NON-CROSSING PARTITIONS AND MILNOR FIBERS

THOMAS BRADY, MICHAEL FALK *, AND COLUM WATT

ABSTRACT. For a finite real reflection group $W$ we use non-crossing partitions of type $W$ to construct finite cell complexes with the homotopy type of the Milnor fiber of the associated $W$-discriminant $\Delta_W$ and that of the Milnor fiber of the defining polynomial of the associated reflection arrangement. These complexes support natural cyclic group actions realizing the geometric monodromy. Using the shellability of the non-crossing partition lattice, this cell complex yields a chain complex of homology groups computing the integral homology of the Milnor fiber of $\Delta_W$.

1. INTRODUCTION

Suppose $g \in \mathbb{C}[z_1, \ldots, z_n]$ is a quasi-homogeneous polynomial, defining the hypersurface $V = g^{-1}(0)$ in $\mathbb{C}^n$. Then $g$ restricts to a locally trivial fibration $g: \mathbb{C}^n - V \to \mathbb{C}^*$, the global Milnor fibration, with fiber $g^{-1}(1)$, the Milnor fiber of $g$ [23]. The topology of $g^{-1}(1)$ and the monodromy of the bundle are invariants of the singularity type of $g$ at the origin. Of special interest is the case where $g = Q_W$ is a product of complex linear forms defining the arrangement $A = A_W$ of reflecting hyperplanes in $\mathbb{C}^n$ of a finite real or complex reflection group $W$, see [25, 26, 21, 17].

In this setting $W$ acts on $\mathbb{C}^n$ preserving $V = \bigcup_{H \in A} H$, the quotient $W \setminus \mathbb{C}^n$ is homeomorphic to $\mathbb{C}^n$, and under this homeomorphism $W \setminus V$ is carried to a hypersurface $\Delta_W$ in $\mathbb{C}^n$. This hypersurface is the zero locus of a quasi-homogeneous polynomial, $P_W$, well-defined up to polynomial automorphism of $\mathbb{C}^n$, called the discriminant associated with $W$ (see Section 3).

The fundamental group of $\mathbb{C}^n - \Delta_W$ is the generalized braid group (or Artin group), $B(W)$, associated with $W$. If $W$ has type $A_{d-1}$ then $P_W$ is the classical discriminant for polynomials of degree $d$ and $B(W)$ is isomorphic to the classical braid group on $d$ strands. In this paper, we construct a non-crossing partition (NCP) model for the Milnor fiber $F_p = P_W^{-1}(1)$ and study its structure, including the monodromy action, in the case where $W$ is a real reflection group. We also construct an NCP model for the Milnor
fiber $F_Q = Q_W^{-1}(1)$ of the reflection arrangement $\mathcal{A}_W$. Both models arise as subcomplexes of appropriate covering spaces of a finite $K(B(W),1)$ which is defined in terms of non-crossing partitions, see [4, 7, 8].

The NCP model for $F_P$ has a natural filtration by subcomplexes, which are seen to be homotopy equivalent to bouquets of spheres using the lexicographic shellability of the non-crossing partition lattice. This yields a chain complex computing $H_*(F_P,\mathbb{Z})$ whose terms are homology groups of truncations of this lattice. The homology of $H_*(F_P,\mathbb{Z})$ has been computed in most cases, see [13, 12].

2. NCP MODELS FOR SUBGROUPS OF $B(W)$

2.1. Background. Let $W$ be a finite, irreducible, real reflection group of rank $n$. For background on finite reflection groups see [6, 19]. Equip $W$ with the total reflection length function $w \mapsto |w|$ and with the partial order $\leq$ given by $u \leq w$ whenever $|u| + |u^{-1}w| = |w|$ (see [2]). We will use the notation $u \prec w$ for the case where $w$ covers $u$. Fix a specific Coxeter element $\gamma$ in $W$ and define the non-crossing partitions to be the elements in the interval $[e,\gamma]$ in the poset $(W,\leq)$. The poset of $W$-non-crossing partitions is a lattice, $L$ (see [9]), whose order complex is denoted $|L|$. When $W$ is of type $A_n$, $W$ is isomorphic to the group of permutations of $\{1, \ldots, n+1\}$ and the non-crossing partitions are those elements whose cycle structure gives a classical non-crossing partition, see [7].

We define $B(W)$ to be the group with generating set

$$\{[w] \mid w \in L, w \neq e\}$$

subject to the relations

$$[w_1][w_1^{-1}w_2] = [w_2] \text{ whenever } w_1 \leq w_2.$$ 

It is shown in [4, 7, 8], that $B(W)$ is isomorphic to the generalized braid group of type $W$.

We recall from [4, 7, 8], the contractible, $n$-dimensional, simplicial complex, $X$, whose $k$-simplices are ordered $(k+1)$-tuples from $B(W)$ of the form $(g_0,g_1,\ldots,g_k)$ with $g_i = g_0[w_i]$ for some chain $e < w_1 < w_2 < \cdots < w_k$ in $L$. It is convenient to use the notation $(g_0,e < w_1 < \cdots < w_k)$ for such a simplex. Thus the simplices of $X$ are identified with pairs $(g,\sigma)$ where $g \in B(W)$ and $\sigma$ is an initialized chain in $L$, that is, $\sigma$ is a chain of the form $e < w_1 < \cdots < w_k$. As $B(W)$ acts freely on $X$, the quotient $K := B(W)\backslash X$ is a $K(B(W),1)$ and $X$ is its universal cover. The action of $B(W)$ on $X$ is given by

$$g \cdot (g_0,g_1,\ldots,g_k) = (gg_0,gg_1,\ldots,gg_k),$$

or, in terms of the pair notation,

$$g \cdot (g_0,e < w_1 < \cdots < w_k) = (gg_0,e < w_1 < \cdots < w_k).$$

(2.1)
It is immediate that the simplex \((g_0, e < w_1 < \cdots < w_k)\) has \(k\) faces of the form \((g_0, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k)\), for \(1 \leq i \leq k\), each obtained by deleting one of \(w_1, w_2, \ldots, w_k\) from \(\sigma\). The remaining face is obtained by deleting \(e\) from \(\sigma\) and hence is given by the ordered set
\[
(g_0[w_1], g_0[w_2], \ldots, g_0[w_k]) = (g_0[w_1]) \cdot (e, [w_1]^{-1}[w_2], \ldots, [w_1]^{-1}[w_k]).
\]
In pair notation, this is denoted \((g_0[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\).

2.2. Quotients of \(X\). If \(H\) is a normal subgroup of \(B(W)\) then we can form a CW complex \(X_H\), whose cells are of the form \((Hg, \sigma)\), where \(\sigma\) is an initialized chain in \(L\) and the first component is a right \(H\) coset. If \(\sigma = e < w_1 < \cdots < w_k\), then this cell has \(k\) boundary faces of the form
\[
(Hg, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k), \quad 1 \leq i \leq k,
\]
with the remaining face given by \((Hg[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\). It will be convenient to refer to \((Hg[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\) as the top face of \((Hg, \sigma)\) and to \((Hg, e < w_1 < \cdots < < w_{k-1})\) as the bottom face of \((Hg, \sigma)\). Since \(X\) is contractible, \(X_H\) is a \(K(H, 1)\). The action of the quotient group \(H \backslash B(W)\) on \(X_H\) is given by \((Hg_1)(Hg_2, \sigma) = (Hg_1g_2, \sigma)\).

We now highlight a particular feature of these complexes \(X_H\).

**Lemma 2.1.** In \(X_H\), each \(k\)-cell of the form
\[
c_k = (Hg, e < w_1 < w_2 < \cdots < w_k)
\]
with \(|w_k| < n\) is incident on precisely two \((k + 1)\)-cells of the form
\[
c_{k+1} = (Hg', e < u_1 < \cdots < u_k < \gamma).
\]

**Proof.** Suppose that the cell
\[
c_k = (Hg, e < w_1 < w_2 < \cdots < w_k),
\]
with \(|w_k| < n\), is incident on a \((k + 1)\)-cell of form
\[
c_{k+1} = (Hg', e < u_1 < \cdots < u_k < \gamma).
\]
Since the chain of \(c_k\) does not contain \(\gamma\), \(c_k\) must be obtained by deleting either \(e\) or \(\gamma\) from the chain of \(c_{k+1}\). In the latter case, \(c_k\) is the bottom face of \(c_{k+1}\), forcing \(Hg' = Hg\) and \(u_i = w_i\) for \(i = 1, \ldots, k\). In the former case, \(c_k\) is the top face
\[
(Hg'[u_1], e < u_1^{-1}u_2 < \cdots < u_1^{-1}u_k < u_1^{-1}\gamma),
\]
so that \(Hg' = Hg[w_1]^{-1}\) and \(u_1 = \gamma w_k^{-1}, u_2 = \gamma w_k^{-1}w_1, \ldots, u_k = \gamma w_k^{-1}w_{k-1}\)
\(\square\)

In our examples, \(H\) will arise as the kernel of a specific homomorphism \(\varphi\) with domain \(B(W)\). In this case we will denote \(X_{\ker(\varphi)}\) by \(X_{\varphi}\). We can then identify the coset \(Hg\) with the element \(\varphi(g)\) and denote the cells of \(X_H\) as pairs \((\varphi(g), \sigma)\) where \(\sigma\) is an initialized chain in \(L\). If \(\sigma = e < w_1 < \cdots < w_k\),
then the cell \((\varphi(g), \sigma)\) has top face given by 
\((\varphi(g|w_1)), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\) or 
\((\varphi(g)\varphi([w_1]), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\).

**Example 2.2.** If \(\varphi\) is the trivial homomorphism, then \(H = B(W)\) and 
\(X_\varphi = K\) is the \(K(B(W), 1)\) introduced earlier. The cells of \(K\) are of the form 
\((e, \sigma)\), where \(\sigma\) is an initialized chain in \(L\). If \(\sigma = e < w_1 < \cdots < w_k\), 
then this cell has top face given by 
\((e\varphi([w_1]), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k) = (e, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k).\)

Thus \(X = X_\varphi\) can be identified with the quotient of \(|L|\) under the equivalence 
relation generated by identifying \(w_1 < w_2 < \cdots < w_k\) with \(e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\).

**Example 2.3.** The standard projection \(s: B(W) \to W: [w] \mapsto w\), which 
takes each NCP generator of \(B(W)\) to the corresponding NCP in \(W\), is 
a homomorphism by our presentation of \(B(W)\). The kernel of \(s\) is the pure 
braid group \(PB(W)\) associated to \(W\) and \(X_s\) is a \(K(\pi, 1)\) for \(\pi = PB(W)\). 
(\text{By [11, 16],} \(X_s\) is homotopy equivalent to the complement \(M = \mathbb{C}^n - \bigcup_{H \in A}H\), 
where \(A\) is the complexification of the associated real reflection 
arrangement.) The cells of \(X_s\) can be identified with pairs \((w, \sigma)\), for \(w \in W\) 
and \(\sigma\) an initialized chain in \(L\). The top face of the cell \((w, e < w_1 < \cdots < w_k)\) is 
\((ww_1, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\).

We will consider two further examples of this construction in sections 4 
and 5. These will give the models for the fibers mentioned in the introduction. 
The next section establishes the homotopy types of these fibers.

### 3. Discriminants and Milnor fibers

In this section we recall some definitions and basic facts about 
discriminants and Milnor fibers.

#### 3.1. The W-discriminant

Recall that \(W\) is a real reflection group whose 
action on \(\mathbb{R}^n\) has been complexified to an action on \(\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}\). Let 
\(\{f_1, \ldots, f_n\}\) be a set of basic invariants for the ring of \(W\)-invariant, 
complex polynomials [14]. The function 
\(f = (f_1, \ldots, f_n): \mathbb{C}^n \to \mathbb{C}^n\) induces a 
homeomorphism of \(W \setminus \mathbb{C}^n\) with \(\mathbb{C}^n\). Let \(T\) denote the set of reflections in 
\(W\) and, for each \(t \in T\), let \(H_t \subset \mathbb{C}^n\) denote its fixed complex hyperplane 
and \(\lambda_t: \mathbb{C}^n \to \mathbb{C}\) be a complex linear form with kernel \(H_t\). The polynomial 
\(Q = \prod_{t \in T} \lambda_t: \mathbb{C}^n \to \mathbb{C}\) has the property that 
\(Q(wx) = \det(w)Q(x)\) for all \(w \in W, x \in \mathbb{C}^n\) and hence \(Q^2\) is invariant under the action of \(W\). It follows that 
\(Q^2 = P(f_1, \ldots, f_n)\) for some quasi-homogeneous polynomial 
\(P \in \mathbb{C}[z_1, \ldots, z_n]\) whose weights are equal to the degrees of the \(f_i\). The polynomial \(P\) is called the discriminant of \(W\). It is unique up to polynomial 
automorphism of \(\mathbb{C}^n\). The action of \(W\) on \(\mathbb{C}^n\) leaves the affine algebraic hypersurface 
\(V = Q^{-1}(0) = \bigcup_{t \in T}H_t\) invariant and its quotient \(\Delta_W = W \setminus V\) 
is identified with the affine algebraic hypersurface \(\Delta := P^{-1}(0)\). The space 
\(M = \mathbb{C}^n - V\) is a \(K(PB(W), 1)\), and \(W\) acts freely on \(M\), so the space
momorphisms Q

The characteristic homomorphism. \((\mathbb{Z}, Y, \pi)

Proof. (ii): By proposition 3.1, \(PB(W)\) with \(\pi_1(M, z_0)\) and \(B(W)\) with \(\pi_1(f(M), f(z_0))\).

3.2. Milnor fibers of \(P\) and \(Q\). The restriction to \(\mathbb{C}^n - g^{-1}(0)\) of any quasi-homogeneous polynomial \(g: \mathbb{C}^n \rightarrow \mathbb{C}\) is a locally trivial fibration whose fiber \(g^{-1}(1)\) is called the Milnor fiber of \(g\) [23]. The Milnor fibers of \(P\) and \(Q\) respectively will be denoted by \(FP\) and \(FQ\). These spaces are determined up to polynomial diffeomorphism by \(W\). Then \((Q^2)^{-1}(1) = Q^{-1}(1) \cup Q^{-1}(-1)\) is invariant under the action of \(W\) and \(FP = W\setminus((Q^2)^{-1}(1)) \cong W^+\setminus FQ\), where \(W^+ = \{w \in W \mid \det(w) = 1\}\). The space \(FQ\) is a connected, regular, \(W^+\)-cover of \(FP\) since the action of \(W^+\) on \(FQ\) is free.

We will show that each of \(FP\) and \(FQ\) is homotopy equivalent to a complex of the form \(X_\varphi\). Our proofs will make use of the following general result.

**Proposition 3.1.** Suppose \(g: E \rightarrow \mathbb{C}^*\) is a fibration with fiber \(F\), where each of \(E\) and \(F\) has the homotopy type of a connected CW complex. Then \(F\) is homotopy equivalent to the cover of \(E\) corresponding to the kernel of \(g*: \pi_1(E) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}\).

**Proof.** Let \(i\) denote the inclusion of \(F\) into \(E\). The exact sequence of the fibration implies that \(i_*: \pi_1(F) \rightarrow \pi_1(E)\) is an injection whose image is \(\ker(g_*)\). Let \(p: F' \rightarrow E\) denote the connected cover of \(E\) corresponding to \(\ker(g_*)\). Then the inclusion \(i\) lifts to a map \(h: F \rightarrow F'\) with \(p \circ h = i\). We show that \(h\) is a homotopy equivalence. Since \(p_*\) and \(i_*\) are injections, it follows that \(h_*: \pi_1(F) \rightarrow \pi_1(F')\) is an injection and hence is an isomorphism. Since \(p_*: \pi_k(F') \rightarrow \pi_k(E)\) and \(i_*: \pi_k(F) \rightarrow \pi_k(E)\) are isomorphisms for all \(k \geq 2\) (the latter by the exact sequence of the fibration), so also is \(h_*: \pi_k(F) \rightarrow \pi_k(F')\). As each of \(F\) and \(F'\) has the homotopy type of a CW complex, \(h\) is a homotopy equivalence. \(\Box\)

Since algebraic sets are homotopy equivalent to CW complexes, applying Proposition 3.1 yields the following corollary.

**Corollary 3.2.**

(i) The Milnor fiber \(FP\) of \(P\) is homotopy equivalent to the cover of \(f(M) = W\setminus M\) corresponding to the kernel of the map \(P_*: B(W) \rightarrow \mathbb{Z}\).

(ii) The Milnor fiber \(FQ\) of \(Q\) is homotopy equivalent to the cover of \(f(M) = W\setminus M\) corresponding to the group \(f_*(\ker(Q_*))\).

**Proof.** (ii): By proposition 3.1, \(FQ\) is homotopy equivalent to the connected domain, \(Y\), of a covering map \(\rho: Y \rightarrow M\) for which \(P_*((\pi_1(Y, y_0)) = \ker(Q_*)\). Since \(f: M \rightarrow W\setminus M\) is a finite cover, the map \(f \circ \rho\) is a covering map with \((f \circ \rho)_*(\pi_1(Y, y_0)) = f_*(\ker(Q_*))\), as required. \(\Box\)

3.3. The characteristic homomorphism. It remains to identify the homomorphisms \(Q_*: PB(W) = \pi_1(M) \rightarrow \mathbb{Z}\) and \(P_*: B(W) = \pi_1(W\setminus M) \rightarrow \mathbb{Z}\). First we describe convenient generating sets for their respective domains.
Fix $\epsilon > 0$. For each reflection $t \in T$, choose a point $z_t \in H_t - \bigcup_{t' \in T, t' \neq t} H_{t'}$ and let $D_t$ be the closed disc of radius $\epsilon$ centred at $z_t$ in the complex line $L_t$ which passes through $z_t$ and is orthogonal to $H_t$. The complex structure induces a natural orientation on each $L_t$. By shrinking $\epsilon$, if necessary, we may assume that $D_t \cap \bigcup_{t' \in T, t' \neq t} (H_{t'} \cup D_{t'}) = \emptyset$ for all $t \in T$. Now choose a basepoint $z_0$ in $M$ and for each $t \in T$ choose a path $\sigma_t$ in $M$ which starts at $z_0$ and ends on the boundary of $D_t$. Let $\gamma_t$ be the loop which travels along $\sigma_t$, then around the boundary of $D_t$ in a positive orientation and finally back to $z_0$ along $\sigma_t$. The set of homotopy classes $\pi_t$ for $t \in T$ generates $\pi_1(M, z_0)$ [10]; see [1] for a recent generalization.

Since $W$ acts freely on $M$, the restriction of $f$ to $M$ is a regular covering map onto $f(M) = W \setminus M$. We specify a set of elements of $\pi_1(W \setminus M, f(z_0))$ which corresponds to the set of reflections in $W \cong \pi_1(W \setminus M, f(z_0))/f_\ast \pi_1(M, z_0)$ Under this isomorphism, if $\delta$ is any path in $M$ which starts at $z_0$ and ends at $w(z_0)$ (or which starts at $w(z_0)$ and ends at $w(w(z_0)))$, then the homotopy class of $f \circ \delta$ corresponds to $w \in W$. If $t \in T$ is a reflection, let $\delta_t$ be the path in $M$ which travels first along $\sigma_t$, then half way around the boundary of $D_t$ in the positive sense and finally along the reverse of $t \sigma_t$ to $t(z_0)$. Similarly, let $\delta'_t$ be the path from $t(z_0)$ to $z_0$ in $M$ which travels first along $t \sigma_t$, then around the other half of the boundary of $D_t$ in the positive sense and finally along the reverse of $\sigma_t$ to $t(t(z_0)) = z_0$. Note that $f \circ \delta_t$ and $f \circ \delta'_t$ represent the same homotopy class, $\Gamma_t$ say, and that the composition $\Gamma_t$ followed by $\Gamma_t$ is represented by the path $f \circ \gamma_t$. From the short exact sequence

$$\pi_1(M, z_0) \to \pi_1(W \setminus M, f(z_0)) \to W$$

the group $\pi_1(W \setminus M, f(z_0))$ is generated by $\{f \circ \gamma_t, \Gamma_t : t \in T\}$ and, in fact, by the smaller set $\{\Gamma_t : t \in T\}$, since $f \circ \gamma_t = \Gamma_t^2$. For $t \in T$, the generator $\Gamma_t \in \pi_1(W \setminus M, f(z_0))$ corresponds to the generator $[t]$ in the presentation for $B(W)$ from subsection 2.1.

**Proposition 3.3.**

(i) The map $Q_\ast : \pi_1(M, z_0) \to \pi_1(C^\ast, 1) \cong \mathbb{Z}$ is given by $Q_\ast(\gamma_t) = 1$ for each $t \in T$.

(ii) The map $P_\ast : \pi_1(W \setminus M, f(z_0)) \to \pi_1(C^\ast, 1) \cong \mathbb{Z}$ is given by $P_\ast(\Gamma_t) = 1$ for each $t \in T$.

**Proof.** (i) By scaling each $\lambda_t$ if necessary, we may assume that $\lambda_t(z_0) = 1$. Then $Q_\ast = \left( \prod_{t \in T} \lambda_t \right)_\ast = \sum_{t \in T} (\lambda_t)_\ast$ (since $C^\ast$ is a topological group).

The result now follows since our choices ensure that the winding number of $$(\lambda_t)_\ast(\pi_t)$$ about the origin is one if $t = t'$ and zero otherwise.

(ii) First note that

$$P_\ast(\Gamma_t^2) = P_\ast(f \circ \gamma_t) = (Q_\ast)^2(\gamma_t) = Q_\ast(\gamma_t) + Q_\ast(\gamma_t) = 2.$$ Since $P_\ast$ is a group homomorphism, it follows that $P_\ast(\Gamma_t) = 1$. \qed
Remark 3.4. Our calculation of $P_*$ from $Q_*$ can be modified to apply when $W$ is a Shephard group, the symmetry group of a complex polytope. The construction of a model for the Milnor fiber of the $W$-discriminant (in the next section) will also be valid in this case.

4. NCP model for the Milnor fiber of the discriminant

Consider the homomorphism $P_* : B(W) \to \mathbb{Z}$ and the cover of $K$ given by $X_{P_*} := \ker(P_*) \setminus X$.

Proposition 4.1. $X_{P_*}$ is homotopy equivalent to the Milnor fiber of the discriminant $P_*$.

Proof. This follows from Corollary 3.2(i) and Proposition 3.3, since corresponding covers of $K$ and $W \setminus M$ are homotopy equivalent.

Remark 4.2. By Section 2.2, $X_{P_*}$ can be identified with the CW-complex whose cells are pairs, $(m, \sigma)$ for $m \in \mathbb{Z}$ and $\sigma$ an initialized chain in $L$. Since $P_*([t]) = 1$ for each reflection generator $[t]$ of $B(W)$, it follows that $P_*([w]) = |w|$, the reflection length of $w$, for each NCP $w \in W$. Hence, the top face of the cell $(m, e < w_1 < \cdots < w_k)$ is the cell $(m + |w_1|, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$.

Definition 4.3. We define the small NCP model $\hat{F}_P$ of the Milnor fiber of $P_W$ to be the finite subcomplex of $X_{P_*}$ consisting of the cells of the form $(m, e < w_1 < \cdots < w_k)$ with $0 \leq m < n - |w_k|$.

Remark 4.4. We observe that $\hat{F}_P$ is the union of cells of the form $(0, e < w_1 < \cdots < w_{n-1})$ together with their faces. In particular, $\hat{F}_P$ is $(n - 1)$-dimensional.

Proposition 4.5. The subcomplex $\hat{F}_P$ is a strong deformation retract of $X_{P_*}$.

Proof. We construct an acyclic matching (see chapter 11 of [20]) which pairs cells of $X_{P_*} - \hat{F}_P$. Suppose $c_{k+1} = (m, \sigma)$ and that the chain $\sigma$ ends in $\gamma$. Then this matching pairs $c_{k+1}$ with its top (resp. bottom) face if $m \geq 0$ (resp. $m < 0$). In particular, the matching pairs cells whose chains end in $\gamma$ with cells whose chains do not end in $\gamma$.

To show that this matching is acyclic, consider an alternating path

$$(l_1, \sigma_1) >_m (l_2, \sigma_2) < (l_3, \sigma_3) >_m (l_4, \sigma_4) < \ldots$$

in this matching, where $(l_i, \sigma_i) < (l_{i+1}, \sigma_{i+1})$ means that $(l_i, \sigma_i)$ is a facet of $(l_{i+1}, \sigma_{i+1})$ and $(l_j, \sigma_j) >_m (l_{j+1}, \sigma_{j+1})$ means that $(l_j, \sigma_j)$ is matched with its facet $(l_{j+1}, \sigma_{j+1})$. By definition of the matching, if $i$ is odd, the chain $\sigma_i$ ends in $\gamma$ and $\sigma_{i+1}$ does not end in $\gamma$. By Lemma 2.1, each such $(l_{i+1}, \sigma_{i+1})$
is incident on precisely two \((k+1)\)-cells with chains ending in \(\gamma\). These must be \((l_i, \sigma_i)\) and \((l_i+2, \sigma_{i+2})\). If \(l_i < 0\) it follows that \((l_{i+1}, \sigma_{i+1})\) is the bottom face of \((l_i, \sigma_i)\) and hence \(l_{i+2} > l_i\) for all \(i\). Similarly, if \(l_i \geq 0\) it follows that \((l_{i+1}, \sigma_{i+1})\) is the top face of \((l_{i+2}, \sigma_{i+2})\) and hence \(l_{i+2} < l_i\) for all \(i\). In particular, the path cannot form a cycle and the matching is acyclic.

It remains to show that the set of critical cells is precisely the set of cells of \(\tilde{F}_P\). Let \(c_k = (m, e < w_1 < \cdots < w_k)\) be a cell of \(X_{P_*}\). When \(w_k = \gamma\), \(c_k\) is matched with its top or bottom face according as \(m \geq 0\) or \(m < 0\) and, hence, is not critical. When \(w_k \neq \gamma\), two cases arise. If \(m < 0\) then \(c_k\) is matched as the bottom face of \((m, e < w_1 < \cdots < w_k < \gamma)\) and is not critical. On the other hand, if \(m \geq 0\), then \(c_k\) is matched as the top face of \((m - n + |w_k|, e < \gamma w_k^{-1} < \gamma w_k^{-1} w_1 \cdots < \gamma w_k^{-1} w_{k-1} < \gamma)\)
if and only if \(m - n + |w_k| \geq 0\).

\[\square\]

**Corollary 4.6.** The finite complex \(\tilde{F}_P\) is homotopy equivalent to the Milnor fiber \(F_P\) and is a \(K(\pi, 1)\) for \(\pi = \ker(P_*)\).

**Example 4.7.** Let \(W\) be a dihedral group acting on \(\mathbb{R}^2\) with \(t\) reflections, \(R_1, R_2, \ldots, R_t\). Thus the NCPs are \(\{e, R_1, R_2, \ldots, R_t, \gamma\}\), where \(\gamma\) can be taken to be a rotation through twice the angle between adjacent lines of symmetry. Here \(n = 2\) and \(\tilde{F}_P\) is 1-dimensional with 2 vertices, namely \((0, e)\) and \((1, e)\). \(\tilde{F}_P\) has a 1-cell \((0, e < R_i)\) for each chain \(e < R_i\) and the endpoints of this 1-cell are \((0, e)\) and \((1, e)\). Thus \(\tilde{F}_P\) has the homotopy type of the suspension of a 0-dimensional subcomplex on \(t\) points and \(\ker(P_*)\) is free of rank \(t - 1\). This agrees with [22, Theorem 1]; \(P\) has weights 2 and \(t\).

**Example 4.8.** Let \(W\) be the group \(A_3 \cong \Sigma_4\), the symmetric group; the polynomial \(P\) is the classical discriminant for univariate quartics. Choose \(\gamma\) to be the four-cycle \((1, 2, 3, 4)\). Here \(n = 3\) and \(\tilde{F}_P\) is 2-dimensional. (See Figure 1.) \(\tilde{F}_P\) has 3 vertices, namely \((0, e)\), \((1, e)\) and \((2, e)\). Each transposition \(R\) contributes a 1-cell \((0, e < R)\) with endpoints \((0, e)\) and \((1, e)\) together with a 1-cell \((1, e < R)\) with endpoints \((1, e)\) and \((2, e)\). Each length \(\alpha \in \{(123), (124), (134), (234), (12)(34), (14)(23)\}\) contributes a single 1-cell \((0, e < \alpha)\) with endpoints \((0, e)\) and \((2, e)\). Finally, for each of the 16 chains of the form \(e < R < \alpha\) corresponding to factorisations of \(\gamma\) by transpositions, we have a 2-cell \((0, e < R < \alpha)\) whose boundary is glued along the three 1-cells \((0, e < R), (0, e < \alpha)\) and \((1, e < R^{-1}\alpha)\).

**Remark 4.9.** The cells of \(\tilde{F}_P\) are simplices, but \(\tilde{F}_P\) is not a simplicial complex; rather it is a \(\Delta\)-complex in the sense of [18]. It can be realized as the nerve of the germ (in the sense of category theory, see [15]) with objects \(\{0, 1, \ldots, n - 1\}\) and morphisms \(i \xrightarrow{\alpha} j\) where \(\alpha\) is an NCP of length \(j - i\). The composition, when defined, is determined by multiplication of NCPs.

**Remark 4.10.** The monodromy action on \(\tilde{F}_P\) is obtained by composing the monodromy action on \(X_{P_*}\) with the retraction defined by the acyclic
matching. It is more convenient to describe the action of \(-1 \in \mathbb{Z}\). Explicitly a cell of form
\[(m, e < w_1 < w_2 < \cdots < w_k)\]
for \(0 < m < n - |w_k|\) is taken to
\[(m - 1, e < w_1 < w_2 < \cdots < w_k)\]
while
\[(0, e < w_1 < w_2 < \cdots < w_k)\]
for \(|w_k| < n\) is taken to
\[(|w_1| - 1, e < w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_k < w_1^{-1}\gamma).\]

The second case is explained as follows. The action of \(-1\) on the \(k\)-cell \((0, e < w_1 < w_2 < \cdots < w_k)\) as a cell in \(X_{P_*}\) takes it to the \(k\)-cell \((-1, e < w_1 < w_2 < \cdots < w_k)\). This last \(k\)-cell is the bottom face of the \((k + 1)\)-cell \((-1, e < w_1 < w_2 < \cdots < w_k < \gamma)\), which in turn has top face \((|w_1| - 1, e < w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_k < w_1^{-1}\gamma)\).

5. NCP model for the Milnor fiber of the arrangement

The homomorphisms \(s\) and \(P_*\) from Example 2.3 and Section 4 combine to give the homomorphism
\[\psi = (P_*, s) : B(W) \to \mathbb{Z} \times W : x \mapsto (P_*(x), s(x)).\]
The space \(X_{\psi}\) (constructed as in subsection 2.2) is then a \(K(\pi, 1)\) for \(\pi = \ker(\psi)\).

**Proposition 5.1.** The complex \(X_{\psi}\) is homotopy equivalent to the Milnor fiber \(F_Q\) of \(Q\).
However, this latter cover is homotopy equivalent to $F$. It follows that $X_f(\psi)$ determines $P_{\psi}$ subgroup inclusion $\ker(\psi)$. Remark 5.5. The complex $\hat{\psi}$ is homotopy equivalent to the cover of $K$ corresponding to $f_*(\ker(Q_*))$ and hence to the cover of $W\setminus M$ corresponding to $f_*(\ker(Q_*))$. However, this latter cover is homotopy equivalent to $F_Q$ by Corollary 3.2(ii). \qed

Remark 5.2. The cover $X_\psi$ is a simplicial complex. The vertices of the $k$-cell $c_k = ((P_s(x), s(x)), e < w_1 < \cdots < w_k)$ are $((P_s(x), s(x)), e)$ and $((P_s(x) + w_i, s(xw_i)), e)$, $1 \leq i \leq k$. This set of $k+1$ vertices uniquely determines $c_k$.

Remark 5.3. The map $\psi = (P_s, s)$ is not onto. For each $x \in B(W)$, the integer $P_s(x)$ is even if and only if $s(x)$ belongs to the subgroup $W+ < W$ of Section 3. Thus $X_\psi$ can be identified with the CW-complex whose cells are triples, $(m, w, \sigma)$, where $\sigma$ is an initialized chain in $L$ and the parity of $m \in \mathbb{Z}$ is the same as that of $w \in W$. The top face of $(m, w, e < w_1 < \cdots < w_k)$ is $(m + |w_1|, ww_1, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$, while the other faces are given by $(m, w, e < w_1 < \cdots < \tilde{w}_i < \cdots < w_k)$ for $1 \leq i \leq k$.

Definition 5.4. We define the small NCP model $\hat{F}_Q$ of the Milnor fiber of $Q$ to be the finite subcomplex of $X_\psi$ which is the preimage of $\hat{F}_P$ under the covering $X_\psi \rightarrow X_{P_s}$ determined by the subgroup inclusion $\ker(\psi) \subseteq \ker(P_s)$.

Remark 5.5. The complex $\hat{F}_Q$ is the union of all cells of the form 

$$(0, w, e < w_1 < w_2 < \cdots < w_{n-1})$$

where $w \in W^+$ together with their faces.

Proposition 5.6. The subcomplex $\hat{F}_Q$ is a strong deformation retract of the cover $X_\psi$.

Proof: The simplicial complex $X_{\psi}$ is the cover of $X_{P_s}$ determined by the subgroup inclusion $\ker(P_s, s) < \ker(P_s)$. By the homotopy lifting property, the strong deformation retraction of $X_{P_s}$ onto $\hat{F}_P$ (proposition 4.5) is covered by a strong deformation retraction of $X_\psi$ onto $\hat{F}_Q$.

Corollary 5.7. The finite simplicial complex $\hat{F}_Q$ is homotopy equivalent to the Milnor fiber $F_Q$ and is a $K(\pi, 1)$ for the group $\ker((P_s, s))$.

Example 5.8. When $W$ is the dihedral group acting on $\mathbb{R}^2$ with $t$ reflections, $R_1, R_2, \ldots, R_t$, the subcomplex $\hat{F}_Q$ has the following description. Each
vertex \((0, w, e)\) has \(w \in W^+\), the rotation subgroup of \(W\), while each vertex \((1, w, e)\) has \(w \in W - W^+\), the set of reflections of \(W\). Furthermore, for each rotation \(w \in W^+\) and each reflection \(R\) there is another reflection \(S\) with \(w = RS\) giving an edge in \(\hat{F}_Q\) labelled \((0, w, e < S)\), starting at \((0, w, e)\) and ending at \((1, R, e)\). Thus \(\hat{F}_Q\) is the complete bipartite graph \(K_{t,t}\), which is homotopy equivalent to a bouquet of \((t-1)^2\) circles. This is consistent with the calculations in [22, Theorem 1].

**Remark 5.9.** The monodromy action on \(\hat{F}_Q\) is obtained as in Remark 4.10 by composing the monodromy action on \(X_\psi\) with the retraction defined by the acyclic matching. In this case, the action on \(X_\psi\) is by shifting the height of cells by multiples of 2. However, since \(\hat{F}_Q\) is a simplicial complex, it is sufficient to compute the action on 0-cells. We describe the action of \(-2 \in \mathbb{Z}\).

Explicitly, we define the action of \(-2\) on 0-cells by

\[
((m, w), e) \mapsto \begin{cases} 
(m - 2, w), e & 2 \leq m \leq n - 1 \\
(n - 1, w\gamma), e & m = 1 \\
(n - 2, w\gamma), e & m = 0.
\end{cases}
\]

The second case is explained as follows. The action of \(-2\) on the 0-cell \(((1, w), e)\) as a cell in \(X_\psi\) takes it to the 0-cell \(((1, w), e)\). This last 0-cell is the bottom face of the 1-cell \(((1, w), e < \gamma)\), which in turn has top face \(((n - 1, w\gamma), e)\). The third case is similar.

### 6. Structure and Homology of \(\hat{F}_P\)

Although our model \(\hat{F}_P\) of the Milnor fiber of \(P\) is not a simplicial complex, it does have a combinatorial description as a sequence of mapping cones. The domains of the mappings in question are complexes of truncations of the non-crossing partition lattice \(L\) and the lexicographic shellability of \(L\) yields a chain complex which computes \(H_*(\hat{F}_P, \mathbb{Z})\).

Let \(L_{[i,j]} = \{w \in L : i \leq |w| \leq j\}\) and let \(A_i = \{(m, \sigma) \in \hat{F}_P | \ m \geq n - i - 1\}\) where \(i, j\) are integers with \(0 \leq i \leq j \leq n\). Note that \(A_i\) has dimension \(i\) and that \(A_0 \subset A_1 \subset \cdots \subset A_{n-1} = \hat{F}_P\). Define \(g_i : |L_{[1,n-i-1]}| \rightarrow A_{n-i-2}\) by

\[w_1 < w_2 < \cdots < w_k \mapsto (i + |w_1|, e < w_1^{-1}w_2 < \cdots < w_k^{-1}w_k).\]

**Proposition 6.1.** The mapping cone of \(g_i\) is cellurally isomorphic to \(A_{n-i-1}\).

**Proof.** Note that \(|L_{[0,k]}|\) is simplicially isomorphic to the cone on \(|L_{[1,k]}|\) since \(L\) has a unique minimal element \(e\). Under this identification, the subcomplex \(|L_{[1,k]}|\) of \(|L_{[0,k]}|\) corresponds to \(|L_{[1,k]}| \times \{1\} \subset |L_{[1,k]}| \times [0,1]\). The map \(|L_{[0,n-i-1]}| \rightarrow A_{n-i-1}\) given by

\[e < w_1 < w_2 < \cdots < w_k \mapsto (i, e < w_1 < w_2 < \cdots < w_k).\]

combines with the inclusion of \(A_{n-i-2}\) into \(A_{n-i-1}\) to give a map

\[\tilde{g}_i : (|L_{[1,n-i-1]}| \times [0,1]) \coprod A_{n-i-2} \rightarrow A_{n-i-1}.\]
One can show that \( \hat{g}_i \) is an identification map which identifies precisely the pairs \((x, 0)\) with \((x', 0)\) for each \(x, x' \in |L_{[1,n-i-1]}|\) and \((x, 1)\) with \(g_i(x)\).

\[ \text{(6.1)} \]

**Lemma 6.2.** For all \( q \geq 1 \), \( H_q(A_p, A_{p-1}) \cong \tilde{H}_{q-1}(|L_{[1,p]}|) \).

**Proof.** The filtration \( A_0 \subset A_1 \subset \cdots \subset A_{n-1} = \hat{F}_p \) yields

\[
H_q(A_p, A_{p-1}) \cong \tilde{H}_q(A_p/A_{p-1}) \\
\cong \tilde{H}_q(\Sigma(|L_{[1,p]}|)) \\
\cong \tilde{H}_{q-1}(|L_{[1,p]}|),
\]

where \( \Sigma \) denotes suspension and the second isomorphism follows from Proposition 6.1.

**Definition 6.3.** For each \( i \), we define the \( i \)th face map \( d_i : C_{p-1}(|L_{[1,p]}|) \to C_{p-2}(|L_{[1,p]}|) \) by

\[
d_i(w_1 < w_2 < \cdots < w_p) = (-1)^{i-1}(w_1 < w_2 < \cdots < \hat{w}_i < \cdots < w_p)
\]

and the top face map \( \Omega : C_{p-1}(|L_{[1,p]}|) \to C_{p-2}(|L_{[1,p]}|) \) by

\[
\Omega(w_1 < w_2 < \cdots < w_p) = (w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_p).
\]

**Theorem 6.4.** The homology of \( \hat{F}_p \) is isomorphic to the homology of the chain complex whose \( p \)th group is \( \tilde{H}_{p-1}(|L_{[1,p]}|) \) and whose boundary homomorphism is given, at the level of chains, by \( \sum a_\sigma \sigma \mapsto \sum a_\sigma \Omega(\sigma) \).

**Proof.** First note that

\[
H_q(A_p, A_{p-1}) \cong \tilde{H}_{q-1}(|L_{[1,p]}|) \cong \begin{cases} \mathbb{Z}^{n_p} & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}
\]

where the last equality uses the lexicographical shellability of the NCP lattices [3, 5]. By Theorem 39.4 of [24], the homology of \( \hat{F}_p \) is isomorphic to the homology of the chain complex with \( p \)th group given by \( H_p(A_p, A_{p-1}) \) and boundary homomorphism given by the connecting homomorphism of the exact sequence of the triple \((A_p, A_{p-1}, A_{p-2})\).

It remains to compute the boundary homomorphism from \( \tilde{H}_{p-1}(|L_{[1,p]}|) \) to \( \tilde{H}_{p-2}(|L_{[1,p-1]}|) \). The isomorphism from \( \tilde{H}_{p-1}(|L_{[1,p]}|) \) to \( H_p(A_p, A_{p-1}) \) of Lemma 6.2 is induced by \( b_p : |L_{[1,p]}| \to A_p : \sigma \mapsto (n-p-1, e \ast \sigma) \), where \( e \ast \sigma \) means the simplex represented by \( e < w_1 < \cdots < w_l \) when \( \sigma \) is the simplex represented by \( w_1 < \cdots < w_l \). Let \( \sum a_\sigma \sigma \in C_{p-1}(|L_{[1,p]}|) \) be a cycle so that

\[
0 = \partial \left( \sum a_\sigma \sigma \right) = \sum a_\sigma \partial(\sigma) = \sum a_\sigma \sum_i d_i(\sigma).
\]

Since each \( \sigma \) is maximal, the \((p-1)\)-chains \( d_i \sigma \) and \( d_j \sigma \) have different length distributions whenever \( i \neq j \). (The length distribution of \( w_1 < w_2 < \cdots < w_p \) is \((|w_1|, |w_2|, \ldots, |w_p|)\).) It follows that

\[
(6.1) \quad \sum a_\sigma d_i(\sigma) = 0 \text{ for each } 1 \leq i \leq p.
\]

\[ \text{(6.1)} \]
Using the fact that the connecting homomorphism is defined on the level of chains by the boundary map in $\hat{F}_P$, we get

$$
\partial \left( b_p \left( \sum a_\sigma \sigma \right) \right) = \partial \left( \sum a_\sigma (n - 1 - p, e \ast \sigma) \right) = \sum a_\sigma \partial (n - 1 - p, e \ast \sigma) = \sum a_\sigma \left( (n - p, e \ast \Omega(\sigma)) + \sum_i (n - 1 - p, e \ast d_i \sigma) \right) = \sum a_\sigma (n - p, e \ast \Omega(\sigma)) \text{ by (6.1)} = b_{p-1} \left( \sum a_\sigma \Omega(\sigma) \right).
$$

□

Acknowledgements. This project began while the second named author was in residence at Dublin City University in Spring, 2012, under a Fulbright U.S. Scholars grant. He expresses gratitude to DCU and the Fulbright Commission in Ireland for support and hospitality.

References

[1] D. Allcock and T. Basak. Geometric generators for braid-like groups. Geom. Topol., 20(2):747–778, 2016.
[2] D. Armstrong. Generalized non-crossing partitions and combinatorics of Coxeter groups. Memoirs of AMS, 202(949), 2009.
[3] Christos A. Athanasiadis, Thomas Brady, and Colum Watt. Shellability of noncrossing partition lattices. Proc. Amer. Math. Soc., 135(4):939–949 (electronic), 2007.
[4] D. Bessis. The dual braid monoid. Ann. Sci. École Norm. Sup. (4), 36(5):647–683, 2003.
[5] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. Trans. Amer. Math. Soc., 260(1):159–183, 1980.
[6] N. Bourbaki. Lie groups and Lie algebras. Chapters 7–9. Elements of Mathematics. Springer-Verlag, Berlin, 2005. translated by A. Pressley.
[7] T. Brady. A partial order on the symmetric group and new $K(\pi,1)$’s for the braid groups. Adv. Math., 161(1):20–40, 2001.
[8] T. Brady and C. Watt. $K(\pi,1)$’s for Artin groups of finite type. In Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000), volume 94, pages 225–250, 2002.
[9] T. Brady and C. Watt. Non-crossing partition lattices in finite real reflection groups. Trans. Amer. Math. Soc., 360(4):1983–2005, 2008.
[10] E. Brieskorn. Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Invent. Math., 12:57–61, 1971.
[11] E. Brieskorn. Sur les groupes de tresses, volume 317 of Lecture Notes in Mathematics, pages 21–44. Springer-Verlag, Berlin, Heidelberg, New York, 1973.
[12] F. Callegaro. The homology of the Milnor fiber for classical braid groups. Algebr. Geom. Topol., 6:1903–1923 (electronic), 2006.
[13] F. Callegaro and M. Salvetti. Integral cohomology of the Milnor fibre of the discriminant bundle associated with a finite Coxeter group. C. R. Math. Acad. Sci. Paris, 339(8):573–578, 2004.
[14] C. Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955.

[15] P. Dehornoy. *Foundations of Garside theory*, volume 22 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2015. With François Digne, Eddy Godelle, Daan Krammer and Jean Michel, Contributor name on title page: Daan Kramer.

[16] P. Deligne. Les inmeubles des groupes de tresses généralisés. *Inventiones mathematicae*, 17:273–302, 1972.

[17] A. Dimca and G. Lehrer. On the cohomology of the milnor fiber of a hyperplane arrangement. In F. Callegaro et al., editor, *Configuration Spaces*, volume 14 of *Springer INdAM Series*, pages 319–360, 2016.

[18] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[19] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[20] D. Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.

[21] A. Măcinic and Ş. Papadima. On the monodromy action on Milnor fibers of graphic arrangements. *Topology Appl.*, 156(4):761–774, 2009.

[22] John Milnor and Peter Orlik. Isolated singularities defined by weighted homogeneous polynomials. *Topology*, 9:385–393, 1970.

[23] J.W. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[24] J. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[25] S. Settepanella. A stability-like theorem for cohomology of pure braid groups of the series $A$, $B$ and $D$. *Topology Appl.*, 139(1-3):37–47, 2004.

[26] S. Settepanella. Cohomology of pure braid groups of exceptional cases. *Topology Appl.*, 156(5):1008–1012, 2009.

School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland
E-mail address: tom.brady@dcu.ie

Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, AZ 86011, U.S.A.
E-mail address: michael.falk@nau.edu

School of Mathematical Sciences, Dublin Institute of Technology, Dublin 8, Ireland
E-mail address: colum.watt@dit.ie