The phase-space structure of the Klein–Gordon field

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Abstract

The formalism based on the equal-time Wigner function of the two-point correlation function for a quantized Klein–Gordon field is presented. The notion of the gauge-invariant Wigner transform is introduced and equations for the corresponding phase-space calculus are formulated. The equations of motion governing the Wigner function of the Klein–Gordon field are derived. It is shown that they lead to a relativistic transport equation with electric and magnetic forces and quantum corrections. The governing equations are much simpler than in the fermionic case which has been treated earlier. In addition the newly developed formalism is applied towards the description of spontaneous symmetry breakdown.
1 Introduction

We consider the time evolution of quantum fields interacting with a classical (external) field. A useful example of such a system is the particle creation by strong but slowly varying electromagnetic fields. In this case the particle fields require full quantum treatment while the electromagnetic field may be considered classically. It is sometimes necessary to treat these classical fields as fully dynamical objects rather than as a fixed background. The study of the back reaction problem is such a case. Quite recently this type of dynamical description has been successfully applied [1] towards the description of the electron–positron vacuum in QED. Eventually some of these approaches may serve as crude models for the formation stage of the quark-gluon plasma.

To achieve their goals the authors of [1] had to simplify the notion of the vacuum by truncating the infinite hierarchy of correlation functions. While the full description of a vacuum state at a given instant of time is provided by the set of all possible $n$-point equal-time correlation functions of the quantum field, two-point functions are sufficient to construct the electric current and therefore to describe the influence of the electron–positron vacuum on a classical electromagnetic field. The authors of [1] make use of the Fourier-transformed two-point equal-time correlation function (an analog of the Wigner function) for fermionic fields. The resulting equations describe the evolution of the vacuum from a phase-space point of view. Our goal here is to extend the basic results presented in the cited work to the case of scalar electrodynamics. The relative simplicity of the scalar result helps in the interpretation of the physical features of the interacting system.

The use of the Wigner functions to the description of the QFT vacuum is not new [3, 4, 5]. Neither is the use of the two-point correlation function to describe the interaction with a classical but dynamical electromagnetic field [6, 7, 8]. The main difference is that we use equal-time correlation functions and do not perform the Fourier transform with respect to the time variable. (The equal-time approach is also used in Ref. [9].) The equation of motion is therefore a true evolution equation that can be numerically solved as an initial-value problem.

In section 2 we introduce the gauge-invariant Wigner transform and show how the noncommutative operator algebra of quantum mechanics is implemented in the space of phase-space transforms. The Wigner transform of the equal-time two-point correlation function of the Klein–Gordon field in
Feshbach-Villars representation is defined and discussed in section 3, which will, incidentally, illustrate the power of the methods developed in section 2. Finally, in section 4 we apply this formalism to spontaneous symmetry breakdown.

2 Wigner function calculus

The Wigner transform associates a quasiprobability distribution in classical phase space to every quantum-mechanical density matrix in such a way that quantum-mechanical averages can be computed like classical statistical averages. The equation of motion of the quasiprobability distribution derives from the quantum-mechanical Hamiltonian and corresponds to the classical Liouville equation with quantum corrections which can be given as an expansion in \( \hbar \). Since the Liouville equations is easier to solve than the full quantum treatment, the Wigner function provides a tool to simplify calculations when semiclassical approximations are valid.

Similarly, the two-point function of a quantum field theory can be interpreted as the matrix element of a density matrix. The resulting phase-space quasiprobability distributions reflect the many-particle content of the quantum field theory. Their equation of motion follows analogously from the operator equation of motion, i.e., the wave equation, and the corresponding Hamiltonian.

Mathematically, the Wigner transform provides a realization of the abstract Heisenberg operator algebra of quantum mechanics on the space of functions over phase space, thus translating every operator equation—like the Heisenberg equation of motion—into a differential-equation on phase space—like the Liouville equation. Especially, it induces a noncommutative multiplication on this function space that is characteristic of the quantum-mechanical content of the theory. Once the representation of the basic operators and this multiplication is known, every operator equation can easily and without explicit calculation be translated into a phase-space equation.

In the following, we discuss this calculus for a Wigner function exhibiting local U(1)-gauge invariance. The notational conventions are from the review article [10], where the gauge issue was not considered. We set \( c = 1 \) but keep \( \hbar \) in this section since expansions around the classical limit are frequent.
2.1 Definition and elementary properties

Let us consider a $D$-dimensional configuration space of vectors $q \in \mathbb{R}^D$ and the corresponding Hilbert space $H$, spanned by basis vectors $\langle q \rangle$. Suppose that the operator $\hat{A}$ transforms covariantly under a local $\text{U}(1)$-gauge transformation $\Lambda(q)$

$$
\langle q | \hat{A} | q' \rangle \rightarrow e^{ie\{\Lambda(q)-\Lambda(q')\}} \langle q | \hat{A} | q' \rangle.
$$

(1)

The corresponding transformation of the vector potential $A(q) \in \mathbb{R}^D$ is

$$
A(q) \rightarrow A(q) + \frac{\partial \Lambda}{\partial q}(q).
$$

(2)

Every operator $\hat{A}$ may be associated with a function in phase-space $(q,p) \in \mathbb{R}^D \times \mathbb{R}^D$

$$
P_{\hat{A}}(q,p) = \int d^Dy \langle q - \frac{1}{2}y | \hat{A} | q + \frac{1}{2}y \rangle \exp \left( \frac{i}{\hbar} p \cdot y + \frac{ie}{\hbar} \int_{q-y/2}^{q+y/2} A(x) \cdot dx \right).
$$

(3)

This is an obvious generalization of the well-known Wigner transformation, and we will use this name here. Indeed, $P_{\hat{\rho}}(q,p)$ ($\hat{\rho}$ being a density matrix) is the usual Wigner phase-space distribution. The line integral in Eq. (3) is taken over a straight line, i.e.,

$$
\int_{q-y/2}^{q+y/2} A(x) \cdot dx = \int_{-1/2}^{+1/2} d\lambda y \cdot A(q + \lambda y)
$$

(4)

where the dot indicates the scalar product in $D$ dimensions. This factor makes the resulting function $P_{\hat{A}}(q,p)$ a gauge-invariant realization of the abstract operator $\hat{A}$ in phase space. Every operator equation can thus be translated into a phase-space equation. In the following we discuss the required formalism and apply it to the Heisenberg equation of motion for a density matrix.

The inverse of the transformation (3) is

$$
\langle q - y | \hat{A} | q + y \rangle = \int \frac{d^Dk}{(2\pi\hbar)^D} P_{\hat{A}}(q,k) \exp \left( -2\frac{i}{\hbar} k \cdot y - \frac{ie}{\hbar} \int_{q-y}^{q+y} A(x) \cdot dx \right).
$$

(5)

The matrix representation and the phase-space representation of the operator $\hat{A}$ are therefore equivalent.
The Wigner transform (3) conserves the linear structure of the underlying operator algebra

\[ P_{A+B}(q,p) = P_A(q,p) + P_B(q,p), \quad P_{\alpha A} = \alpha P_A. \] (6)

If the operators carry internal degrees of freedom, the Wigner function will acquire a matrix structure, and the second equation in (6) holds for constant matrices \( \alpha \). Because of noncommutativity, multiplication of operators translates into a more complex phase-space operation considered below.

The Wigner transforms of the position operator \( \hat{q} \) and the momentum operator \( \hat{p} \) can be determined directly from their matrix elements,

\[
\langle q|\hat{q}|q'\rangle = \delta(q' - q), \quad \langle q|\hat{p}|q'\rangle = -i\hbar \delta(q' - q) \frac{\partial}{\partial(q' - q)}. \] (7)

For the position operator, insertion into (3) yields

\[ P_{\hat{q}}(q,p) = q. \] (8)

This relation may be generalized to analytic functions \( W(q) \): \( P_{W(q)}(q,p) = W(q) \) by a Taylor expansion. In the case of the momentum operator, the derivative in (7) gives an additional term from its action on the line integral, which yields

\[ P_{\hat{p}}(q,p) = p + eA(q). \] (9)

Therefore, the kinetic momentum \( \hat{\pi} = \hat{p} - eA(q) \) has the Wigner transform

\[ P_{\hat{\pi}}(q,p) = p. \] (10)

This allows us to interpret the coordinate \( p \) as a true (i.e., kinetic) momentum.

### 2.2 Noncommutative multiplication in phase space

The noncommutative multiplication of the operator algebra translates into a ‘noncommutative multiplication’ of their Wigner transforms. Consider

\[
P_{\hat{A}\hat{B}}(q,p) = \int d^Dy \langle q - \frac{1}{2}y|\hat{A}\hat{B}|q + \frac{1}{2}y\rangle \\
\times \exp \left( \frac{i}{\hbar} p \cdot y + \frac{ie}{\hbar} \int_{q-y/2}^{q+y/2} A(x) \cdot dx \right). \] (11)
Inserting a complete set of intermediate states and using (5) gives us—after some shifts of integration variables—the multiplication formula in its integral form:

\[ P^A \hat{A} \hat{B}(q,p) = (\pi \hbar)^{-2D} \int d^D y \, d^D y' \, d^D k \, d^D k' \]

\[ \times P_A(q + y, p + k') P_B(q + y', p + k) \]

\[ \times \exp \left\{ \frac{2i}{\hbar} (yk - y'k') \right\} \]

\[ \times \exp \left\{ \frac{ie}{\hbar} \left( \int_{q+y-y'}^{q+y' + y'} dDk' \right) \cdot A(x) \cdot d\bar{x} \right\} \].

With \( A(x) \) put to zero one gets the multiplication formula given by \([10]\).

Let us now consider the usual case \( D = 3 \) with \( A(x) \) being a three-dimensional vector potential. The closed-line integral in (12) is related to the surface integral over the magnetic field \( B(q) \):

\[ \int_{\partial \Delta} A(x) \cdot d\bar{x} = \int_{\Delta} d^2 x \, n \cdot B(x), \quad B(q) = \text{curl} \, A(q), \] (13)

where \( \Delta \) represents the integration region (here a triangle) and \( \partial \Delta \) its border, \( n \) is the normal to the triangle plane, \( d\bar{x} \) the line element of the border, and \( d^2 x \) the area element of the triangle. We parametrize the surface of the triangle according to

\[ x = q + \lambda_1 y + \lambda_2 y', \quad -1 \leq \lambda_1, \lambda_2 \leq 1, \quad \lambda_1 + \lambda_2 \geq 0. \] (14)

The surface integral in question may be rewritten:

\[ \int_{\Delta} d^2 x \, n \cdot \text{curl} \, A(x) = \int_{\lambda_1}^{1} d\lambda_1 \int_{-\lambda_1}^{1} d\lambda_2 (y \times y') \cdot B(q + \lambda_1 y + \lambda_2 y'). \] (15)

Inserting the above formulae into (12) we get

\[ P_{AB}(q,p) = (\pi \hbar)^{-6} \int d^3 y \, d^3 y' \, d^3 k \, d^3 k' \, P_A(q + y, p + k') \]

\[ \times \exp \left\{ \frac{2i}{\hbar} (yk - y'k') + \frac{ie}{\hbar} (y \times y') \cdot \int_{\Delta} d\lambda_1 d\lambda_2 B(q + \lambda_1 y + \lambda_2 y') \right\} \]

\[ \times \exp \left( y' \cdot \frac{\partial}{\partial \bar{q}} + k \cdot \frac{\partial}{\partial \bar{p}} \right) \]

\[ \times P_B(\bar{q}, \bar{p}) \bigg|_{\bar{q}=q, \bar{p}=p}, \] (16)
where we made use of the ‘translation’ formula

\[
P_B(q + y', p + k') = \exp \left( y' \cdot \frac{\partial}{\partial \tilde{q}} + k \cdot \frac{\partial}{\partial \tilde{p}} \right) P_B(\tilde{q}, \tilde{p}) \Bigg|_{\tilde{q}=q, \tilde{p}=p}. \tag{17}\]

Now the integration over \(k\) can be performed, yielding a (formal) delta function

\[
\delta \left( y + \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{p}} \right). \tag{18}\]

Subsequent integration over \(dy\) results in the expression:

\[
P_{AB}(q, p) = \pi^3 \int d^3y' d^3k' P_A \left( q - \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{p}}, p + k' \right) \times \exp \left\{ ie \epsilon_{ijk} \frac{1}{2i} \frac{\partial}{\partial \tilde{p}_i} y'_j \int_{\Delta} d\lambda_1 d\lambda_2 B_k \left( q - \lambda_1 \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{p}} + \lambda_2 y' \right) \right\} \times \exp \left\{ - \frac{2i}{\hbar} y' \cdot \left( k' - \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{q}} \right) \right\} P_B(\tilde{q}, \tilde{p}) \Bigg|_{\tilde{q}=q, \tilde{p}=p}. \tag{19}\]

We can now get rid of the \(y'\) in the first exponential by substituting it by

\[
y' \rightarrow - \frac{\hbar}{2i} \frac{\partial}{\partial k'}, \tag{20}\]

where the derivative acts on the last exponential. This leads to the expression

\[
P_{AB}(q, p) = \int d^3k' P_A \left( q - \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{p}}, p + k' \right) \times \exp \left\{ - \frac{e\hbar}{4i} \epsilon_{ijk} \frac{\partial}{\partial \tilde{p}_i} \frac{\partial}{\partial k'_j} \int_{\Delta} d\lambda_1 d\lambda_2 B_k \left( q - \lambda_1 \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{p}} - \lambda_2 \frac{\hbar}{2i} \frac{\partial}{\partial k'} \right) \right\} \times \delta \left( k' - \frac{1}{2i} \frac{\partial}{\partial \tilde{q}} \right) P_B(\tilde{q}, \tilde{p}) \Bigg|_{\tilde{q}=q, \tilde{p}=p}. \tag{21}\]

After partial integration, this operator acts on \(P_A\) and can be replaced by \(-\hat{\nabla} / \partial p\), where the arrow indicates that the momentum derivative acts on the function to its left yielding the final expression

\[
P_{AB}(q, p) = P_A \left( q - \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{p}}, p + \frac{\hbar}{2i} \frac{\partial}{\partial \tilde{q}} \right)\]
In this way the left multiplication with an operator \( A \) was translated into a phase-space operation. For the right multiplication we get

\[
P_{\hat{B} \hat{A}}(q,p) = P_{\hat{A}} \left( q + \frac{\hbar}{2i} \frac{\partial}{\partial p}, p - \frac{\hbar}{2i} \frac{\partial}{\partial q} \right)
\]

\[
\times \exp \left\{ \frac{\hbar}{4i} \epsilon_{ijk} \frac{\partial}{\partial \bar{p}_i} \frac{\partial}{\partial p_j} \int_{-1}^{1} d\lambda_1 \int_{-\lambda_1}^{1} d\lambda_2 \mathcal{B}_k \left( q - \lambda_1 \frac{\hbar}{2i} \frac{\partial}{\partial \bar{p}} + \lambda_2 \frac{\hbar}{2i} \frac{\partial}{\partial p} \right) \right\}
\]

\[
\times P_{\hat{B}}(\tilde{q},\tilde{p})|_{\tilde{q}=q,\tilde{p}=p}.
\] (23)

### 2.3 Kinetic energy and the flow term

In the following we will derive equations of motion for the Wigner transform of the density matrix \( \hat{\rho} \). The time evolution of the latter is governed by the Heisenberg equation of motion

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}].
\] (24)

Therefore we have to find the phase-space analog of \([\hat{H}, \hat{\rho}]\). We do not assume any special properties of \( \hat{\rho} \), and the following would apply to a general Heisenberg operator that has no explicit time dependence. (Allowing for an explicit time dependence does not pose any problem.)

We begin with the kinetic momentum \( \hat{\pi} = \hat{p} - eA(\tilde{q}) \). Its Wigner transform is \( P_k(q,p) = p \). In this case only first-order derivatives \( \frac{\partial}{\partial p} \) in (22) and (23) lead to a nonzero result. Therefore only the first two terms in the Taylor expansion of the exponential are to be left

\[
\exp \left\{ \frac{\hbar}{4i} \epsilon_{ijk} \frac{\partial}{\partial \bar{p}_i} \frac{\partial}{\partial p_j} \int_{-1}^{1} d\lambda_1 \int_{-\lambda_1}^{1} d\lambda_2 \mathcal{B}_k \left( q + \lambda_1 \frac{\hbar}{2i} \frac{\partial}{\partial \bar{p}} + \lambda_2 \frac{\hbar}{2i} \frac{\partial}{\partial p} \right) \right\}
\]

\[
\rightarrow 1 + \frac{\hbar}{4i} \epsilon_{ijk} \frac{\partial}{\partial \bar{p}_i} \frac{\partial}{\partial p_j} \int_{-1}^{1} d\lambda_1 \int_{-\lambda_1}^{1} d\lambda_2 \mathcal{B}_k \left( q + \lambda_1 \frac{\hbar}{2i} \frac{\partial}{\partial \bar{p}} \right)
\] (25)
are to be left. Employing this we find

\[
P_{\hat{\pi}\hat{\rho}}(q,p) = \left\{ p + \frac{\hbar}{2i} \frac{\partial}{\partial q} - \frac{i e \hbar}{4} \int_{-1/2}^{1/2} d\lambda \int_{-\lambda}^{\lambda} d\lambda_2 B \left( q - \frac{\lambda_1}{2i} \frac{\partial}{\partial p} \right) \times \frac{\partial}{\partial p} \right\} P_{\hat{\rho}}(q,p),
\]

(26)

\[
P_{\hat{\rho}\hat{\pi}}(q,p) = \left\{ p - \frac{\hbar}{2i} \frac{\partial}{\partial q} + \frac{i e \hbar}{4} \int_{-1/2}^{1/2} d\lambda \int_{-\lambda}^{\lambda} d\lambda_2 B \left( q - \frac{\lambda_1}{2i} \frac{\partial}{\partial p} \right) \times \frac{\partial}{\partial p} \right\} P_{\hat{\rho}}(q,p).
\]

(27)

From this one easily obtains the following formulae for the commutator and anticommutator of \( \hat{\pi} \) and \( \hat{\rho} \):

\[
P_{[\hat{\pi},\hat{\rho}]}(q,p) = \left\{ -i \hbar \frac{\partial}{\partial q} - \frac{i e \hbar}{2} \int_{-1/2}^{1/2} d\lambda B \left( q - \frac{\lambda_1}{2i} \frac{\partial}{\partial p} \right) \times \frac{\partial}{\partial p} \right\} P_{\hat{\rho}}(q,p),
\]

(28)

\[
P_{\{\hat{\pi},\hat{\rho}\}}(q,p) = \left\{ 2p - \frac{i e \hbar}{2} \int_{-1/2}^{1/2} d\lambda \lambda B \left( q + i \hbar \lambda \frac{\partial}{\partial p} \right) \times \frac{\partial}{\partial p} \right\} P_{\hat{\rho}}(q,p).
\]

(29)

It is convenient to define two phase-space operators \( \hat{D} \) and \( \hat{P} \):

\[
\hat{D} = \left\{ \frac{\partial}{\partial q} + e \int_{-1/2}^{1/2} d\lambda B \left( q + i \hbar \lambda \frac{\partial}{\partial p} \right) \times \frac{\partial}{\partial p} \right\}
\]

\[
\approx \frac{\partial}{\partial q} + e B(q) \times \frac{\partial}{\partial p},
\]

(30)

\[
\hat{P} = \left\{ p - i e \hbar \int_{-1/2}^{1/2} d\lambda \lambda B \left( q + i \hbar \lambda \frac{\partial}{\partial p} \right) \times \frac{\partial}{\partial p} \right\} \approx p,
\]

(31)

where the expansion to lowest order in \( \hbar \) is given. These operators act on the Wigner transforms in a nonlocal manner. They are generalizations of the \( q \)-derivative and the \( p \)-multiplication in phase-space and their use simplifies our notation:

\[
P_{[\hat{\pi},\hat{\rho}]}(q,p) = -i \hbar \hat{D} P_{\hat{\rho}}(q,p), \quad P_{\{\hat{\pi},\hat{\rho}\}}(q,p) = 2 \hat{P} P_{\hat{\rho}}(q,p).
\]

(32)

We illustrate his technique using a simple example. Let us consider the Hamiltonian of a nonrelativistic particle coupled to an external electromagnetic field

\[
\hat{H} = \hat{H}_{\text{kin}} + e A(q),
\]

(33)

\[
\hat{H}_{\text{kin}} = \frac{\hat{p}^2 - e A(q)}{2m} = \frac{\hat{\pi}^2}{2m} \rightarrow P_{\hat{H}_{\text{kin}}}(q,p) = \frac{p^2}{2m}.
\]

(34)
where $A^0(q)$ is the time-like part of the electromagnetic four-potential.

To study the time evolution of the Wigner function $P_\rho(q,p)$, we must evaluate the commutator and, in the case of a more complex Hamiltonian with matrix structure, the anticommutator. Using the relations found above this is straightforward. We rewrite both as

$$[\hat{\pi}^2, \hat{\rho}] = \{\hat{\pi}, [\hat{\pi}, \hat{\rho}]\},$$
(35)

$$\{\hat{\pi}^2, \hat{\rho}\} = \frac{1}{2} \{\pi, \{\pi, \rho\}\} + \frac{1}{2} [\pi, [\pi, \rho]].$$
(36)

The double (anti)commutators correspond to a repeated application of the operators (30) and (31):

$$P[\hat{\pi}^2/2m, \hat{\rho}](q,p) = -i\hbar \frac{\hat{P} \cdot \hat{D}}{m} P_\rho(q,p),$$
(37)

$$P\{\hat{\pi}^2/2m, \hat{\rho}\}(q,p) = 2 \left( \hat{P}^2 - \frac{\hbar^2}{4} \hat{D}^2 \right) P_\rho(q,p).$$
(38)

In the expression for the commutator we note the classical Liouville flow term with the minimal coupling to a magnetic field,

$$\frac{\hat{P} \cdot \hat{D}}{m} \approx \frac{p}{m} \left( \frac{\partial}{\partial q} + eB(q) \times \frac{\partial}{\partial p} \right),$$
(39)

where only the lowest order term in $\hbar$ was left.

The commutator involving a scalar potential $U(q)$ can also be rewritten as a differential operator in phase space

$$P[U, \hat{\rho}](q,p) = i \int_{-1/2}^{1/2} \frac{\partial U}{\partial q} \left( q + i\hbar \lambda \frac{\partial}{\partial p} \right) P_\rho(q,p) \approx i\hbar \frac{\partial U}{\partial q} P_\rho(q,p) + O(\hbar^2).$$
(40)

The corresponding expression for the anticommutator is

$$P[U, \hat{\rho}](q,p) = U \left( q - \frac{\hbar}{2i} \frac{\partial}{\partial p} \right) P_\rho(q,p) + U \left( q + \frac{\hbar}{2i} \frac{\partial}{\partial p} \right) P_\rho(q,p) \approx U(q) P_\rho(q,p) + O(\hbar^2).$$
(41)

The time evolution of a Wigner function is not only determined by the change of the underlying operator, which is usually given by the Heisenberg
equation of motion, but also by the change in the line integral due to a time-dependent vector potential. The additional term can be rewritten in a manner similar to that used above:

$$\frac{\partial}{\partial t}P_{\hat{\rho}}(q,p) = P_{\partial t}\hat{\rho} + e \int_{-1/2}^{1/2} \partial_t A \left( q + i\hbar \lambda \frac{\partial}{\partial p} \right) \cdot \frac{\partial}{\partial p} P_{\hat{\rho}}. \quad (42)$$

Inserting the Heisenberg equation of motion (24) with the Hamilton operator of Eq. (33), we obtain

$$D_t P_{\hat{\rho}}(q,p) = -\hat{P} \cdot \hat{D} P_{\hat{\rho}}(q,p) \quad (43)$$

with the generalized time derivative

$$D_t = \frac{\partial}{\partial t} + \int_{-1/2}^{1/2} eE \left( q + i\hbar \lambda \frac{\partial}{\partial p} \right) \cdot \frac{\partial}{\partial p} \approx \frac{\partial}{\partial t} + eE(q) \cdot \frac{\partial}{\partial p} + \cdots \quad (44)$$

The derivatives of $A$ resulting from (41) and (42) were combined to yield an electric field

$$E(q) = -\frac{\partial A}{\partial t} - \frac{\partial A^0}{\partial q}. \quad (45)$$

To lowest order in $\hbar$ this is the nonrelativistic Liouville equation of a charged particle in an external electromagnetic field.

### 2.4 Calculation of observables

The Wigner function is normalized to give

$$\int d^D q \frac{d^D p}{(2\pi \hbar)^D} P_A(q,p) = \int d^D p \left( q | \hat{A} | q \right) \equiv \text{Tr} \hat{A}. \quad (46)$$

In the case of a density matrix it is just one. Observables are computed by taking the trace over the product of density matrix and observable operator: Using the multiplication formula (12) one can prove that

$$\langle \hat{O} \rangle \equiv \text{Tr} \left( \hat{O} \hat{\rho} \right) = \int d^D q \frac{d^D p}{(2\pi \hbar)^D} \text{Tr} \left( P_{\hat{O}}(q,p) P_{\hat{\rho}}(q,p) \right), \quad (47)$$

where Tr is the trace over the matrix structure of $P(q,p)$ (if necessary). This formula allows us to compute any observable from the phase-space representation of the density matrix; it further suggests the identification of the phase-space function of an observable operator and the classical observable function.
3 The Wigner function of the Klein–Gordon field

We will now treat the equal-time Wigner function of a Klein–Gordon field in an external electromagnetic field. Instead of the matrix elements of a density matrix, the object to consider will be the symmetrized two-point correlation function. Its properties are very similar to a quantum-statistical density matrix in that it describes a many-particle system. All single-particle observables can be computed from the two-point function, and thus from its Wigner transform.

The time evolution of the two-point function is derived from the Klein–Gordon equation for the Heisenberg field operators. Similarly to the nonrelativistic problem treated above the Klein-Gordon equation can be translated into a differential equation in phase-space with its leading contribution being just the classical relativistic flow equation. Two complications arise: The Klein–Gordon equation contains a second-order derivative in time. Its translation into phase-space would also contain a second-order derivative which does not lead to a sensible evolution equation. The reason for this is that the Klein–Gordon field has an internal charge degree of freedom which can be made explicit in the two-component formalism of Feshbach and Villars [11]. In this representation, the time evolution is first-order in time and governed by a nonhermitian Hamilton operator with a $2 \times 2$-matrix structure. As a second complication in a fully interacting case (scalar electrodynamics) the equation of motion of the two-point function involves higher correlation functions. Here we consider the simpler problem of the interaction with a classical albeit dynamical electromagnetic field in which case the equation of motion for the correlation function is closed and the electromagnetic field is governed by the Maxwell equations.

3.1 Definition

The Klein–Gordon field which obeys the wave equation ($\hbar = c = 1$)

$$\left((\partial_{\mu} - ieA_{\mu})(\partial^{\mu} - ieA^{\mu}) + m^2\right) \phi(x) = 0,$$

(48)
can be expressed by a two-component wave function

$$\Phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$  \hfill (49)

transforming regularly under U(1)-gauge transformations where

$$\psi = \frac{1}{2} \left( \phi + \frac{i}{m} \frac{\partial \phi}{\partial t} - eA^0 \right), \quad \chi = \frac{1}{2} \left( \phi - \frac{i}{m} \frac{\partial \phi}{\partial t} + eA^0 \right).$$  \hfill (50)

The equation of motion for the two-component field operator is:

$$i\frac{\partial}{\partial t} \Phi = \begin{pmatrix} \frac{1}{2m} \left( -i \frac{\partial}{\partial q} - eA \right)^2 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + eA^0 \mathbb{I} \end{pmatrix} \Phi.$$  \hfill (51)

The advantage of this equation over (48) is that the time derivative appears in the first-order giving rise to a Schrödinger type evolution equation. The right-hand side can then be interpreted as a Hamiltonian operator acting on $\Phi$ and the formalism of the previous section can be utilized to derive a phase-space equation. The fact that the resulting Hamiltonian operator is not hermitian does not cause any severe problems, since the nonhermiticity is confined to the matrix structure.

In the following, we consider the time evolution of the symmetrized correlation function

$$C_{\alpha\beta}^+(q, q'; t) = \langle \Omega \left| \{ \Phi_\alpha(q, t), \Phi_\beta^+(q', t) \} \right| \Omega \rangle$$  \hfill (52)

which is a $2 \times 2$ matrix. The matrix element is taken in the Hilbert space of quantum field theory with respect to some state $\Omega$. Later on this state will be identified with the vacuum. We may consider $C^+(q, q')$ to be the matrix element of some density matrix $\hat{\rho}$ in the abstract Hilbert space spanned by position eigenvectors $\langle q |$

\[
\langle q | \hat{\rho} | q' \rangle = C^+(q, q').
\]  \hfill (53)

By Eq. (51), the density matrix $\hat{\rho}$ evolves under the Heisenberg equation of motion

$$i\frac{\partial \hat{\rho}}{\partial t} = \hat{H} - \hat{\rho} \hat{H}^+$$  \hfill (54)
with the two-component nonhermitian Hamiltonian

\[
\hat{H} = \frac{(\hat{p} - eA)^2}{2m} \left( \begin{array}{cc}
1 & 1 \\
-1 & -1 \\
\end{array} \right) + m \left( \begin{array}{cc}
1 & 0 \\
0 & -1 \\
\end{array} \right) + eA^0 \mathbf{1},
\]

(55)

where \( \hat{p} \) denotes momentum operator \(-i\partial/\partial q\).

The energy, momentum, charge, and current of the Klein–Gordon field are bilinear forms in the field operators that can be computed from the two-point correlation function or from the Wigner function. For example, the energy and momentum of free particles are given by the elements of the energy-momentum tensor,

\[
\Theta^{00} = \frac{1}{2m} \left( \frac{\partial \phi}{\partial t} \right)^* \left( \frac{\partial \phi}{\partial t} \right) + (\nabla \phi^*) (\nabla \phi) + m^2 \phi^* \phi
\]

(56)

\[
\Theta^{0i} = \frac{e}{2m} \mathbf{1} (\nabla \phi^*) \left( \begin{array}{c}
1 \\
0 \\
1 \\
-1 \\
\end{array} \right) \mathbf{1} (\nabla \phi) + m \left( \begin{array}{cc}
1 & 1 \\
0 & -1 \\
\end{array} \right) \mathbf{1} (\nabla \phi)
\]

(57)

The sum of these quantities and the corresponding quantities of the electromagnetic field are conserved. Charge and current density operators read

\[
\rho_e = \frac{ie}{2m} \mathbf{1} \left( \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) - \frac{e^2 A^0}{m} \phi^* \phi = \Phi^+ \left( \begin{array}{c}
1 \\
0 \\
-1 \\
\end{array} \right) \mathbf{1} (\nabla \phi^*) \left( \begin{array}{c}
1 \\
0 \\
1 \\
-1 \\
\end{array} \right) \mathbf{1} (\nabla \phi)
\]

(60)

\[
j = \frac{e}{2ma} \mathbf{1} (\phi^* \nabla \phi - \phi \nabla \phi^*) - \frac{e^2 A}{m} \phi^* \phi = \Phi^+ \left( \begin{array}{c}
e \\
0 \\
\frac{e}{m} \hat{p}
\end{array} \right) \mathbf{1} (\nabla \phi)
\]

(61)

The expressions in brackets constitute the observable operators in the sense of Eq. (47).

### 3.2 The Wigner function and its time evolution

The Wigner transform associates a matrix-valued phase-space function \( P(q, p) \) with the density matrix \( \hat{\rho} \). Its evolution is governed by Eq. (54) where
the nonhermiticity of the Hamiltonian (55) has been taken into account. This may be translated into a phase-space evolution equation. Defining

\[ a = (\sigma_3 + i\sigma_2) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad b = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

and using (42) one obtains:

\begin{align*}
P_{\hat{H}_0 - \hat{H}_+}(q, p) &= \frac{1}{4m} \left( a \cdot P_{\hat{p}^2, \hat{\rho}}(q, p) + P_{\hat{p}^2, \hat{\rho}}(q, p) \cdot a^+ \right) \\
&\quad + \frac{1}{4m} \left( a \cdot P_{\hat{p}^2, \hat{\rho}}(q, p) - P_{\hat{p}^2, \hat{\rho}}(q, p) \cdot a^+ \right) \\
&\quad + m \left( b \cdot P(q, p) - P(q, p) \cdot b \right).
\end{align*}  

(63)

Following the general procedure outlined in the previous section we get the phase-space equation of motion:

\begin{align*}
i\hat{D}_t P(q, p) &= -i \frac{\hat{P}}{2m} \cdot \hat{D} \left( a \cdot P(q, p) + P(q, p) \cdot a^+ \right) \\
&\quad + \frac{1}{m} \left( \hat{P}^2 - \frac{1}{4} \hat{D}^2 \right) \left( a \cdot P(q, p) - P(q, p) \cdot a^+ \right) \\
&\quad + m \left( b \cdot P(q, p) - P(q, p) \cdot b \right).
\end{align*}  

(64)

It is convenient to expand \( P(q, p) \) (which is a 2 \times 2 matrix) over the Pauli matrices \( \sigma_i \) and the unit matrix, assembling the coefficients into a four-component object \( \vec{f} \),

\[ P(q, p) = f_3(q, p) \mathbb{1} + \sum_{i=1}^{3} f_{3-i}(q, p, t) \sigma_i. \]  

(65)

The equations of motion take the form

\begin{align*}
\hat{D}_t f_0 &= -\frac{\hat{P}}{m} \cdot \hat{D} (f_2 + f_3), \\
\hat{D}_t f_1 &= -\left( \frac{1}{4m} \hat{D}^2 - \frac{1}{m} \hat{P}^2 \right) (f_2 + f_3) + 2mf_2, \\
\hat{D}_t f_2 &= \left( \frac{1}{4m} \hat{D}^2 - \frac{\hat{P}^2}{m} \right) f_1 + \frac{\hat{P}}{m} \cdot \hat{D} f_0 - 2mf_1, \\
\hat{D}_t f_3 &= -\left( \frac{1}{4m} \hat{D}^2 - \frac{\hat{P}^2}{m} \right) f_1 - \frac{\hat{P}}{m} \cdot \hat{D} f_0.
\end{align*}  

(66)
The components of \( f \) may be interpreted as the phase-space densities of simple observables:

- **charge**: \( e f_0 \), \( \text{(67)} \)
- **energy**: \( \frac{p^2}{2m} (f_2 + f_3) + m f_3 \), \( \text{(68)} \)
- **current**: \( \frac{p}{m} e (f_2 + f_3) \), \( \text{(69)} \)
- **momentum**: \( p(f_0 - f_1) \). \( \text{(70)} \)

It may be interesting to note the conservation law that follows from the preservation of the norm

\[
\frac{\partial}{\partial t} \int \frac{d^D q \, d^D p}{(2\pi\hbar)^3} (f_0^2 - f_1^2 - f_2^2 + f_3^2) = 0. \quad \text{(71)}
\]

The Feshbach-Villars Hamiltonian \( \text{(55)} \) does not give rise to unitary evolution but to \( \tau \)-unitary evolution \( \text{[12]} \) so that two \( f \)s come in with the opposite sign. The conservation laws for energy-momentum and currents may be easily derived and we do not state them here.

The free vacuum solution can be derived from the explicit expression for the correlation function \( C^+ \):

\[
\begin{align*}
    f_0 &= 0, \quad f_1 = 0, \\
    f_2 + f_3 &= \frac{m}{E_p}, \quad f_3 - f_2 = \frac{E_p}{m}.
\end{align*} \quad \text{(72)}
\]

It is the stationary solution of \( \text{(66)} \). The vacuum has a constant energy density (which must be subtracted when computing observables) but no charge.

### 3.3 Diagonalization of local oscillations

For a further discussion of these equations we consider the case of a electric field slowly varying in space. This will allow us to drop the higher-order corrections in \( \hat{P} \) and \( \hat{D} \). Nevertheless, the equation of motion \( \text{(60)} \) contains self-couplings of the components that make its solutions nonstationary even in the absence of a gradient. The resulting oscillations are related to the internal charge degree of freedom. The eigenfrequencies of these oscillations
are (in the absence of an electromagnetic field) just \( \pm 2E_p \), showing clearly that they are related to relativistic effects and therefore to the existence of antiparticles. Let us expand the Wigner function \( P(q, p) \) into eigenmodes \( \tilde{P}_i \) of these oscillations:

\[
P(q, p) = \sum_{i=1}^{4} \tilde{f}_i \tilde{P}_i, \tag{73}
\]

with

\[
\begin{align*}
\tilde{P}_1 &= \frac{1}{4} \left( \begin{array}{cc}
m/E_p + E_p/m & m/E_p - E_p/m \\
m/E_p - E_p/m & m/E_p + E_p/m \end{array} \right), \\
\tilde{P}_2 &= \frac{1}{2} \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \end{array} \right), \\
\tilde{P}_{3/4} &= \frac{1}{4} \left( \begin{array}{cc}
m/E_p - E_p/m & m/E_p + E_p/m \pm 1/2 \\
m/E_p - E_p/m \pm 1/2 & m/E_p - E_p/m \end{array} \right). \tag{74}
\end{align*}
\]

The first two matrices are connected with the eigenfrequency zero while the last two correspond to \( \pm 2E_p \). Each of these matrices can be constructed from plane-wave solutions of the Klein–Gordon equation. Note that the basis matrices \( \tilde{P}_i \) depend on the energy \( E_p \) and are therefore different in different parts of the phase space. A particle whose inner degree of freedom is oscillating in an eigenmode of these equations will be out of phase when it gains or loses momentum.

With the approximation

\[
p^2 + m^2 + \frac{1}{4} \frac{\partial^2}{\partial^2 q} \approx p^2 + m^2, \tag{75}
\]

which is justified in the semiclassical case, the equations of motion take on the form

\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{f}_1 + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} &= -\frac{p}{E_p} \cdot \frac{\partial}{\partial q} \tilde{f}_2 + e \frac{\mathcal{E}(q) \cdot p}{E_p^2} (\tilde{f}_3 + \tilde{f}_4), \\
\frac{\partial}{\partial t} \tilde{f}_2 + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} &= -\frac{p}{E_p} \cdot \frac{\partial}{\partial q} (\tilde{f}_1 + \tilde{f}_3 + \tilde{f}_4), \\
\frac{\partial}{\partial t} \tilde{f}_3 + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} &= -\frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} \tilde{f}_2 \\
\frac{\partial}{\partial t} \tilde{f}_4 + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} &= \frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} \tilde{f}_2
\end{align*}
\]
\[
\left( \frac{\partial}{\partial t} \tilde{f}_4 + e \mathcal{E}(q) \cdot \frac{\partial}{\partial p} \right) = -\frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} \tilde{f}_2 + e \mathcal{E}(q) \cdot \frac{p}{E_p} \frac{\tilde{f}_1}{2} - 2iE_p \tilde{f}_4. \tag{79}
\]

Here \( \tilde{f}_1, \tilde{f}_2 \) and the linear combination \( \tilde{f}_3 + \tilde{f}_4 \) are real, while \( \tilde{f}_3 - \tilde{f}_4 \) is purely imaginary.

These variables have the meaning of the following phase-space densities:

| Charge         | \( e \tilde{f}_2 \)     |
|----------------|--------------------------|
| Energy         | \( E_p \tilde{f}_1 \)   |
| Current        | \( \frac{pe}{E_p} (\tilde{f}_1 + \tilde{f}_3 + \tilde{f}_4) \) |
| Momentum       | \( -p \tilde{f}_2 + ip(\tilde{f}_3 - \tilde{f}_4) \) |

The Wigner transform of a positive- or negative-energy plane-wave solution with the momentum \( p_0 \) of the Klein–Gordon equation is

\[
\tilde{f}_1 = 1 \delta(p - p_0), \quad \tilde{f}_2 = \pm 1 \delta(p - p_0), \quad \tilde{f}_3 = \tilde{f}_4 = 0. \tag{84}
\]

Wigner transforms with nonzero \( \tilde{f}_3 \) and \( \tilde{f}_4 \) involve interferences between positive- and negative-energy modes ("Zitterbewegung"). The correlation function of a free vacuum may be expressed as the superposition of all positive- and negative-energy contributions (with the same sign, as opposed to the Feynman propagator). The free vacuum solution is therefore

\[
\tilde{f}_1 = 1, \quad \tilde{f}_2 = \tilde{f}_3 = \tilde{f}_4 = 0. \tag{85}
\]

In Eq. (78)–(79), the classical limit is exhibited in a more elegant way. The terms on the right-hand sides of Eqs. (78) and (79) are comparable only if the variation of \( f_2 \) is on a scale of \( E_p \), i.e. the Compton wavelength. In a classical limit, where this is not the case, the last two equations decouple since \( \tilde{f}_3 \) and \( \tilde{f}_4 \) will vary so fast that their contribution to \( \tilde{f}_2 \) is averaged out. By introducing two new quantities

\[
\begin{align*}
    f_+ &= \frac{1}{2}(\tilde{f}_1 + \tilde{f}_2), & \tilde{f}_1 = f_+ + f_- \\
    f_- &= \frac{1}{2}(\tilde{f}_1 - \tilde{f}_2), & \tilde{f}_2 = f_+ - f_-,
\end{align*}
\tag{86}
\]
which obey

\[
\frac{\partial}{\partial t} f_+ = -\frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_+, \tag{87}
\]

\[
\frac{\partial}{\partial t} f_- = \frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_-, \tag{88}
\]

one obtains two decoupled Vlasov equations of motion. As can be seen from (80), \( f_+ \) and \( f_- \) carry different charges and thus describe the phase-space densities of positive and negative particles. In this case we have a collisionless relativistic gas consisting of positively and negatively charged particles.

Going back to the equations (76)–(79) we can give some meaning to the individual terms. As we have seen, \( f_1 \) and \( f_2 \) are the particle and charge number density and therefore represent the ‘substantial’ part of the Wigner function. Quantities \( f_3 \) and \( f_4 \) arise from interferences between these two parts. On the right-hand side, the local derivatives involving \( \tilde{f}_1 \) and \( \tilde{f}_2 \) correspond to flow while those involving \( \tilde{f}_3 \) and \( \tilde{f}_4 \) implement local interference (together with the neglected second derivative). Most important, the terms proportional to

\[
e \frac{e E(q) \cdot p}{E_p^2} \tag{89}
\]

stem solely from the momentum dependence of the basis (74). They give rise to pair creation [13]. This will be discussed in detail in a future publication [14].

### 4 Symmetry breaking

As mentioned in section [1], the phase-space description may be useful to study the back reaction problems. We would like to illustrate this point with a relatively simple example. Namely, we now show that the formalism developed above provides a useful tool for the dynamical description of spontaneous symmetry breaking. The Klein–Gordon equation coupled to a scalar potential reads

\[
(i\partial_t)^2 \phi = \left(-i \frac{\partial}{\partial q}\right)^2 \phi + m^2 \phi + U(q, t)\phi. \tag{90}
\]
This leads to the interaction Hamiltonian (in Feshbach-Villars form)

\[ \hat{H}_{\text{int}} = \frac{U(q,t)}{2m} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]  

Using (40) and (41) and expanding as in (65), we get

\[
\frac{\partial}{\partial t} f_0 = -\frac{p}{m} \cdot \frac{\partial}{\partial q} (f_2 + f_3) + \frac{1}{2m} U_-(q) (f_2 + f_3),
\]

\[
\frac{\partial}{\partial t} f_1 = - \left( \frac{1}{4m} \frac{\partial^2}{\partial q^2} - \frac{p^2}{m} \right) (f_2 + f_3) + 2mf_2 + \frac{1}{2m} U_+(q) (f_2 + f_3),
\]

\[
\frac{\partial}{\partial t} f_2 = \left( \frac{1}{4m} \frac{\partial^2}{\partial q^2} - \frac{p^2}{m} \right) f_1 + \frac{p}{m} \cdot \frac{\partial}{\partial q} f_0 - 2mf_1 - \frac{1}{2m} (U_-(q)f_0 + U_+(q)f_1),
\]

\[
\frac{\partial}{\partial t} f_3 = - \left( \frac{1}{4m} \frac{\partial^2}{\partial q^2} - \frac{p^2}{m} \right) f_1 - \frac{p}{m} \cdot \frac{\partial}{\partial q} f_0 + \frac{1}{2m} (U_-(q)f_0 + U_+(q)f_1),
\]

with the potentials \( U_+ \) and \( U_- \) given by

\[
U_+(q,t) = U \left( q + \frac{i}{2} \frac{\partial}{\partial p} , t \right) + U \left( q - \frac{i}{2} \frac{\partial}{\partial p}, t \right) \approx 2U(q,t) + O(h^2),
\]

\[
U_-(q,t) = -i \left[ U \left( q + \frac{i}{2} \frac{\partial}{\partial p} , t \right) - U \left( q - \frac{i}{2} \frac{\partial}{\partial p} , t \right) \right] \approx 0 + O(h).
\]

We shall consider the case when the potential \( U \) is generated by the scalar field itself. To illustrate this point let us consider the charged field \( \phi \) described by the Klein–Gordon equation (90) with

\[
U = -M^2 + \lambda |\phi|^2
\]

where the positive constant \( \lambda \) defines the strength of \( |\phi|^4 \) self-coupling.
It is quite easy to find all time (and coordinate) independent c-number solutions to this equation. For $M^2 < m^2$ the only possibility is
\[
\phi = 0, \quad (99)
\]
while for $M^2 > m^2$ there is an additional set of solutions
\[
|\phi| = \sqrt{(M^2 - m^2)/\lambda}. \quad (100)
\]
The solutions (100) are known as the symmetry-breaking solutions.

It is interesting to calculate the Wigner function corresponding to these classical solutions. The case $\phi = 0$ leads to the null Wigner function and is of no special interest. On the other hand the symmetry breaking solution gives rise to
\[
f_2(q, p) + f_3(q, p) = \frac{(M^2 - m^2)}{\lambda} \delta^3(p), \quad (101)
\]
while all other components of the Wigner function are equal to zero. It is easy to show that the above Wigner function satisfies our basic equations (92)–(95) with the scalar potential given by
\[
U = -M^2 + \lambda \int \frac{d^3p}{(2\pi)^3} \left( f_2 + f_3 \right). \quad (102)
\]
which follows from Eq. (98) and the definitions of $f_2$ and $f_3$. Note that this form of potential introduces self-coupling of a mean-field type.

The appearance of the Dirac delta function reflects the infinite-range correlation of the classical solution (100). The infinite-range correlation (of quantum fields) indicates symmetry breaking.

The classical considerations presented above do not allow for the dynamical description of a symmetry-breaking phase transition. Let us consider a simple—albeit rather abstract—situation in which one is allowed to switch on the $-M^2$ in (98) at a given time $t = t_0$. (We will assume that it was switched on instantly but other possibilities are not excluded.) Initially the quantum field $\phi$ is in its symmetric vacuum state with mass equal to $m$. After the external potential $-M^2$ is switched on this will no longer be a stable state (for sufficiently large $M$. Classically one has to provide a small and quite arbitrary deviation from $\phi = 0$ to initialize the fall towards the “true” vacuum. There is also no way to damp the resulting oscillations.
Therefore it is not possible to predict the average time necessary to complete the phase transition from purely classical considerations. On the other hand quantum theory provides the fluctuations necessary to drive the transition. Indeed, equations (92)–(95) contain the necessary quantum fluctuations and it is possible to study the dynamics of symmetry breaking using them.

For simplicity we concentrate on the spatially uniform and isotropic solutions. In that case functions $f$ depend only on $|p|$; they do depend neither on $q$ nor on the direction of the momentum $p$. The potential $U$ is also uniform and our evolution equations (92)–(95) reduce to:

\begin{align*}
\frac{\partial}{\partial t} f_0 &= 0, \\
\frac{\partial}{\partial t} f_1 &= \frac{p^2}{m} (f_2 + f_3) + 2mf_2 + \frac{U(t)}{m} (f_2 + f_3), \\
\frac{\partial}{\partial t} f_2 &= -\frac{p^2}{m} f_1 - 2mf_1 - \frac{U(t)}{m} f_1, \\
\frac{\partial}{\partial t} f_3 &= \frac{p^2}{m} f_1 + \frac{U(t)}{m} f_1.
\end{align*}

(103)–(106)

Note that $f_0$ decouples completely from the rest of the system and does not undergo any evolution.

The time dependent scalar potential is chosen to be:

\begin{equation}
U(t) = -M^2 \theta(t - t_0) + \lambda \int \frac{d^3 p}{(2\pi)^3} (f_2 + f_3) + C.
\end{equation}

(107)

The first term is some external potential that is switched on (rapidly) at $t = t_0$. If $M^2 > m^2$ it will change the sign of a "mass squared term" inducing the transition towards new equilibrium. The second term describes a mean field self-coupling of $|\phi^4|$ type. It is well known that for $\lambda > 0$ this type of interaction will tend to stabilise the system. In addition we have a constant $C$; its value will be specified in a moment.

As an initial condition we assume the free vacuum state with distribution functions given by Eq. (72) i.e.,

\begin{align*}
f_0^{t=-\infty} &= 0, & f_1^{t=-\infty} &= 0, \\
f_2^{t=-\infty} + f_3^{t=-\infty} &= \frac{m}{E_p}, & f_3^{t=-\infty} - f_2^{t=-\infty} &= \frac{E_p}{m}.
\end{align*}

(108)
were superscript $t = -\infty$ tells us that we assumed this form of Wigner function in a distant past. Note that due to the self-coupling term this state is no longer stable (in general) and will undergo some evolution even for $t < t_0$ i.e., in absence of the external potential $-M^2 \theta(t - t_0)$. This means that the self-interaction tries to rearrange the free vacuum which was imposed as an initial state. It is possible to prevent this rearrangement if we choose constant $C$ to be:

$$ C = -\lambda \int \frac{d^3 p}{(2\pi)^3} \frac{m}{E_p}. \quad (109) $$

This constant cancels exactly the self-interaction term and the initial free vacuum becomes a stationary solution to (103-106) and stays unchanged as long as $t < t_0$. Note that $C$ is defined by the initial vacuum and stays fixed even when we switch on an external potential.

The only problem is that the integrals defining $C$ and self-interaction diverge and one has to introduce a finite cutoff $\Lambda$ to make the subtraction meaningful. We work with an isotropic system so the angular integration is not a problem:

$$ \int d^3 p \rightarrow 4\pi \int_0^\infty d|p| \cdot p^2 \rightarrow 4\pi \int_0^\Lambda d|p| \cdot p^2. \quad (110) $$

The last substitution is just the desired regularization. The presence of divergences is not a strange here. In fact the constant $C$ may be traced back to the “mass renormalization” counterterm in the Lagrangian. (There may be some confusion related to the fact that “mass” appears not only in $U$ but also in other parts of our expressions. This is due to Feshbach-Villars transformation. The mass term plays an important role in this transformation but we are free to use only a part of it – the rest may be attached to a scalar potential. Note, however, that our wave functions and subsequently the Wigner function are normalized with respect to the mass $m$ i.e., the one that participates in the Feshbach-Villars transformation. This must be kept in mind when interpreting the results.)

We solved evolution equations (104)-(106) numerically for a finite volume of phase space $0 \leq |p| < \Lambda$. This automatically introduces the cutoff. We have chosen it to be rather large so that we did not observe its effects.

The non-trivial evolution starts at $t = t_0$ when we switch on the $-M^2$ term. It was chosen large enough to override the initial positive $m^2$ and the system rolls down towards the new ground state. The results are shown
in Fig. 1. The maximum at low momenta indicates the development of long-range correlation as shown by Eq. (101). (It is not exactly a delta function because we are dealing with finite times.) A more detailed analysis shows that the ‘classical component’ of the field reaches the value given by (100) for large times when the oscillations around the new minimum die out. Both the initial decay and damping of oscillations are purely quantum processes (the latter results from particle creation by a time-dependent scalar potential). We should stress that this numerical analysis was intended only as an illustration and there are still some questions left. One of them is the stability of the procedure described here although we observe the good numerical stability of our results.

The methods developed here allow for the study of far more complicated transitions in presence of abelian gauge fields, spatial inhomogeneities etc.

Our formalism does not include the temperature effects which are of great importance for the phase transitions (e.g., in cosmology). This is an obvious shortcoming. We think that thermal effects can be incorporated at moderate expense.

5 Summary

Starting from the definition of the Wigner transform as a realization of the Heisenberg algebra in phase space, we have developed a calculus to translate operator equations into phase-space equations that have an expansion in powers of $\hbar$ and clearly exhibit the classical limit. Applying this formalism to the symmetrized correlation function of the Klein–Gordon field, we have derived the equation of motion for the equal-time Wigner transform of the Klein–Gordon field and given a meaning to its components. The resulting set of equations has been shown to possess an interpretation in terms of a relativistic gas of negatively and positively charged particles. Interference terms gave rise to quantum corrections and, especially, Schwinger pair creation. This formulation of the Klein–Gordon field is especially suited to semiclassical nonperturbative calculations as is demonstrated in the case of spontaneous symmetry breaking.

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Figures

Figure 1: The decay of the false vacuum. Initial mass $m = 1$, $\lambda = 1.0$. The sign of the mass squared term is reversed at $t = t_0 = 0$ by switching on $M^2 = 2$. (Units defined by $c = \hbar = 1$.)