ISOPERIMETRIC RESIDUES AND A MESOSCALE
FLATNESS CRITERION FOR HYPERSURFACES WITH
BOUNDED MEAN CURVATURE

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ABSTRACT. We obtain a full resolution result for minimizers in the exterior isoperimetric problem with respect to a compact obstacle in the large volume regime \( v \to \infty \). This is achieved by the study of a Plateau-type problem with free boundary (both on the compact obstacle and at infinity) which is used to identify the first obstacle-dependent term (called \textit{isoperimetric residue}) in the energy expansion, as \( v \to \infty \), of the exterior isoperimetric problem. A crucial tool in the analysis of isoperimetric residues is a new \textit{“mesoscale flatness criterion”} for hypersurfaces with bounded mean curvature, which we obtain as a development of ideas originating in the theory of minimal surfaces with isolated singularities.

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1. Introduction

1.1. Overview. Given a compact set \( W \subset \mathbb{R}^{n+1} \) \( (n \geq 1) \), we consider the classical \textbf{exterior isoperimetric problem} associated to \( W \), namely

\[
\psi_W(v) = \inf \left\{ P(E; \Omega) : E \subset \Omega = \mathbb{R}^{n+1} \setminus W, |E| = v \right\}, \quad v > 0, \quad (1.1)
\]

in the large volume regime \( v \to \infty \). Here \( |E| \) denotes the volume (Lebesgue measure) of \( E \), and \( P(E; \Omega) \) the (distributional) perimeter of \( E \) relative to \( \Omega \), so that \( P(E; \Omega) = \mathcal{H}^n(\Omega \cap \partial E) \) whenever \( \partial E \) is locally Lipschitz. Relative isoperimetric problems are well-known for their analytical [Maz11, Sections 6.4-6.6] and geometric [Cha01, Chapter V] relevance. They are
Figure 1.1. Quantitative isoperimetry gives no information on how $W$ affects $\psi_W(v)$ for $v$ large.

also important in physical applications: beyond the obvious example of capillarity theory [Fin86], exterior isoperimetry at large volumes provides an elegant approach to the Huisken-Yau theorem in general relativity, see [EM13].

When $v \to \infty$, we expect minimizers $E_v$ in (1.1) to closely resemble balls of volume $v$. Indeed, by minimality and isoperimetry, denoting by $B^{(v)}(x)$ the ball of center $x$ and volume $v$, and with $B_v = B^{(v)}(0)$, we find that

$$\lim_{v \to \infty} \frac{\psi_W(v)}{P(B_v)} = 1. \quad (1.2)$$

Additional information can be obtained by combining (1.2) with quantitative isoperimetry [FMP08, FMP10]: if $0 < |E| < \infty$, then

$$P(E) \geq P(B^{(|E|)}) \{ 1 + c(n) \inf_{x \in \mathbb{R}^{n+1}} \left( \frac{|E \Delta B^{(|E|)}(x)|}{|E|} \right)^2 \}. \quad (1.3)$$

The combination of (1.2) and (1.3) shows that minimizers $E_v$ in $\psi_W(v)$ are close in $L^1$-distance to balls. Based on that, a somehow classical argument exploiting the local regularity theory of perimeter minimizers shows the existence of $v_0 > 0$ and of a function $R_0(v) \to 0^+$, $R_0(v) v^{1/(n+1)} \to \infty$ as $v \to \infty$, both depending on $W$, such that, if $E_v$ is a minimizer of (1.1) with $v > v_0$, then (see Figure 1.1)

$$(\partial E_v) \setminus B_{R_0(v)^{1/(n+1)}} \subset \text{a } C^1\text{-small normal graph over } \partial B^{(v)}(x), \quad (1.4)$$

for some $x \in \mathbb{R}^{n+1}$ with $|x| = (v/\omega_{n+1})^{1/(n+1)} + o(v^{1/(n+1)})$ as $v \to \infty$;

here $\omega_m$ stands for the volume of the unit ball in $\mathbb{R}^m$, $B_r(x)$ is the ball of center $x$ and radius $r$ in $\mathbb{R}^{n+1}$, and $B_r = B_r(0)$. The picture of the situation offered by (1.2) and (1.4) is thus incomplete under one important aspect: it offers no information related to the specific “obstacle” $W$ under consideration – in other words, two different obstacles are completely unrecognizable from (1.2) and (1.4) alone.
The first step to obtain obstacle-dependent information on $\psi_W$ is studying $L^1_{loc}$-subsequential limits $F$ of exterior isoperimetric sets $E_v$ as $v \to \infty$. Since the mean curvature of $\partial E_v$ has order $v^{-1/(n+1)}$ as $v \to \infty$ in $\Omega$, each $\partial F$ is easily seen to be a minimal surface in $\Omega$. A finer analysis leads to establish a more useful characterization of such limits $F$ as minimizers in a “Plateau’s problem with free boundary on the obstacle and at infinity”, whose negative is precisely defined in (1.10) below and denoted by $\mathcal{R}(W)$. We call $\mathcal{R}(W)$ the isoperimetric residue of $W$ because it captures the “residual effect” of $W$ in (1.2), as expressed by the limit identity
\[
\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) = -\mathcal{R}(W).
\] (1.5)

The study of the geometric information about $W$ stored in $\mathcal{R}(W)$ is particularly interesting: roughly, $\mathcal{R}(W)$ is close to an $n$-dimensional sectional area of $W$, although its precise value is elusively determined by the behavior of certain “plane-like” minimal surfaces with free boundary on $W$. The proof of (1.5) itself requires proving a blowdown result for such exterior minimal surfaces, and then extracting sharp decay information towards hyperplane blowdown limits. In particular, in the process of proving (1.5), we shall prove the existence of a positive $R_2$ (depending on $n$ and $W$ only) such that for every maximizer $F$ of $\mathcal{R}(W)$, $(\partial F) \setminus B_{R_2}$ is the graph of a smooth solution to the minimal surfaces equation. An application of Allard’s regularity theorem [All72] leads then to complement (1.4) with the following “local” resolution formula: for every $S > R_2$, for $v$ large in terms of $n, W$ and $S$,
\[
(\partial E_v) \cap (B_S \setminus B_{R_2}) \subset C^1\text{-small normal graph over } \partial F,
\] (1.6)
where $F$ is optimal for the isoperimetric residue $\mathcal{R}(W)$ of $W$.

Interestingly, this already fine analysis gives no information on $\partial E_v$ in the mesoscale region $B_{R_0(v)} \setminus B_S$ between the resolution formulas (1.4) and (1.6). To address this issue, we are compelled to develop what we have called a mesoscale flatness criterion for hypersurfaces with bounded mean curvature. This kind of statement is qualitatively novel with respect to the flatness criteria typically used in the study of blowups and blowdowns of minimal surfaces – although it is clearly related to those tools at the mere technical level – and holds promise for applications to other geometric variational problems. In the study of the exterior isoperimetric problem, it allows us to prove the existence of positive constants $v_0$ and $R_1$, depending on $n$ and $W$ only, such that if $v > v_0$ and $E_v$ is a minimizer of $\psi_W(v)$, then
\[
(\partial E_v) \cap (B_{R_1} \setminus B_{R_2}) \subset C^1\text{-small normal graph over } \partial F,
\] where $F$ is optimal for the isoperimetric residue $\mathcal{R}(W)$ of $W$. (1.7)

The key difference between (1.6) and (1.7) is that the domain of resolution given in (1.7) overlaps with that of (1.4): indeed, $R_0(v) \to 0^+$ as $v \to \infty$ implies that $R_0 v^{1/(n+1)} < R_1 v^{1/(n+1)}$ for $v > v_0$. As a by-product of this overlapping and of the graphicality of $\partial F$ outside of $B_{R_2}$, we deduce that
boundaries of exterior isoperimetric sets, outside of $B_{R_2}$, are diffeomorphic to $n$-dimensional disks. Finally, when $n \leq 6$, and maximizers $F$ of $\mathcal{R}(W)$ have locally smooth boundaries in $\Omega$, (1.7) can be propagated up to the obstacle itself; see Remark 1.7 below.

Concerning the rest of this introduction: In section 1.2 we present our analysis of isoperimetric residues, see Theorem 1.1. In section 1.3 we gather all our results concerning exterior isoperimetric sets with large volumes, see Theorem 1.6. Finally, we present the mesoscale flatness criterion in section 1.4 and the organization of the paper in section 1.5.

1.2. Isoperimetric residues. To define $\mathcal{R}(W)$ we introduce the class

$$
F
$$

of those pairs $(F, \nu)$ with $\nu \in S^n (= \text{the unit sphere of } \mathbb{R}^{n+1})$ and $F \subset \mathbb{R}^{n+1}$ a set of locally finite perimeter in $\Omega$ (i.e., $P(F; \Omega') < \infty$ for every $\Omega' \subset \subset \Omega$), contained in slab around $\nu^\perp = \{x : x \cdot \nu = 0\}$, and whose boundary (see Remark 1.5 below) has full projection over $\nu^\perp$ itself: i.e., for some $\alpha, \beta \in \mathbb{R}$,

$$
\partial F \subset \{x : \alpha < x \cdot \nu < \beta\},
$$

(1.8)

$$
p_{\nu^\perp}(\partial F) = \nu^\perp := \{x : x \cdot \nu = 0\},
$$

(1.9)

where $p_{\nu^\perp}(x) = x - (x \cdot \nu)\nu$, $x \in \mathbb{R}^{n+1}$. In correspondence to $W$ compact, we define the residual perimeter functional, $\text{res}_W : F \to \mathbb{R} \cup \{\pm \infty\}$, by

$$
\text{res}_W(F, \nu) = \lim_{R \to \infty} \omega_n R^n - P(F; C_R^\nu \setminus W), \quad (F, \nu) \in F,
$$

where $C_R^\nu = \{x \in \mathbb{R}^{n+1} : |p_{\nu^\perp}(x)| < R\}$ denotes the (unbounded) cylinder of radius $R$ with axis along $\nu^\perp$ and where the limsup is actually a monotone decreasing limit thanks to (1.8) and (1.9) (see (4.9) below for a proof).

For a reasonably “well-behaved” $F$, e.g. if $\partial F$ is the graph of a Lipschitz function over $\nu^\perp$, $\omega_n R^n$ is the (obstacle-independent) leading order term of the expansion of $P(F; C_R^\nu \setminus W)$ as $R \to \infty$, while $\text{res}_W(F, \nu)$ is expected to capture the first obstacle-dependent “residual perimeter” contribution of $P(F; C_R^\nu \setminus W)$ as $R \to \infty$. The isoperimetric residue of $W$ is then defined by maximizing $\text{res}_W$ over $F$, so that

$$
\mathcal{R}(W) = \sup_{(F, \nu) \in F} \text{res}_W(F, \nu);
$$

(1.10)

see Figure 1.2. Clearly $\mathcal{R}(\lambda W) = \lambda^n \mathcal{R}(W)$ if $\lambda > 0$, and $\mathcal{R}(W)$ is trapped between the areas of the largest hyperplane section and directional projection of $W$, see (1.11) below. In the simple case when $n = 1$ and $W$ is connected, $\mathcal{R}(W) = \text{diam}(W)$ by (1.17) and (1.18) below, although, in general, $\mathcal{R}(W)$ does not seem to admit a simple characterization, and it is finely tuned to the near-to-the-obstacle behavior of “plane-like” minimal surfaces with free boundary on $W$. Our first main result collects these (and other) properties of isoperimetric residues and of their maximizers.
Theorem 1.1 (Isoperimetric residues). If \( W \subset \mathbb{R}^{n+1} \) is compact, then there are \( R_2 \) and \( C_0 \) positive and depending on \( W \) with the following property.

(i): If \( S(W) = \sup \{ \mathcal{H}^n(W \cap \Pi) : \Pi \text{ is a hyperplane in } \mathbb{R}^{n+1} \} \) and \( P(W) = \sup \{ \mathcal{H}^n(p_{\nu \perp}(W)) : \nu \in \mathbb{S}^n \} \), then we have
\[
S(W) \leq \mathcal{R}(W) \leq P(W),
\]
(1.11)

(ii): The family \( \max \{ \mathcal{R}(W) \} \) of maximizers of \( \mathcal{R}(W) \) is non-empty. If \( (F, \nu) \in \max \{ \mathcal{R}(W) \} \), then \( F \) is a perimeter minimizer with free boundary in \( \Omega = \mathbb{R}^{n+1} \setminus W \), i.e.
\[
P(F; \Omega \cap B) \leq P(G; \Omega \cap B), \quad \forall F \Delta G \subset B, \text{ } B \text{ a ball};
\]
and if \( \mathcal{R}(W) > 0 \), then \( \partial F \) is contained in the smallest slab \( \{ x : \alpha \leq x \cdot \nu \leq \beta \} \) containing \( W \), and there are \( a, b, c \in \mathbb{R}, c \in \nu^\perp \) with \( \max \{|a|, |b|, |c|\} \leq C_0 \) and \( f \in C^\infty(\nu^\perp) \) such that
\[
(\partial F) \setminus C_{R_2}^\nu = \{ x + f(x) \nu : x \in \nu^\perp, |x| > R_2 \},
\]
(1.13)
\[
f(x) = a, \quad (n = 1)
\]
\[
|f(x) - \left( a + \frac{b}{|x|^{n-2}} + \frac{c \cdot x}{|x|^n} \right) | \leq \frac{C_0}{|x|^n}, \quad (n \geq 2)
\]
(1.14)
\[
\max \{|x|^{n-1} |\nabla f(x)|, |x|^n |\nabla^2 f(x)|\} \leq C_0, \quad \forall x \in \nu^\perp, |x| > R_2.
\]

(iii): At fixed diameter, isoperimetric residues are maximized by balls, i.e.
\[
\mathcal{R}(W) \leq \omega_n (\text{diam } W/2)^n = \mathcal{R}(\text{cl}(B_{\text{diam } W/2})) ,
\]
(1.15)
where \( \text{cl}(X) \) denotes topological closure of \( X \subset \mathbb{R}^{n+1} \). Moreover, if equality holds in (1.15) and \( (F, \nu) \in \max \{ \mathcal{R}(W) \} \), then (1.14) holds with \( b = 0 \) and \( c = 0 \), and setting \( \Pi = \{ y : y \cdot \nu = a \} \), we have
\[
(\partial F) \setminus W = \Pi \setminus \text{cl}(B_{\text{diam } W/2}(x)),
\]
(1.16)
for some \( x \in \Pi \). Finally, equality holds in (1.15) if and only if there is a hyperplane \( \Pi \) and a point \( x \in \Pi \) such that
\[
\partial B_{\text{diam } W/2}(x) \cap \Pi \subset W,
\]
(1.17)
Figure 1.3. The obstacle $W$ (depicted in grey) is obtained by removing a cylinder $C_{r}^{n+1}$ from a ball $B_{d/2}$ with $d/2 > r$. In this way $d = \text{diam} (W)$ and $B_{d/2}$ is the only ball such that (1.17) can hold. Hyperplanes $\Pi$ satisfying (1.17) are exactly those passing through the center of $B_{d/2}$, and intersecting $W$ on a $(n-1)$-dimensional sphere of radius $d/2$. For every such $\Pi$, $\Omega \setminus (\Pi \setminus B_{d/2})$ has exactly one unbounded connected component, and (1.18) does not hold.

\[
i.e., \ W \text{ contains an } (n-1)\text{-dimensional sphere of diameter } \text{diam} (W), \text{ and } \Omega \setminus (\Pi \setminus \text{cl} (B_{\text{diam} W/2}(x))) \text{ has exactly two unbounded connected components.} \tag{1.18}\]

Remark 1.2. The assumption $\mathcal{R}(W) > 0$ is quite weak: indeed, if $\mathcal{R}(W) = 0$, then $W$ is purely $\mathcal{H}^n$-unrectifiable; see Proposition C.1 in the appendix. For the role of the topological condition (1.18), see Figure 1.3.

Remark 1.3 (Regularity of isoperimetric residues). In the physical dimension $n = 2$, and provided $\Omega$ has boundary of class $C^{1,1}$, maximizers of $\mathcal{R}(W)$ are $C^{1,1/2}$-regular up to the obstacle, and smooth away from it. More generally, condition (1.12) implies that $M = \text{cl} (\Omega \cap \partial F)$ is a smooth hypersurface with boundary in $\Omega \setminus \Sigma$, where $\Sigma$ is a closed set such that $\Sigma \cap \Omega$ is empty if $1 \leq n \leq 6$, is locally discrete in $\Omega$ if $n = 7$, and is locally $\mathcal{H}^{n-7}$-rectifiable in $\Omega$ if $n \geq 8$; see, e.g. [Mag12, Part III], [NV20]. Of course, by (1.13), $\Sigma \setminus B_{R_2} = \emptyset$ in every dimension. Moreover, justifying the initial claim concerning the case $n = 2$, if we assume that $\Omega$ is an open set with $C^{1,1}$-boundary, then $M$ is a $C^{1,1/2}$-hypersurface with boundary in $\mathbb{R}^{n+1} \setminus \Sigma$, with boundary contained in $\partial \Omega$, $\Sigma \cap \partial \Omega$ is $\mathcal{H}^{n-3+\varepsilon}$-negligible for every $\varepsilon > 0$, and Young’s law $\nu_{F} \cdot \nu_{\Omega} = 0$ holds on $(M \cap \partial \Omega) \setminus \Sigma$; see, e.g. [Griß87, GJ86, DPM15, DPM17].

Remark 1.4. An interesting open direction is finding additional geometric information on $\mathcal{R}(W)$, e.g. in the class of convex obstacles.

Remark 1.5 (Normalization of competitors). We adopt the convention that any set of locally finite perimeter $F$ in $\Omega$ open is tacitly modified on and by a set of zero Lebesgue measure so to entail $\Omega \cap \partial F = \Omega \cap \text{cl} (\partial^{*} F)$, where $\partial^{*} F$ is the reduced boundary of $F$ in $\Omega$; see [Mag12, Proposition 12.19]. Under this normalization, local perimeter minimality conditions like (1.12) (or (3.1) below) imply that $F \cap \Omega$ is open in $\mathbb{R}^{n+1}$; see, e.g. [DPM15, Lemma 2.16].
1.3 Resolution of exterior isoperimetric sets. Denoting the family of minimizers of \( \psi_W(v) \) by Min[\( \psi_W(v) \)], our second main result is as follows:

**Theorem 1.6** (Resolution of exterior isoperimetric sets). If \( W \subset \mathbb{R}^{n+1} \) is compact, then Min[\( \psi_W(v) \)] \( \neq \emptyset \) \( \forall v > 0 \). Moreover, if \( R(W) > 0 \), then

\[
\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) = -R(W),
\]

and, depending on \( n \) and \( W \) only, there are \( v_0, C_0, R_1, \) and \( R_2 \) positive, and \( R_0(v) \) with \( R_0(v) \to 0^+ \), \( R_0(v)^{1/(n+1)} \to \infty \) as \( v \to \infty \), such that, if \( E_v \in \text{Min}[\psi_W(v)] \) and \( v > v_0 \), then:

(i): \( E_v \) determines \( x \in \mathbb{R}^{n+1} \) and \( u \in C^\infty(\partial B^{(1)}) \) such that

\[
\frac{|E_v\Delta B^{(v)}(x)|}{v} \leq \frac{C_0}{v^{1/(2(n+1))}},
\]

\[
(\partial E_v) \setminus B_{R_0(v)^{1/(n+1)}},
\]

\[
= \left\{ y + v^{1/(n+1)} u \left( \frac{y - x}{v^{1/(n+1)}} \right) \nu_B^{(v)}(x) : y \in \partial B^{(v)}(x) \right\} \setminus B_{R_0(v)^{1/(n+1)}},
\]

where, for any \( G \subset \mathbb{R}^{n+1} \) with locally finite perimeter, \( \nu_G \) is the outer unit normal to \( G \);

(ii): \( E_v \) determines \( (F,\nu) \in \text{Max}[\mathcal{R}(W)] \) and \( f \in C^\infty((\partial F) \setminus B_{R_2}) \) with

\[
(\partial E_v) \cap A_{R_2}^{R_1 v^{1/(n+1)}} = \left\{ y + f(y) \nu_F(y) : y \in \partial F \right\} \cap A_{R_2}^{R_1 v^{1/(n+1)}},
\]

(iii): \( (\partial E_v) \setminus B_{R_2} \) is diffeomorphic to an \( n \)-dimensional disk;

(iv): Finally, with \( (x,u) \) as in (1.21) and \( (F,\nu,f) \) as in (1.22),

\[
\lim_{v \to \infty} \sup_{E_v \in \text{Min}[\psi_W(v)]} \left\{ \left| \frac{|x|}{v^{1/(n+1)}} - \frac{1}{\omega_{n+1}} \right| : \left| \nu - \frac{x}{|x|} \right|, \|u\|_{C^1(\partial B^{(1)})} \right\} = 0,
\]

\[
\lim_{v \to \infty} \sup_{E_v \in \text{Min}[\psi_W(v)]} \|f\|_{C^1(B_M \cap \partial F)} = 0, \quad \forall M > R_2.
\]

**Remark 1.7** (Resolution up to the obstacle). By Remark 1.3 and a covering argument, if \( n \leq 6, \delta > 0, \) and \( v > v_0(n,W,\delta) \), then (1.22) holds with \( B_{R_1 v^{1/(n+1)}} \setminus I_\delta(W) \) in place of \( B_{R_1 v^{1/(n+1)} \setminus B_{R_2}} \), where \( I_\delta(W) \) is the open \( \delta \)-neighborhood of \( W \). Similarly, when \( \partial \Omega \in C^{1,1} \) and \( n = 2 \) (and thus \( \Omega \cap \partial F \) is regular up to the obstacle), we can find \( v_0 \) (depending on \( n \) and \( W \) only) such that (1.22) holds with \( B_{R_1 v^{1/(n+1)} \cap \Omega} \) in place of \( B_{R_1 v^{1/(n+1)} \setminus B_{R_2}} \), that is, graphicality over \( \partial F \) holds up to the obstacle itself.

**Remark 1.8.** If \( W \) is convex and \( J \) is an half-space, then \( \psi_W(v) \geq \psi_J(v) \) for every \( v > 0 \), with equality for \( v > 0 \) if and only if \( \partial W \) contains a flat facet supporting an half-ball of volume \( v \); see [CGR07, FM21]. Since \( \psi_J(v) = P(B^{(v)})/2^{1/(n+1)} \) and \( \psi_W(v) - P(B^{(v)}) = -\mathcal{R}(W) \) as \( v \to \infty \), the bound \( \psi_W(v) \geq \psi_J(v) \) is far from optimal if \( v \) is large. Are there stronger global bounds than \( \psi_W \geq \psi_J \) on convex obstacles? Similarly, it would be interesting to quantify the convergence towards \( \mathcal{R}(W) \) in (1.19), or even that of \( \partial E_v \) towards \( \partial B^{(v)} \) and \( \partial F \) (where (1.20) should not to be sharp).
1.4. **The mesoscale flatness criterion.** We work with with hypersurfaces $M$ whose mean curvature is bounded by $\Lambda \geq 0$ in an annulus $B_{1/\Lambda} \setminus B_R$, $R \in (0, 1/\Lambda)$. Even without information on $M$ inside $B_R$ (where $M$ could have a non-trivial boundary, or topology, etc.) the classical proof of the monotonicity formula can be adapted to show the monotone increasing character on $r \in (R, 1/\Lambda)$ of

$$
\Theta_{M,R,\Lambda}(r) = \frac{\mathcal{H}^n(M \cap (B_r \setminus B_R))}{r^n} + \frac{R}{n} \int_{M \cap \partial B_R} \frac{|x^{TM}|}{|x|} d\mathcal{H}^{n-1}
+ \Lambda \int_R^r \frac{\mathcal{H}^n(M \cap (B_\rho \setminus B_R))}{\rho^n} d\rho,
$$

(1.23)

(here $x^{TM} = \text{proj}_{T_xM}(x)$); moreover, if $\Theta_{M,R,\Lambda}$ is constant over $(a, b) \subset (R, 1/\Lambda)$, then $M \cap (B_b \setminus B_a)$ is a cone. Since the constant density value corresponding to $M = H \setminus B_R$, $H$ an hyperplane through the origin, is $\omega_n$ (as a result of a double cancellation which also involves the “boundary term” in $\Theta_{H \setminus B_R, R, 0}$), we consider the **area deficit**

$$
\delta_{M,R,\Lambda}(r) = \omega_n - \Theta_{M,R,\Lambda}(r), \quad r \in (R, 1/\Lambda),
$$

(1.24)

which defines a decreasing quantity on $(R, 1/\Lambda)$. Here we use the term “deficit”, rather than the more usual term “excess”, since $\delta_{M,R,\Lambda}$ does not necessarily have non-negative sign (which is one of the crucial property of “excess quantities” typically used in $\varepsilon$-regularity theorems, see, e.g., [Mag12, Lemma 22.11]). Recalling that $A^s_R = B_s \setminus \text{cl}(B_r)$ if $s > r > 0$, we are now ready to state the following “smooth version” of our mesoscale flatness criterion (see Theorem 2.1 below for the varifold version).

**Theorem 1.9** (Mesoscale flatness criterion (smooth version)). If $n \geq 2$, $\Gamma \geq 0$, and $\sigma > 0$, then there are $M_0$ and $\varepsilon_0$ positive and depending on $n, \Gamma$ and $\sigma$ only, with the following property. Let $\Lambda \geq 0$, $R \in (0, 1/\Lambda)$, and $M$ be a smooth hypersurface with mean curvature bounded by $\Lambda$ in $A^{1/\Lambda}_R$, and with

$$
\mathcal{H}^{n-1}(M \cap \partial B_R) \leq \Gamma R^{n-1}, \quad \sup_{\rho \in (R, 1/\Lambda)} \frac{\mathcal{H}^n(M \cap (B_\rho \setminus B_R))}{\rho^n} \leq \Gamma.
$$

(1.25)

If there is $s > 0$ such that

$$
\max\{M_0, 64\} R < s < \frac{\varepsilon_0}{4\Lambda},
$$

(1.26)

and

$$
|\delta_{M,R,\Lambda}(s/8)| \leq \varepsilon_0,
$$

(1.27)

and if, setting,

$$
R_* = \sup\left\{\rho \geq \frac{s}{8} : \delta_{M,R,\Lambda}(\rho) \geq -\varepsilon_0\right\}, \quad S_* = \min\left\{R_*, \frac{\varepsilon_0}{\Lambda}\right\},
$$

(1.28)
we have $R_s > 4 s$ (and thus $S_s > 4 s$), then

$$M \cap A_{s/32}^{S_s/16} = \{ x + f(x) \nu_K : x \in K \} \cap A_{s/32}^{S_s/16},$$

(1.29)

$$\sup \{|x|^{-1} |f(x)| + |\nabla f(x)| : x \in K \} \leq C(n) \sigma,$$

for a hyperplane $K$ with $0 \in K$ and unit normal $\nu_K$, and for $f \in C^1(K)$.

**Remark 1.10** (Structure of the statement). The first condition in (1.26) implicitly requires $R$ to be sufficiently small in terms of $1/\Lambda$, as it introduces a mesoscale $s$ which is both small with respect to $1/\Lambda$ and large with respect to $R$. The condition in (1.27) express the flatness of $M$ at the mesoscale $s$ in terms of its area deficit. The final key assumption, $R_s > 4 s$, express the requirement that the area deficit does not decrease too abruptly, and stays above $-\varepsilon_0$ at least up to the scale $4 s$. Under these assumptions, graphicality with respect to a hyperplane $K$ is inferred on an annulus whose lower radius $s/32$ has the order of the mesoscale $s$, and whose upper radius $S_s/16$ can be as large as the decay of the area deficit allows (potentially up to $\varepsilon_0/16 \Lambda$ if $R_s = \infty$), but in any case not too large with respect to $1/\Lambda$.

**Remark 1.11** (Relationship to other flatness criteria). If $M$ is a hypersurface containing the origin, so that, formally speaking, $R = 0$, and the tangent cone of $M$ there is a plane, Theorem 1 reduces to Allard’s theorem [All72]. Similarly, if $\Lambda = 0$ and the exterior minimal hypersurface $M$ has a planar tangent cone at infinity, we recover the exterior blow-down results stated in [Sim85, Sim87]. In particular, although the motivation for Theorem 1 comes from scenarios where both $R$ and $\Lambda$ are positive, it can also be viewed as a general framework containing as special cases the blow-up and blow-down flatness criteria for hypersurfaces with planar tangent cones.

**Remark 1.12** (Sharpness of the statement). The statement is sharp in the sense that for a surface “with bounded mean curvature and non-trivial topology inside a hole”, flatness can only be established on a mesoscale which is both large with respect to the size of the hole and small with respect to the size of the inverse mean curvature. An example is provided by unduloids $M_\varepsilon$ with waist size $\varepsilon$ and mean curvature $n$ in $\mathbb{R}^{n+1}$; see Figure 1.4. A “half-period” of $M_\varepsilon$ is the graph $\{ x + f_\varepsilon(x) \varepsilon_{n+1} : x \in \mathbb{R}^n, \varepsilon < |x| < R_\varepsilon \}$ of

$$f_\varepsilon(x) = \int_{\varepsilon}^{|x|} \left\{ \left( \frac{r^{n-1}}{r^n - \varepsilon^n + \varepsilon^{n-1}} \right)^2 - 1 \right\}^{-1/2} dr,$$

(1.30)

where $\varepsilon$ and $R_\varepsilon$ are the only solutions of $r^{n-1} = r^n - \varepsilon^n + \varepsilon^{n-1}$. Clearly $f_\varepsilon$ solves $-\text{div}(\nabla f_\varepsilon/\sqrt{1 + |\nabla f_\varepsilon|^2}) = n$ with $f_\varepsilon = 0$, $|\nabla f_\varepsilon| = +\infty$ on $\{|x| = \varepsilon\}$, and $|\nabla f_\varepsilon| = +\infty$ on $\{|x| = R_\varepsilon\}$, where $R_\varepsilon = 1 - O(\varepsilon^{n-1})$; moreover, min $|\nabla f_\varepsilon|$ is achieved at $r = O(\varepsilon^{(n-1)/n})$, and if $r \in (a \varepsilon^{(n-1)/n}, b \varepsilon^{(n-1)/n})$ for some $b > a > 0$, then $|\nabla f_\varepsilon| = O_b(\varepsilon^{2(n-1)/n})$. Thus, the horizontal flatness of $M_\varepsilon$ is no smaller than $O(\varepsilon^{2(n-1)/n})$, and has that exact order on a scale which is both very large with respect to the hole $(\varepsilon^{(n-1)/n} \gg \varepsilon)$ and very small with respect to the inverse mean curvature $(\varepsilon^{(n-1)/n} \ll 1)$. 

Figure 1.4. A half-period of an unduloid with mean curvature \( n \) and waist size \( \varepsilon \) in \( \mathbb{R}^{n+1} \). By (1.30), the flatness of \( M_\varepsilon \) is no smaller than \( O(\varepsilon^{2(n-1)/n}) \), and is exactly \( O(\varepsilon^{2(n-1)/n}) \) on an annulus sitting in the mesoscale \( O(\varepsilon^{(n-1)/n}) \). This mesoscale is both very large with respect to waist size \( \varepsilon \), and very small with respect to the size of the inverse mean curvature, which is order one.

Remark 1.13 (On the application to \( \psi_W(v) \)). Exterior isoperimetric sets \( E_v \) with large volume \( v \) have small constant mean curvature of order \( \Lambda_0(n,W)/v^{1/(n+1)} \). We will work with “holes” of size \( R = R_3(n,W) \), for some \( R_3 \) sufficiently large with respect to the radius \( R_2 \) appearing in Theorem 1.1-(ii), and determined through the sharp decay rates (1.14). The decay properties of \( F \) towards \( \{ x : x \cdot \nu = a \} \) when \((F,\nu)\) is a maximizer of \( \mathcal{R}(W) \), the \( C^1 \)-proximity of \( \partial E \) to \( \partial B^{(v)}(x) \) for \( |x| \approx (\omega_{n+1}/v)^{1/(n+1)} \), and the \( C^1 \)-proximity of \( \partial E \) to \( \partial F \) for some optimal \((F,\nu)\) on bounded annuli of the form \( A^{2R_3} \) are used in checking that (1.25) holds with \( \Gamma = \Gamma(n,W) \), that \( E_v \) is flat in the sense of (1.27), and, most importantly, that the area deficit \( \delta_{M,R,\Lambda} \) of \( M = (\partial E_v) \setminus B_{R_3} \) lies above \( -\varepsilon_0 \) up to scale \( r = O(v^{1/(n+1)}) \) (which is the key information to deduce \( R_s \approx 1/\Lambda \)), and thus obtain overlapping domains of resolutions in terms of \( \partial B^{(v)}(x) \) and \( \partial F \).

Remark 1.14. While Theorem 1.9 seems clearly applicable to other problems, there are situations where one may need to develop considerably finer “mesoscale flatness criteria”. For example, consider the problem of “resolving” almost CMC boundaries undergoing bubbling [CM17, DMMN18, DM19]. When the oscillation of the mean curvature around a constant \( \Lambda \) is small, such boundaries are close to finite unions of mutually tangent spheres of radius \( n/\Lambda \), and can be covered by \( C^1 \)-small normal graphs over such spheres away from their tangency points up to distance \( \varepsilon/\Lambda \), with \( \varepsilon = \varepsilon(n) \), and provided the mean curvature oscillation is small in terms of \( \varepsilon \). For propagating flatness up to a distance directly related to the oscillation of the mean curvature, one would need a version of Theorem 1.9 for “double” spherical graphs; in the setting of blowup/blowdown theorems, this would be similar to passing to the harder case of multiplicity larger than one.

Remark 1.15 (Comparison with blowup/blowdown results). From the technical viewpoint, Theorem 1.9 fits into the framework set up by Allard and Almgren in [AA81] for the study of blowups and blowdowns of minimal
surfaces with tangent integrable cones. At the same time, as exemplified by Remark 1.12, Theorem 1.9 really points in a different direction, since it pertains to situations where neither blowup or blowdown limits make sense. Another interesting point is that, in [AA81], the area deficit \( \delta_{M,R,\Lambda} \) is considered with a sign, non-positive for blowups, and non-negative for blowdowns, see [AA81, Theorem 5.9(4), Theorem 9.6(4)]. A key insight here is that for hypersurfaces where the deficit changes sign, graphicality obtained through small negative (or positive) deficit nevertheless persists past the scale where \( \delta_{M,R,\Lambda} \) vanishes, and possibly much farther depending on the surface in question; this is actually crucial for obtaining overlapping domains of resolutions in statements like (1.4) and (1.7).

Remark 1.16 (Extension to general minimal cones). Proving Theorem 1.9 in higher codimension and with arbitrary integrable minimal cones should be possible with essentially the same proof presented here. We do not pursue this extension because, first, only the case of hypersurfaces and hyperplanes is needed in studying \( \psi_W(v) \); and, second, in going for generality, one should work in the framework set up by Simon in [Sim83a, Sim85, Sim96], which, at variance with the simpler Allard–Almgren’s framework used here, allows one to dispense with the integrability assumption. In this direction, we notice that Theorem 1.9 with \( \Lambda = 0 \) and \( R_* = +\infty \) is a blowdown result for exterior minimal surfaces (see also Theorem 2.1-(ii, iii)). A blowdown result for exterior minimal surfaces is outside the scope of [AA81, Theorem 9.6] which pertains to entire minimal surfaces, but it is claimed, with a sketch of proof, on [Sim85, Page 269] as a modification of [Sim85, Theorem 5.5, \( m < 0 \)]. It should be mentioned that, to cover the case of exterior minimal surfaces, an additional term of the form \( C \int_{\Sigma} (\dot{u}(t))^2 \) should be added on the right side of assumption [Sim85, 5.3, \( m < 0 \)]. This additional term seems not to cause difficulties with the rest of the arguments leading to [Sim85, Theorem 5.5, \( m < 0 \)]. Thus Simon’s approach, in addition to giving the blowdown analysis of exterior minimal surfaces, should also be viable for generalizing our mesoscale flatness criterion.

1.5. Organization of the paper. In section 2 we prove Theorem 1.9 (actually, its generalization to varifolds, i.e. Theorem 2.1). In section 3 we prove those parts of Theorem 1.6 which follow simply by quantitative isoperimetry (i.e., they do not require isoperimetric residues nor our mesoscale flatness analysis); see Theorem 3.1. Section 4 is devoted to the study of isoperimetric residues and of their maximizers, and contains the proof Theorem 1.1. We also present there a statement, repeatedly used in our analysis, which summarizes some results from [Sch83]; see Proposition 4.1. Finally, in section 5, we prove the energy expansion (1.19) and those parts of Theorem 1.6 left out in section 3 (i.e., statements (ii, iii, iv)). This final section is, from a certain viewpoint, the most interesting part of the paper: indeed, it is only the detailed examination of those arguments that clearly illustrates the degree of fine tuning of the preliminary analysis of exterior isoperimetric
sets and of maximizers of isoperimetric residues which is needed in order to allow for the application of the mesoscale flatness criterion.

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2. A mesoscale flatness criterion for varifolds

In section 2.1 we introduce the class $\mathcal{V}_n(\Lambda, R, S)$ of varifolds used to reformulate Theorem 1.9, see Theorem 2.1. In sections 2.2-2.3 we present two reparametrization lemmas (Lemma 2.3 and Lemma 2.5) and some “energy estimates” (Theorem 2.6) for spherical graphs; in section 2.4 we state the monotonicity formula in $\mathcal{V}_n(\Lambda, R, S)$ and some energy estimates involving the monotonicity gap; in section 2.5, we prove Theorem 2.1.

2.1. Statement of the criterion. Given an $n$-dimensional integer rectifiable varifold $V = \var M$, defined by a locally $H^n$-rectifiable set $M$, and by a multiplicity function $\theta : M \to \mathbb{N}$ (see [Sim83b]), we denote by $\|V\| = \theta H^n\lceil M$ the weight of $V$, and by $\delta V$ the first variation of $V$, so that $\delta V(X) = \int \text{div}^T X(x) dV(x,T) = \int_\mathbb{M} \text{div}^M X(x) \theta dH^0_x$ for every $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$. Given $S > R > 0$ and $\Lambda \geq 0$, we consider the family $\mathcal{V}_n(\Lambda, R, S)$, of those $n$-dimensional integral varifolds $V$ with spt $V \subset \mathbb{R}^{n+1} \setminus B_R$ and

$$\delta V(X) = \int X \cdot \vec{H} d\|V\| + \int X \cdot \nu^\circ d bdV, \quad \forall X \in C^1_c(B_S; \mathbb{R}^{n+1}),$$

holds for a Radon measure $bdV$ in $\mathbb{R}^{n+1}$ supported in $\partial B_R$, and Borel vector fields $\vec{H} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\|\vec{H}\| \leq \Lambda$ and $\nu^\circ : \partial B_R \to \mathbb{R}^{n+1}$ with $|\nu^\circ| = 1$. We let $\mathcal{M}_n(\Lambda, R, S) = \{V \in \mathcal{V}_n(\Lambda, R, S) : V = \var (M, 1) \text{ for } M \text{ smooth} \}$, that is, $M \subset \mathbb{R}^{n+1} \setminus B_R$ is a smooth hypersurface with boundary in $A^S_R$, bdry $(M) \subset \partial B_R$, and $|H_M| \leq \Lambda$. If $V \in \mathcal{M}_n(\Lambda, R, S)$, then $\vec{H}$ is the mean curvature vector of $M$, $bdV = H^n\lceil \partial B_R$, and $\nu^\circ$ is the outer unit normal to $M$ along $\partial B_R$. Given $V \in \mathcal{V}_n(\Lambda, R, S)$, we define $\Theta_{V,R,\Lambda}(r)$ as

$$\frac{\|V\|(B_r \setminus B_R)}{r^n} - \frac{1}{n} \int \frac{x \cdot \nu^\circ d bdV}{r^n} + \Lambda \int_R^r \frac{\|V\|(B_{\rho} \setminus B_R)}{\rho^n} d\rho; \quad (2.1)$$

$\Theta_{V,R,\Lambda}(r)$ is increasing for $r \in (R, S)$ (Theorem 2.7-(i) below), and equal to (1.23) when $V \in \mathcal{M}_n(\Lambda, R, S)$. The area deficit of $V$ is then defined as in (1.24), while given a hyperplane $H$ in $\mathbb{R}^{n+1}$ with $0 \in H$ we call the quantity

$$\int_{A^r_\rho} \omega_H(y)^2 \|V\|_y, \quad \omega_H(y) = \arctan \left( \frac{y \cdot H}{|y||H|} \right),$$

the angular flatness of $V$ on the annulus $A^r_\rho = B_\rho \setminus \text{cl } (B_r)$ with respect to $H$. (See (2.10) for the notation concerning $H$.).
Theorem 2.1 (Mesoscale flatness criterion). If $n \geq 2$, $\Gamma \geq 0$, and $\sigma > 0$ then there are positive constants $M_0$ and $\varepsilon_0$, depending on $n$, $\Gamma$ and $\sigma$ only, with the following property. If $\Lambda \geq 0$, $R \in (0, 1/\Lambda)$, $V \in V_n(\Lambda, R, 1/\Lambda)$,

$$\|\partial V\|((\partial B_R) \leq \Gamma R^{n-1}, \sup_{\rho \in (R, 1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \leq \Gamma.$$  \hfill (2.3)

and for some $s > 0$ we have

$$\frac{\varepsilon_0}{4\Lambda} > s > \max\{M_0, 64\} R,$$  \hfill (2.4)

$$|\delta_{V, R, \Lambda}(s/8)| \leq \varepsilon_0,$$  \hfill (2.5)

$$R_* := \sup \left\{ \rho \geq \frac{s}{8} : \delta_{V, R, \Lambda}(\rho) \geq -\varepsilon_0 \right\} \geq 4s,$$  \hfill (2.6)

then (i): if $S_* = \min\{R_*, \varepsilon_0/\Lambda\} < \infty$, then there is an hyperplane $K \subset \mathbb{R}^{n+1}$ with $0 \in K$ and $u \in C^1((K \cap \mathbb{S}^n) \times (s/32, S_* /16))$ with

$$(spt \ V) \bigcap A_{s/32}^{S_* /16} = \left\{ r \frac{\omega + u(r, \omega) \nu_K}{\sqrt{1 + u(r, \omega)^2}} : \omega \in K \cap \mathbb{S}^n, r \in (s/32, S_* /16) \right\}$$

$$\sup_{(K \cap \mathbb{S}^n) \times (s/32, S_* /16)} \left\{ |u| + |\nabla_{K \cap \mathbb{S}^n} u| + |r \partial_r u| \right\} \leq C(n) \sigma;$$  \hfill (2.7)

(ii): if $\Lambda = 0$ and $\delta_{V, R, 0} \geq -\varepsilon_0$ on $(s/8, \infty)$, then $\delta_{V, R, 0} \geq 0$ on $(s/8, \infty)$, (2.7) holds with $S_* = \infty$, and one has decay estimates, continuous in the radius, of the form

$$\delta_{V, R, 0}(r) \leq C(n) \left( \frac{s}{r} \right) \delta_{V, R, 0}(\frac{s}{8}), \quad \forall r > \frac{s}{4},$$  \hfill (2.8)

$$\frac{1}{r^n} \int_{A^2_r} \omega_K^2 d\|V\| \leq C(n) (1 + \Gamma) \left( \frac{s}{r} \right) \delta_{V, R, 0}(\frac{s}{8}), \quad \forall r > \frac{s}{4},$$  \hfill (2.9)

for some $\alpha(n) \in (0, 1)$.

Remark 2.2. In Theorem 2.1, graphicality is formulated in terms of the notion of spherical graph (see section 2.2) which is more natural than the usual notion of “cylindrical graph” in setting up the iteration procedure behind Theorem 2.1. Spherical graphicality in terms a $C^1$-small $u$ as in (2.7) translates into cylindrical graphicality in terms of $f$ as in (1.29) with $f(x)/|x| \approx u(|x|, \hat{x})$ and $\nabla_x f(x) - (f(x)/|x|) \approx |x| \partial_r u(|x|, \hat{x})$ for $x \neq 0$ and $\hat{x} = x/|x|$; see, in particular, Lemma B.1 in appendix B.

2.2. Spherical graphs. We start setting up some notation. We denote by

$$\mathcal{H}$$

the family of the oriented hyperplanes $H \subset \mathbb{R}^{n+1}$ with $0 \in \mathcal{H}$, so that choice of $H \in \mathcal{H}$ implies that a unit normal vector $\nu_H$ to $H$. Given $H \in \mathcal{H}$, we set

$$\Sigma_H = H \cap \mathbb{S}^n, \quad p_H : \mathbb{R}^{n+1} \rightarrow H, \quad q_H : \mathbb{R}^{n+1} \rightarrow H^\perp,$$  \hfill (2.10)
for the equatorial sphere defined by $H$ on $\mathbb{S}^n$ and for the orthogonal projections of $\mathbb{R}^{n+1}$ onto $H$ and onto $H^\perp = \{ t \nu_H : t \in \mathbb{R} \}$. We set
\[
X_\sigma(\Sigma_H) = \{ u \in C^1(\Sigma_H) : \|u\|_{C^1(\Sigma_H)} < \sigma \}, \quad \sigma > 0.
\]
Clearly there is $\sigma_0 = \sigma_0(n) > 0$ such that if $H \in \mathcal{H}$ and $u \in X_{\sigma_0}(\Sigma_H)$, then
\[
f_u(\omega) = \frac{\omega + u(\omega) \nu_H}{\sqrt{1 + u(\omega)^2}}, \quad \omega \in \Sigma_H,
\]
defines a diffeomorphism of $\Sigma_H$ into an hypersurface $\Sigma_H(u) \subset \mathbb{S}^n$, namely
\[
\Sigma_H(u) = f_u(\Sigma_H) = \left\{ \frac{\omega + u(\omega) \nu_H}{\sqrt{1 + u(\omega)^2}} : \omega \in \Sigma_H \right\}. \quad (2.11)
\]
We call $\Sigma_H(u)$ a spherical graph over $\Sigma_H$. Exploiting the fact that $\Sigma_H$ is a minimal hypersurface in $\mathbb{S}^n$ and that if $\{\tau_i\}$ is a local orthonormal frame on $\Sigma_H$ then $\nu_H \cdot \nabla_{\tau_i} \tau_j = 0$, a second variation computation (see, e.g., [ESV19, Lemma 2.1]) gives, for $u \in X_{\sigma}(\Sigma_H)$,
\[
|\mathcal{H}^{n-1}(\Sigma_H(u)) - n \omega_n - \frac{1}{2} \int_{\Sigma_H} |\nabla^{\Sigma_H} u|^2 - (n-1) u^2 | \leq C(n) \sigma \int_{\Sigma_H} u^2 + |\nabla^{\Sigma_H} u|^2,
\]
(where $n \omega_n = \mathcal{H}^{n-1}(\Sigma_H) = \mathcal{H}^{n-1}(\Sigma_H(0))$). We recall that $u \in L^2(\Sigma_H)$ is a unit norm Jacobi field of $\Sigma_H$ (i.e., a zero eigenvector of $\Delta^{\Sigma_H} + (n-1) \text{Id}$ with unit $L^2(\Sigma_H)$-norm) if and only if there is $\tau \in \mathbb{S}^n$ with $\tau \cdot \nu_H = 0$ and $u(\omega) = c_0(n) (\omega \cdot \tau) (\omega \in \Sigma_H)$ for $c_0(n) = (n/\mathcal{H}^{n-1}(\Sigma_H))^{1/2}$. We denote by $E^0_{\Sigma_H}$ the orthogonal projection operator of $L^2(\Sigma)$ onto the span of the Jacobi fields of $\Sigma_H$. The following lemma provides a way to reparameterize spherical graphs over equatorial spheres so that the projection over Jacobi fields is annihilated.

**Lemma 2.3.** There exist constants $C_0$, $\varepsilon_0$ and $\sigma_0$, depending on the dimension $n$ only, with the following properties:

*(i)*: if $H, K \in \mathcal{H}$, $|\nu_H - \nu_K| \leq \varepsilon < \varepsilon_0$, and $u \in X_\sigma(\Sigma_H)$ for $\sigma < \sigma_0$, then the map $T^K_u : \Sigma_H \rightarrow \Sigma_K$ defined by
\[
T^K_u(\omega) = \frac{p_K(\omega) p_K(\omega)}{|p_K(\omega) p_K(\omega)|} \frac{p_K(\omega) p_K(\omega) p_K(\omega) p_K(\omega) p_K(\omega)}{|p_K(\omega) p_K(\omega) p_K(\omega) p_K(\omega) p_K(\omega)|}, \quad \omega \in \Sigma_H,
\]
is a diffeomorphism between $\Sigma_H$ and $\Sigma_K$, and $v^K_u : \Sigma_K \rightarrow \mathbb{R}$ defined by
\[
v^K_u(T^K_u(\omega)) = \frac{q_K(\omega) q_K(\omega)}{|q_K(\omega) q_K(\omega)|} \frac{q_K(\omega) q_K(\omega) q_K(\omega) q_K(\omega) q_K(\omega)}{|q_K(\omega) q_K(\omega) q_K(\omega) q_K(\omega) q_K(\omega)|}, \quad \omega \in \Sigma_H, \quad (2.12)
\]
is such that
\[
v^K_u \in X_{C(\varepsilon)(\sigma+\varepsilon)}(\Sigma_K), \quad \Sigma_H(u) = \Sigma_K(v^K_u), \quad (2.13)
\]
\[
\left| \int_{\Sigma_K} (v^K_u)^2 - \int_{\Sigma_H} u^2 \right| \leq C(n) \left\{ |\nu_H - \nu_K|^2 + \int_{\Sigma_H} u^2 \right\}. \quad (2.14)
\]
(ii): if \( H \in \mathcal{H} \) and \( u \in \mathcal{X}_{\sigma_0}(\Sigma_H) \), then there exist \( K \in \mathcal{H} \) with \( |\nu_H - \nu_K| < \varepsilon_0 \) and \( v \in \mathcal{X}_{C_0\sigma_0}(\Sigma_K) \) such that
\[
\Sigma_H(u) = \Sigma_K(v),
\]
\[
E^0_{\Sigma_K}[v] = 0,
\]
\[
|\nu_K - \nu_H|^2 \leq C_0(n) \int_{\Sigma_H} (E^0_{\Sigma_H}[u])^2,
\]
\[
\left| \int_{\Sigma_K} v^2 - \int_{\Sigma_H} u^2 \right| \leq C_0(n) \int_{\Sigma_H} u^2.
\]

**Remark 2.4.** It may seem unnecessary to present a detailed proof of Lemma 2.3, as we are about to do, given that, when \( \Sigma_H \) is replaced by a generic integrable minimal surface \( \Sigma \) in \( \mathbb{S}^n \), similar statements are found in the first four sections of [AA81, Chapter 5]. However, two of those statements, namely [AA81, 5.3(4), 5.3(5)], seem not to be correct; and the issue requires clarification, since those statements are used in the iteration arguments for the blowup/blowdown theorems [AA81, Theorem 5.9/Theorem 9.6]; see, e.g., the second displayed chain of inequalities on [AA81, Page 254]. To explain this issue we momentarily adopt the notation of [AA81]. In [AA81, Chapter 5] they consider a family of minimal surfaces \( \{M_t\}_{t \in U} \) in \( \mathbb{S}^n \) obtained as diffeomorphic images of a minimal surface \( M = M_0 \). The parameter \( t \) ranges in an open ball \( U \subset \mathbb{R}^j \), where \( j \) is the dimension of the space of Jacobi fields of \( M \). Given a vector field \( Z \) in \( \mathbb{S}^n \), defined on and normal to \( M_t \), they denote by \( F_t(Z) \) the diffeomorphism of \( M_t \) into \( \mathbb{S}^n \) obtained by combining \( Z \) with the exponential map of \( \mathbb{S}^n \) (up to lower than second order corrections in \( Z \), this is equivalent to taking the graph of \( Z \) over \( M_t \), and then projecting it back on \( \mathbb{S}^n \), which is what we do, following [Sim83a], in (2.11)). Then, in [AA81, 5.2(2)], they define \( \Lambda_t \) as the family of those \( Z \) such that \( \text{Image}(F_t(Z)) = \text{Image}(F_0(W)) \) for some vector field \( W \) normal to \( M \), and, given \( t, u \in U \) and \( Z \in \Lambda_t \), they define \( F_t^u : \Lambda_t \to \Lambda_u \) as the map between such classes of normal vector fields with the property that \( \text{Image}(F_t(Z)) = \text{Image}(F_u(F_t^u(Z))) \); in particular, \( F_t^u(Z) \) is the vector field that takes \( M_u \) to the same surface to which \( Z \) takes \( M_t \). With this premise, in [AA81, 5.3(5)] they say that if \( t, u \in U \), and \( Z \in \Lambda_t \), then
\[
\left| \int_{M_u} |F_t^u(Z)|^2 - \int_{M_t} |Z|^2 \right| \leq C |t - u| \int_{M_t} |Z|^2,
\]
for a constant \( C \) depending on \( M \) only. Testing this with \( Z = 0 \) (notice that \( 0 \in \Lambda_t \) by [AA81, 5.3(1)]) one finds \( F_t^u(0) = 0 \), and thus \( M_t = \text{Image}(F_t(0)) = \text{Image}(F_u(F_t^u(0))) = \text{Image}(F_u(0)) = M_u \). In particular, \( M_u = M_t \) for every \( t, u \in U \), that is, \( \{M_t\}_{t \in U} \) consists of a single surface, \( M \) itself. But this is never the case since \( \{M_t\}_{t \in U} \) always contains, to the least, every sufficiently small rotation of \( M \) in \( \mathbb{S}^n \). An analogous problem is contained in [AA81, 5.3(4)]. Coming back to our notation, the analogous estimate to (2.19) in our setting would mean that, for every \( H, K \in \mathcal{H} \) with
\[|\nu_K - \nu_H| < \varepsilon_0 \text{ and } u \in \mathcal{X}_{\sigma_0}(\Sigma_H), \ v^K_u \text{ defined in (2.12)} \text{ satisfies}
\]
\[
\left| \int_{\Sigma_K} (v^K_u)^2 - \int_{\Sigma_H} u^2 \right| \leq C(n) |\nu_H - \nu_K| \int_{\Sigma_H} u^2,
\]
\[
(2.20)
\]
which again gives a contradiction if \( u = 0 \). A correct estimate, analogous in spirit to (2.20) and still sufficiently precise to be used in iterations, is (2.14) in Lemma 2.3. There should be no difficulty in adapting our proof to the more general context of integrable cones, and then in using the resulting generalization of (2.14) to implement the iterations needed in [AA81, Theorem 5.9, Theorem 9.6].

**Proof of Lemma 2.3.** The constants \( \varepsilon_0 \) and \( \sigma_0 \) in the statement will be such that \( \sigma_0 = \varepsilon_0/C_* \) for a sufficiently large dimension dependent constant \( C_* \).

**Step one:** To prove statement (i), let \( H, K \in \mathcal{H}, |\nu_H - \nu_K| \leq \varepsilon < \varepsilon_0 \) and \( u \in \mathcal{X}_\sigma(\Sigma_H) \) with \( \sigma < \sigma_0 \). Setting (for \( \omega \in \Sigma_H \) and \( x \in \mathbb{R}^{n+1} \setminus \{0\} \))
\[
g^K_u(\omega) = p_K \omega + u(\omega) p_K \nu_H, \quad \Phi(x) = x/|x|,
\]
we have \( T^K_u = \Phi \circ g^K_u \), and, if \( u \) is identically 0,
\[
g_0^K(\omega) = p_K \omega, \quad T_0^K(\omega) = \frac{p_K \omega}{|p_K \omega|}, \quad \forall \omega \in \Sigma_H.
\]
By \( |p_K \nu_H|^2 = 1 - (\nu_H \cdot \nu_K)^2 \leq 2 (1 - (\nu_H \cdot \nu_K)) = |\nu_H - \nu_K|^2 \),
\[
|g^K_u - g_0^K| = |u| |p_K \nu_H| \leq |u| |\nu_H - \nu_K|,
\]
\[
|\nabla_{\Sigma_H} g^K_u - \nabla_{\Sigma_H} g_0^K| \leq |\nabla_{\Sigma_H} u| |\nu_H - \nu_K|.
\]
In particular, \( |g^K_u| \geq 1 - \sigma_0 \varepsilon_0 \geq 1/2 \), and since \( \Phi \) and \( \nabla \Phi \) are Lipschitz continuous on \( \{|x| \geq 1/2\} \), we find
\[
\max \left\{ \|g^K_u - g_0^K\|_{C^1(\Sigma_H)}, \|T^K_u - T_0^K\|_{C^1(\Sigma_H)} \right\} \leq C(n) \|u\|_{C^1(\Sigma_H)} |\nu_H - \nu_K|.
\]
\[
(2.21)
\]
Similarly, since \( \omega \cdot \nu_K = \omega \cdot (\nu_K - \nu_H) \) for \( \omega \in \Sigma_H \), we find that
\[
\|g^K_0 - \text{id}\|_{C^1(\Sigma_H)} \leq C(n) |\nu_H - \nu_K|, \quad \|T^K_0 - \text{id}\|_{C^1(\Sigma_H)} \leq C(n) |\nu_H - \nu_K|,
\]
\[
(2.22)
\]
and we thus conclude that \( T^K_u \) is a diffeomorphism between \( \Sigma_H \) and \( \Sigma_K \). As a consequence, the definition (2.12) of \( v^K_u \) is well-posed, and (2.13) immediately follows (in particular, \( \Sigma_H(u) = \Sigma_K(v^K_u) \) is deduced easily from (2.12) and (2.11)). Finally, if we set \( F^K_u(\omega) = v^K_u(T^K_u(\omega))^2 J_{\Sigma_H} T^K_u(\omega) \) \( (\omega \in \Sigma_H) \), then
\[
\int_{\Sigma_K} (v^K_u)^2 - \int_{\Sigma_H} u^2 = \int_{\Sigma_H} \left( \frac{\nu_K \cdot (\omega + u \nu_H)}{|g^K_u(\omega)|} \right)^2 J_{\Sigma_H} T^K_u(\omega) - u^2,
\]
where, using again $|\omega \cdot \nu_K| \leq |\nu_H - \nu_K|$ for every $\omega \in \Sigma_H$, we find
\[
|J^{\Sigma_H} T^K_u(\omega) - 1| \leq C(n) \| T^K_u - \text{id} \|_{C^1(\Sigma_H)} \leq C(n) |\nu_H - \nu_K|,
\]
\[
1 - |g^K_u(\omega)|^2 \leq 1 - |p_K \omega|^2 + |p_K \nu_H| u^2 + 2 |u| |p_K \nu_H| |p_K \omega| 
\leq C \left(|\nu_H - \nu_K|^2 + u^2\right),
\]
\[
(\nu_K \cdot (\omega + u \nu_H))^2 - u^2 
\leq |\nu_K \cdot \omega|^2 + u^2 (1 - (\nu_H \cdot \nu_K)^2) + 2 |u| |\nu_H \cdot \nu_K| |\omega \cdot \nu_K| 
\leq |\nu_K - \nu_K|^2 + 2 u^2 |\nu_H - \nu_K| + 2 |u| |\nu_H - \nu_K| \leq C \left(|\nu_H - \nu_K|^2 + u^2\right)
\]
and thus, (2.14), thanks to
\[
\left| \int_{\Sigma_K} (v^K_u)^2 - \int_{\Sigma_H} u^2 \right| \leq \int_{\Sigma_H} |J^{\Sigma_H} T^K_u - 1| u^2 + 2 \frac{|(\nu_K \cdot (\omega + u \nu_H))^2 - u^2|}{|g^K_u|^2} 
+ 2 \int_{\Sigma_H} \left| 1 - \frac{1}{|g^K_u|^2} \right| u^2 \leq C(n) \left(|\nu_H - \nu_K|^2 + \int_{\Sigma_H} u^2 \right).
\]

**Step two:** We prove (ii). If $E_{\Sigma_H}^0[u] = 0$, then we conclude with $K = H$, $\nu = u$. We thus assume $\gamma^2 = \int_{\Sigma_H} (E_{\Sigma_H}^0[u])^2 > 0$, and pick an orthonormal basis $\{\phi_H^0\}_{i=1}^n$ of $L^2(\Sigma_H) \cap E_{\Sigma_H}^0[u] = \gamma \phi_H^0$ and $\gamma = \int_{\Sigma_H} u \phi_H^0 \neq 0$. This corresponds to choosing an orthonormal basis $\{\tau_H^i\}_{i=1}^n$ of $H$ such that
\[
\phi_H^0(\omega) = c_0(n) \omega \cdot \tau_H^i, \quad \omega \in \Sigma_H, \tag{2.23}
\]
for $c_0(n) = (n/\mathcal{H}^{n-1}(\Sigma_H))^{1/2}$. For each $K \in \mathcal{H}$ with $\text{dist}_{\Sigma_0}(\nu_H, \nu_K) < \varepsilon_0$ we define an orthonormal basis $\{\tau_K^i\}_{i=1}^n$ of $K$ by parallel transport of $\{\tau_H^i\}_{i=1}^n \subset H \equiv T_{\nu_H} S^n$ to $K \equiv T_{\nu_H} S^n$. The maps $\nu \mapsto \tau^i(\nu) := \tau_K^i(\nu)$ define an orthonormal frame $\{\tau^i\}_{i=1}^n$ of $S^n$ on the open set $A = B_{\mathcal{H}}^{\mathcal{H}}(\nu_H) = \{\nu \in S^n : \text{dist}_{\Sigma_0}(\nu, \nu_H) < \varepsilon_0\}$. We denote by $\rho^K_H$ the rotation of $\mathbb{R}^{n+1}$ which takes $H$ into $K$ by setting $\rho^K_H(\tau_H^i) = \tau_K^i$ and $\rho^K_H(\nu_H) = \nu_K$. By the properties of parallel transport we have that
\[
\|\rho^K_H - \text{id}\|_{C^0(\Sigma_K)} \leq C(n) \text{dist}_{\Sigma_0}(\nu_H, \nu_K) \leq C(n) \varepsilon_0. \tag{2.24}
\]
Finally, we define an $L^2(\Sigma_K)$-orthonormal basis $\{\phi_K^i\}_{i=1}^n$ of $L^2(\Sigma_K) \cap E_{\Sigma_K}^0 = 0$ by setting $\phi_K^i(\omega) = c_0(n) \omega \cdot \tau_K^i (\omega \in \Sigma_K)$, and correspondingly consider the map $\Psi_u : A \rightarrow \mathbb{R}^n$ defined by setting
\[
\Psi_u(\nu) = \left( \int_{\Sigma_K(\nu)} v^K_u(\nu) \phi_K^1(\nu), \ldots, \int_{\Sigma_K(\nu)} v^K_u(\nu) \phi_K^n(\nu) \right), \quad \nu \in A,
\]
where $v^K_u(\nu)$ is well-defined for every $\nu \in A$ thanks to step one. **We now claim** the existence of $\nu_* \in A$ such that $\Psi_u(\nu_*) = 0$. By the area formula,
which defines an homotopy between \( \Psi \)
the unit-speed length minimizing geodesic from
see that if
By (2.24), (2.25) and (2.26) we conclude that
\[ h \]
so that (2.21) gives

\[
\| \Psi_u - \Psi_0 \|_{C^1(\mathcal{A})} \leq C(n) \sigma_0 , \quad \text{where}
\]

(2.25)

\[
e_j \cdot \Psi_0(\nu) = c_0(n) \int_{\Sigma_H} \nu \cdot \omega \left( \rho_H^{K(\nu)} \frac{\partial_j}{|\partial_j|} \right) \cdot \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|^2} J_{\Sigma_H} \left[ \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|} \right] d\mathcal{H}_n^{n-1}.
\]

By definition of \( A \) and by (2.22) and (2.24),

\[
\sup_{\nu \in A} \sup_{\omega \in \Sigma_H} | \tau_j^i \cdot \omega - \left( \rho_H^{K(\nu)} \frac{\partial_j}{|\partial_j|} \right) \cdot \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|^2} J_{\Sigma_H} \left[ \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|} \right] | \leq C(n) \varepsilon_0 ,
\]

and thus \( \| \Psi_0 - \Psi_* \|_{C^1(\mathcal{A})} \leq C(n) (\sigma_0 + \varepsilon_0) \),

(2.26)

where \( \Psi_* : A \to \mathbb{R}^n \) is defined by

\[
e_j \cdot \Psi_*(\nu) = c_0(n) \int_{\Sigma_H} \nu \cdot \omega \left( \rho_H^{K(\nu)} \frac{\partial_j}{|\partial_j|} \right) \cdot \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|^2} J_{\Sigma_H} \left[ \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|} \right] d\mathcal{H}_n^{n-1}
\]

(\( \nu \in A \)). Recalling that \( \{ \tau^i_j \}_{i=1}^n \) is an orthonormal frame of \( S^n \) on \( A \), with

\[
\nabla_{\tau^i} \nu = \tau^i(\nu) = \rho_H^{K(\nu)} \frac{\partial_j}{|\partial_j|},
\]

we find

\[
e_j \cdot \nabla_{\tau^i} \Psi_*(\nu) = c_0(n) \int_{\Sigma_H} \tau_j^i \cdot \omega J_{\Sigma_H} \left[ \frac{\mathbf{P}_{K\omega}}{|\mathbf{P}_{K\omega}|} \right] d\mathcal{H}_n^{n-1},
\]

(2.24), (2.25) and (2.26) we conclude that

\[
\| \Psi_u - \Psi_* \|_{C^1(\mathcal{A})} \leq C(n) (\sigma_0 + \varepsilon_0) , \quad \text{and}
\]

(2.27)

\[
\| \nabla_{S^n} \Psi_u - c_0(n)^{-1} \sum_{j=1}^n e_j \otimes \tau^j \|_{C^0(\mathcal{A})} \leq C(n) (\sigma_0 + \varepsilon_0).
\]

(2.28)

Let us finally consider the map \( h : A \times [0, 1] \to \mathbb{R}^n \),

\[
h(\nu, t) = h_t(\nu) = t \Psi_*(\nu) + (1 - t) \Psi_u(\nu) \quad (\nu, t) \in A \times [0, 1] ,
\]

which defines an homotopy between \( \Psi_u \) and \( \Psi_0 \). By (2.27) and (2.28) we see that if \( \nu \in \partial A \), that is, if \( \text{dist}_{S^n}(\nu, \nu_H) = \varepsilon_0 \), then, denoting by \( [\nu_H, \nu]_s \) the unit-speed length minimizing geodesic from \( \nu_H \) to \( \nu \), considering that \( [\nu_H, \nu]_s \in A \) for every \( s \in (0, \varepsilon_0) \), and that \( S^n \) is close to be flat in \( A \), we find

\[
|h_t(\nu)| \geq \left| \int_0^{\varepsilon_0} \frac{d}{ds} h_t([\nu_H, \nu]_s) \right| ds - |h_t(\nu_H)| \geq \left( \frac{1}{c_0(n)} - C(n) (\varepsilon_0 + \sigma_0) \right) \varepsilon_0 - C(n) \sigma_0 \geq \frac{\varepsilon_0}{2 c_0(n)} ,
\]

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provided \( \sigma_0 = \varepsilon_0 / C_* \) is small enough with respect to \( \varepsilon_0 \) (i.e., provided \( C_* \) is large), \( \varepsilon_0 \) is small in terms of \( c_0 \), and where we have used \( \Psi_s(\nu_H) = 0 \) and
\[
|\Psi_u(\nu_H)| = |\gamma| = \left| \int_{\Sigma_H} u \phi_H^1 \right| \leq C(n) \sigma_0 , \tag{2.29}
\]
to deduce \( |h_t(\nu_H)| \leq C(n) \sigma_0 . \) This proves that \( 0 \not\in \partial h_t(\partial A) \) for every \( t \in [0,1] \), so that \( \deg(h_t, A, 0) \) is independent of \( t \in [0,1] \). In particular, \( h_0 = \Psi_u \) and \( h_1 = \Psi_s \) give \( \deg(\Psi_u, A, 0) = \deg(\Psi_s, A, 0) = 1 \), where we have used \( \Psi_s(\nu_H) = 0 \) and the fact that, up to decreasing the value of \( \varepsilon_0 \), \( \Psi_s \) is injective on \( A \). By \( \deg(\Psi_u, \rho) = 1 \), there is \( \nu_* \in A \) such that \( \Psi_u(\nu_*) = 0 \), as claimed. With \( K = K(\nu_*) \) and \( v = v^K_u \) we deduce (2.15) from (2.13) and (2.16) from \( \Psi_u(\nu_*) = 0 \). By (2.28) and (2.29), if \( \eta = \text{dist}_{\Sigma_s}(\nu_*, \nu_H) \), then
\[
\left( \int_{\Sigma_H} (E^0_{\Sigma_H}[u])^2 \right)^{1/2} = |\gamma| = |\Psi_u(\nu_H)| = |\Psi_u(\nu_H) - \Psi_u(\nu_*)| = \left| \int_0^\eta ds \Psi_u([\nu_H, \nu_*)] ds \right| \geq \left( \frac{1}{c_0(n)} - C(n) (\varepsilon_0 + \sigma_0) \right) \eta \geq \left| \frac{\nu_* - \nu_H}{2c_0(n)} \right| ,
\]
that is (2.17). Finally, (2.18) follows from (2.17) and (2.14). \( \square \)

2.3. Energy estimates for spherical graphs over annuli. Given \( H \in \mathcal{H} \) and \( 0 < r_1 < r_2 \) we let \( \mathcal{X}_0(\Sigma_H, r_1, r_2) \) be the class of those \( u \in C^1(\Sigma_H \times (r_1, r_2)) \) such that, setting \( u_r = u(\cdot, r) \), one has \( u_r \in \mathcal{X}_0(\Sigma_H) \) for every \( r \in (r_1, r_2) \) and \( |r \partial_r u| \leq \sigma \) on \( \Sigma_H \times (r_1, r_2) \). If \( u \in \mathcal{X}_0(\Sigma_H, r_1, r_2) \), then the spherical graph of \( u \) over \( \Sigma_H \times (r_1, r_2) \), given by
\[
\Sigma_H(u, r_1, r_2) = \left\{ r \omega + u_r(\omega) \nu_H \sqrt{1 + u_r(\omega)^2} : \omega \in \Sigma_H, r \in (r_1, r_2) \right\} ,
\]
is an hypersurface in \( A^n_H \). It is useful to keep in mind that \( \Sigma_H(0, r_1, r_2) = \{ r \omega : \omega \in \Sigma, r \in (r_1, r_2) \} = H \cap A^n_H \) is a flat annular region of area \( \omega_n (r_2^n - r_1^n) \), and that if \( \sigma < \sigma_1 - \sigma_1(n) \), then
\[
\frac{1}{C(n)} \int_{\Sigma_H(u, r_1, r_2)} \omega^2_H dH^n \leq \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} u^2 \leq C(n) \int_{\Sigma_H(u, r_1, r_2)} \omega^2_H dH^n . \tag{2.30}
\]

Lemma 2.5. There are \( \varepsilon_0, \sigma_0, C_0 \) positive, depending on \( n \) only, such that:

(i): if \( H, K \in \mathcal{H}, \nu_H \cdot \nu_K > 0, |\nu_H - \nu_K| = \varepsilon < \varepsilon_0, u \in \mathcal{X}_0(\Sigma_H, r_1, r_2), \) and \( \sigma < \sigma_0 \), then there is \( v \in \mathcal{X}_{C_0(\varepsilon)}(\Sigma_K, r_1, r_2) \) such that \( \Sigma_K(v, r_1, r_2) = \Sigma_H(u, r_1, r_2) \).

(ii): if \( H \in \mathcal{H}, u \in \mathcal{X}_{C_0(\varepsilon)}(\Sigma_H, r_1, r_2), \) and \( (a, b) \subset (r_1, r_2), \) then there exist \( K \in \mathcal{H}, v \in \mathcal{X}_{C_0(\varepsilon)}(\Sigma_K, r_1, r_2), \) and \( r_* \in (a, b) \) such that
\[
\Sigma_H(u, r_1, r_2) = \Sigma_K(v, r_1, r_2) , \tag{2.31}
\]
\[
E^0_{\Sigma_K}(v, r_*) = 0 , \tag{2.32}
\]
\[
|\nu_H - \nu_K|^2 \leq C(n) \min_{\rho \in [a, b]} \int_{\Sigma_H} (E^0_{\Sigma_H}[u])^2 . \tag{2.33}
\]
Moreover, for every \( r \in (r_1, r_2) \),
\[
\left| \int_{\Sigma_K} (v_r)^2 - \int_{\Sigma_H} (u_r)^2 \right| \leq C_0(n) \left\{ \min_{\rho \in [a, b]} \int_{\Sigma_H} (u_\rho)^2 + \int_{\Sigma_H} (u_r)^2 \right\}. \tag{2.34}
\]

**Proof.** We prove statement (i). If \( |\nu_H - \nu_K| = \varepsilon < \varepsilon_0 \), since \( u_r \in X_\sigma(\Sigma_H) \) for every \( r \in (r_1, r_2) \), by Lemma 2.3-(ii) we see that \( T_r : \Sigma_H \to \Sigma_K \),
\[
T_r(\omega) = |p_K[\omega + u_r(\omega) \nu_H]|^{-1} p_K[\omega + u_r(\omega) \nu_H] \quad \omega \in \Sigma_H, \tag{2.35}
\]
is a diffeomorphism between \( \Sigma_H \) and \( \Sigma_K \), and \( v_r : \Sigma_K \to \mathbb{R} \),
\[
v_r(T_r(\omega)) = \frac{\nu_K \cdot (\omega + u_r(\omega) \nu_H)}{|p_K[\omega + u_r(\omega) \nu_H]|}, \quad \omega \in \Sigma_H, \tag{2.36}
\]
satisfies \( v_r \in X_{C_0(\sigma + \varepsilon)}(\Sigma_K) \), \( \Sigma_H(u_r) = \Sigma_K(v_r) \) for every \( r \in (r_1, r_2) \), and
\[
\left| \int_{\Sigma_K} (v_r)^2 - \int_{\Sigma_H} (u_r)^2 \right| \leq C(n) \left\{ |\nu_H - \nu_K|^2 + \int_{\Sigma_H} (u_r)^2 \right\}. \tag{2.37}
\]
Since \( u \in X_\sigma(\Sigma_H, r_1, r_2) \), and \( T_r \) and \( v_r \) depend smoothly on \( u_r \), setting \( v(\omega, r) := v_r(\omega) \) we have \( \Sigma_H(u, r_1, r_2) = \Sigma_K(v, r_1, r_2) \) (by \( \Sigma_H(u_r) = \Sigma_K(v_r) \) for every \( r \in (r_1, r_2) \)), and \( v \in X_{C_0(\sigma + \varepsilon)}(\Sigma_H, r_1, r_2) \) \(|r \partial_r v_r| \leq C_0(\sigma + \varepsilon)\) is deduced by differentiation in (2.35) and (2.36), and by \(|u_r|, |r \partial_r u_r| < \sigma\).

**Step two:** We prove (ii). Let \( \gamma = \min_{\rho \in [a, b]} \int_{\Sigma_H} (E_{\Sigma_H}^0[u_\rho])^2 \), and let \( r_* \in [a, b] \) be such that the minimum \( \gamma \) is achieved at \( r = r_* \). If \( \gamma = 0 \), then we set \( K = H \) and \( v = u \). If \( \gamma > 0 \), then we apply Lemma 2.3-(ii) to \( u_{r_*} \in X_{C_0(\Sigma_H)} \), and find \( K \in \mathcal{H} \) with \( |\nu_K - \nu_H| < \varepsilon_0 \) and \( v_{r_*} \in X_{C_0(\Sigma_K)} \) such that \( \Sigma_H(u_{r_*}) = \Sigma_K(v_{r_*}) \) and
\[
E_{\Sigma_K}^0[v_{r_*}] = 0, \tag{2.38}
\]
\[
|\nu_K - \nu_H|^2 \leq C_0(n) \int_{\Sigma_H} (E_{\Sigma_H}^0[u_{r_*}])^2 = C_0(n) \gamma, \tag{2.39}
\]
\[
\left| \int_{\Sigma_K} (v_{r_*})^2 - \int_{\Sigma_H} (u_{r_*})^2 \right| \leq C_0(n) \int_{\Sigma_H} (u_{r_*})^2. \tag{2.40}
\]
Since \( v_{r_*} = v(\cdot, r_*) \) for \( v \) constructed in step one starting from \( u, H \) and \( K \), we deduce (2.34) by (2.37) and (2.39), while (2.38) is (2.32). \( \square \)

We will use two basic “energy estimates” for spherical graphs over annuli. To streamline the application of these estimates to diadic families of annuli we consider intervals \((r_1, r_2)\) and \((r_3, r_4)\) are \((\eta, \eta_0)\)-related, meaning that
\[
r_2 = r_0(1 + \eta_0), \quad r_1 = r_0(1 - \eta_0), \quad r_4 = r_0(1 + \eta), \quad r_3 = r_0(1 - \eta), \tag{2.41}
\]
for some \( \eta_0 > \eta > 0 \), and with \( r_0 = (r_1 + r_2)/2 = (r_3 + r_4)/2 \); in particular, \((r_3, r_4)\) is contained in, and concentric to, \((r_1, r_2)\). The case \( \Lambda = 0 \) of the following statement is the codimension one, equatorial spheres case of [AA81, Lemma 7.14, Theorem 7.15].
Theorem 2.6 (Energy estimates for spherical graphs). If \( n \geq 2 \) and \( \eta_0 > \eta > 0 \), then there are \( \sigma_0 = \sigma_0(n, \eta_0, \eta) \) and \( C_0 = C_0(n, \eta_0, \eta) \) positive, with the following property. If \( H \in \mathcal{H}, \Lambda \geq 0 \), and \( u \in \mathcal{X}_r(\Sigma_H, r_1, r_2) \) is such that \( \max\{1, \Lambda r_2\} \sigma \leq \sigma_0 \) and \( \Sigma_H(u, r_1, r_2) \) has mean curvature bounded by \( \Lambda \) in \( \mathcal{A}_2^2 \), then, whenever \((r_1, r_2)\) and \((r_3, r_4)\) are \((\eta, \eta_0)\)-related as in (2.41),

\[
\left| \mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4)) \right| \leq C_0 \int_{\mathcal{H} \times (r_1, r_2)} r^{n-1} (u^2 + \Lambda r |u|).
\]

Moreover, if there is \( r \in (r_1, r_2) \) s.t. \( E^0_{\Sigma_H} u_r = 0 \) on \( \Sigma_H \), then we also have

\[
\int_{\mathcal{H} \times (r_3, r_4)} r^{n-1} u^2 \leq C(n) \Lambda r_2 (r_2^n - r_1^n) + C_0 \int_{\mathcal{H} \times (r_1, r_2)} r^{n-1} (r \partial_r u)^2.
\]

Proof. See appendix A. \( \square \)

2.4. Monotonicity for exterior varifolds with bounded mean curvature. The following theorem states the monotonicity of \( \Theta_{V,R,A} \) for \( V \in \mathcal{V}_n(\Lambda, R, S) \), and provides, when \( V \) corresponds to a spherical graph, a quantitative lower bound for the gap in the associated monotonicity formula; the case \( \Lambda = 0, R = 0 \) is contained in [AA81, Lemma 7.16, Theorem 7.17].

Theorem 2.7. (i): If \( V \in \mathcal{V}_n(\Lambda, R, S) \), then

\[
\Theta_{V,R,A} \text{ is increasing on } (R, S).
\] (2.42)

(ii): There is \( \sigma_0(n) \) such that, if \( V \in \mathcal{V}_n(\Lambda, R, S) \) and, for some \( H \in \mathcal{H}, \) \( u \in \mathcal{X}_r(\Sigma, r_1, r_2) \) with \( \sigma \leq \sigma_0(n) \), and \((r_1, r_2) \subset (R, S)\), we have \( V \) corresponds to \( \Sigma_H(u, r_1, r_2) \) in \( \mathcal{A}_1^2 \),

\[
\int_{\mathcal{H} \times (r_1, r_2)} r^{n-1} (r \partial_r u)^2 \leq C(n) r_2^n \left\{ \Theta_{V,R,A}(r_2) - \Theta_{V,R,A}(r_1) \right\}.
\] (2.44)

(iii): Finally, given \( \eta_0 > \eta > 0 \), there exist \( \sigma_0 \) and \( C_0 \) depending on \( n, \eta_0, \) and \( \eta \) only, such that if the assumptions of part (i) and part (ii) hold and, in addition to that, we also have \( \max\{1, \Lambda r_2\} \sigma \leq \sigma_0 \) and

\[
\exists r \in (r_1, r_2) \text{ s.t. } E^0_{\Sigma_H} u_r = 0 \text{ on } \Sigma_H,
\] (2.45)

then, whenever \((r_1, r_2)\) and \((r_3, r_4)\) are \((\eta, \eta_0)\)-related as in (2.41), we have

\[
\left| \mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4)) \right| \leq C_0 r_2^n \left\{ \Theta_{V,R,A}(r_2) - \Theta_{V,R,A}(r_1) + (\Lambda r_2)^2 \right\}.
\] (2.46)

Proof. We give details of the proof of (i) when \( V \in \mathcal{M}_n(\Lambda, R, S) \) (whereas the general case is addressed as in [Sim83b, Section 17]). By the coarea
thus proving (ii). To prove (iii), we set a statement should be compared to [AA81, Lemma 5.7].

The result to the proof of Theorem 2.1, we prove the following lemma, where this proves (2.42). Assuming now (2.43), by using [AA81, Lemma 3.5(6)] as

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We find (2.46) by (2.44) and \( C(n, r_2) \int_{r_1}^{r_2} C(n, r_2) d\rho \geq \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r u)^2 \), thus proving (ii). To prove (iii), we set \( a = r_0 (1 - (\eta + \eta_0)/2) \) and \( b = r_0 (1 + (\eta + \eta_0)/2) \), so that \( (a, b) \) and \( (r_3, r_4) \) are \( (\eta, (\eta + \eta_0)/2) \)-related, and \( (r_1, r_2) \) and \( (a, b) \) are \( ((\eta + \eta_0)/2, \eta_0) \)-related (in particular, \( (r_3, r_4) \subset (a, b) \subset (r_1, r_2) \)). By suitably choosing \( \sigma_0 \) in terms of \( n, \eta \) and \( \eta_0 \), we can apply Theorem 2.6 with \( (r_3, r_4) \) and \( (a, b) \), so to find (with \( C = C(n, \eta_0, \eta) \))

\[
\|H^n(\Sigma(u, r_3, r_4)) - H^n(\Sigma(0, r_3, r_4))\| \leq C \int_{\Sigma_H \times (a,b)} r^{n-1}(u^2 + \Lambda r |u|) \\
\leq C \left\{ (A b^2 (b^n - a^n) + \int_{\Sigma_H \times (a,b)} r^{n-1} u^2 \right\}.
\]

Thanks to (2.45) we can apply Theorem 2.6 with \( (a, b) \) and \( (r_1, r_2) \) to find

\[
\int_{\Sigma_H \times (a,b)} r^{n-1} u^2 \leq C \left\{ (A r_2^2 (r_2^n - r_1^n) + \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r u)^2 \right\}.
\]

We find (2.46) by (2.44) and \( (A b^2 (b^n - a^n) \leq (A r_2^2 r_2^n \right) \). \( \square \)

2.5. Proof of the mesoscale flatness criterion. As a final preliminary result to the proof of Theorem 2.1, we prove the following lemma, where Allard’s regularity theorem is combined with a compactness argument to provide the basic graphicality criterion used throughout the iteration. The statement should be compared to [AA81, Lemma 5.7].
Lemma 2.8 (Graphicality lemma). Let \( n \geq 2 \). For every \( \sigma > 0 \), \( \Gamma \geq 0 \), \( (\lambda_3, \lambda_4) \subset \subset (\lambda_1, \lambda_2) \subset \subset (0, 1) \), and \( (\eta_1, \eta_2) \subset \subset (0, 1) \), there are positive constants \( \varepsilon_1 \) and \( M_1 \), depending only on \( n \), \( \sigma \), \( \Gamma \), \( (\lambda_1, \lambda_2) \), \( (\lambda_3, \lambda_4) \), and \( (\eta_1, \eta_2) \), and \( \varepsilon_2 \) and \( M_2 \), depending only on \( n \), \( \sigma \), \( \Gamma \), \( \lambda_1 \), and \( (\eta_1, \eta_2) \), with the following properties.

(i): If \( \Lambda \geq 0 \), \( R \in (0, 1/\Lambda) \), \( V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda) \),

\[
\|\bdy V\|((\partial B_R) \leq \Gamma R^{n-1}, \quad \sup_{\rho \in (R, 1/\Lambda)} \frac{\|V\|(B_\rho \setminus B_R)}{\rho^n} \leq \Gamma ,
\]

there exists \( r > 0 \) such that

\[
\max \{M_1, 64\} R \leq r \leq \frac{\varepsilon_1}{\Lambda},
\]

\[
|\delta_{V, R, \Lambda}(r)| \leq \varepsilon_1 ,
\]

\[
\|V\|(A_{\lambda_3}^4 r) > 0 ,
\]

and if, for some \( K \in \mathcal{H} \), we have

\[
\frac{1}{r^n} \int_{A_{\lambda_3}^4 r} \omega_K^2 d\|V\| \leq \varepsilon_1 ,
\]

then there exists \( u \in X_\sigma(\Sigma_K, \eta_1 r, \eta_2 r) \) such that

\( V \) corresponds to \( \Sigma_K(u, \eta_1 r, \eta_2 r) \) on \( A_{\eta_2 r}^n \).

(ii): If \( \Lambda \), \( R \), and \( V \) are as in (i), (2.48) holds, and there exists \( r \) such that

\[
\max \{M_2, 64\} R \leq r \leq \frac{\varepsilon_2}{\Lambda},
\]

\[
\max \{|\delta_{V, R, \Lambda}(\lambda_1 r)|, |\delta_{V, R, \Lambda}(r)|\} \leq \varepsilon_2 ,
\]

then there exists \( K \in \mathcal{H} \) and \( u \in X_\sigma(\Sigma_K, \eta_1 r, \eta_2 r) \) such that

\( V \) corresponds to \( \Sigma_K(u, \eta_1 r, \eta_2 r) \) on \( A_{\eta_2 r}^n \).

Proof. Step one: As a preliminary, we first show that if \( V \) is a stationary, \( n \)-dimensional, integer rectifiable varifold in \( B_1 \) such that

\[
\|V\|(B_1) \leq \omega_n , \quad \text{spt} V \cap A_{\beta_2}^{\beta_1} \subset K , \quad \text{and spt} V \cap A_{\beta_1}^{\beta_2} \neq \emptyset ,
\]

for some \( K \in \mathcal{H} \) and \( 0 < \beta_1 < \beta_2 \leq 1 \), then \( V = \text{var} \,(K \cap B_1, 1|_{K \cap B_1}) \).

Let \( \beta' \subset (\beta_1, \beta_2) \) and \( \varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^{n+1}; [0, 1]) \) be such that \( \text{spt} \varphi_1 \subset B_{\beta_2}, \varphi_1|_{B_{\beta_2}} \equiv 1 \), and \( \varphi_1 + \varphi_2 \equiv 1 \). As a consequence of (2.55) and the stationarity of \( V \) in \( B_{\beta_2} \), for \( X \in C_c^1(\mathbb{R}^{n+1} \setminus (K \cap (\overline{B_{\beta_2}} \setminus B_{\beta'}))) \), we have

\[
\delta(V \cdot B_{\beta'}) (X) = \int_{B_{\beta'}} \div^M (\varphi_1 X) + \div^M (\varphi_2 X) d\|V\|
\]

\[
= \int_{B_{\beta_2}} \div^M (\varphi_1 X) d\|V\| = 0 .
\]

Then by the convex hull property [Sim83b, Theorem 19.2], \( \text{spt} (V \cdot B_{\beta'}) \subset K \).

By the constancy theorem [Sim83b, Theorem 41.1], \( V \cdot B_{\beta_2} = \text{var} \,(K \cap B_{\beta_2}, \theta) \) for some constant \( \theta \). Furthermore, since \( V \) assigns non-trivial mass
to $B_{\beta}$ by (2.55) and is integer rectifiable, $\theta \geq 1$. Therefore $0 \in \text{spt} \|V\|$, and the monotonicity formula gives $\omega_n \leq \lim_{\gamma \to 0^+} \|V\|(B_\gamma) r^{-n} \leq \|V\|(B_1) \leq \omega_n$. Thus $V$ is a stationary, $n$-dimensional, integer rectifiable varifold in $B_1$ with constant area ratios $\omega_n$ and $\text{spt}V \cap A_{\beta_1}^{\beta_2} \subset K$, so $V = \text{var}(K \cap B_1, 1|_{K \cap B_1})$.

Step two: We prove item (i) by contradiction. If it were false, we could find $\sigma > 0$, $\Gamma \geq 0$, $(\lambda_3, \lambda_4) \subset (\lambda_1, \lambda_2) \subset (0, 1)$, $(\eta_1, \eta_2) \subset (0, 1)$, with $K_j \in \mathcal{H}$, positive numbers $R_j$, $A_j < 1/R_j$, and $W_j \in \mathcal{V}_n(\Lambda_j, R_j, 1/\Lambda_j)$ such that $\|W_j\|(A_{\lambda_j}^{\lambda_j} \Lambda_j) > 0$, $\|\text{bd} W_j\|((\partial B_{R_j}) \setminus B_{R_j}) \geq \Gamma R_j^{-n-1}$, $\|W_j\|(B_\rho \setminus B_{R_j}) \leq \Gamma \rho^\eta$ for every $\rho \in (R_j, 1/\Lambda_j)$, and $\rho_j = R_j/r_j \to 0$, $r_j A_j \to 0$, $\delta_{W_j, R_j, \Lambda_j}(r_j) \to 0$, and $r_j^{-n} \int_{B_{\lambda_j} \setminus B_{\lambda_j}} \omega_{\lambda_j}^2 d\|W_j\| \to 0$, but there is no $u \in \mathcal{V}_n(\Sigma_{K_j}, \eta_1 r_j, \eta_2 r_j)$ with the property that $W_j$ corresponds to $\Sigma_{K_j}(u, \eta_1 r_j, \eta_2 r_j)$ on $A_{\eta_1}^{\eta_2}$, $r_j$. Hence, setting $V_j = W_j/r_j$, no $u \in \mathcal{V}_n(\Sigma_{K_j}, \eta_1, \eta_2)$ can exist such that $V_j$ corresponds to $\Sigma_{K_j}(u, \eta_1, \eta_2)$ on $A_{\eta_1}^{\eta_2}$, despite the fact that each $V_j$ belongs to $\mathcal{V}_n(\rho A_j, \rho, 1/(r_j A_j))$ and satisfies

$$\|V_j\|(A_{\lambda_j}^{\lambda_j}) > 0, \quad \frac{\|\text{bd} V_j\|((\partial B_{\rho_j}) \setminus B_{\rho_j})}{\rho_j^{n-1}} \leq \Gamma, \quad \sup_{\rho \in (\rho_j, 1/(\Lambda_j r_j))} \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^\eta} \leq \Gamma,$$

$$\lim_{j \to \infty} \max \{\delta_{V_j, \rho_j, r_j, \Lambda_j}(1), \int_{A_{\lambda_j}^{\lambda_j}} \omega_{\lambda_j}^2 d\|V_j\|\} = 0. \quad (2.56)$$

Clearly we can find $K \in \mathcal{H}$ such that, up to extracting subsequences, $K_j \cap B_1 \to K \cap B_1$ in $L^1(\mathbb{R}^n)$. Similarly, by (2.56), we can find an $n$-dimensional integer rectifiable varifold $V$ such that $V_j \to V$ as varifolds in $B_1 \setminus \{0\}$. Since the bound on the distributional mean curvature of $V_j$ on $B_1/(\Lambda_j r_j) \setminus B_{\rho_j}$ is $r_j A_j$, and since $\rho_j \to 0^+$ and $r_j A_j \to 0^+$, it also follows that $V$ is stationary in $B_1 \setminus \{0\}$, and thus, by a standard argument and since $n \geq 2$, on $B_1$. By $\|V_j\|(A_{\lambda_j}^{\lambda_j}) > 0$, for every $j$ there is $x_j \in A_{\lambda_j}^{\lambda_j} \cap \text{spt} V_j$, so that, up to extracting subsequences, $x_j \to x_0$ for some $x_0 \in \overline{A_{\lambda_j}^{\lambda_j}} \cap \text{spt} V$. By $(\lambda_3, \lambda_4) \subset (\lambda_1, \lambda_2)$, there is $\rho > 0$ such that $B_{\rho}(x_0) \subset A_{\lambda_j}^{\lambda_j}$, hence

$$\|V\|(A_{\lambda_j}^{\lambda_j}) \geq \|V\|(B_\rho(x_0)) \geq \omega_n \rho^\eta > 0, \quad (2.57)$$

thus proving $V \cap A_{\lambda_j}^{\lambda_j} \neq \emptyset$. By this last fact, by $\omega_K = 0$ on $(\text{spt} V) \cap A_{\lambda_j}^{\lambda_j}$, and by the constancy theorem [Sim83b, Theorem 4.1.1], we have

$$A_{\lambda_j}^{\lambda_j} \cap \text{spt} V = A_{\lambda_j}^{\lambda_j} \cap K. \quad (2.58)$$

At the same time, since $\|\text{bd} V_j\|((\partial B_{\rho_j}) \setminus B_{\rho_j}) \leq \Gamma \rho_j^{n-1}$ and $\|V_j\|(B_\rho \setminus B_{\rho_j}) \leq \Gamma \rho^\eta$ for every $\rho \in (\rho_j, 1/(\Lambda_j r_j)) \supset (\rho_j, 1)$, by (2.56),

$$\omega_n = \lim_{j \to \infty} \|V_j\|(B_1 \setminus B_{\rho_j}) - \frac{\rho_j}{n} \|\text{bd} V_j\|((\partial B_{\rho_j}) \setminus B_{\rho_j}) + \Lambda_j r_j \int_{\rho_j}^{1} \frac{\|V_j\|(B_\rho \setminus B_{\rho_j})}{\rho^\eta} d\rho \geq \|V\|(B_1) - \Gamma \lim_{j \to \infty} (\rho_j^\eta + \Lambda_j r_j) = \|V\|(B_1). \quad (2.59)$$
Since \( V \) is stationary in \( B_1 \) and integer rectifiable, and since (2.57) and (2.59) imply (2.55) with \( \lambda_1 = \beta_1 \) and \( \lambda_2 = \beta_2 \), the first step yields \( V = \text{var} (K \cap B_1, 1|_K \cap B_1) \). By Allard’s regularity theorem and by \( V_j \rightharpoonup V \) as \( j \to \infty \) we deduce the existence of a sequence \( \{u_j\}_j \), with \( u_j \in \mathcal{X}_{\sigma_j} (\Sigma_K, \eta_1, \eta_2) \) for some \( \sigma_j \to 0 \) as \( j \to \infty \), such that \( V_j \) corresponds to \( \Sigma_K (u_j, \eta_1, \eta_2) \) in \( A^n_{\eta_2} \) for \( j \) large enough. As soon as \( j \) is large enough to give \( \sigma_j < \sigma \), we have reached a contradiction.

**Step three:** For item (ii), we again argue by contradiction. Should the lemma be false, then we could find \( \sigma > 0 \), \( \Gamma \geq 0 \), \( \lambda_1 \in (0, 1) \), positive numbers \( R_j, \Lambda_j \), \( \Lambda_j < 1/R_j, r_j \), and, by the same rescaling as in step two, \( V_j \in \mathcal{V}_n (r_j \Lambda_j, \rho_j, (r_j \Lambda_j)) \) with

\[
\frac{\| \text{bd} V_j \| (\partial B_{\rho_j})}{\rho_j^{n-1}} \leq \Gamma, \quad \sup_{\rho \in (\rho_j, 1/(\Lambda_j r_j))} \frac{\| V_j \| (B_{\rho} \setminus B_{\rho_j})}{\rho^n} \leq \Gamma, \quad (2.60)
\]

\[
\lim_{j \to \infty} \sup \left\{ \rho_j = \frac{R_j}{r_j}, r_j \Lambda_j, \| \delta V_j, \rho_j, r_j \Lambda_j (1) \|, \rho_j, r_j \Lambda_j (\Lambda_1) \right\} = 0, \quad (2.61)
\]

such that there exists no \( u \in \mathcal{X}_c (\Sigma K_j, \eta_1, \eta_2) \) on \( A^n_{\eta_1} \). As in step two, we can find an \( n \)-dimensional integer rectifiable varifold \( V = \text{var} (M, \theta) \) such that \( V_j \rightharpoonup V \) as varifolds in \( B_1 \setminus \{0\} \) and \( V \) is stationary on \( B_1 \). If for some \( K \in \mathcal{H} \), \( V = \text{var} (K \cap B_1, 1|_K \cap B_1) \), then using Allard’s theorem as in the proof of (i), we have a contradiction. So we prove \( V = \text{var} (K \cap B_1, 1|_K \cap B_1) \).

For every \( r \in [\lambda_1, 1] \), using \( \rho_j \to 0^+ \) and \( r_j \Lambda_j \to 0^+ \) in conjunction with (2.60), and then the monotonicity of \( \delta V_j, \rho_j, r_j \Lambda_j \) and (2.61), we have

\[
\lim_{j \to \infty} \left| \omega_n - \frac{\| V_j \| (B_r \setminus B_{\rho_j})}{r^n} \right| = \lim_{j \to \infty} \left| \delta V_j, \rho_j, r_j \Lambda_j (r) \right| \leq \lim_{j \to \infty} \max_{r \in \{\lambda_1, 1\}} \left| \delta V_j, \rho_j, r_j \Lambda_j (r) \right| = 0.
\]

Thus the convergence \( V_j \rightharpoonup V \) and the monotonicity of \( \| V \| (B_r)/r^n \) yield

\[
\| V \| (B_r) = \omega_n r^n \quad \forall r \in (\lambda_1, 1) \quad \text{and} \quad \| V \| (B_1) = \omega_n. \quad (2.62)
\]

By (2.62), \( V \subseteq (B_1 \setminus \overline{B}_{\lambda_1}) = \text{var} (C, \theta_C) \subseteq (B_1 \setminus \overline{B}_{\lambda_1}) \) for some locally \( \mathcal{H}^n \)-rectifiable cone \( C \subset \mathbb{R}^{n+1} \) and zero homogeneous \( \theta_C : C \to \mathbb{N} \). Now since the integer rectifiable varifold cone \( \text{var} (C, \theta_C) \) is stationary in \( B_1 \setminus \overline{B}_{\lambda_1} \), it is stationary in \( \mathbb{R}^{n+1} \) by \( n \geq 2 \), and due to (2.62), satisfies \( \int_{C \cap B_1} \theta_C d\mathcal{H}^n = \omega_n \). Therefore \( C = K \) for some \( K \in \mathcal{H} \), and \( \theta_C \equiv 1 \). From the definition of \( C \), it follows that

\[
\text{spt} V \cap (B_1 \setminus \overline{B}_{\lambda_1}) \subset K. \quad (2.63)
\]

Finally, (2.62) and (2.63) give (2.55) with \( \beta_1 = \lambda_1, \beta_2 = 1 \). The result of step one then completes the proof that \( V = \text{var} (K \cap B_1, 1|_K \cap B_1) \). \( \Box \)

**Proof of Theorem 2.1.** The proof proceeds in four steps, which we outline here. Precise statements can be found at the beginning of each step. First,
we assume that \( \delta_{V,R,\Lambda}(s/8) \geq 0 \), and prove that \( C^1 \)-graphicality can be propagated from \( s/32 \) to an upper radius \( S_+/16 \leq S_+/16 \) as long as \( \delta_{V,R,\Lambda}(S_+) \) remains non-negative and \( S_+ \leq \varepsilon_0/\Lambda \). This is then enough to prove the exterior blow-down result in part (ii) of Theorem 2.1 in step two. In the third step, we argue that if \( \delta_{V,R,\Lambda}(s/8) \leq 0 \), then \( C^1 \)-graphicality can be propagated inwards from \( S_+/2 \) down to \( s/32 \). The details in this step are quite similar to the first, so we summarize them. Finally, the first and third steps are combined in step four to conclude the proof Theorem 2.1-(i), in which there are no sign restrictions on the deficit.

**Step one:** In this step, given \( n \geq 2, \Gamma \geq 0, \) and \( \sigma > 0, \) we prove the existence of \( \varepsilon_0 \) and \( M_0 \) (specified below in (2.73) and (2.74), and depending on \( n, \Gamma, \) and \( \sigma \)) such that if (2.3), (2.4), (2.5) and (2.6) hold with \( \varepsilon_0 \) and \( M_0, \) and in addition,

\[
0 \leq \delta_{V,R,\Lambda}(s/8) \leq \varepsilon_0, \quad (2.64)
\]

then there exists \( K_+ \in \mathcal{H} \) and \( u_+ \in \mathcal{X}_\sigma(\Sigma_{K_+}, s/32, S_+/16) \) such that

\[
V \text{ corresponds to } \Sigma_{K_+}(u_+, s/32, S_+/16) \text{ on } A_{s/32}^{S_+/16}, \quad (2.65)
\]

where

\[
R_+ = \max \\left\{ \sup \left\{ \rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq 0 \right\}, 4s \right\}, \quad S_+ = \min \left\{ R_+, \frac{\varepsilon_0}{\Lambda} \right\} \geq 4s. \quad (2.66)
\]

We start by imposing some constraints on the constants \( \varepsilon_0 \) and \( M_0. \) For the finite set

\[
J = \left\{ \left( \frac{1}{3}, \frac{1}{6} \right), \left( \frac{2}{3}, \frac{1}{3} \right) \right\} \subset \left\{ (\eta_0, \eta) : \eta_0 > \eta > 0 \right\}, \quad (2.67)
\]

we let \( \sigma_0 = \sigma_0(n) \) be such that Lemma 2.5-(ii), Theorem 2.6, and Theorem 2.7-(ii,iii) hold for every \( (\eta_0, \eta) \in J, \) Lemma 2.5-(i) holds for \( \sigma < \sigma_0, \) and

\[
\sigma_0 \leq \frac{\sigma_1}{C_0} \quad \text{for } \sigma_1(n) \text{ as in (2.30), and } C_0(n) \text{ as in Lemma 2.5-(ii)}; \quad (2.68)
\]

we shall henceforth assume, without loss of generality, that

\[
\sigma < \sigma_0.
\]

Moreover, for \( \varepsilon_1 \) and \( M_1 \) as in Lemma 2.8-(i) and \( C_0 \) as in Lemma 2.5, we let

\[
M_0' \geq \max \left\{ M_1 \left( n, \frac{\sigma}{2C_0}, \Gamma, \left( \frac{1}{8}, \frac{1}{2} \right), \left( \frac{1}{6}, \frac{1}{4} \right), \left( \frac{1}{32}, \frac{1}{2} \right) \right),
M_1 \left( n, \frac{\sigma}{2C_0}, \Gamma, \left( \frac{1}{16}, \frac{1}{8} \right), \left( \frac{3}{32}, \frac{7}{64} \right), \left( \frac{1}{32}, \frac{1}{2} \right) \right) \right\}, \quad (2.69)
\]

\[
\varepsilon_0' \leq \min \left\{ \varepsilon_1 \left( n, \frac{\sigma}{2C_0}, \Gamma, \left( \frac{1}{8}, \frac{1}{2} \right), \left( \frac{1}{6}, \frac{1}{4} \right), \left( \frac{1}{32}, \frac{1}{2} \right) \right), \varepsilon_1 \left( n, \frac{\sigma}{2C_0}, \Gamma, \left( \frac{1}{16}, \frac{1}{8} \right), \left( \frac{3}{32}, \frac{7}{64} \right), \left( \frac{1}{32}, \frac{1}{2} \right) \right) \right\}. \quad (2.70)
\]
We also assume that
\[ C(n, \Gamma)(\varepsilon')^{1/2} \leq \min \left\{ \varepsilon_0, \frac{\sigma}{2 C_0} \right\}, \tag{2.71} \]
where \( C(n, \Gamma) \) is specified in (2.107), \( C_0 \) is as in Lemma 2.5, and \( \varepsilon_0 \) is smaller than both of the \( n \)-dependent \( \varepsilon_0 \)'s appearing in Lemma 2.3 and Lemma 2.5.

Lastly, we choose \( \sigma > 0 \) such that
\[ \sigma \leq \min \left\{ \sigma^2 C_0, \sqrt{\varepsilon'/\omega_n} \right\}, \tag{2.72} \]
and then, for \( \varepsilon_2, M_2 \) as in Lemma 2.8-(ii), we choose \( \varepsilon_0 \) and \( M_0 \) so that
\[ \varepsilon_0 \leq \min \left\{ {\varepsilon_0}', \varepsilon_2 \left( n, \sigma, \Gamma, \frac{1}{8}, \left( \frac{1}{32}, \frac{1}{2} \right) \right) \right\}, \tag{2.73} \]
\[ M_0 \geq \max \left\{ M_0', M_2 \left( n, \sigma, \Gamma, \frac{1}{8}, \left( \frac{1}{32}, \frac{1}{2} \right) \right) \right\}. \tag{2.74} \]

Let us now recall that, by assumption, \( V \in \mathcal{V}_n(\Lambda, R, 1/\Lambda) \) is such that
\[ \| \partial B_R \| \leq \Gamma R^{n-1}, \quad \sup_{\rho \in (R, 1/\Lambda)} \frac{\| V \|_{(B_\rho \setminus B_R)}}{\rho^n} \leq \Gamma; \tag{2.75} \]
in particular, by Theorem 2.7-(i),
\[ \delta_{V,R,\Lambda} \text{ is decreasing on } (R, 1/\Lambda). \tag{2.76} \]
Moreover, we are assuming the existence of \( s \) with \( \max \{ 64, M_0 \} \leq s < \varepsilon_0/4 \Lambda \) such that
\[ |\delta_{V,R,\Lambda}(s/8)| \leq \varepsilon_0, \tag{2.77} \]
\[ R_s = \sup \left\{ \rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq -\varepsilon_0 \right\} \geq 4s, \tag{2.78} \]
so that the latter inequality, together with (2.66), implies
\[ R_s \geq R_+. \tag{2.79} \]

By (2.76), (2.77) and (2.79) we have
\[ |\delta_{V,R,\Lambda}(r)| \leq \varepsilon_0, \quad \forall r \in [s/8, R_+]. \tag{2.80} \]

By (2.75), the specification of \( s \) satisfying (2.4), and (2.80), the assumptions (2.48), (2.53), and (2.54), respectively, of Lemma 2.8-(ii) with \( r = s, \lambda_1 = 1/8, \eta_1 = (1/32, 1/2) \) are satisfied due to our choices (2.73) and (2.74). Setting \( H_0 = H \), where \( H \in \mathcal{H} \) is from the application of Lemma 2.8-(ii), we thus find \( u_0 \in \mathcal{X}_{\Sigma H_0}(s/32, s/2) \) such that
\[ V \text{ corresponds to } \Sigma_{H_0}(u_0, s/32, s/2) \text{ on } A^{s/2}_{s/32}. \tag{2.81} \]

If it is the case that \( S_+ = 4s \), we are in fact done with the proof of (2.65), since then \( s/2 \geq S_+/16 \). We may for the rest of this step assume then that \( S_+ > 4s \), so that
\[ R_+ = \sup \left\{ \rho \geq \frac{s}{8} : \delta_{V,R,\Lambda}(\rho) \geq 0 \right\} \geq S_+ > 4s. \tag{2.82} \]
First, we observe that thanks to (2.81) and then (2.72),
\[
T_0 := \frac{1}{(s/4)^n} \int_{s/8}^{s/4} r^{n-1} dr \int_{\Sigma_{H_0}} |u_0|^2 \leq \omega_n \sigma^2 \leq \varepsilon_0'.
\tag{2.83}
\]
We let \( s_j = 2^{j-3} s \) for \( j \in \mathbb{Z}_{\geq -1} \). By (2.82) and by \( s < \varepsilon_0/4 \Lambda \leq \varepsilon_0'/4 \Lambda \) there exists \( N \in \{ j \in \mathbb{N} : j \geq 2 \} \cup \{ +\infty \} \) such that
\[
\{0, 1, ..., N\} = \left\{ j \in \mathbb{N} : s_j \leq S_+ = \min \left\{ R_+, \frac{\varepsilon_0'}{\Lambda} \right\} \right\}.
\tag{2.84}
\]
Notice that if \( \Lambda > 0 \) then it must be \( N < \infty \). We are now in the position to make the following:

**Claim:** There exist \( \tau = \tau(n) \in (0, 1) \) and \( \{(H_j, u_j)\}_{j=0}^{N-2} \) with \( H_j \in \mathcal{H} \) such that, setting
\[
T_j = \frac{1}{s_j^{n+1}} \int_{s_j}^{s_j+1} r^{n-1} dr \int_{\Sigma_{H_j}} |u_j|^2 r^n,
\]
for every \( j = 0, ..., N-2 \),
\[
u_H - \nu_{H_{j-1}}^2 \leq C(n) T_{j-1},
\tag{2.89}
\]
\[
|\delta_{V,R,\Lambda}(s_j)| \leq \varepsilon_0',
\tag{2.87}
\]
\[
T_j \leq C(n) \varepsilon_0';
\tag{2.88}
\]
additionally, for every \( j = 1, ..., N-2 \),
\[
|\delta_{V,R,\Lambda}(s_j) - \delta_{V,R,\Lambda}(s_{j-1})| \leq \tau \left\{ \delta_{V,R,\Lambda}(s_{j-1}) + (1 + \Gamma) \Lambda s_{j-1} \right\},
\tag{2.90}
\]
\[
T_j \leq C(n) \left\{ \delta_{V,R,\Lambda}(s_{j-1}) - \delta_{V,R,\Lambda}(s_{j+2}) + \Lambda s_{j-1} \right\}.
\tag{2.91}
\]

**Proof of the claim:** We argue by induction. Clearly (2.85)\(_{j=0} \), (2.86)\(_{j=0} \), (2.87)\(_{j=0} \) and (2.88)\(_{j=0} \) are, respectively, (2.81), (2.77) and (2.83). This concludes the proof of the claim if \( N = 2 \), therefore we shall assume \( N \geq 3 \) for the rest of the argument. To set up the inductive argument, we consider \( \ell \in \mathbb{N} \) such that: either \( \ell = 0 \); or \( 1 \leq \ell \leq N-3 \) and (2.85), (2.86), (2.87), and (2.88) hold for \( j = 0, ..., \ell \); and (2.89) and (2.90) hold for \( j = 1, ..., \ell \); and prove that all the conclusions of the claim hold with \( j = \ell + 1 \).

The validity of (2.87)\(_{j=\ell+1} \) is of course immediate from (2.80) and (2.84). Also, after proving (2.91)\(_{j=\ell+1} \), we will be able to combine with (2.87)\(_{j=\ell+1} \) and (2.84) to deduce (2.88)\(_{j=\ell+1} \). We now prove, in order, (2.89), (2.85), (2.86), (2.87), (2.88), and (2.91) with \( j = \ell + 1 \).

**To prove (2.89)\(_{j=\ell+1} \):** Let \( [a, b] \subset (s_\ell, s_{\ell+1}) \) with \( (b - a) = (s_{\ell+1} - s_\ell)/2 \), so that
\[
\frac{1}{C(n)} \min_{r \in [a, b]} \int_{\Sigma_{H_\ell}} |u_r|^2 r^n \leq \frac{1}{s_{\ell+1}^2} \int_{s_{\ell+1}}^{s_{\ell+1}} r^{n-1} dr \int_{\Sigma_{H_\ell}} |u_r|^2 r^n = T_\ell.
\tag{2.92}
\]
Keeping in mind (2.85)_{j=1}, (2.86)_{j=1}, we can apply Lemma 2.5-(ii) with \((r_1, r_2) = (s/32, 4s_\Delta)\) and \([a, b] \subset (s_\Delta, s_{\ell+1})\) to find \(H_{\ell+1} \in \mathcal{H}\),

\[ u_{\ell+1} \in \mathcal{X}_{C_0\sigma_0}(\Sigma_{H_{\ell+1}}, s/32, 4s_\Delta) \tag{2.93} \]

(with \(C_0\) as in Lemma 2.5-(ii)) and

\[ s^*_\ell \in [a, b] \subset (s_\Delta, s_{\ell+1}), \tag{2.94} \]

such that, thanks also to (2.92),

\[ \Sigma_{H_\ell}(u_\ell, s/32, 4s_\Delta) = \Sigma_{H_{\ell+1}}(u_{\ell+1}, s/32, 4s_\Delta), \tag{2.95} \]

\[ E^0_{\Sigma_{H_{\ell+1}}}([u_{\ell+1}]_{s^*_\ell}) = 0, \tag{2.96} \]

\[ |\nu_{H_\ell} - \nu_{H_{\ell+1}}|^2 \leq C(n)T_\ell, \tag{2.97} \]

\[ \int_{\Sigma_{H_{\ell+1}}} [u_{\ell+1}]_{s^*_\ell}^2 \leq C(n) \left( T_\ell + \int_{\Sigma_{H_\ell}} [u_\ell]_{s^*_\ell}^2 \right), \quad \forall r \in (s/32, 4s_\Delta). \tag{2.98} \]

In particular, (2.97) is (2.89)_{j=\ell+1}.

To prove (2.85)_{j=\ell+1} and (2.86)_{j=\ell+1}: Notice that (2.93), (2.95) do not imply (2.85)_{j=\ell+1} and (2.86)_{j=\ell+1}, since, in (2.86)_{j=\ell+1}, we are claiming the graphicality of \(V\) inside \(A^4_{s/32}\) (which is strictly larger than \(A^4_{s/32}\)), and in (2.85)_{j=\ell+1} we are claiming that \(u_{\ell+1}\) has \(C^1\)-norm bounded by \(\sigma\) or \(\sigma/2C_0\) (depending on the radius), and not just by \(C_0\sigma_0\) (with \(C_0\) as in Lemma 2.5-(ii)).

We want to apply Lemma 2.8-(i) with \(K = H_{\ell+1}\) and

\[ r = 8s_{\ell+1}, \ (\lambda_1, \lambda_2) = \left( \frac{1}{16}, \frac{1}{8} \right), \ (\lambda_3, \lambda_4) = \left( \frac{3}{32}, \frac{7}{64} \right), \ (\eta_1, \eta_2) = \left( \frac{1}{32}, \frac{1}{2} \right). \tag{2.99} \]

We check the validity of (2.49), (2.50), (2.51), and (2.52) with \(\varepsilon_1 = \varepsilon_0^\prime\) and \(M_1 = M_0^\prime\) for these choices of \(r, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \eta_1, \eta_2,\) and \(K\). Since \(r = 8s_{\ell+1} \geq 8 \geq \max\{M_0, 64R\} \geq \max\{M_0^\prime, 64R\}\), and since (2.84) and \(\ell + 1 \leq N\) give \(r = 8s_{\ell+1} \leq \varepsilon_0/\Lambda \leq \varepsilon_0^\prime/\Lambda\), we deduce the validity of (2.49) with \(r = 8s_{\ell+1}\). The validity of (2.50) with \(r = 8s_{\ell+1}\) is immediate from (2.80) by our choice (2.70) of \(\varepsilon_0^\prime\). Next we notice that

\[ \|V\|(A^{\lambda_1\Gamma}_{\Lambda_3}) = \|V\|(A_r^7/32) = \|V\|(A_{3/2}) > 0 \]

thanks to (2.86)_{j=\ell}, so that (2.51) holds for \(r, \lambda_3\) and \(\lambda_4\) as in (2.99). Finally, by (2.30) (which can be applied to \(u_{\ell+1}\) thanks to (2.68)), (2.95) and (2.85)_{j=\ell}, and, then by (2.98), we have

\[ \frac{1}{r^n} \int_{A^{\lambda_2}_{\Lambda_1}} \omega_{H_{\ell+1}}^2 \|V\| \leq \frac{C(n)}{s_{\ell+1}^n} \int_{s_{\ell}}^{s_{\ell+1}} r^{n-1} \, dr \int_{\Sigma_{H_{\ell+1}}} [u_{\ell+1}]_{s^*_\ell}^2 \]

\[ \leq C(n)T_\ell + \frac{C(n)}{s_{\ell+1}^n} \int_{s_{\ell}}^{s_{\ell+1}} r^{n-1} \, dr \int_{\Sigma_{H_\ell}} [u_\ell]_{s^*_\ell}^2 \]

\[ \leq C(n)T_\ell \leq C(n)\varepsilon_0^\prime, \]
where in the last inequality we have used (2.88) \(_{j=\ell}^\prime\). Again by our choice (2.70) of \(\varepsilon_0\), we deduce that (2.52) holds with \(r, \lambda_1\) and \(\lambda_2\) as in (2.99). We can thus apply Lemma 2.8-(i), and find \(v \in \mathcal{X}_{\sigma/2} C_0(\Sigma_{H_{\ell+1}}, s_{\ell+1}/4, 4 s_{\ell+1})\) such that
\[
V \text{ corresponds to } \Sigma_{H_{\ell+1}}(v, s_{\ell+1}/4, 4 s_{\ell+1}) \text{ on } A_{s_{\ell+1}/4}^{4 s_{\ell+1}}. \tag{2.100}
\]
By (2.95), (2.86) \(_{j=\ell}\), and (2.100), \(v = u_{\ell+1} \in \Sigma_{H_{\ell+1}}(s_{\ell+1}/4, 4 s_{\ell+1})\). We can thus use \(v\) to extend \(u_{\ell+1}\) from \(\Sigma_{H_{\ell+1}} \times (s_{\ell+1}/4, 4 s_{\ell+1})\) to \(\Sigma_{H_{\ell+1}} \times (s/32, 4 s_{\ell+1})\), and, thanks to (2.95), (2.86) \(_{j=\ell}\) and (2.100), the resulting extension is such that
\[
u_{\ell+1} \in \mathcal{X}_{\sigma/2} C_0(\Sigma_{H_{\ell+1}}, s_{\ell+1}/4, 4 s_{\ell+1}) \quad \text{and} \quad V \text{ corresponds to } \Sigma_{H_{\ell+1}}(u_{\ell+1}, s/32, 4 s_{\ell+1}) \text{ on } A_{s/32}^{4 s_{\ell+1}}. \tag{2.101}
\]
The bound (2.101) is part of (2.85) \(_{j=\ell+1}\), and (2.102) is (2.86) \(_{j=\ell+1}\), so in order to complete the proof of (2.85) \(_{j=\ell+1}\) and (2.86) \(_{j=\ell+1}\), it remains to show that the \(C^1\)-norm of \(u\) is bounded by \(\sigma\) in between \(s/32\) and \(4 s_{\ell}\).

Towards this end, we record the following consequence of taking square roots in (2.90) \(_{j=m}\) (using \(\delta_{V,R,A} \geq 0\) from (2.84)) and summing over \(m = 1, \ldots, i\) for any \(1 \leq i \leq \ell\): for \(\alpha = \sum_{k=0}^{\infty} 2 \cdot 2^{k/2}\) and \(\tilde{C}(n, \Gamma) = \tau^{1/2}(1 + \Gamma)\),
\[
S_i := \sum_{m=0}^{i} \delta_{V,R,A}(s_m)^{1/2} \leq \tau^{1/2} \sum_{m=0}^{i-1} \delta_{V,R,A}(s_m)^{1/2} + \alpha \tilde{C}(n, \Gamma) (\lambda s_m)^{1/2}
\]
\[
= \tau^{1/2} S_{i-1} + \alpha \tilde{C}(n, \Gamma) (\lambda s_m)^{1/2}
\]
\[
\leq \tau^{1/2} S_{i-1} + (1 + \alpha \tilde{C}(n, \Gamma)) (\varepsilon_0')^{1/2}, \tag{2.103}
\]
where in the last line we have used (2.84) and (2.80). By induction, utilizing (2.64), (2.73) for the base case and (2.103) for the induction step we have
\[
S_i \leq \frac{(1 + \alpha \tilde{C}(n, \Gamma)) (\varepsilon_0')^{1/2}}{1 - \tau^{1/2}} \quad \forall 0 \leq i \leq \ell. \tag{2.104}
\]
Now by the positivity of \(\delta_{V,R,A}\) and (2.91) \(_{j=\ell}\), for all \(m = 1, \ldots, \ell\),
\[
T_{m}^{1/2} \leq C(n) \delta_{V,R,A}(s_{m-1})^{1/2} + C(n) (\lambda s_{m-1})^{1/2}, \tag{2.105}
\]
In turn, by (2.89) \(_{j=\ell+1}\), (2.83) and (2.105), then (2.84) and (2.104) \(_{i=\ell-1}\),
\[
\frac{1}{C(n)} \sum_{m=1}^{\ell+1} |\nu_{H_m} - \nu_{H_{m-1}}| \leq \sum_{m=0}^{\ell} T_{m}^{1/2}
\]
\[
\leq (\varepsilon_0')^{1/2} + C(n) S_{\ell-1} + \alpha C(n) (\lambda s_{\ell-1})^{1/2}
\]
\[
\leq C(n, \Gamma) (\varepsilon_0')^{1/2}, \tag{2.106}
\]
for suitable \(C(n, \Gamma)\). Combining constants, we use (2.106) to see
\[
|\nu_{H_i} - \nu_{H_{i+1}}| \leq C(n, \Gamma) (\varepsilon_0')^{1/2} \quad \forall i = 0, \ldots, \ell. \tag{2.107}
\]
Now \( u_i \in \mathcal{X}_{\nu/2} C_0(\Sigma_{H_i}, s_i/4, 4 s_i) \) by (2.85)\(_{j=i}\), and \( \sigma/2 C_0 \) and \( |\nu_{H_i} - \nu_{H_{i+1}}| \) are small enough to apply Lemma 2.5-(i) by our choice of \( \sigma \) above (2.68) and (2.107) with (2.71), respectively. Then we obtain \( w_i \) corresponding to \( V \) on \( A^4 s_i/4 \) and in \( \mathcal{X}_{\nu/2+2 C_0|\nu_{H_i} - \nu_{H_{i+1}}|}(\Sigma_{H_{i+1}}, s_i/4, 4 s_i) \), and by (2.107), (2.71),

\[
\frac{\sigma}{2} + C_0 |\nu_{H_i} - \nu_{H_{i+1}}| \leq \frac{\sigma}{2} + C_0 \frac{\sigma}{2 C_0} = \sigma,
\]

so \( w_i \in \mathcal{X}_0(\Sigma_{H_{i+1}}, s_i/4, 4 s_i) \). Finally, since they represent the same surface over \( \Sigma_{H_{i+1}} \), \( w_i = u_{\ell+1} \) on \( A^4 s_i/4 \). Gathering these estimates for \( i = 0, \ldots, \ell \), we have \( u_{\ell+1} \in \mathcal{X}_0(\Sigma_{H_{\ell+1}}, s/32, 4 s_\ell) \), which finishes the proof of (2.85)\(_{j=\ell+1}\).

To prove (2.90)\(_{j=\ell+1}\): We set \( r_0 = (s_\ell + s_{\ell+1})/2 \) and notice that for \( \eta_0 = 1/3 \),

\[
r_1 = r_0 (1 - \eta_0) = s_\ell, \quad r_2 = r_0 (1 + \eta_0) = s_{\ell+1}.
\]

For \( \eta = 1/6 \) we correspondingly set

\[
r_3 = r_0 (1 - \eta) =: s_\ell^-, \quad r_4 = r_0 (1 + \eta) =: s_\ell^+,
\]

and notice that \( (\eta_0, \eta) \in J \), see (2.67). With the aim of applying Theorem 2.7-(iii) to these radii, we notice that (2.86)\(_{j=\ell+1}\) implies that assumption (2.43) holds with \( H = H_{\ell+1} \) and \( u = u_{\ell+1} \), while, by (2.96), \( r = s_\ell^+ \in (s_\ell, s_{\ell+1}) \) is such that (2.45) holds. By \( \Lambda s_{\ell+1} \leq \varepsilon_0 \leq 1, (2.84), \) and (2.46), with \( C(n) = C_0(n, 1/6, 1/3) \) for \( C_0 \) as in Theorem 2.7-(iii), we have

\[
s_{\ell+1}^{-n} \left\| \frac{\partial V}{\partial \rho} (B_{s_\ell^+} \setminus \Sigma_{s_\ell^-}) \right\| \leq \omega_n \left( (s_\ell^+)^n - (s_\ell^-)^n \right)
\]

\[
= s_{\ell+1}^{-n} \left| \mathcal{H}^n_{\Sigma_{H_{\ell+1}}}(u_{\ell+1}, s_{\ell}^-, s_\ell^+) \right| - \mathcal{H}^n_{\Sigma_{H_{\ell+1}}}(0, s_{\ell}^-, s_\ell^+)
\]

\[
\leq C(n) \left( \Lambda s_{\ell+1}^2 + \Theta_{V,R,A}(s_{\ell+1}) - \Theta_{V,R,A}(s_\ell) \right).
\]

Setting for brevity \( \delta = \delta_{V,R,A} \) and \( \Theta = \Theta_{V,R,A} \), and recalling that

\[
r^n \delta(r) = \omega_n r^n - \Theta(r) r^n
\]

\[
= \omega_n r^n - \left\| \frac{\partial V}{\partial \rho} (B_\rho \setminus B_R) \right\| - \Lambda n \int_{R} \left\| \frac{\partial V}{\partial \rho} (B_\rho \setminus B_R) \right\| d\rho + \frac{R \left\| \partial V \right\| (\partial B_R)}{n}
\]

we have

\[
s_{\ell}^{-n} \left( (s_\ell^-)^n \delta(s^-_\ell) - (s_\ell^+)^n \delta(s^+_\ell) \right) \leq C(n) \left( \Lambda s_\ell^2 + \Theta(s_{\ell+1}) - \Theta(s_\ell) \right)
\]

\[
+ C(n) \Lambda s_{\ell}^{-n} \left\{ (s_\ell^+)^n \int_R^{s_\ell^+} \left\| \frac{\partial V}{\partial \rho} (B_\rho \setminus B_R) \right\| d\rho - (s_\ell^-)^n \int_R^{s_\ell^-} \left\| \frac{\partial V}{\partial \rho} (B_\rho \setminus B_R) \right\| d\rho \right\}
\]

\[
\leq C(n) \left( \Lambda s_\ell^2 + \Theta(s_{\ell+1}) - \Theta(s_\ell) \right) + C(n) \Lambda \int_R^{s_\ell^+} \left\| \frac{\partial V}{\partial \rho} (B_\rho \setminus B_R) \right\| d\rho.
\]

By \( \Lambda s_\ell \leq 1 \) and since \( s_\ell^+ \leq s_\ell \leq \varepsilon_0/8 \Lambda \) thanks to \( \ell < N \), we can use the upper bound \( \left\| \frac{\partial V}{\partial \rho} (B_\rho \setminus B_R) \right\| \leq \Gamma \rho^n \) with \( \rho \in (R, s_\ell^+) \subset (R, 1/\Lambda) \), to find that

\[
\left| \frac{(s_\ell^-)^n}{s_\ell^-} \delta(s^-_\ell) - \frac{(s_\ell^+)^n}{s_\ell^+} \delta(s^+_\ell) \right| \leq C_*(n) \{ \delta(s_\ell) - \delta(s_{\ell+1}) \} + C_*(n) (\Gamma + 1) \Lambda s_\ell,
\]
for a constant $C_*(n)$. By rearranging terms and using the monotonicity of $\delta$ on $(R, \infty)$ and $(s^-, s^+) \in (s_-, s_{\ell+1})$ we find that
\[
(C_*(n) + (s^+/s^0)\delta(s_{\ell+1}) \leq C_*(n)\delta(s_{\ell+1}) + ((s^+/s^0))\delta(s^+) \\
\leq C_*(n)\delta(s_\ell) + ((s^+/s^0))\delta(s^-) + C_*(n)(1+\Gamma)\Lambda s_\ell \\
\leq (C_*(n) + (s^-/s^0))\delta(s_\ell) + C_*(n)(1+\Gamma)\Lambda s_\ell.
\]
(2.111)

We finally notice that by (2.108), (2.109), $\eta_0 = 1/3$, and $\eta = 1/6$, we have
\[
\frac{s^-}{s_\ell} = \frac{r_0(1-\eta)}{r_0(1-\eta_0)} = \frac{5}{4}, \quad \frac{s^+}{s_\ell} = 2 \frac{s^+}{s_{\ell+1}} = 2 \frac{1+\eta}{1+\eta_0} = \frac{7}{4},
\]
so that we find $\delta(s_{\ell+1}) \leq \tau\delta(s_\ell) + (1+\Gamma)\Lambda s_\ell$ (i.e. (2.90)$_{j=\ell+1}$) with
\[
\tau = \tau(n) = \frac{C_*(n) + (5/4)n}{C_*(n) + (7/4)n}, \quad \tau = \tau_*(n) = \frac{C_*(n)}{C_*(n) + (7/4)n} < \tau.
\]

To prove (2.91)$_{j=\ell+1}$: We finally prove (2.91)$_{j=\ell+1}$, i.e.
\[
\frac{1}{s_{\ell+1}}\int_{s_{\ell+1}}^{2s_{\ell+1}} \int_{\Sigma_{H_{\ell+1}}} [u_{\ell+1}]^2 r \leq C(n)\{\delta_{V,R,A}(s_\ell) - \delta_{V,R,A}(s_{\ell+3}) + \Lambda s_\ell\}.
\]
(2.112)

By (2.86)$_{j=\ell+1}$ we know that $V$ corresponds to $\Sigma_{H_{\ell+1}}(u_{\ell+1}, s/32, 4s_{\ell+1})$ on $A^{4s_{\ell+1}}$.
(2.113)

Now, (2.41) holds with $r_0 = 3s_\ell$ and $(\eta_0, \eta) = (2/3, 1/3) \in J$, see (2.67), if
\[
\begin{align*}
1 = s_\ell &= 3s_\ell - 2s_\ell, \\
2 = s_\ell &= 3s_\ell + 2s_\ell, \\
3 = s_{\ell+1} &= 3s_\ell - s_\ell, \\
4 = s_{\ell+1} &= 3s_\ell + s_\ell.
\end{align*}
\]
Since $s^*_{\ell} \in (s_\ell, s_{\ell+1}) \subset (r_1, r_2)$, by (2.113), (2.96) and $(r_1, r_2) \subset (s/32, 4s_{\ell+1})$ we can apply Theorem 2.6 to deduce that
\[
\int_{s_{\ell+1}}^{2s_{\ell+1}} \int_{\Sigma_{H_{\ell+1}}} [u_{\ell+1}]^2 r \leq C(n)\int_{s_\ell}^{5s_\ell} \int_{\Sigma_{H_{\ell+1}}} (\partial_r u_{\ell+1})^2 r^n + C(n)\Lambda (s_\ell)^{n+1}.
\]
Again by (2.113), Theorem 2.7-(ii) with $(r_1, r_2) = (s_\ell, 8s_\ell)$ gives
\[
\frac{s_\ell^{-n}}{s_\ell} \int_{s_\ell}^{5s_\ell} \int_{\Sigma_{H_{\ell+1}}} (\partial_r u_{\ell+1})^2 r \leq s_\ell^{-n} \int_{s_\ell}^{8s_\ell} \int_{\Sigma_{H_{\ell+1}}} (\partial_r u_{\ell+1})^2 r^n
\]
\[
\leq C(n)\{\Theta_{V,R,A}(8s_\ell) - \Theta_{V,R,A}(s_\ell)\} \leq C(n)\{\delta_{V,R,A}(s_\ell) - \delta_{V,R,A}(s_{\ell+3})\}.
\]

The last two estimates combined give (2.112), which finishes the **claim**.

**Proof of (2.65):** We assume $S_+ < \infty$ (that is either $\Lambda > 0$ or $R_+ < \infty$), and recall that we have already proved (2.65) if $S_+ = 4 s$. Otherwise, $N$ (as defined in (2.84)) is finite, with $2^N \leq \frac{S_+}{S} < 2^{N+1}$. By (2.85)$_{j=N-2}$ and (2.86)$_{j=N-2}$, we have that $u_{N-2} \in X_\sigma(\Sigma_{H_{N-2}}; s/32, 4s_{N-2})$ and $V$ corresponds to $\Sigma_{H_{N-2}}(u_{N-2}, s/32, 4s_{N-2})$ on $A^{4s_{N-2}}$. Since $4s_{N-2} = 2^{N+1} s/16 > S_+/16$, we deduce (2.65) with $K_+ = H_{N-2}$ and $u_+ = u_{N-2}$.
Step two: In this step we prove statement (ii) in Theorem 2.1. We assume that \( \Lambda = 0 \) and that
\[
\delta(r) \geq -\varepsilon_0 \quad \forall r \geq \frac{s}{8},
\]
where we have set for brevity \( \delta = \delta_{V,R,0} \). We must first show that
\[
\delta(r) \geq 0 \quad \forall r \geq \frac{s}{8},
\]
(2.115)
Since \( \delta \) is decreasing in \( r \), it has a limit \( \lim_{r \to \infty} \delta(r) =: \delta_{\infty} \geq -\varepsilon_0 \). Next, we know that for any sequence \( R_i \to \infty \), \( V/R_i \) converges locally in the varifold sense to a limiting integer rectifiable varifold cone \( W \). By the local varifold convergence and \( n \geq 2 \), \( W \) is stationary in \( \mathbb{R}^{n+1} \), and it is the case that
\[
\delta_{W,0,0}(r) = \delta_\infty \geq -\varepsilon_0 \quad \forall r > 0.
\]
Up to decreasing \( \varepsilon_0 \) if necessary (and recalling that \( \delta_{W,0,0} \) is the usual area excess multiplied by \(-1\)), Allard’s theorem and the fact that \( W/r = W \) imply that \( W \) corresponds to a multiplicity one plane. In particular, it must be that \( \delta_{\infty} = 0 \), which together with the monotonicity of \( \delta \) yields (2.115).

By (2.115), \( S_+ = S_\ast = \infty \), and so by (2.85) and (2.86), there is a sequence \( \{ (H_j, u_j) \}_{j=0}^{N} \) but with \( N = \infty \) now, satisfying
\[
V \text{ corresponds to } \Sigma_{H_j}(u_j, s/32, 4 s_j) \text{ on } A_{s/32}^{4 s_j} \quad \forall j \geq 0, \quad (2.116)
\]
\[
|\nu_{H_j} - \nu_{H_{j-1}}|^2 \leq C(n) T_{j-1}, \quad \text{if } j \geq 1, \quad (2.117)
\]
\[
\delta(s_j) \leq \left\{ \begin{array}{ll}
\varepsilon_0, & \text{if } j = 0, \\
\tau \delta(s_{j-1}), & \text{if } j \geq 1,
\end{array} \right. \quad (2.118)
\]
\[
T_j \leq \left\{ \begin{array}{ll}
C(n) \varepsilon_0, & \text{if } j = 0, \\
C(n) \delta(s_{j-1}), & \text{if } j \geq 1.
\end{array} \right. \quad (2.119)
\]
Notice that, in asserting the validity of (2.119) with \( j \geq 1 \), we have used (2.115) to estimate \(-\delta(s_{j+2}) \leq 0 \) in (2.91)_j. By iterating (2.118) we find
\[
\delta(s_j) \leq \tau^j \delta(s/8) \leq \tau^j \varepsilon_0, \quad \forall j \geq 1, \quad (2.120)
\]
which, combined with (2.119) and (2.117), gives, for every \( j \geq 1 \),
\[
T_j \leq C(n) \min\{1, \tau^{-j}\} \delta(s/8) \leq C(n) \tau^j \delta(s/8), \quad (2.121)
\]
\[
|\nu_{H_j} - \nu_{H_{j-1}}|^2 \leq C(n) \min\{1, \tau^{-j}\} \delta(s/8) \leq C(n) \tau^j \delta(s/8), \quad (2.122)
\]
thanks also to \( \tau = \tau(n) \) and, again, to (2.115). By (2.122), for every \( j \geq 0 \), \( k \geq 1 \), we have
\[
|\nu_{H_{j+k}} - \nu_{H_j}| \leq C(n) \sqrt{\delta(s/8)} \sum_{h=1}^{k+1} (\sqrt{\tau})^{j-1+h},
\]
so that there exists \( K \in \mathcal{H} \) such that
\[
\varepsilon_j^2 := |\nu_K - \nu_{H_j}|^2 \leq C(n) \tau^j \delta(s/8), \quad \forall j \geq 1, \quad (2.123)
\]
In particular, for \( j \) large enough, we have \( \varepsilon_j < \varepsilon_0 \), and thus, by Lemma 2.5-(i) and by (2.116) we can find \( v_j \in X_{C(n)(\sigma+\varepsilon_j)}(\Sigma_K; s/32, 4 s_j) \) such that
\[
V \text{ corresponds to } \Sigma_K(v_j, s/32, 4 s_j) \text{ on } A_{s/32}^{4 s_j}, \quad (2.124)
\]
By (2.124), \( v_{j+1} = v_j \) on \( \Sigma_K \times (s/32, 4s_j) \). Since \( s_j \to \infty \) we have thus found \( u \in \mathcal{X}_{C(n)}(\Sigma_K; s/32, \infty) \) such that
\[
V \text{ corresponds to } \Sigma_K(u, s/32, \infty) \text{ on } A_{s/32}^\infty, \tag{2.125}
\]
which is (2.7) with \( S_* = \infty \).

To prove (2.8), we notice that if \( r \in (s_j, s_{j+1}) \) for some \( j \geq 1 \), then, setting \( \tau = (1/2)^\alpha \) (i.e., \( \alpha = \log_{1/2}(\tau) \in (0, 1) \)) and noticing that \( r/s \leq 2^{j+1-3} \), by (2.76) and (2.120) we have
\[
\delta(r) \leq \delta(s_j) \leq \tau^j \delta(s/8) = 2^{-j\alpha} \delta(s/8) = 4^{-\alpha} 2^{-(j-2)\alpha} \delta(s/8) \leq C(n) (s/r)^\alpha \delta(s/8), \tag{2.126}
\]
where in the last inequality (2.114) was used again; this proves (2.8). To prove (2.9), we recall that \( \omega_K(y) = \arctan(|\nu_K \cdot \hat{y}|/|\mathbf{p}_K \hat{y}|) \), provided arctan is defined on \( \mathbb{R} \cup \{-\infty, \infty\} \), and where \( \hat{y} = y/|y|, y \neq 0 \). Now, by (2.125),
\[
y = |y| \frac{\mathbf{p}_K \hat{y} + u(\mathbf{p}_K \hat{y}, |y|)\nu_K}{\sqrt{1 + u(\mathbf{p}_K \hat{y}, |y|)^2}}, \quad \forall y \in \text{spt} V \setminus B_{s/32},
\]
so that \( |\mathbf{p}_K \hat{y}| \geq 1/2 \) for \( y \in \text{spt} V \setminus B_{s/32} \); therefore, by (2.123), up to further decreasing the value of \( \varepsilon_0 \), and recalling \( \delta(s/8) \leq \varepsilon_0 \), we conclude
\[
|\mathbf{p}_{H_j} \hat{y}| \geq \frac{1}{3}, \quad \forall y \in \text{spt} V \setminus B_{s/32}, \tag{2.127}
\]
for every \( j \in \mathbb{N} \cup \{+\infty\} \) (if we set \( H_\infty = K \)). By (2.127) we easily find
\[
|\omega_K(y) - \omega_{H_j}(y)| \leq C |\nu_{H_j} - \nu_K|, \quad \forall y \in \text{spt} V \setminus B_{s/32}, \forall j \geq 1,
\]
from which we deduce that, if \( j \geq 1 \) and \( r \in (s_j, s_{j+1}) \), then
\[
\frac{1}{r^n} \int_{A^2_r} |\Lambda^{2r} \omega_K^2 d\|V\| | \leq C(n) \left\{ \frac{1}{s_j^n} \int_{A_{s_j}^j} \omega_K^2 d\|V\| + \frac{1}{s_{j+1}^n} \int_{A_{s_{j+1}}^{j+1}} \omega_K^2 d\|V\| \right\} \\
\leq C(n) \left\{ \frac{1}{s_j^n} \int_{A_{s_j}^j} \omega_{H_j}^2 d\|V\| + \frac{1}{s_{j+1}^n} \int_{A_{s_{j+1}}^{j+1}} \omega_{H_{j+1}}^2 d\|V\| \right\} \\
+ C(n) \frac{\Gamma \left\{ |\nu_K - \nu_{H_j}|^2 + |\nu_K - \nu_{H_{j+1}}|^2 \right\}}{\sigma^n},
\]
where (2.75) was used to bound \( \|V\|(A_p^2 \rho)^2 \leq \Gamma (2 \rho)^n \) with \( \rho = s_j, s_{j+1} \in (R, 1/\Lambda) \). By (2.116) we can exploit (2.30) on the first two integrals, so that taking (2.123) into account we find that, if \( j \geq 1 \) and \( r \in (s_j, s_{j+1}) \), then
\[
r^{-n} \int_{A^{2r}} \omega_K^2 d\|V\| \leq C(n) \left\{ T_j + T_{j+1} \right\} + C(n) \frac{\Gamma \tau^j \delta(s/8)}{\sigma} \leq C(n) \left( 1 + \Gamma \right) \tau^j \delta(s/8),
\]
where in the last inequality we have used (2.121). Since \( \tau^j \leq C(n) (s/r)^\alpha \), we conclude the proof of (2.9), and thus, of Theorem 2.1-(ii).

Step three: In this step, given \( n \geq 2, \Gamma \geq 0, \) and \( \sigma > 0 \), we claim the existence of \( \varepsilon_0 \) and \( M_0 \), depending only on \( n, \Gamma, \) and \( \sigma \), such that if (2.3), (2.4), (2.5) and (2.6) hold with \( \varepsilon_0 \) and \( M_0 \), and in addition,
\[
-\varepsilon_0 \leq \delta_{V, R, \Lambda}(s/8) \leq 0, \tag{2.128}
\]
then there exists \( K_- \in \mathcal{H} \) and \( u_- \in \mathcal{X}_\sigma(\Sigma_{K_-}, s/32, S_*/2) \) such that

\[
V \text{ corresponds to } \Sigma_{K_-}(u_+, s/32, S_*/2) \text{ on } A_{s/32}^{S_*/2};
\]

where \( S_* \) and \( R_* \) are as in Theorem 2.1. The argument is quite similar to that of the first step, with minor differences due to the opposite sign of the deficit. The first is that the iteration instead begins at the outer radius \( S_* \) and proceeds inwards via intermediate radii \( s_j = 2^{-j} S_* \), and the second is that, in the analogue of the graphicality propagation claims (2.85) \( j=\ell+1 \) and (2.86) \( j=\ell+1 \), the negative sign on \( \delta_{V,R,H} \) is used to sum the “tilting” between successive planes \( H_j \) and \( H_{j+1} \).

**Step four:** Finally, we combine steps one and three to prove statement (i) in Theorem 2.1. Before choosing the parameters \( \varepsilon_0 \) and \( M_0 \), we need a preliminary result. We claim that for any \( \varepsilon' > 0 \), there exists \( \sigma'(\varepsilon') > 0 \) such that if \( r_1 < r_2 \), \( K_1, K_2 \in \mathcal{H} \) with \( \nu_{K_1} \cdot \nu_{K_2} \geq 0 \) and accompanying \( u_i \in \mathcal{X}_{\sigma'}(\Sigma_{K_i}, r_i, r_2) \), and \( M \) is a smooth hypersurface such that \( M \cap A_{r_1}^\sigma \) corresponds to \( \Sigma_{K_i}(u_i, r_1, r_2) \) for \( k = 1, 2 \), then

\[
|\nu_{K_1} - \nu_{K_2}| < \varepsilon'.
\]

This is immediate from \( \nu_{K_1} \cdot \nu_{K_2} \geq 0 \) the fact that the \( L^\infty \)-bounds on \( u_i \) imply that \( M \) is contained in the intersection of two cones containing \( K_1 \) and \( K_2 \), whose openings become arbitrarily narrow as \( \sigma' \to 0 \).

Fix \( n \geq 2, \Gamma \geq 0, \) and \( \sigma > 0 \); we assume without loss of generality that \( \sigma < \sigma_0 \), where \( \sigma_0 \) is the dimension-dependent constant from Lemma 2.5. We choose \( \varepsilon' \) with corresponding \( \sigma' \) according to (2.130) such that, up to decreasing \( \sigma' \) if necessary,

\[
\varepsilon' < \varepsilon_0, \quad C_0(\sigma' + \varepsilon') \leq \sigma,
\]

where \( \varepsilon_0, C_0 \) are as in Lemma 2.5. Next, we choose \( \varepsilon_0 = \varepsilon_0(n, \Gamma, \sigma) \) and \( M_0 = M_0(n, \Gamma, \sigma) \) to satisfy several restrictions: first, \( \varepsilon_0 \) is smaller than the \( \varepsilon_0 \) from Lemma 2.5 and each \( \varepsilon_0(n, \Gamma, \sigma') \) from steps one and three, and \( M_0 \) is larger than \( M_0(n, \Gamma, \sigma') \) from those steps; second, with \( \varepsilon_2 \) and \( M_2 \) as in Lemma 2.8-(ii), we also assume that

\[
\varepsilon_0 \leq \min \left\{ \varepsilon', \varepsilon_2 \left( n, \sigma', \Gamma, \frac{1}{16}, \left( \frac{1}{128}, \frac{1}{2} \right) \right) \right\}, \quad M_0 \geq M_2 \left( n, \sigma', \Gamma, \frac{1}{16}, \left( \frac{1}{128}, \frac{1}{2} \right) \right)
\]

In the remainder of this step, we suppose that

\[
V \text{ satisfies (2.3), (2.4), (2.5) and (2.6) at mesoscale } s.
\]

In proving Theorem 2.1-(i), there are three cases depending on whether \( \delta_{V,R,H} \) changes sign on \([s/8, S_*] \).

**Case one:** \( \delta_{V,R,H}(r) \geq 0 \) for all \( r \in [s/8, S_*] \). If the deficit is non-negative, then in particular

\[
0 \leq \delta_{V,R,H}(s/8) \leq \varepsilon_0
\]

and \( S_* = S_+ \), where \( S_+ \) was defined in (2.66). By our choice of \( \varepsilon_0 \) and \( M_0 \) at the beginning of this step and the equivalence of (2.134) and (2.64),
step one applies and the conclusion (2.65) is (2.7). Thus Theorem 2.1-(i) is proved.

Case two: \( \delta_{V,R,A}(r) \leq 0 \) for all \( r \in [s/8, S_s] \). Should the deficit be non-positive in this interval, then in particular, (2.128) holds in addition to (2.3), (2.4), (2.5) and (2.6). Therefore, by our choice of \( \varepsilon_0 \) and \( M_0 \), step three applies. The conclusion (2.129) is (2.7) (in fact with larger upper radii \( S_/2 \)), and Theorem 2.1-(i) is proved.

Case three: \( \delta_{V,R,A} \) changes sign in \([s/8, S_s] \). By the monotonicity of \( \delta_{V,R,A} \), 
\[
\delta_{V,R,A}(s/8) > 0 > \delta_{V,R,A}(S_s) \quad (2.135)
\]
First, by (2.135), (2.64) is satisfied, so (2.65) gives \( K_+ \in H \) and \( u_+ \in X'(\Sigma_{K_+}, s/32, S_/16) \) such that 
\[
V \text{ corresponds to } \Sigma_{K_+}(u_+, s/32, S_/16) \text{ on } A_{s/32}^{S_/16}, \quad (2.136)
\]
where 
\[
R_+ = \max \left\{ \sup \left\{ \rho \geq s/8 : \delta_{V,R,A}(\rho) \geq 0 \right\}, 4s \right\}, \quad S_+ = \min \left\{ R_+, \frac{\varepsilon_0}{\Lambda} \right\} \quad (2.137)
\]
If \( S_+ = S_s \), then (2.136) is (2.7) and we are done. So we assume for the rest of this case that \( S_+ < S_s \), which implies \( S_+ \neq \varepsilon_0/\Lambda \) and thus 
\[
4s \leq R_+ = S_+ < S_s \quad (2.138)
\]
Next, we make the following
Claim: There exists \( K_- \in H \) and \( u_- \in X'(\Sigma_{K_-}, R_+/2, S_/2) \) such that 
\[
V \text{ corresponds to } \Sigma_{K_-}(u_-, R_+/2, S_/2) \text{ on } A_{R_+/2}^{S_/2} \quad (2.139)
\]
Proof of the claim: There are two subcases.
Subcase one: \( 16R_+ < \varepsilon_0/4\Lambda \) and \( 64R_+ < R_+ \). We claim the conditions of step three are verified at \( s' = 16R_+ \). First, (2.3) holds from (2.133), and
\[
\max \{64, M_0\} R < 16R_+ < \frac{\varepsilon_0}{4\Lambda}
\]
(which is (2.4)) holds due to the assumption of the subcase and \( 16R_+ \geq s > \max \{64, M_0\} R \). Next, \( 2R_+ < R_+/4 \) by the assumption of the subcase, which combined with the monotonicity of \( \delta_{V,R,A} \) and (2.137) gives \(-\varepsilon_0 \leq \delta_{V,R,A}(2R_+) \leq 0 \). This implies (2.128) and (2.5) with \( s' = 16R_+ \). Lastly, (2.6) holds at \( s' = 16R_+ \), since \( 64R_+ < R_+ \) and \( R_+ \) is defined in (2.139).
Subcase two: One or both of \( 16R_+ \geq \varepsilon_0/4\Lambda \), \( 64R_+ \geq R_+ \) hold. In this case, 
\[
64R_+ \geq \min \{\varepsilon_0/\Lambda, R_+\} = S_s \quad (2.140)
\]
We wish to apply Lemma 2.8-(ii) with \( r = S_s, \lambda_1 = \frac{1}{16}, (\eta_1, \eta_2) = \left( \frac{128}{\Lambda}, \frac{1}{2} \right) \). By (2.133), (2.48) holds for \( V \), and by (2.132), (2.133), and \( S_s \geq 4s \), 
\[
\max \{M_2, 64\} R \leq s \leq \frac{S_s}{4} \leq S_s \leq \frac{\varepsilon_2}{\Lambda}
\]
which is (2.53). Finally, we have \( R_s \geq S_s/16 \geq s/8 \), so that by the definition of \( R_s \), (2.5), the monotonicity of \( \delta_{V,R,A} \), and (2.132),
\[
\max \left\{ \left| \delta_{V,R,A}(\frac{S_s}{16}) \right|, \left| \delta_{V,R,A}(S_s) \right| \right\} \leq \varepsilon_0 \leq \varepsilon_2,
\]
which is (2.54). By the choices (2.132), Lemma 2.8-(ii) applies and yields the existence of \( K_\in \mathcal{H} \) and \( u_\in \mathcal{X}_\sigma^o(\Sigma_{K_\in}, S_s/128, S_s/2) \) such that
\[
V \text{ corresponds to } \Sigma_{K_\in}(u_+, S_s/128, S_s/2) \text{ on } A_{S_s/128}^{S_s/128}. \tag{2.141}
\]
By (2.140), \( S_s/128 \leq R_+/2 \), so (2.141) implies (2.139). The proof of the claim is complete.

Returning to the proof of Theorem 2.1-(i) under the assumption (2.135), we recall (2.138) and choose \( R' \in (R_+, \min\{2R_+, S_s\}) \). Again, we want to apply Lemma 2.8-(ii), this time with \( r = R' \), \( \lambda_1 = \frac{1}{16} \), and \( (\eta_1, \eta_2) = (1/128, 1/2) \). To begin with, \( V \) satisfies (2.48) as usual from (2.133). Second, (2.53) holds at \( R' \) by \( s \leq R' \leq S_s \), (2.133), and the choices (2.132). By the monotonicity of \( \delta_{V,R,A} \) and \( [R'/16, R'] \subset [R_+/16, S_s] \subset [s/4, S_s] \), (2.54) is valid by our choice (2.132) of \( \varepsilon_0 \). The graphicality result from Lemma 2.8-(ii) therefore yields \( K \in \mathcal{H} \) and \( u \in \mathcal{X}_\sigma^o(\Sigma_{K}, R'/128, R'/2) \) such that
\[
V \text{ corresponds to } \Sigma_K(u_+, R'/128, R'/2) \text{ on } A_{R'/128}^{R'/128}. \tag{2.142}
\]
Now \( s/32 \leq R'/128 < R_+/64 < S_s/16 \) by \( R' < 2R_+ \) and (2.138), and \( R_+/2 < R'/2 < S_s/2 \), so by (2.136) and (2.139), respectively, we have
\[
V \text{ corresponds to } \Sigma_{K_\in}(u_+, R'/128, S_s/16) \text{ on } A_{R'/128}^{S_s/16}, \tag{2.143}
\]
\[ V \text{ corresponds to } \Sigma_{K_\in}(u_-, R_+/2, R'/2) \text{ on } A_{R_+/2}^{R'/2}, \tag{2.144} \]
where \( u_+ \in \mathcal{X}_\sigma^o(\Sigma_{K_\in}, R'/128, S_s/16) \) and \( u_- \in \mathcal{X}_\sigma^o(\Sigma_{K_\in}, R_+/2, R'/2) \). Furthermore, up multiplying \( \nu_{K_+} \) or \( \nu_{K_-} \) by minus one, we may assume \( \nu_K \cdot \nu_{K_\pm} \geq 0 \). Thus \( V \) is represented by multiple spherical graphs on nontrivial annuli. By combining (2.142), (2.143) and (2.144), \( \nu_K \cdot \nu_{K_\pm} \geq 0 \) and \( \sigma' = \sigma'(\varepsilon') \), (2.130) applies and gives
\[
|\nu_K - \nu_{K_+}| < \varepsilon', \quad |\nu_K - \nu_{K_-}| < \varepsilon'. \tag{2.145}
\]
But \( \varepsilon' \) was chosen according to (2.131) so that Lemma 2.5-(i) is applicable; that is, since \( \varepsilon' < \varepsilon_0 \) and \( \sigma' < \sigma_0 \) from that lemma, we may reparametrize (2.136) and (2.139), respectively, as
\[
V \text{ corresponds to } \Sigma_K(w_+, s/32, S_s/16) \text{ on } A_{s/32}^{S_s/16}, \tag{2.146}
\]
\[ V \text{ corresponds to } \Sigma_K(w_-, R_+/2, S_s/2) \text{ on } A_{R_+/2}^{S_s/2}, \tag{2.147} \]
where
\[
w_+ \in \mathcal{X}_{C_0(\sigma'+\varepsilon')}(\Sigma_K, s/32, S_s/16), \quad w_- \in \mathcal{X}_{C_0(\sigma'+\varepsilon')}(\Sigma_K, R_+/2, S_s/2). \]
By (2.131), \( C_0(\sigma'+\varepsilon') \leq \sigma \), and by \( R'/128 < S_s/16 < R_+/2 < R'/2 \), (2.146) and (2.147), we may extend the \( u \) defined in (2.142) onto \( \Sigma_K \times (s/32, S_s/2) \)
using $w_+$ and $w_-$ with $C^1$-norm bounded by $\sigma$. The resulting extension is such that (2.7) holds, so the proof of Theorem 2.1 is finished.

$$\square$$

3. Application of Quantitative Isoperimetry

Here we apply quantitative isoperimetry to prove Theorem 1.6-(i) and parts of Theorem 1.6-(iv).

Theorem 3.1. If $W \subset \mathbb{R}^{n+1}$ is compact, $v > 0$, then $\min[\psi_W(v)] \neq \emptyset$. Moreover, depending on $n$ and $W$ only, there are $v_0$, $C_0$, $\Lambda_0$ positive, $s_0 \in (0,1)$, and $R_0(v)$ with $R_0(v) \to 0^+$ and $R_0(v)^{1/(n+1)} \to \infty$ as $v \to \infty$, such that, if $v > v_0$ and $E_v$ is a minimizer of $\psi_W(v)$, then:

(i): $E_v$ is a $(\Lambda_0/v^{1/(n+1)}, s_0 v^{1/(n+1)})$-perimeter minimizer with free boundary in $\Omega$, that is

$$P(E_v; \Omega \cap B_r(x)) \leq P(F; \Omega \cap B_r(x)) + \frac{\Lambda_0}{v^{1/(n+1)}} |E_v \Delta F|, \quad (3.1)$$

for every $F \subset \Omega = \mathbb{R}^{n+1} \setminus W$ with $E_v \Delta F \subset \subset B_r(x)$ and $r < s_0 v^{1/(n+1)}$;

(ii): $E_v$ determines $x \in \mathbb{R}^{n+1}$ such that

$$|E_v \Delta B^{(v)}(x)| \leq C_0 v^{-1+1/[2(n+1)]}; \quad (3.2)$$

if $\mathcal{R}(W) > 0$, then $E_v$ also determines $u \in C^\infty(\partial B^{(1)})$ such that

$$(\partial E_v) \setminus B_{R_0 v^{1/(n+1)}} = \left \{ y + v^{1/(n+1)} u \left( \frac{y - x}{v^{1/(n+1)}} \right) \nu_{B^{(v)}(x)}(y) : y \in \partial B^{(v)}(x) \right \} \setminus B_{R_0 v^{1/(n+1)}}, \quad (3.3)$$

(iii): if $\mathcal{R}(W) > 0$ and $x$ and $u$ depend on $E_v$ as in (3.2) and (3.3), then

$$\lim_{v \to \infty} \sup_{E_v \in \min[\psi_W(v)]} \max \left \{ ||x|| v^{-1/(n+1)} - \omega_{n+1}^{-1/(n+1)} \right \} \leq \max_{\partial B^{(1)}} u \rangle = 0. \quad (3.4)$$

Remark 3.2. (Improved convergence). We will repeatedly use the following fact (see, e.g. [FM11, FFM+15, CL12, CLM16]): If $\Omega$ is an open set, $\Lambda \geq 0$, $s > 0$, if $\{F_j\}$ are $(\Lambda, s)$-perimeter minimizers in $\Omega$, i.e. if it holds that

$$P(F_j; B_r(x)) \leq P(G_j; B_r(x)) + \Lambda |F_j \Delta G_j|, \quad (3.5)$$

whenever $G_j \Delta F_j \subset \subset B_r(x) \subset \subset \Omega$ and $r < s$, and if $F$ is an open set with smooth boundary in $\Omega$ such that $F_j \to F$ in $L^1_{\text{loc}}(\Omega)$ as $j \to \infty$, then for every $\Omega' \subset \subset \Omega$ there is $j(\Omega')$ such that

$$(\partial F_j) \cap \Omega' = \left \{ y + u_j(y) \nu_F(y) : y \in \Omega \cap \partial F \right \} \cap \Omega', \quad \forall j \geq j(\Omega'),$$

for a sequence $\{u_j\} \subset C^1(\Omega \cap \partial F)$ with $\|u_j\|_{C^1(\Omega \cap \partial F)} \to 0$. Compare the terminology used in (3.1) and (3.5): when we add “with free boundary”, the “localizing balls” $B_r(x)$ are not required to be compactly contained in $\Omega$, and the perimeters are computed in $B_r(x) \cap \Omega$. 

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Proof of Theorem 3.1. **Step one:** We prove \( \min[\psi_W(v)] \neq \emptyset \) for all \( v > 0 \). Since \( W \) is compact, \( B^{(v)}(x) \subset \Omega \) for \( |x| \) large. Hence there is \( \{E_j\}_j \) with
\[
E_j \subset \Omega, \quad |E_j| = v, \quad P(E_j; \Omega) \leq \min \left\{ P(B^{(v)}), P(F; \Omega) \right\} + \frac{1}{j}, \quad (3.6)
\]
for every \( F \subset \Omega \) with \( |F| = v \). Hence, up to extracting subsequences, \( E_j \to E \) in \( L^1_{\text{loc}}(\mathbb{R}^{n+1}) \) with \( P(E; \Omega) \leq \lim_{j \to \infty} P(E_j; \Omega) \), where \( E \subset \Omega \) and \( |E| \leq v \). We now make three remarks concerning \( E \):

(a): If \( \{\Omega_i\}_{i \in I} \) are the connected components of \( \Omega \), then \( \Omega \cap \partial^* E = \emptyset \) if and only if \( E = \bigcup_{i \in I_0} \Omega_i \) (\( I_0 \subset I \)). Indeed, \( \Omega \cap \partial^* E = \emptyset \) implies \( \text{cl} (\partial^* E) \cap \Omega = \partial E \cap \Omega \), hence \( \partial E \subset \partial \Omega \) and \( E = \bigcup_{i \in I_0} \Omega_i \). The converse is immediate.

(b): If \( \Omega \cap \partial^* E \neq \emptyset \), then we can construct a system of “volume-fixing variations” for \( \{E_j\}_j \). Indeed, if \( \Omega \cap \partial^* E \neq \emptyset \), then there are \( B_{S_0}(x_0) \subset \subset \Omega \) with \( P(E; \partial B_{S_0}(x_0)) = 0 \) and a vector field \( X \in C^\infty(\Omega_{S_0}(x_0); \mathbb{R}^{n+1}) \) such that \( \int_E \text{div} \, X = 1 \). By [Mag12, Theorem 29.14], there are constants \( C_0, C_1 > 0 \), depending on \( E \) itself, with the following property: whenever \( |(F \Delta E) \cap B_{S_0}(x_0)| < c_0 \), then there is a smooth function \( \Phi^F : \mathbb{R}^n \times (-c_0, c_0) \to \mathbb{R}^n \) such that, for each \( |t| < c_0 \), the map \( \Phi^F_t = \Phi^F(\cdot, t) \) is a smooth diffeomorphism with \( \{\Phi^F_t \neq \text{id}\} \subset B_{S_1}(x_0), \text{ and } |\Phi^F_t(F)| = |F| + t, \) and \( P(\Phi^F_t(F); B_{S_0}(x_0)) \leq (1 + C_0|t|) P(F; B_{S_0}(x_0)) \).

(c): If \( \Omega \cap \partial^* E \neq \emptyset \), then \( E \) is bounded. Since \( |E| \leq v < \infty \), it is enough to prove that \( \Omega \cap \partial^* E \) is bounded. In turn, taking \( x_0 \in \Omega \cap \partial^* E \), and since \( W \) is bounded and \( |E| < \infty \), the boundedness of \( \Omega \cap \partial^* E \) descends immediately by the following density estimate: there is \( r_1 > 0 \) such that
\[
|E \cap B_r(x)| \geq c(n) r^{n+1} \quad \forall x \in \Omega \cap \partial^* E, \quad r < r_1, B_r(x) \subset \subset \mathbb{R}^{n+1} \setminus (I_{r_1}(W) \cup B_{S_0}(x_0)). \quad (3.7)
\]

To prove (3.7), let \( r_1 > 0 \) be such that \( |B_{r_1}| < c_0 \), let \( x \) and \( r \) be as in (3.7), and set \( F_t = (\Phi^F_t(E_j) \cap B_{S_0}(x_0)) \cup [E \setminus (B_r(x) \cup B_{S_0}(x_0))] \) for \( t = |E_j \cap B_r(x)| \) (which is an admissible value of \( t \) by \( |B_{r_1}| < c_0 \)). In this way, \( |F_j| = |E_j| = v \), and thus we can exploit (3.6) with \( F = F_j \). A standard argument (see, e.g. [Mag12, Theorem 21.11]) leads then to (3.7).

Now, since \( \partial \Omega \subset W \) is bounded, every connected component of \( \Omega \) with finite volume is bounded. Thus, by (a), (b) and (c) above, there is \( R > 0 \) such that \( W \cup E \subset \subset B_R \). Since \( E \cap \partial B_{R+1} \setminus B_R \) is compact, we can pick \( T \in (R, R+1) \) such that \( \mathcal{H}^n(E_j \cap \partial B_T) \to 0 \) and \( P(E_j \setminus B_T) = \mathcal{H}^n(E_j \cap \partial B_T) + P(E_j \setminus \Omega \setminus B_T) \), and consider the sets \( F_j = (E_j \cap B_T) \cup B_{\rho_j}(y) \) corresponding to \( \rho_j = \frac{|E_j \setminus B_T|}{\omega_{n+1}^{1/(n+1)}} \) and \( y \in \mathbb{R}^{n+1} \) which is independent of \( j \) and \( \rho_j \geq T \) (notice that \( \sup_j \rho_j \leq C(n) v^{1/(n+1)} \)). Since \( |F_j| = |E_j| = v \),
By (3.2) there are $\varepsilon_B$ factors $v > v$ denoting by $U_v > v$ where we have used that, if $G_{1} \subset \subset B_{r^{*}}$ and $T^{*}$ independent of $j$. We conclude by the Direct Method.

**Step two**: We prove (3.2). If $E_v$, a minimizer of $\psi_W(v)$ and $R > 0$ is such that $W \subset B_R$, then by $P(E_v; \Omega) \leq P(B(v))$ we have, for $v > v_0$, and $v_0$ and $C_0$ depending on $n$ and $W$,

$$P(E_v \setminus B_R) \leq P(E_v; \Omega) + n \omega_n R^n \leq P(B(v)) + C_0$$

(3.8)

where we have used that, if $v > 2b > 0$ and $\alpha = n/(n + 1)$, then

$$P(B(v)) P(B(v-b)^{-1} - 1 = (v/(v-b))^\alpha - 1 \leq \alpha b/(v-b) \leq 2 \alpha b v^{-1}.$$  

By combining (1.3) and (3.8) we conclude that, for some $x \in \mathbb{R}^{n+1}$,

$$c(n) \left( \frac{|E_v \setminus B_R| \Delta B(\{|E_v \setminus B_R| \})}{|E_v \setminus B_R|} \right)^2 \leq \frac{P(E_v \setminus B_R)}{P(B(|E_v \setminus B_R|))} - 1 \leq \frac{C_0}{v^{n/(n+1)}},$$

provided $v > v_0$. Hence we deduce (3.2) from

$$|E_v \Delta B(v)(x)| = 2 |E_v \setminus B(v)(x)| \leq C_0 + 2 |(E_v \setminus B_R) \setminus B(v)(x)| \leq C_0 + |E_v \setminus B_R| C_0 v^{-n/2(n+1)}.$$

**Step three**: We prove the existence of $v_0, \lambda_0,$ and $s_0$ such that every $E_v \in \text{Min}[\psi_W(v)]$ with $v > v_0$ satisfies (3.1). Arguing by contradiction, we assume the existence of $v_j \rightarrow \infty$, $E_j \in \text{Min}[\psi_W(v_j)]$, $F_j \subset \Omega$ with $|F_j \Delta E_j| > 0$ and $F_j \Delta E_j \subset \subset B_{r_j}(x_j)$ for some $x_j \in \mathbb{R}^{n+1}$ and $r_j = v_j^{1/(n+1)}$, such that

$$P(E_j; \Omega \setminus B_r(x_j)) \geq P(F_j; \Omega \setminus B_{r_j}(x_j)) + j v_j^{-1/(n+1)} |E_j \Delta F_j|.$$  

Denoting by $E_j^*, F_j^*$ and $\Omega_j$ the sets obtained by scaling $E_j, F_j$ and $\Omega$ by a factor $v_j^{-1/(n+1)}$, we find that $F_j^* \Delta E_j^* \subset \subset B_{1/j}(y_j)$ for some $y_j \in \mathbb{R}^{n+1}$, and

$$P(E_j^*; \Omega_j \setminus B_{1/j}(y_j)) \geq P(F_j^*; \Omega_j \setminus B_{1/j}(y_j)) + j |E_j^* \Delta F_j^*|.$$  

(3.9)

By (3.2) there are $z_j \in \mathbb{R}^{n+1}$ such that $|E_j^* \Delta B^{(1)}(z_j)| \rightarrow 0$. We can therefore use the volume-fixing variations of $B^{(1)}$ to find diffeomorphisms $\Phi_{t}^{j} : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and constants $c(n)$ and $C(n)$ such that, for every $|t| < c(n)$, one has $\{\Phi_{t}^{j} \neq \text{id}\} \subset \subset U_j$ for some open ball $U_j \subset \subset \Omega_j \setminus B_{1/j}(y_j)$, $|\Phi_{t}^{j}(E_j^*) \setminus U_j| = |E_j^* \setminus U_j| + t$, and $P(\Phi_{t}^{j}(E_j^*) \setminus U_j) \leq (1 + C(n)|t|) P(E_j^*; U_j)$. Since $F_j^* \Delta E_j^* \subset \subset B_{1/j}(y_j)$ implies $||E_j^* - |E_j^*|| < c(n)$ for $j$ large, if $t = |E_j^*| - |F_j^*|$, then $G_j^* = \Phi_{t}^{j}(F_j^*)$ is such that $|G_j^*| = |E_j^*|$, and by $E_j \in \text{Min}[\psi_W(v_j)],$

$$P(E_j^*; \Omega_j) \leq P(G_j^*; \Omega_j) \leq P(E_j^*; \Omega_j \setminus (U_j \cup B_{1/j}(y_j))) + P(F_j^*; \Omega_j \setminus B_{1/j}(y_j)) + P(E_j^*; U_j) + C(n) P(E_j^*; U_j) |E_j^* \Delta F_j^*|.$$


Taking into account $P(E_j^*; U_j) \leq \psi_W(v_j)/v_j^{n/(n+1)} \leq C(n)$, we thus find
\[ P(E_j^*; \Omega_j \cap B_{1/j}(y_j)) \leq P(F_j^*; \Omega_j \cap B_{1/j}(y_j)) + C(n) \left| E_j^* \Delta F_j^* \right|, \]
which, by (3.9), gives $j \left| E_j^* \Delta F_j^* \right| \leq C(n) \left| E_j^* \Delta F_j^* \right|$, since $\left| E_j^* \Delta F_j^* \right| > 0$, this is a contradiction for $j$ large enough.

**Step four:** We now prove that, if $\mathcal{R}(W) > 0$, then
\[ \lim_{v \to \infty} \sup_{E_v \in \text{Min} \{\psi_W(v)\}} \left| |x| v^{-1/(n+1)} - \omega_{n+1}^{-1/(n+1)} \right| = 0, \tag{3.10} \]
where $x$ is related to $E_v$ by (3.2). In proving (3.10) we will use the assumption $\mathcal{R}(W) > 0$ and the energy upper bound
\[ \lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) \leq -\mathcal{R}(W). \tag{3.11} \]
A proof of (3.11) is given in step one of the proof of Theorem 1.6, see section 5; in turn, that proof is solely based on the results from section 4, where no part of Theorem 3.1 (not even the existence of minimizers in $\psi_W(v)$) is ever used. This said, when $|W| > 0$, and thus $\mathcal{S}(W) > 0$, one can replace (3.11) in the proof of (3.10) by the simpler upper bound
\[ \lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) \leq -\mathcal{S}(W), \tag{3.12} \]
where, we recall, $\mathcal{S}(W) = \sup \{\mathcal{H}^n(W \cap \Pi) : \Pi \text{ is a hyperplane in } \mathbb{R}^{n+1}\}$. To prove (3.12), given $\Pi$, we construct competitors for $\psi_W(v)$ by intersecting $\Omega$ with balls $B(v')(x_v)$ with $v' > v$ and $x_v$ such that $|B(v')(x_v)| \setminus W = v$ and $\mathcal{H}^n(W \cap \partial B(v')(x_v)) \to \mathcal{H}^n(W \cap \Pi)$ as $v \to \infty$. Hence, $\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) \leq -\mathcal{H}^n(W \cap \Pi)$, thus giving (3.12). The proof of (3.11) is identical in spirit to that of (3.12), with the difference that to glue a large ball to $(F, \nu) \in \text{Max} \{\mathcal{R}(W)\}$ we will need to establish the decay of $\partial F$ towards a hyperplane parallel to $v^\perp$ to the high degree of precision expressed in (1.14).

Now to prove (3.10): by contradiction, consider $v_j \to \infty$, $E_j \in \text{Min} \{\psi_W(v_j)\}$, and $x_j \in \mathbb{R}^{n+1}$ with $\inf_{x \in \mathbb{R}^{n+1}} |E_j \Delta B^{(v_j)}(x)| = |E_j \Delta B^{(v_j)}(x)|$, such that
\[ \lim_{j \to \infty} \left| |x_j| v_j^{-1/(n+1)} - \omega_{n+1}^{-1/(n+1)} \right| > 0, \tag{3.13} \]
and set $\lambda_j = v_j^{-1/(n+1)}$, $E_j^* = \lambda_j (E_j - x_j)$, $W_j^* = \lambda_j (W - x_j)$, and $\Omega_j^* = \lambda_j (\Omega - x_j)$. By (3.1), each $E_j^*$ is a $(\Lambda_0, s_0)$-perimeter minimizer with free boundary in $\Omega_j^*$. By (3.2) and the defining property of $x_j$, $E_j^* \to B^{(1)}$ in $L^1(\mathbb{R}^{n+1})$. Moreover, $\text{diam} (W_j^*) \to 0$ and, by (3.13),
\[ \lim_{j \to \infty} \text{dist} (W_j^*, \partial B^{(1)}) > 0. \tag{3.14} \]
Thus there is $z_0 \notin \partial B^{(1)}$ such that, for every $\rho < \text{dist}(z_0, \partial B^{(1)})$, there is $j(\rho)$ such that $\{E_j\}_{j \geq j(\rho)}$ is a sequence of $(\Lambda_0, s_0)$-perimeter minimizers in $\mathbb{R}^{n+1} \setminus B_{\rho/2}(z_0)$. By Remark 3.2, up to increasing $j(\rho)$, $(\partial E_j^*) \setminus B_\rho(z_0)$ is contained in the normal graph over $\partial B^{(1)}$ of $u_j$ with $\|u_j\|_{C^1(\partial B^{(1)})} \to 0$; in...
particular, by (3.14), \((\partial E^*_j) \setminus B_\rho(z_0)\) is disjoint from \(W^*_j\). By the constant mean curvature condition satisfied by \(\Omega \cap \partial E^*_j\), and by Alexandrov’s theorem [Ale62], \((\partial E^*_j) \setminus B_\rho(z_0)\) is a sphere \(M^*_j\) for \(j \geq j(\rho)\). Let \(B^*_j\) be the ball bounded by \(M^*_j\). Since \(M^*_j \cap W^*_j = \emptyset\), we have either one of the following:

**Case one:** \(W^*_j \subset B^*_j\). We have \(\partial (B^*_j \cup E^*_j) \subset M^*_j \cup [(\partial E^*_{j'}) \setminus \text{cl}(B^*_j)] \subset (\partial E^*_{j'}) \setminus W^*_j\), so that, by \(|B^*_j \cup E^*_j| \geq |E^*_j| + |W^*_j| \geq 1\), we find \(P(E^*_j; \Omega^*_j) \geq P(B^*_j \cup E^*_j) \geq P(B^*_j)\), that is, \(\psi_W(v_j) \geq P(B^*_j)\), against (3.11).

**Case two:** \(W^*_j \cap B^*_j = \emptyset\). In this case, \(E^*_j = B^*_j \cup G^*_j\), where \(G^*_j\) is the union of the connected components of \(E^*_j\) whose boundaries have non-empty intersection with \(W^*_j\); in other words, we are claiming that \(B^*_j\) is the only connected component of \(E^*_j\) whose closure is disjoint from \(W^*_j\). Indeed, if this were not the case, we could recombine all the connected components of \(E^*_j\) with closure disjoint from \(W^*_j\) into a single ball of same total volume, centered far away from \(W^*_j\), in such a way to strictly decrease \(P(E^*_j; \Omega^*_j)\), against \(E_j \in \text{Min}[\psi_W(v_j)]\). Let us now set \(G_j = x_j + v_j/(n+1) \cdot G^*_j\) and \(U_j = x_j + v_j^{1/(n+1)} \cdot B^*_j\), so that \(E_j = G_j \cup U_j\) and \(\text{dist}(G_j, U_j) > 0\).

If we start sliding \(U_j\) from infinity towards \(G_j \cup W\) along arbitrary directions, then at least one of the resulting “contact points” \(z_j\) belongs to \(\Omega \cap \partial G_j\): if this were not the case, then \(G_j\) would be contained in the convex envelope of \(W\), so that \(|B_j| = |E_j| - |G_j| \geq v_j - C(W)|\), and thus, by \(\psi_W(v_j) = P(E_j; \Omega) \geq P(B_j; W) = P(B_j)\), and by \(P(B_j) \geq P(B_j - C(W)) \geq P(B_j) - C(W) \cdot v_j^{1/(n+1)}\), against with (3.11) for \(j\) large.

By construction, there is a half-space \(H_j\) such that \(G_j \subset H_j\), \(z_j \in (\partial G_j) \cap (\partial H_j)\), and \(G_j\) is a perimeter minimizer in \(B_r(z_j)\) for some small \(r > 0\). By the strong maximum principle, see, e.g. [DPM15, Lemma 2.13], \(G_j\) has \(H_j - z_j\) as its unique blowup at \(z_j\). By De Giorgi’s regularity theorem, see e.g. [Mag12, Part III], \(G_j\) is an open set with smooth boundary in a neighborhood of \(z_j\). Therefore, if we denote by \(U_j^j\) the translation of \(U_j\) constructed in the sliding argument, then, \(E_j^* = G_j \cup U_j^j \in \text{Min}[\psi_W(v)]\) and, in a neighborhood of \(z_j\), \(E_j^*\) is the union of two disjoint sets with smooth boundary which touch tangentially at \(z_j\). In particular, \(|E_j^* \cap B_r(z_j)|/|B_r| \to 1\) as \(r \to 0^+\), against volume density estimates implied by (3.1), see, e.g. [Mag12, Theorem 21.11].

**Step five:** We finally show the existence of \(v_0\) and \(R_0(v)\) with \(R_0(v) \to 0^+\) and \(R_0(v) v^{1/(n+1)} \to \infty\), such that each \(E_n \in \text{Min}[\psi_W(v)]\) with \(v > v_0\) determines \(x\) and \(u \in C^\infty(\partial B(1))\) such that (3.3) holds and \(\|u\|_{C^1(\partial B(1))} \to 0\) as \(v \to \infty\). To this end, let us consider \(v_j \to \infty\), \(E_j \in \text{Min}[\psi_W(v_j)]\), and define \(x_j, E^*_j\) and \(W^*_j\) as in step four. Thanks to (3.10), there is \(z_0 \in \partial B(1)\) s.t. \(\text{dist}(z_0, W^*_j) \to 0\). In particular, for every \(\rho > 0\), we can find \(j(\rho) \in \mathbb{N}\) such that if \(j \geq j(\rho)\), then \(E^*_j\) is a \((\lambda_0, \delta_0)\)-perimeter minimizer in \(\mathbb{R}^{n+1} \setminus B_\rho(z_0)\), with \(E^*_j \to B(1)\). By Remark 3.2, there are \(u_j \in C^1(\partial B(1))\) such that

\[
(\partial E^*_j) \setminus B_{2\rho}(z_0) = \{y + u_j(y) v_{B(1)}(y) : y \in \partial B(1)\} \setminus B_{2\rho}(z_0), \quad \forall j \geq j(\rho),
\]
and \( \|u_j\|_{C^1(\partial B(1))} \to 0 \). By the arbitrariness of \( \rho \) and by a contradiction argument, (3.3) holds with \( R_0(v) \to 0^+ \) such that \( R_0(v)v^{1/(n+1)} \to \infty \) as \( v \to \infty \), and with the uniform decay of \( \|u\|_{C^1(\partial B(1))} \). \( \square \)

4. Properties of isoperimetric residues

Here we prove Theorem 1.1. It will be convenient to introduce some notation for cylinders and slabs in \( \mathbb{R}^{n+1} \): precisely, given \( r > 0, \nu \in S^n \) and \( I \subset \mathbb{R} \), and setting \( \mathbf{p}_\nu^\perp(x) = x - (x \cdot \nu) \nu \ (x \in \mathbb{R}^{n+1}) \), we let

\[
\begin{align*}
\mathbf{D}_R^\nu &= \{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_\nu^\perp x| < r, x \cdot \nu = 0 \}, \\
\mathbf{C}_R^\nu &= \{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_\nu^\perp x| < r \}, \\
\mathbf{C}_{R,I}^\nu &= \{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_\nu^\perp x| < r, x \cdot \nu \in I \}, \\
\partial_i \mathbf{C}_{R,I}^\nu &= \{ x \in \mathbb{R}^{n+1} : |\mathbf{p}_\nu^\perp x| = r, x \cdot \nu \in I \}, \\
\mathbf{S}_R^\nu &= \{ x \in \mathbb{R}^{n+1} : x \cdot \nu \in I \}.
\end{align*}
\]

In each case, given \( x \in \mathbb{R}^{n+1} \), we also set \( \mathbf{D}_R^\nu(x) = x + \mathbf{D}_R^\nu, \mathbf{C}_R^\nu(x) = x + \mathbf{C}_R^\nu, \) etc. We premise the following proposition, used in the proof of Theorem 1.1 and Theorem 1.6, and based on [Sch83, Proposition 1 and Proposition 3].

**Proposition 4.1.** Let \( n \geq 2, \nu \in S^n \), and let \( f \) be a Lipschitz solution to the minimal surface equation on \( \nu^\perp \setminus \text{cl}(\mathbf{D}_R^\nu) \). If \( n = 2 \), assume in addition that \( M = \{ x + f(x) \nu : |x| > R \} \) is stable and has natural area growth, i.e.

\[
\int_M |\nabla^M \varphi|^2 - |A|^2 \varphi^2 \geq 0, \quad \forall \varphi \in C_c^1(\mathbb{R}^3 \setminus B_R), \quad (4.2)
\]

\[
\mathcal{H}^2(M \cap B_r) \leq Cr^2, \quad \forall r > R. \quad (4.3)
\]

Then there are \( a, b \in \mathbb{R} \) and \( c \in \nu^\perp \) such that, for every \( |x| > R \),

\[
\begin{align*}
|f(x) - (a + b |x|^{2-n} + (c \cdot x) |x|^{2-n})| &\leq C|x|^{-n}, \quad (n \geq 3) \quad (4.4) \\
|f(x) - (a + b \log |x| + (c \cdot x) |x|^{-2})| &\leq C|x|^{-2}, \quad (n = 2) \quad (4.5) \\
\max \left\{ |x|^{n-1} |\nabla f(x)|, |x|^{n} |\nabla^2 f(x)| : |x| > R \right\} &\leq C, \quad (\text{every } n). \quad (4.6)
\end{align*}
\]

**Proof.** If \( n \geq 3 \), the fact that \( \nabla f \) is bounded allows one to represent \( f \) as the convolution with a singular kernel which, by a classical result of Littman, Stampacchia, and Weinberger [LSW63], is comparable to the Green’s function of \( \mathbb{R}^n \); (4.4) is then deduced starting from that representation formula. For more details, see [Sch83, Proposition 3]. In the case \( n = 2 \), by (4.2) and (4.3), we can exploit a classical “logarithmic cut-off argument” to see that \( M \) has finite total curvature, i.e. \( \int_M |K|d\mathcal{H}^2 < \infty \), where \( K \) is the Gaussian curvature of \( M \). As a consequence, see, e.g., [PR02, Section 1.2], the compactification \( \overline{M} \) of \( M \) is a Riemann surface with boundary, and \( M \) is conformally equivalent to \( \overline{M} \setminus \{ p_1, \ldots, p_m \} \), where \( p_i \) are interior points of \( \overline{M} \). One can thus conclude by the argument in [Sch83, Proposition 1] that \( M \) has \( m \)-many ends satisfying the decay (4.5), and then that \( m = 1 \) thanks to the fact that \( M = \{ x + f(x) \nu : |x| > R \} \). \( \square \)
Proof of Theorem 1.1. Step one: Given a hyperplane $\Pi$ in $\mathbb{R}^{n+1}$, if $F$ is a half-space with $\partial F = \Pi$ and $\nu$ is a unit normal to $\Pi$, then $\text{res}_W(F, \nu) = \mathcal{H}^n(W \cap \Pi)$. Therefore the lower bound in (1.11) follows by
\[ \mathcal{R}(W) \geq S(W) = \sup \{ \mathcal{H}^n(\Pi \cap W) : \Pi \text{ an hyperplane in } \mathbb{R}^{n+1} \}. \] (4.7)

Step two: We notice that, if $(F, \nu) \in \mathcal{F}$, then by (1.8), (1.9), and the divergence theorem (see, e.g., [Mag12, Lemma 22.11]), we can define a Radon measure on the open set $\nu^\perp \setminus p_{\nu^\perp}(W)$ by setting
\[ \mu(U) = P(F; (p_{\nu^\perp})^{-1}(U)) - \mathcal{H}^n(U), \quad U \subset \nu^\perp \setminus p_{\nu^\perp}(W). \] (4.8)
In particular, setting $R' = \inf \{ \rho : W \subset C_\rho^\nu \}, \mu(D_R^\nu \setminus p_{\nu^\perp}(W)) \geq 0$ gives
\[ P(F; C_R^\nu \setminus W) \geq \omega_n R^n - \mathcal{H}^n(p_{\nu^\perp}(W)), \quad \forall R > R', \]
while the identity
\[ \omega_n R^n - P(F; C_R^\nu \setminus W) = -\mu(D_R^\nu \setminus D_{R'}^\nu) + \omega_n (R')^n - P(F; C_{R'}^\nu \setminus W) \]
(which possibly holds as $-\infty = -\infty$ if $P(F; C_{R'}^\nu \setminus W) = +\infty$) gives that
\[ R \in (R', \infty) \mapsto \omega_n R^n - P(F; C_R^\nu \setminus W) \]
is decreasing on $(R', \infty)$. (4.9)

In particular, the limsup defining $\text{res}_W$ always exists as a limit.

Step three: We prove the existence of $(F, \nu) \in \text{Max}[\mathcal{R}(W)]$ and (1.12). We first claim that if $\{(F_j, \nu_j)\}_j$ is a maximizing sequence for $\mathcal{R}(W)$, then, in addition to $p_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp$, one can modify $(F_j, \nu_j)$, preserving the optimality in the limit $j \to \infty$, so that (setting $X \subset \mathcal{L}^{n+1} Y$ for $|X \setminus Y| = 0$)
\[ \partial F_j \subset S_{(B_j, \nu_j)}^\nu \subset F_j , \quad S_{[A_j, B_j]}^\nu \subset \mathcal{L}^{n+1} \setminus F_j, \] (4.10)
where $[A_j, B_j] = \bigcap \{ (\alpha, \beta) : W \subset S_{(\alpha, \beta)}^\nu \}$. (4.11)

Indeed, since $(F_j, \nu_j) \in \mathcal{F}$, for some $\alpha_j < \beta_j \in \mathbb{R}$ we have
\[ \partial F_j \subset S_{[\alpha_j, \beta_j]}^\nu , \quad p_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp. \] (4.12)

Would it be that either $S_{(-\infty, \alpha_j]}^\nu \cup (\beta_j, \infty) \subset \mathcal{L}^{n+1} F_j$ or $S_{(\alpha_j, \beta_j]}^\nu \subset \mathcal{L}^{n+1} \setminus F_j$; then, by the divergence theorem and by $p_{\nu_j^\perp}(\partial F_j) = \nu_j^\perp$,
\[ P(F_j; C_R^\nu \cap \Omega) \geq 2 (\omega_n R^n - \mathcal{H}^n(p_{\nu_j^\perp}(W))), \quad \forall R > 0, \]
and thus $\text{res}_W(F_j, \nu_j) = -\infty$; in particular, $(F_j, \nu_j) \in \mathcal{F}$ being a maximizing sequence, we would have $\mathcal{R}(W) = -\infty$, against (4.7). This proves the validity (up to switching $F_j$ with $\mathbb{R}^{n+1} \setminus F_j$), of the inclusions
\[ S_{(-\infty, \alpha_j]}^\nu \subset \mathcal{L}^{n+1} F_j , \quad S_{(\beta_j, \infty)}^\nu \subset \mathcal{L}^{n+1} \setminus F_j . \] (4.13)

Thanks to (4.13) (and by exploiting basic set operations on sets of finite perimeter, see, e.g., [Mag12, Theorem 16.3]), we see that
\[ F_j^* = (F_j \cup S_{(-\infty, A_j-1/j]}^\nu) \cap S_{(-\infty, B_j+1/j]}^\nu \]
with $F_j^*, \nu_j \in \mathcal{F}$, $P(F_j^*; C_R^\nu \setminus W) \leq P(F_j; C_R^\nu \setminus W), \quad \forall R > 0;$. (4.14)
in particular, \( \{(F_j^*; \nu_j)\}_j \) is also a maximizing sequence for \( \mathcal{R}(W) \). By standard compactness theorems there are \( F \) of locally finite perimeter in \( \mathbb{R}^{n+1} \) and \( \nu \in \mathbb{S}^n \) such that \( F_j \to F \) in \( L^1_{\text{loc}}(\mathbb{R}^{n+1}) \) and \( \nu_j \to \nu \). If \( A \subset \subset C_R^\nu \setminus W \) is open, then, for \( j \) large enough, \( A \subset \subset C_R^{\nu_j} \setminus W \), and thus
\[
P(F; C_R^{\nu_j} \setminus W) = \sup_{A \subset \subset C_R^\nu \setminus W} P(F; A) \leq \lim_{j \to \infty} P(F_j; C_R^{\nu_j} \setminus W). \tag{4.15}
\]
By (4.9), \( R \mapsto \omega_n R^n - P(F_j; C_R^{\nu_j} \setminus W) \) is decreasing on \( R > R_j = \inf\{ \rho : W \subset C_\rho^\nu \} \). By sup, \( R_j \leq C(W) < \infty \) and (4.15) we have
\[
\omega_n R^n - P(F; C_R^\nu \setminus W) \geq \lim_{j \to \infty} \omega_n R^n - P(F_j; C_R^{\nu_j} \setminus W) \geq \lim_{j \to \infty} \res_W(F_j, \nu_j),
\]
for every \( R > C(W) \); in particular, letting \( R \to \infty \),
\[
\res_W(F, \nu) \geq \lim_{j \to \infty} \res_W(F_j, \nu_j) = \mathcal{R}(W). \tag{4.16}
\]
By \( F_j \to F \) in \( L^1_{\text{loc}}(\mathbb{R}^{n+1}) \), \( \partial F = \text{cl}(\partial^* F) \) is contained in the set of accumulation points of sequences \( \{x_j\}_j \) with \( x_j \in \partial F_j \), so that (4.10) gives
\[
\partial F \subset S_{[A, B]}^\nu, \quad S_{(-\infty, A]}^\nu \subset L^{n+1} F, \quad S_{(B, \infty)}^\nu \subset L^{n+1} \mathbb{R}^{n+1} \setminus F, \tag{4.17}
\]
if \( [A, B] = \bigcap\{ (\alpha, \beta) : W \subset S_{(\alpha, \beta)}^\nu \} \). Therefore \( (F, \nu) \in \mathcal{F} \), and thus, by (4.16), \( (F, \nu) \in \text{Max} [\mathcal{R}(W)] \). We now show that (4.16) implies (1.12), i.e.
\[
P(F; \Omega \cap B) \leq P(G; \Omega \cap B), \quad \forall F \Delta G \subset B, B \text{ a ball}. \tag{4.18}
\]
Indeed, should (4.18) fail, we could find \( \delta > 0 \) and \( G \subset \mathbb{R}^{n+1} \) with \( F \Delta G \subset B \) for some ball \( B \), such that \( P(G; B \setminus W) + \delta \leq P(F; B \setminus W) \). For \( R \) large enough to entail \( B \subset \subset C_R^\nu \) we would then find
\[
\res_W(F, \nu) + \delta \leq \omega_n R^n - P(F; C_R^\nu \setminus W) + \delta \leq \omega_n R^n - P(G; C_R^\nu \setminus W),
\]
which, letting \( R \to \infty \), would violate the maximality of \( (F, \nu) \) in \( \mathcal{R}(W) \).

**Step four:** We show that if \( \mathcal{R}(W) > 0 \) and \( (F, \nu) \in \text{Max} [\mathcal{R}(W)] \), then \( \partial F \subset S_{[A, B]}^\nu \) for \( A, B \) as in (4.17). Otherwise, by the same truncation procedure leading to (4.14) and by \( (F, \nu) \in \text{Max} [\mathcal{R}(W)] \), we would find
\[
\omega_n R^n - P(F; C_R^{\nu_j} \setminus W) \geq \omega_n R^n - P(F; C_R^{\nu_j} \setminus W) \geq \mathcal{R}(W) \quad \forall R > 0,
\]
so that \( (F^*, \nu) \in \text{Max} [\mathcal{R}(W)] \) too. Now \( P(F; C_R^{\nu_j} \setminus W) - P(F^*; C_R^{\nu_j} \setminus W) \) is increasing in \( R \), and since \( \res_W(F, \nu) = \res_W(F^*, \nu) \), it follows that \( P(F; C_R^{\nu_j} \setminus W) = P(F^*; C_R^{\nu_j} \setminus W) \) for large \( R \). But this can hold only if \( \partial F \cap \Omega \) is an hyperplane disjoint from \( W \), in which case \( \mathcal{R}(W) = \res_W(F, \nu) = 0 \).

**Step five:** Still assuming \( \mathcal{R}(W) > 0 \), we complete the proof of statement (ii) by proving (1.14). By (4.17), if \( (F, \nu) \in \text{Max} [\mathcal{R}(W)] \), then \( F/R \to H^- = \{ x \in \mathbb{R}^{n+1} : x \cdot \nu < 0 \} \) in \( L^1_{\text{loc}}(\mathbb{R}^{n+1}) \) as \( R \to \infty \). By (4.18) and by improved convergence (i.e., Remark 3.2 – notice carefully that \( \partial F \) is bounded in the direction \( \nu \) thanks to step four), we find \( R_F > 0 \) and functions \( \{ f_R \}_{R > R_F} \subset C^1(D_2^\nu \setminus D_1^\nu) \) such that
\[
(C_2^\nu \setminus C_1^\nu) \cap \partial(F/R) = \{ x : f_R(x) \nu : x \in D_2^\nu \setminus D_1^\nu \}, \quad \forall R > R_F.
\]
with \( \|f_R\|_{C^1(D_R^c \setminus D_R^1)} \to 0 \) as \( R \to \infty \). Scaling back to \( F \) we deduce that

\[
(\partial F) \setminus C_{RF}^\nu = \{ x + f(x) \nu : x \in \nu^\perp \setminus D_{RF}^\nu \},
\]  

(4.19)

for a (necessarily smooth) solution \( f \) to the minimal surfaces equation with

\[
\|f\|_{C^0(\nu^\perp \setminus D_{RF}^\nu)} \leq B - A, \quad \lim_{R \to \infty} \|\nabla f\|_{C^0(D_{2R}^c \setminus D_R^\nu)} = 0,
\]  

(4.20)

thanks to the fact that \( f(x) = R f_R(x/R) \) if \( x \in D_{2R}^c \setminus D_R^\nu \). When \( n \geq 3 \), (1.14) follows by (4.19) and Proposition 4.1. When \( n = 2 \), (4.2) holds by (4.18). To check (4.3), we deduce by \( \text{res}_W(F, \nu) \geq 0 \) the existence of \( R' > R_F \) such that \( \omega_n \ R'' \overset{\text{max}}{=} P(F; C_{RF}^\nu \setminus W) - 1 \) if \( R > R' \). In particular, setting \( M = (\partial F) \setminus B_{RF} \), for \( R > R' \) we have

\[
\mathcal{H}^2(M \cap B_R) \leq \mathcal{H}^2(M \cap W) + P(F; C_{RF}^\nu \setminus W) \leq \omega_n \ R'' + 1 + \mathcal{H}^2(M \cap W) \leq C \ R^n,
\]

provided \( C = \omega_n + [(1 + \mathcal{H}^2(M \cap W))/(R')^n] \); while if \( R \in (R_F, R') \), then \( \mathcal{H}^2(M \cap B_R) \leq C \ R^n \) with \( C = \mathcal{H}^2(M \cap B_R)/R_F^n \). This said, we can apply Proposition 4.1 to deduce (4.5). Since \( \partial F \) is bounded in a slab, the logarithmic term in (4.5) must vanish (i.e. (4.5) holds with \( b = 0 \)), and thus (1.14) is proved. Finally, when \( n = 1 \), by (4.19) and (4.20) there are \( a_1, a_2 \in \mathbb{R}, x_1 < x_2, x_1, x_2 \in \nu^\perp \equiv \mathbb{R} \) such that \( f(x) = a_1 \) for \( x \in \nu^\perp \), \( x < x_1 \), and \( f(x) = a_2 \) for \( x \in \nu^\perp \), \( x > x_2 \). Now, setting \( M_1 = \{ x + a_1 \nu : x \in \nu^\perp, x < x_1 \} \) and \( M_2 = \{ x + a_2 \nu : x \in \nu^\perp, x > x_2 \} \), we have that

\[
P(F; C_{RF}^\nu \setminus W) = \mathcal{H}^n(C_{RF}^\nu \cap (\partial F) \setminus (W \cup M_1 \cup M_2)) + 2 R - |x_2 - x_1|;
\]

while, if \( L \) denotes the line through \( x_1 + a_1 \nu \) and \( x_2 + a_2 \nu \), then we can find \( \nu_L \in S^1 \) and a set \( F_L \) such that \( (F_L, \nu_L) \in \mathcal{F} \) with \( \partial F_L = \left( ((\partial F) \setminus (M_1 \cup M_2)) \cup (L_1 \cup L_2) \right) \), where \( L_1 \) and \( L_2 \) are the two half-lines obtained by removing from \( L \) the segment joining \( x_1 + a_1 \nu \) and \( x_2 + a_2 \nu \). In this way, \( P(F_L; C_{RF}^\nu \setminus W) = \mathcal{H}^n(C_{RF}^\nu \cap (\partial F) \setminus (W \cup M_1 \cup M_2)) + 2 R - |(x_1 + a_1 \nu) - (x_2 + a_2 \nu)| \), so that \( \text{res}_W(F_L, \nu_L) - \text{res}_W(F, \nu) = |(x_1 + a_1 \nu) - (x_2 + a_2 \nu)| - |x_2 - x_1| > 0 \) against \( (F, \nu) \in \text{Max}[\mathcal{R}(W)] \) if \( a_1 \neq a_2 \). Hence, \( a_1 = a_2 \).

We are left to prove that (4.19) holds with \( R_2 = R_2(W) \) in place of \( R_F \), and the constants \( a, b, c \) and \( C_0 \) appearing in (1.14) can be bounded in terms of \( W \) only. To this end, we notice that the argument presented in step one shows that \( \text{Max}[\mathcal{R}(W)] \) is pre-compact in \( L^1_{\text{loc}}(\mathbb{R}^{n+1}) \). Using this fact and a contradiction argument based on improved convergence (Remark 3.2), we conclude the proof of statement (ii).

Step six: We complete the proof of statement (i) and begin the proof of statement (iii) by showing that, setting for brevity \( d = \text{diam}(W) \), it holds

\[
\mathcal{H}^n(W \cap \Pi) \leq \mathcal{R}(W) \leq \sup_{\nu \in S^n} \mathcal{H}^n(p_{\nu} \perp (W)) \leq \omega_n (d/2)^n,
\]  

(4.21)

whenever \( \Pi \) is a hyperplane in \( \mathbb{R}^{n+1} \). We have already proved the first inequality in step one. To prove the others, we notice that, if \( (F, \nu) \in \mathcal{F}, \)
We first prove that the two conditions are sufficient. Let \( \nu \) be the two open half-spaces bounded by \( \Pi \) and let \( \Pi \) be the two conditions \( \Pi \cap \Omega \) and let \( \Pi \) gives (4.26) whenever \( x \in (\partial F) \setminus W \) and \( \rho < \text{dist}(x, W) \), so that \( P(F; c_{d/2}(x)) \setminus W = 0 \). Hence, using also (4.26), we find (\( \partial F \) \( \setminus W \) is a minimal surface and \( W \subset C'_{d/2}(x) \), by analytic continuation we find that

\[
\sup_{R > R'} P(F; C'_{d/2}(x)) - H^n(\partial F \setminus W) = 0, \tag{4.23}
\]

By (4.24) and the discussion of the equality cases for the isodiametric inequality (see, e.g. [MPP14]), we see that, for some \( x_0 \in \nu \),

\[
p_{\nu}(W) = \text{cl} (D_{d/2}(x_0)), \quad \text{so that } W \subset C'_{d/2}(x_0). \tag{4.25}
\]

Condition (4.23) implies that (1.14) holds with \( u \equiv a \) for some \( a \in [A, B] = \bigcap \{(\alpha, \beta) : W \subset S_{(\alpha, \beta)} \} \); in particular, since \( \partial F \setminus W \) is a minimal surface and \( W \subset C'_{d/2}(x_0) \), by analytic continuation we find that

\[
(\partial F) \setminus C'_{d/2}(x_0) = \Pi \setminus C'_{d/2}(x_0), \quad \Pi = \{ x : x \cdot \nu = a \}. \tag{4.26}
\]

By (4.26), we have that for \( R > R' \),

\[
P(F; C'_{d/2}(x_0) \setminus W) = \omega_n R^n = P(F; C'_{d/2}(x_0) \setminus W) - \omega_n (d/2)^n.
\]

Going back to (4.22), this implies \( P(F; C'_{d/2}(x_0) \setminus W) = 0. \) However, since \( \partial F \setminus W \) is (distributionally) a minimal surface, \( P(F; B_\rho(x) \setminus W) \geq \omega_n \rho^n \) whenever \( x \in (\partial F) \setminus W \) and \( \rho < \text{dist}(x, W) \), so that \( P(F; C'_{d/2}(x_0) \setminus W) = 0 \) gives (\( (\partial F) \setminus W) \cap C'_{d/2}(x_0) = \emptyset \). Hence, using also (4.26), we find (\( \partial F \) \( \setminus W = \Pi \setminus \text{cl} (B_{d/2}(x)) \)) for some \( x \in \Pi \), that is (1.16).

**Step eight:** We finally prove that \( \mathcal{R}(W) = \omega_n (d/2)^n \) if and only if there are a hyperplane \( \Pi \) and a point \( x \in \Pi \) such that

\[
\Pi \cap \partial B_{d/2}(x) \subset W, \tag{4.27}
\]

\[
\Omega \setminus (\Pi \setminus B_{d/2}(x)) \text{ has two unbounded connected components}. \tag{4.28}
\]

We first prove that the two conditions are sufficient. Let \( \nu \) be a unit normal to \( \Pi \) and let \( \Pi^+ \) and \( \Pi^- \) be the two open half-spaces bounded by \( \Pi \). The condition \( \Pi \cup \partial B_{d/2}(x) \subset W \) implies \( W \subset C'_{d/2}(x) \), and thus

\[
\Omega \setminus \text{cl} [C'_{d/2,(-d,d)}(x)] = (\Pi^+ \cup \Pi^-) \setminus \text{cl} [C'_{d/2,(-d,d)}(x)].
\]
In particular, $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ has a connected component $F$ which contains

$$\Pi^+ \setminus \text{cl} \left[ C_{d/2,(-d,d)}^\nu(x) \right];$$

and since $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ contains exactly two unbounded connected components, it cannot be that $F$ contains also $\Pi^- \setminus \text{cl} [C_{d/2,(-d,d)}^\nu(x)]$, therefore

$$\Pi^+ \setminus \text{cl} \left[ C_{d/2,(-d,d)}^\nu(x) \right] \subset F, \quad \Pi^- \setminus \text{cl} \left[ C_{d/2,(-d,d)}^\nu(x) \right] \subset \mathbb{R}^{n+1} \setminus \text{cl} (F).$$

As a consequence $\partial F$ is contained in the slab $\{y : |(y - x) \cdot \nu| < d\}$, and is such that $p_{\nu}^\perp(\partial F) = \nu^\perp$, that is, $(F, \nu) \in \mathcal{F}$. Moreover, (4.29) implies

$$\Omega \setminus \text{cl} (B_{d/2}(x)) \subset \Omega \cap \partial F,$$

while the fact that $F$ is a connected component of $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ implies $\Omega \cap \partial F \subset \Pi \setminus \text{cl} (B_{d/2}(x))$. In conclusion, $\Omega \cap \partial F = \Pi \setminus \text{cl} (B_{d/2}(x))$, hence

$$\omega_n (d/2)^n = \lim_{r \to \infty} \omega_n r^n - P(F; C_r^\nu \setminus W) \leq \mathcal{R}(W) \leq \omega_n (d/2)^n, \quad (4.30)$$

and $\mathcal{R}(W) = \omega_n (d/2)^n$, as claimed. We prove that the two conditions are necessary. Let $(F, \nu) \in \text{Max}[\mathcal{R}(W)]$. As proved in step seven, there is a hyperplane $\Pi$ and $x \in \Pi$ such that $\Omega \cap \partial F = \Pi \setminus \text{cl} (B_{d/2}(x))$. If $z \in \Pi \cap \partial B_{d/2}(x)$ but $z \in \Omega$, then there is $\rho > 0$ such that $B_\rho(z) \subset \Omega$, and since $\partial F$ is a minimal surface in $\Omega$, we would obtain that $\Pi \cap B_\rho(z) \subset \Omega \cap \partial F$, against $\Omega \cap \partial F = \Pi \setminus \text{cl} (B_{d/2}(x))$. So it must be $\Pi \cap \partial B_{d/2}(x) \subset W$, and the necessity of (4.27) is proved. To prove the necessity of (4.28), we notice that since $\Pi^+ \setminus \text{cl} [C_{d/2,(-d,d)}^\nu(x)]$ and $\Pi^- \setminus \text{cl} [C_{d/2,(-d,d)}^\nu(x)]$ are both open, connected, and unbounded subsets of $\Omega \setminus (\Pi \setminus B_{d/2}(x))$, and since the complement in $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ of their union is bounded, it must be that $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ has at most two unbounded connected components: therefore we just need to exclude that it has only one. Assuming by contradiction that this is the case, we could then connect any point $x^+ \in \Pi^+ \setminus \text{cl} [C_{d/2,(-d,d)}^\nu(x)]$ to any point $x^- \in \Pi^- \setminus \text{cl} [C_{d/2,(-d,d)}^\nu(x)]$ with a continuous path $\gamma$ entirely contained in $\Omega \setminus (\Pi \setminus B_{d/2}(x))$. Now, recalling that $\Omega \cap \partial F = \Pi \setminus \text{cl} (B_{d/2}(x))$, we can pick $x_0 \in \Pi \setminus \text{cl} (B_{d/2}(x))$ and $r > 0$ so that

$$B_r(x_0) \cap \Pi^+ \subset F, \quad B_r(x_0) \cap \Pi^- \subset \mathbb{R}^{n+1} \setminus \text{cl} (F), \quad (4.31)$$

and $B_r(x_0) \cap \text{cl} [C_{d/2,(-d,d)}^\nu(x)] = \emptyset$. We can then pick $x^+ \in B_r(x_0) \cap \Pi^+$, $x^- \in B_r(x_0) \cap \Pi^-$, and then connect them by a path $\gamma$ entirely contained in $\Omega \setminus (\Pi \setminus B_{d/2}(x))$. By (4.31), $\gamma$ must intersect $\partial F$, and since $\gamma$ is contained in $\Omega$, we see that $\gamma$ must intersect $\Omega \cap \partial F = \Pi \setminus \text{cl} (B_{d/2}(x))$, which of course contradicts the containment of $\gamma$ in $\Omega \setminus (\Pi \setminus B_{d/2}(x))$. We have thus proved that $\Omega \setminus (\Pi \setminus B_{d/2}(x))$ has exactly two unbounded connected components. \hfill \Box

5. Resolution theorem for exterior isoperimetric sets

The notation set in (4.1) is in use. Given $v_j \to \infty$, we set $\lambda_j = v_j^{1/(n+1)}$. 48
Proof of Theorem 1.6. We recall that, throughout the proof, $\mathcal{R}(W) > 0$. Theorem 1.6-(i) and the estimate for $|v^{-1/(n+1)}| x - \omega_{n+1}^{-1/(n+1)}$ in Theorem 1.6-(iv), have already been proved in Theorem 3.1-(ii, iii).

Step one: We prove that

$$
\lim_{v \to \infty} \psi_W(v) - P(B(v)) \leq -\mathcal{R}(W).
$$

(5.1)

To this end, let $(F, \nu) \in \text{Max}[\mathcal{R}(W)]$, so that by (1.13) and (1.14), we have

$$
F \setminus C^r_{K,2} = \{x + t \nu : x \in \nu^\perp, |x| > R_2, t < f(x)\},
$$

(5.2)

for a function $f \in C^1(\nu^\perp)$ satisfying

$$
|f(x) - (a + b |x|^{2-n} + (c \cdot x) |x|^{-n})| \leq C_0 |x|^{-n},
$$

(5.3)

$$
\max \{|x|^{n-1} |\nabla f(x)|, |x|^n |\nabla^2 f(x)|\} \leq C_0, \quad \forall x \in \nu^\perp, |x| > R_2,
$$

and for some $a, b \in \mathbb{R}$ and $c \in \nu^\perp$ such that $\max\{|a|, |b|, |c|\} \leq C(W) < \infty$ (moreover, we can take $b = 0$, $c = 0$ and $C_0 = 0$ if $n = 1$). We are going to construct competitors for $\psi_W(v)$ with $v$ large by gluing a large sphere $S$ to $\partial F$ along $\partial C_{r,v}^\nu$ for $r > R_2$. This operation comes at the price of an area error located on the cylinder $\partial C_{r,v}^\nu$. We can make this error negligible thanks to the fact that (5.3) determines the distance (inside of $\partial C_{r,v}^\nu$) of $\partial F$ from a hyperplane (namely, $\partial G_r$ for the half-space $G_r$ defined below) up to $\alpha(r^{1-n})$ as $r \to \infty$. Thus, the asymptotic expansion (1.14) is just as precise as needed in order to perform this construction, i.e. our construction would not be possible with a less precise information.

We now discuss the construction in detail. Given $r > R_2$, we consider the half-space $G_r \subset \mathbb{R}^{n+1}$ defined by the condition that

$$
G_r \cap \partial C_{r,v}^\nu = \{x + t \nu : x \in \nu^\perp, |x| = r, t < a + b r^{2-n} + (c \cdot x) r^{-n}\},
$$

(5.4)

so that $G_r$ is the “best half-space approximation” of $F$ on $\partial C_{r,v}^\nu$ according to (5.3). Denoting by $\text{hd}(X,Y)$ the Hausdorff distance between $X,Y \subset \mathbb{R}^{n+1}$, for every $r > R_2$ and $v > 0$ we can define $x_{r,v} \in \mathbb{R}^{n+1}$ in such a way that $v \mapsto x_{r,v}$ is continuous and

$$
\lim_{v \to \infty} \text{hd}(B(v)_r(x_{r,v}) \cap K, G_r \cap K) = 0 \quad \forall K \subset \subset \mathbb{R}^{n+1}.
$$

(5.5)

Thus, the balls $B(v)_r(x_{r,v})$ have volume $v$ and are locally converging in Hausdorff distance, as $v \to \infty$, to the optimal half-space $G_r$. Finally, we notice that by (5.3) we can find $\alpha < \beta$ such that

$$
((\partial F) \cup (\partial G_r) \cup (G_r \Delta F)) \cap C_{r,v}^\nu \subset C_{r,1,\beta-1}^\nu,
$$

(5.6)

and then define $F_{r,v}$ by setting

$$
F_{r,v} = (F \cap C_{r,1,\beta-1}^\nu) \cup (B(v)_r(x_{r,v}) \setminus \text{cl}[C_{r,1,\beta-1}^\nu]),
$$

(5.7)

see Figure 5.1. We claim that, by using $F_{r,v}$ as comparisons for $\psi_W(|F_{r,v}|)$,
Figure 5.1. The competitors $F_{r,v}$ constructed in (5.7). A maximizer $F$ in the isoperimetric residue $\mathcal{R}(W)$ is joined to a ball of volume $v$, whose center $x_{r,v}$ is determined by looking at best hyperplane $\partial G_r$ approximating $\partial F$ on the “lateral” cylinder $\partial C_r$. To ensure the area error made in joining this large sphere to $\partial F$ is negligible, the distance between $\partial F$ and the sphere inside $\partial C_r$ must be $o(r^{-n})$ as $r \to \infty$. The asymptotic expansion (5.3) gives a hyperplane $\partial G_r$ which is close to $\partial F$ up to $O(r^{-n})$, and is thus just as precise as needed to perform the construction.

and then sending first $v \to \infty$ and then $r \to \infty$, one obtains (5.1). We first notice that by (5.5) and (5.6) (see, e.g. [Mag12, Theorem 16.16]), we have

\[
P(F_{r,v}; \Omega) = P(F; C^\nu_{r,(\alpha,\beta)} \setminus W) + P(B(v)(x_{r,v}); \mathbb{R}^{n+1} \setminus \text{cl} [C^\nu_{r,(\alpha,\beta)}]) + \mathcal{H}^n((F \Delta B(v)(x_{r,v})) \cap \partial_r C^\nu_{r,(\alpha,\beta)}),
\]

where the last term is the “gluing error” generated by the mismatch between the boundaries of $\partial F$ and $\partial B(v)(x_{r,v})$ along $\partial_r C^\nu_{r,(\alpha,\beta)}$. Now, thanks to (5.3) we have

\[
\text{hd (} G_r \cap \partial C^\nu_r, F \cap \partial C^\nu_r \text{)} \leq C_0 r^{-n},
\]

so that

\[
\mathcal{H}^n((F \Delta G_r) \cap \partial C^\nu_r) \leq n \omega_n r^{-n-1} \text{hd (} G_r \cap \partial C^\nu_r, F \cap \partial C^\nu_r \text{)} \leq C(n,W)/r.
\]

At the same time, by (5.5),

\[
\lim_{v \to \infty} \mathcal{H}^n((G_r \Delta B(v)(x_{r,v})) \cap \partial_r C^\nu_{r,(\alpha,\beta)}) = 0,
\]

and thus we have the following estimate for the gluing error,

\[
\lim_{v \to \infty} \mathcal{H}^n((F \Delta B(v)(x_{r,v})) \cap \partial_r C^\nu_{r,(\alpha,\beta)}) \leq \frac{C(n,W)}{r}, \quad \forall r > R_2.
\]

Again by (5.5), we find

\[
\lim_{v \to \infty} P(B(v)(x_{r,v}); C^\nu_{r,(\alpha,\beta)}) = P(G_r; C^\nu_{r,(\alpha,\beta)})
\]

1 \leq (\omega_n r^n)^{-1} P(G_r; C^\nu_{r,(\alpha,\beta)}) = \int_{D_r^v} \sqrt{1 + (c/r^n)^2} \leq 1 + C_0 r^{-2n},
\]

so that, by (5.12) and by the lower bound in (5.13), for every $r > R_2$,

\[
\lim_{v \to \infty} P(B(v)(x_{r,v}); \mathbb{R}^{n+1} \setminus \text{cl} [C^\nu_{r,(\alpha,\beta)}]) - P(B(v)) \leq -\omega_n r^n.
\]
Combining (5.11) and (5.14) with (5.8) and the fact that \( C_{r,\alpha}^\nu \cap \partial F = C_{r}^\nu \cap \partial F \) (see (5.6)), we find that for every \( r > R_2 \),
\[
\lim_{v \to \infty} P(F, r, v; \Omega) - P(\Omega) \leq P(F; C_{r}^\nu \cap W) - \omega_r r^n + C(n, W)/r
\]
\[
\leq -\text{res}_W(F, \nu) + C(n, W)/r = -\mathcal{R}(W) + C(n, W)/r.
\] (5.15)
where (4.9) has been used. Now, combining the elementary estimates
\[
\max \{ \| F, r, v \| - v, v^{-1/(n+1)} \| P(B(\nu)) - P(B(\Omega)) \} \leq C(n) r^{n+1}
\] (5.16) with (5.15), we see that
\[
\lim_{v \to \infty} \psi_W(\| F, r, v \|) - P(B(\Omega)) \leq -\mathcal{R}(W) + C(n, W)/r, \ \forall r > R_2.
\] (5.17)

Again by (5.16) and since \( v \mapsto \| F, r, v \| \) is a continuous function, we see that
\[
\lim_{v \to \infty} \psi_W(\| F, r, v \|) - P(B(\Omega)) = \lim_{v \to \infty} \psi_W(v) - P(B(\nu)).
\]
This last identity combined with (5.17) implies (5.1) in the limit \( r \to \infty \).

**Step two:** Now let \( E_j \in \text{Min}\{\psi_W(v_j)\} \) for \( v_j \to \infty \). By (3.1) and a standard argument (see, e.g. [Mag12, Theorem 21.14]), there is a local perimeter minimizer with free boundary \( F \) in \( \Omega \) such that, up to extracting subsequences,
\[
E_j \to F \text{ in } L^1_{\text{loc}}(\mathbb{R}^{n+1}), \ \mathcal{H}^n \cap \partial E_j \to \mathcal{H}^n \cap \partial F \text{ as Radon measures in } \Omega,
\]
\[
\text{ind}(K \cap \partial E_j; K \cap \partial F) \to 0 \quad \text{for every } K \subset \subset \Omega.
\] (5.18)

Notice that it is not immediate to conclude from \( E_j \in \text{Min}\{\psi_W(v_j)\} \) that (for some \( \nu \in S^n \) \( F, \nu \in \text{Max}\{\mathcal{R}(W)\} \) (or even that \( \nu, F \in \mathcal{F} \)), nor that \( P(E_j; \Omega) - P(B(\nu)) \) is asymptotically bounded from below by \( -\text{res}_W(F, \nu) \).

In this step we prove some preliminary properties of \( F \), and in particular we exploit the blowdown result for exterior minimal surfaces contained in Theorem 2.1-(ii) to prove that \( F \) satisfies (5.2) and (5.3) (see statement (c) below). Then, in step three, we use the decay rates (5.3) to show that \( E_j \) can be “glued” to \( F \), similarly to the construction of step one, and then derive from the corresponding energy estimates the lower bound matching (5.1) and the optimality of \( F \) in \( \mathcal{R}(W) \).

(a) \( \Omega \cap \partial F \cap \partial B_\rho \neq \emptyset \) for every \( \rho \) such that \( W \subset \subset B_\rho \): If not there would be \( \varepsilon > 0 \) such that \( W \subset \subset B_{\rho-\varepsilon} \) and \( \Omega \cap \partial F \cap A_{\rho-\varepsilon} = \emptyset \) (recall that \( A_\rho = \{ x : s > |x| > r \} \)). By (5.18) and the constant mean curvature condition satisfied by \( \Omega \cap \partial E_j \), we would then find that each \( E_j \) (with \( j \) large enough) has a connected component of the form \( B(\nu_j)(x_j) \), with \( B(\nu_j)(x_j) \subset \subset \mathbb{R}^{n+1} \setminus B_{\rho+\varepsilon} \) and \( w_j \geq v_j - C(n) (\rho + \varepsilon)^{n+1} \). In particular, against \( \mathcal{R}(W) > 0 \),
\[
\psi_W(v_j) = P(E_j; \Omega) \geq P(B_{\nu_j} - C(\rho + \varepsilon)^{n+1}) \geq P(B(v_j)) - C \lambda_j^{-1} (\rho + \varepsilon)^{n+1}.
\]

(b) **Sharp area bound:** We combine the upper energy bound (5.1) with the perimeter inequality for spherical symmetrization, to prove
\[
P(F; \Omega \cap B_r) \leq \omega_r r^n - \mathcal{R}(W), \quad \text{for every } r \text{ s.t. } W \subset \subset B_r.
\] (5.19)
(Notice that (5.19) does not immediately imply the bound for \( P(F; \Omega \cap C_{r}^\nu) \) which would be needed to compare \( \mathcal{R}(W) \) and \( \text{res}_W(F, \nu) \).) To prove (5.19)
we argue by contradiction, and consider the existence of $\delta > 0$ and $r$ with $W \subset B_r$ such that $P(F; \Omega \cap B_r) \geq \omega_n r^n - \mathcal{R}(W) + \delta$. In particular, for $j$ large enough, we would then have
\[ P(E_j; \Omega \cap B_r) \geq \omega_n r^n - \mathcal{R}(W) + \delta. \] (5.20)
Again for $j$ large, it must be $\mathcal{H}^n(\partial E_j \cap \partial B_r) = 0$: indeed, by (3.1), $\Omega \cap \partial E_j$ has mean curvature of order $O(\lambda_j^{-1})$, while of course $\partial B_r$ has constant mean curvature equal to $n/r$. Thanks to $\mathcal{H}^n(\partial E_j \cap \partial B_r) = 0$,
\[ P(E_j; \Omega) = P(E_j; \Omega \cap B_r) + P(E_j; \mathbb{R}^{n+1} \setminus \text{cl}(B_r)). \] (5.21)
If $E_j^n$ denotes the spherical symmetral of $E_j$ such that $E_j^n \cap \partial B_r$ is a spherical cap in $\partial B_r$, centered at $\rho e_{n+1}$, with area equal to $\mathcal{H}^n(E_j \cap \partial B_r)$, then we have the perimeter inequality
\[ P(E_j; \mathbb{R}^{n+1} \setminus \text{cl}(B_r)) \geq P(E_j^n; \mathbb{R}^{n+1} \setminus \text{cl}(B_r)); \] (5.22)
see [CPS20]. Now, we can find a half-space $J$ orthogonal to $e_{n+1}$ and such that $\mathcal{H}^n(J \cap \partial B_r) = \mathcal{H}^n(E_j \cap \partial B_r)$. In this way, using that $|E_j^n \setminus B_r| = |E_j \setminus B_r|$ (by Fubini’s theorem in spherical coordinates), and that $\mathcal{H}^n(B_r \cap \partial J) \leq \omega_n r^n$ (by the fact that $\partial J$ is a hyperplane), we find
\[ P(E_j^n; \mathbb{R}^{n+1} \setminus \text{cl}(B_r)) = P((E_j^n \setminus \text{cl}(B_r)) \cup (J \cap B_r)) - \mathcal{H}^n(B_r \cap \partial J) \geq P(B^{(\nu)}) - C(n) r^{n+1} \lambda_j^{-1} - \omega_n r^n \]
which, with (5.20), (5.21) and (5.22), finally gives $P(E_j; \Omega) - P(B^{(\nu)}) > -\mathcal{R}(W) + \delta - C(n) r^{n+1} \lambda_j^{-1}$ for $j$ large, against (5.1).

(c) Asymptotic behavior of $\partial F$: We prove that there are $\nu \in S^n$, $f \in C^\infty(\nu^\perp)$, $a, b \in \mathbb{R}$, $c \in \nu^\perp$, $R' > \sup\{\rho : W \subset C^\nu_\rho\}$ and $C$ positive, with
\[ \partial F \setminus C_\rho^\nu = \{x + f(x) \nu : x \in \nu^\perp, |x| > R'\}, \] (5.23)
\[ f(x) = a, \quad (n = 1) \]
\[ |f(x) - (a + b |x|^{2-n} + (c \cdot x)|x|^{-n})| \leq C |x|^{-n}, \quad (n \geq 2), \] (5.24)
\[ \max \{|x|^{n-1} |\nabla f(x)|, |x|^n |\nabla^2 f(x)|\} \leq C_0, \quad \forall x \in \nu^\perp, |x| > R'. \]
To this end, by a standard argument exploiting the local perimeter minimality of $F$ in $\Omega$, given $r_j \to \infty$, then, up to extracting subsequences, $F/r_j \overset{\text{loc}}{\to} J$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$, where $J$ is a perimeter minimizer in $\mathbb{R}^{n+1} \setminus \{0\}$, $0 \in \partial J$ (thanks to property (a)), $J$ is a cone with vertex at 0 (thanks to Theorem 2.7 and, in particular to (2.47)), and $P(J; B_1) \leq \omega_n$ (by (5.19)). If $n \geq 2$, then $\partial J$ has vanishing distributional mean curvature in $\mathbb{R}^{n+1}$ (as points are removable singularities for the mean curvature operator when $n \geq 2$), thus $P(J; B_1) \geq \omega_n$ by upper semicontinuity of area densities, and, finally, by $P(J; B_1) = \omega_n$ and Allard’s regularity theorem, $J$ is a half-space. If $n = 1$, then $\partial J$ is the union of two half-lines $\ell_1$ and $\ell_2$ meeting at $\{0\}$. If $\ell_1$ and $\ell_2$ are not opposite (i.e., if $J$ is not a half-space), then we can find
Moreover, we claim that setting \( (J \cap J^\star) \Delta J \subset B \subset \mathbb{R}^2 \setminus \{0\} \) for some ball \( B \), and \( P(J \cap J^\star; B) < P(J; B) \), thus violating the fact that \( J \) is a perimeter minimizer in \( \mathbb{R}^{n+1} \setminus \{0\} \).

If \( n = 1 \) it is immediate from the above information that, for some \( R' > 0 \), \( F \setminus B_{R'} = J \setminus B_{R'} \); this proves (5.23) and (5.24) in the case \( n = 1 \). To prove (5.23) and (5.24) when \( n \geq 2 \), we let \( M_0 \) and \( \varepsilon_0 \) be as in Theorem 2.1-(ii) with \( (n, \Gamma, \varepsilon, \sigma) = 1 \). Since \( J \) is a half-space, by using Remark 3.2 and \( F \setminus r J \) on the annulus \( A_{1/2}^L \), for some \( L > \max\{M_0, 64\} \) to be chosen later on depending also on \( \varepsilon_0 \), we find that

\[
(\partial F) \cap A_{r_j/2}^{4Lr_j} = \{ x + r_j f_j(x/r_j) : x \in \nu^\perp \cap A_{r_j/2}^{4Lr_j}, \quad \nu^\perp = \partial J \}, \quad (5.25)
\]

for \( f_j \in C^1(\nu^\perp) \) with \( \|f_j\|_{C^1(\nu^\perp)} \to 0 \). By (5.25), \( V_j = \text{var}(\partial F \setminus B_{r_j}, 1) \in V_n(0, r_j, \infty) \), with \( (o(1) \to 0 \) as \( j \to \infty \)

\[
r_j^{-n} \int x \cdot \nu_{V_j}^\perp \, d \text{bd}V_j = -n \omega_n + o(1) \quad (5.26)
\]

\[
\sup_{r \in (r_j, 3Lr_j)} \|V_j\|((B_r \setminus B_{r_j}) - \omega_n) = o(1). \quad (5.28)
\]

By our choice of \( \Gamma \), by (5.19) and (5.26) we see that, for \( j \) large, we have

\[
\|\text{bd}V_j\|((\partial B_{r_j}) \leq \Gamma r_j^{-n}, \quad \|V_j\|(B_r \setminus B_{r_j}) \leq \Gamma r^n, \quad \forall r > r_j. \quad (5.29)
\]

Moreover, we claim that setting

\[
s_j = 2L r_j
\]

so that, in particular, \( s_j > \max\{M_0, 64\} r_j \), then

\[
|\delta_{V_j, r_j, 0}(s_j/8)| \leq \varepsilon_0, \quad \inf_{r > s_j/8} \delta_{V_j, r_j, 0}(r) \geq -\varepsilon_0, \quad (5.30)
\]

provided \( j \) and \( L \) are taken large enough depending on \( \varepsilon_0 \). To check the first inequality in (5.30) we notice that, by (5.26) and (5.28),

\[
\delta_{V_j, r_j, 0}(s_j/8) = \omega_n - \frac{\|V_j\|(B_{s_j/8} \setminus B_{r_j})}{(s_j/8)^n} + \frac{1}{n (s_j/8)^n} \int x \cdot \nu_{V_j}^\perp \, d \text{bd}V_j
\]

\[
= \omega_n - \frac{\omega_n + o(1)}{(s_j/8)^n} \frac{r_j^n}{(s_j/8)^n} - \frac{\omega_n r_j^n}{(s_j/8)^n} \left( 1 + o(1) \right)
\]

\[
= o(1) \left( 1 + (r_j/s_j)^n \right) = o(1),
\]

so that \( |\delta_{V_j, r_j, 0}(s_j/8)| \leq \varepsilon_0 \) as soon as \( j \) is large with respect to \( \varepsilon_0 \). Similarly, if \( r > s_j/8 = (Lr_j)/4 \), then by (5.26), (5.28), (5.19), and \( r_j/r \leq 4/L \),

\[
\delta_{V_j, r_j, 0}(r) = \omega_n - \frac{\|V_j\|(B_r \setminus B_{2r_j})}{r^n} - \frac{\|V_j\|(B_{2r_j} \setminus B_{r_j})}{r^n} - \frac{\omega_n r_j^n}{r^n} \left( 1 + o(1) \right)
\]

\[
\geq \omega_n - \frac{\omega_n r^n - \mathcal{R}(W)}{r^n} - \frac{\omega_n + o(1)}{r^n} \frac{2 r_j^n - r_j^n}{r^n} - \frac{\omega_n r_j^n}{r^n} \left( 1 + o(1) \right)
\]

\[
\geq r^{-n} \mathcal{R}(W) - 2(4/L)^n \omega_n + o(1) - (4/L)^n o(1) \geq -3(4/L)^n \omega_n,
\]

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provided \( j \) is large; hence the second inequality in (5.30) holds if \( L \) is large in terms of \( c_0 \). By (5.29) and (5.30), Theorem 2.1-(ii) can be applied to \( (V, R, \Lambda, s) = (V_j, r_j, 0, s_j) \) with \( j \) large. As a consequence, passing from spherical graphs to cylindrical graphs with the aid of Lemma B.1, we find that, for some large \( j \),

\[
(\partial F) \setminus B_{s_j/16} = \{ x + f(x) \nu : x \in \nu \} \setminus B_{s_j/16}, \tag{5.31}
\]

where \( f : \nu \to \mathbb{R} \) is a smooth function which solves the minimal surfaces equation on \( \nu \setminus B_{s_j/16} \). Since \( \partial F \) admits at least one sequential blowdown limit hyperplane (namely, \( \nu = \partial J \)), by a theorem of Simon [Sim87, Theorem 2] we find that \( \nabla f \) has a limit as \( |x| \to \infty \); in particular, \( |\nabla f| \) is bounded. Moreover, by (5.31) (or by the fact that \( F \) is a local perimeter minimizer in \( \Omega \)), \( \partial F \) is a stable minimal surface in \( \mathbb{R}^{n+1} \setminus B_{s_j/16} \), which, thanks to (5.19), satisfies an area growth bound like (4.3). We can thus apply Proposition 4.1 to deduce the validity of (5.24) when \( n \geq 3 \), and of

\[
\sum |f(x) - [a + b \log|\nu| + c] + |x|^2| \leq C|x|^{-2} \quad \text{for all } |x| > R
\]

when \( n = 2 \) (with \( R > s_j \)). Recalling that \( F \) is a local perimeter minimizer with free boundary in \( \Omega \) (that is, \( P(F; \Omega \cap B) \leq P(F'; \Omega \cap B) \) whenever \( F \Delta F' \subset B \subset \subset \mathbb{R}^3 \)) it must be that \( b = 0 \), as it can be seen by comparing \( F \) with the set \( F' \) obtained by changing \( F \) inside \( C_r \) with the half-space \( G_r \) bounded by the plane \( \{ x + t \nu : x \in \nu, t = a + b \log(r) + c \cdot x/r^2 \} \) and such that \( \mathcal{H}^2((F \Delta G_r) \cap \partial C_r) \leq C/r^2 \) (we omit the details of this standard comparison argument). Having shown that \( b = 0 \), the proof is complete.

**Step three:** We prove that

\[
\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) \geq -R(W) \tag{5.35}
\]
For \( v_j \to \infty \) achieving the liminf in (5.35), let \( E_j \in \text{Min}[\psi_W(v_j)] \) and let \( F \) be a (sub-sequential) limit of \( E_j \), so that properties (a), (b), (c) and (d) in step two hold for \( F \). In particular, properties (5.23) and (5.24) from (c) are entirely analogous to properties (5.2) and (5.3) exploited in step one: therefore, the family of half-spaces \( \{G_r\}_{r > R'} \) defined by (5.4) is such that

\[
\left((\partial F) \cup (\partial G_r) \cup (G_r, \Delta F)\right) \cap C^u_{r,(\alpha+1,\beta-1)} , \tag{5.36}
\]

\[
\mathcal{H}^n((F \Delta G_r) \cap \partial C^u_r) \leq r^{-1} C(n, W) , \tag{5.37}
\]

\[
\left| P(G_r; C^u_{r,(\alpha,\beta)}) - \omega_n r^n \right| \leq r^{-n} C(n, W) , \tag{5.38}
\]

(compare with (5.6), (5.9), and (5.13) in step one). By (5.38) we find

\[
-\text{res}_W(F', \nu) = \lim_{r \to \infty} P(F; C^u_r \setminus W) - P(G_r; C^u_{r,(\alpha,\beta)}). \tag{5.39}
\]

In order to relate the residue of \((F', \nu)\) to \( \psi_W(v_j) - P(B^{(v_j)}) \) we consider the sets \( Z_j = (G_r \cap C^u_{r,(\alpha,\beta)}) \cup (E_j \setminus C^u_{r,(\alpha,\beta)}) \), which, by isoperimetry, satisfy

\[
P(Z_j) \geq P(B^{\{E_j\} \cup C^u_{r,(\alpha,\beta)}}) \geq P(B^{(v_j)}) - C(n) r^n (\beta - \alpha) \lambda_j^{-1} . \tag{5.40}
\]

Since for a.e. \( r > R' \) we have

\[
P(Z_j) = P(E_j; \mathbb{R}^{n+1} \setminus C^u_{r,(\alpha,\beta)}) + P(G_r; C^u_{r,(\alpha,\beta)}) + \mathcal{H}^n((E_j \Delta G_r) \cap \partial C^u_{r,(\alpha,\beta)}) ,
\]

we conclude that

\[
\psi_W(v_j) - P(B^{(v_j)}) = P(E_j; C^u_{r,(\alpha,\beta)} \setminus W) + P(E_j; \mathbb{R}^{n+1} \setminus C^u_{r,(\alpha,\beta)}) - P(B^{(v_j)})
\]

\[
= P(E_j; C^u_{r,(\alpha,\beta)} \setminus W) + P(Z_j) - P(B^{(v_j)})
\]

\[
- P(G_r; C^u_{r,(\alpha,\beta)}) - \mathcal{H}^n((E_j \Delta G_r) \cap \partial C^u_{r,(\alpha,\beta)})
\]

so that \( E_j \to F \) in \( L^1_{\text{loc}}(\mathbb{R}^{n+1}) \) and (5.40) give, for a.e. \( r > R' \),

\[
\lim_{j \to \infty} \psi_W(v_j) - P(B^{(v_j)}) \geq P(F; C^u_{r,(\alpha,\beta)} \setminus W) - P(G_r; C^u_{r,(\alpha,\beta)})
\]

\[
- \mathcal{H}^n((F \Delta G_r) \cap \partial C^u_{r,(\alpha,\beta)}) \geq P(F; C^u_r \setminus W) - P(G_r; C^u_r) - C(n, W)/r ,
\]

thanks to (5.37) and \( (F \Delta G_r) \cap \partial C^u_r = (F \Delta G_r) \cap \partial C^u_{r,(\alpha,\beta)} \). Letting \( r \to \infty \), recalling (5.39), and by \( (F', \nu) \in \mathcal{F} \), we find

\[
\lim_{j \to \infty} \psi_W(v_j) - P(B^{(v_j)}) \geq - \text{res}_W(F', \nu) \geq -\mathcal{R}(W) .
\]

This completes the proof of (5.35), which in turn, combined with (5.1), gives (1.19), and also shows that \( L^1_{\text{loc}} \)-subsequential limits \( F \) of \( E_j \in \text{Min}[\psi_W(v_j)] \) for \( v_j \to \infty \) are such that, for some \( \nu \in \mathbb{S}^n \), \( (F \cup W, \nu) \in \mathcal{F} \) and \( F' = F \cup W \in \text{Max}[\mathcal{R}(W)] \).

**Step four:** Moving towards the proof of (1.22), we prove the validity, uniformly among varifolds associated to maximizers of \( \mathcal{R}(W) \), of estimates analogous to (5.29) and (5.30). For a constant \( \Gamma > 2\omega_n \) to be determined later on (see (5.52), (5.53), and (5.54) below) in dependence of \( n \) and \( W \), and for \( \sigma > 0 \), we let \( M_0 = M_0(n, 2\Gamma, \sigma) \) and \( \varepsilon_0 = \varepsilon_0(n, 2\Gamma, \sigma) \) be determined
by Theorem 2.1. If \((F, \nu) \in \text{Max}[\mathcal{R}(W)]\), then by Theorem 1.1-(ii) we can find \(R_2 = R_2(W) > 0\), \(f \in C^\infty(\nu^\perp)\) such that
\[
(\partial F) \setminus C_{R_2}^\nu = \{ x + f(x) \nu : x \in \nu^\perp, |x| > R_2 \},
\]
and such that (1.14) holds with \(\max\{|a|, |b|, |c|\} \leq C(W)\) and \(|\nabla f(x)| \leq C_0/|x|^{n-1}\) for \(|x| > R_2\). Thus \(\|\nabla f\|_{C^0(\nu^\perp \setminus D_{\nu}^r)} \to 0\) as \(r \to \infty\) uniformly on \((F, \nu) \in \text{Max}[\mathcal{R}(W)]\), and there is \(R_3 > \max\{2R_2, 1\}\) (depending on \(W\)) such that, if \(V_F = \text{var} ((\partial F) \setminus B_{R_3}, 1)\), then \(V_F \in \mathcal{V}_n(0, R_3, \infty)\), and
\[
\|\partial B_{R_3}\| \leq \Gamma R_3^{n-1}, \quad \|V_F\|(B_\rho \setminus B_{R_3}) \leq \Gamma \rho^n \quad \forall \rho > R_3,
\]
(compare with (5.29)). Then, arguing as in step three-(c), or more simply by exploiting (5.41) and the decay estimates (1.14), we see that there is \(L > \max\{M_0, 64\}\), depending on \(n, W\) and \(\sigma\) only, such that, setting
\[
s_W(\sigma) = 2L R_3
\]
we have for some \(c(n) > 0\) (compare with (5.30))
\[
|\partial V_{F, R_3, 0}(s_W(\sigma)/8)| \leq \varepsilon_0/2, \quad \text{inf}_{r > s_W(\sigma)/8} \partial V_{F, R_3, 0}(r) \geq -\varepsilon_0/2.
\]

**Step five:** Given \(E_j \in \text{Min}[\psi_W(v_j)]\) for \(v_j \to \infty\), we prove the existence of \((F, \nu) \in \text{Max}[\mathcal{R}(W)]\) and \(h_j \in C^\infty((\partial F) \setminus B_{R_2})\) such that
\[
(\partial E_j) \cap A_{4 R_2}^{R_1 \lambda_j} = \{ y + h_j(y) \nu_F(y) : y \in \partial F \} \cap A_{4 R_2}^{R_1 \lambda_j},
\]
\[
\lim_{j \to \infty} \|h_j\|_{C^1((\partial F) \cap A_{4 R_2}^M)} = 0, \quad \forall M < \infty;
\]
and that if \(x_j\) satisfies \(|E_j \Delta B(v_j)(x_j)| = \inf_x |E_j \Delta B(v_j)(x)|\), then
\[
\lim_{j \to \infty} \|x_j\|^{-1} x_j - \nu = 0;
\]
finally, we prove statement (iii) (i.e., \((\partial E_j) \setminus B_{R_2}\) is diffeomorphic to an \(n\)-dimensional disk). By step three, there is \((F, \nu) \in \text{Max}[\mathcal{R}(W)]\) such that, up to extracting subsequences, (5.18) holds. By (5.18) and (5.41), and with \(s_W(\sigma)\) defined as in step four (see (5.43)) starting from \(F\), we can apply Remark 3.2 to find \(f_j \in C^\infty(\nu^\perp)\) such that
\[
(\partial E_j) \cap A_{2 R_2}^{s_W(\sigma)} = \{ x + f_j(x) \nu : x \in \nu^\perp \} \cap A_{2 R_2}^{s_W(\sigma)},
\]
for \(j\) large enough (in terms of \(\sigma, n, W,\) and \(F\)), and such that \(f_j \to f\) in \(C^1(D_{s_W(\sigma)}^\nu \setminus D_{2 R_2}^\nu)\). With \(R_3\) as in step four and with the goal of applying Theorem 2.1 to the varifolds \(V_j = \text{var} ((\partial E_j) \setminus B_{R_3}, 1)\), we notice that \(V_j \in \mathcal{V}_n(\Lambda_j, R_3, \infty)\), for some \(\Lambda_j \leq \Lambda_0 \lambda_j^{-1}\) (thanks to (3.1)). In particular, by (5.43), \(s_W(\sigma)\) satisfies the “mesoscale bounds” (compare with (2.4))
\[
\varepsilon_0 (4 \Lambda_j)^{-1} > s_W(\sigma) > \max\{M_0, 64\} R_3
\]
(5.49)
provided $j$ is large. Moreover, by $R_3 > 2 R_2$ and $s_W(\sigma)/8 > 2 R_2$, by (5.41), (5.48) and $f_j \to f$ in $C^1$, we exploit (5.42) and (5.44) to deduce

$$
\|b d V_j\| (\partial B_{R_3}) \leq (2 \Gamma) R_3^{-1}, \tag{5.50}
$$

$$
|\delta_{V_j, R_3, 0}(s_W(\sigma)/8)| \leq (2/3) \varepsilon_0. \tag{5.51}
$$

We claim that, up to increasing $\Gamma$ (depending on $n$ and $W$), we can entail

$$
\|V_j\|(B_\rho \setminus B_{R_3}) \leq \Gamma \rho^n, \quad \forall \rho > R_3. \tag{5.52}
$$

Indeed, by Theorem 3.1-(i), for some positive $\Lambda_0$ and $s_0$ depending on $W$ only, $E_j$ is a $(\Lambda_0 \lambda_j^{-1}, s_0 \lambda_j)$-perimeter minimizer with free boundary in $\Omega$. Comparing $E_j$ to $E_j \setminus B_r$ by (3.1), for every $r < s_0 \lambda_j$,

$$
P(E_j; \Omega \cap B_r) \leq C(n) \left( r^n + \Lambda_0 \lambda_j^{-1} r^{n+1} \right) \leq C(n, W) r^n; \tag{5.53}
$$

since, at the same time, if $r > s_0 \lambda_j$, then

$$
P(E_j; \Omega \cap B_r) \leq P(E_j; \Omega) = \psi_W(v_j) \leq P(B^{(v_j)}) \leq C(n) s_0^{-n} r^n, \tag{5.54}
$$

by combining (5.53) and (5.54) we find (5.52). With (5.50) and (5.52) at hand, we can also show that

$$
|\delta_{V_j, R_3, \lambda_j}(s_W(\sigma)/8)| \leq \varepsilon_0. \tag{5.55}
$$

Indeed, by $s_W(\sigma) = 2 L R_3$ and by $\Lambda_j \leq \Lambda_0 \lambda_j^{-1}$,

$$
|\delta_{V_j, R_3, \lambda_j}(s_W(\sigma)/8) - \delta_{V_j, R_3, 0}(s_W(\sigma)/8)|
\leq (\Lambda_0/\lambda_j) \int_{R_3}^{s_W(\sigma)/8} \rho^{-n} \|V_j\|(B_\rho \setminus B_{R_3}) d\rho
\leq \frac{\Lambda_0 R_3 \Gamma}{\lambda_j} \left( \frac{L}{4} - 1 \right) \leq \frac{\varepsilon_0}{3},
$$

provided $j$ is large enough. To complete checking that Theorem 2.1 can be applied to every $V_j$ with $j$ large enough, we now consider the quantities

$$
R_{*j} = \sup \{ \rho > s_W(\sigma)/8 : \delta_{V_j, R_3, \lambda_j}(\rho) \geq -\varepsilon_0 \},
$$

and prove that, for a constant $\tau_0$ depending on $n$ and $W$ only, we have

$$
R_{*j} \geq \tau_0 \lambda_j; \tag{5.56}
$$

in particular, provided $j$ is large enough, (5.56) implies immediately

$$
R_{*j} \geq 4 s_W(\sigma), \tag{5.57}
$$

which was the last assumption in Theorem 2.1 that needed to be checked. To prove (5.56), we pick $\tau_0$ such that

$$
|\tau_0 n H^n(B_{\tau_0}(z) \cap \partial B^{(1)}) - \omega_n| \leq \varepsilon_0/2, \quad \forall z \in \partial B^{(1)}. \tag{5.58}
$$

(Of course this condition only requires $\tau_0$ to depend on $n$.) By definition of $x_j$ and by (3.4), and up to extracting a subsequence, we have $x_j \to z_0$ for some $z_0 \in \partial B^{(1)}$. In particular, setting $\rho_j = \tau_0 \lambda_j$, we find

$$
\rho_j^{-n} \|V_j\|(B_{\rho_j} \setminus B_{R_3}) = \tau_0^{-n} P((E_j - x_j)/\lambda_j; B_{\tau_0}(-x_j) \setminus B_{R_3/\rho_j}(-x_j))
\to \tau_0^{-n} H^n(B_{\tau_0}(-z_0) \cap \partial B^{(1)}) \leq \omega_n + (\varepsilon_0/2),
$$

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thus proving that, for $j$ large enough,

$$\delta_{V_j, R_3, \Lambda_j}(\rho_j) \geq -\frac{\varepsilon_0}{2} + \frac{1}{n \rho_j^2} \int x \cdot \nu_{V_j}^\infty \, d\nu_{V_j} - \Lambda_j \int_{R_3} \frac{[V_j\| (B_0 \setminus B_{R_3})]}{\rho^n} \, d\rho$$

$$\geq -\frac{\varepsilon_0}{2} - \frac{2 \Gamma R_3}{n \tau_0 \lambda_j} - \Lambda_0 \frac{\rho_j - R_3}{\lambda_j} \geq -\frac{\varepsilon_0}{2} - \frac{C_s(n, W)}{\tau_0 \lambda_j} - C_{**}(n, W) \tau_0,$$

where we have used (5.50), spt $\nu_{V_j} \subset \partial B_{R_4}$, and (5.52). Therefore, provided we pick $\tau_0$ depending on $n$ and $W$ so that $C_{**} \tau_0 \leq \varepsilon_0/4$, and then we pick $j$ large enough to entail $(C_s(n, W)/\tau_0^\infty) \lambda_j^{-1} \leq \varepsilon_0/4$, we conclude that if $r \in (R_3, \rho_j]$, then $\delta_{V_j, R_3, \Lambda_j}(r) \geq \delta_{V_j, R_3, \Lambda_j}(\rho_j) \geq -\varepsilon_0$, where in the first inequality we have used Theorem 2.7-(i) and the fact that $V_j \in V_n(\Lambda_j, R_3, \infty)$. In summary, by (5.50) and (5.52) (which give (2.3)), by (5.49) (which gives (2.4) with $s = s_W(\sigma)/8$, and by (5.55) and (5.57) (which imply, respectively, (2.5) and (2.6)) we see that Theorem 2.1-(i) can be applied with $V = V_j$ and $s = s_W(\sigma)/8$ provided $j$ is large in terms of $\sigma, n, W$ and the limit $F$ of the $E_j$'s. Thus, setting

$$S_{s_j} = \min \{ R_s, \varepsilon_0/\Lambda_j \}, \quad (5.59)$$

and noticing that by (5.56) and $\Lambda_j \leq \Lambda_0 \lambda_j^{-1}$ we have

$$S_{s_j} \geq 16 R_1 \lambda_j, \quad (5.60)$$

(for $R_1$ depending on $n$ and $W$ only) we conclude that, for $j$ large, there are $K_j \in H$ and $u_j \in X_\sigma(\Sigma_{K_j}, \sigma_W(\sigma)/32, R_1 \lambda_j)$, such that

$$(\partial E_j) \cap A^{R_1 \lambda_j}_{s_W(\sigma)/32} = \Sigma_{K_j}(u_j, s_W(\sigma)/32, R_1 \lambda_j). \quad (5.61)$$

Similarly, by (5.42) and (5.44), thanks to Theorem 2.1-(ii) we have

$$(\partial F) \cap (\mathbb{R}^{n+1} \setminus B_{s_W(\sigma)/32}) = \Sigma_{\nu} (u, s_W(\sigma)/32, \infty), \quad (5.62)$$

for $u \in X_{\sigma'}(\Sigma_{\nu}, s_W(\sigma)/32, \infty)$ for every $\sigma' > \sigma$. Now, by $E_j \to F$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$, (5.61) and (5.62) can hold only if $|\nu_{K_j} - \nu| \leq \zeta(\sigma)$ for a function $\zeta$, depending on $n$ and $W$ only, such that $\zeta(\sigma) \to 0$ as $\sigma \to 0^+$. In particular (denoting by $\sigma_0^\sigma, \varepsilon_0^\sigma$ and $C_0^\sigma$ the dimension dependent constants originally introduced in Lemma 2.5 as $\sigma_0, \varepsilon_0$ and $C_0$) we can find $\sigma_1 = \sigma_1(n, W) \leq \sigma_0^\sigma$ such that if $\sigma < \sigma_1$, then $\varepsilon_0^\sigma \geq \zeta(\sigma) \geq |\nu_{K_j} - \nu|$, and correspondingily, Lemma 2.5-(i) can be used to infer the existence of $u_j^\sigma \in X_{C_0(\sigma + \zeta(\sigma))}(\Sigma_{\nu}, s_W(\sigma)/32, 2 R_1 \lambda_j)$ such that, for $j$ large,

$$\Sigma_{\nu} (u_j^\sigma, s_W(\sigma)/32, 2 R_1 \lambda_j) = \Sigma_{K_j}(u_j, s_W(\sigma)/32, 2 R_1 \lambda_j)$$

$$= (\partial E_j) \cap A^{2 R_1 \lambda_j}_{s_W(\sigma)/32}. \quad (5.63)$$
By (5.48) and Lemma B.1, (5.63) implies cylindrical graphicality: more precisely, provided $\sigma_1$ is small enough, there are $g_j \in C^1(\nu^\perp)$ such that
\begin{equation}
\sup_{x \in \nu^\perp} \{|g_j(x)| |x|^{-1}, |\nabla g_j(x)|\} \leq C (\sigma + \zeta(\sigma)), \tag{5.64}
\end{equation}
\begin{equation}
(\partial E_j) \cap A_{2R_2}^{R_1 \lambda_j} = \{x + g_j(x) \nu : x \in \nu^\perp\} \cap A_{2R_2}^{R_1 \lambda_j}. \tag{5.65}
\end{equation}
At the same time, by (5.41), (1.14), and up to further increasing $R_2$ and decreasing $\sigma_1$, we can exploit Lemma B.2 in the appendix to find $h_j \in C^1(G(f))$, $G(f) = \{x + f(x) \nu : x \in \nu^\perp\}$, such that
\begin{equation}
\{x + g_j(x) \nu : x \in \nu^\perp\} \setminus B_{4R_2} = \{z + h_j(z) \nu_F(z) : z \in G(f)\} \setminus B_{4R_2}
\end{equation}
which, combined with (5.41) and (5.65) shows that
\begin{equation}
(\partial E_j) \cap A_{4R_2}^{R_1 \lambda_j} = \{z + h_j(z) \nu_F(z) : z \in \partial F\} \cap A_{4R_2}^{R_1 \lambda_j}
\end{equation}
that is (5.45). By $E_j \to F$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$, we find $h_j \to 0$ in $L^1((\partial F) \cap A_{4R_2}^M)$ for every $M < \infty$, so that, by elliptic regularity, (5.46) follows. We now recall that, by Theorem 3.1-(ii), $(\partial E_j) \setminus B_{R_0(v_j)} \lambda_j$ coincides with
\begin{equation}
\{y + \lambda_j w_j((y - x_j)/\lambda_j) \nu_{B_1(1)}(y) : y \in \partial B(1)\} \setminus B_{R_0(v_j)} \lambda_j
\end{equation}
for $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$ and $R_0(v_j) \to 0$. \tag{5.66}
The overlapping of (5.65) and (5.66) (i.e., the fact that $R_0(v_j) < R_1$ if $j$ is large enough) implies statement (iii). Finally, combining (5.64) and (5.65) with (5.66) and $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$ we deduce the validity of (5.47). More precisely, rescaling by $\lambda_j$ in (5.64) and (5.65) and setting $E_j^* = E_j/\lambda_j$, we find $g_j^* \in C^1(\nu^\perp)$ such that, for every $j \geq j_0(\sigma)$ and $\sigma < \sigma_1$,
\begin{equation}
\sup_{x \in \nu^\perp} \{|g_j^*(x)| |x|^{-1}, |\nabla g_j^*(x)|\} \leq C (\sigma + \zeta(\sigma)), \tag{5.67}
\end{equation}
\begin{equation}
(\partial E_j^*) \cap A_{2R_2}^{R_1 \lambda_j} = \{x + g_j^*(x) \nu : x \in \nu^\perp\} \cap A_{2R_2}^{R_1 \lambda_j}, \tag{5.68}
\end{equation}
while rescaling by $\lambda_j$ in (5.66) and setting $z_j = x_j/\lambda_j$ we find
\begin{equation}
(\partial E_j^*) \setminus B_{R_0(v_j)} = \{z_j + z + w_j(z) \nu : y \in \partial B^{(1)}(z_j)\} \setminus B_{R_0(v_j)} \tag{5.69}
\end{equation}
where $|z_j| - \omega_{n+1}^{1/2} \to 0$ thanks to (3.4). Up to subsequences, $z_j \to z_0$, where $|z_0| = \omega_{n+1}^{1/2}$. Should $z_0 \neq |z_0| \nu$, then picking $\sigma$ small enough in terms of $|\nu - (z_0/|z_0|)| > 0$ and picking $j$ large enough, we would then be able to exploit (5.67) to get a contradiction with $\|w_j\|_{C^1(\partial B^{(1)})} \to 0$.

**Conclusion:** Theorem 3.1 implies Theorem 1.6-(i), and (1.19) was proved in step three. Should Theorem 1.6-(ii), (iii), or (iv) fail, then we could find a sequence $\{(E_j, v_j)\}_j$ contradicting the conclusions of either step five or Theorem 3.1. We have thus completed the proof of Theorem 1.6. \qed
Appendix A. Proof of Theorem 2.6

We assume $H \in \mathcal{H}$, $\Lambda \geq 0$, $\eta_0 > \eta > 0$, $(r_1, r_2)$ and $(r_3, r_4)$ are $(\eta, \eta_0)$-related as in (2.41), and $u \in X_\sigma(\Sigma_H, r_1, r_2)$ is such that $\Sigma_H(u, r_1, r_2)$ has mean curvature bounded by $\Lambda$ in $A_{r_2}^2$. We want to find $\sigma_0$ and $C_0$, depending on $n$, $\eta_0$, and $\eta$ only, such that, if $\max\{1, \Lambda r_2\} \sigma \leq \sigma_0$, then

$$\left| \mathcal{H}^n(\Sigma_H(u, r_3, r_4)) - \mathcal{H}^n(\Sigma_H(0, r_3, r_4)) \right| \leq C_0 \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} \left( u^2 + \Lambda r |u| \right); \quad (A.1)$$

and such that, if there is $r \in (r_1, r_2)$ s.t. $E_{\Sigma_H}^0[u_r] = 0$ on $\Sigma_H$, then

$$\int_{\Sigma_H \times (r_1, r_2)} r^{n-1} u^2 \leq C(n) \Lambda r (r_2^2 - r_1^2) + C_0 \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r u)^2. \quad (A.2)$$

We make three preliminary considerations: (i): By [AA81, 4.5(8)]

$$\left| \mathcal{H}^n(\Sigma_H(u, r_1, r_2)) - \mathcal{H}^n(\Sigma_H(0, r_1, r_2)) \right| \leq C(n) \sigma \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (u^2 + |\nabla \Sigma_H u|^2 + (r \partial_r u)^2). \quad (A.3)$$

Similarly, by the last displayed formula on [AA81, Page 236] and by [AA81, Lemma 4.9(1)], if $\varphi = \psi^2 w$, $w \in C^1(\Sigma_H \times (r_1, r_2))$ and $\psi \in C^1(r_1, r_2)$, then

$$\left| \frac{d}{dt} \bigg|_{t=0} \mathcal{H}^n(\Sigma_H(u + t \varphi, r_1, r_2)) \right| \leq C(n) \sigma \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} \left\{ |\nabla \Sigma_H u|^2 + |\nabla \Sigma_H w|^2 + (r \partial_r w)^2 + (r \partial_r u)^2 + u^2 + \psi^2 w^2 \right\}, (A.4)$$

which is the second order expansion of the first variation of the area at $\Sigma_H(u, r_1, r_2)$ along outer variations in spherical coordinates of the form $\varphi = \psi^2 w$, $\psi = \psi(r)$. (ii): For the sake of brevity, given $\zeta : (r_1, r_2) \to \mathbb{R}$ a radial function, $u, v : \Sigma_H \times (r_1, r_2) \to \mathbb{R}$, $X, Y : \Sigma_H \times (r_1, r_2) \to \mathbb{R}^m$, we set

$$Q_\zeta(u, v) = \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} \zeta(r)^2 u v, \quad Q_\zeta(X, Y) = \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} \zeta(r)^2 X \cdot Y,$$

and $Q_\zeta(u) = Q_\zeta(u, u)$, $Q_\zeta(X) = Q_\zeta(X, X)$. (iii): The following two estimates (whose elementary proof is contained in [AA81, Lemma 7.13]) hold: whenever $v \in C^1(\Sigma_H \times (r_1, r_2))$, we have

$$\int_{\Sigma_H \times (r_1, r_2)} r^{n-1} v^2 \leq C(n, \eta, \eta_0) \left\{ \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r v)^2 + \int_{\Sigma_H \times (r_3, r_4)} r^{n-1} v^2 \right\}, (A.5)$$
and, provided there is \( r \in [r_1, r_2] \) such that \( v_r = 0 \) on \( \Sigma_H \), we have
\[
\int_{\Sigma_H \times (r_1, r_2)} r^{n-1} v^2 \leq C(n, \eta_0) \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r v)^2 . \tag{A.6}
\]
We are now ready for the proof. Compared to [AA81, Chapter 4], the main difference is that we replace [AA81, Lemma 4.10] with (A.7).

**Step one:** We prove that there is \( h : \Sigma_H \times (r_1, r_2) \to [-\Lambda, \Lambda] \) such that for every \( w \in C^1(\Sigma_H \times (r_1, r_2)) \) and \( \psi \in C^1(r_1, r_2) \) we have
\[
|T_\psi(u, w) - \int_{\Sigma_H \times (r_1, r_2)} r^n \psi^2 w \, h| \leq C(n) \sigma_0 (Q_\psi(u) + Q_\psi(w) + Q_\psi(\nabla^{\Sigma_H} u)
+ Q_\psi(\nabla^{\Sigma_H} w) + Q_r(\partial_r u) + Q_r(\partial_r w) + Q_r(\psi)) . \tag{A.7}
\]
where \( T_\psi(u, w) = Q_\psi(\nabla^{\Sigma_H} u, \nabla^{\Sigma_H} w) + Q_r(\partial_r u, \partial_r(\psi^2 w)) - (n-1) Q_\psi(u, w) \).

We start rewriting (A.4) as
\[
\left| T_\psi(u, w) - \frac{d}{dt} \right|_{t=0} H^n(\Sigma(u + t \psi^2 w, r_1, r_2)) \right| \leq C(n) \sigma (Q_\psi(u) + Q_\psi(w) + Q_\psi(\nabla^{\Sigma_H} u) + Q_\psi(\nabla^{\Sigma_H} w)
+ Q_r(\partial_r u) + Q_r(\partial_r w) + Q_r(\psi)) . \tag{A.8}
\]
If \( F_{u+t\varphi} : \Sigma_H \times (r_1, r_2) \to \Sigma_H(u+t\varphi, r_1, r_2) \), \( \varphi = \psi^2 w \), is given by
\[
F_{u+t\varphi}(\omega, r) = r \frac{\omega + (u(\omega, r) + t\varphi(\omega, r)) \nu_H}{\sqrt{1 + (u(\omega, r) + t\varphi(\omega, r))^2}} ,
\]
then \( \{ \Phi_t = F_{u+t\varphi} \circ (F_u)^{-1} \}_{t \in [0, 1]} \) are diffeomorphisms on \( \Sigma_H(u, r_1, r_2) \), with \( \Phi_t(\Sigma_H(u, r_1, r_2)) = \Sigma_H(u+t\varphi, r_1, r_2) \) and \( \Phi_0 = (d/dt)_{t=0} \Phi_t \). Since \( \Sigma_H(u, r_1, r_2) \) has mean curvature bounded by \( \Lambda \) in \( A_{r_1}^2 \), for some bounded function \( h : \Sigma_H \times (r_1, r_2) \to [-\Lambda, \Lambda] \) we have
\[
\frac{d}{dt} \left| H^n(\Sigma(u + t\varphi, r_1, r_2)) \right| = \Lambda \int_{\Sigma_H(u, r_1, r_2)} h(F_u^{-1}) \Phi_0 \cdot \nu_{\Sigma_H(u, r_1, r_2)}
= \Lambda \int_{\Sigma_H \times (r_1, r_2)} h \Phi_0(F_u) \cdot (\partial_r F_u \wedge \wedge_{i=1}^{n-1} \partial_i F_u) ,
\]
where \( \partial_i = \nabla_{\tau_i} \) for a local orthonormal frame \( \{ \tau_i \}_{i=1}^{n-1} \) in \( \Sigma_H \), and where \( \ast \) is the Hodge star operator (so that \( \ast (v_1 \wedge v_2 \ldots \wedge v_n) \) is a normal vector to the hyperplane spanned by the \( v_i \)'s, with length equal to the \( n \)-dimensional volume of the parallelogram defined by the \( v_i \)'s, and whose orientation depends on the ordering of the \( v_i \)'s themselves). We can compute the initial velocity \( \Phi_0(F_u) \) of \( \{ \Phi_t \}_{t \in [0, 1]} \) by noticing that \( \Phi_t(F_u(\omega, r)) = r (1 + (u + t \varphi)^2)^{-1/2} (\omega + (u + t \varphi) \nu_H) \), so that,
\[
\Phi_0(F_u) = \frac{d}{dt} \bigg|_{t=0} r \frac{\omega + (u + t \varphi) \nu_H}{\sqrt{1 + (u + t \varphi)^2}} = r \frac{-u \varphi \omega + \varphi \nu_H}{(1 + u^2)^{3/2}}
= r \left( -u \varphi \omega + \varphi \nu_H \right) + r \sigma O \left( \psi^2(u^2 + w^2) \right).
\]
Step three:

\[ \sigma \{ \text{Plugging this estimate into (A.8), and by max } \}

\text{At the same time}

\[ \tau = \frac{-(u^2/2)}{1 + u^2} \rho + (u + \rho \partial_t u) \omega + \sigma O(u^2 + (\rho \partial_t u)^2) \]

\[ = A \omega + B \nu_H + \sigma O(u^2 + (\partial_t u)^2) \]

\[ \frac{\partial_t F_u}{r} = \partial_t \left( \frac{\omega + u \nu_H}{\sqrt{1 + u^2}} \right) = \frac{\tau_i}{\sqrt{1 + u^2}} - \frac{u \partial_t u}{(1 + u^2)^{3/2}} \omega + \frac{\partial_t u}{(1 + u^2)^{3/2}} \nu_H \]

\[ = (1 - (u^2/2)) \tau_i - u \partial_t u \omega + \partial_t u \nu_H + \sigma O(u^2 + (\partial_t u)^2) \]

\[ = C \tau_i + E_i \omega + F_i \nu_H + \sigma O(u^2 + (\partial_t u)^2) \]

so that, with \( \Xi = \wedge_{i=1}^{n-1} \tau_i, \tilde{\tau}_i = \wedge_{j \neq i} T_j, \) and \( P(u)^2 = u^2 + |\nabla \Sigma u|^2 + (r \partial_r u)^2, \)

\[ \frac{\partial_t F_u \wedge \wedge_{i=1}^{n-1} \partial_t F_u}{r^{n-1}} = (A \omega + B \nu_H) \wedge \wedge_{i=1}^{n-1} (C \tau_i + E_i \omega + F_i \nu_H) + \sigma O(P(u)^2) \]

\[ = A C^{n-1} \omega \wedge \Xi + B C^{n-1} \nu_H \wedge \Xi + G_i (\omega \wedge \nu_H \wedge \tilde{\tau}_i) + \sigma O(P(u)^2) \]

for a coefficient \( G_i \) which we do not need to compute. Indeed, \( \ast (\omega \wedge \nu_H \wedge \tilde{\tau}_i), \)

being parallel to \( \tilde{\tau}_i, \) is orthogonal to \( \omega \) and \( \nu_H, \) so that

\[ r^{-n} \Phi_0(F_u(r, \omega)) \ast (\partial_t F_u \wedge \wedge_{i=1}^{n-1} \partial_t F_u) \]

\[ = \left[ (-u \varphi \omega + \varphi \nu_H) + \sigma O(\psi^2(u^2 + \nu^2)) \right] \cdot \left[ A C^{n-1} \nu_H - B C^{n-1} \omega + \sigma O(P(u)^2) \right] \]

\[ = C^{n-1} \left[ (1 - \frac{u^2}{2}) - u \partial_r u \varphi + (u + \rho \partial_r u) u \varphi \right] + \sigma O(\psi^2(w^2 + P(u)^2)) \]

\[ = \varphi + \sigma O(\psi^2(w^2 + P(u)^2)) \]

In particular, since \( |h| \leq \Lambda, \)

\[ \frac{d}{dt} \bigg|_{t=0} H^n(S(u + t \varphi, r_1, r_2)) = \int_{\Sigma \times (r_1, r_2)} h \Phi_0(F_u) \ast (\partial_t F_u \wedge \wedge_{i=1}^{n-1} \partial_t F_u) \]

\[ = \int_{\Sigma \times (r_1, r_2)} r^n \psi^2 w h + \sigma \Lambda r_2 O(Q_\varphi(u) + Q_\varphi(w) + Q_\psi(\nabla \Sigma u) + Q_\psi(\partial_r u)) \]

Plugging this estimate into (A.8), and by max \{1, \Lambda r_2 \} \( \sigma \leq \sigma_0, \) we find (A.7).

**Step two:** We prove that

\[ Q_\varphi(\nabla \Sigma u) + Q_\varphi(\partial_r u) \leq Q_\varphi(|u|, \Lambda \sigma) + C(n) (Q_\varphi(u) + Q_\varphi(\psi(u))) \]  \( \text{(A.9)} \)

By \( Q_\varphi(\psi \partial_r u, \psi' u) \leq Q_\varphi(\partial_r u)/4 + C Q_\varphi(u) \) and by (A.7) \( w = u \) we find

\[ Q_\varphi(\nabla \Sigma u) + Q_\varphi(\partial_r u) \leq Q_\varphi(|u|, \Lambda \sigma) + C(n) (Q_\varphi(u) + Q_\varphi(\psi(u))) \]

\[ + C(n) \sigma_0 (Q_\psi(u) + Q_\psi(\psi(u) + Q_\psi(\nabla \Sigma u) + Q_\psi(\partial_r u)) \]

which implies (A.9) provided \( \sigma_0 \) is small enough.

**Step three:** We prove that, if \( w : \Sigma H \times (r_1, r_2) \rightarrow \mathbb{R} \) is slice-wise orthogonal to \( u - w, \) in the sense that \( \int_{\Sigma H} w_r (u_r - w_r) = 0, \int_{\Sigma H} \rho w_r (\partial_r u_r - \partial_r w_r) = 0, \)

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and \( \int_{\Sigma_H} \nabla_{\Sigma_H} w_r \cdot (\nabla_{\Sigma_H} u_r - \nabla_{\Sigma_H} w_r) = 0 \) for every \( r \in (r_1, r_2) \), then
\[
|T_\psi(u, w)| \leq Q_\psi(|w|, \Lambda r) + C(n) \sigma_0 \left( Q_\psi(u) + Q_\psi(|u|, \Lambda r) \right).
\]
(A.10)

Indeed, by slice-wise orthogonality, we find that \( Q_\zeta(w) \leq Q_\zeta(u) \), \( Q_\zeta(\partial_r w) \leq Q_\zeta(\partial_r u) \) and \( Q_\zeta(\nabla_{\Sigma_H} w) \leq Q_\zeta(\nabla_{\Sigma_H} u) \) whenever \( \zeta : (r_1, r_2) \to \mathbb{R} \) is radial. Therefore (A.7) gives \( |T_\psi(u, w)| \leq Q_\psi(|w|, \Lambda r) + C(n) \sigma_0 R_\psi(u) \), with \( R_\psi(u) = Q_\psi(u) + Q_\psi(\nabla_{\Sigma_H} u) + Q_\psi(\partial_r u) + Q_\psi(\partial_r u) \). Combining this with (A.9) we get (A.10).

**Step four:** We prove (A.1). Let now \( \psi \) be a cut-off function between \((r_3, r_4)\) and \((r_1, r_2)\), so that with \( Z_\psi(u) = Q_\psi(\nabla_{\Sigma_H} u) + Q_\psi(u) + Q_\psi(\partial_r u) \),
\[
\left| \int_{\Sigma_H \times (r_3, r_4)} r^{n-1} \left( |\nabla_{\Sigma_H} u|^2 - (n-1) u^2 + |r \partial_r u|^2 \right) \right| \leq Z_\psi(u).
\]
If \( A(u) = H^n(\Sigma_H(u, r_3, r_4)) - H^n(\Sigma_H(0, r_3, r_4)) \), then by (A.3) with \((r_3, r_4)\) in place of \((r_1, r_2)\), we find
\[
|A(u)| \leq Z_\psi(u) + C(n) \sigma Z_\psi(u) \leq Q_\psi(|u|, \Lambda r) + C(n) \left( Q_\psi(u) + Q_\psi(\partial_r u) \right)
\]
\[
+ C(n) \sigma \left\{ Q_\psi(u) + Q_\psi(|u|, \Lambda r) + C(n) \left( Q_\psi(u) + Q_\psi(\partial_r u) \right) \right\},
\]
where in the last inequality we have used (A.9). We deduce
\[
|A(u)| \leq C(n) \left( Q_\psi(|u|, \Lambda r) + Q_\psi(u) + Q_\psi(\partial_r u) \right),
\]
(A.11)
and (A.1) follows (with \( C_0 = C_0(n, \eta_0, \eta) \) by the properties of \( \psi \).

**Step five:** We finally prove that, if \( E_{\Sigma_H}^0[u_{r*}] = 0 \) for some \( r* \in (r_1, r_2) \), then (A.2) holds, that is
\[
\int_{\Sigma_H \times (r_3, r_4)} r^{n-1} u^2 \leq C(n) \Lambda r_2 \left( r_2^n - r_1^n \right) + C(n, \eta_0, \eta) \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r u)^2. \]
(A.12)

Define \( u^+, u^-, u^0 : \Sigma_H \times (r_1, r_2) \to \mathbb{R} \) by setting, for \( r \in (r_1, r_2) \), \((u^+)_r = E_{\Sigma_H^+}^0[u_{r*}], (u^-)_r = E_{\Sigma_H^-}^0[u_{r*}] \) and \((u^0)_r = E_{\Sigma_H^0}^0[u_{r*}] \), where \( E_{\Sigma_H^0}^0 \) denote the \( L^2(\Sigma_H) \)-orthogonal projections on the spaces of positive/negative eigenvectors of the Jacobi operator of \( \Sigma_H \), and where \( E_{\Sigma_H}^0 \) is the \( L^2(\Sigma_H) \)-orthogonal projection onto the space of the Jacobi fields of \( \Sigma_H \). Since \((u^0)_{r*} = 0 \), we can directly apply (A.6) with \( v = u^0 \) and deduce that
\[
\int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (u^0)^2 \leq C(n, \eta_0) \int_{\Sigma_H \times (r_1, r_2)} r^{n-1} (r \partial_r u^0)^2.
\]
(A.13)

By the orthogonality relations between \( u^0, u^+ \) and \( u^- \) we have that
\[
\int_{\Sigma_H \times (r_3, r_4)} r^{n-1} u^2 = \int_{\Sigma_H \times (a, b)} r^{n-1} \left( (u^0)^2 + (u^+)^2 + (u^-)^2 \right)
\]
(A.14)
\[
\int_{\Sigma_H \times (r_1, r_2)} r^{n+1} (\partial_r u)^2 = \int_{\Sigma_H \times (r_1, r_2)} r^{n+1} \left( |(\partial_r u)^0|^2 + (\partial_r u^+)^2 + (\partial_r u^-)^2 \right).
\]
(A.15)
By the spectral theorem, for every \(r \in (r_1, r_2)\) we have
\[
C_1(n)^{-1} \int_{\Sigma_H} (u^-)^2 \leq \int_{\Sigma_H} (n-1) (u^-)^2 - |\nabla^2 H (u^-)|^2 ,
\]
which, multiplied by \(r^{-1} \psi^2\), gives
\[
C_1(n)^{-1} Q_\psi (u^-) \leq (n-1) Q_\psi (u^-) - Q_\psi (\nabla^2 H u^-) = (n-1) Q_\psi (u^-) - Q_\psi (\nabla^2 H u^-) \\
\] \\
\[
= -T_\psi (u^-) + Q_r (\partial_r u, \partial_r (\psi^2 u^-)),
\]
where in the second to last identity we have used that \(w = u^-\) is slice-wise orthogonal to \(w - u;\) in particular, by (A.10) with \(w = u^-\), we find
\[
C_1(n)^{-1} Q_\psi (u^-) \leq Q_\psi (|u^-|, \Lambda r) + Q_r (\partial_r u, \partial_r (\psi^2 u^-)) + C(n) \sigma_0 (Q_\psi (u) + Q_r \psi (u) + Q_\psi (|u|, \Lambda r)).
\]
Again by slice-wise orthogonality of \(w = u^-\) to \(w - u\), we have
\[
Q_r (\partial_r u, \partial_r (\psi^2 u^-)) = Q_r (\partial_r u^-, \partial_r (\psi^2 u^-)) = Q_r (\partial_r u^-)
\]
\[
+ 2 Q_r (\psi' \partial_r u^-, \psi u^-) \leq Q_r (\partial_r u^-) + \frac{Q_\psi (u^-)}{2 C_1(n)} + C(n) Q_r \psi (\partial_r u^-),
\]
which combined into (A.16) gives
\[
(2 C_1(n)^{-1}) Q_\psi (u^-) \leq Q_\psi (|u^-|, \Lambda r) + Q_r (\partial_r u^-, \partial_r (\psi^2 u^-)) + C(n) Q_r \psi (\partial_r u^-) + C(n) \sigma_0 (Q_\psi (u) + Q_r \psi (u) + Q_\psi (|u|, \Lambda r)) .
\]
Using Hölder inequality again we have
\[
Q_\psi (|u^-|, \Lambda r) \leq \frac{Q_\psi (u^-)}{4 C_1(n)} + C(n) \Lambda r_2 (r_2^n - r_1^n),
\]
\[
Q_\psi (|u|, \Lambda r) \leq 2 Q_\psi (u) + C(n) \Lambda r_2 (r_2^n - r_1^n),
\]
so that
\[
\frac{Q_\psi (u^-)}{4 C_1(n)} \leq Q_r (\partial_r u^-) + C(n) (Q_r \psi (\partial_r u^-) + \Lambda r_2 (r_2^n - r_1^n)) + C(n) \sigma_0 (Q_\psi (u) + Q_r \psi (u) + \Lambda r_2 (r_2^n - r_1^n))
\]
Taking \(\psi\) to be a cut-off function between \((r_3, r_4)\) and \((r_1, r_2)\), we find
\[
\int_{\Sigma \times (r_3, r_4)} r^{n-1} (u^-)^2 \leq C(n) \Lambda r_2 (r_2^n - r_1^n)
\]
\[
+ C(n, \eta, \eta_0, \eta) \left\{ \int_{\Sigma \times (r_1, r_2)} r^{n-1} (r \partial_r u^-)^2 + \sigma_0 \int_{\Sigma \times (r_1, r_2)} r^{n-1} u^2 \right\}.
\]
By combining (A.13), (A.17), and the analogous estimate to (A.17) for \(u^+\) with (A.14) and (A.15) we find that (A.17) holds with \(u\) in place of \(u^-\); this latter estimate, thanks to (A.5), finally gives (A.12).

**APPENDIX B. SPHERICAL AND CYLINDRICAL GRAPHS**

We state here for the reader’s convenience two technical lemmas concerning spherical and cylindrical graphs. They are both used in the last step of the proof of Theorem 1.6. The elementary proofs are omitted.
Lemma B.1 (Spherical graphs as cylindrical graphs). There are dimension independent positive constants $C$ and $\eta_0$ with the following property. If $n \geq 1$, $H \in \mathcal{H}$ and $u \in X_{\eta}(\Sigma_H, r_1, r_2)$ with $\eta < \eta_0$, then we have

$$D_{(1-C \eta^2)r_2}^H \setminus D_{r_1}^H \subset p_H(\Sigma_H(u, r_1, r_2)) \subset D_{(1-C \eta^2)r_1}^H \setminus D_{(1-C \eta^2)r_2}^H,$$

and there is $g \in C^1(H)$ such that $\sup \{ |x|^{-1} |g(x)| + |\nabla g(x)| : x \in H \} \leq C \eta$ and $\Sigma_H(u, r_1, r_2) = \{ x + g(x) \nu_H : x \in p_H(\Sigma_H(u, r_1, r_2)) \}$. Moreover, if $(p_1, p_2) \subset ((1 + C \eta)r_1, (1 - C \eta^2)r_2)$, then $\Sigma_H(u, p_1, p_2) = \{ x + g(x) \nu_H : x \in H \} \cap A^2_{p_1}$.  

Lemma B.2. There is $\eta \in (0, 1)$ with the following property. If $H \in \mathcal{H}$, $R > 1$, $f \in C^2(H)$, and $g \in C^1(H)$ are such that

$$\max \{ |f(x)|, |x| |\nabla f(x)|, |x| |\nabla^2 f(x)| : x \in H, |x| > R \} < \eta, \quad \text{(B.1)}$$

$$\max \{ |x|^{-1} |g(x)|, |\nabla g(x)| : x \in H \} < \eta, \quad \text{(B.2)}$$

then there is $h \in C^1(G_H(f))$ such that

$$G_H(g) \setminus B_{4R} = \{ z + h(z) \nu_f(z) : z \in G_H(f) \} \setminus B_{4R}, \quad \text{(B.3)}$$

where $G_H(f) = \{ x + f(x) \nu_H : x \in H \}$ and, for $z = x + f(x) \nu_H$, we have set $\nu_f(z) = (1 + |\nabla f(x)|^2)^{-1/2} (-\nabla f(x) + \nu_H).$

Appendix C: Obstacles with zero isoperimetric residue

Proposition C.1. If $W$ is compact and $\mathcal{R}(W) = 0$, then $\psi_W(v) - P(B^{(v)}) \to 0$ as $v \to \infty$ and $W$ is purely $\mathcal{H}^n$-unrectifiable, in the sense that $W$ cannot contain an $\mathcal{H}^n$-rectifiable set of $\mathcal{H}^n$-positive measure. In a partial converse, if $W$ is purely $\mathcal{H}^n$-unrectifiable and $\mathcal{H}^n(W) < \infty$, then $\mathcal{R}(W) = 0$.

Proof. Step one: Let $\mathcal{R}(W) = 0$. Comparing with balls, $\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) \leq 0 = \mathcal{R}(W)$. To prove the matching lower bound, we argue by contradiction and consider $E_j \in \min \{ \psi_W(v_j) \}$ with $v_j \to \infty$ such that

$$\lim_{v \to \infty} \psi_W(v) - P(B^{(v)}) = \lim_{j \to \infty} P(E_j; \Omega) - P(B^{(v_j)}) < 0. \quad \text{(C.1)}$$

With (C.1) replacing $\mathcal{R}(W) > 0$, one can repeat verbatim step two-(a) of the proof of Theorem 1.1; we thus derive the asymptotic expansion for $F$ as in step two-(c), which is then the key fact used in step three to derive that $\lim_{j \to \infty} P(E_j; \Omega) - P(B^{(v_j)}) \geq -\text{res}_W(F \cup W, \nu) \geq -\mathcal{R}(W)$; the latter inequality is of course in contradiction with (C.1) if $\mathcal{R}(W) = 0$. Next, arguing again by contradiction, we assume the existence of an $\mathcal{H}^n$-rectifiable set $S$ with $\mathcal{H}^n(W \cap S) > 0$. By [Sim83b, Lemma 11.1], without loss of generality, $S$ is a $C^1$-embedded hypersurface in $\mathbb{R}^{n+1}$. Let $x$ be a point of tangential differentiability for $W \cap S$, so that $\mathcal{H}^n(W \cap S \cap B_\rho(x)) = \omega_n \rho^n + o_\rho(\rho^n)$ as $\rho \to 0^+$. Since $S$ is a $C^1$-embedded hypersurface, there is $\nu \in \mathbb{S}^n$ such that for every $\varepsilon > 0$ there is $\rho_\varepsilon = \rho_\varepsilon(x, \varepsilon) > 0$ with $S \cap C_{\rho_\varepsilon}(x) = \{ y + g(y) \nu : y \in D^\nu_{\rho_\varepsilon}(x) \}$, where $g \in C^1(x + \nu)$ with $g(x) = 0$.  


where we have used $f$ as a set of locally finite perimeter in $\Omega$. Now, \( \omega_n R^n > 0 \), and thus deduce $\mathcal{R}(W) > 0$.

**Step two:** Let $W$ be purely $\mathcal{H}^n$-unrectifiable with $\mathcal{H}^n(W) < \infty$, and let $(F, \nu) \in \text{Max}[\mathcal{R}(W)]$. Since $F$ is a local perimeter minimizer in $\Omega$, $F$ is open in $\Omega$ with $\Omega \cap \partial F = \text{cl}(\partial^* F)$ ($\partial^* F$ is the reduced boundary of $F$ as a set of locally finite perimeter in $\Omega$). Now, \( \omega_n R^n - P(F; C^\nu_R \setminus W) \geq \omega_n (2 \rho_*)^n - P(F; C^\nu_{2 \rho_*} (x) \setminus W) \)
\[
= \int_{D^\nu_{2 \rho_*}(x)} 1 - \sqrt{1 + |\nabla f|^2} + P(F; C^\nu_{2 \rho_*} (x) \cap W)
\]
\[
\geq -\omega_n (2 \rho_*)^n \varepsilon^2 + (1 - \varepsilon) \omega_n \rho_*^n
\]
where we have used $f = 0$ on $\nu^\perp \setminus D^\nu_{2 \rho_*}(x)$, (C.2) and $\sqrt{1 + \varepsilon^2} \leq 1 + \varepsilon^2$. Up to taking $\varepsilon < \varepsilon(n)$, we thus find $\text{res}_W(F, \nu) > 0$, and thus $\mathcal{R}(W) > 0$.

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