We describe the present status of the pion distribution amplitude as it originated from two sources: (i) a nonperturbative approach, based on QCD sum rules with nonlocal condensates and (ii) a NLO QCD analysis of the CLEO data on $F_{\gamma^*\pi}(Q^2)$, supplemented by the recent high-precision lattice calculations of the second moment of the pion distribution amplitude.

Keywords: Pion Distribution Amplitude; QCD Sum Rules; Lattice QCD; CLEO data.

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1. Pion distribution amplitude from QCD sum rules

The pion distribution amplitude $\phi_\pi$, describes the transition of a pion $\pi(P)$ to a pair of valence quarks $u$ and $d$, separated by the (straight) Fock–Schwinger connector $E$, with corresponding momentum fractions $xP$ and $\bar{x}P$, $(\bar{x} \equiv 1 - x)$. In order to obtain the pion DA we use a QCD sum rule (SR) approach with non-local condensates (NLC), employing for the scalar and vector condensates the same minimal model as in the QCD SRs and on the lattice:

$$\langle \bar{q}(0)q(z) \rangle = \langle \bar{q}q \rangle e^{-|z^2|\lambda^2_\varphi/8} ; \quad \langle \bar{q}(0)\gamma_\mu q(z) \rangle = i\frac{z^2}{4} \frac{2\alpha_s\pi \langle \bar{q}q \rangle^2}{81} e^{-|z^2|\lambda^2_\varphi/8}. \quad (2)$$

The nonlocality parameter $\lambda^2_\varphi = \langle k^2 \rangle$ characterizes the average momentum of quarks in the QCD vacuum and has been estimated in QCD SRs and on the lattice.

$$\lambda^2_\varphi = 0.45 \pm 0.1 \text{ GeV}^2.$$ For the quark-gluon-antiquark condensates we use

$$\langle \bar{q}(0)\gamma_\mu (-g \hat{A}_\nu(y))q(x) \rangle = (y_\mu x_\nu - g_\mu_\nu(yx))M_1(x^2, y^2, (y - x)^2),$$

$$+ (y_\mu y_\nu - g_\mu_\nu y^2)M_2(x^2, y^2, (y - x)^2),$$

$$\langle \bar{q}(0)\gamma_5 \gamma_\mu (-g \hat{A}_\nu(y))q(x) \rangle = i\varepsilon_{\mu\nuyx}M_3(x^2, y^2, (y - x)^2),$$

with $(A_{1,2,3} = A_0 (-\frac{3}{2}, 2, \frac{3}{2}))$

$$M_i(x^2, y^2, z^2) = A_i \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma f_i(\alpha, \beta, \gamma) e^{(\alpha x^2 + \beta y^2 + z^2)/4}. \quad (3)$$
Fig. 1. (a) “Bunch” of pion DAs extracted from NLC QCD sum rules. \(F^\gamma_{\gamma \pi}(Q^2)\) for comparison we show here also the asymptotic DA (dotted line) and Chernyak–Zhitnitsky (CZ) DA (dashed red line). (b) Allowed values of the pion DA parameters \(a_2\) and \(a_4\) are bounded by the solid blue line. Region bounded by the dotted red line represents results obtained in the minimal model. Both panels show results for the value \(\lambda^2 = 0.4\) GeV.

The minimal model of nonlocal QCD vacuum suggests the following Ansätze

\[
f_i(\alpha, \beta, \gamma) = \delta(\alpha - \Lambda) \delta(\beta - \Lambda) \delta(\gamma - \Lambda)
\]

with \(\Lambda = \frac{1}{2}\lambda_\pi^2\) and faces problems with QCD equations of motion and gauge invariance of 2-point correlator of vector currents. In order to fulfill QCD equations of motion exactly and minimize non-transversity of \(V^\pi_{\gamma \gamma}\) correlator we suggest the improved model of QCD vacuum with

\[
f_i^{\text{imp}}(\alpha, \beta, \gamma) = (1 + X_i \partial_x + Y_i \partial_y + Z_i \partial_z) \delta(\alpha - x\Lambda) \delta(\beta - y\Lambda) \delta(\gamma - z\Lambda),
\]

where \(x = y, \Lambda = \frac{1}{2}\lambda_\pi^2\) and

\[
X_1 = +0.082; \quad X_2 = -1.298; \quad X_3 = +1.775; \quad x = 0.788;
\]

\[
Y_1 = -2.243; \quad Y_2 = -0.239; \quad Y_3 = -3.166; \quad y = 0.212.
\]

Then the NLC sum rules for the pion DA produce a “bunch” of 2-parameter models at \(\mu^2 = 1.35\) GeV\(^2\) (with \(\varphi^{as}(x) \equiv 6x(1-x)\))

\[
\varphi_\pi(x) = \varphi^{as}(x) \left[1 + a_2 C_2^{3/2}(2x - 1) + a_4 C_4^{3/2}(2x - 1)\right],
\]

shown in Fig. 1b. Allowed values of this bunch parameters \(a_2\) and \(a_4\) are shown in Fig. 1b with coordinates of the central point to be \(a_2 = 0.268\) and \(a_4 = -0.186\). These values correspond to \(\langle x^{-1}\rangle_{\text{bunch}} = 3.24 \pm 0.20\), which is in agreement with the result of an independent sum rule, viz., \(\langle x^{-1}\rangle_{\text{SR}} = 3.40 \pm 0.34\).

We emphasize here that BMS model 4, shown in Fig. 1b by symbol ✱, is inside the allowed region dictated by the improved QCD vacuum model. This means that all the characteristic features of the BMS bunch are valid also for the improved bunch: one can see in Fig. 1b that in comparison with CZ model (dashed red line, \(a_2 = 0.56\) and \(a_4 = 0\) at \(\mu^2 = 1\) GeV\(^2\)) NLC-dictated models are much more end-point suppressed, although are double-humped.

2. NLO light-cone sum rules (LCSR), CLEO data and lattice QCD

The CLEO experimental data on \(F^{\gamma\gamma}_{\gamma \pi}(Q^2)\) allow one to obtain direct constraints on \(\varphi_\pi(x)\). Applying the LCSR approach, one can effectively account for...
for the long-distance effects of a real photon by using quark-hadron duality in the vector channel and a dispersion relation in $q^2$.

In our CLEO data analysis\cite{14}, we also used the relation between $\lambda_q^2$ and the twist-4 magnitude $\delta_{\text{Tw-4}}^2 \approx \lambda_q^2/2$ and estimated $\delta_{\text{Tw-4}}^2 = 0.19 \pm 0.02$ at $\lambda_q^2 = 0.4$ GeV$^2$. We used, following the approach of\cite{12,13}, the asymptotic model for the twist-4 contribution. We found that even with a 20\% uncertainty in $\delta_{\text{Tw-4}}^2$, the Chernyak–Zhitnitsky (CZ) DA\cite{10} was excluded at least at the 4\(\sigma\)-level, whereas the asymptotic DA was off the 3\(\sigma\)-level, while the BMS “bunch” was inside the 1\(\sigma\)-region\cite{14}, shown in Fig. 2 as a solid ovals around the best-fit point (✚).

Another possibility, suggested in\cite{15}, to obtain constraints on the pion DA in the LCSR analysis of the CLEO data – to use for the twist-4 contribution renormalon-based model, relating it then to parameters $a_2$ and $a_4$ of the pion DA. Using this method we obtain\cite{16} the renormalon-based constraints for the parameters $a_2$ and $a_4$, shown in Fig. 2 in a form of 1\(\sigma\)-ellipses (dashed contours) around the corresponding best-fit point (○).

Recently, high-precision lattice measurements of the second moment $\langle \xi^2 \rangle_\pi = \int_0^1 (2x - 1)^2 \varphi_\pi(x) dx$ of the pion DA were reported by two different collaborations\cite{17,18}. Both groups extracted from their respective simulations, values of $a_2$ at the Schmedding–Yakovlev scale $\mu_{\text{SY}}^2$ around 0.24, but with different error bars. It is remarkable that these lattice results are in striking agreement with the previous\cite{4} and improved\cite{9} estimates of $a_2$ both from NLC QCD SRs and also from the CLEO-data analyses—based on LCSR—\cite{13,14}, as illustrated in Fig. 2, where the lattice results of\cite{18} are shown in the form of a vertical strip, containing the central value with associated errors.

Noteworthily, the value of $a_2$ of the displayed lattice measurements (middle line of the strip) is very close to the CLEO best fit in\cite{14} (✚), whereas almost all the bunch, dictated by the improved NLC QCD SRs\cite{9} is inside the strip. Moreover, this bunch is completely inside the standard CLEO 1\(\sigma\)-ellipse and partially inside the renormalon-based CLEO 1\(\sigma\)-ellipse.

![Fig. 2. Results of the LCSR-based CLEO-data analysis on $F_{\pi\gamma\gamma}^\ast(Q^2)$ in comparison with the lattice results of\cite{18}, shown as shaded area. 1\(\sigma\)-ellipse of\cite{14} is enclosed by the solid line, while the renormalon-based one\cite{16} – by the dashed line. Panel (a) shows comparison with predictions of the minimal NLC model, whereas panel (b) – with those of the improved NLC model, displayed in both cases as slanted shaded rectangles. The displayed models are: ✚ – the asymptotic DA; ✶ – BMS model\cite{4}; ✦ – the central point of our new bunch\cite{9}; ■ – CZ model\cite{10}. All results are evaluated at $\mu_{\text{SY}}^2 = 5.76$ GeV$^2$ after NLO ERBL evolution.](image-url)
3. Conclusions

So, we can conclude that the two-humped and endpoint-suppressed profile of the pion DA emerging from the CLEO-data analysis is consistent with that we have determined independently from QCD sum rules with nonlocal condensates.\cite{1} The improvement of the NLC model, suggested in \cite{9}, shifts the allowed region and puts it just in the intersection of the CLEO-data 1\(\sigma\)-regions, obtained using the asymptotic and the renormalon-based models for twist-4 contribution. Remarkably, this intersection lies almost in the center of the recent lattice-QCD strip.

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