Scalarized-charged wormholes in Einstein-Gauss-Bonnet gravity

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Abstract

The Einstein-Maxwell-Klein-Gordon Lagrangian is supplemented by a non-minimal coupling of the real scalar field to the Gauss-Bonnet invariant. The non-minimal coupling function is chosen as a general second degree polynomial in the scalar field for which the system is known to admit hairy black holes. The new interaction leads naturally to a violation of the null energy condition, allowing for wormholes to exist without the need of exotic matter. Spherically symmetric, charged wormholes are constructed and their domain of existence is determined in terms of the different choices of the non-minimal coupling constants and of the electric charge. A special emphasis is set to the case of the purely quadratic coupling function. A phenomenon reminiscent to spontaneously scalarised black holes occurs for wormholes. The interaction with the electromagnetic field leads to new families of wormholes supported by a non-vanishing, large enough, electric charge.

1 Introduction

In the last decades the Gauss-Bonnet (GB) term has played an important role in the construction of several extensions of General Relativity (GR). Apart from the mathematical aspects, the interest for this geometric invariant (which, by itself is a total divergence in four space-time dimensions) is related to the fact that it emerges in effective theories describing the low energy limit of some string theories [1], [2]. It is a coupling involving some extra field – e.g. a scalar field like the dilaton – that leads to a non-trivial gravitational interaction. The general theory extending gravity by mean of a scalar field was elaborated by Horndeski [3]. This theory is very rich and characterized by a Lagrangian density containing large arbitrariness (see e.g. [4]). A few specific truncations of the theory were considered by several authors. One of them consists in selecting a non minimal term of the form $H(\phi)L_{GB}$ where the scalar field $\phi$ interacts with geometry through the Gauss-Bonnet term $L_{GB}$ and a function $H(\phi)$.

Apart from cosmological implications, one interesting issue of tensor-scalar gravities is that they allow for new kinds of compact objects to exist, e.g. hairy black holes and wormholes, see e.g. [5], [6], [7] for recent reviews. One example is the family of hairy black holes obtained in [8] with General Relativity minimally coupled to a massive, complex scalar field. In this case, the No-hair theorems for black holes related to scalar hair [9, 10] are evaded by constructing a rotating black hole and the synchronisation of the spin of the black hole with the angular frequency of the scalar field.

Recently Horndeski theory was been studied thoroughly in the context of Galileon theory [11] and some generalizations of the latter [12]. Galileon theory with a shift symmetric scalar field was considered in [13] and still leads to a large family of models. Considering $H(\phi)$ to be a linear function, hairy black holes have been constructed numerically and perturbatively [14]. Dropping the requirement of shift symmetry, a coupling function of the form $H(\phi) = \gamma \phi^2$ has been used in [15], [16] (and some other forms in [17], [18]). In these latter models, the existence of hairy black holes results from an unstable mode associated to the linearized equation of the scalar field in the background of a Schwarschild geometry and sourced by the non-minimal coupling term. The coupling constant $\gamma$ plays the role of a spectral parameter of the linear...
equation and black holes get *spontaneous scalarized* at a critical value of this coupling constant.

Next to black holes, another interesting class of solutions appearing in gravity theories are wormholes first discussed in [19] and interpreted in [20]. It is well known that viable (meaning: traversable) wormholes need the existence of exotic matter, i.e. matter that violates the null energy condition [21]. However, as has been demonstrated in [22], the energy-momentum tensor associated to a GB term violates this energy condition, opening the possibility for constructing wormholes in Einstein-Gauss-Bonnet models coupled to *normal* matter fields. Such objects where indeed constructed in [23] and in [24] where the properties, the domain of existence and the stability have been studied in details for some choices of the coupling function $H(\phi)$.

In the present paper we will consider a scalar-tensor gravity model with $H(\phi) = \alpha \phi + \gamma \phi^2$ where $\alpha$, $\gamma$ are independent constants. With such a combination, used first in [26], the features of the model containing spontaneously scalarised black holes (for $\alpha = 0$, $\gamma \neq 0$) and of the model containing shift symmetry (for $\alpha \neq 0$, $\gamma = 0$) appear together. Following [27], we supplement the model by a Maxwell term in order to study the influence of an electromagnetic field on the solutions. In the following, we will present strong numerical evidences that wormholes exist in the model and study how the presence of an electric potential affects their pattern. Our results reveal in particular that the presence of an electric field leads to charged wormholes for both signs of the coupling constant $\gamma$. Only a subset of these solutions persists while the electric is suppressed progressively. The paper is organized as follow: in section 2 we present the model, discuss the Ansatz and the boundary conditions. Section 3 contains the presentation of several properties of the wormholes and the determination of the domain of existence of these solutions in terms of the charge and of the coupling constants of the model. A summary of the results and perspectives are given in section 4.

2 The model

We are interested in wormhole-solutions associated with Einstein-Maxwell-Klein-Gordon lagrangian extended by a non-minimal coupling. The action considered is the form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + H(\phi) I(g) \right]$$

(2.1)

where $R$ is the Ricci scalar (in the following we will pose $8\pi G = 1$), $F_{\mu\nu}$ in the electromagnetic field strength and $\phi$ represents a real scalar field. The gravity sector is supplemented by a non-minimal coupling of the scalar field to a geometrical invariant $I(g)$. In this paper, we choose this invariant as the Gauss-Bonnet-scalar:

$$I = L_{GB} \equiv R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}$$

This combination is well known to be a total derivatives in four dimensions but it contributes non trivially to the equations of motion through its interaction with the scalar field.

Several forms of the function $H(\phi)$ have been emphasized in the litterature to construct hairy black holes and/or neutron stars in scalar tensor gravity. The purely linear case $H(\phi) = \alpha \phi$ corresponds to a shift-symmetric theory adressed in [13]. A quadratic choice $H(\phi) = \gamma \phi^2$ was considered in [15] and [16]. Several other choices of the function $H(\phi)$, (e.g. $H(\phi) = \gamma \exp(-\phi^2)$) have been used in [17], [18]. Wormholes were constructed in [24], [23] for a dilaton coupling function $H(\phi) = \alpha \exp(-\phi)$ and in [25] for several monomials forms of the function $H(\phi)$.

In this paper, we choose

$$H(\phi) = \alpha \phi + \gamma \phi^2$$

(2.2)

where $\alpha$, $\gamma$ are independent coupling constants (without loosing generality we can assume $\alpha > 0$). The choice (2.2) can be considered as a truncation of a general analytic coupling function; in addition it leaves
the possibility to construct wormholes in the two limits $\alpha = 0$ and $\gamma = 0$ where hairy black holes are known to exist (as explained in the introduction). The behavior of the solutions with the mixed coupling will be studied as well as the influence of the electromagnetic field. Let us add that the construction of hairy black holes with this combination was addressed in [26].

2.1 Ansatz

We will be interested in spherically symmetric solutions and adopt a metric of the form

$$ ds^2 = -f(r)a^2(r)dt^2 + \frac{1}{f(r)}dr^2 + (r^2 + r_0^2)d\Omega^2 $$

(2.3)

(note the change of notation: $g_{rr} = 1/f(r)$ with respect to [24]). This is completed by the scalar field and vector fields of the form

$$ \phi(x^\mu) = \phi(r) , \ A_0(x^\mu) = V(r) , \ A_1 = A_2 = A_3 = 0. $$

(2.4)

Substituting the ansatz in the field equations, the system can be reduced to a set of three non linear differential equations (plus a constaint) in the functions $f(r), a(r)$ and $\phi(r)$. The potential $V(r)$ is easily eliminated by using the Maxwell equation, leading to

$$ V'(r) = Qa/(r^2 + r_0^2) $$

(2.5)

where $Q$ is an integration constant which, for instance, is proportional to the electric charge of the solution. The final equations are of the first order for the functions $f(r), a(r)$ and of the second order for $\phi(r)$; they are to be solved on the interval $r \in [0, \infty]$, the throat of the wormhole corresponds to the limit $r \to 0$.

2.2 Boundary conditions

For a fixed couple of parameters $(\alpha, \gamma)$ and of the constant $Q$, four conditions on the boundary need to be fixed to specify a solution. Because we look for localized, asymptotically flat solutions, we require

$$ \alpha(r \to \infty) = 1 \ , \ \phi(r \to \infty) = 0 . $$

(2.6)

The equations are singular in the limit $r \to 0$, and obtaining regularity require a very specific relation between the functions and their derivatives at $r = 0$. To specify this relation (and for later use) it is convenient to write the Taylor expansion of the fields

$$ f(r) = f_0 + f_1 r + o(r^2) , \ a(r) = a_0 + a_1 r + o(r^2) \ , \ \phi(r) = \phi_0 + \phi_1 r + o(r^2) $$

(2.7)

The relevant regularity condition is :

$$ \phi_1^2 = \frac{2(1 - f_0) - Q^2(1 + 16(1 - f_0)(\alpha + 2\gamma\phi_0))}{16f_0(\gamma - 2(1 - f_0)(\alpha + 2\gamma\phi_0)^2)} $$

(2.8)

In the limit $\alpha = 0 , |\gamma| \ll 1$, the condition simplifies to $\phi_1^2 = (1 - Q^2/2 - f_0)/(8f_0\gamma)$. We see that for $\gamma > 0$, the condition implies $f_0 < 1 - Q^2/2$; by contrast, for $\gamma < 0$, the parameter $f_0$ is not bounded by the regularity condition.

Finally, the fourth boundary condition (necessary to specify a boundary value problem) is imposed by fixing by hand the value $f(0)$ (or alternatively $\phi(0)$). This value somehow serves as a control parameter; as we will see in the next section its variation is generally limited to a specific interval with bounds depending on $\alpha, \gamma$ and of $Q$.

The different coefficients in (2.7) can be computed recursively in terms of $f_0, a_0, \phi_0$, the charge $Q$ and the constants $\alpha, \beta$; the final expressions are quite involved and not illuminating.
2.3 Physical parameters

Along black holes, the wormholes solutions can be characterized by their mass $M$, the electromagnetic charge $Q$ (if $V \neq 0$) and a charge, say $D$, characterizing the scalar field. These charges are related respectively to the asymptotic decay of the functions $f(r)$, $V(r)$ and $\phi(r)$

$$f(r) = 1 - \frac{2M}{r} + O(1/r^2), \quad V(r) = V_0 - \frac{Q}{r} + O(1/r^2), \quad \phi(r) = - \frac{D}{r} + O(1/r^2), \quad a(r) = 1 - \frac{1 + D^2}{2r^2} + O(1/r^3).$$

The area of the throat is given by $A_{th} = \pi r_0^2$ and the curvature at the throat, say $R_0$ is given by $R_0 = r_0/f(0)$ (see e.g. [24]) for more details). The surface gravity at the throat, say $\kappa$, and temperature $T_H$ are defined as in the case of black holes. With our choice of the metric, we find $\kappa = (f'(0)a(0) + 2f(0)a'(0))/2$ and $T_H = \kappa/(2\pi)$. In terms of the Taylor expansion (and setting $\alpha = 0$ for simplicity), the surface gravity takes the form

$$\kappa = \frac{a_0(2f_0\phi_1^2 + 2 - Q^2)}{16\phi_1(\alpha + 2\gamma\phi_0)} \quad (2.10)$$

The classification of the solutions can be simplified by exploiting the following scale invariance of the equations

$$r \to \lambda r, \quad r_0 \to \lambda r_0, \quad \gamma \to \lambda^2 \gamma, \quad \alpha \to \lambda^2 \alpha. \quad (2.11)$$

The throat radius $r_0$ of the wormhole can therefore be normalized to one without losing generality. This normalization has been used throughout the numerical construction.

3 Numerical results

We now discuss the pattern of solutions in the space of the parameters $\alpha, \gamma$ and $Q$. On a suitable domain of this triplet, a branch of solutions exist which can be labeled by the value $f(0)$ (or alternatively by $\phi(0)$).

Along [24], we find that the various branches exist on a finite interval of these parameters and approach critical configurations of three types

- Type A : The wormhole approaches a black hole as $f(0) \to 1$.
- Type B : The limiting configuration presents a singularity at the throat.
- Type C : The limiting configuration presents a singularity at an intermediate value $r_s$, with $0 < r_s < \infty$.

Since the non-linear equations do not admit closed form solutions, we solved the system by using the numerical routine COLSYS [28]. It is based on a collocation method for boundary-value differential equations and on damped Newton-Raphson iterations. The equations are solved with a mesh of a few hundred points and relative errors of the order of $10^{-6}$. The variation of four independent quantities, namely $\alpha, \gamma, Q$ and $f(0)$, renders the determination of the domain quite cumbersome; we therefore limit our investigation to two sub-cases, namely :

- (i) We vary $\gamma, Q$ for the quadratic coupling (i.e. $\alpha = 0$). The influence of the charge of spontaneous scalarisation can then be appreciated.
- (ii) Varying the constants $\alpha, \gamma$ in the uncharged, i.e. with $Q = 0$.

3.1 Pure quadratic coupling

3.1.1 Uncharged solutions

We first discuss the case $Q = 0, \alpha = 0$. The corresponding theory admits a family of hairy black holes which bifurcates from the Schwarschild solution at a critical value of the coupling constant $\gamma$. To see this, the the Klein-Gordon equation sourced by the scalar-Gauss-Bonnet interaction term is considered in the
Schwarzschild background. It turns out that a regular, localized solution of the linear equation exists for a specific value of the coupling constant $\gamma$, say for $\gamma_c \sim 0.1814$ (solutions with nodes exist as well, with different values of $\gamma$, but are not studied here). From the critical value, a branch solutions of the system of coupled equations exist for $\gamma \in [0.172, 0.1814]$; they have $\phi(r) \neq 0$ and then constitutes hairy black holes \footnote{Similar solutions with other choices of the function $H(\phi)$ were constructed in \footnote{15}).

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Up: The values of $a(0)$ and $\phi(0)$ (dashed and solid curves respectively) as functions of $f(0)$ for several values of $\gamma$. Down: The values of $f(0)$ and of $\phi(0)$ (dashed and solid curves respectively) as functions of the dimensionless parameter $D/M$.}
\end{figure}

Turning to the wormhole equations reveals that solutions with $\phi(r) \neq 0$ exist for $\gamma \gtrsim 0.172$ as well. For a fixed value of $\gamma$ slightly larger than this threshold, a family of wormholes exists with a finite extend of the parameters $\phi(0)$ and $f(0)$. For the discussion we use $f(0)$ as a control parameter, the counting of the branches of solutions will refer to $f(0)$.

Two different patterns occur according to the value of $\gamma$ as revealed by Fig. 1. In each case the phenomenon limiting the branches has different characteristics.

- (i) For $\gamma \in [0.172, 0.1814]$ a single branch of solutions exist with $f(0) \in [c_1, 1]$ (the value $c_1$ depends on $\gamma$). The solutions approach a hairy black holes for $f(0) \rightarrow 1$. In the limit $f(0) \rightarrow c_1$, a configuration presenting a singularity at some intermediate radius is approached.

- (ii) For $\gamma > 0.1814$ no hairy black hole that can be approached do exist. Instead two branches of solutions exist respectively with $f(0) \in [0, c_2]$ (say branch I) and with $f(0) \in [c_1, c_2]$ (branch II). The
branch I ends up into a singular configuration at the throat since \( f(0) \to 0 \). The limit of branch II for \( f(0) \to c_1 \) is a configuration presenting a singularity at an intermediate radius.

On the upper part of Fig. 1 the dependence of \( \phi(0) \) on \( f(0) \) is shown by the solid lines for several values of \( \gamma \). The transition between the one-branch and the two-branches regimes clearly appears. The corresponding values of \( a(0) \) are plotted by the dashed lines. The same quantities are reported in terms of the dimensionless ratio \( D/M \) on the lower part of Fig. 1. The scalar function \( \phi(r) \) is negative, presenting a local maximum at some intermediate radius and then increases, explaining the positivity of the charge \( D \). Completing the data, Fig. 3 (left side) reveals that, when two branches are present, the surface gravity \( \kappa \) of the solutions of the main branch depends only a little from \( f(0) \); by contrast \( \kappa \) becomes large and strongly dependent of \( f(0) \) for the solutions of the second branch.

Let us point out that, like in Ref. [24], we managed to construct uncharged wormholes for \( \gamma > 0 \) and found no evidence of solutions for \( \gamma < 0 \).

### 3.1.2 Influence of the charge

From the regularity condition (2.8), different patterns of charged solutions can be expected depending on the sign of the constant \( \gamma \). It turns out, indeed, that solving the equations for \( Q \neq 0 \) leads to new families of solutions that have no limit for \( Q \to 0 \).

**Positive \( \gamma \).** As pointed out already, in the absence of an electric potential (i.e. \( Q = 0 \)), wormholes can be constructed for positive values of \( \gamma \) only. Increasing the charge parameter \( Q \) progressively, these uncharged solutions get continuously deformed leading to families of charged wormhole exist. These exist up to a maximal value of the parameter, say \( Q_c \), of the charge parameter. In all cases that considered, the limiting configuration presents a singular geometry at the throat as the metric function \( a(0) \) indeed approaches zero for \( Q \to Q_c \). Since these results are somehow expected, we do not present details and concentrate on the solutions available for negative values of \( \gamma \).

**Negative \( \gamma \).** Interestingly, the analysis of the equations with \( Q \neq 0 \) reveals that, for sufficiently large \( Q \), new branches of solutions exist for \( \gamma < 0 \). These solutions do not have a regular limit for \( Q \to 0 \). For the numerous cases that we considered, the numerical results indicate that both parameters \( \phi(0), a(0) \) tends to zero for \( Q \to Q_c \) and we found the critical value of the charge parameter to be typically \( Q_c \sim 0.7 \). All wormholes of this type that we constructed have \( \phi(r) > 0 \) and \( \phi'(r) < 0 \); as a consequence the scalar charge \( D \) is negative (contrasting with the \( \gamma > 0 \) solutions).

The understanding of the full pattern of these solutions is quite demanding. We obtained families of wormholes for several values of \( Q \) but, for definiteness, we discuss the results for the case \( Q = 1 \); they are illustrated by Fig. 2.
Figure 3: Left: The surface gravity $\kappa$ versus $f(0)$ for several values of $\gamma$ and $Q = 0$. Right: Idem for the solutions with $\gamma < 0$ and $Q = 1$.

Figure 4: Left: Profile of $f, a, \phi$ for the solution with $f(0) = 3, \gamma = -0.015$ and $Q = 1$. Right: The corresponding Ricci and Kretschmann invariants.

For a fixed negative value of $\gamma$ it turns out that wormholes exist for $f(0) \in [f_a, f_b]$ where $f_a, f_b$ depend on $\gamma$. The minimal $f_a$ depends only a little from $\gamma$: we find $f_a \sim 0.55$. In this limit the solutions approach a singular configuration as the parameter $a(0)$ approaches zero while the scalar field approaches uniformly the null function.

The evolution of the solutions obtained when the parameter $f(0)$ is increases is more involved and depends strongly of the magnitude of $|\gamma|$. As illustrated by Fig.2 two scenarii clearly occur:

- For $|\gamma| \ll 1$, solutions can be constructed up to a maximal value $f(0) = f_b$, then another branch of solutions exist. The second branch, back bending from the main one, terminates at an intermediate value, say $f_c$ with $f_a < f_c < f_b$ in a configuration presenting a singularity at an intermediate value of $r$. The profile of a solution close to the critical value is presented on Fig. 4 where the Ricci and Kretschmann invariants reveals the existence of a singularity at a finite radius. Two solutions coexist for $f(0) \in [f_c, f_b]$ and have clearly different masses (see the lower part of the figure).

- For $|\gamma| > 0.035$ only one branch of solutions exist, stopping for some $f(0) = f_b$. Again, the limiting configuration presents a singularity at some intermediate radius.

The dependance of $\phi(0)$ on the parameter $f(0)$ is shown on the upper part of Fig. 2 for several values of $\gamma$. The lower part of the figure shows the corresponding mass. The extend of the solution in the parameter $f(0)$ decreases progressively while increasing $|\gamma|$. The solutions reported on the figure are for $|\gamma| \leq 0.5$ but
solutions exist up to $\gamma \sim -5.0$.

Interestingly, the range of the parameter $f(0)$ for the solutions discussed in this section is much larger that for solutions available for $\gamma > 0$. In particular, charged wormholes can be constructed for $f(0) \gg 1$, implying that their radius of curvature at the throat, $R_0 = r_0/f(0)$ can be very small; this contrasts with uncharged wormholes that have $R_0 \geq 1$. Finally, the surface gravity corresponding to the solutions the solutions of Figs. 1 and 2. is shown on Fig. 3 (left and right side respectively).

### 3.2 Mixed coupling

We now discuss the solutions available with the mixed coupling. For simplicity, we limit to the uncharged case. We found no solution for $\gamma \leq 0$; therefore the relevant range of parameters is $\alpha \geq 0, \gamma \geq 0$. We will sketch the influence of these parameters on the pattern of solutions by presenting results respectively for $\alpha$ fixed and $\gamma$ varying and for $\gamma$ fixed and $\alpha$ varying.

**Case $\alpha$ fixed, $\gamma$ varying.**

Perhaps one of the striking features in the case $\alpha > 0$ is the fact that wormholes exist for $\gamma > 0$ : contrasting with the pure quadratic case (see section 3.1) there is no threshold in the coupling constant $\gamma$. We therefore put the emphasis on families of solutions occurring for $\alpha \ll 1$. Some new features of the solutions are sketched on Fig. 5 for several values of $\gamma$ and where we set for definiteness $\alpha = 0.02$. A single branch of solutions occurs for $0 < \gamma < 0.175$; it is labeled by $f(0)$ (see left side of Fig. 5) and extend for $f(0) \in [0, 1]$. The central value of the scalar field $\phi(0)$ is positive or negative along the branch. For $\gamma > 0.175$ two families of wormholes exist and coincide at a maximal value of $f(0)$ with $f(0) < 1$. All these solutions have a positive scalar charge: $D > 0$ as seen on the right side of Fig. 5.

![Figure 5](image)

**Figure 5:** Left: The values $\phi(0)$ as function of $f(0)$ for several values of $\gamma$ and $\alpha = 0.02$. Right: The values $f(0)$ and $\phi(0)$ versus $D/M$; The curves from the left to the right are for $\gamma = 0.02, 0.04, 0.1, 0.2, 0.25, 0.26$.

**Case $\gamma$ fixed, $\alpha$ varying.**

We finally analyze the influence of the increase of the linear coupling constant $\alpha$ on a solution with a fixed $\gamma$; for definiteness we concentrate on solutions corresponding to $\gamma = 0.1$ (remember: they have no $\alpha = 0$-limit). The deformation of the physical data by the increase of $\alpha$ is illustrated by Figs. 6, 7. We see in particular that the increase of $\alpha$ allows for solutions with positive central density of the scalar field : solutions with $\phi(0) > 1$ typically exist while all solutions for $\alpha = 0$ have $\phi(0) < 0$. Plotting the data as function of the dimensionless parameter $D/M$ also reveals new features. It turns out that the interval of $D/M$ where solutions are available considerably while $\alpha$ increases. A large fraction of wormholes with mixed coupling have a negative scalar charge $D$. Typical solutions have $D/M \in [-1, 0.5]$; again contrasting with all $\alpha = 0$ solutions.
4 Summary

The Einstein-Hilbert-Maxwell-Klein-Gordon action considered in this paper is extended by a non-minimal interaction involving the Gauss-Bonnet term coupled to a specific function $H(\phi)$ of the scalar field $\phi$. The choice $H(\phi) = \alpha \phi + \gamma \phi^2$ is motivated by the fact that hairy black holes are known to exist in the two limits $\alpha = 0$ and $\gamma = 0$. It is therefore natural to emphasize the existence of wormholes separately in both cases and to further study how these solutions evolve in the mixed case. It was first demonstrated numerically that wormholes appear spontaneously in the case $\alpha = 0$ at a critical value of $\gamma$. By continuity it was then shown that, for a large domain of the coupling constants $\alpha$ and $\gamma$, the model possesses wormholes solutions crucially supported by the non-trivial scalar field. Let us stress that this scalar field has a conventional kinetic term in the Lagrangian.

The influence of the electromagnetic field, characterized by the electric charge $Q$, was also taken into account leading to families of charged wormholes solutions. Thereby, new classes of solutions have been constructed which present different features from the wormholes constructed with the help of the Gauss-Bonnet term. Namely: (i) they present very small curvature radius at the throat, (ii) they have no smooth limit for $Q \to 0$, (iii) they exist with both signs of the scalar field at the throat. Nevertheless, work is still needed to examine further properties of these new wormholes, namely their stability and their analytic continuation in the $r < 0$ region.
References

[1] D. J. Gross, J. H. Sloan, Nucl. Phys. B291 (1987) 41.
[2] R. R. Metsaev, A. A. Tseytlin, Nucl. Phys. B293 (1987) 385.
[3] G. W. Horndeski, Int. J. Theor. Phys. 10 (1974) 363.
[4] A. Maselli, H. O. Silva, M. Minamitsuji and E. Berti, Phys. Rev. D 92 (2015) no.10, 104049
[5] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 24 (2015) no.09, 1542014 [arXiv:1504.08209 [gr-qc]].
[6] T. P. Sotiriou, Class. Quant. Grav. 32 (2015) no.21, 214002 [arXiv:1505.00248 [gr-qc]].
[7] M. S. Volkov, arXiv:1601.08230 [gr-qc].
[8] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. 112 (2014) 221101. [arXiv:1403.2757 [gr-qc]].
[9] J. D. Bekenstein, Phys. Rev. Lett. 28 (1972) 452; C. Teitelboim, Lett. Nuovo Cim. 382 (1972) 397.
[10] J. D. Bekenstein, Phys. Rev. D 51 (1995) no.12, R6608.
[11] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79 (2009) 064036.
[12] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D 84 (2011) 064039 [arXiv:1103.3260 [hep-th]].
[13] T. P. Sotiriou and S. Y. Zhou, Phys. Rev. Lett. 112 (2014) 251102 [arXiv:1312.3622 [gr-qc]].
[14] T. P. Sotiriou and S. Y. Zhou, Phys. Rev. D 90 (2014) 124063 [arXiv:1408.1698 [gr-qc]].
[15] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou and E. Berti, Phys. Rev. Lett. 120 (2018) no.13, 131104 [arXiv:1711.02080 [gr-qc]].
[16] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. Lett. 120 (2018) no.13, 131102 [arXiv:1711.03390 [hep-th]].
[17] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120 (2018) no.13, 131103 [arXiv:1711.01187 [gr-qc]].
[18] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. D 97 (2018) no.8, 084037 [arXiv:1711.07431 [hep-th]].
[19] A. Einstein, N Rosen, Phys. Lett. 48 (1935) 73.
[20] J. A. Wheeler, Annals Phys. 2 (1957) 604.
[21] H. G. Ellis, J. Math. Phys. 14 (1973) 104.
[22] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, Phys. Rev. D 54 (1996) 5049 [hep-th/9511071].
[23] P. Kanti, B. Kleihaus and J. Kunz, Phys. Rev. Lett. 107 (2011) 271101 [arXiv:1108.3003 [gr-qc]].
[24] P. Kanti, B. Kleihaus and J. Kunz, Phys. Rev. D 85 (2012) 044007 [arXiv:1111.4049 [hep-th]].
[25] G. Antoniou, A. Bakopoulos, P. Kanti, Phys. Rev. D 97 (2018) no.8, 084037 [arXiv:1711.07431 [hep-th]].
[26] Y. Brihaye and L. Ducobu, Phys. Lett. B 795 (2019) 135
[27] Y. Brihaye and B. Hartmann, Phys. Lett. B 792 (2019) 244, [arXiv:1902.05760 [gr-qc]].
[28] U. Ascher, J. Christiansen, R. D. Russell, Math. Comp. 33 (1979) 659;
U. Ascher, J. Christiansen, R. D. Russell, ACM Trans. 7 (1981) 209.