Residual Representations of Spacetime

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Abstract

Spacetime is modelled by binary relations - by the classes of the automorphisms $GL(\mathbb{C}^2)$ of a complex 2-dimensional vector space with respect to the definite unitary subgroup $U(2)$. In extension of Feynman propagators for particle quantum fields representing only the tangent spacetime structure, global spacetime representations are given, formulated as residues using energy-momentum distributions with the invariants as singularities. The associated quantum fields are characterized by two invariant masses - for time and position - supplementing the one mass for the definite unitary particle sector with another mass for the indefinite unitary interaction sector without asymptotic particle interpretation.
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1 Introduction

Quantum theory starts with operations\(^3\). An experiment for quantum structures probes a ‘diagonalization’ of the operator under question, e.g. of a time and position translation or of a rotation or of a charge transformation, with the eigenvalues as possible experimental results, e.g. energy and momenta and mass or spin or a charge number resp. Therewith, I shall take the radical point of view that all relevant mathematical structures and tools used in quantum theories have to have an interpretation in terms of operations, of monoids, groups and algebras, especially of real Lie groups and Lie algebras, realized and represented as acting upon sets, especially upon complex vector spaces with a reality defining conjugation. Representation theory gives the irreducible and - for linear structures - also the nondecomposable action spaces. Almost all functions, relevant for physics, can be interpreted as arising from representation structures\(^3\).

Physical events represent spacetime operations, e.g. translations, rotations and boosts. A quantum mechanical dynamics, implemented by \(iH\) (Hamiltonian \(H\) with eigenvalues \(E \in \mathbb{R}\)) as basis for the time translation Lie algebra \(\mathbb{R}\), is a representation of the causal time group \(D(1) = \exp \mathbb{R}\), irreducible for \(e^{iE} \in U(1)\), e.g. for the harmonic oscillator or for creation and annihilation operators in quantum particle fields. In the Schrödinger picture the time representations in \(U(1)\) are realized on a Hilbert space with the scalar product (probability amplitudes) induced by the time representing \(U(1)\). The wave functions come as position translation representation matrix elements, e.g. the scattering and bound state wave functions \(\psi(r)\) in rotation symmetric problems with \(r\psi(r) \sim e^{\pm ir|Q|}\), \(e^{-r|Q|}\) as compact \(U(1)\) and noncompact \(D(1)\)-representations resp. of the radial translation monoid \(\mathbb{R}^+\). In quantum mechanics the time translation eigenvalue \(iE\) (energy \(E\)) and the position translation eigenvalue \(Q\) are in a unique correspondence: E.g., for a constant potential \(V_0\) with \(-\frac{Q^2}{2} = E - V_0\) the scattering case is given by \(E > V_0\) with imaginary eigenvalues \(\pm i|Q|\) and momentum \(|Q|\) whereas the bound states come with \(E < V_0\) - there \(|Q|\) cannot be interpreted as momentum.

In analogy to the dynamics for time \(D(1) = \exp \mathbb{R}\) the representations\(^2\) of the globally symmetric manifold \(D(2) = \exp \mathbb{R}^4\) as spacetime model\(^19, 20\) (discussed in more detail below) with the Minkowski translations as tangent space \(\mathbb{R}^4\) will be considered as possible candidates for a spacetime dynamics:

- **time dynamics**: \(\text{rep} D(1)\) with \(D(1) = \text{GL}(\mathbb{C})/U(1)\)
- **spacetime dynamics**: \(\text{rep} D(2)\) with \(D(2) \cong \text{GL}(\mathbb{C}^2)/U(2)\)

The spacetime manifold \(D(2) = D(1) \times SD(2)\) contains as factor for the causal group \(D(1)\) the rank 1 position manifold \(SD(2) \cong SO_0(1,3)/SO(3)\) with another Cartan subgroup \(SO_0(1,1) \cong \exp \mathbb{R}\). An independent realization of both factors in the Cartan subgroups \(D(1) \times SO_0(1,1)\) of the rank 2 spacetime manifold \(D(2)\) is characterized by two continuous invariants.

For particles with mass \(m\), the energy-momenta \((q_0, \vec{q})\) as eigenvalues for spacetime translations \((x_0, \vec{x})\) are on shell, i.e. \(q^2 = m^2\). With Wigner\(^14\),

\(^2\text{irrep} G\) and \(\text{rep} G\) denotes the (irreducible) representation classes of a group \(G\).

particle quantum fields implement definite unitarily the Poincaré Lie algebra with the mass \( m^2 = q_0^2 - \mathbf{q}^2 \) as the translation eigenvalue. In the following the off shell structures of a propagator, i.e. for \( q^2 \neq m^2 \), will be extended for a complete realization of rank 2 spacetime \( \mathbf{D}(2) \) with its two noncompact invariants.

Representation matrix elements\(^3\) of a real Lie group are analytic functions on this group
\[
D : G \longrightarrow \mathbb{C}, \quad g(x) \longmapsto D(x)
\]
e.g. \( \frac{2}{r} \sin r \) for compact spin \( \text{SU}(2) \) or \( \cos xm \) for compact axial rotations \( \text{U}(1) \) or both \( \cos xm \) and \( t \cosh xm \) for noncompact time \( \text{D}(1) \). According to the Peter-Weyl theorem\(^1, 4\), the span of the irreducible representation matrix elements of a compact Lie group is dense in the continuous functions on this group.

In a harmonic analysis, representation matrix elements of a group can be written as Fourier transforms of distributions of their Lie algebra forms, e.g. of energies or angular momenta values, where the representation characterizing invariants come as singularities, i.e. as poles of the distributions. This defines the concept of residual representations. In the following, familiar algebraic representation concepts\(^\ref{fn:1} \) like weights, invariants and Lie algebras are translated into the language of residual representations.

In analogy to Lie groups like the compact \( \text{U}(n) \) or the noncompact \( \text{D}(1) \) also symmetric spaces like the noncompact position manifold \( \text{SD}(2) \) and spacetime \( \text{D}(2) \) have linear representations which will be considered in analogy to the representations of the time group \( \text{D}(1) \). To construct residual representations of the rank 2 spacetime manifold \( \text{D}(2) \) distributions of the energy-momenta \( q \in \mathbb{R}^4 \) (tangent space forms) are used, supported by two invariant masses \( q^2 \in \{ m_0^2, m_3^2 \} \) characterizing the Cartan subgroup \( \text{D}(12) \times \text{SO}_0(1,1) \)-representations for time and position.

### 2 Quantum Representations of Time

A dynamics is a representation of time, realized in quantum mechanics by the quantization (anti-) commutators of the quantum algebra generating operators. In the simplest cases of a harmonic oscillator with Hamiltonian \( H = \frac{p^2}{2M} + m^2 M \frac{x^2}{2} \) for mass \( M \) and frequency \( m \) or of a free mass point with \( H = \frac{p^2}{2M} \) for frequency \( m \to 0 \) the time dependent commutation relations of the dual quantum algebra generating position-momentum pair \( (x, p) \) give the time representation matrix elements

\[
\text{D}(1) \ni e^t \longmapsto D(t) = \begin{pmatrix} [i \mathbf{p}, x] & [x, x] \\ [\mathbf{p}, p] & [x, -i \mathbf{p}] \end{pmatrix} (t) = \begin{pmatrix} \cos tm & iMm \sin tm & \frac{\sin tm}{\sin tm} \\ iMm \sin tm & \cos tm & \frac{-\sin tm}{\sin tm} \\ 0 & 0 & 1 \end{pmatrix} \in \text{SO}(2) \\
\end{pmatrix}
\]

with the shorthand notation \([a(s), b(t)]_\epsilon = [a, b]_\epsilon (t - s), \epsilon = \pm 1\), valid for all matrix elements. Those representations arise from the complex irreducible and

\(^3\)In the following the short ‘representation’ can stand for the more correct ‘representation matrix element(s)’.\(^1\)
nondecomposable time representations with creation and annihilation operator \((u, u^*)\) and nil- and eigenoperators\([15]\) (\(b, g, b^x, g^x\)) resp.

\[
D(1) \ni e^t \longmapsto \begin{cases} 
[u^*, u_c](t) = e^{tim} 
\begin{pmatrix}
1 & tiv \\
0 & 1 
\end{pmatrix} e^{tim} \in U(1) \\
[b^x, b_c](t) = e^{tim} 
\begin{pmatrix}
1 & tiv \\
0 & 1 
\end{pmatrix} e^{tim} \in U(1, 1)
\end{cases}
\]

The quantization opposite commutators implement the Lie algebra of the basic space endomorphisms, e.g. the Hamiltonians above. For the harmonic oscillator the \(U(1)\)-induced Fock form \(⟨...,⟩_φ\) of the time dependent anticommutators arises as time derivative of the quantization

\[
\begin{pmatrix}
⟨i[p, x]⟩_φ \\
⟨i[p, p]⟩_φ \\
⟨[x, i[p]]⟩_φ
\end{pmatrix}(t) = \begin{pmatrix}
i \sin tm \\
M m \cos tm \\
M m \sin tm
\end{pmatrix} = \begin{pmatrix}
\frac{1}{im} \cos tm \\
\frac{1}{im} \sin tm \\
\frac{1}{im} \sin tm
\end{pmatrix}
\]

For the general quantum mechanical case with \(iH = i\frac{[p^2]}{2M} + V(x)\) as basis for the represented Lie algebra\([\log D(1) \cong \mathbb{R}]\) the time \(D(1)\)-representation matrix elements as the ground state values \(⟨[a(s), b(t)]⟩_φ = ⟨[a, b]⟩_φ(t - s)\) of the position-momentum commutators can be computed from the imaginary and time translation antisymmetric position commutator

\[
⟨[x, x]⟩_φ(t) = \int_0^∞ dm^2 \mu(m^2)i \frac{\sin tm}{M m} (M m)^2
\]

with a spectral measure \(\mu(m^2)\) for the time translation eigenvalues \(m \in \mathbb{R}\) (frequencies, energies), e.g. \(\mu(m^2) = δ(m^2 - m_0^2)\) with \(m_0 > 0\) for oscillator and \(m_0 = 0\) for free mass point, and \(p = M \frac{d x}{d t}\)

\[
⟨\begin{pmatrix}
[p, x] \\
[x, p] \\
[x, -i[p]]
\end{pmatrix}⟩_φ(t) = \int_0^∞ dm^2 \mu(m^2)\begin{pmatrix}
\cos tm \\
\frac{1}{im} \cos tm \\
\frac{1}{im} \cos tm
\end{pmatrix} \in rep \mathbb{SO}(2)
\]

In the case of a compact time development, i.e. representations in \(U(1)\) or \(\mathbb{SO}(2)\), where there exists a basis of normalizable energy eigenvectors (for the oscillator build by the monomials of creation and annihilation operator), the energy measure is definite \(\mu(m^2) ≥ 0\).

## 3 Time and Position Translations

### 3.1 The Lie Groups for the Translations

Translations are formalized by additive groups (vector spaces) \(\mathbb{R}^n\). It will be convenient to introduce a distinguishing notation for the Lie group and the Lie algebra involved which have an isomorphic Abelian Lie group structure

\[
\begin{align*}
\text{Lie group} & \quad D(1) = \exp \mathbb{R} = \{e^x \mid x \in \mathbb{R}\} \\
\text{Lie algebra} & \quad \mathbb{R} = \log D(1) = \{x \mid x \in \mathbb{R}\}, \quad \exp \mathbb{R} \cong \mathbb{R}
\end{align*}
\]

The noncompact group \(D(1)\) as universal covering group is locally isomorphic to the compact one \(e^{iα} \in U(1) = \exp i\mathbb{R} \cong \mathbb{R}/\mathbb{Z}\) with Lie algebra \(\log U(1) = i\mathbb{R}\).

---

\(^4\)The Lie group to Lie algebra transition \(G \longrightarrow \log G\) is denoted with the logarithm \(\log\) as covariant functor.
The groups $U(1)$ and $D(1)$ are - as real 1-dimensional Lie groups - isomorphic to the axial rotations $SO(2)$ and the Procrustes$^5$ dilatation group $SO_0(1,1)$ resp., i.e. the 1-dimensional boosts

$$
\begin{align*}
\text{compact } U(1) & \cong SO(2) = \left\{ \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} \\
\text{noncompact } D(1) & \cong SO_0(1,1) = \left\{ \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \mid x \in \mathbb{R} \right\}
\end{align*}
$$

Those orthogonal groups with invariant bilinear forms of the 2-dimensional vector space they are acting upon, will be called selfdual representations$^6$ of $U(1)$ and $D(1)$ resp. with the obvious isomorphy (for $SO(2)$ only in the complex)

$$
\begin{align*}
\text{definite unitary: } SO(2) & \ni \begin{pmatrix} \cos \alpha & i \sin \alpha \\ i \sin \alpha & \cos \alpha \end{pmatrix} \cong \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \in SU(2) \\
\text{indefinite unitary: } SO_0(1,1) & \ni \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \cong \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \in SU(1,1)
\end{align*}
$$

### 3.2 Real Operations have Unitary Representations

The algebraic and topological completeness of the complex field $\mathbb{C}$ allows the definition of the transcendental number $\exp e$ involving ‘exponential completeness’ $\exp \mathbb{C} = \mathbb{C} \setminus \{0\}$ and, therewith, the exponential transition from local linear structures (tangent vector spaces, Lie algebras) to global possibly nonlinear structures (symmetric spaces, Lie groups). Therefore, I will consider representations on complex vector spaces only. The complex representations of the physically arising only real Lie groups or Lie algebras have to be unitary - definite $U(n)$ or indefinite $U(p,q)$, in order to recognize the realness also in the representation. Therewith, the complex numbers are always used together with the canonical conjugation, i.e. as the doubled real field $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$.

Only for one complex dimension unitarity is unique, characterized by the real Lie group $U(1) = \exp i\mathbb{R}$. The $n$ unitarities for $n$ complex dimensions go with the signature: E.g., in two dimensions the $U(2)$-conjugation of $2 \times 2$-matrices can be written as the familiar conjugate transposition which exchanges the elements of the skewdiagonal whereas the $U(1,1)$-conjugation can be written with an exchange of the diagonal elements

$$
\begin{align*}
\text{U(2)-conjugation: } & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \leftrightarrow \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} \\
\text{U(1,1)-conjugation: } & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
$$

### 3.3 NilDimensions for Noncompact Groups

Noncompact groups have reducible, but nondecomposable representations$^5$ where the representation space cannot be spanned by eigenvectors only

---

$^5$Procrustes in the Greek mythology either shranked or stretched his visitors - tall or short resp. - to death.

$^6$For a group and a Lie algebra dual representations on finite dimensional dual vector spaces are related to each other by inverse and negative transposition resp.
- there occur also nilvectors, i.e. principal vectors which are not eigenvectors. The linear operators involved have a Jordan triangular form with nontrivial off-diagonal entries.

The situation is characterized by the nondecomposable representations of the group $D(1)$ with an eigenvalue $m$ for $e^x \mapsto e^{xim}$ which comes multiplied with an automorphism of the representation space $V \cong \mathbb{C}^{1+N}$ and can be written with a nilcyclic matrix $M_N$ (nil-Hamiltonian), nilpotent to the power $N + 1$

$$D(1) \ni e^x \mapsto e^{xi(m+MN)} \cong e^{xim} \begin{pmatrix} 0 & \frac{(ix)^2}{2!} & \ldots & \frac{(ix)^N}{N!} \\ 0 & 0 & \ldots & \frac{(ix)^{N-1}}{(N-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & ix \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in GL(\mathbb{C}^{1+N})$$

$$(M_N)^N \neq 0, \quad (M_N)^{N+1} = 0, \quad M_N \cong \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

The natural number $N$ is called the nildimension with $1 + N$ the dimension of the nondecomposable representation. Irreducible representations have trivial nildimension $N = 0$ and $M_0 = 0$. For $N \geq 1$ the conjugation is indefinite, i.e. the group image is a subgroup of $U(1, 1), U(2, 1), U(2, 2)$ etc.

An example for nontrivial nildimensions in quantum mechanics is the radial part $\psi_{nL}$ of the bound state wave functions in the hydrogen atom: It is a linear combination of matrix elements $r \mapsto r e^{-r/k}$ of noncompact representations of the radial translations with eigenvalue $-\frac{1}{k}, k = n + L + 1$

$$\mathbb{R}^+ \ni r \mapsto D_{nL}(r) = r \psi_{nL}(r) \sim \left(\frac{2}{k} \right)^{L+1} \mathcal{L}_n^{2L+1}(\frac{2}{k} r) e^{-r/k}$$

with the Laguerre polynomials $\mathcal{L}$ as combinations of radial powers $r^N$.

An example for nontrivial nildimensions in quantum field theory is quantum electrodynamics where the nonparticle components of the $U(1)$-gauge field which come in addition to the left and right circularly polarized particle degrees of freedom (photons), i.e. the Coulomb force inducing degree of freedom and the so called gauge degree of freedom, are spacetime translation nilvectors\cite{16, 17}, i.e. principal vectors which are no eigenvectors. The dichotomy between particles and interaction degrees of freedom in the electromagnetic potential reflects the compact and noncompact Cartan subgroups in the Lorentz group $SO(2) \times SO_0(1, 1) \subset SO_0(1, 3)$, represented definite unitarily $SO(2) \rightarrow U(2)$ for the photons and indefinite unitarily $SO_0(1, 1) \rightarrow U(1, 1)$ for Coulomb and gauge degree of freedom. The nilpotency of the BRS-generator\cite{1} with power $N + 1 = 2$ has its origin in the time translation representation $D(1) \rightarrow U(1, 1)$ for the two nonparticle degrees of freedom with $M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ the nil-Hamiltonian which fulfills $M_1^2 = 0$. 
4 The Spacetime Representation Structure of Quantum Particle Fields

Particle fields are appropriate to describe free particles, they implement definite unitary representations of the Poincaré Lie algebra \[ \mathbb{SO}_0(1,3) \oplus \mathbb{R}^4 \]. A particle field, in the simplest case a hermitian scalar massive field \( \Phi \), \( m > 0 \), with creation and annihilation operators \((u, u^*)\)

\[
\Phi(x) = \int \frac{d^3q}{(2\pi)^3} \frac{\delta(q^2 - m^2)}{\sqrt{q^0}} e^{iqx} u(q) + e^{-iq} u^*(q), \quad q_0 = \sqrt{m^2 + q^2}
\]

\[
[u^*(\vec{p}), u(\vec{q})] = (2\pi)^3 \delta(\vec{q} - \vec{p}) = \{u^*(\vec{p}), u(\vec{q})\} = \langle u^*(\vec{p}) u(\vec{q}) \rangle
\]

is characterized by its quantization\[\footnote{The linear Minkowski spacetime parametrization is used in the notation for (anti)commutators \([A(y), B(x)]_{\pm} = [A, B]_{\pm}(x-y)\).}\], causally supported and on shell

\[
\frac{[\Phi, \Phi^\dagger](x)}{m} = i \frac{s(x|m)}{m} = \int \frac{d^3q}{(2\pi)^3} \epsilon(q_0) \delta(q^2 - m^2) e^{ixq} = 0 \text{ for } x^2 < 0
\]

and its Feynman propagator adding up the Fock form value of the quantization-opposite commutator, also on shell

\[
\frac{\{\Phi, \Phi^\dagger\}(x)}{m} = \frac{C(x|m)}{m} = \int \frac{d^3q}{(2\pi)^3} \delta(q^2 - m^2) e^{ixq}
\]

and the \( \epsilon(x_0) \)-multiplied quantization\[\footnote{The linear Minkowski spacetime parametrization is used in the notation for (anti)commutators \([A(y), B(x)]_{\pm} = [A, B]_{\pm}(x-y)\).}\] which has also off shell contributions, i.e. for \( q^2 \neq m^2 \)

\[
\frac{\epsilon(x_0) s(x|m)}{m} = \pm \frac{1}{\pi} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^0 + m^2} e^{ixq} \quad \text{(principal value P)}
\]

\[
\frac{\{\Phi, \Phi^\dagger\}(x) \pm \epsilon(x_0) [\Phi, \Phi^\dagger](x)}{m} = \pm \frac{1}{\pi} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^0 + m^2} e^{ixq}
\]

The harmonic contributions in the quantization

\[
i \frac{s(x|m)}{m} = \int \frac{d^3q}{(2\pi)^3} e^{ix_0 q_0} \epsilon(q_0) \vartheta(q_0^2 - m^2) \frac{\sin r \sqrt{q_0^2 - m^2}}{r}
\]

and in the Feynman propagator

\[
\frac{C(x|m)}{m} = \pm \frac{\epsilon(x_0) s(x|m)}{m} = \int \frac{d^3q}{(2\pi)^3} e^{ix_0 q_0} \left[ \vartheta(q_0^2 - m^2) \frac{\sin r \sqrt{q_0^2 - m^2}}{r} \pm i \vartheta(q_0^2 - m^2) \frac{\cos r \sqrt{q_0^2 - m^2}}{r} \pm i \vartheta(m^2 - q_0^2) e^{-r \sqrt{q_0^2 - m^2}} \right]
\]

show irreducible (definite unitary) time translation representation matrix elements

\[
\mathbb{R} \ni x_0 \mapsto e^{\pm i x_0 q_0} \in \mathbb{U}(1)
\]

With the polar coordinate position translation decomposition

\[
x \in \mathbb{R}^3 \cong \mathbb{R}^+ \times \mathbb{SO}(3)/\mathbb{SO}(2)
\]

and the geometrical Kepler factor \( \frac{1}{r} \) for the sphere surface \( \mathbb{SO}(3)/\mathbb{SO}(2) \)-distribution, the position radial translation monoid \( r \in \mathbb{R}^+ \) is represented by
(spherical Bessel function) with \( \sin r|\vec{q}| \) as matrix element of a compact group for the quantization \( s(x|m) \) and the Fock form function \( C(x|m) \). In the propagator contribution \( \epsilon(x_0)s(x|m) \) there arise the \( r = 0 \)-singular spherical Neumann function \( \cos r|\vec{q}| \) which contains \( \cos r|\vec{q}| \) as a compact position translation representation matrix element. The additional off shell induced Yukawa contributions displays a representation matrix element of the radial position translations in a noncompact (indefinite unitary) group

\[
\mathbb{R}^+ \ni r \mapsto \begin{cases} 
    e^{\pm r|\vec{q}|} & \in \text{SO}(2) \\
    e^{-r|\vec{Q}|} & \in \text{SO}_0(1,1)
\end{cases}
\]

The off shell contributions with the Yukawa interactions in the Feynman propagator are no definite unitary Poincaré Lie algebra representation matrix elements.

The time projection \( \int d^3x \) of quantization and Feynman propagator gives matrix elements for the representation of time translations in the rest system of a massive particle

\[
x_0 \mapsto \int d^3x \begin{pmatrix} C(x|m) \\ i s(x|m) \end{pmatrix} = \int dE \epsilon(E) \begin{pmatrix} E \\ m \end{pmatrix} \delta(E^2 - m^2) e^{x_0iE} = \begin{pmatrix} \cos x_0m \\ i \sin x_0m \\ i \sin |x_0|m \end{pmatrix}
\]

The analogue position projection \( \int dx_0 \)

\[
\vec{x} \mapsto 2\pi \int dx_0 \begin{pmatrix} i s(x|m) \\ C(x|m) \end{pmatrix} = \int \frac{dQ}{2} \begin{pmatrix} 0 \\ 0 \\ -r^2 \end{pmatrix} \delta(Q^2 - m^2) e^{-r|\vec{Q}|} = \begin{pmatrix} 0 \\ \cos r|\vec{Q}| \\ -r^2 \end{pmatrix}
\]

is nontrivial only for the off shell contributions with radial translation representation matrix element \( e^{-r|\vec{Q}|} \) in a noncompact group.

Particle fields display in the quantization \( s(x|m) \) and the Fock form \( C(x|m) \), both on shell \( q^2 = m^2 \), matrix elements of definite unitary representations for the translations. The off shell contributions in \( \epsilon(x_0)s(x|m) \) involve matrix elements for indefinite unitary representation matrix elements for position translations \( \mathbb{R}^3 \).

5 Homogeneous Models for Time, Position and Spacetime

5.1 Exponentiating Time Translations

The time translations as a real 1-dimensional vector space \( x_0 = \frac{\vec{x}_0}{|\vec{x}_0|} \in \mathbb{R} \) are isomorphic - as Lie group - to its exponent \( D(1) = \exp \mathbb{R} \), the time group. They constitute the noncompact part (modulus) of the full complex group, given by the phase classes

\[
\text{time: } \text{GL}(\mathbb{C})/\text{U}(1) = D(1) = \exp \mathbb{R} \cong \mathbb{R}
\]
5.2 Exponentiating Position Translations

In the semidirect Euclidean group \( \text{SO}(3) \times \mathbb{R}^3 \) the position translations as a real 3-dimensional vector space \( \mathbb{R}^3 \) are isomorphic - as vector space with rotation action - to the Lie algebra of the rotations \( \log \text{SO}(3) \cong \mathbb{R}^3 \). In the \( \text{SU}(2) \)-formulation, the rotations \( \text{SO}(3) \) are represented by the adjoint action of its covering group \( \text{SU}(2) \)

\[
\text{SO}(3) \times \mathbb{R}^3 \cong \text{SU}(2) \times \mathbb{R}^3,
\text{O.} \vec{x} \sim u \circ \vec{\sigma} \circ u^{-1}
\]

with \( \vec{\sigma} = \begin{pmatrix} x_3 & x_1 + ix_2 & x_1 - ix_2 \\ x_1 + ix_2 & x_3 & -x_3 \\ x_1 - ix_2 & -x_3 & x_3 \end{pmatrix} \)

\[u \in \text{SU}(2) \Rightarrow O^q_{\vec{a}} = \frac{1}{2} \text{tr} \sigma^a u \sigma^b u^{-1}, \quad O \in \text{SO}(3) \cong \text{SU}(2)/\{\pm 1_2\}\]

In the Pauli representation, the position translations are hermitian \( 2 \times 2 \)-matrices, i.e. representatives\(^\text{8}\) of the classes of all complex matrices \( \log \text{SL}(\mathbb{C}^2) \cong \mathbb{R}^3 \oplus (i\mathbb{R})^3 \) with respect to the special unitary ones \( \log \text{SU}(2) \cong (i\mathbb{R})^3 \)

\[\vec{\sigma} = (\vec{\sigma})^* \in \log \text{SL}(\mathbb{C}^2)/ \log \text{SU}(2)\]

The global position manifold arises by exponentiation, isomorphic as symmetric space to the classes of the Lorentz covering group \( \text{SL}(\mathbb{C}^2) \) with respect to the rotation covering group \( \text{SU}(2) \)

\[
\text{position: } \text{SL}(\mathbb{C}^2)/\text{SU}(2) \cong \text{SD}(2) = \exp \mathbb{R}^3 \cong \mathbb{R}^3
\]

The global symmetric space position \( \text{SD}(2) \) and its tangent vector space \( \mathbb{R}^3 \) have a manifold isomorphy only, \( \exp \mathbb{R}^3 \neq (\exp \mathbb{R})^3 \).

5.3 Exponentiating Spacetime Translations

In the Poincaré group \( \text{SO}_0(1, 3) \times \mathbb{R}^4 \) the translations \( \mathbb{R}^4 \) are not isomorphic to the Lie algebra of the Lorentz group \( \log \text{SO}_0(1, 3) \cong \mathbb{R}^6 \). In the \( \text{SL}(\mathbb{C}^2) \)-formulation, the Lorentz transformations \( \text{SO}_0(1, 3) \) are represented by the conjugate adjoint action of its covering group \( \text{SL}(\mathbb{C}^2) \)

\[
\text{SO}_0(1, 3) \times \mathbb{R}^4 \cong \text{SL}(\mathbb{C}^2) \times \mathbb{R}^4,
\Lambda. \vec{x} \sim s \circ \vec{x} \circ s^*
\]

with \( x = x_k \sigma^k = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \)

\[s \in \text{SL}(\mathbb{C}^2) \Rightarrow \Lambda^s_{\vec{x}} = \frac{1}{2} \text{tr} \sigma^k s \vec{\sigma}^j s^* \quad \Lambda \in \text{SO}_0(1, 3) \cong \text{SL}(\mathbb{C}^2)/\{\pm 1_2\}\]

with Weyl matrices \( \sigma^k = (1_2, \vec{\sigma}) = \vec{\sigma}_k \). In the Cartan representation, the spacetime translations are hermitian \( 2 \times 2 \)-matrices, i.e. representatives of the classes of all complex matrices \( \log \text{GL}(\mathbb{C}^2) \cong \mathbb{R}^4 \oplus (i\mathbb{R})^4 \) with respect to the unitary ones \( \log \text{U}(2) \cong (i\mathbb{R})^4 \)

\[x = x^* \in \log \text{GL}(\mathbb{C}^2)/ \log \text{U}(2)\]

Global spacetime arises by exponentiation and is given by the classes of the full group \( \text{GL}(\mathbb{C}^2) \) with respect to the unitary phases \( \text{U}(2) \), the moduli of \( \text{GL}(\mathbb{C}^2) \)

\[
\text{spacetime: } \text{GL}(\mathbb{C}^2)/\text{U}(2) \cong \text{D}(2) = \exp \mathbb{R}^4 \cong \mathbb{R}^4
\]

\(^8\text{The funny double element symbol means a representative of a coset, i.e. } g \in G/H \iff g \in gH \in G/H.\)
The causal structure of spacetime is the spectral order[1] of the $C^*$-algebra $\log GL(\mathbb{C}^2)$.

The noncompact symmetric space $D(2)$ has - analogue to its compact counterpart $U(2)$ with $U(2) = U(1_2) \circ SU(2)$ - a product decomposition into Abelian causal time group $D(1_2)$ and real 3-dimensional position (boost) manifold $SD(2)$

$$D(2) = D(1_2) \times SD(2), \quad SD(2) \cong SL(\mathbb{C}^2)/SU(2)$$

Both symmetric spaces have real rank 2 - also indicated in the notation $U(2)$ and $D(2)$ - which reflects both the number of independent invariants and the dimension of a maximal Abelian Cartan subgroup (flat submanifold[7]), arising as factor of the 2-sphere $SO(3)/SO(2)$ in the polar decomposition

$$U(2) = U(1_2) \circ SU(2) \cong U(1) \circ SO(2) \times SO(3)/SO(2)$$
$$D(2) = D(1_2) \times SD(2) \cong D(1) \times SO_0(1, 1) \times SO(3)/SO(2)$$

For the decomposition of the real 4-dimensional tangent spaces (Lie algebra for $U(2)$) with the Lie algebra of the Cartan subgroup the sphere factor remains unchanged

$$\log U(2) = \log U(1_2) \oplus \log SU(2)$$
$$\cong \log U(1) \oplus [\log SO(2) \times SO(3)/SO(2)]$$

$$\log D(2) = \log D(1_2) \oplus \log SD(2)$$
$$\cong \log D(1) \oplus [\log SO_0(1, 1) \times SO(3)/SO(2)]$$

The representations of noncompact spacetime $D(2)$ and compact internal group $U(2)$ are characterized by two invariants from a continuous spectrum for a Cartan subgroup $D(1) \times SO_0(1, 1)$ and from a discrete spectrum for a Cartan subgroup $U(1) \circ SO(2)$ resp. Minkowski spacetime $\mathbb{R}^4$ in the Cartan representation by $U(2)$-hermitian $2 \times 2$-matrices has the familiar conjugate adjoint $GL(\mathbb{C}^2)$-transformation behaviour to be compared with the adjoint action of the compact group $U(2)$ on its Lie algebra $\log U(2) \cong (i\mathbb{R})^4$

$$g \in GL(\mathbb{C}^2), \quad x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \in \log D(2) \quad \Rightarrow \quad x \longmapsto g \circ x \circ g^*$$
$$u \in U(2), \quad i\alpha = i\begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \in \log U(2) \quad \Rightarrow \quad i\alpha \longmapsto u \circ i\alpha \circ u^*$$
$$u^* = u^{-1}$$

However, in contrast to the decomposition of the $U(2)$-Lie algebra into Abelian $U(1_2)$ and simple $SU(2)$-contribution, compatible with the adjoint $U(2)$-action, the decomposition of spacetime $D(2)$ and its tangent space into time and position is not compatible with the action of the Lorentz group

$$u \in U(2), \quad \log U(2) \ni i\alpha = i\alpha_0 1_2 + i\tilde{\alpha} \bar{\sigma}, \quad \left\{ \begin{array}{l} u \circ i\alpha_0 1_2 \circ u^* \in \log U(1_2) \\ u \circ i\tilde{\alpha} \bar{\sigma} \circ u^* \in \log SU(2) \end{array} \right.$$
Both symmetric spaces are parametrizable by exponentiating the tangent space, e.g. in the polar Cartan decomposition

\[
\log U(2) \ni i\alpha = u(\frac{\alpha}{|\alpha|}) \circ i(\alpha_0 1_2 + |\alpha|\sigma_3) \circ u^*(\frac{\alpha}{|\alpha|}) \\
\Rightarrow \quad \exp i\alpha = u(\frac{\alpha}{|\alpha|}) \circ e^{i(\alpha_0 1_2 + |\alpha|\sigma_3)} \circ u^*(\frac{\alpha}{|\alpha|}) \in U(2)
\]

\[
\log D(2) \ni x = u(\frac{x}{|x|}) \circ (x_0 1_2 + r \sigma_3) \circ u^*(\frac{x}{|x|}) \quad r = |\vec{x}|
\Rightarrow \quad \exp x = u(\frac{x}{|x|}) \circ e^{x_0 1_2 + r \sigma_3} \circ u^*(\frac{x}{|x|}) \in D(2)
\]

The diagonalization of \(D(2)\) and \(U(2)\) with the sphere operations

\[
u(\frac{x}{r}) = \left( \begin{array}{cc}
\cos \frac{\theta}{2} & -e^{-i\varphi} \sin \frac{\theta}{2} \\
 e^{i\varphi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{array} \right) \in SU(2)/U(1) \cong SO(3)/SO(2)
\]
defines \(\{i\alpha_0, i|\alpha|\}\) as Cartan coordinates for the internal group and \(\{x_0, r\}\) (time and radial translations) as Cartan coordinates for spacetime.

Similar to the local-global group isomorphism for time \(\mathbb{R} \cong \exp \mathbb{R} = D(1)\) one has the manifold isomorphism for spacetime \(\mathbb{R}^4 \cong \exp \mathbb{R}^4 = D(2)\). Via their embedding as future cones \(D(1)\) and \(D(2)\) are parametrizable with tangent space \(\mathbb{R}\) and \(\mathbb{R}^4\) coordinates

\[
t \in \mathbb{R} \Rightarrow D(1) \ni e^t = e(s)s \quad \in \mathbb{R}^+ \quad \text{with} \quad s \in \mathbb{R}, \quad s^2 = e^{2t}
\]

\[
x \in \mathbb{R}^4 \Rightarrow D(2) \ni e^x = e(y_0)\vartheta(y^2)y \quad \in (\mathbb{R}^4)^+ \quad \text{with} \quad y \in \mathbb{R}^4, \quad \left\{ \begin{array}{c}
y_0^2 = e^{2x_0} \\
|\vec{y}| = e^{r}
\end{array} \right.
\]

### 5.4 Time in Spacetime

A dynamics in quantum mechanics arises from representations of the time group \(D(1) \cong \exp \mathbb{R}\) whose representation spaces are realized in the Schrödinger picture by wave functions depending on position translations. The quantum mechanical relevant time structure is a proper substructure of spacetime, modeled by the homogeneous space \(D(2) \cong GL(\mathbb{C}^2)/U(2)\) and represented by quantum fields. The quantum mechanical energy eigenstates for compact \(D(1)\)-representations are embedded as spacetime particles. The strict future cone with dimension four in flat spacetime being isomorphic to nonlinear spacetime \(D(2)\) contains not only the totally ordered 1-dimensional causal subgroup \(D(1)\), it leaves room for a 3-dimensional position submanifold \(SD(2)\) whose noncompact dilatations \(SO_0(1,1)\) characterize spacetime interactions. The particle contributions, unitarily representing \(D(1)\), have to be supplemented in relativistic quantum theories by nonparticle ones to implement genuine \(SO_0(1,1)\)-representations. The nonparticle contributions are a genuine intrinsic feature of spacetime \(D(2)\) without analogue in quantum mechanics. There the interactions, e.g. the Coulomb potential for atoms, have to be put in by hand.
$$\begin{array}{|c|c|c|}
\hline
\text{time} & \text{spacetime} \\
\hline
\text{D}(1) = \text{GL}(\mathbb{C})/U(1) & \text{D}(2) \cong \text{GL}(\mathbb{C}^2)/U(2) \\
\hline
\text{quantum theory} & \text{quantum fields} \\
\text{quantum mechanics} & \text{quantum fields} \\
\hline
\text{Cartan subgroup} & \text{full group} \\
\text{D}(1) & \text{GL}(\mathbb{C}) \\
\hline
\text{tangent space} & \text{tangent space} \\
\text{(translations)} & \text{tangent space} \\
\mathbb{R} & \mathbb{R}^4 \\
\hline
\text{future} & \text{future} \\
\mathbb{R}^+ & \mathbb{R}^4 \\
\hline
\text{particles} & \text{particles} \\
\text{(states)} & \text{particles} \\
\text{D}(1) \rightarrow U(1) & \text{D}(1) \rightarrow U(1) \\
\hline
\text{interactions} & \text{interactions} \\
\text{not intrinsic} & \text{SO}_0(1, 1) \rightarrow U(1, 1) \\
\hline
\end{array}$$

6 Two Continuous Invariants for Spacetime Representations

Since Yukawa, the unification of a causal time development, characterized by a particle mass $m_0 \geq 0$ with a position interaction, characterized by a range $1/m_3$, $m_3 \geq 0$, in one spacetime Klein-Gordon equation for an $\epsilon(x_0)$-multiplied quantization distribution with one mass $m \geq 0$

$$\left\{ \begin{array}{l}
\frac{d^2}{dt^2} + m_0^2 = 2\delta(t) \\
-\frac{\partial^2}{\partial \vec{x}^2} + m_3^2 = 2\delta(\vec{x})
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
(\partial^2 + m^2)\epsilon(x_0)\frac{s(x|m)}{m} = 2\delta(x)
\end{array} \right\}$$

seems to be an obvious relativistic bonus - all interactions can be interpreted as particle induced.

Particle fields with a Dirac energy-momentum distribution in their quantization

$$i\mathbf{s}(x|m) = \int \frac{d^3q}{(2\pi)^3} \epsilon(q_0)m\delta(q^2 - m^2)e^{xq}$$

give by position integration representation matrix elements of the Abelian time group $\text{D}(1) \cong \exp \mathbb{R}$ in $\text{SO}(2)$

$$\begin{array}{c}
\text{D}(1) \rightarrow \mathbb{C} \\
e^{x_0} \mapsto \int d^3x \ i\mathbf{s}(x|m) = \int dE \ m\epsilon(E)\delta(E^2 - m^2) \ e^{x_0E} = i \sin x_0m \\
\end{array}$$

The appropriate distribution for a representation of the position symmetric space $\text{SD}(2) \cong \exp \mathbb{R}^3$ arises from a derived energy-momentum Dirac distribution

$$\int \frac{d^3q}{m} \frac{i\mathbf{s}(x|m)}{m} = \int \frac{d^3q}{m} \epsilon(q_0)\delta(q^2 - m^2) e^{xq}$$

Time integration leads to a Dirac distribution for the invariant and to $\text{SD}(2)$-representation matrix elements in $\text{SO}_0(1, 1)$

$$\begin{array}{c}
\text{SD}(2) \rightarrow \mathbb{C} \\
e^{-\pi} \mapsto 4\pi \int dx_0 \epsilon(x_0)G(x|m) = \int dQ \ m\delta(Q^2 - m^2) \ e^{-r|Q|} = e^{-r\pi}
\end{array}$$
The Dirac energy-momentum distribution for time with characterizing 2nd order differential equation in contrast to the derived distribution for position with characterizing 4th order differential equation
\[
\left(\frac{d^2}{dt^2} + m^2\right)e^{i\tau t} = 2\delta(t), \quad \left(-\frac{\partial^2}{\partial x^2} + m^2\right)^2 e^{-r x} = 2\delta(\vec{x})
\]
reflect the different dimensions 1 and 3 of the time group \(D(1)\) and the position manifold \(SD(2)\) resp.

The association of energy-momentum singularities to representation invariants for \(D(1)\) (time) and \(SD(2)\) (position) resp. is blurred since a decomposition of the spacetime tangent Minkowski translations \(\mathbb{R}^4 \ni x = 1_2 x_0 + \vec{\sigma} \vec{x}\) into time and position translations is not compatible with the action of the Lorentz group \(SO_0(1,3)\). The Dirac distribution has also a nontrivial projection for the position \(SD(2)\) structure

\[
2\pi \int dx_0 \epsilon(x_0)s(x|m) = m e^{-r m}\frac{1}{\tau}
\]
and the derived Dirac distribution a nontrivial projection for time \(D(1)\) representations

\[
\int d^3x \text{is} \text{dip}(x|m) = i \frac{\sin x_0 m - x_0 m \cos x_0 m}{2m^2}
\]
The position projection of the Dirac distribution leads to a Yukawa force which is no matrix element of an \(SD(2)\)-representation - only of its tangent position translations \(\mathbb{R}^3\). The time projection of the derived Dirac distribution leads to matrix elements of reducible nondecomposable \(D(1)\)-representations.

Related to the two Cartan coordinates \(\{x_0, r\}\) which reflect the rank 2 of the noncompact homogeneous manifold \(D(2)\), i.e. two Abelian subgroups \(D(1_2)\) (time) and \(SO_0(1,1)\) as a dilatation subgroup of the position manifold \(SD(2)\), two invariants \(\{m_0^2, m_3^2\}\) have to characterize the \(D(2)\)-representations. The definite unitary representations \(D(1_2) \ni e^{x_0 1_2} \mapsto e^{\pm x_0 i m_0} \in U(1)\) are characterized by a particle mass \(m_0^2\). A second mass \(m_3^2\) characterizes the indefinite unitary representation \(SO_0(1,1) \ni e^{\pm r} \mapsto e^{\mp r m_3} \in SU(1,1)\) with an interaction range \(\frac{1}{m_3}\) and without particle asymptotics. There is no group theoretical reason to identify both scales \(m_0^2 = m_3^2\) - in general, the representations of spacetime \(D(2)\) come with two different scales whose ratio \(\frac{m_3^2}{m_0^2}\) is a representation characteristic physically important constant. The ratio of the characterizing invariants for particle and interaction should be seen in analogy to the relative normalization of time and position translations \(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) as given with the speed of light \(c^2 = \frac{\ell^2}{r}\).

7 Residual Representations

Before the definition of residual representations in general their structure will be exemplified in the familiar example of the compact and noncompact abelian groups \(U(1)\) and \(D(1)\).
7.1 Residual U(1) × D(1)-Representations

An irreducible representation of the complex Abelian group exp C can be written as residue of its eigenvalue by using the complex Lie algebra forms Q ∈ I C exp I C ∋ ez → eζ = ∫ dQ 2iπ \frac{1}{Q - \zeta} e^{\zeta Q}, \zeta \in \text{irrep exp C} \cong C.

which, with the canonical conjugation, gives for the irreducible U(1) and D(1)-representations, necessarily in U(1)

\[ U(1) \ni e^{i \alpha} \mapsto e^{i \alpha Z} = \int dq (q - Z) e^{i \alpha q} \in U(1) \quad Z \in \text{irrep U(1)} \cong \mathbb{Z} \]

\[ D(1) \ni e^t \mapsto e^{tim} = \int dq (q - m) e^{ti q} \in U(1) \quad im \in \text{irrep D(1)} \cong i \mathbb{R} \]

with the neutral representations for Z = 0 and m = 0 resp. The integrations for the compact and noncompact group are related to each other via the Lie algebras and their forms by multiplication with the imaginary unit i

for compact U(1) \((i \alpha, q) \leftrightarrow (t, iq)\) for noncompact D(1)

Measures of the integer winding numbers Z as invariants of the compact group U(1) lead to Fourier series as measured U(1)-representations

\[ \mu : \text{irrep U(1)} \rightarrow \mathbb{R}, \quad Z \mapsto \mu(Z) \]

\[ \text{meas irrep U(1)} \ni \mu \mapsto D^\mu \in \text{rep U(1)} \]

\[ U(1) \ni e^{i \alpha} \mapsto D^\mu(\alpha) = \sum_{Z \in \mathbb{Z}} \mu(Z) e^{i \alpha Z} \]

The continuous irreducible representation classes for D(1) characterized by imaginary numbers im have Lebesque measure dm based real valued measures giving rise to Fourier integrals as measured D(1)-representations

\[ \mu : \text{irrep D(1)} \rightarrow \mathbb{R}, \quad m \mapsto \mu(m) \]

\[ \text{meas irrep D(1)} \ni \mu \mapsto D^\mu \in \text{rep D(1)} \]

\[ D(1) \ni e^t \mapsto D^\mu(t) = \int dm \mu(m) e^{tim} \]

where also matrix elements of reducible nondecomposable representations may occur by using derivatives with respect to the invariant

\[ \mu(m) = \sum_{N=0,1,...} \mu_N(m) \left( \frac{d}{dm} \right)^N \]

7.2 The Definition of Residual Representations

Residual representations are complex functions on a real finite dimensional symmetric space G, e.g. a Lie group, with tangent space (Lie algebra) log G \cong \mathbb{R}^n, as above for U(1) and D(1) and in the following for SU(2) and SL(\mathbb{C}^2)
and generalized to the position manifold \( \text{SD}(2) \) and the spacetime manifold \( D(2) \).

The equivalence classes \( \text{irrep} G \) of the irreducible \( G \)-representations are characterizable by invariants, taken from a rational spectrum for a compact and from an also continuous spectrum for a noncompact Cartan subgroup. The weights (eigenvalues) for the symmetric space \( G \) are a submodule of the linear forms \( q \in (\log G)^T \) of the tangent space \( x \in \log G \). The invariants \( \{ I_1, \ldots, I_r \} \), characterizing an irreducible representation, are related to multilinear tangent space forms (monomials in the weights). Appropriate measures \( d^n q I(q) \) of the linear forms, which can be written with a Lebesgue measure basis and a distribution of the tangent space forms \( (\log G)^T \approx \mathbb{R}^n \) lead to matrix elements of irreducible symmetric space representations

\[
I : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad q \mapsto I(q) \\
D : \text{meas} \mathbb{R}^n \longrightarrow \text{irrep} G, \quad I \mapsto D^I \\
D^I : G \longrightarrow \mathbb{C}, \quad g(x) \mapsto D^I(x) = \int d^n q I(q) e^{ixq}
\]

The complex generalized functions \( I(q) \) have poles at the values for the invariants characterizing an irreducible representation, the distributions come as quotients of two polynomials \( I(q) = \frac{P_N(q)}{P_D(q)} \). \( D^I \) is called a residual representation of \( G \) with \( I(q) \) a residual group distribution.

Measured representations for a symmetric space (Lie group) \( G \) integrate irreducible \( G \)-representations with a measure \( d^r \mu(\mu) \) of the invariants

\[
\mu : \text{irrep} G \longrightarrow \mathbb{R}, \quad I \mapsto \mu(I) \\
D : \text{meas} \text{irrep} G \longrightarrow \text{rep} G, \quad \mu \mapsto D^\mu \\
D^\mu : G \longrightarrow \mathbb{C}, \quad g(x) \mapsto D^\mu(x) = \int d^r \mu(\mu) D^I(x)
\]

The product in the algebra of the representation classes \( \text{rep} G \) is implemented via the convolution of the distributions for the matrix elements of the product representation

\[
D^{I_1} \otimes D^{I_2} = D^{I_1*I_2}
\]

In the following, these general structures will be concretized for the groups and symmetric spaces relevant for the spacetime model \( D(2) \).

### 7.3 Residual \( \text{SO}(2) \times \text{SO}_0(1,1) \)-Representations

The real Abelian group \( \text{SO}(2) \times \text{SO}_0(1,1) \) has its irreducible selfdual complex representations in the two types of 2-dimensional unitary groups, the definite unitary \( \text{SU}(2) \) or the indefinite unitary \( \text{SU}(1,1) \)

\[
\text{SO}(2) \times \text{SO}_0(1,1) \longrightarrow \begin{cases} 
\text{SO}(2) \subset \text{SU}(2) \\
\text{SO}_0(1,1) \subset \text{SU}(1,1) 
\end{cases} \\
e^{(i\alpha + x)\sigma^3} \longmapsto e^{(i\alpha Z + x\delta)\sigma^3}
\]

\(^9\)The linear forms (dual space) of a vector space \( V \) are denoted by \( V^T \).
The unitary groups $\text{SU}(2)$ and $\text{SU}(1,1)$ define the weights $(Z, \delta)$ of the principal (compact) and supplementary (noncompact) representations resp.

The principal $\text{SO}(2) \times \text{SO}_0(1,1)$-weights coincide with the $\text{U}(1) \times \text{D}(1)$-weights $\mathbb{Z} \times i\mathbb{R}$. An integer eigenvalue pair $\{\pm Z\}$ characterizes a selfdual $\text{SO}(2)$-representation

$$\text{SO}(2) \ni \begin{pmatrix} \cos \alpha & i \sin \alpha \\ i \sin \alpha & \cos \alpha \end{pmatrix} \mapsto \begin{pmatrix} \cos \alpha Z & i \sin \alpha Z \\ i \sin \alpha Z & \cos \alpha Z \end{pmatrix} \cong \begin{pmatrix} e^{i\alpha Z} & 0 \\ 0 & e^{-i\alpha Z} \end{pmatrix} \in \text{SU}(2)$$

leading to a quadratic natural number valued invariant $Z^2$. An imaginary continuous eigenvalue pair $\{\pm im\}$ characterizes a selfdual compact $\text{SO}_0(1,1)$-representation

$$\text{SO}_0(1,1) \ni \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \mapsto \begin{pmatrix} \cosh x m & \sinh x m \\ \sinh x m & \cosh x m \end{pmatrix} \cong \begin{pmatrix} e^{x m} & 0 \\ 0 & e^{-x m} \end{pmatrix} \in \text{SU}(1,1)$$

with a continuous positive invariant $m^2 \geq 0$

weights $\text{SO}(2) = \{Z\} \cong \mathbb{Z}$, irrep $\text{SU}(2) = \{|Z|\} \cong \mathbb{N}_0$

weights $^{(2,0)}\text{SO}_0(1,1) = \{im\} \cong i\mathbb{R}$, irrep $^{(2,0)}\text{SO}_0(1,1) = \{m^2\} \cong \mathbb{R}^+$

The new real $\text{SO}_0(1,1)$-weights $m \in \mathbb{R}$ (supplementary) in contrast to the imaginary principal weights $im \in i\mathbb{R}$ above come for dimensions $n \geq 2$ with the possibility of indefinite unitary groups. A supplementary $\text{SO}_0(1,1)$-representation is characterized by a real continuous eigenvalue pair $\{\pm m\}$

$$\text{SO}_0(1,1) \ni \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \mapsto \begin{pmatrix} \cosh x m & \sinh x m \\ \sinh x m & \cosh x m \end{pmatrix} \cong \begin{pmatrix} e^{x m} & 0 \\ 0 & e^{-x m} \end{pmatrix} \in \text{SU}(1,1)$$

with a continuous negative definite invariant

weights $^{(1,1)}\text{SO}_0(1,1) = \{m\} = \mathbb{R}$, irrep $^{(1,1)}\text{SO}_0(1,1) = \{-m^2\} \cong \mathbb{R}^-$

Residual representations in $\text{SO}(2)$ (principal) with invariants $m^2 \in \mathbb{R}^+$ can be formulated by distributions with the $q$-integration deformed as prescribed by $q^2 \mp io$ which for an undeformed integration gives singularities at $m^2 \pm io = (|m| \pm io)^2$

$$e^{\pm i|tm|} = \int d^3 q \ [m^2]_\pm^0(q) e^{itq}, \ [m^2]_\pm^0(q) = \pm \frac{1}{i\pi} \frac{|m|}{q^2 \pm io - m^2}$$

for $\text{SO}(2), \text{SO}_0(1,1) \longrightarrow \text{SU}(2), \ m \in (\mathbb{Z}, \mathbb{R})$

Residual representations in $\text{SU}(1,1)$ (supplementary) with invariants $-m^2 \in \mathbb{R}^-$ are obtained from residual representations in $\text{SU}(2)$ (principal) by the real-imaginary exchange $(it, q) \leftrightarrow (x, iq)$

$$e^{-|xm|} = \int d^3 q \ [-m^2]_\pm^0(q) e^{-xiq}, \ [-m^2]_\pm^0(q) = \pm \frac{1}{i\pi} \frac{|m|}{q^2 \mp io - m^2}$$

for $\text{SO}_0(1,1) \longrightarrow \text{SU}(1,1), \ m \in \mathbb{R}$
In the transition from the compact to the noncompact representation structure the invariant $±i|m|$ has to be replaced by $−|m|$ for $SO(2) ⊂ SU(2)$ $±i|m| ↔ −|m|$ for $SO_0(1, 1) ⊂ SD(2)$

for $SO(2) ⊂ SU(2)$

The matrix elements for the representations in $SO(2)$ and $SO_0(1, 1)$ fulfill the 2nd order differential equations

$$(\frac{d^2}{dt^2} + m^2)e^{±i|m| t} = ±2i|m|δ(t), \quad (\frac{d^2}{dx^2} - m^2)e^{−|m|x} = −2|m|δ(x)$$

The product representations arise by convolution - for $SO(2)$ with equal type, either $+io$ or $−io$ - with the supindices $\{1, 0\}$ adding up modulo 2, e.g.

$$\begin{align*}
[\frac{m^2}{1}]_{±} & \cdot [\frac{m^2}{2}]_{±} = [\frac{m^2}{1}]_{±} \\
[−\frac{m^2}{1}]_{±} & \cdot [−\frac{m^2}{2}]_{±} = [−\frac{m^2}{1}]_{±}
\end{align*}$$

$$|m_+| = |m_1| + |m_2|$$

With the convolution the distributions

$$\begin{align*}
\text{irrep } SO(2) & = \{q \mapsto [Z^2]_{±} (q) = \frac{1}{2π} \frac{q}{q^2 ± 10 − Z^2} | Z \in \mathbb{Z}\} \\
\text{irrep }^{(2,0)} SO_0(1, 1) & = \{q \mapsto [m^2]_{±} (q) = \frac{1}{2π} \frac{q}{q^2 ± 10 − m^2} | m \in \mathbb{R}\} \\
\text{irrep }^{(1,1)} SO_0(1, 1) & = \{q \mapsto [−m^2]_{±} (q) = \frac{1}{2π} \frac{q}{q^2 ± m^2} | m \in \mathbb{R}\}
\end{align*}$$

generate the compact and noncompact selfdual Abelian representations resp. The neutral representations arise for trivial invariant.

7.4 Residual Representations for Spin $SU(2)$

If the compact group $SO(2)$ comes as Cartan subgroup in the special group $e^{−i\vec{q}.\vec{x}} \in SU(2)$ with the Cartan polar decomposition

$$SU(2) \cong SO(2) \times SO(3)/SO(2)$$

residual representations employ the forms $\vec{q} \in \mathbb{R}^3$ of the tangent Lie algebra log$SU(2)$ (angular momenta) with the singularities of the distributions determined by the values of the invariant bilinear Killing form $q^2$ as singularity location of a dipole

$$\text{for } SO(2) \subset SU(2) : \quad e^{±ir|m|} = \int d^3q \quad [0, m^2]_{±}(\vec{q})e^{−i\vec{q}.\vec{x}}$$

$$\quad [0, m^2]_{±}(\vec{q}) = \frac{1}{2πi} \frac{|m|}{(q^2 ± io − m^2)^2} , \quad m \in \mathbb{R}$$

This scalar representation and similar integrals can be obtained by derivations with respect to the invariant $m^2$ and the Lie parameter $\vec{x}$ from the in- and outgoing spherical waves

$$\frac{d^3q}{2π^2q^2 ± io − m^2}e^{−i\vec{q}.\vec{x}} = \frac{e^{±ir|m|}}{r} , \quad m \in \mathbb{R}, \quad \vec{x} ≠ 0$$

$$\frac{∂}{∂m^2} = \frac{1}{2|m|\frac{∂}{∂m}}, \quad \frac{∂}{∂x} = \frac{1}{r} \frac{∂}{∂x}, \quad \left(\frac{∂}{∂x^2} + m^2\right)\frac{e^{±ir|m|}}{r} = 4πδ(\vec{x})$$

which, however, are no $SU(2)$-representation matrix elements because of the Lie parameter $\vec{x} = 0$ singularity.
The scalar matrix elements fulfill 4th order differential equations

\[(\frac{\partial^2}{\partial x^2} + m^2)^2 e^{\pm ir|m|} = \mp 8\pi i |m| \delta(\vec{x})\]

Vector valued distributions represent nontrivially the 2-sphere \(SO(3)/SO(2)\)

\[-\frac{\vec{x}}{r} e^{\pm ir|m|} = \int d^3q [1, m^2]_\pm (\vec{q}) e^{-i\vec{q} \vec{x}} = \int d^3q \frac{\vec{q}}{r} \frac{q}{(q^2 + io - m^2)^2} e^{-i\vec{q} \vec{x}}, \quad m \in \mathbb{R}
\]

leading to the matrix elements of the defining Pauli representation

\[
\begin{cases}
\int d^3q [0, 1]_+ (\vec{q}) e^{-i\vec{q} \vec{x}} = e^{\pm ir} \\
\int d^3q [1, 1]_+ (\vec{q}) e^{-i\vec{q} \vec{x}} = -\frac{\vec{x}}{r} e^{\pm ir}
\end{cases}
\]

\(\longleftrightarrow e^{-i\vec{q} \vec{x}} = 1_2 \cos r - \frac{\vec{x}}{r} \sin r\)

The spherical dependence \(\frac{\vec{x}}{r}\) replaces the \(\epsilon(x)\)-dependence for \(SO(2)\).

With the Lie algebra additive convolution product of the distributions for the irreducible residual \(SU(2)\)-representations

\[
\text{irrep } SU(2) = \{ \vec{q} \mapsto [1, m^2]_\pm (\vec{q}) = \frac{1}{i\pi^2} \frac{\vec{q}}{r^2 - m^2} | |m| = 2J \in \mathbb{N}_0 \}
\]

involving the neutral representation for trivial invariant \(m = 0\) one can combine the matrix elements for all other representations, e.g. the scalar ones with \(|m_1| + |m_2| = |m_+|\)

\[
\frac{x^a}{r} e^{\pm ir|m_1|} \delta_{ab} \frac{x^b}{r} e^{\pm ir|m_2|} = e^{\pm ir|m_+|}
\]

\[
= (\frac{1}{i\pi^2})^2 \int d^3q_1 d^3q_2 \frac{q^a_1}{(q^2 + io - m_1^2)^2} \delta(\vec{q}_1 + \vec{q}_2 - \vec{x}) \frac{q^b_2}{(q^2 + io - m_2^2)^2} = \pm \frac{1}{(q^2 + io - m_+^2)^2}
\]

or for the adjoint representation

\[
\delta_{ab} 2r + \frac{x^a x^b}{r^2} (1 - \cos 2r) + \epsilon_{abc} \frac{x^c}{r} \sin 2r
\]

which arises for \(|m_+| = |m_1| + |m_2| = 2\)

\[
\frac{x^a}{r} e^{\pm ir|m_1|} \frac{x^b}{r} e^{\pm ir|m_2|} = \frac{x^a x^b}{r^2} e^{\pm ir|m_+|}
\]

\[
[1, m^2]_\pm \quad \quad \quad [1, m^2]_\pm (\vec{q}) = \left(\frac{1}{i\pi^2}\right)^2 \int d^3q_1 d^3q_2 \frac{q^a_1}{(q^2 + io - m_1^2)^2} \delta(\vec{q}_1 + \vec{q}_2 - \vec{q}) \frac{q^b_2}{(q^2 + io - m_2^2)^2}
\]

In general, the matrix elements of \(SU(2)\)-representations come as products of a homogeneous polynomial (spherical harmonics) of degree \(2J\) for the sphere \(SO(3)/SO(2)\)-representation and an exponential for the Cartan subgroup \(SU(2)\) with winding numbers \(\pm 2J\)

\[
\{ [\vec{x}]^{2J} e^{\pm ir2J} | 2J' \in \mathbb{N}_0, \ 2J \in \mathbb{N}_0 \}
\]

\[
[\vec{x}]^0 = \{ 1 \}, \quad [\vec{x}]^1 = \{ \frac{x^a}{r} | a = 1, 2, 3 \}, \quad [\vec{x}]^2 = \{ \frac{x^a x^b}{r^2} - \frac{\delta_{ab}}{3} \}, \ldots
\]

Matrix elements of measured \(SU(2)\)-representations use real measures of the irreducible representations classes

\[
\mu : \text{irrep } SU(2) \rightarrow \mathbb{R}, \quad 2J \mapsto \mu(4J^2)
\]

\[
\text{meas irrep } SU(2) \ni \mu \mapsto D^\mu_{\pm} \in \text{rep } SU(2)
\]

with the functions on the spin group \(SU(2)\)

\[
SU(2) \ni e^{i\vec{q} \vec{x}} \mapsto D^\mu_{\pm}(\vec{x}) = \sum_{2J = 0, 1, \ldots} \mu(4J^2) \int \frac{d^3q}{i\pi^2} \frac{q}{(q^2 + 4J^2)^2} e^{-i\vec{q} \vec{x}} = -\frac{\vec{x}}{r} \sum_{2J = 0, 1, \ldots} \mu(4J^2) e^{\pm ir2J}
\]
7.5 Residual Representations for Position SD(2)

For the position manifold \( e^{-\vec{x}\vec{\sigma}} \in \text{SD}(2) \) with the Cartan polar decomposition \( \text{SD}(2) \cong \text{SO}_0(1,1) \times \text{SO}(3)/\text{SO}(2) \) residual representations use the tangent space forms (momenta \( \vec{q} \in \mathbb{R}^3 \)) and, in comparison to \( \text{SU}(2) \), the tangent space real-imaginary exchange for compact-noncompact for \( \text{SU}(2) \):

\[
\{ \text{Lie algebra and forms } (i\vec{x},\vec{q}) \leftrightarrow (\vec{x},i\vec{q}) \} \quad \text{invariant } \pm i|m| \leftrightarrow -|m|
\]

As for the Cartan subgroup \( \text{SO}_0(1,1) \) there exist two types: The compact representations \( \text{SD}(2) \rightarrow \text{SU}(2) \) (principal) with \( \text{SO}_0(1,1) \rightarrow \text{SO}(2) \) and the noncompact ones \( \text{SD}(2) \rightarrow \text{SU}(1,1) \) (supplementary) with faithful representations \( \text{SO}_0(1,1) \rightarrow \text{SO}_0(1,1) \). Both are representations of the homogeneous position manifold in a unitary group, definite or indefinite.

From the Yukawa potential

\[
\int \frac{d^3q}{2\pi^2} \frac{1}{q^2 + m^2} e^{-\vec{x}\vec{q}} = \frac{e^{-r|m|}}{r}, \quad m \in \mathbb{R}, \quad \vec{x} \neq 0
\]

\[
\left( \frac{\partial^2}{\partial \vec{x}^2} - m^2 \right) e^{-r|m|} = -4\pi \delta(\vec{x})
\]

which, by itself, is no \( \text{SD}(2) \)-representation matrix element because of the \( \vec{x} = 0 \) singularity, one obtains by derivations \( \frac{\partial}{\partial m^2} \) and \( \frac{\partial}{\partial \vec{x}} \) the scalar matrix elements, trivially representing the sphere \( \text{SO}(3)/\text{SO}(2) \) for \( \text{SO}_0(1,1) \subset \text{SD}(2) \):

\[
e^{-r|m|} = \int d^3q \left[ 0, -m^2 \right] (\vec{q}) e^{-\vec{x}\vec{q}}
\]

\[
[0, -m^2] (\vec{q}) = \frac{1}{\pi^2} \frac{|m|}{(q^2 + m^2)^2}, \quad m \in \mathbb{R}
\]

and the fundamental noncompact residual \( \text{SD}(2) \)-representations using a vector valued distribution

\[
-\frac{\vec{x}}{r} e^{-r|m|} = \int d^3q \left[ 1, -m^2 \right] (\vec{q}) e^{-\vec{x}\vec{q}} = \int \frac{d^3q}{i\pi^2(q^2 + m^2)^2} e^{-\vec{x}\vec{q}}, \quad m \in \mathbb{R}
\]

This has to be compared with the elements in the defining representation

\[
e^{-\vec{x}\vec{\sigma}} = 1_2 \cosh r - \frac{\vec{x}}{r} \sinh r
\]

The scalar matrix elements fulfill 4th order differential equations

\[
\left( \frac{\partial^2}{\partial \vec{x}^2} - m^2 \right)^2 e^{-r|m|} = 8\pi|m|\delta(\vec{x})
\]

\[
\left( \frac{\partial^2}{\partial \vec{x}^2} + m^2 \right)^2 e^{\pm ir|m|} = \mp 8\pi i|m|\delta(\vec{x}), \quad m \in \mathbb{R}
\]

In contrast to the spin group \( \text{SU}(2) \) where the representations of the compact Cartan subgroup \( \text{SO}(2) \) and the sphere \( \text{SO}(3)/\text{SO}(2) \) go both with discrete invariants \( 2J', 2J \in \mathbb{N}_0 \) - arising as degree of the spherical harmonics and as winding numbers, the continuous invariant \( m^2 \in \mathbb{R}^+ \) of the noncompact Cartan group \( \text{SO}_0(1,1) \)-representation in the case of the position manifold.
SD(2) is taken from a different spectrum as the discrete invariant $2J' \in \mathbb{N}_0$ for the sphere $SO(3)/SO(2)$-representations. Again the convolution products of the distributions for the fundamental residual $SD(2)$-representations

\[
\text{irrep}^{(1,1)} SD(2) = \{ \tilde{q} \mapsto [1, -m^2](\tilde{q}) = \frac{1}{i\pi^2} \frac{\tilde{q}}{(q^2 + m^2)^2} \mid m \in \mathbb{R} \}
\]

\[
\text{irrep}^{(2,0)} SD(2) = \{ \tilde{q} \mapsto [1, m^2](\tilde{q}) = \frac{1}{i\pi^2} \frac{\tilde{q}}{(q^2 + m^2)^2} \mid m \in \mathbb{R} \}
\]

define the matrix elements of the $SD(2)$-representations. The representations for trivial invariant $m = 0$ will be called neutral.

Measured $SD(2)$-representations use real measures of the continuous invariants

\[
\mu : \text{irrep SD}(2) \rightarrow \mathbb{R}, \quad m^2 \mapsto \mu(m^2)
\]

\[
\text{meas irrep SD}(2) \ni \mu \mapsto D^\mu \in \text{rep SD}(2)
\]

with the functions on the position manifold $SD(2)$

\[
SD(2) \ni e^{-\vec{x} \vec{q}} \mapsto D^\mu(\vec{x}) = \begin{cases} 
\int_0^\infty dm^2 \mu(m^2) \frac{d^3 q}{i\pi^2} \frac{q}{(q^2 + m^2)^2} e^{-i\vec{q} \cdot \vec{x}} & \text{for rep}^{(1,1)} SD(2) \\
= -\frac{2}{\tau} \int_0^\infty dm^2 \mu(m^2) e^{-r|m^2|} & \\
\text{and} \\
\int_0^\infty dm^2 \mu(m^2) \frac{d^3 q}{i\pi^2} \frac{q}{(q^2 + m^2)^2} e^{-i\vec{q} \cdot \vec{x}} & \text{for rep}^{(2,0)} SD(2) \\
= -\frac{2}{\tau} \int_0^\infty dm^2 \mu(m^2) e^{+ir|m^2|} & 
\end{cases}
\]

The two integrations in measured representation matrix elements go over the tangent space forms $\int d^3 q$ and the invariants $\int_0^\infty dm^2$ with the dimensions 3 and 1 of the symmetric space $SD(2)$ and a Cartan subgroup $SO_0(1, 1)$ resp. Matrix elements of reducible nondecomposable representations occur by using derivatives with respect to the invariant

\[
\mu(m^2) = \sum_{N=0,1,\ldots} \mu_N(m^2) \left(\frac{d}{dm^2}\right)^N
\]

8 Residual Representations of Spacetime

Matrix elements of representations of a symmetric space (Lie group) can be formulated as residues for characterizing invariant singularities of their tangent translation (Lie algebra) forms. For the groups $U(1), D(1), SU(2)$ and the position manifold $SD(2)$, as done in the former sections, this is only a reformulation of known structures. Residual representations constitute a genuine formulation for the rank 2 symmetric spacetime $D(2)$. Two values for the Lorentz invariant energy-momentum square $q^2$ characterize the action of the causal group $D(1)$ and the position manifold $SD(2)$.

Representations of spacetime

\[
D(2) = D(1_2) \times SD(2) \cong D(1_2) \times SO_0(1, 3)/SO(3)
\]

\[
\cong D(1_2) \times SO_0(1, 1) \times SO(3)/SO(2)
\]

will be formulated as Fourier transforms of energy-momentum distributions, compatible with the action of the Lorentz group $SO_0(1, 3)$ on the tangent
Minkowski spacetime. The two invariant masses characterizing the representations are implemented via singularities.

The irreducible residual representation matrix elements of spacetime $D(2)$, parametrizable with causal vectors $x \vartheta(x^2) \in \mathbb{R}^4$ in tangent Minkowski spacetime where the two reflected points $\{\pm x \vartheta(x^2)\}$ and, equally, their representation images have to be identified

$$D(2) \ni x \vartheta(x^2) \mapsto (m_0^2; 1, -m_3^2)(x) = \int d^4q \left[m_0^2; 1, -m_3^2\right](q)e^{iq}$$

involve a two factorial energy-momentum distribution

$$\text{irrep } D(2) = \{q \mapsto [m_0^2; 1, -m_3^2](q) = \frac{1}{i\pi^2(q_0^2 - m_0^2)(q_0^2 - m_3^2)} \mid m_0, m_3 \in \mathbb{R}\}$$

It describes the Lorentz compatible embedding for the representation of the two $D(2)$-factors and involves a simple pole (particle singularity) for the compact representation of a Cartan subgroup time

$$\text{for } D(1_2) \rightarrow SU(2) : \frac{1}{q_0^2 - m_0^2}$$

with pole location for $\vec{q} = 0$: $q_0^2 = m_0^2$

and a dipole (interaction singularity) for the noncompact representation of the position symmetric space $SD(2)$ with Cartan subgroup $SO_0(1, 1)$

$$\text{for } SD(2) \ni SO_0(1, 1) \rightarrow SU(1, 1) : \frac{1}{(q_0^2 - m_3^2)^2}$$

with dipole location for $q_0 = 0$: $q_0^2 = -m_3^2$

The 2-sphere $SO(3)/SO(2)$ nontrivially representing factor is given by $\{q^j\}_{j=0}^3$ in the numerator.

The Fourier transform of a principal value distribution is the causal Fourier transform of a Dirac distribution and vice versa

$$\int \frac{d^4q}{i\pi} \frac{1}{q_0^2 - m^2} \left(\frac{e^{iq}}{e^{iq}}\right) = \int d^4q \: \epsilon(q_0)\delta(q^2 - m^2) \left(\frac{\epsilon(x_0)e^{iq}}{e^{iq}}\right)$$

The matrix elements of the measured spacetime representations as $D(2)$-functions involve a measure for the two continuous invariants

$$\mu : \text{irrep } D(2) \rightarrow \mathbb{R}, \quad (m_0^2, m_3^2) \mapsto \mu(m_0^2, m_3^2)$$

$$\text{meas irrep } D(2) \ni \mu \mapsto D^\mu \in \text{rep } D(2)$$

$$D(2) \ni x \vartheta(x^2) \mapsto D^\mu(x) = \int_0^\infty dm_0^2 dm_3^2 \: \mu(m_0^2, m_3^2) \int d^4q \frac{2q_0}{i\pi^2(q_0^2 - m_0^2)(q_0^2 - m_3^2)} e^{iq}$$

The $D(2)$-representations are different from the Lorentz compatible position distributions of time representations as used for the quantization of the tangent Minkowski spacetime particle fields (Källen-Lehmann representations [3]), e.g. for a spin $\frac{1}{2}$ massive particle in a Dirac field

$$\text{particle fields: } \int_0^\infty dm^2 \: \mu(m^2) \int d^4q \frac{i\gamma^\mu [m]}{i \gamma^\nu \gamma^\mu \gamma^\nu - m^2 + i\epsilon} e^{iq}, \quad \mu(m^2) \geq 0$$

with probability related spectral measure $\mu(m^2)$ for the invariant of the definite unitary representations of the spacetime translations.
One obtains from the Lorentz scalar spacetime distribution with a simple energy-momentum pole
\[(\partial^2 + m^2) \int \frac{d^4q}{i\pi^3} \frac{1}{(q^2 - m^2)^2} e^{xq} = -16\pi\delta(x)\]
the derivatives \(\frac{\partial}{\partial m^2}\) with respect to the invariant
\[
\int \frac{d^4q}{i\pi^3} \frac{\Gamma(2+N)}{(q^2 - m^2)^3+\pi^2} e^{xq} = \begin{cases} \frac{\partial}{\partial m^2} & \frac{\partial}{\partial m^2} \vartheta(x^2)e_0(\frac{x^2 m^2}{4}), \ N = -1 \\ \frac{\partial}{\partial m^2} & \vartheta(x^2)e_0(\frac{x^2 m^2}{4}), \ N = 0, 1, \ldots \end{cases}
\]
and the derivative \(\frac{\partial}{\partial x}\) with respect to the Lie parameter
\[
\int \frac{d^4q}{i\pi^3} \frac{q \Gamma(3+N)}{(q^2 - m^2)^3+\pi^2} e^{xq} = \begin{cases} \frac{\partial}{\partial m^2} & \frac{\partial}{\partial m^2} \vartheta(x^2)e_0(\frac{x^2 m^2}{4}), \ N = -1, -2 \\ \frac{\partial}{\partial m^2} & \vartheta(x^2)e_0(\frac{x^2 m^2}{4}), \ N = 0, 1, \ldots \end{cases}
\]
which involve the Bessel functions \(J_N\) with \(\xi \in \mathbb{R}\)
\[
\vartheta(x^2) = \frac{J_N(\xi)}{(\frac{x^2}{4})^N} = (-\frac{\partial}{\partial \frac{x^2}{4}})^N J_0(\xi) = \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{4})^n}{n!(N+n)!}.
\]

The distributions for strictly negative nildimension \(N\), i.e. with a Dirac distribution on the light cone \(x^2 = 0\), are no spacetime \(D(2)\)-representation matrix elements. One obtains for the irreducible spacetime representations the \(D(2)\)-functions
\[
x\vartheta(x^2) \mapsto (m_0^2; 1, -m_3^2)(x) = \int \frac{d^4q}{i\pi^3} \frac{2q}{(q^2 - m_0^2)(q^2 - m_3^2)} e^{xq} = x\vartheta(x^2) \left[ \frac{m_0^2 e_2(\frac{x^2 m_0^2}{4}) - m_3^2 e_2(\frac{x^2 m_3^2}{4})}{(m_0^2 - m_3^2)^2} + \frac{m_3^2 e_1(\frac{x^2 m_3^2}{4})}{m_3^2 - m_0^2} \right]
\]
The neutral elements, either for \(SD(2)\) or \(D(1)\), are defined by trivial masses
\[
\begin{align*}
(m_0^2, 1, 0)(x) &= \int \frac{d^4q}{i\pi^3} \frac{2q}{(q^2 - m_0^2)(q^2 - m_3^2)} e^{xq} = x\vartheta(x^2) e_2(\frac{x^2 m_0^2}{4}) \\
(0; 1, -m_3^2)(x) &= \int \frac{d^4q}{i\pi^3} \frac{2q}{q^2 (q^2 - m_3^2)} e^{xq} = x\vartheta(x^2) [-e_2(\frac{x^2 m_3^2}{4}) + e_1(\frac{x^2 m_3^2}{4})] \\
(0; 1, 0)(x) &= \int \frac{d^4q}{i\pi^3} \frac{2q}{q^2} e^{xq} = \frac{\xi}{2} \vartheta(x^2)
\end{align*}
\]


## 9 Associated Residual Distributions

Given a residual distribution \( I(q) = I_0(q) \) for an irreducible representation of a symmetric space (Lie group) \( G \), singular distributions \( \{ I_N(q) \} \) using the same pole locations, but with possibly different orders, are called \( I \)-associated residual distributions. The possibly different singularity orders of the associated distributions will be characterized by integer nildimensions \( N \in \mathbb{Z} \).

Associated to the Dirac distribution for an irreducible Abelian group representation are its derivatives

\[
\{ \delta^{(N)}(m - q) \mid N = 0, 1, \ldots \} \quad \begin{cases} 
   m \in \mathbb{Z} & \text{for } \text{irrep } U(1) \\
   m \in \mathbb{R} & \text{for } \text{irrep } D(1)
\end{cases}
\]

\[
\frac{dq}{i\pi} \frac{\Gamma(1+N)}{(q^2 + \omega^2 m^2)^{1+N}} e^{-i\omega q} = \frac{dq}{2i\pi (q - m)^{1+N}} e^{itq} = (it)^N e^{itm}
\]

For the selfdual Abelian groups one has as associated distributions for the compact representations (always only where the \( \Gamma \)-functions are defined)

\[
\left\{ \frac{1}{i\pi} \frac{q}{(q^2 + \omega^2 m^2)^{1+N}} \right\} \quad \begin{cases} 
   m \in \mathbb{Z} & \text{for } \text{irrep } SO(2) \\
   m \in \mathbb{R} & \text{for } \text{irrep } (2,0) \text{SO}_0(1,1)
\end{cases}
\]

\[
\frac{dq}{i\pi} \frac{\Gamma(1+N)}{(q^2 + \omega^2 m^2)^{1+N}} e^{-i\omega q} = -\epsilon(x) \left( \frac{\partial}{\partial m} \right)^N e^{\pm i|mx|}
\]

\[
= \begin{cases} 
   -\epsilon(x) e^{\pm i|xm|} , & + i \frac{x}{2|m|} e^{\pm i|xm|} , \\
   \text{for } N = 0, 1, \ldots 
\end{cases}
\]

\[
\pm \frac{dq}{i\pi} \frac{|m| \Gamma(1+N)}{(q^2 + \omega^2 m^2)^{1+N}} e^{-i\omega q} = |m| \left( \frac{\partial}{\partial m^2} \right)^N e^{\pm i|mf|}
\]

\[
= \begin{cases} 
   e^{\pm i|mx|} , & - \frac{1 + |mx|}{2m^2} e^{\pm i|mx|} , \\
   \text{for } N = 0, 1, \ldots 
\end{cases}
\]

and for the noncompact \( \text{SO}_0(1,1) \)-representations

\[
\left\{ \frac{1}{i\pi} \frac{q}{(q^2 + m^2)^{1+N}} \right\} \quad m \in \mathbb{R} \text{ for } \text{irrep } (1,1) \text{SO}_0(1,1)
\]

\[
\frac{dq}{i\pi} \frac{\Gamma(1+N)}{(q^2 + m^2)^{1+N}} e^{-i\omega q} = -\epsilon(x) \left( - \frac{\partial}{\partial m^2} \right)^N e^{-|xm|}
\]

\[
= \begin{cases} 
   -\epsilon(x) e^{-|xm|} , & - \frac{x}{2|m|} e^{-|xm|} , \\
   \text{for } N = 0, 1, \ldots 
\end{cases}
\]

\[
\frac{dq}{i\pi} \frac{|m| \Gamma(1+N)}{(q^2 + m^2)^{1+N}} e^{-i\omega q} = |m| \left( - \frac{\partial}{\partial m^2} \right)^N e^{-|mx|}
\]

\[
= \begin{cases} 
   e^{-|mx|} , & 1 + |mx| \frac{1}{2m^2} e^{-|mx|} , \\
   \text{for } N = 0, 1, \ldots 
\end{cases}
\]

With respect to the sign of the nildimension \( N \) the residual distributions are used for

- nondecomposable group representations \( \iff N \geq 0 \)
- irreducible group representations \( \iff N = 0 \)
- tangent representations (below) \( \iff N \leq 0 \)

For compact groups strictly positive nildimensions cannot occur, they define no functions on the group

\[
\text{for compact groups } N \leq 0
\]
Residual distributions with strictly negative nildimensions $N = -1, -2, \ldots$ do not lead to $G$-representation matrix elements. They arise only for groups where the rank is strictly smaller than the dimension.

Associated to the dipole distribution of an irreducible representation of the spin group $\text{SU}(2)$ and - for compact representations - of the position manifold $\text{SD}(2)$ are the following distributions

$$
\left\{ \frac{1}{i \pi^2} \frac{q \Gamma(2+N)}{(q^2 + i \alpha - m^2)^{2+N}} \right\} \text{ with } m \in \mathbb{Z} \text{ for irrep } \text{SU}(2)
$$

$$
\int \frac{d^3q}{i \pi^2} \frac{q \Gamma(2+N)}{(q^2 + i \alpha - m^2)^{2+N}} e^{-i \vec{x} \vec{q}} = 2 \frac{\vec{x}}{r} \frac{\partial}{\partial m^2} \left( \frac{\partial}{\partial \alpha^2} \right) e^{1+N} e^{i \pm \alpha \pm \alpha m} \text{ for } N = -1, 0, \ldots
$$

$$
\pm \int \frac{d^3q}{i \pi^2} \frac{|m| \Gamma(2+N)}{(q^2 + m^2)^{2+N}} e^{-i \vec{x} \vec{q}} = \mp 2i |m| \left( \frac{\partial}{\partial m^2} \right) e^{1+N} e^{i \pm \alpha \pm \alpha m} \text{ for } N = -1, 0, 1, \ldots
$$

and to an irreducible noncompact position $\text{SD}(2)$-representation

$$
\left\{ \frac{1}{i \pi^2} \frac{q \Gamma(2+N)}{(q^2 + m^2)^{2+N}} \right\} \text{ with } m \in \mathbb{R} \text{ for irrep } (1,1) \text{SD}(2)
$$

$$
\int \frac{d^3q}{i \pi^2} \frac{q \Gamma(2+N)}{(q^2 + m^2)^{2+N}} e^{-i \vec{x} \vec{q}} = 2 \frac{\vec{x}}{r} \frac{\partial}{\partial m^2} \left( \frac{\partial}{\partial \alpha^2} \right) e^{-r} e^{i \pm \alpha \pm \alpha m} \text{ for } N = -1, 0, \ldots
$$

$$
\int \frac{d^3q}{i \pi^2} \frac{|m| \Gamma(2+N)}{(q^2 + m^2)^{2+N}} e^{-i \vec{x} \vec{q}} = 2 |m| \left( \frac{\partial}{\partial m^2} \right) e^{-r} e^{i \pm \alpha \pm \alpha m} \text{ for } N = -1, 0, 1, \ldots
$$

The distributions with $N = -1$ lead to spherical waves $\frac{e^{i \pm \alpha \pm \alpha m}}{r}$ and Yukawa potentials $\frac{e^{-r}}{r}$ and their derivatives which are no $\text{SU}(2)$ and $\text{SD}(2)$-representation matrix elements.

The distributions associated to an irreducible $D(2)$-representation include as negative nildimension distributions

$$
\frac{q \Gamma(3+N_0+N_3)}{(q_0 - m_0)^{3+N_0} (q_0 - m_3)^{3+N_3} (q_0^2 - m_0^2 - m_3^2)} \Rightarrow \left\{ \frac{q}{q_0^2 - m_0^2}, \frac{q}{q_0^2 - m_3^2}, \frac{q}{q_0^2 - m_0^2 - m_3^2}, \frac{q}{q_0^2 - m_0^2 - m_3^2} \right\}, \quad N_0 + N_3 = -1 \quad N_0 + N_3 = -2
$$

### 10 Residual Subrepresentations

A representation of a symmetric space (Lie group) $G$ contains representations of subspaces (subgroups) $H$. How does this look for residual representations?

A residual $G$-representation with tangent space (Lie algebra) parameters $x = (x_H, x_\perp)$

$$
D^I : G \rightarrow \mathfrak{g}, \quad g(x) \mapsto D^I(x) = \int d^n q \, I(q) e^{iqx}
$$
is projected to a residual $H$-representation by integration $\int d^{n-s}x_\perp$ over the complementary space $\log G/H$

$$D^I_H : H \longrightarrow \mathbb{C}, \quad h(x_H) \longmapsto D^I_H(x_H)$$

with $D^I_H(x_H) = \int \frac{d^{n-s}x_\perp}{(2\pi)^{n-s}} \int d^sq \, I(q)e^{ixq} = \int d^sq_H \, I(q_H, 0)e^{ixHq_H}$

With the integration one picks up the Fourier components for trivial tangent space forms (momenta) $q_\perp = 0$ of $\log G/H$.

### 10.1 $\text{SO}(2) \times \text{SO}_0(1, 1)$-Subrepresentations in Spin-Position-Representations

The $\text{SO}(2)$-subrepresentations in spin $\text{SU}(2)$-representations are given as follows

$$\text{irrep } \text{SU}(2) \longrightarrow \text{rep } \text{SO}(2), \quad d^2x_\perp = d^2x_{1,2}$$

$$\int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{4\pi} \, \frac{q \, \Gamma(2+N)}{(q^2+\omega^2-\omega_0^2)^{1+\kappa+N}}e^{ixq} = \int \frac{dq}{4\pi} \, \frac{q \, \Gamma(2+N)}{(q^2+\omega^2)^{1+\kappa+N}}e^{-ixq}$$

$$= \epsilon(x_3)(\frac{\partial}{\partial m^2})^{1+N}e^{ixq}$$

$$\pm \int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{4\pi} \, \frac{|m| \, \Gamma(2+N)}{(q^2+\omega^2-\omega_0^2)^{1+\kappa+N}}e^{-ixq} = \pm \int \frac{dq}{4\pi} \, \frac{|m| \, \Gamma(2+N)}{(q^2+\omega^2)^{1+\kappa+N}}e^{-ixq}$$

$$= |m|(\frac{\partial}{\partial m^2})^{1+N}e^{-ixq}$$

and the $\text{SO}_0(1, 1)$-subrepresentations of noncompact position $\text{SD}(2)$-representations

$$\text{irrep } \text{SD}(2) \longrightarrow \text{rep } (1^{1})\text{SO}_0(1, 1)$$

$$\int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{4\pi} \, \frac{q \, \Gamma(2+N)}{(q^2+\omega^2)^{1+\kappa+N}}e^{-ixq} = \int \frac{dq}{4\pi} \, \frac{q \, \Gamma(2+N)}{(q^2+\omega^2)^{1+\kappa+N}}e^{-ixq}$$

$$= \epsilon(x_3)(\frac{\partial}{\partial m^2})^{1+N}e^{-ixq}$$

$$\int \frac{d^2x_\perp}{4\pi} \int \frac{d^3q}{4\pi} \, \frac{|m| \, \Gamma(2+N)}{(q^2+\omega^2)^{1+\kappa+N}}e^{-ixq} = \int \frac{dq}{4\pi} \, \frac{|m| \, \Gamma(2+N)}{(q^2+\omega^2)^{1+\kappa+N}}e^{-ixq}$$

$$= |m|(\frac{\partial}{\partial m^2})^{1+N}e^{-ixq}$$

The vector dependence $\vec{\tau}$ for the sphere is projected to two values $\epsilon(x_3) \in \{\pm 1\}$ for the hemispheres.

### 10.2 Time and Position Subrepresentations in Spacetime Representations

The energy-momentum distribution used in the residual spacetime representations is the principal value part in the decomposition of a complex distribution into imaginary and real part

$$\pm \frac{1}{4\pi} \frac{q}{(q^2+\omega_0^2)(q^2-m_0^2)} = \pm \frac{1}{4\pi} \frac{q}{(q^2-m_0^2)(q^2-m_3^2)} + \frac{1}{(m_0^2-m_3^2)}q\delta(q^2-m_0^2)$$

which is also the decomposition for the representation matrix elements of spacetime $D(2)$ and its tangent space $\mathbb{R}^4$. The integrated principal value part has causal support whereas the integrated Dirac distribution for the particle
pole gets both spacelike and causal support. The decomposition with respect to the two singularities

\[
\frac{1}{(q^2-m_0^2)(q^2-m_9^2)} = \frac{1}{(m_0^2-m_9^2)^2} \left[ \frac{1}{q^2-m_0^2} - \frac{q^2-m_9^2}{(q^2-m_9^2)^2} \right]
\]

is not parallel with the representation of the factors in \( D(2) = D(1_2) \times SD(2) \). The projections to representation matrix elements of the manifold factors are given by position integration for the causal group \( D(1_2) \) and by time integration for the position manifold \( SD(2) \) with Cartan subgroup \( SO_0(1,1) \), i.e. by the Fourier transforms for trivial momenta \( \vec{q} = 0 \) and trivial energy \( q_0 = 0 \) resp.

\[
\begin{align*}
\int d^3x : & \quad \text{irrep} \ D(2) \quad \rightarrow \quad \text{rep} \ D(1) \\
\int dx_0 : & \quad \text{irrep} \ D(2) \quad \rightarrow \quad \text{rep} \ SD(2) \\
\int d^2x_\perp : & \quad \text{rep} \ SD(2) \quad \rightarrow \quad \text{rep} \ SO_0(1,1)
\end{align*}
\]

where one uses

\[
\left( \begin{array}{c}
\int \frac{d^3x}{8\pi} \\
\int \frac{d^2x_\perp}{4\pi} \\
\int \frac{dx_0}{2}
\end{array} \right) \int \frac{dq}{\pi^3} \frac{\Gamma(3+N)}{(q^2-m_9^2)^{3+N}} e^{xiq} = \left( \frac{\partial}{\partial m^2} \right)^{2+N} \left\{ \begin{array}{c}
\frac{\epsilon(x_0) \cos x_0 m_0}{2(1+r|m|) e^{-r|m|}} \\
-\epsilon(x_3) e^{-x_3 m_3} 
\end{array} \right\}
\]

This leads for irreducible spacetime representations to

\[
\begin{align*}
\int \frac{d^3x}{8\pi} (m_0^2; 1, -m_9^2)(x) &= \int \frac{dq_0}{\pi} \frac{q_0}{(q_0^2-m_0^2)^2} e^{x_0 q_0} \\
&= \epsilon(x_0) \left( \frac{\cos x_0 m_0 - \cos x_0 m_3}{(m_0^2-m_3^2)^2} \right) \\
&= \epsilon(x_0) \left( \frac{\cos x_0 m_0 - \cos x_0 m_3}{(m_0^2-m_3^2)^2} \right) \\
\int \frac{dx_0}{2} (m_0^2; 1, -m_9^2)(x) &= \int \frac{dq}{\pi} \frac{q}{(q^2+m_0^2)^2} e^{-x_0 q} \\
&= -\frac{x_3}{r} \left[ \frac{1}{2} \frac{(1+r|m_0|e^{-r|m_0|}-(1+r|M|m_3|e^{-r|m|})}{r^2(m_0^2-m_3^2)^2} \right] \\
&= -\frac{x_3}{r} \left[ \frac{1}{2} \frac{(1+r|M|m_3|e^{-r|m_3|}-(1+r|M|m_3|e^{-r|m|})}{r^2(m_0^2-m_3^2)^2} \right] \\
\int \frac{d^2x_\perp}{4\pi} \int \frac{dx_0}{2} (m_0^2; 1, -m_9^2)(x) &= \int \frac{dq}{\pi} \frac{q}{(q^2+m_0^2)^2} e^{-x_3 q} \\
&= -\epsilon(x_3) \left( \frac{e^{-x_3 m_0}-e^{-x_3 m_3}}{(m_0^2-m_3^2)^2} \right) \\
&= -\epsilon(x_3) \left( \frac{e^{-x_3 m_0}-e^{-x_3 m_3}}{(m_0^2-m_3^2)^2} \right)
\end{align*}
\]

The measure of the invariants for an irreducible spacetime representation

\[
\rho(M_0^2, M_3^2) = \delta(M_0^2-m_0^2)\delta(M_3^2-m_3^2)
\]

is projected to measures for the representation of the two factors. The time \( D(1_2) \)-subrepresentation with the measure

\[
\rho_0(m_0^2) = \frac{\delta(m_0^2-m_0^2)}{(m_0^2-m_3^2)^2} + \frac{\delta'(m_0^2-m_3^2)}{m_0^2-m_3^2}
\]

contains matrix elements of reducible nondecomposable representations for the nonparticle dipole at \( m_3^2 \).
The linear combinations occurring in the position SD(2)-projections of spacetime D(2)-representations are matrix elements of measured SD(2)-representations involving the difference of two Yukawa potentials

\[
2e^{-|m_2^2|} \int dm^2 \frac{e^{-|m|}}{|m|} = \int_0^{m_0^2} dm^2 \frac{e^{-|m|}}{|m|} = \int_0^\infty dm^2 \vartheta(m^2 - m_0^2) \vartheta(m_0^2 - m^2) e^{-|m|}
\]

The measure for the SD(2)-subrepresentation reads

for SD(2) : \( \rho_3(m^2) = -\frac{\vartheta(m_0^2 - m^2)}{(m_0^2 - m^2)^2} + \frac{\vartheta(m^2 - m_0^2)}{m_0^4 - m^4} \)

11 Residual Tangent Distributions

The residual tangent distributions for an irreducible symmetric space (group) representation will be defined by the associated distributions with a simple pole, i.e. for minimal negative nildimension \( N \leq 0 \), and a trivial invariant. They arise as the inverse differential operators in the Lie algebra action representing differential equations of motions.

The tangent \( \mathbb{R} \) distributions for the Abelian groups have trivial nildimensions \( N = 0 \) - for the non-selfdual

\[
\log U(1) : \ \delta(q) \approx \frac{1}{2\pi i} \frac{i}{q}, \quad \int dq \frac{1}{2\pi i} e^{tiq} = 1
\]

for the selfdual compact representations

\[
\log SO(2) : \ \log SO_0(1, 1) : \ \int dq \frac{q}{q^2 + i\alpha} e^{tiq} = \epsilon(t)
\]

and for the selfdual noncompact \( SO_0(1, 1) \)-representations

\[
\log SO_0(1, 1) : \ \frac{1}{2\pi i} \frac{q}{q^2 + i\alpha} e^{-xiq} = -\epsilon(x)
\]

For the nonabelian rank 1 spaces the residual tangent \( \mathbb{R}^3 \) distributions come with nildimension \( N = -1 \)

\[
\log SU(2) : \ \log SD(2) : \ \frac{1}{2\pi i} \frac{q}{q^2 + i\alpha + m^2}, \quad \int dq \frac{q}{q^2 + i\alpha + m^2} e^{-xiq} = -2 \frac{x}{r^3}
\]

and in the noncompact case

\[
\log SD(2) : \ \int dq \frac{q}{q^2 + i\alpha + m^2} e^{-xiq} = -2 \frac{r}{r^3}
\]

They lead both to the Coulomb force with the Cartan subalgebra projection

\[
\int dq \frac{q}{q^2 + i\alpha} e^{-xiq} = \epsilon(x_3)
\]

The residual tangent spacetime \( \mathbb{R}^4 \) distributions have nildimension \( N_0 + N_3 = -2 \)

\[
\log D(2) : \ \frac{1}{2\pi i} \frac{q}{q^2}, \quad \int dq \frac{q}{q^2} e^{xiq} = -2 \frac{r}{2(x_0 + x_3)} \\
\]

with projections \( \left( \begin{array}{c} \int dq \frac{q}{q^2} \\ \int dq \end{array} \right) \right) \int dq \frac{q}{q^2} e^{xiq} = \left( \begin{array}{c} \epsilon(x_0) \\ \epsilon(x_3) \end{array} \right) \)
12 Defining Representations for Time, Position and Spacetime

Spacetime, particles and interactions cannot be taken as separate concepts. Spacetime is known via interacting particles and the interactions of particles can be understood only in spacetime.

This connection will be translated into the mathematical language with the concept of a defining representation, familiar from Lie groups. E.g., the Lie group \( SU(n) \) is defined by the automorphisms of a vector space \( V \cong \mathbb{C}^n \) compatible with a scalar product - the linear space and the operating group merge in the concept of the defining representation.

In addition to one defining representation for some Lie groups there exist fundamental representations which reflect the rank and the number of independent invariants. E.g., the Lie symmetry \( SU(r + 1) \) one has \( r \) fundamental representations whose highest weights are basic vectors for the \( \mathbb{Z} \)-module with all weights. The products of a defining representation may build the fundamental ones, e.g. in the case of \( SU(n) \) via the totally antisymmetric Grassmann powers of the defining vector space.

12.1 The Harmonic Oscillator - Defining a Compact Time

The irreducible time \( D(1) \) representation in the group \( U(1) \) as seen in the quantization for creation and annihilation operators \((u, u^*)\) of a harmonic Fermi or Bose oscillator with frequencies \( \pm m \in \mathbb{R} \):

\[
D(1) \ni e^t \mapsto e^{tim} = [u^*, u]_\pm \in U(1)
\]
defines a compact model for time with the invariant \( \frac{1}{m} \) as characteristic time unit.

The adjoint action with the Hamiltonian as the represented Lie algebra basis defines the time translations in the equations of motion

\[
H = m \frac{[u, u^*]_+}{2} \Rightarrow \left\{ \begin{array}{l}
\frac{du}{dt} = [iH, u] = imu, \\
\frac{du^*}{dt} = [iH, u^*] = -imu^*, \\
u(t) = e^{tim}u \\
u^*(t) = e^{-tim}u^*
\end{array} \right.
\]

The operators are \( U(1) \)-isomorphic time orbits in the \( \mathbb{C} \)-isomorphic representation spaces

\( u, u^* : D(1) \rightarrow V, V^T \cong \mathbb{C} \)

The product representations \( e^{tim_1}e^{tim_2} = e^{ti(m_1+m_2)} \) generate the familiar equidistant time weights (eigenvalues, frequencies) for the quantum oscillator - \( \{Zm \mid Z \in \mathbb{Z}\} \) for Bose and \( \{Zm \mid Z = 0, \pm 1\} \) for Fermi which - for the states - are projected on the positive values.

\( ^{10}u \) without argument means \( u(0) \), i.e. for the trivial translation.
12.2 The Exponential Potential - Defining a Noncompact Position

An indefinite unitary representation of the noncompact Procrustes dilatation group \( \text{SO}_0(1, 1) \) for dual operators \((d, d^*)\) of Fermi or Bose type with eigenvalues \( \pm m \in \mathbb{R} \)

\[
\text{SO}_0(1, 1) \ni \left( \begin{array}{cc} e^{-x} & 0 \\ 0 & e^x \end{array} \right) \longmapsto \left( \begin{array}{cc} e^{-xm} & 0 \\ 0 & e^{xm} \end{array} \right) = \left( \begin{array}{cc} [d^*, d]_{\pm} & [d, d^*]_{\pm} \\ [d^*, d^*]_{\pm} & [d, d]_{\pm} \end{array} \right) (x) \in \text{SU}(1, 1)
\]

defines a faithful model for the position space Cartan subgroup \( \text{SO}_0(1, 1) \) with the invariant \( \frac{1}{|m|} \) as characteristic length unit.

The translations are implemented with the basis

\[
D = im \frac{[d, d^*]}{2} \Rightarrow \left\{ \frac{d}{dx}, \frac{d^*}{dx} \right\} = [iD, d] = -md, \quad d(x) = e^{-xm} d \quad \frac{d}{dx}, \quad d^*(x) = e^{xm} d^*
\]

The operators are noncompact \( D(1) \)-isomorphic dilatation orbits in the \( \mathbb{C} \)-isomorphic representations spaces \( \text{SO}_0(1, 1) \rightarrow V, V^T \cong \mathbb{C} \).

The product representations (convolutions) lead to exponentials with the eigenvalues \( \{zm | z = 0, \pm 1\} \) for Fermi and \( \{zm | z \in \mathbb{Z}\} \) for Bose.

A representation matrix element of the symmetric space position model \( \text{SD}(2) \cong \text{SL}(\mathbb{C}^2)/\text{SU}(2) \)

\[
\text{SD}(2) \ni e^{-\vec{x} \vec{\sigma}} \longmapsto -\frac{\vec{\sigma}}{r} e^{-r|m|} = \int \frac{d^4q}{i\pi^2 (q^2+m^2)^2} e^{-\vec{x} \vec{q}} = \{ \psi^*, \psi \}(\vec{x})
\]

with Pauli matrices \( \vec{\sigma} \) defines a noncompact position with a characteristic length \( \frac{1}{|m|} \) (interaction range), implemented by \( \mathbb{C}^2 \)-valued Pauli spinor fields on the position manifold

\[
\psi^A, \psi^*_A : \text{SD}(2) \rightarrow V, V^T \cong \mathbb{C}^2, \quad A = 1, 2
\]

The Cartan subgroup \( \text{SO}_0(1, 1) \) is represented by an indefinite unitary \( \text{SU}(1, 1) \)-representation matrix element \( e^{-r|m|} \).

The product representations (convolutions) add up the noncompact invariants \( \{n|m| | n = 1, 2, \ldots\} \) in the exponential and are multiplied with spherical harmonics of degree \( \{2J | 2J = 0, 1, 2, \ldots\} \) for the representation of the sphere \( \text{SO}(3)/\text{SO}(2) \).

12.3 Defining Spacetime with Two Invariants

The representation matrix element

\[
\text{D}(2) \ni \vartheta(x^2) x \longmapsto \int \frac{d^3q}{i\pi^3 (q^2-m_0^2)(q^2-m_3^2)} e^{xq} = \epsilon(x_0) \{ \Psi^*, \Psi \}(x)
\]
defines symmetric spacetime. The two invariants $m_0^2$ and $m_3^2$ characterize time and position and give units for particle masses and interaction lengths. The representation is implemented by $\mathbb{C}^2$-valued Weyl spinor fields

$$\Psi_A, \Psi^*_A : D(2) \rightarrow V, V^T \cong \mathbb{C}^2, \quad A = 1, 2$$

It involves two conjugations - a definite $U(2)$-conjugation for the time $D(1)$-representation and an indefinite $U(1, 1)$-conjugation for the position $SD(2)$-representation. Therefore only the particle pole can be endowed with an additional asymptotic positive unitary spacetime translation representation structure by adding a real on shell contribution via $\pm \frac{1}{\sqrt{q^2 + \Theta}}$. A parametrization with creation and annihilation operators has to take care of the indefinite conjugation involved.

The product representations of the defining spacetime representation will give rise to product invariants which - in the case of an accompanying definite unitary conjugation, can be identified with particle masses for bound states. To carry out explicitly such a program, i.e. to compute a mass spectrum from the spacetime defining two invariants, the representation characteristic ratio $\frac{m_0^2}{m_3^2}$ has to be determined as well as the relevant normalization factors to be used in the eigenvalue equations for the product representation invariants.
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