Single-machine scheduling with an external resource

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Abstract

This paper studies the complexity of single-machine scheduling with an external resource, which is rented for a non-interrupted period. Jobs that need this external resource are executed only when the external resource is available. There is a cost associated with the scheduling of jobs and a cost associated with the duration of the renting period of the external resource. We look at three classes of problems with an external resource: a class of problems where the renting period is budgeted and the scheduling cost needs to be minimized, a class of problems where the scheduling cost is budgeted and the renting period needs to be minimized, and a class of two-objective problems where both, the renting period and the scheduling cost, are to be minimized. We provide a thorough complexity analysis (NP-hardness proofs and pseudo-polynomial algorithms) for different members of these three classes.

Keywords: single-machine scheduling, external resource, complexity, pseudo-polynomial algorithm

1. Introduction

We study single-machine scheduling problems with an external resource. We assume that the machine is available throughout the planning horizon and can process job at a time. However, some of the jobs require an external (and relatively expensive) resource (such as a crane, loader, human expert, etc.) and only one job can use the external resource at each moment in time. The external resource can be rented only once for an uninterrupted period. Both, scheduling of jobs and renting the external resource, incur costs. The renting cost is a linear function of the renting period. We investigate three variants of problems.

- A variant where the renting cost is budgeted and the scheduling cost is to be minimized.
- A variant where the scheduling cost is budgeted and the renting cost is to be minimized.
- A two-objective variant where both, scheduling cost and renting cost, are to be minimized.
For the sake of simplicity, the two terms ‘external resource’ and ‘resource’ are used interchangeably in the remainder of this paper.

This paper is devoted to study the complexity of the above problems. Below, we review the complexity of a number of relevant classical scheduling problems. The single machine scheduling problem to minimize the total weighted completion time is polynomially solvable even with a serial-parallel precedence graph (Lawler, 1978) or with two-dimensional partial orders (Ambühl and Mastrolilli, 2009). On the other hand, single machine scheduling to minimize the total weighted completion time becomes strongly NP-hard as soon as release dates are present (Lenstra et al., 1977). Similarly, the single machine scheduling problem to minimize the maximum lateness is polynomially solvable even in the presence of precedence constraints (Lawler, 1973) but strongly NP-hard with release dates (Lenstra et al., 1977). Moreover, the single machine scheduling problem with an objective function of weighted number of tardy jobs is known to be weakly NP-hard (Karp, 1972; Lawler and Moore, 1969) whereas the problem with an objective function of total weighted tardiness is already strongly NP-hard (Lenstra et al., 1977).

The literature on scheduling with external resources is rather scarce. The most relevant problem in the project scheduling literature is perhaps the resource renting problem (RRP). The RRP as initially proposed by Nübel (2001) aims to minimize the costs associated with renting resources throughout a project. These costs include fixed handling or procurement cost and variable renting cost. Unlike the setting in this paper where the external resource is rented for an uninterrupted period, the RRP considers renting resources that can be rented for as many as needed disjoint intervals. The problem has been recently extended by Vandenheede et al. (2016) who combined the RRP and the total adjustment cost problem and by (Kerkhove et al., 2017) who studied a variant of the RRP with overtime.

The RRP is closely associated with the resource availability cost problem where resources are no longer to be rented but to be utilized. The assumption is that the resources, once utilized, are available for the whole duration of the project. The decision variables are the resource utilizations and the starting times. The utilization of a resource imposes some expenses in the resource availability cost problem, which needs to be minimized (Rodrigues and Yamashita, 2010). The problem is also known as the resource investment problem (Drexl and Kimms, 2001). To the best of our knowledge, the problem was first introduced by Möhring (1984) motivated by a bridge construction project.

A budget imposed on the length of the renting period can be seen as a type of maximum delay constraints: For any pair of jobs $i, j$ that require the resource, the time between the start of $i$ and the end of $j$ must not exceed the budget. Such maximum delay constraints have been investigated by Wikum et al. (1994) in the context of the single-machine generalized precedence-constrained scheduling problem. However, in this problem, maximum delay constraints are always combined with a non-negative minimum delay, thus enforcing an order among the two jobs. Thus, complexity results for this problem do not extend to external resource renting.

In this paper, we discuss the complexity of the three classes of single-machine scheduling problems with an external resource and different objective functions. The remainder of this text is structured as follows: we formally define different variants of our problem in Section 2, discuss the complexity of these variants in Section 3 and Section 4 and finally summarize the results and discuss future research possibilities in Section 5.
2. Problem definition

We consider a set \( J \) of \( n \) jobs with \( p_j \), \( d_j \), and \( w_j \) representing the processing time, the due date, and the weight of job \( j \in J \), respectively. Throughout the paper we assume \( p_j \) and \( w_j \) to be integer for each job \( j \in J \). We let \( P = \sum_{j \in J} p_j \) and \( W = \sum_{j \in J} w_j \) denote the sum of processing times and the sum of weights, respectively. There is a subset \( J' \subseteq J \) of jobs that require an external resource. We refer to jobs in \( J' \) as r-jobs (resource jobs) and to jobs in \( J'' = J \setminus J' \) as o-jobs (ordinary jobs). We assume that the external resource must be rented from the start of the first r-job to the completion of the last r-job. Let \( C_j \) be the completion time of job \( j \). The length of the renting period, which is denoted by \( er \), is \( er = \max_{j \in J'} C_j - \min_{j \in J'} \{C_j - p_j\} \). Now, for any single-machine scheduling problem \( 1||\gamma \) with objective function \( \gamma \), there are three natural counter-part problems with an external resource:

- Problem 1\(|er|\gamma\) is to find a sequence of jobs that minimizes \( \gamma \) among all sequences with a length of the renting period of at most \( K^r \).
- Problem 1\(|\gamma|er\) is to find a sequence of jobs that minimizes the length of the renting period among all sequences with a scheduling cost of at most \( K^r \).
- Problem 1\(|(\gamma, er)|\) is to find the sequences in the Pareto-front with respect to minimization of both, scheduling cost and the length of the renting period.

For a given sequence \( \sigma \) of jobs we denote by \( C_j^\sigma \) the completion time of job \( j \), by \( L_j^\sigma = C_j^\sigma - d_j \) its lateness and we let \( U_j^\sigma \) be the indicator for tardiness, i.e., \( U_j = 1 \) if \( C_j > d_j \) and \( U_j = 0 \) otherwise. We omit the superscript \( \sigma \) whenever the sequence is clear from the context.

In what follows, we explore the complexity of the above problems when \( \gamma \) is one of the following objective functions:

- Total completion time (\( \sum C_j \))
- Total weighted completion time (\( \sum w_j C_j \))
- Maximum lateness (\( \max L_j \))
- Weighted number of tardy jobs (\( \sum w_j U_j \))

All these problems are shown to be NP-hard but allow for pseudo-polynomial algorithms with the running time depending on the total processing time \( P \) of all jobs. These results are described in Section 3 and Section 4. A summary of the complexity orders for the algorithms is given in Table 1.

Note that these results suggest that all variants are polynomially solvable whenever the total processing time \( P \) is polynomial in the number \( n \) of jobs. Note, furthermore, that the case with identical processing times, that is \( p_j = p \) for each job \( j \), can be easily reduced to the case with \( p = 1 \). The latter is solvable in polynomial time as \( P = n \) in this case. Hence, each problem is solvable in polynomial time under identical processing times.
Table 1: Summary of results

| Problem | \(\gamma\) | Complexity |
|---------|-------------|------------|
| \(1|\epsilon|\gamma\) | \(\sum C_j\) | \(O(n^2P)\) |
| | \(\sum \omega_j C_j\) | \(O(nP\min\{W, P\})\) |
| | \(\max L_j\) | \(O(nP)\) |
| | \(\sum \omega_j U_j\) | \(O(nP^4)\) |
| \(1|\gamma|\epsilon\) | \(\sum C_j\) | \(O(nP^2)\) |
| | \(\sum \omega_j C_j\) | \(O(nP\min\{W, P\})\) |
| | \(\max L_j\) | \(O(nP)\) |
| | \(\sum \omega_j U_j\) | \(O(nP\log P)\) |
| \(1||(\gamma, \epsilon)\) | \(\sum C_j\) | \(O(nP^2)\) |
| | \(\sum \omega_j C_j\) | \(O(nP^2)\) |
| | \(\max L_j\) | \(O(nP^2)\) |
| | \(\sum \omega_j U_j\) | \(O(nP^5)\) |

3. Complexity results for \(1|\epsilon|\gamma\)

In this section, we discuss the complexity of \(1|\epsilon|\gamma\). Throughout this section, we sometimes refer to a sequence \(\sigma\) as feasible which means it respects the resource budget.

Without loss of generality, we assume \(J = \{1, \ldots, n\}\) (later, we will assume this numbering to reflect an ordering of the jobs according to some attribute) and define \(J[a, b] := \{j \in J : a \leq j \leq b\}\) for \(a, b \in J\).

We denote the total processing time of a job set \(S\) by \(p(S) = \sum_{j \in S} p_j\) and its total weight by \(w(S) = \sum_{j \in S} w_j\). Also, we use \(\text{TWC}(\sigma)\) as the total weighted completion time and \(L_{\max}(\sigma)\) as the maximum lateness for sequence \(\sigma\). Note that, to avoid excess of notations, we let \(\sigma\) not only represent a sequence, but also imply the sequence’s set of jobs. Thus, \(p(\sigma)\) denotes to total processing time of jobs present in \(\sigma\).

3.1. Total weighted completion time

In this section, we review the complexity of \(1|\epsilon|\sum w_j C_j\). We first prove that even the unweighted problem \(1|\epsilon|\sum C_j\) is already NP-hard (Theorem 1) and then we propose a pseudo-polynomial algorithm for \(1|\epsilon|\sum w_j C_j\) (Theorem 2).

Theorem 1. \(1|\epsilon|\sum C_j\) is NP-hard.

Proof. We prove the NP-hardness of \(1|\epsilon|\sum C_j\) by a reduction from EVEN-ODD-PARTITION which is known to be NP-hard, see (Garey et al., 1988).

**EVEN-ODD-PARTITION:** Given integers \(a_1, \ldots, a_{2m}\) with \(a_{k-1} < a_k\) for \(k = 2, \ldots, 2m\) and with total value \(2B\), is there a subset of \(m\) of these numbers with total value of \(B\) such that for each \(k = 1, \ldots, m\) exactly one of the pair \(\{a_{2k-1}, a_{2k}\}\) is in the subset?

In the following, we assume \(B \geq 2m(m + 1) - 2\) for our instance. Note that we can always avoid \(B < 2m(m + 1) - 2\) by increasing the value of each integer \(a_k\), \(k = 1, \ldots, 2m\), by \(2(m + 1)\) and increasing the value of \(B\) by \(2m(m + 1)\), accordingly.

Given such an instance of **EVEN-ODD-PARTITION**, we construct an instance of \(1|\epsilon|\sum C_j\) with \(2m + 2\) jobs as follows:

- \(J = \{1, \ldots, 2m + 2\}\) and \(J' = \{2m + 1, 2m + 2\}\);
\( \sigma(m + 1) = 2m + 1 \)

\( C + D + 1 \)

\( D = 2mB^2 + 2B \)

\( 0 \)

\( mB^2 + B \)

\( mB^2 + B \)

\( 2mB^2 + 2B \)

\( C + 2D + 1 \)

\begin{align*}
\sigma(1) & \quad \ldots \quad \sigma(2m) \\
\sigma(2m + 1) & \quad \ldots \quad \sigma(2m + 2)
\end{align*}

\textbf{Figure 1:} The schedule for sequence \( \sigma \) in the proof of Theorem 1

- \( p_j = B^2 + a_j \) for each \( j = 1, \ldots, 2m, \)
- \( p_{2m+1} = 0 \) and \( p_{2m+2} = C + D + 1, \) and
- \( K' = p_{2m+2} + mB^2 + B \)

where

\[
C = \sum_{k=1}^{m} (m + 1 - k)(p_{2k-1} + p_{2k}) + (mB^2 + B)(m + 1)
\]

and

\[
D = \sum_{j=1}^{2m} p_j = 2mB^2 + 2B.
\]

We claim that there is a feasible schedule with total completion time of no more than

\[ 2(C + D) + 1 \]

if and only if the answer to the instance of \textsc{Even-Odd-Partition} is yes.

First, consider a job sequence \( \sigma \) with total completion time of no more than \( 2(C + D) + 1. \)

**Claim 1.** Job \( 2m + 2 \) is the last job in \( \sigma \) and job \( 2m + 1 \) is not started before \( mB^2 + B. \)

**Proof.** Assume that job \( 2m + 2 \) is not the last job. Then, at least two jobs have a completion time of at least \( p_{2m+2} = C + D + 1 \) and, thus, total completion time is at least \( 2(C + D) + 2. \)

Due to \( C'_{2m+2} = \sum_{j=1}^{2m+2} p_j \) and due to feasibility of \( \sigma, \) job \( 2m + 1 \) is not started before

\[
D + p_{2m+1} + p_{2m+2} - (p_{2m+2} + mB^2 + B) = D - mB^2 - B = mB^2 + B.
\]

\[
\square
\]

**Claim 2.** Exactly \( m \) jobs are scheduled between \( 2m + 1 \) and \( 2m + 2 \) in \( \sigma. \)
Proof. On the one hand, no more than \( m \) jobs can be scheduled between \( 2m + 1 \) and \( 2m + 2 \) since total processing time of the jobs following job \( 2m + 1 \) for any \( B > 1 \) amounts to at least
\[
p_{2m+2} + (m+1)B^2 > p_{2m+2} + mB^2 + B = K^*.
\]
On the other hand, if less than \( m \) jobs are scheduled between \( m + 1 \) and \( m + 2 \), the total processing time \( TC(\sigma) \) exceeds \( 2(C + P) + 1 \) since
\[
TC(\sigma) > \sum_{j=1}^{2m} jB^2 + 2B + (m+1)B^2 + C_{\text{max}}
\]
\[
= \sum_{j=1}^{m} (m + 1 - j)B^2 + \sum_{j=1}^{m} (m + m + 1 - j)B^2 + (m + 1)B^2
\]
\[
+ 2B + C_{\text{max}}
\]
\[
= \sum_{j=1}^{m} 2(m + 1 - j)B^2 + (mB^2)(m + 1) + B^2 + 2B + C_{\text{max}}
\]
\[
= \sum_{j=1}^{m} 2(m + 1 - j)(B^2 + 2B) + (mB^2)(m + 1)
\]
\[
+ B^2 + 2B - 2m(m + 1)B + C_{\text{max}} \geq 0 \quad (\text{since } B \geq 2m(m+1)-2)
\]
\[
\geq \sum_{j=1}^{m} 2(m + 1 - j)(B^2 + 2B) + (mB^2)(m + 1) + C_{\text{max}}
\]
\[
= \sum_{j=1}^{m} 2(m + 1 - j)(B^2 + B) + (mB^2 + mB)(m + 1) + C_{\text{max}} \geq C
\]
\[
> C + C_{\text{max}} = C + D + p_{2m+2} = C + D + C + D + 1
\]
\[
= 2(C + D) + 1.
\]

Following the above two claims, we conclude that \( \sigma(m + 1) = 2m+1 \) and \( \sigma(2m + 2) = 2m + 2 \). The schedule for sequence \( \sigma \) is depicted in Figure 1. We derive the total completion time \( TC(\sigma) \) of \( \sigma \) as follows:
\[
TC(\sigma) = \sum_{k=1}^{m} C_{\sigma(k)} + mB^2 + B + \delta + \sum_{k=m+1}^{2m+1} C_{\sigma(k)} + C_{\text{max}}
\]
where
\[
\delta = \sum_{k=1}^{m} p_{\sigma(k)} - (mB^2 + B)
\]
is the difference between the starting time of job $2m + 1$ according to $\sigma$ and its earliest starting time. Note that $\delta \geq 0$ due to feasibility of $\sigma$. Since $C_{\sigma(k)} = \sum_{s=1}^{k} p_{\sigma(s)}$, $\text{TC}(\sigma)$ can be rewritten as

$$\text{TC}(\sigma) = \sum_{k=1}^{m} (2m + 1 - k)p_{\sigma(k)} + \sum_{k=1}^{m} (m + 1 - k)p_{\sigma(k+m+1)} + mB^2 + B + \delta + C_{\text{max}}$$

$$= \sum_{k=1}^{m} (m + 1 - k)p_{\sigma(k)} + \sum_{k=1}^{m} (m + 1 - k)p_{\sigma(k+m+1)} + (m + 1)(mB^2 + B + \delta) + C_{\text{max}}$$

$$= \sum_{k=1}^{m} (m + 1 - k)(p_{\sigma(k)} + p_{\sigma(k+m+1)}) + (m + 1)\delta + (m + 1)(mB^2 + B) + C + 2D + 1.$$  

We observe that $\text{TC}(\sigma) \leq 2(C + D) + 1$ only if

$$\sum_{k=1}^{m} (m + 1 - k)(p_{\sigma(k)} + p_{\sigma(k+m+1)}) + (m + 1)\delta \leq \sum_{k=1}^{m} (m + 1 - k)(p_{2k-1} + p_{2k})$$

holds. This inequality holds only if $\delta = 0$ and for each $k = 1, \ldots, m$, one of the jobs $2k$ or $2k - 1$ is assigned to position $k$ and the other to position $k + m + 1$ in $\sigma$ (recall that numbers are ordered increasingly in \textsc{Even-Odd-Partition}). Thus, we conclude that the subsets of jobs before and after job $2m + 1$ constitute a yes-certificate for the corresponding instance of \textsc{Even-Odd-Partition}.

Second, if a yes-certificate for the instance of \textsc{Even-Odd-Partition} is given we can construct a sequence with the structure discussed above and, thus, yielding total completion time of at most $2(C + D) + 1$. This completes the proof. 

We now show that $|1|er| \sum w_j C_j$ can be solved in pseudo-polynomial time. We show, in Lemma 1, that there always exists an optimal sequence with a special structure consisting of five blocks that are internally ordered according to the weighted shortest processing time (WSPT) rule and then we exploit this structure to find an optimal sequence using dynamic programs (DPs) in Lemma 2 and Lemma 3.

Without loss of generality, we assume the jobs to be numbered according to WSPT (i.e., $w_1/p_1 \geq \cdots \geq w_n/p_n$). We let $\alpha = \min J^r$, $\beta = \max J^r$, and $H = J[\alpha, \beta] \setminus J^r$. Also, we let $t_k = p(J[1, k-1])$ for each $k \leq n$.

Let $X, Y \subseteq H$ with $\max X < \min Y$. We consider a sequence $\sigma_{XY}$ as follows. The sequence consists of five blocks and each block is sorted internally according to WSPT. The first block is $J[1, \alpha - 1]$; the second block is $X$; the third block is $J[\alpha, \beta] \setminus (X \cup Y)$; the fourth block is $Y$; the fifth block is $J[\beta + 1, n]$. Figure 2 depicts such a sequence.

**Lemma 1.** For each instance of $|1|er| \sum w_j C_j$ there exists $X^*, Y^* \subseteq H$ with $\max X^* < \min Y^*$ such that $\sigma_{X^*, Y^*}$ is optimal.
Proof. Let $\sigma$ be an optimal feasible sequence. Let $i_1 := \min\{i : \sigma(i) \in J^r\}$ and $i_2 := \max\{i : \sigma(i) \in J^r\}$ be the first and last occurrence, respectively, of an r-job in the sequence. Let $S = \{\sigma(i_1), \ldots, \sigma(i_2)\}$. Note that $\sum_{j \in S} p_j \leq K^r$ by feasibility of $\sigma$. Hence, any sequence that schedules the jobs of $S$ consecutively is feasible. In particular, rearranging the jobs within $S$ according to WSPT maintains feasibility of $\sigma$ without increasing total weighted completion time. Therefore, we can assume, without loss of generality, $\alpha = \sigma(i_1) < \cdots < \sigma(i_2) = \beta$ (i.e., the jobs in $S$ are scheduled according to WSPT and, in particular, $S \subseteq J[\alpha, \beta]$).

Now consider the job set $J' := J \setminus S \cup \{j'\}$ where the jobs of $S$ are merged into the single job $j'$ with processing time $p_{j'} = p(S)$ and weight $w_{j'} = w(S)$. Let

$$X^* := \{j \in H \setminus S : w_j/p_j \geq w_{j'}/p_{j'}\}$$
and

$$Y^* := H \setminus (S \cup X^*).$$

Note that $\max X^* < \min Y^*$ by construction and that $\sigma_{X^*, Y^*}$ is a feasible sequence for $J$. Further note that both $\sigma_{X^*, Y^*}$ and $\sigma$ induce sequences $\sigma'_{X^*, Y^*}$ and $\sigma'$ for $J'$, respectively. In particular, $\sigma'_{X^*, Y^*}$ orders jobs in $J'$ according to WSPT and therefore

$$\text{TWC}(\sigma'_{X^*, Y^*}) \leq \text{TWC}(\sigma').$$

Moreover,

$$\text{TWC}(\sigma_{X^*, Y^*}) = \text{TWC}(\sigma'_{X^*, Y^*}) - \sum_{j \in S} w_j \cdot p(S[j + 1, n])$$
$$\leq \text{TWC}(\sigma') - \sum_{j \in S} w_j \cdot p(S[j + 1, n])$$
$$= \text{TWC}(\sigma),$$

which establishes that $\sigma_{X^*, Y^*}$ is also an optimal sequence for $J$. \hfill \square

Lemma 2. $1|\text{er}| \sum w_j C_j$ can be solved in $O(nP^2)$-time.

Proof. For each $\kappa, \rho \in \mathbb{N}$ with $\alpha < \kappa \leq \beta$ and $\rho \leq K^r$, let us define

$$X_{\kappa, \rho} = \{X \subseteq H : \max X < \kappa, p(X) = \rho\},$$
$$Y_{\kappa, \rho} = \{Y \subseteq H : \min Y \geq \kappa, p(Y) = \rho\},$$
$$f_\kappa(X) = \sum_{j=\alpha}^{\kappa-1} w_j C_j^{\sigma_{X,Y}}$$
and
$$g_\kappa(Y) = \sum_{j=\kappa}^{\beta} w_j C_j^{\sigma_{X,Y}}.$$
Also let

\[ X_{\kappa, \rho} \in \arg \min_{X \in \mathcal{X}_{\kappa, \rho}} \{ f_\kappa(X) \} \quad \text{and} \quad X_{\kappa, \rho} = J[\alpha, \kappa] \setminus X_{\kappa, \rho}, \]

\[ Y_{\kappa, \rho} \in \arg \min_{Y \in \mathcal{Y}_{\kappa, \rho}} \{ g_\kappa(Y) \} \quad \text{and} \quad Y_{\kappa, \rho} = J[\kappa, \beta] \setminus Y_{\kappa, \rho}. \]

Based on Lemma 1, it suffices to find \( X^*, Y^* \subseteq H \) with \( \max X^* < \min Y^* \) such that \( \sigma_{X^*, Y^*} \) is optimal. The first step is thus to compute \( X_{\kappa, \rho} \) and \( Y_{\kappa, \rho} \) for all pairs \((\kappa, \rho)\) and then compute \( X^* = X_{\kappa^*, \rho^1} \) and \( Y^* = Y_{\kappa^*, \rho^2} \), where

\[ (\kappa^*, \rho^1, \rho^2) \in \arg \min \{ f_\kappa(X_{\kappa, \rho}) + g_\kappa(Y_{\kappa, \rho}) \} \]

and

\[ \Xi = \{ (\kappa, \rho_1, \rho_2) \mid X_{\kappa, \rho_1}, Y_{\kappa, \rho_2} \neq \emptyset, p(J[\alpha, \beta]) - \rho_1 - \rho_2 \leq K^* \}. \]

Given a tuple \((\kappa, \rho_1, \rho_2)\), Figure 3 depicts the associated sequence \( \sigma_{X_{\kappa, \rho_1}, Y_{\kappa, \rho_2}} \).

We propose two dynamic programs to obtain \( X_{\kappa, \rho} \) and \( Y_{\kappa, \rho} \) for each pair \((\kappa, \rho)\). The first dynamic program (DP1) computes for fixed \( \rho \leq K^* \) the corresponding sets \( X_{\kappa, \rho} \) for each choice of \( \kappa \). The DP is based on the following observation: Since, within each block of the sequence, jobs are ordered according to WSPT, the completion time \( C_j \) of each job \( j \) is determined entirely by the fact whether or not \( j \in X_{\kappa, \rho} \) and by the total processing time \( g = p(X_{\kappa, \rho} \cap J[\alpha, j]) \) of jobs with index at most \( j \) in \( X_{\kappa, \rho} \). If \( j \in X_{\kappa, \rho} \), then \( C_j = t_\alpha + g \) (see Figure 4b), otherwise \( C_j = p(J[1, j]) + p - g \) (see Figure 4c). Thus, iterating over the jobs in WSPT order, for each \( j \in J[\alpha, \beta - 1] \) and each \( g \leq \rho \), the DP constructs a set \( X \subseteq J[\alpha, j] \) with \( p(X) = g \) so as to minimize the total weighted completion time of the jobs in \( J[\alpha, j] \).

Formally, the DP considers states \((j, g)\) with \( j \in J[\alpha, \beta - 1] \) and \( g \leq \rho \). We introduce a cost function \( \theta_{1, \rho}(j, g) \) which denotes the total weighted completion time of job sequences so far (i.e., jobs in \( J[\alpha, j] \)). This cost function \( \theta_{1, \rho}(j, g) \) is computed recursively as follows:

\[
\theta_{1, \rho}(\alpha - 1, g) = \begin{cases} 0 & \text{if } g = 0 \\ \infty & \text{otherwise} \end{cases}
\]

\[
\theta_{1, \rho}(j, g) = \min \left\{ \begin{array} {l} \theta_{1, \rho}(j - 1, g - p_j) + w_j \cdot (t_\alpha + g) & \text{if } j \in J^o \\ \infty & \text{if } j \in J^r \\ \theta_{1, \rho}(j - 1, g) + w_j \cdot (p(J[1, j]) + p - g) & \end{array} \right\}.
\]

This recursion runs in \( O(nP) \). We immediately see that \( f_\beta(X_{\beta, \rho}) = \theta_{1, \rho}(\beta, \rho) \) and the corresponding set \( X_{\beta, \rho} \) can be retrieved, in \( O(n) \) time, by traversing the state space.
backward starting from state \((\beta - 1, \rho)\) and each time choosing the state leading to the minimum associated cost. Interestingly, as a byproduct of the above DP, we obtain \(X_{\kappa, \rho}\) for all \(\kappa\) with \(\alpha < \kappa \leq \beta\) simply by traversing the state space backward starting from \((\kappa - 1, \rho)\). This works since the cost values for states do not depend on \(\kappa\). However, note that the cost function \(\theta_{1, \rho}\) does depend on the target processing time \(\rho\) for the jobs to be included in \(X\). Thus, we must run DP1 for each choice of \(\rho \leq K^r\). Therefore, all subsets \(X_{\kappa, \rho}\) are obtained in \(O(n^2)\) time.

By a symmetric argument we can design DP2 to compute \(Y_{\kappa, \rho}\) for all \(\kappa\) and \(\rho\) in time \(O(n^2)\). Figure 5a to Figure 5c support the intuition about how completion time of job \(j\) is determined by the fact whether or not \(j \in Y_{\kappa, \rho}\) and by the total processing time \(\bar{\rho} = p(J[\alpha, \beta]) - \rho - \bar{\rho}\leq K^r\).

Finally, we show that searching over all \((\kappa, \rho_1, \rho_2) \in \Xi\) to find \(X^*\) and \(Y^*\) can be done in \(O(nP)\) time. We say \(X_{\kappa, \rho}\) dominates \(X_{\kappa, \rho'}\) if \(\rho > \rho'\) and \(f(X_{\kappa, \rho}) \leq f(X_{\kappa, \rho'})\) and \(Y_{\kappa, \rho}\) dominates \(Y_{\kappa, \rho'}\) if \(\rho > \rho'\) and \(g(Y_{\kappa, \rho}) \leq g(Y_{\kappa, \rho'})\). For each \(\kappa\), we compile a set \(X'_\kappa\) of non-dominated sets \(X_{\kappa, \rho}\) and a set \(Y'_\kappa\) of non-dominated sets \(Y_{\kappa, \rho}\), both of which are sorted in decreasing order of \(\rho\). Then for each \(\kappa\), we scan through \(X'_\kappa\), each time choose \(X_{\kappa, \rho} \in X'_\kappa\) and only pair it with \(Y_{\kappa, \rho'} \in Y'_\kappa\) with

\[
\rho' = \min \{\bar{\rho} : Y_{\kappa, \rho} \in Y'_\kappa, p(J[\alpha, \beta]) - \rho - \bar{\rho} \leq K^r\}.
\]

Among the pairs, we choose the one which minimizes \(f(X_{\kappa, \rho}) + g(Y_{\kappa, \rho'})\). Generating and scanning through the dominating sets both are done in \(O(nP)\) time.

**Lemma 3.** \(|cr| \sum w_j C_j\) can be solved in \(O(nPW)\)-time.
Proof. We define $X_{\kappa, \rho}, Y_{\kappa, \rho}, f_\kappa(X), g_\kappa(Y)$ and compute $X_{\kappa, \rho}, Y_{\kappa, \rho}, \bar{X}_{\kappa, \rho}, \bar{Y}_{\kappa, \rho}, X^*$ and $Y^*$ similarly to the proof of Lemma 2.

We propose two DPs to obtain $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$. The first DP (DP3) computes the corresponding sets $X_{\kappa, \rho}$ for each choice of $\kappa$ and for each choice of $\rho$. In DP3, we use states $(j, \varrho, \omega)$ that stores the current $j$, the total processing time of jobs added to $X_{\kappa, \rho}$ so far, and the total weight of jobs in $\bar{X}_{\kappa, \rho}$. We check jobs in $J[\alpha, \kappa - 1]$ one by one in WSPT order and decide whether to add job $j \in H$ to $X_{\kappa, \rho}$ or not (see Figure 6a). If we decide to add job $j$ to $X_{\kappa, \rho}$, then $C_j = t_\alpha + \varrho$ (see Figure 6b), otherwise job $j$ is temporarily set to be completed at $p(J[1, j])$ (see Figure 6c) but could be shifted to the right if more jobs are to be added to $X_{\kappa, \rho}$. The extent of such a shift depends on the jobs in $J[j + 1, \kappa - 1]$ that will be eventually added to $X_{\kappa, \rho}$. However, since such information is not available at state $(j, \varrho, \omega)$, when adding job $j$ to $\bar{X}_{\kappa, \rho}$, we only consider its temporary completion time while computing its cost $w_j \cdot p(J[1, j])$ and later when more information is available, we add extra costs: whenever a job $j$ is added to $X_{\kappa, \rho}$, for which a cost of $w_j \cdot (t_\alpha + \varrho)$ is incurred, jobs in $\bar{X}_{\kappa, \rho}$ also move $p_j$ time units to the right that induces an extra cost of $\omega p_j$ (recall that $\omega$ is the weight of jobs added to $\bar{X}_{\kappa, \rho}$ so far). We introduce a cost function $\theta_2(j, \varrho, \omega)$ which is the total weighted completion time of jobs sequenced so far (i.e., $J[\alpha, j]$). This cost function is computed recursively as follows:

$$\theta_2(\alpha - 1, \varrho, \omega) = \begin{cases} 0 & \text{if } \varrho = 0, \omega = 0 \\ \infty & \text{otherwise} \end{cases},$$

$$\theta_2(j, \varrho, \omega) =$$

**Figure 5:** Deciding on the position of job $j \in H$ in DP2 and DP6
(a) Before sequencing job $j$

(b) Job $j$ is assigned to $X_{\kappa,\rho}$

(c) Job $j$ is assigned to $\bar{X}_{\kappa,\rho}$

Figure 6: Deciding on the position of job $j \in H$ in DP3

\[
\min \left\{ \begin{array}{ll}
\theta_2(j-1, g - p_j, \omega) + w_j(t_\alpha + g) + \omega p_j & \text{if } j \in J^o \\
\theta_2(j-1, g, \omega - w_j) + w_j p(J[1,j]) & \text{if } j \in J^r
\end{array} \right. .
\]

This recursion runs in $O(nPW)$ time. We see that $f_\beta(X_{\beta,K^r}) = \theta_2(\beta, K^r)$ and the corresponding set $X_{\beta,K^r}$ can be retrieved, in $O(n)$ time, by traversing the state space backward starting from $(\beta - 1, K^r, w^*)$ with

\[ w^* := \arg \min_{w \in [0,w(J[\alpha,j])]} \{ \theta_2(\beta - 1, K^r, w) \} , \]

each time choosing the state with minimum cost. Interestingly, as a byproduct of the above DP, we obtain $X_{\kappa,\rho}$ for all $\kappa$ with $\alpha < \kappa \leq \beta$ and all $\rho$ with $0 < \rho \leq K^r$ simply by traversing the state space backward starting from $(\kappa - 1, \rho)$. This works since the cost values for states do not depend on $\kappa$ and $\rho$. Therefore, all subsets $X_{\kappa,\rho}$ combined are obtained in $O(nPW)$ time.

By a symmetric argument we can design DP4 to compute $Y_{\kappa,\rho}$ for each $\alpha \leq \kappa < \beta$ and $1 \leq \rho < K^r$ in time $O(nPW)$. Figure 7a to Figure 7c support the intuition about how completion time of job $j$ is determined.

Finally, we argue that searching over all $(\kappa, \rho_1, \rho_2) \in \Xi$ to find $X^*$ and $Y^*$ can be done in $O(nP)$ (see the final paragraph in the proof of Lemma 2), the proof is concluded. \qed

From Lemma 2 and Lemma 3 we infer the following theorem.

**Theorem 2.** $1|err| \sum w_j C_j$ can be solved in $O(nP \min\{P, W\})$-time.

The following corollary is immediate.

**Corollary 1.** $1|err| \sum C_j$ can be solved in $O(n^2 P)$-time.
3.2. Maximum lateness

We first show the NP-hardness of $1\text{\mid}e\text{\mid} \max L_j$ and then we propose a pseudo-polynomial time approach to solve $1\text{\mid}e\text{\mid} \max L_j$.

**Theorem 3.** $1\text{\mid}e\text{\mid} \max L_j$ is NP-hard.

**Proof.** We prove the NP-hardness of $1\text{\mid}e\text{\mid} \max L_j$ by a reduction from PARTITION which is known to be NP-hard, see (Garey and Johnson, 1979).

PARTITION: Given integer numbers $a_1, \ldots, a_m$, is there a subset of $\{a_1, \ldots, a_m\}$ with total value of $B = \frac{1}{2} \sum_{i=1}^{m} a_i$?

Given an instance of PARTITION, we construct an instance of $1\text{\mid}e\text{\mid} \max L_j$ with $m+2$ jobs as follows:

- $J = \{1, \ldots, m+2\}$ and $J^r = \{m+1, m+2\},$
- $p_j = a_j$ and $d_j = 2B + 1$ for each $j = 1, \ldots, m,$
- $p_{m+1} = 1, d_{m+1} = B + 1, p_{m+2} = 1$ and $d_{m+2} = 2B + 2,$ and
- $K^r = B + 2.$

We claim that there is a feasible schedule with maximum lateness of at most zero if and only if the answer to the instance of PARTITION is yes. Notice that zero is also a lower bound to maximum lateness since no due date exceeds the makespan of $2B + 2$.

Let us consider a schedule with lateness zero. Job $m+2$ is scheduled last since it is the only job with due date $2B + 2$. Job $m+1$ is started exactly at $B$ since it cannot
be started before $B$ due to feasibility and it cannot be started after $B$ without being tardy. Hence, we conclude that the subsets of jobs before and after job $m + 1$ both have a total processing time of $B$ and, thus, constitute a yes-certificate for the corresponding instance of PARTITION. Figure 8 depicts the structure of the schedule.

Second, if a yes-certificate for the instance of PARTITION is given we can construct a sequence with the structure discussed above and, thus, yielding maximum lateness of at most zero. This completes the proof.

We now show that $1|\text{er}|\max L_j$ can be solved in pseudo-polynomial time. First, we will show that there always exists an optimal sequence with a special structure consisting of five blocks that are internally ordered according to the earliest due date (EDD) rule.

Without loss of generality, we assume the jobs to be numbered according to EDD (i.e., $d_1 \leq \cdots \leq d_n$). We also let $\alpha = \min J', \beta = \max J'$, and $H = J[\alpha, \beta] \setminus J'$. Let $X, Y \subseteq H$ with $\max X < \min Y$. We construct a sequence $\sigma_{X,Y}$ as outlined in Section 3.1, but now with the jobs within each block being ordered according to EDD.

**Lemma 4.** For each instance of $1|\text{er}|\max L_j$ there exists $X^*, Y^* \subseteq H$ with $\max X^* < \min Y^*$ such that $\sigma_{X^*, Y^*}$ is optimal.

**Proof.** Let $\sigma$ be an optimal sequence. Let $i_1 := \min \{i : \sigma(i) \in J'\}$, $i_2 := \max \{i : \sigma(i) \in J'\}$ and $S = \{\sigma(i_1), \ldots, \sigma(i_2)\}$. Similarly to the proof of Lemma 1, we see that $\sum_{j \in S} p_j \leq K'$ by feasibility of $\sigma$ and any sequence that schedules jobs in $S$ consecutively is feasible. Therefore, rearranging the jobs within $S$ according to EDD maintains feasibility of $\sigma$ without increasing maximum lateness. We can thus assume, without loss of generality, $\alpha = \sigma(i_1) < \cdots < \sigma(i_2) = \beta$ (i.e., the jobs in $S$ are scheduled according to EDD and, in particular, $S \subseteq J[\alpha, \beta]$).

Now consider the job set $J' := J \setminus S \cup \{j'\}$ where jobs in $S$ are merged into a single job $j'$ with due date

$$d_{j'} = \min \left\{ d_{\sigma(i_1 + k)} + \sum_{i=i_1+k+1}^{i_2} p_{\sigma(i)} \mid k = 0, \ldots, i_2 - i_1 \right\}.$$ 

Thus, the lateness of job $j'$ captures the maximum lateness among jobs $\sigma(i_1), \ldots, \sigma(i_2)$ if they are scheduled consecutively in EDD. Furthermore, we let

$$X^* := \{j \in H \setminus S : d_j \leq d_{j'}\}$$

and

$$Y^* := H \setminus (S \cup X^*).$$
Note that $\max X^* < \min Y^*$ by construction and that $\sigma_{X^*, Y^*}$ is a feasible sequence for $J$. Further, note that both $\sigma_{X^*, Y^*}$ and $\sigma$ induce sequences $\sigma'_{X^*, Y^*}$ and $\sigma'$ for job set $J'$, respectively. In particular, $\sigma'_{X^*, Y^*}$ orders jobs in $J'$ according to EDD and, therefore,

$$\frac{L_{\max}(\sigma_{X^*, Y^*})}{L_{\min}(\sigma')} \leq \frac{L_{\max}(\sigma')}{L_{\max}(\sigma)},$$

which shows the optimality of $\sigma_{X^*, Y^*}$.

**Theorem 4.** $|\{\varepsilon\}| \max L_j$ can be solved in $O(nP)$-time.

**Proof.** Based on Lemma 4, it suffices to find $X^*, Y^* \subseteq H$ with $\max X^* < \min Y^*$ such that $\sigma_{X^*, Y^*}$ is optimal. As before, we define

$$X_{\kappa, \rho} = \{X \subseteq H : \max X < \kappa, p(X) = \rho\}$$

and

$$Y_{\kappa, \rho} = \{Y \subseteq H : \min Y \geq \kappa, p(Y) = \rho\}.$$

Furthermore, we define

$$f'_\kappa(X) = \max_{j \in J_{[\kappa, \kappa - 1]}} \{C_{j}^{\kappa, \rho} - d_j\} \quad \text{and} \quad g'_\kappa(Y) = \max_{j \in J_{[\kappa, \rho]}} \{C_{j}^{\kappa, \rho} - d_j\}$$

and let

$$X_{\kappa, \rho}, Y_{\kappa, \rho} \in \arg \min_{X \in X_{\kappa, \rho}} f'_\kappa(X) \quad \text{and} \quad Y_{\kappa, \rho} \in \arg \min_{Y \in Y_{\kappa, \rho}} g'_\kappa(Y)$$

for each $\kappa \in J_{[\alpha, \beta]}$ and $\rho \leq P$.

Let $\theta_3(\kappa, \rho) = \min_{X \in X_{\kappa, \rho}} f'_\kappa(X)$ and $\theta_4(\kappa, \rho) = \min_{Y \in Y_{\kappa, \rho}} g'_\kappa(Y)$. We show that $\theta_3$ and $\theta_4$ can again be expressed by simple recursions, giving ways to dynamic programs for computing $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$ for all pairs $(\kappa, \rho)$, respectively. Once $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$ for all pairs $(\kappa, \rho)$ are computed, $\sigma_{X^*, Y^*}$ is obtained as $X^* = X_{\kappa^*, \rho^*}$ and $Y^* = Y_{\kappa^*, \rho^*}$, where

$$(\kappa^*, \rho_1^*, \rho_2^*) \in \arg \min_{(\kappa, \rho_1, \rho_2) \in \Xi} \{\max\{f'_\kappa(X_{\kappa, \rho_1}), g'_\kappa(Y_{\kappa, \rho_2})\}\}$$

and

$$\Xi = \{(\kappa, \rho_1, \rho_2) | X_{\kappa, \rho_1}, Y_{\kappa, \rho_2} \neq \emptyset, p(J_{[\kappa, \beta]}) - \rho_1 - \rho_2 \leq K^*\}.$$

The sequence $\sigma_{X^*, Y^*}$ minimizes maximum lateness among all such sequences because $C_{j}^{\kappa^*, \rho^*} = C_{j}^{\kappa^*, \rho^*}$ for all $j \in J_{[\kappa^*, \rho^*]}$, and the completion time of job $j \in J \setminus J_{[\alpha, \beta]}$ is independent of the choice of $X^*$ and $Y^*$.

Note that $\kappa^*, \rho_1^*, \rho_2^*$ can again be determined in time $O(nP)$ (see the final paragraph in the proof of Lemma 2)

In order to obtain the recursion of $\theta_3$, we first prove the following two claims on the structure of the function $f'_\kappa$.

**Claim 3.** For all $X \subseteq H$ and $\kappa \in J_{[\alpha, \beta]}$, we have

$$f'_\kappa(X) = \max_{j \in J_{[\alpha, \kappa - 1] \setminus X}} \{C_{j}^{\kappa, \rho} - d_j\}.$$
Proof. Note that for all \( j \in X \), we have \( C_{j}^{X, \kappa} \leq C_{\alpha}^{X, \kappa} \) and \( d_\alpha \leq d_j \). Thus \( C_{\alpha}^{X, \kappa} - d_\alpha \geq C_{j}^{X, \kappa} - d_j \) for any \( j \in X \).

**Claim 4.** Let \( X \subseteq H \) and \( \kappa > \max X \). Then \( f'_{\kappa+1}(X \cup \{\kappa\}) = f'_{\kappa}(X) + p_\kappa \) and \( f'_{\kappa+1}(X) = \max\{f'_{\kappa}(X), \ p(J[1, \kappa]) - d_\kappa\} \).

Proof. The first equality follows immediately from the preceding claim because \( C_{j}^{X \cup \{\kappa\}, \kappa} = C_{\beta}^{X, \kappa} + p_\kappa \) for all \( j \in J[\alpha, \kappa - 1] \setminus X \). The second equality follows from the definition of \( f'_{\kappa} \) and the fact that \( \kappa > \max X \), and hence job \( \kappa \) precedes each job \( j > \kappa \) in \( \sigma_{X, \varphi} \).

From Claim 4 we can deduce the following recursion for \( \theta_3 \):

\[
\theta_3(\alpha + 1, \rho) = \begin{cases} \rho(J[1, \alpha]) - d_\alpha & \text{if } \rho = 0 \\ \infty & \text{otherwise} \end{cases},
\]

\[
\theta_3(\kappa + 1, \rho) = \min \left\{ \begin{array}{l}
\max \left\{ \theta_3(\kappa, \rho), \ p(J[1, \kappa]) - d_\kappa \right\}, \\
\{ \theta_3(\kappa, \rho - p_\kappa) + p_\kappa \text{ if } \kappa \in J^o \}
\end{array} \right\}
\]

In order to obtain the recursion of \( \theta_4 \), we first prove the following claim on the structure of the function \( g' \).

**Claim 5.** Let \( Y \subseteq H \) and \( \kappa < \min Y \). Then

\[
g'_{\kappa}(Y) = \max\{g'_{\kappa+1}(Y), \ p(J[1, \kappa]) - d_\kappa\} \quad \text{and} \quad g'_{\kappa}(Y \cup \{\kappa\}) = \max\{g'_{\kappa+1}(Y), \ p(J[1, \beta]) - p(Y) - d_\kappa\}.
\]

Proof. The first identity follows from the definition of \( g'_{\kappa} \) and the fact that all jobs \( j < \kappa \) precede \( \kappa \) in \( \sigma_{\beta, Y} \) because \( \kappa < \min Y \).

Now let \( j^* \in J[\kappa, \beta] \) be such that \( g'_{\kappa}(Y \cup \{\kappa\}) = C_{j^*}^{\sigma_{\beta, Y \cup \{\kappa\}}} - d_{j^*} \). Note that \( d_{j^*} \geq d_\kappa \) because jobs are ordered according to EDD. Therefore, our choice of \( j^* \) implies \( C_{j^*}^{\sigma_{\beta, Y \cup \{\kappa\}}} \geq C_{\kappa}^{\sigma_{\beta, Y \cup \{\kappa\}}} \). We can thus conclude that \( j^* \in Y \cup \{\kappa\} \). The second identity then follows from the observation that \( C_{j^*}^{\sigma_{\beta, Y \cup \{\kappa\}}} = C_{j^*}^{\sigma_{\beta, Y}} \) for all \( j \in Y \) and \( C_{\kappa}^{\sigma_{\beta, Y \cup \{\kappa\}}} = p(J[1, \kappa] \setminus Y) \), by construction of \( \sigma_{\beta, Y \cup \{\kappa\}} \) and \( \kappa < \min Y \).

Claim 5 implies the following recursion for \( \theta_4 \):

\[
\theta_4(\beta, \rho) = \begin{cases} \rho(J[1, \beta]) - d_\beta & \text{if } \rho = 0 \\ \infty & \text{otherwise} \end{cases},
\]

\[
\theta_4(\kappa, \rho) = \min \left\{ \begin{array}{l}
\max \left\{ \theta_4(\kappa + 1, \rho), \ p(J[1, \kappa]) - d_\kappa \right\}, \\
\{ \theta_4(\kappa + 1, \rho - p_\kappa) + p_\kappa \text{ if } \kappa \in J^o \}
\end{array} \right\}
\]

From the above recursions for \( \theta_3 \) and \( \theta_4 \), it is easy to see that we can compute \( X_{\kappa, \rho} \) and \( Y_{\kappa, \rho} \) for all \( \kappa \in J[\alpha, \beta] \) and all \( \rho \leq P \) combined in time \( O(nP) \). This concludes the proof of the theorem. \( \square \)
3.3. Weighted number of tardy jobs

Theorem 3 implies that even minimizing the unweighted number of tardy jobs is NP-hard.

Corollary 2. \(1\text{|er} \sum w_jU_j\) is NP-hard.

In the following, we describe a pseudo-polynomial algorithm for solving \(1\text{|er} \sum w_jU_j\). Again, we start with an observation on the structure of an optimal solution.

As in the previous section, we assume the jobs to be numbered according to EDD, i.e., \(d_1 \leq \cdots \leq d_n\). For disjoint sets \(X, Y, Z \subseteq J\), let \(\sigma_{X,Y,Z}\) be the sequence consisting of the following five blocks with each block internally ordered according to EDD: The first block consists of the jobs in \(X\); the second block consists of the jobs in \(Y\); the third block consists of the jobs in \(J^r \setminus Y\); the fourth block consists of the jobs in \(Z\); the fifth block consists of all remaining jobs.

Lemma 5. For each instance of \(1\text{|er} \sum w_jU_j\) there exist disjoint sets \(X^*, Y^*, Z^* \subseteq J\) such that the sequence \(\sigma_{X^*,Y^*,Z^*}\) is optimal and, moreover,

1. all jobs in \(X^* \cup Y^* \cup Z^* = E\) are non-tardy in \(\sigma_{X^*,Y^*,Z^*}\),
2. \((X^* \cup Z^*) \cap J^r = \emptyset\), and
3. \(\max(X^* \cup (Y^* \cap J^o)) < \min Z^*\).

Proof. Let \(\sigma\) be an optimal feasible sequence. Let \(i_1 := \min\{i : \sigma(i) \in J^r\}\) and \(i_2 := \max\{i : \sigma(i) \in J^r\}\) be the first and last occurrence, respectively, of an \(r\)-job in the sequence. Let \(E = \{j \in J \mid U_j^r = 0\}\) be the set of non-tardy jobs in \(\sigma\). We define \(X = \{j \in E \mid \sigma^{-1}(j) < i_1\}\), \(Y = \{j \in E \mid i_1 \leq \sigma^{-1}(j) \leq i_2\}\), and \(Z = \{j \in E \mid i_2 < \sigma^{-1}(j)\}\), with \(\sigma^{-1}(j)\) being the position of job \(j\) in sequence \(\sigma\).

It is easy to see that \(X, Y, Z\) are disjoint and fulfill property 2, and that \(\sigma_{X,Y,Z}\) is feasible because \(Y \cap J^o\) is a subset of \(o\)-jobs processed between the first and the last \(r\)-job in \(\sigma\), which means

\[
p(Y \cup (J^r \setminus Y)) = p(J^r \cup (Y \cap J^o)) = p(J^r) + p(Y \cap J^o) \\
\leq p(J^r) + p(\{j \in J^o \mid i_1 \leq \sigma^{-1}(j) \leq i_2\}) \\
\leq K^r
\]

by feasibility of \(\sigma\). We now show the following claim, which immediately implies that \(X, Y, Z\) fulfills property 1 and that \(\sigma_{X,Y,Z}\) is optimal.

Claim 6. Every job in \(X \cup Y \cup Z = E\) is non-tardy in \(\sigma_{X,Y,Z}\).

Proof. We obtain \(\sigma_{X,Y,Z}\) from \(\sigma\) by applying the following three modifications.

First, we delay all tardy jobs in \(J^o\) to the end of the schedule (keeping their relative order). No non-tardy job is delayed by this. Let \(i_3\) be the last slot holding a job in \(J^r\) after this modification.
Lemma 5, it is sufficient to identify appropriate sets \( \sigma \) considering the sequence \( k \sigma \).

Proof. By the lemma. 1

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We now have five blocks in the current sequence holding jobs from \( X, Y, J^r \setminus Y, Z, \) and \( J^o \setminus (X, Y, Z) \). Finally, by having each block in EDD we do not cause any currently non-tardy job to be tardy.

To establish property 3, let \( k = \min Z \) and \( S = \{ j \in J^o \cap (X \cup Y) \mid j > k \} \). Now consider the sequence \( \sigma' \) that arises from \( \sigma_{X,Y,Z} \) by moving all jobs in \( S \) to the position right before job \( k \) (in arbitrary order). Note that \( C_j^* \leq C_j^{x,y,z} \) for all \( j \in J \setminus S \), and \( C_j^* \leq C_j^{x,y,z} \) for all \( j \in S \). Furthermore, resorting the jobs of \( S \cup Z \) in \( \sigma' \) according to EDD does not cause any job to become tardy. The resulting sequence is \( \sigma_{X',Y',Z'} \). For \( X = X \setminus S, Y = Y \setminus S, \) and \( Z = Z \cup S \) and fulfills all requirements of the lemma.

Theorem 5. \( 1|\sigma| \sum w_j U_j \) can be solved in \( O(nP^4) \) time.

Proof. By Lemma 5, it is sufficient to identify appropriate sets \( X^*, Y^*, Z^* \) as described in the lemma. We do so using three dynamic programs: one for constructing candidates for \( X^* \) and a prefix of \( Y^* \), one for constructing candidates for a suffix of \( Y^* \) only containing \( r \)-jobs, and one for constructing candidates for \( Z^* \).

More precisely, let \( \kappa^* = \min Z^\cup \{n+1\}, \rho_1^* = p(X^* \cup (Y^*[1, \kappa - 1])), \) and \( \rho_2^* = p(X^* \cup Y^* \cup J^r) \). We enumerate all possible values \( \kappa, \rho_1, \rho_2 \) for \( \kappa^*, \rho_1^*, \rho_2^* \) and determine candidates \( X_{\kappa,\rho_1,\rho_2} \) for \( X^*, Y_{\kappa,\rho_1,\rho_2} \) for \( Y^*[1, \kappa - 1], Y_{\kappa,\rho_1} \) for \( Y^* \{\kappa, n\}, \) and \( Z_{\kappa,\rho_2} \) for \( Z^* \).

We now describe the three DP's for computing the pair \( (X_{\kappa,\rho_1,\rho_2}, Y_{\kappa,\rho_1,\rho_2}) \), and the sets \( Y_{\kappa,\rho_1} \) and \( Z_{\kappa,\rho_2} \), respectively. Our goal is to make sure that the jobs in the respective sets will not be tardy while maximizing the total weight of jobs contained the set.

For \( \kappa \in \{1, \ldots, n+1\} \) and \( \rho_1, \rho_2 \in \{0, \ldots, P\} \), define

\[
X_{\kappa,\rho_1,\rho_2} = \begin{cases} 
(X, Y) & \text{if } X, Y \subseteq J^o[1, \kappa - 1], X \cap Y = \emptyset = X \cap J^r, \\
 & \sum_{j' \in X \cup Y, j' \leq j} p_{j'} \leq d_j, \forall j \in X \cup Y', \\
 & p(Y' \cap J^o) \leq K^r - p(J^r), \\
 & p(X \cup Y') = \rho_1 + \rho_2 \end{cases}
\]

\[
Y_{\kappa,\rho_1} = \begin{cases} 
Y^* & \text{if } Y^* \subseteq J^r[\kappa, n], \rho_1 + \sum_{j' \in Y^*, j' \leq j} p_{j'} \leq d_j, \forall j \in Y^* 
\end{cases}
\]

\[
Z_{\kappa,\rho_2} = \begin{cases} 
Z & \text{if } Z \subseteq J^o[\kappa, n], \rho_2 + \sum_{j' \in Z, j' \leq j} p_{j'} \leq d_j, \forall j \in Z 
\end{cases}
\]

and let

\[
(X_{\kappa,\rho_1,\rho_2}, Y'_{\kappa,\rho_1,\rho_2}) \in \arg \max_{(X, Y') \in X_{\kappa,\rho_1,\rho_2}} w(X), \\
Y''_{\kappa,\rho_1,\rho_2} \in \arg \max_{Y'' \in Y_{\kappa,\rho_1}} w(Y''), \text{ and} \\
Z_{\kappa,\rho_2} \in \arg \max_{Z \in Z_{\kappa,\rho_2}} w(Z)
\]
Note that $Y''_{\kappa,\rho_1}$ and $Z_{\kappa,\rho_2}$ can be computed in time $O(nP)$ for all choices of $\kappa, \rho_1, \rho_2$ by a dynamic program for the classic problem $1||\sum w_j U_j$, see Sahni (1976).

In order to construct $X_{\kappa,\rho_1,\rho_2}$ and $Y'_{\kappa,\rho_1,\rho_2}$, we guess $t = p(X)$ and construct two sets $X,Y'$ by iterating through the jobs from 1 to $\kappa$ in EDD order, keeping track of the processing time of the jobs added to $X$ so far (denoted by $\varrho$), the processing time of the jobs added to $Y'$ so far ($\varrho'$) and the processing time of the o-jobs added to $Y'$ so far ($\varrho''$). To this end, we define

$$\theta_{5,t}(k, \varrho, \varrho', \varrho'') = \max \left\{ \begin{array}{l} w(X \cup Y') \\ \text{s.t. } X, Y' \subseteq J[1,k], \ X \cap Y' = \emptyset = X \cap J^r, \ \sum_{j \in X} p_j \leq d_j, \forall j \in X, \ t + \sum_{j' \in Y', j' \leq j} p_j \leq d_j, \forall j \in Y', \ p(X) = \varrho, \ p(Y') = \varrho', \ p(Y' \cap O) = \varrho'' \end{array} \right\}$$

and observe that $\theta_{5,t}$ can be computed by the following recursion

$$\begin{align*}
\theta_{5,t}(j, \varrho, \varrho', \varrho'') & = \begin{cases} 0 & \text{if } \varrho, \varrho', \varrho'' = 0 \\ -\infty & \text{otherwise} \end{cases}, \\
\theta_{5,t}(j, \varrho, \varrho', \varrho'') & = \max \left\{ \begin{array}{l} \theta_{5,t}(j - 1, \varrho, \varrho', \varrho'') + w_j \text{ if } \varrho \leq d_j, j \in J^o \\ -\infty \text{ otherwise} \end{array} \right\} \\
\theta_{5,t}(j - 1, \varrho, \varrho', \varrho'' - p_j) + w_j \text{ if } t + \varrho' \leq d_j, j \in J^o \\ -\infty \text{ otherwise} \right\}
\end{align*}$$

We can thus compute the values $\theta_{5,t}(j, \varrho, \varrho', \varrho'')$ for all choices of $j \in J$ and $t, \varrho, \varrho', \varrho'' \in \{0, \ldots, P\}$ in time $O(nP^4)$. Note that

$$\begin{align*}
w(X_{\kappa,\rho_1,\rho_2}) + w(Y'_{\kappa,\rho_1,\rho_2}) &= \max \left\{ \begin{array}{l} \theta_{5,t}(\kappa - 1, \varrho', \varrho'') \text{ if } \varrho'' \leq K^r - p(J^r), \\ \rho_1 + p(J^r) - \varrho' + \varrho'' = \rho_2 \end{array} \right\}, \\
\end{align*}$$

Hence we can obtain $X_{\kappa,\rho_1,\rho_2}$ and $Y'_{\kappa,\rho_1,\rho_2}$ by iterating through all combinations of $t, \varrho' \in \{0, \ldots, P\}$ and $\varrho'' \in \{0, \ldots, K^r - p(J^r)\}$.

After constructing $X_{\kappa,\rho_1,\rho_2}$, $Y'_{\kappa,\rho_1,\rho_2}$, $Y''_{\kappa,\rho_1}$, and $Z_{\kappa,\rho_2}$ for all choices of $\kappa \in \{1, \ldots, n + 1\}$ and $\rho_1, \rho_2 \in \{0, \ldots, P\}$, we can find $\kappa^*, \rho_{1}^*, \rho_{2}^*$ so as to maximize $w(X_{\kappa^*,\rho_{1}^*,\rho_{2}^*}) + w(Y'_{\kappa^*,\rho_{1}^*,\rho_{2}^*}) + w(Y''_{\kappa^*,\rho_{1}^*,\rho_{2}^*}) + w(Z_{\kappa^*,\rho_{2}^*})$ in time $O(nP^2)$. $\square$

4. Complexity results for $1|\gamma|\text{er}$ and $1||\text{(er, }\gamma)$

4.1. Complexity results for $1|\gamma|\text{er}$

**Theorem 6.** $1|\sum C_j|\text{er}$ is NP-hard.
Proof. Suppose we have an algorithm $A^{er}$ that solves $1\{C_j\}|er$ in polynomial time. We, then, can solve $1\{er\}|\sum C_j$ by applying $A^{er}$ in a binary search scheme. Since $\sum_{j \in J} C_j \leq n \sum_{j \in J} p_j$ we have to solve at most $O(\log n + \log(\sum p_j))$ instances of $1\{C_j\}|er$ and, thus, can solve $1\{er\}|\sum C_j$ in polynomial time. However, Theorem 1 states that this is not possible (unless $P = NP$).

**Theorem 7.** $1\{max L_j\}|er$ is NP-hard.

**Proof.** As in the proof of Theorem 6 we can conclude NP-hardness of $1\{max L_j\}|er$ from NP-hardness of $1\{er\}|max L_j$, see Theorem 3.

**Theorem 8.** Given $A^{\gamma}$ as an algorithm that solves $1\{er\}|\gamma$ in $O(T(n, P, W))$, there is an algorithm $A^{r}$ that solves $1\{er\}|\rho$ in $O(T(n, P, W) \log P)$.

**Proof.** Note that we can employ the algorithm $A^{\gamma}$ to check whether there exists a feasible schedule with scheduling cost of at most $K^{\gamma}$ and with a renting period of at most $K^{r}$. Hence, we can perform binary search to determine the minimum length of the renting period such that there exists a feasible schedule with scheduling cost of at most $K^{r}$. As $P$ is a natural upper bound on the renting period the binary search takes at most $\log P$ steps.

In particular, Theorem 8 implies that we can use any pseudopolynomial algorithm for $1\{er\}|\gamma$ to obtain a pseudopolynomial algorithm for $1\{er\}|\rho$ with the runtime increasing only by a factor of $\log P$.

The generic result of Theorem 8 suggests that $1\{|w_j C_j\}|er$ and $1\{max L_j\}|er$ are solvable in $O(nP \min\{W, P\} \log P)$ and $O(nP \log P)$, respectively. We now show that these two problems can be solved more efficiently.

**Theorem 9.** $1\{|w_j C_j\}|er$ can be solved in $O(nP \min\{W, P\})$ time.

**Proof.** Let us obtain subsets $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$ for all $\kappa$ with $\alpha < \kappa \leq \beta$ and all $\rho$ with $0 < \rho \leq K^{r}$, as described in the proofs of Lemma 2 and Lemma 3. Then we compute $X^{*} = X_{\kappa^{*}, \rho^{*}}$ and $Y^{*} = Y_{\kappa^{*}, \rho^{*}}$, where

$$(\kappa^{*}, \rho^{1}, \rho^{2}) \in \arg \min_{(\kappa, \rho^{1}, \rho^{2}) \in \Xi^{'} \gamma} \{p(J[\alpha, \beta]) - \rho_{1} - \rho_{2}\}$$

and

$$\Xi^{'} = \{(\kappa, \rho_{1}, \rho_{2}) \mid X_{\kappa, \rho_{1}}, Y_{\kappa, \rho_{2}} \neq \emptyset, f_{\kappa}(X_{\kappa, \rho_{1}}) + g_{\kappa}(Y_{\kappa, \rho_{2}}) \leq K^{\gamma}\}.$$
Obtaining all subsets $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$ requires $O(n^p)$ (see the proof of Theorem 4) and finding $(\kappa^*, \rho_1^*, \rho_2^*)$ can be done in $O(n^p)$.

4.2. Complexity results for $1\| (er, \gamma)$

**Theorem 11.** If $1\| (er) \gamma$ is NP-hard, then $1\| (er, \gamma)$ is NP-hard.

*Proof.* Suppose we have an algorithm $A^{(er, \gamma)}$ that solves $1\| (er, \gamma)$ in polynomial time. Note that the Pareto set has at most $n$ members. We, thus, can sort its member in $O(n \log n)$ time. Then, we can solve $1\| (er) \gamma$ by applying binary search inspecting the members of the Pareto set.

**Theorem 12.** Given $A^\gamma$ as an algorithm that solves $1\| (er) \gamma$ in $O(T(n, P, W))$, there is an algorithm $A^{r, \gamma}$ that solves $1\| (er, \gamma)$ in $O(T(n, P, W)^n)$.

*Proof.* The proof is similar to the proof of Theorem 8.

The generic result of Theorem 12 suggests that $1\| (er, \sum w_j C_j)$ is solvable in $O(n^p 2^\min\{W, P\})$. In the following, however, we show that this problem only requires $O(n^p)$ to be solved.

**Theorem 13.** $1\| (er, \sum w_j C_j)$ can be solved in $O(n^p)$ time.

*Proof.* Again, let us obtain subsets $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$ for all $\kappa$ with $\alpha < \kappa \leq \beta$ and all $\rho$ with $0 < \rho \leq K^r$, as described in the proof of Lemma 2. Then for each fixed value $L \leq K^r$, we compute $X^*_L = X_{\kappa^*_L, \rho^*_1, \rho^*_2}$ and $Y^*_L = Y_{\kappa^*_L, \rho^*_1, \rho^*_2}$, where

$$(\kappa^*_L, \rho^*_1, \rho^*_2) \in \{ (\kappa, \rho_1, \rho_2) \in \Xi \mid f_\kappa(X_{\kappa, \rho_1}) + g_\kappa(Y_{\kappa, \rho_2}) \}$$

and

$$\Xi_L = \{ (\kappa, \rho_1, \rho_2) \mid X_{\kappa, \rho_1}, Y_{\kappa, \rho_2} \neq 0, \ p(J[\alpha, \beta]) - \rho_1 - \rho_2 = L \}.$$

The Pareto front is obtained by considering all sequences $\sigma_{X^*_L, Y^*_L}$. Obtaining all subsets $X_{\kappa, \rho}$ and $Y_{\kappa, \rho}$ requires $O(n^p)$ and finding $(\kappa^*_L, \rho^*_1, \rho^*_2)$ for each $L \leq K^r$ requires $O(n^p)$ (see the last paragraph of the proof of Lemma 2), which combined requires $O(n^p^2)$ time.

5. Summary and conclusion

We study three classes of single machines scheduling problems with an external resource. A class of problems where the length of the renting period is budgeted and the scheduling cost needs to be minimized, a class of problems where the scheduling cost is budgeted and the length of the renting period needs to be minimized, and finally a class of two-objective problems where both the length of the renting period and the scheduling cost are to be minimized. For each class, we consider total (weighted) completion time,
maximum lateness, or weighted number of tardy jobs as the scheduling cost function. We show that all discussed problems are NP-hard in ordinary sense. Table 1 provides a summary of the complexity of the proposed pseudo-polynomial algorithms in this paper.

A natural generalization considers the case where rental intervals have to be determined for multiple distinct resources and each job can only be scheduled when all its required resources are available. This setting constitutes a generalization of the linear arrangement problem (LAP; Adolphson and Hu (1973); Liu and Vannelli (1995)) to hypergraphs: The jobs correspond to the nodes and each set of jobs requiring a specific resource corresponds to a hyperedge. A schedule corresponds to an ordering of the nodes, where each hyperedge incurs a cost proportional to the difference of the latest completion time and the earliest start time of a job within the hyperedge. The LAP is notorious for being computationally challenging both in theory and practice. Still, devising exponential-time exact methods or efficient approximation algorithms for this setting are interesting directions of future research.

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