Classical Coding Problem from Transversal $T$ Gates

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Abstract—Universal quantum computation requires the implementation of a logical non-Clifford gate. In this paper, we characterize all stabilizer codes whose code subspaces are preserved under physical $T$ and $T^\dagger$ gates. For example, this could enable magic state distillation with non-CSS codes and, thus, provide better parameters than CSS-based protocols. However, among non-degenerate stabilizer codes that support transversal $T$, we prove that CSS codes are optimal. We also show that $T$, we prove that CSS codes are optimal. We also show that

I. INTRODUCTION

Quantum computers have been theoretically shown to provide computational advantages over conventional (classical) computers, which could have impacts in fields as varied as quantum simulation, optimization, chemistry, communications, and metrology. Recently, Google and NASA demonstrated a computational advantage for a random circuit sampling task via a real experiment on their 53-qubit quantum machine [1]. Although the extent of the advantage has been disputed by IBM [2], it is widely accepted that this is a milestone hardware demonstration. However, these computers are still very noisy and algorithms that are sensitive to noise are not within reach. One example is Shor’s algorithm for factoring integers [3], [4], which has huge implications for digital security. A quantum error correcting code (QECC) provides resilience to noise, and in this paper we focus on fault-tolerant implementation of a universal set of gates on the qubits protected by a QECC.

Universality requires one to realize a logical non-Clifford gate and the easiest fault-tolerant realization is a transversal operation, which splits into gates on individual qubits. In other words, given an $[[n, k, d]]$ QECC, we would like to understand the $k$-qubit (logical) gates that can be realized as transversal operations on the $n$ physical qubits of the code.

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addressing this question in full generality is a challenging, in this paper we algebraically characterize all $[[n, k, d]]$ stabilizer QECCs [5], [6] whose code subspaces are preserved by a given pattern of $T$ and $T^\dagger$ gates on the $n$ qubits, i.e., this transversal operation induces some logical operation on the $k$ protected qubits. This characterization encompasses all schemes in the literature that use transversal $T$ gates on stabilizer codes to achieve their objective. For example, [7], [8] use this approach for magic state distillation (MSD).

In particular, for state distillation, almost all existing protocols use Calderbank-Shor-Steane (CSS) codes [9], [10], which form a subclass of stabilizer codes. Our results can be used to construct distillation protocols that utilize transversal gates on non-CSS stabilizer codes. At first look, this points towards the possibility of better parameters than CSS-based protocols. However, we prove that, given any $[[n, k, d]]$ non-degenerate stabilizer code supporting a pattern of $T$ and $T^\dagger$, there exists an $[[n, k, d]]$ CSS code with the same property. Here, by non-degenerate we mean that each stabilizer element acts non-trivially on at least $d$ physical qubits. While the degenerate case remains unsolved, our algebraic approach enables one to reason about CSS optimality for transversal $Z$-rotations, which is an important open problem in quantum error correction.

When our main result (Theorem 2) is specialized to CSS codes we obtain new classical coding problems, and the general case is quite similar. Since this is a self-contained problem that classical coding theorists can analyze, we describe it here. CSS-T Codes: A pair $(C_1, C_2)$ of binary linear codes with parameters $[[n, k_1, d_1]]$ and $[[n, k_2, d_2]]$, respectively, such that $C_2 \subset C_1$ and the following properties hold:

1) $C_2$ is an even code, i.e., $w_H(x) \equiv 0 \pmod{2}$ for all $x \in C_2$, where $w_H(x)$ is the Hamming weight of $x$.
2) For each $x \in C_2$, there exists a dimension $w_H(x)/2$ self-dual code in $C_1$ that is supported on $x$, i.e., there exists $C_x \subseteq C_1$ such that $|C_x| = 2^{w_H(x)/2}$, $C_x = C_x^\perp$, and $z \in C_x \Rightarrow z^\perp \subseteq x$, i.e., $\supp(z) \subseteq \supp(x)$, where $C_1^\perp$ is the code dual to $C_1$ and $\supp(x)$ is the support of $x$.

Open Problem: A $[[n, k_1 - k_2, \min(d_1, d_2^\perp)]]$ family of CSS-T codes such that $\left(\frac{k_1 - k_2}{n}\right) = \Omega(1)$ and (ideally) $\min(d_1, d_2^\perp) \geq \Omega(1)$, where $d_2^\perp$ is the minimum distance of $C_2^\perp$.

This specific code family arises when the $T$ gate is applied transversally, but different patterns of $T$ and $T^\dagger$ gates produce variants of it. A $[[2m, (m/3), 2m/3]]$ Reed-Muller CSS-T family is described by $C_1 = RM(m/3, m)$, $C_2 = RM(m/3-
1, m). However, this family has vanishing rate and distance. It is an important open problem to construct a constant rate CSS-T family with growing distance. For example, this would enable constant overhead MSD, since the ratio of input noisy states to output e-noisy states is $O\left(\frac{\log(n)}{\log(d)}\right)$, where $\gamma \triangleq \frac{\log(a^k)}{\log(n)}$ for an $[n, k, d]$ code [8]. This leads to a tremendous decrease in resource counts for this critical subroutine [9].

Several researchers have worked on constructing codes that support $T$ gates. One of the earliest known codes to support transversal $T$ is the [15, 1, 3] (CSS) quantum Reed-Muller (QRM) code [2, 13, 14]. Subsequently, triorthogonal codes [9] were developed to produce a systematic construction of CSS codes where the logical transversal $T$ can be realized via physical transversal $T$ (up to diagonal Clifford corrections). In Section II-A we will show that this is essentially the only family of CSS codes that satisfies this property. The topological family of 3D color codes [15] has also been shown to support transversal $T$ gates. More recently, quasi-transversality [16] and the implied generalized triorthogonality [17] conditions have been developed to construct CSS codes that support transversal $T$. Finally, quantum pin codes [18] are CSS codes that are inspired by topological codes, but they have a more general abstract construction that intrinsically supports (quasi-)transversal $Z$-rotations.

The approach in this prior work is to analyze the CSS basis states. Our approach is different in that we analyze the operators in the stabilizer, and it is more general, in that it extends beyond CSS codes. For details and proofs see [11].

II. BACKGROUND AND NOTATION

A. Heisenberg-Weyl and Clifford Groups

The 1-qubit Pauli operators are the unitaries $I_2$ (identity),

$$X \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y \triangleq iXZ = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

where $i \triangleq \sqrt{-1}$. They satisfy $X^2 = Z^2 = Y^2 = I_2$. The $n$-qubit Heisenberg-Weyl (or Pauli) group $\HW_n$, $N \triangleq 2^n$, consists of Kronecker products of these single-qubit operators with overall phases $i^a, \kappa \in \mathbb{Z}_4 \triangleq \{0, 1, 2, 3\}$. We represent a Hermitian Pauli matrix via two binary vectors $a = [a_1, \ldots, a_n], b = [b_1, \ldots, b_n] \in \mathbb{Z}_2^n$ with the notation

$$E(a, b) \triangleq \left(\alpha^{a_1b_1} X^{a_1} Z^{b_1}\right) \otimes \cdots \otimes \left(\alpha^{a_nb_n} X^{a_n} Z^{b_n}\right).$$

Two Pauli matrices $E(a, b)$ and $E(c, d)$ commute if the symplectic inner product $\langle [a, b], [c, d] \rangle \triangleq ad^T + be^T \pmod{2} = 0$, and they anti-commute otherwise [19].

Throughout the paper, $\oplus$ denotes modulo-2 addition and $+$ denotes standard integer addition. Also, all binary and integer-valued vectors will be row vectors while complex-valued vectors will be column vectors. For $x = [x_1, \ldots, x_n], y = [y_1, \ldots, y_n] \in \mathbb{Z}_2^n$, we define $x * y \triangleq [x_1y_1, \ldots, x_ny_n]$.

The Clifford group $\Cliff_n$ is the normalizer of $\HW_n$ in $\qty{U}$, the unitary group of $N \times N$ matrices. Hence, for $g \in \Cliff_n$, $gE(a, b)g^\dagger = \pm E([a, b], F_g)$, where $F_g \Omega F_g^T = \Omega \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$.

So $F_g$ is a binary symplectic matrix, i.e., it preserves symplectic inner products $\langle [a, b], [c, d] \rangle = [a, b] \Omega [c, d]^T$. Since, up to scalars, $\Cliff_N$ is a finite subgroup of $\qty{U}_N$, it is insufficient to perform universal quantum computation. It is well-known that $\Cliff_N$ augmented by any non-Clifford unitary can approximate any other unitary operator arbitrarily well. A standard choice is the “$T$” gate $T \triangleq \pi/2 \approx Z^{1/4}$ [20].

B. Quadratic Form Diagonal (QFD) Gates

The Clifford hierarchy is a hierarchy of unitary operators first defined by Gottesman and Chuang [21] to demonstrate universal quantum computation via teleportation. The first level of the hierarchy is $C^{(1)} \triangleq \HW_N$ and the subsequent levels $\ell \geq 2$ are defined recursively by

$$C^{(\ell)} \triangleq \{U \in \mathbb{U}_N : U \in \mathbb{U}_N \forall a, b \in \mathbb{Z}_2^n\}.$$  (4)

From this definition, it is easily seen that $C^{(2)} = \Cliff_N$. Cui et al. [22] described the structure of all diagonal unitaries in this hierarchy. In particular, they showed that the entries in such unitaries have to be of the form $\exp\left(\frac{2\pi i q}{2^n}\right)$, where $q \in \mathbb{Z}_2^n$. In [23], the set of QFD gates is introduced and defined by

$$\tau^{(\ell)}_R \triangleq \sum_{v \in \mathbb{Z}_2^n} \xi^{\ell Rv} \mod 2^\ell \left| v \right\rangle \left\langle v \right|,$$  (5)

where $\xi \triangleq \exp\left(\frac{2\pi i q}{2^n}\right)$. $R \in \mathbb{Z}_2^{n \times n}$ is symmetric, $\left| v \right\rangle = e_v$ is the standard basis vector in $\mathbb{C}_2^{n}$ with a 1 in the entry indexed by $v \in \mathbb{Z}_2^n$, and $\left| v \right\rangle \left\langle v \right|$ is the projector. It is shown that all 1- and 2-local diagonal gates in the hierarchy are QFD, e.g., $T = \tau^{(3)}_1$.

Moreover, their action on Pauli operators is characterized by

$$\tau^{(\ell)}_R E(a, b) (\tau^{(\ell)}_R)^\dagger = \xi^{\ell R(a, b, \ell)} E(a_0, b_0 + a_0 R) \tau^{(\ell - 1)}_{R(R(a, \ell))},$$  (6)

$$\phi(R, a, b, \ell) \triangleq (1 - 2^{\ell - 2}) a_0 R a_0^\dagger + 2^{\ell - 1} (a_1 b_1^T + b_0 a_1^T),$$

$$\hat{R}(R, a, \ell) \triangleq (1 + 2^{\ell - 2}) D_{a_0 R} - (D_{a_0} D_{a_0} R a_0^\dagger + D_{a_0} R D_{a_0} a_0^\dagger + D_{a_0 R} D_{a_0} a_0),$$

Equation (6) naturally extends the action (5) to a large class of diagonal unitaries, e.g., $TXT^\dagger = e^{-i\pi/4}YP, P \triangleq \sqrt{Z}$.

Note that the symplectic matrix in this case is $\Gamma_R = \begin{bmatrix} I_n & R \\ 0 & I_n \end{bmatrix}$ (defined over $\mathbb{Z}_2^n$), which also satisfies $\Gamma_R \Omega \Gamma_R^T = \Omega (\pmod{2})$. Here, $D_x$ represents a diagonal matrix with the diagonal set to the vector $x$, and $\bar{x} = 1 - x$ with $1$ representing the vector whose entries are all 1. We write $a = a_0 + 2a_1 + 4a_2 + \ldots b = b_0 + 2b_1 + 4b_2 + \ldots \in \mathbb{Z}^n$ with $a_i, b_i \in \mathbb{Z}_2$. With this notation, $b_0 + a_0 R$ is an integer sum and the definition of $E(a, b)$ has been suitably generalized to integer vectors $a, b$ (see [23]).

C. Stabilizer Codes

A stabilizer group $S$ is a commutative subgroup of $\HW_N$ with Hermitian elements that does not contain $-I_N$. If $S$ has $r$ generators, then it can be expressed as $S = \langle v_i, E(c_i, d_i); i = 1, \ldots, r \rangle$, where $v_i \in \{\pm 1\}$ and $E(c_i, d_i), E(c_j, d_j)$ commute for all $i, j \in \{1, \ldots, r\}$, i.e., $\langle [c_i, d_i], [c_j, d_j] \rangle \equiv 0 (\pmod{2})$.

Given a stabilizer $S$, the associated $[n, k, d]$ stabilizer code is
defined as \( V(S) \triangleq \{ |\psi\rangle \in \mathbb{C}^N : g |\psi\rangle = |\psi\rangle \) for all \( g \in S \), where \( k \triangleq n - r \) and \( d \) is the distance of the code that is defined as the minimum weight of an undetectable error.

A Calderbank-Shor-Steane (CSS) code has a set of purely X-type and purely Z-type stabilizer generators. Consider two classical binary codes \( C_1, C_2 \) such that \( C_2 \subset C_1 \), and let \( C_1^+, C_2^+ \) represent their respective dual codes. Then, \( C_1^+ \subset C_2^+ \) and the stabilizer for the resulting CSS code is given by \( S \triangleq \{ \nu \varepsilon E(c,0), \nu_\mu E(0,d), c \in C_2, d \in C_1^+ \} \) for some suitable \( \nu, \nu d \in \{ \pm 1 \} \). Let \( C_1 \) be an \([n, k_1]\) code and \( C_2 \) be an \([n, k_2]\) code such that \( C_1 \) and \( C_2^+ \) can correct up to \( t \) errors. Then, \( S \) defines an \([n, k_1 - k_2, 2t + 1]\) CSS code that we will denote by CSS\((X, C_2; Z, C_1^+)\). If \( G_2 \) and \( G_1^+ \) are generator matrices for the codes \( C_2 \) and \( C_1^+ \), respectively, then a generator matrix for the binary representation of stabilizers can be written as

\[
G_S = \begin{bmatrix}
  n \\
  G_2 \\
  G_1^+
\end{bmatrix} - k_1 \quad (9)
\]

For any \( S \), the projector onto the code \( V(S) \) is given by

\[
\Pi_S \triangleq \frac{1}{2} \sum_{i=1}^{r} (|I_N + \nu_i E(c_i, d_i)\rangle \langle I_N + \nu_i E(c_i, d_i)|)
\]

\[
= \frac{1}{2^r} \sum_{j=1}^{2^r} \epsilon_j E(a_j, b_j), \quad \text{(10)}
\]

where \( \epsilon_j \in \{ \pm 1 \} \) in the last equality is determined by the product of signs of the generators of \( S \) that multiply to produce the stabilizer element \( E(a_j, b_j) \).

### III. STABILIZER CODES SUPPORTING QFD GATES

In order to perform universal fault-tolerant quantum computation with stabilizer QECCs, we need to identify fault-tolerant realizations of the necessary logical operators. For logical Pauli operators, there are at least two known algorithms \([5], [24]\) to translate them into the relevant physical Pauli operators for stabilizer codes. At the second level of the Clifford hierarchy, for logical Clifford gates, there have been several works that determine fault-tolerant realizations on specific codes or code families. In \([19], [25]\) we developed a systematic and efficient algorithm using symplectic matrices to translate logical Clifford circuits into physical Clifford circuits for any stabilizer code. Although this Logical Clifford Synthesis (LCS) algorithm currently does not guarantee fault-tolerance of the solutions, a better understanding of the symplectic solution space might help us achieve that objective.

For non-Clifford gates, the lack of a symplectic formalism and the fact that Paulis are not mapped to Paulis under conjugation together make synthesis of logical non-Clifford gates much harder. Therefore, our first goal is to understand the structure required in the stabilizer so that a specified (non-Clifford) gate preserves the code subspace. In this paper we restrict ourselves to physical QFD gates since we have an extension of the symplectic formalism for these gates. We will discuss two steps involved in achieving this goal and solve the transversal \( T \) special case completely. For proofs, refer to \([11]\).

**Step 1:** Express QFD action on Pauli matrices in Pauli basis.

First we expand \( \tau_R^{(f)} = \sum_{x \in \mathbb{Z}_2^n} c_{x, R}^{(f)} x \cdot \frac{1}{\sqrt{2^n}} E(0, x) \), where

\[
c_{x, R}^{(f)} \triangleq \text{Tr} \left[ \frac{E(0, x)}{\sqrt{2^n}} \tau_R^{(f)} \right] = \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_2^n} (-1)^{x^T \xi + \nu R x^T}. \quad (11)
\]

Applying this for \( \tau_R^{(f)} \) in \( \text{Step 1} \) we get, assuming \( a, b \in \mathbb{Z}_2^n \)

\[
\tau_R^{(f)} E(a, b) \left( \tau_R^{(f)} \right)^{\dagger} = \xi^{(R, a, b,\ell)} E(a, b + aR) \left( \tau_R^{(f)} \right)^{\dagger}
\]

\[
= \xi^{(R, a, b,\ell)} E(a, b + aR) \cdot \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_2^n} c_{x, R(a, \ell), x} E(0, x)
\]

\[
= \frac{\xi^{(R, a, b,\ell)}}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_2^n} c_{x, R(a, \ell), x} x^{-a x^T} E(a, b + aR + x). \quad (12)
\]

The primary problem here is to determine which coefficients are non-zero for given \( R, a, \ell \), and to compute their values.

**Lemma 1:** Let \( E(a, b) \in HW_N \), for some \( a, b \in \mathbb{Z}_2^n \). Then the transversal \( T \) gate acts on \( E(a, b) \) as

\[
T^{\otimes n} E(a, b) \left( T^{\otimes n} \right)^{\dagger} = \frac{1}{2^{w_H(a)/2}} \sum_{y \leq a} (-1)^{y^T a} E(a, b \oplus y),
\]

where \( w_H(a) = aa^T \) is the Hamming weight of \( a \), and \( y \leq a \) denotes that support of \( y \) is contained in the support of \( a \).

For the general case where each qubit is acted upon by a possibly different integer power of \( T \), we provide the result in \([11]\). These formulae may be of independent interest.

**Step 2:** Determine conditions on \( S \) for \( \tau_R^{(f)} \Pi_S \left( \tau_R^{(f)} \right)^{\dagger} = \Pi_S \).

We focus on the above equality because this is the necessary and sufficient condition for a (QFD) unitary to preserve the code subspace (see \([11]\) for a simple argument). By expanding the above equality for \( T^{\otimes n} \) using the result in Step 1, we get

\[
T^{\otimes n} \Pi_S \left( T^{\otimes n} \right)^{\dagger} = \frac{1}{2^{w_H(a)/2}} \sum_{y \leq a} (-1)^{y^T a} E(a, b \oplus y),
\]

\[
= \frac{1}{2^r} \sum_{j=1}^{2^r} w_H(a_j) \sum_{y \leq a_j} (-1)^{y^T a_j} E(a_j, b_j \oplus y). \quad (14)
\]

This needs to equal \( \text{Step 1} \) and the following characters that.

**Theorem 2:** Let \( S = \{ \nu_i E(c_i, d_i) : i = 1, \ldots, r \} \) define a stabilizer code, with arbitrary \( \nu_i \in \{ \pm 1 \} \), and denote the elements of \( S \) by \( \epsilon_j E(a_j, b_j), j = 1, 2, \ldots, 2^r \). If the transversal application of the \( T \) gate preserves the code space \( V(S) \) and hence realizes a logical operation on \( V(S) \), then:

1) For any \( \epsilon_j E(a_j, b_j) \in S, w_H(a_j) \) is even, where \( w_H(a_j) \) represents the Hamming weight of \( a_j \in \mathbb{Z}_2^n \).

2) For any \( \epsilon_j E(a_j, b_j) \in S \) with non-zero \( a_j \), define \( Z_j \triangleq \{ z \leq a_j : \epsilon_j E(0, z) \in S \) for some \( \epsilon_z \in \{ \pm 1 \} \}. \) Then \( Z_j \) contains its dual computed only on the support of \( a_j \), i.e., on the ambient dimension \( w_H(a_j) \). Equivalently, \( Z_j \) contains a dimension \( w_H(a_j)/2 \) self-dual code \( A_j \) that is supported on \( a_j \), i.e., there exists a subspace \( A_j \subseteq Z_j \).
such that $yz^T = 0 \pmod{2}$ for any $y, z \in A_j$ (including $y = z$) and $\dim(A_j) = w_H(a_j)/2$.

3) Let $\tilde{Z}_j \subseteq \mathbb{Z}^{w_H(a_j)}$ represent $Z_j$ with all positions outside the support of $a_j$ punctured (dropped). Then, for each $z \in \mathbb{Z}^n$ such that $\tilde{z} \in (\tilde{Z}_j)^\perp$ for some $j \in \{1, \ldots, 2^n\}$, we have $\varepsilon_z = e^{z^T x}$, i.e., $z^T E(0, z) \in S$. Here, $(\tilde{Z}_j)^\perp$ denotes the dual of $Z_j$, taken over this punctured space with ambient dimension $w_H(a_j)$. (Also, $Z_j \supseteq (\tilde{Z}_j)^\perp$ with zeros added outside the support of $a_j$.) Conversely, if the first two conditions above are satisfied, and if the third condition holds for all $z \in A_j$ instead of just the dual of (the punctured) $Z_j$, then transversal $T$ preserves the code space $V(S)$ and hence induces a logical operation. We will illustrate this theorem using a simple CSS example. Example 1: Define a [6, 2, 2] CSS code by the matrix

$$G_S = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$ (15)

The right half of the last 3 rows form the generators of $Z_S$ for this code. Since there is only one non-trivial $a_j$ in this case, we see that $Z_S = A_1$ with $a_1 = [1, 1, 1, 1, 1, 1]$. Hence, the stabilizer generators are $X^{\otimes 6} = X_1X_2 \cdots X_6, -Z_1Z_2, -Z_3Z_4, -Z_5Z_6$, since the generators of $Z_S$ have weight 2. Multiplying $X^{\otimes 6}$ and the product of these three $Z$-stabilizers, we see that $Y^{\otimes 6} \subseteq S$.

We can define the logical $X$ operators for this code to be $\tilde{X}_1 = X_1X_2, \tilde{X}_2 = X_3X_4$, since these are linearly independent and commute with all stabilizers. Then we observe

$$T^{\otimes 6}X_1X_2(T^{\otimes 6})^\dagger = e^{-i2\pi/3}(Y_1P_1)(Y_2P_2)$$

$$= -i(x_1X_1Z_1P_1)(x_2X_2Z_2P_2)$$

$$\equiv -i(x_1x_2)(P_1P_2),$$ (16) (17) (18)

since $-Z_1Z_2 \subseteq S$. We observe that $(P_1P_2)X^{\otimes 6}(P_1P_2)^\dagger = Y_1Y_2X_3X_4X_5X_6 = X^{\otimes 6}$ up to the stabilizer $-Z_1Z_2$, so $P_1P_2$ indeed preserves $V(S)$. But $(P_1P_2)(X_1X_2)(P_1P_2)^\dagger = Y_1Y_2 = (X_1X_2)(-Z_1Z_2) \equiv X_1X_2$, and $P_1P_2$ obviously commutes with $X_1X_2$, so $P_1P_2$ is essentially the logical identity gate. A similar reasoning holds for $P_3P_4$. Therefore, up to a global phase, the transversal $T$ preserves the logical operators $\tilde{X}_1$ and $\tilde{X}_2$, so in this case the transversal $T$ gate realizes just the logical identity (up to a global phase). This can also be checked explicitly by writing the logical basis states.

Given that $S$ has the necessary structure given by Theorem 2, note that we can freely add another $Z$-stabilizer generator that commutes with $X^{\otimes 6}$, e.g., $Z_1Z_3Z_4Z_6 \leftrightarrow [1, 0, 1, 1, 0, 1] \notin Z_S$. This preserves the transversal $T$ property: once $T^{\otimes 6} \Pi_S(T^{\otimes 6})^\dagger = \Pi_S$, mapping $\Pi_S \mapsto \Pi_S \cdot \frac{(I_6 + E(0, z))}{2}$ preserves equality since $E(0, z)$ is diagonal.
Clifford corrections as in [8], if and only if the matrix $G_1$ is triorthogonal and the following condition holds for all $a \in C_2$:

$$x = \bigoplus_{i=1}^{k} c_i x_i, \quad c_i \in \{0,1\} \Rightarrow w_h(x \oplus a) \equiv w_h(c) \pmod{8}.$$  

**Corollary 8**: The triorthogonal construction introduced by Bravyi and Haah [8] is the most general CSS family that realizes logical transversal $T$ from physical transversal $T$.

**Proof**: The strategy is to show that the weight condition in Theorem [7] is equivalent to the condition one obtains by setting the Clifford correction in [8] to be trivial (see [11]).

Note that if the weight condition in Theorem [7] is replaced by the condition that $E(x,0) \in X \Rightarrow \rho_{w_h(x)}E(0,x) \in S$, then the induced logical operator is trivial, i.e., the logical identity $[11]$. Since for CSS-T codes we already have $C_2 \subset C_T^0$, this condition is equivalent to the constraint $C_1 \subset C_T^0$.

**B. Logical Controlled-Controlled-Z Gates from Transversal T**

The gate $CCZ = \text{diag}(1,1,1,1,1,1,1,-1)$ belongs to $C^{(3)}$ and enables universal computation when combined with $C^{(2)}$. One of the simplest codes that realizes logical $CCZ$ from physical transversal $T$ is Campbell’s $[8,3,2]$ (CSS) “smallest interesting color code” [27]. In our notation, this code is described by setting $C_2$ to be the 8-bit repetition code $RM(0,3)$ and $C_1 = C_T^0$ to be the $[8,4,4]$ extended Hamming code, which is also the self-dual Reed-Muller code $RM(1,3)$.

A general class of polynomial evaluation codes, called decreasing monomial codes (DMCs), were introduced by Bardet et al. [28]. While a Reed-Muller code $RM(r,m)$ is generated by all binary $m$-variate monomials of degree up to $r \leq m$, DMCs allow one to include all monomials up to degree $r-1$ and a subset of degree-$r$ monomials according to a partial order. This provides greater design freedom, and we refer to [28] for a description of some code properties.

**Example 2**: Recently, Krishna and Tillich used DMCs to construct triorthogonal codes from punctured polar codes for magic state distillation [29]. We are able to construct a $[16,3,2]$ CSS code from DMCs where transversal $T$ realizes logical $CCZ$. Define the code $C_2$ as the space generated by the monomials $G_2 = \{1, x_1, x_2\}$, and the code $C_1$ as the space generated by $G_1 = G_2 \cup \{x_3, x_4, x_1 x_2\}$. Hence, the logical Z group is generated by $G_X = \{x_3, x_4, x_1 x_2\}$. Using [28] it is easy to see that $G_T^1 = \{1, x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4\}$. So the logical Z group is generated by $G_Z = \{x_1 x_2 x_3, x_1 x_2 x_4, x_3 x_4\}$. To see that this code satisfies Theorem [2] consider for example the X-stabilizer corresponding to the monomial $x_1 \in G_2$. We observe that the elements $x_1, x_1 x_2, x_1 x_3, x_1 x_4 \in G_T^1$ are supported on $x_1$. When we project down to $x_1$, we get the monomials $1, x_1 x_2, x_1 x_3, x_1 x_4$ that precisely generate the code $RM(1,3)$ that is self-dual. A similar analysis can be made for other elements in $C_2$. Moreover, since the elements in $G_T^1$ have weights 4, 8, or 16, the last condition of Theorem [2] does not introduce any negative signs for the $Z$-stabilizers. We believe this is not just one special case but points towards using this formalism for a general construction of CSS codes that support transversal $Z$-rotations. In [11] we also discuss connections to pin codes [18], quasitransversality [16] and the generalized triorthogonality [17] conditions for CSS codes to realize logical CCZs from transversal $T$.

Finally, we describe a $[2^m, (m+1), 2^m]$ quantum Reed-Muller (QRM) family that we generalize to support transversal finer angle $Z$-rotations in [11]. We also characterize the exact induced logical operation through $AX$’s theorem on residue weights of polynomials [30]. For the $T$ case, QRM$(r,m)$ is described by $C_1 = RM(r,m)$ and $C_2 = RM(r-1,m)$, where $\frac{m-1}{3} < r \leq \frac{m}{3}$ ensures that transversal $T$ preserves the code space and induces a non-trivial logical gate. This has close connections to [17]. The argument to show that QRM$(r,m)$ satisfies Theorem [2] is very similar to the $[16,3,2]$ example.

**Example 3**: We use the $[64,15,4]$ code to demonstrate the general form of the logical operation. Here, the logical qubits $v_f \in Z_\mathbb{C}^{15}$ are identified with the degree $r = 2$ monomials that define generators for logical $X$ operators. Hence, we have

$$|v_f\rangle_L = |v_{x_1 x_2}\rangle_L \otimes |v_{x_3 x_4}\rangle_L \otimes \cdots \otimes |v_{x_5 x_6}\rangle_L \in \mathbb{C}^{15}.$$  

(The $f$ will be clarified shortly.) The logical gate induced by $T^\otimes 64$ is described by $U^L|v_f\rangle_L = (-1)^{f(v)}|v_f\rangle_L, q(v_f) = v_{x_1 x_2} v_{x_3 x_4} v_{x_5 x_6} + v_{x_1 x_2} v_{x_3 x_5} v_{x_4 x_6} + v_{x_1 x_2} v_{x_3 x_6} v_{x_4 x_5} + v_{x_1 x_3} v_{x_2 x_4} v_{x_5 x_6} + v_{x_1 x_3} v_{x_2 x_5} v_{x_4 x_6} + v_{x_1 x_3} v_{x_2 x_6} v_{x_4 x_5} + v_{x_1 x_4} v_{x_2 x_3} v_{x_5 x_6} + v_{x_1 x_4} v_{x_2 x_5} v_{x_3 x_6} + v_{x_1 x_4} v_{x_2 x_6} v_{x_3 x_5} + v_{x_1 x_5} v_{x_2 x_3} v_{x_4 x_6} + v_{x_1 x_5} v_{x_2 x_4} v_{x_3 x_6} + v_{x_1 x_5} v_{x_2 x_6} v_{x_3 x_4} + v_{x_1 x_6} v_{x_2 x_3} v_{x_4 x_5} + v_{x_1 x_6} v_{x_2 x_4} v_{x_3 x_5} + v_{x_1 x_6} v_{x_2 x_5} v_{x_3 x_4}.$$  

where each term in the polynomial corresponds to a logical $CCZ$ gate acting on the three logical qubits indexed by the three monomial subscripts, and the sum corresponds to a product of such gates (in the logical unitary space).

Recall that for $v_f \in Z_\mathbb{C}^{15}$ the CSS basis states are given by

$$|v_f\rangle_L = \frac{1}{|C_2|} \sum_{c \in C_2} |v_f \cdot G_{C_1/C_2} + c\rangle.$$  

For QRM$(r,m)$, the rows of $G_{C_1/C_2}$ correspond to degree $r$ monomials, each identifying a logical qubit. So a non-trivial logical $X$ operator is described by a degree $r$ polynomial $f$, but only the degree $r$ terms determine which logical qubits are acted upon. This implies that each degree $r$ term in $f$ sets the corresponding logical qubit to $|1\rangle_L$ if $f(v_f) = 0$ (initially). For this code, the rows of $G_{C_1/C_2}$ are evaluations of the 15 degree 2 monomials, namely $x_1 x_2, x_1 x_3, x_1 x_4, \ldots, x_5 x_6$. So the polynomial $f \in RM(r,m)$ above is a linear combination of degree $r = 2$ monomials, and possibly lower degree monomials (that correspond to just X-type stabilizers). Hence, $v_f \in Z_\mathbb{C}^{15}$ exactly describes which corresponding rows of $G_{C_1/C_2}$ are chosen in this linear combination. Therefore, if $f = x_1 x_2 + x_3 x_4 + x_5 x_6 + (\text{smaller degree terms})$, then $v_{x_1 x_2} v_{x_3 x_4} v_{x_5 x_6} = 1$ and other logical qubits are set to $|0\rangle_L$, so $q(v_f) = 1$. But if $f = x_1 x_2 + x_3 x_4 + x_5 x_6 + x_3 x_5 + x_4 x_6 + (\text{smaller degree terms})$, then $q(v_f) = 0$ as this $f$ corresponds to two CCZs applying the phase $-1$.■
