Closed minimal surfaces of high Morse index in manifolds of negative curvature

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Abstract
We show that compact Riemannian three-manifolds with negative sectional curvature possess closed minimal surfaces of arbitrarily high Morse index.

1 Sources of noncompactness for parametrized minimal surfaces

In this article, we apply the theory of parametrized two-dimensional minimal surfaces in a compact \( n \)-dimensional Riemannian manifold \( M \) to the case in which \( M \) has strictly negative sectional curvature. We begin with a discussion of how this case fits within the general theory of parametrized minimal surfaces in Riemannian manifolds, as presented in [10].

A parametrized minimal surface of genus \( g \) in a curved \( n \)-dimensional Riemannian manifold \( M \) is a critical point of the two-variable Dirichlet energy

\[
E : \text{Map}(\Sigma_g, M) \times \mathcal{T}_g \rightarrow \mathbb{R},
\]

where \( \Sigma_g \) is the compact connected oriented surface of genus \( g \) and \( \mathcal{T}_g \) is the Teichmüller space of marked conformal structures on \( \Sigma_g \). This two-variable energy \( E \) is defined by the Dirichlet integral,

\[
E(f, \omega) = \frac{1}{2} \int_{\Sigma} \left[ \left( \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial x} \right) + \left( \frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial y} \right) \right] dx dy = \frac{1}{2} \int_{\Sigma} |df|^2 dA,
\]

with \((x, y)\) being local \( \omega \)-conformal coordinates on \( \Sigma \), and \( |df| \) and \( dA \) representing the norm of \( df \) and the area element with respect to any metric within the conformal structure \( \omega \) on \( \Sigma \).
We start by assuming that \( g \geq 2 \). Then the mapping class group \( \Gamma_g \), which is defined to be the group of homotopy classes of orientation-preserving diffeomorphisms of \( \Sigma_g \), is a symmetry group for \( E \). This group \( \Gamma_g \) acts on both factors of the domain of the energy \( E \), and the product action preserves \( E \), so \( E \) descends to a continuous map on the quotient,

\[
E : M(\Sigma_g, M) = \frac{\text{Map}(\Sigma_g, M) \times T_g}{\Gamma_g} \longrightarrow \mathbb{R}.
\]  

(1)

The projection \( \pi : \text{Map}(\Sigma_g, M) \times T_g \rightarrow T_g \) induces a map

\[
\pi : M(\Sigma_g, M) \longrightarrow M_g = T_g/\Gamma_g,
\]

(2)

with \( M_g = T_g/\Gamma_g \) being the Riemann moduli space of conformal structures on \( \Sigma_g \). The isotropy group for the \( \Gamma_g \)-action varies from point to point, so that the quotients in (2) should be thought of as orbifolds, not manifolds.

There is a more general approach to minimal surfaces via geometric measure theory, which can be formulated for integral currents of arbitrary dimension within \( M \). But when the minimal surface has dimension two, the parametrized approach sometimes makes more refined results possible because one has a priori control on the genus. A second advantage is that the Dirichlet energy \( E \) admits an \( \alpha \)-energy perturbation which makes Morse theory by perturbation available.

The perturbation we can use is the \( \alpha \)-energy of Sacks and Uhlenbeck [12], [13], which is the function

\[
E_\alpha : \text{Map}(\Sigma_g, M) \times T_g \longrightarrow \mathbb{R}
\]

defined by

\[
E_\alpha(f, \omega) = \frac{1}{2} \int \int \Sigma \left[ (1 + |df|^2)^\alpha - 1 \right] dA,
\]

(3)

for \( \alpha > 1 \). The \( \alpha \)-energy \( E_\alpha \) is dependent on the choice of metric on \( \Sigma_g \), not just the conformal structure, and we choose the metric to have constant curvature \(-1\) when \( g \geq 2 \). Just like the Dirichlet energy \( E \), \( E_\alpha \) is preserved by the action of \( \Gamma_g \), and as \( \alpha \rightarrow 1 \), \( E_\alpha \) approaches \( E \). When \( \text{Map}(\Sigma, M) \) is completed with respect to the \( L^2 \) topology, the function \( E_\alpha \) is \( C^2 \) on the resulting Banach manifold, and it can be proven that the critical points of \( E \) are automatically \( C^\infty \). Moreover, for fixed choice of \( \omega \in T_g \),

\[
E_{\alpha, \omega} : L^2(\Sigma, M) \longrightarrow \mathbb{R}, \quad E_{\alpha, \omega}(f) = E_\alpha(f, \omega),
\]

satisfies Condition C of Palais and Smale, which is what is needed to develop an infinite-dimensional Morse theory on each fiber of (2).

One can formulate minimax constraints for \( \alpha \)-energy via arbitrary elements of the cohomology of the mapping space \( \text{Map}(\Sigma, M) \). Because of the symmetry group \( \Gamma_g \), we should really study \( \Gamma_g \)-orbits, so it is better to consider constraints coming from the \( \Gamma_g \)-equivariant cohomology of \( \text{Map}(\Sigma, M) \times T_g \).

We can relax our hypothesis that \( g \geq 2 \). A similar theory for parametrized minimal surfaces of genus zero or one is given in [10], although the symmetry...
group in these cases contains the group of complex automorphisms of the domain \( \Sigma \), which is positive-dimensional.

We would like to prove existence of minimal surfaces of a specific genus corresponding to a given topological constraint when the perturbation is turned off. Indeed, our goal is to develop a partial Morse theory for parametrized minimal surfaces in compact Riemannian manifolds parallel to the Morse theory for closed geodesics, a theory which should have implications for the existence of closed minimal surfaces. But the parametrized theory encounters three potential problems, which we call \textit{sources of noncompactness}:

1. As \( \alpha \to 1 \), we must allow for bubbling, with the limit being a collection of minimal two-spheres or a base minimal surface of genus \( g \geq 1 \) together with a collection of minimal two-spheres.

2. For a minimizing sequence, the conformal structure on \( \Sigma \) may approach the boundary of moduli space, implying a possible degeneration to a surface of lower genus or a disconnected surface.

3. The sequence may approach a nontrivial branched cover of a prime minimal surface of lower energy. Branched covers count as critical points within a (possibly different) space of functions, although they are not geometrically distinct from the covered surface.

In spite of these difficulties, one can often prove important topological theorems via the parametrized theory. For example, the geometric sphere theorem of Meeks and Yau (see §4.7 of [10]) provides imbedded minimal two-spheres which divide a compact oriented three-manifold \( M \) into its prime decomposition, even though not every component of \( \text{Map}(S^2, M) \) can be represented by an area minimizer.

Because of the sources of noncompactness, we cannot hope to achieve full Morse inequalities for \( E \) in parallel with those achievable in the theory of closed geodesics. The most we might achieve is partial Morse inequalities, involving those minimax constraints which manage to avoid all three sources of noncompactness. This is offset by the fact that when \( \Sigma \) is a surface and \( M \) is simply connected of dimension at least four, the cohomology of the mapping space \( \text{Map}(\Sigma, M) \) is far richer than that of the mapping space \( \text{Map}(S^1, M) \) encountered in the theory of closed geodesics, with an internal structure that can be exploited.

In this article we focus on a simple case in which the sources of noncompactness are controlled, and describe the resulting equivariant Morse inequalities.

\section{Ambient manifolds of negative curvature}

When the ambient manifold \( M \) is compact and has negative sectional curvature, a case first studied by Tromba (see [15] or §6.8 of [2]), all three noncompactness problems are easily controlled making the theory much easier, and full Morse inequalities become achievable.
Indeed, when $M$ is compact and has negative sectional curvature, it follows from the Hadamard-Cartan Theorem that $M$ is an “aspherical manifold,” a $K(\pi, 1)$ for which the original Eells-Sampson theory of harmonic maps via heat flow works with no difficulty. As Tromba points out, this implies that $E$ reduces to a smooth $\Gamma_g$-equivariant function on the finite-dimensional Teichmüller space $T_g$, simplifying the analysis. We will see that although $T_g$ is contractible, equivariant Morse inequalities make it possible to prove existence of closed minimal surfaces of high Morse index.

The curvature hypothesis on $M$ rules out minimal two-spheres and tori by Corollary 4.6.2 of [10], and this focuses attention on the case in which the domain $\Sigma$ is an compact oriented surface of genus $g \geq 2$. Moreover, nonexistence of minimal two-spheres makes bubbling impossible a priori, eliminating the first, and arguably most difficult, of the sources of noncompactness.

We make a further simplification by restricting to only those components of the mapping space $\text{Map}(\Sigma, M)$ which contain incompressible elements, maps $f : \Sigma \to M$ which are injective on the fundamental group. If $\text{Map}_0(\Sigma, M)$ is such a component, it follows from Theorem 4.8.1 of [10] (which is proven using techniques of Schoen and Yau [14]) that

$$E_\alpha : \mathcal{M}_0(\Sigma, M) = \frac{\text{Map}_0(\Sigma, M) \times T_g}{\Gamma_g} \longrightarrow \mathbb{R}$$

satisfies Condition C. Note that it only the quotient by $\Gamma_g$ that satisfies Condition C, but we can think of elements of this quotient as $\Gamma_g$-orbits within $\text{Map}_0(\Sigma, M) \times T_g$. Since we are interested in counting only those minimal surfaces which are “geometrically distinct,” we further restrict to those components which contain no unbranched covers of incompressible maps from surfaces of lower genus.

More precisely, we say that a component $\text{Map}_0(\Sigma, M)$ of $\text{Map}(\Sigma, M)$ is incompressible if any $f \in \text{Map}_0(\Sigma, M)$ induces a monomorphism on fundamental groups. We further call the incompressible component prime if there is no unbranched cover $p : \Sigma \to \Sigma_0$, with the genus of $\Sigma_0$ strictly less than the genus $g$ of $\Sigma$, together with an incompressible map $f_0 : \Sigma_0 \to M$ such that $f = f_0 \circ p$.

We can state these conditions in terms of surface subgroups, a surface subgroup of genus $g \geq 2$ within $\pi_1(M)$ being a subgroup with presentation

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle,$$

and hence isomorphic to the fundamental group of a compact oriented surface of genus $g$. Then incompressible components of $\text{Map}(\Sigma, M)$ correspond to surface subgroups of $\pi_1(M)$ of genus $g$, while prime incompressible components correspond to surface subgroups of genus $g$ which are not contained in any surface subgroup of genus strictly less than $g$.

It might seem at first that restriction to prime incompressible components is stringent, but Kahn and Markovic [7, 8] show that when $M$ is any compact three-manifold with constant negative curvature metric, there are many such components. Indeed, they show that there is a constant $c > 0$ such that there
are at least \((cg)^2\) incompressible components in which the domain has genus \(g\), and the rate of growth implies that most of these must be prime. Hamenstädt [4] further shows that many higher dimensional locally symmetric spaces of rank one also contain incompressible immersed surfaces.

We claim it is possible to establish Morse inequalities for parametrized minimal surfaces in any prime incompressible component \(\text{Map}_0(\Sigma, M)\) when the metric on \(M\) is generic and has negative sectional curvature:

**Theorem 2.1.** Suppose that \(M\) is a compact Riemannian manifold of dimension \(n \geq 3\) with a generic metric of negative sectional curvature, that \(\Sigma\) is a compact oriented surface of genus \(g\) and that \(\text{Map}_0(\Sigma, M)\) is a prime incompressible component of \(\text{Map}(\Sigma, M)\). Then

\[
\text{(the number of } \Gamma \text{-orbits of minimal surfaces in } \text{Map}_0(\Sigma, M) \text{ of index } \lambda) \geq \dim H^\lambda(M_g; \mathbb{Q}),
\]

where \(M_g\) is the moduli space of conformal structures on the oriented surface of genus \(g\).

The rational cohomology of \(M_g\) has been extensively studied by algebraic geometers and topologists. David Mumford made conjectures about the stable cohomology which were verified by Madsen and Weiss, as nicely explained in [16]. With some work, it is possible to define inclusions

\[
\cdots \rightarrow H^\lambda(M_g; \mathbb{Q}) \rightarrow H^\lambda(M_{g+1}; \mathbb{Q}) \rightarrow H^\lambda(M_{g+2}; \mathbb{Q}) \rightarrow \cdots
\]

and for each \(\lambda\) this sequence stabilizes as the genus increases, giving the so-called stable cohomology of moduli space. Madsen and Weiss show that this stable cohomology is a polynomial algebra

\[
H^\ast_{\text{stable}}(M_g; \mathbb{Q}) \cong \mathbb{P}[\kappa_1, \kappa_2, \kappa_3, \ldots], \quad \deg \kappa_i = 2i,
\]

where \(\kappa_1, \kappa_2, \kappa_3, \ldots\) are certain explicit cohomology classes, thus verifying Mumford’s conjecture. This is complemented by Harer’s stability theorem which implies that

\[
H^\lambda(M_g; \mathbb{Q}) \cong H^\lambda_{\text{stable}}(M_g; \mathbb{Q}), \quad \text{for } \lambda \leq (2/3)(g - 1).
\]

Theorem 2.1 and the resolution of Mumford’s conjecture give quite explicit lower bounds on the number of minimal surfaces of genus \(g\) with Morse index \(\lambda\) in hyperbolic three-manifolds when the metric is perturbed to be generic and \(\lambda \leq (2/3)(g - 1)\). Although there are only finitely many geometrically distinct minimal surfaces of each genus \(g \geq 2\), infinitely many minimal surfaces appear as the genus grows, with higher and higher Morse index. We thus obtain an elegant application of the Madsen-Weiss theory to the existence of minimal surfaces of high Morse index in hyperbolic three-manifolds with generic metrics of negative curvature:
Theorem 2.2. Suppose that $M$ is a compact Riemannian manifold of dimension $n \geq 3$ with generic metric of negative sectional curvatures, that $\Sigma$ is a compact oriented surface of genus $g$ and that $\text{Map}_0(\Sigma, M)$ is a prime incompressible component of $\text{Map}(\Sigma, M)$. Then for $\lambda \leq \left(\frac{2}{3}\right)(g - 1)$, the number of minimal immersions in $\text{Map}_0(\Sigma, M)$ of Morse index $\lambda$ is $\geq$ the number of monomials of degree $\lambda$ in the polynomial ring

$$P[\kappa_1, \kappa_2, \kappa_3, \ldots], \quad \text{where} \quad \deg \kappa_i = 2i.$$ 

When the dimension of the ambient manifold $M$ is at least five, it follows from the Transversal Crossing Theorem 5.1.2 of [10] that the minimal immersions whose existence is guaranteed by Theorem 2.2 are imbeddings.

We can compare these results with well-known facts regarding smooth closed geodesics in a compact Riemannian manifold of negative sectional curvature $M$: each component of $\text{Map}(S^1, M)$ has a unique strictly stable smooth closed geodesic. When $M$ is three-dimensional and has negative sectional curvature, the theory of minimal surfaces of high genus is much richer. We will see that Condition C for the function $E_\alpha$ of (4) implies that when the metric is generic, the number of minimal surfaces in a prime incompressible component $\text{Map}_0(\Sigma, M)$ is always finite. But Theorem 2.2 shows that there are closed immersed minimal surfaces in $M$ of arbitrarily high Morse index as we allow the genus to grow.

3 Proof of Theorem 2.1

We first note that the only remaining source of noncompactness within a prime incompressible component $\text{Map}_0(\Sigma, M)$ is the possibility of branched covers with nontrivial branch locus. But suppose that $(f, \omega)$ is a parametrized minimal surface, with $f \in \text{Map}_0(\Sigma, M)$, which is a nontrivial branched cover of a surface of lower genus. Then $f = f_0 \circ p$ where $p : \Sigma \to \Sigma_0$ is a nontrivial branched cover and $f_0 : \Sigma_0 \to M$ is a prime parametrized minimal surface, which must be an immersion by the Bumpy Metric Theorem. We claim that

$$(f_0)_* : \pi_1(\Sigma_0, q_0) \to \pi_1(M, f(q_0))$$

is a monomorphism when $q_0 \in \Sigma_0$ is not in the image of the branch locus. Indeed, any element $\gamma \in \pi_1(\Sigma_0, q_0)$ has infinite order, while some power of $\gamma$ must lift to an element in $\pi_1(\Sigma, q)$ for some $q \in p^{-1}(q_0)$, and no power of that lift can go to zero in $\pi_1(M, f(q_0))$, implying that also no power of $\gamma$ can go to zero. Since $f_*$ is a monomorphism, so is $p_*$, and hence

$$p_* : \pi_1(\Sigma, q) \to \pi_1(\Sigma_0, q_0),$$

is a monomorphism between surface subgroups. Such monomorphisms correspond to coverings of $\Sigma_0$ by $\Sigma$ with empty branch locus. So incompressibility of $f$ implies that the branch locus is empty, or put another way, branched covers
of prime minimal surfaces with nontrivial branch locus can never lie in incompressible components of $\text{Map}(\Sigma, M)$.

Thus negative sectional curvature implies that all sources of noncompactness are eliminated in the prime incompressible components.

Once a prime incompressible component $\text{Map}_0(\Sigma, M)$ has been chosen, it follows from the well-known heat flow theorem of Eells and Sampson [3] together with the uniqueness result of Hartman [6] that for fixed conformal structure $\omega \in T_g$, there is a unique harmonic map $f : \Sigma \to M$ in $\text{Map}_0(\Sigma, M)$ which induces the conformal structure $\omega$, and indeed the Eells-Sampson heat flow provides a contraction of $\text{Map}_0(\Sigma, M)$ to that harmonic map. This defines a section

$$\sigma : T \to \text{Map}_0(\Sigma, M),$$

which takes a conformal structure to the corresponding harmonic map.

To show that this section is smooth, we reason at first formally, computing the first derivative of energy as

$$dE(f, \omega)(X, \dot{\omega}) = \int_{\Sigma} \langle F(f, \omega), X \rangle dA + \text{(term involving } \dot{\omega}),$$

where

$$F : \text{Map}_0(\Sigma, M) \times T_g \to T\text{Map}_0(\Sigma, M)$$

is the first component of an Euler-Lagrange operator for $E$ which gives the variational equations for critical points. We can think of $F$ as a parametrized vector field on $\text{Map}(\Sigma, M)$, the parameter being $\omega \in T_g$, whose zeros are the $\omega$-harmonic maps. A calculation of the second derivative of energy for a fixed choice of conformal structure then yields

$$d^2E(\sigma(\omega), \omega)(X, 0, Y, 0) = d^2E_\omega(\sigma(\omega))(X, Y) = \int_{\Sigma} \langle L_\omega(X), Y \rangle dA,$$

where $L_\omega$ is the second order Jacobi operator for $E_\omega$, which is the linearization of $F$ when $\omega$ is fixed. But a calculation of second variation for $E_\omega$ (Corollary 4.5.2 of [10]) shows that all $\omega$-harmonic maps are strictly stable when the metric has negative sectional curvature. We can therefore apply the implicit function theorem to the appropriate Sobolev completions of the function spaces, and conclude that the parametrized vector field $F$ is transverse to the zero-section of the bundle

$$T\text{Map}(\Sigma, M) \times T_g \to \text{Map}(\Sigma, M) \times T_g.$$

It follows that the intersection of the image of $F$ with the zero-section is smooth, and this intersection is just the image of $\sigma$.

The construction of the preceding paragraph can be extended to apply to the $\alpha$-energy. Thus

$$dE_\alpha(f, \omega)(X, \dot{\omega}) = \int_{\Sigma} \langle F_\alpha(f, \omega), X \rangle dA + \text{(term involving } \dot{\omega}),$$
where $F_\alpha$ is the first component of an Euler-Lagrange map for $E_\alpha$, the zeros of $F_\alpha$ being the critical points for $E_{\alpha,\omega}$. Once again, a second variation formula for $E_{\alpha,\omega}$ (Proposition 4.5.1 of [10]) shows that all critical points for $E_{\alpha,\omega}$ are strictly stable when the metric has negative sectional curvature. But since $E_{\alpha,\omega}$ satisfies Condition C, we can show that any component of the mapping space, say $\text{Map}_0(\Sigma, M)$ contains an element minimizing $E_{\alpha,\omega}$. Moreover, using Liusternik-Schnirelmann theory on Banach manifolds (see [11]) we can apply the mountain pass lemma to conclude that if the component contained two distinct critical points, both strictly stable, there would be a third which is not strictly stable, contradicting strict stability. Thus there is a unique critical point which minimizes $E_{\alpha,\omega}$. As before, the implicit function theorem yields a smooth section

$$
\sigma_\alpha : T \rightarrow \text{Map}_0(\Sigma, M),
$$

which takes a conformal structure $\omega$ to the corresponding critical point for $E_{\alpha,\omega}$.

Since bubbling is ruled out, it follows from Theorem 4.6.6 of [10] that $\sigma_\alpha(\omega)$ converges to $\sigma(\omega)$ uniformly in each $C^k$ norm on $\Sigma$ for each $\omega \in T_g$. In fact we can use this obstruction to give an alternate proof of the existence of a unique energy-minimizing minimum of $\omega$-energy in $\text{Map}_0(\Sigma, M)$ when the domain $\Sigma$ is a compact surface, independent of the Eells-Sampson heat flow.

The section $\sigma$ allows us to reduce the two-variable Dirichlet energy $E$ to a smooth function on finite-dimensional Teichmüller space

$$
\bar{E}_0 : T_g \rightarrow \mathbb{R}, \quad \bar{E}_0(\omega) = E(\sigma(\omega), \omega),
$$

the domain $T_g$ being diffeomorphic to Euclidean space of dimension $6g − 6$ by Teichmüller’s theorem. When we restrict to a prime incompressible component, $\bar{E}_0$ has nondegenerate critical points for generic choice of metric on $M$ by the Bumpy Metric Theorem 5.1.1 of [10]. (Instead of the Bumpy Metric Theorem one can use an argument for this special case like that which Tromba gives in [2].) Note that although these critical points are strict local minima as harmonic maps when the conformal structure on $\Sigma$ is fixed, they can have positive Morse index as minimal surfaces when the conformal structure is allowed to vary.

Similarly, the section $\sigma_\alpha$ allows us to reduce the two-variable function $E_\alpha$ to a smooth function with finite-dimensional range

$$
(\bar{E}_\alpha)_0 : T_g \rightarrow \mathbb{R}, \quad (\bar{E}_\alpha)_0(\omega) = E_\alpha(\sigma(\omega), \omega).
$$

The functions $\bar{E}_0$ and $(\bar{E}_\alpha)_0$ are invariant under the action of the mapping class group $\Gamma$, so they descend to maps on the quotient,

$$
E_0, (E_\alpha)_0 : \mathcal{M}_g = T_g/\Gamma \rightarrow [0, \infty).
$$

(6)

Since the function $E_\alpha$ of [11] satisfies Condition C, $(E_\alpha)_0$ is proper with finite critical set. Since $E_0 \leq (E_\alpha)_0$ and $(E_\alpha)_0$ converges to $E_0$ as $\alpha \rightarrow 1$, $E_0$ is also proper with finite critical set, and in particular there are only finitely many critical values for $E_0$. This correctly suggests that the critical set for $E_0$ can be
analyzed via the equivariant Morse theory of Bott \cite{Bott}, as further developed for closed geodesics by Hingston \cite{Hingston}.

To complete the proof of Theorem 2.1, we need to discuss the key ideas behind equivariant Morse theory. We start by finding a contractible CW complex $E\Gamma$ on which $\Gamma$ acts freely, so that we can take the quotient $B\Gamma$ which is called a classifying space for bundles with structure group $\Gamma$. (The classifying space is unique up to homotopy type.) Then given any space $X$ on which $\Gamma$ acts, the diagonal action of $\Gamma$ on $X \times E\Gamma$ is free, so we can set

$$X_\Gamma = \frac{X \times E\Gamma}{\Gamma} \quad \text{and} \quad H^*_\Gamma(X; \mathbb{Q}) = H^*(X_\Gamma; \mathbb{Q}).$$

We call $X_\Gamma$ the homotopy quotient, and we call $H^*_\Gamma(X; \mathbb{Q})$ the $\Gamma$-equivariant cohomology of $X$. Bott explains how Morse theory can be extended to the case of a proper $\Gamma$-invariant function with nondegenerate critical points on a finite-dimensional manifold, such as $E_0 : M_\gamma \to \mathbb{R}$, the result being equivariant Morse inequalities, which relate the number of critical orbits of $E_0$ to the $\Gamma$-equivariant cohomology of the contractible manifold $T_\gamma$.

To establish these inequalities, we investigate how the $\Gamma$-equivariant topology of the sublevel set

$$T^c_\gamma = \{ \omega \in T_\gamma : \tilde{E}_0(\omega) \leq c \}$$

changes as $c$ increases. Note that we can perturb the metric so that $\tilde{E}_0$ takes on distinct values at the finitely many distinct $\Gamma$-orbits.

To calculate the change in topology as we rise through a critical level which contains an isolated $\Gamma$-orbit of nondegenerate critical points, Bott’s prescription requires calculating the $\Gamma$-equivariant cohomology of the orbit $\Gamma/H$ which is the cohomology of the classifying space $BH$, where $H$ is the isotropy group of this orbit (see formula (4.13) in Bott \cite{Bott}). What we obtain is

$$H^k_\Gamma(\Gamma/H; \mathbb{Q}) \cong H^k(BH; \mathbb{Q}) \cong \text{Ext}^k_{\mathbb{Z}H}(\mathbb{Z}, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{when } k = 0, \\ 0, & \text{when } k \neq 0, \end{cases}$$

which follows from Corollary 5.4 in Chapter 4 of MacLane \cite{MacLane}, since $H$ is a finite group, and the additive group $\mathbb{Q}$ is divisible with every element of infinite order.

As the value of $E_0$ increases through a critical level $c$ which contains an isolated $\Gamma$-orbit of Morse index $\lambda$, the change in the $\Gamma$-equivariant rational cohomology of $T^c_\gamma$ is given by applying the Thom isomorphism theorem to the normal bundle of the orbit. But the orbit is zero-dimensional and the normal bundle is trivial, so according to formula (4.5) in Bott \cite{Bott}, the resulting change in equivariant cohomology is

$$H^k_\Gamma(T^c_\gamma^\varepsilon; T^c_\gamma^{-\varepsilon}; \mathbb{Q}) \cong H^{k+\lambda}_\Gamma(\Gamma/H; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{when } k = \lambda, \\ 0, & \text{when } k \neq \lambda. \end{cases}$$

In other words, the contribution to cohomology over the rationals is no different from what would occur if the action of $\Gamma$ were free, the moduli space $M_\gamma = T_\gamma/\Gamma$
itself were a genuine manifold, and we were calculating the usual Morse inequalities for $E_0$ on a smooth moduli space. Since $E_0$ is proper, the $\Gamma$-equivariant topology of $T^\Gamma_g$ is that of $T_g$ when $c$ is sufficiently large. Moreover, the rational $\Gamma$-equivariant topology of $T_g$ is the rational topology of $M_g$ because the rationalization of $M_g$ is homotopy equivalent to the classifying space $B(\Gamma \otimes \mathbb{Q})$.

Now we simply keep track of the changes as $c$ increases through the finitely many critical values for $E_0$ and the standard inductive procedure, as outlined for example in §2.9 of [10], yields Morse inequalities, but equivariant as explained by Bott [1], which directly imply Theorem 2.1, finishing our argument.

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