RANKS FOR REPRESENTATIONS OF $GL_n$ OVER FINITE FIELDS, THEIR AGREEMENT, AND POSITIVITY OF FOURIER TRANSFORM

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Abstract. In [Frobenius1896], it was shown that many important properties of a finite group could be examined using formulas involving the character ratios of group elements, i.e., the trace of the element acting in a given irreducible representation, divided by the dimension of the representation.

In [Gurevich-Howe15] and [Gurevich-Howe17], the current authors introduced the notion of rank of an irreducible representation of a finite classical group.

One of the motivations for studying rank was to clarify the nature of character ratios for certain elements in these groups.

In fact in the above cited papers, two notions of rank were given. The first is the Fourier theoretic based notion of $U$-rank of a representation, which comes up when one looks at its restrictions to certain abelian unipotent subgroups. The second is the more algebraic based notion of tensor rank which comes up naturally when one attempts to equip the representation ring of the group with a grading that reflects the central role played by the few "smallest" possible representations of the group.

In [Gurevich-Howe17] we conjectured that the two notions of rank mentioned just above agree on a suitable collection called "low rank" representations.

In this note we review the development of the theory of rank for the case of the general linear group $GL_n$ over a finite field $\mathbb{F}_q$, and give a proof of the "agreement conjecture" that holds true for sufficiently large $q$. Our proof is Fourier theoretic in nature, and uses a certain curious positivity property of the Fourier transform of the set of matrices of low enough fixed rank in the vector space of $m \times n$ matrices over $\mathbb{F}_q$.

In order to make the story we are trying to tell clear, we choose in this note to follow a particular example that shows how one might apply the theory of rank to certain counting problems.

Dedicated to the memory of Tonny Springer

0. Introduction

The Fourier theoretic study of a function on a finite abelian group via its expansion as a linear combination of exponentials is by now a classical example for the applications of harmonic analysis to pure and applied mathematics [Auslander-Tolimieri79]. This expansion has a well known generalization to the study of class (i.e., invariant by conjugation) functions on any finite group $G$. Indeed, let us denote by $\widehat{G}$ the set of (isomorphism classes of) complex finite dimensional irreducible representations (irreps
for short) of $G$, and by

$$\chi_\pi, \pi \in \hat{G},$$

(0.0.1)

the associated collection of irreducible characters, with $\chi_\pi(g) = trace(\pi(g))$ for $g \in G$. In [Schur1905] Schur formulated his famous orthogonality relations for the collection (0.0.1), which implies that it forms an orthogonal basis for the space of class functions on $G$, equipped with the natural $G$-bi-invariant inner product on functions on the group. This fact generalizes the abelian setting, and gives birth to the theory of harmonic analysis on $G$, namely the investigation of class functions on the group via their expansion as a linear combination of irreducible characters.

As was already pointed out by Frobenius in [Frobenius1896], for many interesting class functions on $G$ the expansion, as a linear combination of irreducible characters, involves the normalized quantities

$$\frac{\chi_\pi(g)}{\dim(\pi)}, \pi \in \hat{G}, g \in G,$$

(0.0.2)
called character ratios (CRs).

So to make use of Frobenius’s type formulas, it seems that one might benefit from a solution to the following:

**Problem (Core problem of harmonic analysis on $G$).** Estimate the character ratios (0.0.2).

This note will focus on a particular example for the general and special linear groups. We will show how to get precise information about the character ratios for arbitrary representations of these groups for the elements known as transvections. We introduce them now.

0.1. **Example: Generation by Transvections.** Consider the group $G = SL_n(\mathbb{F}_q)$ of $n \times n$ matrices with entries in a finite field $\mathbb{F}_q$ and determinant equal to one. For this example let us assume that $n \geq 3$.

Inside $G$ we look at the conjugacy class $C$ of the transvection

$$T = \begin{pmatrix} 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix},$$

(0.1.1)

with $T_{ii} = 1$ for $i = 1, .., n$; $T_{12} = 1$, and $T_{ij} = 0$ elsewhere.

It is not difficult to show (see [Artin57]) that $C$ generates the group $G$, and for a given element $g \in G$, we would like to understand in how many ways it can be be obtained, i.e., for $\ell \geq 1$ what is the cardinality
RANKS FOR REPRESENTATIONS OF $GL_n$ OVER FINITE FIELDS, THEIR AGREEMENT, AND POSITIVITY

of the set

$$M_{\ell,g} = \left\{ (c_1, \ldots, c_\ell) \in C^\ell; \ c_1 \cdot \ldots \cdot c_\ell = g \right\}?$$ \hspace{1cm} (0.1.2)$$

Let us try to answer the above question for the "typical" elements of $G$. Before we do so, let us recall some information regarding the conjugacy class $C$.

**Facts.** The following hold$^{1,2,3}$:

- **Cardinality:** $\#(C) = q^{2n-2} + o(...)$ [Artin57];

- **Generation:** every element of $G$ can be written as a product of no more than $n$ elements from $C$ [Humphries80];

  Moreover,

- **Most elements:** the "boundary" $\partial(G)$, of members of $G$ that one can’t form by less than $n$ products from $C$, is

  $$\partial(G) = \{ g \in G; \ \ker(g-I) \neq 0 \},$$

  in particular$^4$,

  $$\#(\partial(G)) = #(G) \cdot \left( 1 - O\left( \frac{1}{q} \right) \right).$$

Of course, most of the elements of $\partial(G)$ are regular semi-simple, i.e., are diagonalizable over some field extension of $\mathbb{F}_q$, and have $n$ different eigenvalues there. These are our typical elements for the example we are giving, and we would like to solve for them the following:

**Problem 0.1.3** (Generation). For a regular semi-simple element $g \in \partial(G)$, what is the cardinality of $M_{\ell,g} \ (0.1.2)$?

Note that, because we specialized to the case of a typical $g$, it makes sense to expect (and probably not difficult to prove) that $\#(M_{\ell,g}) \to \#(C^\ell)/\#(G)$, as $\ell \to \infty$. Before we write down a precise statement, let us look at some numerics$^5$ for the ratio of $\#(M_{\ell,g})$ and $\#(C^\ell)/\#(G)$. Figure 1 illustrates, for the group $G = SL_8(\mathbb{F}_3)$, how this quantity is close to being 1 in log$\frac{1}{2}$-scale. Let us elaborate a bit on what you see there. Of course, for $G = SL_8(\mathbb{F}_3)$, and our choice of $g$, the set $M_{\ell,g}$ is empty for $\ell < 8$; but then the numerics shows that a "cutoff phenomenon" occurs, namely, at the $\ell = 8$ step the two quantities $\#(M_{\ell,g})$ and $\#(C^\ell)/\#(G)$ all of a sudden come close, and then at every additional step they come closer by a multiple of $\frac{1}{3}$.

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$^1$We write $\#(X)$ for the number of elements in a finite set $X$.

$^2$The notation $a(q) = o(b(q))$ means that $a(q)/b(q) \to 0$ as $q \to \infty$.

$^3$The notation $c(q) + o(...)$ stands for $c(q) + o(c(q))$.

$^4$We write $a(q) = O(b(q))$ if there is constant $A$ with $a(q) \leq A \cdot b(q)$ for all sufficiently large $q$.

$^5$The numerics in this note were done using the Magma Computational Algebra System.
Figure 1. \( \log_{\frac{3}{2}} \text{-scale of } (1 - \frac{\#(M_{\ell,g})}{\#(C^\ell)/\#(G)}) \text{ vs. } \ell \), for a typical element of \( G = SL_8(\mathbb{F}_3) \).

The numerical observations made just above, can be formulated and proved. Indeed,

**Theorem 0.1.4 (Set-theoretic size).** For a regular semi-simple \( g \in \partial(G) \), we have,

\[
\#(M_{\ell,g}) = \frac{\#(C^\ell)}{\#(G)} \cdot \begin{cases} 1 - O\left(\frac{1}{q}\right), & \text{if } \ell = n; \\ 1 - (\frac{2}{q^2})^{\ell-n} + o(...), & \text{if } \ell > n. \end{cases} 
\]  

(0.1.5)

0.2. **A Geometric Analog of the Generation Problem.** The set \( M_{\ell,g} \) (0.1.2) is in a natural way the set of \( \mathbb{F}_q \)-rational points of an algebraic variety \( M_{\ell,g} \) defined over \( \mathbb{F}_q \), and both objects can teach us something about the other (for basic terminology of algebraic geometry see [Hartshorne77]).

A prototype example for the relation mentioned above—and relevant to our story— is given by the famous "Lang-Weil bound" [Lang-Weil54]. It relates the dimension of an (affine) algebraic variety \( X \) defined over \( \mathbb{F}_q \) (i.e., the set of solutions in an algebraic closure \( \overline{\mathbb{F}_q} \) of \( \mathbb{F}_q \) of a finite set of polynomial equations over \( \mathbb{F}_q \), and topology also given by polynomials over \( \mathbb{F}_q \)) and the cardinality of the set \( X = X(\mathbb{F}_q) \) of its \( \mathbb{F}_q \)-rational points (i.e., the solutions in \( \mathbb{F}_q \) of the polynomials defining \( X \)). Here is a precise formulation that will serve us well.

**Fact 0.2.1 (Lang-Weil bound).** The following are equivalent:

1. **Set-theoretic size:** \( \#(X) = q^d + O(q^{d-\frac{1}{2}}) \), for some integer \( d \geq 0 \).
(2) Geometric size: \(X\), as a variety over \(\overline{\mathbb{F}}_q\), has a unique irreducible component of maximal dimension \(d = \dim(X)\); all other components have smaller dimension.

In our case, we consider the algebraic group \(G = \text{SL}_n\) defined over \(\mathbb{F}_q\), and the conjugacy class \(C \subset G\) of the transvection \(T\) \((0.1.1)\). Then, for any \(g \in G\) we can form the algebraic variety \(M_{\ell,g} \subset C^\ell\) in exactly the same way as in \((0.1.2)\). Moreover, \(M_{\ell,g}\) is defined over \(\mathbb{F}_q\), and, indeed, \(M_{\ell,g} = M_{\ell,g}(\mathbb{F}_q)\). So, in view of the Lang-Weil bound, a reasonable geometric analog of Problem 0.1.3, might be the following:

**Problem 0.2.2** (Generation - geometric version). For regular semi-simple element \(g \in \partial(G)\) \((i.e., no eigenvalue equal to 1)\), compute the dimension of \(M_{\ell,g}\), \(\ell \geq n\), and the number of its irreducible components of maximal dimension.

To solve Problem 0.2.2, note that \(M_{\ell,g}\) is the fiber over \(g\) of the multiplication morphism from \(C^\ell\) to \(G\). Hence, by the general "fiber dimension theorem" \([Hartshorne77]\), for \(\ell \geq n\), all components of \(M_{\ell,g}\) have dimension \(\geq \dim(C^\ell) - \dim(G)\). In particular, looking on Fact 0.2.1, we learn that Theorem 0.1.4 implies the following:

**Corollary 0.2.3** (Geometric size). Assume \(g \in \partial(G)\) is regular semi-simple element and \(\ell \geq n\). Then \(M_{\ell,g}\), as a variety over \(\mathbb{F}_q\), is irreducible of dimension

\[
\dim(M_{\ell,g}) = \dim(C^\ell) - \dim(G) = 2\ell(n - 1) - (n^2 - 1),
\]

where in the last equality we used the fact (verified by a direct computation) that \(\dim(C) = 2(n - 1)\).

In fact, it will be interesting to find also a direct geometric proof of Corollary 0.2.3. However, it seems (compare the "error" term in Part (1) of Fact 0.2.1 with the one appearing in Theorem 0.1.4) that the set-theoretic estimate we obtained is stronger than the geometric information given in Corollary 0.2.3.

It still left for us to explain why the counting statement appearing in Theorem 0.1.4, i.e., identity \((0.1.5)\), is valid. For this, we propose to use harmonic analysis.

0.3. **Harmonic Analysis of the Generation Problem.** As a function of \(g \in G = \text{SL}_n(\mathbb{F}_q)\), the cardinality \(#(M_{\ell,g})\), is a class function. The harmonic analytic expansion of this function in irreducible characters can be computed explicitly. Indeed,

**Proposition 0.3.1.** We have,

\[
#(M_{\ell,g}) = \frac{#(C^\ell)}{#(G)} \cdot \left(1 + \sum_{1 \neq \pi \in \hat{G}} \dim(\pi) \left(\frac{\chi_\pi(T)}{\dim(\pi)}\right)^\ell \chi_\pi(g^{-1})\right), \tag{0.3.2}
\]
where $T$ is the transvection (0.1.1).

Formula (0.3.2) is well known [Arad-Herzog-Stavi85, Frobenius1896]; however, for the convenience of the reader we give another verification in Appendix B.1.1.

The Formula (0.3.2), suggests proving Theorem 0.1.4 by estimating the sum over the non-trivial representations,

$$S_{\ell, g} = \sum_{1 \neq \pi \in \hat{G}} \dim(\pi) \left( \frac{\chi_\pi(T)}{\dim(\pi)} \right) \chi_\pi(g^{-1}),$$  
(0.3.3)

and show that it is as small as the required "error" term in (0.1.5).

Recall that the element $g \in G$, appearing in the sum $S_{\ell, g}$ (0.3.3), is regular semi-simple. For such generic elements the following is known by [Lusztig84] (and maybe can be deduced already from the work [Green55]):

**Fact 0.3.4.** Suppose $g \in G$, is a regular semi-simple element. Then, there is a constant $c$, independent of $q$, such that for every irrep $\pi \in \hat{G}$, we have,

$$|\chi_\pi(g)| \leq c.$$  

Moreover, one can take $c = n!$, the cardinality of the Weyl group $W_G = S_n$ of $G$.

Looking back on (0.3.3) we see that, a possible approach for getting the desired bound on $S_{\ell, g}$ will be to have strong estimates on the dimensions $\dim(\pi)$, and, most importantly, on the character ratios $\frac{\chi_\pi(T)}{\dim(\pi)}$ of the irreps $\pi$ of $G = SL_n(\mathbb{F}_q)$ at the transvection $T$ (0.1.1).

In recent years we have been developing a method that attempts to produce this piece of information for the irreps of classical groups over finite fields, and probably for character ratios of many other elements of interest.

0.4. **Rank of a Representation.** We want to estimate character ratio $\frac{\chi_\pi(T)}{\dim(\pi)}$, on the transvection $T$ (0.1.1), for arbitrary irrep $\pi$ of $G = SL_n(\mathbb{F}_q)$.

The group $G$ is a member of the family of reductive groups over finite and local fields. The most popular method that people use to analyze representations of such groups is the philosophy of cusp forms [Harish-Chandra70] put forward by Harish-Chandra in the 60s. In this approach one studies the irreps of the group by means of certain basic objects called cuspidal representations. It turns out that cuspidality is a generic property, i.e., these irreps constitute a major part of all irreps, and most of them are, in some sense, among the "largest".
The philosophy of cusp forms has had enormous success in establishing the Plancherel formula for reductive groups over local fields [Harish-Chandra84], and leads to Lusztig’s classification [Lusztig84] of the irreps of reductive groups over finite fields. However, analysis of character ratios (CRs) seems to benefit from a different approach.

The attempts to estimate the CRs motivated us to introduce, in [Gurevich-Howe15, Gurevich-Howe17], a new way to think on the irreps of the classical groups; a way in which the building blocks are the very few “smallest” representations; in fact representations that may seem to be anomalies in the cusp form approach.

In [Gurevich-Howe15, Gurevich-Howe17] we explained that the choice of looking on the irreps of a given classical group through the lens of its smallest ones, reveals the existence of a pair of related invariants, which we refer to by the label of ”rank”. Specifically, we have defined ”$U$-rank” and ”tensor rank”. For example, for the case of the group $GL_n(\mathbb{F}_q)$, the $U$-rank of an irrep is an integer between 0 and $\frac{n}{2}$ or $\frac{n-1}{2}$, depending if $n$ is even or odd, respectively, while tensor rank is an integer between 0 and $n$. In [Gurevich-Howe17] we conjectured that in the case of $GL_n(\mathbb{F}_q)$ for values in the range 0 and $\frac{n}{2} - 1$, for $n$ even, or $\frac{n-1}{2} - 1$, for $n$ odd, these two invariants coincide.

**Remark 0.4.1** (General agreement conjecture and its meaning). The conjectural agreement between $U$-rank and tensor rank for irreps of $GL_n$, is part of a general conjecture for all classical groups. Indeed, in [Gurevich-Howe17] we have defined these two invariants in the mentioned generality, and conjectured that they agree on the collection of “low” $U$-rank irreps. This would mean that tensor rank, which is defined in a formal way using the representation (aka Grothendieck) ring, has a concrete, down-to-earth meaning in terms of harmonic analysis on $G$ and its subgroups. A future goal should be to extend this kind of interpretation to representations of higher tensor rank.

The first main goal of this note is to establish that agreement conjecture for the group $GL_n(\mathbb{F}_q)$, for sufficiently large $q$. The value of these two notions of rank is that they provide, in some sense, two very different (to some extent complementary) reasons why certain analytic properties of a representation, such as dimension an character ratio are, in principle, what they are. So it is interesting and valuable to know that these two notions in fact agree in the relevant range.

In particular, we show that,

**Theorem.** Fix $0 \leq k \leq n$. Then for an irrep $\rho$ of $GL_n(\mathbb{F}_q)$ of rank $k$, we have an estimate:
\[ \frac{\chi_{\rho}(T)}{\dim(\rho)} = \begin{cases} 
\frac{1}{q^k} + o(...), & \text{if } k < \frac{n}{2}; \\
\frac{c_{\rho}}{q^k} + o(...), & \text{if } \frac{n}{2} \leq k \leq n - 1; \\
\frac{-1}{q^{n-1}}, & \text{if } k = n, 
\end{cases} \quad (0.4.2) \]

where \( c_{\rho} \) is a certain integer (independent of \( q \)) combinatorially associated with \( \rho \).

**Remark.** For irreps \( \rho \) of tensor rank \( \frac{n}{2} \leq k \leq n - 1 \), the constant \( c_{\rho} \) in (0.4.2) might be equal to zero. In this case, the estimate on \( \frac{\chi_{\rho}(T)}{\dim(\rho)} \) is simply \( o\left(\frac{1}{q^k}\right) \). However, it is typically non-zero, and in many cases it is 1.

The estimates in (0.4.2) induce similar results for the irreps of \( G = SL_n(\mathbb{F}_q) \). In particular, using some additional analytic information, Theorem 0.1.4 on the cardinality of the set \( M_{\ell,g} \) (0.1.2) follows, and our introductory story is complete.

A first proof of the estimates (0.4.2) appeared in [Gurevich-Howe19]. However, the fact that irreps of the same rank have essentially the same character ratio on the transvection (despite the fact that their dimensions might differ by multiple of a large power of \( q \)) remains somewhat of a surprise. In this note we clarify this phenomenon for low rank irreps (i.e., of rank \( k < \frac{n}{2} \) or \( \frac{n-1}{2} \), depending, respectively, if \( n \) is odd or even). We give a clear picture why this is so, using the \( U \)-rank realization of the notion of rank. This clarification is the second main contribution of this note.

0.5. **Fourier Transform of Sets of Matrices of Fixed Rank.** The third and final contribution of this note is an explicit formula for the value of the Fourier transform of the set \( (M_{n,n})_k \), of \( n \times n \) matrices of rank \( k \leq n \), over a finite field \( \mathbb{F}_q \), evaluated at a rank one matrix \( T \). The formula leads to an observation that for \( k < n \), this value is **positive**, which turns out to be a significant ingredient in our proof of the agreement conjecture mentioned above.

Let us write down the formula. Fixing an additive character \( \psi \neq 1 \) of \( \mathbb{F}_q \), we have in the standard manner the associated Fourier transform \( f \mapsto \hat{f} \) on the space of complex valued function on \( M_{n,n} \), given by

\[ \hat{f}(B) = \sum_{A \in M_{n,n}} f(A)\psi(-\text{trace}(B^t \circ A)), \quad B \in M_{n,n}. \]

Consider now the characteristic function \( 1_{(M_{n,n})_k} \) of the set \( (M_{n,n})_k \). It is easy to see that its Fourier transform takes only real values. Denote by \( \Gamma_{n,k} \) the Grassmannian of all \( k \)-dimensional subspaces of \( \mathbb{F}_q^n \).
Theorem. The value of $\hat{1}_{(M_n,n)_k}$ on a rank one matrix $T \in M_{n,n}$, is an integer that satisfies,

$$\hat{1}_{(M_n,n)_k}(T) = \sum_{A \in (M_n,n)_k} \psi(\text{trace}(T^t \circ A)) = (q^{2n-k} - 2q^n + 1)\left(\frac{\#(\Gamma_{n,k})^2 \#(GL_k)}{(q^n - 1)^2}\right),$$

and in particular it is positive if $k < n$, and negative if $k = n$.

After the acknowledgements and table of contents part, we proceed to the body of the note, and start with a detailed discussion on character ratios and the notion of $U$-rank for irreps of $GL_n(\mathbb{F}_q)$.

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Contents

0. Introduction 1
0.1. Example: Generation by Transvections 2
0.2. A Geometric Analog of the Generation Problem 4
0.3. Harmonic Analysis of the Generation Problem 5
0.4. Rank of a Representation 6
0.5. Fourier Transform of Sets of Matrices of Fixed Rank 8
1. Character Ratios and $U$-Rank 11
1.1. Numerics for Character Ratios vs. Dimension 11
1.2. $U$-Rank: Motivation, Intuition, and Formal Definition 12
1.3. Numerics for Character Ratios vs. $U$-Rank 16
2. Analytic Information on $U$-Rank $k$ Irreps of $GL_n$ 17
2.1. Character Ratios on the Transvection 17
2.2. Dimensions of Irreps 18
2.3. Cardinality of the Set of $U$-rank $k$ Irreps 19
3. The eta Correspondence and $U$-Rank 20
1. Character Ratios and $U$-Rank

In Section 0.1 we described an example that motivated the need to extract information on irreps of $SL_n(F_q)$. However, let us start with a slightly better behaved group, namely the group $GL_n = GL_n(F_q)$ of $n \times n$ invertible matrices with entries in $F_q$. Moreover, for this group\(^6\) let us concentrate for a while only on the problem of estimating the character ratios (CRs) on the transvection $T (0.1.1),$\(^{1.0.1}\)

$$\frac{\chi_{\rho}(T)}{\dim(\rho)}, \quad \rho \in \widehat{GL}_n.$$ \hfill (1.0.1)

We want now to develop some intuition for how the quantity (1.0.1) behaves.

1.1. Numerics for Character Ratios vs. Dimension. Let us look at the numerics—appearing in Figure 2—for the group $GL_8(F_3)$. Let us explain a bit what appears there. For each irrep $\rho$ of this group\(^6\) we want to plot its CR vs. its dimension. It can be deduced from [Deligne-Lusztig76, Green55] that, the

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\(^6\)In this note, for clarity, we denote irreps of $GL_n$ mostly by $\rho$ and of $SL_n$ mostly by $\pi$. 
dimensions of the irreps of groups like $GL_n(\mathbb{F}_q)$ are certain polynomials in $q$, and as such have degrees. This integer can be computed numerically, i.e., if $\rho \in \hat{GL}_8(\mathbb{F}_3)$, then this degree is approximately (the nearest integer to) $\log_3(\dim(\rho))$, and this is what appears\(^7\) on the horizontal axis of Figure 2. Next, looking closely at the numerics for the CRs (1.0.1), one learns that they tend to come in quantities which are powers of $\frac{1}{q}$, so it makes sense to plot the (nearest integer of the) absolute value of them in $\log_{\frac{1}{q}}$-scale, and this is what appears on the vertical axis of Figure 2.

What we can learn from Figure 2? Let’s read together part of the data presented. The group $GL_8(\mathbb{F}_q)$ has around $q$ irreps of dimension 1, and of course their character ratios are of size around 1, this is the blue dot at $(0,0)$ in the figure. After that, we have the around $q^2$ (more details later on why this is the cardinality) irreps of dimension around $q^7$, all of them seem to have $\text{CR} \approx \frac{1}{q}$. Next, we have around $q^3$ irreps of dimensions (already some variation) from around $q^{12}$ to $q^{14}$, but CR "exactly" $\frac{1}{q^2}$. Let us read one more layer, we have the black circles in Figure 2, of around $q^4$ irreps, and dimensions vary (by multiple of a quite large power of $q$) from around $q^{15}$ to $q^{19}$, but the CRs of these irreps are nearly the same, of order of magnitude $\frac{1}{q^3}$. Finally, another look at Figure 2 reveals a collection of irreps (see the black, green, and red circles above 16 there) all of them have the same dimension, around $q^{16}$, but for some reason they have very different CRs (respectively, $\frac{1}{q^4}$, $\frac{1}{q^7}$, and $\frac{1}{q^{19}}$).

In summary, based on the numerics appearing in Figure 2, we can make the following reasonable:

**Observation 1.1.1.** *In general, the character ratios (1.0.1) are not strictly controlled by the dimensions of the irreps.*

In the literature we are aware of, the CRs are estimated using information on the dimensions of the irreps (for example see [Bezrukavnikov-Liebeck-Shalev-Tiep18]); so in general (e.g., on elements like the transvection $T$ (0.1.1)) these estimates cannot be optimal.

In [Gurevich-Howe15, Gurevich-Howe17] we initiated the study of a pair of related invariants that seem to do better job than dimension in controlling the CRs (1.0.1). We proceed to discuss the first of these invariants.

1.2. *U-Rank: Motivation, Intuition, and Formal Definition.* Look again on Figure 2. What makes the irreps of $GL_8(\mathbb{F}_3)$ with character ratio of order of magnitude $\frac{1}{3^k}$, for $k < 4$, a family?, i.e., what puts them together?

More generally, we want an invariant that in some sense "knows" which irrep $\rho$ of $GL_n = GL_n(\mathbb{F}_q)$ has

$$\left| \frac{\chi_\rho(T)}{\dim(\rho)} \right| = \frac{1}{q^k} + o(...), \quad \text{for } k < \frac{n - 1}{2},$$

\(^7\)We denote by $\lfloor x \rfloor$ the nearest integer to the real number $x$. 
We want to give some intuition for the invariant we propose for such a job. Let us start with some data (for basic notions from the theory of algebraic groups see [Borel69]). Consider the vector space $\mathbb{F}_q^n$, and the "first $m$-coordinates" subspace

$$\mathbb{F}_q^n \supset X_m = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid x_j \in \mathbb{F}_q \right\}. \quad (1.2.1)$$

This allows to define three subgroups of $GL_n$, that play an important role in our story. The first is the stabilizer of $X_m$, i.e., the parabolic subgroup

$$P_m = \text{Stab}_{GL_n}(X_m) = \{ g \in GL_n; g(X_m) = X_m \}, \quad (1.2.2)$$

of elements of $GL_n$, that take the subspace $X_m$ to itself. Note that,

$$P_m = \left\{ \begin{pmatrix} C & A \\ 0 & D \end{pmatrix} \right\},$$

where $A \in M_{m,(n-m)}$—the space of $m \times (n-m)$ matrices—and $C \in GL_m$, $D \in GL_{n-m}$.

In particular, we have a (split) short exact sequence of groups (with obvious maps):

$$1 \to \left\{ \begin{pmatrix} I_m & A \\ 0 & I_{n-m} \end{pmatrix} \right\} \hookrightarrow \left\{ \begin{pmatrix} C & A \\ 0 & D \end{pmatrix} \right\} \to \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right\} \twoheadrightarrow 1, \quad (1.2.3)$$

where $U_m$ and $L_m$, are called, respectively, the unipotent radical and Levi component of $P_m = U_m \cdot L_m$. Here $I_m$ and $I_{n-m}$, are the identity matrices of order $m$ and $n-m$, respectively.

Note that the group $U_m$ is commutative and naturally isomorphic to the vector space $M_{m,(n-m)}$, using the map

$$\left( \begin{pmatrix} I_m & A \\ 0 & I_{n-m} \end{pmatrix} \right) \mapsto A.$$  

In particular, we might, and in many cases will, think of elements of $U_m$ as $m \times (n-m)$ matrices, and write $A \in U_m$, for a matrix $A \in M_{m,(n-m)}$.

Next, let us denote by $\hat{U}_m$ the Pontryagin dual of the commutative group $U_m$, consisting of all of its characters (one-dim reps). Then, in the standard manner, fixing a non-trivial additive character $1 \neq \psi$
of the field $\mathbb{F}_q$, one gets an isomorphism

$$\begin{cases}
U_m \overset{\sim}{\longrightarrow} \hat{U}_m; \\
B \mapsto \psi_B,
\end{cases}$$

where for $B \in U_m$, with transpose $B^t$, we define $\psi_B(A) = \psi(\text{trace}(B^tA))$, $A \in U_m$.

In particular, we have a notion of rank for every representation of $U_m$ as follows:

**Definition 1.2.4.** We define,

1. the rank of a character $\psi_B \in \hat{U}_m$, to be $\text{rank}(B)$;

and,

2. the rank of a representation $\varrho$ of $U_m$, to be the maximum over the ranks of characters that appear in $\varrho$.

For simplicity of exposition, let us now restrict (however, see Remark 1.2.13 below) our attention to the case,

$$U = U_{\left\lfloor \frac{n}{2} \right\rfloor}, \text{ with } \left\lfloor x \right\rfloor = \text{the largest integer below } x. \quad (1.2.5)$$

**Example 1.2.6** (Fourier transform of rank $k$ matrices). Fix integer $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and consider the collection

$$\mathcal{O}_k \subset U,$$

of rank $k$ matrices in $U$, and by

$$\varrho_{\mathcal{O}_k} = \sum_{B \in \mathcal{O}_k} \psi_B, \quad (1.2.7)$$

the representation of $U$, which is the direct sum of all characters corresponding to the members of $\mathcal{O}_k$.

Then,

- $\text{rank}(\varrho_{\mathcal{O}_k}) = k$.
- $\text{dim}(\varrho_{\mathcal{O}_k}) = \#(\mathcal{O}_k)$.
- The character $\chi_{\varrho_{\mathcal{O}_k}}$ of $\varrho_{\mathcal{O}_k}$, satisfies, as a function on $U$,

$$\chi_{\varrho_{\mathcal{O}_k}} = \sum_{B \in \mathcal{O}_k} \psi_B = \hat{1}_{\mathcal{O}_k}, \quad (1.2.8)$$

where $\hat{1}_{\mathcal{O}_k}$ is the Fourier transform (with respect to the, previously fixed, additive character $\psi$ of $\mathbb{F}_q$) of the characteristic function of $\mathcal{O}_k$.

- The transvection $T$ (0.1.1) has a $\text{GL}_n$-conjugate in $U$, that for simplicity we will also denote by $T$. Later, in Appendix A, we will show that, the value of the character sum (1.2.8) at $T$ is:
(1) positive, if \( k < \left\lfloor \frac{n}{2} \right\rfloor \); in fact, we obtain an explicit formula for \( \hat{1}_{O_k}(T) \) that implies,
\[
\frac{\hat{1}_{O_k}(T)}{\#(O_k)} = \frac{1}{q^k} + o(...). \tag{1.2.9}
\]

and,

(2) negative, if \( k = \left\lfloor \frac{n}{2} \right\rfloor \); in fact we get,
\[
\frac{\hat{1}_{O_{\left\lfloor \frac{n}{2} \right\rfloor}}(T)}{\#(O_{\left\lfloor \frac{n}{2} \right\rfloor})} = -\frac{1}{q^{\left\lfloor \frac{n}{2} \right\rfloor}} + o(...). \tag{1.2.10}
\]

In particular, we see that if \( \rho \) is a rep of \( GL_n \) which is "supported" just on one orbit \( O_k \subset U \), i.e., \( \rho \) restricted to \( U \) satisfies \( \rho|_U = m_k \cdot \varrho_{O_k} \) for some integer \( m_k > 0 \), then,

(a) \( \dim(\rho) = m_k \cdot \#(O_k) \);

and, using (1.2.9), for \( k < \left\lfloor \frac{n}{2} \right\rfloor \),

(b) \( \frac{\chi_{\rho}(T)}{\dim(\rho)} = \frac{\hat{1}_{O_k}(T)}{\#(O_k)} = \frac{1}{q^k} + o(...) \).

The discussion above suggests an invariant that might explain the behavior of the character ratio at the transvection. Indeed, take a rep \( \rho \) of \( GL_n \), and look at its restriction \( \rho|_U \) to \( U \). This has the following description:

**Proposition 1.2.10.** Characters of \( U \) of the same rank, appear in \( \rho|_U \) with the same multiplicity, i.e.,
\[
\rho|_U = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} m_r \cdot \varrho_{O_r}, \tag{1.2.11}
\]
for some non-negative integers \( m_r \), where \( \varrho_{O_r} \) is given by Formula (1.2.7).

For a proof of Proposition 1.2.10 see Appendix B.2.1.

So, motivated by (b) in Example 1.2.6 above, we introduce the key notion:

**Definition 1.2.12.** The **\( U \)-rank** of a representation \( \rho \) of \( GL_n \), is the maximal \( k, 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), such that \( m_k \neq 0 \), in (1.2.11).

We will write \( U\text{-rank}(\rho) = k \), or \( rank_U(\rho) = k \), to denote that a rep \( \rho \) of \( GL_n \), has \( U \)-rank \( k \), and will use the notation \( (\hat{GL}_n)_{U,k} \), for the set of all irreps with this property.

Sometimes we will call representations of \( U \)-rank less than \( \left\lfloor \frac{n}{2} \right\rfloor \), **low \( U \)-rank representations.**
Remark 1.2.13. It was shown in [Gurevich-Howe17] that for a low $U$-rank representation $\rho$ of $GL_n$, the value $\operatorname{rank}_U(\rho)$ is "independent" of $U$, in the following sense. Consider in $GL_n$ a general parabolic subgroup of block upper triangular matrices:

$$
\left( \begin{array}{cccc}
C_1 & A_{12} & A_{13} & \cdots & A_{1l} \\
C_2 & A_{23} & \cdots & A_{2l} \\
C_3 & \cdots & A_{3l} \\
\vdots & \vdots & \ddots & \vdots \\
C_l & \cdots & \cdots & \cdots \\
\end{array} \right)
$$

If the matrices $C_j$ are of size $m_j \times m_j$, then for a fixed $1 \leq i < j \leq l$, the collection of matrices $\{A_{ij}\}$ forms a (unipotent) subgroup of $GL_n$ isomorphic to $m_i \times m_j$ matrices. Let us call such subgroups of $GL_n$ standard matrix subgroups. The unipotent radicals $U_m$, introduced above, are examples of such subgroups, and in particular the group $U$ (1.2.5). The point is that you can develop the theory discussed above using the restrictions to each of the standard matrix subgroups. In particular, we have an induced notion of rank for representations of $GL_n$ which is associated with each of these subgroups, and a definition of what does it mean for a representation to be of low rank in each case. In [Gurevich-Howe17] we showed that if we have a representation $\rho$ of $GL_n$, and two standard matrix subgroups $U_{i,m} \simeq M_{i,m}, U_{i',m'} \simeq M_{i',m'} \subset GL_n$, such that $\rho$ is of low $U_{i,m}$-rank $k$, i.e., $\operatorname{rank}_{U_{i,m}}(\rho) = k < l, m$, and $U_{i',m'}$ is big enough, i.e., $l \leq l'$ and $m \leq m'$, then

$$
\operatorname{rank}_{U_{i,m}}(\rho) = \operatorname{rank}_{U_{i',m'}}(\rho).
$$

In particular, for a $U$-rank $k < \left\lfloor \frac{n}{2} \right\rfloor$ irrep, any large enough standard matrix subgroup can be used to detect this invariant. The meaning of this might be that, from the point of view of standard matrix subgroups the notion of rank developed above seems somehow canonical.

Let us look at some numerics that provide further evidence that we are on the right track.

1.3. Numerics for Character Ratios vs. $U$-Rank. As we saw in Part (b) of Example 1.2.6 above, if $\rho$ is an irrep of $GL_n$, for which its restriction to $U$, is supported solely on the rank $k$ matrices in $U$, for $k < \left\lfloor \frac{n}{2} \right\rfloor$, then its character ratio at the transvection (0.1.1) is around $\frac{1}{q^k}$. This leads us to define the notion of $U$-rank $k$, with the hope that if $\rho$ is a representation of that rank, then the contribution of the lower orbits in $U$, on which it might also be supported, will not contribute much to its CR on the transvection. The numerics done for $GL_8(\mathbb{F}_3)$ and appear in Figure 3, illustrate the fact that this is probably true for all (and probably only for) the irreps of rank $k$, as long as $k < \left\lfloor \frac{n}{2} \right\rfloor$. In particular,
Figure 3 suggests that for the irreps of $GL_n$ of rank less than $\left\lfloor \frac{n}{2} \right\rfloor$, the $U$-rank invariant does a better job than (compare with Figure 2) dimension in controlling the CRs at the transvection. In that range it puts (compare the black circles in Figures 3 and 2) the irreps of the "same" CRs together.

The above numerical observations can be quantified precisely and proved. This is part of what we do next.

2. Analytic Information on $U$-Rank $k$ Irreps of $GL_n$

To bound the sum (0.3.3)—discussed in Section 0.3—we would like to formulate statements on certain analytic properties of irreps of $GL_n$. In particular, on their character ratios at the transvection $T$ (0.1.1), and on their dimensions. We give now precise information on these quantities—in fact sharp estimates in term of the rank—for the irreps of $U$-rank $k$, with $k < \left\lfloor \frac{n}{2} \right\rfloor$. In addition, we calculate the number of such irreps.

The results follow from the analog results for tensor rank $k$ irreps formulated in Theorem 5.1.1, and the fact, given in Theorem 8.1.1, that if $k < \left\lfloor \frac{n}{2} \right\rfloor$, then for an irrep of $GL_n$ being of $U$-rank $k$ is the same thing as being of tensor rank $k$.

2.1. Character Ratios on the Transvection. For this quantity we have,
Theorem 2.1.1. Suppose $k < \lfloor \frac{n}{2} \rfloor$, and $\rho$ is an irrep of $GL_n$ of $U$-rank $k$. Then,

$$\frac{\chi_\rho(T)}{\dim(\rho)} = \frac{1}{q^k} + o(...).$$

(2.1.2)

Note that (2.1.2) is a formal validation to some of the phenomena that Figure 3 illustrates.

2.2. Dimensions of Irreps. As we already remarked earlier, although the CRs (2.1.2) of the irreps of $GL_n$ of $U$-rank $k$, $k < \lfloor \frac{n}{2} \rfloor$, are approximately the same, their dimensions might vary by multiple of a large power of $q$.

Figure 4 illustrates (compare with Figures 3 and 2) the distribution of the dimensions of the irreps of $GL_8(\mathbb{F}_q)$ within each given $U$-rank $k$. What you see there, can be formulated and proved in general for low rank irreps of $GL_n$. Indeed, we have the following sharp lower and upper bounds in term of the $U$-rank.

Theorem 2.2.1. Suppose $k < \lfloor \frac{n}{2} \rfloor$, and $\rho$ is an irrep of $GL_n$ of $U$-rank $k$. Then,

$$q^{k(n-k)} + o(...) \leq \dim(\rho) \leq q^{k(n-k) + \frac{k(k-1)}{2}} + o(...).$$

(2.2.2)

Moreover, the upper and lower bounds in (2.2.2) are attained.
Looking on the upper and lower bounds appearing in (2.2.2) and comparing with Estimate (2.1.2), we get a more quantitative form of the general pattern that was hinted before when we looked on Figure 2. In particular, for irreps of $U$-rank $k < \left\lfloor \frac{n}{2} \right\rfloor$:

- the dimensions vary by a multiple of $q^{k(k-1)/2}$, although their CRs are practically the same, of size around $\frac{1}{q^k}$;

and,

- for $n > \frac{(k+1)(k+2)}{2}$, the upper bound for the dimension of $U$-rank $k$ irreps is (for sufficiently large $q$) smaller than the lower bound for rank $k + 1$.

But,

- when $n < \frac{(k+1)(k+2)}{2}$, the range of dimensions for $U$-rank $k$ irreps overlaps (for large enough $q$) the range for $k + 1$, and the overlap grows with $k$. For $k$ in this range, representations of the same dimension can have different character ratios, which are accounted for by looking at rank.

2.3. Cardinality of the Set of $U$-rank $k$ Irreps. The cardinality of the set $(\hat{GL}_n)_{U,k}$, of irreps of $GL_n$ of $U$-rank $k$ can be estimated explicitly. Figure 5 illustrates how the cardinality of the set $(\hat{GL}_8(\mathbb{F}_3))_{U,k}$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{log$_q$-scale of the number of $U$-rank $k$ irreps of $GL_8(\mathbb{F}_q)$, $q = 3$.}
\end{figure}
of $U$-rank $k$ irreps of $GL_8(\mathbb{F}_3)$, grows with $k$.

In general, the following hold:

**Theorem 2.3.1.** We have,

$$\#((\widehat{GL}_n)_{U,k}) = \begin{cases} q^{k+1} + o(...), & \text{if } k < \left\lfloor \frac{n}{2} \right\rfloor; \\ q^n + o(...), & \text{if } k = \left\lfloor \frac{n}{2} \right\rfloor; \end{cases}$$

Our main tool to construct and analyze irreps of $GL_n$ of each given $U$-rank $k < \left\lfloor \frac{n}{2} \right\rfloor$ is what we discuss next.

3. **The eta Correspondence and $U$-Rank**

We want to answer the following:

**Question:** How to get the information (e.g., the results of Section 2) on $U$-rank $k < \left\lfloor \frac{n}{2} \right\rfloor$ irreps?

In fact, till this point in our story, it is a priori not clear why in general for each $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ there are at all irreps of $U$-rank $k$?

To answer the above questions for all classical groups we discussed in [Gurevich-Howe15, Gurevich-Howe17] the eta correspondence (EC). In the case of the general linear group $GL_n$, it first led in [Gurevich-Howe17] to an explicit parametrization of the members of the set $(\widehat{GL}_n)_{U,k}$ for $k < \left\lfloor \frac{n}{2} \right\rfloor$, enabling the computation of the cardinality of that set, and in particular to an answer for the second question above. Secondly, the EC leads to an explicit Harish-Chandra’s "philosophy of cusp forms" type formula (developed in [Gurevich-Howe17], and which will be recalled in Section 9) for a general member of $(\widehat{GL}_n)_{U,k}$ for $k < \left\lfloor \frac{n}{2} \right\rfloor$.

This particular formula allowed us in [Gurevich-Howe19] to obtain the CRs on the transvection and the dimensions for the irreps of $U$-rank $k < \left\lfloor \frac{n}{2} \right\rfloor$, announced in the previous section.

Next we go over some of the details of the basic construction of the EC for $GL_n$ given in [Gurevich-Howe17].

3.1. **The eta Correspondence.** Consider the vector space $L^2(M_{n,k})$ of complex valued functions on the set $M_{n,k}$ of $n \times k$ matrices over $\mathbb{F}_q$. The natural actions of $GL_n$ (from the left) and $GL_k$ (from the right) on such matrices induce the pair of commuting actions

$$GL_n \curvearrowright L^2(M_{n,k}) \curvearrowleft GL_k,$$

that form a single representation

$$\omega_{n,k} \text{ of } GL_n \times GL_k \text{ on } L^2(M_{n,k}), \quad (3.1.1)$$

given by $\left[\omega_{n,k}(g,h)f\right](m) = f(g^{-1}mh)$, for every $h \in GL_n$, $m \in M_{n,k}$, $g \in GL_n$, and $f \in L^2(M_{n,k})$. 

The pair \((GL_n, GL_k)\) is an example of a dual pair in the language of [Howe73], and \(\omega_{n,k}\) is sometime called its oscillator (aka Weil) representation.

Let us decompose \(\omega_{n,k}\) into a direct sum of isotypic components for the irreps of \(GL_k\),

\[
\omega_{n,k} \simeq \sum_{\tau \in \hat{GL}_k} M(\tau) \otimes \tau, \tag{3.1.2}
\]

where \(M(\tau)\) denotes the multiplicity space \(\text{Hom}_{GL_k}(\tau, \omega_{n,k})\) which is a representation of \(GL_n\).

What can be said about \(M(\tau)\)?

It turns out that for \(k \leq \left\lfloor \frac{n}{2} \right\rfloor\), although it might be reducible, each \(M(\tau)\) has a unique big irreducible chunk which is (in a quantitative sense) most of it, and that can be effectively analyzed. In fact, the notion of \(U\)-rank gives a way to distinguish it inside \(M(\tau)\). All of this is the content of the following:

**Theorem 3.1.3.** Assume \(k \leq \left\lfloor \frac{n}{2} \right\rfloor\). We have,

1. \(U\)-rank \(k\) piece: Each \(M(\tau)\) contains a unique irreducible component \(\eta(\tau)\) of \(U\)-rank \(k\), and it appears with multiplicity one, in addition to irreps of lower \(U\)-rank, i.e.,

\[
M(\tau) = \underbrace{\eta(\tau)}_{U\text{-rank } k} + \text{lower } U\text{-rank irreps.}
\]

   Moreover,

2. eta correspondence: The assignment \(\tau \mapsto \eta(\tau)\), defines a one-to-one map

\[
\eta : \hat{GL}_k \hookrightarrow (\hat{GL}_n)_{U,k}. \tag{3.1.4}
\]

We call the map (3.1.4) the eta correspondence.

A proof of Parts (1) and (2) of Theorem 3.1.3 appeared in [Gurevich-Howe17].

We proceed to show that, in the case \(k < \left\lfloor \frac{n}{2} \right\rfloor\), we can say a bit more.

### 3.2. Exhaustivity of the eta Correspondence

Suppose \(\rho\) is a representation of \(GL_n\) and \(\chi\) a character (i.e., one dimensional rep) of this group. We will call the representation \(\chi \otimes \rho\), the twist of \(\rho\) by \(\chi\). Note that since every character of \(GL_n\) is trivial on \(U\), then the set \((\hat{GL}_n)_{U,k}\) is preserved under twists by characters.

**Example 3.2.1** (All \(U\)-rank \(k = 1\) irreps?). The oscillator representation \(\omega_{n,1}\) of \(GL_n \times GL_1\) is given by the natural actions on the space \(L^2(\mathbb{F}_q^n)\) of complex valued functions on the set of column vectors of
length \( n \) over \( \mathbb{F}_q \). Let us assume that \( n \geq 4 \). In this case the decomposition (3.1.2) is

\[
\omega_{n,1} = \sum_{\lambda \in \hat{GL}_1} \mathcal{M}(\lambda),
\]

where \( \mathcal{M}(\lambda) = \{ f : \mathbb{F}_q^n \to \mathbb{C}^*; f(a v) = \lambda(a) f(v), \ a \in \mathbb{F}_q^*, \ v \in \mathbb{F}_q^n \} \).

It is not difficult to see using direct calculations that,

(1) for \( \lambda \neq 1 \) the space \( \mathcal{M}(\lambda) \) is irreducible as a \( GL_n \)-representation, it has dimension around \( q^{n-1} \), and its CR on \( T \) (0.1.1) is around \( \frac{1}{q} \).

In particular, each multiplicative character \( \lambda \neq 1 \) of \( GL_1 = \mathbb{F}_q^* \), is assigned by the EC (3.1.4) to the \( U \)-rank \( k = 1 \) irrep

\[ \eta(\lambda) = \mathcal{M}(\lambda). \]

and,

(2) The space \( \mathcal{M}(1) = (2 \times \text{trivial rep}) \oplus \mathcal{M}(1)_0 \), where \( \mathcal{M}(1)_0 = \{ f \in \mathcal{M}(1); f(0) = 0 \text{ and } \sum_{v \in \mathbb{F}_q^n} f(v) = 0 \} \) is irreducible as a \( GL_n \)-representation, it has dimension around \( q^{n-1} \), and its CR on \( T \) is around \( \frac{1}{q} \).

In particular, we have the \( U \)-rank \( k = 1 \) irrep

\[ \eta(1) = \mathcal{M}(1)_0. \]

It can be shown that, if we twist the above irreps by characters we obtain overall a collection of pairwise non-isomorphic \( U \)-rank \( k = 1 \) irreps, and the question is whether we exhausted the set \( \hat{GL}_n \)?

In [Gurevich-Howe15, Gurevich-Howe17] we conjectured that the answer to the above question is yes, and formulated the following:

**Conjecture 3.2.2 (Exhaustion).** Suppose \( k < \left[ \frac{n}{2} \right] \). Then, up to twist by a character, every irrep of \( U \)-rank \( k \) of \( GL_n \), is in the image of eta correspondence (3.1.4).

In Section 8, we will show that for sufficiently large \( q \), Conjecture 3.2.2 holds true.

### 3.3. Concluding Remarks on eta and the Analytic Information.

We would like to remark that,

(A) **Concerning CRs:** it was shown in [Gurevich-Howe19] (see also Section 9) that the description of the irrep \( \eta(\tau) \)'s appearing in Part (1) of Theorem 2.3.1 can be made effective so one can compute their CRs on the transvection and obtain Theorem 2.1.1.
Overall, note for any irrep $\rho \in \hat{GL}_n(U,k)$, $k < \lfloor \frac{n}{2} \rfloor$, we indeed have
\[
\frac{\chi_\rho(T)}{\dim(\rho)} = \frac{\mathbf{1}_{\mathcal{O}_k}(T)}{\#(\mathcal{O}_k)} + o(...),
\]
supporting the intuition we had in the process of giving the formal definition of $U$-rank in Section 1.2.

(B) Concerning dimensions: in [Gurevich-Howe19] we gave an effective description of the $\eta(\tau)$’s, that appears in the eta correspondence, which in particular, implies,
\[
\dim(\eta(\tau)) = \dim(\tau) \#(\mathcal{O}_k) + o(...),
\]
and so, together with the fact (see Lemma B.4.2) that $\#(\mathcal{O}_k) = q^{k(n-k)} + o(...)$, we see that
\[
\dim(\eta(\tau)) = \dim(\tau) \cdot q^{k(n-k)} + o(...).
\]
Moreover, the irreps of $GL_k$ all have dimensions in the range $1$ to $q^{k(n-k)} + o(...)$ [Green55], and we find that Theorem 2.2.1 follows. In Section 9, we recall another argument from [Gurevich-Howe19] that verifies Theorem 2.2.1.

(C) Concerning cardinality: Part (2) of Theorem 3.1.3 combined with the surjectivity of $\eta$ (3.1.4) gives the cardinality of $(\hat{GL}_n)_{U,k}$ announced in Theorem 2.3.1. Indeed, the number of irreps of $GL_k$ (the size of the set of conjugacy classes of that group) is $q^k + o(...)$, and in [Gurevich-Howe19] we showed that after we twist the members in the image of $\eta$ by the $q-1$ characters of $GL_n$, you get $q^{k+1} + o(...)$ non-isomorphic irreps, as claimed.

4. Character Ratios and Tensor Rank

Most of the irreps of $GL_n$ are of the maximal possible $U$-rank, i.e., $\lfloor \frac{n}{2} \rfloor$. Although the CRs of these irreps might be relatively small—maybe even too small to contribute to the harmonic analysis sums, such as (0.3.3), that come up in important counting problems—it is still the case that we need to estimate them.

To say that an irrep of $GL_n$ is of $U$-rank $\lfloor \frac{n}{2} \rfloor$ does an injustice to it from the analytic perspective. For example, look on the numerical data collected for the group $GL_8(F_3)$ and appear in Figure 3. It shows a large variation of the CRs at the transvection for the irreps of $U$-rank equal to $\lfloor \frac{8}{2} \rfloor = 4$.

So, we want an extension of the notion of $U$-rank in order to control the CRs $\frac{\chi_\rho(T)}{\dim(\rho)}$ also within the irreps of $U$-rank $k = \lfloor \frac{n}{2} \rfloor$.

In [Gurevich-Howe17] we proposed such an extension, called tensor rank, for representations of all classical groups. In fact it appeared with different terminology already in the unpublished notes [Howe73].
We proceed to discuss this notion in the case of $GL_n$, where we showed in [Gurevich-Howe19] that it does a pretty good job—see Figure 6.

4.1. Tensor Rank: Formal Definition and Agreement with $U$-Rank. The definition of tensor rank will be given using an extension of the way we realized the set $(\hat{GL}_n)_{U,k}$, for $k < \lfloor \frac{n}{2} \rfloor$.

Consider the oscillator representation $\omega_n$ of $GL_n$ given by its right action on the space of complex valued functions $L^2(\mathbb{F}_q^n)$ on $\mathbb{F}_q^n$, or more generally consider its $k$-fold tensor product $GL_n^{\otimes k} \rightrightarrows L^2(M_{n,k})$, given using the right action of this group on $n \times k$ matrices. Note that $\omega_n^{\otimes k}$ is just the restriction of $\omega_{n,k}$ (3.1.1) to $GL_n$.

Denote by $\hat{GL}_n(\omega_n^{\otimes k})$ the set of irreps of $GL_n$ that appear in $\omega_n^{\otimes k}$, and by $1$ the trivial representation. In [Gurevich-Howe19] we showed that,

**Proposition 4.1.1.** We have a sequence of proper containments

$$\{1\} \subsetneq \hat{GL}_n(\omega_n^{\otimes 1}) \subsetneq \ldots \subsetneq \hat{GL}_n(\omega_n^{\otimes n}) = \hat{GL}_n.$$  

(4.1.2)

Now, looking at (4.1.2) and taking into account the action of characters (i.e., 1-dim representations) on irreps, we introduce in [Gurevich-Howe17] the following:

**Definition 4.1.3 (Tensor rank).** We will say that $\rho \in \hat{GL}_n$ is of tensor rank $k$, if the minimal $\ell$ that we can write it as a tensor product of a character and an irrep from $\hat{GL}_n(\omega_n^{\otimes \ell})$ is $\ell = k$.

We may use the notations $\otimes\text{-}rank(\rho) = k$, or $\text{rank}_{\otimes}(\rho) = k$, to indicate that a representation $\rho$ of $GL_n$ has tensor rank $k$, and denote the set of all such irreps by $(\hat{GL}_n)_{\otimes,k}$.

**Remark 4.1.4.** The notion of tensor rank induces (and is defined by) a filtration on the representation ring

$$R(GL_n) = \mathbb{Z}[\hat{GL}_n],$$

generated from the set $\hat{GL}_n$ using the operations of addition and multiplication given, respectively, by direct sum $\oplus$ and tensor product $\otimes$. Indeed, let us extend the definition of tensor rank to arbitrary (not necessarily irreducible) representation of $GL_n$ and say it is of tensor rank $k$ if it contains irreps of tensor rank $k$ but not of higher tensor rank. In particular, we have the tensor rank filtration which is obtained

---

8Up to a sign, $\omega_n$ is the restriction of the oscillator representation of $Sp_{2n}$ to $GL_n$ [Gerardin77, Howe73, Weil64].
by letting $F_{\otimes,k}$ be the collection of elements of $R(G)$ that are sums of irreps of tensor rank less or equal to $k$, and it satisfies

- $F_{\otimes,(k-1)} \subset F_{\otimes,k}$, $F_{\otimes,i} \otimes F_{\otimes,j} \subset F_{\otimes,i+j}$ for every $i, j, k$;

and

- $F_{\otimes,n} = R(G)$.

Finally, note that Part (1) of Theorem 3.1.3 implies that

$$(\hat{GL}_n)_{\otimes,k} \subset (\hat{GL}_n)_{U,k}, \quad \text{for } k < \left\lfloor \frac{n}{2} \right\rfloor ,$$

and Conjecture 3.2.2 can be restated as follows:

**Conjecture 4.1.6 (Agreement).** The inclusion (4.1.5) should be replaced by equality.

In particular, we conclude that, tensor rank is a natural extension of the notion of $U$-rank. But, is it going to split nicely the collection of $U$-rank $k = \left\lfloor \frac{n}{2} \right\rfloor$ irreps of $GL_n$?

4.2. **Numerics for Character Ratios vs. Tensor Rank**. The answer to the above question seems to be yes and, before we write down formal statements, we would like to see this with the aid of some supportive numerical data collected for the group $GL_8(\mathbb{F}_3)$ which appears in Figures 6, 3, and 2.

![Figure 6. log\(_3\)-scale of CRs vs. \(\otimes\)-rank for irreps \(\rho\) of \(GL_8(\mathbb{F}_q)\), \(q = 3\).](image)

Recall that Figure 3 illustrates the general fact that for irreps of $GL_n$ of $U$-rank $k < \left\lfloor \frac{n}{2} \right\rfloor$, the CRs at the transvection $T$ are all essentially of the same size $\frac{1}{q^k}$, despite the fact that the dimensions of the
The irreps involved might vary by a multiple of large power of $q$. Due to the agreement conjecture this should also holds—and illustrated in Figure 6—for low $\otimes$-rank irreps, i.e., these of $\otimes$-rank $k < \lfloor \frac{n}{2} \rfloor$.

The next thing that Figure 6 illustrates is that indeed (compare with Figure 3) the tensor rank invariant splits further the collection of irreps of $U$-rank of $k = \lfloor \frac{n}{2} \rfloor$, and that the splitting seems to do more or less the job we wanted the tensor rank to do. Specifically, Figure 6 illustrates the fact that for tensor rank $\frac{n}{2} \leq k \leq n - 1$, the CRs at the transvection are of the order of magnitude of $\frac{1}{q_k}$ time a constant (independent of $q$), and that for all tensor rank $n$ irreps the CRs are exactly $\frac{1}{q_{n-1}}$ in absolute value.

Finally, a look at the black-green-red circles above 16 in Figure 2, and how they appear in Figure 6, illustrates the fact that tensor rank provides a reason for why irreps of the same order of magnitude of dimension can have very different CRs at $T$, namely, the answer is that these irreps have different tensor ranks.

The above numerical results can be quantified precisely and proved. This is part of what we do next.

5. Analytic Information on Tensor Rank $k$ Irreps of $GL_n$

In this section we present results obtained in [Gurevich-Howe19] concerning the character ratios and dimensions of the irreps of $\otimes$-rank $k$, i.e., the members of $\widehat{(GL_n)}_{\otimes,k}$, including the cardinality of that set. Although we will not prove these results in this note, for the benefit of the reader we explain in Section 9 what are the main sources of informations that enable us to derive them.

5.1. Character Ratios on the Transvection. For the CRs on the transvection $T(0.1.1)$, the following, essentially sharp, estimate in term of the tensor rank holds:

**Theorem 5.1.1.** Fix $0 \leq k \leq n$. Then, for $\rho \in \widehat{(GL_n)}_{\otimes,k}$, we have an estimate:

\[
\frac{\chi_\rho(T)}{\dim(\rho)} = \begin{cases} 
\frac{1}{q_k} + o(...), & \text{if } k < \frac{n}{2}; \\
\frac{c_\rho}{q_k} + o(...), & \text{if } \frac{n}{2} \leq k \leq n - 1; \\
\frac{1}{q_{n-1}}, & \text{if } k = n,
\end{cases}
\]

(5.1.2)

where $c_\rho$ is a certain integer (independent of $q$) combinatorially associated with $\rho$.

**Remark 5.1.3.** For irreps $\rho$ of tensor rank $\frac{n}{2} \leq k \leq n - 1$, the constant $c_\rho$ in (5.1.2) might be equal to zero. In this case, the estimate on $\frac{\chi_\rho(T)}{\dim(\rho)}$ is simply of $\frac{1}{q_k}$. However, the possibility of $c_\rho = 0$ is fairly rare, and (at least for $k \neq n - 1$) we are not sure if it happens at all.

Note that (5.1.2) is a formal validation to some of the phenomena that Figure 6 illustrates.
5.2. **Dimensions of Irreps.** We proceed to present information on the dimensions of the irreps of tensor rank $k$. Figure 7 gives a numerical illustration for the distribution of the dimensions of the irreps of $GL_8(\mathbb{F}_q)$ within each given tensor rank.

![Figure 7. log_q-scale of dimension vs. ⊗-rank for irreps $\rho$ of $GL_8(\mathbb{F}_q)$, $q = 3$.](image)

The following are the sharp lower and upper bounds obtained in [Gurevich-Howe19] (and that formally explain Figure 7; the black-green-red dots were discussed in Section 1.1) on the dimensions of the ⊗-rank $k$ irreps:

**Theorem 5.2.1.** Fix $0 \leq k \leq n$. Then, for $\rho \in (\hat{GL}_n)_{\otimes k}$, we have an estimate:

$$q^{k(n-k)} + o(...) \geq \dim(\rho) \geq \begin{cases} 
q^{(n-k)(3k-n)} + o(...), & \text{if } \frac{n}{2} \leq k < \frac{2n}{3}; \\
q^{k(n-k)+\frac{k^2}{4}+3(k-2)} + o(...), & \text{if } \frac{2n}{3} \leq k \leq n, \text{ odd}; \\
\end{cases} \quad (5.2.2)$$

Moreover, the upper and lower bounds in (5.2.2) are attained.

5.3. **The Number of Irreps of Tensor Rank $k$ of $GL_n$.** Finally, we present information concerning the cardinality of the set of irreps of ⊗-rank $k$—see Figure 8 for illustration.
Figure 8. \(\log_q\)-scale of the number of \(\otimes\)-rank \(k\) irreps of \(GL_8(\mathbb{F}_q)\), \(q = 3\).

In this aspect, we have the following essentially sharp estimate:

**Theorem 5.3.1.** Fix \(0 \leq k \leq n\). Then, we have,

\[
\#((\widehat{GL}_n)_{\otimes,k}) = \begin{cases} 
q^{k+1} + o(\ldots), & \text{if } k \leq n - 2; \\
c_k q^n + o(\ldots), & \text{if } n - 2 < k,
\end{cases}
\]

where \(0 < c_{n-1}, c_n < 1, c_{n-1} + c_n = 1\).

5.4. **Some Remarks.** We would like to make several remarks concerning the analytic information given just above, that extend in a bit more detailed way similar remarks given for \(U\)-rank in Section 2.2.

5.4.1. **Tensor Rank vs. Dimension as Indicator for Size of Character Ratio.** Looking back on the analytic information presented in the Sections 5.1 and 5.2, we observe the following:

(A) For irreps in a given tensor rank.

A comparison of (5.2.2) and (5.1.2) demonstrates—see Figure 9 for a summary—what we illustrated in Sections 1.1 and 4.2: Within a given tensor rank \(k\) the dimensions may vary by a large factor (around \(q^{k(k-1)/2}\) for rank \(k < \frac{n}{2}\), and between \(q^{k^2/4}\) to \(q^{k^2/2}\) for \(\frac{n}{2} \leq k\) - quantities are given in approximate order of magnitude of power of \(q\)) but the CRs at the transvection are practically the same, of size around \(\frac{1}{q^2}\) (for \(\frac{n}{2} \leq k \leq n - 1\) a multiple of \(\frac{1}{q^2}\) by a constant independent of \(q\)).
RANKS FOR REPRESENTATIONS OF $GL_n$ OVER FINITE FIELDS, THEIR AGREEMENT, AND POSITIVITY

| Tensor rank | Number of irreps | Dimension varies by factor | Character ratio at $T$ |
|-------------|------------------|----------------------------|-----------------------|
| $k < \frac{n}{2}$ | $q^{k+1}$ | $q^{\frac{k(k+1)}{2}}$ | $\frac{1}{q^k}$ |
| $\frac{n}{2} \leq k \leq n - 1$ | $q^{k+1}$ | $q^{\frac{k^2}{4}}$ to $q^{\frac{k^2}{2}}$ | $\frac{c}{q^k}$ |
| $k = n$ | $q^n$ | $q^{\frac{k^2}{4}}$ | $\frac{1}{q^{n+1}}$ |

Figure 9. CRs vs. variation in dimensions (in order of magnitude of power of $q$) for $\hat{GL}_n$.

(B) For irreps of different tensor ranks.

Looking on (5.2.2) we notice that:

- for $n > \frac{(k+1)(k+2)}{2}$, the upper bound for the dimension of $\otimes$-rank $k$ irreps is (for sufficiently large $q$) smaller than the lower bound for rank $k + 1$.

  But,

- when $n < \frac{(k+1)(k+2)}{2}$, the range of dimensions for $\otimes$-rank $k$ irreps overlaps (for large enough $q$) the range for $k + 1$, and the overlap grows with $k$. For $k$ in this range, representations of the same dimension can have different character ratios, which are accounted for by looking at rank.

In conclusion, it seems that tensor rank of a representation is a better indicator than dimension for the size of its character ratio, at least on elements such as the transvection.

5.4.2. Comparison with Existing Formulations in the Literature. In most of the literature on character ratios that we are aware of (see, e.g., [Bezrukavnikov-Liebec-Shalev-Tiep18] and the papers cited there.), estimates on character ratios are given in terms of the dimension of representations.

Although the dimension is a standard invariant of representations, as we have seen in Parts (A) and (B) of Section 5.4.1, the dimensions of representations with a given tensor rank can vary substantially (i.e., by large powers of $q$), while the character ratio at the transvection stays more or less constant (at least for $k < \frac{n}{2}$). Thus, using only dimension to bound character ratio might lead to non-optimal estimates.

In particular, the estimates in this note for the character ratio on the transvection are optimal (in terms of the tensor rank), and are, in general, stronger than the corresponding estimates in the paper cited above. For example, for $k < \frac{n}{2}$, rather than the bound of $\frac{1}{q^k}$, the paper [Bezrukavnikov-Liebec-Shalev-Tiep18] gives bounds of the order of magnitude of $\frac{q^{k(k-1)}}{q^k}$, and the exponent $\frac{k(k-1)}{n-1}$ can be fairly large when $n$ is large and $k$ is near $\frac{n}{2}$. The table in Figure 10 gives some examples of the relationship between the results of this note, and of the literature cited above.
Figure 10. Bounds on CRs: Current literature vs. this note (in order of magnitude).

Recall that for the motivational example described in the introduction (see Section 0.3) we wanted to have information on irreps of $SL_n = SL_n(\mathbb{F}_q)$, $n \geq 3$. These can be deduced from the one we just formulated above for $GL_n$, as $SL_n$ is a very big subgroup of $GL_n$.

6. Analytic Information on Tensor Rank $k$ Irreps of $SL_n$

In this section we formulate the analytic results obtained in [Gurevich-Howe19] for the tensor rank $k$ irreps of $SL_n$, $n \geq 3$. The case of $SL_2$ is somewhat special and [Gurevich-Howe18] was devoted to its description.

6.1. Tensor Rank for Representations of $SL_n$. First we introduce the following terminology. We assume $n \geq 3$.

**Definition 6.1.1.** We will say that an irreducible representation $\pi$ of $SL_n$ has tensor rank $k$ if it appears in the restriction of a tensor rank $k$ (and not less) irrep of $GL_n$.

As before, we denote by $(\hat{SL}_n)_{\otimes, k}$ the set of irreps of $SL_n$ of $\otimes$-rank $k$.

**Remark 6.1.2.** Note that the condition that $\pi$ should satisfy in Definition 6.1.1 is equivalent to the requirement that (replacing $GL_n$ by $SL_n$) in (4.1.2) it will appear in $\hat{SL}_n(\omega_{n}^{\otimes \ell})$ for $\ell = k$, but not for $\ell < k$.

The main technique used in [Gurevich-Howe19] to get information on irreps of $SL_n$ is through the way they appear inside irreps of $GL_n$. This can be understood using the Clifford-Mackey’s theory.
We will not repeat the analysis in this note, but the "intuitive picture" is that nearly every irrep of $GL_n$ stays irreducible after restriction to $SL_n$, and hence—using some additional favorite facts—the results for $SL_n$ are the "same" as for $GL_n$.

We start with the estimates on the character ratios.

6.2. Character Ratios on the Transvection. We have the following sharp estimates in term of the tensor rank:

\begin{equation}
\frac{\chi_\pi(T)}{\dim(\pi)} = \begin{cases} 
\frac{1}{q^k} + o(...), & \text{if } k < \frac{n}{2}; \\
\frac{c_\pi}{q^k} + o(...), & \text{if } \frac{n}{2} \leq k \leq n - 1; \\
\frac{-1}{q^{n-1}}, & \text{if } k = n,
\end{cases}
\end{equation}

(6.2.2)

where $c_\pi$ is a certain integer (independent of $q$) combinatorially associated with $\pi$.

Remark 6.2.3. For irreps $\pi$ of tensor rank $\frac{n}{2} \leq k \leq n - 1$, the constant $c_\pi$ in (6.2.2) might be equal to zero. In this case, the estimate on $\frac{\chi_\pi(T)}{\dim(\pi)}$ is simply $o(\frac{1}{q^k})$.

6.3. Dimensions of Irreps. It turns out that most of the irreps of $GL_n$ that give the lower and upper bounds on dimensions of tensor rank $k$ irreps, stays irreducible as representations of $SL_n$. As a consequence, from the corresponding results for $GL_n$, we obtain,

\begin{equation}
q^{k(n-k) + \frac{k(k-1)}{2}} + o(...) \geq \dim(\pi) \geq \begin{cases} 
q^{k(n-k)} + o(...), & \text{if } k < \frac{n}{2}; \\
q^{(n-k)(3k-n)} + o(...), & \text{if } \frac{n}{2} \leq k < \frac{2n}{3}; \\
q^{k(n-k) + \frac{k^2}{4}} + o(...), & \text{if } \frac{2n}{3} \leq k \leq n, \text{ even}; \\
q^{k(n-k) + \frac{(k-3)^2}{4} + 3(k-2)} + o(...), & \text{if } \frac{2n}{3} \leq k \leq n, \text{ odd};
\end{cases}
\end{equation}

(6.3.2)

Moreover, the upper and lower bounds in (6.3.2) are attained.
6.4. The Number of Irreps of Tensor Rank $k$ of $SL_n$. The fact, mentioned earlier, that most tensor rank $k$ irreps of $GL_n$ stay irreducible after restricting them to $SL_n$, is the core fact used in [Gurevich-Howe19] to deduce (see estimates (5.3.2)) the following:

**Proposition 6.4.1.** Fix $n \geq 3$, and $0 \leq k \leq n$. Then, we have,

$$\#(\tilde{SL}_n \otimes k) = \begin{cases} 
q^k + o(...), & \text{if } k \leq n - 2; \\
ckq^{k-1} + o(...), & \text{if } n - 2 < k,
\end{cases}$$

(6.4.2)

where $0 < c_{n-2}, c_n < 1$, with $c_{n-1} + c_n = 1$.

7. Back to the Generation Problem

Having the analytic information on the irreps of $SL_n$, $n \geq 3$, we can address the generation problem discussed in the introduction. In particular, we can derive Theorem 0.1.4.

7.1. Setting and Statement. We considered (see the introduction) the conjugacy class $C \subset SL_n$ of the transvection $T$ (0.1.1), and for $\ell \geq n$, we looked at the set

$$M_{\ell,g} = \{(c_1, \ldots, c_\ell) \in C^\ell; \ c_1 \cdot \ldots \cdot c_\ell = g\},$$

(7.1.1)

where $g$ is an element of $SL_n$ which one can’t form by less than $n$ products from $C$, and is regular semi-simple, i.e., all its eigenvalues over an algebraic closure of $\mathbb{F}_q$ are different. We explained that this means that our $g$ is semi-simple regular and 1 is not one of its eigenvalues. Let us denote the set of all such $g$’s by $(\partial(G))_{reg}$.

We wanted to compute the cardinality of the set $M_{\ell,g}$ (7.1.1), and to show:

**Theorem 0.1.4 (Set-theoretic size - restated).** For an element $g \in (\partial(G))_{reg}$, we have,

$$\#(M_{\ell,g}) = \frac{\#(C^\ell)}{\#(SL_n)} \cdot \begin{cases} 
1 - O\left(\frac{1}{q}\right), & \text{if } \ell = n; \\
1 - \left(\frac{2}{q}\right)\frac{\ell-n + o(...)}{q^\ell}, & \text{if } \ell > n.
\end{cases}$$

(7.1.2)

7.2. A Proof of the Set-Theoretic Size Theorem. To proof Theorem 0.1.4, we follow the strategy proposed in Section 0.3 and invoke harmonic analysis for our purpose. We have the Frobenius type formula

$$\#(M_{\ell,g}) = \frac{\#(C^\ell)}{\#(SL_n)} \cdot \sum_{1 \neq \pi \in \hat{SL}_n} \dim(\pi) \left(\frac{\chi_\pi(T)}{\dim(\pi)}\right)^\ell \chi_\pi(g^{-1}),$$

(7.2.1)

and so we just need to show the sum $S_{\ell,g}$ in (7.2.1) is of the size of the error term in statement (7.1.2).
At this point we already know about tensor rank and can further split $S_{\ell,g}$ over the various ranks

$$S_{\ell,g} = \sum_{k=1}^{n} \sum_{\pi \in (\hat{S}L_{n})_{\otimes,k}} \dim(\pi) \left( \frac{\chi_{\pi}(T)}{\dim(\pi)} \right)^{\ell} \chi_{\pi}(g),$$

and analyze each of the sub-sums $(S_{\ell,g})_{k}, k = 1,\ldots,n$, in (7.2.2).

**Claim 7.2.3.** Suppose, $g \in (\partial(G))_{reg}$. Then, for $\ell \geq n$,

1. For $k = 1$, $(S_{\ell,g})_{1} = -\frac{2}{q} (\frac{1}{q})^{\ell-n} + o(...)$;
2. For $k = 2$, $(S_{\ell,g})_{2} = O(\frac{1}{q^{2}} (\frac{1}{q})^{\ell-n})$;
3. and more generally,
   
   For $2 \leq k \leq n$, $(S_{\ell,g})_{k} = \begin{cases} 
   O\left( \frac{1}{q^{2}} \left( \frac{1}{q} \right)^{\ell-n} \right), & \text{if } k \leq n-2; \\
   O\left( \frac{1}{q^{n-2}} \left( \frac{1}{q} \right)^{\ell-n} \right), & \text{if } n-2 < k. 
   \end{cases}$

For a proof of Claim 7.2.3, see Appendix B.3.1.

Finally, the estimates (7.1.2) follows from Claim 7.2.3. This completes the proof of Theorem 0.1.4.

### 8. A Proof of the Agreement Conjecture

In this section we propose a Fourier theoretic proof of the agreement conjecture that uses the CRs estimates (5.1.2), and certain curious positivity results for the Fourier transform of the collection of matrices of a fixed low enough rank.

#### 8.1. The Statement

We know that for $k < \left\lfloor \frac{n}{2} \right\rfloor$, we have $(\hat{GL}_{n})_{\otimes,k} \subset (\hat{GL}_{n})_{U,k}$.

We will show that the following is true:

**Theorem 8.1.1 (Agreement).** Suppose $k < \left\lfloor \frac{n}{2} \right\rfloor$. Then, for sufficiently large $q$, we have,

$$(\hat{GL}_{n})_{\otimes,k} = (\hat{GL}_{n})_{U,k}.$$
Fact 8.2.1 is a straightforward combination of Theorem A.2.3, and Part (1) of Lemma 1, in Appendices A.2, and B.4.2, respectively.

8.3. **Proof of the Agreement Theorem.** To prove Theorem 8.1.1, take \( \rho \in (\hat{GL}_n)_{U,k} \), \( k < \lfloor \frac{n}{2} \rfloor \); and compute,

\[
\frac{\chi_\rho(T)}{\dim(\rho)} = \frac{\sum_{r=0}^{k} m_r \cdot \hat{1}_{\mathcal{O}_r}(T)}{\sum_{r=0}^{k} m_r \cdot \#(\mathcal{O}_r)} = \frac{\sum_{r=0}^{k} m_r \cdot (\frac{1}{q^r} + o(...)) \cdot \#(\mathcal{O}_r)}{\sum_{r=0}^{k} m_r \cdot \#(\mathcal{O}_r)} \geq \frac{\sum_{r=0}^{k} m_r \cdot \#(\mathcal{O}_r)}{\sum_{r=0}^{k} m_r \cdot \#(\mathcal{O}_r)} = \frac{1}{q^k} + o(...),
\]

where, the first equality is the expansion, of the restriction of \( \rho \) to \( U \), discussed in Section 1.2 (see Formulas (1.2.11), (1.2.8), and (1.2.7)), the second equality is by Formula (8.2.2), and finally, the inequality at the bottom is due to the positivity of \( \hat{1}_{\mathcal{O}_r}(T) \) for \( r < \lfloor \frac{n}{2} \rfloor \).

In particular, we see by the CRs estimates (5.1.2) that \( \rho \) must be in \( (\hat{GL}_n)_{\otimes,k} \), as we wanted to show. This completes the proof of Theorem 8.1.1.

We proceed to the last section of this note, where we give some details on how one might obtain the analytic results for the irreps of \( GL_n \).

9. **The eta Correspondence, the Philosophy of Cusp Forms, and Tensor Rank**

We will wrap up the body of this note by giving some indications on how we derived in [Gurevich-Howe17, Gurevich-Howe19] the analytic results described in Section 5. In this way or another, this means to address the following:

**Question:** How to get information on the (e.g., CRs, dimensions, and cardinality of the set of) \( \otimes \)-rank \( k \) irreps of \( GL_n \)?

One way to answer this question was carried out in [Gurevich-Howe17, Gurevich-Howe19]. It used the philosophy of cusp forms, and developed criteria for representations to be of tensor rank \( k \) in terms of their appearance in representations induced from parabolic subgroups.
In more detail, the process of getting the information on the tensor rank \( k \) irreps includes the following three steps:

1. **Eta correspondence (EC).** To some extent the EC might be considered as giving you a convenient place where to search for a formula for the irreps of tensor rank \( k \). Moreover, it allows one to count the number of such irreps.

2. **Philosophy of cusp forms (P-of-CF).** This is a method, put forward in the 60s by Harish-Chandra [Harish-Chandra70], that allows one to write formulas for irreps of groups like \( GL_n \). In particular, in our case the EC from Step (1) above lead us to find certain P-of-CF formulas that seems to be effective for the analysis we want to do for the irreps of tensor rank \( k \).

3. **Derivation of the analytic information.** Having the formulas from Step (2) above, one, in principal, does explicit calculations and derives the analytic results.

Let us go over the main statement of Step (1) above, then write down the P-of-CF formulas of Step (2) as they applied to tensor rank \( k \) irreps, and finally give the main computations done in Step (3) in order to derive: the CRs at the transvection and dimensions for the \( \otimes \)-rank \( k \) irreps of \( GL_n \), and the cardinality of the set of all these irreps.

### 9.1. The eta Correspondence and Strict Tensor Rank.

Recall (see Section 4.1, in particular Definition 4.1.3) that an irrep of \( GL_n \) is of tensor rank \( k \), \( 0 \leq k \leq n \), if up to twist by a character (one dim irrep) it appears in \( \omega_{n,k} = \omega_n^{\otimes k} \) and not in \( \omega_{n,(k-1)} \), where \( \omega_{n,k} \) denotes the permutation representation of \( GL_n \) on the space \( L^2(M_{n,k}) \). Let us introduce the following terminology:

**Definition 9.1.1.** We say that an irrep \( \rho \) of \( GL_n \) is of **strict tensor rank** \( k \), \( 0 \leq k \leq n \), if it appears in \( \omega_{n,k} \), but not in \( \omega_{n,(k-1)} \).

Let us denote the set of all irreps of \( GL_n \) of strict tensor rank \( k \) by \( (\hat{GL}_n)^\otimes_{\otimes,k} \).

Since every irrep of tensor rank \( k \) is up to twist by a character in \( (\hat{GL}_n)^\otimes_{\otimes,k} \), and this twist does not affect the dimension or the CR of the transvection, so it might be beneficial for us to get information on the members of \( (\hat{GL}_n)^\otimes_{\otimes,k} \). Since they all appear inside \( L^2(M_{n,k}) \), we want to zoom into this space and locate them. To do this, in [Gurevich-Howe17, Gurevich-Howe19] we followed [Howe73] and (as in Section 3.1, but now for any value of \( k \)) use the concept of \( GL_n\)-\( GL_k \) dual pair.

Consider the oscillator rep \( \omega_{n,k} \) as the joint action (3.1.1) of \( GL_n \times GL_k \) on \( L^2(M_{n,k}) \), and decompose it as in (3.1.2) to a direct sum of \( GL_k\)-isotypic components

\[
\omega_{n,k} \simeq \sum_{\tau \in GL_k} \mathcal{M}(\tau) \otimes \tau,
\]

(9.1.2)
where each multiplicity space $\mathcal{M}(\tau)$ is a rep of $GL_n$.

It turns out that (similar to the $k < \lfloor \frac{n}{2} \rfloor$ case discussed in Section 3.1) for "most" $\tau$'s the space $\mathcal{M}(\tau)$ contains a distinguished irrep $\eta(\tau)$, which is in fact from $(\widehat{GL}_n)^*_{\otimes,k}$. To describe it more closely, let us consider the parabolic subgroup $P_{k,n-k} \subset GL_n$ stabilizing the first $k$ coordinates subspace of $\mathbb{F}_q^n$ (we denoted it by $P_k$ previously, see Formulas (1.2.1) and (1.2.2)), and recall that it has a natural projection onto its Levi component $GL_k \times GL_{n-k}$. Then, to each irrep $\tau \in \widehat{GL}_k$ we can consider the rep $\tau \otimes 1_{n-k}$ of $GL_k \times GL_{n-k}$, pull it back to $P_{k,n-k}$ and look at the induced representation

$$I_\tau = \text{Ind}^{GL_n}_{P_{k,n-k}}(\tau \otimes 1_{n-k}).$$ (9.1.3)

Now we can write down, with some more details, the natural extension given in [Gurevich-Howe17, Gurevich-Howe19] for the eta correspondence described in Theorem 3.1.3.

**Theorem 9.1.4 (eta correspondence).** Take $\tau \in \widehat{GL}_k$, $0 \leq k \leq n$, and look at the decomposition (9.1.2). We have,

1. **Existence.** The representation $\mathcal{M}(\tau)$ contains a strict tensor rank $k$ component if and only if $\tau$ is of strict tensor rank $\geq 2k - n$.

Moreover, if the condition of Part (1) is satisfied, then,

2. **Uniqueness.** the representation $\mathcal{M}(\tau)$ has a unique constituent $\eta(\tau)$ of strict tensor rank $k$, and it appears with multiplicity one.

and,

3. **Approximate formula.** the constituent $\eta(\tau)$ satisfies $\eta(\tau) < I_\tau < \mathcal{M}(\tau)$, and we have,

$$I_\tau = \eta(\tau) + \sum \rho,$$

where the sum is multiplicity free, and over certain irreps $\rho$ which are of strict tensor rank less than $k$ and dimension smaller than $\eta(\tau)$.

Finally, the mapping

$$\tau \mapsto \eta(\tau),$$ (9.1.5)

gives an explicit bijective correspondence

$$(\widehat{GL}_k)^*_{\otimes,\geq 2k-n} \rightarrow (\widehat{GL}_n)^*_{\otimes,k},$$

between the collection $(\widehat{GL}_k)^*_{\otimes,\geq 2k-n}$ of irreps of $GL_k$ of strict tensor rank $\geq 2k - n$, and the set $(\widehat{GL}_n)^*_{\otimes,k}$ of strict tensor rank $k$ irreps of $GL_n$. 

Next, we want to analyze further the $\eta(\tau)$'s mentioned just above, and find more about them.

9.2. The Philosophy of Cusp Forms Formula and Rank. Using the eta correspondence and in particular the observation that each of the $\eta(\tau)$ (9.1.5) is by Part (3) of Theorem 9.1.4 "not too far" from being the induced representation $I_\tau$ (9.1.3), we were lead in [Gurevich-Howe17, Gurevich-Howe19] to an explicit Harish-Chandra type formula for irreps of $GL_n$ from which one can easily read off their strict tensor rank and tensor rank.

In this note we will just write down the above mentioned formula and explain how to get the rank invariants from it. We will leave the details of how we arrived to that expression (as well as the details of the relevant Harish-Chandra P-of-CF theory) to [Gurevich-Howe17, Gurevich-Howe19].

9.2.1. The P-of-CF Formula. The formula mentioned just above will be given in terms of representations induced from certain representations of parabolic subgroups of $GL_n$ that contain the standard Borel subgroup $B$ of upper-triangular matrices [Borel69].

Recall that to every ordered partition

$$D = \{d_1 \geq ... \geq d_\ell\}, \quad (9.2.1)$$

of $n$, we can associate the standard flag $F_D$ of subspaces of $\mathbb{F}_q^n$:

$$F_D : \quad 0 = X_{m_0} \subset X_{m_1} \subset ... \subset X_{m_\ell} = \mathbb{F}_q^n, \quad (9.2.2)$$

where for each $1 \leq j \leq \ell$, we have $m_j - m_{j-1} = d_j$, and $X_{m_j}$ is the first $m_j$-coordinates subspace of $\mathbb{F}_q^n$ (see Formula (1.2.1)).

In particular, we can attach to $D$ (9.2.1) the parabolic parabolic subgroup

$$P_D = Stab_{GL_n}(F_D), \quad (9.2.3)$$

of all elements $g \in GL_n$ that stabilize the flag $F_D$ (9.2.2), i.e., satisfy $g(X_{m_j}) = X_{m_j}$, for all $j$.

Note that the subgroup $P_D$ has the following structure of a block upper triangular matrix:

$$P_D = \left\{ \begin{pmatrix} C_1 & * & * & * \\ \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & * \\ C_\ell \end{pmatrix} \right\}; \quad C_j \in GL_{d_j}, \quad j = 1, ..., \ell,$$
and in particular admits a natural projection

\[
\begin{array}{c}
\left\{ \begin{array}{ccc}
C_1 & \ast & \ast \\
\vdots & \ast & \ast \\
& \ast & \\
\end{array} \right\}
\end{array}
\rightarrow
\begin{array}{c}
\left\{ \begin{array}{ccc}
C_1 & \ast & \ast \\
\vdots & \ast & \\
& & \\
\end{array} \right\}
\end{array}
\] (9.2.4)

onto its Levi component \( L_D \simeq GL_{d_1} \times \cdots \times GL_{d_\ell} \).

**Split Representations.** The first type of representations we will need for our formula have been called in [Gurevich-Howe17, Gurevich-Howe19] split principal series. They are the constituents of the induced representations \( Ind_B^{GL_n}(\chi) \) from characters of the Borel subgroup \( B \). To write efficient expressions for them, let us first look at the spherical principal series (SPS) representations, which are the constituents of the induced representation \( Ind_B^{GL_n}(1) \) from the trivial representation 1 of \( B \).

The SPS irreps can be realized nicely using the parabolic subgroups \( P_D \) (9.2.3). Indeed, consider the induced representation

\[ I_D = Ind_{P_D}^{GL_n}(1). \] (9.2.5)

Recall that,

**Definition 9.2.6.** If, in addition to \( D \) (9.2.1), we have another partition \( D' = \{d'_1 \geq d'_2 \geq \cdots \geq d'_{\ell'} \} \) of \( n \), then we say that \( D' \) dominates \( D \), and write \( D \preceq D' \), if \( \ell' \leq \ell \) and

\[
\sum_{j=1}^{\ell'} d'_j \leq \sum_{j=1}^{\ell} d_j, \quad \text{for } j = 1, \ldots, \ell'.
\]

With the above terminology one can show [Gurevich-Howe17, Gurevich-Howe19],

**Fact 9.2.7.** The representation \( I_D \) contains a constituent

\[ \rho_D < I_D, \] (9.2.8)

with multiplicity one, and with the property that it is not contained in any \( I_{D'} \) with \( D' \succeq D \) in the dominance order.

**Remark 9.2.9.** The representation \( \rho_D \) can also be distinguished by its dimension: it is the only constituent of \( I_D \) whose dimension, as a polynomial in \( q \), has the same degree as the cardinality of \( GL_n/P_D \).

The irreps \( \rho_D \) (9.2.8), where \( D \) runs over all ordered partitions of \( n \), are pairwise non-isomorphic and exhaust the collection of SPS irreps.
Using the $\rho_D$’s we can describe the split principal series representations. Indeed, consider the parabolic subgroup $P_S$ attached to the partition $S = \{s_1 \geq \ldots \geq s_\ell\}$ of $n$, and its Levi component $L_S \simeq GL_{s_1} \times \ldots \times GL_{s_\ell}$. Suppose $D_1, \ldots, D_\ell$, are ordered partitions of $s_1, \ldots, s_\ell$, respectively, and that $\chi_1, \ldots, \chi_\ell$, are $\ell$ distinct characters of $F_q^\times$. Then we can consider the representation $\bigotimes_{j=1}^{\ell} [(\chi_j \circ \det) \otimes \rho_{D_j}]$ of $L_S$, pull it back to $P_S$, and induce to form, $\rho_S = \text{Ind}_{P_S}^{GL_n} (\bigotimes_{j=1}^{\ell} [(\chi_j \circ \det) \otimes \rho_{D_j}]). \quad (9.2.10)$

It is not difficult to check that (up to order of the inducing factors that correspond to $s_j$’s of the same size) the irreps $\rho_S$’s 9.2.10 are irreducible, pairwise non-isomorphic, and exhausts the collection of split principal series representations.

**Unsplit Representations.** The second type of representations we use in our formula are the irreps we called in [Gurevich-Howe17] *unsplit*. To define them, let us first recall the following basic objects in Harish-Chandra’s P-of-CF [Harish-Chandra70]:

**Definition 9.2.11.** A representation $\kappa$ of $GL_n$ is called cuspidal if it does not contain a non-trivial fixed vector for the unipotent radical of any parabolic subgroup stabilizing a flag in $F_q^n$.

In the above definition, it is enough to consider parabolic subgroups of the form $P_D$ (9.2.3), and their unipotent radicals, i.e., the kernels of the projections (9.2.4).

In this note we will not explicitly discuss the cuspidal representations (for this see [Gel’fand70, Howe-Moy86, Zelevinsky81]), but only use them and some of their properties as needed. In particular, if $U = \{u_1 \geq \ldots \geq u_m\}$ is an ordered partition of $n$, and $P_U$ is the corresponding parabolic subgroup, with Levi component $L_U \simeq GL_{u_1} \times \ldots \times GL_{u_m}$, we can take cuspidal irreps $\kappa_1, \ldots, \kappa_m$, of $GL_{u_1}, \ldots, GL_{u_m}$, respectively, then form the representation $\bigotimes_{i=1}^{m} \kappa_i$ of $L_U$, and induce to get $\text{Ind}_{P_U}^{GL_n} (\bigotimes_{i=1}^{m} \kappa_i)$. If in the partition $U$ above, we have that $u_i \geq 2$, for every $i = 1, \ldots, m$, and

$$\rho_U < \text{Ind}_{P_U}^{GL_n} (\bigotimes_{i=1}^{m} \kappa_i), \quad (9.2.12)$$

is an irreducible component, then we call $\rho_U$ *unsplit representation*.

**General Representations.** The P-of-CF formula for general irreps of $GL_n$, is obtained by a combination of the formulas for split and unsplit representations defined just above.

Suppose $n = u + s$, with integers $u, s \geq 0$. Then we can consider the parabolic subgroup $P_{u,s} \subset GL_n$, stabilizing the first $u$-coordinates subspace of $F_q^n$, and its Levi subgroup $L_{u,s} \simeq GL_u \times GL_s$. Take an unsplit irrep $\rho_U$ of $GL_u$ in addition to a split irrep $\rho_S$ of $GL_s$, then pullback the rep $\rho_U \otimes \rho_S$ of $L_{u,s}$ to
$P_{u,s}$ and induce to get

$$
\rho_{U,S} = \text{Ind}^{GL_n}_{P_{u,s}}(\rho_U \otimes \rho_S).
$$

(9.2.13)

Now, the philosophy of cusp forms [Bump04, Howe-Moy86, Harish-Chandra70] tells us that

(a) $\rho_{U,S}$ is irreducible; and

(b) the map $(\rho_U, \rho_S) \mapsto \rho_{U,S}$ is an injection from the relevant subsets of the unitary dual of $GL_u \times GL_s$ into the unitary dual of $GL_n$; and

(c) all irreducible representations of $GL_n$ arise in this way (including the cuspidal representations, which are included in the situation when $P_{n,0} = GL_n$).

For a later use, we want to make Formula (9.2.13) a bit more explicit. Indeed, suppose, in addition, that our $S = \{s_1 \geq \ldots \geq s_\ell\}$ is an ordered partition of $s$, then we can consider the standard parabolic subgroup $P_{u,s_1,...,s_\ell} \subset GL_n$ with blocks of sizes $u, s_1, ..., s_\ell$, and pullback to it the rep (with distinct $\chi_j$’s) $\rho_U \otimes \bigotimes_{j=1}^\ell (\chi_j \circ \det) \otimes \rho_{D_j}$ of its Levi component $L_{u,s_1,...,s_\ell} \simeq GL_u \times GL_{s_1} \times \ldots \times GL_{s_\ell}$, and induce to get

$$
\rho_{U,S} = \text{Ind}^{GL_n}_{P_{u,s_1,...,s_\ell}} \left( \rho_U \otimes \bigotimes_{j=1}^\ell (\chi_j \circ \det) \otimes \rho_{D_j} \right).
$$

(9.2.14)

We will call (9.2.14) the P-of-CF formula.

### 9.2.2. Reading Ranks from the P-of-CF Formula

In [Gurevich-Howe17, Gurevich-Howe19] we showed that one can compute the strict tensor rank and tensor rank of a representation from its P-of-CF Formula (9.2.14), more precisely directly from its split principal series component. To state this, and similar results, it is convenient to use the notions of tensor co-rank and strict tensor co-rank, by which we mean, respectively, $n$ minus the tensor rank and $n$ minus the strict tensor rank.

**Fact 9.2.15.** We have,

1. For an ordered partition $D = \{d_1 \geq \ldots\}$ of $n$, the tensor co-rank of the SPS representation $\rho_D$ (9.2.3) is the same as its strict tensor co-rank and is equal to $d_1$.

2. The tensor co-rank of the representation $\rho_{U,S}$ of $GL_n$ described by Formula (9.2.14), is the maximum of the tensor co-ranks of the SPS representations $\rho_{D_j}, j = 1, \ldots, \ell$, that appear in description of the split part of $\rho_{U,S}$. The strict tensor co-rank of $\rho_{U,S}$ is the strict tensor rank of the SPS representation $\rho_{D_j}$ that is twisted in (9.2.14) by the trivial character.

### 9.3. Deriving the Analytic Information for Tensor Rank $k$ irreps of $GL_n$

In this last section we want to remark briefly on how the eta correspondence (Section 9.1) and the P-of-CF Formula for tensor rank $k$ irreps of $GL_n$, enable us in [Gurevich-Howe19] to obtain the analytic information for these irreps given in Section 5.
9.3.1. **Character Ratios on The Transvection.** It is not difficult to see [Gurevich-Howe19] that one just need to estimate the CR at $T$ (0.1.1) for irreps of tensor rank $k$ of the form

$$\rho_{U,D} = \text{Ind}_{P_{u,d}}^{GL_n} (\rho_U \otimes \rho_D),$$

(9.3.1)

where

- $P_{u,d}$ is the parabolic subgroup with blocks of sizes $u$ and $d$, $u + d = n$;
- $\rho_U$ is an unsplit irrep (see (9.2.12)) of $GL_u$;
- $D = \{d_1 \geq \ldots \}$ is a partition of $d$, with longest row of length $d_1 = n - k$; and
- $\rho_D$ the SPS irreps (9.2.3) attached to $D$.

In particular, using the standard formula [Fulton-Harris91] for character of induced representation, the estimate is reduced in [Gurevich-Howe19] to the known cardinality of $GL_n/P_{u,d}$, and the following two specific cases of CR estimates:

**SPS case.** Consider a SPS irrep $\rho_D$ (9.2.3) of $GL_n$, where $D = \{d_1 \geq \ldots \geq d_\ell\}$ is a partition of $n$. Then, the tensor rank of $\rho_D$ is equal to $k = n - d_1$ (see Fact 9.2.15), and,

$$\frac{\chi_{\rho_D}(T)}{\dim(\rho_D)} = \begin{cases} \frac{1}{q^{n-d_1}} + o(...), & \text{if } d_1 > d_2; \\ \frac{c_D}{q^{n-d_1}} + o(...), & \text{otherwise}, \end{cases},$$

where $c_D$ is a certain integer depending only on $D$ (and not on $q$).

**Unsplit case.** Consider an unsplit irrep $\rho_U$ (9.2.12) of $GL_n$. Then, the tensor rank of $\rho_U$ is equal to $n$, and

$$\frac{\chi_{\rho_U}(T)}{\dim(\rho_U)} = -1 \frac{q^{n-1}}{q^{n-1}-1}.$$  

9.3.2. **Dimensions of Irreps.** Again, it is enough to compute the dimensions of irreps of the form $\rho_{U,D}$ (9.3.1), and for such irrep it boils down to the following two known [Gurevich-Howe19] cases:

**SPS case.** The dimension of the SPS irrep $\rho_D$ (9.2.8) attached to a partition $D = \{d_1 \geq d_2 \geq \ldots \geq d_\ell\}$ of $n$, satisfies,

$$\dim(\rho_D) = q^{d_D} + o(...),$$

where $d_D = \sum_{1 \leq i < j \leq \ell} d_id_j$.

**Cuspidal case.** The dimension of any cuspidal representation of $GL_n$ is $q^{\frac{n(n-1)}{2}} + o(...)$ [Bump04, Gel’fand70].
9.3.3. The Number of Irreps of Tensor Rank $k$. The P-of-CF formula for the SPS irreps shows that the cardinality of $(\hat{GL}_n)_{\otimes, n-1}$ is $c_{n-1}q^n + o(...)$, for some constant $c_{n-1} > 0$, independent of $q$, and the well known [Bump04, Gel’fand70] cardinality of the collection of cuspidal irreps, implies that $\#((\hat{GL}_n)_{\otimes})$ is also of the form $c_nq^n + o(...)$, for some constant $c_n > 0$, independent of $q$, and using more or less the same knowledge one can show that $c_{n-1} + c_n \geq 1$, and hence $= 1$, since the size of $\hat{GL}_n$ is $q^n + o(...)$.

For $k < n-1$, the estimation given in [Gurevich-Howe19] use the eta correspondence described by Theorem 9.1.4. For such $k$, the domain of the eta map (9.1.5)

$$\eta : (\hat{GL}_k)_{\otimes, 2k-n} \rightarrow (\hat{GL}_n)_{\otimes, k},$$

has cardinality $q^k + o(...)$, for example because it contains the the tensor rank $k-1$, and tensor rank $k$ irreps of $GL_k$. Moreover, using the P-of-CF formula one can show (see [Gurevich-Howe19]) that for $q^k + o(...)$ of the irreps $\tau \in ((\hat{GL}_k)_{\otimes, 2k-n}^*)$, the irrep $\eta(\tau)$ is of tensor rank $k$, and stays like this even after twists by the $q - 1$ characters $GL_n$, and, moreover, all these are pairwise non-isomorphic. This shows that $\#((\hat{GL}_n)_{\otimes, k}) = q^{k+1} + o(...)$, for $k < n-1$, as claimed.

This completes the story we were trying to give in the body of this note.

Appendix A. Fourier Transform of Sets of Matrices of Fixed Rank

For $k \leq m \leq n$, we look at the Fourier transform (FT) of the set $(M_{m,n})_k$, of $m \times n$ matrices of rank $k$ over $\mathbb{F}_q$, and want to evaluate it on a rank one matrix.

Let us identify, in the standard way, the space $M_{m,n}$ of $m \times n$ matrices over $\mathbb{F}_q$ with its vector space dual via the trace map, and then to its Pontryagin dual in the standard way, after fixing an additive character $\psi$ of $\mathbb{F}_q$. In particular, we can associate with a function $f$ on $M_{m,n}$ its Fourier transform $\hat{f}$ given by

$$\hat{f}(B) = \sum_{A \in M_{m,n}} f(A)\psi(-\text{trace}(B^t \circ A)), \ B \in M_{m,n}. \quad (A.0.1)$$

A.1. A Formula for the Fourier Transform of $(M_{m,n})_k$. By the FT of $(M_{m,n})_k$, we mean the FT of the characteristic function $1_{(M_{m,n})_k}$ of that set. Using (A.0.1), it is given by

$$\hat{1}_{(M_{m,n})_k}(B) = \sum_{A \in (M_{m,n})_k} \psi(\text{trace}(B^t \circ A)), \ B \in M_{m,n}. \quad (A.1.1)$$

In particular, $\hat{1}_{(M_{m,n})_k}(B)$ is a real number.

Next, denote by $(M_{m,n})_k^k(B)$ the set of matrices $A \in (M_{m,n})_k$ such that trace$(B^t \circ A) = 0$. 
Claim A.1.2. We have,
\[ \hat{1}_{(M_{m,n})_k}(B) = -\left( \frac{1}{q-1} \right) \#((M_{m,n})_k) + \left( \frac{q}{q-1} \right) \#((M_{m,n})_k^0(B)). \]  
(A.1.3)

For the calculations leading to Formula (A.1.3), see Appendix B.4.1.

Note that, for a non-zero \( B \in M_{m,n} \), the map \( A \mapsto \text{trace}(B^t \circ A) \) is a linear functional on the space \( M_{m,n} \), so its kernel is a hyperplane, i.e., it has co-dimension 1. So \( (M_{m,n})_k(B) \) is some sort of hypersurface in \( (M_{m,n})_k \), and it should be reasonable to compare the two numbers appearing as the two terms of the sum on the right side of (A.1.3), and see if it is negative or positive.

We proceed to do just this in the case of a rank one matrix.

A.2. The Value of the FT of \( (M_{m,n})_k \) on a Rank One Matrix. Let \( T \in M_{m,n} \) be a rank one matrix.

The value of \( \hat{1}_{(M_{m,n})_k} \) at \( T \) is of course independent of \( T \), and it can be computed explicitly. To write it down and use it, it will be useful for us to recall (see also [Artin57]) that the cardinality of the group \( GL_k = GL_k(\mathbb{F}_q) \) is
\[ \#(GL_k) = q^k \cdot (q^k-1)(q^k-q)(q^n-q^2) \cdots (q^n-q^{n-1}) = q^{k(k-1)} \prod_{a=1}^{k} (q^n-1) = q^{n^2} + o(...) , \]
and that the cardinality of \( \Gamma_{n,k} \), the Grassmannian of all \( k \)-dimensional subspaces of \( \mathbb{F}_q^n \), is
\[ \#(\Gamma_{n,k}) = \frac{\#(GL_n)}{q^{k(n-k)} \cdot \#(GL_k) \cdot \#(GL_{n-k})} \]
\[ = \frac{\prod_{a=1}^{k} (q^{n-k+a}-1)}{\prod_{a=1}^{k} (q^n-1)} = q^{k(n-k)} + o(...) . \]

Remark A.2.2. In order for certain formulas to include all cases, we may sometime use the notation \( \Gamma_{n,k} \) also in the case \( k > n \), by this we mean the empty set with \( \#(\Gamma_{n,k}) = 0 \).

Now we can write an explicit expression,

Theorem A.2.3 (FT of \( M_{(m,n)}_k \) on rank one matrix). Assume \( k \leq m \leq n \). Then, the value of the FT of \( (M_{m,n})_k \) at a rank one matrix \( T \in M_{m,n} \), is
\[ \hat{1}_{(M_{m,n})_k}(T) = (q^{n+m-k} - q^n - q^m + 1)(\frac{\#(\Gamma_{n,k}) \#(\Gamma_{m,k}) \#(GL_k)}{(q^n-1)(q^m-1)}) . \]
In particular, it is positive if $k < m$, and negative if $k = m$.

For a proof of Theorem A.2.3, see Appendix B.4.2.

APPENDIX B. Proofs

B.1. Proofs for Section 0.

B.1.1. Proof of Proposition 0.3.1.

Proof. For the basic notions and facts from representation theory of finite groups see [Serre77].

The result we need to prove holds for any finite group $G$ and any conjugacy class $C \subset G$.

For a finite group $G$, consider the regular representation $\pi_G$ on the space $L(G)$ of complex-valued functions on $G$.

Associated with the conjugacy class $C \subset G$, we have a summation operator $A_C : L(G) \to L(G)$, given by

$$A_C = \sum_{c \in C} \pi_G(c).$$

It is easy to check that

$$\#(M_{\ell,g}) = \left[ (A_C)^\ell \delta_1 \right](g),$$

where $\delta_1$ denotes the Dirac delta function at the identity element of $G$.

Next, for each irrep $\pi \in \hat{G}$, denote by $Pr_\pi$ the projection of $L(G)$ onto the $\pi$-isotypic component. Then we have:

\[ (*) \sum_{\pi \in \hat{G}} Pr_\pi = Id, \text{ the identity operator on } L(G); \]

and,

\[ (**) \ (A_C)^\ell \circ Pr_\pi = \#(C^\ell) \cdot \left( \frac{\chi_\pi(C)}{\dim(\pi)} \right)^\ell \cdot Pr_\pi. \]

Fact (**) above follows by a direct computation using Schur’s lemma, that holds here since $A_C$ intertwines the action of $G$ on each $\pi \in \hat{G}$.

Now, the projectors $Pr_\pi, \pi \in \hat{G}$, have an explicit formula [Serre77],

$$Pr_\pi = \frac{\dim(\pi)}{\#(G)} \cdot \sum_{h \in G} \chi_\pi(h^{-1}) \pi_G(h). \tag{B.1.1}$$
Concluding, we get,

\[
\#(M_{\ell,g}) = \left[(AC)^\ell \delta_1\right](g)
\]

\[
= \sum_{\pi \in \hat{G}} \left[(AC)^\ell \circ Pr_\pi(\delta_1)\right](g)
\]

\[
= \frac{\#(C^\ell)}{\#(G)} \left(\sum_{\pi \in \hat{G}} \dim(\pi) \left(\frac{\chi_\pi(C)}{\dim(\pi)}\right)^\ell \chi_\pi(g^{-1})\right),
\]

where for the second equality we used Fact (*) above, and the third equality is obtained using Fact (**), and Identity (B.1.1). This completes the derivation of Formula (0.3.2), as needed. \(\square\)

B.2. Proofs for Section 1.

B.2.1. **Proof of Proposition 1.2.10.**

**Proof.** Take a rep \(\rho\) of \(GL_n\). Then for every element \(l\) in the Levi subgroup \(L_{\lfloor \frac{n}{2} \rfloor}\) (1.2.3), we have the representation \(\rho^l\) of \(GL_n\), given by \(\rho^l(g) = \rho(l \cdot g \cdot l^{-1}), g \in GL_n\), which is of course isomorphic to \(\rho\). Now, the fact that \(L_{\lfloor \frac{n}{2} \rfloor}\) normalizes \(U = U_{\lfloor \frac{n}{2} \rfloor}\), and acts transitively on each collection \(O_r\) of all matrices in \(U\) of a fixed rank, completes the verification of the proposition. \(\square\)

B.3. Proofs for Section 7.

B.3.1. **Proof of Claim 7.2.3.**

**Proof.** Parts (2), and (3), of the claim follow using a direct substitution of the numerical data given by Formulas (6.2.2), (6.3.2), and (6.4.2).

Concerning Part (1). First we know that for each member \(\pi \in (SL_n)_{\otimes,1}\), we have,

- \(\dim(\pi) = q^{n-1} + o(...);\)

  and,

- \(\frac{\chi_\pi(T)}{\dim(\pi)} = \frac{1}{q} + o(...).\)

So we can write

\[
(S_{\ell,g})_1 = \sum_{\pi \in (SL_n)_{\otimes,1}} \dim(\pi) \left(\frac{\chi_\pi(T)}{\dim(\pi)}\right)^\ell \chi_\pi(g) \tag{B.3.1}
\]

\[
= \left(\frac{1}{q} \left(\frac{1}{q}\right)^{\ell-n} + o(...)\right) \sum_{\pi \in (SL_n)_{\otimes,1}} \chi_\pi(g).
\]

Now, to understand the right-hand side factor in (B.3.1), recall that, the irreps of \(SL_n\) of tensor rank \(k = 1\), are given in Example 3.2.1. In particular, they are members of the space \(L^2(\mathbb{P}_q^n)\) of complex valued
functions on $\mathbb{F}_q^n$. In more detail, each of them appears there with multiplicity one, in addition to the trivial rep $1$ which shows up twice, i.e., we have,

$$L^2(\mathbb{F}_q^n) = 2 \cdot 1 + \sum_{\rho \in (\hat{SL}_n)\otimes 1} \rho.$$  

Recall that we denoted the permutation representation of $GL_n$ on that space by $\omega_{n,1}$. In particular, we conclude that

$$\sum_{\rho \in (\hat{SL}_n)\otimes 1} \chi_\rho(g) = \text{trace}(\omega_{n,1}(g) \rtimes L^2(\mathbb{F}_q^n)) - 2 = -2, \quad (B.3.2)$$

taking into account that the element $g$ has no eigenvalue equal to 1.

Combining (B.3.1) and (B.3.2), we see that Part (1) holds true. This completes the verification of the claim.

\[\square\]

### B.4. Proofs for Appendix A.

#### B.4.1. Proof of Claim A.1.2.

**Proof.** Each element of the set $(M_{m,n})^o_k(B)$ contributes 1 to the FT (A.1.1) of $1_{(M_{m,n})_k}$ evaluated at $B$. On the other hand the multiplicative group $\mathbb{F}_q^*$ acts naturally by scaling on the complement $(M_{m,n})_k \setminus (M_{m,n})^o_k(B)$, and (by one-dimensional harmonic analysis, i.e., orthogonality of characters of $\mathbb{F}_q$) each orbit contributes $-1$ to the FT. Overall we have,

$$\hat{1}_{(M_{m,n})_k}(B) = -\left(\frac{1}{q-1}\right)\left(\#((M_{m,n})_k) - \#((M_{m,n})^o_k(B))\right) + \#((M_{m,n})^o_k(B)) \quad (B.4.1)$$

as claimed. \[\square\]

#### B.4.2. Proof of Theorem A.2.3. We want to use Formula (B.4.1) above. For the rank one matrix $T$ we denote $(M_{m,n})^o_k = (M_{m,n})^o_k(T)$, and we note that,

**Lemma B.4.2.** We have,

1. $\#((M_{m,n})_k) = \#(\Gamma_{n,k})\#(\Gamma_{m,k})\#(GL_k)$;

and,

2. $\#((M_{m,n})^o_k) = (q^{m-1} - 1)(q^n - 1) + (q^{n-k} - 1)(q - 1)q^{m-1} \left(\frac{\#(\Gamma_{n,k})\#(\Gamma_{m,k})\#(GL_k)}{(q^n - 1)(q^{m-n} - 1)}\right)$.

We will verify Lemma B.4.2 below.
Proof. (of Theorem A.2.3) We compute
\[
\widehat{1}_{(M_{m,n})_k}(T) = -(\frac{1}{q-1})\#((M_{m,n})_k) + (\frac{q}{q-1})\#((M_{m,n})_k^0)
\]
\[
= \left( q^{n+m-k} - q^n - q^m + 1 \right) \frac{\#((\Gamma_{n,k})\#(\Gamma_{m,k})\#(GL_k))}{(q^n - 1)(q^m - 1)},
\]
where in the second equality we used Lemma B.4.2. □

Proof. (of Lemma B.4.2)

Part 1. This is a consequence of the rank-nullity theorem. Indeed, for \( A \) in \((M_{m,n})_k\), the kernel is an \( n - k \) dimensional subspace of \( \mathbb{F}_q^n \), while the image is a \( k \)-dimensional subspace of \( \mathbb{F}_q^m \), and \( A \) defines an invertible transformation from \( \mathbb{F}_q^n/\ker(A) \) to \( \text{im}(A) \). The assertion follows.

Part 2. First let's give some standard formula for a general matrix of rank one in \( M_{n,m} \). Fix a vector \( 0 \neq v \in \mathbb{F}_q^n \), and a linear functional \( 0 \neq \lambda \) in the dual space \( (\mathbb{F}_q^m)^* \). Then
\[
T_{v,\lambda} : \mathbb{F}_q^m \rightarrow \mathbb{F}_q^n, \quad T_{v,\lambda}(w) = \lambda(w)v,
\]
is a rank one operator, and any rank one matrix in \( M_{n,m} \) is of this form.

We are trying to compute the cardinality of \( (M_{m,n})_k \) = \( \{ A \in (M_{m,n})_k; \text{trace}(T_{v,\lambda} \circ A) = \lambda(A(v)) = 0 \} \).

We will count and find (justification below) that the number of the matrices \( A \in (M_{m,n})_k \) that,

(a) satisfy \( A(v) = 0 \), is \( \#(\Gamma_{n-1,k})\#(\Gamma_{m,k})\#(GL_k) \);

(b) satisfy \( \text{im}(A) \subset \ker(\lambda) \), is \( \#(\Gamma_{n,k})\#(\Gamma_{m-1,k})\#(GL_k) \);

(c) satisfy both (a) and (b) above, is \( \#(\Gamma_{n-1,k})\#(\Gamma_{m-1,k})\#(GL_k) \);

(d) satisfy \( \lambda(A(v)) = 0 \), but neither (a) nor (b), is \( \#(\Gamma_{n,k})\#(\Gamma_{n-1,k})\#(GL_k) \).

In addition, Formula (A.2.1) implies that
\[
\#(\Gamma_{n-1,k}) = (\frac{q^{n-k} - 1}{q^n - 1})\#(\Gamma_{n,k}). \tag{B.4.3}
\]

Now, note that
\[
\#((M_{m,n})_k^0) = (a)+(b)-(c)+(d),
\]
and so, a direct calculation using the explicit cardinalities presented in (a)-to-(d), including Identity (B.4.3), produce the assertion made in Part 2.

Let us now finish the proof, by justifying (a)-to-(d).

We will frequently use the fact that \( \#(\Gamma_{n,n-1}) = \#(\Gamma_{n,k}) \).
To justify (a), note that if $A(v) = 0$, then $\ker(A)$ defines $(n - k - 1)$-dimensional subspace of $\mathbb{F}_q^n/\text{span}(v) \cong \mathbb{F}_{q}^{n-1}$, and there are $\#(\Gamma_{n-1,n-k-1}) = \#(\Gamma_{n-1,k})$ such subspaces. On the other hand, $\text{im}(A)$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$, and each of these $\#(\Gamma_{m,k})$ is a legitimate one. After these choices were made, $A$ can be viewed as an element of $GL_k$, and again each such is possible. Overall, Part (a) follows.

The derivation of Parts (b) and (c) is very similar to that of Part (a), so let us omit it.

Finally, let us justify Part (d). For the $A$’s there the kernel is $n - k$ dimensional subspace, so a member of $\Gamma_{n,n-k}$, and there are $\#(\Gamma_{n,n-k}) = \#(\Gamma_{n,k})$ such. But $v$ is not allowed to be in $\ker(A)$, for such $A$, so is not one of the subspaces that are sent to an $n - k - 1$ dimensional subspace of $\mathbb{F}_q^n/\text{span}(v) \cong \mathbb{F}_{q}^{n-1}$, and there are $\#(\Gamma_{n-1,n-k-1}) = \#(\Gamma_{n-1,k})$ of such. After one is making one of these $\Gamma_{n,k} \setminus \Gamma_{n-1,k}$ choices, we should look on the constraints on the image of $A$. The first one is that $A(v) \in \ker(\lambda) \setminus \{0\}$, and we have $q^{m-1} - 1$ such options. After this choice, we need to make sure $A$ has rank $k$. For this the number of options is

$$
(q^m - q) \cdot \ldots \cdot (q^m - q^{k-1}) = \frac{1}{q^m - 1}(M_{m,n})_k
= \frac{1}{q^m - 1}\#(\Gamma_m.k)\#(GL_k).
$$

And, lastly, we need to make sure that $\text{im}(A)$ is not contained in $\ker(\lambda)$, so we need to subtract $\#(\Gamma_{m-1,k})$ options.

Overall, when we multiply the number of options for the domain of $A$ with that for its range, we get what is claimed in Part (d). This completes the proof of Lemma B.4.2.

References

[Arad-Herzog-Stavi85] Arad Z., Herzog M., and Stavi J., Powers and products of conjugacy classes in groups. LNM 1112, Springer-Verlag, Berlin, (1985) 6-51.

[Artin57] Artin E., Geometric Algebra. Interscience, New York (1957).

[Auslander-Tolimieri79] Auslander L. and Tolimieri R., Is computing with the finite Fourier transform pure or applied mathematics? Bulletin of the AMS Vol. 5 (1981), 263-312.

[Bezrulkavnikov-Liebeck-Shalev-Tiep18] Bezrulkavnikov R., Liebeck M., Shalev A., and P. H. Tiep. Character bounds for finite groups of Lie type. Acta Math. 221 (2018) 1-57.

[Borel69] Borel A., Linear algebraic groups. GTM 126, Springer-Verlag (1969).

[Bump04] Bump D., Lie Groups. Springer, New York (2004).

[Clifford37] Clifford A. H., Representations induced in an invariant subgroup. Annals of Math 38 (1937) 533–550.
[Deligne-Lusztig76] Deligne P. and Lusztig G., *Representations of reductive groups over finite fields*. Annals of Math. 103 (1976), 103-161.

[Frobenius1896] Frobenius F.G., Über Gruppencharaktere. Sitzber. Preuss. Akad. Wiss. (1896) 985–1021.

[Fulton-Harris91] Fulton W. and Harris J., Representation theory: A first course. GTM 129, Springer (1991).

[Gel’fand70] Gel’fand S.I., Representations of the full linear group over a finite field. Math. USSR-Sb., 12 (1970) 13-39.

[Gerardin77] Gérardin P., Weil representations associated to finite fields. J. Alg. 46 (1977), 54-101.

[Green55] Green J.A., The characters of the finite general linear groups. TAMS 80 (1955) 402–447.

[Gurevich-Howe15] Gurevich S. and Howe R., Small Representations of finite classical groups. Proceedings of Howe’s 70th birthday conference, New Haven (2015).

[Gurevich-Howe17] Gurevich S. and Howe R., Rank and Duality in Representation Theory. The 19th Takagi Lectures - July 2017, Jpn J. Math. 15 (2020) 223-309.

[Gurevich-Howe18] Gurevich S. and Howe R., A look on representations of $SL_2(F_q)$ through the lens of size. Joe Wolf’s 80th Birthday Volume, São Paulo J. of Math. Sci. 12 (2018) 252–277.

[Gurevich-Howe19] Gurevich S. and Howe R., Harmonic Analysis on $GL_n$ over Finite Fields. The Bertram Kostant Memorial Volume. Pure and Applied Mathematics Quarterly (2019) 62p. Accepted.

[Harish-Chandra70] Harish-Chandra., Eisenstein series over finite fields. Functional analysis and related fields, Springer (1970) 76–88.

[Harish-Chandra84] Harish-Chandra., Collected papers IV - 1970-83. Springer-Verlag (1984).

[Hartshorne77] Hartshorne R., Algebraic Geometry. Springer-Verlag, GTM 52 (1977).

[Howe73] Howe R., Invariant theory and duality for classical groups over finite fields with applications to their singular representation theory. Preprint, Yale University (1973).

[Howe-Moy86] Howe R. and Moy A., Harish-Chandra homomorphisms for p-adic groups. CBMS Regional Conference Series in Mathematics 59 (1986).

[Humphries80] Humphries S.P., Generation of special linear groups by transvections. J. of Alg. 99 (1986) 480-495.

[Lang-Weil54] Lang S. and Weil A. Number of points of varieties in finite fields. Amer. J. Math. 76 (1954), 819–827.

[Lusztig84] Lusztig G., Characters of Reductive Groups over a Finite Field. Annals of Math. Studies, Princeton University Press (1984).

[Mackey49] Mackey G.W., Imprimitivity for representations of locally compact groups I. PNAS 35 (1949) 537–545.

[Schur1905] Schur I., Neue Begründung der Theorie der Gruppencharaktere. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1905) 406-432.
[Serre77] Serre J.P., Linear Representations of Finite Groups. Springer (1977).
[Weil64] Weil A., Sur certains groupes d’opérateurs unitaires. Acta Math. 111 (1964), 143-211.
[Zelevinsky81] Zelevinsky A., Representations of finite classical groups: A Hopf algebra approach. LNM 869. Springer (1981).

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