p-adic discrete dynamical systems and their applications in physics and cognitive sciences

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Abstract

This review is devoted to dynamical systems in fields of p-adic numbers: origin of p-adic dynamics in p-adic theoretical physics (string theory, quantum mechanics and field theory, spin glasses), continuous dynamical systems and discrete dynamical systems. The main attention is paid to discrete dynamical systems - iterations of maps in the field of p-adic numbers (or their algebraic extensions): ergodicity, behaviour of cycles, holomorphic dynamics. We also discuss applications of p-adic discrete dynamical systems to cognitive sciences and psychology.

1 Introduction

The fields $\mathbb{Q}_p$ of p-adic numbers (where $p = 2, 3, \ldots, 1999, \ldots$ are prime numbers) were introduced by German mathematician K. Hensel at the end of 19th century, [1]. Hensel started with the following question:

Is it possible to expend a rational number $x \in \mathbb{Q}$ in a power series of the form

$$x = \sum_{n=k}^{\infty} \alpha_n p^n, \alpha_n = 0, \ldots, p - 1,$$

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where \( k = 0, \pm 1, \pm 2, \ldots \) depends on \( x \). Of course, this question was motivated by the existence of real expansions of rational numbers with respect to a \( p \)-scale:

\[
x = \sum_{n=-\infty}^{k} \alpha_n p^n, \alpha_n = 0, \ldots, p - 1.
\]  

(2)

Hensel knew, for example, that

\[
\frac{4}{3} = \sum_{n=-\infty}^{0} 2^{2n}.
\]

He studied the possibility to expand \( x = \frac{4}{3} \) in a series with respect to positive powers of \( p = 2 \):

\[
\frac{4}{3} = \sum_{n=0}^{\infty} \alpha_n 2^n, \alpha_n = 0, 1.
\]

Such rather innocent manipulations with rational numbers and series generated the idea that there exists some algebraic structure similar to the system of real numbers \( \mathbb{R} \). K. Hensel observed that it is possible to introduce algebraic operations (addition, subtraction, multiplication, division) on the set \( \mathbb{Q}_p \) of all formal series \( \mathbb{Q}_p \). Thus each \( \mathbb{Q}_p \) has the structure of a number field. The field of rational numbers \( \mathbb{Q} \) is a subfield of \( \mathbb{Q}_p \). In fact, the fields of \( p \)-adic numbers \( \mathbb{Q}_p \) were first examples of infinite fields that differs from \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \) and corresponding fields of rational functions.

It is possible to work in more general framework, namely to consider not only prime numbers \( m \), but all natural numbers \( p \) as bases of expansions. In principle, we can do this. However, the corresponding number system is not in general a field. If \( m = p_1 p_2 \), where \( p_j \) are distinct prime numbers, then \( \mathbb{Q}_m \) is not a field (there exists divisors of zero), but only a ring, i.e., division is not well defined. The field structure is very important to develop analysis. Therefore the main part of investigations is performed for prime \( p \). In particular, we consider only this case in this paper.

The construction of new fields \( \mathbb{Q}_p \) induced strong interest in number theory and algebra. Practically one hundred years \( p \)-adic numbers were intensively used only in pure mathematics, mainly in number theory, see, for example, the classical book of Borevich and Schafarevich [2]. In particular, \( p \)-adic numbers are useful in investigations of some number-theoretical problems in the field of rational numbers \( \mathbb{Q} \). Typically if we can prove some fact for the field of real numbers \( \mathbb{R} \) as well as for all fields of \( p \)-adic numbers \( \mathbb{Q}_p, p = 2, 3, \ldots \), then we get the corresponding result for the field of rational numbers \( \mathbb{Q} \), see [2] for the details.
The presence of the structure of a topological field on $\mathbb{Q}_p$ gives the possibility to develop analysis for functions $f : \mathbb{Q}_p \to \mathbb{Q}_p$. In particular, the derivative of such a function is defined in the usual way:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, x, h \in \mathbb{Q}_p.$$ 

Of course, to perform limit-procedure, we need a topology on $\mathbb{Q}_p$. It is possible to define on $\mathbb{Q}_p$ a $p$-adic absolute value, valuation, $x \to |x|_p$ that has properties similar to the properties of the ordinary absolute value on $\mathbb{R}$. The topology on the field $\mathbb{Q}_p$ is defined by the metric $\rho_p(x, y) = |x - y|_p$. The $\mathbb{Q}_p$ is a locally compact topological field (i.e., the unit ball of $\mathbb{Q}_p$ is a compact set and all algebraic operations are continuous). The field of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{Q}_p$. Thus $\mathbb{Q}_p$ (as well as $\mathbb{R}$) is a completion of the field of rational numbers $\mathbb{Q}$.

This review is devoted to applications of $p$-adic numbers in various fields. However, we start with an extended historical review on $p$-adic (and more general non-Archimedean or ultrametric) analysis and related fields. Such a review can be useful for researchers working in applications of $p$-adic numbers.

The general notion of an absolute valued field was introduced by J. Kürschak [3] in 1913 and a few years later A. Ostrowski [4] described absolute values on some classes of fields, especially on the rationals. In 1932 L. S. Pontryagin [5] proved that the only locally compact and connected topological division rings are the classical division rings and N. Jacobson [6] began the systematic study of the structure of locally compact rings. We should also mention the theory of Krull valuations [7] and the paper by S. MacLane [8] that began the study of valuations on polynomial rings. We recall that I. Kaplansky initiated the general theory of topological rings, see e.g. [9].

In 1943 I. R. Shafarevich [10] found necessary and sufficient conditions for a topological field to admit an absolute value preserving the topology and D. Zelinsky [11] characterized the topological fields admitting a non-Archimedean valuation. We also mention the work [12] of H.-J. Kowalsky, who described locally compact fields.

First fundamental investigations in $p$-adic analysis were done by K. Mahler [13] (differential calculus, differential and difference equations), M. Krasner [14], [15] (topology on $\mathbb{Q}_p$, the notion of an ultrametric space \(^1\), $p$-adic an-

\(^1\)One of the important features of the $p$-adic metric $\rho_p$ is that it is an ultrametric. It satisfies to the strong triangle inequality, see section 2. Metric spaces in that the strong triangle inequality holds true (ultrametric or non-Archimedean spaces) were actively used in analysis, since Krasner’s work [14]. In topology these spaces were actively studied, since the works of F. Hausdorff [16].

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alytic functions), E. Motzkin and Ph. Robba [17], [18] (analytic functions), Y. Amice [19] (analytic functions, interpolation, Fourier analysis), M. Lazard [20] (zeros of analytic functions), A. Monna [21] (topology, integration), T. A. Springer [22] (integration, theory of $p$-adic Hilbert spaces).

In the period between 1960 and 1987 $p$-adic analysis developed intensively due to pure mathematical self-motivations. Main results of these investigations are collected in the nice book of W. Schikhof [23] that is really a fundamental encyclopedia on $p$-adic analysis, recently there was published another excellent book on $p$-adic analysis – namely the book [24] of A. M. Robert, see also the book [25] of A. Escassut on $p$-adic analytic functions and the book [26] of P.-C. Hu and C.-C. Yang on non-Archimedean theory of meromorphic functions. An important approach to the non-Archimedean analysis is based on the theory of rigid analytic spaces, see J. T. Tate [27], S. Bosch, L. Gerritzen, H. Grauert, U. Güntzer, Ya. Morita, R. Remmert, [28]–[37]. We also mention works of Kubota, Leopoldt, Iwasawa, Morita [38]–[43] (investigations on $p$-adic $L$ and $\Gamma$-functions).

This $p$-adic analysis is analysis for maps $f : \mathbb{Q}_p \to \mathbb{Q}_p$ (or finite or infinite algebraic extensions of $\mathbb{Q}_p$). And the present paper is devoted to the theory of dynamical systems based on such maps.

However, not only maps $f : \mathbb{Q}_p \to \mathbb{Q}_p$, but also maps $f : \mathbb{Q}_p \to \mathbb{C}$ are actively used in $p$-adic mathematical physics. Analysis for complex valued functions of the $p$-adic variable was intensively developed in general framework of the Fourier analysis on locally compact groups. There was developed a theory of distributions on locally compact disconnected fields (and, in particular, fields of $p$-adic numbers). There were obtained fundamental results on theory of non-Archimedean representations. Fundamental investigations in this domain were performed in early 60th by I. M. Gelfand, M. I. Graev and I. I. Pjatetski-Shapiro [44]–[48] (see also papers of M. I. Graev and R. I. Prohorova [49], [50], A. A. Kirillov and R. R. Sundcheleev [51] and A. D. Gvishiani [52], P. M. Gudivok [53], A. V. Zelevinskii [54], A. V. Trusov [55], [56]).

The first (at least known to me) publication on the possibility to use the $p$-adic space-time in physics was article [57] of A. Monna and F. van der Blij. Then E. Beltrametti and G. Cassinelli tried to use $p$-adic numbers in quantum logic [58]. But they obtained the negative result: $p$-adic numbers could not be used in quantum logic. The next fundamental step was the discussion on $p$-adic dimensions in physics started by Yu. Manin [59].

The important event in the $p$-adic world took place in 1987 when I. Volovich published paper [60] on applications of $p$-adic numbers in string theory. The string theory was new and rather intriguing attempt to reconsider foundations of physics by using space extended objects, strings, instead
of the pointwise objects, elementary particles. The scenarios of string spectacle is performed on fantastically small distances, so called Planck distances, $l_P \approx 10^{-34}\text{cm}$. Physicists have (at least) the feeling that space-time on Planck distances has some distinguishing features that could not be described by the standard mathematical model based on the field of real numbers $\mathbb{R}$. In particular, there are ideas (that also are strongly motivated by cosmology\footnote{We remark that one of the aims of string theory was to provide a new approach to general relativity. Therefore string investigations are closely connected to investigations on fields of gravity and cosmology.}) that on Planck distances we could not more assume that there presents a kind of an order structure on the real line $\mathbb{R}$. We remark that there is no order structure on $\mathbb{Q}_p$ (this is a disordered field).

Another argument to consider a $p$-adic model of space-time on Planck distances is that $\mathbb{Q}_p$ is a non-Archimedean field. We do not plan to discuss here Archimedean axiom on the mathematical level of rigorousness.

From the physical point of view this axiom can be interpreted in the following way. If we have some unit of measurement $l$, then we can measure each interval $L$ by using $l$. By addition of this unit: $l, l+l, l+l+l, \ldots, l+\ldots+l$, we obtain larger and larger intervals that, finally, will cover $L$. The precision of such a measurement is equal to $l$. The process of such a type we can be realized in the field of real numbers $\mathbb{R}$. Therefore all physical models based on real numbers are Archimedean models. However, Archimedean axiom does not hold true in $\mathbb{Q}_p$. Here successive addition does not increase the quantity. And there were (long before $p$-adic physics) intuitive cosmological ideas that space-time on Planck distances has non-Archimedean structure.

In 80th and 90th there was demonstrated large interest to various $p$-adic physical models, see, for example, papers on $p$-adic string theory of Aref’eva, Brekke, Dragovich, Framton, Freund, Parisi, Vladimirov, Volovich, Witten and many others, \cite{60}-\cite{64} $^3$, $p$-adic quantum mechanics and field theory \cite{65}-\cite{70}, $p$-adic models for spin glasses \cite{71}, \cite{72}. These $p$-adic physical investigations stimulated the large interest to dynamical systems in fields of $p$-adic numbers $\mathbb{Q}_p$ and their finite and infinite extensions (and, in particular, in the field of complex $p$-adic numbers $\mathbb{C}_p$).

Continuous dynamical systems, namely $p$-adic differential equations, were studied by purely mathematical reasons. We can mention investigations of B. Dwork, P. Robba, G. Gerotto, F. J. Sullivan \cite{73}-\cite{76}, G. Christol \cite{77}, A. Escassut \cite{25} on $p$-adic ordinary differential equations. However, $p$-adic physics stimulated investigations on new classes of continuous dynamical sys-

\footnote{We remark that these investigations in $p$-adic string theory were strongly based on the results of mathematical investigations of I. M. Gelfand, M. I. Graev and I. I. Pjatetskii-Shapiro \cite{44} on distributions on $p$-adic fields.}
tems; in particular, partial differential equations over \( \mathbb{Q}_p \), see, for example, [78]-[83] (\( p \)-adic Schrödinger, heat, Laplace equations, Cauchy problem, distributions). We do not consider continuous \( p \)-adic dynamical systems in this review, see, e.g. book [69].

This review is devoted to discrete \( p \)-adic dynamical systems, namely iteration

\[
x_{n+1} = f(x_n)
\]

of functions \( f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p \) or \( f : \mathbb{C}_p \rightarrow \mathbb{C}_p \).

The development of investigations on \( p \)-adic discrete dynamical systems is the best illustration of how physical models can stimulate new mathematical investigations. Starting with the paper on \( p \)-adic quantum mechanics and string theory [67] (stimulated by investigations of Vladimirov and Volovich) E. Thiren, D. Verstegen and J. Weyers performed the first investigation on discrete \( p \)-adic dynamical systems [84] (iterations of quadratic polynomials). After this paper investigations on discrete \( p \)-adic dynamical systems were proceeded in various directions [84]-[111], e.g. : 1) conjugate maps [84]-[86] and [87] – general technique based on Lie logarithms, [96], [97] - problem of small denominators in \( \mathbb{C}_p \); 2) ergodicity [98], [99], [97], [110]; 3) random dynamical systems [100], [108]\(^4\); 4) behaviour of cycles [101], [102], [106], [107], [111]; 5) dynamical systems in finite extensions of \( \mathbb{Q}_p \) [103], [104], [111]; 6) holomorphic and meromorphic dynamics [90], [26], [93]-[95].

Recently discrete \( p \)-adic dynamical systems were applied to such interesting and intensively developed domains as cognitive sciences and psychology. [91], [113]-[122]. These cognitive applications are based on coding of human ideas by using branches of hierarchic \( p \)-adic trees and describing the process of thinking by iterations of \( p \)-adic dynamical system. \( p \)-adic dynamical cognitive models were applied to such problems as memory recalling, depression, stress, hyperactivity, unconscious and conscious thought and even Freud’s psychoanalysis. These investigations stimulated the development of \( p \)-adic neural networks [116].\(^5\) Recently \( p \)-adic numbers were applied to problems of image recognition and compression of information, see [123], [124].

## 2 \( p \)-adic numbers

The field of real numbers \( \mathbb{R} \) is constructed as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the metric \( \rho(x, y) = |x - y| \), where \( | \cdot | \) is

\(^4\)See L. Arnold [112] for the general theory of random dynamical systems.

\(^5\)We remark that in cognitive models there naturally appear \( m \)-adic trees for nonprime \( m \). Therefore we also have to develop analysis and theory of dynamical systems in such a general case.
the usual valuation given by the absolute value. The fields of $p$-adic numbers $\mathbb{Q}_p$ are constructed in a corresponding way, but using other valuations. For a prime number $p$, the $p$-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number $n$ can be represented as the product of prime numbers, $n = 2^{r_2}3^{r_3} \cdots p^{r_p} \cdots$, and we define $|n|_p = p^{-r_p}$, writing $|0|_p = 0$ and $|-n|_p = |n|_p$. We then extend the definition of the $p$-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p = |n|_p/|m|_p$ for $m \neq 0$. The completion of $\mathbb{Q}$ with respect to the metric $\rho_p(x, y) = |x - y|_p$ is the locally compact field of $p$-adic numbers $\mathbb{Q}_p$.

The number fields $\mathbb{R}$ and $\mathbb{Q}_p$ are unique in a sense, since by Ostrovsky’s theorem [2] $|\cdot|$ and $|\cdot|_p$ are the only possible valuations on $\mathbb{Q}$, but have quite distinctive properties. The field of real numbers $\mathbb{R}$ with its usual valuation satisfies $|n| = n \to \infty$ for valuations of natural numbers $n$ and is said to be Archimedean. By a well know theorem of number theory [2] the only complete Archimedean fields are those of the real and the complex numbers. In contrast, the fields of $p$-adic numbers, which satisfy $|n|_p \leq 1$ for all $n \in \mathbb{N}$, are examples of non-Archimedean fields. Here the Archimedean axiom is violated. We could not get larger quantity by successive addition. Let $l$ be any element of $\mathbb{Q}_p$. There does not exist such a natural number $n$ that $|nl|_p \leq 1$.

The field of real numbers $\mathbb{R}$ is not isomorphic to any $\mathbb{Q}_p$. Fields $\mathbb{Q}_s$ and $\mathbb{Q}_t$ are not isomorphic for $s \neq t$. Thus starting with the field of rational numbers $\mathbb{Q}$ we get an infinite series of locally compact non-isomorphic fields: $\mathbb{Q}_2, \mathbb{Q}_3, \ldots, \mathbb{Q}_{1997}, \mathbb{Q}_{1999}, \ldots$

Unlike the absolute value distance $|\cdot|$, the $p$-adic valuation satisfies the strong triangle inequality

$$|x + y|_p \leq \max[|x|_p, |y|_p], \quad x, y \in \mathbb{Q}_p. \tag{4}$$

Consequently the $p$-adic metric satisfies the strong triangle inequality

$$\rho_p(x, y) \leq \max[\rho_p(x, z), \rho_p(z, y)], \quad x, y, z \in \mathbb{Q}_p, \tag{5}$$

which means that the metric $\rho_p$ is an ultrametric.

Write $U_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}$ and $U^{-}_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$ where $r = p^n$ and $n = 0, \pm 1, \pm 2, \ldots$. These are the “closed” and “open” balls in $\mathbb{Q}_p$ while the sets $S_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}$ are the spheres in $\mathbb{Q}_p$ of such radii $r$. These sets (balls and spheres) have a somewhat strange topological structure from the viewpoint of our usual Euclidean intuition: they are both open and closed at the same time, and as such are called clopen sets. Another interesting property of $p$-adic balls is that two balls have nonempty intersection if and only if one of them is contained in the
other. Also, we note that any point of a \( p \)-adic ball can be chosen as its center, so such a ball is thus not uniquely characterized by its center and radius. Finally, any \( p \)-adic ball \( U_r(0) \) is an additive subgroup of \( \mathbb{Q}_p \), while the ball \( U_1(0) \) is also a ring, which is called the ring of \( p \)-adic integers and is denoted by \( \mathbb{Z}_p \).

Any \( x \in \mathbb{Q}_p \) has a unique canonical expansion (which converges in the \( | \cdot |_p \)-norm) of the form

\[
x = a_{-n}/p^n + \cdots + a_0 + \cdots + a_k p^k + \cdots
\]

where the \( a_j \in \{0, 1, \ldots, p-1\} \) are the “digits” of the \( p \)-adic expansion and \( n \) depend on \( x \).

This expansion is similar to the standard expansion of a real number \( x \) in the \( p \)-adic scale (e.g. binary expansion, \( p = 2 \)):

\[
x = \cdots + a_{-n}/p^n + a_0 + \cdots + a_k p^k
\]

In the \( p \)-adic case the expansion is finite in the direction of negative powers of \( p \) and infinite in the direction of positive powers of \( p \). In the real case the expansion is infinite in the direction of negative powers of \( p \) and finite in the direction of positive powers of \( p \). In the \( p \)-adic case the expansion is unique; in the real case - not. The elements \( x \in \mathbb{Z}_p \) have the expansion \( x = a_0 + \cdots + a_k p^k + \cdots \) and can thus be identified with the sequences of digits

\[
x = (a_0, \ldots, a_k, \ldots).
\]

We remark that, as \( \mathbb{Q}_p \) is a locally compact additive group, there exists the Haar measure \( dm \) on the \( \sigma \)-algebra of Borel subsets of \( \mathbb{Q}_p \).

If, instead of a prime number \( p \), we start with an arbitrary natural number \( m > 1 \) we construct the system of so-called \( m \)-adic numbers \( \mathbb{Q}_m \) by completing \( \mathbb{Q} \) with respect to the \( m \)-adic metric \( \rho_m(x, y) = |x - y|_m \) which is defined in a similar way to above. However, this system is in general not a field as there may exist divisors of zero; \( \mathbb{Q}_m \) is only a ring. Elements of \( \mathbb{Z}_m = U_1(0) \) can be identified with sequences \( \mathbb{N} \) with the digits \( a_k \in \{0, 1, \ldots, m-1\} \). We can also use more complicated number systems corresponding to non-homogeneous scales \( M = (m_1, m_2, \ldots, m_k, \ldots) \), where the \( m_j > 1 \) are natural numbers, to obtain the number system \( \mathbb{Q}_M \). The elements \( x \in \mathbb{Z}_M = U_1(0) \) can be represented as sequences of the form \( \mathbb{N} \) with digits \( a_j \in \{0, 1, \ldots, m_j-1\} \). The situation here becomes quite complicated mathematically. In general, the number system \( \mathbb{Q}_M \) is not a ring, but \( \mathbb{Z}_M \) is always a ring.

We shall use a \( p \)-adic analogue of complex numbers. As we know, the field of complex numbers \( \mathbb{C} \) is the quadratic extension of \( \mathbb{R} \) with respect to the
root of the equation \( x^2 + 1 = 0 \): \( C = \mathbb{R}(i) \), \( i = \sqrt{-1} \), \( z = x + iy \), \( x, y \in \mathbb{R} \).

In this case we have a very simple algebraic structure, because this quadratic extension is at the same time the algebraic closure of the field of real numbers (every polynomial equation has a solution in \( C \)). In the \( p \)-adic case the structure of algebraic extensions is more complicated. A quadratic extension is not unique. If \( p = 2 \) then there are seven quadratic extensions and if \( p \neq 2 \), then there are three quadratic extensions. Thus if we consider the fixed quadratic extension \( \mathbb{Q}_p(\sqrt{r}) \) of \( \mathbb{Q}_p \) then there exist \( p \)-adic numbers for which it is impossible to find a square root in \( \mathbb{Q}_p(\sqrt{r}) \). All quadratic extensions are not algebraically closed. Extensions of any finite order (i.e., corresponding to roots of polynomials of any order) are not algebraically closed. The algebraic closure \( \mathbb{Q}_p^a \) of \( \mathbb{Q}_p \) is constructed as an infinite chain of extensions of finite orders. Therefore this is an infinite-dimensional vector space over \( \mathbb{Q}_p \). This algebraic closure is not a complete field. Thus we must consider the completion of this field. It is the final step of this long procedure, because this completion is an algebraically closed field (so we are lucky!), Krasner’s theorem, see e.g. [25]. Let us denote this field by \( \mathbb{C}_p \). This field is called the field of complex \( p \)-adic numbers.

### 3 Roots of Unity

The roots of unity in \( \mathbb{C}_p \) will play an important role in our considerations. To find fixed points and cycles of monomial functions \( f(x) = x^n \), we have to find the roots of unity.

As usual in arithmetics, \((n,k)\) denotes the greatest common divisor of two natural numbers. Denote the group of \( m \)th roots of unity, \( m = 1, 2, \ldots, \), by \( \Gamma^{(m)} \). Set

\[
\Gamma = \bigcup_{m=1}^{\infty} \Gamma^{(m)}, \quad \Gamma_m = \bigcup_{j=1}^{\infty} \Gamma^{(mj)}, \quad \Gamma_u = \bigcup_{(m,p)=1} \Gamma_m,
\]

By elementary group theory we have \( \Gamma = \Gamma_u \cdot \Gamma_p \), \( \Gamma_u \cap \Gamma_p = \{1\} \).

Denote the \( k \)th roots of unity by \( \theta_{j,k} \), \( j = 1, \ldots, k \); \( \theta_{1,k} = 1 \).

We remark that \( \Gamma_u \subset S_1(1) \) and \( \Gamma_p \subset U^{-1}_1(1) \).

The following estimate plays the important role in \( p \)-adic analysis and theory of dynamical systems. We also present the proof to demonstrate the rules of working in the framework of \( p \)-adic analysis. We denote binomial coefficients by symbols

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k \leq n.
\]

**Lemma 3.1.** \( |\binom{n}{k}|_{p} \leq 1/p \) for all \( j = 1, \ldots, p^k - 1 \).
Proof. Let \( j = ip + q, \) \( q = 0, 1, \ldots, p - 1. \) First consider the case \( q = 0 : \)
\[
|C^j_{p^k}|_p = |p^{p^k} \cdots (p^k - ip + p)|_p
\]
\[
= \left| \frac{p^k}{ip} \cdots (\frac{p^k - ip + p}{ip - p}) \right|_p = \left| \frac{p^{k-1}}{i} \right|_p \leq \frac{1}{p},
\]
as \( i < p^{k-1}. \) Now let \( q \neq 0 : \)
\[
|C^j_{p^k}|_p = \left| \frac{p^k (p^k - p) \cdots (p^k - ip)}{ip} \right|_p = |p^k|_p \leq \frac{1}{p}.
\]

To find fixed points and cycles of functions \( f(x) = x^n \) in \( \mathbb{Q}_p, \) we have to know whether the roots of unity belong to \( \mathbb{Q}_p. \) We present the corresponding result. Denote by \( \xi_l, \) \( l = 1, 2, \ldots, \) a primitive \( l \)th root of 1 in \( \mathbb{C}_p. \) We are interested in whether \( \xi_l \in \mathbb{Q}_p. \)

**Proposition 3.1.** (Primitive roots) If \( p \neq 2 \) then \( \xi_l \in \mathbb{Q}_p \) if and only if \( l \mid (p - 1). \) The field \( \mathbb{Q}_2 \) contains only \( \xi_1 = 1 \) and \( \xi_2 = -1. \)

To prove this proposition we have to prove the same result for the field \( F_p = \{0, 1, \ldots, p - 1\} \) of mod \( p \) residue classes and apply Hensel’s lemma [2] (\( p \)-adic variant of Newton method, see appendix 1). This is one of the most powerful methods to get results for \( \mathbb{Q}_p : \) first get such a result for \( F_p, \) and try to find conditions to apply Hensel’s lemma.

**Corollary 3.1.** The equation \( x^k = 1 \) has \( g = (k, p - 1) \) different roots in \( \mathbb{Q}_p. \)

### 4 Dynamical Systems in Non-Archimedean Fields

To study dynamical systems in fields of \( p \)-adic numbers \( \mathbb{Q}_p \) and complex \( p \)-adic numbers \( \mathbb{C}_p, \) as well as finite extensions of \( \mathbb{Q}_p, \) it is convenient to consider the general case of an arbitrary non-Archimedean field \( K. \)

Let \( K \) be a field (so all algebraic operations are well defined). Recall that a **non-Archimedean valuation** is a mapping \( | \cdot |_K : K \to \mathbb{R}_+ \) satisfying the following conditions: \( |x|_K = 0 \iff x = 0 \) and \( |1|_K = 1; \) \( |xy|_K = |x|_K |y|_K; \) \( |x + y|_K \leq \max(|x|_K, |y|_K). \) The latter inequality is the well known strong triangle axiom. We often use in non-Archimedean investigations the following property of a non-Archimedean valuation:

\[
|x + y|_F = \max(|x|_F, |y|_F), \text{ if } |x|_F \neq |y|_F.
\]
The field $K$ with the valuation $|\cdot|_K$ is called a non-Archimedean field. The fields of $p$-adic numbers $\mathbb{Q}_p$ and complex $p$-adic numbers $\mathbb{C}_p$ as well as finite extensions of $\mathbb{Q}_p$ are non-Archimedean fields.

Thus all triangles in a non-Archimedean fields (in particular, in fields of $p$-adic numbers) are isosceles.

 Everywhere below $K$ denotes a complete non-Archimedean field with a nontrivial valuation $|\cdot|_K$; $U_r(a), U_r^-(a)$ and $S_r(a)$ are respectively balls and spheres in $K$. We always consider $r \in |K| = \{s = |x|_K : x \in K\}$ for radii of balls $U_r(a)$ and spheres $S_r(a)$. In particular, in the $p$-adic case $r = p^l, l = 0, \pm 1, \pm 2, \ldots$ and in the case of $\mathbb{C}_p - r = p^n, q \in \mathbb{Q}$.

A function $f : U_r(a) \to K$ is said to be analytic if it can be expanded into a power series $f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n$ with $f_n \in K$ which converges uniformly on the ball $U_r(a)$.

Let us study the dynamical system:

$$U \to U, \ x \to f(x), \quad (9)$$

where $U = U_R(a)$ or $K$ and $f : U \to U$ is an analytic function. First we shall prove a general theorem about a behaviour of iterations $x_n = f^n(x_0)$, $x_0 \in U$. As usual $f^n(x) = f \circ \ldots \circ f(x)$. Then we shall use this result to study a behaviour of the concrete dynamical systems $f(x) = x^n, \ n = 2, 3, \ldots$, in the fields of complex $p$-adic numbers $\mathbb{C}_p$.

We shall use the standard terminology of the theory of dynamical systems. If $f(x_0) = x_0$ then $x_0$ is a fixed point. If $x_n = x_0$ for some $n = 1, 2, \ldots$ we say that $x_0$ is a periodic point. If $n$ is the smallest natural number with this property then $n$ is said to be the period of $x_0$. We denote the corresponding cycle by $\gamma = (x_0, x_1, \ldots, x_{n-1})$. In particular, the fixed point $x_0$ is the periodic point of period 1. Obviously $x_0$ is a fixed point of the iterated map $f^n$ if $x_0$ is a periodic point of period $n$.

A fixed point $x_0$ is called an attractor if there exists a neighborhood $V(x_0)$ of $x_0$ such that all points $y \in V(x_0)$ are attracted by $x_0$, i.e., $\lim_{n \to \infty} y_n = x_0$. If $x_0$ is an attractor, we consider its basin of attraction $A(x_0) = \{y \in K : y_n \to x_0, n \to \infty\}$. A fixed point $x_0$ is called repeller if there exists a neighborhood $V(x_0)$ of $x_0$ such that $|f(x) - x_0|_K > |x - x_0|_K$ for $x \in V(x_0), x \neq x_0$. A cycle $\gamma = (x_0, x_1, \ldots, x_{n-1})$ is said to be an attractor (repeller) if $x_0$ is attractor (repeller) of the map $f^n$.

We have to be more careful in defining a non-Archimedean analogue of a Siegel disk. In author's book [91] non-Archimedean Siegel disk was defined in the following way. Let $a \in U$ be a fixed point of a function $f(x)$. The ball $U_r^-(a)$ (contained in $U$) is said to be a Siegel disk if each sphere $S_\rho(a)$, $\rho < r$, is an invariant sphere of $f(x)$, i.e., if one takes an initial point on one of the
spheres $S_\rho(a)$, $\rho < r$, all iterated points will also be on it. The union of all Siegel disks with center in $a$ is said to be a maximal Siegel disk. Denote the maximal Siegel disk by $SI(a)$.

**Remark 4.1.** In complex geometry the center of a disk is uniquely determined by the disk. Hence it does not happen that different fixed points have the same Siegel disk. But in non-Archimedean geometry centers of a disk are nothing but the points which belong to the disk. And in principle different fixed points may have the same Siegel disk (see the next section).

In the same way we define a Siegel disk with center at a periodic point $a \in U$ with the corresponding cycle $\gamma = \{a, f(a), \ldots, f^{n-1}(a)\}$ of the period $n$. Here the spheres $S_\rho(a)$, $\rho < r$, are invariant spheres of the map $f^n(x)$.

As usual in the theory of dynamical systems, we can find attractors, repellers, and Siegel disks using properties of the derivative of $f(x)$. Let $a$ be a periodic point with period $n$ of $C^1$-function $g : U \to U$. Set $\lambda = \frac{dg^n(a)}{dx}$. This point is called: 1) attractive if $0 \leq |\lambda|_K < 1$; 2) indifferent if $|\lambda|_K = 1$; 3) repelling if $|\lambda|_K > 1$.

**Lemma 4.1.** [91] Let $f : U \to U$ be an analytic function and let $a \in U$ and $f'(a) \neq 0$. Then there exist $r > 0$ such that

$$s = \max_{1 \leq n < \infty} \left| \frac{d^n f}{n! dx^n}(a) \right|_K r^{n-1} < |f'(a)|_K.$$  \hspace{1cm} (10)

If $r > 0$ satisfies this inequality and $U_r(a) \subset U$ then

$$|f(x) - f(y)|_K = |f'(a)|_K|x - y|_K$$  \hspace{1cm} (11)

for all $x, y \in U_r(a)$.

By using the previous Lemma we prove:

**Theorem 4.1.** Let $a$ be a fixed point of the analytic function $f : U \to U$. Then:

1. If $a$ is an attracting point of $f$ then it is an attractor of the dynamical system $[\bullet]$. If $r > 0$ satisfies the inequality:

$$q = \max_{1 \leq n < \infty} \left| \frac{d^n f}{n! dx^n}(a) \right|_K r^{n-1} < 1,$$  \hspace{1cm} (12)

and $U_r(a) \subset U$ then $U_r(a) \subset A(a)$.

2. If $a$ is an indifferent point of $f$ then it is the center of a Siegel disk. If $r > 0$ satisfies the inequality $[\bullet]$ and $U_r(a) \subset U$ then $U_r(a) \subset SI(a)$.

3. If $a$ is a repelling point of $f$ then $a$ is a repeller of the dynamical system $[\bullet]$.

We note that (in the case of an attracting point) the condition $[\bullet]$ is less restrictive than the condition $[\bullet]$. 

To study dynamical systems for nonanalytic functions we can use the following theorem of non-Archimedean analysis [23]:

**Theorem 4.2.** (Local injectivity of $C^1$-functions) Let $f : U_r(a) \to K$ be $C^1$ at the point $a$. If $f'(a) \neq 0$ there is a ball $U_s(a)$, $s \leq r$, such that (11) holds for all $x, y \in U_s(a)$.

However, Theorem 4.1 is more useful for our considerations, because Theorem 4.2 is a so-called ‘existence theorem’. This theorem does not say anything about the value of $s$. Thus we cannot estimate a volume of $A(a)$ or $SI(a)$. Theorem 4.1 gives us such a possibility. We need only to test one of the conditions (12) or (10).

Moreover, the case $f'(a) = 0$ is ‘a pathological case’ for nonanalytic functions of a non-Archimedean argument. For example,

*there exist functions $g$ which are not locally constant but $g' = 0$ in every point.*

In our analytic framework we have no such problems.

A **Julia set** $J_f$ for the dynamical system (3) is defined as the closure of the set of all repelling periodic points of $f$. The set $F_f = U \setminus J_f$ is called a **Fatou set.** These sets play an important role in the theory of real dynamical systems. In the non-Archimedean case the structures of these sets were investigated in [91], [93], [95].

We shall also use an analogue of Theorem 4.1 for periodic points. There we must apply our theorem to the iterated function $f^n(x)$.

## 5 Dynamical Systems in the Field of Complex $p$-adic Numbers

As an application of Theorem 4.1 we study the simplest discrete dynamical systems, namely monomial systems:

$$f(x) = \psi_n(x) = x^n, \ n = 2, 3, ..., $$

in fields of complex $p$-adic numbers $\mathbb{C}_p$. We shall see that behaviour of $\psi_n$ crucially depends on the prime number $p$, the base of the corresponding field. Depending on $p$ attractors and Ziegel disks appear and disappear transforming one into another. Especially complex dependence on $p$ will be studied in section 6 devoted to dynamical systems in $\mathbb{Q}_p, p = 2, 3, ..., 1997, 1999, ...$

It is evident that the points $a_0$ and $a_{\infty}$ are attractors with basins of attraction $A(0) = U_1^{-1}(0)$ and $A(\infty) = \mathbb{C}_p \setminus U_1(0)$, respectively. Thus the main scenario is developed on the sphere $S_1(0)$. Fixed points of $\psi_n(x)$ belonging to this sphere are the roots $\theta_{j,n-1}$, $j = 1, ..., n - 1$, of unity of degree $(n - 1)$. There are two essentially different cases: 1) $n$ is not divisible by $p$; 2) $n$ is
for all \( j \).

1. Let \((n, p) = 1\). There all these points are centers of Siegel disks and \(\text{SI}(a_j) = U_1^-(a_j)\). If \(n - 1 = p^l, l = 1, 2, ..., \) then \(\text{SI}(a_j) = \text{SI}(1) = U_1^-(1)\) for all \( j = 1, ..., n - 1 \). If \((n - 1, p) = 1\), then \(a_j \in S_1(1), j = 2, ..., n - 1, \) and \(\text{SI}(a_j) \cap \text{SI}(a_i) = \emptyset, i \neq j\). For any \( k = 2, 3, ... \) all \( k \)-cycles are also centers of Siegel disks of unit radius.

2. If \((n, p) \neq 1\), then these points are attractors and \(U_1^-(a_j) \subset A(a_j)\). For any \( k = 2, 3, ... \) all \( k \)-cycles are also attractors and open unit balls are contained in basins of attraction.

Corollary 5.1. Let \((n, p) = 1\). Let \( n^k - 1 = p^l, l = 1, 2, ... \) Any \( k \)-cycle \( \gamma = (a_1, ..., a_k) \) for such a \( k \) is located in the ball \( U_1^-(1) \); it has the behaviour of a Siegel disk with \( \text{SI}(\gamma) = \bigcup_{j=1}^{k} U_1^-(a_j) = U_1^-(1) \). During the process of the motion the distances \( c_j = \rho_p(x_0, a_j), j = 1, ..., k, \) where \( x_0 \in U_1^-(1) \) is an arbitrary initial point, are changed according to the cyclic law: \((c_1, c_2, ..., c_{n-1}, c_n) \rightarrow (c_n, c_1, ..., c_{n-2}, c_{n-1}) \rightarrow \ldots \).

Thus in the case \((n, p) = 1\) the motion of a point in the ball \( U_1^-(1) \) is very complicated. It moves cyclically (with different periods) around an infinite number of centers. Unfortunately, we could not paint a picture of such a motion in our Euclidean space - it is too restrictive for such images.

Theorem 5.1 does not completely describe the case \((n, p) = 1, (n - 1, p) \neq 1\). Let us consider the general case: \((n, p) = 1, n - 1 = mp^l, (m, p) = 1\) and \( l \geq 0 \). Set \( a_i = \xi_{m^i}, i = 0, 1, ..., m - 1, \) \( b_j = \xi_{p^j}, j = 0, 1, ..., p^l - 1, \) and \( c_{ij} = a_i b_j \). Then these points \( c_{ij}, i = 0, 1, ..., m - 1, j = 0, 1, ..., p^l - 1, \) are centers of Siegel disks and \( \text{SI}(c_{ij}) = U_1^-(c_{ij}) \). For each \( i \) we have \( \text{SI}(c_{0i}) = \text{SI}(c_{1i}) = \ldots = \text{SI}(c_{(p^l - 1)i}) \). If \( i \neq 0 \) then all these disks \( \text{SI}(c_{ij}) \) are in \( S_1(0) \cap S_1(1) \). Further, \( \text{SI}(c_{ij}) \cap \text{SI}(c_{ik}) = \emptyset \) if \( i \neq k \). We can formulate the same result for \( k \)-cycles.

Now we find the basins of attraction \( A(a_j), j = 1, ..., n - 1, (n, p) \neq 1, \) exactly. We begin from the attractor \( a_1 = 1 \).

Let \( n = mp^k, (m, p) = 1, \) and \( k \geq 1 \).

Lemma 5.1. The basin of attraction \( A(1) = \bigcup_{\xi} U_1^-(\xi) \) where \( \xi \in \Gamma_m \); these balls have empty intersections for different points \( \xi \).

Corollary 5.2. Let \( n = p^l, l \geq 1 \). Then \( S_1(1) \) is an invariant sphere of the dynamical system \( \psi_n(x) \).

Examples. 1. Let \( n = p^l, l \geq 1 \). Then \( A(1) = U_1^-(1) \).

2. Let \( p \neq 2 \) and \( n = 2p^l, l \geq 1 \). Then \( A(1) = \bigcup U_1^-(\xi) \) where \( \xi \in \Gamma_2 \).

Theorem 5.2. The basin of attraction \( A(a_k) = \bigcup_{\xi} U_1^-(\xi a_k) \) where \( \xi \in \)
$\Gamma_m$. These balls have empty intersections for different points $\xi$.

The dynamical system $\psi_n(x)$ has no repelling points in $\mathbb{C}_p$ for any $p$. Thus the Julia set $J_{\psi_n} = \emptyset$ and the Fatou set $\mathcal{F}_{\psi_n} = \mathbb{C}_p$, cf. [93], [95].

6 Dynamical Systems in the Fields of $p$-adic Numbers

Here we study the behaviour of the dynamical system $\psi_n(x) = x^n$, $n = 2, 3, \ldots$, in $\mathbb{Q}_p$. In fact, this behaviour can be obtained on the basis of the corresponding behaviour in $\mathbb{C}_p$. We need only to apply the results of section 3 about the roots of unity in $\mathbb{Q}_p$.

**Proposition 6.1.** The dynamical system $\psi_n(x)$ has $m = (n-1, p-1)$ fixed points $a_j = \theta_{j,m}$, $j = 1, \ldots, m$, on the sphere $S_1(0)$ of $\mathbb{Q}_p$. The character of these points is described by Theorem 5.1. Fixed points $a_j \neq 1$ belong to the sphere $S_1(1)$.

We remark that a number of attractors or Siegel disks for the dynamical system $\psi_n(x)$ on the sphere $S_1(0)$ is $\leq (p-1)$.

To study $k$-cycles in $\mathbb{Q}_p$, we use the following numbers:

$m_k = (k, p-1), k = 1, 2, \ldots$, with $l_k = n^k - 1$.

**Proposition 6.2.** The dynamical system $\psi_n(x)$ has $k$-cycles ($k \geq 2$) in $\mathbb{Q}_p$ if and only if $m_k$ does not divide any $m_j$, $j = 1, \ldots, k-1$. All these cycles are located on $S_1(1)$.

In particular, if $(n, p) = 1$ (i.e., all fixed points and $k$-cycles are centers of Siegel disks) there are no such complicated motions around a group of centers as in $\mathbb{C}_p$.

**Corollary 6.1.** The dynamical system $\psi_n(x)$ has only a finite number of cycles in $\mathbb{Q}_p$ for any prime number $p$.

**Example.** Let $n = p^l$, $l \geq 1$. Then $m_1 = p-1$ and there are $p-1$ attractors $a_j = \theta_{j,p-1}$, $j = 1, \ldots, p-1$, with the basins of attraction $A(a_j) = U_{1/p}(a_j)$ and there is no $k$-cycle for $k \geq 2$. As we can choose $a_j = j \mod p$, then $U_{1/p}(a_j) = U_{1/p}(j)$. In particular, if $p = 2$ then all points of the sphere $S_1(0)$ are attracted by $a_1$.

To study the general case $n = qp^l$, $l \geq 1$, $(q, p) = 1$, we use the following elementary fact.

**Lemma 6.1.** Let $n = qp^l$, $l \geq 1$, $(q, p) = 1$. Then $m_k = (l_k, p-1) = (q^{k-1}, p-1)$, $k = 1, 2, \ldots$.

**Examples.** 1). Let $n = 2p$, $p \neq 2$. There is only one attractor $a_1 = 1$ on $S_1(0)$ for all $p$. To find $k$-cycles, $k \geq 2$, we have to consider the numbers $m_k$, $k = 2, \ldots$. However, by Lemma 6.1 $m_k = (2^{k-1}, p-1)$. Thus the number
of $k$-cycles for the dynamical system $\psi_{2p}(x)$ coincides with the corresponding number for the dynamical system $\psi_2(x)$. An extended analysis of the dynamical system $\psi_2(x)$ will be presented after Proposition 6.3. Of course, it should be noted that the behaviours of $k$-cycles for $\psi_{2p}(x)$ and $\psi_2(x)$, $p \neq 2$, are very different. In the first case these are attractors; in the second case these are centers of Siegel disks.

2). Let $n = 3p$, $p \neq 2$. Then there are two attractors $a_1 = 1$ and $a_2 = -1$ on $S_1(0)$ for all $p$.

3). Let $n = 4p$. Here we have a more complicated picture: 1 attractor for $p = 2, 3, 5, 11, 17, 23, \ldots$; 3 attractors for $p = 7, 13, 19, 29, 31, \ldots$.

4). Let $n = 5p$. Here we have: 1 attractor for $p = 2$; 2 attractors for $p = 3, 7, 11, 23, 31, \ldots$; 4 attractors for $p = 5, 13, 17, \ldots$.

We now study basins of attraction (in the case $n = qp^l$, $l \geq 1$, $(q, p) = 1$). As a consequence of our investigations for the dynamical system in $C_p$, we find that $A(1) = \bigcup_\xi U_{1/p}(\xi)$ where $\xi \in \Gamma_q \cap Q_p$. We have $\Gamma_q \cap Q_p \neq \{1\}$ iff $(q, p - 1) \neq 1$.

**Examples.** 1). Let $p = 5$ and $n = 10$, i.e., $q = 2$. As $(q^2, p - 1) = 4$, then $\Gamma_2 \cap Q_5 = \Gamma(4)$ and $A(1) = \bigcup_{j=1}^4 U_{1/5}(\theta_{j,4})$. Thus $A(1) = S_1(0)$. All points of the sphere $S_1(0)$ are attracted by $a_1 = 1$.

2). Let $p = 7$ and $n = 21$, i.e., $q = 3$. There $m_1 = (q - 1, p - 1) = 2$. Hence there are two attractors; these are $a_1 = 1$ and $a_2 = -1$. As all $m_j = (q^k - 1, p - 1) = 2$, $j = 1, 2, \ldots$, then there are no $k$-cycles for $k \geq 2$.

3). Let $p = 7$ and $n = 14$, i.e., $(q, p - 1) = 2$. Thus $\Gamma_2 \cap Q_7 = \Gamma(2)$ and as $m_2 = 3$, there exist 2-cycles. It is easy to see that the 2-cycle is unique and $\gamma = (b_1, b_2)$ with $b_1 = 2, b_2 = 4 \mod 7$. This cycle generates a cycle of balls on the sphere $S_1(1): \gamma^j = (U_{1/7}(2), U_{1/7}(4))$ (‘fuzzy cycle’). Other two balls on $S_1(1): U_{1/7}(3), U_{1/7}(5)$ are attracted by $\gamma^j$ (by the balls $U_{1/7}(2)$ and $U_{1/7}(4)$, respectively).

The last example shows that sometimes it can be interesting to study not only cycles of points but also cycles of balls. We propose the following general definition, see [91]:

Let $x \rightarrow g(x)$, $x \in Q_p$, be a dynamical system. If there exist balls $U_r(a_j)$, $j = 1, \ldots, n$, such that iterations of the dynamical system generate the cycle of balls $\gamma^{(j)} = (U_r(a_1), \ldots, U_r(a_n))$, $(r = p^l, l = 0, \pm 1, \ldots)$ then it is called a fuzzy cycle of length $n$ and radius $r$. Of course, we assume that the balls in the fuzzy cycle do not coincide.

**Proposition 6.3.** There is a one to one correspondence between cycles and fuzzy cycles of radius $r = 1/p$ of the dynamical system $\psi_n(x)$ in $Q_p$.

The situation with fuzzy cycles of radius $r < 1/p$ is more complicated, see the following example and appendix 2.

**Examples.** Let $p = 3, n = 2$. There exist 2-cycles of radius $r = 1/9$ which
define $\psi_2(x) = x^2$. This is the simplest among monomial dynamical systems. However, even here we observe very complicated dependence on $p$.

**Examples.** Let $n = 2$ in all the following examples.

1). Let $p = 2$. There is only one fixed point $a_1 = 1$ on $S_1(0)$. It is an attractor and $A(1) = U_{1/2}(1) = S_1(0)$.

2). As $l_k$ are odd numbers, then $m_k$ must also be an odd number. Therefore there are no any $k$-cycle ($k > 1$) for $p = 3, 5, 17$ and for any prime number which has the form $p = 2^k + 1$.

3). Let $p = 7$. Here $m_k$ can be equal to 1 or 3. As $m_2 = 3$ there are only 2-cycles. It is easy to show that the 2-cycle is unique.

4). Let $p = 11$. Here $m_k = 1$ or 5. As $m_2 = m_3 = 1$ and $m_4 = 5$ there exist only 4-cycles. There is only one 4-cycle: $\gamma(\xi_5)$.

5). Let $p = 13$. Here $m_k = 1$ or 3. As $m_2 = 3$ there exists only the (unique) 2-cycle.

6). Let $p = 19$. Here $m_k = 1$ or 3, or 9. As $m_2 = 3$ there is the (unique) 2-cycle. However, although $m_4 = 3$ there are no 4-cycles because $m_4$ divides $m_3$. As $m_6 = 9$ does not divide $m_2,...,m_5$ there exist 6-cycles and there are no $k$-cycles with $k > 6$. There is only one 6-cycle: $\gamma(\xi_5)$.

7). Let $p = 23$. Here $m_k = 1$ or 11. The direct computations show that there are no $k$-cycles for the first $k = 2,...,8$. Further computations are complicated. We think that an answer to the following question must be known in number theory: Does there exist $k$ such that 11 divides $l_k$?

8). Let $p = 29$. Here $m_k = 1$ or 7. As $m_3 = 7$ and $m_2 = 1$ there exist only 3-cycles. It is easy to show that there are two 3-cycles: $\gamma(\xi_7)$ and $\gamma(\xi_7^2)$.

9). Let $p = 31$. Here $m_k = 1, 3, 5, 15$. As $m_2 = 3$ there exists an (unique) 2-cycle. As $m_4 = 15$ and $m_3 = 1$ there exist 4-cycles: $\gamma(\xi_{15}), \gamma(\xi_{15}^3), \gamma(\xi_{15}^7)$. There are no $k$-cycles with $k \neq 2, 4$.

10). Let $p = 37$. Here $m_k = 1, 3, 9$. As $m_2 = 3$ there exists an (unique) 2-cycle. As $m_6 = 9$ and $m_2 = m_4 = 3, m_3 = m_5 = 1$ there exist 6-cycles. It is easy to show that there is an unique 6-cycle: $\gamma(\xi_9)$. There are no $k$-cycles for $k \neq 2, 6$.

11). Let $p = 41$. Here $m_k = 1, 5$. As $m_4 = 5$ and all previous $m_j = 1$ there exist 4-cycles. It is easy to show that this cycle is unique: $\gamma(\xi_5)$. There are no $k$-cycles with $k \neq 4$.

Thus, even for the simplest $p$-adic dynamical system, $x \to x^2$, the structure of cycles depends in a very complex way on the parameter $p$. In general case this dependence was studied in [101], [102], [106]. It was demonstrated
that a number $N_m$ of cycles of the fixed length, $m$, depends randomly on $p$. So $N_m(p)$ is a random variable defined on the set of all prime numbers. We found mean value and covariance of this random variable [101], [102], [106]. There we used essentially classical results on the distribution of prime numbers, see e.g. [125], [126]. We hope that the connection between the theory of $p$-adic dynamical systems and the classical theory on distributions of prime numbers established in [101], [102], [106] will have further applications.

## 7 $p$-adic ergodicity

In this section we study in details ergodic behavior of $p$-adic monomial dynamical systems. As we have already seen in previous section, behaviour of $p$-adic dynamical systems depends crucially on the prime parameter $p$. The main aim of investigations performed in paper [98] was to find such a $p$-dependence for ergodicity.

Let $\psi_n$ be a (monomial) mapping on $\mathbb{Z}_p$ taking $x$ to $x^n$. Then all spheres $S_p^{-l}(1)$ are $\psi_n$-invariant iff $n$ is a multiplicative unit, i.e., $(n,p) = 1$.

In particular $\psi_n$ is an isometry on $S_p^{-l}(1)$ if and only if $(n,p) = 1$. Therefore we will henceforth assume that $n$ is a unit. Also note that, as a consequence, $S_p^{-l}(1)$ is not a group under multiplication. Thus our investigations are not about the dynamics on a compact (abelian) group.

We remark that monomial mappings, $x \mapsto x^n$, are topologically transitive and ergodic with respect to Haar measure on the unit circle in the complex plane. We obtained [98] an analogous result for monomial dynamical systems over $p$–adic numbers. The process is, however, not straightforward. The result will depend on the natural number $n$. Moreover, in the $p$–adic case we never have ergodicity on the unit circle, but on the circles around the point 1.

### 7.1 Minimality

Let us consider the dynamical system $x \mapsto x^n$ on spheres $S_p^{-l}(1)$. The result depends crucially on the following well known result from group theory. We denote the multiplicative group of the ring $F_k$ of mod k residue classes by the symbol $F_k^*$; we also set $< n > = \{n^N : N = 0, 1, 2, ... \}$ for a natural number $n$.

**Lemma 7.1.** Let $p > 2$ and $l$ be any natural number, then the natural number $n$ is a generator of $F_p^*$ if and only if $n$ is a generator of $F_{p^2}^*$. $F_{p^2}^*$ is noncyclic for $l \geq 3$.

Recall that a dynamical system given by a continuous transformation $\psi$ on a compact metric space $X$ is called *topologically transitive* if there exists a dense orbit $\{\psi^n(x) : n \in \mathbb{N}\}$ in $X$, and (one-sided) *minimal*, if all orbits for
ψ in X are dense. For the case of monomial systems \( x \mapsto x^n \) on spheres \( S_{p-1}(1) \) topological transitivity means the existence of an \( x \in S_{p-1}(1) \) s.t. each \( y \in S_{p-1}(1) \) is a limit point in the orbit of \( x \), i.e. can be represented as

\[
y = \lim_{k \to \infty} x^{nN_k},
\]

for some sequence \( \{N_k\} \), while minimality means that such a property holds for any \( x \in S_{p-1}(1) \). Our investigations are based on the following theorem.

**Theorem 7.1.** For \( p \neq 2 \) the set \( \langle n \rangle \) is dense in \( S_1(0) \) if and only if \( n \) is a generator of \( F_{p^2} \).

**Proof.** We have to show that for every \( \epsilon > 0 \) and every \( x \in S_1(0) \) there is a \( y \in \langle n \rangle \) such that \( |x - y|_p < \epsilon \). Let \( \epsilon > 0 \) and \( x \in S_1(0) \) be arbitrary. Because of the discreteness of the \( p \)-adic metric we can assume that \( \epsilon = p^{-k} \) for some natural number \( k \). But (according to Lemma 7.1) if \( n \) is a generator of \( F_{p^2} \), then \( n \) is also a generator of \( F_{p^l} \) for every natural number \( l \) (and \( p \neq 2 \)) and especially for \( l = k \). Consequently there is an \( N \) such that \( n^N = x \) mod \( p^k \).

From the definition of the \( p \)-adic metric we see that \( |x - y|_p < p^{-k} \) if and only if \( x \) equals to \( y \) mod \( p^k \). Hence we have that

\[
| x^{nN} - y |_p < p^{-k}.
\]

Let us consider \( p \neq 2 \) and for \( x \in U_{p-1}(1) \) the \( p \)-adic exponential function \( t \mapsto x^t \), see, for example [23]. This function is well defined and continuous as a map from \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \). In particular, for each \( a \in \mathbb{Z}_p \), we have

\[
x^a = \lim_{k \to a} x^k, \quad k \in \mathbb{N}.
\]

We shall also use properties of the \( p \)-adic logarithmic function, see, for example [23]. Let \( z \in U_{p-1}(1) \). Then \( \log z \) is well defined. For \( z = 1 + \lambda \) with \( |\lambda|_p \leq 1/p \), we have:

\[
\log z = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \Delta^k}{k} = \lambda(1 + \lambda \Delta), \quad |\Delta|_p \leq 1.
\]

By using (15) we obtain that \( \log : U_{p-1}(1) \to U_{p-1}(0) \) is an isometry:

\[
| \log x_1 - \log x_2 |_p = | x_1 - x_2 |_p, \quad x_1, x_2 \in U_{1/p}(1).
\]

**Lemma 7.2.** Let \( x \in U_{p-1}(1), x \neq 1, a \in \mathbb{Z}_p \) and let \( \{m_k\} \) be a sequence of natural numbers. If \( x^{m_k} \rightarrow x^a, \) \( k \rightarrow \infty \), then \( m_k \rightarrow a \) as \( k \rightarrow \infty \), in \( \mathbb{Z}_p \).

This is a consequence of the isometric property of log.

**Theorem 7.2.** Let \( p \neq 2 \) and \( l \geq 1 \). Then the monomial dynamical system \( x \mapsto x^n \) is minimal on the circle \( S_{p-1}(1) \) if and only if \( n \) is a generator of \( F_{p^2} \).
Proof. Let \( x \in S_{p^{-1}}(1) \). Consider the equation \( x^a = y \). What are the possible values of \( a \) for \( y \in S_{p^{-1}}(1) \)? We prove that \( a \) can take an arbitrary value from the sphere \( S_1(0) \). We have that \( a = \frac{\log x}{\log y} \). As \( \log : U_{p^{-1}}(1) \to U_{p^{-1}}(0) \) is an isometry, we have \( \log(S_{p^{-1}}(1)) = S_{p^{-1}}(1) \). Thus \( a = \frac{\log x}{\log y} \in S_1(0) \) and moreover, each \( a \in S_1(0) \) can be represented as \( \frac{\log x}{\log y} \) for some \( y \in S_{p^{-1}}(1) \).

Let \( y \) be an arbitrary element of \( S_{p^{-1}}(1) \) and let \( x^a = y \) for some \( a \in S_1(0) \). By Theorem 7.1 if \( n \) is a generator of \( F_p^* \), then each \( a \in S_1(0) \) is a limit point of the sequence \( \{n^N\}_{N=1}^\infty \). Thus \( a = \lim_{k \to \infty} n^{N_k} \) for some subsequence \( \{N_k\} \).

By using the continuity of the exponential function we obtain (13).

Suppose now that, for some \( n \), \( x^{n^{N_k}} \to x^a \). By Lemma 7.2 we obtain that \( n^{N_k} \to a \) as \( k \to \infty \). If we have (13) for all \( y \in S_{p^{-1}}(1) \), then each \( a \in S_1(0) \) can be approximated by elements \( n^N \). In particular, all elements \( \{1, 2, ..., p-1, p+1, ..., p^2-1\} \) can be approximated with respect to mod \( p^2 \). Thus \( n \) is a \( \psi \)-generator of \( F_p^* \).

Example. In the case that \( p = 3 \) we have that \( \psi_n \) is minimal if \( n = 2, 2 \) is a generator of \( F_9^* = \{1, 2, 4, 5, 7, 8\} \). But for \( n = 4 \) it is not; \( \langle 4 \rangle \mod 3^2 = \{1, 4, 7\} \). We can also see this by noting that \( S_{1/3}(1) = U_{1/3}(4) \cup U_{1/3}(7) \) and that \( U_{1/3}(4) \) is invariant under \( \psi_4 \).

Corollary 7.1. If \( a \) is a fixed point of the monomial dynamical system \( x \mapsto x^a \), then this is minimal on \( S_{p^{-1}}(a) \) if and only if \( n \) is a generator of \( F_p^* \).

Example. Let \( p = 17 \) and \( n = 3 \). In \( \mathbb{Q}_{17} \) there is a primitive 3rd root of unity, see for example [2]. Moreover, 3 is also a generator of \( F_{17^2}^* \). Therefore there exist \( n \)-th roots of unity different from 1 around which the dynamics is minimal.

7.2. Unique ergodicity. In the following we will show that the minimality of the monomial dynamical system \( \psi_n : x \mapsto x^n \) on the sphere \( S_{p^{-1}}(1) \) is equivalent to its unique ergodicity. The latter property means that there exists a unique probability measure on \( S_{p^{-1}}(1) \) and its Borel \( \sigma \)-algebra which is invariant under \( \psi_n \). We will see that this measure is in fact the normalized restriction of the Haar measure on \( \mathbb{Z}_p \). Moreover, we will also see that the ergodicity of \( \psi_n \) with respect to Haar measure is also equivalent to its unique ergodicity. We should point out that – though many results are analogous to the case of the (irrational) rotation on the circle, our situation is quite different, in particular as we do not deal with dynamics on topological subgroups.

Lemma 7.3. Assume that \( \psi_n \) is minimal. Then the Haar measure \( m \) is the unique \( \psi_n \)-invariant measure on \( S_{p^{-1}}(1) \).

Proof. First note that minimality of \( \psi_n \) implies that \( (n, p) = 1 \) and hence
that $\psi_n$ is an isometry on $S_{p^{-1}}(1)$. Then, as a consequence of Theorem 27.5 in [23], it follows that $\psi_n(U_r(a)) = U_r(\psi_n(a))$ for each ball $U_r(a) \subset S_{p^{-1}}(1)$. Consequently, for every open set $U \neq \emptyset$ we have $S_{p^{-1}}(1) = \bigcup_{N=0}^{\infty} \psi_n^N(U)$. It follows for a $\psi_n$–invariant measure $\mu$ that $\mu(U) > 0$.

Moreover we can split $S_{p^{-1}}(1)$ into disjoint balls of radii $p^{-(l+k)}$, $k \geq 1$, on which $\psi_n$ acts as a permutation. In fact, for each $k \geq 1$, $S_{p^{-1}}(1)$ is the union, $S_{p^{-1}}(1) = \bigcup_{l \geq 1} \bigcup_{b \in \{0, 1, \ldots, p-1\}} U_{p^{-l-k}}(1 + b \cdot p^{l-k}),$ (17)

where $b \in \{0, 1, \ldots, p-1\}$ and $b_1 \neq 0$.

We now show that $\psi_n$ is a permutation on the partition (17). Recall that every element of a $p$–adic ball is the center of that ball, and as pointed out above $\psi_n(U_r(a)) = U_r(\psi_n(a))$. Consequently we have for all positive integers $k$, $\psi_n^k(a) \in U_r(a) \Rightarrow \psi_n^k(U_r(a)) = U_r(\psi_n^k(a)) = U_r(a)$ so that $\psi_n^{Nk}(a) \in U_r(a)$ for every natural number $N$. Hence, for a minimal $\psi_n$ a point of a ball $B$ of the partition (17) must move to another ball in the partition.

Furthermore the minimality of $\psi_n$ shows indeed that $\psi_n$ acts as a permutation on balls. By invariance of $\mu$ all balls must have the same positive measure. As this holds for any $k$, $\mu$ must be the restriction of Haar measure $m$.

The arguments of the proof of Lemma 7.3 also show that Haar measure is always $\psi_n$–invariant. Thus if $\psi_n$ is uniquely ergodic, the unique invariant measure must be the Haar measure $m$. Under these circumstances it is known [127] that $\psi_n$ must be minimal.

**Theorem 7.3.** The monomial dynamical system $\psi_n : x \mapsto x^n$ on $S_{p^{-1}}(1)$ is minimal if and only if it is uniquely ergodic in which case the unique invariant measure is the Haar measure.

Let us mention that unique ergodicity yields in particular the ergodicity of the unique invariant measure, i.e., the Haar measure $m$, which means that

$$\frac{1}{N} \sum_{i=0}^{N-1} f(x^{ni}) \to \int f \, dm \quad \text{for all } x \in S_{p^{-1}}(1),$$ (18)

and all continuous functions $f : S_{p^{-1}}(1) \to \mathbb{R}$.

On the other hand the arguments of the proof of Lemma 7.3, i.e., the fact that $\psi_n$ acts as a permutation on each partition of $S_{p^{-1}}(1)$ into disjoint balls if and only if $\langle n \rangle = F^*_p$, proves that if $n$ is not a generator of $F^*_p$ then the system is not ergodic with respect to Haar measure. Consequently, if $\psi_n$ is ergodic then $\langle n \rangle = F^*_p$ so that the system is minimal by Theorem 7.2, and hence even uniquely ergodic by Theorem 7.3. Since unique ergodicity implies ergodicity one has the following.
Theorem 7.4. The monomial dynamical system $\psi_n : x \mapsto x^n$ on $S_{p^{-l}(1)}$ is ergodic with respect to Haar measure if and only if it is uniquely ergodic.

Even if the monomial dynamical system $\psi_n : x \mapsto x^n$ on $S_{p^{-l}(1)}$ is ergodic, it never can be mixing, especially not weak-mixing. This can be seen from the fact that an abstract dynamical system is weak-mixing if and only if the product of such two systems is ergodic. If we choose a function $f$ on $S_{p^{-l}(1)}$ and define a function $F$ on $S_{p^{-l}(1)} \times S_{p^{-l}(1)}$ by $F(x, y) := f(\log x / \log y)$ (which is well defined as log does not vanish on $S_{p^{-l}(1)}$), we obtain a non-constant function satisfying $F(\psi_n(x), \psi_n(y)) = F(x, y)$. This shows, see [127], that $\psi_n \times \psi_n$ is not ergodic, and hence $\psi_n$ is not weak-mixing with respect to any invariant measure, in particular the restriction of Haar measure.

Let us consider the ergodicity of a perturbed system

$$\psi_q = x^n + q(x),$$

for some polynomial $q$ such that $q(x)$ equals to $0 \mod p^{l+1}$, $(|q(x)|_p < p^{-(l+1)})$. This condition is necessary in order to guarantee that the sphere $S_{p^{-l}(1)}$ is invariant. For such a system to be ergodic it is necessary that $n$ is a generator of $F_{p^2}^\star$. This follows from the fact that for each $x = 1 + a_i p^l + ...$ on $S_{p^{-l}(1)}$ (so that $a_l \neq 0$) the condition on $q$ gives

$$\psi_q^N(x) \text{ equals to } 1 + n^N a_i \mod p^{l+1}.$$  

Appendix 1: Newton’s Method (Hensel’s Lemma)

Here we present a $p$-adic analogue of the Newton procedure to find the roots of polynomial equations, see e.g. [2]:

**Theorem.** Let $F(x), x \in \mathbb{Z}_p$, be a polynomial with coefficients $F_i \in \mathbb{Z}_p$. Let there exists $\gamma \in \mathbb{Z}_p$ such that

$$F(\gamma) = 0 \text{ (mod } p^{2\delta+1}) \text{ and } F'(\gamma) = 0 \text{ (mod } p^\delta), \ F'(\gamma) \neq 0 \text{ (mod } p^{\delta+1}),$$

where $\delta$ is a natural number. Then there exists a $p$-adic integer $\alpha$ such that

$$F(\alpha) = 0 \text{ and } \alpha = \gamma \text{ (mod } p^{\delta+1}).$$

**Corollary (Hensel Lemma).** Let $p(x)$ be a polynomial with $p$-adic integer coefficients and let there exist $\gamma \in \mathbb{Z}_p$ such that:

$$F(\gamma) = 0 \mod p, \ F'(\gamma) \neq 0 \mod p.$$
Then there exists $\alpha \in \mathbb{Z}_p$ such that

$$F(\alpha) = 0 \quad \text{and} \quad \alpha = \gamma \pmod{p}.$$

**Appendix 2: Computer Calculations for Fuzzy Cycles**

The following results were obtained with the aid of the complex of $p$-adic programs, $p$-ADIC, which was created by De Smedt, see e.g. [92], using the standard software package MATHEMATICA.

**Example.** Consider the function $\psi_3(x) = x^3$ in $\mathbb{Q}_5$. Then we found among others the following fuzzy cycles. Cycles of length 2:

- $U_{125}^1(2) - U_{125}^1(3)$; $U_{25}^1(7) - U_{25}^1(18)$; $U_{125}^1(57) - U_{125}^1(68)$.

Cycles of length 4:

- $U_{25}^1(6) - U_{25}^1(16) - U_{25}^1(21) - U_{25}^1(11)$;
- $U_{25}^1(2) - U_{25}^1(8) - U_{25}^1(12) - U_{25}^1(3)$;
- $U_{25}^1(22) - U_{25}^1(23) - U_{25}^1(17) - U_{25}^1(13)$;
- $U_{25}^1(9) - U_{25}^1(4) - U_{25}^1(14) - U_{25}^1(19)$;
- $U_{125}^1(7) - U_{125}^1(93) - U_{125}^1(107) - U_{125}^1(43)$;
- $U_{125}^1(26) - U_{125}^1(76) - U_{125}^1(101) - U_{125}^1(51)$;
- $U_{125}^1(18) - U_{125}^1(82) - U_{125}^1(112) - U_{125}^1(32)$;
- $U_{125}^1(24) - U_{125}^1(74) - U_{125}^1(99) - U_{125}^1(49)$.

Cycles of length 20:

- $U_{125}^1(6) - U_{125}^1(91) - U_{125}^1(71) - U_{125}^1(36) - U_{125}^1(31)$
- $-U_{125}^1(41) - U_{125}^1(46) - U_{125}^1(86) - U_{125}^1(56) - U_{125}^1(116)$
- $-U_{125}^1(21) - U_{125}^1(11) - U_{125}^1(81) - U_{125}^1(66)$
- $-U_{125}^1(121) - U_{125}^1(61) - U_{125}^1(106)$
- $-U_{125}^1(16) - U_{125}^1(96) - U_{125}^1(111)$

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\[ p \text{-ADIC gives the possibility of studying more complicated dynamical systems. However, we cannot find exact cycles with the aid of a computer. As a consequence of a finite precision we can find only fuzzy cycles. Therefore we shall study fuzzy cycles and their behaviour. We define fuzzy attractors and fuzzy Siegel disks by direct generalization of the corresponding definitions for point cycles. Let } f(x) = x^2 + x. \text{ The following fuzzy cycles were found in the case where } p \text{ is a prime less than 100.}
\]

- Cycles of length 2 for \( p = 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97 \).
- Moreover, we have proved the following general statement.

**Proposition.** Let \( p = 1 (\text{mod } 4) \). Then the dynamical system \( f(x) = x^2 + x \) has fuzzy cycles of length 2. In case \( p = 5 \) these are fuzzy cyclic attractors, and in the other case these are Siegel disks.

- Cycles of length 3 for \( p = 11, 41, 43, 59, 67, 89 \) (twice), 97. In the case \( p = 89 \) one of the fuzzy cycles is an attractor, all others are Siegel disks.
- Cycles of length 4 for \( p = 19, 43, 47, 71 \) (all Siegel disks).
- Cycles of length 5 for \( p = 23, 41, 71, 73 \) (all Siegel disks).
- Cycles of length 6 for \( p = 47, 83, 89 \) (all Siegel disks).
- Cycles of length 7 for \( p = 29, 53, 59, 67 \) (cyclic attractors in the case \( p = 29 \)).
- Cycles of length 8 for \( p = 61 \) (all Siegel disks).
- Cycles of length 9 for \( p = 31 \) (all Siegel disks).

Remark that for some primes we have fuzzy cycles of different lengths. There are fuzzy cycles of length 2, 3 and 6, for example, for \( p = 89 \). There are fuzzy cycles of length 2, 3 and 5 and for \( p = 41 \).

Some of these cycles (we think all of them, but we have not proved it) contain subcycles. For example, in the case \( p = 11 \) we have the cycle of length 3: \( U_{11}(2) - U_{11}(6) - U_{11}(9) \), which contains subcycles of length 15:

\[
\begin{align*}
U_{121}(112) & \quad - U_{121}(72) - U_{121}(53) - U_{121}(79) - U_{121}(28) \\
- U_{121}(86) & \quad - U_{121}(101) - U_{121}(17) - U_{121}(64) - U_{121}(46) \\
- U_{121}(105) & \quad - U_{121}(119) - U_{121}(2) - U_{121}(6) - U_{121}(42)
\end{align*}
\]

and

\[
\begin{align*}
U_{121}(35) & \quad - U_{121}(50) - U_{121}(9) - U_{121}(90) - U_{121}(83) \\
- U_{121}(75) & \quad - U_{121}(13) - U_{121}(61) - U_{121}(31) - U_{121}(24) \\
- U_{121}(116) & \quad - U_{121}(20) - U_{121}(57) - U_{121}(39) - U_{121}(108)
\end{align*}
\]

and so on.
In the case \( p = 13 \) we have the cycle of length 2: \( U_{1/13}(4) - U_{1/13}(7) \) which contains, amongst others, the subcycle of length 8:

\[
U_{1/169}(4) - U_{1/169}(20) - U_{1/169}(82) - U_{1/169}(36) \\
- U_{1/169}(134) - U_{1/169}(7) - U_{1/169}(56) - U_{1/169}(150)
\]

which contains, amongst others, subcycles of length 104.

One of the problems which arise in our computer investigations of \( p \)-adic dynamical systems is that we cannot propose a reasonable way of creating \( p \)-adic pictures which can illustrate our numerical results. However, this is a general problem of the \( p \)-adic framework because the human brain can understand only pictures in real space.

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