Trace and extension operators for fractional Sobolev spaces with variable exponent

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Abstract Assume that \( \Omega \subset \mathbb{R}^n \) is a bounded domain. We show that under certain regularity assumptions on \( \Omega \) there exists a linear extension operator from the space \( W^{s,p(\cdot)}(\Omega) \) to \( W^{s,p(\cdot)}(\mathbb{R}^n) \). As an application we study complemented subspaces in \( W^{s,p(\cdot)} \) via the trace operator.

Keywords extension operator · trace operator · fractional variable exponent Sobolev spaces · complemented subspace problem.

1 Introduction

Sobolev spaces are very interesting mathematical structures in their own right, but their principal significative lies in the central role they, and their numerous generalizations and applications. So it is necessary to develop a reasonable abstract body related to these spaces. In particular the problem of how to extend Sobolev functions was recognized early in the development of the theory of the Sobolev spaces. In this direction many people were interested in determining the exention of the Sobolev functions we mention in particular the works of Sobolev ([18], [19]), of Deny and Lions ([8]), of Gagliardo ([11]) and of the authors in ([17], [3], [16], [20]). Calderón ([6]), Stein ([21]) and Jones ([13]) studied the problem of extension in \( W^{k,p}(\Omega) \) for \( k \in \mathbb{N} \) and \( 1 < p < \infty \). For the fractional Sobolev Spaces \( W^{s,p}(\Omega) \) where \( 0 < s < 1 \) we cite ([10]). For variable exponent Sobolev functions \( W^{k,p(\cdot)}(\Omega) \) there exists a extension...
operator $E$ from $W^{k,p(\cdot)}(\Omega)$ to $W^{k,p(\cdot)}(\mathbb{R}^n)$ (see [9]). All these previous results are holds under certain crucial regularity assumptions on the domain $\Omega$. The objective of this paper is to study the problem of extension in the fractional Sobolev spaces with variable exponent $W^{s,p(\cdot)}(\Omega)$ and its relation with the trace operator. Precisely we show that the existence of an extension operator implies that:

- the trace operator is surjective,
- the kernel of the trace operator is complemented in $W^{s,p(\cdot)}(\mathbb{R}^n)$.

First motivation of this paper is the following extension problem: if $\Omega \subset \mathbb{R}^n$ is a domain with regular geometry of $\partial \Omega$ and $u \in W^{s,p(\cdot)}(\Omega)$, can be extend $u$ to the whole of $\mathbb{R}^n$? Second motivation of studying these spaces is that solutions of partial differential equations belong naturally to Sobolev spaces, and in particular for the study of partial differential equations related to the $p(\cdot)$-Laplacian operator, it can be much more fruitful to study weaker forms of equations in Sobolev spaces $W^{k,p(\cdot)}$, where notions of smoothness are relaxed to require only the existence of weak derivatives. For the nonlocal fractional $p(\cdot)$-Laplacian operator, we constrain functions in a fractional variable exponent Sobolev spaces $W^{s,p(\cdot)}(\Omega)$ where $s \in (0,1)$. Variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$ with $k \in \mathbb{N}$ have been studied in many papers; for surveys see ([9],[7],[15]).

This paper consists of four sections. After an introduction, we give a notation and preliminaries used throughout this paper. In section three under certain regularity assumptions on the domain $\Omega$ we prove the existence of the linear extension operator $E : W^{s,p(\cdot)}(\Omega) \to W^{s,p(\cdot)}(\mathbb{R}^n)$ such that for all $u \in W^{s,p(\cdot)}(\Omega)$, $Eu |_{\Omega} = u$. Firstly we prove the existence when the function $u$ is identically zero in a neighborhood of the boundary $\partial \Omega$, after that we prove the extension theorem for any domain satisfying certain regularity assumptions. In the last section, we define a trace operator in fractional Sobolev spaces with variable exponent and we prove that his kernel is complemented in $W^{s,p(\cdot)}(\mathbb{R}^n)$.

2 Notation and preliminaries

Throughout of this paper we will use the following notation: $\mathbb{R}^n$ is the real Euclidean $n$-space. $\Omega$ is a open subset of $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$ measurable we denote by $|E|$ the Lebesgue measure of $E$. If $A \subset \mathbb{R}^n$ we denote by $\partial A$ the topological boundary of $A$, by $\overline{A}$ the closure of $A$ and $d = diam(A) = \sup\{|x-y| : x, y \in A\}$ denote the diameter of a set $A$. $C$ will denote a constant which may change even in single string of an estimate. By $supp f$ we denote the support of the function $f$.

We say that a closed subspace $Y$ of a Banach space $X$ is complemented if there is another closed subspace $Z$ of $X$ such that $X = Y \oplus Z$. That is, $Y \cap Z = \{0\}$ and every element $x \in X$ can be written as $x = y + z$, with $y \in Y$ and $z \in Z$. By $Y^\perp$ we denote the orthogonal of $Y$. For an operator $L$ we denote by $Ker L$ the kernel of $L$. 
We say that an open set $D \subset \mathbb{R}^n$ is of class $C^1$ if for every $x \in \partial D$ there exist a neighborhood $U$ of $x \in \mathbb{R}^n$ and a bijective map $H : Q \to U$ such that:

$$H \in C^1 (\overline{Q}), H^{-1} \in C^1 (\overline{U}), H (Q_\alpha) = U \cap Q, \text{ and } H (Q_\beta) = U \cap \partial D,$$

where for given $x \in \mathbb{R}^n$ write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x' = (x_1, \ldots, x_{n-1})$,

$$|x'| = \left( \sum_{i=1}^{n-1} x_i^2 \right)^{\frac{1}{2}},$$

$$\mathbb{R}_+^n := \left\{ x = (x', x_n) : x_n > 0 \right\},$$

$$Q := \left\{ x = (x', x_n) : |x'| < 1 \text{ and } |x_n| < 1 \right\},$$

$$Q_+ := Q \cap \mathbb{R}_+^n,$$

$$Q_0 := \left\{ x = (x', 0) : |x'| < 1 \right\}. \text{ The map } H \text{ is called a local chart.}$$

For an open set $D \subset \mathbb{R}^n$, we denote by $\mathcal{C}^k (D)$ the set of functions with $k$-th derivative is continuous for $k$ positive integer, $\mathcal{C}^\infty (D) = \cap_{k \geq 1} \mathcal{C}^k (D)$ and by $\mathcal{C}^\infty_0 (D)$ the set of all functions in $\mathcal{C}^\infty (D)$ with compact support.

We will next introduce variable exponent Lebesgue spaces and fractional exponent Sobolev spaces; note that we nevertheless use the standard definitions of the spaces $L^p (D)$ and $W^{s,p} (D)$ in the fixed exponent case with open $D \subset \mathbb{R}^n$.

Let $(\mathcal{A}, \sum, \mu)$ be a $\sigma$–finite complete measure space and $p : \mathcal{A} \to [1, \infty)$ be a $\mu$–measurable function (called the variable exponent on $\mathcal{A}$). We define $p^+ := \sup_{x \in \mathcal{A}} p(x)$, $p^- := \inf_{x \in \mathcal{A}} p(x)$, $\alpha := \min \{ 1, \min \{ p^+, p^- \} \}$ and $\beta := \max \{ 1, \max \{ p^+, p^- \} \}$. We say that $p$ is a bounded variable exponent if $p^+ < \infty$. By $\mathcal{M} (\mathcal{A}, \mu)$ we denote the space of all measurable $\mu$–function $u : \mathcal{A} \to \mathbb{R}$. The variable exponent *Lebesgue* $L^{p^+} (\mathcal{A}, \mu)$ is defined as follows:

$$L^{p^+} (\mathcal{A}, \mu) := \left\{ u \in \mathcal{M} (\mathcal{A}, \mu) : \rho_{p^+} (\lambda u) = \int_\mathcal{A} |\lambda u|^p (x) \, dx < \infty, \right\},$$

where the function $\rho_{p^+} : L^{p^+} (\mathcal{A}, \mu) \to [0, \infty)$ is called the modular of the space $L^{p^+} (\mathcal{A}, \mu)$. We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{p^+} = \inf \left\{ \lambda > 0 : \rho_{p^+} (\lambda u) \leq 1 \right\},$$

which makes $L^{p^+} (\mathcal{A}, \mu)$ a Banach space. In the classical case when $\mu$ is the $n$–Lebesgue measure, $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$, $\sum$ is the $\sigma$–algebra of $\mu$–measurable subsets of $\Omega$ and $p \in \mathcal{M} (\Omega, \mu)$ is bounded variable exponent, we simply denote $L^{p^+} (\Omega, \mu)$ by $L^{p^+} (\Omega)$. We denote by $L^{p} (\Omega)$ the conjugate space of $L^{p^+} (\Omega)$ where $\frac{1}{p^+} + \frac{1}{p} = 1$, we have the following theorem where his proof is in ([9]):
Theorem 1 (Hölder’s inequality) For any \( u \in L^p(\Omega) \) and \( v \in L^q(\Omega) \), we have
\[
\left| \int_{\Omega} uv dx \right| \leq 2 \| u \|_{L^p(\Omega)} \| v \|_{L^q(\Omega)}.
\]

The variable exponent Sobolev space \( W^{1,p(\cdot)}(\Omega) \) is the space of all measurable function \( u : \Omega \to \mathbb{R} \) such that \( u \) and the absolute value of distributional gradient \( \nabla u = (\partial_1 u, ..., \partial_n u) \) is in \( L^{p(\cdot)}(\Omega) \). The norm \( \| u \|_{W^{1,p(\cdot)}(\Omega)} = \| u \|_{L^p(\Omega)} + \| \nabla u \|_{L^p(\Omega)} \) makes \( W^{1,p(\cdot)}(\Omega) \) a Banach space.

From now let \( \Omega \) be a fixed smooth bounded domain in \( \mathbb{R}^n \) and \( 0 < s < 1 \). Let \( p \) be a bounded continuous variable exponent in \( \overline{\Omega} \times \overline{\Omega} \), \( q \) be a bounded continuous variable exponent in \( \overline{\Omega} \) and \( p^*(x) := \frac{np(x)}{n-sp(x)} \) the fractional critical variable Sobolev exponent with \( p^*(x) > p^*q(x) \) and \( q(x) > p(x,x) \) for \( x \in \overline{\Omega} \). We want to define the fractional Sobolev spaces with variable exponent \( W^{s,p(\cdot)}(\Omega) \), for this we extend the definition given in (10) to the case of variable exponent. So we define the fractional Sobolev spaces with variable exponent \( W^{s,p(\cdot)}(\Omega) \) as follows:
\[
W^{s,p(\cdot)}(\Omega) := \left\{ u \in L^q(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{n+sp(x,y)}} \in L^{p(\cdot)}(\Omega \times \Omega) \right\};
\]
i.e., an intermediary Banach space between two Banach spaces \( L^q(\Omega) \) and \( W^{1,p(\cdot)}(\Omega) \), endowed with the natural norm
\[
\| u \|_{W^{s,p(\cdot)}(\Omega)} := \| u \|_{L^q(\Omega)} + [u]_{W^{s,p(\cdot)}(\Omega)},
\]
where
\[
[u]_{W^{s,p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} dxdy \leq 1 \right\},
\]
is the semi norm of \( u \).

For general theory of classical Sobolev spaces we refer the reader to (2, 1, 22, 5, 10, 12, 4) and for the Lebesgue and Sobolev with variable exponent to (24, 9, 15).

Theorem 2 Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain. Assume that \( sp(x,y) < n \) for \( (x,y) \in \overline{\Omega} \times \overline{\Omega} \), \( q(x) > p(x,x) \) for \( x \in \overline{\Omega} \) and \( r : \overline{\Omega} \rightarrow (1, \infty) \) is a continuous function such that \( p^*(x) > r(x) \geq r^- > 1 \), for \( x \in \overline{\Omega} \). Then the space \( W^{s,p(\cdot)}(\Omega) \) is continuously embedded in \( L^{r(\cdot)}(\Omega) \) for any \( r \in (1, p^*) \).

Proof we find it in (14). \( \square \)

Lemma 1 Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \). Then there exists a suitable positive constant \( C \) such that
\[
\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} dxdy \leq C \left( \| u \|_{W^{s,p(\cdot)}(\Omega)}^{p^+} + \| u \|_{W^{s,p(\cdot)}(\Omega)}^{-} \right)
\]
for all \( u \in W^{s,p(\cdot)}(\Omega) \).
Proof Let \( u \in W^{k,p} \cdot (\Omega) \), then:

\[
\begin{align*}
&\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \, dx \, dy \\
\leq& \ 2^{p+1} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{p(x,y)} + |u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \, dx \, dy \\
\leq& \ 2^{p+1} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \, dx \, dy \\
+& \ 2^{p+1} \int_{\Omega} \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{|u(x)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \, dx \, dy \\
\leq& \ 2^{p+1} \int_{\Omega} \left( \int_{\{|y| \geq 1\}} \frac{dy}{|x-y|^{n+sp(x,y)}} \right) |u(x)|^{p(x,y)} \, dx \\
+& \ 2^{p+1} \int_{\Omega} \left( \int_{\{|y| < 1\}} \frac{dy}{|x-y|^{n+sp(x,y)}} \right) |u(x)|^{p(x,y)} \, dx \\
\leq& \ 2^{p+1} \int_{\Omega} \left( \int_{\{|y| \geq 1\}} \frac{dy}{|x-y|^{n+sp(x,y)}} \right) |u(x)|^{p(x,y)} \, dx \\
+& \ 2^{p+1} \int_{\Omega} \left( \int_{\{|y| < 1\}} \frac{dy}{|x-y|^{n+sp(x,y)}} \right) |u(x)|^{p(x,y)} \, dx \\
\leq& \ 2^{p+1} \int_{\Omega} \left( \int_{\{|y| \geq 1\}} \frac{dy}{|x-y|^{n+sp(x,y)}} \right) \left( |u(x)|^{p^+} + |u(x)|^{p^-} \right) \, dx + \\
2^{p+1} \max \left( d^{p^+}, d^{p^-} \right) \int_{\Omega} \left( \int_{\{|y| < 1\}} \frac{dy}{|x-y|^{n+(s-1)p(x,y)}} \right) \times \\
&\left( |u(x)|^{p^+} + |u(x)|^{p^-} \right) \, dx \\
\leq& \ 2C \left( n, s, p^+ \right) \|1\|_{L^n^p(\Omega)} \left( \|u\|^{p^+}_{L^{n^+p(u)}(\Omega)} + \|u\|^{p^-}_{L^{n^-p(u)}(\Omega)} \right) \\
+& \ 2C \left( n, s, p^+, p^-, d \right) \|1\|_{L^n^p(\Omega)} \left( \|u\|^{p^+}_{L^{n^+p(u)}(\Omega)} + \|u\|^{p^-}_{L^{n^-p(u)}(\Omega)} \right) \\
\leq& \ 2C \left( n, s, p^+, \Omega \right) \left( \|u\|^{p^+}_{L^{n^+p(u)}(\Omega)} + \|u\|^{p^-}_{L^{n^-p(u)}(\Omega)} \right) \\
+& \ 2C \left( n, s, p^+, p^-, d, \Omega \right) \left( \|u\|^{p^+}_{L^{n^+p(u)}(\Omega)} + \|u\|^{p^-}_{L^{n^-p(u)}(\Omega)} \right) \\
\leq& \ C \left( \|u\|^{p^+}_{W^{k,p}(\Omega)} + \|u\|^{p^-}_{W^{k,p}(\Omega)} \right) \\
\leq& \ C \left( \|u\|^{p^+}_{W^{k,p}(\Omega)} + \|u\|^{p^-}_{W^{k,p}(\Omega)} \right)
\end{align*}
\]
where $C := \max \{2C(n,s,p^+,|\Omega|), 2C(n,s,p^-,d,|\Omega|)\}$. Note that this inequality follows in order from the fact that the kernel $\frac{1}{|x-y|^{n+s+p}}$ is summable with respect to $y$ if $|x-y| \geq 1$ since $n+sp(x,y) > n$, on the other hand, the kernel $\frac{1}{|x-y|^{n+(s-1)p(x,y)}}$ is summable when $|x-y| < 1$ since $n+(s-1)p(x,y) < n$, by using the Hölder’s inequality, and finally by using theorem 2. So for all $u \in W^{s,p}(\Omega)$:

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \, dx \, dy \leq C \left( \|u\|_{W^{s,p}(\Omega)}^{p^+} + \|u\|_{W^{s,p}(\Omega)}^{p^-} \right).$$

\[\square\]

### 3. Sobolev Extension Operators

To study the properties of the fractional sobolev spaces with variable exponent $W^{s,p}(\Omega)$ it is often preferable to beginning with the case $\Omega = \mathbb{R}^n$. It is therefore useful to be able to extend a function $u \in W^{s,p}(\Omega)$ to a function $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$.

We start with some preliminary lemmas, in which we will construct the extension to the whole of $\mathbb{R}^n$ of a function $u$ defined on $\Omega$.

**Lemma 2** Let $\Omega$ be an open set in $\mathbb{R}^n$, $u$ a function in $W^{s,p}(\Omega)$. If there exists a compact subset $K \subseteq \Omega$ such that $u \equiv 0$ in $\Omega \setminus K$, then the extension function $\tilde{u}$ defined as

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega; \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

belongs to $W^{s,p}(\mathbb{R}^n)$ and

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq \begin{cases} C \|u\|_{W^{s,p}(\Omega)}^{p^+} & \text{if } \|u\|_{W^{s,p}(\Omega)} \geq 1, \\ C \|u\|_{W^{s,p}(\Omega)}^{p^-} & \text{if } \|u\|_{W^{s,p}(\Omega)} \leq 1, \end{cases}$$

where $C$ is a suitable positive constant.

**Proof** By construction we have $\tilde{u} \in L^{p(\cdot)}(\mathbb{R}^n)$ and

$$\|\tilde{u}\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|u\|_{L^{p(\cdot)}(\Omega)} \leq \|u\|_{W^{s,p}(\Omega)}.$$

Hence we show that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} \, dx \, dy < \infty, \text{ for some } \lambda > 0.$$
\[
\begin{align*}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dx \, dy &= \\
= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dx \, dy \\
+ 2 \int_{\Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dy \right) \, dx,
\end{align*}
\]

since \( u \in W^{s,p} (\Omega) \), then
\[
\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dx \, dy < \infty.
\]

For all \( y \in \mathbb{R}^n \setminus K \),
\[
\begin{align*}
\frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} &= \frac{\chi_K(x) |u(x)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \\
&\leq \min \left\{ \lambda^{p^+}, \lambda^{p^-} \right\} \sup_{x \in K} \frac{1}{|x - y|^{n + sp(x,y)}} \chi_K(x) |u(x)|^{p(x,y)} \\
&\leq \min \left\{ \lambda^{p^+}, \lambda^{p^-} \right\} \frac{1}{\text{dist} (y, \partial K)^{n + sp(x,y)}} \chi_K(x) |u(x)|^{p(x,y)} \\
&\leq \min \left\{ \lambda^{p^+}, \lambda^{p^-} \right\} \frac{1}{\text{dist} (y, \partial K)^{n + sp(x,y)}} \left( |u(x)|^{p^+} + |u(x)|^{p^-} \right)
\end{align*}
\]

so
\[
\int_{\Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dy \right) \, dx \\
\leq \frac{1}{\min \left\{ \lambda^{p^+}, \lambda^{p^-} \right\}} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{\chi_K(x)}{\text{dist} (y, \partial K)^{n + sp(x,y)}} \left( |u(x)|^{p^+} + |u(x)|^{p^-} \right) \, dx \, dy,
\]
using the Hölder inequality, we get
\[
\int_{\Omega} \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^p(x,y)}{\lambda^{p(x,y)} |x - y|^{n+sp(x,y)}} \, dy \right) \, dx
\]
\[
\leq 2 \min \left\{ \lambda^{p^-}, \lambda^{p^+} \right\} \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{\operatorname{dist} (y, \partial K)^{n+sp(x,y)}} \, dy \right)
\]
\[
\times \left( \|u\|^{p^+}_{L^{p^+(\cdot)}(\Omega)} + \|u\|^{p^-}_{L^{p^-(\cdot)}(\Omega)} \right)
\]
\[
\leq 2 \max \left\{ 1, |K| \right\} \min \left\{ \lambda^{p^-}, \lambda^{p^+} \right\} \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{\operatorname{dist} (y, \partial K)^{n+sp(x,y)}} \, dy \right)
\]
\[
\times \left( \|u\|^{p^+}_{L^{p^+(\cdot)}(\Omega)} + \|u\|^{p^-}_{L^{p^-(\cdot)}(\Omega)} \right),
\]
by theorem 2 we know that there exists a positive constant $C$ such that
\[
\|u\|^{p^+}_{L^{p^+(\cdot)}(\Omega)} \leq C \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} \quad \text{and} \quad \|u\|^{p^-}_{L^{p^-(\cdot)}(\Omega)} \leq C \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)}.
\]
On the other hand
\[
\left( \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{\operatorname{dist} (y, \partial K)^{n+sp(x,y)}} \, dy \right) < \infty,
\]
since $n + sp(x,y) > n$ and $\operatorname{dist}(y, \partial K) > 0$. Using the lemma 1 we get
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\tilde{u}(x) - \tilde{u}(y)|^p(x,y)}{\lambda^{p(x,y)} |x - y|^{n+sp(x,y)}} \, dx \, dy
\]
\[
\leq \min \left\{ \lambda^{p^-}, \lambda^{p^+} \right\} \left( \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} + \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)} \right)
\]
\[
+ 2C \max \left\{ 1, |K| \right\} \min \left\{ \lambda^{p^-}, \lambda^{p^+} \right\} \left( \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} + \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)} \right)
\]
\[
\leq \max \left\{ C, 2C \max \left\{ 1, |K| \right\} \right\} \min \left\{ \lambda^{p^-}, \lambda^{p^+} \right\} \left( \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} + \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)} \right)
\]
\[
\leq C \left( \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} + \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)} \right),
\]
which implies that
\[
\left[ \tilde{u} \right]_{W^{s,p(\cdot)}(\mathbb{R}^n)} \leq C \left( \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} + \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)} \right),
\]
so
\[
\|\tilde{u}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left[ \tilde{u} \right]_{W^{s,p(\cdot)}(\Omega)(\mathbb{R}^n)}
\]
\[
\leq \max \left\{ 1, C \right\} \left( \|u\|^{p^+}_{W^{s,p(\cdot)}(\Omega)} + \|u\|^{p^-}_{W^{s,p(\cdot)}(\Omega)} \right),
\]
Consequently
\[ \| \widetilde{u} \|_{W^{s,p}(-)}(\mathbb{R}^n) \leq \begin{cases} C \| u \|_{W^{s,p}(-)}^2(\Omega) & \text{if } \| u \|_{W^{s,p}(-)}(\Omega) \geq 1, \\ C \| u \|_{W^{s,p}(-)}^2(\Omega) & \text{if } \| u \|_{W^{s,p}(-)}(\Omega) \leq 1, \end{cases} \]
where \( C \) is a suitable positive constant.

**Lemma 3** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), symmetric with respect to the coordinate \( x_n \), and consider the sets \( \Omega_+ = \{ x \in \Omega : x_n > 0 \} \) and \( \Omega_- = \{ x \in \Omega : x_n \leq 0 \} \). Let \( u \in W^{s,p}(-) (\Omega_+) \). We define the function \( \widetilde{u} \) extended by reflection as
\[ \widetilde{u} = \begin{cases} u(x', x_n) & \text{if } x_n \geq 0, \\ u(x', -x_n) & \text{if } x_n < 0. \end{cases} \]
Then \( \widetilde{u} \in W^{s,p}(-) (\Omega) \).

**Proof** By splitting the integrals and changing variable \( \hat{x} = (x', -x_n) \), we get
\[
\int_{\Omega} |\hat{u}|^{q(x)} \, dx = \int_{\Omega_+} |\hat{u}|^{q(x)} \, dx + \int_{\Omega_-} |\hat{u}|^{q(x)} \, dx \\
= \int_{\Omega_+} |u|^{q(x)} \, dx + \int_{\Omega_+} |u(x', \hat{x}_n)|^{q(x)} \, d\hat{x} \\
= 2 \int_{\Omega_+} |u|^{q(x)} \, dx.
\]
Since \( u \in W^{s,p}(-) (\Omega_+) \) then \( \int_{\Omega_+} |\hat{u}|^{q(x)} \, dx < \infty \), and therefore \( \widetilde{u} \in L^{q(x)} (\Omega) \).

Also, we have
\[
\int_{\Omega} \int_{\Omega} \frac{|\hat{u}(x) - \hat{u}(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}} \, dxdy \\
= \int_{\Omega_+} \int_{\Omega_+} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}} \, dxdy \\
+ 2 \int_{\Omega_+} \int_{\mathbb{R}^n \setminus \Omega_+} \frac{|u(x) - u(y', -y_n)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}} \, dxdy \\
+ \int_{\mathbb{R}^n \setminus \Omega_+} \int_{\mathbb{R}^n \setminus \Omega_+} \frac{|u(x', -x_n) - u(y', -y_n)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}} \, dxdy
\]
< \infty.

This concludes the proof. \( \square \)

**Remark 1** Note that this lemma gives a very simple construction of extension operators for certain open sets that are not necessarily smooth.
Lemma 4 Let $\Omega$ be an open set in $\mathbb{R}^n$. Let us consider $u \in W^{s,p}(...)(\Omega)$ and $\psi \in C^0(\Omega)$, $0 \leq \psi \leq 1$. Then $\psi u \in W^{s,p}(...)(\Omega)$ and

$$
\|\psi u\|_{W^{s,p}(\Omega)} \leq \begin{cases} 
C \|u\|_{W^{s,p}(\Omega)}^q & \text{if } \|u\|_{W^{s,p}(\Omega)} \geq 1, \\
C \|u\|_{W^{s,p}(\Omega)} & \text{if } \|u\|_{W^{s,p}(\Omega)} \leq 1,
\end{cases}
$$

where $C$ is a suitable positive constant.

Proof Since $|\psi| \leq 1$, it follows that

$$
\int_{\Omega} \frac{|\psi u|^{q(x)}}{\lambda} \, dx \leq \int_{\Omega} \frac{|u|^{q(x)}}{\lambda} \, dx
$$

and consequently:

$$
\inf \left\{ \lambda > 0, \int_{\Omega} \frac{|\psi u|^{q(x)}}{\lambda} \, dx \leq 1 \right\} \leq \inf \left\{ \lambda > 0, \int_{\Omega} \frac{|u|^{q(x)}}{\lambda} \, dx \leq 1 \right\}
$$

so

$$
\|\psi u\|_{L^{q(\cdot)}(\Omega)} \leq \|u\|_{L^{q(\cdot)}(\Omega)} \leq \|u\|_{W^{s,p}(\cdot)(\Omega)}.
$$

On the other hand

$$
\int_{\Omega} \int_{\Omega} \frac{|\psi(x) u(x) - \psi(y) u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dx \, dy
\leq \int_{\Omega} \int_{\Omega} \frac{|\psi(x) u(x) - \psi(y) u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + sp(x,y)}} \, dx \, dy
+ \int_{\Omega} \int_{\Omega} \frac{|\psi(x) u(y) - \psi(y) u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n - sp(x,y)}} \, dx \, dy
\leq \frac{1}{\min \{\lambda^{p-}, \lambda^{p+}\}} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)}} \, dx \, dy
+ \frac{1}{\min \{\lambda^{p-}, \lambda^{p+}\}} \int_{\Omega} \int_{\Omega} \frac{|u(y)|^{p(x,y)} |\psi(x) - \psi(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)}} \, dx \, dy.
$$

Since $\psi \in C^{\infty}_0(\Omega)$, we have

$$
\frac{|\psi(x) - \psi(y)|^{p(x,y)}}{|x - y|^{n + sp(x,y)}} \leq k^{\max\{p^+, p^-\}} |x - y|^{(1-s)p(x,y) - n}
\leq k^{\max\{p^+, p^-\}} \max\{d(1-s)p^+ - n, d(1-s)p^- - n\},
$$

where $k$ denotes the Lipschitz constant of $\psi$. So
Now combining this last inequality with the lemma (1) we can find a suitable positive constant $C$ such that

$$
\int_\Omega \int_\Omega \frac{|u(y)|^{p(x,y)} |\psi(x) - \psi(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}} dxdy
\leq C \max(p^+,p^-) \max \left\{ d(1-s)p^+ - n, d(1-s)p^- - n \right\} \int_\Omega \int_\Omega |u(y)|^{p(x,y)} dxdy
\leq C \max(p^+,p^-) \max \left\{ d(1-s)p^+ - n, d(1-s)p^- - n \right\}
\times \int_\Omega \int_\Omega \left( |u(y)|^{p^+} + |u(y)|^{p^-} \right) dxdy
\leq C \max(p^+,p^-) \max \left\{ d(1-s)p^+ - n, d(1-s)p^- - n \right\} 2 |\Omega| \|1\|_{L^{p^+}(\Omega)}
\times (\|u\|_{L^{p^+}(\Omega)} + \|u\|_{L^{p^-}(\Omega)})
\leq C (k,p^+,p^-,d,s,|\Omega|) \left(\|u\|_{W^{s,p}(\Omega)}^{p^+} + \|u\|_{W^{s,p}(\Omega)}^{p^-}\right).
$$

Now combining this last inequality with the lemma (1) we can find a suitable positive constant $C$ such that

$$
\int_\Omega \int_\Omega |\psi(x) u(x) - \psi(y) u(y)|^{p(x,y)} dxdy
\leq C \left(\|u\|_{W^{s,p}(\Omega)}^{p^+} + \|u\|_{W^{s,p}(\Omega)}^{p^-}\right),
$$

which implies that

$$
[\psi u]_{W^{s,p}(\Omega)} \leq C \left(\|u\|_{W^{s,p}(\Omega)}^{p^+} + \|u\|_{W^{s,p}(\Omega)}^{p^-}\right) \tag{2}
$$

combining (1) with (2) we obtain

$$
\|\psi u\|_{W^{s,p}(\Omega)} \leq \begin{cases} C \|u\|_{W^{s,p}(\Omega)}^{\alpha} & \text{if } \|u\|_{W^{s,p}(\Omega)} \geq 1, \\
C \|u\|_{W^{s,p}(\Omega)} & \text{if } \|u\|_{W^{s,p}(\Omega)} \leq 1. 
\end{cases}
$$

Now, we are ready to state and prove the extension theorem for any domain $\Omega$ satisfying certain regularity assumptions.

**Theorem 3** Suppose that $\Omega$ is of classe $C^1$ with $\partial \Omega$ bounded. Then there exists a linear extension operator

$$
\mathcal{E} : W^{s,p}(\Omega) \rightarrow W^{s,p}(\mathbb{R}^n)
$$

such that for all $u \in W^{s,p}(\Omega)$,

- $\mathcal{E} u | \Omega = u$, 

- $\mathcal{E} u | \partial \Omega = 0$. 

\[ \|E u\|_{W^{s,p}(\Omega)} \leq \begin{cases} C \|u\|_{W^{s,p}(\Omega)}^\beta & \text{if } \|u\|_{W^{s,p}(\Omega)} \geq 1, \\ C \|u\|_{W^{s,p}(\Omega)}^\alpha & \text{if } \|u\|_{W^{s,p}(\Omega)} \leq 1, \end{cases} \]

where \( C \) is a suitable positive constant.

To show this theorem we need the following lemma.

**Lemma 5** (partition of unity). Let \( \Gamma \) be a compact subset of \( \mathbb{R}^n \) and \( U_1, \ldots, U_k \) be a open covering of \( \Gamma \). Then there exist functions \( \theta_0, \theta_1, \ldots, \theta_k \in C^\infty (\mathbb{R}^n) \) such that:

- \( 0 \leq \theta_i \leq 1, \quad \forall i = 0, 1, \ldots, k \) and \( \sum_{i=0}^k \theta_i = 1 \) on \( \mathbb{R}^n \),
- \( \text{supp} \theta_i \) is compact, \( \text{supp} \theta_i \subset U_i \) for all \( i = 1, 2, \ldots, k \) and \( \text{supp} \theta_0 \subset \mathbb{R}^n \setminus \Gamma \).

If \( \Omega \) is an open bounded set and \( \Gamma = \partial \Omega \), then \( \theta_0 \mid \Omega \in C^\infty_c (\Omega) \).

**Proof** This lemma is classical; similar statements can be found, for example, in [1]. \( \Box \)

**Proof** of theorem 3

We rectify \( \partial \Omega \) by local charts and use a partition of unity. Precisely, since \( \partial \Omega \) is compact of class \( C^1 \), we can find a finite number of balls \( B_i \) such that:

\( \partial \Omega \subset \bigcup_{i=1}^k B_i \), \( \mathbb{R}^n = \bigcup_{i=1}^k B_i \cup (\mathbb{R}^n \setminus \partial \Omega) \) and bijective maps: \( H_i : Q \to B_i \) such that: \( H_i \in C^1 (Q), H_i^{-1} \in C^1 (\overline{B_i}), H_i (Q_+) = B_i \cap \Omega, \) and \( H_i (Q_0) = B_i \cap \partial \Omega \).

There exist \( k \) smooth functions \( \psi_0, \psi_1, \ldots, \psi_k \) such that \( \text{supp} \psi_0 \subset \mathbb{R}^n \setminus \partial \Omega \), \( \text{supp} \psi_i \subset B_i \) for any \( i \in \{ 1, 2, \ldots, k \} \), \( 0 \leq \psi_1 \leq 1 \) for any \( i \in \{ 0, 1, 2, \ldots, k \} \) and \( \sum_{i=0}^k \psi_i = 1 \).

Given \( u \in W^{s,p}(\Omega) \), then:

\[ u = \sum_{i=0}^k \psi_i u \]

by lemma 4 we know that \( \psi_0 u \in W^{s,p}(\Omega) \). Since \( \psi_0 u \equiv 0 \) in a neighborhood of \( \partial \Omega \), then by lemma 2 we can extend it to the whole of \( \mathbb{R}^n \), by setting:

\[ \tilde{\psi}_0 u (x) = \begin{cases} \psi_0 u (x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \]

and

\[ \|\tilde{\psi}_0 u\|_{W^{s,p}(\Omega)} \leq \begin{cases} C \|u\|_{W^{s,p}(\Omega)}^\beta & \text{if } \|u\|_{W^{s,p}(\Omega)} \geq 1, \\ C \|u\|_{W^{s,p}(\Omega)}^\alpha & \text{if } \|u\|_{W^{s,p}(\Omega)} \leq 1, \end{cases} \]

where \( C \) is a suitable positive constant.
Now, we extend $u_i$, $1 \leq i \leq k$ where $u_i = \psi_i u$. Consider the restriction of $u$ to $B_i \cap \Omega$ and transfer this function to $Q_+$ with the help of $H_i$.

For any $i \in \{1, ..., k\}$, let us consider $u \mid B_i \cap \Omega$ and set: $v_i(y) := u(H_i(y))$ for all $y \in Q_+$.

Show that $v_i \in W^{s,p}(\cdot)(Q_+)$, by setting $x = H_i(\tilde{x})$, we have:

\[
\int_{Q_+} \int_{Q_+} \frac{|v(\tilde{x}) - v(\tilde{y})|^{p(H_i(\tilde{x}),H_i(\tilde{y}))}}{\lambda^p(H_i(\tilde{x}),H_i(\tilde{y}))} d\tilde{x}d\tilde{y} = \int_{Q_+} \int_{Q_+} \frac{|u(H_i(\tilde{x})) - u(H_i(\tilde{y}))|^{p(H_i(\tilde{x}),H_i(\tilde{y}))}}{\lambda^p(H_i(\tilde{x}),H_i(\tilde{y}))} d\tilde{x}d\tilde{y} = \int_{B_i \cap \Omega} \int_{B_i \cap \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^p(x,y)} \det \left( H_i^{-1} \right) dx dy
\]

since $\det \left( H_i^{-1} \right) \in L^\infty(B_i \cap \Omega)$ and $H_i$ is a bi-lipschitz map, then:

\[
\int_{Q_+} \int_{Q_+} \frac{|v(\tilde{x}) - v(\tilde{y})|^{p(H_i(\tilde{x}),H_i(\tilde{y}))}}{\lambda^p(H_i(\tilde{x}),H_i(\tilde{y}))} d\tilde{x}d\tilde{y} \leq C \int_{B_i \cap \Omega} \int_{B_i \cap \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^p(x,y)} dx dy.
\]

Note that this integral is finite since $u \in W^{s,p}(\cdot)(B_i \cap \Omega)$. Now by lemma, we can extend $v_i$ to all $Q$ so that the extension $v_i^* \in W^{s,p}(\cdot)(Q)$. Retract $v_i^*$ to $B_i$ using $H_i^{-1}$ and we set:

\[
v_i^* = v_i \left( H_i^{-1}(x) \right), \text{ for any } x \in B_i.
\]

Since $H_i$ is a bi-lipschitz map, by arguing as above it follows that $w_i \in W^{s,p}(\cdot)(B_i)$, $w_i = u$ on $B_i \cap \Omega$, and consequently $\psi_i w_i = \psi_i u$ on $B_i \cap \Omega$.

By definition $\psi_i w_i$ has compact support in $B_i$ and therefore, as done for $\psi_i u$, we can consider the extension $\tilde{\psi_i} w_i$ to all $\mathbb{R}^n$ a way that $\tilde{\psi_i} w_i \in W^{s,p}(\cdot)(\mathbb{R}^n)$, we set for $x \in \mathbb{R}^n$,

\[
\tilde{\psi_i} w_i(x) = \begin{cases} 
\psi_i w_i(x) & \text{if } x \in B_i, \\
0 & \text{if } x \in \mathbb{R}^n \setminus B_i.
\end{cases}
\]

Finally, let the operator:

\[
\mathcal{E} u = \tilde{\psi_0} u + \sum_{i=1}^k \tilde{\psi_i} w_i
\]

be the extension of $u$ defined on all $\mathbb{R}^n$. By construction we have $\mathcal{E} u \mid \Omega = u$ and $\mathcal{E}$ is a linear extension operator. On the other hand

\[
\| \mathcal{E} u \|_{W^{s,p}(\cdot)(\mathbb{R}^n)} = \left\| \tilde{\psi_0} u + \sum_{i=1}^k \tilde{\psi_i} w_i \right\|_{W^{s,p}(\cdot)(\mathbb{R}^n)} \leq \left\| \tilde{\psi_0} u \right\|_{W^{s,p}(\cdot)(\mathbb{R}^n)} + \sum_{i=1}^k \left\| \tilde{\psi_i} w_i \right\|_{W^{s,p}(\cdot)(\mathbb{R}^n)}.
\]
If \( \|u\|_{W^{s,p}(\Omega)} \geq 1 \), then

\[
\|\mathcal{E}u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)}^\beta + \sum_{i=1}^{k} C_i \|\psi_i w_i\|_{W^{s,p}(B_i)}^\beta
\]

\[
\leq C \|u\|_{W^{s,p}(\Omega)}^\beta + \sum_{i=1}^{k} C_i \|w_i\|_{W^{s,p}(B_i)}^\beta
\]

\[
\leq C \|u\|_{W^{s,p}(\Omega)}^\beta + \sum_{i=1}^{k} C_i \|v_i^*\|_{W^{s,p}(Q)}^\beta
\]

\[
\leq C \|u\|_{W^{s,p}(\Omega)}^\beta + \sum_{i=1}^{k} C_i \|v_i\|_{W^{s,p}(Q^+)}^\beta
\]

\[
\leq C \|u\|_{W^{s,p}(\Omega)}^\beta + \sum_{i=1}^{k} C_i \|v_i\|_{W^{s,p}(\Omega \cap B_i)}^\beta
\]

\[
\leq C \|u\|_{W^{s,p}(\Omega)}^\beta.
\]

By the same way we get

\[
\|\mathcal{E}u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)}^\alpha
\]

if \( \|u\|_{W^{s,p}(\Omega)} \leq 1 \). Finally the operator:

\[
\mathcal{E}u = \bar{\psi}_0u + \sum_{i=1}^{k} \bar{\psi}_i w_i
\]

possesses all the desired properties. \( \square \)

4 Complemented subspaces in \( W^{s,p}(\Omega) \)

The complemented subspace problem plays a key role in the development of the Banach space theory. In the case of Hilbert space \( H \) it is known that every closed subspace \( Y \subset H \) is complemented; the orthogonal complement \( Y^\perp \) is a closed subspace of \( H \) and we have

\[
H = Y \oplus Y^\perp.
\]

The lack of the Hilbert structure of the space \( W^{s,p}(\Omega) \) makes the complemented subspace problem in this space very difficult. In this section we will study this problem by using the previous extension theorem.

The trace of a function is in some sense a restriction of the function to a subset of the original set of definition.
**Definition 1** For any \( u \in W^{s,p} (\mathbb{R}^n) \), define the trace operator \( T \) by

\[
T: W^{s,p} (\mathbb{R}^n) \rightarrow W^{s,p} (\Omega), \quad Tu = u \mid _{\Omega}.
\]

Note that if \( E \) is an extension operator, then \( T \circ E \) is the identity on \( W^{s,p} (\Omega) \).

**Corollary 1** The operator \( T \) is surjective.

**Proof** by theorem 3 every function \( u \) in \( W^{s,p} (\Omega) \) admits an extension to \( W^{s,p} (\mathbb{R}^n) \) then the trace operator \( T \) is surjective. \( \square \)

**Theorem 4** Suppose that \( \Omega \) is of classe \( C^1 \) with \( \partial \Omega \) bounded. Then the subspace \( Ker T \) is complemented in \( W^{s,p} (\mathbb{R}^n) \).

**Proof** By theorem 3 there exist a linear extension operator

\[
\mathcal{E} : W^{s,p} (\Omega) \rightarrow W^{s,p} (\mathbb{R}^n)
\]

such that

\[
\forall x \in \Omega : \mathcal{E}u (x) = u (x).
\]

We have

\[
\mathcal{E} \left( W^{s,p} (\Omega) \right) \subset W^{s,p} (\mathbb{R}^n)
\]

is a closed subspace, and

\[
\forall u \in W^{s,p} (\mathbb{R}^n) : u = u - \mathcal{E} (Tu) + \mathcal{E} (Tu).
\]

Since

\[
(u - \mathcal{E} (Tu)) \in Ker T, \quad \mathcal{E} (Tu) \in \mathcal{E} \left( W^{s,p} (\Omega) \right),
\]

and

\[
\text{Ker } T \cap \mathcal{E} \left( W^{s,p} (\Omega) \right) = \{0\},
\]

we conclude that

\[
W^{s,p} (\mathbb{R}^n) = \text{Ker } T \oplus \mathcal{E} \left( W^{s,p} (\Omega) \right).
\]

and this complete the proof. \( \square \)

**Corollary 2** If \( p (x,y) = 2 \), then :

\[
W^{s,2} (\mathbb{R}^n) = H^s (\mathbb{R}^n) = \text{Ker } T \oplus (\text{Ker } T)^\perp.
\]

**Proof** In this case, theorem 4 is a direct consequence of the Hilbert structure of the Space \( W^{s,2} (\mathbb{R}^n) \). Indeed \( \text{Ker } T \) is a closed subspace of \( W^{s,2} (\mathbb{R}^n) \), so

\[
W^{s,2} (\mathbb{R}^n) = \text{Ker } T \oplus (\text{Ker } T)^\perp.
\]

\( \square \)
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