Incidence Bounds on Edge Partitions of $K_n$

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Abstract

We solve a problem conjectured by Cheriyan, giving sharp bounds for incidence of certain edge partitions of the connected graph on $n$-vertices. We briefly discuss the history of the problem and relation to node connectivity of strongly regular graphs. We show that the bound cannot be made sharper.

1 Introduction

Several characterizations of edge-connectivity are known within the literature. See, for instance, [5], [3], or [6]. In recent years there has been a push to understand an analogous notion of connectivity for nodes [2]. One earlier result within this area is found in [4].

Theorem 1.

In an effort to extend this to a larger family of strongly regular graphs the Johnson $J(n, 2)$ (the line graph of $K_n$) was considered. It was remarked that this problem is equivalent to proving the main theorem within this work. Since then, the analogue for $J(n, 2)$ on $J(n, 2)$ was proven in [1] but the incidence bound remained an open question.

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2 The main result

Let $K_n$ be the connected graph on $n$ vertices. Partition the edges of $K_n$ into 3 sets $S,T,$ and $Z$ with $|Z| = n - 3$. We prove that the incidence of $S$ and $T$ is at least the minimum of the incidence of either $Z$ and $T$ or $Z$ and $S$. 
Label the vertices \( v_1, \ldots, v_n \) and let the degree in \( S, T, Z \) of \( v_i \) be \( s_i, t_i, z_i \), respectively. We prove

**Theorem 2.**

\[
\sum_{i=1}^{n} s_i t_i \geq \min\{\sum_{i=1}^{n} z_i t_i, \sum_{i=1}^{n} z_i s_i\}.
\]

We begin with an elementary observation, since the degree of each node is \( n - 1 \), it follows that \( s_i + t_i + z_i = n - 1 \). Suppose \( s_1, \ldots, s_p \) are the nodes with degree in \( S \) equal to 0, let \( t_{p+1}, \ldots, t_{p+q} \) be the nodes with degree in \( T \) being 0. We let \( P = \{v_1, \ldots, v_p\}, Q = \{v_{p+1}, \ldots, v_{p+q}\}, R = V \setminus (P \cup Q) \).

Assume for contradiction that \( \sum_{i=1}^{n} s_i t_i < \sum_{i=1}^{n} z_i t_i \) and \( \sum_{i=1}^{n} s_i t_i < \sum_{i=1}^{n} z_i s_i \). We first prove a few quick lemmas:

**Lemma 1.** We have:

\[
2(n-1)(n-3) - \sum_{i=1}^{n} z_i^2 = (n-1) \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} z_i^2 > \sum_{i=1}^{n} (n-1)t_i - \sum_{i=1}^{n} t_i^2,
\]

\[
2(n-1)(n-3) - \sum_{i=1}^{n} z_i^2 = (n-1) \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} z_i^2 > \sum_{i=1}^{n} (n-1)s_i - \sum_{i=1}^{n} s_i^2.
\]

And:

\[
\frac{\sum_{i=1}^{n} z_i^2}{2} + \sum_{i=1}^{n} s_i t_i < (n-1)(n-3) \tag{1}
\]

**Proof.** From

\[
\sum_{i=1}^{n} s_i t_i < \sum_{i=1}^{n} z_i t_i
\]

Write \( t_i = n - 1 - s_i - z_i \) to get

\[
(n-1) \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} z_i^2 > \sum_{i=1}^{n} (n-1)s_i - \sum_{i=1}^{n} s_i^2.
\]

And by symmetry we have the other inequality. For the last inequality simply sum the 2 incidence inequalities (note that \( \sum z_i = 2(n-3) \):
\[
2 \sum_{i=1}^{n} s_i t_i < \sum_{i=1}^{n} z_i(t_i + s_i)
\]
\[
2 \sum_{i=1}^{n} s_i t_i - \sum_{i=1}^{n} z_i^2 < \sum_{i=1}^{n} z_i(n - 1) = 2(n - 1)(n - 3)
\]
And divide through by 2. \hfill \Box

**Lemma 2.**

\[p + q \leq 2\sqrt{n - 3}.\]

**Proof.** We have exactly \(n - q\) non-zero degree in \(S\) vertices so for each \(s_i\) we have \(s_i \leq n - 1 - q\) and similarly \(t_i \leq n - 1 - p\). Then for \(v_i \in P, Q\) we must have \(z_i \geq p, q\). Using this we bound the sum \(p + q\), since \(p^2 + q^2 \leq \sum_{P} z_i + \sum_{Q} z_i \leq \sum_{i=1}^{n} z_i = 2(n - 3)\). The maximum over \(p\) and \(q\) is achieved when 1 term dominates, so \(p + q\) is at most \(2\sqrt{n - 3}\). \hfill \Box

**Lemma 3.**

\[\sum_{i=1}^{n} \frac{z_i^2}{2} + 3 \geq \sum_{R} z_i\]

**Proof.** The sum of squares is smallest when they are evenly distributed, in this case this is achieved when \(z_i = 2\) for \(n - 6\) of the vertices and 1 for the remaining 6. We also know \(\sum_{R} z_i \leq \sum_{i=1}^{n} z_i = 2(n - 3)\):

\[\sum_{i=1}^{n} \frac{z_i^2}{2} \geq \frac{2^2}{2}(n - 6) + \frac{6}{2} + 3 = 2(n - 3) \geq \sum_{R} z_i\]

\hfill \Box

Now for \(p + q \leq 2\), using \(2\) and \(3\)

\[
\sum_{i=1}^{n} s_i t_i + \sum_{i=1}^{n} \frac{z_i^2}{2} = \sum_{R} s_i t_i + \sum_{i=1}^{n} \frac{z_i^2}{2}
\]
\[
\geq \sum_{R} (n - 2 - z_i) + \sum_{i=1}^{n} \frac{z_i^2}{2}
\]
\[
= (n - p - q)(n - 2) - \sum_{R} z_i + \sum_{i=1}^{n} \frac{z_i^2}{2}
\]
\[
\geq (n - p - q)(n - 2) - 3 \geq (n - 2)(n - 2) \geq (n - 1)(n - 3)
\]

\[(2)\]
We can WLOG assume $p \geq q$. Assume $p \geq 4$. Then by summing the first two lines of (1)

$$4(n - 1)(n - 3) - 2 \sum_{i=1}^n z_i^2 > (n - 1) \sum_{i=1}^n (n - 1 - z_i) - \sum_{i=1}^n (t_i^2 + s_i^2)$$

$$= n(n - 1)^2 - 2(n - 1)(n - 3) - \sum_{P,Q} (t_i^2 + s_i^2) - \sum_R (t_i^2 + s_i^2)$$

$$> n(n - 1)^2 - 2(n - 1)(n - 3) - \sum_{P,Q} (n - 1 - z_i)^2$$

$$- (n - p - q)((n - 1 - p)^2 + p^2)$$

$$= n(n - 1)^2 - 2(n - 1)(n - 3) - \sum_{P,Q} z_i^2 + 2(n - 1)\left(\sum_{P,Q} z_i\right)$$

$$- (p + q)(n - 1)^2 - (n - p - q)((n - 1 - p)^2 + p^2)$$

(3)

Rearranging and simplifying terms now gives us:

$$6(n - 1)(n - 3) > 2 \sum_{i=1}^n z_i^2 + 2(n - 1)\left(\sum_{P,Q} z_i\right) - \sum_{P,Q} z_i^2 + 2pn^2 + 2p(-1-2p-q)n + 2p(p^2+q+pq)$$

The goal is to prove the RHS is larger than the LHS for contradiction. Notice:

$$2 \sum_{i=1}^n z_i^2 + 2(n - 1)\left(\sum_{P,Q} z_i\right) - \sum_{P,Q} z_i^2 \geq 2 \sum_{i=1}^n z_i + \sum_{P,Q} (2(n - 1) - z_i) z_i \geq 4(n - 3) + (n + 1)q.$$ 

The last $(n + 1)q$ is from at least $z_i = 1$ for all $i \in Q$, otherwise $z_i = 0$ implies there is some vertex with all edges in $S$ meaning $p = 0$. And bounding $2(n - 1) - z_i \geq n + 1$.

Substituting the above in and collecting $n$ yields

$$0 > n^2(2p - 6) + n(2p(2p - q - 1) + q + 28) + 2p(p^2 + pq + q) + q - 30.$$ 

We assumed that $p$ is at least 4. The worst case for the above is when
\[ q = p \] with the largest zero of the RHS occurring at
\[ n = \frac{1}{4(p-3)} \left( \sqrt[ ]{-32p^3(p-1) + 4p^2(p^2 + 12p + 57) - 4p(p^2 + 19p - 32) + p^2 + 80p + 64} \\
- 4p^2 + 2p(p + 1) - p - 28 \right) \quad (4) \]

Recall the bound \( p + q < 2\sqrt{n - 3} \). This gives \( (\frac{n}{2})^2 + 3 < n \). Substitute this for \( n \) in the above. What one obtains is a polynomial that is positive for all \( p \geq 4 \). Our contradictive assumption was that this expression is negative, contradiction.

We have only a few cases left to consider: \( (p, q) \in \{(2, 1), (2, 2), (3, 0), (3, 1), (3, 2), (3, 3)\} \). One can verify by computer that the conjecture is true for \( n \leq 15 \) for the given \( (p, q) \). For larger \( n \) we use the following technique.

By symmetry we can assume that \( p \geq q \) and that \( p \) is at least 2. Consider the 2 vertices, \( v_1, v_2 \) in \( P \). Let \( P_2 \) be the subset of vertices in \( R \) connected to both \( v_1 \) and \( v_2 \) via an edge in \( T \). Use \( \sigma \) to denote \( |P_2| \). We must then have \( z_1 + z_2 = 2(n - 1) - t_1 - t_2 \) by degree considerations on \( P \). Also, \( \sigma \geq t_1 + t_2 - 2p + 3 - (n - q - p - 2) = t_1 + t_2 - p + q - n + 5 \) by pigeonhole principle. This comes from the realization that in the worst case we have \( t_1 + t_2 - 2p + 3 \) \((−2p+3 \text{ comes from edges contained in } P)\) for the case where edges going to the \( n - p - q - 2 \) vertices in \( R \). A double count occurs at least \( t_1 + t_2 - 2p + 3 - (n - q - p - 2) \) many times.

We intend to show
\[ \frac{1}{2} \sum_{i=1}^{n} z_i^2 + \sum_{i} s_i t_i \geq (n - 1)(n - 3). \]

Indeed
\[ \frac{1}{2} \sum_{i} z_i^2 + \sum_{i} s_i t_i = \frac{1}{2} \sum_{i} z_i^2 + \sum_{P_2} s_i t_i + \sum_{R \setminus P_2} s_i t_i \]
\[ = \frac{1}{2} \sum_{i=1}^{n} z_i^2 + \sum_{P_2} 2(n - 3 - z_i) + \sum_{R \setminus P_2} (n - 2 - z_i) \]
\[ \geq \frac{1}{2} \sum_{i=1}^{n} z_i^2 - \sum_{R} z_i - \sum_{P_2} z_i + 2(n - 3)\sigma + (n - p - q - \sigma)(n - 2) \]
\[ = C_z + n^2 + n(\sigma - p - q - 2) - 4\sigma + 2(p + q). \]

(5)
Where \( C_z := \frac{1}{2} \sum_{i=1}^{n} z_i^2 - \sum_{R} z_i - \sum_{P_2} z_i \). Comparing the above to \((n-1)(n-3)\) gives a difference of
\[
\frac{n(\sigma - p - q + 2) + C_z + 2(p + q) - 3 - 4\sigma}{\sigma - p - q + 2}
\]
That is, we wish to show
\[
n > \frac{(4\sigma + 3 - 2(p + q) - C_z)}{\sigma - p - q + 2}
\]
for a contradiction.

**Lemma 4.**
\[
\sigma - p - q + 2 \geq 1
\]
**Proof.** From the above we have \( \sigma \geq t_1 + t_2 - p + q - n + 5 \) or
\[
\sigma + z_1 + z_2 \geq n + 3 - p + q.
\]
And since \( z_1 + z_2 \) is at most \( n - 2 \) (we can have 1 adjacent edge among the \( n - 3 \) in \( Z \))
\[
\sigma - p - q + 2 \geq n + 3 - 2p + 2 - z_1 - z_2 \geq 7 - 2p \geq 1.
\]

**Lemma 5.** The penultimate step is using this bound on \( C_z \)
\[
-C_z = \sum_{P_2} 2z_i - \frac{1}{2} z_i^2 + \sum_{R \setminus P_2} z_i - \frac{1}{2} z_i^2 \leq 2\sigma + \frac{1}{2}(n - p - q - \sigma).
\]
**Proof.** We can finally simplify the fraction above \([6]\) to
\[
\frac{4\sigma + 3 - 2(p + q) - C_z}{\sigma - p - q + 2} \leq \frac{1}{2} n + \frac{15}{2}
\]
by finding common denominators and reducing.

So we are left with \( n > \frac{1}{2} n + \frac{15}{2} \), the theorem is true for \( n > 15 \), as required.

To summarize, we are left with

**Theorem 3.** Suppose \( S, T, Z \) is an edge partition of \( K_n \) with \( |Z| = n - 3 \). Then
\[
\sum_{i=1}^{n} s_i t_i \geq \min\{ \sum_{i=1}^{n} z_i t_i, \sum_{i=1}^{n} z_i s_i \}
\]
Where \( s_i, t_i, z_i \) are the respective degrees of vertex \( v_i \) in \( S, T, Z \).
3 Sharp Bounds

We now demonstrate that the bound obtained on the incidence is in some sense the best possible. Suppose we strengthened the condition on the size of $Z$ to be $n - 2$ instead of $n - 3$. In this case we can construct a family of counterexamples.

Let $n > 5$ and consider the edge partition

\[ S = \{v_1, v_2\}, \{v_2, v_3\} \]
\[ Z = \{v_1, v_3\}, \{v_2, v_i\} \quad \forall i \in \{4, \ldots, n\} \]
\[ T = \text{the remaining nodes.} \]

The incidence of $S$ and $T$ is then $2(n - 3)$ while the incidence of $S$ and $Z$ is $2(n - 3) + 2$. The incidence between $T$ and $Z$ is at least $(n - 3)(n - 1)$. Therefore, the bound from 3 fails.

We've added an illustration of the case $n = 5$ below.

The blue edges are in $S$, the black, $T$, and red, $Z$. 

![Diagram of the case n = 5](image.png)
References

[1] Florian Hoersch and Z. Szigeti. “Eulerian orientations and vertex-connectivity”. In: 2019.

[2] Tamás Király and Lap Chi Lau. “Approximate minâĂŞmax theorems for Steiner rooted-orientations of graphs and hypergraphs”. In: Journal of Combinatorial Theory, Series B 98.6 (2008), 1233âĂŞ1252. issn: 0095-8956. doi: 10.1016/j.jctb.2008.01.006. url: http://dx.doi.org/10.1016/j.jctb.2008.01.006.

[3] Zoltán Király and Zoltán Szigeti. “Simultaneous well-balanced orientations of graphs”. In: Journal of Combinatorial Theory, Series B 96.5 (2006), 684âĂŞ692. issn: 0095-8956. doi: 10.1016/j.jctb.2006.01.002. url: http://dx.doi.org/10.1016/j.jctb.2006.01.002.

[4] Maxwell Levit, L. Sunil Chandran, and Joseph Cheriyan. “On Eulerian orientations of even-degree hypercubes”. In: Operations Research Letters 46.5 (2018), 553âĂŞ556. issn: 0167-6377. doi: 10.1016/j.orl.2018.09.002. url: http://dx.doi.org/10.1016/j.orl.2018.09.002.

[5] C. ST. J. A. Nash-Williams. “On Orientations, Connectivity and Odd-Vertex-Pairings in Finite Graphs”. In: Canadian Journal of Mathematics 12 (1960), 555âĂŞ567. issn: 1496-4279. doi: 10.4153/cjm-1960-049-6. url: http://dx.doi.org/10.4153/cjm-1960-049-6.

[6] Carsten Thomassen. “Strongly 2-connected orientations of graphs”. In: Journal of Combinatorial Theory, Series B 110 (2015), 67âĂŞ78. issn: 0095-8956. doi: 10.1016/j.jctb.2014.07.004. url: http://dx.doi.org/10.1016/j.jctb.2014.07.004.