KHINTCHINE’S THEOREM WITH RANDOM FRACTIONS

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ABSTRACT. We prove a version of Khintchine’s Theorem for approximations by rational numbers whose numerators have been randomly chosen, and we explore the extent to which its monotonicity assumption can be removed. This leads to a natural question analogous to the Duffin–Schaeffer Conjecture in this setting, and several other related questions.

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1. Introduction

Let $\psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$ be some function, and for $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. In metric Diophantine approximation we are concerned with properties of the set

$$\mathcal{W}(\psi) = \left\{ x \in [0, 1] : \left| x - \frac{a}{n} \right| < \frac{\psi(n)}{n} \text{ for infinitely many } n \in \mathbb{N}, a \in [n] \right\}$$

of $\psi$-approximable numbers. One of the foundational results in this area is Khintchine’s Theorem [Khi24].

Khintchine’s Theorem (1924). With $\psi$ and $\mathcal{W}$ as above,

$$|\mathcal{W}(\psi)| = \begin{cases} 1 & \text{if } \psi(n) \text{ is non-increasing and } \sum_{n \in \mathbb{N}} \psi(n) = \infty \\ 0 & \text{if } \sum_{n \in \mathbb{N}} \psi(n) < \infty \end{cases}$$

where $|\cdot|$ denotes Lebesgue measure.

It is often most useful to see $\mathcal{W}(\psi)$ as the set $\mathcal{W}(\psi) = \limsup_{n \to \infty} \mathcal{A}_n = \bigcap_{k \geq 1} \bigcup_{n \geq k} \mathcal{A}_n$, where

$$\mathcal{A}_n = \bigcup_{a \in [n]} \left( \frac{a - \psi(n)}{n}, \frac{a + \psi(n)}{n} \right) \pmod{1} \subset [0, 1].$$

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(In other words, $W(\psi)$ is the set of points that lie in infinitely many of the sets $A_n$.) This way the “convergence part” of Khintchine’s Theorem is easily seen as a simple consequence of the Borel–Cantelli Lemma. We will focus most of our attention on the “divergence part.”

R. J. Duffin and A. C. Schaeffer showed by counterexample that the monotonicity assumption cannot be removed from Khintchine’s Theorem [DS41]. They also formulated what is now one of the foremost open problems in Diophantine approximation, the Duffin–Schaeffer Conjecture. Let $[n]' := \{a \in [n] : (a, n) = 1\}$. The idea is to consider instead the set

$$W'(\psi) = \left\{ x \in [0, 1] : \left| x - \frac{a}{n} \right| < \frac{\psi(n)}{n} \text{ for infinitely many } n \in \mathbb{N}, a \in [n]' \right\}$$

of numbers that are $\psi$-approximable by reduced fractions, and to modify the divergence condition accordingly.

**The Duffin–Schaeffer Conjecture (1941).** With $\psi$ and $W'$ as above,

$$|W'(\psi)| = \begin{cases} 1 & \text{if } \sum_{n \in \mathbb{N}} \frac{\phi(n)\psi(n)}{n} = \infty \\ 0 & \text{if } \sum_{n \in \mathbb{N}} \frac{\phi(n)\psi(n)}{n} < \infty \end{cases}$$

where $\phi(n)$ is the number of integers from 1 to $n$ that are coprime to $n$.

Again, the convergence part follows easily from the Borel–Cantelli Lemma, so what is really conjectured is the divergence part. Part of the challenge comes from the fact that the sets

$$A_n' = \bigcup_{a \in [n]'} \left( \frac{a - \psi(n)}{n}, \frac{a + \psi(n)}{n} \right) \mod 1 \subset [0, 1]$$

are not in general pairwise independent (nor quasi-independent, nor quasi-independent on average), so one cannot directly apply any of the partial converses of the Borel–Cantelli Lemma. In their paper, Duffin and Schaeffer proved what is known as the Duffin–Schaeffer Theorem, namely, that the conjecture holds under the additional assumption that

$$\limsup_{N \to \infty} \left( \frac{\sum_{n=1}^{N} \phi(n)\psi(n)}{n} \right) \left( \sum_{n=1}^{N} \psi(n) \right)^{-1} > 0.$$ 

Though it may not be obvious at first sight, the extra assumption in the Duffin–Schaeffer Theorem plays the role of independence for the sets $A_n'$.

Most subsequent attacks on the Duffin–Schaeffer Conjecture have revolved around this issue of independence of sets. For example, A. Pollington and R. Vaughan were able to show that the higher-dimensional analogue of the conjecture is true [PV90]. In their paper, they established an important bound on the measures of intersections $A_m' \cap A_n'$. This bound was later used by V. Beresnevich, A. Haynes, G. Harman, A. Pollington, and S. Velani to prove the Duffin–Schaeffer Conjecture under “extra divergence” assumptions [HPV12, BHHV13], and by C. Aistleitner under a “slow divergence” assumption [Ais14].
2. Problem

With the above discussion in mind, we propose the following problem. Suppose that instead of restricting numerators to the set of integers in \([n]\) that are coprime to \(n\), we simply choose \(\varphi(n)\) numbers from \([n]\) at random. Or, more generally, let us choose \(f(n)\) numbers from \([n]\), where \(f : \mathbb{N} \rightarrow \mathbb{N}\) is some fixed function such that \(f(n) \leq n\). Denoting this collection of subsets by \(\mathcal{P} = \{\mathcal{P}_n\}_{n \in \mathbb{N}}\), where each \(\mathcal{P}_n \subseteq [n]\) has cardinality \(f(n)\), what can be said about the set

\[
W^\mathcal{P}(\psi) = \left\{ x \in [0, 1] : \left| x - \frac{a}{n} \right| < \frac{\psi(n)}{n} \text{ for infinitely many } n \in \mathbb{N}, a \in \mathcal{P}_n \right\}
\]

of numbers that are \(\psi\)-approximable by rationals with numerators prescribed by \(\mathcal{P}\)? Of course, any result should depend on \(\mathcal{P}\), and it makes most sense to formulate questions with respect to a probability measure \(\mu\) on the space

\[
\Omega_f := \prod_{n \in \mathbb{N}} \left( \left( \frac{n}{f(n)} \right) \right)
\]

from which \(\mathcal{P}\) is chosen, and to consider the set

\[
W^f(\psi) = \left\{ (\mathcal{P}, x) : x \in W^\mathcal{P}(\psi) \right\} \subset \Omega_f \times [0, 1].
\]

The hope of course is that this extra randomness will make it easier to establish the kind of independence that is called for when showing that a lim sup set has full measure.

3. Results

We offer the following “random fractions” version of Khintchine’s Theorem, and first step toward the above questions.

**Theorem 1.** For any \(f(n) \leq n\) and \(\psi : \mathbb{N} \rightarrow \mathbb{R}^+ \cup \{0\}\),

\[
|W^f(\psi)| = \begin{cases} 1 & \text{if } \frac{f(n)\psi(n)}{n} \text{ is non-increasing and } \sum_{n \in \mathbb{N}} \frac{f(n)\psi(n)}{n} = \infty, \\ 0 & \text{if } \sum_{n \in \mathbb{N}} \frac{f(n)\psi(n)}{n} < \infty. \end{cases}
\]

Notice that Theorem 1 returns Khintchine’s Theorem upon setting \(f(n) \equiv n\). Fubini’s Theorem allows us to deduce the following corollary.

**Corollary 2.** With \(f, \mathcal{P}, \psi, \mathcal{W}\) as above,

\[
|W^\mathcal{P}(\psi)| = \begin{cases} 1 & \text{almost surely, if } \frac{f(n)\psi(n)}{n} \text{ is non-increasing and } \sum_{n \in \mathbb{N}} \frac{f(n)\psi(n)}{n} = \infty, \\ 0 & \text{if } \sum_{n \in \mathbb{N}} \frac{f(n)\psi(n)}{n} < \infty. \end{cases}
\]

The “almost surely” in the divergence part of Corollary 2 refers to the probability space \((\Omega_f, \mu)\) from which \(\mathcal{P}\) is chosen (see §4). It is easy to produce examples of \(\mathcal{P} \in \Omega_f\) for which the divergence condition holds, yet \(|W^\mathcal{P}(\psi)| = 0\), so it is clear that the divergence part of Corollary 2 cannot hold for every \(\mathcal{P}\). On the other hand, the convergence part is again a standard consequence of the Borel–Cantelli Lemma, and does hold for every \(\mathcal{P}\).

The challenge now is to remove the monotonicity assumption. Duffin and Schaeffer already provided a counterexample to the resulting statement in the case \(f(n) \gg n\), which automatically serves the same purpose whenever \(f(n) \gg n\). We give a counterexample
whose rate of divergence we can track, and which immediately shows that monotonicity cannot be removed from Theorem 1 if \( f(n) \) grows faster than \( n / \log \log n \).

**Theorem 3** (Duffin–Schaeffer Counterexample). Suppose that

\[
\lim_{n \to \infty} \frac{f(n) \log \log n}{n} = \infty.
\]

Then there exists a function \( \psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\} \) such that \( \sum_{n \in \mathbb{N}} f(n)\psi(n)/n \) diverges, yet \( |W(\psi)| = 0 \).

This modification of Duffin and Schaeffer’s example yields a number of simple corollaries that are relevant to our problem. Since for any \( P \in \Omega_f \) and \( \psi \), we have \( W^p(\psi) \subset W(\psi) \), it is clear that \( W^p(\psi) \) is null whenever \( W(\psi) \) is. Therefore, Theorem 3 gives us the following.

**Corollary 4.** If \( (f(n) \log \log n)/n \to \infty \), then there exist functions \( \psi \) such that \( \sum_{n \in \mathbb{N}} f(n)\psi(n)/n \) diverges, yet \( |W^p(\psi)| = 0 \) for every \( P \in \Omega_f \).

Theorem 3 implies that monotonicity cannot be removed from Theorem 1 (or Corollary 2) if \( f(n) \) grows strictly faster than \( n / \log \log n \). But it is well-known that

\[
(1) \quad \varphi(n) > \frac{n}{e^\gamma \log \log n + \frac{3}{\log \log n}}
\]

for all \( n > 2 \), where \( \gamma \) is Euler’s constant. Therefore Theorem 3 leads to the following immediate corollary, which shows that, in a sense, \( \varphi \) is the fastest growing function for which the Duffin–Schaeffer Conjecture could possibly hold.

**Corollary 5.** If \( f(n)/\varphi(n) \to \infty \), then there exists a function \( \psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\} \) such that \( \sum_{n \in \mathbb{N}} f(n)\psi(n)/n \) diverges, yet \( |W(\psi)| = 0 \), hence \( |W'(\psi)| = 0 \).

Still, we expect the Duffin–Schaeffer Conjecture itself to be true, so in view of (1) we should expect that Theorem 3 would become false if we replaced the condition

\[
\lim_{n \to \infty} \frac{f(n) \log \log n}{n} = \infty
\]

with the more relaxed condition

\[
\frac{f(n) \log \log n}{n} \gg 1 \quad \text{as} \quad n \to \infty.
\]

In fact, since it is also known that

\[
(2) \quad \varphi(n) < \frac{n}{e^\gamma \log \log n}
\]

for infinitely many \( n \), we may be tempted to suppose that if a function \( f \) has the following two things in common with \( \varphi \):

- \( f(n) \gg n/\log \log n \) for all \( n \), and
- \( f(n) \ll n/\log \log n \) on some sequence of \( n \)’s,

then this is enough to guarantee that we will never find a function \( \psi \) like the one in Theorem 3. But the next corollary shows that that is not necessarily true.
Corollary 6. There exist functions $f(n) \leq n$ such that

\[
\begin{cases}
\lim_{i \to \infty} \frac{f(k_i) \log \log k_i}{k_i} = \infty & \text{for some subsequence } \{k_i\} \subseteq \mathbb{N}, \text{ and} \\
\frac{f(n) \log \log n}{n} & \text{is bounded uniformly over all } n \not\in \{k_i\},
\end{cases}
\]

and such that there exists a function $\psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$ such that $\sum_{n \in \mathbb{N}} f(n)\psi(n)/n$ diverges, yet $|W(\psi)| = 0$, hence $|W^P(\psi)| = 0$ for every $P \in \Omega_f$.

The natural goal is to prove that monotonicity can be removed from Theorem 1 for $f = \varphi$, because this would turn Theorem 1 into a random version of the Duffin–Schaeffer Conjecture, asserting that “the conjecture is almost surely true,” in the sense of Corollary 2. In view of Theorem 3 and its corollaries, it is clear that doing this would require us to consider more than just the growth properties of $\varphi$. It may be worth pursuing the case $f(n) \ll n/\log \log n$ first.

4. Preliminaries

Let $\mu$ be the probability measure on $\Omega_f$ assigning uniform weights in each factor. A cylinder set of length $\ell$ is a set

$$
\mathcal{C} := \mathcal{C}(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\ell) = \left\{ P \in \Omega_f : P_k = \tilde{P}_k, k = 1, \ldots, \ell \right\}
$$

whose first $\ell$ entries have been prescribed. The measure of this cylinder set is

$$
\mu(\mathcal{C}) = \prod_{n \leq \ell} \left( \frac{n}{f(n)} \right)^{-1}.
$$

One can more generally define cylinder sets where finitely-many not-necessarily-consecutive entries have been prescribed, and the measure would be defined similarly. In fact, the measure $\mu$ is defined on the $\sigma$-algebra generated by these cylinders.

Let

$$
\Psi(N) := \sum_{n \leq N} \frac{f(n)\psi(n)}{n}.
$$

Let $1_n : \mathbb{R} \to \{0, 1\}$ denote the indicator function of the interval $\left(-\frac{\psi(n)}{n}, \frac{\psi(n)}{n}\right)$, and put

$$
\gamma^p_n(x) = \sum_{a \in \mathcal{P}'} 1_n \left(x - \frac{a}{n}\right)
$$

where $\mathcal{P}'_n = \{ a \in \mathcal{P}_n : (a, n) = 1 \}$. Let

$$
\Gamma_N(P, x) = \sum_{n \leq N} \gamma^p_n(x)
$$

be the number of solutions to $|nx - a| < \psi(n)$ with $a \in \mathcal{P}'_n, n \leq N$. The reason we restrict to the case $(a, n) = 1$ is essentially because we will use the following fact.

Lemma 7. Whenever $m \neq n$,

$$
\sum_{(a, b) \in [m']^2 \times [n']'} \int_{\mathbb{R}} 1_m \left(x - \frac{a}{m}\right) 1_n \left(x - \frac{b}{n}\right) \, dx \leq 4\psi(m)\psi(n).
$$
Lemma 7 is the heart of the proof of Khintchine’s Theorem, and many other similar results have a similar heart. This is because any partial converse of the Borel–Cantelli Lemma involves some form of independence among sets. Lemma 7 provides it in the proof of Khintchine’s Theorem, as well as in our proof of Theorem 1. (See Lemma 14.)

5. Proof of Theorem 1

As mentioned before, the convergence part of Theorem 1 follows from the Borel–Cantelli Lemma, which states that the measure of the “limsup” set of a sequence of subsets of a probability space is zero if the sum of the measures of the sequence of subsets converges.

Proof of the convergence part of Theorem 1. Notice that

\[ W^p(\psi) = \limsup_{n \to \infty} \left( \bigcup_{a \in P_n} \left( \frac{a - \psi(n)}{n}, \frac{a + \psi(n)}{n} \right) \right). \]

By assumption, the sum

\[ \sum_{n \in \mathbb{N}} \left| \bigcup_{a \in P_n} \left( \frac{a - \psi(n)}{n}, \frac{a + \psi(n)}{n} \right) \right| = 2 \sum_{n \in \mathbb{N}} f(n)\psi(n) \]

converges. Therefore, by the Borel–Cantelli Lemma, \( |W^p(\psi)| = 0. \)

The proof of the divergence part of Theorem 1 is an adaptation of the proof of Khintchine’s Theorem, following the one found in [Cas57].

Lemmas 8, 9, and 10 show that in order to prove the divergence part of Theorem 1, we only need to prove that the set

\[ W^f(\psi) = \{ (P, x) : x \in W^p(\psi) \} \]

has positive measure.

Lemma 8. If \( x \in W^p(\psi) \) for every \( P \) in some positive-measure subset of \( \Omega_f \), then \( x \in W^p(\psi) \) for almost every \( P \in \Omega_f \).

Proof. Notice that for a fixed \( x \in [0, 1] \), \( x \in W^p(\psi) \) is a tail event for the sequence of random variables \( \gamma^p_n(x) \), seen as measurable functions of \( P \in \Omega_f \). Notice also that as random variables on \( \Omega_f \) they are independent. Therefore, the lemma holds by Kolmogorov’s zero-one law.

Lemma 9. If \( \Omega \times \{ x \} \subset W^f(\psi) \subset \Omega_f \times [0, 1] \) for some positive-measure subset \( \Omega \subset \Omega_f \), then for any \( k \in \mathbb{N} \),

\[ \Omega^{(k)} \times \{ kx \ (\text{mod} \ 1) \} \subset W^f(k\psi), \]

where

\[ \Omega^{(k)} = \{ \Omega \in \Omega_f : \text{for every } n \in \mathbb{N} \text{ there exists some } P \in \Omega \text{ such that } kP_n \subseteq \Omega_n \}. \]

The set \( \Omega^{(k)} \) has positive measure.
**Proof.** The first statement follows from the observation that if \(|nx - a| < \psi(n)\) with \(n \in \mathbb{N}\) and \(a \in \mathcal{P}_n\), then \(|k \psi(n)| = ka| < k\psi(n)|\), and that we will have \(ka \in k\mathcal{P}_n\). It is only left to show that the set \(\Omega^{(k)}\) has positive measure. This is easily seen, since any cylinder \(\mathcal{C} \subset \Omega_f\) is taken to a finite union of cylinders \(\mathcal{C}^{(k)}\), each having the same measure as \(\mathcal{C}\). So the operation can only increase the measure of a set. \(\square\)

**Lemma 10.** In order to prove Theorem 1 it is enough to show that \(WF(\psi) \subset \Omega_f \times [0, 1]\) has positive measure.

**Proof.** Let us assume that we have proved that under the conditions of Theorem 1, the set \(WF(\psi)\) has positive measure, and let \(\psi\) be given as in the divergence part of the theorem statement. We may replace \(\psi\) with \(\psi_\tau(n) = \tau(n)\psi(n)\), where \(0 < \tau(n) \leq 1\) decreases to 0, in such a way that \(\psi_\tau\) satisfies the same conditions, namely that \(\frac{f(n)\psi_\tau(n)}{n}\) is non-increasing and the corresponding series diverges. Then, by our assumption, the set \(WF(\psi_\tau) \subset \Omega_f \times [0, 1]\) has positive measure, and in particular there is some positive-measure subset \(\Omega \times \mathcal{E} \subset \Omega_f \times [0, 1]\) such that \(x \in WF(\psi_\tau)\) for every \(x \in \mathcal{E}, \mathcal{P} \in \Omega\). But by Lemma 8, we actually have \(x \in WF(\psi_\tau)\) for almost every \(\mathcal{P} \in \Omega_f\). Next, by Lemma 9 and then Lemma 8, for any \(k \in \mathbb{N}\) and \(x \in k\mathcal{E}_{\text{mod 1}}\), we have \(x \in WF(k\psi_\tau)\) for almost every \(k\mathcal{P} \in \Omega_f\). But notice that \(WF(k\psi_\tau) \subset WF(\psi)\): if we know that \(|nx - a| < k\psi_\tau(n)|\) for infinitely many \(n \in \mathbb{N}\) and \(a \in \mathcal{P}_n\), then we can certainly guarantee that \(|nx - a| < \psi(n)|\) for infinitely many \(n \in \mathbb{N}\), \(a \in \mathcal{P}_n\), just by taking \(n\) large enough that \(k\psi_\tau(n) \leq \psi(n)\). So we have shown that every \(x \in \bigcup_{k \in \mathbb{N}} k\mathcal{E}_{\text{mod 1}}\) is in \(WF(\psi)\) for almost every \(\mathcal{P} \in \Omega_f\). Finally, since \(|\mathcal{E}| > 0\), we know that the union \(\bigcup_{k \in \mathbb{N}} k\mathcal{E}_{\text{mod 1}}\) of all its integer dilations has full measure, so we have shown that almost every \(x \in [0, 1]\) is in \(WF(\psi)\) for almost every \(\mathcal{P} \in \Omega_f\). This implies that \(WF(\psi) \subset \Omega_f \times [0, 1]\) has full measure. \(\square\)

Lemmas 11–16 will be used to prove that \(WF(\psi)\) has positive measure. The approach is classical.

The following lemma is a standard fact about the Euler \(\varphi\)-function.

**Lemma 11.** There is some \(C > 0\) such that for any non-increasing sequence \(g(n)\) of positive real numbers,

\[
\sum_{n \leq N} \frac{\varphi(n)g(n)}{n} \geq C \sum_{n \leq N} g(n)
\]

as \(N \to \infty\).

**Lemma 12.** Let \(A \subset \{1, \ldots, n\}\) be a fixed subset of cardinality \(a \leq n\). Then the expected size of the intersection of \(A\) with another subset \(B \subset \{1, \ldots, n\}\) of size \(b \leq n\) is \(ab/n\).

**Proof.** Label the elements \(B = \{\beta_1, \ldots, \beta_b\}\), and let \(X_i\) be the random variable taking the value 1 if \(\beta_i \in A\), and 0 otherwise. We must show that \(\mathbb{E}(X_1 + \cdots + X_b) = ab/n\). But \(\mathbb{E}(X_1 + \cdots + X_b) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_b)\), and \(\mathbb{E}(X_i) = a/n\) for every \(i\), so the result follows. \(\square\)

**Lemma 13.**

\[
\|\Gamma_N\|_{L^1(\Omega_f \times [0, 1])} \geq C \psi(N) \quad \text{as} \quad N \to \infty.
\]
Proof. We simply calculate

\[
\int_{\Omega_f} \int_{[0,1]} \Gamma_N(P, x) \, dx \, d\mu(P) = \int_{\Omega_f} \sum_{n \leq N} \int_{[0,1]} \gamma_n^P(x) \, dx \, d\mu(P) \\
= \int_{\Omega_f} \sum_{n \leq N} \sum_{a \in \mathcal{P}_n'} \int_R 1_n \left(x - \frac{a}{n}\right) \, dx \, d\mu(P) \\
= \int_{\Omega_f} \sum_{n \leq N} |\mathcal{P}_n'| \frac{2\psi(n)}{n} \, d\mu(P) \\
\overset{\text{Lem. 12}}{=} \sum_{n \leq N} \varphi(n) \frac{2f(n)\psi(n)}{n} \geq C\Psi(N)
\]

as \( N \to \infty \), by Lemma 11.

Lemma 13 implies that for the “average” \( P \) in \( \Omega_f \), and for a fixed large \( N \), there is some positive-measure set of \( x \)'s where \( \Gamma_N(P, x) \geq C\Psi(N) \), where \( C > 0 \) is the implied constant in the lemma. If we knew what the measure of that set was, and if this measure did not shrink too badly with \( N \to \infty \), then we might find that the “lim sup” of these sets had positive measure, and since \( \Psi(N) \to \infty \) as \( N \to \infty \), this would imply that \( |\mathcal{W}^P(\psi)| > 0 \) for the average \( P \in \Omega_f \). In the next few lemmas we do exactly this.

Lemma 14. Whenever \( m \neq n \),

\[
\int_{\Omega_f} \int_{[0,1]} \gamma_m^P(x)\gamma_n^P(x) \, dx \, d\mu(P) \leq 4\frac{f(m)f(n)}{mn} \psi(m)\psi(n).
\]

Proof. We calculate

\[
\int_{\Omega_f} \int_{[0,1]} \gamma_m^P(x)\gamma_n^P(x) \, dx \, d\mu(P) = \int_{\Omega_f} \sum_{a \in \mathcal{P}_m', b \in \mathcal{P}_n'} \int_R 1_m \left(x - \frac{a}{m}\right) 1_n \left(x - \frac{b}{n}\right) \, dx \, d\mu(P) \\
= \frac{f(m)f(n)}{mn} \sum_{(a,b) \in [m'] \times [n']} \int_R 1_m \left(x - \frac{a}{m}\right) 1_n \left(x - \frac{b}{n}\right) \, dx \\
\overset{\text{Lem. 7}}{\leq} 4\frac{f(m)f(n)}{mn} \psi(m)\psi(n),
\]

proving the lemma.

Lemma 15.

\[
\|\Gamma_N\|_{L^2(\Omega_f \times [0,1])}^2 \leq 5\Psi(N)^2 \quad \text{for } N \text{ sufficiently large.}
\]
Proof. We have
\[
\int_{\Omega_f} \int_{[0,1]} \Gamma_N(\mathcal{P}, x)^2 \, dx \, d\mu(\mathcal{P}) = \int_{\Omega_f} \int_{[0,1]} \left( \sum_{n \leq N} \gamma_n^{\mathcal{P}}(x) \right)^2 \, dx \, d\mu(\mathcal{P})
\]
\[
= \int_{\Omega_f} \sum_{n \leq N} \int_{[0,1]} \gamma_n^{\mathcal{P}}(x)^2 \, dx \, d\mu(\mathcal{P})
\]
\[
+ \int_{\Omega_f} \sum_{m,n \leq N} \int_{[0,1]} \gamma_m^{\mathcal{P}}(x) \gamma_n^{\mathcal{P}}(x) \, dx \, d\mu(\mathcal{P})
\]
\[
\leq \sum_{n \leq N} 2f(n)\Psi(n) n + \sum_{m,n \leq N} 4f(m)f(n) mn \psi(m)\psi(n)
\]
\[
\leq 2\Psi(N) + 4\Psi(N)^2 \leq 5\Psi(N)^2
\]
for large enough N, since \(\Psi(N) \to \infty\).

\[\square\]

Lemma 16 (Paley–Zygmund [PZ32]). Let \((X, m)\) be a probability space, and \(f : X \to \mathbb{R}_{\geq 0}\) a measurable function such that \(\|f\|_{L^1(X)} \geq C\|f\|_{L^2(X)}\) for some \(0 < C < 1\). Then for any \(c < C\),
\[
m \left\{ x \in X : f(x) \geq c\|f\|_{L^1(X)} \right\} \geq (C - c)^2.
\]

Proof of the divergence part of Theorem 1. By Lemma 10, it is enough to prove that \(\mathcal{W}^f(\psi) \subset \Omega_f \times [0, 1]\) has positive measure.

Lemmas 13 and 15 imply that
\[
\|\Gamma_N\|_{L^1(\Omega_f \times [0, 1])} \geq C \|\Gamma_N\|_{L^2(\Omega_f \times [0, 1])}
\]
for \(N\) sufficiently large, where \(C > 0\) is some universal constant (in fact, it is the same constant from Lemmas 11 and 13, divided by \(\sqrt{5}\)). By Lemma 16, for any \(0 < c < C\) the measure of the subset of \(\Omega_f \times [0, 1]\) where
\[
\Gamma_N(\mathcal{P}, x) \geq c \|\Gamma_N\|_{L^1(\Omega_f \times [0, 1])}
\]
is bounded below by \((C - c)^2\). Combining with Lemma 13, we have found that for every \(N\) sufficiently large, there is some set \(S_N \subset \Omega_f \times [0, 1]\) of measure at least \((C - c)^2\) such that
\[
\Gamma_N(\mathcal{P}, x) \geq \Psi(N)
\]
for every \((\mathcal{P}, x) \in S_N\). Notice that the constant \((C - c)^2\) does not depend on \(N\). Since \(\Omega_f \times [0, 1]\) has finite measure, the set \(S = \limsup_{N \to \infty} S_N\) must have positive measure. Notice that for every \((\mathcal{P}, x) \in S\), we have \(\Gamma_N(\mathcal{P}, x) \to \infty\). Therefore \(S \subset \mathcal{W}^f(\psi)\) and so \(\mathcal{W}^f(\psi)\) has positive measure and Lemma 10 implies that it has full measure, proving the theorem.

\[\square\]

6. Proof of Theorem 3

Clearly, if \(f(n) \gg n\), the Duffin–Schaeffer Counterexample will serve its purpose for Theorem 1 as well. In this section we investigate how much slower \(f(n)\) can grow. In a sense, finding counterexamples for all \(f(n) \gg n/\log \log n\) is already out of the question,
because \( \varphi(n) \gg n/\log \log n \) and we do not expect any counterexamples in this case. We also should not expect to find a counterexample every time there is a sequence of \( n \)'s on which \( f(n) \ll n/\log \log n \) fails, because this is true for \( \varphi \), on the sequence of primes. We can, however, prove:

- Theorem 3, which provides counterexamples whenever \( f(n) = n/o(\log \log n) \), and
- Corollary 6, which shows that there exist functions such that \( f(n) \ll n/\log \log n \) on a sequence of \( n \)'s, for which there are counterexamples.

These show that \( \varphi \) is in a sort of a “sweet spot” for this problem.

**Proof of Theorem 3.** Since

\[
\frac{f(n) \log \log n}{n} \to \infty,
\]

we can find some non-increasing sequence \( \tau(n) \to 0 \) such that

\[
f(n) \geq \frac{n}{\tau(n) \log \log n}
\]

for all \( n \) sufficiently large. Now, let \( \sum_j c_j \) be a convergent series of positive real numbers. Since \( \tau(n) \downarrow 0 \), we can find a strictly increasing sequence \( \{M_j\}_{j=0}^{\infty} \) of integers with \( M_0 = 0 \), such that

\[
\frac{c_j}{\tau((M_j - 1)!)} \gg 1
\]

holds for infinitely many \( j \geq 1 \).

Let us further stipulate that

\[
\text{(5)} \quad \text{any two consecutive terms in the sequence} \{M_j\} \text{ differ by at least 2.}
\]

Now set \( K_j = M_j! \) for \( j \geq 1 \), and let

\[
k_i^{(j)} = \frac{K_j}{i}
\]

for \( i = 1, \ldots, M_j \). Notice that

\[
\text{(6)} \quad M_j! = k_1^{(j)} > k_2^{(j)} > \cdots > k_{M_j}^{(j)} = (M_j - 1)!
\]

so that the \( k_i^{(j)} \) are pairwise distinct. (The case \((M_j - 1)! = M_{j-1}! \) is ruled out by stipulation (5).) Define

\[
\psi(k_i^{(j)}) = \frac{c_j k_i^{(j)}}{K_j},
\]

and \( \psi(k) = 0 \) for all \( k \notin \{k_i^{(j)}\} \). Then we have \( A_{k_i^{(j)}}(\psi) \subset A_{k_i^{(j)}}(\psi) \) for all \( j \geq 1 \) and \( i = 1, \ldots, M_j \). In particular,

\[
W(\psi) = \limsup_{n \to \infty} A_n(\psi) \subset \limsup_{j \to \infty} A_{k_i^{(j)}}(\psi).
\]

But \( \sum_j |A_{k_i^{(j)}}(\psi)| \leq \sum_j 2c_j < \infty \), so the Borel–Cantelli Lemma implies that \( |W(\psi)| = 0 \).

It is only left to show that

\[
\sum_{n \geq 1} \frac{f(n)\psi(n)}{n} = \infty.
\]
For some large $j_0$ we have
\[
\sum_{n \geq 1} \frac{f(n)\psi(n)}{n} \geq \sum_{j \geq j_0} \sum_{i=1}^{M_j} \frac{f(k_i^{(j)})c_jk_i^{(j)}}{k_i^{(j)}k_j} \geq \sum_{j \geq j_0} \sum_{i=1}^{M_j} \frac{c_jk_i^{(j)}}{\tau(k_i^{(j)})K_j \log\log k_i^{(j)}} \geq \sum_{j \geq j_0} \frac{c_j \log M_j}{\tau((M_j - 1)!) \log\log(M_j)!}.
\]

Notice that
\[
\frac{c_j \log M_j}{\tau((M_j - 1)!) \log\log(M_j)!} \sim \frac{c_j}{\tau((M_j - 1)!) \log\log(M_j)!} \quad (j \to \infty).
\]

Therefore, by (4), the sum (8) must diverge, and this proves the theorem. \qed

**Remark.** To see that this could not possibly work for $f = \varphi$, we compute
\[
\sum_{n \geq 1} \frac{\varphi(n)\psi(n)}{n} = \sum_{j \geq 1} \sum_{i=1}^{M_j} \frac{\varphi(k_i^{(j)})c_j}{k_i^{(j)}k_j} \leq \sum_{j \geq 1} \frac{c_j}{k_j} \sum_{k|k_i^{(j)}} \varphi(k) = \sum_{j \geq 1} c_j,
\]
which converges.

**Proof of Corollary 6.** We re-create the previous proof, starting with a convergent series $\sum_j c_j$ of positive real numbers. Let $\tau(n)$ be a decreasing function that converges to 0, and such that
\[
\tau(n) \geq \frac{1}{\log\log n}
\]
for all sufficiently large $n$. We find a strictly increasing sequence $\{M_j\}_{j \geq 0}$ of integers with $M_0 = 0$, such that
\[
\frac{c_j}{\tau((M_j - 1)!) \log\log(M_j)!} \gg 1 \quad \text{holds for infinitely many} \quad j \geq 1.
\]

We further stipulate that
\[
\text{any two consecutive terms in the sequence } \{M_j\} \text{ differ by at least 2.}
\]

We define $K_j, k_i^{(j)}, \psi$ as before. Again, for the same reasons as before, we have $|W(\psi)| = 0$. Now suppose $f$ is such that
\[
\begin{cases}
  f(n) \geq \frac{n}{\tau(n) \log\log n} & \text{if } n = k_i^{(j)} \text{ for some } i, j, \text{ and} \\
  f(n) \ll \frac{n}{\log\log n} & \text{otherwise}.
\end{cases}
\]

The previous proof will now show that $\sum_{n \geq 1} \frac{f(n)\psi(n)}{n}$ diverges. \qed
7. Questions

Theorem 3 tells us cases in which we cannot remove the monotonicity assumption from Theorem 1. But when can we remove it? Can we remove it if \( f(n) \equiv 1 \)? If \( f(n) \ll 1 \)? If \( f(n) \ll n / \log \log n \)?

It seems likely that any successful approach would work by showing that the resulting sets \( A_n^P \) almost surely exhibited some sort of independence. For example, we may try to find a way to apply the following lemma, quoted from [Har98].

**Lemma 17.** Let \((X, \lambda)\) be a probability space, and \( A_n \subset X \) a sequence of measurable subsets such that \( \sum_{n \in \mathbb{N}} \lambda(A_n) \) diverges. Then

\[
\lambda(\limsup_{n \to \infty} A_n) \geq \limsup_{N \to \infty} \left( \sum_{n \leq N} \lambda(A_n) \right)^2 \left( \sum_{m, n \leq N} \lambda(A_m \cap A_n) \right)^{-1}.
\]

**Remark.** The Duffin–Schaeffer Theorem is proved using this lemma.

We will be able to apply Lemma 17 to our problem if we can show that for almost every \( P \in \Omega_f \) there is some increasing sequence \( \{N_k := N^P_k\} \subset \mathbb{N} \) such that

\[
\sum_{m,n \leq N_k, m \neq n} |A_m^P \cap A_n^P| \leq C_P \sum_{m,n \leq N_k} |A_m^P| |A_n^P|
\]

for all \( k \in \mathbb{N} \), where \( C_P > 0 \) is some constant that may depend on \( P \). This is easier than trying to show that whenever \( m \neq n \) we have \( |A_m^P \cap A_n^P| \ll |A_m^P| |A_n^P| \) on average over \( \Omega_f \); for one, it might actually be true for some functions \( f \). Although it is obviously not true for all functions, as \( f(n) \equiv n \) and the functions in §6 attest. Then again, it seems almost obvious for \( f(n) \equiv 1 \), because there are so many opportunities for \( A_m^P \cap A_n^P = \emptyset \). Therefore, the problem becomes: what do we need to assume about \( f \) to be able to prove that for almost every \( P \in \Omega_f \), there exists some \( C_P > 0 \) such that (11) holds for infinitely many \( N \in \mathbb{N} \)?

We end with a tempting problem. If in Theorem 1 we make the additional assumption that

\[
\limsup_{N \to \infty} \left( \frac{\sum_{n=1}^{N} f(n) \psi(n)}{n} \right) \left( \sum_{n=1}^{N} \psi(n) \right)^{-1} > 0,
\]

can we then remove monotonicity? That is, can one prove a random fractions version of the Duffin–Schaeffer Theorem?

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