A TQFT OF TURAEV–VIRO TYPE ON SHAPED TRIANGULATIONS

RINAT KASHAEV, FENG LUO, AND GRIGORY VARTANOV

ABSTRACT. A shaped triangulation is a finite triangulation of an oriented pseudo three
manifold where each tetrahedron carries dihedral angles of an ideal hyperbolic tetrahe-
dron. To each shaped triangulation, we associate a quantum partition function in the form
of an absolutely convergent state integral which is invariant under shaped $3 - 2$ Pachner
moves and invariant with respect to shape gauge transformations generated by total dihe-
dral angles around internal edges through the Neumann–Zagier Poisson bracket. Similarly
to Turaev–Viro theory, the state variables live on edges of the triangulation but take their
values on the whole real axis. The tetrahedral weight functions are composed of three hy-
perbolic gamma functions in a way that they enjoy a manifest tetrahedral symmetry. We
conjecture that for shaped triangulations of closed 3-manifolds, our partition function is
twice the absolute value squared of the partition function of Techmüller TQFT defined by
Andersen and Kashaev. This is similar to the known relationship between the Turaev–Viro
and the Witten–Reshetikhin–Turaev invariants of three manifolds. We also discuss inter-
pretations of our construction in terms of three-dimensional supersymmetric field theories
related to triangulated three-dimensional manifolds.

1. INTRODUCTION

Topological Quantum Field Theories were discovered and axiomatized by Atiyah [2],
Segal [38] and Witten [51]. First examples in $2 + 1$ dimensions were constructed by
Reshetikhin and Turaev [35, 36, 47] by using the combinatorial framework of Kirby cal-
culus, and by Turaev and Viro [48] by using the framework of triangulations and Pachner
moves. The algebraic ingredients of both constructions come from the finite dimensional
representation category of the quantum group $U_q(sl(2))$ at roots of unity. For example,
the basic building elements in Turaev–Viro construction are tetrahedral weight functions
given by $6j$-symbols. These theories have been the subject of much subsequent investi-
gation in the works of Blanchet, Habegger, Masbaum, Vogel, Barrett, Westbury, Turaev,
Virelizier, Balsam, Kirillov and others [7, 8, 6, 49, 4]. A related but somewhat different
line of development was initiated by Kashaev in [27] where a state sum invariant of links
in three manifolds was defined by using the combinatorics of charged triangulations where
the charges are algebraic versions of dihedral angles of ideal hyperbolic tetrahedra in finite
cyclic groups. This approach has been subsequently developed by Baseilhac, Benedetti,
Geer, Kashaev, Turaev [3, 20]. The common feature of all these theories is that the partition
functions are always given by finite state sums.

On the other hand, the idea of partition functions of Turaev–Viro type originates from
the work of Ponzano and Regge [35] where, based on $SU(2) 6j$-symbols, a lattice version
of quantum $2 + 1$ gravity was suggested, but this theory was not complete and remained
of restricted use because of problems of convergence of infinite sums. Similar problems of
convergence appear when one tries to construct combinatorial versions of quantum Chern–
Simons theories with non-compact gauge groups. For example, a connected component
of $PSL(2, \mathbb{R})$ Chern–Simons theory is identified with Teichmüller space, and its quantum

Supported in part by Swiss National Science Foundation and United States National Science Foundation.
theory corresponds to specific class of unitary mapping class group representations in infinite dimensional Hilbert spaces \cite{26,9}. Based on quantum Teichmüller theory, formal state-integral partition functions of triangulated three manifolds were defined by Hikami, Dimofte, Gukov, Lenells, Zagier, Dijkgraaf, Fuji, Manabe \cite{22,23,14,10,11}, mostly for the purposes of quasi classical expansions, but the question of convergence remained largely open until a mathematically rigorous version of Teichmüller TQFT was suggested in \cite{11}. The convergence property of Teichmüller TQFT is due its specific underlying combinatorial setting: it is not just triangulations but shaped triangulations where each tetrahedron carries dihedral angles of an ideal hyperbolic tetrahedron. Moreover, the role of dihedral angles is two-fold: they not only provide absolute convergence of state integrals but they also implement the complete symmetry with respect to change of edge orientations. Although, shaped triangulations are similar to charged triangulations of \cite{27}, the positivity condition of dihedral angles imposes important restrictions on construction of topologically invariant partition functions.

The purpose of this paper is to suggest yet another TQFT based on combinatorics of shaped triangulations. As its basic building block is defined in terms of Faddeev’s quantum dilogarithm \cite{17} and the absolute convergence of partition functions relies on the positivity of dihedral angles, it is similar to the Teichmüller TQFT. As a consequence, we are still restricted in our abilities of constructing topologically invariant partition functions in the sense that the 2–3 shaped Pachner move is not always applicable. On the other hand, unlike the Teichmüller TQFT, our tetrahedral weight functions enjoy manifest tetrahedral symmetry and the partition function is well defined on any shaped triangulation without any extra topological restrictions.

Let us now describe our construction in precise terms.

1.1. States, state potentials, and state gauge invariance. Let $Y$ be a CW-complex. Denote by $\Delta_i(Y)$ the set of $i$-dimensional cells of $Y$. A state of $Y$ is a map $s: \Delta_1(Y) \to \mathbb{R}$. A state potential is a map $g: \Delta_0(Y) \to \mathbb{R}$. Define a linear state gauge map

$$b: \mathbb{R}^{\Delta_0(Y)} \to \mathbb{R}^{\Delta_1(Y)}, \quad bg(e) = g(\partial_0 e) + g(\partial_1 e), \quad \text{(1)}$$

where $\partial_i e, i \in \{0, 1\}$, are the two end points of $e$ (they coincide if the edge is a loop). A state is called pure gauge if it finds itself in the image of the state gauge map. The pure gauge states constitute a vector subspace of the state space.

Let $S$ be a set. A function $f: \mathbb{R}^{\Delta_1(Y)} \to S$ is called state gauge invariant at state $s$ if $f(s + bg) = f(s)$ for any state potential $g$.

A (state) gauge fixing at vertex $v \in \Delta_0(Y)$ is a linear form $\lambda$ on the vector space of states $\mathbb{R}^{\Delta_1(Y)}$ such that

$$\langle \lambda, bg \rangle = g(v), \quad \forall g \in \mathbb{R}^{\Delta_0(Y)}, \quad \text{(2)}$$

Note that a gauge fixing at a vertex may not exist if the state gauge map is not injective.

In what follows, a real valued function defined on only a subset of vertices will always be thought of as a state potential having zero values on the vertices where initially it was not defined.

1.2. Shaped tetrahedra and their Boltzmann weights. Let $T$ be an oriented tetrahedron embedded into $\mathbb{R}^3$ together with its standard CW-complex structure. Let $\Box(T)$ be the set of normal quadrilateral types (to be called quads) in $T$ which is in bijection with the set of pairs of opposite edges of $T$. We fix the action of $\mathbb{Z}/3\mathbb{Z} = \{1, \tau, \tau^2\}$ on $\Box(T)$ so that the images of a quad $q$ under the action are $q, q' = \tau(q)$ and $q'' = \tau^2(q)$ corresponding to the clockwise cyclic order of three edges around a vertex (as seen from the outside
of the tetrahedron). We say $T$ is shaped tetrahedron if it is provided with a dihedral angle map $\alpha : \square(T) \to [0, \pi]$, such that $\alpha(q) + \alpha(q') + \alpha(q'') = \pi$. Associated to $\alpha$, the complex shape variables entering Thurston’s hyperbolicity equations are given by a map $z_\alpha : \square(T) \to \mathbb{C} \setminus \{0, 1\}$ defined by the formula

$$z_\alpha(q) = e^{i\alpha(q)} \sin \alpha(q'') / \sin \alpha(q').$$  \hspace{1cm} (3)

Any state $s : \Delta_1(T) \to \mathbb{R}$ induces a map $\tilde{s} : \square(T) \to \mathbb{R}$ defined by the formula $\tilde{s}(q) = s(e) + s(e')$, where the $e$ and $e'$ are the opposite edges separated by $q$. To each pair $(T, s)$ consisting of a shaped tetrahedron $T$ and a state $s$ of $T$, we associate the following Boltzmann weight

$$B(T, s) := \prod_{q \in \square(T)} \gamma(2) \left( \frac{\omega_1 + \omega_2}{\pi} \alpha(q) + \sqrt{\omega_1 \omega_2} (\tilde{s}(q) - \tilde{s}(q'')) ; \omega_1, \omega_2 \right).$$  \hspace{1cm} (4)

where function $\gamma(2)(z; \omega_1, \omega_2)$ is defined below in \(22\) with $\omega_1, \omega_2 \in \mathbb{C}$ and $\omega_1 / \omega_2 \notin (-\infty, 0]$. It is easily verified that this Boltzmann weight is state gauge invariant at any state.

1.3. Shaped triangulations and their Boltzmann weights. A triangulation is an oriented pseudo 3-manifold obtained from finitely many tetrahedra in $\mathbb{R}^3$ by gluing them along triangular faces through orientation reversing affine $CW$-homeomorphisms. Any triangulation $X$ is naturally a $CW$-complex and its boundary $\partial X$ is the $CW$-subcomplex composed of unglued triangular faces. We will use the following notation:

$$\Delta_i(X) := \Delta_i(X) \setminus \Delta_i(\partial X).$$  \hspace{1cm} (5)

A shaped triangulation is a triangulation where all tetrahedra are shaped. Similarly to the case of one shaped tetrahedron, to each pair $(X, s)$ consisting of a shaped triangulation $X$ and a state $s$ of $X$, we associate a Boltzmann weight

$$B(X, s) := \prod_{T \in \Delta(s)} B(T, s|_{\Delta_i(T)}).$$  \hspace{1cm} (6)

Again, this Boltzmann weight is state gauge invariant at any state.

1.4. The partition function of shaped triangulations. A boundary state of a triangulation $X$ is a state of its boundary. We have the natural linear restriction map from the vector space of states of $X$ to the vector space of its boundary states

$$\partial : \mathbb{R}^{\Delta_1(X)} \to \mathbb{R}^{\Delta_1(\partial X)},$$

and for any boundary state $s$, we have a canonical identification of the preimage $\partial^{-1}(s)$ with the linear space $\mathbb{R}^{\Delta_1(X)}$ of real valued functions on the interior edges of $X$.

A state gauge fixing in the interior of a triangulation $X$ is a collection

$$\lambda = \{\lambda_v \}_{v \in \Delta_0(X)}$$  \hspace{1cm} (7)

defined at all interior vertices. Notice that for any triangulation the state gauge map is injective and state gauge fixings exist at any vertex.

To any triple $(X, s, \lambda)$, where $X$ is a shaped triangulation, $s$ is a boundary state of $X$, and $\lambda$ is a state gauge fixing in the interior of $X$, we associate a partition function

$$W_b(X, s, \lambda) := \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda, t \rangle) dt,$$  \hspace{1cm} (8)
where
\[
\delta(\langle \lambda, t \rangle) := \prod_{v \in \Delta_0}(\hat{X}) \delta(\langle \lambda_v, t \rangle), \quad dt := \prod_{v \in \Delta_1}(\hat{X}) dt(e) \tag{9}
\]
and \(b = \sqrt{\frac{\omega_1}{\omega_2}}\). The main result of this paper is the following theorem where we use the notions of shaped \(3 - 2\) Pachner moves and shape gauge transformations considered in \([1]\).

**Theorem 1.** The partition function \(W_b(X, s, \lambda)\) is an absolutely convergent integral independent of the choice of the state gauge fixing \(\lambda\), invariant under shaped \(3 - 2\) Pachner moves, and invariant under the shape gauge transformations induced by interior edges.

Several examples of explicit calculations make us to believe that for shaped triangulations of closed 3-manifolds, when the Teichmüller TQFT is defined as well, our partition function is twice the absolute value squared of the partition function of the Teichmüller TQFT. This is similar to the known relationship between the Turaev–Viro and the Witten–Reshetikhin–Turaev invariants of three manifolds.

**Conjecture 1.** Let \((X, \ell_X)\) be an admissible shaped levelled branched triangulation of a closed oriented compact three manifold in the sense of \([1]\). Then the following equality holds true

\[
2 |F_h(X, \ell_X)|^2 = W_b(X, \lambda) \tag{10}
\]

where \(h = (b + b^{-1})^{-2} \in \mathbb{R}_{>0}\).

The rest of this paper is organized as follows. Section 2 contains the proof of the main Theorem 1. In Section 3, we derive the pentagon identity which underlies the invariance of our partition function with respect to shaped \(3 - 2\) Pachner move from the elliptic beta-integral. In Section 4 we provide examples of concrete calculations which justify Conjecture 1. Section 5 is devoted to some considerations from the perspective of 3d supersymmetric field theories. Namely, based on our construction we get a class of 3d supersymmetric field theories defined on a squashed three-sphere \(S_3^b\) related to triangulated three-dimensional manifolds. The latter relation is known as 3d/3d correspondence which is the topic of recent study \([44, 12, 13, 45]\). Appendices contain some technical information on the special functions used.

**Acknowledgements.** We would like to thank the organizers of following events during which our collaboration in this work took place: the conference “Groupes de difféotopie et topologie quantique”, Strasbourg, 25–29 June 2012; the “International congress on mathematical physics”, Aalborg, 6–11 August 2012; the workshop “New Perspectives in Topological Field Theories”, Hamburg, 27–31 August, 2012. We also appreciate the supports by Swiss National Science Foundation and ITGP (Interactions of Low-Dimensional Topology and Geometry with Mathematical Physics), an ESF RNP, and US National Science Foundation.

2. **Proof of Theorem 1**

**Lemma 1.** Let \(X\) be a shaped triangulation, and let \(s\) and \(s'\) be states of \(X\) such that \(B(X, s' + ts) = B(X, s')\) for any \(t \in \mathbb{R}\). Then the state \(s\) is in the image of the state gauge map.
Proof. By a straightforward verification, the statement of the lemma is true if $X$ is a disjoint union of unglued tetrahedra. Thus, it suffices to prove that if triangulation $X$ is obtained from a triangulation $Y$ by identification of two triangular faces $f$ and $f'$, and the statement of the lemma is true for $Y$, then it is also true for $X$.

Denote by $p: Y \rightarrow X$ the identification projection, and by $p^* : \mathbb{R}^{\Delta_0(X)} \rightarrow \mathbb{R}^{\Delta_0(Y)}$ the corresponding pull-back maps. Let $s$ and $s'$ be states of $X$ such that

$$B(X, s' + ts) = B(X, s'), \quad \forall t \in \mathbb{R}. \quad (11)$$

Using the fact that $B(X, r) = B(Y, p^*(r))$ for any state $r$ of $X$, equation (11) is equivalent to

$$B(Y, p^*(s') + tp^*(s)) = B(Y, p^*(s')) , \quad \forall t \in \mathbb{R}. \quad (12)$$

As we assume that the statement of the lemma is true for $Y$, there exists $g \in \mathbb{R}^{\Delta_0(Y)}$ such that $p^*(s) = bg$. Let us show that there exists $g' \in \mathbb{R}^{\Delta_0(X)}$ such that $g = p^*(g')$. Indeed, let triangles $f$ and $f'$ have respective vertices $v_i$ and $v'_i$ and edges $e_i$ and $e'_i$ for $i \in \{1, 2, 3\}$ such that

$$\partial e_i = \{v_j, v_k\}, \quad \partial e'_i = \{v'_j, v'_k\}, \quad \{i, j, k\} = \{1, 2, 3\}, \quad (13)$$

and

$$p(e_i) = p(e'_i), \quad p(v_i) = p(v'_i), \quad i \in \{1, 2, 3\}. \quad (14)$$

That means that when applied to edges $e_i$ and $e'_i$, the equality $p^*(s) = bg$ gives

$$g(v_i) + g(v_k) = g(v'_i) + g(v'_k) \iff g(v_i) - g(v'_i) = \xi := \sum_{m=1}^{3} (g(v_m) - g(v'_m)). \quad (15)$$

Taking sum over $i$ in the last equation, we obtain $\xi = 3\xi \iff \xi = 0$ which implies that $g(v_i) = g(v'_i)$ for any $i \in \{1, 2, 3\}$, i.e. $g = p^*(g')$. Thus, we have the equality $p^*(s) = bp^*(g') = p^*(bg')$, and as $p^*$ is injective, we conclude that $s = bg'$.

Proof of Theorem 1\footnote{Similarly to QED, our system is linear and the Faddeev–Popov determinant is trivial so that no ghosts are needed.}. By injectivity of the state gauge map in the case of triangulations and Lemma 1 the state gauge map image of the group $\mathbb{R}^{\Delta_0(X)}$ is the maximal translation subgroup of the state space of $X$ which leaves invariant the boundary state $s$ and the Boltzmann weight $B(X, t)$. On the other hand, the product of delta functions $\delta(\lambda, t)$ restricts the integral to a hyperplane in the space $\mathbb{R}^{\Delta_0(X)} \simeq b^{-1}(s)$ which intersects any orbit of this group action in a unique point, while the Boltzmann weight exponentially decays along any direction in this hyperplane. This implies that the integral in (8) is absolutely convergent.

Independence on the choice of the state gauge fixing $\lambda$ easily follows through the use of a simplest finite-dimensional version of the Faddeev–Popov trick in path integrals for gauge invariant systems\footnote{Indeed, if $t$ is a state of $X$ and $\lambda'$ a state gauge fixing in the interior of $X$, then we have the identity

$$1 = \int_{\mathbb{R}^{\Delta_0(X)}} \delta((\lambda', t + bg)) \, dg,$$

where

$$dg := \prod_{v \in \Delta_0(X)} dg(v). \quad (17)$$} [29]. Indeed, if $t$ is a state of $X$ and $\lambda'$ a state gauge fixing in the interior of $X$, then we have the identity

$$1 = \int_{\mathbb{R}^{\Delta_0(X)}} \delta((\lambda', t + bg)) \, dg,$$
Inserting (16) into (8), exchanging the order of integrations, shifting the integration state variables, using the gauge invariance of the Boltzmann weight, again exchanging the order of integrations, and again using identity (16) with $\lambda'$ replaced by $\lambda$, we obtain

$$\begin{align*}
W_b(X, s, \lambda) &= \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda, t \rangle) \, dt = \\
&= \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda, t \rangle) \left( \int_{\mathbb{R}^2(X)} \delta (\langle \lambda', t + bg \rangle) \, dg \right) \, dt \\
&= \int_{\mathbb{R}^2(X)} \left( \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda, t \rangle) \delta (\langle \lambda', t + bg \rangle) \, dt \right) \, dg \\
&= \int_{\mathbb{R}^2(X)} \left( \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda', t \rangle) \delta (\langle \lambda, t - bg \rangle) \, dt \right) \, dg \\
&= \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda', t \rangle) \left( \int_{\mathbb{R}^2(X)} \delta (\langle \lambda, t - bg \rangle) \, dg \right) \, dt \\
&= \int_{\partial^{-1}(s)} B(X, t) \delta (\langle \lambda', t \rangle) \, dt = W_b(X, s, \lambda'). \quad (18)
\end{align*}$$

Invariance under 3 - 2 shaped Pachner moves is a consequence of the shaped pentagon identity for the tetrahedral Boltzmann weights, which in turn is equivalent to identity (11), provided the relevant integration variable does not enter the product of delta-functions $\delta (\langle \lambda, t \rangle)$. This condition can always be satisfied by appropriate choice of $\lambda$.

Finally, the gauge transformation in the space of dihedral angles induced by an edge $e$, see (1), is equivalent to an imaginary shift of the integration variable $s(e)$, which, by using the holomorphicity of the Boltzmann weights, can be compensated by an imaginary shift of the integration path in the complex $s(e)$-plane.

\section{Pentagon identities from elliptic beta-integral}

Let us start from Spiridonov’s elliptic beta-integral (39)

$$\kappa \int_T \prod_{i=1}^6 \frac{\Gamma(s_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \, dz \, \frac{dz}{2\pi iz} = \prod_{1 \leq i < j \leq 6} \Gamma(s_i s_j; p, q), \quad (19)$$

where parameters $z = \{s_1, \ldots, s_6\}$ satisfy the so-called balancing condition $\prod_{i=1}^6 s_i = pq$. Here

$$\kappa = \frac{(p; p)_\infty (q; q)_\infty}{2},$$

where $(z; p)_\infty = \prod_{i=0}^\infty (1 - z p^i)$. Also in (19) the building block is the elliptic gamma function defined as

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \quad (20)$$

with $|z| < 1$ and two basis parameters $|p|, |q| < 1$. Here we use the following useful conventions

$$\Gamma(a, b; p, q) = \Gamma(a; p, q) \Gamma(b; p, q), \quad \Gamma(a z^{\pm 1}; p, q) = \Gamma(a z; p, q) \Gamma(a z^{-1}; p, q).$$
The elliptic gamma function has the following limit when all its parameters and the two basis parameters simultaneously go to unity \(37\)
\[
\Gamma(e^{2\pi i z}; e^{2\pi i \omega_1}, e^{2\pi i \omega_2}) = e^{-\pi i (2z - \omega_1 - \omega_2)/12r} \gamma(2)(z; \omega_1, \omega_2),
\]
where on the right hand side one has the so-called hyperbolic gamma function
\[
\gamma(2)(u; \omega_1, \omega_2) = e^{-\pi i B_{2,2}(u; \omega_1, \omega_2)/2} \frac{(e^{2\pi i u/\omega_1} q; q)_\infty}{(e^{2\pi i u/\omega_2}; q)_\infty},
\]
with the redefined basis parameters
\[
q = e^{2\pi i \omega_1/\omega_2}, \quad \bar{q} = e^{-2\pi i \omega_2/\omega_1},
\]
and \(B_{2,2}(u; \omega_1, \omega_2)\) denoting the second order Bernoulli polynomial,
\[
B_{2,2}(u; \omega_1, \omega_2) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.
\]
The conventions,
\[
\gamma(2)(a, b; \omega_1, \omega_2) \equiv \gamma(2)(a; \omega_1, \omega_2) \gamma(2)(b; \omega_1, \omega_2),
\]
and
\[
\gamma(2)(a \pm u; \omega_1, \omega_2) \equiv \gamma(2)(a + u; \omega_1, \omega_2) \gamma(2)(a - u; \omega_1, \omega_2),
\]
are applied further.

Also we are going to use the following reduction of the hyperbolic gamma function \(37\)
\[
\gamma(2)(z; \omega_1, \omega_2) = \lim_{\omega_2 \to \infty} \left( \frac{\omega_2}{2\pi \omega_1} \right)^{1/4} \frac{1}{\sqrt{\pi}} \Gamma(z/\omega_1),
\]
where \(\Gamma(u)\) is the usual gamma function. The relation between the hyperbolic gamma function and Faddeev’s quantum dilogarithm is presented in the Appendix where we collect all the definitions and properties of the special functions.

### 3.1. The First Pentagon

Let us start from elliptic beta-integral \(19\) and reparametrize parameters as
\[
s_i = e^{2\pi i \omega_i}, i = 1, \ldots, 6; \quad z = e^{2\pi i \omega_1}; \quad p = e^{2\pi i \omega_1}; \quad q = e^{2\pi i \omega_2},
\]
and use the limit of elliptic gamma function \(21\) to get
\[
\frac{1}{2} \int_{-1/\infty}^{1/\infty} \prod_{i=1}^{6} \gamma(2)(\alpha_i; \omega_1, \omega_2) du = \prod_{1 \leq i < j \leq 6} \gamma(2)(\alpha_i + \alpha_j; \omega_1, \omega_2),
\]
where the balancing condition becomes \(\sum_{i=1}^{6} \alpha_i = \omega_1 + \omega_2\).

To get the new form of the pentagon one should proceed as follows. Let us take reparameterization \(41\)
\[
\alpha_i = \mu + a_i, \quad \alpha_{i+3} = -\mu + b_i, i = 1, 2, 3,
\]
which preserves the balancing condition and consider the limit \(\mu \to \infty\). We use the inversion relation
\[
\gamma(2)(z, \omega_1 + \omega_2 - z; \omega_1, \omega_2) = 1
\]
and the asymptotic formulas
\[
\lim_{u \to \infty} e^{-\frac{\pi}{2} B_{2,2}(u; \omega)} \gamma(2)(u; \omega) = 1, \quad \text{for } \arg \omega_1 < \arg u < \arg \omega_2 + \pi,
\]
\[
\lim_{u \to \infty} e^{-\frac{\pi}{2} B_{2,2}(u; \omega)} \gamma(2)(u; \omega) = 1, \quad \text{for } \arg \omega_1 - \pi < \arg u < \arg \omega_2,\]


and shifting the integration variable $u \to u + \mu$ to get
\[
\int_{-\infty}^{\infty} \prod_{i=1}^{3} \frac{\gamma(2)(a_i - u, b_i + u; \omega_1, \omega_2)}{\sqrt{\omega_1 \omega_2}} \, du = \prod_{i,j=1}^{3} \gamma(2)(a_i + b_j; \omega_1, \omega_2),
\] (29)
with \( \sum_{i=1}^{3} (a_i + b_i) = \omega_1 + \omega_2 \). Let us introduce now the following function
\[
\mathcal{B}(x, y) = \frac{\gamma(2)(x, y; \omega_1, \omega_2)}{\gamma(2)(x + y; \omega_1, \omega_2)},
\] (30)
after which we rewrite (29) as
\[
\int_{-\infty}^{\infty} \prod_{i=1}^{3} \mathcal{B}(a_i - u, b_i + u) \frac{du}{\sqrt{\omega_1 \omega_2}} = \mathcal{B}(a_2 + b_1, a_3 + b_2) \mathcal{B}(a_1 + b_2, a_3 + b_1),
\] (31)
where we used the inversion relation for the hyperbolic gamma function. The geometric meaning of the pentagon relation (31) can be seen in the figure below. Note that by the inversion formula (27), we have
\[
\mathcal{B}(x, y) = \frac{\gamma(2)(x; \omega_1, \omega_2)}{\gamma(2)\omega_1 + \omega_2 - x - y; \omega_1, \omega_2}\gamma(2)(y; \omega_1, \omega_2)\gamma(2).\]
Therefore, the right hand-side of (31) is the Boltzmann weight for the union of two tetrahedra and the left-hand-side of (31) is the integration of the Boltzmann weight of three tetrahedra.

**Figure 1.** 2-3 moves

From (31) we see that the function \( \mathcal{B} \) satisfies pentagon identity. It will be natural to suggest that the original elliptic beta-integral should also satisfy some kind of pentagon identity, but to our knowledge it is not realized so far. Recently in papers [5] it was realized that the elliptic beta-integral satisfies the Yang-Baxter star-triangle relation (see also [41]) with the Boltzman weight \( W_{\alpha}(x, y) = \Gamma(e^{\alpha x}x^\pm 1 y_{\pm 1}^\pm 1; p, q) \). So, instead of \( 2 - 3 \) Pachner move the elliptic beta-integral (19) satisfies \( 3 - 3 \) Pachner move [28] which might be relevant for construction of quantum invariants of four-dimensional manifolds. Moreover, the elliptic hypergeometric integrals describe specific partition functions of 4d \( \mathcal{N} = 1 \) SYM theories known as superconformal indices [15, 42]. Combining these facts together one expects that triangulations of four-dimensional manifolds can be connected to four-dimensional \( \mathcal{N} = 1 \) supersymmetric field theories.
One has the following orthogonality relations for the $B$ function:

$$
\int_{\mathbb{R}} B(a - i\nu, b + i\nu) B(-a - i\nu, -b + i\nu) \frac{du}{\sqrt{\omega_1\omega_2}} = 2i\sqrt{\omega_1\omega_2}\delta(a - b),
$$

$$
\int_{\mathbb{R}} B(a - i\nu, a + i\nu) B(-a - i(u + b), -a + i(u + b)) \frac{du}{\sqrt{\omega_1\omega_2}} = 2\sqrt{\omega_1\omega_2}\delta(b).
$$

### 3.2. The second pentagon.

Let us rewrite (31) as

$$
\int_{-i\infty}^{i\infty} \prod_{i=1}^{3} \gamma^{(2)}(a_i - u; \omega_1, \omega_2) \prod_{i=1}^{2} \gamma^{(2)}(b_i + u; \omega_1, \omega_2) \frac{du}{i\sqrt{\omega_1\omega_2}} = \prod_{i,j=1}^{3} \gamma^{(2)}(a_i + b_j; \omega_1, \omega_2),
$$

Applying the limit $\omega_2 \to \infty$ to (33) and using (24) we get

$$
\int_{-i\infty}^{i\infty} B(a_1 + u, b_1 - u) B(a_2 + u, b_2 - u) B(a_3 + u, a_1 + a_2 + b_1 + b_2) \frac{du}{2\pi i} = B(a_2 + b_1, a_3 + b_2) B(a_1 + b_2, a_3 + b_1),
$$

where $B(x, y)$ is the usual beta-function

$$
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
$$

Taking the limit $\omega_2 \to \infty$ in (32) one gets analogous orthogonality relations for the beta-integral.

### 4. Examples of calculations

We will use the following notation

$$
\Delta := (\omega_1 + \omega_2)/\pi, \quad \nabla := \sqrt{\omega_1\omega_2},
$$

and also

$$
u(x) := c_b \left(1 - \frac{x}{\pi}\right).
$$

Further we will use the $\psi$ function defined as

$$
\psi(x, y) := \Psi(x, -x, y) = \int_{\mathbb{R}} \frac{\Phi_b(t + x)}{\Phi_b(t - x)} e^{2\pi i yt} dt
$$

see also (101). We also have the equality

$$
B(\Delta \alpha + i\nabla x, \Delta \beta + i\nabla y) = \psi \left(\frac{x}{2}, y - 2c_b \frac{\beta}{\pi}\right)
$$

which is equivalent to (100).
4.1. One vertex H-triangulation of \((S^3, 3_1)\). Following [1], we have one tetrahedron \(T\) with linearly ordered vertices (enumerated by 0, 1, 2, 3) and with the face identifications
\[
\partial_i T \simeq \partial_{3-i} T, \quad i \in \{0, 1\},
\]
represented by diagram

\[\text{Diagram}
\]

The quotient space \(X\) is a triangulation of \(S^3\) with only one vertex \(v\) and two edges: 
\(e_1\) knotted like trefoil and having as preimage the only edge \(03\) of \(T\), and \(e_2\) having as preimages all other five edges of \(T\). The Boltzmann weight reads
\[
B(X, s) = B(\Delta \alpha_0, \Delta \alpha_1 + i \nabla (s_2 - s_1))
\]
where \(\alpha_i := \alpha(q_i)\) with \(q \in \square(T)\) being the quad corresponding to the opposite edge pair \((03, 12)\) of \(T\), and \(s_1 := s(e_1)\). Choosing the gauge fixing map \(\lambda\) so that \(\langle \lambda, e \rangle = s_1/2\), we obtain the following integral for the partition function:
\[
W_b(S^3, 3_1) = \int_{\mathbb{R}^2} B(\Delta \alpha_0, \Delta \alpha_1 + i \nabla (s_2 - s_1)) \delta(s_1/2) ds_1 ds_2
\]
\[
= 2 \int_{\mathbb{R}} B(\Delta \alpha_0, \Delta \alpha_1 + i \nabla s_2) ds_2
\]
which under substitution of (38) is calculated as follows
\[
W_b(S^3, 3_1) = 2 \int_{\mathbb{R}} \psi \left( u(\alpha_0), s_2 - 2c_b \frac{\alpha_1}{\pi} \right) ds_2
\]
\[
= 2 \int_{\mathbb{R}^2} \frac{\Phi_b(t + u(\alpha_0))}{\Phi_b(t - u(\alpha_0))} e^{2\pi i (s_2 - 2c_b \frac{\alpha_1}{\pi})} dt ds_2
\]
\[
= 2 \int_{\mathbb{R}} \frac{\Phi_b(t + u(\alpha_0))}{\Phi_b(t - u(\alpha_0))} \delta(t) e^{-4icb \alpha_1 t} dt
\]
\[
= 2 \frac{\Phi_b(u(\alpha_0))}{\Phi_b(-u(\alpha_0))} = 2 |\Phi_b(u(\alpha_0))|^2.
\]
As in [1], the partition function diverges in the H-balanced limit \(\alpha_0 \to 0\), so that it makes sense only to consider the ratios of partition functions. Thus, we define the renormalized partition function \(\hat{W}_b(S^3, 3_1) = 1\).

4.2. One vertex H-triangulation of \((S^3, 4_1)\). In the graphical notation of [1], let \(X\) be given by the diagram where the figure-eight knot is represented by the edge of the cen-

\[\text{Diagram}
\]

tral tetrahedron \(T\) connecting the maximal and the next to maximal vertices. This H-triangulation of \((S^3, 4_1)\) consists of two positive \(T\) and \(T_L\) to the left from the central tetrahedron and one negative \(T_R\) to the right. One has the following identification of the faces
\[
\partial_0 T \simeq \partial_1 T, \quad \partial_2 T \simeq \partial_1 T_L, \quad \partial_3 T \simeq \partial_0 T_R,
\]
\[
\partial_0 T_L \simeq \partial_2 T_R, \quad \partial_2 T_L \simeq \partial_0 T_R, \quad \partial_3 T_L \simeq \partial_1 T_R.
\]
Identifying the corresponding edges, one gets

\[
\begin{align*}
z & \equiv x_{13} = x_{03} = x_{23}^L = x_{03}^L = x_{13}^R, \\
y & \equiv x_{12} = x_{02} = x_{01}^L = x_{12}^R = x_{12}^R, \\
x & \equiv x_{01} = x_{02}^L = x_{12}^L = x_{13}^R = x_{03}^R = x_{23}^R,
\end{align*}
\]

and one has also the edge \( x' \equiv x_{23} \). Then for the partition function we have

\[
W_b(S^3, 4_1) = \int_{\mathbb{R}^4} B(\Delta \alpha_1, \Delta \alpha_2 + i\nabla(y + z - x - x'))
\]

\[
\times B(\Delta \beta_1 + i\nabla(x - z), \Delta \beta_2 + i\nabla(x - y))
\]

\[
\times B(\Delta \gamma_1 + i\nabla(x - z), \Delta \gamma_2 + i\nabla(x - y)) \delta(x/2) dx dy dz dx'
\]

\[
= 2 \int_{\mathbb{R}^3} B(\Delta \alpha_1, \Delta \alpha_2 + i\nabla(y + z - x'))
\]

\[
\times B(\Delta \beta_1 - i\nabla z, \Delta \beta_2 - i\nabla y) B(\Delta \gamma_1 - i\nabla z, \Delta \gamma_2 - i\nabla y) dy dz dx'
\]

\[
= 2 \tilde{W}_b(S^3, 4_1) \int_{\mathbb{R}} B(\Delta \alpha_1, \Delta \alpha_2 + i\nabla t) dt = 2 \tilde{W}_b(S^3, 4_1) |\Phi_b(u(\alpha_1))|^2
\]

where

\[
\tilde{W}_b(S^3, 4_1) := \frac{W_b(S^3, 4_1)}{2|\Phi_b(u(\alpha_1))|^2}
\]

\[
= \int_{\mathbb{R}^3} B(\Delta \beta_1 - i\nabla z, \Delta \beta_2 - i\nabla y) B(\Delta \gamma_1 - i\nabla z, \Delta \gamma_2 - i\nabla y) dy dz.
\]

Now, using (38) and (101) we continue as follows:

\[
\tilde{W}_b(S^3, 4_1) = \int_{\mathbb{R}^2} \psi \left( u(\beta_1) - \frac{z}{2}, -2c_b \frac{\beta_2}{\pi} - y \right) \psi \left( u(\gamma_1) - \frac{z}{2}, -2c_b \frac{\gamma_2}{\pi} - y \right) dy dz
\]

\[
= \int_{\mathbb{R}^4} \Phi_b \left( u(\beta_1) - \frac{z}{2} + s \right) \Phi_b \left( u(\gamma_1) - \frac{z}{2} + t \right) e^{-2\pi i(s+t)\nu-4ic_o(s\beta_2+\tau_2)} ds dt dy dz
\]

\[
= \int_{\mathbb{R}^3} \Phi_b \left( u(\beta_1) - \frac{z}{2} + s \right) \Phi_b \left( u(\gamma_1) - \frac{z}{2} + t \right) e^{-4ic_o(s\beta_2+\tau_2)} \delta(s + t) ds dt dz
\]

\[
= \int_{\mathbb{R}^2} \Phi_b \left( u(\beta_1) - \frac{z}{2} - t \right) \Phi_b \left( u(\gamma_1) - \frac{z}{2} + t \right) e^{4ic_o(t(\beta_2-\gamma_2))} dt dz
\]

\[
= \{ t \mapsto t + \frac{z}{2} \} \int_{\mathbb{R}^2} \Phi_b \left( -u(\beta_1) - t \right) \Phi_b \left( u(\gamma_1) + t \right) e^{4ic_o(t+\frac{z}{2})(\beta_2-\gamma_2)} dt dz
\]

\[
= \{ z \mapsto z - t \} = \int_{\mathbb{R}^2} \Phi_b \left( -u(\beta_1) - z \right) \Phi_b \left( u(\gamma_1) + z \right) e^{2ic_o(t+z)(\beta_2-\gamma_2)} dt dz
\]

As the complete balancing conditions take the form \( \beta_1 = \gamma_1 \) and \( \beta_2 = \gamma_2 \), we finally obtain

\[
\tilde{W}_b(S^3, 4_1) = \left| \int_{\mathbb{R} \setminus i0} \frac{\Phi_b(-z)}{\Phi_b(z)} dz \right|^2.
\]
4.3. **One vertex H-triangulation of \((S^3, 5_2)\).** In the graphical notation of [1], let \(X\) be given by the diagram

![Diagram](image)

One vertex H-triangulation of \((S^3, 5_2)\) consists of 4 tetrahedra: \(T\) which is a negative tetrahedron and is sitting in the center of the above picture, and \(T_1, T_2, T_3\) are positive tetrahedra \((T_3\) is on the left from the central tetrahedron, \(T_2\) on the right and \(T_3\) is on top). One has to identify the following faces of four tetrahedra:

\[
\partial_0 T \simeq \partial_1 T, \quad \partial_2 T \simeq \partial_0 T_2, \quad \partial_3 T \simeq \partial_0 T_1, \quad \partial_0 T_1 \simeq \partial_2 T_3, \quad \partial_2 T_1 \simeq \partial_0 T_2, \quad \partial_1 T_2 \simeq \partial_0 T_3, \quad \partial_2 T_2 \simeq \partial_1 T_3.
\]  

(48)

From the identification of the faces we get the following equalities for the edges

\[
x_{01} = x_{12} = x_{01}^{(2)} = x_{02}^{(1)} = x_{13}^{(3)} = x_{13}^{(1)}; \\
x_{02} = x_{12} = x_{01}^{(2)} = x_{03}^{(1)} = x_{13}^{(2)}; \\
x_{03} = x_{13} = x_{23}^{(3)} = x_{23}^{(2)}; \\
x_{01}^{(1)} = x_{02}^{(2)} = x_{03}^{(3)} = x_{12}^{(1)} = x_{13}^{(2)} = x_{13}^{(3)} = x_{03}^{(1)},
\]

(49)

and just \(x_{23}\). Here, the superscripts \((1), (2)\) and \((3)\) refer to tetrahedra \(T_i, i = 1, 2, 3\).

The partition function for this one vertex H-triangulation of \((S^3, 5_2)\) is equal to the integral of the product of four Boltzmann weights of four tetrahedra

\[
W_b(S^3, 5_2) = \int_{\mathbb{R}^5} B(\Delta \alpha_1, \Delta \alpha_2 + i \nabla(x_1 + x_2 - 3 - x_2)) \\
\times B(\Delta \beta_1 - i \nabla(x_3' - x_1), \Delta \beta_2 - i \nabla(x_1 - x_2)) \\
\times B(\Delta \gamma_1 - i \nabla(x_3 - x_3), \Delta \gamma_2 - i \nabla(x_3 - x_3')) \\
\times B(\Delta \delta_1 - i \nabla(x_3 - x_1), \Delta \delta_2 - i \nabla(x_2 + x_3 - 3x_3')) \delta(x_3'/2dx_2dx_1dx_2dx_3dx_3') \\
= 2\hat{W}_b(S^3, 5_2) \int_\mathbb{R} B(\Delta \alpha_1, \Delta \alpha_2 + i \nabla t)dt
\]

(50)

where \(x_i = x_{0i}, i = 1, 2, 3\) and also we denote \(x_3' = x_{03}^{(i)}\). Here one has

\[
\hat{W}_b(S^3, 5_2) = \int_{\mathbb{R}^3} dx_1dx_2dx_3 B(\Delta \beta_1 + i \nabla x_1, \Delta \beta_2 - i \nabla(x_1 - x_2)) \\
\times B(Q - \Delta \gamma_1 - \Delta \gamma_2 + i \nabla x_1, \Delta \gamma_2 - i \nabla x_3) \delta(x_3/2dx_2dx_1dx_2dx_3dx_3') \\
= \int_{\mathbb{R}^3} \psi \left(u(\beta_1) + \frac{x_1}{2}, -x_1 + x_2 - 2c_\beta \frac{\beta_2}{\pi}\right) \psi \left(u(\pi - \gamma_1 - \gamma_2) + \frac{x_1}{2}, -x_3 - 2c_\gamma \frac{\gamma_2}{\pi}\right) \\
\times \psi \left(u(\delta_1) + \frac{x_1}{2}, -x_2 - x_3 - 2c_\delta \frac{\delta_2}{\pi}\right) dx_1dx_2dx_3,
\]

(51)
then writing the definition for $\psi(a, b)$ one gets

$$\tilde{W}_b(S^3, 5_2) = \int_{R^6} dx_1 dx_2 dx_3 ds dt du \quad \times \quad \frac{\Phi_b(u(\beta_1) + \frac{x_1}{2} + s) \Phi_b(u(\pi - \gamma_1 - \gamma_2) + \frac{x_2}{2} + s) \Phi_b(u(\delta_1) + \frac{x_1}{2} + u)}{\Phi_b(-u(\beta_1) - \frac{x_1}{2} - s) \Phi_b(-u(\pi - \gamma_1 - \gamma_2) - \frac{x_2}{2} - s) \Phi_b(-u(\delta_1) - \frac{x_1}{2} - u)}$$

$$\times e^{2\pi i(-x_1 + x_2)x - 4ic_b\beta_2s} e^{-2\pi i x_1 - 4ic_b\gamma_2t} e^{-2\pi i x_2 - 4ic_b\delta_2u}$$

$$= \int_{R^4} dx_1 ds dt du \quad \frac{\Phi_b(u(\beta_1) + \frac{x_1}{2} + s)}{\Phi_b(-u(\beta_1) - \frac{x_1}{2} - s)} \quad \frac{\Phi_b(u(\pi - \gamma_1 - \gamma_2) + \frac{x_2}{2} + s)}{\Phi_b(-u(\pi - \gamma_1 - \gamma_2) - \frac{x_2}{2} - s)}$$

$$\times \frac{\Phi_b(u(\delta_1) + \frac{x_1}{2} + u)}{\Phi_b(-u(\delta_1) - \frac{x_1}{2} - u)} e^{-2\pi i x_1 s - 4ic_b\beta_2s} e^{-4ic_b\gamma_2t} e^{-4ic_b\delta_2u}$$

$$= \int_{R^2} dx_1 du \quad e^{-2\pi i x_1 u - 4ic_b(\beta_1 - \gamma_1 - \gamma_2)}$$

$$\times \frac{\Phi_b(u(\beta_1) + \frac{x_1}{2} + u) \Phi_b(u(\pi - \gamma_1 - \gamma_2) + \frac{x_2}{2} + u) \Phi_b(u(\delta_1) + \frac{x_1}{2} + u)}{\Phi_b(-u(\beta_1) - \frac{x_1}{2} + u) \Phi_b(-u(\pi - \gamma_1 - \gamma_2) - \frac{x_2}{2} + u) \Phi_b(-u(\delta_1) - \frac{x_1}{2} + u)}$$

Changing the integration variables

$$x_1 \rightarrow \frac{x_1}{2} + u, \quad x_2 = u - \frac{x_1}{2},$$

we get

$$\tilde{W}_b(S^3, 5_2) = \int_{R^2} dx_1 dx_2 e^{-\pi i(x_1^2 - x_2^2)} e^{-2ic_b(\beta_2 - \gamma_2 + \delta_2)(x_1 + x_2)}$$

$$\times \frac{\Phi_b(u(\beta_1) + x_1)}{\Phi_b(-u(\beta_1) + x_1)} \quad \frac{\Phi_b(u(\pi - \gamma_1 - \gamma_2) + x_2)}{\Phi_b(-u(\pi - \gamma_1 - \gamma_2) - x_2)} \quad \frac{\Phi_b(u(\delta_1) + x_1)}{\Phi_b(-u(\delta_1) + x_1)}.$$ (52)

Finally using the inversion relation (83) and shifting integration variables

$$x_1 \rightarrow x_1 - u(\delta_1), \quad x_2 \rightarrow x_2 + u(\delta_1),$$

$$\tilde{W}_b(S^3, 5_2)$$

$$= \int_{R^2} dx_1 \quad e^{\pi ix_1^2} e^{-2ic_b(\pi - \beta_1 - \beta_2 - \gamma_2 + \delta_2)x_1}$$

$$\times \quad \frac{\Phi_b(x_1) \Phi_b(u(\beta_1) - u(\delta_1) + x_1) \Phi_b(u(\pi - \gamma_1 - \gamma_2) - u(\delta_1) + x_1)}{\Phi_b(x_1) \Phi_b(u(\beta_1) + x_1) \Phi_b(u(\pi - \gamma_1 - \gamma_2) - x_1) \Phi_b(-u(\delta_1) + x_1)}$$

$$= \left| \int_{R^2} dx \quad e^{\pi ix^2} e^{-2ic_b(\pi - \beta_1 - \beta_2 - \gamma_2 + \delta_2)x} \right|^2.$$ (53)
In the complete balancing case \( \delta_2 = \beta_3 + \gamma_2, \delta_1 = \gamma_3 = \beta_1, \) one gets
\[
\left| \int_{\mathbb{R} - i 0} dy \frac{e^{\pi y^2}}{\Phi_b(y)^3} \right|^2.
\]

### 4.4. One vertex H-triangulation of \((S^3, 6_1)\)

First of all we start from describing of an H-triangulation of \((S^3, 6_1)\). In the graphical notation of [1], let \(X\) be given by the diagram

![Diagram](image-url)

This one vertex H-triangulation of \((S^3, 6_1)\) consists of 5 tetrahedra: \(T_1\) and \(T_3\) which are positive tetrahedra and \(T_2, T_4, T_5\) - negative tetrahedra. In the above picture, tetrahedra \(T_1, T_3\) and \(T_5\) are situated on the bottom row in the same order from left to right and tetrahedra \(T_2, T_4\) lie on the upper row from left to right. According to the picture one has to identify the following faces of five tetrahedra:

\[
\begin{align*}
\partial_0 T_1 & \simeq \partial_5 T_2, \quad \partial_1 T_1 \simeq \partial_2 T_3, \quad \partial_2 T_1 \simeq \partial_3 T_1, \quad \partial_1 T_2 \simeq \partial_2 T_4, \quad \partial_3 T_2 \simeq \partial_5 T_3, \\
\partial_2 T_2 & \simeq \partial_5 T_3, \quad \partial_1 T_3 \simeq \partial_5 T_3, \quad \partial_3 T_3 \simeq \partial_1 T_4, \quad \partial_1 T_4 \simeq \partial_3 T_5, \quad \partial_3 T_4 \simeq \partial_2 T_5.
\end{align*}
\]

(54)

From the identification of faces we get the following equalities for the edges

\[
\begin{align*}
x^{(1)}_{23} &= x^{(2)}_{23} = x^{(4)}_{13} = x^{(5)}_{02} = x^{(4)}_{02} = x^{(2)}_{03} = x^{(3)}_{03}, \\
x^{(1)}_{13} &= x^{(2)}_{13} = x^{(3)}_{12} = x^{(2)}_{12} = x^{(1)}_{12}, \\
x^{(1)}_{03} &= x^{(3)}_{03} = x^{(1)}_{02} = x^{(3)}_{01} = x^{(4)}_{02} = x^{(5)}_{03}, \\
x^{(2)}_{02} &= x^{(4)}_{01} = x^{(5)}_{12} = x^{(1)}_{12}, \\
x^{(2)}_{01} &= x^{(3)}_{12} = x^{(4)}_{23} = x^{(5)}_{12}.
\end{align*}
\]

(55)

and just \(x^{(1)}_{01}\). Here, the superscripts \(1, 2, 3, 4, \) and \(5\) refer to tetrahedra \(T_i, i = 1, \ldots, 5\).

The partition function for this one vertex H-triangulation of \((S^3, 6_1)\) is equal to the integral of five Boltzmann weights of five tetrahedra

\[
\begin{align*}
W_b(S^3, 6_1) &= \int_{\mathbb{R}^6} B(\Delta \alpha_1, \Delta \alpha_2 + i \nabla (x^{(1)}_{01} + x^{(1)}_{13} - x^{(1)}_{03} - x^{(1)}_{12}) / 2) \\
&\times B(\Delta \beta_1 + i \nabla (x^{(1)}_{03} - x^{(2)}_{01}), \Delta \beta_2 + i \nabla (x^{(2)}_{01} - x^{(1)}_{13})) \\
&\times B(\Delta \gamma_1 - i \nabla (x^{(1)}_{03} + x^{(2)}_{01} - 2x^{(1)}_{23}), \Delta \gamma_2 - i \nabla (x^{(1)}_{13} - x^{(2)}_{01})) \\
&\times B(\Delta \delta_1 + i \nabla (x^{(1)}_{03} + x^{(2)}_{01} - x^{(1)}_{23} - x^{(2)}_{02}), \Delta \delta_2 + i \nabla (x^{(2)}_{02} + x^{(1)}_{13} - x^{(1)}_{03} - x^{(2)}_{01})) \\
&\times B(\Delta \rho_1 + i \nabla (x^{(2)}_{02} - x^{(1)}_{03}), \Delta \rho_2 + i \nabla (x^{(1)}_{03} - x^{(2)}_{02}) dx^{(1)}_{01} dx^{(1)}_{02} dx^{(1)}_{13} dx^{(1)}_{03} dx^{(2)}_{01}) \\
&= 2W_b(S^3, 6_1) \int_{\mathbb{R}} B(\Delta \alpha_1, \Delta \alpha_2 + i \nabla t) dt \\
&= 2W_b(S^3, 6_1) |\Phi_b(u(\alpha_1))|^2,
\end{align*}
\]

(56)
where we fixed the edge \(x_2^{(1)}\) and 

\[
\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^4} B(\Delta \beta_1 - i \nabla x_{02}^{(2)}, \Delta \beta_2 + i \nabla (x_{01}^{(2)} - x_{13}^{(1)})) \\
\times B(\Delta \gamma_1 - i \nabla (x_{03}^{(1)} + x_{01}^{(2)}), \Delta \gamma_2 - i \nabla (x_{13}^{(1)} - x_{01}^{(2)})) \\
\times B(\Delta \delta_1 + i \nabla (x_{03}^{(1)} + x_{02}^{(2)}), \Delta \delta_2 + i \nabla (x_{13}^{(1)} - x_{03}^{(2)})) \\
\times B(\Delta \rho_1 + i \nabla (x_{03}^{(2)} - x_{01}^{(1)}), \Delta \rho_2 + i \nabla x_{01}^{(2)}) dx_{02} dx_{13} dx_{03} dx_{01},
\]  
(57)

now we can change variables \(x_{02}^{(2)} \rightarrow x_{02}^{(2)} + x_{03}^{(1)} + x_{01}^{(2)}, x_{03}^{(1)} \rightarrow x_{03}^{(1)} - x_{01}^{(2)}\) and \(x_{13}^{(1)} \rightarrow x_{13}^{(1)} + x_{01}^{(2)}\) and one gets

\[
\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^4} B(\Delta \beta_1 - i \nabla (x_{02}^{(2)} + x_{03}^{(1)}), \Delta \beta_2 - i \nabla x_{13}^{(1)}) \\
\times B(\Delta \gamma_1 - i \nabla x_{03}^{(1)}, \Delta \gamma_2 - i \nabla x_{13}^{(1)} - i \nabla (x_{01}^{(2)} + x_{13}^{(1)})) \\
\times B(\Delta \delta_1 + i \nabla x_{02}^{(2)}, \Delta \delta_2 + i \nabla x_{01}^{(2)}) dx_{02} dx_{13} dx_{03} dx_{01},
\]  
(58)

where \(\delta_3 = \Delta (\pi - \delta_1 - \delta_2)\). Using now relation (100) we rewrite the latter expression as

\[
\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^4} \psi \left( u(\beta_2) - \frac{x_{03}^{(1)} - x_{01}^{(2)} - 2c\beta_1}{\pi} \right) \\
\times \psi \left( u(\gamma_2) - \frac{x_{03}^{(1)} - x_{01}^{(2)} - 2c\beta_1}{\pi} \right) \psi \left( u(\rho_2) + \frac{x_{01}^{(2)}}{2}, x_{02}^{(2)} + x_{01}^{(2)} - 2c\rho_1 \right) \\
\times \psi \left( u(\delta_3) - \frac{x_{01}^{(2)} + x_{13}^{(1)}}{2}, x_{02}^{(2)} - 2c\rho_1 \right) dx_{02} dx_{13} dx_{03} dx_{01},
\]  
(59)

and using the definition for \(\psi\) function we get

\[
\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^4} \Phi_b(\beta_2) - \frac{x_{03}^{(1)} - x_{01}^{(2)} - u}{\pi} \Phi_b(\gamma_2) - \frac{x_{03}^{(1)} + u + v}{\pi} \\
\times \Phi_b(\rho_2) + \frac{x_{02}^{(2)}}{2} + t \Phi_b(\delta_3) - \frac{x_{01}^{(2)} + x_{13}^{(1)}}{2} + t \\
\times \Phi_b(-u(\beta_2) + \frac{x_{01}^{(2)}}{2} + s) \Phi_b(u(\gamma_2) - \frac{x_{03}^{(1)}}{2} + v) \\
\times e^{-2\pi i (x_{02}^{(2)} + x_{03}^{(1)}) u - 4c\beta_1} e^{2\pi i x_{02}^{(2)} v - 4c\gamma_1} e^{2\pi i x_{03}^{(1)} v} e^{-4c\rho_1} e^{-4ic\delta_3} t \delta(-u - s - t) \delta(-v - u) dx_{13} dx_{01} dx_{02} dx_{03} dudvdwdsdt,
\]  
(60)
taking the integrals over $t$ and $v$ one gets

$$\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^2} \frac{\Phi_b(u(\beta_2) - \frac{x_{(1)}'}{2} + u)}{\Phi_b(-u(\beta_2) + \frac{x_{(1)}'}{2} + u)} \Phi_b(u(\gamma_2) - \frac{x_{(1)}'}{2} - u)$$
$$\times \frac{\Phi_b(u(\rho_2) + \frac{x_{(2)}'}{2} + s)}{\Phi_b(-u(\rho_2) - \frac{x_{(2)}'}{2} + s)} \Phi_b(u(\delta_3) - \frac{x_{(3)}'}{2} + s - u)$$
$$\times \Phi_b(-u(\rho_2) + w) \Phi_b(-u(\delta_1) + z - x)$$
$$\times e^{-2ic_b(\beta_1 - \gamma_1 - \delta_1)(z+y)e^{x(2-w^2)}e^{-2ic_b(\rho_1 + \delta_1)(z+w)}dx_{(1)}dx_{(2)}duds.}$$ (62)

Let us consider reparametrization $x = u - \frac{1}{2}x_{(1)}', y = u + \frac{1}{2}x_{(1)}', z = s + \frac{1}{2}x_{(2)}', w = s - \frac{1}{2}x_{(2)}'$

$$\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^2} \frac{\Phi_b(u(\beta_2) + x)}{\Phi_b(-u(\beta_2) + x)} \Phi_b(u(\gamma_2) - y)$$
$$\times \Phi_b(u(\rho_2) + z) \Phi_b(u(\delta_3) + w - y)$$
$$\times \Phi_b(-u(\rho_2) + w) \Phi_b(-u(\delta_1) + z - x)$$
$$\times e^{-2ic_b(\beta_1 - \gamma_1 - \delta_1)(x+y)e^{x(2-w^2)}e^{-2ic_b(\rho_1 + \delta_1)(z+w)}dx_{(1)}dx_{(2)}duds.}$$ (63)

which is equal to

$$\tilde{W}_b(S^3, 6_1) = \int_{\mathbb{R}^2} \frac{\Phi_b(u(\beta_2) + x)\Phi_b(u(\rho_2) + z)}{\Phi_b(-u(\gamma_2) - x)\Phi_b(-u(\delta_1) + z - x)} e^{-2ic_b((\beta_1 - \gamma_1 - \delta_1)x + (\rho_1 + \delta_1)z) + \pi iz^2} dxdz^2,$$ (64)

demonstrating the factorization for the H-triangulation of $(S^3, 6_1)$ and which is consistent with Conjecture [11].

5. Application to 3d supersymmetric field theories

5.1. 3d supersymmetric theories living on a squashed three-sphere. Following the work of Pestun [32], the partition functions of 3d $\mathcal{N} = 2$ supersymmetric theories, defined on a squashed three-sphere $S^3_b$, were calculated in the papers [25, 24, 21] by using the localization method. These partition functions are given in the form of integrals with the integrands composed of hyperbolic gamma functions [50, 16]. For any 3d $\mathcal{N} = 2$ supersymmetric theory defined on $S^3_b$ with a gauge group $G$ and a flavour group $F$, the corresponding partition function has the following structure

$$Z(f) = \int_{-\infty}^{\infty} \prod_{j=1}^{\text{rank}G} du_j \int_{\mathbb{C}} J(\bar{u}) Z^{\text{pert}}(\bar{u}) \prod_{i} Z^{\text{chir}}(f_i, \bar{u}).$$ (66)

Here the integral is taken over $u_j$-variables which are associated with the Weyl weights for the Cartan subalgebra of the gauge group $G$ and the $f_i$’s denote the chemical potentials for
the flavor symmetry group $H$. For CS theory one has $J(u) = e^{-\pi k \sum_{i=1}^{N} u_i^2}$, where $k$ is the level of the CS-term, while for SYM theories one has $J(u) = e^{2\pi \lambda \sum_{i=1}^{N} u_i}$, where $\lambda$ is the Fayet–Illiopoulos term. The terms $Z^{\text{vec}}(u)$ and $Z^{\text{chir}}(f, y)$ in (66) come from the vector superfield and the matter fields, respectively, and are given in terms of the hyperbolic gamma function.

The result of localization allows us to relate the physical theory with some matrix integral of the form (66). Also we can invert the logic: having some matrix integral of the type (66) one can find a supersymmetric field theory whose partition function is given by this matrix integral (16). Thus, all the partition functions which we get by considering (3) can be interpreted as partition functions for some 3d $\mathcal{N} = 2$ supersymmetric field theories. Moreover, as the expression (3) corresponds to some triangulation of a 3-dimensional manifold $M$, we obtain a link between 3-manifolds and 3d $\mathcal{N} = 2$ supersymmetric field theories defined on $S_3^3$. This is known as a 3d/3d duality considered recently in [44, 12, 13, 45] (see also [43] for the relation of the objects to four-dimensional supersymmetric field theories).

In [12], the state variables live in the faces, while in our case the state variables live on the edges. To get a 3d theory from 3d manifold $M$ one has to triangulate this manifold and calculate its partition function (3) and then interpret this expression as a partition function (66). One should notice that every common edge corresponds to abelian gauge group.

Let us start from our building block: the tetrahedral Boltzmann weight composed of three hyperbolic gamma functions, each corresponding to the contribution coming from an abelian gauge group.

$$B(T, x) = \prod_{i=1}^{3} \gamma^{(2)}(\Delta \alpha_i + i \nabla(x_{i+1} + x_{i+1}' - x_{i-1} - x_{i-1}'); \omega_1, \omega_2),$$

(67)

corresponds to three chiral superfields $Q_i$, $i = 1, 2, 3$, with $SU(3)$ global symmetry group (since $\sum_{i=1}^{3} i(x_{i+1} + x_{i+1}' - x_{i-1} - x_{i-1}') = 0$) and a superpotential

$$W \sim Q_1 Q_2 Q_3,$$

which has a correct $R$-charge. This can be easily seen from the fact that the dihedral angles $\alpha_i$, $i = 1, 2, 3$ correspond to $R$-charges of three chiral superfields. And since $\sum_{i=1}^{3} \alpha_i = \pi$ then the $R_W$-charge of the superpotential $W$ is given as $R_W = \sum_{i=1}^{3} R_{Q_i} = \sum_{i=1}^{3} 2\alpha_i/\pi = 2$.

The first non-trivial case of a 3d theory with a non-trivial gauge group is the pentagon identity (31) (once again we stress that it is a 2 + 3 Pachner move) when we take two positive tetrahedra and glue them together over the common face. The partition function of two glued tetrahedra having vertices $(0, 1, 2, 4)$ and $(0, 2, 3, 4)$ is

$$W_{b, A} = B(\Delta \alpha_1 + i \nabla(x_{02} + x_{34} - x_{03} - x_{24}), \Delta \alpha_2 + i \nabla(x_{03} + x_{24} - x_{04} - x_{23})),$$

$$\times B(\Delta \beta_1 + i \nabla(x_{01} + x_{24} - x_{02} - x_{14}), \Delta \beta_2 + i \nabla(x_{02} + x_{14} - x_{04} - x_{12}),$$

(68)

where $\sum_{i=1}^{3} (a_i + b_i) = \omega_1 + \omega_2$ which is the partition function for a theory $A$ which consists of six 3d $\mathcal{N} = 2$ free chiral hypermultiplets with $F = SU(3) \times SU(3) \times U(1)$ global symmetry group. Here we have $\alpha_1 = a_2 + b_1, \alpha_2 = a_3 + b_2, \beta_1 = a_1 + b_2$ and $\beta_2 = a_3 + b_1$. Here each $SU(3)$ corresponds to separate tetrahedron and $U(1)$ group distinguishes the two tetrahedra. At the same time, using 2 – 3 Pachner move, two glued

\footnote{From physical point of view, $f_i’s$ are linear combinations of the $R$-charge, the masses of the hypermultiplets, and the Fayet–Illiopoulos terms associated to the additional Abelian global symmetries.}
tetrahedra can be considered as three tetrahedra with the vertices \((0, 1, 2, 3), (0, 1, 3, 4)\) and \((1, 2, 3, 4)\) having a common edge \(x_{04}\) whose partition function is

\[
W_{b,B} = \int B(\Delta a_1 + i\nabla(x_{01} + x_{23} - x_{02} - x_{13}), \Delta b_1 + i\nabla(x_{02} + x_{13} - x_{03} - x_{12}))
\times B(\Delta a_2 + i\nabla(x_{12} + x_{34} - x_{13} - x_{24}), \Delta b_2 + i\nabla(x_{13} + x_{24} - x_{14} - x_{23}))
\times B(\Delta a_3 + i\nabla(x_{01} + x_{34} - x_{03} - x_{14}), \Delta b_3 + i\nabla(x_{03} + x_{14} - x_{04} - x_{13})) \, dx_{13},
\]

where \(\sum_{i=1}^{3}(a_i + b_i) = \omega_1 + \omega_2\).

Expression (69) gives a partition function for 3d \(\mathcal{N} = 2\) SQED theory \(B\) (which has \(U(1)\) gauge group) with 3 flavors and overall \(F = SU(3) \times SU(3) \times U(1)\) global symmetry group and 2 singlet baryons. There are three tetrahedra in this picture so one can think of \(SU(3)\) global symmetry group but the part of this, namely, \(U(1)\) becomes a gauge group leaving \(SU(3) \times SU(3) \times U(1)\) global symmetry group.

Since \((68) = (69)\) the partition functions for theories \(A\) and \(B\) are the same which suggests the duality between these theories. Generally, different triangulations of 3-manifolds produce different phases of the same theory, in other words, we get dual descriptions for 3d supersymmetric field theories related to a given 3-dimensional manifold.

One can continue further and construct triangulations for other 3-manifolds and relate them to 3d supersymmetric field theories. As a next example, we consider four tetrahedra built from vertices \((0, 1, 2, 5), (0, 2, 3, 5), (0, 3, 4, 5), (0, 1, 4, 5)\) glued together over a common edge \(x_{05}\) to form an octahedron. We have four tetrahedra: three positive \(T_1, T_2, T_3\) and one negative \(T_4\). These tetrahedra have the following vertices: \(T_1 = \{0, 1, 2, 5\}, T_2 = \{0, 2, 3, 5\}, T_3 = \{0, 3, 4, 5\}, T_4 = \{0, 1, 4, 5\}\). Identifying the faces, we get

\[
\partial_1 T_1 \simeq \partial_2 T_2, \quad \partial_2 T_2 \simeq \partial_3 T_3, \quad \partial_3 T_3 \simeq \partial_4 T_4, \quad \partial_2 T_1 \simeq \partial_4 T_4,
\]

from which we get

\[
\begin{align*}
(1) & \quad x_{05}^{(1)} = x_{05}^{(2)} = x_{05}^{(3)} = x_{05}^{(4)}, \quad x_{25}^{(1)} = x_{25}^{(2)} = x_{25}^{(4)} = x_{25}^{(3)}, \quad x_{02}^{(1)} = x_{02}^{(2)}, \quad x_{35}^{(2)} = x_{35}^{(3)}, \\
(3) & \quad x_{03}^{(2)} = x_{03}^{(3)}, \quad x_{45}^{(3)} = x_{45}^{(4)} = x_{04}^{(3)} = x_{04}^{(4)}, \quad x_{15}^{(1)} = x_{15}^{(4)}, \quad x_{01}^{(1)} = x_{01}^{(4)}.
\end{align*}
\]

so that the partition function is equal to

\[
W_{b,\text{octahedron}} = \int dx_{05}^{(1)}
\times B(\Delta a_1 + i\nabla(x_{02}^{(1)} + x_{15}^{(1)} - x_{12}^{(1)} - x_{05}^{(1)}), \Delta b_1 + i\nabla(x_{12}^{(1)} - x_{01}^{(1)} - x_{25}^{(1)} + x_{05}^{(1)}))
\times B(\Delta a_2 + i\nabla(x_{03}^{(2)} + x_{23}^{(2)} - x_{23}^{(2)} - x_{05}^{(2)}), \Delta b_2 + i\nabla(x_{23}^{(2)} - x_{02}^{(2)} - x_{25}^{(2)} + x_{05}^{(2)}))
\times B(\Delta a_3 + i\nabla(x_{04}^{(3)} + x_{34}^{(3)} - x_{34}^{(3)} - x_{05}^{(3)}), \Delta b_3 + i\nabla(x_{34}^{(3)} - x_{03}^{(3)} - x_{35}^{(3)} + x_{05}^{(3)}))
\times B(\Delta a_4 + i\nabla(x_{14}^{(4)} - x_{04}^{(4)} - x_{45}^{(4)} + x_{05}^{(4)}), \Delta b_4 + i\nabla(x_{04}^{(4)} + x_{15}^{(4)} - x_{14}^{(4)} - x_{05}^{(4)}))
\]

which corresponds to the partition function of 3d \(\mathcal{N} = 2\) SQED theory with 4 flavors and four singlet baryons with the overall global symmetry \(SU(3) \times SU(3) \times U(1)\). Octahedron can be also represented by gluing five tetrahedra which is not so obvious from geometrical point of view and will be much easier to see from the next subsection using the Bailey tree technique. As we will show the triangulation with five tetrahedra gives a dual description of the starting theory in terms of a quiver gauge theory with \(U(1) \times U(1)\) gauge group.

Continuing further and gluing more tetrahera one gets a class of 3d supersymmetric field theories corresponding to a given triangulation of a 3-manifold. For example, gluing \(F\) tetrahedra along one common edge one gets the partition function for 3d \(\mathcal{N} = 2\) SQED
theory with $F$ flavors and $F$ additional singlet baryons which has $SU(3)^{F-1} \times U(1)$ global symmetry group (since $U(1)$ becomes a gauge group).

5.2. Bailey tree technique. There is an alternative way to see the results of the previous subsection based on the application of Bailey tree technique for hyperbolic integrals (very much in the spirit of [40]). This approach gives an algebraic way of getting the partition functions and relates the triangulated 3-dimensional manifolds from one hand side, and 3d supersymmetric field theories defined on a squashed three sphere, from the other. The Bailey tree technique is useful for tracking different triangulations related to each other by 2 − 3 Pachner move from the algebraic viewpoint.

**Definition 1.** We say that two functions $\alpha(z, t)$ and $\beta(z, t)$, $z, t \in \mathbb{C}$ form an integral hyperbolic Bailey pair (the hyperbolic level) with respect to the parameter $t$ if

$$\beta(w, t) = \int B(t + w - z, t - w + z) \alpha(z, t) dz. \quad (73)$$

**Theorem 2** (follows from Theorem 1 [40]). Whenever two functions $\alpha(z, t)$ and $\beta(z, t)$ form an integral hyperbolic Bailey pair with respect to $t$, the new functions

$$\alpha'(w, s + t) = B(t + u + w, 2s) \alpha(w, t)$$

and

$$\beta'(w, s + t) = \int B(s + w - x, u + x) B(s + 2t + u + w, s - w + x) \beta(x, t) dx, \quad (75)$$

form an integral hyperbolic Bailey pair with respect to parameter $s + t$.

**Proof.** The proof is similar to the proof in the elliptic case [40]. We start from the definition for $\beta'(w, s + t)$:

$$\beta'(w, s + t) = \int B(s + w - x, u + x) B(s + 2t + u + w, s - w + x) \beta(x, t) dx,$$

where we substitute $\beta(x, t)$ from equation (73):

$$\beta'(w, s + t) = \int B(s + w - x, u + x) B(s + 2t + u + w, s - w + x)$$

$$\times B(t + x - y, t - x + y) \alpha(y, t) dy dx. \quad (76)$$

In the latter expression we can apply formula

$$\int \prod_{i=1}^{3} \gamma(2)(a_i - u; \omega_1, \omega_2) \gamma(2)(b_i + u; \omega_1, \omega_2) du = \prod_{i,j=1}^{3} \gamma(2)(a_i + b_j; \omega_1, \omega_2), \quad (77)$$

where $\sum_{i=1}^{3}(a_i + b_i) = \omega_1 + \omega_2$, so that we get

$$\beta'(w, s + t) = \int B(s + t + w - x, s + t - w + x) \alpha'(x, s + t) dx. \quad (78)$$

□

From identity (77) one gets the following Bailey pair

$$\alpha(z, t) = \prod_{i=1}^{2} B(\alpha_i - z, \beta_i + z), \quad (79)$$
where $2t + \sum_{i=1}^{2}(\alpha_i + \beta_i) = \omega_1 + \omega_2$, and
\[
\beta(w; t) = \prod_{i=1}^{2} B(t + w + \alpha_i, t - w + \beta_{3-i}).
\]  
(80)

The pentagon identity permits us to define a particular Bailey pair thus giving the definition for Bailey pairs a topological interpretation in terms of the 2-3 Pachner move. In other words, the construction of new Bailey pairs through Theorem 2 corresponds to changing a triangulation by a 2-3 Pachner move so that
\[
\sum_{i=1}^{2}(\alpha_i + \beta_i) = \omega_1 + \omega_2.
\]
On the other hand, this expression is equal to (76)
\[
Z_{3\Delta_3} = \int B(s + t + w - x, s + t - w + x)B(t + u + x, 2s) \prod_{i=1}^{2} B(\alpha_i - x, \beta_i + x)dx,
\]  
(81)

where $2t + \sum_{i=1}^{2}(\alpha_i + \beta_i) = \omega_1 + \omega_2$. The invariant
\[
\sum_{i=1}^{2}(\alpha_i + \beta_i)
\]
inducing the same angle structure on the orientated triangulated closed pseudo 3-manifold, i.e., $\partial X = \emptyset$, where $M$ is the underlying pseudo 3-manifold and $\mathcal{T}$ is the triangulation. Let $\square(\mathcal{T})$ be the set of all quads in $\mathcal{T}$. Recall that $\mathbb{Z}/3\mathbb{Z} = \{1, \tau, \tau^2\}$ acts on $\square(\mathcal{T})$ corresponding to the cyclic order of three edges around each vertex. For $q \in \square(\mathcal{T})$, we will use $q'$ and $q''$ to denote $\tau(q)$ and $\tau^2(q)$ below. A shaped structure on $X$ (or $\mathcal{T}$) is a function $\alpha : \square(\mathcal{T}) \to (0, \pi)$ so that $\alpha(q) + \alpha(q') + \alpha(q'') = \pi$ for all $q \in \square(\mathcal{T})$. The weight of a shape structure $\alpha$ is the function $f : \Delta_3(\mathcal{T}) \to \mathbb{R}$ sending each edge $e$ to $f(e) = \sum_{q \sim e} \alpha(q)$ where $q \sim e$ means the quad $q$ faces the edge $e$. In particular, an angle structure is a shaped structure whose weight at each edge is $2\pi$.

The invariant $W_6(X)$ in Theorem 1 is defined for each shaped triangulation, i.e., $W_6(X) = W_6(M; \mathcal{T}, \alpha)$. Theorem 1 implies that $W_6(M; \mathcal{T}_3, \alpha) = W_6(M; \mathcal{T}_2, \beta)$ if $\mathcal{T}_3$ is obtained from $\mathcal{T}_2$ by a 2-3 Pachner move so that $\beta$ is the angle structure on $\mathcal{T}_2$ induced by the angle structure $\alpha$ on $\mathcal{T}_3$. (The equation for defining $\beta$ from $\alpha$ is indicated in figure 1.) In general, there are many different (or may be non) angle structures on $\mathcal{T}_3$ inducing the same angle structure $\alpha$ on $\mathcal{T}_3$. These different angle structures are related by a gauge transformation induced by the degree 3 edge in $\mathcal{T}_3$. Theorem 2 says that $W_6(M; \mathcal{T}, \alpha)$ depends only on the (edge type) gauge equivalence class of the shaped structure $\alpha$.

We will describe briefly the edge type gauge equivalence class now. Recall that a tangential angle structure on $\mathcal{T}$ (see [30]) is a map $x : \square(\mathcal{T}) \to \mathbb{R}$ so that for each $q \in \square(\mathcal{T})$,
$x(q) + x(q') + x(q'') = 0$ and for each edge $e \in \Delta_1(T)$, $\sum_{q \sim e} x(q) = 0$. Thus the space of all tangential angle structures is a vector space, denoted by $TAS(T)$. For any shape structure $\alpha, v \in TAS(T)$ and small $t, \beta = \alpha + tv$ is still a shape structure so that $\beta$ and $\alpha$ have the same weight. A generating set of vectors for $TAS(T)$ was well known and can be described as follows. Consider the vertex link $lk(v)$ of a vertex $v \in \Delta_0(T)$. Let $s$ be an edge loop in the dual CW-decomposition of the triangulated surface $lk(v)$. The loop $s$ can be described as a sequence of triangles $\{t_1, ..., t_n\}$ and edges $\{\epsilon_1, ..., \epsilon_n\}$ in $lk(v)$ so that $\epsilon_i$ is adjacent to $t_i$ and $t_{i+1}$ ($t_{n+1} = t_1$). Since each $t_i$ corresponds to a tetrahedron $T_i$ and each $\epsilon_i$ corresponds to a co-dimension-1 face $F_i$ in $T$, each $T_i$ contains a unique quad $q_i$ facing the edge $F_i \cap F_i$ in $T_i$.

**FIGURE 2.** Edge loop in a vertex link

Define a map $g_s : \partial(T) \to \mathbb{R}$ by $g_s(q_1) = 1$, $g_s(q_i') = -1$ and $g_s(q) = 0$ for all other $q$’s. One checks easily that $g_s \in TAS(T)$. In particular, if $s$ is the loop around a vertex $u$ in $lk(v)$, then $g_s$ is the gauge transformation associated to the edge $e$ corresponding to $u$. Two shaped structures $\alpha$ and $\beta$ on $T$ are edge type gauge equivalent if their difference $\alpha - \beta$ is a linear combinations of $g_s$’s for edge loops $s$ which are around vertices in vertex links. Theorem 1 says that $W_\beta(M; \mathcal{T}, \alpha)$ depends only on the edge type gauge equivalence classes. A theorem in [46] shows $TAS(T)$ is generated by vectors $g_{q_i}$. Define the angle holonomy $\alpha(s)$ of a shaped structure $\alpha$ along an edge loop $s$ in $lk(v)$ to be $\sum_{i=1}^n \alpha(q_i)$. The work of [46] and [1] show two shaped structures are edge type gauge equivalent if and only if they have the same angle holonomy along any edge loop $s$ in vertex links. This suggests a way to represent the edge type gauge equivalence class of shaped structures using volume optimization. Namely, given a shaped structure $\alpha$, let $A_\alpha$ be the set of all shaped structures on $T$ edge type gauge equivalent to $\alpha$. The volume of a shape structure is the sum of the volume of the hyperbolic tetrahedra determined by the shape. It is well known that volume is a strictly concave function of shape structure $\alpha$. In particular, there is at most one shape structure $\beta \in A_\alpha$ which has the maximum volume. Note that it may not exist in $A_\alpha$, i.e., the maximum volume point may appear in the boundary of the closure of $A_\alpha$. Suppose now that $\alpha$ is an angle structure and the maximum volume $\beta$ exists in $A_\alpha$. Then by the standard volume optimization method (see [44], [19], or [80]), one sees that the complex shape parameter $z_\beta$ given by (4) is associated to $\beta$ satisfies Thurston’s gluing equation. Therefore, it produces a representation $\rho$ of $\pi_1(M - \Delta_0(T))$ to $PSL(2, \mathbb{C})$ so that for any edge loop $s$ in $lk(v)$, the eigenvalues of $\rho(s)$ are of the form $re^{\pm \sqrt{-1} \beta(s)/2}$ for $r \in \mathbb{R}_{>0}$. This shows if there exists an angle structure of the maximum volume edge type gauge equivalent to $\alpha$, one can assigns the invariant $W_\beta(M; \mathcal{T}, \alpha)$ to the representation $\rho$, i.e., the invariant $W_\beta(M; \mathcal{T}, \alpha)$ may be an invariant of a pair $(M, \rho)$. The
precise conjectural picture of $W_b(M; T, \alpha)$ is: if two angle structures $(T_i, \alpha_i)$ ($i = 1, 2$) are associated to the same representation $\rho$, then $W_b(M; T_1, \alpha_1) = W_b(M; T_2, \alpha_2)$.

6.2. **Relationship with Simplicial $PSL(2, \mathbb{R})$ Chern–Simons theory.** In [31], we proposed a variational principle for finding real valued solutions of Thurston’s equation on a triangulated oriented closed pseudo 3-manifold $(M; T)$. Given $(M; T)$, we introduce the homogeneous Thurston’s equation (HTE) as follows. A map $x : \square(T) \to \mathbb{R}$ is said to solve HTE if for each $q \in \square(T)$, $x(q) + x(q') + x(q'') = 0$ and for each edge $e$ in $T$,

$$\prod_{q \sim e} x(q') = \prod_{q \sim e} (-x(q'')).$$

It can be proved that solutions to Thurston’s equation over the real numbers on $(M, T)$ correspond to nowhere zero solutions to HTE. The main observation in [31] is that critical points of an entropy function of the form $\sum_{i=1}^n x_i \ln(|x_i|)$ are nowhere zero solutions to HTE. The converse also holds if $M$ is a closed 3-manifold.

Our pentagon relation (33) implies the following pentagon relation for the entropy. Namely, given five positive numbers $a_1, a_2, b_1, b_2, b_3$ so that $\sum_{i=1}^2 a_i + \sum_{j=1}^3 b_i = 1$ and $a_1 a_2 = b_1 b_2 b_3$, then

$$\sum_{i,j} (a_i + b_j) \ln(a_i + b_j) = 2 \sum_{i=1}^2 (a_i \ln(a_i) + (1 - a_i) \ln(1 - a_i)) + 3 \sum_{j=1}^3 b_j \ln(b_j). \quad (83)$$

Identity (83) suggests there should exist a non-quantum topological invariant for 3-manifold from simplicial $SL(2, \mathbb{R})$ Chern-Simons theory. Furthermore, this invariant is the semi-classical limit of $W_b(M; T, \alpha)$ when $b$ degenerates.

**APPENDIX A. SPECIAL FUNCTIONS**

A.1. **Faddeev’s quantum dilogarithm.** Faddeev’s quantum dilogarithm $\Phi_b(z)$ is defined by the integral

$$\Phi_b(z) \equiv \exp \left( \int_{\mathbb{R} + i0} \frac{e^{-2izw} dw}{4 \sinh(wb) \sinh(w/b) w} \right), \quad (84)$$

in the strip $|\text{Im}z| < |\text{Im}c_b|$, where

$$c_b = i(b + b^{-1})/2.$$

It is useful to define

$$\zeta_{inv} \equiv e^{\pi i (1 + 2c_b^2)/6} = e^{\pi i c_b^2} \zeta_0, \quad \zeta_0 \equiv e^{\pi i (1 - 4c_b^2)/12}. \quad (85)$$

- **symmetry**: $\Phi_b(z) = \Phi_{-b}(z) = \Phi_{1/b}(z)$,
- **functional equations**: $\Phi_b(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z}) \Phi_b(z + ib^{\pm 1}/2)$, (87)
- **inversion property**: $\Phi_b(z) \Phi_b(-z) = \zeta_{inv} e^{\pi i z^2}$, (88)
- **zeros**: $z \in \{ c_b + mib + nib^{-1}; m, n \in \mathbb{Z}_{\geq 0} \}$, (89)
- **poles**: $z \in \{ -c_b - mib - nib^{-1}; m, n \in \mathbb{Z}_{\geq 0} \}$, (90)
- **unitarity**: $\overline{\Phi_b(z)} = 1/\Phi_b(\overline{z})$. (91)
A.2. **The elliptic Gamma function.** The elliptic gamma function is defined by the formula

\[ \Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \]  

and it satisfies the following properties

- **symmetry** \( \Gamma(z; p, q) = \Gamma(z; q, p) \),
- **functional equations** \( \Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q) \), \( \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q) \),
- **reflection property** \( \Gamma(z; p, q) \Gamma(pqz; p, q) = 1 \),
- **zeros** \( z \in \{p^{i+1}q^{j+1}; i, j \in \mathbb{Z} \geq 0 \} \),
- **poles** \( z \in \{p^{-i}q^{-j}; i, j \in \mathbb{Z} \geq 0 \} \),
- **residue** \( \text{Res}_{z=1} \Gamma(z; p, q) = -\frac{1}{(p; p)\infty(q; q)\infty} \).

Here \( \theta(z; p) \) is a theta-function \( \theta(z; p) = (z; p)\infty(pz; p)\infty \).

A.3. **Some useful formulas.** Faddeev’s quantum dilogarithm and the hyperbolic gamma functions are related via formula

\[ \gamma^{(2)}(-i\sqrt{\omega_1\omega_2}(x+c_b); \omega_1, \omega_2) = e^{i\pi x^2/2} \sqrt{\sin \Phi_b(x)} \]

where \( b := \sqrt{\omega_1\omega_2} \).

Recall that the inversion relation (27) for \( \gamma^{(2)}(x) \) is of the form

\[ \gamma^{(2)}(x; \omega_1, \omega_2)\gamma^{(2)}(\omega_1 + \omega_2 - x; \omega_1, \omega_2) = 1 \]

and the complex conjugation property

\[ \gamma^{(2)}(z) = \gamma^{(2)}(\bar{z}). \]

If we define

\[ B(u, v) := \frac{\gamma^{(2)}(u; \omega_1, \omega_2)\gamma^{(2)}(v; \omega_1, \omega_2)}{\gamma^{(2)}(u + v; \omega_1, \omega_2)} \]

then it is easy to see that

\[ B\left(\sqrt{-\omega_1\omega_2}x, \sqrt{-\omega_1\omega_2}y\right) = \Psi\left(\frac{x}{2} + c_b, \frac{-x}{2} - c_b, y\right) \]

where

\[ \Psi(u, v, w) := \int_{\mathbb{R}} \frac{\Phi_b(u + x)}{\Phi_b(v + x)} e^{i2\pi wx} dx, \]

which is calculated as follows [18]

\[ \Psi(u, v, w) = \zeta e\frac{\Phi_b(u - v - c_b)\Phi_b(w + c_b)}{\Phi_b(u - v + w - c_b)} e^{-2\pi i (v + c_b)}. \]
REFERENCES

[1] J. E. Andersen and R. Kashaev, A TQFT from quantum Teichmüller theory, arXiv:1109.6295 [math.QA].
[2] M. Atiyah, Topological quantum field theories, Inst. Hautes Études Sci. Publ. Math. No. 68 (1988), 175–186.
[3] S. Baseilhac, B. Benedetti, Quantum hyperbolic geometry, Algebr. Geom. Topol. 7 (2007), 845–917.
[4] B. Balsam and A. Kirillov Jr., Turaev-Viro Invariants as an Extended TQFT, arXiv:1004.1533.
[5] V. V. Bazhanov and S. M. Sergeev, Elliptic gamma-function and multi-spin solutions of the Yang-Baxter equation, Nucl. Phys. B 856 (2012) 475–496.
[6] John W. Barrett; Bruce W. Westbury, Invariants of piecewise-linear 3-manifolds, Trans. Amer. Math. Soc. 348 (1996), no. 10, 3997–4022.
[7] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, Three-manifold invariants derived from the Kauffman Bracket. Topology 31 (1992), 685–699.
[8] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, Topological Quantum Field Theories derived from the Kauffman bracket. Topology 34 (1995), 885–927.
[9] L. O. Chekhov, V. V. Fock, Quantum Teichmüller spaces, Theor. Math. Phys. 120 (1999), 1245–1259.
[10] D. Futer, David and F. Guéritaud, Algebr. Geom. Topol. 7 (2007), 845–917.
[11] F. A. Dolan and H. Osborn, Hypergeometric Identities to the Kauffman Bracket. Commun. Math. Phys. 348 (1996), no. 10, 3997–4022.
[12] F. A. H. Dolan, V. P. Spiridonov, and G. S. Vartanov, From 4d superconformal indices to 3d partition functions, Phys. Lett. B704 (2011), no. 3, 234–241.
[13] L. D. Faddeev, Discrete Heisenberg-Weyl group and modular group, Lett. Math. Phys. 34 (3) (1995), 249–254.
[14] L. D. Faddeev, R. M. Kashaev and A. Y. Volkov, Strongly coupled quantum discrete Liouville theory. 1. Algebraic approach and duality, Commun. Math. Phys. 219 (2001) 199–219.
[15] D. Futer, David and F. Guéritaud, From angled triangulations to hyperbolic structures. Interactions between hyperbolic geometry, quantum topology and number theory, 159182, Contemp. Math., Amer. Math. Soc., Providence, RI, 2011.
[16] N. Geer, R. Kashaev, V. Turaev, Tetrahedral forms in monoidal categories and 3-manifold invariants, arXiv:1008.3315, to appear in Crelle’s Journal.
[17] N. Hama, K. Hosomichi, and S. Lee, Notes on SUSY gauge theories on three-sphere, JHEP 1103 (2011), 127; SUSY gauge theories on squashed three-spheres, JHEP 1105 (2011), 014.
[18] K. Hikami, Hyperbolicity of Partition Function and Quantum Gravity, Nucl.Phys. B616 (2001) 537–548.
[19] K. Hikami, Generalized Volume Conjecture and the A-Polynomials – the Neumann-Zagier Potential Function as a Classical Limit of Quantum Invariant, J. Geom. Phys. 57 (2007), 1895–1940.
[20] D. L. Jafferis, The exact superconformal R-symmetry extremizes Z, JHEP 1205 (2012) 159.
[21] A. Kapustin, B. Willett, and I. Yaakov, Exact results for Wilson loops in superconformal Chern–Simons theories with matter, JHEP 1003 (2010) 089.
[22] R. M. Kashaev, Quantumization of Teichmüller spaces and the quantum dilogarithm, Lett. Math. Phys. 43 (1998), 105–115.
[23] R. M. Kashaev, Quantum dilogarithm as a 6j-symbol, Modern Phys. Lett. A9 (1994), no. 40, 3757–3768.
[24] Korepanov I.G., Euclidean 4-simplices and invariants of four-dimensional manifolds: I. Moves 3 → 3, Theor. Math. Phys. 131 2002, no. 3, 765-774; Pochner Move 3 → 3 and Affine Volume-Preserving Geometry in R³, SIGMA 1 (2005), 021.
[25] L. D. Faddeev and V.N. Popov, Feynman Diagrams for the Yang–Mills Field, Phys. Lett. B25 (1967) 29.
[26] F. Luo, Volume optimization, normal surfaces and Thurston’s equation on triangulated 3-manifolds, to appear, J. Diff. Geometry.
[27] F. Luo, Simplicial SL(2, R) Chern-Simons theory and Boltzmann entropy, in preparation.
[28] V. Pestun, Localization of gauge theory on a four-sphere and supersymmetric Wilson loops, Commun. Math. Phys. 313 (2012) 71–129.
[33] G. Ponzano; T. Regge, *Semiclassical limit of Racah coefficients*, in: Spectroscopic and group theoretical methods in physics, 1–58, ed. F. Bloch, North-Holland Publ. Co., Amsterdam, 1968.

[34] I. Rivin, *Euclidean structures on simplicial surfaces and hyperbolic volume*, Ann. of Math. (2) 139 (1994), no. 3, 553–580.

[35] N. Reshetikhin & V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990), 1–26.

[36] N. Reshetikhin & V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991), 547–597.

[37] S. N. M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. 38 (1997), 1069–1146.

[38] G.B. Segal, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), 165–171, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 250, Kluwer Acad. Publ., Dordrecht, 1988.

[39] V. P. Spiridonov, *On the elliptic beta function*, Uspekhi Mat. Nauk 56 (1) (2001) 181–182 (Russian Math. Surveys 56 (1) (2001) 185–186).

[40] V. P. Spiridonov, *A Bailey tree for integrals*, Theor. Math. Phys. 139 (1) (2004), 536–541.

[41] V. P. Spiridonov, *Elliptic beta integrals and solvable models of statistical mechanics*, Contemp. Math. 563 (2012) 181–211.

[42] V. P. Spiridonov and G. S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities*, Commun. Math. Phys. 304 (2011), 797–874.

[43] V. P. Spiridonov and G. S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices*, [arXiv:1107.5788 [hep-th]].

[44] Y. Terashima and M. Yamazaki, *SL(2, R) Chern-Simons, Liouville, and Gauge Theory on Duality Walls*, JHEP 1108 (2011) 135.

[45] J. Teschner and G. S. Vartanov, *6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories*, [arXiv:1202.4693 [math-ph]].

[46] Tillmann, Stephan *Normal surfaces in topologically finite 3-manifolds*, Enseign. Math. (2) 54 (2008), no. 3-4, 329–380.

[47] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

[48] V.G. Turaev; O.Ya. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology 31 (1992), no. 4, 865–902.

[49] V. G. Turaev and A. Virelizier, *On two approaches to 3-dimensional TQFTs*, [arXiv:1006.3501 [math.QA]].

[50] B. Willett and I. Yaakov, *N = 2 Dualities and Z Extremization in Three Dimensions*, [arXiv:1104.0487 [hep-th]].

[51] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. 117, no. 3 (1988), 353–386.

**University of Geneva, 2-4 Rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland**
E-mail address: rinat.kashaev@unige.ch

**Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA**
E-mail address: fluo@math.rutgers.edu

**DESY Theory, Notkestrasse 85, 22603 Hamburg, Germany**
E-mail address: grigory.vartanov@desy.de