POINTWISE DECAY FOR THE WAVE EQUATION ON NONSTATIONARY SPACETIMES

SHI-ZHUO LOOI

ABSTRACT. This first article in a two-part series (the second article being [arXiv:2205.13197]) assumes a weak local energy decay estimate holds and proves that solutions to the linear wave equation with variable coefficients in \( \mathbb{R}^{1+3} \), first-order terms, and a potential decay at a rate depending on how rapidly the vector fields of the metric, first-order terms, and potential decay at spatial infinity. We prove results for both stationary and nonstationary metrics. The proof uses local energy decay to prove an initial decay rate, and then uses the one-dimensional reduction repeatedly to achieve the full decay rate.

1. Introduction

In this paper, we examine pointwise decay for linear wave equations on asymptotically flat, nonstationary and stationary backgrounds in 1 + 3 dimensions and show how, given certain weak local energy decay estimates, the decay rate of the solution depends on the relative rates of the radial decay of the potential, the first-order coefficients and the background geometry. More on this assumption can be found in Definition 1.2.

Let

\[
P := \partial_\alpha g^{\alpha \beta}(t,x) \partial_\beta + g^{\omega}(t,x) \Delta_\omega + \partial_\alpha A^\alpha(t,x) + B^\alpha(t,x) \partial_\alpha + V(t,x)
\]

where the conditions on the potential \( V \), the coefficients \( A, B, g^{\omega} \) and the Lorentzian metric \( g \) are given in the main result, Theorem 1.8. \( \Delta_\omega \) denotes the Laplace operator on the unit sphere. We let \( \alpha, \beta \) range across 0, \ldots, 3. We consider the linear Cauchy problem

\[
P \phi = f, \quad (\phi(0), \tilde{N} \phi(0)) = (\phi_0, \phi_1)
\]

where \( \tilde{N} \) denotes the unit normal derivative to the hypersurface \( \{ t = 0 \} \).

In the next three definitions, we consider a few types of bounds for (1.2) as a precursor to the main result, Theorem 1.8. Given \( x \in \mathbb{R}^3 \), let \( r := |x| \) and \( \langle r \rangle := (1 + r^2)^{1/2} \).

We will use the following norms throughout the paper. In \( (1+3)\)-dimensions, we define

\[
A_R := \{ x \in \mathbb{R}^3 : R < |x| < 2R \} \quad (R > 2), \quad A_{R=1} := \{ |x| < 2 \}.
\]

Given a subinterval \( I \) of \([0, \infty)\),

\[
\begin{align*}
\| \phi \|_{LE(I)} &:= \sup_R \| \langle r \rangle^{-\frac{1}{2}} \phi \|_{L^2(I \times A_R)}, \\
\| \phi \|_{LE^1(I)} &:= \| \nabla_{t,x} \phi \|_{LE(I)} + \| \langle r \rangle^{-1} \phi \|_{LE(I)}, \\
\| f \|_{LE^{-1}(I)} &:= \sum_R \| \langle r \rangle^{\frac{1}{2}} f \|_{L^2(I \times A_R)}. 
\end{align*}
\]

(1.3)
We also define

\[ \|\phi\|_{LE^{1,k}(I)} = \sum_{|\alpha| \leq k} \|\partial^\alpha \phi\|_{LE^1(I)} \]
\[ \|\phi\|_{LE^{0,k}(I)} = \sum_{|\alpha| \leq k} \|\partial^\alpha \phi\|_{LE(I)}, \]
\[ \|f\|_{LE^{*,k}(I)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{LE^*(I)}. \]

For any norm, an omission of \( I \) will denote \( I := [0, \infty) \).

**Definition 1.1** (Local energy decay). We say that the solution to (1.2) satisfies the local energy decay estimate if the following estimate holds in \([0, \infty) \times \mathbb{R}^3\):

\[ \|\phi\|_{LE^{1,k}} \lesssim_k \|\nabla_{t,x} \phi(0)\|_{H^k} + \|f\|_{LE^*,k}, \quad k \geq 0 \]

(1.4)

Let \( \chi(x) \) be a compactly supported and smooth function equalling 1 in a neighbourhood of the trapped set. We define a weaker version each of the \( LE^1 \) norm that excises the trapped set region when evaluating \( \nabla_{t,x} \phi \) in \( LE \) norm. We also define the attendant dual weak norm.

\[ \|\phi\|_{LE^1_k(I)} := \|(1 - \chi)\nabla_{t,x} \phi\|_{LE(I)} + \|\langle r \rangle^{-1} \phi\|_{LE(I)}, \quad \|\phi\|_{LE^1_k(I)} := \sum_{|\alpha| \leq k} \|\partial^\alpha \phi\|_{LE^1_k(I)} \]
\[ \|f\|_{LE^*_{k}(I)} := \|f\|_{LE^*_{k}(I)} + \|\chi \nabla_{t,x} f\|_{L^2(I) L^2}, \quad \|f\|_{LE^*_{k}(I)} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{LE^*_{k}(I)} \]

We assume that (1.2) satisfies the following weak version of local energy decay **Definition 1.2**, as expressed by the following bounds:

**Definition 1.2** (Weak local energy decay). We say (1.2) satisfies the weak local energy decay estimate if for any real \( T_0 \geq 0 \) and any integer \( k \geq 0 \)

\[ \|\phi\|_{LE^{1,k}[T_0, \infty)} \lesssim_k \|\nabla_{t,x} \phi(T_0)\|_{H^k} + \|f\|_{LE^{*,k}[T_0, \infty)}. \]

(1.5)

**Remark 1.3** (Loss of two derivatives in the inhomogeneity). Combining the \( k \) and \( k + 1 \) cases of (1.5) implies

\[ \|\phi\|_{LE^{1,k}[T_0, \infty)} \lesssim_k \|\nabla_{t,x} \phi(T_0)\|_{H^{k+1}} + \|f\|_{LE^{*,k+2}[T_0, \infty)}. \]

(1.6)

Notice that the right-hand side must have \( k + 2 \) derivatives falling on \( f \), since the weak dual norm loses one derivative (at least on supp \( \chi \)), and we have applied the \( k + 1 \) case.

**Definition 1.4** (Commuting vector fields and function classes \( S^Z \)). In \( \mathbb{R}^{1+3} \), we consider the three (ordered) sets

\[ \partial := (\partial_t, \partial_1, \partial_2, \partial_3), \quad \Omega := (x^i \partial_j - x^j \partial_i), \quad S := t \partial_t + \sum_{i=1}^3 x^i \partial_i, \]

which are, respectively, the generators of translations, rotations and scaling. We set

\[ Z := (\partial, \Omega, S) \]

and we define the function class

\[ S^Z(f) \]
to be the collection of functions $g : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$|Z^J g(t,x)| \lesssim_J |f|$$

whenever $J$ is a multiindex. We will frequently use $f = \langle r \rangle^k$ for some real $k$.

**Remark 1.5** (Instances in which weak local energy decay holds). Weak local energy decay is known to hold in the Schwarzschild space-time and the Kerr space-time with small $0 \leq |a| \ll M$, where the parameter $M$ denotes the mass of the black hole and the parameter $a$ denotes the angular momentum per unit mass (thus $aM$ denotes the angular momentum of the black hole); more can be found in Section 1.1.

Examples where the assumptions we make on $\phi$ in this paper are actually satisfied include the following situations:

- the case with small (meaning $O(\epsilon)$ in all compact regions for the function and all its derivatives) and asymptotically flat perturbations $h \in S^2(\epsilon \langle r \rangle^{-1-\sigma})$ and a small potential $V \in S^2(\epsilon \langle r \rangle^{-2-\delta})$ for arbitrary real numbers $\delta, \sigma > 0$. See [24].
- The situation analysed in [32], which proves local energy estimates for solutions to scalar wave equations on nontrapping, asymptotically flat space-times (in particular large perturbations of Minkowski space-time).

**Remark 1.6** (Relation between weak local energy decay and stationary local energy decay). The problem (1.7) would be said to satisfy stationary local energy decay estimates (for derivatives) if for any interval $[T_1, T_2]$ and any integer $k \geq 0$, we have

$$\|\phi\|_{LE_v^k[T_1, T_2]} \lesssim_k \sum_{j=1}^2 \|\nabla_{t,x} \phi(T_j)\|_{H^k} + \|f\|_{LE_w^k[T_1, T_2]} + \|\partial_0 \phi\|_{LE_w^{0,k}[T_1, T_2]}.$$

Notice that we allow for $f$ in the weak dual norm $LE_w^*$, rather than in the usual dual norm $LE^*$. This differs from the definition of stationary local energy decay in [33].

There is an analogous version of the stationary local energy decay estimates for commuting vector fields (see Definition 1.4; they are also simply called vector fields) of $\phi$.

One can prove the following: if the weak local energy decay Definition 1.2 for derivatives of $\phi$ holds and $\partial_t$ is timelike on any trapped region that may exist, then $\phi$ in fact also satisfies stationary local energy decay estimates for vector fields of $\phi$. In this sense, the weak local energy decay is a weaker assumption than stationary local energy decay.

In this paper, we do not make the assumption of stationary local energy decay, and we do not assume that $\partial_t$ is timelike on any trapped region that may exist. This is in contrast to the assumptions made in [33], where the authors do make these assumptions. Our argument in this paper generalises the argument in [33] in this regard, and also in the decay rates of our coefficients, the kinds of coefficients considered in the wave operator $P$, and the support of the initial data. We consider solutions with non-compactly-supported initial data, in fact solutions with initial data in a weighted $L^2$ space (see Theorem 1.8 for the main theorem and assumptions).

**Definition 1.7.** We define $S_{\text{cone}}^Z(f)$ to be the collection of $g$ such that $|Z^J g| \lesssim |f|$ in $\{ t/2 \leq r \leq 3t/2 \}$. Thus $S^Z(f) \subsetneq S_{\text{cone}}^Z(f)$. We define $S_{\text{int}}^Z(f)$ to be the collection of $g$ such that $|Z^J g| \lesssim |f|$ in $\{ r < t/2 \}$. We define $S_{\text{radial}}^Z(f) := \{ g \in S^Z(f) : g$ is spherically symmetric $\}$. 

3
Let \( \| \cdot \| \) be any norm used in this paper. Given any nonnegative integer \( N \geq 0 \), we write \( \| g_{\leq N} \| \) to denote \( \sum_{|j| \leq N} \| g_j \| \). (See also **Definition 2.1**.)

Our main result, **Theorem 1.8**, is a pointwise decay estimate for the solution to the following equation:

\[
\begin{aligned}
P \phi(t, x) &= 0 \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \quad P \text{ given in (1.1)} \\
(\phi(0, x), \vec{N} \phi(0, x)) &= (\phi_0(x), \phi_1(x))
\end{aligned}
\]  

(1.7)

where \( g \) is a Lorentzian metric and \( g^\omega, A, B, V \) are functions satisfying the following conditions:

**Theorem 1.8 (Main theorem).** Let \( m \geq 0 \) be an integer and let \( N \) be a sufficiently large integer relative to \( m \), or \( N \gg m \). Let \( g^{\alpha \beta}(t, x) \) be a Lorentzian metric such that for all \( t_0 \geq 0 \) the level sets \( \{ t = t_0 \} \) are space-like, and let \( h := g - m \) with \( m \) denoting the Minkowski metric. Let \( u := t - r \) and \( v := t + r \). Assume that \( \phi \) solving (1.7) satisfies the weak local energy decay (1.5), and that \( \phi_0 \in L^2(\mathbb{R}^3) \).

1. Suppose that for some real \( 0 < \sigma, \delta, \delta' < \infty \),
   \[
   \begin{aligned}
   h &\in S^Z(\langle r \rangle^{-1-\sigma}) \\
   A &\in S^Z(\langle r \rangle^{-1-\sigma}) \\
   \partial_t A &\in S^Z(\langle \langle v \rangle \langle u \rangle^{-1} \langle r \rangle^{-1} \rangle^{-1-\sigma}) \cap S^Z_{cone}(\langle \langle r \rangle^{-1} \rangle^{-1-\sigma}) \\
   \partial A &\in S^Z_{int}(\langle \langle r \rangle^{-2} \rangle^{-1-\sigma}) \cap S^Z_{cone}(\langle \langle r \rangle^{-2} \rangle^{-1-\sigma}) \\
   B &\in S^Z(\langle r \rangle^{-1-\sigma}) \\
   \partial_t B &\in S^Z(\langle r \rangle^{-2-\delta}) \\
   V &\in S^Z(\langle r \rangle^{-2-\delta}) \\
   g^\omega &\in S^Z_{radial}(\langle \langle r \rangle^{-2-\delta} \rangle^{-1-\sigma})
   \end{aligned}
\]

Then

\[
| \phi_{\leq m}(t, x) | \lesssim \frac{1}{\langle v \rangle \langle u \rangle^{1+\min(\sigma, \delta, \delta')} } \| \langle r \rangle^{\frac{1}{2} + \min(\sigma, \delta, \delta')} \nabla_{t, x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)}
\]  

(1.8)

\[
| \partial \phi_{\leq m}(t, x) | \lesssim \frac{1}{\langle r \rangle \langle v \rangle^{2+\min(\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2} + \min(\sigma, \delta, \delta')} \nabla_{t, x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)}
\]  

(1.9)

\[
| \partial_t \phi_{\leq m}(t, x) | \lesssim \frac{1}{\langle v \rangle \langle u \rangle^{2+\min(\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2} + \min(\sigma, \delta, \delta')} \nabla_{t, x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)}
\]  

(1.10)

\[
| \partial^2 \phi_{\leq m}(t, x) | \lesssim \frac{\langle v \rangle}{\langle v \rangle^2 \langle u \rangle^{3+\min(\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2} + \min(\sigma, \delta, \delta')} \nabla_{t, x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)}
\]  

(1.11)

2. If in addition to the assumptions in part (1),
   \[
   \begin{aligned}
   \partial_t h &\in S^Z(\langle \langle r \rangle^{-2-\sigma} \rangle^{-1-\sigma}) \\
   \partial_t^2 h &\in S^Z_{cone}(\langle \langle u \rangle^{-2} \rangle^{-1-\sigma}) \\
   A &\in S^Z(\langle \langle r \rangle^{-2-\sigma} \rangle^{-1-\sigma}) \\
   \partial_t A &\in S^Z(\langle \langle v \rangle^{-1} \langle u \rangle^{-1} \langle r \rangle^{-2-\sigma} \rangle^{-1-\sigma}) \\
   B &\in S^Z(\langle \langle r \rangle^{-2-\sigma} \rangle^{-1-\sigma}) \\
   \partial_t B &\in S^Z(\langle \langle r \rangle^{-3-\sigma} \rangle^{-1-\sigma})
   \end{aligned}
\]
then the solution to (1.7) satisfies

\[ |\phi_{\leq m}(t, x)| \lesssim \frac{1}{\langle v \rangle \langle u \rangle^{1+\min(1+\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2}+\min(\sigma, \delta, \delta')} \nabla_{t,x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)} \]  

(1.12)

\[ |\partial_t \phi_{\leq m}(t, x)| \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{2+\min(1+\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2}+\min(\sigma, \delta, \delta')} \nabla_{t,x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)} \]  

(1.13)

\[ |\partial_t \phi_{\leq m}(t, x)| \lesssim \frac{1}{\langle v \rangle \langle u \rangle^{2+\min(1+\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2}+\min(\sigma, \delta, \delta')} \nabla_{t,x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)} \]  

(1.14)

\[ |\partial^2 \phi_{\leq m}(t, x)| \lesssim \frac{1}{\langle v \rangle \langle u \rangle^{3+\min(1+\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2}+\min(\sigma, \delta, \delta')} \nabla_{t,x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)} \]  

(1.15)

**Remark 1.9.** We make some remarks supplementing the main theorem.

- Second order operators that have spherically symmetric coefficients of the form $1/r$, at least away from the origin, are covered, i.e. included, by the definition of our operator $P$ in (1.1). This appears in some equations of physical interest, such as in general relativity.

- The argument shown in this paper straightforwardly yields a proof of a more general version of Theorem 1.8 which assumes more general decay rates on $A$ and $B$. Namely, given any real $\sigma', \sigma'' > 0$, for part (1) of Theorem 1.8 (and similarly for part (2)), if

\[
A \in S^{2}(\langle r \rangle^{-1-\sigma'})
\]

\[
\partial_t A \in S^Z \left( \langle v \rangle \langle u \rangle^{-1} \langle r \rangle^{-1-\sigma'} \right) \cap S^Z_{\text{cone}} (\langle r \rangle^{-1-\sigma'})
\]

\[
B \in S^{2}(\langle r \rangle^{-1-\sigma''})
\]

\[
\partial_t B \in S^{2}(\langle r \rangle^{-2-\sigma''})
\]

(in addition to the assumptions on $h$, $g^\alpha\beta$ and $V$ in part (1), as well as the assumption on the generic derivative $\partial A$) then the same arguments in this paper automatically give, for instance, with part (1) assumptions,

\[
|\phi_{\leq m}(t, x)| \lesssim \frac{1}{\langle v \rangle \langle u \rangle^{1+\min(\sigma, \delta, \delta')}} \| \langle r \rangle^{\frac{1}{2}+\min(\sigma, \delta, \delta')} \nabla_{t,x} \phi_{\leq N}(0) \|_{L^2(\mathbb{R}^3)},
\]

and the corresponding bounds also hold for $\partial_t \phi_{\leq m}$, $\partial \phi_{\leq m}$, and so on.

For simplicity of presentation, in this paper we restrict to the case $\sigma = \sigma' = \sigma''$.

- In item (2) of Theorem 1.8, one class of examples of metrics $g^\alpha\beta$ satisfying the conditions given are the stationary metrics $g$, that is, those with stationary component

\[
h = h(x).
\]

By substituting the natural number values $\delta \geq 1, \delta \in \mathbb{N}$ and $\sigma \geq 2, \sigma \in \mathbb{N}$, this special case of item (2) of Theorem 1.8 recovers a similar result as the main theorem in [34].

- If $h^\alpha\beta \in S^Z(\langle r \rangle^{-q})$ for some $q > 0$, then $\sqrt{|g|} h^\alpha\beta \in S^Z(\langle r \rangle^{-q})$. This is a consequence of the product rule and the assumption that $-q < 0$. Thus Theorem 1.8 also holds if $\partial_\alpha g^\alpha\beta \partial_\beta$ is replaced by the geometric wave operator

\[
\Box_g = \frac{1}{\sqrt{|g|}} \partial_\alpha \sqrt{|g|} g^\alpha\beta \partial_\beta, \quad |g| := |\det g^\alpha\beta|.
\]

\[1\text{ Indeed, (1.1) covers coefficients that have the following form away from the origin: } 1/r^a, \ a \in \mathbb{R}_{>0}.\]
- One statement this theorem says is that if local energy estimates even with derivative loss is assumed (see (1.6)), then one can obtain the pointwise bounds in Theorem 1.8.
- For a first reading, since $S^Z(\langle v \rangle \langle u \rangle^{-1} \langle r \rangle^{-1} \langle r \rangle^{-1} - \sigma) \cap S^Z_{cone}(\langle r \rangle^{-2} - \sigma) \subset S^Z(\langle r \rangle^{-2} - \sigma)$, the reader may wish to keep in mind that $\partial_t A \in S^Z(\langle r \rangle^{-2} - \sigma)$ for part 1 of Theorem 1.8.

Remark 1.10 (Black hole spacetimes). All the arguments in this paper can be adapted to the exterior of a ball and hence the proofs in this paper can be applied in the case of black hole spacetimes.

1.1. Local energy decay estimates. The first instance of a local energy estimate was obtained by Morawetz for the Klein-Gordon equation in [28]. Some other work on local energy decay estimates and their applications can be found in, for instance, [1, 20, 21, 29, 31, 43, 45, 46]. For local energy decay estimates for small and time dependent long range perturbations of the Minkowski space-time, see for instance [1], [30], [31] for time dependent perturbations, as well as, e.g., [10], [9], [44] for time independent, non-trapping perturbations. There is a related family of local energy decay estimates for the Schrödinger equation as well.

For Schwarzschild metrics, trapping at the event horizon was shown to be trivial due to an effect guaranteeing energy decay along the trapped rays called the redshift effect. On the other hand, for Kerr metrics, a local energy estimate with derivative loss on the trapped set is often introduced. Definition 1.2 includes this loss.

For large perturbations of the Minkowski metric, if one assumes the absence of trapping then local energy estimates can still hold; see for instance [9, 32]. For weak enough trapping, Definition 1.2 has been established; see for instance [8, 11, 37, 51]. If one assumes absence of trapping, then Definition 1.1 holds; with trapping Definition 1.1 cannot hold, see [40, 41]. With sufficiently strong trapping, even Definition 1.2 fails, see [16].

Weak local energy decay for the Schwarzschild metric was established in [10, 14, 27]. For the Kerr metric with low angular momenta, weak local energy decay estimates were proved in [10, 14, 15].

The local energy estimate for Kerr spacetimes with small angular momenta was proven in [50] (see also [4] and [12] for related work), for large angular momentum $|a| < M$ in [15], and for extremal Kerr $|a| = M$ in [5].

1.2. Pointwise decay and asymptotic behaviour. It is well-understood that local energy decay in a compact region on an asymptotically flat region implies pointwise decay rates that are related to how rapidly the metric coefficients decay to the Minkowski metric; see, for example, the works [2, 3, 19, 23–25, 33–36, 38, 49]. Similar results as this paper, for stationary spacetimes, were shown in [35] concurrently using spectral theory techniques, with particular focus on analyzing the resolvent at low frequencies.

Local energy decay is also involved in proving scattering, another type of asymptotic behaviour, on variable-coefficient backgrounds. In particular, they imply Strichartz estimates on certain variable-coefficient backgrounds, see [29]. The article [26] used local energy decay to prove scattering for the version of the problem (1.7) without the potential
V and first-order terms A and B, although the argument extends straightforwardly to the problem including V, A and B defined above.

In the case of the Schwarzschild metric, Price [39] conjectured that the solution to the wave equation decays at the rate $t^{-3}$ within any compact region; this rate was shown to hold for a variety of spacetimes, including Schwarzschild and Kerr spacetimes with small angular momenta—see [17, 33, 49].

1.3. The main ideas of the proof. Aside from the standard tools of Sobolev embedding, albeit exploited primarily in dyadic conical subregions, when proving pointwise bounds we take advantage of the reduction to $1 + 1$ dimensions in spherical symmetry—called the “one-dimensional reduction”—and the positivity of the fundamental solution to the $1 + 3$ dimensional wave equation. This not only provides a simple setting for the analysis but also allows us to “absorb” pointwise decay from the vector fields of the coefficients $h, V,$ and so forth, and transfer them to the decay of the solution $\phi$ or its vector fields. In this way, gradual improvements, starting from an initial decay estimate (4.6)—obtained from only Sobolev embedding and integrated local energy decay—are possible, with the improvements arising from the positivity of $\sigma, \delta$ and $\delta'.$

A little more precisely, for components of the wave equation that contain a derivative structure, we analyse them separately: in a neighbourhood of the light cone $\{r = t\}$ (see Section 10); and in all other regions. However, for components of the wave equation that do not contain a derivative structure, we need not make this distinction.

1.4. Summary of sections. In Section 2, we define notation and conventions for the rest of the paper.

In Section 3, we commute $P$ with vector fields and prove (weak) local energy estimates for vector fields.

In Section 4, we prove Sobolev embedding estimates and obtain an initial pointwise decay estimate. We connect pointwise bounds to $L^2$ estimates and norms, thereby connecting local energy decay to pointwise bounds.

In Section 5, we prove that derivatives of vector fields of the solution decay better at the cost of applying more vector fields.

In Section 6, we define more notation that will be used for the pointwise decay iteration, which occupies the remainder of the paper. We also prove certain lemmas used in the iteration.

In Section 7, we prove the upper bound in $\{r > t + 1\}$ for components of the solution away from the cone.

In Section 8, we show how to convert a decay rate of $\langle r \rangle^{-p}$ for the solution $\phi$ and its vector fields to $\langle t + r \rangle^{-p}$ for $p \leq 1$.

In Section 9, we prove the upper bound in $\{r < t\}$ for components of the solution away from the cone.
In Section 10, we prove the upper bound for components of the solution near the cone.

2. Notation and conventions

2.1. Notation for dyadic numbers and conical subregions. We work only with dyadic numbers that are at least 1. We denote dyadic numbers by capital letters for that variable; for instance, dyadic numbers that form the ranges for radial (resp. temporal and distance from the cone \( \{ |x| = t \} \)) variables will be denoted by \( R \) (resp. \( T \) and \( U \)); thus

\[ R, T, U \geq 1. \]

We choose dyadic integers for \( T \) and a power \( a \) for \( R, U \)—thus \( R = a^k \) for \( k \geq 1 \) but not much larger than 2, for instance in the interval \((2, 5]\), such that for every \( j \in \mathbb{N} \), there exists \( j' \in \mathbb{N} \) with

\[ a^{j'} = \frac{3}{8} 2^j. \]  

(2.1)

2.1.1. Dyadic decomposition. We decompose the region \( \{ r \leq t \} \) based on either distance from the cone \( \{ r = t \} \) or distance from the origin \( \{ r = 0 \} \). We fix a dyadic number \( T \).

\[ C_T := \begin{cases} 
\{(t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T, \ r \leq t \} & T > 1 \\
\{(t, x) \in [0, \infty) \times \mathbb{R}^3 : 0 < t < 2, \ r \leq t \} & T = 1 
\end{cases} \]

\[ C_T^R := \begin{cases} 
C_T \cap \{ R < r < 2R \} & R > 1 \\
C_T \cap \{ 0 < r < 2 \} & R = 1 
\end{cases} \]

\[ C_T^U := \begin{cases} 
\{(t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T \} \cap \{ U < |t - r| < 2U \} & U > 1 \\
\{(t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T \} \cap \{ 0 < |t - r| < 2 \} & U = 1 
\end{cases} \]

If a need arises to distinguish between the \( R = 1 \) and \( U = 1 \) cases, we shall write \( C_T^{R=1} \) and \( C_T^{U=1} \) respectively. Note that \( |C_T^{R}| \sim (R^3 T)^{1/2} \) and \( |C_T^{U}| \sim (T^3 U)^{1/2} \). We define

\[ C_T^{<3T/4} := \bigcup_{R<8T/8} C_T^R. \]

Now letting \( R > T \), we define

\[ C_R^T := \{(t, x) \in [0, \infty) \times \mathbb{R}^3 : r \geq t, T \leq t \leq 2T, R \leq r \leq 2R, R \leq |r - t| \leq 2R \} \]

Note that \( |C_R^T| \sim R^2 \), as can be seen in the \( |r - t| \) and \( r \) directions.

\( C_T^R, C_T^U \) and \( C_R^T \) are where we shall apply Sobolev embedding, which allows us to obtain pointwise bounds from \( L^2 \) bounds.

2.1.2. Enlargements of sets. Given any subset of these conical regions, a tilde atop the symbol \( C \) will denote a slight enlargement of that subset; for example, \( \tilde{C}_T^R \) denotes a slightly larger set containing \( C_T^R \).

2.2. More notation for vector fields. Beyond Definition 1.4, we now define more notation for vector fields.
2.2.1. Subscripts on functions will denote vector fields. Given a nonnegative integer \( m \) and a triplet \( J = (i, j, k) \) of multi-indices \( i, j \) and \( k \) for \((\partial, \Omega, S)\)—by this we mean \( \partial^i \Omega^j S^k \)—we denote \(|J| = |i| + 4|j| + 10k\). See (2.2).

The coefficient 10 in front of \( k \) arises because of the fact \([P, S] − 2P − \sigma S \in \mathcal{C} \), where \( \mathcal{C} \) is the class of operators defined in (3.2). In particular the presence of \( \Omega^2 \) as well as loss of derivative considerations (a price of losing two derivatives if one wants to control the full \( LE^1 \) norm—see (1.6)) for the inhomogeneity in the weak local energy decay Definition 1.2 leads to the count \( 10 = 2 + 2 \cdot 4 \). If \( g^\omega = 0 \), then we would count each \( S \) in the same way we would count \( \partial^2 \), i.e. two derivatives. We put in place these differences in these numerical weights for \( i, j, \) and \( k \) (respectively: 1, 4, and 10) because of the trapped set.

We denote
\[
\phi_J := Z^J \phi := \partial^i \Omega^j S^k \phi,
\]
(2.2)
\[
\phi_{\leq m} := (\phi_J)_{|J| \leq m}, \quad \phi_{m_1 \leq m_2} := (\phi_J)_{m_1 \leq |J| \leq m_2}, \quad \phi_m := (\phi_J)_{|J| = m}.
\]

Furthermore, by \( Z^m \phi \) we mean \( \phi_{\leq m} \), and so on. We write \( J_1 \leq J_2 \) to mean
\[
i_1 \leq i_2, \quad j_1 \leq j_2, \quad k_1 \leq k_2,
\]
and \( J_1 < J_2 \) if at least one of the inequalities above is strict. If \( I \) is a multiindex of order \( \ell \) and \( n \) an integer, by \( I + n \) we mean
\[
\{I + J : |J| = n, J \text{ is an } \ell\text{-multiindex}\}.
\]

Given a multiindex \( K \), we define
\[
\phi_{\leq K} := (\phi_J)_{|J| \leq K}.
\]

2.3. Notation for the symbols \( n \) and \( N \). Throughout the paper the integer \( N \) will denote a fixed and sufficiently large positive numbersignifying the highest total number of vector fields that will ever be applied to the solution \( \phi \) to (1.7) in the paper.

We use the convention that the value of \( n \) may vary by line.

2.4. The use of the tilde symbol. If \( \Sigma \) is a set, we shall use \( \tilde{\Sigma} \) to indicate a slight enlargement of \( \Sigma \), and we only perform a finite number of slight enlargements in this paper to dyadic subregions. The symbol \( \tilde{\Sigma} \) may vary by line.

If \( f \) is a function, we shall typically use \( \tilde{f} \) to denote commuting vector fields applied to \( f \).

2.5. Notation for implicit constants. We write \( X \lesssim Y \) to denote \(|X| \leq CY \) for an implicit constant \( C \) which may vary by line. Similarly, \( X \ll Y \) will denote \(|X| \leq cY \) for a sufficiently small constant \( c > 0 \). In this paper, all implicit constants are allowed to depend on the dimension and the initial data \( \phi_{\leq N}[0] \), for a fixed \( N \in \mathbb{N} \) that is sufficiently large.
**Definition 2.1.** Given a real number \( t \geq 0 \), a norm or square of a norm \( \mathcal{F}[\cdot](t) \) (including the absolute value), a function \( f \), a multiindex \( J \), and an integer \( k \), we let

\[
\mathcal{F}[f_{\leq J}](t) := \sum_{\text{multiindices } I \leq J} \mathcal{F}[f_I](t)
\]

\[
\mathcal{F}[f_{\leq k}](t) := \sum_{\text{multiindices } |I| \leq k} \mathcal{F}[f_I](t)
\]

2.6. **Other notation.** If \( t \geq 0 \) is a real number, let

\[
\Sigma_t := \{ (t, x) : x \in \mathbb{R}^3 \}
\]

denote the constant time \( t \) slice.

If \( x = (x^1, x^2, x^3) \in \mathbb{R}^3 \), we write

\[
\begin{align*}
    r &:= \left( \sum_{i=1}^{3} (x^i)^2 \right)^{1/2}, \\
u &:= t - r, \\
v &:= t + r.
\end{align*}
\]

We write \( \Box := -\partial_t^2 + \Delta \).

We write \( s_q \) to denote element of \( S^2(\langle r \rangle^{-q}) \). \( q \) will denote a nonnegative number.

3. **Commuting with vector fields, and weak local energy decay for vector fields**

**Remark 3.1.** Let \( w \) be a sufficiently smooth function. Then

\[
\partial w \in S^2(\langle r \rangle^{-1})\bar{Z} w + \mu S^2(1)|\partial t w| \text{ if } r \geq t/2
\]

with \( \mu = 0, \bar{Z} = \Omega \) for angular derivatives \( \partial_\omega w \) on the left-hand side, and \( \mu = 1, \bar{Z} = S \) for the radial derivative \( \partial_r w \) on the left-hand side.

We define \( C \) to be the collection of real linear combinations of the operators

\[
\partial s_{1+q'} \partial, \ s_{1+q'} \partial \partial, \ s_{2+q'}, \ \partial s_{1+q'}, \ s_{1+q'} \partial
\]

where \( q' > 0 \) is a number which depends on the assumptions made about the coefficients \( h, g^\omega, V, A, \text{ and } B \) in **Theorem 1.8**. That is, schematically, \( C = \{ \partial s_{1+q'} \partial + s_{1+q'} \partial \partial + s_{2+q'} + \partial s_{1+q'} + s_{1+q'} \partial \} \).

**Lemma 3.2.** Let \( w \) be a sufficiently smooth function. Given \( J \) and \( k \geq 0 \), there are some operators \( C \in C \) such that

\[
\Omega^J(S + 2)^k P w = P \Omega^J S^k w + \hat{C} w_{\leq 4(|J|-1)+10k}
\]

where we adopt the following conventions: we interpret \( \hat{C} w_{\leq 4(|J|-1)+10k} \) as a sum, and subscripts with negative real value denote the zero multiindex.
Proof (sketch). By the assumptions in the main theorem,
\[
[P, \partial] \in \mathcal{C}, \quad (3.4)
\]
\[
[P, \Omega] \in \mathcal{C}, \quad (3.5)
\]
\[
[P, S] - 2P - s_{2+q} \Omega^2 \in \mathcal{C}. \quad (3.6)
\]
One uses (3.4) to (3.6) and proves the result by mathematical induction. We omit the details of the proof, except for the following observation. Starting from $\Omega^j(S+2)^kP$ and then commuting the vector fields with $P$, then other than $P\Omega^jS^k$, the terms with the highest vector field count (assuming $g^\omega$ is not the zero function) are those of the form
\[
\dot{C}\tilde{Z}=|J|+k^{-1}w, \quad \tilde{Z} \in \{\Omega, S\}, \quad \dot{C} \in \mathcal{C};
\]
more specifically, those of the form $\dot{C}\Omega^{|J|-1}S^k$. This explains the subscript $4(|J|-1)+10k$.

Lemma 3.3. Given the assumptions in either part 1 or part 2 of Theorem 1.8, there exists a positive real number $q' > 0$ such that for any multiindex $J$,
\[
|P\phi| \lesssim \frac{|\phi|_{|J|-1}}{\langle r \rangle^{2+q'}} + \frac{|\nabla_{t,x}\phi|_{|J|}}{\langle r \rangle^{1+q'}} + |(P\phi)|_{|J|}.
\]

Proof. There is a constant $q' > 0$ such that the operator $P$ can be written schematically as $P = \Box + (s_1+q')\partial + s_1+q'\partial^2 + s_2+q' + s_3+q'\partial + (s_1+q')\partial$. We have $[Z, \partial] = c\partial$ schematically, for some real number $c$ depending on $Z$.

For terms of the form $(\partial A)\dot{\phi}$, where $A, \dot{\phi}$ denote possible vector fields of $A, \phi$, we apply the assumption
\[
\partial A \in S^Z_{\text{int}}(\langle r \rangle^{-2}) \cap S^Z_{\text{cone}}(\langle r \rangle^{-2})
\]
on generic derivatives $\partial A$ from part 1 of Theorem 1.8 in $\{r < 3t/2\}$, and the assumption on $\partial_t A$ and (3.1) in $\{r \geq 3t/2\}$, giving a contribution of the form $\langle r \rangle^{-2-q'}|\dot{\phi}|_{|J|}$. For part 2, on the other hand, we in fact need not look at $r < 3t/2$ and $r \geq 3t/2$ separately, because the statement $\partial A \in S^Z(\langle r \rangle^{-2})$ is already trivially satisfied for any $(t, r)$-pair given the assumption on $A$.

We include the terms arising from $g^\omega \Delta_\omega$, together with the $\langle r \rangle^{-3-}|\nabla_{t,x}\phi|_{|J|}$ term. The rest is clear, and the claim follows.

We recall the weak local energy decay estimate $\|\phi\|_{LE^{1,k}} \lesssim_k \|\nabla_{t,x}\phi(T_0)\|_{H^k} + \|f\|_{LE^{k,k}}$, which can be rephrased as
\[
\sum_{|\alpha|=k+1} \|\partial^\alpha \phi\|_{LE(T_0, \infty)} + \|\langle r \rangle^{-1} \phi\|_{LE^{k,k},T_0, \infty} \lesssim_{k,x} \|\nabla_{t,x}\phi(T_0)\|_{H^k} + \|f\|_{LE^{k,k},T_0, \infty}.
\]

Proposition 3.4 (Weak local energy decay for vector fields). Let $\phi$ be any smooth-enough function solving (1.7) and satisfying Definition 1.2. Then for any natural number $m \geq 0$,
\[
\|\phi\|_{LE^1} \lesssim \|\nabla_{t,x}\phi_m(0)\|_{L^2} + \|f\|_{LE^m}.
\]

Proof. We prove (3.7) by induction.

The base case
\[
\|\phi\|_{LE^1} \lesssim \|\nabla_{t,x}\phi_1(0)\|_{L^2} + \|f\|_{LE^1}.
\]
is simply given by combining Definition 1.2 at \( k = 0 \) and \( k = 1 \), which yields
\[
\| \phi \|_{LE^1} \lesssim \| \nabla_{t,x} \phi(0) \|_{H^1} + \| \partial^{\leq 1} f \|_{LE^*} + \| \chi \partial^{\geq 2} f \|_{L^2},
\]
which is clearly bounded by
\[
\| \nabla_{t,x} \phi(0) \|_{H^1} + \| \partial^{\leq 2} f \|_{LE^*} \leq \| \nabla_{t,x} \phi_{\leq 1}(0) \|_{L^2} + \| f_{\leq 2} \|_{LE^*}.
\]

Next, we use Lemma 3.2. Let \(| I, J, k | = m \).
\[
\| \phi_{(I,J,k)} \|_{LE^1} \lesssim \| \nabla_{t,x} \Omega^J S^k \phi(0) \|_{H^{I+1}} + \| \Omega^J S^k \|_{LE^*,|I|+2} + \| P, \Omega^J S^k \|_{LE^*,|I|+2}
\]
\[
\lesssim \| \nabla_{t,x} \phi_{\leq m+1}(0) \|_{L^2} + \| f_{\leq m+2} \|_{LE^*} + \| P, \Omega^J S^k \|_{LE^*,|I|+2}
\]
\[
\lesssim \| \nabla_{t,x} \phi_{\leq m+1}(0) \|_{L^2} + \| f_{\leq m+2} \|_{LE^*} + \| \phi_{\leq m-2} \|_{LE^*}
\]
\[
\lesssim \| \nabla_{t,x} \phi_{\leq m+1}(0) \|_{L^2} + \| f_{\leq m+2} \|_{LE^*} + \| \phi_{\leq m-2} \|_{LE^1}
\]
\[
\lesssim \| \nabla_{t,x} \phi_{\leq m+1}(0) \|_{L^2} + \| f_{\leq m+2} \|_{LE^*}.
\]

In transitioning from the second line to the third line, we used (3.3). The third line follows by the assumption that \( \Omega \) counts for four partial derivatives. The final line follows by the induction hypothesis. \(\square\)

**Remark 3.5.** The above proof extends to time intervals \([T_1, \infty)\), \(T_1 \geq 0\). (The proof above assumes \(T_1 = 0\).) The estimate is
\[
\| \phi_{\leq m} \|_{LE^1[T_1, \infty)} \lesssim \| \nabla_{t,x} \phi_{\leq m+1}(T_1) \|_{L^2} + \| f_{\leq m+2} \|_{LE^*[T_1, \infty)}.
\]

### 4. Initial \(L^\infty\) estimates

We now state the Sobolev embedding estimates localised to our selected conical regions.

**Lemma 4.1.** Let \( w \in C^4 \).

- **For all** \( T \geq 1 \) and \( 1 \leq U \leq 3T/8 \), we have
  \[
  \| w \|_{L^\infty(C^U_T)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{(T^3U)^{1/2}} \| S^i \Omega^j w \|_{L^2(C^U_T)} + \left( \frac{U}{T^3} \right)^{1/2} \| r \partial_s S^i \Omega^j w \|_{L^2(C^U_T)}; \tag{4.1}
  \]

- **For all** \( T \geq 1 \) and \( R > T \), we have
  \[
  \| w \|_{L^\infty(C^R_T)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{(R^3T)^{1/2}} \| S^i \Omega^j w \|_{L^2(C^R_T)} + \frac{1}{(RT)^{1/2}} \| r \partial_s S^i \Omega^j w \|_{L^2(C^R_T)}; \tag{4.2}
  \]

- **For all** \( T \geq 1 \) and \( 1 \leq R \leq 3T/8 \), we have
  \[
  \| w \|_{L^\infty(C^R_T)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{(R^3T)^{1/2}} \| S^i \Omega^j w \|_{L^2(C^R_T)} + \frac{1}{(RT)^{1/2}} \| r \partial_s S^i \Omega^j w \|_{L^2(C^R_T)}; \tag{4.3}
  \]

**Proof.** In \( C^U_T \) we make the change of coordinates \( t = e^s \) and \( |r - t| = e^{s+\rho} \). With this change of coordinates, we are now dealing with a region of size 1 in spherical coordinates including \( s \). We have \( \partial_s = t \partial_t + r \partial_r = S \) and \( \partial_\rho = (r - t) \partial_r \). Then we apply the fundamental theorem of calculus in \( s \) and also in \( \rho \). Finally, we rescale to \( C^U_T \), obtaining (4.1).
For $C^T_R$, we let $r = e^s$ and $r - t = e^{s+\rho}$. Thus $\partial_s = S$ and $\partial_\rho = (t-r)\partial_t$. We get

$$\|w\|_{L^\infty(C^T_R)} \lesssim \sum_{i\leq j\leq 2} \frac{1}{(R^3T)^{1/2}} \|S^i\Omega^j w\|_{L^2(C^T_R)} + \frac{R - T}{(R^3T)^{1/2}} \|\partial_t S^i\Omega^j w\|_{L^2(C^T_R)}.$$  

This implies (4.2) since $R - T \leq R$.

For $C^R_T$, we let $t = e^s$ and $r = e^{s+\rho}$. We obtain $\partial_s = S$ and $\partial_\rho = r\partial_t$ and (4.3).

**Corollary 4.2.**

$$\|\phi\|_{L^\infty_{t,\rho}(C^{<3T/4}_T)} \lesssim \sum_{i\leq j\leq 2} \frac{1}{T^{1/2}} \|S^i \Omega^j \phi\|_{LE^{1,\rho}_{t,\rho}(C^{<3T/4}_T)}.$$  

**Proof.** By rewriting (4.3) in the local energy norm by shifting the $R$ weights around, we obtain (4.4).

**Lemma 4.3.** If $f \in C^1([0,\infty)_t \times \mathbb{R}^3_\rho)$, then

$$\int_{t/2}^{3t/2} \frac{f(t, x)^2}{\langle u \rangle^2} dx \lesssim \left( \int_{t/4}^{7t/4} |\partial_r f(t, x)|^2 dx + \frac{1}{t^2} \left( \int_{t/4}^{7t/4} f(t, x)^2 dx + \int_{3t/2}^{7t/4} f(t, x)^2 dx \right) \right).$$  

**Proof.** Let $\chi : [0, \infty) \to [0, 1]$ be a cutoff such that $\chi(s) = 1$ for $1/2 \leq s \leq 3/2$ and 0 when $s \leq 1/4$ and $s \geq 7/4$. We will show that, if $\gamma > -1/2$, and $\gamma \neq 1/2$, then

$$\int \langle u \rangle^{-2-2\gamma} \chi(r/t) f(r, \omega)^2 r^2 dr \lesssim \int \langle u \rangle^{-2\gamma} |\partial_r f(r, \omega)\chi(r/t)|^2 r^2 dr + \frac{1}{t^2} \int \langle u \rangle^{-2\gamma} |f(r, \omega)\chi(r/t)|^2 r^2 dr.$$  

The conclusion follows if we take $\gamma = 0$ and integrate over $\omega$.

We have

$$f(r, \omega)^2 \chi(r/t) - f(7t/4, \omega)^2 \chi((7t/4)/t) = -2 \int_{r}^{7t/4} f(\rho, \omega) \chi(\rho/t) \cdot \partial_r (f(\rho, \omega) \chi(\rho/t)) d\rho.$$  

Hence

$$f(r, \omega)^2 \chi(r/t) r^2 \lesssim f(7t/4, \omega)^2 \chi(3t/2)^2 t^2 + 2 \int_{r}^{7t/4} |f(\rho, \omega) \chi(\rho/t) \cdot \partial_r (f(\rho, \omega) \chi(\rho/t))| d\rho.$$  

Recall that $\chi(7t/4) = 0$. We multiply by $\langle u \rangle^{-2-2\gamma}$ and integrate $r$ from $t/4$ to $7t/4$. This yields

$$\int_{t/4}^{7t/4} \langle u \rangle^{-2-2\gamma} \chi(r/t) f(r, \omega)^2 r^2 dr \lesssim \int_{t/4}^{7t/4} \langle u \rangle^{-1-2\gamma} |f(r, \omega)\chi(r/t)|^2 r^2 dr.$$  

By the chain rule, $\partial_r (\chi(r/t)) \lesssim \chi'(r/t) \cdot \frac{1}{t}$. Thus by Cauchy-Schwarz and the chain rule

$$\int_{t/2}^{3t/2} \langle u \rangle^{-2-2\gamma} f(r, \omega)^2 r^2 dr \lesssim \int_{t/4}^{7t/4} \langle u \rangle^{-2\gamma} |\partial_r f(r, \omega)\chi(r/t)|^2 r^2 dr + \frac{1}{t^2} \int_{t/4}^{7t/4} \langle u \rangle^{-2\gamma} |f(r, \omega)\chi'(r/t)|^2 r^2 dr.$$  

□
The following result is an analogue of Theorem 5.3 in [22].

**Lemma 4.4.** Let $T$ be fixed and $\phi$ solve (1.7) for the times $t \in [T, 2T]$. There is a fixed positive integer $k$ such that for any multi-index $J$ with $|J| + k \leq N$, we have:

$$|\phi_J| \lesssim_{|J|} \|\phi_{|J| \leq |J| + k}\|_{LE^1[T, 2T]} \frac{\langle u \rangle^{1/2}}{\langle v \rangle}.$$  \hspace{1cm} (4.6)

**Proof.** We prove this by looking separately at $(t, x)$-pair values in $C^R_T, C^R_R$ and $C^U_T$.

- (The $C^U_T$ regions, with $1 \leq U \leq 3T/8$) In contrast to the “near” region $C^R_T$ and the “far” region $C^R_R$, the regions close to the cone will proceed differently: we utilise a Hardy-like inequality adapted to the cone, namely (4.5).

  Let $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a smooth cutoff function with $\chi(s) = 1, s \geq 1/2$ and $\chi(s) = 0, s \leq 1/4$. For any smooth-enough function $w$,

$$\|w\|_{L^2(C^U_T)} \lesssim \|\chi(T)w\|_{L^2[T, 2T]} \lesssim \|\partial_r(\chi(T)w)\|_{L^2[T, 2T]} + T^{-1}\|\chi(T)w\|_{L^2_{r,t}(T/8 \leq r \leq 15/8T)} \lesssim T^{1/2}\|w\|_{LE^1[T, 2T]}$$  \hspace{1cm} (4.7)

where the second line follows by (4.5).

Thus

$$\|\phi_J\|_{L^\infty(C^U_T)} \lesssim \sum_{i \leq |J| \leq 2} \frac{1}{(T^3U)^{1/2}} \|S^i \Omega^j \phi_J\|_{L^2(C^U_T)} + \left(\frac{U}{T^3}\right)^{1/2} \|\partial_r S^i \Omega^j \phi_J\|_{L^2(C^U_T)}$$

$$\lesssim \left(\frac{U}{T^3}\right)^{1/2} T^{1/2} \sum_{i \leq |J| \leq 2} \|S^i \Omega^j \phi_J\|_{LE^1[T, 2T]}$$

$$\lesssim \frac{U^{1/2}}{T} \|\phi_{|J| \leq |J| + k}\|_{LE^1[T, 2T]}.$$  

- (The $C^R_T$ regions, for $R$ values sufficiently small relative to $T$) This is essentially Corollary 4.2: apply the Sobolev embedding estimate (4.3) to $\phi_J$

$$\|\phi_J\|_{L^\infty(C^R_T)} \lesssim \sum_{i \leq |J| \leq 2} \frac{1}{(R^3T)^{1/2}} \|S^i \Omega^j \phi_J\|_{L^2(C^R_T)} + \frac{1}{R^{1/2}T^{1/2}} \|\partial_r S^i \Omega^j \phi_J\|_{L^2(C^R_T)}$$

$$\lesssim \frac{1}{T^{1/2}} \|\phi_{|J| \leq |J| + k}\|_{LE^1[T, 2T]}.$$  

and take the supremum over, say, $R < 3T/8$. The second inequality comes from commuting $S^i \Omega^j$ with $Z^j$ in a way that will put it in the form (2.2). This is where the integer $k$ arises.
• (The $C^T_R$ regions) (4.2) implies
\[
\|\phi_j\|_{L^\infty(C^T_R)} \lesssim \frac{1}{R^{1/2}} \sum_{i \leq 1, j \leq 2} \|R^{-3/2} S^{-1/2} \Omega^j \phi_j\|_{L^2[\hat{\mathcal{C}}^T_R]} + \|R^{-1/2} \partial_t S^{-1/2} \Omega^j \phi_j\|_{L^2[\hat{\mathcal{C}}^T_R]}
\]
\[
\lesssim \frac{1}{R^{1/2}} \sum_{i \leq 1, j \leq 2} \|S^j \Omega^j \phi_j\|_{L^1[T, 2T]}
\]
\[
\lesssim \frac{1}{R^{1/2}} \|\phi_j|_{L, L} \lesssim |j| + k\|_{L^1[T, 2T]}
\]

Then we take the supremum over the relevant $R$ values. In $C^T_R$, we have $v \sim r$ and $u \sim r$.

5. Derivative estimates in $L^2$

**Lemma 5.1.** Suppose that $\sigma$ and $\delta$ from (1.1) are nonnegative real numbers, and $\delta' \in [-1, \infty)$. Let $L, L'$ denote dyadic numbers of the form (2.1), with $L, L' = 1$ when $h = 0$ and, in general, $L, L' \gg h, g^2$ are appropriately large relative to 1, depending on $h$ and $g^2$.

- If $L \leq U, R \leq 3T/8$, then
\[
R \|\nabla_{t, x} w_{\leq m}\|_{L^2(C^R_T)} \lesssim \|w_{\leq m}\|_{L^2(\mathcal{C}^R_T)} + \|S_{\leq m}\|_{L^2(\hat{\mathcal{C}}^R_T)} + R^2 \|(Pw)_{\leq m}\|_{L^2(\hat{\mathcal{C}}^R_T)}
\]  
(5.1)

- Let $C^U_{T, 1} := C^U_T \cap \{r < t\}$ and $C^U_{T, 2} := C^U_T \cap \{r > t\}$.

\[
U \|\nabla_{t, x} w_{\leq m}\|_{L^2(C^U_{T, 1})} \lesssim \|w_{\leq m}\|_{L^2(\mathcal{C}^U_{T, 1})} + \|S_{\leq m}\|_{L^2(\hat{\mathcal{C}}^U_{T, 1})} + UT \|(Pw)_{\leq m}\|_{L^2(\hat{\mathcal{C}}^U_{T, 1})}
\]  
(5.2)

\[
U \|\nabla_{t, x} w_{\leq m}\|_{L^2(C^U_{T, 2})} \lesssim \|w_{\leq m}\|_{L^2(\mathcal{C}^U_{T, 2})} + \sum_{Z \in \Theta(S)} \|\bar{Z} w_{\leq m}\|_{L^2(\hat{\mathcal{C}}^U_{T, 2})} + UT \|(Pw)_{\leq m}\|_{L^2(\hat{\mathcal{C}}^U_{T, 2})}
\]  
(5.3)

- If $L' \leq T < R$, i.e. $L' \leq T \leq 3R/8$, then
\[
R \|\nabla_{t, x} w_{\leq m}\|_{L^2(C^R_T)} \lesssim \|w_{\leq m}\|_{L^2(\mathcal{C}^R_T)} + \sum_{Z \in \Theta(S)} \|\bar{Z} w_{\leq m}\|_{L^2(\mathcal{C}^R_T)} + R^2 \|(Pw)_{\leq m}\|_{L^2(\mathcal{C}^R_T)}
\]  
(5.4)

**Proof.** We begin by proving (5.1). Let $w$ denote a reasonably smooth function. We shall first prove that for $1 \ll R \leq 3T/8$,
\[
R \|\nabla_{t, x} w\|_{L^2(C^R_T)} \lesssim \|w\|_{L^2(\mathcal{C}^R_T)} + \|S w\|_{L^2(\hat{\mathcal{C}}^R_T)} + R^2 \|P w\|_{L^2(\hat{\mathcal{C}}^R_T)}
\]  
(5.5)

Let $\chi(t, r)$ be a radial cutoff function on $\mathbb{R}^{1+3}$ with $\text{supp} \chi \subset \mathcal{C}^R_T$ and $\chi = 1$ on $C^R_T$; a further fixing of $\chi$ will come later in the proof. Two observations are in order:

1. If $r < t$ then for a sufficiently large constant $C'$, we have
\[
\chi \left(\frac{u}{t} |\nabla_{t, x} w(t, x)|^2\right) \leq \chi \left(\nabla_{t, x} w^2 - w_t^2 + \frac{C'}{ut} |S w|^2\right)
\]  
(5.6)

\footnote{For example, if $h \in S^2(\epsilon(r)^{-1})$ for a sufficiently small $\epsilon > 0$, then $L = 1$.}

\[\]
which holds without the multiplication by $\chi$ as well) as an expansion of the terms $|Sw|^2, |\nabla_{t,x}w|^2$ reveals; the values $C' \geq 3$ work for every $(r,t)$ such that $0 \leq r < t$.

(2) By integration by parts,

$$\int \chi(|\nabla_{t,x}w|^2 - w_t^2) \, dxdt = \int \chi w(\partial_t^2 - \Delta)w \, dxdt - \int \frac{1}{2}(\partial_t^2 - \Delta)\chi w^2 \, dxdt. \quad (5.7)$$

There are no boundary terms in either time or space because of the compact support of $\chi(t,r)$ in both time and space.

Integrating (5.6) in spacetime, we have via (5.7)

$$\int \chi \frac{u}{t} |\nabla_{t,x}w|^2 \, dxdt \leq \int \chi w(\partial_t^2 - \Delta)w + O(|\Box \chi|w^2) + \frac{C'}{ut} \chi |Sw|^2 \, dxdt. \quad (5.8)$$

The proof of (5.5) will be complete once we incorporate $Pw$ into (5.8):

- Let $\Box_h$ denote the second order operator $\Box_h := \partial_\alpha h^{\alpha\beta} \partial_\beta$.

For $\int (\chi w)(\Box_h w) \, dxdt$, we integrate by parts and use Cauchy-Schwarz. A term arises, and for this term we use the hypothesis that $L \gg h$ for $h \neq 0$.

Similarly, $\int (\chi w)(g^2 \Delta w) \, dxdt$ is treated by integration by parts and Cauchy-Schwarz. We use the smallness of $\langle r \rangle^{-2 - \delta'}$ (which is $O(\langle r \rangle^{-1})$ since $\delta' \in [-1, \infty)$) for sufficiently large $R$.

- We use the bound $V \lesssim \langle r \rangle^{-2}$.

- For $\int \chi w B \partial w$ we use Cauchy-Schwarz. For $\int \chi w \partial(Aw)$ we integrate by parts and use Cauchy-Schwarz; it is also possible to bound this using information on $\partial A$ if one does not integrate by parts, but we integrate by parts in order to use fewer assumptions. The bounds we obtain are sufficient to prove the claim (5.5) even when $\sigma = 0$, and we only assume $A, B \in S^2(\langle r \rangle^{-1})$ in this part.

Assuming $\Box \chi \lesssim \langle r \rangle^{-2}$, separating $|\chi w Pw| \lesssim \chi[(R^{-1}w)^2 + (RPw)^2]$ in the right-hand side of (5.8), and using the reasoning in the bullet points (along with the triangle inequality) to deal with $\int (\chi w)((\Box - P)w) \, dxdt$, this proves the claim (5.5) for $C^R_T$.

The same proof shows the analogue of (5.5) for the $C^{U}_T \cap \{r < t\}$ region,

$$U \|\nabla_{t,x}w\|_{L^2(C^U_T \cap \{r < t\})} \lesssim \|w\|_{L^2(C^U_T \cap \{r < t\})} + \|Sw\|_{L^2(C^U_T \cap \{r < t\})} + UT \|Pw\|_{L^2(C^U_T \cap \{r < t\})} \quad (5.9)$$

if we choose a $\chi$ adapted to $C^{U}_T \cap \{r < t\}$ (rather than $C^R_T$) that satisfies $\Box \chi \lesssim \frac{1}{(t+r)(t-r)}$ (rather than $\Box \chi \lesssim 1/\langle r \rangle^2$). \footnote{Note that if $T$ is sufficiently large, then we may even take $L = 1$ for $C^{U}_T$ and $L' = 1$ for $C^T_R$.}
Similar arguments show the result for vector fields, (5.1) and (5.2). The only new thing one has to deal with is \( \int \chi w_{\leq m}[P, Z^{\leq n}] w \, dx \, dt \) and similar arguments involving integration by parts and Cauchy-Schwarz establish the claims (5.1) and (5.2).

Next, we prove
\[
R \| \nabla_{t,x} w \|_{L^2(C_R^\ast)} \lesssim \| w \|_{L^2(C_R^\ast)} + \sum_{\tilde{Z} \in \{\Omega, S\}} \| \tilde{Z} w \|_{L^2(C_R^\ast)} + R^2 \| P w \|_{L^2(C_R^\ast)},
\]
(5.10)
The proof for the region \( \{r > t\} \) is essentially a switching of the \( r \) and \( t \) variables in what has been done for the \( C_R^T \) and \( C_T^\ast \cap \{r < t\} \) regions. For any point \((t, x)\) such that \(|x| > t\),
\[
|\nabla_{t,x} w(t, x)|^2 \leq \frac{r}{r-t}(w_t^2 - w_x^2) + \frac{C'}{(r-t)^2}(Sw)^2 + \frac{(\Omega w)^2}{r^2}
\]
for some sufficiently large constants \( C, C' > 0 \). For the angular derivatives, this follows because \( \partial_w = \sum_j c_j \Omega_j \) for some coefficients \( c_j \) such that \(|c_j| \lesssim 1/r\). We shall only use the weaker estimate
\[
|\nabla_{t,x} w(t, x)|^2 \leq \frac{r}{r-t}(w_t^2 - |\nabla_x w|^2) + \frac{C'}{(r-t)^2}(Sw)^2 + \frac{(\Omega w)^2}{r(r-t)}.
\]
(5.12)
We use this because it makes (5.13) conceptually cleaner; and because using (5.11) would lead to no gain in the final derivative estimates for \( C_R^T \), due to the presence of the \((r-t)^{-2}\) coefficient of \( (Sw)^2 \).

- (Bound in \( C_R^T \)) Let \( \chi(t, r) \) be a radial cutoff function adapted to \( C_R^T \). By (5.12),
\[
\int \chi |\nabla_{t,x} w|^2 \, dx \, dt \leq \int \frac{r}{r-t}\chi(w_t^2 - |\nabla_x w|^2) + C \frac{r}{r-t}\chi|\Omega w|^2 \frac{1}{r(r-t)} + \frac{C'}{(t-r)^2}|Sw|^2 \, dx \, dt.
\]
(5.13)
The analysis henceforth is similar to the three bullet points above. Assuming \( \Box \chi \lesssim (r^{-2}) \), we end up with
\[
\| \nabla_{t,x} w \|_{L^2(C_R^\ast)} \lesssim R^{-1} \left( \| w \|_{L^2(C_R^\ast)} + \sum_{\tilde{Z} \in \{\Omega, S\}} \| \tilde{Z} w \|_{L^2(C_R^\ast)} \right) + R \| P w \|_{L^2(C_R^\ast)},
\]
i.e., (5.10).
- (Bound in \( C_T^T \cap \{r > t\} \)) We adapt \( \chi \) to \( C_T^T \cap \{r > t\} \) with \( \Box \chi \lesssim ((t+r)\langle t-r \rangle)^{-1} \). Then by Cauchy-Schwarz,
\[
\| \nabla_{t,x} w \|_{L^2(C_T^T \cap \{r > t\})} \lesssim U^{-1} \left( \| w \|_{L^2(C_T^T \cap \{r > t\})} + \sum_{\tilde{Z} \in \{\Omega, S\}} \| \tilde{Z} w \|_{L^2(C_T^T \cap \{r > t\})} \right) + T \| P w \|_{L^2(C_T^T \cap \{r > t\})}.
\]
The full results for vector fields \( w_{\leq m} \) again follow simply by similar integration by parts and Cauchy-Schwarz arguments.

We will need to bound the second derivative of vector fields in \( L^2 \) when proving \( L^\infty \) estimates for vector fields of a function. Hence we present Corollary 5.2 immediately.

**Corollary 5.2.** Assume the hypotheses of Lemma 5.1. Then
\[ R \| \nabla_{t,x}^2 w \|_{L^2(C_{T}^R)} \lesssim \| \nabla_{t,x} w \|_{L^2(C_{T}^R)} + R^2 \| \nabla_{t,x} (Pw) \|_{L^2(C_{T}^R)} \] (5.14)

\[ U \| \nabla_{t,x}^2 w \|_{L^2(C_{T}^R)} \lesssim \| \nabla_{t,x} w \|_{L^2(C_{T}^R)} + UT \| \nabla_{t,x} (Pw) \|_{L^2(C_{T}^R)} \] (5.15)

\[ R \| \nabla_{t,x}^2 w \|_{L^2(C_{T}^R)} \lesssim \| \nabla_{t,x} w \|_{L^2(C_{T}^R)} + R^2 \| \nabla_{t,x} (Pw) \|_{L^2(C_{T}^R)} \] (5.16)

**Proof.** Fixing any \( \alpha \in \{0,1,2,3\} \) and denoting \( \partial_{\alpha} \) by \( \partial \), we substitute \( \partial w \leq m \) for the function \( w \) in the proof of Lemma 5.1.

A new type of term arises, which is

\[ \int \chi \partial w \leq m P \partial w \leq m = \int \chi \partial w \leq m (\partial f \leq m + \partial [P, Z^{\leq m}] w + [P, \partial] w \leq m) . \]

We can handle the first term on the right-hand side by Cauchy-Schwarz.

For the \( \square, g^w \Delta_w \) and \( V \) contributions to \( P \), similar arguments as before using Cauchy-Schwarz and integration by parts work. For the contributions of the \( \partial_{\alpha} A^a \) and \( B^a \partial_{\alpha} \) components to \( P \) in both \( \partial [P, Z^{\leq m}] w \) and \( [P, \partial] w \leq m \), we also use integration by parts and Cauchy-Schwarz, and the fact that \( \partial A \in S^Z(r^{-2}) \), i.e., this bound holds for all \( (r,t) \) (and hence all three dyadic regions), which follows from the assumptions \( \partial A \in S^Z_{\text{int}}(r^{-2}) \cap S^Z_{\text{cone}}(r^{-2}) \) and \( \partial_{\alpha} A \in S^Z(v(w)^{-1}r^{-1}(r)^{-1}r^{-1}r^{-1}) \) because of (3.1). More concretely, we have the schematic equalities

\[ \int \chi \partial w \leq m \partial [B^a \partial_{\alpha}, Z^{\leq m}] w = \int (\chi \partial^2 w \leq m + \chi \partial^2 w \leq m) \tilde{B} \partial w \leq m \]

\[ \int \chi \partial w \leq m \partial [\partial_{\alpha} A^a, Z^{\leq m}] w = \int (\chi \partial^2 w \leq m + \chi \partial^2 w \leq m) \partial \tilde{A} \cdot w \leq m \]

where tildes denote vector fields. We apply the aforementioned assumptions \( \partial A \in S^Z(r^{-2}) \) and \( B \in S^Z(r^{-1}) \).

\[ \square \]

**Corollary 5.3 (\( L^\infty \) estimates for derivatives).** Assume the hypotheses of Corollary 5.2. Hence \( \sigma \) and \( \delta \) from (1.1) are nonnegative real numbers.

1. If \( 1 \ll U \leq 3T/8 \), we have

\[ \| \partial w \leq m \|_{L^\infty(C_{T}^R)} \lesssim \frac{1}{\sqrt{UT^3}} \left( U^{-1} \| w \leq m \|_{L^2(C_{T}^R)} + T(\| (Pw) \leq m \|_{L^2(C_{T}^R)} + \| U \partial (Pw) \leq m \|_{L^2(C_{T}^R)}) \right) . \] (5.17)

2. Let \( 1 \ll R \leq 3T/8 \). Then we have:

\[ \| \partial w \leq m \|_{L^\infty(C_{T}^R)} \lesssim \frac{1}{\sqrt{TR^3}} \left( R^{-1} \| w \leq m \|_{L^2(C_{T}^R)} + R(\| (Pw) \leq m \|_{L^2(C_{T}^R)} + \| R \partial (Pw) \leq m \|_{L^2(C_{T}^R)}) \right) . \]

3. Let \( 1 \ll T \leq 3R/8 \). Then we have:

\[ \| \partial w \leq m \|_{L^\infty(C_{T}^R)} \lesssim \frac{1}{\sqrt{TR^3}} \left( R^{-1} \| w \leq m \|_{L^2(C_{T}^R)} + R(\| (Pw) \leq m \|_{L^2(C_{T}^R)} + \| R \partial (Pw) \leq m \|_{L^2(C_{T}^R)}) \right) . \]
Proof. Let \( v = w_{\leq m} \). The main idea in this proof is to

- first use the initial \( L^\infty \) estimates proved in Section 4 on derivatives \( \partial v \), and to commute this \( \partial \) with the vector fields \( S^i \Omega^j \) in both terms of the majorizer in the estimates (4.1) (4.3). This results in

\[
\|\partial v\|_\infty \lesssim \sum_{i,j \leq 2} (W^3W')^{-1/2}\|S^i \Omega^j \partial v\|_2 + (\tilde{W})((W^3W')^{-1/2})\|\nabla_{t,x} S^i \Omega^j \partial v\|_2
\]

\[
\lesssim (W^3W')^{-1/2}\|\partial v_{\leq n}\|_2 + (\tilde{W})((W^3W')^{-1/2})\|\partial^2 v_{\leq n}\|_2
\]

\[
= (W^3W')^{-1/2}\left(\|\partial v_{\leq n}\|_2 + \tilde{W}\|\partial^2 v_{\leq n}\|_2\right)
\]

for dyadic weights \( W, W' \) and \( \tilde{W} \in \{W, W'\} \), where the choices of \( W, W' \) and \( \tilde{W} \) all depend on the region in question.

- And secondly to use the derivative estimates just proved in Lemma 5.1 and Corollary 5.2, in order to control \( \|\nabla_{t,x} v_{\leq n}\|_2 \) and \( \tilde{W}\|\nabla_{t,x}^2 v_{\leq n}\|_2 \) respectively.

In \( C^u_{\rho} \), one has \( W = T \) and \( W' = \tilde{W} = U \). Let \( k \geq 0 \) be any integer. Then

\[
\|\partial v_{\leq k}\|_2 + \tilde{W}\|\partial^2 v_{\leq k}\|_2 = \|\partial v_{\leq k}\|_2 + U\|\partial^2 v_{\leq k}\|_2
\]

\[
\lesssim \|\partial v_{\leq k+1}\|_2 + UT\|\partial (P v)_{\leq k}\|_2
\]

\[
\lesssim U^{-1}\|v_{\leq k+2}\|_2 + T\|(P v)_{\leq k+1}\|_2 + UT\|\partial (P v)_{\leq k}\|_2
\]

This proves (5.17). For the other two regions, the proof is similar. 

\[\square\]

6. Setup for pointwise decay iteration

**Definition 6.1.** Let

\[ R_1 := \{\text{dyadic numbers } R : R \geq 1, R < \frac{t-r}{8}\} \]

denote the collection of dyadic numbers we call Region 1, and let

\[ R_2 := \{\text{dyadic numbers } R : R \geq 1, \frac{t-r}{8} \leq R < t+r\} \]

denote the collection we call Region 2.

**Definition 6.2.** Let \( \mathbb{R}_+ := [0, \infty) \).

- Let \( D_{tr} \) denote

\[ D_{tr} := \{(\rho, s) \in \mathbb{R}_+^2 : -(t+r) \leq s - \rho \leq t-r, |t-r| \leq s + \rho \leq t+r\}. \]

When we work with \( D_{tr} \), we shall use \((\rho, s)\) as variables, and \( D_{tr}^R \) is short for \( D_{tr}^{\rho < R} \).

- For \( R > 1 \), let

\[ D_{tr}^{R_1} := D_{tr} \cap \{(\rho, s) : R < \rho < 2R\} \]

and let

\[ D_{tr}^{R=1} := D_{tr} \cap \{(\rho, s) : \rho < 2\}. \]

**Lemma 6.3** (Maximal vertical length within \( D_{tr} \cap \{(\rho, s) \in \mathbb{R}_+^2 : \rho \leq s\} \)). Uniformly in the set of \( r, t \) values lying in \( \{(r, t) : 0 \leq r \leq t\} \), we have that for any point \((\rho', s') \in D_{tr} \subset \mathbb{R}_+^r \times \mathbb{R}_+^s\),
If $r \leq t/3$, then
\[ |D_{tr} \cap \{ (\rho', s') : \rho = \rho' \}| \leq \min\{2\rho, 2r\} \]

If $t \geq r \geq t/3$, then
\[ |\{ s' \geq \rho' \geq 0 \} \cap D_{tr} \cap \{ (\rho', s') : \rho = \rho' \}| \leq t - r \]
where $| \cdot |$ denotes the length.

**Proof.** We split the proof into two cases.

(1) Let $r \leq t/3$; then for each $\rho$, the maximal vertical length within $D_{tr}$ is $2r$ and occurs when $r \leq \rho \leq \frac{t-r}{2}$; by symmetry, this length, $2r$, is maximal. When $0 \leq \rho \leq r$, the maximal vertical length of $D_{tr}$ is $2\rho$, which implies that this value of this length is sharp if and only if $0 \leq \rho \leq r$.

(2) Let $t \geq r \geq t/3$; then for each $\rho$, the maximal vertical length within $D_{tr} \cap \{ s \geq \rho \}$ is $r - t$ and occurs when $\frac{r-t}{2} \leq \rho \leq r$ and by symmetry once more, this length, $t - r$, is maximal. Furthermore, in a manner precisely analogous to the $r \leq t/3$ case, we once more have that when $0 \leq \rho \leq \frac{t-r}{2}$, the bound $2\rho$ is sharp if and only if $\rho$ lies in this small region.

$\square$

**Definition 6.4.** Given $\lambda \in \mathbb{R}$,
\[
\kappa(\lambda, t-r) := \begin{cases} 
1 & \lambda > 1 \\
\log\langle t-r \rangle & \lambda = 1 \\
\langle t-r \rangle^{1-\lambda} & \lambda < 1
\end{cases}
\]

In this paper, this function arises either as
\[
\sum_{1 \leq R \leq \langle t-r \rangle} \frac{1}{R^{\lambda-1}} \quad \text{or} \quad \int_0^{\langle t-r \rangle} \frac{1}{\langle v \rangle^\lambda} dv.
\]

**Lemma 6.5.** Let $m \geq 0$ be an integer and suppose that $\psi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ solves
\[ \square \psi(t,x) = g(t,x), \quad (t > 0, x \in \mathbb{R}^3) \]
with vanishing initial data, with
\[
g \lesssim \frac{\log^m \langle t-r \rangle}{\langle r \rangle^\alpha \langle t-r \rangle^\beta}.
\]
where the values of $\alpha, \beta, \eta$ will be specified below.

- **(The case $r \leq t$)** Assume that $\beta \geq 0$ and $\eta \in \mathbb{R}$. Assume also that $|x| \leq t$.
  Assume that $\text{supp } g$ is contained within $\{ r \leq t \}$.
  If $1 < \alpha < 3$, then
  \[
  \frac{\langle r \rangle \psi(t,x)}{\log^m \langle t-r \rangle} \lesssim \min \left( \frac{\kappa(\alpha-1, t-r)}{\langle t-r \rangle^{\beta+\eta-1}}, \frac{1}{\langle t-r \rangle^{\beta+\eta+\alpha-3}} \right) + \frac{\kappa(\eta, t-r)}{\langle t-r \rangle^{\alpha+\beta-2}} \tag{6.1}
  \]
  If $\alpha > 3$ (we will not be needing the cases $\alpha = 3$ or $\alpha \leq 1$),
  \[
  \frac{\langle r \rangle \psi(t,x)}{\log^m \langle t-r \rangle} \lesssim \frac{\kappa(\alpha-1, t-r)}{\langle t-r \rangle^{\beta+\eta-1}} + \frac{\kappa(\eta, t-r)}{\langle t-r \rangle^{\alpha+\beta-2}} \tag{6.2}
  \]
• (The case \( r > t + 1 \)) Let \( \alpha > 1, \eta \in \mathbb{R} \). Suppose that \( r > t + 1 \), and

\[
g \lesssim \frac{1}{\langle r \rangle^\alpha (t-r)^\eta}.
\]

Then

\[
\langle r \rangle \psi \lesssim \begin{cases} 
\frac{1}{\langle t-r \rangle^{\alpha-2}} & \eta > 1 \\
\frac{1}{\langle t-r \rangle^{\alpha-1}} & \eta < 1
\end{cases}
\]

(6.3)

For the case \( r > t \): in this paper we will only need \( \alpha > 1, \eta \neq 1 \); a full explanation is given in the proof of Proposition 7.2.

Proof. (1) (The case \( r \leq t \)) We write

\[
\int_{D_{rt}} \rho \sup_{S^2} |\Box \psi| d\rho = \sum_{R_1} \int_{D_{rt}^R} \rho \sup_{S^2} |\Box \psi| d\rho + \sum_{R_2} \int_{D_{rt}^R} \rho \sup_{S^2} |\Box \psi| d\rho
\]

and bound \( \Box \psi \) pointwise by the bound in the hypotheses. Throughout \( D_{tr} \), we have

\[
\frac{1}{s} \lesssim \frac{1}{t-r}
\]

and we will use this repeatedly below.

We begin with the first bound in (6.1), namely,

\[
\sum_{R_1} \int_{D_{tr}^R} \rho \sup_{S^2} |\Box \psi| d\rho \lesssim \frac{\kappa(\alpha-1, t-r)}{\langle t-r \rangle^{\beta+\eta-1}}.
\]

In the region \( R_1 := \{ 1 \leq R < \frac{t-r}{8} \} \) defined at the beginning of this section, we have

\[
s - \rho \sim t - r.
\]

Therefore, for \( R \in R_1 \) and any \( \beta \geq 0, \eta \in \mathbb{R} \),

\[
\int_{D_{tr}^R} \rho \sup_{S^2} |\Box \psi| d\rho \lesssim \int_{D_{tr}^R} \frac{\log^m(v)}{\langle \rho \rangle^{\alpha-1} \langle s \rangle^\beta \langle v \rangle^\eta} d\rho \lesssim \frac{\log^m(t-r)}{\langle t-r \rangle^{\beta+\eta-1}}
\]

where \( v := s - \rho \). Thus

\[
\sum_{1 \leq R \leq \frac{t-r}{8}} \frac{\log^m(t-r)}{\langle t-r \rangle^{\beta+\eta-1}} = \frac{\log^m(t-r)}{\langle t-r \rangle^{\beta+\eta-1}} \kappa(\alpha-1, t-r).
\]

Next, we prove that when \( \alpha < 3 \), and \( \beta \geq 0, \eta \in \mathbb{R} \), we have

\[
\sum_{R_1} \int_{D_{tr}^R} \rho |\Box \psi| d\rho \lesssim \frac{\log^m(t-r)}{\langle t-r \rangle^{\beta+\eta+\alpha-3}}.
\]
This is shown as follows: since \( \beta \geq 0 \), we have \( \langle s \rangle^{-\beta} \lesssim \langle t - r \rangle^{-\beta} \), and
\[
\log^{-m}(t - r) \sum_{R_1} \int_{D_{R_1}} \rho|\square\psi|d\sigma \rho \lesssim (t - r)^{-\beta - \eta} \sum_{R_1} R_1^{-\alpha} \int \int d\sigma \rho \\
\lesssim (t - r)^{-\beta - \eta} \sum_{R_1} R_1^{3 - \alpha} \\
\lesssim (t - r)^{-\beta - \eta + 3 - \alpha}
\]
where the last line follows by the hypothesis \( \alpha < 3 \).

Finally, we show that when \( \alpha > 1 \) and \( \beta \geq 0 \), then
\[
\int_{\bigcup_{R \in R_2} D_{R_1}} \rho|\square\psi|d\sigma \rho \lesssim \log^{-m}(t - r) \frac{\kappa(\eta, t - r)}{(t - r)^{\alpha + \beta - 2}}
\]
which will complete the proof. For \( R \in R_2 \), we employ the fact that when \( \beta \geq 0 \) we have
\[
\langle \rho \rangle^{-\beta} \lesssim \langle t - r \rangle^{-\beta}
\]
to find that
\[
\log^{-m}(t - r) \int_{D_{R_1}} \rho|\square\psi|d\sigma \rho \lesssim (t - r)^{-\beta} \int \langle s \rangle^{\beta} \langle \psi \rangle^{-\eta} d\sigma \rho \\
\lesssim (t - r)^{-\beta} \int ds \int_{0}^{t-r} \langle \psi \rangle^{-\eta} d\nu \\
\lesssim \frac{1}{(t - r)^{\beta + \alpha - 2}} \kappa(\eta, t - r)
\]
with the last line following by Lemma 6.3.

(2) (The case \( r > t \)) We now prove (6.3). Assume that \( \alpha > 1 \). A straightforward integration shows that
\[
\int_{D_{R_1}} \rho \frac{d\sigma \rho}{\langle \rho \rangle^{\alpha}(s - \rho)^{\eta}} \lesssim \frac{1}{(t - r)^{\alpha - 2}} \left\{ \begin{array}{ll}
\frac{1}{(t - r)^{\eta}} & \eta > 1 \\
\ln \frac{(r + t)}{(t - r)} & \eta = 1 \\
\frac{1}{(r + t)^{\eta - 1}} & \eta < 1
\end{array} \right.
\]
which shows (6.3).

Next, in Remark 6.6 we apply Lemma 6.5 in a setting that is relevant to our current problem:

Remark 6.6 (\( P \)'s coefficients in the wave zone determine final decay rate). In this remark we show how the pointwise decay rate for the solution (and its vector fields) improves indefinitely in the subset of \( D_{R} \) closest to \( \{ \rho = 0 \} \). We also show that it is the pointwise decay rate of the coefficients in the wave zone (i.e., close to \( \{ \rho = s \} \)) that determines the final decay rate of the solution.

Let \( \phi_{\leq m+n} \) satisfy the bounds
\[
\phi_{\leq m+n}(t, x) \lesssim \frac{1}{\langle t \rangle^{\langle u \rangle_\alpha}}, \quad \alpha \in \mathbb{R}.
\]
Then the contribution of the integral over \( D_{tr}^{\leq u} := D_{tr} \cap \{ \rho \ll u \} \) is bounded by \( \langle u \rangle^{-\alpha - \sigma} \); thus this integral always gains \( \langle u \rangle^{-\sigma} \) (no matter what the value of \( \alpha \) is), in contrast to the integration that takes place over the wave zone. In the rest of this remark, we will show this.

Suppose that
\[
\Box \phi_1 = V_1(t, x) \phi_{\leq m+n}
\]  
(6.5)
where \( V_1 \in S^2(\langle r \rangle^{-2-\sigma}) \). Note that by the procedure outlined in (6.6), essentially the entire problem (1.2) can be written in this form (6.5).\(^4\) Then by (6.4), if given a function \( G \) we let
\[
H_G := \sum_{k=0}^2 \| \Omega^k(G) \|_{L^2(S^2)}
\]
then
\[
\int_{D_{tr}^{\leq u} \cap \{ R < \rho < 2R \}} \rho H_{\Box \phi_1} \, dA \lesssim \int_{D_{tr}^{\leq u} \cap \{ R < \rho < 2R \}} \langle s \rangle^{-1} \langle s - \rho \rangle^{-\alpha} \rho H_{V_1} \, dA \quad \text{by (6.4)}
\]
\[
\lesssim \langle u \rangle^{-1} \int_{D_{tr}^{\leq u} \cap \{ R < \rho < 2R \}} \langle s - \rho \rangle^{-\alpha} \rho H_{V_1} \, dA
\]
\[
\sim \langle u \rangle^{-1-\alpha} \int_{D_{tr}^{\leq u} \cap \{ R < \rho < 2R \}} \rho H_{V_1} \, dA
\]
\[
\lesssim \langle u \rangle^{-1-\alpha} R^{1-\sigma}.
\]
For the lowest \( R \) value we integrate over \( D_{tr}^{\leq u} \cap \{ \rho \lesssim 1 \} \). Summing in \( \{ R : R \ll u \} \), we obtain for all sufficiently small \( \sigma \), namely \( \sigma < 1 \) (or, if \( \sigma \) is large, we can truncate to values less than 1), the following bound
\[
\langle u \rangle^{-\sigma - \alpha}.
\]
In conclusion, we have indefinite improvement in the “deep interior” \( \{ \rho \ll u \} \) of \( D_{tr} \) of the pointwise decay rate of \( \phi \), no matter the starting decay rate (i.e. no matter the value of \( \alpha \)).

Let us now bound the integral in the wave zone. If \( \alpha > 1 \), then by Lemma 6.5
\[
\int_{D_{tr} \setminus D_{tr}^{\leq u}} \rho H_{\Box \phi_1} \, dA \lesssim \langle u \rangle^{-1-\sigma}
\]
Thus, once \( \alpha > 1 \), it is \( V_1 \)'s decay rate near the cone that dictates the decay rate for \( \phi_1 \) of
\[
\langle r \rangle \phi_1 \lesssim \langle u \rangle^{-1-\sigma}.
\]
Using a later tool (Theorem 8.4), this bound can be improved to
\[
\phi_1 \lesssim \langle u \rangle^{-1} \langle u \rangle^{-1-\sigma}.
\]

**Definition 6.7** (Cutoff functions). Let \( \chi_{exte}(t, x) \) denote a smooth radial cutoff function adapted to \( \{ r \geq t, r - t \sim r \} \).

Let \( \chi_{inte}(t, x) \) denote a smooth radial cutoff function adapted to \( \{ r \leq t, t - r \sim t \} \).

\(^4\) The only exception is an extra consideration near the light cone \( \{ r = t \} \), which is explained in Section 10.
Let \( \chi^\text{cone}(t, x) \) be a smooth radial cutoff function equalling \( 1 - (\chi^\text{inte} + \chi^\text{ext}) \). We also assume \( \operatorname{supp} \chi^\text{cone} \subseteq \{ r/2 \leq t \leq 3t/2 \} \).

Thus: in \( C_T \), for instance, \( \chi^\text{inte} \) and \( \chi^\text{cone} \) sum to 1, while in \( ([T, 2T] \times \mathbb{R}^3) \setminus C_T \), \( \chi^\text{ext} \) and \( \chi^\text{cone} \) sum to 1.

In the following sections, we shall finish the proof of Theorem 1.8; by the product rule, and also (3.1), it will suffice to prove pointwise decay for

\[
\Box \phi_{\leq m} = (\tilde{V} + \partial \tilde{B}) \phi_{\leq m} + \partial(\tilde{A} + \tilde{B}) \phi_{\leq m} + \partial(\tilde{h} \partial \phi_{\leq m}) + \tilde{g}^\nu \partial^2 \phi_{\leq m},
\]

(6.6)

where \( \sim \) denotes vector fields.

Before commencing the pointwise decay iteration in the next section, we note that:

- By (4.6), the desired decay rate in Theorem 1.8 already holds in the region \( \{|u| \leq 1\} \). Henceforth in this article, we shall assume that \( |u| > 1 \), i.e., \( |t - r| > 1 \). Thus we work away from the light cone \( \{ r = t \} \).
- Due to the domain of dependence properties of the wave equation, we shall first complete the iteration in \( \{ r > t + 1 \} \), which is the content of Section 7. For the iteration in \( \{ r < t - 1 \} \), the decay rates obtained from the fundamental solution are insufficient in the region \( \{ r < t/2 \} \). To remedy this, we prove Theorem 8.4. With the new decay rates obtained from Theorem 8.4, we are then able to obtain new decay rates for the solution and its vector fields. At every step of the iteration, Lemma 6.5 is used to turn the decay gained at previous steps into new decay rates.

7. THE UPPER BOUND IN \( \{ r > t + 1 \} \)

Before embarking on the pointwise decay iteration for the equation in (1.2), we first explain in Remark 7.1 how we deal with the initial data in (1.2).

Remark 7.1 (The initial data). Let \( w := S(t, 0)(\phi_0, \phi_1) \) denote the solution to the free wave equation with initial data \( (\phi_0, \phi_1) \) at time 0. Thus

\[
w_J(t, x) = \frac{1}{|\partial B(x, t)|} \int_{\partial B(x, t)} (\phi_0)_J(y) + \nabla_y (\phi_0)_J(y) \cdot (y - x) + t(\phi_1)_J(y) \, dS(y).
\]

Let \( \alpha - 1 \in \{ 1 + \min(\sigma, \delta, \delta'), 1 + \min(\sigma + 1, \delta, \delta') \} \) with the first (resp. second) one as the value of \( \alpha - 1 \) assuming hypotheses from part 1 (resp. part 2) of Theorem 1.8. For any multiindex \( J \), we now show that

\[
w_J \lesssim \frac{1}{\langle v \rangle^{\alpha - 1}}
\]

by the Kirchhoff formula and the weighted \( L^2 \) decay assumption on the initial data. We use Cauchy-Schwarz and Sobolev embedding to control the free wave pointwise by the weighted \( L^2 \) bound assumed on the initial data. When \( r \gg t \) and \( y \in \partial B(x, t) \),

\[
| (\phi_0)_J(y) | + | \nabla (\phi_0)_J(y) \cdot (y - x) | + | t(\phi_1)_J(y) | \lesssim \langle r \rangle^{-\alpha}
\]

so that \( w_J \lesssim \langle r \rangle^{-\alpha} \lesssim \langle v \rangle^{-1} \langle u \rangle^{-(\alpha - 1)} \). Similarly, when \( r \ll t \) and \( y \in \partial B(x, t) \),

\[
| (\phi_0)_J(y) | + | \nabla (\phi_0)_J(y) \cdot (y - x) | + | t(\phi_1)_J(y) | \lesssim \langle t \rangle^{-\alpha}
\]
so that $w_J \lesssim \langle t \rangle^{-\alpha} \lesssim \langle v \rangle^{-1} (\langle u \rangle^{-(\alpha-1)}$. When $r \sim t$, we have $w_J \lesssim \langle v \rangle^{-1}$.

Recalling (6.6), in this section we prove that the solution to

$$\Box w_{(m)} = O(\langle r \rangle^{-2 - \min(\delta', \delta, \sigma + 1)}) \phi_{\leq m+n}$$

obeys the full rate bound

$$w_{(m)} \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(\sigma, \delta, \delta')}}$$

in $\{ r > t + 1 \}$ assuming that $B \in S\langle \langle r \rangle^{-\delta} \rangle$, $\partial_t B \in S\langle \langle r \rangle^{-\delta} \rangle$. We used the results from Section 5 in transitioning from (6.6) to (7.1).

If $B \in S\langle \langle r \rangle^{-\delta} \rangle$, $\partial_t B \in S\langle \langle r \rangle^{-\delta} \rangle$, then we instead have

$$\Box w_{(m)} = O(\langle r \rangle^{-2 - \min(\delta', \delta, \sigma)}) \phi_{\leq m+n}$$

and the final bound

$$w_{(m)} \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(\sigma, \delta, \delta')}}$$

the argument shown below proving Proposition 7.2 covers this case equally well. For the sake of simplicity and concreteness, we pick and fix the assumption (7.1).

(7.1) includes all the terms in (6.6) except for the parts of the right-hand side of (6.6) that are supported near the cone; we prove estimates for those parts in Section 10.

**Proposition 7.2.** Assume that $r > t + 1$. Assuming the hypotheses of part 2 of Theorem 1.8,

$$w_{(m)} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1+\min(1+\delta', \delta, \sigma)}}.$$ 

Assuming the hypotheses of part 1 of Theorem 1.8,

$$w_{(m)} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1+\min(\sigma, \delta, \delta')}}.$$ 

**Proof.** We only prove the case assuming the hypotheses of part 2 (thus $\Box w_{(m)} = O(\langle r \rangle^{-2 - \min(\delta', \delta, \sigma + 1)}) \phi_{\leq m+n}$) since the other case is similar.

Given $\Box w_{(m)} = \tilde{G} \phi_{\leq m+n}$ with $\tilde{G} = O(1/\langle r \rangle^{1+\beta})$ (here, $\beta = 1 + \min(\sigma + 1, \delta, \delta')$), the first step is to use (4.6) and Lemma 6.5, which yields

$$w_{(m)} \lesssim \frac{1}{\langle v \rangle^{-1/2} \langle u \rangle^{\beta}}.$$

Then, a second application of Lemma 6.5 yields

$$w_{(m)} \lesssim \left\{ \begin{array}{ll}
\frac{1}{\langle v \rangle^{\beta - \delta/2}} & \beta - \frac{1}{2} > 1 \text{ i.e. } \min(\delta', \delta, \sigma + 1) > \frac{1}{2} \\
\frac{1}{\langle v \rangle^{\beta - 1/2} \langle u \rangle^{\delta'}} & \beta - \frac{1}{2} < 1 \text{ i.e. } \min(\delta', \delta, \sigma + 1) < \frac{1}{2}.
\end{array} \right.$$ 

Note that the sum of exponents in the denominator, call it $i_n$ if we are at step $n$, has increased by $\min(\sigma + 1, \delta, \delta')$.

The case $\eta = 1$ in Lemma 6.5, whenever $r > t$, arises if $n \min(\delta, \delta') = 1$ for some integer $n \geq 1$, but in this case we incur an arbitrarily small polynomial loss in $\langle t-r \rangle$;
Lemma 6.5

The function $w_{(m)}$ is bounded by one of

$$
\min \left( \frac{1}{\langle u \rangle^{\frac{1}{2} + (N+1)a}}, \frac{1}{\langle u \rangle^{1+a}} \right).
$$

This iteration continues until $Na > 1$. Then respectively

$$
\langle r \rangle w_{(m)} \lesssim \frac{1}{\langle r \rangle (t-r)^{-\frac{1}{2} + 2(N+1)a}}, \quad \langle r \rangle w_{(m)} \lesssim \frac{1}{\langle r \rangle (t-r)^{-\frac{1}{2} + 2(N+1)a}}.
$$

For the minimal integer $N$ satisfying $Na > 1$, by Lemma 6.5 we have

$$
\langle r \rangle w_{(m)} \lesssim 1/\langle t-r \rangle^{1+a-\delta}, \quad \langle r \rangle w_{(m)} \lesssim 1/\langle t-r \rangle^{1+a-\delta}.
$$

Suppose that $a > 1/2$. After one iteration, by Lemma 6.5, $w_{(m)} \lesssim \langle v \rangle^{1/2}/\langle u \rangle^{1+\min(1+\sigma,\delta,\delta')}$.

After the second iteration,

$$
\langle r \rangle w_{(m)} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1+2a}}.
$$

In the third iteration, one obtains $1/\langle u \rangle^{1+\min(1+\sigma,\delta,\delta')}$ from the $\rho$ integration alone, and for $a$ which is big enough ($a > 3/4$, more precisely), the iteration halts here. For $1/2 < a \leq 3/4$, continuing as many times as necessary, one eventually gets

$$
\langle r \rangle w_{(m)} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1+\min(1+\sigma,\delta,\delta')}}.
$$

\[\square\]

8. Converting $r$ decay to $t + r$ decay

In this section, we convert radial decay $1/\langle r \rangle^p$, $p \leq 1$ in $\{r < t/2\}$ to $1/\langle v \rangle^p$. In particular, the fundamental solution to the wave equation gives a $1/\langle r \rangle$ decay rate, which we can now convert to $1/\langle v \rangle$. This holds for both $\phi$ and vector fields of $\phi$. 

26
Lemma 8.1. We have
\[ \|\phi_{\leq m}\|_{L^\infty(C_T^{<3T/4})} \lesssim \frac{1}{T^{3/2}} \|\langle r\rangle \phi_{\leq m+n}\|_{LE^1(C_T^{<3T/4})}. \]

Proof. This estimate will follow as a consequence of Corollary 4.2 and from proving that
\[ \|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim \frac{1}{T} \|\langle r\rangle \phi_{\leq m+n}\|_{LE^1(C_T^{<3T/4})}, \] (8.1)
The statement (8.1) hints at the fact that we will transfer (a limited amount of) \( \langle r\rangle \) decay into \( \langle v\rangle \) decay in the \( LE^1 \) local energy norm. From the \( LE^1 \) norm, we can recover pointwise bounds simply by explicit computation.

Remark 8.2 (It suffices to look at \( \phi \) supported in \( C_T^{<3T/4} \)). In this proof we can assume that \( \phi \) is supported in \( C_T^{<3T/4} \) because we can control the commutator \([P, \chi_{C_T^{<3T/4}}] \) adequately where \( \chi_{C_T^{<3T/4}} \) is a cutoff function adapted to the region \( C_T^{<3T/4} \). Henceforth, we will assume that \( \phi \) is supported in \( C_T^{<3T/4} \), although support in \( \langle r\rangle \leq \lambda T \) for any fixed \( \lambda > 0 \) would also be fine.

Let \( m \geq 0 \). Let \( \gamma_{(T,x)}(t') \) denote an integral curve of \( S \), parametrized by unit speed, such that \( t'=0 \) corresponds to time \( t=T \) and spatial position \( x \). That is, it corresponds to the point \( (T, x) \). By the fundamental theorem of calculus and Cauchy-Schwarz, we have
\[ |\nabla_{t,x}\phi_{\leq m+1}(T, x)|^2 \lesssim \frac{1}{T} \int_0^T |(\nabla_{t,x}\phi_{\leq m+1})(\gamma_{(T,x)}(t'))|^2 + |(S\nabla_{t,x}\phi_{\leq m+1})(\gamma_{(T,x)}(t'))|^2 \, dt'. \] (8.2)
(This bound clearly works for any smooth-enough function other than \( \phi_{\leq m+1} \) as well.)

Next, integrating (8.2) on \( \{x : r < \lambda t\} \) for some \( \lambda > 0 \), say, \( \{x : r \leq t\} \),
\[ \int_{C_T \cap \{t=T\}} |\nabla_{t,x}\phi_{\leq m+1}(T, x)|^2 \, dx \lesssim \frac{1}{T} \int_{C_T} |\nabla_{t,x}\phi_{\leq m+1}|^2 + |S\nabla_{t,x}\phi_{\leq m+1}|^2 \, dx \, dt \]
\[ \lesssim \frac{1}{T} \int_{C_T} |\nabla_{t,x}\phi_{\leq m+2}|^2 \, dx \, dt. \]

A similar bound holds for \( t = 2T \), where we now average over \([0, T]\) again but this time over the integral curves of \( -S \), using \( \gamma_{(2T,x)}(t') \) as the argument for the function \( \nabla_{t,x}\phi_{\leq m+1} \), with \( t'=0 \) corresponding to time \( t=2T \). Thus
\[ |\nabla_{t,x}\phi_{\leq m+1}(2T, x)|^2 \lesssim \frac{1}{T} \int_0^T |(\nabla_{t,x}\phi_{\leq m+1})(\gamma_{(2T,x)}(t'))|^2 + |(S\nabla_{t,x}\phi_{\leq m+1})(\gamma_{(2T,x)}(t'))|^2 \, dt'. \]
Then, integrating over \( \{x : r \leq t\} \), we obtain the same upper bound \( T^{-1}\|\nabla_{t,x}\phi_{\leq m+2}\|^2_{L^2(C_T)} \).

Hence by the solution \( \phi \) satisfying Proposition 3.4 (weak local energy decay for vector fields),
\[ \|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim \|\nabla_{t,x}\phi_{\leq m+1}(T)\|_{L^2} \lesssim \frac{1}{T^{1/2}} \|\nabla_{t,x}\phi_{\leq m+2}\|_{L^2(C_T)}. \] (8.3)
Next, we bound $\|\nabla_{t,x} \phi_{\leq m}\|_{L^2(C_T)}$ using Lemma 8.3. Intuitively, Lemma 8.3 “multiplies” or “boosts” all integrands in Proposition 3.4 by $\langle r \rangle^{1/2}$. A first naive thought that comes to mind is to multiply the equation by $r \partial_r \phi$ to achieve this boost. This works, if we add a zeroth-order correction term $\phi$ to the multiplier. Unlike the unweighted multiplier, this weighted multiplier leads to unsigned constant-time boundary terms, hence we put both energy terms in the majorizer. Lemma 8.3 adds new information beyond Proposition 3.4 only for sufficiently large values of $r$.

We will sometimes use the notation $C_T := [T_1, T_2] \times \{x : r \leq t\}$.

**Lemma 8.3 (\langle r \rangle^{1/2}-weighted Weak Local Energy Decay for Vector Fields).** Suppose that the solution to $P\phi = f$ satisfies the weak local energy decay for vector fields, as proved in Proposition 3.4. For all $0 \leq T_1 \leq T_2$, we have

$$\|\nabla_{t,x} \phi_{\leq m}\|_{L^2(C_T)} \lesssim \sum_{j=1}^{2} \|\langle r \rangle^{1/2} \nabla_{t,x} \phi_{\leq m+1}(T_j)\|_{L^2} + \|\langle r \rangle f_{\leq m+2}\|_{L^2[T_1, T_2]L^2}. \quad (8.4)$$

**Proof.** We will take as assumptions those stated in part (1) of Theorem 1.8 and prove this result. This implies that this result also holds for part (2), because the assumptions in part (2) are stronger than those for part (1).

- (The zero multiindex case) We demonstrate the case $m = 0$ first for simplicity. In this proof we shall need $\sigma$ and $\delta$ to be strictly positive real numbers, as well as $\delta' > -1$, in contrast to the situation in Lemma 5.1.

  We multiply $P\phi = f$ by $r \partial_r \phi + \phi$ and integrate by parts in $[T_1, T_2] \times \mathbb{R}^3$. There is a number $q' > 0$ such that

$$\int |\nabla_{t,x} \phi|^2 + O(\langle r \rangle^{-q'})|\nabla_{t,x} \phi|^2 + O(\langle r \rangle^{-1-q'})|\partial_r \phi|^2 + O(\langle r \rangle^{-2-q'})|\phi|^2 \, dxdt$$

$$\lesssim \sum_{j=1}^{2} \int_{\mathbb{R}^3} O(\langle r \rangle) |\nabla_{t,x} \phi(T_j, x)|^2 + O(\langle r \rangle^{-1})|\phi(T_j, x)|^2 \, dx + \int |rf \partial_r \phi| + |f \phi| \, dxdt$$

$$\lesssim \sum_{j=1}^{2} \int_{\mathbb{R}^3} O(\langle r \rangle) |\nabla_{t,x} \phi(T_j, x)|^2 \, dx + \int |rf \partial_r \phi| + |f \phi| \, dxdt \quad (8.5)$$

with the last statement following by a version of Hardy’s inequality.

For instance, with the term $\partial_r A^\mu$,

$$\partial_r A^\mu \phi)(r \partial_r \phi) \lesssim O(\langle r \rangle^{-\sigma})|\nabla_{t,x} \phi|^2 + O(\langle r \rangle^{-2-\sigma})\phi^2.$$

This follows by combining the assumptions on $A, \partial A$ and $\partial_r A$ as stated in part (1) of Theorem 1.8. Only an arbitrarily small $\sigma > 0$ is needed.

Next,

$$\iint |f||r \partial_r \phi| = \iint |fr| |\partial_r \phi| \leq \|rf\|_{L^2L^2} \|\partial_r \phi\|_{L^2L^2} \leq \frac{1}{\epsilon} \|rf\|_{L^2L^2}^2 + \epsilon \|\partial_r \phi\|_{L^2L^2}^2$$

and we then bring $\epsilon \|\partial_r \phi\|_{L^2L^2}$ onto the other side together with $|\nabla_{t,x} \phi|^2_{L^2L^2}$. Similarly

$$\iint |f||\phi| = \iint |fr| |\phi/ r| \leq \frac{1}{\epsilon} \|rf\|_{L^2L^2}^2 + \epsilon \|\partial_r \phi\|_{L^2L^2}^2.$$
We remark that it is possible to place \( rf \) in \( L^1 L^2 \) if we place \( \partial_t \phi \) in \( L^\infty L^2 \) (we can use Hardy’s inequality for the zero order term). This alternate route leads to \( \| r f \|_{L^1 L^2} \) instead of \( \| r f \|_{22}^2 \) on the right-hand side.

For small \(|x|\) values, our assumption of Proposition 3.4 implies the bound on

\[
\| \nabla_{t,x} \phi \|_{L^2[T_1,T_2]L^2(r \leq 1)}.
\]

By using the positivity of \( q' \) on the left-hand side of (8.5) for large \(|x|\) values, we can then obtain

\[
\| \nabla_{t,x} \phi \|_{L^2[T_1,T_2]L^2} \lesssim \sum_{j=1}^2 \| \langle r \rangle^{1/2} \nabla_{t,x} \phi(T_j) \|_{L^2} + \min \left( \| \langle r \rangle \|_{L^1[T_1,T_2]L^2}^{1/2}, \| \langle r \rangle \|_{L^2[T_1,T_2]L^2} \right).
\]

(8.6) implies (8.4) for \( m = 0 \).

- (The higher multiindex case) Next, we prove (8.6) but for \( \phi_j, J \neq 0 \). By Lemma 3.2, we have

\[
P \phi_j = f_j + O((\langle r \rangle^{-1-q'}) \nabla_{t,x} \phi_{\leq |J|} + O((\langle r \rangle^{-2-q'}) \phi_{\leq |J|-1}.
\]

We multiply this by \( r \partial_t \phi_j + \phi_j \). Then we integrate in \([T_1, T_2] \times \mathbb{R}^3\).

- For small \( r \), the estimate (8.4) is implied by the weak local energy decay estimate for vector fields proved in Proposition 3.4, so to prove the desired conclusion (8.4) it suffices to restrict attention to the case of large \( r \).

- For large \( r \), owing to the positivity of \( q' > 0 \), we may use the triangle inequality, the triangle inequality for integrals, Cauchy-Schwarz, and Hardy’s inequality to absorb the terms

\[
\int \int \left( r \partial_t \phi_j + \phi_j \right) \left( O((\langle r \rangle^{-1-q'}) \nabla_{t,x} \phi_{\leq |J|} + O((\langle r \rangle^{-2-q'}) \phi_{\leq |J|}) \right)
\]

into the left-hand side, namely into

\[
\| \nabla_{t,x} \phi_{\leq m} \|_{L^2([T_1,T_2] \times \{r \leq t\})}.
\]

The positiveness of \( q' \) provides the necessary smallness for the absorption.

We have explained how to take care of the extra terms arising from commutators in the higher multiindex case, namely the terms \( O((\langle r \rangle^{-1-q'}) \nabla_{t,x} \phi_{\leq |J|} + O((\langle r \rangle^{-2-q'}) \phi_{\leq |J|-1} \). For the remaining part of the equation, namely \( P \phi_j = f_j + (\text{taken care of}) \), we can just apply precisely the same procedure used to prove (8.6) to \( \phi_j \)—that is, the first bullet point. Then we sum over \(|J| \leq m\).

\[\square\]

Applying Lemma 8.3 for \( \phi \), we have

\[
\| \nabla_{t,x} \phi_{\leq m+2} \|_{L^2(\mathbb{R}^3)} \lesssim \sum_{i=1}^2 \| \langle r \rangle^{1/2} \nabla_{t,x} \phi_{\leq m+3} \|_{L^2}.
\]

The next step will be to bound these weighted energy terms by \( LE^1 \) norms, picking up appropriate \( T \) weights along the way.

By the fundamental theorem of calculus and Cauchy-Schwarz once more,

\[
\int \langle r \rangle |\nabla_{t,x} \phi_{\leq m+n}(T)|^2 dx \lesssim \frac{1}{T^{1/2}} \int \langle r \rangle^{1/2} |\nabla_{t,x} \phi_{\leq m+n}|^2 + \frac{1}{T} \langle r \rangle^{3/2} |S \nabla_{t,x} \phi_{\leq m+n}|^2 dxdt.
\]
By Remark 8.2, we assume that \( \langle r \rangle \lesssim T \), which lets us bound
\[
\frac{1}{T^{3/2}} \int \langle r \rangle^{3/2} |S\nabla_{t,x} \phi_{\leq m+n}|^2 \, dx \, dt \lesssim \frac{1}{T} \int \langle r \rangle |S\nabla_{t,x} \phi_{\leq m+n}|^2 \, dx \, dt \lesssim \frac{1}{T} \| \langle r \rangle \phi_{\leq m+n+n'} \|_{L^2(C_T)}^2
\]
for some \( n' \).

We are not able to directly bound \( T^{-1/4} \| \langle r \rangle^{1/4} \nabla_{t,x} \phi_{\leq m} \|_{L^2(C_T)} \) by \( T^{-1/2} \| \langle r \rangle \phi_{\leq m+n} \|_{LE^1(C_T)} \). Instead, we treat this term perturbatively for small \( r \), and for a fixed finite number of large \( R \) regions, where \( r \sim R \), we can make this bound. Thus let us decompose
\[
T^{-1/4} \| \langle r \rangle^{1/4} \nabla_{t,x} \phi_{\leq m} \|_{L^2(C_T)} = \sum_R T^{-1/4} \| \langle r \rangle^{1/4} \nabla_{t,x} \phi_{\leq m} \|_{L^2([T,2T] \times A_R)}.
\]
When \( R \ll T \) we absorb this term into the left hand side. For all values of \( R \) with \( R \sim T \), we are able to directly bound by \( T^{-1/2} \| \langle r \rangle \phi_{\leq m+n} \|_{LE^1(C_T)} \). Thus from
\[
\| \nabla_{t,x} \phi_{\leq m+2} \|_{L^2(C_T)} \lesssim T^{-1/4} \| \langle r \rangle^{1/4} \nabla_{t,x} \phi_{\leq m+n} \|_{L^2(C_T)} + T^{-1/2} \| \langle r \rangle \phi_{\leq m+n} \|_{LE^1(C_T)}
\]
we are able to conclude
\[
\| \nabla_{t,x} \phi_{\leq m+2} \|_{L^2(C_T)} \lesssim T^{-1/2} \| \langle r \rangle \phi_{\leq m+n} \|_{LE^1(C_T)}
\]
so that
\[
\| \phi_{\leq m} \|_{LE^1(C_T^{<3T/4})} \lesssim T^{-1} \| \langle r \rangle \phi_{\leq m+4} \|_{LE^1(C_T)},
\]
which proves Lemma 8.1. Note that we could actually have obtained the upper bound in \( LE^1(C_T^{<3T/4}) \), rather than \( LE^1(C_T) \), i.e.
\[
\| \phi_{\leq m} \|_{LE^1(C_T^{<3T/4})} \lesssim T^{-1} \| \langle r \rangle \phi_{\leq m+4} \|_{LE^1(C_T^{<3T/4})},
\]
\( \Box \)

Theorem 8.4. Let \( \phi \) solve the main equation (1.7). If
\[
\phi_{\leq M} \lesssim \langle r \rangle^{-p} \langle t \rangle^{-q} \langle t - r \rangle^{-\eta}
\]
for some real \( p \leq 1 \), \( q, \eta \in \mathbb{R} \) and a (sufficiently large) fixed \( M \in \mathbb{N} \), then
\[
\phi \lesssim \langle t \rangle^{-p-q} \langle t - r \rangle^{-\eta}.
\]

Proof. For all \( (t, r) \) pairs with \( r \) sufficiently large relative to \( t \), say \( r > t/2 \), the conclusion follows since \( \langle r \rangle \sim \langle t \rangle \).

For the other region, \( C_T^{<3T/4} \), this follows from the proof of Lemma 8.5, because in \( C_T^{<3T/4} \), \( \langle t - r \rangle \sim \langle t \rangle \).

Lemma 8.5. Let \( \phi \) solve the main equation (1.7). If
\[
\phi_{\leq M} \lesssim r^{-p} \langle t \rangle^{-q}
\]
for some real \( p, q \in \mathbb{R} \) and a (sufficiently large) fixed \( M \in \mathbb{N} \) where \( p \leq 1 \), then
\[
\phi \lesssim \langle t \rangle^{-p-q}.
\]

Proof of Lemma 8.5. We compute the norms involved on the right-hand side in Lemma 8.1. The rest of this proof works for not only \( C_T^{<3T/4} \), which is the region we compute in, but
actually in \( [T, 2T] \times \{ r \leq \lambda t \} \) for any fixed \( \lambda > 0 \). The right-hand side norm of Lemma 8.1 is
\[
\| \langle r \rangle \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} = \| \nabla_{t,x} (\langle r \rangle \phi_{\leq m+n}) \|_{L^1(C_T^{< 3T/4})} + \| \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} \\
\lesssim \| \langle r \rangle \nabla_{t,x} \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} + \| \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} \\
\lesssim \| \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})}
\]
where the last line is a consequence of Corollary 5.3 applied uniformly across the collection \( \{ C_T^R : 1 \leq R < 3T/8 \} \) of dyadic regions. Thus \( \| \phi_{\leq m} \|_{L^1(C_T^{< 3T/4})} \lesssim \frac{1}{T} \| \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} \).

Next, we bound \( \| \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} \) and finish the proof. We shall use pointwise bounds on \( |\phi_{\leq m}| \), and not just on \( |\phi| \), here:

- For \( R > 1 \) we have
\[
\sup_{1 < R < 3T/8} \left( \int_T^{2T} \int_R^{2R} \frac{1}{\langle r \rangle} \langle t \rangle^{-2q} r^2 \, dr \, dt \right)^{1/2} \lesssim \sup_{1 < R < 3T/8} \left( T^{-2q} \int_T^{2T} \int_R^{2R} \frac{1}{\langle t \rangle} r^{-2p} r^2 \, dr \, dt \right)^{1/2} \\
\lesssim T^{1/2 - q} \sup_{1 < R < 3T/8} \frac{1}{R^{p-1}} \lesssim T^{1/2 - q} \frac{1}{T^{p-1}} \text{ since } p \leq 1.
\]
- For \( R = 1 \) we have
\[
\left( \int_0^2 \frac{1}{\langle r \rangle} r^{-2p-1} \, dr \right)^{1/2} \lesssim_p 1
\]
for any \( p \in \mathbb{R} \).

Thus
\[
\frac{1}{T^{3/2}} \| \phi_{\leq m+n} \|_{L^1(C_T^{< 3T/4})} \lesssim \frac{1}{T^{3/2}} \frac{1}{T^{\min(0,p-1)+q-1/2}} = \frac{1}{T^{\min(1,p)+q}}
\]
hence \( \| \phi_{\leq m} \|_{L^\infty(C_T^{< 3T/4})} \lesssim T^{-p-q} \) if \( p \leq 1 \).

This establishes the proof of Theorem 8.4.

Corollary 8.6. Let \( k \geq 1 \) be an integer. If \( \phi \) solves \( P\phi = f \) and \( \phi_{\leq M} \lesssim \sum_{j=1}^k \langle r \rangle^{-p_j} \langle t \rangle^{-q_j} \langle t-r \rangle^{-\eta_j} \) and the conditions on the exponents \( p_j, q_j, \eta_j \) and \( M \) in Theorem 8.4 above are satisfied, then \( \phi \lesssim \sum_{j=1}^k \langle t \rangle^{-p_j} \langle t-r \rangle^{-q_j} \).

Proof. The proof is a straightforward consequence of what has already been done. One can use elementary inequalities to handle sums instead of single summands in the computations above, and the estimates still hold.

9. The upper bound in \( \{ r < t \} \)

We consider (7.1) with \( r < t \). We now show the desired final decay rate in Theorem 1.8, namely
\[
w(m) \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(1+\sigma,\delta,\theta)}}.
\]
Proposition 9.1. Assume that \( r < t \). Assuming the hypotheses of part 2 of Theorem 1.8,

\[
    w(m) \lesssim \frac{1}{\langle r \rangle \langle t - r \rangle^{1 + \min(1 + \sigma, \delta, \delta')}}.
\]

Assuming the hypotheses of part 1 of Theorem 1.8,

\[
    w(m) \lesssim \frac{1}{\langle r \rangle \langle t - r \rangle^{1 + \min(\sigma, \delta, \delta')}}.
\]

Proof. In \( \{\rho \geq s\} \), the argument has essentially been done in Section 7; when integrating in \( \{\rho > s\} \), one plugs in the final pointwise decay rates for vector fields of \( \phi \) obtained in Section 7, only to get the final pointwise decay rates as output.

In \( \{\rho < s\} \), we let \( \nu := \min(1 + \sigma, \delta, \delta', 1 - \epsilon) \). Note that \( 2 + \nu < 3 \), allowing us to apply Lemma 6.5’s Region 1 bound \( \frac{1}{(t - r)^{\beta + \eta + \alpha - 3}} \) if we put \( \langle v \rangle \) as \( \langle t \rangle \) in (9.1) when applying Lemma 6.5. For the rest of the proof, the strategy will be to improve by increments \( \nu \) which are strictly less than 1. Below in the proof, we split into the cases where \( \min(\delta, \delta') \) is either < 1 or \( \geq 1 \), but the main idea in either case is really the same, since in the latter case we simply introduce an artificial decrement \( \tilde{\epsilon} \ll 1 \) to make \( \nu \), which equals \( 1 - \tilde{\epsilon} \) in that case, smaller than 1.

By Lemma 5.1, we have

\[
    \Box w(m) \lesssim \langle r \rangle^{2 - \nu} \langle v \rangle^{-1} \langle t - r \rangle^{1/2}.
\] (9.1)

By Lemma 6.5,

\[
    \langle r \rangle w(m) \lesssim \langle t - r \rangle^{1/2 - \nu}.
\]

We have gained \( \langle t - r \rangle^{-\nu} \). Hence by Theorem 8.4,

\[
    \Box w(m) \lesssim \langle r \rangle^{-2 - \nu} \langle v \rangle^{-1} \langle t - r \rangle^{1/2 - \nu},
\]

and this process can be continued as long as the uppermost case thresholds in the definition of \( \kappa \) are not met. Suppose \( n' > 0 \) is an integer for which this threshold is not met; then after performing this procedure \( n' \) times,

\[
    \Box w(m) \lesssim \langle r \rangle^{-2 - \nu} \langle t \rangle^{-1} \langle t - r \rangle^{1/2 - n'\nu}.
\]

Now we define \( n' \) to be

\[
    n' := \max\{n \in \mathbb{N} : \frac{1}{2} + n\nu < 1\}.
\]

There are two cases:

(1) If \( \frac{1}{2} + (n' + 1)\nu < 1 + \nu \)

then write \( \frac{1}{2} + (n' + 1)\nu = 1 + \lambda \nu \)

where \( 0 < \lambda < 1 \); thus

\[
    w(m) \lesssim \langle r \rangle^{-1} \langle t - r \rangle^{-1 - \lambda \nu}.
\]

5we need \( \nu < 1 \) because we will be using the fact that in \( \mathcal{R}_1 \), we have

\[
    \sum_{R \in \mathcal{R}_1} \int_{D^n} \rho |\Box w(m)| ds d\rho \lesssim \frac{1}{\langle t - r \rangle^{\beta + \eta + \alpha - 3}}
\]

(using the notation from Lemma 6.5).
Then
\[ \square w_{(m)} \lesssim \langle r \rangle^{-2-\nu} \langle v \rangle^{-1} \langle t - r \rangle^{-1-\lambda \nu} \leq \min \{ \langle r \rangle^{-2-\nu} \langle v \rangle^{-1} \langle t - r \rangle^{-1-\lambda \nu}, \langle r \rangle^{-3-\nu} \langle t - r \rangle^{-1-\lambda \nu} \} =: \min \{ a, b \}. \]

**Lemma 6.5** implies that in Region $\mathcal{R}_1$, we have the bound by
\[ 1/\langle t - r \rangle^{\beta + \eta + \alpha - 3}. \]
We use $a$ to get the bound in $\mathcal{R}_1$ by
\[ \langle t - r \rangle^{-1-\nu-\lambda \nu}. \]
On the other hand, we use $b$, with $\alpha = 3 + \nu$ and $\beta + \eta = 1 + \lambda \nu$ to get a Region $\mathcal{R}_2$ bound by
\[ \langle t - r \rangle^{-1-\nu}. \]
Thus
\[ \langle r \rangle w_{(m)} \lesssim 1/\langle t - r \rangle^{\beta + \eta + \alpha - 3} + \langle t - r \rangle^{-(\beta + \alpha - 2) \kappa(\eta, t - r)} = \langle t - r \rangle^{-1-\nu-\lambda \nu} + \langle t - r \rangle^{-1-\nu} \lesssim \langle t - r \rangle^{-1-\nu}. \]

(2) If
\[ 1/2 + (n' + 1)\nu = 1, \]
thus $w_{(m)} \lesssim \langle t \rangle^{-1} \langle t - r \rangle^{-1}$, we have
\[ \square w_{(m)} \lesssim \langle r \rangle^{-2-\nu} \langle t \rangle^{-1} \langle t - r \rangle^{-1}. \]
Hence
\[ \langle r \rangle w_{(m)} \lesssim \langle t - r \rangle^{-(\beta + \eta)} \kappa(\alpha - 1, t - r) + \langle t - r \rangle^{-(\alpha + \beta - 2) \kappa(\eta, t - r)} = \langle t - r \rangle^{-(\beta + \eta) + (t - r)^{-(\alpha + \beta - 2) \log(t - r)}} = \langle t - r \rangle^{-2} + \langle t - r \rangle^{-1-\nu} \log(t - r) \leq 2 \langle t - r \rangle^{-1-\nu} \log(t - r) \lesssim \langle t - r \rangle^{-1-\lambda \nu} \]
for any $0 < \lambda < 1$, which now puts us in case (1).

The proof is complete when $\min(\delta, \delta') < 1$.

**Part two of the proof:** The case where $\min(\delta, \delta') \geq 1$, that is, all three parameters $\delta, \delta'$ and $1 + \sigma$ are at least 1. Suppose that $\min(1 + \sigma, \delta, \delta') \geq 1$. We shall still work with an increment $\nu$ that is less than 1. Rather than using "1−" in the definition of $\nu$, we write $\nu$ as the definite number $\nu := 1 - \tilde{\epsilon}$ where $\tilde{\epsilon} > 0$ is a small number. Then
\[ \square w_{(m)} \lesssim \langle v \rangle^{-1} \langle t - r \rangle^{-1-\nu} \min \{ \langle r \rangle^{-2-\nu}, \langle r \rangle^{-2-\min(1+\sigma,\delta,\delta')} \} = \langle v \rangle^{-1} \langle t - r \rangle^{-1-(1-\tilde{\epsilon})} \min \{ \langle r \rangle^{-2-(1-\tilde{\epsilon})}, \langle r \rangle^{-2-\min(1+\sigma,\delta,\delta')} \} \]
where we wrote down the trivial minimum of the two powers of $\langle r \rangle$ to emphasise the fact that we will be using $\alpha = 2 + \min(1 + \sigma, \delta, \delta')$ for $\mathcal{R}_2$ but $\alpha = 2 + \nu$ for $\mathcal{R}_1$. Thus by Lemma 6.5,

$$\langle r \rangle w(m) \lesssim \langle t - r \rangle^{-(\beta + \eta + \alpha - 3)} + \frac{\kappa(\eta, t - r)}{\langle t - r \rangle^{\alpha + \beta - 2}}$$

$$\lesssim \langle t - r \rangle^{-(\beta + \eta + \alpha - 3)} + \frac{1}{\langle t - r \rangle^{\alpha + \beta - 2}}$$

$$= \langle t - r \rangle^{-(\beta + \eta + \alpha - 3)} + \frac{1}{\langle t - r \rangle^{(2 + \min(1 + \sigma, \delta, \delta')) + 1} + 1}$$

$$= \langle t - r \rangle^{-1 - 2\nu} + \frac{1}{\langle t - r \rangle^{1 + \min(1 + \sigma, \delta, \delta')}}.$$ 

It remains to prove the desired bound in Region $\mathcal{R}_1$, and it is safe to ignore the $\mathcal{R}_2$ portion of the bound henceforth because the $\beta$ and $\alpha$ exponent components of $\Box w(m)$ remain stable while $\eta > 1$ will stay larger than 1, and in $\mathcal{R}_2$ we use the bound $\kappa(\eta, t - r)/\langle t - r \rangle^{\alpha + \beta - 2}$.

We note that no more improvement is possible in $\mathcal{R}_2$ using Lemma 6.5.

In $\mathcal{R}_1$, this iteration continues until

$$w(m) \lesssim \langle r \rangle^{-1} \langle t - r \rangle^{-1 - n''\nu}$$

where

$$n'' := \max\{n \in \mathbb{N} : n(1 - \tilde{\epsilon}) < \min(1 + \sigma, \delta, \delta')\},$$

e.g., $n'' = 1$ if the two numbers $\min(1 + \sigma, \delta, \delta')$ and $1 - \tilde{\epsilon}$ are both close to 1.

One way to view this situation is that there are two cases:

1. If

$$(n'' + 1)(1 - \tilde{\epsilon}) > \min(1 + \sigma, \delta, \delta')$$

then we obtain the bound $\langle t - r \rangle^{-1 - n''(1)} = \langle t - r \rangle^{-1 - (n'' + 1)(1 - \tilde{\epsilon})}$ in $\mathcal{R}_1$ by using

$$\frac{1}{\langle t - r \rangle^{\beta + \eta + \alpha - 3}} = \frac{1}{\langle t - r \rangle^{1 + (n'' + 1)\nu}}$$

$$\leq \frac{1}{\langle t - r \rangle^{1 + \min(1 + \sigma, \delta, \delta')}}.$$ 

2. If

$$(n'' + 1)(1 - \tilde{\epsilon}) = \min(1 + \sigma, \delta, \delta')$$

then we obtain the final display in item (1) but with equality rather than inequality, and we halt.

This completes the proof for $w(m)$ when $r < t$. \qed

Remark 9.2 (Lockstep). If $\nu := \min(\sigma, \delta, \delta', 1 - \tilde{\epsilon})$ then essentially an identical proof follows for proving $w(m) \lesssim \frac{1}{\langle r \rangle^{(t - r)^{1 + \min(1 + \sigma, \delta, \delta')}}}$. The case partition is then (a) part one: $\min(\sigma, \delta, \delta') < 1$, (b) part two: $\min(\sigma, \delta, \delta') \geq 1$. Everything else follows when one replaces $1 + \sigma$ in the appropriate locations in the proof above by $\sigma$. 34
In this section we show how we prove the final decay rate in the main theorem for the terms involving the metric coefficients $h^{\alpha \beta}$ that are supported near the cone $\{r = t\}$.

Recall that we write $\tilde{B} \partial \phi_{\leq m} = \partial(\tilde{B} \phi_{\leq m}) - (\partial \tilde{B}) \phi_{\leq m}$. Let $j = 0$ (respectively $j = 1$) correspond to the hypotheses of part 1 (respectively part 2) of Theorem 1.8. Near the cone, we rewrite (6.6) as

$$\square \phi_{\leq m} = (|\tilde{V}| + |\partial \tilde{B}| + |\tilde{g}^{\tilde{\omega}}|)\phi_{\leq m} + \partial_t (\chi^{\text{cone}}(\tilde{h} \partial_t + \tilde{A} + \tilde{B}) \phi_{\leq m}),$$

(10.1)

$$(\phi_{\leq m}(0), \tilde{N} \phi_{\leq m}(0)) = (0, 0).$$

(10.2)

and use (3.1). Note that $|\tilde{V}| + |\partial \tilde{B}| + |\tilde{g}^{\tilde{\omega}}| \lesssim |\tilde{V}| + |\partial \tilde{B}| + (r)^{-1} |\tilde{B}_{\leq n}| + |\tilde{g}^{\tilde{\omega}}| = O(1/(r)^{2 + \min(\sigma + (j-1), \delta, \delta')))$ assuming the hypotheses of part $j$ of Theorem 1.8, $j = 1, 2$.

It suffices to prove pointwise decay estimates for

$$\square \nu_{(m,1)} = \chi^{\text{cone}} \tilde{h} \partial_t \phi_{\leq m}, \quad \square \nu_{(m,2)} = \chi^{\text{cone}} (\tilde{A} + \tilde{B}) \phi_{\leq m},$$

Let $\tilde{v} \in \{\nu_{(m,j)} : j = 1, 2\}$. We now prove

**Proposition 10.1.** We have

$$\partial_t \tilde{v} \lesssim \frac{1}{(r)^{1 + \min(\sigma, \delta, \delta')}}$$

under the assumptions of part 1 of the main theorem.

**Proof.** If $\chi := \chi^{\text{cone}}$ and $f \in \{\chi(\tilde{A} + \tilde{B}) \phi_{\leq m}, \chi \tilde{h} \partial_t \phi_{\leq m}\}$, then by Corollary 5.3 and assumptions on $h$ and $A$ we have

$$|f(s, \rho)| + |Sf| + |(\rho - s) \partial_\rho f| \lesssim \frac{1}{(\rho)^{1 + \sigma}} |\phi_{\leq m+n}|$$

The iteration for $\tilde{v}$ is as follows. Note that $\text{supp} \, \chi \subset \{|s - \rho| \lesssim |t - r|\}$. One simplifying observation is that $\rho \geq c(t - r)$ in all $f$ cases with $c \geq 1/4$; in $r < t$, supp $\chi^{\text{cone}}$ for instance, $\rho \geq |t - r|/4$, which has smallest $c$ value amongst all cases (for example, if $r > t$ then $c = 1$ and if $r < t$ then in supp $\chi^{\text{cone}}, c = 1/2$). This and the fact that the horizontal (i.e. $\rho$) diameter of supp $\chi$ is $O(|t - r|)$ leads to simpler integrations in $\rho$.

We begin with the bound (4.6) for the functions $\phi_{\leq m+n}$. By Corollary 5.3 (used to handle terms that have the operator $(s - \rho) \partial_\rho$ in the integration inside $D_{tr}$) and Lemma 6.5,

$$\partial_t \tilde{v} \lesssim \frac{(r - t)^{1/2 - \sigma}}{(r)^{1 + \sigma}}.$$ 

Thus we run the iteration with exponent $a := \min(\sigma, \delta, \delta', 1-)$—see Remark 9.2. By Lemma 6.5, after $N$ steps one gets

$$\langle r \rangle ^{\partial_t \tilde{v}} \lesssim \frac{1}{\langle t - r \rangle^{1+\sigma}} |t - r|^{3/2 - \theta}$$

where $\theta = Na < 3/2$ is the gain at the $N$-th step of the lockstep. The procedure is similar to $w_{(m)}$’s case, and in the end we get

$$\partial_t \tilde{v} \lesssim \frac{1}{(r)^{1 + \min(\sigma, \delta, \delta')}}.$$
We have \( \partial(\hat{A}\phi) + \hat{B}\partial\phi = \partial([\hat{A} + \hat{B}]\phi) - (\partial\hat{B})\phi \). For \( \tilde{v} \) solving
\[ \Box \tilde{v} = \chi \hat{A}\phi, \]
the bound on \( \partial\tilde{v} \) is just an argument that is an application of Lemma 6.5 similar to what has been done. We write \( B\partial\phi = \partial(B\phi) - (\partial B)\phi \); then the arguments already shown (along with the assumptions on \( B \) in Theorem 1.8) give the bound on \( \partial W \) for \( \Box W = \chi \hat{B}\phi \). This concludes the proof. \( \Box \)

**Proposition 10.2.** We have
\[ \partial \tilde{v} \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(1+\sigma,\delta,\delta')}}, \]
under the hypotheses of part 2 of the main theorem.

**Proof.** We now prove
\[ \partial \tilde{v} \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(1+\sigma,\delta,\delta')}} \]
assuming more on the time derivatives of our coefficients and also a little more on \( A \) and \( B \); see part (2) of Theorem 1.8. For the first-order terms, we again write \( \partial(\hat{A}\phi) + \hat{B}\partial\phi = \partial([\hat{A} + \hat{B}]\phi) - (\partial\hat{B})\phi \); since \( A \) and \( B \), respectively \( \partial A \) and \( \partial B \), belong to the same \( S^2 \) class, with one higher rate of \( \langle r \rangle \) decay relative to the hypotheses of part 1 of the main theorem, we are done by the previous proof and we henceforth focus on the metric coefficients. By the product rule,
\[ \partial_t(\chi \hat{h}\phi) = \partial_t^2(\chi \hat{h}\phi) - \partial_t(\partial_t(\chi \hat{h})\phi) \]
(10.3)
where \( \partial_t(\chi \hat{h}) = O(\langle r \rangle^{-2-\sigma}) \) since \( \partial_t h \in S^{1}_{cone}(\langle r \rangle^{-2-\sigma}) \). For \( \Box U = -\partial_t(\partial_t(\chi \hat{h})\phi) \), the pointwise decay rates for \( U \) follow from techniques already shown.

For \( \Box \partial_t^2 u = \partial_t^2(\chi \hat{h}\phi) \), we have
\[ \langle r-t \rangle u_{tt} \lesssim |Lu_t| + |Su_t| \lesssim |\partial_t(Lu)| + |\partial_t(Su)| + \frac{1}{\langle r \rangle} \sum_{Z \in \{(n,s)\}} |Z u| \]
where \( L \) denotes the Lorentz boosts. It suffices to bound the first three terms on the right hand side by
\[ O(\frac{(t-r)^{3/2-\theta-q}}{\langle r \rangle}) \]
if
\[ h \in S^Z(\langle r \rangle^{-\gamma}). \]

We now bound \( |\partial_t u| \). Writing \( \Box u = \chi \hat{h}\phi \) with simplified notation henceforth as \( \chi h \phi \) or \( \chi h\phi \),
\[
\langle r \rangle \langle t-r \rangle u_{tt} \lesssim \langle r \rangle (|Su_t| + |Lu_t|) \\
\lesssim \int_{D_{(r)}} |\chi h\phi| + |S(\chi h\phi)| + (s-r)|\partial_\rho(\chi h\phi)| \rho dsd\rho \\
\lesssim \frac{1}{(t-r)^q} \langle t-r \rangle^{5/2-\theta}, \quad q = 1 + \sigma \quad (10.4)
\]
where the last line follows from Lemma 6.5 and Corollary 5.3.
The same calculation shows that \((Su)_t\) is also bounded by this, since by replacing \(u\) by \(Su\) above we still find the same upper bounds for the integrand; this is because the three functions \(S^j(\chi h\phi), j = 0, 1, 2\) satisfy the same bounds, and the analogous integral is

\[
\langle r \rangle (t - r)(Su)_t \lesssim \int_{D_{tr}} \left| \square Lu \right| + \left| \left| \square S^2 u \right| \right| \rho dsd\rho
\]

\[
\lesssim \int \langle s - \rho \rangle \sum_{k=0}^{1} |\partial_{\rho} S^k(\chi h\phi)| + \left( \sum_{j=0}^{2} |S^j(\chi h\phi)| \right) \rho dsd\rho
\]

\[
\lesssim \frac{1}{\langle t - r \rangle^q} (t - r)^{5/2 - \theta}.
\]

The function \((Lu)_t\) also obeys the same bounds as \(u_t\), and the analogous integral is

\[
\langle r \rangle (t - r)(Lu)_t \lesssim \int_{D_{tr}} \left| \square SLu \right| + \left| \left| \square LLu \right| \right| \rho dsd\rho.
\]

We have

\[
\int_{D_{tr}} \left| \square LLu \right| \rho dsd\rho \leq \int (2|\square Lu|) + |LLu| \rho dsd\rho
\]

\[
\lesssim \int \langle S(\chi h\phi) \rangle + \langle s - \rho \rangle |\partial_{\rho}(\chi h\phi)| + |LLu| \rho dsd\rho
\]

\[
\lesssim \int \langle S(\chi h\phi) \rangle + \langle s - \rho \rangle |\partial_{\rho}(\chi h\phi)|
\]

\[
\lesssim \int (S^2(\chi h\phi)) + S(\langle s - \rho \rangle |\partial_{\rho}(\chi h\phi)|) + \langle s - \rho \rangle |\partial_{\rho} S(\chi h\phi)| + \langle s - \rho \rangle |\partial_{\rho} S(\chi h\phi)| \rho dsd\rho
\]

\[
\lesssim \frac{1}{\langle t - r \rangle^q} (t - r)^{5/2 - \theta}, \quad q = 1 + \sigma
\]

where we changed \(\partial_{\rho}\) to \(\partial_{\eta}\) by \((3.1)\) and used the assumption on \(\partial_{\eta}^2 h\). The final line follows by Lemma 6.5. The final integrand term (and specifically, when the two derivatives both fall on \(h\)) is the sole instance the extra assumption on \(\partial_{\eta}^2 h\) in Theorem 1.8 is used.

For \(\square SLu\) we have

\[
\square SLu = S \square Lu + 2 \square Lu
\]

\[
= SL\square u + 2 \square Lu
\]

\[
= LS\square u + O(t \partial_{\eta}(\chi h\phi)) + 2 \square Lu
\]

where \(O(t \partial_{\eta}(\chi h\phi))\) arises from \([S, L]\) and can be broken into three cases: this function takes one of the three forms \(t \partial_{\eta}(\chi h\phi) = \partial_{\eta}(t \chi h\phi), x_i \partial_{\eta}(\chi h\phi) = \partial_{\eta}(x_i \chi h\phi),\) and \(t \partial_{\eta} \partial_{\eta}(\chi h\phi) = \partial_{\eta}(t \partial_{\eta} \chi h\phi)\). We may then replace \(\partial_{\eta}, \partial_{\eta}\) in the first and third cases by \(\partial_{\eta}\) via \((3.1)\). Then all these terms on the right hand side yield the upper bound \(\langle t - r \rangle^{5/2 - \theta - (1 + \sigma)}\) via Lemma 6.5; to see this, it suffices to consider a solution of \(\square w = t \chi h\phi\) and prove bounds for \(\partial_{\eta}w\).

For \(LS\square u\) on the other hand,

\[
\int_{D_{tr}} |S^2(\chi h\phi)| + \langle s - \rho \rangle |\partial_{\rho} S(\chi h\phi)| \rho dsd\rho \lesssim \frac{1}{\langle t - r \rangle^q} (t - r)^{5/2 - \theta}, \quad q = 1 + \sigma
\]
also. For $\Box Lu$, this upper bound was proved earlier. Thus
\[
\langle r \rangle \langle t - r \rangle (Lu)_t \lesssim \frac{1}{\langle t - r \rangle^q} (t - r)^{5/2 - \theta}.
\]

In summary, $(Lu)_t$ and $(Su)_t$ obey the same bound as $u_t$, because $\Box (Lu)_t$ and $\Box (Su)_t$ obey the same bounds as $\Box u_t$ and the claim then follows from Lemma 6.5. Thus $u_{tt} \lesssim \frac{1}{\langle r \rangle} (t - r)^{1/2 - \theta - q}$, $q = 1 + \sigma$ and the iteration finishes with $\partial_t \tilde{v} \lesssim \frac{1}{(\rho) (t - r)^{1 + \min(1 + \sigma, \delta, \delta')}}$. Notice that this argument works for any positive value of $q$. \hfill \Box

**ACKNOWLEDGEMENTS**

I am grateful to Mihai Tohaneanu for suggesting this problem and for discussions related to this work, and to Katrina Morgan and Jared Wunsch for discussing their paper [35].

**References**

[1] Serge Alinhac: *On the Morawetz-Keel-Smith-Sogge inequality for the wave equation on a curved background*. Publ. Res. Inst. Math. Sci. 42(3) (2006), 705-720

[2] Y. Angelopoulos, S. Aretakis, and D. Gajic: *Late-time asymptotics for the wave equation on spherically symmetric stationary backgrounds*, Advances in Mathematics 323 (2018), 529–621.

[3] Y. Angelopoulos, S. Aretakis, and D. Gajic: *Late-time asymptotics for the wave equation on extremal Reissner-Nordström backgrounds*, Adv. Math. 375 (2020), 107363, 139 pp.

[4] L. Andersson, P. Blue: *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, Annals of Mathematics, Volume 182 (2015), Issue 3, 787–853.

[5] S. Aretakis: *Decay of axisymmetric solutions of the wave equation on extremal Kerr backgrounds*, Journal of Functional Analysis 263, no. 9 (2012), 2770–2831.

[6] P. Bizon, T. Chmaj, A. Rostworowski, N. Szpak: *Linear and nonlinear tails II: Exact decay rates in spherical symmetry*. J. Hyper. Diff. Equa. 6 (2009) 107-125.

[7] P. Blue and A. Soffer: *Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates*. Adv. Differential Equations Volume 8, Number 5 (2003), 595-614.

[8] R. Booth, H. Christianson, J. Metcalfe, and J. Perry: *Localized Energy for Wave Equations with Degenerate Trapping*. Math. Res. Lett. 26 (2019), 991–1025.

[9] J.F. Bony and D. Häfner: *The semilinear wave equation on asymptotically Euclidean manifolds*, Comm. Partial Differential Equations 35 (2010), 23–67.

[10] N. Burq, Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge: “Global Strichartz estimates for nontrapping perturbations of the Laplacian” [Comm. Partial Differential Equation 25 (2000), no. 11-12 2171–2183; MR1789924 (2001j:35180)]. Comm. Partial Differential Equations, 28 (9-10):1675–1683, 2003.

[11] H. Christianson: *Dispersive estimates for manifolds with one trapped orbit*, Comm. Partial Differential Equations 33 (2008), 1147–1174.

[12] M. Dafermos, I. Rodnianski: *Lectures on black holes and linear waves*, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[13] M. Dafermos and I. Rodnianski: *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*. XVIth International Congress on Mathematical Physics, 421-432. World Sci. Publ., Hackensack, NJ, 2010.

[14] M. Dafermos and I. Rodnianski: *The red-shift effect and radiation decay on black hole spacetimes*, Comm. Pure Appl. Math. 62 (2009), 859-919.

[15] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman: *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$*, Annals of Math. 183 (2016), no. 3, 787-913.

[16] K. Datchev, J. Metcalfe, J. Shapiro, M. Tohaneanu: *On the interaction of metric trapping and a boundary*, arXiv:2008.05408
[17] R. Donninger, W. Schlag, and A. Soffer: On pointwise decay of linear waves on a Schwarzschild black hole background, Comm. Math. Phys., 309, (2012), 51–86.
[18] M. Grillakis: Regularity and Asymptotic Behavior of the Wave Equation with a Critical Nonlinearity, Annals of Math. 132, 3, 1990, 485–509.
[19] P. Hintz: A sharp version of Price’s law for wave decay on asymptotically flat spacetimes. arXiv:2004.01664.
[20] C. E. Kenig, G. Ponce, L. Vega: On the Zakharov and Zakharov-Schulman systems, J. Funct. Anal. 127 (1995), 204–234.
[21] M. Keel, H. Smith, C. D. Sogge: Almost global existence for some semilinear wave equations, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. 87 (2002), 265–279.
[22] H. Lindblad and M. Tohaneanu Global existence for quasilinear wave equations close to Schwarzschild, Comm. PDE. 43 (2018), 6, 893-944.
[23] H. Lindblad and M. Tohaneanu: A local energy estimate for wave equations on metrics asymptotically close to Kerr, Ann. Henri Poincare 21 (2020), no. 11, 3659-3726.
[24] S.-Z. Looi. Pointwise decay for the energy-critical nonlinear wave equation, preprint 2022 arXiv:2205.13197.
[25] S.-Z. Looi. Global existence and pointwise decay for some nonlinear wave equations, preprint 2022.
[26] S.-Z. Looi and M. Tohaneanu Scattering for critical wave equations with variable coefficients, Proceedings of the Edinburgh Mathematical Society, 1-19. doi:10.1017/S0013091521000158.
[27] J. Marzuola, J. Metcalfe, D, Tataru, and M. Tohaneanu. Strichartz Estimates on Schwarzschild Black Hole Backgrounds Comm. Math. Phys., 293 (2010), 37.
[28] C. Morawetz: Time decay for the nonlinear Klein-Gordon equations. Proc. Roy. Soc. Ser. A. 306 (1968), 291–296.
[29] J. Metcalfe and D. Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. Math. Ann. 353 (2012), 4, 1183-1237.
[30] J. Metcalfe and D. Tataru. Decay estimates for variable coefficient wave equations in exterior domains, Advances in Phase Space Analysis of Partial Differential Equations, 201-216.
[31] J. Metcalfe and C. Sogge: Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods. SIAM J. Math. Anal. 38(1)(2006), 188-209 (electronic)
[32] J. Metcalfe, J. Sterbenz, and D. Tataru: Local energy decay for scalar fields on time dependent non-trapping backgrounds, Amer. Math. 142 (2020), no. 3, 821-883.
[33] J. Metcalfe, D. Tataru, and M. Tohaneanu. Price’s law for nonstationary spacetimes. Adv. in Math. 230 (2012), 3, 995-1028.
[34] K. Morgan: The effect of metric behavior at spatial infinity on pointwise wave decay in the asymptotically flat stationary setting, preprint.
[35] K. Morgan and J. Wunsch: Generalized Price’s Law on fractional-order asymptotically flat stationary spacetimes, preprint.
[36] G. Moschidis: The $r^p$-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications, Annals of PDE 2:6 (2016), 1–194.
[37] S. Nonnenmacher and M. Zworski: Semiclassical resolvent estimates in chaotic scattering. Appl. Math. Res. Express. AMRX, (1) (2009):74–86.
[38] J. Oliver and J. Sterbenz: A Vector Field Method for Radiating Black Hole Spacetimes. Analysis and PDE. Vol. 13 (2020) no. 1, 29–92.
[39] R. Price: Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. Phys. Rev. D (3) 5 (1972), 2419–2438.
[40] J. V. Ralston: Solutions of the wave equation with localized energy. Comm. Pure Appl. Math. 22 (1969), 807–823.
[41] J. Shbierski: Characterisation of the Energy of Gaussian Beams on Lorentzian Manifolds - with Applications to Black Hole Spacetimes. Analysis & PDE, Vol. 8 (2015), No. 6, 1379–1420.
[42] T. C. Sideris, Nonresonance and global existence of prestressed nonlinear elastic waves, Ann. of Math. (2) 151 (2000), no. 2, 849-874.
[43] H. F. Smith and C. D. Sogge: Global Strichartz estimates for nontrapping perturbations of the Laplacian. Comm. Partial Differential Equations 25 (2000), 2171–2183.
[44] Christopher D. Sogge and Chengbo Wang. Concerning the wave equation on asymptotically Euclidean manifolds Journal d’Analyse Mathematique volume 112, 1732 (2010)
[45] J. Sterbenz: Angular regularity and Strichartz estimates for the wave equation. With an appendix by I. Rodnianski. Int. Math. Res. Not. 2005, 187–231.
[46] W. Strauss: *Dispersal of waves vanishing on the boundary of an exterior domain*. Comm. Pure Appl. Math. **28** (1975), 265–278.

[47] J. Sterbenz and I. Rodnianski. *Angular Regularity and Strichartz Estimates for the Wave Equation*, IMRN, **4** (2005), 187-231.

[48] N. Szpak. *Simple proof of a useful pointwise estimate for the wave equation*. [arXiv:0708.2801](https://arxiv.org/abs/0708.2801).

[49] D. Tataru: *Local decay of waves on asymptotically flat stationary space-times*. AJM Volume 135, No. 2, 361–401

[50] D. Tataru and M. Tohaneanu: *Local energy estimate on Kerr black hole backgrounds*. Int. Math. Res. Not. (2011), no. 2, 248–292.

[51] J. Wunsch, and M. Zworski: *Resolvent estimates for normally hyperbolic trapped sets*. Ann. Henri Poincaré (2011), 12(7):1349–1385.

[52] Shiwu Yang: *Global behaviors of defocusing semilinear wave equations*. [arXiv:1908.00606](https://arxiv.org/abs/1908.00606)

[53] Shiwu Yang: *Pointwise decay for semilinear wave equations in $\mathbb{R}^{3+1}$*. [arXiv:1908.00607](https://arxiv.org/abs/1908.00607)

[54] Shiwu Yang: *Uniform bound for solutions of semilinear wave equations in $\mathbb{R}^{1+3}$*. [arXiv:1910.02230](https://arxiv.org/abs/1910.02230)

Department of Mathematics, University of Kentucky, Lexington, KY 40506

*Email address: Shizhuo.Looi@uky.edu*