Loop transfer matrix
and
gonihedric loop diffusion

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Abstract

We study a class of statistical systems which simulate 3D gonihedric system on euclidean lattice. We have found the exact partition function of the 3D-model and the corresponding critical indices analysing the transfer matrix $K(P_i, P_f)$ which describes the propagation of loops on a lattice. The connection between 3D gonihedric system and 2D-Ising model is clearly seen.

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1 Introduction

In the articles [1] the authors formulated a model of random surfaces with an action which is proportional to the linear size of the surface. The model has a number of properties which make it very close to the Feynman path integral for a point-like relativistic particle. In the limit when the surface degenerates into a single world line, the action becomes proportional to the length of the path and the classical equation of motion for the gonihedric string is reduced to the classical equation of motion for a free relativistic particle.\(^1\)

In addition to the formulation of the theory in the continuum space the system allows an equivalent representation on Euclidean lattices where a surface is associated with a collection of plaquettes [4, 5]. In these lattice spin systems the interface energy coincides with the linear-gonihedric action for random surfaces. This gives an opportunity for analytical investigations [6, 7, 8, 9, 12, 14] and numerical simulations [15, 16, 17, 18, 14] of the corresponding statistical systems.

Additional understanding of the physical behaviour of the system comes from the analysis of the transfer matrix [6] which describes the propagation of the closed loops-strings in time direction with an amplitude which is proportional to the sum of the length of the string and of the total curvature. In this article we shall study the physical picture of string propagation which was suggested in the transfer matrix approach [6].

The partition function of the system is defined as [1]

\[
Z_{\text{gonihedric}}(\beta) = \sum_{\{M\}} \exp\{-\beta A(M)\}, \quad A(M) = \sum_{<ij>} \lambda_{ij} \cdot |\pi - \alpha_{ij}|, \quad (1)
\]

where \(\lambda_{ij} = a\) is the length of the edge <ij> which is equal to the lattice spacing \(a\) on the cubic lattice, \(\alpha_{ij}\) is the dihedral angle between two neighbouring plaquettes of the singular surface \(M\) sharing a common edge <ij>, \(\alpha_{ij} = 0, \pi/2, \pi\). \(^2\) In \(\{M\}\) denote the set of closed singular surfaces on the three-dimensional toroidal lattice \(T^3\) of size \(N \times N \times N\). A singular surface \(M\) is a collection of plaquettes in \(T^3\) such that every link is contained in 0, 2 or 4 plaquettes and every plaquette of the lattice \(T^3\) can be occupied only once [4, 5]. The surfaces are closed because only an even number of plaquettes meet at a given lattice link. The singular surfaces of interface which describe the states of arbitrary three-dimensional spin system can be viewed as a set of surfaces \(\{M\}\) [4, 10].

In [6] it has been proven that the partition function [1] can be represented in the form

\[
Z(\beta) = \sum_{\{P_1,P_2,...,P_N\}} K\beta(P_1,P_2) \cdots K\beta(P_N,P_1) = Tr K_N^\beta, \quad (2)
\]

where \(K\beta(P_1,P_2)\) is the transfer matrix of size \(\gamma \times \gamma\), defined as (see formulas (15),(16) of [6])

\[
K_{\beta_{\text{gonihedric}}}(P_1,P_2) = \exp\{-\beta \left[k(P_1) + 2l(P_1 \triangle P_2) + k(P_2)\right]\}, \quad (3)
\]

\(^1\)The problems of spiky instability and the convergence of the partition function have been studied in [1, 2, 3].

\(^2\)Usually we take \(a = 1\).
where $P_1$ and $P_2$ are closed polygons on a two-dimensional toroidal lattice $T^2$ of size $N \times N$ and $\gamma$ is the total number of polygon-loops on a toroidal lattice $T^2$. Closed polygons $\{P\} \equiv \Pi$ are associated with the collection of links on $T^2$ with the restriction that only an even number of links can intersect at a given vertex of the lattice and that the links can be occupied only once.

The transfer matrix (3) can be viewed as describing the propagation of the polygon-loop $P_1$ at time $\tau$ to another polygon-loop $P_2$ at the time $\tau + 1$. The functional $k(P)$ is the total curvature of the polygon-loop $P$ which is equal to the number of corners of the polygon (the vertices with self-intersection are not counted) and $l(P)$ is the length of $P$ which is equal to the number of its links. The length functional $l(P_1 \triangle P_2)$ is defined as (see formula (12) of [6])

$$l(P_1 \triangle P_2) = l(P_1) + l(P_2) - 2l(P_1 \cap P_2), \quad (4)$$

where the polygon-loop $P_1 \triangle P_2 \equiv P_1 \cup P_2 \setminus P_1 \cap P_2$ is a union of links $P_1 \cup P_2$ without common links $P_1 \cap P_2$. The operation $\triangle$ maps two polygon-loops $P_1$ and $P_2$ into a polygon-loop $P = P_1 \triangle P_2$. Note that the operations $\cup$ and $\cap$ do not have this property. These operations acting on a polygon-loops can produce link configurations which do not belong to $\Pi$. The length functional $l(P_1 \triangle P_2)$ defines a distance between two polygon-loops $P_1$ and $P_2$. It is natural that the transition amplitude $K_{\beta}(P_1, P_2)$ (3) depends only on the functional which measures the distance between the two subsequent polygon-loop configurations, because transition amplitude decreases when the distance (4) between two configurations increases. This can be seen also from the inequality

$$l(P_1 \triangle P_2) \geq |l(P_1) - l(P_2)|, \quad (5)$$

therefore

$$K^{gonihedric}_{\beta}(P_1, P_2) \leq \exp\{-2\beta |l(P_1) - l(P_2)|\}. \quad (6)$$

Algebraically one can construct many functionals of that kind, but what is important here is that this distance functional appears naturally from the geometrical action (1) of the original theory. The expression for the transition amplitude $K_{\beta}(P_1, P_2)$ in the polygon-loop space $\Pi$ is very close in its form with the transfer matrix for the random walks

$$K_{\beta}(X, Y) = \exp\{-\beta |X - Y|\}, \quad (7)$$

which depends only on the distance between initial and final position of the point particle. Statistical mechanics of paths with curvature-dependent action is also well known $\Pi$. The aim of this work is to study spectral properties of the transfer matrix $K_{\beta}(P_1, P_2)$ and critical behaviour of the statistical system (2).

### 2 Free energy and correlation functions

The eigenvalues of the transfer matrix $K_{\beta}(P_1, P_2)$ define all statistical properties of the system and can be found as a solution of the following integral equation in the

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3 The polygon-loops $P$ appear as the intersection of the singular surfaces $M$ with the planes between coordinate planes in $T^3$.

4 The symmetric difference of sets $P_1 \triangle P_2$ is an important concept in functional analysis.
where $\Psi(P)$ is a function on loop space. The Hilbert space of complex functions $\Psi(P)$ on $\Pi$ will be denoted as $H = L^2(\Pi)$.

The eigenvalues define the partition function

$$Z(\beta) = \Lambda_0^N + ... + \Lambda_N^N,$$

and in the thermodynamical limit the free energy is equal to

$$-\beta f(\beta) = \lim_{N \to \infty} \frac{1}{N^3} \ln Z(\beta).$$

The correlation lengths are defined by the ratios of eigenvalues $\Lambda_i(\beta)/\Lambda_0(\beta)$

$$\xi_i(\beta) = \frac{1}{-\ln \frac{\Lambda_i(\beta)}{\Lambda_0(\beta)}},$$

and grow if the eigenvalues $\Lambda_i(\beta)$ approach the eigenvalue $\Lambda_0(\beta)$ at some critical temperature $\beta_c$. By the Frobenius-Perron theorem $\Lambda_0(\beta)$ is simple and we have

$$\Lambda_0(\beta) > \Lambda_1(\beta) \geq \Lambda_2(\beta) \geq ...$$

Finite time propagation amplitude of an initial loop $P_i$ to a final loop $P_f$ for the time interval $t = M/\beta$ can be defined as

$$K(\beta)(P_i, P_f) = \Lambda_0^{-M} \sum_{\{P_1, P_2, ..., P_{M-1}\}} K_\beta(P_i, P_1) \cdots K_\beta(P_{M-1}, P_f),$$

where we have introduced natural normalization to the biggest eigenvalue $\Lambda_0$ and $M \leq N$. The trace of the operator $K(P_i, P_f)$ is equal to

$$TrK = (1 + (\Lambda_1/\Lambda_0)^M + \cdots + (\Lambda_N/\Lambda_0)^M)$$

and depends on the ratio $\Lambda_i/\Lambda_0$.

## 3 Connection between 3D gonihedric system and 2D-Ising model

We consider below a transfer matrix which is less complicated than the original matrix and depends only on the distance functional $l(P_1 \Delta P_2)$

$$K_\beta(P_1, P_2) = \exp\{-2\beta \left( l(P_1 \Delta P_2) - N^2 \right) \}.$$
The largest eigenvalue appearing in the equation

\[ \sum_{\{P_2\}} \exp\{-2\beta l(P_1 \triangle P_2) + 2\beta N^2 \} \cdot \Psi(P_2) = \Lambda(\beta) \Psi(P_1) \]  

(16)
can be found because the corresponding eigenfunction is a constant function \( \Psi_0(P) = 1 \) (see Appendix). Therefore

\[ \Lambda_0 = \sum_{\{P\}} \exp\{-2\beta l(P) + 2\beta N^2 \}, \]  

(17)
The sum (17) does not depend on \( P_1 \). To prove this we note that the loop \( P = P_1 \triangle P_2 \) runs over all loops in \( \Pi \) as \( P_2 \) runs over \( \Pi \), i.e. the mapping \( P_1 \rightarrow P_1 \triangle P_2 \) is one to one for any \( P_2 \). This change of the variable proves that

\[ \Lambda_0 = \sum_{\{P\}} \exp\{-2\beta l(P) + 2\beta N^2 \}, \]  

(18)
so \( \Lambda_0 \) is the partition function of the 2D-Ising ferromagnet. Indeed [19, 20, 21],

\[ Z_{2D-I} = \sum_{\{P\}} \exp\{-2\beta l(P) + 2\beta N^2 \} = e^{-\beta f(\beta)N^2}, \]  

(19)
where \( f(\beta) \) is the free energy of the 2D-Ising model. Therefore

\[ \Lambda_0 = Z_{2D-I} = \lambda_0^N + \ldots + \lambda_2^N, \]  

(20)
where \( \lambda_i \) are the eigenvalues of the transfer matrix of 2D-Ising model [20, 21, 22]. Thus the largest eigenvalue of the 3D-system (15) is equal to the partition function of the 2D-Ising ferromagnet.

The free energy of the 2D-Ising ferromagnet in the thermodynamical limit is given by [20, 21, 22]

\[ -\beta f(\beta) = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\xi d\eta}{(2\pi)^2} \ln\left( (1 + w^4)^2 - 2(w^2 - w^6)(\cos\xi + \cos\eta) \right), \]  

(21)
where \( w = e^{-\beta} \). Therefore the free energy of the three-dimensional system which is defined by the transfer matrix (15) is given by \( f(\beta) \) and coincides with the one of 2D-Ising ferromagnet.

From this result we can deduce that the critical temperature of the three-dimensional system (15) is equal to the one for the 2D-Ising ferromagnet \( 2\beta_c = \ln(\sqrt{2} - 1) \), that the specific heat exponent \( \alpha = 0 \) and from the hyperscaling law \( \nu d = 2 - \alpha \) that \( \nu = 2/3 \).

We recall [6] that in the alternative approximation when the intersection term \( 2k(P_1 \cap P_2) \) is ignored in the original transfer matrix (3)

\[ K_{0\beta}(P_1, P_2) = \exp\{-\beta [k(P_1 \triangle P_2) + 2l(P_1 \triangle P_2) - 2N^2] \} \]  

(22)
the free energy can also be computed [6]
\[ + 4w^4(1 - \omega^2) \cos \xi \cos \eta - 2(w^2 - 2w^6\omega^2 + w^6)(\cos \xi + \cos \eta) \], \quad (23) \]

where \( \omega^2 = w \) is the contribution from the curvature term \( k(P) \) and exhibits the same critical behavior as the 2D-Ising ferromagnet. Thus the original gonihedric system \((2), (3)\) is bounded by two close statistical systems

\[- \beta f_0(\beta) \leq - \beta f_{\text{gonihedric}}(\beta) \leq - \beta f(\beta), \quad (24)\]

because

\[ K_{0\beta}(P_1, P_2) \leq K_{\beta}^{\text{gonihedric}}(P_1, P_2) \leq K_{\beta}(P_1, P_2) \quad (25) \]

This confirms the conjecture \([6]\) that 3D gonihedric system should have statistical properties close to the ones of 2D-Ising ferromagnet. Earlier numerical simulations \([16, 17, 18]\) support this dimensional ”reduction”. It is also consistent with the analytical estimate of the entropy factor of the random surfaces on a cubic lattice \([8, 9]\).

Layer-to-layer transfer matrices for three-dimensional statistical systems, whose elements are the product of all Boltzmann weight functions of cubes between two adjacent layers have been considered in the literature \([23, 24, 25, 26]\). Using Yang-Baxter and Tetrahedron equations one can compute the spectrum of the transfer matrix in a number of interesting cases \([23, 24]\). In the given case the transfer matrix \((3)\) has geometrical interpretation which helps to compute the spectrum.

We have to remark that if the loops \(\{P\}\) are not restricted to be closed the system is essentially simplified. Indeed in the model where all subsets of links from \(T^2\) are allowed as configurations of the system and the transfer matrix is defined by the same formula the model becomes trivial, the transfer matrix is the \(2N^2\)th tensor product of 1D-Ising model transfer matrix

\[ K_{\beta} = \begin{pmatrix} 1 & e^{-2\beta} \\ e^{-2\beta} & 1 \end{pmatrix}. \quad (26) \]

Using this observation one can derive inequality

\[ \Lambda_1 \leq \frac{(1 + e^{-2\beta})^{2N^2-1}(1 - e^{-2\beta})}{Z_{2D-\text{Ising}}} . \]

4 Loop space and eigenfunctions

To proceed it is convenient to introduce some notation. An invariant product in \(\Pi\) can be defined as

\[ < P_1 | P_2 > = l(P_1 \triangle \bar{P}_2) - l(P_1 \triangle P_2) \quad (27) \]

where \( \bar{P} = T^2 \triangle P = T^2 \setminus P \) and \( T^2 \) is the loop which contains all links of the toroidal lattice \( T^2 \) (\( l(T^2) = 2N^2 \)). The invariant product \((27)\) is also equal to

\[ < P_1 | P_2 > = 2N^2 - 2l(P_1 \triangle P_2) \quad (28) \]
and is odd with respect to $\bar{P}$-the complement operation $< P_1|P_2 > = - < P_1|\bar{P}_2 >$. In particular the energy functional of the system is equal to

$$E_P = < 0 | P > = 2N^2 - 2l(P)$$

The product (27) is invariant under the simultaneous rotations of the loops $P_1$ and $P_2$, defined as

$$P \to P \triangle \delta,$$

where $\delta$ is an arbitrary loop

$$< P_1 \triangle \delta | P_2 \triangle \delta > = < P_1 | P_2 > .$$

This group of transformations in the loop space $\Pi$ is Abelian because for its representations on $H$

$$R_\delta \Psi(P) = \Psi(P \triangle \delta),$$

we have

$$[R_{\delta_1}, R_{\delta_2}] = 0 \quad R_\delta^2 = 1. \quad (33)$$

The product $< P_1|P_2 >$ is invariant also under translations. The group of translations is defined as a rigid translation of the loop $P$ in $x$ and $y$ directions on $T^2$ by same units of lattice spacing $a$

$$P \to P + a_x e_x + a_y e_y. \quad (34)$$

Together these two groups form a Nonabelian group which acts on the loop space $\Pi$.

With this notation the transfer matrix (15) takes the form

$$K_\beta(P_1, P_2) = exp\{ \beta < P_1|P_2 > \}. \quad (35)$$

and the integral equation (16) the form

$$\sum_{\{P_2\}} e^{\beta < P_1|P_2 >} \cdot \Psi(P_2) = \Lambda(\beta) \Psi(P_1). \quad (36)$$

Note that (36) is invariant under rotations (30). This suggests that we search for eigenfunctions of the operator (35) in the form of invariant polynomials in $H = L^2(\Pi)$. These polynomials can be constructed in the form of powers of the invariant $< P|Q >$ as follows:

$$\Psi_Q^{(0)}(P) = 1,$$
$$\Psi_Q^{(1)}(P) = < P|Q >,$$
$$\Psi_Q^{(2)}(P) = < P|Q >^2 - \rho_2,$$
$$\Psi_Q^{(3)}(P) = < P|Q >^3 - \rho_3 < P|Q >,$$

..........................................................

$$\Psi_Q^{(n)}(P) = < P|Q >^n - \rho_n < P|Q >^{n-2} -... \quad (37)$$
where the coefficients
\[
\rho_2 = \frac{\sum_{\{P\}} < 0 | \sum_{\{P\}} 1 >}{\sum_{\{P\}} 1}, \quad \rho_3 = \frac{\sum_{\{P\}} < 0 | \sum_{\{P\}} 2 >}{\sum_{\{P\}} 2},
\]
and so on are chosen so that \(\Psi^{(i)}_Q\) is orthogonal to \(\Psi^{(j)}_Q\) if \(i \neq j\). The set of functions \(\{\Psi^{(n)}_Q(P)\}\) in \(H\) form an invariant subset \(H_n\) of the level \(n\). The index \(Q\) numerates functions inside the level. We introduce general Legendre loop polynomials as
\[
L_n(x) = x^n - \rho_n x^{n-2} - 
\]
so \(L_n(<P|Q>) = \Psi^{(n)}_Q(P)\). This set of functions in \(H\) is appropriate for solving the integral equation (36) and in its form is very similar with the ones which are used for the random paths with curvature-dependent action [11].

Let us begin by proving that \(\Psi^{(1)}_Q(P)\) is the eigenfunction of (36). Because (see the next section)
\[
\sum_{\{P\}} <Q_1|P>^n \cdot <P|Q_2> = \mu_{n1} <Q_1|Q_2>
\]
where
\[
\mu_{n1} = \frac{1}{2N^2} \sum_{\{P\}} <0|P>^{n+1}, \quad \mu_{2k} = 0
\]
we have
\[
\sum_{\{P_2\}} e^\beta <P_1|P_2> \cdot <P_2|Q> = \Lambda_1(\beta) <P_1|Q>
\]
where
\[
\Lambda_1 = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \mu_{n1} = \frac{1}{2N^2} \sum_{\{P\}} \exp\{ \beta <0|P> \} <0|P>
\]
and thus
\[
\Lambda_1 = \frac{1}{2N^2} \frac{\partial}{\partial \beta} \Lambda_0.
\]
For the ratio \(\Lambda_1/\Lambda_0\) one can get
\[
-(\Lambda_1/\Lambda_0) = -\frac{1}{2N^2} \frac{\partial}{\partial \beta} \ln \Lambda_0 = u(\beta)
\]
which is the internal energy of the 2D-Ising system. Thus the second eigenvalue of the three-dimensional system coincides with internal energy of the two-dimensional Ising system.

Using equation (40) one can also address the question of degeneracy of the eigenvalues. The scalar product of the loop functions can be defined as
\[
\Psi_{Q_1} \cdot \Psi_{Q_2} = \sum_{\{P\}} \Psi_{Q_1}(P) \Psi_{Q_2}(P)
\]
then for the zero and first level functions (37) we have
\[
\Psi^{(0)}_{Q_1} \cdot \Psi^{(1)}_{Q_2} = 0,
\]
\[ \Psi_{Q_1}(1) \cdot \Psi_{Q_2}(1) = \mu_{11} < Q_1|Q_2 > . \] (47)

The rank of the matrix \( G_{Q_1,Q_2} = < Q_1|Q_2 > \) defines the number \( \epsilon_1 \) of linearly independent functions on the first level \( H_1 \) and thus the degeneracy of the eigenvalue \( \Lambda_1 \). The number \( \epsilon_1 \) is bigger or equal to the number of functions \( \Psi_{Q_i}^{(1)} \) which are orthogonal to each other. The orthogonality condition follows from (47)

\[ < Q_i|Q_j > = 0 \quad i, j = 1, 2, \ldots, 2N^2 \]

The last equation can be rewritten also in the form

\[ l(Q_i \triangle Q_j) = N^2 \]

and its solutions provide \( 2N^2 \) linearly independent functions on the first level. The solutions will be presented in a separate place.

In the following we will search for approximate solutions since the sums

\[ \sum_{\{P\}} < Q_1|P >^n \cdot < P|Q_2 >^k, \quad k \geq 2 \]

depend not only on invariant \( < Q_1|Q_2 > \), but also on the shape of \( Q_1 \) and \( Q_2 \). However the sum depends only weakly on the shape of \( Q' \)s and the leading behaviour involves only the length. In this approximation we have (see also the next section)

\[ \sum_{\{P\}} < Q_1|P >^n \cdot < P|Q_2 >^2 = \mu_{n2} < Q_1|Q_2 >^2 + \mu_{n0} \] (48)

where

\[ \mu_{n2} = \frac{1}{(2N^2)^2 - \rho_2} \sum_{\{P\}} (< 0|P >^{n+2} - \rho_2 < 0|P >^n) \]

and

\[ \mu_{n0} = -\frac{\rho_2}{(2N^2)^2 - \rho_2} \sum_{\{P\}} (< 0|P >^{n+2} - (2N^2)^2 < 0|P >^n) \]

so that the next eigenvalue can be found in the same way

\[ \Lambda_2 = \frac{1}{(2N^2)^2 - \rho_2} \sum_{\{P\}} \exp\{ \beta < 0 \mid P > \} [ < 0 \mid P >^2 - \rho_2], \] (49)

and for the third one we have

\[ \Lambda_3 = \frac{1}{(2N^2)^3 - \rho_3(2N^2)} \sum_{\{P\}} \exp\{ \beta ( < 0 \mid P > ) \} [ < 0 \mid P >^3 - \rho_3 < 0 \mid P >], \] (50)

and so on. All eigenvalues can be expressed through the generalised Legendre polynomials \( L_n \)

\[ \Lambda_n = \frac{1}{L_n(2N^2)} \sum_{\{P\}} \exp\{ \beta < 0 \mid P > \} L_n( < 0 \mid P >), \] (51)

\[ \text{The degeneracy of the zero level is one } \epsilon_0 = 1. \]
or as corresponding derivative of the Ising partition function
\[ \Lambda_n(\beta) = \frac{1}{L_n(2N^2)} L_n(\frac{\partial}{\partial \beta}) \Lambda_0(\beta), \] (52)

This completes the computation of the eigenfunctions and eigenvalues of the integral equation (16), (36).

5 Loop correlation functions

The equality (40) can be proven by using the fact that the l.h.s. of (40) is equal to
\[ \sum_{\{P\}} <0|P>^n \cdot <P|Q> = \sum_{\{P\}} <0|P>^{n+1} - 2l(Q) \sum_{\{P\}} <0|P>^n + 4 \sum_{\{P\}} <0|P>^n \cdot l(P \cap Q). \] (53)
The last term in (53) is a linear function of \( l(Q) \). If \( Q = Q_1 \cup Q_2 \) and \( Q_1 \cap Q_2 = 0 \) then \( l(P \cap Q) = l(P \cap Q_1) + l(P \cap Q_2) \) and we have
\[ \sum_{\{P\}} <0|P>^n \cdot l(P \cap Q) = \eta_{n1} l(Q). \] (54)
The normalization constant can be computed at the point \( Q = T^2 \)
\[ \eta_{n1} = \frac{1}{2N^2} \sum_{\{P\}} <0|P>^n \cdot l(P). \] (55)
The linearity of the last term in (53) with respect to \( l(Q) \) follows also from the decomposition of the functional \( l(P \cap Q) \) into the sum over the links of \( Q \)
\[ l(P \cap Q) = \sum_{i \in Q} \chi_i(P) \]
where \( \chi_i(P) = 1 \) if \( i \in P \) and zero otherwise. By homogeneity
\[ \sum_{\{P\}} <0|P>^n \cdot \chi_i(P) \]
is independent of \( i \). Hence the summation over \( i \in Q \) gives the expected result (54).

To proceed we have to compute the sum
\[ \sum_{\{P\}} <0|P>^n \cdot l(P \cap Q_1) \cdot l(P \cap Q_2). \] (57)
The last correlation function is a sum over $i \in Q_1$ and $j \in Q_2$ of the two-link correlation function
\[ \sum_{\{P\}} <0| P >^n \cdot \chi_i(P) \chi_j(P). \] (58)

In the approximation when the two-link correlation function (58) factorises, then the correlation function (57) is equal to
\[ \eta_{n_2} l(Q_1)l(Q_2) + \eta_{n_0} l(Q_1 \cap Q_2) \]
and we recover the expression (48).

For the high powers we have to compute a many-set correlation function
\[ \sum_{\{P\}} <0| P >^n \prod_a l(P \cap Q_a) = \eta_{n_a} \prod_a l(Q_a) + \cdots \] (59)
which can be expressed as a function of lengths in the approximation when many-link correlation function
\[ \sum_{\{P\}} <0| P >^n \cdot \chi_i(P) \chi_j(P) \chi_k(P) \cdots, \] (60)
factorises. Because the link variable $\chi_i(P)$ is a quadratic function of Ising spins it follows that many-link correlation function (60) is nothing else than many-spin correlation function of the 2D-Ising model.

6 Acknowledgement

One of the authors (T.J.) is indebted to the National Research Center Demokritos for hospitality. This work was supported in part by the EEC Grant no. ERBFM-BICT972402.

7 Appendix

It is elementary to verify that the function $l(P_1 \triangle P_2)$ on $\Pi \times \Pi$ is a distance functional $\rho(P_1, P_2) \equiv l(P_1 \triangle P_2)$ and that if $P_1, P_2, P \in \Pi$ are such that $l(P_1 \triangle P_2) = l(P_1 \triangle P) + l(P \triangle P_2)$, then $P_1 \cap P_2 \subseteq P \subseteq P_1 \cup P_2$. Indeed we have
\[ l(P) = l(P \cap P_1) - 2l(P_1 \cap P_2) + l(P_2 \cap P), \]
hence $l(P \setminus P_1 \cup P_2) = l(P \setminus P_1 \cap P_2) - l(P_1 \cap P_2)$ from which we conclude that $P \subseteq P_1 \cup P_2$ and $P_1 \cap P_2 \subseteq P$.

We complete our arguments by showing that all eigenvalues of the normalized transfer matrix $\Lambda_0^{-1} K_\beta(P_1, P_2)$ are smaller than or equal to 1. In order to prove this statement let $\chi_P$ be a ”delta function” in $\Pi$, i.e. $\chi_P(Q) = 1$ if $P = Q$ and $\chi_P(Q) = 0$ otherwise. The set of functions $\{\chi_P\}$ form a basis in $H = L^2(\Pi)$. Note that for the propagator (13)
\[ K(P_i, P_f) = \langle \chi_{P_i} | \Lambda_0^{-M} K_\beta^M | \chi_{P_f} \rangle \]
we have (13)

\[ K(P_i, P_f) = \Lambda_0^{-M} \sum_{\{P_1, P_2, \ldots, P_{M-1}\}} K_\beta(P_i, P_1)K_\beta(P_1, P_2)\cdots K_\beta(P_{M-1}, P_f) \leq \]

\[ \leq \Lambda_0^{-M} \sum_{\{P_1, P_2, \ldots, P_{M-1}\}} K_\beta(P_1, P_1)K_\beta(P_2, P_3)\cdots K_\beta(P_{M-1}, P_f). \]

We can sum successively over \( P_i \)'s to obtain

\[ K(P_i, P_f) \leq \Lambda_0^{-1} \]

for any \( P_i, P_f \) provided \( M > 1 \). If \( K(P_i, P_f) \) had an eigenvalue \( \Lambda \geq 1 \) then \( K(P_i, P_f) \) would grow as \( \Lambda^M \) as \( M \to \infty \) for same \( P_f \). Since \( K_\beta(P_1, P_2) > 0 \) it follows from the Frobenius-Perron theorem that 1 is a simple eigenvalue.

It follows that for \( M > 1 \), \( K(P, P) \leq \Lambda_0^{-1} \), thus

\[ \frac{\ln \Lambda_0(\beta)}{M} \leq m(\beta), \]

where the mass \( m(\beta) \) is defined as

\[ K(P, P) \approx \exp\{-M m(\beta)\}. \]

for sufficiently large \( M \). From the other side if in the sum which defines \( K(P, P) \) we restrict all intermediate loops to be \( P \) then \( \Lambda_0^{-M}(\beta) \leq K(P, P) \) and thus

\[ \frac{\ln \Lambda_0(\beta)}{M} \leq m(\beta) \leq \ln \Lambda_0(\beta). \]

The operator \( K(P, P) \) is bounded from below also by positive number \( 1/\gamma \). Indeed, because

\[ K(P, P) = < \chi_P | \Lambda_0^{-M} K_\beta^M | \chi_P > = \sum_{n=0} < \chi_P | \Lambda_0^{-M} K_\beta^M | \Psi^{(n)}> < \Psi^{(n)} | \chi_P > = \]

\[ = \sum_{n=0} \left( \frac{\Lambda_n}{\Lambda_0} \right)^M < \chi_P | \Psi^{(n)}>^2 \geq | < \chi_P | \Psi^{(0)}> |^2. \]

Since for the normalized constant function on \( \Pi \)

\[ < \chi_P | \Psi^{(0)}> = 1/\sqrt{\gamma}, \]

where \( \gamma \) is total number of loops on a toroidal lattice \( T^2 \), thus

\[ 1/\gamma \leq K(P, P). \]

For the mass \( m(\beta) \) one can get

\[ m(\beta) \leq \frac{\ln \gamma}{M}. \]
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