Newton’s gravitational coupling constant from a quantum of area

F.R. Klinkhamer
Institute for Theoretical Physics, University of Karlsruhe,
Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
frans.klinkhamer@kit.edu

A previous calculation of Newton’s gravitational coupling constant $G$ is generalized. This generalization makes it possible to have “atoms of two-dimensional space” with an integer dimension $d_{\text{atom}}$ of the internal space, where the case $d_{\text{atom}} = 1$ is found to be excluded. Given the quantum of area $l^2$, the final formula for $G$ is inversely proportional to the logarithm of the integer $d_{\text{atom}}$. The generalization used may be interpreted as a modification of the energy equipartition law of the microscopic degrees of freedom responsible for gravity, suggesting some form of long-range interaction between these degrees of freedom themselves.

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1. Introduction

It has been argued$^{1}$ that the fundamental length scale of quantum spacetime need not be given by the Planck length, $l_P \equiv \left(\frac{h}{G}\right)^{1/2}/c^{3/2} \approx 1.6 \times 10^{-35}$ m, but may correspond to a new fundamental constant of nature, $l$. This would then suggest that Newton’s gravitational coupling constant $G$ becomes calculable in terms of the fundamental constants $c$ (velocity of light in vacuum), $\hbar$ (Planck’s quantum of action), and $l$ (the hypothetical quantum of length).

Stimulus for a calculation of $G$ was provided by the approach of Verlinde$^{2}$ to consider the Newtonian gravitational attraction as a type of entropic force, with the fundamental microscopic degrees of freedom living on a two-dimensional screen,
in line with the so-called holographic principle. Following this approach and using the Bekenstein–Hawking black-hole entropy as input, a single transcendental equation can be derived, which fixes the numerical factor $f$ entering the $G$ expression.

Now, it is possible to make further progress by combining two recent suggestions. The first is by Sahlmann (last paragraph in Ref. 9) that the internal Hilbert-space dimension of the “atom of two-dimensional space,” corresponding to the “quantum of area” $l^2$, may very well need to be integer and that this places further restrictions on the microscopic theory. The second is by Neto 10 that the microscopic degrees of freedom on the holographic screen may have a modified energy equipartition law. Such a behavior may result from a generalization of the standard Boltzmann-Gibbs statistics, but it may also have an entirely different origin. (The important role of the equipartition law has previously been emphasized in, e.g., Ref. 14.)

Prompted by these two suggestions, a new calculation of $G$ is presented in this paper. In addition, a physical interpretation of the result can be given, which is based on the Verlinde approach to the origin of gravity.

2. Combinatorial calculation

This section gives a purely combinatorial calculation of the numerical factor $f$ entering the general expression for the gravitational coupling constant:

$$G = f \frac{e^3 l^2}{\hbar},$$  \hspace{1cm} (1)

where $l^2$ is considered to be a new fundamental constant of nature with the dimension of area. Incidentally, the notation $G_N$ will be kept for the experimental value of Newton’s gravitational coupling constant.

Following Sec. 4 of Ref. 8 there are two steps for the combinatorial calculation of $f$. First, consider the event horizon of a large nonrotating (Schwarzschild) black-hole and write the number of degrees of freedom on this surface (with area $A$) as the product of two dimensionless numbers,

$$N_{\text{dof}} = d_{\text{atom}} N_{\text{atom}}.$$  \hspace{1cm} (2)
Here, $N_{\text{atom}}$ is interpreted as the number of distinguishable “atoms of two-dimensional space” making up the area ($l^2$ being the quantum of area) and $d_{\text{atom}}$ as the dimension of the internal space of an individual atom:

$$N_{\text{atom}} \equiv A/l^2 \in \mathbb{N}_1 \equiv \{1, 2, 3, \ldots\}, \quad (3a)$$
$$d_{\text{atom}} \equiv f^{-1} I_1^{-1} \in \mathbb{N}_1. \quad (3b)$$

Compared to the analysis of Ref. [8] there are two new ingredients in (3b): $d_{\text{atom}}$ is no longer equal to $f^{-1}$ and $d_{\text{atom}}$ is demanded to be a positive integer. The factor $I_1^{-1}$ in definition (3b) simply parameterizes the difference of $d_{\text{atom}}$ and $f^{-1}$.

For the moment, the origin and meaning of $I_1^{-1}$ is left open (one possible physical interpretation will be given in Sec. 3). From now on, abbreviate “atoms of two-dimensional space” as “atoms of space” or even “atoms.” One such “atom” will be said to contribute one “quantum of area” $l^2$ to a macroscopic surface.

Second, take as input the Bekenstein–Hawking formula [15] for the entropy of a large (macroscopic) black-hole

$$S_{\text{BH}}/k_B = c^3 A/(4\hbar G) = (1/4) f^{-1} A/l^2 = (1/4) I_1 d_{\text{atom}} N_{\text{atom}}, \quad (4)$$

where [1] has been used in the second step and (3a) and (3b) in the third step. Equating the number of configurations of the distinguishable atoms of space from (2) with the exponential of the Bekenstein–Hawking entropy (4) gives the following set of conditions:

$$\left( d_{\text{atom}} \right)^{N_{\text{atom}}} = \exp \left[ \left(1/4\right) I_1 d_{\text{atom}} N_{\text{atom}} \right], \quad (5)$$

for positive integers $N_{\text{atom}} \gg 1$ (there may be significant corrections to the black-hole entropy for $N_{\text{atom}} \sim 1$; see, e.g., Ref. [10] and references therein). The infinite set of conditions (5) reduces, for given $I_1$, to a single transcendental equation for $d_{\text{atom}}$,

$$\ln d_{\text{atom}} = (1/4) I_1 d_{\text{atom}}. \quad (6a)$$

In addition, there is still the condition that the dimension of the internal space be a positive integer [9]

$$d_{\text{atom}} \in \mathbb{N}_1. \quad (6b)$$
Table 1. Selected $I_1$ values required for having integer $d_{\text{atom}}$ values, according to \textit{(6a)} and \textit{(6b)}. Also shown is the corresponding $q$ value from \textit{(12)}. The value $d_{\text{atom}} = 1$ is nonphysical, because $\hat{I}_1$ is found to vanish. If \textit{(12)} holds, the values $d_{\text{atom}} \geq 27$ are, most likely, also nonphysical, because the values $\hat{q}$ turn out to be negative.\textsuperscript{11}

\begin{table}[h]
\begin{tabular}{ccc}
\hline
$d_{\text{atom}} = \hat{d} \equiv n$ & $\hat{I}_1 \equiv (4 \ln n)/n$ & $\hat{q} \equiv 2 - 1/\hat{I}_1$
\hline
1 & (0) & ($-\infty$) \\
2 & 1.3863 & 1.2787 \\
3 & 1.4648 & 1.3173 \\
4 & 1.3863 & 1.2787 \\
5 & 1.2876 & 1.2233 \\
8 & 1.0397 & 1.0382 \\
9 & 0.9765 & 0.9760 \\
26 & 0.5012 & 0.0050 \\
27 & 0.4883 & $-0.0480$
\hline
\end{tabular}
\end{table}

Note that \textit{(6a)} has precisely the same form as Eq. (13) of Ref. \textit{S} except for the additional factor $I_1$ on the right-hand side. Similar modifications can be expected for the generalized models of Ref. \textit{9}.

Table 1 gives the required $I_1$ values (indicated by hats) from \textit{(6a)} to make for integer $d_{\text{atom}}$ values. Three remarks are in order. First, having a solution of \textit{(6a)} demands a small enough numerical factor $(1/4)I_1$ on the right-hand side, corresponding to $I_1 \leq 4/e \approx 1.47152$, with $e \approx 2.71828$ the base of the natural logarithm. Second, the required $I_1$ values for $d_{\text{atom}} = 2$ and $d_{\text{atom}} = 4$ are equal, but it is not clear if this carries over to generalized models (for example, those of Ref. \textit{9}). Third, the value $d_{\text{atom}} = 1$ is physically not allowed, as $I_1 = 0$ from \textit{(6a)} implies a vanishing black-hole entropy \textsuperscript{14} for $d_{\text{atom}} = 1$ and finite $N_{\text{atom}}$.

With the solutions of \textit{(6a)} and \textit{(6b)}, the final formula for Newton’s gravitational coupling constant $G$ from \textit{(11)} reads

\begin{equation}
G = \left(1/4\right) \left(\ln \hat{d} \right)^{-1} c^3 l^2/h, \tag{7a}
\end{equation}

\begin{equation}
\hat{d} \in \mathbb{N}_1 \setminus \{1\} = \{2, 3, 4, \ldots \}, \tag{7b}
\end{equation}
where $\hat{d}$ is the internal dimension of an atom of space with quantum of area $l^2$. The fundamental microscopic theory will have to decide which value of $\hat{d}$ appears in \[7\]. Observe that, given $l^2$, the maximal value of $G$ is obtained for the minimal value of the integer $\hat{d}$, namely, $\hat{d} = 2$.

From the experimental value $G_N = 6.6743(7) \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$ (see, e.g., Chap. X of Ref. [15] for further discussion), the following numerical estimate of the smallest possible quantum of area is obtained:

$$l^2 \bigg|_{\hat{d}=2} = 4 \ln 2 \left(l_P\right)^2 \approx 7.2423 \times 10^{-70} \text{ m}^2,$$

with $l_P \equiv (\hbar G_N)^{1/2}/c^{3/2} \approx 1.6162 \times 10^{-35} \text{ m}$. Not surprisingly, this particular value of the quantum of area has already been given in an earlier article by 't Hooft.

Here, there is the further result that the dimension $d_{\text{atom}} = \hat{d} = 1$ is ruled out on physics grounds, leaving $\hat{d} = 2$ as the lowest possible value and allowing for the storage of information ("bits") on a holographic screen.

Expression \[7\] for $G$ is the main result of this paper. The crucial Eqs. \[6\] and \[6b\] for its derivation rely only on the definitions \[1\]–\[3\], the interpretation of $N_{\text{dof}}$ mentioned in the lines under \[2\], and the input \[4\] corresponding to the entropy of a Schwarzschild black hole. The really new ingredient, here, is the extra factor $I_1^{-1}$ in definition \[3b\]. The rest of this paper is devoted to one possible physical interpretation of $I_1$, but there may, of course, be other interpretations.

3. Modified equipartition law and entropic gravity

As mentioned in the Introduction, it has been suggested\[10\] that the microscopic degrees of freedom responsible for gravity obey a modified equipartition law which can be written as:\[13\]

$$E = N_{\text{dof}} \frac{1}{2} I_1 k_B T,$$

where, for the moment, the real factor $I_1 > 0$ is considered to be unrelated to the quantity $I_1^{-1}$ appearing in \[3b\]. At first, it may be best to remain agnostic.
as to the possible origin of the nonstandard equipartition law with \( I_1 \neq 1 \). A particular calculation of \( I_1 \neq 1 \) from nonstandard statistics will, however, be considered in Sec. 4.

Taking (9) for granted, return to the derivation (4) in Ref. 8 of the Newtonian gravitational acceleration \( \mathbf{A}_{\text{grav}} \) on a test mass arising from a spherical holographic screen \( \Sigma_{\text{sph}} \) with area \( A = 4\pi R^2 \):

\[
|\mathbf{A}_{\text{grav}}| \overset{1}{=} 2\pi c \left( k_B T / \hbar \right) \\
\overset{2}{=} 4\pi f c \left( N_{\text{dof}} \frac{1}{2} I_1 k_B T / \hbar \right) \left( f^{-1} I_1^{-1} / N_{\text{dof}} \right) \\
\overset{3}{=} 4\pi f c \left( E / \hbar \right) \left( l^2 / A \right) \\
\overset{4}{=} f c \left( M c^2 / \hbar \right) \left( l^2 / R^2 \right) \\
\overset{5}{=} \left( f c^3 l^2 / \hbar \right) M / R^2. 
\tag{10}
\]

Step 1 in the above derivation relies on the Unruh temperature (but with the logic reversed, temperature giving rise to acceleration). Step 2′ uses straightforward mathematics and prepares the way for the next move. Step 3′, then, relies on (9) and the following relation between the number \( N_{\text{dof}} \) of degrees of freedom on the holographic screen and the area \( A \) of the screen:

\[
N_{\text{dof}} = f^{-1} I_1^{-1} A / l^2. 
\tag{11}
\]

Step 4 depends on the well-known relation of energy and mass from special relativity. Step 5, finally, separates the fundamental microscopic constants of nature (indicated by lower-case letters) from the macroscopic variables of the experimental setup (upper-case letters).

The last expression in (10) gives the Newtonian gravitational coupling constant \( G \) in the form (11). That is, the classical constant \( G \) is obtained as a ratio of the two quantum constants \( l^2 \) and \( \hbar \) (this point has already been emphasized in Ref. 1). Furthermore, (11) concords with the previous definitions (2), (3a), and (3b). The derivation (10) identifies, therefore, the number \( I_1^{-1} \) entering definition (3b) as the inverse of the modification factor \( I_1 \) of the equipartition law (9). The numerical values of \( I_1 \) and \( f \) are determined by the calculation of Sec. 2.
Hence, the suggestion is that the atoms of space have some type of long-range interaction or long-time memory, which results in a modification of the energy equipartition law. The numerical values for $I_1$ in Table 1 show that, for the simplest atom with $\hat{d} = 2$, the standard equipartition law must be modified by some +40%. Note also that the $\hat{I}_1$ values in Table 1 are larger than 1 for dimensions $2 \leq \hat{d} \leq 8$ and smaller than 1 for $\hat{d} \geq 9$.

4. Generalized statistics

Now, consider one possible explanation of the nonstandard equipartition law, namely, the generalization of the standard Boltzmann-Gibbs statistics along the lines suggested by Tsallis. This allows for an explicit calculation of the modification factor $I_1$ in the equipartition law, as a function of the nonextensive entropy index $q \in \mathbb{R}$ of Tsallis.

For a quadratic classical Hamiltonian, the modified equipartition law has been derived in Eq. (32) of Ref. 13, with $I_1$ defined by Eq. (47) of that same reference. Specifically, for a generalized Maxwell velocity distribution in two-dimensional Euclidean space, the following result holds:

$$I_1 = \frac{\int_0^1 du \ u \left[1 - u^2\right]^{1/(1-q)}}{\int_0^1 du \ u \left[1 - u^2\right]^{q/(1-q)}} = \frac{1}{2 - q}, \quad (12)$$

for $0 < q < 2$. The standard Boltzmann–Gibbs statistics ($q = 1$) gives $I_1 = 1$.

The index $q$ enters the generalized entropy relation for two independent systems, $L$ and $R$, in the following way:

$$s(L + R)/k_q = s(L)/k_q + s(R)/k_q + (1 - q) \ s(L)/k_q \ s(R)/k_q, \quad (13)$$

where, for clarity, the nonstandard entropy is denoted by a lower-case letter ‘$s$’ and $k_q$ is a new Boltzmann-type constant with the only requirement that $k_1 = k_B$.

From the numerical values for $q$ in Table 1, a system of atoms of space with internal dimensions $2 \leq \hat{d} \leq 8$ then has a subadditive entropy $s$ and a system of atoms with dimensions $9 \leq \hat{d} \leq 26$ a superadditive entropy $s$. Systems of atoms with $\hat{d} \geq 27$ have unusual (most likely, unacceptable) thermodynamics, with, for example, a convex entropy.
If the modified equipartition law \( (9) \) is indeed due to a form of nonstandard statistics as suggested in the previous paragraph, then the following question arises: how does the nonextensive entropy \( s \) of a relatively small number of atoms of space combine into the extensive Bekenstein–Hawking entropy \( S_{\text{BH}} \) of a macroscopic black hole? Somehow, this may involve a form of collective behavior of a subset of the atoms, “monatomic molecules,” perhaps even collective behavior of combinations of different types of atoms, “hetero-atomic molecules.” (For a related discussion of entropic modifications of Newton’s law in the Verlinde framework, see, e.g., Refs. 17, 18 and references therein.)

Elaborating on the discussion of the previous paragraph, there may be special mixtures of different types of atoms, which give an effective index \( q_{\text{eff}} = 1 \) for the entropy but an effective factor \( I_{1, \text{eff}} \neq 1 \) for the energy equipartition law. This may not be altogether impossible, as the following simple argument shows. Imagine an equal mixture of two hypothetical types of noninteracting atoms, \( a \) and \( b \), with \( q_a = 1 - \Delta q \) and \( q_b = 1 + \Delta q \), for \( 0 < |\Delta q| < 1 \), and \( k_{qa} = k_{qb} = k_B \). Mathematically, there is \( (1 - q_a) + (1 - q_b) = 0 \) and \( 1/(2 - q_a) + 1/(2 - q_b) = 2/(1 - (\Delta q)^2) \neq 1 + 1 \).

Using a short-hand notation and setting \( k_B = 1 \), two independent systems, \( L \) and \( R \), each with approximately equal numbers of \( a \)- and \( b \)-type atoms (\( N_{La} \sim N_{Lb} \), \( s_{La} \sim s_{Lb} \), and similarly for \( R \)), then have the following total entropy from (13) and total energy from (9) and (12):

\[
s_{L+R} = s_{La} + s_{Ra} + (1 - q_a) s_{La} s_{Ra} + s_{Lb} + s_{Rb} + (1 - q_b) s_{Lb} s_{Rb}
\sim s_{La} + s_{Lb} + s_{Ra} + s_{Rb} + (1 - q_a + 1 - q_b) s_{La} s_{Rb}
\sim s_{La} + s_{Lb} + s_{Ra} + s_{Rb}
\sim s_L + s_R, \quad (14a)
\]

\[
E_{L+R}/T = \left( \frac{1}{2} \left( N_{La}/(2 - q_a) + N_{Lb}/(2 - q_b) \right) + \frac{1}{2} \left( N_{Ra}/(2 - q_a) + N_{Rb}/(2 - q_b) \right) \right)
\sim N_{La}/(1 - (\Delta q)^2) + N_{Rb}/(1 - (\Delta q)^2)
\sim \frac{1}{2} \left( N_{La} + N_{Lb} + N_{Ra} + N_{Rb} \right) 1/(1 - (\Delta q)^2). \quad (14b)
\]

The above results effectively show a standard (extensive) entropy and a modified
energy equipartition law, at least, for the hypothetical types of atoms of this simple argument. It remains to be seen if a similar result holds for a mixture of atoms from Table 1 or atoms obtained from a more advanced calculation.

5. Conclusion

It may be helpful to give a brief summary of what has been achieved in this paper. Consider, for simplicity, the case of “bits” building up the macroscopic surface (black-hole horizon), each bit contributing a quantum of area $l^2$ and having an internal (Hilbert-space) dimension $d_{\text{atom}} = \hat{d} = 2$. The main result, then, is the simple formula (7a) for Newton’s gravitational coupling constant $G$, with $\hat{d} = 2$ for the case of bits.

One possible physical interpretation uses the framework of Verlinde.2 In that framework,2,3 a finite-temperature system of bits on a holographic screen gives rise to Newtonian gravity (10) with the above-mentioned coupling constant $G$. However, in order to obtain an integer internal dimension $\hat{d}$, these bits must obey a nonstandard energy equipartition law (9), which may perhaps trace back to a type of nonstandard statistics,11 Most likely, the origin of this nonstandard behavior is some form of long-range interaction between the bits themselves.

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References

1. F.R. Klinkhamer, “Fundamental length scale of quantum spacetime foam,” JETP Lett. 86, 73 (2007), arXiv:gr-qc/0703009.
2. E.P. Verlinde, “On the origin of gravity and the laws of Newton,” JHEP 1104, 029 (2011) arXiv:1001.0785.
3. G. ’t Hooft, “The holographic principle: Opening lecture,” in: A. Zichichi (ed.), Basics and Highlights in Fundamental Physics, (World Scientific, Singapore, 2001), arXiv:hep-th/0003004.
4. J.D. Bekenstein, “Black holes and entropy,” Phys. Rev. D 7, 2333 (1973).
5. S.W. Hawking, “Particle creation by black holes,” Commun. Math. Phys. 43, 199 (1975); Erratum-ibid. 46, 206 (1976).
6. W.G. Unruh, “Notes on black hole evaporation,” Phys. Rev. D 14, 870 (1976).
7. S.W. Hawking, “Quantum black holes” and “Quantum cosmology” in: S.W. Hawking and R. Penrose, The Nature of Space and Time, (Princeton University Press, Princeton, 1996), Chaps. 3 and 5.
8. F.R. Klinkhamer, “Newtonian gravity as an entropic force: Towards a derivation of $G$,” Class. Quantum Grav. 28, 125003 (2011), arXiv:1006.2094
9. H. Sahlmann, “Newton’s constant from a minimal length: Additional models,” Class. Quantum Grav. 28, 015006 (2011), arXiv:1010.2650
10. J.A. Neto, “Fundamental constants, entropic gravity and nonextensive equipartition theorem,” arXiv:1101.2927v1.
11. C. Tsallis, “Possible generalization of Boltzmann–Gibbs statistics,” J. Statist. Phys. 52, 479 (1988).
12. C. Tsallis, “Nonextensive statistics: Theoretical, experimental and computational evidences and connections,” Braz. J. Phys. 29, 1 (1999).
13. A.R. Plastino and J.A.S. Lima, “Equipartition and virial theorems within generalized thermostatistical formalisms,” Phys. Lett. A 260, 46 (1999).
14. T. Padmanabhan, “Equipartition of energy in the horizon degrees of freedom and the emergence of gravity,” Mod. Phys. Lett. A 25, 1129 (2010), arXiv:0912.3165
15. P.J. Mohr, B.N. Taylor, and D.B. Newell, “CODATA recommended values of the fundamental physical constants: 2006,” Rev. Mod. Phys. 80, 633 (2008), arXiv:0801.0028
16. J. Engle, K. Noui, A. Perez and D. Pranzetti, “Black hole entropy from an $SU(2)$–invariant formulation of Type I isolated horizons,” Phys. Rev. D 82, 044050 (2010), arXiv:1006.0634
17. L. Modesto and A. Randono, “Entropic corrections to Newton’s law,” arXiv:1003.1998v1.
18. P. Nicolini, “Entropic force, noncommutative gravity and ungravity,” Phys. Rev. D 82, 044030 (2010), arXiv:1005.2996