Topologies induced by group actions

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Abstract

We introduce some canonical topologies induced by actions of topological groups on groups and rings. For $H$ being a group [or a ring] and $G$ a topological group acting on $H$ as automorphisms, we describe the finest group [ring] topology on $H$ under which the action of $G$ on $H$ is continuous. We also study the introduced topologies in the context of Polish structures. In particular, we prove that there may be no Hausdorff topology on a group $H$ under which a given action of a Polish group on $H$ is continuous.

0 Introduction

The main motivation for this paper is the following general problem. Suppose $G$ is a topological group acting on $X$, where $X$ is a set, possibly equipped with some algebraic structure preserved by the action of $G$. When does there exist a “nice” topology on $X$, such that the action of $G$ on $X$ is continuous, and the topology is compatible with the structure on $X$? Clearly, if there is such a topology which is at least $T_1$, then, for every element $x \in X$, its stabilizer $G_x$ is closed in $G$. On the other hand, if the latter is satisfied, then, by Remark 0.1 below, we can equip $X$ with a topology under which the action is continuous and which inherits many properties of the given topology on $G$.

Now, suppose that $X$ is equipped with a group structure (preserved by the action of $G$). Then, the topology $\tau$ defined below usually fails to be a group topology. In Theorem 1.2 we give a description of the finest group topology on $X$, under which the action of $G$ on $X$ is continuous. Using this description, we give an example of an action of the polish group $Homeo([0,1])$ on a certain group $H$, such that there is no Hausdorff group topology on $H$ under which the action is continuous.

Also, we give in Theorem 1.9 a description of the finest compatible topology in the case of $X$ being a ring.

By a topological group we will mean a group equipped with a topology, such that the multiplication and the inversion are continuous functions (we do not assume that the topology is Hausdorff).

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Remark 0.1 Suppose $G$ is a topological group acting on a set $X$. Define $U := \{U \cdot x : U \subseteq G, U \text{ is open, } x \in X\}$. Then, we have:

1. The family $U$ is a basis of a topology on $X$. Denote this topology by $\tau$.
2. The action of $G$ on $(X, \tau)$ is continuous.
3. All $G$-orbits in $X$ are clopen in $\tau$; moreover, for every $x \in X$, $G/G_x \approx G \cdot x$
   (where the orbit $G \cdot x$ is equipped with the topology $\tau$).
4. $(\forall x \in X)(G_x \text{ is closed in } G) \iff \tau \text{ is } T_1 \iff \tau \text{ is Hausdorff.}$

Proof.
(1) Take any $x_1, x_2 \in X$, open sets $U_1, U_2 \subseteq G$ and a point $y \in U_1 x_1 \cap U_2 x_2$. Then,
   $y = u_1 x_1 = u_2 x_2$ for some $u_1 \in U_1$ and $u_2 \in U_2$. For $i = 1, 2$, let $W_i$ be
   an open neighbourhood of $e$ in $G$ such that $W_i u_i \subseteq U_i$. Put $W = W_1 \cap W_2$. Then,
   $Wy = W u_i x_i \subseteq U_i x_i$ for $i = 1, 2$, so we are done.
(2) Take any $g \in G$, $x, y \in X$ and an open set $U \subseteq G$ such that $gx \in U y$. Choose
   $u \in U$ so that $gx = uy$. Let $U_1$ be an open neighbourhood of $e$ in $G$ such that
   $U_1 u \subseteq U$, and let $V_1, V_2 \subseteq G$ be open neighbourhoods of $e$ such that
   $V_1 g V_2 \subseteq U_1 g$. Then, $(V_1 g)(V_2 x) \subseteq U_1 g x = U_1 u y \subseteq U y$, which
   yields the continuity of the action.
(3) By the definition of $\tau$, every $G$-orbit is open, so also clopen. It is straightforward
   to check that $f_x : G/G_x \to Gx$ given by $aG_x \mapsto ax$ is a homeomorphism.
(4) First condition implies third by (3) and the fact that the quotient of any topological
   group (not necessarily Hausdorff) by a closed subgroup is Hausdorff. Third
   condition implies second trivially, and second implies first by (2). \qed

We will denote the topology $\tau$ from the above remark by $\tau(X, G)$.

1 Group actions

In this section, we will describe the finest group [ring] topology on $H$ (where $H$ is a
   group or a ring, respectively) under which a given action of a topological group on
   $H$ (as automorphisms) is continuous.

   First, let us recall a recent result of Bergman, which we use in our construction. Let $G$
   be a group. We will denote the neutral element of $G$ by $e$, and for $S \subseteq G$
   we will write $S^* = S \cup \{e\} \cup S^{-1}$. Following the notation from [I], given any $Q$-tuple
   $(S_q)_{q \in Q}$, put
   $$U((S_q)_{q \in Q}) = \bigcup_{n<\omega} \bigcup_{q_1 < \cdots < q_n \in Q} S_{q_1}^* \cdots S_{q_n}^*.$$ 

Below, we will omit the symbol $\bigcup_{n<\omega}$ in similar expressions. For a family $F$
   of subsets of $G$, $F^g$ will denote the collection of all subsets of $G$ of the form
   $\bigcup_{g \in F} g S g^{-1}$; for $G$-tuples $(S_q)_{q \in G}$ of members of $F$. We say that a filter on $G$
   converges to $e$ in a given topology, if every neighbourhood of $e$ contains a member of the filter. By Lemma 14
   and Proposition 15 from [I], we have:

Fact 1.1 Let $F$ be a downward directed family of nonempty subsets of $G$. Then the
   sets $U((S_q)_{q \in Q})$, where $(S_q)_{q \in Q}$ ranges over all $Q$-tuples of members of $F^G$, form a
basis of open neighbourhoods of $e$ in a group topology $\mathcal{T}_F$, which is the finest group topology on $G$ under which $F$ converges to $e$.

When $\rho$ is a topology on $G$, we will denote by $\rho^*$ the topology $\mathcal{T}_F$, where $F$ consists of $\rho$-open neighbourhoods of $e$. In particular, if $\rho$ is a group topology, then $\rho^* = \rho$.

Let $G$ be a topological group equipped with a topology $\sigma$.

**Theorem 1.2** Suppose $G$ acts on a group $H$ as automorphisms. We identify $G$ and $H$ with $\{e\} \times G < H \times G$ and $H \times \{e\} < H \times G$, respectively. Put $T = (D \times \sigma)^*$, where $D$ is the discrete topology on $H$. Denote by $T_H$ and $T_G$ the topologies induced by $T$ on the subgroups identified with $H$ and $G$, respectively. Then:

1. $T_G = \sigma$.
2. $T_H$ is a group topology on $H$ under which the action of $G$ on $H$ is continuous.
3. If $\rho$ is another group topology on $H$ under which the action of $G$ on $H$ is continuous, then $T_H$ is finer than $\rho$.

We will denote the topology $T_H$ by $T(H, G)$.

**Proof.**

(1) We will show that $T_G = \sigma^*$ (using the description of $T = (D \times \sigma)^*$ and $\sigma^*$ given by Fact 1.1), which suffices since $\sigma = \sigma^*$. It is easy to see that $T_G$ is a group topology on $G$. Hence, it is enough to show that the neighbourhoods of $e$ in $T_G$ are the same as in $\sigma^*$.

Take a $T_G$-open neighbourhood of $e$ of the form

$$V = G \cap \bigcup_{q_1 < \cdots < q_n \in \mathbb{Q}} \left( \bigcup_{(h, g) \in H \times G} \left(\{e\} \times U_{q_i}^{q_i(h, g)}\right) \right) \cdots \left( \bigcup_{(h, g) \in H \times G} \left(\{e\} \times U_{q_n}^{q_n(h, g)}\right) \right) =$$

$$= G \cap \bigcup_{q_1 < \cdots < q_n \in \mathbb{Q}} \left( \bigcup_{(h, g) \in H \times G} \left\{(h(u^g h^{-1}), u^g) : u \in U_{q_i}^{q_i(h, g)}\right\} \right) \cdots$$

$$\cdots \left( \bigcup_{(h, g) \in H \times G} \left\{(h(u^g h^{-1}), u^g) : u \in U_{q_n}^{q_n(h, g)}\right\} \right),$$

where each $U_{q_i}^{q_i(h, g)}$ is a symmetric $\sigma$-open neighbourhood of $e$ in $G$. Then,

$$\bigcup_{q_1 < \cdots < q_n} \left( \bigcup_{g \in G} g U_{(e, g) \cdot g^{-1}}^{q_1} \right) \cdots \left( \bigcup_{g \in G} g U_{(e, g) \cdot g^{-1}}^{q_n} \right)$$

is a $\sigma^*$-open neighbourhood of $e$ contained in $V$.

Conversely, take a $\sigma^*$-open neighbourhood of $e$ of the form

$$W = \bigcup_{q_1 < \cdots < q_n} \left( \bigcup_{g \in G} g U_{q_i}^{q_i g^{-1}} \right) \cdots \left( \bigcup_{g \in G} g U_{q_n}^{q_n g^{-1}} \right),$$

3
where each $U^q_e$ is a symmetric neighbourhood of $e$ in $G$. For any $(h, g) \in G$ and $q \in \mathbb{Q}$, find a $\sigma$-open, symmetric neighbourhood $U^q_{(h, g)}$ of $e$, whose conjugate by $g$ is contained in $U_g$ (it can be chosen independently from $h$). Then,

$$G \cap \bigcup_{q_1<\cdots<q_n\in\mathbb{Q}} \left( \bigcup_{(h, g)\in H\times G} \left( \{e\} \times U^q_{(h, g)} \right)^{(h, g)} \right) \cdots \left( \bigcup_{(h, g)\in H\times G} \left( \{e\} \times U^q_{(h, g)} \right)^{(h, g)} \right)$$

is a $T_G$-open neighbourhood of $e$ contained in $W$.

(2) $T_H$ is a group topology since $H$ is a subgroup of $H \times G$, and $T$ is a group topology on $H \times G$ by Fact 1.1. For the continuity of the action, take any $g \in G$, $h \in H$ and a $T$-open set $W$, such that $gh \in U \cap H$. This means that, in $H \times G$, $(e, g)(h, e)(e, g)^{-1} \in U$, so we can choose $T$-open sets $U_1$ and $U_2$, such that $(e, g) \in U_1$, $(h, e) \in U_2$, and $U_1U_2U_1^{-1} \subseteq U$. Then, for any $g_1 \in U_1 \cap G$ and $h_1 \in U_2 \cap H$, we have that $(e, g_1)(h_1, e)(e, g_1)^{-1} = (g_1h_1, e)$ belongs to $U$, so $g_1h_1 \in U \cap H$. This proves the continuity of the action.

(3) Suppose $\rho$ is a group topology on $H$ under which the action of $G$ on $H$ is continuous. Then, the product topology $\rho \times \sigma$ is a group topology on $H \times G$, which is coarser than $D \times \sigma$, so, by the choice of $T$, we have that $T$ is finer than $\rho \times \sigma$. In particular, $T_H$ is finer than $\rho$. \hfill \Box

Using the above theorem we obtain an explicit formula describing the topology $T(H, G)$:

**Corollary 1.3** With the notation from the above theorem, $T(H, G)$ has a basis of open neighbourhoods of $e$ consisting of the sets:

$$\bigcup_{q_1<\cdots<q_n\in\mathbb{Q}} \{h_1(u_1h_1^{-1})u_1(h_2(u_2h_2^{-1}))u_1u_2(h_3(u_3h_3^{-1}))\cdots u_1u_2\cdots u_{n-1}(h_n(u_nh_n^{-1})) : h_i \in H, u_i \in U^{q_i}_{h_i}, u_1\cdots u_n = e\},$$

where $(U^q_H)_{h\in H, q\in \mathbb{Q}}$ range over all $H \times \mathbb{Q}$-tuples of $\sigma$-open symmetric neighbourhoods of $e$ in $G$.

**Proof.** By the description of the topology $T_H$ given in Fact 1.1 we get that it has a basis of open neighbourhoods of $e$ consisting of the sets:

$$\bigcup_{q_1<\cdots<q_n\in\mathbb{Q}} \{h_1(v_1^{q_1}h_1^{-1})v_1^{q_1}(h_2(v_2^{q_2}h_2^{-1}))\cdots v_1^{q_1}\cdots v_{n-1}^{q_{n-1}}(h_n(v_n^{q_n}h_n^{-1})) : h_i \in H, g_i \in G, v_i \in U^{q_i}_{(h_i, g_i)}, v_1^{q_1}\cdots v_n^{q_n} = e\} =$$

$$= \bigcup_{q_1<\cdots<q_n\in\mathbb{Q}} \{h_1(u_1h_1^{-1})u_1(h_2(u_2h_2^{-1}))\cdots (u_1\cdots u_{n-1})(h_n(u_nh_n^{-1})) : h_i \in H, \}.$$
\[ g_i \in G, u_i \in (U_{(h_i, g_i)}^q)^{g_i}, u_1 \ldots u_n = e \],

where \((U_{(h,g)}^q)_{(h,g)} \in H \times G, q \in \mathbb{Q}\) range over all \((H \times G) \times \mathbb{Q}\)-tuples of \(\sigma\)-open symmetric neighbourhoods of \(e \in G\). Since the tuples \((U_{(h,g)}^q)_{(h,g)} \in H \times G, q \in \mathbb{Q}\) range over the same set, we can omit the conjugations in the formula. Now, if we replace each \(U_{(h,g)}^q\) by \(U_{(h,e)}^q\) in a tuple \((U_{(h,g)}^q)_{(h,g)} \in H \times G, q \in \mathbb{Q}\), then the corresponding neighbourhood of \(e \in H\) will be contained in the original one. Thus, we obtain the same topology when we restrict ourselves to tuples in which \(U_{(h,g)}^q = U_h^q\) does not depend on \(g\). This gives the conclusion.

\[ \Box \]

In Section 2 we will use the description of \(T(H, G)\) that we have obtained to prove the absence of a compatible Hausdorff topology for some classes of Polish group structures (see Proposition 2.8).

Let us keep the notation from above and define \(\lambda(H, G)\) to be the topology on \(H\) in which a set \(U\) is open if for each \(h_1, h_2 \in H\), the sets \(h_1Uh_2, h_1U^{-1}h_2\) are open in the topology \(\tau(H, G)\) (defined after Remark 1.1). It is easy to see that if we equip \(H\) with \(\lambda(H, G)\), then the action of \(G\) on \(H\) is separately continuous, the inversion on \(H\) is continuous and the multiplication on \(H\) is separately continuous. Moreover, \(\lambda(H, G)\) is the finest topology on \(H\) with these properties. Indeed, let \(\xi\) be any other such topology. Take any \(\xi\)-open set \(V\). Then, for any \(h_1, h_2 \in H\), \(h_1Uh_2, h_1U^{-1}h_2\) are \(\xi\)-open, so also \(\tau\)-open. Hence, \(V\) is \(\lambda(H, G)\)-open.

**Remark 1.4** In Theorem 1.2, we can replace the discrete topology \(D\) by any topology on \(H\) which is finer than all group topologies under which the action of \(G\) on \(H\) is continuous. Examples of such topologies are \(\tau(H, G)\) and \(\lambda(H, G)\). However, the simplest description of \(T(H, G)\) we obtain starting from the discrete topology on \(H\).

Let us formulate a remark about the topology \(\lambda(H, G)\) defined above.

**Remark 1.5** If the topology of \(G\) and \(\lambda(H, G)\) are metrizable and Baire, then \(\lambda(H, G) = T(H, G)\).

**Proof.** Let us equip \(H\) with the topology \(\lambda(H, G)\). Since the multiplication on \(H\) is separately continuous, the inversion on \(H\) is continuous and the action of \(G\) on \(H\) is separately continuous, we get by Theorem 9.14 from [5] that \(H\) is a topological group with the topology \(\lambda(H, G)\), and the action of \(G\) on \(H\) is continuous. So, \(\lambda(H, G)\) is coarser than \(T(H, G)\). But \(\lambda(H, G)\) is always finer than \(T(H, G)\) (see the discussion preceding Remark 1.3), so these topologies are equal. \(\Box\)

The above remark can by illustrated by the following example.

**Example 1.6** Let \(G = S_\omega\) be the group of all permutations of \(\omega\), considered with the product topology (which is Polish, so, in particular, metrizable and Baire). Consider the action of \(G\) on \(H := 2^\omega\), given by \(g \cdot h = h \circ g^{-1}\). Then, \(\lambda(H, G)\) is the product topology on \(2^\omega\), so it coincides with \(T(H, G)\).
Proof. Clearly \( \lambda(H,G) \) is finer than the product topology. For the converse, let \( U \) be any \( \lambda(H,G) \)-open neighbourhood of \( 0 \in H \). We will show that it contains an open neighbourhood of \( 0 \in H \) in the sense of the product topology. Let \( \omega = A \cup B \cup C \) be a partition of \( \omega \) into three infinite sets. For \( i, j, k \in \{0, 1\} \) define \( h_{i,j,k} \in H \) to be equal to \( i \) on \( A \), equal to \( j \) on \( B \), and equal to \( k \) on \( C \). For \( i, j, k \in \{0, 1\} \), \( h_{i,j,k} + U \) contains a \( \tau(H,G) \)-open neighbourhood of \( h_{i,j,k} \) of the form \([\alpha_{i,j,k}] \cdot h_{i,j,k} \), where \( \alpha_{i,j,k} : \omega \to \omega \) is a partial function with a finite domain, and \( [\alpha] = \{ \eta \in S_\omega : \alpha \subseteq \eta \} \). We finish by the following claim:

**Claim 1** Put \( I = \bigcup_{i,j,k \in \{0,1\}} (\text{dom}(\alpha_{i,j,k}) \cup \text{rng}(\alpha_{i,j,k})) \). Then, \( \{ x \in H : x|_I = 0 \} \subseteq U \).

**Proof of Claim** Take any \( x \in H \) such that \( x|_I = 0 \). Notice that we can choose \( i, j, k \in \{0, 1\} \) so that \((h_{i,j,k} + x)^{-1}[\{0\}]\) and \((h_{i,j,k} + x)^{-1}[\{1\}]\) are both infinite, and \( i, j, k \) are not all equal. Then, since \( h_{i,j,k}^{-1}[\{0\}] \) and \( h_{i,j,k}^{-1}[\{1\}] \) are also both infinite, and \( h_{i,j,k} + x \) agrees with \( h_{i,j,k} \) on \( I \), we can find a permutation \( \eta \in [\alpha_{i,j,k}] \), such that 

\[
\eta \cdot h_{i,j,k} = h_{i,j,k} + x.
\]

Thus, \( h_{i,j,k} + x \in h_{i,j,k} + U \), so \( x \in U \). \( \square \)

Now, we aim towards a description of the finest ring topology on \( R \) under which a given action of a topological group on \( R \) is continuous. First, we give a variant of Fact 1.1 in which we are interested in semigroup topologies (i.e. topologies under which the multiplication is continuous) rather than group topologies on \( G \), but still we assume that \( G \) is a group. For a subset \( S \) of \( G \), we will write \( S^* = S \cup \{ e \} \). We define

\[
U'(\{(S_q)_{q \in \mathbb{Q}}\}) = \bigcup_{q_1 < \cdots < q_n \in \mathbb{Q}} S_{q_1} \cdots S_{q_n}.
\]

Then, by a straightforward modification (which is just replacing expressions of the form \( S^* \) by \( S^* \)) of the proof of Lemma 14 and Proposition 15 from [1], we obtain:

**Fact 1.7** Let \( F \) be a downward directed family of nonempty subsets of \( G \). Then, the sets \( U'(\{(S_q)_{q \in \mathbb{Q}}\}) \), where \( (S_q)_{q \in \mathbb{Q}} \) ranges over all \( \mathbb{Q} \)-tuples of members of \( F^G \), form a basis of open neighbourhoods of \( e \) in a semigroup topology \( T'_F \), which is the finest semigroup topology on \( G \) under which \( F \) converges to \( e \).

When \( \rho \) is a topology on \( H \), we will denote by \( \rho^* \) the topology \( T'_F \), where \( F \) consist of \( \rho \)-open neighbourhoods of \( e \).

Let \( \sigma \) be a fixed group topology on a group \( G \). Repeating the proof of Theorem 1.2, we obtain:

**Proposition 1.8** Suppose \( G \) acts on a group \( H \) as automorphisms. We identify \( G \) and \( H \) with \( \{e\} \times G < H \times G \) and \( H \times \{e\} < H \times G \), respectively. Put \( T' = (D \times \sigma)^* \), where \( D \) is the discrete topology on \( H \). Denote by \( T'_H \) and \( T'_G \) the topologies induced by \( T' \) on the subgroups identified with \( H \) and \( G \), respectively. Then, \( T'_G = \sigma \) and \( T'_H \) is the finest semigroup topology on \( H \) under which the action of \( G \) on \( H \) is continuous. We will denote it by \( T'(H,G) \).

Now, we are in a position to give a description of the finest topology in the ring case.
Theorem 1.9 Suppose $G$ is a group equipped with a group topology $\sigma$, acting as automorphisms on a ring $R$. Put $R_1 = R \times \mathbb{Z}$, and define $+\text{ and } \cdot$ on $R_1$ by $(a, k) + (b, l) = (a + b, k + l)$ and $(a, k) \cdot (b, l) = (ab + l \times a + k \times b, k \cdot l)$. Clearly, $G$ acts on $R_1$ as automorphisms by $g(a, k) := (g(a), k)$. Consider the induced action of $G$ on $GL_3(R_1)$. We identify $R$ with a subset $\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in R \}$ of $GL_3(R_1)$. Denote by $T^\sigma(R, G)$ the topology induced on $R$ by $T'(GL_3(R_1), G)$. Then, $T^\sigma(R, G)$ is the finest ring topology on $R$ such that the action of $G$ on $R$ is continuous.

Proof. Let $T_1$ be the topology induced on $R_1$ by $T'(GL_3(R_1), G)$ (we identify $R_1$ with a subset of $GL_3(R_1)$ in the same manner as we do with $R$).

Claim 1 $T_1$ is the finest ring topology on $R_1$ under which the action of $G$ on $R_1$ is continuous.

First, suppose the claim is proved and let us see that the conclusion of the theorem follows.

By the claim, $T^\sigma(R, G)$ is a ring topology on $R$, and the action of $G$ on $R$ equipped with $T^\sigma(R, G)$ is continuous (as the restriction of the action on $R_1$).

Suppose $\rho$ is another topology on $R$ such that the action of $G$ on $R$ is continuous. Consider $R_1$ equipped with the product of $\rho$ and the discrete topology $E$ on $\mathbb{Z}$. Then, the action $G$ on $R_1$ is also continuous and $R_1$ is a topological ring, so, by the claim, $T_1$ is finer than $\rho \times E$. Hence, $T^\sigma(R, G)$ (which is equal to the topology induced on $R$ by $T_1$) is finer than $\rho$, so we are done.

Proof of Claim 1. First, we will check that $T_1$ is finer than every ring topology on $R_1$ under which the action of $G$ on $R_1$ is continuous. Let $\chi$ be any such topology. Let us equip $GL_3(R_1)$ with the topology $Z$ induced from the product topology $\chi^9$ on $R_1^9$. Then, $GL_3(R_1)$ becomes a topological semigroup, and the action of $G$ on it is continuous. Thus, $T'(GL_3(R_1), G)$ is finer that $Z$, so $T_1$ is finer than the topology induced by $Z$ on $R_1$, i.e. $T_1$ is finer than $\chi$.

Now, consider $R_1$ equipped with the topology $T_1$. The action of $G$ on $R_1$ is continuous (as a restriction of the action on $GL_3(R_1)$) and the addition in $R$ is continuous (as a restriction of the multiplication in $GL_3(R_1)$). Moreover, the additive inversion in $R$ is continuous, as it is given by the map

$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

which is continuous with respect to $T'(GL_3(R_1), G)$.

It remains to show that the multiplication on $R_1$ is continuous. So, we will be done if we show that the map

$\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
is continuous with respect to $T'(GL_3(R_1), G)$. The latter follows, since

$$
\begin{pmatrix}
1 & 0 & xy \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & y \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -y \\
0 & 0 & 1 \\
\end{pmatrix},
$$

and maps

$$
\begin{pmatrix}
1 & 0 & x \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \mapsto \begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & x \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & x \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$

$$
\begin{pmatrix}
1 & 0 & y \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & y \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$

are continuous.

The proof of the theorem has been completed.

**Remark 1.10** In the context of Theorem 1.9, we obtain the same topology on $R$ if we identify $R$ with

$$
\{ \begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} : x \in R \}.
$$

**Proof.** This follows from the fact that

$$
\begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & x \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & -x & x \\
0 & -1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix} = \left( \begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix} \right)^2
$$

and

$$
\begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & -z \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = \left( \begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix} \right)^2
$$

8
is a continuous function (the continuity of the inverse to that map follows from the calculations made at the end of the proof of Theorem 1.9).

2 Topologies on Polish structures

In this section, we will study the topologies introduced above in the context of Polish structures, which were introduced in [6], and studied also in [2, 3, 4, 7].

Definition 2.1 A Polish structure is a pair \((X, G)\), where \(G\) is a Polish group acting faithfully on a set \(X\) so that the stabilizers of all singletons are closed subgroups of \(G\). If \(X\) is equipped with a structure of a group, which is preserved under the action of \(G\), then we call \((X, G)\) a Polish group structure. We say that \((X, G)\) is small if for every \(n < \omega\), there are only countably many orbits on \(X^n\) under the action of \(G\).

The class of small Polish structures contains examples of the form \((X, \text{Homeo}(X))\) (where \(\text{Homeo}(X)\) is considered with the compact-open topology) for \(X\) being one of the spaces \([0,1]^n, S^n\) (\(n\)-dimensional sphere), \((S^1)^n\) for \(n \in \omega \cup \{\omega\}\), as well as various other examples, see [6, Chapter 4].

By Remark 0.1, we have:

Corollary 2.2 (1) If \(G\) is a Polish group acting on a set \(X\), then \((X, G)\) is a Polish structure iff \(\tau(X, G)\) is \(T_1\) iff \(\tau(X, G)\) is completely metrizable.

(2) If \((X, G)\) is a small Polish structure, then \(\tau(X, G)\) is a Polish topology. In particular, \((X, G)\) is a Polish \(G\)-space if we equip \(X\) with the topology \(\tau(X, G)\).

Let \((X, G)\) be a Polish structure. For any finite \(C \subseteq X\), by \(G_C\) we denote the pointwise stabilizer of \(C\) in \(G\), and for a finite tuple \(a\) of elements of \(X\), by \(o(a/C)\) we denote the orbit of \(a\) under the action of \(G_C\) (and we call it the orbit of \(a\) over \(C\)).

A fundamental concept for [6] is the relation of \(nm\)-independence in an arbitrary Polish structure.

Definition 2.3 Let \(a\) be a finite tuple and \(A, B\) finite subsets of \(X\). Let \(\pi_A : G_A \rightarrow o(a/A)\) be defined by \(\pi_A(g) = ga\). We say that \(a\) is \(nm\)-independent from \(B\) over \(A\) (written \(a \downarrow_{nm} B\)) if \(\pi_A^{-1}[o(a/AB)]\) is non-meager in \(\pi_A^{-1}[o(a/A)]\). Otherwise, we say that \(a\) is \(nm\)-dependent on \(B\) over \(A\) (written \(a \downarrow_{nm} B\)).

By [6, Theorem 2.14], under some assumptions, \(nm\)-dependence in a \(G\)-group \((H, G)\) can be expressed in terms of the topology on \(H\):

Fact 2.4 Let \((X, G)\) be a Polish structure such that \(G\) acts continuously on a Hausdorff space \(X\). Let \(a, A, B \subseteq X\) be finite. Assume that \(o(a/A)\) is non-meager in its relative topology. Then, \(a \downarrow_{nm} B \iff o(a/AB) \subseteq_{nm} o(a/A)\).
Using the above fact, we now express the relation of \(nm\)-independence in terms of a family of topologies on \(X\), without assuming anything about the Polish structure \((X, G)\).

**Remark 2.5** Let \((X, G)\) be a Polish structure and let \(a, A, B \subseteq X\) be finite. Then \(a \downarrow_{A}^{nm} B \iff o(a/B) \subseteq_{nm} o(a/A)\), where \(X\) is equipped with the topology \(\tau(G_A, X)\) (and the action of \(G_A\) on \(X\) is the restriction of the action of \(G\) on \(X\)).

**Proof.** The conclusion follows from Fact 2.4 and Corollary 2.2(2). \(\square\)

If \(A\) is a finite subset of \(X\) (where \((X, G)\) is a Polish structure), we define the algebraic closure of \(A\) (written \(Acl(A)\)) as the set of all elements of \(X\) with countable orbits over \(A\). If \(A\) is infinite, we define \(Acl(A) = \bigcup \{ Acl(A_0) : A_0 \subseteq A\text{ is finite}\}\). By Theorems 2.5 and 2.10 from \([6]\), \(nm\)-independence has some nice properties corresponding to those of forking independence in stable first-order theories:

**Fact 2.6** In any Polish structure \((X, G)\), \(nm\)-independence has the following properties:

1. (Invariance) \(a \downarrow_{A}^{nm} B \iff g(a) \downarrow_{g[A]}^{nm} g[B]\) whenever \(g \in G\) and \(a, A, B \subseteq X\) are finite.
2. (Symmetry) \(a \downarrow_{C}^{nm} b \iff b \downarrow_{C}^{nm} a\) for every finite \(a, b, C \subseteq X\).
3. (Transitivity) \(a \downarrow_{B}^{nm} C\) and \(a \downarrow_{A}^{nm} B\) iff \(a \downarrow_{A}^{nm} C\) for every finite \(A \subseteq B \subseteq C \subseteq X\) and \(a \subseteq X\).
4. For every finite \(A \subseteq X\), \(a \in Acl(A)\) iff for all finite \(B \subseteq X\) we have \(a \downarrow_{A}^{nm} B\). If additionally \((X, G)\) is small, then we also have:
   5. (Existence of \(nm\)-independent extensions) For all finite \(a \subseteq X\) and \(A \subseteq B \subseteq X\) there is \(b \in o(a/A)\) such that \(b \downarrow_{A}^{nm} B\).

Using Remark 2.5 we can slightly simplify some of the arguments from \([6]\). For example, we reprove the existence of non-forking extensions in small Polish structures (point 4 of Fact 2.6):

Let \(a \subseteq X\) and \(A \subseteq B \subseteq X\) be all finite. Since \(\tau(X, G_A)\) is Polish, and there are countably many orbits over \(B\), we can find, by the Baire category theorem, an element \(b \in o(a/A)\), such that \(o(b/B)\) is non-meager in \(\tau(X, G_A)\). Then, by Remark 2.5 we get that \(b \downarrow_{A}^{nm} B\).

We will now apply Corollary 1.3 to some of the structures constructed in \([3]\) Chapter 2. First, we outline the construction of those structures.

Suppose \((X, G)\) is a Polish structure. Let \(H\) be an arbitrary group. For any \(x \in X\) we consider an isomorphic copy \(H_x = \{ h_x : h \in H \}\) of \(H\). By \(H(X)\) we will denote the group \(\bigoplus_{x \in X} H_x\). Although \(H(X)\) is not necessarily commutative, we will denote its group action by \(+\). For any \(y \in H(X)\) there are \(h_1, \ldots, h_n \in H \setminus \{e\}\) and pairwise distinct \(x_1, \ldots, x_n \in X\) such that \(y = (h_1)_{x_1} + \cdots + (h_n)_{x_n}\). We will then write \(h(y) = \{ x_i : h_i = h \}\).

The group \(G\) acts as automorphisms on \(H(X)\) by

\[
g((h_1)_{x_1} + \cdots + (h_n)_{x_n}) = (h_1)_{gx_1} + \cdots + (h_n)_{gx_n}.
\]
It was proved in [3] that with this action $(H(X), G)$ is a Polish structure, and that if $H$ is countable, and $(X, G)$ is small, then also $(H(X), G))$ is small. Moreover, it was proved there that if, additionally, $X$ is uncountable, then these structures do not possess any $nm$-generic orbits (the notion of an $nm$-generic orbit was introduced in [6, Definition 5.3]). On the other hand, [6, Theorem 5.5] states:

**Fact 2.7** Suppose $(H, G)$ is a small Polish group structure, where $H$ is equipped with a topology in which $H$ is not meager in itself (and the action of $G$ on $H$ is continuous). Then, at least one $nm$-generic orbit in $H$ exists, and an orbit is $nm$-generic in $H$ iff it is non-meager.

From the above theorem and the absence of generics it was concluded that for any non-trivial countable group $H$ and any small Polish structure $(X, G)$, if $X$ is uncountable then there is no non-meager in itself Hausdorff group topology on $H(X)$, such that the action of $G$ on $H(X)$ is continuous (in particular, there is no such Polish topology). We strengthen this observation in some cases:

**Proposition 2.8** Let $H$ be any non-trivial group and let $X$ be a compact Hausdorff space containing an open subset homeomorphic to $(0, 1)^n$ for some non-zero $n \in \omega \cup \{\omega\}$ (notice that the examples listed after Definition 2.1 satisfy this assumption). Then, there is no Hausdorff group topology on $H(X)$ under which the action of Homeo($X$) on $H(X)$ is continuous (where Homeo($X$) is considered with the compact-open topology).

**Proof.** Suppose first that $X = [0, 1]$. It is enough to show that the topology $\rho := T(H([0, 1]), \text{Homeo}([0, 1]))$ is not Hausdorff. Take any $a \in H \setminus \{e\}$. We will show that any $\rho$-open neighbourhood of $e \in H([0, 1])$ contains the element $a_{1/3} - a_{2/3}$. Let $W$ be any such neighbourhood and choose (by Corollary 13) a $\rho$-open set

$$V = \bigcup_{q_1, \ldots, q_n \in \mathbb{Q}} \{h_1(u_1 h_1^{-1}) u_1 (h_2 (u_2 h_2^{-1})) u_1 u_2 (h_3 (u_3 h_3^{-1})) \ldots u_1 u_2 \ldots u_{n-1} (h_n (u_n h_n^{-1})) : h_i \in H, u_i \in U_{h_i}^\rho, u_1 \ldots u_n = e\},$$

such that $V + V \subseteq W$. Let $B_e(id) \subseteq \text{Homeo}([0, 1])$ be a ball (in the supremum metric) contained in $U_0^\rho \cap U_0^3$, and choose $n < \omega$ such that $1/3n < \epsilon$. Put

$$h = a_{n/3n} + a_{(n+1)/3n} + \cdots + a_{(2n-1)/3n}, h' = -a_{(n+1)/3n} - a_{(n+2)/3n} - \cdots - a_{2n/3n}$$

and $U = U_0^1 \cap U_{h'}^1 \cap U_0^3$. Notice that $\{u_0(h - u_1 h), u_0(h' - u_1 h') : u_0 \in B_e(id), u_1 \in U\} \subseteq V$ (to see this, choose $q_j = j$ for $j = 0, 1, 2, 3$, $u_2 = u_1^{-1}, u_3 = u_0^{-1}, h_0 = h_2 = h_3 = 0$ and $h_1$ equal to either $h$ or $h'$). Since $U$ is open, we can find $u_1 \in U$ such that $u_1(k/3n) \in (k/3n, (k + 1)/3n)$ for $k = n, n + 1, \ldots, 2n - 1$. Then, there is some $u_0 \in B_e(id)$ such that $u_0(k/3n) = k/3n$ and $u_0 u_1(k/3n) = (2k + 1)/6n$ for
k = n, n + 1, . . . , 2n − 1. So, we get that \( \sum_{k=n}^{2n-1}(a_{k/3n} - a_{(2k+1)/6n}) \in V \). Similarly, we obtain using \( h' \) that \( \sum_{k=n+1}^{2n}(-a_{k/3n} + a_{(2k-1)/6n}) \in V \). Thus, \( a_{1/3} - a_{2/3} \in V + V \subseteq W \).

Now, suppose \( X \) is any space as in the statement. Then, we can find a copy \( F \) of \( [0, 1]^n \) contained in \( (0, 1]^n \subseteq X \), and an isometric (with respect to a fixed metric on \( F \)) copy \( I \) of \( [0, 1] \) contained in \( F \), such that every homeomorphism of \( I \) preserving its endpoints can be extended to a homeomorphism of \( F \) having the same distance from the identity (with respect to the supremum metric) and equal to the identity on the border of \( F \) in \( (0, 1]^n \). Furthermore, since \( (0, 1]^n \) is open in \( X \), we can extend such a homeomorphism of \( F \) to a homeomorphism of \( X \) equal to the identity on \( X \setminus F \).

Notice that any open neighbourhood of \( \text{id} \in \text{Homeo}(X) \) contains \( \{ f \in \text{Homeo}(X) : f|_{X \setminus \text{int}(F)} = \text{id}, d(\text{id}_F, f|_F) < \epsilon \} \) for some \( \epsilon > 0 \), where \( d \) is the supremum metric.

Indeed, by the definition of the compact-open topology, such a neighbourhood is of the form \( \{ f \in \text{Homeo}(X) : f[K_i] \subseteq W_1, \ldots, f[K_i] \subseteq W_i \} \) where \( W_i \)'s are open, and each \( K_i \) is a compact subset of \( W_i \). Then, it is enough to choose \( \epsilon \) such that for each \( i \) and \( x \in K_i \cap F \), \( B_F(x, \epsilon) \subseteq F \cap U_i \). Now, we can repeat the proof that we gave in the case of \( X = [0, 1] \). Namely, choosing \( V \) as above (but for an arbitrary \( X \)), we define \( \epsilon \) to be such that \( \{ f \in \text{Homeo}(X) : f|_{X \setminus \text{int}(F)} = \text{id}, d(\text{id}_F, f|_F) < \epsilon \} \subseteq U^0_0 \cap U^3_0 \) and define \( h, h' \) in the same way as above (identifying \( [0, 1] \) with a subset of \( F \)). Since \( U_0 \) and \( U_1 \) can be chosen to preserve endpoints of \( [0, 1] \), the choice of \( F \) and of the copy of \( [0, 1] \) inside it allows us to repeat the argument. \( \square \)

The only known examples of small Polish group structures without \( nm \)-generic orbits are of the form \( H(X) \). For those of them for which we were able to compute the finest compatible topology, it turned out that it is not Hausdorff. This may suggest that there could be a topological property of a group \( H \) other than being non-meager in itself, which guarantees the existence of \( nm \)-generic orbits in a structure \((H, G)\).

**Problem 2.9** Characterize the existence of \( nm \)-generic orbits in a Polish group structure \((H, G)\) in terms of topological properties of \( H \).

In particular, we can ask:

**Question 2.10** Does the existence of a Hausdorff group topology on a group \( H \) such that the action of a Polish group \( G \) on \( H \) is continuous imply that the structure \((H, G)\) has an \( nm \)-generic orbit?

Also, we do not know whether the converse is true.

**Question 2.11** Does the existence of \( nm \)-generic orbits in a Polish group structure \((H, G)\) imply the existence of a compatible Hausdorff topology on \( H \)?

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