Gauge Freedom in the Path Integral Formalism

Seiji Sakoda

Department of Applied Physics, National Defense Academy, Yokosuka 239-8686, Japan

We investigate ‘t Hooft’s technique of changing the gauge parameter of the linear covariant gauge from the point of view of the path integral with respect to the gauge freedom. Extension of the degrees of freedom allows us to formulate a system with extended gauge symmetry. The gauge fixing for this extended symmetry yields the ‘t Hooft averaging as a path integral over the additional degrees of freedom. Another gauge fixing is found as a non-abelian analogue of the type II gaugeon formalism of Yokoyama and Kubo. In this connection, the ‘t Hooft average can be viewed as the analogue of the type I gaugeon formalism. As a result, we obtain gauge covariant formulations of non-abelian gauge theories, which allow us to understand ‘t Hooft’s technique also from the canonical formalism.

§ 1. Introduction

In the quantization of non-abelian gauge fields, the path integral (PI) formalism is a powerful and efficient tool. In particular, in this formalism changing gauge fixing conditions can be carried out in a very simple manner. For instance, the change of gauges from the physical Coulomb gauge to the Landau gauge first presented by Faddeev and Popov (FP)\(^1\) and its generalization\(^2\) formulated by ‘t Hooft to any linear covariant gauge cannot be treated in the canonical quantization, because the latter in a specific gauge requires its own Hilbert space and there are no gauge transformations which generate transitions between different Hilbert spaces. Therefore, to clarify the equivalence of two or more theories in different gauges in the canonical formalism, we must compare the results obtained from each of the quantum theories. In the path integral formalism, by contrast, we can transform between different gauges in a relatively simple manner, by inserting an identity and carrying out a change of variables.

As far as the canonical formulation of quantum electrodynamics is concerned, there is an elegant prescription, the gaugeon formalism of Yokoyama and Kubo,\(^3\) to carry out the change of the gauge parameter \(\alpha\) in linear covariant gauges. By taking the concept of the form invariance of the Lagrangian in this formalism as a principle, generalizations of Yang-Mills theory have also been explored.\(^4,5\) Unfortunately, however, the concept of the form invariance does not seem to be compatible with the gauge condition of linear covariant gauges for the case of non-abelian gauge theories, and for this reason, the gauge parameter \(\alpha\) cannot be controlled by means of the \(q\)-number gauge transformation, which is a global transformation defined on the extension of the degrees of freedom for the gaugeon field and its associated field; a very complicated Lagrangian is necessary in order to implement the form invariance for the case of a non-abelian theory, and the \(q\)-number gauge transformation then generates changes in parameters of the Lagrangian in a manner different from...
that in the case of the abelian theory. Contrastingly, in the case of the path integral formalism, 't Hooft formulated a very simple method\(^2\) of changing the gauge parameter \(\alpha\). This method consists of averaging with respect to an unknown \(c\)-number function that enters the PI when the condition of the Landau gauge is slightly modified. In comparison with the complexity of the gaugeon formalism of Yang-Mills theory, the simplicity of this method represents a great advantage of the path integral formalism, and due to this convenience the path integral formalism is more practical than the canonical formalism in formulating a quantization of the Yang-Mills theory. In this paper, we investigate the meaning of such an advantage of the path integral method, first considering the ‘t Hooft averaging\(^2\) from our own point of view. (Throughout this paper, we refer to the averaging over a \(c\)-number function in the path integral formalism performed to change a gauge condition in this way as “‘t Hooft averaging.”) Here, similarly to Yokoyama’s gaugeon formalism,\(^3\) a pair of fields plays a role in finding a new interpretation of this technique. Our proposal here is to view the ‘t Hooft averaging as a PI obtained through gauge fixing for an extended gauge invariance.\(^6\) As a technical tool for changing the gauge condition, the ‘t Hooft averaging shares much with the gaugeon formalism. If our interpretation of the ‘t Hooft average is correct, it will allow us to treat such an extended system in terms of a canonical quantization similar to that in the case of the gaugeon formalism. As mentioned above, however, a gaugeon formalism suitable for handling the linear covariant gauge is known only for the abelian gauge theory. In this paper, we attempt to realize the above-stated goal for non-abelian gauge theories.

We first introduce a change of variable for the \(c\)-number function in the ‘t Hooft average to reformulate the PI with extended degrees of freedom in the next section. Then, in § 3, an extended gauge transformation is introduced to make the system invariant under this extended transformation. Using the usual FP trick for such an invariant PI, we obtain the ‘t Hooft averaging as a result of the gauge fixing. Because we utilize the FP trick for the gauge fixing, a PI possessing invariance under BRST transformations\(^7\) is naturally obtained. It is shown that the extended BRST transformation consists of two anti-commuting generators. On the basis of these generators, we carry out analyses of the BRST invariance and the gauge structure of the total Hilbert space for the extended system. In the derivation of the ‘t Hooft average discussed above, however, we still need an averaging with respect to another \(c\)-number field. To avoid this, we formulate another prescription for the gauge fixing. The difference between the two gauge fixing procedures is found to be similar to that between the type I and type II gaugeon formalisms of Yokoyama and Kubo.\(^3\) Despite the difference in their Lagrangians, the structure of the Hilbert space and the BRST invariance of the system obtained with this second method are quite similar to those of the first method. In this way, we obtain non-abelian analogues of type I and type II gaugeon formalism. This is the contents of § 3. Comparison of our method for the non-abelian case with the derivation of the PI presented by Koseki, Sato and Endo\(^8\) for the BRST invariant version of Yokoyama’s gaugeon formalism is made in § 4. We also explain there how to change the gauge parameter in our formalism, confirming that our method can be regarded as a non-abelian generalization of the gaugeon formalism suitable for the linear covariant gauge. It is known, of course,
that there exist different approaches for gauge covariant formulations of non-abelian
gauge theories. However, they are not suitable to study the linear covariant
gauge. Therefore we do not go into the details of these approaches. The final section
contains a summary. In the appendix, we present an explanation of how our method
can also be understood from the point of view of the non-abelian generalization of
the model considered by Kashiwa in Ref. 10). Some results given there are found to
be useful in the understanding we describe in the main text, in particular the BRST
invariance of the extended system.

§2. ’t Hooft average as a path integral over gauge degrees of freedom

The Faddeev-Popov path integral\(^\star\) (FPPI) in the Landau gauge is given by

\[
Z = \int DA \delta(\partial^\mu A_\mu) \Delta[A] e^{iS[A]},
\]

where \(S[A]\) is the gauge invariant action\(^\star\star\) of the (non-)abelian gauge field and \(\Delta[A]\)
represents the explicit form of the FP determinant \(\Delta_{\text{FP}}[A, f]\) \((f = 0\) in the present
case), defined by

\[
1 = \Delta_{\text{FP}}[A, f] \int D\theta \delta(\partial^\mu A^\theta_\mu - f), \quad A^\theta_\mu = \theta A_\mu \theta^{-1} + i \theta \partial^\mu \theta^{-1}.
\]

Because of the existence of \(\delta(\partial^\mu A_\mu)\) in Eq. (2.1), it is evaluated to be \(\Delta[A] = |\det(\partial^\mu D_\mu)|\). For the reason explained below, we assume periodic boundary condition
(PBC) for all variables in the PI given in Eq. (2.1).

The above FPPI of the Landau gauge can be transformed to that for other
covariant gauges by making use of the ’t Hooft averaging: (i) replacing the constraint
\(\delta(\partial^\mu A_\mu)\) by \(\delta(\partial^\mu A_\mu - f)\) with an arbitrary \(c\)-number function \(f\), and (ii) inserting
the Gaussian identity by regarding \(f\) introduced in the first step as identical to the
integration variable in

\[
1 = \int Df \exp \left( -\frac{i}{2\alpha} \int d^4x f(x)^2 \right).
\]

Then, the FPPI in Eq. (2.1) is rewritten as that suitable for the \(\alpha\)-gauge,

\[
Z = \int DA \Delta[A] e^{iS[A] - i \int d^4x (\partial^\mu A_\mu)^2/(2\alpha)}.
\]

To elucidate the meaning of ’t Hooft’s technique, let us suppose that the \(c\)-number function \(f\) in the constraint is related to a gauge transformation through

\(^\star\) As for the use of continual representation of PI’s in this paper, it should be understood that
we define them by their Euclidean version with the initial and final times, \(t_i\) and \(t_f\), and then taking
the limits \(t_i \to -\infty\) and \(t_f \to +\infty\) in the end.

\(^\star\star\) We employ matrix notation for quantities that take values in a group and its Lie algebra
without expressing the group indices explicitly.
\[ \partial^\mu A_\mu - f = \partial^\mu A^g_\mu, \text{ where } A^g_\mu \text{ is given by} \]

\[ A^g_\mu = gA_\mu g^{-1} + ig\partial_\mu g^{-1} = A_\mu + igD_\mu g^{-1}, \]

\[ D_\mu g^{-1} = \partial_\mu g^{-1} - i[A_\mu, g^{-1}]. \]  

(2.5)

We thus set \( f = -i\partial^\mu(gD_\mu g^{-1}) \) in the procedure of the 't Hooft average above.

Because we fix the gauge field \( A_\mu \) by the variation \( \delta f = -i\partial^\mu(D^g_\mu(g\delta g^{-1})) \), in which \( D^g_\mu(g\delta g^{-1}) = \partial_\mu(g\delta g^{-1}) - i[A^g_\mu, (g\delta g^{-1})] \), and the invariant measure on the group manifold is obtained from the volume form, which is defined by the wedge product that consists of \( gdg^{-1} \), we can rewrite the Gaussian identity utilized above as

\[ 1 = \int Dg \Delta[A^g] \exp \left( -\frac{i}{2\alpha} \int d^4x \left\{ -i\partial^\mu(gD_\mu g^{-1}) \right\}^2 \right), \]  

(2.6)

where \( \Delta[A^g] = |\det(\partial^\mu D^g_\mu)| \). We therefore find

\[ Z = \int DA DB Dg \Delta[A] \Delta[A^g] \exp \left( iS[A] + i \int d^4x B\partial^\mu A^g_\mu \right) \]

\[ \times \exp \left( -\frac{i}{2\alpha} \int d^4x \left\{ -i\partial^\mu(gD_\mu g^{-1}) \right\}^2 \right), \]  

(2.7)

which, of course (if the integrations with respect to \( g \) are carried out first), results in a PI for the \( \alpha \)-gauge with the Nakanishi-Lautrup (NL) field \( B \). Although the PI given in Eq. (2.7), obtained by setting \( f = -i\partial^\mu(gD_\mu g^{-1}) \) in Eq. (2.3), is merely a re-expression of 't Hooft's original prescription, as we see below, this expression makes it clear that the 't Hooft average can be understood as a PI over gauge degrees of freedom.

Here we comment on the boundary conditions of the PI given in Eq. (2.7). In the original formulation of 't Hooft's technique, the identity of the functional Gaussian integral does not require any boundary conditions, because it can be defined as a product of functional integrals over variables on a surface of constant \( x_0 \),

\[ 1 = \int \prod_{x \in \mathbb{R}^3} \left[ \sqrt{\frac{i \Delta x_0}{2\pi\alpha}} \right] df(x, x_0) \exp \left( -\frac{i \Delta x_0}{2\alpha} \int d^3x f(x)^2 \right), \]  

(2.8)

where \( \Delta x_0 \) is the infinitesimal increment of \( x_0 \) in the discretized PI. If the number of equal-time surfaces in such a PI is \( n+1 \), specified by \( t_1 = x_0(0), x_0(1), \ldots, x_0(n-1), \) and \( x_0(n) = t_f \), in this order, we have \( n \) integrals with respect to \( f(x, x_0(j)) \) (\( j = 1, 2, \ldots, n \)), corresponding to \( n \) delta functions. Then, setting \( f = -i\partial^\mu(gD_\mu g^{-1}) \), the initial and final values of \( g \) enter the integrand, because \( f \) includes a time derivative. They must be connected by some condition, as otherwise the numbers of the variables \( f \) and \( g \) do not match. For a PI with PBC, we can simply set the two boundary values equal. In this way, we can change the gauge conditions in a relatively simple way in the trace formulae. The reason for this simplicity is that the value of a trace is merely a number, and therefore we can add any expression to it, provided that the value of this expression vanishes. The cancellation mechanism
Gauge Freedom in the Path Integral Formalism

resulting from the BRST invariance\(^7\) ensures that contributions from unphysical degrees of freedom to a PI with PBC satisfy this condition. In contrast to this simple case, the situation becomes much more complicated for PIs with other boundary conditions. Dealing with such a delicate issue is not the aim of this paper. We thus restrict our inquiry here to the case of PIs with PBC. Note, however, that naive use of the PBC may introduce complications due to the existence of zero modes in cases of massless fields. Hence, we must treat them carefully. (See Ref. 13), for example, for details regarding this point.)

The Gaussian identity (2.6) can be regarded as a non-abelian generalization of the BRST invariant version of the Froissart model\(^14\) studied by Kashiwa in Ref. 10) in detail. The map given by 
\[ f = -i \partial^\mu (g D_\mu g^{-1}) \]
then considered to be a Nicolai map.\(^15\) (See Appendix A of this paper and discussions in Ref. 10).)

§3. Extended gauge symmetry

3.1. The system with the extended gauge symmetry in 't Hooft’s path integral

Although the equivalence of the expression in Eq. (2.7) with that in Eq. (2.4) is evident, our method of obtaining Eq. (2.7) may seem heuristic. In particular, the FP determinant \( \Delta[A,f] \) in Eq. (2.2) must be equal to \( \Delta[A,g] \) if we obtain the constraint \( \delta(\partial^\mu A_\mu - f) = \delta(\partial^\mu A_\mu^g) \) through a gauge transformation from Eq. (2.1). However, we have set 
\[ f = -i \partial^\mu (g D_\mu g^{-1}) \]
simply leaving the determinant \( \Delta[A] \) unchanged in the derivation of the PI of Eq. (2.7). Therefore we need to verify the derivation of Eq. (2.7) above. For this purpose, below we present another derivation of this PI. However, before doing so, it is useful to clarify the advantageous features of the PI in Eq. (2.7). For this purpose, here we examine the \( \alpha \) dependence of this PI. If we wish to obtain a PI, as explained above, for the \( \alpha \)-gauge, we first complete the square of \( f + \alpha B \) before carrying out the PI of \( g \) with the measure \( Df = Dg \Delta[A^g] \). Alternatively, we may first perform the gauge transformation \( A_\mu^g \mapsto A_\mu \), under which we have 
\[ f \mapsto i \partial^\mu (g^{-1} D_\mu g) \].
We then find

\[ Z = \int DADB Dg \Delta[A^{g^{-1}}] \Delta[A] \exp \left( iS[A] + i \int d^4x B \partial^\mu A_\mu \right) \]
\[ \times \exp \left( -\frac{i}{2\alpha} \int d^4x \left\{ i \partial^\mu (g^{-1} D_\mu g) \right\}^2 \right). \]  
(3.1)

Then computing the PI of \( g \), we find that the quantity in Eq. (3.1) is equal to the original FPPI of Eq. (2.1). In other words, we can consider the 't Hooft average as the insertion of the identity

\[ 1 = \int Dg \Delta[A^{g^{-1}}] \exp \left( -\frac{i}{2\alpha} \int d^4x \left\{ i \partial^\mu (g^{-1} D_\mu g) \right\}^2 \right), \]  
(3.2)

which is, of course, equivalent to Eq. (2.6), into the original FPPI Eq. (2.1) for the Landau gauge. We have thus confirmed that \( Z \) is actually independent of \( \alpha \). It is interesting that the PI in Eq. (2.7) simultaneously represents PIs for both the Landau
gauge and the \( \alpha \)-gauge, provided that we first integrate over the additional degrees of freedom. Because \( \alpha \) is arbitrary, we expect that the PI given by Eq. (2.7) with extended degrees of freedom contains all linear covariant gauges, given by \( \partial^\mu A_\mu + \alpha B = 0 \), for the original variables.

We have performed the gauge transformation of \( A_\mu \) above in order to return to the PI for the Landau gauge from Eq. (2.7). If we extend the gauge transformation to include that for \( g \) in addition to \( A_\mu \rightarrow A_\mu^h \), in order for \( A_\mu^g \) to be invariant, we observe that the action in the first line of Eq. (2.7), except for the gauge-variant factor \( \Delta[A] \), possesses gauge invariance under the extended transformations

\[
A \mapsto A^h, \quad g \mapsto gh^{-1}, \quad (3.3)
\]

where \( h \) takes values in the gauge group. Then, the second line of Eq. (2.7) combined with \( \Delta[A] \) is recognized as a gauge fixing for this gauge invariance. We thus realize that the procedure of the 't Hooft averaging consists essentially of the following two steps: (i) an extension of the degrees of freedom that compensates for the gauge degrees of the gauge field through the above gauge invariant implementation and (ii) a gauge fixing for this extended gauge symmetry.

We now attempt to extend the observation above to the formulation of the PI in Eq. (2.7). We first rewrite Eq. (2.1) by replacing \( A_\mu \) with \( A_\mu^g \) and define the formally divergent PI

\[
Z_{\text{div}} = \int \mathcal{D}A \mathcal{D}B \mathcal{D}g I[A, B, g] \left( = \int \mathcal{D}g \int \mathcal{D}A \mathcal{D}B I[A, B, 1] \right), \quad (3.4)
\]

in which the gauge invariant functional \( I[A, B, g] \) is given by

\[
I[A, B, g] = \Delta[A^g] \exp \left( iS[A] + i \int d^4x \, B \partial^\mu A^\mu_\mu \right). \quad (3.5)
\]

The FP trick for this PI with gauge invariance is implemented by inserting the identity

\[
1 = \Delta_{\text{FP}}[A, g, C] \int \mathcal{D}h \delta(f[g, A^h, A^h^{-1}]) \quad (3.6)
\]

where \( C \) is an arbitrary \( \ast \)-number function, and we have written \( f = -i\partial^\mu (gD_\mu g^{-1}) \) as \( f[g, A] \), taking its functional dependence into account. By making use of the invariance of \( I[A, B, g] \) and \( \Delta_{\text{FP}}[A, g, C] \) under Eq. (3.3), we can factorize \( \int \mathcal{D}h (= \infty) \) as usual to obtain

\[
Z = \int \mathcal{D}A \mathcal{D}B \mathcal{D}g I[A, B, g] \Delta_{\text{FP}}[A, g, C] \delta(f[g, A] - C). \quad (3.7)
\]

Then, because we have

\[
\Delta_{\text{FP}}[A, g, C] \delta(f[g, A] - C) = \Delta[A] \delta(f[g, A] - C) \quad (3.8)
\]

and Eq. (3.7) is independent of \( C \), we may insert the Gaussian identity for \( C \). We then find that this PI results in Eq. (2.7). We have thus completed the explanation of Eq. (2.7) from our new point of view.
Extending the degrees of freedom for the extended gauge symmetry, we introduce new fields, which require an extension of the Hilbert space from that of the original degrees of freedom. To examine the structure of this extended Hilbert space, let us Fourier transform the second line of Eq. (2.7). This gives

\[ Z = \int \mathcal{D} A \mathcal{D} B \mathcal{D} g \mathcal{D} \Phi \Delta[A] \Delta[A^g] \exp \left( i S[A] + i \int d^4 x B \partial^\mu A_\mu^g \right) \]
\[ \times \exp \left( i \int d^4 x \left\{ -i \Phi \partial^\mu (g D_\mu g^{-1}) + \frac{\alpha}{2} \Phi^2 \right\} \right). \]  \tag{3.9}

Then, if we set \( g = e^{i \Theta} \) and shift the field \( \Phi \) to \( \Phi + B \), we find that the quadratic part of the Lagrangian is given by

\[ \mathcal{L}_G^{(0)} = -\Phi \partial^\mu \partial_\mu \Theta + \frac{\alpha}{2} (\Phi + B)^2 \] \tag{3.10}

for these new variables. Because both \( \Phi \) and \( B \) are subject to the d’Alembert equation at the tree level, \( \Theta \) is regarded as a dipole ghost, except in the case \( \alpha = 0 \). Hence we need an indefinite metric for the sector of \( \Theta \) and \( \Phi \). This unphysical sector is accompanied by that for ghost fermions coming from\(^{\text{\footnote{We assume that } \Delta[A] \text{ and } \Delta[A^g] \text{ do not vanish, so that sign changes do not occur. In this case, the Gribov problem\(^{\text{\footnote{\cite{Gribov}}}} \text{ can be avoided in the perturbative definition of these quantities.}}} \text{\footnote{\cite{Gribov}}}}\)\).
corresponding to the method of obtaining Eq. (2.1) from Eq. (2.7) demonstrated above. The important point here is that the PI given by Eq. (2.7) simultaneously contains these two systems, described by Lagrangians Eq. (3.14) and Eq. (3.15). It is noteworthy that such a structure, i.e. a hybrid of the Landau gauge and the α-gauge, of the PI in Eq. (2.7) can be constructed only after our identification of the unknown c-number function $f$ in the 't Hooft average with $f[g, A] = -i\partial^\mu (gD_\mu g^{-1})$, which allows us to interpret the system being equipped with the extended gauge transformation (3.3).

3.2. BRST invariance and the structure of the total Hilbert space

As we saw in the preceding section, the 't Hooft average can be interpreted as a PI with gauge fixing for the system with extended gauge invariance. Since we have formulated this PI by means of the FP trick, as usual, it is natural to conjecture the BRST invariance for this gauge fixing procedure. We show in this section that this is indeed the case, and, in addition to the usual BRST symmetry, there exists another BRST symmetry for the system described by the Lagrangian (3.13).

By replacing the c-number function $\theta(x)$ in the gauge transformations $A_\mu \rightarrow A_\mu + D_\mu \theta$ and $g \rightarrow g - ig\theta$ with $\lambda c(x)$, we observe that the Lagrangian in Eq. (3.13) is invariant under the BRST transformation

\[
\delta A_\mu = \lambda D_\mu c, \quad \bar{\delta} c = i\lambda \bar{\phi}, \quad -ig\delta g^{-1} = \lambda g cg^{-1}, \quad \delta c = i\lambda c^2, \quad \delta \eta = i\lambda \{c, \eta\}, \quad \delta \bar{\eta} = \delta B = \delta \bar{\phi} = 0,
\]

(3.16)

where $\lambda$ is a Grassmann parameter. Apparently, the BRST invariance mentioned above corresponds to the gauge fixing of the extended gauge symmetry. In addition to this usual BRST invariance, there exists the transformation

\[
-ig\tilde{\delta} g^{-1} = \tilde{\lambda} g \eta g^{-1} = \tilde{\lambda} \eta^2, \quad \tilde{\delta} \bar{\eta} = i\tilde{\lambda} (\bar{\phi} - B), \quad \tilde{\delta} \eta = i\tilde{\lambda} \eta^2,
\]

(3.17)

under which the system remains invariant. This additional BRST invariance originates from that given in the second line of Eq. (3.15), and it is a consequence of the trivial nature of the Gaussian identity (2.6) [or (3.2)] when expressed as a PI with ghost fermions. (See Appendix A for details.)

Setting $\delta = \lambda \delta$ and $\tilde{\delta} = \lambda \tilde{\delta}$ in Eqs. (3.16) and (3.17), respectively, we can write the Lagrangian as

\[
\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu + i\bar{\eta} \partial^\mu D_\mu \eta - i\delta \left[ \bar{c} \left( f[g, A] + \frac{\alpha}{2} \bar{\phi} \right) \right]
\]

(3.18)

or as

\[
\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu + i\bar{\delta} \partial^\mu D_\mu \eta + \frac{\alpha}{2} B^2 - i\delta \left[ \bar{\eta} \left( f[g, A] + \frac{\alpha}{2} (\Phi + B) \right) \right].
\]

(3.19)

Furthermore, if we write $\delta_B = \delta + \tilde{\delta}$, the Lagrangian can be expressed as

\[
\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - i\delta_B \left[ \bar{c} \left( \partial^\mu A_\mu + \frac{\alpha}{2} \bar{\phi} \right) - \bar{\eta} \partial^\mu A_\mu \right].
\]

(3.20)
Note that we have defined $\delta$ and $\tilde{\delta}$ to be nilpotent and also to anti-commute with each other. Therefore $\delta_B$ is also nilpotent. Next, we can define the BRST charges $Q$, $\tilde{Q}$ and $Q_B$ corresponding to $\delta$, $\tilde{\delta}$ and $\delta_B$, respectively. Then the state vectors destroyed by multiplying these charges will be physical states. Here, the meaning of physical needs to be explained. In view of Eq. (3.20), it is evident that the subspace specified by the condition $Q_B|\text{phys}\rangle = 0$ is equivalent to that defined by the physical state condition proposed by Kugo and Ojima$^{12}$ for the original degrees. This is the meaning of the term physical in application to $Q_B$. Then the expression of the Lagrangian given by Eq. (3.19) reveals that there exists a local decomposition of the total Hilbert space into subspaces, that specified by $\tilde{Q}|\text{phys}\rangle; \alpha\rangle = 0$ and the rest. Because the PI of the Lagrangian $L_{\text{ex}}$ given below with respect to $g$, $\Phi$, $\eta$ and $\bar{\eta}$ is trivial owing to the quartet mechanism, only the vacuum of these extended degrees of freedom can be a positive normed and physical state with respect to $\tilde{Q}$ in the subspace that describes the system defined by

$$L_{\text{ex}} = -i\tilde{\delta} \left[ \bar{\eta} \left( f[g, A] + \alpha/2 (\Phi + B) \right) \right]$$

for a fixed configuration (i.e. the $\alpha$-gauge in the present case) of $A_\mu$ and $B$. Therefore, $\tilde{Q}$ defines the Hilbert space for the $\alpha$-gauge of the original degrees of freedom as its invariant subspace. For this reason, we have written the condition for $\tilde{Q}$ as $\tilde{Q}|\text{phys}; \alpha\rangle = 0$ above. Then, integrating out the extended degrees of freedom, we obtain the reduced system for the original variables in the $\alpha$-gauge. Because, in the course of this reduction, the BRST charge $Q$ is reduced to $Q_KO$ (the BRST charge of Kugo and Ojima), we observe that the total BRST charge $Q_B = Q + \tilde{Q}$ is the proper extension of $Q_KO$ needed to fit the extended gauge symmetry. We note here that the structure of the total Hilbert space for the extended system is quite similar to that found by Koseki, Sato and Endo in Ref. 8) for the BRST invariant version of Yokoyama’s gaugeon formalism. This similarity is discussed in more detail in § 4.

In the same way, $Q$ defines its own invariant subspace, according to the decomposition of the Lagrangian given by Eq. (3.18). However, it is impossible to carry out the integrations with respect to the additional degrees of freedom in this form because the subspace is the Hilbert space of $A_\mu^g$ and $B$ (along with their ghosts) for the Landau gauge. We therefore need to perform the same transformations as in the case that we obtained Eq. (3.15) in order to separate the additional degrees of freedom from the original ones. Then the corresponding decomposition of the Lagrangian becomes

$$L' = \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu + i\bar{c} \partial^\mu D_\mu c - i\tilde{\delta}' \left[ \bar{\eta} \left( -f[g^{-1}, A] + \alpha/2 \Phi \right) \right]$$

in which $\tilde{\delta}'$ differs from $\tilde{\delta}$ due to the change of the rule for $\bar{\eta}$ in Eq. (3.17) to $\tilde{\delta}'\bar{\eta} = i\Phi$, but is, of course, nilpotent and anti-commutes with $\delta_B$. Note that the positions of the two pairs of ghost fermions, $(\bar{c}, c)$ and $(\bar{\eta}, \eta)$, are exchanged under the change of variables that brings $L$ to $L'$. With this change of variables, the BRST charge $Q$ is transformed to $\tilde{Q}'$, corresponding to $\tilde{\delta}'$ above. The transformed charge $\tilde{Q}'$ then defines the Hilbert space for the Landau gauge of the original variables as its invariant
subspace. As the counterpart to the transformation \( Q \mapsto \tilde{Q}, \tilde{Q} \) of the original system transforms to \( Q' \), which is such that the relation \( Q_B = Q + \tilde{Q} = Q' + \tilde{Q}' \) holds. Hence we again observe the hybrid of the Landau gauge and the \( \alpha \)-gauge of the original degrees of freedom in the structure of the total Hilbert space.

3.3. Gauge fixing without averaging over the \( c \)-number function

To this point, we have presented our understanding of the ’t Hooft average from the point of view of extended gauge symmetry and gauge fixing for it. In the derivation of the PI given in Eq. \((2.7)\), however, we have used the same technique (averaging with respect to a \( c \)-number function) again in Eq. \((3.7)\) to find the Gaussian weight in the second line of Eq. \((2.7)\). In this sense, we have not yet realized our entire goal for this paper. Here we show that we can avoid the use of this technique and discuss the difference between this new prescription and that utilized in previous sections.

Returning to the divergent PI \((3.4)\) considered in §3.1, let us reconsider the use of the FP trick for \( Z_{\text{div}} \). By making use of the facts that (i) the identity \((3.6)\) holds for any \( c \)-number function \( C \) and (ii) the FP determinant \( \Delta_{\text{FP}}[A, g, C] \) becomes \( \Delta[A] \) in front of \( \delta(f[g, A] - C) \), we have multiplied \((3.7)\) by a Gaussian identity of \( C \), regarding \( C \) as identical to the integration variable of the Gaussian identity. These facts also ensure the validity of using

\[
1 = \Delta_{\text{FP}}[A, g, -B/2] \int D_h \delta \left( f[gh, A^{h^{-1}}] + \frac{\alpha}{2} B \right) \tag{3.23}
\]

instead of Eq. \((3.6)\) and the Gaussian averaging with respect to \( C \) afterward. Factorizing out the gauge volume from \( Z_{\text{div}} \) again, we obtain

\[
Z = \int DA DB Dg I[A, B, g] \Delta[A] \delta \left( f[g, A] + \frac{\alpha}{2} B \right). \tag{3.24}
\]

Then, because by integrating \( g \) out with the measure \( Df = Dg \Delta[A^g] \), we immediately return to Eq. \((2.4)\) for the \( \alpha \)-gauge, we can easily check that \( Z \) given above is also equivalent to the PI for the original degrees of freedom. We are therefore convinced that the PI given by Eq. \((3.24)\), with the extended degrees of freedom, is useful for another gauge fixing procedure of the extended gauge invariance, and we can avoid using the ’t Hooft average with this new prescription of the gauge fixing.

Let us now consider the Lagrangian in the PI given in Eq. \((3.24)\) from the point of view of the BRST invariance. If we express the Fourier transform of the delta function in Eq. \((3.24)\) as a functional integral with respect to \( \Phi \), this Lagrangian reads

\[
L_\delta = -\frac{1}{4} F^\mu \nu F_{\mu \nu} + B \partial^\mu A^g_\mu + i\bar{\eta} \partial^\mu D_\mu \eta^g + \Phi f[g, A] + \frac{\alpha}{2} \Phi B + i\bar{c} \partial^\mu D_\mu c. \tag{3.25}
\]

As will become evident, the difference between this Lagrangian and that in Eq. \((3.13)\) appears only in the term proportional to \( \alpha \). Hence, we can define BRST transformations that are identical to those given in Eqs. \((3.16)\) and \((3.17)\). Accordingly, we obtain similar decompositions of this Lagrangian, given by

\[
L_\delta = -\frac{1}{4} F^\mu \nu F_{\mu \nu} + B \partial^\mu A^g_\mu + i\bar{\eta} \partial^\mu D_\mu \eta^g - i\delta \left( \bar{c} \left( f[g, A] + \frac{\alpha}{2} B \right) \right) \tag{3.26}
\]
and

\[ \mathcal{L}_\delta = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu + i \bar{c} \partial^\mu D_\mu c + \frac{\alpha}{2} B^2 - i \tilde{\delta}' \left[ \bar{\eta} \left( -f[g, A] + \frac{\alpha}{2} B \right) \right], \]  

(3.27)

corresponding to Eqs. (3.18) and (3.19), respectively. In the same way, in accordance with the nilpotency of the total BRST transformation \( \delta_B = \delta + \tilde{\delta} \), we can rewrite the Lagrangian as

\[ \mathcal{L}_\delta = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + i \delta_B \left[ \bar{c} \left( \partial^\mu A_\mu + \frac{\alpha}{2} B \right) - \bar{\eta} \partial^\mu A_\mu \right]. \]  

(3.28)

In analogy to the change of the Lagrangian \( \mathcal{L} \) to \( \mathcal{L}' \), we can make a change of variables in \( \mathcal{L}_\delta \) appearing in Eq. (3.26) to obtain

\[ \mathcal{L}_\delta' = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu + i \bar{c} \partial^\mu D_\mu c - i \tilde{\delta}' \left[ \bar{\eta} \left( -f[g^{-1}, A] + \frac{\alpha}{2} B \right) \right], \]  

(3.29)

where \( \tilde{\delta}' \) is the same as that defined above. Therefore we observe that the total Hilbert space of the extended system described by the Lagrangian (3.25) has exactly the same structure as that described by the Lagrangian (3.16).

We have thus obtained another method of gauge fixing for the extended gauge symmetry. As explained at the beginning of this section, the advantage of this second method of gauge fixing is that, with it, we never need to carry out the averaging according to the Gaussian weight with respect to an unknown \( c \)-number field. Hence we have realized the main goal of this paper by formulating this new method. It is important, however, to note that the same prescription cannot be applied to the case of the original form of the ’t Hooft average; it can be applied only after the extension of the degrees of freedom needed to obtain the extended gauge symmetry. In this regard, it is interesting that, as was pointed out by Nakanishi\(^{17}\) and also shown by Yokoyama\(^3\) for the NL formalism\(^{11}\) of QED, we cannot change the gauge parameter \( \alpha \) in a consistent way without introducing additional degrees of freedom (a gaugeon and its associated field) for the quantum gauge degree of freedom. Therefore, our formulation of the PI with additional degrees of freedom may have some connection to the gaugeon formalism\(^3,8,18,19\). This is the subject of the next section. However, before closing this section, we give some discussion of the PIs in Eqs. (2.7) and (3.24). Interestingly, the resemblance of the decomposition of the Lagrangian in Eq. (3.29) to that in Eq. (3.22) suggests yet another way of deriving these PIs. As we have already shown for the case of Eq. (2.7), when we write the Lagrangian of the PI in the form of Eq. (3.22), the corresponding PI can be regarded as a product of \( Z \) in Eq. (2.1) and the identity

\[ 1 = \int Dg \ D\Phi \ D\bar{\eta} \ D\eta \exp \left( \int d^4x \ \tilde{\delta}' \left[ \bar{\eta} \left( -f[g^{-1}, A] + \frac{\alpha}{2} B \right) \right] \right), \]  

(3.30)

provided that we perform the gauge transformation \( A_\mu \rightarrow A_\mu^0 \) afterward. In the same sense, the identity

\[ 1 = \int Dg \ D\Phi \ D\bar{\eta} \ D\eta \exp \left( \int d^4x \ \delta' \left[ \bar{\eta} \left( -f[g^{-1}, A] + \frac{\alpha}{2} B \right) \right] \right), \]  

(3.31)
can be multiplied by \( Z \) of Eq. (2.1) to obtain the PI in Eq. (3.24). We are, therefore, convinced that the procedure of implementing the extended gauge symmetry and the gauge fixing by means of the FP trick is equivalent to the multiplication of an identity that is given by the product of the bosonic and fermionic functional determinant from a PI of the extended degrees of freedom. With this understanding, we can avoid the procedure of averaging over an unknown \( c \)-number function, even for the case of the PI given in Eq. (2.7). If we start our formulation with PI in Eq. (2.4) for the \( \alpha \)-gauge and multiply Eq. (3.30) or (3.31) after replacing \( \alpha \) in these identities with \( \delta \alpha = \alpha' - \alpha \), we obtain the PI of the extended system as a hybrid of the \( \alpha \)-gauge and \( \alpha' \)-gauge. This extended PI then reduces to that suitable for the \( \alpha' \)-gauge of the original system by integrating out the additional degrees of freedom after the gauge transformation \( A_\mu \mapsto A_\mu^{\alpha'} \).

§4. Relation to the gaugeon formalism

We start this section with a brief review of the essence of Yokoyama’s gaugeon formalism\(^3\) for QED. The gauge fixing term \( B \partial_\mu A_\mu \) is extended to \( B \partial_\mu A_\mu - \varphi \partial_\mu \partial_\mu \theta \) and then the extended Lagrangian should be invariant under the \( q \)-number gauge transformation given by

\[
A_\mu \mapsto A_\mu + a \partial_\mu \theta, \quad \varphi \mapsto \varphi + a B,
\]

leaving \( B \) and \( \theta \) intact. (This is a global symmetry with the global parameter \( a \). The resemblance to the BRST invariance should be noted.) Because \( \theta \) describes the quantum gauge degree of freedom of \( A_\mu \), it is called a gaugeon field and appears in the extended Lagrangian with its partner field \( \varphi \). They add a term to the Lagrangian that breaks the symmetry above, and we obtain

\[
\mathcal{L}_I = \mathcal{L}_0 + B \partial_\mu A_\mu - \varphi \partial_\mu \partial_\mu \theta + \frac{\epsilon_1}{2} (\varphi + a_0 B)^2,
\]

where \( \mathcal{L}_0 \) represents the gauge invariant Lagrangian of genuine QED, for type I gaugeon formalism. For the case of the type II gaugeon theory, the Lagrangian is given by

\[
\mathcal{L}_{II} = \mathcal{L}_0 + B \partial_\mu A_\mu - \varphi \partial_\mu \partial_\mu \theta + \frac{\epsilon_2}{2} (\varphi + a_0 B) B.
\]

If we integrate out \( \theta \) with \( \varphi \), the Lagrangians of both systems reduce to that of the original degrees of freedom for the \( \alpha \)-gauge; \( \alpha = \epsilon_1 a_0^2 \) for a type I system and \( \alpha = \epsilon_2 a_0 \) for a type II system. However, if we carry out the \( q \)-number gauge transformation given by Eq. (4.1) first, the gauge parameter \( \alpha \) of the resulting reduced system becomes \( \alpha' = \epsilon_1 (a + a_0)^2 \) for the type I case and \( \alpha' = \epsilon_2 (a + a_0) \) for the type II case. Therefore we can change the gauge parameter for the reduced system by performing the \( q \)-number gauge transformation before integrating out the additional degrees of freedom. The difference between the type I and type II formalisms is in their rules

\(^3\) The corresponding notation for the fields \((\theta, \varphi, B)\) in the original paper by Yokoyama\(^3\) is \((B, B_2, B_1)\). Koseki et al. adopt the notation \((Y, Y_\ast, B)\) in Ref. 8).
for changing $\alpha$ to $\alpha'$ via Eq. \[4.1\]. For the case of a type I system, the rule is $\alpha' = (1 + a/a_0)^2 \alpha$, while for a type II system, it is $\alpha' = (1 + a/a_0) \alpha$.

The BRST invariant formulation of the gaugeon formalisms above were given by Izawa for the type II case\(^8\) and by Koseki, Sato and Endo\(^8\) for both types of gaugeon formalism. The generalization of the type I theory discussed above, in which the rule for changing the gauge parameter in order to admit any real values for $\alpha$, was also formulated by Endo\(^9\). Let us examine here whether we can generalize the derivation of the PI carried out by Koseki et al. in Ref. 8) to non-abelian gauge theories. The key to their derivation seems to be multiplication by unity expressed by the right-hand side of an identity of the form $1 = \det \Box^{-1} \cdot \det \Box$, which should further be rewritten as a PI with BRST invariance. The corresponding identity in our case is given by Eq. \[3.30\]. Then, we can follow the procedure employed in Ref. 8) to obtain a non-abelian generalization of the $\mathcal{L}_{\text{YK}}$ that includes three parameters. For the case of an abelian theory, Koseki et al. observed that gaugeon formalism can be reproduced by setting these three parameters as $\alpha_1 = \pm 1 = \varepsilon$, $\alpha_2 = \varepsilon a$, $\alpha_3 = \varepsilon a^2$ for the type I formalism and as $\alpha_1 = 0$, $\alpha_2 = 1/2$, $\alpha_3 = a$ for the type II theory, respectively. The corresponding Lagrangians in our formulation are those appearing in Eqs. \[3.13\] and \[3.25\]. In these Lagrangians, however, we do not have any free parameter that represents $a$ in the gaugeon formalism. Rather, we have to set $a = 1$ as well as $\alpha_1 = \alpha$ for Eq. \[3.13\] and $\alpha_2 = \alpha/2$ for Eq. \[3.25\]. Thus, our results partially generalize those of Ref. 8). The reason why the three parameters in gaugeon formalism of Yokoyama and Kubo are not useful in non-abelian cases is explained below, but it is important to keep in mind that we can always perform the $q$-number gauge transformation \[4.1\] to eliminate the parameters $a_0$ in Eqs. \[4.2\] and \[4.3\]. From the point of view of form invariance in the gaugeon formalism, the three parameters are fundamental. But as seen below, the concept of form invariance cannot be regarded as a generic one for linear covariant gauges when we extend the formalism to non-abelian systems.

If the global symmetry under the $q$-number gauge transformation \[4.1\] observed above generalizes to non-abelian cases as well, our formulation of the PIs given in Eqs. \[2.7\] and \[3.21\] can be regarded as gaugeon formalisms for non-abelian gauge theories. However, this is not the case: Due to the fact that we need to change $A_\mu$ in $gD_\mu g^{-1}$, the possible form given by $B \partial^\mu A_\mu + i \Phi \partial^\mu (gD_\mu g^{-1})$ for the generalization of $B \partial^\mu A_\mu - \varphi \partial^\mu \partial_\mu \theta$ cannot be invariant under any finite gauge transformation of $A_\mu$ in combination with a shift in $\Phi$ proportional to $B$. To resolve this problem, there can exist only one possibility, that is, that we stipulate $g = e^{i\Theta}$ to be infinitesimal and consider the $q$-number gauge transformation within this infinitesimal one-dimensional subgroup,

\[ A_\mu \rightarrow A_\mu + a D_\mu \Theta, \quad \Phi \rightarrow \Phi + a B, \quad (4.4) \]

while disregarding the change of $A_\mu$ in $D_\mu \Theta$. Then we find that $B \partial^\mu A_\mu - \Phi \partial^\mu D_\mu \Theta$ is invariant under this $q$-number gauge transformation. (Again, the similarity to the BRST invariance should be noted.) Therefore, if we accept this restriction, our formulation of PIs developed in this paper can be regarded as the gaugeon formalism for non-abelian gauge theories. Despite the restriction stated above, it is useful for
treating gauge fields in perturbation theory. In this case, our PI becomes that with the Lagrangian
\[
\mathcal{L}_I = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu - \Phi \partial^\mu D_\mu \Theta + \frac{\alpha}{2} \Phi^2 + i \bar{\eta} \partial^\mu D_\mu \eta + i \bar{c} \partial^\mu D_\mu c \tag{4.5}
\]
in the case of the type I formalism, and the Lagrangian
\[
\mathcal{L}_{II} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu A_\mu - \Phi \partial^\mu D_\mu \Theta + \frac{\alpha}{2} \Phi B + i \bar{\eta} \partial^\mu D_\mu \eta + i \bar{c} \partial^\mu D_\mu c, \tag{4.6}
\]
in the case of the type II formalism, corresponding to those given in Eqs. (2.7) and (3.24), respectively.

Turning now to the formulation for finite gauge transformations, we consider the possibility of changing the gauge parameter through some change of variables in our PIs. In a related context, Yokoyama, Takeda and Monda\textsuperscript{5}) have formulated a gauge covariant canonical quantization of non-abelian gauge theories. Its BRST symmetric version was then constructed by Abe and also by Koseki, Sato and Endo.\textsuperscript{9}) In their formulation, there exists a parameter that can be changed under a $q$-number gauge transformation, but this parameter cannot be identified with $\alpha$ in the linear covariant gauge, $\partial^\mu A_\mu + \alpha B = 0$. Due to this discrepancy, the propagator of the gauge field becomes highly complicated and different from that of the standard Lorentz-covariant formulation. This sharply contrasts with the simplicity of 't Hooft’s technique in the path integral formalism. To inquire further into this matter would lead us into a specialized area that is irrelevant to the main subject here, and such digression would undoubtedly obscure the outline of our argument. We thus continue to examine the possibility of formulating our PIs in a gauge covariant way.

Since, if we begin with the Landau gauge, as we have done throughout this paper, the gauge parameter $\alpha$ enters the PI from the Gaussian identity (3.30) [being equivalent to Eq. (3.2)] or from Eq. (3.31) for the second type of gauge fixing, we may change $\alpha$ to an arbitrary value by hand, using the $\alpha$-independence of these formulae. However, we may also change $\alpha$ by means of some change of variables in these identities. This can be done, as shown in Appendix A for the Gaussian identity, by solving the equation $f[g[g', A], A] = \gamma f[g', A]$ for a given constant $\gamma$. We therefore seek a change of variables from $g$ to $g'$ such that the scaling of $f[g, A]$ is generated. If the solution is given by $g = g[g', A]$ as a functional of $g'$ and $A_\mu$, the PI in terms of $g$ with the gauge parameter $\alpha$ is transformed to that of $g'$ with $\alpha' = \alpha/\gamma^2$ for the case of Eq. (3.30) and $\alpha' = \alpha/\gamma$ for the identity (3.31).

Although the change of variables from $g$ to $g'$ needed to satisfy the scaling of $f[g, A]$ is quite complicated, we can confirm the validity of our prescription as follows. Setting $f = -f[g^{-1}, A]$, the identities (3.30) and (3.31) are simplified as
\[
1 = \int \mathcal{D}f \mathcal{D}\Phi \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left( \int d^4x \delta' \left[ \bar{\eta} \left( f + \frac{\alpha}{2} V_i \right) \right] \right), \quad V_1 = \Phi, \ V_{II} = B, \tag{4.7}
\]
in which the BRST transformation becomes
\[
\delta' f = \eta, \ \delta' \bar{\eta} = i \Phi, \ \delta' \eta = \bar{\delta} V_i = 0. \tag{4.8}
\]
Since these transformations and the measure of the integration (4.7) are invariant under the scaling of these variables given by

\[ f \mapsto e^{\rho} f, \ \Phi \mapsto e^{-\rho} \Phi, \ \bar{\eta} \mapsto e^{-\rho} \bar{\eta}, \ \eta \mapsto e^{\rho} \eta, \]

(4.9)
in addition to \( B \mapsto B \), we can rewrite Eq. (4.7) as

\[ 1 = \int Df \ D\Phi \ D\bar{\eta} \ D\eta \exp \left( \int d^4x \tilde{\delta} \left[ \bar{\eta} \left( f + \frac{\alpha_1^{(\rho)}}{2} V_i \right) \right] \right), \]

(4.10)
in which \( \alpha_1^{(\rho)} = e^{-2\rho} \alpha \) and \( \alpha_II^{(\rho)} = e^{-\rho} \alpha \), corresponding to the definition of \( V_i \) above. Thus, we confirm the \( \alpha \)-independence of these identities. (Though this was evident from very beginning.) The important point here is that the BRST transformation, given by Eq. (4.8), commutes with the scaling in Eq. (4.9). The change of variables from \( g \) to \( g' \) considered above for changing \( \alpha \) is identical to the scaling (4.9) when expressed in terms of \( f, \Phi \) and their ghosts. Furthermore, the BRST transformation \( \tilde{\delta} \) given above returns to that for \( g, \Phi \) and their ghosts if we go back to the expression in terms of these variables. Hence, our method of changing \( \alpha \) described above also commutes with the BRST transformation. It is thus clear that the structure of the total Hilbert space is preserved under such a change of the gauge parameter \( \alpha \). In view of these facts, we conclude that the method presented here can be regarded as a gauge covariant formulation that is useful even in the case of finite gauge transformations for non-abelian theories.

§5. Summary

We have proposed to regard the ’t Hooft average\(^2\) as a PI over additional unphysical degrees of freedom. This allows us to formulate a PI with extended gauge symmetry.\(^6\) Extension of the gauge invariance and the reduction obtained by integrating the additional degrees out from the extended PI after the gauge fixing for this extended gauge invariance are the keys to understanding ’t Hooft’s technique. The remarkable feature of the extended PI thus obtained is that it can be viewed as a hybrid of two systems for different values of the gauge parameter of the original degrees of freedom; different methods of integrating the additional degrees of freedom, that is, different arrangements of the Lagrangian and the gauge transformation of gauge fields in combination with extended degrees of freedom, result in PIs for different gauges of the original degrees of freedom.

We have also carried out analyses of the BRST invariance for the extended system, finding as the symmetry of the extended system two types of BRST transformations, that associated with the extended gauge invariance and that resulting from the trivial nature of the systems behind the identities useful for gauge fixing. This latter BRST invariance can be regarded as a condition for specifying the Hilbert space of the original degrees of freedom for a specific value of the gauge parameter. Accordingly, by integrating the additional degrees of freedom out of the extended

\(^{2}\) It seems that this was partly recognized by Koseki et al. in Ref. 8).
PI, we observe that the total BRST charge reduces to that of Kugo and Ojima\cite{12} and obtain the BRST invariant Lagrangian of the original degrees of freedom for this specific gauge. In this sense, the BRST invariance of the extended system can be regarded as a proper extension of the BRST symmetry in the original system. Such a structure of the total Hilbert space, viewed in the light of the BRST invariance, is very similar to that found by Koseki et al. in Ref. 8) for the BRST invariant formulation of Yokoyama’s gaugeon formalism.\cite{3} In accordance with the hybrid structure of the extended PI, the Hilbert space of the extended system has the same property, and this allows us to write the total BRST charge in two ways.

The observation that a PI formulated by means of a gauge fixing procedure from a gauge invariant one can be rewritten as a product of a PI of the original degrees of freedom in a specific gauge with another PI of the additional degrees of freedom which represents an identity [i.e., \((3.30)\) or \((3.31)\)], as a cancellation of bosonic and fermionic functional determinants with each other, provides another way of finding the gauge fixed PIs of the extended system. We can regard the identity mentioned above as a generalization of similar one, utilized to formulate a PI by Koseki et al. in Ref. 8), to non-abelian theories. Furthermore, the use of such an identity plays the central role in formulating a PI with extended degrees without averaging over any \(c\)-number function. (Avoiding the average over an unknown \(c\)-number function is important when we consider the canonical quantization of the system, as there exists no nice prescription for understanding such a procedure in the operator formalism.) It is noteworthy that there exist systems with BRST invariance underlying ’t Hooft’s Gaussian identity and also in a trivial relation from the functional integration of a delta function. With this observation, we have noted that setting the \(c\)-number function \(f\) in the ’t Hooft average to \(f = -i\partial^\mu(gD_\mu g^{-1})\) can be regarded as a Nicolai map.\cite{15} We then recognize that the systems possessing these identities are the non-abelian counterparts of the Froissart model\cite{14} with the BRST symmetry discussed in detail by Kashiwa in Ref. 10).

The invariance under the scaling of the variables, which commutes with the BRST transformation, in the identities \((3.30)\) and \((3.31)\) allows us to change the gauge parameter with this change of variables. Taking this fact into account, in conjunction with the hybrid structure seen in the PIs, and also in the total Hilbert space of the extended system, we recognize that the total Hilbert space involves all linear covariant gauges of the original system. Since we can move freely in this total Hilbert space to change the resulting gauge parameter, we conclude that the ’t Hooft average, viewed from the point of view of our formulation of PIs with extended degrees of freedom, is a generalization of Yokoyama’s gaugeon formalism to non-abelian gauge theories.

Acknowledgements

The author would like to thank Professor T. Kashiwa for fruitful discussions. He is also grateful to a referee for many valuable suggestions. In particular, the analyses of the BRST invariance of the extended system could not have been completed without the referee’s advice.
Appendix A

BRST Invariant Formulation of a Gaussian Identity

In this appendix we show that we can find a BRST invariance in a Gaussian identity. Although a thorough explanation of the relation between a Gaussian identity and the BRST invariance can be found in Ref. 10, here we present our own description, which is useful for understanding presented in the main text, in particular the additional BRST invariance of the PI in Eq. (2.7).

Let us consider the identity

$$1 = \left( \frac{i}{2\pi \alpha} \right)^{n/2} \int d^n \varphi \exp \left( -\frac{i}{2\alpha} \varphi^2 \right), \quad \varphi^2 = \sum_{a=1}^{n} \varphi_a^2. \tag{A.1}$$

If we regard $\varphi$ as a set of functions of $n$ independent variables $x$, this can be rewritten as

$$1 = \left( \frac{i}{2\pi \alpha} \right)^{n/2} \int d^n x \left| \frac{\partial \varphi}{\partial x} \right| \exp \left( -\frac{i}{2\alpha} \varphi^2(x) \right), \tag{A.2}$$

where $|\partial \varphi/\partial x|$ is the Jacobian, assumed to be positive definite hereafter, of the change of variables through $\varphi = \varphi(x)$. Apparently, this integral is invariant under the change of variables $x \mapsto x'$, though the integrand undergoes a change of functional form through $\varphi(x(x')) = \varphi'(x')$ and the Jacobian $|\partial x/\partial x'|$. This invariance can be seen as the BRST symmetry of the exponent in the integrand of

$$1 = \int d^n x d^n k \left( \frac{2\pi}{i} \right)^n (d\bar{c} dc)^n \exp \left( ik\varphi(x) + \frac{i\alpha}{2} k^2 - \bar{c}D(x)c \right), \quad D(x) = \frac{\partial \varphi}{\partial x}, \tag{A.3}$$

where $c$ and $\bar{c}$ are a set of Grassmann variables, under

$$x \mapsto x' = x + \lambda c, \quad \bar{c} \mapsto \bar{c}' = \bar{c} + i\lambda k, \quad k \mapsto k, \quad c \mapsto c \tag{A.4}$$

with $\lambda$ a Grassmann parameter. If we set $\delta x = c$ and $\delta \bar{c} = ik$ in the above definition of the BRST transformation, we can express the trivial nature of the Gaussian identity as

$$1 = \int \frac{d^n x \, d^n k}{(2\pi)^n} (d\bar{c} dc)^n \exp \left( \delta \left[ \bar{c} \left\{ \varphi(x) + \frac{\alpha}{2} k \right\} \right] \right). \tag{A.5}$$

This is identical to the BRST invariance underlying the original Gaussian identity (A.1).

Because the right-hand side of Eq. (A.1) is actually independent of $\alpha$, we can change its value by hand to $\alpha'$ without affecting the above argument. It is useful, however, to note that the change in $\alpha$ can be generated by a change of variables. To see this, let us suppose that $x$ is connected to the new variable $y$ through the relation $x = x(y)$, so that $\varphi(x(y)) = \gamma \varphi(y)$ holds for a constant $\gamma$. Then scaling $k$ as $\gamma k \mapsto k$, we find

$$k\varphi(x) + \frac{\alpha}{2} k^2 \mapsto k\varphi(y) + \frac{\alpha}{2\gamma^2} k^2. \tag{A.6}$$
Hence, we can carry out the change $\alpha \rightarrow \alpha' = \alpha / \gamma^2$ through this change of variables, though it is equivalent, as explained above, to replacing $\alpha$ with $\alpha'$ in Eq. (A.1) by hand.

Let us now consider the Gaussian identity (2.6) from the point of view of the argument above. As is now clear, the original form given by Eq. (2.3) corresponds to the identity (A.1), and setting $f = f[g, A] = -i\partial^\mu(gD_\mu g^{-1})$ is interpreted as the analogue of $\varphi = \varphi(x)$ above. By exponentiating the Jacobian $\Delta[A^0]$ in terms of a fermionic PI, we obtain

$$1 = \int Dg \, D\Phi \, D\eta \, D\bar{\eta} \, \exp \left( i \int d^4x \left\{ \Phi f[g, A] + \frac{\alpha}{2} \Phi^2 + i\bar{\eta} \partial^\mu D_\mu \eta \right\} \right),$$

where $\eta^0 = g\eta^{-1}$, and the Fourier transform of the Gaussian weight has also been carried out. If we parametrize $g$ as $g = e^{i \theta}$, we find that the quadratic part of the Lagrangian in this PI is given by

$$L^{0}_{\text{BRST}} = -\Phi \partial^\mu \partial_\mu \theta + \frac{\alpha}{2} \Phi^2 + i\bar{\eta} \partial^\mu \partial_\mu \eta.$$  (A.8)

Because this Lagrangian is simply the BRST invariant version of the Froissart model, we recognize that the Lagrangian in the PI (A.7) can be understood as the non-abelian generalization of the system discussed in the appendix of Ref. 10). Then, the change of variables from $f$ to $g$ introduced in §2 can be regarded as the Nicolai map corresponding to the trivial nature of this system. As a consequence, the system is invariant under the BRST transformation

$$-i\gamma \delta g^{-1} = \lambda g \eta^{-1} = \lambda \eta^0, \quad \delta \bar{\eta} = i \lambda \Phi, \quad \delta \eta = i \lambda \eta^2.$$  (A.9)

By setting $\delta = \lambda \delta$ in Eq. (A.9), we can rewrite (A.7) as

$$1 = \int Dg \, D\Phi \, D\eta \, D\bar{\eta} \, \exp \left( \int d^4x \, \delta \left( \bar{\eta} \left( f[g, A] + \frac{\alpha}{2} \Phi \right) \right) \right).$$  (A.10)

This explains the trivial nature of the Gaussian identity (2.6) in terms of the BRST invariance. Finally, we note that we can scale the gauge parameter $\alpha$ by transforming from $g$ to $g'$, so that $f[g'[g', A] = \gamma f[g', A]$, as done above in Eq. (A.6).

References

1) L. D. Faddeev and V. N. Popov, Phys. Lett. B 25 (1967), 29.
2) L. D. Faddeev, Theor. Math. Phys. 1 (1970), 1.
3) G. ’t Hooft, Nucl. Phys. B 33 (1971), 173.
4) K. Yokoyama, Prog. Theor. Phys. 51 (1974), 1956.
5) K. Yokoyama and R. Kubo, Prog. Theor. Phys. 52 (1974), 290.
6) K. Yokoyama, Prog. Theor. Phys. 59 (1978), 1699.
7) K. Yokoyama, M. Takeda and M. Monda, Prog. Theor. Phys. 60 (1978), 927.
8) K. Yokoyama, Prog. Theor. Phys. 60 (1978), 1167.
9) K. Yokoyama, M. Takeda and M. Monda, Prog. Theor. Phys. 64 (1980), 1412.
10) O. Baberon, F. Schaposnik and C. Viallet, Phys. Lett. B 177 (1986), 385.
11) K. Harada and I. Tsutsui, Phys. Lett. B 183 (1987), 311.
12) K. Harada and I. Tsutsui, Prog. Theor. Phys. 78 (1987), 878.
C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42 (1975), 127; Ann. of Phys. 98 (1976), 287.
I. V. Tyutin, Lebedev preprint FIAN No. 39 (1975), unpublished.
M. Koseki, M. Sato and R. Endo, Prog. Theor. Phys. 90 (1993), 1111.
M. Abe, “The Symmetries of the Gauge-Covariant Canonical Formalism of Non-Abelian Gauge Theories”, Master Thesis, Kyoto University, 1985.
M. Koseki, M. Sato and R. Endo, Bull of Yamagata Univ. Nat. Sci. 14 (1996), 15; hep-th/9510102.
T. Kashiwa, Prog. Theor. Phys. 70 (1983), 1124.
N. Nakanishi, Prog. Theor. Phys. 35 (1966), 1111; Prog. Theor. Phys. 49 (1973), 640; Prog. Theor. Phys. 52 (1974), 1929.
B. Lautrup, K. Dan. Vidensk. Selsk. Mat. -Fys. Medd. 35 (1967), No. 11.
T. Kugo and I. Ojima, Phys. Lett. B 73 (1978), 459; Prog. Theor. Phys. Suppl. No. 66 (1979), 1.
T. Kashiwa, Prog. Theor. Phys. 66 (1981), 1858.
T. Kashiwa and M. Sakamoto, Prog. Theor. Phys. 67 (1982), 1927.
M. Froissart, Nuovo Cim. Suppl. 14 (1959), 197.
H. Nicolai, Phys. Lett. B 89 (1980), 341; Nucl. Phys. B 176 (1980), 419.
V. N. Gribov, Nucl. Phys. B 139 (1978), 1.
I. M. Singer, Commun. Math. Phys. 60 (1978), 7.
K. Fujikawa, Nucl. Phys. B 223 (1983), 218.
N. Nakanishi, Prog. Theor. Phys. Suppl. No. 51 (1974), 952.
K. Izawa, Prog. Theor. Phys. 88 (1992), 759.
R. Endo, Prog. Theor. Phys. 90 (1993), 1121.