Singular Limits and String Solutions

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Abstract

It is shown that new leading (α') as well as all-order solutions of String theory can be obtained by taking appropriate singular limits of the known solutions. We give several leading order solutions for the bosonic as well as the heterotic string. We then present all-order forms of the previously known two dimensional cosmological solutions. An all-order form for the cosmological solution in three dimensions is also predicted. The physical implications of our results are discussed.
Classical solutions of string theory and their physical implications have been explored extensively recently. The search for the solutions of string theory has been done mainly along two lines. First, several leading order ($\alpha'$) solutions have been obtained by solving the equations of motion of the string effective action or the vanishing of the one-loop $\beta$-function equations[1, 2]. However this has the drawback of ignoring important stringy effects. The exact string solutions in the curved background, on the other hand, are derived from the Lagrangians of several exact conformal field theories, such as WZW and coset conformal field theories[3, 4].

In this paper we show that appropriate singular limits of the above backgrounds, both exact as well as effective (leading order solution), give rise to new consistent solutions of string theory. The backgrounds generated in this way are not equivalent to the original ones. For example, by writing the two dimensional black hole in an appropriate coordinate system, the two dimensional cosmological solution[5] and the Liouville theory can be obtained by taking two different limits. Inspired by this result, we take singular limits in various other physical circumstances and obtain new solutions. A set of three dimensional cosmological backgrounds are obtained in this way from the 3-D black string[6]. Similarly, a set of new solutions are derived from the two dimensional charged black hole of the heterotic string theory[7]. These solutions are characterized by the vanishing of the corresponding cosmological constant.

One can also extract all-order solutions of string theory in this manner. In this paper we have given the all-order extension of the 2-D cosmological solution[3]. The singular limit of the exact three dimensional black string[8] also generates cosmological backgrounds which we conjecture to be exact to all orders.

We believe that our results can also be used in further classification of the solutions of string theory. Earlier, it has been shown that a number of such solutions
can be classified by the $O(d, d)$ symmetry transformations [9]. However these transformations correspond, in the language of the conformal field theory (CFT), to the deformations by the marginal operators [10] and therefore leave the cosmological constant, a measure of the central charge, unchanged. But as we mentioned above, the solutions generated by singular limits correspond to zero cosmological constant. As a result we may be able to show new connections among the classical vacua of string theory. A possible CFT interpretation of our results probably requires deformations by relevant operators.

Now, in table-1 below we present our results for the leading order solutions. We use the notations and conventions of [11]. In column-1, we give the original solutions. The final solutions are presented in column-2.

We now discuss the main points about the solutions presented in table-1. The first column for the solution (I) represents the dual of the 2-D black hole of ref. [3]. The corresponding cosmological constant is $-4b^2$. For this case, there are in fact more than one ways to reach to the solution of column-2. First, one can directly take the limit $A \to 0$, $b \to 0$ such that $\frac{A}{b^2} = 1$, redefine $r \to t$, $t \to x$ and reach to the final solution with zero cosmological constant. Alternatively, by a coordinate transformation, one can rewrite the 2-D black hole background of column-1 in the form:

$$ds^2 = dt^2 + \frac{1}{(at + bx)^2} dx^2 \quad \text{and} \quad \phi = - \log (at + bx) + \text{const.},$$

which for $b \to 0$ reproduces the solution of column-2. From eqn. (1), we also see that in another limit $a \to 0$ one obtains the Liouville theory together with an extra free string coordinate. The final solution of column-2 was discussed in ref. [5] and describes the string moving in a superinflationary universe for $t < 0$. Later on we will present an extension of this solution to all-orders in $\alpha'$. 

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Table 1: Leading order solutions

| Original solution | Final solution |
|-------------------|----------------|
| $g_{rr} = 1$      | $g_{tt} = 1$   |
| $g_{tt} = A \coth^2 br$ | $g_{xx} = \frac{1}{t^2}$ |
| $\phi = -\frac{1}{4} \log \sinh^2 br + \text{const.}$ | $\phi' = -\log t + \text{const.}$ |

| Original solution | Final solution |
|-------------------|----------------|
| $g_{tt} = A \tanh^2 br$ | $g_{tt} = (1 + \frac{q^2}{t^2})^{-1}$ |
| $g_{xx} = (1 - \frac{Q^2}{Mr})$ | $g_{xx} = (1 + \frac{q^2}{t^2})$ |
| $g_{rr} = (1 - \frac{M}{r})^{-1}(1 - \frac{Q^2}{Mr})^{-1}k \frac{k}{Mr}$ | $g_{yy} = -\frac{1}{t^2}$ |
| $B_{tx} = -\frac{Q}{r}$ | $B_{xy} = -\frac{q}{t^2}$ |
| $\phi = -\frac{1}{4} \log r + \text{const.}$ | $\phi = -\log t + \text{const.}$ |

Solution III.1

| Original solution | Final solution |
|-------------------|----------------|
| $g_{rr} = 1$      | $g_{zz} = 1$ |
| $g_{tt} = (1 - 2\alpha' t^2)$ | $g_{xy} = (1 + \frac{q^2}{t^2})^{-1}$ |
| $g_{zz} = (1 - r^2)(1 - 2\alpha' t^2)$ | $g_{xy} = (1 + \frac{q^2}{t^2})$ |
| $B_{tx} = \sqrt{2\alpha'} r$ | $F_{tx} = -\frac{2\sqrt{2\alpha'}}{t^2}$ |
| $\phi = -\frac{1}{4} \log r + \text{const.}$ | $\phi = -\log t + \text{const.}$ |

Table 2: All-Order Solutions

| Original solution | Final solution |
|-------------------|----------------|
| $g_{rr} = 1$      | $g_{tt} = 1$   |
| $g_{tt} = A \coth^2 br(1 - \frac{2\alpha' b^2}{\sinh^2 br})^{-1}$ | $g_{xx} = \frac{1}{(t^2 - 2\alpha')} \log (t^4 - 2\alpha' t^2) + \text{const.}$ |
| $\phi = -\frac{1}{4} \log (\sinh^4 br - 2\alpha' \sinh^2 br) + \text{const.}$ | $\phi = -\frac{1}{4} \log (t^4 - 2\alpha' t^2) + \text{const.}$ |

| Original solution | Final solution |
|-------------------|----------------|
| $g_{tt} = (1 - \frac{r}{t})$ | $g_{tt} = (1 + \frac{q^2 + 2\alpha'}{t^2})^{-1}$ |
| $g_{xx} = (1 - \frac{r}{t})$ | $g_{xx} = (1 + \frac{q^2}{t^2 + 2\alpha'})$ |
| $g_{rr} = (1 - \frac{r}{t})^{-1}(1 - \frac{r}{t})^{-1}$ | $g_{yy} = -\frac{1}{t^2}$ |
| $B_{tx} = \sqrt{\frac{r - r_q}{r + r_q}} \frac{r - r_q}{r - r_q}$ | $B_{xy} = -\frac{q}{t^2 + 2\alpha'}$ |
| $\phi = -\frac{1}{4} \log (r(r - r_q)) - \text{const.}$ | $\phi = -\frac{1}{4} \log (t^4 + 2\alpha' t^2)$ |
The next solution (II) in table-1 is dual\[9,5\] to (I) and describes the string dynamics in Milne space-time. The metric in (II) can in fact be transformed to a Minkowski metric by a general coordinate transformation\[5\]. Therefore it is already an exact solution of string theory to all-orders.

The background written in the first column of solution (III.1) is the charged black string of ref.\[6\]. Here we treat $k$ as a parameter in the solution of the effective action which gives the cosmological constant $\Lambda = -\frac{8}{k}$. By a coordinate transformation $r = -M \sinh^2 b\rho$, and scaling $t \to \sqrt{A}t$ it can be written as

$$ds^2 = -A \coth^2 bpd^t^2 + (1 + \frac{Q^2}{M^2 \sinh^2 b\rho}) dx^2 + \frac{b^2 k}{2} (1 + \frac{Q^2}{M^2 \sinh^2 b\rho})^{-1} d\rho^2,$$

$$B_{tx} = \frac{Q\sqrt{A}}{M \sinh^2 b\rho}, \quad \text{and} \quad \phi = -\frac{1}{2} \log \sinh^2 b\rho + \text{const.}.$$  (2)

Then by renaming the coordinates as, $t \to y, \rho \to t$, we get in the limits, $b \to 0, A \to 0, k \to \infty, \frac{Q}{M} \to 0$ with $\frac{Q}{M} = q, \frac{A}{\rho} = 1$, and $b^2 k = 1$, the background (III.1). It gives the cosmological evolution in the presence of a nontrivial antisymmetric tensor. We have verified that they satisfy the one-loop beta function equations and are basically of the type given in ref.\[5\]. We will later on present an all-order extension of this background also.

The above three examples conclusively show that the limiting procedure outlined in the introduction can generate new solutions belonging to a different class than the original ones. We now present its usefulness by generating several new solutions. The solution (III.2) has been generated from (III.1) by a coordinate transformation $t \to q \sinh b\sqrt{t}$, and by appropriate limits and redefinitions. Once again it describes the cosmological evolution in the presence of an antisymmetric tensor field. But unlike (III.1), the charge of the antisymmetric tensor is fixed to a definite value. The singular limits taken in this case do not leave any free parameter.
The solution (IV) has been generated from the charged black hole of the heterotic string\cite{7} in the same manner as outlined in detail for the 3-D black string. This now describes the cosmology in two dimensions in the presence of a nontrivial gauge field. We have again verified that these backgrounds satisfy the one-loop beta function equations for the 2-D heterotic strings. In the limit $q \to 0$ this solution reduces to (I). By a coordinate transformation, $T^2 = t^2 + q^2$, the curvature scalar is given by $R = -4[(T^2 + 3q^2)/(T^2 - q^2)^2]$. The singularities are therefore at the points $T = \pm q$. The metric describes an expanding space-time in the regions $-\infty < T < -q$ and $0 < T < q$ and a contracting one in the regions $\infty > T > q$ and $0 > T > -q$.

We now come to the discussions of the exact solutions presented in table-2. The first one (V) is the all-order extension of the cosmological solution (I). It is obtained from the exact black hole solution presented in \cite{12,13} by taking the same limit as for the solution (I). We have by now already shown, through a number of examples, that singular limits give consistent string solutions. Therefore we expect that the field equations satisfied by the exact black hole will also be satisfied by the new solution (V). In another context in string theory, it has been pointed out that singular limits, known as the Inönu-Wigner contractions\cite{14}, generate consistent conformal algebra starting form a known one.

We have explicitly verified that the backgrounds (V) satisfy the beta function equations upto two-loops\cite{13,14,15}:

\begin{align}
R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi + \frac{\alpha'}{2} R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma}_\nu &= 0 \\
(\nabla \phi)^2 - \frac{1}{2} \nabla^2 \phi + \frac{\alpha'}{16} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} &= 0
\end{align}

At higher loop orders, the beta functions are not uniquely defined\cite{13,15}. They depend on the renormalization scheme chosen to regularize the theory. But we do
expect that there exists a scheme in which the backgrounds (V) satisfy the beta function equations to all-orders. As a further check, we observe that the backgrounds (V) are related to the leading order solution (I) by a field redefinition:

\[ g'_{\mu\nu} = g_{\mu\nu} - 2\alpha' \partial_{\mu}\phi\partial_{\nu}\phi \frac{1}{1 + \frac{\alpha'}{2} R} + 2\alpha' g_{\mu\nu}(\partial\phi)^2 \frac{1}{1 + \frac{\alpha'}{2} R} \]

\[ \phi' = \phi - \frac{1}{4} \log[1 + \frac{\alpha'}{2} R] \]  

(5)

These are the same redefinitions given in ref.[16] which connect the exact and the leading order black hole solutions.

The curvature scalar for the solution (V) is given by \( R = -4[(t^2 + \alpha')/(t^2 - 2\alpha')^2] \) and the Hubble parameter[17] is given by \( H = -[t/(t^2 - 2\alpha')] \). The curvature singularities are at the points \( t = \pm \sqrt{2\alpha'} \). One can verify that the geometric structure of the space-time is similar to the one for solution (IV).

As we have mentioned above, the solution (II) being flat is already exact. This can also be seen by taking the appropriate singular limits in the exact black hole solution. Also, since in (II) \( R = 0 \) and \( \phi = \text{const.} \), the redefinitions (5) now give back the original solution.

The solution (VI) in table-2 has been generated from the exact three dimensional charged black string presented in eqns. (7.13) - (7.16) of ref.[8]. It matches with (III.1) for \( \alpha' = 0 \). Alternatively, for \( q = 0 \), but \( \alpha' \neq 0 \), we get the all-order solution (V), together with a free string coordinate, by the coordinate transformation, \( t^2 \to t^2 + 2\alpha' \).

We conjecture that (VI) is in fact an all-order solution. Unfortunately, in the presence of a nontrivial \( B \) field, the beta function becomes dependent on the renormalization scheme even at the two loop level [18] and the explicit verification of the field equations becomes complicated. Alternatively, to show that it is an all-order solution a field redefinition of the type given in eqn.(5) has to be found between the solutions (III.1) and (VI). We hope to report on this issue in future.
There are other important aspects of the exact solutions (V) and (VI) which should be examined in detail. For example, it will be interesting to get an exact conformal field theory for these backgrounds. Since these solutions correspond to zero cosmological constant, we can think of two ways to describe them as exact conformal field theories. One can hope to obtain a solution with two dimensional space-time supersymmetry which will ensure the vanishing of the cosmological constant. However this is expected to keep the leading order solution unchanged even at higher orders. A more likely scenario for obtaining the CFT for our solution may be the construction of an exact string action with underlying $N = 2$ local worldsheet supersymmetry. For such a theory, in target space dimension $D = 2$, the tree level cosmological constant vanishes. Along with these aspects, we are also planning to present a large class of solutions based on the techniques of this paper in a forthcoming publication.

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