Abstract

There has recently been work by multiple groups in extracting the properties associated with cardinal invariants of the continuum and translating these properties into similar analogous combinatorial properties of computational oracles. Each property yields a highness notion in the Turing degrees. In this paper we study the highness notions that result from the translation of the evasion number and its dual, the prediction number, as well as two versions of the rearrangement number. When translated appropriately, these yield four new highness notions. We will define these new notions, show some of their basic properties and place them in the computability-theoretic version of Cichoń’s diagram.

1 Introduction

Recent work of Rupprecht [17] and, with some influence of Rupprecht but largely independently, Brendle, Brooke-Taylor, Ng, and Nies [5] developed...
and showed a process for extracting the combinatorial properties of cardinal characteristics and translating them into highness properties of oracles with related combinatorial properties. Some of the analogs so derived are familiar computability-theoretic properties, some are new characterizations of existing notions, and some are completely new. It is interesting to notice that that many of the proofs of relationships between the cardinals in the set-theoretic setting translate to the effective setting. The work so far has mostly focused on the cardinal characteristics of Cichoń’s diagram.

The nodes in Cichoń’s diagram, figure 1, have the usual notation and meaning as define in [13], we will work with most of them in this paper. It is important to notice that the arrows in figure 1 stand for inequalities, with $A \rightarrow B$ in the diagram indicating $A \leq B$.

There is a purely semantic formulation of the translation scheme to an effective notion where all of these characteristics can be viewed as either an unbounding number or a dominating number along the lines of $b$ and $d$ for a different relationship between two spaces. They can then be semantically converted to the appropriate highness notion. For all the details of the semantic scheme, see [17] or [5].

An alternative, somewhat intuitive way to think about this translation scheme is to frame it as follows: when working with cardinal characteristics
on the set theory side, it is common to build models by forcing extensions that have specific properties, one way to do this is to force a characteristic to be larger by building an extension which has a new object that negates the desired property for a specific collection from the ground model. If we reinterpret the ground model as the computable objects, and the extension as adding those things computable from an oracle, the degree corresponding to the characteristic will be exactly the combinatorial definition needed to negate the characteristic property for the collection of computable objects. Among other things, this means that the highness notions actually end up looking like the negations of the characteristics that they were derived from.

For example, let us take the unbounding number $b$. In building a forcing extension to make $b$ larger, we would want to add a function which does bound a collection of functions from the ground model. When translated to a computability-theoretic highness notion, this becomes an oracle which computes a function dominating every computable function. This is exactly the set of oracles of high degree. Similarly, for the domination number $d$, in building a forcing extension to make $d$ larger, we would want to add a function which is not dominated by any function from the ground model. When translated to the computability side, this becomes an oracle which computes a function not dominated by any computable function, i.e. of hyperimmune degree. Some of the analogs, like these, are well-studied, and some were introduced by Rupprecht in §17.
Figure 2 is a summary of the results known in these areas. Here, arrows actually do mean implication, where the lower-left highness properties are generally stronger than the upper-right. It is possible to find all the definitions in [17], but it is important to remark that some of the Rupprecht terminology is different.

In this paper, we will expand on this work by looking at four different cardinal characteristics not appearing in Cichoń’s diagram. First, we will examine the evasion number, a cardinal characteristic first introduced by Blass in [2], as well as its less-studied dual, the prediction number. We will also look at two forms of the so-called rearrangement number, as introduced by Blass et al. in [3]. In all these cases, we will give the correct effective analogs, and prove relationships between these new highness notions and their relationships with other properties which are analogous to well-studied cardinal characteristics.

The questions in this paper were independently studied by Noam Greenberg, Gregory Igusa, Rutger Kuyper, Menachem Magidor and Dan Turetsky. There is significant overlap between their results and those we present below.

2 Prediction and Evasion

2.1 Definitions

**Definition 2.1** (Blass [2]). A predictor is a pair $P = (D, \pi)$ such that $D \in [\omega]^{\omega}$ (infinite subsets of $\omega$) and $\pi$ is a sequence $\langle \pi_n : n \in D \rangle$ where each $\pi_n : \omega^n \rightarrow \omega$. By convention, we will sometimes refer to $\pi_n(\sigma)$ by simply $\pi(\sigma)$. This predictor $P$ predicts a function $x \in \omega^\omega$ if, for all but finitely many $n \in D, \pi_n(x|_n) = x(n)$. Otherwise $x$ evades $P$. The evasion number $e$ is the smallest cardinality of any family $E \subseteq \omega^\omega$ such that no single predictor predicts all members of $E$.

We will also make use of the dual to $e$, which is explored by Brendle and Shelah in [6].

**Definition 2.2.** The prediction number, which we will call $\sigma$, is the smallest cardinality of any family $O$ of predictors such that every function is predicted by a member of $O$.

The known results for $e$ and $\sigma$ position them as illustrated in figure 3 relative to Cichoń’s diagram.

In order to effectivize our prediction-related cardinal characteristics, we must first effectivize the definition of a predictor.
Figure 3: Cichoń’s diagram including $\mathfrak{e}$ and $\mathfrak{o}$.

**Definition 2.3.** A **computable predictor** is a pair $P = (D, (\pi_n : n \in D))$ where $D \subseteq \omega$ is infinite and computable and each $\pi_n : \omega^n \to \omega$ is a computable function.

Similarly, we define an $A$-**computable predictor** as the relativized version where all objects are computable relative to some oracle $A$.

Finally, we define an oracle $A$ to be of **evasion degree** if there is a function $f \leq_T A$ which evades all computable oracles, and $A$ is of **prediction degree** if there is a predictor $P \leq_T A$ which predicts all computable functions.

Because of the fact that we negate the original statements of the definitions of cardinal characteristics, under our scheme the evasion number $\mathfrak{e}$ is an analog to being a prediction degree, and the prediction number $\mathfrak{o}$ is an analog to being an evasion degree.

We present below known facts about $\mathfrak{e}$ and $\mathfrak{o}$ represented by Cichoń’s diagram with $\mathfrak{e}$ and $\mathfrak{o}$ included, as well as their translations into effective analogs.

**Theorem 2.4.** The following relationships are known for $\mathfrak{e}$.
2.2 Prediction Degrees

**Theorem 2.5.** If \( A \in 2^\omega \) is high, then it is of prediction degree.

**Proof.** Let \( A \) be high and set \( D = \omega \). We will use the fact that if \( A \) is high, then \( A \) can enumerate a list of indices for the total computable functions. A proof of this fact can be found in [8]. Using this, we simply enumerate all the computable functions. Then to define \( \pi_n \), for each finite string \( f \in \omega^n \), we go through the list of computable functions \( \{ \varphi_e \} \) until we find one such that \( \varphi_e|n = f \). Then we define \( \pi_n(f) = \varphi_e(n) \). This predictor is computable in \( A \) and predicts all computable functions. \( \square \)

**Lemma 2.6.** For any predictor \( P \), there is an effectively-in-\( P \) meager set covering all functions predicted by \( P \).

**Proof.** The collection

\[
C_i = \{ f : |\{ n \in D : \pi(f|n) \neq f(n)\}| < i \}
\]

is nowhere dense and \( \Pi^0_1 \) in \( P \), and the collection of functions predicted by \( P \) is exactly \( \bigcup_{i \in \mathbb{N}} C_i \). \( \square \)

**Theorem 2.7.** If \( A \) is a prediction degree, then \( A \) is weakly meager engulfing.

\[
\text{CON}(\text{e < add}(M)) \Rightarrow \text{prediction degree} \Rightarrow \text{meager engulfing} \Rightarrow \text{weakly meager engulfing} \Rightarrow \text{weakly 1-generic} \Rightarrow \text{evasion degree} \Rightarrow \text{not low for Schnorr tests} \Rightarrow \text{not low for 1-generics} \Rightarrow \text{open}
\]

Similarly, for \( \omega \) (all results can be found in [6]).

This results can be seen in figure 4.
Figure 4: Effective Cichoń’s diagram including prediction and evasion degrees. Dotted lines are open questions.
Proof. Assume $A$ is a prediction degree, then there is a predictor $P$ computable from $A$ which predicts all computable functions. In particular, we just need a predictor which predicts all 0,1-valued computable functions.

Then, by Lemma 2.6 one can, using $P$, effectively find a meager set covering every function predicted by $P$. Thus there is an $A$-effectively meager set covering all 0,1-valued computable functions, and hence covering all computable reals, as desired. \hfill $\square$

**Theorem 2.8.** If $A \in 2^\omega$ is of prediction degree, then $A$ is weakly 1-generic.

We will actually prove the equivalent statement that if $A$ is a prediction degree, then $A$ has hyperimmune degree. This is an analog of the characteristic inequality $e \leq d$. The above theorem is the analog of the strictly stronger cardinal relation $e \leq \text{cov}(\mathcal{M})$. However, these notions are one of the places where a relationship that is separable in the set-theoretic case collapses in the computability-theoretic analog, so the theorems are equivalent. The proof follows one of Blass from [2].

**Proof.** Given $A \in 2^\omega$ which is not weakly 1-generic, by a result of Kurtz, $A$ is hyperimmune-free. In particular we will use the fact that for all $f \leq_T A$ with $f : \omega \times \omega \to \omega$, there is a function $g \leq_T 0$ such that $g > f$.

Let $P = (D_P, \{\pi_n\}) \leq T A$ be a predictor, and define $f : \omega \times \omega \to \omega$ by

$$f(n,k) = \begin{cases} \max \{\pi_n(t) : t \in k^n\} & \text{if } n \in D_P \\ 0 & \text{otherwise.} \end{cases}$$

We note that $f \leq_T A$. Then, by assumption, there is a computable function $g$ such that $g(n,k) > f(n,k)$ for all $n,k$. Then we define

$$x(n) = g(n,1 + \max\{x(p) : p < n\}).$$

Now, let $n \in D_P$ and $k = 1 + \max\{x(p) : p < n\}$. We note that $x|_n$ is of length $n$ and has all values less than $k$, and so is an admissible $t$ from the definition of $f(n,k)$, so $f(n,k) \geq \pi_n(x|_n)$. On the other hand, by definition of $x$ and the choice of $g$, we also have $x(n) = g(n,k) > f(n,k)$. Thus, we have $x(n) > \pi_n(x|_n)$. Since $n$ was arbitrary, it follows that $x$ evades $P$, and so $A$ is not a prediction degree. \hfill $\square$

**Theorem 2.9.** There is an $A \in 2^\omega$ which computes a Schnorr random but is not of prediction degree.
Proof. This follows immediately from the fact that there is $A \in 2^{<\omega}$ which computes a Schnorr random, but is hyperimmune-free. See, e.g. [3] §4.2 (2).

The next theorem is an effectivization of the proof of the consistency of $b < c$ done by Brendle and Shelah in [4]. In their forcing, in order to show that all functions in the extension are bounded by a ground model function they rely on a claim that is analogous to 2.12.

The proof of the claim is also long in their paper due to the fact that each forcing condition has continuum many compatible conditions but we want to encode all maximum values that the function can take in only countably many functions. In our case, we also face two extra problems: the function that we are working with might not be total and, in order to keep using computable information, we need to find extensions that are hyperimmune-free. The way we solve both problems is to rely on the hyperimmune-free basis theorem in an specific compact space.

Theorem 2.10. There is an $A$ which is of prediction degree but does not compute any $B$ which is high.

Proof. We will force with conditions $\langle d, \pi, F \rangle = p \in P$ where $d \in 2^{<\omega}$ is a finite partial function, $\pi = \{\pi^n : n \in d\}$ and $\pi^n : \omega^\omega \to \omega$ is a finite partial function, $F \subseteq \omega^\omega$ is a finite collection of functions with the property $f, g \in F, f \neq g \Rightarrow f|_d \neq g|_d$. Here, the $d$ and $\pi$ can be thought of as partial approximations of $D$ and $\pi$ in the eventual predictor we are constructing, and $F$ as the collection of functions that we are committed to predicting correctly for the rest of the construction.

We define $(d', \pi', F')$ as an extension of $(d, \pi, F)$ by

$$(d', \pi', F') \leq (d, \pi, F) \iff d' \supseteq d, \pi' \supseteq \pi, F' \supseteq F \text{ and } f \in F, n \in \text{dom}(d') \setminus \text{dom}(d) \Rightarrow \pi'(f|_n) = f(n).$$

Due to the use of various indexes it is important to make a notation comment:

- Given $q$ a forcing condition, we will express it as $q = \langle p_d, \pi, q, F \rangle$, unless otherwise stated.

- We will do a construction by stages, so the condition that is selected at each stage will be $p_s = \langle d_s, \pi_s, F_s \rangle$.

- $\pi^m$, as mention above, will denote the function of $\pi$ corresponding to $m \in d^{-1}(1)$. We will not use the superscript for anything else.
• The left subscript will only be use in case we need to enumerate something, in that case \( q = \langle d, \pi, iF \rangle \).

To initialize the construction, we let \( d_0 = \langle \rangle, \pi_0 = \{\}, F_0 = \{\} \). We will maintain the property that the joint \( \bigoplus F_s = \bigoplus_{f \in F_s} f \) is hyperimmune-free and we will extend by the following rules:

\( P_e \): The goal of this requirement will be to ensure that we predict \( \varphi_e \).

At stage \( s = 3e \), we simply set \( F_s = F_{s-1} \cup \{\varphi_e\} \) and \( d_s = d_{s-1} \sim 0^n \) with \( n \) least such that for \( f \in F_{s-1}, \) if \( \varphi_e \neq f \), then \( \varphi_e|_{|d_s|} \neq f|_{|d_s|} \). Additionally, if \( \pi_s(\varphi_e|_n) \) is undefined, we define it to be \( \varphi_e(n) \).

\( I_e \): The goal of this requirement is to ensure that \( D \) is infinite.

At stage \( s = 3e + 1 \), \( D_s = D_{s-1} \sim 1 \), and \( \pi_s = \pi_{s-1} \cup \{\pi^m\} \) with \( m = |D_{s-1}| \) where \( \pi^m : \omega^m \to \omega \) with \( \pi^m(f|m) = f(m) \) for all \( f \in F_{s-1} \), \( \pi^m(\sigma) \) undefined for all other \( \sigma \), and \( F_s = F_{s-1} \).

\( E_{e,n} \): The goal of this requirement will be to ensure that \( \varphi^A_e \) is not total or that there is a computable function \( h_e \) such that \( \exists^\infty n(\varphi^A_e(n) \leq h_e(n)) \).

In order to create the function \( h_e(n) \), we have to make a guess depending on every forcing extension below \( p_s \). Because of that we define:

**Definition 2.11.** A collection of functions indexed by \( d, \pi, \vec{F} \) where \( d, \pi \) are as in \( P \) and \( \vec{F} \) is a finite sequence of finite initial segments of functions, we will call the elements \( h^{e,d,\pi,F} \), is seer if and only if they have the property that for any collection \( \hat{F} \) of total functions extending the \( \vec{F} \), the forcing condition \( \langle d, \pi, F_s \cup \hat{F} \rangle \) can be extended by \( q = \langle qd, \pi, \hat{F} \rangle \) such that \( \varphi^{qd,\pi}_e \) is below our function. Syntactically, this is

\[
\left\{ h^{e,d,\pi,F} \in \omega^\omega : \langle d, \pi, F_s \rangle \leq \langle d_s, \pi_s, F_s \rangle, \vec{F} = \langle f^*_1, \vec{f} \rangle, |\vec{f}| = l \in \omega, f^*_i \in \omega^{|d|} \right\}
\]

such that

\[
h^{e,d,\pi,F}(n) \geq \min\{m : \forall q = \langle qd, \pi, F_s \cup \hat{F} \rangle \text{ with } \hat{F} = \{f_i \in \omega^\omega \}_{i < l} \text{ and } f_i|_{|d|} = f^*_i (\exists q \leq p \ \varphi^{qd,\pi}_e(n) \downarrow < m)\}.
\]

**Claim 2.12.** We claim that either

1. There is \( n \in \omega \) and \( p \leq \langle d, \pi, F_s \rangle \) such that for any \( q \leq p, \varphi^{qd,\pi}_e(n) \uparrow \) and \( \bigoplus pF \) is hyperimmune free

or,

2. There is a uniformly \( \Sigma^0_1 \bigoplus F_s \) collection of functions indexed by \( d, \pi, \vec{F} \) that is seer.
At stage \( s = 3(e, 0) + 2 \), we will use the following claim in the following way:

If (1), then we define \( \langle d_{s+1}, \pi_{s+1}, F_{s+1} \rangle \) to be such a \( p \) and we do nothing for stages of the form \( s = 3(e, n) + 2 \). This will make \( \varphi_e^{(d, \pi)} \) not total.

If (2), then we can find \( h_e^e \leq T \bigoplus F_s \) such that \( h_e^{d, \pi, \mathcal{T}^*} \leq \hat{h}^e \) for all such functions. However, since \( \bigoplus F_s \) is hyperimmune-free, it follows that there is a computable function \( h_e^e \) for which \((\forall n)h_e^e(n) \geq \hat{h}^e \). We then resume the construction.

At stage \( s = 3(e, n + 1) + 2 \) we can find \( j > n \) so that \( h_e^e(j) \geq h_e^{d, \pi, \mathcal{T}^*}(j) \) where \( \mathcal{T}^* \) are the restrictions of the functions in \( F_s \setminus F_{3(e,0) + 2} \) to \( |d_s| \) and such that \( \varphi_e^{(d_s, \pi_s)}(j) \) is not yet defined.

In this situation, we can find \( p_{s+1} = \langle d_{s+1}, \pi_{s+1}, F_{s+1} \rangle \) such that
\[
\varphi_e^{(d_{s+1}, \pi_{s+1})}(j) \downarrow \leq h_e^{e(d_s, \pi_s, \mathcal{T}^*)}(j) \leq h_e^e(j),
\]
however, we note that this property of the \( p_{s+1} \) only depends on finite initial segments of the the members of \( F_{s+1} \setminus F_s \), and so there actually is such a condition with \( \bigoplus F_{s+1} \) hyperimmune-free. We pick a condition with this property.

Verification: By construction, the predictor \( P = (\bigcup d_s, \bigcup \pi_s) \) has the desired properties. \( P_e \) ensures our predictor predicts all computable functions, \( I_e \) ensures that \((\bigcup d_s)^{-1}(1) \) is infinite, and \( E_{e,n} \) ensures that the computational strength of the predictor cannot compute a total function dominating the computable functions, specifically, \( h_e^e \not\leq \varphi_e^P \), so \( P \) is not high.

Proof of Claim 2.12

Proof. Before doing the technical work to show the claim, we will explain the idea of the upcoming proof. As we see above, we want – if possible – to define the function \( h_{d,\pi,g_i^*}^e \) in such a way that, given \( \langle d, \pi, F_s \cup G \rangle \leq \langle d_s, \pi_s, F_s \rangle \) with \( G = \{g_i : i < l + 1\} \) and \( g_i|_{|d|} = g_i^* \) then we can find \( q \leq \langle d, \pi, F_s \cup G \rangle \) such that \( \varphi_e^{(d, \pi, q)}(n) \) is smaller than \( h_{d,\pi,g_i^*}^e(n) \). In other words, \( h_{d,\pi,g_i^*}^e(n) \) represents the minimal value that we can force \( \varphi_e^{(D, \pi)}(n) \) to take given that we already committed to \( d, \pi, g_i^* \).

In order to do this, we try to find all the possible extensions \( q \) of the node \( \langle d, \pi, F_s \cup G \rangle \) that make \( \varphi_e^{(d, q, \pi)} \) small. In general this is not necessarily possible, but our best chance to find them is if we restrict ourselves to a compact space (there, we will only have finitely many extensions that are compatible with everything that we consider).
The conversion from the whole $\omega^\omega$ to a compact space is possible thanks to the following observation: $q = \langle q^d, q^\pi, q^F \rangle \leq \langle d, \pi, F \rangle$ and $\langle d, \pi, \{g\} \rangle$ are compatible if $g|d$ is different from $f|d$ for all $f \in F$ and $g|d)$ is bigger than the $|d|$th index of all strings in the domain of any function in $q^\pi$ (more formally, it is bigger than $\sigma(|d|)$ for all $\sigma \in dom(q^{\pi n})$ with $d(n) = 1$).

This observation hints at the possibility of only worrying about functions of certain growth while we are looking for our small convergences.

In our proof, we will ask $h_{d,\pi,g_i}^e(n)$ to not only be bigger than the minimal value that $\varphi_{e(D,\pi)}$ can take, but also to be bigger than the values taken by strings in the domain of functions from $\pi$. In that way, we make $h_{d,\pi,g_i}^e(n)$ carry some information of compatibility. To define the compact space where we will work, we will define functions $B_i$ that combine nicely the information needed.

Now, for the proof, we will show this by induction on $l = |F|$. Our induction hypothesis is slightly stronger than the statement of the claim. Case (1) remains unchanged, but we add to case (2) the additional requirement:

(2a) For all $k \in \omega$ and all $f \in \omega^\omega$ with $f(n) > h_{d,\pi,F}^e(n)$ with $n < |d|$
then $\langle d, \pi, F_s \cup \{f\} \rangle$ is compatible with an extension $r \leq \langle d, \pi, F_s \cup \widehat{F} \rangle$ with $\varphi_e^{d,\pi,r}(k) \downarrow < h_{d,\pi,F}^e(r)(k)$, $\widehat{F} = \{f_i \in \omega^\omega \}_{i \leq l}$, and $f_i|d| = f_i^*$. Furthermore, $r$ does not depend on $f$ (but, most likely, it would depend on $k$ and $n$).

Case $l = 0$:
If there is $n \in \omega$ and $\langle d, \pi, F_s \rangle \leq \langle d_s, \pi_s, F_s \rangle$ such that for all $q$ extending $\langle d, \pi, F_s \rangle$ we have that $\varphi_e^{q(d,\pi,F)}(n)$ diverges then $\langle d, \pi, F_s \rangle$ satisfy (1). Otherwise, fix $\langle d, \pi, F_s \rangle \leq \langle d_s, \pi_s, F_s \rangle$. We will define a function $h_{d,\pi,\emptyset}^e$ computable from $\bigoplus F_s$ with the desired properties.

Fix $n \in \omega$. We begin searching for extensions $q \leq \langle d, \pi, F_s \rangle$ with $\varphi_e^{q(d,\pi,F)}(n) \downarrow$. As soon as we find a convergence to a value $m$, we let

$$h_{d,\pi,\emptyset}^e(n) = \max\{m + 1, \min\{k : \forall i \in q^d \forall \sigma \in dom(q^\pi^i) (\sigma \in k^i)\} + 1\}.$$  

Notice that the first part of the max ensures (2), and the second part ensures that (2a) is satisfied, as the only way there is no such extension, is if $q^\pi$ incorrectly predicts $f$ for some $n \in ([d], |q^d|)$, but this is impossible, as

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1The compatibility is true because $g|k$, with $k > |d|$, is not defined in any function from $q^\pi$, therefore, we can create $\pi'$ which always predicts $g$ correctly after $|d|$ such that $q^\pi \leq \pi'$. In this way $\langle q^d, \pi', F \cup \{g\} \rangle$ is below $q$ and $\langle d, \pi, \{g\} \rangle$. 

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f takes a value at some $m < |d|$ which is larger than anything that shows up in the domain of any of the functions from $^q \pi$, by definition.

Case $l = 1$:

If (1) already has already happened, we are done. Otherwise, fix $\langle d, \pi, F_s \rangle$ extending $\langle d_s, \pi_s, F_s \rangle$ and $g^* \in \omega^{[d]}$ such that for all $f \in F_s$, $f_{|d|} \neq g^*$.

We will define a function $h_{d,\pi,\langle g^* \rangle}$ computable from $\bigoplus F_s$ with the desired properties.

Now, let $h_{d,\pi,\emptyset}$ be as in the $l = 0$ case. We define

$$B_1(j) = \begin{cases} 0 & j < |d| \\ h_{d,\pi,\emptyset}(j) & |d| \leq j \end{cases}$$

Notice that, given $f \in \omega^\omega$ with $f_{|d|} = g^*$, if there is $j \geq |d|$ such that $f(j) > B_1(j)$ then, given a $t \in \omega$, $\langle d, \pi, F_s \cup \{f\} \rangle$ is compatible with an extension $r \leq \langle d, \pi, F_s \cup \emptyset \rangle$ with $\varphi_{^q \sigma}^{(d,\pi)}(t) \downarrow < h_{d,\pi,\emptyset}^{r}(t)$.

Since $B_1$ is computable from $\bigoplus F_s$ we have that the space

$$C_1 = \{ f \in \omega^\omega : f_{|d|} = g^* \land \forall j \geq |d| \ f_i(j) \leq B_1(j) \}$$

is effectively compact with respect to $\bigoplus F_s$.

Fix $n \in \omega$. Then we can define open sets in $C_1$ representing bounded convergence. We define these sets as

$$U^n_m = \{ h \in C_1 : \exists q \leq \langle d, \pi, F_s \cup \{h\} \rangle \ \varphi_{^q \sigma}^{(d,\pi)}(n) \downarrow < m \}.$$ 

Notice that $U^n_m \subseteq U^n_t$ as long as $m \leq t$, and that $U^n_m$ is a $\Sigma_1^0 \bigoplus F_s$ set of functions.

Furthermore, if we call $A^n = \bigcup_{m \in \omega} U^n_m$, we have that $C \setminus A^n$ is a $\Pi_1^0 \bigoplus F_s$ class that can be express as follows:

$$\{ h \in C_1 : \forall q \leq \langle d, \pi, F_s \cup \{h\} \rangle \ \varphi_{^q \sigma}^{(d,\pi)}(n) \uparrow \}.$$ 

If $C_1 \setminus A^n \neq \emptyset$, using the hyperimmune-free basis theorem, we can find an $h$ which is hyperimmune-free relative to $\bigoplus F_s$, but since this join is hyperimmune-free, it follows that $h$ is hyperimmune-free, and we can satisfy (1) with $p = \langle d, \pi, F_s \cup \{h\} \rangle$.

Otherwise, $C_1 = A^n = \bigcup_{m \in \omega} U^n_m$, so, by compactness there is $m^*$, which can be found in an effective way from $\bigoplus F_s$, such that $C_1 = U^n_{m^*}$. This $m^*$ will help us satisfy (2).

Now, in order to satisfy (2a), take $d', \pi'$. Notice that the set of functions in $C^1$ that can be add to $F_s$ and have a small convergence using $d', \pi'$ is

$$O_{d',\pi'} = \{ f \in C_1 : \exists \langle d', \pi', F \rangle \leq \langle d, \pi, F_s \cup \{f\} \rangle \ \varphi_{^q \sigma}^{(d',\pi')}(n) \downarrow < m^* \}.$$
This set is $\Sigma^0_1 \oplus F_s$ and we have that

$$C_1 = U^m_{\mathcal{P}} = \bigcup_{(d', \pi', \emptyset) \in \mathcal{P}} O_{d', \pi'}.$$  

By effective compactness we can find $\alpha \in \omega$ and $(\xi d, \xi \pi)$ for all $\xi \leq \alpha$, such that $C_1 \subseteq \bigcup_{\xi=1}^{\alpha} O_{\xi d, \xi \pi}$. In other words, this gives us finitely many $(\xi d, \xi \pi, \emptyset)$ forcing a convergence less than $m^*$ compatible with $(d, \pi, F_s \cup \{f\})$ for all $f \in C_1$. Let

$$h^c_{d, \pi, (\xi d)}(n) = \max \left\{ m^*, B_1(n), \min \{ k : \forall \xi < \alpha \forall i \in (\xi d)^{-1}\{1\}\} \sigma \in \text{dom}(\xi \pi^i)(\sigma \in k^i) \right\}.$$

Each of these satisfies a different condition. $h$ bigger than $m^*$ ensures that (2) holds, the last line ensures that (2a) holds, and being bigger than $B_1$ satisfies a technical requirement we will need later for the induction step.

**Case $l + 1$:**

Fix $(d, \pi, F_s) \leq (d_s, \pi_s, F_s)$ and $g^i_s \in \omega^{|d|}$ for $i \in [0, \ldots, l]$ such that for all $f \in F_s$, and all $i < l + 1$, $f \upharpoonright |d| \neq g^i_s$ and $g^i_s \neq g^j_s$ if $i \neq j$. Then, by our inductive hypothesis, we have that for all $A \subseteq \overline{g}'$ with $|A| \leq l$, either case (2) and (2a) hold or case (1) holds. If for any such subset, we see that (1) holds, then by definition, (1) holds of $\overline{f}'$, and we are done. Otherwise, we will define a function $h^c_{d, \pi, (g^i_s : i < l + 1)}$ computable from $\bigoplus F_s$ with the desired properties.

Now, define

$$B_{l+1}(j) = \begin{cases} 0 & \text{if } j < |d| \\ \max \left\{ h^c_{d, \pi, (g^i_s : i < k)}(j) : |\{ f^i_s \upharpoonright |d| : i < k \}| = k < l + 1, \right. \\ \left. |\{ f^i_s \upharpoonright |d| \in \overline{f}' : f^i_s(t) < B_{l+1}(t) \text{ for } t \geq |d| \} \right\} \} & \text{if } |d| \leq j. \end{cases}$$

In order for our proof to work, following the idea of case $l = 1$, we will define a compact space in $(\omega^\omega)^{|l+1|}$ such that each coordinate is bounded by $B_{l+1}$. Restricting to the functions in this compact space is sufficient, given that for all $G \subseteq \omega^\omega$ with $G = \{ g_i : i < l + 1 \}$, $g_i \upharpoonright |d| = g^i_s$, if there is $g \in G$ and $j \in \omega$ with $g(j) > B_{l+1}(j)$, then we can find an extension that will make a small convergence.

This is, indeed, true. Fix $k \in \omega$. Assume that there is a function in $G$ exceeding $B_{l+1}$. Assume that $g(j) > B_{l+1}(j)$ and that, for all $i < l + 1$, $m < j$, $g_i(m) \leq B_{l+1}(m)$ (so, $g(j)$ is the first time we are above $B_{l+1}$). Let
\[ G = G_0 \cup G_1 \text{ such that for all } f \in G_0, f(j) > B_{l+1}(j) \text{ and for all } a \in G_1, a(j) \leq B_{l+1}(j). \] Since \( |G_1| < l + 1 \), and we know that for all \( f \in G_0, f(j) > B_{l+1}(j) \) and so by definition of \( B_{l+1} \),

\[ f(j) > B_{l+1}(j) \geq h_{d, \pi, \{a| j + 1, a \in G_1\}}(j). \]

By our inductive hypothesis (specifically, by (1a)) we have that for all \( f \in G_0, \langle d, \pi, F_\mathbf{s} \cup \{f\} \rangle \) is compatible with an extension \( r \leq \langle d, \pi, F_\mathbf{s} \cup G_1 \rangle \) with \( \varphi^{(d, \pi)}(k) \downarrow h_{d, \pi, \{h| j + 1, h \in G_1\}}(k) \), and \( r \) does not depend on \( f \). This means that \( \langle d, \pi, F_\mathbf{s} \cup G_0 \rangle \) is compatible with that \( r \leq \langle d, \pi, F_\mathbf{s} \cup G_1 \rangle \). Therefore, \( \langle d, \pi, F_\mathbf{s} \cup G_1 \cup G_0 \rangle = \langle d, \pi, F_\mathbf{s} \cup G \rangle \) is compatible with that \( r \).

In order to make everything work we just need to make sure that

\[ h_{d, \pi, \{g^*_i : i < l + 1\}}(t) \geq h_{d, \pi, \{h| j + 1, h \in G_1\}}(t) \]

for all \( t \geq j \) to do it, we just need to ask for \( h_{d, \pi, \{g^*_i : i < l + 1\}} \) to be bigger than \( B_{l+1} \). This was the technical requirement necessary in our previous step.

Now that we know that our function \( B_{l+1} \) works as we want. We will create the compact space.

Since \( B_{l+1} \) is computable from \( \bigoplus F_\mathbf{s} \) we have that the space of collections of functions agreeing with \( g^*_i \) up to \( |d| \) and bounded by \( B_{l+1} \) thereafter, defined by

\[ C_{l+1} = \{ (f_i : i < l + 1) : f_i \in \omega^\omega, f_i| |d| = g^*_i \& \forall j \geq |d| f_i(j) \leq B_{l+1}(j) \} \]

is effectively compact with respect to \( \bigoplus F_\mathbf{s} \).

Furthermore, fixing \( n \), we define the sets

\[ U^n = \{ (h_i : i < l + 1) \in C_{l+1} : \exists q \leq \langle d, \pi, F_\mathbf{s} \cup \{h_i : i < l + 1\} \rangle \varphi^{(s, \pi)}(n) \downarrow m \}. \]

We can do the same as the case \( l = 1 \). If the compact space is not the union of \( U^n \) then we can satisfy (1). Otherwise, we can satisfy (2) as we did in \( l = 1 \). To satisfy (2a), we do the same as in \( l = 1 \), i.e., we work with \( O_{d', \pi'} \) and ask that \( h_{d', \pi', \{g^*_i : i < l + 1\}}(t) \geq B_{l+1}(t) \) for all \( t \geq |d| \).

## 2.3 Evasion Degrees

Now we will look at the results relating evasion degrees to the the rest of the nodes in the computable version of Cichoń’s diagram.

**Theorem 2.13.** If \( A \) computes a weakly \( 1 \)-generic, then \( A \) is an evasion degree.
Proof. If $A$ computes a weakly 1-generic, then it computes a function escaping all computably meager sets. Furthermore, the collection of sets predicted by any computable predictor is a computably meager set by Lemma 2.6 and so $A$ computes a function evading any computable predictor.

Theorem 2.14. If $A$ is DNC, then $A$ is an evasion degree.

Proof. Let $\{P_e = \langle D_e, \pi_e \rangle\}$ be a list of the partial computable predictors by index $e$. We note that by a result of Jockusch in [9], $A$ computes a DNC function if and only if it computes a strongly DNC function—that is, a function $f \leq_T A$ such that for all $n$, and $\forall e \leq n$ $f(n) \neq \varphi_e(e)$. Then we can define $g(m) = f(n_m)$ for $n_m$ large enough that $f(n_m) \neq \pi_e(g|_m)$ for all $e \leq m$. We can effectively find $n_m$ large enough by a simple coding argument.

Corollary 2.15. If $A$ is weakly meager engulfing, then $A$ is an evasion degree.

Proof. By a result of Rupprecht in [17] $A$ is weakly meager engulfing if and only if it is high or DNC. If $A$ is high, then it has hyperimmune degree, and so is an evasion degree by Theorem 2.13 and the fact that hyperimmune degrees compute weakly 1-generics. If $A$ is DNC, then it is an evasion degree by Theorem 2.14. This completes the proof.

Surprisingly, we actually get an even stronger result, which differs greatly from the analogous case on the set theoretic side:

Corollary 2.16. If $A$ is not low for weak 1-generics, then $A$ is an evasion degree.

Proof. By a result of Stephan and Yu in [19], $A$ is not low for weak 1-generics if and only if $A$ is hyperimmune or DNC. Combining this with Theorem 2.13 and Theorem 2.14 we have the desired result.

Definition 2.17. We define a trace to be a function $g : \omega \to [\omega]^{<\omega}$ with $|g(n)| = n$. A computable trace will simply have $g$ computable.

We define $A \in 2^\omega$ to be computably traceable if for all $f \in \omega^\omega$ with $f \leq_T A$, there is a computable trace $g$ such that $f(n) \in g(n)$ for all $n$.

Theorem 2.18. If $A$ is an evasion degree then $A$ is not low for Schnorr tests.
Proof. Let $A$ be low for Schnorr tests. Then, by a result of Terwijn and Zambrina in [20], it follows that $A$ is computably traceable. Let $f \leq_T A$ be a total function. Then we define $g$ by $g(n) = f|_{I_n}$ where $I_n = \left[\frac{n(n-1)}{2}, \frac{n(n+1)}{2}\right)$. (Any computable partition of $\omega$ into disjoint sets with $|I_n| = n$ works here.) Note that since $g \leq_T f \leq_T A$, it follows that $g$ is computably traceable. Then, by assumption, there is a computable trace $T$ where $T(n) \subset \omega^n$, $|T(n)| = n$, and $g|_{I_n} \in T(n)$. However, for any $n$, there are at most $n - 1$ values on which a first difference between members of $T(n)$ is witnessed. Put another way, there are at most $n - 1$ many values $i$ such that there are $\sigma, \tau \in T(n)$ with $\sigma|_i = \tau|_i$, but $\sigma(i) \neq \tau(i)$. So there must be $j \in I_n$ where for all $\sigma, \tau \in T(n)$, $\sigma|_j = \tau|_j \Rightarrow \sigma(j) = \tau(j)$. Then, we can computably build a predictor which predicts $f$ by adding $j$ to $D$, and accurately predicting all the elements of the trace.

To prove the next theorem we will use the notion of clumpy trees introduced by Downey and Greenberg in [7]. A necessary lemma and definitions are reproduced here. $K$ will be used to refer to prefix-free Kolmogorov complexity.

**Lemma 2.19.** There is a computable mapping $(\sigma, \varepsilon) \mapsto n_\varepsilon(\sigma)$ which maps a finite binary string $\sigma \in 2^{<\omega}$ and a rational $\varepsilon > 0$ to a natural number $n$ such that there is some binary string $\tau$ of length $n$ such that

$$\frac{K(\sigma\tau)}{|\sigma\tau|} \geq 1 - \varepsilon.$$

**Definition 2.20.** A perfect function tree is a function $T : 2^{<\omega} \rightarrow 2^{<\omega}$ that preserves extension and compatibility.

Let $T$ be a perfect function tree, $\sigma \in \text{im } T$, the image of $T$, and let $\varepsilon$ be a positive rational. We say that $T$ contains an $\varepsilon$-clump above $\sigma$ if for all binary strings $\tau$ of length $n_\varepsilon(\sigma)$, $\sigma\tau = T(\rho\tau)$, where $\sigma = T(\rho)$. We further define $T$ to be $\varepsilon$-clumpy if for all $\sigma \in T$, $T$ contains an $\varepsilon$-clump above $\sigma$.

**Definition 2.21** (Athreya, et al.[1]). Given $A \in 2^\omega$, the effective packing dimension of $A$ is given by

$$\limsup_{n \to \infty} \frac{K(A|_n)}{n}.$$

**Theorem 2.22.** There is an $A \in 2^\omega$ which is not an evasion degree, but has positive packing dimension.
Proof. The idea of this proof will be to use forcing with computable trees with some specific properties. First, at the \( e \)th stage, we will be pruning to a tree consisting entirely of paths \( A \) for which \( \varphi^A_e \) is computably predictable. We will use this to ensure that the result of our forcing does not compute an evading function. Second, the trees will be clumpy, allowing us to choose extensions which occasionally have high relative complexity. This will mean our resulting set has positive packing dimension.

Given an initial segment \( A_{e-1} \) and a computable tree \( T_{e-1} \) extending this initial segment, we will prune our tree to \( T_e \), so that there is a single predictor that always predicts \( \varphi^A_e(n) \) for every remaining path \( A \in T_e \) while maintaining the clumpiness requirement.

At every stage in our construction, we will assume that there is no initial segment \( \sigma \) in our current tree \( T_{e-1} \) such that \( \varphi^A_e \) is non-total for all paths \( A \succ \sigma \). Additionally, we will assume that for any \( \sigma \in T_e \), there exist \( \tau_1, \tau_2 \succ \sigma \) such that \( \varphi^{\tau_1}_e \neq \varphi^{\tau_2}_e \). If either of these fail, we define \( A_e = \sigma \) and \( T_e \) is the portion of \( T_{e-1} \) extending \( \sigma \). In either case, the clumpiness condition is preserved for the next stage. In the case that the first assumption fails, \( \varphi^A_e \) is not total for all \( A \succ \sigma \), and so we need not predict it accurately. In the case that the latter assumption fails, \( \varphi^A_e \) is computable for all \( A \succ \sigma \), and so can be predicted easily.

Each run of the construction will go as follows: We will rotate through 3 distinct goals. We can think of them as clumping, differentiating, and predicting.

First, we will add clumps. Given a collection \( \{\sigma_i\} \) of initial segments in the tree, each of length \( n \), we will search for \( m > n \) such that \( T_{e-1}|_m \) contains a 1/2-clump above \( \sigma_i \) for each \( \sigma_i \). Then, the collection given by \( T_{e-1}|_m \) will be the \( \{\tau_i\} \) for the next stage.

Next, we will differentiate. We look for \( j > m \) so that each \( m \)-length \( \tau_i \) has an extension \( \gamma_i \) of length \( j \) such that \( \varphi^\gamma_i_e \) is distinct for each such \( \gamma_i \). We are guaranteed to find these by our previous assumption about splitting.

In the final step, we predict. We now look for \( d \in \omega \) such that \( \varphi^\gamma_i_e(d) \) is undefined for all \( \gamma_i \) previously defined. We add this \( d \) to \( D \) for the predictor we are building, and for each \( \gamma_i \) we look for a further extension \( \sigma_i \succ \gamma_i \) such that \( \varphi^{\sigma_i}_e(k) \downarrow \) for all \( k \leq d \). Then we define \( \pi(\varphi^{\sigma_i}_e|_d) = \varphi^{\sigma_i}_e(d) \). For all other strings \( a \) of length \( d \), we can define \( \pi(a) = 0 \). Now, finally, these \( \sigma_i \) become the initial segments of the tree that we start with for the next pass through these three steps. We repeat the process indefinitely.

Finally, once \( T_e \) is defined, we will pick \( A_e \succ A_{e-1} \) with \( |A_e| > 2|A_{e-1}| \) and \( \frac{K(A_e)}{|A_e|} > \frac{1}{2} \). Such a string is guaranteed to exist because of the clumpi-
ness condition on our tree.

Then, $A = \bigcup A_e$ is the desired degree, as it is a path through each $T_e$, and so $\varphi^A_e$ is computably predictable, but by construction, $A$ has packing dimension $\geq 1/2$.

Note that there is nothing special about $1/2$ in our construction, and a small alteration in the proof can give us $A$ with effective packing dimension of 1.

**Lemma 2.23** (Downey and Greenberg[7]). If $A \in 2^\omega$ is computably traceable then $A$ has effective packing dimension 0.

Indeed, this is true of c.e. traceable sets as well.

**Corollary 2.24.** There is a degree which is not computably traceable, but not an evasion degree.

**Proof.** This is an immediate result of Lemma 2.23 and Theorem 2.22.

In our finished diagram including prediction and evasion (Figure 4), we have included some of the alternate characterizations of nodes we used that include properties of and relations to the computable functions.

### 3 Rearrangement

The rearrangement number was recently introduced in [3] by Blass, Brendle, Brian, Hamkins, Hardy, and Larson. All results and definitions about this characteristic can be found there.

#### 3.1 Definitions

**Definition 3.1.** The *rearrangement number* $\text{rr}$ is defined as the smallest cardinality of any family $C$ of permutations of $\omega$ such that, for every conditionally convergent series $\sum a_n$ of real numbers, there is a permutation $p \in C$ for which

$$\sum a_{p(n)} \neq \sum a_n.$$

A priori, there are a few different ways of making this happen, namely making the permuted series diverge to infinity, making the permuted series oscillate, and making the permuted series sum to a different finite sum than the original series. In practice, oscillation is easier to achieve than the other two, and so it only makes sense to isolate the other two possibilities, giving a few additional characteristics, where the variation requirement is stronger.
Definition 3.2. We present three additional refinements, giving slightly different characterizations:

- $\mathbf{rr}_f$ is defined the same way as $\mathbf{rr}$, but where the sum is required to converge to a different finite number.

- $\mathbf{rr}_i$ is defined the same way, but the sum is required to diverge to infinity.

- $\mathbf{rr}_{fi}$ is defined the same way, but the sum is required to either diverge to infinity or converge to a different finite number.

Simply by definition, one can easily see that $\mathbf{rr} \leq \mathbf{rr}_{fi} \leq \mathbf{rr}_f, \mathbf{rr}_i$. The authors in [3] were able to show that it is consistent that $\mathbf{rr} < \mathbf{rr}_{fi}$, but were unable to conclusively show whether or not the latter three characteristics were separable from each other. Similarly, on the effective side, we have been unable to separate the finite case, the infinite case, or the case allowing either from each other, and so here we will only present the highness notions analogous to $\mathbf{rr}$ and $\mathbf{rr}_{fi}$ (although it should be clear what the other two would look like).

Definition 3.3. We define a conditionally convergent series of rationals $\sum a_n$ to be computably imperturbable if, for all computable permutations $p$, we have that

$$\sum a_n = \sum a_{p(n)}.$$
Also, we define $\sum a_n$ to be *weakly computably imperturbable* if no computable permutation $p$ has that either
\[
\sum a_{p(n)} = B \neq A = \sum a_n \quad \text{or} \quad \sum a_{p(n)} = \pm \infty.
\]
Equivalently, we can define a series to be weakly computably imperturbable if the only way we get inequality of series under computable permutation is by oscillation, that is
\[
\sum a_n \neq \sum a_{p(n)} \Rightarrow \sum a_{p(n)} \text{ fails to converges by oscillation}.
\]
Finally, we define a real $X \in 2^\omega$ as (weakly) computably imperturbable if it computes a series with the corresponding property.

We present here known facts about $rr$ and $rr_{fi}$ along with their computable analogs. All results can be found in [3].

**Theorem 3.4.** The following relationships are known for $rr$ and $rr_{fi}$.

| Cardinal Char. | Highness Properties | Theorem |
|---------------|---------------------|---------|
| $b \leq rr$   | high $\Rightarrow$ imperturbable | 3.5     |
| $d \leq rr_{fi}$ | weak 1-gen $\Rightarrow$ weakly imperturbable | 3.6     |
| $\text{non}(\mathcal{N}) \leq rr$ | computes a Schnorr random $\Rightarrow$ imperturbable | 3.15    |
| $rr \leq \text{cov}(M)$ | imperturbable $\Rightarrow$ weakly meager engulfing | 3.16    |
| $\text{CON}(\text{non}(\mathcal{N}) < rr)$ | imperturbable $\not\Rightarrow$ computes a Schnorr random | Open    |
| $\text{CON}(b < rr)$ | imperturbable $\not\Rightarrow$ high | 3.17    |
| $\text{CON}(rr < rr_{fi})$ | weakly imperturbable $\not\Rightarrow$ imperturbable | 3.18    |
| $\text{CON}(d < rr_{fi})$ | weakly imperturbable $\not\Rightarrow$ hyperimmune | 3.19    |

*This results can be seen in figure 6.*

### 3.2 Imperturbability results

The following is an adaptation of Theorems 15 and 16 in [3].

**Theorem 3.5.** If $X$ is high, then it is imperturbable.

*Proof.* Let $X \in 2^\omega$ be high and $\sum a_n$ be any computable conditionally convergent series. By a classic result of Martin in [14], this means that there is a (strictly increasing) function $f \leq_T X$ such that $f$ dominates all computable functions. Let $\sum a_n$ be any computable conditionally convergent series. Define the sequence $\{b_k\}$ by
\[
b_k = \begin{cases} a_n & k = f^n(0) \\ 0 & \text{otherwise} \end{cases},
\]
using the convention that $f^n$ is the $n$-times application of $f$, that is
\[ f^n(a) = f(\cdots f(f(a))). \]

We claim that $\sum b_{p(n)} = \sum a_n$ for all computable permutations $p$. To see that this is true, for each $e \in \omega$, we will define a computable function $g_e$ such that if $\varphi_e$ is a permutation, it follows that $\varphi_e(i) \leq n, g_e(n) \leq \varphi_e(j) \Rightarrow i \leq j$ for all $i, j \in \omega$. Clearly, given such computable functions, we can see that the series $\sum b_k$ defined above has the desired property, as $f$ dominates all of the $g_e$, and so no computable permutation alters the order of any more than finitely many non-zero elements, leaving the sum unchanged.

In order to define $g_e(n)$, we first assume $\varphi_e$ is a permutation, if it isn’t, nothing that we do matters, as we do not have to defeat it. We begin searching computably for $A_n = \{l \in \omega : \varphi_e(l) \leq n\}$. At some finite stage in our computation, we will have found $l_k$ such that $\varphi_e(l_k) = k$ for all $k \leq n$. This follows from the fact that $\varphi_e$ is a permutation. Then, let $a = \max\{l_k : k \leq n\}$. Finally, we can define $g_e(n) = \max\{\varphi_e(m) : m \leq a\}$. This $g_e$ has the desired property by construction. \qed
The following is an adaptation of Theorem 18 in [3].

**Theorem 3.6.** If $X$ is of hyperimmune degree, then $X$ is weakly imperturbable.

**Proof.** This proof will be very similar to that of Theorem 3.5. Here, let $X$ be of hyperimmune degree. Then, in particular, there is some $f \leq_T X$ such that $f \varphi_e$ infinitely often for any $e$. That is, for every $e$, there are infinitely many $n$ with $f(n) > \varphi_e(n)$. Here, we will also require that $f$ is strictly increasing. Again, for $\sum a_n$ some computable conditionally convergent series, we define the sequence $\{b_k\}$ by

$$b_k = \begin{cases} a_n & k = f^n(0) \\ 0 & \text{otherwise} \end{cases}.$$

We claim that for all $\varepsilon > 0$ and $e \in \omega$, if $\varphi_e$ is a permutation, then there are infinitely-many distinct pairs $i, j \in \omega$ such that

$$\left| \sum_{n=0}^{i} b_{\varphi_e(n)} - \sum_{n=0}^{j} a_n \right| < \varepsilon.$$

To see that this is true, we can use exactly the same $g_e$ as we used in Theorem 3.5. Remember, if $\varphi_e$ is a computable permutation, then $g_e$ is total computable. Since $f$ is not dominated by any computable function, it follows that $f(n) > g_e(n)$ infinitely often. In particular, since $f$ is monotone increasing, there must be infinitely-many $n$ so that $f^{n+2}(0) \geq g_e(f^n(0))$. For each such $n$, there is an initial partial sum of the $b_{\varphi_e(k)}$ which differs from $\sum_{n=0}^{j} a_n$ by at most $|a_{j+1}|$. These pairs have the desired property. Then, since $|a_n| \to 0$ for $n$ large, the initial partial sums of the $b_{\varphi_e(k)}$ are infinitely often arbitrarily close to those of the $a_n$. It follows that $\sum b_{\varphi_e(k)}$ can neither converge to a different limit than $\sum a_n$, nor diverge to infinity. Thus we have that $\sum b_k$ is a weakly imperturbable sum, as desired.

For the next lemma we will need the following definitions and facts from [18]:

**Definition 3.7.** A *computable metric space* is a triple $\mathbb{X} = (X, d, S)$ such that

1. $X$ is a complete metric space with metric $d : X \times X \to [0, \infty)$.
(2) $S = \{a_i\}_{i \in \omega}$ is a countable dense subset of $X$.

(3) The distance $d(a_i, a_j)$ is computable uniformly from $i$ and $j$.

A point $x \in X$ is said to be computable if there is a computable function $h : \omega \to \omega$ such that for all $m > n$, we have $d(a_{h(m)}, a_{h(n)}) \leq 2^{-n}$ and $x = \lim_{n \to \infty} a_{h(n)}$. The sequence $(a_{h(m)})$ is the Cauchy-name for $x$.

**Definition 3.8.** Let $Y = (\mathbb{Y}, S, d_Y)$ be a computable metric space. The space of measurable functions from $(2^\omega, \lambda)$ to $Y$ is a computable metric space under the metric $d_{\text{meas}}(f, g) = \int \min(d_Y(f, g), 1) \, d\lambda$ and test functions of the form $\varphi(x) = c_11_{[\sigma_1]}$ when $x \in [\sigma_1]$ (prefix-free $\sigma_0, \ldots, \sigma_{k-1} \in 2^\omega; c_0, \ldots, c_{k-1} \in S$). The computable points in this space are called effectively measurable functions.

**Lemma 3.9 (Rute).** Suppose $f : (X, \mu) \to Y$ is effectively measurable with Cauchy-name $(\varphi_n)$ in $d_{\text{meas}}$. The limit $\lim_{n \to \infty} \varphi_n(x)$ exists on all Schnorr randoms $x$.

**Lemma 3.10 (Kolmogorov).** Let $X_0, \ldots, X_n$ be independent random variables with expected value $E[X_i] = 0$ and finite variance. Then for each $\epsilon > 0$

$$P \left[ \max_{0 \leq k \leq n} \left( \sum_{i=0}^{k} X_i \right) \geq \epsilon \right] \leq \frac{1}{\epsilon^2} \sum_{i=0}^{n} \text{Var}(X_i).$$

This collection of lemmas will be used to prove the following result which is an effectivization of a theorem of Rademacher [16].

**Lemma 3.11.** If the sequence of rationals $\{a_n\}$ is computable with the limit $\sum a_n^2 < \infty$ also computable, and $X \in 2^\omega$ is a Schnorr random, then $\sum a_n(-1)^X(n)$ converges.

**Proof.** To see this, we will find a Cauchy-name for the function $f(x) = \sum a_n(-1)^x(n)$ in the metric $d_{\text{meas}}$. Then we need only apply Lemma 3.9 to get the desired result.

Given a computable sequence of rationals $\{a_n\}$ with $\sum a_n^2 < \infty$ computable, and $m \in \omega$ we define $\varphi_m(x) = \sum_{n=0}^{i_m} a_n(-1)^x(n)$ where $i_m$ is least such that

$$\sum_{n=i_m}^{\infty} a_n^2 < \frac{1}{8^m+1}.$$
To see that this is a Cauchy-name, given \( j > m \), if we define

\[
A_{j,m} = \left\{ x \in 2^\omega : |\varphi_j(x) - \varphi_m(x)| \leq \frac{1}{2^{m+1}} \right\}
\]

we have that

\[
d_{\text{meas}}(\varphi_j, \varphi_m) \leq \int_{A_{j,m}} |\varphi_j(x) - \varphi_m(x)| \, d\lambda + \int_{2^\omega \setminus A_{j,m}} 1 \, d\lambda \\
\leq \frac{1}{2^{m+1}} + \mu \left\{ x \in 2^\omega : \left| \sum_{n=i_m+1}^{i_j} a_n(-1)^{x(n)} \right| > \frac{1}{2^{m+1}} \right\}.
\]

However, we can effectively bound the measure of the set in this inequality by

\[
\left\{ x \in 2^\omega : \left| \sum_{n=i_m+1}^{i_j} a_n(-1)^{x(n)} \right| > \frac{1}{2^{m+1}} \right\} \subseteq \\
\bigcup_{k=0}^{\infty} \left\{ x \in 2^\omega : \left| \sum_{j=i_m}^{i_m+k} a_j(-1)^{x(j)} \right| > \frac{1}{2^{m+1}} \right\}.
\]

Then, applying Lemma 3.10 we have

\[
\mu \left( \bigcup_{k=0}^{\infty} \left\{ x \in 2^\omega : \left| \sum_{j=i_m}^{i_m+k} a_j(-1)^{x(j)} \right| > \frac{1}{2^{m+1}} \right\} \right) \leq \frac{1}{(1/2^{m+1})^2} \sum_{j=i_m}^{\infty} a_j^2 \\
< \frac{1}{2^{m+1}}
\]

and so \( d_{\text{meas}}(\varphi_m, \varphi_j) \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} = \frac{1}{2^m} \), as desired. Thus, \( \varphi_m \) is a Cauchy name, as desired. Then, by Lemma 3.9 it must converge on all Schnorr randoms.

**Lemma 3.12 (Folklore).** A computable permutation of a Schnorr Random is Schnorr Random.

The following is an adaptation of Theorem 11 in [3].

**Lemma 3.13.** Given a computable permutation \( p \), there is a computable permutation \( q \) with the property that there are infinitely many \( i \) such that \( \{q(n) : n \leq i\} = \{p(n) : n \leq i\} \) and infinitely many \( j \) such that the same happens with the identity, i.e., \( \{q(n) : n \leq j\} = \{0, \ldots, j\} \).
Proof. We can essentially just build this. Let $p$ be a computable permutation, then we alternate between conditions. We define $q_0(0) = 0$, and then we build $q$ in stages such that the domain of $q_s$ will always be an initial segment of $\omega$. For each $s > 0$, we do the following:

If $s$ is odd, we aim to add an $i$ so that $\{q(n) : n \leq i\} = \{p(n) : n \leq i\}$. To do this, we begin to search computably for $m_k \in \omega$ for $k$ on which $q_{s-1}$ has already been defined such that $p(m_k) = q_{s-1}(k)$ for each $k \in \text{dom}(q_{s-1})$. Then we will define $q_s$ up to $\max\{m_k\}$ by simply building a bijection between $\{0, \ldots, \max\{m_k\}\}$ and $\{p(0), \ldots, p(\max\{m_k\})\}$ picking one element at a time while respecting $q_{s-1}$. This is simple, as the collection is computable, and $q_{s-1}$ is already a bijection with a subset, and so we can simply extend. Then, $\max\{m_k\}$ will be the desired $i$.

If $s$ is even, we aim to add a $j$ so that $\{q(n) : n \leq j\} = \{0, \ldots, j\}$. This is even more straightforward. The $j$ we choose will be $j = \max(\text{range}(q_{s-1}))$, and we can simply build a bijection between the finite, computable, same-size sets, $\{0, \ldots, j\} \setminus \text{range}(q_{s-1})$ and $\{0, \ldots, j\} \setminus \text{dom}(q_{s-1})$ in order to extend $q_{s-1}$ to $q_s$.

It is straightforward to see that, from the construction, $q = \bigcup q_s$ is a bijection, and $\text{range}(q) = \text{dom}(q) = \omega$. Thus, $q$ is a computable permutation, and has the desired property. \qed

Note, this result can actually be extended so that, given any two permutations $p_1, p_2$, there is a permutation $q \leq_T p_1 \oplus p_2$ such that there are infinitely many $i, j$ such that $\{q(n) : n \leq i\} = \{p_1(n) : n \leq i\}$ and $\{q(n) : n \leq i\} = \{p_2(n) : n \leq i\}$.

The following is an adaptation of Theorem 6 in [3].

Lemma 3.14. If $\sum a_n$ is not computably imperturbable, then there is a computable permutation $p$ such that $\sum a_{p(n)}$ fails to converge due to oscillation.

Proof. Let $\sum a_n$ be a series which is not computably imperturbable. That is, there is a computable permutation $p$ such that

$$\sum a_n \neq \sum a_{p(n)}.$$  

We can assume that $\sum a_{p(n)} = \pm \infty$ or $\sum a_{p(n)} = B \neq A = \sum a_n$, otherwise there is nothing to show. Now let $q$ be as in Lemma 3.13. This $q$ has the desired property. If $\sum a_{p(n)} = \infty$, then for $i$ as in the lemma, we have that

$$\sum_{n=0}^{i} a_{q(n)} = \sum_{n=0}^{i} a_{p(n)},$$

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thus we can see that these partial sums grow without bound, but simultaneously, for \( j \) as in the lemma, we have that

\[
\sum_{n=0}^{j} a_{q(n)} = \sum_{n=0}^{j} a_n,
\]

and so these partial sums tend towards \( A = \sum a_n \). Thus, the whole series must be non-convergent due to oscillation. A similar argument shows that if \( \sum a_{p(n)} = B \neq A \), then there are infinite subsequences of initial sums of \( \sum a_{q(n)} \) converging to both \( A \) and \( B \), which also means that \( \sum a_{q(n)} \) must be non-convergent due to oscillation.

**Theorem 3.15.** If \( X \) computes a Schnorr Random, then \( X \) is imperturbable.

**Proof.** Let \( X \in 2^\omega \) and \( A \leq_T X \) be Schnorr Random. Then, we claim that if we define \( a_n = \frac{(-1)^{A(n)}}{n} \), the series \( \sum a_n \) is imperturbable. To see this, let \( p \) be a computable permutation, then \( \sum a_{p(n)} \) converges by Lemma 3.11 and Lemma 3.12. Namely, the sequence \( \left\{ \frac{1}{p(n)} \right\} \) is a computable sequence by construction,

\[
\sum \left( \frac{1}{p(n)} \right)^2 = \sum \frac{1}{n^2} = \frac{\pi^2}{6}
\]

is computably converging to a computable sum, and the indices of negative entries of our sequence is Schnorr Random by Lemma 3.12. Thus, we can apply Lemma 3.11 and so the series converges for all computable permutations. Further, since this series must converge for all computable permutations, it follows from Lemma 3.14 that it must be imperturbable.

**Theorem 3.16.** If \( X \) is imperturbable, then \( X \) is weakly meager engulfing.

**Proof.** We will actually show that \( X \) is weakly meager engulfing in the space of permutations, but there is a computable bijection between \( \omega^\omega \) and the space of permutations. Let \( X \) imperturbable, then there is a conditionally convergent imperturbable series \( \sum a_n \leq_T X \). We claim that the set of permutations leaving this sum unchanged is contained in an \( X \)-effectively meager set. In particular, the set of permutations which do not make the sum \(+\infty\) is exactly the set

\[
E = \bigcup_{k \in \omega} \bigcap_{m \geq k} \left\{ p : \sum_{n=0}^{m} a_{p(n)} \leq k \right\}.
\]
Now, we simply observe that the intersection

\[ E_k = \bigcap_{m \geq k} \left\{ p : \sum_{n=0}^{m} a_{p(n)} \leq k \right\} \]

is \( \Pi^0_1 \) in \( X \), additionally, it is nowhere dense, as any initial segment which falls in the appropriate range can then have all terms of the same sign for long enough to escape the interval.

Thus, \( E \) is an \( X \)-effectively meager set of permutations containing all computable permutations, as desired.

We can immediately see that almost all of the foregoing implications are not reversible. This follows from the theorems plus existing known cuts of the computable Cichoń’s diagram. These cuts are cataloged in \[5\] §4.2.

**Corollary 3.17.** There is an \( X \) which is imperturbable but not high.

**Proof.** This is a direct result of Theorem 3.15 plus the fact that there is a Schnorr random which is not high. In fact, there is a low ML-random, which we can see from the low basis theorem plus the existence of a universal ML-test. See e.g. \[15\] Theorem 1.8.37.

**Corollary 3.18.** There is an \( X \) which is weakly imperturbable but not imperturbable.

**Proof.** We will use the fact that weakly meager engulfing is equivalent to high or DNC, a proof of which can be found in \[11\]. The corollary follows directly from Theorems 3.16 and 8.10 plus the existence of a set of hyperimmune degree which is not weakly meager engulfing. Any nonrecursive low c.e. set suffices. Obviously, being of hyperimmune degree means that it is also weakly computably imperturbable. Additionally, by Arslanov’s completeness criterion (\[15\], 4.1.11), such a set cannot be DNC, and is not high by definition. Thus, the set is also not weakly meager engulfing.

**Corollary 3.19.** There is an \( X \) which is weakly imperturbable and is also hyperimmune-free.

**Proof.** This follows directly from Theorem 3.15 plus the fact that imperturbable implies weakly imperturbable and the existence of a Schnorr random which is hyperimmune-free. The fact follows by taking a set \( A \) of hyperimmune-free PA degree (see e.g. \[15\] 1.8.32 and 1.8.42).

In figure 6 you can see were imperturbability stands in the effective Chichoń’s diagram.
4 Questions

Question 4.1. Is there an $A \in 2^\omega$ of prediction degree which does not compute a Schnorr random?

Question 4.2. Is an $A$ which is an evasion degree and low for weak 1-generics?

Question 4.3. Is imperturbable equivalent to weakly meager engulfing?

Question 4.4. Does weakly imperturbable imply any known highness notion?

Question 4.5. Can we separate the finite case and the infinite case of weakly imperturbable from each other or from the combined notion?

Question 4.6. Is there an $A \in 2^\omega$ which is non-computable and not weakly imperturbable?
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