Average of Hardy’s function at Gram points

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Abstract

Let
\[ Z(t) = \chi^{-1/2}(1/2 + it)\zeta(1/2 + it) = e^{i\theta(t)}\zeta(1/2 + it) \]
be Hardy’s function and \( g(n) \) be the \( n \)-th Gram points defined by \( \theta(g(n)) = \pi n \). Titchmarsh proved that
\[
\sum_{n \leq N} Z(g(2n)) = 2N + O(N^{3/4} \log^{3/4} N)
\]
and
\[
\sum_{n \leq N} Z(g(2n + 1)) = -2N + O(N^{3/4} \log^{3/4} N).
\]
We shall improve the error terms to \( O(N^{1/4} \log^{3/4} N \log \log N) \).

1 Introduction

Let \( s = \sigma + it \) be a complex variable and let \( \zeta(s) \) be the Riemann zeta-function. It satisfies the functional equation
\[
\zeta(s) = \chi(s)\zeta(1 - s),
\]
where
\[
\chi(s) = 2^s\pi^{s-1/2} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}
\]
and the asymptotic behavior of \( \chi(s) \) is given by
\[
\chi(s) = \left(\frac{|t|}{2\pi}\right)^{1/2 - \sigma - it} e^{\frac{it(1+\pi)}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right), \quad (|t| \geq 1)
\]
with \( t \pm \frac{\pi}{4} = t + \text{sgn}(t)\frac{\pi}{4} \) (see Ivić [2] (1.25)). Hardy’s function \( Z(t) \) is defined by
\[
Z(t) = \chi^{-1/2}\left(\frac{1}{2} + it\right) \zeta\left(\frac{1}{2} + it\right),
\]
From (1.1), (1.2) and (1.4), it follows that $Z(t)$ is a real-valued even function for real $t$ and $|Z(t)| = |\zeta(1/2 + it)|$. Thus the real zeros of $Z(t)$ coincide with the zeros of $\zeta(s)$ on the critical line $\text{Re} \ s = 1/2$. Furthermore we have an equivalent expression of (1.4)

$$Z(t) = e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right),$$

where

(1.5)  
$$\theta(t) = -\frac{1}{2i} \log \chi \left( \frac{1}{2} + it \right) = \text{Im} \left( \log \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) \right) - \frac{t}{2} \log \pi \in \mathbb{R}.$$  

It is well-known that the function $\theta(t)$ is strictly monotonic increasing for $t \geq 6.5$. (cf. (2.4) below.)

Historically, Hardy proved the infinity of numbers of zeros of $\zeta(s)$ on the critical line in 1914. A little later Hardy and Littlewood gave another proof by showing that $\int_0^T |Z(t)| dt \ll T^{7/8}$ and $\int_0^T |Z(t)| dt \gg T$ (see Chandrasekharan [3] Chapter II, § 4 and Notes on Chapter II or Titchmarsh [12, 10.5]).

It must be mentioned that the mean value estimate of $Z(t)$ was improved extensively to $\int_0^T Z(t) dt \ll T^{1/4+\varepsilon}$ by Ivić [6] in 2004, where $\varepsilon$ is an arbitrary small positive number. See Ivić’s monograph [7] for the recent development of the theory of Hardy’s function.

Before Hardy, Gram calculated zeros of $\zeta(1/2 + it)$ and observed that the points $t$ such that $\text{Re} \ \zeta(1/2 + it) \in \mathbb{R}$ and the zeros of $\zeta(1/2 + it)$ are distributed alternately. There is also tendency that $\text{Re} \ \zeta(1/2 + it)$ takes positive values and $\text{Im} \ \zeta(1/2 + it)$ takes positive and negative values regularly. See e.g. the graphs of $\zeta(1/2 + it)$ in Akiyama and Tanigawa [1].

For $n \geq -1$, let $g(n) > 7$ be the $n$-th Gram point defined by

$$\theta(g(n)) = \pi n.$$  

Obviously

$$\zeta \left( \frac{1}{2} + ig(n) \right) = (-1)^n Z(g(n)).$$

Gram’s law is stated that there exists a zero of $Z(t)$ for some $t \in [g(n), g(n+1)]$. The first twelve Gram points are (Haselgrove and Miller [4])

- $g(-1) = 9.6$,  
- $g(0) = 17.8$,  
- $g(1) = 23.1$,  
- $g(2) = 27.6$,  
- $g(3) = 31.7$,  
- $g(4) = 35.4$,  
- $g(5) = 38.9$,  
- $g(6) = 42.3$,  
- $g(7) = 45.5$,  
- $g(8) = 48.7$,  
- $g(9) = 51.7$,  
- $g(10) = 54.7$.  

At present it is known that there is a positive proportion of failures of Gram’s law (see e.g. Trudgian [13] and Ivić [7, p. 112]).
As for the distribution of $Z(g(n))$ on the average, Titchmarsh showed that, for a fixed large integer $M$,

\[(1.6) \sum_{M+1}^{N} Z(g(2n)) = 2(N - M) + O(N^{3/4} \log^{3/4} N),\]

\[(1.7) \sum_{M+1}^{N} Z(g(2n + 1)) = -2(N - M) + O(N^{3/4} \log^{3/4} N)\]

[12] 10.6, where he used the approximation

\[(1.8) Z(g(n)) = 2(-1)^{n} \sum_{m \leq \sqrt{g(n)/2\pi}} m^{-1/2} \cos(g(n) \log m) + O(g(n)^{-1/4})\]

obtained by the classical approximate functional equation of the Riemann zeta-function due to Hardy and Littlewood. In Ivić [7, Theorem 6.5], the error terms of (1.6) and (1.7) are improved to $O(N^{3/4} \log^{1/4} N)$. In [11] Titchmarsh also proved

\[(1.9) \sum_{\nu=M+1}^{N} \left( \frac{\sqrt{g(\nu)/2\pi}}{\nu} \cos(g(\nu) \log n) \right) = N + O(N^{1/4} \log^{1/4} N),\]

\[\sum_{n \leq N} Z(g(n))Z(g(n + 1)) = -2(\gamma + 1)N + o(N),\]

where $\gamma$ is Euler’s constant and conjectured

\[\sum_{n \leq N} Z(g(n))^2 Z(g(n + 1))^2 \ll N \log^A N\]

with some positive constant $A$. This conjecture was proved by Moser in [10] (See Ivić [7, Notes of Chapter 6]).

The purpose of the present paper is to show the following theorem.

**Theorem 1.** We have

\[(1.10) \sum_{n \leq N} Z(g(2n)) = 2N + O\left( N^{1/4} \log^{3/4} N \log \log N \right) ,\]

\[(1.11) \sum_{n \leq N} Z(g(2n + 1)) = -2N + O\left( N^{1/4} \log^{3/4} N \log \log N \right) .\]

**Remark 1.** Note that there is not sign $(-1)^n$ in the sum on the left hand side of (1.9). It seems that our Theorem 1 does not follow from (1.9), since when we consider $g(2n)$ and $g(2n + 1)$ separately we may not be able to use the first derivative test directly (cf. Trudgian [13]).

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*Titchmarsh uses the notation $t_\nu$ for the Gram points $g(\nu)$. In [12, Theorem 10.6, p. 263] he asserted that $\sum_{\nu=\nu_0}^{\nu} Z(t_\nu) \sim 2N$ and $\sum_{\nu=\nu_0}^{\nu} Z(t_\nu + 1) \sim -2N$, but in fact he obtained the error terms as in (1.9) and (1.7), see p.264 of [12].*
2 The function $\theta(t)$

As in (1.24)–(1.26) in Ivić [7], the function $\theta(t)$ defined by (1.5) and its derivatives have asymptotic expansions. In particular

\begin{align}
\theta(t) &= \frac{t}{2} \log \frac{t}{2 \pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{7}{48t} + \frac{7}{5760t^3} + O(t^{-5}), \\
\theta'(t) &= \frac{1}{2} \log \frac{t}{2 \pi} + O(t^{-2}), \\
\theta''(t) &= \frac{1}{2} + O(t^{-3}),
\end{align}

(2.1) and

\begin{align}
\theta'''(t) &\ll \frac{1}{t^2}, \quad \theta^{(4)}(t) \ll \frac{1}{t^3}.
\end{align}

For $\theta'(t)$ and $\theta''(t)$ we shall need more precise formulas in the proof Theorem 1.

Lemma 1. For $t \geq 6$ we have

\begin{align}
\theta'(t) &= \frac{1}{2} \log \frac{t}{2 \pi} - \frac{1}{48t^2} + V_1(t) \quad \text{with} \quad |V_1(t)| \leq 0.07 t^{-3}
\end{align}

and

\begin{align}
\theta''(t) &= \frac{1}{2t} + V_2(t) \quad \text{with} \quad |V_2(t)| \leq 0.46 t^{-3}.
\end{align}

Proof. Let $B_n(x)$ be the $n$-th Bernoulli polynomial defined by $\frac{e^x - 1}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n$, \((|t| < 2\pi)\) and $B_n = B_n(0)$ be the $n$-th Bernoulli number. It is well known that for $|\arg z| < |\pi|$,\n
$$
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \left( z - \frac{1}{2} \right) \log 2\pi + \sum_{r=1}^{n} \frac{B_{2r}}{2r(2r-1)} z^{-2r+1}
\quad - (2n)! \int_0^\infty \frac{P_{2n+1}(x)}{(x+z)^{2n+1}} dx,
$$

where $P_n(x)$ is defined by $P_n(x) = \frac{B_n(x-z)}{n!}$, see Wang and Guo [14, p.114 (8)]. Differentiating the above formula we have

\begin{align}
\psi(z) &= \log z - \frac{1}{2z} - \sum_{r=1}^{n} \frac{B_{2r}}{2r} z^{-2r} + (2n+1)! \int_0^\infty \frac{P_{2n+1}(x)}{(x+z)^{2n+2}} dx.
\end{align}

Here $\psi(z)$ denotes the digamma function: $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Let $z = \sigma + it$, \((0 < \sigma < 1 \text{ and } t > 0)\). Then using $|P_m(x)| \leq \frac{4}{(2\pi)^m} \text{ for } m \geq 1$ (14, p. 11...
(15)], we have
\[
\left| \int_0^\infty \frac{P_{2n+1}(x)}{(x + z)^{2n+2}} \, dx \right| \leq \frac{4}{(2\pi)^{2n+1}} \int_0^\infty \frac{1}{((x + \sigma)^2 + t^2)^{n+1}} \, dx \\
\leq \frac{4}{(2\pi)^{2n+1}} \int_0^\infty \frac{1}{(u^2 + 1)^{n+1}} \, du \\
= \frac{4}{(2\pi)^{2n+1}} \frac{(2n - 1)!!}{(2n)!!} t^{-2n-1}.
\]

Take \( n = 1 \) in (2.6), then for \( z = \sigma + it, \) \((0 < \sigma < 1, t > 0)\) we have
\[
\psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + K_1, \quad |K_1| \leq \frac{3}{4\pi^2} t^{-3}.
\]

Now we differentiate both sides of (1.5) and get
\[
\theta'(t) = \frac{1}{4} \left( \psi \left( \frac{1}{4} + \frac{it}{2} \right) + \psi \left( \frac{1}{4} - \frac{it}{2} \right) \right) - \frac{1}{2} \log \pi \\
= \frac{1}{4} \left( \log \left( \frac{1}{16} + \frac{t^2}{4} \right) - \frac{1}{4} \frac{1}{\frac{1}{16} + \frac{t^2}{4}} - \frac{1}{24} \left( \frac{t^2}{2} + \frac{1}{\frac{1}{16} + \frac{t^2}{4}} \right) + K_1 + \tilde{K}_1 \right) - \frac{1}{2} \log \pi.
\]

Since
\[
- \frac{1}{32t^4} < \log \left( \frac{1}{16} + \frac{t^2}{4} \right) - 2 \log \frac{t}{2} - \frac{1}{4t^2} < 0,
\]
\[-\frac{1}{t^4} < \frac{1}{\frac{1}{16} + \frac{t^2}{4}} - \frac{4}{t^2} < 0, \quad 0 < -\frac{t^2 + \frac{3}{4}}{(\frac{1}{16} + \frac{t^2}{4})^2} + \frac{16}{t^2} < \frac{12.4}{t^4},
\]
we have
\[
\theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} - \frac{1}{48t^2} + V_1 \quad \text{with} \quad -0.061 t^{-3} < V_1 < 0.0485 t^{-3}.
\]

This proves the assertion (2.4).

The assertion (2.3) is proved in a similar way.

In connection with the definition of the Gram point \( g(n) \), we define a function \( g(x) \) for real variable \( x \geq -1 \) by
\[
(2.7) \quad \theta(g(x)) = \pi x.
\]
The function \( g(x) \) is uniquely determined and monotonic increasing. When \( x \) is an integer, \( g(x) \) coincides with the definition of Gram points. For the order of \( g(n) \) it is known that
\[
g(n) = \frac{2\pi n}{\log n} \left\{ 1 + \frac{1 + \log \log n}{\log n} + O \left( \frac{(\log \log n)^2}{\log n} \right) \right\}
\]
\[
n = \frac{g(n)}{2\pi} \log g(n) \left\{ 1 - \frac{\log 2\pi e}{\log g(n)} - \frac{\pi}{4g(n) \log g(n)} + O \left( \frac{1}{g(n)^2 \log g(n)} \right) \right\}.
\]
See Ivić [7, Theorem 6.1], also Bruijin [2]. Note that they hold for any positive numbers $n$.

From this definition of $g(x)$, we have
\[
g'(x) = \frac{\pi}{\theta'(g(x))},
g''(x) = -\frac{\pi^2 \theta''(g(x))}{\theta'(g(x))^3},
g'''(x) = -\frac{\pi^3 \theta'''(g(x)) \theta'(g(x)) - 3 \theta''(g(x))^2}{\theta'(g(x))^5},
g^{(4)}(x) = -\frac{\pi^4 \theta^{(4)}(g(x)) \theta'(g(x))^2 - 10 \theta'''(g(x)) \theta''(g(x)) \theta'(g(x)) + 15 \theta''(g(x))^3}{\theta'(g(x))^7}.
\]

3 Some Lemmas

The error term of (1.8) is rather big for our purpose. So we apply an approximation of $Z(t)$ with smooth weight which is due to Ivić [7].

**Lemma 2.** We have
\[
Z(t) = 2 \sum_{m=1}^{\infty} \frac{1}{m^2} \rho \left( m \sqrt{\frac{2\pi}{t}} \right) \cos(\theta(t) - t \log m) + O(t^{-5/6}),
\]
where $\rho(t)$ is a real-valued function such that

(i) $\rho(t) \in C^\infty(0, \infty),$

(ii) $\rho(t) + \rho(1/t) = 1$ for $t > 0,$

(iii) $\rho(t) = 0$ for $t \geq 2.$

**Proof.** This lemma is obtained from the definition of $Z(t)$ and Theorem 4.16 of Ivić [7]. See also (4.81) of [7].

In [7, Lemma 4.15], Ivić constructed a function $\rho(t)$ in more general form in such a way that, instead of (iii), it satisfies $\rho(t) = 0$ for $t \geq b$ for any fixed $b > 1$. But the choice $b = 2$ is sufficient for our purpose. More explicitly it is given as follows with the choices $\alpha = 3/2, \beta = 1/2$ in [7, Lemma 4.15]. Let
\[
\varphi(t) = \exp((t^2 - 1/4)^{-1}) \left\{ \int_{-1/2}^{1/2} \exp((u^2 - 1/4)^{-1}) du \right\}^{-1}
\]
if $|t| < 1/2$ and $\varphi(t) = 0$ if $|t| \geq 1/2,$ and let
\[
f(x) = \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} \varphi(t) dt = \int_{-\infty}^{x} (\varphi(t+3/2) - \varphi(t-3/2)) dt.
\]
The function $f(x)$ is infinitely differentiable in $(-\infty, \infty)$, $f(x) \geq 0$ and

$$f(x) = \begin{cases} 0 & \text{for } |x| \geq 2 \\ 1 & \text{for } |x| \leq 1 \end{cases}$$

(see [7, p.88] for details). Then the function

$$\rho(x) = \frac{1}{2} \left( 1 + f(x) - f \left( \frac{1}{x} \right) \right)$$

satisfies (i), (ii) and (iii) of Lemma 2.

**Lemma 3.** Let $\rho(t)$ be the function as above. When $x \to 0$, we have

$$\rho(1 + x) = \frac{1}{2} + O(|x|^C),$$

where $C$ is any positive large constant.

**Proof.** This property depends on the construction as above.

When $x$ is positive and small, we have

$$f(1 + x) = \int_{-\frac{1}{x}}^{\frac{1}{x}} \varphi(t) dt.$$ 

Using $\int_{-\frac{1}{x}}^{\frac{1}{x}} \varphi(t) dt = 1$ we find that

$$1 - f(1 + x) = \int_{-\frac{1}{x}}^{0} \varphi(t) dt \leq x \max \left\{ \varphi(t) \mid -\frac{1}{2} \leq t \leq -\frac{1}{2} + x \right\} \ll x^C,$$

where $C$ is any positive large constant. On the other hand $f(\frac{1}{1+x}) = 1$. Thus from (3.2) it follows that

$$\rho(1 + x) = \frac{1}{2} + O(x^C)$$

when $x \to 0^+$. It is proved by the same way when $x \to 0^-$. \qed

We need the following two lemmas on exponential sums.

**Lemma 4.** Let $f(x)$ and $\varphi(x)$ be real-valued functions which satisfy the following conditions on the interval $[a, b]$.

1. $f'(x)$ is continuous and monotonic on $[a, b]$ and $|f'(x)| \leq \delta < 1$.
2. $\varphi(x)$ is positive monotonic and $\varphi'(x)$ is continuous, and there exist numbers $0 < H$, $0 < b - a \leq U$ such that

$$\varphi(x) \ll H, \quad \varphi'(x) \ll HU^{-1}.$$
Then we have
\[ \sum_{a < n \leq b} \varphi(n) \exp(2\pi i f(n)) = \int_{a}^{b} \varphi(x) \exp(2\pi i f(x)) \, dx + O \left( \frac{H}{1 - \delta} \right). \]  

Furthermore, if \( 0 < \delta_1 < f'(x) \leq \delta < 1 \) on \([a, b]\), then we have
\[ \sum_{a < n \leq b} \varphi(n) \exp(2\pi i f(n)) \ll \frac{H}{\delta_1} + \frac{H}{1 - \delta}. \]

**Proof.** The assertion (3.4) is obtained by Lemma 1.2 of Ivić [5] and partial summation. The second assertion (3.5) is obtained by applying (2.3) of Ivić [5] (the so-called first derivative test) on the integral of (3.4). See also Karatsuba and Voronin [9, p. 70 Corollary 1].

**Lemma 5** (Karatsuba and Voronin [9, Chapter III, Theorem 1]). Suppose that the real-valued function \( \varphi(x) \) and \( f(x) \) satisfy the following conditions on the interval \([a, b]\):

1. \( f^{(4)}(x) \) and \( \varphi''(x) \) are continuous;
2. there exist numbers \( H, U, A \), \( 0 < H, 1 \ll A \ll U, 0 < b - a \leq U \), such that
   \[ A^{-1} \ll f''(x) \ll A^{-1}, \quad f^{(3)}(x) \ll A^{-1}U^{-1}, \quad f^{(4)}(x) \ll A^{-1}U^{-2} \]
   \[ \varphi(x) \ll H, \quad \varphi'(x) \ll HU^{-1}, \quad \varphi''(x) \ll HU^{-2}. \]

Suppose that the numbers \( x_n \) are determined from the equation
\[ f'(x_n) = n. \]

Then we have
\[ \sum_{a < x \leq b} \varphi(x) \exp(2\pi i f(x)) = \sum_{f'(a) \leq n \leq f'(b)} c(n)W(n) + R, \]
where
\[ R = O \left( H(A(b - a)^{-1} + T_a + T_b + \log(f'(b) - f'(a) + 2)) \right); \]
\[ T_\mu = \begin{cases} 0, & \text{if } f'(\mu) \text{ is an integer}, \\ \min \left( \| f'(\mu) \|^{-1}, \sqrt{A} \right), & \text{if } \| f'(\mu) \| \neq 0; \end{cases} \]
\[ c(n) = \begin{cases} 1, & \text{if } f'(a) < n < f'(b), \\ 1/2, & \text{if } n = f'(a) \text{ or } n = f'(b); \end{cases} \]
\[ W(n) = \frac{1 + i}{\sqrt{2}} \frac{\varphi(x_n)}{\sqrt{f''(x_n)}} \exp(2\pi i(f(x_n) - nx_n)). \]
Remark 2. For the assertion of Lemma 5, the condition $A \ll U$ in (2) is not necessary. See also Jia [8, Lemma 5].

Remark 3. When $f''(x)$ is negative and $A^{-1} \ll -f''(x) \ll A^{-1}$, the sum on the right hand side of (3.6) should be replaced by

$$\sum_{f'(b) \leq n \leq f'(a)} c(n)W(n),$$

where, instead of (3.9), $W(n)$ is given by

$$W(n) = \frac{1 - i}{\sqrt{2}} \frac{\varphi(x_n)}{\sqrt{|f''(x_n)|}} \exp(2\pi i(f(x_n) - nx_n)).$$

4 Proof of Theorem 1

Instead of (1.8) we shall use the expression of $Z(t)$ containing a smooth weight $\rho(t)$. This is because that the error term in (1.8) is too big for our purpose. If we use (3.1) of Lemma 2 we get

$$Z(g(n)) = 2(-1)^n \sum_{m=1}^{\infty} m^{-1/2} \rho \left( m \sqrt{\frac{2\pi}{g(n)}} \right) \cos(g(n) \log m) + O(g(n)^{-5/6}).$$

As for the sum of $Z(g(n))$ we consider the sum over even $n$ and odd $n$ separately. First we consider the even $n$ case. From (4.1) it follows that

$$\sum_{0 \leq n \leq N} Z(g(2n)) = 2 \sum_{m=1}^{\infty} \frac{1}{m^{1/2}} \sum_{0 \leq n \leq N} \rho \left( m \sqrt{\frac{2\pi}{g(2n)}} \right) \cos(g(2n) \log m)$$

$$+ O(N^{1/6} \log^{5/6} N).$$

The sum over $m$ is actually a finite sum, in fact $m$ runs over from 1 to $2\sqrt{g(2N)/2\pi}$ and for such $m$, $n$ runs over under the condition $\frac{m^2}{4} \leq g(2n)/2\pi \leq \frac{g(2N)}{2\pi}$. The contribution from $m = 1$ becomes

$$2 \sum_{0 \leq n \leq N} \rho \left( \sqrt{\frac{2\pi}{g(2n)}} \right) = 2 \left( N + \rho \left( \sqrt{\frac{2\pi}{g(0)}} \right) \right)$$

(note that $\sqrt{2\pi/g(0)} = 0.594\ldots$).

To consider the sum from $m \geq 2$, let

$$S_m = \sum_{0 \leq n \leq N} \frac{\rho \left( m \sqrt{\frac{2\pi}{g(2n)}} \right) e^{2\pi i f_m(n)}}{\frac{m^2}{4} \leq g(2n) \leq \frac{g(2N)}{2\pi}},$$
where

\[(4.2)\]
\[f_m(x) = \frac{1}{2\pi} g(2x) \log m.\]

Thus we get

\[\sum_{0 \leq n \leq N} Z(g(2n)) = 2N + \sum_{m=2}^{\infty} \frac{1}{m^{1/2}} (S_m + \bar{S}_m) + O(1).\]

From (2.7) the first and second derivatives of \(f_m(x)\) are given by

\[(4.3)\]
\[f'_m(x) = \frac{\log m}{\theta'(g(2x))}\]

and

\[(4.4)\]
\[f''_m(x) = -\frac{2\pi (\log m) \theta''(g(2x))}{\theta'(g(2x))^3},\]

respectively. Since \(\theta'(t)\) is positive and increasing for \(t \geq 6.5\) by (2.4) and (2.5), \(f'_m(x)\) is positive and decreasing for \(x\) such that \(g(2x) \geq 6.5\).

Let \(M_0 = M_0(m) = m^2/4\) and \(M_j = 32^j M_0\). For \(M \geq M_0\) we put

\[S_m(M) = \sum_{M \leq 2g(2n) < 32M} \rho \left( m \sqrt{\frac{2\pi}{g(2n)}} \right) e^{2\pi i f_m(n)}.\]

Then we get the decomposition

\[S_m = S_m(M_0) + S_m(M_1) + S_m(M_2) + \cdots + S_m(M'_J),\]

where the last sum is taken over the range \(M_j \leq g(2n) \leq g(2N)/2\pi\). \(J\) is the largest integer such that \(32^J M_0 \leq g(2N)/2\pi\) and in fact \(J = O(\log N)\).

Now we consider the sum \(S_m(M_j)\) in more details. For \(m = 2\) and \(M_0 = 1\), \(S_2(1)\) consists of finite Gram points, hence \(S_2(1) = O(1)\).

To treat other cases, let \(h(y)\) be the inverse function of \(g(2x)/2\pi = y\), namely, \(h(y) = x\). We have

\[(4.5)\]
\[\frac{g(2h(y))}{2\pi} = y \quad \text{and} \quad h\left( \frac{g(2x)}{2\pi} \right) = x.\]

So the summation condition of \(S_m(M)\) is converted to

\[h(M) \leq n < h(32M).\]
We also note that \( h(y) \sim \frac{1}{2} \log y \). For the cases other than \( m = 2 \) and \( M_0 = 1 \), \( f'_m(x) \) is positive and decreasing, and from (4.3), (4.5) and (2.4) we have

\[
(4.6) \quad f'_m(h(M)) = \frac{\log m}{\Theta'(g(2h(M)))} = \frac{\log m}{\Theta'(2\pi M)} \log m = \frac{1}{2} \log M - \frac{1}{192\pi^2 M^2} + V_1(2\pi M).
\]

First we consider the sum \( S_m(M_j) \) for \( j \geq 1 \). For \( x \in [h(M_j), h(32M_j)] \), we see that

\[
0 < f'_m(h(32M_j)) \leq f'_m(x) \leq f'_m(h(M_j)).
\]

From (4.6) and (2.4) we find that

\[
(4.7) \quad f'_m(h(M_j)) = \frac{\log m}{\frac{1}{2} \log(8M_{j-1}) + \log 2 - \frac{1}{192\pi^2 M_j^2} + V_1(2\pi M_j)} < \frac{\log m}{\frac{1}{2} \log(8M_{j-1})} < 1
\]

and

\[
(4.8) \quad f'_m(h(32M_j)) = \frac{\log m}{\frac{1}{2} \log(32M_j) - \frac{1}{192\pi^2 (32M_j)^2} + V_1(64\pi M_j)} \geq \frac{2 \log m}{\log 32M_j}.
\]

Here we have used the inequalities

\[
\log 2 - \left( \frac{1}{192\pi^2 M_j^2} + |V_1(2\pi M_j)| \right) > 0
\]

and

\[
\frac{1}{192\pi^2 (32M_j)^2} - |V_1(64\pi M_j)| > 0
\]

for \( m \geq 2 \) and \( j \geq 1 \). Since the conditions of Lemma 4 are satisfied we can apply (4.5) to the sum \( S_m(M_j) \) with \( \delta = \frac{2 \log m}{\log 8M_{j-1}} \) and \( \delta_1 = \frac{2 \log m}{\log 32M_j} \). As a result we get

\[
S_m(M_j) \ll \frac{\log 32M_j}{2 \log m} + \frac{\log(M_j/4)}{(5j - 4) \log 2} \ll \frac{j}{\log m} + \frac{\log m}{j} + 1
\]

for \( j \geq 1 \). Hence putting \( l_N = \left[ 2\sqrt{g(2N)/2\pi} \right] \), the contribution of these terms to the sum \( \sum_{2 \leq m \leq l_N} m^{-1/2} S_m \) becomes

\[
(4.9) \quad \sum_{2 \leq m \leq l_N} \frac{1}{\sqrt{m}} \sum_{1 \leq j \leq J} S_m(M_j) \ll \sum_{2 \leq m \leq l_N} \frac{1}{\sqrt{m}} \left( \frac{\log^2 N}{\log m} + \log N + \log m \cdot \log \log N \right) \ll N^{1/4} \log^{3/4} N \log \log N.
\]
For \( S_m(M_0) \) \((m \geq 3)\), we apply Lemma 5 with \( \varphi(x) = \rho(m, \sqrt{2\pi/g(2x)}) \) and \( f(x) = f_m(x) \) for \( x \in [h(32M_0), h(M_0)] \). For this we have to check the assumptions in Lemma 5. By (4.3), (2.2) and (2.3) we have \( |f''(x)| \lesssim \frac{\log m}{M_0 \log^4 M_0} \). Furthermore we have easily that 

\[
|f'''(x)| \ll \frac{\log m}{M_0 \log^5 M_0}, \quad |f^{(4)}(x)| \ll \frac{\log m}{M_0 \log^5 M_0}.
\]

Hence we can take 

\[
A = \frac{M_0 \log^3 M_0}{\log m}, \quad U = M_0 \log M_0.
\]

As we remarked in Remark 2, the assertion of Lemma 5 is valid though the condition \( A \ll U \) is not satisfied. On the other hand, we have 

\[
\varphi'(x) = -\rho' \left( m \sqrt{\frac{2\pi}{g(2x)}} \right) \sqrt{2\pi/m} g(2x)^{-3/2} g'(2x).
\]

Since \( g(2x) \approx M_0 \), we have \( \varphi'(x) \ll \frac{1}{M_0 \log M_0} = U^{-1} \). Similarly we see that \( \varphi''(x) \ll U^{-2} \) holds. (Here we can take \( H = c \max\{\rho(x), |\rho'(x)|, |\rho''(x)|\} \ll 1 \), where \( c \) is a constant.) Furthermore we have to note that \( f''(x) \) is negative in our case, hence the sum on the right hand side of (3.6) should be taken in the range \( f_m'(h(32M_0)) \leq \nu \leq f_m'(h(M_0)) \). Now by (4.6) we have 

\[
(4.10) \quad f'_m(h(M_0)) = \frac{\log m}{\log m - \left( \log 2 + \frac{1}{12\pi^2 m} - V_1(\frac{\pi m^2}{2}) \right)}.
\]

Similarly as above, if we use (2.4) in Lemma 1 we find that 

\[
2 < f'_m(h(M_0)) < 3 \quad \text{for } m = 3, 4
\]

and 

\[
1 < f'_m(h(M_0)) < 2 \quad \text{for all } m \geq 5.
\]

On the other hand we have already seen in (4.7) that 

\[
f'(h(32M_0)) < \frac{\log m}{\log m + \frac{1}{2} \log 2} < 1.
\]

Thus we get 

\[
S_m(M_0) = W_m(1) + R_m \quad \text{for } m \geq 5,
\]

\[
S_m(M_0) = W_m(1) + W_m(2) + R_m \quad \text{for } m = 3, 4,
\]

where 

\[
W_m(\nu) = \frac{1 - i}{\sqrt{2}} \cdot \frac{\rho \left( m \sqrt{2\pi/g(2x)} \right)}{\sqrt{|f''(x)|}} \exp \left( 2\pi i (f_m(x) - \nu x) \right)
\]

\[(4.11)\]

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(3.10) must be used since \( f''_m(x) \) is negative) and

\[
R_m \ll \log m + T_h(32M_0) + T_h(M_0) + \log(f'_m(h(M_0)) - f'_m(h(32M_0)) + 2)
\]

(see (3.7) and (3.8) for the definitions of \( R_m \) and \( T_h \)). The contribution of \( S_m(M_0) \) to the sum \( \sum_{2 \leq m \leq lN} m^{-1/2} S_m \) is

(4.12)

\[
\sum_{3 \leq m \leq lN} \frac{W_m(1)}{\sqrt{m}} + \sum_{3 \leq m \leq lN} \frac{R_m}{\sqrt{m}} + O(1),
\]

which we shall calculate now.

First we shall determine the explicit form of \( W_m(1) \) given by (4.11). Define \( x_1 \) so as to satisfy \( f'_m(x_1) = 1 \), which is equivalent to \( \theta'(g(2x_1)) = \log m \). Using (2.2) we obtain

(4.13)

\[
\frac{g(2x_1)}{2\pi} \left( 1 + O \left( \frac{1}{g(2x_1)^2} \right) \right) = m^2.
\]

On the other hand, by the definition of \( g(2x) \) we have \( \theta(g(2x_1)) = 2\pi x_1 \), hence

\[
\theta \left( 2\pi m^2 + O \left( \frac{1}{g(2x_1)} \right) \right) = 2\pi x_1.
\]

Using (2.1) on the left hand side of the above formula, we obtain

(4.14)

\[
x_1 = m^2 \log m - \frac{1}{2} m^2 - \frac{1}{16} + O \left( \frac{\log m}{m^2} \right).
\]

Therefore from (4.13) and (4.14) we get

\[
f_m(x_1) - x_1 = \frac{g(2x_1)}{2\pi} \log m - x_1
\]

\[
= \frac{1}{2} m^2 + \frac{1}{16} + O \left( \frac{\log m}{m^2} \right).
\]

Furthermore from (2.2), (2.3) and (4.13) we have

\[
\theta'(g(2x_1)) = \log m + O \left( \frac{1}{m^4} \right)
\]

and

\[
\theta''(g(2x_1)) = \frac{1}{4\pi m^2} + O \left( \frac{1}{m^6} \right).
\]

Hence

\[
f''_m(x_1) = -\frac{2\pi \log m \theta''(g(2x_1)) \theta'(g(2x_1))^3}{\theta'(g(2x_1))^4} = -\frac{1}{2} \left( \frac{1}{m \log m} \right)^2 \left( 1 + O \left( \frac{1}{m^2} \right) \right).
\]
Combining these, we get
\[
W_m(1) = e^{-\frac{\pi}{8}} \rho \left( m \sqrt{\frac{2\pi}{g(2x_1)}} \right) \sqrt{2} m \log m \left( 1 + O \left( \frac{1}{m^4} \right) \right) \\
\times e^{2\pi i \left( \frac{m^2}{2} + \frac{1}{16} + O \left( \frac{\log m}{m^4} \right) \right)} \\
= \sqrt{2} e^{-\frac{\pi}{8}} \rho \left( m \sqrt{\frac{2\pi}{g(2x_1)}} \right) (-1)^m m \log m \left( 1 + O \left( \frac{\log m}{m^2} \right) \right).
\]

Now from (4.13) and (3.3) of Lemma 3, we have
\[
\rho \left( m \sqrt{\frac{2\pi}{g(2x_1)}} \right) = \rho (1 + O(m^{-2})) = \frac{1}{2} + O(m^{-C})
\]
for any large \( C > 0 \). Hence we get
\[
W_m(1) = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{8}} (-1)^m m \log m + O \left( \frac{\log m}{m} \right).
\]

Therefore we find that
\[
\sum_{3 \leq m \leq l_N} W_m(1) = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{8}} \sum_{3 \leq m \leq l_N} (-1)^m m^{1/2} \log m + O(1)
\]
\[
= \frac{1}{2\sqrt{2}} e^{-\frac{\pi}{8}} (-1)^{l_N} l_N^{1/2} \log l_N + O(1)
\]
\[
\ll g(2N)^{1/4} \log g(2N)
\]
\[
\ll N^{1/4} \log^{3/4} N.
\]

The second equality is obtained as follows. Suppose first that \( l_N = 2L \) (even integer). Then we have
\[
U := \sum_{1 \leq m \leq 2L} (-1)^m m^{1/2} \log m
\]
\[
= \sum_{k=1}^{L} \left( (2k)^{1/2} \log 2k - (2k-1)^{1/2} \log (2k-1) \right).
\]

Since
\[
(2k)^{1/2} \log 2k - (2k-1)^{1/2} \log (2k-1)
\]
\[
= \left( \frac{1}{\sqrt{2}} + \log \frac{2k}{\sqrt{2}} \right) \frac{1}{k^{1/2}} + \frac{1}{\sqrt{2}} \frac{\log k}{k^{1/2}} + O \left( \frac{\log k}{k^{3/2}} \right),
\]
\[
\sum_{1 \leq k \leq L} \frac{1}{k^{1/2}} = 2L^{1/2} + O(1)
\]
and
\[ \sum_{1 \leq k \leq L} \frac{\log k}{k^{1/2}} = 2L^{1/2} \log L - 4L^{1/2} + O(1), \]
we get
\[ U = \frac{1}{2}(2L)^{1/2} \log(2L) + O(1). \]
The case \( l_N = 2L + 1 \) (odd integer) is similar.

Next we treat the contribution from \( R_m \) in (4.12). It is enough to consider the sum for \( m \geq 5 \). For such \( m \), we know that \( f'(h(32M_0)) < 1 < f'(h(M_0)) < 2 \) and from (4.8) (which is also true for \( j = 0 \)) and (4.10) we have
\[ T_{f'(h(32M_0))}, T_{f'(h(M_0))} \ll \log m \]
and hence
\[ R_m \ll \log m. \]
Therefore we find that
\[ \sum_{3 \leq m \leq l_N} \frac{R_m}{\sqrt{m}} \ll N^{1/4} \log^{3/4} N. \]
From (1.9), (1.15) and (1.16) we get the assertion (1.10).

The assertion (1.11) is proved similarly.

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