Effects of DC voltage on initiation of whirling motion of nanowire oscillators

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Abstract

In-plane driven nanowire oscillators are susceptible to undergoing planar to whirling motion transition due to coupling between flexural modes of vibration. This letter reports analytical modeling of initiation of whirling motion of an electrostatically actuated nanowire oscillator which is pre-deflected by applied DC voltage. A nonlinear coupled oscillator problem has been formulated and solved using second-order averaging method. Planar to whirling motion transition has been investigated by studying bifurcation diagrams. We have quantified the effect of DC voltage on nature of whirling dynamics of the nanowire oscillator.

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Investigations of nanoelectromechanical system (NEMS) oscillators have received considerable attention in recent years due to their potential applications as sensors and signal-processing devices.\textsuperscript{1,2} NEMS oscillators have rich nonlinear dynamic characteristics which can be used in designing of efficient devices.\textsuperscript{3,4} In case of nanowire (NW) and carbon nanotube (CNT) oscillators, two flexural orthogonal modes of vibration have nearly identical natural frequencies.\textsuperscript{5} There is a nonlinear coupling between these two flexural modes of vibration, which may interact during resonance. A few investigations demonstrated that planar (plane of electrostatic actuation X-Z in Fig. 1) motion of NW and CNT oscillators may exhibit a change to three-dimensional whirling motion due to the nonlinear coupling.\textsuperscript{5,6}

Conley et al.\textsuperscript{5} first investigated whirling dynamics in electrostatically actuated nano-oscillators for providing theoretical understanding of experiments of Sazonova et al.\textsuperscript{7} The authors modeled a CNT oscillator, after neglecting initial deflection due to DC voltage, and provided an analytical criteria for the initiation of whirling motion using first-order averaging method. The same group, in a later investigation, also performed numerical analysis to study the role of DC voltage in whirling dynamics of CNT oscillators.\textsuperscript{8} In another study, Chen et al.\textsuperscript{6} have numerically investigated the chaotic response in whirling motion of NW oscillators. Interaction between planar and nonplanar modes of vibration due to internal resonance has been also experimentally observed in CNT oscillators by Eichler et al.\textsuperscript{9}

A simple analytical model is desirable which provides the criteria for planar to whirling motion transition and thus help to identify the nature of resonance characteristics of NW and CNT oscillators. In this work, we have accounted for the effects of DC voltage, a prime component of electrostatic actuation, and an analytical criteria for initiation of whirling provided in the form of coupled algebraic equations. DC voltage induces static pre-deflection, which breaks down the symmetry of beam displacement and becomes the source of geometric quadratic nonlinearity.\textsuperscript{10} In addition, DC voltage also mistunes the planar and nonplanar natural frequencies of oscillation. We show in this paper that DC voltage can qualitatively change the whirling dynamics of nano-oscillators.

Figure 1 shows the schematic diagram of a doubly-clamped NW oscillator of radius $\hat{R}$ and length $\hat{L}$ and is placed on a side of an electrode plate at a gap $\hat{g}$. The nanowire is electrostatically actuated by applying bias voltage $\hat{V}$ which is a combination DC voltage $\hat{V}_{DC}$ and AC voltage of amplitude $\hat{V}_{AC}$ at frequency $\hat{\omega}_f$. The governing differential equations of motion for non-dimensional planar $u(x, t)$ and nonplanar $v(x, t)$ displacements are coupled
Integro-partial differential equations of the form of four-order differential equations:

\[
\begin{align*}
    u'''' + \ddot{u} + c \dot{u} &= \left[ \alpha_1 \int_0^1 \left( u'^2 + v'^2 \right) dx \right] u'' + \alpha_2 V^2 F_e(u), \\
    v'''' + \ddot{v} + c \dot{v} &= \left[ \alpha_1 \int_0^1 \left( u'^2 + v'^2 \right) dx \right] v''.
\end{align*}
\] (1)

Here,

\[
\begin{align*}
    \alpha_1 &= \hat{A} \hat{g}^2 / 2I, \quad \alpha_2 = \frac{\varepsilon_0 \hat{L}^4 V_{DC}^2}{\hat{g}^2 \hat{E} \hat{I}}, \quad R_0 = \frac{\hat{R}}{\hat{g}}, \\
    F_e(u) &= \frac{\varepsilon_0 \hat{L}^4 V_{AC}^2}{\sqrt{(1 + R_0 - u)^2 - R_0^2} \left[ \cosh^{-1} \left( \frac{1 + R_0 - u}{R_0} \right) \right]^2}, \\
    V &= (1 + V_{AC} \cos(\omega_f t)) \quad V_{AC} = \frac{V_{AC}}{V_{DC}}, \quad \text{and} \quad \omega_f = \hat{\omega}_f \hat{T}.
\end{align*}
\] (2)

The dimensional form of the planar displacement \( \hat{u}(\hat{x}, \hat{t}) \), nonplanar displacement \( \hat{v}(\hat{x}, \hat{t}) \), spatial coordinate \( \hat{x} \), and time \( \hat{t} \) are related to their non-dimensional form as:

\[
\begin{align*}
    u = \frac{\hat{u}}{\hat{g}}, \quad v = \frac{\hat{v}}{\hat{g}}, \quad x = \frac{\hat{x}}{\hat{L}}, \quad t = \frac{\hat{t}}{\hat{T}}, \quad \hat{T} = \sqrt{\frac{\rho \hat{A} \hat{L}^4}{\hat{E} \hat{I}}}.\n\end{align*}
\]

The non-dimensional parameter \( c \) is damping coefficient, \( \alpha_1 \) quantifies the nonlinearity due to nanowire stretching because of boundary condition constraint, and \( \alpha_2 \) represents strength of electrostatic actuation nonlinearity. In Eq. (2), Young’s modulus and mass density of the nanowire are represented by \( \hat{E} \) and \( \hat{\rho} \) respectively, whereas cross-sectional area and moment of inertia are given by \( \hat{A} \) and \( \hat{I} \) respectively; \( \hat{T} \) is a time constant.

To investigate the whirling dynamics of an electrostatically actuated NW oscillator, we have developed a two degree of freedom (DOF) reduced order model using Galerkin method. Such a reduced order formalism has been applied earlier in many investigations of nano-oscillators. In this formalism, for our problem, in-plane motion is assumed as \( u(x, t) = (u_s + u_d(t)) \phi_1(x) \) and nonplanar displacement is represented as \( v(x, t) = v_d(t) \phi_1(x) \); \( \phi_1(x) \) is the first mode shape, with natural frequency \( \omega_1 \), of a doubly-clamped straight beam. The variable \( u_s \) is the non-dimensional static deflection component due to DC voltage and can be calculated by solving Eq. (1) after neglecting AC voltage and time derivative terms, whereas \( u_d(t) \) and \( v_d(t) \) are planar and nonplanar vibration components respectively. After substituting the assumed solutions for \( u \) and \( v \) in Eq. (1), multiplying the outcome with \( \phi_1(x) \), and integrating the equations from \( x = 0 \) to \( x = 1 \), we have obtained the two DOF model.
FIG. 1. Schematic diagram of an electrostatically actuated nanowire oscillator.

\[
\ddot{u}_d + c \dot{u}_d + k_{1u} u_d + k_{2u} u_d^2 + k_{2v} v_d^2 + k_{3u} u_d^3 + k_{3uv} u_d v_d^2 = 2V_{AC} C_0 \cos(\omega f t),
\]

\[
\ddot{v}_d + c \dot{v}_d + k_{1v} v_d + k_{2uv} u_d v_d + k_{3v} v_d^3 + k_{3uv} u_d^2 v_d = 0.
\]

Various parameters of Eq. 3 are defined as

\[
k_{1u} = \omega_1^2 + 3k_{3u}u_s^2 - C_1, \quad k_{2u} = 3k_2 - C_2, \quad k_{3u} = k_3 - C_3,
\]

\[
k_{1v} = \omega_1^2 + k_{3v}u_s^2, \quad k_{2v} = k_2, \quad k_{2uv} = 2k_2, \quad k_{3v} = k_{3uv} = k_3,
\]

where, \(k_2 = k_3u_s\), \(k_3 = \alpha_1 \left( \int_0^1 \phi_i^2 \, dx \right)^2\), and

\[
C_i = \frac{1}{n} \alpha_2 \int_0^1 F_e^{(i)}(u_s \phi_i) \phi_i^{i+1} \, dx, \quad i = 0, 1, 2, 3.
\]

For deriving Eq. 3 we have expressed electrostatic forcing function \(F_e(u)\) from Eq. 2 using Taylor series of third order, and retained only the first harmonic \(2V_{AC} C_0 \cos(\omega f t)\) term because primary resonance is the main scope of this investigation. Note that Eq. 3 contains both symmetric odd order and asymmetric even order nonlinearities. The parameters \(C_i\)s arise due to nonlinear electrostatic actuation force, whereas \(k_2\) and \(k_3\) arise due to beam stretching. An important point to note here is that \(k_2 = k_3u_s\) arises due to the static deflection by applied DC voltage and is the source of geometric quadratic nonlinearity in the coupled oscillator.

To solve Eq. 3 we assume it as a weakly nonlinear problem by rewriting coefficient of quadratic nonlinear terms of the order \(\epsilon\) and remaining terms (cubic nonlinearity, damping coefficient, and harmonic forcing) of the order \(\epsilon^2\) – here \(\epsilon\) is a small book-keeping parameter.
To analyze this problem, second-order averaging technique has been applied\(^\text{12}\), in contrast to to first-order averaging applied by Conley et al.\(^\text{5}\) It can be shown that first-order perturbation is sufficient to account for symmetric cubic nonlinearity effects on resonance curves, whereas second-order perturbation is indispensable to properly incorporate asymmetric quadratic nonlinearity.\(^\text{13}\) To make the perturbation problem suitable for averaging, we have expressed the displacement and velocity components as \([u_d \, \dot{u}_d]^T = A [x_1 \, x_2]^T\) and \([v_d \, \dot{v}_d]^T = A [x_3 \, x_4]^T\), where \(A\) is a matrix defined as

\[
A = \begin{bmatrix}
\cos(\omega_f t) & \sin(\omega_f t) \\
-\omega_f \sin(\omega_f t) & \omega_f \cos(\omega_f t)
\end{bmatrix}.
\]

With this substitution, our dynamical equation has the form \(\dot{x} = \epsilon f_1(x, t) + \epsilon^2 f_2(x, t)\). The central idea of second-order averaging is to obtain a near identity transformation \(x(t) = y(t) + \epsilon b_1(y, t)\), such that the averaged equation \(\ddot{y} = \epsilon g_1(y) + \epsilon^2 g_2(y)\) is an autonomous system.\(^\text{12}\) The averaged equations which determine amplitude and phase evolution are

\[
\dot{y}_1 = -\mu_0 y_1 + (\Omega_1 + \gamma_1 r_u^2 + \gamma_3 r_v^2) y_2 + \gamma_4 y_3 r_{uv}, \\
\dot{y}_2 = -\mu_0 y_2 - (\Omega_1 + \gamma_1 r_u^2 + \gamma_3 r_v^2) y_1 + \gamma_4 y_3 r_{uv} + p_0, \\
\dot{y}_3 = -\mu_0 y_3 + (\Omega_2 + \gamma_2 r_u^2 + \gamma_3 r_v^2) y_4 - \gamma_4 y_1 r_{uv}, \\
\dot{y}_4 = -\mu_0 y_4 - (\Omega_2 + \gamma_2 r_u^2 + \gamma_3 r_v^2) y_3 - \gamma_4 y_2 r_{uv},
\]

\[r_u^2 = y_1^2 + y_2^2, \quad r_v^2 = y_3^2 + y_4^2, \quad r_{uv} = y_1 y_4 - y_2 y_3,\]

where, different coefficients are

\[
\mu_0 = \frac{c}{2} p_0 = \frac{2V_{AC} C_0}{2\omega_f}, \quad \Omega_1 = \frac{\omega_u^2 - \omega_f^2}{2\omega_f}, \quad \Omega_2 = \frac{\omega_v^2 - \omega_f^2}{2\omega_f},
\]

\[
\gamma_1 = \frac{3 k_{3u}}{8 \omega_f} - \frac{5 k_{2u}^2}{12 \omega_f^2}, \quad \gamma_3 = \frac{3 k_{3uv}}{8 \omega_f} + \frac{5 k_{2u} k_{2v}}{24 \omega_f^2} - \frac{5 k_{2uv} k_{2u}}{12 \omega_f^2}, \\
\gamma_2 = \frac{3 k_{3v}}{8 \omega_f} - \frac{5 k_{2v}^2}{12 \omega_f^2}, \quad \gamma_4 = \frac{k_{3uv}}{4 \omega_f} + \frac{k_{2uv} k_{2u}}{12 \omega_f^2} - \frac{k_{2uv} k_{2v}}{2 \omega_f^2}.
\]

We have introduced two detuning parameters \(\Omega_1\) and \(\Omega_2\) in Eq. 5 which quantify the difference of \(\omega_f\) from planar natural frequency \(\omega_u = \sqrt{k_{1u}}\) and \(\omega_f\) from nonplanar natural frequency \(\omega_v = \sqrt{k_{1v}}\) respectively. Steady state solutions or equilibrium points of Eq. 5 provide periodic solution of the coupled oscillator problem \(\text{[3]}\), where the planar and nonplanar amplitudes are \(r_u = \sqrt{y_1^2 + y_2^2}\) and \(r_v = \sqrt{y_3^2 + y_4^2}\). Effect of DC voltage can be observed from the expressions for parameters \(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \Omega_1,\) and \(\Omega_2\) in Eqs. 4 and 6. \(\hat{V}_{DC}\) modifies these parameters, ultimately affecting the dynamics of NW oscillators.
FIG. 2. (i) and (ii) are primary planar and nonplanar resonance curves respectively for $\hat{V}_{DC} = 10$ V and $\hat{V}_{AC} = 0.80$ V. Here, solid lines represent stable solutions and dashed lines represent unstable solutions of Eq. Unfilled circles and stars denote saddle-node and branch bifurcation points, and corresponding bifurcation diagrams are shown in (iii) and (iv) respectively. Continuous lines in (iii) and (iv) are bifurcation points obtained by solving averaged Eq. for $\hat{V}_{DC} = 10$ V, whereas unfilled circles and stars are bifurcation points obtained by numerically solving Eq. 

In this paper, we have studied the dynamics of a typical silicon nanowire oscillator of length $\hat{L} = 3000$ nm and radius $\hat{R} = 25$ nm. The gap $\hat{g}$ between nanowire and electrode plate has been chosen as 300 nm and quality factor $Q = 100$ has been taken to account for damping effects. As in a linear harmonic oscillator, quality factor $Q$ is related to the damping coefficient $c$ as $Q = \omega_1/c$. Figure 2 shows planar to whirling motion transition in the nanowire oscillation for $\hat{V}_{DC} = 10$ V. We have solved Eq. using nonlinear dynamics software XPPAUT to obtain planar and nonplanar resonance curves which are shown in Figs. (i) and (ii). Variation of planar and nonplanar mid-point amplitude ($r_u$ and $r_v$ multiplied by $\phi_1(0.5)$) with $\hat{\omega}_f$ are termed here as planar and nonplanar resonance curve respectively. Two distinct branches of resonance curves are shown in each of the Figs. (i) and (ii): planar (P) branch and whirling (W) branch. In the figures, continuous lines represent stable solutions, whereas dashed lines represent unstable solutions of the resonance curves. When nanowire oscillation is characterized by P branch, the nanowire oscillates only in the plane of excitation i.e., nonplanar amplitude is zero. The W branch indicates
finite magnitude of both planar and nonplanar amplitudes and three-dimensional whirling motion can exist on this branch only. In Fig. 2(i), unfilled circles denote two saddle-node bifurcation of Eq. 5 which characterize the nonlinear nature of resonance curves. The P branch of resonance curves display hysteresis behavior, i.e., multiple solutions exist between these two saddle-node bifurcation points. Two more bifurcation points denoted by unfilled stars also reside on the P branch. These bifurcation points are termed here as branch bifurcation points and the W branch emanates from these two points. When a NW oscillator is operated in a forward sweep of forcing frequency, the oscillatory motion shows a planar to whirling transition at the branch bifurcation point. However, as forcing frequency further increases, periodic whirling motion becomes unstable. Such observation of existence of both stable and unstable solutions on W branch is consistent with findings of Johnson and Bajaj in their investigation of sinusoidally forced string.

The bifurcation points of P branch of resonance curves are of main interest for the current study. We have derived simple conditions to obtain saddle-node and branch bifurcation points from Eq. 5. A nonlinear algebraic equation has been derived by substituting $y_1 = ru \cos(\theta_u)$, $y_2 = -ru \sin(\theta_u)$, $y_3 = 0$, and $y_4 = 0$ in Eq. 5 to give

$$ (p_0)^2 = r^2 u \left( (\mu_0)^2 + (\Omega_1 + \gamma_1 r^2_u)^2 \right),$$

which can be used to compute P branch of resonance curves.

Equation 5 can be re-written in the vector form as $\dot{y} = G(y)$, where $y = [y_1 y_2 y_3 y_4]^T$. The bifurcation of P branch of resonance curves occurs when the determinant of the Jacobian matrix of $G(y)$ vanishes. It is interesting to note that the Jacobian matrix, say $B$, is a four by four block diagonal matrix for $y_3 = y_4 = 0$. The saddle node bifurcation occurs when the determinant of first block matrix containing elements $B_{11}$, $B_{12}$, $B_{21}$, and $B_{22}$ vanishes. Hence the condition for saddle-node bifurcation is

$$ \mu_0^2 + (\Omega_1 + \gamma_1 r^2_u) \left( \Omega_1 + 3\gamma_1 r^2_u \right) = 0.$$  

We solve Eqs. 7 and 8 simultaneously to obtain the bifurcation points. Equation 8 is only concerned with planar amplitude $r_u$ and is similar to the condition for saddle-node bifurcation of the averaged equation of Duffing oscillator. Figure 2(iii) is the saddle-node bifurcation diagram in $\hat{V}_{AC} - \hat{\omega}_F$ plane. To demonstrate the effectiveness of our analytical model, numerically obtained bifurcation points of Eq. 5 are correspondingly plotted in Fig. 7.
FIG. 3. Planar resonance curves for $\hat{V}_{DC} = 17.5$ V and $\hat{V}_{AC} = 0.40$ V. (i) Planar (P) branch (ii) Whirling (W) branch. As in Fig. 2(i), solid lines represent stable solutions and dashed lines represent unstable solutions.

(ii) as unfilled circles. One can observe from the figure that our analytical model is capable of obtaining the bifurcation points with reasonable accuracy. The bifurcation diagram shows that below a threshold $\hat{V}_{AC}$, there is no hysteresis in resonance curves.

Branch bifurcation corresponding to initiation of whirling motion occurs when the determinant of second block matrix, containing elements $B_{33}$, $B_{34}$, $B_{43}$, and $B_{44}$ vanishes.\(^\text{17}\) The condition for branch bifurcation is

$$\mu_0^2 + (\Omega_2 + \gamma_3 r_u^2) (\Omega_2 + (\gamma_3 - \gamma_4) r_u^2) = 0.$$  \hspace{1cm} (9)

Figure 2(iv) is the branch bifurcation diagram in $\hat{V}_{AC} - \hat{\omega}_f$ plane which has been obtained by solving Eqs. 7 and 9. Again, numerically obtained bifurcation points of Eq. 9 are correspondingly plotted in Fig. 2(iv) as unfilled stars; both results show good agreement. It can be further see there is a threshold magnitude of AC voltage, below which, no whirling motion exists. So far the characteristics of the NW oscillator (Fig. 2) are similar to that of a symmetric oscillator.\(^\text{5,16}\) However, upon increasing the magnitude of DC voltage, qualitatively different dynamic behavior of the NW oscillator has been observed.

Figure 3 shows resonance curves of the NW oscillator for $\hat{V}_{DC} = 17.5$ V and $\hat{V}_{AC} = 0.40$ V; these curves are the solution of the averaging Eq. 5. In this figure, P and W branches of planar resonance curve are plotted in sub-figures (i) and (ii). The qualitatively distinct features of resonance curves for higher DC voltage can be observed by comparing Figs. 2(i) and 3. There are four branch bifurcation points corresponding to two W branches in Fig. 3 as compared to only two bifurcation points corresponding to one W branch in Fig. 2(i). Consequently, the branch bifurcation diagram for $\hat{V}_{DC} = 17.5$ V in Fig. 4 is qualitatively different from its counterpart in Fig. 2(iv). Furthermore, though, as can be seen from
Fig. 4. Branch bifurcation diagram for $\hat{V}_{DC} = 17.5$ V. As in Fig. 2(iv), Continuous lines are solutions of averaged Eq. 5 whereas unfilled stars are bifurcation points obtained by numerically solving Eq. 3.

As seen in Fig. 4, whirling motion is initiated around $\hat{\omega}_f = 47$ MHz at threshold magnitude of $\hat{V}_{AC} = 0.29$ V. The whirling initiation frequency decreases with the increment of AC voltage and finally becomes nearly invariable around 43.5 MHz for large magnitude of AC voltage. The unfilled stars in Fig. 4 are corresponding bifurcation points obtained by numerically solving Eq. 3; there is overall qualitative agreement between the two solutions. This qualitative variability in whirling dynamics can be attributed to change in parameters $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, $\Omega_1$, and $\Omega_2$ of Eq. 6 due to modification in the magnitude of applied DC voltage.

In conclusion, we have presented a simple analytical model – developed using second-order averaging method – which takes into account the effect of DC voltage on the initiation pattern of whirling motion in nanowire oscillators. We have observed that DC voltage can qualitatively change the whirling behavior of nanowire oscillators.

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