REGULARIZATION FOR THE SUPERCRITICAL QUASI-GEOSTROPHIC EQUATION.

B. BARRIOS

Abstract. Motivated by the De Giorgi type argument used in a recent paper by Caffarelli and Vasseur, we prove Hölder regularity for weak solutions of the supercritical quasi-geostrophic equation with minimal assumptions on the initial datum.

1. Introduction and Motivation.

1.1. Introduction. In this work we study the regularity properties of solutions \( \theta : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R} \) to the quasi-geostrophic equation (SQG or 2D QG), with initial datum \( \theta(x,0) = \theta_0(x) \in L^2(\mathbb{R}^2) \) given by

\[
\partial_t \theta(x,t) + (u_\theta \cdot \nabla \theta)(x,t) + (-\Delta)^{\alpha/2} \theta(x,t) = 0.
\]

Here \( \alpha \in (0,2] \) is a fixed parameter and \((-\Delta)^{\alpha/2} \theta = \Lambda^\alpha \theta\) represents the fractional laplacian in the \( x \) variable. The velocity \( u_\theta \) is divergence free and is determined by the Riesz transforms of the potential temperature \( \theta \), that is,

\[ u = (-R_2 \theta, R_1 \theta) = R_\perp \theta, \]

where \( R_i \) are the Riesz transforms given by

\[ R_i \theta(x) = c \text{ P.V.} \int_{\mathbb{R}^2} \frac{(y_i - x_i) \theta(y)}{|y - x|^3} dy. \]

Equation (1.1) is an important model in geophysical fluid dynamics. The equation is physically motivated and is, perhaps, the simplest equation of fluid dynamics for which the question of global existence of smooth solutions is still poorly understood. Mathematically the equation has also been considered to be a 2D model of the 3D incompressible Navier-Stokes equations (NS). Indeed the pioneering works by Constantin, Majda and Tabak \[6\] and Constantin and Wu \[8\] revealed close relations between dissipative/non dissipative 2D QG and the 3D NS/Euler equations. It is therefore an interesting model for investigating existence issues on genuine 3D Navier-Stokes equations. This equation has recently been studied by many authors (see \[10\], \[5\], \[7\], \[8\], \[14\], \[15\]).

The global existence of a weak solution to (1.1) follows from Resnick \[19\]. The cases \( \alpha > 1 \), \( \alpha = 1 \), \( \alpha < 1 \) are called subcritical, critical and supercritical, respectively. The subcritical case is well understood. Wu established in \[23\] the global existence of a unique regular solution to (1.1) with initial datum \( \theta_0 \in L^p(\mathbb{R}^2) \) for \( p > 2/(\alpha - 1) \). With initial datum in the space \( L^{2/(\alpha - 1)} \), the proof of the global well posedness can be found in a recent article \[3\], where the asymptotic behavior of the solutions is also studied. By using a Fourier splitting method, Constantin and Wu \[5\] showed the global existence of a regular solution on the torus with periodic boundary conditions and also a sharp \( L^2 \) decay estimate for weak solutions with datum in \( L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \). Very recently, Dong and Li in \[12\] estimated the higher order derivatives of the solution and proved that it is actually spatial analytic.

However the critical and supercritical cases still have unsolved problems. Note that in these cases there is a higher derivative in the flow term \( u_\theta \cdot \nabla \theta \) than in the dissipation term \((-\Delta)^{\alpha/2} \theta\).

A general understanding is that the first term tends to make the smoothness of \( \theta \) worse, while
the second tends to make it better. Very recently, there are two important papers \cite{1} and \cite{16} that show the global regularity for the SQG equation in the critical case. In \cite{16} the global well-posedness with periodic $C^\infty$ datum was established by Kiselev, Nazarov and Volberg by proving certain non-local maximum principle. In \cite{1} Caffarelli and Vasseur constructed a global regular solution with $L^2$ initial datum. The proof in \cite{16} is certainly simpler than the one in \cite{1} but in this article the full structure of the nonlinearity in \eqref{eq:SQG} is not used, so the result is somewhat more general. To the best of our knowledge, the uniqueness of such weak solution is still open.

For the supercritical case, several small initial data results have been obtained. More specifically, global existence and uniqueness have been shown for small initial datum in the critical Besov space $B^{2,1}_{2,1}$. Also global regularity has been obtained when the initial datum is small in $H^r$ with $r > 2$, \cite{10}, or in $B^2_{2,\infty}, r > 2 - 2 \alpha$, \cite{24}. There are some partial results assuming some extra regularity. In \cite{9}, Constantin and Wu showed that if the velocity $u_\theta$ is $C^\delta$ then the solution $\theta$ is $C^\delta([\mathbb{R}^n \times [t_0, \infty)])$, for some $\delta > 0$. Observe that in the equation \eqref{eq:SQG} we do not have this regularity condition in the velocity function. Recently in \cite{20}, Silvestre has studied the regularization for the slightly supercritical equation, that is, $\alpha = 1 - \epsilon, \epsilon \ll 1$, and he concluded, using a De Giorgi type argument, that weak solutions, for initial datum in $L^2$, become smooth for large time.

In this work we prove that this result is valid for any $\alpha \in (0, 5, 1], \not\text{necessarily for } \alpha \text{ very close to } 1. \text{ In other words, we show that for any initial datum } \theta_0 \in L^2(\mathbb{R}^2) \text{ there is a time } t_0 \text{ after which the solution of the supercritical quasi-geostrophic equation } \theta \text{ becomes smooth if } \alpha > 0.5. \text{ So, we prove that for } \alpha \in (0, 5, 1], \text{ the dissipation is still strong enough to balance the nonlinear term. Moreover we present a different proof of the second technical lemma that appears in \cite{1} and that we use to get a oscillation lemma. For the case } \alpha \in (0, 0.5] \text{ we have to make some changes in the Energy Lemma. The result of the regularity for this equation can be found in the last section.}

1.2. The extension problem. The fractional laplacian may be naturally introduced in the Fourier space. Indeed, one has that

\[
(\partial_j f) = 2\pi i \omega_j \hat{f},
\]

and therefore

\[
((-\Delta) f) = 2\pi |\omega|^2 \hat{f}.
\]

Thus, it is natural to define

\[
((-\Delta)^{\alpha/2} f)(\omega) = 2\pi |\omega|^\alpha \hat{f}(\omega), \alpha \in (0, 2).
\]

It is known \cite{17, 21} that the $\alpha$-fractional laplacian, on a given function $f$, may also be represented as the principal value

\[
(-\Delta)^\alpha f(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(x + y)}{|y|^{n+2\alpha}} dy,
\]

where the $C_{n,\alpha}$ are normalizing constants. This is a more useful form to represent this operator. Fractional laplacians can also be defined via extension. In the case $\alpha = 1$ it is a well-known technique of harmonic extension to the upper plane of the space adding one more dimension and then taking the boundary normal derivative. For $\alpha \neq 1$ the method has been recently developed in \cite{2} by L. Caffarelli and L. Silvestre. They showed that any fractional power of the laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. Indeed, they prove that

\[
((-\Delta)^{\alpha/2} \theta^\ast(x, 0, t) = \lim_{z \to 0} z^\ast \partial_z \theta^\ast(x, z, t),
\]

where $\theta^\ast(x, z, t)$ is the extension function of the temperature whose expression is given in \cite{13} below. For $n \geq 1$ and $x \in \mathbb{R}^n$, we can rewrite the (extended) equation \eqref{eq:SQG} as follows

\[
(P_1) = \begin{cases}
\text{div}(z^\ast \nabla \theta^\ast) = 0, & z > 0, \\
\partial_t \theta^\ast(x, 0, t) + u_\theta \cdot \nabla \theta^\ast(x, 0, t) + \lim_{z \to 0} z^\ast \partial_z \theta^\ast(x, z, t) = 0 & x \in \mathbb{R}^n,
\end{cases}
\]
where $\epsilon = 1 - \alpha$. Although we still have the non-local nature of $u_0$, the nonlocal behaviour has been replaced by a local equation in one more variable. Now, unlike the case $\alpha = 1$, the function $\theta^*$ is not harmonic but rather $\alpha$-harmonic, to wit, it solves the equation

$$(P_2) = \left\{ \begin{array}{l}
\frac{\partial\theta^*(x, z, t)}{\partial z} + \frac{\partial}{\partial x} \theta^*(x, z, t) + \Delta_{\alpha} \theta^*(x, z, t) = 0 \\
\theta^*(x, 0, t) = \theta(x, t)
\end{array} \right. \quad z > 0, \quad x \in \mathbb{R}^n.$$  

Applying the Fourier transform, we can consider the next problem equivalent to $(P_2)$,

$$(P_3) = \left\{ \begin{array}{l}
y(0) = \delta_0 \\
y'' + \frac{\xi}{y} |\omega|^2 y = 0 \\
y = \theta^*(\omega, z, \cdot), \omega \in \mathbb{R}^n,
\end{array} \right. \quad \text{where } \delta_0 \text{ is the Dirac delta in } x = 0$$

(See 22.) Hence, 

$$Q^\epsilon(\omega, z) = C_\epsilon z^{1-\alpha} \int_0^\infty e^{-|\omega|^2 t} e^{-z^2/4t} \frac{dt}{t^{1+\frac{\alpha}{2}}}.$$ 

Therefore, 

$$Q^\epsilon(\omega, z) \approx \hat{P}^\alpha(\omega, z),$$

so 

$$\theta^*(x, z, t) = P^\alpha(x, z) * \theta(x, t),$$

is a solution of $(P_2)$. To simplify the notation we are going to write $P^\alpha(x, z) = P^\alpha_z(x)$.

1.3. Scaling. As in 20 we are going to consider 

$$B_r = \{x \in \mathbb{R}^n : |x| \leq r\} = [-r, r]^n,$$ 

$$B^*_r = B_r \times [0, r),$$ 

$$Q^*_r = B_r \times [0, r) \times (1 - r^\alpha, 1].$$

We are interested in the case $n = 2$ but the results that we are going to present in section 3 can be applied to any dimension $n \geq 2$. Note that there is nothing special about the translation in the time domain by 1, that is, we can consider $t \in (m - r^\alpha, m]$, $m \geq 0$.

1.4. Maximum principle for $\alpha$-harmonic functions. Let the function 

$$\psi(x) = \chi_{(-4-\omega, -4+\omega)^n}(x) + \chi_{(4-\omega, 4+\omega)^n}(x), \quad x \in \mathbb{R}^n,$$

where $\omega \in \mathbb{R}$ is a small parameter that will be chosen later. Set now 

$$F(x, z) = (P^\alpha_{z_0} \cdot \psi)(x), \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}.$$ 

If we define 

$$f(x, y, z) = \frac{z^\alpha}{(z^2 + |y|^2)^{1+\frac{\alpha}{2}}}.$$

then 

$$F(x) = \int_{(-4-\omega, -4+\omega)^n} f(x, y, z) dy + \int_{(4-\omega, 4+\omega)^n} f(x, y, z) dy.$$
We want to know how is the behavior of this function in the domain \( B^n \). First as \( P^n_z \) is a summability kernel we have that

\[
F(x, z) \xrightarrow{z \to 0} \psi(x),
\]

so

\[
F(x, 0) = 0, \quad x \in [-4 + \omega, 4 - \omega]_n,
\]

and

\[
F(x, 0) = 1, \quad x \in [-4, -4 + \omega]_n \cup (4 - \omega, 4]_n.
\]

It is clear that

\[
F(x, z) \geq \inf_{\partial B^n_4 \setminus \{z = 0\}} (P^n_z(\cdot) * \psi(\cdot))(x), \quad (x, z) \in \partial B^n_4 \setminus \{z = 0\}.
\]

Therefore using the Lebesgue differentiation theorem we obtain that

\[
F(x, z) = \frac{1}{K_n}(P^n_z(\cdot) * \psi(\cdot))(x),
\]

it follows that

\[
F(x, z) \geq 1, \quad (x, z) \in \partial B^n_4 \setminus \{z = 0\},
\]

\[
F(x, z) = 0, \quad (x, z) \in [-4 + \omega, 4 - \omega]_n \times \{z = 0\},
\]

and

\[
F(x, z) = \frac{1}{K_n} > 0, \quad (x, z) \in \{-4, -4 + \omega\}_n \cup (4 - \omega, 4]_n \times \{z = 0\}.
\]

Let \( c_0 \) satisfy

\[
1 > c_0 > \frac{32}{641^\alpha}.
\]

Then

\[
\sup_{B^n_\alpha} F(x, z) = \frac{1}{K_n} \sup_{B^n_\alpha} \left( \int_{(-4 - \omega, -4 + \omega)_n} f(x, y, z) dy + \int_{(4 - \omega, 4 + \omega)_n} f(x, y, z) dy \right)
\]

\[
\geq \frac{1}{K_n} \sup_{B^n_\alpha \cap \{z \geq 0\}} \int_{(4 - \omega, 4 + \omega)_n} f(x, y, z) dy
\]

\[
\leq \frac{1}{K_n} \sup_{B^n_\alpha \cap \{z \geq 0\}} \int_{(4 - \omega, 4 + \omega)_n} \frac{z^\alpha}{|x - y|^{n + \alpha}} dy
\]

\[
\leq \frac{c_0^\alpha |\omega|^n}{K_n n^\frac{\alpha}{2} (4 - \frac{\omega}{2} - c_0)^{n + \alpha}} < c_0^\alpha, \quad n \in \mathbb{N}^n,
\]

with

\[
\omega < 2(1 - c_0) < 2(1 - \frac{32}{641^\alpha}).
\]

**Remark 1.1.** The condition \((1.4)\) it is neccessary to apply Theorem 2.5.
Consider now the function
\[
(P_g) \begin{cases}
\Delta_{x,z} g(x,z) + \frac{1}{\alpha} g(x,z) = 0, & (x,z) \in B^*_4, \\
g(x,z) = 1, & (x,z) \in \partial B^*_4 \setminus \{z = 0\}, \\
g(x,0) = 0, & (x,0) \in \partial B^*_4.
\end{cases}
\]
By the maximum principle for \(\alpha\)-harmonic functions we know that there exists \(\lambda > 0\) such that
\[
g < 1 - \lambda \text{ in } B_{c_0}^c.
\]
On the other hand we can affirm that \(g \leq F\) because \(F\) is an \(\alpha\)-harmonic function with boundary values greater than those of \(g\). Hence, by (1.5), it follows that
\[
1 - \lambda \approx c_0^\alpha,
\]
so we know the behavior of \(\lambda\) in terms of \(\alpha\).

2. Principal Result.

In their famous paper, Leray [18] and Hopf [13] constructed a weak solution \(\theta\) of N-S equations for an initial datum \(\theta_0\). The solution is called the Leray-Hopf weak solution. In the general case the problem on uniqueness and regularity of this type of solutions are still open questions for a lot of equations like Navier-Stokes equations.

By a solution of (1.1) with initial datum \(\theta_0\), we mean a weak solution \(\theta\) (in the sense of distributions) that is also a Leray-Hopf’s weak solution, so that
\[
\theta \in L^\infty((0,T),L^2(\mathbb{R}^n)) \cap L^2((0,T),H^{\alpha/2}(\mathbb{R}^n)).
\]
This type of solution can be found also in [7].

The first result obtained after adapting the arguments of [1] to the equation (1.1), is the following

Theorem 2.1 (L^2 to L^\infty). Let \(\theta\) the solution of (1.1). Then
\[
\sup_{x \in \mathbb{R}^n} |\theta(x,t)| \leq C \frac{||\theta_0||_{L^2}}{t_0^{2+2\alpha}}.
\]
To prove this theorem we proceed as in [1] verifying that if \(\theta\) is a solution of (1.1) then, using a corollary that we can find in [10], we obtain the next level set energy inequality:
\[
\int_{\mathbb{R}^n} (\theta^2_\lambda(t_2, x) - \theta^2_\lambda(t_1, x))dx - 2\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |\Lambda^{\alpha/2}\theta_\lambda|^2dxdt \leq 0, \lambda > 0,
\]
where \(\theta_\lambda := (\theta - \lambda)_+\) and \(0 < t_1 < t_2\). Next we proceed as in [1].

The next theorem is the key result that leads to Hölder continuity in [1].

Theorem 2.2 (Oscillation Lemma). Let \(\theta\) be the solution of
\[
\partial_t \theta + u_\theta \cdot \nabla \theta + (-\Delta)^{\alpha/2}\theta = 0, \ x \in \mathbb{R}^n, \ \alpha < 1,
\]
for a vector function \(u_\theta\) with zero divergence such that
\[
||u_\theta||_{L^\infty([0,1],L^{2n/\alpha}(B_1))} \leq C_u.
\]
Then there exists \(\eta \approx \lambda > 0\), such that
\[
\text{osc}_{Q^*_2} \theta^*(x) \leq (1 - \eta) \text{osc}_{Q^*_1} \theta^*
\]
where \(a\) is given in Lemma 3.3 and with \(\lambda\) as in Lemma 3.2.

The proof of the claim presented above relies mainly on a local energy inequality and the De Giorgi’s isoperimetric lemma and was given in [1] for the case \(\alpha = 1\). We are going to prove it in detail in the last section for an arbitrary \(n \geq 2\). Remember that we only need this result for the case \(n = 2\).

Our main result is
Theorem 2.3 (L∞ to Cα). Let the function \( \theta : [0, \infty) \to \mathbb{R} \), the solution of (1.1) with initial data \( \theta_0 \in L^2(\mathbb{R}^2) \). Suppose that \( \alpha = 1 - \epsilon, \epsilon > 0 \), such that \( \alpha > \epsilon \). Then for every \( T > 0 \)

\[
|\theta(x, T) - \theta(y, T)| \leq C|x - y|^{\alpha},
\]

where \( C \) is a constant that depends on \( \alpha, ||\theta_0||_{L^2} \) and \( T \).

Proof

We are going to prove that \( \theta \) is Hölder continuous at the point \( (0, T) \). By a slight abuse of language, only in this proof, we rename \( \theta^n(x, z, t) \) by \( \theta(x, z, t) \) and \( Q^\alpha_r \) by \( Q_r \), \( r > 0 \). There is nothing special about \( x = 0 \) and the arguments seen in the proof can be extended to prove the regularity of the solution at any point \((x_0, T)\). Indeed, it is enough to make the change of variable given by

\[
\tilde{\theta}(x, z, t) = \lambda^{-\epsilon}\theta(x_0 + \lambda x, \lambda z, \lambda^\alpha t), \lambda > 0,
\]

since the function \( \tilde{\theta} \) continues to satisfy \( (P_1) \).

It is clear that if for any \( r \in (0, 1/s) \subseteq (0, 1), s \geq 1 \), we prove that

\[
\text{osc}_{Q_r} \theta \leq Cr^\alpha,
\]

then \( \theta \) is \( \alpha \)-Hölder continuous in \( x = 0 \). Hence defining

\[
\theta_\alpha(x, z, t) = r^{-\alpha}\theta(rx, rz, 1 - r^\alpha(1 - t)),
\]

our objective is to show that \( \text{osc}_{Q_{r^{1/\alpha}}(z)} \theta_\alpha \leq C \) since it implies immediately (2.1).

Let \( 0 < \epsilon < 0.5 \) and \( \theta \) the solution of (1.1) for \( \alpha = 1 - \epsilon \). By the Theorem 2.1, it follows that \( \theta \in L^\infty_{[0,1]}(B_1) \) and consequently \( u = R^\epsilon \theta \in BMO_\alpha(B_1) \) for any \( t > 0 \). In this way we obtain that \( ||u||_{L^\infty([0,1],BMO_\alpha(B_1))} < \infty \), and therefore \( ||u||_{L^\infty([0,1],L^{2\alpha/\alpha}(B_1))} < \infty \). We can consider, without loss of generality, that \( ||\theta||_{L^\infty_{[0,1]}(B_1)} = 1 \), so applying Theorem 2.2, we get that exists \( \eta > 0 \) such that

\[
\text{osc}_{Q_{r^{1/\alpha}}} \theta \leq (1 - \eta) \text{osc}_{Q_1} \theta \leq 2(1 - \eta).
\]

Let \( \alpha > \epsilon \). If we consider \( r_0 < 1 \) such that \( (1 - \eta) < 128r_0^\alpha \) it follows that

\[
\text{osc}_{Q_{r^{1/\alpha}}} \theta \leq 256r_0^\alpha.
\]

Now let \( \theta_1(x, z, t) = r_0^{-\alpha}\theta(rx_0, rz_0, 1 - r_0^\alpha(1 - t)) \). One can clearly see that if \( (x, z, t) \in Q_r \) then \( (rx_0, rz_0, 1 - r_0^\alpha(1 - t)) \in Q_{r_0} \), with \( Q_{r_0} = [-r_0, 0] \times [0, r_0] \times [1 - r_0^\alpha, 1] \). It follows that

\[

\text{and}

\[

u_{\theta_1}(x, t) = (R^\epsilon \theta_1)(x, 0, t) = \int_{\mathbb{R}^2} \theta_1(y, 0, t) \frac{(y - x)^\perp}{|y - x|^3} \, dy
\]

Hence, since \( \theta \) satisfies (1.1), we have for \( \theta_1 \)

\[
\partial_t \theta_1(x, 0, t) + r_0^{-\alpha}(u_{\theta_1}, \nabla \theta_1)(x, 0, t) + \lim_{z \to 0} (z^\epsilon \partial_z \theta_1)(x, z, t) = 0.
\]
Let $a$ given as in Lemma 3.3. Note that, defining
\[ m_0 = (255 \sup_{Q_{c^{2a}/128}} \theta + \inf_{Q_{c^{2a}/128}} \theta)/256, \]
it follows that
\[ \theta(x, z, t) - m_0 \leq \sup_{Q_{c^{2a}/128}} \theta - m_0 = \frac{\text{osc}_{Q_{c^{2a}/128}} \theta}{256} \leq r_0^\alpha, (x, z, t) \in Q_{c^{2a}/128}. \]
This fact yields that
\[ r_0^{-\alpha}(\theta(x, z, t) - m_0) \leq 1, (x, z, t) \in Q_{c^{2a}/128}. \]
It is clear that if $(x, z, t) \in Q_1$ and we choose $r_0 < c^2a/128$, then $(r_0 x, r_0 z, 1 - r_0^\alpha(1 - t)) \in Q_{c^{2a}/128}$, so we have that
\[ r_0^{-\alpha}(\theta(r_0 x, r_0 z, 1 - r_0^\alpha(1 - t)) - m_0) \leq 1, (x, z, t) \in Q_1. \]
Renaming
\[ \theta_1(x, z, t) = r_0^{-\alpha}(\theta(r_0 x, r_0 z, 1 - r_0^\alpha(1 - t)) - m_0), \]
we obtain
\[ (2.3) \begin{align*}
\partial_t \theta_1(x, 0, t) + r_0^{-\alpha}(u_\theta \cdot \nabla \theta_1)(x, 0, t) &+ \lim_{z \to 0}(z^\alpha \partial_z \theta_1)(x, z, t) = 0, \\
\theta_1(x, z, t) &\leq 1, (x, z, t) \in Q_1.
\end{align*} \]
Since $||u_\theta||_{L^\infty([0,1], L^{2n/\alpha}(B_1))} < \infty$ if we choose $r_0^{-\alpha} < C_1$, where $C_1$ will be chosen later, we get that $u_\theta \in L^\infty([0,1], L^{2n/\alpha}(B_1))$. Indeed, since $r_0 < c^2a/128 < 1/2$ it follows that
\[ \sup_{0 < t < 1} \left( \int_{B_1} |\partial_t \theta_1(x, t)|^{2n/\alpha} dx \right)^{\alpha/2n} = \sup_{0 < t < 1} \left( \int_{B_1} |r_0^{-\alpha} u_\theta(r_0 x, 1 - r_0^\alpha(1 - t))^{2n/\alpha} dx \right)^{\alpha/2n} \leq \sup_{1 - r_0^{\alpha/2} < h < 1} \left( \int_{r_0 B_1} |C_1 u_\theta(y, h)|^{2n/\alpha} r_0^{-\alpha} dy \right)^{\alpha/2n} \leq \frac{C}{r_0^{\alpha/2}} \sup_{0 < h < 1} \left( \int_{B_1} |u_\theta(y, h)|^{2n/\alpha} dy \right)^{\alpha/2n} < \infty. \]
Then, since $\alpha > \epsilon$, from Theorem 2.2 and (2.3) we deduce that
\[ (2.4) \quad \text{osc}_{Q_{c^{2a}/128}} \theta_1 \leq (1 - \eta) \text{osc}_{Q_1} \theta_1 < 256r_0^\alpha. \]
If we define
\[ \theta_{k+1} = r_0^{-\alpha}(\theta_k(r_0 x, r_0 z, 1 - r_0^\alpha(1 - t)) - m_k), k \geq 0 \]
where
\[ m_k = (255 \sup_{Q_{c^{2a}/128}} \theta + 255 \inf_{Q_{c^{2a}/128}} \theta)/256, \]
it is clear that
\[ \begin{align*}
&\partial_t \theta_{k+1}(x, 0, t) + (r_0^{-\alpha})^{k+1} (u_{\theta_{k+1}} \cdot \nabla \theta_{k+1})(x, 0, t) + \lim_{z \to 0}(z^\alpha \partial_z \theta_{k+1})(x, z, t) = 0, \\
&\theta_{k+1}(x, z, t) \leq 1, (x, z, t) \in Q_1.
\end{align*} \]
By induction, if we suppose that $u_{\theta_k} \in L^\infty([0,1], L^{2n/\alpha}(B_1))$ and we proceed as in the case $k = 0$, we conclude that $u_{\theta_{k+1}} \in L^\infty([0,1], L^{2n/\alpha}(B_1))$. Therefore, applying Theorem 2.2 it follows that
\[ \text{osc}_{Q_{c^{2a}/128}} \theta_{k+1} \leq (1 - \eta) \text{osc}_{Q_1} \theta_{k+1} < 256r_0^\alpha. \]
This allows us to conclude that
\[ (2.5) \quad \text{osc}_{Q_{c^{2a}/128}} \theta_k \leq 256r_0^\alpha, k \in \mathbb{N}, \]
where $r_0$ is such that
\[ \begin{align*}
&2(1 - \eta) < 256r_0^\alpha, \eta > 0, \alpha > \epsilon, \\
r_0 < c^2a/128, & \quad r_0^{-\alpha} < C_1.
\end{align*} \]
Note that we can affirm that there exists \( r_0 < 1 \) verifying these conditions above because the relations (1.6), (3.9) and (3.11) are satisfied. Indeed we can find \( r_0 \) such that
\[
\frac{c_0^2}{2^{1/\alpha}32} > r_0 > \frac{c_0}{128^{1/\alpha}}
\]
and
\[
(2.6) \quad r_0^{-\alpha} < C_1.
\]
We can choose for example
\[
(2.7) \quad r_0 = \frac{c_0}{64^{1/\alpha}},
\]
and
\[
(2.8) \quad c_0 > \frac{32}{32^{1/\alpha}}.
\]
In that case the constant \( C_1 \) would be \( 64c_0^{\sup\{\alpha\}} = 64^{-1} \). Note that (2.8) does not contradict the condition (1.4), so finally we select \( c_0 \in (32^{1-\frac{1}{\alpha}}, 1) \). Observe that
\[
\theta_k(x, z, t) = r_0^{-\alpha}(\theta_{k-1}(r_0x, r_0z, 1 - r_0(1 - t)) - m_{k-1})
\]
and
\[
\vdots
\]
\[
= r_0^{-k\alpha}(\theta_r^k x, r_0^k z, 1 - r_0^{k\alpha}(1 - t)) - (m_{k-1} + m_{k-2} + \ldots + m_0), \quad k \in \mathbb{N}.
\]
Hence by (2.6) and (2.7) we have that
\[
(2.9) \quad 256r_0^{\alpha} \geq \text{osc}_{Q_{r_0^{1/2}}^{2/128}} \theta_k(x, z, t) = \text{osc}_{Q_{r_0^{1/2}}^{2/128}} r_0^{-k\alpha}(\theta_{r_0^k x, r_0^k z, 1 - r_0^{k\alpha}(1 - t)}) = \text{osc}_{Q_{r_0^{1/2}}^{2/128}} \theta_{r_0^k x}, \quad k \in \mathbb{N}.
\]
Now let \( r \in (0, 1) \). We can assert that there exists \( k \in \mathbb{N} \cup \{0\} \) such that \( r_0^{k+1} \leq r \leq r_0^k \). Note that (2.9) is equivalent to \( \text{osc}_{Q_{r_0^{1/2}}^{2/128}} \theta \leq 256r_0^{k\alpha} r_0^k \), so we have that
\[
\text{osc}_{Q_{r_0^{1/2}}^{2/128}} \theta \leq 256(r_0^{k+1})^{\alpha} \leq 256r_0^{\alpha} \leq C_\alpha \left( \frac{ac_0^2}{128} \right)^{\alpha},
\]
where
\[
C_\alpha = 256 \left( \frac{128}{ac_0^2} \right)^{\alpha}.
\]
Then it is clear that
\[
\text{osc}_{Q_r} \theta \leq C_\alpha r^\alpha, \quad r \in \left( 0, \frac{ac_0^2}{128} \right),
\]
so \( \theta \in C_\alpha^\infty \). Note that we have obtained (2.4) for \( s = \frac{128}{ac_0^2} > 1 \). \( \Box \)

**Corollary 2.4.** Let \( \theta \) be a solution of (1.1), \( \alpha > \epsilon \) and \( 0 < t_0 < t < \infty \). Then
\[
\theta \in C^\infty((t_0, t) \times \mathbb{R}^2).
\]

**Proof**

The \( C^\infty \) regularity follows from the theorem above and [7]. \( \Box \)

### 3. Proof of the Oscillation Lemma

In this section we are going to prove the Theorem 2.2 using the ideas given in [1] and [20]. Before showing this result we need two auxiliary lemmas and an energy estimate adapted to the problem \( (P_1) \) associated to the equation (1.1).
3.1. Local energy inequality.

**Theorem 3.1 (Energy Lemma).** Let \( 0 < t_1 < t_2 \) and let \( \theta \) be the solution of

\[
\partial_t \theta(x,t) + (u_\theta \cdot \nabla \theta)(x,t) + (-\Delta)^{\alpha/2} \theta(x,t) = 0, \quad x \in \mathbb{R}^n, \quad \alpha > 1, \quad t \in (t_1, t_2),
\]

such that \( \text{div} \ u_\theta = 0 \) and

\[
\|u_\theta\|_{L^\infty(t_1, t_2, L^{2n/\alpha}(B_{c_0})}) < C_u, \quad C_u > 0.
\]

Then there exists \( C_u \) such that for any \( t \in [t_1, t_2] \) and cut-off function \( \eta(x, z) \) with \( \eta^2 \theta^* \) of compact support in \( B_{c_0} \times [-c_0, c_0] \) one has the next local energy inequality:

\[
\begin{aligned}
\int_{t_1}^{t_2} \int_{B_{c_0}} \eta |\nabla \theta|^2 \, dz \, dx \, dt + \int_{B_{c_0}} \eta \theta_{t_1}^2(t,x) \, dz \, dx \\
\leq 2 \int_{t_1}^{t_2} \int_{B_{c_0}} \eta^2 |((\nabla \eta) \theta_{t_1}^2| \, dz \, dx \, dt + C_u \int_{t_1}^{t_2} \int_{B_{c_0}} |((\nabla \eta) \theta_{t_1}^2| \, dz \, dx \, dt.
\end{aligned}
\]

**Proof.**

Let, by abuse of notation, \( u_{\theta} = u \). The proof of this result is very similar to the proof presented in [1] adding the weight \( z^\epsilon \) each time we integrate on the variable \( z \). The main difference comes when we consider the term

\[
\int_{t_1}^{t_2} \int_{B_{c_0}} \eta |\nabla \eta u_{\theta}^2| \, dx \, dt.
\]

Here we are going to specify this estimate. It is clear that for any \( \epsilon > 0 \) we have that

\[
\begin{aligned}
\int_{t_1}^{t_2} \int_{B_{c_0}} \eta |\nabla \eta u_{\theta}^2| \, dx \, dt \\
\leq \int_{t_1}^{t_2} \left( \int_{B_{c_0}} \eta |\nabla \eta u_{\theta}^2| \, dx \right) \frac{\alpha}{\alpha+2} \left( \int_{B_{c_0}} |\nabla \eta u_{\theta}^2| \, dx \right) \frac{\alpha+2}{\alpha} \, dt \\
\leq \int_{t_1}^{t_2} \epsilon \eta^2 |\nabla \eta u_{\theta}^2| \,
\int_{B_{c_0}} \left( \int_{B_{c_0}} \eta \theta_{t_1}^2 \, dz \, dx \right) \, dt \\
= I_1 + I_2.
\end{aligned}
\]

Using Hölder’s inequality with \( p = (n+\alpha)/\alpha > 1 \) and the hypothesis on the velocity function \( u \), it follows that

\[
I_2 \leq \frac{1}{\epsilon} C_u \int_{t_1}^{t_2} \int_{B_{c_0}} |((\nabla \eta) \theta_{t_1}^2| \, dx \, dt.
\]

To estimate \( I_1 \) first we apply the Sobolev inequality getting that

\[
|((\nabla \eta u_{\theta}^2)| \, dx \leq C_u |\nabla^{\alpha/2} (\eta u_{\theta}^2)| \, |\nabla |_{L^2(B_{c_0})} \, dx \leq C_u \int_{B_{c_0}} (\eta u_{\theta}^2) \, dx.
\]

We claim that

\[
\begin{aligned}
\int_{\mathbb{R}^n} (\eta u_{\theta}^2) \, dx = C \int_{\mathbb{R}^n} z^\epsilon |(P_{\alpha}^* \ast \eta u_{\theta} + P_{\alpha}^* \ast \eta u_{\theta})| \, dx.
\end{aligned}
\]

Indeed, renaming

\[
H(x, z, t) = \eta u_{\theta} + \chi_{B_{c_0}},
\]

we have to show that

\[
\int_{\mathbb{R}^n} |\nabla^{\alpha/2} H|^2 \, dx = C \int_{\mathbb{R}^n} z^\epsilon |(P_{\alpha}^* \ast H)|^2 \, dx.
\]

Applying the Fourier transform we obtain that

\[
\int_{\mathbb{R}^n} |\nabla^{\alpha/2} H|^2 \, dx = \int_{\mathbb{R}^n} |\omega |^\alpha |\hat{H}(\omega)|^2 \, d\omega.
\]

In the same way using (1.2) it follows that

\[
\int_{\mathbb{R}^n} z^\epsilon |\nabla_{x} (P_{\alpha}^* \ast H)|^2 \, dx \, dz = \int_{\mathbb{R}^n} z^\epsilon |\omega |^\alpha |\hat{H}(\omega)|^2 \, d\omega \, d\omega.
\]

\[
= \int_{\mathbb{R}^n} |\omega |^\alpha |\hat{H}(\omega)|^2 \left( \int_{\mathbb{R}^n} z^\epsilon e^{2 \alpha t} \left( \int_{0}^{\infty} e^{-\alpha t} \frac{d\omega}{t^{1+\alpha/2}} \right)^2 \right)\]
Thus we get (3.3). Using the same argument this yield that
\[
\int_0^\infty \int_{\mathbb{R}^n} z^t |\partial_\tau \mathcal{P}_z^\alpha \mathcal{H}(\omega)|^2 d\omega dz = C \int_{\mathbb{R}^n} |\omega|^{-\gamma} |\mathcal{H}(\omega)|^2 d\omega.
\]
Then, since the function \( \theta^* = P_\alpha^\bullet + \theta \) minimizes the functional \( \int_0^\infty |\nabla \theta^*|^2 z^t dz \), we conclude that
\[
I_1 \leq C \epsilon \int_0^\infty \int_{\mathbb{R}^n} z^t |\nabla_x z_\tau \left( P_\alpha^\bullet + \eta \theta_+ \chi_{B_\epsilon_0} \right) |^2 dxdz
\]
\[
\leq C \epsilon \int_0^\infty \int_{\mathbb{R}^n} z^t |\nabla_x (\eta \theta_+^*)|^2 dxdz
\]
\[
= C \epsilon \int_{B_{\epsilon_0}^*} z^t |\nabla_x (\eta \theta_+^*)|^2 dxdz.
\]
Finally proceeding in the same way as in the case \( \alpha = 1 \), (3.4), we obtain (3.1). \( \Box \)

3.2. Two auxiliary lemmas. The results presented in [11] are based on the De Giorgi’s ideas in this classical proof of the Hölder continuity of solutions to elliptic equations (see [11]). We are going to follow this type of arguments to prove the next two auxiliary lemmas.

As Caffarelli and Vasseur explain in [1], if \( |\theta_+| \|_{L^2} \) is very small, then the local \( L^2 \) to \( L^\infty \) bound mentioned in the section 1, will imply that (in a small domain) \( \theta_+ \) is very small. In particular we prove that \( \theta_+, |Q_1| \leq 1 - \lambda \), reducing the oscillations of \( \theta \) by \( \lambda \) (see the first technical lemma). But we do not know a priori that \( |\theta_+| \|_{L^2} \) is very small. We only know that in \( Q_4 \), \( \theta \) is at least half of the time positive or negative, say negative. We then have to reproduce a version of De Giorgi’s isoperimetric inequality for \( z^t d\omega \) (see [21]) that says that to go from zero to one \( \theta := 2\theta \) needs “some room”. Therefore the set \( \{ \theta \leq 1 \} \) is “strictly larger” than the set \( \{ \theta \leq 0 \} \) (see the second technical lemma). Repeating this arguments at truncation levels we get, after a finite number of steps \( K^+ \), diminishing the oscillations of \( \theta \) by \( \lambda 2^{-(K^+)} \). This implies Hölder continuity (see the Oscillation Lemma).

In the first lemma we are going to present how to control the \( L^\infty \) norm of \( \theta \) from the \( L^2 \) norms of both \( \theta \) and \( \theta^* \) locally, under suitable conditions on \( u_0 \) (see the hypothesis of the Energy Lemma).

**Lemma 3.2** (First technical lemma). Let \( \theta \) satisfy the assumptions of the Energy Lemma. Then, there exist \( \epsilon_0 > 0 \) and \( \lambda > 0 \), that depend only on \( C, \alpha \) and the dimension \( n \), such that whenever we have that

\[
\theta^*(x, z, t) \leq 1, (x, z, t) \in Q_4^*,
\]

and

\[
\int_{Q_4^*} z^t (\theta_+^*)^2 dxdzdt + \int_{Q_4^*} \theta_+^2 dxdzdt \leq \epsilon_0,
\]

we get

\[
\theta_+(x, t) \leq 1 - \lambda, (x, t) \in Q_4^*.
\]

**Proof**

The proof of this result is essentially the same as in Lemma 6 in [11]. The main differences are that now we have to consider \( b_1 \) and \( b_2 \) two \( \alpha \)-harmonic barrier functions and that our estimates are in terms of \( P_\alpha^\bullet \). As we said in Section 1, for this class of function we also have a maximum principle so we can bound the barrier functions as in [11]. Namely, \( b_1 \) is bounded by \( 1 - \lambda, \lambda > 0 \), and \( b_2 \) by a function with exponential decay. The principal and important difference is that the domain of the maximum principle has changed. Let \( b_1(x, z) = g(x, z), \) where \( g \) was defined in (P_\alpha), then we have the relation (1.6) between the constants \( \lambda \) and \( \epsilon_0 \). Another difference is that we have modified the time domain, so we have to apply Lemma 3.1 in a different space. Finally using the Sobolev inequality for \( \Lambda^\bullet \), we can conclude the proof of this lemma. To complete the details of the proof see [11] changing the domain \( Q_4^* \) to \( Q_4^\bullet \). \( \Box \)

**Lemma 3.3** (Second technical lemma). Let \( \theta \) satisfy the assumptions of the Energy lemma such that \( \theta^* \leq 1 \) in \( Q_4^* \) and

\[
|(x, z, t) \in Q_4^*: \theta^* \leq 0| \geq \frac{|Q_4^*|}{2}.
\]
Then
\[ \int_{A^*} z^\epsilon |(\theta^* - \frac{1}{2})_+|^2 dx dz dt + \int_A (\theta - \frac{1}{2})_+^2 dx dt \leq C S^{0.05}, \]
where
\[ A = B_4 \times [1 - a^\alpha, 1], \]
\[ A^* = B_4^* \times [1 - a^\alpha, 1], \quad a < \frac{4}{21\alpha}, \]
and
\[ S = \min\{|(x, z, t) \in Q_4^* : 0 < \theta^* < \frac{1}{2}|z^\epsilon, \frac{1}{100}\}. \]

Remark 3.4. There is not a deep reason for the choice of the number $1/100$ in the above lemma. We need a number, say $1/100$, which is much smaller than $1$ to make the inequality $S^{m_2} \leq S^{m_1}, \quad m_1 \leq m_2$ hold. Similarly the number $0.05$ can be replaced by another one, say $m/2$, that satisfies
\[ \frac{1}{14} < m < \min\{1, \frac{1}{6}\}. \]

Proof
First we are going to rename $\theta^* = 2\theta^*$ and $\vartheta = 2\vartheta$. Note that
\[ S = \min\{|((x, z, t) \in Q_4^* : \theta^* < 1)|z^\epsilon, \frac{1}{100}\}. \]
Using (3.1) and the hypothesis $\theta^* \leq 2$ in $Q_4^*$ we get
\[ \int_{1-4a}^1 \int_{B_4^*} z^\epsilon |\nabla \theta^*_+|^2 dx dz dt \leq C^*. \tag{3.5} \]
Let us set $b = 0.1$ and
\[ K = \frac{4}{S^6} \int_{1-4a}^1 \int_{B_4^*} z^\epsilon |\nabla \theta^*_+|^2 dx dz dt \leq \frac{4C^*}{S^6} \approx S^{-b}. \]
The De Giorgi isoperimetric lemma adapted to the measure $z^\epsilon dz$, whose proof can be found in [20], gives that
\[ C^{**} |C(t)|_{z^\epsilon} \geq |A(t)|_{z^\epsilon}^2, \quad |B(t)|_{z^\epsilon}^2, \]
where
\[ A(t) = \{(x, z, t) \in B_4^* \times \{t\} : \theta^* \leq 0\}, \]
\[ B(t) = \{(x, z, t) \in B_4^* \times \{t\} : \theta^* \geq 1\}, \]
\[ C(t) = \{(x, z, t) \in B_4^* \times \{t\} : 0 < \theta^* < 1\}, \]
\[ \int_{B_4^*} z^\epsilon |\nabla \theta^*_+|^2 dx dz \leq C^{**}. \]
Now let
\[ I = \{t \in [1 - 4a, 1] : \int_{B_4^*} z^\epsilon |\nabla \theta^*_+|^2 dx dz \leq K, \quad |C(t)|_{z^\epsilon} \leq S^{1/2}\}. \]
Observe that
\[ |\{t \in [1 - 4a, 1] : |C(t)|_{z^\epsilon} > S^{1/2}\}| \leq \int_{1-4a}^1 |C(t)|_{z^\epsilon} dt \leq S^{1/2}, \]
and
\[ |\{t \in [1 - 4a, 1] : \int_{B_4^*} z^\epsilon |\nabla \theta^*_+|^2 dx dz > K\}| \leq \int_{1-4a}^1 \int_{B_4^*} z^\epsilon |\nabla \theta^*_+|^2 dx dz dt \approx S^b. \]
Thus we obtain that
\[ |I^c|_{z^\epsilon} \lesssim S^{1/2} + S^b \approx S^b. \tag{3.6} \]
Let
\[ A^* = B_4^* \times [1 - a^\alpha, 1], \]
where $a$ will be chosen later. If $|A(t)|_{z^\epsilon} \geq 1/4$, $t \in I \cap [1 - a^\alpha, 1]$, then
\[ |B(t)|_{z^\epsilon} \leq \frac{K^{1/2} |C(t)|_{z^\epsilon}^{1/2}}{|A(t)|_{z^\epsilon}} \lesssim S^{1/2 - b}, \quad t \in I \cap [1 - a^\alpha, 1]. \tag{3.7} \]
and, by (3.5), it follows that
\[ |\{(x, z, t) \in A^* : \theta^* \geq 1\}|_{z^*} \lesssim S^b + S^{1/2 - b} \approx S^b. \]

Hence, using the boundedness hypothesis, we get
\[ 4 \int_{A^*} \left( \theta - \frac{1}{2} \right)^2 + z^* dz dt = \int_{A^*} \left( \theta^* - 1 \right)^2 + z^* dz dt \lesssim S^b = S^{0.1}, \]

Moreover, since
\[ (\theta - 1) - (\theta^* - 1) = -\int_0^z \partial_z \theta^* dz, \]

using (3.6) and Cauchy-Schwartz inequality, it is clear that
\[ 4 \int_A \left( \theta - \frac{1}{2} \right)^2 dx dt = \int_A (\theta - 1)^2 dx dt \]
\[ \leq \int_A (\theta^* - 1)^2 dx dt + C^t \int_A \int_0^z \frac{1}{z} dz dx dt \]
\[ = \int_A (\theta^* - 1)^2 dx dt + C z^{1-\epsilon}, \]

where
\[ A = B_4 \times [1 - a^\alpha, 1]. \]

So, integrating in \([0, S^{b/2}]\) respect to the measure \(z^* dz\), we conclude that
\[ 4 \int_A \left( \theta - \frac{1}{2} \right)^2 dx dt \leq \int_A (\theta - 1)^2 dx dt \leq C(S^{b/2})^{1-\epsilon} \approx S^{0.05a} = S^{0.05a}. \]

That is
\[ \int_A \left( \theta^* - 1 \right)^2 + z^* dz dt + \int_A \left( \theta - \frac{1}{2} \right)^2 dx dt \lesssim \frac{S^{0.05a}}{4} \leq S^{0.005a}, \]

and the lemma will be proved.

Therefore, our next objective is to see that \(|A(t)|_{z^*} \geq 1/4\), \(t \in I \cap [1 - a^\alpha, 1]\). Note that if this is true then
\[ \int_{B_4} z^* (\theta^*)^2 dx dz \leq 4(|B(t)|_{z^*} + |C(t)|_{z^*}) \]
\[ \leq C(S^{1/2 - b} + S^{1/2}) \lesssim S^{1/2 - b}. \]

Moreover, since
\[ \int_{B_4} (\theta^*)^2 dx = \int_{B_4^*} (\theta^*)^2 dx - \int_{B_4} \int_0^z \partial_z (\theta^*)^2 dz dx, \]

integrating in \(z \in [0, 4]\) respect to the measure \(z^* dz\), we have that
\[ C \int_{B_4} (\theta^*)^2 dx \leq \int_{B_4^*} (\theta^*)^2 z^* dx + 2 \left( \int_0^4 \int_{B_4} \int_0^z z |\nabla \theta^*|^2 dz dx dz \right)^{1/2} \left( \int_0^4 \int_{B_4^*} (\theta^*)^2 z^* dz dx dz \right)^{1/2} \]
\[ \leq CS^{1/2 - b} + 2\sqrt{KS^{1/2 - b}} \]
\[ \approx S^{1/2 - b}. \]

Note that, as
\[ \beta := |\{(t, x, z) \in [1 - 4^\alpha, 1] \times B_4^* : \theta^* \leq 0\}|_{z^*} \geq \frac{|Q_2|_{z^*}}{2}, \]

then there exists \(t_0 < 1 - a^\alpha\), where \(a\) will be choosen later, such that
\[ \gamma := |\{(t_0, x, z) \in t_0 \times B_4^* : \theta^* \leq 0\}|_{z^*} \geq \frac{1}{4}. \]

Indeed
\[ \gamma \geq \beta - \left| [-4, 4]^n \times [0, 4] \times [1 - 4^\alpha, 1] \right| \left| [-4, 4]_x \times [0, 4] \times [1 - 4^\alpha, 1 - a^\alpha] \right|_{z^*} \]
\[ \geq \frac{8^n 4^{\alpha + 1}}{2(\epsilon + 1)} (4^\alpha - 2a^\alpha) \]
\[ \geq \frac{1}{4}. \]
Then, applying the energy inequality (3.1), in the same form as in [1], we obtain that
\[ \int_{B_4} (\vartheta^+)^2(x,t_0)dx \leq S^{1/2 - 3\kappa}. \]

Applying the energy inequality (3.1), in the same form as in [1], we obtain that
\[ \int_{B_4} (\vartheta^+)^2(x,t)dx \leq S^{1/2 - 3\kappa}, \quad t - t_0 \leq \delta^* \approx S^{1/2 - 3\kappa}, \quad t \in I. \]

Since
\[ \vartheta^* = \vartheta + \int_0^z \partial_z \vartheta^* d\tilde{z}, \]
applying Cauchy-Schwartz inequality, we have that
\[ \int_{B_4} (\vartheta^*_+)^2 dx \leq 2 \int_{B_4} \partial^2 \vartheta dx + 2 \int_{B_4} \left( \int_0^z \partial_z \vartheta^* d\tilde{z} \right)^2 dx \]
\[ \leq \int_{B_4} \partial^2 \vartheta dx + \int_{B_4} \left( \int_0^z \nabla \vartheta^* |d\tilde{z}| \right)^2 dx \]
\[ \leq \int_{B_4} \partial^2 \vartheta dx + \left( \int_{B_4} \int_0^z |\nabla \vartheta^* |(\tilde{z})^d d\tilde{z} dx \right) \left( \int_{B_4} \int_0^z \frac{1}{(\tilde{z})^d} d\tilde{z} dx \right) \]
\[ \leq S^{1/2 - 3\kappa} + C^* z^{1-\epsilon}. \]

Hence
\[ |\{ x \in B_4 : (\partial^*_+)^2 \geq 1 \}| \leq S^{1/2 - 3\kappa} + C^* z^{1-\epsilon}. \]

Set
\[ c = \frac{0.01}{1 - \epsilon} = \frac{0.01}{\alpha}. \]

Then,
\[ |\{ x \in B_4, z \in [0, S^\alpha] : \vartheta^*_+ \geq 1 \}|_{z^\alpha} \leq |\{ x \in B_4, z \in [0, S^\alpha] : (\vartheta^*_+)^2 \geq 1 \}|_{z^\alpha} \]
\[ \leq \int_{B_4} \int_0^{S^\alpha} (\partial^*_+)^2 z^\epsilon dx dz \]
\[ \leq CS^{1/2 - 3\kappa + \epsilon(1 + \epsilon)} + CS^2 c \]
\[ \approx S^{2c} = S^{0.02} \leq S^{0.01}, \quad t \in I \cap [t_0, t_0 + \delta^*]. \]

This implies that
\[ |\{ x \in B_4, z \in [0, S^{0.01}] : \vartheta^*_+ \leq 0 \}|_{z^\alpha} \geq S^{0.01(1 + \epsilon)} - S^{1/2} - S^{0.01} \approx S^{0.01}, \quad t \in I \cap [t_0, t_0 + \delta^*]. \]

Whences
\[ |B(t)|_{z^\alpha} \leq \frac{K^{1/2} C(t)|_{z^\alpha}^{1/2}}{|\{ x \in B_4, z \in [0, S^{0.01}] : \vartheta^*_+ \leq 0 \}|_{z^\alpha}} \leq S^{-0.01 - \frac{3}{4} + \frac{1}{4}} = S^{0.19}, \]
and, therefore,
\[ |A(t)|_{z^\alpha} \geq 4^n \frac{4^{n+1}}{\epsilon + 1} - S^{1/2} - S^{0.19} > \frac{1}{4}, \quad t \in I \cap [t_0, t_0 + \delta^*]. \]

Let’s see now that this property is spread by iteration. In fact, since
\[ \frac{\delta^*}{4} \approx S^{1/2 - 3\kappa} - S^b \approx S^b, \]
then exists \( t_1 = \delta^* = \left[ t_0 + \frac{3\delta^*}{4}, t_0 + \delta^* \right] \subseteq I \), and \( |A(t)|_{z^\alpha} \geq 1/4, \quad t \in I_1 \). Choosing \( t_0 = t_1 \) in the next iteration, and repeating this idea we conclude that
\[ |A(t)|_{z^\alpha} \geq \frac{1}{4}, \quad t \in I \cap [1 - a^n, 1], \]
3.3. Oscillation Lemma. We are going to prove Theorem 2.2. Before that, we present the next theorem from which we will deduce immediately the Oscillation Lemma.

**Theorem 3.5.** Let \( \theta \) satisfy the assumptions of the Energy Lemma such that \( \theta^* \leq 1 \) in \( Q^*_1 \) and

\[
|(x, z, t) \in Q^*_1 : \theta^* \leq 0|z^*| \geq \frac{|Q^*_1|z^*}{2}.
\]

Then

\[
\theta^* \leq 1 - \lambda^* \text{ in } Q^*_1 \cap \lambda^*,
\]

where \( a \) was given in (3.9) and \( \lambda^* = \lambda^*(n, C_a) > 0 \). Moreover, \( \lambda^* \) satisfies \( \lambda^* \approx \lambda \) with \( \lambda \) as in Theorem 3.2.

**Proof**

Let \( K^+ = \left[ \frac{1}{2}, 1 \right] |Q^*_1|z^* \), where \( S \) was defined in Lemma 3.3. For \( k \leq K^+ \) we define

\[
\theta_k = 2 \left( \theta_{k-1} - \frac{1}{2} \right) = 2^k \left( \theta_{k-3} - \frac{1}{2} \right) + 1,
\]

So

\[
\theta_k = 2^k \left( \theta_{k-3} - \frac{1}{2} \right) + 1 = 2^k \left( \theta_{k-3} - \frac{1}{2} \right) + \ldots + 2^{k-1} \left( \theta_{k-1} - \frac{1}{2} \right) + 1 \leq 1 \text{ in } Q^*_1.
\]

It is clear that \( |Q^*_1 \cap \theta_k^* \leq 0|z^*| > |Q^*_1|z^*/2 \) and \( \theta_k \) satisfies (3.11). If for all \( k \leq K^+ \) we have that

\[
S_k := \{|Q^*_1 \cap 0 < \theta_k^* < \frac{1}{2} |z^*| \} \geq S
\]

Then

\[
|Q^*_1 \cap \theta_k^* \leq 0|z^*| = |Q^*_1 \cap 0 < \theta_k^* \leq \frac{1}{2} |z^*| > S + |Q^*_1 \cap \theta_{k-1}^* \leq 0|z^*| > \ldots > kS + \frac{|Q^*_1|z^*}{2}.
\]

Hence, choosing \( k = K^+ \), we obtain that

\[
\{|\theta_{k+}^* \leq 0|z^*| > |Q^*_1|z^*,
\]

so \( \theta^*_{K^+} = 0 \) a.e. and this is a contradiction with the assumption. Therefore we claim that exists \( k_0 < K^+ \) such that \( S_{k_0} < S \). Observe that \( a < 4 \) for every \( \alpha < 1 \), so \( [1 - a^\alpha, 1] \subseteq [1 - 4^\alpha, 1] \) and applying the Lemma 3.3 it follows that

\[
\int_{Q^*_1} \left( \theta_{k_0}^* - \frac{1}{2} \right)^2 + a^\alpha |z|^2 dx dt dz + \int_{Q^*_1} \left( \theta_{k_0} - \frac{1}{2} \right)^2 + \alpha |z|^2 dx dt dz \leq \int_{Q^*_1} \left( \theta_{k_0} - \frac{1}{2} \right)^2 + \alpha |z|^2 dx dt dz \leq C S_{k_0}^{0.05a} < S^{0.05a},
\]

where \( a \) was given in (3.13). Then

\[
\int_{Q^*_1} \left( \theta_{k_0+1}^* \right)^2 + a^\alpha |z|^2 dx dt dz + \int_{Q^*_1} \left( \theta_{k_0+1} \right)^2 + \alpha |z|^2 dx dt dz \leq S^{0.05a} / 4,
\]

hence, applying Lemma 3.2, we get that

\[
\theta_{k_0+1}^* \leq 1 - \lambda \text{ in } Q_{c0^a/8},
\]

or, equivalently,

\[
\theta_k \leq 1 - \frac{\lambda}{2k^a} \leq 1 - \frac{\lambda}{2k^a} = 1 - \lambda^* \text{ in } Q_{c0^a/8}.
\]

Finally considering \( b_3 \) an \( \alpha \)-harmonic function such that \( b_3 = 0 \) in \( \partial B^*_{c0^a/8} \setminus \{z = 0\} \) and \( b_3 = 1 - \lambda^* \) in \( \{z = 0\} \), set

\[
\tilde{b}_3(x, z) = \begin{cases} \Delta_{x,z}b_3(x, z) + \frac{a}{2}b_3(x, z) = 0, & (x, z) \in B^*_1, \\ b_3(x, z) = 0, & (x, z) \neq B^*_1 \setminus \{z = 0\}, \\ b_3(x, 0) = c(1 - \lambda^*), & (x, 0) \in \partial B^*_1. \end{cases}
\]

where \( c \) will be chosen later. It is clear that
Choosing $c$ such that $c(1 - \lambda^*) = 1$ we obtain that $\tilde{b}_3(x, z)$ satisfies the same system of the equations than the barrier function $b_1$ defined in Lemma 3.2. Hence

$$\tilde{b}_3(x, z) \leq 1 - \lambda, \ (x, z) \in B_{c^0}^*$.$$

It follows that

$$b_3(x, z) \leq (1 - \lambda^*)(1 - \lambda), \ (x, z) \in B_{c^0/32}^*.$$

Also using the maximum principle we know that

$$b_3(x, z) \leq 1 - \lambda^{**} \text{ in } B_{c^2/32}^*, \lambda^{**} \in (0, 1).$$

Therefore we obtain the relation

$$(1 - \lambda)(1 - \lambda^*) = 1 - \lambda^{**},$$

so, as $0 < \lambda^* < \lambda < 1$, we get

$$\lambda^{**} = \lambda(1 - \lambda^*) + \lambda^* < 2\lambda,$$

and

$$\lambda^{**} = \lambda^* + \lambda - \lambda\lambda^* > -\lambda\lambda^* > -\lambda^2 > \lambda.$$

This yields that

$$\lambda^{**} \approx \lambda,$$

so, we conclude that

$$\theta^*(x, z, t) \leq 1 - \lambda^{**}, \ (x, z, t) \in Q_{c^2/32}^*,$$

where

$$\lambda^{**} \approx \lambda. \Box$$

Proof of the Oscillation Lemma

Note that by Theorem 2.1 we have obtained a bound for $|\theta^*|$, that is, $|\theta^*| \leq C_2$, $C_2 > 0$. Without loss of generality we can consider that $C_2 = 1$. Therefore, if we have

$$|\{\theta^* \leq 0 \cap Q_1^*\}|_{\bar{z}} > \frac{|Q_1^*|_{\bar{z}}}{2},$$

or

$$|\{-\theta^* \leq 0 \cap Q_1^*\}|_{\bar{z}} > \frac{|Q_1^*|_{\bar{z}}}{2},$$

then, by the theorem above, it is clear that

$$\text{osc}_{Q_{c^2/32}^*} \theta^* \leq 2 - \lambda^{**} = (1 - \eta)\text{osc}_{Q_1^*} \theta^*,$$

where

(3.11) $$\eta = \lambda^{**}/2 \approx \lambda.$$ 

So we have obtained Theorem 2.2. $\square$

4. Case $\alpha < 0.5$

In this section we are going to explain how the results presented in the previous section can be extended to $\alpha \in (\alpha_0, 0.5]$ where $\alpha_0$ is an arbitrary value greater than zero. In the proof of Theorem 2.3 we had to choose $\alpha > \epsilon$ to obtain (2.4). In this section we are going to show how we can obtain it without requiring that $\alpha > \epsilon$, that it, we are going to get the Oscillation Lemma for $\alpha_0 < \alpha < \epsilon$. To verified that it is possible we have to adapt the Energy Lemma and first and second technical lemmas to the case $\alpha < \epsilon$. From now on we are going to consider de modified QGE

(4.1) $$\partial_t \theta(x, t) + r_0^\alpha + (u_0 \cdot \nabla \theta)(x, t) + (-\Delta)^{\alpha/2} \theta(x, t) = 0,$$

where $r_0$ was given in (2.6) and $\alpha \in (\alpha_0, 0.5]$. 
4.1. Modification of the Energy Lemma. We will to obtain an energy inequality for the equation (4.1). Indeed, following the proof of Theorem [3.1], for the new equation, we get that
\[ I_1 \leq C\tilde{\epsilon}r_0^{\alpha_0-\epsilon} \int_{t_1}^{t_2} \int_{B_{c0}} z'|\nabla_{x_t}(\eta\theta^*_t)|^2dzdt, \]
and
\[ I_2 \leq C_u r_0^{\alpha_0-\epsilon} \tilde{\epsilon} \int_{t_1}^{t_2} \int_{B_{c0}} |(\nabla\eta)\theta_+|^2dzdt, \]
where \( \tilde{\epsilon} \) is the parameter that we choose in (5.2). Therefore it follows that
\[
(1 - C\tilde{\epsilon}r_0^{\alpha_0-\epsilon}) \int_{t_1}^{t_2} \int_{B_{c0}} z'|\nabla_{x_t}(\eta\theta^*_t)|^2dzdt + \frac{1}{2} \int_{B_{c0}} (\eta\theta_+)^2|t_{12}|^2dx \leq \\
\int_{t_1}^{t_2} \int_{B_{c0}} z'|\nabla\eta\theta_+|^2dzdt + C_u r_0^{\alpha_0-\epsilon} \tilde{\epsilon} \int_{t_1}^{t_2} \int_{B_{c0}} |(\nabla\eta)\theta_+|^2dzdt.
\]
Let
\[ A = 1 - C\tilde{\epsilon}r_0^{\alpha_0-\epsilon} \]
and
\[ B = \frac{C_u r_0^{\alpha_0-\epsilon}}{\tilde{\epsilon}}. \]
We have to choose \( \tilde{\epsilon} \) such that \( A > 0 \), that is, \( C\tilde{\epsilon}r_0^{\alpha_0-\epsilon} < 1 \) and \( B < \infty \). By (2.7), we can take
\[
\tilde{\epsilon} < \frac{1}{C} \frac{c_0^{\alpha_0-\epsilon}}{64^{(\alpha-\epsilon)\frac{n}{2}}} < \frac{1}{C} \frac{1}{2^6(\alpha-\epsilon)\frac{n}{2}}.
\]
Hence we define
\[
(4.2) \quad \tilde{\epsilon} = \frac{1}{C} \frac{1}{2^6(\alpha-\epsilon)\frac{n}{2}}.
\]
As \( \alpha > \alpha_0 > 0 \), it follows that
\[
C\tilde{\epsilon}r_0^{\alpha_0-\epsilon} = (2^{\frac{6}{n}}c_0)^{\alpha_0-\epsilon} < (2^{\frac{6}{n}}c_0)^{2\alpha_0-1},
\]
that is
\[ A > 1 - (2^{\frac{6}{n}}c_0)^{2\alpha_0-1}. \]
Note that \( A > 0 \) if and only if
\[
(4.3) \quad c_0 > 2^{\frac{6}{n}}.
\]
By (2.7), (2.8) and (1.2) we get
\[
B = C_u c_0^{\alpha_0-\epsilon} 2^{12(\alpha-\epsilon)\frac{n}{2}} \leq C_u c_0^{2\alpha_0-1} 2^{\frac{32}{n}}
\leq C_u 32^{\frac{4}{n}} 2^{\frac{32}{n}} = C_u 2^{\frac{64}{n}}.
\]
We conclude that the next local energy inequality is verified
\[
(1 - (2^{\frac{6}{n}}c_0)^{2\alpha_0-1}) \int_{t_1}^{t_2} \int_{B_{c0}} z'|\nabla(\eta\theta^*_t)|^2dzdt + \frac{1}{2} \int_{B_{c0}} (\eta\theta_+)^2|t_{12}|^2dx \leq \\
\int_{t_1}^{t_2} \int_{B_{c0}} z'|\nabla(\eta\theta_+)|^2dzdt + C_u 2^{\frac{64}{n}} \int_{t_1}^{t_2} \int_{B_{c0}} |(\nabla(\eta\theta_+)|^2dzdt.
\]
4.2. Modification of auxiliary lemmas. In the proof of Lemma [3.2], we have to make some modifications to adapt this result to the new equation. To follow the details of this modifications see the original proof of the first auxiliary lemma in [1]. We define
\[
A_k = (1 - (2^{\frac{6}{n}})^{2\alpha_0-1}) \int_{\frac{2}{3}2^{-k}}^{\frac{2}{3}2^{-k}} \int_{\frac{2}{3}2^{-k}}^{\frac{2}{3}2^{-k}} |\nabla(\eta_k\theta^*_t)|^2z^2dzdt + \sup_{t \in [-2^{-k-2}, 0]} \int_{\mathbb{R}^n} (\eta_k\theta_k)^2dx,
\]
and, as in the proof of Lemma [5.2], the goal is to show that \( A_k \to 0 \).
Since \( \alpha_0 \) is an arbitrary but fixed value, as in [1], it follows that
\[
A_k \leq C_u 2^{24n+2\alpha_0-\epsilon_0} \epsilon_0 + \epsilon_0 \leq \epsilon_0, \quad k \leq 12n,
\]
where $n$ is the dimension of the domain. Moreover we get
\[ A_{k-3} \geq (1 - (2^{\frac{n}{2}} c_0)^{2^{\alpha_0}-1} C(n))(\|\eta_{k-2} \hat{\theta}_{k-1}\|^2_{L^2} + \|\eta_{k-2} \hat{\theta}_{k-1}^*\|^2_{L^2}). \]

Then
\[ A_k \lesssim C^k A_{k-3}^{\frac{n}{2(n-2)}} , \quad k \leq 12n + 3, \]

where
\[ C = \left( \frac{2}{n-2} \right)^{\frac{n}{2(n-2)}} (1 - (2^{\frac{n}{2}} c_0)^{2^{\alpha_0}-1}). \]

Choosing, in Lemma 7 of [1],
\[ M = \sup \left( 1, C^{\frac{4(n-2)}{n+2}} \right), \]

we obtain Lemma [5.2] for the case $\alpha < \epsilon$.

In Lemma 3.3 we only have to take care about the energy of the solution $\theta$ of the [10]. Indeed by (4.1), we obtain
\[ \int_{1-4\epsilon}^{1} \int_{B_4^*} z^* |\nabla \theta^*_+|^2 dx dz dt \leq \frac{C^*}{1 - (2^{\frac{n}{2}} c_0)^{2^{\alpha_0}-1}} \]

instead of (3.5). Observe that, as $\alpha_0$ is arbitrary (but fixed) we can consider
\[ \int_{1-4\epsilon}^{1} \int_{B_4^*} z^* |\nabla \theta^*_+|^2 dx dz dt \lesssim C^* \]

and the proof of the second auxiliary lemma for $\alpha < \epsilon$ follows as in Lemma [5.3].

4.3. Modification of the Oscillation Lemma and Theorem “$L^\infty$ to $C^\alpha$”. Once Lemma [5.2] and Lemma [3.3] are known to hold for the case $\alpha \in (\alpha_0, 0.5]$, Theorem [3.5] and the Oscillation Lemma can be obtained in the same way as in the case $\alpha \in (0.5, 1)$. Therefore we deduce that (2.4) is satisfied in the case $\alpha < \epsilon$ too. In this way we get

**Theorem 4.1** ($L^\infty$ to $C^\alpha$). Let the function $\theta : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ be the solution of (1.1) with initial datum $\theta_0 \in L^2(\mathbb{R}^2)$. Suppose that $\alpha = 1 - \epsilon$, $\epsilon > 0$. Then for every $T > 0$
\[ |\theta(x, T) - \theta(y, T)| \leq C|x - y|^\alpha, \]

with $C$ is a constant that depends on $\alpha$, $\|\theta_0\|_{L^2}$ and $T$.

**Proof**

The proof of this result is similar to that of Theorem [2.3] using, where appropriate, the modified auxiliary and energy lemmas instead of the original ones presented in Section 3. □

**Acknowledgements:**

I would to thanks my advisor, Fernando Soria, for proposing me this problem and to help me in the preparation of this work. Without his help this paper would not have been possible. Also I am grateful with the professor Ireneo Peral because he has always shown a lot of interest and has trusted that this work would be achieved.

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Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
E-mail address: bego.barrios@uam.es