Research on Satellite Positioning Based on Total Least Squares Algorithm

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Abstract. In the technology of satellite positioning, positioning observation equation is constructed by measuring pseudo distance between the satellite and the receiver and the satellite position coordinates from navigation message, then the equation is solved and positioning results are obtained. The algorithm proposed in this paper is to use total least squares method to calculate initial value, and then someone carry out the Taylor series expansion on the basis of it, and then use weighted least squares in the iterative process to get the calculation results. Finally, user positioning is achieved. The algorithm is simulated by using observation series of Fangshan GNSS station in Beijing, and the simulation results are compared with the results obtained by the least squares algorithm. The results show that the satellite positioning algorithm based on total least squares is more accurate and faster in positioning calculation.

1. Introduction

Satellite positioning technology plays an important role in people's daily life and even in the military field, the accurate position of the user can be obtained through the radio navigation signal sent by the navigation satellite. After receiving the satellite signal, the user demodulates the navigation message, and then obtains the navigation information, and finally calculates the position coordinates of the receiver [1]. As the satellite signal reaching the ground is weak, it is easily disturbed by noise. At the same time, considering the actual measurement error, it is of great theoretical and practical significance to improve the positioning accuracy and speed up the operation speed. The improved least squares algorithm is applied to astronomical positioning, which combines the least squares algorithm and the total least squares algorithm effectively to form the hybrid least squares algorithm, and carries out robust weighting processing [2]. The iterative solution of total least squares is proposed, and the formula of its iterative approximation is derived [3]. A Taylor series localization algorithm based on constrained total least squares is proposed, and its feasibility in two-dimensional coordinate system is proved in the simulation environment [4]. This paper takes into account that when the software receiver conducts positioning calculation after receiving satellite navigation signals, there are not only measurement errors caused by interference signals, but also errors in the observation matrix. In order to achieve better real-time performance and higher positioning accuracy, the total least squares algorithm is used to estimate the initial value of the receiver position [5-6], and a reasonable weighting matrix is set for the least
squares algorithm in the iterative process, and the measurement value with high SNR is used as much as possible.

2. Observation Equation of Satellite Positioning

In satellite positioning technology, users obtain navigation information from the satellite signals, then measure the pseudo distance between the satellite and the receiver, and correct its error [7] to obtain the corrected pseudo distance measurement value. After neglecting the measurement error, the following nonlinear positioning observation equation can be established [8]

\[
P_i = \sqrt{(X_{si} - X)^2 + (Y_{si} - Y)^2 + (Z_{si} - Z)^2} + cT \quad (i = 1, 2, \ldots, N)
\]  

(1)

Where \((X_{si}, Y_{si}, Z_{si})\) is position coordinates of \(i\)th satellite and \(N\) is the number of satellites participating in positioning calculation, \((X, Y, Z)\) is position coordinate of the receiver, Typically \(C\) is the propagation speed of electromagnetic wave in vacuum, \(T\) is the clock difference between the receiver clock and the satellite clock.

3. The Initial Value from Total Least Squares

In the process of satellite positioning, when the least square algorithm is used for iterative calculation, it is necessary to assume that there is an initial value of a receiver position, and make it approach the optimal solution continuously through iterative cycle. Thus, it can be seen that the closer the selected initial position is to the final target position, the fewer times of iteration, the faster the convergence speed and the more accurate the positioning will be when the least square algorithm is used for iterative solution [9]. In this work, the total least squares algorithm is used to obtain more accurate initial value of receiver position.

By combining the positioning observation equations of \(N\) satellites, the following equations can be obtained.

\[
\begin{align*}
P_1 &= \sqrt{(X_{s1} - X)^2 + (Y_{s1} - Y)^2 + (Z_{s1} - Z)^2} + cT \\
P_2 &= \sqrt{(X_{s2} - X)^2 + (Y_{s2} - Y)^2 + (Z_{s2} - Z)^2} + cT \\
&\vdots \\
P_N &= \sqrt{(X_{sN} - X)^2 + (Y_{sN} - Y)^2 + (Z_{sN} - Z)^2} + cT
\end{align*}
\]

(2)

The linear equation is obtained by summing the square variances of Equation (2).

\[
\frac{1}{2}(P_i^2 - P_f^2 + K_i^2 - K_f^2) = (X_{si} - X_s)X + (Y_{si} - Y_s)Y + (Z_{si} - Z_s)Z + (P_i - P_f)(-cT)
\]

(3)

Where \(K_i = X_{si}^2 + Y_{si}^2 + Z_{si}^2\) \(i = 1, 2, \ldots, N\), \(N\) is the number of satellites participating in positioning calculation, the linear matrix equation becomes

\[
P = GB
\]

(4)

Where

\[
P = \frac{1}{2} \begin{bmatrix}
P_2^2 - P_1^2 + K_1^2 - K_2^2 \\
P_3^2 - P_1^2 + K_1^2 - K_3^2 \\
\vdots \\
P_N^2 - P_1^2 + K_1^2 - K_N^2
\end{bmatrix}
\]
\[
G = \begin{bmatrix}
X_{s1} - X_{s2} & Y_{s1} - Y_{s2} & Z_{s1} - Z_{s2} & P_2 - P_1 \\
X_{s1} - X_{s3} & Y_{s1} - Y_{s3} & Z_{s1} - Z_{s3} & P_3 - P_1 \\
\vdots & \vdots & \vdots & \vdots \\
X_{s1} - X_{sN} & Y_{s1} - Y_{sN} & Z_{s1} - Z_{sN} & P_N - P_1
\end{bmatrix}, \quad B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.
\]

Considering that there are errors in both the actual observed quantity \( P \) and the coefficient matrix \( G \), the total least squares algorithm is used to solve the matrix. Here, the augmented matrix is constructed first \( C = [-P; G] \). In this case, \( \text{rank} \{ C \} = 5 \), singular value decomposition was performed to obtain
\[
C = U \Sigma V^T = \Sigma_k \sigma_k u_k v_k^T (k = 1, 2, 3, 4, 5)
\]

Where, \( u_k \) and \( v_k \) are the left and right singular vectors of the augmented matrix respectively. \( \sigma_k \) is the singular value corresponding to \( C \). There is a unique solution to a matrix
\[
\hat{C} = \begin{bmatrix} v_4(2) & v_4(3) & v_4(4) & v_4(5) \\ v_4(1) & v_4(1) & v_4(1) & v_4(1) \end{bmatrix}
\]

Here taking the estimate as the initial value of the receiver position coordinates \((X_0, Y_0, Z_0)\).

4. **Weighted Least Squares Algorithm**

It is linearized first, since the positioning equation is a nonlinear equation. The Taylor series expansion of Equation (1) is performed at the initial value, and the higher-order terms are omitted. The final linear equation becomes
\[
P_1 = F(X_0, Y_0, Z_0) - \frac{X_{s1} - X_0}{F(X_0, Y_0, Z_0)} (X - X_0) - \frac{Y_{s1} - Y_0}{F(X_0, Y_0, Z_0)} (Y - Y_0) - \frac{Z_{s1} - Z_0}{F(X_0, Y_0, Z_0)} (Z - Z_0) + cT
\]

Where, \( P_1 \) is the pseudo-distance measurement value after error correction between the satellite and the receiver; \( F(X_0, Y_0, Z_0) = \sqrt{(X_{s1} - X_0)^2 + (Y_{s1} - Y_0)^2 + (Z_{s1} - Z_0)^2} \). (\( X_{s1}, Y_{s1}, Z_{s1} \)) is position coordinates of \( i \)th satellite \( (i = 1, 2, \ldots, N) \), \( N \) is the number of satellites in positioning calculation.

The matrix equation becomes
\[
\Delta P = H \Delta B + E_i
\]

Where
\[
\Delta P = \begin{bmatrix}
F(X_0, Y_0, Z_0) - P_1 \\
F(X_0, Y_0, Z_0) - P_2 \\
\vdots \\
F(X_0, Y_0, Z_0) - P_N
\end{bmatrix}, \quad H = \begin{bmatrix}
X_{s1} - X_0 & Y_{s1} - Y_0 & Z_{s1} - Z_0 \\
X_{s2} - X_0 & Y_{s2} - Y_0 & Z_{s2} - Z_0 \\
\vdots & \vdots & \vdots \\
X_{sN} - X_0 & Y_{sN} - Y_0 & Z_{sN} - Z_0
\end{bmatrix}, \quad \Delta B = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \\ -cT \end{bmatrix}
\]

Here \( (\Delta X, \Delta Y, \Delta Z) \) is offset between the estimated value and the initial value, \( \Delta X = X - X_0, \Delta Y = Y - Y_0, \Delta Z = Z - Z_0 \). \( E_i \) is an unknown amount of error in a measurement.

In the calculation process, weights can be selected according to needs [10]. In this paper, the signal-to-noise ratio at each moment is selected as the weight value, and its weighting matrix is
\[
W = \begin{bmatrix}
\omega_1 & 0 & 0 \\
0 & \omega_2 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & \omega_N
\end{bmatrix}
\]
When the number of positioning satellites is greater than 4, the least square algorithm is used to solve the problem, and the estimated value of the offset is obtained

$$\Delta B = (H^TWH)^{-1}H^TWP$$

The root mean square of the iteration result is compared with the threshold value. When the requirement is not met, the current offset is added to the initial value to get the initial value of the next iteration. If the requirements are met, the iterative operation is stopped and the estimated value of the current receiver position is obtained.

5. Experiment and Performance Analysis

In this paper, the pseudo-range observation values of 24 hours received by Fangshan GNSS station in Beijing on October 26 2020 and the satellite ephemeris data are used as the experimental data. The number of available satellites varies between 8 and 10, and the receiver position is calculated every 30 seconds. In this paper, the measured SNR was selected to construct the weighted matrix, the least square method and the method based on total least squares algorithm were used to solve the location respectively. Figures 1-3 show the comparison of the upper errors of the two positioning algorithms in the X direction, Y direction and Z direction in the Earth Centered Earth Fixed system, respectively. In the figures, the results of the least squares algorithm and the algorithm based on total least squares are shown with red asterisk and blue circle respectively. Figure 4 shows the actual measured SNR value. Table 1 shows the standard deviations of the positioning results of the least squares algorithm and the algorithm based on total least squares in the X direction, Y direction and Z direction.

Figure 1. Comparison of X direction position errors using the least squares algorithm and the total least squares in the X direction.
Figure 2. Comparison of position errors using the least squares algorithm and the algorithm based on total least squares in the Y direction.

Figure 3. Comparison of position errors using the least squares algorithm and the algorithm based on total least squares in the Z direction.
Figure 4. The actual measured SNR value.

Table 1. Standard deviations of positioning results in X, Y, and Z directions.

| Method                        | X direction (m) | Y direction (m) | Z direction (m) |
|-------------------------------|-----------------|-----------------|-----------------|
| Least squares algorithm       | 1.0317          | 0.5580          | 1.5320          |
| Algorithm based on total least squares | 1.0139          | 1.5305          | 1.5070          |

Judging from the positioning error in figure 1, the results of the least squares algorithm and the algorithm based on total least squares are almost the same. Combined with the comparison of the standard deviations of the positioning results in Table 1, it can be concluded that the positioning results of the algorithm based on total least squares are only improved by centimeters compared with the positioning results of the least squares algorithm. From figure 1, it can be seen that based on the total least squares algorithm of positioning results than the least squares algorithm of positioning results closer to the center, in the discrete points of the error is larger than 2 m, combining SNR values in figure 2, it can be seen that these moments of signal-to-noise ratio is low, the error is caused by the external environment factors, led to the decrease of the positioning accuracy of reason, in these moments calculated results based on total least squares algorithm accuracy is higher and the stability is better.

The average number of iterations of the least squares algorithm and the algorithm based on total least squares in the overlapping process is shown in Table 2.

Table 2. The average number of iterations of the two algorithms

| Algorithms                        | The average number of iterations |
|-----------------------------------|----------------------------------|
| the least squares algorithm       | 5                                |
| the algorithm based on total least squares | 2                                |

It can be seen that the algorithm based on total least squares can reduce the number of iterations in the localization process and accelerate the convergence speed. The above results show that the algorithm based on total least squares can effectively improve the accuracy and speed of satellite positioning.

6. Conclusion

In the process of satellite positioning, the choice of positioning algorithm has an important impact on the real-time and accuracy of satellite positioning. In this paper, the total least squares algorithm is used to solve the initial value, and the weighted least squares method is used to get the iterative result, which
improves the efficiency of operation and the accuracy of positioning results. Combined with the actual observation data of Fangshan GNSS station in Beijing, the simulation analysis shows that the solution results based on the total least squares algorithm are closer to the actual coordinates and the calculation speed is faster.

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