Towards a lattice calculation of the coefficients of the QCD chiral Lagrangian

A.R. Levi, V.Lubicz and C.Rebbi
Department of Physics, Boston University, 590 Commonwealth Avenue, Boston, MA 02215, USA

We discuss a general strategy to compute the coefficients of QCD chiral Lagrangian by using the lattice regularization of QCD with Wilson fermions. This procedure requires the introduction of an effective Lagrangian for lattice QCD as an intermediate step in the calculation. The continuum QCD chiral Lagrangian can be then obtained by expanding the lattice effective Lagrangian in increasing powers of the external momenta. A suitable renormalization procedure is required to account for the chiral symmetry breaking introduced by the Wilson term in the lattice action. In anticipation of a numerical simulation, the lattice effective Lagrangian is computed analytically and investigated in the strong coupling and large $N$ limit.

1. Introduction

A first principle calculation of the coefficients of the QCD chiral Lagrangian is a task of fundamental theoretical interest. Besides representing an important test of QCD in the low-energy sector, such a calculation would also improve our knowledge of the chiral coefficients. At present, these coefficients are mainly determined from phenomenological constraints and are known, in some cases, with relative uncertainties larger than 100%.

Being only a function of the fundamental scale $\Lambda_{\text{QCD}}$ and the heavy quark masses, the coefficients of chiral Lagrangian are in principle directly calculable from QCD. This requires the use of lattice QCD, which represents today the only viable computational method for calculating non-perturbative QCD observables from first principles. The aim of this talk is to discuss a general strategy to perform such a calculation.

A standard approach to compute the coefficients of an effective theory consists in performing a matching between the effective and the underlying fundamental theory. Specifically, one computes a sufficient numbers of physical amplitudes, both in the effective and the original theory, and derives the values of the coefficients from a comparison of the results. For the QCD chiral Lagrangian, the calculation in the fundamental theory, being non-perturbative, should be performed on the lattice.

However, such an approach would be significantly affected by the existence, on the lattice, of an infra-red cut-off. In current numerical simulations, the minimum allowed value of momentum, $p_{\text{min}} = 2\pi/La$, is typically of the order of the rho meson mass. This prevents the possibility of performing the calculation in the low-energy region, where the predictions of the effective theory can be reliably obtained. In order to overcome this difficulty, one should then consider lattices of very large volumes, which unfortunately entails substantial losses in efficiency and, therefore, in statistical accuracy.

An alternative approach to the calculation of the coefficients of QCD chiral Lagrangian on the lattice has been proposed in ref. [4]. The basic observation there is that the separation between effective and non-effective degrees of freedom, occurring in the continuum theory, must be mirrored by an equivalent distinction in the theory regularized on the lattice. Therefore, one can define an effective theory on the lattice which is equivalent, for any value of the lattice spacing or the lattice bare coupling constant, to the fundamental lattice QCD theory. The coefficients of the continuum QCD chiral Lagrangian can be then computed through the following steps: i) Define the lattice effective theory by assuming a

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sufficiently large set of couplings, with strengths determined by arbitrary coefficients. ii) Fix these coefficients through the matching of an overcomplete set of expectation values, computed both in the full and the effective lattice theory. The calculation in the full theory requires a numerical simulation. iii) Expand the effective lattice Lagrangian in increasing powers of the external momenta. The result of such an expansion is the continuum QCD chiral Lagrangian.

The basic feature of this procedure is the introduction of the effective lattice Lagrangian as an intermediate step in the calculation. The main advantage is that this Lagrangian can be organized in terms of the distance of couplings rather than in powers of momenta, as in the corresponding continuum effective theory. Therefore, in performing the matching between the effective and the fundamental theory, the existence of an infrared cut-off in the calculation does not represent a problem anymore.

In addition, in this procedure both the full and the effective theory are defined in the same regularization scheme. Therefore, the finite ultraviolet cut-off effects, affecting the numerical calculation, can be better kept under control, since these effects are exactly predicted by the lattice effective theory.

Since the effective Lagrangian contains explicitly the collective fields responsible for the long distance behavior of the fundamental lattice theory, the calculations of its coefficients should only involve shorter scale of length, and should be feasible, therefore, on lattices of moderate size.

2. The lattice effective Lagrangian in the strong coupling and large $N$ limit

A useful feature of the strategy discussed above is that, in the limit of strong coupling and large number of colors $N$, the integration of the non-effective degrees of freedom in the QCD lattice Lagrangian can be performed analytically \[ \mathcal{L}_{\text{eff}}(U) \]

\[ = C_{\Delta} \text{Tr} \left[ (\Delta_{\mu} U)^\dagger (\Delta_{\mu} U) \right] + C_{m} \text{Tr} \left[ \chi^\dagger U + U^\dagger \chi \right] \]

\[ + C_{W} r^2 \text{Tr} \left[ \xi^\dagger U \xi U + U^\dagger \xi U^\dagger \xi \right] + \ldots \]

where the dots represent an infinite series of additional terms. In eq. (1), $U$ is the effective field, $\Delta_{\mu} U$ represents a lattice version of the chiral covariant derivative, $\chi$ is the external source including also the light quark mass matrix and $\xi$ is an additional source which is coupled, in the lattice fermionic action, to the Wilson term. This source has no correspondent in the continuum theory, and it is introduced in the lattice action in order to achieve a local chiral invariance with respect to combined transformations of the quark fields and the external sources.

The first two terms on the right-hand side of eq. (1) formally reduce in the continuum limit to the operators appearing in the QCD chiral Lagrangian at $\mathcal{O}(p^2)$. In contrast, the last term in eq. (1) does not have any correspondent in the continuum effective theory. This term is an example of an infinite number of couplings which are originated by the Wilson term. They mainly appear, in the effective Lagrangian, in order to reproduce the effects of the explicit chiral symmetry breaking induced by the Wilson term in the lattice theory.

The role of these “Wilson terms” in the effective Lagrangian can be studied explicitly. A clear example is given, in this respect, by the well known equation which relates, in the continuum, the pseudoscalar mass and decay constant to the quark mass and condensate. By using the effective theory, in the strong coupling and large $N$ limit, we find that on the lattice this relation has the form:

\[ Z_{A}^{2} F_{\pi}^{2} M_{\pi}^{2} = (m - \hat{m}) \langle \bar{\psi} \psi \rangle \]
where \( Z_A = 1 + r^2 \) and \( \hat{m} \) is a function of the bare quark mass \( m \) and the Wilson parameter \( r \). This function defines the critical value \( m_c \) of the bare quark mass. In the limit \( r = 0 \), chiral symmetry prevents an additive renormalization of the quark mass; the function \( \hat{m} \) identically vanishes and \( m_c = 0 \). In the case \( r = 1 \), one finds that the pion mass vanishes for \( m_c = -2 \), corresponding to the critical value \( k_c = 1/4 \) of the Wilson hopping parameter.

The constant \( Z_A \) in eq. (2) is also a consequence of the Wilson terms in the effective Lagrangian. Despite the strong coupling approximation, this constant can be consistently interpreted as a renormalization of the point-split axial current \( A_\mu \), from which \( F_\pi \), in eq. (2), has been calculated. In the limit of vanishing Wilson term, this current is partially conserved and \( Z_A \) reduces to 1. More generally, we find that, once the bare quark mass and the lattice operators have been properly renormalized, the lattice correlation functions satisfy, even in the strong coupling approximation, the Ward identities of the continuum theory, as predicted by recovered chiral symmetry. These identities can be then used to derive the values of the renormalization constants of the lattice operators. For instance, we find that the lattice conserved vector current has indeed \( Z_V = 1 \), and the local scalar and pseudoscalar densities renormalize with \( Z_S/Z_P = 1 \). The way in which this scenario sets up is exactly the one outlined in ref. [7] for the weak coupling regime.

### 3. The renormalized effective Lagrangian

In the continuum limit \( g_0 \to 0 \) the effects of the chiral symmetry breaking induced by the Wilson term are expected to vanish. However, for the typical values of couplings used in current numerical simulations, these effects are still quite relevant, and the Wilson terms in the lattice effective Lagrangian cannot be neglected.

The presence of these terms complicate the task of deriving the continuum QCD chiral Lagrangian from the lattice calculation. First, these terms significantly increase the number of couplings to be introduced in the lattice effective theory. Secondly, they do not have a direct continuum limit in the QCD chiral Lagrangian, and, because of their presence, the external sources in the lattice effective theory do not reduce, directly, to their continuum counterparts. Indeed, some of the effects of the Wilson terms in the effective theory can be interpreted as a renormalization of the lattice external sources.

In order to overcome these difficulties, one can follow a different procedure, which makes it possible to discard the Wilson terms in the lattice effective theory. The basic idea is to consider a “renormalized” effective Lagrangian, which reproduces the correlation functions of properly renormalized operators in the fundamental lattice theory. Because these correlation functions satisfy all the Ward identities predicted by continuum current algebra, the renormalized effective Lagrangian does not contain the Wilson terms at all. This can be shown explicitly in the strong coupling and large \( N \) limit. In addition, because the renormalized lattice theory exhibits the same chiral structure of its continuum counterpart, this procedure may significantly simplify the task of deriving the QCD chiral Lagrangian from the lattice effective theory.

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