Braided Morita equivalence for finite-dimensional semisimple and
cosemisimple Hopf algebras

Dedicated to Professor Masahiko Suzuki on the occasion of his 65th birthday

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Abstract

Braided Morita invariants of finite-dimensional semisimple and cosemisimple Hopf algebras with braidings are constructed by refining the polynomial invariants introduced by the author. The invariants are computed for the duals of Suzuki’s braided Hopf algebras, and as an application of that, the braided Morita equivalence classes over the 8-dimensional Kac-Paljutkin algebra are determined. This paper also includes the modified results and proofs on determination of the coribbon elements of Suzuki’s braided Hopf algebras, that are discussed and given in [23].

1 Introduction

On the classification of Hopf algebras over a field \( k \) two problems are now actively progressed. One is the classification up to isomorphism under some restriction like dimension fixed, or semisimple, or pointed. Another is the classification up to monoidal Morita equivalence, that is based on a categorical point of view. Two Hopf algebras \( A \) and \( B \) are called a \( k \)-linear monoidal Morita equivalent if their module categories \( \mathcal{A} \mathcal{M} \) and \( \mathcal{B} \mathcal{M} \) are equivalent as \( k \)-linear monoidal categories. In this paper we take the later stance, and consider some classification problem on quasitriangular Hopf algebras, namely, Hopf algebras with braiding structures. A braiding structure on a Hopf algebra \( A \) is determined by some element \( R \in A \otimes A \) called a universal \( R \)-matrix, which is introduced by Drinfeld [3]. We write \( c^R \) for the braiding structure and \((A,R)^\mathcal{M}\) for the braided monoidal category \((A\mathcal{M},c^R)\). Two quasitriangular Hopf algebras \((A,R)\) and \((B,R')\) are called braided Morita equivalent if the braided categories \((A,R)^\mathcal{M}\) and \((B,R')^\mathcal{M}\) are equivalent as \( k \)-linear braided monoidal categories. There are a few results of classification of quasitriangular Hopf algebras up to braided Morita equivalence [6, 17].

The eigenvalues of \( S \)-matrices and the Brauer groups in a braided monoidal category are well-known as braided Morita invariants [12, 22]. In [24] the author introduced some monoidal Morita invariant of semisimple and cosemisimple Hopf algebras of finite dimension. It is given as a polynomial in one variable, which constructed from the data of the braidings and the absolutely simple modules. By refining the invariant on braidings we have braided Morita invariants of semisimple and cosemisimple quasitriangular Hopf algebras of finite dimension. In this paper
we compute these braided Morita invariants for the duals of Suzuki’s braided Hopf algebras \[21\], which fit into a Hopf algebra extension \(1 \rightarrow (kC_2)^* \rightarrow K \rightarrow kD_{2L} \rightarrow 1\), where \(C_2\) is the cyclic group of order 2, and \(D_{2L}\) is the dihedral group of order \(2L\). In particular, the 8-dimensional Kac-Paljutkin algebra \[9, 14\], denoted by \(H_8\), is contained in the family of Suzuki’s Hopf algebras. As an application of the computation results of our polynomial invariants, we determine the braided Morita equivalence classes over \(H_8\).

In closely connection with the above consideration, the coribbon elements of Suzuki’s braided Hopf algebras are determined. Actually, although they have studied in \[23\] by the author, the proof of Lemma 8 and the statement of Theorem 5 in \[23\] contain several mistakes. I noticed them by a detailed note \[20\] sent from Sommerhäuser. We modify arguments in \[23\] and show the correct results on that with thanks to him. Another proof of the revised version of Theorem 5 in \[23\] is also given by using the spherical structures of Suzuki’s Hopf algebras.

This paper is organized as follows. In Section 2 we review the definition of (co)ribbon Hopf algebras, and introduce braided Morita invariants of semisimple and cosemisimple quasitriangular Hopf algebras of finite dimension. In Section 3 we review the definition of Suzuki’s braided Hopf algebras and some basic results on that obtained by Satoshi Suzuki \[21\]. We give the revised results on determination of the coribbon elements of Suzuki’s braided Hopf algebras. In Section 4 we compute the polynomial invariants defined in Section 2 for the duals of Suzuki’s braided Hopf algebras. In the final section we compute the Hopf algebra automorphism group for \(H_8\), and determine the braided Morita equivalence classes of \(H_8\). In Appendix we give a list of corrigenda in my paper \[23\].

Throughout this paper \(k\) denotes a field. For a bialgebra or a Hopf algebra \(A\), denoted by \(\Delta\), \(\varepsilon\) and \(S\) the comultiplication, the counit and the antipode of \(A\), respectively. We use Sweedler’s notation such as \(\Delta(x) = \sum x_{(1)} \otimes x_{(2)}\) for \(x \in A\). For general facts on Hopf algebras or monoidal categories, refer to Montgomery’s book \[16\] and Kassel’s book \[10\].

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2 Braided Morita invariants of quasitriangular Hopf algebras

2.1 Definitions of braided and coribbon Hopf algebras

The notion of a quasitriangular bialgebra or a quasitriangular Hopf algebra is introduced by Drinfeld \[3\]. It is a pair of a bialgebra or a Hopf algebra \(A\) over \(k\) and an invertible element \(R \in A \otimes A\) satisfying some suitable conditions. Such an \(R\) is called a universal \(R\)-matrix of \(A\).

Lemma 2.1 (Drinfeld \[3\], Radford \[18\]). Let \((A, R)\) be a quasitriangular Hopf algebra. Then

(1) the antipode \(S\) is bijective,
(2) \(R^{-1} = (S \otimes id)(R)\),
(3) \(R = (S \otimes S)(R)\),
(4) \((\varepsilon \otimes id)(R) = 1 = (id \otimes \varepsilon)(R)\).
Furthermore, if we write \( R \) in the form \( R = \sum R^{(1)} \otimes R^{(2)} \), and set \( u := \sum S(R^{(2)})R^{(1)} \in A \), then the following conditions are satisfied.

\begin{align*}
(\text{DE1}) \quad & u \text{ is invertible, and } S^2(a) = uau^{-1} \text{ for all } a \in A, \\
(\text{DE2}) \quad & \Delta(u) = (u \otimes u)(R_{21}R)^{-1} = (R_{21}R)^{-1}(u \otimes u), \\
(\text{DE3}) \quad & \varepsilon(u) = 1, \\
(\text{DE4}) \quad & u^{-1} = \sum R^{(2)}S(R^{(1)}).
\end{align*}

Here, \( R_{21} = \sum R^{(2)} \otimes R^{(1)} \). The element \( u \) is called the Drinfeld element of \((A, R)\). \( \square \)

An element \( v \in A \) is called a ribbon element of a quasitriangular bialgebra \((A, R)\) and the triplet \((A, R, v)\) is called a ribbon bialgebra \cite{ribbon1} if the following conditions are satisfied:

\begin{align*}
(\text{Rib1}) \quad & v \in Z(A), \text{ where } Z(A) \text{ denotes the center of } A, \\
(\text{Rib2}) \quad & \Delta(v) = (v \otimes v)(R_{21}R)^{-1}, \\
(\text{Rib3}) \quad & \varepsilon(v) = 1.
\end{align*}

In the case where \((A, R)\) is a quasitriangular Hopf algebra, the condition

\( (\text{Rib4}) \ S(v) = v \)

is also required in addition to the above three conditions. Then, the triplet \((A, R, v)\) is called a ribbon Hopf algebra. By definition any ribbon element \( v \) is invertible, and if \( A \) is of finite dimension, then the condition

\( (\text{Rib0}) \ v^2 = uS(u) \)

is automatically satisfied \cite{ribbon2}, where \( u \) is the Drinfeld element of \((A, R)\).

A ribbon element is characterized by a special group-like element as follows \cite{ribbon3}.

**Lemma 2.2.** Let \((A, R)\) be a quasitriangular Hopf algebra over \( k \). For an element \( v \in A \) the following conditions (1) and (2) are equivalent.

(1) \( v \) is a ribbon element of \((A, R)\).

(2) there is an element \( g \in G(A) \) such that

\( \begin{align*}
(a) \quad & v = g^{-1}u, \\
(b) \quad & S(u) = g^{-2}u, \\
(c) \quad & g^{-1}u \in Z(A).
\end{align*} \)

Here, \( G(A) \) denotes the set of the group-like elements of \( A \). \( \square \)

Although the Drinfeld element is not necessary to be a ribbon element, in the semisimple and cosemisimple case the following holds.

**Proposition 2.3** (Gelaki \cite[Lemma 2.1.1]{ribbon4}). Let \((A, R)\) be a quasitriangular Hopf algebra over \( k \), and \( u \) be its Drinfeld element. If \( A \) is semisimple and cosemisimple, then \( u \in Z(A) \) and \( u = S(u) \). Therefore, the Drinfeld element \( u \) of \((A, R)\) is a ribbon element of \((A, R)\). \( \square \)

Using Lemma \ref{lem:ribbon_element} and Proposition \ref{prop:ribbon_element} we have:
Proposition 2.4. Let \((A, R)\) be a finite-dimensional quasitriangular Hopf algebra over \(k\), and \(u\) be its Drinfeld element. If \(A\) is semisimple and cosemisimple, then the set of all ribbon elements \(\text{Rib}(A, R)\) is given by \(\text{Rib}(A, R) = \{ gu \mid g \in G(A) \cap Z(A), \, g^2 = 1 \}\). \(\square\)

In order to know the ribbon elements of a finite-dimensional semisimple and cosemisimple quasitriangular Hopf algebra \((A, R)\), it is enough to determine the set \(\text{Sph}(A) := \{ g \in G(A) \cap Z(A) \mid g^2 = 1 \}\) by Proposition 2.4.

Let us recall the definitions of braided Hopf algebras and coribbon Hopf algebras that are the dual notions of quasitriangular Hopf algebras and ribbon Hopf algebras, respectively. The former and the latter are introduced by Doi [2] and Hayashi [4, 5], respectively. Let \(A\) be a bialgebra \(A\) over \(k\). A linear functional \(\sigma : A \otimes A \rightarrow k\) is called a braiding of \(A\), if it is convolution-invertible, and the following conditions are satisfied:

\[
\begin{align*}
(B1) \quad & \sum \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = \sum \sigma(x_{(2)}, y_{(2)})y_{(1)}x_{(1)}, \\
(B2) \quad & \sigma(xy, z) = \sum \sigma(x, z_{(1)})\sigma(y, z_{(2)}), \\
(B3) \quad & \sigma(x, yz) = \sum \sigma(x_{(1)}, z)\sigma(x_{(2)}, y)
\end{align*}
\]

for all \(x, y, z \in A\). The pair \((A, \sigma)\) is called a braided bialgebra. In a braided bialgebra \((A, \sigma)\) the following equation holds:

\[
(B4) \quad \sigma(1_A, x) = \sigma(x, 1_A) = \varepsilon(x) \text{ for all } x \in A.
\]

An invertible element \(\theta \in A^*\) is said to be a coribbon element of a braided bialgebra \((A, \sigma)\) if the following conditions are satisfied:

\[
\begin{align*}
(CR1) \quad & \sum \theta(x_{(1)})x_{(2)} = \sum \theta(x_{(2)})x_{(1)}, \\
(CR2) \quad & \theta(xy) = \sum \sigma^{-1}(x_{(1)}, y_{(1)})\theta(x_{(2)})\theta(y_{(2)})\sigma^{-1}(y_{(3)}, x_{(3)}), \\
(CR3) \quad & \theta(1) = 1
\end{align*}
\]

for all \(x, y \in A\). The triplet \((A, \sigma, \theta)\) is called a coribbon bialgebra. Furthermore, if \(A\) is a Hopf algebra and the condition

\[
(CR4) \quad \theta \circ S = \theta
\]

is satisfied, then the triplet \((A, \sigma, \theta)\) is called a coribbon Hopf algebra.

Remark 2.5. If a Hopf algebra \(A\) is of finite dimension, then a braiding \(\sigma\) of \(A\) is a universal \(R\)-matrix of \(A^*\) via the usual isomorphism \((A \otimes A)^* \cong A^* \otimes A^*\). This construction gives a one-to-one correspondence between the braidings of \(A\) and the universal \(R\)-matrices of \(A^*\). Furthermore, an element \(\theta \in A^*\) is a coribbon element of a braided Hopf algebra \((A, \sigma)\) if and only if it is a ribbon element of the quasitriangular Hopf algebra \((A^*, \sigma)\).

Dualizing Proposition 2.4 we have:
Corollary 2.6. Let \((A, \sigma)\) be a finite-dimensional braided Hopf algebra over \(k\), and \(\mathcal{Y} \in A^*\) be its Drinfeld element:
\[
\mathcal{Y}(a) = \sum \sigma(a(2), S(a(1))) \quad (a \in A).
\]

If \(A\) is semisimple and cosemisimple, then the Drinfeld element \(\mathcal{Y}\) is a coribbon element of \((A, \sigma)\), and the set of all coribbon elements of \((A, \sigma)\), written by \(\text{CRib}(A, \sigma)\), is given by
\[
\text{CRib}(A, \sigma) = \{ p \mathcal{Y} | p \in G(A^*) \cap Z(A^*), \ p^2 = \varepsilon \}.
\]

Remark 2.7. For the dual Hopf algebra \(A^*\),
\[
Z(A^*) = \{ p \in A^* | \forall a \in A, \ \sum p(a(1))a(2) = \sum p(a(2))a(1) \},
\]
\[
G(A^*) = \{ p \in A^* | \forall a, b \in A, \ p(ab) = p(a)p(b), \ p(1) = 1 \}.
\]

2.2 Polynomial invariants of quasitriangular Hopf algebras

In [24] the author introduced some invariant of a finite-dimensional semisimple and cosemisimple Hopf algebra defined by using braiding structures and given as a polynomial. This invariant is a monoidal Morita invariant for such a Hopf algebra. In this subsection we consider a braided refinement of the invariant.

Let \(A\) be a finite-dimensional semisimple and cosemisimple Hopf algebra over \(k\). By Etingof and Gelaki [11 Corollary 1.5], the set of universal \(R\)-matrices \(\text{Braid}(A)\) is finite. Let us consider a quasitriangular Hopf algebra \((A, R)\). For an element \(a \in A\) and a finite-dimensional left \(A\)-module \(M\), let \(\varrho_M : M \to M\) denote the left action of \(a\) on \(M\), and \(u \in A\) is the Drinfeld element of \((A, R)\). Then, we set
\[
\dim_R M = \text{Tr}(\varrho_M),
\]
and call it the categorical dimension of \(M\) [13]. We note that if \(A\) is a finite-dimensional semisimple and cosemisimple Hopf algebra over \(k\), then for any absolutely simple left \(A\)-module \(M\), \((\dim M)1_k \neq 0\) by [11], and the following equation holds [24 Lemma 3.2]:
\[
\left(\frac{\dim_R M}{\dim M}\right)^{\dim A^3} = 1.
\]

So, \(\dim_R M / \dim M\) is a root of unity in \(k\).

Let \(d\) be a positive integer, and \(\{M_1, \ldots, M_t\}\) be a complete system of the absolutely simple left \(A\)-modules of dimension \(d\). Then we define a polynomial \(P_{A, R}^{(d)}(x) \in k[x]\) by
\[
P_{A, R}^{(d)}(x) = \prod_{i=1}^t \left(x - \frac{\dim_R M_i}{d}\right).
\]

If there is no absolutely simple left \(A\)-module of dimension \(d\), then we define \(P_{A, R}^{(d)}(x) := 1\).

For a quasitriangular Hopf algebra \((A, R)\) we denote the braided monoidal category \((\tilde{A}\tilde{M}, c^R)\) by \((A, R)^\tilde{M}\). Here, \(c^R\) is the braiding associated to \(R = \sum R^{(1)} \otimes R^{(2)},\) that is, for \(M, N \in \tilde{A}\tilde{M}\)
\[
(c^R)_{M,N}(m \otimes n) = \sum (R^{(2)} \cdot n) \otimes (R^{(1)} \cdot m) \quad (m \in M, \ n \in N).
\]

Two quasitriangular Hopf algebras \((A, R)\) and \((B, R')\) over \(k\) are said to be braided Morita equivalent if the braided monoidal categories \((A, R)^\tilde{M}\) and \((B, R')^\tilde{M}\) are equivalent as \(k\)-linear
braided monoidal categories. By using the same technique in the proof of [24, Theorem 2.6] it can be verified that the above polynomial \( P^{(d)}_{A,R}(x) \) is a braided Morita invariant, that is, \((A, R)\) and \((B, R')\) are braided Morita equivalent, then \( P^{(d)}_{(A, R)}(x) = P^{(d)}_{(B, R')}(x) \) for all positive integers \( d \). By using \( P^{(d)}_{A,R}(x) \), the polynomial invariant \( P^{(d)}_A(x) \) defined in [24] can be written by \( P^{(d)}_A(x) = \prod_{R \in \text{Braid}(A)} P^{(d)}_{A,R}(x) \).

Another braided Morita invariant can be constructed by using ribbon structures. Let \((A, R)\) be a quasitriangular Hopf algebra, and \( v \in A \) be its ribbon element. Then any ribbon element \( v \) of \((A, R)\) induces a twist \( \theta^v = \{ \theta^v_M \}_{M \in \mathcal{M}} \) for the braided category \((A, R)\mathcal{M}\), where \( \theta^v_M : M \rightarrow M \) is an \( A \)-linear isomorphism defined by

\[
(\theta^v_M)(m) = v^{-1} \cdot m \quad (m \in M).
\]

Suppose that \( A \) is finite-dimensional semisimple and cosemisimple. For a positive integer \( d \) a polynomial \( \tilde{P}^{(d)}_{A,R}(x) \) can be defined as follows.

\[
\tilde{P}^{(d)}_{A,R}(x) := \prod_{v \in \text{Rib}(A, R)} \prod_{i=1}^t (x - \xi_v(M_i)),
\]

where \( \{M_1, \ldots, M_t\} \) is a complete system of the absolutely simple left \( A \)-modules of dimension \( d \), and \( \xi_v(M_i) \) is a scalar determined by \( \sum_{i=1}^t \xi_v(M_i) = \frac{1}{n} \). This polynomial \( \tilde{P}^{(d)}_{A,R}(x) \) is also a braided Morita invariant. By Proposition 2.8 and Lemma 2.10 given in the next subsection we have:

**Proposition 2.8.** Let \((A, R)\) be a semisimple and cosemisimple quasitriangular Hopf algebra of finite dimension. For a positive integer \( d \), \( \tilde{P}^{(d)}_{A,R}(x) \) can be divided by \( P^{(d)}_{A,R}(x) \) in \( k[x] \). So, a polynomial \( Q^{(d)}_{A,R}(x) = \tilde{P}^{(d)}_{A,R}(x)/P^{(d)}_{A,R}(x) \in k[x] \) is defined, and it is also a braided Morita invariant.

**Example 2.9.** Let \( C_n \) denote the cyclic group of order \( n \) which is generated by \( a \), and \( \omega \in \mathbb{C} \) be a primitive \( n \)th root of unity. The universal R-matrices of the group Hopf algebra \( \mathbb{C}C_n \) are

\[
R_d = \sum_{k,l=0}^{n-1} \omega^{dk} E_k \otimes E_l \quad (d = 0, 1, \ldots, n-1),
\]

where \( E_k = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-ik} a^i \) for each \( k \in \mathbb{Z} \). The Drinfeld element \( u_d \) of \( (\mathbb{C}C_n, R_d) \) is \( u_d = \sum_{k=0}^{n-1} \omega^{-dk^2} E_k \). We have

\[
\text{Rib}(\mathbb{C}C_n, R_d) = \begin{cases} 
\{u_d\} & \text{if } n \text{ is odd}, \\
\{u_d, a \} & \text{if } n \text{ is even}.
\end{cases}
\]

If we set \( M_k = \mathbb{C}E_k \), then \( \{M_0, M_1, \ldots, M_{n-1}\} \) forms a complete system of simple \( \mathbb{C}C_n \)-modules. Then \( \xi_{u_d}(M_k) = \dim R_d M_k = \text{Tr}(u_d M_k) = \omega^{-dk^2} \), and if \( n \) is even, then \( \xi_{u_d}(M_k) = (-1)^k \dim R_d M_k = (-1)^k \omega^{-dk^2} \) for \( \mathfrak{m}_d := u_d a \) by using \( a^j E_k = \omega^j E_k \) \((j, k \in \mathbb{Z}) \). Therefore

\[
P^{(1)}_{(\mathbb{C}C_n, R_d)}(x) = \prod_{k=0}^{n-1} (x - \omega^{-dk^2}),
\]

\[
Q^{(1)}_{(\mathbb{C}C_n, R_d)}(x) = \begin{cases} 
n! & \text{if } n \text{ is odd}, \\
1 & \text{if } n \text{ is even}, \\
\prod_{k=0}^{n-1} (x - (-1)^k \omega^{-dk^2}) & \text{if } n \text{ is even}.
\end{cases}
\]
By comparing $P_{(CC_n, R_d)}^{(1)}(x)$ we see that $(CC_n, R_d)$ ($d = 0, 1, \ldots, n - 1$) are not mutually braided Morita equivalent for $n = 2, 3, 4$. In the case of $n = 5$

\[ P_{(CC_5, R_0)}^{(1)}(x) = (x - 1)^5, \]
\[ P_{(CC_5, R_1)}^{(1)}(x) = P_{(CC_5, R_4)}^{(1)}(x) = (x - 1)(x - \omega^{-1})^2(x - \omega)^2, \]
\[ P_{(CC_5, R_2)}^{(1)}(x) = P_{(CC_5, R_3)}^{(1)}(x) = (x - 1)(x - \omega^3)^2(x - \omega^2)^2. \]
So, $(CC_5, R_d)$ ($d = 0, 1, 2$) are not mutually braided Morita equivalent.

### 2.3 Relationship between ribbon and pivotal structures

It is known that any ribbon category has a pivotal structure [25]. In the case where a ribbon category is the module category $A_{\text{fd}}$ of finite-dimensional left $A$-modules over a ribbon Hopf algebra $(A, R, v)$, the associated pivotal structure $\tau$ is given by

\[ \tau_M(m) = u^{-1} \cdot m \quad (m \in M) \]

for each object $M \in A_{\text{fd}}$. Therefore the left and right pivotal dimensions of $M$ in the ribbon category are

\[ \text{pdim}^l_{u^{-1}} M = \text{pdim}^r_{u^{-1}} M = \text{Tr}(uu^{-1}M). \quad (2.4) \]

Suppose that $M$ is absolutely simple. Since $v$ is invertible, it follows that $\xi_v(M) \neq 0$, and hence

\[ \text{pdim}^l_{u^{-1}} M = \text{pdim}^r_{u^{-1}} M = \xi_v(M)^{-1} \text{Tr}(uM) = \xi_v(M)^{-1} \dim_R M. \]

From this, we also have

\[ \frac{\text{pdim}^r_{u^{-1}} M}{\dim M} = \xi_v(M)^{-1} \frac{\dim_R M}{\dim M}. \quad (2.5) \]

If $A$ is semisimple and cosemisimple, then the pivotal structures of $A_{\text{fd}}$ are uniquely determined by the group $Z(A) \cap G(A)$. Thus, $\eta := uu^{-1}$ has finite order, and it follows that $\text{pdim}^r_{u^{-1}} M/\dim M$ is a root of unity in $k$. By (2.1), $\dim_R M/\dim M$ is also a root of unity, and so by (2.5), $\xi_v(M)^{-1}$ is, too.

**Lemma 2.10.** Let $(A, R)$ be a semisimple and cosemisimple quasitriangular Hopf algebra of finite dimension, and $M$ be an absolutely simple left $A$-module. Then $\xi_u(M) = \frac{\dim uM}{\dim M}$ for the Drinfeld element $u$ of $(A, R)$.

**Proof.** The pivotal element $\eta$ corresponding to $u$ is $\eta = uu^{-1} = 1_A$. Since $\text{pdim}^r_M$ coincides with the trace of $\eta_M$ by (2.4), we see that $\text{pdim}^r_M = (\dim M)1_k$. Now, the desired equation follows from (2.5).

Let $C$ be a coalgebra over $k$. The dual space $C^*$ has a $k$-algebra structure, and any finite-dimensional right $C$-comodule $M$ can be regarded as a left $C^*$-module with the action

\[ p \cdot m = \sum p(m_{(1)})m_{(0)} \quad (p \in C^*, \ m \in M), \]

where we write the right $C$-coaction $\rho : M \rightarrow M \otimes C$ in the form $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$. This construction gives rise to an identical category equivalence between $k$-linear monoidal categories
of finite-dimensional right $C$-comodules and of finite-dimensional left $C^*$-modules. For a finite-dimensional right $C$-comodule $M$, an element $\text{ch}(M) \in C$ called the character of $M$ is defined by

$$\text{ch}(M) := \sum_{i=1}^{d} (e_i^* \otimes \text{id}_C)(\rho(e_i)),$$

where $\{e_i\}_{i=1}^{d}, \{e_i^*\}_{i=1}^{d}$ are mutually dual bases of $M, M^*$, respectively. The above element does not depend on the choice of bases.

**Lemma 2.11.** Let $A$ be a finite-dimensional Hopf algebra over $k$, and $M$ be a finite-dimensional right $A$-comodule.

1. $\eta \in A^*$ is a pivotal element of the dual Hopf algebra $A^*$, then $\text{pdim}^\eta M = \eta(\text{ch}(M))$, where the left-hand side is the right pivotal dimension of $M$ viewed as a left $A^*$-module as usual.

2. Assume that $M$ is absolutely simple with $(\dim M)1_k \neq 0$. Then $\xi_p(M) = \frac{p(\text{ch}(M))}{\text{dim} M}$ for any $p \in Z(A^*)$.

**Proof.** Let $\{e_i\}_{i=1}^{d}$ and $\{e_i^*\}_{i=1}^{d}$ be mutually dual bases of $M$ and $M^*$, respectively, and $\rho$ be the right coaction on $A$. We write $\rho(e_i) = \sum_{j=1}^{d} e_j \otimes a_{ji} (a_{ji} \in A)$. Then $\text{ch}(M) = \sum_{i=1}^{d} a_{ii}$, and hence $\text{pdim} M = \sum_{i=1}^{d} p(a_{ii})$. On the other hand, $\rho_M(e_i) = \sum_{j=1}^{d} p(a_{ji})e_j$. Thus we have $\text{Tr}(\rho_M) = \sum_{j=1}^{d} p(a_{ji}) = p(\chi_M)$. Since $\text{Tr}(\eta \chi_M) = \text{pdim}^\eta M$ for a pivotal element $\eta$, Part (1) is proved. Assume that $M$ is absolutely simple with $(\dim M)1_k \neq 0$. Then there is an element $\xi_p(M) \in k$ such that $\rho_M = \xi_p(M)\text{id}_M$. Taking the trace of this map we have the formula in Part (2).

3 **The coribbon elements of Suzuki’s Hopf algebras**

In this section we review the definition of Suzuki’s Hopf algebras, and describe the braiding structures of them in accordance with Suzuki’s paper [21]. The correct results on coribbon elements of Suzuki’s braided Hopf algebras described in [23] are also given.

Suzuki’s Hopf algebras are given as a family of finite-dimensional cosemisimple Hopf algebras generated by a comatrix basis of the $2 \times 2$-matrices. Suppose that $k$ is an algebraically closed field whose characteristic is not 2, and let $C = (C, \Delta, \varepsilon)$ be the comatrix coalgebra of degree 2 over $k$, that is, there is a basis $\{X_{11}, X_{12}, X_{21}, X_{22}\}$ of $C$ such that

$$\Delta(X_{ij}) = X_{11} \otimes X_{1j} + X_{12} \otimes X_{2j}, \quad \varepsilon(X_{ij}) = \delta_{ij}.$$  

Let $I$ be a coideal of the tensor algebra $\mathcal{T}(C)$ defined by

$$I = k(X_{11}^2 - X_{22}^2) + k(X_{12}^2 - X_{21}^2) + \sum_{i-j \neq l-m \text{ (mod 2)}} k(X_{ij}X_{lm}).$$

We set $B := \mathcal{T}(C)/I$, and denote by $x_{ij}$ the image of $X_{ij}$ under the natural projection $\mathcal{T}(C) \rightarrow B$. For $i, j = 1, 2$, $m \geq 1$ we define an element $\chi_{ij}^{m}$ in $B$ by

$$\chi_{11}^{m} := x_{11}x_{22}x_{11} \cdots x_{11}, \quad \chi_{22}^{m} := x_{22}x_{11}x_{22} \cdots x_{22},$$

$$\chi_{12}^{m} := x_{12}x_{21}x_{12} \cdots x_{12}, \quad \chi_{21}^{m} := x_{21}x_{12}x_{21} \cdots x_{21}.$$  

(3.1)
Then we have $\Delta(\chi_{ij}^m) = \chi_{11}^m \otimes \chi_{12}^m + \chi_{21}^m \otimes \chi_{22}^m$.

Let $N \geq 1$, $L \geq 2$, $\nu, \lambda = \pm 1$, and consider the following subset of $B$:

$$J^{\lambda}_N := k(x_{11}^{2N} + \nu x_{12}^{2N} - 1) + k(\chi_{11}^L - \chi_{22}^L) + k(-\lambda \chi_{12}^L + \chi_{21}^L).$$

Then $J^{\lambda}_N$ is a coideal of $B$, and $A^{\lambda}_N := B/\langle J^{\lambda}_N \rangle$ is a bialgebra. We also denote the image of $x_{ij}$ by the same symbol. It can be easily shown that

$$\{ x_{11}^s \chi_{22}^t, x_{12}^s \chi_{21}^t \mid 1 \leq s \leq 2N, 0 \leq t \leq L - 1 \}$$

is a basis of $A^{\lambda}_N$ over $k$, and $\dim A^{\lambda}_N = 4NL$. The bialgebra $A^{\lambda}_N$ actually is a cosemisimple Hopf algebra, whose structure maps are given by

$$\Delta(x_{ij}) = x_{11} \otimes x_{1j} + x_{i2} \otimes x_{2j}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}^4N^{-1}.$$  \hspace{1cm} (3.2)

**Remark 3.1.** The description of the antipode of $A^{\lambda}_N$ in [23] is wrong in the case when $\nu = -$. If $\text{ch}(k) \nmid NL$, then the cosemisimple Hopf algebra $A^{\lambda}_N$ is also semisimple [21, Theorem 3.1 vii], and $A_{1L}^{+\nu}, A_{1L}^{-\nu}$ coincide with $A_{4L}B_{4L}$, respectively, which are introduced by Masuoka [15] and generalized in [1]. In particular, the Hopf algebra $H_8 = A_{12}^{+\nu}$ is the unique Hopf algebra which is an 8-dimensional non-commutative and non-cocommutative Hopf algebra up to isomorphism. This Hopf algebra is called the Kac-Paljutkin algebra [14]. By uniqueness we see that $H_8$ is self-dual, that is the dual Hopf algebra is isomorphic to itself.

By (3.2), we see that for any integers $s, t \geq 0$ satisfying with $s + t \geq 1$,

$$\Delta(x_{11}^s \chi_{22}^t) = x_{11}^s \chi_{22}^t \otimes x_{11}^s \chi_{22}^t + x_{12}^s \chi_{21}^t \otimes x_{21}^s \chi_{12}^t,$$

$$\Delta(x_{12}^s \chi_{21}^t) = x_{11}^s \chi_{22}^t \otimes x_{12}^s \chi_{21}^t + x_{21}^s \chi_{12}^t \otimes x_{22}^s \chi_{11}^t,$$

$$S(x_{11}^s \chi_{22}^t) = \begin{cases} x_{11}^{(4N-2)(t+s)+s} \chi_{11}^t & \text{if } s + t \text{ is even}, \\ x_{11}^{(4N-2)(t+s)+s} \chi_{22}^t & \text{if } s + t \text{ is odd}, \end{cases}$$

$$S(x_{12}^s \chi_{21}^t) = \begin{cases} x_{12}^{(4N-2)(t+s)+s} \chi_{21}^t & \text{if } s + t \text{ is even}, \\ x_{12}^{(4N-2)(t+s)+s} \chi_{12}^t & \text{if } s + t \text{ is odd}. \end{cases}$$

It is known by Suzuki [21] that the group $G(A_{N\lambda}^\nu)$ is given by

$$G(A_{N\lambda}^\nu) = \{ x_{11}^{2s} \pm x_{12}^{2s}, x_{12}^{s+1} \chi_{22}^{L-1} \pm \sqrt{\lambda} x_{12}^{s+1} \chi_{21}^{L-1} \mid 1 \leq s \leq N \},$$ \hspace{1cm} (3.3)

that is of order $4N$, and the set

$$\{ k g \mid g \in G(A_{N\lambda}^\nu) \} \cup \{ k x_{11}^{2s} \chi_{22}^t + k x_{12}^{2s} \chi_{21}^t \mid 0 \leq s \leq N - 1, 1 \leq t \leq L - 1 \}$$

gives a complete system of absolutely simple right $A_{N\lambda}^\nu$-comodules, where the coactions of all subspaces above are induced from the comultiplication $\Delta$ of $A_{N\lambda}^\nu$.

Let $N$ be odd, and set $\lambda = +$ or $-$ if $L$ is odd or even, respectively. Then $A_{N\lambda}^\nu$ is isomorphic to the group algebra of the following finite group [24]:

$$G_{NL} = \langle h, t, w \mid t^2 = h^{2N} = 1, w^L = h^N, tw = w^{-1}t, ht = th, hw = wh \rangle.$$

In fact, an algebra isomorphism $\varphi : k[G_{NL}] \rightarrow A_{N\lambda}^\nu$ is given by
\[ \varphi(h) = x_{11}^2 - x_{12}^2, \quad \varphi(t) = x_{12}^N + x_{22}^N, \quad \varphi(w) = x_{11}^{2N-1}x_{22} - x_{21}^{2N-1}x_{12}. \]

Suzuki [24] also determined the all braidings of \( A_{NL}^{\alpha \beta} \). The construction of \( A_{NL}^{\alpha \beta} \) and the method of determination of its braidings are closely related to the universality for quadratic bialgebras (see [2] for a detailed statement and also [23] for the above fact).

The following lemma is partially proved in [23, p.341]. The equation \( C \) is generated by \( C \) as an algebra. Thus, a braiding \( \sigma \) of \( A_{NL}^{\alpha \beta} \) is determined by the values on \( C \) by (B2), (B3).

The following lemma is partially proved in [23] p.341. The equation \( \sigma_{\alpha \beta}^{-1}(x_{21}^m, x_{22}) = \sigma_{\alpha \beta}^{-1}(x_{22}, x_{21}^{m-1}) = \alpha \frac{m-1}{m} \beta - \frac{m-1}{m} \) for an odd integer \( m \geq 3 \) is added. In particular, these values are not equal to 0. Hereinafter, we treat the indices of Kronecker’s delta \( \delta_{ij} \) as modulo 2.

**Theorem 3.2 (S.Suzuki [21]).** (1) For \( \alpha, \beta \in k \), let \( \sigma_{\alpha \beta} : C \otimes C \to k \) be a \( k \)-linear map whose values \( \sigma_{\alpha \beta}(x_{ij}, x_{kl}) (i, j, k, l = 1, 2) \) are given by the left table above. Then \( \sigma_{\alpha \beta} \) is extended to a braiding of \( A_{NL}^{\alpha \beta} \) if and only if \( \alpha, \beta \in k^\times \), \( (\alpha \beta)^N = \nu, (\alpha \beta)^{-1}L = \lambda \).

(2) Consider the case \( L = 2 \). For \( \gamma, \delta \in k \), let \( \tau_{\gamma \delta}^\lambda : C \otimes C \to k \) be a \( k \)-linear map whose values \( \tau_{\gamma \delta}^\lambda(x_{ij}, x_{kl}) (i, j, k, l = 1, 2) \) are given by the right table above. Then, \( \tau_{\gamma \delta}^\lambda \) is extended to a braiding of \( A_{NL}^{\alpha \beta} \) if and only if \( \gamma, \delta \in k^\times \), \( \gamma^2 = \delta^2, \gamma^{2N} = 1 \).

(3) If \( L \geq 3 \), then the braidings of \( A_{NL}^{\alpha \beta} \) are given by

\[ \{ \sigma_{\alpha \beta} | \alpha, \beta \in k^\times, (\alpha \beta)^N = \nu, (\alpha \beta)^{-1}L = \lambda \}. \]

If \( L = 2 \), then the braidings of \( A_{NL}^{\alpha \beta} \) are given by

\[ \{ \sigma_{\alpha \beta} | \alpha, \beta \in k^\times, (\alpha \beta)^N = \nu, (\alpha \beta)^{-1} = \lambda \} \cup \{ \tau_{\gamma \delta}^\lambda | \gamma, \delta \in k^\times, \gamma^2 = \delta^2, \gamma^{2N} = 1 \}. \]

We note that there is a natural embedding \( C \subset A_{NL}^{k^\times} \), and therefore \( A_{NL}^{k^\times} \) is generated by \( C \) as an algebra. Thus, a braiding \( \sigma \) of \( A_{NL}^{k^\times} \) is determined by the values on \( C \) by (B2), (B3).

The following lemma is partially proved in [23] p.341. The equation \( \sigma_{\alpha \beta}^{-1}(x_{21}^m, x_{22}) = \sigma_{\alpha \beta}^{-1}(x_{22}, x_{21}^{m-1}) = \alpha \frac{m-1}{m} \beta - \frac{m-1}{m} \) holds if \( m \) is odd, \( \alpha \frac{m}{m} \beta + \frac{m}{m} \) if \( m \) is even.

**Lemma 3.3.** In the braided Hopf algebra \( (A_{NL}^{k^\times}, \sigma_{\alpha \beta}) \) the following holds.

\[ \sigma_{\alpha \beta}^\pm(x_{ij}^m, x_{kl}) = \sigma_{\alpha \beta}^\pm(x_{kl}, x_{ij}^m) = \begin{cases} \delta_{i+j,1} \delta_{k+l,1} (\alpha \pm \frac{m}{m} \beta \pm \frac{m}{m}) \delta_{j,k} & \text{if } m \text{ is odd}, \\ \delta_{i+j,1} \delta_{k,l} (\alpha \pm \frac{m}{m} \beta \pm \frac{m}{m}) \delta_{j,k} & \text{if } m \text{ is even}. \end{cases} \]

**Lemma 3.4.** In the braided Hopf algebra \( (A_{NL}^{k^\times}, \tau_{\gamma \delta}^\lambda) \) the following holds:

\[ (\tau_{\gamma \delta}^\lambda)^\pm(x_{ij}^m, x_{kl}) = (\tau_{\gamma \delta}^\lambda)^\pm(x_{ij}, x_{kl}^m) = \begin{cases} \gamma^m & \text{if } i = j = k = l, \\ \delta^m & \text{if } i = j = 1, k = l = 2, \\ (\lambda \delta)^m & \text{if } i = j = 2, k = l = 1, \\ 0 & \text{otherwise.} \end{cases} \]

The Drinfeld elements of Suzuki’s braided Hopf algebras are given by the following lemma.
Lemma 3.5. Suppose that $k$ contains a $4NL$th root of unity.

1. The Drinfeld element $\varrho_{\alpha\beta}$ of $(A_{NL}^{\alpha\beta}, \sigma_{\alpha\beta})$ is given by $\varrho_{\alpha\beta}(x_{1j}) = \delta_{ij}^{-1}$. 
2. The Drinfeld element $\varrho_{\gamma\delta}$ of $(A_{NL}^{\gamma\delta}, \tau_{\gamma\delta})$ is given by $\varrho_{\gamma\delta}(x_{ij}) = \delta_{ij}^{-1}$. 

Proof. (1) $\varrho_{\alpha\beta}(x_{1j}) = \sigma_{\alpha\beta}(x_{1j}, S(x_{1j})) = \sigma_{\alpha\beta}(x_{1j}, x_{1j}^{4N-1}) + \sigma_{\alpha\beta}(x_{2j}, x_{2j}^{4N-1})$.

By Lemma 3.3 and $\gamma^2N = 1$, we have $\varrho_{\gamma\delta}(x_{1j}) = \tau_{\gamma\delta}(x_{1j}, x_{1j}^{4N-1}) + \tau_{\gamma\delta}(x_{2j}, x_{2j}^{4N-1}) = \delta_{ij}\delta_{ij}^{-1} = \delta_{ij}^{-1}$. 

Lemma 3.6. The Yang-Baxter form $\sigma_{\alpha\beta}$ on $C$ given in Theorem 3.2 (1) can be extended to a braiding of the bialgebra $B = T(C)/(I)$. We denote it by the same symbol $\sigma_{\alpha\beta}$. For an element $\omega \in k^\times$, the $k$-linear functional $\theta_\omega : C \rightarrow k$ defined by $\theta_\omega(x_{ij}) = \delta_{ij}\omega$ $(i, j = 1, 2)$ can be extended to a coribbon element of the braided bialgebra $(B, \sigma_{\alpha\beta})$. We denote the coribbon element by the same symbol $\theta_\omega$. Suppose that $\alpha$ and $\beta$ satisfy $(\alpha\beta)^N = \nu$, $(\alpha\beta)^{-1}L = \lambda$. Then, 

1. $\theta_\omega$ induces a coribbon element of the braided bialgebra $(A_{NL}^{\alpha\beta}, \sigma_{\alpha\beta})$ if and only if $\omega^2N = \alpha^2N$. 
2. For $\omega$ with $\omega^2N = \alpha^2N$, $\theta_\omega \circ S = \theta_\omega$ if and only if $\omega = \pm \omega^{-1}$. 

Lemma 3.7. The Yang-Baxter form $\tau_{\gamma\delta}$ on $C$ given in Theorem 3.2 (2) can be extended to a braiding of the bialgebra $B^{(\lambda)} = T(C)/(I(\lambda))$, where

$$I(\lambda) = k(X_{11}X_{22} - X_{22}X_{11}) + k(X_{12}X_{21} - \lambda X_{21}X_{12}) + \sum_{i-j\neq m \text{ (mod 2)}} k(X_{ij}X_{im}).$$

We denote this braiding of $B^{(\lambda)}$ by the same symbol $\tau_{\gamma\delta}$. For an element $\omega \in k^\times$, the $k$-linear functional $\theta_\omega : C \rightarrow k$ by the same formula in Lemma 3.6 can be extended to a coribbon element of the braided bialgebra $(B^{(\lambda)}, \tau_{\gamma\delta})$. We denote the coribbon element by the same symbol $\theta_\omega$. Suppose that $\gamma$ and $\delta$ satisfy $\gamma^2 = \delta^2$, $\gamma^2N = 1$. Then,

1. $\theta_\omega$ induces a coribbon element of the braided bialgebra $(A_{NL}^{\lambda\delta}, \tau_{\gamma\delta})$ if and only if $\omega^2N = 1$.
2. For $\omega \in k$ with $\omega^2N = 1$, $\theta_\omega \circ S = \theta_\omega$ if and only if $\omega = \pm \omega^{-1}$. 

Combining Lemmas 3.6 and 3.7, we have the following correct version of [23] Theorem 5].

Theorem 3.8. Let $k$ be an algebraically closed field whose characteristic does not divide $2NL$.

For each element $\omega \in k$, let $\theta_\omega : C \rightarrow k$ be the $k$-linear functional defined by the same formula in Lemma 3.6. Then for a braided Hopf algebra $(A_{NL}^{\alpha\beta}, \sigma)$ the following statements hold.

1. Let $\alpha, \beta$ be elements in $k^\times$ satisfying $(\alpha\beta)^N = \nu$, $(\alpha\beta)^{-1}L = \lambda$. Then, $\theta_\omega$ is extended to a coribbon element of the braided bialgebra $(A_{NL}^{\alpha\beta}, \sigma_{\alpha\beta})$ if and only if $\omega^2N = \alpha^2N$, and any coribbon element of the braided bialgebra $(A_{NL}^{\alpha\beta}, \sigma_{\alpha\beta})$ is given by the form $\theta_\omega$. In addition, $\theta_\omega$ is a coribbon element of the braided Hopf algebra $(A_{NL}^{\lambda\delta}, \sigma_{\alpha\beta})$ if and only if $\omega = \pm \omega^{-1}$. Therefore, there are exactly two coribbon elements of the braided Hopf algebra $(A_{NL}^{\lambda\delta}, \sigma_{\alpha\beta})$.

2. Let $\gamma, \delta$ be elements in $k^\times$ satisfying $\gamma^2 = \delta^2$, $\gamma^2N = 1$. Then, $\theta_\omega$ is extended to a coribbon element of the braided bialgebra $(A_{NL}^{\lambda\delta}, \tau_{\gamma\delta})$ if and only if $\omega^2N = 1$, and any coribbon
element of the braided bialgebra $(A_{N^2}^{\lambda}, \tau_{\gamma}^{\lambda})$ is given by the form $\theta_{\omega}$. In addition, $\theta_{\omega}$ is a coribbon element of the braided Hopf algebra $(A_{N^2}^{\lambda}, \tau_{\gamma}^{\lambda})$ if and only if $\omega = \pm \gamma^{-1}$. Therefore, there are exactly two coribbon elements of the braided Hopf algebra $(A_{N^2}^{\lambda}, \tau_{\gamma}^{\lambda})$.

**Proof.** (1) Let $\theta$ be a coribbon element of the braided bialgebra $(B/\langle J_N^{\lambda} \rangle, \sigma_{\omega, \beta})$. By $\Delta(x_{1j}) = x_{11} \otimes x_{1j} + x_{12} \otimes x_{2j}$ for $j = 1, 2$ and (CR1) we have

$$
\begin{align*}
\theta(x_{11})x_{11} + \theta(x_{12})x_{21} &= \theta(x_{11})x_{11} + \theta(x_{21})x_{12}, \\
\theta(x_{11})x_{12} + \theta(x_{12})x_{22} &= \theta(x_{12})x_{11} + \theta(x_{22})x_{12}.
\end{align*}
$$

Since $x_{11}, x_{12}, x_{21} = \nu x_{11}^N x_{21}, x_{22} = x_{11}^N x_{22}$ are linearly independent, it follows that $\theta(x_{12}) = \theta(x_{21}) = 0$, $\theta(x_{11}) = \theta(x_{22})$. One can set $\theta(x_{11}) = \theta(x_{22}) = \omega$ for some $\omega \neq 0$ since $\theta$ is convolution-invertible. So, $\theta$ is obtained by $\theta = \theta_{\omega} \circ \pi$, where $\pi : B \to B/\langle J_N^{\lambda} \rangle$ is the natural projection, and $\theta_{\omega}$ is the coribbon element of the braided bialgebra $(B, \sigma_{\omega, \beta})$ determined by $\theta_{\omega}(x_{ij}) = \delta_{ij}\omega$ for all $i, j = 1, 2$. Thus, by Lemma 3.6(1) it is required that $\omega = \xi \alpha$ for some $\xi^2N = 1$. It can be easily shown that the converse is true. By Lemma 3.6(2) a necessary and sufficient condition for that $\theta_{\omega}$ is a coribbon element of $(A_{N^2}^{\lambda}, \sigma_{\omega, \beta})$ is $\omega = \pm \gamma^{-1}$.

(2) Let $\theta$ be a coribbon element of the braided bialgebra $(B/\langle J_N^{\lambda} \rangle, \tau_{\gamma}^{\lambda})$. As the same manner with the proof of Part (1) we see that $\theta(x_{12}) = \theta(x_{21}) = 0$, $\theta(x_{11}) = \theta(x_{22})$, and $\theta(x_{11}) = \theta(x_{22}) = \omega$ is not 0. Hence $\theta$ is given by $\theta = \theta_{\omega} \circ \pi'$, where $\pi' : B^{(\lambda)} \to B^{(\lambda)}/\langle J_N^{\lambda} \rangle \cong B/\langle J_N^{\lambda} \rangle$ is the natural projection, $J_N^{(\omega)} = k(x_{11}^2 - x_{22}^2) + k(x_{21}^2 - x_{22}^2) + k(x_{12}^2 + \nu x_{12}^2 - 1)$, and $\theta_{\omega}$ is the coribbon element of the braided bialgebra $(B^{(\lambda)}, \tau_{\gamma}^{\lambda})$. Thus, $\omega^2N = 1$ by Lemma 3.7, and $\theta$ is needed to be the form in Part (2). The converse is also true. Furthermore, by Lemma 3.7(2) a necessary and sufficient condition for that $\theta_{\omega}$ is a coribbon element of $(A_{N^2}^{\lambda}, \tau_{\gamma}^{\lambda})$ is $\omega = \pm \gamma^{-1}$.

To determine the coribbon elements of a braided Hopf algebra $(A_{N^2}^{\lambda}, \sigma)$ one can apply Corollary 2.6. This fact gives us an alternative proof of Theorem 3.8 as follows.

Suppose that $k$ contains a 4Nth root of unity. If $L$ is odd, then $G((A_{N^2}^{\lambda})^*) = \{ p_{\omega}, q_{\eta} | \omega, \eta \in k, \omega^{2N} = 1, \eta^{2N} = \nu \}$. Here, $p_{\omega}, q_{\eta} \in (A_{N^2}^{\lambda})^*$ are defined by $p_{\omega}(x_{ij}) = \delta_{ij}{\omega}, q_{\eta}(x_{ij}) = \delta_{i,j-1}{\eta}$, and the products between them are given by $p_{\omega'}p_{\omega''} = p_{\omega'}, q_{\eta'}q_{\eta''} = p_{\lambda\eta'}, p_{\omega}q_{\eta} = q_{\eta}p_{\omega} = q_{\omega \eta}$. If $L$ is even, then

$$
G((A_{N^2}^{\lambda})^*) = \{ p_{\omega, \epsilon} | \omega, \epsilon \in k, \omega^{2N} = 1, \epsilon = 0, 1 \},
$$

$$
G((A_{N^2}^{\lambda})^*) = \{ p_{\omega, \epsilon}, q_{\eta, \epsilon} | \omega, \eta, \epsilon \in k, \omega^{2N} = 1, \eta^{2N} = \nu, \epsilon = 0, 1 \}.
$$

Here, $p_{\omega, \epsilon}, q_{\eta, \epsilon} \in (A_{N^2}^{\lambda})^*$ are given by $p_{\omega, \epsilon}(x_{ij}) = \delta_{ij}(-1)^{\epsilon(i-1)}{\omega}, q_{\eta, \epsilon}(x_{ij}) = \delta_{i,j+1}(-1)^{\epsilon(i-1)}{\eta}$, and products between them are given by $p_{\omega, \epsilon}p_{\omega', \epsilon'} = p_{\omega'}, q_{\eta, \epsilon}q_{\eta', \epsilon'} = p_{\lambda \eta'}, p_{\omega, \epsilon}q_{\eta, \epsilon'} = q_{\omega \eta, \epsilon + \epsilon'}, q_{\eta, \epsilon}p_{\omega, \epsilon} = q_{(-1)^{\epsilon} \omega \eta, \epsilon + \epsilon'}$, where the indices of the right-hand sides are treated as modulo 2.

**Proposition 3.9.** Suppose that $k$ contains a 4Nth root of unity. Then, $Sph((A_{N^2}^{\lambda})^*) = \{ \varepsilon, p_{-1} \}$, where $\varepsilon$ is the counit of $A_{N^2}^{\lambda}$, and $p_{-1}$ is the algebra map defined by $p_{-1}(x_{ij}) = -\delta_{ij}$ (i.e., $i, j = 1, 2$).

**Proof.** An element $p \in G((A_{N^2}^{\lambda})^*)$ belongs to the center of $(A_{N^2}^{\lambda})^*$ if and only if $p(x_{11}) = p(x_{22}), p(x_{12}) = p(x_{21}) = 0$. Thus, whereas $p_{\omega} \in Z((A_{N^2}^{\lambda})^*), q_{\eta} \notin Z((A_{N^2}^{\lambda})^*)$ since $q_{\eta}(x_{12}) = 0$. Since $(A_{N^2}^{\lambda})^*$ is a Hopf algebra with counit the form $\varepsilon$, it follows that $p_{\omega} = \varepsilon, q_{\eta} = 0$. Therefore, $\varepsilon$ is the unique 4Nth root of unity, and $p_{-1}$ is the algebra map defined by $p_{-1}(x_{ij}) = -\delta_{ij}$.
η ≠ 0. Furthermore, it follows from $p_\omega^2 = p_{\alpha^2}$ that $p_\omega^2 = \varepsilon$. This implies that $\omega^2 = 1$, that is, $\omega = \pm 1$. Since $p_1 = \varepsilon$, it follows that $\text{Sph}(A^{(\nu)}_{NL}) = \{\varepsilon, -\varepsilon\}$. \hfill \qed

**Alternative proof of Theorem 3.8** By Corollary [2,4] and Proposition [3.3] we have $\text{CRib}(A^{\nu}_{NL}, \sigma_{\alpha\beta}) = \{\mathcal{Y}_{\alpha\beta}, p_1 \mathcal{Y}_{\alpha\beta}\}$. Here, $\mathcal{Y}_{\alpha\beta}$ is the Drinfeld element of $(A^{\nu}_{NL}, \sigma_{\alpha\beta})$, and it is given by $\mathcal{Y}_{\alpha\beta}(x_{ij}) = \delta_{ij} \beta^{-1}$ by Lemma [3.3(1)]. Thus, (1) is proved. Similarly, it can be shown that $\text{CRib}(A^{\nu}_{NL}, \mathcal{Y}^*_{\gamma\delta}) = \{\mathcal{Y}^*_{\gamma\delta}, p_1 \mathcal{Y}^*_{\gamma\delta}\}$, and hence (2) is also proved. \hfill \qed

4 Polynomial invariants for duals of Suzuki’s braided Hopf algebras

In this section we assume that $N \geq 1$, $L \geq 2$, $\lambda, \nu = \pm 1$, and $k$ is an algebraically closed field which contains a $4NL$th root of unity. We also assume that $\alpha, \beta \in k^\times$ satisfy $(\alpha \beta)^N = \nu$, $(\alpha \beta^{-1})^L = \lambda$, and $\gamma, \delta \in k^\times$ satisfy $\gamma^2 = \delta^2$, $\gamma^{2N} = 1$.

By Lemma [2.11] we have:

**Lemma 4.1.** (1) Let us consider the coribbon elements $\mathcal{Y}_{\alpha\beta}$ and $\mathcal{Y}^*_{\alpha\beta} := p_1 \mathcal{Y}_{\alpha\beta}$ of the braided Hopf algebra $(A^{\nu}_{NL}, \sigma_{\alpha\beta})$.

(i) for the simple right $A^{\nu}_{NL}$-comodule $kg$ ($g \in G(A^{\nu}_{NL})$)

$$\xi_{\mathcal{Y}_{\alpha\beta}}(kg) = \dim_{\alpha\beta} kg = \begin{cases} \xi_{\mathcal{Y}_{\alpha\beta}}(kg) = (\alpha \beta)^{-2s^2} & \text{if } g = x_{11}^2 \pm x_{12}^2, \\ (-1)^L \xi_{\mathcal{Y}_{\alpha\beta}}(kg) = (\alpha \beta)^{-2s^2 - 2sL - L^2 \alpha^2} & \text{if } g = x_{11}^{2s+1} x_{22}^{L-1} \pm \sqrt{\lambda} x_{12}^{2s+1} x_{22}^{L-1} \end{cases}$$

(ii) for the simple right $A^{\nu}_{NL}$-comodule $V_{st} = k x_{11}^2 x_{22} + k x_{12}^2 x_{21}$

$$\xi_{\mathcal{Y}_{\alpha\beta}}(V_{st}) = \frac{\dim_{\alpha\beta} V_{st}}{2} = (\alpha \beta)^{-2s^2 - 2sL - \ell^2} \alpha^2 = (-1)^L \xi_{\mathcal{Y}_{\alpha\beta}}(V_{st}).$$

(2) Let us consider the coribbon elements $V_{\gamma\delta}$ and $V^*_{\gamma\delta} := p_1 V_{\gamma\delta}$ of the braided Hopf algebra $(A^{\nu}_{NL}, \sigma_{\gamma\delta})$.

(i) for the simple right $A^{\nu}_{NL}$-comodule $kg$ ($g \in G(A^{\nu}_{NL})$)

$$\xi_{V_{\gamma\delta}}(kg) = \xi_{V^*_{\gamma\delta}}(kg) = \dim_{\gamma\delta} kg = \begin{cases} \gamma^{-4s^2} & \text{if } g = x_{11}^{2s} \pm x_{12}^{2s}, \\ \gamma^{-4(s+1)^2} \lambda & \text{if } g = x_{11}^{2s+1} x_{22} \pm \sqrt{\lambda} x_{12}^{2s+1} x_{21}. \end{cases}$$

(ii) for the simple right $A^{\nu}_{NL}$-comodule $V_{s1} = k x_{11}^2 x_{22} + k x_{12}^2 x_{21}$

$$\xi_{V_{\gamma\delta}}(V_{s1}) = -\xi_{V^*_{\gamma\delta}}(V_{s1}) = \frac{\dim_{\gamma\delta} V_{s1}}{2} = \gamma^{-(2s+1)^2}.$$
needs to remove $\nu^t$ from that formula. So, we obtain the formulas for $\xi_{\alpha,\beta}$ and $\xi_{\gamma,\delta}$ in the proposition. Other equations can be derived as follows.

(1) (i) First, we note that $\text{ch}(kg) = g$ for $g \in G(A_N^{1/2})$. By Lemma 2.11(2), if $g = x_{11}^{2k} \pm x_{12}^{2s}$, then $\xi_{\alpha,\beta}(kg) = \xi_{\alpha,\beta}(g) = p-1(g) \xi_{\alpha,\beta}(g) = \xi_{\alpha,\beta}(kg)$, and similarly if $g = x_{11}^{2s+1}L^{-1} \pm \sqrt{x_{12}^{2s+1}L^{-1}}$, then $\xi_{\alpha,\beta}(kg) = (-1)^t \xi_{\alpha,\beta}(kg)$.

(ii) Since $\text{ch}(kg) = x_{11}^{2k} \pm x_{12}^{2s}$, we have the equations of (2).

By a similar computation we have the equations of (2).

By Lemma 2.11(2) we have:

**Theorem 4.2.** (1) For $i \in \{0,1,\ldots,N-1\}$ and $j \in \{0,1,\ldots,L-1\}$, set

$$
\alpha_{ij} := \pm \omega^{L(2i+1)\frac{1+\lambda}{2}} N(2j+1+\lambda) = \pm \omega^{L(2i+1)\frac{1+\lambda}{2}} = \pm \omega^{L(2i+1)\frac{1+\lambda}{2}}.
$$

Then $I_{\nu,\lambda} := \{ (\alpha, \beta) \in k \times k | (\alpha, \beta)^N = \nu, (\alpha\beta)^{-1}L = \lambda \}$ is represented as $I_{\nu,\lambda} = \{ (\alpha_{ij}+, \beta_{ij}+), (\alpha_{ij}-, \beta_{ij}-) | i = 0,1,\ldots,N-1, j = 0,1,\ldots,L-1 \}$, and by setting $\epsilon_P = \epsilon_Q = 1, \epsilon_P = (-1)^t$ and $X = P, Q$ we have:

$$
X_{(1)}^{(A_N^{1/2})^*} \sigma_{(1)\alpha_{ij}, (1)\beta_{ij}}(x) = \prod_{s=1}^{N} (x - \omega^{-4s^2}L(2i+1+\lambda))
$$

$$
\cdot \prod_{s=1}^{N} (x - \epsilon_X (\pm 1)^{L(2s+t)})(x - \omega^{-2(t+1)L}N(2j+1+\lambda)),
$$

$$
X_{(2)}^{(A_N^{1/2})^*} \sigma_{(2)\alpha_{ij}, (2)\beta_{ij}}(x) = \prod_{s=0}^{N-1} \prod_{t=1}^{L-1} (x - \epsilon_X (\pm 1)^{L(2s+t)})(x - \omega^{-2(t+1)L}N(2j+1+\lambda)),
$$

(2) For $i \in \{0,1,\ldots,2N-1\}$, define $\gamma_{i,\pm}, \delta_{i,\pm} \in k^\times$ by $\gamma_{i,\pm} = (\omega^{4i}, \pm \omega^{4i})$. Then $J := \{ (\gamma, \delta) \in k \times k | \gamma^2 = \delta^2, \gamma^{2N} = 1 \}$ is represented as $J = \{ (\gamma_{i}, \delta_{i}), (\gamma_{i}, \delta_{i}) | i = 0,1,\ldots,2N-1 \}$, and

$$
P_{(1)}^{(A_N^{1/2})^*} (x) = Q_{(A_N^{1/2})^*}^{(1)} \sigma_{(1)\alpha_{ij}, (1)\beta_{ij}}(x) = \prod_{s=1}^{N} (x - \omega^{-16s^2})(x - \omega^{-16(s+1)^2}),
$$

$$
P_{(2)}^{(A_N^{1/2})^*} (x) = \prod_{s=0}^{N-1} (x - \omega^{-4s(2s+1)^2}),
$$

$$
Q_{(A_N^{1/2})^*}^{(2)} (x) = \prod_{s=0}^{N-1} (x + \omega^{-4s(2s+1)^2}).
$$
Example 4.3. Let $\omega \in k$ be a primitive 8th root of unity. For a braiding $\sigma$ of $H_8$ we set $P^{(2)}_\sigma(x) := f^{(2)}_{H_8,\sigma}(x)$. Then

$$P^{(2)}_{\omega^{\pm}}(x) = x \mp \omega,$$

$$P^{(2)}_{\omega^{3},\pm}(x) = x \mp \omega^3,$$

$$P^{(2)}_{\tau_{1,\pm}}(x) = x - 1,$$

$$P^{(2)}_{\tau_{1,\pm}^{-1}}(x) = x + 1.$$

It follows that all pairs of $(H_8,\sigma_{\omega,\omega^{-1}})$, $(H_8,\sigma_{\omega^{-1},\omega})$, $(H_8,\sigma_{\omega^3,\omega^{-3}})$, $(H_8,\tau_{1,1})$, $(H_8,\tau_{1,1}^{-1})$ are not braided Morita equivalent.

5 The braided Morita equivalence classes of $H_8$

In this section we compute the automorphism group of the 8-dimensional Kac-Paljutkin algebra $H_8$, and determine its braided Morita equivalence classes.

Since the finite group $G_{12}$ is isomorphic to the dihedral group $D_8 = \langle t, w | t^2 = w^4 = 1, tw = w^{-1}t \rangle$ of order 8, it follows that $H_8$ is isomorphic to the group algebra $kD_8$ as an algebra. An algebra isomorphism $\varphi : kD_8 \rightarrow H_8$ is given by $\varphi(t) = x_{12} + x_{22}$, $\varphi(w) = x_{11}x_{22} - x_{21}x_{12}$. The induced Hopf algebra structure of $kD_8$ that $\varphi$ is a Hopf algebra map is as follows [24].

\[
\begin{align*}
\Delta(t) &= w^{-1}t \otimes e_1t + t \otimes e_0t, \\
\varepsilon(t) &= 1, \\
S(t) &= (e_0 - e_1w)t, \quad S(w) = w,
\end{align*}
\]

where $e_0 := \frac{1-w^2}{2}$, $e_1 := \frac{1-w^2}{2}$. They are central orthogonal idempotents, and satisfy $\Delta(e_0) = e_0 \otimes e_0 + e_1 \otimes e_1$, $\Delta(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0$. Via the map $\varphi$ we identify $H_8 = kD_8$. Then

\[
\begin{align*}
x_{11} &= \frac{w + w^{-1}}{2}t, \\
x_{12} &= \frac{1 - w^2}{2}t, \\
x_{21} &= \frac{w^{-1} - w}{2}t, \\
x_{22} &= \frac{1 + w^2}{2}t.
\end{align*}
\]

By [33] we see that the group-like elements of $H_8$ are given by

\[G(H_8) = \{ 1, w^2, w(e_0 + \sqrt{-1}e_1), w(e_0 - \sqrt{-1}e_1) \} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.
\]

We set $a := w^2$ and $b := w(e_0 + \sqrt{-1}e_1)$.

Let $f$ be a Hopf algebra automorphism of $H_8$. Then we see that

\[
\begin{align*}
f(e_i) &= \frac{1}{2}(1 + (-1)^if(a)) \quad (i = 0, 1), \\
f(w^{\pm 1}) &= \frac{1 \mp \sqrt{-1}}{2}f(b) + \frac{1 \pm \sqrt{-1}}{2}f(ab), \\
f(e_0 - e_1w) &= \frac{1}{2}(1 + f(a)) + \frac{\sqrt{-1}}{2}(f(b) - f(ab)).
\end{align*}
\]

Now, we write $x := f(t) \in H_8$ as $x = \sum_{i,j=0,1} a_{ij}w^ie_j + \sum_{i,j=0,1} b_{ij}w^ie_jt \quad (a_{ij}, b_{ij} \in k)$. Then,

\[\varepsilon(x) = 1 \iff a_{00} + a_{10} + b_{00} + b_{10} = 1,
\]
Lemma 5.1. If \( f \) is a Hopf algebra automorphism on \( H_8 \), then \( f \) is one of the Hopf algebra automorphisms \( \text{id}_{H_8}, f_+, f_- \) where \( f_\pm \) are defined by \( f_\pm(w) = w^{-1}, f_\pm(t) = w^{\pm 1} t \). Therefore, the group \( \text{Aut}(H_8) \) of the Hopf algebra automorphisms is
\[
\text{Aut}(H_8) = \{ \text{id}_{H_8}, f_+, f_- \} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.
\]

Since \( f_\pm(x_{11}) = x_{22}, f_+(x_{12}) = -x_{12}, f_+(x_{21}) = -x_{21}, f_+(x_{22}) = x_{11} \), it follows that \( \tau_{1, \pm 1} \circ (f_+ \otimes f_+) = \tau_{1, \mp 1} \), and this implies the following result.

Corollary 5.2. As braided Hopf algebras \((H_8, \tau_{1,1}) \cong (H_8, \tau_{1,-1})\), \((H_8, \tau_{-1,1}) \cong (H_8, \tau_{-1,-1})\). In particular, there are isomorphisms \((H_8, \tau_{1,1})M \cong (H_8, \tau_{1,-1})M\) and \((H_8, \tau_{-1,1})M \cong (H_8, \tau_{-1,-1})M\) as \( k \)-linear braided monoidal categories.

By Corollary 4.2 and Example 4.3 we have:

Theorem 5.3. Let \( k \) be an algebraically closed field whose characteristic is not 2. For two braidings \( \sigma, \sigma' \) of the 8-dimensional Kac-Paljutkin algebra \( H_8 \) over \( k \), the braided Hopf algebras \((H_8, \sigma)\) and \((H_8, \sigma')\) are braided Morita equivalent if and only if one of the following is satisfied:

1. \( \sigma = \sigma' \),
2. \( \{ \sigma, \sigma' \} = \{ \tau_{1,1}, \tau_{1,-1} \} \),
3. \( \{ \sigma, \sigma' \} = \{ \tau_{-1,1}, \tau_{-1,-1} \} \).

Therefore, there are exactly 6 braided Morita equivalence classes for \( H_8 \).

Appendix: List of corrigenda in [23] with correct statements.

- p.333 in the abstract and p.334, l.6-7; the following sentence should be deleted:

  As a consequence, we see that such a Hopf algebra has a coribbon structure if and only if it is of Kac-Paljutkin type (see Theorem 5).

- p.339, the statements of Theorem 5 should be changed as follows.

  THEOREM 5. (1) The set of coribbon elements of the braided Hopf algebra \((A_{NL}^0, \sigma_{\alpha\beta})\) is \{ \( \theta_{\beta-1}, \theta_{-\beta-1} \) \}. 

16
(2) The set of coribbon elements of the braided Hopf algebra \( (A_{N\lambda}^N, \tau_{\alpha\beta}^N) \) is \( \{ \theta_{\beta^{-1}}, \theta_{\alpha^{-1}} \} \).

Here, \( \theta_{\pm\beta^{-1}} \) are the elements of \( (A_{N\lambda}^N)^* \), that are determined by the condition (iii) in Definition 3 and the equations \( \theta_{\pm\beta^{-1}}(x_{ij}) = \pm \delta_{ij} \beta^{-1} \) \( (i, j = 1, 2) \).

- p.340, the conclusion part of Lemma 8 (1) should be changed as follows: Then, \( \theta_\omega \) induces a coribbon element of the bialgebra \( (A_{N\lambda}^N, \sigma_{\alpha\beta}) \) if and only if \( \omega^{2N} = \alpha^{2N} \).

- p.340, the symbols \( \chi_1^L, \chi_2^L, \eta_1^L, \eta_2^L \) should be replaced by \( \chi_{11}^L, \chi_{22}^L, \chi_{12}^L, \chi_{21}^L \), respectively, and \( \text{(3.1)} \) should be added.

- p.341, the parts from the fourth line to the 16th line should be modified as follows:

If \( m \) is even, then
\[
\sigma_{\alpha\beta}^{-1}(x_{12}^{m-1}, x_{12}) = \sigma_{\alpha\beta}^{-1}(x_{21}, x_{21}^{m-1}) = \alpha^{-\frac{m}{2}} \beta^{-(m-1)}, \\
\sigma_{\alpha\beta}^{-1}(x_{12}^{m-1}, x_{11}) = \sigma_{\alpha\beta}^{-1}(x_{11}, x_{21}^{m-1}) = 0.
\]

If \( m \geq 3 \) is odd, then
\[
\sigma_{\alpha\beta}^{-1}(x_{21}^{m-1}, x_{21}) = \sigma_{\alpha\beta}^{-1}(x_{12}, x_{12}^{m-1}) = 0, \\
\sigma_{\alpha\beta}^{-1}(x_{21}^{m-1}, x_{22}) = \sigma_{\alpha\beta}^{-1}(x_{22}, x_{12}^{m-1}) = \alpha^{-\frac{m+1}{2}} \beta^{-\frac{m+1}{2}}.
\]

Hence, if \( m \) is even, then
\[
\theta_\omega(x_{11}^{m}) = \sigma_{\alpha\beta}^{-1}(x_{12}^{m-1}, x_{12}) \theta_\omega(x_{22}^{m-1}) \theta_\omega(x_{22}) \sigma_{\alpha\beta}^{-1}(x_{21}, x_{21}^{m-1}) \\
= \omega \tilde{\theta}_\omega(x_{22}^{m-1}) (\alpha^{-1})^m (\beta^{-1})^{m-2},
\]

and if \( m \) is odd, then
\[
\theta_\omega(x_{22}^{m}) = \sigma_{\alpha\beta}^{-1}(x_{11}^{m-1}, x_{21}) \theta_\omega(x_{11}^{m-1}) \theta_\omega(x_{11}) \sigma_{\alpha\beta}^{-1}(x_{12}, x_{12}^{m-1}) \\
= \omega \tilde{\theta}_\omega(x_{11}^{m-1}) (\alpha^{-1})^m (\beta^{-1})^{m-1}.
\]

Thus, we have
\[
\theta_\omega(x_{11}^{2N} + \nu x_{12}^{2N}) = \omega^{2N} \beta^{-(2N-2)} \theta_\omega(x_{22}^{2N-1}) \\
= \omega^{2N} \alpha^{-1} (2N+(2N-2)(\beta^{-1}(2N-2)) \theta_\omega(x_{11}^{2N-1}) \\
= \cdots \cdots \\
= \omega^{2N} \alpha^{-2N}.
\]

It follows that
\[
(i) \iff \omega^{2N} = \alpha^{2N}.
\]

- p.342, the equations in the 14th and 16th lines should be modified as follows, respectively:
\[
\tilde{\theta}_\omega(x_{ij}^{2}) = (\tau_{\alpha\beta}^N)^{-1}(x_{ii}, x_{ii}) \tilde{\theta}_\omega(x_{ij}) \tilde{\theta}_\omega(x_{ij}) (\tau_{\alpha\beta}^N)^{-1}(x_{jj}, x_{jj}) = \tilde{\theta}_\omega(x_{ij})^2 \beta^{-2},
\]
\[
(\tau_{\alpha\beta}^N)^{-1}(x_{jj}^{m-1}, x_{jj}) = (\tau_{\alpha\beta}^N)^{-1}(x_{jj}, x_{jj}^{m-1}) = \alpha^{-(m-1)}
\]
p.342, the equation \(= \omega^{2(m-1)}\theta_\omega(x_{m-1}^m)\theta_\omega(x_{m-1}^m)\) in the 19th line should be modified as \(= \omega^{2(m-1)}\theta_\omega(x_{m-1}^m)\), and the equation in the 21st line should be modified as \(= \omega^{2(m-1)}\theta_\omega(x_{m-1}^m)\).

p.342, the equations in the 25th and 27th lines should be modified as follows, respectively:

\[
\theta_\omega(x_{11}^m) = \theta_\omega(x_{22}^m) = \omega^m \alpha^{-2(m-1)} \theta_\omega(x_{11}^m) = \omega^m \alpha^{-m(m-1)}.
\]

p.342, in the fourth line from the bottom the sentence \((A_{N_L}^{\alpha}, \sigma_{\alpha\beta})\) should be modified as \((A_{N_L}^{\alpha}, \sigma_{\alpha\beta})\) as a braided bialgebra.

p.343, 1.9; the statement “By Lemma 8, it follows that \(N = 1\) and \(\omega = \pm \alpha\)” should be corrected as follows: By Lemma 8, it follows that \(\omega^{2N} = \alpha^{2N}\). Since \(\theta_\omega(S(x_{ij})) = \delta_{ij}\omega^{-1}\beta^{-2}\), the condition \(\theta_\omega \circ S = \theta_\omega\) implies \(\omega = \pm \beta^{-1}\).

References

[1] C. Călinescu, S. Dăscălescu, A. Masuoka and C. Menini, Quantum lines over non-cocommutative cosemisimple Hopf algebras, J. Algebra 273 (2004), 753–779.

[2] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra 21 (1993), 1731–1749.

[3] V.G. Drinfel’d, Quantum groups. In Proceedings of the International Congress of Mathematics, Berkeley, CA., 1987, 798–820.

[4] P. Etingof and S. Gelaki, On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, I.M.R.N. no.16 (1998), 851–864.

[5] S. Gelaki, On the classification of finite-dimensional triangular Hopf algebras, in: ‘New directions in Hopf algebras’ edited by S. Montgomery and H.-J. Schneider, MSRI Publications 43, 2002, 69–116.

[6] C. Goff, G. Mason and S.-H. Ng, On the gauge equivalence of twisted quantum doubles of elementary abelian and extra-special 2-groups, J. Algebra 312 (2007), 849–875.

[7] T. Hayashi, Quantum groups and quantum determinants, J. Algebra 152 (1992), 146–165.

[8] T. Hayashi, Coribbon Hopf (face) algebras generated by lattice models, J. Algebra 233 (2000), 614–641.

[9] G.I. Kac and V.G. Paljutkin, Finite ring groups, in: Transactions of the Moscow Mathematical Society for the year 1966, AMS, 1967, 251–294 (original Russian paper: Trudy Moscov. Math. Ob. 15 (1966), 224–261).
[10] C. Kassel, *Quantum Groups*, G.T.M. 155, Springer-Verlag, New York, 1995.

[11] L.H. Kauffman, *Gauss codes, quantum groups and ribbon Hopf algebras*, Reviews in Math. Phys. 5 (1993), 735–773.

[12] A.A. Kirillov, Jr., *On an inner product in modular tensor categories*, J. Amer. Math. Soc. 9 (1996), 1135–1169.

[13] S. Majid, *Representation-theoretic rank and double Hopf algebras*, Comm. Algebra 18 (1990), 3705–3712.

[14] A. Masuoka, *Semisimple Hopf algebras of dimension 6, 8*, Israel J. Math. 92 (1995), 361–373.

[15] A. Masuoka, *Cocycle deformations and Galois objects for some cosemisimple Hopf algebras of finite dimension*, Contemp. Math. 267 (2000), 195–214.

[16] S. Montgomery, *Hopf algebras and their action on rings*, C.B.M.S. 82, American Mathematical Society, 1993.

[17] D. Naidu and D. Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, Commun. Math. Phys. 279 (2008), 845–872.

[18] D.E. Radford, *On the antipode of a quasitriangular Hopf algebra*, J. Algebra 151 (1992), 1–11.

[19] N.Yu. Reshetikhin and V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. 127 (1990), 1–26.

[20] Y. Sommerhäuser, *Remarks on ‘M. Wakui: The coribbon structures of some finite dimensional braided Hopf algebras generated by 2 × 2-matrix coalgebras’*, a private note, 2009.

[21] S. Suzuki, *A family of braided cosemisimple Hopf algebras of finite dimension*, Tsukuba J. Math. 22 (1998), 1–29.

[22] F. Van Oystaeyen and Y. Zhang, *The Brauer group of a braided monoidal category*, J. Algebra 202 (1998), 96–128.

[23] M. Wakui, *The coribbon structures of some finite dimensional braided Hopf algebras generated by 2 × 2-matrix coalgebras*, Banach Center Publ. 61, Noncommutative geometry and quantum groups, 2003, 333–344.

[24] M. Wakui, *Polynomial invariants for a semisimple and cosemisimple Hopf algebra of finite dimension*, J. Pure Appl. Algebra 214 (2010), 701–728.

[25] D.N. Yetter, *Framed tangles and a theorem of Deligne on braided deformations of Tannakian categories*, Contemp. Math. 134 (1992), 325–349.