On the Spectral Singularities and Spectrality of the Hill Operator

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Abstract
First we study the spectral singularity at infinity and investigate the connections of the spectral singularities and the spectrality of the Hill operator. Then we consider the spectral expansion when there is not the spectral singularity at infinity.

Key Words: Hill operator, Spectral singularities, Spectral operator, Spectral expansion.

AMS Mathematics Subject Classification: 34L05, 34L20.

1 Introduction

In this paper we investigate the one dimensional Schrödinger operator $L(q)$ generated in $L^2(-\infty, \infty)$ by the differential expression

$$l(y) = -y''(x) + q(x)y(x),$$

where $q$ is 1-periodic, Lebesgue integrable on $[0, 1]$ and complex-valued potential. Without loss of generality we assume that the integral of $q$ over $[0, 1]$ is 0. It is well-known [1,5-9] that the spectrum $\sigma(L)$ of the operator $L$ is the union of the spectra $\sigma(L_t)$ of the operators $L_t(q)$ for $t \in (-\pi, \pi]$ generated in $L^2[0, 1]$ by the expression (1) and the boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0).$$

The eigenvalues of $L_t$ are the roots of the characteristic equation

$$F(\lambda) = 2 \cos t,$$

where $F(\lambda) =: \varphi'(1, \lambda) + \theta(1, \lambda)$ is the Hill discriminant $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ are the solutions of the equation $l(y) = \lambda y$ satisfying the following initial conditions

$$\theta(0, \lambda) = \varphi'(0, \lambda) = 1, \quad \theta'(0, \lambda) = \varphi(0, \lambda) = 0.$$

In this paper we study the spectral singularity of $L(q)$ at infinity, investigate the connections of the spectral singularities and the spectrality of $L(q)$ and consider the spectral expansion of $L(q)$ when there is not the spectral singularity at infinity. Note that the spectral singularities of the operator $L(q)$ are the points of its spectrum in neighborhoods of which the projections of $L(q)$ are not uniformly bounded (see [9] and [10]). McGarvey [6] proved that $L(q)$ is a spectral operator if and only if the projections of the operators $L_t(q)$ are bounded uniformly with respect to $t$ in $(-\pi, \pi]$. Tkachenko proved in [9] that the non-self-adjoint operator $L$ can be reduced to triangular form if all eigenvalues of the operators $L_t$ for $t \in (-\pi, \pi]$ are simple. Gesztesy and Tkachenko [3,4] proved two versions of a criterion for the Hill operator $L(q)$ with $q \in L^2[0, 1]$ to be a spectral operator of scalar type, in
sense of Danford, one analytic and one geometric. The analytic version was stated in term of the solutions of Hill’s equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems. In paper [12, 13] we found the conditions on the potential \( q \) such that \( L(q) \) has no spectral singularity at infinity and it is an asymptotically spectral operator.

Now let us recall the precise definition of the spectral singularities and asymptotic spectrality of \( L(q) \). Following [4, 10], we define the projections and the spectral singularities of \( L \) as follows. By Definition 2.4 of [4], a closed arc

\[
\gamma = \{ z \in \mathbb{C} : z = \lambda(t), t \in [\alpha, \beta] \}
\]

(4)

with \( \lambda(t) \) continuous on the closed interval \([\alpha, \beta]\), analytic in an open neighborhood of \([\alpha, \beta]\) and

\[
F(\lambda(t)) = 2 \cos t,
\]

\[
\frac{\partial F(\lambda(t))}{\partial \lambda} \neq 0, \quad \forall t \in [\alpha, \beta], \quad \lambda'(t) \neq 0, \quad \forall t \in (\alpha, \beta)
\]

is called a regular spectral arc of \( L(q) \). The projection \( P(\gamma) \) corresponding to the regular spectral arc \( \gamma \) is defined by

\[
P(\gamma)f = \frac{1}{2\pi} \int_{\gamma} \left[ (\Phi_+(x, \lambda)F_-(\lambda, f) + \Phi_-(x, \lambda)F_+(\lambda, f)) \frac{\varphi(1, \lambda)}{p(\lambda)} \right] d\lambda,
\]

(5)

where

\[
\Phi_{\pm}(x, \lambda) = \theta(x, \lambda) + (\varphi(1, \lambda))^{-1}(e^{\pm it} - \theta(1, \lambda))\varphi(x, \lambda)
\]

are the Floquet solution and

\[
F_{\pm}(\lambda, f) = \int_{\mathbb{R}} f(x)\Phi_{\pm}(x, \lambda)dx, \quad p(\lambda) = \sqrt{4 - F^2(\lambda)}.
\]

**Definition 1** We say that \( \lambda \in \sigma(L(q)) \) is a spectral singularity of \( L(q) \) if for all \( \varepsilon > 0 \) there exists a sequence \( \{\gamma_n\} \) of the regular spectral arcs \( \gamma_n \subset \{ z \in \mathbb{C} : |z - \lambda| < \varepsilon \} \) such that

\[
\lim_{n \to \infty} \| P(\gamma_n) \| = \infty.
\]

(6)

In the similar way, we defined in [12] the spectral singularity at infinity.

**Definition 2** We say that the operator \( L \) has a spectral singularity at infinity if there exists a sequence \( \{\gamma_n\} \) of the regular spectral arcs such that \( d(0, \gamma_n) \to \infty \) as \( n \to \infty \) and (6) holds, where \( d(0, \gamma_n) \) is the distance from the point \((0, 0)\) to the arc \( \gamma_n \).

The asymptotic spectrality of the operator \( L(q) \) was defined in [12] as follows. Let \( e(t, \gamma) \) be the spectral projection defined by contour integration of the resolvent of \( L(t) \), where \( \gamma \in \mathcal{R} \) and \( \mathcal{R} \) is the ring consisting of all sets which are the finite union of the half closed rectangles. In [6] it was proved Theorem 3.5 (for the differential operators of arbitrary order with periodic coefficients rather than for \( L(q) \)) which can be written in the form:

\( L(q) \) is a spectral operator if and only if

\[
\sup_{\gamma \in \mathcal{R}} \left( \sup_{t \in [-\pi, \pi]} \| e(t, \gamma) \| \right) < \infty.
\]

(7)

According to this theorem, in [12] we gave the following definition of the asymptotic spectrality.
Definition 3. The operator \( L(q) \) is said to be an asymptotically spectral operator if there exists a positive constant \( C \) such that

\[
\sup_{\gamma \in R(C)} (\text{ess sup}_{t \in (-\pi, \pi]} || e(t, \gamma) ||) < \infty,
\]

where \( R(C) \) is the ring consisting of all sets which are the finite union of the half closed rectangles lying in \( \{ \lambda \in \mathbb{C} : |\lambda| > C \} \).

In [12] and [13] we obtained the following results about asymptotic spectrality under the following conditions.

Condition 1. Let \( q \in W^r_p[0, 1] \), \( q^{(k)}(0) = q^{(k)}(1) \), for \( k = 0, 1, ..., s - 1 \) and \( q^{(s)}(0) \neq q^{(s)}(1) \) for some \( s \leq p \). Suppose that \( q_n \sim q_{-n} \), \( | q_n | > cn^{-s-1} \) and at least one of the following inequalities

\[
\Re q_n q_{-n} \geq 0, \quad | \Im q_n q_{-n} | \geq \varepsilon | q_n q_{-n} |
\]

hold for some \( c > 0 \) and \( \varepsilon > 0 \) and for large value of \( n \), where \( q_n \) is the Fourier coefficient of the potential \( q \) and \( q_n \sim q_{-n} \) means that \( q_n = O(q_{-n}) \) and \( q_{-n} = O(q_n) \).

Condition 2. Suppose that

\[
q(x) = ae^{i2\pi x} + be^{i2\pi x},
\]

\[
| a | = | b |, \quad \inf_{q, p \in \mathbb{N}} \{ | q_0 - (2p - 1) | \} \neq 0,
\]

where \( a \) and \( b \) are the complex numbers and \( \alpha = \pi^{-1} \arg(ab) \).

In [12] we proved that if Condition 1 holds then \( L(q) \) is an asymptotically spectral operator and has no spectral singularity at infinity. It was proven in [13] that the operator \( L(q) \) with the potential (8) is an asymptotically spectral operator and has no spectral singularity at infinity if and only if (9) holds.

2 Spectrum, Spectral Singularity and Spectrality

First, let us discuss the spectrum of \( L(q) \) by using some results of [11, 12] about the uniform with respect to \( t \) in \( (-\pi, \pi] \) asymptotic formulae for eigenvalues of the operator \( L_t(q) \). Note that, the formula \( f(k, t) = O(h(k)) \) is said to be uniform with respect to \( t \) in a set \( I \) if there exist positive constants \( M \) and \( N \), independent of \( t \), such that

\[
| f(k, t) | < M | h(k) |
\]

for all \( t \in I \) and \( | k | \geq N \).

In the case \( q = 0 \) the eigenvalues and eigenfunctions of \( L_t(q) \) are \( (2\pi n + t)^2 \) and \( e^{i(2\pi n + t)x} \) for \( n \in \mathbb{Z} \). In [11] we proved that the large eigenvalues of the operators \( L_t(q) \) for \( t \neq 0, \pi \) consist of the sequence \( \{ \lambda_n(t) : | n | \gg 1 \} \) satisfying

\[
\lambda_n(t) = (2\pi n + t)^2 + O\left( \frac{\ln |n|}{n} \right).
\]

This asymptotic formula is uniform with respect to \( t \) in \( [\rho, \pi - \rho] \), where \( \rho \in (0, \frac{\pi}{2}) \). There exists a positive number \( N(\rho) \), independent of \( t \), such that the eigenvalues \( \lambda_n(t) \) for \( t \in [\rho, \pi - \rho] \) and \( | n | > N(\rho) \) are simple and hence is analytic function in the neighborhood of \( [\rho, \pi - \rho] \).
In the paper [12] we proved that there exists a positive integer \( N(0) \) such that the disk

\[
U(n,t,\rho) = \{ \lambda \in \mathbb{C} : |\lambda - (2\pi n + t)| \leq 15\pi n \rho \}
\]

for \( t \in [0,\rho] \), where \( 15\pi \rho < 1 \), and \( n > N(0) \) contains two eigenvalues (counting with multiplicities) denoted by \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) and these eigenvalues can be chosen as continuous function of \( t \) on the interval \([0,\rho]\). In addition to these eigenvalues, the operator \( L_t(q) \) for \( t \in [0,\rho] \) has only \( 2N(0) + 1 \) eigenvalues denoted by \( \lambda_k(t) \) for \( k = 0, \pm 1, \pm 2, \ldots, \pm N \) (see Remark 2.1 of [12]). Similarly, there exists a positive integer \( N(\pi) \) such that the disk \( U(n,t,\rho) \) for \( t \in [\pi - \rho, \pi] \) and \( n > N(\pi) \) contains two eigenvalues (counting with multiplicities) denoted again by \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) that are continuous function of \( t \) on the interval \([\pi - \rho, \pi]\). In addition to these eigenvalues, the operator \( L_t(q) \) for \( t \in [\pi - \rho, \pi] \) has only \( 2N(\pi) + 1 \) eigenvalues. Thus for \( n > N := \max \{ N(\rho), N(0), N(\pi) \} \), the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) are continuous on \([0,\rho] \cup [\pi - \rho, \pi] \) and for \( n > N \) the eigenvalue \( \lambda_n(t) \), defined by (10), is analytic function in the neighborhood of \([\rho, \pi - \rho] \). Moreover, by (10) there exist only two eigenvalues \( \lambda_{n,1}(\rho) \) and \( \lambda_{n,2}(\rho) \) of the operator \( L_\rho(q) \) lying in the disk \( U(n,\rho,\rho) \). Therefore these 2 eigenvalues coincides with the eigenvalues \( \lambda_{n,1}(\rho) \) and \( \lambda_{n,2}(\rho) \). By (10) \( \text{Re}(\lambda_{-n}(\rho)) < \text{Re}(\lambda_n(\rho)) \). Let \( \lambda_{n,2}(\rho) \) be the eigenvalue whose real part is larger. Then

\[
\lambda_{n,1}(\rho) = \lambda_{-n}(\rho), \quad \lambda_{n,2}(\rho) = \lambda_n(\rho), \quad \forall n > N
\]

In the same way we obtain that

\[
\lambda_{n,1}(\pi - \rho) = \lambda_n(\pi - \rho), \quad \lambda_{n,2}(\pi - \rho) = \lambda_{-(n+1)}(\pi - \rho), \quad \forall n > N
\]

if \( \lambda_{n,2}(\pi - \rho) \) is the eigenvalue whose real part is larger. Let \( \Gamma_n \) be the union of the following continuous curves \( \{ \lambda_{n,1,2}(t) : t \in [\pi - \rho, \pi] \} \cup \{ \lambda_{-n}(t) : t \in [\rho, \pi - \rho] \} \cup \{ \lambda_{n,1}(t) : t \in [0, \rho] \} \). By (12) and (13) these curves are connected and \( \Gamma_n \) is a continuous curve. Similarly, the curve \( \Gamma_n \) which is the union of the curves \( \{ \lambda_{n,2}(t) : t \in [0, \rho] \} \cup \{ \lambda_n(t) : t \in [\rho, \pi - \rho] \} \cup \{ \lambda_{n,1}(t) : t \in [\pi - \rho, \pi] \} \) is a continuous curve. For \( t \in [0, \rho] \) redenote \( \lambda_{n,1}(t) \) by \( \lambda_{-n}(t) \) and \( \lambda_{n,2}(t) \) by \( \lambda_n(t) \), where \( n > N \). In the same way we put \( \lambda_n(t) =: \lambda_{n,1}(t), \lambda_{-(n+1)}(t) =: \lambda_{n,2}(t) \) for \( t \in [\pi - \rho, \pi] \) and \( n > N \). In this notation we have

\[
\Gamma_n = \{ \lambda_n(t) : t \in [0, \pi] \}.
\]

The eigenvalues of \( L_{-t}(q) \) coincides with the eigenvalues of \( L_t(q) \), because they are roots of equation (3) and \( \cos(-t) = \cos t \). We define the eigenvalue \( \lambda_n(-t) \) of \( L_{-t}(q) \) by \( \lambda_n(-t) = \lambda_n(t) \) for all \( t \in (0, \pi) \). Then \( \lambda_n(t) \) for \( n > N \) is an continuous function on \((-\pi, \pi)\), \( \Gamma_n = \{ \lambda_n(t) : t \in (-\pi, \pi) \} \) and

\[
\sigma(L(q)) = \bigcup_{t \in (-\pi, \pi)} \sigma(L_t(q)) \supset \bigcup_{|n| > N} \Gamma_n.
\]

Thus the spectrum \( \sigma(L) \) of the operator \( L \) contains the continuous curves \( \Gamma_n \) for \( n > N \). The remaining part of \( \sigma(L) \) consist of finite simple analytic arcs \( \gamma_1, \gamma_2, \ldots, \gamma_m \) whose endpoints are the eigenvalues of \( L_0(q) \) and \( L_\pi(q) \) and the roots of the equations \( \frac{dF(\lambda)}{d\lambda} = 0 \) lying in the spectrum (see [9]). On the other hand the above arguments show that \( \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_m \) is the union of \( 2N + 1 \) eigenvalues (counting multiplicity and denoted by \( \lambda_n(t) \) for \( n = 0, \pm 1, \ldots, \pm N \) of \( L_t(q) \) for \( t \in (-\pi, \pi) \). Moreover if \( \lambda_n(t) \) is a root of (3) of multiplicity \( k \), then it is the end point of \( k \) curves \( \gamma_{n_1}, \gamma_{n_2}, \ldots, \gamma_{n_k} \), that is, these \( k \) curves are joined by \( \lambda_n(t) \). Therefore one can numerate the eigenvalues so that

\[
\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_m = \bigcup_{|n| \leq N} \Gamma_n
\]
where $\Gamma_n = \{ \lambda_n(t) : t \in (-\pi, \pi) \}$ for $|n| \leq N$ are continuous curves. Thus we have

$$\sigma(L(q)) = \bigcup_{n \in \mathbb{Z}} \Gamma_n. \quad (16)$$

Using (10), (11) and the definition of $\lambda_n(t)$ one can readily see that

$$|\lambda_n(t) - (2\pi k \pm t)^2| \geq |(n - k)| |n + k| \quad (17)$$

for $k \neq \pm n, \pm(n + 1)$ and $t \in (-\pi, \pi]$, where $|n| > N$. In [11], [12] to write the asymptotic formulas for the eigenfunctions $\Psi_{n,t}(x)$ corresponding to the eigenvalue $\lambda_n(t)$ we used the following relations

$$\left(\lambda_n(t) - (2\pi k + t)^2\right)(\Psi_{n,t}, e^{i(2\pi k + t)x}) = (q\Psi_{n,t}, e^{i(2\pi k + t)x}), \quad (18)$$

$$\left|(q\Psi_{n,t}, e^{i(2\pi k + t)x})\right| < 2M \quad (19)$$

for $t \in (-\pi, \pi]$, $|n| > N$ and $k \in \mathbb{Z}$, where

$$M = \sup_{n \in \mathbb{Z}} |q_n|, \quad q_n = \int_0^1 q(x)e^{-i2\pi nx} \, dx.$$  

From (17)-(19) we obtain the following, uniform with respect to $t$ in $(-\pi, \pi]$, asymptotic formulas

$$\sum_{k \in \mathbb{Z}\setminus\{\pm n, \pm(n + 1)\}} \left|(\Psi_{n,t}, e^{i(2\pi k + t)x})\right|^2 = O(n^{-2})$$

and

$$\sum_{k \in \mathbb{Z}\setminus\{\pm n, \pm(n + 1)\}} \left|(\Psi_{n,t}, e^{i(2\pi k + t)x})\right| = O\left(\frac{\ln n}{n}\right)$$

Therefore $\Psi_{n,t}(x)$ has an expansion of the form

$$\Psi_{n,t}(x) = \sum_{k \in \{\pm n, \pm(n + 1)\}} u_{n,k}(t)e^{i(2\pi k + t)x} + h_{n,t}(x), \quad (20)$$

where

$$\|h_n\| = O(n^{-1}), \quad \sup_{x \in [0,1], t \in (-\pi, \pi]} |h_{n,t}(x)| = O\left(\frac{\ln |n|}{n}\right), \quad (h_{n,t}, e^{i(2\pi k + t)x}) = 0 \quad (21)$$

for $k \in \{\pm n, \pm(n + 1)\}$ and

$$u_{n,k}(t) = (\Psi_{n,t}, e^{i(2\pi k + t)x}). \quad (22)$$

Let $\{\chi_{n,t} : n \in \mathbb{Z}\}$ be biorthogonal to $\{\Psi_{n,t} : n \in \mathbb{Z}\}$ and $\Psi_{n,t}^*(x)$ be the normalized eigenfunction of $(L_t(q))^*$ corresponding to $\lambda_n(t)$. Since the boundary condition (2) is self-adjoint we have $(L_t(q))^* = L_t(\bar{q})$. Therefore, we have

$$\Psi_{n,t}^*(x) = \sum_{k \in \{\pm n, \pm(n + 1)\}} u_{n,k}^*(t)e^{i(2\pi k + t)x} + h_{n,t}^*(x), \quad (23)$$
have two eigenfunctions corresponding to one eigenvalue $\lambda$ and

$$\|h_{n,t}^*\| = O(n^{-1}), \quad \sup_{x \in [0,1], t \in [-\pi, \pi]} |h_{n,t}(x)| = O\left(\frac{\ln |n|}{n}\right), \quad \langle h_{n,t}^*, e^{i(2\pi^2 t^2)x} \rangle = 0$$

for $k \in \{\pm n, \pm(n+1)\}$.

Introduce the functions

$$\alpha_n(t) = (\Psi_{n,t}(x), \Psi_{n,t}^*(x)_{(0,1)}), \quad \chi_{n,t}(x) = \frac{1}{\alpha_n(t)} \Psi_{n,t}^*(x),$$

where $(\cdot, \cdot)_{(a,b)}$ denotes the inner product in $L_2(a, b)$. One can easily verify that

$$\Psi_{n,t}(x) = \frac{\Phi_+(x, \lambda_n(t))}{|\Phi_+(x, \lambda_n(t))|} \chi_{n,t}(x) = \frac{1}{\alpha_n(t)} \frac{\Phi_-(x, \lambda_n(t))}{|\Phi_-(x, \lambda_n(t))|},$$

$$\Psi_{n,t}(x+1) = e^{it} \Psi_{n,t}(x), \quad \Phi_+^*(x+1) = e^{it} \Phi_+^*(x), \quad \chi_{n,t}(x+1) = e^{it} \chi_{n,t}(x),$$

where $\Phi_+$ and $\Phi_-$ are defined in (5). Using these formulas and the equalities

$$d\lambda = -p(\lambda) \left(\frac{dF}{d\lambda}\right)^{-1} dt, \quad dF(\lambda_n(t)) = -\varphi(1, \lambda_n(t))(\Phi_+(x, \lambda_n(t)), \Phi_-(x, \lambda_n(t)))$$

and changing the variable $\lambda$ to the variable $t$ in the integral (5) we get

$$P(\gamma) f = \frac{1}{2\pi i} \int_{C} (L_t - \lambda I)^{-1} f dt,$$

where $\delta = \{t \in (-\pi, \pi] : \lambda_n(t) \in \gamma\}$ and $\gamma \in \Gamma_n$.

It is very natural that the projection of the operator $L(q)$ is connected with the projection

$$e(t, C) := P_t(\lambda_n(t)) f = \frac{1}{2\pi i C} \int (L_t - \lambda I)^{-1} f dt$$

of the operator $L_t(q)$, where $C$ is a closed contour containing the eigenvalue $\lambda_n(t)$ but no other eigenvalues.

In this paper we use the following proposition which was proved in [10].

**Proposition 1** Let $\gamma \subset \Gamma_n$ be regular spectral arc and $\delta = \{t : \lambda_n(t) \in \gamma\}$. Then the operators $P(\gamma)$ and $P_t(\lambda_n(t))$ for $t \in \delta$ are projections and

$$\|P_t(\lambda_n(t))\| = |\alpha_n(t)|^{-1}, \quad \|P(\gamma)\| = \sup_{t \in \delta} \|P_t(\lambda_n(t))\| = \sup_{t \in \delta} |\alpha_n(t)|^{-1}.$$  

The eigenvalue $\lambda_n(t)$, where $t \in (-\pi, \pi]$, is a spectral singularity of $L(q)$ if and only if the operator $L_t(q)$ has an associated function corresponding to the eigenvalue $\lambda_n(t)$.

Note that it is well-known and easily checkable that, for $t \neq 0, \pi$, operator $L_t$ cannot have two eigenfunctions corresponding to one eigenvalue $\lambda$. Indeed if both solution $\varphi(x, \lambda)$ and $\theta(x, \lambda)$ satisfy the boundary condition (2) then

$$\varphi'(1, \lambda) = e^{it}, \quad \theta(1, \lambda) = e^{it}.$$
which contradicts (3) for $t \neq 0, \pi$. Therefore Proposition 1 at once implies the following

**Proposition 2.** A number $\lambda \in \sigma(L_t(q))$ for $t \in (0, \pi)$ is a spectral singularity of $L(q)$ if and only if it is a multiple eigenvalue of $L_t(q)$. Moreover, $\Gamma_n$ does not contain the spectral singularities if and only if there exists $\beta > 0$ such that $|\alpha_n(t)|^{-1} < \beta$ for all $t \in (0, \pi)$.

Now we are ready to prove the main results of this chapter.

**Theorem 1.** The following statements are equivalent

(a) The operator $L(q)$ has no spectral singularity at infinity.

(b) $L(q)$ is an asymptotically spectral operator.

(c) There exists $N$ such that the following are satisfied: (i) The operator $L(q)$ has no spectral singularity on $\Gamma_n$ for $n > N$ and hence $L(q)$ may have at most finitely many spectral singularities. (ii) For $|n| > N$ and $t \in (0, \pi)$ the eigenvalues $\lambda_n(t)$ are simple. (iii) The algebraic multiplicity of $\lambda_n(0)$ and $\lambda_n(\pi)$ for $|n| > N$ are equal to their geometric multiplicities, that is, there are not associated functions corresponding to those eigenvalues.

(iii) There exists a constant $d$ such that

$$|\alpha_n(t)|^{-1} < d$$

for all $|n| > N$ and $t \in (-\pi, 0) \cup (0, \pi)$.

**Proof.** First let us show that (c) implies (a). From (32) and (33) it follows that $\|P(\gamma)\| < d$ for all regular spectral arcs $\gamma \in \Gamma_n$ whenever $|n| > N$. Therefore, by Definition 2, (a) holds.

Now we prove that (a) implies (c). Suppose that (a) holds. If (i) does not hold then there exist a sequence of pairs $\{(n_k, t_k)\}$ such that $|n_k| \to \infty$ and $\lambda_{n_k}(t_k)$ for $k = 1, 2, \ldots$, are spectral singularities of $L(q)$. Then by Definition 1 one can choose a sequence of regular arc $\gamma_{n_k} \subset \Gamma_{n_k}$ such that

$$\|P(\gamma_{n_k})\| > k$$

which contradicts (a) (see Definition 2). To complete the proof of (i) it remains to note that the spectral singularities of $L(q)$ are contained in the set

$$\{\lambda : \frac{dF(\lambda)}{d\lambda} = 0, \ \lambda \in \sigma(L(q))\}$$

and the entire function $\frac{dF(\lambda)}{d\lambda}$ has at most finite number of roots on the compact set $\cup_{|n| \leq N} \Gamma_n$.

If (ii) does not hold then there exist a sequence of pairs $\{(n_k, t_k)\}$ such that $t_k \in (0, \pi)$ and $\lambda_{n_k}(t_k)$ is a multiple eigenvalue. Then by Proposition 2 the numbers $\lambda_{n_k}(t_k)$ for $k = 1, 2, \ldots$, are spectral singularities of $L(q)$ which contradicts (i).

Similarly, if (iii) does not hold then there exist infinitely many associated functions and hence by Proposition 1 there exist infinitely many spectral singularities which again contradicts (i).

If (iii) does not hold then there exist a sequence of pairs $\{(n_k, t_k)\}$, where $|n_k| \to \infty$ and $t_k \in (-\pi, 0) \cup (0, \pi)$, such that

$$|\alpha_{n_k}(t_k)|^{-1} > k.$$

Moreover by (ii) the eigenvalues $\lambda_{n_k}(t_k)$ for $|n_k| > N$ are simple. Therefore using the continuity of $\alpha_{n_k}(t)$ at $t_k$ and (32) we obtain that there exists a sequence of regular arc $\gamma_{n_k} \subset \Gamma_{n_k}$ such that $\|P(\gamma_{n_k})\| \to \infty$ which contradicts (a) (see Definition 2).

Now we prove that (a) and (b) are equivalent. If (a) does not hold then it is clear that (b) also does not holds. Suppose that (a) holds. Then (33) holds too. Let $C$ be a positive
constant such that if \( \lambda_n(t) \in \{ \lambda \in \mathbb{C} : |\lambda| > C \} \), then \(|n| > N\) for all \( t \in (-\pi, \pi) \), where \( N \) is defined in (e). If \( \gamma \in R(C) \), then \( \gamma \) encloses finite number of the simple eigenvalues of \( L_t(q) \). Thus, there exists a finite subset \( J(t, \gamma) \) of \( \{ n \in \mathbb{Z} : |n| > N \} \) such that the eigenvalue \( \lambda_k(t) \) lies inside \( \gamma \) if and only if \( k \in J(t, \gamma) \). It is well-known that these eigenvalues are the simple poles of the Green function of \( L_t(q) \) and the projection \( e(t, \gamma) \) has the form

\[
e(t, \gamma)f = \sum_{n \in J(t, \gamma)} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \Psi_{n,t}.
\]  

(36)

Therefore the proof of the theorem follows from the following lemma □

**Lemma 1** If (33) holds then there exists a positive constant \( D \) such that

\[
\| \sum_{n \in J} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \Psi_{n,t} \|^2 < D \| f \|^2
\]  

(37)

for all \( t \in (-\pi, 0) \cup (0, \pi) \) and for all subset \( J \) of \( \{ n \in \mathbb{Z} : |n| > N \} \).

**Proof.** First let us prove that there exists a positive constant \( c \) such that

\[
\| \sum_{n \in J} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \Psi_{n,t} \|^2 < c \sum_{n : |n| > N} |(f, \Psi_{n,t}^*)|^2
\]  

(38)

for all \( t \in (-\pi, 0) \cup (0, \pi) \). By (20) and (21) we have

\[
\| \sum_{n \in J} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \Psi_{n,t} \|^2 = S_1 + S_2^2
\]  

(39)

where

\[
S_1 = \| \sum_{n \in J} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \left( \sum_{k \in \{ \pm n, \pm (n+1) \}} u_{n,k}(t) e^{i(2\pi k + t)x} \right) \|^2,
\]

\[
S_2 = \| \sum_{n \in J} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) h_{n,t} \|
\]

Since \( \{ e^{i(2\pi n + t)x} : n \in \mathbb{Z} \} \) is an orthonormal basis and \(|u_{n,k}(t)| \leq 1 \) (see (22)) using (33) and the Bessel inequality one can easily verify that

\[
S_1 \leq 16d^2 \sum_{n : |n| > N} |(f, \Psi_{n,t}^*)|^2
\]  

(40)

It follows from (33) and (21) that

\[
S_2 < c_1 \sum_{n : |n| > N} |(f, \Psi_{n,t}^*)| \left| \frac{1}{n} \right|
\]

for some constant \( c_1 \). Now using the Schwarz inequality for \( l_2 \) we obtain

\[
S_2 < c_1 \left( \sum_{n : |n| > N} |(f, \Psi_{n,t}^*)|^2 \right)^{1/2} \left( \sum_{n : |n| > N} \left| \frac{1}{n} \right|^2 \right)^{1/2} \leq \frac{c_1}{\sqrt{N}} \left( \sum_{n : |n| > N} |(f, \Psi_{n,t}^*)|^2 \right)^{1/2}.
\]

Thus (38) follows from (39) and (40). Therefore to prove (37) it is enough show that there
exists a positive constant $c_2$ such that

$$\sum_{n:|n|>N} |(f, \Psi_{n,t}^{*})|^2 \leq c_2 \|f\|^2. \quad (41)$$

It can be proved arguing as in the proof of (38) and using (23) instead of (20). \hfill \blacksquare

**Theorem 2** The operator $L(q)$ is a spectral operator if and only if it has no spectral singularities at $\sigma(L(q))$ and at infinity.

**Proof.** If $L(q)$ is a spectral operator, then by (7), (30) and Proposition 1 there exists a positive constant $c_3$ such that $\| P(\gamma) \|< c_3$ for all regular spectral arcs $\gamma$. Therefore, by Definition 1 and Definition 2, the operator $L(q)$ has no spectral singularities at $\sigma(L(q))$ and at infinity. If $L(q)$ has no spectral singularities at $\sigma(L(q))$ and at infinity then (33) holds for all $n \in \mathbb{Z}$ and $t \in (-\pi, 0) \cup (0, \pi)$. Using this and arguing as in the proof of the implication $(a) \implies (b)$ we obtain that $L(q)$ is a spectral operator. \hfill \blacksquare

**Definition 4** The component $\Gamma_n$, defined by (14), of the spectrum $\sigma(L(q))$ of the operator $L(q)$ is said to be separated if $\Gamma_n \cap \Gamma_m = \emptyset$ for all $m \neq n$. Thus all component $\Gamma_n$ of the spectrum $\sigma(L(q))$ are separated if and only if all eigenvalues of the operators $L_t$ for $t \in (-\pi, \pi]$ are simple.

**Corollary 1** The Mathieu operator $L(2a \cos x)$, where $a$ is a complex number, is a spectral operator if and only if

$$\inf_{q,p \in \mathbb{N}} \{|2qa - (2p - 1)|\} \neq 0,$$

and all eigenvalues of the operators $L_t$ for $t \in (-\pi, \pi]$ are simple.

**Proof.** Using the result of [13] mentioned in the end of introduction which states that $L(2a \cos x)$ has no spectral singularities at infinity if and only if (42) holds, we prove the corollary as follows. If (42) holds and all eigenvalues are simple then $L$ has no spectral singularity at infinity and in spectrum and hence, by Theorem 2, $L$ is a spectral operator. Now suppose that $L$ is a spectral operator. Then, by Theorem 2, $L$ has no spectral singularities at infinity and hence (42) holds. It remains to prove that all eigenvalues are simple. By Proposition 2 the eigenvalues $\lambda_n(t)$ for $t \in (0, \pi)$ and for all $n$ are simple, since $L$ has no spectral singularity in spectrum (see Theorem 2). Now to complete the proof of the corollary it remains to prove that the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi)$ for $n \in \mathbb{Z}$ are simple. By Proposition 1 the operators $L_0$ and $L_\pi$ have no associated functions. On the other hand, it is well-known that, the geometric multiplicity of $\lambda_n(0)$ and $\lambda_n(\pi)$ for all $n$ is 1 (see p. 34-35 of [1]). Note that in [1] this theorem was proved for the real $a$. However, the proof pass through for the complex $a$ without any change). Therefore $\lambda_n(0)$ and $\lambda_n(\pi)$ for all $n$ are simple. \hfill \blacksquare

### 3 On the Spectral Expansion of the Asymptotically Spectral Hill Operators

In this section we examine the spectral expansion theorem in the case when $L(q)$ has no spectral singularity at infinite by using Section 2 and the results of [2] where it was proved that every function $f \in L_2(-\infty, \infty)$ can be represented in the form

$$f(x) = \frac{1}{2\pi} \int_{0}^{2\pi} f_t(x)dt, \quad (43)$$
In this case, by Theorem 1, the root of \( dF \) exists for almost all \( x \). Denote these roots (if exists) by \( \lambda \)

Therefore by Fubini theorem

where the series converges in the norm of \( L_2(0, 1) \). This with (43) implies that

Since in this section it is assumed that \( L(q) \) has no spectral singularity at infinite, by Theorem 1 there are at most finite number of spectral singularities denoted by \( \lambda_1, \lambda_2, ... \lambda_s \), where \( s \geq m \), since by Proposition 2, \( \lambda_1, \lambda_2, ... \lambda_m \) are spectral singularities. Let \( S \) be the set of integers such that \( \Gamma_n \) contains spectral singularities for \( n \in \mathbb{Z} \). Note that \( S \) is a finite subset of \( \mathbb{Z} \). By Proposition 2 and (45), (44) if \( n \notin S \) then

Therefore by Fubini theorem

exists for almost all \( x \) when \( k \notin S \). Similarly by (33) the integral (49) exists for \( |k| > N \).

**Theorem 3** If the operator \( L(q) \) has no spectral singularity at infinity, then every function \( f \in L_2(-\infty, \infty) \) has the spectral decomposition

where the series in (50) converges in the norm of \( L_2(a, b) \) for every \( a, b \in \mathbb{R} \).
Proof. First let us consider the series
\[ \sum_{k>N} a_k(t)\Psi_{k,t}(x), \tag{51} \]
where \( N \) is defined in Theorem 1 and \( t \in E \). Let \( R_n(x,t) \) be remainder of (51)
\[ R_n(x,t) = \sum_{k>n} a_k(t)\Psi_{k,t}(x), \]
where \( n > N \). Since the series (51) converges in the norm of \( L_2(0,1) \) by (45) and (27) we have
\[ R_n(x+1,t) = e^{it}R_n(x,t), \quad R_n(x,t) \in L_2(-m,m) \tag{52} \]
for \( t \in E \) and for all \( m \in \mathbb{N} \). Repeating the proof of (38) and using (52) we obtain
\[ \| R_n(.,t) \|_{(−m,m)}^2 \leq 2mc_4 \sum_{|k|>n} |(f_t,\Psi_{k,t}^*)(0,1) |^2 \tag{53} \]
for some constant \( c_4 \), where \( \| f \|_{(−m,m)} \) is the \( L_2(−m,m) \) norm of \( f \). On the other hand, it follows from (33), (23) and (24) that
\[ \sum_{|k|>n} |(f_t,\Psi_{k,t}^*)(0,1)|^2 \leq c_5 \sum_{|k|>n} |(f_t,e^{i(2\pi k+t)x})(0,1)|^2 + c \| f_t \|_{(0,1)}^2 n^{-1} \tag{54} \]
for some constant \( c_5 \). Now using the Parseval equality for \( L(q) \) when \( q = 0 \) (see [2]) we obtain
\[ \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \int_E |(f_t,e^{i(2\pi k+t)x})(0,1)|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_E \| f_t \|_{(0,1)}^2 \]
Therefore by (54) and (53) the following integral exists and
\[ I_n =: \int_E \int_{(−m,m)} |R_n(x,t)|^2 dx dt \to 0 \tag{55} \]
as \( n \to \infty \). Thus by Fubini theorem \( R_n(x,t) \) is integrable with respect to \( t \) for almost all \( x \).

Now using the inequality
\[ \left| \int_E f(t)dt \right|^2 \leq 2\pi \int_E |f(t)|^2 dt, \]
Fubini theorem and then (55), we obtain
\[ \| \sum_{k>n} a_k(t)\Psi_{k,t} dt \|_{(−m,m)}^2 \leq 2\pi \int_{(−m,m)} \int_E \| \sum_{k>n} a_k(t)\Psi_{k,t}(x) \|^2 dx dt = I_n \to 0 \tag{56} \]
as \( n \to \infty \). Hence the series (51) is integrable and
\[ \int_E \left( \sum_{k>N} a_k(t)\Psi_{k,t}(x) \right) dt = \sum_{k>N} \int_E a_k(t)\Psi_{k,t}(x) dt, \tag{57} \]
where the last series converges in the norm of \( L_2(−m,m) \) for every \( m \in \mathbb{N} \). In the same way
we prove that
\[ \int_E \left( \sum_{k<N} a_k(t) \Psi_{k,t}(x) \right) dt = \sum_{k<N} \int_E a_k(t) \Psi_{k,t}(x) dt. \] (58)

Therefore using (57), (58) and (47) and taking into account that the integral in (49) exists for \( k \notin S \) we get the proof of the theorem.

Now changing the variable to \( \lambda \) in (50) by using (28) and taking into account that \( \lambda_n(-t) = \lambda_n(t) \) we obtain

**Theorem 4** If the operator \( L(q) \) has no spectral singularity at infinity, then every function \( f \in L_2(-\infty, \infty) \) has the spectral decomposition

\[ f(x) = \frac{1}{\pi} \int_{\gamma(s)} (\phi(x, \lambda)) \frac{1}{p(\lambda)} d\lambda + \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus S} \left( \int_{\mathbb{R}_k} (\phi(x, \lambda)) \frac{1}{p(\lambda)} d\lambda \right), \] (59)

where

\[ \phi(x, \lambda) = \theta'(1, \lambda) h(\lambda) \varphi(x, \lambda) + \frac{1}{2} (\theta(1, \lambda) - \varphi'(1, \lambda))(h(\lambda) \theta(x, \lambda) + g(\lambda) \varphi(x, \lambda)) - \varphi(1, \lambda) g(\lambda) \theta(x, \lambda), \]

\[ h(\lambda) = \int_{-\infty}^{\infty} \varphi(x, \lambda) f(x) dx, \quad g(\lambda) = \int_{-\infty}^{\infty} \theta(x, \lambda) f(x) dx, \quad p(\lambda) = \sqrt{4 - F^2(\lambda)}, \]

\( \gamma(s) =: \bigcup_{k \in S} \Gamma_k \) is the part of the spectrum that contains the spectral singularities and the series in (59) converges in the norm of \( L_2(a, b) \) for every \( a, b \in \mathbb{R} \).

The results of [12] and [13] mentioned in the end of the introduction with Theorem 3 and Theorem 4 imply

**Corollary 2** If the potential \( q \) satisfies one of the Condition 1 and Condition 2 then the spectral expansions of \( L(q) \) have the forms (50) and (59).

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