Spectral Representation and Analysis of the KBR Precision Based on the Earth Gravity Spectrum

LUO Jia   NING Jinsheng   LUO Zhicai

ABSTRACT Satellite-to-Satellite Tracking in low-low model (SST-II) is a new technique to resolve the series of problems met in the determination of the earth's gravity field. As the key technique of SST-II, KBR can get SST-II measurements directly. So the KBR performance analysis is the first step in SST-II design. In this paper, assuming that the satellite pairs of SST-II are in near circle polar orbits, the spectrum relationship between the earth gravity field and KBR is established using analytic method. And then some examples are analyzed, the suggestions and conclusions are drawn from these examples. The research results could be taken as a reference for future satellite gravity project of China.

KEY WORDS SST-II; KBR; earth gravity field; spectrum

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Introduction

The GRACE (gravity recovery and climate experiment) mission, twin satellites flying in formation, which carries several key payloads including KBR (K-band ranging) system, was launched in 2002. The unique design of GRACE mission is expected to lead to an improvement in several orders of magnitude in these gravity measurements and allow much improved resolution of the Earth gravity field of finer scale over both land and ocean to study a lot of geosciences phenomena[1]. Though the GRACE higher level data have been released since the middle of 2004, the low-level data are inaccessible and the development of the China SST-II mission needs the analysis of the impact of the Earth gravity field and the ranging between satellites in detail. As a rule, one science satellite mission is supported by a self-contained simulation system to assess and analyze if a appointed configuration meet the mission requirements. For especial reasons, the material of the SST-II simulation is scarce in public resource and opening organizations. This paper is focus on the precise K-band inter-satellite ranging system, which is abbreviated with the KBR to simplify the expression, and analyzing the relationship between the KBR precision and the Earth gravity signal power in the satellite altitude.

1 Spectral representation of KBR precision with regard to the Earth gravity signal

To contract the polar blank region, without satellite footprint, the near polar satellite orbit is selected in the SST mission. In this case, the coverage of observation is well proportioned and dense. To simplify the analysis, the orbit inclination is regarded as 90°, and we can investigate the along-track inter-satellite ranging observation to estimate the performance of the SST-II
system. See Fig. 1, the two satellites, flying in formation and separated by several hundred kilometers, fly in the same near polar circle orbit and the radii of the orbit is \( r \), so the mean line velocity of the two satellites is \( v_0 \). If the sample interval of KBR is \( \delta t \), the sample number of the SST observation is 
\[
\frac{2\pi r}{v_0 \delta t}
\]
per cycle. According to the Nyquist ruler, the maximum degree and order of the Earth gravity field model \( N \) can be expressed directly as below:

\[
N = \frac{\pi r}{v_0 \delta t}
\]  

According to Kepler’s laws, the mean angle velocity of a satellite—\( \dot{n} \) can be expressed as:

\[
\dot{n}^2 \cdot a^3 = GM  
\]  

And the mean line velocity of the satellite \( v_0 \) is:

\[
v_0 = r \cdot \dot{n}
\]  

Eq. (1) can be rewritten as below:

\[
N = \frac{\pi r}{v_0 \delta t}  
\]  

where \( a \) is the length of the orbit semi-major axis, if circular orbit \( a \approx r \); \( r \) is the geocentric distance of satellite; \( v_0 \) is the mean line velocity of satellite; \( \delta t \) is the sample interval of SST; \( GM \) is the geocentric constant of gravitation.

Because the orbit is the near circular orbit, the SST inter-satellite observation, which includes line-of-sight velocity (\( V_{\text{los}} \)) and line-of-sight acceleration (\( V_{\text{los}}' \), shows the kinetic energy difference of the satellite pair. In the light of the description of the Reference [2], the SST spectral preference expression can be drawn with reasonable approximating.

The energy of one satellite of a SST-11 satellite pair can be expressed as below:

\[
\frac{1}{2} \sum \left( \frac{dX_i}{dt} \right)^2 = V_i - V_0 + C
\]  

where \( dX_i/dt \) is the velocity of three orthodoxy directions; \( V_i - V_0 \) is the satellite potential difference of two epoch; \( C \) is the integral constant.

The left part of the Eq. (5) is the kinetic energy difference of the satellite pair. If the satellite pair fly in a circular orbit, the kinetic energy expressions of the two satellites is:

\[
E_1 = \frac{1}{2} \left[ v_1^2 + v_2^2 \right]  
\]

\[
E_2 = \frac{1}{2} \left[ (v_1 + v_2)^2 + (v_1 + v_2)^2 + (v_1 + v_2)^2 \right]  
\]

where \( v_1, v_2, v_3 \) are the RTN (radial, along-track and normal) velocity of satellite 1; \( v_1, v_2, v_3 \) are the small velocity increments along RTN directions.

The kinetic energy difference of the satellite pair is:

\[
\delta E = E_2 - E_1 = \frac{1}{2} \left[ 2v_1 v_2 + 2v_1 v_1 + v_1^2 + 2v_2 v_3 + v_3^2 \right]
\]  

In the circular orbit case, from the second and to sixth items of the right part of above equation are less than the first item greatly. So the equation can be approximated as:

\[
\delta E \approx v_1 v_2
\]  

Because of

\[
\delta E = V_1 - V_0
\]  

and

\[
V_1 - V_0 = \frac{\partial V}{\partial M} \cdot \frac{M}{dt} \cdot \Delta t = \frac{\partial V}{\partial M} \cdot M \cdot \Delta t
\]

\[
V_s \cdot \Delta t\]

\[
= \frac{GM}{r^2} \cdot \frac{GM}{r^2} \approx \frac{GM}{r}
\]

where \( M \) is the mean anomaly of the satellite; \( M \cdot \Delta t \) is the geocentric angle of the satellite pair.

The approximating relationship is obtained as:

\[
v_1 v_2 \approx \frac{\partial V}{\partial M} \cdot M \cdot \Delta t
\]  

According to the Eqs. (2) and (3),

\[
v_s = \frac{GM}{a^2} \approx \frac{GM}{r}
\]  

And the definition of the Earth gravity field spectrum is described in many references[2,3,4]. In a similar way, the power spectrum of \( v_s \) is deducted with Eqs. (11) and (12):

\[
V_s = \frac{r}{GM} \left( \frac{\partial V}{\partial M} \cdot M \cdot \Delta t \right)
\]
Because the along-track variation of gravitational potential is caused by the perturbation potential \(-R\) (see Eq. (15)), and the Eq. (14) can be deducted in the circular orbit case, the Eq. (16) is drawn.

\[
\frac{r}{GM(M \cdot \Delta t)^2} \left| \frac{\partial V}{\partial M} \right| = \frac{r}{GM(M \cdot \Delta t)^2} V \left| \frac{\partial V}{\partial \phi} \right|^2 \tag{13}
\]

According to the definition of \(Y_{\text{ben}}\), the \(\partial Y_{\text{ben}}/\partial \phi\) does not change above equation spectrum property, so the \(\partial V/\partial \phi\) and the \(R\) have the same power spectrum feature:

\[
\left| \frac{\partial V}{\partial M} \right| = \left| \frac{\partial V}{\partial \phi} \right| \tag{17}
\]

According to above equation and Eq. (13), the power spectral expression is as below:

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{r}{GM(M \cdot \Delta t)^2} V \right) (R) \tag{18}
\]

And according to the Kaula ruler, the power spectrum of the perturbation potential is:

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} = \left( \frac{GM}{a_r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{19}
\]

Because of

\[
\rho = r \cdot M \cdot \Delta t \tag{20}
\]

we can further obtain the left part of Eq. (18) from Eqs. (19) and (20):

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{21}
\]

In this paper, the SST observation types \(-D_{\text{isd}}, V_{\text{isv}}, \text{and } A_{\text{isv}}\) are import directly \([1,4]\). If the distance between the satellite pair is adequately small, the along-track direction is almost same as the line-of-sight direction. The power spectrum of \(V_{\text{isv}}\) is written as below.

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{22}
\]

where \(a_r\) is the equatorial radius of the Earth; \(r\) is the the geocentric distance of satellite; \(\rho\) is the the distance between the satellite pair; \(n\) is the degree.

In view of the error of the inter-satellite distance measurement, the actual maximum degree \(N\) submits to below inequation.

\[
N < \frac{\pi r}{\sqrt{\rho/g}} \tag{23}
\]

If the error is white error and the variance of LSV is \(\sigma_{\text{isv}}^2\), the measuring error spectrum is:

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \sigma_{\text{isv}}^2 \tag{24}
\]

The maximum degree and order \(-n\) reached by the KBR system is obtained, in which the gravitational signal power is same as the measuring error power. The \(n\) meets below equation:

\[
\left( \frac{GM}{a_r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} = \sigma_{\text{isv}}^2 \tag{25}
\]

According to the relationship among the Earth gravity signal, geoid height and gravity anomaly, the spectrum of geoid height \(-N\) is connected with \(V_{\text{isv}}\):

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{26}
\]

Combining Eqs. (26) and (24), the error spectrum of geoid height is expressed as:

\[
\sigma_{\text{isv}}^2 \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{a_r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{27}
\]

In the same way, the spectral representation of gravity anomaly with \(V_{\text{isv}}\) can be given as follows:

\[
V_z \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{28}
\]

Combining Eqs. (28) and (24), the error spectrum of gravity anomaly is expressed as:

\[
\sigma_{\text{isv}}^2 \left| \frac{\partial V}{\partial \phi} \right|^2 = \left( \frac{GM}{a_r} \right) \left( \frac{a_r}{r} \right)^2 1.6 \times 10^{-10} \cdot n^{-3} \tag{29}
\]

On the other hand, the \(D_{\text{isd}}\) and \(A_{\text{isv}}\) are expressed with \(V_{\text{isv}}\), as below:

\[
V_{\text{isv}} = \frac{D_{\text{isd}} - D_{\text{isd}}}{g} \tag{30}
\]

\[
\frac{A_{\text{isv}}}{\text{isv}} = \frac{V_{\text{isv}} - V_{\text{isv}}}{g} \tag{31}
\]

Assuming that the precision of KBR and the
time is not related to the time, and then the precision of $D_{lsd}$ and $A_{lsd}$ are calculated with:

$$\sigma_{lsd}^2 = \frac{\tau^2 \cdot \sigma_{lsd}^2}{2}$$

$$\sigma_{lsd}^2 = \frac{2 \cdot \sigma_{lsd}^2}{\tau^2}$$

where $\sigma_{lsd}$, $\sigma_{lsd}$, $\sigma_{lsd}$ are the SST observation precisions.

## 2 Simulation result

By the analytical method as above, the performance of an appointed SST-II configuration can be analyzed conveniently. For example, a SST-II system contains a KBR system whose ranging precision is $1.0 \times 10^{-6}$ m/s with 10 s interval. The simulation results of the altitude vs. degree and order are listed in Table 1.

### Table 1 Distance analysis of satellite pair

| Altitude | Degree | 90  | 120 | 150 |
|----------|--------|-----|-----|-----|
| 300 km   | 221,625| 1,354,789 | 7,517,637|
| 400 km   | 853,910 | 8,152,963 | 70,660,307|
| 500 km   | 3,225,714 | 47,790,702 | 642,714,186|

From Table 1, we can draw a conclusion that the SST system performance can be improved slightly with increasing the distance between satellite pair properly, if the KBR system has same precision level.

If the KBR performance is known, which is same as in former case, some SST system configurations can recover Earth gravity field model of lower degree and order.

From Table 2, if the satellite pair fly at the 300 km orbit with about 300 km separated distance and the ranging precision is $1.0 \times 10^{-6}$ m/s (GRACE-like), the system can recover almost 100 degree and order Earth gravity field.

### Table 2 Capability of SST (sample rate: 10 s)

| Altitude | Degree |
|----------|--------|
|          | 100 km | 200 km | 300 km |
| 300 km   | 77     | 88     | 95     |
| 400 km   | 63     | 71     | 76     |
| 500 km   | 54     | 61     | 64     |

For the other applications, the geoid height and gravity anomaly expression is convenient. We can get the ability to recover geoid height with the configuration which has 300 km altitude, $1.0 \times 10^{-6}$ m/s KBR precision and 300 km separated distance. See Table 3.

### Table 3 The Geoid recovery analysis of satellite pair (sample rate: 10 s) /km

| Altitude | Degree |
|----------|--------|
| 300 km   | 5.717  | 0.936  | 0.236  |
| 400 km   | 30.590 | 4.933  | 0.795  |
| 500 km   | 251.586| 26.144 | 2.717  |

At similar way, we can be able to recover gravity anomaly with the configuration which has 300 km altitude, $1.0 \times 10^{-6}$ m/s KBR precision and 300 km separated distance. See Table 4.

### Table 4 The Gravity anomaly analysis of satellite pair (sample rate: 10 s) /mGal

| Altitude | Degree |
|----------|--------|
| 300 km   | 79.243 | 15.668 | 2.869 |
| 400 km   | 663.781| 84.393 | 9.963 |
| 500 km   | 5,173.343| 453.244| 34.631|

Because the analysis is based on one-coverage on the global, the simulation results only show the lower performance, which includes precision and resolution, than the actual GRACE data.

## 3 Conclusions

1) The SST system performance can be improved slightly with increasing the distance between satellite pair properly, if the KBR system has same precision level.

2) If the satellite pair fly at the 300 km orbit with about 300 km separated distance and the ranging precision is $1.0 \times 10^{-6}$ m/s (GRACE-like), the system can recover about 100 degree and order Earth gravity field with one-coverage observation.

3) Because the actual orbit inclination is not equate 90° in Eq. (14), there is a need for stricter deducting for the more precise analysis. Prof.