THEORY OF SMALL $x$ INCLUSIVE PHOTON SCATTERING, I

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ABSTRACT.-

In the early eighties, López, González-Arroyo and the present author proved that, if at a given $Q^2_0$ large enough for perturbative QCD to be valid, structure functions behave as a power of $x$ for $x \to 0$, then for all larger $Q^2$ one has

$$F_2(x,Q^2) \simeq B_S[\alpha_s(Q^2)]^{-d_+} x^{-\Lambda} + B_{NS}[\alpha_s(Q^2)]^{-D_{11}} x^{0.5},$$

$$F_G(x,Q^2) \simeq B_G[\alpha_s(Q^2)]^{-d_+} x^{-\Lambda},$$

$$R(x,Q^2) = \frac{r_0 \alpha_s(Q^2)}{\pi},$$

with $D_{11}$, $d_+$, $B_G$, $r_0$ calculable in terms of $B_S$, $\Lambda$. Moreover, it was suggested that the “hard” part of the scattering cross section for real photons (Compton scattering) obeys a similar law, so that

$$\sigma_{\gamma p} \simeq B_{\gamma p} s^\Lambda + A_{\gamma p} \hat{\sigma}_P,$$

with a value of $\Lambda$ comparable to that in the expression for the structure functions, and where $\hat{\sigma}_P \sim \log^2 s$ is a universal, Pomeron-type cross section, and $A_{\gamma p}$, $B_{\gamma p}$ are constants. In the present paper it is shown that the recent HERA measurements may be described by these formulas, with a chi-squared/d.o.f. substantially less than unity, and with values of the parameters compatible with those of the old fits of the ’80s. Moreover, further discussions are presented both on the low $Q^2$ limit, and the transition between Compton and deep inelastic scattering, in particular in connection with possible saturation of the coupling constant $\alpha_s(Q^2)$ at small $Q^2$; and on the ultra high energy limit, and how one might test the so-called BFKL conjecture,

$$\lim_{x \to 0} \lim_{Q^2 \to \infty} F_2(x,Q^2) \sim x^{-c_0 \alpha_s}.$$

With respect to the last we find some evidence against it, at least at the HERA energies.

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1.-INTRODUCTION

We consider in this note high energy inclusive scattering of (virtual or real) photons off protons:

\[ \gamma^*(q) + p(p) \to \text{all}. \quad (1.1) \]

In the case of real photons we will call the process Compton scattering; for virtual photons we have DIS (=deep inelastic scattering). In this last situation we consider the region of very small \( x \) with

\[ x = Q^2/\nu, \ \nu = p \cdot (p + q), \ Q^2 = -q^2. \quad (1.2) \]

Compton scattering may be thought of as the limit of DIS when \( Q^2 \to 0, \ x \to 0 \) in such a way that

\[ Q^2/x \approx s, \ s = (p + q)^2. \quad (1.3) \]

To a large extent the present work may be considered as an aggiornamento of the old analysis of C. López, A. González-Arroyo and the present author, applying it to the recent HERA DIS and Compton data. In this respect it is shown that the new data are fitted very well by the formulas derived from QCD in the small \( x \) region, and that the expressions

\[ F_2(x, Q^2) \approx B_{S}[\alpha_s(Q^2)]^{-d_+}x^{-\lambda} + B_{NS}[\alpha_s(Q^2)]^{-D_{11}}x^{0.5}, \]

\[ F_G(x, Q^2) \approx B_G[\alpha_s(Q^2)]^{-d_+}x^{-\lambda} \quad (1.4) \]

\[ R(x, Q^2) = \frac{r_0\alpha_s(Q^2)}{\pi}, \]

with \( B_G, D_{11}, d_+ , r_0 \) calculable in terms of \( B_S, \lambda \), give an excellent approximation to the data for \( x < 10^{-2} \) (Sect.2). Likewise (Sect.3) we find that the very high energy Compton cross section is still correctly described by the formula

\[ \sigma_{\gamma p} \approx B_{\gamma p}s^\lambda + A_{\gamma p}\hat{\sigma}^P, \quad (1.5) \]

with a value of \( \lambda \) similar to the \( \lambda \) in (1.4), and where \( \hat{\sigma}^P \sim \log^2 s \) is a universal, Pomeron-type cross section. \( A_{\gamma p} \) and \( B_{\gamma p} \) are constants.

The quality of the fits is so good, with chi-squared of less than one by d.o.f., that one may consider studying the corrections to (1.4), and the interpolation between (1.4) and (1.5); this we do in Sect.4. In Sect.5 we discuss the ultra high momentum limit of our formulas, in particular in connection with the BFKL approach which suggests the asymptotic behaviour

\[ F_2(x, Q^2) \sim x^{\alpha_s}. \quad (1.6) \]

We study the region of validity of (1.4), limited for fixed \( x \) by a certain \( Q^2(x) \). We get indications that, for \( x \lesssim 10^{-2}, \ Q^2(x) \approx 200 - 300 \ \text{GeV}^2 \).

In what regards the transition from (1.4) to (1.6) we get (not conclusive, however) evidence against it at least in the region covered by the HERA data, i.e., below \( Q^2 \approx 800 \ \text{GeV}^2 \).

* More details on notation may be found in ref.1
2. DEEP INELASTIC SCATTERING AT $x \to 0$

2.1. General considerations

For DIS the relevant quantities are the structure functions,

$$ F_2(x, Q^2), \ F_G(x, Q^2); \ R(x, Q^2) = \frac{F_2(x, Q^2) - F_1(x, Q^2)}{F_2(x, Q^2)}. \quad (2.1) $$

$F_G$ is the gluon structure function, and $R$ is (proportional to) the longitudinal one.

It is convenient to use the singlet function $F_S$, normalized so that the momentum sum rule reads

$$ \int_0^1 dx [F_S(x, Q^2) + F_G(x, Q^2)] \to 1. \quad (2.2) $$

The relation between $F_2$ and $F_S$ is as follows: one has

$$ F_2(x, Q^2) = \langle e_2^2 \rangle [F_S(x, Q^2) + F_{NS}(x, Q^2)], \quad (2.3a) $$

where $F_{NS}$ is the so-called nonsinglet structure function. Since in the limit $x \to 0$ it decreases very fast compared to $F_S$, we may consider the approximate relation

$$ F_2(x, Q^2) \approx x \to 0 \langle e_2^2 \rangle F_S(x, Q^2). \quad (2.3b) $$

The average charge of the excited flavours in (2.3) is $\langle e_2^2 \rangle = \frac{5}{18}$ for $n_f = 4$ or $\langle e_2^2 \rangle = \frac{11}{45}$ for $n_f = 5$. For the values of $Q^2$, $x$ with $8 \text{ GeV}^2 \lesssim Q^2 \lesssim 65 \text{ GeV}$ we are in a mixed situation in which bottom is excited in the $s$ variable but not in the $Q^2$ variable. We will thus present results for both $n_f = 4$ and 5: as will be seen they differ very little. For $Q^2 \lesssim 9 \text{ GeV}^2$ only three flavours are excited in the $Q^2$ variable. Both for this and other reasons this energy region requires a specific treatment.

The evolution equations for the structure functions may be written in two ways. One has the so called Altarelli-Parisi, or DGLAP equations\cite{2,3], that we write only for the singlet structure functions,

$$ \frac{\partial F_i(x, Q^2)}{\partial t} = \sum_j \int_x^1 \mathrm{d}z P_{ij}(z) F_j(x/z, Q^2), \quad (2.4) $$

and the explicit form of the splitting functions $P_{ij}$ may be found in refs. 1, 3. Alternatively, one may define the moments of the structure functions,

$$ \mu_S(n, Q^2) = \int_0^1 \mathrm{d}x x^{n-2} F_S(x, Q^2), \quad \mu_G(n, Q^2) = \int_0^1 \mathrm{d}x x^{n-2} F_G(x, Q^2) \quad (2.5) $$

and then the integro-differential equations (2.4) may be solved for the $\mu_i$:

$$ \begin{pmatrix} \mu_S(n, Q^2) \\ \mu_G(n, Q^2) \end{pmatrix} = \begin{pmatrix} \alpha_s(Q_0^2) \\ \alpha_s(Q_0^2) \end{pmatrix} \begin{pmatrix} \alpha_s(Q^2) \\ \alpha_s(Q^2) \end{pmatrix} \begin{pmatrix} \mu_S(n, Q_0^2) \\ \mu_G(n, Q_0^2) \end{pmatrix}, \quad (2.6a) $$

where the anomalous dimension matrix $D(n)$ is\cite{4]
\[ D(n) = \frac{16}{33 - 2n_f} \times \left( \begin{array}{c} 1 \frac{3}{4} - S_1(n) \\ \frac{n^2 + n + 2}{2n(n^2 - 1)} \frac{9}{4n(n - 1)} + \frac{3n_f}{8} \frac{n^2 + n + 2}{n(n + 1)(n + 2)} + \frac{9}{4(n + 1)(n + 2)} + \frac{33 - 2n_f}{16} - \frac{9S_1(n)}{4} \end{array} \right). \]  

The function \( S_1(z) = \psi(z + 1) + \gamma_E \), \( \psi(z) = d\log \Gamma(z)/dz \), \( \gamma_E = 0.5772 \ldots \).

It should perhaps be stressed that eqs. (2.4) and (2.6) are strictly equivalent, as known from the very first works on the subject\(^8\); (2.6) follows from (2.4) by taking moments and integrating, and (2.4) from (2.6) by inverting the Mellin transform (2.5) and differentiating.

Eqs (2.6), in the limit \( x \to 0 \), were solved long ago by López and the present author for \( F_{NS}, F_2 \) and \( F_G \) to leading order (LO) in ref.5; including next to leading corrections in ref.6 and, for the longitudinal structure function by the quoted authors and González-Arroyo in ref.7 [where the interested reader may find the extension of eqs. (2.4), (2.6) to \( R(x; Q^2) \)]. The comparison with experiment was carried to leading order in refs. 5, 7; the comparison with experimental data of the calculation to next to leading order (NLO) was given in ref.8.

The LO results are remarkably simple. It was proved in ref.5 that, under certain conditions, the structure functions are given by the very explicit expressions, which therefore solve the evolution equations, *

\[ F_S(x; Q^2) \approx \frac{B_S[\alpha_s(Q^2)]^{-d_+(1+\lambda)} x^{-\lambda}}{(1+\lambda)}, \]

\[ F_G(x; Q^2) \approx \frac{B_G(1+\lambda)}{\langle e_q^2 \rangle B_S} F_S(x; Q^2), \]

\[ F_{NS}(x; Q^2) \approx \frac{B_{NS}[\alpha_s(Q^2)]^{-D_{11}(1-\rho)} x^\rho}{\gamma_E}. \]

We will also use the quantities

\[ B_S \equiv \langle e_q^2 \rangle B_S, B_{NS} \equiv \langle e_q^2 \rangle^2 B_{NS}. \]

\( \lambda, B_{NS} \) and \( B_S \) are free parameters, independent of \( x, Q^2 \), although they may have a dependence on \( n_f \) (expected to be slight; it will be discussed in Sect.6). \( \rho \) is known from Regge theory to be the intercept of the \( f^0 - \rho \) trajectory, \( \rho \approx 0.5 \). The quantities \( B_G, r_0 \) are obtainable in terms of \( \lambda, B_S \):

\[ \frac{B_G(1+\lambda)}{B_S} = d_+(1+\lambda) - D_{11}(1+\lambda), \]

\[ r_0(1+\lambda) = \frac{4}{3(2+\lambda)} \left[ 1 + \frac{3n_f}{2} \frac{d_+(1+\lambda) - D_{11}(1+\lambda)}{3(2+\lambda) D_{12}(1+\lambda)} \right]. \]

Here the \( D_{11}(1-\rho), D_{12}(1+\lambda) \) and the \( d_\pm(1+\lambda) \),

\[ d_\pm(1+\lambda) = \frac{1}{2} \left\{ D_{11}(1+\lambda) + D_{22}(1+\lambda) \pm \sqrt{[D_{11}(1+\lambda) - D_{22}(1+\lambda)]^2 + 4D_{12}(1+\lambda)D_{21}(1+\lambda)} \right\} \]

are the matrix elements and eigenvalues of the matrix \( D(n) \) evaluated (respectively) at \( n = 1 - \rho, n = 1 + \lambda \) (the eigenvalues ordered so that \( d_+ > d_- \)). We note that eqs. (2.5), (2.6) are valid for arbitrary (even complex) values of \( n \). Thus, eqs. (2.8) give an explicit expression for all three structure functions \( F_S, F_G \)

* For ease of reference we will reproduce the proof in Sect.6 here.
and $R$ in terms of the only two free parameters $\lambda$ and $B_S$ ($B_{NS}$ also intervenes if we include $F_{NS}$ in the analysis).

Surprisingly enough, eqs. (2.8) and, more generally, the work of refs. 5-8 seem to have been ignored by recent publications\cite{9,10}. Even more surprising, the HERA physicists have not used eqs. (2.8) in spite of their simplicity to analyse the experimental data, relying instead on a painstaking numerical integration of the Altarelli-Parisi equations (2.4) –a procedure not only infinitely more complicated than use of the explicit solutions (2.8), but that, as will be shown, produces larger errors.

2.2.-Low $Q^2$ and medium $x$ analysis, and higher $Q^2$ and very small $x$ predictions

In the 1980 paper, López and the present author analyzed DIS data in the range

$$3.25 \text{ GeV}^2 \leq Q^2 \leq 22.5 \text{ GeV}^2, \quad x > 2 \times 10^{-2}. \quad (2.9)$$

We found that the parameter $\lambda$ could be well determined while $B_S$ (which gives the overall normalization) was more uncertain. This uncertainty was due basically to the strong contribution of subleading effects, especially those due to the nonsinglet structure function, at the (relatively) large values of $x$ used. One had

$$\lambda = 0.37 \pm 0.07, \quad 0.02 \geq B_S \geq 0.001. \quad (2.10a)$$

[Actually the value and error take into account also the analysis of $R$; $F_2$ alone would produce a slightly smaller $\lambda$, $\lambda = 0.36$]. The results of the evaluation were subsequently shown to be essentially unaltered by inclusion of NLO corrections\cite{8}. As for $B_{NS}$, the fits based on functions containing both singlet and nonsinglet parts fix it only within large errors. Fortunately, however, the $W_3$ structure function in neutrino scattering is purely nonsinglet and allows a reasonably precise determination: $B_{NS} \sim 1.4$ to 1.8 or\cite{8},

$$B_{NS} \simeq 0.3 \text{ to } 0.6. \quad (2.10b)$$

With these numbers it is a trivial matter to predict the very low $x$ measurements of $F_2$ at HERA in terms of the only parameter $B_S$ that has to be taken as essentially free because, as explained before, it is not well fixed by the old, larger $x$ data. In a more precise determination we will of course also allow $\lambda$, $B_{NS}$ and even $\Lambda$ to vary.

We will perform the analysis in two steps. In the first we will consider the old, 1993 data\cite{11}, whose errors were typically $O(10\%)$, and with points in the range $x < 10^{-2}$, $8.5 \text{ GeV}^2 \leq Q^2 \leq 65 \text{ GeV}^2$; discussing then the more recent, as yet unpublished set of data which have much smaller errors, and cover a wider range. Later in Sects. 4, 5 we will consider the theoretical corrections to the formulas used.

The results of the analysis of the 1993 data are presented in the following two tables. There the value of $\lambda$ choosen is 0.38, as it gives the best fit. The approximation of neglecting $F_{NS}$, i.e., Eq. (2.3b) was used. For the QCD parameter $\Lambda$ we first take it such that we reproduce the value of $\alpha_s(m^2) = 0.32$, i.e.,

$$\Lambda(n_f = 4, \text{1 loop}) = 0.200 \text{ GeV}, \quad \Lambda(n_f = 5, \text{1 loop}) = 0.165 \text{ GeV}.$$  

| $\lambda$  | $d_s(1 + \lambda)$ | $B_S$     | $B_G(1 + \lambda)/B_S$ | $r_0(1 + \lambda)$ | $\chi^2 / \text{d.o.f.}$ |
|------------|---------------------|-----------|------------------------|--------------------|-----------------|
| 0.38 ± 0.01| 2.406 ± 0.100       | (2.70 ± 0.22) $\times 10^{-3}$ | 20.56 ± 0.54        | 6.24 ± 0.24        | 9.13 / (32 - 2) |

Table Ia.- $n_f = 4$; $\Lambda(\text{1 loop, } n_f = 4) = 0.200 \text{ GeV}$; $\alpha_s(m^2) = 0.32$.

| $\lambda$  | $d_s(1 + \lambda)$ | $B_S$     | $B_G(1 + \lambda)/B_S$ | $r_0(1 + \lambda)$ | $\chi^2 / \text{d.o.f.}$ |
|------------|---------------------|-----------|------------------------|--------------------|-----------------|
| 0.38 ± 0.01| 2.595 ± 0.110       | (2.19 ± 0.11) $\times 10^{-3}$ | 18.53 ± 0.57        | 6.20 ± 0.29        | 9.34 / (32 - 2) |

Table Ib.- $n_f = 5$; $\Lambda(\text{1 loop, } n_f = 5) = 0.165 \text{ GeV}$; $\alpha_s(m^2) = 0.32$.  


One can consider fitting also the QCD parameter, $\Lambda$. In this case one discovers that, due to the slow, logarithmic variation of $\alpha_s$ with $\Lambda$ and the large size of the experimental errors, the effect of altering $\Lambda$ may be largely compensated by a change in $B_S$. As an indication, we give the results of an evaluation with a small $\Lambda$:

$$F_2(x, Q^2) = (2.70 \times 10^{-3}) [\alpha_s(Q^2)]^{-2.406} \times -0.38, \quad \alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f)\log Q^2/(0.2 \text{ GeV})^2}. \quad (2.11a)$$

Moreover, the error bars represent the statistical plus systematic errors composed quadratically.

We find the phenomenon already encountered in the analysis of the old, low energy data: $\lambda$ is well determined, but $B_S$ is less precisely fixed. Nevertheless, the errors have diminished substantially, and the consistency between the present analysis and that of refs. 5–8 is, to say the least, remarkable.
The results obtained from the fit to $F_2$ permit us to calculate also, and without any extra parameter, the two remaining structure functions, which for the central values of the parameters are,

$$F_G(x, Q^2) = (55.5 \times 10^{-3}) [\alpha_s(Q^2)]^{-2.406} x^{-0.38},$$

(2.11b)

$$R = 6.24 \frac{\alpha_s(Q^2)}{\pi}.$$  

(2.11c)

with $\alpha_s(Q^2)$ still as in (2.11a).

We do not plot $R$ as there are unfortunately no experimental data available. The gluon structure function is depicted in Fig. 2, where for the sake of comparison we also show the results based on a numerical evaluation\cite{11} using experimental data for $F_2$ and the DGLAP equations with the method of ref.10. The superior accuracy and simplicity of our formulas should be evident: note that the largest source of error in our evaluation is the uncertainty in the number of flavours between 4 and 5.
2.3.- The new HERA results.

We will here consider the comparison of our predictions with the more recent HERA data (as yet unpublished). If one replaces, *tels quels*, the formulas used for the fits to the old data, then a very large chi-squared is produced. The reasons for this, rather obvious, are two. First of all, the precision of the data is such that the approximation of neglecting the $F_{NS}$ piece is no more justified. Secondly, the range is now such that the one loop evolution of $\alpha_s$ is not sufficiently accurate. If we restrict our analysis to $x < 10^{-2}$, we have 63 data points with $Q^2$ varying in the bounds $12 \text{ GeV}^2 \leq Q^2 \leq 350 \text{ GeV}^2$. We do not consider in the same fit smaller values of $Q^2$ because this would imply a large variation of $n_f$, on which $\lambda$, $B_S$ are expected to depend, and because the NLO corrections would certainly play an important role.

We consider two possibilities: a restricted fit, only in the region

$$12 \text{ GeV}^2 \leq Q^2 \leq 90 \text{ GeV}^2,$$

where it is sufficient to include the NS contribution, and an evaluation in the full range where the two loop expression for $\alpha_s$ has to be taken into account as well.

| $\lambda$ | $B_{NS}$ | $B_S$ | $r_0$ | $\chi^2$/d.o.f. |
|-----------|-----------|-------|-------|-----------------|
| $0.39 \pm 0.01$ | $1.03$ | $2.73 \times 10^{-3}$ | $6.03$ | $44.2/(48 - 3)$ |

**Table III.** - $n_f = 4$; $x < 10^{-2}$; $12 \text{ GeV}^2 \leq Q^2 \leq 90 \text{ GeV}^2$; $\Lambda(1 \text{ loop}, n_f = 4) = 0.2 \text{ GeV}$

The results of the fits are again excellent. For the first case they are summarized in Table III, where we also give the parameters pertinent to the NS contribution, and $\Lambda$ is considered fixed. For the full range, the results are given in Table IV. Note that the value we obtained of $\Lambda$, taken now as a free parameter in the fit, cannot be considered as a true determination, as we have not included second order effects (that we discuss below).

| $\lambda$ | $\Lambda(2 \text{ loops}, n_f = 4)$ | $d_s(1 + \lambda)$ | $B_{NS}$ | $B_S$ | $r_0$ | $\chi^2$/d.o.f. |
|-----------|-------------------------------|-----------------|-------|-------|-------|-----------------|
| $0.38 \pm 0.01$ | $0.10 \pm 0.01 \text{ GeV}$ | $2.406$ | $0.772$ | $(1.0 \pm 0.22) \times 10^{-3}$ | $6.24$ | $50.41/(63 - 4)$ |

**Table IV.** - $n_f = 4$; $x < 10^{-2}$; $12 \text{ GeV}^2 \leq Q^2 \leq 350 \text{ GeV}^2$

We do not give the ensuing expressions for $F_G$ or $R$. If we take the values of the parameters, and the range of the variables pertaining to Table III, they are like those in eqs. (2.11), *mutatis mutandi*. If we consider the situation in which we take $\alpha_s$ to two loops, Table IV, then the expressions for $F_G$, $R$ would not be more reliable than before because the two loop corrections to these quantities differ from the ones to $F_S$. Thus we give only the expression for the structure function $F_2$. We have,

$$F_2(x, Q^2) = (1.0 \pm 0.22) \times 10^{-3}[\alpha_s(Q^2)]^{-2.406} x^{-0.38} + 0.77[\alpha_s(Q^2)]^{-0.514} x^{0.5}, \quad (2.12a)$$

where now

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f)} \left\{ 1 - \frac{153 - 19n_f}{(33 - 2n_f)^2} \frac{\log \log Q^2/A^2}{\log Q^2/A^2} \right\}, \quad n_f = 4, \; A = 0.10 \pm 0.01. \quad (2.12b)$$

The comparison with experiment of the results from the restricted fit using the values of the parameters recorded in Table III would give something very much like one what sees in the Figures 1 and 2. The comparison of the theoretical evaluation for $F_2$ using eq. (2.12) and experiment is shown in Fig.3.
To end this section we fit separately the low $Q^2$ data points, with a two loop expression for $\alpha_s$. We then find the results of Table V.

We do only give errors on $\lambda$. In fact the results in Table V should be taken as representing effective estimates. The reason is that the values of $Q^2$ are so low that an exact treatment of subleading corrections would be essential to get realistic values of the parameters. This includes not only $O(\alpha_s)$ corrections, but diffractive ones as well. This may easily alter substantially the results reported here; one should realize, for example, that even without the corrections a chi-squared/d.o.f. of less than one may be obtained provided $\lambda \lesssim 0.35$, with $B_S \sim 0.6 \times 10^{-3}$ and $B_{NS} \sim 1.1$. Note also that the compatibility of the results reported in Tables IV and V is made more apparent if we compare values of the $\hat{B}_S$, $\hat{B}_{NS}$, i.e., we extract the factor $\langle q^2 \rangle = 5/18$ ($n_f = 4$) or $2/9$ (for $n_f = 3$). Then we get,

$$\hat{B}_S = 3.6 \times 10^{-3} \ (Q^2 > 10 \text{ GeV}^2); \ (1.4 \text{ to } 2.7) \times 10^{-3} (Q^2 < 10 \text{ GeV}^2);$$

$$\hat{B}_{NS} = 3.11 \ (Q^2 > 10 \text{ GeV}^2); \ 3.4 \text{ to } 4.4 \ (Q^2 < 10 \text{ GeV}^2).$$

**Table V.** - $n_f = 3$; $x < 10^{-2}$; $1.5 \text{ GeV}^2 \leq Q^2 \leq 8.5 \text{ GeV}^2$

| $\lambda$ | $A(2 \text{ loops, } n_f = 3)$ | $B_{NS}$ | $B_S$ | $\chi^2$/d.o.f. |
|-----------|-----------------|--------|--------|---------------|
| $0.29 \pm 0.02$ | $0.14 \text{ GeV}$ | $0.75$ | $0.3 \times 10^{-3}$ | $18/(45 - 3)$ |

*Figure 3.* Fit to the latest HERA data with eq. (2.12) and $Q^2$ from 12 to 350 GeV$^2$. The errors of the experimental points, of the order of the size of the dots, are not shown.
3. HIGH ENERGY COMPTON SCATTERING

The results of the previous section indicate that a behaviour

\[ F_2(x,Q^2) \simeq f_S(Q^2)s^\lambda + f_{NS}(Q^2)s^{0.5} + \ldots \]  

(3.1)

where \( s \) is the c.m. energy squared, and we set the scale so that it is measured in GeV\(^2\), may hold beyond the region of applicability of perturbative QCD; note that \( s \simeq Q^2/x \). This was first suggested in ref.12 for Compton scattering, \( \gamma + p \rightarrow \gamma + p \). The supposed subleading terms, denoted by dots in (3.1) were identified with diffractive effects – the Pomeron. To be precise, one may suppose that the photon has a “hard”, or pointlike component to which (3.1) applies (without the dots); and a “soft”, hadronic or Pomeron component, which in old fashioned photon physics was identified as connected with the probability of finding the rho resonance in the photon. One thus writes

\[ \sigma_{\gamma p}(s) = \sigma_{\gamma p}^h(s) + \sigma_{\gamma p}^P(s), \]  

(3.2a)

where

\[ \sigma_{\gamma p}^h(s) = B \gamma p(s/1\text{GeV})^\lambda, \]  

(3.2b)

\[ \sigma_{\gamma p}^P(s) = A_{\gamma p} \hat{\sigma}^P(s) \]  

(3.2c)

and \( \hat{\sigma}^P(s) \) is a universal, Pomeron hadronic cross section [into which, for reasons of convenience, we have also added the \( \rho - f^0 \) trajectory contribution, corresponding to the term \( f_{NS}(Q^2)s^{0.5} \) in (3.1)].

In view of its universality, \( \hat{\sigma}^P(s) \) may be obtained, up to a constant, from any hadronic scattering cross sections, say

\[ \sigma_{\pi p}(s) = C_{\pi p} \hat{\sigma}^P(s); \quad \sigma_{\bar{p}p}(s) = C_{\bar{p}p} \hat{\sigma}^P(s). \]  

(3.3)

The formulas employed for fitting the various hadronic cross sections are thus,

\[ \sigma_{hp}(s) = C_{hp} \hat{\sigma}(s), \]  

(3.3)

where the constant \( C_{hp} \) depends on the process and \( \hat{\sigma}(s) \) is obtained saturating the Froissart bound in the improved version of ref.14* (but with the overall constant and an additive constant left as free parameters), plus a Regge pole contribution corresponding to the \( f^0 \) trajectory, which as explained we have found it convenient to incorporate into \( \hat{\sigma}^P \):

\[ \hat{\sigma}(s) = A_F \log \left( \frac{s}{m_T^2} \right) + 1 + A_{f^0}(s/1\text{GeV})^{-0.5}. \]  

(3.4)

The values of \( A_F, A_{f^0} \) and \( C_{hp} \) are obtained fitting the cross sections \( \sigma_{\pi p} \equiv \frac{1}{2}[\sigma_{\pi^+ p} + \sigma_{\pi^- p}] \) and \( \sigma_{N p} \equiv \frac{1}{2}[\sigma_{\bar{p}p} + \sigma_{pp}] \). For energies above 500 GeV we assume \( \sigma_{\bar{p}p} = \sigma_{pp} + \text{“Regge”} \), where the piece “Regge” is obtained extrapolating the low energy difference \( \sigma_{\bar{p}p} - \sigma_{pp} \) to high energy with a Regge pole formula, \( \sigma_{\bar{p}p} - \sigma_{pp} \approx C s^{\alpha_p} \). This gives a minute correction, 0.6 to 0.2 mb, for \( s_T^2 \simeq 500 \) to 2000 GeV. One then finds the universal parameters

\[ A_F = 0.0116, \quad A_{f^0} = 0.69 \]  

(3.5a)

and moreover

\[ C_{\pi p} = 23.0 \text{ mb}; \quad C_{N p} = 39.6 \text{ mb}. \]  

(3.5b)

Note that the ratio \( C_{N p}/C_{\pi p} = 1.72 \) is reasonably close to the value \( 3/2 \) predicted by the naive quark model.

It should be stressed that the present paper is not about hadronic cross sections. If a set of experimental measurements of \( \pi p, \bar{p}p, pp \) existed from “low” energies \( (s_T^2 \sim 10 \text{ GeV}) \) to the large values reached in \( \gamma p \) scattering, \( s_T^2 \sim 200 \text{ GeV} \), a fit like that in eqs.(3.4,5) would be unnecessary; and indeed, one could also

* This corrects an error in ref.14, where the scale factor is wrongly given as \( \log^7(s/m_T^2) \) instead of the correct value \( \log^2(s/m_T^2) \), cf. (3.4) below.
have used the phenomenological interpolations provided by the Particle Data Group\cite{15}. We choose (3.4) because of a theoretical prejudice in its favour, which is substantiated by the quality of the fits, with only two parameters, shown in Fig.4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{Fit to the $\pi p$ (indented) and $\frac{1}{2}\bar{p}p + pp$ cross sections with eqs. (3.4) and (3.5).}
\end{figure}

With this we write $\sigma_{\gamma p}$ as in eqs. (3.2). The values of the constants one obtains are, for a $\chi^2$/d.o.f. = 11.3/12 (with statistical errors only),

\[ C_{\gamma p} = 102.0 \pm 2.0 \mu b; \quad B_{\gamma p} = 1.64 \pm 0.30 \mu b; \quad \lambda = 0.23 \pm 0.03. \]  \hspace{1cm} (3.6)

The ratio $C_{\pi p}/C_{\gamma p} = 225$ is close to the value 220 obtained at low energy from experiment and vector meson dominance arguments\cite{12}. Finally, the parameters $\lambda = 0.23$ and $B_{\gamma p} = 1.64$ are reasonably similar to those obtained at “low energy” ($s_{\gamma p} \lesssim 20$ GeV) in ref.12, that is, $\lambda = 0.40$ and $B_{\gamma p} = 1.40$, while $\lambda$ is also close to the value deduced in the previous subsection (2.3) from small $x$ in deep inelastic scattering, $\lambda \simeq 0.3$. One should not take the small discrepancies very seriously; the values recorded above for the parameters were obtained taking only statistical errors into account in the recent, very high energy HERA data\cite{16}. If we include also systematic errors, then we get $\lambda = 0.26$ and a chi-squared/d.o.f. less than one provided $0.22 \leq \lambda \leq 0.29$, perfectly compatible with what we found before, modulo the (expected; see Sect.6) dependence of $\lambda$ on the number of excited flavours.
The quality of the fit should be apparent from Fig. 5, where only statistical errors are shown for the high energy HERA data; also impressive is the stability of the extrapolation by one order of magnitude in energy from the 1980 analysis to the present one.

Figure 5.- Fit to the $\gamma p$ cross section with eqs.(3.2)
In the indent, a blowup of the low energy region.
4. SUBLEADING CORRECTIONS

4.1. Two loop QCD corrections

The equations we have used take only into account LO (one loop) QCD, except for the running of $\alpha_s(Q^2)$. A proper analysis should fully take into account NLO (two loop) corrections. These are known since the work of ref.6, and amount to replacing (2.8a-d) by

$$F_S(x, Q^2) \simeq \hat{B}_S \left[ 1 + \frac{\nu_s(1 + \lambda)\alpha_s(Q^2)}{4\pi} \right] \left[ \alpha_s(Q^2) \right]^{-d_s(1 + \lambda)} x^{-\lambda}, \quad (4.1a)$$

$$F_G(x, Q^2) \simeq B_G(1 + \lambda) \left[ 1 + \frac{\nu_G(1 + \lambda)\alpha_s(Q^2)}{4\pi} \right] F_S(x, Q^2), \quad (4.1b)$$

$$F_{NS}(x, Q^2) \simeq \hat{B}_{NS} \left[ 1 + \frac{\nu_{NS}(1 - \rho)\alpha_s(Q^2)}{4\pi} \right] \left[ \alpha_s(Q^2) \right]^{-d_{NS}(1 - \rho)} x^\rho, \quad (4.1c)$$

where the $\nu_i(n)$ are known, complicated combinations of one- and the two-loop anomalous dimension matrix elements and the one loop Wilson coefficients whose explicit expression may be found in ref.8. Because of this structure, the bulk of the NLO effect may be approximated for a single structure function, by just leaving $A$ as a free parameter – precisely as we did in subsect. 2.3. The full NLO analysis, with eqs. (4.1) taken into account, requires more precision, in particular incorporating also other subleading effects and will be presented in a separate paper.

4.2. The joining of low and high $Q^2$

In Subsect. 2.3 and Sect. 3 we have shown that on can reproduce the cross sections

$$\sigma_{\gamma^*(-Q^2)p}(s)$$

for, respectively, off shell (but with fairly small $Q^2$, down to 1.5 GeV$^2$) or on shell photons, $Q^2 = 0$, with formulas with similar functional form for the structure function and the “hard” piece of the Compton cross section. It is then tempting to try and join the two approaches. This presents a number of problems, and rises some very interesting questions, that we now briefly discuss, leaving the full analysis for a future publication.

First of all, it is clear that an equation like (2.8a) (say) cannot hold for $Q^2$ too small. Not only, if taken literally, it would become singular for $Q^2 \sim A^2$, but, from the relation between $F_2$ and $\sigma_{\gamma^*(-Q^2)p}(s)$ it follows that, as $Q^2 \to 0$,

$$\sigma_{\gamma^*(-Q^2)p} \simeq \frac{4\pi^2\alpha}{Q^2} F_2(x, Q^2), \quad (4.2)$$

so $F_2(x, Q^2)$ must vanish proportional to $Q^2$ for $Q^2$ small.

There are three ways in which one may look at eqs. (2.8), (4.2). First of all comes the question of the joining of the two. If we take literally the values of the parameters we found in Sect.2, and the hard part of the photon-proton cross section, this occurs for a very reasonable value of $Q^2$, $Q^2_{\text{crit}} \sim 1$ GeV. It would not be difficult to write interpolation formulas, provided one tackles first the two remaining questions: the divergence of $\alpha_s(Q^2)$ for $Q^2 \sim A^2$, and what happens to the soft, or Pomeron contribution to $\sigma_{\gamma^*(-Q^2)p}$, numerically dominant when $Q^2 = 0$ and disappearing for $Q^2 \gg A^2$.

In what regards the matter of the divergence of $\alpha_s(Q^2)$ an interesting possibility is that, as has been suggested recently,$^{[17]}$, $\alpha_s(Q^2)$ freezes at low $Q^2$. Specifically, if this idea is correct, one would have an expression for $\alpha_s$ like

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f)\log((Q^2 + M^2)/A^2)} \quad (4.3)$$
with $M^2$ a parameter that one expects to be of the order of a typical hadronic mass (some estimates give $M \sim 0.96 \text{ GeV}$). With respect to the remains of the Pomeron at large $Q^2$ one may return to its interpretation as the probability of finding a real rho inside a photon\textsuperscript{[13]} and parametrize it with something like

$$\sigma^P_{\gamma^*(-Q^2)p} \simeq \left[ \frac{m^2_{\rho}}{2m^2_{\rho} + Q^2} \right]^2 \sigma^P_{\gamma^*(Q^2=0)p},$$

or perhaps adscribe the Pomeron to a diffractive effect and use instead an exponential form-factor:

$$\sigma^P_{\gamma^*(-Q^2)p} \simeq e^{-Q^2/b^2} \sigma^P_{\gamma^*(Q^2=0)p}.$$

We will say no more about this, leaving the matter for the announced future publication.
5. VERY HIGH MOMENTA AND CONNECTION WITH THE BFKL CONJECTURE

The questions discussed in the previous section affect mostly the medium and low \( Q^2 \) behaviour of structure functions. We will now discuss a very important matter that becomes relevant at ultra high energies.

According to our equations we have (Table IV),

\[
F_2(x, Q^2) \simeq B_S \langle \alpha_s(Q^2) \rangle ^{-d_+(1+\lambda)} x^{-\lambda}, \quad (5.1a)
\]

\[
\lambda = 0.38 \pm 0.01, \quad d_+(1+\lambda) = 2.406 \; (n_f = 4).
\]

On the other hand, the so-called BFKL equations\textsuperscript{[18]} imply

\[
F_2(x, Q^2) \simeq \int_{Q^2}^{\infty} C_2 x^{\omega_0 \alpha_s(Q^2)}, \quad \omega_0 = \frac{4C_A \log 2}{\pi}, \quad C_A = 3.
\]  (5.2)

Eqs. (5.1) and (5.2) are, on the face of it, incompatible. What is more, while (as we have seen) (5.1) agrees very well with experiment, (5.2) deviates from experimental data by dozens of standard deviations. Since eq.(5.2) is only valid in the leading approximation and, at least to the author’s knowledge, there is no control of the subleading corrections, one may be tempted to conclude that it should be rejected.

The situation, however, is not as clear as may appear at first sight. First of all, the behaviour of the anomalous dimension matrix \( D(n) \) for large \( n \) imply that, for any fixed \( x \) the structure functions must tend to zero for very large \( Q^2 \), contrarily to the behaviour following from eqs.(5.1) which make them grow with \( Q^2 \). Therefore, the behaviour (5.1) must stop at ultra large \( x \)-dependent values \( Q^2 \geq Q^2(x) \) where \( F_2(x, Q^2) \) should start decreasing, and perhaps turn into something like (5.2). To be precise, the conditions of validity of (5.1) (for whose proof we refer to Sect.6) are as follows. Assume that, at a given \( Q^2_0 \) sufficiently large for perturbation theory to apply, we have

\[
F_2(x, Q^2_0) = f(Q^2_0)x^{-\lambda_0(Q^2_0)} + O(x^{-\mu}), \quad \mu < \lambda_0(Q^2_0).
\]

Then, for all \( x, Q^2 \geq Q^2_0 \), we have that, for sufficiently small \( x \)

\[
F_2(x, Q^2) = \text{(Constant)} \langle \alpha_s(Q^2) \rangle ^{-d_+(1+\lambda)} + O(x^{-\mu}), \quad \lambda = \lambda_0(Q^2_0) \text{ independent of } Q^2.
\]

What the results of refs.5,6 do not say is starting from which \( x = x(Q^2) \) is this behaviour valid. The analysis of experimental data carried so far indicates that, when \( Q^2 < 350 \text{ GeV}^2 \), the behaviour (5.1) holds for \( x < 10^{-2} \). In the remainder of this section we will address the question of what happens above such values of \( x \) and \( Q^2 \).

To clarify the situation, and make more quantitative the analysis, we will consider the momentum sum rule [eq.(2.2)]. Separating the quark and gluon contributions one has (see e.g. ref.1),

\[
\lim_{Q^2 \to \infty} \int_0^1 dx \ F_2(x, Q^2) = \langle e_q^2 \rangle \frac{3n_f}{16 + 3n_f}.
\]  (5.3)

If we saturate the sum rule with the asymptotic behaviour we find that (5.2) is compatible with (5.3) provided \( C_2 = \langle e_q^2 \rangle [3n_f/(16 + 3n_f)] \); but if we substitute (5.1), assuming it to be valid up to \( x = x_0 \), we get

\[
\int_0^{x_0} dx \ F_2(x, Q^2) \simeq B_S x_0^{-\lambda} \frac{\langle \alpha_s(Q^2) \rangle ^{-d_+(1+\lambda)}}{1-\lambda}
\]

which violates (5.3) for \( Q^2 > Q^2_{\text{lim}}(x_0) \) with \( Q^2_{\text{lim}}(x_0) \) defined by

\[
\alpha_s(Q^2_{\text{lim}}(x_0)) = \left\{ \frac{B_S x_0^{-\lambda}}{1-\lambda} \frac{16 + 3n_f}{3n_f \langle e_q^2 \rangle} \right\}^{1/d_+}.
\]
For \( n_f = 4 \) and with the values of \( B_S, \lambda \) we have found, say
\[
B_S = 10^{-3}, \quad \lambda = 0.38; \quad \Lambda = 0.10
\]
we get that sizeable (~ a few percent) corrections to (5.1) must occur when \( x \sim 10^{-2} \) already for
\[
Q_{\text{fin}}^2 \sim 100 \text{ GeV}^2.
\]
This means that the precision of the more recent HERA data may be able to reveal the corrections to (5.1), and even to discriminate against, or for, (5.2).

In order to be more quantitative about this, we start by assuming that at a fixed \( Q_0^2 \) one had, exactly,
\[
F_S(x, Q_0^2) = \hat{B}_S[\alpha_s(Q_0^2)]^{-d_+(1+\lambda)} x^{-\lambda}.
\]
(5.5)
Then the moments at \( Q_0^2 \) are
\[
\mu_S(n, Q_0^2) = \frac{\hat{B}_S}{n - 1 - \lambda}[\alpha_s(Q_0^2)]^{-d_+(1+\lambda)},
\]
and, in view of the evolution equations (2.6) we get the result, valid at all \( Q^2 \),
\[
\mu_S(n, Q^2) = \frac{\hat{B}_S}{n - 1 - \lambda}[\alpha_s(Q_0^2)]^{-d_+(1+\lambda)} \left[ \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{d_+(n)},
\]
(5.6)
an evaluation that holds to corrections of relative order \( \alpha_s^{d_+ - d_-} \). In particular, (5.6) is certainly compatible with the momentum sum rule in that, for \( n = 2, d_+(2) = 0 \) and thus
\[
\mu_S(2, Q^2) = \int_0^1 dx F_S \to \text{constant}.
\]
Next we get the behaviour implied by (5.6) for \( x \to 0 \). The leading behaviour is still given by eqs. (2.8) [or (5.1)]. Because, however now (5.6) is assumed to be exact, we may find the corrections. These are dominated by the first singularity of \( d_+(n) \), which occurs at \( n = 1 \). After a simple calculation we find,
\[
F_S(x, Q^2) = \hat{B}_S[\alpha_s(Q^2)]^{-d_+(1+\lambda)} x^{-\lambda} - \Delta(x, Q^2) + O(x),
\]
(5.7a)
and \( \Delta(x, Q^2) \) is such that, for \( n = 1 + \epsilon, \epsilon \to 0 \), it is given by the leading singularity of \( d_+(1+\epsilon) \approx \frac{36}{(33 - 2n_f)\epsilon} \):
\[
\frac{\hat{B}_S}{\lambda}[\alpha_s(Q_0^2)]^{-d_+(1+\lambda)} \left[ \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{\frac{36}{(33 - 2n_f)\epsilon}} = \int_0 dx x^{-1} \Delta(x, Q^2).
\]
At large \( Q^2 \) and small \( x \) we have the asymptotic expression for the correction,
\[
\Delta(x, Q^2) \approx \frac{B_S}{\lambda\alpha_s(Q_0^2)^{d_+(1+\lambda)}} \left[ \frac{9 \log[(\log Q^2/A^2)/(\log Q_0^2/A^2)]}{(33 - 2n_f)\pi \log x} \right]^{1/2} \log \frac{144 \log x}{(33 - 2n_f) \log Q_0^2}.
\]
(5.7b)
We see how the momentum sum rule and the behaviour as \( x \to 0 \) get reconciled. For \( x \to 0 \), the piece \( \hat{B}_S[\alpha_s(Q_0^2)]^{-d_+(1+\lambda)} x^{-\lambda} \) dominates over \( \Delta(x, Q^2) \); but, for fixed \( x \), \( \Delta \) dominates when \( Q^2 \to \infty \) and in fact (5.7) cease to be valid. It is to be noted, however, that this mechanism excludes the BFKL conjecture (5.2) which is never attained.

We may generalize this last result, as the analysis depends only on the first singularity of \( D(n) \), located at \( n = 1 \), occurring not after the exponent of the first correction to (5.5). So we get essentially the same result if, at a given \( Q^2 \) one only assumes
\[
F_S(x, Q_0^2) = \hat{B}_S[\alpha_s(Q_0^2)]^{-d_+(1+\lambda)} x^{-\lambda} + O(x^n).
\]
(5.8)
The conditions that one has the behaviour (5.2) then are that the correction to (5.1) be of the form (const.) $x^{-\mu}$, $\mu > 0$. In fact, it is not difficult to get convinced that would need an infinite set of terms so that one had,

$$F_S(x, Q^2) = \sum_{j=0}^{\infty} \hat{B}_{Sj}[\alpha_s(Q^2)]^{-d_s(1+\lambda_j)} x^{-\lambda_j} - \Delta(x, Q^2),$$

$$\Delta = \sum \Delta_j.$$  

(5.9a)

(5.9b)

$\lambda_0 = \lambda > \lambda_1 > \lambda_2 > \ldots > 0$.

In order to discriminate between the two possibilities, we note that, when, for a fixed $x$, the subleading terms become sizeable, they are negative, if they are like in eq. (5.7); while if they are as in (4.11) they will start by giving a positive contribution (until, in the end, the term $\Delta$ also here starts being important). We may thus employ the following strategy. We fit data points including slightly larger values of $x$ than before. To be precise, we choose to add to the fit the values of $F_2(x, Q^2)$ with $x = 1.3 \times 10^{-2}$ and still we take $12 \text{ GeV}^2 \leq Q^2 \leq 350 \text{ GeV}^2$.

We keep $\lambda = 0.38$, $A = 0.10$ GeV fixed. We get a reasonably good fit, with $\chi^2$/d.o.f. $= 85.1/(73 - 2)$, $B_S = 2.14 \times 10^{-3}$, $B_{NS} = 0.54$. The comparison of the result, for $x = 1.3 \times 10^{-2}$ and experiment is shown in Fig.5a where we have added two points with $Q^2 = 500$ and 650 GeV$^2$ not included in the fit; and, for $x = 2.0 \times 10^{-2}$ in Fig. 5b, whose experimental points, however, were also not included in the fit.

Although the results are not conclusive (one should have had more care in the treatment of other subleading effects, in particular those mentioned in the previous section) it would appear that data lie below theory, thus suggesting (5.7)* and disfavouring the BFKL conjecture, at least at the values of $x, Q^2$ attained at HERA.

* In fact one may fit the difference between the experimental points and the dominating contribution by a formula like (5.7b), although one should not attach too much meaning to this.
6.-PROOF OF EQS.(2.8), (5.1), SUBLEADING CORRECTIONS AND DISCUSSION OF ALTERNATIVES

We repeat here the proof of the asymptotic behaviour (2.8), (5.1) for ease of reference and with a view to the study of subleading corrections, and to discuss the alternatives. Our assumption is that at a certain, fixed \( Q_0^2 \) sufficiently large for perturbation theory to be valid we have

\[
F_i(x, Q_0^2) = f_i(Q_0^2) x^{-\lambda_i(Q_0^2)} + O(x^{-\mu_i}), \quad \mu_i < \lambda_i(Q_0^2),
\]

for \( i = S, G \). Taking moments with \textit{continuous} index \( n \) as in eq.(2.5) we find that the first singularity of the \( \mu_i(n, Q_0^2) \) as functions of \( n \) occur when the integrals diverge, i.e., for \( n = 1 + \lambda_i(Q_0^2) \). \textit{Conversely}, if for any other \( Q^2 \) we have moments diverging,

\[
\mu_i(n, Q^2) \sim \frac{1}{n - \lambda_i - 1}
\]

then necessarily one must have

\[
F_i(x, Q^2) \xrightarrow{x \to 0} f_i(Q^2) x^{-\lambda_i}.
\]

Because of the evolution equations for the moments,

\[
\begin{pmatrix}
\mu_S(n, Q^2) \\
\mu_G(n, Q^2)
\end{pmatrix} = \begin{pmatrix}
\alpha_s(Q_0^2) \\
\alpha_s(Q^2)
\end{pmatrix}^D(n) \begin{pmatrix}
\mu_S(n, Q_0^2) \\
\mu_G(n, Q_0^2)
\end{pmatrix},
\]

the singularities of the \( \mu_i(n, Q^2) \) must originate either in those of \( \mu_i(n, Q_0^2) \) or in those of the anomalous dimension matrix \( D(n) \). By inspection, the last is seen to be analytic to the right of \( n = 1 \): so the \textit{leading} singularity of the \( \mu_i(n, Q^2) \) is identical to that of \( \mu_i(n, Q_0^2) \). Thus we may write

\[
F_i(x, Q^2) \xrightarrow{x \to 0} f_i(Q^2) x^{-\lambda_i},
\]

with \( \lambda_i \equiv \lambda_i(Q_0^2) \) and thus independent of \( Q^2 \). The rest of the formulas (2.8) (say) follow by diagonalizing the matrix \( D(n) \) and thus reducing the case to the evolution of a combination of the structure functions: actually, the coefficient relating \( F_S \) and \( F_G \) in (2.8f) is just the corresponding element of this diagonalizing matrix.

Let us next turn to the corrections. Subtracting from e.g. \( F_S \) the leading singularity we get

\[
F_S(x, Q^2) = B_S[\alpha_s(Q^2)]^{d_a(1+\lambda)} x^{-\lambda} + F_S^{(1)}(x, Q^2).
\]

We can repeat the analysis for \( F_S^{(1)} \) provided it behaves, at a fixed \( Q_0^2 \), as \( \sim x^{-\lambda(0)} \), \( \lambda(0) > 0 \). In this way we get a series like that in (5.9). If, however, the first singularity of \( \mu_i(n, Q_0^2) \) (once removed that at \( n = 1 + \lambda \)) lies to the left of \( n = 1 \), then the dominating singularity is that of \( D(n) \), and we get the result reported in (5.7).

From the preceding analysis it follows that \( \lambda, B_S \) cannot have a \textit{perturbative} dependence on \( Q^2 \); but they may, and likely will, have an indirect dependence via a dependence on the number of flavours excited at a given \( Q^2 \). This has been shown to occur by Witten\cite{19} in the case of DIS on \textit{photon} targets, where one can \textit{calculate} \( \lambda \), and one finds a slight dependence of \( \lambda \) on \( n_f \). This dependence (quite generally now) comes about as follows. As discussed before, the behaviour of the structure functions is related to the singularities of the moments at a given \( Q^2 \). Because of the operator product expansion, one has

\[
\mu(n, Q_0^2) = C_n \langle p | O_n(Q_0^2) | p \rangle,
\]

and

\[
\mu(n, Q^2) = C_n \left[ \alpha_s(Q_0^2) \over \alpha_s(Q^2) \right]^{D(n)} \langle p | O_n(Q_0^2) | p \rangle,
\]

(6.5a)
where \( C_n \) are the Wilson coefficients, and the \( O_n(Q_0^2) \) combinations of quark and gluon operators, renormalized at \( Q_0^2 \), that we rather sketchily write as

\[
O_n \sim G \partial \ldots \partial G; \bar{q}_f \partial \ldots \partial q_f,
\]

and \( f = u, d, s, \ldots \) runs over the various flavors excited. By inspection, the singularities of \( C_n, D(n) \) are seen to start at \( n = 1 \), both at leading and next to leading order. [For the singlet case; for the nonsinglet the relevant quantity is \( D_{11}(n) \) whose first singularity lies at \( n = 0 \). We send to refs. 5, 6 for the details of study of this case]. Therefore the leading singularities are those of the operator matrix elements \( \langle p|O_n|p \rangle \), depending on the number of quark flavors that intervene. Because the more operators intervene, the more poles one is likely to get, this analysis also shows that one must have

\[
\lambda(n_f = 5) \geq \lambda(n_f = 4) \geq \lambda(n_f = 3) \geq \lambda(n_f = 2),
\]

an expectation confirmed by our findings in this work.

This argument also shows that the number of flavors to be consider should depend mostly on \( Q^2 \) (and not on the energy, \( \nu \)). In fact, the integral that gives the moments in terms of the structure function recieves contributions from all \( x \), and not only from \( x \sim 0 \).

To finish this section we will discuss what happens if the assumption (6.1) fails. That is to say, if at all \( Q^2 \leq Q_0^2 \) with \( Q_0^2 \) a momentum at which one may join the nonperturbative Pomeron regime, and perturbative QCD (assumed to exist) one has

\[
F_i(x, Q_0^2) \sim O(x^0).
\]

This was discussed long ago by De Rújula et al.\cite{20} and has been revived recently (see e.g. ref. 21). As explained above, this means that the singularities in \( n \) of the \( \langle p|O_n(Q_0^2)|p \rangle \) lie at, or to the left of \( n = 1 \) [else we would have had (6.1)]. Then the behaviour of the \( F_i \) will be dictated by the singularities of \( D(n) \), for \( i = S, G \) or \( D_{11}(n) \) for the nonsinglet case. The analysis is essentially like for the corrections \( \Delta \) to \( F_2 \) discussed in the previous section and we immediately obtain,

\[
F_2(x, Q_2^2) \sim \exp \sqrt{b (\log (\log Q^2/ \log Q_0^2))} |\log x|, \tag{6.6a}
\]

and, for the nonsinglet,

\[
F_{NS}(x, Q_2^2) \sim \exp \sqrt{b' (\log (\log Q^2/ \log Q_0^2))} |\log x|. \tag{6.6b}
\]

The constants \( b, b' \), and the more detailed form of the behaviour depend on the assumption one makes at \( Q^2 = Q_0^2 \); see e.g. ref. 21. For example, if we assume that \( F_S(x, Q_0^2) \sim C \), then the formula for \( F_S(x, Q_0^2) \) becomes like (5.7b), with the obvious changes and in particular,

\[
b = \frac{144}{33 - 2n_f}.
\]

It is perhaps not idle to emphasize that (6.6) and (6.1) are mutually exclusive. If, for any \( Q^2 \) one has (6.1) with \( \lambda \) strictly positive, one necessarily gets (2.8) and, conversely. Only experiment may discriminate between the two alternatives since the value of \( \lambda \) is not fixed by perturbative QCD. Thus either (2.8) or (6.6), but not both, will fit experiment if the last is sufficiently precise.

The fits with (2.8) have been described in the previous sections. In what regards eqs.(6.6), I have found it impossible to fit actual experimental data with them. Eyeball inspection shows that all the NS structure functions behave as \( x^{1/2} \) near \( x = 0 \); and even nonbiased fit, including those in refs.10, 11 (not to mention ours in the present paper!) give an \( F_2 \) behaving also as a power \( x^{-\lambda} \); with similar exponents \( \lambda \) for all \( Q^2 \) (between 0.23 and 0.40) in the region of HERA data. On the other hand, any fit with eqs. (6.6), performed fixing \( x \) to get rid of the large uncertainties in (6.6), give impossibly large values of \( F_2 \) as \( Q^2 \) grows beyond 100 GeV\(^2\), or impossibly small for \( Q^2 \lesssim 40 \) GeV\(^2\).
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