Simplifying coefficients in differential equations for generating function of Catalan numbers

Feng Qi and Yong-Hong Yao

Institute of Mathematics, Henan Polytechnic University, Jiaozuo, Henan, People's Republic of China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia, People's Republic of China

ABSTRACT
In the paper, by the Faà di Bruno formula, several identities for the Bell polynomials of the second kind, and an inversion theorem, the authors simplify coefficients in two families of nonlinear ordinary differential equations for the generating function of the Catalan numbers.

1. Motivation
The Catalan numbers $C_n$ for $n \geq 0$ form a combinatorial sequence of natural numbers that occur in enumeration problems [1, 2]. The Catalan numbers $C_n$ can be explicitly expressed by

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

and can be formally generated by

$$G(x) = \frac{2}{1 + \sqrt{1 - 4x}} = \sum_{n=0}^{\infty} C_n x^n$$

has a solution $G(x) = 2/(1 + \sqrt{1 - 4x})$, where $a_1(n) = 2^{n-1} (2n - 3)!!$ and

$$a_i(n) = 2^{n-i} \prod_{k=1}^{i-1} \sum_{j=0}^{n-i-k_{j-1}} \sum_{k_{j-1}=0}^{n-i-k_j} \cdots \sum_{k_1=0}^{n-i-k_{i-2}} \left( 2n - 2 \sum_{j=1}^{i-1} k_j - 2i + 1 - \ell \right)!!$$

with $\langle x; \alpha \rangle_0 = 1$ and $\langle x; \alpha \rangle_n = x(x - \alpha) \cdots [x - (n - 1)\alpha]$ for $n \in \mathbb{N}$.

In [3, Theorem 2.1], by similar argument as in the proof of [3, Theorem 3.1], they found that the family of differential equations

$$G^{(n)}(x) = \sum_{i=1}^{n} a_i(n) (1 - 4x)^{-i/2} G^{i+1}(x), \quad n \in \mathbb{N}$$

and

$$n! G^{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i(n) (1 - 4x)^{n/2-i} G^{(n-i)}(x), \quad n \in \mathbb{N}$$

has a solution $G(x) = 2/(1 + \sqrt{1 - 4x})$, where $a_1(n) = 2^{n-1} (2n - 3)!!$ and

$$a_i(n) = 2^{n-i} \prod_{k=1}^{i-1} \sum_{j=0}^{n-i-k_{j-1}} \sum_{k_{j-1}=0}^{n-i-k_j} \cdots \sum_{k_1=0}^{n-i-k_{i-2}} \left( 2n - 2 \sum_{j=1}^{i-1} k_j - 2i + 1 - \ell \right)!!$$

with $\langle x; \alpha \rangle_0 = 1$ and $\langle x; \alpha \rangle_n = x(x - \alpha) \cdots [x - (n - 1)\alpha]$ for $n \in \mathbb{N}$.

ARTICLE HISTORY
Received 17 May 2019
Revised 20 August 2019
Accepted 30 August 2019

KEYWORDS
Simplifying; coefficient; nonlinear ordinary differential equation; generating function; Catalan number; Faà di Bruno formula; Bell polynomial of the second kind; inversion theorem

2010 MATHEMATICS SUBJECT CLASSIFICATION.
Primary: 05A15; Secondary: 11B65; 11B75; 11B37; 15A09; 15B36; 34A05; 34A34; 40E99

CONTACT
Feng Qi, qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, People’s Republic of China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, People’s Republic of China; School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, People’s Republic of China

© 2019 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group
This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
has a solution $G(x) = 2/(1 + \sqrt{1 - 4x})$, where $\lfloor t \rfloor$ denotes the floor function whose value is the largest integer less than or equal to $t$, the coefficients $b_0(n) = 1$ and
\begin{equation}
\label{eq5}
b_i(n) = (-2)^i S_{n+1-2i}, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor
\end{equation}
with $S_{n,1} = n + (n - 1) + \cdots + 1$ and
\[ S_{nj} = nS_{n+1,j-1} + (n - 1)S_{n-1,j-1} + \cdots + 1S_{2,j-1}, \quad j \geq 2. \]
In [3, Theorems 2.2 and 3.2, [3, Remark]], Kim and Kim also used the coefficients $a_i(n)$ and $b_i(n)$ respectively defined in (3) and (5) to express their other results in [3]. In other words, the quantities $a_i(n)$ and $b_i(n)$ are the core of the paper [3].

It is obvious that the coefficients $a_i(n)$ and $b_i(n)$ respectively defined in (3) and (5) cannot be easily remembered, possibly understood, and simply computed.

The aim of this paper is the same one as in the papers [4–18] and closely related references therein. Concretely speaking, our aim in this paper is to discover simple, significant, meaningful, easily remembered, possibly understood, readily computed expressions for the coefficients $a_i(n)$ and $b_i(n)$ in the families (2) and (4) respectively.

2. Lemmas

To reach our aim in this paper, we recall the following lemmas.

**Lemma 2.1 ([19, p. 134 and 139]):** The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ by
\begin{equation}
\label{eq6}
d^n f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k} \times \left( h'(t), h''(t), \ldots, h^{(n-k+1)}(t) \right)
\end{equation}
for $n \geq 0$, where the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ for $n \geq k \geq 0$, are defined by
\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq \ell_1 \leq \cdots \leq \ell_k \leq n-k+1} \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i}.
\]

**Lemma 2.2 ([19, p. 135]):** For $n \geq k \geq 0$, we have
\begin{equation}
\label{eq7}
B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}),
\end{equation}
where $a$ and $b$ are any complex numbers.

**Lemma 2.3 ([20, Theorem 5.17, [21, Theorem 1.2]):** For $n \geq k \geq 0$, we have
\begin{equation}
\label{eq8}B_{n,k}((-1)^{n-k}!, 1, 1, \ldots, 1, (2(n-k)-1)!!) = [2(n-k)-1]!! \left( \frac{2n-k-1}{2(n-k)} \right),
\end{equation}
where the double factorial of negative odd integers $-(2n+1)$ is defined by
\[
(-2n-1)!! = \frac{(-1)^n (2n)!}{(2n-1)!!}, \quad n = 0, 1, \ldots.
\]

**Lemma 2.4 ([17, Theorem 4.3 and Remark 6.2]):** For $n \geq k \geq 1$, let $\{s_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ be two sequences which are independent of $n$. Then
\[
s_n = \sum_{k=1}^{n} \left( \begin{array}{c} k \\ n-k \end{array} \right) s_k
\]
if and only if
\[
(-1)^n n s_n = \sum_{k=1}^{n} \left( \begin{array}{c} 2n-k-1 \\ n-1 \end{array} \right) (-1)^k k s_k.
\]

**Remark 2.1:** Every inversion theorem in combinatorics corresponds to a lower triangular invertible matrix and its inverse. Conversely, every lower triangular invertible matrix and its inverse correspond to an inversion theorem. Generally, it is not easy to compute the inverse of a lower triangular invertible matrix.

Lemma 2.4 is equivalent to that the lower triangular integer matrices $A_n = (a_{ij})_{1 \leq i \leq n}$ and $B_n = (b_{ij})_{1 \leq i \leq n}$ with
\[
a_{ij} = \begin{cases} 0, & i < j \\ \left( \begin{array}{c} j \\ i-j \end{array} \right), & 1 \leq j \leq 2i \\ 0, & i > 2j \end{cases}
\]
and
\[
b_{ij} = \begin{cases} 0, & 1 \leq i < j \\ (-1)^{i-j} \left( \begin{array}{c} 2i-j-1 \\ i-1 \end{array} \right), & i \geq j \geq 1 \end{cases}
\]
are inverses to each other. See [17, Theorem 4.1].

Lemma 2.4 has been cited and applied in the papers [4, 15, 22] and closely related references therein.

3. Main results and their proofs

Now we are in a position to state our main results and to prove them simply.

**Theorem 3.1:** For $n \in \mathbb{N}$, the $n$th derivative and the powers of the generating function $G(x)$ defined in (1) satisfy
\begin{equation}
\label{eq9}G^{(n)}(x) = \frac{(n-1)!}{(1-4x)^n} \sum_{k=1}^{n} k \left( \begin{array}{c} 2n-k-1 \\ n-1 \end{array} \right) \times (1-4x)^{k/2} G^{k+1}(x).
\end{equation}
Proof: This proof is a slight modification of the first part in the second proof of [21, Theorem 1.1].

Taking \( f(u) = 2/(1 + u) \) and \( u = h(x) = \sqrt{1 - 4x} \) in the formula (6) and utilizing the identity (7) yield

\[
G^{(n)}(x) = 2 \sum_{k=0}^{n} (-1)^k \frac{k!}{(1 + u)^{k+1}} B_{n,k} \left( \frac{2}{(1 - 4x)^{1/2}} - \frac{2^2}{(1 - 4x)^{3/2}} \cdots \right.
\]

\[
- \frac{2^{n-k+1}[2(n - k + 1) - 3]!!}{(1 - 4x)^{2n-k+1}}
\]

\[
= 2 \sum_{k=0}^{n} (-1)^k \frac{k!}{(1 + \sqrt{1 - 4x})^{k+1}} (-1)^k 2^n
\]

\[
\times \frac{(1 - 4x)^{k/2}}{(1 - 4x)^n}
\]

\[
\times B_{n,k} \left( (-1)!!, 1!!, \ldots, [2(n - k) - 1]!! \right)
\]

\[
= \frac{2^{n+1}}{(1 - 4x)^n} \sum_{k=0}^{n} \frac{k!}{(1 + \sqrt{1 - 4x})^{k+1}}
\]

\[
\times B_{n,k} \left( (-1)!!, 1!!, \ldots, [2(n - k) - 1]!! \right)
\]

for \( n \in \mathbb{N} \). Further making use of the formula (8) and simplifying arrive at

\[
G^{(n)}(x) = \frac{2^{n+1}}{(1 - 4x)^n} \sum_{k=1}^{n} k! [2(n - k) - 1]!!
\]

\[
\times \frac{(2n - k - 1)}{2(n - k)} \left( \frac{1}{2} \right)^{k/2} \left( \frac{1}{1 + \sqrt{1 - 4x}} \right)^{k+1}
\]

\[
= \frac{2^{n+1}}{(1 - 4x)^n} \sum_{k=1}^{n} k \left( \begin{array}{c}
2n - k - 1 \\
2n - k - 1
\end{array} \right)
\]

\[
\times \frac{(1 - 4x)^{k/2}}{(1 + \sqrt{1 - 4x})^{k+1}}
\]

\[
= \frac{(n - 1)!}{(1 - 4x)^n} \sum_{k=1}^{n} k \left( \begin{array}{c}
2n - k - 1 \\
n - k
\end{array} \right)
\]

\[
\times (1 - 4x)^{k/2} G^{(k+1)}(x)
\]

\[
= \frac{(n - 1)!}{(1 - 4x)^n} \sum_{k=1}^{n} k \left( \begin{array}{c}
2n - k - 1 \\
n - 1
\end{array} \right)
\]

\[
\times (1 - 4x)^{k/2} G^{(k+1)}(x)
\]

The proof of Theorem 3.1 is complete.

Remark 3.1: Comparing (2) with (9) derives

\[
a_k(n) = (n - 1)! \left[ \frac{2n - k - 1}{n - 1} \right]
\]

\[
= \frac{(2n - k - 1)!}{(n - k)!}, \quad n \geq k \geq 1.
\]

This expression is quite simpler, more easily remembered, more possibly understood, more readily computed, more significant, and more meaningful than the one in (3).

Theorem 3.2: For \( n \in \mathbb{N} \), the power to \( n \) and the derivatives of the generating function \( G(x) \) defined in (1) satisfy

\[
G^{n+1}(x) = \frac{(-1)^n}{(1 - 4x)^{n/2}} \sum_{k=1}^{n} \frac{(-1)^k}{k!}
\]

\[
\times \left( \frac{k}{n-k} \right) (1 - 4x)^{k} G^{(k)}(x).
\]

Proof: The derivative formula (9) can be rearranged as

\[
(-1)^n \left( \frac{4x - 1}{n!} \right) G^{(n)}(x)
\]

\[
= \sum_{k=1}^{n} \frac{(2n - k - 1)}{n - 1} (-1)^k k
\]

\[
\times \left[ (-1)^k (1 - 4x)^{k/2} G^{(k+1)}(x) \right], \quad n \in \mathbb{N}.
\]

Considering Lemma 2.4 leads straightforwardly to

\[
(-1)^n (1 - 4x)^{n/2} G^{n+1}(x)
\]

\[
= \sum_{k=1}^{n} \frac{k}{n-k} (4x - 1)^k G^{(k)}(x), \quad n \in \mathbb{N}.
\]

The proof of Theorem 3.2 is complete.

Remark 3.2: Comparing (4) with (10) reveals

\[
b_i(n) = (-1)^i \frac{n!}{(n-i)!} \left( \begin{array}{c}
 n - i \\
 i
\end{array} \right), \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]

This expression is rather simpler, more easily remembered, more possibly understood, more readily computed, more significant, and more meaningful than the one in (5)!!

Remark 3.3: This paper is a shortened version of the preprint [23].

Acknowledgments

The authors are grateful to Dr. Professor Taekyun Kim (Kwangwoon University, South Korea) for his sending an electronic copy of his paper [3] to the first author on November 15, 2017 through e-mail. The authors are thankful to anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

Feng Qi  http://orcid.org/0000-0001-6239-2968
Yong-Hong Yao  http://orcid.org/0000-0002-0452-785X
References

[1] Koshy T. Catalan numbers with applications. Oxford: Oxford University Press; 2009.

[2] Stanley RP. Catalan numbers. New York: Cambridge University Press; 2015. Available from: https://doi.org/10.1017/CBO9781139871495

[3] Kim DS, Kim T. A new approach to Catalan numbers using differential equations. Russ J Math Phys. 2017;24(4):487–496. Available from: https://doi.org/10.1134/S1061920817040070

[4] Qi F. A simple form for coefficients in a family of ordinary differential equations related to the generating function of the Legendre polynomials. Adv Appl Math Sci. 2018;17(11):693–700.

[5] Qi F. Simplifying coefficients in a family of nonlinear ordinary differential equations. Acta Comment Univ Tartu Math. 2018;22(2):293–297. Available from: https://doi.org/10.12697/ACUTM.2018.22.24

[6] Qi F. Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Laguerre polynomials. Appl Appl Math. 2018;13(2):750–755.

[7] Qi F. Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Mittag–Leffler polynomials. Korean J Math. 2019;27(2):417–423. Available from: https://doi.org/10.11568/kjm.2019.27.2.417

[8] Qi F, Guo B-N. A diagonal recurrence relation for the Stirling numbers of the first kind. Appl Anal Discrete Math. 2018;12(1):153–165. Available from: https://doi.org/10.2298/AADM170405004Q

[9] Qi F, Guo B-N. Explicit formulas and recurrence relations for higher order Eulerian polynomials. Indag Math (NS). 2017;28(4):884–891. Available from: https://doi.org/10.1016/j.indag.2017.06.010

[10] Qi F, Lim D, Guo B-N. Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations. Rev R Acad Cienc Exactas Fis Nat Ser A Mat RACSAM. 2019;113(1):1–9. Available from: https://doi.org/10.1007/s13398-017-0427-2

[11] Qi F, Lim D, Guo B-N. Some identities related to Eulerian polynomials and involving the Stirling numbers. Appl Anal Discrete Math. 2018;12(2):467–480. Available from: https://doi.org/10.2298/AADM171008014Q

[12] Qi F, Liu A-Q, Lim D. Explicit expressions related to degenerate Cauchy numbers and their generating function. In: Singh J, Kumar D, Dutta H, Baleanu D, Purohit S, editors. Mathematical Modelling, Applied Analysis and Computation, ICMMAAC 2018. Springer Proceedings in Mathematics & Statistics, Vol. 272, Chapter 2, Singapore: Springer; p. 41–52. Available from https://doi.org/10.1007/978-981-13-9608-3_2

[13] Qi F, Niu D-W, Guo B-N. Simplification of coefficients in differential equations associated with higher order Frobenius–Euler numbers. Tatra Mt Math Publ. 2018;72:67–76. Available from: https://doi.org/10.2478/tmj-2018-0022

[14] Qi F, Niu D-W, Guo B-N. Simplifying coefficients in differential equations associated with higher order Bernoulli numbers of the second kind. AIMS Math. 2019;4(2):170–175. Available from: https://doi.org/10.3934/Math.2019.2.170

[15] Qi F, Niu D-W, Guo B-N. Some identities for a sequence of unnamed polynomials connected with the Bell polynomials. Rev R Acad Cienc Exactas Fis Nat Ser A Mat RACSAM. 2019;113(2):557–567. Available from: https://doi.org/10.1007/s13398-018-0494-z

[16] Qi F, Zhao J-L. Some properties of the Bernoulli numbers of the second kind and their generating function. Bull Korean Math Soc. 2018;55(6):1909–1920. Available from: https://doi.org/10.4134/BKMS.b180039

[17] Qi F, Zou Q, Guo B-N. The inverse of a triangular matrix and several identities of the Catalan numbers. Appl Anal Discrete Math. 2019;13(2). Available from: https://doi.org/10.22436/jnsa.010.04.06

[18] Comtet L. Advanced combinatorics: the art of finite and infinite expansions. Revised and Enlarged Edition. Dordrecht and Boston: D. Reidel Publishing Co.; 1974. Available from: https://doi.org/10.1007/978-94-010-2196-8

[19] Qi F, Guo B-N. Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers. In: Ruzhansky Michael, Dutta Hemen, Agarwal Ravi P, editors. Mathematical analysis and applications: selected topics. 1st ed. Published by John Wiley & Sons, Inc. 2018. Chapter 5; p. 101–133. Available from: https://doi.org/10.1002/9781119414421.ch5

[20] Qi F, Shi X-T, Liu F-F, et al. Several formulas for special values of the Bell polynomials of the second kind and applications. J Appl Anal Comput. 2017;7(3):857–871. Available from: https://doi.org/10.11948/2017054

[21] Qi F, Liu A-Q, Lim D. Notes on explicit and inversion formulas for the Chebyshev polynomials of the first two kinds. Miskolc Math Notes. 2019.2.170–175. Available from: https://doi.org/10.3934/Math.2019.2.170

[22] Qi F, Qin X-L, Yao Y-H. The generating function of the Legendre polynomials. Adv Appl Math Sci. 2019;27(2):417–423. Available from:https://doi.org/10.22436/jnsa.010.04.06

[23] Qi F, Qin X-L, Yao Y-H. The generating function of the Frobenius–Euler numbers. Tatra Mt Math Publ. 2018;72:67–76. Available from: https://doi.org/10.2478/tmj-2018-0022

[24] Qi F, Niu D-W, Guo B-N. Simplifying coefficients in differential equations associated with higher order Bernoulli numbers of the second kind. AIMS Math. 2019;4(2):170–175. Available from: https://doi.org/10.3934/Math.2019.2.170