General solution of asymptotic conditions for electromagnetic form factors of hadrons represented by VMD model

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Abstract

General solution of asymptotic conditions, derived previously for $n$ vector-meson parametrization of electromagnetic form factor of any strongly interacting particle with the asymptotics $\sim_{|t|\to\infty} t^{-m}(m < n)$ and combined with a form factor normalization condition, is presented. The special case of $m = n$ and the solution of asymptotic conditions without any form factor normalization are discussed too.

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1 Introduction

Recently, starting with different properties of the electromagnetic (EM) form factor (FF) $F_h(t)$ of a strongly interacting particle to be saturated by $n$ vector-mesons and possessing the asymptotic behaviour $\sim |t|^{-m}$ ($m \leq n$), two dissimilar systems of $(m - 1)$ linear homogeneous algebraic equations for coupling constant ratios of vector-mesons to hadron under consideration were derived [1]. Though they really look differently, in [1] it has been demonstrated explicitly that both systems are exactly equivalent.

In this paper we are concerned with a more simple one, derived by means of the superconvergent sum rules for the imaginary part of the EM FF, in which the coefficients are simply even powers of the corresponding vector-meson masses. In more detail, we look for its general solution, which leads to the VMD representation of $F_h(t)$ with the required asymptotics.

There are interesting three cases appearing in various physical situations, and all of them are discussed in this paper.

The first one appears in the construction of the unitary and analytic model of EM structure [2] of any strongly interacting particle with a number of building quarks $n_q > 2$, when at the first stage one has need for the VMD parametrization of FF under consideration with the required asymptotics and normalization. The latter is found by a combination of the $(m - 1)$ asymptotic conditions with the FF normalization condition and by a general solution of the obtained $m$ linear algebraic equations for $n$ coupling constant ratios. As a result, the FF depends then on the $(n - m)$ coupling constant ratios as free parameters of the model.

The second case is obtained from the previous one for $m \equiv n$ and it leads to expressions of all coupling constant ratios through the vector-meson masses. If the latter are known, numerical values of the coupling constant ratios are found, like in [3], for tensor coupling constants of vector-mesons to nucleons.

The third case appears naturally in the determination of strange FF behaviours of strongly interacting particles with the spin $s > 0$ from the isoscalar parts of the corresponding EM FF’s. For instance, the value of the strangeness nucleon magnetic moment $\mu_s$ is unknown in advance and thus, the corresponding strange magnetic FF (as a consequence also the strange Pauli FF’s) model is constructed without the normalization [4]. In order to keep some inner analytic structure of the corresponding EM form factor model, one has to construct it also without any normalization, though in the electromagnetic case it is exactly known experimentally to be equal to the magnetic moment of the nucleon. So in such a situation one has to solve the asymptotic conditions in the form of $(m - 1)$ linear homogeneous algebraic
equations for \( n \) coupling constant ratios. The resultant solutions express the 
\((m - 1)\) coupling constant ratios through the rest \((n - m + 1)\) ones which are 
then free parameters of the model.

More details about the general solutions of asymptotic conditions and 
their consequences for all three specific cases can be found in the next section. 
The last section is devoted to conclusions and discussion.

2 General solution of asymptotic conditions

First, we look for a general solution of the asymptotic conditions to be com-
bined with the FF norm when FF is saturated by more vector-meson reso-
nances than the power determining the FF asymptotics.

If we assume that EM FF of any strongly interacting particle is well 
approximated by a finite number \( n \) of vector-meson exchange tree Feynman 
diagrams, one finds the VMD pole parametrization

\[
F_h(t) = \sum_{i=1}^{n} \frac{m_i^2}{m_i^2 - t}(f_{ihh}/f_i) \tag{1}
\]

where \( t = -Q^2 \) is the momentum transfer squared of the virtual photon, \( m_i \) 
are the masses of vector mesons, and \( f_{ihh} \) and \( f_i \) are the coupling constants of 
the vector-meson to hadron and vector-meson-photon transition, respectively.

Furthermore, let us assume that EM FF in (1) has the asymptotic behaviour

\[
F_h(t)|_{t \to \infty} \sim t^{-m} \tag{2}
\]

and it is normalized at \( t = 0 \) as follows:

\[
F_h(0) = F_0. \tag{3}
\]

The requirement for the conditions (3) and (2) to be satisfied by (1) 
(including also the results of ref. [1]) leads to the following system of \( m \) 
linear algebraic equations:

\[
\sum_{i=1}^{n} a_i = F_0 \tag{4}
\]

\[
\sum_{i=1}^{n} m_i^{2r} a_i = 0, \quad r = 1, 2, \ldots, m - 1
\]

for \( n \) coupling constant ratios \( a_i = (f_{ihh}/f_i) \). Therefore, a solution of (4) 
will be looked for \( m \) unknowns \( a_1, \ldots, a_m \) and \( a_{m+1}, \ldots, a_n \) will be considered as
free parameters of the model. Then, the system (4) can be rewritten in the matrix form

$$\mathbf{Ma} = \mathbf{b}, \quad (5)$$

with the $m \times m$ Vandermonde matrix $\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\frac{m_1^2}{m_2} & \frac{m_2^2}{m_3} & \ldots & \frac{m_m^2}{m_{m+1}} \\
\frac{m_1^4}{m_2^2} & \frac{m_2^4}{m_3^2} & \ldots & \frac{m_m^4}{m_{m+1}^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{m_1^{2(m-1)}}{m_2^{2(m-1)}} & \frac{m_2^{2(m-1)}}{m_3^{2(m-1)}} & \ldots & \frac{m_{m-1}^{2(m-1)}}{m_m^{2(m-1)}}
\end{pmatrix}, \quad (6)$$

and the column vectors

$$\mathbf{a} = \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
\vdots \\
a_m
\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}
F_0 - \sum_{k=m+1}^{n} a_k \\
- \sum_{k=m+1}^{n} m_k^2 a_k \\
- \sum_{k=m+1}^{n} m_k^4 a_k \\
\vdots \\
- \sum_{k=m+1}^{n} m_k^{2(m-1)} a_k
\end{pmatrix}. \quad (7)$$

The Vandermonde determinant of the matrix (6) is different from zero

$$\det \mathbf{M} = \prod_{j=1, \ j < l}^{m} (m_j^2 - m_l^2). \quad (8)$$

This has been proved explicitly by reducing the matrix (6) to the triangular form and then taking into account the fact that the determinant of a triangular matrix is the product of its main diagonal elements.

As a consequence of (8) a nontrivial solution of (5) exists. To find the latter we use Cramer’s Rule despite the fact that computationally Cramer’s Rule for $m > 3$ offers no advantages over the Gaussian elimination method. However, in our case (as one can see further) all calculations are for the most part reduced to a calculation of the Vandermonde type determinants, and there is no problem to come to the explicit solutions.

So, the corresponding solutions of (5) for $i = 1, \ldots, m$ are

$$a_i = \frac{\det \mathbf{M}_i}{\det \mathbf{M}}, \quad (9)$$
Since any determinant is an additive function of each column, for each scalar $C$, $\det(A_1, ..., C A_i, ..., A_n) = C \det(A_1, ..., A_i, ..., A_n)$ and $\det(A_1, ..., A_{i-1}, \sum_k x_k A_k, A_{i+1}, ..., A_n) = \sum_k x_k \det(A_1, ..., A_{i-1}, A_k, A_{i+1}, ..., A_n)$. As a result, for a determinant of the matrix $M_i$ one can write the decomposition

$$
\text{det} M_i = \begin{vmatrix}
1 & 1 & \cdots & F_0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n & n & \cdots & n & \cdots & n \\
\end{vmatrix}
$$

(11)

- $\sum_{k=m+1}^n a_k$

from where, if in the first determinant the Laplace expansion by the entries of the column $i$ is used, the explicit form is obtained

$$
\text{det} M_i = F_0 (-1)^{1+i} \prod_{j=1, j\neq i}^m m_j^2 \prod_{j,l=1, j,l\neq i}^m (m_i - m_j^2) -
\sum_{k=m+1}^n a_k \prod_{j,l=1, j,l\neq i}^m (m_i - m_j^2).
$$

(12)
Now, substituting (8) and (12) into (9), one gets the solutions of (5) as follows

\[ a_i = \frac{F_0(-1)^{i-1} \prod_{j=1}^{m} m_j^2 \prod_{j<l,j\neq i}^{m} (m_i^2 - m_j^2)}{\prod_{j<l,j\neq i}^{m} (m_i^2 - m_j^2)} - \]

\[ (-1)^{i-1} \prod_{j=1}^{m} \sum_{j<l,j\neq i}^{m} (m_i^2 - m_j^2) \sum_{k=m+1}^{n} a_k \prod_{j=1}^{m} (m_i^2 - m_j^2) \]

\[ \prod_{j=l}^{m} (m_i^2 - m_j^2). \]

In order to find, by means of \((13)\), an explicit form of \(F_h(t)\) to be automatically normalized with the required asymptotic behaviour, let us separate the sum in \((4)\) into two parts with the subsequent transformation of the first one into a common denominator as follows:

\[ F_h(t) = \sum_{i=1}^{m} \frac{m_i^2 a_i}{m_i^2 - t} + \sum_{k=m+1}^{n} \frac{m_k^2 a_k}{m_k^2 - t} = \]

\[ = \frac{\sum_{i=1}^{m} \prod_{j=i}^{m} (m_i^2 - t) m_i^2 a_i}{\prod_{j=1}^{m} (m_i^2 - t)} + \sum_{k=m+1}^{n} \frac{m_k^2 a_k}{m_k^2 - t}. \]

Then \((13)\) together with \((14)\) gives

\[ F_h(t) = F_0 \frac{\sum_{i=1}^{m} (-1)^{i-1+i} m_i^2 \prod_{j=i}^{m} m_j^2 \prod_{j<l,j\neq i}^{m} (m_i^2 - m_j^2)}{\prod_{j=l}^{m} (m_i^2 - t) \prod_{j=1}^{m} (m_i^2 - m_j^2)} - \]

\[ - \frac{\sum_{i=1}^{m} (-1)^{i-1-i} m_i^2 \prod_{j=i}^{m} (m_i^2 - t) \prod_{j<l,j\neq i}^{m} (m_i^2 - m_j^2) \sum_{k=m+1}^{n} a_k \prod_{j=i}^{m} (m_i^2 - m_j^2)}{\prod_{j=1}^{m} (m_i^2 - t) \prod_{j<l}^{m} (m_i^2 - m_j^2)} + \]

\[ + \sum_{k=m+1}^{n} \frac{m_k^2 a_k}{m_k^2 - t}. \]

The first term in \((15)\) can be rearranged into the form

\[ F_0 \frac{\prod_{j=1}^{m} m_j^2}{\prod_{j=1}^{m} (m_j^2 - t)} \sum_{i=1}^{m} (-1)^{i-1+i} \prod_{j=1}^{m} (m_j^2 - t) \prod_{j<l,j\neq i}^{m} (m_i^2 - m_j^2) \]

\[ \prod_{j=1}^{m} (m_i^2 - m_j^2) \]

\[ \prod_{j=1}^{m} (m_i^2 - m_j^2) \]

in which one can prove explicitly the identity

\[ \sum_{i=1}^{m} (-1)^{i+1} \prod_{j=1}^{m} (m_j^2 - t) \prod_{j=l,j\neq i}^{m} (m_i^2 - m_j^2) \equiv \prod_{j=1}^{m} (m_i^2 - m_j^2) \]

\[ \prod_{j=1}^{m} (m_i^2 - m_j^2) \]
leading to remarkable simplification of the term under consideration as follows:

\[ F_0 \frac{\prod_{j=1}^{m} m_j^2}{\prod_{j=1}^{m} (m_j^2 - t)}. \] (18)

One could prove (17) by rewriting its left-hand side into the following form

\[
\sum_{i=1}^{m} (-1)^{1+i} \times
\begin{vmatrix}
(m_1^2 - t) \ldots & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
(m_1^2 - t) & (m_1^2 - t) & (m_1^2 - t) & \ldots & (m_1^2 - t) \\
\end{vmatrix}
\]

and then by using various basic properties of the determinants decomposing it into the sum of large number of various determinants of the same order with their subsequent explicit calculations. Since this procedure seems to be, from the calculational point of view, not simple, with the aim of a proving (17) let us define the new matrix

\[
D(t) =
\begin{pmatrix}
1 & 1 \ldots & 1 \ldots & 1 \\
(m_1^2 - t) & (m_1^2 - t) \ldots & (m_1^2 - t) \ldots & (m_1^2 - t) \\
(m_1^2 - t)^2 & (m_1^2 - t)^2 & \ldots & (m_1^2 - t)^2 \\
(m_1^2 - t)^3 & (m_1^2 - t)^3 & \ldots & (m_1^2 - t)^3 \\
\ldots & \ldots & \ldots & \ldots \\
(m_1^2 - t)^{m-2} & (m_1^2 - t)^{m-2} & \ldots & (m_1^2 - t)^{m-2} \\
(m_1^2 - t)^{m-1} & (m_1^2 - t)^{m-1} & \ldots & (m_1^2 - t)^{m-1} \\
\end{pmatrix}
\]

(20)

Denoting \( (m_1^2 - t) = x_i \) one gets the Vandermonde matrix

\[
D(t) =
\begin{pmatrix}
1 & 1 \ldots & 1 \ldots & 1 \\
x_1 & x_2 \ldots & x_i \ldots & x_m \\
x_1^2 & x_2^2 \ldots & x_i^2 \ldots & x_m^2 \\
x_1^3 & x_2^3 \ldots & x_i^3 \ldots & x_m^3 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
x_1^{m-2} & x_2^{m-2} & \ldots & x_i^{m-2} & \ldots & x_m^{m-2} \\
x_1^{m-1} & x_2^{m-1} & \ldots & x_i^{m-1} & \ldots & x_m^{m-1} \\
\end{pmatrix}
\]

(21)
the determinant of which is equal just to the right-hand side of (17)

$$det D(t) = \prod_{j,l=1 \atop j < l}^m (x_l - x_j) \equiv \prod_{j,l=1 \atop j < l}^m (m_i^2 - t - m_j^2) = \prod_{j,l=1 \atop j < l}^m (m_i^2 - m_j^2). \quad (22)$$

On the other hand, if in the determinant of the matrix (20) the Laplace expansion by the entries of the first row with a subsequent pulling out of common factors in all columns of the subdeterminants is carried out, one gets the expression

$$det D(t) = \sum_{i=1}^m (-1)^{1+i} \prod_{j=1 \atop j \neq i}^m (m_j^2 - t) \times \left| \begin{array}{cccc} 1 & \ldots & 1 & \ldots \\ (m_i^2 - t) & \ldots & (m_{i-1}^2 - t) & \ldots \\ (m_i^2 - t)^2 & \ldots & (m_{i-1}^2 - t)^2 & \ldots \\ (m_i^2 - t)^3 & \ldots & (m_{i-1}^2 - t)^3 & \ldots \\ \vdots & \ddots & \vdots & \ddots \\ (m_i^2 - t)^{m-3} & \ldots & (m_{i-1}^2 - t)^{m-3} & \ldots \\ (m_i^2 - t)^{m-2} & \ldots & (m_{i-1}^2 - t)^{m-2} & \ldots \end{array} \right| \quad (23)$$

Then calculating explicitly the determinant in (23) by using again the denotation $x_k = (m_k^2 - t)$ for $k = 1, ..., i - 1, i + 1, ..m$, one finally obtains

$$det D(t) = \sum_{i=1}^m (-1)^{1+i} \prod_{j=1 \atop j \neq i}^m (m_j^2 - t) \prod_{j,l=1 \atop j < l, j \neq i}^m (m_i^2 - m_j^2) \quad (24)$$

just the left-hand side of (17) and in this way the identity under consideration is clearly proved.

The second and third term in (15), transforming them to a common denominator, can be unified into one following term:

$$\sum_{k=m+1}^n \left\{ \frac{m_k^2 \prod_{j=1}^m (m_j^2 - t) \prod_{j,l=1 \atop j < l}^m (m_i^2 - m_j^2)}{(m_k^2 - t) \prod_{j=1}^m (m_j^2 - t) \prod_{j,l=1 \atop j < l}^m (m_i^2 - m_j^2)} + \frac{(m_k^2 - t) \sum_{i=1}^m (-1)^i m_i^2 \prod_{j=1}^m (m_j^2 - m_k^2) \prod_{j,l=1 \atop j < l, j \neq i}^m (m_i^2 - m_j^2) \prod_{j=1}^m (m_j^2 - t)}{(m_k^2 - t) \prod_{j=1}^m (m_j^2 - t) \prod_{j,l=1 \atop j < l, j \neq i}^m (m_i^2 - m_j^2)} \right\} a_k \quad (25)$$
the numerator of which is exactly the Laplace expansion by the entries of
the first row of the determinant of the matrix of the \((m + 1)\) order

\[
N(t) = \begin{pmatrix}
  m_k^2 & m_1^2 & \ldots & m_m^2 \\
  (m_k^2 - t) & (m_1^2 - t) & \ldots & (m_m^2 - t) \\
  (m_k^2 - t)^2 & (m_1^2 - t)^2 & \ldots & (m_m^2 - t)^2 \\
  (m_k^2 - t)^3 & (m_1^2 - t)^3 & \ldots & (m_m^2 - t)^3 \\
  \vdots & \vdots & \ddots & \vdots \\
  (m_k^2 - t)^{m-1} & (m_1^2 - t)^{m-1} & \ldots & (m_m^2 - t)^{m-1} \\
  (m_k^2 - t)^m & (m_1^2 - t)^m & \ldots & (m_m^2 - t)^m
\end{pmatrix}.
\] (26)

If we define the new matrix of the \((m + 1)\) order

\[
R(t) = \begin{pmatrix}
  1 & 1 & \ldots & 1 \\
  (m_k^2 - t) & (m_1^2 - t) & \ldots & (m_m^2 - t) \\
  (m_k^2 - t)^2 & (m_1^2 - t)^2 & \ldots & (m_m^2 - t)^2 \\
  (m_k^2 - t)^3 & (m_1^2 - t)^3 & \ldots & (m_m^2 - t)^3 \\
  \vdots & \vdots & \ddots & \vdots \\
  (m_k^2 - t)^{m-1} & (m_1^2 - t)^{m-1} & \ldots & (m_m^2 - t)^{m-1} \\
  (m_k^2 - t)^m & (m_1^2 - t)^m & \ldots & (m_m^2 - t)^m
\end{pmatrix},
\] (27)

then for the determinant of both matrices, (26) and (27), the equation

\[
det N(t) - t \cdot det R(t) \equiv det S(t) = 0
\] (28)

is fulfilled under the assumption that \(det S(t)\) is obtained by multiplication
of the first row of \(det R(t)\) by \(t\), and the subtractions of the resultant determinants from \(N(t)\) is carried out explicitly.

There is valid also a relation

\[
det N(0) = 0
\] (29)

as in \(det N(0)\) (like in \(det S(t)\)) the first two rows are identical.

Now, in order to arrange the numerator of (23) conveniently, we write \(det N(t)\) in the form

\[
det N(t) = t \cdot det R(0) - det N(0),
\] (30)

taking into account (28), (29) and the identity

\[
det R(t) \equiv det R(0).
\] (31)
In (30) we apply the Laplace expansion by entries of the first row to \( \det \mathbf{R}(0) \) and \( \det \mathbf{N}(0) \), separately. As a result, one gets

\[
\det \mathbf{N}(t) = t \left| \begin{array}{cccc}
  m_1^2 & m_2^2 & \ldots & m_m^2 \\
  m_1^4 & m_2^4 & \ldots & m_m^4 \\
  m_1^{2(m-1)} & m_2^{2(m-1)} & \ldots & m_m^{2(m-1)} \\
  m_1^{2m} & m_2^{2m} & \ldots & m_m^{2m}
\end{array} \right| + \sum_{i=1}^{m} (-1)^i t \left| \begin{array}{cccc}
  m_1^2 & m_2^2 & \ldots & m_i^2 \\
  m_1^4 & m_2^4 & \ldots & m_i^4 \\
  m_1^{2(m-1)} & m_2^{2(m-1)} & \ldots & m_i^{2(m-1)} \\
  m_1^{2m} & m_2^{2m} & \ldots & m_i^{2m}
\end{array} \right| - \left| \begin{array}{cccc}
  m_1^2 & m_2^2 & \ldots & m_m^2 \\
  m_1^4 & m_2^4 & \ldots & m_m^4 \\
  m_1^{2(m-1)} & m_2^{2(m-1)} & \ldots & m_m^{2(m-1)} \\
  m_1^{2m} & m_2^{2m} & \ldots & m_m^{2m}
\end{array} \right| - \sum_{i=1}^{m} (-1)^i m_i^2 \left| \begin{array}{cccc}
  m_k^2 & m_l^2 & \ldots & m_{i-1}^2 \\
  m_k^4 & m_l^4 & \ldots & m_{i-1}^4 \\
  m_k^{2(m-1)} & m_l^{2(m-1)} & \ldots & m_{i-1}^{2(m-1)} \\
  m_k^{2m} & m_l^{2m} & \ldots & m_{i-1}^{2m}
\end{array} \right|
\]

or calculating explicitly the corresponding subdeterminants

\[
\det \mathbf{N}(t) = (t - m_k^2) \prod_{j=1}^{m} m_j^2 \prod_{j,l=1}^{m} (m_j^2 - m_l^2) + \sum_{i=1}^{m} (-1)^i (t - m_i^2) \prod_{j=1}^{m} m_j^2 \prod_{j=1}^{m} (m_j^2 - m_k^2) \prod_{j=1}^{m} (m_j^2 - m_l^2).
\]

Substituting the latter into (31) one obtains

\[
\sum_{k=m+1}^{n} \left\{ -\frac{\prod_{j=1}^{m} m_j^2}{\prod_{j=1}^{m} (m_j^2 - t)} + \sum_{i=1}^{m} \frac{m_k^2}{m_j^2 - t} \prod_{j=1}^{m} m_j^2 \prod_{j=1}^{m} (m_j^2 - m_k^2) \prod_{j=1}^{m} (m_j^2 - m_l^2) \right\} a_k
\]

(34)
and combining this result with (18), one gets the form factor $F_h(t)$ to be saturated by $n$-vector mesons ($n > m$) in the form suitable for the unitarization

$$F_h(t) = F_0 \frac{\prod_{j=1}^{m} m_j^2}{\prod_{j=1}^{m} (m_j^2 - t)} +$$

$$+ \sum_{k=m+1}^{n} \left\{ \sum_{i=1}^{m} \frac{m_k^2}{(m_k^2 - t)} \frac{\prod_{j \neq i}^{m} m_j^2}{\prod_{j \neq i}^{m} (m_j^2 - t)} \prod_{j=1}^{m} (m_j^2 - m_k^2) - \prod_{j=1}^{m} (m_j^2 - t) \right\} a_k$$

for which the asymptotic behaviour (2) and for $t = 0$ the normalization (3) are fulfilled automatically.

The asymptotic behaviour in (35) is transparent. However, for the normalization (3) the following identity:

$$\sum_{i=1}^{m} \frac{\prod_{j \neq i}^{m} (m_j^2 - m_k^2)}{\prod_{j \neq i}^{m} (m_j^2 - m_i^2)} = 1$$

has to be valid in the second term of (35) generally.

For $m = 2, 3, 4, 5$ it can be proved explicitly. And for an arbitrary finite $m$ it follows directly from (25), the numerator of which is exactly the Laplace expansion by the entries of the first row of the determinant of the matrix (26). Then, just relation (29) causes the term (25) and also (34) at $t = 0$ for arbitrary nonzero values of $a_k$ to be zero. Hence, every term in the wavebrackets of (34) for $t = 0$ has to be zero and this is true if and only if identity (36) is fulfilled.

Now we consider the case of equations (4) for $n = m$. Then it can also be rewritten into the matrix form (5) with the $m \times m$ Vandermonde matrix (6) and the same column vector $a$, but with the $b$ vector of the following form:

$$b = \begin{pmatrix} F_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

(37)

So the corresponding solutions are again looked for in the form

$$a_i = \frac{\det M_i}{\det M}$$
but with the matrix $M_i$

$$M_i = \begin{pmatrix}
1 & \ldots & 1 & F_0 & 1 & \ldots & 1 \\
m^2 & \ldots & m^2_{i-1} & 0 & m^2_{i+1} & \ldots & m^2_m \\
m^4 & \ldots & m^4_{i-1} & 0 & m^4_{i+1} & \ldots & m^4_m \\
\vdots & & \vdots & & \vdots & & \vdots \\
m^2_{i-1} & \ldots & m^2_{i-1} & 0 & m^2_{i+1} & \ldots & m^2_m \\
m^2_{i-1} & \ldots & m^2_{i-1} & 0 & m^2_{i+1} & \ldots & m^2_m \\
\end{pmatrix}, \quad (38)$$

and as result

$$\det M_i = F_0(-1)^{1+i} \prod_{j=1}^{m} m^2_j \prod_{j,l=1}^{m} (m^2_i - m^2_j) \quad (39)$$

and the solutions

$$a_i = F_0 \frac{(-1)^{1+i} \prod_{j=1}^{m} m^2_j \prod_{j,l=1}^{m} (m^2_i - m^2_j)}{\prod_{j<l}^{m}(m^2_l - m^2_j)} = \quad (40)$$

$$= F_0 \frac{\prod_{j=1}^{m} m^2_j (-1)^{1+i}}{\prod_{j=1}^{m} (m^2_j - m^2_i) (-1)^{i-1}}$$

are completely expressed only through the masses of $m$ vector-mesons by means of which the considered FF is saturated.

The third case with the $(m-1)$ linear homogeneous algebraic equations for the $n$ ($n > m$) coupling constant ratios without any normalization of FF appears naturally in the determination of strange FF behaviours of a strongly interacting particles with the spin $s > 0$ from the isoscalar parts of the corresponding EM FF’s, as we have mentioned in Introduction.

Then, we have only the equations

$$\sum_{i=1}^{n} m^2_i a_i = 0, \quad r = 1, 2, \ldots m - 1 \quad (41)$$

which can be rewritten in the matrix form (3) with the $(m - 1) \times (m - 1)$ matrix $M$

$$M = \begin{pmatrix}
m^2_1 & m^2_2 & \ldots & m^2_{m-1} \\
m^4_1 & m^2_2 & \ldots & m^2_{m-1} \\
\vdots & & \vdots & & \vdots \\
m^2_{m-1} & m^2_{m-1} & \ldots & m^2_{m-1} \\
\end{pmatrix}, \quad (42)$$
and the column vectors

\[
\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{m-1} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -\sum_{k=m}^{n} m_k^2 a_k \\ -\sum_{k=m}^{n} m_k^2 a_k \\ -\sum_{k=m}^{n} m_k^2 a_k \\ \vdots \\ -\sum_{k=m}^{n} m_k^{2(m-1)} a_k \end{pmatrix}.
\]

The determinant of the matrix \( M \)

\[
det M = \prod_{j=1}^{m-1} \prod_{j<l}^{m-1} (m_i^2 - m_j^2)
\]

is different from zero, and thus, a nontrivial solution of (41) exists

\[
a_i = \frac{det M_i}{det M}
\]

where the matrix \( M_i \) takes the form

\[
M_i = \begin{pmatrix} m_1^2 & \cdots & m_i^2 & -\sum_{k=m}^{n} m_k^2 a_k & m_{i+1}^2 & \cdots & m_{m-1}^2 \\ m_1^4 & \cdots & m_i^4 & -\sum_{k=m}^{n} m_k^4 a_k & m_{i+1}^4 & \cdots & m_{m-1}^4 \\ m_1^6 & \cdots & m_i^6 & -\sum_{k=m}^{n} m_k^6 a_k & m_{i+1}^6 & \cdots & m_{m-1}^6 \\ \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ m_1^{2(m-1)} & \cdots & m_i^{2(m-1)} & -\sum_{k=m}^{n} m_k^{2(m-1)} a_k & m_{i+1}^{2(m-1)} & \cdots & m_{m}^{2(m-1)} \end{pmatrix}.
\]

Then employing the basic properties of the determinants one gets

\[
det M_i = -\sum_{k=m}^{n} a_k \prod_{j=1}^{m-1} m_j^2 \prod_{j \neq i}^{n} 1 \\
= -\sum_{k=m}^{n} a_k m_k^2 \prod_{j=1}^{m-1} m_j^2 \prod_{j \neq i}^{n} (m_i^2 - m_j^2) (-1)^{i-1} \prod_{j=1}^{n} (m_j^2 - m_k^2).
\]
Now substituting (47) into or finally,

\[ a_i = -\sum_{k=m}^{n} \frac{m_k^2 a_k \prod_{j=1}^{m-1} (m_j^2 - m_k^2) (-1)^{i-1} \prod_{j \neq i}^{m-1} (m_j^2 - m_i^2)}{\prod_{j=1}^{m-1} m_j^2 \prod_{j \neq i}^{m-1} (m_j^2 - m_i^2)} \]

(47)

or finally,

\[ a_i = -\sum_{k=m}^{n} \frac{m_k^2 \prod_{j=1}^{m-1} (m_j^2 - m_k^2)}{\prod_{j=1}^{m-1} (m_j^2 - t)} a_k, \quad i = 1, 2, \ldots, m - 1. \]  

(48)

Now substituting (47) into

\[ F_h(t) = \sum_{k=m}^{n} \frac{\prod_{j=1}^{m-1} (m_j^2 - t) m_i^2 a_i}{\prod_{j=1}^{m-1} (m_j^2 - t)} + \sum_{k=m}^{n} \frac{m_k^2 a_k}{m_k^2 - t} \]

(49)

and transforming both terms into a common denominator one gets the relation

\[ F_h(t) = \sum_{k=m}^{n} \left\{ \prod_{j=1}^{m-1} (m_j^2 - t) \prod_{j \neq i}^{m-1} (m_j^2 - m_i^2) \prod_{j<l}^{m-1} (m_j^2 - m_l^2) \prod_{j<l, j \neq l}^{m-1} (m_j^2 - m_l^2) \right\} \times \]

\[ \frac{(m_k^2 - t) \sum_{i=1}^{m-1} (-1)^i \prod_{j=1}^{m-1} (m_j^2 - t) \prod_{j \neq i}^{m-1} (m_j^2 - m_i^2) \prod_{j<l}^{m-1} (m_j^2 - m_l^2) \prod_{j<l, j \neq l}^{m-1} (m_j^2 - m_l^2)}{\prod_{j=1}^{m-1} (m_j^2 - t) \prod_{j \neq i}^{m-1} (m_j^2 - m_i^2) \prod_{j<l}^{m-1} (m_j^2 - m_l^2) \prod_{j<l, j \neq l}^{m-1} (m_j^2 - m_l^2)} \] 

(50)

in which the numerator of the first term under the sum is just the Laplace expansion by the entries of the first row of the determinant of the matrix \( T(t) \) of the \( m \) order

\[ T(t) = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
(m_k^2 - t) & (m_1^2 - t) & \ldots & (m_k^2 - t) \\
(m_k^2 - t)^2 & (m_1^2 - t)^2 & \ldots & (m_k^2 - t)^2 \\
\vdots & \vdots & \ddots & \vdots \\
(m_k^2 - t)^{m-1} & (m_1^2 - t)^{m-1} & \ldots & (m_k^2 - t)^{m-1}
\end{pmatrix}. \]  

(51)
One can immediately prove that
\[ \text{det} \mathbf{T}(t) \equiv \text{det} \mathbf{T}(0). \]  
(52)

Then calculating \( \text{det} \mathbf{T}(0) \) explicitly
\[ \text{det} \mathbf{T}(0) = \prod_{j=1}^{m-1} (m_j^2 - m_k^2) \prod_{j,l=1}^{m-1} (m_i^2 - m_j^2) \]  
(53)

and substituting the result into (50) instead of the numerator of the term in the wave-brackets, one finally obtains the parametrization
\[ F_{\text{h}}(t) = \sum_{k=m}^{n} \frac{\prod_{j=1}^{m-1} (m_j^2 - m_k^2)}{\prod_{j=1}^{m-1} m_j^2} \frac{\prod_{j=1}^{m-1} m_j^2}{\prod_{j=1}^{m-1} (m_j^2 - t)} \frac{m_k^2}{m_k^2 - t} \alpha_k \]  
(54)

for which the asymptotic behaviour (2) is fulfilled automatically.

### 3 Conclusions

General solutions of asymptotic conditions for EM FF’s of hadrons represented by the VMD model, derived by means of the superconvergent sum rules for the imaginary part of FF under consideration, in which the coefficients are simply even powers of the corresponding vector-meson masses, have been found.

We have distinguished three cases appearing in various physical situations:

i) in the construction of unitary and analytic models of the EM structure of any strongly interacting particle with a number of building quarks \( n_q > 2 \) when at the first stage one has need for the VMD parametrization of FF to be saturated with \( n \) different vector mesons, but with the required asymptotics (2) and normalization (3), under the assumption \( m < n \)

ii) in the same problem, however, when \( m = n \)

iii) in the determination of the strange form factor behaviours of a strongly interacting particle from the isoscalar parts of the corresponding electromagnetic form factors.
In the first case, we have found the explicit form (35) of EM FF for which the asymptotic behaviour (3) and for \( t = 0 \) the normalization (3) are fulfilled automatically. Such a form is the starting point in the construction of the unitary and analytic model of the EM structure of any strongly interacting particle in which a superposition of vector-meson poles and continuum contributions are considered at the same time.

In the second case, the explicit expressions (40) of all considered coupling constant ratios are found to be expressed through the masses of saturated vector mesons and the norm \( F_0 \) of FF. The direct application of (40) to nucleons \([5]\) gives surprising coincidence with the values obtained by a fit \([3]\) of existing experimental data by means of the modified VMD model.

In the third case, the explicit form (54) of the isoscalar part of EM FF of the strongly interacting particle with the spin \( s > 0 \) was obtained, which can be used to predict the behaviour of the strange magnetic FF.

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