Three pearls of Bernoulli numbers

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Abstract

The Bernoulli numbers are fascinating and ubiquitous numbers; they occur in several domains of Mathematics like Number theory (FLT), Group theory, Calculus and even in Physics. Since Bernoulli’s work, they are yet studied to understand their deep nature \cite{9}, \cite{6} and particularly to find relationships between them. In this paper, we give, firstly, a short response \cite{15} to a problem stated, in 1971, by Carlitz \cite{4} and studied by many authors like Prodinger \cite{10}; the second pearl is an answer to a question raised, in 2008, by Tom Apostol \cite{1}. The third pearl is another proof of a relationship already given in 2011, by the authors \cite{14}.

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1 Introduction

The aim of this work is to give original proofs of three relationships involving Bernoulli numbers. In the first section, we give a short proof to a problem stated, in 1971, by Carlitz \cite{4} and studied by many authors like Prodinger \cite{10}. In the second section, we give a response to a question raised by Apostol in 2008 in his relevant paper \cite{1}. In the third section, we expose a different proof of a relationship already given by us in 2011 \cite{15}.

2 Pearl #1: Carlitz’s Problem

In Mathematics Magazine, Vol. 44, No. 2 (Mar., 1971), pp. 105-114+101, Carlitz states the following problem:

\[ \sum_{k=0}^{n} \binom{n}{k} B_k = B_n \]

show that for arbitrary \( m, n > 0 \)

\[ (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k} \]

This identity was firstly proved by Shannon \cite{11} in 1971, by Gessel \cite{7} in 2003, by Wu, Sun and Pan \cite{13} in 2004, by Vassilev-Missana \cite{12} in 2005, by Chen and Sun \cite{5} in 2009, by Gould and J. Quaintance \cite{8} in 2014 and by Prodinger \cite{10} in 2014. The Prodinger’s proof is very short and uses a two variables formal series. In fact, one can see that Carlitz’s problem can be easily deduced from the following relationship already proved in 2012 by Benchérif and Garici in 2012 \cite{3}:

\[ (-1)^m \sum_{k=0}^{m+q} \binom{m+q}{k} \binom{n+q+k}{q} B_{n+k} = (-1)^{n+q} \sum_{k=0}^{n+q} \binom{n+q}{k} \binom{m+q+k}{q} B_{m+k} = 0. \]

Hereafter, we give a proof different from that was given by Prodinger.
Proof. We consider the linear functional $L$ defined on $\mathbb{Q}[x]$ by $L(x^n) = B_n$ for $n \geq 0$, which gives

$$L\left(\left(x + \frac{1}{2}\right)^{2n+1}\right) = B_{2n+1} \left(\frac{1}{2}\right) = 0,$$

see $[\Pi]$, p.182, then the polynomial defined by :

$$P(x) = (-1)^{m+q}x^{n+q}(1 + x)^{m+q} - (-1)^{n}x^{m+q}(1 + x)^{n+q}$$

satisfies

$$P\left(\frac{1}{2} + x\right) + (-1)^q P\left(-\frac{1}{2} - x\right) = 0$$

and

$$P^{(q)}\left(-\frac{1}{2} + x\right) + P^{(q)}\left(-\frac{1}{2} - x\right) = 0$$

Now, with use of the equality :

$$L\left(\left(x + \frac{1}{2}\right)^{2n+1}\right) = B_{2n+1} \left(\frac{1}{2}\right) = 0.$$

And as $P^{(q)}$ is an even polynomial, we get :

$$L\left(P^{(q)}(x)\right) = 0$$

Thus

$$\frac{1}{q!} P^{(q)}(x) = (-1)^{m+q} \sum_{k=0}^{n+q} \binom{m+q}{k} \binom{n+q+k}{q} x^{n+k} - (-1)^n \sum_{k=0}^{n+q} \binom{n+q}{k} \binom{m+q+k}{q} x^{m+k} = 0$$

and finally:

$$(-1)^{m} \sum_{k=0}^{n+q} \binom{m+q}{k} \binom{n+q+k}{q} B_{n+k} - (-1)^n \sum_{k=0}^{n+q} \binom{n+q}{k} \binom{m+q+k}{q} B_{m+k} = 0$$

which yields the identity wanted by Carlitz, by taking $q = 0$. \hfill \Box

3 Pearl #2: APOSTOL’S PROBLEM

In his relevant paper published in 2008, Tom Apostol writes: we leave it as a challenge to the reader to find another proof of (42) as a direct consequence of (3) without the use of integration. In Apostol’s paper, (42) denotes the relationship:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{(n+2-k)} = \frac{B_{n+1}}{n+1}, \quad n \geq 1$$

and (3) is one of the six definitions of the Bernoulli numbers he recalls to show his relationship and which is:

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad \text{for } n \geq 2$$

As he said it, Apostol uses integration method to deduce his (42)-numbered relation from the Bernoulli numbers’s definition that he has chosen. To take up the challenge he has launched, we expose a proof without use of integration method.
Proof. (Answer to Apostol’s problem)

Let’s define the sequence \( (u_n) \) by:

\[
 u_n := \sum_{k=0}^{n} \binom{n+1}{k} B_k
\]

We can see that \( u_n = 0 \) for \( n \geq 1 \). Writing:

\[
 \binom{n}{k} \frac{1}{n+2-k} = \frac{1}{n+1} \binom{n+1}{k} - \frac{1}{(n+1)(n+2)} \binom{n+2}{k}
\]

we get:

\[
 \sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{n+2-k} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k - \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \binom{n+2}{k} B_k
\]

which yields:

\[
 \sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{n+2-k} = \frac{1}{n+1} u_n - \frac{1}{(n+1)(n+2)} \left( u_{n+1} - \frac{n+2}{n+1} B_{n+1} \right)
\]

As if \( n \geq 1 \), we have \( u_n = u_{n+1} = 0 \) and as \( \binom{n+2}{n+1} = n+2 \), we get:

\[
 -\frac{1}{(n+1)(n+2)} \left( - \frac{n+2}{n+1} B_{n+1} \right) = \frac{B_{n+1}}{n+1}
\]

which gives the asked relation:

\[
 \sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{n+2-k} = \frac{B_{n+1}}{n+1}, \quad n \geq 1
\]

Finally, Apostol’s relationship is proved without use of integration method.

\[\square\]

4 Pearl #3: New proof of a relationship

In our paper [15], we proved the following relationship:

\[
 \sum_{k=0}^{n+q} \binom{n+q}{k} \left( \prod_{j=1}^{q} (n+k+j) \right) B_{n+k} = 0
\]

where \( q \) is an odd number. For this, we showed that the two well-suited polynomials:

\[
 H_n(x) = \frac{1}{2} x^{n+q}(x-1)^{n+q}
\]

and

\[
 K_n(x) = \sum_{k=0}^{n+q} \frac{\varepsilon_{n+k}}{n+q+k+1} \binom{n+q}{k} B_{n+q+k+1}(x) - B_{n+q+k+1}
\]

Are equal, where \( B_n(x) \) and \( B_n \) are respectively the Bernoulli polynomials and the Bernoulli numbers defined by the generating function:

\[
 \frac{x}{e^x - 1} e^{xz} = \sum_{n=0}^{+\infty} B_n(x) \frac{x^n}{n!}
\]

knowing that \( B_n = B_n(0) = B_n(1), n \geq 2; B_{2n+1} = 0, n \geq 1 \), see [1], relations (11), (12) (13) and (15). Furthermore, we shall use the well-known equalities:

\[
 B_n(x+1) - B_n(x) = nx^{n-1}, \quad B_n'(x) = nB_{n-1}(x), \quad \int_{x}^{x+1} B_n(t) dt = x^n, \quad n \geq 0
\]
for \( n \geq 1 \), see e.g. [1], relations (14), (27) and (30).

Now, to give another proof of the relation already proved in [15], we consider the polynomials:

\[
P_n(x) := \frac{1}{2} x^{n+1} (x-1)^{n+1}
\]

\[
K_n(x) := \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(x)
\]

\[
H_n(x) := \frac{1}{2} (n+1)x^n (x-1)^n (2x-1)
\]

and the automorphism of the \( \mathbb{Q} \)-space vector \( \mathbb{Q}[x] \) defined by \( f(P(x)) = \int_x^{x+1} P(t)dt \).

First of all, let’s prove the

**Theorem 4.1.** The two polynomials \( K_n(x) \) and \( H_n(x) \) are equal, i.e. \( K_n(x) = H_n(x) \).

**Proof.**

\[
\int_x^{x+1} P'_n(t)dt = P_n(x+1) - P_n(x)
\]

\[
= \frac{1}{2} x^{n+1}(x+1)^{n+1} - \frac{1}{2} x^{n+1}(x-1)^{n+1}
\]

\[
= \frac{1}{2} x^{n+1}((x+1)^{n+1} - (x-1)^{n+1})
\]

\[
= x^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} x^k
\]

\[
= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} x^{n+1+k}
\]

\[
= \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} \int_x^{x+1} B_{n+1+k}(t)dt
\]

\[
= \int_x^{x+1} \left( \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(t) \right)
\]

Thus, we can see that

\[
f(P'_n(t)) = f \left( \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(t) \right)
\]

As \( f \) is bijective, we get:

\[
P'_n(t) = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1 - (-1)^{n+1-k}}{2} B_{n+1+k}(t)
\]

i.e.

\[
P'_n(t) = K_n(t)
\]

Let’s compute \( P'_n(x) \):

\[
P'_n(x) = \frac{1}{2} (x^2 - x)^{n+1}
\]

\[
= \frac{1}{2} (n+1)(x^2 - x)^n (2x - 1)
\]

\[
= \frac{1}{2} (n+1)x^n (x-1)^n (2x-1)
\]

\[
= H_n(x)
\]
Theorem 4.2. The following identity holds:

\[ \sum_{k=0}^{n+q} \binom{n+q}{k} \left( \prod_{j=1}^{q} (n+k+j) \right) B_{n+k} = 0 \]

Proof. To get this, let’s replace \( n \) by \( n + q - 1, \) \( q \geq 1, \) \( q \) odd. Then we compute the coefficient of \( x^q \) in the equality: \( K_{n+q-1}(x) = H_{n+q-1}(x). \) The coefficient of \( x^q \) in the polynomial \( K_n(x) \)

\[ K_n(x) := \sum_{k=0}^{n+q} \binom{n+q}{k} \left( \prod_{j=1}^{q} (n+k+j) \right) \frac{1 - (-1)^{n+q-k}}{2} B_{n+1+k}(x) \]

is :

\[ C_q := [x]^q B_{n+q-1}(x), \]

so that \( C_q \) has the value

\[ C_q = \sum_{k=0}^{n+q} \binom{n+q}{k} \left( \prod_{j=1}^{q} (n+k+j) \right) \frac{1 - (-1)^{n+q-k}}{2} B_{n+1+k} \]

On the other hand, the coefficient of \( x^q \) in \( H_{n+q-1}(x) \) is :

\[ \begin{cases} 
0 & \text{if } n \geq 1 \\
(-1)^{q+1} q & \text{if } n = 0
\end{cases} \]

As we have :

\[ \frac{1 + (-1)^m}{2} B_m = \begin{cases} 
B_m & \text{if } m \neq 1 \\
0 & \text{if } m = 1
\end{cases} \]

We get the aimed relationship.

Remark 4.3. I would like to dedicate this modest contribution to the memory of Tom Mike Apostol who passed away on 8 May of this year 2016.(APO2)

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