y_1 - a\ell) - 1^-(y_1 - a\ell))P(y) \, dy,$$

which implies, in view of the symmetry of $P$, that we must have $a \leq 0$. \hfill \Box

We continue now with the

Proof of Lemma 3.2. To keep the ideas clear we present a formal proof using Dirac masses etc. Everything can, of course, be made rigorous considering smooth approximations of $1^+$ and passing to the limit. We leave it up to the reader to do so.

For $\rho > 0$ we have

$$f'(\rho) = \int_{\mathbb{R}^N} \delta(y_1 + F(y, \rho)) \partial_\rho F(y, \rho) P(y) \, dy \quad \text{if} \quad h = 1. \quad \text{The reason is that the former is more or less straightforward, while the latter requires a bit more delicate analysis due to the border-line integrability properties of the kernel } p_{\alpha}.$$

Let

$$C_{\alpha} = \left[ 2 \int_{\mathbb{R}^N} P_{\alpha}(0, y') \, dy' \right]^{-1} \int y_2^2 P_{\alpha}(0, y') \, dy'.$$

We have:

Proposition 3.3. Assume $\alpha \in (1, 2)$ and set $\sigma_{\alpha}(h) = h^{\alpha/2}$. If (2.5) holds, then, at $(x_0, t_0)$, we have (1.1) with $C_{\alpha}$ given by (3.14) if $|D\phi| \neq 0$ or $\phi \leq 0$ if $D\phi = 0$ and $D^2 \phi = 0$.

Proof. As in Proposition 3.1 to simplify the notation we write $\sigma$ for $\sigma_{\alpha}(h)$, $t_h$ for $nh$, and $\phi_{h}(y, t)$ for $\phi(x_h + y, t) - \phi(x_h, t_h)$. 

The starting point is again \( [2.5] \) which, using \( [2.3] \) and after a rescaling, implies

\[
0 \leq \int (1^+(\phi_h(\sigma^{1/\alpha}y, t_h - h)) - 1^- (\phi_h(\sigma^{1/\alpha}y, t_h - h)))P(y) \, dy .
\]

Expanding \( \phi_h \), using that \( \phi_h(0, t_h) = 0, \phi_h \in C^{2,1} \) and \( \sigma = h^{\alpha/2} \), we find

\[
\phi_h(\sigma^{1/\alpha}y, t_h - h) = \sigma^{1/\alpha}(p_h, y) + \sigma^{2/\alpha}((A_h y, y) - a_h + O(\sigma^{1/\alpha}|y| + \sigma^{2/\alpha}(|y|^2 + \sigma^{2/\alpha})) ,
\]

where

\[
p_h = D\phi_h(x_h, t_h), \quad A_h = \frac{1}{2}D^2\phi_h(x_h, t_h) \quad \text{and} \quad a_h = \phi_{ht}(x_h, t_h) .
\]

After a rotation and a change of variables, we may assume that \( p_h = \beta_h e_1 \) with \( \beta_h = |p_h| \). We denote by \( \tilde{A} \) the matrix we obtain from \( A_h \) after the rotation. The integration in \((3.15)\) is taking place over the sets \( C_h \) and \( C^c_h \), where

\[
C_h = \{ y \in \mathbb{R}^N : \sigma^{1/\alpha}\beta_h y_1 + \sigma^{2/\alpha}((\tilde{A}_h y, y) - a_h + O((\sigma^{1/\alpha}|y| + \sigma^{2/\alpha}(|y|^2 + \sigma^{2/\alpha})) \geq 0 \} .
\]

We argue now as in the proof of Proposition 3.1 and \([BG]\). The difference with the former is that \( \alpha \in (1, 2) \) gives integrability at the origin. The difference with the latter is that here the kernel only has algebraic decay while in \([BG]\) it is an exponential.

Here we only present the argument if \( |D\phi(x_0, t_0)| \neq 0 \). If \( |D\phi(x_0, t_0)| = 0 \) and \( D^2\phi(x_0, t_0) = 0 \), it is necessary to look again at different cases as in the proof of Proposition 3.1 and \([BG]\). The argument is considerably simpler than the one presented in Proposition 3.1 since we do not need to consider special balls, etc.. We leave the details to the reader.

Since, as \( h \to 0, \beta_h \to |D\phi(x_0, t_0)| \neq 0 \), the \( \beta_h \)'s are strictly positive for sufficiently small \( h \). Therefore

\[
C_h = \{ y \in \mathbb{R}^N : y_1 + \sigma^{1/\alpha}\beta_h((\tilde{A}_h y, y) - a_h + O((\sigma^{1/\alpha}|y| + \sigma^{2/\alpha}(|y|^2 + \sigma^{2/\alpha})) \geq 0 \} .
\]

Finally using that, as \( h \to 0, \tilde{A}_h \to \tilde{A} \) and \( a_h \to a \), where \( \tilde{A} \) is the rotated matrix \( A \) and \( a = \phi_t(x_0, t_0) \), we find

\[
C_h = \{ y \in \mathbb{R}^N : y_1 + \sigma^{1/\alpha}\beta_h((\tilde{A}_h y, y) - a + O((\sigma^{1/\alpha}|y| + \sigma^{2/\alpha}(|y|^2 + \sigma^{2/\alpha})) + o(1)(|y|^2 + 1) \geq 0 \} .
\]

After all the above reductions we are left with the inequality

\[
0 \leq \int (1^+(\Psi_h(y)) - 1^- (\Psi_h(y)))P(y) \, dy ,
\]

where, for \( y \in \mathbb{R}^N ,
\]

\[
\Psi_h(y) = y_1 + \sigma^{1/\alpha}(\beta_h)^{-1}[(\tilde{A}_h y, y) - a + O((\sigma^{1/\alpha}|y| + \sigma^{2/\alpha}(|y|^2 + \sigma^{2/\alpha}) + o(1)(|y|^2 + 1)] .
\]

Let \( R = \sigma^{\theta/\alpha} \) for some \( \theta > 0 \). Then

\[
\int_{\mathbb{R}^N} 1^+(\Psi_h(y))P(y) \, dy = \int_{B_R} 1^+(\Psi_h(y))P(y) \, dy + \int_{B^c_R} 1^+(\Psi_h(y))P(y) \, dy .
\]
Using (1.9), we find, for some \( C_{N,\alpha} > 0 \), that
\[
\int_{\mathbb{R}^N \setminus B_R} 1^\pm (\Psi_h(y)) P(y) \, dy \leq C_{N,\alpha} \int_{|y| \geq R} (1 + |y|^2)^{-\frac{N+\alpha}{2}} \, dy \leq C_{N,\alpha} R^{-\alpha}.
\]

Fix \( \gamma > 0 \). For \( h \) small we have \( \Psi_h \leq \Psi^h \) in \( B_R \), where
\[
\Psi^h(z) = z_1 + \sigma^{1/\alpha} \beta_h^{-1}(\frac{1}{2}(\tilde{A} + \gamma \text{Id}) - a).
\]

Hence
\[
\int_{B_R} 1_{\{\Psi_h \geq 0\}}(y) P(y) \, dy \leq \int_{\mathbb{R}^N} 1_{\{\Psi_h \geq 0\}}(y) P(y) \, dy.
\]

We summarize the above, using that
\[
\int \{ (\Psi_h + \rho \beta_h^{-1}\psi(y)) - (\Psi_h + \rho \beta_h^{-1}\psi(y)) \} P(y) \, dy
\]

in the inequality
\[
(3.17) \quad 0 \leq \int \{ (\Psi_h + \rho \beta_h^{-1}\psi(y)) - (\Psi_h + \rho \beta_h^{-1}\psi(y)) \} P(y) \, dy,
\]
where \( \rho = \sigma^{1/\alpha} \) and
\[
\psi(y) = (\tilde{A} + \gamma \text{Id}) - a.
\]

Let
\[
f(\rho) = \int \{ (\Psi_h + \rho \beta_h^{-1}\psi(y)) - (\Psi_h + \rho \beta_h^{-1}\psi(y)) \} P(y) \, dy.
\]

The properties of \( P \) yield that \( f(0) = 0 \), therefore, as in the proof of Proposition 3.1, we use Lemma 3.2 to find \( f'(0) \).

It is a straightforward computation to see that it yields to the inequality
\[
a \leq C_{\alpha}[\text{tr}(\tilde{A} - \tilde{A}_{1,1}) - 2^{-1}(N + 1)\gamma].
\]

An elementary linear algebra calculation yields
\[
\text{tr}(\tilde{A} - \tilde{A}_{1,1}) = \frac{1}{2} \text{tr}(I - D\phi(x_0, t_0) \otimes D\phi(x_0, t_0)) D^2\phi(x_0, t_0),
\]
and, hence, after letting \( \gamma \to 0 \), to the desired inequality.

We continue with the case \( \alpha = 1 \). The argument is very similar to the one above. There is, however, a technical complication due to the logarithmic integrability of the kernel \( p_1 \). To deal with this difficulty, it is necessary to choose \( \sigma_1 \) differently.

Let
\[
(3.18) \quad C_1 = (2 \int_{\mathbb{R}^{N-1}} P_1(0, y') dy')^{-1} \omega_{N-1} \lim_{R \to \infty} (R^{N+1} P_1(0, R)),
\]
where \( \omega_{N-1} \) is the area of the unit sphere in \( \mathbb{R}^{N-1} \) and, as always, \( P_1 \) is defined by (1.8).

We have:
Proposition 3.4. Assume \( \alpha = 1 \) and choose \( \sigma \) so that \( h = \sigma^2(\ln \sigma) \). If (2.5) holds, then, at \((x_0, t_0)\), (11) holds with \( C_1 \) given by (3.18) if \( |D\phi| \neq 0 \) or \( \phi_t \leq 0 \), if \( D\phi = 0 \) and \( D^2\phi = 0 \).

Proof. We only discuss the case \( |D\phi(x_0, t_0)| \neq 0 \). The argument when \( |D\phi(x_0, t_0)| = 0 \) and \( D^2\phi(x_0, t_0) = 0 \), is similar to the one in Propositions 3.1 and 3.3, hence we omit the details. The proof follows very closely the one presented for the case \( \alpha \in (1, 2) \), hence, again we only sketch the main steps. The main difference/difficulty is the logarithmic integrability of the kernel. To circumvent this potential problem it is necessary to make a “more involved” scaling.

To this end for each \( \delta > 0 \), arguing as before, we reach the inequality

\[
0 \leq \int_{|y| \leq \delta \sigma^{-1}} \left[ I^+(y_1 + F(y, \rho)) - I^-(y_1 + F(y, \rho)) \right] P(y) dy + C_N \delta^{-1} \sigma
\]

where, for \( \rho = \sigma |\ln \sigma| \),

\[
F(y, \rho) = \sigma((\tilde{A}_h + \gamma_h \text{Id})y, y) - \rho a_h
\]

with \( \tilde{A}_h = \beta_h^{-1} \tilde{A}, \gamma_h = \beta_h^{-1} \gamma \) and \( a_h = \beta_h^{-1} \gamma \).

Define

\[
f(\rho) = \int_{|y| \leq \delta \sigma^{-1}} (1^+(y_1 + F(y, \rho)) - 1^-(y_1 + F(y, \rho))) P(y) dy.
\]

Using the decay properties of \( P \) and the dominated convergence theorem we find easily that \( f(0) = 0 \).

As before we need to calculate \( f'(0) \). Arguing formally—the calculation can be justified rigorously using regularizations of \( I^+ \) and \( I^- \), etc.,—we get

\[
f'(\rho) = 2 \int_{|y| \leq \delta \sigma^{-1}} \delta(y_1 + F(y, \rho)) \partial_\rho(F(y, \rho)) P(y) dy
\]

\[
+ \int_{|y| = \delta \sigma^{-1}} (1^+(y_1 + F(y, \rho)) - 1^-(y_1 + F(y, \rho))) P(y) d\Sigma'(\delta \sigma^{-1})
\]

where \( d\Sigma' \) is the surface measure on \( \partial \tilde{B}_{\delta \sigma^{-1}} \) and \( t \) denotes differentiation with respect to \( \rho \).

Then

\[
f(\rho) - f(0) = I_\rho + II_\rho
\]

with

\[
I_\rho = 2 \int_0^\rho \int_{|y| \leq \delta \sigma^{-1}(\lambda)} \delta(y_1 + F(y, \lambda)) \partial_\lambda(F(y, \lambda)) P(y) dy d\lambda
\]

and

\[
II_\rho = \int_0^\rho \int_{|y| = 1} \left[ (1^+(\delta \sigma^{-1}(\lambda))y_1 + F(\delta \sigma^{-1}(\lambda)y, \lambda))
\right.

\[
- 1^-(\delta \sigma^{-1}(\lambda))y_1 + F(\delta \sigma^{-1}(\lambda)y, \lambda))] P(\delta \sigma^{-1}(\lambda)y)(\delta \sigma^{-1})^{N-1} d\Sigma d\lambda
\]

where \( d\Sigma \) is the surface measure on \( \partial B_1 \) and \( \lambda = \sigma(\lambda)|\ln \sigma(\lambda)|.$$
The growth of $P$ and the fact that $\sigma'(\lambda) = (|\ln \sigma| - 1)^{-1}$ yield, for some $C' > 0$, the estimate
\[ \rho^{-1} |\mathbb{I}_\rho| \leq \rho^{-1} C' \int_0^\rho \frac{(\delta \sigma^{-1})^{N-1}}{(1 + (\delta \sigma^{-1})^2)^{N+1}} \left( \frac{\delta \sigma'}{\sigma^2} \right) d\lambda \leq \frac{C}{\delta} \rho^{-1} \int_0^\rho (|\ln \sigma| - 1)^{-1} \]
and, hence,
\[ \lim_{\rho \to 0} \rho^{-1} |\mathbb{I}_\rho| = 0. \]

Next we analyze $I_\rho$. We begin with the observation that
\[ \partial_\lambda F(y, \lambda) = (\tilde{A}h y, y)' - a. \]

Hence
\[ I_\rho = I_1^\rho + I_2^\rho, \]
with
\[ I_1^\rho = -2a \int_0^\rho \int_{|y| \leq \sigma^{-1}} \delta(y_1 + F(y, \lambda)) P(y) dy d\lambda \]
and
\[ I_2^\rho = \int_0^\rho (|\ln \sigma| - 1)^{-1} \int_{|y| \leq \delta \sigma^{-1}} \delta(y_1 + F(y, \lambda))(\tilde{A}h y, y) P(y) dy d\lambda. \]

It is immediate that
\[ \lim_{\rho \to 0} \rho^{-1} I_1^\rho = -2a \int_{\mathbb{R}^{N-1}} P(0, y') dy', \]
while
\[ \rho^{-1} I_2^\rho = \int_0^\rho \ln(\delta \sigma^{-1})(|\ln \sigma| - 1)^{-1} \int_{|y| \leq \delta \sigma^{-1}} \delta(y_1 + F(y, \lambda))(\tilde{A}h y, y) P(y) dy d\lambda, \]
and, hence,
\[ \lim_{\rho \to 0} \rho^{-1} I_2^\rho = \frac{1}{(N - 1)} \lim_{R \to \infty} \frac{1}{\ln R} \int_{|y| \leq R} |y'|^2 P(0, y') dy \text{ tr}(\tilde{A} - \tilde{A}_{1,1}). \]

Returning now to (3.19) we find
\[ 0 \leq \rho^{-1} \int_{|y| \leq \delta \sigma^{-1}} (1^+(y_1 + F(y, \rho)) - 1^-(y_1 + F(y, \rho))) P(y) dy + C_{N,1} \frac{1}{\delta |\ln \sigma|}. \]

Letting $\rho \to 0$ yields
\[ 2a \int_{\mathbb{R}^{N-1}} P(0, y') dy \leq \frac{1}{(N - 1)} \lim_{R \to \infty} \frac{1}{\ln R} \int_{|y| \leq R} |y'|^2 P(0, y') dy \]
\[ = \frac{1}{N - 1} \omega_{N-1} \lim_{R \to \infty} (R^{N+1} P_1(0, R)). \]

We may now conclude as in Proposition 3.2. \qed
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