Equidistribution of primitive vectors in $\mathbb{Z}^n$

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Abstract

We prove effective equidistribution of several natural parameters associated to primitive vectors in $\mathbb{Z}^n$. These parameters include the direction, the orthogonal lattice, and the length of the shortest solution to the associated gcd equation. We show that the first two parameters equidistribute w.r.t. the Haar measure on the corresponding spaces, which are the unit sphere and the space of unimodular codimension one lattices in $\mathbb{R}^n$ respectively. Our main theorem concerns the shortest solutions to the gcd equations. It states that, when normalized by the covering radius of the orthogonal lattice, the lengths of these solutions equidistribute in the interval $[0, 1]$ w.r.t. a measure that is Lebesgue only when $n = 2$, and non-Lebesgue otherwise.

These equidistribution results are deduced from effectively counting lattice points in domains which are defined w.r.t. a generalization of the Iwasawa decomposition in simple algebraic Lie groups, and we establish several such counting results. The asymptotics in the counting is w.r.t. a height function that depends on the projection to the Cartan subgroup.

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1 Introduction

1.1 Background and results

An integral vector $v = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ is called primitive if $\gcd(a_1, \ldots, a_n) = 1$. In this paper we study counting and equidistribution of various quantities associated with primitive vectors, as well as of Iwasawa components of lattices in simple Lie groups. Let $v^\perp$ denote the hyperplane orthogonal to $v$, and let $\Lambda_v := \mathbb{Z}^n \cap v^\perp$ denote the $(n - 1)$-dimensional lattice, to whom we refer as the orthogonal lattice of $v$. We view $\Lambda_v$ as an oriented lattice, which means it is equipped with a choice of orientation. Without loss of generality, the chosen orientation on $\Lambda_v$ is such that a basis $B$ is positively oriented if the basis $\{B, v\}$ for $\mathbb{R}^n$ is positively oriented. The Diophantine equation

$$a_1x_1 + \cdots + a_nx_n = 1$$

is referred to as the gcd equation of $v$, where its set of integer solutions is a grid $w + \Lambda_v$ in $\mathbb{Z}^n$. There is no canonical way to select a specific representative $w$ of this grid, yet it is natural to choose a representative which possesses some extremal property; in this paper we consider $w = w_v$ which has the minimal $L^2$ norm. The length $\|w_v\|$ is unbounded as $\|v\| \to \infty$, but it can be normalized into a bounded quantity, where perhaps the first guess would be to normalize $w_v$ by the norm of $v$, and consider the quotients $\|w_v\| / \|v\|$. Indeed, for $n = 2$ these quotients were shown [RR09] to uniformly distribute in
the interval $[0, 1/2]$. However, it turns out that for $n > 2$ the quotients $\frac{\|w_v\|}{\|v\|}$ tend to concentrate at zero, suggesting that this is not the natural normalization in higher dimensions. To describe the appropriate normalization for general $n$ (which indeed coincides with $\frac{\|v\|}{\|v\|}$ in dimension $2$), we consider the Dirichlet domain $\text{Dir}(\Lambda_v)$ of $\Lambda_v$, which is the set consisting of the shortest representative from every coset of the lattice. $\text{Dir}(\Lambda_v)$ is a closed polyhedron that is symmetric around the origin, and we let $\rho_v$ denote the radius of a bounding sphere for $\text{Dir}(\Lambda_v)$, namely the covering radius of the lattice $\Lambda_v$. It turns out that the lengths $\|w_v\|$ should be normalized by $\rho_v$, and our first theorem concerns with the equidistribution of the quotients $\frac{\|w_v\|}{\rho_v}$. To describe it, we consider the Lie group

$$P_{n-1} := \{ \text{group of upper triangular matrices of order } n-1 \}$$

with determinant $1$ and positive diagonal entries. It is clearly diffeomorphic with the symmetric space $\text{SO}_{n-1}(\mathbb{R}) \setminus \text{SL}_{n-1}(\mathbb{R})$, and therefore $\text{SL}_{n-1}(\mathbb{Z})$ acts on it by right multiplication. We let $F_{n-1}$ denote a fundamental domain for this action, contained in the set of matrices $z$ whose columns form a Siegel reduced basis for the lattices spanned by their columns; for $z \in P_{n-1}$, we denote this lattice by $\Lambda_z$. We recall the construction of $F_{n-1}$ in Section 2, and in the meanwhile we remark that $F_2$ is the well-known fundamental domain of $\text{SL}_2(\mathbb{Z})$ in the hyperbolic half-plane depicted in Figure 1.

In the theorem below, $B_\alpha$ is a ball in $\mathbb{R}^{n-1}$ that is centered at the origin and has radius $\alpha$; a BCS is a set that satisfies the following (not too strict) regularity condition:

**Definition 1.1.** A bounded subset of a manifold $\mathcal{M}$ will be called boundary controllable set, or a BCS, if its (topological) boundary consists of finitely many subsets of embedded $C^1$ submanifolds, whose dimension is strictly smaller than $\dim \mathcal{M}$.

**Theorem A.** Assume that $\Phi' \subseteq S^{n-1}$ is a BCS. For primitive vectors $v \in \mathbb{Z}^n$ such that $\hat{v} := v/\|v\|$ lies in $\Phi'$, the quotients $\|w_v\|/\rho_v$ equidistribute as $\|v\| \to \infty$ in the interval $[0, 1]$ w.r.t. the probability measure $\nu_v$ given by

$$\nu_v([0, \alpha]) := \frac{\int_{\mathcal{F}_{n-1}} \text{EucVol}(\text{Dir}(\Lambda_z) \cap B_{\alpha \rho_v(\Lambda_z)}) \frac{d\mu_{\text{det}(z)}}{\mu_{\text{det}(z)}}(z)}{\mu_{\text{det}(z)}(F_{n-1})}.$$

---

1 A subset of a topological space is bounded if there exists a compact set which contains it.
The equidistribution is at rate $O\left(\frac{\|v\|^n}{\kappa_n} - \frac{\|v\|}{\kappa_n^2} n^2 + 2\|v\| + \frac{\|v\|}{\kappa_n^2} + \epsilon\right)$ for every $\epsilon > 0$, where $\kappa_n \in \left[\frac{1}{n^3}, \frac{1}{2n^2}\right]$ is a parameter defined in Formula 7.3 and Theorem 7.4.

In the case of $n = 2$, the measure $\nu_2$ is the Lebesgue measure on the interval $[0, 1]$, and the covering radius $\rho_v$ equals $\|v\|$. Thus, Theorem A for $n = 2$ states that the quotients $2\|w_v\|/\|v\|$ uniformly distribute in $[0, 1]$, which (as mentioned above) agrees with existing results. This fact was first proved in [RR09], and effective versions were later established in [Tru13] and [HN16], in which the error term coincides with the one of Theorem A for $n = 2$. This theorem is therefore new only for $n > 2$, where it is worth noting that the measure $\nu_n$ is very different from the Lebesgue measure; see Figure 2 for the density function of $\nu_3$.

We now proceed to discuss equidistribution of further parameters associated with primitive vectors. One of these parameters is the shape of the orthogonal lattice $\Lambda_v$, which is the equivalence class of all the $\left(n - 1\right)$-dimensional lattices that can be obtained from $\Lambda_v$ by rotation and rescaling. The space of shapes of $\left(n - 1\right)$-dimensional lattices is

$$\mathcal{X}_{n-1} := \text{SO}_{n-1}(\mathbb{R}) \setminus \text{SL}_{n-1}(\mathbb{R}) / \text{SL}_{n-1}(\mathbb{Z}),$$

which can clearly be parameterized by the domain $F_{n-1} \subset P_{n-1}$ considered above (see Section 2.4 for an extensive discussion on the properties of this parameterization).

Another parameter associated with a primitive vector $v$ is the lattice $\Lambda_v$ itself (as opposed to its shape); we note that these lattices are also primitive, where the notion of primitiveness is extended from integral vectors to integral sublattices as follows:

**Definition 1.2.** A primitive sub-lattice $\Lambda \subset \mathbb{Z}^n$ is a discrete subgroup whose bases can be completed to a basis of $\mathbb{Z}^n$; equivalently, it is of the form $\Lambda = \mathbb{Z}^n \cap V$ where $V \subset \mathbb{R}^n$ is a linear subspace and $\text{span}_\mathbb{R}(\Lambda) = V$.

The lattices $\Lambda_v$ are elements in the space of $\left(n - 1\right)$-dimensional lattices in $\mathbb{R}^n$. To describe this space, we view its elements as spanned over $\mathbb{Z}$ by the first $n - 1$ columns of matrices in $\text{SL}_n(\mathbb{R})$, implying that this space is the quotient $\text{SL}_n(\mathbb{R})/Q$ where

$$Q := \left[\begin{array}{c|c}
I_{n-1} & \mathbb{R}^{n-1} \\
0 & 1
\end{array}\right] \times \left[\begin{array}{c|c}
\text{SL}_{n-1}(\mathbb{Z}) & 0 \\
0 & 0 \\
0 & 1
\end{array}\right].$$

To see this, note that $Q$ is the stabilizer of $\text{span}_\mathbb{Z}\{e_1, \ldots, e_{n-1}\}$ in the transitive action of $\text{SL}_n(\mathbb{R})$ on $n - 1$ dimensional lattices in $\mathbb{R}^n$ induced from its action on itself by right multiplication. The space
$SL_n(\mathbb{R})/Q$ has infinite (Haar) measure, and in order to obtain a finite measure space one should mod out by the scalar matrices and consider the space of normalized $(n-1)$-dimensional lattices in $\mathbb{R}^n$, which is

$$\mathcal{L}_{n-1,n} := SL_n(\mathbb{R}) \left/ \left( Q \times \left\{ \left[ \alpha^{-n-1} I_{n-1}, 0 \right] : \alpha > 0 \right\} \right. \right).$$

This is in analogy with the fact that the space of all lattices in $\mathbb{R}^n$, $GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, has infinite measure, whereas the space of unimodular lattices,

$$\mathcal{L}_n := SL_n(\mathbb{R})/SL_n(\mathbb{Z}),$$

has finite measure. Indeed, $\mathcal{L}_{n-1,n}$ should be thought of as the space of unimodular lattices in $\mathbb{R}^n$. We let $[\Lambda_v]$ denote the class of lattices $\alpha \Lambda_v$ with $\alpha > 0$, also referred to as the normalized $\Lambda_v$. A choice of orientation is still assumed, without being mentioned explicitly. We remark that there is a one-to-one correspondence

$$v \leftrightarrow \Lambda_v \leftrightarrow [\Lambda_v]$$

between primitive vectors in $\mathbb{Z}^n$, primitive codimension 1 (oriented) sublattices in $\mathbb{Z}^n$ and primitive elements (i.e., equivalence classes of oriented primitive lattices) in $\mathcal{L}_{n-1,n}$.

In the theorem below we study equidistribution of several parameters associated with primitive vectors, one of them is the normalized lattice $[\Lambda_v]$ in the space $\mathcal{L}_{n-1,n}$. Unlike their shapes, equidistribution of the normalized orthogonal lattices has not been studied before, and is indeed stronger than the equidistribution of shapes.

We remark that equidistribution here (and anywhere else in this paper) is w.r.t. sets that are BCS.

**Theorem B.** For every $\epsilon > 0$ and primitive vectors $v \in \mathbb{Z}^n$, the following parameters effectively equidistribute as $\|v\| \to \infty$:

1. The directions $\hat{v} = v/\|v\|$ on the unit sphere $\mathbb{S}^{n-1}$:

   (a) when there is no restriction on the orthogonal lattices, at rate $O_\epsilon \left( \|v\|^{-n^2/6 + \frac{3}{4} n \kappa_n + \epsilon} \right)$;

   (b) when the shapes of $\Lambda_v$ are restricted to a BCS $E \subset X_{n-1}$, at rate $O_\epsilon \left( \|v\|^{-n \kappa_n + \epsilon} \right)$.

2. The shapes of $\Lambda_v$, where $\hat{v}$ is restricted to the BCS $\Phi' \subset \mathbb{S}^{n-1}$, equidistribute in $X_{n-1}$ at rate $O_\epsilon \left( \|v\|^{-n \kappa_n + \epsilon} \right)$.

3. The pairs $(\hat{v}, \text{shape of the lattice } \Lambda_v)$ equidistribute in $\mathbb{S}^{n-1} \times X_{n-1}$, at rate $O_\epsilon \left( \|v\|^{-n \kappa_n + \epsilon} \right)$.

4. The normalized orthogonal lattices $[\Lambda_v]$, where $\hat{v}$ lies in any BCS $\Phi' \subset \mathbb{S}^{n-1}$, equidistribute in $\mathcal{L}_{n-1,n}$ at rate $O_\epsilon \left( \|v\|^{-n \kappa_n + \epsilon} \right)$.

Here $\kappa_n \in \left[ \frac{1}{n^2}, \frac{1}{2n^2} \right]$ is the parameter that appears in Theorem A.

The equidistribution of shapes of primitive sub-lattices of any co-dimension was established in [Sch98]; the case of co-dimension 1 was also obtained in [Mar10], using a dynamical approach. Theorem B adds an error term (i.e., rate of convergence) to the aforementioned results, as well as the consideration of the normalized sub-lattices themselves, and not just their shapes. Our method can be used to consider the case of general co-dimension as well, which we will do in a forthcoming paper. We remark that for general (not necessarily primitive) sub-lattices of $\mathbb{Z}^n$, a quantitative result was proved in [Sch98]. Indeed, as is often the case with counting integral points in $\mathbb{R}^n$ (e.g. the Gauss
circle problem), improvements on the error term are harder to achieve in the primitive case than in the non-primitive case.

We also remark that the equidistribution of pairs discussed in part 3 of Theorem [B] has been studied in the case where the primitive vectors $v$ are restricted to a large sphere $\|v\| = e^T$, as opposed to a large ball $\|v\| \leq e^T$, the latter being the case considered in Theorem [B]. The sphere case is of course much more delicate, and this is the reason why all current results (to the best of our knowledge), e.g. [AES16b, AES16a, EMSS16], do not include an error term.

As we shall explain below, Theorems A and B are closely related to the Iwasawa decomposition (and generalizations of it) in $SL_n(\mathbb{R})$, and to counting $SL_n(\mathbb{Z})$ points inside families defined w.r.t. this decomposition. Such counting questions can be formulated in the more general context of a lattice $\Gamma$ in a simple algebraic Lie group $G$, where by ‘simple’ we mean that the Lie algebra is simple.

Consider the Iwasawa decomposition $G = KAN$, and a splitting of $A$ into a product of two subgroups $A = A' \times A''$ defined in the following manner. Let $a$ denote the Lie algebra of $A$ and choose a vector space decomposition of $a = a' \oplus a''$ where $a'$ is spanned by a basis whose elements $H_1', \ldots, H_l'$ lie in the closure of the positive Weil chamber defined w.r.t. $N$. Set $A' = \exp a', A'' = \exp a''$ and

$$A_T' := \left\{ \exp \left( \sum t_i \cdot H_i' \right) \mid t_1, \ldots, t_l \in [0,T] \right\}.$$ 

Note that $A'$ can also be the whole of $A$. Our third theorem is as follows.

**Theorem C.** (Theorem 9.11) Let $G$ be a simple algebraic Lie group with Haar measure $\mu$ and $\Gamma < G$ a lattice. Denote

$$B_T = \Phi A_T' \Pi \Psi;$$

where $\Phi \subseteq K$, $\Psi \subseteq N$ and $\Pi \subseteq A''$ are BCS. There exists a parameter $\kappa (\Gamma) \in (0,1)$ (defined in Theorem 7.4) such that for every $0 < \epsilon < \kappa (\Gamma)$ and for $T$ large enough,

$$\left| \# (B_T \cap \Gamma) - \frac{\mu (B_T)}{\mu (G/\Gamma)} \right| \leq \epsilon, \mu (B_T)^{(1-\kappa (\Gamma) + \epsilon)}.$$

Theorem [C] is concerned with counting in subsets of $G$ that grow only in the $A$-directions that are included in $A'$. Depending on the chosen parametrization of $A$ (i.e. basis for $a$), $A'$ can include any number of $A$-axes: from one to $\dim (A)$. We also prove counting results in which the growth is simultaneous in both the $A'$ and the $A''$ directions, e.g. in Corollary 9.15 in this case, where the $A''$ component also grows, we are forced to compromise the quality of the error term.

When $G$ is of rank one, then necessarily $A = A'$; counting lattice points w.r.t. Iwasawa decomposition in general simple real rank one Lie groups was studied in [HN16]; see also [MO15, Thm 1.4] for more general discrete subgroups.

### 1.2 Techniques and the outline of the paper

The proofs of Theorems A and B consist of two main ideas, and the paper is divided accordingly:

1. **A reduction to a problem of counting lattice points in the group** $SL_n(\mathbb{R})$ (Part I), which is done by establishing a correspondence between primitive vectors and integral matrices that lie in some subset of $SL_n(\mathbb{R})$, translating Theorems A and B into counting problems.

2. **Solving the counting problems** (Part II) via a method due to A. Gorodnik and A. Nevo, for which we develop a small theory that characterizes families of sets that satisfy a strong regularity property called “Lipschitz well-roundedness”.

Let us briefly describe the content of these two parts.
1.2.1 **Reduction to counting lattice points in** $\text{SL}_n(\mathbb{R})$ (Part I).

The idea is to obtain a correspondence between primitive vectors in $\mathbb{Z}^n$ and certain integral matrices in $\text{SL}_n(\mathbb{R})$, such that the parameters of the primitive vectors that we are interested in — norm, direction, orthogonal lattice etc. — are represented by the components of these matrices w.r.t. a decomposition of $\text{SL}_n(\mathbb{R})$ that we now turn to define.

**Generalized Iwasawa (GI) decomposition of** $\text{SL}_n(\mathbb{R})$. Set $G := \text{SL}_n(\mathbb{R})$ and let $K := \text{SO}_n(\mathbb{R})$, $A$ the diagonal subgroup in $G$, and $N$ the subgroup of upper unipotent matrices. Then, $G = KAN$ is the Iwasawa decomposition of $G$.

Consider yet another subgroup of $G$, $G'' := \begin{bmatrix} \text{SL}_{n-1}(\mathbb{R}) & \vdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$, which is clearly an isomorphic copy of $\text{SL}_{n-1}(\mathbb{R})$ inside $G$. Write $G'' = K''A''N''$ for the Iwasawa decomposition of $G''$, i.e.

- $K'' := K \cap G'' = \begin{bmatrix} \text{SO}_{n-1}(\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$,
- $A'' := A \cap G'' = \text{diag}(\alpha_1, \ldots, \alpha_{n-1}, 1)$ with $\alpha_1 \cdots \alpha_{n-1} = 1$,
- $N'' := N \cap G'' = \begin{bmatrix} \text{upper unipotent} & 0 \\ 0 & 1 \end{bmatrix}$.

The crux of the GI decomposition is that it completes the Iwasawa decomposition of $G''$ to the Iwasawa decomposition of $G$. For this we define $K', A', N'$ that complete $K'', A'', N''$ to $K$, $A$ and $N$ respectively. Define

- $N' := \begin{bmatrix} I_{n-1} & \mathbb{R}^{n-1} \\ 0 & 1 \end{bmatrix}$, $A' := \begin{bmatrix} a^{-\frac{1}{n-1}} I_{n-1} & 0 \\ 0 & a \end{bmatrix}$

and note that $N = N''N'$, $A = A''A'$, and that $A'$ is a one-parameter subgroup of $A$ which commutes with $G''$. Fix a transversal $K'$ of the diffeomorphism $K/K'' \to \mathbb{S}^{n-1}$ with the following property:

**Condition 1.3.** If $\Phi' \subseteq \mathbb{S}^{n-1}$ and $\Phi'' \subseteq K''$ are BCS, then so does $\Phi''K_{\Phi'} \subseteq K$, where $K_{\Phi'}$ is the inverse image of $\Phi'$ in $K'$.

The existence of such a transversal $K'$ is proved in Lemma 2.22. The GI decomposition is given by

$$G = K'K''A''A'N''N' = K'G''A''N', \quad \text{and we also let } P'' := A''N''.$$  

**Correspondence between** $\text{SL}_n(\mathbb{Z})$ **matrices and primitive vectors (Section 3).** The action of $Q$ (Formula 1.1) on $g \in \text{SL}_n(\mathbb{R})$ preserves the lattice spanned by the first $n-1$ columns, hence the action of

$$Q(\mathbb{Z}) = \begin{bmatrix} I_{n-1} & \mathbb{Z}^{n-1} \\ 0 & 1 \end{bmatrix} \rtimes \begin{bmatrix} \text{SL}_{n-1}(\mathbb{Z}) & 0 \\ 0 & 1 \end{bmatrix}$$

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preserves also the lattice spanned by all the columns of \( g \). Thus, the orbit \( g \cdot Q(\mathbb{Z}) \) is a pair:
\[
\left( \text{lattice spanned by first } n - 1 \text{ columns of } g , \text{ lattice spanned by all } n \text{ columns of } g \right).
\]

When the right-hand lattice is \( \mathbb{Z}^n \), which is equivalent to \( g \) lying in \( SL_n(\mathbb{Z}) \), then the left-hand lattice is primitive. It follows that the primitive \( n - 1 \) dimensional lattices are in bijection with the elements in \( SL_n(\mathbb{Z}) \backslash Q(\mathbb{Z}) \), which are the integral points in the space \( SL_n(\mathbb{R}) \backslash Q(\mathbb{Z}) \). Indeed, in Section 3.1 we establish a one-to-one correspondence between primitive \( (n - 1) \) dimensional sublattices (which are themselves in correspondence with primitive \( \mathbb{Z}^n \) vectors) to integral matrices lying inside a fundamental domain of \( Q(\mathbb{Z}) \) in \( SL_n(\mathbb{R}) \). In Section 3.2 we describe a general construction for such a fundamental domain; this construction is via the GI coordinates on \( SL_n(\mathbb{R}) \), as can be expected from the fact that \( Q(\mathbb{Z}) = N'(\mathbb{Z}) \times G''(\mathbb{Z}) \). In Section 3.3 we focus on fundamental domains in which, in addition to the correspondence between their integral points to primitive vectors, there is also a correspondence between the GI components of these integral matrices to parameters of the primitive vectors:

**Proposition 1.4** (Corollaries 4.6 and 6.2). There exists a fundamental domain \( \Omega \) short for \( Q(\mathbb{Z}) \) for which there is a bijection \( v \leftrightarrow \gamma_v \) between primitive integral vectors in \( \mathbb{R}^n \) and integral matrices in \( \Omega \) such that: the \( A' \) component of \( \gamma_v \) corresponds to the norm of \( v \), the \( K' \) component corresponds to the direction of \( v \), the \( K'' \) component corresponds to the shape of \( \Lambda_v \), and the \( P'' \) component corresponds to the shortest solution \( w_v \).

Following Proposition 1.4 above, in Sections 5 and 6 we formulate the counting questions to which the proofs of Theorems A and B (respectively) amount. For example, estimating the number of integral matrices inside the domain
\[
\Omega \cap \{ g = k' g'' a' n' : k' \in K' \cdot K'' \subset K', a' = a'_t \text{ with } t \leq T \}
\]
is equivalent to estimating the number of primitive integral vectors in \( \mathbb{R}^n \) whose direction \( \hat{v} \) lies in \( \Phi' \subset S^{n-1} \) and whose norm is bounded by \( e^T \). This counting question (and more) is dealt with in Part II of the paper.

### 1.2.2 Solving the counting problem (Part II).

This is the technical part of the paper, and it is dedicated entirely to proving counting lattice points results in simple Lie group \( G \), and in \( SL_n(\mathbb{R}) \) in particular. The main ingredient is a method due to A. Gorodnik and A. Nevo [GN12], henceforward referred to as the GN method, which concerns counting lattice points in increasing families \( \{ B_T \}_{T>0} \) inside non-compact algebraic simple Lie groups. This method establishes that the number of lattice points inside \( B_T \) is asymptotic to (a constant times) the volume of the set, and even allows to estimate the error term, provided the the family \( \{ B_T \}_{T>0} \) satisfies a regularity condition called *Lipschitz well roundedness* (Definition 7.1). In the case of a constant bounded family \( B_T = B \), this condition reduces to requiring that \( B \) is a BCS (Proposition 7.10). The estimation provided for (an upper bound on) the error term is responsible for the parameter \( \kappa(\Gamma) \) appearing in Theorem C. This parameter measures the spectral gap of the unitary representation \( L^2_0(G/\Gamma) \) of \( G \).

Our counting results are for two types of families, both defined w.r.t. the Iwasawa or Generalized Iwasawa coordinates on the group. A family of the first type consists of sets that are product of subsets in the subgroups appearing in the decomposition, as is the case of Theorem C. Such sets are referred to as *product sets*. A family of the second type consists of sets that have the structure of a fiber bundle over a subset of \( G'' \) or \( P'' \), and they will be referred to as *fibered sets*. Counting in a family of the second type is more difficult, but crucial for the proof of Theorem A. In each of these two types of families the \( K \) and the \( N \) components are fixed and bounded, and the \( A' \) component grows. As for the \( A'' \) component, we will make the distinction between two different situations: one in which the
projections of the sets in family to $A''$ are uniformly bounded, and one in which these projections grow simultaneously with the $A'$ component. This distinction is significant, because in the first situation the family is Lipschitz well rounded, while in the second situation it is not. Counting in the second situation is deduced from counting in the first situation, and balancing the error caused by the lack of regularity with the error term provided by the GN method. For this reason, it is crucial for our needs to use a counting method that produces an error term.

The organization of Part II is as follows.

In Section 7 we introduce the GN counting method and develop some tools to simplify the usage of it, namely to show that a given family is indeed well rounded. The main such tool would be group maps that we call *roundomorphisms*, with the property that they pull back well rounded families to well rounded families.

In Section 8 we show that the Iwasawa diffeomorphism $G \to K \times A \times N$ is a roundomorphism, enabling us to reduce the question of well roundedness of a family in $G$ to the one of well roundedness of its projections to the subgroups $K, A$ and $N$. A similar result holds for the GI decomposition.

In Section 9 we use the results of Section 8 in order to establish counting results in product sets w.r.t. the Iwasawa and GI decompositions, and in particular prove Theorem C.

In Section 10 we formulate a condition concerning the Lipschitzity of the fibers of a set whose structure is a fibration over $P''$; in Section 11 we use this condition to prove counting results in families of fibered sets in $\text{SL}_n(\mathbb{R})$, completing the proofs of Theorems A and B.

1.3 Further related work

With regards to Theorem C

Counting lattice orbit points in homogeneous spaces is a well studied field. For general affine spaces, we mention [DRS93], and then (via a dynamical approach) [EM93] followed by [EMS90] and [LO12]. The case of rank one symmetric spaces has received considerable attention; the most well known problem, also refereed to as the *hyperbolic circle problem*, is the one of counting lattice (orbit) points in increasing Riemannian balls. This question was raised by Selberg [Sel88], and studied by many authors, e.g. [Pat75, LP82, PR94]. It is by now well established that the number of lattice points inside Riemannian balls is asymptotic to the volume of the balls, but the problem of reducing the error term is still open. Counting lattice orbit points in balls and sectors inside rank one symmetric spaces is closely related to the question of counting lattice points in the associated Lie group, inside sets that are product sets w.r.t. the Cartan decomposition. Indeed, while Theorem C discusses counting lattice points in Iwasawa product sets, one can also consider counting lattice points in product sets w.r.t. other decompositions of $G$, e.g. the Cartan or the Bruhat decompositions; in both cases it is possible to define families in which (in analogy with Theorem C) the $A$ component grows and the remaining components lie in fixed compact sets. The first case was considered in [GON12], and the latter in [MMO14]. We note that (in the rank one setting) counting in Bruhat product sets also carries a geometric interpretation — as observed in [MMO14], it relates to counting closed geodesics in hyperbolic manifolds.

With regards to Theorem B

In the context of counting lattice points, perhaps the most natural lattice to consider is the one of integral points [BHC62]. This is where the topic of counting and equidistribution in homogeneous spaces meets the realm of counting integral points in algebraic varieties, in which the dynamical approach has become dominant in recent decades. A similar approach to the one of the present paper is utilized in [NS10, GGN13, GGN14]. Counting integral points is naturally connected to arithmetic (and in particular, to geometry of numbers) results, which is the case of Theorem B where equidistribution of primitive sublattices is deduced from effective lattice point counting for the lattice $\text{SL}_n(\mathbb{Z})$. We remark that an effective counting of primitive sublattices was established already in [Sch15], but the generality of the subsets considered was not enough to deduce equidistribution. We also remark that Part 1 of Theorem B which concerns equidistribution of directions of integral points on the unit sphere, is related to a family of questions referred to as “Linnik type problems” (e.g. [Duk07]). Such questions concern counting $\mathbb{Z}^n$ points on the sphere...
$dS^{n-1}$ as $d \in \mathbb{Z}$ satisfying necessary congruence conditions tends to $\infty$, and certain generalizations of this question.

1.4 Notations and conventions

Projections and measures on the GI components. For every $S \subset G$ appearing as a component in the Iwasawa or Generalized Iwasawa decompositions of $G$, we let $\pi_S(g)$ denote the projection of $g \in G$ to $S$. Also, for every such component $S$ we let $\mu_S$ denote a measure on $S$ as follows. $\mu_K, \mu_N$ are Haar measures, and so do $\mu_{K''}, \mu_{N''}$ and $\mu_N$. In the case of $G = SL_n(\mathbb{R})$, where $N'$ is measure equivalent and diffeomorphic to $\mathbb{R}^{n-1}$ and $K'$ is parameterizes $S^{n-1}$, we choose $\mu_{N'}$ to be the Lebesgue measure on $\mathbb{R}^{n-1}$ and $\mu_K$ to be the pullback Lebesgue measure on the sphere. Finally, $\mu_A, \mu_{A'}, \mu_{A''}$ are non Haar measures defined such that

$$\mu_K \times \mu_A \times \mu_N = \mu_{K''} \times \mu_{K'} \times \mu_A' \times \mu_{N''} \times \mu_{N'} \times \mu_{N''}$$

is the Haar measure on $G$. They are described explicitly in Definition 9.2 and computed for the case of $SL_n(\mathbb{R})$ in Examples 9.4 and 9.10.

Inequalities. We will use the following conventions for inequalities. If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two $n$-tuples of real numbers, we denote $a \leq b$ if $a_i \leq b_i$ for every $i = 1, \ldots, n$. If $f$ and $g$ are two non-negative functions then we denote

$$f \prec g$$

if there exists a positive constant $C$ such that $f \leq Cg$ , and denote

$$f \preceq g$$

if $f + O(1) \leq g$.

Parameterizations. As was already mentioned, $K'$ parameterizes the unit sphere $S^{n-1}$, and a subset $F_{n-1}$ in $P_{n-1}$ (the group of determinant 1 upper triangular matrices with positive diagonal entries) parameterizes the space of shapes $X_{n-1}$. In general, when a space $X$ is parameterized by a (subset of) manifold $\mathcal{M}$, we will use the notation $\mathcal{M}_B$ for the set in $\mathcal{M}$ that corresponds to a set $B \subset X$. The parameterizations that will be considered are by manifolds $\mathcal{M}$ that appear as components in the GI decomposition of $SL_n(\mathbb{R})$, as follows.

1. $SL_{n-1}(\mathbb{R}), \mathbb{R}^{n-1}$ and $P_{n-1}$ by $G'', N'$ and $P''$ respectively. These parameterizations are the obvious diffeomorphisms $G'' \simeq SL_{n-1}(\mathbb{R}), N' \simeq \mathbb{R}^{n-1}$ and $P'' \simeq P_{n-1}$.

2. $\mathbb{R}^{\dim A}, \mathbb{R}^{\dim A'}$ and $\mathbb{R}^{\dim A''}$ by $A, A'$ and $A''$ respectively. Indeed, $\mathbb{R}^{\dim A} \simeq a, \mathbb{R}^{\dim A'} \simeq a'$ and $\mathbb{R}^{\dim A''} \simeq a''$; for any of these cases, a choice of a diffeomorphism sets a parameterization of $\mathbb{R}^{\dim A}$ (resp. $\mathbb{R}^{\dim A'}, \mathbb{R}^{\dim A''}$) by $A$ (resp. $A', A''$) which is the diffeomorphicism composed with the exponent map. This parameterization is in fact a choice of axes in $A$. If $S = (S_1, \ldots, S_{\dim A}) > 0$, then the set $A\prod_{i=1}^{\dim A} [0.S_i]$ (a product of intervals in the chosen axes on $A$) will be denoted for short $A(S_1, \ldots, S_{\dim A}) = A_S$.

3. These parameterizations are the diffeomorphisms given by the exponent map.

4. The unit sphere $S^{n-1}$ by $K'$; this parameterization is described in Condition 1.3 and Section 2.5.

5. The space of shapes $X_{n-1}$ by a subset $F_{n-1}$ of the manifold $P'' \simeq P_{n-1}$ which is a fundamental domain (constructed in Section 2.3) for the action of $SL_{n-1}(\mathbb{Z})$ on $P_{n-1}$. Similarly, the space $\mathcal{L}_{n}$ of unimodular lattices is parameterized by a fundamental domain (also constructed in Section 2.3) for the action of $SL_{n-1}(\mathbb{Z})$ on $SL_{n-1}(\mathbb{R})$, and the space $\mathcal{L}_{n-1,n}$ of normalized $(n-1)$ dimensional sublattices is parameterized by a fundamental domain for the action of $Q(\mathbb{Z})$ on $KP''$. All these parameterizations by fundamental domains are formalized and justified in Section 2.4.
When $B = \{b\}$ is a singleton, we will denote the point $x \in \mathcal{M}$ that corresponds to $b$ by $x_b$. In particular: the point in $K'$ that corresponds to a unit vector $u \in S^{n-1}$ is $k'_u$; the element in $P''$ that corresponds to $z \in P_{n-1}$ (i.e., $\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$) is $p''_z$; the element in $N'$ that corresponds to $x \in \mathbb{R}^{n-1}$ (i.e., $\begin{bmatrix} 1 & -1 & x \\ x & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$) is $n'_x$; etc.

We will also allow the reverse notation, at least in the case of the $P''$ component: we denote by $z_g \in P_{n-1}$ the point that corresponds to the $P''$ component of an element $g$ (i.e. $\pi_{P''}(g) = \begin{bmatrix} z_g & 0 \\ 0 & 1 \end{bmatrix}$).
Part I
From \( \mathbb{Z}^n \) to \( SL_n(\mathbb{Z}) \)

2 Fundamental domains of \( SL_m(\mathbb{Z}) \)

In this section we recall several spaces related to lattices in \( \mathbb{R}^m \); space of unimodular lattices, space of shapes of lattices and space of codimension one lattices. All of these spaces are quotients of \( SL_m(\mathbb{R}) \), and the main goal of this section is to construct fundamental domains in \( SL_m(\mathbb{R}) \) for these quotients. A secondary goal is to establish a correspondence between BCS’s in the lattice spaces and BCS’s in the associated fundamental domains.

2.1 Spaces of lattices and organization of the section

Let \( \Lambda \) be a (full rank) lattice inside \( \mathbb{R}^m \) having the columns of \( M \in GL_m(\mathbb{R}) \) as an ordered basis. It is well known that any other basis of \( \Lambda \) is the columns of a matrix obtained by multiplying \( M \) from the right by a matrix from \( GL_m(\mathbb{Z}) \). As a result, the space of \( m \)-lattices can be defined as \( GL_m(\mathbb{R})/GL_m(\mathbb{Z}) \). One can also consider a more crude space which is the space of \textit{shapes} of lattices: two lattices \( \Lambda_1 = M_1 \cdot GL_m(\mathbb{Z}) \) and \( \Lambda_2 = M_2 \cdot GL_m(\mathbb{Z}) \) have the same shape if \( \Lambda_1 \) differs from \( \Lambda_2 \) by an orthogonal transformation and rescaling, namely there are \( k \in O_m(\mathbb{R}) \) and \( c > 0 \) such that \( ckM_1 \cdot GL_m(\mathbb{Z}) = M_2 \cdot GL_m(\mathbb{Z}) \). As a result, the space of shapes can be defined as

\[
\mathcal{X}_m = PO_m(\mathbb{R}) \backslash PGL_m(\mathbb{R}) / PGL_m(\mathbb{Z}) \cong SO_m(\mathbb{R}) \backslash SL_m(\mathbb{R}) / SL_m(\mathbb{Z}) .
\]

Notice that in the right hand side we consider unimodular lattices (i.e. lattices with covolume 1), since clearly every lattice can be rescaled to a unimodular lattice. We let

\[
\mathcal{L}_m := SL_m(\mathbb{R}) / SL_m(\mathbb{Z})
\]

denote the space of unimodular lattices in \( \mathbb{R}^m \). Denote by \( P_m \) the group of upper triangular matrices of determinant 1 with positive diagonal entries, and write \( P_m = A_m N_m \) where \( A_m \) is the subgroup of diagonal matrices, and \( N_m \) the subgroup of unipotent matrices. The majority of this section is devoted to constructing fundamental domains \( F_m \) and \( F_m \) for the (right) actions of \( SL_m(\mathbb{Z}) \) on \( SL_m(\mathbb{R}) \) and \( P_m \), which then parameterize the spaces \( \mathcal{L}_m \) and \( \mathcal{X}_m \) respectively. This construction is well known due to the work of Siegel (e.g., [MBB00]), Schmidt [Sch98], and Grenier [Gre93].

In Section 2.2 we introduce a variant of Siegel sets inside \( SL_m(\mathbb{R}) \), which contain a finite number of representatives from every orbit, and therefore a fundamental domain; in Section 2.3 we define the domain \( F_m \subset SL_m(\mathbb{R}) \) inside the Siegel set, as well as the resulting fundamental domain \( F_m \subset P_m \); Section 2.4 is devoted to a fundamental domain parameterizing the space \( \mathcal{L}_{m-1,n} \) of lattices of codimension one in \( \mathbb{R}^n \), and to the connection between BCS’s in the aforementioned fundamental domains, and in the lattice spaces that they parameterize.

2.2 Siegel-reduced bases

Let \( \Lambda \) be a lattice in \( \mathbb{R}^m \) (not necessarily unimodular). We describe an inductive method to construct an ordered basis \( \{v_1, \ldots, v_m\} \) for \( \Lambda \) as follows. Let \( v_1 \) be a shortest nonzero element of \( \Lambda \). For future reference, we denote its length by \( a_1 \) and its direction \( v_1/a_1 \) by \( \phi_1 \). Next, write \( V_1 \) for \( \text{span}_\mathbb{R} \{v_1\} \) and consider the projection of \( \Lambda \) to \( V_1^\perp \), which is a lattice of dimension \( m-1 \). One can find a vector \( v_2 \in \Lambda \) whose projection to \( V_1^\perp \) is of nonzero minimal length \( a_2 \). Since actually all the elements in \( \{v_2 + nv_1 : n \in \mathbb{Z}\} \) share this property of having their projection to \( V_1^\perp \) be of length \( a_2 \), we may assume that \( v_2 \) also satisfies that its projection to \( V_1 \) is \( n_{1.2} a_1 \cdot \phi_1 \) with \( |n_{1.2}| \leq \frac{1}{2} \). We proceed by induction:

\textbf{Definition 2.1 (and notations).} A \textit{Siegel-reduced (SR)} basis for a lattice \( \Lambda \) is a basis \( \{v_1, \ldots, v_m\} \) in which for all \( j \in \{1, \ldots, m\} \), the basis element \( v_j \) is chosen such that:
1. The projection of \( v_j \) to \( V_{j-1}^\perp = \text{span}_\mathbb{R} \{v_1, \ldots, v_{j-1}\}^\perp \) has minimal non-zero length \( a_j \) (here \( V_0 = \{0\} \)); denote this projection by \( a_j \phi_j \), where \( \phi_j \) is a unit vector.

2. The projection of \( v_j \) to \( V_{j-1} = \text{span}_\mathbb{R} \{v_1, \ldots, v_{j-1}\} = \text{span}_\mathbb{R} \{\phi_1, \ldots, \phi_{j-1}\} \) is

\[
\sum_{i=1}^{j-1} (n_{i,j} a_i) \phi_i \text{ with } |n_{ij}| \leq \frac{1}{2} \text{ for all } i = 1, \ldots, j - 1.
\]

The matrix \( M = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} \) is called a Siegel-reduced (SR) matrix of \( \Lambda \).

We note that in the case of unimodular bases (bases of co-volume 1), one may need to replace \( v_1 \) by \(-v_1\) in order for the SR matrix \( M \) to have determinant 1 (and not \(-1\)).

The parameters \( \{a_j\}, \{n_{i,j}\} \) and \( \{\phi_j\} \) involved in the process of constructing an SR basis \( \{v_1, \ldots, v_m\} \) are interpreted via the KAN coordinates of the associated SR matrix as follows. Let

\[
a = \text{diag} (a_1, \ldots, a_m), \quad k = [\phi_1 \cdots \phi_m]
\]

and

\[
n = \begin{pmatrix}
1 & n_{1,1} & \cdots & n_{1,m} \\
1 & \ddots & & \\
& \ddots & 1 \\
& & 1
\end{pmatrix},
\]

Then, since the \( i \)-th column of \( ka \) is \( a_i \phi_i \) = the projection of \( v_i \) to \( V_{i-1}^\perp \), and the \( i \)-th column of \( n \) is exactly the coordinates of \( v_i \) w.r.t. the orthogonal set \( \{a_1 \phi_1, \ldots, a_m \phi_m\} \), we obtain that the SR matrix is

\[
M = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} = \text{kan}.
\]

**Lemma 2.2.** Suppose \( M = k_M a_M n_M \) is Siegel-reduced w.r.t some lattice \( \Lambda \), where \( k_M, a_M \) and \( n_M \) are as above.

1. \( n_M \) is a unipotent upper triangular matrix whose entries are bounded in \([\frac{-1}{2}, \frac{1}{2}]\) (in particular, \( \|n_M^{\pm 1}\|, \|n_M^{\pm t}\| < 1 \)).
2. \( a_M = \text{diag} (a_1, \ldots, a_m) \) is a diagonal matrix which satisfies that \( a_1 \prec \cdots \prec a_m \). Specifically, \( \frac{\sqrt{2}}{2} a_j \leq a_{j+1} \).
3. If \( \lambda \in \Lambda \) (i.e. \( \lambda = M v \) for some \( v \in \mathbb{Z}^m \)) satisfies \( \lambda \notin V_{j-1} \), then

\[
||\lambda|| \geq \text{dist} (\lambda, V_{j-1}) \geq \text{dist} (v_j, V_{j-1}) = a_j.
\]
4. If \( x \in V_j \), then \( ||a_M x|| < a_j ||x|| \).

**Proof.** Parts 1 and 3 are immediate from the construction of \( M \). For part 2, recall that \( \{a_1 \phi_1, \ldots, a_m \phi_m\} \) is an orthogonal set in \( \mathbb{R}^n \) and that \( v_{j+1} = a_{j+1} \phi_{j+1} + \sum_{i=1}^{j} n_{j,i} a_i \phi_i \). Then,

\[
a_j^2 \leq \text{dist} (v_{j+1}, V_{j-1})^2 = \|n_{j,j+1} (a_j \phi_j) + (a_{j+1} \phi_{j+1})\|^2 = a_j^2 |n_{j,j+1}|^2 + a_{j+1}^2.
\]

Now, since \( |n_{j,j+1}| \leq \frac{1}{2} \) (by part 1), we obtain:

\[
\frac{1}{4} a_j^2 + a_{j+1}^2 \geq a_j^2
\]
and therefore
\[ a_{j+1} \geq \frac{\sqrt{3}}{2} a_j. \]

As for part 4
\[ \| a_M x \| = \left\| \begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \end{pmatrix} \right\| \leq \max_{1 \leq i \leq j} |a_i| \| x \| \quad \text{part 2} \]
\[ \leq 2 a_j \| x \|. \]
\[ \Box \]

**Definition 2.3.** We refer to the sets
\[ \left\{ M \in \text{GL}_m(\mathbb{R}) \quad | \quad M \text{ Siegel reduced for } \right. \\
\left. \left( \text{or SL}_m(\mathbb{R}), \text{if unimodular} \right) \right. \]
\[ \left. \text{the lattice spanned by its columns} \right\} \]
as **reduced Siegel sets**.

**Remark 2.4.** The reduced Siegel sets are contained in the well-known Siegel sets (e.g., [MBB00, Rag72, Chapter X]).

We note that parts 1 and 3 of Lemma 2.2 are the defining conditions of the Reduced Siegel sets (the inequalities in parts 2 and 4 are redundant, since they follow from part 3). Observe that these defining inequalities depend only on the entries of the \( N \) and \( A \) components of the matrix. Indeed, let
\[ M = \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} = k \begin{bmatrix} n \end{bmatrix} \; \text{span} \{ e_1, \ldots, e_m \}; \]
the inequalities in 1 are on the entries of \( n \), and the inequalities in 3 translate into
\[ a_j \leq \left\| \text{projection of } zv \right\|, \quad (2.1) \]
for every \( v = (\alpha_1, \ldots, \alpha_m)^t \in \mathbb{Z}^n \) and \( j = 1, \ldots, m \). This is because:

\[ a_j = \text{dist} (v_j, V_{j-1}) \leq \text{dist} \left( \sum_{i=j}^m \alpha_i v_i, V_{j-1} \right) \text{ rotation by } k \quad \text{dist} \left( \sum_{i=j}^m \alpha_i z_i, E_{j-1} \right) \]
\[ = \text{dist} (zv, E_{j-1}) = \left\| \text{projection of } zv \right\|. \]

We also note that the number of the defining inequalities for these reduced Siegel sets is infinite: indeed, every \( v \in \mathbb{Z}^n \) yields an inequality in formula 2.1 (resp. part 3 of Lemma 2.2). However, it is shown in [Sch98, p.49] that it is actually sufficient to consider the inequalities 2.1 for only finitely many \( v \in \mathbb{Z}^n \). We state it for future reference:

**Proposition 2.5.** The reduced Siegel sets are defined by a finite number of inequalities in (the entries of) the \( N \) and \( A \) components of a matrix.

**Remark 2.6.** The finitely many integral vectors \( u \) that imply the sufficient inequalities from 2.1 are as follows. For any \( j = 1, \ldots, m \) one considers the \( v \in \mathbb{Z}^n \) which satisfy
\[ \max (|\alpha_j|, \ldots, |\alpha_m|) \leq \frac{C}{a_j} \left\| \text{projection of } zv \right\| \]
where \( C \) is some constant that depends only on \( m \) and can be computed explicitly from [Sch98]; clearly, there is only a finite number of integral vectors \( u \) which satisfy this condition. In particular, the reduced Siegel sets can be computed explicitly.
2.3 Fundamental domains of $\text{SL}_m(\mathbb{Z})$ inside $\text{SL}_m(\mathbb{R})$ and $P_m$

From now on we shall only consider unimodular lattices (resp. bases) in $\mathbb{R}^m$. Since these unimodular lattices are identified with cosets in $\text{SL}_m(\mathbb{R}) / \text{SL}_m(\mathbb{Z})$, where a representative for a coset is a choice of a basis for the corresponding lattice, it follows that a fundamental domain for (the right action of) $\text{SL}_m(\mathbb{Z})$ inside $\text{SL}_m(\mathbb{R})$ consists of a unique choice of a basis for every unimodular lattice $\Lambda \subset \mathbb{R}^m$.

We know (by the construction presented in the previous section) that every lattice has a Siegel-reduced basis, and therefore the reduced Siegel set contains a fundamental domain for $\text{SL}_m(\mathbb{Z})$. We turn to describe a closure of such a fundamental domain, namely a choice of a unique SR basis for almost every unimodular lattice $\Lambda \subset \mathbb{R}^m$.

**Remark 2.7.** It is shown in [Sch98] that the number of SR bases for a lattice $\mathbb{R}^m$ is finite, where a bound on this number depends only on $m$, and not on the lattice. Intuitively, this is due to the fact that a given lattice has only a finite number of shortest vectors (where the bound on this number depends only on the dimension), and a Siegel-reduced basis is constructed such that in every step, one chooses a shortest vector from some lattice.

Given an SR basis $\{v_1, \ldots, v_m\}$, one can clearly obtain further SR bases for the same lattice by alternating the signs of the elements $v_j$. Note that the corresponding SR matrices $M$ will have the same $A$ components, and in fact they will vary from each other only by the signs of the entries of $n$ and $k$ as follows:

- replacing $\{v_1, \ldots, v_m\}$ by $\{-v_1, \ldots, -v_m\}$ is done by multiplying from the left by $-I$ (replacing $k$ by $-k$);
- replacing $v_j$ by $-v_j$ for $j = 2, \ldots, m$ is done by changing the sign of the $j$-th row and column of $n$ (above the diagonal) and changing the sign of the $j$-th column of $k$, $\phi_j$.

In order to preserve the property $\det(M) = 1$, one is only allowed to alternate the sign of an *even* number among the $v_j$’s. In particular, one is allowed to change all signs simultaneously if and only if $-I \in K = \text{SO}_m(\mathbb{R})$.

**Definition 2.8.** We let $\tilde{F}_m \subset \text{SL}_m(\mathbb{R})$ denote the closed subset of the reduced Siegel set (Definition 2.3) which satisfies the following conditions on the $N$ and $K$ components:

1. Condition on the sign of the first row of $n$:
   
   $n_1,j \geq 0$ for $j > 2 \implies n_1,j \in \left[0, \frac{1}{2}\right]$ for $j > 2$.
   
   $n_1,j \geq 0$ for $j > 1 \implies n_1,j \in \left[0, \frac{1}{2}\right]$ for $j > 1$.

2. Condition on the $K$-components: they lie in a closure of a fundamental domain of the lattice $Z(K)$ in $K$ (Notation 2.9).

**Notation 2.9.** We denote a fundamental domain of $Z(K)$ in $K = \text{SO}_m(\mathbb{R})$ by $K$, and let

$$\iota(m) = [\text{SO}_m(\mathbb{R}) : Z(\text{SO}_m(\mathbb{R}))] = \begin{cases} 2 & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{cases}.$$ 

Indeed, $Z(\text{SO}_m(\mathbb{R})) \cong \begin{cases} \mathbb{Z}_2 & \text{if } m \text{ is even} \\ \text{id}_K & \text{if } m \text{ is odd} \end{cases}$.

Clearly, $\mu_K(K) = \mu_K(K)/\iota(m)$.

We also denote by $F_m \subseteq P_m$ the projection of $\tilde{F}_m$ mod $K$, namely the set of upper triangular matrices whose columns are SR bases for the lattice spanned by their columns, and whose $N$ components satisfy condition [1].

We state the following for future reference:
Corollary 2.10. The boundary of \( \tilde{F}_m \) (resp. \( F_m \)) is contained in a finite union of lower-dimensionnal manifolds in \( \text{SL}_m(\mathbb{R}) \) (resp. \( P_m \)).

Proof. According to Proposition 2.5 and Definition 2.8, \( \tilde{F}_m \) and \( F_m \) are defined by a finite number of inequalities.

The significance of \( \tilde{F}_m \) and \( F_m \) stems from the following:

Theorem 2.11. \( \tilde{F}_m \subseteq \text{SL}_m(\mathbb{R}) \) and \( F_m \subseteq P_m \) are the closures of fundamental domains for the right actions of \( \text{SL}_m(\mathbb{Z}) \) on \( \text{SL}_m(\mathbb{R}) \) and on \( P_m \) respectively.

Proof. Since \( \tilde{F}_m \) is a subset of the reduced Siegel set, the latter containing a basis for every lattice, defined by the additional conditions from Definition 2.8 which merely impose a choice of signs for the basis elements — we conclude that \( \tilde{F}_m \) contains a representative (basis) for every unimodular lattice in \( \mathbb{R}^m \).

It is now sufficient to show that every \( M \in \text{int} (\tilde{F}_m) \) is the unique representative for the lattice spanned by its columns. In other words, if a given lattice has more than one representative in \( \tilde{F}_m \), then these representatives (that are SR matrices for the lattice) lie in the boundary of \( \tilde{F}_m \).

Let \( M \in \text{int} (\tilde{F}_m) \). Then, \( M \) satisfies a strict version of the inequalities in Formula 2.1 and in Definition 2.8. Write \( M = \text{kan} \). Using induction and (the strict version of) inequality 2.1, it is clear that \( v_1 \) is uniquely determined up to a sign; \( v_2 \) is uniquely determined up to a sign and modulo \( V_1 \); and so forth, every \( v_j \) is uniquely determined up to a sign and modulo \( V_{j-1} \). As a result, \( a \) is uniquely determined, and the columns of \( k, \phi_1, \ldots, \phi_m \), are determined up to a sign. Since \( v_j = \sum_{i=1}^{j} n_{ij} (a_i \phi_i) \), one can show using reverse induction (from \( i = j-1 \) to \( i = 1 \)) that there are unique \( n_{ij} \) satisfying the strict version of condition 1 from Definition 2.8 so that \( v_1, \ldots, v_m \) are determined up to a sign. According to the explanation in the beginning of this section, the inequalities in Definition 2.8 impose a unique choice of signs, and therefore a unique representative in the interior of \( F_m \). □

Remark 2.12 (and a notation convention). As mentioned in Theorem 2.11, \( \tilde{F}_m \) and \( F_m \) are closures of fundamental domains; in order to obtain actual fundamental domains, one should remove parts of their boundaries, leaving a unique representative for every lattice. However, we will abuse notation and denote \( \tilde{F}_m \) and \( F_m \) for the actual fundamental domains, and not their closures.

Notation 2.13. Let \( \Lambda \) be a lattice. Denote by \( K_\Lambda \) a fundamental domain of the finite group \( \text{Sym}(\Lambda) \cap \text{SO}_m(\mathbb{R}) \) (the stabilizer of \( \Lambda \) w.r.t. the left action of \( K = \text{SO}_m(\mathbb{R}) \), sometimes called the “point group” of \( \Lambda \). If \( M \) (resp. \( z \)) is a matrix representing a lattice \( \Lambda \) (resp. the shape of a lattice \( \Lambda \)) then we write \( K_M \) (resp. \( K_z \)) for \( K_\Lambda \).

Proposition 2.14. The relation between the fundamental domains \( \tilde{F}_m \) and \( F_m \) is given by

\[
\tilde{F}_m = \bigcup_{z \in F_m} K_z \cdot z,
\]

and when \( z \in \text{int} (F_m) \) it holds that \( K_z = K \).

Remark 2.15. In fact, it is shown in [Sch98, p.50] that the interior of any fundamental domain of \( \text{SL}_m(\mathbb{Z}) \) consists of lattices for which \( \text{Sym}(\Lambda) \cap \text{SO}_m(\mathbb{R}) \) is \( \mathbb{Z}(K) \).

We end this section with two technical notes. The first is regarding the regularity of \( K \), and the second one concerns lattices that are in \( K \cdot F_m \) but not in \( \tilde{F}_m \).

Fact 2.16. It is possible to choose \( K \) to be BCS (Definition 1.1).

Proof. See Lemma 2.22 □
Fact 2.17. The number of \( m \)-dimensional sub-lattices \( \Lambda < \mathbb{Z}^n \) of covolume \( < X \) with shape in \( \partial F_m \) is \( O\left(X^{n-\frac{1}{2}}\right) \). In particular, the number of \( m \)-dimensional sub-lattices \( \Lambda \) for which \( K_\Lambda \neq K \), i.e. \( \text{Sym}(\Lambda) \cap \text{SO}(\text{span}\Lambda) \) is not \( \{\pm I\} \cap \text{SO}_m(\mathbb{R}) \), is \( O\left(X^{n-\frac{1}{2}}\right) \).

Proof. For the number of sub-lattices projected to \( \partial F_m \), see [Sch98, Lemma 6 and Theorem 7]. These sub-lattices include the ones for which \( K_\Lambda \neq K \), according to Remark 2.15. \( \square \)

2.4 Relation between fundamental domains and quotient spaces

Having defined the fundamental domains \( \widetilde{F}_m \) and \( F_m \) in \( \text{SL}_m(\mathbb{R}) \) and \( P_m \) respectively, we would like to identify them with the spaces \( \mathcal{L}_m \) and \( \mathcal{X}_m \). In general, it is natural to identify “reasonable” fundamental domains for properly discontinuous actions with the manifolds obtained by gluing their boundaries; e.g., the interval \([0,1]\) and the unit circle \( S^1 \). The quotient map restricted to the fundamental domain is clearly a bijection, and it identifies the fundamental domain (if measurable) with the quotient space in terms of measure. However, one should be more cautious with regards to the topological properties that this map carries, as the fundamental domains are usually not even manifolds. This is the aspect that we focus on in this subsection, which aims at proving that the quotient map reduced to the fundamental domain carries the (topological) property of a set being a BCS. We begin with defining the type of fundamental domains for which this map will (be shown to) satisfy this property.

Definition 2.18 (Spread model). Assume that a group \( \Gamma \) acts properly discontinuously and isometrically on a Riemannian manifold \( \mathcal{M} \), and let \( \pi : \mathcal{M} \to \mathcal{M}/\Gamma \) denote the associated quotient map. A fundamental domain \( F \subset \mathcal{M} \) for this action is called a spread model for the quotient space \( \mathcal{M}/\Gamma \) if its boundary satisfies the following definiteness assumption:

1. it is contained in a finite union of lower dimensional submanifolds of \( \mathcal{M} \);
2. the quotient map restricted to \( F \) is proper, namely it pulls back compact sets to compact sets.

We will denote \( F \overset{\sim}{=} \mathcal{M}/\Gamma \).

We note that (as we shall see in the proof of Lemma 2.21) when \( F \) is a spread model for \( \mathcal{M}/\Gamma \), the quotient map \( \pi : \mathcal{M} \to \mathcal{M}/\Gamma \) restricted to \( \text{int}(F) \) is a measure preserving diffeomorphism.

Since one of the spaces for which we want to consider a spread model is the \( \mathcal{L}_{n-1,n} \), we shall need the following fact.

Fact 2.19. \( K'G''_{\widetilde{F}_{n-1}} \) (defined in Section 1.2) is a fundamental domain for the right action of \( QA' \) on \( \text{SL}_n(\mathbb{R}) \). As result,

\[
\mu(\mathcal{L}_{n-1,n}) = \mu_{K'G''=KP''} \left(K'G''_{\widetilde{F}_{n-1}}\right) = \mu_{KP''} \left(K'K_{\mathcal{L}_{n-1,n}}\right) = \frac{\mu(K)}{l(n-1)} \mu_{P''}(F_{n-1})
\]

and the measure on \( \mathcal{L}_{n-1,n} \) is the product of Haar measures

\[
\mu_{\mathcal{L}_{n-1,n}} = \mu_{K'K''} \times \mu_{P''}|_{F_{n-1}}.
\]

We omit the proof, which is almost identical to the one of Corollary 3.4 in the next section. Intuitively, this fact is true because the following spaces are diffeomorphic:

\[
\text{SL}_n(\mathbb{R})/QA' \simeq K'G''A'N'/G''(\mathbb{Z}) N'A' = K'G''A'N'/G''(\mathbb{Z}) A'N' \simeq K'G''/G''(\mathbb{Z})
\]

and a fundamental domain for \( G''/G''(\mathbb{Z}) \) is \( C_{\mathcal{L}_{n-1}}^{\mu} \).

In this paper we will use the following spaces and their spread models:
Proposition 2.20. The following pairs consist of quotient spaces and their spread models in the corresponding manifolds:

1. \( \mathcal{L}_m = \text{SL}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z}) \), and \( \tilde{F}_m \) inside \( \mathcal{M} = \text{SL}_m(\mathbb{R}) \);

2. \( \mathcal{X}_m = \text{SO}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z}) \cong P_m/\text{SL}_m(\mathbb{Z}) \), and \( F_m \) inside \( \mathcal{M} = P_m \) (in particular, \( \mathcal{X}_{n-1} \cong \mathcal{P}_{F_{n-1}} \));

3. \( \mathcal{L}_{n-1,n} = \text{SL}_n(\mathbb{R})/\text{Q}A' \cong K\mathcal{P}''/\text{Q}, \) and \( K'\mathcal{G}''_{\tilde{F}_{n-1}} \) inside \( \mathcal{M} = K\mathcal{P}'' \).

**Proof.** Parts 1 and 2. The spaces \( \mathcal{L}_m \) and \( \mathcal{X}_m \) are dealt in the same manner: the boundaries of \( \tilde{F}_m \) and \( F_m \) are contained in a finite union of lower dimensional submanifolds according to Corollary 2.10 and the fact that the quotient map restricted to the closure is proper is a consequence of the Mahler compactness criterion, as we now explain. Assume that \( B \subseteq \mathcal{L}_m \) is compact, which by Mahler’s criterion means that there exists a positive constant \( \beta \) such that for every \( \Lambda \in B \), the length of the shortest vector in \( B \) is at least \( \beta \). Let \( g = \pi|_{\tilde{F}_m}^{-1}(\Lambda) \in \tilde{F}_m \) and write \( g = \text{kan} \) where \( a = \text{diag}(a_1, \ldots, a_m) \). By construction of \( \tilde{F}_m \), \( a_1 \) is the length of a shortest vector in \( \Lambda \), and therefore \( \beta \leq a_1 \). Also by construction of \( \tilde{F}_m \), the columns of \( g \) are a Siegel reduced basis to \( \Lambda \); hence by part 2 Lemma 2.2

\[
0 < \beta \leq a_1 \leq \frac{\sqrt{3}}{2} a_2 \leq \cdots \leq \left( \frac{\sqrt{3}}{2} \right)^{m-1} a_m = \left( \frac{\sqrt{3}}{2} \right)^{m-1} \frac{1}{a_1 \cdots a_{m-1}} \leq \left( \frac{\sqrt{3}}{2} \right)^{\frac{(m-1)m}{2}} \frac{1}{\beta^{m-1}}.
\]

We conclude that \((a_1, \ldots, a_{m-1})\) lies in a bounded set in \( \mathbb{R}^{m-1} \), namely the \( A \) component of \( g \) lies in a compact set. The \( N \) and \( K \) component of the elements of \( \tilde{F}_m \) (\( g \) included) are bounded uniformly, so we conclude that \( \pi|_{\tilde{F}_m}^{-1}(B) \) is a compact set in \( \tilde{F}_m \). The compactness for the case of \( F_m \) is identical.

Part 3. As for the space \( \mathcal{L}_{n-1,n} \), it is stated in Fact 2.10 that \( K'\mathcal{G}''_{\tilde{F}_{n-1}} \) is indeed a fundamental domain, which clearly lies in the manifold \( K'\mathcal{G}'' = K\mathcal{P}'' \). The second condition of being a spread model holds for \( K'\mathcal{G}''_{\tilde{F}_{n-1}} \), since it does in \( \tilde{F}_{n-1} \), so it remains to verify the first condition. First

\[
\partial \left( K'\mathcal{G}''_{\tilde{F}_{n-1}} \right) = \partial \left( K'\mathcal{K}'' \mathcal{P}''_{\tilde{F}_{n-1}} \right) \cup \left( \partial z \in \partial \mathcal{P}_{\mathcal{F}_{n-1}} K'K_z z \right)
\]

\[
\subseteq \partial \left( K'\mathcal{K}'' \mathcal{P}''_{\tilde{F}_{n-1}} \right) \cup \partial \left( \partial z \in \partial \mathcal{P}_{\mathcal{F}_{n-1}} K'K_z z \right) \subseteq \partial \left( K'\mathcal{K}'' \mathcal{P}''_{\tilde{F}_{n-1}} \right) \cup K\mathcal{P}''_{\partial (\mathcal{F}_{n-1})}.
\]

Now, if \( \varphi : K \times \mathcal{P}'' \rightarrow K\mathcal{P}'' \) is the diffeomorphism \( (k, p') \mapsto kp'' \) then the above equals

\[
\partial \varphi^{-1} \left( K'\mathcal{K}'' \times \mathcal{P}''_{\tilde{F}_{n-1}} \right) \cup \varphi^{-1} \left( K \times \mathcal{P}''_{\partial (\mathcal{F}_{n-1})} \right)
\]

\[
= \varphi^{-1} \left( \partial \left( K'\mathcal{K}'' \times \mathcal{P}''_{\tilde{F}_{n-1}} \right) \cup \left( K \times \mathcal{P}''_{\partial (\mathcal{F}_{n-1})} \right) \right).
\]

The claim follows since both \( \partial \left( K'\mathcal{K}'' \times \mathcal{P}''_{\tilde{F}_{n-1}} \right) \) and \( K \times \mathcal{P}''_{\partial (\mathcal{F}_{n-1})} \) are contained in a finite union of lower dimensional submanifolds of \( K\mathcal{P}'' \); the second is obvious, and to see the first note that

\[
\partial \left( K'\mathcal{K}'' \times \mathcal{P}''_{\tilde{F}_{n-1}} \right) = \left( \partial K'\mathcal{K}'' \times \mathcal{P}''_{\tilde{F}_{n-1}} \right) \cup \left( K'\mathcal{K}'' \times \partial \mathcal{P}''_{\tilde{F}_{n-1}} \right)
\]

and recall that \( \partial \mathcal{P}''_{\tilde{F}_{n-1}} \) is contained in a finite union of submanifolds (Remark 2.10), and so does \( K'\mathcal{K}'' \) (since it is BCS).

We now turn to prove that for fundamental domains \( F \) that are spread models, the map \( \pi|_F \) preserves the BCS property of a set.
Lemma 2.21. Assume that a group \( \Gamma \) acts properly discontinuously and isometrically on a Riemannian manifold \( \mathcal{M} \), and that \( F \subset \mathcal{M} \) is a spread model for this action. Then if \( B \subseteq \mathcal{M}/\Gamma \) is a BCS, so does \( \mathcal{M}_B := \pi|_{F}^{-1}(B) \), where \( \pi : \mathcal{M} \to \mathcal{M}/\Gamma \) is the natural projection.

Proof. We first prove that \( \mathcal{M}_B \) is compact. This is because \( \mathcal{M}_B \subseteq \mathcal{M}_{\overline{B}} \) and \( \mathcal{M}_{\overline{B}} \) is compact since \( \overline{B} \) is compact and \( \pi|_{\overline{B}} \) is proper. It remains to show that \( \partial \mathcal{M}_B \) is contained in a finite union of lower dimensional submanifolds of \( \mathcal{M} \). Since \( \Gamma \) acts properly discontinuously on \( \mathcal{M} \), \( \pi \) is a submersion, and it clearly remains a submersion when restricted to the full dimensional submanifold \( \text{int}(F) \). Since \( \pi|_{\text{int}(F)} \) is also a submersion, then it is a diffeomorphism onto its image. Now

\[
\partial \mathcal{M}_B \subseteq \pi|_{\text{int}(F)}^{-1}(\partial B) \cup \partial F
\]

where \( \partial F \) is a finite union of lower dimensional submanifolds (by definition of a spread model) and so does \( \pi|_{\text{int}(F)}^{-1}(\partial B) \) — because \( \partial B \) has this property and \( \pi|_{\text{int}(F)} \) is a diffeomorphism. \( \square \)

2.5 Pulling boundary controllable sets from the sphere to \( \text{SO}_n(\mathbb{R}) \)

Let \( K, A, N \) be such that \( G = KAN \) is the Iwasawa decomposition of \( G \). While in the case of \( S \in \{ A, N \} \) the definition of \( S' \) is explicit, and the decomposition \( S = S'S'' \) is into a product of subgroups, this is not the case for \( S = K \). Here \( K' \) is not a subgroup, but rather a set of representatives for \( K'' \simeq \text{SO}_{n-1}(\mathbb{R}) \) in \( K = \text{SO}_n(\mathbb{R}) \), which is then parameterized by the sphere of the corresponding dimension. The topic of this subsection is to specify such a parameterization, and more specifically to construct a transversal \( K' \) for \( K'' \) in \( K \) which satisfies Condition 1.3 from the Introduction. This subsection completes the task initiated in the previous subsection, which is to define a manner in which certain quotient spaces (\( \mathcal{L}_n, \mathcal{X}_{n-1}, \mathcal{L}_{n-1,n} \) from Proposition 2.20 and now \( -\mathbb{S}^{n-1} \)) parameterize subsets of the GI components of \( \text{SL}_n(\mathbb{R}) \) in a way that a BCS in the space corresponds to a BCS in the component.

Lemma 2.22. Let \( K \) be a Lie group. Assume that \( K'' < K \) a closed subgroup such that the quotient space \( K/K'' \) is compact. There exists subset \( K' \subseteq K \) which is a BCS such that:

1. \( \pi|_{K'} : K' \to K/K'' \) is a bijection;
2. if \( \Phi' \subseteq K/K'' \) and \( \Phi'' \subseteq K'' \) are BCS, then the product \( \Phi'' \Phi' \) in \( K' \) is also a BCS.

For the proof, we need the following observation.

Remark 2.23. Since \( \partial (A \cup B), \partial (A \cap B) \subseteq \partial A \cup \partial B \), the union, intersection and subtraction of BCSs are in themselves BCS. Also, a Cartesian product of BCSs is a BCS in the Cartesian product of the manifolds, and a diffeomorphic image of a BCS is a BCS.

Proof of Lemma 2.22. Since \( \pi : K \to K/K'' \) is a fiber bundle with a fiber \( K'' \), there exists a covering \( \{ U_\alpha \} \) of \( K/K'' \) with diffeomorphisms

\[
\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times K''.
\]

We can assume that the sets \( U_\alpha \) are BCS (e.g., by reducing to contained open balls); by compactness, we may also assume that this covering is finite. Finally, by replacing every \( U_\alpha \) with \( U_\alpha \setminus \bigcup_{i=1}^{n-1} U_i \), we may assume that the sets \( U_\alpha \) are disjoint, maintaining the BCS property (Remark 2.23).

Set

\[
K' = \sqcup_{\alpha} \tau_\alpha^{-1}(U_\alpha \times \text{id}_{K''})
\]

(note that the interior is a manifold). Since the union is disjoint, \( \pi|_{K'} : K' \to K/K'' \) is a bijection. Moreover, since \( U_\alpha \) is a BCS, then so does \( U_\alpha \times \text{id}_{K''} \), and then so does \( \tau_\alpha^{-1}(U_\alpha \times \text{id}_{K''}) \); by Remark 2.23 \( K' \) is a BCS.
Finally, by definition of $K'$ one has that $k' \in U_\alpha \cap K'$ maps under $\tau_\alpha$ to $(\pi (k'), \text{id}_{K''})$. Since $\pi : K \to K/K''$ is a principle $K''$-bundle (meaning that $K''$ acts on every fiber and $\tau_\alpha$ can be chosen to be equivariant with this action),

$$\tau_\alpha (k' \cdot k'') = (\pi (k'), k'').$$

In particular, if $\Phi' \subseteq K/K''$ and $\Phi'' \subseteq K''$ then

$$\pi|_{K'}^{-1} (\Phi' \cap U_\alpha) \cdot \Phi' = \tau_\alpha^{-1} ((\Phi' \cap U_\alpha) \times \Phi'').$$

where by Remark 2.23 the right-hand side is a BCS; then $\pi|_{K'}^{-1} (\Phi') \cdot \Phi''$ is a BCS, as a finite union of such.

3 Integral matrices representing primitive vectors

In Section 3.1 we describe domains $\Omega$ in $\text{SL}_n (\mathbb{R})$ such that there exists a one-to-one correspondence between the $\text{SL}_n (\mathbb{Z})$ elements inside them and primitive vectors in $\mathbb{Z}^n$, which is part of the claim of Proposition 1.4 from the Introduction. These domains are fundamental domains of a discrete subgroup of $\text{SL}_n (\mathbb{R})$, which is introduced in Section 3.2.

3.1 Correspondence between primitive vectors and matrices in $\text{SL}_n (\mathbb{Z})$

Let $0 \neq v \in \mathbb{R}^n$ and define:

$$G_v = \left\{ g = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \text{SL}_n (\mathbb{R}) : v_1 \wedge \cdots \wedge v_{n-1} = v \right\}. \quad (3.1)$$

Lemma 3.1. For $g \in \text{SL}_n (\mathbb{R})$ as above, the following are equivalent:

1. $g \in G_v$.
2. The columns $\{v_1, \ldots, v_{n-1}\}$ form a basis of co-volume $\|v\|$ to $v^\perp$ such that $\{v_1, \ldots, v_{n-1}, v\}$ is a positively oriented basis w.r.t. the standard basis of $\mathbb{R}^n$.
3. $\langle v_n, v \rangle = 1$.

Proof. 1 $\iff$ 2 by definition. The equivalence 2 $\iff$ 3 follows from

$$1 = \det (g) = (v_1 \wedge \cdots \wedge v_{n-1}) \cdot v_n = v \cdot v_n.$$ 

The sets $G_v$ can easily seen to be orbits of the group

$$Q (\mathbb{R}) = N' \rtimes G''$$

(see Formula 1.1 for definition of $Q$) acting by right multiplication on $G = \text{SL}_n (\mathbb{R})$, where

$$G'' = \begin{bmatrix} \text{SL}_{n-1} (\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}, \ N' = \begin{bmatrix} I_{n-1} & \mathbb{R}^{n-1} \\ 0 & 1 \end{bmatrix}. $$

The subgroup $Q (\mathbb{Z})$ is a discrete (infinite covolume) subgroup of $\text{SL}_n (\mathbb{R})$, whose significance to our problem follows from the fact that there is a bijection between primitive $(n - 1)$-dimensional subgroups of $\mathbb{Z}^n$ (alternatively, primitive vectors $v \in \mathbb{Z}^n$) to integral points in fundamental domains of $Q (\mathbb{Z})$. Therefore, the problem of counting primitive $(n - 1)$-dimensional subgroups of $\mathbb{Z}^n$ (or primitive vectors) reduces to counting $\text{SL}_n (\mathbb{Z})$ points in fundamental domains of $Q (\mathbb{Z})$. 

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Proposition 3.2. If $\Omega \subset \text{SL}_n(\mathbb{R})$ is a fundamental domain for the right action of $\mathbb{Q}(\mathbb{Z})$, then there exists a bijection that depends on $\Omega$

\[
\left\{ \text{primitive oriented} \atop \{(n-1)\text{-dim subgroups of } \mathbb{Z}^n) \right\} \leftrightarrow \left\{ \text{primitive vectors} \atop \text{in } \mathbb{Z}^n \right\} \leftrightarrow \left\{ \text{integral elements} \atop \text{in } \Omega \right\}
\]
given by

\[(\mathbb{Z}^n \cap v^\perp) \leftrightarrow v \leftrightarrow \gamma_v(\Omega),\]

where $\mathbb{Z}^n \cap v^\perp$ is oriented as in Lemma 3.1(2), and

\[\gamma_v(\Omega) := \text{the unique element in } \Omega \cap \text{SL}_n(\mathbb{Z}).\]

Proof. The correspondence $(\mathbb{Z}^n \cap v^\perp) \leftrightarrow v$ is obvious, and it suffices to show the correspondence $v \leftrightarrow \gamma_v(\Omega)$. We first claim that

\[G_v \cap \text{SL}_n(\mathbb{Z}) \neq \{\emptyset\} \iff v \in \mathbb{Z}^n \text{ primitive}. \quad (3.2)\]

The direction $\Rightarrow$ is a consequence of part 3 of Lemma 3.1. Conversely, if $v$ is primitive, then there exists $w \in \mathbb{Z}^n$ such that $(v,w) = 1$, and since $w$ is also primitive, it can be completed to a basis \{\(v_1,\ldots,v_{n-1},w\)\} of $\mathbb{Z}^n$. The resulting matrix \[
\begin{bmatrix}
    v_1 & \cdots & v_{n-1} & w
\end{bmatrix}
\]
is therefore in $\text{SL}_n(\mathbb{Z})$, and by Lemma 3.1 it is also in $G_v$.

Now, according to 3.2, $v \in \mathbb{Z}^n$ is primitive if and only if there exists an integral $\gamma$ in $G_v$. This is equivalent to all the points in the orbit $\gamma \cdot \mathbb{Q}(\mathbb{Z})$ being integral. Since $\Omega$ is a fundamental domain for $\mathbb{Q}(\mathbb{Z})$, the coset $\gamma \cdot \mathbb{Q}(\mathbb{Z})$ intersects $\Omega$ in a single point \{\(\gamma_v\)\} = $\Omega \cap (\gamma \cdot \mathbb{Q}(\mathbb{Z}))$. We claim that $\gamma \cdot \mathbb{Q}(\mathbb{Z}) = G_v(\mathbb{Z})$; indeed,

\[G_v(\mathbb{Z}) = G_v \cap \text{SL}_n(\mathbb{Z}) = \gamma \cdot \mathbb{Q}(\mathbb{R}) \cap \text{SL}_n(\mathbb{Z}) = \gamma \cdot \mathbb{Q}(\mathbb{Z}),\]

where the right equality holds since

\[\gamma \cdot q \in \text{SL}_n(\mathbb{Z}) \iff q \in \gamma^{-1} \cdot \text{SL}_n(\mathbb{Z}) \iff q \in \text{SL}_n(\mathbb{Z}).\]

\[\square\]

3.2 Fundamental domains for the action of $\mathbb{Q}(\mathbb{Z})$ on $\text{SL}_n(\mathbb{R})$

The following proposition allows us to define a fundamental domain for the action $\text{SL}_n(\mathbb{R}) \actson \mathbb{Q}(\mathbb{Z})$.

Proposition 3.3. Assume $G$ acts from the right on $X \times Y$.

1. If the action $X \times Y \actson G$ is such that $(x,y) \cdot g = (x,y \cdot g)$ then every fundamental domain for this action is of the form

\[\bigcup_{x \in X} \{x\} \times F_y\]

where $F_y \subset Y$ is a fundamental domain for the action $Y \actson G$.

2. If $A,B$ are subgroups of $G$ such that $1 \rightarrow B \rightarrow G \rightarrow A \rightarrow 1$, $B$ acts freely on $Y$ and the action $X \times Y \actson G$, is such that $(x,y) \cdot g = (x \cdot \pi(g), y \cdot g)$ then the following is fundamental domain for this action:

\[\bigcup_{x \in F_A} \{x\} \times F_x\]

where $F_y \subset Y$ is a fundamental domain for the action $Y \actson B$, and $F_A$ is a fundamental domain for the action $X \actson A$. 

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Proof. The first part is trivial. As for the second part, the domain $\bigcup_{x \in F_A} \{x\} \times F_x$ contains a representative from every orbit since: if $(x, y) \in X \times Y$, then, since $F_0$ is a fundamental domain for the action $X \curvearrowright A$, there exist $x' \in F_A$ and $a \in A$ such that $x \cdot a = x'$. Since $F_x$ is a fundamental domain for the action $Y \curvearrowright B$, there exist $y' \in F_x$ and $b \in B$ such that $(y \cdot a) \cdot b = y'$. Then for $g = ab \in G$ it holds that

$$(x, y) \cdot g = (x, y) \cdot (x \cdot a, (y \cdot a) \cdot b) = (x', y') \in \bigcup_{x \in F_A} \{x\} \times F_x.$$  

Moreover, the domain $\bigcup_{x \in F_0} \{x\} \times F_x$ does not contain two different representatives from the same orbit, since: if both $(x, y)$ and $(x', y')$ are in this domain and $(x, y) \cdot g = (x', y')$, then (by definition of the action) $(x \cdot \pi(g), y \cdot g) = (x', y')$. But $\pi(g) \in A$ and $x, x'$ are both in $F_0$, hence $(F_A$ is a fundamental domain for the action $X \curvearrowright A) \ x = x'$. In particular, due to the action of $A$ on $X$ being free, $\pi(g) = \text{id}_A$, i.e. $g = b \in B$. Then

$$(x', y') = (x, y) \cdot g = (x, y \cdot b),$$

namely $y' = y \cdot b$, where both $y, y'$ are in $F_{x'}$. Since $F_{x'}$ is a fundamental domain for the action $Y \curvearrowright B$, it must be that $y' = y$. □

Corollary 3.4 (and definition of $\Omega$). Let $\mathcal{D} = \{\mathcal{D}(z)\}_{z \in \mathbb{F}_{n-1}}$ be a family of fundamental domains for $\mathbb{Z}^{n-1}$ in $\mathbb{R}^{n-1}$. Then

$$\Omega = \Omega(\mathcal{D}) := \bigcup_{g'' \in \mathbb{F}_{n-1}} K' \cdot g'' \cdot A' \cdot N'_\mathcal{D}(z_{g''})$$

(see Section 1.2 for the definition of $K'$, $A'$, $N'$ and $G''$ and Section 1.4 for the notations $N'_\mathcal{D}$ and $z_{g''}$) is a fundamental domain for the action of $Q(\mathbb{Z})$ on $\text{SL}_n(\mathbb{R})$ by multiplication from the right.

Remark 3.5. In the above Corollary, $\mathbb{F}_{n-1}$ can be replaced with any fundamental $\mathbb{F} \subset \text{SL}_{n-1}(\mathbb{R})$ domain for $\text{SL}_{n-1}(\mathbb{Z})$.

Proof. Let $q \in Q$ and write $q = ab$ where $b \in N'$ and $a \in G''$. Recall that $h$ normalizes $N'$ and commutes with $A'$. Let $g = k'g''a'n'$. The GI coordinates of $g \cdot q$ are given by:

$$g \cdot q = g \cdot ab = k'g''a'n' \cdot ab = k'g''a' \cdot aa^{-1} \cdot n' \cdot ab = k' \cdot (g''a) \cdot a' \cdot (a^{-1} \cdot n' \cdot ab). \quad (3.3)$$

Since $Q(\mathbb{Z})$ is the (semi-direct) product of $G''(\mathbb{Z}) \cong \text{SL}_{n-1}(\mathbb{Z})$ and $N'(\mathbb{Z}) \cong \mathbb{Z}^{n-1}$, then the group homomorphism

$$Q(\mathbb{Z}) \xrightarrow{\pi} G''(\mathbb{Z})$$

$$g''n' \mapsto g''$$

is onto and has kernel $N'$. Identifying $\text{SL}_n(\mathbb{R})$ (as a set) with $K' \times G'' \times A' \times N'$, then according to 3.3

$$(k', g'', a', n') \cdot ab = (k', g''a, a^{-1}a \cdot b).$$

Thus, if $\{\mathcal{D}(g'')\}_{g'' \in \mathbb{F}_{n-1}}$ is a family of fundamental domains for $\mathbb{Z}^{n-1}$ in $\mathbb{R}^{n-1}$ (meaning that $\{N'_{\mathcal{D}(g'')}\}_{g'' \in \mathbb{F}_{n-1}}$ is a family of fundamental domains for $N'(\mathbb{Z})$ in $N'$), then Proposition 3.3 implies that

$$\bigcup_{g'' \in \mathbb{F}_{n-1}} K' \cdot g'' \cdot A' \cdot N'_{\mathcal{D}(g'')}$$

is a fundamental domain for $Q(\mathbb{Z})$. □

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is a fundamental domain for the action of $Q(\mathbb{Z})$ on $K' \times G'' \times A' \times N'$. If $\overline{D}(g'') = D(zg'')$, then

$$\left\{ \overline{D}(g'') \right\}_{g'' \in F_{n-1}} = \{ D(z) \}_{z \in F_{n-1}},$$

which completes the proof.

**Remark 3.6.** Clearly, if all the domains $D(z)$ are the same domain $D$, then $\Omega$ is the product set $K'G''_{F_{n-1}}A'N'_D$.

## 4 Properties of primitive vectors represented by the GI components

After having established the correspondence between primitive vectors in $\mathbb{Z}^n$ and integral matrices in $\Omega$, we turn to verify the rest of Proposition 1.4 from the Introduction and show how the different parameters of these primitive vectors are represented by the GI components of the corresponding integral matrices in $\Omega$. Consequently, counting primitive vectors whose parameters are restricted to a certain range — their norm to a certain interval, their direction to some subset of the sphere, etc. — is reduced to counting integral matrices in subsets of $\Omega$ that are defined by restricting the GI coordinates. These subsets are defined in Section 4.2.

### 4.1 Explicit GI components of $g \in \text{SL}_n(\mathbb{R})$, and their interpretation

For $g \in \text{SL}_n(\mathbb{R})$, let $\Lambda^j_g$ denote the lattice spanned by the first $j$ columns of $g$ and set $\Lambda_g := \Lambda_g^{n-1}$. Recall that the projection of $g \in \text{SL}_n(\mathbb{R})$ to the component $S$ in the GI decomposition is denoted $\pi_S(g)$. The explicit GI coordinates of $g$ are specified in the coming proposition, for which it will be convenient to set the following notations:

**Definition 4.1.** We let

$$\Upsilon : S^{n-1} \to \text{Mat}_{n \times (n-1)}(\mathbb{R})$$

denote the map that corresponds to the choice of transversal $K'$ such that the element $k' \in K'$

associated to $\tilde{v} \in S^{n-1}$ is of the form

$$k' = \left[ \Upsilon(\tilde{v}) \tilde{v} \right].$$

**Notation 4.2.** For $t \in \mathbb{R}$ and $s = (s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-2}$, let

$$a^t_s = \text{diag} \left( e^{\frac{s_1}{2}}, e^{\frac{s_2}{2}}, \ldots, e^{\frac{s_{n-2}}{2}}, 1 \right),$$

$$a^0_s = \text{diag} \left( e^{-\frac{s_1}{2}}, e^{-\frac{s_2}{2}}, \ldots, e^{-\frac{s_{n-2}}{2}}, 1 \right).$$

**Proposition 4.3.** Let $g \in \text{SL}_n(\mathbb{R})$. Recall that $z_g$ denotes $(n-1) \times (n-1)$ upper triangular matrix such that $\pi_{vu}(g) = \begin{bmatrix} z_g & 0 \\ 0 & 1 \end{bmatrix}$. The GI components of $g$ are as follows:

$$g = \begin{bmatrix} v_1 & \ldots & v_{n-1} & v_n \\ 1 & \ldots & 1 & 1 \end{bmatrix} \in G_v$$

has GI coordinates

$$g = k'_e k'' e^t a^u e^{\frac{\|v\|}{\text{covol}(\Lambda_t^i)}} a^\mid \tilde{v} \mid \mid \left( v_n \right) v''_{B_g}$$

$$= k'_e g^{-\frac{\|v\|}{\text{covol}(\Lambda_t^i)}} a^u e^{\frac{\|v\|}{\text{covol}(\Lambda_t^i)}}$$

$$\Rightarrow \begin{cases} e^t &= \|v\| \\ e^{-\frac{\|v\|}{\text{covol}(\Lambda_t^i)}} &= \text{covol}(\Lambda_t^i) \\ u &= \tilde{v} := \left( v_n \right) v''_{B_g} \\ x &= \left( x \right) \left( v_n \right) v''_{B_g} \\ \text{shape of } \Lambda_t^i &= \text{shape of } \Lambda_t^i g' \\ e^{-\frac{\|v\|}{\text{covol}(\Lambda_t^i)}} \Lambda_t^i &= \Lambda_{k'_e g''} \end{cases}, \quad (4.1)$$

where $B_g = \{ v_1, \ldots, v_{n-1} \}$. In particular:
1. \( \pi_A(g) = \text{diag} \left( \frac{\text{covol}(\Lambda_1^1)}{\text{covol}(\Lambda_2^2)}, \ldots, \frac{\text{covol}(\Lambda_n^n)}{\text{covol}(\Lambda_{n-1}^{n-1})} \right) \).

2. The \( A' \)-component of \( g \), \( \pi_{A'}(g) \), equals \( \begin{bmatrix} \|v\|^{-\frac{1}{p_k}} I_{n-1} \end{bmatrix} \).

3. \( \pi_{A''}(g) = \text{diag} \left( \frac{\|v\|}{\pi_k} \text{covol}(\Lambda_1^1), \frac{\|v\|}{\pi_k} \text{covol}(\Lambda_2^2), \ldots, \frac{\|v\|}{\pi_k} \text{covol}(\Lambda_n^n) \right) ; \) the columns of \( \pi_{A''}(g) \) are an orthonormal basis for the hyperplane \( v^\perp \).

4. \( \pi_K(g) = k'_e = [\kappa(v) \hat{v}] \), where \( \hat{v} = v/\|v\| \) and the columns of \( \kappa(v) \) are an orthonormal basis for the hyperplane \( v^\perp \).

5. \( \pi_N(g) = \begin{bmatrix} 1 & \cdots & 0 & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \alpha_{n-1} \end{bmatrix} \) such that \( v_n = \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + \frac{1}{\|v\|^2} v \) (namely, \( \sum_{i=1}^{n-1} \alpha_i v_i \) is the orthogonal projection of \( w \) to the hyperplane \( v^\perp \)).

6. The columns of \( k'g'' = kp'' \) span the lattice \( e^{-\frac{1}{\pi_k}} \Lambda_g \); the columns of \( p'' \) span a rotation of this lattice to span \( \{e_1, \ldots, e_{n-1}\} \).

The proof of this proposition requires a short lemma.

**Lemma 4.4.** If \( g = [B \mid w] \) is in \( G_v \), and \( w^\perp \) is the orthogonal projection of \( w \) on the hyperplane \( v^\perp \), then

\[
\frac{1}{\|v\|^2} v.
\]

**Proof.** Write \( w = w^\perp + \alpha v \). By part 3 of Lemma 3.1

\[
1 = \langle w, v \rangle = \langle w^\perp + \alpha v, v \rangle = \langle \alpha v, v \rangle
\]

hence \( \alpha = \frac{1}{\|v\|^2} \).

**proof of Proposition 4.3.** Write \( g = pk \) according to the Gram-Schmidt decomposition:

\[
\begin{bmatrix}
\phi_1 & \cdots & \phi_n \\
\rho_1 & \cdots & \rho_n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\langle v_1, \phi_1 \rangle & \langle v_2, \phi_1 \rangle & \cdots & \langle v_n, \phi_1 \rangle \\
0 & \langle v_2, \phi_2 \rangle & \cdots & \langle v_n, \phi_2 \rangle \\
0 & 0 & \cdots & \langle v_n, \phi_n \rangle
\end{bmatrix}.
\]

The columns of \( k \) are \( \phi_1, \ldots, \phi_n \), and since they are obtained by the Gram-Schmidt algorithm on the columns of \( g \), \( \{v_1, \ldots, v_n\} \), then

\[
\text{span} \{\phi_1, \ldots, \phi_{n-1}\} = \text{span} \{v_1, \ldots, v_{n-1}\} = v^\perp.
\]

Since \( \{\phi_1, \ldots, \phi_n\} \) are orthonormal,

\[
\phi_n = \hat{v} = v/\|v\|.
\]

Since \( \pi_{K'}(g) \) and \( \pi_K(g) \) have the same last column, then \( \pi_{K'}(g) = k'_e = [\kappa(v) \hat{v}] \), which proves part 4.

The entries of \( \pi_A(g) \) are the diagonal entries of \( p \). Then, it is sufficient to show that

\[
\|v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i\| = \text{product of the entries } 1, \ldots, i \text{ of } \pi_A(g)
\]

\[
= \text{product of the diagonal entries } 1, \ldots, i \text{ of } p.
\]
This is indeed the case, since the columns of \( p = [p_1 \cdots p_n] \) differ from the columns of \( g = [v_1 \cdots v_n] \) by a rotation; thus
\[
\| v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i \| = \| p_1 \wedge \cdots \wedge p_{i-1} \wedge p_i \|.
\]

But, since \( p \) is upper triangular,
\[
\| p_1 \wedge \cdots \wedge p_{i-1} \wedge p_i \| = \text{product of the diagonal entries 1, \ldots, } i \text{ of } p.
\]

This proves part 1.

Since \( g = [v_1 \cdots v_n] \in \text{SL}_n(\mathbb{R}) \), then \( \| v_1 \wedge \cdots \wedge v_{n-1} \wedge v_n \| = 1 \), and since \( g \in G_v \) then \( \| v_1 \wedge \cdots \wedge v_{n-1} \| = \| v \| \); thus, the last diagonal entry of \( \pi_A(g) \) is \( \frac{1}{\| v \|} \). Then by definition of the decomposition \( A = A''A' \),
\[
\pi_A'(g) = \text{diag} \left( \frac{\| v \|}{\| v \|^{-1}}, \ldots, \frac{\| v \|}{\| v \|^{-1}}, \frac{\| v \|}{\| v \|^{-1}} \right),
\]
which proves parts 2 and 3.

Since \( g = k'g''a'n' \) then
\[
g(n')^{-1} = k'g''a';
\]
i.e.
\[
\left[ \begin{array}{cccc}
v_1 & \cdots & v_{n-1} & v_n
\end{array} \right] \cdot \left[ \begin{array}{cccc}
1 & \cdots & 0 & -\alpha_1 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & -\alpha_{n-1} \\
0 & \cdots & \cdots & 1
\end{array} \right] = \left[ \begin{array}{cccc}
\gamma(v) \hat{v} \\
\vdots \\
0 \\
1
\end{array} \right] \cdot \left[ \begin{array}{cccc}
g'' \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1
\end{array} \right] \cdot \left[ \begin{array}{cccc}
\frac{\| v \|}{\| v \|^{-1}} \\
\vdots \\
\frac{\| v \|}{\| v \|^{-1}} \\
\frac{\| v \|}{\| v \|^{-1}}
\end{array} \right],
\]
If \( \tilde{v}_n = v_n - \alpha_1v_1 - \cdots - \alpha_{n-1}v_{n-1} \), then:
\[
\left[ \begin{array}{cccc}
v_1 & \cdots & v_{n-1} & \tilde{v}_n
\end{array} \right] = \left[ \begin{array}{cccc}
\frac{\| v \|}{\| v \|^{-1}} \\
\vdots \\
\frac{\| v \|}{\| v \|^{-1}} \\
\frac{\| v \|}{\| v \|^{-1}}
\end{array} \right] \cdot \left[ \begin{array}{cccc}
g'' \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1
\end{array} \right] \cdot \left[ \begin{array}{cccc}
\gamma(v) \hat{v} \\
\vdots \\
0 \\
1
\end{array} \right],
\]
which proves 6.

We turn to prove part 5. Write \( v_n \) as the sum of its projections to the orthogonal spaces span \( \{ v \} \) and \( v^\perp \):
\[
v_n = (v_n)^v + (v_n)^{v^\perp}.
\]
The claim in part 5 is that \( \alpha_1v_1 + \cdots + \alpha_{n-1}v_{n-1} = (v_n)^{v^\perp} \), namely that
\[
v_n - \alpha_1v_1 - \cdots - \alpha_{n-1}v_{n-1} = (v_n)^v.
\]
To see this, observe that
\[
g(n')^{-1} = k'p''a'
\]
(part 2) =
\[
\left[ \begin{array}{cccc}
\phi_1 & \cdots & \phi_{n-1} & \phi_n
\end{array} \right] \cdot \left[ \begin{array}{cccc}
p_0 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1
\end{array} \right] \cdot \left[ \begin{array}{cccc}
\gamma(v) \hat{v} \\
\vdots \\
0 \\
1
\end{array} \right],
\]
Namely,
\[
\begin{bmatrix}
v_1 \\
\vdots \\
v_n-1 \\
v_n - \alpha_1 v_1 - \cdots - \alpha_{n-1} v_{n-1}
\end{bmatrix}^t = \begin{bmatrix}
\ast \\
\ast \\
\ast \\
\|v\|^t \phi_n
\end{bmatrix}^t;
\]
the desired equality is now in the bottom row of these equal matrices; indeed, according to Lemma 4.4, \((v_n)^v = v/\|v\|^2\), and according to the observation in the beginning of this proof, \(\phi_n = v/\|v\|\); we conclude that
\[(v_n)^v = \phi_n/\|v\|\cdot\]
In order to establish Equation 4.1, it is left to observe the following.
According to part 2 and to the definition of \(a'_i\),
\[
\operatorname{diag} \left( e^{\frac{\pi i}{\|v\|}}, \ldots, e^{\frac{\pi i}{\|v\|}}, e^{-\frac{\pi i}{\|v\|}} \right) = a'_i = \operatorname{diag} \left( \|v\|^{\frac{1}{\pi i}}, \ldots, \|v\|^{\frac{1}{\pi i}}, \|v\|^{-\frac{1}{\pi i}} \right),
\]
namely \(e'^i = \|v\|\).
According to part 3 and to the definition of \(a''_z\),
\[
\operatorname{diag} \left( e^{-\frac{\pi i}{\|v\|}}, e^{\frac{\pi i}{\|v\|}}, e^{-\frac{\pi i}{\|v\|}}, \ldots, e^{\frac{\pi i}{\|v\|}}, e^{-\frac{\pi i}{\|v\|}} \right) = a''_x
\]
\[= \operatorname{diag} \left( \|v\|^{-\frac{1}{\pi i}} \operatorname{covol} (\Lambda_1^n), \|v\|^{-\frac{1}{\pi i}} \operatorname{covol} (\Lambda_2^n), \ldots, \|v\|^{-\frac{1}{\pi i}} \operatorname{covol} (\Lambda_2^n) \right)\]
and therefore
\[
(\|v_1 \wedge v_2 \wedge \cdots \wedge v_n\|) = e^{-\frac{\pi i}{\|v\|}} \cdot \|v\|^{\frac{1}{\pi i}} = e^{-\frac{\pi i}{\|v\|} + \frac{1}{\pi i}}.
\]

4.2 Subsets of \(\Omega\) defined w.r.t. GI coordinates

We are now at a point where we have established that the GI components of matrices \(g\) in \(\text{SL}_n(\mathbb{R})\) encode information regarding the vectors \(v\) such that \(g \in G_v\) (Proposition 4.3); moreover, that there is a bijection \(v \leftrightarrow \gamma_v\) (Proposition 3.2) between primitive vectors in \(\mathbb{Z}^n\) and integral matrices in \(\Omega\), in which every primitive \(v\) corresponds to the unique integral matrix in \(\Omega \cap G_v\). This implies that the GI components of the integral matrices in \(\Omega\) encode certain parameters of the associated primitive vectors. Consequently, subsets of \(\Omega\) defined by restricting the GI coordinates capture the integral matrices \(\gamma_v\) associated with primitive vectors whose parameters are restricted to a given range. These subsets of \(\Omega\) are of the following form:

**Definition 4.5.** Let every \(T, S > 0\), \(\Phi' \subseteq \mathbb{S}^{n-1}\), a family \(\mathcal{D} = \{\mathcal{D}(z)\}_{z \in F_{n-1}}\) of fundamental domains for \(\mathbb{Z}^{n-1}\) in \(\mathbb{R}^{n-1}\) and a family \(\mathcal{D}_0 = \{\mathcal{D}_0(z)\}_{z \in F_{n-1}}\) such that \(\mathcal{D}_0 \subseteq \mathcal{D}\) (meaning that \(\mathcal{D}_0(z) \subseteq \mathcal{D}(z)\) for every \(z \in F_{n-1}\)). Let \(\Omega\) be a fundamental domain for \(Q(\mathbb{Z})\) in \(\text{SL}_n(\mathbb{R})\) as in Corollary 3.4 and define
\[
\Omega_T^S(\mathcal{D}_0, \Phi') : = \Omega \cap \left\{ g : \begin{array}{l}
a'' = a''_x \leq s_i \leq S_i, \\
a' = a'_i \text{ with } t \in [0, T], \\
k' \in K'_{\Phi'}, n' \in N'_{\mathcal{D}_0(z)}
\end{array} \right\}
\]
where \(g = k'k''a''n''a'n').

We set some further notations. For a primitive \(v \in \mathbb{Z}^n\), let \(\Lambda_v := \Lambda_{\gamma_v}\) (the lattice spanned by the \(n-1\) first columns of \(\gamma_v\)). This coincides with the notation \(\Lambda_v = \mathbb{Z}^n \cap v^1\) defined in the Introduction, according to Lemma 3.1. We also let \(\Lambda_j := \Lambda_{\gamma_v}\) denote the sub-lattice spanned by the first \(j\) columns of \(\gamma_v\). Finally for every \(\Sigma = (S_1, \ldots, S_{n-2}) > 0\) let
\[
\hat{F}_{n-1}(\Sigma) := \hat{F}_{n-1} \cap \left\{ g'' \in G'' : \pi_{\Lambda''} (g) \leq S_i \right\}
\]

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The integral matrices inside the sets \( \Omega_S^{\underline{\Phi}}(\mathcal{D}_0, \Phi') \) are characterized as follows.

**Corollary 4.6.** Consider the correspondence \( v \leftrightarrow \gamma_v \) associated with a fundamental domain \( \Omega \) of \( Q(\mathbb{Z}) \), i.e. \( \gamma_v = \gamma_v(\Omega) = (G_v(\mathbb{Z})) \cap \Omega \). Let \( \Phi' \subseteq S^{n-1} \), \( \mathcal{D}_0 \subseteq \mathcal{D} \) and \( T > 0 \). Denote

\[
\zeta_v := z_{\gamma_v} \quad \text{(see 1.4 for the notation of } \zeta_g) \text{, and}
\]

\[
\sigma_{top} := 2 \left( \frac{1}{n-1}, \ldots, \frac{n-2}{n-1} \right).
\]

1. For \( \mathcal{E} \subseteq X_{n-1} \cong F_{n-1} \),

\[
\Omega_T(\mathcal{Q}_0, \Phi') \cap K P_{\mathcal{E}}' A' N' \cap SL_n(\mathbb{Z}) = \left\{ \gamma_v : \|v\| \leq e^T, \text{shape}(\Lambda_v) \in \mathcal{E}, \hat{\nu} \in \Phi', \left[ w^{v+} \right]_B \in \mathcal{D}_0(z_v) \right\},
\]

where \( \gamma_v = (B | w) \) and \( B \) is an SR basis for the lattice \( \Lambda_v \). In particular, for \( \mathcal{E} = F_{n-1}^{(S)} \) where \( S = (S_1, \ldots, S_{n-2}) \) such that \( S_i > 0 \) for every \( i \):

\[
\Omega_T^{\underline{S}}(\mathcal{Q}_0, \Phi') \cap SL_n(\mathbb{Z}) = \left\{ \gamma_v : \|v\| \leq e^T, \frac{\text{covol}(\Lambda_v)}{\|v\|^{(n-1)/2}} \geq e^{-S_i/2}, \hat{\nu} \in \Phi', \left[ w^{v+} \right]_B \in \mathcal{D}_0(z_v) \right\}.
\]

2. For \( S \subseteq L_{n-1,n} \cong K'G_{n-1}^{\mathcal{F}} \),

\[
\Omega_T(\mathcal{Q}_0, K') \cap (K'G_n')_S A' N' \cap SL_n(\mathbb{Z}) = \left\{ \gamma_v : \|v\| \leq e^T, [\Lambda_v] \in S, \left[ w^{v+} \right]_B \in \mathcal{D}_0(z_v) \right\}
\]

and the case of \( S = K'F_{n-1}^{\mathcal{F}} \) is analogous to part 1.
3. For $S = \sigma^{\top} T$, the domain $\Omega_T^{\top} \cap SL_n(\mathbb{Z})_n$ captures all the primitive vectors up to norm $e^T$, namely

$$\Omega_T^{\top} (\mathcal{B}_0, \Phi') \cap SL_n(\mathbb{Z}) = \\{ \gamma_v : \|v\| \leq e^T, \hat{v} \in \Phi', \exists \{ u^{n+1} \} \in D_0 (z_v) \}. $$

**Proof.** The corollary follows from Proposition 4.3 (Equation 4.1). Parts 1, 2 are a direct consequence, and part 3 follows by observing that $\Lambda_i^T < \mathbb{Z}^n$, and therefore $\text{covol}(\Lambda_i^T) \geq 1$ for every $i = 1, \ldots, n - 2$. Then, according to Equation 4.1

$$1 \leq \text{covol}(\Lambda_i^T) = e^{-s_i/2\pi + \frac{1}{n}}$$

which implies that

$$s_i \leq \frac{2it}{n-1} \leq \frac{2iT}{n-1}.$$

**Remark 4.7.** The upper bound $S = \sigma^{\top} T$ in part 3 cannot be lowered in any of the coordinates of $S$, as shown in the following Example:

**Example 4.8.** Consider the following family of $n - 1$ dimensional subgroups of $\mathbb{Z}^n$:

$$\Lambda (m) = \text{span}_\mathbb{Z} (e_1, \ldots, e_{n-2}, u (m))$$

where

$$u (m) = (0, \ldots, 0, 1, m) = e_{n-1} + me_n.$$

Then

$$\text{covol}(\Lambda (m)) = ||e_1 \wedge \cdots \wedge e_{n-2} \wedge u (m)|| = ||(0, \ldots, 0, m, 1)|| = \sqrt{m^2 + 1},$$

and for $1 \leq i < n - 1$

$$\text{covol}(\Lambda_i (m)) = ||e_1 \wedge \cdots \wedge e_i|| = 1.$$

Thus, if $\gamma = \begin{bmatrix} e_1 \cdots e_{n-2} u (m) \end{bmatrix} \in SL_n(\mathbb{Z})$, then by Proposition 4.3 (Equation 4.1) $\gamma$ is determined by

$$e^t = \text{covol}(\Lambda (m)) = \sqrt{m^2 + 1},$$

and

$$e^{s_i} = \left( \frac{\text{covol}(\Lambda (m))}{\text{covol}(\Lambda_i (m))} \right)^{i/(n-1)} = \left( \frac{(e^t)^{i/(n-1)}}{1} \right)^2 = e^{2it/n}. $$

5 Counting and equidistribution of primitive vectors of $\mathbb{Z}^n$

The goal of this section is to prove Theorem B. We first phrase it in terms of counting primitive vectors as follows.

**Theorem 5.1.** (Counting formulation of Theorem B) Let $i (n-1)$, $\kappa_n$ be the parameters defined in Notation 5.1 and in Formula 7.3 respectively. Assume that $\Phi' \subseteq S^{n-1}$, $\mathcal{E} \subseteq \mathcal{X}_{n-1}$ and $S \subseteq \mathcal{L}_{n-1,n}$ are BCS’s. Then, for every $\epsilon > 0$:

1. **(Counting primitive codimension 1 sublattices in a subset $S \subseteq \mathcal{L}_{n-1,n}$)**

$$\# \left\{ \left. \text{primitive sublattices of } \mathbb{Z}^n \text{ in } S \right| \begin{array}{l} \text{with co-volume } \leq e^T \\ \mu_{K' \mathcal{G}^{(n)}_S} (K' \mathcal{G}^{(n)}_S) \end{array} \right\} = \# \left\{ \left. v \in \mathbb{Z}^n \text{ primitive : } ||v|| \leq e^T, [\Lambda_v] \in S \right\}$$

$$= \frac{\mu_{\mathcal{L}_{n-1,n}} (S)}{\mu (SL_n(\mathbb{R}) / SL_n(\mathbb{Z}))} \cdot \frac{e^{nt}}{n} + O_{S, \epsilon} \left( e^{nT(1-\kappa_n+\epsilon)} \right).$$

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2. (counting primitive codimension 1 sublattices with restricted direction and shape)

\[
\# \left\{ \text{primitive sublattices of } \mathbb{Z}^n \text{ with } \langle \Phi', \text{shape in } \mathcal{E}, \text{ and co-volume } \leq e^T \rangle \right\} = \# \left\{ v \in \mathbb{Z}^n \text{ primitive: } \|v\| \leq e^T, \hat{v} \in \Phi' \right\}
\]

\[
= \frac{\mu_{\mathcal{N}}(P')}{{\mu}_{\mathcal{N}}(\mathbb{Z}^n)} \cdot \frac{\mu_{\mathcal{K}}(K_{\Phi'})}{{\mu}_{\mathcal{K}}(\mathbb{Z}^n)} \cdot \frac{e^{nT}}{n} + O_{\Phi', \mathcal{E}} \left( e^{nT(1-\kappa_n + \epsilon)} \right).
\]

3. (counting primitive codimension 1 sublattices with restricted direction only)

\[
\# \left\{ \text{primitive sublattices of } \mathbb{Z}^n \text{ with } \langle \Phi', \text{ and co-volume } \leq e^T \rangle \right\} = \# \left\{ v \in \mathbb{Z}^n \text{ primitive: } \|v\| \leq e^T, \hat{v} \in \Phi' \right\}
\]

\[
= \frac{\mu_{\mathcal{N}}(\mathcal{D})}{{\mu}_{\mathcal{N}}(\mathbb{Z}^n)} \cdot \frac{\mu_{\mathcal{K}}(K_{\Phi'})}{{\mu}_{\mathcal{K}}(\mathbb{Z}^n)} \cdot \frac{e^{nT}}{n} + O_{\Phi', \mathcal{E}} \left( e^{nT(1-\kappa_n + \epsilon)} \right).
\]

where \(\Phi' \times \mathcal{D} = \text{the image of } K_{\Phi'} G_{\mathcal{E}}' \text{ in } \mathcal{L}_{n-1,n} \) (see part 3 of Proposition 2.20).

We remark that the difference in the error term in part 3 of the theorem is due to the fact that the \(A''\) component is not bounded there.

The proof of Theorem 5.1 relies on the correspondence between primitive vectors (and therefore their primitive orthogonal sub-lattices \(v^\perp \cap \mathbb{Z}^n\)) and integral matrices in a fundamental domain of the group \(Q(\mathbb{Z})\), which was discussed in Corollary 4.6. It also relies on Theorem 9.11, which is a generalization of Theorem C.

**proof of parts 1 and 2.** Part 2 is an incident of Part 1 for \(S = K_{\Phi'} G_{\mathcal{E}}' \) where \(G_{\mathcal{E}}'\) is the lift of \(P_{\mathcal{E}}'\) to \(G_{F_{n-1}}'\). Indeed,

\[
\mu_{\mathcal{L}_{n-1,n}} (K_{\Phi'} G_{\mathcal{E}}') \overset{\text{Prop. 2.20}}{=} \mu_{\mathcal{K}} P'' \left( K_{\Phi'} G_{\mathcal{E}}' \right) = \mu_{\mathcal{K}} P'' \left( K_{\Phi'} K'' P_{\mathcal{E}}' \right) \overset{\text{Fact 2.19}}{=} \frac{\mu_{\mathcal{G}}(\mathcal{E}) \cdot \mu_{\mathcal{K}}(K_{\Phi'})}{\mu_{\mathcal{L}_{n-1,n}}}.
\]

It remains to prove part 1. Set \(\Omega = K' G''_{F_{n-1}} A' N'_{D}\),

where \(D\) is some BCS fundamental domain of \(\mathbb{Z}^{n-1}\) in \(\mathbb{R}^{n-1}\), e.g., a unit cube. According to Corollary 3.4 (see also Remark 3.6), \(\Omega\) is a fundamental domain for \(Q(\mathbb{Z})\). Then, by part 2 of Corollary 4.6, the quantity we seek to estimate is

\[
\# \{ \Omega_T \cap (K' G'')_S A' N' \cap \mathbb{Z}^n \} = \# \{ (K' G'')_S A' N'_{D} \cap \mathbb{Z}^n \}.
\]

Since \((K' G'')_S\) and \(N'_{D}\) are BCS (according to Proposition 2.20 and Lemma 2.21), then by Theorem 9.11 the statement follows.

We now turn to prove Theorem B

**Proof of Theorem B based on Theorem 5.1.** The equidistribution in Theorem B is deduced from the counting in Theorem 5.1. E.g., for part 3, we consider the quotient

\[
\frac{\# \left\{ \text{primitive codimension 1 sublattices of } \mathbb{Z}^n \text{ in } S \right\}}{\# \left\{ \text{primitive codimension 1 sublattices of } \mathbb{Z}^n \right\}}.
\]
We use parts 1 and 3 (with $\Phi' = S^{n-1}$) of Theorem 5.1 to estimate the nominator and denominator (respectively), and conclude that this quotient equals

$$\mu_{\mathcal{L}_{n-1,n}}(\mathcal{S}) \frac{O_{S,n}(e^{T(\kappa_n - \epsilon)}).}{\mu_{\mathcal{L}_{n-1,n}}(\mathcal{L}_{n-1,n}) + O_{S,n}(e^{T(\kappa_n - \epsilon)})} + O_{S,n}(e^{T(\kappa_n - \epsilon)}).$$

The remaining parts of the theorem are proved similarly. The amount of lattice points appearing in the denominators is estimated via part 3 of Theorem 5.1, the nominator in part B1a is also estimated by part 3, and the nominators in B1b, B2 and B3 are estimated by part 2 of Theorem 5.1.

The rest of Section 5 is devoted to the proof of (the third part of) Theorem 5.1.

5.1 Counting primitive subgroups of $\mathbb{Z}^n$ whose shape is high up the cusp

The goal of this subsection is to prove the following:

**Corollary 5.2.** Let $\Omega$ a fundamental domain for $Q(\mathbb{Z})$ in $G$, and $\sigma = (\sigma_1, \ldots, \sigma_{n-2})$ where $0 < \sigma_i < 1$ \forall $i$. Denote $\Omega_T^{[\sigma_T, \sigma_{top} T]} := \Omega_T^{top} \setminus \Omega_T^{\sigma_T T}$. Then for every $\epsilon > 0$

$$\# \left\{ v \in \mathbb{Z}^n \text{ primitive : } \|v\| \leq e^T, e^{\frac{1}{2}T} \leq \text{covol}(\Lambda_{v}) \leq e^{\frac{1}{n+1-T}} \right\} \leq \# \left( \Omega_T^{[\sigma_T, \sigma_{top} T]} \cap \text{SL}_n(\mathbb{Z}) \right) = O_{\Phi, \epsilon}(e^{T(n - \sigma_{\min} + \epsilon)}),$$

where $\sigma_{\min} = \min(\sigma_1, \ldots, \sigma_{n-1})$.

Two auxiliary claims are requires for the proof.
Lemma 5.3. Let there be \( n \) intervals \([\alpha_i, \beta_i]\) with

\[
0 \leq \alpha_i < \beta_i \leq \frac{i}{n-1}.
\]

The number of \((n-1)\)-dimensional subgroups of \(\mathbb{Z}^n\) whose co-volume is \(\leq X\) and who satisfy that 
\(\text{covol}(\Lambda^i) \subseteq [X^{\alpha_i}, X^{\beta_i}]\) (for some Siegel reduced basis \(\{v_1, \ldots, v_{n-1}\}\) such that \(\Lambda^i := \text{span}_\mathbb{Z}\{v_1, \ldots, v_i\}\)) is \(O\left( X^{e(\alpha, \beta)} \right) \), where

\[
e(\alpha, \beta) = 2 + 2 \sum_{i=1}^{n-2} \beta_i + \sum_{i=1}^{n-2} (n-i-1) (\beta_i - \alpha_i).
\]

Proof. Let \(\Lambda < \mathbb{Z}^{n-1}\) be an \((n-1)\)-dimensional subgroup, and write \(\Lambda = \text{span}_\mathbb{Z}\{v_1, \ldots, v_{n-1}\}\) where \(\{v_1, \ldots, v_{n-1}\}\) is a SR basis for \(\Lambda\). We use the notations introduced at the beginning of Section 2.2 assume that \(\{\phi_1, \ldots, \phi_{n-1}\}\) is the Gram-Schmidt basis obtained from \(\{v_1, \ldots, v_{n-1}\}\), and let \(a_i\) denote the projection of \(v_i\) on the line orthogonal to span \(\{v_1, \ldots, v_{i-1}\}\) inside the space span \(\{v_1, \ldots, v_i\}\), where span \(\{0\}\) is set to be the trivial subspace \(\{0\}\). In other words, \(a_i\) is the distance of \(v_i\) from the subspace span \(\{v_1, \ldots, v_{i-1}\}\) (Figure 3a). If \(\Lambda\) is such that \(\text{covol}(\Lambda^i) \subseteq [X^{\alpha_i}, X^{\beta_i}]\), then \(a_i \leq R_i = X^{\beta_i - \alpha_i - 1}\). Denote the number of possibilities for choosing \(v_i\) given that \(\Lambda^{i-1}\) is known by \(\#v_i|_{\Lambda^{i-1}}\).

We first claim that for every \(1 \leq i \leq n-1\)

\[
\#v_i|_{\Lambda^{i-1}} = O\left( (R_i)^{n-i+1} \cdot \text{covol}(\Lambda^{i-1}) \right).
\]

Indeed, for \(i = 1\), the number \(\#v_1|_{\Lambda^{0}}\) is simply the number of possibilities for choosing an integral vector \(v_1\) inside a ball of radius \(a_1 = \|v_1\|\) in \(\mathbb{R}^n\), and therefore

\[
\#v_1|_{\Lambda^{0}} = \# |\mathbb{Z}^n \cap B_{R_1}| = O\left( R_1^n \right).
\]

For \(i > 1\), the orthogonal projection of \(v_i\) to the subspace span \(\{v_1, \ldots, v_{i-1}\}\) must lie inside a Dirichlet domain of the lattice \(\Lambda^{i-1} := \text{span}_\mathbb{Z}\{\phi_1, \ldots, a_{i-1}\phi_{i-1}\}\). Thus, \(v_i\) has to be chosen from the set of integral points which are of distance \(\leq a_i \leq R_i\) from the Dirichlet domain for \(\Lambda^{i-1}\) in \(\text{span}_\mathbb{Z}(\Lambda^{i-1})\). These are the integral points that lie in a domain which is the product of the Dirichlet domain for \(\Lambda^{i-1}\) (in \(\text{span}_\mathbb{Z}\{\phi_1, \ldots, \phi_{i-1}\}\)) with a ball of radius \(R_i\) in the \(n - (i-1)\) dimensional subspace span \(\{\phi_i, \ldots, \phi_{n}\}\) (Figure 3b). Denote this ball by \(B_{R_i}^{n-(i-1)}\), and then

\[
\#v_i|_{\Lambda^{i-1}} \leq \# |\mathbb{Z}^n \cap \left\{B_{R_i}^{n-(i-1)} \times \text{Dirichlet domain for } \Lambda^{i-1}\right\}|
\]

\[
= O\left( \text{vol}\left( B_{R_i}^{n-(i-1)} \right) \cdot \text{covol}\left( \Lambda^{i-1} \right) \right)
\]

\[
= O\left( (R_i)^{n-i+1} \cdot \text{covol}\left( \Lambda^{i-1} \right) \right).
\]

This establishes Equation 5.1. Now, the number of possibilities for \(\Lambda\) is given by:

\[
\#(\Lambda) = \prod_{i=1}^{n-1} \#v_i|_{\Lambda^{i-1}}
\]

\[
\leq \prod_{i=1}^{n-1} \left( (R_i)^{n-i+1} \cdot \text{covol}\left( \Lambda^{i-1} \right) \right)
\]

\[
= \prod_{i=1}^{n-1} \left( (X^{\beta_i - \alpha_{i-1}})^{n-i+1} \cdot X^{\beta_{i-1}} \right)
\]
where \( \alpha_0 = 0 \) and \( \beta_{n-1} = 1 \) (as \( \text{covol} (\Lambda^1) = \|v_1\| \geq X^0 \)), and \( \text{covol} (\Lambda^{i-1}) = \text{covol} (\Lambda) \leq X^1 \). Since

\[
\sum_{i=1}^{n-1} (n-i+1)(\beta_i - \alpha_{i-1}) + \beta_{i-1} = 2 + \sum_{i=1}^{n-2} (n-i)(\beta_i - \alpha_i) + 2 \sum_{i=1}^{n-2} \beta_i
\]

then

\[
\# (\Lambda) \leq \prod_{i=1}^{n-1} X^{(n-i+1)\beta_i - (n-i)\alpha_{i-1}} = X^{2+\sum_{i=1}^{n-2} (n-i)(\beta_i - \alpha_i) + 2 \sum_{i=1}^{n-2} \beta_i}.
\]

\[\square\]

**Proposition 5.4.** Let \( \omega_1, \ldots, \omega_{n-2} \) such that \( 0 < \omega_i < \frac{1}{n-1} \) for all \( i = 1, \ldots, n-2 \). For every \( \epsilon > 0 \), the number of \( n-1 \) dimensional subgroups of \( \mathbb{Z}^n \) with covolume \( \leq X \) who satisfy \( \text{covol} (\Lambda_i) \in [1, X^{\omega_i}] \) is \( O_{\epsilon} \left( X^{2+2\omega + \left( \frac{(n-2)(n+1)}{2} \right) \epsilon} \right) \), where \( \omega := \sum_{i=1}^{n-2} \omega_i \).

**Proof.** Divide every interval \([0, \omega_i]\) into \( N_i = N_i (\omega_i) \) sub-intervals

\[0 = \beta_0^i < \beta_1^i < \ldots < \beta_{N_i}^i = \omega_i\]

such that \( |\beta_j^i - \beta_{j-1}^i| \leq \epsilon \) for every \( j = 1, \ldots, N_i \). By refining these partitions, we may assume without loss of generality that \( N_1 = \ldots = N_{n-2} := N \). Fix \( j \in \{1, \ldots, N\} \); according to Lemma \[5.3\] the number of \( (n-1) \)-dimensional subgroups \( \Lambda \) of \( \mathbb{Z}^n \) with \( \text{covol} (\Lambda) \leq X \) and \( \text{covol} (\Lambda_i) \in [X^{\beta_{i-1}^j}, X^{\beta_j^i}] \) for every \( i = 1 \ldots n-2 \) is of order \( X \) to the power of

\[
2 + 2 \sum_{i=1}^{n-2} \frac{\beta_j^i + \sum_{i=1}^{n-2} (n-i)(\beta_j^i - \beta_{j-1}^i)}{\omega_i} \leq 2 + 2 \sum_{i=1}^{n-2} \omega_i + \sum_{i=1}^{n-2} \omega_i \cdot (n-i) \cdot \epsilon = 2 + 2 \omega + \epsilon \cdot \frac{(n-2)(n+1)}{2},
\]

where we have used \( |\beta_j^i - \beta_{j-1}^i| \leq \epsilon \) and \( \beta_j^i \leq \omega_i \).

Let \( \Lambda \subset \mathbb{Z}^n \) as in the statement. Since \( \text{covol} (\Lambda_i) \) lies in \([X^0, X^{\omega_i}]\) for every \( i = 1, \ldots, n-2 \), then for every \( i \) there exist \( j_1^i, \ldots, j_{n-2}^i \) such that

\[\text{covol} (\Lambda_i) \in [X^{\beta_{j-1}^i}, X^{\beta_j^i}]\,.
\]

It follows that

\[
\# \Lambda = O \left( \sum_{\{j_1, \ldots, j_{n-2} \} \subset \{1, \ldots, N\}} X^{2+2\omega + \epsilon \left( \frac{(n-2)(n+1)}{2} \right)} \right) = O_{\epsilon} \left( X^{2+2\omega + \epsilon \left( \frac{(n-2)(n+1)}{2} \right)} \right).
\]

\[\square\]

**Proof of Corollary 5.2.** Let

\[
\gamma = \begin{bmatrix} v_1 & \cdots & v_{n-1} & w \end{bmatrix} = k a_t' a_t'n.
\]
According to Proposition 4.3 (Equation 4.1)

\[ \text{covol} \left( \Lambda_i^\prime \right) = e^{\frac{n}{n-1}-\frac{\omega}{2}}. \]

If \( \gamma \in \Omega_T^{[\sigma T, \sigma^{top} T]} \) then \( t \in [0, T] \) and \( s_i \in \left[ \sigma_i T, \frac{2}{n-1} T \right] \) for some \( i \), hence the last equation implies

\[ \text{covol} \left( \Lambda_i^\prime \right) \leq e^{\left( \frac{n}{n-1} - \frac{\omega}{2} \right) T}. \]

By assuming further that \( \gamma \in \text{SL}_n(\mathbb{Z}) \) we have that \( v_1, \ldots, v_{n-1} \in \mathbb{Z}^n \), and in particular

\[ 1 \leq \text{covol} \left( \Lambda_i^\prime \right) \leq e^{\left( \frac{n}{n-1} - \frac{\omega}{2} \right) T}. \]

Thus, the number of \( \text{SL}_n(\mathbb{Z}) \)-elements \( \gamma \in \Omega_T^{[\sigma T, \sigma^{top} T]} \) is bounded by the number of \( (n-1) \)-dimensional subgroups \( \Lambda_c \) of \( \mathbb{Z}^n \) of co-volume \( \leq e^T := X \), for which there exists \( i \in \{1, \ldots, n-2\} \) such that

\[ \text{covol} \left( \Lambda_i^\prime \right) \in \left[ 1, X^{\frac{n}{n-1} - \frac{\omega}{2}} \right]. \]

In other words,

\[
\begin{align*}
\# \left[ \Omega_T^{[\sigma T, \sigma^{top} T]} \cap \text{SL}_n(\mathbb{Z}) \right] &= \# \left[ \bigcup_{\nu=(u_1, \ldots, u_{n-2}) \in \{0,1\}^{n-2} - \{0\}} \{ \Lambda_c : \forall i, \text{covol} \left( \Lambda_i^\prime \right) \in \left[ 1, X^{\frac{n}{n-1} - \frac{\omega}{2}} \right] \} \right] \\
&= \sum_{\nu \in \{0,1\}^{n-2} - \{0\}} O \left( X^{2+\frac{\omega}{n} - \sum_{i=1}^{n-2} \sigma_i u_i} \right) \end{align*}
\]

which by Proposition 5.4 with \( e^T = X \) and \( \frac{\omega}{n-1} - \frac{\sigma_{\min}}{2} = \omega_i \) equals to

\[ \sum_{\nu \in \{0,1\}^{n-2} - \{0\}} O \left( X^{2+\frac{\omega}{n} - \sum_{i=1}^{n-2} \sigma_i u_i} \right) = O \left( X^{n-\sigma_{\min}+\epsilon} \right) \]

where \( \sigma_{\min} = \min \{ \sigma_i \} \).

\[ \square \]

### 5.2 Counting primitive vectors in \( \mathbb{Z}^n \): proof of Theorem 5.1

For the proof of (the second part of) Theorem 5.1 we introduce the following notation: for every \( \mathcal{S}_2 \geq \mathcal{S}_1 > 0 \)

\[ F_{n-1}^{(\mathcal{S}_1, \mathcal{S}_2)} := F_{n-1}^{(\mathcal{S}_1)} - F_{n-1}^{(\mathcal{S}_2)}. \]

In the proof we will use the following Proposition which is a consequence of Proposition 6.3 and Lemma 10.2.

**Proposition 5.5.** Let \( \epsilon > 0, \delta \in [0, \kappa_n - \epsilon) \), \( \Psi' \subseteq N' \) and \( \Phi \subseteq K \) BCS’s. For \( T \geq 0 \) and \( \mathcal{S}(T) = (S_1(T), \ldots, S_{n-2}(T)) \) such that \( \sum S_i(T) \leq n \delta \lambda_n T \) for \( \lambda_n = \frac{n^2}{(n^2 - 1)} \).

\[ \# \left( \Phi P_{\mathcal{S}(T)}^{n-1} A_T' \Psi' \cap \text{SL}_n(\mathbb{Z}) \right) = \frac{\mu \left( \Phi P_{\mathcal{S}(T)}^{n-1} A_T' \Psi' \right)}{\mu (\text{SL}_n(\mathbb{Z}))} + O_{\Psi', \Phi, \epsilon} \left( \mu \left( \Phi P_{\mathcal{S}(T)}^{n-1} A_T' \Psi' \right)^{(1-\kappa_n+\delta+\epsilon)} \right). \]
Proof of part 3 of Theorem 5.1. For a fundamental domain $\Omega$ of $Q(\mathbb{Z})$ and the obtained correspondence $\nu \leftrightarrow \gamma_3(\Omega) = \gamma_3$ (Proposition 3.2) between primitive vectors and integral points in $\Omega$, it holds according to statement 3 in Corollary 4.6 that

$$\# \{ v \in \mathbb{Z}^n \text{ primitive : } \| v \| \leq \epsilon^T, \hat{v} \in \Phi' \} = \# \{ \gamma_3 : \| \gamma_3 \| \leq \epsilon^T, \hat{\gamma}_3 \in \Phi' \}$$

$$= \# \left| \bigcap_{\nu} \Omega^T_{\nu} \cap \text{SL}_n(\mathbb{Z}) \right| .$$

The above holds for any fundamental domain $\Omega$ of $Q(\mathbb{Z})$; for convenience, we set

$$\Omega = K' \sigma'_{F_{n-1}} A' N'_D,$$

used in the proof of parts 1 and 2 of Theorem 5.1. Then

$$\Omega^T_{\nu} = K' \sigma'_{F_{n-1}} A' N'_D.$$  

1. Reduction to counting in simpler sets, with $F_{n-1}$ instead of $\sigma$. Let $K'$ be a BCS fundamental domain (see Lemma 2.16) of $Z(K')$ in $K''$. Recall that

$$K''/\text{SL}_n(\mathbb{Z}) = \begin{cases} K'' & \text{if } n \text{ is even} \\ K''/\{\pm \text{id}\} & \text{if } n \text{ is odd} \end{cases}.$$  

We claim that the problem of counting $\text{SL}_n(\mathbb{Z})$ points in $\Omega^T_{\nu} (\Phi')$ can be reduced to counting $\text{SL}_n(\mathbb{Z})$ points in

$$\Delta^T_{\nu} := K' F_{n-1} G''_{F_{n-1}} A' N'_D,$$

in which the component $G''_{F_{n-1}}$ was replaced by $\tilde{K''} P''_{F_{n-1}}$, and which has the same measure as $\Omega^T_{\nu} (\Phi')$. To see this, notice that

$$\Delta^T_{\nu} \setminus \Omega^T_{\nu} = K' \left( K'' F_{n-1} \setminus G''_{F_{n-1}} \right) A' N'_D$$

(Prop. 2.14)

Since $K''$ is a BCS fundamental domain, the number of $(n-1)$-dimensional lattice points in $\mathbb{R}^n$ with non-trivial point group and of covolume at most $\epsilon^T$ is bounded by $O\left(e^{(n-1)\epsilon^T}\right)$. Therefore,

$$\# \left( \left( \Delta^T_{\nu} \setminus \Omega^T_{\nu} \right) \cap \text{SL}_n(\mathbb{Z}) \right) \leq O\left(e^{(n-1)\epsilon^T}\right).$$

2. Counting in the sets $\Delta^T_{\nu} (K'')$ by splitting $F_{n-1}$ into two subsets. Denote $g_n := \frac{n \lambda_n}{n^2 \lambda_n^2} \cdot 1$, where $\lambda_n = \frac{n^2}{2 \pi n^2}$. Note that the sum of the coordinates of $g_n$ is $n \lambda_n$. Let $\epsilon > 0$ and $\delta \in [0, \kappa_n - \epsilon]$, and write

$$\Delta^T_{\nu} (\Phi') = K' K'' \left( P''_{F_{n-1}} \cup P''_{F_{n-1}, \sigma'_{F_{n-1}}} \right) A' N'_D$$

$$= K' K'' P''_{F_{n-1}} A' N'_D \bigcup K' K'' P''_{F_{n-1}} A' N'_D.$$
Here $\Delta_T^{\text{top}, T}(\Phi')$ is presented as the union of two sets, and we count $\text{SL}_n(\mathbb{Z})$ points in each of these sets separately.

We begin with the first summand. By Proposition 5.3 applied for $\Phi = K\Phi, K''$ (which is BCS by Lemma 2.22) and $\Psi' = N_D$

$$\# \left| K\Phi, K'' P'_{(n-1)} A_T N_D \cap \text{SL}_n(\mathbb{Z}) \right| = \frac{\mu_K, (K\Phi')} \mu \left( F(\sigma T) \right) \cdot e^{nT} + O_D, \phi' \left( e^{nT(1-\kappa_n + \delta + \epsilon)} \right).$$

As for the second summand: by Corollary 5.2 and step 1 of the proof,

$$\# \left| K\Phi, K'' P''_{(n-1)} A_T N_D \cap \text{SL}_n(\mathbb{Z}) \right| = \# \left| \left( \Omega_T^{\text{top}, T} \setminus \Omega_T^{\sigma T} \right) \cap \text{SL}_n(\mathbb{Z}) \right| + O \left( e^{(n-\frac{1}{nT})T} \right) = O_D, \epsilon \left( e^{nT(1-\delta(\mathbb{Z})_{\min} + \epsilon)} \right).$$

where $\delta(\mathbb{Z})_{\min} = \delta \cdot \frac{n^2}{n-2} = \frac{n^3}{2(n-2)(n-1)}$ is smaller than $\frac{1}{n-1}$, since $\kappa_n \leq \frac{1}{2n}$. Thus the above quantity is in $O_D, \epsilon \left( e^{nT(1-\delta(\mathbb{Z})_{\min} + \epsilon)} \right)$ in which case one obtains an error term of order $e^{nT(1-\frac{1}{2n^3-n^2-2n+4})}$. 

Finally, $1 - \kappa_n + \delta = 1 - \frac{n^2}{2n^3-4n^2-2n+4}$ if and only if $\delta = \kappa_n \cdot \left( 1 - \frac{n^2}{2n^3-3n^2-2n+4} \right)$; in which case one obtains an error term of order $e^{nT(1-\frac{n^2\kappa_n}{2n^3-3n^2-2n+4})}$. 

\[\Box\]

6 Equidistribution of the shortest solution to the gcd equation

Recall that for every primitive $v \in \mathbb{Z}^n$, we refer to the equation

$$\langle v, w \rangle = 1$$

as the \textit{gcd equation} of $v$. The goal of this Section is to prove Theorem A up to a technical proposition 6.3 which will be proved in Section 6. This theorem concerns the equidistribution of the (normalized) lengths of the shortest solutions to the gcd equations of primitive vectors in $\mathbb{Z}^n$, as their norm grows to $\infty$.

The plan of this section is as follows:

1. In Section 6.1 we define a fundamental domains $\Omega_{\text{short}}$ for $Q(\mathbb{Z})$ in which every integral point corresponds to a shortest solution for the gcd equation associated with a primitive vector. This domain is of the form $\Omega(D)$ (Corollary 5.4) where $D$ is a family Dirichlet domains of $n - 1$ dimensional lattices.

2. In Section 6.2 we define subsets of $\Omega_{\text{short}}$ which restrict the lengths of the normalized shortest solutions to sub-intervals of $[0, 1]$.

3. In Section 6.3 we state a counting result (Theorem 6.3) for the aforementioned subsets of $\Omega_{\text{short}},$ from which we deduce Theorem A.

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6.1 A fundamental domain for $Q(\mathbb{Z})$ that captures the shortest solutions

We first introduce the following notation.

Notation 6.1. If $g = [B \mid w]$ is in $G_v$ (Formula 3.1), we write that $g = (v, w, B)$. Notice that although $g$ is already determined by $w$ and $B$, we include $v$ for convenience.

In this section, we define a fundamental domain $\Omega(\mathcal{D})$ for $Q(\mathbb{Z})$ (see Corollary 3.4) in which the family $\mathcal{D}$ consists of the Dirichlet domains of the $(n-1)$ dimensional lattices spanned by the first $n-1$ columns of $SL_n(\mathbb{R})$ matrices, and more accurately, the images of these Dirichlet domains under linear maps $L$ that map the hyperplanes spanned by these lattices to $\mathbb{R}^{n-1}$. As a result, every $g = (v, w, B)$ in this fundamental domain will be the representative in the coset $g \cdot Q(\mathbb{Z})$ for which its upper row $w$ is the shortest possible.

For $g = (v, w, B)$ in $SL_n(\mathbb{R})$, recall the notations

$$\Lambda_g = \text{span}_\mathbb{Z}(B) = \text{lattice spanned by first } n-1 \text{ columns of } g$$

and consider the linear map

$$L_g : v^\perp \to \mathbb{R}^{n-1}$$

If $B = \{v_1, \ldots, v_{n-1}\}$, then $L_g : \{v_1, \ldots, v_{n-1}\} \to \{e_1, \ldots, e_{n-1}\}$; note that the map $L_g$ is a composition of two invertible linear maps

$$(\text{first } n-1 \text{ columns of } g) \sqsupset \text{rotation} \sqsupset \text{projection} \sqsupset \text{rescaling} \quad \text{columns of } z_g \quad \text{columns of } I_{n-1}$$

The first (rotation and normalization) part of $L_g$ forgets the information regarding the covolume and direction of $\Lambda_g$, and preserves only the shape of $\Lambda_g$. According to Proposition 4.3, this is also the shape of $\Lambda_{z_g}$. In particular, a Dirichlet domain for $\Lambda_g$ in $v^\perp$ maps under the first part of $L_g$ into a Dirichlet domain for $\Lambda_{z_g}$ in $\mathbb{R}^{n-1}$, which maps under the second part $L_{z_g}$ into a fundamental (non Dirichlet) domain of $\mathbb{Z}^{n-1}$ in $\mathbb{R}^{n-1}$.

We let

$$Y(z) := L_z(\text{Dir}(\Lambda_z)) \quad \text{if } z = z_g \quad L_g(\text{Dir}(\Lambda_g))$$

and

$$\mathcal{Y}_{F_{n-1}} = \{Y(z)\}_{z \in F_{n-1}}.$$

As explained, every $Y(z)$ is a fundamental domain for $\mathbb{Z}^{n-1}$ in $\mathbb{R}^{n-1}$, hence by Corollary 3.4 and Notation 2.13 for $K_z^{\mathcal{D}}$

$$\Omega_{\text{short}} := \Omega(\mathcal{Y})$$

$$= \bigcup_{g'' \in F_{n-1}} K' \cdot \tau_{g''(z)} \cdot A'N_y''(z)$$

$$= \bigcup_{z \in F_{n-1}} K' \cdot K'' \cdot p_z' \cdot A'N_y'(z)$$

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Thus, based on Proposition 4.3 we may conclude this discussion with

\[ g = (v, w, B) \in \Omega(\mathcal{Y}) \implies w^\perp \in L_{Y}^{-1}(Y(z_g)) \iff w^\perp \in \text{Dir}(\Lambda_g), \]

namely \( w^\perp \) (such that \( w = \frac{1}{\|v\|^2} + w^\perp \), see Lemma 4.4) is the shortest representative of the coset \( w^\perp + \Lambda_g \) in the hyperplane \( v^\perp \). This means that \( w \) is the shortest representative of the coset \( w + \Lambda_g \) (which is in the hyperplane \( \{ u : \langle u, v \rangle = 1 \} \)). As a result, for every primitive vector \( v \), the representative \( \gamma_v = G_v(\mathbb{Z}) \cap \Omega_{\text{short}} \), has upper row \( w \) which is

\[ w_v = \text{the shortest integral } w \text{ which satisfies } \langle w, v \rangle = 1. \]

### 6.2 Subsets of \( \Omega_{\text{short}} \) that restrict the norm of \( w \)

Consider the covering radius of a lattice:

\[ \rho(\Lambda_g) = \text{radius of bounding circle for } \text{Dir}(\Lambda_g). \]

Clearly, the norm \( \|w^\perp\| \) lies in the interval \([0, \rho(\Lambda_g)]\), i.e.

\[ \frac{\|w^\perp\|}{\rho(\Lambda_g)} \in [0, 1]. \]

We consider sub-families \( \mathcal{Y}_0 \subseteq \mathcal{Y} \) for which \( w^\perp \) (such that \( g = (v, w, B) \) is in \( \Omega_{\text{short}} \)) lies in an origin-centered \( n-1 \) dimensional ball in \( v^\perp \). Let \( B_r \) denote such a ball with radius \( r \). For \( \alpha \in [0, 1] \), let

\[ Y \alpha (z) := L_{z} (B_{\alpha \rho(\Lambda_g)} \cap \text{Dir}(\Lambda_g)) \text{ if } z = z_g, \]

\[ \mathcal{Y}_0^{\alpha} := \{ Y \alpha (z) \}_{z \in F_{n-1}}. \]

Recall notation 4.3 for \( \Omega_f^S(\mathcal{Y}_0, \Phi') \subseteq \Omega \) where \( \Omega \) is a fundamental domain for \( Q(\mathbb{Z}) \), and consider

\[ (\Omega_{\text{short}})^S_T(\mathcal{Y}_0^{\alpha}, \Phi') = (\Omega_{\text{short}}) \cap \left\{ g: \pi_{A'}(g) \in [0, T], \pi_{A''}(g) \leq S_t, \pi_{K'}(g) \in \Phi', \pi_{N'}(g) \in Y \alpha (z_g) \right\} \]

\[ = \bigcup_{z \in F_{n-1}^{(2)}} \left\{ g: g'' \in F_{n-1}^{(2)} \right\} \]

\[ = \bigcup_{z \in F_{n-1}^{(2)}} \left\{ g: g'' \in F_{n-1}^{(2)} \right\} \]

\[ \subseteq \bigcup_{z \in F_{n-1}^{(2)}} \left\{ g: g'' \in F_{n-1}^{(2)} \right\} \]

Thus, based on Proposition 4.3 we may conclude this discussion with

\[ (\Omega_{\text{short}})^S_T(\mathcal{Y}_0^{\alpha}, \Phi') = \left\{ g = (v, w, B): \right. \]

\[ \left. \begin{array}{l}
\|v\| \leq e^{T}, \hat{v} \in \Phi', \\
B \text{ an SR basis for } \Lambda_g, \\
\frac{\|v\|}{\rho(\Lambda_g)} \in [0, \alpha] \end{array} \right\}. \]

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6.3 Counting shortest solutions

For a primitive vector \( v \), we use the notations (introduced in Section 1)
\[
\begin{align*}
\Lambda_v & := \Lambda_{\gamma_v}, \\
\rho_v & := \rho (\Lambda_v).
\end{align*}
\] (6.3)

Corollary 4.6 established the correspondence between integral matrices in \( \Omega_T^0 (\mathcal{D}_0, \Phi') \), and primitive vectors with norm less than \( \epsilon^T \), direction in \( \Phi' \) and last column that is determined by the family \( \mathcal{D}_0 \).

For the case of \( \Omega = \Omega_{\text{short}} \), we obtain the following:

**Corollary 6.2.** Consider the correspondence \( v \mapsto \gamma_v \) where \( \gamma_v = (G_v (\mathbb{Z})) \cap \Omega_{\text{short}} \). For \( T > 0 \), \( \Phi' \subseteq \mathbb{S}^{n-1} \) and \( \alpha \in [0,1] \):
\[
(\Omega_{\text{short}})_T \cap \Phi' \subseteq \mathbb{S}^{n-1} \mathcal{D}_0 \{ v : v \in \mathbb{Z}^n \text{ primitive} , \|v\| \leq \epsilon^T , \hat{v} \in \Phi', \|v_{\hat{v}}\| \in [0,\alpha] \}
\]
which is in correspondence \( v \mapsto \gamma_v \) with the elements of
\[
\left\{ v \in \mathbb{Z}^n \text{ primitive} : \|v\| \leq \epsilon^T , \hat{v} \in \Phi', \|v_{\hat{v}}\| \in [0,\alpha] \right\}.
\]

**Proof.** Immediate from Corollary 4.6 and Equation 6.2 \( \square \)

**Theorem 6.3.** For \( T > 0 \), \( \Phi' \subseteq \mathbb{S}^{n-1} \) BCS and \( \alpha \in [0,1] \):
\[
\# \left\{ v \in \mathbb{Z}^n \text{ primitive} : \|v\| \leq \epsilon^T , \hat{v} \in \Phi', \|v_{\hat{v}}\| \in [0,\alpha] \right\} = \frac{1}{\iota (n-1)} \cdot \mu (\mathbb{S}^n (\mathbb{R}) / \mathbb{S}^n (\mathbb{Z})) \cdot \frac{e^{nT}}{n} + O_{\epsilon} \left( e^{nT} \left( \frac{1}{2n} \right) \right)
\]
where \( \iota (n-1) \in \{1,2\} \) is as defined in Notation 2.9.

**Remark 6.4.** It is easy to check that the main term above is equal to
\[
\frac{\mu (\Omega_{\text{short}})_T \cap \Phi' \subseteq \mathbb{S}^{n-1} \mathcal{D}_0 \{ v : v \in \mathbb{Z}^n \text{ primitive} , \|v\| \leq \epsilon^T , \hat{v} \in \Phi', \|v_{\hat{v}}\| \in [0,\alpha] \}}{\iota (n-1)} \cdot \mu (\mathbb{S}^n (\mathbb{R}) / \mathbb{S}^n (\mathbb{Z})) \cdot e^{nT}.
\]

The proof requires the following proposition, which is a direct consequence of Propositions 11.1 and 11.16

**Proposition 6.5.** Let \( \mathcal{Y}_{F_{n-1}} = \{ Y_{\alpha r} (z) \}_{z \in F_{n-1}} \) as defined in Formula 6.1. \( S = (S_1, \ldots, S_{n-2}) \geq 1 \) and \( \Phi \subseteq K \) a BCS. For
\[
\Delta^S_{\Phi} (\mathcal{Y}_{F_{n-1}}, \Phi) := \bigcup_{z \in F_{n-1}(\mathbb{Z})} K_{\Phi'} \cdot \mu^\prime \cdot A_T N'_{\alpha^r (z)},
\]
\( \epsilon > 0, \delta \in [0, \kappa_n - \epsilon], T \geq 0 \) and \( S \) such that \( \sum_{i} S_i \leq \frac{n^3 \delta}{2(\kappa_n - 1)} T \),
\[
\# \left( \left( \Delta^S_{\Phi} (\mathcal{Y}_{F_{n-1}}, \Phi) \right) \cap \mathbb{S}^n (\mathbb{Z}) \right) = \mu \left( \Delta^S_{\Phi} (\mathcal{Y}_{F_{n-1}}, \Phi) \right) + O_{\Phi, \epsilon, \delta} \left( \mu \left( \Delta^S_{\Phi} (\mathcal{Y}_{F_{n-1}}, \Phi) \right) \right)^{(1 - \kappa_n + \delta + \epsilon)}.
\]
proof of Theorem 6.3. The proof goes along the lines of the proof of (part 3 of) Theorem 5.1, with the necessary adjustments of using Corollary 6.2 instead of Corollary 4.6, and Proposition 6.5 instead of Proposition 5.5.

Theorem A is a consequence of Theorem 6.3 as follows.

Proof. We will prove that the quotients \[ \frac{\|w_\rho v\|}{\rho_v} \] equidistribute in \([0, 1]\) w.r.t. to \(\nu_n\) as \(\|v\| \to \infty\); this is sufficient, since it is shown in \([RR09, HN16]\) that

\[ \frac{\|w_\rho v\|}{\rho_v} - \frac{\|w_\rho v\|}{\rho_v} = O \left( \frac{1}{\|v\|^2} \right). \]

For abbreviation, we write EV instead of EucVol. By definition of \(\nu_n\) and dominated convergence (DCT),

\[
\nu_n ([0, \alpha]) = \frac{\int_{F_{n-1}} \mathop{EV} (Y^{\alpha T} (z)) \, d\mu_{P''} (z)}{\mu_{P''} (F_{n-1})}
\]

(by DCT) \[
= \lim_{T \to \infty} \frac{\int_{F_{n-1}^{(\sigma_{top})T}} \mathop{EV} (Y^{\alpha T} (z)) \, d\mu_{P''} (z)}{\mu_{P''} (F_{n-1}^{(\sigma_{top})T})} \cdot \mu_{K^T} (\Phi')
\]

\[
= \lim_{T \to \infty} \frac{\mu \left( \mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \right)}{\mu \left( \mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \right)}
\]

(by Thm 6.3) \[
= \lim_{T \to \infty} \frac{\# (\mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \cap \mathop{\mathcal{SL}_n} (Z))}{\# (\mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \cap \mathop{\mathcal{SL}_n} (Z))}
\]

(by Cor 6.2) \[
= \lim_{T \to \infty} \frac{\# \left( \{ \|v\| \leq e^T, \hat{v} \in \Phi', \|w_{\rho \hat{v}}\| \rho_v \in [0, \alpha] \} \cap \mathop{\mathcal{SL}_n} (Z) \right)}{\# \left( \{ \|v\| \leq e^T, \hat{v} \in \Phi' \} \cap \mathop{\mathcal{SL}_n} (Z) \right)}
\]

The rate of convergence is obtained from Theorem 6.3 since according to this theorem

\[
\frac{\# (\mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \cap \mathop{\mathcal{SL}_n} (Z))}{\# (\mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \cap \mathop{\mathcal{SL}_n} (Z))} - \frac{\mu \left( \mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \right)}{\mu \left( \mathop{\sigma_{top}\sigma_{\alpha T}} (\mathcal{Y}^{\alpha T}, \Phi') \right)} = O \left( \frac{1}{\|v\|^2} \right).
\]
Part II
Counting lattice points

This second part is the technical part of the paper, where we prove all the counting statements that appeared so far and several more (Theorem 9.11 and Corollary 9.15). Here we extend our discussion from SL$_n$($\mathbb{R}$) to a general non-compact algebraic simple Lie group $G$, refocusing on SL$_n$($\mathbb{R}$) only in the last section.

7 Counting lattice points in well rounded families of sets inside Lie groups

Our main tool for counting lattice points in SL$_n$($\mathbb{R}$) is a method introduced in [GN12] for counting lattice points in increasing families of sets inside semisimple Lie groups. The advantages of this method is that it produces an error term, and that it allows counting in quite general families, requiring only that these families are well rounded, which is a regularity condition. The cost of this generality is that the property of well roundedness is often hard to verify. In this section we develop a machinery to somewhat simplify this process, mainly by allowing us to replace the underlying simple group $G = KAN$ with the much-easier-to-work-in Cartesian product $K \times A \times N$.

7.1 A method for lattice points counting in Well-rounded families

In this subsection we briefly describe the counting method developed in [GN12], which we refer to as the GN method. This approach, aimed at counting lattice points in increasing families of sets inside non-compact algebraic simple Lie groups, consists of two ingredients: a regularity condition on the sets involved, and a spectral estimate concerning the unitary $G$ representation $\pi_0^G : G \rightarrow L^2(G/\Gamma)$ (the orthogonal complement of the $G$ invariant $L^2$ functions). Before stating the counting theorem from [GN12], we describe the two ingredients, starting with the regularity condition.

Definition 7.1. Let $G$ be a Lie group with a Borel measure $\mu$, and let $\{O_\epsilon\}_{\epsilon > 0}$ be a family of identity neighborhoods in $G$. Assume $\{B_T\}_{T > 0} \subset G$ is a family of measurable domains and denote

$$B_T^+(\epsilon) := O_\epsilon B_T O_\epsilon = \bigcup_{u,v \in O_\epsilon} u B_T v,$$

$$B_T^-(\epsilon) := \bigcap_{u,v \in O_\epsilon} u B_T v$$

(see Figure 5). The family $\{B_T\}$ is Lipschitz well-rounded (LWR) with (positive) parameters $(C, T_0, \epsilon_0)$ if for every $0 < \epsilon < \epsilon_0$ and $T > T_0$:

$$\mu (B_T^+(\epsilon)) \leq (1 + C\epsilon) \mu (B_T^-(\epsilon)). \quad (7.1)$$

The parameter $C$ is called the Lipschitz constant of the family $\{B_T\}$.

The definition above allows any family $\{O_\epsilon\}_{\epsilon > 0}$ of identity neighborhoods; in this paper we shall work with the images under the exponent map of $\epsilon$-balls in the Lie algebra, and this choice is the topic of the next subsection.

Remark 7.2. We allow the case of a constant family $\{B_T\} = B$: we say that $B$ is a Lipschitz well rounded set (as apposed to a Lipschitz well rounded family) with parameters $(C, \epsilon_0)$ if

$$\mu (B^+(\epsilon)) \leq (1 + C\epsilon) \mu (B^- (\epsilon))$$

for every $0 < \epsilon < \epsilon_0$. 
In what follows it will often be the case that \(\epsilon_0\) is close to equal \(1/C\), hence it will be convenient to introduce the following terminology:

**Notation 7.3.** When we write that a family \(\{B_T\}\) is LWR with parameters \((C, T_0)\) it should be understood that \(C\) is an upper bound on the Lipschitz constant \(C\) and a lower bound on \(1/\epsilon_0\) (i.e. \(\epsilon_0 \leq 1/C\)).

We now turn to describe the second ingredient, which is the spectral estimation. In certain Lie groups, among which algebraic simple Lie groups \(G\), there exists \(p \in \mathbb{N}\) for which the matrix coefficients \(\langle \pi^0_{G/\Gamma}, u, v \rangle\) are in \(L^{p+\epsilon}(G)\) for every \(\epsilon > 0\), with \(u, v\) lying in a dense subspace \(L^2(G/\Gamma)\) (see [GN09, Thm 5.6]). Let \(p(\Gamma)\) be the smallest among these \(p\)’s, and denote

\[
m(\Gamma) = \begin{cases} 1 & \text{if } p = 2, \\ 2 \left[ p(\Gamma) / 4 \right] & \text{otherwise.} \end{cases}
\]

The parameter \(m(\Gamma)\) appears in the error term exponent of the counting theorem below, which is the cornerstone of the counting results in this paper.

**Theorem 7.4 ([GN12, Theorems 1.9, 4.5, and Remark 1.10]).** Let \(G\) be an algebraic simple Lie group with Haar measure \(\mu\), and let \(\Gamma < G\) be a lattice. Assume that \(\{B_T\} \subset G\) is a family of finite-measure domains which satisfy \(\mu(B_T) \to \infty\) as \(T \to \infty\). If the family \(\{B_T\}\) is Lipschitz well-rounded with parameters \((C_B, T_0)\), then for every \(\delta > 0\):

\[
\left| \# (B_T \cap \Gamma) - \frac{1}{\mu(G/\Gamma)} \mu(B_T) \right| \lesssim_{G,\Gamma,\delta} C_B^{\frac{\dim G}{1 + \dim G}} \cdot \mu(B_T)^{1 - \kappa(\Gamma) + \delta} \quad \text{as } T \to \infty,
\]

where \(\mu(G/\Gamma)\) is the measure of a fundamental domain of \(\Gamma\) in \(G\) and

\[
1 - \kappa(\Gamma) = 1 - \frac{1}{2m(\Gamma) (1 + \dim G)} \in (0, 1).
\]

The parameter \(T_1\) is such that \(T_1 \geq T_0\) and for every \(T \geq T_1\)

\[
\mu(B_T)^{\kappa(\Gamma)} \gtrsim_{G,\Gamma} C_B^{\frac{\dim G}{1 + \dim G}}. \tag{7.2}
\]

Bounds on the parameter \(p(\Gamma)\) (i.e. on \(m(\Gamma)\)) clearly imply bounds on the parameter \(\kappa(\Gamma)\) appearing in the error term exponent. We refer to [Li95], [LZ96] and [Sca90] for upper bounds on \(p(\Gamma)\) in simple Lie groups. Specifically for the group \(SL_n(\mathbb{R})\), the current known bound for \(n > 2\) and any lattice \(\Gamma\) is \(2 \leq p(\Gamma) \leq n\) [Sca90], and therefore \(1 \leq m(\Gamma) \leq n/2\) and \(1/\sqrt{n} \leq \kappa(\Gamma) \leq \frac{1}{\sqrt{n}}\). As for
there is no known upper bound which holds for any lattice, but for the lattice $\text{SL}_2(\mathbb{Z})$ it is known that $p(\text{SL}_2(\mathbb{Z})) = 2$,\footnote{We refer to [Sha00] for the connection between the decay of matrix coefficients and the eigenvalues of the Laplacian in $\text{SL}_2(\mathbb{R})/\Gamma$.} and therefore $\kappa(\text{SL}_2(\mathbb{Z})) = \frac{1}{8}$. We denote for future reference

$$\kappa_n := \kappa(\text{SL}_2(\mathbb{Z})) \in \left[ \frac{1}{n^3}, \frac{1}{2n^2} \right].$$  \hfill (7.3)

### 7.2 Coordinate balls

Well roundedness is a regularity condition that was first introduced in [EM93] (see also [GW07]) for families of subsets of symmetric spaces, and later rephrased in [GN12] in the setting of Lie groups. We use the latter Definition 7.1, which is w.r.t. a nested family $\{O_{\epsilon}\}_{\epsilon > 0}$ of identity neighborhoods in the group, where by “nested” we mean that $\epsilon_1 < \epsilon_2$ implies $O_{\epsilon_1} \subset O_{\epsilon_2}$. While the definition of well roundedness allows any nested family of identity neighborhoods, we shall work only with families that are the images of small balls in the Lie algebra under the exponent map — this is Assumption 7.11, which concludes the current subsection. The advantages of this choice follow from the fact that it is a special case of coordinate balls (Definition 7.6), and this subsection is devoted to investigating the properties of neighborhoods of this sort.

**Definition 7.5 (Equivalence of identity neighborhoods).** Let $G$ be a Lie group and consider two families $\{O_{\epsilon}\}_{\epsilon > 0}, \{O'_{\epsilon}\}_{\epsilon > 0}$ of nested and symmetric identity neighborhoods. We say that these families are equivalent if there exist $\epsilon_1, c, C > 0$ such that for every $0 < \epsilon < \epsilon_1$

$$O_{c\epsilon} \subseteq O'_{\epsilon} \subseteq O_{C\epsilon}.$$

**Definition 7.6 (Coordinate balls).** A family $\{O_{\epsilon}\}_{\epsilon > 0}$ of identity neighborhoods inside a Lie group $G$ will be called a family of coordinate balls if there exist a ball $B_{\epsilon} = \{x \in \mathbb{R}^{\dim(G)} : \|x\| < \epsilon\}$ inside $\mathbb{R}^{\dim(G)}$, and a $C^1$ chart

$$\phi : \bigcup_{1 \geq i \in \mathbb{Z}} \rightarrow \mathbb{R}^m$$

of the identity, such that $\{\phi^{-1}(B_{\epsilon})\}_{\epsilon > 0}$ is equivalent to $\{O_{\epsilon}\}_{\epsilon > 0}$.

**Remark 7.7.** All coordinate balls of a given Lie group are equivalent. Indeed, if $\phi_1$ and $\phi_2$ are two charts, then $\phi_2 \phi_1^{-1}|_{B_{\epsilon}}$ is a bi-Lipschitz map. Hence,

$$\phi_2^{-1}(B_{c\epsilon}) \subseteq \phi_2^{-1}(\phi_2 \phi_1^{-1}(B_{\epsilon})) \subseteq \phi_2^{-1}(B_{C\epsilon})$$

for some $c, C > 0$ and $\epsilon < 1$.

The following Lemma specifies two useful features of coordinate balls.

**Lemma 7.8.** Let $\{O_{\epsilon}\}_{\epsilon > 0}$ be a family of coordinate balls inside a Lie group $G$. Then for small enough $\epsilon$ and $\delta$, the following two properties hold:

- **(Connectivity)** $O_{\epsilon}$ is a connected subset of $G$.

- **(Additivity)** There exists $c > 0$ such that:

$$O_{\epsilon} O_{\delta} \subseteq O_{\epsilon(\epsilon + \delta)}.$$

2The content of the Ramanujan-Selberg $1/4$ conjecture is that $p = 2$ not only for $\text{SL}_2(\mathbb{Z})$, but also for its congruence subgroups, and the bound in Selberg’s $3/16$ Theorem (see also [LRS05]) implies $p = 4$.
Proof. Connectivity holds since \( \phi^{-1} \) (\( \phi \) being the associated chart) is continuous. Additivity holds for Riemannian left \( G \)-invariant balls with \( c = 1 \) (triangle inequality); these Riemannian balls are indeed coordinate balls, where the implied chart is the Riemannian exponential map. Since all families of coordinate balls are equivalent (Remark \( \ref{sec:coordinates} \)), the statement follows.

One last property of coordinate balls is the following.

**Proposition 7.9.** Let \( \{ \mathcal{O}_\epsilon \}_{\epsilon > 0} \) be a family of coordinate balls inside a Lie group \( G \), and assume

\[
\phi : \bigcup_{g \in U} \mathcal{O}_\epsilon \to \mathbb{R}^m
\]

is a chart that contains an element \( g \). Then, there exist an open set \( g \in V \subseteq U \) and positive \( \epsilon(g), c(g) \) such that for \( \epsilon \leq \epsilon(g) \):

\[
\mathcal{O}_\epsilon V \mathcal{O}_\epsilon \subseteq U
\]

and for every \( h \in V \)

\[
\phi(\mathcal{O}_\epsilon h \mathcal{O}_\epsilon) \subseteq \phi(h) + B_{c(g)\epsilon}.
\]

The proof requires an auxiliary lemma:

**Lemma 7.10** ([HN16]). Let \( G \) be a Lie group with Lie algebra \( g \). For \( \mathcal{O}_\epsilon = \exp(\mathcal{B}_\epsilon) \) and every \( g \in G \),

\[
\exp^{-1} \mathcal{O}_\epsilon g \subseteq \mathcal{O}_\epsilon \exp(\mathcal{B}_\epsilon)_{op} = \exp \left\{ \mathcal{Z} \in g : \| \mathcal{Z} \| \leq \epsilon \cdot \| \text{Ad}_h \|_{op} \right\},
\]

where \( \| \cdot \| \) is any euclidean norm on \( g \) and \( \| \cdot \|_{op} \) is the norm on the space of linear operators on \( g \).

**Proof of Proposition 7.9.** Observe that by the previous lemma and the additivity property in Lemma \( \ref{sec:coordinates} \) for every \( h \in G \) there is a constant \( c_1(h) \) such that for \( 0 < \epsilon \leq c_1(h) \):

\[
\phi(\mathcal{O}_\epsilon h \mathcal{O}_\epsilon) = \phi \left( h \cdot \mathcal{O}_\epsilon \right) \subseteq \phi \left( h \cdot \mathcal{O}_{c_1(h)\epsilon} \right).
\]

where \( L_h : G \to G \) is the left translation by \( h \). By compactness of \( D_g := \overline{\mathcal{O}_\delta g \mathcal{O}_\delta} \) and continuity of \( \| \text{Ad} (\cdot) \|_{op} \), there exist \( c_0(g) \) and \( c_0 \) such that for \( 0 < \epsilon \leq c_0(g) \):

\[
\phi(\mathcal{O}_\epsilon h \mathcal{O}_\epsilon) \subseteq \psi_h(B_{c_0(g)\epsilon}),
\]

We choose \( \epsilon(g) > 0 \) and \( 0 < \delta < 1 \) small enough so that (using the additivity property again) for \( V := \mathcal{O}_\delta g \mathcal{O}_\delta \) and \( 0 < \epsilon \leq c(g) \) we have

\[
\mathcal{O}_\epsilon V \mathcal{O}_\epsilon \subseteq \mathcal{O}_\epsilon g \mathcal{O}_\epsilon g \mathcal{O}_\epsilon \subseteq \mathcal{O}_{c_0(g)\epsilon} g \mathcal{O}_{c_0(g)\epsilon}.
\]

We also assume \( c_0(g) \) is small enough such that \( \mathcal{O}_{c_0(g)\epsilon} g \mathcal{O}_{c_0(g)\epsilon} \subseteq U \).

Since \( \psi(h, x) = \psi_h(x) \) is a differentiable map defined on a compact domain \( U \times \overline{B_{c_0(g)\epsilon}} \), there exists \( c(g) = c(D_g) > 0 \) such that for every \( h \in U \) and \( x \in B_{c_0(g)\epsilon} \):

\[
\| \psi_h(x) - \psi_h(0) \| \leq c(g) \| x - 0 \|.
\]

Hence,

\[
\psi_h(B_{c_0(g)\epsilon}) \subseteq \psi_h(0) + B_{c(g)\epsilon} = \phi(h) + B_{c(g)\epsilon}.
\]

Finally, we fix a choice of coordinate balls that will be used from now on.

**Assumption 7.11.** Unless specified otherwise we will assume that \( \mathcal{O}_\epsilon = \exp(\mathcal{B}_\epsilon) \), where \( \exp \) is the Lie exponent.
7.3 Well rounded sets - criteria and properties

This section is devoted to investigating the concept of well-roundedness. Among other things, we will show that finite unions and intersections of well-rounded sets are also well-rounded. Since the property of well roundedness is critical to the usage of the counting method described in Section 7.1, one needs to be able to verify it in given families. However, this property has the disadvantage that it is not easy to prove in concrete examples, due to the fact that the sets $B^\pm_\epsilon (\epsilon)$ from Definition 7.1 are (easy to state but) hard to compute. Our goal in this section is to reformulate well-roundedness as an easier-to-check boundary condition, and mainly to establish that a large class of (fixed) sets are well rounded. These sets are the BCS’s defined in the introduction.

The following lemma is useful in verifying that a certain set (or a family) is well rounded.

**Lemma 7.12.** Suppose $\{O_\epsilon\}_{\epsilon>0}$ is a family of coordinate balls, and let $B \subseteq G$. Then $B^+ (\epsilon) \setminus B^- (\epsilon) = O_\epsilon \partial B O_\epsilon$, or equivalently:

$$B^+ (\epsilon) = B \cup (O_\epsilon \partial B O_\epsilon) =: B^{(+\epsilon)}$$

and

$$B^- (\epsilon) = B \setminus (O_\epsilon \partial B O_\epsilon) =: B^{(-\epsilon)}.$$  

**Remark 7.13.** In fact, Lemma 7.12 applies for any family $\{O_\epsilon\}_{\epsilon>0}$ of connected identity neighborhoods.

**Proof.** We first show that

$$B^+ (\epsilon) \setminus B^- (\epsilon) = O_\epsilon \partial B O_\epsilon.$$  

For the inclusion $\supseteq$, we must show that $B^+ (\epsilon) \supseteq O_\epsilon \partial B O_\epsilon$ and that $(O_\epsilon \partial B O_\epsilon) \cap B^- (\epsilon) = \emptyset$. For the first, assume $g \in O_\epsilon \partial B O_\epsilon$. By symmetry of $O_\epsilon$, the open set $O_\epsilon \cdot g \cdot O_\epsilon$ intersects $\partial B$ non-trivially, and therefore meets $B$, say in a point $h$. Then (again by symmetry) $g \in O_\epsilon \cdot h \cdot O_\epsilon \subset O_\epsilon \cdot B O_\epsilon$. For the latter, note that $h \in B^- (\epsilon)$ if and only if $h \in uBv$ for all $u, v \in O_\epsilon$, i.e. if and only if $u^{-1}h v^{-1} \in B$ for all $u, v \in O_\epsilon$, which by symmetry of $O_\epsilon$ is equivalent to $O_\epsilon \cdot h \cdot O_\epsilon \subset B$. Now if $g \in O_\epsilon \partial B O_\epsilon$ then as before the open set $O_\epsilon \cdot g \cdot O_\epsilon$ intersects $\partial B$ non-trivially, and in particular meets $B^-$; then $O_\epsilon \cdot g \cdot O_\epsilon \not\subset B$, namely $g \notin B^- (\epsilon)$.

For the inclusion $\subseteq$, let $g \notin O_\epsilon \partial B O_\epsilon$, and we show that $g \notin B^+ (\epsilon) \setminus B^- (\epsilon)$. Namely, that either $g \in B^- (\epsilon)$ or that $g \in B^+ (\epsilon)^c$. Indeed, $g \notin O_\epsilon \partial B O_\epsilon$ implies that $(O_\epsilon \cdot g O_\epsilon) \cap \partial B = \emptyset$, and since $O_\epsilon \cdot g O_\epsilon$ is connected it follows that either $O_\epsilon \cdot g O_\epsilon \subseteq B$ or $O_\epsilon \cdot g O_\epsilon \subseteq B^c$. The first implies (by the equivalence established in the first inclusion) that $g \in B^- (\epsilon)$. The latter implies that $g \notin O_\epsilon \partial B O_\epsilon = B^+ (\epsilon)$.

The statement of the lemma now follows:

$$B^+ (\epsilon) = B^- (\epsilon) \cup O_\epsilon \partial B O_\epsilon \subseteq B \cup O_\epsilon \partial B O_\epsilon$$

where the opposite inclusion holds as $B^+ (\epsilon) \supseteq O_\epsilon \partial B O_\epsilon$. Furthermore,

$$B^- (\epsilon) = B^+ (\epsilon) \setminus O_\epsilon \partial B O_\epsilon = (B \cup O_\epsilon \partial B O_\epsilon) \setminus O_\epsilon \partial B O_\epsilon =$$

$$= ((B \setminus O_\epsilon \partial B O_\epsilon) \cup O_\epsilon \partial B O_\epsilon) \setminus O_\epsilon \partial B O_\epsilon = B \setminus O_\epsilon \partial B O_\epsilon.$$  

$\Box$

From Lemma 7.12 we deduce the following simple criterion for the Lipschitz well roundedness of a (fixed) set.

**Lemma 7.14.** Let $G$ be a Lie group with a Borel measure $\mu$. If a subset $B \subseteq G$ satisfies that $0 < \mu (B) < \infty$ and that there exists $c > 0$ such that

$$\mu (O_\epsilon \partial B O_\epsilon) \leq c$$

for every $0 < \epsilon < \frac{\mu(B)}{2c}$, then $B$ is LWR with

$$C = \frac{2c}{\mu (B)}.$$  

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The converse also holds: suppose \( B \) is LWR with positive measure and parameter \( C \). Then for \( \epsilon < C^{-1} \),
\[
\mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon) \leq C\mu(B)\epsilon.
\]

**Proof.** According to Lemma 7.12, Lipschitz well-roundedness can be verified with \( B^{(\pm \epsilon)} \) instead of \( B^{\pm} (\epsilon) \). By our assumption, for \( \epsilon < \frac{\mu(B)}{2c} \):

\[
\mu(B^{(\epsilon)}) = \mu(O_\epsilon B O_\epsilon) \\
= \mu(B^{(-\epsilon)}) + \mu(O_\epsilon \cdot \partial B \cdot O_\epsilon) \\
\leq \mu(B^{(-\epsilon)}) + c\epsilon
\]

and

\[
\mu(B^{(-\epsilon)}) = \mu(B \setminus (O_\epsilon \cdot \partial B \cdot O_\epsilon)) \\
\geq \mu(B) - \mu(O_\epsilon \cdot \partial B \cdot O_\epsilon) \\
\geq \mu(B) - c\epsilon
\]

As a result, for \( \epsilon < \frac{\mu(B)}{2c} \),

\[
\frac{\mu(B^{(+\epsilon)}) - \mu(B^{(-\epsilon)})}{\mu(B^{(-\epsilon)})} \leq \frac{c\epsilon}{\frac{\mu(B)}{2}} = \frac{2c}{\mu(B)} \cdot \epsilon.
\]

Regarding the opposite direction, our assumption is that for \( \epsilon < C^{-1} \),

\[
\frac{\mu(B^{(+\epsilon)}) - \mu(B^{(-\epsilon)})}{\mu(B^{(-\epsilon)})} \leq C\epsilon.
\]

Hence,

\[
\frac{\mu(O_\epsilon \cdot \partial B \cdot O_\epsilon)}{\mu(B)} \leq \frac{\mu(B^{(+\epsilon)}) - \mu(B^{(-\epsilon)})}{\mu(B^{(-\epsilon)})} \leq C\epsilon.
\]

In other words,

\[
\mu(O_\epsilon \cdot \partial B \cdot O_\epsilon) \leq C\epsilon \cdot \mu(B).
\]

One consequence of Lemma 7.14 is that finite unions and intersections of LWR sets are in themselves LWR.

**Lemma 7.15.** Let \( G \) be a Lie group with a Borel measure \( \mu \). If two subsets \( B \) and \( B' \) of \( G \) such that \( 0 < \mu(B \cap B') \) are LWR, then \( B \cap B' \) and \( B \cup B' \) are also LWR with Lipschitz constant

\[
C_{B \cap B'} = 2 \max \{C, C'\} \cdot \frac{\mu(B) + \mu(B')}{\mu(B \cap B')}; \quad C_{B \cup B'} = 2 \max \{C, C'\} \cdot \frac{\mu(B) + \mu(B')}{\mu(B \cup B')}.
\]

**Proof.** We prove the lemma only for the intersection \( B \cap B' \); the proof for the union \( B \cup B' \) is similar. By Lemma 7.14 for \( \epsilon < \frac{1}{C_{B \cap B'}} \) (so \( \epsilon < C^{-1}, C'^{-1} \)): \( \mu(O_\epsilon \cdot \partial B \cdot O_\epsilon) \leq C\mu(B)\epsilon \), \( \mu(O_\epsilon \cdot \partial B' \cdot O_\epsilon) \leq C'\mu(B')\epsilon \).

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Hence, by using the fact that the boundary of an intersection is contained in the union of the boundaries, we obtain that for $\epsilon < \frac{1}{c_{BCS}}$,

$$
\mu(\mathcal{O}_\epsilon \cdot \partial (B \cap B') \cdot \mathcal{O}_\epsilon) \leq \mu(\mathcal{O}_\epsilon \cdot \partial B \cdot \mathcal{O}_\epsilon) + \mu(\mathcal{O}_\epsilon \cdot \partial B' \cdot \mathcal{O}_\epsilon) \\
\leq \max\{c, c'\} \cdot (\mu(B) + \mu(B')) \cdot \epsilon
$$

The first direction of Lemma 7.14 yields the desired conclusion.

The LWR criterion in Lemma 7.14 will not be used directly in order to verify the LWR for a given set; instead, it will be used to show that BCS’s are LWR. This will suffice for our needs, as the only (fixed) sets we will work with are BCS's. The rest of the section is therefore devoted to showing that a BCS is LWR.

The main result of this section is the following.

**Proposition 7.16.** Let $G$ be a Lie group. Assume that $\mu$ is a measure on $G$ that is absolutely continuous w.r.t. Haar measure, and has density that is bounded on compact sets. If $B$ is BCS with $\mu(B) > 0$, then $B$ is Lipchitz well-rounded.

**Proof.** The strategy is to apply Lemma 7.14. This will be done by showing that for a subset $Y$ of a manifold $M$ which is compact and consists of a finite union of subsets of embedded submanifolds of strictly smaller dimension (e.g. the boundary of $B$) there exist $c = c(Y), \epsilon(Y) > 0$ such that

$$
\mu(\mathcal{O}_\epsilon Y \mathcal{O}_\epsilon) \leq c\epsilon \tag{7.4}
$$

for some $0 < \epsilon < \epsilon(Y)$.

It is clearly sufficient to assume that $Y$ is contained in one submanifold. For each point $g \in Y$, there is some chart $\phi_g : U_g \to \mathbb{R}^m$ for which $g \in U_g$ and $\phi(U_g \cap Y) \subseteq \mathbb{R}^{m-1} \times \{0\}$. Let $V_g$ be the open sets from Proposition 7.9 which satisfy: $g \in V_g \subseteq U_g$. By compactness, there are $g_1, \ldots, g_r \in Y$ for which $V_{g_1}, \ldots, V_{g_r}$ cover $Y$ entirely. In order to establish the inequality in Formula (7.4), it is sufficient to prove it for each $Y \cap V$, separately. Consequently, we may assume that $r = 1$: $g_1 = g$, $V_{g_1} = V$, $Y_0 = Y \cap V$ and $\phi_{g_1} = \phi$.

By Proposition 7.9, there exist $c(g), \epsilon(g) > 0$ such that for $\epsilon < \epsilon(g)$ and $h \in V$, $\phi(\mathcal{O}_\epsilon h \mathcal{O}_\epsilon) \subseteq \phi(h) + B_{c(g)\epsilon}$. In particular

$$
\phi(\mathcal{O}_\epsilon Y_0 \mathcal{O}_\epsilon) \subseteq \phi(Y_0) + B_{c(g)\epsilon}.
$$

Hence it is sufficient to show that $\phi_* \mu \left( \phi(Y_0) + B_{c(g)\epsilon} \right) \leq c\epsilon$.

Let $\omega \in L^1(\mathbb{R}^m)$ be such that $\phi_* \mu = \omega \cdot \mu_{\mathbb{R}^m}$ where $\mu_{\mathbb{R}^m}$ is the Lebesgue measure on $\mathbb{R}^m$. Then, since $\omega$ is bounded on compact sets (and in particular on $\phi(\mathcal{O}_{c(g)} Y_0 \mathcal{O}_{c(g)})$), it is sufficient to show that

$$
\mu_{\mathbb{R}^m}(\phi(Y_0) + B_c) \leq c\epsilon.
$$

Indeed, since $Y_0$ is an embedded submanifold, there exists a bounded set $E \subseteq \mathbb{R}^{m-1}$ such that $\phi(Y_0) + B_c \subseteq E \times [-c_2\epsilon, c_2\epsilon]$, which implies the desired result.

### 7.4 Roundomorphisms

In order to count lattice points in families of sets via the GN method (Theorem 7.4), one needs to establish that the family in question is Lipschitz well rounded. This presents a technical difficulty, since the well roundedness, which is essentially a multiplicative property, is hard to verify in simple Lie groups. Nevertheless, simple Lie groups have several known decompositions — Cartan, Iwasawa, etc. — which allow them to be written as the product of more “convenient” subgroups. E.g., in the case of the Iwasawa decomposition, the subgroups $K, A, N$ are compact, abelian and nilpotent respectively, which makes it considerably easier to prove well roundedness inside them. The goal of this section is to reduce the question of whether a family $B_T \subseteq G$ is LWR, to verifying LWR of the projections of
$\mathcal{B}_T$ to each of the components of $G$ w.r.t. a given decomposition. E.g. when considering the Iwasawa decomposition, the well roundedness of $\mathcal{B}_T$ is reduced to the question of well roundedness of the image of $\mathcal{B}_T$ in the Cartesian product $K \times A \times N$. This can be achieved if the Iwasawa diffeomorphism $G \to K \times A \times N$ preserves well roundedness; maps with this property are the topic of the following definition.

**Definition 7.17 (Roundomorphism).** Let $G$ and $Y$ be two topological groups with measures $\mu_G$ and $\mu_Y$, and let $(\mathcal{O}^G_\epsilon)_{\epsilon>0}$ and $(\mathcal{O}^Y_\epsilon)_{\epsilon>0}$ be two families of identity neighborhoods in $G$ and $Y$ respectively. A Borel measurable map $r : G \to Y$ will be called an $f$-roundomorphism if it is:

1. **Measure preserving:** $r_\ast(\mu_G) = \mu_Y$.

2. **Locally Lipschitz:** $r(\mathcal{O}^G_\epsilon g\mathcal{O}^G_\epsilon) \subseteq \mathcal{O}^Y_{f_\epsilon r(g)}\mathcal{O}^Y_{f_\epsilon}$ for some continuous $f = f(g) : G \to \mathbb{R}_{>0}$ and for every $0 < \epsilon < \frac{1}{f}$.

The content of the following proposition is that the pre-image of an LWR family under a roundomorphism, is also an LWR family.

**Proposition 7.18.** Let $r : G \to Y$ be an $f$-roundomorphism. Assume that $\{\mathcal{B}_T\}_{T>0}$ is a family of measurable subsets of $Y$ such that $f$ is bounded uniformly on $r^{-1}(\mathcal{B}_T)$ by a constant $F$. If $\{\mathcal{B}_T\}$ is LWR with parameters $(T_0, C_0)$, then $r^{-1}(\mathcal{B}_T)$ is LWR with parameters $(T_0, F \cdot \max\{C_0, 1\})$.

**Proof.** The strategy of the proof is to show that for $\epsilon < F^{-1}$,

$$\mu_G\left(\left(r^{-1}(\mathcal{B}_T)\right)^\epsilon\right) \leq \mu_Y\left(\mathcal{B}_T^\epsilon(Fe)\right)$$

and

$$\mu_Y\left(\mathcal{B}_T^\epsilon(Fe)\right) \leq \mu_G\left(\left(r^{-1}(\mathcal{B}_T)\right)^\epsilon\right).$$

(7.5) (7.6)

It will then follow that for $T > T_0$ and $\epsilon < \frac{1}{F \cdot \max\{C_0, 1\}}$ (so that both $\epsilon < F^{-1}$ and $\epsilon < (FC_0)^{-1}$: the first for inequalities (7.5) and (7.6) to hold, and the second for the LWR of $\{\mathcal{B}_T\}$),

$$\mu_G\left(\left(r^{-1}(\mathcal{B}_T)\right)^\epsilon\right) \leq \mu_Y\left(\mathcal{B}_T^\epsilon(Fe)\right) \leq \frac{1}{F} + FC_0 \epsilon.$$

Inequalities (7.5) and (7.6) follow from measure preservation of $r$, along with the following inclusions:

$$\left(r^{-1}(\mathcal{B}_T)\right)^\epsilon \subseteq r^{-1}(\mathcal{B}_T^\epsilon(Fe)),$$

$$\left(r^{-1}(\mathcal{B}_T)\right)^-\epsilon \supseteq r^{-1}(\mathcal{B}_T^\epsilon(Fe)),$$

that we now justify. For the first, note that by definition of a roundomorphism, $O^G_\epsilon gO^G_\epsilon \subseteq r^{-1}\left(O^Y_{f_\epsilon r(g)}O^Y_{f_\epsilon}\right)$. Hence, $O^G_\epsilon \cdot r^{-1}(\mathcal{B}_T) \cdot O^G_\epsilon \subseteq r^{-1}(O^Y_{f_\epsilon} \mathcal{B}_T O^Y_{f_\epsilon})$. For the second inclusion, suppose $g \in r^{-1}(\mathcal{B}_T^\epsilon(Fe))$. We want to show that if $u, v \in O^G_\epsilon$, then $ugv \in r^{-1}(\mathcal{B}_T)$. Put differently, $r(ugv) \in \mathcal{B}_T$. This is indeed the case, since $r(ugv) = u' r(g) v'$ for some $u', v' \in O^Y_{f_\epsilon}$ (local Lipschitzity of $r$), and $u' r(g) v' \in \mathcal{B}_T$ since $r(g) \in \mathcal{B}_T^\epsilon(Fe)$.

The most useful incident of Proposition 7.18 is when $Y$ (such that $r : G \to Y$ is a roundomorphism) is a direct product of groups.

**Corollary 7.19.** Let $r : G \to Y = Y_1 \times \cdots \times Y_q$ be an $f$-roundomorphism and let $\mathcal{B}_T = \mathcal{B}_T^1 \times \cdots \times \mathcal{B}_T^q \subseteq Y$. Set

1. $\mu_Y = \mu_{Y_1} \times \cdots \times \mu_{Y_q}$
2. \( O_Y^r = O_{\epsilon_1}^{Y_1} \times \cdots \times O_{\epsilon_q}^{Y_q} \)

and assume that:

1. For \( j = 1, \ldots, q \): \( B_j^T \subseteq Y_j \) is LWR w.r.t. the parameters \((T_j, C_j)\);
2. \( f \) is bounded uniformly by \( F \) on the sets \( r^{-1}(B_T) \).

Then \( r^{-1}(B_T) \) is LWR, w.r.t. the parameters

\[
T = \max \{ T_1, \ldots, T_q \}, \quad C \asymp q \cdot F \cdot \max \{ C_1, \ldots, C_q, 1 \}.
\]

**Proof.** It is sufficient to prove the claim for \( q = 2 \), where one then proceeds by induction. According to the previous proposition we only need to show that \( B_T^T \) is Lipchitz well-rounded w.r.t. the parameters \((T, C/F)\). Indeed, since

\[
\mu_Y(B_T^T(\epsilon)) = \mu_{Y_1}((B_1^T)^\pm(\epsilon)) \cdot \mu_{Y_q}((B_q^T)^\pm(\epsilon)),
\]

we obtain

\[
\frac{\mu_Y(B_T^T(\epsilon))}{\mu_Y(B_T(\epsilon))} \leq (1 + C_1 \epsilon)(1 + C_2 \epsilon) \leq (1 + \max \{ C_1, C_2 \} \epsilon)^2 \leq 1 + 3 \max \{ C_1, C_2 \} \epsilon
\]

for \( \epsilon \asymp \frac{1}{\max \{ C_1, C_2 \}} \). \( \Box \)

**Remark 7.20.** One consequence of Corollary 7.19 is that a Cartesian product of LWR families

\[
B_1^T \times \cdots \times B_q^T \subseteq Y_1 \times \cdots \times Y_q
\]

is LWR. To see this, take \( G = Y_1 \times \cdots \times Y_q \) and \( r \) that is the identity map on \( G \); it is a roundomorphism with \( f \equiv 1 \).

The content of the following lemma is that a composition of roundomorphisms is a roundomorphism.

**Lemma 7.21.** Suppose that \( r_1 : G_1 \to G_2 \) is an \( f_1 \)-roundomorphism and \( r_2 : G_2 \to G_3 \) is an \( f_2 \)-roundomorphism. Then, \( r_2 \circ r_1 \) is an \( f = (f_2 \circ r_1) \cdot f_1 \)-roundomorphism.

**Proof.** Clearly we only need to check that \( r_2 \circ r_1 \) is locally Lipchitz:

\[
r_2 r_1 \left( O_{\epsilon_1}^{G_1} \cdot g \cdot O_{\epsilon}^{G_1} \right) \subseteq r_2 \left( O_{f_1 \epsilon}^{G_2} \cdot r_1 (g) \cdot O_{f_1 \epsilon}^{G_2} \right) \subseteq O_{f_2 \epsilon}^{G_3} \cdot r_2 r_1 (g) \cdot O_{f_2 \epsilon}^{G_3}.
\]

\( \Box \)

**8 Lipschitzity of Iwasawa and generalized Iwasawa decomposition**

In the previous section we defined maps called roundomorphisms, for which the pre-image of a well rounded family is in itself well rounded. In this section we show that the map \( G \to K \times A \times N \) projecting to the Iwasawa coordinates of a semisimple group is a roundomorphism, allowing us to reduce the well roundedness of families in \( G \) to well roundedness of their projections to \( K, A \) and \( N \).
8.1 Effective Iwasawa decomposition

In what follows we let $G$ denote a real semi-simple Lie group with finite center and Iwasawa decomposition $G = KAN$. The subgroups $K$, $A$ and $N$ are equipped with measures $\mu_K$, $\mu_A$ and $\mu_N$ respectively, such that for a given Haar measure $\mu_G$ of $G$,

$$\mu_G = \mu_K \times \mu_A \times \mu_N.$$ 

Note that while $\mu_K$ and $\mu_N$ are Haar measures of their corresponding group, $\mu_A$ is not (see Section 9.1 for more details regarding $\mu_A$).

Let $a = \text{Lie algebra of } A$, $n = \text{Lie algebra of } N$, $\Sigma = \text{restricted roots (w.r.t. } a)$, $\Sigma^+ = \text{positive (restricted) roots w.r.t. } n$.

We denote $\Sigma^+ = \{\phi_1, \ldots, \phi_p\} \subset a^*$ where the $\phi_i$’s are not necessarily different, but with multiplicities.

**Notation 8.1.** For $a = \exp (H) \in A$ define

$$m(H) = \max_i \{-\phi_i(H), 0\}$$

and

$$\text{err}(a) := C_{\text{norm}}^2 e^{m(H)},$$

where $C_{\text{norm}} \geq 1$ is a constant which depends on the specific choice of norm $\|\cdot\|$ on $n$ in the following manner: $\frac{1}{C_{\text{norm}}} \|Z\|_{\infty} \leq \|Z\| \leq C_{\text{norm}} \|Z\|_{\infty}$ for every $Z \in n$.

**Remark 8.2.** Notice that $\text{err}(\cdot)$ is sub-multiplicative:

$$\text{err}(a_1 a_2) \leq \text{err}(a_1) \text{err}(a_2).$$

The goal of this section is to prove the following proposition.

**Proposition 8.3** (Effective Iwasawa decomposition). Let $G$ be a semisimple Lie group. The diffeomorphism defining the Iwasawa decomposition

$$r : G \rightarrow K \times A \times N$$

$$r(g) = (k,a,n)$$

is a $f$-roundomorphism w.r.t. $O_N\epsilon \trianglelefteq O_{N\epsilon}^{\text{err}(a)}$,

$$O^G_e = \exp_G(B_e), \ O^{K\times A\times N}_e = \exp_{K\times A\times N}(B_e)$$

and

$$f(g) \prec C(n) \cdot \text{err}(a)^2,$$

where $C(n) = \|Adn\|_{op}$.

The proof requires the following auxiliary lemma.

**Lemma 8.4.** Let $N^\perp := \Theta(N)$, where $\Theta$ is a global Cartan involution compatible with the given Iwasawa decomposition. Then $A$ acts on both $N, N^\perp$ by conjugation such that the following holds:

$$a^{-1}O^N_e a \subseteq O^{N^{\text{err}(a)e^*}}_e,$$

$$aO^{N^\perp}_e a^{-1} \subseteq O^{N^\perp}_{\text{err}(a)e^*}.$$
Proof. First we introduce some notations. Let \( Z_1, \ldots, Z_p \) be the corresponding linearly independent eigenvectors in \( g \) of \( \phi_1, \ldots, \phi_p \) respectively. Denote
\[
n_\mathbb{R} = n_{[x_1, \ldots, x_p]} := \exp \left( \sum_{i=1}^{p} x_i Z_i \right).
\]
Then
\[
N = \{ n_\mathbb{R} : x \in \mathbb{R}^p \} ; N^- = \left\{ n_\mathbb{R} : x \in \mathbb{R}^p \right\}.
\]
For every \( H \in a \) and \( Z \in n \) the action of \( a^{-1} = \exp (-H) \) on \( \exp (Z) \) is given by
\[
\text{Conj}_{\exp(-H)}(\exp(Z)) = \exp(\text{Ad}_{a^{-1}}(Z)) = \exp \left( e^{a\phi(-H)} \cdot Z \right).
\]
In particular, if \( Z = \sum_{i=1}^{p} x_i Z_i \) then (since \( \text{ad}_{-H}(Z_i) = [-H, Z_i] = \phi_i(-H) \cdot Z_i \) and therefore \( e^{a\phi(-H)} \cdot Z_i \):
\[
\text{Conj}_{\exp(-H)} \left( \exp \left( \sum_{i=1}^{p} x_i Z_i \right) \right) = \exp \left( \text{Ad}_{a^{-1}} \left( \sum_{i=1}^{p} x_i Z_i \right) \right) = \exp \left( \sum_{i=1}^{p} x_i \text{Ad}_{a^{-1}}(Z_i) \right) = \exp \left( \sum_{i=1}^{p} x_i \cdot e^{a\phi(-H)} \cdot Z_i \right).
\]
As a result,
\[
a^{-1} \cdot n_\mathbb{R} \cdot a = \exp (-H) \cdot n_\mathbb{R} \cdot \exp (H) = n_{[x_1 e^{\phi_1(-H)}, \ldots, x_p e^{\phi_p(-H)}]} = n_\mathbb{R} (e^{-\phi_i(H)})^{p}_{\cdot i=1}.
\]
If \( a^{-1} \cdot n_\mathbb{R} \cdot a = n_\mathbb{R} \), then for \( n_\mathbb{R} \in O^N \) and \( ||x|| < \epsilon \) it holds for \( \mathbb{R} \) that
\[
||y|| = \left\| x \cdot \left( e^{-\phi_i(H)} \right)^{p}_{\cdot i=1} \right\| \leq C_{\text{norm}} \left\| x \right\| \left\| e^{-\phi_i(H)} \right\|^{p}_{\cdot i=1} \leq C_{\text{norm}} \left\| x \right\| \left\| e^{-\phi_i(H)} \right\|^{p}_{\cdot i=1} \leq \epsilon \cdot \text{err} (a).
\]
Thus,
\[
a^{-1} O^N a \subseteq O^n_{\text{err}(a)\epsilon}.
\]
The second part follows from the first part by applying \( \Theta \) (the global Cartan involution) to the above. \( \square \)

Proof of Proposition 8.3. Clearly, we only need to show that
\[
r \left( O^{G} g O^{G} \right) \subseteq O^{K \times A \times N}_{f} \cdot r(g) O^{K \times A \times N},
\]
where \( f \) is as in the statement. This will be accomplished in three steps.

Step 1: Left perturbations. Since the family of coordinate balls \( O^{G} \) is equivalent to the family \( O^{K} \cdot O^{A} \cdot O^{N} \) (Remark 7.7), we obtain by Lemma 7.10 that for \( \epsilon < \epsilon_1 \) (\( \epsilon_1 \) is the constant arising from the equivalence)
\[
O^{G} \cdot \text{kan} = k (k^{-1} O^{G} k) \cdot \text{an} \subseteq k O^{G} \cdot \text{an} \subseteq k O^{K} \cdot O^{A} \cdot O^{N} \cdot \text{an}.
\]
By Lemma 8.4 \( a^{-1} O^{N} a \subseteq O^{N}_{\text{err}(a)\epsilon} \), hence
\[
r \left( O^{G} g \right) \subseteq O^{K \times A \times N}_{e_{2\epsilon} \text{err}(a)\epsilon} \cdot r(g) O^{K \times A \times N}_{e_{2\epsilon}}.
\]
Step 2: Right perturbations. Using Bruhat coordinates on identity neighborhood in $G$, the family $O_e^g$ is equivalent to the family $O^M_e O^N_e O^A_e O^N_e$, where $M = (Z_K(A))_0$. We may assume that the parameter $\epsilon_1$ arising from the equivalence is the same as in Step 1. By Lemma 7.10

$$\text{kan} O^g_e \subseteq \text{kan} O^M_{c_3 e} O^N_{c_3 e} O^A_{c_3 e} O^N_{c_3 e} = \text{kan} \left( O^M_{c_3 e} O^N_{c_3 e} O^A_{c_3 e} O^N_{c_3 e} \right)$$

$$\subseteq \text{kan} \left( O^M_{c_3 C(n)e} O^N_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e} \right)$$

$$= \text{kan} O^M_{c_3 C(n)e} \cdot a \cdot O^N_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e}$$

$$= \text{kan} O^M_{c_3 C(n)e} \left( a O^N_{c_3 C(n)e} a^{-1} \right) a O^A_{c_3 C(n)e} O^N_{c_3 C(n)e}$$

By Lemma 8.4, $a O^N_{c_3 C(n)e} a^{-1} \subseteq O^N_{c_3 C(n)}(\text{err}(a))e \subseteq O^G_{c_3 C(n)}(\text{err}(a))e$. Moreover, for

$$\epsilon \leq \frac{\epsilon_1}{c_3 C(n) \text{err}(a)} \cdot$$

we have

$$O^M_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e} \subseteq O^K_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e} O^N_{c_3 C(n)e}$$

$$\subseteq O^K_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e} O^N_{c_3 C(n)e}$$

As a result,

$$\text{kan} O^g_e \subseteq \text{kan} O^K_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e} O^N_{c_3 C(n)e} a O^A_{c_3 C(n)e} O^N_{c_3 C(n)e}$$

Let $a_e \in O^A_{c_3 C(n)e}$. Write $a_1 = a a_e$. By sub-multiplicativity of $\text{err} (\cdot)$ (Remark 8.2) we get,

$$O^N_{c_3 C(n)e} a_1 = a_1 a^{-1} O^N_{c_3 C(n)e} a_1 \subseteq a_1 O^N_{c_3 C(n)e} a_1 \subseteq a_1 O^N_{c_3 C(n)e} a_1 \leq a_1 O^N_{c_3 C(n)e} \text{err}(a)^2 e$$

Combining all of the above, we conclude

$$\text{kan} O^g_e \subseteq \text{kan} O^K_{c_3 C(n)e} O^A_{c_3 C(n)e} O^N_{c_3 C(n)e} O^N_{c_3 C(n)e} \cdot$$

In other words,

$$r \left( g O^g_e \right) \subseteq O^K_{c_3 C(n)e} \cdot$$

Step 3: Combining left and right perturbations. Finally, using the additivity property (Lemma 7.8) on $O^K A N$ we conclude that

$$r \left( O^g_e g O^g_e \right) \subseteq O^K_{c_3 C(n)e} \cdot$$

for

$$\epsilon \leq \frac{1}{f(g)} \epsilon$$

and $f (g) \propto C(n) \cdot \text{err}(a)^2$.

8.2 Effective Generalized Iwasawa decomposition

After having established that the map $G \to K \times A \times N$ projecting to the $KAN$ coordinates is a roundomorphism, we deduce it for the $KA'A'N$ and GI decompositions as well (Corollary 8.6).

Lemma 8.5. Let $N$ be a connected nilpotent Lie group with Haar measure $\mu_N$. Suppose that $N = N_1 \times N_2$, where $N_1$ and $N_2$ are two closed subgroups of $N$ equipped with Haar measures $\mu_{N_1}$ and $\mu_{N_2}$. Then each element in $N$ can be decomposed in a unique way as $n = n_1 n_2$, and the map

$$r (n) = (n_1, n_2) \in N_1 \times N_2$$

is a $f$-roundomorphism for some continuous $f : N \to \mathbb{R}^{\geq 0}$. If $N$ is abelian, then $f$ is the constant function $f \equiv 1$.  

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Proof. Since the first condition of roundomorphisms is taken care of by the nilpotency assumption (see [Kna02 Corollary 8.31, Theorem 8.32]), we will focus on the second one. For \( \mathbf{x} = (x_1, \ldots, x_p) \) and \( \mathbf{Z} = (Z_1, \ldots, Z_p) \in \mathbb{N}^p \) we write \( \mathbf{x} \cdot \mathbf{Z} \) for \( \sum_{i=1}^p x_i Z_i \) (see notation 9.1). The families \( O_{\mathcal{N}}^N n_1 n_2 O_{\mathcal{N}}^N \subseteq O_{\mathcal{N}}^N n_1 \mathcal{O}_{f_{\mathcal{N}}}^N \mathcal{O}_{f_{\mathcal{N}}}^N n_2 O_{\mathcal{N}}^N \).

We need to show that
\[
O_{\mathcal{N}}^N n_1 n_2 O_{\mathcal{N}}^N \subseteq O_{\mathcal{N}}^N n_1 \mathcal{O}_{f_{\mathcal{N}}}^N \mathcal{O}_{f_{\mathcal{N}}}^N n_2 O_{\mathcal{N}}^N.
\]

Since \( \exp : \mathfrak{n} \to N \) is onto, the above will follow if we show that for \( n_1 = n_x \) and \( n_2 = n_y \) there is \( c = c(n_1, n_2) > 0 \) and \( f = f(n) > 0 \)
\[
\exp \left( B_{x}^N \right) \exp \left( \mathbf{x} \cdot \mathbf{Z} \right) \exp \left( \mathbf{y} \cdot \mathbf{Z} \right) \exp \left( B_{y}^N \right) \subseteq \exp \left( (\mathbf{x} + \mathbf{y}) \cdot \mathbf{Z} + B_{ce} \right)
\]

and
\[
\exp \left( ((\mathbf{x} + \mathbf{y}) \cdot \mathbf{X} + B_{ce}) \right) \subseteq \exp \left( B_{f_{\mathcal{N}}}^N \right) \exp \left( \mathbf{x} \cdot \mathbf{Z} \right) \exp \left( B_{f_{\mathcal{N}}}^N \right) \exp \left( \mathbf{y} \cdot \mathbf{Z} \right) \exp \left( B_{f_{\mathcal{N}}}^N \right).
\]

Indeed, this is a direct consequence of the Baker–Campbell–Hausdorff formula which states that
\[
\exp (X_1) \exp (X_2) = \exp (X_1 + X_2 + Z_{\text{error}}),
\]
where \( Z_{\text{error}} \) (which is given explicitly) has a norm smaller than \( c_0(n) \epsilon \), and \( c_0(n) \) is a positive continuous function. In the Abelian case one clearly has \( Z_{\text{error}} = 0. \)

**Corollary 8.6 (Effective GI decomposition).** Let \( N' \) and \( N'' \) be closed subgroups of \( N \) equipped with Haar measures \( \mu_{N'}, \mu_{N''} \) such that \( N = N'' \ltimes N' \) and \( \mu_N = \mu_{N'} \times \mu_{N''} \). Similarly, let \( A' \) and \( A'' \) be closed subgroups of \( A \) such that \( A = A' \times A'' \) and \( \mu_A = \mu_{A'} \times \mu_{A''} \).

The GI decompositions
\[
r_1 : G \to K \times A' \times A'' \times N
\]
\[
r_1 (g) = (k, a', a'', n)
\]

and
\[
r_2 : G \to K \times A' \times A'' \times N'' \times N'
\]
\[
r_2 (g) = (k, a', a'', n'', n')
\]
are \( f_i \)-roundomorphisms w.r.t.
\[
f_1 (g) \preceq c_1 (n) \cdot \text{err} (a)^2 \text{ for } r_1
\]
\[
f_2 (g) \preceq c_2 (n', n'') \cdot \text{err} (a)^2 \text{ for } r_2
\]

where \( c_1 (n) = \| \text{Ad} n \|_{op} \) and \( c_2 (n', n'') \) are continuous functions on \( N \) and \( N' \times N'' \) respectively.

**Proof.** This follows from Proposition 8.3 combined with Lemmas 8.5 and 7.21.

9 Counting in Iwasawa product sets of algebraic simple Lie groups

The goal of this section is to prove Theorem C and several similar results. Theorem C is concerned with counting lattice points in families of product sets inside an algebraic simple Lie group \( G \), where the product is w.r.t. the Iwasawa decomposition of the group. The families are of the form \( \mathcal{B}_T = \Phi A'_T \Pi \Psi \) with \( \Phi \subseteq K, \Pi \subseteq A'' \) and \( \Psi \subseteq N \), namely the \( K, A' \) and \( N \) components lie in fixed sets, and the \( A' \) component grows with a parameter \( T \). The counting will be performed using the method introduced in Section 7.1, for which one needs to prove that the family \( \mathcal{B}_T \) is well rounded.
Due to the results of Sections 7.4 and 8, we know that it is sufficient to show well roundedness of the projections to each of the components $K$, $A'$, $A''$ and $N$; due to Proposition 7.16 we know that the projections $\Phi \subseteq K$, $\Pi \subseteq A''$ and $\Psi \subset N$ (which are assumed to be BCS) are well rounded. It is therefore left to verify well roundedness of the family $\{A'_T\}$, and indeed Section 9.1 is concerned with studying well roundedness of such families in $A$. Section 9.2 is devoted to proving Theorem C and several further results about counting lattice points in Iwasawa and Generalized Iwasawa product sets.

9.1 Well roundedness in subgroups of $A$

We consider subgroups of $A$ that are the image of subspaces in $\mathfrak{a}$ under the exponent map. To introduce them, we first set some notations.

**Notation 9.1.** For vectors $H_1, \ldots, H_q \in \mathfrak{a}$, we write $H := (H_1, \ldots, H_q) \in \mathfrak{a}^q$.

If $\underline{s} = (s_1, \ldots, s_q) \in \mathbb{R}^q$ we let $\underline{s} \cdot H = \sum_{i=1}^{q} s_i H_i$ and if $\phi \in \mathfrak{a}^*$, we let $\phi(H) = \sum_{i=1}^{q} \phi(H_i)$.

We say that $H$ is linearly independent if $H_1, \ldots, H_q$ are.

Throughout this section we use the standard notation for the sum of positive roots: $2\rho = \sum_{i=1}^{p} \phi_i \in \mathfrak{a}^*$.

**Definition 9.2.** Given linearly independent $H = (H_1, \ldots, H_q)$, we define the subgroup $A_H < A$ to be $A_H := \{ \exp(\underline{s} \cdot H) : \underline{s} \in \mathbb{R}^q \}$, and endow it with the measure (that is not Haar!)

$\mu_{A_H} := e^{2\rho(H_1)s_1} \cdots e^{2\rho(H_q)s_q} ds_1 \cdots ds_q$.

When $q = 1$, we omit the underline: $H = \underline{H}$ and $s = \underline{s}$.

**Remark 9.3.** Every closed connected subgroup of $A$ is of the form $A_H$. Furthermore, $A_H \cap A_{H'} = \{1_A\}$ if and only if $H$ is linearly independent of $H'$. In that case, $A_H \times A_{H'} = A_H \times A_{H'}$ as both groups and measure spaces. In particular, if $H$ is a basis for $\mathfrak{a}$, then $A_H = A$ and $\mu_{A_H} = \mu_A$.

**Example 9.4.** In the case of $G = \text{SL}_n(\mathbb{R})$, $N = \begin{bmatrix} 1 & & & \vdots & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \ddots \\ & & & 0 & \vdots \\ & & & & 1 \end{bmatrix}$ and $A = \begin{bmatrix} e^{\alpha_1} & 0 & & & \\ & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots \\ & & & e^{\alpha_{n-1}} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & & & e^{\alpha_n} \end{bmatrix}$, where $\sum \alpha_i = 0$. The roots $\phi_{i,j} \in \mathfrak{a}^*$ are defined via

$$\phi_{i,j} \left( \sum_{k=1}^{n} \alpha_k e_{k,k} \right) = \alpha_j - \alpha_i,$$

where the positive roots (w.r.t. which $N$ is defined) are the ones with $j < i$. For $H = \sum_{k=1}^{n} \alpha_k e_{k,k} \in \mathfrak{a}$,

$$2\rho(H) = 2\rho \left( \sum_{k=1}^{n} \alpha_k e_{k,k} \right) = \sum_{k=1}^{n} \left( n + 1 - 2k \right) \alpha_k.$$
For \( i = 1, \ldots, n - 1 \), the vectors
\[
H_i = \frac{1}{2} (-e_{i,i} + e_{i+1,i+1})
\]
form a basis for \( a \) such that \( 2\rho(H_i) = -1 \) for all \( i \). As a result (Remark 9.3), when \( A \) is parameterized by \( H_1, \ldots, H_{n-1} \):

\[
A = \left\{ \begin{array}{ccc}
\begin{pmatrix}
e^{-\frac{2\rho(H)}{2}} & \cdots & 0 \\
e^{s_1 - s_2} & \cdots & 0 \\
0 & \cdots & e^{s_{n-2} - s_{n-1}}
\end{pmatrix} : (s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-1}\end{array} \right\},
\]

then
\[
\mu_{AH} = \mu_A = \frac{ds_1 \cdots ds_{n-1}}{e^{s_1 + \cdots + s_{n-1}}}.\]

**Definition 9.5.** We consider the following subsets of \( A \):

1. For \( S = (S_1, \ldots, S_q) \),
   \[
   A^H_S := \left\{ \exp \left( s \cdot H : s \in \prod_{i=1}^q [0, S_i] \right) \right\} \subseteq A^H.\]

2. When all \( S_i \) are equal to \( T \), we simply write \( A^H_T \subseteq A^H \).

The goal of this subsection is to prove the following:

**Proposition 9.6.** The family \( \{ A^H_T \}_{T > 0} \) is LWR with parameters which depend only on \( H \), and the fixed set \( A^H_S \) is well-rounded with parameters which depend only on \( H \), if \( S_1, \ldots, S_q \) are large enough to satisfy:

\[
S_i \geq \begin{cases} 
\frac{4}{2\rho(H)} & \text{if } 2\rho(H_i) \neq 0 \\
1 & \text{otherwise}
\end{cases}
\]

**Remark 9.7.** Notice that the sets \( A^H_S \) are clearly BCS and therefore (Proposition 7.16) LWR, hence the content of the proposition for these sets is that their LWR parameters are uniform (i.e., not depend on \( S \)).

**Proof.** We only prove the proposition for the family \( \{ A^H_T \}_{T > 0} \) since the proof for the set \( A^H_S \) is identical. Moreover, it is sufficient to consider the case of \( q = 1 \), and then the general case follows from Corollary 7.19.

Notice that
\[
\ln \left( A^H_T^{(+\epsilon)} \right) = [-\epsilon, T + \epsilon],
\]
\[
\ln \left( A^H_T^{(-\epsilon)} \right) = [\epsilon, T - \epsilon].
\]

We shall prove LWR of \( \{ A^H_T \}_{T > 0} \) computationally, by splitting to different cases according to the sign of \( \rho(H) \). Assume first that \( 2\rho(H) \neq 0 \), and then
\[
\mu_{AH} \left( A^H_T^{(+\epsilon)} \right) = \int_{t=-\epsilon}^{t=T+\epsilon} e^{2\rho(H)t} dt = \frac{1}{2\rho(H)} \left( e^{2\rho(H)T + \epsilon} - e^{-2\rho(H)\epsilon} \right),
\]

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and

\[ \mu_{A_H} \left( \left( A_H^T \right)^{(-\epsilon)} \right) = \int_{t=\epsilon}^{t=T-\epsilon} e^{2\rho(H)t} dt = \frac{1}{2\rho(H)} \left( e^{2\rho(H)(T-\epsilon)} - e^{2\rho(H)\epsilon} \right). \]

It follows that,

\[ \frac{\mu_{A_H} \left( \left( A_H^T \right)^{(\epsilon)} \right) - \mu_{A_H} \left( \left( A_H^T \right)^{(-\epsilon)} \right)}{\mu_{A_H} \left( \left( A_H^T \right)^{(-\epsilon)} \right)} = \frac{(e^{2\rho(H)(T+\epsilon)} - e^{-2\rho(H)\epsilon}) - (e^{2\rho(H)(T-\epsilon)} - e^{2\rho(H)\epsilon})}{e^{2\rho(H)(T-\epsilon)} - e^{2\rho(H)\epsilon}}. \]

- If \( 2\rho(H) > 0 \) we continue in the following way

\[ \frac{e^{2\rho(H)T} + 1}{e^{2\rho(H)T}} \ \frac{e^{2\rho(H)\epsilon} - e^{-2\rho(H)\epsilon}}{e^{-2\rho(H)\epsilon} - e^{2\rho(H)\epsilon}}. \]

For \( \epsilon \leq \frac{1}{2e^{2\rho(H)}} \) and \( T \geq \frac{4}{2\rho(H)} \) it holds that \( e^{2\rho(H)\epsilon} - e^{-2\rho(H)\epsilon} \leq 3 \cdot 2\rho(H) \epsilon \) and \( e^{-2\rho(H)\epsilon} \leq \frac{1}{2} \); then,

\[ \frac{\mu_{A_H} \left( \left( A_H^T \right)^{(\epsilon)} \right) - \mu_{A_H} \left( \left( A_H^T \right)^{(-\epsilon)} \right)}{\mu_{A_H} \left( \left( A_H^T \right)^{(-\epsilon)} \right)} \leq 2 \cdot \frac{3 \cdot 2\rho(H) \epsilon}{1/2} = 12 \cdot 2\rho(H). \]

- If \( 2\rho(H) < 0 \), we have

\[ \frac{(e^{-2\rho(-H)\epsilon} - e^{2\rho(-H)(T+\epsilon)}) - (e^{2\rho(-H)\epsilon} - e^{2\rho(-H)(T-\epsilon)})}{e^{2\rho(-H)\epsilon} - e^{2\rho(-H)(T-\epsilon)}} \]

\[ = \frac{(e^{-2\rho(-H)\epsilon} - e^{2\rho(-H)(T-\epsilon)}) + (e^{-2\rho(-H)(T+\epsilon)} - e^{-2\rho(-H)(T+\epsilon)})}{e^{2\rho(-H)\epsilon} - e^{2\rho(-H)(T-\epsilon)}} \]

\[ = \frac{1 + e^{-2\rho(-H)\epsilon}}{e^{-2\rho(-H)\epsilon} - e^{-2\rho(-H)(T+\epsilon)}}. \]

So, the same computation as in the previous case shows that the last expression is \( \leq 2 \cdot \frac{3|\rho(H)|\epsilon}{1/2} = 12 \cdot 2|\rho(H)|\epsilon \) when \( \epsilon \leq \frac{1}{2e^{2\rho(H)}} \) and \( T \geq \frac{4}{2\rho(H)} \).

Finally, when \( 2\rho(H) = 0 \),

\[ \frac{\mu_{A_H} \left( \left( A_H^T \right)^{(\epsilon)} \right)}{\mu_{A_H} \left( \left( A_H^T \right)^{(-\epsilon)} \right)} = \frac{T + 2\epsilon}{T - 2\epsilon} = 1 + \frac{4}{T - 2\epsilon} \leq 1 + 4\epsilon, \]

when \( T - 2\epsilon > 1 \), which holds when for \( \epsilon < 1/4 \) and \( T > 1 \).

\[ \square \]

### 9.2 Counting results for families of product sets

Denote \( L := \dim(A) \). Let \( H_1', \ldots, H_l', H_1'', \ldots, H_{l-l}' \) be a basis for \( A \) such that \( H_1', \ldots, H_l' \in \mathcal{C} \setminus \{0\} \), where \( \mathcal{C} \) is the positive Weil chamber w.r.t. \( N \). Denote

\[ A' = A^{H_1'}, A'' = A^{H_2''}, \]

where \( H_1' = (H_1', \ldots, H_l') \) and \( H_2'' = (H_1'', \ldots, H_{l-l}'') \). We will assume the following situation, which occurs in the parameterization we will use in the proofs of Theorems \([A] \) and \([B] \).
Assumption 9.8. for every $i = 1, \ldots, L - l$,  
$$2p(H''_i) < 0 \text{ for every } i = 1, \ldots, L - l.$$  
This assumption can be achieved, for example, by requiring that $H''_i \in -\mathcal{C} \setminus \{0\}$ for every $i$.

Notation 9.9. For $H = (H_1, \ldots, H_q)$ and $m(H_j)$ as defined in Notation 8.1, denote  
$$m_H = \max_j \{ m(H_j) \} = \max_i \{ -\phi_i(H_j), 0 \}.$$  

Example 9.10. We continue Example 9.4 for $G = \mathrm{SL}_n(\mathbb{R})$. Choose, $H' = \left( \frac{1}{n-1}, \ldots, \frac{1}{n-1}, -1 \right)$ and $H''_i = -\frac{1}{2} e_i, \frac{1}{2} e_{i+1,i+1}$ for $i = 1, \ldots, n - 2$. In this case,  
$$A' = \begin{bmatrix}
  e^{\frac{t}{n-1}} & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 0 \\
\end{bmatrix}, \quad A'' = \begin{bmatrix}
  e^{-\frac{4t}{n-1}} & \cdots & 0 & 0 \\
  e^{-\frac{t}{2}} & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & e^{-\frac{s_{n-3} - s_{n-2}}{2}} & 0 \\
  0 & \cdots & 0 & 1 \\
\end{bmatrix}.$$  

The subgroups involved carry the following measures:  
$$\mu_{A'} = \mu_{A'_H} = e^{nt} dt,$$
$$\mu_{A''_i} = \frac{ds_i}{e^{s_i}},$$
$$\mu_{A''} = \frac{ds_1 \cdots ds_{n-2}}{e^{s_1 + \cdots + s_{n-2}}} = \prod_{i=1}^{n-2} \frac{ds_i}{e^{s_i}},$$
$$\mu_{A} = \frac{e^{nt}}{e^{s_1 + \cdots + s_{n-2}}} dt ds_1 \cdots ds_{n-2}.$$  

Finally, $m_{H''} = 1$ (since $m_{H''_j} = 1$ for all $j$).

The following two theorems are concerned with counting in product sets in an algebraic simple Lie group $G$, where the growth of the sets is only in the $A'$ component, and the remaining components are restricted to compact sets. The key to the proof of these two theorems is the simple observation that since the elements $H'_{ij}$ are in $\mathcal{C} \setminus \{0\}$, then  
$$\begin{vmatrix}
  m \left( B_T \cdot H' + \mathcal{S} \cdot H'' \right) = m \left( \mathcal{S} \cdot H'' \right) \leq m_{H''} S, (9.1)
\end{vmatrix}$$

where  
$$S := \mathcal{S} \cdot (1, \ldots, 1) = \sum S_i.$$  

Theorem 9.11 (generalizes Theorem C). Let $G$ be an algebraic simple non-compact Lie group and let $\Gamma < G$ be a lattice. Set  
$$B_T = \{ g = ka''n'a'n' \in G : (k, a'', n'', n') \in B \text{ and } a' \in A'_T \}.$$  

If $B \subseteq K \times A'' \times N'' \times N'$ is BCS, then  
$$\# (B_T \cap \Gamma) \leq \frac{\mu \left( B_T \right)}{\mu \left( G / \Gamma \right)} \left\ll \mu \left( B_T \right)^{1 - \kappa(\Gamma) + \epsilon} \right\ll$$

for $0 < \epsilon < \kappa(\Gamma)$ and $T$ large enough.
Proof. The claim will follow from Theorem 7.4 once established that the family \( \mathcal{B}_T \) is LWR. According to Remark 7.20, a (direct) product of well-rounded sets is well-rounded. According to Corollary 8.6 the map \( G \to K \times A^* \times A'' \times \Lambda' \times N'' \) of the GI decomposition is an \( f \)-roundomorphism, with \( f \) that (according to the observation in Formula 9.1) is bounded when the \( A'' \), \( N'' \) and \( \Lambda' \) components are restricted to a bounded set. The set \( \mathcal{B} \) is BCS, hence well-rounded according to Proposition 7.20 and the set \( \mathcal{A} \) is well-rounded according to Proposition 9.6. The projections of \( \mathcal{B} \) to \( A'' \), \( \Lambda' \) and \( N'' \) are bounded, since \( \mathcal{B} \) is.

We now proceed to consider a special case of Theorem 9.11, which is a family of increasing product sets w.r.t. the \( KA'A''N \) coordinates. Let

\[
\mathcal{B}^S_T (\Psi; \Phi) := \Phi A'_T A''_S \Phi,
\]

(9.2)

where \( S \in \mathbb{R}^{\dim A''} \) is fixed. The next theorem is a counting statement for this family \( \mathcal{B}^S_T \), which adds to Theorem 9.11 the effect of \( S \) on the error term and on the bound for \( T \). To state it, we introduce the following notation:

**Notation 9.12.** For \( m_{H''} \) defined in Notation 9.9 let

\[
\lambda_{H''} = \lambda_{H''} (G) := \frac{1 + \dim G}{2m_{H''} \dim G}.
\]

**Example 9.13.** In the case of \( G = SL_n (\mathbb{R}) \) and \( H'' \) as in Example 9.10 let \( \lambda_n := \lambda_{H''} (SL_n (\mathbb{R})) \) denote the parameter defined in Notation 9.12. Then,

\[
\lambda_n = \frac{n^2}{2 (n^2 - 1)}.
\]

**Theorem 9.14.** Let \( G \) an algebraic simple non-compact Lie group, \( \Gamma < G \) a lattice and \( \lambda_{H''} \) as in Notation 9.12. Assuming that \( \Psi \subseteq N \) and \( \Phi \subseteq K \) are BCS and that \( H', H'' \) satisfy Assumption 9.8, then for \( S > 0 \),

\[
\left| \left( \mathcal{B}^S_T (\Psi; \Phi) \cap \Gamma \right) \right| \sim \frac{\mu_K (\Phi) \mu_N (\Psi) \mu_{A''} (A''_S)}{\mu (G / \Gamma)} \cdot \left( \frac{e^{2\rho (H')} T}{\mu (\mathcal{B}^S_T susceptibility \mu (\mathcal{B}^S_T)} \cdot \left( \frac{e^{2\rho (H')} T}{\mu (\mathcal{B}^S_T)} \right)^{(1 - \kappa (\Gamma) + \varepsilon)}\right)
\]

for \( 0 < \varepsilon < \kappa (\Gamma) \) and for \( T \)

\[
\frac{S}{\delta_{\Phi; \Psi}} \geq 2\rho (H') \kappa (\Gamma) \lambda_{H''}.
\]

where \( S = \sum S_i \).

**Proof.** Due to Assumption 9.8 we have that \( \mu (\mathcal{B}^S_T ) \sim_{\Phi, \Psi} e^{2\rho (H') T} \). We first claim that the family \( \mathcal{B}^S_T (\Psi; \Phi) \) is LWR with \( C \sim_{\Phi, \Psi} e^{2\rho (H') S} \) and \( T_0 \) that is independent of \( \Phi, \Psi \) and \( S \). To see this, we apply the same considerations as in Theorem 9.11, while paying attention to the LWR parameters. We then get that \( T_0 \) is the same as in Proposition 9.6 regarding the regularity of \( A''_W \), and in particular independent of \( \Phi, \Psi \) and \( S \). Moreover, that the parameter \( C \) is proportional to the maximum on \( \mathcal{B}^S_T (\Psi; \Phi) \) of the function \( f \) such that \( G \to K \times A^* \times A''_W \times N \) is an \( f \)-roundomorphism, which by Corollary 8.6 is \( \sim_{\Phi, \Psi} e^{2\rho (H') S} \).

Now Theorem 9.14 is a direct consequence of Theorem 7.4 where it is only left to verify the condition on \( T \). The latter is obtained by substituting the parameter \( C \) into the condition 7.2 in Theorem 7.4. Indeed, using the notation of Theorem 7.4 this condition is equivalent to

\[
\kappa (\Gamma) \ln (\mu (\mathcal{B}_T )) \gtrsim \frac{\dim G}{1 + \dim G} \ln C_B.
\]

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Since \( \mu(B_T) \asymp \phi, \Psi e^{2\rho(H')T} \) and \( C \leq C_{G} \prec_{\phi, \Phi} e^{2m_{H'}S} \), this condition holds if
\[
\kappa(\Gamma) \cdot 2\rho(H')T \geq \dim G \frac{\dim H - \dim G}{1 + \dim G} \cdot 2m_{H'}S = \frac{S}{\lambda_{H'}}.
\]
Put differently, if
\[
T \geq \frac{S}{\phi, \Psi} \frac{2\rho(H')}{\kappa(\Gamma)} \lambda_{H'}.
\]

The content of the following corollary is that at the cost of compromising the error term, we can allow the sum of \( S_i \)s to grow proportionally to \( T \). Namely, we pass from counting in sets where the growth is only in the \( A' \) axes, to counting in sets where the growth is in all of the \( A \) axes.

**Corollary 9.15.** Let \( G, \Gamma \) and \( \lambda_{H'} \) be as in Theorem 9.14. Then for every \( 0 < \epsilon < \kappa(\Gamma), \delta \in (0, \kappa(\Gamma) - \epsilon) \) and \( S(T) = (S_1(T), \ldots, S_{L(r)}(T)) \) such that \( S(T) = \sum S_i(T) < \lambda_{H'} \delta \cdot 2\rho(H')T \),
\[
\#(B_T^{S(T)} \cap \Gamma) - \frac{\mu(\Phi) \mu_{\Psi}(\Psi)}{2\rho(H')} e^{2\rho(H')T}\left(e^{2\rho(H')T} \right)^{1-(\kappa(\Gamma)+\delta+\epsilon)} \ll_{\phi, \Phi} \epsilon
\]
for \( T \geq 0 \).

**Proof.** Let \( S = S(T) > 0 \) and \( \lambda = \lambda_{H'} \). In order for the main term in Theorem 9.14 to be of lower order than the main term, we require the existence of a parameter \( \gamma \in (0, 1) \) for which
\[
\frac{S}{\lambda} + (1 - \kappa(\Gamma) + \epsilon) \cdot 2\rho(H')T < \kappa(\Gamma) \cdot 2\rho(H')T.
\]
This is equivalent to
\[
S < \lambda \cdot (\gamma + \kappa(\Gamma) - \epsilon - 1) \cdot 2\rho(H')T.
\]
Hence, if we denote by \( \delta \) the number \( \gamma + \kappa(\Gamma) - \epsilon - 1 \), we must require that \( \delta > 0 \) and that \( \gamma = \delta + (1 + \epsilon - \kappa(\Gamma)) \) lies in \((0, 1)\). If \( 0 < \epsilon < \kappa(\Gamma) \), then clearly \( 0 < 1 + \epsilon - \kappa(\Gamma) < 1 \), so the condition on \( \delta \) becomes \( \delta \in (0, \kappa(\Gamma) - \epsilon) \).

The counting in Theorem 9.14 requires \( T \geq \frac{S}{\phi, \Psi} \), which is equivalent to
\[
S \leq 2\rho(H') \lambda \kappa(\Gamma) \cdot T,
\]
i.e.
\[
S \leq \min \{2\rho(H') \lambda \delta \cdot T, 2\rho(H') \lambda \kappa(\Gamma) \cdot T \} = 2\rho(H') \lambda \delta \cdot T
\]
for \( T \) large enough and \( \delta \in (0, \kappa(\Gamma) - \epsilon) \). \( \square \)

### 10 Counting in fibered sets with Lipschitz fibers

The goal of Sections 10 and 11 is to prove Proposition 5.5, which is the missing element in the proof of Theorems A and B. This proposition is concerned with counting in the family of sets:
\[
\Delta_\Phi^S \left( \mathcal{Y}_{F_n-1}, \Phi \right) \subset SL_n(\mathbb{R}).
\]

These sets are defined w.r.t. the GI coordinates on \( SL_n(\mathbb{R}) \), but they do not have the form of product sets as considered in Section 9. Rather, they carry a slightly more complicated structure of a *fibration*, so proving that they are indeed LWR presents a technical challenge. This challenge will be overcome by
a similar strategy to the one we used for proving LWR in product sets: first we prove well roundedness of the images in \(K \times A' \times A'' \times N' \times N\), and then we pull back to SL\(_n\) (\(\mathbb{R}\)) with a roundomorphism.

In the current section we discuss well roundedness of families of fibered sets in a group that is a Cartesian product; we consider sets \(B_T\) inside \(P \times \mathbb{R}^m\) which consist of slices of the form \(z \times D_z\) with \(z \in P\) and \(D_z \subseteq \mathbb{R}^m\). The goal of this section is to prove Proposition \(\text{[10.6]}\) which ensures that such families are indeed LWR under certain regularity conditions on the fibers \(D_z\) and on the “base set” (the projection to \(P\)).

### 10.1 Bounded Lipschitz Continuous families

In this subsection we formulate a regularity condition on a family \(\{D_z\}\) of subsets of \(\mathbb{R}^n\).

**Definition 10.1.** Let \(P\) be a Lie group and \(O\) a family of coordinate balls. Let \(E\) be a subset of \(P\), and consider the family \(\mathcal{D}_E = \{D_z\}_{z \in E}\), where \(D_z \subseteq \mathbb{R}^m\) (\(m\) is uniform for all \(z\)). We say that the family \(\mathcal{D}_E\) is bounded Lipschitz continuous (or BLC) w.r.t \(O\) if there exists \(C > 0\) such that for every \(0 < \epsilon < C^{-1}\) the following hold:

1. For a norm ball \(B_\epsilon \subseteq \mathbb{R}^m\) of radius \(\epsilon\), \(D_z + B_\epsilon \subseteq (1 + C\epsilon)D_z\).
2. If \(z' \subseteq O\) and \(O\) for \(z, z' \in E\), then \(D_{z'} \subseteq (1 + C\epsilon)D_z\).
3. The volume of \(D_z\) (w.r.t. Lebesgue measure) is bounded uniformly from below by a positive constant \(V_{\min}\).
4. \(D_z \subseteq B_R\) for some uniform \(R > 0\) and every \(z \in E\).

The following Lemma shows that in \(\mathbb{R}^n\), the notion of BLC extends the notion of BCS from convex sets to families of sets.

**Lemma 10.2.** Assume \(D \subseteq \mathbb{R}^n\) is a BCS that is convex and contains the origin as an internal point. Then the constant family \(\mathcal{D}_E = \{D_z\}_{z \in E}\) with \(D_z = D\) for every \(z\) is BLC.

**Proof.** The second property of BLC is trivial since \(D\) is constant, and the third and fourth properties hold since \(D\) is bounded and of positive measure. It remains to show that \(D\) satisfies the first property of BLC. Let \(\alpha > 0\) be such that \(D\) contains a ball of radius \(\alpha\) around the origin; we show that \(D + B_\epsilon \subseteq (1 + \alpha^{-1}\epsilon)D\). Indeed, let \(x \in D\) and \(v \in \mathbb{R}^n\) such that \(\|v\| = 1\). Then

\[
\begin{align*}
x + \epsilon v &= (x + \epsilon v) \left(1 + \frac{\epsilon}{\alpha}\right) \frac{1}{1 + \frac{\epsilon}{\alpha}} = \left(1 + \frac{\epsilon}{\alpha}\right) \cdot \frac{x + \epsilon v}{1 + \frac{\epsilon}{\alpha}} \\
&= \left(1 + \frac{\epsilon}{\alpha}\right) \left[\frac{1}{1 + \frac{\epsilon}{\alpha}} \cdot x + \frac{\epsilon}{\alpha} \cdot v\right] \\
&= \left(1 + \frac{\epsilon}{\alpha}\right) \cdot v
\end{align*}
\]

where \((*)\) lies in \(D\), as a convex combination of the two points \(x, \alpha v\) in \(D\). \(\square\)

The following proposition and corollary are concerned with certain manipulations that can be performed on fibered sets, while maintaining the BLC property of the fibers. These manipulations include pulling back the fibers by a locally-Lipschitz map, and enlarging the basis set by taking a product with another set.

**Proposition 10.3.** Let \(P, P'\) be Lie groups and suppose that \(r : P' \to P\) is an \(f\)-locally Lipschitz map (Definition \(7.17\)). Let \(E \subseteq P\) and \(E' := r^{-1}(E) \subseteq P'\). If the family

\[
\mathcal{D}_E = \{D_z\}_{z \in E}
\]

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is BLC with parameters \((C, V_{\text{min}}, R)\), then the family

\[ \mathcal{D}_{\mathcal{E}} = \{ D_{r(z)} \}_{z \in \mathcal{E}} \]

is BLC with parameters \((FC, V_{\text{min}}, R)\), where \(F = \sup_{g \in r^{-1}(\mathcal{E})} f(g) < \infty\).  

**Proof.** Since \(\mathcal{E} \subseteq \mathcal{E}'\), properties 1, 3, and 4 of BLC hold automatically in \(\mathcal{D}_{\mathcal{E}'}\), and it is only left to verify the second property. Indeed, if \(z' \in O_{\mathcal{E}'}^\rho(z) O_{\mathcal{E}'}^\rho\), then by local Lipschitzity and definition of \(\mathcal{F}\), \(D_{r(z')} \subseteq (1 + C \cdot F \epsilon) D_{r(z)}\).

**Corollary 10.4.** Let \(P \times Q\) be a product of Lie groups and let \(\mathcal{E} \subseteq P, \mathcal{E}' = \mathcal{E} \times Q\). If \(\mathcal{D}_{\mathcal{E}} = \{D_z\}_{z \in \mathcal{E}}\) is BLC w.r.t. \(O_{\mathcal{E}}^\rho\), then \(\mathcal{D}_{\mathcal{E}'} = \{D_{(z,q)}\}_{(z,q) \in \mathcal{E}'}\) such that \(D_{(z,q)} = D_z\ \forall q \in Q\) is BLC with the same parameters and w.r.t. \(O_{\mathcal{E}}^\rho \times O_{\mathcal{Q}}^\rho\).

**Remark 10.5.** Clearly we can replace the group \(Q\) in the definition of \(\mathcal{E}'\) with any subset \(B \subseteq Q\).

**Proof.** This follows from Proposition 10.3 using the projection map \(r: P \times Q \to P\) which is an \(f\)-local Lipschitz map with \(f \equiv 1\).  

### 10.2 Well roundedness of fibered sets in product of groups

The goal of this subsection is to prove the following proposition:

**Proposition 10.6.** Let \(\{\mathcal{E}_T\}_{T > 0}\) be an increasing family inside a Lie group \(P\), and \(\mathcal{E} := \cup_{T > 0} \mathcal{E}_T\). Let \(\mathcal{D}_{\mathcal{E}} = \{D_z\}_{z \in \mathcal{E}}\) where \(D_z \subset \mathbb{R}^m\), and consider the family

\[ \mathcal{B}_T = \bigcup_{z \in \mathcal{E}_T} z \times D_z \subseteq P \times \mathbb{R}^m. \]

If \(\{\mathcal{E}_T\}_{T > 0}\) is LWR with parameters \((T_0, C_{\mathcal{E}})\), and \(\mathcal{D}_{\mathcal{E}}\) is BLC w.r.t. a family \(\{O_{\mathcal{E}}^\rho\}_{\rho > 0}\) of coordinate balls and with parameters \((C_{\mathcal{E}}, V_{\text{min}}, R)\), then \(\mathcal{B}_T\) is LWR w.r.t the coordinate balls \(O_{\mathcal{E}}^\rho \times B_{\epsilon/2} \subset P \times \mathbb{R}^m\) and with parameters \((T_0, C_{\mathcal{B}})\) where

\[ C_{\mathcal{B}} < C_{\mathcal{E}} + \frac{V_{\text{max}}}{V_{\text{min}}} C_{\mathcal{E}} + 1 \]

and \(V_{\text{max}} = \mu_{\mathbb{R}^m}(B_R)\).

**Proof.** **Step 1:** estimation of \(\mathcal{B}_T^{(+\epsilon)}\). We claim that for \(\epsilon < \frac{1}{C_{\mathcal{E}} + 1}\) (so \(\epsilon < 1, C_{\mathcal{E}}^{-1}\)),

\[ \mathcal{B}_T^{(+\epsilon)} \subseteq \left( \bigcup_{z \in \mathcal{E}_T} \left( z \times (1 + C_{\mathcal{E}} \epsilon)^2 D_z \right) \right) \bigcup \{\Delta \mathcal{E}_T \times B_{\epsilon + 1}\} =: Y^++. \]

where

\[ \Delta \mathcal{E}_T := O_{\epsilon}^\rho \mathcal{E}_T O_{\epsilon}^\rho \setminus \mathcal{E}_T. \]
We shall first bound the affect of $O_\epsilon^P$ perturbations. For that recall that $D_z \subseteq B_R$ for all $z \in E$. As a result, for $u, v \in O_\epsilon^P$ we have

$$(v, 0) B_T (u, 0) = \bigcup_{z \in \mathcal{E}_T} (vzu \times D_z) \subseteq \left( \bigcup_{z \in \mathcal{E}_T \cap \mathcal{E}_T u} (z \times D_{v^{-1}zu^{-1}}) \right) \bigcup (\Delta \mathcal{E}_T \times B_R).$$

By the second property of BLC, for $\epsilon < \frac{1}{C_\varphi}$, this is contained in

$$\bigcup_{z \in \mathcal{E}_T} (z \times (1 + C_\varphi \epsilon) D_z) \bigcup (\Delta \mathcal{E}_T \times B_R).$$

We will now address the $B_{\epsilon/2}$ perturbations. To this end, note that by the first property of BLC, for $\epsilon < \frac{1}{C_\varphi}$

$$(1 + C_\varphi \epsilon) D_z + B_\epsilon = (1 + C_\varphi \epsilon) \left( D_z + B_{\epsilon(1 + 2C_\varphi \epsilon)} \right) \subseteq (1 + C_\varphi \epsilon) (D_z + B_\epsilon) \subseteq (1 + C_\varphi \epsilon)^2 D_z.$$

Combining $O_\epsilon^P$ and $B_{\epsilon/2}$ perturbations together we obtain,

$$B_{T}(\epsilon^+) = O_\epsilon B_T O_\epsilon \subseteq \bigcup_{z \in \mathcal{E}_T} \left( z \times (1 + C_\varphi \epsilon)^2 D_z \right) \bigcup (\Delta \mathcal{E}_T \times B_{R+1}) = Y^+$$

(where we have used $\epsilon < 1$).

**Step 2: estimation of $B_{T}(-\epsilon)$**. We claim that for $\epsilon < \frac{1}{2C_\varphi}$,

$$B_{T}(-\epsilon) \supseteq \bigcup_{z \in \mathcal{E}_T} \left( z \times \frac{1}{1 + 2C_\varphi \epsilon} D_z \right) =: Y^-.$$

Indeed, for $u, v \in O_\epsilon^P$ we have

$$(v, 0) Y^- (u, 0) = \bigcup_{z \in \mathcal{E}_T} \left( vzu \times \frac{1}{1 + 2C_\varphi \epsilon} D_z \right) \subseteq \bigcup_{z \in \mathcal{E}_T} \left( z \times \frac{1}{1 + 2C_\varphi \epsilon} D_{v^{-1}zu^{-1}} \right).$$

By the second property of BLC, for $\epsilon < \frac{1}{C_\varphi}$ this is contained in

$$\bigcup_{z \in \mathcal{E}_T} \left( z \times \frac{1}{1 + 2C_\varphi \epsilon} D_z \right).$$

For $B_{\epsilon/2}$ perturbations, by the first property of BLC and for $\epsilon < \frac{1}{2C_\varphi}$,

$$\frac{1}{1 + 2C_\varphi \epsilon} D_z + B_\epsilon = \frac{1}{1 + 2C_\varphi \epsilon} \left( D_z + B_{\epsilon(1 + 2C_\varphi \epsilon)} \right) \subseteq \frac{1}{1 + 2C_\varphi \epsilon} (D_z + B_{2\epsilon}) \subseteq D_z.$$

Combining $O_\epsilon^P$ and $B_{\epsilon/2}$ perturbations together we obtain that $O_\epsilon Y^- O_\epsilon \subseteq B_T$, proving the claim.

**Step 3: estimation of $\mu \left( B_{T}(\epsilon^+) \right) / \mu \left( B_{T}(-\epsilon) \right)$**. Notice that for $\epsilon < \frac{1}{C_\varphi}$

$$\mu_G \left( Y^+ \right) = (1 + C_\varphi \epsilon)^2 \mu_G (B_T) + \mu_P (\Delta \mathcal{E}_T) \mu_R (B_{R+1}) \leq (1 + 3C_\varphi \epsilon) \mu_G (B_T) + \mu_P (\Delta \mathcal{E}_T) V_{\max}$$

(10.1)
and that
\[
\mu_G(Y^-) = \frac{1}{(1 + 2CG\epsilon)^2} \mu_G \left( \bigcup_{z \in E_T^{-\epsilon}} (z \times D_z) \right)
\geq (1 - 4CG\epsilon) \mu_G \left( \bigcup_{z \in E_T^{-\epsilon}} (z \times D_z) \right)
\] (10.2)

Combining what we have shown in the previous steps with estimations (10.1) and (10.2), we obtain that for \( \epsilon < \frac{1}{2C_G + 1} \) (so that both \( \epsilon < \frac{1}{2C_G} \) and \( \epsilon < 1 \)):

\[
\frac{\mu_G(B_T^{(\epsilon)})}{\mu_G(B_T^{(-\epsilon)})} \leq \frac{\mu_G(Y^+)}{\mu_G(Y^-)}
\leq \frac{1}{1 - 4CG\epsilon} \frac{\mu_G(B_T)}{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)} + \frac{V_{\max}}{1 - 4CG\epsilon} \cdot \frac{\mu_P(\Delta E_T)}{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)}
\]

where:

- for \( \epsilon < \frac{1}{32C_G} \),
  \[
  1 + 3C_G\epsilon \leq 1 + 8C_G\epsilon
  \]
- for \( \epsilon < \frac{1}{8C_G} \),
  \[
  \frac{V_{\max}}{1 - 4CG\epsilon} \leq 2V_{\max}
  \]
- for \( \epsilon < C_G^{-1} \) and \( T > T_0 \)
  \[
  \frac{\mu_G(B_T)}{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)} = 1 + \frac{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right) - \mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)}{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)} = 1 + \frac{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right) - \mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)}{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)}
  \leq 1 + \mu_P \left( \mathcal{E}_T \setminus \mathcal{E}_T^{(-\epsilon)} \right) V_{\max} \leq 1 + \mu_P \left( \mathcal{E}_T \setminus \mathcal{E}_T^{(-\epsilon)} \right) V_{\min} \leq 1 + V_{\max} V_{\min} C_G \epsilon
  \]
- and for \( \epsilon < C_G^{-1} \) and \( T > T_0 \),
  \[
  \frac{\mu_P(\Delta E_T)}{\mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right)} = \mu_P \left( \mathcal{O}_x^P \mathcal{E}_T \setminus \mathcal{O}_x^P \mathcal{E}_T \right) \mu_G \left( \bigcup_{z \in E_T^{(-\epsilon)} (z \times D_z)} \right) \leq \mu_P \left( \mathcal{E}_T^{(\epsilon)} \setminus \mathcal{E}_T^{(-\epsilon)} \right) V_{\min} \leq \frac{C_G}{V_{\min}} \epsilon.
  \]

All in all, for \( \epsilon < \frac{1}{32C_G + 1} \) (so that \( \epsilon \leq (32C_G)^{-1} C_G^{-1} \)) and for \( T > T_0 \):

\[
\frac{\mu_G(Y^+)}{\mu_G(Y^-)} \leq (1 + 8C_G\epsilon) \cdot \left( 1 + \frac{V_{\max}}{V_{\min}} C_G \epsilon \right) + 2V_{\max} \cdot \frac{C_G}{V_{\min}} \epsilon
\leq 1 + \left( 4 \frac{V_{\max}}{V_{\min}} C_G + 8C_G \right) \epsilon.
\]

In order to have that LWR holds for \( \epsilon < C_B^{-1} \), we let \( C_B = 4 \frac{V_{\max}}{V_{\min}} C_G + 8C_G + 1. \)

\[\square\]

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11 Counting in fibered sets in $\text{SL}_n(\mathbb{R})$

The aim of this section is to prove Proposition 6.5 thus end the proof of Theorems A and B. This proposition deals with counting lattice points in the sets

$$\Delta_T^{S} \left( \mathcal{Y}_{F_n^{-1}}^{\alpha}, \Phi \right) = \bigcup_{z \in F_{n-1}^{(S)}} \Phi \cdot p_z^{r'} \cdot A_T' \cdot N'$$

(11.1)

(where $\mathcal{Y}_{F_n^{-1}}^{\alpha}$, $Y^{\alpha \tau}(z)$ were defined in Formula 6.1), which have the structure of a fiber bundle over $F_{n-1}^{(S)} \subset P'$. The plan of this section is as follows.

In Section 11.1 we prove a counting result for families of sets in $\text{SL}_n(\mathbb{R})$ that are fibered over $F_{n-1}^{(S)}$, provided that the fibers satisfy the regularity condition (BLC family) defined in Section 10.1; in Section 11.2 we prove that the family $\mathcal{Y}_{F_n^{-1}}^{\alpha} = \{ Y^{\alpha \tau}(z) \}$ indeed satisfies this regularity condition.

We note that we now abandon the discussion about a general non-compact algebraic simple Lie group, and focus solely on $\text{SL}_n(\mathbb{R})$.

11.1 Fibered sets in $\text{SL}_n(\mathbb{R})$

In this subsection we establish a counting lattice points result in families of sets in $\text{SL}_n(\mathbb{R})$ that have the structure of a fiber bundle over the truncated fundamental domain $F_{n-1}^{(S)} \subset P''$. These are the sets $\Delta_T^{S} \left( \mathcal{Y}_{F_n^{-1}}^{\alpha}, \Phi \right)$ defined in Proposition 6.5 and recalled in Formula 11.1. The goal is to prove Proposition 11.1 below; to state it, we first set the necessary background.

We focus on the $KA''N''A'N'$ decomposition of $\text{SL}_n(\mathbb{R})$, introduced in Section 1 and recalled in Examples 9.4, 9.10 and 9.13. Here $N' = \left[ \begin{array}{c} 1_{n-1} \\ 0_{n-1} \end{array} \right] \cong \mathbb{R}^{n-1}$, $N'' = \left[ \begin{array}{c} 1 \cdots 1 \\ 0 \cdots 0 \end{array} \right]$ and $K = \text{SO}_n(\mathbb{R})$. Also,

$$A' = \left\{ \text{diag} \left( e^{s_i T}, e^{-t} \right) : t \in \mathbb{R} \right\} = A'' \text{ for } H'' = \left( \begin{array}{c} \frac{1}{n-1} \cdots 1 \end{array} \right),$$

and

$$A'' = \left\{ \text{diag} \left( e^{-s_{i-1}}, e^{s_{i-1}}, \cdots, e^{s_{n-2}}, 1 \right) : s_1, \cdots, s_{n-2} \in \mathbb{R} \right\}$$

$$= \prod_{i=1}^{n-2} A''_{H''} \text{ for } \mu''_{H''} = \frac{1}{2} e^{s_i + 1} + \frac{1}{2} e^{s_{i+1}}.$$

The Haar measure on $\text{SL}_n(\mathbb{R})$ is $\mu_K \times \mu_{A''} \times \mu_{N''} \times \mu_{A'} \times \mu_{N'}$ where $\mu_K, \mu_{N''}, \mu_{N'}$ are Haar measures (in particular $\mu_{N'}$ is the Lebesgue measure on $\mathbb{R}^{n-1}$), $\mu_{A'} = e^{nt} dt$ and $\mu_{A''} = \prod_{A''}$ with $\mu_{A''} = \frac{2}{k''}$.

As a result, the volume of $\Delta_T^{S} \left( \mathcal{Y}_{F_n^{-1}}^{\alpha}, \Phi \right)$ is given by

$$\mu \left( \Delta_T^{S} \left( \mathcal{Y}_{F_n^{-1}}^{\alpha}, \Phi \right) \right) = \frac{1}{n} \left( e^{nt} - 1 \right) \cdot I \left( \mathcal{Y}_{F_n^{-1}}^{\alpha} \right) \cdot \mu_K (\Phi)$$

where

$$I \left( \mathcal{Y}_{F_n^{-1}}^{\alpha} \right) := \int_{z \in F_{n-1}} \left( \int_{Y^{\alpha \tau}(z)} d\mu_{P''} \right) d\mu_{P''} = \int_{z \in F_{n-1}} \text{EucVol} \left( Y^{\alpha \tau}(z) \right) d\mu_{P''}$$

(here $\mathcal{Y}_{F_n^{-1}}^{\alpha}$, $Y^{\alpha \tau}(z)$ were defined in Formula 6.1).
Finally, we recall the map
\[ r_2 : \text{SL}_n(\mathbb{R}) \to K \times A'' \times N'' \times A' \times N' \]
\[ g \mapsto (k, a'', n'', a', n') \]
defined in Corollary [8.6]. We denote the restriction of \( r_2 \) to the subgroup \( P'' = A''N'' \) of \( \text{SL}_n(\mathbb{R}) \) by \( r_{P''} \), namely
\[ r_{P''} : P'' \to A'' \times N'' \]
\[ a''n'' \mapsto (a'', n''). \]

**Proposition 11.1.** Assume that
\[ \mathcal{D} = \{ \mathcal{D}_{P''}(F_{n-1}) \} \]
is a BLC family of subsets of \( \mathbb{R}^n \) and that \( \Phi \subseteq K \) is a BCS. Set
\[ \lambda_n = \frac{n^2}{2(n-1)} \] (Example [9.13]). Let \( \Gamma < \text{SL}_n(\mathbb{R}) \) be a lattice and \( \kappa = \kappa(\Gamma) \).

1. For \( 0 < \epsilon < \kappa, \sum S = (S_1, \ldots, S_{n-2}) \), \( S = \sum_{i=1}^{n-2} S_i \) and every \( T \geq \frac{\sum_{i=1}^{n-2} S_i}{\lambda_n \kappa(\Gamma)} \),
\[ \# \left( \Delta_{\mathcal{S}}(\Phi, \mathcal{D}_{P''}(F_{n-1})) \cap \Gamma \right) = \frac{I \left( \mathcal{D}_{P''}(F_{n-1}) \right) \mu_K(\Phi)}{\mu(\Gamma \backslash G)} \frac{e^{\kappa T}}{n} \leq \mathcal{S}, \Phi, \epsilon e^{S/\lambda_n} e^{nT(1-\kappa+\epsilon)}. \]

2. For \( 0 < \epsilon < \kappa, \delta \in (0, \kappa - \epsilon) \), \( S(T) = (S_1(T), \ldots, S_{n-2}(T)) \) such that \( S(T) = \sum S_i(T) < n\delta \lambda_n T \) and every \( T \geq 0 \),
\[ \# \left( \Delta_{\mathcal{S}(T)}(\Phi, \mathcal{D}_{P''}(F_{n-1})) \cap \Gamma \right) = \frac{I \left( \mathcal{D}_{P''}(F_{n-1}(T)) \right) \mu_K(\Phi)}{\mu(\Gamma \backslash G)} \frac{e^{\kappa T}}{n} + O_{\mathfrak{S}, \Phi, \epsilon} \left(e^{nT(1-\kappa+\delta+\epsilon)}\right). \]

As in our counting results for product sets (Section [9]), our proof strategy is to show well roundedness of the image of \( \Delta_{\mathcal{S}}(\Phi) \) in the Cartesian product \( K \times A'' \times N'' \times A' \times N' \), and then pull back to \( \text{SL}_n(\mathbb{R}) \) via the map \( r_2 \), which is a roundomorphism (Corollary [8.6]). In order to prove well roundedness in the Cartesian product, we need the following Lemma:

**Lemma 11.2.** The set \( r_{P''}(F_{n-1}) \) is LWR in \( A'' \times N'' \). As a result, \( r_{P''}(F_{n-1}(S)) \) is LWR with parameters that do not depend on \( \sum S \).

**Remark 11.3.** The set \( F_{n-1} \) itself is not LWR in \( P'' \).

Since Lemma [11.2] is about counting in a group that is a Cartesian product, it is proved by working in each of the components separately. Among the two components \( A'' \) and \( N'' \), the problematic one is of course \( A'' \); the role of the following two lemmas is to handle this component.

**Lemma 11.4.** The projection to the \( A'' \) component of \( F_{n-1} \) is bounded from below for every \( i = 1, \ldots, n-2 \).

**Proof.** We need to show that for every \( H \in a'' \) such that \( \exp(H) \cdot n \in F_{n-1} \), it holds that the coefficients of \( H \) in its presentation of a linear combination of \( \{ H_j \} \) are bounded from below. These coefficients are given by linear functionals: \( H = \sum_{j=1}^{n-2} \psi_j(H) H_j'' \) (actually, \( \{ \psi_j \}_{j=1}^{n-2} \subset (a'')^* \) is the dual basis to
\( \{ H''_j \}_{j=1}^{n-2} \subset a'' \). Denote \( \phi_i := \phi_{i+1,i} \) where \( \{ \phi_{i,j} \} \) are the roots for \( \text{SL}_n (\mathbb{R}) \) defined in Example 9.4. Clearly \( \{ \phi_i \} \) form a basis to \( (a'')^* \), and by Lemma 2.2 they satisfy that
\[
\phi_i (H) \geq \ln \left( \frac{\sqrt{3}}{2} \right)
\]
for every \( i = 1, \ldots, n - 2 \) and \( H \) as above. It is therefore sufficient to show that in the presentation of every \( \psi \), as a linear combination of \( \{ \phi_i \} \), the coefficients are non-negative. Write \( \psi_i = 2 \sum_{j=1}^{n-2} x_{i,j} \phi_j \) and evaluate at each of \( H''_1, \ldots, H''_{n-2} \) we obtain the following system of linear equations
\[
\begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 2 & -1 & 2
\end{pmatrix}
\begin{pmatrix}
x_{i,1} \\
x_{i,2} \\
x_{i,3} \\
\vdots \\
x_{i,n-2}
\end{pmatrix}
= e_i.
\]
A computation shows that the solution \( (x_{i,j})_{j=1}^{n-2} \) is indeed non-negative. \( \square \)

To see how the following lemma concerns the \( A'' \) component, notice that the group \( (A''_i, d\mu_{A''_i}) \) is measure preserving isomorphic to \( (\mathbb{R}^{\geq 0}, d\mu_{\mathbb{R}^{\geq 0}}) \) for every \( i = 1, \ldots, n - 2 \).

**Lemma 11.5.** The map
\[
\psi : (\mathbb{R}^{\geq 0}, d\frac{dx}{x^2}) \to (\mathbb{R}, +, 1_{(0, \infty)} (x) \cdot dx)
\]
\[
\psi (x) = \frac{1}{x}
\]
is a \( f \)-roundomorphism with
\[
f (x) = \frac{2}{x}.
\]

**Proof.** A standard computation shows that \( \varphi \) pushes \( d\frac{dx}{x^2} \) to \( 1_{(0, \infty)} (x) \cdot dx \). Moreover, for \( \epsilon < \frac{1}{12} \),
\[
\psi (O_{\epsilon}^{x \geq 0} xO_{\epsilon}^{x \geq 0}) \subseteq \psi (x \cdot [1 - 3\epsilon, 1 + 3\epsilon]) \subseteq \frac{1}{x} [1 - 4\epsilon, 1 + 4\epsilon] = \psi (x) + 2f (x) [-\epsilon, \epsilon] = O_{\epsilon f} \psi (x) O_{\epsilon f}.
\]
\( \square \)

**Proof of Lemma 11.2.** Consider the map
\[
\varphi : A'' \times N'' \to (\mathbb{R}, +, 1_{(0, \infty)} (x) \cdot dx)^{x(n-2)} \times N''
\]
induced by the map given in the previous Lemma. It is an \( f \)-roundomorphism with \( f (x_1, \ldots, x_{n-2}, n'') = \frac{1}{x_1 \cdots x_{n-2}}. \)

Since, by Lemma 11.4, \( \pi_{A''_i} (r_{p''_i} (F_{n-1})) \) is bounded from below for every \( i \), we conclude that \( \varphi (r_{p''_i} (F_{n-1})) \) is a bounded set. By Corollary 2.10 \( \partial \varphi (r_{p''_i} (F_{n-1})) = \varphi (r_{p''_i} \partial (F_{n-1})) \) is contained in a finite union of lower dimensional embedded submanifolds, so, according to Proposition 7.16 \( \varphi (r_{p''_i} (F_{n-1})) \) is LWR. Finally, since \( f |_{r_{p''_i}(F_{n-1})} \) is bounded, then by Corollary 7.19 we conclude that \( r_{p''_i} (F_{n-1}) \) is LWR.

We now turn to prove that the set \( r_{p''_i} (F_{n-1}) \) is LWR; this set is the intersection of \( r_{p''_i} (F_{n-1}) \) with the set \( A''_i \times \pi_{N''_i} (F_{n-1}) \). According to Lemma 7.13 LWR property is maintained under intersections, and so it is sufficient to show that \( A''_i \times \pi_{N''_i} (F_{n-1}) \) is LWR. This is indeed the case since \( A''_i \) is LWR with a parameter independent of \( S '' \) by proposition 9.6. \( \pi_{N''_i} (F_{n-1}) \) is LWR since it is a BCS (see Lemma 2.2) and LWR is maintained under taking Cartesian products by Remark 7.20. \( \square \)
Proof of Proposition 11.1. Consider the image of $\Delta^S_T(\Phi, \mathcal{D}_{r_{p''}(F_{n-1})})$ under $r_2$:

$$r_2 \left( \Delta^S_T \right) = \bigcup_{(a'',n'') \in r_{p''}(F_{n-1})} K_\Phi \times (a'',n'') \times A_T' \times N_{D(a'',n'')}$$

$$= \bigcup_{(k',n'',d') \in K_\Phi \times r_{p''}(F_{n-1})} (k',a'',n'',d') \times N_{D(a'',n'')}.$$  

We claim that it is a well rounded family (with increasing parameter $T$) in the group $K \times A'' \times N'' \times A' \times N'$. First, since the family $\mathcal{D}_{r_{p''}(F_{n-1})}$ is assumed to be BLC, then so is the family $\mathcal{D}_{r_{p''}(F_{n-1})}$ and by Corollary 10.4 so is the family $\mathcal{D}_{K \times r_{p''}(F_{n-1})}$.  

As for the base set, $r_{p''}(F_{n-1}) \subset A'' \times N''$ is LWR according to Lemma 11.2, with parameters that do not depend on $S$. The BCS $\Phi$ is LWR by Proposition 7.16 and $A_T'$ is LWR according to Proposition 9.6. Thus, the family $\Phi \times r_{p''}(F_{n-1}) \times A_T'$ inside $K \times A'' \times N'' \times A'$ is LWR (Remark 7.20), which implies by Proposition 10.6 that the family $r_2 \left( \Delta^S_T \right)$ is LWR with Lipschitz constant that is independent of $0$.

Since by Corollary 8.6 $r_2$ is an $f_2$-roundomorphism with $f_2 \left( ka''n''d' \right) \prec e^{2a}$, it follows from Corollary 7.19 that $\Delta^S_T \subset \text{SL}_n(\mathbb{R})$ is LWR with

$$C \prec_{\Phi, \mathcal{D}} e^{2S}$$

and $T_0$ that is independent of $\Phi$, $S$ and the family $\mathcal{D}$. The first part of the proposition now follows from Theorem 7.4 and we refer to Theorem 9.14 for the calculation of the lower bound for $T$.

The second part of the proposition follows from the first part in the same way Corollary 9.15 follows from Theorem 9.14. \qed

Remark 11.6. Clearly, the domain $r_{p''}(F_{n-1})\subset A'' \times N''$ over which $\Delta^S_T$ is fibered can be replaced by any other domain $D$ that is LWR in $A'' \times N''$, and the result of Proposition 11.1 would still hold.

11.2 The family $\mathcal{D}_{F_{n-1}}$ is BLC

The goal of this subsection is to show that the family $\mathcal{D}_{F_{n-1}}$ is BLC, concluding the proof of Proposition 6.5 and therefore of Theorems A and B.

The domain $F_{n-1}$, over which the family $\mathcal{D}_{F_{n-1}}$ is fibered, is a subset of $P_{n-1}$, which is the group of $(n-1) \times (n-1)$ upper triangular matrices with positive diagonal entries and determinant 1. To simplify the notation, we consider the situation in general dimension with $F_m \subset P_m$, and write $P_m = A_m N_m$ where $A_m$ is the diagonal subgroup of $\text{SL}_m(\mathbb{R})$ and $N_m$ is the subgroup of upper triangular unipotent matrices. In particular, we abandon the notations of $P''$, $A''$, $N''$.

Let us recall some notations that were introduced in Section 6. For every $z \in F_m$ such that $z = \begin{bmatrix} z_1 & \cdots & z_m \end{bmatrix}$, we denote the lattice spanned by the columns of $z$ by

$$\Lambda_z = \text{span}_z \{ z_1, \ldots, z_m \}.$$  

We also consider the linear map

$$L_z : \mathbb{R}^m \to \mathbb{R}^m$$

$$z_j \mapsto e_j$$

for every $j = 1, \ldots, n$, where $\{ e_1, \ldots, e_n \}$ is the standard basis (in column vectors). Note that $L_z$ maps $\Lambda_z$ to $\mathbb{Z}^n$.  

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Remark 11.7. $L_z^{-1}(x) = zx$ for every $x \in \mathbb{R}^m$ (i.e., the linear map $L_z^{-1}$ is given by the matrix $z$). Hence, $L_z(zx) = x$, namely the image under $L_z$ of a vector is its coordinates w.r.t. the basis $\{z_1, \ldots, z_m\}$ (which is also clear from the definition of $L_z$).

We remind the notation

$$\text{Dir}(z) = \text{Dirichlet domain of } \Lambda_z,$$

and consider the families

$$\mathcal{Y}_F = \{Y(\cdot)\}_{\mathcal{Z} \in F_m} = \{L_z(\text{Dir}(z))\}_{z \in F_m}$$

and

$$\mathcal{Y}_F^\alpha = \{Y^\alpha(\cdot)\}_{\mathcal{Z} \in F_m} = L_z\left(B_{\alpha \rho}(\Lambda_z) \cap \text{Dir}(\Lambda_z)\right).$$

Recalling the diffeomorphism

$$r^{\alpha \rho} : P_m \to A_m \times N_m$$

$$z = an \mapsto (a, n)$$

from the previous subsection, the families $\mathcal{Y}_F$ and $\mathcal{Y}_F^\alpha$ can be viewed as fibered over $r^{\alpha \rho}(F_m) \subset A_m \times N_m$. As we said, our goal in this subsection is to show that the latter family (fibered over $r^{\alpha \rho}(F_m)$) is BLC for all $0 < \alpha \leq 1$; as a first step, we prove this fact for the first family, which is an instance of the latter one for $\alpha = 1$.

**Proposition 11.8.** The family $\mathcal{Y}_F^{r^{\alpha \rho}(F_m)} = \{Y(\cdot)\}_{(a, n) \in r^{\alpha \rho}(F_m)} = \{Y(\cdot)\}_{r^{\alpha \rho}(z) \in r^{\alpha \rho}(F_m)}$ is BLC w.r.t. any family $\mathcal{O}$ of coordinate balls (Definition 7.6) in $P_m = A_m \times N_m$.

Before we begin the proof, we recall Lemma 11.9 which will play a key role.

**Lemma 11.9.** Let $z = [z_1 \cdots z_n] \in F_m$ such that $z = a_z n_z$.

1. $n_z$ is a unipotent upper triangular matrix whose entries are bounded in $[-\frac{1}{2}, \frac{1}{2}]$ (in particular, $\|n_z\|, \|n_z^{-1}\| < 1$).
2. $a_z = \text{diag}(a_1, \ldots, a_m)$ is a diagonal matrix which satisfies that $a_1 \prec \cdots \prec a_m$. Specifically, $\sqrt{3} \leq a_j \leq 1$.
3. If $\lambda \in \Lambda_z$ (i.e. $\lambda = zv$ with $v \in \mathbb{Z}^m$) satisfies $\lambda \notin E_{j-1}$ where $E_i = \text{span}_{\mathbb{R}}\{e_1, \ldots, e_i\}$, then $\|\lambda\| \geq \text{dist}(\lambda, E_{j-1}) \geq \text{dist}(z_j, E_{j-1}) = a_j$.
4. If $x \in E_j$, then $\|a_z x\| \prec a_j \|x\|$.

The last part of the Lemma 11.9 implies shrinking property of conjugation of upper triangular matrices by elements of $F_m$.

**Corollary 11.10.** Let $\left[\begin{array}{c}z_1 \\ \vdots \\ z_m\end{array}\right] = z = a_z n_z \in F_m$. Then for any upper triangular matrix $p$,

1. $\|a_z p a_z^{-1}\| \prec \|p\|$;
2. $\|z p^{-1}\|, \|z p^{-1}\| \prec \|p\|$.

**Proof.** Part 1 follows from the fact that if $i \leq j$ then $a_i \prec a_j$ and therefore $|a_i p_{i,j} a_j^{-1}| = |a_i| |p_{i,j}| |a_j|^{-1} \prec |a_j| |p_{i,j}| |a_j|^{-1} = |p_{i,j}|$.
Proof. Clearly
\[ \|az^{-1}\| < \|az^{-1}\|_1 < \|p\|_1 \leq \|p\| . \]

For the second part notice that:
\[ \|zp^{-1}\| = \|az_nz^{-1}a_z^{-1}\| \leq \left(\|az_nz^{-1}\| \|az^{-1}\| \|z^{-1}\|\right) < \|p\| . \]
and
\[ \|zp^{-1}\| = \left|\|n_z az^{-1}n_z^{-1}\|\right| \leq \left(\|n_z\| \|az^{-1}\| \|n_z^{-1}\|\right) < \|p\| . \]

Recall that \( \rho_z \) is the covering radius of the lattice \( \Lambda_z \).

**Fact 11.11.** Let \( \left[z_1, \ldots, z_m\right] = z = az_nz \) in \( F_m \).

1. For \( j = 1, \ldots, m, \|z_j\| \simeq a_j \).
2. \( \rho_z \simeq a_m \times \|z\| . \)

**Proof.** According to Corollary [11.10]
\[ a_i = \text{dist}(z_i, E_{i-1}) \leq \|z_i\| = \|az_nz_i\| = \|az_nz^{-1}a_z e_i\| \leq \left(\|az_nz^{-1}\| \|az e_i\|\right) \leq a_i, \]
which proves the first part. As for the second part, we have from the one hand that (by Lemma [11.9] parts 1 and 2)
\[ \|z\| = \|az_nz\| \simeq \|a_z\| \simeq a_m \]
and from the other hand that
\[ a_m \simeq \|az\| = \|az_nz^{-1}\| \times \|az_nz\| = \|z\| . \]

The fact that \( a_m \simeq \rho_z \) is proved in [MG02, Theorem 7.9].

**Lemma 11.12.** Let \( (a', n') \in \mathcal{O}_\epsilon^{A_m \times N_m} (a, n) \mathcal{O}_\epsilon^{A_m \times N_m} \). If \( z = an, z' = a'n' \) and \( z \in F_m \), then
\[ \|z z^{-1}\|, \|z z^{-1}\| \leq 1 + C_1 \epsilon \] for some \( C_1 > 0 \).

**Proof.** Clearly \( (a', n') \in \mathcal{O}_\epsilon^{A_m \times N_m} (a, n) \mathcal{O}_\epsilon^{A_m \times N_m} \) is equivalent to \( z' \in \mathcal{O}_\epsilon^{A_m} a \mathcal{O}_\epsilon^{A_m} n \mathcal{O}_\epsilon^{N_m} \mathcal{O}_\epsilon^{N_m} \). Using the fact that \( \mathcal{O}_\epsilon^{P_m} \) is equivalent to \( \mathcal{O}_\epsilon^{A_m} \mathcal{O}_\epsilon^{N_m} \) and Corollary [11.10] we obtain,
\[ \mathcal{O}_\epsilon^{A_m} a \mathcal{O}_\epsilon^{A_m} n \mathcal{O}_\epsilon^{N_m} = an \left(n^{-1} \mathcal{O}_\epsilon^{A_m} \mathcal{O}_\epsilon^{N_m} n\right) \mathcal{O}_\epsilon^{N_m} \subseteq an \mathcal{O}_\epsilon^{P_m} n \mathcal{O}_\epsilon^{N_m} \subseteq an \mathcal{O}_\epsilon^{P_m} \mathcal{O}_\epsilon^{N_m} \subseteq z \mathcal{O}_\epsilon^{P_m} . \]
Again using Corollary [11.10] one also obtains
\[ z \mathcal{O}_\epsilon^{P_m} = \left(z \mathcal{O}_\epsilon^{P_m} z^{-1}\right) = \mathcal{O}_\epsilon^{P_m} z . \]
Finally, fix \( C_1 > 0 \) such that
\[ \mathcal{O}_\epsilon^{P_m} \subseteq \{p \in P_m : \|p\| \leq 1 + C_1 \epsilon\} . \]

The following lemma is the technical core of the proof of Proposition[11.8]
Lemma 11.13. Suppose $z, z' \in F_m$ and that $r_{p^n}(z') \in O_{c_p^n}(z) O_c$. Let $v \in \mathbb{Z}^m$ and write $\lambda = zv, \lambda' = z'v$. Then the following hold:

1. $\|z^t \lambda\| < \|\lambda\|^2$.
2. $\|\lambda'\| \leq (1 + C_1 \epsilon) \|\lambda\|$ for the constant $C_1 > 0$ from Remark 11.12.
3. $\|z^t \lambda - z'^t \lambda'\| < \epsilon \|\lambda\|^2$.

Proof. For the first part, recall that $L_z^{-1} (x) = zx$ and then

$$\|L_z^{-1} (\lambda)\| = \|z^t \lambda\| = \|n_z^t a_z \lambda\| \leq \|n_z^t\| \|a_z \lambda\| < a_z \|\lambda\|.$$  

Next, let $j \in \{1, \ldots, m\}$ such that $\lambda \in E_j \setminus E_{j-1}$. By parts (3) and (4) of Lemma 11.9:

$$\|a \lambda\| < a_j \|\lambda\| \leq \|\lambda\|^2.$$  

All in all, $\|L_z^{-1} (\lambda)\| < \|\lambda\|^2$.

For the second part, use Lemma 11.12

$$\|\lambda'\| = \|z'v\| = \|z'z^{-1} zv\| \leq \|z'z^{-1}\| \|zv\| \leq (1 + C_1 \epsilon) \|\lambda\|.$$  

For the third part, it is clear that

$$\|z^t \lambda - z'^t \lambda'\| \leq \|z^t (\lambda - \lambda')\| + \|\left( z^t - z'^t \right) \lambda'\|$$  

and we shall bound each of these two summands. The first one is bounded by

$$\|z^t (\lambda - \lambda')\| = \|z^t (z - z') v\| = \|z^t \left( I - z'z^{-1} zv \right) \| \leq \|z^t p z^{-1} \| \|z^t \lambda\|$$  

where by Corollary 11.10, Lemma 11.12, and the first part of the current Lemma,

$$< \|p\| \|z^t \lambda\| \leq \epsilon \|\lambda\|^2.$$  

The second summand is bounded by

$$\left\|\left( z^t - z'^t \right) \lambda'\right\| = \left\|\left( z^t z'^{-t} - I\right) z^t \lambda'\right\| = \left\|\left( (z'^{-1} z)^t - I\right) z'^t \lambda'\right\| \leq \left\|(z'^{-1} z)^t - I\right\| \|z'^t \lambda'\|.$$  

By Lemma 11.12 and the first part of this Lemma

$$< (C_1 \epsilon) \left( \epsilon \|\lambda'\|^2 \right),$$  

and by its second part

$$\leq (C_1 \epsilon) \cdot \epsilon \cdot ((1 + C_1 \epsilon) \|\lambda\|)^2 < \epsilon \|\lambda\|^2.$$  

Towards proving Proposition 11.8 stating that the family $\mathcal{Y}_m$ is BLC, we prove that this family satisfies the fourth property of BLC.

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Lemma 11.14. The family \( \mathcal{W} \) is bounded uniformly from above. Namely, there exists \( R > 0 \) that depends only on \( m \) such that \( Y(z) = L_z(\text{Dir}(z)) \) is contained in \( B_R \) for every \( z \in F_m \).

We introduce a notation, to be used in the proofs of Lemma 11.14 and Proposition 11.8. For \( \lambda \in \Lambda_z \), write \( H_{[\lambda]} \) for the strip

\[
H_{[\lambda]} := \left\{ x : |\langle x, \lambda \rangle| \leq \frac{1}{2} \| \lambda \|^2 \right\}.
\]

It is easy to check that it consists of all the vectors in \( \mathbb{R}^m \) which are closer to the origin than to \( \pm \lambda \). As a result,

\[
\text{Dir}((z)) = \bigcap_{0 \neq \lambda \in \Lambda_z} H_{[\lambda]}.
\]

Proof. According to equation (11.2) and definition of \( H_{[\lambda]} \), an element \( x \in \text{Dir}(z) \) satisfies that \(|\langle \lambda, x \rangle| \leq \frac{1}{2} \| \lambda \|^2 \), for every \( 0 \neq \lambda \in \Lambda_z \). In particular, this holds for \( \lambda \in \{ z_1, \ldots, z_m \} \subset \Lambda_z \) (the columns of \( z \)).

Recall that by Remark 11.7, \( x = zL_z(x) \). The inequality \(|\langle z_j, x \rangle| \leq \frac{1}{2} \| z_j \|^2 \) therefore translates into the inequality \(|\langle \frac{z_j}{\| z_j \|^2}, zL_z(x) \rangle| \leq \frac{1}{2} \), i.e.

\[
\left| \langle \frac{z_j}{\| z_j \|^2}, zL_z(x) \rangle \right| \leq \frac{1}{2}
\]

or

\[
\left| \frac{\| z_j \|^2}{\| z \|^2} \langle z_j, zL_z(x) \rangle \right| \leq \frac{1}{2}.
\]

considering all \( m \) inequalities, we obtain

\[
\left[ \begin{array}{c}
- \| z_1 \|^2 z_1^t \\
\vdots \\
- \| z_m \|^2 z_m^t
\end{array} \right] \cdot z \cdot L_z(x) \leq \left( \frac{1}{2}, \ldots, \frac{1}{2} \right)^t
\]

(whr one should understand \( \leq \) and \( | \) as referring to the components), namely

\[
\left[ \begin{array}{c}
\text{diag} \left( \| z_j \|^2 \right)_{j=1}^m \end{array} \right] \cdot z^t z \cdot L_z(x) \leq \left( \frac{1}{2}, \ldots, \frac{1}{2} \right)^t.
\]

Let \( g := \left[ \text{diag} \left( \| z_j \|^2 \right)_{j=1}^m \right] \cdot z^t z \); based on the last inequality, in order to show that \( \| L_z(x) \| \) is bounded by some constant \( R = R(m) \), it is sufficient to prove that \( \| g^{-1} \| < 1 \) where the implied constant depends only on \( m \). Indeed,

\[
\| g^{-1} \| = \left\| z^{-1} z^{-t} \left[ \begin{array}{c}
\text{diag} \left( \| z_j \|^2 \right)_{j=1}^m
\end{array} \right] \right\| \leq \left\| z^{-1} z^{-t} \left[ \begin{array}{c}
\text{diag} (a_j^2)_{j=1}^m
\end{array} \right] \right\| = \left\| z^{-1} z^{-t} a_z^2 \right\|
\]

\[
= \left\| n_z^{-1} a_z^2 n_z^{-1} a_z^2 \right\| \leq \| n_z^{-1} \| \cdot \| a_z^2 n_z^{-1} a_z^2 \| \leq 1
\]

\( \prec 1 \)

where the estimation \( \| a_z^2 n_z^{-1} a_z^2 \| \prec \| n_z^{-1} \| \) is also due to Corollary 11.10. \( \square \)
We are now ready to prove Proposition 11.8.

**Proof of Proposition 11.8.** We begin by verifying property BLC (I). According to Equation 11.2, it is sufficient to prove that this property holds for each strip $\mathcal{H}_{[\lambda]}$ separately, namely that

$$L_z (\mathcal{H}_{[\lambda]}) + B_c \subseteq (1 + C \epsilon) L_z (\mathcal{H}_{[\lambda]}).$$

Since (Remark 11.7)

$$L_z (\mathcal{H}_{[\lambda]}) = \left\{ y : \left| \langle L_z^{-t} (\lambda) , y \rangle \right| \leq \frac{\| \lambda \|^2}{2} \right\} = \left\{ y : \left| \langle z^t \lambda , y \rangle \right| \leq \frac{\| \lambda \|^2}{2} \right\},$$

and

$$L_z (\mathcal{H}_{[\lambda]}) + B_c \subseteq \left\{ x : \left| \langle x , L_z^{-t} (\lambda) \rangle \right| \leq \frac{\| \lambda \|^2}{2} + \| L_z^{-t} (\lambda) \| \cdot \epsilon \right\},$$

the desired inclusion is equivalent to

$$\frac{1}{2} \| \lambda \|^2 + \epsilon \left\| L_z^{-t} (\lambda) \right\| \leq (1 + C \epsilon) \frac{1}{2} \| \lambda \|^2.$$

This indeed holds, since by part 1 of Lemma 11.13

$$\left\| L_z^{-t} (v) \right\| = \| z^t v \| \leq \| v \|^2.$$  

We turn to prove property BLC (II). As with property BLC (I), it is sufficient to verify it for each strip $\mathcal{H}_{[\lambda]}$ separately. Assume that $r_{p^{v'}} (z') \in \mathcal{O}, r_{p^{v'}} (z) \in \mathcal{O}$. Let $y \in \text{Dir} ((z')) \subset \mathbb{R}^m$, namely

$$\left| \langle z^t \lambda', y \rangle \right| \leq \frac{\| \lambda' \|^2}{2}$$

for every $0 \neq \lambda' \in \Lambda_{z'}$. We need to prove that $y \in (1 + C \epsilon) L_z (\mathcal{H}_{[\lambda]}), \text{ for all } 0 \neq \lambda \in \Lambda_z,$ namely that

$$\left| \langle z^t \lambda, y \rangle \right| \leq (1 + C \epsilon) \frac{\| \lambda \|^2}{2}.$$

Now,

$$\left| \langle z^t \lambda, y \rangle \right| \leq \left| \langle z^t \lambda', y \rangle \right| + \left| \langle z^t \lambda - z^t \lambda', y \rangle \right| \leq \frac{\| \lambda' \|^2}{2} + \| y \| \cdot \left\| z^t \lambda - z^t \lambda' \right\|.$$  

According to Lemma 11.14

$$\leq \frac{\| \lambda' \|^2}{2} + R \cdot \left\| z^t \lambda - z^t \lambda' \right\| = \frac{\| \lambda \|^2}{2} \left( \frac{\| \lambda' \|^2}{\| \lambda \|^2} + 2R \left( \frac{\| z^t \lambda - z^t \lambda' \|}{\| \lambda \|^2} \right) \right)$$

and according to parts 2 and 3 of Lemma 11.13

$$= \frac{\| \lambda \|^2}{2} \left( \frac{\| \lambda' \|^2}{\| \lambda \|^2} + 2R \left( \frac{\| z^t \lambda - z^t \lambda' \|}{\| \lambda \|^2} \right) \right) \leq \frac{\| \lambda \|^2}{2} (1 + C \epsilon).$$

The BLC (III) is trivial since $Y (z) = L_z (\text{Dir} (z))$ are fundamental domains for $\mathbb{Z}^m$ in $\mathbb{R}^m$, hence their volume is exactly 1. Property BLC (IV) for the family $\mathcal{M}_{p^{v}} (F_m)$ is the content of Lemma 11.14. □
For the proof of the next proposition, notice that $\rho_z$, the covering radius of $\Lambda_z$, is equal to the maximal distance between the origin and a point in $\text{Dir}(z)$. In other words, $\rho_z$ is equal to the minimal radius of a ball centered at the origin and covers $\text{Dir}(z)$.

**Proposition 11.15.** For every $\alpha > 0$ the family

$$\mathcal{Y}^\alpha = \{ Y(z) \cap L_z(B_{\alpha \rho_z}) \}_{r \in \mathcal{P}'(F_m)}$$

defined in Formula 6.1 is BLC w.r.t. $\mathcal{O}_\epsilon$ as in Proposition 11.8.

**Proof.** To prove the first property, it is sufficient to show that for some $C > 0$,

$$B_{\alpha \rho_z} + L_z^{-1}(B_\epsilon) \subseteq (1 + C\epsilon) B_{\alpha \rho_z}.$$  

By Fact 11.11 there is a constant $C > 0$ such that:

$$L_z^{-1}(B_\epsilon) = z(B_\epsilon) \subseteq B_{\|z\|\epsilon} \subseteq B_{C(\alpha \rho_z)\epsilon}.$$  

As a result,

$$B_{\alpha \rho_z} + L_z^{-1}(B_\epsilon) \subseteq B_{\alpha \rho_z} + B_{C\alpha \rho_z \epsilon} \subseteq B_{\alpha \rho_z (1 + C\epsilon)} = (1 + C\epsilon) B_{\alpha \rho_z}.$$  

As for the second property, since it is maintained under intersections, it is sufficient to prove that

$$L_{z'}(B_{\alpha \rho_{z'}}) \subseteq (1 + C\epsilon) L_z(B_{\alpha \rho_z}).$$

Or in other words,

$$L_z^{-1} L_{z'}(B_{\alpha \rho_{z'}}) \subseteq (1 + C\epsilon) B_{\alpha \rho_z}.$$  

To this end, we first claim that

$$\rho_{z'} \leq (1 + C_1\epsilon)(1 + C_2\epsilon) \rho_z;$$  

indeed, by property BLC (II) for $\mathcal{Y}^\alpha$ (Proposition 11.8), we have that

$$L_{z'}(\text{Dir}(\Lambda_{z'})) \subseteq (1 + C_2\epsilon) \cdot L_z(\text{Dir}(\Lambda_z))$$

and therefore

$$\text{Dir}(\Lambda_{z'}) \subseteq (1 + C_2\epsilon) \cdot L_z^{-1} L_{z'}(\text{Dir}(\Lambda_z))$$

(Rmk. 11.7) \subseteq (1 + C_2\epsilon) \cdot z'z^{-1} \cdot \text{Dir}(\Lambda_z) \subseteq (1 + C_2\epsilon) \cdot \|z'z^{-1}\| \cdot \text{Dir}(\Lambda_z).$$

(Rmk. 11.12) \subseteq (1 + C_2\epsilon) \cdot (1 + C_1\epsilon) \text{Dir}(\Lambda_z).$$

Now,

$$L_z^{-1} L_{z'}(B_{\alpha \rho_{z'}}) \subseteq \|z'z^{-1}\| \cdot B_{\alpha \rho_{z'}}.$$  

by Rmk. 11.12 \subseteq (1 + C_1\epsilon) \cdot B_{\alpha \rho_{z'}}$$  

by eq. 11.3 \subseteq (1 + C_1\epsilon)^2 \cdot (1 + C_2\epsilon) \cdot B_{\alpha \rho_z}$$

which establishes that $L_z^{-1} L_{z'}(B_{\alpha \rho_{z'}}) \subseteq (1 + C\epsilon) B_{\alpha \rho_z}$ and completes the proof of the second property.

Property BLC (IV) is a direct consequence of Lemma 11.14 and so we turn to prove the third property. First, we claim that for $z = a_z n_z \in F_m$, the vectors

$$\pm a_j := \frac{1}{2} a_z e_j = \frac{1}{2} a_j e_j$$  

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lie in $\text{Dir}(\Lambda_z)$. Indeed, suppose otherwise that there exists $\lambda \in \Lambda_z$ such that $\|a_j + \lambda\| < \|a_j\|$. Then $\lambda$ cannot lie inside $E_{j-1} = \text{span}\{z_1, \ldots, z_{j-1}\}$, because if it did then it would be orthogonal to $a_j$ which implies
\[
\|a_j\|^2 + \|\lambda\|^2 = \|a_j + \lambda\|^2 < \|a_j\|^2,
\]
a contradiction. Hence $\lambda \notin E_{j-1}$, implying $\lambda = \lambda_{j-1} + \lambda_j$ with $0 \neq \lambda_j \in E_{j-1}$. Now,
\[
a_j = \text{dist}(z_j, E_{j-1}) \leq \text{dist}(\lambda, E_{j-1}) = \|\lambda_j\| \leq \|\lambda\| \leq 2\|a_j\| = a_j.
\]
This is clearly a contradiction, establishing that the vectors $\pm a_j$ indeed lie inside $\text{Dir}(\Lambda_z)$.

Let $c > 0$ such that $\|ca_j\| = \frac{1}{c}ca_j \leq \alpha \rho_z$ for every $j = 1, \ldots, m$; such $c$ exists and is independent of $z$ because $a_1 \prec \cdots \prec a_m \prec \rho_z$ (according to Fact 11.11 and part 2 of Lemma 11.9). We may assume that $c \leq 1$ and therefore (since $\text{Dir}(\Lambda_z)$ is convex and contains the origin and the points $a_j$), that the points $ca_j$ are also contained in $\text{Dir}(\Lambda_z)$. They are obviously contained in $B_{\alpha \rho_z}$, hence by convexity
\[
[-c, c]a_1 \times \cdots \times [-c, c]a_m = c^m \cdot \prod_{j=1}^m \left[ -\frac{a_j}{2}, \frac{a_j}{2} \right] \subseteq \text{Dir}(\Lambda_z) \cap B_{\alpha \rho_z}.
\]
The above shape has volume $c^m \cdot \prod_{j=1}^m a_j = c^m \cdot \det(z)$; its image under $L_z = z^{-1}$ has therefore volume $c^m$. It follows that the volume of $L_z (\text{Dir}(\Lambda_z) \cap B_{\alpha \rho_z})$ is bounded from below by $c^m$, which does not depend on $z$.

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