The magnonic spin Seebeck effect is a key element of spin caloritronic, a field that exploits thermal effects for spintronic applications. Early studies were focused on investigating the steady-state nonequilibrium magnonic spin Seebeck current, and the underlying physics of the magnonic spin Seebeck effect is now relatively well established. However, the initial steps of the formation of the spin Seebeck current are in the scope of recent interest. To address this dynamical aspect theoretically we propose here a new approach to the time-resolved spin Seebeck effect. Our method exploits the supersymmetric theory of stochastics and Ito - Stratonovich integration scheme. We found that in the early step the spin Seebeck current has both nonzero transversal and longitudinal components. As the magnetization dynamics approaches the steady-state, the transversal components decay through dephasing over the dipole-dipole reservoir. The time scale for this process is typically in the sub-nanoseconds pointing thus to the potential of an ultrafast control of the dynamical spin Seebeck during its buildup.

I. INTRODUCTION

Irradiating magnetic samples with electromagnetic fields may result in a variety of phenomena, including subpicosecond magnetic order breakdown, electron-phonon spin-flip scattering [1], electron-magnon scattering in non-equilibrium [2], and superdiffusive spin transport [3]. These observations are related to ultrafast spin dynamics [4, 5] and depend on the parameters of the driving fields such as their intensity, duration, and frequencies, we well as on the inherent properties of the magnetic sample. Our interest here is devoted to a particular aspect namely to the non-equilibrium magnonic current generated by a temperature gradient due to local heating of the sample by a laser pulse [6]. This means that we concentrate on the regime where the phonon temperature profile has already been established and consider the nonequilibrium dynamics of magnons. We note that magnon dynamics is of a particular importance for applications, as magnons are low energy excitations that can carry information over long distances and can be utilized for logic operations. To deal with non-equilibrium processes under the influence of irregular forces and thermal fluctuations the FokkerPlanck (FP) equation is the method of choice [7–12].

In general, FP applies also to nonlinear (chaotic) systems with positive Lyapunov exponents [13]. Treating the thermally activated magnetization dynamics and the steady-state magnonic spin current, the FP equation allows obtaining results beyond the linear response theory [14, 15]. However, the corresponding nonstationary case has not yet been treated with FP equation. In fact, FP equation is a nonlinear partial differential equation which admits exact analytical time-dependent solution only in a few limited cases. Our aim to describe the ultrafast spin dynamics entails access to the time-dependent solution of the FP equation. The available procedures and analytical tools for solving the time-dependent FP equation are limited basically to 1D systems. As an alternative, one can consider a supersymmetric theory of stochastics and the Stratonovich-Ito integration scheme. In this work, we apply the Stratonovich-Ito integration scheme to the system below the Curie temperature.

Here we present an analytical FP-based approach to study of thermally activated ultrafast magnonic spin current. Specifically, we focus on the behavior of the nonequilibrium spin current, generated at the interface of ferromagnetic insulator and normal metal [16, 17] and calculate how it approaches the nonequilibrium (steady) state. To this end the evaluation of the correlation functions in the nonequilibrium state is needed, and as we show here, this can be achieved by using the FP equation and the Stratonovich-Ito integration scheme for the stochastic noise.

Our choice of the sample is motivated by the recent experiments uncovering the early stage of the spin Seebeck effect [18]. Using terahertz spectroscopy applied to bilayers of ferrimagnetic yttrium iron garnet (YIG) and platinum, the spin Seebeck current is shown to arise on the ~100 fs time scale.

The work is organized as follows: In section II, we define the magnonic spin current. In section III, we describe the theoretical methods used afterward. In section IV we present results and conclude the work.
The spin pumping current flowing from the ferromagnetic state is broken down or not yet established is the latter we assume that the temperature $T$ to nanoseconds). As we are interested in the dynamics of the mismatch between magnon temperature $T_F$ and the phonon temperature. The same applies to the temperature of the normal metal $T_N$. The total spin current thus reads

$$
\langle I_{\text{tot}} \rangle = \frac{M_s V}{\gamma} \alpha \langle \mathbf{m} \times \mathbf{m} \rangle - \langle \mathbf{m} \times \mathbf{\zeta} \rangle.
$$

The temperature-dependent magnetization dynamics is governed by the stochastic LLG equation

$$
\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times (H_{\text{eff}} + \mathbf{h}) + \alpha \mathbf{m} \times \mathbf{m},
$$

where $\alpha$ is the Gilbert damping constant, $H_{\text{eff}}$ is the effective field, and the time-dependent random magnetic field in the ferromagnet is described by $\mathbf{h}$. This effective field $H_{\text{eff}}$ contains the anisotropy field $H_A$ and the external magnetic field $H_{0z}$ oriented along the $z$-axis. The total random magnetic field $\mathbf{h}(t)$ has two contributions from independent noise sources: the thermal random field $\mathbf{h}_0(t)$, and the random field $\mathbf{\zeta}(t)$. The former is related to the finite temperature in the ferromagnetic insulator and the second to the fluctuations in the normal metal. The correlators of the statistically independent noise sources are additive leading to the effective (enhanced) magnetic damping constant $\alpha = \alpha_0 + \alpha'$ ( $\alpha_0$ is the damping parameter of the ferromagnetic material, meaning without the contributions from the pumping currents),

$$
\langle \zeta_i(t) \zeta_j(0) \rangle = \frac{2\alpha' \gamma k_B T_N}{M_s V} \delta_{ij} \delta(t) = \sigma^2 \delta_{ij} \delta(t),
$$

where $\zeta(t) = \gamma \mathbf{h}(t)$, and $\alpha T_F = \alpha_0 T_F + \alpha' T_N$.

III. THEORETICAL METHOD

To find the total spin current (Eq.(4)) we use the FP equation for the distribution function of the magnetization $P(m_z, t)$ which is related to the stochastic equation

$\frac{dP}{dt} = -\nabla \cdot (V F \nabla P) - \alpha \frac{dP}{dt} - \gamma \frac{dP}{dt} + \alpha' \frac{dP}{dt}$,

where $V F \nabla P$ is the variation of the free energy with respect to $P$, and $\alpha$, $\gamma$, and $\alpha'$ are the damping constants due to spin pumping, gyromagnetic effect, and stochastic LLG equation, respectively.
of the magnetic dynamics (Eq.(5))

\[
\frac{\partial P(m_z, t)}{\partial t} = \frac{\partial}{\partial m_z} \left[ \frac{\partial}{\partial m_z} + \beta U'(m_z) \right] P(m_z, t),
\]

where \( U(m_z) = 2\alpha \left( \omega_0 m_z - \frac{\omega_p m^2_z}{2} \right) \) is the potential, \( \beta = 1/\sigma^2 \) is the effective inverse temperature, \( \omega_p = \gamma H_A \), and \( H_A \) is the anisotropy field. As detailed above, in our case \( U(m_z) \) is time-independent. The stationary solution of Eq. (7) is given by [14]

\[
P_0(m_z) = Z^{-1} \exp \left[ -\beta U(m_z) \right], \quad Z = \int \exp \left[ -\beta U(m_z) \right] d^2m.
\]

For the time-dependent distribution one makes the Ansatz

\[
P(m_z, t) = \psi(m_z, t) \exp \left[ -\beta \frac{U(m_z)}{2} \right].
\]

Using Eq.(9) and Eq.(7) we find that \( \psi(m_z, t) \) is a solution of the Schrödinger equation for imaginary time,

\[
\frac{\partial \psi(m_z, t)}{\partial t} = -\hat{H} \psi(m_z, t),
\]

with the Hamiltonian

\[
\hat{H} = -\frac{d^2}{dm_z^2} + \left( \frac{U'(m_z)}{2\sigma^2} \right)^2 - \frac{U''(m_z)}{2\sigma^2}.
\]

As \( U \) is time independent, the general solution of Eq. (11) is \( \psi(m_z, t) = \sum_n C_n \exp(-\lambda_n t) \psi_n(m_z) \), where \( \psi_n(m_z) \) and \( \lambda_n \) are the eigenfunctions and eigenvalues of the stationary equation, \( \hat{H} \psi_n(m_z) = \lambda_n \psi_n(m_z) \). Hence, the problem of solution of the time-dependent FP equation reduces to the determination of \( \lambda_n \) and the corresponding eigenfunctions of the Hamiltonian \( \hat{H} \). The calculations can be substantially simplified due to the hidden supersymmetry of this problem [22]. Indeed, one can introduce the supersymmetric Hamiltonian (for the supersymmetry see [23][28]) \( \hat{H}_{suss} = Q^\dagger Q + QQ^\dagger = \text{diag} (H_+, H_-) \), where

\[
Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}
\]

and \( A = \frac{1}{2} \beta U'(m_z) - \partial / \partial m_z \). Note that \( Q \) and \( Q^\dagger \) are nilpotent operators, \( Q^2 = (Q^\dagger)^2 = 0 \), and the commutator \( [Q, \hat{H}_{suss}] = 0 \). As a result, \( \hat{H}_+ \) and \( \hat{H}_- \) have common eigenfunctions: if \( \Psi_n \) is an eigenfunction of \( \hat{H}_+ \) then \( Q \Psi_n \) is the eigenfunction of \( \hat{H}_- \) (except for the ground state corresponding to \( \lambda_0 = 0 \)). The operator \( \hat{H}_{suss} \) acts in the space of Bose and Fermi fields. Namely, \( \hat{H}_+ = d^2/dm_z^2 + V_+(m_z) \) is the operator for bosons and \( \hat{H}_- = d^2/dm_z^2 + V_-(m_z) \) for fermions, where \( V_\pm(m_z) = (U'/2\sigma^2)^2 \pm U''/2\sigma^2 \) are the corresponding potentials. The operator \( Q \) transforms bosons to fermions and vice versa.

\( \hat{H}_+ \) coincides with Hamiltonian (11). It is however more convenient to solve the Schrödinger equation with fermionic Hamiltonian \( \hat{H}_- \) because the corresponding potential \( V_-(m_z) \) is close to the parabolic form. Owing to the supersymmetry, the eigenfunctions and eigenvalues are the same. Using this approach we find \( \lambda_1 = \frac{\sigma^2}{2\pi} \exp(-\alpha\omega_p/\sigma^2) \), and in the limit of strong anisotropy

\[
\lambda_n \approx 4\alpha\omega_p (n - 1) / \sigma^2.
\]

The first non-vanishing \( 1/\lambda_2 \) defines the characteristic relaxation time scale. For more details on the supersymmetry theory of stochastics we refer to [29][33].

To explore the time dependence of the non-equilibrium spin current \( \langle \mathbf{I}(t)_{\text{tot}} \rangle \) we utilize the Stratonovich-Ito integration scheme [30][33] and construct a reductive perturbation theory valid in the low-temperature limit (specified below). We briefly recall the main concepts of the stochastic Ito - Stratonovich integration. The time integral from the stochastic noise is equal to the function \( W(t) \) which has no time derivative \( W(t) \) is not a smooth function \( \int_0^\infty \xi(t) d\tau = W(t) \). Therefore, the stochastic integration is performed using the mean-square (ms) convergence of the sequence of the random variable \( X_n(\omega) \), meaning that

\[
\text{ms} \left\{ \lim_{n \to \infty} X_n \right\} = X,
\]

is equivalent to

\[
\text{ms} \left\{ \lim_{n \to \infty} \left( (X_n - X)^2 \right) \right\} = 0.
\]

stochastic integral is defined as follows:

\[
\int_{t_0}^{t} G(\tau) dW(\tau) = \text{ms} \left\{ \lim_{n \to \infty} \sum_{i=1}^{n} G(t_{i-1}) [W(t_i) - W(t_{i-1})] \right\},
\]
where \(G(t)\) is an arbitrary function of time. We assume that the magnon temperature in the system is low, which means that the thermal energy is smaller than the anisotropy barrier. Therefore, the appropriate ansatz for the solution of the stochastic LLG equation is

\[
m(t) = m_0(t) + \varepsilon m_1(t), \tag{17}
\]

where \(m_0(t)\) is the deterministic solution and \(m_1(t)\) is the correction due to the stochastic field. The equation for the stochastic part reads

\[
dm_1(t) = -A \langle m_0(t) \rangle m_1(t) \, dt + \int_{0}^{T} \, B \langle m_0(t) \rangle \xi(t) \, dW(t), \tag{18}
\]

where \(dW(t) = \xi(t) \, dt\) and for brevity we introduced the notations

\[
A[m_0(t)] = \begin{pmatrix}
0 & \omega_{\text{eff}}(t) & 0 \\
-\omega_{\text{eff}}(t) & 0 & 0 \\
A & 0 & 0
\end{pmatrix}, \quad B[m_0(t)] = \begin{pmatrix}
0 & m_{0z}(t) & -m_{0y}(t) \\
-m_{0z}(t) & 0 & m_{0x}(t) \\
m_{0y}(t) & -m_{0x}(t) & 0
\end{pmatrix}, \tag{19}
\]

with \(\omega_{\text{eff}}(\omega_0 + \omega_p m_z)\). Taking into account Eq. (14)-Eq. (19), after relatively involved analytical calculations for the correlation functions and the non-equilibrium spin current we deduce

\[
\langle I_x(t) \rangle = 2\alpha' k_B \varepsilon^2 m_0(t) \left( T_m^m - T_N \right),
\]

\[
\langle m_{1z}(t) \xi(t) \rangle = \sigma^2 \varepsilon_{i j k} m_{0k}(t),
\]

\[
\langle m_{1z}(t) \xi(t) \rangle = \sigma^2 \varepsilon_{i j k} m_{0k}(t). \tag{20}
\]

In the case of a weak anisotropy Eq. (20) simplifies and for the non-equilibrium magnonic spin current components we obtain:

\[
\langle I_x'(t) \rangle = 2\alpha' k_B \varepsilon^2 \cos(\varphi_0 + \omega_0 t) \left( T_m^m - T_N \right),
\]

\[
\langle I_y'(t) \rangle = 2\alpha' k_B \varepsilon^2 \sin(\varphi_0 + \omega_0 t) \left( T_m^m - T_N \right),
\]

\[
\langle I_z'(t) \rangle = 2\alpha' k_B \varepsilon^2 \tanh(\alpha \omega_0 t) \left( T_m^m - T_N \right). \tag{21}
\]

From Eq. (21) follows that in the asymptotic, long-time limit the only component of the magnonic spin current that survives is \(I_x'(t)\), and we recover the classical result of Xiao et al. [16]. For short times, however, the other components are sizable and even dominant and thus can thus be exploited for ultrafast picosecond magnonics.

We applied the Stratonovich-Ito integration scheme to the system below the Curie temperature. Nevertheless, our method can be extended to the Landau-Lifshitz-Bloch equation as well. Note that Eq. (18), in the coefficient \(A[m_0(t)]\) and \(B[m_0(t)]\), contains the solution of the deterministic Landau-Lifshitz-Gilbert equation. One can replace the solution of the deterministic LLG equation \(m_0(t)\) by the solution of the deterministic Landau-Lifshitz-Bloch equation with extra longitudinal damping parameter [39]. After this replacement, we can again perform Stratonovich-Ito integration.

The result Eq. (21) is obtained in the single macrospin approximation but can be generalized to an extended system using the ensemble averaging over the dipole-dipole reservoir. We note that the transversal spin current components in Eq. (21) contain the rotating terms. In the case of extended systems, each spin rotates with a slightly different frequency due to the broadening of the resonance frequency \(\omega_0\). Precession with different frequencies leads to the dephasing of the signal in time. We assume that the dephasing of the transversal magnetization and current components have the same nature. Following [40], we write down the equation for the transversal magnetization component

\[
-\hbar i \frac{dm_x(t)}{dt} = \left[ H_d(t), m_x(t) \right], \tag{22}
\]

or in the matrix form

\[
-\hbar i \frac{d(m_x(t))_{nn'}}{dt} = \hbar \Delta \omega(t) \langle m_x(t) \rangle_{nn'}. \tag{23}
\]

The Hamiltonian \(H_d(t)\) in Eq. (22), (23) describes the dipole-dipole reservoir, and the time dependence of the Heisenberg operators is governed through the Zeeman Hamiltonian \(H_Z\). (see [40] for more details). Let us quantitatively the fluctuations of the local field through the function

\[
\langle \Delta \omega(t) \Delta \omega(t + \tau) \rangle = M_2 \Psi(\tau), \tag{24}
\]

where

\[
M_2 = -\frac{Tr \left[ \left[ \hat{H}_d, m_x \right]^2 \right]}{\hbar^2 Tr \left[ m_x^2 \right]} - \omega_0^2, \tag{25}
\]

is the second moment of the transversal component. We assume that the dephasing mechanism of the transversal spin current components is the same. Taking into account Eq. (22)-Eq. (25) for the ensemble averaged dephasing transversal spin currents we infer
\[
\langle I_x^z(t) \rangle = 2\alpha'k_B\varepsilon^2 \frac{\cos(\varphi_0 + \omega_0 t)}{\cosh \alpha \omega_0 t} \exp \left[ -M_2 \int_0^t (t - \tau)\Psi(\tau)d\tau \right] (T_F^m - T_N).
\]

In the limit of the white noise the dephasing exponent takes the simpler form: \( \langle I_x^z(t) \rangle \approx \langle I_x^z(0) \rangle \exp (-t/T_2) \), where the transversal relaxation time is given by \( T_2 = -M_2 \int_0^\infty \Psi(\tau)d\tau \). Thus, the decay of the transversal magnonic spin current components in the ultrafast spin Seebeck effect is solely determined by the dipole-dipole interactions.

IV. RESULTS AND DISCUSSIONS

![Graph showing time dependent non-equilibrium transversal and longitudinal magnonic spin current components I_x^z(t), I_y^z(t) and I_z^z(t). The magnon temperature is equal to T_F^m = 5 K and temperature of the normal metal T_N = 0. The external magnetic field H_{0z} = 2 \times 10^5 A/m is applied in +z direction.](image)

In the numerical simulation, the motion of \( \mathbf{m} \) is governed by the LLG equation [7]. The adopted numerical parameters are \( M_s = 1.4 \times 10^5 \) A/m, the damping constant \( \alpha = 0.001 \), the external magnetic field \( H_{0z} = 2 \times 10^5 \) A/m, and the spin-mixing conductance \( g_r = 3 \times 10^{15} \) 1/m². In the equilibrium state, the local magnetization points along the +z direction. We set the temperature \( T_F^m = 5K \) and \( T_N = 0 \). The time-dependent magnonic spin pumping currents \( I_x^z(t) \), \( I_y^z(t) \) and \( I_z^z(t) \) are plotted in Fig.2. The spin current is calculated using Eq. 1. For the transversal components \( I_x^z(t) \) and \( I_y^z(t) \) we consider the averaging procedure through the exponential factors \( \langle I_x^z(t) \rangle = e^{-t/T_2} \langle I_x^z(0) \rangle \) and \( \langle I_y^z(t) \rangle = e^{-t/T_2} \langle I_y^z(0) \rangle \), with the transversal relaxation time \( T = N/\omega_0 \), \( N = 50 \), and the values of \( \langle I_{x,0}^z \rangle \) and \( \langle I_y^z(0) \rangle \) are calculated from Eq. F2. The numerical solution plotted in Fig. 2 is in good agreement with the analytical results expressed by Eqs. 20 and 26. Calculations done for the anisotropy field \( H_z = 2\mu_0 M_r \) (along z axis) with constant \( K_z = 1.8 \times 10^4 \) J/m³ (not shown) leads to similar conclusions. To explore the dephasing problem for an extended sample, we performed micromagnetic simulations. The results of the simulations are presented in the supplementary information. We performed numerical simulations for an extended ferromagnetic sample. Our simulations include the effects of the dipole-dipole and exchange interactions as well as the magnetic anisotropy. The geometry of the ferromagnetic sample is as follows: the length is 350 nm (along z axis), the width 50 nm (along y axis), and the thickness is 5 nm (along x axis). In this case, the equilibrium magnetization points in the in-plane along +z direction. The magnon temperature \( T_F^m = 5K \) and the temperature of the normal metal is set to zero \( T_N = 0 \). The time-dependent magnonic spin pumping currents \( I_x^z(t) \), \( I_y^z(t) \) and \( I_z^z(t) \) are plotted in Fig.3. For extended samples, the dephasing of the transversal spin current components is faster and can hardly be captured through the micromagnetic simulations. As evident, the transversal components of the current oscillate randomly close to the zero value leading to the \( \langle I_x^z(t) \rangle = 0 \) and \( \langle I_y^z(t) \rangle = 0 \). The longitudinal component \( I_z^z(t) \) increases in time and saturates in the equilibrium regime.

Summarizing, we proposed a theoretical approach to the time evolution of the spin Seebeck current. The approach is based on the time-dependent Fokker-Planck equation and supersymmetry arguments. We managed to derive the analytical formula for the initial step of the buildup of the spin current in the spin Seebeck effect. The results are confirmed by full numerical calculations. The current experimental interest shows that ultrafast spin dynamics will play an increasingly significant role in spin caloritronics in the foreseeable future. Analytical tools for the time-dependent FP equation are quite limited. Therefore, the alternative method proposed in our work should be useful for spin caloritronic studies.

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FIG. 3. Time dependent, non-equilibrium transversal and longitudinal magnonic spin current components $I_x(t)$, $I_y(t)$ and $I_z(t)$ for a finite ferromagnetic sample. The short time scale longitudinal magnonic spin currents are important, whereas in the long-time limit $I_z(t)$ is dominant.
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