INVARIANT RICCI-FLAT KÄHLER METRICS ON TANGENT
BUNDLES OF COMPACT SYMMETRIC SPACES

P. M. GADEA, J. C. GONZÁLEZ-DÁVILA, AND I. V. MYKYTYUK

Abstract. We give a description of all $G$-invariant Ricci-flat Kähler metrics on the canonical complexification of any compact Riemannian symmetric space $G/K$ of arbitrary rank, by using some special local $(1,0)$ vector fields on $T(G/K)$. As the simplest application, we obtain the explicit description of the set of all complete SO(3)-invariant Ricci-flat Kähler metrics on $TS^2$, which includes the well-known Eguchi-Hanson-Stenzel metrics and a new one-parameter family of metrics.

1. Introduction

As it is well known, the existence of Ricci-flat Kähler metrics on either compact or non-compact Kähler manifolds is very different. Given a compact Kähler manifold whose first Chern class is zero, by Yau’s solution of Calabi’s conjecture, there is a unique Ricci-flat Kähler metric in the original Kähler class. If the Kähler manifold is not compact, the situation is completely different, and it could in principle admit many of such metrics, even complete metrics. There is not for now a general existence theorem for Ricci-flat Kähler metrics on non-compact Kähler manifolds.

Over the latest decades there has been considerable interest in Ricci-flat Kähler metrics whose underlying manifold is diffeomorphic to the tangent bundle $T(G/K)$ of a rank-one compact Riemannian symmetric space $G/K$. For instance, a remarkable class of Ricci-flat Kähler manifolds of cohomogeneity one was discovered by M. Stenzel [1]. This has originated an extensive series of papers. To cite but a few: M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope [2] studied certain harmonic forms on these manifolds and found an explicit formula for the Stenzel metrics in terms of hypergeometric functions. Earlier, T. C. Lee [3] gave an explicit formula of the Stenzel metrics for classical spaces $G/K$ but in another vein, using the approach of G. Patrizio and P. Wong [4]. J. M. Baptista [5] used the Stenzel metrics on $SL(2, \mathbb{C}) \cong T(SU(2))$ for holomorphic quantization of the classical symmetries of the metrics. A.S. Dancer and I.A.B. Strachan [6] gave a much more elementary and concrete treatment in the case that $G/K$ is the round sphere $\mathbb{S}^n = SO(n + 1)/SO(n)$, exploiting the fact that the Stenzel metrics

1991 Mathematics Subject Classification. 53C30, 53C35, 53C55,
Key words and phrases. Invariant Ricci-flat Kähler structures, compact Riemannian symmetric spaces, restricted roots.
Research supported by the Ministry of Economy, Industry and Competitiveness, Spain, under Project MTM2016-77093-P.
are of cohomogeneity one with respect to the natural action of the Lie group $G$ on $T(G/K)$. Remark also that in the case of the standard sphere $S^2$, the Stenzel metrics coincide with the well-known Eguchi-Hanson metrics \[7\].

The natural question arises on a construction of $G$-invariant Ricci-flat Kähler metrics (all metrics in as many cases as possible) on the tangent bundles of compact Riemannian symmetric spaces $G/K$ of any rank or, equivalently, on the complexification $G^C/K^C$ (for the latter spaces, see G. D. Mostow \[8\] and R. Bielawski \[11\]). These results are non-constructive in nature and rely on non-linear analysis. At this moment, explicit expressions for such metrics have been found only when $G/K$ is an Hermitian symmetric space (see O. Biquard and P. Gauduchon \[12, 13\], where these metrics are hyper-Kählerian, thus automatically Ricci-flat). Note that for the simplest case, $G/K = \mathbb{C}P^n$, there is also an explicit formula for these metrics by E. Calabi \[14\] giving the Kähler form of $T(G/K)$ as the sum of the pull-back of the Kähler form on $\mathbb{C}P^n$ and a term given by an explicit potential.

In the present paper we obtain a description, reached in our main theorem (Theorem 5.1) of such metrics for compact Riemannian symmetric spaces of any rank, as we outline with some more detail in the next paragraph.

Let $G/K$ be a homogeneous manifold, $G$ being a connected, compact Lie group. The tangent bundle $T(G/K)$ has a canonical complex structure $J^K_c$ coming from the $G$-equivariant diffeomorphism $T(G/K) \to G^C/K^C$. The latter space is the complexification of $G/K$ mentioned above. Our approach is based on the explicit algebraic description of some special local $(1,0)$ vector fields defined on an open subset of $T(G/K)$ (see Lemma 3.5). These vector fields determine, for each $G$-invariant Kähler metric $g$ on $T(G/K)$ associated to $J^K_c$, a $G$-invariant function $S: T(G/K) \to \mathbb{C}$ so that the Ricci form $\text{Ric}(g)$ of $g$ can be expressed (Proposition 3.6) as $\text{Ric}(g) = -i \partial \bar{\partial} \ln S$.

Then, using the root theory of symmetric spaces, we can describe, for $G/K$ being any Riemannian symmetric space of compact type, all $G$-invariant Kähler and Ricci-flat Kähler structures $(g, J^K_c)$ which moreover are Ricci-flat on an open dense subset $T^+(G/K)$ of $T(G/K)$. Here, $T^+(G/K)$ is the image of $G/H \times W^+$ under a certain $G$-equivariant diffeomorphism, where $W^+$ is some Weyl chamber and $H$ denotes the centralizer of a (regular) element of $W^+$ in $K$. Such $G$-invariant Kähler and Ricci-flat Kähler structures are determined uniquely by a vector-function $a: W^+ \to \mathfrak{g}_H$ satisfying certain conditions (Theorem 5.1), $\mathfrak{g}_H$ being the subalgebra of $\text{Ad}(H)$-fixed points of the Lie algebra of $G$.

We also give (in Section 6) its simplest application; namely, we describe, in terms of our vector-functions $a: \mathbb{R}^+ \to \mathfrak{so}(3)$, the set of all $G$-invariant Ricci-flat Kähler metrics $(g, J^K_c)$ on the punctured tangent bundle $T^+S^2 = TS^2\setminus\{\text{zero section}\}$ of $S^2 = \text{SO}(3)/\text{SO}(2)$ and, among them, those that extend to smooth complete metrics on the whole tangent bundle. This family of $\text{SO}(3)$-invariant Ricci-flat Kähler metrics on $TS^2$ includes the well-known $\partial \bar{\partial}$-exact Eguchi-Hanson (hyper-Kählerian) metrics \[7\] reopened by M. Stenzel \[1\] and a new family of metrics which are not $\partial \bar{\partial}$-exact.
In our next paper [15] we give an explicit description, by using the technique of this article and the main Theorem 5.1, of the set of all $G$-invariant Ricci-flat Kähler metrics $(g, J^K_c)$ on $T(G/K)$, where $G/K$ is a compact rank-one symmetric space. It is also shown that this set contains a new family of metrics which are not $\partial\bar{\partial}$-exact if $G/K \in \{\mathbb{C}P^n, n \geq 1\}$, and coincides with the set of Stenzel metrics for any of the latter spaces $G/K$.

2. Preliminaries

2.1. Invariant polarizations. Let $\tilde{M}$ be a smooth real manifold such that two real Lie groups $G$ and $K$ act on it and suppose that these actions commute and the action of $K$ on $\tilde{M}$ is free and proper. Then the orbit space $M = \tilde{M}/K$ is a well-defined smooth manifold and the projection mapping $\pi : \tilde{M} \to M$ is a principal $K$-bundle. Since the actions of $G$ and $K$ on $\tilde{M}$ commute, there exists a unique action of $G$ on $M$ such that the mapping $\pi$ is $G$-equivariant.

Let $\mathcal{K} \subset T\tilde{M}$ be the kernel of the tangent map $\tilde{\pi}_* : T\tilde{M} \to TM$. Then $\mathcal{K}$ is an involutive subbundle (of rank $\dim K$) of the tangent bundle $T\tilde{M}$. Since the actions of $G$ and $K$ commute, the subbundle $\mathcal{K}$ is $G$-invariant.

Suppose that $(M, \omega)$ is a (smooth) symplectic manifold with a $G$-invariant symplectic structure $\omega$. Let $J$ be a $G$-invariant almost complex structure on $M$ and let $F(J) \subset T^CM$ be its complex subbundle of $(1,0)$-vectors, that is, $\Gamma F(J) = \{Y - iJY, Y \in \Gamma(TM)\}$. The pair $(\omega, J)$ is a Kähler structure on $M$ if and only if the subbundle $F(J)$ is a positive-definite polarization, i.e. (a) the complex subbundle $F(J)$ of rank $\frac{1}{2} \dim_\mathbb{R} M$ is involutive; (b) $F(J) \cap \overline{F(J)} = \emptyset$; (c) $\omega(F(J), F(J)) = 0$ (it is Lagrangian); and (d) $i\omega_x(Z, Z) > 0$ for all $x \in M$, $Z \in F_x(J) \setminus \{0\}$ (see V. Guillemin and S. Sternberg [16 Lemma 4.3]).

In this case, the 2-form $\omega$ is invariant with respect to the automorphism $J$ of the real tangent bundle $TM$ and the bilinear form $g = g(\omega, J)$, where $g(Y, Z) \overset{\text{def}}{=} \omega(JY, Z)$, for all vector fields $Y, Z$ on $M$, is symmetric and positive-definite. It is clear that each positive-definite polarization $F$ on $(M, \omega)$ determines the Kähler structure $(g, \omega, J)$ with complex tensor $J$ such that $F = F(J)$ and $g = g(\omega, J)$.

Since $F$ is an involutive subbundle of $T^CM$, it is determined by the differential ideal $\mathcal{I}(F) \subset \Lambda T^CM$ (closed with respect to exterior differentiation). Then $\tilde{\pi}^*(\mathcal{I}(F)) \subset \Lambda T^C\tilde{M}$ is also a differential ideal and, consequently, its kernel $\mathcal{F}$ is an involutive subbundle of $T^C\tilde{M}$. We will denote $\mathcal{F}$ also by $\mathcal{F}$ (F). This subbundle is uniquely determined by two conditions: (1) $\dim_\mathbb{C} \mathcal{F} = \frac{1}{2} \dim_\mathbb{R} M + \dim K = \frac{1}{2}(\dim \tilde{M} + \dim K)$; (2) $\tilde{\pi}_*(\mathcal{F}) = F$. It is evident that $(\tilde{\pi}^*\omega)(\mathcal{F}, \mathcal{F}) = 0$ and the subbundle $\mathcal{F}$ contains $\mathcal{K}$. Moreover, $\mathcal{F}$ is $K$-invariant.

We can substantially simplify matters by working on the manifold $\tilde{M}$ with the subbundle $\mathcal{F}$ rather than on the manifold $M$ with the polarization $F$.

Lemma 2.1. Let $\tilde{M}$ be a manifold with two commuting actions of the Lie groups $G$ and $K$. Suppose that the action of $K$ on $\tilde{M}$ is free and proper and let $\tilde{\pi} : \tilde{M} \to M$, where $M = \tilde{M}/K$, be the corresponding $G$-equivariant projection. Let $\omega$ be
a $G$-invariant symplectic structure on $M$. Let $\mathcal{F}$ be a $G$-invariant involutive complex subbundle of $T^*\tilde{M}$ such that

1. $\mathcal{F}$ is $K$-invariant;
2. $\mathcal{K}^C = \mathcal{F} \cap \mathcal{F}$;
3. $\dim_{\mathbb{C}} \mathcal{F} = \frac{1}{2} \dim_{\mathbb{R}} M + \dim K$;
4. $(\tilde{\pi}^*\omega)(\mathcal{F}, \mathcal{F}) = 0$;
5. $i(\tilde{\pi}^*\omega)_{\tilde{x}}(Z, \overline{Z}) > 0$ for all $\tilde{x} \in \tilde{M}$, $Z \in \mathcal{F}|_{\tilde{x}} \setminus \mathcal{K}^C|_{\tilde{x}}$.

Then $F = \tilde{\pi}_*(\mathcal{F})$ is a positive-definite polarization on $(M, \omega)$, i.e., there exists a unique Kähler structure $(g, \omega, J)$ on $M$ such that $F = F(J)$ and $g = g(\omega, J)$.

Conversely, any positive-definite $G$-invariant polarization $F$ on $(M, \omega)$ determines a unique $G$-invariant involutive subbundle $\mathcal{F} = \tilde{\pi}_*^{-1}(F)$ with properties (1)-(5).

Proof. The proof coincides up to some simple changes with that of Mykytyuk [17, Lemma 3]. Since $\mathcal{F}$ is $K$-invariant and the kernel $\mathcal{K}$ of $\tilde{\pi}_*$ is contained in $\mathcal{F}$, the image $F = \tilde{\pi}_*(\mathcal{F})$ of $\mathcal{F}$ is a well-defined subbundle of $T^*M$ of rank $\frac{1}{2} \dim_{\mathbb{R}} M$. We have $F \cap C = 0$ because $\mathcal{K}^C = \mathcal{F} \cap \mathcal{F}$. It then immediately follows from (4) that the subbundle $F$ is Lagrangian. To prove the smoothness and involutiveness of $F$ we notice that $\tilde{\pi}$ is a submersion, i.e. for any point $\tilde{x} \in \tilde{M}$ there exists a convex neighborhood $\tilde{U}$ of $\tilde{x}$, coordinates $x_1, \ldots, x_N$ on $\tilde{U}$ and coordinates $x_1, \ldots, x_N$ on the open subset $U = \tilde{\pi}(\tilde{U})$ such that $x_j(\tilde{x}) = 0$, $j = 1, \ldots, N$, and in these coordinates $\tilde{\pi}|_U$ is of type $\pi : (x_1, x_2, \ldots, x_N) \mapsto (x_1, x_2, \ldots, x_N)$. We can choose $\tilde{U}$ such that $(x_1, \ldots, x_N, 0, \ldots, 0) \in \tilde{U}$ if $(x_1, \ldots, x_N) \in \tilde{U}$. Let $Y(x_1, \ldots, x_N) = \sum_{j=1}^N a_j(x_1, \ldots, x_N) \partial/\partial x_j$ be any section of $\mathcal{F}|_U$. The subbundle $\mathcal{K}$ is spanned on $\tilde{U}$ by $\partial/\partial x_j$, $j = N + 1, \ldots, \tilde{N}$, and $\mathcal{F}|_U$ is preserved by these $\partial/\partial x_j$. Therefore, the smooth vector field

$$Y_0(x_1, \ldots, x_N) = \sum_{j=1}^N a_j(x_1, \ldots, x_N, 0, \ldots, 0) \frac{\partial}{\partial x_j}$$

is also a section of $\mathcal{F}|_U$ ($\mathcal{F}$ is preserved by $\partial/\partial x_j$ if and only if $\mathcal{F}$ is preserved by the corresponding local one-parameter group [18, Prob. 2.56, p. 124]). Thus, $\tilde{\pi}_*(Y_0(x_1, \ldots, x_N)) = \sum_{j=1}^N a_j(x_1, \ldots, x_N, 0, \ldots, 0) \partial/\partial x_j$ is a smooth section of $F|_{\tilde{U}}$. The involutiveness of $F$ follows easily from (2.1). Now, it follows from (5) that $F$ is a positive-definite polarization. \qed

2.2. Invariant Ricci-flat Kähler metrics. Let $G$ be a Lie group acting on the manifold $M$. Let $J$ be some $G$-invariant complex structure on $M$. To substantially simplify matters we will work on $M$ with the $G$-invariant Kähler (symplectic) form $\omega$ rather than with the metric $g$:

$$g(X, Y) = \omega(JX, Y), \quad \omega(X, Y) = -g(JX, Y), \quad \forall X, Y \in \Gamma(TM).$$

Let $\dim M = 2n$ and $z_1, \ldots, z_n$ be some local complex coordinates on $(M, J)$. Then $\omega = \sum_{1 \leq j, s \leq n} w_{js} dz_j \wedge d\overline{z}_s$. In particular, $w_{js} = \omega(\partial/\partial z_j, \partial/\partial \overline{z}_s)$ and $w_{js} = \ldots$
Remark 3.1. \( \overline{\mathfrak{m}_j} \) that is, the matrix \( (\omega_{js}) \) is skew-Hermitian. The Ricci form \( \text{Ric}(\mathfrak{g}) \) (corresponding to the Ricci curvature) of the metric \( \mathfrak{g} \) is the (global) form given in the local coordinates \( z_1, \ldots, z_n \) (see [19] Ch. IX, §5) by \( \text{Ric}(\mathfrak{g}) = -i\partial\bar{\partial}\ln \det(w_{js}) \). The right-hand side does not depend on the choice of local complex coordinates. The Ricci form \( \text{Ric}(\mathfrak{g}) \) is \( G \)-invariant because so are the complex structure \( J \) and the form \( \omega \).

Let \( X_1, \ldots, X_n \) be some linearly independent \( J \)-holomorphic vector fields defined on some open dense subset \( O \) of \( M \). Using these holomorphic (possibly non-global) vector fields \( X_j, j = 1, \ldots, n \), we can calculate the function \( \det(w_{js}) \) on the subset \( O \subset M \) up to multiplications by some holomorphic and some anti-holomorphic functions. Indeed, putting \( W_{js} = \omega(X_j, X_s) \), we obtain that locally \( \det(W_{js}) = h \cdot \det(w_{js}) \cdot \overline{\mathfrak{m}}_j \), where \( h \) is some non-vanishing local holomorphic function (specifically, some determinant). Thus

\[
\text{Ric}(\mathfrak{g}) = -i\partial\bar{\partial}\ln \det(W_{js}).
\]

3. The canonical complex structure on \( T(G/K) \)

Consider a homogeneous manifold \( G/K \), where \( G \) is a compact connected Lie group and \( K \) is some closed subgroup of \( G \). Let \( \mathfrak{g} \) and \( \mathfrak{k} \) be the Lie algebras of \( G \) and \( K \). There exists a positive-definite \( \text{Ad}(G) \)-invariant form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \).

Denote by \( \mathfrak{m} \) the \( \langle \cdot, \cdot \rangle \)-orthogonal complement to \( \mathfrak{k} \) in \( \mathfrak{g} \), that is, \( \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k} \) is the \( \text{Ad}(K) \)-invariant vector space direct sum decomposition of \( \mathfrak{g} \). Consider the trivial vector bundle \( G \times \mathfrak{m} \) with the two Lie group actions (which commute) on it: the left \( G \)-action, \( l_h: (g, w) \mapsto (hg, w) \) and the right \( K \)-action \( r_k: (g, w) \mapsto (gk, \text{Ad}_k^{-1} w) \). Let

\[
\pi: G \times \mathfrak{m} \to G \times_K \mathfrak{m}, \quad (g, w) \mapsto [(g, w)],
\]

be the natural projection for this right \( K \)-action. This projection is \( G \)-equivariant. It is well known that \( G \times_K \mathfrak{m} \) and \( T(G/K) \) are diffeomorphic. The corresponding \( G \)-equivariant diffeomorphism

\[
\phi: G \times_K \mathfrak{m} \to T(G/K), \quad [(g, w)] \mapsto \frac{d}{dt} \bigg|_{t=0} g \exp(tw)K,
\]

and the projection \( \pi \) determine the \( G \)-equivariant submersion \( \Pi = \phi \circ \pi: G \times \mathfrak{m} \to T(G/K) \). It is clear that there exists a sufficiently small neighborhood \( O_\mathfrak{m} \subset \mathfrak{m} \) of zero in \( \mathfrak{m} \) and an open subset \( O \subset T(G/K) \) containing the whole linear subspace \( T_0(G/K), o = \{K\} \), such that the restriction of the map \( \Pi \)

\[
\Pi|_{\exp(O_\mathfrak{m}) \times \mathfrak{m}} : \exp(O_\mathfrak{m}) \times \mathfrak{m} \to O
\]

is a diffeomorphism. We will use this special local section \( \exp(O_\mathfrak{m}) \times \mathfrak{m} \subset G \times \mathfrak{m} \) of the projection \( \Pi \) in our calculations below.

**Remark 3.1.** In the case when we consider simultaneously different homogeneous manifolds \( G/K \) with the same Lie group \( G \) we will denote the mappings \( \Pi, \pi \) and \( \phi \) by \( \Pi_K, \pi_K \) and \( \phi_K \), respectively, and the \( K \)-orbit \( [(g, w)]_K \) of the element \( (g, w) \in G \times \mathfrak{m} \) by \( [(g, w)]_K \).
Let $G^C$ and $K^C$ be the complexifications of the Lie groups $G$ and $K$. In particular, $K$ is a maximal compact subgroup of the Lie group $K^C$ and the intersection of $K$ with each connected component of $K^C$ is not empty (note that $G^C$, $K^C$, $G$ and $K$ are algebraic groups). Let $\mathfrak{g}^C = \mathfrak{g} \oplus i\mathfrak{g}$ and $\mathfrak{k}^C = \mathfrak{k} \oplus i\mathfrak{k}$ be the complexifications of the compact Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$.

We denote by $G^R$ and $K^R$ the groups $G^C$ and $K^C$, respectively, considered as real Lie groups. Denote by $\xi^l$, $\xi \in \mathfrak{g}$ (resp. $\xi^r$) the left (resp. right) $G^C$-invariant (holomorphic) vector fields on $G^C$. The natural (canonical) complex structure $J_c$ on $G^R = G^C$ is defined by the right $G^R$-invariant $(1,0)$ vector fields $\xi^r = \xi^c - i(I\xi)^r$, $\xi \in \mathfrak{g}$, where $I$ is a complex structure (in particular, $I$ can be taken as the multiplication by $i$) on $\mathfrak{g}^C$ and $\xi^r$ and $(I\xi)^r$ are the right $G^R$-invariant vector fields on the real Lie group $G^R$ obtained from $\xi$ and $I\xi$, respectively. In turn, we denote by $\xi^l$ the corresponding left $G^R$-invariant vector field on $G^R$. Note here that, when dealing with vector fields on the Lie group $G$, we will use the same notation, i.e. $\xi^l$, for the left $G$-invariant vector field on $G$ corresponding to $\xi \in \mathfrak{g}$.

Consider the complex homogeneous manifold $G^C/K^C$ and the canonical holomorphic projection $p_h : G^C \to G^C/K^C$. Since the vector field $\xi^l_h$ on $G^C$ is right $K^C$-invariant and $p_h$ is a holomorphic submersion, its image $p_h(\xi^l_h)$ is a well-defined holomorphic vector field on the complex manifold $G^C/K^C$.

Identifying $G^C/K^C$ naturally with the real homogeneous manifold $G^R/K^R$ we obtain on $G^R/K^R$ the canonical left $G^R$-invariant complex structure $J^K_c$. This structure is defined by the global $(1,0)$ vector fields $p_h(\xi^l_h) = p_*(\xi^r) - ip_*(I\xi)^r$, $\xi \in \mathfrak{g}$, where $p$ is the canonical projection $p : G^R \to G^R/K^R$.

Since $G$ and $K$ are maximal compact Lie subgroups of $G^C$ and $K^C$, respectively, by a result of Mostow [5, Theorem 4], we have that $K^C = K \exp(i\mathfrak{k})$, $G^C = G \exp(i\mathfrak{m}) \exp(i\mathfrak{t})$, and the mappings

$$
\begin{align*}
G \times \mathfrak{m} \times \mathfrak{t} & \to G^C, \quad (g, w, \zeta) \mapsto g \exp(iw) \exp(i\zeta), \\
K \times \mathfrak{t} & \to K^C, \quad (k, \zeta) \mapsto k \exp(i\zeta),
\end{align*}
$$

are diffeomorphisms. Then the map

$$
(3.4) \quad f^K_* : G^C/K^C \to G \times_K \mathfrak{m}, \quad g \exp(iw) \exp(i\zeta)K^C \mapsto [(g, w)],
$$

is a $G$-equivariant diffeomorphism [9, Lemma 4.1]. It is clear that

$$
(3.5) \quad f_K : G^C/K^C \to T(G/K), \quad g \exp(iw) \exp(i\zeta)K^C \mapsto \Pi(g, w),
$$

is also a $G$-equivariant diffeomorphism. The diffeomorphism $f_K$ supplies the manifold $T(G/K)$ with the $G$-invariant complex structure $J^K_c$. Moreover, this structure is determined by the set of global holomorphic vector fields $X^K_c, \xi \in \mathfrak{g}$, on $T(G/K)$ which are images of the holomorphic vector fields $p_*(\xi^r) - ip_*(I\xi)^r$, under the tangent map $f^K_*$. 

**Lemma 3.2.** Let $G/K$ be a homogeneous manifold, where $G$ is a connected compact Lie group and $K$ is a closed subgroup of $G$. Then for every $w \in \mathfrak{m}$ there
exists a unique pair \((B^m_w, B^\xi_w)\) of \(\mathbb{R}\)-linear mappings \(B^m_w: \mathfrak{g} \to \mathfrak{m}\) and \(B^\xi_w: \mathfrak{g} \to \mathfrak{k}\) such that

\[
(3.6) \quad \text{Id} = \frac{\sin \text{ad}_w}{\text{ad}_w} \circ B^m_w + (\cos \text{ad}_w) \circ B^\xi_w \quad \text{on the space } \mathfrak{g}.
\]

The operator-functions

\[
B^m: \mathfrak{m} \to \text{End}(\mathfrak{g}, \mathfrak{m}), \quad w \mapsto B^m_w, \quad \text{and} \quad B^\xi: \mathfrak{m} \to \text{End}(\mathfrak{g}, \mathfrak{k}), \quad w \mapsto B^\xi_w,
\]

are smooth on \(\mathfrak{m}\). The global holomorphic vector field \(X^\xi\) on \(T(G/K) = \Pi(G \times m)\), \(\xi \in \mathfrak{g}\), is the \(\Pi\)-image of the following global vector field \(X^\xi\) on \(G \times m\),

\[
(3.7) \quad X^\xi(g, w) = \left( \left( \xi' - i \frac{1 - \cos \text{ad}_w}{\text{ad}_w} \right)(B^m_w \xi') - i \frac{\sin \text{ad}_w}{\text{ad}_w} (B^\xi_w \xi') \right)(g), -i(B^m_w \xi'),
\]

where \(\xi' = \text{Ad}_{g^{-1}} \xi\) and \(T_{(g, w)}(G \times m)\) is identified naturally with the space \(T_g G \times m\).

If in addition the homogeneous manifold \(G/K\) is a symmetric space, we have the following exact solutions of \((3.6)\):

\[
(3.8) \quad B^m_w = \frac{\text{ad}_w}{\sin \text{ad}_w} \circ P_m, \quad B^\xi_w = \frac{1}{\cos \text{ad}_w} \circ P_t,
\]

where \(P_m: \mathfrak{g} \to \mathfrak{m}\) and \(P_t: \mathfrak{g} \to \mathfrak{k}\) are the natural projections determined by the splitting \(\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}\), and therefore \(X^\xi = \Pi_\xi(X^\xi)\), where

\[
X^\xi(g, w) = \left( \left( \xi' - i \frac{1 - \cos \text{ad}_w}{\text{ad}_w} \right)(P_m \xi') - i \frac{\sin \text{ad}_w}{\text{ad}_w} (P_t \xi') \right)(g), -i \frac{\text{ad}_w}{\text{ad}_w} (P_m \xi').
\]

Proof. To prove the lemma we calculate the components of the image of the right \(G^G\)-invariant vector field \(\xi' - i(I\xi)'\), \(\xi \in \mathfrak{g}\) on \(G^G\), under the diffeomorphism \((3.3)\) we obtain that

\[
(3.9) \quad \exp(t \varepsilon \xi)g \exp(iw) = gg_{\varepsilon}(t) \exp(i\varepsilon(t)) \exp(iK_{\varepsilon}(t)),
\]

where \(\varepsilon \in \{1, i\}\) (hereafter this means that either \(\varepsilon = 1\) or \(\varepsilon = i\) all times in the formula), \(g_{\varepsilon}(0) = e, v_{\varepsilon}(0) = w\) and \(g_{\varepsilon}(t) = 0 = g_{\varepsilon}(t), v_{\varepsilon}(t)\) and \(k_{\varepsilon}(t)\) are the (unique) smooth curves in \(G, m\) and \(k\), respectively. Hence, \(X^\xi(g, w) \in T^{C}_{(g, w)}(G \times m) \cong T^G_{g} G \times m^C\) is given by

\[
(3.10) \quad X^\xi(g, w) = \left( \left( g^{l}_{1}(0) - ig^{l}_{1}(0) \right)(g), v^{l}_{1}(0) - iv^{l}_{1}(0) \right).
\]

Moreover, transforming the curves \((3.9)\) to the curves

\[
\exp(-iw)g^{-1} \exp(t\varepsilon \xi)g \exp(iw)
\]

\[
= \left( \exp(-iw)g^{-1}g_{\varepsilon}(t) \exp iw \right) \left( \exp(-iw) \exp(i\varepsilon(t)) \right) \exp(iK_{\varepsilon}(t))
\]
through the identity in $G^C$ and calculating their tangent vectors at $e \in G^C$, we obtain the following equation in $g^C = g \oplus ig$ for the tangent vectors $g'_\varepsilon(0) \in g$, $v'_\varepsilon(0) \in m$, $k'_\varepsilon(0) \in \mathfrak{t}$:

$$e^{-i \text{ad} w} (e \text{Ad}_{g^{-1}} \xi) = e^{-i \text{ad} w} g'_\varepsilon(0) + \frac{1 - e^{-i \text{ad} w}}{i \text{ad} w} i v'_\varepsilon(0) + i k'_\varepsilon(0),$$

because $\left. \frac{d}{dt} \right|_{t=0} \exp(-X) \exp(X + i Y) = \frac{1 - e^{-\text{ad} X}}{\text{ad} X}(Y)$ (see [20, Ch. II, Theorem 1.7]). Since the map (3.3) is a diffeomorphism, there exists a unique solution $(g'_\varepsilon(0), v'_\varepsilon(0), k'_\varepsilon(0)) \in g \times m \times \mathfrak{t}$ of Equation (3.11) in $g^C$, for each $\varepsilon \in \{1, i\}$. If $\varepsilon = 1$, one directly gets that $(g'(0), v'(0), k'(0)) = (\text{Ad}_{g^{-1}} \xi, 0, 0)$. If $\varepsilon = i$, we obtain one equation for $g^C$:

$$i \text{Ad}_{g^{-1}} \xi = g'(0) + \frac{e^{i \text{ad} w} - 1}{\text{ad} w} (v'(0)) + i e^{i \text{ad} w} k'(0),$$

or two equations for $g$: \[
\begin{aligned}
\text{Ad}_{g^{-1}} \xi &= \frac{\sin \text{ad}_w}{\text{ad}_w} v'(0) + \cos \text{ad}_w k'(0), \\
0 &= g'(0) + \cos \text{ad}_w -\frac{1}{\text{ad}_w} v'(0) - \sin \text{ad}_w k'(0).
\end{aligned}
\]

As we remarked above, these equations possess a unique solution. It is easy to see that the first of them defines the operator-functions $B^m: m \to \text{End}(g, m)$, $w \mapsto B^m_w$, and $B^\mathfrak{t}: m \to \text{End}(g, \mathfrak{t})$, $w \mapsto B^\mathfrak{t}_w$, by $B^m_w(\text{Ad}_{g^{-1}} \xi) = v'(0)$ and $B^\mathfrak{t}_w(\text{Ad}_{g^{-1}} \xi) = k'(0)$. Since the mapping $G \times m \times \mathfrak{t} \to G^C$ in (3.3) is a diffeomorphism, these operator-functions are smooth functions on $m$.

To complete the proof of the first part of our lemma, we substitute in (3.10) the two triples $(g'_\varepsilon(0), v'_\varepsilon(0), k'_\varepsilon(0))$ for $\varepsilon \in \{1, i\}$ calculated above.

To prove the second part of the lemma it is sufficient to note that $g = m \oplus \mathfrak{t}$, $w \in m$, and in the symmetric case the subspaces $m$ and $\mathfrak{t}$ are invariant subspaces of the operators $(\text{ad}_w)^{2p}$, $p = 0, 1, 2, ...$

**Remark 3.3.** Since the map $m \to \text{End}(g)$, $w \mapsto \text{ad}_w$, is $\text{Ad}(K)$-equivariant, i.e. $\text{Ad}_k \circ \text{ad}_w \circ \text{Ad}_{k^{-1}} = \text{ad}_{\text{Ad}_k w}$, for all $k \in K$, $\text{Ad}(K)(m) = m$, $\text{Ad}(K)(\mathfrak{t}) = \mathfrak{t}$, from the uniqueness of the splitting (3.6) we obtain the $\text{Ad}(K)$-equivariance of the maps $w \mapsto B^m_w$ and $w \mapsto B^\mathfrak{t}_w$:

$$\text{Ad}_k \circ B^m_w \circ \text{Ad}_{k^{-1}} = B^m_{\text{Ad}_k w}, \quad \text{Ad}_k \circ B^\mathfrak{t}_w \circ \text{Ad}_{k^{-1}} = B^\mathfrak{t}_{\text{Ad}_k w}.$$  

**Remark 3.4.** Let $K_0 \subset K$ be the identity component of the group $K$. Then its complexification $K^C_0$ is also connected. According to Stenzel [11, Lemma 2], there exists a $G^C$-invariant non-vanishing holomorphic form $\Theta_h$ of maximal rank on the complex homogeneous space $G^C / K^C_0$; that is, the canonical bundle $\Lambda^{\langle n,0 \rangle}(G^C / K^C_0)$, $n = \dim G / K^C_0$, is holomorphically trivial. The existence of the form $\Theta_h$ relies on the fact that as a group of transformations of $m^C$ the group $\text{Ad}(K^C)|_{m^C}$ is a subgroup of the complex orthogonal group $O(m^C)$ but $\text{Ad}(K^C_0)|_{m^C} \subset \text{SO}(m^C)$. The $G$-equivariant diffeomorphism $f_{K_0}$ given in (3.5) endows the manifold $T(G / K_0)$ with the complex structure $J_{cK_0}$ and a $G$-invariant nowhere-vanishing $J_{cK_0}$-holomorphic $n$-form, which we denote also by $\Theta_h$. 
Fix some orthonormal basis $\xi_1, \ldots, \xi_n$ (with respect to the form $\langle \cdot, \cdot \rangle$) of the space $\mathfrak{m}$. By definition, the holomorphic vector fields $X^\xi_1, \ldots, X^\xi_n$ are linearly independent at the point $\Pi(e,0) \in T(G/K)$ and therefore they are linearly independent on some open dense subset of $T(G/K)$ (since the holomorphic vector fields $p_h^*(\xi_1)_h, \ldots, p_h^*(\xi_n)_h$ are linearly independent at the point $p_h(e) = \{K^C\} \in G^C/K^C$). However, these global $J^C$-holomorphic (specifically, $(1,0)$) vector fields on $T(G/K)$ are not $G$-invariant.

We will construct now certain global $G$-invariant smooth vector fields on $G \times \mathfrak{m}$, which in turn determine certain local $(1,0)$ vector fields on $T(G/K)$, the latter having the important property that the form $\Theta_h$ is a nonzero constant on them (see (3.14) below), whenever $K$ is connected. To this end, we consider the special local section $\exp(O_{\mathfrak{m}}) \times \mathfrak{m} \subset G \times \mathfrak{m}$ of the projection $\Pi$ and the corresponding open subset $O = \Pi(\exp(O_{\mathfrak{m}}) \times \mathfrak{m})$ of $T(G/K)$ (see (3.2)).

**Lemma 3.5.** We retain the notation of Lemma 3.2. The (complex) vector fields $Y^\xi, \xi \in \mathfrak{g}$, on the manifold $G \times \mathfrak{m}$ defined by

\[
Y^\xi(g, w) = \left( \frac{\cos \text{ad}_w - 1}{\text{ad}_w \cos \text{ad}_w} B_w^m(\sin \text{ad}_w \xi) + \frac{1}{\cos \text{ad}_w} \xi \right)^t(g, B_w^m(\cos \text{ad}_w \xi)) - i \left( \frac{\cos \text{ad}_w - 1}{\text{ad}_w \cos \text{ad}_w} B_w^m(\cos \text{ad}_w \xi) \right)^t(g, B_w^m(\cos \text{ad}_w \xi)),
\]

are smooth and $G$-invariant. The vector fields $Y^\xi_O$ on the open subset $O \subset T(G/K)$, $Y^\xi_O(\Pi(g, w)) = \Pi^*(g, w)Y^\xi(g, w)$, $(g, w) \in \exp(O_{\mathfrak{m}}) \times \mathfrak{m}$, are smooth $(1,0)$-vector fields. If the subgroup $K$ is connected we have

\[
\Theta_h(Y^\xi_O, \ldots, Y^\xi_n_O) = \text{const} \neq 0,
\]

where $\xi_1, \ldots, \xi_n$ is the given orthonormal basis of $\mathfrak{m}$.

If, in addition, the homogeneous manifold $G/K$ is a symmetric space ($K$ is not necessarily connected), then for each $\xi \in \mathfrak{m}$ we have

\[
Y^\xi(g, w) = \left( \frac{1}{\cos \text{ad}_w} \xi - i \frac{\cos \text{ad}_w - 1}{\sin \text{ad}_w} \xi \right)^t(g, -i \frac{\text{ad}_w \cos \text{ad}_w \xi}{\sin \text{ad}_w \xi}).
\]

**Proof.** Suppose that $K$ is connected and consider the canonical holomorphic projection $p_h : G^C \to G^C/K^C$. Since the form $\Theta_h$ on $G^C/K^C$ is $G^C$-invariant and holomorphic, its lift $p_h^*(\Theta_h)$ is also a $G^C$-invariant and holomorphic form on $G^C$. It is clear that

\[
p_h^*(\Theta_h)((\xi_1)_h, \ldots, (\xi_n)_h) = \text{const} \neq 0,
\]

where $(\xi_1)_h, \ldots, (\xi_n)_h$ are the left $G^C$-invariant (global holomorphic) vector fields on the complex Lie group $G^C$ corresponding to the vectors $\xi_1, \ldots, \xi_n \in \mathfrak{m}$.

But, as we remarked above, the group $G^C$ is diffeomorphic to $G \times \mathfrak{m} \times \mathfrak{k}$, $G^C = G \exp(\text{im}) \exp(\text{i}\mathfrak{k})$. Therefore for any $g \in G$, $w \in \mathfrak{m}$, $\xi \in \mathfrak{g}$, we have

\[
\xi_h(g \exp(\text{i}w)) = Y^\xi(g \exp(\text{i}w)) + \mathcal{A}^\xi(g \exp(\text{i}w)) \in T^C(G^R),
\]
where the first component is tangent to the (real) submanifold $G \exp(iw) \subset \mathbb{R}^k$ at $g \exp(iw)$ and the second one is tangent to the submanifold $g \exp(iw) \exp(it) \subset \mathbb{R}^k$ at $g \exp(iw)$. Since the image $p_h(g \exp(iw) \exp(it))$ is a one-point subset $p_h(g \exp(iw))$ of $G^C/K^C$, $p_h(A^C(g \exp iw)) = 0$ and, consequently,

$$p_h(\Theta_h)(Y^{\xi_1}, \ldots, Y^{\xi_n})|_{G \exp im} = p_h(\Theta_h)((\xi_1)_h, \ldots, (\xi_n)_h)|_{G \exp im} = \text{const} \neq 0.$$  

Taking into account that the restriction $(f_K \circ p_h)|_{\exp(O_m) \exp(im)}$ of the map (submersion) $f_K \circ p_h$ (see Definitions 3.2, 3.4 and 3.5),

$$G^C \overset{p_h}{\to} \frac{G^C}{K^C} \overset{f_K}{\to} T(G/K), \quad g \exp(iw) \exp(i\xi) \overset{p_h}{\mapsto} g \exp(iw) \exp(i\xi) \frac{K^C}{f_K} \Pi(g, w)$$

is a diffeomorphism onto the open subset $O \subset T(G/K)$, we obtain relation (3.14) — note that — for the vector fields $(g \exp(iw))$, $(\xi_1'(0))$ in (3.13) under the natural identification of the submanifold $G \exp(im) \subset \mathbb{R}^k$ with the manifold $G \times m$. By such identification, the vector field $Y^\xi$ on $G \times m$ is $G$-invariant and smooth.

Since the left $G^C$-invariant vector field $\xi'_h$, for $\xi \in g$, is given by $\xi'_1 = \xi'_1 - i(I_1 \xi)'_1$, $\xi \in g$, we have to calculate the components of the $G^C$-invariant vector fields $\xi'_1$ and $(I_1 \xi)'_1$, tangent to the submanifold $G \exp(im)$.

To this end it is sufficient to consider the two curves $g \exp(iw) \exp(t \xi)$ and $g \exp(iw) \exp(t i \xi)$, $t \in \mathbb{R}$, in the group $G^C = G^R$ through the point $g \exp(iw)$ with tangent vectors $\xi'(g \exp(iw))$, $(I_1 \xi)'(g \exp(iw))$, respectively. By (3.3),

$$g \exp(iw) \exp(t \varepsilon \xi) = gg_\varepsilon(t) \exp(i v_\varepsilon(t)) \exp(i k_\varepsilon(t)),$$

where $\varepsilon \in \{1, i\}$, $g_\varepsilon(0) = e$, $v_\varepsilon(0) = w$, $k_\varepsilon(0) = 0$ and $g_\varepsilon(t)$, $v_\varepsilon(t)$ and $k_\varepsilon(t)$ are smooth curves in $G$, $m$ and $\mathfrak{k}$, respectively. Hence, $Y^\xi(g, w) \in T_{(g, w)}^C(G \times m) \cong T_g^C G \times m^C$ is given by

$$Y^\xi(g, w) = ((g'_1(0) - ig'_1(0))'(g), v'_1(0) - iv'_1(0)).$$

From the equation

$$\exp(t \varepsilon \xi) = (\exp(-iw)g_\varepsilon(t) \exp(iw)) (\exp(-iw) \exp(i v_\varepsilon(t)) \exp(i k_\varepsilon(t))$$

in $G^C$ we obtain the following equation in $g^C = g \oplus ig$ for the tangent vectors $g'_\varepsilon(0) \in g$, $v'_\varepsilon(0) \in m$, $k'_\varepsilon(0) \in \mathfrak{k}$:

$$\varepsilon \xi = e^{-i \text{ad}_w} g'_\varepsilon(0) + \frac{1}{i \text{ad}_w} iv'_\varepsilon(0) + ik'_\varepsilon(0).$$

Since the map (3.3) is a diffeomorphism, there exists a unique solution $(g'(0), v'(0), k'(0)) = (g'_\varepsilon(0), v'_\varepsilon(0), k'_\varepsilon(0)) \in g \times m \times \mathfrak{k}$ of Equation (3.17) in $g^C$.

If $\varepsilon = 1$ we obtain one equation for $g^C$:

$$\xi = e^{-i \text{ad}_w} g'(0) + \frac{1}{i \text{ad}_w} v'(0) + ik'(0),$$
or two equations for $g$:

$$\xi = \cos \text{ad}_w g'(0) + \frac{1 - \cos \text{ad}_w}{\text{ad}_w} v'(0), \ 0 = - \sin \text{ad}_w g'(0) + \frac{\sin \text{ad}_w}{\text{ad}_w} v'(0) + k'(0),$$

which are equivalent to the pair of equations

$$\sin \text{ad}_w \xi = \frac{\sin \text{ad}_w}{\text{ad}_w} v'(0) + \cos \text{ad}_w k'(0), \ g'(0) = \frac{\cos \text{ad}_w - 1}{\text{ad}_w \cos \text{ad}_w} v'(0) + \frac{1}{\cos \text{ad}_w} \xi.$$

Thus,

$$v'_1(0) = B^m_w(\sin \text{ad}_w \xi), \ k'_1(0) = B^\ell_w(\sin \text{ad}_w \xi),$$

$$g'_1(0) = \frac{\cos \text{ad}_w - 1}{\text{ad}_w \cos \text{ad}_w} B^m_w(\sin \text{ad}_w \xi) + \frac{1}{\cos \text{ad}_w} \xi,$$

where the operator-functions $B^m_w: m \to \text{End}(g, m)$ and $B^\ell_w: m \to \text{End}(g, \ell)$, $w \mapsto B^\ell_w$, are determined in Lemma 3.2. Similarly,

$$v'_1(0) = B^m_w(\cos \text{ad}_w \xi), \ k'_1(0) = B^\ell_w(\cos \text{ad}_w \xi), \ g'_1(0) = \frac{\cos \text{ad}_w - 1}{\text{ad}_w \cos \text{ad}_w} B^m_w(\cos \text{ad}_w \xi).$$

Now, substituting the two triples $(g'_1(0), v'_1(0), k'_1(0))$, for $\varepsilon \in \{1, i\}$, in (3.16) we obtain expressions (3.13). Note here that because the mappings $w \mapsto \text{ad}_w$, $w \mapsto B^m_w$, and $w \mapsto B^\ell_w \ (w \in m)$, are $\text{Ad}(K)$-equivariant (see Remark 3.3 and (3.12)), for any $w \in m$, $\xi \in g$, $k \in K$, it follows that

$$\text{(3.18) if } Y^\xi(e, w) = (\eta, v) \in g \times m \text{ then } Y^{\text{Ad}_k \xi}(e, \text{Ad}_k w) = (\text{Ad}_k \eta, \text{Ad}_k v).$$

To prove the last part of the lemma it is sufficient to note that in the symmetric case the subspaces $m$ and $\ell$ are invariant subspaces of the operators $(\text{ad}_w)^{2p}$, $p = 0, 1, 2, \ldots$ and use (3.5).

Given a $G$-invariant Kähler structure $(g, \omega, J^K_c)$ on $T(G/K)$, where $J^K_c$ is its canonical complex structure, it follows from (2.2) that the Ricci form $\text{Ric}(g)$ is given by $\text{Ric}(g) = -i \partial \bar{\partial} \ln \det \left( \omega(X^\xi_h, X^\xi_h) \right)$. Since the vector fields $X^\xi_h$ in (3.7) are not $G$-invariant, the calculation of the function $\det(\omega(X^\xi_h, X^\xi_h))$ is not simple. To substantially simplify this calculation we will prove that this function is equal, up to a non-zero complex factor, to $S \cdot h$, where $S$ is some global $G$-invariant function and $h$ is locally expressed as $h = h \cdot \bar{\theta}$ for some local $J^K_c$-holomorphic function $h$ on $T(G/K)$. Specifically, we obtain the following result.

**Proposition 3.6.** Let $G$ be a compact Lie group. Let $g$ be a $G$-invariant Kähler metric on $T(G/K)$ associated with the canonical complex structure $J^K_c$ and let $\omega$ be its fundamental form. Then the function $\tilde{S}: G \times m \to \mathbb{C}$ given by

$$\text{(3.19) } \tilde{S}(g, w) = \det \left( \left( \Pi^\ast \omega \right)_{(g, w)}(Y^\xi_h, \overline{Y^\xi_h}) \right),$$

is left $G$-invariant and right $K$-invariant and therefore determines a unique $G$-invariant function $\tilde{S}: T(G/K) \to \mathbb{C}$ such that $\tilde{S} = \Pi^\ast \tilde{S}$. We have

$$\text{(3.20) } \text{Ric}(g) = -i \partial \bar{\partial} \ln \tilde{S}.$$
The function $\tilde{S}$ is $G$-invariant because so are the vector fields $\{Y^\xi\}$. Let us show that moreover $\tilde{S}$ is right $K$-invariant or, equivalently,

$$\det\left((\Pi^*\omega)(e,w)(Y^\xi,\overline{Y^\xi})\right) = \det\left((\Pi^*\omega)(e,\Ad_k w)(Y^\xi,\overline{Y^\xi})\right),$$  \hspace{1cm} (3.21)

for all $k \in K$. Indeed, since the form $\omega$ is $G$-invariant and the projection $\Pi: G \times m \to T(G/K)$ is an equivariant submersion with respect to the natural left $G$-actions on $G \times m$ and $T(G/K)$, it follows that $\Pi^*\omega$ is a left $G$-invariant and right $K$-invariant form on $G \times m$. Therefore, for any $g \in G$, $\xi_1, \xi_2 \in \mathfrak{g}$, $u_1, u_2 \in m$, $k \in K$, we have

$$\left((\Pi^*\omega)(e,w)((\xi_1, u_1), (\xi_2, u_2))\right) = \left((\Pi^*\omega)(e,\Ad_k w)((\Ad_k \xi_1, \Ad_k u_1), (\Ad_k \xi_2, \Ad_k u_2))\right),$$ \hspace{1cm} (3.22)

because (exp $t\xi \cdot k^{-1}$, $\Ad_k (w + tv)) = (k^{-1} \cdot \exp(\Ad_k \xi), \Ad_k (w + t \Ad_k v))$.

Putting $Y^\xi (e,w) = (\eta_j, v_j) \in T_e^G \times T_m^G$, we obtain that

$$\left((\Pi^*\omega)(e,w)(Y^\xi,\overline{Y^\xi})\right) \overset{3.22}{=} \left((\Pi^*\omega)(e,\Ad_k w)((\Ad_k \eta_j, \Ad_k v_j), (\Ad_k \eta_s, \Ad_k v_s))\right).$$ \hspace{1cm} (3.23)

Since by [3.13] the map $m \to \mathfrak{g}^C \times \mathfrak{m}^C$, $\xi \mapsto Y^\xi (e,w)$, is linear and the endomorphism $\Ad_k: m \to m$ is orthogonal (det $\Ad_k = \pm 1$), we obtain

$$\det\left((\Pi^*\omega)(e,w)(Y^\xi,\overline{Y^\xi})\right) \overset{3.22}{=} \det\left((\Pi^*\omega)(e,\Ad_k w)(Y^{\Ad_k \xi},\overline{Y^{\Ad_k \xi}})\right) = \det(\Ad_k)^t \cdot \det(\Ad_k) \cdot \det\left((\Pi^*\omega)(e,\Ad_k w)(Y^\xi,\overline{Y^\xi})\right).$$

This proves (3.21). Now it is easy to see that the function $S$ with $\tilde{S} = \Pi^*S$ is well defined. This function is smooth and $G$-invariant because the mapping $\Pi$ is a $G$-equivariant submersion.

In order to prove (3.20), suppose first that the subgroup $K \subset G$ is connected. Consider the holomorphic form $\Theta_h$ on $T(G/K)$. Then there exists a constant $\varepsilon_n \in \mathbb{C}$ such that $\varepsilon_n \Theta_h \wedge \overline{\Theta_h}$ is a volume form compatible with the orientation defined by the symplectic structure $\omega$ on $T(G/K)$. Thus there is a positive, real analytic function $S_1$ on $T(G/K)$ such that

$$\omega^n = S_1 \cdot \varepsilon_n \Theta_h \wedge \overline{\Theta_h},$$ \hspace{1cm} (3.24)

The function $S_1$ is $G$-invariant because so are the forms $\omega$ and $\Theta_h$.

On the other hand, putting $(e_1, \ldots, e_{2n}) = (X^\xi_1, \ldots, X^\xi_n, \overline{X^\xi_1}, \ldots, \overline{X^\xi_n})$ and using the fact that $\omega$, considered as a complex form, is of degree $(1, 1)$, we obtain, in particular, $\omega(X^\xi_h, X^\xi_h) = 0$, $\omega(X^\xi_h, \overline{X^\xi_h}) = 0$. Hence, we can deduce from
equation (3.24) that for some local holomorphic function $h_1$ on $T(G/K)$,
\[
S_1 \cdot \xi_n | h_1 |^2 = \omega^n(X_{h_1}^{\xi_1}, \ldots, X_{h_1}^{\xi_n}, \overline{X}_{h_1}^{\xi_1}, \ldots, \overline{X}_{h_1}^{\xi_n})
\]
\[
def = 2^{-n} \sum_{\sigma \in S_{2n}} \varepsilon(\sigma) \omega(e_{\sigma(1)}, e_{\sigma(2)}) \cdots \omega(e_{\sigma(2n-1)}, e_{\sigma(2n)})
\]
\[
= \sum_{\sigma \in S_n} \varepsilon(\sigma) \omega(X_{h_1}^{\xi_1}, X_{h_1}^{\xi_{\sigma(1)}}) \cdots \omega(X_{h_1}^{\xi_n}, X_{h_1}^{\xi_{\sigma(n)}}) \ast = \det\left(\omega(X_{h_1}^{\xi_j}, \overline{X}_{h_1}^{\xi_j})\right).
\]

Here $\ast$ means “equal up to a non-zero constant complex factor.”

Similarly, by Lemma 3.3 and since $\omega(Y_{\omega}^{\xi_j}, Y_{\omega}^{\xi_j}) = 0$, $\omega(\overline{Y}_{\omega}^{\xi_j}, \overline{Y}_{\omega}^{\xi_j}) = 0$, we have from (3.24) and (3.14) that $S_1$ is locally expressed as
\[
S_1 = \omega^n(Y_{\omega}^{\xi_1}, \ldots, Y_{\omega}^{\xi_n}, \overline{Y}_{\omega}^{\xi_1}, \ldots, \overline{Y}_{\omega}^{\xi_n}) \ast = \det\left(\omega(Y_{\omega}^{\xi_j}, \overline{Y}_{\omega}^{\xi_j})\right).
\]

Then, using Lemma 3.3 again we obtain that locally
\[
\Pi^* S_1 = \Pi^* \left(\det\left(\omega(Y_{\omega}^{\xi_j}, \overline{Y}_{\omega}^{\xi_j})\right)\right) = \det\left(\Pi^* \omega(Y_{\omega}^{\xi_j}, \overline{Y}_{\omega}^{\xi_j})\right) = \overline{S} = \Pi^* S.
\]

Hence, using that $S_1$ is $G$-invariant, $G \cdot O = T(G/K)$, we obtain that $S(x) \ast S_1(x)$ and from (3.25) we have
\[
\det\left(\omega(X_{h}^{\xi_j}, \overline{X}_{h}^{\xi_j})\right) = S \cdot c h,
\]
where $c \in \mathbb{C} \setminus \{0\}$ and $h$ is a function on $T(G/K)$ such that locally $h = |h_1|^2$. Thus (3.20) holds.

Finally, suppose that $K \neq K_0$, that is, $K$ is not connected.

Since below we will consider the manifolds $T(G/K_0)$ and $T(G/K)$ simultaneously, we will use the notation introduced above for objects on $T(G/K)$ but with indexes $K_0$ and $K$ respectively (if they exist). To complete the proof of the lemma, consider the natural covering map $\Psi : G^C/K_0^C \rightarrow G^C/K^C$. This map is holomorphic and $G^C$-equivariant. There exists a unique map $\psi : T(G/K_0) \rightarrow T(G/K)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
G^C/K_0^C & \xrightarrow{\Psi} & G^C/K^C \\
\downarrow f_{K_0}^C & & \downarrow f_{K}^C \\
G \times K_0 \ m & \xrightarrow{\phi_{K_0}} & G \times K \ m \\
\psi & & \phi_K \\
T(G/K_0) & \xrightarrow{\psi} & T(G/K) \\
\phi_{K_0}(\{g, w\}_K) & \rightarrow & \phi_K(\{g, w\}_K)
\end{array}
\]

By definition, the map $\psi$ is holomorphic and $G$-equivariant. Here the maps (diffeomorphisms) $\phi_K, \phi_{K_0}$ and $f_{K}^C, f_{K_0}^C$ are defined by (3.1) and (3.4).

Since the global vector fields $Y^{\xi_j} : \xi_j \in \mathfrak{m}$, in (3.14) on $G \times \mathfrak{m}$ are determined only in terms of the pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$, the function $\mathcal{S}$ in (3.19) is the same for the spaces $T(G/K)$ and $T(G/K_0)$, that is, $\mathcal{S} = \Pi^*_K \mathcal{S}_K = \Pi^*_K \mathcal{S}_K_0$. Thus $\mathcal{S}_{K_0} = \psi^* \mathcal{S}_K$. 

RICHIC-FAT FLAT KÄHLER STRUCTURES 13
Now, the map $\psi$ is a local holomorphic diffeomorphism, hence $\bar{\partial} \partial \ln S_{K_0} = \psi^*(\bar{\partial} \partial \ln S_K)$. Moreover, the form $\omega_{K_0} = \psi^* \omega_K$ ($\omega_K = \omega$) is the fundamental form of the Kähler metric $g_{K_0}$ on $T(G/K_0)$ associated with the canonical complex structure $J_{K_0}$. Since $\text{Ric}(g_{K_0}) = \psi^*(\text{Ric}(g_K))$ and as we proved above, $\text{Ric}(g_{K_0}) = -i \bar{\partial} \partial \ln S_{K_0}$, we obtain (3.20). □

4. Invariant Ricci-flat Kähler metrics on tangent bundles of compact Riemannian symmetric spaces

We continue with the previous notations but in this section and the following one it is assumed in addition that $G/K$ is a rank-$r$ Riemannian symmetric space of a connected, compact (possibly with nontrivial center) Lie group $G$.

4.1. Root theory of Riemannian symmetric spaces and reduced symmetric spaces of maximal rank. Here we will review a few facts about Riemannian symmetric spaces [20, Ch. VII, §2, §11]. We have

\begin{equation}
\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}, \quad \text{where} \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad \mathfrak{k} \perp \mathfrak{m}.
\end{equation}

In other words, there exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that

\begin{equation}
\mathfrak{k} = (1 + \sigma)\mathfrak{g}, \quad \text{and} \quad \mathfrak{m} = (1 - \sigma)\mathfrak{g}.
\end{equation}

Moreover the scalar product $\langle \cdot, \cdot \rangle$ is $\sigma$-invariant.

Let $\mathfrak{a} \subset \mathfrak{m}$ be some Cartan subspace of the space $\mathfrak{m}$. There exists a $\sigma$-invariant Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing the commutative subspace $\mathfrak{a}$, i.e.

\begin{equation}
\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{t}_0, \quad \text{where} \quad \mathfrak{a} = (1 - \sigma)\mathfrak{t}, \quad \mathfrak{t}_0 = (1 + \sigma)\mathfrak{t}.
\end{equation}

Then the complexification $\mathfrak{t}^\mathbb{C}$ is a Cartan subalgebra of the reductive complex Lie algebra $\mathfrak{g}^\mathbb{C}$ and we have the root space decomposition

\begin{equation}
\mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \sum_{\alpha \in \Delta} \tilde{\mathfrak{g}}_{\alpha},
\end{equation}

Here $\Delta$ is the root system of $\mathfrak{g}^\mathbb{C}$ with respect to the Cartan subalgebra $\mathfrak{t}^\mathbb{C}$. For each $\alpha \in \Delta$ we have

\begin{equation}
\tilde{\mathfrak{g}}_{\alpha} = \{ \xi \in \mathfrak{g}^\mathbb{C} : \text{ad}_{\tilde{t}}\xi = \alpha(\tilde{t})\xi, \ \tilde{t} \in \mathfrak{t}^\mathbb{C} \} \quad \text{and} \quad \dim_{\mathbb{C}} \tilde{\mathfrak{g}}_{\alpha} = 1.
\end{equation}

It is evident that the centralizer $\tilde{\mathfrak{g}}_0$ of the space $\mathfrak{a}^\mathbb{C}$ in $\mathfrak{g}^\mathbb{C}$ is the subalgebra

\begin{equation}
\tilde{\mathfrak{g}}_0 = \mathfrak{t}^\mathbb{C} \oplus \sum_{\alpha \in \Delta_0} \tilde{\mathfrak{g}}_{\alpha},
\end{equation}

where $\Delta_0 = \{ \alpha \in \Delta : |\alpha|_{\mathfrak{a}^\mathbb{C}} = 0 \}$ is the root system of the reductive Lie algebra $\tilde{\mathfrak{g}}_0$ with respect to its Cartan subalgebra $\mathfrak{t}^\mathbb{C}$.

Since the algebra $\tilde{\mathfrak{g}}_0$ coincides with the centralizer of some (regular) element $x_\Pi \in \mathfrak{a}$ in $\mathfrak{g}^\mathbb{C}$, there exists a basis $\Pi$ of $\Delta$ (a system of simple roots) such that $\Pi_0 = \Pi \cap \Delta_0$ is a basis of $\Delta_0$. Indeed, the element $-ix_\Pi \in \mathfrak{i}t$ belongs to the closure of some Weyl chamber in $\mathfrak{i}t$ determining the basis $\Pi$. Then $\Pi_0 = \{ \alpha \in\}$.
\( \Pi : \alpha(-ix_\Pi) = 0 \). The bases \( \Pi \) and \( \Pi_0 \) determine uniquely the subsets \( \Delta^+ \) and \( \Delta_0^+ \) of positive roots of \( \Delta \) and \( \Delta_0 \), respectively. It is evident that

\[ \Delta^+ \setminus \Delta_0^+ = \{ \alpha \in \Delta : \alpha(-ix_\Pi) > 0 \}. \]

The set \( \Sigma = \{ \lambda \in (a^C)^* : \lambda = \alpha|_{a^C}, \alpha \in \Delta \setminus \Delta_0 \} \) is the set of restricted roots of the triple \((g, \mathfrak{t}, a)\), which is independent of the choice of the \(\sigma\)-invariant Cartan subalgebra \( t \) containing the Cartan subspace \( a \). The following decomposition

\[ g^C = \tilde{g}_0 \oplus \sum_{\lambda \in \Sigma^+} (\tilde{g}_\lambda \oplus \tilde{g}_{-\lambda}), \quad \text{where} \quad \tilde{g}_\lambda = \sum_{\alpha \in \Delta \setminus \Delta_0, \alpha|_{a^C} = \lambda} \tilde{g}_\alpha, \]

and \( \Sigma^+ \) denotes the subset of positive restricted roots in \( \Sigma \) determined by the set of positive roots \( \Delta^+ \), gives us the simultaneous diagonalization of \( \text{ad}(a^C) \) on \( g^C \). Denote by \( m_\lambda \) the multiplicity of the restricted root \( \lambda \in \Sigma^+ \), that is, \( m_\lambda = \text{card}\{\alpha \in \Delta : \alpha|_{a^C} = \lambda\} \).

The set \( \Sigma \) is an abstract (not necessarily reduced) root system and its subset

\[ \Pi_\Sigma = \{ \lambda \in (a^C)^* : \lambda = \alpha|_{a^C}, \alpha \in \Pi \setminus \Pi_0 \} \]

is a basis of \( \Sigma \) containing \( \dim a \) elements [20, Ch. VII, Theorem 2.19].

**Lemma 4.1.** Let \( \Delta' = \{ \alpha \in \Delta : \alpha(t_0) = 0 \} \). Then \( \Delta' \subset \Delta \) is a root subsystem of the root system \( \Delta \) and \( \Delta' \subset \Delta \setminus \Delta_0 \). If \( \alpha \in \Delta' \), \( \hat{\alpha} \in \Delta \), with \( \alpha \neq \hat{\alpha} \) and \( \alpha|_{a} = \hat{\alpha}|_{a} \), then \( \alpha - \hat{\alpha} \in \Delta_0 \). In particular, for any root \( \alpha \in \Delta' \) the following conditions are equivalent:

1. \( \alpha + \beta \notin \Delta \) for all \( \beta \in \Delta_0 \);
2. the restricted root \( \lambda = \alpha|_{a^C} \) has multiplicity 1 (as an element of the restricted root system \( \Sigma \)).

**Proof.** The set \( \Delta' \) is an (abstract) root subsystem of \( \Delta \) because the subset \( \Delta' \subset \Delta \) is symmetric (\( \Delta' = -\Delta' \)) and closed (if \( \alpha_1, \alpha_2 \in \Delta' \), \( \alpha_1 + \alpha_2 \notin \Delta \) then \( \alpha_1 + \alpha_2 \in \Delta' \)). We have \( \Delta' \cap \Delta_0 = 0 \) because \( a \oplus t_0 = t \).

We now look at the standard scalar product on the real subspace \( V \subset (t^C)^* \) spanned by the set \( \Delta \subset (t^C)^* \). We can suppose that the Lie algebra \( g \) is semisimple. Consider on \( g^C \) the Killing form \( \langle \cdot, \cdot \rangle_K \) (which up to multiplication by a non-zero scalar coincides with our form \( \langle \cdot, \cdot \rangle \) on each (real) simple ideal of \( g \)). For each \( \mathbb{C}\)-linear form \( \mu \) on the Cartan subalgebra \( t^C \) let \( A_\mu \in t^C \) be determined by \( \mu(A) = \langle A_\mu, A \rangle_K \) for all \( A \in t^C \) and put \( \langle \mu_1, \mu_2 \rangle_K \) \( \overset{\text{def}}{=} \langle A_{\mu_1}, A_{\mu_2} \rangle_K \) for any two elements \( \mu_1, \mu_2 \in (t^C)^* \). It is well known that for each \( \mu \in \Delta \) the vector \( A_\mu \in \mu \) and that the restriction of the Killing form \( \langle \cdot, \cdot \rangle_K \) to it is positive-definite.

Since \( a \perp t_0 \), \( \alpha|_{t_0} = 0 \) and \( \alpha|_{a} = \hat{\alpha}|_{a} \), we obtain that \( A_\alpha \in i\mathbb{R}, A_\alpha - A_{\hat{\alpha}} \in i t_0 \) and, consequently, \( \langle \alpha, \alpha - \hat{\alpha} \rangle_K = 0 \). Thus

\[ \langle \alpha, \hat{\alpha} \rangle_K = \langle \alpha, \alpha - (\alpha - \hat{\alpha}) \rangle_K = \langle \alpha, \alpha \rangle_K > 0. \]

By the well-known property of root systems (see for example [21, Ch. 4, §1, Theorem 1]) if \( \langle \alpha, \hat{\alpha} \rangle_K > 0 \) then \( \alpha - \hat{\alpha} \notin \Delta \) unless \( \alpha = \hat{\alpha} \). Since \( \langle \alpha - \hat{\alpha} \rangle(a) = 0 \), \( \alpha - \hat{\alpha} \in \Delta_0 \) by definition of the root subsystem \( \Delta_0 \). Hence \( m_\lambda > 1 \) if and only if there exists \( \beta \in \Delta_0 \) such that \( \alpha + \beta \in \Delta \) (because \( \langle \alpha + \beta \rangle|_{a} = \alpha|_{a} \)).
For each linear form \( \lambda \) on \( a^C \) put
\[
\mathfrak{m}_\lambda \overset{\text{def}}{=} \{ \eta \in \mathfrak{m} : \text{ad}^2_w(\eta) = \lambda^2(w)\eta, \ \forall w \in a \},
\]
(4.4)
\[
\mathfrak{t}_\lambda \overset{\text{def}}{=} \{ \zeta \in \mathfrak{t} : \text{ad}^2_w(\zeta) = \lambda^2(w)\zeta, \ \forall w \in a \}.
\]

Then \( \mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda}, \mathfrak{t}_\lambda = \mathfrak{t}_{-\lambda}, \mathfrak{m}_0 = a \) and \( \mathfrak{t}_0 \) equals \( \mathfrak{h} \), the centralizer of \( a \) in \( \mathfrak{k} \).

It is clear that \( \mathfrak{m}_\lambda^C \oplus \mathfrak{t}_\lambda^C = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} \) for \( \lambda \in \Sigma^+ \) and \( \mathfrak{g}_0 = \mathfrak{m}_0^C \oplus \mathfrak{t}_0^C = a^C \oplus \mathfrak{h}^C \) (the Cartan subspace \( a^C \) is a maximal commutative subspace of \( \mathfrak{m}^C \)). Note also here that by (4.2) the subspace \( t_0 = (1 + \sigma)t \) is a Cartan subalgebra of the centralizer \( \mathfrak{h} \) and \( t = a \oplus t_0 \).

By [20] Ch. VII, Lemma 11.3, the following decompositions are direct and orthogonal:
\[
\mathfrak{m} = a \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda, \quad \mathfrak{t} = \mathfrak{h} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{t}_\lambda.
\]
(4.5)

We shall put
\[
\mathfrak{m}^+ \overset{\text{def}}{=} \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda, \quad \mathfrak{t}^+ \overset{\text{def}}{=} \sum_{\lambda \in \Sigma^+} \mathfrak{t}_\lambda.
\]

Since the Lie algebra \( \mathfrak{g} \) is compact then \( \lambda(a) \subset i\mathbb{R} \). Define the linear function \( \lambda' : a \to \mathbb{R}, \lambda \in \Sigma^+ \), by the relation \( i\lambda' = \lambda \). We need the following lemma, which is a generalization of [20] Ch. VII, Lemma 2.3.3.

Lemma 4.2. For any vector \( \xi_\lambda \in \mathfrak{m}_\lambda, \lambda \in \Sigma^+ \), there exists a unique vector \( \zeta_\lambda \in \mathfrak{t}_\lambda \) such that
\[
[w, \xi_\lambda] = -\lambda'(w)\zeta_\lambda, \quad [w, \zeta_\lambda] = \lambda'(w)\xi_\lambda \quad \text{for all } w \in a.
\]
(4.6)

In particular, \( \dim \mathfrak{m}_\lambda = \dim \mathfrak{t}_\lambda = m_\lambda \) and there exists a unique endomorphism \( T : \mathfrak{m}^+ \oplus \mathfrak{t}^+ \to \mathfrak{m}^+ \oplus \mathfrak{t}^+ \) such that
\[
\text{ad}_w |_{\mathfrak{m}_\lambda \oplus \mathfrak{t}_\lambda} = \lambda'(w)T |_{\mathfrak{m}_\lambda \oplus \mathfrak{t}_\lambda}, \quad T(\mathfrak{m}_\lambda) = \mathfrak{t}_\lambda, \quad T(\mathfrak{t}_\lambda) = \mathfrak{m}_\lambda, \quad \forall \lambda \in \Sigma^+.
\]
(4.7)

This endomorphism is orthogonal and \( T^2 = -\text{Id}_{\mathfrak{m}^+ \oplus \mathfrak{t}^+} \).

Proof. As we remarked above,
\[
(\mathfrak{m}_\lambda \oplus \mathfrak{t}_\lambda)^C = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}.
\]
(4.8)

It is well known that the \( \mathbb{C} \)-linear extension of \( \sigma \) is an involutive automorphism of the complex Lie algebra \( \mathfrak{g}^C \). We denote this involution also by \( \sigma \). Then
\[
\sigma(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda} \quad \text{and} \quad \sigma(\mathfrak{g}_{-\lambda}) = \mathfrak{g}_\lambda.
\]
(4.9)

Indeed, for any \( w \in a \) and \( E \in \mathfrak{g}_\lambda \) we have \( [w, E] = \lambda(w)E \) and, consequently, \( [-w, \sigma(E)] = \lambda(w)\sigma(E) \) because \( \sigma(w) = -w \). Thus \( \sigma(\mathfrak{g}_\lambda) \subset \mathfrak{g}_{-\lambda} \). Similarly, we can show that \( \sigma(\mathfrak{g}_{-\lambda}) \subset \mathfrak{g}_\lambda \). Taking into account that \( \sigma \) is nondegenerate, we obtain (4.9).

Since \( \mathfrak{g}_\lambda \cap \mathfrak{g}_{-\lambda} = 0 \) and \( \sigma(\xi) = -\xi \) for \( \xi \in \mathfrak{m}_\lambda^C \), \( \sigma(\zeta) = \zeta \) for \( \zeta \in \mathfrak{t}_\lambda^C \), from (4.9) it follows that
\[
\mathfrak{g}_\lambda \cap \mathfrak{m}_\lambda^C = 0, \quad \mathfrak{g}_\lambda \cap \mathfrak{t}_\lambda^C = 0, \quad \mathfrak{g}_{-\lambda} \cap \mathfrak{m}_\lambda^C = 0, \quad \mathfrak{g}_{-\lambda} \cap \mathfrak{t}_\lambda^C = 0.
\]
(4.10)
From (1.8), (4.10) and dimensional arguments it follows that for the subspace 
\[ \tilde{\mathfrak{g}}_\lambda \subset \mathfrak{m}_\lambda^C \oplus \mathfrak{t}_\lambda^C \] the natural projections \( \tilde{\mathfrak{g}}_\lambda : \mathfrak{m}_\lambda \to \mathfrak{m}_\lambda^C \) and \( \tilde{\mathfrak{g}}_\lambda : \mathfrak{t}_\lambda \to \mathfrak{t}_\lambda^C \) are isomorphisms. Therefore for any \( \xi_\lambda \in \mathfrak{m}_\lambda \subset \mathfrak{m}_\lambda^C \) there exists a unique vector \( E \in \tilde{\mathfrak{g}}_\lambda \) such that \( E = \xi_\lambda + \zeta \), \( \zeta \in \mathfrak{t}_\lambda^C \). But \( [w, E] = \lambda(w)E \) for each \( w \in \mathfrak{a} \), that is, \( [w, \xi_\lambda + \zeta] = \lambda(w)(\xi_\lambda + \zeta) \). By the relations (4.11), \( [w, \xi_\lambda] = \lambda(w)\zeta \) and \( [w, \zeta] = \lambda(w)\xi_\lambda \). Taking into account that \( \lambda(w) \in i\mathbb{R} \) and \( [w, \xi_\lambda] \in \mathfrak{k} \), putting \( \zeta_\lambda = i\zeta \) we obtain (4.6).

Relations (4.6) and (4.7) determine a unique endomorphism \( T \) for which \( T(\xi_\lambda) = -\zeta_\lambda, \quad T(\zeta_\lambda) = \xi_\lambda, \quad \forall \lambda \in \Sigma^+ \). Now the latter assertion of the lemma is evident excepting orthogonality.

To prove the orthogonality of \( T \) it is sufficient to note that, by (1.5), for different \( \lambda, \mu \in \Sigma^+ \), the subspaces \( \mathfrak{m}_\lambda \oplus \mathfrak{t}_\lambda \) and \( \mathfrak{m}_\mu \oplus \mathfrak{t}_\mu \) are orthogonal and for two arbitrary pairs \( \{ \xi_\lambda, \zeta_\lambda \} \) and \( \{ \xi_\mu, \zeta_\mu \} \) for which Conditions (4.6) hold, we have by Definition (4.4) that

\[
(L'(w))^2 \langle \xi_\lambda, \zeta_\lambda \rangle = \langle \text{ad}_w \xi_\lambda, \text{ad}_w \zeta_\lambda \rangle = -\langle \text{ad}_w^2 \xi_\lambda, \xi_\lambda^* \rangle = (L'(w))^2 \langle \xi_\lambda, \xi_\lambda^* \rangle = (L'(w))^2 \langle T\xi_\lambda, T\xi_\lambda^* \rangle.
\]

From this and the similar relation for \( \{ \xi_\lambda, \xi_\lambda^* \} \), the orthogonality of \( T \) follows. \( \square \)

Each restricted root \( \lambda \) defines a hyperplane \( \lambda(w) = 0 \) in the vector space \( \mathfrak{a} \). These hyperplanes divide the space \( \mathfrak{a} \) into finitely many connected components, called Weyl chambers. These are open, convex subsets of \( \mathfrak{a} \) (see [20, Ch. VII, \$2,\$11]). Fix the Weyl chamber \( \mathbb{W}^+ \) in \( \mathfrak{a} \), containing the element \( x_\Pi \):

\[ \mathbb{W}^+ = \{ w \in \mathfrak{a} : L'(w) > 0 \text{ for each } \lambda \in \Sigma^+ \}. \]

The subspace \( \mathfrak{m} \subset \mathfrak{g} \) is \( \text{Ad}(K) \)-invariant. Each \( \text{Ad}(K) \)-orbit in \( \mathfrak{m} \) intersects the Cartan subspace \( \mathfrak{a} \), that is, \( \text{Ad}(K)(\mathfrak{a}) = \mathfrak{m} \). The open connected subset \( \mathfrak{m}^R = \text{Ad}(K)(\mathbb{W}^+) \) of \( \mathfrak{m} \) is called the set of regular points in \( \mathfrak{m} \).

Since the centralizer of a (regular) element \( w \in \mathbb{W}^+ \subset \mathfrak{a} \) in the space \( \mathfrak{m} \) coincides with the Cartan subspace \( \mathfrak{a} \subset \mathfrak{m} \), the centralizer of the element \( w \) coincides with the centralizer of the Cartan subspace \( \mathfrak{a} \) in \( \text{Ad}(K) \), i.e.

\[
H = \{ k \in K : \text{Ad}_k w = w \} = \{ k \in K : \text{Ad}_k u = u \text{ for all } u \in \mathfrak{a} \}
\]

because by [20, Ch. VII, Lemma 2.14], one has \( G_w = G_\mathfrak{a} \), where

\[
G_w = \{ g \in G : \text{Ad}_g w = w \} \quad \text{and} \quad G_\mathfrak{a} = \{ g \in G : \text{Ad}_g u = u \text{ for all } u \in \mathfrak{a} \}.
\]

Note also that the space \( \mathfrak{h} \) (see (1.5)),

\[
\mathfrak{h} = \{ u \in \mathfrak{k} : [u, \mathfrak{a}] = 0 \} = \{ u \in \mathfrak{k} : [u, w] = 0 \} = \mathfrak{k}_w
\]

is the Lie algebra of \( H \). In particular, the subalgebra \( \mathfrak{a} \oplus \mathfrak{h} = \mathfrak{g}_w ((\mathfrak{a} \oplus \mathfrak{h})^C = \tilde{\mathfrak{g}}_0) \) is a subalgebra of \( \mathfrak{g} \) of maximal rank.

Recall that a subalgebra \( \mathfrak{b} \subset \mathfrak{g} \) is said to be regular if its normalizer \( \mathfrak{n}(\mathfrak{b}) \) in \( \mathfrak{g} \) has maximal rank, that is, \( \text{rank} \mathfrak{n}(\mathfrak{b}) = \text{rank} \mathfrak{g} \). In other words, \( \mathfrak{b} \) is regular if and only if \( \mathfrak{b} \) is normalized by some Cartan subalgebra of the algebra \( \mathfrak{g} \).
Our interest now centers on what will be shown to be an important subalgebra of \( g \). Let \( g_H \subset g \) be the subalgebra of fixed points of the group \( \text{Ad}(H) \), i.e. 

\[
g_H \overset{\text{def}}{=} \{ u \in g : \text{Ad}_h u = u \text{ for all } h \in H \}.
\]

It is evident that \( g_H \subset g_0 \), where

\[
g_0 \overset{\text{def}}{=} \{ u \in g : [u, \zeta] = 0 \text{ for all } \zeta \in h \}
\]

is the centralizer of the algebra \( h \) in \( g \). Note that in the general case one has \( g_H \neq g_0 \) (see Example 4.6 below).

To understand the structure of the algebra \( g_H \) we consider more carefully the centralizer \( g_h \). Since \( h \) is a compact Lie algebra, \( h = z(h) \oplus [h, h] \), where \( z(h) \) is the center of \( h \) and \([h, h]\) is a maximal semisimple ideal of \( h \). It is clear that

\[
z(h) \subset g_h \text{ and } g_h \cap [h, h] = 0 \text{ because } \langle g_h, [h, h] \rangle = \langle [g_h, h], h \rangle = 0.
\]

Thus \( g_h \oplus [h, h] \) is a subalgebra of \( g \).

By its definition, \( z(h) \) is a subspace of the center of the algebra \( g_h \). Moreover, by (4.12), \( a \subset g_h \). The space \( a \oplus z(h) \subset g_h \) is a Cartan subalgebra of \( g_h \). Indeed, as we remarked above, the centralizer \( a^C = g_0 = a^C \oplus h^C \) (by its definition, \( t_0 = h \)) is a subalgebra of \( g^C \) of maximal rank. Now since \( a_0 = a \oplus z(h) \oplus [h, h] \) with \( a \oplus z(h) \subset g_h \) and \( g_h \oplus [h, h] \) is a subalgebra of \( g \), then rank \( g_h - \text{rank}[h, h] = \text{rank } g_h = \text{dim}(a \oplus z(h)) \).

Since \( a \oplus t_0 \) is a Cartan subalgebra of the algebras \( g \) and \( g_0 = a \oplus h \), the algebra \( t_0 \) is a Cartan subalgebra of the algebra \( h \) and, consequently, \( z(h) \subset t_0 \). Moreover, since \([t_0, m] \subset m \), \([t_0, t] \subset t \), \([a, t_0] = 0 \), from Definitions (4.4) and (4.7) we obtain that

\[
[t_0, m_\lambda] \subset m_\lambda \text{ and } [t_0, t_\lambda] \subset t_\lambda \text{ for each } \lambda \in \Sigma^+,
\]

\[
[\text{ad}_x, T] = 0 \text{ on } m^+ \oplus t^+ \text{ for each } x \in t_0.
\]

By definition, \([g_h, h] = 0 \). Hence the space \( g_h + h \) is a subalgebra of \( g \). Since \( a \subset g_h \) and \( t_0 \subset h \), then \( a \oplus t_0 \subset g_h + h \). But \( a \oplus t_0 = t \) is a Cartan subalgebra of \( g \). This means that the complex reductive Lie algebras \((g_h + h)^C \), \( g_h^C \) and \( h^C \) are \( \text{ad}(t^C)\)-invariant (regular) subalgebras of \( g^C \). Taking into account that \( t \cap g_h = a \oplus z(h) \) and \( t \cap h = t_0 \), we obtain the following direct sum decompositions:

\[
g^C_h = a^C \oplus z(h)^C \oplus \sum_{\alpha \in \Delta_h} \tilde{g}_\alpha \text{ and } h^C = t_0^C \oplus \sum_{\alpha \in \Delta_h} \tilde{g}_\alpha,
\]

where \( \Delta_h \) is some subset of the root system \( \Delta \). Since the spaces \( a \oplus z(h) \subset t \) and \( t_0 \subset t \) are Cartan subalgebras of the algebras \( g_h \) and \( h \) respectively, the decompositions above are the root space decompositions of \((g_h^C, (a \oplus z(h))^C) \) and \((h^C, t_0^C) \), respectively. In particular, the subset \( \Delta_h \subset \Delta \) is the root system of \((g_h^C, (a \oplus z(h))^C) \).

**Proposition 4.3.** The algebra \( g_h \) is a \( \sigma \)-invariant regular compact subalgebra (possibly with nontrivial center) of \( g \), in particular,

\[
g_h = m_h \oplus t_h, \quad \text{where } m_h = g_h \cap m, \quad t_h = g_h \cap t,
\]

where \( m_h \subset m \), \( t_h \subset t \).
and \((\mathfrak{g}_h, \mathfrak{t}_h)\) is a symmetric pair. The space \(a\) is a Cartan subspace of \(m_h \subset \mathfrak{g}_h\) and \(a \oplus \mathfrak{z}(h)\) is a Cartan subalgebra of \(\mathfrak{g}_h\). The root subsystem \(\Delta \subset \Delta\) in \((4.16)\) of the reductive complex Lie algebra \(\mathfrak{g}_h^C\) is defined by the following relation

\[(4.18)\quad \Delta_h = \{ \alpha \in \Delta : \alpha(t_0) = 0, \alpha + \beta \notin \Delta \text{ for all } \beta \in \Delta_0 \}.
\]

The set \(\Sigma_h = \{ \lambda \in (a^C)^* : \lambda = \alpha|_{a^C}, \alpha \in \Delta_0 \} \subset \Sigma\) is the set of restricted roots of the triple \((\mathfrak{g}_h, \mathfrak{t}_h, a)\). Each element \(\lambda \in \Sigma_h \subset \Sigma\) has multiplicity 1, that is, \(\dim m_\lambda = \dim \mathfrak{t}_\lambda = 1\), and the following decompositions are direct and orthogonal:

\[
\begin{align*}
\mathfrak{m}_h &= a \oplus \sum_{\lambda \in \Sigma_0} m_\lambda, \\
\mathfrak{t}_h &= \mathfrak{z}(h) \oplus \sum_{\lambda \in \Sigma_0} \mathfrak{t}_\lambda.
\end{align*}
\]

Proof. Since \(\mathfrak{h} \subset \mathfrak{t}\), then \(\sigma(h) = \mathfrak{h}\) and the centralizer \(\mathfrak{g}_h\) of \(\mathfrak{h}\) in \(\mathfrak{g}\) is \(\sigma\)-invariant, i.e. \((4.17)\) holds.

As we proved above, \(a \oplus \mathfrak{z}(h) \subset \mathfrak{g}_h\) is a Cartan subalgebra of \(\mathfrak{g}_h\). Since \(a \subset m_h = \mathfrak{g}_h \cap m\), this subspace is a Cartan subspace of \(m_h\) as it is a maximal commutative subspace of \(m\).

The root system \(\Delta_h \subset \Delta\) of the algebra \(\mathfrak{g}_h^C\) in \((4.16)\) is a subset of the root system \(\Delta'\) (see Lemma 1.11). Indeed, since \([\mathfrak{g}_h, \mathfrak{h}] = 0\) and \(t_0 = (1 + \sigma)t\) is a subspace of \(\mathfrak{h} = \mathfrak{a} \cap \mathfrak{t}\), we obtain that \(\alpha(t_0) = 0\) for all \(\alpha \in \Delta_h\), that is, \(\Delta_h \subset \Delta'\).

Now to prove the relation \((4.18)\) describing the root system \(\Delta_h\) it is sufficient to recall that for any roots \(\alpha, \beta \in \Delta\), \(\alpha + \beta \neq 0\), the commutator \([\mathfrak{g}_a, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha + \beta}\) if \(\alpha + \beta \in \Delta\) and \([\mathfrak{g}_a, \mathfrak{g}_\beta] = 0\) otherwise \([20]\) Ch. III, Theorem 4.3]. Taking into account the second relation in \((4.16)\) we obtain \((4.18)\). By Lemma 4.1 each restricted root \(\lambda = \alpha|_{a^C}, \alpha \in \Delta_h \subset \Delta\) has multiplicity 1.

Now all the latter assertions of the proposition follow from \((4.13)\), \((4.14)\), \((4.15)\) and the first decomposition in \((4.16)\).

To describe the algebra \(\mathfrak{g}_H\) we consider now in more detail the subgroup \(H \subset K\). By its definition, \(H = K \cap G_a\). By \([20]\) Ch. VII, Corollary 2.8], the centralizer \(G_a\) of the commutative subalgebra \(a\) is connected (it is the union of all maximal tori containing the torus \(\exp a \subset G\)). Since \(a = a \oplus \mathfrak{h}\) is the Lie algebra of the compact Lie group \(G_a\), we have that \(G_a = \exp(a \oplus \mathfrak{h})\). But \(\exp(a \oplus \mathfrak{h}) = \exp(a) \exp(\mathfrak{h})\) because \([a, \mathfrak{h}] = 0\). The set \(H_0 = \exp \mathfrak{h}\) is the identity component of the Lie group \(H\) and \(H_0 \subset K\) because \(\mathfrak{h} \subset \mathfrak{t}\). Therefore \(H = G_a \cap K = (\exp(a) \cap K)H_0\).

**Proposition 4.4.** The subalgebra \(\mathfrak{g}_H \subset \mathfrak{g}_h \subset \mathfrak{g}\) is determined by the relation

\[(4.19)\quad \mathfrak{g}_H = \{ u \in \mathfrak{g}_h : \text{Ad}(D_\mathfrak{a})u = u \},
\]

where \(D_\mathfrak{a}\) stands for the commutative finite group

\[(4.20)\quad D_\mathfrak{a} = \exp(a) \cap K = \exp \{ \{ v \in a : \exp v = \exp(-v) \} \}.
\]

The algebra \(\mathfrak{g}_H \subset \mathfrak{g}_h\) is a \(\sigma\)-invariant regular compact subalgebra of \(\mathfrak{g}\). In particular,

\[
\mathfrak{g}_H = m_H \oplus \mathfrak{t}_H, \quad \text{where} \quad m_H = \mathfrak{g}_H \cap m, \quad \mathfrak{t}_H = \mathfrak{g}_H \cap \mathfrak{t}.
\]
and \((g_H, \mathfrak{k}_H)\) is a symmetric pair. The space \(\mathfrak{a}\) is a Cartan subspace of \(m_H \subset g_H\) and the space \(\mathfrak{a} \oplus \mathfrak{z}(h)\) is a Cartan subalgebra of \(g_H\). For each \(\lambda \in \Sigma^+\) and \(g \in D_\lambda\) we have that \(\text{Ad}_g(m_\lambda \oplus \mathfrak{k}_\lambda) = m_\lambda \oplus \mathfrak{k}_\lambda\). The set

\[
\Sigma_H = \{\lambda \in \Sigma_h : \text{Ad}_g |_{m_\lambda \oplus \mathfrak{k}_\lambda} = \text{Id}_{m_\lambda \oplus \mathfrak{k}_\lambda} \text{ for all } g \in D_\lambda\}
\]

is the set of restricted roots of the triple \((g_H, \mathfrak{k}_H, \mathfrak{a})\). Each element \(\lambda \in \Sigma_H \subset \Sigma_h \subset \Sigma\) has multiplicity 1, that is, \(\dim m_\lambda = \dim \mathfrak{k}_\lambda = 1\).

The following decompositions are direct and orthogonal:

\[
m_H = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma_H \cap \Sigma^+} m_\lambda, \quad \mathfrak{k}_H = \mathfrak{z}(h) \oplus \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \mathfrak{k}_\lambda.
\]

**Proof.** Since \([h, g_h] = 0\), the connected Lie group \(\text{Ad}(H_0)\) with Lie algebra \(\text{ad}(\mathfrak{h})\) acts trivially on \(g_h\). Taking into account that \(H = D_\lambda H_0\) we obtain \((4.19)\).

Since \(K\) is a subgroup of the group of fixed points of certain involutive automorphism on \(G\) acting by \(v \mapsto \exp(-v)\) on \(\exp(\mathfrak{a})\), we obtain the second relation in \((4.20)\). The group \(D_\lambda\) is a commutative finite group because \(\exp(\mathfrak{a}) \subset G\) is a toral subgroup, \(K\) is compact, and the intersection \(\exp(\mathfrak{a}) \cap K\) is a group of dimension 0 \((\mathfrak{a} \cap \mathfrak{k} = 0)\).

The algebra \(g_H\) is \(\sigma\)-invariant because by Definition \((4.11)\), \(\sigma \text{Ad}(H)\sigma = \text{Ad}(H)\). Since: \(D_\lambda = \{v_1, \ldots, v_s\}\), where \(v_j \in \mathfrak{a}\); \(j = 1, \ldots, s; [\mathfrak{a}, \mathfrak{a} \oplus \mathfrak{z}(h)] = 0\); and \(\text{Ad}_{\exp v_j} = \exp(\text{ad}v_j)\), then from relations \((4.6)\) we obtain that \(\text{Ad}_{\exp v_j} v = v = v\) for all \(v \in \mathfrak{a} \oplus \mathfrak{z}(h)\) and

\[
\begin{align*}
\text{Ad}_{\exp v_j} \xi_\lambda &= \cos(\lambda'(v_j)) \xi_\lambda - \sin(\lambda'(v_j)) \xi_\lambda, \\
\text{Ad}_{\exp v_j} \xi_\lambda &= \cos(\lambda'(v_j)) \xi_\lambda + \sin(\lambda'(v_j)) \xi_\lambda
\end{align*}
\]

for arbitrary \(\xi_\lambda \in m_\lambda\) and \(\xi_\lambda \in \mathfrak{k}_\lambda\). Let \(\lambda \in \Sigma^+\) satisfy condition \((4.6)\). But \(\exp v_j = \exp(-v_j)\) and, consequently, \(\sin(\lambda'(v_j)) = 0\). Then \(\cos(\lambda'(v_j)) \in \{1, -1\}\). Now it is clear that \(g_H = \mathfrak{a} \oplus \mathfrak{z}(h) \oplus \sum_{\lambda \in \Sigma_h \cap \Sigma^+} (m_\lambda \oplus \mathfrak{k}_\lambda)\), where

\[
\Sigma_H \cap \Sigma^+ = \{\lambda \in \Sigma_h \cap \Sigma^+ : \cos(\lambda'(v_j)) = 1 \text{ for all } j = 1, \ldots, s\}.
\]

The last assertion of the proposition follows from Proposition \((4.3)\). \(\square\)

**Remark 4.5.** Put \(m_H^+ = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} m_\lambda\) and \(\mathfrak{k}_H^+ = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \mathfrak{k}_\lambda\). Consider the orthogonal decompositions: \(m^+ = m_H^+ \oplus m_H^+\) and \(\mathfrak{k}^+ = \mathfrak{k}_H^+ \oplus \mathfrak{k}_H^+\), where \(m_h^+ = \sum_{\lambda \in \Sigma^+ \cap \Sigma_H} m_\lambda\) and \(\mathfrak{k}_H^+ = \sum_{\lambda \in \Sigma^+ \cap \Sigma_H} \mathfrak{k}_\lambda\). Since the following decompositions are orthogonal

\[
\begin{align*}
g_H &= \mathfrak{a} \oplus m_H^+ \oplus \mathfrak{k}_H^+ \oplus \mathfrak{z}(h), \\
g &= \mathfrak{a} \oplus m_H^+ \oplus \mathfrak{k}_H^+ \oplus m_H^+ \oplus \mathfrak{k}_H^+ \oplus \mathfrak{z}(h) = g_H \oplus (m_H^+ \oplus \mathfrak{k}_H^+) \oplus [h, h]
\end{align*}
\]

and \([g_H, h] = 0\), one sees that \(g_H \oplus [h, h]\) is a subalgebra of \(g\). Then

\[
[g_H \oplus [h, h], m_H^+ \oplus \mathfrak{k}_H^+] \subset m_H^+ \oplus \mathfrak{k}_H^+ \\
\text{and } [\mathfrak{a} \oplus h, m_H^+ \oplus \mathfrak{k}_H^+] \subset m_H^+ \oplus \mathfrak{k}_H^+.
\]

Moreover, because by its definition, \(T(m_\lambda) = \mathfrak{k}_\lambda, T(\mathfrak{k}_\lambda) = m_\lambda\), for all restricted roots \(\lambda \in \Sigma^+,\) we have that

\[
T(m_H^+) = \mathfrak{k}_H^+, \quad T(\mathfrak{k}_H^+) = m_H^+ \quad \text{and} \quad T(m_H^+) = \mathfrak{k}_H^+, \quad T(\mathfrak{k}_H^+) = m_H^+.
\]
Example 4.6. Let $G/K = SU(n)/SO(n)$, $n \geq 2$. Let $\mathfrak{g} = \mathfrak{su}(n)$ and $\mathfrak{k} = \mathfrak{so}(n)$ be the Lie algebras of $G$ and $K$, i.e. the spaces of traceless skew-Hermitian and skew-symmetric real $n \times n$ matrices, respectively. It is clear that the space $\mathfrak{a} = \{ \text{diag}(it_1, \ldots, it_n), t_j \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \}$, is a Cartan subspace of the space $\mathfrak{m} \subset \mathfrak{g}$. Since $\mathfrak{a}$ is a Cartan subalgebra of the algebra $\mathfrak{g}$, the centralizer $\mathfrak{h} = \mathfrak{g}_\mathfrak{a} \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{k} = 0$, that is, the Lie algebra of the group $H$, is trivial and $\mathfrak{g}_\mathfrak{h} = \mathfrak{g}$.

Then by Proposition 4.4, $H = D_\mathfrak{a}$ and $D_\mathfrak{a} \overset{\text{def}}{=} \exp(\mathfrak{a}) \cap K = \{ \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \}$, where $\varepsilon_j = \pm 1$ and $\prod_{j=1}^n \varepsilon_j = 1$. It is easy then, on account of (4.14), to verify that $\mathfrak{g}_H = \mathfrak{a}$ if $n \geq 3$ (in this case for any $k, j \leq n$, $k \neq j$ there exists an element $g \in D_\mathfrak{a}$ for which $\varepsilon_k \varepsilon_j = -1$) and $\mathfrak{g}_H = \mathfrak{g} = \mathfrak{su}(2)$ if $n = 2$. In the latter case the group $H$ coincides with the center of the Lie group SU(2), i.e. the action of $\text{Ad}(H)$ on $\mathfrak{g}$ is trivial.

Fix in each subspace $\mathfrak{m}_\lambda$, $\lambda \in \Sigma^+$, some basis $\{ \xi^j_\lambda, j = 1, \ldots, m_\lambda \}$, orthonormal with respect to the form $(\cdot, \cdot)$. In the case that $\lambda \in \Sigma^+ \cap \Sigma^+$, $m_\lambda = 1$, we have a unique vector $\xi^1_\lambda$. By Lemma 4.2 for each $\lambda \in \Sigma^+$ there exists a unique basis $\{ \zeta^j_\lambda, j = 1, \ldots, m_\lambda \}$ of $\mathfrak{t}_\lambda$ such that for each pair $\{ \xi^j_\lambda, \zeta^j_\lambda, j = 1, \ldots, m_\lambda \}$, the condition (4.6) holds. The basis $\{ \zeta^j_\lambda, j = 1, \ldots, m_\lambda \}$, $\lambda \in \Sigma^+$, of $\mathfrak{t}_\lambda$, is also orthonormal due to the orthogonality of the operator $T$ (see Lemma 4.2). Fix also some orthonormal basis $\{ X_1, \ldots, X_r \}$ of the Cartan subspace $\mathfrak{a}$ and some orthonormal basis $\{ \zeta^k_0, k = 1, \ldots, \dim \mathfrak{h} \}$ of the centralizer $\mathfrak{h}$ of $\mathfrak{a}$ in $\mathfrak{k}$. We will use the orthonormal basis

$$X_1, \ldots, X_r; \xi^j_\lambda, j = 1, \ldots, m_\lambda, \lambda \in \Sigma^+; \zeta^k_0, k = 1, \ldots, \dim \mathfrak{h},$$

of the algebra $\mathfrak{g}$ in our calculations below.

4.2. The canonical complex structure on $G/H \times W^+$. Each element $w \in W^+$ is regular in $\mathfrak{m}$. Therefore if for some $k \in K$, $\text{Ad}_k w \in W^+$, then $\text{Ad}_k(\mathfrak{a}) = \mathfrak{a}$ because $0 = [w, \mathfrak{a}] = [\text{Ad}_k w, \text{Ad}_k \mathfrak{a}]$. Since the Weyl group of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is simply transitive on the set of Weyl chambers in $\mathfrak{a}$ (see [20, Ch. VII, Theorem 2.12]), $\text{Ad}(K)w \cap W^+ = \{ w \}$. Then by definition (4.11) of the group $H$, the map

$$K/H \times W^+ \rightarrow \mathfrak{m}^R, \quad (kH, w) \mapsto \text{Ad}_k w,$$

is a well-defined diffeomorphism. Thus the map

$$f^+: G/H \times W^+ \rightarrow G \times_K \mathfrak{m}^R, \quad (gH, w) \mapsto [(g, w)]_K,$$

is a well-defined $G$-equivariant diffeomorphism of $G/H \times W^+$ onto the subset $D^+ = G \times_K \mathfrak{m}^R$, which is an open dense subset of $G \times_K \mathfrak{m}$.

It is clear that the following diagram is commutative

$$
\begin{array}{cccc}
G \times W^+ & \overset{\text{id}}{\rightarrow} & G \times \mathfrak{m}^R & \overset{\text{id}}{\rightarrow} & (g, w) \\
\downarrow{\pi_H \times \text{id}} & & \pi & & (g, w) \\
G/H \times W^+ & \overset{f^+}{\rightarrow} & G \times_K \mathfrak{m}^R & \overset{f^+}{\rightarrow} & [(g, w)]_K \\
\end{array}
$$

where $\pi_H: G \rightarrow G/H$ is the canonical projection.
The submersion (projection) $\pi: G \times m \to G \times_K m$ is (left) $G$-equivariant. Therefore, the kernel $\mathcal{K} \subset T(G \times m)$ of the tangent map $\pi_*: T(G \times m) \to T(G \times_K m)$ is generated by the global (left) $G$-invariant vector fields $\zeta^L$, for $\zeta \in \mathfrak{g}$, on $G \times m$,

\begin{equation}
\zeta^L(g, w) = (\zeta^l(g), [w, \zeta]) \in T_g G \times T_w m,
\end{equation}

where the tangent space $T_w m$ is identified canonically with the space $m$.

Note also here that the tangent space $T_o(G/H)$ at $o = \{H\} \subset G/H$ can be identified naturally with the space $m \oplus \mathfrak{t}^+ = \mathfrak{a} \oplus \mathfrak{m}^+ \oplus \mathfrak{t}^+$, because by definition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}^+$ and $\mathfrak{h}$ is the Lie algebra of the group $H$.

We can rewrite the expression for the vector field $Y^\xi, \xi \in \mathfrak{m}$, in (3.15) on the product $G \times m$ in a simpler way using Lemma 4.2 and the basis (4.21) of the algebra $\mathfrak{g}$. Indeed, for $w \in W^+ \subset \mathfrak{a}$, by (3.15) we have

\begin{equation}
Y^X_j(g, w) = (X^j(g), -iX_j), \quad j = 1, \ldots, r,
\end{equation}

\begin{equation}
Y^\xi_j(g, w) = \left( \frac{1}{\cosh \lambda'_w} \cdot \xi^j - i \frac{\cosh \lambda'_w - 1}{\sinh \lambda'_w} \cdot \xi^j \right)^l(g), \quad j = 1, \ldots, r,
\end{equation}

where $j = 1, \ldots, m_\lambda, \lambda \in \Sigma^+$, and $\lambda'_w \overset{\text{def}}{=} \lambda(w) \in \mathbb{R}$.

By Lemma 4.2, for any (regular) element $w \in W^+ \subset \mathfrak{a} \subset \mathfrak{m}$, the map $\text{ad}_w: \mathfrak{t}^+ \to \mathfrak{m}$, $\zeta \mapsto [w, \zeta]$, is nondegenerate and thus $\mathfrak{a} \oplus \text{ad}_w(\mathfrak{t}^+) = \mathfrak{m}$. Therefore

\[ T_{(e,0)}(G \times m) = \mathfrak{g} \times m = (\mathfrak{m} \times m) \oplus \mathcal{K}(e, 0) = ((\mathfrak{m} \oplus \mathfrak{t}^+) \times \mathfrak{a}) \oplus \mathcal{K}(e, 0), \]

where, as we remarked above, $\mathfrak{m} \oplus \mathfrak{t}^+$ is identified naturally with the tangent space of $G/H$ at the point $\{H\}$.

Using Lemma 4.2 again (note that in (4.25), $\coth \lambda'_w \cdot \xi^j \in \mathfrak{g}$ and the second component is $\cosh \lambda'_w \cdot [w, \xi^j]$) and from the expression (4.23) of the kernel $\mathcal{K}(g, w)$ of $\Pi_*(g, w)$, we have for all $w \in W^+$ that

\begin{equation}
\Pi_*(g, w)(Y^\xi_j(g, w)) = \Pi_*(g, w) \left( \left( \frac{1}{\cosh \lambda'_w} \cdot \xi^j - i \frac{1}{\sinh \lambda'_w} \cdot \xi^j \right)^l(g), 0 \right).
\end{equation}

To describe the $G$-invariant Ricci-flat Kähler metrics on $T(G/K)$ associated to the canonical complex structure $J^K_c$, we first attempt to describe such metrics on the $G$-invariant open and dense subset

\[ T^+(G/K) \overset{\text{def}}{=} (\phi \circ f^+)(G/H \times W^+) \subset T(G/K), \]

isomorphic to the direct product $G/H \times W^+ \subset G/H \times \mathbb{R}^r$, where the action of the group $G$ is natural on the first component and trivial on the second component (see the commutative diagram (4.22)). Since this diffeomorphism is $G$-equivariant, we denote the corresponding complex structure on $G/H \times W^+$ also by $J^K_c$. Consider the coordinates $(x_1, \ldots, x_r)$ on $W^+$ associated with the basis $(X_1, \ldots, X_r)$ of $\mathfrak{a}$, that is, $w = x = x_1X_1 + \cdots + x_rX_r$. By the $G$-invariance it suffices to describe the operators $J^K_c$ only at the points $(o, x) \in G/H \times W^+$,
where \( o = \{ H \} \). Then from (4.24), (4.26) and the commutative diagram (4.22) we see that
\[
J^K_c(o,x)(X_j,0) = \left( 0, \frac{\partial}{\partial x_j} \right), \quad j = 1, \ldots, r,
\]
and (4.27)
\[
J^K_c(o,x)(\xi^j,0) = \left( \frac{-\cosh \lambda'_x}{\sinh \lambda'_x} \cdot \zeta^j_\lambda, 0 \right), \quad j = 1, \ldots, m_\lambda, \lambda \in \Sigma^+,
\]
where, recall, \( \lambda'_x = \lambda(x) \in \mathbb{R} \). Here \( T_o(G/H) \) is identified naturally with the space \( a \oplus \sum_{\lambda \in \Sigma^+} m_\lambda \oplus \sum_{\lambda \in \Sigma^+} t_\lambda \) and, in the first equation, we use naturally the usual basis \( \{ \partial / \partial x_j \} \) of \( T_o \mathbb{R}^r \) \((W^+ \text{ is an open subset of } \mathbb{R}^r)\).

Often we will use the second relation in (4.27) in the more general form:
\[
J^K_c(o,x)(\xi,0) = \left( \frac{-\cos \text{ad}_x}{\sin \text{ad}_x} \xi, 0 \right), \quad \text{where } \xi \in m^+.
\]

Let \( F = F(J^K_c) \) be the subbundle of \((1,0)\)-vectors of the structure \( J^K_c \) on the manifold \( G/H \times W^+ \). We can substantially simplify calculations by working on the manifold \( G \times W^+ \) with the subbundle \( \mathcal{F} = (\pi_H \times \text{id})^{-1}(F) \) rather than on the manifold \( G/H \times W^+ \) with \( F \) (see Subsection 2.1). From (4.27) (see also (4.24) and (4.26)) it follows that the subbundle \( \mathcal{F} \) of \( T^c(G \times W^+) \) is generated by the kernel \( \mathcal{H} \) of the submersion \( \pi_H \times \text{id} \),
\[
\mathcal{H}(g,x) = \{ (\zeta^j(g),0), \zeta \in h \}, \quad g \in G, \quad x \in W^+,
\]
and the left \( G \)-invariant vector fields
\[
Z^{X_j}(g,x) = \left( X^j(g), -i \frac{\partial}{\partial x_j} \right), \quad j = 1, \ldots, r,
\]
\[
Z^{\xi^j}(g,x) = \left( \left( \frac{1}{\cosh \lambda'_x} \cdot \xi^j_\lambda - i \frac{-1}{\sinh \lambda'_x} \cdot \zeta^j_\lambda \right)^l(g), 0 \right),
\]
where \( j = 1, \ldots, m_\lambda, \lambda \in \Sigma^+ \), and \( \lambda'_x \overset{\text{def}}{=} \lambda(x) \in \mathbb{R} \).

To simplify calculations in the forthcoming subsection, we will use for the vector fields of the second family the following more general expression,
\[
Z^{\xi}(g,x) = \left( (R_x \xi - i S_x \xi^l(g), 0 \right), \quad \xi \in m^+,
\]
in terms of the two operator-functions \( R: W^+ \to \text{End}(g) \) and \( S: W^+ \to \text{End}(g) \) on the set \( W^+ \) such that
\[
R_x \eta = \frac{1}{\cos \text{ad}_x} \eta \quad \text{if } \eta \in m^+ \oplus t^+, \quad R_x \eta = 0 \quad \text{if } \eta \in a \oplus h,
\]
\[
S_x \eta = -\frac{1}{\sin \text{ad}_x} \eta \quad \text{if } \eta \in m^+ \oplus t^+, \quad S_x \eta = 0 \quad \text{if } \eta \in a \oplus h,
\]
where, recall, \( x = \sum_{j=1}^r x_j X_j \in W^+ \). Remark also that \( \frac{1}{\cos \text{ad}_x} \eta = \eta \) if \( \eta \in a \oplus h \) but \( R_x \eta = 0 \) in this case. Since the operator \( \text{ad}_x \) is skew-symmetric with respect to the scalar product on \( g \), each operator \( R_x \) is symmetric and \( S_x \) is skew-symmetric:
\[
\langle R_x \eta_1, \eta_2 \rangle = \langle \eta_1, R_x \eta_2 \rangle, \quad \langle S_x \eta_1, \eta_2 \rangle = \langle \eta_1, -S_x \eta_2 \rangle, \quad x \in W^+, \quad \eta_1, \eta_2 \in g.
\]
Moreover, it is clear that for all $x \in W^+$, the restrictions $R_x|_{m^+ \oplus t^+}$ and $S_x|_{m^+ \oplus t^+}$ are nondegenerate and by Remark 4.5 the following relations hold:

\[(4.33) \quad R_x(m^+_s) = m^+_s, \quad R_x(t^+_s) = t^+_s, \quad S_x(m^+_s) = t^+_s, \quad S_x(t^+_s) = m^+_s, \quad s \in \{H, *\}.
\]

It is clear also that

\[(4.34) \quad R_x|_{m^+ \oplus t^+} = \frac{1}{\cosh \lambda^+_x} \Id_{m^+ \oplus t^+}, \quad S_x|_{m^+ \oplus t^+} = \frac{1}{\sinh \lambda^+_x} \Id_{m^+ \oplus t^+} \cdot T|_{m^+ \oplus t^+}
\]

for all $\lambda \in \Sigma^+$, and $[R_x, T] = [S_x, T] = 0$ on $m^+ \oplus t^+$ for all $x \in W^+$, where, recall that the operator $T$ is defined by expression (4.7).

**Proposition 4.7.** Assume that the group $G$ is semisimple. Let $f_h: G/H \times W^+ \to \mathbb{C}$ be a G-invariant $J^K_c$-harmonic function, that is, $\partial \bar{\partial} f_h = 0$. Then $f_h = \text{const}$.

**Proof.** It is clear that

\[(\pi_H \times \text{id})^*(\partial \bar{\partial} f_h) = (\pi_H \times \text{id})^*(\partial \bar{\partial} f_h) = d((\pi_H \times \text{id})^* \bar{\partial} f_h) = 0.
\]

Let us calculate the 1-form $\alpha_h = (\pi_H \times \text{id})^* \bar{\partial} f_h$ on $G \times W^+$. By its definition,

\[\alpha_h|_F = 0 \quad \text{and} \quad \alpha_h|_{T^+} = d((\pi_H \times \text{id})^* f_h)|_{T^+}.
\]

Since the function $f_h$ is $G$-invariant, $f_h$ is determined uniquely by some smooth function $f: W^+ \to \mathbb{C}$, $(x_1, \ldots, x_r) \mapsto f(x_1, \ldots, x_r)$. Taking into account the description of the vector fields $Z^X_j, Z^\xi_j$ generating the subbundle $\mathcal{F}$ of $T(G \times W^+)$ (see (4.29)), we obtain that $\alpha_h = \frac{1}{2} \sum_{j=1}^r \frac{\partial f}{\partial x_j} (i \theta^X_j + dx_j)$, where $\theta^X_j$ is the left $G$-invariant 1-form on $G$ (considered as a form on $G \times W^+$) such that $\theta^X_j(e) (\xi) = \langle X_j, \xi \rangle$, $\xi \in \mathfrak{g}$. But

\[2d\alpha_h = i \sum_{j=1}^r d\left( \frac{\partial f}{\partial x_j} \right) \wedge \theta^X_j + i \sum_{j=1}^r \frac{\partial f}{\partial x_j} d\theta^X_j = 0.
\]

It is clear that the first and second summands above vanish (they are independent as differential two-forms on $G \times W^+$). Since the left-invariant forms $\{\theta^X_j\}_{j=1}^r$ are independent on $G$, we obtain that $\frac{\partial f}{\partial x_j} = c_j$, $c_j \in \mathbb{C}$. Taking into account that $\sum_{j=1}^r c_j d\theta^X_j |_{e} (\xi, \eta) = -\sum_{j=1}^r c_j X_j (\xi, \eta)$ and the algebra $\mathfrak{g}$ is semisimple ($[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), we obtain that $c_j = 0$ for all $j = 1, \ldots, r$. \(\Box\)

4.3. **Invariant Ricci-flat Kähler metrics on $G/H \times W^+$.** Let $\mathcal{K}(G/H \times W^+) = \{(\mathfrak{g}, \omega, J^K_c)\}$ (resp. $\mathcal{R}(G/H \times W^+) = \{(\mathfrak{g}, \omega, J^L_c)\}$) be the set of all $G$-invariant Kähler (resp. Ricci-flat Kähler) structures on $G/H \times W^+$, identified also with the set $\mathcal{K}(T^+(G/K))$ (resp. $\mathcal{R}(T^+(G/K))$) of all $G$-invariant Kähler (resp. Ricci-flat Kähler) structures on the open dense subset $T^+(G/K)$ of $T(G/K)$, associated with $J^K_c$, via the $G$-equivariant diffeomorphism $\phi \circ f^+: G/H \times W^+ \to T^+(G/K)$. Put

\[\{T_1, \ldots, T_n\} = \{Z^X_1, \ldots, Z^X_r\} \cup \{Z^\xi_\lambda, \lambda \in \Sigma^+, j = 1, \ldots, m_\lambda\}.
\]
Theorem 4.8. Let $\mathcal{K}(G \times W^+) = \{ \bar{\omega} \}$ be the set of all 2-forms $\bar{\omega}$ on $G \times W^+$ such that

1. the form $\bar{\omega}$ is closed;
2. the form $\bar{\omega}$ is left $G$-invariant and right $H$-invariant;
3. the kernel of $\bar{\omega}$ coincides with the subbundle $\mathcal{H} \subset T(G \times W^+)$ in (4.28);
4. $\bar{\omega}(T_j, T_k) = 0$, $j, k = 1, \ldots, n$;
5. $i \bar{\omega}(T, \overline{T}) > 0$ for each $T = \sum_{j=1}^n c_j T_j$, where $(c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\}$.

Let $\mathcal{R}(G \times W^+) = \{ \bar{\omega} \}$ be the subset of the set $\mathcal{K}(G \times W^+) = \{ \bar{\omega} \}$ such that for its elements $\bar{\omega}$ (in addition) the following condition holds:

6. $\det(\bar{\omega}(T_j, \overline{T_k})) = \text{const}$ on $G \times W^+$.

Then (i) For any 2-form $\bar{\omega} \in \mathcal{K}(G \times W^+)$ there exists a unique 2-form $\omega$ on $G/H \times W^+$ such that $(\pi_H \times \text{id})^* \omega = \bar{\omega}$. The map $\bar{\omega} \mapsto \omega$ is a one-to-one map from $\mathcal{K}(G \times W^+)$ onto $\mathcal{K}(G/H \times W^+) \cong \mathcal{K}(T^+(G/K))$.

(ii) If the group $G$ is semisimple then the restriction of this map to $\mathcal{R}(G \times W^+)$ is a one-to-one map from $\mathcal{R}(G \times W^+)$ onto $\mathcal{R}(G/H \times W^+) \cong \mathcal{R}(T^+(G/K))$.

Proof. From (1)–(3), $\omega$ becomes into a $G$-invariant symplectic structure on $G/H \times W^+$. Then item (i) of the theorem follows from Lemma 2.1, using (4) and (5) and taking $\mathcal{F} = (\pi_H \times \text{id})^{-1}(F)$, where $F = F(J^K_c)$ is the subbundle of $(1, 0)$-vectors of the structure $J^K_c$ on the manifold $G/H \times W^+$. To prove assertion (ii) of the theorem, let $\omega \in \mathcal{K}(G/H \times W^+)$ and $\bar{\omega} = (\pi_H \times \text{id})^* \omega$. By Proposition 3.6, the form $\omega = ((\phi \circ f^+)^{-1})^* \omega \in \mathcal{R}(T^+(G/K))$ if and only if the $G$-invariant function $\ln S$ ($S = S(\omega)$, see (3.20)) on $T^+(G/K) \cong G/H \times W^+$ is a $J^K_c$-harmonic function. In this case, by Proposition 4.7, $S = \text{const}$. Now to complete the proof of the theorem it is sufficient to remark that, by (3.19), $\Pi^* S = \det[(\Pi^* \omega)(Y^\xi, \overline{Y^\xi})]$; by the commutative diagram (4.22), $(\Pi^* \omega)|_{G \times W^+} = \bar{\omega}$; and by the definition of the vector fields $Z^\xi, \xi \in m$, the difference $Z^\xi - Y^\xi$ belongs to the kernel of the tangent map $\Pi_*$:

$$\text{const} = \det[(\Pi^* \omega)(Y^\xi, \overline{Y^\xi})|_{G \times W^+}] = \det[(\Pi^* \omega)(Z^\xi, \overline{Z^\xi})|_{G \times W^+}] = \det(\bar{\omega}(Z^\xi, \overline{Z^\xi})).$$

Remark 4.9. Note that condition (5) of the previous theorem is equivalent to the following condition: the Hermitian matrix-function $w$ on $W^+$ with entries $w_{jk}(x) = i\bar{\omega}(T_j, \overline{T_k})(e, x)$, $j, k = 1, \ldots, n$, is positive-definite.

Corollary 4.10. Let $\omega \in \mathcal{K}(G/H \times W^+)$ and $\bar{\omega} = (\pi_H \times \text{id})^* \omega$. Then the form $\omega = ((\phi \circ f^+)^{-1})^* \omega \in \mathcal{K}(T^+(G/K))$. Suppose that there exists a smooth form (extension) $\omega_0$ on the whole tangent bundle $T(G/K)$ such that $\omega_0 = \omega$ on $T^+(G/K)$. Then the form $\omega_0$ determines a $G$-invariant Kähler structure on $T(G/K)$ (associated to the canonical complex structure $J^K_c$) if and only if for each limit point $x \in W^+ \setminus W^+ \subset a$ and some sequence $x_m \in W^+$, $m \in \mathbb{N}$, such that $\lim_{m \to \infty} x_m = x$, the Hermitian matrix $w(x)$ with entries $w_{jk}(x) = \lim_{m \to \infty} w_{jk}(x_m) = \lim_{m \to \infty} i\bar{\omega}(T_j, \overline{T_k})(e, x_m)$, $j, k = 1, \ldots, n$, is positive-definite.
Proof. The form $\tilde{\omega}$ on $G \times W^+$ is left $G$-invariant, right $H$-invariant and ker $\tilde{\omega} = \mathcal{K}$. By the commutative diagram (4.22) there exists a unique 2-form $\tilde{\omega}$ on the space $G \times m^R$ which is left $G$-invariant, right $K$-invariant, ker $\tilde{\omega} = \mathcal{K}|_{G \times m^R}$, $\tilde{\omega}|_{G \times W^+} = \bar{\omega}$ and $\Pi^* \omega = \bar{\omega}$. Here, recall, $\mathcal{K} \subset T(G \times m)$ is the kernel of the tangent map $\pi_*: T(G \times m) \to T(G \times K \cdot m)$. By Lemma 2.1 Item (5), the form (extension) $\tilde{\omega}_0$ determines a $G$-invariant Kähler structure on $T(G/K)$ if and only if the Hermitian matrix $\nu(x)$ for each $x \in m \setminus m^R$ with entries $v_{jk}(x) = i(\Pi^* \omega_0) (Y^{\xi_j}, Y^{\xi_k}) (e, x)$ is positive-definite. Remark that $\text{Ad}(K)(\bar{\omega}^+) = m$ and $\text{Ad}(K)(\bar{\omega}^+) = m^R$. Since the form $\Pi^* \omega_0$ on $G \times m$ is smooth,

$$v_{jk}(x) = \lim_{m \to \infty} v_{jk}(x_m) = \lim_{m \to \infty} i(\Pi^* \omega_0) (Y^{\xi_j}, Y^{\xi_k}) (e, x_m).$$

But as we remark above, at each point $(e, x_m) \in G \times W^+$

$$(\Pi^* \omega_0)(e, x_m) = (\Pi^* \omega)(e, x_m) = \tilde{\omega}(e, x_m) = \bar{\omega}(e, x_m)$$

restricted to the subspace $T(e, x_m)(G \times W^+) \subset T(e, x_m)(G \times m)$. Taking into account again (as in the proof of Theorem 4.5) that the difference $(Z^\xi - Y^\xi)(e, x_m), \xi \in m$, belongs to the kernel of the tangent map $\Pi\mathfrak{L}(e, x_m)$ we obtain that $v_{jk}(x) = \lim_{m \to \infty} i\omega(Z^{\xi_j}, Z^{\xi_k})(e, x_m)$. Now all the other required properties of the form $\omega_0$ follow by continuity. \hfill $\Box$

5. Description of the Space $\mathcal{R}(G \times W^+)$

For any vector $a \in \mathfrak{g}$, denote by $\theta^a$ the left $G$-invariant 1-form on the group $G$ such that $\theta^a(\xi) = \langle a, \xi \rangle$. Since $r_g^* \theta^a = \theta^{\text{Ad}_g a}$, where $g \in G$, the form $\theta^a$ is right $H$-invariant if and only if $\text{Ad}_h a = a$ for all $h \in H \subset G$. Because

$$d\theta^a(\xi, \eta) = -\theta^a([\xi, \eta]) = -\langle a, [\xi, \eta] \rangle,$$

the $G$-invariant form $\omega^a$ on $G$,

$$\omega^a(\xi, \eta) \overset{\text{def}}{=} \langle a, [\xi, \eta] \rangle, \quad \xi, \eta \in \mathfrak{g},$$

is a closed 2-form on $G$.

Let $\text{pr}_1: G \times W^+ \to G$ and $\text{pr}_2: G \times W^+ \to W^+$ be the natural projections. Choosing some orthonormal basis $\{e_1, \ldots, e_N\}$ of the Lie algebra $\mathfrak{g}$, where $e_j = X_j, j = 1, \ldots, r$, put $\tilde{\theta}^e_k \overset{\text{def}}{=} \text{pr}_1^* \theta_{e_k}$ and $\tilde{\omega}^e_k \overset{\text{def}}{=} \text{pr}_1^* \omega_{e_k}$. For any vector-function $a: W^+ \to \mathfrak{g}, a(x) = \sum_{k=1}^N a^k(x)e_k$, denote by $\tilde{\theta}^a$ (resp. $\tilde{\omega}^a$) the $G$-invariant 1-form $\sum_{k=1}^N a^k \cdot \tilde{\theta}^e_k$ (resp. 2-form $\sum_{k=1}^N a^k \cdot \tilde{\omega}^e_k$). Then we have

**Theorem 5.1.** Let $\tilde{\omega}$ be a 2-form belonging to $\mathcal{K}(G \times W^+)$, where the compact Lie group $G$ is semisimple. Then there exists a unique (up to a real constant) smooth function $f: W^+ \to \mathbb{R}, x \mapsto f(x)$, and a unique smooth vector-function
\( a: W^+ \to \mathfrak{g}_H \) given by

\[
a(x) = \sum_{j=1}^{r} \frac{\partial f}{\partial x_j}(x)X_j + z_h + a^\lambda(x) + a^m(x), \quad \text{where} \quad z_h \in \mathfrak{z}(\mathfrak{h}),
\]

(5.3)

\[
a^\lambda(x) = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \frac{c^\lambda_1}{\cosh \lambda(x)} \xi_\lambda \in \mathfrak{t}_H^+, \quad a^m(x) = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} \frac{c^m_\lambda}{\sinh \lambda(x)} \xi_\lambda \in \mathfrak{m}_H^+,
\]

c^\lambda_1, c^m_\lambda \in \mathbb{R}, \text{ such that } \bar{\omega} \text{ is the exact form expressed in terms of } a \text{ as}

(5.4)

\[
\bar{\omega} = d\bar{\theta}^a = \sum_{j=1}^{r} dx_j \wedge \bar{\theta}^{a[j]} - \bar{\omega}^a, \quad \text{where} \quad a_{[j]} = \frac{\partial a}{\partial x_j}.
\]

Moreover, for all points \( x \in W^+ \), the following conditions (1)–(3) hold:

1. the components \( a^\lambda(x) + z_h \) and \( a^m(x) \) of the vector-function \( a(x) \) in (5.3)
   satisfy the following commutation relations

(5.5)

\[
\begin{align*}
(R_x \cdot \text{ad}_{a^\lambda(x)} - \sum_{k \in \mathbb{N}} S_k \cdot \text{ad}_{a^\lambda(x)} \cdot (R^2_x + S^2_x) \text{ad}_{a^\mu(x)})(m^+) &= 0, \\
(R_x \cdot \text{ad}_{a^m(x)} - \sum_{k \in \mathbb{N}} S_k \cdot \text{ad}_{a^m(x)} \cdot (R^2_x + S^2_x) \text{ad}_{a^\mu(x)})(m^+) &= 0;
\end{align*}
\]

\( z_h = 0 \) if \( a^\lambda(x) \equiv 0 \) and \( G/K \) is an irreducible Riemannian symmetric space;

2. the Hermitian \( p \times p \)-matrix-function \( w_H(x) = (w_{k|j}(x)), p = \dim \mathfrak{m}_H = \dim a + \text{card}(\Sigma_H \cap \Sigma^+), \text{ with indices } k, j \in \{1, \ldots, r\} \cup \{1, \lambda \in \Sigma_H \cap \Sigma^+\} \)
   and entries

\[
w_{k|j}(x) = 2 \frac{\partial^2 f}{\partial x_k \partial x_j}(x), \quad k, j \in \{1, \ldots, r\},
\]

\[
w_{k|\lambda}(x) = 2 \lambda'(X_k) \left( \frac{c^\lambda_1}{\cosh^2 \lambda' x} - \frac{c^m_\lambda}{\sinh^2 \lambda' x} \right), \quad k \in \{1, \ldots, r\}, \lambda \in \Sigma_H \cap \Sigma^+,
\]

\[
w_{\lambda|\mu}(x), \lambda, \mu \in \Sigma_H \cap \Sigma^+, \text{ determined by (5.6)},
\]

is positive-definite;

3. if \( \mathfrak{m}_s^+ \neq 0 \) then the Hermitian \( s \times s \)-matrix \( w_s(x) = (w_{\lambda|\mu}(x)), \)
   \( s = \dim \mathfrak{m}_s^+ = \Sigma_{\lambda \in \Sigma^+ \setminus \Sigma_H, \mu}, \text{ with indices } \lambda \in \{1, \lambda \in \Sigma^+ \setminus \Sigma_H, j = 1, \ldots, m\} \) and entries

(5.6)

\[
w_{\lambda|\mu}(x) = -\frac{2i}{\sinh \lambda' x \sinh \mu' x} \langle (\text{ad}_{a^\lambda(x)} + z_h) \zeta_\lambda, \zeta_\mu \rangle
\]

\[
- \frac{2}{\cosh \lambda' x \sinh \mu' x} \langle (\text{ad}_{a^\lambda(x)} + a^m(x)) \zeta_\lambda, \zeta_\mu \rangle
\]

is positive-definite.

If in addition

4. either \( \det w_H(x) \cdot \det w_s(x) = \text{const} \) when \( \mathfrak{m}_s^+ \neq 0 \) or \( \det w_H(x) \equiv \text{const} \) otherwise,
then \( \tilde{\omega} \in \mathcal{K}(G \times W^+) \).

Conversely, any 2-form as in \((5.4)\) determined by a vector-function \( a: W^+ \to \mathfrak{g}_H \) as in \((5.3)\) for which conditions (1)–(3) hold, belongs to \( \mathcal{K}(G \times W^+) \) and if in addition (4) holds, it belongs to \( \mathcal{R}(G \times W^+) \).

**Proof.** The following lemma is crucial for our proof.

**Lemma 5.2.** Suppose that the Lie group \( G \) is semisimple. Let \( \omega_0 \) be a \( G \)-invariant closed 2-form on \( G \). Then there exists a unique vector \( a \in \mathfrak{g} \) such that \( \omega_0 = \omega^a \). The form \( \omega^a \), \( a \in \mathfrak{g} \), on \( G \) is right \( H \)-invariant if and only if \( \text{Ad}(H)(a) = a \). The kernel of such a right \( H \)-invariant form \( \omega^a \) contains the vector fields \( \xi^l \), for all \( \xi \in \mathfrak{h} \).

The map \( a \mapsto \omega^a, \ a \in \mathfrak{g}, \) is an injection.

**Proof.** (Of the lemma.) Since the \( G \)-invariant form \( \omega_0 \) is closed, we have

\[
0 = \omega(\xi_1, \xi_2, \xi_3) = -\omega_0(\xi_1, \xi_2) - \omega_0(\xi_2, \xi_3) - \omega_0(\xi_1, \xi_3),
\]

i.e. the map \( c: \mathfrak{g} \times \mathbb{R} \to \mathbb{R}, \ c(\xi, \eta) = \omega(\xi, \eta)(e) \), is a cocycle on the Lie algebra \( \mathfrak{g} \). The cocycle \( c \) determines the central extension of \( \mathfrak{g} \) (i.e. the linear space \( \mathfrak{g} \oplus \mathbb{R}, \) which equipped with the commutator \( [[\xi, \eta], 0] = ([\xi, \eta], c(\xi, \eta)) \) and \([\xi, 0], 0, z]) = 0 \) is a Lie algebra).

Since any central extension of a semisimple Lie algebra \( \mathfrak{g} \) is trivial, this cocycle is exact. Indeed, by the Malcev theorem for the radical \( \mathbb{R} \) there exists some complement which is an algebra, evidently isomorphic to \( \mathfrak{g} \). This complement has the basis \( \{\xi, a(\xi)\} \), where \( a \) is a linear function on \( \mathfrak{g} \). Since this complement is a subalgebra, \( c(\xi, \eta) = a(\xi, \eta) \). In other words, \( c(\xi, \eta) = \langle a, [\xi, \eta] \rangle \) for some vector \( a \in \mathfrak{g} \). But \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \) (the algebra is semisimple) and, consequently, such a vector \( a \) is unique. Now by the \( G \)-invariance, \( \omega_0 = \omega^a \).

It is easy to see (using Definition \(5.2\)) that \( r^*_h \omega^a = \omega^{\text{Ad}_h a} \) for any \( h \in H \subseteq G \). Using the relation \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \) again we obtain that the map \( a \mapsto \omega^a, \ a \in \mathfrak{g}, \) is an injection. Therefore \( \omega^a \) is a right \( H \)-invariant form on \( G \) if and only if \( \text{Ad}(H)(a) = a \).

By the invariance of the form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \), that is,

\[
(5.7) \quad \langle a, [\xi, \eta] \rangle = \langle [a, \xi], \eta \rangle,
\]

the kernel of \( \omega^a \) is generated by the vector fields \( \xi^l \), where \( \xi \) is an element of the centralizer \( \mathfrak{g}_a \) of \( a \) in \( \mathfrak{g} \). But if \( \text{Ad}(H)(a) = a \) then \([h, a] = 0, \) that is, \( h \subset \mathfrak{g}_a \). \( \square \)

An arbitrary \( G \)-invariant 2-form \( \tilde{\omega} \) on \( G \times W^+ \) reads

\[
\tilde{\omega} = \text{pr}_2^*(q) + \sum_{k=1}^N \sum_{j=1}^r b^k_j \cdot dx_j \wedge \tilde{\theta}^e_k + \sum_{1 \leq s < k \leq N} b^s_k \cdot \tilde{\theta}^e_s \wedge \tilde{\theta}^e_k,
\]

where \( q \) is a 2-form on \( W^+ \), and \( b^k_j, b^s_k \) are smooth real-valued functions on \( W^+ \).

Suppose that \( \tilde{\omega} \in \mathcal{K}(G \times W^+) \), that is, the form \( \tilde{\omega} \) satisfies conditions (1)–(5) of Theorem \(4.3\). It is easy to verify that if the form \( \tilde{\omega} \) is closed then the form \( q \) is closed and each form \( \tilde{\Delta}(x) = \sum_{1 \leq s < k \leq N} b^s_k(x) \cdot \tilde{\theta}^e_s \wedge \tilde{\theta}^e_k \) for arbitrary but fixed \( x \in W^+ \) also is closed as a form on \( G \). Then by Lemma \(5.2\) there exists a
unique (smooth) vector-function \( b: W^+ \to \mathfrak{g} \), \( b(x) = \sum_{k=1}^N b^k(x)e_k \), such that

\[
\bar{\omega} = \text{pr}_2^*(q) + \sum_{k=1}^N \sum_{j=1}^r b^k_j \cdot dx_j \wedge \bar{\omega}^e_j + \sum_{k=1}^N b^k \cdot \bar{\omega}^e_k, \quad dq = 0.
\]

The form \( \bar{\omega} \) is closed if and only if

\[
d\bar{\omega} = \sum_{k=1}^N \sum_{j=1}^r \left( \frac{\partial b^k_j}{\partial x_j} \cdot dx_j \wedge \bar{\omega}^e_j + \sum_{k=1}^N b^k \cdot \bar{\omega}^e_k \right) \wedge \bar{\omega}^e_k + \sum_{k=1}^N \left( \left( \sum_{j=1}^r b^k_j \cdot dx_j \right) \wedge \bar{\omega}^e_k + \left( \sum_{j=1}^r \frac{\partial b^k_j}{\partial x_j} \right) \wedge \bar{\omega}^e_k \right) = 0,
\]

because \( d(dx_j \wedge \bar{\omega}^e_k) = -dx_j \wedge d\bar{\omega}^e_k \) and by (5.1), (5.2) we have \( d\theta^e_k = -\omega^e_k \). Since the (closed) 2-forms \( \{\omega^e_k\} \) and 1-forms \( \{\theta^e_k\} \) are linearly independent forms on \( G \) (see Lemma 5.2), we see that \( \sum_{j=1}^r b^k_j dx_j + dB^k = 0 \) and \( \sum_{j=1}^r \frac{\partial b^k_j}{\partial x_j} = 0 \) for all \( k = 1, \ldots, N \). However the second relation is the differential of the first one, i.e. these two sets of relations are equivalent to the relations

\[
b^k_j(x) = -\frac{\partial b^k_j}{\partial x_j}(x), \quad j = 1, \ldots, r, \quad k = 1, \ldots, N; \quad x \in W^+.
\]

Then

\[
\bar{\omega} = \text{pr}_2^*(q) + \sum_{k=1}^N \sum_{j=1}^r \frac{\partial b^k_j}{\partial x_j} \cdot dx_j \wedge \bar{\omega}^e_j + \sum_{k=1}^N b^k \cdot \bar{\omega}^e_k = \text{pr}_2^*(q) + d\left( -\sum_{k=1}^N b^k \cdot \bar{\omega}^e_k \right).
\]

Since \( \bar{\omega} \in \mathcal{K}(G \times W^+) \), \( \bar{\omega}(Z^{X_s}, Z^{X_p}) = 0 \), where, recall, \( Z^{X_j} = (X_j^l, -i \partial/\partial x_j) \) and \( (X_1, \ldots, X_r) \) is the given basis of \( a \). Now, the subalgebra \( a \subset \mathfrak{g} \) is commutative. Thus the restriction \( \omega(b(x))|_a \) vanishes (see (5.2)) and, consequently, \( \left( \sum_{k=1}^N b^k \cdot \bar{\omega}^e_k \right)(Z^{X_s}, Z^{X_p}) = 0 \). Taking into account that \( \text{pr}_2^*(q)(Z^{X_s}, Z^{X_p}) \in \mathbb{R} \) and \( (dx_j \wedge \bar{\omega}^e_k)(Z^{X_s}, Z^{X_p}) \in i\mathbb{R} \), we obtain that

\[
\text{pr}_2^*(q)(Z^{X_s}, Z^{X_p}) = -q\left( \frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_p} \right) = 0,
\]

\[
\left( \sum_{k=1}^N \sum_{j=1}^r -\frac{\partial b^k_j}{\partial x_j} \cdot dx_j \wedge \bar{\omega}^e_k \right)(Z^{X_s}, Z^{X_p}) = 0.
\]

Therefore, the form \( q \) vanishes on \( W^+ \) and \( \frac{\partial b^p_i}{\partial x_s}(x) = \frac{\partial b^s_i}{\partial x_p}(x) \) for all \( x \in W^+ \), \( s, p \in \{1, \ldots, r\} \). Since the domain \( W^+ \subset \mathbb{R}^r \) is convex, there exists a unique (up to a real constant) smooth function \( f: W^+ \to \mathbb{R} \) such that

\[
b^k(x) = -\frac{\partial f}{\partial x_k}(x), \quad k = 1, \ldots, r, \quad x \in W^+.
\]
In other words, a $G$-invariant 2-form $\tilde{\omega}$ on $G \times W^+$ is closed and $\tilde{\omega}(Z^{X_s}, Z^{X_p}) = 0$, $s, p \in \{1, \ldots, r\}$, if and only if

$$\tilde{\omega} = d\tilde{\theta}^a, \quad a(x) = \sum_{k=1}^r \frac{\partial f}{\partial x_k}(x) X_k + a^\perp(x),$$

where $a^\perp : W^+ \to m^+ \oplus \mathfrak{k}$ is a smooth vector-function. It is clear that

$$\tilde{\omega} = \sum_{j=1}^r dx_j \wedge \tilde{\theta}^{a[j]} - \tilde{\omega}^a, \quad \text{where} \quad a_{[j]} = \frac{\partial a}{\partial x_j} = \sum_{k=1}^r \frac{\partial^2 f}{\partial x_j \partial x_k} X_k + \frac{\partial a^\perp}{\partial x_j}. \quad (5.8)$$

This form is right $H$-invariant if $\tilde{\omega} = \tilde{\omega}^h = \sum_{j=1}^r dx_j \wedge \tilde{\theta}^{Ad_h a_{[j]}} - \tilde{\omega}^{Ad_h a}$ for all $h \in H$. Since the maps $a \mapsto \tilde{\theta}^a$ and $a \mapsto \omega^a$, $a \in \mathfrak{g}$, are injections (cf. Lemma 5.2), we obtain that $a_{[j]}(x), \ldots, a_{[r]}(x), a(x) \in \mathfrak{g}_H$ (see Definitions (4.13) and (4.14)).

In this case by Lemma 5.2 the kernel of the form $\tilde{\omega}^a$ contains the subbundle $\mathcal{H} \subset TG \times TW^+$. It is easy now to verify that the kernel of $\tilde{\omega}$ contains $\mathcal{H}$ if and only if $(a_{[j]}(x), h) = 0$ for all $x \in W^+$ and $j = 1, \ldots, r$. Thus $a_{[j]}(x) \perp \mathfrak{z}(h)$ for all $x \in W^+$ because $\mathfrak{g}_H \cap \mathfrak{h} = \mathfrak{z}(h)$. This means that the $\mathfrak{z}(h)$-component of the vector $a(x)$ is a constant. Taking into account Proposition 4.4 and Remark 4.5 we obtain that $a(x) = a^a(x) + z_h + a^\xi(x) + a^m(x)$, where

$$a^a(x) = \sum_{j=1}^r \frac{\partial f}{\partial x_j}(x) X_j, \quad z_h \in \mathfrak{z}(h), \quad (5.9)$$

$$a^\xi(x) = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} a^\xi_\lambda(x) \xi_\lambda^+ \in \mathfrak{k}(h), \quad a^m(x) = \sum_{\lambda \in \Sigma_H \cap \Sigma^+} a^m_\lambda(x) \xi_\lambda^+ \in m^+_H.$$

It is convenient now, using (5.5), to calculate $i\tilde{\omega}(Z^{X_k}, Z^{X_j}) = i\tilde{\omega}(Z^{X_k}, Z^{X_j} - Z^{X_j})$:

$$i\tilde{\omega}(Z^{X_k}, Z^{X_j}) = i\tilde{\omega}\left(\left(\frac{X_j}{h}, -i \frac{\partial}{\partial x_k}\right), (0, 2i \frac{\partial}{\partial x_j})\right) = -i\langle a_{[j]}, X_k \rangle 2i = 2 \frac{\partial^2 f}{\partial x_k \partial x_j}. \quad (5.10)$$

Using (4.30) for the vector field $Z^{\xi}$, $\xi \in m^+$ we obtain, for all $(g, x) \in G \times W^+$,

$$\tilde{\omega}(g, x)(Z^{X_k}, Z^{X_j}) = -i\langle a_{[k]}, x \rangle, R_x \xi - iS_x \xi \rangle - \langle a(x), [X_k, R_x \xi - iS_x \xi]\rangle \quad (5.11)$$

$$= -i\langle (R_x + iS_x)a_{[k]}(x), \xi\rangle + \langle (R_x + iS_x)ad_{X_k} a(x), \xi\rangle \quad (5.12)$$

$$= -i\langle R_x a_{[k]}(x) - S_x ad_{X_k} a(x), \xi\rangle + \langle S_x a_{[k]}(x) + R_x ad_{X_k} a(x), \xi\rangle. \quad (5.13)$$

Thus $\tilde{\omega}(Z^{X_k}, Z^{\xi}) = 0$ for all $k = 1, \ldots, r$, and $\xi \in m^+$ if and only if

$$R_x a^m_{[k]}(x) - S_x ad_{X_k} a^m(x) = 0 \quad (5.14)$$

$$S_x a^\xi_{[k]}(x) + R_x ad_{X_k} a^\xi(x) = 0 \quad (5.15)$$

for all $x \in W^+$, because $[a, a \oplus \mathfrak{z}(h)] = 0$ and (4.33) hold. In other words, we obtain the following differential relations:

$$\frac{\partial a^m_\lambda(x)}{\partial x_k} = -\frac{\lambda'(X_k) \cdot \cosh \lambda'_x}{\sinh \lambda'_x} a^m_\lambda(x) \quad \text{and} \quad \frac{\partial a^\xi_\lambda(x)}{\partial x_k} = -\frac{\lambda'(X_k) \cdot \sinh \lambda'_x}{\cosh \lambda'_x} a^\xi_\lambda(x).$$

The differential relations are also satisfied by $a^a(x)$ and $z_h$.
with solutions

\[
ax^m(x) = \frac{c^m_o}{\sinh \lambda^x}, \quad ax^f(x) = \frac{c^f_o}{\cosh \lambda^x}, \quad c^m_o, c^f_o \in \mathbb{R}, \quad \lambda \in \Sigma_H \cap \Sigma^+.
\]

In this place it is convenient to calculate also \(\tilde{\omega}(Z^{X_k}, \overline{Z^\xi})(g, x), \xi \in \mathfrak{m}^+\):

\[
\tilde{\omega}(g, x)(Z^{X_k}, \overline{Z^\xi}) = i(-i\langle a_{[k]}(x), 2iS_x\xi \rangle - \langle a(x), [X_k, 2iS_x\xi]\rangle)
\]

\[
= 2i\langle -S_xa^m_{[k]}(x), \xi \rangle + 2\langle a^m(x), [X_k, S_x\xi]\rangle
\]

\[
= 2i(R_x \operatorname{ad}_{X_k} a^f(x), \xi) + 2\langle a^m(x), (\operatorname{ad}_{X_k} S_x)\xi\rangle
\]

\[
= -2i(a^f(x), (\operatorname{ad}_{X_k} R_x)\xi) + 2\langle a^m(x), (\operatorname{ad}_{X_k} S_x)\xi\rangle.
\]

Thus, \(\tilde{\omega}(Z^{X_k}, \overline{Z^\xi}) = 0 \) if \(\xi \in \mathfrak{m}^+\) and for vectors \(\xi^1 = \mathfrak{m}^+_{\xi}, \lambda \in \Sigma_H \cap \Sigma^+\):

\[
i\tilde{\omega}(g, x)(Z^{X_k}, \overline{Z^\xi}) = \frac{\lambda(X_k)}{\cosh \lambda^x} \langle a^f(x), \xi^1 \rangle - \frac{2\lambda(X_k)}{\sinh \lambda^x} \langle a^m(x), \xi^1 \rangle.
\]

Using the invariance (5.7) of the scalar product, the properties (4.32), (4.33) of the operator-functions \(R, S\) and the commutation relations (4.1), we calculate \(\tilde{\omega}(Z^{\xi}, Z^\eta) = -\tilde{\omega}^{a(x)}(Z^{\xi}, Z^\eta), \xi, \eta \in \mathfrak{m}^+\), putting \(A(x) = \operatorname{ad}_{a(x)}, A^m(x) = \operatorname{ad}_{a(x) + a^m(x)}, A^f(x) = \operatorname{ad}_{a^f(x) + a^m(x)}\):

\[
\tilde{\omega}(Z^{\xi}, Z^\eta) = -\langle a, [R\xi - iS\xi, R\eta - iS\eta]\rangle
\]

\[
= -\langle a, [R\xi, R\eta] - [S\xi, S\eta]\rangle + i\langle a, [R\xi, S\eta] + [S\xi, R\eta]\rangle
\]

\[
= -\langle [a, R\xi], R\eta\rangle + \langle [a, S\xi], S\eta\rangle + i\langle [a, R\xi], S\eta\rangle + i\langle [a, S\xi], R\eta\rangle
\]

\[
= -\langle (R\xi + SAS)\xi, \eta\rangle + i\langle (R\xi - SAR)\xi, \eta\rangle
\]

\[
= -\langle (R\xi + SAS)\xi, \eta\rangle + i\langle (R\xi + SAR)\xi, \eta\rangle.
\]

Since the algebra \(a\) is commutative, \([R_x, S_x] = [R_x, \operatorname{ad}_{a(x)}] = [S_x, \operatorname{ad}_{a(x)}] = 0\) on \(g\) for any \(x \in W^+\). Similarly, from \([a, \mathfrak{z}(\mathfrak{h})] = 0\) it follows that \([R_x, \operatorname{ad}_{\mathfrak{z}}] = [S_x, \operatorname{ad}_{\mathfrak{z}}] = 0\) on \(g\) for any \(x \in W^+\). Thus \(\tilde{\omega}(Z^{\xi}, Z^\eta) = 0\) for all \(\xi, \eta \in \mathfrak{m}^+\) if and only if for all \(x \in W^+\), Equations (5.5) hold, because relations (4.1), (4.33) hold and the space \(a \oplus \mathfrak{z} (a \subset \mathfrak{m}, \mathfrak{h} \subset \mathfrak{k})\) is the kernel of \(R_x\) and \(S_x\) \((R_x(m) = \mathfrak{m}^+, S_x(\mathfrak{k}) = \mathfrak{m}^+, \) that is, \(R_x(a) = 0, S_x(h) = 0\).

If \(a^f(x) = 0\) then the condition (5.5) implies \((R_x^2 + S_x^2) \operatorname{ad}_{\mathfrak{z}}(m^+) = 0\). Since for each \(x \in W^+\) by (4.13), \(\operatorname{ad}_{\mathfrak{z}}(\lambda_m(x)) \subset \mathfrak{m}_x\) and \(\cosh^{-2} \lambda'(x) = \sinh^{-2} \lambda'(x) \neq 0\), we obtain that \(\operatorname{ad}_{\mathfrak{z}}(\mathfrak{m}) = 0\). But \(\operatorname{ad}_{\mathfrak{z}}(a) = 0\) by its definition, that is, \(\{\mathfrak{z}, \mathfrak{m}\} = 0\). If \(G/K\) is an irreducible symmetric space then the algebra \(g\) is generated by \(m\) and therefore \(\mathfrak{z}_h = 0\).
Next we calculate also the value $\tilde{\omega}(Z^\xi, \overline{Z^\eta}) = i\tilde{\omega}(Z^\xi, \overline{Z^\eta})$:

\[
\begin{align*}
\tilde{\omega}(Z^\xi, \overline{Z^\eta}) &= -i\langle a, [R\xi - iS\xi, 2iS\eta]\rangle \\
&= -2i\langle a, [S\xi, S\eta]\rangle + 2\langle a, [R\xi, S\eta]\rangle \\
&= -2i\langle a, S\xi\rangle + 2\langle a, R\xi\rangle, \\
&= 2i\langle (S(\mathfrak{ad}_{\eta}^{+}+\lambda a))\xi, \eta\rangle - 2\langle (S(\mathfrak{ad}_{\eta}^{+}+\lambda a))R\xi, \eta\rangle.
\end{align*}
\]

(5.11)
or well, for any vectors $\xi, \eta \in \mathfrak{g}$, \(\mathfrak{g}\)-invariant Kähler metric \(g\) on \(G/H\), \(\xi\) is the unique smooth vector-function \(f\) on \(G/H\) such that \(\pi_H \times \text{id})^*\omega = d\tilde{\theta}^a\), where \(a\) is the unique smooth vector-function \(a: \mathbb{R}^+ \to \mathfrak{g}_H\) in (5.3) satisfying Conditions (1)-(3) of Theorem 5.1. If, in addition, condition (4) of Theorem 5.1 holds, this metric \(g\) is Ricci-flat.

Corollary 5.3. Let \(G/K\) be a Riemannian symmetric space of compact type. Each \(G\)-invariant Kähler metric \(g\), associated with the canonical complex structure \(J^K\) on \(G/H \times W^+ \cong T^+(G/K)\) \((T^+(G/K)\) is an open dense subset of \(T(G/K)\)), is determined precisely by the Kähler form \(\omega(\cdot, \cdot) = g(-J^K\cdot, \cdot)\) on \(G/H \times W^+\) given by

\[
(\pi_H \times \text{id})^*\omega = d\tilde{\theta}^a,
\]

where \(a\) is the unique smooth vector-function \(a: \mathbb{R}^+ \to \mathfrak{g}_H\) in (5.3) satisfying Conditions (1)-(3) of Theorem 5.1. If, in addition, condition (4) of Theorem 5.1 holds, this metric \(g\) is Ricci-flat.

Corollary 5.4. The \(G\)-invariant function \(Q: G/H \times W^+ \to \mathbb{R}, Q(gH,x) = 2f(x), \) where \(f \in C^\infty(W^+, \mathbb{R})\), is a potential function of the Kähler structure \((\omega, J^K)\) on \(G/H \times W^+\) (equivalently \((\pi_H \times \text{id})^*\omega \in \mathcal{K}(G \times W^+)\)) if and only if

\[
(\pi_H \times \text{id})^*\omega = d\tilde{\theta}^a,
\]

where \(a: W^+ \to W^+, a(x) = \sum_{k=1} \frac{\partial^2 f}{\partial x_j \partial x_k}(x), \)

it is a \(W^+\)-valued vector-function such that for all \(x \in W^+\) the matrix \(\left(\frac{\partial^2 f}{\partial x_j \partial x_k}(x)\right)\) is positive-definite.

This Kähler structure with \(G\)-invariant potential function \(Q\) is Ricci-flat Kähler (equivalently \((\pi_H \times \text{id})^*\omega \in \mathcal{R}(G \times W^+)\)) if and only if

\[
\det\left(\frac{\partial^2 f}{\partial x_j \partial x_k}(x)\right) \prod_{\lambda \in \Sigma^+} \left(\frac{2\lambda(a)}{\sinh 2\lambda(a)}\right)^{m_{\lambda}} \equiv \text{const.}
\]

(5.12)

Proof. Let \(f \in C^\infty(W^+, \mathbb{R})\) be an arbitrary function. Consider the form \(i\tilde{\theta}\partial Q\). By definition, \(\partial Q|_F = dQ|_F\) and \(\partial Q|_F = 0\). Denote by \(\tilde{\Delta}\) the 1-form \((\pi_H \times \text{id})^*\partial Q\) on \(G \times W^+.\) By (3.29), the form \(\tilde{\Delta}\) is the unique 1-form on \(G \times W^+\) such that

\[
\tilde{\Delta}_{(g,x)}(X^i_k(g), -i \frac{\partial}{\partial x_k}(x)) = -2i \frac{\partial f(x)}{\partial x_k}(x), \quad \tilde{\Delta}_{(g,x)}(X^i_k(g), i \frac{\partial}{\partial x_k}(x)) = 0,
\]

\[
\tilde{\Delta}_{(g,x)}(\eta^j(g), 0) = 0 \quad \text{for all} \; \eta \in \mathfrak{g}, \; \langle \eta, a \rangle = 0, \; k = 1, \ldots, r.
\]
It is easy to verify that $\Delta = -i\bar{\omega} + \frac{1}{2}dQ$:

$$
\Delta(g,x)(\xi \ell(g), \sum_{k=1}^{s} t_k \frac{\partial f}{\partial x_k}(x)) = -i(\sum_{k=1}^{s} \frac{\partial f}{\partial x_k}(x)X_k, \xi) + \sum_{k=1}^{s} t_k \frac{\partial f}{\partial x_k}(x),
$$

for all $\xi \in \mathfrak{g}$, $t_k \in \mathbb{R}$. Thus $i \cdot \Delta = d\bar{\omega}$. The form $d\bar{\omega} = \sum_{k=1}^{s} dx_k \wedge \bar{\omega}^{(k)} - \bar{\omega}^{a}$ is right $H$-invariant because $a(x) \in \mathfrak{a} \subset \mathfrak{g}_H$. Its kernel contains kernel (4.28) of the submersion $\pi_H \times \text{id}$ because $\langle a, h \rangle = 0$ and $[a, h] = 0$. Therefore there exists a unique 2-form $\omega$ on $G/H \times W^+$ such that $d\bar{\omega} = (\pi_H \times \text{id})^{*}\omega$. Since

$$d\bar{\omega} = i \cdot \Delta \overset{\text{def}}{=} i \cdot d((\pi_H \times \text{id})^{*}(d\bar{Q})) = i(\pi_H \times \text{id})^{*}(d(\bar{Q})) = (\pi_H \times \text{id})^{*}(i\delta(\partial Q))$$

and $\pi_H \times \text{id}$ is a submersion, we obtain that $i\delta\partial Q = \omega$.

To prove that the form $\omega$ is a Kähler form note that our form $d\bar{\omega}$ is a special case of the form considered in Theorem 5.1.

Indeed, choosing the vector-function $a$ as in Theorem 5.1 such that its components $a^i$, $a^m$ vanish identically on $W^+$ so that $a(x) = \sum_{k=1}^{s} \frac{\partial f}{\partial x_k}(x)X_k$, we obtain from (5.10) for this function $a$ that: (1) the $p \times p$-matrix-function $w_H(x)$, $p = \dim \mathfrak{m}_H$, is diagonal except for the first $r \times r$-block $2\frac{\partial^2 f}{\partial x_0 \partial x_j}(x)$, $k, j \in \{1, \ldots, r\}$; (2) the $s \times s$-matrix $w_s(x)$, $s = \dim \mathfrak{m}_s^+$, is diagonal.

Then the Hermitian matrices $w_H(x)$, $w_s(x)$ are positive-definite if and only if the matrix $\left(\frac{\partial^2 f}{\partial x_j \partial x_k}(x)\right)$ is positive-definite and the condition

$$w_{\lambda,j}(x) = 2\lambda'(a(x))/(\cosh \lambda'(x) \cdot \sinh \lambda'(x)) > 0$$

is satisfied for all restricted roots $\lambda \in \Sigma^+$ and $j = 1, \ldots, m_{\lambda}$. Since $x \in W^+$, one has $\sinh \lambda'(x) > 0$, $\cosh \lambda'(x) > 0$, so the previous condition can be simplified to $\lambda'(a(x)) > 0$ for all $\lambda \in \Sigma^+$, which amounts to the fact that $a(W^+) \subset W^+$.

Taking into account that condition (5.12) is condition (4) of Theorem 5.1 in our special case, we obtain the last statement of the corollary.

6. New complete invariant Ricci-flat Kähler metrics on $T\mathbb{S}^2$

Let $\mathfrak{g}$ be a compact Lie algebra and let $\sigma$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\Sigma$, etc. be as in Subsection 4.1. We continue with the previous notations but in this section it is assumed in addition that $G/K$ is the rank-one Riemannian symmetric space $\mathbb{S}^2$, that is $G/K = \text{SO}(3)/\text{SO}(2)$ (also $\mathbb{S}^2 \cong \mathbb{C}P^1 \cong \text{SU}(2)/\text{SU}(1) \times \text{U}(1)$). In this case the Lie algebra $\mathfrak{g}$ is the algebra $\mathfrak{so}(3)$ of skew-symmetric $3 \times 3$ real matrices. Denote by $E_{jk}$ the elementary $3 \times 3$ matrix with 1 in the entry in the $j$th row and the $k$th column and 0 elsewhere. Then the set of vectors $\{X, Y, Z\}$, where

$$X \overset{\text{def}}{=} E_{12} - E_{21}, \quad Y \overset{\text{def}}{=} E_{13} - E_{31} \quad \text{and} \quad Z \overset{\text{def}}{=} E_{23} - E_{32},$$

is a basis of the three-dimensional Lie algebra $\mathfrak{g}$. The compact Lie subalgebra $\mathfrak{k} = \mathbb{R}Z$ of the semisimple Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ is a Cartan subalgebra of $\mathfrak{g}$.

Fix on $\mathfrak{g} = \mathfrak{so}(3)$ the invariant trace form given by $\langle B_1, B_2 \rangle = -\frac{1}{2} \text{tr} B_1 B_2$, $B_1, B_2 \in \mathfrak{so}(3)$. Then all three vectors $X, Y, Z$ have the same length equal to 1 and the space $\mathfrak{m} = \mathbb{R}X \oplus \mathbb{R}Y$ is the orthogonal complement of $\mathfrak{k} = \mathbb{R}Z$ in $\mathfrak{g}$. Since $G/K$ is a rank-one symmetric space, each nonzero vector from the subspace $\mathfrak{m}$
generates a Cartan subspace of \( m \). Fix the Cartan subspace \( a = \mathbb{R}X \) of \( m \). It is easy to verify that
\[
(6.1) \quad [X, Y] = -Z, \ [X, Z] = Y, \ [Z, Y] = X \quad \text{and} \quad [Z, [Z, w]] = -w, \ \forall w \in m.
\]
From (6.1) it follows that the restricted root system \( \Sigma = \{ \pm \varepsilon \} \), where \( \varepsilon \in (a^C)^* \) and \( \varepsilon'(X) = 1 \), where, recall, \( \varepsilon = i \varepsilon' \). Also by (6.1), \( m^+ = m_\varepsilon = \mathbb{R}Y, \ t^+ = t_\varepsilon = \mathbb{R}Z \) and the algebra \( h = 0 \) (\( h \) is the centralizer of \( a = \mathbb{R}X \) in \( t = \mathbb{R}Z \)). Therefore the centralizer \( g_h \) of \( h = 0 \) in \( g \) coincides with the whole Lie algebra \( g \). Remark also that the domain \( W^+ = \{ xX : x \in \mathbb{R}, x > 0 \} \) can be naturally identified with \( \mathbb{R}^+ \). From (6.1) we have \( \text{Ad}_{\exp tZ} X = e^{t \text{ad}Z}(X) = \cos tX - \sin tY \).

But \( K = \{ \exp tZ, t \in \mathbb{R} \} \) and, as it is easy to verify, \( \exp tZ = E_{11} + \cos t(E_{22} + E_{33}) + \sin tZ \). Thus the map \( K \to m, \exp tZ \mapsto \text{Ad}_{\exp tZ} X \), is a one-to-one map and therefore, \( H = \{ e \} \), \( m^R = m \setminus \{ 0 \} \) and \( g_H = g \). Moreover, one obtains
\[
G \times W^+ \cong D^+ = G \times_K m^R = (G \times_K m) \setminus (G \times_K \{ 0 \}) \cong T^+ S^2,
\]
where \( T^+ S^2 \) is the punctured tangent bundle \( T^+ S^2 = TS^2 \setminus \{ \text{zero section} \} \) of \( S^2 \).

**Theorem 6.1.** Let \( G/K = \text{SO}(3)/\text{SO}(2) = \mathbb{S}^2 \). A 2-form \( \omega \) on the punctured tangent bundle \( G \times W^+ \cong T^+ S^2 \) of \( S^2 \) defines a \( G \)-invariant Kähler structure, associated to the canonical complex structure \( J^K_c \), and the corresponding metric \( \omega(J^K_c \cdot, \cdot) \) is Ricci-flat, if and only if \( \omega \) on \( G \times W^+ \) is expressed as \( \omega = d\theta^a \), where the vector-function \( a(x) = f'(x)X + \frac{c_Z}{\cosh x}Z, \ c_Z \) being an arbitrary real number and
\[
(6.2) \quad f'(x) = \sqrt{C \sinh^2 x + c_Z^2 \sinh^2 x \cosh^2 x + C_1},
\]
for some real constants \( C > 0 \) and \( C_1 \geq 0 \).

The corresponding \( G \)-invariant Ricci-flat Kähler metric \( g = g(C, C_1, c_Z) \) on \( T^+ S^2 \cong G \times W^+ \) is uniquely extendable to a smooth complete metric on the tangent bundle \( TS^2 \) if and only if \( C_1 = 0 \) (that is, \( \lim_{x \to 0} f'(x) = 0 \)).

**Proof.** By Theorem 5.1 we have to describe all vector-functions \( a : \mathbb{R}^+ \to g \) \( (g_H = g) \) satisfying Conditions (1)-(4) of that theorem. Then the 2-form \( \tilde{\omega} = d\theta^a \) belongs to the space \( \mathcal{R}(G \times W^+) \). Remark here that since \( H = \{ e \} \), one has \( G/H = G \) and \( \tilde{\omega} = \omega \).

By their definitions, for \( R_x \overset{\text{def}}{=} R_xX \) and \( S_x \overset{\text{def}}{=} S_xX, \ xX \in W^+ \),
\[
(6.3) \quad R_x|_{m_\varepsilon \oplus t_\varepsilon} = \frac{1}{\cosh x} \text{Id}_{m_\varepsilon \oplus t_\varepsilon} \quad \text{and} \quad S_x|_{m_\varepsilon \oplus t_\varepsilon} = \frac{1}{\sinh x} \text{ad}_X|_{m_\varepsilon \oplus t_\varepsilon}.
\]

Put \( \xi_\varepsilon = Y \in m_\varepsilon \). In the notation of the previous subsection, \( \xi_\varepsilon = Z \in t_\varepsilon \). Now we have to verify Conditions (1)-(4) of Theorem 5.1 for the vector-function
\[
a(x) = a^a(x) + a^\varepsilon(x) + a^{m}(x) = f'(x)X + f_Z(x)Z + f_Y(x)Y,
\]
where
\[
f \in C^\infty(\mathbb{R}^+, \mathbb{R}), \quad f_Y(x) = \frac{c_Y}{\sinh x}, \quad f_Z(x) = \frac{c_Z}{\cosh x}, \quad c_Y, c_Z \in \mathbb{R}.
\]
Remark here that \(h = 0\) and, consequently, the center \(z(h) = 0\). Consider now Conditions (5.5). We have \(m^+ = \mathbb{R}Y\). Using (6.3), we can rewrite the first condition in (5.5) for the vector \(Y = \xi^1\) as
\[
(6.4) \quad \frac{1}{\cosh x} \cdot R_x[f_Z(x)Z, Y] + \frac{1}{\sinh x} \cdot S_x[f_Z(x)Z, \text{ad}_X Y] = 0.
\]
The first term in (6.4) vanishes because \([Z, Y] = X \in \mathfrak{a}\) and \(R_x(\mathfrak{a}) = 0\); the second term vanishes because \(\text{ad}_X Y = -Z\).

Consider now the second condition in (5.5). Using (6.3) again, we can rewrite this condition for the vector \(Y \in m_\varepsilon\) as
\[
(6.5) \quad \frac{1}{\sinh x} \cdot R_x[f_Y(x)Y, \text{ad}_X Y] - \frac{1}{\cosh x} \cdot S_x[f_Y(x)Y, Y] = 0.
\]
The first term vanishes because \(\text{ad}_X Y = -Z\). By Theorem 5.1, (4), the corresponding form \(\tilde{\omega} = \tilde{d}^\mathfrak{a}\) belongs to the space \(R(G \times W^+)\) if and only if
\[
(6.6) \quad w_{11}(x) = 2f''(x), \quad w_{11}(x) = 2\left(\frac{c_Z}{\cosh^2 x} - \frac{c_Y}{\sinh^2 x}\right), \quad w_{11}(x) = -\frac{2f'(x)}{\cosh x \sinh x}.
\]
Calculating the determinant of the Hermitian matrix \(w_H(x)\) (as \(m^+ = 0\) in our case) we obtain that by Theorem 5.1(4), the corresponding form \(\tilde{\omega} = \tilde{d}^\mathfrak{a}\) belongs to the space \(R(G \times W^+)\) if and only if
\[
(6.7) \quad f''(x) > 0 \quad \text{and} \quad f''(x)f'(x) = \left(C + \frac{c_Z^2}{\cosh^4 x} + \frac{c_Y^2}{\sinh^4 x}\right) \cosh x \sinh x
\]
for all \(x \in \mathbb{R}^+\) and for some constant \(C \in \mathbb{R}^+\). Then
\[
(6.8) \quad f'(x) = \sqrt{C \cosh^2 x - c_Z^2 \cosh^2 x - c_Y^2 \sinh^2 x + C_2},
\]
where \(C_2 \in \mathbb{R}\). By (6.7), \(f''(x) > 0\) if and only if \(f'(x) > 0\). Therefore there exists a solution of (6.7) on the whole semi-axis if and only if \(c_Y = 0\). Putting \(C_2 = c_Z^2 - C + C_1\) one can rewrite (6.8) (with \(c_Y = 0\)) in the form (6.2).

Let us prove the last statement of the theorem. By its definition, \(\tilde{\omega} = \tilde{d}^\mathfrak{a}\). Since \(\text{ad}(x) = f'(x)X + \frac{c_Z}{\cosh x}Z\), by the expression (5.4) at the point \((g, x) \in G \times W^+ (W^+ = \mathbb{R}^+)\) we have
\[
(6.9) \quad \tilde{\omega}_{(g, x)}((\xi^1(g), t_1 \frac{\partial}{\partial x}), (\xi_2^1, t_2 \frac{\partial}{\partial x})) = -\langle f'(x)X + \frac{c_Z}{\cosh x}Z, [\xi_1, \xi_2]\rangle
\]
\[+ f''(x) \left(t_1 \langle X, \xi_2 \rangle - t_2 \langle X, \xi_1 \rangle \right) - \frac{c_Z \sinh x}{\cosh^3 x} \left(t_1 \langle Z, \xi_2 \rangle - t_2 \langle Z, \xi_1 \rangle \right),
\]
where \(\xi_1, \xi_2 \in g = T_x G\) and \(t_1, t_2 \in \mathbb{R}\). Our aim is to find the expression for the form \(\omega^R = ((f^+)^{-1})^*\omega\) on the space \(G \times K m^R \cong T^+(G/K)\) where, recall, \(f^+: G/H \times W^+ \rightarrow G \times K m^R\) is a \(G\)-equivariant diffeomorphism. But by the diagram (4.12) there exists a unique form \(\tilde{\omega}^R\) on \(G \times m^R\) such that
\[
(6.10) \quad \tilde{\omega}^R = \pi^*\omega^R \quad \text{and} \quad \tilde{\omega} = \text{id}^*\tilde{\omega}^R.
\]
Thus it is sufficient to calculate the form \(\tilde{\omega}^R\) on the space \(G \times m^R\). By the second expression in (6.10),
\[
\tilde{\omega}^R_{(g, x)}((\xi^1(g), t_1 X), (\xi^2_2(g), t_2 X)) = \tilde{\omega}_{(g, x)}((\xi^1(g), t_1 \frac{\partial}{\partial x}), (\xi^2_2(g), t_2 \frac{\partial}{\partial x}))
\]
on the space \(G \times m^R\). By (6.6)
and because $\langle X, X \rangle = 1$, one gets

$$\tilde{\omega}^R_{(g,x)}(\langle \xi_1^1(g), t_1 X \rangle, (\xi_2^1(g), t_2 X))$$

(6.11)

$$= -\langle \frac{f'}{|w|} u_1 + \frac{f'(|w|)}{|w|} t_1 X, \xi_1 \rangle + \langle f''(x)t_1 X, \xi_2 \rangle - \langle f''(x) \rangle t_2 X, \xi_1 \rangle$$

$$- \frac{c_g}{|w| \cosh x} \left( \langle t_1 X, x X \rangle \langle Z, \xi_2 \rangle - \langle t_2 X, x X \rangle \langle Z, \xi_1 \rangle \right).$$

Consider on the whole tangent space $T_{(g,w)}(G \times m^R)$ ($w \neq 0$), the following bilinear form $\Delta$,

$$\Delta_{(g,w)}(\langle \xi_1^1(g), u_1 \rangle, (\xi_2^1(g), u_2)) = -\langle \frac{f'(|w|)}{|w|} u_1 + \frac{f'(|w|)}{|w|} t_1 X, \xi_2 \rangle$$

(6.12)

$$- \langle \frac{f'(|w|)}{|w|} u_2 + \frac{f'(|w|)}{|w|} t_2 X, \xi_1 \rangle$$

$$- \frac{c_g}{|w| \cosh |w|} \left( \langle u_1, w \rangle \langle Z, \xi_2 \rangle - \langle u_2, w \rangle \langle Z, \xi_1 \rangle + \langle Z, [u_1, u_2] \rangle \right),$$

where $\xi_1, \xi_2 \in g = T_e G, u_1, u_2 \in m = T_w m^R$. Here $|w|^2 = \langle w, w \rangle (|x| = x)$. It is clear that this form is skew-symmetric. Since $\left( \frac{f''(|w|)}{|w|} \right) = \frac{f''(|w|)}{|w|} - \frac{f''(|w|)}{|w|^2}$ and $[t_1 X, t_2 X] = 0$, it is easy to verify that

$$\Delta_{(g,x)}(\langle \xi_1^1(g), t_1 X \rangle, (\xi_2^1(g), t_2 X)) = \tilde{\omega}^R_{(g,x)}(\langle \xi_1^1(g), t_1 X \rangle, (\xi_2^1(g), t_2 X)),$$

i.e. the restrictions of $\tilde{\omega}^R$ and $\Delta$ to $G \times W^+$ coincide. Now to prove that the differential forms $\tilde{\omega}^R$ and $\Delta$ coincide on the whole space $G \times m^R$ it is sufficient to show that the form $\Delta$ is left $G$-invariant, right $K$-invariant and its kernel contains (and therefore coincides with) the subbundle $\mathcal{K}$ defined by relation (1.24).

Since for each $k \in K$ the scalar product $\langle \cdot, \cdot \rangle$ is $Ad_k$-invariant, $Ad_k$ is an automorphism of $g$ and $Ad_k(Z) = Z$ ($k = e^{tZ}$ for some $t \in \mathbb{R}$) whence (3.22) holds, that is, $\Delta$ is left $G$-invariant and right $K$-invariant. We now prove that $\mathcal{K} \subset \ker \Delta$. Taking into account that by definition $\langle Z, m \rangle = 0$, $\langle Z, Z \rangle = 1$, by the invariance of the scalar product, $\langle \xi, [\xi, \eta] \rangle = 0, \forall \xi, \eta \in g$, and by (6.11)

$$\langle Z, [u_1, [w, Z]] \rangle = \langle Z, [[Z, w], u_1] \rangle = \langle [Z, [Z, w]], u_1 \rangle = -\langle w, u_1 \rangle,$$

we obtain that

$$\Delta_{(g,w)}(\langle \xi_1^1(g), u_1 \rangle, (Z^1(g), [w, Z])) = -\langle \frac{f'(|w|)}{|w|} u_1 + \frac{f'(|w|)}{|w|} t_1 X, \xi_1 \rangle$$

$$+ \left\langle \frac{f'(|w|)}{|w|} u_1 + \frac{f'(|w|)}{|w|} t_1 X, [w, Z] \right\rangle - \langle f''(|w|) \rangle [w, Z]$$

$$+ \left\langle \frac{f'(|w|)}{|w|} u_1 + \frac{f'(|w|)}{|w|} t_1 X, \xi_1 \right\rangle$$

$$- \frac{c_g}{|w| \cosh |w|} \left( \langle u_1, w \rangle \langle Z, \xi_2 \rangle - \langle [w, Z], w \rangle \langle Z, \xi_1 \rangle + \langle Z, [u_1, [w, Z]] \rangle \right)$$

$$= -\langle \frac{f'(|w|)}{|w|} u_1 + \frac{f'(|w|)}{|w|} t_1 X, \xi_1 \rangle - \langle f''(|w|) \rangle [w, Z], \xi_1 \rangle - \frac{c_g}{|w| \cosh |w|} \left( \langle u_1, w \rangle - \langle w, u_1 \rangle \right) \right) = 0.$$
whole axis $\mathbb{R}$, such that $(f'(x))^2$ is the restriction to $\mathbb{R}^+$ of the function

$$
(6.13) \quad \psi(x) = C_1 + (C + c_Z^2)x^2 + \psi_4(x)x^4, \quad \psi(x) > C_1, \ x \in \mathbb{R} \setminus \{0\}
$$

(see (6.2)). Expression (6.11) determines a smooth 2-form at $(g, 0) \in G \times m$ if and only if $\lim_{x \to 0} f'(x) = 0$, that is, if and only if $C_1 = 0$. In this case, expression (6.12) (which, possibly, is not the unique expression representing the form $\tilde{\omega}^R$) determines a smooth 2-form on the whole space $G \times m$ because $\frac{\sqrt{\psi(x)}}{x}$ and $\frac{1}{x} \left(\frac{\sqrt{\psi(x)}}{x}\right)'$ are even real analytic functions on the whole axis. We will denote this form (extension) on $G \times m$ by $\tilde{\omega}^R_0$. There exists a unique 2-form $\omega^R_0$ on $G \times K, m \cong T(G/K)$ such that $\tilde{\omega}^R_0 = \pi^* \omega^R_0$. The forms $\omega^R_0$ and $\omega^R$ coincide, by construction, on the open submanifold $G \times K, m \cong T^+(G/K)$, i.e. $\omega^R_0$ is a smooth extension of $\omega^R$. But by (6.6) and (6.13) for $C_1 = 0$

$$
\lim_{x \to 0} w_{11}(x) = 2\sqrt{C + c_Z^2}, \quad \lim_{x \to 0} w_{11}'(x) = 2icZ, \quad \lim_{x \to 0} w_{11}'(x) = 2\sqrt{C + c_Z^2},
$$

i.e. the corresponding limit $2 \times 2$ Hermitian matrix $\lim_{x \to 0} \omega_H(x)$ is positive-definite. Thus by Corollary 4.10, $\omega^R_0$ is the Kähler form of the metric $g_0$ (the extension of $g$) on $G \times K, m \cong T(G/K)$, for $G/K = S^2$.

Next, we show that the metric $g_0$ on $G \times K, m \cong T^2$ is complete. To prove this, we consider again the description of the form $\omega$ in (6.9) on the space $G \times W^+ \cong G \times K, m^R \cong T^+S^2$ ($G = SO(3)$, $H = \{e\}$ and $\tilde{\omega} = \omega$). For our aim it is sufficient to calculate the distance $\text{dist}(b, c)$ between the compact subsets $G \times \{b\}$ and $G \times \{c\}$, where $\text{dist}(b, c) = \inf\{d(p_b, p_c), p_b \in G \times \{b\}, p_c \in G \times \{c\}\}$. Since the sets $G \times \{x\}$ are compact, it is clear that the metric $g_0$ is complete if and only if for some $b > 0$ one has $\lim_{c \to \infty} \text{dist}(b, c) = \infty$.

To calculate the function $\text{dist}(b, c)$ note that the tangent bundle $T(G/K) \cong G \times K, m$ is a cohomogeneity-one manifold, i.e. the Lie group $G$ acts on this manifold with a codimension-one orbit. We will use only one fundamental fact on the structure of these manifolds [22]: A unit smooth vector field $U$ on a $G$-invariant domain $D \subset T(G/K)$ which is $g_0$-orthogonal to each $G$-orbit in $D$ is a geodesic vector field, i.e. its integral curves are geodesics of the metric $g_0$.

We now describe such a vector field $U$ on $G \times W^+ \cong T^+(G/K)$. Put

$$
(6.14) \quad f_U(x) = \left(\frac{f'(x) \cosh^3 x}{f''(x) \cosh^3 x - c_Z^2 \sinh x}\right)^{1/2}, \ x \in \mathbb{R}^+.
$$

**Lemma 6.2.** Such a unit vector field $U$ on $G \times W^+$ is determined by the expression

$$
U(g, x) = f_U(x) \cdot \left(\frac{c_Z \sinh x}{f'(x) \cosh^2 x}Y'(g), \frac{\partial}{\partial x}\right).
$$

For the coordinate function $x$ on $G \times W^+$ the following inequality holds

$$
(6.15) \quad |d_x(g, x)(\xi^l(g), t_\frac{\partial}{\partial x})| \leq f_U(x) \cdot \|\xi^l(g), t_\frac{\partial}{\partial x}\|_{(g, x)},
$$

where $(\xi^l(g), t_\frac{\partial}{\partial x}) \in T(g, x)(G \times W^+)$ and $\| \cdot \|$ is the norm determined by the metric $g$. 

Proof. (Of the Lemma.) Let us rewrite the expression (6.9) as

\[ \omega_{(g,x)}((\xi_1^i(g), t_1 \frac{\partial}{\partial x}), (aX + bY + cZ, t_2 \frac{\partial}{\partial x})) = t_1 \left( f''(x) \langle X, \xi_2 \rangle - \frac{cZ \sinh x}{\cosh^2 x} \langle Z, \xi_2 \rangle \right) + \langle [f'(x)X + \frac{cZ}{\cosh x}Z, \xi_2] - f''(x)t_2X + \frac{cZ \sinh x}{\cosh^2 x} \rangle t_2Z, \xi_1 \rangle. \]

Therefore for \( \xi_2 = aX + bY + cZ, a, b, c \in \mathbb{R} \), by the commutation relations (6.1) we have

\[ \omega_{(g,x)}((\xi_1^i(g), t_1 \frac{\partial}{\partial x}), ((aX + bY + cZ)^l(g), t_2 \frac{\partial}{\partial x})) = t_1 \left( f''(x)a - \frac{cZ \sinh x}{\cosh^2 x} b \right) + \langle [f'(x)X - f''(x)t_2]X, \xi_1 \rangle + \langle [f'(x) - a \frac{cZ}{\cosh x}]Y + (\frac{cZ \sinh x}{\cosh^2 x} - b f'(x))Z, \xi_1 \rangle. \]

Since the vector field \( U \) is \( g \)-orthogonal to the subbundle \( V \subset T(G \times W^+) \) generated by the vector fields \( (\xi_1^i, 0), \xi_1 \in g \), then \( U \) is \( \omega \)-orthogonal to the subbundle \( J_c^K(V) \) generated by \( (\xi_1^i, t_1 \frac{\partial}{\partial x}) \), \( \xi_1 \in g \), \( (\xi, X) = 0, t_1 \in \mathbb{R} \), because of (6.1)

\[ J_c^K(X^l, 0) = (0, \frac{\partial}{\partial x}) \quad \text{and} \quad J_c^K(Y^l, 0) = (-\frac{\cosh x}{\sinh x} Z^l, 0). \]

Putting \( U = ((aX + bY + cZ)^l, \tau \frac{\partial}{\partial x}) \), where \( a, b, c, \tau \) are functions of \( x \), we obtain the following orthogonality conditions

\[ a f'' - \frac{cZ \sinh x}{\cosh^2 x} = 0, \quad cf' - a \frac{cZ}{\cosh x} = 0, \quad \tau \frac{cZ \sinh x}{\cosh^2 x} = b f' = 0, \]

with the solution: \( a = 0, c = 0 \) and \( b = \tau \frac{cZ \sinh x}{f' \cosh^2 x} \). Thus \( U = \tau \left( \frac{cZ \sinh x}{f' \cosh^2 x} Y^l, \frac{\partial}{\partial x} \right) \).

Since \( \|U\|_{\omega} \equiv \omega(J_c^K(U), U) \equiv 1 \), then by (6.17) and (6.18)

\[ \tau^2 \omega \left( \left( -\frac{cZ}{f' \cosh x} Z^l - X^l, 0 \right), \left( \frac{cZ \sinh x}{f' \cosh^2 x} Y^l, \frac{\partial}{\partial x} \right) \right) = \tau^2 \left( f'' - \frac{cZ \sinh x}{f' \cosh^2 x} \right) \equiv 1. \]

Thus \( U(g, x) = \left( \frac{f'(x) \cosh^3 x}{f'(x) \cosh^2 x - cZ \sinh x} \right)^{1/2} \left( \frac{cZ \sinh x}{f'(x) \cosh^2 x} Y^l(g), \frac{\partial}{\partial x} \right) \).

To prove the inequality in the statement let us find the Hamiltonian vector field \( H^x \) of the \( (G\text{-invariant}) \) function \( x \). Putting \( H^x = ((a_0X + b_0Y + c_0Z)^l, \tau_0 \frac{\partial}{\partial x}) \), where \( a_0, b_0, c_0, \tau_0 \) are functions of \( x \), we obtain the following relation

\[ t_1 = dx(\xi_1^l, t_1 \frac{\partial}{\partial x}) \overset{\text{def}}{=} \omega((\xi_1^l, t_1 \frac{\partial}{\partial x}), H^x) = \omega((\xi_1^l, t_1 \frac{\partial}{\partial x}), \langle (a_0X + b_0Y + c_0Z)^l, \tau_0 \frac{\partial}{\partial x} \rangle), \]

for arbitrary \( t_1 \in \mathbb{R}, \xi_1 \in g \). Using (6.17) again we obtain the following equations:

\[ a_0 f'' - \frac{cZ \sinh x}{cZ} = 1, \quad b_0 \frac{cZ}{\cosh x} = \tau_0 f'', \quad c_0 f' = a_0 \frac{cZ}{\cosh x}, \quad \tau_0 \frac{cZ \sinh x}{\cosh^2 x} = b_0 f'. \]

with the following solution: \( b_0 = 0, \tau_0 = 0 \) and

\[ a_0 = c_0 \cdot \frac{f' \cosh x}{cZ}, \quad c_0 = \frac{cZ \cosh^2 x}{f' \cosh^2 x - cZ \sinh x}. \]
Thus \( H^x = (a_0X + c_0Z)^t, 0 \). Since \( J_c^K (H^x) = (c_0 \sinh x Y^t, a_0 \frac{\partial}{\partial x}) \), we have that

\[
\|H^x\|^2 \overset{\text{def}}{=} \omega \left( \left( c_0 \sinh x Y^t, a_0 \frac{\partial}{\partial x} \right), \left( (a_0X + c_0Z)^t, 0 \right) \right) = c_0^2 f' \sinh x - 2a_0c_0 \frac{\partial}{\partial x} \sinh x + a_0^2 f''.
\]

Taking into account (6.19) we obtain that \( \|H^x\|^2 = \frac{f' \cosh^3 x}{f'' \cosh x - c_0^2 \sinh x} \). Hence \( \|H^x\| = f_U \). Now, by the Cauchy-Schwarz inequality for metrics one has

\[
\left| dx(\xi_1', t_1), t_1 \right| = \left| g((\xi_1', t_1), H^x) \right| = \left| g((\xi_1', t_1), J_c^K (H^x)) \right| \leq \left\| J_c^K (H^x) \right\| : \left\| (\xi_1'(g), t_1, t_1) \right\| = \|H^x\| : \left\| (\xi_1'(g), t_1) \right\|,
\]

that is, we obtain (6.15).

Using now the vector field \( U \) we shall calculate the distance between the level sets \( G \times \{b\} \) and \( G \times \{c\} \) in \( G \times W^+ \) with respect to the metric \( g \). Let \( \gamma(t) = (\hat{g}(t), \hat{x}(t)), t \in [0, T] \), be the integral curve of the vector field \( U \) with initial point \( p_b \) in \( G \times \{b\} \), that is, \( \hat{x}(0) = b \). There exists a function \( h \) on \( \mathbb{R}^+ \) such that the function \( h(\hat{x}(t)) \) is linear in \( t \). It is easy to verify that

\[
\frac{d}{dt} h(\hat{x}(t)) = h'(\hat{x}(t)) \cdot \hat{x}'(t) = h'(\hat{x}(t)) \cdot dx(\gamma(t)) = h'(\hat{x}(t)) \cdot (f_U(\hat{x}(t))) = 1.
\]

Suppose that \( p_c \in G \times \{c\} \), where \( p_c = \gamma(t_c) \), \( t_c \in [0, T] \). Since the curve \( \gamma \) is a geodesic, the length of the curve \( \gamma(t), t \in [0, t_c] \), from \( p_b \) to \( p_c \) is \( t_c = h(x(p_c)) - h(x(p_b)) = h(c) - h(b) \). Thus \( \text{dist}(b, c) \geq h(c) - h(b) \).

For any other curve \( \gamma_1(t) = (\hat{g}_1(t), \hat{x}_1(t)) \), with \( \|\gamma_1'(t)\| = 1 \), starting at the point \( p_b \), and ending at a point \( p^1_c \in G \times \{c\} \), \( p^1_c = \gamma_1(t^1_c) \) (of length \( t^1_c \)), we have by Lemma 6.2

\[
\frac{d}{dt} h(\hat{x}_1(t)) = h'(\hat{x}_1(t)) \cdot dx(\gamma_1(t)) \leq \frac{1}{f_U(\hat{x}_1(t)))} : f_U(\hat{x}_1(t))) \cdot \|\gamma_1'(t)\| = 1.
\]

Thus \( h(c) - h(b) \leq t^1_c \) and the length \( t^1_c \) of the curve \( \gamma_1 \) from \( p_b \) to \( p^1_c \) is not less than the length of the curve \( \gamma(t), t \in [0, t_c] \). So the distance between the level surfaces \( G \times \{b\} \) and \( G \times \{c\} \) is \( |h(c) - h(b)| \).

Now, since by (6.12) and (6.7) for \( C_1 = 0 \)

\[
f'(x) = \sqrt{C \sinh^2 x + c^2 \sinh^2 x \cosh^2 x}, \quad f''(x) = (C \sinh x \cosh x + c^2 \sinh x \cosh^3 x) / f'(x),
\]

we see that \( f'(x) \sim \sqrt{C} \sinh x, f''(x) \sim \sqrt{C} \sinh x \) and, by (6.11), \( f_U(x) \sim \left( \sqrt{C} \sinh x \right)^{1/2} \) as \( x \to \infty \). Therefore \( \lim_{x \to \infty} h(x) = \infty \). Hence the metric \( g_0 = g_0(C, c_Z, 0) \) (that is, for \( C_1 = 0 \) on the tangent bundle \( T(G/K) \cong G \times_K \mathfrak{m} \) is complete for any \( C > 0, c_Z \in \mathbb{R} \).\]
The proof of Theorem 6.1 above and Corollary 5.4 immediately imply the following

**Corollary 6.3.** Let \( G/K = \text{SO}(3)/\text{SO}(2) = \mathbb{S}^2 \). A 2-form \( \omega \) on the punctured tangent bundle \( G \times W^+ \cong T^+ \mathbb{S}^2 \) of \( \mathbb{S}^2 \) determines a \( G \)-invariant Kähler structure, associated to the canonical complex structure \( J_e^K \) if and only if \( \omega \) on \( G \times W^+ \) is expressed as \( \omega = d\theta^a \), for the vector-function \( a(x) = f''(x)x + \frac{c_f}{\cosh x}Z + \frac{c_Y}{\sinh x}Y \), where \( c_Z, c_Y \in \mathbb{R}, f \in C^\infty(\mathbb{R}^+, \mathbb{R}) \) and

\[
f''(x) > 0, \quad \forall x \in \mathbb{R}^+, \quad \frac{f''(x)f'(x)}{\cosh x \sinh x} - \frac{c_Z^2}{\cosh^4 x} - \frac{c_Y^2}{\sinh^4 x} > 0, \quad \forall x \in \mathbb{R}^+.
\]

In particular, if \( c_Z = c_Y = 0 \) then the function \( (g, x) \mapsto 2f(x) \) on \( G \times W^+ \) is a potential function of the Kähler structure \((\omega, J_e^K)\).

Finally, we relate the metrics \( g(C, c_Z, C_1) \) of Theorem 6.1 with the Eguchi-Hanson and Stenzel metrics. We will show that our metrics coincide with the well-known (hyper-Kähler) Eguchi-Hanson metrics if \( C_1 = 0 \) and \( c_Z = 0 \). To prove it, let us rewrite the metrics \( g(C, c_Z, C_1) \) in terms of the left \( G \)-invariant forms \( \theta^X, \theta^Y \) and \( \theta^Z \) on the Lie algebra \( G = \text{SO}(3) \). Indeed, taking into account the commutation relations (6.1) for any \( \xi, \eta \in \mathfrak{g} = T_eG \), we have

\[
[\xi, \eta] = -((\theta^X \wedge \theta^Z)(\xi, \eta) \cdot X + (\theta^Y \wedge \theta^Z)(\xi, \eta) \cdot Y + (\theta^Y \wedge \theta^X)(\xi, \eta) \cdot Z).
\]

Taking into account expression (6.9) for the Kähler form \( \omega = \bar{\omega} \), we obtain that

\[
\omega = f'(x)\theta^Y \wedge \theta^Z + \frac{c_f}{\cosh x}\theta^X \wedge \theta^Y + f''(x)dx \wedge \theta^X - \frac{c_Z \sinh x}{\cosh^2 x}dx \wedge \theta^Z.
\]

But \( g(\cdot, \cdot) = \omega(J_e^K \cdot, \cdot) \) and therefore by (6.18),

\[
ds^2 = f''(x)(dx^2) + f'(x)\sinh x \cdot \theta^Z dx^2 + f'(x)\cosh x \cdot (\theta^Y)^2 + f''(x)(\theta^X)^2 - c_Z \left( \frac{\sinh x}{\cosh x} \cdot (\theta^X \theta^Z + \theta^Z \theta^X) + \frac{1}{\cosh x}(dx \cdot \theta^Y + \theta^Y dx) \right),
\]

where the functions \( f'(x) \) and \( f''(x) \) are described by expressions (6.20). Putting \( c_Z = 0 \) (then \( f'(x) = \sqrt{C} \sinh x \)) we obtain the “diagonal” Stenzel metric

\[
(1/\sqrt{C})ds^2 = \cosh x(dx)^2 + \sinh x \tanh x(\theta^Z)^2 + \cosh x(\theta^Y)^2 + \cosh x(\theta^X)^2,
\]

which for \( C = 1 \) after the change of variable \( \cosh x = (t/\ell)^2 \) becomes the Eguchi-Hanson metric with parameter \( \ell \) (see Gibbons and Pope [23] (4.17)).

**References**

[1] M. Stenzel, Ricci-flat metrics on the complexification of a compact rank one symmetric space, Manuscripta Math. 80 (1993) 151–163.

[2] M. Cvetić, G.W. Gibbons, H. Lü, C.N. Pope, Ricci-flat metrics, harmonic forms and brane resolutions, Comm. Math. Phys. 232 (2003) 457–500. [doi:10.1007/s00220-002-0730-3](https://doi.org/10.1007/s00220-002-0730-3)

[3] T.C. Lee, Complete Ricci-flat Kähler metric on \( M^1_4, M^2_4, M_{11}^4 \), Pacific J. Math. 185 (2) (1998) 315–326. [doi:10.2140/pjm.1998.185.315](https://doi.org/10.2140/pjm.1998.185.315)

[4] G. Patrizio, P. Wong, Stein manifolds with compact symmetric center, Math. Ann. 289 (1991) 355–382.

[5] J.M. Baptista, Some special Kähler metrics on \( SL(2,\mathbb{C}) \) and their holomorphic quantization, J. Geom. Phys. 50 (1) (2004) 1–27. [doi:10.1016/j.geomphys.2003.10.012](https://doi.org/10.1016/j.geomphys.2003.10.012)
[6] A.S. Dancer, I.A.B. Strachan, Einstein metrics on tangent bundles of spheres, Classical Quantum Gravity 19 (18) (2002) 4663–4670. doi:10.1088/0264-9381/19/18/303

[7] T. Eguchi, A.J. Hanson, Asymptotically flat self-dual solutions to Euclidean gravity, Phys. Lett. B74 (1978) 249–251.

[8] G.D. Mostow, Some new decomposition theorems for semisimple groups, Mem. Amer. Math. Soc. 14 (1955) 31–54, ISSN: 0065-9266.

[9] G.D. Mostow, On covariant fiberings of Klein spaces, Amer. J. Math. 77 (1955) 247–278. doi:10.2307/2372530.

[10] H. Azad, R. Kobayashi, Ricci flat Kähler metrics on symmetric varieties, Miramare-Trieste (1994).

[11] R. Bielawski, Prescribing Ricci curvature on complexified symmetric spaces, Math. Res. Lett. 11 (2004) 435–441. doi:10.4310/MRL.2004.v11.n4.a3

[12] O. Biquard, P. Gauduchon, Hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces, in: P. H. Andersen J., Dupont J., S. A. (Eds.), Geometry and Physics, Vol. 184 of Lect. Notes Pure Appl. Math., CRC Press, 1996, pp. 287–298. ISBN 9780824797911.

[13] O. Biquard, P. Gauduchon, Géométrie hyperkählérienne des espaces hermitiens symétriques complexifiés, Séminaire de Théorie spectrale et Géométrie 16 (1998) 127–173.

[14] E. Calabi, Métriques kähleriennes et fibrés holomorphes, Ann. Sci. Ec. Norm. Super. 12 (1979) 269–294.

[15] P.M. Gadea, J.C. González-Dávila, I.V. Mykytyuk, Complete Ricci-flat Kähler metrics on compact rank-one symmetric spaces, preprint.

[16] V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (3) (1982) 515–538. doi:10.1007/BF01398934

[17] I.V. Mykytyuk, Invariant Kähler structures on the cotangent bundles of compact symmetric spaces, Nagoya Math. J. 169 (2003) 191–217. doi:10.1017/S00277630000008497

[18] P.M. Gadea, J.Muñoz Masqué, I.V. Mykytyuk, Analysis and Algebra on Differentiable Manifolds: A Workbook for Students and Teachers, Springer-Verlag, London, 2013, ISBN: 978-94-007-5951-0; 978-94-007-5952-7.

[19] S. Kobayashi, K. Nomizu, Foundations of differential geometry, Vol. 2, Interscience Publishers, New York, London, 1969.

[20] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Pure and Applied Mathematics, a Series of Monographs and Textbooks, Academic Press, New York, San Francisco, London, 1978, ISBN:0-12-338460-5.

[21] N. Bourbaki, Lie Groups and Lie Algebras. Chapters 4-6, Springer, 2002.

[22] A.V. Alekseevsky, D.V. Alekseevsky, Riemannian G-manifolds with one-dimensional orbit space, Ann. Global Anal.Geom. 11 (3) (1993) 197–211. doi:10.1007/BF00773366

[23] G.W. Gibbons, C.N. Pope, The positive action conjecture and asymptotically Euclidean metrics in quantum gravity, Comm. Math. Phys. 66 (3) (1979) 267–290. doi:10.1007/BF01197188

Instituto de Física Fundamental, CSIC, Serrano 113 bis, 28006-Madrid, Spain.
E-mail address: p.m.gadea@csic.es

Departamento de Matemáticas, Estadística e Investigación Operativa, University of La Laguna, 38200 La Laguna, Tenerife, Spain.
E-mail address: jcgonza@ull.es

Institute of Mathematics, Cracow University of Technology, Warszawska 24, 31155, Cracow, Poland.
Institute of Applied Problems of Mathematics and Mechanics, Naukova Str. 3b, 79601, Lviv, Ukraine.
E-mail address: mykytyuk_i@yahoo.com