D-branes as GMS Solitons in Vacuum String Field Theory

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Abstract

In this paper we map the D-brane projector states in the vacuum string field theory to the noncommutative GMS solitons based on the recently proposed map of Witten’s star to Moyal’s star. We find that the singular geometry conditions of Moore and Taylor are associated with the commutative modes of these projector states in our framework. The properties of the candidate closed string state and the wedge state are also discussed, and the possibility of the non-GMS soliton in VSFT is commented.
1 Introduction

Vacuum string field theory (VSFT) [9] is a very good tool to understand the open string tachyon condensation around the non-perturbative vacuum, around which the BRST operator by assumption consists only of ghost fields. In this context, the sliver states and its generalization are algebraically constructed and identified as the solitonic D25-branes and the lower dimensional tachyonic Dp-branes [11, 12]. Moreover, the descent relation of the D-brane tension has been verified in the various cases [11, 20, 21, 22]. This is the strong support to the VSFT ansatz.

Since the D-branes are the solitonic objects in the VSFT, it is natural to see if they could be noncommutative GMS solitons [18] because the Witten’s star product of the cubic string field theory is noncommutative [2]. Recently, there is an explicit identification of Witten’s star as Moyal’s star [4, 5] so that the cubic string field theory can be reformulated as an infinite copies of the noncommutative field theory. (See also [26] for a simpler formulation of this map in the higher energy limit.)

In the perturbative string theory, the D-brane geometry is characterized by the boundary condition on the open string endpoints, which could be Neumann, Dirichlet or the mixing one. On the other hand, in Witten’s string field theory it is the midpoint not the endpoint being selected for the joining and splitting. So a pressing question is that: how to characterize the D-brane geometry in Witten’s string field theory? This question is partly answered in [11] by utilizing the translational invariance. In [1] Moore and Taylor provide a sharper distinction between the longitudinal and the transverse directions to the D-brane by the so-called singular geometry constraints. These constraints say that the sliver states are the eigenstates of the half-string momenta and the midpoint coordinates, in a form analogous to the Neumann and Dirichlet conditions for the endpoints. In this paper we will see that these eigen-conditions are more transparent in the Moyal language, and they are characterized by the algebra for the commutative mode.

In this paper, we will rewrite the sliver state and its higher rank cousins as the GMS solitons with respect to the Moyal product and map back to the form with respect to the Witten’s star, and verified that they are indeed the projectors with respect to Witten’s star product. Our reformulation also gives a more transparent derivation of the singular geometry conditions of these projectors than the ones given in [11] and [12], the commutative mode play an essential role in yielding the singular geometry.

In the next section we summarize the map of Witten’s star to Moyal’s star for both the Neumann matrices with and without the zero mode, especially we emphasize the physical meaning of the commutative mode. The main results of this paper are contained in section 3. In section 3.1 we map the sliver state to Witten’s star into the GMS soliton $\langle 0 \rangle \langle 0 \rangle$ state to Moyal star. In section 3.2 we discuss the relation between the commutative mode and the singular geometry and reproduce the eigen-conditions of Moore and Taylor. In section 3.3 and 3.4, we generalize the above to the shifted sliver state and the Dp-brane projector, also their corresponding singular geometry conditions. In section 3.5, the higher rank projectors for the Moyal’s star are mapped to the one for the Witten’s star, and their properties are discussed. The details of demonstrating the projector condition is given in the Appendix.
In section 4 we discuss the candidate state for the closed string based on the “no-endpoint” algebraic condition. In section 5, we briefly discuss the wedge state in the Moyal basis. Finally we conclude our paper in section 6 with few comments, especially on the possibility of the non-GMS soliton in VSFT.

2 Map of Witten’s star to Moyal’s star

In [1] it is found that the pointwise commutative product can be mapped into Witten’s star product for the zero mode of the string coordinate in the zero slope limit [1] with the following 3-string vertex in terms of the oscillator $a^\dagger$

$$|V_3\rangle = \left(\frac{2}{3\pi}\right)^{1/2} \exp\sum_{i=1}^{3} \frac{1}{6} a_i^\dagger a_i^\dagger - \frac{2}{3} a_i^\dagger a_{i+1}^\dagger |0\rangle_{123},$$  \hspace{1cm} (1)

where $i \in Z_3$ is the string labels, so that

$$(f \cdot g)(x) = \langle f| \otimes \langle g| \otimes \langle x| |V_3\rangle,$$  \hspace{1cm} (2)

and $f(x) = \langle x| f\rangle$, and $\langle x|$ is the position state of $x = \frac{i}{\sqrt{2}}(a - a^\dagger)$, or

$$\langle x| = \langle 0| \exp\left\{\frac{-1}{2}x^2 + i\sqrt{2}ax + \frac{1}{2}a^2\right\}. \hspace{1cm} (3)$$

Based on the star spectroscopy [7, 8], this map is generalized to the one for all the higher stringy modes in [1], also in [6], and it turns out that Witten’s star product is a continuous Moyal product with the following correspondence: the 3-string vertex for Witten’s star is

$$|V_3\rangle = \mathcal{N}^{26} \exp \left\{ \int_0^\infty dk \sum_{i=1}^{3} \left[ -\frac{1}{2} \left( \frac{-4 + \theta^2}{12 + \theta^2} \right) (e_i^\dagger e_i^\dagger + o_i^\dagger o_i^\dagger) - \left( \frac{8}{12 + \theta^2} \right) (e_i^\dagger e_{i+1}^\dagger + o_i^\dagger o_{i+1}^\dagger) \right. \right. \left. \right.$$

$$\left. \left. - \left( \frac{4i\theta}{12 + \theta^2} \right) (e_i^\dagger o_{i+1}^\dagger - o_i^\dagger e_{i+1}^\dagger) \right] \right\} |0\rangle_{123},$$  \hspace{1cm} (4)

for pairs of the creation and annihilation operators $(e(k), e^\dagger(k), o(k), o^\dagger(k))$ labelled by the continuous parameter $0 \leq k < \infty$, and $\mathcal{N}$ is some normalization factor to be fixed later. The Moyal conjugate pairs are

$$x(k) = \frac{i}{\sqrt{2}}(e(k) - e^\dagger(k)), \hspace{1cm} y(k) = \frac{i}{\sqrt{2}}(o(k) - o^\dagger(k)),$$  \hspace{1cm} (5)

with the Moyal bracket

$$[x(k), y(k')] = i\theta(k)\delta(k - k'),$$  \hspace{1cm} (6)

\footnote{In [22] we have generalized this map to the case with constant B-field background and get the noncommutative product for the zero mode.}
where the continuous noncommutativity parameter

\[ \theta(k) = 2 \tanh \left( \frac{\pi k}{4} \right), \]

so that

\[ (f * g)(x, y) = \langle x, y | \otimes f | \otimes g | V_3 \rangle. \]

In the above \( f(x, y) = \langle x, y | f \rangle \), and \( \langle x, y | = \langle x | \otimes \langle y | \) with \( \langle x | \) (\( \langle y | \) being associated with \( x(k) (y(k)) \). Note that \( \theta(k = 0) = 0 \) and \( \theta(k = \pm \infty) = \pm 2 \).

Conventionally [3], the 3-string vertex is written in the oscillator basis

\[ \hat{X}(\sigma) = \frac{i}{2} \sum_{n=0}^{\infty} \hat{x}_n \cos n\sigma, \quad \pi \hat{P}(\sigma) = \hat{p}_0 + \sqrt{2} \sum_{n=1}^{\infty} \hat{p}_n \cos n\sigma, \]

with the creation and annihilation operators \( a_n^\dagger, a_n \) given by

\[ \hat{x} = \frac{i}{2} E \cdot (a - a^\dagger), \quad \hat{p} = E^{-1} \cdot (a + a^\dagger), \quad E_{mn}^{-1} = \sqrt{n/2} \delta_{m,n} + \frac{1}{\sqrt{b}} \delta_{m,0} \delta_{n,0}, \]

where \( b > 0 \) is an arbitrary parameter. The above Moyal basis \((e(k), o(k))\) is related to the oscillator basis by

\[ e(k) = \sqrt{2} \sum_{n=0}^{\infty} V_{2n}(k) a_{2n}, \quad o(k) = -i \sqrt{2} \sum_{n=0}^{\infty} V_{2n+1}(k) a_{2n+1}, \]

with respect to the BPZ conjugate property

\[ \text{bpz } e(k) = -e^\dagger(k) \quad \text{bpz } o(k) = -o^\dagger(k), \]

where the vector \( V(k) \) labelled by \( k \) is the eigenvector of the Neumann matrices of the 3-string vertex.

In [4] the 3-string vertex ([4]) is the one for the zero-momentum sector with zero mode excluded, and it is the star product used to construct the D25-brane sliver [11]. In this case, \( V_0(k) \equiv 0 \) and

\[ CV(k) = -V(-k), \quad C = (-1)^m \delta_{m,n}, \quad m, n \geq 1 \]

where \( C \) is the twist operator. From this fact, each eigenvalue is doubly degenerate for \( \pm k \) except at \( k = 0 \) where \( e(k = 0) = 0 \) thus \( x(k = 0) = 0 \) but

\[ y(k = 0) = \frac{i}{\sqrt{2}} (o(k = 0) - o^\dagger(k = 0)) = -\sqrt{2} \int_0^{\pi/2} \pi \hat{P}(\sigma) d\sigma = -\sqrt{2} \pi \hat{P}_L, \]

which is the momentum carried by the left half part of a string [4]. Note that \( P_L = -P_R \) in the zero-momentum sector.

Note that the map ([8]) of the noncommutative products for \( k \neq 0 \) modes reduces to the map ([3]) of the commutative products for \( y(k = 0) \) at \( k = 0 \).
In [6] the 3-string vertex of nonzero momentum with zero mode included is also shown to be in the form of (4) with the same \( \theta(k) \) but with respect to the different eigenvectors denoted by \( \tilde{V}(k) \) and

\[
\tilde{C} \tilde{V}(k) = \tilde{V}(-k), \quad \tilde{C}_{m,n} = (-1)^m \delta_{m,n}, \quad m, n \geq 0
\]

so that the nonzero non-degenerate commutative coordinate at \( k = 0 \) is not \( y(k = 0) \) as in the above but \( x(k = 0) \). Moreover, it is shown in [6]

\[
x(k = 0) = \frac{i}{\sqrt{2}}(e(k = 0) - e^\dagger(k = 0)) = \frac{1}{\sqrt{2\pi}} X\left(\frac{\pi}{2}\right)
\]

which is the midpoint of the string.

Besides the continuous spectrum labelled by \( k \), there exists a discrete spectrum for the Neumann matrices with zero mode. And it has been shown in [6] the supplemented 3-string vertex associated with the discrete mode can be also put in the Moyal form (4), however, with a theta parameter \( \theta_d(b) \in (2, \infty) \) determined by the parameter \( b \) associated with the zero mode given in (10). In other words, the three-string vertex is the tensor product of the continuous one with the discrete one:

\[
|V_3\rangle_{p \neq 0} = |V_3\rangle_c \otimes |V_3\rangle_d.
\]

Both \(|V_3\rangle_c\) and \(|V_3\rangle_d\) could take the Moyal form (4).

Before ending this section, we would like to point out that the normalization factor \( N \) is 1 in the usual SFT for Witten’s star, but for (8) to hold true for the the Moyal product defined by

\[
(f * g)(\vec{x}_3) = \int d\vec{x}_1 d\vec{x}_2 K(\vec{x}_1, \vec{x}_2, \vec{x}_3) f(\vec{x}_1)g(\vec{x}_2)
\]

with the kernel

\[
K(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{\pi^2 \det \Theta} \exp\left(-2i(\vec{x}_1 - \vec{x}_3)\Theta^{-1}(\vec{x}_2 - \vec{x}_3)\right),
\]

where \( \vec{x} = (x, y) \) and \( \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \), we need to set\[2\]

\[
\mathcal{N} = \exp\left\{ \log L \int_0^\infty dk \log(\frac{8}{12 + \theta^2}) \right\},
\]

with the factor \( \frac{\log L}{2\pi} \) as the spectral measure with \( L \) the level regulator, i.e. approximating the infinite \( K_1 \) matrix by a finite one of the size \( L \times L \)\[11, 14, 4\]. Therefore, the 3-string vertex of Witten’s star is related to the one of Moyal’s star by

\[
|V_3\rangle_M = \mathcal{N}^{26}|V_3\rangle_W,
\]

and the corresponding projectors are related in an opposite way.

\[2\]We will drop the overall \( \pi \) factors from now on, which can be compensated by proper re-scalings.
3 Sliver state as GMS solitons

In this section, it will be shown that the sliver state is a GMS soliton in the continuous Moyal basis, and the \( k = 0 \) mode gives the singular geometry in [1]. Its higher rank cousins will be constructed and the projector condition with respect to the 3-string vertex of Moyal product will be checked. This establishes the correspondence between the GMS soliton and the projectors of VSFT. Some properties of these projectors as multiple D-branes will also be discussed.

3.1 Sliver state

The sliver state of a D25-brane of VSFT constructed in [10, 11] in the oscillator basis has the form

\[
|\Xi\rangle = \left[ \det(1 - M) \det(1 + T) \right]^{13} \exp\left( -\frac{1}{2} \sum_{n,m} a_n^\dagger S_{mn} a_m^\dagger \right) |p = 0\rangle
\]

where

\[
T = \frac{1}{2M} \left( 1 + M - \sqrt{(1 + 3M)(1 - M)} \right), \quad S = CT,
\]

(23)

where \( C_{mn} = (-1)^m \delta_{m,n} \) is the twist operator, and \( M = CV^{11} \) with \( V^{11} \) the one of the Neumann matrices in the 3-string vertex. The eigenvalues of \( M \) [7] is

\[
\mu(k) = \frac{\theta^2 - 4}{12 + \theta^2},
\]

(24)

with \( \theta = 2 \tanh\left( \frac{\pi k}{4} \right) \).

Rewriting the sliver state in the Moyal basis \((e(k), o(k))\) we have

\[
|\Xi\rangle = N^{26} \exp\left( -\frac{1}{2} \int_0^\infty dk \frac{\theta - 2}{\theta + 2} (e^\dagger e^\dagger + o^\dagger o^\dagger) \right) |0\rangle
\]

(25)

where \( N \) is the normalization factor,

\[
N \equiv \sqrt{\det(1 - M) \det(1 + T)} = \exp\left( \frac{\log L}{2\pi} \int_0^\infty dk \log \frac{32\theta}{(\theta + 2)(12 + \theta^2)} \right).
\]

(26)

Note that we have taken into account the double degeneracy of each eigenvalue at \( k \neq 0 \) for the normalization factor. The divergence of the normalization in the limit of \( L \to \infty \) is quite similar to the discussion in [1].

From the above, we can obtain the string field functional of the sliver state in terms of the Moyal conjugate variables for \( k \neq 0 \) modes as

\[
\Psi_W^{(s)}(x(k), y(k)) \equiv \langle x, y|\Xi\rangle = \tilde{N}^{26} \exp\left( -\int_0^\infty dk \frac{x^2(k) + y^2(k)}{\theta(k)} \right),
\]

(27)
where

\[ \tilde{N} = \exp \left( \frac{\log L}{2\pi} \int_0^\infty dk \, \log \frac{16}{12 + \theta^2} \right). \]  

(28)

The sliver state with respect to Moyal’s star is

\[ \Psi^{(s)}_M = \mathcal{N}^{-26} \Psi^{(s)}_W, \]  

(29)

which is the continuous version of the GMS soliton of the lowest rank

\[ 2 \exp \left\{ -\frac{x^2 + y^2}{\theta} \right\}. \]  

(30)

In summary, the sliver state obeys the projector condition for the usual GMS soliton, that is,

\[ (\Psi^{(s)}_W \star \Psi^{(s)}_W)(x(k), y(k)) = \Psi^{(s)}_W(x(k), y(k)), \]  

(31)

this also holds if we replace the sub-index W by M. Moreover, the norm of the sliver state is

\[ \langle \Xi | \Xi \rangle = \exp \left\{ \frac{26\log L}{2\pi} \int_0^\infty dk \, \log \frac{128\theta}{(12 + \theta^2)^2} \right\} \]  

(32)

which could also be obtained by calculating \( \int Dx(k)Dy(k) |\Psi^{(s)}_W(x(k), y(k))|^2 \).

3.2 Commutative mode and singular geometry

The algebraic operations in Witten’s string field theory are defined by gluing the left and right half-strings, so the mid-point is expected to be special because there one loses the distinction between the left and the right. In [1] it is shown that the sliver states obey some singular geometry conditions with respect to the mid-point.

In the Moyal basis, the mid-point is associated with \( k = 0 \) mode as implied by (14) and (16), and this mode is commutative unlike the other noncommutative \( k \neq 0 \) modes. It is easy to see the relation between the singular geometry conditions and the commutative \( k = 0 \) mode as following. By acting the Moyal coordinate \( y(k) \) defined in (5) on the sliver state (25), we get

\[ y(k')|\Xi\rangle = \{ N^{26} \exp \int_0^\infty dk \left( -\frac{s_1}{2} (e^\dagger e^\dagger + o^\dagger o^\dagger) \right) \} \frac{i}{\sqrt{2}} \left[ -\left(s_1(k') + 1\right) o^\dagger(k') \right] |0\rangle, \]  

(33)

and the last factor in [\cdots] vanishes only if \( k' = 0 \) since \( s_1(k = 0) = -1 \). This leads to the result that the sliver state is the eigenstate of the half-string momentum \( \hat{P}_L = -y(k = 0)/(\sqrt{2}\pi) \), that is,

\[ \hat{P}_L |\Xi\rangle = 0. \]  

(34)

This is nothing but the singular geometry condition for the D25-brane as discussed in [4]. One can translate the above eigen-condition into the one with respect to the conjugate coordinate of \( \hat{P}_L \), we will not do it here.
After knowing that the singular geometry is associated with the \( k = 0 \) commutative mode, we can single out the mode and translate it into the coordinate basis and examine its behavior in this basis. However, if we naively continue the normalization factor (26) to the \( k = 0 \) mode, we find that it is zero so that we need to introduce a small cutoff for it, that is, the regularized \( k = 0 \) projector is

\[
|\psi_\epsilon\rangle = \mathcal{N}_\epsilon^{26} \exp\left\{ \frac{1}{2} o_0 \dagger o_0 \right\} |0\rangle
\]  

(35)

where \( o_0 \dagger \equiv o \dagger (k = 0) \), and the regularized normalization constant

\[
\mathcal{N}_\epsilon = \sqrt{\frac{4\epsilon}{3}}
\]  

(36)

which is the square root of (26) for small \( \theta = \epsilon \), the square root is taken because of no double degeneracy as for the noncommutative modes, i.e. \( e(k = 0) = 0 \) as mentioned in section 2.

From the fact that the integrand of (27) is singular at \( k = 0 \), we know that the transformation of \( k = 0 \) mode in the \( o_0 \dagger \)-basis to the one in the \( y(k = 0) \equiv y_0 \)-basis is singular unless we introduce a small cutoff such that

\[
e_0 \langle y_0 | = \langle 0 | \exp\left\{ \frac{-1}{2} y_0^2 + i\sqrt{2} ay_0 + \frac{(1 - \epsilon_0)}{2} a^2 \right\} .
\]  

(37)

so that

\[
e_0 \langle y_0 | \psi_\epsilon \rangle = (\mathcal{N}_\epsilon \frac{1}{\sqrt{\epsilon_0}})^{26} \exp\left\{ -\left( \frac{1 + 2\epsilon_0}{\epsilon_0} \right) y_0^2 \right\} .
\]  

(38)

If we choose \( \epsilon_0 = \frac{4\epsilon}{3} \) and take the limit of \( \epsilon \to 0 \), we obtain

\[
\lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon_0} \langle y_0 | \psi_\epsilon \rangle = \begin{cases} 0, & y_0 \neq 0 \\ 1, & y_0 = 0 \end{cases}.
\]  

(39)

This is a projector with unit height due to the choice of the ratio between \( \epsilon \) and \( \epsilon_0 \). This result is also discovered by [1] in the oscillator basis. Note that the integrand of (27) suggests a delta function for \( k = 0 \) mode, this is not case because of the vanishing normalization, and these two effects match to produce a unit-height function on a point, which is of zero measure and thus singular in the sense of functions.

The most general projector for a commutative coordinate is the unit-height function with support on discrete points or some finite intervals. The choice of the commutative projector with support on finite intervals yields regular mid-point geometry. However, the above limiting procedure yields only the single-point support function which implies the zero half-string momentum condition \( P_L = 0 \). This can be seen as some intrinsic feature of the sliver state originally derived in the oscillator basis, and its singular geometry manifests in the commutative mode of the Moyal basis.

It might be easy to mix up the singular geometry and the issue of the midpoint singularity as discussed in [14, 15]. Although \( k = 0 \) mode is associated with both the singular geometry
of the mid-point and the midpoint issue, we shall emphasize that our way of deriving singular geometry condition is independent of the midpoint singularity issue. The issue concerns how to consistently extract the finite physical quantities from the infinite dimensional Neumann coefficient matrix, whose $-1/3$ eigenvalue break the twist symmetry of the Neumann matrix [4, 8] and make the physical observables behave singularly there. The $-1/3$ eigenvalue is exactly the $k = 0$ in our formulation. Similar issue happens when one includes the ghost sectors of the slivers with pure ghost BRST operator inserted at the midpoint [13, 14, 15, 20]. On the other hand, the derivation of the eigen-condition (34) concerns only the gaussian part of (25) and has nothing to do with the twist anomaly or any cut-off of the size of the Neumann matrix. The irrelevance of the cutoff issue to our derivation can be also seen from the existence of the well-defined projector (39) in the limit of $\epsilon \to 0$. The name of singular geometry is mainly due to the unit-height function of zero measure for the commutative mode.

3.3 Shifted D25-brane sliver

Instead of starting from the sliver state in the oscillator basis and then transforming it to the GMS soliton, we can start with the following general ansatz of the projector in the Moyal basis

$$|\Xi_s\rangle = N^{26}_2 \exp \int_0^{\infty} dk \left( \alpha e^\dagger + \beta o^\dagger - \frac{s}{2} (e^\dagger e^\dagger + o^\dagger o^\dagger) \right) |0\rangle,$$  \hspace{1cm} (40)

and require the projector condition

$$|\Xi_s\rangle_3 = 1 \langle \Xi_s | 2 \langle \Xi_s | V_3 \rangle.$$  \hspace{1cm} (41)

Since the inner product is defined with respect to BPZ conjugation, we have to take $bpz(e^\dagger) = -e, bpz(o^\dagger) = -o$, so that

$$\langle \Xi_s | = N^{26}_2 \langle 0 | \exp \int'_0 dk \left( -\alpha e - \beta o - \frac{s}{2} (ee + oo) \right).$$  \hspace{1cm} (42)

After some calculation, the projector condition imposes the following condition on $s$

$$\left( \mu_2 + (s^{-1} - \mu)^{-1} (\mu_2^2 + \mu_3^2 \right)^2 - (\mu_3 - 2\mu_2\mu_3(s^{-1} - \mu)^{-1})^2
= \left( s - [\mu + (s^{-1} - \mu)^{-1}(\mu_2^2 - \mu_3^2)] \right) \left( s^{-1} - [\mu + (s^{-1} - \mu)^{-1}(\mu_2^2 - \mu_3^2)] \right)$$  \hspace{1cm} (43)

where

$$\mu = \frac{\theta^2 - 4}{12 + \theta^2}, \quad \mu_2 = \frac{8}{12 + \theta^2}, \quad \mu_3 = \frac{4\theta}{12 + \theta^2}. \hspace{1cm} (44)$$

The solutions of (43) are

$$s_1 = \frac{\theta - 2}{\theta + 2}, \quad s_2 = \frac{\theta + 2}{\theta - 2}, \quad s_3 = 1.$$  \hspace{1cm} (45)
This agrees with the 3 solutions found in the oscillator basis [11]: the \( s_1 \) solution gives the sliver state as we can see from the discussion in the last subsection; The \( s_2 \) solution gives the non-normalizable projectors as in [11], which has the form

\[
T = \frac{1}{2M}(1 + M + \sqrt{(1 + 3M)(1 - M)}).
\]  

(46)

The functional form of this state is proportional to \( \exp\left(\frac{x^2 + y^2}{2}\right) \), which is non-normalizable so it is unphysical; the \( s_3 \) solution gives the identity string field state.

When \( s = s_3 = 1 \), the projection condition on the linear terms requires that \( \alpha = \beta = 0 \). That means there is no projector state with a linear term in the exponential acting on the identity string state. Nevertheless, the states with a linear term in the exponential acting on the identity string state have been used as gauge transformations in [19].

However, when \( s = s_1 \), there is no constraint on \( \alpha(k) \) and \( \beta(k) \) from the projection condition (41), while the normalization factor (for Witten’s star) is required to be

\[
N_{\parallel} = \exp\left\{ \frac{\log L}{2\pi} \int_0^\infty dk \left( \frac{\theta + 2}{2}(\alpha^2 + \beta^2) + \log \frac{4\theta}{\theta + 2} \right) \right\}.
\]  

(47)

Its string field in the Moyal variables is

\[
\langle x, y | \Xi \rangle_{s=s_1} = \exp\left\{ 26\frac{\log L}{2\pi} \int_0^\infty dk \log 2 \right\} 
\exp \int_0^\infty dk \left( \frac{-1}{\theta} \left\{ \left( x - \frac{i(\theta + 2)\alpha}{2\sqrt{2}} \right)^2 + \left( y - \frac{i(\theta + 2)\beta}{2\sqrt{2}} \right)^2 \right\} \right).
\]  

(48)

This is the shifted GMS soliton with the same norm as the unshifted one. In the language of the VSFT, this means that the tension of the shifted sliver is the same as the one for the unshifted sliver.

We can obtain the singular geometry condition for the shifted sliver by acting \( y(k) \) on it, the result is

\[
\hat{P}_L|\Xi_{s=s_1} = -\frac{i\beta_0}{2\pi}|\Xi_{s=s_1}. \]  

(49)

Alternatively, we can get the singular projector of the commutative mode as that in (39) but with the support at \( y_0 = i\beta_0/\sqrt{2} \). Unlike the singular geometry condition for the unshifted sliver, it is more clear that (49) is an eigen-condition because of non-zero eigenvalue.

3.4 Lower dimensional Dp-brane projector

The lower dimensional Dp-brane projectors can be constructed in a similar way for the sliver as a GMS soliton by using the Moyal basis for the Neumann matrices with zero mode. The corresponding state for the continuous spectrum is the same as the continuous shifted
sliver given by (10), we should also supplement the part for the discrete spectrum with the noncommutativity parameter \( \theta_d(b) \). Explicitly, the Dp-brane projector is

\[
|\Xi_s(p)\rangle = N_2^p \exp \int' dk \left( \alpha e^\dagger + \beta o^\dagger - \frac{1}{2} (e^\dagger e^\dagger + o^\dagger o^\dagger) \right) |0\rangle \\
\otimes \left( \frac{32 \theta_d}{(\theta_d + 2)(12 + \theta_d^2)} \right)^{25-p} \exp \left( - \frac{1}{2} \frac{\theta_d - 2}{\theta_d + 2} (\epsilon_b^\dagger \epsilon_b^\dagger + \omega_b^\dagger \omega_b^\dagger) \right) |0\rangle ,
\]

(50)

where in the first factor the Moyal pairs \((e^\dagger, o^\dagger)\) along the \(25-p\)-dimensional transverse directions are related to the oscillator basis by the eigenvectors of the Neumann matrices with zero mode, while the rest along the longitudinal directions to the ones without zero mode.

The derivation of the singular geometry condition along the transverse directions is technically the same as the one for the one of the D25-brane sliver, it gives

\[
\hat{X}^\perp(\frac{\pi}{2}) |\Xi_s(p)\rangle_{s=s_1} = \frac{i \alpha_0^\perp}{2 \sqrt{\pi}} |\Xi_s(p)\rangle_{s=s_1} .
\]

(51)

Note that \(x^\perp(k = 0) = \frac{1}{\sqrt{2\pi}} \hat{X}^\perp(\frac{\pi}{2})\) and \(\alpha_0^\perp \equiv \alpha^\perp(k = 0)\), where the superscript \(\perp\) denotes the transverse directions.

Or, we get the following limiting singular projector

\[
\lim_{\epsilon \to 0} \langle x_0^\perp | N^{25-p}_\epsilon \exp \left\{ \alpha_0^\perp e_0^\dagger + \frac{1}{2} e_0^\dagger e_0^\dagger \right\} |0\rangle = \begin{cases} \\
0, & x_0^\perp \neq \frac{i \alpha_0^\perp}{\sqrt{2}} \\
1, & x_0^\perp = \frac{i \alpha_0^\perp}{\sqrt{2}},
\end{cases}
\]

(52)

where the definition of the regularized bra \(\epsilon_0 \langle x_0^\perp\rangle\) is the same as \(\epsilon_0 \langle y_0\rangle\) given in (17). These results are also discovered in [1] for the D-instanton in the oscillator basis. Here we explicitly demonstrate that it is associated with the commutative \(k = 0\) mode.

### 3.5 Higher rank projectors and multiple D-branes

In this subsection we would like to use the map of Witten’s star to Moyal’s star to construct the new projector states with respect to \(|V_3\rangle\).

We start with the generating function of the projection operators [23, 18] on the noncommutative plane defined by its Moyal pairs. It is

\[
G(\lambda, \bar{\lambda}, x, y) = 2e^{-\lambda \bar{\lambda} + \bar{\lambda} \sqrt{\frac{\pi}{3}} r e^{i \phi} + \lambda \sqrt{\frac{\pi}{3}} r e^{-i \phi} - \frac{r^2}{4}} = \sum_{n,m} \frac{\lambda^n \bar{\lambda}^m}{\sqrt{n!} \sqrt{m!}} f_{nm}(x, y) ,
\]

(53)

where \(x + iy = re^{i \phi}\), and \(f_{nm}(x, y)\) is the Weyl transform of the operator \(|n\rangle\langle m|\):

\[
f_{nm}(x, y) = \int dp \ e^{-ipy} \langle x + \frac{p}{2} | n \rangle \langle m | x - \frac{p}{2} \rangle .
\]

(54)

Then we look for the state \(|G\rangle\) with respect to \(|V_3\rangle\) for a fixed \(k \neq 0\) mode such that

\[
G(\lambda, \bar{\lambda}, x, y) = \langle x, y | G \rangle .
\]

(55)
After some manipulations we obtain the projector generating state

\[ |G\rangle = \frac{4\theta}{\theta + 2} e^{-s\lambda\bar{\lambda} - it[\lambda(e^\dagger + i\sigma^x) + \lambda(e^\dagger - i\sigma^x)] - \frac{1}{2}s(e^{i\sigma^2 + o^{12}})|0\rangle, \] (56)

where

\[ s = \frac{\theta - 2}{\theta + 2}, \quad t = \frac{2\sqrt{\theta}}{\theta + 2}. \] (57)

Now it is easy to get the projector state for \(|n\rangle\langle n| \) by differentiating \(|G\rangle \) \(n\) times with respect to \(\lambda, \bar{\lambda}\) and then set \(\lambda, \bar{\lambda}\) to zero. For example, the \(|0\rangle\langle 0| \) corresponds to

\[ |G_{00}\rangle = \frac{4\theta}{\theta + 2} e^{-\frac{1}{2}s(e^{i\sigma^2 + o^{12}})|0\rangle, \] (58)

and the corresponding function form is

\[ f_{00}(x, y) = 2e^{-x^2}; \] (59)

and the \(|1\rangle\langle 1| \) corresponds to

\[ |G_{11}\rangle = -\frac{4\theta}{\theta + 2} [s + t^2(e^{i\sigma^2} + o^{12})]e^{-\frac{1}{2}s(e^{i\sigma^2 + o^{12}})|0\rangle, \] (60)

and its corresponding function form is

\[ f_{11}(x, y) = 2(-1 + \frac{2}{\theta}x^2)e^{-x^2}. \] (61)

which could be obtained directly from \(\langle x, y|G_{11}\rangle\). One can explicitly check that (51) is indeed a projector state with respect to \(|V_3\rangle\) although it is not in the gaussian form as the sliver state, the details are given in the Appendix.

It is straightforward to generalize to the continuous Moyal basis, for example, the generating projector state is

\[ |G\rangle_W = N^{26} \exp\left\{ \int_0^\infty dk \left(-s\lambda\bar{\lambda} - it[\lambda(e^\dagger + i\sigma^x) + \lambda(e^\dagger - i\sigma^x)] - \frac{1}{2}s(e^{i\sigma^2 + o^{12}}) \right) \right\} |0\rangle \] (62)

where \(N\) is the same normalization factor for the sliver state as defined in (26), and \(s\) and \(t\) are now functions of \(k\) as understood. Similarly one can then obtain the projectors of the definite rank in the continuous Moyal basis.

One can calculate the norm of the \(|1\rangle\langle 1| \) projector, from \(\langle G_{11}|G_{11}\rangle\) or using the functional integration over \(x(k), y(k)\). It turns out that it is the same as the norm for the sliver state.

One can also check the orthogonality between the sliver state and the above projector state by considering their inner product, which could be defined either on the Moyal basis or in functional forms. As expected, it is not hard to find that

\[ \langle G_{00}|G_{11}\rangle = 0 = \int dx dy f_{00}(x, y)f_{11}(x, y). \] (63)
One may wonder if $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ obey the same singular geometry condition. This is indeed the case and can be verified straightforwardly by acting the coordinates $x(k)$ and $y(k)$ on $|G\rangle$.

In principle, one could obtain all the projectors from the projector generating state $|G\rangle$, whose “wavefunctional” is the generating function of the usual GMS soliton projectors. We expect the projector states obtained from it have the same norm and orthogonal to each other. Similarly, the algebraic eigen-constraints (59) and (61) should also hold for the higher rank projectors. This implies that we can form the multiple D-branes state by the linear combinations of these projectors to form a general GMS soliton. Symbolically that is

$$O_c = \sum_j c_j |j\rangle\langle j|.$$  \hspace{1cm} (64)

In the context of VSFT, the string fields are assumed to factorize into the ghost and matter parts as

$$\Phi = \Phi_g \otimes \Phi_m,$$ \hspace{1cm} (65)

if it also satisfies

$$Q\Phi_g + r\Phi_g \ast g \Phi_g = 0, \quad Q\Phi_m = 0,$$ \hspace{1cm} (66)

where $Q$ is the BRST operator around the closed string vacuum, then the cubic action of the string field theory can also be factorize into the ghost and the matter parts, with the matter action taking the form

$$S_m = \int \left( \Phi_m \ast \Phi_m - \frac{r}{3} \Phi_m \ast \Phi_m \ast \Phi_m \right).$$ \hspace{1cm} (67)

Note that we can set $r = 1$ by properly re-scaling $\Phi_m$.

If we treat the above Witten’s star as a continuous Moyal’s star, then (67) is the action of infinite copies of the noncommutative field theory for the GMS soliton, where the kinetic term is suppressed in the large $\theta$ limit [18]. On the other hand, the kinetic term is also dropped here due to the assumption of VSFT. Follow the discussions in [18], the absence of the kinetic term makes the action (67) possess a global $U_k(\infty)$ symmetry for each mode labelled by $k$. Moreover, a level (the number of nonzero $c_j$’s in defining $O_c$) $m$ soliton spans an infinite dimensional Grassmanian moduli space parameterized by $\otimes_{k \in [0,\infty)} U_k(\infty) U_{k(\infty-m)}$. The partial isometry [24] of the Hilbert space for a fixed $k$ relating different projectors $|j\rangle\langle j|$ is also the symmetry of the matter action. This partial isometry could be important when one tries to take into account the kinetic part of the string field theory action.

4 Candidate state for closed string

In the last section we see that the stringy midpoint in VSFT plays the similar role of the zero mode $\hat{x}_0$ in the perturbative string theory for the Dp-brane’s boundary conditions. It is then natural to expect that the “stringy endpoints” $\hat{X}(0)$ and $\hat{X}(\pi)$ in VSFT will play the same role as in the perturbative string theory. Based on the expectation, a candidate state
\(|\Gamma\rangle\) for the closed string is proposed in \([1]\) by imposing the following “no-endpoint” algebraic condition
\[
[\hat{X}(0) - \hat{X}(\pi)]|\Gamma\rangle = 0 .
\] (68)

In terms of the Moyal basis, this condition becomes
\[
\int_0^\infty dk \, V_o(k)[o(k) - o^\dagger(k)]|\Gamma\rangle = 0 , \quad V_o(k) \equiv \sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}} V_{2n-1}(k) .
\] (69)

By taking the gaussian ansatz for \(|\Gamma\rangle\) as
\[
|\Gamma\rangle = \exp\left\{ -\frac{1}{2} \int_0^\infty dk \, dk' \, \Gamma(k, k') o^\dagger(k) o^\dagger(k') \right\} |0\rangle \otimes |\Gamma_e\rangle ,
\] (70)

the “no-endpoint” condition (69) becomes
\[
\int_0^\infty dk' \, \Gamma(k, k') V_o(k') = -V_o(k) ,
\] (71)

while there is no condition on \(|\Gamma_e\rangle\) which by definition consists of only \(e^\dagger(k)\)’s. It is interesting to note that we will arrive the same condition (71) even we change the condition (68) to \(\hat{X}(0) - \hat{X}(\pi) = 2\pi R\) for the compact direction with radius \(R\).

The condition (71) says that \(V_o(k)\) is the eigenvector of \(\Gamma(k, k')\) with eigenvalue \(-1\). Since \(V_o(k)\) is a fixed vector, this condition alone will not fix a unique \(\Gamma(k, k')\). Moreover, a specific solution \(\Gamma(k, k')\) is equivalent to the ones different by the kernel of \(V_o(k)\),
\[
\{K_V | \int_0^\infty dk' \, K_V(k, k') V_o(k') = 0 \} .
\] (72)

This suggests that \(K_V\) generates the gauge symmetry around the candidate closed string state, which is similar to the quotient condition for the BRST constraint. It deserves further study to see if the gauge symmetry generated by \(K_V\) is related to the gauge symmetry of the closed string field theory.

In this paper we are not trying to examine the full solution space of (71) but mention a particular solution
\[
\Gamma(k, k') = -\delta(k, k') .
\] (73)

This gives the above mentioned \(s = -1\) state for all modes not just for \(k = 0\). As commented in subsection 3.2, \(s = -1\) gives the singular wavefunctional and we get
\[
\langle x(k), y(k)|\Gamma\rangle = \left\{ \begin{array}{ll} 0 , & y(k) \neq 0 \quad \forall \, k \\ \langle x(k)|\Gamma_e\rangle , & y(k) = 0 \quad \forall \, k \end{array} \right. \] (74)

It is not a projector and thus not a solution of the matter sector in VSFT for arbitrary \(\langle x(k)|\Gamma_e\rangle \neq 1\), and its geometry is highly singular because it has a single point support for all modes. This implies that the candidate closed string state is a highly singular state in VSFT.

\(^3\)Need to use \([o(l), \int dk d\lambda' \, \Gamma(k, k') V_o(q') o^\dagger(k) + \int dk \, V_o(k) o^\dagger(k)] = 0\) and \(\Gamma(k, k') = \Gamma(k', k)\).

\(^4\)Some proper normalization factor is chosen to arrive the well-defined singular functional (74).
5 Wedge states

It has been pointed out that the sliver state can be taken as the limit of wedge state\textsuperscript{[16, 17]}. The wedge state is defined as
\[ |n\rangle = (|0\rangle)^{n-1}. \] (75)

Here the star product is with respect to $|V_3\rangle$. As we has seen in the above sections, the string product could be understood as the continuous Moyal product. Here, we just assume that the wedge state is “diagonal”, which means that we define the wedge state for any fixed $k$. Let us take the form of the three vertex as in (4) without integral over $k$, up to a normalization factor. Then it’s not hard to find that the wedge state has the form
\[ |n\rangle = N_n \exp \left\{ -\frac{1}{2} \left( \frac{s + (−s)^{n-1}}{1−(−s)^n} \right) (e^{i2} + o^{i2}) \right\} |0\rangle. \] (76)

The normalization factor $N_n$ satisfies the recursive relation
\[ N_{n+1} = \frac{1}{1−\mu t_n} N_n \mathcal{N} \] (77)
where $\mathcal{N}$ is the normalization factor of $|V_3\rangle$ and
\[ s = \frac{θ−2}{θ+2}, \quad t_n = \frac{s + (−s)^{n-1}}{1−(−s)^n}. \] (78)

Taking the large $n$ limit, if $θ \neq 0$, the exponential term has exactly the form of the sliver state. This fact support the original argument\textsuperscript{[10]} that the sliver state is taken as the large $n$ limit of the wedge state. However the normalization is subtle. From the above relation on normalization factor, it is easy to notice that the $\mathcal{N}$ determine the normalization $N_n$. If we take Witten’s vertex, then the normalization factor $N_n$ diverge as $n \to \infty$. On the contrary, the choice of Moyal product makes $N_\infty$ vanish. It seems that the right choice is
\[ \mathcal{N} = \frac{1−s}{1−s + s^2}, \] (79)
which correspond to the choice in\textsuperscript{[17]}. However, it is the form of the projector that we are interested in. One can fix the normalization of the sliver state with respect to different $|V_3\rangle$’s by projector condition and simple rescaling. But the form of the exponential part is always the same. In this sense, the sliver state could be taken as the large $n$ limit of the wedge states.

One can write down the wavefunction of the wedge state, up to the normalization factor, it is
\[ \psi^n(x, y) = \langle x, y|n\rangle = N_n \frac{1}{1+t_n} \exp \left( -\frac{1}{2} \frac{1−t_n}{1+t_n} (x^2 + y^2) \right). \] (80)

The generalization to the continuous Moyal product is straightforward.
6 Discussions and Conclusions

In this paper we map the sliver state and its higher rank cousins into the GMS solitons in the continuous Moyal basis. Although their forms in the operator and coordinate bases look different, they are surprisingly to be the same thing.

We also notice that the midpoint singular geometry of the lump sliver state is related to projector representation of the $k = 0$ commutative algebra, they can be summarized as the following eigen-conditions with respect to the string midpoint

$$\hat{P}_L^{||}|Ξ_s⟩^{(p)} = p_m|Ξ_s⟩^{(p)} .$$

(81)

and

$$\hat{X}_\bot^{(\frac{π}{2})}|Ξ_s⟩^{(p)} = x_m|Ξ_s⟩^{(p)} .$$

(82)

where $||$ and $\bot$ denote the longitudinal and transverse directions to the Dp-brane. These conditions are the analogies to the Neumann and Dirichlet condition for string endpoints.

In [21], it has been argued that the singular property of the midpoint disappears in the VSFT with NS $B$ field. It is interesting to understand the smearing feature in the continuous basis with $B$ field, and see if the projector functional with constant $B$ field can satisfy some eigen-condition by mixing up the (81) and (82) in analogy to the mixing Neumann and Dirichlet conditions in the perturbative string theory with constant $B$ field.

We also find that a specified candidate closed string state defined by the “no-endpoint” condition is a highly singular state of VSFT with the gauge symmetry defined by some null condition, and its geometry is singular for all modes. Its properties deserve further study to understand the emergence of closed string after tachyon condensation.

The matter sector of VSFT in the continuous Moyal basis can be seen as an continuous tensor product of the non-propagating noncommutative scalar field theory with noncommutativity $\theta(k)$. So are the sliver and lump projector as the continuous tensor product of the corresponding GMS solitons for each $k$. For such a projector it seems that we have the freedom to take different GMS solitons for different $k$ because the matter action (67) has an $\bigotimes_{k\in[0,\infty)} U_k(\infty)$ symmetry and a symmetry of the partial isometry. The former transforms the radially symmetric solution to the general ones [18], and the latter relates different ranks of the GMS solitons [24]. However, just like in the noncommutative scalar field theory, we expect that the situation could be different after taking into account the kinetic terms dictated by the pure ghost BRST operator of the VSFT. It will be important to analyze the BRST operator and find the classical solution to the ghost sector in the continuous basis [6]. It is also interesting to investigate Witten’s string field from this new angle.

In the above discussions, we have been focusing on the projectors, which are just continuous tensor product of the projectors in the respective Moyal basis. We call such kind of projectors as the “GMS” projectors since it is nothing but the tensor product of the usual GMS solitons. It is a very interesting question to find out the non-GMS projectors, the ones with mixings among different modes. One simple choice is

$$|P⟩ = N_m \exp \left( -\int dk dk' M(k, k')(e^\dagger(k)e^\dagger(k') + o^\dagger(k)o^\dagger(k')) \right) |0⟩ .$$

(83)
where the mixing function $\mathcal{M}(k,k')$ will be determined by the projector condition. However, due to the fact that the Neumann coefficients are $k$-dependent, the projector condition is too complicated to be solved.

To bypass the technical difficulty and have some flavor of the (im)possibility of the non-GMS soliton, we use the fact to be pointed out in [26] that: there is a simple and natural Moyal basis on the half string $0 \leq \sigma < \pi/2$ so that the noncommutativity is $\theta = 2$ for all stringy modes except for the commutative string midpoint.

We then try to construct the non-GMS solitons in the $\theta = 2$ Moyal basis. The 3-string vertex in this basis is simple

$$|V_3\rangle = \exp\left\{ \sum_{i=1,2,3} \int_0^{\frac{\pi}{2}} d\sigma \left[ -\frac{1}{2}(c^\dagger_i(\sigma)c^\dagger_{i+1}(\sigma) + o^\dagger_i(\sigma)o^\dagger_{i+1}(\sigma)) - \frac{i}{2}(c^\dagger_i(\sigma)o^\dagger_{i+1}(\sigma) - o^\dagger_i(\sigma)c^\dagger_{i+1}(\sigma)) \right] \right\} |0\rangle.$$  

(84)

Now the noncommutative parameter is the same for any $\sigma \in (0,\pi/2]$. In this case, the issue on the non-GMS projector is workable. We assume the non-GMS projector take the form:

$$|S\rangle = \exp\left\{ \int_0^{\frac{\pi}{2}} d\sigma \frac{-s(\sigma)}{2} [(c^\dagger(\sigma)c^\dagger(\sigma) + o^\dagger(\sigma)o^\dagger(\sigma)] + \int_0^{\frac{\pi}{2}} d\sigma d\sigma' x(\sigma,\sigma')(c^\dagger(\sigma)c^\dagger(\sigma') + o^\dagger(\sigma)o^\dagger(\sigma')) \right\}. $$

(85)

Since the 3-string vertex takes the same form for all $\sigma \in (0,\pi/2]$, the projector condition is far more simple and it turns out to be

$$\begin{cases} 2sx = x \\ \frac{1}{4}(2s + 3s^2 + 3x^2 + i(2s - s^2 - x^2)) = s \end{cases} \quad (86)$$

The only solution to the above two equations is $x = 0, s = 0$, which is exactly the sliver state solution in $\theta = 2$ algebra. The non-GMS projector seems to be impossible in this case. This suggests that it is hard to find non-GMS projectors in the continuous Moyal basis of [4]. More efforts should be made to have the full answers to the possibility of the non-GMS soliton.

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7 Appendix

In this section, we will outline the step in checking the projector state (60) with respect to $|V_3\rangle$ in the form (34). The calculation is straightforward although it is tedious. The relevant formulae have the following form: for creation and annihilation operator $a^\dagger$, $a$ we have

$$\langle 0|e^{-\frac{\alpha}{2}a^2} a^n a^\dagger m e^{-\frac{\beta}{2}a^2 + \gamma a^\dagger}|0\rangle$$
\[ \langle 0 | e^{-\frac{d^2}{2}a^\dagger e^{-\frac{d^2}{2}a^2 + \gamma a^\dagger}} | 0 \rangle = \frac{1}{\sqrt{1 - \alpha \beta}} \left[ -\gamma (1 - \alpha \beta)^{-1} \alpha \exp \left( -\frac{1}{2} \gamma^2 (1 - \alpha \beta)^{-1} \alpha \right) \right] \] (88)

\[ \langle 0 | e^{-\frac{d^2}{2}a^\dagger e^{-\frac{d^2}{2}a^2 + \gamma a^\dagger}} | 0 \rangle = \frac{1}{\sqrt{1 - \alpha \beta}} \left[ -\gamma (1 - \alpha \beta)^{-1} \beta + (1 - \alpha \beta)^{-2} \gamma^2 \right] \exp \left( -\frac{1}{2} \gamma^2 (1 - \alpha \beta)^{-1} \alpha \right) \] (89)

\[ \langle 0 | e^{-\frac{d^2}{2}a^\dagger e^{-\frac{d^2}{2}a^2 + \gamma a^\dagger}} | 0 \rangle = \frac{1}{\sqrt{1 - \alpha \beta}} \left[ -\gamma (1 - \alpha \beta)^{-1} \alpha + \gamma^2 (1 - \alpha \beta)^{-2} \beta^2 \right] \exp \left( -\frac{1}{2} \gamma^2 (1 - \alpha \beta)^{-1} \alpha \right) \] (90)

\[ \langle 0 | e^{-\frac{d^2}{2}a^\dagger e^{-\frac{d^2}{2}a^2 + \gamma a^\dagger}} | 0 \rangle = \frac{1}{\sqrt{1 - \alpha \beta}} \left[ -\gamma (1 - \alpha \beta)^{-1} \beta - (1 - \alpha \beta)^{-2} \gamma^3 \right] \exp \left( -\frac{1}{2} \gamma^2 (1 - \alpha \beta)^{-1} \alpha \right) \] (91)

\[ \langle 0 | e^{-\frac{d^2}{2}a^\dagger e^{-\frac{d^2}{2}a^2 + \gamma a^\dagger}} | 0 \rangle = \frac{1}{\sqrt{1 - \alpha \beta}} \left[ -\gamma (1 - \alpha \beta)^{-1} \beta - (1 - \alpha \beta)^{-3} \alpha \gamma^2 (1 - \alpha \beta + 5) \right] + (1 - \alpha \beta)^{-4} \alpha^2 \gamma^4 \right] \exp \left( -\frac{1}{2} \gamma^2 (1 - \alpha \beta)^{-1} \alpha \right) \] (92)

We would like to prove

\[ |G_{11}\rangle \times |G_{11}\rangle = |G_{11}\rangle \] (93)

i.e.

\[ |G_{11}\rangle_{3} = \frac{1}{2} \langle G_{11}| |G_{11}\rangle |V_3\rangle_{123} \] (94)

From the contraction between \(2\langle G_{11}| |V_3\rangle\), we obtain

\[ 2\langle G_{11}| |V_3\rangle = \mathcal{N} \left( -\frac{4\theta}{\theta + 2} \right) (1 - s\mu_1)^{-1} (c_1 + c_2(e_1^{i2} + o_1^{i2}) + d_1 e_1^\dagger + d_2 o_1^\dagger) \]
\[ \cdot \exp \left\{ -\frac{1}{2} s(e_1^{i2} + o_1^{i2}) - \gamma_1 e_1^\dagger - \gamma_2 o_1^\dagger \right\} \exp \left( -\frac{1}{2} s(e_3^{i2} + o_3^{i2}) \right) |0\rangle_{13} \] (95)

where the integration over \(k\) is omitted, and

\[ \gamma_1 = \frac{4 \theta}{(\theta + 2)^2} (e_3^\dagger - i o_3^\dagger) \] (96)

\[ \gamma_2 = \frac{4 \theta}{(\theta + 2)^2} (o_3^\dagger + i e_3^\dagger) \] (97)
\[ c_1 = s - 2t^2(1 - s\mu_1)^{-1}\mu_1 + t^2(1 - s\mu_1)^{-2}(\mu_2^2 - \mu_3^2)(\varepsilon_3^1 + \omega_3^2) \]  
\[ c_2 = t^2(1 - s\mu_1)^{-2}(\mu_2^2 - \mu_3^2) \]  
\[ d_1 = t^2(1 - s\mu_1)^{-2}[2(\mu_2^2 + \mu_3^2)\varepsilon_3^1 + 4i\mu_2\mu_3\omega_3^2] \]  
\[ d_2 = t^2(1 - s\mu_1)^{-2}[2(\mu_2^2 + \mu_3^2)\omega_3^1 - 4i\mu_2\mu_3\varepsilon_3^1] \]

Furthermore, the contraction \( \langle G_{11}\rangle_2 \langle G_{11}\rangle_3 | V_3 \rangle \) can be obtained from the above relations (88) to (92). Note that due to the relations \( \gamma_1^2 + \gamma_2^2 = 0 \), the result could be simplified and the end result depends only on \( c_2 \) and the combination (\( \gamma_1 d_1 + \gamma_2 d_2 \)):

\[ \langle G_{11}\rangle_2 \langle G_{11}\rangle_3 | V_3 \rangle = N \left( \frac{4\theta}{\theta + 2}(1 - s^2)^{-1}(\gamma_1 d_1 + \gamma_2 d_2 - 2c_2) \right) \]
\[ -[s^2(1 - s^2)^{-1} - 2t^2(1 - s^2)^{-2}(s^2 + 1)] \exp\left(-\frac{1}{2}s(\varepsilon_3^2 + \omega_3^2)\right) |0\rangle \]
\[ = N\left( \frac{12 + \theta^2}{8} \right) |G_{11}\rangle_3 = |G_{11}\rangle_3 \]

where (20) is used in the last step.

**References**

[1] G. Moore and W. Taylor, “The Singular Geometry of the Sliver” [hep-th/0111069].

[2] E. Witten, “Noncommutative geometry and string field theory”, *Nucl. Phys. B* 268 (1986) 253.

[3] D. Gross and A. Jevicki, “Operator formulation of interacting string field theory”, *Nucl. Phys. B* 287 (1986) 1. “Operator formulation of interacting string field theory. 2”, *Nucl. Phys. B* 287 (1987) 225.

[4] M. Douglas, H. Liu, G. Moore and B. Zwiebach, “Open String star as a continuous Moyal Product”, [hep-th/0202087].

[5] I. Bars, “Map of Witten’s \( \star \) to Moyal’s \( \star \)”, *Phys.Lett. B* 517 (2001) 436-444, [hep-th/0106157].

[6] D.M. Belov, ”Diagonal representation of open string star and Moyal product”, [hep-th/0204164].

[7] L. Rastelli, A. Sen, B. Zwiebach, “Star Algebra Spectroscopy”, *JHEP* 0203 (2002) 029, [hep-th/0111281].

[8] B. Feng, Y.-H. He and N. Moller, “The Spectrum of the Neumann Matrix with Zero Modes”, [hep-th/0202176].

[9] L. Rastelli, A. Sen, B. Zwiebach, “String field theory around the tachyon vacuum”, [hep-th/0012251]. “Vacuum String Field Theory”, [hep-th/0106010].

[10] A. Kostelecky, R. Potting, “Analytical construction of a nonperturbative vacuum for the open bosonic string”, *Phys. Rev. D* 63 (2001) 046007, [hep-th/0008252].
[11] L. Rastelli, A. Sen and B. Zwiebach, “Classical Solutions in String Field Theory Around the Tachyon Vacuum”, hep-th/0102112.

[12] L. Rastelli, A. Sen, B. Zwiebach, “Half-strings, Projectors, and Multiple D-branes in Vacuum String Field Theory”, JHEP 0111 (2001) 035, hep-th/0105053.

[13] D. Gross, W. Taylor, “Split string field theory I”, JHEP 0108 (2001) 009, hep-th/0105059; “Split string field theory II”, JHEP 0108 (2001) 010, hep-th/0106036.

[14] H. Hata, S. Moriyama, “Observables as Twist Anomaly in Vacuum String Field Theory”, JHEP 0201 (2002) 042, hep-th/0111034; “Exact Results on Twist Anomaly”, JHEP 0202 (2002) 036, hep-th/0201177.

[15] D. Gaiotto, L. Rastelli, A. Sen, B. Zwiebach, “Ghost Structure and Closed Strings in Vacuum String Field Theory”, ep-th/0111129; “Patterns in Open String Field Theory Solutions”, JHEP 0203 (2002) 003, hep-th/0201159. L. Rastelli, A. Sen, B. Zwiebach, “A Note on a Proposal for the Tachyon State in Vacuum String Field Theory”, JHEP 0202 (2002) 034, hep-th/0111153.

[16] L. Rastelli, A. Sen, B. Zwiebach, “Boundary CFT Construction of D-branes in Vacuum String Field Theory”, JHEP 0111 (2001) 045, hep-th/0105168.

[17] K. Furuuchi and K. Okuyama, “Comma Vertex and String Field Algebra”, JHEP 0109 (2001) 035, hep-th/0107101.

[18] R. Gopakumar, S. Minwalla and A. Strominger, “Noncommutative Solitons”, JHEP 0005 (2000) 020, hep-th/0003160.

[19] Y. Imamura, “Gauge Transformations on a D-brane in Vacuum String Field Theory”, hep-th/0204031.

[20] K. Okuyama, “Ghost Kinetic Operator of Vacuum String Field Theory”, JHEP 0201 (2002) 027, hep-th/0201013; “Ratio of Tensions from Vacuum String Field Theory”, JHEP 0203 (2002) 050, hep-th/0201136.

[21] L. Bonora, D. Mamone and M. Salizzoni, “Vacuum String Field Theory with B field”, hep-th/0203188.

[22] Bin Chen and Feng-Li Lin, “Star Spectroscopy in the Constant B field Background”, hep-th/0203204.

[23] J. Harvey, “Komaba Lectures on Noncommutative Solitons and D-Branes”, hep-th/0102076.

[24] J. Harvey, “Topology of the Gauge Group in Noncommutative Gauge Theory”, hep-th/0105242. J. A. Harvey, P. Kraus, F. Larsen, “Exact Noncommutative Solitons”, JHEP 0012 (2000) 024, hep-th/0010060.

[25] T. Kawano, K. Okuyama, “Open String Fields as Matrices”, JHEP 0106 (2001) 061, hep-th/0105129.

[26] C.-S. Chu, P.-H. Ho, F.-L. Lin, “Cubic String Field Theory in pp-wave Background and Background Independent Moyal Structure”, hep-th/0205218.