PARAMETER ESTIMATION OF NON-ERGODIC ORNSTEIN-UHLENBECK PROCESSES DRIVEN BY GENERAL GAUSSIAN PROCESSES

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Abstract. In this paper, we consider the statistical inference of the drift parameter $\theta$ of non-ergodic Ornstein-Uhlenbeck (O-U) process driven by a general Gaussian process $(G_t)_{t \geq 0}$. When $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ the second order mixed partial derivative of $R(t, s) = E[G_t G_s]$ can be decomposed into two parts, one of which coincides with that of fractional Brownian motion (fBm), and the other of which is bounded by $|ts|^{H-1}$. This condition covers a large number of common Gaussian processes such as fBm, sub-fractional Brownian motion and bi-fractional Brownian motion. Under this condition, we verify that $(G_t)_{t \geq 0}$ satisfies the four assumptions in references $[1]$, that is, noise has Hölder continuous path; the variance of noise is bounded by the power function; the asymptotic variance of the solution $X_T$ in the case of ergodic O-U process $X$ exists and strictly positive as $T \to \infty$; for fixed $s \in [0, T)$, the noise $G_s$ is asymptotically independent of the ergodic solution $X_T$ as $T \to \infty$, thus ensure the strong consistency and the asymptotic distribution of the estimator $\hat{\theta}_T$ based on continuous observations of $X$. Verify that $(G_t)_{t \geq 0}$ satisfies the assumption in references $[2]$, that is, the variance of the increment process $\{\zeta_i - \zeta_{i-1}, i = 1, ..., n\}$ is bounded by the product of a power function and a negative exponential function, which ensure that $\hat{\theta}_n$ and $\tilde{\theta}_n$ are consistent and the sequences $\sqrt{T_n}(\hat{\theta}_n - \theta)$ and $\sqrt{T_n}(\tilde{\theta}_n - \theta)$ are tight based on discrete observations of $X$.

Keywords: Ornstein-Uhlenbeck process; Gaussian process; least squares estimation; consistency; asymptotic distribution.

1. Introduction

In this paper, we consider the statistical inference of the nonergodic O-U process $X = \{X_t, t \in [0, t]\}$ defined by the following stochastic differential equation:

$$dX_t = \theta X_t dt + dG_t, \quad X_0 = 0,$$

(1.1)

Where $G = (G_t)_{t \geq 0}$ is a one-dimensional zero-mean Gaussian process, and $\theta > 0$ is an unknown parameter.

First, we review the statistical inference of the drift coefficient $\theta$ under the condition that $(\theta < 0)$ is traversed by the model (1.1). When $G$ is a fractional Gaussian process, citation $[3, 6, 4, 7, 5]$ and its references study the strong consistency and asymptotic normality of least-squares estimators (LSE), moment estimators and maximum likelihood estimators based on continuous time observations of $X$ respectively. $[8, 9]$ and their references study the asymptotic behavior of LSE based on discrete-time observations of $X$. $[10]$ and $[11]$ have studied $H \in (\frac{1}{2}, 1)$ and $H \in (0, \frac{1}{2})$ strong consistency, asymptotic normal Berry-esséen bound of $X$ continuous time...
observation for LSE sum and moment estimation, Where [11] gives the Berry-essén bound which requires $H \in (0, \frac{3}{8})$.

For the case where model (1.1) is not ergodic ($\theta > 0$). Based on continuous time observations of $X$, [12] and [13] respectively study that when $G$ is a fBm and sub-fractional Browne motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, the strong consistency and asymptotic distribution of the LSE of the drift coefficient $\theta$ [1] generalize the results to the general form, and give sufficient conditions based on the properties of $G$. Three examples of fBm with general Hurst parameter, subfractional Brownian motion and double fractional Brownian motion are given. There are also many studies based on discrete time observations of $X$. For example, [14] and [15] respectively study the weighted fractional Brownian motion when $G$ is $H \in \left(\frac{1}{2}, 1\right)$ and the parameters are $-1 < a < 0, -a < b < a + 1$. The strong consistency and random sequence compactness of two LSEs of the drift coefficient $\theta$, then the results in [2] are generalized to the general form, and sufficient conditions based on the properties of $G$ are given. Three examples of fBm with general Hurst parameter, subfractional Brownian motion and double fractional Brownian motion are given. [16] is studied based on the continuous and discrete time $X$ observation, and [15] will the result in promoting to the parameters for $a > 1, |b| < 1$ and $|b| < a + 1$.

For a given noise $G$, prove the strong consistency and asymptotic distribution of the LSE of the drift coefficient $\theta$ based on the continuous time observation of $X$, and prove the strong consistency and random sequence compactness of the two LSEs based on the discrete time observation of $X$. It is necessary to test the four hypotheses in [1] and the third hypothesis in [2]. If the noise $G$ is a Gaussian process listed below, it needs to be tested separately. This work is undoubtedly tedious and adds a lot of unnecessary calculations to the study of drift coefficient estimation for non-ergodic O-U processes. Therefore, this paper presents a more concise method, that is, including a large number of Hypothesis 1.1 of Gaussian processes.

**HYPOTHESIS 1.1.** For $H \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}, 1\right)$, the covariance function $R(s, t) = E(G_t G_s)$ satisfies that

1. for any $s \geq 0$, $R(s, 0) = 0$.
2. for any fixed $s \in (0, T)$, $R(t, s)$ is a continuous function on $[0, T]$ which is differentiable with respect to $t(0, s) \cup (s, T)$, such that $\frac{\partial}{\partial t} R(s, t)$ is absolutely integrable.
3. for any fixed $t \in (0, T)$, the difference

$$\frac{\partial R(s, t)}{\partial t} - \frac{\partial R_H(s, t)}{\partial t}$$

is a continuous function on $[0, T]$ which is differentiable with respect to $s$ in $(0, T)$ such that $\Phi(s, t)$, the partial derivative with respect to $s$ of the difference, satisfies

$$|\Phi(s, t)| \leq C_H^\prime |t|^H - 1,$$

where the constant $C_H^\prime \geq 0$ do not dependent on $T$, and $R_H(s, t)$ is the covariance function of the fBm.

Examples of fBm, subfractional Brownian motion, and other Gaussian processes that satisfy the Hypothesis 1.1 are given below, as well as some that do not.
Parameter estimation of non-ergodic Ornstein-Uhlenbeck

Based on continuous and discrete time observations of $X$, the least square technique is used to construct the estimators of the drift coefficient $\theta$. Firstly, reference [1] studies the LSE of the drift coefficient $\hat{\theta}$ based on continuous time observations of $X$, which is defined as follows

$$\tilde{\theta} = \frac{\int_0^T X_s dX_s}{\int_0^T X_s^2 ds} = \frac{X_T^2}{2 \int_0^T X_s^2 ds}. \quad (1.3)$$

Based on the discrete time observation of $X$, it is assumed that the process given by the model $(1.1)$ $X_t$ is observed at equal distance in time with step size $\Delta_n$: $t_i = i\Delta_n, I = 0, \cdots, n$, and $T_n = n\Delta_n$ represents the length of "observation window", where when $n \to \infty$, $\Delta_n \to 0$ and $n\Delta_n \to \infty$. Then, based on the sampled data $X_{t_i}, I = 0, \cdots, n$, consider the following two LSEs

$$\hat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}, \quad (1.4)$$

and

$$\tilde{\theta}_n = \frac{X_{T_n}^2}{2 \Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}. \quad (1.5)$$

It is worth noting that $X_t$ in model $(1.1)$ can be expressed as follows

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dG_s, \quad t \in [0, T]. \quad (1.6)$$

The integral about $G$ is interpreted as Young meaning. Suppose the lemma 3.1 holds. By using (a-1) of the reference [1], we can obtain

$$X_t = G_t + \theta e^{\theta t} Z_t, \quad t \in [0, T], \quad (1.7)$$

where

$$Z_t := \int_0^t e^{-\theta s} G_s ds, \quad t \in [0, T]. \quad (1.8)$$

In addition, the following process is given:

$$\zeta_t := \int_0^t e^{-\theta s} dG_s, \quad t \in [0, T]. \quad (1.9)$$

The strong consistency and asymptotic distribution of the estimator $\tilde{\theta}_T$ for continuous time observation based on $X$ are given below, and for discrete time observation based on $X$, estimators $\hat{\theta}_n$ and $\tilde{\theta}_n$ the strong consistency and the random sequence of $\sqrt{T_n}(\hat{\theta}_n - \theta)$ and $\sqrt{T_n}(\tilde{\theta}_n - \theta)$ is tight.

**Theorem 1.2.** When Hypothesis 1.1 is satisfied, the estimator $\tilde{\theta}_T$ is given by the formula $(1.3)$. Then, when $T \to \infty$,

$$\tilde{\theta}_T \longrightarrow \theta \quad a.s.. \quad (1.10)$$
In addition, when \( T \to \infty \)
\[
e^{\theta T} (\hat{\theta}_T - \theta) \xrightarrow{\text{law}} \frac{2\sigma_G}{\sqrt{E(Z_\infty^2)}} \mathcal{C}(1),
\]  
(1.11)
where \( \sigma_G = \sqrt{\mathbb{H}^{(2H)}_0(2\pi)} \), integral \( Z_\infty = \int_0^\infty e^{-\theta s} G_s ds \) is well-defined (see Lemma 2.1 of [1]), and \( \mathcal{C}(1) \) is the standard Cauchy distribution with the probability density function \( \frac{1}{\pi(1+x^2)} : x \in \mathbb{R} \).

**Theorem 1.3.** When Hypothesis 1.1 is satisfied, the estimators \( \hat{\theta}_n \) and \( \hat{\theta}_n \) are given by the formulas (1.4) and (1.5). If when \( n \to \infty \), \( \Delta_n \to 0 \) and for some \( \alpha > 0 \), \( n\Delta_n^{1+\alpha} \to \infty \), then,
\[
\hat{\theta}_n \longrightarrow \theta, \quad \hat{\theta}_n \longrightarrow \theta \quad \text{a.s.}
\]
(1.12)
And, for any \( q \geq 0 \),
\[
\Delta_n^q e^{\alpha T_n} (\hat{\theta}_n - \theta) \quad \text{and} \quad \Delta_n^q e^{\alpha T_n} (\hat{\theta}_n - \theta) \quad \text{is not tight.}
\]
(1.13)
In addition, when \( n \to \infty \), \( n\Delta_n^3 \to 0 \), in a sense, the estimators \( \hat{\theta}_n \) and \( \hat{\theta}_n \) is \( T_n \)-consistent, the sequence
\[
\sqrt{T_n}(\hat{\theta}_n - \theta) \quad \text{and} \quad \sqrt{T_n}(\hat{\theta}_n - \theta) \quad \text{is tight.}
\]
(1.14)
Next, we give some procedures that satisfy the Hypothesis 1.1.

**Example 1.4.** The fBm \( \{B^H_t, t \geq 0\} \) with parameter \( H \in (0,1) \) has the covariance function
\[
R_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).
\]
(1.15)

**Example 1.5.** [17] The sub-fractional Brownian motion \( \{S^H_t, t \geq 0\} \) with parameter \( H \in (0,1) \) has the covariance function
\[
R(s,t) = s^{2H} + t^{2H} - \frac{1}{2}((s+t)^{2H} + |s-t|^{2H}).
\]

**Example 1.6.** [18, 19] The bi-fractional Brownian motion \( \{B^{H,K}_t, t \geq 0\} \) with parameter \( H \in (0,1), K \in (0,2) \) and \( HK \in (0,1) \) has the covariance function
\[
R(s,t) = \frac{1}{2K}((s^{2H} + t^{2H})^K - |s-t|^{2HK}).
\]

**Example 1.7.** [20] The generalized sub-fractional Brownian motion \( \{S^{H,K}_t, t \geq 0\} \) with parameter \( H \in (0,1), K \in (0,2) \) and \( HK \in (0,1) \) has the covariance function
\[
R(s,t) = (s^{2H} + t^{2H})^K - \frac{1}{2}[(s+t)^{2HK} + |t-s|^{2HK}].
\]

**Example 1.8.** [21] The gaussian process \( \{G_t, t \geq 0\} \) with parameter \( H \in (0,1) \) has the covariance function
\[
R(s,t) = (t+s)^{2H} - |t-s|^{2H}.
\]

**Example 1.9.** [22] The gaussian process \( \{G_t, t \geq 0\} \) with parameter \( \gamma \in (0,1) \) has the covariance function
\[
R(s,t) = (\max(s,t))^{\gamma} - |t-s|^{\gamma}.
\]
Example 1.10. [23] The gaussian process \( \{G_t, t \geq 0\} \) with parameter \( H \in (0, 1), K \in (0, 1) \) has the covariance function
\[
R(s, t) = \frac{1}{2} [t^H + s^H - K(t + s)^H - (1 - K)|t - s|^H].
\]

Below are some procedures that do not satisfy the Hypothesis 1.1.

Example 1.11. [21] The weighted fractional Brownian motion \( \{B_t^a, t \geq 0\} \) with parameter \( a > -1, |b| < 1 \) and \( |b| < a + 1 \) has the covariance function
\[
R(s, t) = \int_0^{s \wedge t} u^a [(t - u)^b + (s - u)^b] \, du.
\]
It is known from [16] that can be reduced to
\[
R(s, t) = \beta(a + 1, b + 1)[a^{a+b+1} + s^{a+b+1}] - \int_{s \land t}^{a \vee t} u^a(t \land s - u)^b \, du,
\]
where \( \beta(\cdot, \cdot) \) denotes the beta function. \((s \land t)^{-a}\) of its second mixed partial derivative coincides with fBm, so it does not satisfy Hypothesis 1.1; [16] gives a hypothesis to prove the statistical inference of LSE based on continuous and discrete time observations of \( X \), which is different from the hypothesis in [1] and [2].

Example 1.12. [18, 24] The gaussian process \( \{G_t, t \geq 0\} \) with parameter \( H \in (0, 1), K \in (0, 2) \) has the covariance function
\[
X_t^K = \int_0^\infty (1 - e^{-rt}) e^{-\frac{t}{K}} \, dW_r, \quad t \geq 0,
\]
where \( \{W_t, t \geq 0\} \) is the standard Brownian motion. The covariance function is
\[
R_K(s, t) = \text{Cov}(X_s^K, X_t^K) = \begin{cases} 
\frac{1}{K} [t^K + s^K - (t + s)^K], & K \in (0, 1); \\
\frac{1}{K(K - 1)} [(t + s)^K - K - s^K], & K \in (1, 2). 
\end{cases}
\]
The second order mixed partial derivative is only to \( |s|^{H-1} \) s bounded part, so don’t say to meet 1.1; and when \( T \to \infty \), the asymptotic variance of the solution \( X_T \) in the case of the O-U process \( X \) traversal is 0, which does not satisfy the \( \sigma_G > 0 \) of [1](H3). However, it satisfies three hypotheses in [2].

Example 1.13. [22] The gaussian process \( \{G_t, t \geq 0\} \) with parameter \( \gamma \in (0, 1) \) has the covariance function
\[
R(s, t) = (t + s)^\gamma - (\max(s, t))^{\gamma}.
\]
The second order mixed partial derivative is only to \( |s|^{H-1} \) is bounded part, so don’t say to meet Hypotheses 1.1; and when \( T \to \infty \), the asymptotic variance of the solution \( X_T \) in the case of the O-U process \( X \) traversal is 0, which does not satisfy the \( \sigma_G > 0 \) of [1](H3). However, it satisfies three hypotheses in [2].

Remark 1.14. Through the examples listed above, found that based on \( X \) discrete observations, even if the covariance function of second order mixed partial derivative is only to \( |s|^{H-1} \) is bounded part, also meet [2] in the three hypotheses. However, based on the continuous observation of \( X \), we do not find any Gaussian process satisfying the four assumptions in [1] but not the
Hypothesis 1.1. In addition, if the model (1.1) is driven by a linear combination of independent zero-mean Gaussian processes, each of which satisfies the Hypothesis 1.1, the estimator \( \tilde{\theta}_T \) based on continuous time observation \( X \) has strong consistency and asymptotic distribution. And based on the discrete time \( X \) observation estimator \( \hat{\theta}_n \) and \( \tilde{\theta}_n \) has strong consistency and random sequence \( \sqrt{T_n}(\tilde{\theta}_n - \theta) \) and \( \sqrt{T_n}(\hat{\theta}_n - \theta) \) is tight.

2. Preliminary

Denote \( G = \{G_t, t \in [0, T]\} \) as a continuous centered Gaussian process with covariance function
\[
E(G_tG_s) = R(s, t), \quad s, t \in [0, T],
\]
defined on a complete probability space \((\Omega, F, P)\). The filtration \( F \) is generated by the Gaussian family \( G \). Suppose, in addition, that the covariance function \( R \) is continuous. Let \( \mathbb{E} \) denote the space of all real valued step functions on \([0, T]\). The Hilbert space \( \mathcal{H} \) is defined as the closure of \( \mathbb{E} \) with the inner product
\[
\langle 1_{[a,b)}, 1_{[c,d]\rangle}_\mathcal{H} = E((G_b - G_a)(G_d - G_c)).
\]
Where \( 1_{[a,b)} \) represents the indicator function of \([a, b)\). We denote \( G = \{G(h), h \in \mathcal{H}\} \) as the isonormal Gaussian process on the probability space \((\Omega, F, P)\), indexed by the elements in the Hilbert space \( \mathcal{H} \). In other words, \( G \) is a Gaussian family of random variables such that
\[
E(G(h)) = 0, \quad E(G(g)G(h)) = \langle g, h \rangle_{\mathcal{H}}.
\]
for any \( g, h \in \mathcal{H} \).

Notation 1. Let \( R_H(s, t) \) be the covariance function of the fractional Brownian motion as in (1.15). \( \mathcal{H}_1 \) is the separable Hilbert space associated with fBm. \( V_{[0, T]} \) denote the set of bounded variation function on \([0, T]\). For function \( f, g \in V_{[0, T]} \), we define two products as
\[
\langle f, g \rangle_{\mathcal{H}_1} = - \int_{[0, T]^2} f(t) \frac{\partial R_H(s, t)}{\partial t} d\nu_g(ds), \quad (2.1)
\]
\[
\langle f, g \rangle_{\mathcal{H}_2} = C'_H \int_{[0, T]^2} |f(t)g(s)|(ts)^{H-1} dt ds, \quad (2.2)
\]
where \( \nu_g \) is given below.

The following proposition is an extension of Theorem 2.3 of [25] and from Proposition 2.1 of [26], which gives the products representation of the Hilbert space \( \mathcal{H} \):

Proposition 2.1. \( V_{[0, T]} \) denote the set of bounded variation function on \([0, T]\). Then, \( V_{[0, T]} \) is dense in \( \mathcal{H} \) and we have
\[
\langle f, g \rangle_{\mathcal{H}} = \int_{[0, T]^2} R(s, t) \nu_f(dt) \nu_g(ds), \quad \forall f, g \in V_{[0, T]}, \quad (2.3)
\]
where \( \nu_g \) is the restriction to \(([0,T], B([0,T])) \) of the Lebesgue-Stieltjes signed measure associated with \( g^0 \) defined as

\[
g^0(x) = \begin{cases} 
g(x), & \text{if } x \in [0,T]; \\
0, & \text{otherwise}.
\end{cases}
\]

In addition, if the covariance function \( R(s,t) \) satisfies the Hypothesis \( 1.1 \), then

\[
\langle f, g \rangle_{\delta} = - \int_{[0,T]^2} f(t) \frac{\partial R(s,t)}{\partial t} \dd \nu_g(ds), \tag{2.4}
\]

and

\[
\| \langle f, g \rangle_{\delta} - \langle f, g \rangle_{\delta_\alpha} \| \leq \langle f, g \rangle_{\delta_\alpha}. \tag{2.5}
\]

**Corollary 2.2.** Denote by \( \delta_a(.) \) the Dirac \( \delta \) function centered at a point \( a \). Let \( f = h_1 \cdot 1_{[a,b]}(.) \), \( g = h_2 \cdot 1_{(c,d)}(.) \), with \( h_1 \) and \( h_2 \) are continuously differentiable function. Then we have

\[
\langle f, g \rangle_{\delta} = - \int_{[0,T]^2} h_1(t) 1_{[a,b]}(t) h_2'(s) 1_{[c,d]}(s) \frac{\partial R(s,t)}{\partial t} \dd t \dd s \\
+ \int_{[0,T]^2} h_1(t) 1_{[a,b]}(t) h_2(s) \frac{\partial R(s,t)}{\partial t} [\delta_b(s) - \delta_a(s)] \dd t \dd s. \tag{2.6}
\]

**Remark 2.3.** The inequality (2.5) is the starting point of the present paper, which, when \( H \in (\frac{1}{2}, 1) \), Hypothesis 1.1 (3) imply that the identity (2.4) can be rewritten as

\[
\langle f, g \rangle_{\delta} = \int_{[0,T]^2} f(t) g(s) \frac{\partial^2 R(s,t)}{\partial t \partial s} \dd t \dd s \quad \forall f, g \in V_{[0,T]}.
\]

In the case, inequality (2.5) has been obtained in [10]. when \( H \in (0, \frac{1}{2}) \), it is well known that both \( \frac{\partial^2 R(u,v)}{\partial u \partial v} \) and \( \frac{\partial^2 R(u,v)}{\partial u \partial v} \) are not absolutely integrable. But the absolute integrability of their difference makes the key inequality (2.5) still valid.

However, if \( f \) and \( g \) support on the disjoint interval, the formula (2.7) also holds in \( H \in (0, \frac{1}{2}) \) when the process \( g \) is Bm. The lemma given below is for Lemma 2.1 of [27] extension, proof method is the same as formula (2.5).

**Lemma 2.4.** Let \( f, g \) be bounded variances supported on \([a,b] \) and \([c,d] \) respectively, when \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \) and \( 0 \leq a < b \leq c < d < \infty \), we have

\[
\langle f, g \rangle_{\delta} = \int_c^d \int_a^b f(t) g(s) \frac{\partial^2 R(s,t)}{\partial t \partial s} \dd t \dd s.
\]

In addition, if the covariance function \( R(s,t) \) satisfies the Hypothesis 1.1, then

\[
\| \langle f, g \rangle \| \leq \int_c^d \int_a^b |f(t) g(s)| \left[ C_H(s-t)^{2H-2} + C_H'(ts)^{H-1} \right] \dd t \dd s,
\]

where \( C_H = |H(2H-1)| \).

The following theorem is a description of the main results in the references [1] and [2].

**Theorem 2.5.** In model \((1.1), \) the following conditions are given for process \( G \):

1. The process \( G \) has Hölder continuous paths of order \( \delta \in (0,1] \).
for every $t \geq 0$, $E(G_t^2) \leq ct^{2\gamma}$ for some positive constants $c$ and $\gamma$.

(3) The limiting variance of $e^{-\theta T} \int_0^T e^{\theta s} dG_s$ exist and strictly positive as $T \to \infty$, i.e., there exists a constant $\sigma_G > 0$ such that

$$\lim_{T \to \infty} E \left[ \left( e^{-\theta T} \int_0^T e^{\theta s} dG_s \right)^2 \right] = \sigma_G^2.$$

(4) For all fixed $s \in [0, T)$,

$$\lim_{T \to \infty} E \left( G_s e^{-\theta T} \int_0^T e^{\theta r} dG_r \right) = 0.$$

(5) For each $i = 1, \ldots, n$ and $n \geq 1$, existence of normal number $\gamma$, have

$$E \left[ \left( \zeta_{t_i} - \zeta_{t_{i-1}} \right)^2 \right] \leq C(\gamma) \Delta_{t_i}^{\gamma} e^{-2\theta t_i}.$$

If the process $G$ satisfy conditions (1) and (2), the strong consistency (1.10) holds; If the process $G$ satisfy conditions (1) – (4), the asymptotic distribution (1.11) holds; If the process $G$ satisfy conditions (1), (2) and (5), then the strong consistency (1.12) and the random sequence compactness (1.13), (1.14) holds.

The following inequality comes from Lemma 3.3 of [10].

**Lemma 2.6.** Assuming $\beta \in (0, 1)$, there is a constant $C > 0$, for any $t \geq 0$, we have

$$\int_0^t e^{-\theta(t-r)} r^{\beta-1} dr \leq C(1 \land t^{\beta-1}).$$

(2.8)

For the rest of this article, $C$ will be a normal number that does not depend on $T$, and its value may vary from line to line.

3. PROOF OF THEOREM 1.2 AND THEOREM 1.3

First, define some functions that will be used in the proof. Let

$$h(\cdot) = 1_{[s, T]}(\cdot), \quad f(\cdot) = e^{-\theta(T-\cdot)} 1_{[0,T]}(\cdot),$$

$$g(\cdot) = 1_{[0,s]}(\cdot), \quad m(\cdot) = e^{-\theta(\cdot)} 1_{[t_{i-1},t_i]}(\cdot).$$

The following Lemmas 3.1, 3.2 and 3.3 show that the assumptions in [1] are valid if 1.1 is valid. Lemmas 3.1 and 3.5 indicates that if 1.1 holds, the assumptions in [2] holds.

**Lemma 3.1.** Under Hypothesis 1.1, there exists a constant $C > 0$ that does not depend on $T$, for all $s, t \geq 0$, there is

$$E[|G_t - G_s|^2] \leq C|t - s|^{2H}.$$

(3.1)

Specially, when $s = 0$, $E(G_t^2) \leq Ct^{2H}$.

**Proof.** Suppose $t \geq s$. According to Itô isometric and inequality (2.5), there is

$$E[|G_t - G_s|^2] = \langle h, h \rangle_{B_t} \leq \langle h, h \rangle_{B_1} + \langle h, h \rangle_{B_2}.$$
where \((h,h)_{\mathbb{H}}\) is shown in equation (2.1). Because \(R_H(t,s)\) is covariance function of fBm, so \((h,h)_{\mathbb{H}}\) value is \(|t-s|^{2H}\). The formula (2.2) can be obtained

\[
(h,h)_{\mathbb{H}} = C_H' \left( \int_s^t u^{H-1} du \right)^2 = C(t^H - s^H)^{2} \leq C|t-s|^{2H},
\]

We’re using an inequality \(1-x^H \leq (1-x)^H\) here, where \(x \in [0,1]\) and \(H \in (0,1)\). Therefore, we have

\[
E[|G_t - G_s|^2] \leq C|t-s|^{2H},
\]

and when \(s = 0\), we have \(E(G_t^2) \leq Ct^{2H}\). □

**Lemma 3.2.** The constant \(\sigma_G^2 = \sqrt{\frac{H^2(2H)}{\pi}}\). Under Hypothesis 1.1, we have that

\[
\lim_{T \to \infty} E\left[ \left( e^{-\theta T} \int_0^T e^{\theta u} dG_u \right)^2 \right] = \sigma_G^2.
\]  \(\text{(3.2)}\)

**Proof.** According to Itô isometric, there is

\[
E\left[ \left( e^{-\theta T} \int_0^T e^{\theta u} dG_u \right)^2 \right] = \langle f, f \rangle_{\mathbb{H}}.
\]

When \(H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)\), Lemma 2.3 of [28] gives the limit when \(T \to \infty\):

\[
\langle f, f \rangle_{\mathbb{H}} \to \sigma_G^2.
\]

In addition, by the formula (2.2) and Lemma 2.6, we have

\[
\langle f, f \rangle_{\mathbb{H}} = C_H' \int_{[0,T]^2} e^{-\theta (T-u)} e^{-\theta (T-v)} (uv)^{H-1} du dv \leq C T^{2H-2}.
\]

Therefore, according to the inequality (2.5) and the triangle inequality, when \(T \to \infty\), we have

\[
|\langle f, f \rangle_{\mathbb{H}} - \sigma_G^2| \leq |\langle f, f \rangle_{\mathbb{H}} - \langle f, f \rangle_{\mathbb{H}_1}| + |\langle f, f \rangle_{\mathbb{H}_1} - \sigma_G^2| \\
\leq \langle f, f \rangle_{\mathbb{H}_2} + |\langle f, f \rangle_{\mathbb{H}_1} - \sigma_G^2| \\
\to 0.
\]  □

**Lemma 3.3.** Under Hypothesis 1.1. For any fixed \(s \in [0,T]\), we have

\[
\lim_{T \to \infty} E\left( G_s e^{-\theta T} \int_0^T e^{\theta u} dG_u \right) = 0.
\]  \(\text{(3.3)}\)

**Proof.** For fixed \(s \in [0,T]\), divide the function \(f(\cdot)\) into the following form

\[
f(\cdot) = e^{-\theta (T-s)} \mathbf{1}_{[0,s)}(\cdot) + e^{-\theta (T-s)} \mathbf{1}_{[s,T)}(\cdot) := f_1(\cdot) + f_2(\cdot).
\]
According to Itô isometric, there is
\[
E \left( G_s e^{-\theta T} \int_0^T e^{\theta r} dG_r \right) = \langle f, g \rangle_5 = \langle f_1, g \rangle_5 + \langle f_2, g \rangle_5. \tag{3.4}
\]

First calculate the limit of \( \langle f_1, g \rangle_5 \). According to formulas (2.6) and (2.2), monotonicity indicate the presence of normal number \( C_{\theta, H, s} \) and \( C'_{\theta, H, s} \), have
\[
0 < \langle f_1, g \rangle_5 = e^{-\theta T} \int_0^s e^{\theta v} 1_{[0,s)}(v) (\delta_s(u) - \delta_0(u)) \frac{\partial R_H(u, v)}{\partial v} dudv
\]
\[
= H e^{-\theta T} \int_0^s e^{\theta v} (v^{2H-1} + (s - v)^{2H-1}) dv
\]
\[
\leq H e^{-\theta T} e^{\theta s} \int_0^s (v^{2H-1} + (s - v)^{2H-1}) dv
\]
\[
= C_{\theta, H, s} e^{-\theta T},
\]
and
\[
0 < \langle f_1, g \rangle_5 = C'_{\theta} e^{-\theta T} \int_0^s \int_0^s e^{\theta (uv)} H^{-1} dudv
\]
\[
\leq C'_{\theta} e^{-\theta T} e^{\theta s} \left( \int_0^s u^{H-1} du \right)^2
\]
\[
= C'_{\theta, H, s} e^{-\theta T}.
\]

Therefore, when \( T \to \infty \), inner products \( \langle f_1, g \rangle_5 \) and \( \langle f_1, g \rangle_5 \) converge to 0. By substituting it into (2.5), it can be obtained by forced convergence
\[
\langle f_1, g \rangle_5 \to 0 \quad (T \to \infty).
\]

Next, calculate the limit of \( \langle f_2, g \rangle_5 \). Because of the function \( f_2 \) and \( g \) were support on \([s, T)\) and \([0, s)\), and \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), so according to Lemma 2.4, we have
\[
|\langle f_2, g \rangle_5| \leq \int_s^T \int_0^s e^{-\theta (T - v)} \left[ C_H(v - u)^{2H-2} + C'_{H} (uv)^{H-1} \right] dudv.
\]

Using the L’Hôpital rule
\[
\lim_{T \to \infty} |\langle f_2, g \rangle_5| \leq \lim_{T \to \infty} e^{-\theta T} \int_s^T \int_0^s e^{\theta v} \left[ C_H(v - u)^{2H-2} + C'_{H} (uv)^{H-1} \right] dudv
\]
\[
= \lim_{T \to \infty} e^{-\theta T} \int_0^s \left[ C_H(T - u)^{2H-2} + C'_{H} (Tu)^{H-1} \right] du
\]
\[
= 0.
\]

Finally, combining the limits of \( \langle f_1, g \rangle_5 \) and \( \langle f_2, g \rangle_5 \) with formula (3.4), the desired result (3.3) is obtained.

\textit{proof of Theorem 1.2} According to the Kolmogorov continuity criterion and the Lemma 3.1, we know that for all \( \varepsilon \in (0, H) \), the process \( G \) has \( H - \varepsilon \) order Hölder continuous path, and the strong consistency of the estimator \( \hat{\theta}_T \) is deduced from the Theorem 2.5. In addition, combining
the results of Lemma 3.1, Lemma 3.2 and Lemma 3.3, the Theorem 2.5 can derive its asymptotic distribution.

**Lemma 3.4.** Assuming \( \gamma \in (0, 1) \), \( t_i = i \Delta_n \), and \( \Delta_n \to 0 \). Then, for any \( i = 1, \ldots, n \), there is

\[
e^{t_i} \int_{t_{i-1}}^{t_i} e^{-\theta u} (t_i - u)^{2\gamma - 1} du \leq C(\gamma) \Delta_n^{2}\gamma e^{-2\theta t_{i-1}},
\]

\[
e^{t_i} \int_{t_{i-1}}^{t_i} e^{-\theta u} (u - t_{i-1})^{2\gamma - 1} du \leq C(\gamma) \Delta_n^{2}\gamma e^{-2\theta t_{i-1}}.
\]

**Proof.** The proof process is similar to Lemma 4.2 of [2], which will not be proved in detail here. \( \square \)

**Lemma 3.5.** Under Hypothesis 1.1, for each \( i = 1, \ldots, n \) and \( n \geq 1 \), we have that

\[
E \left[ (\zeta_{t_i} - \zeta_{t_{i-1}})^2 \right] \leq C(H) \Delta_n^{2H} e^{-2\theta t_{i-1}}.
\]

**Proof.** According to Itô isometric and inequality (2.5), there is

\[
E \left[ (\zeta_{t_i} - \zeta_{t_{i-1}})^2 \right] = \langle m, m \rangle_{B_1} \leq \langle m, m \rangle_{B_1} + \langle m, m \rangle_{B_2}.
\]

Again, first calculate the value of \( \langle m, m \rangle_{B_1} \). When the functions \( f, g \) in the Corollary 2.2 are equal, there is

\[
\langle m, m \rangle_{B_1} = \int_{[t_{i-1}, t_i]^2} e^{-\theta(u+v)} \frac{\partial R_H(u,v)}{\partial u} [\theta + (\delta_{t_i}(v) - \delta_{t_{i-1}}(v))] dudv
\]

\[
= \theta H \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} e^{-\theta(u+v)} (u^{2H-1} - |u - v|^{2H-1} \text{sgn}(u - v)) dudv
\]

\[
+ H e^{-\theta t_i} \int_{t_{i-1}}^{t_i} e^{-\theta u} (u^{2H-1} - (t_i - u)^{2H-1}) du
\]

\[
- H e^{-\theta t_{i-1}} \int_{t_{i-1}}^{t_i} e^{-\theta u} (u^{2H-1} - (u - t_{i-1})^{2H-1}) du
\]

\[
= H e^{-\theta t_i} \int_{t_{i-1}}^{t_i} e^{-\theta u} (t_i - u)^{2H-1} du + H e^{-\theta t_{i-1}} \int_{t_{i-1}}^{t_i} e^{-\theta u} (u - t_{i-1})^{2H-1} du
\]

\[
\leq e^{-2\theta t_{i-1}} \Delta_n^{2H}.
\]

The Lemma 3.4 holds when \( H = \gamma \). Next, calculate the value of \( \langle m, m \rangle_{B_2} \). By the formula (2.2), we have

\[
\langle m, m \rangle_{B_2} = C_H' \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} e^{-\theta(u+v)} (uv)^{H-1} dudv
\]

\[
= C_H' \left( \int_{t_{i-1}}^{t_i} e^{-\theta u} u^{H-1} du \right)^2
\]

\[
\leq \frac{C_H'}{H^2} e^{-2\theta t_{i-1}} \Delta_n^{2H}.
\]

The desired result is obtained by substituting the above two inequalities into (3.8). \( \square \)
proof of the Theorem 1.3 Combining the results of the Lemma 3.1 and the Lemma 3.5, the strong consistency of $\hat{\theta}_n$ and $\tilde{\theta}_n$ is deduced by 2.5, which is based on $X$ discrete time observation. And random sequence $\sqrt{T_n}(\hat{\theta}_n - \theta)$ and $\sqrt{T_n}(\tilde{\theta}_n - \theta)$ is tight.

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