Optimal Teleportation Based on Bell Measurement

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January 18, 2022

Abstract

We study optimal teleportation based on the Bell measurements. An explicit expression for the quantum channel associated with the optimal teleportation with an arbitrary mixed state resource is presented. The optimal transmission fidelity of the corresponding quantum channel is calculated and shown to be related to the fully entangled fraction of the quantum resource, rather than the singlet fraction as in the standard teleportation protocol.

PACS: 03.67.Hk; 89.70.+c; 03.65.-w
AMS classification: 81P68; 68P30

Quantum teleportation protocols play an important role in quantum information processing. In terms of a classical communication channel and a quantum resource (a nonlocal entangled state like an EPR-pair of particles), the teleportation protocol gives ways to transmit an unknown quantum state from a sender traditionally named “Alice” to a receiver “Bob” who are spatially separated. These teleportation processes can be viewed

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as quantum channels. The nature of a quantum channel is determined by the particular protocol and the state used as a teleportation resource [1, 2, 3]. The standard teleportation protocol $T_0$ proposed in [1] uses Bell measurements and Pauli rotations. When the maximally entangled pure state $|\Phi\rangle = \frac{1}{\sqrt n} \sum_{i=0}^{n-1} |ii\rangle$ is used as the quantum resource, it provides an ideal noiseless quantum channel $\Lambda^{(|\Phi\rangle\langle\Phi|)}_{T_0}(\rho) = \rho$. However in realistic situation, instead of the pure maximally entangled states, Alice and Bob usually share a mixed entangled state due to the decoherence. Teleportation using mixed state as an entangled resource is, in general, equivalent to having a noisy quantum channel. Recently, an explicit expression for the output state of the quantum channel associated with the standard teleportation protocol $T_0$ with an arbitrary mixed state resource has been obtained [4, 5].

In this paper, we consider the following problem. Alice and Bob previously only share a pair of particles in an arbitrary mixed entangled state $\chi$. In order to teleport an unknown state to Bob, Alice first performs a joint Bell measurement on her particles (particle 1 and particle 2) and tell her result to Bob by the classical communication channel. Then Bob, instead of the Pauli rotation like in the standard teleportation protocol [1], tries his best to choose an particular unitary transformation which depends on the quantum resource $\chi$, so as to get the maximal transmission fidelity. We call our teleportation protocol the optimal teleportation based on the Bell measurement. We derive an explicit expression for the quantum channel associated with the optimal teleportation with an arbitrary mixed state resource. The transmission fidelity of the corresponding quantum channel is given in term of the fully entangled fraction of the quantum resource.

Let $\{|i\rangle, i = 0, \ldots, n-1\}$, $n < \infty$, be an orthogonal normalized basis of an $n$-dimensional Hilbert space $\mathcal{H}$. Any linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ can be represented by an $n \times n$-matrix as follow:

$$A(|i\rangle) = \sum_{j=0}^{n-1} A_{ij} |j\rangle, \quad A_{ij} \in \mathbb{C}$$

We shall only consider the following three-tensor Hilbert space: $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ where Alice has the first and the second Hilbert space, and the third one belongs to Bob. Let $h$ and $g$ be $n \times n$ matrices such that $h|j\rangle = |(j+1) \mod n\rangle$, $g|j\rangle = \omega^j |j\rangle$, with $\omega = \exp\left\{-\frac{2\pi i}{n}\right\}$. We can introduce $n^2$ linear-independent $n \times n$-matrices $U_{st} = h^t g^s$, which satisfy

$$U_{st} U_{s't'} = \omega^{s't'-ts'} U_{s't'} U_{st}, \quad tr(U_{st}) = n \delta_{s0} \delta_{t0}. \quad (1)$$

One can also check that $\{U_{st}\}$satisfy the condition of bases of the unitary operators in the
sense of [4], i.e.

\[
\begin{align*}
\text{tr} \left( U_{st} U_{s't'}^+ \right) &= n \delta_{tt'} \delta_{ss'}, \\
U_{st} U_{s't}^+ &= I_{n \times n},
\end{align*}
\]

where \( I_{n \times n} \) is the \( n \times n \) identity matrix. \( \{ U_{st} \} \) form a complete basis of \( n \times n \)-matrices, namely, for any \( n \times n \) matrix \( W \), \( W \) can be expressed as

\[
W = \frac{1}{n} \sum_{s,t} \text{tr}(U_{st}^+ W) U_{st}.
\]

From \( \{ U_{st} \} \), we can introduce the generalized Bell-states,

\[
| \Phi_{st} > = (1 \otimes U_{st}^*) | \Phi > = \frac{1}{\sqrt{n}} \sum_{i,j} (U_{st})_{ij}^* |ij>, \quad \text{and} \quad | \Phi_{00} > = | \Phi > ,
\]

\( | \Phi_{st} > \) are all maximally entangled states and form a complete orthogonal normalized basis of \( \mathcal{H} \otimes \mathcal{H} \) shared by Alice and Bob. For any state \( \chi \) shared by Alice and Bob, let us introduce the singlet fraction [5]:

\[
F(\chi) = \max \left\{ < \Phi | (1 \otimes U^+) \chi (1 \otimes U) | \Phi > \right\}, \quad \text{for all} \quad UU^+ = U^+ U = I_{n \times n}. \]

Since the group of unitary transformations in \( n \)-dimensions is compact, there exists an unitary matrix \( W_\chi \) such that

\[
F(\chi) = < \Phi | (1 \otimes W_{\chi}^+) \chi (1 \otimes W_\chi) | \Phi > .
\]

Suppose now Alice and Bob previously share a pair of particles in an arbitrary mixed entangled state \( \chi \). To transform an unknown state to Bob, Alice first performs a joint Bell measurement based on the generalized Bell-states Eq. (4) on her parties. According to the measurement results of Alice, Bob chooses particular unitary transformations \( \{ T_{st} \} \) to act on his particle.

**Theorem 1** The teleportation protocol defined by \( \{ T_{st} \} \), when used with an arbitrary mixed state with density matrix \( \chi \) as a resource, acts as a quantum channel

\[
\Lambda(\chi)(\{ T \})(\rho) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{s't'} > \left\{ \sum_{\gamma \beta} T_{\gamma \beta}^+ U_{\gamma \beta}^+ \rho U_{\gamma \beta} U_{s't'}^+ T_{\gamma \beta} \right\}.
\]
Proof. The proof can be given in two steps:

Step 1. Pure entangled state as a resource. Each entangled pure state $|\Psi>$ shared by Alice and Bob has the form

$$|\Psi> = \sum_{i,j=0}^{n-1} a_{ij} |ij>,$$

with $n = \sum_{i,j=0}^{n-1} |a_{ij}|^2 = 1$, $a_{ij} \in \mathbb{C}$ (8).

Let $A$ be the $n \times n$ matrix with elements $(A)_{ij} = a_{ij}$, $a_{ij} \in \mathbb{C}$. Suppose Alice wishes to teleport the unknown pure state $|\phi> = \sum_{i=1}^{n} \alpha_i |i>$. The initial state Alice and Bob have is then given by

$$|\phi> \otimes |\Psi> = \sum_{i,j,k=0}^{n-1} \alpha_i a_{jk} |ijk> \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}. (9)$$

To transform the state $|\phi>$ to Bob, Alice first performs a joint Bell measurement based on the generalized Bell-states Eq.(4) on her party. After her measurement with outcome in the state $|\Phi_{st}>$, Bob’s particle gets into an (unnormalized) state

$$|\phi> \rightarrow \frac{1}{\sqrt{n}} AU_{st}|\phi>.$$

Once Bob learns from Alice that she has obtained the result $st$, he performs on his previously entangled particle (particle 3) a unitary transformation $T_{st}$. Then the final state becomes $\frac{1}{\sqrt{n}} T_{st}^+ AU_{st} |\phi>$. In terms of the density matrix, the teleportation based on the unitary matrices $\{T_{st}\}$, with quantum resource being a pure state $|\Psi>$, is a quantum channel with the output

$$\Lambda(|\Psi><\Psi|)\{\{T\}\} (\rho) = \frac{1}{n} \sum_{st} T_{st}^+ AU_{st} \rho U_{st}^+ A T_{st}. (10)$$

Step 2. An arbitrary mixed entangled state as a resource. Let $\chi$ be a mixed state,

$$\chi = \sum_{\alpha} p_{\alpha} |\Psi_{\alpha}> <\Psi_{\alpha}|, \quad 0 \leq p_{\alpha} \leq 1 \quad \text{and} \quad \sum_{\alpha} p_{\alpha} = 1, \quad |\Psi_{\alpha}> = \sum_{i,j} a_{ij}^{(\alpha)} |ij>.$$

Applying the teleportation protocol $T$ with a mixed state $\chi$, the final state of Bob becomes

$$\Lambda(\chi)\{\{T\}\} (\rho) = \frac{1}{n} \sum_{s,t} \sum_{\alpha} p_{\alpha} T_{st}^+ A^{(\alpha)} U_{st} \rho U_{st}^+ (A^{(\alpha)})^+ T_{st}. (10)$$

Since each matrix $A^{(\alpha)}$ can be decomposed in the basis of $\{U_{st}\}$ by $(A^{(\alpha)})_{ij} = \sum_{s,t} a_{ij}^{(\alpha)} (U_{st})_{ij}$, (10) becomes

$$\Lambda(\chi)\{\{T\}\} (\rho) = \frac{1}{n} \sum_{s,t} \sum_{s',t'} \left( \sum_{\alpha} p_{\alpha} a_{s't'}^{(\alpha)} (a_{st}^{(\alpha)})^* \right) \sum_{\gamma,\beta} T_{\gamma\beta}^+ U_{\gamma\beta} U_{\gamma\beta}^+ \rho U_{\gamma\beta}^+ U_{s't'}^+ T_{\gamma\beta}. (10)$$
Using the definition of generalized Bell-states $\{|\Phi_{st}\rangle\}$ in Eq.(4), after a lengthy calculation, we arrive at

$$\sum_{\alpha} p_{\alpha} A_{st}^{(\alpha)} A_{s',t'}^{(\alpha)*} = < \Phi_{st} | \chi | \Phi_{s't'} > .$$

Substituting the above results into Eq.(10), one obtains Eq.(7). Using Eq.(2) and the identity

$$\sum_{s,t} U_{st}^+ A U_{st} = n \text{tr}(A) I_{n\times n}, \text{ for any } n \times n \text{ matrix } A,$$

the trace-preserving property of the quantum channel can be proved by

$$\text{tr} \left( \Lambda^{(\chi)}(\{T\}) (\rho) \right) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{s't'} > \left\{ \sum_{\gamma\beta} \text{tr} \left( U_{\gamma\beta} U_{s't'}^+ U_{st} U_{\gamma\beta} \rho \right) \right\}$$

$$= \frac{1}{n} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{s't'} > \text{tr} \left( U_{st} U_{s't'}^+ \right) \times \text{tr}(\rho)$$

$$= \sum_{s,t} < \Phi_{st} | \chi | \Phi_{st} > = \text{tr}(\chi) = 1.$$

\[\Box\]

The fidelity of the teleportation is given by

$$f(\chi) = \frac{1}{n+1} \sum_{\gamma\beta} < \Phi_{in} | \Lambda^{(\chi)}(\{T\}) (|\phi_{in}\rangle < |\phi_{in}\rangle) | \Phi > ,$$

averaged over all pure input states $\phi_{in}$.

In order to calculate the transmission fidelity Eq.(11), we need an irreducible $n$-dimensional representation of the unitary group $U(n)$, denoted by $G$. Let $U(g)$ be the unitary matrix representation of the element $g$ of $G$. Recalling Schur’s Lemma, one has the identity

$$\int_{G} dg \left( U^+(g) \otimes U^+(g) \right) \sigma \left( U(g) \otimes U(g) \right) = \alpha_1 I \otimes I + \alpha_2 P,$$

$$\alpha_1 = \frac{n^2 \text{tr}(\sigma) - n \text{tr}(\sigma P)}{n^2(n^2-1)}, \quad \alpha_2 = \frac{n^2 \text{tr}(\sigma P) - n \text{tr}(\sigma)}{n^2(n^2-1)},$$

for any operator $\sigma$ acting on the tensor space, where $P$ is the flip operator such that $P|ij\rangle = |ji\rangle$. The invariant (Haar) measure $dg$ on $G$ is normalized by $\int_{G} dg = 1$.

**Theorem 2** The transmission fidelity of the teleportation protocol defined by $\{T_{st}\}$ with arbitrary mixed state $\chi$ as a resource is given by

$$f(\chi) = \frac{1}{n(n+1)} \sum_{\gamma\beta} < \Phi | \left( 1 \otimes (T_{\gamma\beta} U_{\gamma\beta}^+) \right) \chi \left( 1 \otimes T_{\gamma\beta} U_{\gamma\beta}^+ \right) | \Phi > + \frac{1}{n+1}.$$
Proof. From Theorem 1 and Eq. (12), one has

\[
f(\chi) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{st'} > \sum_{\gamma\beta} < \phi_{in}| T_{\gamma\beta} U_{st} U_{\gamma\beta}\phi_{in} > < \phi_{in}| U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}\phi_{in} >
\]

or

\[
f(\chi) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{st'} > \sum_{\gamma\beta} < \phi_{in}| \otimes < \phi_{in}| (T_{\gamma\beta} U_{st} U_{\gamma\beta} \otimes U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta})| \phi_{in} > \otimes |\phi_{in} >
\]

or

\[
f(\chi) = \frac{1}{n(n+1)} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{st'} > \sum_{\gamma\beta} \left\{ tr \left( (T_{\gamma\beta} U_{st} U_{\gamma\beta} \otimes U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}) (U(g) \otimes U(g))| 00 > \right) \right.
\]

\[
\times \left( (T_{\gamma\beta} U_{st} U_{\gamma\beta} \otimes U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}) (U(g) \otimes U(g))| 00 > \right)
\]

or

\[
= \frac{1}{n(n+1)} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{st'} > \sum_{\gamma\beta} \left\{ tr \left( (T_{\gamma\beta} U_{st} U_{\gamma\beta}) \otimes (U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}) \right) \right.
\]

\[
\times \left( (T_{\gamma\beta} U_{st} U_{\gamma\beta} \otimes U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}) (U(g) \otimes U(g))| 00 > \right)
\]

or

\[
= \frac{1}{n(n+1)} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{st'} > \sum_{\gamma\beta} \left\{ tr \left( (T_{\gamma\beta} U_{st} U_{\gamma\beta}) \otimes (U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}) \right) \right.
\]

\[
\times \left( (T_{\gamma\beta} U_{st} U_{\gamma\beta} \otimes U_{\gamma\beta}^+ U_{st'}^+ T_{\gamma\beta}) (U(g) \otimes U(g))| 00 > \right)
\]

where the identity \( tr_{12} ((A \otimes B)P) = tr(AB), \) Eq. (12) and Eq. (13) have been used. □

Obviously when the term \( < \Phi | \left( 1 \otimes (T_{\gamma\beta} U_{\gamma\beta}^+)^1 \right) \chi \left( 1 \otimes (T_{\gamma\beta} U_{\gamma\beta}^+)^1 \right) | \Phi > \) is maximized, i.e., \( T_{\gamma\beta} U_{\gamma\beta}^+ = W_{\chi}, \) one gets the maximal fidelity. Recalling the definition of the fully entangled fraction Eq. (8) and Eq. (9), we arrive at our main result:

**Theorem 3** The optimal teleportation based on the Bell measurements, when used with an arbitrary mixed state with density matrix \( \chi \) as a resource, acts as a general trace-preserving quantum channel

\[
A^{(\chi)} (\rho) = \frac{1}{n^2} \sum_{s,t} \sum_{s',t'} < \Phi_{st} | \chi | \Phi_{st'} > \left\{ \sum_{\gamma\beta} U_{\gamma\beta}^+ W_{\chi}^+ U_{st} U_{\gamma\beta} \rho U_{\gamma\beta}^+ U_{st'}^+ W_{\chi} U_{\gamma\beta} \right\}.
\]

The corresponding transmission fidelity is given by

\[
f_{\text{max}}(\chi) = \frac{n F(\chi)}{n + 1} + \frac{1}{n + 1}.
\]

where \( F(\chi) \) is the fully entangled fraction Eq. (8) and \( W_{\chi} \) is the unitary matrix which fulfills such a fully entangled fraction Eq. (14).

Our results show that the maximally transmission fidelity of the teleportation based on the Bell measurement depends on the fully entangled fraction only, whereas that of a standard teleportation depends on the singlet fraction \( \tilde{\chi} \). Our result also agrees with the fidelity formula of the general optimal teleportation given by the Horodecki family \( \tilde{\chi} \).

Summarizing, we obtain the explicit expression of the output state of the optimal teleportation, with arbitrary mixed entangled state as resource, in terms of some noisy...
quantum channel. This allow us to calculate the transmission fidelity of the quantum channel. It is shown that the transmission fidelity depends only on the fully entangled fraction of the quantum resource shared by the sender and the receiver. The fidelity in our optimal teleportation protocol is in general greater than the one in standard teleportation protocol [1, 4, 5].

Acknowledgments. W.-L. Yang would like to thank Prof. von Gehlen for his continuous encouragements. He has been supported by the Alexander-von-Humboldt Foundation.

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