Abstract. In this paper we extend the recent work of C.A. Braumann [1] to the case of stochastic differential equation with random coefficients. Furthermore, the relationship of the Itô-Stratonovich stochastic calculus to studies of random population growth is also explained.

1. Introduction

As mentioned in the paper by Carlos A. Braumann [1], there is the issue of the Itô-Stratonovich controversy. That is, the issue of which stochastic calculus, Itô or Stratonovich, to adopt in the SDE models of the population growth. It has been shown that Itô and Stratonovich calculus give different results and do not yield the same solutions to the SDE models, leading to this controversy on which calculus is more appropriate when modelling population growth in a random environment. Hence, creating an obstacle on the use of this modelling approach. Carlos A. Braumann clears up the confusion concerning this controversy by showing that the apparent difference between the Itô and Stratonovich calculus is due to the confusion based on the assumption that both Itô and Stratonovich employ the same type of mean rates, i.e. interpreting the mean rate as an unspecified “average” per capita growth rate. In fact, Itô and Stratonovich calculus will yield exactly the same results when coupled with the appropriate mean rate. It is proven that, when using Itô calculus, \( b(N) \) is the arithmetic average growth rate \( R_a(x, t) \), and when using Stratonovich calculus, \( b(N) \) is the geometric average growth rate \( R_g(x, t) \).

2. The Model

Let \( N = N(t) \) denote the population size (number of individuals, density) at time \( t \) of a closed population (no migrations) and assume that the initial population size \( N(0) = N_0 > 0 \) is known. In a randomly varying environment, we shall refer to \( \frac{dN}{dt} \) as the total population growth rate and to the per capita growth rate, \( \frac{1}{N} \frac{dN}{dt} \), simply by the growth rate.

We can model the dynamics by assuming that the growth rate \( \frac{1}{N} \frac{dN}{dt} \) is the sum of an “average” growth rate \( b(N) \) and perturbations caused by random environmental fluctuations. We can approximate these perturbations by a white noise \( \sigma \xi(t) \). Where the growth rate \( \frac{1}{N} \frac{dN}{dt} \) is for some \( N > 0 \), some density-dependent function \( b(N) \) (which is the growth rate of the population) having a continuous derivative.

We can model the dynamics using the stochastic differential equation (SDE):

\[
\frac{1}{N} \frac{dN}{dt} = b(N) + \sigma \xi(t), \quad N(0) = N_0 > 0, \tag{2.1}
\]

where we write the per capita growth rate \( \frac{1}{N} \frac{dN}{dt} \) as an “average” density-dependent rate \( b(N) \) perturbed by a white noise \( \sigma \xi(t) \), to take into account the effect of random environmental fluctuations.

The \( b(N) \) represents the “average” growth rate in the population size. The \( \sigma \xi(t) \) represents the random and uncertain movement in the population size which can be attributed to random environmental fluctuations that perturb the per capita growth rate. This perturbation is assumed...
to be a stationary stochastic process and can be reasonably approximated by a white noise process. \( \sigma(t) > 0 \) is the volatility and can be regarded as adding noise or variability to the fluctuations in the population size, and \( \xi(t) \) is a standard Gaussian white noise process. Furthermore, \( \xi(t) \) is a generalised derivative of the standard Brownian motion process \( W(t) \) and is therefore equal to \( dW(t) \).

The SDE given in (2.1) can be rewritten as the following:

\[
\frac{dN}{dt} = b(t, \omega)N(t) + \sigma(t, \omega)N(t)\xi(t). \tag{2.2}
\]

Here, \( \omega \in \Omega \) represents the random environmental scenario (event) in the set \( \Omega \) of all possible environmental scenarios, on the probability space structure, \( (\Omega, \mathcal{F}, P) \). By a scenario \( \omega \), we mean a specific combination of environmental conditions that a population might be subjected to.

We can rewrite this differential equation (2.2) in terms of a Brownian motion process, where we simply substitute \( \xi(t)dt \) with \( dW_t \), since \( \int_0^t \xi(s)ds = W(t) = \int_0^t dW(s) \). Hence, the basic model of the population growth in a random environment is given by:

\[
dN(t) = N(t)\left[ b(t, \omega)dt + \sigma(t, \omega)dW_t \right], \quad N(0) = N_0 > 0, \tag{2.3}
\]

or in integral form

\[
N(t) = N_0 + \int_0^t N(s)b(s)ds + \int_0^t N(s)\sigma(s)dW_s, \tag{2.4}
\]

where \( W \) is a standard one-dimensional Brownian motion. \( b(\cdot, \omega) \) and \( \sigma(\cdot, \omega) \) are assumed to be adapted and satisfy the integrability condition:

\[
\int_0^T (|b(t)| + \sigma(t)^2)dt < \infty
\]

almost surely, for every \( T \in (0, \infty) \). This integrability condition ensures that \( N(t) < \infty \) \( \forall t \in \mathbb{R}^+ \).

Since \( N(t) \) is continuous, the first integral in Equation (2.4), can be defined as a Riemann integral. However, problems now arise with the definition of the second integral of Equation (2.4), in which \( W(t) \) oscillates too rapidly to be defined in the usual Riemann-Stieltjes sense (which follows ordinary calculus rules). The second integral in Equation (2.4) contains the Brownian motion component which defines this integral as a stochastic integral. This integral cannot usually be defined as a classical Riemann-Stieltjes integral due to the fact that the limits of Riemann-Stieltjes sums differ according to the choice of intermediate points in the integrand function. As a result, there are many alternative definitions of these stochastic integrals according to the choice of such intermediate points.

The most commonly used integrals in the literature are the \textbf{Itô} and \textbf{Stratonovich} integrals. The Itô integral has nice probabilistic properties, which includes, apart from being a martingale, the property of zero expectation as well as having a convenient expression for its variance. However, it follows non-ordinary calculus rules. Stratonovich calculus, on the other hand, follows ordinary calculus rules. We will examine the problems that arise in the interpretation of (2.3) when the SDE is taken in the Itô and Stratonovich sense.

In the next two sections, we will introduce the important concepts of the Itô and Stratonovich calculus respectively. Thereafter, in Section 5, we will discuss and represent the relationship between Itô and Stratonovich calculus. In Section 6, the controversy itself will be discussed along with its resolution.
3. Itô Calculus

In Itô calculus we can express the Itô SDE in the following form:
\[ dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \]
or equivalently in integral form as
\[ X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s. \]

**Definition 3.1. The Itô Integral**

Suppose that \( W(t) \) is a Brownian motion process and that \( X(t) \) is a stochastic process. Consider a partition of \([0, T]\), \( 0 = t_0 < t_1 < \cdots < t_n = T \), then the Itô integral of \( X \) w.r.t. \( W \) is a random variable
\[ \int_0^T X_t dW_t := \lim_{n \to \infty} \sum_{j=0}^{n-1} X_{t_j} (W(t_{j+1}) - W(t_j)). \]

Notice, in the summation, the function \( X \) is defined at the left-hand point, i.e. the value of \( X \) at the beginning of each timestep is used, this is of crucial importance.

**Theorem 3.2. Itô’s Lemma**

Let \( X(t) \) be a generalised Brownian motion process or an Itô process. That is, let \( X(t) \) have the following dynamics
\[ dX(t) = a(X_t)dt + b(X_t,dW_t), \]
where \( W_t \) is a Brownian motion process.

Let \( F(X_t, t) \) be a function with continuous second derivatives, where \( F \) and \( X \) have a functional dependence.

Then \( F(X_t, t) \) is also an Itô process and has the following dynamics
\[ dF(X_t, t) = \frac{\partial F}{\partial t}(x, t)dt + \frac{\partial F}{\partial x}(x, t)dX_t(t) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(x, t)(dX_t(t))^2 \]
\[ = \left( \frac{\partial F}{\partial t}(x, t) + a(x, t) \frac{\partial F}{\partial x}(x, t) + \frac{1}{2} b^2(x, t) \frac{\partial^2 F}{\partial x^2}(x, t) \right) dt + b(x, t) \frac{\partial F}{\partial x}(x, t)dW_t. \]

Hence, \( F \) is also an Itô process, but with adjusted drift rate given by \( \frac{\partial F}{\partial t}(x, t) + a(x, t) \frac{\partial F}{\partial x}(x, t) + \frac{1}{2} b^2(x, t) \frac{\partial^2 F}{\partial x^2}(x, t) \) and a scaled variance, \( b^2(x, t) \frac{\partial F}{\partial x}(x, t) \).

We may use Itô Lemma to solve (2.3), in the form
\[ d\ln N(t) = \gamma(t)dt + \sigma(t)dW_t, \quad N(0) = N_0 > 0, \]
where
\[ \gamma(t) := b(t) - \frac{1}{2} \sigma^2(t), \quad (3.1) \]
or equivalently:
\[ N(t) = N_0 \exp \left\{ \int_0^t \gamma(s)ds + \int_0^t \sigma(s)dW_s \right\}. \quad (3.2) \]

We note from (3.2) that \( N(t) > 0 \) for all \( t > 0 \) provide that \( N_0 > 0 \).

We shall refer to the quantity of (3.1) as the rate of growth of the population \( N \), because of the a.s. relationship
\[ \lim_{T \to \infty} \frac{1}{T} \left( \ln N(T) - \int_0^T \gamma(s)ds \right) = 0, \quad (3.3) \]
valid when the variance \( a(\cdot) = \sigma^2(\cdot) \) is bounded, uniformly in \((t, \omega)\); this follows from the strong law of large numbers and from the representation of (local) martingales as time-changed Brownian motions.
4. Stratonovich Calculus

In probability theory, the Stratonovich integral is a stochastic integral, the most common alternative to the Itô integral. The appeal of Stratonovich calculus is that in certain circumstances, integrals in the Stratonovich definition are easier to manipulate. Unlike the Itô integral counterpart, it is defined such that the chain rule of ordinary calculus holds for the stochastic integrals. Perhaps the most common situation in which these are encountered is as the solution to SDEs. These Stratonovich SDEs are equivalent to Itô SDEs, apart from the notation \( \sigma(X_t, t) \circ dW_t \), where the “\( \circ \)” simply indicates that we are working in the Stratonovich sense. Furthermore, it is possible to convert between the two whenever one definition is more convenient for our purposes.

In Stratonovich calculus we can express the Stratonovich SDE in the following form

\[
dX_t = \mu(X_t, t)dt + \sigma(X_t, t) \circ dW_t,
\]

or equivalently in integral form as

\[
X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s) \circ dW_s.
\]

**Definition 4.1. The Stratonovich Integral**

It is defined in a similar manner to the Riemann integral, i.e. as a limit of Riemann sums. Suppose that \( W(t) \) is a Brownian motion process and that \( X(t) \) is a stochastic process, consider a partition of \([0, T]\), \( 0 = t_0 < t_1 < \cdots < t_n = T \), then the Stratonovich integral of \( X \) w.r.t. \( W \) is a random variable

\[
\int_0^T X_t \circ dW_t := \lim_{n \to \infty} \sum_{j=0}^{n-1} X_{(t_{j+1}+t_j)/2} (W(t_{j+1}) - W(t_j)).
\]

Here, the function \( X \) is evaluated in the middle of each timestep (i.e. choose value of process at midpoint of each subinterval). In the definition of the Itô integral, the same procedure is used except for choosing the value of the process \( X \) at the left-hand point of each subinterval, i.e. \( X_{t_j} \) in place of \( X_{(t_{j+1}+t_j)/2} \).

In a similar fashion to Itô’s Lemma, Stratonovich satisfies

\[
dF(X_t, t) = \left( \frac{\partial F}{\partial t}(x, t) + a(x, t) \frac{\partial F}{\partial x}(x, t) \right) dt + b(x, t) \frac{\partial F}{\partial x}(x, t) dW_t.
\]

5. The Itô-Stratonovich Relationship

**Theorem 5.1. Conversion Formula**

Let \( X \) be a stochastic process, in particular, an Itô process satisfying the SDE, \( dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t \). Let \( \sigma \) be some function of \( X \) and \( t \). Furthermore, to make apparent the distinction between the two integrals, we will adopt the subscript \( I \) to indicate an Itô integral and the subscript \( S \) to indicate a Stratonovich integral.

Then conversion between the Itô and Stratonovich integrals may be performed using the formula

\[
\int_0^T \sigma_S(X_t, t)dW_t = \int_0^T \sigma_I(X_t, t)dW_t + \frac{1}{2} \int_0^T \frac{\partial \sigma}{\partial x}(x, t) \sigma(X_t, t)dt. \tag{5.1}
\]

**Proof.**

\[
\int_0^T \sigma_S(X_t, t)dW_t - \int_0^T \sigma_I(X_t, t)dW_t = \lim_{n \to \infty} \sum_{j=0}^{n-1} \left( \sigma \left( \frac{X(t_j) + X(t_{j+1})}{2}, t \right) - \sigma \left( \frac{X(t_j) + X(t_{j+1})}{2}, t \right) \right) (W(t_{j+1}) - W(t_j)).
\]
By the Mean Value Theorem, we obtain
\[
\lim_{n \to \infty} \frac{1}{n-1} \sum_{j=0}^{n-1} \frac{\partial \sigma}{\partial x}(X, t) \left( \frac{X(t_j) + X(t_{j+1})}{2} - X(t_j) \right) (W(t_{j+1}) - W(t_j))
\]
\[
= \frac{1}{2} \lim_{n \to \infty} \frac{1}{n-1} \sum_{j=0}^{n-1} \frac{\partial \sigma}{\partial x}(X, t) (X(t_j + 1) - X(t_j)) (W(t_{j+1}) - W(t_j))
\]
\[
= \frac{1}{2} \lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{\partial \sigma}{\partial x}(X, t) \Delta X_t \Delta W_t
\]
\[
= \frac{1}{2} \lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{\partial \sigma}{\partial x}(X, t) \sigma(X, t) \Delta t
\]
\[
= \frac{1}{2} \lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{\partial \sigma}{\partial x}(X, t) \sigma(X, t)(t_{j+1} - t_j).
\]

Which is the Riemann sum of
\[
\frac{1}{2} \int_0^T \frac{\partial \sigma}{\partial x}(x, t) \sigma(X_t, t) dt.
\]

Giving us the required result. □

From Theorem 5.1, it is evident that the Itô SDE,
\[
dX(t) = a(X_t, t)dt + \sigma(X_t, t)dW_t
\]
(5.2)
is equivalent to the Stratonovich SDE,
\[
dX(t) = \left[ a(X_t, t) - \frac{1}{2} \sigma(X_t, t) \frac{\partial \sigma}{\partial x}(X_t, t) \right] dt + \sigma(X_t, t)dW_t
\]
(5.3)
and that the Stratonovich SDE,
\[
dX(t) = \alpha(X_t, t)dt + \beta(X_t, t)dW_t
\]
(5.4)
is equivalent to the Itô SDE
\[
dX(t) = \left[ \alpha(X_t, t) + \frac{1}{2} \beta(X_t, t) \frac{\partial \beta}{\partial x}(X_t, t) \right] dt + \beta(X_t, t)dW_t.
\]
(5.5)

In the population dynamics context, the Stratonovich SDE, given by
\[
dN(t) = b(t)N(t)dt + \sigma(t)N(t)dW(t).
\]
(5.6)
Using the Itô-Stratonovich conversion formula (5.5), we have \(a(N_t, t) = b(t)N(t)\) and \(\sigma(N_t, t) = \sigma(t)N(t)\), where \(\frac{\partial \sigma}{\partial x}(N_t, t) = \sigma(t)\). Hence, (5.6) is equivalent to the Itô SDE
\[
dN(t) = \left[ b(t)N(t) - \frac{1}{2} \sigma(N_t, t) \sigma(t) \right] dt + \sigma(N_t, t)N(t)dW(t)
\]
(5.7)
\[
= \left[ b(t)N(t) - \frac{1}{2} \sigma^2(t)N(t) \right] dt + \sigma(t)N(t)dW(t).
\]
(5.8)

In a similar fashion we can obtain the reverse conversion formula.
6. THE CONTROVERSY

Many qualitative differences have been uncovered between Itô and Stratonovich calculus. In particular, there are instances in which Stratonovich calculus predicts, for the population, non-extinction and the existence of a stochastic equilibrium, whereas, at the same time, Itô calculus will predict population extinction. So, it seems, which calculus one uses does have important consequences. This fact has resulted in there being much controversy over which calculus is more appropriate to employ when finding a solution to the SDE.

Considering the dramatic differences in predictions concerning important issues like extinction, which calculus should one trust? This is a major obstacle to the use of these stochastic models.

Braumann (2003) resolved the issue of the Itô-Stratonovich controversy for the density-independent growth model, where \( b(N) \equiv b \) is identically constant, in a random population environment. Braumann then extended these results in a random environment for the general density-dependent population growth model. It is revealed that the possible reason for this controversy, is the subtle fact that the same per capita “average” growth rate, ‘\( b \)’ is used in both the Itô and Stratonovich calculus. Therefore, the issue here regards the meaning and interpretation of this average, since it is not elucidated what type of “average” is being referred to. Furthermore, it is inherently assumed that both the Itô and Stratonovich calculus make use of the same average, this is of course an incorrect assumption!

Hence, this issue of the “average” needs to be addressed and clarification needs to be made of what type of average each method uses. In fact, it is found that the interpretation of \( b \) is different when considering population dynamics. When one decides to use Itô calculus in obtaining a solution to the SDE, \( b \) is interpreted as the arithmetic average growth rate. However, if ones chooses to implement Stratonovich calculus, \( b \) is interpreted as the geometric average growth rate. The differences between these two types of averages results in the dramatic differences between the Itô and Stratonovich calculus to disappear, yielding exactly the same solutions in both instances.

Thus the differences are merely due to the absence of clarification of the meaning of \( b \). So, all that is required is to match the appropriate average with the correct type of calculus, and exact same results will be obtained, putting to rest the Itô-Stratonovich controversy.

6.1. Types of Averages. We will denote by \( E_{t,x}[\cdot] := \mathbb{E}[\cdot|N(t) = x] \), as the expectation conditioned on the knowledge that at time \( t \) the population size \( N(t) \) is \( x \).

6.1.1. The Arithmetic Average. The first type of average we have already mentioned is the arithmetic average. It is simply given by the usual expected value, conditioned on the knowledge that \( N(t) = x \), and given by \( \frac{1}{n} \sum_{i=1}^{n} x_i = \mathbb{E}[X] \).

**Definition 6.1.** We define the arithmetic average growth rate at time \( t \), when the population size at time \( t \) is \( x \), as

\[
R_{a}(x,t) := \frac{1}{x} \lim_{\Delta t \to 0} \frac{E_{t,x}[N(t+\Delta t)] - x}{\Delta t}.
\]  

(6.1)

**Theorem 6.2.** Let \( R_{a}(x,t) \) be the arithmetic average growth rate as defined above, we have equivalently

\[
R_{a}(x,t) = E_{t,x}[b(t)].
\]  

(6.2)

**Proof.** We have

\[
N(t+\Delta t) = N(t) \exp \left\{ \int_t^{t+\Delta t} \gamma(s)ds + \int_t^{t+\Delta t} \sigma(s)dW_s \right\}.
\]
It follows
\[
R_a(x,t) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{t,x}\left[ N(t + \Delta t) \right] - x}{\Delta t} = \frac{1}{x} \mathbb{E}_{t,x} \left[ x \exp \left\{ \int_t^{t + \Delta t} \gamma(s) ds + \int_t^{t + \Delta t} \sigma(s) dW_s \right\} - x \right] \Delta t
\]
\[
= \lim_{\Delta t \downarrow 0} \mathbb{E}_{t,x} \left[ \exp \left\{ \int_t^{t + \Delta t} b(s) ds \right\} \right] \Delta t
\]
\[
= \mathbb{E}_{t,x} [b(t)].
\]
\]

6.1.2. The Geometric Average.

Definition 6.3. The geometric mean of a positive random variable \(X\) is defined as \(e^{\mathbb{E}[\ln(X)]}\)

Since
\[
\mu_{\text{geom}} = \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} = e^{\ln\left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}} = e^{\frac{1}{n} \ln\left( \prod_{i=1}^{n} x_i \right)} = e^{\frac{1}{n} \sum \ln(x_i)} = e^{\mathbb{E}[\ln(X)]}
\]

Definition 6.4. We define the geometric average growth rate at time \(t\), when the population size at time \(t\) is \(x\), as
\[
R_g(x,t) := \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{t,x}\left[ \ln(N(t + \Delta t)) \right] - \ln x}{\Delta t}.
\] (6.3)

Hence, the geometric average is obtained by transforming the quantities to be averaged to log scale then taking an ordinary arithmetic average and then revert to the initial scale by inverting the algorithm.

Proposition 6.5. Let \(R_g(x,t)\) be the geometric average growth rate as defined above, we have equivalently
\[
R_g(x,t) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{t,x}\left[ \ln(N(t + \Delta t)) \right] - \ln x}{\Delta t} = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{t,x}\left[ \ln(N(t + \Delta t)/x) \right]}{\Delta t}.
\] (6.4)

Proof. This follows from the fact that when \(z \to 0\) we have \((e^z - 1)/z \to 1\). We apply this result to \(z := \mathbb{E}_{t,x}\left[ \ln(N(t + \Delta t)/x) \right]\) which tends to 0 when \(\Delta t \downarrow 0\) as follows:
\[
\lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{t,x}\left[ \ln(N(t + \Delta t)/x) \right]}{\Delta t} = \lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{t,x}\left[ \ln(N(t + \Delta t)/x) \right]}{\Delta t} \frac{\Delta t}{\Delta t}
\]
\[
= \lim_{\Delta t \downarrow 0} \frac{(e^z - 1)}{z} \times \frac{z}{\Delta t}
\]
\[
= \lim_{\Delta t \downarrow 0} \frac{(e^z - 1)}{\Delta t}
\]
\[
= \lim_{\Delta t \downarrow 0} \frac{(xe^z - x)}{\Delta t}
\]
\[
= R_g(x,t).
\]
Theorem 6.6. Let $R_g(x, t)$ be the geometric average growth rate as defined above, we have equivalently

$$R_g(x, t) = E_{t,x} [\gamma(t)].$$

where $\gamma(t)$ is as defined in Equation (3.1).

Proof. We have

$$N(t + \Delta t) = N(t) \exp \left\{ \int_t^{t+\Delta t} \gamma(s) ds + \int_t^{t+\Delta t} \sigma(s) dW_s \right\}.$$ 

By Proposition 6.5 above we have

$$R_g(x, t) = \lim_{\Delta t \to 0} \frac{E_{t,x} [\ln N(t + \Delta t)] - \ln x}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{E_{t,x} \left[ \int_t^{t+\Delta t} \gamma(s) ds + \int_t^{t+\Delta t} \sigma(s) dW_s \right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_t^{t+\Delta t} \gamma(s) ds \right]$$

$$= E_{t,x} [\gamma(t)].$$

$\square$

Corollary 6.7. When $\gamma(t, \omega) = b(N_t) - \frac{1}{2} \sigma^2$.

Let $R_g(x)$ be the geometric average growth rate as defined above, we have equivalently

$$R_g(x, t) = b(x) - \frac{1}{2} \sigma^2.$$ 

It seems as though the arithmetic and geometric averages are equivalent, where the geometric average substitutes the process $N(t)$ by the process $\ln N(t)$. These definitions give an indication on how these two rates can be estimated from observed data. For example, to determine an estimate of $R_a(x, t)$, we look at all the instances $t$ for which $N(t)$ is close to $x$ and then we take the average of those $N(t + \Delta t)$ as an approximation of $E_{t,x} [N(t + \Delta t)]$.

6.1.3. Other Types of Averages. These two averages are not the only averages we can consider, there are many other types of averages that are possible, so when we refer to an “average” it is of extreme importance to specify which particular average is being referred to. Some other possible averages are the: harmonic, median and quadratic averages.

Median \[ \hat{\mu} = med(x_1, \ldots, x_n) \]

Quadratic \[ \hat{\mu} = \left( \frac{1}{n} \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = (E[X^2])^{\frac{1}{2}} \]

6.2. Density-Independent Growth. Recall that there are two main ways in which to interpret SDEs: Itô calculus and Stratonovich calculus. They usually lead to different solutions. Let us first consider the density-independent growth rate model in a random environment, this model corresponds to a constant “average” growth rate, $b(N) \equiv b$.

$$dN(t) = bN dt + \sigma(t) N(t) dW(t).$$

To make the distinction between the two approaches more apparent, we will use the notation $b_I$, to denote the “average” growth rate under Itô calculus, and $b_S$ to denote the “average” growth rate under Stratonovich calculus.
6.2.1. *Itô Model*. Let us consider the density-independent Itô calculus model:

\[ dN(t) = b_I N dt + \sigma(t)N(t)dW(t). \]  

(6.6)

It is convenient to work in the logarithmic scale by making the change of variables, \( Y(t) = \ln N(t), \quad y = \ln x. \)

Applying Itô’s Lemma to Equation (6.6), where \( F(N(t)) = \ln N(t) \), we obtain

\[
d\ln N(t) = \frac{\partial F(N)}{\partial x} dN(t) + \frac{1}{2} \frac{\partial^2 F(N)}{\partial x^2} (dN(t))^2
\]

(6.7)

\[
= \frac{1}{N(t)} dN(t) + \frac{1}{2} \left( -\frac{1}{N(t)^2} \right) (dN(t))^2
\]

(6.8)

\[
= \left( b_I - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t)dW(t).
\]

(6.9)

Therefore, \( Y(t) = \ln N(t) \), satisfies the SDE in Equation (6.6).

This can alternatively be expressed in the equivalent integral form as

\[
Y(t) = Y_0 + \int_0^t \left( b_I - \frac{1}{2} \sigma^2 \right) ds + \int_0^t \sigma dW_s
\]

(6.10)

\[
= Y_0 + \left( b_I - \frac{1}{2} \sigma^2 \right) t + \sigma W_t.
\]

(6.11)

From Equation (6.11), we can conclude that \( Y(t) \sim N(Y_0 + (b_I - \frac{1}{2} \sigma^2)t, \sigma^2 t) \). From this we obtain a solution to Equation (6.6), which is represented as

\[
N(t) = N_0 \exp \left[ \left( b_I - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right].
\]

Hence, \( N(t) \) has a lognormal distribution with expected value the knowledge that \( \mathbb{E}[e^{\sigma W_t}] = e^{\frac{1}{2} \sigma^2 t} \), given by:

\[
\mathbb{E}[N(t)] = N_0 \exp[b_I t].
\]

From Equation (6.11), one obtains the asymptotic result \( Y(t) \sim (b_I - \frac{1}{2} \sigma^2)t \) as \( t \to +\infty \). Therefore, as \( t \to \infty \), \( N(t) \to \infty \) or \( N(t) \to 0 \) according to whether the “average” growth rate \( b_I \) is larger than \( \sigma^2 \) or smaller than \( \sigma^2 \).

6.2.2. *Stratonovich Model*. Let us consider the density-independent Stratonovich calculus model

\[ dN(t) = b_S N dt + \sigma(t)N(t)dW(t). \]  

(6.12)

Since Stratonovich calculus obeys the ordinary calculus rules, we have by the ordinary chain rule of differentiation

\[
dY(t) = d\ln N(t) = \frac{d\ln N(t)}{dN(t)} dN(t)
\]

(6.13)

\[
= \frac{1}{N(t)} dN(t)
\]

(6.14)

\[
= b_S dt + \sigma dW(t).
\]

(6.15)

Therefore, \( Y(t) = \ln N(t) \) satisfies the SDE given in Equation (6.15).

This can alternatively be expressed in the equivalent integral form as

\[
Y(t) = Y_0 + \int_0^t b_S ds + \int_0^t \sigma dW_s.
\]

(6.16)

From which one immediately obtains the solution

\[
Y(t) = Y_0 + b_S t + \sigma W_t.
\]

(6.17)
Since the integrand is constant, in this density-independent case, the Itô and Stratonovich integrals coincide. For both approaches, we have \( \int_0^t \sigma W_t = \sigma (W_t - W_0) = \sigma W_t \) since \( W_0 = 0 \). Since \( W(t) \sim N(0, t) \) (i.e. normally distributed with mean zero and variance \( t \)), we conclude that \( Y(t) \sim N(Y_0 + b_S, \sigma^2 t) \). From this we obtain a solution to Equation (6.12), which is given by

\[
N(t) = N_0 \exp \left[ b_S t + \sigma W_t \right].
\]

Hence, \( N(t) \) has a lognormal distribution with expected value given by:

\[
E[N(t)] = N_0 \exp \left[ \left( b_S + \frac{1}{2} \sigma^2 \right) t \right].
\]

From Equation (6.17), one obtains the asymptotic result \( Y(t) \sim b_S t \) as \( t \to +\infty \) (since \( \frac{W_t}{t} \to 0 \) as \( t \to \infty \)). Therefore, as \( t \to \infty \), \( N(t) \to \infty \) (i.e. growth without bound) or \( N(t) \to 0 \) (extinction) according to whether the “average” growth rate \( b_S \) is positive or negative.

**6.2.3. Conclusion.** The long-term behaviour of \( N(t) \) for both interpretations of the SDE can be further analysed by examining the trajectory of \( N(t) \) in probability. Since, \( \frac{W_t}{t} \to 0 \) a.s. when \( t \to \infty \), one easily notices that under Stratonovich calculus, \( N(t) \to \infty \) when \( b_S > 0 \) (probability of extinction is zero and there is a stochastic equilibrium) and \( N(t) \to 0 \) when \( b_S < 0 \) (i.e. population extinction occurs with probability one).

In a similar fashion, under Itô calculus, \( N(t) \to \infty \) when \( b_I > \frac{\sigma^2}{2} \) and \( N(t) \to 0 \) when \( b_I < \frac{\sigma^2}{2} \).

The differences between the Itô and Stratonovich approaches are now apparent. The behaviour appears to be different from the Stratonovich calculus. Hence, if one employs Itô instead of Stratonovich, the conditions for non-extinction and existence of a stochastic equilibrium are qualitatively different. This illustrates the consequences of the two approaches for the population behaviour. Using Stratonovich calculus, extinction would occur a.s. if the “average” growth rate \( b < 0 \), but with Itô calculus, one can have extinction a.s. even for positive values of the “average” growth rate \( b \) if \( b < \frac{\sigma^2}{2} \).

Furthermore, the Itô calculus obtains different results compared to the deterministic model, this makes Itô calculus quite popular in modelling, hence, avoiding the issue of ignoring random environmental fluctuations.

The approach taken here works for all density-dependent models and completely and exactly elucidates the difference between the two interpretations. It also exactly solves the problem of which calculus to use and how to use it.

\[
\text{Itô SDE: } \quad N(t) = b_I N dt + \sigma N dW_t
\]
\[
d \ln N(t) = \left( b_I - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t
\]

\[
\text{Stratonovich SDE: } \quad dN(t) = b_S N dt + \sigma N dW_t
\]
\[
d \ln N(t) = b_S dt + \sigma dW_t
\]

**6.2.4. Resolution of which Average to use.**

**6.2.4.1 Itô**

Let us compute the two averages for the Itô SDE (6.6), we obtain from (6.17),

\[
Y(t + \Delta t) = \ln x + \left( b_I - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (W(t + \Delta t) - W(t)).
\]
Therefore, $Y(t + \Delta t)$ is normally distributed with mean $\ln x + (b_I - \frac{\sigma^2}{2})\Delta t$ and variance $\sigma^2\Delta t$. The conditional expectation is

$$
\mathbb{E}_{t,x}[Y(t + \Delta t)] = \mathbb{E}_{t,x}[\ln N(t + \Delta t)]
= \ln x + \left( b_I - \frac{\sigma^2}{2} \right) \Delta t.
$$

Replacing into equation (6.3), we obtain

$$
R_g(x, t) = \frac{1}{x} \lim_{\Delta t \downarrow 0} \frac{\exp(\ln x + (b_I - \frac{\sigma^2}{2})\Delta t) - x}{\Delta t}
= \lim_{\Delta t \downarrow 0} \frac{\exp[(b_I - \frac{\sigma^2}{2})\Delta t] - 1}{\Delta t}
\equiv b_I - \frac{1}{2}\sigma^2 \equiv b_S.
$$

We also notice that $N(t + \Delta t) = \exp(Y(t + \Delta t))$ is lognormal with parameters $\ln x + (b_I - \frac{\sigma^2}{2})\Delta t$ and $\sigma^2\Delta t$, and so its conditional expectation is

$$
\mathbb{E}_{t,x}[N(t + \Delta t)] = \mathbb{E}_{t,x}[\exp(Y(t + \Delta t)]
= \mathbb{E}_{t,x}\left[ \exp\left( \ln x + \left( b_I - \frac{\sigma^2}{2} \right) \Delta t + \sigma(W(t + \Delta t) - W(t)) \right) \right]
= \exp \left( \ln x + \left( b_I - \frac{\sigma^2}{2} \right) \Delta t + \frac{1}{2}\sigma^2\Delta t \right)
= x \exp(b_I \Delta t).
$$

Replacing into equation (6.1), we obtain

$$
R_a(x, t) = \frac{1}{x} \lim_{\Delta t \downarrow 0} \frac{x \exp(b_I \Delta t) - x}{\Delta t}
\equiv b_I.
$$

The conclusion is that

$$
R_g(x, t) \equiv b_I - \frac{\sigma^2}{2},
R_a(x, t) \equiv b_I.
$$

Hence, when using Itô calculus, the “average” growth rate $b_I$ is specified as the arithmetic average growth rate.

### 6.2.4.2 Stratonovich

Let us compute these two averages for the Stratonovich SDE model (6.12). Since $N(t) = x$, we obtain from (6.17),

$$
Y(t + \Delta t) = \ln x + b_S\Delta t + \sigma(W(t + \Delta t) - W(t)).
$$

Therefore, $Y(t + \Delta t) \sim N(\ln x + b_S\Delta t, \sigma^2\Delta t)$ and so its conditional expectation is

$$
\mathbb{E}_{t,x}[Y(t + \Delta t)] = \mathbb{E}_{t,x}[\ln N(t + \Delta t)] = \ln x + b_S\Delta t.
$$

Replacing into equation (6.3), we obtain
\[ R_g(x, t) = \lim_{\Delta t \to 0} x \frac{\exp(\ln x + b_S \Delta t) - x}{\Delta t} = \lim_{\Delta t \to 0} x \frac{e^{b_S \Delta t} - x}{\Delta t} = \lim_{\Delta t \to 0} x (e^{b_S \Delta t} - 1) \Delta t \]

\[ N(\Delta t) = \exp(Y(t + \Delta t)) \]

is lognormally distributed with parameters \( \ln x + b_S \Delta t \) and \( \sigma^2 \Delta t \), hence its conditional expectation is

\[ E_{t,x}[N(t + \Delta t)] = E_{t,x}[\exp(Y(t + \Delta t))] = E_{t,x}[\exp(\ln x + b_S \Delta t + \sigma(W(t + \Delta t) - W(t)))] = \exp\left( \ln x + b_S \Delta t + \frac{1}{2} \sigma^2 \Delta t \right) = x \exp\left( (b_S + \frac{1}{2} \sigma^2) \Delta t \right), \]

where \( E_{t,x}[\exp(W(t + \Delta t) - W(t))] = e^{\frac{1}{2} \sigma^2 \Delta t} \), since \( W(t + \Delta t) - W(t) \sim N(0, \Delta t) \).

Replacing this into equation (6.1), we obtain

\[ R_a(x, t) = \lim_{\Delta t \to 0} x \frac{\exp((b_S + \frac{1}{2} \sigma^2) \Delta t) - x}{\Delta t} = \lim_{\Delta t \to 0} \frac{\exp((b_S + \frac{1}{2} \sigma^2) \Delta t) - 1}{\Delta t} \Delta t = b_S + \frac{1}{2} \sigma^2 \Delta t. \]

The conclusion is that

\[ R_g(x, t) \equiv b_S, \]

\[ R_a(x, t) \equiv b_S + \frac{1}{2} \sigma^2. \]

Hence, when using Stratonovich calculus, the “average” growth rate \( b_S \) is specified as the geometric average growth rate.

6.2.4.3 Conclusion

The conclusion is that when using Itô calculus, the “average” growth rate, \( b_I \) is specified as the arithmetic average growth rate, and when using Stratonovich calculus, the “average” growth rate, \( b_S \) is specified as the geometric average growth rate.

This fact instructs us to replace the unspecified growth rate \( b \) by the specified average it truly represents. It is only in this manner that the results acquire meaning. Since \( b_S \) is indeed the geometric average growth rate \( R_g \), we can conclude that the solution of the Stratonovich SDE density-independent growth model is

\[ N(t) = N_0 \exp[R_g t + \sigma W(t)]. \]

Since \( b_I \) is indeed the arithmetic average growth rate \( R_a \), we can conclude that the solution of the Itô SDE density-independent growth model is
5.3 Density-Dependent Growth. We can interpret (2.3) as an Itô SDE and it can be written as:

$$dN(t) = N(t)\left[b_I(t)dt + \sigma(t)\,dW_t\right].$$

(6.19)

By the Itô-Stratonovich conversion formula given in Equation (5.3), where \(a(N_t, t) = b(t)N(t)\), \(\sigma(N_t, t) = \sigma(t)N(t)\) and \(\frac{\partial a}{\partial x}(N_t, t) = \sigma(t)\).

Therefore the Itô SDE is equivalent to the Stratonovich SDE

$$dN(t) = N(t)\left[b_I(t)N(t) - \frac{1}{2}\sigma^2(t)N(t)\right] \,dt + \sigma(t)N(t)\,dW_t,$$

(6.20)

$$dN(t) = N(t)\left[b_I(t) - \frac{\sigma^2(t)}{2}\right] \,dt + \sigma(t)N(t)\,dW_t$$

(6.21)

$$= b_S N(t)dt + \sigma(t)N(t)\,dW_t.$$  

(6.22)

This can be written in terms of the per capita average growth rate as

$$\frac{1}{N} \frac{dN(t)}{dt} = \left(b_I(t) - \frac{\sigma^2(t)}{2}\right) + \sigma(t)\xi(t)$$

$$= b_S + \sigma(t)\xi(t).$$

This is similar to Equation (2.3) but simply interpreted as a Stratonovich SDE, where \(b(t) \equiv b_I(t)\) is replaced by \(b(t) - \frac{\sigma^2(t)}{2} \equiv b_S\). Whether we interpret equation (2.3) as an Itô or a Stratonovich SDE, the solution is a homogeneous diffusion process with diffusion coefficient (variance rate), \(N^2\sigma^2\) (which is the same in both the Itô and Stratonovich SDEs). The drift coefficient is, however, different; it is respectively for Itô and Stratonovich:

$$N(t) = N_0 \exp\left[\left(R_a - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right],$$

or, alternatively it was shown that \(R_a - \frac{\sigma^2}{2} = R_g\), yielding

$$N(t) = N_0 \exp[R_g t + \sigma W(t)].$$  

(6.18)

Hence, we can conclude that the two interpretations yield exactly the same solutions in terms of a specific average growth rate. Therefore, it does not matter which average we choose as long as it is clearly specified. With regard to the conditions under which extinction occurs, we conclude that both approaches predict population extinction or a stochastic equilibrium according to whether the geometric average growth rate is negative or positive. So, we can use either calculus indifferently as long as we are careful to use \(b\) for the appropriate average for that calculus.

Once \(R_a(x, t)\) or \(R_g(x, t) = R_a(x, t) - \frac{\sigma^2(t)}{2}\) have been estimated, one can choose to estimate \(b_I(t) = R_a(x, t) = R_g(x, t) + \frac{\sigma^2(t)}{2}\) and use Itô calculus or choose to estimate \(b_S(t) = R_g(x, t) = R_a(x, t) - \frac{\sigma^2(t)}{2}\) and use Stratonovich calculus.

It does not matter what choice one makes, the solution one obtains is the same. Therefore, the Itô and Stratonovich SDEs can be written in terms of these estimated quantities, \(R_a(x, t)\) and \(R_g(x, t)\):

\[
\begin{align*}
\text{Itô SDE:} & \quad \frac{1}{N} \frac{dN}{dt} = R_a(x, t) + \sigma(t)\xi(t) = R_g(x, t) + \frac{\sigma^2}{2} + \sigma(t)\xi(t) \\
\text{Stratonovich SDE:} & \quad \frac{1}{N} \frac{dN}{dt} = R_g(x, t) + \sigma(t)\xi(t) = R_a(x, t) - \frac{\sigma^2}{2} + \sigma(t)\xi(t)
\end{align*}
\]

which are equivalent.
Itô: \[ \mu(N) = N(t)b_I(t) \]

Stratonovich: \[ \mu(N) = N(t) \left[ b_I(t) + \frac{\sigma^2(t)}{2} \right] = N(t)b_S(t). \]

To conclude, we have:

**Itô SDE:**  \[ \frac{dN(t)}{N(t)} = b_I(t) + \sigma(t)\xi(t) dt \]

**Stratonovich SDE:**  \[ \frac{dN(t)}{N(t)} = b_S(t) + \sigma(t)\xi(t) dt \]

6.3.1. **Deterministic Model.** We need to clarify what the growth rate and “average” growth rate mean in terms of the observed population dynamics \( N(t) \). Let us consider the deterministic model, where \( \sigma = 0 \),

\[ dN(t) = b_d(t)N(t) dt. \]  \hspace{1cm} (6.23)

We can define the growth rate (i.e. the per capita growth rate) at time \( t \) when the observed population size at time \( t \) is \( x \), i.e. \( N(t) = x \), as

\[ b_d(x) := \frac{1}{x} \frac{dN(t)}{dt} \]

\[ = \frac{1}{x} \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t} = \frac{1}{x} \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - x}{\Delta t}. \]  \hspace{1cm} (6.24)

The limit part represents the total growth rate, \( \frac{dN(t)}{dt} \), at time \( t \), which is then divided by the population size \( x \) to obtain the per capita growth rate. In this density-independent case, \( b_d(x) \) does not depend on the population size \( x \), however, we adopt the notation \( b_d(x) \) since the dependence will be present in the more general density-dependence case.

Alternatively, we could have obtained the solution of (6.23) as \( N(t) = N_0 \exp(b_d t) \), where this gives \( N(t + \Delta t) = N(t) \exp(b_d \Delta t) \), and substituting into equation (6.26) to obtain

\[ \lim_{\Delta t \to 0} x^b_d \Delta t - 1 \Delta t. \]

Furthermore, we know that as \( \Delta t \to 0 \), \( x^b_d \Delta t - 1 \to b_d \), to arrive at the same solution.

However, in the stochastic model (2.3), i.e. \( \sigma \neq 0 \), \( N(t + \Delta t) \) is a random variable and it is then necessary to take some kind of average of \( N(t + \Delta t) \), to obtain a possible estimate of the population size at time \( t + \Delta t \). One possible way is to take the limit and determine the average afterwards, however, this is shown not to work since the limit itself is a generalised stochastic process and does not exist in the ordinary sense. So, instead, we follow the other approach by first taking the average and then computing the limit afterwards. But we have to be precise on what type of average we are using.
6.3.2. Itô Model. Let us see what these two averages turn out to be, under the Itô model. It is sometimes more convenient to work with the log transformed stochastic process \( Y(t) = \ln N(t) \) and with \( y = \ln(x) \). Using Itô’s rule of calculus, one obtains, using the fact that in the limit as \( dt \) tends to 0 \((dt \to 0), (dt)^2 = 0, (dt)(dW_t) = 0 \) and \((dW_t)^2 = dt\):

\[
dN(t) = b_I(t)N(t)dt + \sigma(t)N(t)dW(t). \tag{6.27}
\]

This Itô model has drift coefficient \( b_I(t)N(t) \) and diffusion coefficient \( \sigma^2(t)N^2(t) \), where the growth rate \( b_I(t) \) is given by:

\[
b_I(t) = \frac{1}{x} \lim_{\Delta t \to 0} \frac{E_{t,x}[N(t + \Delta t)] - x}{\Delta t} = R_a(x, t). \tag{6.28}
\]

The arithmetic average growth rate is then given by \( R_a(x, t) \equiv b_I(t) \).

To compute the geometric average growth rate, let us consider the log scale \( Y(t) = \ln N(t) \), then using Itô’s lemma, we obtain the SDE \((6.30)\):

\[
dY = d\ln N(t) = \frac{d\ln N(t)}{dN(t)}dN(t) + \frac{1}{2} \frac{d^2\ln N(t)}{dN(t)^2} (dN(t))^2 = \frac{1}{N(t)}[N(t)b_I(t)dt + \sigma(t)N(t)dW(t)] + \frac{1}{2} \left( \frac{-1}{N(t)^2} \right) \left[ \sigma^2(t)N^2(t)dt \right] \tag{6.29}
\]

\[
= \left[ b_I(t) - \frac{1}{2} \sigma^2(t) \right] dt + \sigma(t)dW(t). \quad \tag{6.30}
\]

In terms of \( Y(t) = \ln N(t) \), \( y = \ln(x) \), the solution of equation \((6.30)\) has drift coefficient \( b_I(t) - \frac{1}{2} \sigma^2(t) \) and diffusion coefficient \( \sigma^2(t) \), where the growth rate \( b_I(t) - \frac{1}{2} \sigma^2(t) \) is given by:

\[
b_I(t) - \frac{1}{2} \sigma^2(t) = \lim_{\Delta t \to 0} \frac{E_{t,x}[Y(t + \Delta t)] - y}{\Delta t} = R_g(x, t). \tag{6.31}
\]

Therefore, the geometric average growth rate is given by

\[
R_g(x, t) = \lim_{\Delta t \to 0} \frac{E_{t,x}[\ln N(t + \Delta t)] - \ln x}{\Delta t} = b_I(t) - \frac{1}{2} \sigma^2(t) = b_S(t). \tag{6.32}
\]

Hence, for the Itô SDE, we have that

\[
R_a(x, t) = b_I(t), \quad R_g(x, t) = b_I(t) - \frac{\sigma^2(t)}{2} = b_S(t). \tag{6.33}
\]

These are respectively the arithmetic average growth rate (the expected value of average \( w.r.t. \) the process \( N(t) \) as per the definition of the average \((6.1)\)) and the geometric average growth rate defined by \((6.3)\) for the solution of the Itô SDE.

By the definition \((6.3)\) of the geometric average, \( R_g(x, t) \) is the average \( w.r.t. \) the process \( \ln N(t) \), so by determining the dynamics of \( \ln N(t) \) we can obtain the drift rate which gives the per capita geometric growth average.

To conclude \( b_I(x, t) \) is the arithmetic average growth rate

\[
R_a(x, t) := \frac{1}{x} \lim_{\Delta t \to 0} \frac{E_{t,x}[N(t + \Delta t)] - x}{\Delta t};
\]

and \( b_I(t) - \frac{\sigma^2(t)}{2} = b_S \) is the geometric average growth rate

\[
R_g(x, t) := \lim_{\Delta t \to 0} \frac{E_{t,x}[\ln N(t + \Delta t)] - \ln x}{\Delta t}.
\]

Therefore, \( R_g(x, t) = R_a(x, t) - \frac{\sigma^2(t)}{2} \).
6.3.3. Stratonovich Model. For the Stratonovich model, we make use of an easier approach which is to convert the Stratonovich SDE to an equivalent Itô SDE using the Itô-Stratonovich conversion formula (5.5):

Stratonovich SDE: \[
\frac{1}{N} \frac{dN(t)}{dt} = b_S(t) + \sigma(t)\xi(t),
\]

\[dN(t) = N(t)b_S(t)dt + \sigma(t)N(t)dW(t)\]

equivalent Itô SDE: \[
\frac{1}{N(t)} \frac{dN(t)}{dt} = b_S(t) + \frac{\sigma^2(t)}{2} + \sigma(t)\xi(t)
\]

\[dN(t) = N(t)b_S(t)dt + N(t)\left(\frac{\sigma^2(t)}{2}\right) dt + \sigma(t)N(t)dW(t),\]

\[= N(t) \left[b_S(t) + \frac{\sigma^2(t)}{2}\right] dt + \sigma(t)N(t)dW(t).\]

The solution of the Stratonovich SDE is a diffusion process with drift coefficient \(b_S(t)\) and diffusion coefficient \(\sigma^2(t)N^2(t)\), which is identical to the diffusion coefficient of the Itô SDE.

If we now consider the transformation \(Y(t) = \ln N(t)\), which is instrumental in these deductions. Using Itô’s Lemma we obtain

\[dY(t) = \frac{d \ln N(t)}{dN(t)} dN(t) + \frac{1}{2} \frac{d^2 \ln N(t)}{dN(t)^2} (dN(t))^2,\]

\[= \frac{1}{N(t)} \left[N(t) \left(\sigma^2(t)\right) dt + \sigma(t)N(t)dW(t)\right] + \frac{1}{2} \left(-\frac{1}{N(t)}\right) \left[\sigma^2(t)N^2(t)dt\right],\]

\[= b_S(t)dt + \sigma(t)dW(t).\]

The solution of equation (6.33) has drift \(b_S(t)\) and diffusion coefficient \(\sigma^2(t)\).

Alternatively, we could compute the geometric average growth rate by starting from the Stratonovich SDE and using the ordinary chain rule of differentiation, we obtain the SDE:

\[d\ln N(t) = \frac{d \ln N(t)}{dN(t)} dN(t)\]

\[= \frac{1}{N(t)} \left[N(t)b_S(t)dt + \sigma(t)N(t)dW(t)\right] = b_S(t)dt + \sigma(t)dW(t).\]

This can indifferently be interpreted as an Itô or a Stratonovich SDE. Since, the stochastic term has a constant coefficient, the correction term in the conversion method is now zero and the two approaches coincide.

Therefore, using a similar reasoning as explained with the Itô SDE, for the Stratonovich SDE, we have

\[R_g(x, t) = b_S(t),\]

\[= b_I(t) - \frac{1}{2}\sigma^2\]

\[R_a(x, t) = b_S(t) + \frac{\sigma^2}{2}\]

\[= b_I(t).\]

These are respectively the geometric average growth rate and the arithmetic average growth rate for the solution of the Stratonovich SDE. This proves that the average used in Stratonovich calculus
is the geometric average and the average used in Itô calculus is the arithmetic average. Hence, from these results, we reach the following final conclusions:

To conclude the arithmetic average growth rate is given by

\[ R_a(x, t) := \frac{1}{x} \lim_{\Delta t \downarrow 0} \frac{E_{t,x}[N(t + \Delta t)] - x}{\Delta t} = b_S(t) + \frac{\sigma^2}{2}, \]

and the geometric average growth rate is given by

\[ R_g(x, t) := \lim_{\Delta t \downarrow 0} \frac{E_{t,x}[\ln N(t + \Delta t)] - \ln x}{\Delta t} = b_S(t). \]

Again, for the Stratonovich SDE, we also have that \( R_g(x, t) = R_a(x, t) - \frac{\sigma^2}{2} \).

7. Conclusion

Under Itô calculus, we interpret the growth rate as the arithmetic average growth rate, \( R_a(x, t) \) defined by equation (6.1).

Under Stratonovich calculus, we interpret the growth rate as the geometric average growth rate, \( R_g(x, t) \) defined by equation (6.3).

Therefore, for Itô calculus, \( b(t) \) really means the arithmetic average growth rate, \( R_a(x, t) \) or equivalently \( R_g(x, t) + \frac{\sigma^2}{2} \) and for Stratonovich calculus, \( b(t) \) really means the geometric average growth rate, \( R_g(x, t) \).

It is shown, and finally concluded, in Braumann (2007) that both calculus lead to the exact same conclusions in terms of the conditions under which population extinction or the existence of a stochastic equilibrium occur. Since if one takes into account the difference \( \frac{\sigma^2}{2} \) between the two averages, the solutions under both approaches coincide.

Hence, after clearing the confusion, Itô and Stratonovich calculus yield the same results. It is now possible to easily tackle the major obstacle to the use of these SDE models, however, care must be taken in using the appropriate type of average for each calculus. If that care is indeed taken, Itô and Stratonovich will give the same results and draw the same conclusions.

References

[1] Braumann C. A., (2007). *Harvesting in a random environment: Itô or Stratonovich calculus?* J. Theoret. Biol. 244, no. 3, 424–432.

[2] Fernholz, R., and B. Shay, (1982). *Stochastic portfolio theory and stock market equilibrium* Journal of Finance 37, 615–624.