THE LEAST PRIME PRIMITIVE ROOT AND THE SHIFTED SIEVE

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1. Introduction

If \( p \) is a prime, we define \( g^*(p) \) to be the least prime that is a primitive root (mod \( p \)), and similarly for prime powers \( p^r \). The problem of establishing a bound for \( g^*(p) \) uniformly in \( p \) is quite difficult, comparable with establishing a uniform upper bound for the least prime in an arithmetic progression. Indeed, there do not exist any uniform upper bounds for \( g^*(p) \) that improve upon the current bounds for the least prime in an arithmetic progression. However, much more can be said if we exclude a very small set of primes. The purpose of this paper is to improve existing bounds for \( g^*(p) \) which hold for almost all primes \( p \), and to establish analogous results for all composite moduli.

Elliott [2] had first given a bound for \( g^*(p) \) for all but \( O(Y^\varepsilon) \) primes \( p \) up to \( Y \), of the form

\[
g^*(p) \leq (\log p)^{O(\log_3 p)}.
\]

(Here we have defined \( \log_1 x = \max\{\log x, 1\} \) and \( \log_n x = \max\{\log(\log_{n-1} x), 1\} \) for any integer \( n \geq 2 \).) This was subsequently improved by Nongkynrih [6] to \( g^*(p) \leq (\log p)^{O(\log_3 p/\log_4 p)} \). We are able to establish the following bound. Write \( \omega(n) \) for the number of distinct prime factors of \( n \).

**Theorem 1.** Let \( Y, \varepsilon, \) and \( \eta \) be positive real numbers with \( \varepsilon \leq 20/21 \), and define \( B = B(\varepsilon, \eta) = \frac{3}{\varepsilon} + \frac{5}{4} + \eta \). The number of odd prime powers \( p^r \) not exceeding \( Y \) for which the estimate

\[
g^*(p^r) \ll \varepsilon, \eta \left( \omega(p - 1) \log p \right)^B
\]

fails is \( O_{\varepsilon, \eta}(Y^\varepsilon) \).

Since \( \omega(n) \ll \log n \) for all integers \( n \), it is apparent that the bound for \( g^*(p^r) \) given in Theorem 1 is no larger than a fixed (depending on \( \varepsilon \) and \( \eta \)) power of \( \log p \). We see that this is an improvement over the existing bounds, where the exponent of \( \log p \) tends to infinity with \( p \). We remark that Theorem 1 may easily be extended to include all moduli which admit primitive roots, i.e., to include moduli of the form \( 2p^r \).

To extend this type of result to composite moduli, we use the following definition. Given an integer \( q \geq 2 \), we say that a \( \lambda \)-root (mod \( q \)) is an integer, coprime to \( q \), whose multiplicative order is maximal among all integers coprime to \( q \). We see that the \( \lambda \)-root is an extension of the primitive root to all moduli, and we extend the notation \( g^*(q) \) to mean the least prime \( \lambda \)-root (mod \( q \)).

**Theorem 2.** Let \( \varepsilon \) be a positive real number. For almost all integers \( q \geq 2 \), we have

\[
g^*(q) \ll \varepsilon \omega(\phi(q))^{44/5+\varepsilon}(\log q)^{22/5}.
\]

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The approach to establishing these theorems is through Proposition 3 below, which gives a bound for $g^*(q)$ based on the assumption of a zero-free rectangle for characters (mod $q$). This is the same approach taken in earlier work on this subject; the improvement lies in the use of the “shifted sieve”, a version of the linear sieve with very good error terms, rather than Brun’s sieve.

For any integer $n$, let $s(n)$ denote the largest squarefree divisor of $n$. For any integer $q \geq 2$, let $E(q)$ denote the exponent of the group $\mathbb{Z}_q^*$ of reduced residue classes (mod $q$), let $\Phi(q)$ be the group of Dirichlet characters (mod $q$), and define

$$\Phi_*(q) = \{ \chi^{E(q)/s(\phi(q))} : \chi \in \Phi(q) \}.$$  

Only the characters in $\Phi_*(q)$ are relevant to detecting $\lambda$-roots, as we show in Section 2. Let $c_0$ be the probability that a randomly chosen element of $\mathbb{Z}_q^*$ is a $\lambda$-root. Also, given real numbers $\sigma$ and $T$ with $1/2 \leq \sigma < 1$ and $T > 0$, define $Q(\sigma, T)$ to be the set of integers $q \geq 2$ such that, for some nonprincipal $\chi \in \Phi_*(q)$, the corresponding $L$-function $L(s, \chi)$ has a zero $\beta + i\gamma$ with $\beta > \sigma$ and $|\gamma| < T$.

**Proposition 3.** Let $q \geq 2$ be an integer and $\sigma$ a real number satisfying $1/2 \leq \sigma < 1$, and set

$$f(q, \sigma) = \left( \frac{\omega(\phi(q))^2 \log_1 \omega(\phi(q))}{c_0^{-1} \log q} \right)^{1/(1-\sigma)}.$$  

If $q \notin Q(\sigma, f(q, \sigma))$, then $g^*(q) \ll \sigma f(q, \sigma)$.

We remark that $f(q, \sigma) \ll_{\sigma, \theta} q^\theta$ for every $\theta > 0$. We also remark that $c_0^{-1} \ll \log_1 \omega(\phi(q))$ (see Section 2) and that the generalized Riemann hypothesis implies that $Q(1/2, T)$ is empty for every $T > 0$. Thus the following corollary of Proposition 3 is immediate.

**Corollary 3.1.** If the generalized Riemann hypothesis holds for (certain) characters (mod $q$), then

$$g^*(q) \ll \left( \frac{\omega(\phi(q)) \log_1 \omega(\phi(q))}{\log q} \right)^4 (\log q)^2.$$  

In the case where $q$ is a prime, this has already been shown by Shoup in an earlier result of Wang in which $(\omega(\phi(q)) \log_1 \omega(\phi(q)))^4$ is replaced by $\omega(\phi(q))^6$. Although both authors state their bounds only for primitive roots, the bounds actually hold for prime primitive roots as well.

To deduce Theorems 1 and 2 from Proposition 3, we need bounds on the size of $Q(\sigma, T)$. To this end, we define $Q'(Y; \sigma, T)$ to be the number of elements of $Q(\sigma, T)$ not exceeding $Y$, and $Q'(Y; \sigma, T)$ to be the number of elements of $Q(\sigma, T)$ which are odd prime powers not exceeding $Y$. The following lemmas, when combined with Proposition 3, imply Theorems 1 and 2.

**Lemma 4.** Let $Y$, $\varepsilon$, $\eta$, and $B$ be as in Theorem 1. There exists $\theta = \theta(\varepsilon, \eta) > 0$ such that

$$Q'(Y; 1 - B^{-1}, Y^\theta) \ll_{\varepsilon, \eta} Y^\varepsilon.$$  

**Lemma 5.** We have $Q(Y; \frac{17}{22}, Y^{1/20}) = o(Y)$.

Lemma 4 follows directly from existing zero-density estimates for Dirichlet $L$-functions, but Lemma 3 is somewhat more complicated due to the prevalence of imprimitive characters in $\Phi_*(q)$ for composite moduli $q$ (see Section 3).
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2. Preliminaries

We begin by developing some notation and simple facts relating to the characters (mod $q$) which are relevant to detecting $\lambda$-roots. Let $G$ be a finite abelian group with exponent $E$. For every prime $\ell$ that divides $E$, let $\alpha(\ell)$ be the largest integer such that $\ell^{\alpha(\ell)}$ divides $E$.

There exist integers $m(\ell)$ for which we can write

$$G \cong \left( \bigoplus_{\ell | E} \left( \mathbb{Z}_{\ell^{\alpha(\ell)}} \right)^{m(\ell)} \right) \bigoplus H$$

for some subgroup $H$ whose exponent divides $E/s(E)$. For each prime $p$ dividing $E$, we define subgroups $G_p$ of $G$ by

$$G_p = (p\mathbb{Z}_{p^{\alpha(p)}})^{m(p)} \bigoplus \left( \bigoplus_{\ell | E, \ell \neq p} \left( \mathbb{Z}_{\ell^{\alpha(\ell)}} \right)^{m(\ell)} \right) \bigoplus H,$$

the set of all elements of $G$ whose order divides $E/p$. We see that the index of $G_p$ in $G$ is $p^{m(p)}$. We extend this notation to all squarefree divisors $d$ of $E$ by defining subgroups $G_d$ by

$$G_d = \bigcap_{p|d} G_p,$$

and (abusing notation somewhat) we define $m(d)$ to be the real number which satisfies

$$d^{m(d)} = \prod_{p|d} p^{m(p)},$$

so that $d^{m(d)}$ is a multiplicative function of $d$. By convention, we let $G_1 = G$ and $m(1) = 1$. We note that $m(d) \geq 1$ for all squarefree divisors $d$ of $E$, and that the index of $G_d$ in $G$ is $d^{m(d)}$.

Let $\gamma(g)$ be the characteristic function of elements of maximal order in $G$. Then, by definition (1) of the $G_p$, we have

$$\{ g \in G : \gamma(g) = 1 \} = G \setminus \bigcup_{p | E} G_p.$$

If we define $\nu(g)$ to be the product of all primes $p$ dividing $E$ such that $g \in G_p$ (or equivalently, the largest squarefree divisor $d$ of $E$ such that $g \in G_d$), then we see from equation (2) that for any $g \in G$, we have

$$\gamma(g) = \begin{cases} 1 & \text{if } \nu(g) = 1, \\ 0 & \text{if } \nu(g) > 1. \end{cases}$$

We may also detect these elements of maximal order using group characters. Let $\Phi$ be the group of homomorphisms from $G$ into $\mathbb{C}$. For each squarefree $d$ dividing $E$, define subgroups
\( \Phi_d \) of the character group \( \Phi \) by

\[ \Phi_d = \{ \chi^{E/d} : \chi \in \Phi \}. \]

For convenience we write \( \Phi^* \) for \( \Phi_{s(E)} \). Let \( h_d \) be the characteristic function of \( G_d \). By the standard properties of group characters, for any \( g \in G \) we have

\[ h_d(g) = \frac{1}{|\Phi_d|} \sum_{\chi \in \Phi_d} \chi(g). \]

By summing this over all \( g \in G \) we see that

\[ |\Phi_d| = |G|/|G_d| = d^{m(d)}, \]

and in fact we can treat this as the definition of the real numbers \( m(d) \). Finally, we define \( c_0 \) to be the probability that a randomly chosen element of \( \mathbb{Z}_q^\times \) is a \( \lambda \)-root. From equation (2) and the definition (1) of the \( G_p \), we can easily calculate that

\[ c_0 = \prod_{p|\phi(q)} \left( 1 - \frac{1}{p^{m(p)}} \right). \]

We note in particular that

\[ c_0^{-1} \leq q/\phi(q) \ll \log_1 \omega(\phi(q)). \]

In the course of applying the sieve, it will be important to understand the behavior of the sum \( \psi_1(x, \chi) \) defined by

\[ \psi_1(x, \chi) = \sum_{n<x} \chi(n)\Lambda(n)(x-n). \]

The following lemma provides the necessary bound, for the moduli \( q \) for which Proposition 3 will be established.

**Lemma 6.** Let \( q \geq 2 \) be an integer, and let \( x, \sigma, \) and \( T \) be real numbers satisfying \( 1/2 \leq \sigma < 1 \) and \( 1 \leq x \ll T \ll q \). If \( q \notin \mathcal{Q}(\sigma, T) \), then for all nonprincipal \( \chi \in \Phi^*(q) \), we have

\[ \psi_1(x, \chi) \ll x^{1+\sigma} \log q. \]

**Proof:** We begin by writing

\[ \psi_1(x, \chi) = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L'(s, \chi) \frac{x^{s+1} ds}{s(s+1)} \]

and pulling the contour leftwards towards \( \Re s = -\infty \) to see that

\[ \psi_1(x, \chi) = -\sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} + O(x \log x), \]

where the sum runs over all nontrivial zeros \( \rho = \beta+i\gamma \) of \( L(s, \chi) \) (see for instance [1, Chapter 19]). Because \( q \) is not in \( \mathcal{Q}(\sigma, T) \), every zero of \( L(s, \chi) \) has either \( \beta \leq \sigma \) or \( |\gamma| \geq T \), and thus we can write

\[ \psi_1(x, \chi) \ll \sum_{\beta \leq \sigma} \frac{x^{1+\beta}}{\gamma^2} + \sum_{|\gamma| \geq T} \frac{x^{1+\beta}}{\gamma^2} + x \log x. \]

However, the number of zeroes of \( L(s, \chi) \) up to height \( T \) is \( \ll T \log qT \), and so \( \sum_{|\gamma| \geq T} \gamma^{-2} \ll T^{-1} \log qT \) by partial summation. Therefore

\[ \psi_1(x, \chi) \ll x^{1+\sigma} \log q + x^2 T^{-1} \log qT + x \log x. \]

Since \( x \ll T \ll q \), the first term is dominant, and the lemma is established. \( \square \)
3. The shifted sieve: Proof of Proposition 3

Let $\mathcal{A}$ be a finite sequence, $\nu$ a map from $\mathcal{A}$ to the positive integers, and $w$ a function from $\mathcal{A}$ to the nonnegative reals. Let $\Upsilon$ be a squarefree integer, put

$$ S(\mathcal{A}, \Upsilon) = \sum_{a \in \mathcal{A}} w(a), $$

and, for all $d$ dividing $\Upsilon$, put

$$ A_d = \sum_{a \in \mathcal{A}} w(a). $$

Lemma 7. Suppose that $X$ and $R$ are positive numbers and $f(d)$ a multiplicative function such that for all $d$ dividing $\Upsilon$, we have $f(d) \geq d$ and

$$ \left| A_d - \frac{X}{f(d)} \right| \leq R. \quad (5) $$

Then there exists an absolute positive constant $C_1$ such that

$$ S(\mathcal{A}, \Upsilon) \geq \frac{C_1 X}{\log_1 \omega(\Upsilon)} \prod_{p | \Upsilon} \left(1 - \frac{1}{f(p)}\right) + O(R(\omega(\Upsilon))^2). $$

Proof: Let $p_j$ denote the $j$th prime, and put $z = p_{\omega(\Upsilon)}$ and $P = \prod_{p \leq z} p$. Also let $\{\lambda_d^-\}$ be a sequence of real numbers such that $\lambda_1^- \leq 1$ and, if we define

$$ \sigma_n = \sum_{d | n} \lambda_d^-, $$

then $\sigma_n \leq 0$ for all integers $n \geq 2$. We begin by citing the lower bound

$$ S(\mathcal{A}, \Upsilon) \geq X \prod_{p | \Upsilon} \left(1 - \frac{1}{f(p)}\right) \sum_{d | P} \frac{\sigma_d}{\prod_{p | d}(p - 1)} - R \sum_{d | P} |\lambda_d^-|. \quad (6) $$

This is a special case of the shifted sieve of Iwaniec \cite[Lemma 1]{Iwaniec2000}, where we have specified that $Q = \Upsilon$, $A = R$, $B = 1$, and $g(d) = d$ for all $d$ dividing $P$, and that the correspondence $l$ sends the smallest prime factor of $\Upsilon$ to $p_1$, the next smallest to $p_2$, and so on. We now take $\{\lambda_d^-\}$ to be Rosser’s weights for the linear sieve, whose definition depends on a positive parameter $y$ as follows. If $d$ is not squarefree, define $\lambda_d^- = 0$. If $d = q_1 \cdots q_r$ for primes $q_1 > \cdots > q_r$, define

$$ \lambda_d^- = \begin{cases} (-1)^r & \text{if } q_1 \cdots q_{2l-1} q_{2l}^2 < y \text{ for all } 0 \leq l \leq r/2, \\ 0 & \text{otherwise.} \end{cases} $$

We will need the following facts about the sequence $\{\lambda_d^-\}$ \cite[Lemma 2]{Iwaniec2000}: if $4 \leq z^2 \leq y \leq z^4$, then

$$ \sum_{d | P} |\lambda_d^-| \ll y(\log y)^{-2} $$

and

$$ \sum_{d | P} \frac{\sigma_d}{\prod_{p | d}(p - 1)} = 2e^z \frac{\log(s - 1)}{s} + O\left(\frac{1}{\log y}\right), \quad (7) $$
where \( s = (\log y)/(\log z) \). Applying this with \( y = C_2z^2 \) for \( C_2 \) a positive constant gives us

\[
2e^\gamma \frac{\log (s-1)}{s} + O\left( \frac{1}{\log y} \right) = e^\gamma \frac{\log C_2}{\log z} \left( 1 + O\left( \frac{\log C_2}{\log z} \right) \right) + O\left( \frac{1}{\log z} \right) \geq \frac{C_1}{\log z} \tag{8}
\]

for some positive constant \( C_1 \), if \( C_2 \) and \( z \) are sufficiently large. With these estimates, the lower bound (6) becomes

\[
S(A, \Upsilon) \geq \frac{C_1 X}{\log z} \prod_{p \mid \Upsilon} \left( 1 - \frac{1}{f(p)} \right) + O\left( \frac{RC_2z^2}{(\log z)^2} \right).
\]

We note that \( C_2 \) is an absolute constant, since it depends only on the \( O \)-constant in equation (7), and thus \( C_1 \) is absolute as well, since it depends only on \( C_2 \) and the \( O \)-constants in equation (8). It remains only to note that \( z \sim \omega(\Upsilon) \log \frac{1}{\omega(\Upsilon)} \) to establish the lemma.

We may now establish Proposition 3. Let \( q \geq 2 \) be an integer and \( x > 1 \) and \( 1/2 \leq \sigma < 1 \) real numbers. We will apply Lemma 7 with \( A \) being the set of positive integers less than \( x \). Let \( \Upsilon = s(\phi(q)) \), let \( \nu(n) \) be defined as in Section 2 before equation (3), and let \( w(n) = \Lambda(n)(x-n) \). From the relation (3), we see that

\[
S(A, \Upsilon) = \sum_{n<x} \gamma(n)\Lambda(n)(x-n)
\]

counts only prime powers which are \( \lambda \)-roots (mod \( q \)). Using the form (4) for \( h_d \) and the definition of the \( \psi_1(x, \chi) \), we also have

\[
A_d = \sum_{\substack{n<x \\text{d}}\nu(n)} w(n) = \sum_{\substack{n<x \\text{d}}\nu(n)} h_d(n)w(n)
\]

\[
= \frac{1}{|\Phi_d|} \sum_{\chi \in \Phi_d} \sum_{n<x} \chi(n)w(n)
\]

\[
= \frac{1}{d^{m(d)}} \psi_1(x, \chi_0) + \frac{1}{|\Phi_d|} \sum_{\chi \in \Phi_d, \chi \neq \chi_0} \psi_1(x, \chi).
\]

If we write \( \psi_1(x) = \sum_{n<x} \Lambda(n)(x-n) \), then

\[
\psi_1(x) - \psi_1(x, \chi_0) = \sum_{\substack{n<x \\text{(n,q)>1}}} \Lambda(n)(x-n) \ll x \sum_{p\mid q} \sum_{r \geq 1} \log p \ll (x \log x) \log q,
\]

since \( \omega(q) \ll \log q \). Moreover, if we assume that \( q \notin Q(\sigma, x) \), then we may apply Lemma 4 (with \( T = x \)) to bound the terms in the last sum of equation (3); we obtain

\[
A_d = \frac{1}{d^{m(d)}} \psi_1(x) + O\left( x^{1+\sigma} \log q \right).
\]
Thus if we take $X = \psi_1(x)$ and $f(d) = d^{m(d)}$ for all $d$ dividing $s(\phi(q))$, we see that we can take $R \ll x^{1+\sigma} \log q$. Applying Lemma 4, we see that

$$S(\mathcal{A}, \mathcal{Y}) \geq \frac{C_1 \psi_1(x)}{\log_1 \omega(\phi(q))} c_0 + O\left( (x^{1+\sigma} \log q \omega(\phi(q))^2 \right)
$$

$$= \frac{C_1 \psi_1(x)}{\log_1 \omega(\phi(q))} c_0 \left( 1 + O\left( x^{-1+\sigma} \left( \omega(\phi(q))^2 \log_1 \omega(\phi(q)) \right) c_0^{-1} \log q \right) \right)
$$

$$= \frac{C_1 \psi_1(x)}{\log_1 \omega(\phi(q))} c_0 \left( 1 + O\left( \left( x^{-1} f(q, \sigma) \right)^{1-\sigma} \right) \right),$$

since the bound $\psi_1(x) \gg x^2$ follows from Chebyshev's bound for $\psi(x)$. Assuming that $x$ exceeds a sufficiently large (in terms of $\sigma$) multiple of $f(q, \sigma)$, we obtain a positive lower bound for $S(\mathcal{A}, \mathcal{Y})$. Therefore, there exists a prime power $p^r \ll f(q, \sigma)$ which is a $\lambda$-root (mod $q$). But if $p^r$ is a $\lambda$-root, we must have $(r, \phi(q)) = 1$, in which case $p$ itself is also a $\lambda$-root which is $\ll f(q, \sigma)$. This establishes the proposition.

4. Proof of Lemmas 4 and 5

To establish Lemma 4, we introduce the notation $Q'(\sigma, T)$ to denote the subset of $Q(\sigma, T)$ consisting of the odd prime powers, and we recall that $Q'(Y; \sigma, T)$ denotes the number of elements of $Q'(\sigma, T)$ not exceeding $Y$. Given an odd prime power $p^r$, every character in $\Phi_*(p^r)$ is induced by a character (mod $p^2$) [5, Lemma 6]. The proof of this fact is similar to the proof that any primitive root (mod $p^2$) is also a primitive root (mod $p^r$) for every odd prime $p$ and integer $r \geq 3$.

Consequently, for every prime power $p^r \in Q'(\sigma, T)$, there is a character $\chi$ which is primitive to one of the moduli $p$ or $p^2$ such that $L(s, \chi)$ has a zero $\beta + i\gamma$ with $\beta > \sigma$ and $|\gamma| < T$. On the other hand, every such character will account for $\ll f(q, \sigma)$ prime powers in $Q'(\sigma, T)$ which do not exceed $Y$, and so

$$Q'(Y; \sigma, T) \ll (\log Y) \sum_{q < Y^\sigma} \sum_{\chi \pmod{q}}^{*} N(\sigma, T, \chi), \quad (10)$$

where $N(\sigma, T, \chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ satisfying $\beta > \sigma$ and $|\gamma| < T$, and $\sum^{*}$ denotes a summation over primitive characters only. Zhang [9] has established the following zero-density estimate for Dirichlet $L$-functions: for any real numbers $Y$, $\delta > 0$ and $\frac{17}{22} \leq \sigma \leq 1$, we have

$$\sum_{q < Y^\sigma} \sum_{\chi \pmod{q}}^{*} N(\sigma, T, \chi) \ll_{\delta} (Y^2 T)^{6(1-\sigma)/(5\sigma-1)+\delta}. \quad (11)$$

We apply this estimate with $T = Y^\theta$ and $\sigma = 1 - B^{-1}$, where $B$ is as in Theorem 4. Together with the bound (10), this gives us $Q'(Y; \sigma, T) \ll_{\varepsilon, \eta} Y^{\varepsilon}$, as long as $\delta = \delta(\varepsilon, \eta)$ and $\theta = \theta(\varepsilon, \eta)$ are small enough with respect to $\varepsilon$ and $\eta$. This establishes Lemma 4.

Unfortunately, a given character can in general induce characters in $\Phi_*(q)$ for many more moduli $q$ if we do not restrict to prime powers, and so we must work harder to establish Lemma 5. Given positive integers $m$ and $n$ such that $m$ divides $n$, we say that $n$ is an admissible multiple of $m$ if there exists a character in $\Phi_*(n)$ which is induced by a primitive character (mod $m$).
Lemma 8. Let \( q \geq 2 \) be an integer, and set \( t = \omega(q) \). Let \( p_1, \ldots, p_t \) be the primes dividing \( q \) and \( r_1, \ldots, r_t \) positive integers. Then for every admissible multiple \( nq \) of \( q \), either:

(i) \( p_i^{r_i} \) divides \( n \) for some \( 1 \leq i \leq t \); or

(ii) \( n \) is not divisible by any prime which is congruent to \( 1 \mod \phi^2(q)p_1^{r_1}\cdots p_t^{r_t} \).

Proof: We use parenthetical superscripts to indicate explicitly the modulus of a character, so that \( \chi^{(q)} \) denotes a character (mod \( q \)), for example. To establish the lemma, it suffices to show that if (i) and (ii) both fail, then any character \( \chi^{(q)} \) which induces an element \( \chi_1^{(nq)} \) of \( \Phi_*(nq) \) is in fact principal (hence imprimitive), contradicting the assumption that \( nq \) is an admissible multiple of \( q \).

Assume the negations of (i) and (ii). Write \( nq = n'q' \), where \( q' \) is the largest divisor of \( nq \) with \( s(q') = s(q) \), so that \( q \) divides \( q' \) and \( (n', q') = 1 \). Then any character (mod \( nq \)) is the product of a character (mod \( n' \)) and a character (mod \( q' \)). Since \( \chi_1^{(nq)} \in \Phi_*(nq) \), we may write

\[
\chi_1^{(nq)} = \left( \chi_2^{(n')} \chi_3^{(q')} \right)^{E(nq)/s(E(nq))}
\]

for some characters \( \chi_2^{(n')} \) and \( \chi_3^{(q')} \). Since \( p_i^{r_i} \) does not divide \( n \) for any \( 1 \leq i \leq t \), we see from the definition of \( q' \) that \( \phi(q') \) divides \( \phi(q)p_1^{r_1-1}\cdots p_t^{r_t-1} \). On the other hand, \( n \) is divisible by a prime which is congruent to \( 1 \mod \phi^2(q)p_1^{r_1}\cdots p_t^{r_t} \) and so \( \phi^2(q)p_1^{r_1}\cdots p_t^{r_t} \) must divide \( E(nq) \). These observations together imply that \( \phi(q') \) divides \( E(nq)/s(E(nq)) \), and thus

\[
\left( \chi_2^{(n')} \chi_3^{(q')} \right)^{E(nq)/s(E(nq))} = \left( \chi_2^{(n')} \right)^{E(nq)/s(E(nq))} \chi_0^{(q')},
\]

where \( \chi_0^{(q')} \) is the principal character (mod \( q' \)). We see that the character \( \chi_1^{(nq)} \) induced by \( \chi^{(q)} \) is also induced by a character (mod \( n' \)). But since \( (q, n') = 1 \), it must be the case that \( \chi^{(q)} \) is principal. This establishes the lemma. \( \square \)

Let \( A(x; q) \) be the number of admissible multiples of \( q \) not exceeding \( x \).

Lemma 9. Let \( \delta > 0 \) be a real number and \( x, y = y(x) \), and \( z = z(x) \) real parameters satisfying \( x, y, z > 1 \) and

\[
z^3 y^{\log z} \ll (\log x)^{1-\delta}.
\]

Then for all integers \( q \) with \( 2 \leq q \leq z \), we have

\[
A(xq; q) \ll_{\delta} \frac{x \log z}{y} + \frac{x}{\exp(\log_2 x/z^3 y^{\log z})}.
\]

Proof: Set \( t = \omega(q) \), and choose integers \( r_i \) such that

\[
p_i^{r_i-1} \leq y \leq p_i^{r_i} \quad (1 \leq i \leq t).
\]

By applying Lemma 8, we see that the number of admissible multiples \( nq \) of \( q \) with \( n < x \) is bounded by

\[
\sum_{i=1}^{t} \frac{x}{p_i^{r_i}} + \# \{ n < x : p | n \Rightarrow p \not\equiv 1 \mod \phi^2(q)p_1^{r_1}\cdots p_t^{r_t} \}.
\]
In the first term, we use the estimate $t \leq \log z$ for $z$ sufficiently large, and the choice (14) of the $r_i$, to see that

$$
\sum_{i=1}^{t} \frac{x}{p_i^{r_i}} \leq \frac{x \log z}{y}.
$$

(16)

We treat the second term using a simple upper bound sieve. Notice that by the choice (14) of the $r_i$, we have

$$
\phi^2(q) p_1^{r_1} \cdots p_t^{r_t} \leq q^2 \left( \prod_{i=1}^{t} y p_i \right) \leq q^2 (y^t z) \leq z^3 y \log z.
$$

(17)

The prime number theorem for arithmetic progressions states that given $\delta > 0$, we have

$$
\psi(x; d, 1) = \frac{x}{\phi(d)} + O_\delta \left( x \exp(-C_3(\log x)^{1/2}) \right)
$$

for some positive constant $C_3$, uniformly for all $d \ll (\log x)^{1-\delta}$ [1, equations (10)–(11) of Section 20]. By partial summation, this implies that

$$
\sum_{p<x} p^{-1} = \frac{\log_2 x}{\phi(d)} + O_\delta(1),
$$

(18)

again uniformly for $d$ in the above range, which includes $d = \phi^2(q) p_1^{r_1} \cdots p_t^{r_t}$ due to equation (17) and the restriction (12). The formula (18) allows us to apply an upper bound sieve from Halberstam–Richert [3, Corollary 2.3.1] to deduce that

$$
\# \{ n<x \colon p \mid n \Rightarrow p \not\equiv 1 (\text{mod } \phi^2(q) p_1^{r_1} \cdots p_t^{r_t}) \} \ll \delta x (\log x)^{-1/\phi^2(q) p_1^{r_1} \cdots p_t^{r_t}}.
$$

We rewrite this using the bound (17) as

$$
\# \{ n<x \colon p \mid n \Rightarrow p \not\equiv 1 (\text{mod } \phi^2(q) p_1^{r_1} \cdots p_t^{r_t}) \} \ll \delta \frac{x}{\exp(\log_2 x / z^3 y \log z)}.
$$

Using this bound together with the bound (16) in equation (15) establishes the lemma. 

Lemma 10. For all real $x > 1$, we have

$$
\sum_{q<x} \frac{1}{q^{100}} \ll x^{0.997} \quad \text{and} \quad \sum_{q<x} \frac{1}{q^{100}} \ll x^{-0.003}.
$$

(19)

Proof: The right-hand side of the zero-density estimate (14) is certainly an upper bound for the first sum in (19) as well. Taking $Y = x$, $T = x^{1/20}$, and $\theta = \frac{1}{100}$ in (14), we see that

$$
\sum_{q<x} \frac{1}{q^{100}} \ll x^{41861/42000},
$$

and $41861/42000 < .997$. This establishes the first bound in (19), and the second bound follows directly by partial summation. 

\[ \square \]
We are now ready to prove Lemma 4. We note that every element of $Q(\sigma, T)$ is an admissible multiple of some element of $R(\sigma, T)$. Therefore,

$$Q(Y; \sigma, T) \leq \sum_{q \in R(\sigma, T)} A(Y; q).$$

(20)

For $q \leq \log_3 Y$, we bound $A(Y; q)$ by applying Lemma 9 with $z = \log_3 Y$ and $y = (\log_2 Y)^{1/(2 \log z)}$, which satisfy the condition (12) with any $\delta < 1$. Of the two terms in equation (13), the first term is dominant, giving

$$A(Y; q) \leq A(Y q; q) \ll \frac{Y \log_4 Y}{\exp(\log_3 Y/2 \log_4 Y)}.$$

For the remaining values of $q$, we have the trivial bound $A(Y; q) \leq Y/q$. Therefore equation (20) becomes

$$Q(Y; \sigma, T) \ll \sum_{q < \log_3 Y} \frac{Y \log_4 Y}{\exp(\log_3 Y/2 \log_4 Y)} + \sum_{\log_3 Y \leq q < Y} \frac{Y}{q}.$$

Upon choosing $\sigma = \frac{17}{22}$ and $T = Y^{1/20}$, we apply Lemma 10 to the second sum to obtain

$$Q\left(Y; \frac{17}{22}, Y^{1/20}\right) \ll \frac{Y \log_3 Y \log_4 Y}{\exp(\log_3 Y/2 \log_4 Y)} + \frac{Y}{(\log_3 Y)^{0.003}} = o(Y),$$

which establishes the lemma.

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