Counterexample to Bell’s theorem: Arithmetic loophole in action

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Bell’s theorem is supposed to exclude all local hidden-variable models of quantum correlations. However, an explicit counterexample shows that a new class of local realistic models, based on generalized arithmetic and calculus, can exactly reconstruct quantum probabilities typical of two-electron singlet states. The model is classical in the sense of Einstein, Podolsky, Rosen, and Bell: elements of reality exist and probabilities are computed by means of appropriate integrals over hidden variables. However, the integral here is not the standard one. It has all the standard properties but only with respect to appropriate arithmetic. Certain formal transformations of integral expressions one finds in the usual proofs à la Bell do not work. Standard Bell-type inequalities cannot be proved. The system we consider is deterministic, local-realistic, rotationally invariant, observers have free will, detectors are perfect, so is free of all the canonical loopholes discussed in the literature. The model is quantum enough to fake quantum correlations, but classical enough to allow for eavesdropping. Quantum cryptography has a problem.

I. INTRODUCTION

The problem posed by Einstein, Podolsky, and Rosen [1, 2], and reformulated by Bell [3], is as follows: Can there exist elements of reality whose knowledge would allow to predict in advance results of quantum measurements? The advent of quantum cryptography [4–6] has turned the purely academic debate into a practical one: any loophole in Bell’s reasoning creates a potential threat for security of data transmission.

Bell’s inequality [8] does not apply to systems which do not satisfy at least one assumption needed for its proof. This includes nonlocal hidden variables [7, 8], theories based on detector inefficiency [9], locally incompatible random variables [10–12], observers with limited freedom of choice [13], contextual cognitive models [14, 15]. In each of these cases it is easy to understand why the inequality cannot be derived. Detector inefficiency was used to hack a Bell-type cryptosystem a long time ago [16, 17]. Threats based on nonlocal hidden variables, as well as remedies against them, are less known [18].

At the other extreme are various abstract constructions, involving probability manifolds [19], non-measurable sets [20], non-computable fractals [21], or covering spaces of group manifolds [22]. However, the more abstract the model, the more controversial and obscure its physical and probabilistic interpretation.

What I will discuss is much more down to earth. Quite recently I have identified a new, ‘arithmetic’ loophole in the proof of the theorem [23]. It remained to construct an explicit counterexample that would be simultaneously free of all the other loopholes discussed in the literature. The article shows how to do it. The observers have free will, detectors are ideal, hidden variables are local, and yet the derived probabilities are exactly those implied by quantum mechanics.

The trick is in the unexplored mathematical freedom: the form of hidden-variable arithmetic and calculus. Arithmetic is a natural language of mathematics. It defines the ways we add, subtract, multiply, and divide numbers. Modified arithmetic implies a modified calculus. However, as there are different languages, there exist different arithmetics. The same set of physical variables may be equipped with several coexisting arithmetics. Isomorphic arithmetics are like different languages faithfully expressing the same truth. Of course, in order to understand a sentence, one first has to know the language in which it was formulated, otherwise amusing mistakes can be made. Similarly, a theorem formulated in one arithmetic may or may not be valid in another one. In particular, Bell’s inequality may be satisfied in a hidden-variable calculus, but violated in the calculus used by macroscopic observers.

Locality, the key assumption of Bell, effectively means that certain probabilities have a product form. The notion of a product depends on the choice arithmetic. So, how many arithmetics are there available if we assume that probabilities are represented by non-negative real numbers that sum (in the ordinary sense of the word) to 1? The answer may be surprising: infinitely many! It remains to find a correct hidden-variable calculus and prove that it predicts local-realistic probabilities that are identical with the quantum ones. This is what we will do in the paper.

The result seems to be bad news for quantum cryptography, even in its most ideal device-independent version [24]. Bell’s theorem cannot certify security of quantum protocols.

Since the subject is unknown to a wider audience, we will gradually develop the construction. We will begin with arithmetic of parallel-connected resistors. Although the system is well understood from a physical point of view, its arithmetic aspects may appear paradoxical. In particular, there is a nontrivial relation between addition and multiplication, a fact with consequences for natural numbers.

The next example is related to the problem of dark energy. We will see that accelerated expansion of the Universe can be regarded as a consequence of a mismatch between two arithmetics: the one we normally use, and the one applying to cosmological-scale observers [25].
example is particularly relevant for our discussion. It shows that ‘large’ and ‘small’ systems may be in principle based on different types of arithmetic. In the context of Bell’s theorem it is us, the macroscopic-scale observers who are ‘large’, while the hidden-variables are ‘small’.

Finally, we construct the hidden-variable model of singlet-state correlations. Technically it is based on two elements: The product which defines locality in Clauser-Horne-type probabilities, and the integral which relates hidden variables with observable averages. Our model is further analyzed from a geometric perspective. We will see that it is rotationally symmetric, a property one expects from singlet state correlations, but this rotational symmetry is as hidden as the hidden variables themselves.

The construction is simple, one just has to get used to a more general perspective. I believe the proposed formulation circumvents all the basic limitations imposed by Bell’s theorem. Most importantly, the model is probabilistically quantum enough to fake quantum correlations, and classical enough to allow for eavesdropping in quantum cryptography.

II. THE WORLD ACCORDING TO RESISTOR

Let us begin with the example that is truly down to earth and easy to understand. A parallel configuration of resistors is a resistor whose resistance is computed by

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = f^{-1}(f(R_1) + f(R_2)),$$

and thus $f$ makes the ‘parallel arithmetic’ isomorphic to the standard arithmetic of $\mathbb{R}$ (we only have to be cautious at $0$). $\oplus$ and $\oslash$ are commutative and associative, and $\oslash$ is distributive with respect to $\oplus$. So how come that two mathematically isomorphic structures cannot play the same mathematical roles?

In fact, they can play the same roles. The problem is with the meaning of $n$. The natural number $n$ at the right-hand side of (5) is not a natural number in the sense of the new arithmetic. In order to understand why, we first have to clarify what should be meant by ‘zero’ and ‘one’. Once we define a ‘one’ we can add it several times to itself. The result should be a well defined natural number.

‘Zero’ is an element $0'$ such that $x \oplus 0' = x$ for any $x$. An insulated wire is a parallel configuration of resistors with insulation in the role of an infinitely resistant resistor. Insulation does not influence the wire, $R \oplus \infty = R$, hence $0' = 0$. ‘One’ is an element $1'$ such that $x \oplus 1' = x$, but since multiplication is unchanged we get $1' = 1$.

Greater natural numbers are constructed iteratively,

$$2' = 1' \oplus 1' = f^{-1}(f(1') + f(1')) = f^{-1}(2) = 1/2, \quad (8)$$
$$3' = 2' \oplus 1' = 1' \oplus 1' \oplus 1' = f^{-1}(3) = 1/3, \quad (9)$$
$$\vdots$$
$$n' = (n-1)' \oplus 1' = 1' \oplus \cdots \oplus 1' = f^{-1}(n) = 1/n. \quad (10)$$

Accordingly, $n' = f^{-1}(n) = 1/n$ is the harmonic representation of $n$. More precisely, $n'$ is the natural number from the point of view of the harmonic arithmetic. Alternatively, following Benioff [26][28], we could say that $f^{-1}$ is a value function which maps a natural number $n$ into its value. Benioff’s natural numbers are just abstract elements of a well ordered set and in themselves do not possess concrete values. The latter are produced by value functions. The natural number $n'$ satisfies the consistency condition

$$n' \oslash n' = (n + m)',$$

as one can directly verify by inserting $n' = 1/n$ and $m' = 1/m$ into (11). The same rules apply to $1' = f^{-1}(1)$ and $0' = f^{-1}(0) = \lim_{x \to 0} f^{-1}(x)$. As we can see, the harmonic multiplication actually is a repeated addition:

$$\underbrace{R \oplus \cdots \oplus R}_{n} = n' \oslash R = n'R. \quad (12)$$

Subtraction and division are defined analogously,

$$R_1 \oslash R_2 = f^{-1}(f(R_1) - f(R_2)) = \frac{1}{(1/R_1) - (1/R_2)}.$$ 

$$\ominus R_2 = 0' \oslash R_2 = \frac{1}{0 - (1/R_2)} = -R_2; \quad (15)$$
with the convention that \( R \oplus R = \infty = 0' \);

\[
R_1 \otimes R_2 = f^{-1}(f(R_1)/f(R_2))
\]
\[
= \frac{1}{(1/R_1)/(1/R_2)} = R_1/R_2.
\]

The new arithmetic involves an ordering relation: \( x \leq y \) if and only if \( f(x) \leq f(y) \). In particular, \( r' \leq s' \) if and only if \( r \leq s \). The 6-tuple \( \{\mathbb{R}, \oplus, \ominus, \mathcal{O}, \otimes, \leq\} \) defines an arithmetic which, in the terminology of Burgin [29, 30], is a non-Diophantine projective arithmetic with projection \( f \) and coprojection \( f^{-1} \).

A frequentist definition of probability parallels the standard one (the number \( n' \) of successes divided by the number \( N' \) of trials),

\[
p' = n' \otimes N' = n'/N' = f^{-1}(n)/f^{-1}(N) = N/n = f^{-1}(n/N).
\]

Probabilities sum to one because

\[
n' \otimes N' \otimes (N' \ominus n') \otimes N' = 1' = 1.
\]

Despite appearances, \( p' = N/n \) is not greater than one — not in the new arithmetic. Indeed, \( p'^1' > 1 = 1 \) if and only if \( n/N = f(p') > f(1') = 1 \), which is impossible.

Anyfor whom this paper is a first encounter with non-Diophantine arithmetic should pause here and contemplate the result. The notions of ‘greater’ or ‘smaller’ are local concepts. Just like ‘above’ and ‘below’ in the antipodic cities of Auckland and Seville. There are many analogies between non-Diophantine arithmetics and non-Euclidean geometries. Something which is larger in one arithmetic may appear smaller in another one (e.g. \( 0' = \infty \)). A number which is negative in one arithmetic can be positive in another one (the arithmetic in \( \mathbb{R}_+ \), defined by \( f(x) = \ln x \), implies \( \ominus x = 1/x \in \mathbb{R}_+ \)). However, in order not to confuse the reader, it should be stressed that in the hidden variable model the two arithmetics will involve the same ordering relation, \( \leq' \) will be equivalent to \( \leq \). The loophole will technically follow from Diophantine non-linearity (i.e. non-Diophantine linearity) of hidden-variable integrals.

To make matters worse, the two arithmetics are exactly symmetric with respect to each other: \( x' = f^{-1}(x) = 1/x \) implies \( x = f^{-1}(x') = 1/x' \),

\[
x \oplus y = f^{-1}(f(x) + f(y))
\]

implies

\[
x + y = f^{-1}(f(x) \ominus f(y)).
\]

Which of the natural numbers, \( n' = 1/n \) or \( n = 1/n' \), are those we learned as kids? Everything in one arithmetic is exactly upside-down in the other one. Maybe it is us who live in a Matrix world of wires and resistors? There is absolutely no criterion telling us which of the two arithmetics is Diophantine. This relativity of arithmetics will become essential for the reformulation of the problem of dark energy we will give in the next section.

Non-Diophantine arithmetics imply non-Newtonian calculus [31, 41], in this concrete example a harmonic one. A harmonic derivative of a function \( A : \mathbb{R} \rightarrow \mathbb{R} \) is defined in the usual way by means of the harmonic arithmetic,

\[
\frac{DA(x)}{Dx} = \lim_{\delta \rightarrow 0'} (A(x \oplus \delta) \otimes A(x)) \otimes \delta
\]

implies

\[
\frac{DA(x)}{Dx} = \lim_{\delta \rightarrow \infty} A(x \oplus \delta) \otimes A(x) \div \delta.
\]

The derivative is a linear map and satisfies the Leibniz rule (both properties defined with respect to \( \ominus \) and \( \mathcal{O} \)). One can directly check that \( A(x) = e^{-1/x} \) is the harmonic exponential function, i.e. satisfies

\[
\frac{DA(x)}{Dx} = A(x), \quad A(0') = 1',
\]

and \( A(x \ominus y) = A(x) \otimes A(y) \). Rewriting \( e^{-1/x} \) as

\[
A(x) = f^{-1}(e^{f(x)})
\]

we can understand why \( A(x) \) plays the role of \( e^x \). Continuing in a similar vain, we will arrive at a full calculus, linear algebra, or probability theory. Actually, all of physical theories will have their harmonic analogues.

Before we will formulate a non-Diophantine/non-Newtonian version of sub-quantum hidden variables, let us first have a look at another, in a sense dual problem of cosmological-scale arithmetic.

### III. DARK ENERGY AS A PROBLEM OF ARITHMETIC

Friedman equation for a dimensionless scale factor evolving in a dimensionless time \([45]\),

\[
\frac{da(t)}{dt} = \sqrt{\frac{\Omega_M a^2 + \Omega_L a}{a(t)}} \quad a(t) > 0,
\]

is exactly solvable,

\[
a(t) = \left( \frac{9 \Omega_M}{\Omega_L} \sinh \frac{3 \sqrt{\Omega_L}(t - t_1)}{2} \right)^{2/3}, \quad t > t_1.
\]

The dimensionless time is here expressed in units of the Hubble time \( t_H \approx 13.58 \times 10^9 \) yr. It correctly models the observed cosmological expansion if \( \Omega_M = 0.3, \Omega_L = 0.7 \) [46, 47]. The origin of this concrete value of \( \Omega_L \) is the so-called cosmological constant problem. \( \Omega_L \neq 0 \) is responsible for dark energy. The present time, \( t = t_0 \), satisfies \( a(t_0) = 1 \) and thus

\[
t_0 - t_1 = \frac{2}{3 \sqrt{\Omega_L}} \sinh^{-1} \sqrt{\frac{\Omega_M}{\Omega_L}} \approx 0.96.
\]
I will now show that (27) can be obtained with $\Omega_\Lambda = 0$, if we change the arithmetic of time. To begin with, the standard Diophantine/Newtonian Friedman equation without $\Omega_\Lambda$,

$$\frac{da(t)}{dt} = \sqrt{\frac{\Omega_M}{a(t)}}, \quad a(t) > 0, \quad (29)$$

has to be rewritten in a general non-Diophantine/non-Newtonian form which does not specify the arithmetics of $X \ni t$ and $Y \ni a(t)$, namely

$$\frac{Da(t)}{Dt} = \Omega_M^{(1/2)_Y} \otimes_Y a(t)^{(1/2)_Y}, \quad a(t) \succ Y 0_Y, \quad (30)$$

where $a^{(1/2)_Y} \otimes_Y a^{(1/2)_Y} = a$, i.e.

$$a^{(1/2)_Y} = f^{-1}_Y \left(\sqrt{f_Y(t)}\right). \quad (31)$$

All the arithmetic operations in $X$ and $Y$ are induced from the usual (Diophantine) arithmetic of $\mathbb{R}$ by means of one-to-one maps $f_X : X \rightarrow \mathbb{R}$, $f_Y : Y \rightarrow \mathbb{R}$, in exact analogy to the harmonic arithmetic discussed in the previous section. (30) is solved by (cf. Appendices 1–2 and 24)

$$a(t) = f^{-1}_Y \left(\left(3 f_Y \left(\Omega_M^{(1/2)_Y}\right) f_X(t)/2\right)^{2/3}\right). \quad (32)$$

Its comparison with (27), written as

$$a(t) = \left[\frac{3}{2} \sqrt{\Omega_M} - \frac{2}{3 \sqrt{\Omega_\Lambda}} \sinh \frac{3 \sqrt{\Omega_\Lambda}}{2} (t - t_1)\right]^{2/3}, \quad (33)$$

suggests a linear $f_y(y) = \lambda y$. Inserting $f_Y(\Omega^{(1/2)_Y}) = \sqrt{f_Y(\Omega)} = \sqrt{\lambda} \Omega$ into (32),

$$a(t) = \lambda^{-1} \left(3 \sqrt{\lambda \Omega M} f_X(t)/2\right)^{2/3} \quad (34)$$

we arrive at

$$f_X(t) = \frac{2}{3 \sqrt{\Omega_\Lambda}} \sinh \frac{3 \sqrt{\Omega_\Lambda}}{2} (t - t_1) \quad (36)$$

$$\approx 0.8 \sinh \frac{t - t_1}{0.8}, \quad (37)$$

$$f_X^{-1}(r) = t_1 + \frac{2}{3 \sqrt{\Omega_\Lambda}} \sinh^{-1} \frac{3 \sqrt{\Omega_\Lambda}}{2} r, \quad (38)$$

$$0_X = f_X^{-1}(0) = t_1, \quad (39)$$

$$\lambda = \sqrt{\Omega_M/0.3}. \quad (40)$$

Assuming $\Omega_M = 1$ we find $\lambda = 1.82574$. $\lambda \neq 1$ can be incorporated into a change of units as $a(t)$ is here dimensionless.

Cosmological-scale observers, who employ their own arithmetic related by (27) to the arithmetic we are taught at school, believe the Universe at their scales expands according to Einstein’s general relativity with zero cosmological constant. But they are aware of the dark energy problem: Small objects, such as galaxies or planetary systems, expand with unexplained deceleration...

IV. BELL’S THEOREM AS A PROBLEM OF ARITHMETIC

We are now ready to formulate a local hidden-variable theory of the Einstein-Podolsky-Rosen-Bohm two-electron singlet-state correlations. The resulting model is free of all the known loopholes of the Bell theorem, but is based on the arithmetic loophole which I will now describe in detail. The arithmetic perspective will lead to a product which is in between the classical multiplication from Bell-type proofs, and the tensor product from quantum mechanics. It will be quantum enough to fake quantum probabilities, and still classical enough to allow for eavesdropping in quantum cryptography. The model is meant as a proof-of-principle counterexample to Bell’s theorem, and not as a full hidden-variables alternative to quantum mechanics.

For our purposes it will be enough to assume that $X = \mathbb{R}$. Hidden-variable reals $X$ are equipped with their own non-Diophantine sub-quantum arithmetic, and non-Newtonian sub-quantum calculus determined by a single one-to-one unknown function $f : X \rightarrow \mathbb{R}$. The hidden-variable arithmetic is defined by

$$x \oplus y = f^{-1}(f(x) + f(y)), \quad (41)$$

$$x \odot y = f^{-1}(f(x) - f(y)), \quad (42)$$

$$x \otimes y = f^{-1}(f(x) \cdot f(y)), \quad (43)$$

$$x \otimes y = f^{-1}(f(x)/f(y)). \quad (44)$$

$\oplus$ and $\odot$ are associative and commutative, and $\odot$ is distributive with respect to $\oplus$. $X$ is ordered: $x \preceq y$ if and only if $f(x) \leq f(y)$. The neutral elements of addition and multiplication read, respectively, $0' = f^{-1}(0)$ and $1' = f^{-1}(1)$. For arbitrary real numbers $r \in \mathbb{R}$ we denote $r' = f^{-1}(r)$. We will assume $0' = 0$ and $1' = 1$.

In order to mimic the Bell construction we need the notion of an integral. Its form must be consistent with the arithmetic. We begin with the derivative, which is conceptually simpler, since once we know how to differentiate it becomes clear how to integrate.

The derivative of a function $A : X \rightarrow \mathbb{R}$ is defined by (22) which, due to $0' = 0$, can be written here as

$$\frac{DA(x)}{Dx} = \lim_{\delta \rightarrow 0} \left(A(x \oplus \delta) \ominus A(x)\right) \ominus \delta, \quad (45)$$

A non-Newtonian (Riemann or Lebesgue) integral is defined in a way guaranteeing the fundamental theorem...
of non-Newtonian calculus, linking derivatives and integrals. In particular, under certain technical assumptions paralleling those from the fundamental theorem of Newtonian calculus, if $A$ is a function mapping a given set into itself, $A: \mathbb{R} \to \mathbb{R}$, then (see Appendix 1 and 33–32)

$$\int_{x_1}^{x} \frac{DA(x)}{Dx} \, dx = A(x_2) \ominus A(x_1)$$

(46)

and

$$\frac{D}{Dx} \int_{x_1}^{x} A(y) \, Dy = A(x).$$

(47)

It is easy to show that

$$\int_{x_1}^{x} A(y) \, Dy = f^{-1} \left( \int_{f(x_1)}^{f(x)} f \circ A \circ f^{-1}(r) \, dr \right)$$

(48)

where the integral over $r$ is Newtonian.

Properties (49)–(52) stand in contrast with other calculi one encounters in physical applications, say the fractional ones 45, that typically have great difficulties with fulfilling the fundamental theorem. The power and efficiency of the non-Newtonian approach lies in its low-level starting point — the arithmetic.

We just need to construct $f$. In order to do so, consider two sets of probabilities,

$$p'_{\pm \pm}(\theta) = \frac{1}{2} \cos^2 \frac{\theta}{2},$$

(49)

$$p'_{\pm \pm}(\theta) = \frac{1}{2} \sin^2 \frac{\theta}{2},$$

(50)

$$p_{\pm \pm}(\theta) = \frac{\pi - \theta}{2\pi},$$

(51)

$$p_{\pm \pm}(\theta) = \frac{\theta}{2\pi},$$

(52)

for $0 \leq \theta \leq \pi$. Obviously

$$1 = p'_{+-} + p'_{++} + p'_{-+} + p'_{++}$$

(53)

$$= p_{+-} + p_{++} + p_{-+} + p_{++}.$$  (54)

A classical model leading to joint probabilities $p_{\pm \pm}$, $p_{\pm \mp}$ is illustrated in Fig. 4. Probabilities are determined by ratios of arc lengths on a circle. The hidden variable is here given by a point on the circle or, equivalently, by its polar angle $\lambda$. Once one knows $\lambda$ the results of future measurements are known in advance. The model does not violate Bell-type inequalities.

Our hidden-variable model will be essentially the same. We will only change arithmetic and calculus. The arc length has to be computed by means of a non-Newtonian integral.

Now consider the one-to-one function $f^{-1} : [0, 1/2] \to [0, 1/2]$, defined for $0 \leq \theta \leq \pi$ by

$$p'_{\pm \pm} = \frac{1}{2} \sin^2 \frac{\theta}{2} = f^{-1} \left( \frac{\theta}{2\pi} \right) = f^{-1}(p_{\pm \pm}).$$

(55)

Equivalently,

$$p'_{\pm \mp} = \frac{1}{2} \cos^2 \frac{\theta}{2} = f^{-1} \left( \frac{\pi - \theta}{2\pi} \right) = f^{-1}(p_{\pm \mp}).$$

(56)

Formulas (55)–(56) might seem trivial, expressing the obvious fact that $\sin x$ is a function of $x$. What is non-trivial, however, is that this trivial function may be non-trivially employed to construct a new arithmetic and calculus. This is the key observation of the paper. The arithmetic will allow us to build a rotationally invariant hidden-variables model, although the notion of rotational symmetry will have to be formulated within the language of the new arithmetic.

Since (55)–(56) are equivalent on $[0, \pi]$, (55) can define the restriction to $[0, 1/2]$ of a one-to-one $f^{-1} : \mathbb{R} \to \mathbb{R}$.

$$f^{-1}(0) = 0, \quad f^{-1}(1/2) = \frac{1}{2} \sin^2 \frac{\pi}{2} = 1/2.$$  (57)

For example (Fig. 2),

$$f^{-1}(x) = \frac{n}{2} + \frac{1}{2} \sin^2 \pi \left( x - \frac{n}{2} \right),$$

(57)

$$f(x) = \frac{n}{2} + \frac{1}{\pi} \arcsin \sqrt{2x - n},$$

(58)

for $n/2 \leq x \leq n + 1/2$, $n \in \mathbb{Z}$.  (59)

The function so defined satisfies

$$f^{-1}(n/4) = n/4 = f(n/4), \quad \text{for } n \in \mathbb{Z},$$

(60)

and thus, in particular, $0' = 0$, $(\pm 1)' = \pm 1$, $(1/2)' = 1/2$. All integers are unchanged, so number theory will be unaffected. Sums, differences and products of integers are the usual ones, as opposed to their ratios.

As opposed to the harmonic arithmetic from Section II, the non-Diophantine ordering relation $\leq'$ is here identical to the Diophantine $\leq$ because $f$ is strictly increasing. In consequence, $x \leq y$ if and only if $f(x) \leq f(y)$, which holds if and only if $x \leq y$. Modulus is thus defined in the usual way,

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ \ominus x & \text{if } x \leq 0 \end{cases},$$

(61)

where $\ominus x = -x$, a consequence of $f^{-1}(x) = -f^{-1}(x)$.

The trigonometric identities,

$$p'_{+-} + p'_{++} = p'_{-+} + p'_{++}$$

$$= p_{+-} + p_{++} + p_{-+} + p_{++}.$$  (59)

express the fact that $+$ and $-$ are equally probable. The same is found in the hidden-variables world, although the reasons for that are more subtle, for example,

$$p'_{+-} \oplus p'_{++} = f^{-1}(f(p'_{+-}) + f(p'_{++})) = f^{-1} \left( \frac{\pi - \theta}{2\pi} + \frac{\theta}{2\pi} \right) = f^{-1}\left( \frac{1}{2} \right) = \frac{1}{2},$$

(60)

can be rewritten as
1. \[ \frac{1}{2} \cos^2 \frac{\alpha - \beta}{2} = f^{-1} \left( \frac{f(\pi') - |f(\alpha') - f(\beta')|}{f(2\pi')f(\pi') \prime} \right) \]  
2. \[ \frac{1}{2} \sin^2 \frac{\alpha - \beta}{2} = f^{-1} \left( \frac{|f(\alpha') - f(\beta')|}{f(2\pi')f(\pi') \prime} \right) \]

where \[ 0 \leq |f(\alpha' \ominus \beta')| = |f(\alpha') - f(\beta')| = |\alpha - \beta| \leq \pi, \] and

\[ \pi' = f^{-1}(\pi) = 3 + \frac{1}{2} \sin^2(\pi^2) = 3.09258, \]

\[ (2\pi)' = f^{-1}(2\pi) = 6 + \frac{1}{2} \sin^2(2\pi^2) = 6.30175. \]

Probabilities (65) and (68) are ratios of arc lengths, computed by means of non-Newtonian integrals. Indeed, the non-Newtonian integral

\[ \int_{x_1}^{x_2} D\lambda(x) = \int_{x_1}^{x_2} \frac{Dx}{dx}Dx = x_2 \ominus x_1 \]

can be used to cross-check our construction. The length of the unit circle is

\[ \int_0^{(2\pi)'} D\lambda = (2\pi)' = f^{-1}(2\pi). \]

The length of the arc \[ \alpha' \leq \lambda \leq \beta' \] reads

\[ \int_{\alpha'}^{\beta'} D\lambda = \beta' \ominus \alpha' = f^{-1}(\beta - \alpha). \]

Employing the explicit form of our hidden-variables arithmetic we obtain, for \[ 0 \leq \beta - \alpha \leq \pi, \]

\[ \int_{\alpha'}^{\beta'} D\lambda = \frac{1}{2} \sin^2[\pi(\beta - \alpha)]. \]

The probability of randomly selecting a point belonging to the arc is the ratio of the two lengths,

\[ \left( \int_{\alpha'}^{\beta'} D\lambda \right) \ominus (2\pi)' = \left( \frac{1}{2} \sin^2[\pi(\beta - \alpha)] \right) \ominus (2\pi)' \]

\[ = f^{-1} \left( f \left( \frac{1}{2} \sin^2[\pi(\beta - \alpha)] \right) / f((2\pi)') \right) \]

\[ = f^{-1} \left( \frac{\beta - \alpha}{2\pi} \right) = \frac{1}{2} \sin^2 \frac{\beta - \alpha}{2}. \]
Notice that the ratio of lengths defines a normalized probability density

$$\rho(\lambda) = 1' \otimes (2\pi)' = (1/(2\pi))', \tag{77}$$

with rotationally invariant normalization

$$\int_0^{(2\pi)'} \rho(\lambda)D\lambda = \int_0^{\phi \oplus (2\pi)'} \rho(\lambda)D\lambda = 1, \tag{78}$$

for any $\phi$. Quantum probabilities can be thus written in terms of non-Newtonian integrals of the local-realistic form assumed in the proof of the Clauser-Horne (CH) inequality [43] (see the next Section and Appendix 3),

$$p_{++}' = \frac{1}{2} \sin^2 \frac{\beta - \alpha}{2} = \int_{\alpha'}^{\beta'} \rho(\lambda)D\lambda \tag{79}$$

$$= \int_0^{(2\pi)'} \chi_{\alpha'}(\lambda) \otimes \chi_{\beta'}(\lambda) \otimes \rho(\lambda)D\lambda, \tag{80}$$

$$p_{+-}' = \frac{1}{2} \cos \frac{\beta - \alpha}{2} = \int_{\beta'}^{\alpha'} \rho(\lambda)D\lambda \tag{81}$$

$$= \int_0^{(2\pi)'} \chi_{\beta'}(\lambda) \otimes \chi_{\alpha'}(\lambda) \otimes \rho(\lambda)D\lambda, \tag{82}$$

$$p_{--}' = \frac{1}{2} \sin^2 \frac{\beta - \alpha}{2} = \int_{\alpha'}^{\beta'} \rho(\lambda)D\lambda \tag{83}$$

$$= \int_0^{(2\pi)'} \chi_{\alpha'}(\lambda) \otimes \chi_{\beta'}(\lambda) \otimes \rho(\lambda)D\lambda, \tag{84}$$

$$p_{-+}' = \frac{1}{2} \cos \frac{\beta - \alpha}{2} = \int_{\beta'}^{\alpha'} \rho(\lambda)D\lambda \tag{85}$$

$$= \int_0^{(2\pi)'} \chi_{\beta'}(\lambda) \otimes \chi_{\alpha'}(\lambda) \otimes \rho(\lambda)D\lambda. \tag{86}$$

Here the $\chi$'s are the characteristic functions discussed below.

As required, two normalizations hold simultaneously:

$$p_{++}' + \cdots + p_{--}' = 1 = p'_{++} \oplus \cdots \oplus p'_{--}. \tag{87}$$

The right-hand form follows from the general non-Newtonian formula (see Appendix 1), for integrals of functions $F: \mathbb{X} \to \mathbb{Y}$,

$$\int_a^b F(x)Dx \oplus_\mathbb{Y} \int_c^d F(x)Dx = \int_c^d F(x)Dx, \tag{88}$$

where $\oplus_\mathbb{Y}$ is the addition in $\mathbb{Y}$. The left-hand form guarantees that macroscopic-scale observers can test the probabilities by comparing them with experimentally measured frequencies, which necessarily sum to 1 in the arithmetic used by the observers.

Formulas (80)-(86) pinpoint similarities and differences between our hidden-variable model and those discussed in the literature so far. The difference reduces to $\otimes$ instead of “•”. The properties of the integral are also important but it is hard to say if this is really different from what Bell had in mind. Anyway, what he assumed was that some sort of integration applies to some unspecified hidden variables.

V. PROJECTION POSTULATE

A measurement of a yes-no random variable projects onto a subset of states corresponding to the result ‘yes’. In classical probability the projector is represented by a characteristic function $\chi_+(x)$, equal 1 if $x$ represents ‘yes’, and 0 otherwise. The orthogonal projector is $\chi_-(x) = 1 - \chi_+(x)$. In quantum probability the projection is on a vector subspace spanned by appropriate eigenvectors. Our model is classical, so the projector is represented by a characteristic function $\chi_\pm(\lambda) = 1 \otimes \chi_\pm(\lambda)$,

$$\chi_\pm(\lambda) \otimes \chi_\pm(\lambda) = \chi_\pm(\lambda), \tag{89}$$

$$\chi_\pm(\lambda) \otimes \chi_\mp(\lambda) = 0. \tag{90}$$

More explicitly, we can represent characteristic functions by the diagram

$$\mathbb{X} \xleftarrow{\times} \mathbb{Y} \xrightarrow{f} \mathbb{X} \xleftarrow{\times} \mathbb{Y} \xrightarrow{f}, \tag{91}$$

where,

$$\tilde{x}_+(r) = \begin{cases} 1 & \text{if } r' = f^{-1}(r) \text{ corresponds to 'yes'}, \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}_-(r) = 1 - \tilde{x}_+(r). \tag{92}$$

Measurements reduce probability by projection and renormalization,

$$\rho(\lambda) \mapsto \rho_\pm(\lambda) \tag{93}$$

$$= \chi_\pm(\lambda) \otimes \rho(\lambda) \otimes \int \chi_\pm(x) \otimes \rho(x)Dx. \tag{94}$$

Joint probabilities (79)-(86) provide examples of the construction (see Appendix 3).

VI. BELL-TYPE INEQUALITIES

Bell-type inequalities are simultaneously violated and not violated, depending on the viewpoint. Let us see how it works.

A. CH inequality

The Clauser-Horne inequality [43]

$$0 \leq 3p'_{+-}(\theta) - p'_{--}(3\theta) \leq 1, \tag{95}$$

if true, in our case would be equivalent to

$$0 \leq 3f^{-1} \left(\frac{\pi - \theta}{2\pi}\right) - f^{-1} \left(\frac{\pi - 3\theta}{2\pi}\right) \leq 1. \tag{96}$$
Of course, (95) is violated because it cannot be proved in our hidden-variable model. What can be proved, however, is

\[ 0 \leq 3 \odot p'_{\gamma-} \odot p'_{\gamma-}(3\theta) \leq 1, \]

a fact following from (80), (82), (84), (86), if one follows the steps of Clauser-Horne derivation [13]. One can cross-check:

\[ 3 \odot p'_{\gamma-}(\theta) \odot p'_{\gamma-}(3\theta) \\
= f^{-1} [f(3) f(p'_{\gamma-}(\theta)) - f(p'_{\gamma-}(3\theta))] \\
= f^{-1} \left[ \frac{3\pi - \theta}{2\pi} - \frac{\pi - 3\theta}{2\pi} \right] = f^{-1}(1) = 1. \]

The model is local, deterministic, detectors are ideal, observers have free will. All the standard loopholes are absent, so Bell-type inequalities are not violated... in the non-Diophantine world of the hidden variables. The only modification is that we employ \( \odot \) instead of \( \cdot \), and the integral is non-Newtonian.

**FIG. 3: Comparison of the three averages: \( f^{-1}(-1 + 2/\pi) \) (full), \(-\cos \theta \) (dotted), and \(-1 + 2/\pi \) (dashed). Although the averages differ, the probabilities corresponding to \( f^{-1}(-1 + 2/\pi) \) and \(-\cos \theta \) are identical. Experiments test probabilities.**

Notice that for \( f^{-1}(x) = x \) in (96) we would obtain the identity

\[ 3\frac{\pi - \theta}{2\pi} - \frac{\pi - 3\theta}{2\pi} = 3p_{\gamma-}(\theta) - p_{\gamma-}(3\theta) = 1, \]

valid for any \( \theta \) and consistent with (96). As expected, probabilities (51)-(52) satisfy (95).

Inequality (96) is not valid for a large class of \( f \), but in our concrete case, setting \( \theta = \pi/4 \), we find the maximal violation:

\[ 3f^{-1}\left(\frac{\pi - \pi/4}{2\pi}\right) - f^{-1}\left(\frac{\pi - 3\pi/4}{2\pi}\right) = 3f^{-1}\left(\frac{3}{8}\right) - f^{-1}\left(\frac{1}{8}\right) = 1.20711. \]

Of course, (95) is violated because it cannot be proved in our hidden-variable model. What can be proved, however, is

\[ 0 \leq 3 \odot p'_{\gamma-}(\theta) \odot p'_{\gamma-}(3\theta) \leq 1, \]

a fact following from (80), (82), (84), (86), if one follows the steps of Clauser-Horne derivation [13]. One can cross-check:

\[ 3 \odot p'_{\gamma-}(\theta) \odot p'_{\gamma-}(3\theta) \\
= f^{-1} [f(3) f(p'_{\gamma-}(\theta)) - f(p'_{\gamma-}(3\theta))] \\
= f^{-1} \left( \frac{3\pi - \theta}{2\pi} - \frac{\pi - 3\theta}{2\pi} \right) = f^{-1}(1) = 1. \]

The model is local, deterministic, detectors are ideal, observers have free will. All the standard loopholes are absent, so Bell-type inequalities are not violated... in the non-Diophantine world of the hidden variables. The only modification is that we employ \( \odot \) instead of \( \cdot \), and the integral is non-Newtonian.

**B. CHSH inequality**

The EPR-Bohm-Bell hidden-variables average \( \langle AB \rangle' \) is computed in the hidden-variables world as follows

\[ \langle AB \rangle' = p'_{++} \oplus p'_{+-} \ominus p'_{+-} \odot p'_{-+} \]

valid for any \( \theta \) (full), \(-\cos \theta \) (dotted), and \(-1 + 2/\pi \) (dashed). Although the averages differ, the probabilities corresponding to \( f^{-1}(-1 + 2/\pi) \) and \(-\cos \theta \) are identical. Experiments test probabilities.

The average satisfies the hidden-variables CHSH inequality [24]

\[ |\langle A_1 B_1 \rangle' \oplus \langle A_1 B_2 \rangle' \oplus \langle A_2 B_1 \rangle' \oplus \langle A_2 B_2 \rangle'| \leq 2. \]

The observer-arithmetic average

\[ \langle AB \rangle = p'_{++} + p'_{+-} - p'_{+-} - p'_{-+} \]

nevertheless does violate the observer-arithmetic CHSH inequality.

Fig. 3 shows that (102) is neither the classical average \( 2|\alpha - \beta|/\pi - 1 \) corresponding to the upper part of Fig. 1, nor the quantum one. However, quantum experiments do not measure averages — they measure probabilities which coincide here by construction.

**VII. IMPLICATIONS FOR CRYPTOGRAPHY**

Security of the Ekert protocol [5] is certified by violation of the Bell inequality. Now, which inequality: Diophantine, or non-Diophantine? Why should a model based on Clauser-Horne type expressions \( \leq 1 \) be secure? Knowing \( \lambda \) we know in advance the results of future measurements performed by Alice and Bob. The model of probability we employ is internally consistent although not fully ‘classical’. Probabilities are constructed by means of a non-standard mathematical construction, but the end result is just a real number, an ordinary probability \( 0 \leq p' \leq 1 \) which can be tested in ordinary experiments. Probabilities sum to 1 in two ways, Diophantine and non-Diophantine, but only non-Diophantine Bell-type inequalities have to be satisfied.

There are technical reasons why standard Bell-type inequalities cannot be proved. For example, (88) holds in any non-Newtonian calculus but in general (see Appendix 1)

\[ \int_a^b F(x) \text{D}x + \int_b^c F(x) \text{D}x \neq \int_a^c F(x) \text{D}x, \]

and

\[ \int_a^b [F(x) + G(x)] \text{D}x \neq \int_a^b F(x) \text{D}x + \int_a^b G(x) \text{D}x, \]

so all the proofs à la Bell one finds in the literature will not work. Non-Newtonian integrals are linear maps but with respect to appropriate non-Diophantine arithmetic.
With respect to the Diophantine arithmetic they are non-linear. This type of duality is well known in physics (non-linear waves interfere, n-soliton solutions are formed by Darboux-Bäcklund transformations from 1-soliton solutions. Lax-pair represents a nonlinear system by a linear one, etc.). If one consistently works according to the non-Diophantine/non-Newtonian rules, Bell-type inequalities can be proved, but their correct form is exemplified by \ref{99} and \ref{103}.

It looks like the model we describe is classical enough to allow for eavesdropping, but quantum enough to escape detection by quantum protocols.

**VIII. HIDDEN ROTATIONAL SYMMETRIES**

Two-electron singlet-state correlations are rotationally symmetric. In order to understand why our hidden-variables model is rotationally symmetric as well, we first have to define the action of the rotation group in $\mathbb{X} \times \mathbb{X}$, the Cartesian product of $\mathbb{X}$ with itself. I will illustrate the construction with two suggestive fractal examples.

Trigonometric functions mapping $\mathbb{X}$ into $\mathbb{X}$,

\[
\sin x = f^{-1}(\sin f(x)), \quad (108)
\]

\[
\cos x = f^{-1}(\cos f(x)), \quad (109)
\]

are periodic with the period \((2\pi)^t = f^{-1}(2\pi)\) (e.g. \(\sin(x + (2\pi)^t) = \sin x\)).

They satisfy all the standard trigonometric formulas (with respect to the arithmetic in $\mathbb{X}$), in particular:

\[
\sin(x \oplus y) = \sin x \odot \cos y \odot \cos x \odot \sin y, \quad (110)
\]

\[
\cos(x \oplus y) = \cos x \odot \cos y \odot \sin x \odot \sin y, \quad (111)
\]

\[
1 = \sin^2 x \odot \cos^2 x. \quad (112)
\]

Here $\sin^2 x = \sin x \odot \sin x$, etc. Rotations in the plane $\mathbb{X} \times \mathbb{X}$ are defined in the usual way,

\[
x_1(\alpha) = x_1 \odot \cos \alpha \odot x_2 \odot \sin \alpha, \quad (113)
\]

\[
x_2(\alpha) = x_2 \odot \sin \alpha \odot x_2 \odot \cos \alpha. \quad (114)
\]

Formulas \ref{110}–\ref{111} show that rotations form a Lie group (with group parameters subject to the non-Diophantine arithmetic). Fig. 4 depicts two examples of unit circles generated by \ref{113}–\ref{114}: The one constructed in the Cartesian product of two Cantor sets, and the one in the Cartesian product of two Koch curves. Both circles are homogeneous spaces generated by rotations. The construction works because Cantor sets and Koch curves have the same cardinality as the continuum $\mathbb{R}$. This is why appropriate one-to-one maps $f : \mathbb{X} \rightarrow \mathbb{R}$ exist, and non-Diophantine arithmetics can be constructed \ref{33}–\ref{41}. The rotational symmetries from Fig. 4 are ‘hidden’ in the sense that in order to see them one must plot the curves in coordinate systems based on appropriate arithmetics.

The property is shared by our hidden variables.

**IX. HIDDEN ROTATIONAL SYMMETRY OF THE HIDDEN-VARIABLES MODEL**

Let us return to the hidden-variables arithmetic defined by \ref{57}–\ref{58}. A straight line through the origin is defined by (Fig. 5).

\[
t \mapsto (t \odot \cos \theta, t \odot \sin \theta) \quad (115)
\]

A unit circle is the curve

\[
\phi \mapsto (\cos \phi, \sin \phi), \quad 0 \leq \phi \leq (2\pi)^t, \quad (116)
\]

(i.e. $0 \leq f(\phi) \leq 2\pi$). In order to visualize the rotations let us draw the unit circle together with the straight lines $t \mapsto (X(t), Y(t)) = (t \odot \cos \alpha \odot \beta, t \odot \sin \alpha \odot \beta)$, for $0 \leq f(\alpha) \leq 7\pi/8$, and $f(\beta) = 0$, $\pi/10$, and $\pi/3$ (Fig. 6). Non-Diophantine angular distances $\pi \odot 8$ between the neighboring lines are identical at all the plots. The two ‘deformed’ plots in Fig. 6 are just the rotated versions of the top one. The octagon-shaped curve is the unit circle \ref{116}.

The circle is rotationally invariant in spite of its apparent octagon form. Needless to say, all these deformations are invisible for hidden-variable-level observers who consistently employ their own arithmetic.
X. SUMMARY

Let us summarize our construction. We have constructed singlet-state probabilities (79)–(86) as follows,

\[ p'_{jk}(\beta - \alpha) = \int_0^{(2\pi)'/2} \chi^1_{\alpha j}(\lambda) \odot \chi^2_{\beta k}(\lambda) \odot \rho(\lambda) d\lambda, \quad (117) \]

where the parameters are related by \(\alpha' = f^{-1}(\alpha)\), \(\beta' = f^{-1}(\beta)\), \(\alpha, \beta \in [-\pi, \pi]\), \(\alpha', \beta' \in [-\pi', \pi']\). The probabilities have the Clauser-Horne local-realistic form. Products and integrals at both sides of (79)–(86) are defined by means of arithmetic operations from two different arithmetics, both acting in \(\mathbb{R}\). The observer-level Diophantine arithmetic \(\{\mathbb{R}, +, -, \cdot, \leq\}\), and the hidden-variable-level non-Diophantine arithmetic \(\{\mathbb{R}, \oplus, \ominus, \odot, \leq\}\). Formulas such as (79)–(86) make sense because both arithmetics act in the same set \(\mathbb{R}\). For this reason, there are always two ways of manipulating the numbers that appear in various equations. In particular, since unit elements in both arithmetics are the same, 1’ = 1, the probabilities are normalized in two coinciding ways. The observer-level normalization,

\[ p'_{++} + p'_{+-} + p'_{-+} + p'_{--} = 1, \quad (118) \]

and the hidden-variables normalization,

\[ p'_++ + p'_+ - + p'_- + + p'-- = 1' = 1. \quad (119) \]

Probabilities (79)–(86) have a geometric representation: they represent ratios of arc lengths on the unit circle \(\sin' x \oplus \cos' x = 1\). The set of hidden variables is just the unit circle (which can be identified, if one wishes, in the usual way with its covering space \(\mathbb{R}\) equipped with non-Diophantine arithmetic). Both the circle itself, and the probabilities are rotationally invariant. The latter explicitly follows from

\[ \alpha' \odot \beta' = (\alpha' \odot \phi) \odot (\beta' \odot \phi) \quad (120) \]

for any \(\phi \in \mathbb{R}\).

The model we have constructed is local, detectors are ideal, observers have free will, and yet the probabilities are exactly those implied by quantum mechanics.

Appendix 1: Non-Newtonian differentiation and integration

Consider two sets \(X, Y\), with arithmetics \(\{\oplus_X, \ominus_X, \odot_X, \leq_X\}\) and \(\{\oplus_Y, \ominus_Y, \odot_Y, \leq_Y\}\),
For similar reasons, a function \( a : X \rightarrow \mathcal{Y} \) defines a new function \( \tilde{a} : \mathbb{R} \rightarrow \mathbb{R} \) such that the diagram
\[
\begin{array}{ccc}
X & \overset{a}{\longrightarrow} & \mathcal{Y} \\
\downarrow f_X & & \downarrow f_Y \\
\mathbb{R} & \overset{\tilde{a}}{\longrightarrow} & \mathbb{R}
\end{array}
\]
(121)
is commutative. The derivative of \( a \) is defined as
\[
\frac{D a(x)}{D x} = \lim_{h \to 0} \left( a(x \oplus h) \ominus a(x) \right) \ominus_a hv,
\] (122)
where the limit is appropriately constructed \([40][41]\). One proves that (122) implies
\[
\frac{D a(x)}{D x} = f_Y^{-1} \left( \frac{d \tilde{a}(f_X(x))}{d f_X(x)} \right).
\] (123)
Here \( d\tilde{a}(r)/dr \) is the usual Newtonian derivative of \( \tilde{a} : \mathbb{R} \rightarrow \mathbb{R} \). The form (123) is extremely useful in practical calculations. The non-Newtonian derivative is linear with respect to \( \oplus \) and satisfies the Leibniz rule
\[
\frac{D(a_1 \ominus a_2(x))}{D x} = \left( a_1 \ominus \frac{D a_2(x)}{D x} \right) \ominus \left( \frac{D a_1(x)}{D x} \ominus a_2(x) \right).
\] (124)
Once we have the derivatives we define a non-Newtonian (Riemann, Lebesgue,...) integral of \( \tilde{a} \) by
\[
\int_{x_1}^{x_2} a(x) D x = f_Y^{-1} \left( \int_{f_X(x_1)}^{f_X(x_2)} \tilde{a}(r) dr \right),
\] (125)
i.e. in the terms of the Newtonian (Riemann, Lebesgue,...) integral of \( \tilde{a} \). The two functions \( a \) and \( \tilde{a} \) are related by \([121]\). Under standard assumptions about differentiability and continuity of \( \tilde{a} \) we obtain both fundamental theorems of non-Newtonian calculus. For example,
\[
\frac{D}{D x} \int_{x_1}^{x_2} a(y) D y = f_Y^{-1} \left( \frac{D}{D f_X(x)} \int_{f_X(x_1)}^{f_X(x_2)} \tilde{a}(r) dr \right)
= f_Y^{-1} \left( \tilde{a}(f_X(x)) \right) = a(x).
\] (126)
Formulas \([88],[106]\) follow trivially from
\[
\begin{align*}
\int_{x_1}^{x_2} F(x) D x \oplus \int_{x_2}^{x_3} F(x) D x &= f_Y^{-1} \left( \int_{f_X(x_1)}^{f_X(x_2)} \tilde{a}(r) dr + \int_{f_X(x_2)}^{f_X(x_3)} \tilde{a}(r) dr \right) \\
&\neq \int_{x_1}^{x_2} F(x) D x + \int_{x_2}^{x_3} F(x) D x.
\end{align*}
\] (127)
For similar reasons,
\[
c \ominus \int_{x_1}^{x_2} F(x) D x = \int_{x_1}^{x_2} c \ominus \int_{x_2}^{x_3} F(x) D x
\] (128)
for a constant \( c \in \mathbb{Y} \), and
\[
\int_{x_1}^{x_2} \left[ F(x) \oplus G(x) \right] D x = \int_{x_1}^{x_2} F(x) D x \oplus \int_{x_1}^{x_2} G(x) D x,
\] (129)
but
\[
c \int_{x_1}^{x_2} F(x) D x \neq \int_{x_1}^{x_2} c \cdot F(x) D x,
\] (130)
and
\[
\int_{x_1}^{x_2} \left[ F(x) + G(x) \right] D x \neq \int_{x_1}^{x_2} F(x) D x + \int_{x_1}^{x_2} G(x) D x.
\] (131)
It is now clear that standard Bell-type proofs based on apparently general formulas involving integrals over the space of hidden-variables just cannot work for probabilities \([79][85]\). Further details of non-Newtonian calculus can be found in \([41][42]\).

**Appendix 2: Solution of non-Newtonian Friedman equation**

Employing the diagram (121) we rewrite (30) as
\[
f_Y^{-1} \left( \frac{d \tilde{a}(f_X(t))}{d f_X(t)} \right) = f_Y^{-1} \left( \frac{f_Y(\Omega_M^{(1/2)})}{\tilde{a}(f_X(t))^{1/2}} \right),
\] (132)
so that
\[
\tilde{a}(f_X(t)) = \left( 3 f_Y(\Omega_M^{(1/2)}) f_X(t) / 2 \right)^{2/3},
\] (133)
\[
a(t) = \left( 3 f_Y(\Omega_M^{(1/2)}) f_X(t) / 2 \right)^{2/3}.
\] (134)

**Appendix 3: Clauser-Horne local-realistic form of quantum probabilities**

Let \( f \) be given by \([58]\). Consider the diagram,
\[
\begin{array}{ccc}
X & \overset{\rho}{\longrightarrow} & X \\
\downarrow f & & \downarrow f \\
\mathbb{R} & \overset{\tilde{\rho}}{\longrightarrow} & \mathbb{R}
\end{array}
\]
(135)
where
\[
1 = \int_{0}^{(2\pi)'} \rho(x) D x = f^{-1} \left( \int_{0}^{(2\pi)'} \tilde{\rho}(r) dr \right) = f^{-1}(1)
\] implies \( \int_{0}^{(2\pi)} \tilde{\rho}(r) dr = 1 \). Assuming \( \tilde{\rho}(r) = 1 / (2\pi) \), we get
\[
\rho(x) = f^{-1} \circ \tilde{\rho} \circ f(x) = f^{-1}(1 / (2\pi))
= (1 / (2\pi))' = 1 / 2 \sin^2 \frac{1}{2} = 0.114924.
\] (136)
Obviously,

\[
\int_{\alpha'}^{\beta'} \rho(x) \, dx = f^{-1}\left( \int_{\alpha}^{\beta} \hat{\rho}(r) \, dr \right) = f^{-1}\left( \frac{\beta - \alpha}{2\pi} \right) = \frac{1}{2} \sin^2 \frac{\beta - \alpha}{2}.
\]

Now consider the following non-Newtonian characteristic functions, \( \chi_{\alpha \pm}^j, j = 1, 2, \)

\[
\chi_{\alpha \pm}^j \xrightarrow{f} \chi_{\alpha \pm}^j
\]

where, for any \( k \in \mathbb{Z}, \)

\[
\chi_{\alpha +}^1(r) = \begin{cases} 
1 & \text{for } r \in [\alpha - \frac{\pi}{2} + 2k\pi, \alpha + \frac{\pi}{2} + 2k\pi) \\
0 & \text{otherwise}
\end{cases},
\]

\[
\chi_{\alpha -}^1(r) = 1 - \chi_{\alpha +}^1(r),
\]

\[
\chi_{\alpha \mp}^2(r) = \chi_{\alpha \pm}^1(r).
\]

Our \( f \) satisfies \( f^{-1}(0) = 0 \) and \( f^{-1}(1) = 1, \) hence for any binary-valued function \( \chi \) we find

\[
\chi(x) = f^{-1}\left( \tilde{\chi}(f(x)) \right) = \tilde{\chi}(f(x)),
\]

\[
1 \circ \chi(x) = f^{-1}\left( f(1) - f(\chi(x)) \right) = f^{-1}\left( 1 - \tilde{\chi}(f(x)) \right) = 1 - \tilde{\chi}(f(x)) = (1 \circ \tilde{\chi})(f(x)).
\]

The diagram for characteristic functions can be simplified

\[
X \xrightarrow{\chi^j_{\alpha \pm}} X
\]

as \( f^{-1} \) is here redundant.

Finally, if \( 0 \leq \alpha \leq \beta \leq \pi, \) then

\[
p'_{++} = \int_{\alpha'}^{\beta'} \rho(\lambda) \, d\lambda = \int_0^{2\pi} \chi_{\alpha +}^1(\lambda) \circ \chi_{\beta +}^2(\lambda) \circ \rho(\lambda) \, d\lambda
\]

\[
= f^{-1}\left( \int_0^{2\pi} \chi_{\alpha +}^1(r) \chi_{\beta +}^2(r) \rho(r) \, dr \right)
\]

\[
= f^{-1}\left( \frac{1}{2\pi} \int_0^{2\pi} \chi_{\alpha +}^1(r) \chi_{\beta -}^1(r) \rho(r) \, dr \right)
\]

\[
= f^{-1}\left( \frac{1}{2\pi} \int_0^{\beta - \alpha} \rho(r) \, dr \right) = f^{-1}\left( \frac{\beta - \alpha}{2\pi} \right) = \frac{1}{2} \sin^2 \frac{\beta - \alpha}{2},
\]

as well as the remaining three probabilities in (79)–(86) are of the form assumed by Clauser and Horne in their analysis of Bell’s theorem [33].

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