SCHUR MULTIPLIERS AND MIXED UNITARY MAPS

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\textbf{Abstract.} We consider the tensor product of the completely depolarising channel on $d \times d$ matrices with the map of Schur multiplication by a $k \times k$ correlation matrix and characterise, via matrix theory methods, when such a map is a mixed (random) unitary channel. When $d = 1$, this recovers a result of O’Meara and Pereira, and for larger $d$ is equivalent to a result of Haagerup and Musat that was originally obtained via the theory of factorisation through von Neumann algebras. We obtain a bound on the distance between a given correlation matrix for which this tensor product is nearly mixed unitary and a correlation matrix for which such a map is exactly mixed unitary. This bound allows us to give an elementary proof of another result of Haagerup and Musat about the closure of such correlation matrices without appealing to the theory of von Neumann algebras.

\section{1. Introduction}

For any $k \in \mathbb{N}$, let $M_k = M_k(\mathbb{C})$ denote the set of all $k \times k$ complex-valued matrices. A quantum channel is a completely positive, trace-preserving linear map $\phi : M_m \to M_n$; such a map can be written in its (non-unique) Choi-Kraus decomposition as $\phi(X) = \sum_{i=1}^{M} A_i X A_i^*$, where the $A_i$ are $n \times m$ matrices known as the Kraus operators, and $A^*$ is the complex conjugate transpose of $A$. Trace-preservation yields $\sum_{i=1}^{M} A_i^* A_i = I_m$, where $I_m$ denotes the unit of the algebra $M_m$, i.e., the diagonal matrix with 1’s on the diagonal. The map $\phi$ is unital if $\phi(I_m) = I_n$, which is equivalent to having $\sum_{i=1}^{M} A_i A_i^* = I_n$.

Given $d \in \mathbb{N}$, the completely depolarising channel $\delta_d : M_d \to M_d$ is the unital quantum channel given by $\delta_d(X) = \text{tr}_d(X) I_d$, where we let $\text{tr}_d(X) = \frac{1}{d} \text{Tr}(X)$ denote the normalised trace on $M_d$ with $\text{tr}_d(I_d) = 1$.

Let $\mathcal{U}(d)$ be the set of unitary elements of $M_d$. A map $\phi : M_d \to M_d$ is said to be mixed unitary if it is in the convex hull of the maps of the form $X \mapsto U X U^*$ for $U \in \mathcal{U}(d)$. Mixed unitary maps are unital quantum

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channels, and are sometimes also referred to as random unitary channels in the literature, in particular in the context of quantum cryptography.

Given $C = (c_{i,j})$ and $X = (x_{i,j})$ in $M_k$, their Schur product is the matrix $C \circ X = (c_{i,j}x_{i,j})$. For any $C \in M_k$, the corresponding Schur multiplier is the map $S_C : M_k \to M_k$ given by Schur multiplication: $X \mapsto C \circ X$. The Schur multiplier $S_C$ is a unital quantum channel if and only if $C$ is a correlation matrix, i.e., a positive semidefinite matrix whose diagonal elements are all equal to 1.

O’Meara and Pereira [2] characterised the set of correlation matrices $C$ such that $S_C$ is a mixed unitary map, via matrix theory methods. The work of Haagerup andMusat [3, 4] on factorisation of completely positive maps through von Neumann algebras can be seen as yielding higher order versions of the O’Meara-Pereira result, as well as an important asymptotic version that relates limits of certain correlation matrices to Connes’ embedding problem. The relation to Connes’ embedding problem uses the work of Dykema and Juschenko [1] on matrices of unitary moments. Our work is also motivated by the recent work of Musat and Rørdam [5], which builds on the earlier work of Haagerup and Musat and makes this asymptotic relation to correlation matrices explicit.

The results of Haagerup and Musat rely on the theory of factorisation through von Neumann algebras and properties of ultrapowers of the hyperfinite II$_1$-factor in their proofs. The main goal of this paper is to obtain these higher order analogues of the O’Meara-Pereira result and their asymptotic versions by matrix theoretic methods, without reference to factorisation through von Neumann algebras. Our main new result, which allows us to take this approach, is a theorem that says, roughly, if $C$ is a correlation matrix such that $\delta_d \otimes S_C$ is nearly a mixed unitary map, then $C$ is near to a correlation matrix $\tilde{C}$ such that $\delta_{2d} \otimes S_{\tilde{C}}$ is a mixed unitary map, with explicit bounds.

In Section 2, for each $d$ and $k$, we characterise the $k \times k$ correlation matrices such that $\delta_d \otimes S_C$ is a mixed unitary map. The result we obtain is equivalent to a corresponding result of [3] once one realizes that exact factorisation through $M_d \otimes M_k \otimes L^\infty[0,1]$ reduces to our statement about convex combinations. In Section 3, we prove the “nearly” result and use it give a new proof of the asymptotic results for correlation matrices of [3, 4].

2. Schur multipliers with a finite mixed-unitary ancilla

In this section we obtain “higher order” versions in the spirit of the result of [2].

The following result is likely well known, but we prove it here for completeness. Given a subset $S$ of a vector space, we use $\text{conv}(S)$ to denote the, not necessarily closed, set of convex combinations of elements of $S$. 

Lemma 2.1. Let $A_d$ be a subset of the closed unit ball of $\mathbb{R}^m$ for $d \in \mathbb{N}$. If $\bigcup_d A_d$ is convex, then

$$
\bigcup_d A_d = \bigcup_d \text{conv}(A_d).
$$

Proof. Let $A = \bigcup_d A_d$ and $B = \bigcup_d \text{conv}(A_d)$. Plainly, $A \subseteq B$. Conversely, let $b \in B$. Then $b = \lim_{n \to \infty} b_n$ for some $b_n \in \text{conv}(A_{d_n})$ where $d_n \in \mathbb{N}$. By Carathéodory’s theorem, there is $N \in \mathbb{N}$ (independent of $n$) so that $b_n = \sum_{l=1}^{N} p_{n,l} a_{n,l}$ for some probability distribution $p_{n,1}, \ldots, p_{n,N} \in [0,1]$ and some $a_{n,l} \in A_{d_n}$. By compactness of $[0,1]$ and of $A$, we may pass to a subsequence for which $(p_{n,l})$ and $(a_{n,l})$ are convergent, say to $p_l \in [0,1]$ and $a_l \in A$. So

$$
b = \lim_{n \to \infty} \sum_{l=1}^{N} p_{n,l} a_{n,l} = \sum_{l=1}^{N} p_l a_l \in \text{conv}(A) = A.
$$

By Lemma 2.1, we also have

$$
\mathcal{F}_k = \bigcup_{d \in \mathbb{N}} \mathcal{F}_k(d).
$$

We now define some interesting sets of correlation matrices. For any $d,k \in \mathbb{N}$, let

$$
\mathcal{F}_k(d) = \{(\text{tr}_d(U_i^* U_j))_{i,j=1}^k \in M_k : U_1, \ldots, U_k \in \mathcal{U}(d)\}.
$$

Since the normalised trace $\text{tr}_d : M_d \to \mathbb{C}$ is unital and completely positive, it follows that $\mathcal{F}_k(d)$ is indeed a set of correlation matrices. We remark that since the convex hull of a compact set in $\mathbb{R}^n$ is compact, the sets $\mathcal{F}_k(d)$ and $\text{conv}(\mathcal{F}_k(d))$ are both compact, for any $d \in \mathbb{N}$. Adopting the notation of [1], we define

$$
\mathcal{F}_k = \bigcup_{d \in \mathbb{N}} \mathcal{F}_k(d).
$$

By Lemma 2.1, we also have

$$
\mathcal{F}_k = \bigcup_{d \in \mathbb{N}} \text{conv}(\mathcal{F}_k(d)).
$$

We use $E_{i,j} = |i\rangle \langle j|$ to denote the standard matrix units. Let $d \in \mathbb{N}$ and $\omega = \exp(\frac{2\pi i}{d})$. Let $S = \sum_{j=1}^{d} E_{j+1,j} \in M_d$ be the forward cyclic shift operator and let $D$ be the diagonal operator given by $D = \sum_{j=1}^{d} \omega^j E_{j,j} \in M_d$. The Weyl-Heisenberg unitaries are the $d^2$ unitaries given by

$$
W_{a,b} = S^a D^b, \quad 0 \leq a, b \leq d - 1.
$$

This set is a projective unitary group, i.e., modulo scalars it is a group.

Recall that the completely depolarising channel on $M_d$ is the map

$$
\delta_d : M_d \to M_d, \quad \delta_d(X) = \text{tr}_d(X)I_d.
$$

It is not difficult to show [7, Chapter 4] that

$$
\delta_d(X) = \frac{1}{d^2} \sum_{a,b=0}^{d-1} W_{a,b}XW_{a,b}^*, \quad (1)
$$
so $\delta_d$ is mixed unitary.

We recall a theorem of O’Meara and Pereira:

**Theorem 2.2 ([2, Theorem 5]).** Let $k \in \mathbb{N}$ and let $C \in M_k$ be a correlation matrix. The Schur multiplier $S_C : M_k \to M_k$ is mixed unitary if and only if $C$ is in $\text{conv}(\mathcal{F}_k(1))$, i.e., $C$ lies in the convex hull of the rank-one correlation matrices in $M_k$.

We generalise Theorem 2.2 as follows:

**Theorem 2.3.** Let $C \in M_k$ be a correlation matrix and let $d \in \mathbb{N}$. The map $\delta_d \otimes S_C : M_d \otimes M_k \to M_d \otimes M_k$ is mixed unitary if and only if $C \in \text{conv}(\mathcal{F}_k(d))$.

**Proof.** Suppose $S_C : M_k \to M_k$ is such that $\delta_d \otimes S_C$ is mixed unitary. Then for $A = (A_{i,j}) = \sum_{i,j=1}^{k} A_{i,j} \otimes E_{i,j} \in M_d \otimes M_k$, we have

$$\delta_d \otimes S_C(A) = (c_{i,j} \operatorname{tr}_d(A_{i,j}) I_d) = \sum_{l=1}^{M} p_l V_l A V_l^*$$

for some unitaries $V_l \in \mathcal{U}(dk)$, some probability distribution $p_1, \ldots, p_M > 0$ and some $M \in \mathbb{N}$. For $A = I_d \otimes E_{i,i}$, we have

$$\delta_d \otimes S_C(I_d \otimes E_{i,i}) = I_d \otimes E_{i,i} = \sum_{l=1}^{M} p_l V_l(I_d \otimes E_{i,i}) V_l^*.$$

Write $V_l = (V_{l,i,i})_{i,j=1}^{k} = \sum_{i,j=1}^{k} V_{l,i,i} \otimes E_{i,j} \in M_d \otimes M_k$ where each $V_{l,i,i}$ is in $M_d$.

The right hand side becomes

$$\sum_{l=1}^{M} p_l \sum_{s,t=1}^{k} V_{l,s,i}^* V_{l,t,i} \otimes E_{s,t}.$$

For $s = t \neq i$, we obtain

$$0 = \sum_{l=1}^{M} p_l V_{l,s,i}^* V_{l,s,i}$$

and each term in the latter sum is positive semidefinite. Hence $V_{l,s,i} = 0$ whenever $s \neq i$, and $V_l = \bigoplus_{i=1}^{k} V_{l,i,i}$. In particular, since $V_l \in \mathcal{U}(dk)$, each $V_{l,i,i}$ is in $\mathcal{U}(d)$. On the other hand, taking $A = I_d \otimes E_{i,j}$ where $i \neq j$, we obtain

$$\delta_d \otimes S_C(I_d \otimes E_{i,j}) = c_{i,j} I_d \otimes E_{i,j} = \sum_{l=1}^{M} p_l \sum_{s,t=1}^{k} (V_{l,s,i} V_{l,t,j}^* \otimes E_{s,t})$$

$$= \left( \sum_{l=1}^{M} p_l V_{l,i,i} V_{l,j,j}^* \right) \otimes E_{i,j},$$
hence
\[ c_{i,j} = \text{tr}_d(c_{i,j}I_d) = \sum_{l=1}^{M} p_l \text{tr}_d(V_{l,i,i}V_{l,j,j}^*) \]
and
\[ C = (c_{i,j}) = \sum_{l=1}^{M} p_l (\text{tr}_d(U_{l,i}U_{l,j}^*)) \]
where \( U_{l,i} = V_{l,i,i} \in U(d) \). Therefore, \( C \in \text{conv} \mathcal{F}_k(d) \), as claimed.

Consider the converse. Since the set of mixed unitary maps is convex, so is the set of maps \( \Phi : M_k \to M_k \) such that \( \delta_d \otimes \Phi \) is mixed unitary, for a fixed \( d \). Thus, it suffices to establish the converse in the case where \( C = (c_{i,j}) \) is in \( \mathcal{F}_k(d) \). Then \( C \) is of the form
\[ C = (\text{tr}_d(U_{i}U_{j}^*)) \]
for some \( U_{i} \in U(d) \). Let \( \{W_i\}_{i=1}^{d^2} \) be an enumeration of the Weyl operators in \( U(d) \). Define \( \widetilde{W}_{l,l'}(A_{i,j}) = \bigoplus_{i=1}^{k} W_l U_i W_{l'} \in \mathcal{U}(dk) \). By Equation (1), for \( A = (A_{i,j}) \in M_k(M_d) \) we have
\[ d^{-4} \sum_{l,l'=1}^{d^2} \widetilde{W}_{l,l'}(A_{i,j})\widetilde{W}_{l,l'}^* = d^{-4} \sum_{l,l'=1}^{d^2} (W_l U_i W_l A_{i,j} W_l U_j^* W_l^*) \]
\[ = d^{-2} \sum_{l,l'=1}^{d^2} (W_l U_i \text{tr}_d(A_{i,j})U_j^* W_l^*) \]
\[ = (\text{tr}_d(A_{i,j}) \text{tr}_d(U_j^*)I_d) \]
\[ = (\delta_d \otimes S_C)(A) \]
which shows that the map \( \delta_d \otimes S_C \) is indeed mixed unitary.

**Remark 2.4.** We recover Theorem 2.2 by taking \( d = 1 \).

**Remark 2.5.** In [4] a quantum channel \( T : M_k \to M_k \) is called *factorisable of degree \( d \)* if and only if \( \delta_d \otimes T \) is a mixed unitary. In [4, Proposition 3.4], they prove that a channel is factorisable of degree \( d \) if and only if it has an *exact factorisation through \( M_d \otimes M_k \otimes L[0,1] \)*. By unraveling what this latter property means in the case of a Schur product map and using the fact that the set of mixed unitary channels is closed, one obtains Theorem 2.3.

**Remark 2.6.** Musat and Rørdam [5] prove the remarkable result that \( \bigcup_d \text{conv}(\mathcal{F}_k(d)) \) is not closed for any \( k \geq 11 \).

3. **Asymptotically mixed unitary Schur multipliers**

We write \( \text{MU}(dk) \) or \( \text{MU}(M_d \otimes M_k) \) for the set of mixed unitary maps on \( M_{dk} = M_d \otimes M_k \). Note that mixed unitary maps are closed under several natural operations, including taking convex combinations, tensor products and composition. One goal of this section is to give a matrix theoretic proof
of a result of [4] that shows that $C$ is in the closure of $\bigcup_d \text{conv}(F_k(d))$ if and only if $\|\text{dist}_{cb}(\delta_d \otimes SC; \text{MU}(dk))\| \to 0$.

Since $\delta_d: M_d \to M_d$ is in $\text{MU}(M_d)$, the idempotent map $\Delta_{d,k} := \delta_d \otimes \text{id}_{M_k}$ is in $\text{MU}(M_d \otimes M_k)$, for any $d, k \in \mathbb{N}$.

We now consider the effect of “$\Delta$-compression”.

**Lemma 3.1.** Let $d, k \in \mathbb{N}$ and let $\Phi: M_d \otimes M_k \to M_d \otimes M_k$ be a unital quantum channel, and write $\Delta = \Delta_{d,k}$. Then $\Delta \circ \Phi \circ \Delta = \delta_d \otimes T$ for some unital quantum channel $T: M_k \to M_k$. Moreover, if $\Phi \in \text{MU}(dk)$, then $\Delta \circ \Phi \circ \Delta \in \text{MU}(dk)$.

**Proof.** Let $\Psi = \Delta \circ \Phi \circ \Delta$, which is a unital quantum channel on $M_d \otimes M_k$. Every element of the range of $\Delta$ is of the form $I_d \otimes Y$ for some $Y \in M_k$.

Since $\Psi = \Delta \circ \Psi$, it follows that there exist linear maps $\Psi_i: M_k \to M_k$ for $1 \leq i \leq d$ so that

$$\Psi(E_{i,i} \otimes B) = I_d \otimes \Psi_i(B), \quad B \in M_k.$$  

Define

$$T = \sum_{i=1}^{d} \Psi_i.$$  

Then for $A \in M_d$ and $B \in M_k$, since $\Psi = \Psi \circ \Delta$, we have

$$\Psi(A \otimes B) = \Psi(\Delta(A \otimes B))$$

$$= \text{tr}_d(A)\Psi(I_d \otimes B)$$

$$= \text{tr}_d(A)\sum_{i=1}^{d} \Psi(E_{i,i} \otimes B)$$

$$= \text{tr}_d(A) \sum_{i=1}^{d} I_d \otimes \Psi_i(B)$$

$$= \text{tr}_d(A)I_d \otimes \sum_{i=1}^{d} \Psi_i(B)$$

$$= \delta_d(A) \otimes T(B),$$

so $\Psi = \delta_d \otimes T$. This implies that $T(B) = (\text{tr}_d \otimes \text{id}_{M_k}) \circ \Psi(I_d \otimes B)$; since the maps $\text{tr}_d \otimes \text{id}_{M_k}$ and $\Psi$ are both unital quantum channels, as is the embedding $B \mapsto I_d \otimes B$, so is $T$.

The final assertion follows immediately, since the set of mixed unitary maps is closed under composition. $\square$

Recall that for completely bounded maps [6] we have that $\|\phi \otimes \psi\|_{cb} = \|\phi\|_{cb} \cdot \|\psi\|_{cb}$, and that for a unital completely positive map $\phi$, we have that $\|\phi\|_{cb} = 1$. From these facts, it follows that when $\phi$ is a unital completely positive map and $\phi$ is completely bounded, we have $\|\phi \otimes \psi\|_{cb} = \|\psi\|_{cb}$.
Proposition 3.2. Let \( d \in \mathbb{N} \) and let \( R : M_k \to M_k \) be a quantum channel. Then there is a quantum channel \( T : M_k \to M_k \) such that \( \delta_d \otimes T \) is mixed unitary with
\[
\| R - T \|_{cb} = \text{dist}_{cb}(\delta_d \otimes R, \text{MU}(dk)).
\]

Proof. Since the set of mixed unitaries is a compact set, we may choose a mixed unitary \( \Phi : M_d \otimes M_k \to M_d \otimes M_k \) such that
\[
\| \delta_d \otimes R - \Phi \|_{cb} = \text{dist}_{cb}(\delta_d \otimes R, \text{MU}(dk)).
\]
By the above, \( \Delta \circ \Phi \circ \Delta = \delta_d \otimes T \) for some unital quantum channel \( T : M_k \to M_k \). Then
\[
\| R - T \|_{cb} = \| \delta_d \otimes R - \delta_d \otimes T \|_{cb} = \| \Delta \circ [\delta_d \otimes R - \Phi] \circ \Delta \|_{cb}
\leq \| \delta_d \otimes R - \Phi \|_{cb} = \text{dist}_{cb}(\delta_d \otimes R, \text{MU}(dk)).
\]
However,
\[
\| R - T \|_{cb} = \| \delta_d \otimes R - \delta_d \otimes T \|_{cb} \geq \text{dist}_{cb}(\delta_d \otimes R, \text{MU}(dk)),
\]
and so the result follows. \( \square \)

Our main theorem is an analogue of Proposition 3.2 in the case that \( R \) is a Schur product map, except that we would also like to choose \( T \) to be a Schur product map. In this case we do not get an equality, but we are able to get a bound.

Working towards our main theorem requires a certain averaging by unitary conjugation. Typically, averaging by unitary conjugation is called twirling; in the quantum physics literature, it is said that one is applying a twirling channel. The twirl of a quantum channel is again a quantum channel. Here we do a somewhat different operation, which preserves complete positivity, but does not generally preserve the property of being a quantum channel; this operation was previously used in \([8, \text{Section V}]\). We use \( \{e_j\} \) to denote the standard basis of vectors of a given dimension.

Lemma 3.3. Let \( T : M_k \to M_k \) be completely positive and let \( \gamma : M_k \to M_k \) be the “\( D \)-biaverage” of \( T \), given by
\[
\gamma(X) = \int \int D_1^* T(D_1 XD_2) D_2^* dD_1 dD_2,
\]
where the integrals are taken over the group of diagonal unitary matrices in \( M_k \) with respect to Lebesgue measure on \( T^k \). Then \( \gamma = S_B \) for the positive semidefinite matrix \( B = V^* C_T V \) where \( C_T \) is the Choi matrix of \( T \) given by \( C_T = (T(E_{i,j}))_{i,j=1}^k \in M_k \otimes M_k \) and \( V : \mathbb{C}^k \to \mathbb{C}^k \otimes \mathbb{C}^k \) is the isometry \( e_j \mapsto e_j \otimes e_j \). In particular, \( \gamma \) is completely positive.
Proof. We have
\[
\gamma(E_{i,j}) = \int \int D_i^* T(D_1 E_{i,j} D_2) D_i^* dD_1 dD_2
\]
\[
= \int \int D_i^* d_{1,i} T(E_{i,j}) d_{2,j} D_i^* dD_1 dD_2
\]
\[
= \left( \int D_i^* d_{1,i} dD_1 \right) T(E_{i,j}) \left( \int d_{2,j} D_i^* dD_2 \right)
\]
\[
= E_{i,i} T(E_{i,j}) E_{j,j}
\]
\[
= T(E_{i,j}) E_{i,j} E_{i,j}
\]
So \( \gamma \) is Schur multiplication by the matrix \( B = (b_{i,j}) \), where \( b_{i,j} = T(E_{i,j}) E_{i,j} \).

Since \( T \) is completely positive, its Choi matrix \( C_T = (T(E_{i,j}))_{i,j=1}^k \in M_k \otimes M_k \) is positive, and \( B \) is the compression of \( C_T \) to the subspace spanned by \( \{ e_i \otimes e_i : 1 \leq i \leq k \} \). Hence, \( B = V^* C_T V \) is positive semidefinite. Consequently, \( \gamma = S_B \) is completely positive. \( \square \)

Remark 3.4. The map \( \gamma \) of Lemma 3.3 can also be obtained by averaging over the \( 2^k \) diagonal matrices of \( \pm 1 \)’s.

Remark 3.5. The same proof shows that, if \( G \subseteq U(k) \) is any compact subgroup, \( \phi : M_k \to M_k \) is completely positive, and we set
\[
\gamma(X) = \int \int U_1^* \phi(U_1 X U_2) U_2^* dU_1 dU_2,
\]
where \( dU \) denotes Haar measure on \( G \), then \( \gamma \) is completely positive and \( G \)-covariant, i.e., \( \gamma(U_1 X U_2) = U_1 \gamma(X) U_2 \) for any \( U_1, U_2 \in G \). We shall refer to \( \gamma \) as the \( G \)-biaverage of \( \phi \). Note that the \( G \)-biaverage of a quantum channel need not be a quantum channel, but it can be shown to be trace decreasing for positive elements. Similarly, if \( \phi \) is unital, then \( 0 \leq \gamma(I_k) \leq I_k \). Both of these latter inequalities follow by showing that if \( \{ A_i \} \) is a set of Choi-Kraus operators for \( \phi \), then \( \{ \mathbb{E}(A_i) \} \) is a set of Choi-Kraus operators for \( \gamma \), where \( \mathbb{E} : M_k \to M_k \) is the conditional expectation onto the commutant of \( G \), and using the Cauchy-Schwarz inequality for completely positive maps.

In finite dimensions, we have the following unitary dilation at our disposal.

Lemma 3.6. If \( X \in M_d \) with \( \|X\| \leq 1 \), then there exist \( A, B \in M_d \) such that the \((2d) \times (2d)\) matrix
\[
W = \begin{bmatrix} X & A \\ B & X \end{bmatrix}
\]
is unitary.

Proof. Define \( C = \sqrt{I_d - XX^*} \) and \( D = \sqrt{I_d - X^*X} \), and let \( X = UP \) be the polar decomposition of \( X \), where, since we work in finite dimensions, we may assume that \( U \) is unitary (rather than merely a partial isometry) and
Proof. By hypothesis, there is a mixed unitary $\Phi$:

$$
\begin{bmatrix}
X & C \\
D & -X^*
\end{bmatrix} \in U(2d).
$$

Thus,

$$
W := \begin{bmatrix} I_d & 0 \\ 0 & -I_d \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} X & C \\ D & -X^* \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & U \end{bmatrix} \in U(2d)
$$

since $W$ is a product of unitaries. Since $UX^*U = UPU^*U = UP = X$, we see that $W$ has the desired form. \qed

Remark 3.7. In infinite dimensions, taking $X$ to be the unilateral shift, one sees that there are no operators $A$ and $B$ such that the above operator matrix is a unitary.

Theorem 3.8. Let $C = (c_{i,j})$ be a $k \times k$ correlation matrix and let $\varepsilon > 0$. If $\text{dist}_{cb}(\delta_d \otimes SC, \text{MU}(dk)) < \varepsilon$, then there is $\tilde{C} = (\tilde{c}_{i,j}) \in \text{conv}(\mathcal{F}_k(2d))$ with $\|S_C - S_{\tilde{C}}\|_{cb} < 2\varepsilon$. In particular, $|c_{i,j} - \tilde{c}_{i,j}| < 2\varepsilon$ for all $1 \leq i, j \leq k$.

Proof. By hypothesis, there is a mixed unitary $\Phi : M_{dk} \to M_{dk}$ with

$$
\|\delta_d \otimes SC - \Phi\|_{cb} < \varepsilon.
$$

By Lemma 3.1, the mixed unitary map $\Delta \circ \Phi \circ \Delta$ is of the form $\delta_d \otimes T$ for some unital quantum channel $T : M_k \to M_k$. Since $\Delta \circ (\delta_d \otimes S_C) \circ \Delta = \delta_d \otimes S_C$, we have

$$
\|S_C - T\|_{cb} = \|\delta_d \otimes S_C - \delta_d \otimes T\|_{cb} = \|\Delta \circ (\delta_d \otimes S_C - \Phi) \circ \Delta\|_{cb} < \varepsilon.
$$

Since $\delta_d \otimes T$ is mixed unitary, we may write

$$
\delta_d \otimes T(X) = \sum_{l=1}^M t_l U_l X U_l^*
$$

for some $M \in \mathbb{N}$ and unitaries $U_1, \ldots, U_M \in U(dk)$ and $t_l \geq 0$ satisfying $\sum_{l=1}^M t_l = 1$. By Lemma 3.3, the $D$-biaverage of $T$ is a positive Schur multiplier, say $S_{\tilde{C}} : M_k \to M_k$, for some positive semidefinite $\tilde{C} \in M_k$. For any $X \in M_k$ with $\|X\| \leq 1$, we have

$$
\|S_C(X) - S_{\tilde{C}}(X)\| = \left\| \int \int D_1^* S_C(D_1 XD_2) D_2^* - D_1^* T(D_1 XD_2) D_2^* dD_1 dD_2 \right\|
\leq \int \int \left\| D_1^*(S_C(D_1 XD_2) - T(D_1 XD_2)) D_2^* \right\| dD_1 dD_2
\leq \int \int \|S_C - T\| dD_1 dD_2 \leq \|S_C - T\|_{cb} < \varepsilon.
$$

Hence,

$$
\|S_C - S_{\tilde{C}}\|_{cb} = \|S_C - S_{\tilde{C}}\| < \varepsilon.
$$

This implies that $|c_{i,j} - \tilde{c}_{i,j}| < \varepsilon$ for every $i, j$. In particular, since $c_{i,i} = 1$ for all $i$,

$$
(2) \quad |1 - \tilde{c}_{i,i}| < \varepsilon, \quad 1 \leq i \leq k.
$$
For $X \in M_{dk}$, we have
\[
\delta_d \otimes S_C(X) = \sum_l t_l \int \int (I_d \otimes D_1)^* U_l(I_d \otimes D_1) X(I_d \otimes D_2) U_l^* (I_d \otimes D_2)^* dD_1 dD_2
\]
\[
= \sum_l t_l \left( \int (I_d \otimes D_1)^* U_l(I_d \otimes D_1) dD_1 \right) X \left( \int (I_d \otimes D_2) U_l^* (I_d \otimes D_2)^* dD_2 \right)
\]
\[
= \sum_l t_l X_l XX_l^*
\]
where $X_l = \int (I_d \otimes D)^* U_l(I_d \otimes D) dD \in M_{dk}$. Since $X_l$ lies in the convex hull of $U(dk)$, we have $\|X_l\| \leq 1$.

To see that each $X_l$ is block diagonal, just as in the proof of Theorem 2.3, we calculate
\[
\delta_d \otimes S_C(I_d \otimes E_{i,i}) = I_d \otimes \tilde{c}_{i,i} E_{i,i} = \sum_{l=1}^M t_l X_l(I_d \otimes E_{i,i}) X_l^*.
\]
Write $X_l = (X_{l,i,j})_{i,j=1}^k = \sum_{i,j=1}^k X_{l,i,j} \otimes E_{i,j} \in M_d \otimes M_k$; then we obtain
\[
I_d \otimes \tilde{c}_{i,i} E_{i,i} = \sum_{l=1}^M t_l \sum_{s,t=1}^k X_{l,s,i} X_{l,t,i}^* \otimes E_{s,t}.
\]
For $s = t \neq i$, we have
\[
0 = \sum_{l=1}^M t_l X_{l,s,i} X_{l,s,i}^*.
\]
By positivity, $X_{l,s,i} = 0$ whenever $s \neq i$. Writing $X_{l,i} := X_{l,i,i}$, it follows that $X_l = \bigoplus_{i=1}^k X_{l,i} \otimes E_{i,i}$. Since $\|X_l\| \leq 1$, it follows that $\|X_{l,i}\| \leq 1$ for each $i$. Moreover,
\[
I_d \otimes \tilde{c}_{i,j} E_{i,j} = \delta_d \otimes S_C(I_d \otimes E_{i,j}) = \sum_{l=1}^M t_l X_l(I_d \otimes E_{i,j}) X_l^* = \sum_{l=1}^M t_l X_{l,i} X_{l,j}^* \otimes E_{i,j},
\]
so
\[
\tilde{c}_{i,j} E_{i,j} = \text{tr}_d \otimes \text{id}(I_d \otimes \tilde{c}_{i,j} E_{i,j}) = \text{tr}_d \otimes \text{id} \left( \sum_{l=1}^M t_l X_{l,i} X_{l,j}^* \otimes E_{i,j} \right).
\]
Hence,
\[
(3) \quad \tilde{c}_{i,j} = \sum_{l=1}^M t_l \text{tr}_d(X_{l,i} X_{l,j}^*).
\]
Applying Lemma 3.6 to each $X_{l,i}$, we obtain unitary matrices $W_{l,i} \in U(2d)$ of the form
\[
W_{l,i} = \begin{bmatrix} X_{l,i} & A_{l,i} \\ B_{l,i} & X_{l,j} \end{bmatrix},
\]
for some \( A_{l,i}, B_{l,i} \in M_d \). Now, consider
\[
\hat{C} := (\hat{c}_{i,j}) \in \text{conv}(\mathcal{F}_k(2d))
\]
defined by
\[
\hat{c}_{i,j} := \sum_{l=1}^{M} t_l \text{tr}_d(W_{l,i}W_{l,j}^*)
\]
\[
= \sum_{l=1}^{M} t_l \text{tr}_d\left(X_{l,i}X_{l,j}^* + A_{l,i}A_{l,j}^* B_{l,i}B_{l,j}^* + X_{l,i}X_{l,j}^*\right)
\]
\[
= \sum_{l=1}^{M} t_l \text{tr}_d(X_{l,i}X_{l,j}^*) + \frac{1}{2} \left( \sum_{l=1}^{M} t_l \left( \text{tr}_d\left( A_{l,i}A_{l,j}^* + B_{l,i}B_{l,j}^*\right) \right) \right)
\]
\[
= \tilde{c}_{i,j} + \frac{1}{2} \left( \sum_{l=1}^{M} t_l \left( \text{tr}_d\left( A_{l,i}A_{l,j}^* + B_{l,i}B_{l,j}^*\right) \right) \right)
\]
where the off-diagonal terms denoted by * in the second line may be ignored, as they do not affect the trace.

Since each \( W_{l,i} \) is unitary, we have \( X_{l,i}X_{l,i}^* + A_{l,i}A_{l,i}^* = I_d \). By Equation (3) above, we have
\[
\sum_{l=1}^{M} t_l \text{tr}_d(A_{l,i}A_{l,i}^*) = \sum_{l=1}^{M} t_l \text{tr}_d(I_d - X_{l,i}X_{l,i}^*)
\]
\[
= 1 - \sum_{l=1}^{M} t_l \text{tr}_d(X_{l,i}X_{l,i}^*)
\]
\[
= 1 - \text{tr}_d(\tilde{c}_{i,i}I_d)
\]
\[
= 1 - \tilde{c}_{i,i}.
\]

In particular, by Equation (2),
\[
\left| \sum_{l=1}^{M} t_l \text{tr}_d(A_{l,i}A_{l,i}^*) \right| < \varepsilon.
\]
Define \( y_{i,j} = \sum_{l=1}^{M} t_l \text{tr}_d(A_{l,i}A_{l,j}^*) \) and \( z_{i,j} = \sum_{l=1}^{M} t_l \text{tr}_d(B_{l,i}B_{l,j}^*) \), and set \( Y = (y_{i,j})_{i,j=1}^{k} \) and \( Z = (z_{i,j})_{i,j=1}^{k} \). Then \( |y_{i,i}| < \varepsilon \) for each \( i \). A similar argument shows that \( |z_{i,i}| < \varepsilon \). We have \( \tilde{C} = \tilde{C} + \frac{1}{2}(Y + Z) \), so
\[
S_{\tilde{C}} - S_{\tilde{C}} = \frac{1}{2}(SY + SZ).
\]
We will show that \( \|SY\|_{cb} < \varepsilon \); the argument for \( SZ \) is similar. For each \( 1 \leq l \leq M \), the matrix \( (A_{l,i}A_{l,j}^*)_{i,j=1}^{k} \in M_k \otimes M_d \) is positive. Then \( (\text{tr}_d(A_{l,i}A_{l,j}^*)) = \text{id}_k \otimes \text{tr}_d(A_{l,i}A_{l,j}^*)_{i,j} \) is positive as well. Taking convex combinations, we see that \( Y \) is positive in \( M_k \). In particular, \( 0 \leq y_{i,i} < \varepsilon \) for
each $i$. But since $Y$ is positive, the Schur multiplier map $S_Y$ is completely positive, so that
\[ \|S_Y\|_{cb} = \|S_Y(I_k)\| = \max\{y_{i,i} : 1 \leq i \leq k\} < \varepsilon. \]
Similarly, $\|S_Z\|_{cb} < \varepsilon$, so that
\[ \|S_Z\|_{cb} \leq \frac{1}{2}(\|S_Y\|_{cb} + \|S_Z\|_{cb}) < \frac{1}{2}(\varepsilon + \varepsilon) = \varepsilon. \]
Finally, since $\|S_C - S_{\tilde{C}}\|_{cb} < \varepsilon$, it follows that
\[ \|S_C - S_{\tilde{C}}\|_{cb} \leq \|S_C - S_{\tilde{C}}\|_{cb} + \|S_{\tilde{C}} - S_C\|_{cb} < 2\varepsilon. \]
Hence,
\[ |c_{i,j} - \hat{c}_{i,j}| = \| (S_C - S_{\tilde{C}})(E_{i,j}) \| \leq \|S_C - S_{\tilde{C}}\|_{cb} < 2\varepsilon. \]
\[ \square \]
**Corollary 3.9.** Let $C \in M_k$ be a correlation matrix. The following are equivalent:

1. $C \in F_k$;
2. $\inf_d \text{dist}_{cb}(\delta_d \otimes S_C, \text{MU}(d_k)) = 0$.

**Proof.** Given $C \in F_k$, there exist $d_n \in \mathbb{N}$ and $C_n \in F_k(d_n)$ with $C_n \to C$, entrywise, as $n \to \infty$. Then
\[ \text{dist}_{cb}(\delta_{d_n} \otimes S_C, \text{MU}(d_n k)) \leq \|\delta_{d_n} \otimes S_C - \delta_{d_n} \otimes S_{C_n}\| \leq \|S_C - S_{C_n}\|_{cb} \to 0 \]
as $n \to \infty$, since all norms are equivalent in finite dimensions. Hence the infimum above is 0.

Conversely, suppose the infimum is 0. Given $\varepsilon > 0$, there exists $d_{\varepsilon} \in \mathbb{N}$ and $\tilde{C} = (\tilde{c}_{i,j}) \in \text{conv}(F_k(2d_{\varepsilon}))$ such that $d_{\infty}(C, \tilde{C}) < 2\varepsilon$ by Theorem 3.8, where $d_{\infty}(A, B) = \|A - B\|_{\infty} = \max_{i,j} |(A - B)_{i,j}|$. It follows that
\[ C \in \bigcup_{\varepsilon > 0} \text{conv}(F_k(2d_{\varepsilon})) \subseteq F_k. \]
\[ \square \]
**Remark 3.10.** By [3, (3.15)],
\[ \text{dist}_{cb}(\delta_{d+1} \otimes T, \text{MU}((d + 1)k)) \leq \text{dist}_{cb}(\delta_d \otimes T, \text{MU}(d k)) + \frac{1}{d+1}, \]
so that
\[ \inf_d \text{dist}_{cb}(\delta_d \otimes T, \text{MU}(d k)) = \lim_d \text{dist}_{cb}(\delta_d \otimes T, \text{MU}(d k)). \]
The last corollary should be compared to [4, Theorem 3.6].

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