DYNAMICAL MODELS OF ADIABATIC $N$-SOLITON INTERACTIONS

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Abstract. The adiabatic $N$-soliton train interactions for the scalar nonlinear Schrödinger (NLS) equation and its perturbed versions are well studied. Here we briefly outline how they can be generalized for the higher NLS-type equations and for the multicomponent NLS equations. It is shown that in all these cases the complex Toda chain plays fundamental role.

1. Introduction.

In a number of applications to fiber optics communications (see e.g. [1, 2, 3] and the numerous references therein) it is important to analyze the $N$-soliton train interaction of the (perturbed) nonlinear Schrödinger equation ((p)NLS) and some of its multicomponent versions. An important class of these equations are known to be integrable by applying the inverse scattering method to the generalized Zakharov-Shabat system [4]:

$$L\psi \equiv \left( i \frac{d}{dx} + q(x, t) - \lambda J \right) \psi(x, t, \lambda) = 0, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad q(x, t) = \begin{pmatrix} 0 & \vec{u}^\dagger \\ \vec{u} & 0 \end{pmatrix},$$  

where $q(x, t)$ and $J$ are matrices with compatible block structure. The case when $\vec{u}$ is a $P$-component vector is relevant to the vector NLS known also as the Manakov model [4]; more general multicomponent NLS equations are obtained if $u(x, t)$ is a generic rectangular matrix.

The class of higher multicomponent pNLS equations may be written as [5]:

$$iJq_t + 2f(\Lambda)q(x, t) = iJR[u], \quad R[u] = \begin{pmatrix} 0 & R_1[u] \\ R_2[u] & 0 \end{pmatrix},$$

where $f(\lambda)$ is the dispersion law (for the NLS eq. $f(\lambda) = \lambda^2$), and $\Lambda$ is the recursion operator

$$\Lambda = \Lambda_+ + \Lambda_- \quad \Lambda_{\pm}X = \frac{i}{4} \left[ J, \frac{dX}{dx} \right] - i q(x) \int_{\pm \infty}^x dy \frac{1}{2} \text{tr} \left( [J, q(y)] \right).$$

In what follows we consider only dispersion laws $f(\lambda) = \lambda^2 g(\lambda)$ with polynomial $g(\lambda)$.

2. $N$-soliton trains of the higher scalar NLS equations.

Let us start with a brief analysis of the $N$-soliton train interactions for the higher scalar (i.e., $P = 1$) NLS equations; by $N$-soliton train we mean a solution of the equation satisfying the initial condition

$$u(x, 0) = \sum_{k=1}^N u_k^{1s}(x, 0)$$
where \( u_k^{1s}(x,t) \) is the one-soliton solution of the higher NLS eq. Below following the Karpman-Solov’ev parametrization [2] we write:

\[
\begin{align*}
  u_k^{1s}(x,t) &= \frac{2\nu_k e^{i\phi_k(x,t)}}{\cosh(z_k(x,t))}, \\
  z_k(x,t) &= 2\nu_k(x - \xi_k(t)), \\
  \xi_k(t) &= \frac{f_{1,k}}{\nu_k}t + \xi_{k,0}, \\
  \phi_k(x,t) &= \frac{2\mu_k}{\nu_k}z_k(x,t) + \delta_k(t), \\
  \delta_k(t) &= \frac{2(\mu_k f_{1,k} - \nu_k f_{0,k})}{\nu_k}t + \delta_{k,0},
\end{align*}
\]

(5)

where \( \xi_k(t) \) characterizes the center of mass position of the \( k \)-th soliton and \( \nu_k, \mu_k, \delta_k \) characterize its amplitude, velocity and phase. Here \( f(\lambda_k^\pm) = f_{0,k} \pm i f_{1,k} \) is the value of the dispersion law for \( \lambda_k^\pm = \mu_k \pm i\nu_k \). As it is well known, \( L \) (1) remains isospectral if \( q(x,t) \) satisfies (2) with \( \mathcal{R} = 0 \).

The adiabatic approximation means that the solitons are well separated and have nearly equal amplitudes and velocities. If we denote by \( \epsilon \) the overlap of two neighboring solitons and assume that at \( t = 0 \) we have \( \xi_k(0) = \xi_{k,0} \) and \( \xi_{k+1,0} - \xi_{k,0} \approx r_0 \) then

\[
|\lambda_k^+ - \lambda_0^-|^2 \approx \epsilon \ll 1, \quad \epsilon \approx 16\nu_0^2 r_0 e^{-2\nu_0 r_0}, \quad \nu_0 r_0 \gg 1, \quad |\nu_0 - \nu_{k,0}| r_0 \ll 1.
\]

(6)

where \( \lambda_0^+ = \mu_0 + i\nu_0 = \sum_{k=1}^N (\mu_k + i\nu_k)/N \) and \( \lambda_0^- = (\lambda_0^+)^* \).

In the adiabatic approximation all \( N \) solitons keep their identity; the main part of the energy of the train is related to the \( 2N \) discrete eigenvalues of \( L \) which coincide with \( \lambda_k^\pm \) only in the limit \( r_0 \to \infty \). Since the solitons are well separated we can describe the slow evolution of their parameters by deriving a dynamical system for them. Thus in [2] we derived the generalized Karpman-Solov’ev system (GKS) describing soliton trains with \( N > 2 \). After some additional approximations using (4) the GKS simplifies to the complex Toda chain (CTC) with \( N \) sites:

\[
\frac{d^2 Q_k}{dt^2} = C \left( e^{Q_{k+1} - Q_k} - e^{Q_k - Q_{k-1}} \right) + \mathcal{R}[Q],
\]

(7)

where \( C = 4g_0 h_0, f_0 = f(\lambda_0^+), g_0 = g(\lambda_0^-), h_0 = f(\lambda_0^-) - f(\lambda_0^+) - (\lambda_0^- - \lambda_0^+) f'(\lambda_0) \) and

\[
Q_k = -2\nu_0 \xi_k + i(2\mu_0 \xi_k - \delta_k) + Q(t) + i k \pi, \quad \frac{dQ}{dt} = 2i f_0 + \frac{\lambda_0^+}{\nu_0} h_0.
\]

(8)

\( \mathcal{R}[Q] \) is determined by the perturbative terms \( \mathcal{R}[u] \) in the pNLS equation, see [3]. For \( f(\lambda) = \lambda^2 \) we have \( g_0 = 1, C = 16\nu_0^2; \) besides the linear function \( Q(t) \) can be replaced by \( 1/N \sum_{k=1}^N \delta_k \) and thus we reproduce the results for the \( N \)-soliton trains of the scalar pNLS, see [2] [3] [4] [5].

Note that \( Q(t) \) can be adjusted as convenient. Indeed, it is a linear function of \( t \) so it does nor show up in the l.h.side of (7). At the same time it does not depend on \( k \) so it does not show up in the exponentials in the r.h.side of (7) as well.

The importance of the CTC is based on the following facts:

i) for \( \mathcal{R}[u] = 0 \) we have \( \mathcal{R}[Q] = 0 \) and we deal with a completely integrable system. As a result, given the initial values of the soliton parameters, we are able to predict the asymptotic regime of the corresponding \( N \)-soliton train, [3];

ii) the CTC, in contrast with the well known real Toda chain, possesses much richer sets of asymptotical regimes. These include in addition bound states, mixed states,
singular regimes etc. In [3] we singled out a special subclass of $N$-soliton bound states in which the solitons move quasi-equidistantly.

iii) the CTC is an universal model. Indeed, it describes the $N$-soliton train for all the higher NLS equations. The only dependence on the dispersion law in (5) is in the constant $C$ which can easily be taken away by changing $q_k \to q_k + k \ln C$.

iv) If the coefficients in $R[u]$ are of the order of $\epsilon$ then the corresponding $R[Q]$ becomes a constant of the order of $\epsilon$. As a result the perturbation drives only the center of mass and the total phase of the $N$-soliton train and does not influence the relative coordinates and phases, see [7]. Our final remark here is that there are indications, showing that the CTC may describe $N$-soliton trains also for the NLEE related to more complicated Lax operators, e.g. to ones depending quadratically on $\lambda$, [8].

3. $N$-soliton trains of MNLS equations.

Our main idea here is to show that some of the results in Section 2 can be generalized also to a special type of MNLS equations of the form:

$$
i \frac{du_p}{dt} + \frac{1}{2} \frac{d^2 u_p}{dx^2} + \sum_{s=1}^{P} A_{ps}|u_s|^2 u_p = i R_p[\vec{u}],$$

(9)

where $A_{ps}$ is a $P \times P$ symmetrical matrix. Some of these multicomponent equations with $R[\vec{u}] = 0$ are integrable. They can be written in the form (4) and treated by the inverse scattering problem method applied to (1). Among them is the Manakov model [4] with $A_{ps} = (-1)^{\rho_s}$, where $\rho_s$ takes values 0 and 1. In these cases the soliton solutions and their interactions in the generic case are well known [4, 5, 9].

In what follows we consider somewhat more general case with $A_{sp}$ satisfying the condition $\sum_{s=1}^{P} A_{sp} = a^2 = \text{const}$ for all values of $p$. They are Hamiltonian models with

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dx \left( \left( \frac{\partial \vec{u}^*}{\partial x}, \frac{\partial \vec{u}}{\partial x} \right) - \sum_{s,p=1}^{P} A_{sp}|u_s|^2 |u_p|^2 \right).$$

(10)

A particular case of such MNLS with $P = 2$ and $A = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$ with $\beta \neq 1$ plays important role in nonlinear optics: it describes propagation of light in birefringent fibers, see e.g. [4, 8]. Although for such generic $A$ the MNLS (4) is not integrable it allows a special type of solutions whose behavior is very close to the ones of the scalar NLS equations.

Indeed, let us consider an $N$-soliton train of (4) defined as the solution to (4) satisfying the initial conditions

$$\vec{u}(x, 0) = \sum_{k=1}^{N} \vec{n}^{(k)} u^{1s}_k(x, 0),$$

(11)

where $u^{1s}_k(x, t)$ is the one-soliton solution of the scalar ($P = 1$) NLS equation and $\vec{n}^{(k)}$ is the polarization vector of the $k$-th solitons in the train. If we put $(\vec{n}^{(k)})_s = e^{i \varphi_k} a^{-1}$ then obviously $\sum_{s=1}^{P} A_{sp}|\vec{n}^{(k)}_s|^2 = 1$ and the system (4) reduces to the scalar NLS equation.

Now it remains to take into account that in the scalar case the $N$-soliton interaction is described by the CTC. The important difference as to the scalar case is that in the definition of the corresponding $q_k$ besides the phases $\varphi_k$ there enter also the scalar products
of the corresponding polarization vectors \[10\]:

\[ Q_{k+1} - Q_k = 2i\lambda^+ (\xi_{k+1} - \xi_k) + i \left( \pi - \delta_{k+1} + \delta_k + \ln \left( \frac{\vec{n}_k, \vec{n}_{k+1}}{a^2} \right) \right). \] (12)

As it is the CTC for the center of mass coordinates \(q_k\) is not sufficient to determine the dynamics for the MNLS \(N\)-soliton train. The complete treatment should include also an additional system of equations describing the dynamics of the polarization vectors \(\vec{n}_k\). Work on this is in progress.

4. Conclusions.

The dynamics of the (perturbed) scalar higher NLS equations is described by the (perturbed) CTC. The only thing that depends on the dispersion law is the coefficient \(C\) in (7). If the perturbations coefficients in \(R[u]\) are of the order of \(\epsilon\) then \(R[Q]\) is a constant of the order of \(\epsilon\) whose effect is to drive only the center of mass motion of the \(N\)-soliton train and the total phase. The relative positions \(\xi_{k+1}(t) - \xi_k(t)\) and the phase differences \(\delta_{k+1}(t) - \delta_k(t)\) are not influenced by these perturbations.

In the multicomponent case we showed that some special types of \(N\)-soliton trains of the MNLS equations reduce to an extension of the CTC by additional system of dynamic equations responsible for the dynamics of the polarization vectors \(\vec{n}_k\).

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References

[1] G. P. Agrawal. Nonlinear Fiber Optics. Second ed. Academic Press, 1995.
[2] J. M. Arnold. Phys. Rev. E 60, 979-986 (1999); JOSA A 15, 1450-1458 (1998).
[3] V. S. Gerdjikov, D. J. Kaup, I. M. Uzunov, E. G. Evstatiev, Phys. Rev. Lett. 77, pp. 3943-3946 (1996); V. S. Gerdjikov, I. M. Uzunov, E. G. Evstatiev, G. L. Diankov, Phys. Rev. E 55, pp. 6039-6060 (1997); V. S. Gerdjikov, E. G. Evstatiev, D. J. Kaup, G. L. Diankov, I. M. Uzunov. Phys. Lett. A 241, 323-328 (1998).
[4] S. V. Manakov. JETPh 67 543 (1974); Sov. Phys. JETP, 38, 248 (1974).
[5] P. P. Kulish. Sci. Notes. LOMI seminars 115 126 - 134, (1980).
V. S. Gerdjikov. Inverse Problems 2, n. 1, 51–74, 1986.
[6] V. I. Karpman, V. V. Solov’ev, Physica D 3, 487-502, (1981).
[7] V. S. Gerdjikov, I. M. Uzunov. Physica D (In press) (2000).
[8] E. Doktorov, V. I. Karpman, V. Shcheshnovich. Private communications.
[9] J. Yang. Phys. Rev. E 59, no. 2, 2393-2405 (1999).
[10] V. S. Gerdjikov. Thessaloniki (1998). N-soliton interactions, the complex Toda chain and stability of NLS soliton trains. In: Ed. E. Kriezis. Proc. URSI Symposium on electromagnetic theory, Thessaloniki, May 1998; pp. 307-309.