A Note on Maass-Jacobi Forms

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Abstract. In this paper, we introduce the notion of Maass-Jacobi forms and investigate some properties of these new automorphic forms. We also characterize these automorphic forms in several ways.

1. Introduction

We let $SL_{2,1} (\mathbb{R}) = SL(2, \mathbb{R}) \rtimes \mathbb{R}^{(1,2)}$ be the semi-direct product of the special linear group $SL(2, \mathbb{R})$ of degree 2 and the commutative group $\mathbb{R}^{(1,2)}$ equipped with the following multiplication law

\[(g, \alpha) \ast (h, \beta) = (gh, \alpha^t h^{-1} + \beta), \quad g, h \in SL(2, \mathbb{R}), \quad \alpha, \beta \in \mathbb{R}^{(1,2)},\]

where $\mathbb{R}^{(1,2)}$ denotes the set of all $1 \times 2$ real matrices. We let

$SL_{2,1} (\mathbb{Z}) = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^{(1,2)}$

be the discrete subgroup of $SL_{2,1} (\mathbb{R})$ and $K = SO(2)$ the special orthogonal group of degree 2.

Throughout this paper, for brevity we put

$G = SL_{2,1} (\mathbb{R})$, \quad $\Gamma_1 = SL(2, \mathbb{Z})$ \quad and \quad $\Gamma = SL_{2,1} (\mathbb{Z})$.

Let $\mathbb{H}$ be the Poincaré upper half plane. Then $G$ acts on $\mathbb{H} \times \mathbb{C}$ transitively by

\[(g, \alpha) \circ (\tau, z) = ((d \tau - c)(-b \tau + a)^{-1}, (z + \alpha_1 \tau + \alpha_2)(-b \tau + a)^{-1}),\]

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(1,2)}$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. We observe that $K$ is the stabilizer of this action (1.2) at the origin $(i, 0)$. $\mathbb{H} \times \mathbb{C}$ may be identified with the homogeneous space $G/K$ in a natural way.

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The aim of this paper is to define the notion of Maass-Jacobi forms generalizing that of Maass wave forms and study some properties of these new automorphic forms. For the convenience of the reader, we review Maass wave forms. For \( s \in \mathbb{C} \), we denote by \( W_s(\Gamma_1) \) the vector space of all smooth bounded functions \( f : SL(2, \mathbb{R}) \rightarrow \mathbb{C} \) satisfying the following conditions (a) and (b):

(a) \( f(\gamma g k) = f(g) \) for all \( \gamma \in \Gamma_1, \ g \in SL(2, \mathbb{R}) \) and \( k \in K \).

(b) \( \Delta_0 f = \frac{1 - s^2}{4} f \),

where \( \Delta_0 = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta} + \frac{5}{4} \frac{\partial^2}{\partial \theta^2} \) is the Laplace-Beltrami operator associated to the \( SL(2, \mathbb{R}) \)-invariant Riemannian metric

\[
d s_0^2 = \frac{1}{y^2} (dx^2 + dy^2) + \left( d\theta + \frac{dx}{2y} \right)^2
\]

on \( SL(2, \mathbb{R}) \) whose coordinates \( x, y, \theta \) (\( x \in \mathbb{R}, \ y > 0, \ 0 \leq \theta < 2\pi \)) are given by

\[
g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \ g \in SL(2, \mathbb{R})
\]

by means of the Iwasawa decomposition of \( SL(2, \mathbb{R}) \). The elements in \( W_s(\Gamma_1) \) are called Maass wave forms. It is well known that \( W_s(\Gamma_1) \) is nontrivial for infinitely many values of \( s \). For more detail, we refer to [6], [9], [13], [17] and [20].

The paper is organized as follows. In Section 2, we calculate the algebra of all invariant differential operators under the action (1.2) of \( G \) on \( \mathbb{H} \times \mathbb{C} \) completely. In addition, we provide a \( G \)-invariant Riemannian metric on \( \mathbb{H} \times \mathbb{C} \) and compute its Laplace-Beltrami operator. In Section 3, using the above Laplace-Beltrami operator, we introduce a concept of Maass-Jacobi forms generalizing that of Maass wave forms. We characterize Maass-Jacobi forms as smooth functions on \( G \) or \( SP_2 \times \mathbb{R}^{(1,2)} \) satisfying a certain invariance property, where \( SP_2 \) denotes the symmetric space consisting of all \( 2 \times 2 \) positive symmetric real matrices \( Y \) with \( \det Y = 1 \). In Section 4, we find the unitary dual of \( G \) and present some properties of \( G \). In Section 5, we describe the decomposition of the Hilbert space \( L^2(\Gamma \backslash G) \). In the final section, we make some comments on the Fourier expansion of Maass-Jacobi forms.

Notations. We denote by \( \mathbb{Z}, \mathbb{R} \) and \( \mathbb{C} \) the ring of integers, the field of real numbers and the field of complex numbers respectively. \( \mathbb{Z}^+ \) denotes the set of all positive integers. \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \). For a square matrix \( A \), \( \sigma(A) \) denotes the trace of \( A \). For any \( M \in F^{(k,l)} \), \( tM \) denotes the transpose of \( M \). For \( A \in F^{(k,l)} \) and \( B \in F^{(k,k)} \), we set \( B[A] = tABA \).

We denote the identity matrix of degree \( n \) by \( E_n \). \( \mathbb{H} \) denotes the Poincaré upper-half plane.

2. Invariant Differential Operators on \( \mathbb{H} \times \mathbb{C} \)
We recall that $SP_2$ is the symmetric space consisting of all $2 \times 2$ positive symmetric real matrices $Y$ with $\det Y = 1$. Then $G$ acts on $SP_2 \times \mathbb{R}^{(1,2)}$ transitively by

$$(g, \alpha) \cdot (Y, V) = (gY^t g, (V + \alpha)^t g),$$

where $g \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^{(1,2)}$, $Y \in SP_2$ and $V \in \mathbb{R}^{(1,2)}$. It is easy to see that $K$ is a maximal compact subgroup of $G$ stabilizing the origin $(E_2, 0)$. Thus $SP_n \times \mathbb{R}^{(m,n)}$ may be identified with the homogeneous space $G/K$ as follows:

$$G/K \ni (g, \alpha)K \mapsto (g, \alpha) \cdot (E_2, 0) \in SP_2 \times \mathbb{R}^{(1,2)},$$

where $g \in SL(2, \mathbb{R})$ and $\alpha \in \mathbb{R}^{(1,2)}$.

We know that $SL(2, \mathbb{R})$ acts on $H$ transitively by

$$(g, \alpha) \cdot (\tau, z) = (a\tau + b)(c\tau + d)^{-1}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R}), \quad \tau \in H.$$
\[ \alpha_{Y,V} = V^t g_Y^{-1}. \]

Then we have
\[ T(Y, V) = (g_Y, \alpha_{Y,V}) \circ (i, 0). \]

**Proof.** It is easy to prove the lemma. So we leave the proof to the reader. \( \square \)

Now we give a complete description of the algebra \( D(H \times \mathbb{C}) \) of all differential operators on \( H \times \mathbb{C} \) invariant under the action (1.2) of \( G \). First we note that the Lie algebra \( g \) of \( G \) is given by \( g = \{ (X, Z) \mid X \in \mathbb{R}^{(2,2)}, \sigma(X) = 0, Z \in \mathbb{R}^{(1,2)} \} \)

equipped with the following Lie bracket
\[
\left[X_1, X_2\right] = \left[\begin{array}{cc} X_1^t & 0 \\ 0 & X_2 \end{array}\right], \quad \left[Z_1, Z_2\right] = \left[\begin{array}{cc} 0 & Z_1^t \\ -Z_2 & 0 \end{array}\right],
\]

where \( [X_1, X_2]_0 = X_1X_2 - X_2X_1 \) denotes the usual matrix bracket and \( (X_1, Z_1), (X_2, Z_2) \in g \). And \( g \) has the following decomposition
\[ g = \mathfrak{k} \oplus \mathfrak{p} \] (direct sum),

where \( \mathfrak{k} = \left\{ (X, 0) \in g \mid X = \left[\begin{array}{cc} 0 & x \\ -x & 0 \end{array}\right], \ x \in \mathbb{R} \right\} \) and \( \mathfrak{p} = \left\{ (X, Z) \in g \mid X = \left[\begin{array}{cc} 0 & x \\ -x & 0 \end{array}\right], \sigma(X) = 0, Z \in \mathbb{R}^{(1,2)} \right\} \).

We observe that \( \mathfrak{k} \) is the Lie algebra of \( K \) and that we have the following relations
\[ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \]

Thus the coset space \( G/K \cong H \times \mathbb{C} \) is a reductive homogeneous space in the sense of [12], p. 284. It is easy to see that the adjoint action \( \text{Ad} \) of \( K \) on \( \mathfrak{p} \) is given by
\[ \text{Ad}(k)((X, Z)) = (kX^t k, Z^t k), \]

where \( k \in K \) and \( (X, Z) \in \mathfrak{p} \) with \( X = X^t, \sigma(X) = 0 \). The action (2.9) extends uniquely to the action \( \rho \) of \( K \) on the polynomial algebra \( \text{Pol}(\mathfrak{p}) \) of \( \mathfrak{p} \) given by
\[ \rho : K \rightarrow \text{Aut}(\text{Pol}(\mathfrak{p})). \]

Let \( \text{Pol}(\mathfrak{p})^K \) be the subalgebra of \( \text{Pol}(\mathfrak{p}) \) consisting of all invariants of the action \( \rho \) of \( K \). Then according to [12], Theorem 4.9, p. 287, there exists a canonical linear bijection \( \lambda : (P \rightarrow D_{\lambda(P)}) \) of \( \text{Pol}(\mathfrak{p})^K \) onto \( \mathbb{D}(H \times \mathbb{C}) \). Indeed, if \( \{\xi_k\} (1 \leq k \leq 4) \) is any basis of \( \mathfrak{p} \) and \( P \in \text{Pol}(\mathfrak{p})^K \), then
\[ (D_{\lambda(P)}f)(\hat{g} \circ (i, 0)) = \left[ P \left( \frac{\partial}{\partial t_k} \right) f((\hat{g} \ast \exp(\sum_{k=1}^{4} t_k \xi_k)) \circ (i, 0)) \right]_{(t_k) = 0}, \]

where \( \hat{g} = \exp(\sum_{k=1}^{4} t_k \xi_k) \).
where \( \tilde{g} \in G \) and \( f \in C^\infty(\mathbb{R} \times \mathbb{C}) \).

We put
\[
e_1 = \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0) \right), \quad e_2 = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (0,0) \right)
\]
and
\[
f_1 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (1,0) \right), \quad f_2 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (0,1) \right).
\]
Then \( e_1, e_2, f_1, f_2 \) form a basis of \( \mathfrak{p} \). We write for coordinates \((X, Z)\) by
\[
X = \begin{pmatrix} x \\ y \\ -x \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2)
\]
with real variables \( x, y, z_1 \) and \( z_2 \).

**Lemma 2.2.** The following polynomials
\[
P(X, Z) = \frac{1}{8} \sigma(X^2) = \frac{1}{4} (x^2 + y^2),
\]
\[
\xi(X, Z) = Z^t \tilde{Z} = z_1^2 + z_2^2,
\]
\[
P_1(X, Z) = -\frac{1}{2} ZX^t \tilde{Z} = \frac{1}{2} (z_2^2 - z_1^2) x - z_1 z_2 y \quad \text{and}
\]
\[
P_2(X, Z) = \frac{1}{2} (z_2^2 - z_1^2) y + z_1 z_2 x
\]
are algebraically independent generators of \( \text{Pol} (\mathfrak{p})^K \).

**Proof.** We leave the proof of the above lemma to the reader. \( \square \)

Now we are ready to compute the \( G \)-invariant differential operators \( D, \Psi, D_1 \) and \( D_2 \) corresponding to the \( K \)-invariants \( P, \xi, P_1 \) and \( P_2 \) respectively under the canonical linear bijection (2.11). For real variables \( t = (t_1, t_2) \) and \( s = (s_1, s_2) \), we have
\[
\exp (t_1 e_1 + t_2 e_2 + s_1 f_1 + s_2 f_2) = \left( \begin{pmatrix} a_1(t, s) & a_3(t, s) \\ a_3(t, s) & a_2(t, s) \end{pmatrix}, (b_1(t, s), b_2(t, s)) \right),
\]
where
\[
a_1(t, s) = 1 + t_1 + \frac{1}{2!} (t_1^2 + t_2^2) + \frac{1}{3!} t_1(t_1^2 + t_2^2) + \frac{1}{4!} (t_1^3 + t_2^3)^2 + \cdots
\]
\[
a_2(t, s) = 1 - t_1 + \frac{1}{2!} (t_1^2 + t_2^2) - \frac{1}{3!} t_1(t_1^2 + t_2^2) + \frac{1}{4!} (t_1^3 + t_2^3)^2 - \cdots,
\]
\[
a_3(t, s) = t_2 + \frac{1}{3!} t_2(t_1^2 + t_2^2) + \frac{1}{5!} t_2(t_1^2 + t_2^2)^2 + \cdots,
\]
\[
b_1(t, s) = s_1 - \frac{1}{2!} (s_1 t_1 + s_2 t_2) + \frac{1}{3!} s_1(t_1^2 + t_2^2) - \frac{1}{4!} (s_1 t_1 + s_2 t_2)(t_1^2 + t_2^2) + \cdots,
\]
\[
b_2(t, s) = s_2 - \frac{1}{2!} (s_1 t_2 - s_2 t_1) + \frac{1}{3!} s_2(t_1^2 + t_2^2) - \frac{1}{4!} (s_1 t_2 - s_2 t_1)(t_1^2 + t_2^2) + \cdots.
\]
For brevity, we write $a_j, b_k$ for $a_j(t, s), b_k(t, s)$ ($j = 1, 2, 3, k = 1, 2$) respectively. We now fix an element $(g, \alpha) \in G$ and write

$$
g = \begin{pmatrix} g_1 & g_{12} \\ g_{21} & g_2 \end{pmatrix} \in SL(2, \mathbb{R}) \quad \text{and} \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(1, 2)}.
$$

We put $(\tau(t, s), z(t, s)) = ((g, \alpha) \ast \exp (t_1 e_1 + t_2 e_2 + s_1 f_1 + s_2 f_2)) \circ (i, 0)$ with $\tau(t, s) = x(t, s) + i y(t, s)$ and $z(t, s) = u(t, s) + i v(t, s)$. Here $x(t, s), y(t, s), u(t, s)$ and $v(t, s)$ are real. By an easy calculation, we obtain

$$
x(t, s) = -(\tilde{\alpha} + \tilde{\beta}d)(\tilde{a}^2 + \tilde{b}^2)^{-1},
y(t, s) = (\tilde{a}^2 + \tilde{b}^2)^{-1},
u(t, s) = (\tilde{\alpha} \tilde{\alpha}_2 - \tilde{\beta} \tilde{\alpha}_1)(\tilde{a}^2 + \tilde{b}^2)^{-1},
v(t, s) = (\tilde{\alpha} \tilde{\alpha}_1 + \tilde{\beta} \tilde{\alpha}_2)(\tilde{a}^2 + \tilde{b}^2)^{-1},
$$

where $\tilde{a} = g_1 a_1 + g_{12} a_3, \tilde{b} = g_1 a_3 + g_{12} a_2, \tilde{c} = g_{21} a_1 + g_2 a_3, \tilde{\alpha} = g_1 a_1 + g_{12} a_1 + b_1, \tilde{\alpha}_2 = -a_1 a_3 + a_2 a_1 + b_2$.

By an easy calculation, at $t = s = 0$, we have

$$
\frac{\partial x}{\partial t_1} = 4 g_1 g_{12} (g_1^2 + g_{12}^2)^{-2},
\frac{\partial y}{\partial t_1} = -2 (g_1^2 - g_{12}^2) (g_1^2 + g_{12}^2)^{-2},
\frac{\partial u}{\partial t_1} = 4 g_1 g_{12} (g_1 a_1 + g_{12} a_2) (g_1^2 + g_{12}^2)^{-2},
\frac{\partial v}{\partial t_1} = -2 (g_1 a_1 + g_{12} a_2) (g_1^2 - g_{12}^2) (g_1^2 + g_{12}^2)^2,
\frac{\partial^2 x}{\partial t_1^2} = -16 g_1 g_{12} (g_1^2 - g_{12}^2) (g_1^2 + g_{12}^2)^{-3},
\frac{\partial^2 y}{\partial t_1^2} = 8 (g_1^2 - g_{12}^2)^2 (g_1^2 + g_{12}^2)^{-3} - 4 (g_1^2 + g_{12}^2)^{-1},
\frac{\partial^2 u}{\partial t_1^2} = -16 g_1 g_{12} (g_1 a_1 + g_{12} a_2) (g_1^2 - g_{12}^2) (g_1^2 + g_{12}^2)^{-3},
\frac{\partial^2 v}{\partial t_1^2} = 4 (g_1 a_1 + g_{12} a_2) (g_1^4 + g_{12}^4 - 6 g_1^2 g_{12}^2) (g_1^2 + g_{12}^2)^{-3}
$$

and

$$
\frac{\partial x}{\partial t_2} = -2 (g_1^2 - g_{12}^2) (g_1^2 + g_{12}^2)^{-2},
\frac{\partial y}{\partial t_2} = -4 g_1 g_{12} (g_1^2 + g_{12}^2)^{-2},
\frac{\partial u}{\partial t_2} = -2 (g_1 a_1 + g_{12} a_2) (g_1^2 - g_{12}^2) (g_1^2 + g_{12}^2)^{-2},
$$
\[
\frac{\partial v}{\partial t_2} = -4g_1g_{12}(g_1\alpha_1 + g_{12}\alpha_2)(g_1^2 + g_{12}^2)^{-2},
\]
\[
\frac{\partial^2 x}{\partial t_2^2} = 16g_1g_{12}(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-3},
\]
\[
\frac{\partial^2 y}{\partial t_2^2} = 32g_1^2g_{12}^2(g_1^2 + g_{12}^2)^{-3} - 4(g_1^2 + g_{12}^2)^{-1},
\]
\[
\frac{\partial^2 u}{\partial t_2^2} = 16g_1g_{12}(g_1\alpha_1 + g_{12}\alpha_2)(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-3},
\]
\[
\frac{\partial^2 v}{\partial t_2^2} = -4(g_1\alpha_1 + g_{12}\alpha_2)(g_4^1 + g_{12}^2 - 6g_1g_{12}^2)(g_1^2 + g_{12}^2)^{-3}.
\]

We note that \(\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1\), \(a_1a_2 - a_3^2 = 1\) and \(g_1g_2 - g_{12}g_{21} = 1\).

Using the above facts and applying the chain rule, we can easily compute the differential operators \(D, \Psi, D_1\) and \(D_2\). It is known that the images of generators \(P, \xi, P_1\) and \(P_2\) under \(\lambda\) are generators of \(\mathcal{D}(\mathbb{H} \times \mathbb{C})\) (cf. [11]).

Summarizing, we have the following.

**Theorem 2.3.** The algebra \(\mathcal{D}(\mathbb{H} \times \mathbb{C})\) is generated by the following differential operators

\begin{align*}
(2.12) & \quad D = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2yv \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right), \\
(2.13) & \quad \Psi = y \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right), \\
(2.14) & \quad D_1 = 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} - y^2 \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \left( v \frac{\partial}{\partial v} + 1 \right) \Psi \\
\text{and} & \\
(2.15) & \quad D_2 = y^2 \frac{\partial}{\partial x} \left( \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} \Psi,
\end{align*}

where \(\tau = x + iy\) and \(z = u + iv\) with real variables \(x, y, u, v\). Moreover, we have

\[
[D, \Psi] = D\Psi - \Psi D = 2y^2 \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left( v \frac{\partial}{\partial v} \Psi + \Psi \right).
\]
In particular, the algebra $\mathbb{D}(\mathbb{H} \times \mathbb{C})$ is not commutative. Thus the homogeneous space $\mathbb{H} \times \mathbb{C}$ is not weakly symmetric in the sense of A. Selberg ([19]).

Now we provide a natural $G$-invariant Kähler metric on $\mathbb{H} \times \mathbb{C}$.

**Proposition 2.4.** The Riemannian metric $ds^2$ on $\mathbb{H} \times \mathbb{C}$ defined by

$$ds^2 = \frac{y + v^2}{y^3} (dx^2 + dy^2) + \frac{1}{y} (du^2 + dv^2) - \frac{2v}{y^2} (dx du + dy dv)$$

is invariant under the action (1.2) of $G$ and is a Kähler metric on $\mathbb{H} \times \mathbb{C}$. The Laplace-Beltrami operator $\Delta$ of the Riemannian space $(\mathbb{H} \times \mathbb{C}, ds^2)$ is given by

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (y + v^2) \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2yv \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right).$$

That is, $\Delta = D + \Psi$.

**Proof.** For $Y \in SP_2$ of the form (2.4) and $(v_1, v_2) \in \mathbb{R}^{(1,2)}$, it is easy to see that $dY = \left( -y^{-1} dy \begin{array}{c} dy \\ dx \end{array} + x y^{-2} dy \end{array} \right)$ and $dV = (dv_1, dv_2)$. Then we can show that the following metric $d\tilde{s}^2$ on $SP_2 \times \mathbb{R}^{(1,2)}$ defined by

$$d\tilde{s}^2 = \frac{dx^2 + dy^2}{y^2} + \frac{1}{y} \left\{ (x^2 + y^2) dv_1^2 + 2x dv_1 dv_2 + dv_2^2 \right\}$$

is invariant under the action (2.1) of $G$. Indeed, since

$$Y^{-1} = \begin{pmatrix} y + x^2 y^{-1} & xy^{-1} \\ x y^{-1} & y^{-1} \end{pmatrix},$$

we can easily show that $d\tilde{s}^2 = \frac{1}{2} \sigma(Y^{-1} dY Y^{-1} dY) + dV Y^{-1} t(dV)$. For an element $(g, \alpha) \in G$ with $g \in SL(2, \mathbb{R})$ and $\alpha \in \mathbb{R}^{(1,2)}$, we put

$$(Y^*, V^*) = (g, \alpha) \cdot (Y, V) = (g Y^t g, (V + \alpha)^t g).$$

Since $Y^* = g Y^t g$ and $V^* = (V + \alpha)^t g$, we get $dY^* = g dY^t g$ and $V^* = (V + \alpha)^t g$. Therefore by a simple calculation, we can show that

$$\sigma \left( Y^{*-1} dY^* Y^{*-1} dY^* \right) + dV^* Y^{*-1} t(dV^*) = \sigma(Y^{-1} dY Y^{-1} dY) + dV Y^{-1} t(dV).$$

Hence the metric $d\tilde{s}^2$ is invariant under the action (2.1) of $G$. 
Using this fact and Lemma 2.1, we can prove that the metric $ds^2$ in the above theorem is invariant under the action (1.2). Since the matrix form $(g_{ij})$ of the metric $ds^2$ is given by

$$(g_{ij}) = \begin{pmatrix} (y + v^2) y^{-3} & 0 & -v y^{-2} & 0 \\ 0 & (y + v^2) y^{-3} & 0 & -v y^{-2} \\ -v y^{-2} & 0 & y^{-1} & 0 \\ 0 & -v y^{-2} & 0 & y^{-1} \end{pmatrix}$$

and $\det (g_{ij}) = y^{-6}$, the inverse matrix $(g^{ij})$ of $(g_{ij})$ is easily obtained by

$$(g^{ij}) = \begin{pmatrix} y^2 & 0 & y v & 0 \\ 0 & y^2 & 0 & y v \\ y v & 0 & y + v^2 & 0 \\ 0 & y v & 0 & y + v^2 \end{pmatrix}.$$ 

Now it is easily shown that $D + \Psi$ is the Laplace-Beltrami operator of $(\mathbb{H} \times \mathbb{C}, ds^2)$. □

**Remark 2.5.** We can show that for any two positive real numbers $\alpha$ and $\beta$, the following metric

$$ds^2_{\alpha,\beta} = \alpha \frac{dx^2 + dy^2}{y^2} + \beta \frac{\nu^2(dx^2 + dy^2) + y^2(du^2 + dv^2) - 2 y \nu (dx \, du + dy \, dv)}{y^3}$$

is also a Riemannian metric on $\mathbb{H} \times \mathbb{C}$ which is invariant under the action (1.2) of $G$. In fact, we can see that the two-parameter family of $ds^2_{\alpha,\beta}$ ($\alpha > 0$, $\beta > 0$) provides a complete family of Riemannian metrics on $\mathbb{H} \times \mathbb{C}$ invariant under the action of (1.2) of $G$. It can be easily seen that the Laplace-Beltrami operator $\Delta_{\alpha,\beta}$ of $ds^2_{\alpha,\beta}$ is given by

$$\Delta_{\alpha,\beta} = \frac{1}{\alpha} y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \left( \frac{y}{\beta} + \frac{\nu^2}{\alpha} \right) \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \frac{2 y \nu}{\alpha} \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right)$$

$$= \frac{1}{\alpha} D + \frac{1}{\beta} \Psi.$$ 

**Remark 2.6.** By a tedious computation, we see that the scalar curvature of $(\mathbb{H} \times \mathbb{C}, ds^2)$ is $-3$.

We want to propose the following problem to be studied in the future.

**Problem 2.7.** Find all the eigenfunctions of $\Delta$.

We will give some examples of eigenfunctions of $\Delta$.
(1) \( h(x, y) = y^2 e^{2\pi ia x} (s \in \mathbb{C}, \ a \neq 0) \) with eigenvalue \( s(s - 1) \),
where
\[
K_s(z) := \frac{1}{2} \int_0^\infty \exp \left\{ -\frac{z}{2} (t + t^{-1}) \right\} t^{s-1} dt, \quad Re\ z > 0.
\]

(2) \( y^s, y^s x, y^s u \ (s \in \mathbb{C}) \) with eigenvalue \( s(s - 1) \).

(3) \( y^sv, y^s uv, y^s xv \) with eigenvalue \( s(s + 1) \).

(4) \( x, y, u, v, xv, uv \) with eigenvalue 0.

(5) All Maass wave forms.

3. Maass-Jacobi forms

Let \( \Delta \) be the Laplace-Beltrami operator of the Riemannian metric \( ds^2 \) on \( \mathbb{H} \times \mathbb{C} \) defined in Proposition 2.4. Using this operator, we define the notion of Maass-Jacobi forms.

**Definition 3.1.** A smooth bounded function \( f : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C} \) is called a Maass-Jacobi form if it satisfies the following conditions (MJ1)-(MJ3):

(MJ1) \( f(\bar{\gamma} \circ (\tau, z)) = f(\tau, z) \) for all \( \bar{\gamma} \in \Gamma \) and \( (\tau, z) \in \mathbb{H} \times \mathbb{C} \).

(MJ2) \( f \) is an eigenfunction of the Laplace-Beltrami operator \( \Delta \).

(MJ3) \( f \) has a polynomial growth, that is, \( f \) fulfills a boundedness condition.

For a complex number \( \lambda \in \mathbb{C} \), we denote by \( MJ(\Gamma, \lambda) \) the vector space of all Maass-Jacobi forms \( f \) such that \( \Delta f = \lambda f \). We note that, since \( \Delta f = \lambda f \) is an elliptic partial differential equation, Maass-Jacobi forms are real analytic (see [8]).

Professor Berndt kindly informed me that he also considered such automorphic forms in ([1]) (also see [4], p.82).

Let \( f \in MJ(\Gamma, \lambda) \) be a Maass-Jacobi form with eigenvalue \( \lambda \). Then it is easy to see that the function \( \phi_f : G \rightarrow \mathbb{C} \) defined by
\[
\phi_f(g, \alpha) = f((g, \alpha) \circ (i, 0)), \quad (g, \alpha) \in G
\]
satisfies the following conditions (MJ1)-(MJ3):

(MJ1) \( \phi_f(\gamma xk) = \phi_f(x) \) for all \( \gamma \in \Gamma, \ x \in G \) and \( k \in K \).

(MJ2) \( \phi_f \) is an eigenfunction of the Laplace-Beltrami operator \( \Delta_0 \) of \( (G, ds_0^2) \), where \( ds_0^2 \) is a \( G \)-invariant Riemannian metric on \( G \) induced by \( (\mathbb{H} \times \mathbb{C}, ds^2) \).

(MJ3) \( \phi_f \) has a suitable polynomial growth (cf. [5]).
For any right $K$-invariant function $\phi : G \to \mathbb{C}$ on $G$, we define the function $f_\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ by

\[
(3.2) \quad f_\phi(\tau, z) = \phi(g, \alpha), \quad (\tau, z) \in \mathbb{H} \times \mathbb{C},
\]

where $(g, \alpha)$ is an element of $G$ such that $(g, \alpha) \circ (i, 0) = (\tau, z)$. Obviously it is well defined because (3.2) is independent of the choice of $(g, \alpha) \in G$ such that $(g, \alpha) \circ (i, 0) = (\tau, z)$. It is easy to see that if $\phi$ is a smooth bounded function on $G$ satisfying the conditions (MJ1)\(^0\)-(MJ3)\(^0\), then the function $f_\phi$ defined by (3.2) is a Maass-Jacobi form.

Now we characterize Maass-Jacobi forms as smooth eigenfunctions on $SP_n \times \mathbb{R}^{(m,n)}$ satisfying a certain invariance property.

**Proposition 3.2.** Let $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ be a nonzero Maass-Jacobi form in $MJ(\Gamma, \lambda)$. Then the function $h_f : SP_2 \times \mathbb{R}^{(1,2)} \to \mathbb{C}$ defined by

\[
(3.3) \quad h_f(Y, V) = f((g, V^tg^{-1}) \circ (i, 0)) \quad \text{for some } g \in SL(2, \mathbb{R}) \text{ with } Y = g^t g
\]

satisfies the following conditions:

(MJ1)* $h_f(\gamma Y^t \gamma, (V + \delta)^t \gamma) = h_f(Y, V)$ for all $(\gamma, \delta) \in \Gamma$ with $\gamma \in SL(2, \mathbb{Z})$ and $\delta \in \mathbb{Z}^{(1,2)}$. 

(MJ2)* $h_f$ is an eigenfunction of the Laplace-Beltrami operator $\tilde{\Delta}$ on the homogeneous space $(SP_2 \times \mathbb{R}^{(1,2)}, d\tilde{s}^2)$, where $d\tilde{s}^2$ is the $G$-invariant Riemannian metric on $SP_2 \times \mathbb{R}^{(1,2)}$ induced from $d\tilde{s}^2$.

(MJ3)* $h_f$ has a suitable polynomial growth.

Here if $(Y, V)$ is a coordinate of $SP_2 \times \mathbb{R}^{(1,2)}$ given in Lemma 2.1, then the $G$-invariant Riemannian metric $d\tilde{s}^2$ and its Laplace-Beltrami operator $\tilde{\Delta}$ on $SP_2 \times \mathbb{R}^{(1,2)}$ are given by

\[
d\tilde{s}^2 = \frac{1}{y^2} (dx^2 + dy^2) + \frac{1}{y} \left\{ (x^2 + y^2) dv_1^2 + 2x dv_1 dv_2 + dv_2^2 \right\}
\]

and

\[
\tilde{\Delta} = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{y} \left\{ \frac{\partial^2}{\partial v_1^2} - 2x \frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 + y^2) \frac{\partial^2}{\partial v_2^2} \right\}.
\]

Conversely, if $h$ is a smooth bounded function on $SP_2 \times \mathbb{R}^{(1,2)}$ satisfying the above conditions (MJ1)*-(MJ3)*, then the function $f_h : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ defined by

\[
(3.4) \quad f_h(\tau, z) = h(g^t g, \alpha^t g)
\]

for some $(g, \alpha) \in G$ with $(g, \alpha) \circ (i, 0) = (\tau, z)$ is a Maass-Jacobi form on $\mathbb{H} \times \mathbb{C}$.

**Proof.** First of all, we note that $h_f$ is well defined because (3.3) is independent
of the choice of \( g \) with \( Y = g^t g \). If \((\gamma, \delta) \in \Gamma \) with \( \gamma \in \Gamma_1 \), \( \delta \in \mathbb{Z}^{(1,2)} \) and \((Y, V) \in \mathcal{SP}_2 \times \mathbb{R}^{(1,2)} \) with \( Y = g^t g \) for some \( g \in SL(2, \mathbb{R}) \), then the element \( g_\gamma := \gamma g \) satisfies \( \gamma Y^t \gamma = \gamma g^t (\gamma g) \).

Thus according to the definition of \( h_f \), for all \((\gamma, \delta) \in \Gamma \) and \((Y, V) \in \mathcal{SP}_2 \times \mathbb{R}^{(1,2)} \) with \( Y = g^t g \) for some \( g \in SL(2, \mathbb{R}) \), we have

\[
h_f(\gamma Y^t \gamma, (V + \delta)^t \gamma) = f((\gamma g, (V + \delta)^t g^{-1}) \circ (i, 0)) = f((\gamma g, (V + \delta)^t g^{-1}) \circ (i, 0)) = f((g, V^t g^{-1}) \circ (i, 0)) \quad \text{because } f \text{ is } \Gamma^-\text{-invariant) = h_f(Y, V) \).
\]

Therefore this proves the condition \((\text{MJ}1)^*\). \( d\tilde{s}^2 \) and \( \tilde{\Delta} \) are obtained from Lemma 2.1 and Proposition 2.3. Hence \( h_f \) is an eigenfunction of \( \tilde{\Delta} \). Clearly \( h_f \) satisfies the condition \((\text{MJ}3)^*\).

Conversely we note that \( f_h \) is well defined because \((3.4) \) is independent of the choice of \((g, \alpha) \in G \) with \((g, \alpha) \circ (i, 0) = (\tau, z) \). If \( \tilde{\gamma} = (\gamma, \delta) \in \Gamma \) and \((\tau, z) \in \mathbb{H} \times \mathbb{C} \) with \((g, \alpha) \circ (i, 0) = (\tau, z) \), then we have

\[
f_h(\tilde{\gamma} \circ (\tau, z)) = f_h(\tilde{\gamma} \circ ((g, \alpha) \circ (i, 0))) = f_h((\tilde{\gamma} \circ (g, \alpha)) \circ (i, 0)) = f_h((\gamma g, (V + \delta)^t g^{-1} + \alpha) \circ (i, 0)) = h((\gamma g)^t (\gamma g), (\delta^t g^{-1} + \alpha)^t (\gamma g)) = h((\gamma g)^t \gamma, (\delta + \alpha^t g)^t \gamma) = h(g^t g, \alpha^t g)
\]
\[
= f_h((g, \alpha) \circ (i, 0)) = f_h(\tau, z).
\]

Thus \( f_h \) satisfies the condition \((\text{MJ}1) \). It is easy to see that \( f_h \) satisfies the conditions \((\text{MJ}2) \) and \((\text{MJ}3) \).

\[\square\]

**Definition 3.3.** A smooth bounded function on \( G \) or \( \mathcal{SP}_2 \times \mathbb{R}^{(1,2)} \) is also called a **Maass-Jacobi form** if it satisfies the conditions \((\text{MJ}1)^0 - (\text{MJ}3)^0 \) or \((\text{MJ}1)^* - (\text{MJ}3)^* \).

**Remark 3.4.** We note that Maass wave forms are special ones of Maass-Jacobi forms. Thus the number of \( \lambda \)'s with \( MJ(\Gamma, \lambda) \neq 0 \) is infinite.

**Theorem 3.5.** For any complex number \( \lambda \in \mathbb{C} \), the vector space \( MJ(\Gamma, \lambda) \) is finite dimensional.

**Proof.** The proof follows from \[10\], Theorem 1, p. 8 and \[5\], p. 191. \[\square\]

4. **On the group** \( SL_{2,1}(\mathbb{R}) \)
For brevity, we set $H = \mathbb{R}^{(1,2)}$. Then we have the split exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow SL(2, \mathbb{R}) \rightarrow 1.$$  

We see that the unitary dual $\hat{H}$ of $H$ is isomorphic to $\mathbb{R}^2$. The unitary character $\chi(\lambda, \mu)$ of $H$ corresponding to $(\lambda, \mu) \in \mathbb{R}^2$ is given by

$$\chi(\lambda, \mu)(x, y) = e^{2\pi i(\lambda x + \mu y)}, \quad (x, y) \in H.$$  

$G$ acts on $H$ by conjugation and hence this action induces the action of $G$ on $\hat{H}$ as follows.

$$G \times \hat{H} \rightarrow \hat{H}, \quad (g, \chi) \mapsto \chi^g, \quad g \in G, \chi \in \hat{H},$$  

where the character $\chi^g$ is defined by $\chi^g(a) = \chi(gag^{-1}), \quad a \in H.$

If $g = (g_0, \alpha) \in G$ with $g_0 \in SL(2, \mathbb{R})$ and $\alpha \in H$, it is easy to check that for each $(\lambda, \mu) \in \mathbb{R}^2$,

$$\chi^g_{(\lambda, \mu)} = \chi_{(\lambda, \mu)g_0}.$$  

We see easily from (4.2) that the $G$-orbits in $\hat{H} \cong \mathbb{R}^2$ consist of two orbits $\Omega_0, \Omega_1$ given by

$$\Omega_0 = \{(0, 0)\}, \quad \Omega_1 = \mathbb{R}^2 - \{(0, 0)\}.$$  

We observe that $\Omega_0$ is the $G$-orbit of $(0, 0)$ and $\Omega_1$ is the $G$-orbit of any element $(\lambda, \mu) \neq 0$.

Now we choose the element $\delta = \chi_{(1, 0)}$ of $\hat{H}$. That is, $\delta(x, y) = e^{2\pi i x}$ for all $(x, y) \in \mathbb{R}^2$. It is easy to check that the stabilizer of $\chi_{(0, 0)}$ is $G$ and the stabilizer $G_{\delta}$ of $\delta$ is given by

$$G_{\delta} = \left\{ \left( \begin{array}{cc} 1 & 0 \\ c & 1 \end{array} \right), \alpha \mid c \in \mathbb{R}, \alpha \in \mathbb{R}^{(1,2)} \right\}.$$  

We see that $H$ is regularly embedded. This means that for every $G$-orbit $\Omega$ in $\hat{H}$ and for every $\sigma \in \Omega$ with stabilizer $G_\sigma$ of $\sigma$, the canonical bijection $G_\sigma \backslash G \rightarrow \Omega$ is a homeomorphism.

According to G. Mackey ([18]), we obtain

**Theorem 4.1.** The irreducible unitary representations of $G$ are the following:

(a) The irreducible unitary representations $\pi$, where the restriction of $\pi$ to $H$ is trivial and the restriction of $\pi$ to $SL(2, \mathbb{R})$ is an irreducible unitary representation of $SL(2, \mathbb{R})$. For the unitary dual of $SL(2, \mathbb{R})$, we refer to [7] or [15], p. 123.
(b) The representations $\pi_r = \text{Ind}_{G_\delta}^{G} \sigma_r$ ($r \in \mathbb{R}$) induced from the unitary character $\sigma_r$ of $G_\delta$ defined by

$$\sigma_r \left( \left( \begin{array}{cc} 1 & 0 \\ c & 1 \end{array} \right), (\lambda, \mu) \right) = \delta(rc + \lambda) = e^{2\pi i (rc + \lambda)}, \quad c, \lambda, \mu \in \mathbb{R}.$$ 

Proof. The proof of the above theorem can be found in [22], p. 850.

We put

$$W_1 = \left( \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), (0,0) \right), \quad W_2 = \left( \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), (0,0) \right), \quad W_3 = \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), (0,0) \right)$$

and

$$W_4 = \left( \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), (1,0) \right), \quad W_5 = \left( \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), (0,1) \right).$$

Clearly $W_1, \cdots, W_5$ form a basis of $\mathfrak{g}$.

Lemma 4.2. We have the following relations.

$$[W_1, W_2] = W_3, \quad [W_3, W_1] = 2W_1, \quad [W_3, W_2] = -2W_2,$$

$$[W_1, W_4] = 0, \quad [W_3, W_5] = -W_4, \quad [W_2, W_4] = W_5, \quad [W_2, W_5] = 0,$$

$$[W_3, W_4] = W_4, \quad [W_3, W_5] = -W_5, \quad [W_4, W_5] = 0.$$

Proof. The proof follows from an easy computation. \hfill \Box

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$. We put

$$\mathfrak{k}_\mathbb{C} = \mathbb{C}(W_1 - W_2), \quad \mathfrak{p}_\pm = \mathbb{C}(W_3 \pm i(W_1 + W_2)).$$

Then we have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{p}_+ + \mathfrak{p}_-, \quad [\mathfrak{k}_\mathbb{C}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad \mathfrak{p}_- = \overline{\mathfrak{p}_+}.$$ 

We note that $\mathfrak{k}_\mathbb{C}$ is the complexification of the Lie algebra $\mathfrak{k}$ of $K$.

We set $\mathfrak{a} = \mathbb{R} W_3$. By Lemma 4.2, the roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ are given by $\pm e, \pm 2e$, where $e$ is the linear functional $e : \mathfrak{a} \rightarrow \mathbb{C}$ defined by $e(W_3) = 1$. The set $\Sigma^+ = \{e, 2e\}$ is the set of positive roots of $\mathfrak{g}$ relative to $\mathfrak{a}$. We recall that for a root $\alpha$, the root space $\mathfrak{g}_\alpha$ is defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Then we see easily that

$$\mathfrak{g}_e = \mathbb{R} W_3, \quad \mathfrak{g}_{-e} = \mathbb{R} W_5, \quad \mathfrak{g}_{2e} = \mathbb{R} W_1, \quad \mathfrak{g}_{-2e} = \mathbb{R} W_2.$$
and
\[
\mathfrak{g} = \mathfrak{g}_{-2e} \oplus \mathfrak{g}_{-e} \oplus \mathfrak{a} \oplus \mathfrak{g}_e \oplus \mathfrak{g}_{2e}.
\]

**Proposition 4.3.** The Killing form $B$ of $\mathfrak{g}$ is given by
\[
B((X_1, Z_1), (X_2, Z_2)) = 5 \sigma(X_1 X_2),
\]
where $(X_1, Z_1), (X_2, Z_2) \in \mathfrak{g}$ with $X_1, X_2 \in \mathfrak{sl}(2, \mathbb{R})$ and $Z_1, Z_2 \in \mathbb{R}^{(1,2)}$. Hence the Killing form is highly nondegenerate. The adjoint representation $\text{Ad}$ of $G$ is given by
\[
\text{Ad}((g, \alpha))(X, Z) = (gXg^{-1}, (Z - \alpha^t X)^t g),
\]
where $(g, \alpha) \in G$ with $g \in \text{SL}(2, \mathbb{R}), \alpha \in \mathbb{R}^{(1,2)}$ and $(X, Z) \in \mathfrak{g}$ with $X \in \mathfrak{sl}(2, \mathbb{R}), Z \in \mathbb{R}^{(1,2)}$.

**Proof.** The proof follows immediately from a direct computation. \[\square\]

An Iwasawa decomposition of the group $G$ is given by
\[
G = NAK,
\]
where
\[
N = \left\{ \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right), a \right\} \in G \mid x \in \mathbb{R}, a \in \mathbb{R}^{(1,2)} \right\}
\]
and
\[
A = \left\{ \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right), 0 \right\} \in G \mid a > 0 \right\}.
\]

An Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of $G$ is given by
\[
\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k},
\]
where
\[
\mathfrak{n} = \left\{ \left( \begin{array}{cc} 0 & x \\ 0 & 0 \end{array} \right), Z \right\} \in \mathfrak{g} \mid x \in \mathbb{R}, Z \in \mathbb{R}^{(1,2)} \right\}
\]
and
\[
\mathfrak{a} = \left\{ \left( \begin{array}{cc} x & 0 \\ 0 & -x \end{array} \right), 0 \right\} \in \mathfrak{g} \mid x \in \mathbb{R} \right\}.
\]
In fact, $\mathfrak{a}$ is the Lie algebra of $A$ and $\mathfrak{n}$ is the Lie algebra of $N$.

Now we compute the Lie derivatives for functions on $G$ explicitly. We define the differential operators $L_k, R_k$ $(1 \leq k \leq 5)$ on $G$ by
\[
L_k f(\tilde{g}) = \frac{d}{dt} \bigg|_{t=0} f(\tilde{g} \ast \exp tW_k)
\]
and
\[
R_k f(\tilde{g}) = \left. \frac{d}{dt} \right|_{t=0} f(\exp tW_k \ast \tilde{g}),
\]
where \( f \in C^\infty(G) \) and \( \tilde{g} \in G \).

By an easy calculation, we get
\[
\begin{align*}
\exp tW_1 &= \left( \begin{array}{cc}
1 & t \\
0 & 1 \\
\end{array} \right), \\
\exp tW_2 &= \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
\end{array} \right), \\
\exp tW_3 &= \left( \begin{array}{cc}
\exp t & 0 \\
0 & \exp -t \\
\end{array} \right), \\
\exp tW_4 &= \left( \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\end{array} \right), \\
\exp tW_5 &= \left( \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\end{array} \right).
\end{align*}
\]

Now we use the following coordinates \((g, \alpha)\) in \(G\) given by
\[
(4.6) \quad g = \begin{pmatrix}
1 & x \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
y^{1/2} & 0 \\
0 & y^{-1/2} \\
\end{pmatrix} \begin{pmatrix}
\cos \theta & 0 \\
0 & \sin \theta \\
\end{pmatrix}
\]
and
\[
(4.7) \quad \alpha = (\alpha_1, \alpha_2),
\]
where \( x, \alpha_1, \alpha_2 \in \mathbb{R}, \ y > 0 \) and \( 0 \leq \theta < 2\pi \). By an easy computation, we have
\[
\begin{align*}
L_1 &= y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} + \sin^2 \theta \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1}, \\
L_2 &= y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} - \cos^2 \theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
L_3 &= -2y \sin 2\theta \frac{\partial}{\partial x} + 2y \cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2}, \\
L_4 &= \frac{\partial}{\partial \alpha_1}, \\
L_5 &= \frac{\partial}{\partial \alpha_2}, \\
R_1 &= \frac{\partial}{\partial x}, \\
R_2 &= (y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - y \frac{\partial}{\partial \theta}, \\
R_3 &= 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \\
R_4 &= y^{-1/2} \cos \theta \frac{\partial}{\partial \alpha_1} + y^{-1/2} \sin \theta \frac{\partial}{\partial \alpha_2}, \\
R_5 &= -y^{-1/2} (x \cos \theta + y \sin \theta) \frac{\partial}{\partial \alpha_1} + y^{-1/2} (y \cos \theta - x \sin \theta) \frac{\partial}{\partial \alpha_2}.
\end{align*}
\]
In fact, the calculation for $L_3$ and $R_5$ can be found in [22], p. 837-839.

We define the differential operators $\mathbb{L}_j$ ($1 \leq j \leq 5$) on $\mathbb{H} \times \mathbb{C}$ by

$$\mathbb{L}_j f(\tau, z) = \frac{d}{dt} \bigg|_{t=0} f(\exp tW_j \circ (\tau, z)), \quad 1 \leq j \leq 5,$$

where $f \in C^\infty(\mathbb{H} \times \mathbb{C})$. Using the coordinates $\tau = x + iy$ and $z = u + iv$ with $x, y, u, v$ real and $y > 0$, we can easily compute the explicit formulas for $\mathbb{L}_j$’s. They are given by

$L_1 = (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} + (xu - yv) \frac{\partial}{\partial u} + (yu + xv) \frac{\partial}{\partial v},$

$L_2 = -\frac{\partial}{\partial x},$

$L_3 = -2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v},$

$L_4 = x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v},$

$L_5 = \frac{\partial}{\partial u}.$

5. The decomposition of $L^2(\Gamma \backslash G)$

Let $R$ be the right regular representation of $G$ on the Hilbert space $L^2(\Gamma \backslash G)$. We set $G_1 = SL(2, \mathbb{R})$. Then the decomposition of $R$ is given by

$$(5.1) \quad L^2(\Gamma \backslash G) = L^2_{\text{disc}}(\Gamma_1 \backslash G_1) \bigoplus L^2_{\text{cont}}(\Gamma_1 \backslash G_1) \bigoplus \int_{-\infty}^{\infty} \mathcal{H}(\tau) d\tau,$$

where $L^2_{\text{disc}}(\Gamma_1 \backslash G_1)$ (resp. $L^2_{\text{cont}}(\Gamma_1 \backslash G_1)$) is the discrete (resp. continuous) part of $L^2(\Gamma_1 \backslash G_1)$ (cf. [14], [15]) and $\mathcal{H}(\tau)$ is the representation space of $\pi(\tau)$ (cf. Theorem 4.1. (b)).

We recall the result of Rolf Berndt (cf. [2], [3], [4]). Let $H_{\mathbb{R}}^{(1,1)}$ denote the Heisenberg group which is $\mathbb{R}^3$ as a set and is equipped with the following multiplication

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda').$$

We let $G^J = SL(2, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(1,1)}$ be the semidirect product of $SL(2, \mathbb{R})$ and $H_{\mathbb{R}}^{(1,1)}$, called the Jacobi group whose multiplication law is given by

$$(M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) = (MM', (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda')).$$
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with $M, M' \in SL(2, \mathbb{R}), (\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H^{(1,1)}_{\mathbb{R}}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu) M'$. Obviously the center $Z(G^J)$ of $G^J$ is given by \{$(0, 0, \kappa) \mid \kappa \in \mathbb{R}$\}. We denote

$$H^{(1,1)}_{\mathbb{Z}} = \{ (\lambda, \mu, \kappa) \in H^{(1,1)}_{\mathbb{R}} \mid \lambda, \mu, \kappa \text{ integral} \}.$$ 

We set

$$\Gamma^J = SL(2, \mathbb{Z}) \ltimes H^{(1,1)}_{\mathbb{Z}}, \quad K^J = K \times Z(G^J).$$

R. Berndt proved that the decomposition of the right regular representation $R^J$ of $G^J$ in $L^2(\Gamma^J \backslash G^J)$ is given by

$$L^2(\Gamma^J \backslash G^J) = \bigoplus_{m,n \in \mathbb{Z}} H_{m,n} \bigoplus \left( \bigoplus_{\nu = \pm \frac{1}{2}} \int_{\text{Re} s = 0} \text{Im} s > 0 \mathcal{H}_{m,s,\nu} ds \right),$$

where the $H_{m,n}$ is the irreducible unitary representation isomorphic to the discrete series $\pi_{n,k}$ or the principal series $\pi_{m,s,\nu}$, and the $\mathcal{H}_{m,s,\nu}$ is the representation space of $\pi_{m,s,\nu}$ (cf. [4], p. 47-48). For more detail on the decomposition of $L^2(\Gamma^J \backslash G^J)$, we refer to [4], p. 75-103.

Since $\mathbb{H} \times \mathbb{C} = K^J \backslash G^J = K \times G$, the space of the Hilbert space $L^2(\Gamma \backslash (\mathbb{H} \times \mathbb{C}))$ consists of $K^J$-fixed elements in $L^2(\Gamma^J \backslash G^J)$ or $K$-fixed elements in $L^2(\Gamma \backslash G)$. Hence we obtain the spectral decomposition of $L^2(\Gamma \backslash (\mathbb{H} \times \mathbb{C}))$ for the Laplacian $\Delta$ or $\Delta_{\alpha,\beta}$ (cf. Proposition 2.4 or Remark 2.5).

6. Remarks on Fourier expansions of Maass-Jacobi forms

We let $f : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ be a Maass-Jacobi form with $\Delta f = \lambda f$. Then $f$ satisfies the following invariance relations

$$f(\tau + n, z) = f(\tau, z) \quad \text{for all } n \in \mathbb{Z}$$

and

$$f(\tau, z + n_1 \tau + n_2) = f(\tau, z) \quad \text{for all } n_1, n_2 \in \mathbb{Z}.$$ 

Therefore $f$ is a smooth function on $\mathbb{H} \times \mathbb{C}$ which is periodic in $x$ and $u$ with period 1. So $f$ has the following Fourier series

$$f(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} c_{n,r}(y, v) e^{2\pi i (nx + ru)}.$$ 

For two fixed integers $n$ and $r$, we have to calculate the function $c_{n,r}(y, v)$. For brevity, we put $F(y, v) = c_{n,r}(y, v)$. Then $F$ satisfies the following differential equation

$$\left[ y^2 \frac{\partial^2}{\partial y^2} + (y + v^2) \frac{\partial^2}{\partial v^2} + 2yv \frac{\partial}{\partial v} - \{ (ay + bv)^2 + b^2 y + \lambda \} \right] F = 0.$$
Here $a = 2\pi n$ and $b = 2\pi r$ are constant. We note that the function $u(y) = y^s K_{s-\frac{1}{2}}(2\pi |n|y)$ satisfies the differential equation (6.4) with $\lambda = s(s - 1)$. Here $K_s(z)$ is the $K$-Bessel function defined by (2.16) (see Lebedev [16] or Watson [21]). The problem is that if there exist solutions of the differential equation (6.4), we have to find their solutions explicitly.

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