ON LOCAL LINEAR CONVEXITY GENERALIZED TO COMMUTATIVE ALGEBRAS

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1. Introduction

The notion of linear convexity that is studied in the theory of functions of many complex variables was coined in 1935 by H. Behnke and E. Peschl [3], but it has been actively used only since the 60s due to the works of A. Martineau [5] and L. Aizenberg [1, 2] who defined a linearly convex set in n-dimensional complex space $\mathbb{C}^n$ independently in slightly different ways. Here we give the Aizenberg’s definition, since we take it as the basis. Hereinafter, a neighborhood of a point in a vector space is an open ball centered at this point.

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**Definition 1.** (L. Aizenberg [1]) A domain $D \subset \mathbb{C}^n$ is said to be **locally linearly convex** if for every boundary point $w = (w_1, w_2, \ldots, w_n) \in \partial D$ there is a complex hyperplane 

$$
\Pi_C(w) := \left\{ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} c_j (z_j - w_j) = 0, \right. \\
\left. (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\} \right\}
$$

passing through $w$ but not intersecting $D$ in some neighborhood of the point $w$. If $\Pi_C(w) \cap D = \emptyset$, then $D$ is said to be **(globally) linearly convex**.

However, H. Behnke and E. Peschl in [3] considered linearly convex sets only in the complex plane $\mathbb{C}^2$. They proved that global linear convexity follows the local one for bounded domains with a smooth boundary in $\mathbb{C}^2$. For the case of $\mathbb{C}^n$ this result was obtained in 1971 by A. Yuzhakov and V. Kryvokolesko [10]. Besides, in the work [3], the analytical conditions of local linear convexity of domains with a smooth boundary in $\mathbb{C}^2$ (Behnke-Peschl conditions) were obtained.

In 1971 B. Zinoviev got the following generalization of Behnke-Peschl conditions for the case $\mathbb{C}^n$, $n \geq 2$ [4]. Let domain

$$
D = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \varphi(z) = \varphi(z, \bar{z}) < 0 \} \tag{1}
$$

be defined by the function $\varphi(z) : \mathbb{C}^n \to \mathbb{R}$, where $\varphi \in C^2$ in a neighborhood of the boundary $\partial D = \{ z \in \mathbb{C}^n : \varphi(z) = 0 \}$ of $D$ and $\text{grad}\varphi \neq 0$ everywhere on $\partial D$. Then the following theorem is true.

**Theorem 1.** (4) If a domain $D$ is locally linearly convex, then for every boundary point $w \in \partial D$ and for all vectors $s = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$, $\|s\| = 1$, such that

$$
\sum_{j=1}^{n} \frac{\partial \varphi(w)}{\partial z_j} s_j = 0
$$

the following inequality is true

$$
\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi(w)}{\partial z_j \partial z_k} s_j s_k \geq 0, \quad \text{where} \quad z_{n+j} = \bar{z}_j, \; s_{n+j} = \bar{s}_j, \; j = 1, \ldots, n. \tag{2}
$$

If for every boundary point $w \in \partial D$ and for the same vectors $s$

$$
\sum_{j,k=1}^{2n} \frac{\partial^2 \varphi(w)}{\partial z_j \partial z_k} s_j s_k > 0, \quad \text{where} \quad z_{n+j} = \bar{z}_j, \; s_{n+j} = \bar{s}_j, \; j = 1, \ldots, n, \tag{3}
$$

then domain $D$ is locally linearly convex.
Similar conditions were obtained for the algebra of real quaternions \([6]\), the algebra of real generalized quaternions \([8]\), and Clifford algebras \([7]\). Let us notice here that the listed algebras are noncommutative. Professor A. Pogoruy reviewing these results proposed to obtain similar conditions for arbitrary commutative algebra. The purpose of the present work is to obtain necessary and sufficient conditions of generalized local linear convexity for a commutative associative algebra \(A\) over the field of real numbers with an identity and with some conditions imposed on its basis which are described in chapter 2. In that chapter real linear and quadratic forms are presented in terms of algebra \(A\) numbers and the generalization of the complex formal partial derivatives to algebra \(A\) is obtained. In chapter 3 the notion of linear convexity and the conditions of local linear convexity \([2]\), \([3]\) are generalized to the space \(A^n\) that is the Cartesian product of \(n\) algebras \(A\).

2. REAL LINEAR AND QUADRATIC FORMS IN COMMUTATIVE ALGEBRAS

In what follows, unless otherwise is specified, an \(m \times n\) matrix of elements \(a_{ij} \in \mathbb{R}\), \(i = 1, n\), \(j = 1, n\), will be denoted as \((a_{ij})\) and its determinant as \(\det (a_{ij})\). Let \(A\) be a commutative and associative algebra over the field of real numbers \(\mathbb{R}\) with identity \(e\). We identify \(e\) with 1. Let \(\dim A = m\), elements \(\{e_k\}_{k=0}^{m-1}\) be a basis of \(A\), and \(\gamma^p_{lk} \in \mathbb{R}\) be structure constants \(\gamma^p_{lk} \in \mathbb{R}\) of \(A\) defined as follows:

\[
e_l e_k = \sum_{p=0}^{m-1} \gamma^p_{lk} e_p, \quad l, k = 0, m - 1.
\]

Then each element \(x \in A\) can be presented as

\[
x = \sum_{q=0}^{m-1} x_q e_q, \quad x_q \in \mathbb{R}.
\]

Numbers of algebra \(A\) will be denoted by small Latin letters in bold and the real numbers will be denoted by small Latin or Greek letters in normal font. Since a basis of some algebras includes identity and the other elements of the basis are denoted as \(e_1, e_2, \text{ etc.}\), it is convenient to denote the identity as \(e_0\). So hereinafter, we start the numeration of the basis decomposition of \(A\) numbers from zero. Such a numeration of the basis decomposition requires starting the numeration of the elements of matrixes and other objects within this paper from zero too.

Let the basis satisfy the following conditions:

1) there exist the inverse elements \(e_k^{-1} = \frac{1}{e_k}\), \(k = 0, m - 1\).

2) among the matrixes \(\Gamma^p = (\gamma^p_{lk})\), \(p = 0, m - 1\) there is at least one that is non-degenerate.
The author does not know whether it is possible for any commutative and associative algebra over \( \mathbb{R} \) with an identity to choose a basis that simultaneously satisfies conditions 1) and 2).

Let us consider \( n \)-dimensional vector space

\[
\mathcal{A}^n := A \times A \times \ldots \times A
\]

with elements \( z = (z_1, z_2, \ldots, z_n) \in \mathcal{A}^n \), where

\[
z_j := \sum_{q=0}^{m-1} x_j^q e_q \in A, \quad x_j^q \in \mathbb{R}, \quad j = 1, n.
\]

(6)

We identify points (vectors) \( z \in \mathcal{A}^n \) with points (vectors) \( z = (x_0, x_1^1, \ldots, x_{m-1}^n) \in \mathbb{R}^{mn} \). Herewith, the elements of the space \( \mathcal{A}^n \) are in a bold font and the elements of the space \( \mathbb{R}^{mn} \) are in normal font. Let

\[
\| z \| = \sqrt{\sum_{j=1}^{n} \sum_{q=0}^{m-1} |x_j^q|^2}.
\]

Consider the following matrixes

\[
\begin{align*}
E &= \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & e_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & e_{m-1}
\end{pmatrix}, \\
X^j &= \begin{pmatrix}
x_0^j \\
x_1^j \\
\vdots \\
x_{m-1}^j
\end{pmatrix}, \\
\Gamma &= \begin{pmatrix}
\gamma_{10} & \gamma_{11} & \ldots & \gamma_{1(m-1)} \\
\gamma_{(m-1)0} & \gamma_{(m-1)1} & \ldots & \gamma_{(m-1)(m-1)}
\end{pmatrix}, \text{ where } \gamma_{lq} \in \mathbb{R}.
\end{align*}
\]

and a non degenerate \( m \times m \) matrix

And let

\[
Z_j = \Gamma E X^j,
\]

(7)

where

\[
Z_j = \begin{pmatrix}
z_0^j \\
z_1^j \\
\vdots \\
z_{m-1}^j
\end{pmatrix}, \quad j = 1, n.
\]

Thus,

\[
z_j^l = \sum_{q=0}^{m-1} \gamma_{lq} x_j^q e_q, \quad j = 1, n, \quad \text{where } \gamma_{lq} = 1 \text{ as } l = 0.
\]
From now on, for any number \( x \in \mathcal{A} \) the numbers \( x^l \in \mathcal{A} \), \( l = 0, m - 1 \), with upper index \( l \) in bold are obtained from \( x \) by multiplying the elements of \( l \)th row of matrix \( \Gamma \) by the respective summands \( x_q e_q \) in the basis decomposition \([5]\) of \( x \). As we can see, \( x^0 = x \).

We obtain from \([7]\):
\[
X^j = E^{-1} \Gamma^{-1} Z_j,
\]
where \( \Gamma^{-1} = (\eta_{lp}) \), \( \eta_{lp} \in \mathbb{R} \), \( l, p = 0, m - 1 \). That is to say,
\[
\begin{pmatrix}
  x_0^j \\
  x_1^j \\
  \vdots \\
  x_{m-1}^j
\end{pmatrix}
= \begin{pmatrix}
  e_0^{-1} & 0 & \ldots & 0 \\
  0 & e_1^{-1} & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & e_{m-1}^{-1}
\end{pmatrix}
\begin{pmatrix}
  \eta_{00} & \eta_{01} & \ldots & \eta_{0(m-1)} \\
  \eta_{10} & \eta_{11} & \ldots & \eta_{1(m-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  \eta_{(m-1)0} & \eta_{(m-1)1} & \ldots & \eta_{(m-1)(m-1)}
\end{pmatrix}
\begin{pmatrix}
  z_0^j \\
  z_1^j \\
  \vdots \\
  z_{m-1}^j
\end{pmatrix}.
\]
Hence
\[
x_i^j = e_i^{-1} \sum_{p=0}^{m-1} \eta_{lp} z_p^j, \quad j = \overline{1, n}, \quad l = 0, m - 1. \tag{8}
\]
Consider a real linear form
\[
\sum_{j=1}^{n} \sum_{l=0}^{m-1} a_j^l x_l^j,
\]
where \( a_j^l \in \mathbb{R} \), \( a_j^l = \text{const} \), \( j = \overline{1, n}, \ l = 0, m - 1 \). Substitute \( x_l^j \) for their expressions from \([8]\) and group together the respective components with \( z_p^j \), \( j = \overline{1, n}, \ l = 0, m - 1 \) fixing \( j \) and \( p \). Then we obtain
\[
\sum_{j=1}^{n} \sum_{l=0}^{m-1} a_j^l x_l^j = \sum_{j=1}^{n} \sum_{l=0}^{m-1} a_j^l e_i^{-1} \sum_{p=0}^{m-1} \eta_{lp} z_p^j = \sum_{j=1}^{n} \sum_{l=0}^{m-1} z_p^j \sum_{j=1}^{n} \sum_{p=0}^{m-1} \eta_{lp} a_j^l e_i^{-1} = \sum_{j=1}^{n} \sum_{p=0}^{m-1} z_p^j a_j^p,
\]
where
\[
a_j^p = \sum_{l=0}^{m-1} \eta_{lp} a_j^l e_i^{-1}, \quad j = \overline{1, n}, \ p = 0, m - 1. \tag{9}
\]
Let us rewrite the expression of \( a_j^p \) in terms of indexes \( i, q \).
\[
a_i^q = \sum_{k=0}^{m-1} \eta_{kq} a_i^k e_i^{-1}, \quad i = \overline{1, n}, \ q = 0, m - 1.
\]
Now we consider a real quadratic form
\[
\sum_{j,i=1}^{n} \sum_{l,k=0}^{m-1} a_{lj}^{ji} x_l^j x_k^i,
\]
where \( a_{lj}^{ji} \in \mathbb{R} \) are the elements of symmetric \( nm \times nm \) matrix
\[
(a_{lj}^{ji}), \quad a_{lj}^{ji} = a_{kl}^{ij}, \quad j, i = \overline{1, n}, \ k, l = 0, m - 1. \tag{10}
\]
This matrix is presented as follows:

\[(a_{jk}^{ji}) = \begin{pmatrix} A_{11}^{j} & A_{12}^{j} & \cdots & A_{1n}^{j} \\ A_{21}^{j} & A_{22}^{j} & \cdots & A_{2n}^{j} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}^{j} & A_{n2}^{j} & \cdots & A_{nn}^{j} \end{pmatrix},\]

where

\[A^{ji} = \begin{pmatrix} a_{00}^{ji} & a_{01}^{ji} & \cdots & a_{0(m-1)}^{ji} \\ a_{10}^{ji} & a_{11}^{ji} & \cdots & a_{1(m-1)}^{ji} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(m-1)}^{ji} & a_{(m-1)}^{ji} & \cdots & a_{(m-1)(m-1)}^{ji} \end{pmatrix}, \quad i, j = 1, n.\]

Multiplying \(a_{pq}^{ji}\) with \(a_{ij}^{k}\) and replacing products \(a_{ij}^{k}a_{kl}^{ji}\) by the elements \(a_{jk}^{ji}\) of matrix (10) we get the following numbers of algebra \(A\):

\[a_{pq}^{ji} = \sum_{l, k=0}^{m-1} \eta_{lp}\eta_{kq}a_{lk}^{ji}e_{l}^{-1}e_{k}^{-1}, \quad j, i = 1, n, \quad p, q = 0, m - 1. \quad (11)\]

Then the quadratic form can be expressed in terms of numbers \(z_{j}^{p}, z_{i}^{q}, a_{pq}^{ji}\) as follows:

\[
\sum_{j, i=1}^{n} \sum_{l, k=0}^{m-1} a_{lk}^{ji}x_{l}^{i}x_{k}^{i} = \sum_{j, i=1}^{n} \sum_{l, k=0}^{m-1} a_{lk}^{ji} \left( e_{l}^{-1} \sum_{p=0}^{m-1} \eta_{lp}z_{j}^{p} \right) x_{k}^{i} = \\
= \sum_{j, i=1}^{n} \sum_{l, k=0}^{m-1} a_{lk}^{ji}e_{i}^{-1} \sum_{p=0}^{m-1} \eta_{lp}z_{j}^{p} = \sum_{j, i=1}^{n} \sum_{l, k=0}^{m-1} a_{lk}^{ji}e_{i}^{-1} \sum_{p=0}^{m-1} \eta_{lp} \left( e_{k}^{-1} \sum_{q=0}^{m-1} \eta_{kq}z_{j}^{q} \right) z_{j}^{p} = \\
= \sum_{j, i=1}^{n} \sum_{p, q=0}^{m-1} \eta_{lp}\eta_{kq}a_{lk}^{ji}e_{i}^{-1}e_{k}^{-1}z_{j}^{p}z_{j}^{q} = \sum_{j, i=1}^{n} \sum_{p, q=0}^{m-1} a_{pq}^{ji}z_{j}^{p}z_{j}^{q}.
\]

Let \(\rho(z) = \rho(z) : \mathbb{R}^{mn} \rightarrow \mathbb{R}\) have continuous partial derivatives of the first and the second order at a point \(w \in \mathbb{R}^{mn}\). Then function \(\rho(z)\) is twice continuously differentiable at the point \(w\) and its full differentials of the first and the second order are defined as follows:

\[d\rho(w) = \sum_{j=1}^{n} \sum_{i=0}^{m-1} \frac{\partial \rho(w)}{\partial x_{i}^{j}} dx_{i}^{j}, \quad d^{2}\rho(w) = \sum_{j, i=1}^{n} \sum_{l, k=0}^{m-1} \frac{\partial^{2} \rho(w)}{\partial x_{i}^{j} \partial x_{l}^{k}} dx_{i}^{j} dx_{l}^{k}.
\]

We present \(d\rho(w), d^{2}\rho(w)\) in terms of the elements of algebra \(A\). Let

\[dz_{j}^{p} := \sum_{l=0}^{m-1} \gamma_{pl}dx_{l}^{j}e_{l}, \quad \text{where} \quad \gamma_{pl} = 1 \quad \text{as} \quad p = 0.
\]
Let \( a^p_j = \frac{\partial \rho(w)}{\partial x^p_j} \), \( \alpha_p^j = \frac{\partial^2 \rho(w)}{\partial z^p_j \partial x^p_k} \), \( a^q_{ijl} = \frac{\partial^2 \rho(w)}{\partial z^q_j \partial z^q_l} \), \( a^p_{ijl} = \frac{\partial^2 \rho(w)}{\partial z^p_j \partial z^p_l} \) in (11) and \( a^q_{ijl} = \frac{\partial^2 \rho(w)}{\partial z^q_j \partial z^q_l} \) in (11), \( p, q = 0, m - 1 \). Then

\[
\frac{\partial \rho(w)}{\partial z^p_j} := \sum_{l=0}^{m-1} \eta_p \frac{\partial \rho(w)}{\partial x^l_i} e^{-1}_l, \quad j = 1, n, \quad p = 0, m - 1, \tag{12}
\]

\[
\frac{\partial^2 \rho(w)}{\partial z^p_j \partial z^q_l} := \sum_{i,k=0}^{m-1} \eta_p \eta_q \frac{\partial^2 \rho(w)}{\partial x^l_i \partial x^k_j} e^{-1}_l, \quad j, i = 1, n, \quad p, q = 0, m - 1. \tag{13}
\]

And

\[
d\rho(w) = \sum_{j=1}^{n} \sum_{p=0}^{m-1} \frac{\partial \rho(w)}{\partial z^p_j} dz^p_j, \quad d^2 \rho(w) = \sum_{j,i=1}^{n} \sum_{p,q=0}^{m-1} \frac{\partial^2 \rho(w)}{\partial z^p_j \partial z^q_l} dz^p_j dz^q_l. \tag{14}
\]

On the other hand, substitute the values of \( x^j_l, j = 1, n, \quad l = 0, m - 1 \), from (9) in the expression of function \( \rho(z) = \rho(x^0_1, x^1_1, \ldots, x^m_{m-1}) \). We obtain \( \rho(z) = \rho(x^0_1(z^0_1, z^1_1, \ldots, z^m_{m-1}), x^1_1(z^0_1, z^1_1, \ldots, z^m_{m-1}), \ldots, x^n_{m-1}(z^0_n, z^1_n, \ldots, z^m_{m-1})) \). Formally differentiating function \( \rho \) as a composite function with respect to variables \( z^j_l, j = 1, n, \quad l = 0, m - 1 \), we also obtain formulas (12), (13) for the formal partial derivatives \( \frac{\partial \rho(w)}{\partial z^p_j}, \frac{\partial^2 \rho(w)}{\partial z^p_j \partial z^q_l} \).

In the case when \( \dim A = 2^k, k \in \mathbb{N} \), it is possible to fit matrix \( \Gamma \) such that \( |\gamma_{lq}| = 1 \) and \( \Gamma^{-1} = \frac{1}{2^k} \Gamma \):

\[
\Gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} \Gamma_1 & \Gamma_1 \\ \Gamma_1 & -\Gamma_1 \end{pmatrix}, \quad \ldots
\]

\[
\ldots, \quad \Gamma = \Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \Gamma_{k-1} \\ \Gamma_{k-1} & -\Gamma_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & -1 & \ldots & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & (-1)^{k-1} & (-1)^{k-1} \\ 1 & -1 & \ldots & (-1)^{k-1} & (-1)^{k-1} \end{pmatrix}.
\]

Thus, it is not difficult to see that matrix \( \Gamma_1 \) corresponds to the case of algebra of complex numbers \( \mathbb{C} \) and formula (12) gives a generalization of well known complex formal derivatives \( \frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial \bar{z}}, z \in \mathbb{C} \), to the algebra \( A \).
3. Generalized linear convexity

Let \( s_j = \sum_{l=0}^{m-1} s_l^j e_l, j = 1, n \). We say that a hyperplane

\[
\Pi_A := \left\{ s = (s_1, s_2, \ldots, s_n) \in \mathcal{A}^n : \sum_{j=1}^n c_j s_j = 0, \quad (c_1, c_2, \ldots, c_n) \in \mathcal{A}^n \setminus \{0\} \right\},
\]

lies in a real hyperplane

\[
\Pi_R := \left\{ (s_0, s_1, \ldots, s_{(m-1)}) \in \mathbb{R}^{mn} : \sum_{j=1}^n \sum_{l=0}^{m-1} a_l^j s_l^j = 0, \quad (a_0^1, a_1^1, \ldots, a_{(m-1)}^n) \in \mathbb{R}^{mn} \setminus \{0\} \right\},
\]

if any vector \( s \) satisfying the equation of the hyperplane (15) satisfies the equation of the hyperplane (16).

**Lemma 1.** For any real hyperplane \( \Pi_R \) (16), the hyperplane \( \Pi_A \) such that

\[
c_j = \sum_{k,l=0}^{m-1} \eta_{kl}^p a_l^j e_k, \quad j = 1, n,
\]

where \( (\eta_{kl}^p) = (\gamma_{lk}^p)^{-1} \). Hence

\[
a_l^j = \sum_{k=0}^{m-1} \gamma_{lk}^p c_k^j, \quad l = 0, m-1, j = 1, n.
\]

Let

\[
c_j = \sum_{k=0}^{m-1} c_k^j e_k = \sum_{k,l=0}^{m-1} \eta_{kl}^p a_l^j e_k, \quad j = 1, n.
\]
Substitute the values of $c_j$ from the first equality of (18) in the equation of the hyperplane (15) considering (4):

$$
\sum_{j=1}^{n} c_j s_j = \sum_{j=1}^{n} c_j^e k \sum_{l=0}^{m-1} s^j_l e_l = \sum_{j=1}^{n} c^j e_k e_l = \sum_{j=1}^{n} \sum_{k,l=0}^{m-1} c^j_k s^j_l e_k e_l = \sum_{j=1}^{n} \sum_{k,l=0}^{m-1} \sum_{p=0}^{m-1} \sum_{k,l=0}^{m-1} \gamma^p c^j_k s^j_l e_p = 0.
$$

The last equation is equivalent to simultaneous real equations

$$
\sum_{j=1}^{n} \sum_{k,l=0}^{m-1} \gamma^p c^j_k s^j_l = 0, \ p = 0, m-1.
$$

In particular, for $p = \tilde{p}$ we obtain

$$
\sum_{j=1}^{n} \sum_{k,l=0}^{m-1} \gamma^\tilde{p} c^j_k s^j_l = \sum_{j=1}^{n} \sum_{l=0}^{m-1} a^j_l s^j_l = 0.
$$

Thus, any vector $s = (s_1, s_2, \ldots, s_n) \in \mathcal{A}^n$ satisfying equation (15), where constants $c_j, j = \overline{1,n}$, are defined by (18), satisfies equation in (16).

Let $\Omega$ be a domain in the space $\mathcal{A}^n$. A domain is an open connected set in $\mathbb{R}^{mn}$.

**Definition 2.** A domain $\Omega \subset \mathcal{A}^n$ is said to be **locally $\mathcal{A}$-linearly convex**, if for every boundary point $w = (w_1, w_2, \ldots, w_n) \in \partial \Omega$ there is a hyperplane $\Pi_A$ (15), where $s_j = z_j - w_j, z_j \in \mathcal{A}, j = \overline{1,n}$, not intersecting $\Omega$ in some neighborhood of the point $w$. The hyperplane $\Pi_A$ is called **locally supporting** for $\Omega$ at $w$.

It is obvious that the notion of $\mathbb{C}$-linear convexity is equivalent to the notion of linear convexity.

Now we consider domain $\Omega = \{ z \in \mathcal{A}^n : \rho(z) = \rho(z, z^1, \ldots, z^{m-1}) < 0 \}$, $z^l = (z^l_1, z^l_2, \ldots, z^l_n), l = \overline{1,m-1}$, with the boundary $\partial \Omega = \{ z \in \mathcal{A}^n : \rho(z) = 0 \}$, where function $\rho : \mathcal{A}^n \rightarrow \mathbb{R}$ is twice continuously differentiable in a neighborhood of $\partial \Omega$ with respect to its real variables and such that $\operatorname{grad}\rho \neq 0$ everywhere on $\partial \Omega$.

Let $w \in \partial \Omega, z_j \in \mathcal{A}, s_j = z_j - w_j, j = \overline{1,n}$. We say that vector $s = (s_1, s_2, \ldots, s_n) \in \mathcal{A}^n$ **belongs to the tangent hyperplane** $T_A(w)$ to $\Omega$ at the point $w$ if

$$
\sum_{j=1}^{n} \sum_{k,l=0}^{m-1} \eta^\tilde{p} k l \frac{\partial \rho(w)}{\partial x^l_j} e_k e_j s_j = 0.
$$

Thus, if $s \in T_A(w)$, then by Lemma 1 and considering (14),

$$
\sum_{j=1}^{n} \sum_{l=0}^{m-1} \frac{\partial \rho(w)}{\partial x^j_l} s^j_l = \sum_{j=1}^{n} \sum_{l=0}^{m-1} \frac{\partial \rho(w)}{\partial z^j_l} s^j_l = 0. \quad (19)
$$
Theorem 2. If domain $\Omega$ is locally $\mathcal{A}$-linearly convex and $T_A(w)$ is locally supporting for $\Omega$ at any point $w \in \partial \Omega$, then for any point $w$ and any vector $s \in T_A(w)$, $\|s\| = 1$, the following inequality is true
\[
\sum_{i,j=1}^{n} \sum_{k,l=0}^{m-1} \frac{\partial^2 \rho(w)}{\partial z^k_i \partial z^l_j} s^l_j s^k_i \geq 0.
\] (20)

If for any point $w \in \partial \Omega$ and any vector $s \in T_A(w)$, $\|s\| = 1$,
\[
\sum_{i,j=1}^{n} \sum_{k,l=0}^{m-1} \frac{\partial^2 \rho(w)}{\partial z^k_i \partial z^l_j} s^l_j s^k_i > 0,
\] (21)
then domain $\Omega$ is locally $\mathcal{A}$-linearly convex.

Proof. Sufficiency. Formally write the Taylor series for the function $\rho(z) = \rho(z^0, z^1, \ldots, z^{m-1})$, $z^l = (z^l_1, z^l_2, \ldots, z^l_n)$, $l = 0, m-1$, with respect to variables $z^l_j$ in the neighborhood $U(w)$ of any point $w \in \partial \Omega$:
\[
\rho(z) = \rho(w) + \sum_{j=1}^{n} \sum_{k=0}^{m-1} \frac{\partial \rho(w)}{\partial z^k_j} (z^k_j - w^k_j) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k,l=0}^{m-1} \frac{\partial^2 \rho(w)}{\partial z^k_i \partial z^l_j} (z^l_j - w^l_j) (z^k_i - w^k_i) + o(\|z - w\|^2), \quad z \to w.
\]

Since $\rho(w) = 0$ at any boundary point $w$ and considering condition (19), we get
\[
\rho(z) = \frac{1}{2} \left( \sum_{i,j=1}^{n} \sum_{k,l=0}^{m-1} \frac{\partial^2 \rho(w)}{\partial z^k_i \partial z^l_j} (z^l_j - w^l_j) (z^k_i - w^k_i) \right) \|z - w\|^2 + o(\|z - w\|^2), \quad z \to w, \quad (22)
\]
for any point $z \in U(w) \cap T_A(w)$.

Thus, $\rho(z) \geq 0$ for any point $z \in U(w) \cap T_A(w)$ and any point $w \in \partial \Omega$ by (21) and (22), which means local $\mathcal{A}$-linear convexity of domain $\Omega$.

Necessity. Let domain $\Omega$ be locally $\mathcal{A}$-linearly convex and for a point $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n) \in \partial \Omega$ and for a vector $t = (t_1, t_2, \ldots, t_n) \in T_A(\tilde{w})$ the following inequality is true
\[
\sum_{i,j=1}^{n} \sum_{k,l=0}^{m-1} \frac{\partial^2 \rho(\tilde{w})}{\partial z^k_i \partial z^l_j} t^l_j t^k_i < 0.
\] (23)
On the other hand, for points $z \in U(\tilde{w}) \cap T_A(\tilde{w})$ the expansion (22) is valid. Thus, for the point $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n) \in U(\tilde{w}) \cap T_A(\tilde{w})$, which corresponds to the tangent vector $t$, where correspondence is defined by the relation $t_i = (\tilde{z}_i - \tilde{w}_i)/\|\tilde{z} - \tilde{w}\|$, $i = 1, n$, the inequality $\rho(\tilde{z}) < 0$ is true by (23), which contradicts the fact that hyperplane $T_A(\tilde{w})$ is locally supporting for $\Omega$ at $\tilde{w}$.
\qed
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