Linear Inviscid Damping and Vorticity Depletion for Shear Flows

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Abstract
In this paper, we prove the linear damping for the 2-D Euler equations around a class of shear flows under the assumption that the linearized operator has no embedding eigenvalues. For the symmetric flows, we obtain the same decay estimate of the velocity as the monotone shear flows. Moreover, we confirm a new dynamical phenomena found by Bouchet and Morita: the depletion of the vorticity at the stationary streamlines, which along with the vorticity mixing leads to the damping for the base flows with stationary streamlines.

Keywords Euler equations · Shear flow · Inviscid damping · Vorticity depletion · Rayleigh equation

1 Introduction
We consider the 2-D incompressible Euler equations in a finite channel \( \Omega = \{(x, y) : x \in \mathbb{T}, y \in [-1, 1]\} \):

\[
\begin{align*}
\partial_t V + V \cdot \nabla V + \nabla P &= 0, \\
\nabla \cdot V &= 0, \\
V^2(t, x, -1) &= V^2(t, x, 1) = 0, \\
V|_{t=0} &= V_0(x, y).
\end{align*}
\] (1.1)
where $V = (V^1, V^2)$ and $P$ denote the velocity and the pressure of the fluid respectively. Let $\omega = \partial_x V^2 - \partial_y V^1$ be the vorticity, which satisfies

$$\omega_t + V \cdot \nabla \omega = 0. \quad (1.2)$$

The appearance of large coherent structures is an important phenomena in 2-D flows. The stability of coherent structures has been an active field of fluid mechanics [10,28], which started in the nineteenth century with Rayleigh, Kelvin, Orr, Sommerfeld and many others. Let us mention some of classical results: Rayleigh’s inflection theorem [26] giving a necessary condition of spectral instability, Howard’s semicircle theorem [13], and Arnold’s criterion for the Lyapunov stability [1]. We refer to [2,14,17,18,21,29] and references therein for some recent mathematical studies.

In this paper, we are concerned with the asymptotic stability of the 2-D Euler equations around the shear flow $(u(y), 0)$, which is a steady solution of (1.1). The key step toward this problem is to study the linearized Euler equations around $(u(y), 0)$:

$$\begin{cases} 
\partial_t \omega + L \omega = 0, \\
\omega|_{t=0} = \omega_0(x, y),
\end{cases} \quad (1.3)$$

where $L = u(y)\partial_x + u''(y)\partial_x(-\Delta)^{-1}$. For the Couette flow (i.e., $u(y) = y$), (1.3) is reduced to a passive transport equation

$$\partial_t \omega + y \partial_x \omega = 0, \quad \omega|_{t=0} = \omega_0(x, y).$$

In this case, Orr [25] first observed that the velocity will tend to 0 as $t \to \infty$, which is so-called inviscid damping. Recently, Lin and Zeng [20] proved that if $\int_\Omega w_0(x, y) dx = 0$, then

$$\|V(t)\|_{L^2_{x,y}} \leq \|\omega(t)\|_{L^2_{x}H^{-1}_y} \leq \frac{C}{(t)}\|\omega_0\|_{H^{-1}_xH^1_y}.$$

The mechanism leading to the damping is the vorticity mixing driven by the shear flow. This phenomena is similar to the well-known Landau damping found by Landau in 1946 [15].

Due to possible nonlinear transient growth, it is a challenging task from linear damping to nonlinear damping. Moreover, nonlinear damping is sensitive to the topology of the perturbation. Indeed, Lin and Zeng [20] proved that nonlinear inviscid damping is not true for the perturbation in $H^s$ for $s < \frac{3}{2}$. Motivated by the breakthrough work of Mouhot and Villani on Landau damping [23], Bedrossian and Masmoudi [4] proved nonlinear inviscid damping and asymptotic stability of the 2-D Euler equations around the Couette flow in Gevrey class in the domain $\Omega = T \times \mathbb{R}$. See [5–7] and references therein for more relevant works.

For general shear flow, the linear inviscid damping is also a difficult problem due to the presence of nonlocal operator $u''(y)\partial_x(-\Delta)^{-1}$. In this case, the linear dynamics is associated with the singularities at the critical layers $u = c$ of the solution for the Rayleigh equation

$$(u - c)(\phi'' - \alpha^2 \phi) - u'' \phi = f.$$
In fact, Ran $u$ is just the continuous spectrum of $\mathcal{L}$, whose properties and the non-normality of $\mathcal{L}$ are related to many important phenomena such as transient growth [11], inviscid damping [9] and algebraic instabilities [24].

Based on the Laplace transform and analyzing the singularity of the solution $\phi$ at the critical layer, Case [9] gave a first prediction of linear damping for monotone shear flow. However, there are few rigorous mathematical results. Rosencrans and Sattinger [27] gave $r^{-1}$ decay of the stream function for analytic monotone shear flow. Stepin [30] proved $r^{-1}(v < \mu_0)$ decay of the stream function for monotone shear flow $u(y) \in C^2 + \mu_0$ ($\mu_0 > \frac{1}{2}$) without inflection point. Zillinger [35] proved $r^{-1}$ decay of $\|V(t)\|_{L^2}$ for a class of monotone shear flow, which satisfies $L\|u''\|_{W^{3,\infty}} \ll 1$. In a recent work [31], we removed the smallness assumption in [35] and showed that if $u(y) \in C^4$ is monotone and $\mathcal{L}$ has no embedding eigenvalues, then

$$\|V(t)\|_{L^2} \leq \frac{C}{(t)} \|\omega_0\|_{H^{1-\frac{1}{4},1}^2}, \quad \|V^2(t)\|_{L^2} \leq \frac{C}{(t)^2} \|\omega_0\|_{H^{1-\frac{1}{4},2}^2},$$

for the initial vorticity $\omega_0$ satisfying $\int_{-\infty}^{\infty} \omega_0(x, y) dx = 0$ and $P_{\mathcal{L}} \omega_0 = 0$, where $P_{\mathcal{L}}$ is the spectral projection to $\sigma_d(\mathcal{L})$.

However, many important base flows such as Poiseuille flow $u(y) = y^2$ and Kolmogorov flow $u(y) = \cos y$ are not monotone. Bouchet and Morita [8] conducted the systematic studies for the asymptotic behaviour of the vorticity and the velocity around the base flows with stationary streamlines. Based on Laplace tools and numerical computations, they found a new dynamic phenomena: depletion phenomena of the vorticity at the stationary streamlines. More precisely, they formally proved that for large times,

$$\hat{\omega}(t, \alpha, y) \sim \omega_\infty(y) \exp(-i\alpha u(y)t) + O(t^{-r}),$$

where $\omega_\infty(y_c) = 0$ at stationary points $y_c$ of $u(y)$. Based on this and using stationary phase expansion, they also predicted similar decay rates of the velocity as in the monotonic case.

The goal of this paper is to study the linear damping for general shear flows. In particular, we confirm Bouchet and Morita’s prediction about the linear damping and the depletion phenomena of the vorticity for the base flows with stationary streamlines.

The first result is the linear damping and vorticity depletion for a class of shear flows denoted by $\mathcal{K}$, which consists of the function $u(y)$ satisfying $u(y) \in H^3(-1, 1)$, and $u''(y) \neq 0$ for critical points (i.e., $u'(y) = 0$) and $u'(\pm 1) \neq 0$.

**Theorem 1.1** Assume that $u(y) \in \mathcal{K}$ and the linearized operator $\mathcal{R}_\alpha$ defined by (3.3) has no embedding eigenvalues. Assume that $\hat{\omega}_0(\alpha, y) \in H^1_y(-1, 1)$ and $P_{\mathcal{R}_\alpha} \hat{\psi}_0(\alpha, y) = 0$, where $\psi_0$ is the stream function and $P_{\mathcal{R}_\alpha}$ is the spectral projection to $\sigma_d(\mathcal{R}_\alpha)$. Then it holds that

$$\|\hat{\nu}(\cdot, \alpha, \cdot)\|_{L^2_y L^1_\gamma} + \|\partial_t \hat{\nu}(\cdot, \alpha, \cdot)\|_{L^2_y L^2_\gamma} \leq C_\alpha \|\hat{\omega}_0(\alpha, \cdot)\|_{H^1_y}.$$

In particular, $\lim_{t \to +\infty} \|\hat{V}(t, \alpha, \cdot)\|_{L^2_y} = 0$. 

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Formally, Theorem 1.1 implies that the velocity should have at least $t^{-\frac{1}{2}}$ decay. To obtain the explicit decay rate of the velocity, we consider a class of symmetric shear flow:

$$u(y) = u(-y), \quad u'(y) > 0 \text{ for } y > 0, \quad u'(0) = 0 \text{ and } u''(0) > 0.$$ 

An important example is the Poiseuille flow $u(y) = y^2$.

**Theorem 1.2** Assume that $u(y) \in C^4([-1, 1])$ satisfies (S) and the linearized operator $L$ has no embedding eigenvalues. Assume that $\int_T \omega_0(x, y)dx = 0$ and $P_L\omega_0 = 0$, where $P_L$ is the spectral projection to $\sigma_d(L)$. Then it holds that

1. if $\omega_0(x, y) \in H_x^{-\frac{1}{2}}H_y^1$, then
   $$\|V(t)\|_{L^2} \leq \frac{C}{\langle t \rangle^{\frac{1}{2}}} \|\omega_0\|_{H_x^{-\frac{1}{2}}H_y^1};$$

2. if $\omega_0(x, y) \in H_x^1H_y^2$, then
   $$\|V^2(t)\|_{L^2} \leq \frac{C}{\langle t \rangle^2} \|\omega_0\|_{H_x^1H_y^2};$$

3. if $\omega_0(x, y) \in H_x^{-\frac{1}{2}+k}H_y^k$ for $k = 0, 1$, there exists $\omega_\infty(x, y) \in H_x^{-\frac{1}{2}+k}H_y^k$ such that
   $$\|\omega(t, x + tu(y), y) - \omega_\infty\|_{H_x^{-\frac{1}{2}+k}L_y^2} \to 0 \text{ as } t \to +\infty.$$

As a direct corollary of Theorem 1.1, we have

**Corollary 1.3** Assume that $u(y) \in K$ and the linearized operator $L$ has no embedding eigenvalues. Assume that $\int_T \omega_0(x, y)dx = 0$ and $P_L\omega_0 = 0$, where $P_L$ is the spectral projection to $\sigma_d(L)$. Then there exists a sequence $c_\alpha > 0$ such that if $\sum_\alpha c_\alpha \|\hat{\omega}_0(\alpha, \cdot)\|_{H_y^1}^2 < +\infty$, then

$$\|V\|_{L_x^2L_y^2}^2 + \|\partial_t V\|_{L_x^2L_y^2}^2 \leq \sum_\alpha c_\alpha \|\hat{\omega}_0(\alpha, \cdot)\|_{H_y^1}^2.$$

In particular, $\lim_{t \to +\infty} \|V(t)\|_{L^2} = 0$.

We say that the shear flow satisfies the uniform boundedness property, if the solution of (1.3) satisfies

$$\|\omega(t)\|_{L^2} \leq C \|\omega_0\|_{L^2},$$

here $C$ is independent of $t \in \mathbb{R}, \omega_0$. A typical case is $u'' > 0$ in $[-1, 1]$, in which $\|\omega(t)/(u'')^{1/2}\|_{L^2}$ is conserved. In this case, we have the following linear damping result.
Corollary 1.4 Assume that \( u(y) \in K \) and satisfies the uniform boundedness property, and the linearized operator \( \mathcal{L} \) has no embedding eigenvalues. Then for \( \omega_0 \in L^2 \), we have 
\[
\lim_{t \to +\infty} \| V(t) \|_{L^2} = 0.
\]

Indeed, the uniform boundedness property implies \( \sigma_d(\mathcal{R}_\alpha) \subset \mathbb{R} \), which together with Howard’s semicircle theorem implies \( \sigma_d(\mathcal{R}_\alpha) \subset \text{Ran} \ u \), i.e., \( P_{\mathcal{R}_\alpha} = 0 \). Then Corollary 1.4 follows from a density argument.

Let us give some remarks on our results:

1. If \( u(y) \) has no inflection points, then \( \mathcal{L} \) has no embedding eigenvalues (see Sect. 5).

2. For non-monotone flow, the decay rate of the velocity is unexpected. Formal prediction by Case [9] does not work in this case. If we neglect the nonlocal part \( u''(y)\partial_x(-\Delta)^{-1} \) of \( \mathcal{L} \), one can only obtain \( t^{-\frac{1}{2}} \) decay of the velocity, which will be explained in Sect. 2. On the other hand, whether one can obtain the decay rate of the velocity as in Theorem 1.2 is a very interesting question for shear flows in \( K \).

3. The proof of Theorem 1.1 is based on the limiting absorption principle, which follows from a compactness argument. Hence, it is unclear how the constant \( C_\alpha \) depends on the Fourier modes \( \alpha \). To give a uniform estimate in \( \alpha \), the key point is to prove a stronger version of Proposition 6.1. For example, the following version is expected:

\[
\alpha \| \Phi \|_{L^2(-1, 1)} + \| \partial_y \Phi \|_{L^2(-1, 1)} \leq C \left( \alpha \| \omega \|_{L^2(-1, 1)} + \| \partial_y \omega \|_{L^2(-1, 1)} \right).
\]

However, the proof involves a very complicated argument for general flows. The case of monotone flow is relatively simple, see Lemma 2.4 in [33].

4. In a recent work [22], Lin and Zeng also proved the linear damping for general stable steady flows under a similar spectral assumption in the following weak sense:

\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T \| V(t) \|_{L^2}^2 dt = 0. \tag{1.4}
\]

5. Our method could be applicable to the Kolmogorov flow. In a separate work [32], we will prove the depletion phenomena of the vorticity and the decay rate of the velocity conjectured by Bouchet and Morita [8] for the Kolmogorov flow. Lin and Xu [19] proved the linear damping in the sense of (1.4).

6. Nonlinear inviscid damping should be a challenging problem, even for the Couette flow in a finite channel (see [34] for a related discussions).

2 New Dynamic Phenomena: Vorticity Depletion

If we neglect the nonlocal term in the linearized vorticity equation (1.3), the vorticity will satisfy a passive transport equation

\[
\partial_t \omega + u(y) \partial_x \omega = 0, \quad \omega(0, x, y) = \omega_0(x, y).
\]
If \( u'(y) > 0 \) and \( \int_T \omega_0 dx = 0 \), then we have

\[
\|\omega(t)\|_{L^2_x H^{-1}_y} \leq \frac{C}{\langle t \rangle} \|\omega_0\|_{H^{-1}_x H^1_y}.
\]  

(2.1)

If \( u \) satisfies \((S)\) and \( \int_T \omega_0 dx = 0 \), then we have

\[
\|\omega(t)\|_{L^2_x H^{-1}_y} \leq \frac{C}{\langle t \rangle^{1/2}} \|\omega_0\|_{H^{-1/2}_x H^1_y}.
\]  

(2.2)

The damping in (2.1) and (2.2) is due to vorticity mixing. So, this can not explain the enhanced damping in Theorem 1.2 when \( u \in (S) \). In such case, the mechanism leading to the enhanced damping is due to the vorticity mixing and a new dynamical phenomena found by Bouchet and Morita [8]: the depletion of the vorticity at the stationary streamlines, which is due to the effect of the perturbation velocity on the background vorticity gradient. More precisely, based on the Laplace transform analysis, they formally proved that the solution of (1.3) behaves as \( t \to \infty \)

\[
\hat{\omega}(t, \alpha, y) \sim \omega_\infty(y) e^{-i\alpha u(y)t} + O(t^{-\gamma})
\]  

(2.3)

for \( \gamma > 0 \), where the profile \( \omega_\infty(y) \) vanishes at the stationary points.

In this paper, we confirm the vorticity depletion phenomena for general shear flows in the class \( \mathcal{K} \). Recently, Bedrossian, Zelati and Vicol [3] also confirm similar phenomena for the linearized 2-D Euler equations around a radially symmetric, strictly monotone decreasing vorticity distribution.

**Theorem 2.1** Under the same assumptions as in Theorem 1.1, if \( u'(y_0) = 0 \), then

\[
\lim_{t \to +\infty} \hat{\omega}(t, \alpha, y_0) = 0.
\]

Assuming the ansatz (2.3) and \( \omega_\infty(0) = 0 \), one can obtain the decay estimate as in (2.1) by following the proof of (2.2) as below.

Let us prove (2.2). Taking Fourier transform in \( x \) variable, we obtain

\[
\partial_t \hat{\omega}(t, \alpha, y) + i\alpha u(y) \hat{\omega}(t, \alpha, y) = 0.
\]

Thus, \( \hat{\omega}(t, \alpha, y) = \hat{\omega}_0(\alpha, y) e^{-i\alpha u(y)t} \). Let \( \alpha > 0 \) and \( v \) be given by (4.4). For any smooth function \( \eta(y) \), we have

\[
\left| \int_{-\infty}^{\infty} \hat{\omega}(t, \alpha, y) \eta(y) dy \right| = \left| \int_{-\infty}^{v(1)} \eta(y) \hat{\omega}_0(\alpha, y) e^{-i\alpha v(y)t} dy \right| = \left| \int_{-v(1)}^{v(1)} \eta(v^{-1}(z)) \hat{\omega}_0(\alpha, v^{-1}(z)) e^{-i\alpha z^2 t} (v^{-1}(z))' dz \right|
\]
\[ = \left| \int_{|z|^2 \leq \frac{\pi}{2\alpha t}} \eta(v^{-1}(z))\mathcal{O}_0(\alpha, v^{-1}(z))e^{-i\alpha z^2t} (v^{-1}(z))'dz \right| + \left| \int_{v(1)^2 \geq |z|^2 > \frac{\pi}{2\alpha t}} \eta(v^{-1}(z))\mathcal{O}_0(\alpha, v^{-1}(z))e^{-i\alpha z^2t} (v^{-1}(z))'dz \right|. \]

For the first term, we have
\[ \left| \int_{|z|^2 \leq \frac{\pi}{2\alpha t}} \eta(v^{-1}(z))\mathcal{O}_0(\alpha, v^{-1}(z))e^{-i\alpha z^2t} (v^{-1}(z))'dz \right| \leq C \| \eta \|_{L^\infty} \| \mathcal{O}_0(\alpha, \cdot) \|_{L^\infty_y} \left( \int_{|z|^2 \leq \frac{\pi}{2\alpha t}} \cos(z^2\alpha t)dz + \int_{|z|^2 \leq \frac{\pi}{2\alpha t}} \sin(z^2\alpha t)dz \right) \]
\[ \leq C(\alpha t)^{-\frac{1}{2}} \| \eta \|_{L^\infty} \| \mathcal{O}_0(\alpha, \cdot) \|_{L^\infty_y} \left( \int_{-\frac{\pi}{2\alpha t}}^{\frac{\pi}{2\alpha t}} \cos(y)\sqrt{y}dy \right) \]
\[ \leq C(\alpha t)^{-\frac{1}{2}} \| \eta \|_{H^1} \| \mathcal{O}_0(\alpha, \cdot) \|_{H^1_y}. \]

For the second term, by integration by parts and Lemma 4.1, we get
\[ \left| \frac{2}{\alpha t} \int_{v(1)^2 \geq |z|^2 > \frac{\pi}{2\alpha t}} \eta(v^{-1}(z))\mathcal{O}_0(\alpha, v^{-1}(z)) (v^{-1}(z))' \frac{(v^{-1}(z))'}{z} e^{-i\alpha z^2t}dz \right| \]
\[ \leq \frac{C}{\sqrt{\alpha t}} \| \eta \|_{L^\infty} \| \mathcal{O}_0(\alpha, \cdot) \|_{L^\infty_y} + \frac{2}{\alpha t} \int_{v(1)^2 \geq |z|^2 > \frac{\pi}{2\alpha t}} \partial_z \left( \eta(v^{-1}(z))\mathcal{O}_0(\alpha, v^{-1}(z))z^{-1}(v^{-1}(z))' \right)dz \]
\[ \leq \frac{C}{\sqrt{\alpha t}} \left( \| \eta \|_{L^\infty} \| \mathcal{O}_0(\alpha, \cdot) \|_{L^\infty_y} + \| \eta \|_{H^1} \| \mathcal{O}_0(\alpha, \cdot) \|_{L^2_y} + \| \eta \|_{L^2} \| \mathcal{O}_0(\alpha, \cdot) \|_{H^1_y} \right). \]

This shows that
\[ \left| \int_{-1}^{1} \mathcal{O}(t, \alpha, y)\eta(y)dy \right| \leq C(\alpha t)^{-\frac{1}{2}} \| \eta \|_{H^1} \| \mathcal{O}_0(\alpha, \cdot) \|_{H^1_y}, \]

which implies (2.2).

## 3 Sketch and Key Ideas of the Proof

### 3.1 Resolvent and Rayleigh Equation

In terms of the stream function $\psi$, the linearized Euler equations (1.3) take
\[ \partial_t \Delta \psi + u(y)\partial_x \Delta \psi - u''(y)\partial_x \psi = 0. \]
Taking the Fourier transform in $x$, we get
\[
\left( \partial_y^2 - \alpha^2 \right) \partial_t \hat{\psi} = i \alpha \left( u''(y) - u(y) \left( \partial_y^2 - \alpha^2 \right) \right) \hat{\psi}.
\]

Inverting the operator $(\partial_y^2 - \alpha^2)$, we find
\[
-\frac{1}{i \alpha} \partial_t \hat{\psi} = \mathcal{R}_\alpha \hat{\psi},
\]
where
\[
\mathcal{R}_\alpha \hat{\psi} = - \left( \partial_y^2 - \alpha^2 \right)^{-1} \left( u''(y) - u \left( \partial_y^2 - \alpha^2 \right) \right) \hat{\psi}.
\]

Let $\Omega$ be a simple connected domain including the spectrum $\sigma(\mathcal{R}_\alpha)$ of $\mathcal{R}_\alpha$. We have the following representation formula of the solution to (3.2):
\[
\hat{\psi}(t, \alpha, y) = \frac{1}{2\pi i} \int_{\partial \Omega} e^{-i\alpha t c} (c - \mathcal{R}_\alpha)^{-1} \hat{\psi}(0, \alpha, y) dc.
\]

Then the large time behaviour of the solution $\hat{\psi}(t, \alpha, y)$ is reduced to the study of the resolvent $(c - \mathcal{R}_\alpha)^{-1}$.

Let $\Phi(\alpha, y, c)$ be the solution of the inhomogeneous Rayleigh equation with $f(\alpha, y, c) = \frac{\hat{\omega}_0(\alpha, y)}{i \alpha (u - c)}$ and $c \in \Omega$:
\[
\begin{cases}
\Phi'' - \alpha^2 \Phi - \frac{u''}{u - c} \Phi = f, \\
\Phi(-1) = \Phi(1) = 0.
\end{cases}
\]

Then we find that
\[
(c - \mathcal{R}_\alpha)^{-1} \hat{\psi}(0, \alpha, y) = i \alpha \Phi(\alpha, y, c).
\]

In Sect. 4, we will solve the Rayleigh equation when $u \in (S)$ following the method introduced in [31]. Main difference is that $\frac{1}{u(y) - c}$ is more singular for $c = u(0)$. The price to pay is that we can only establish uniform estimates of the solution of the homogeneous Rayleigh equation in the weighted space (see key Proposition 4.14).

### 3.2 Linear Damping via the Limiting Absorption Principle

An important observation in this paper is that the linear damping in the sense of Theorem 1.1 is closely connected with the following limiting absorption principle: as $\varepsilon \to 0$,
\[
\Phi(\alpha, y, c \pm i \varepsilon) \to \Phi_{\pm}(\alpha, y, c) \quad \text{for} \ c \in \text{Ran} \ u.
\]
If \( u \) has critical points, then the limit is highly nontrivial. Indeed, the authors (in [8], P. 952) pointed out that

*These results (the limit and its properties) are the difficult aspects of the discussion, from a mathematical point of view.*

The arguments in [8] are based on the hypothesis of the limiting absorption principle, which was verified by using numerical computations.

Using blow-up analysis and compactness argument, in Sect. 6, we establish the uniform \( H^1 \) estimate of the solution \( \Phi(\alpha, y, c \pm i \varepsilon) \) for \( c \in \text{Ran}u \) and \( \varepsilon > 0 \) and the limiting absorption principle: there exists \( \Phi_\pm(\alpha, y, c) \in H^1_y \) so that as \( \varepsilon \to 0 \),

\[
\Phi(\alpha, y, c \pm i \varepsilon) \to \Phi_\pm(\alpha, y, c) \quad \text{in} \quad C([-1, 1]).
\]

With these information, in Sect. 7, we prove the linear damping and the vorticity depletion phenomena for a class of shear flows in \( \mathcal{K} \), i.e., Theorem 1.1 and Theorem 2.1.

Let us present a sketch of the proof. Assume that there exists \( \psi_n \in H^1_0(-1, 1), \omega_n \in H^1(-1, 1) \), and \( c_n \) with \( \text{Im} \, c_n > 0 \), such that \( \| \psi_n \|_{H^1(-1, 1)} = 1, \| \omega_n \|_{H^1(-1, 1)} \to 0, \quad c_n \to c \in \text{Ran} \, u \), and

\[
(\psi_n - \alpha^2 \psi_n) - u'' \psi_n = \omega_n.
\]

Then there exists a subsequence of \( \{ \psi_n \} \) (still denoted by \( \{ \psi_n \} \)) and \( \psi \in H^1_0(-1, 1) \) such that \( \psi_n \rightharpoonup \psi \) weakly in \( H^1(-1, 1) \). The key point is to show that for all \( \varphi \in H^1_0(-1, 1) \),

\[
\int_{-1}^1 \left( \psi' \varphi' + \alpha^2 \varphi \psi \right) dy + p.v. \int_{-1}^1 \frac{u'' \psi \varphi}{u - c} dy + i \pi \sum_{y \in u^{-1}[c], u'(y) \neq 0} \frac{(u'' \psi \varphi)(y)}{|u'(y)|} = 0.
\]

Since \( \mathcal{R}_\sigma \) has no embedding eigenvalues, we have \( \psi = 0 \) by Definition 5.1. Moreover, there holds that \( \psi_n \to 0 \) in \( H^1(-1, 1) \), which leads to a contradiction.

In the case when \( y_0 \in u^{-1}[c] \) and \( u'(y_0) \neq 0 \), the compactness used the following important trick: let \( g_n = u'' \psi_n + \omega_n \), then \( g_n \) is uniformly bounded in \( H^1(a, b) \) and

\[
\left( \psi_n - \frac{g_n}{u'_n} \ln u_n \right)' = \alpha^2 \psi_n - \left( \frac{g_n}{u'_n} \right)' \ln u_n,
\]

from which, we can deduce that \( \psi_n - \frac{g_n}{u'_n} \ln u_n \) is uniformly bounded in \( L^2(a, b) \) and \( \tilde{\psi}^{1,1}(a, b) \). See Lemma 6.2 for more details.

In the case when \( u'(y_0) = 0 \), without loss of generality, we may assume that \( y_0 = 0, u''(y_0) = 2, u(y_0) = 0 \). Let \( c_n = r_n e^{2i\theta_n}, \quad r_n \to 0, \quad 0 < \theta_n \to \pi/2 \). We introduce

\[
\tilde{\psi}_n(y) = r_n^{-1} \psi_n(r_n y), \quad \tilde{\omega}_n(y) = r_n^{-1} \omega_n(r_n y), \quad u_n(y) = r_n^{-2}(u(r_n y) - u(0)).
\]

It holds that

\[
(u_n - e^{2i\theta_n}) \left( \tilde{\psi}_n'' - (\alpha r_n)^2 \tilde{\psi}_n \right) - u_n'' \tilde{\psi}_n = \tilde{\omega}_n.
\]
Since $\tilde{\psi}_n$ is bounded in $H^1_{loc}(\mathbb{R})$ and $\tilde{\theta}_n \to 0$ in $H^1_{loc}(\mathbb{R})$, up to a subsequence, we may assume that $\tilde{\psi}_n \to \tilde{\psi}_0$ in $H^1_{loc}(\mathbb{R})$, $\theta_n \to \theta_0$ with $\tilde{\psi}_0' \in L^2(\mathbb{R})$, $\theta_0 \in [0, \pi/2]$, and there holds that

$$\left(y^2 - e^{2i\theta_0}\right)\tilde{\psi}_0'' = 2\tilde{\psi}_0.$$

The next task is to show that $\tilde{\psi}_0 = 0$, which is nontrivial. See Lemma 6.4 for more details.

### 3.3 Decay Estimates of the Velocity for Symmetric Flows

To obtain the explicit decay rate of the velocity, we need to know more precise behaviour of the limit function $\Phi_\pm(\alpha, y, c)$. To this end, we consider a class of symmetric flows. The main advantage is that we can decompose the solution of (3.5) into the odd part and even part due to the symmetry of $u(y)$:

$$\begin{align*}
\begin{cases}
\Phi_o'' - \alpha^2 \Phi_o - \frac{u''}{u - c}\Phi_o = f_o, \\
\Phi_o(0) = \Phi_o(1) = 0,
\end{cases} \\
\begin{cases}
\Phi_e'' - \alpha^2 \Phi_e - \frac{u''}{u - c}\Phi_e = f_e, \\
\Phi_e(0) = \Phi_e(1) = 0,
\end{cases}
\end{align*}$$

where $f_o$ and $f_e$ are the odd part and even part of $f$ respectively.

These two equations can be dealt as in the monotonic case. As mentioned above, the solutions of the homogeneous Rayleigh equation behave more singular due to the degenerate of $u(y) - c$ for $c = u(0)$. To solve the inhomogeneous Rayleigh equation, a key point is to prove that the Wronskian of the solution does not vanish for the homogeneous Rayleigh equations in the odd and even case. In Sect. 5, we show that this key fact is equivalent to our spectral assumption: the linearized operator has no embedding eigenvalues.

For the symmetric flow, we give the precise formula of $\Phi_\pm(\alpha, y, c)$ in Sect. 8. Thus, we obtain the representation formula of the stream function

$$\begin{align*}
\hat{\psi}(t, \alpha, y) &= \frac{1}{2\pi} \int_{u(0)}^{u(1)} \alpha \widetilde{\Phi}_o(y, c)e^{-i\alpha ct} dc + \frac{1}{2\pi} \int_{u(0)}^{u(1)} \alpha \widetilde{\Phi}_e(y, c)e^{-i\alpha ct} dc \\
&= \hat{\psi}_o(t, \alpha, y) + \hat{\psi}_e(t, \alpha, y).
\end{align*}$$

In the monotonic case, we first derive the formula of the vorticity from the stream function. Then we prove that the vorticity is bounded in Sobolev space. Finally, the decay estimate of the velocity is proved by using a dual argument. In the present case,
we find that it is difficult to follow the same procedure. New idea is that we directly derive the decay estimates of the velocity based on the formulation of stream function and the dual argument. More precisely, in Sect. 9, we will derive the following two important formulas: for $f = g'' - \alpha^2 g$ with $g \in H^2(0, 1) \cap H_0^1(0, 1)$,

$$
\int_0^1 \hat{\psi}_{o}(t, \alpha, y) f(y)dy = -\int_{u(0)}^{u(1)} K_o(c, \alpha) e^{-i\alpha c t} dc,
$$

and for $f = g'' - \alpha^2 g$ with $g \in H^2(0, 1)$ and $g'(0) = g(1) = 0$,

$$
\int_0^1 \hat{\psi}_{e}(t, \alpha, y) f(y)dy = -\int_{u(0)}^{u(1)} K_e(c, \alpha) e^{-i\alpha c t} dc,
$$

where

\begin{align*}
K_o(c, \alpha) & = \frac{\Lambda_1(\hat{\omega}_o)(c)\Lambda_2(g)(c)}{(A(c)^2 + B(c)^2)u'(y_c)}, \\
K_e(c, \alpha) & = \frac{\Lambda_3(\hat{\omega}_e)(c)\Lambda_4(g)(c)}{u'(y_c)(A_2^2 + B_2^2)(c)}.
\end{align*}

If $K_o$ and $K_e$ are bounded in $c$ in the Sobolev space $W^{2,1}$, then it is easy to derive the decay estimate of the velocity. However, the fact that the factor $u'(y_c)$ vanishes for $y_c = 0 (c = u(0))$ leads to the essential difficulties. To cancel this singularity, we have to make very precise estimates (especially using the cancellation properties) for some quantities and singular integral operators. For example, for the estimate of $\|\partial_y^2 K_o\|_{L^1_y}$, we have to deal with the term $\|\Lambda_1(\hat{\omega}_o)(c)\Lambda_2(g)(c)/u'(y_c)\|_{L^1_y} = \|\Lambda_1(\hat{\omega}_e)(c)\Lambda_2(g)(c)/u'(y_c)\|_{L^1_y}$, which requires enough vanishing order of $\Lambda_j$, for example $\Lambda_j = O(y_c^2)$. Here we show some formal arguments to explain the behavior near the critical point $y_c = 0$ and the behavior near boundary may regard as the same as the monotonic case. The expression of $\Lambda_j$ is as follows

\begin{align*}
\Lambda_1(\varphi)(c) & = \Lambda_{1,1}(\varphi)(c) + \Lambda_{1,2}(\varphi)(c), \\
\Lambda_2(\varphi)(c) & = \Lambda_{2,1}(\varphi)(c) + \Lambda_{2,2}(\varphi)(c), \\
\Lambda_3(\hat{\omega}_e)(c) & = -\rho_1(c)\Lambda_1(\hat{\omega}_e)(c) + \Lambda_{3,1}(\hat{\omega}_e)(c), \\
\Lambda_4(g)(c) & = -\rho_1(c)\Lambda_2(g)(c) + \Lambda_{4,1}(g)(c),
\end{align*}

where $\rho_1(c) = c - u(0)$, $\rho(c) = (c - u(0))(u(1) - c)$, and

\begin{align*}
\Lambda_{1,1}(\varphi)(c) & = A_1(c)\varphi(y_c) + \rho(c)u''(y_c)\Pi_{1,1}(\varphi)(c), \\
\Lambda_{1,2}(\varphi)(c) & = \rho(c)u''(y_c)\Pi_{1,2}(\varphi) + u'(y_c)\rho(c)\Pi_{3}(c)\varphi(y_c), \\
\Lambda_{2,1}(\varphi)(c) & = A_1(c)\varphi(y_c) + \rho(c)\Pi_{1,1}(u''\varphi)(c), \\
\Lambda_{2,2}(\varphi)(c) & = \rho(c)\Pi_{1,2}(u''\varphi)(c) + u'(y_c)\rho(c)\Pi_{3}(c)\varphi(y_c).
\end{align*}
and

\[ II_{1,1}(\varphi)(c) = \int_0^1 \int_{yc}^y \frac{\varphi(y')}{(u(y) - c)^2} dy', \]

\[ II_{1,2}(\varphi)(c) = \int_0^1 \int_{yc}^y \frac{\varphi(y')}{(u(y) - c)^2} \left( \frac{\phi_1(y', c)}{\phi_1(y, c)^2} - 1 \right) dy' dy. \]

The second term is better by the fact that \(|\phi_1(y, c) - 1| \leq C|y - y_c|\) (one may refer to Sect. 11.2).

For the odd case: \(\varphi \in H^2(0, 1), \varphi(0) = 0\), we have \(\rho \sim y_c^2(1 - y_c)\), \(II_{1,2}(\varphi)(c) = O(1)\) and \(II_3(c) = O(y_c^{-1})\) (see Sect. 10.1), which imply that \(\Lambda_{j, 2}(\varphi)(c) = O(y_c^2), j = 1, 2\), and then we have to show that \(II_{1,1}(\varphi)(c) = O(1)\) at least in the \(L^2\) sense. Formally, the estimate of \(II_{1,1}\) is equivalent to the special case \(u(y) = y^2\). In this case, we rewrite \(II_{1,1}\) as

\[ y_c^2 II_{1,1}(\varphi)(c) = \frac{y_c}{2} \partial_{y_c} \left( \frac{1}{2y_c} p.v. \int_{-1}^1 \frac{\text{Int}(\varphi)(y)}{y - y_c} dy \right) \]

\[ - \frac{y_c \text{Int}(\varphi)(y_c)}{2} \partial_{y_c} \left( \frac{1}{2y_c} p.v. \int_{-1}^1 \frac{1}{z - y_c} dz \right) = I_1(c) + I_2(c). \]

In the case when \(\varphi\) is odd, \(\text{Int}(\varphi)(y) = \int_0^y \varphi(y')dy'\) is an even function in \(H^3\). Then one can prove that both \(I_1(c)\) and \(I_2(c)\) are of high regularity. Moreover, it is easy to show that \(I_2(c) \sim y_c^2\) as \(c \sim 0\), which follows from the fact that \(\frac{1}{2y_c} p.v. \int_{-1}^1 \frac{1}{z - y_c} dz = \frac{1}{2y_c} \ln \frac{1 - y_c}{1 + y_c}\) is an even smooth function near \(y_c \sim 0\). We can write

\[ I_1(c) = (y_c/4) \partial_{y_c} (H(\text{Int}(\varphi))(y_c)/y_c). \]

Now \(H(\text{Int}(\varphi))(y_c)\) is an odd function in \(H^3(-1/2, 1/2)\), \(H(\text{Int}(\varphi))(y_c)/y_c\) is an even function in \(H^2(-1/2, 1/2)\), and then \(\partial_{y_c} (H(\text{Int}(\varphi))(y_c)/y_c)\) is an odd function in \(H^1(-1/2, 1/2)\), and \(I_1(c)/y_c^2 = \partial_{y_c} (H(\text{Int}(\varphi))(y_c)/y_c)/(4y_c)\) is an even function of \(y_c\) in \(L^2(-1/2, 1/2)\).

For the even case: \(\varphi \in H^2(0, 1), \varphi'(0) = 0, \Lambda_{j, 1}(\varphi)(c), j = 1, 2\) may not vanish at critical point, and the behavior of \(\Lambda_{j}(\varphi)(c), j = 3, 4\) at critical point is mainly based on the coefficients \(\rho\) and \(\rho_1\). The fact that \(\varphi'(0) = 0\) plays an important role in the estimate of the normal integral operators \(\Lambda_{j, 1}(\varphi)(c), j = 3, 4\).

The details can be found in Sects. 10, 11 and 12 and the “Appendix”.

### 4 The Homogeneous Rayleigh Equation for Symmetric Flow

In this section, we solve the homogeneous Rayleigh equation on \([-1, 1]\):

\[(u - c)(\phi'' - \alpha^2 \phi) - u'' \phi = 0, \quad (4.1)\]
where the complex constant \( c \) will be taken in four kinds of domains:

\[
D_0 = \{ c \in (u(0), u(1)) \},
\]
\[
D_{\epsilon_0} = \{ c = c_r + i\epsilon, \ c_r \in (u(0), u(1)), 0 < |\epsilon| < \epsilon_0 \},
\]
\[
B_{\epsilon_0}^l = \{ c = u(0) + \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \},
\]
\[
B_{\epsilon_0}^r = \{ c = u(1) - \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \}.
\]

for some \( \epsilon_0 \in (0, 1) \). We denote

\[
\Omega_{\epsilon_0} = D_0 \cup D_{\epsilon_0} \cup B_{\epsilon_0}^l \cup B_{\epsilon_0}^r. \quad (4.2)
\]

For \( c \in \Omega_{\epsilon_0} \), we denote

\[
\begin{align*}
c_r &= \text{Re} \ c \quad \text{for} \ c \in D_{\epsilon_0}, \\
\ c_r &= u(0) \quad \text{for} \ c \in B_{\epsilon_0}^l, \\
\ c_r &= u(1) \quad \text{for} \ c \in B_{\epsilon_0}^r. \quad (4.3)
\end{align*}
\]

Then we define \( y_c \in [0, 1] \) so that \( u(y_c) = c_r \).

In this section, we always assume that \( u \in C^4([-1, 1]) \) satisfies \((S)\).

4.1 Basic Properties of Symmetric Flow

We introduce

\[
\begin{align*}
v(y) &= \sqrt{u(y) - u(0)} \quad \text{for} \ y \in [0, 1] \ \text{and} \\
v(y) &= -v(-y) \quad \text{for} \ y \in [-1, 0]. \quad (4.4)
\end{align*}
\]

Lemma 4.1 It holds that

1. there exists \( c_0 > 0 \) such that

\[
u'(y) \geq c_0 y \quad \text{for} \ |y| \in [0, 1];
\]

2. the function \( v \in C^4([-1, 0) \cup (0, 1]) \) satisfies

\[
v \in C^3([-1, 1]), \ \ yv(y) \in C^4([-1, 1]), \ \ \tilde{v}v^{-1}(\tilde{c}) \in C^4([-v(1), v(1)]),
\]

and there exists \( c_1 > 0 \) so that \( v'(y) \geq c_1; \)

3. there exists \( C > 0 \) so that for any \( 1 \geq y \geq y' \geq 0, \)

\[
C^{-1}(y - y')(y + y') \leq u(y) - u(y') \leq C(y - y')(y + y') \quad \text{or}
\]
\[
C^{-1} \leq \frac{(u'(y) + u'(y'))(y - y')}{u(y) - u(y')} \leq C.
\]
The first property is obvious. Thanks to $u'(0) = 0$, we get

$$v(y)^2 = u(y) - u(0) = \int_0^y u'(y') dy' = \int_0^y \int_0^{y'} u''(y'') dy'' dy' = y^2 m(y),$$

where $m(y) = \int_0^1 (1 - t) u''(ty) dt$. Due to $u''(0) > 0$, there exists $\delta_0 > 0$ and $c_0 > 0$ so that for $y \in [0, \delta_0],

$$m(y) \geq \int_0^1 c_0 (1 - t) dt \geq \frac{1}{2} c_0.$$

For $y \in [\delta_0, 1],

$$m(y) \geq y^2 m(y) = u(y) - u(0) \geq \int_0^{\delta_0} c_0 y dy \geq \frac{c_0 \delta_0^2}{2}.$$

Thus, $m(y) \geq \frac{c_0 \delta_0^2}{2}$ and $v(y) = y m(y)^{\frac{1}{2}} \in C^2([-1, 1])$. So,

$$v'(y) = m(y)^{\frac{1}{2}} + \frac{ym'(y)}{2m(y)^{\frac{1}{2}},}$$

where

$$ym'(y) = \frac{u'(y) - 2ym(y)}{y} = \int_0^1 u''(ty) dt - 2m(y) \in C^2([-1, 1])$$

with the bound $\|ym'\|_{L^\infty} \leq 5\|u\|_{C^2} \leq C$. Thus, we get $v' \in C^2([-1, 1])$, so $v \in C^3([-1, 1])$ Moreover, for $y \in [0, \min \{\delta_0, \frac{c_0 \delta_0^2}{c_0}\}]$,

$$v'(y) \geq \sqrt{\frac{c_0 \delta_0^2}{2}} - \sqrt{\frac{\frac{c_0 y}{2\sqrt{\frac{c_0 \delta_0^2}{2}}}}{2}} \geq \frac{1}{2} \sqrt{\frac{c_0 \delta_0^2}{2}}.$$

For $y \in [\min \{\delta_0, \frac{c_0 \delta_0^2}{c_0}\}, 1]$, we have $u'(y) \geq c_0 \min \{\delta_0, \frac{c_0 \delta_0^2}{c_0}\}$, thus,

$$v'(y) = \frac{u'(y)}{2ym(y)^{\frac{1}{2}}} \geq c_1 > 0$$

for some $c_1 > 0$.

Because of $6(v'')^2 + 8v'v''' + 2vv'''' = u'''$, we have

$$2vy'''(y) = \frac{y}{v(y)} (u''' - 6(v'')^2 - 8v'v''')(y) \in C([-1, 1]),$$

\(\blacksquare\) Springer
which shows that \( vy(y) \in C^4([-1, 1]) \) and \( v^{-1}(\tilde{c}) \in C^4([-v(1), v(1)]) \).

Finally, for \( 1 \geq y \geq y' \geq 0 \), we have

\[
\begin{align*}
  u(y) - u(y') &= (v(y) - v(y'))(v(y) + v(y')) \\
  &= (y - y')(y + y') \left( \int_0^1 v'(y' + t(y - y')) dt \right) \left( \int_0^1 v'(-y' + t(y + y')) dt \right),
\end{align*}
\]

which implies the third property. \( \square \)

### 4.2 Rayleigh Integral Operator

Given \( |\alpha| \geq 1 \), let \( A \) be a constant larger than \( C|\alpha| \) with \( C \geq 1 \) only depending on \( c_0 \) and \( \|u\|_{C^4} \). Unlike the monotonic case, we need to introduce the weighted functional spaces.

**Definition 4.2** For a function \( f(y, c) \) defined on \([0, 1] \times \Omega_{\varepsilon_0} \) and \( k \geq 0 \), we define

\[
\|f\|_{X_0^k}^{\alpha} \overset{\text{def}}{=} \sup_{(y, c) \in [0, 1] \times D_0} \left| \frac{u'(yc)^k f(y, c)}{\cosh(A(y - yc))} \right|,
\]

\[
\|f\|_{X^k}^{\alpha} \overset{\text{def}}{=} \sup_{(y, c) \in [0, 1] \times D_0} \left| \frac{u'(yc)^k f(y, c)}{\cosh(A(y - yc))} \right|,
\]

\[
\|f\|_{X^k_1}^{\alpha} \overset{\text{def}}{=} \sup_{(y, c) \in [0, 1] \times B_0} \left| \frac{u'(yc)^k f(y, c)}{\cosh(Ay)} \right|,
\]

\[
\|f\|_{X_r} \overset{\text{def}}{=} \sup_{(y, c) \in [0, 1] \times B_0} \left| \frac{f(y, c)}{\cosh(Ay - 1)} \right|.
\]

**Definition 4.3** For a function \( f(y, c) \) defined on \([0, 1] \times \Omega_{\varepsilon_0} \setminus D_0 \), we define

\[
\|f\|_{C^{1/2}_{B_0}} \overset{\text{def}}{=} \sup_{(y, c) \in [0, 1] \times [0, 1]} \sup_{\epsilon_1 \neq \epsilon_2, \epsilon_1 \epsilon_2 \geq 0} \left| \frac{f(y, u(y_c) + i\epsilon_1) - f(y, u(y_c) + i\epsilon_2)}{\epsilon_1 - \epsilon_2} \right| \cosh(A(y - yc)),
\]

\[
\|f\|_{C^{1/2}_{\tilde{c}}} \overset{\text{def}}{=} \sup_{(y, \theta) \in [0, 1] \times [\pi/2, 3\pi/2]} \sup_{\epsilon_1 \neq \epsilon_2, \epsilon_1 \epsilon_2 \geq 0} \left| \frac{f(y, u(0) + i\epsilon_1 e^{i\theta}) - f(y, u(0) + i\epsilon_2 e^{i\theta})}{\epsilon_1 - \epsilon_2} \right| \cosh(Ay).
\]

**Definition 4.4** For a function \( f(y, c) \) defined on \([0, 1] \times \Omega_{\varepsilon_0} \), we define

\[
\|f\|_{Y_0} \overset{\text{def}}{=} \sum_{k=0}^2 \sum_{\beta + \gamma = k} A^{-k} \|\partial^\beta_y \partial^\gamma_c f\|_{X_0^\gamma} + A^{-3} \|\partial^2_c \partial_y f\|_{X_1^\gamma},
\]

\[
\|f\|_{Y} \overset{\text{def}}{=} \|f\|_{X^0} + \frac{1}{A} \left( \|\partial_y f\|_{X^1} + \|\partial_c f\|_{X^1} + \|f\|_{C^{1/2}_{B_0}} \right).
\]
\[
\|f\|_{Y_l} \overset{\text{def}}{=} \|f\|_{X_0^l} + \frac{1}{A} \left( \|\partial_y f\|_{X_1^l} + \|\partial_c f\|_{X_1^c} + \|f\|_{C_1^c} \right),
\]
\[
\|f\|_{Y_r} \overset{\text{def}}{=} \|f\|_{X_0^r} + \frac{1}{A} \left( \|\partial_y f\|_{X_1^r} + \|\partial_c f\|_{X_1^c} + \|f\|_{C_1^c} \right),
\]

where

\[
\|f\|_{X_3^1} \overset{\text{def}}{=} \sup_{(y, c) \in [0, 1] \times D_0} \left| \frac{A u'(y') f(y, c)}{\cosh(A(y - y')) (A + y_c/(y + y_c)^2)} \right|.
\]

Now we introduce the Rayleigh integral operator, which will be used to give the solution formula of the homogeneous Rayleigh equation.

**Definition 4.5** The Rayleigh integral operator \( T \) is defined by

\[
T \triangleq T_0 \circ T_{2,2} \overset{\text{def}}{=} \int_{y_c}^{y} \frac{1}{(u(y') - c)^2} \int_{y_c}^{y'} f(z, c)(u(z) - c)^2 \, dz \, dy',
\]

where

\[
T_0 f(y, c) \overset{\text{def}}{=} \int_{y_c}^{y} f(z, c) \, dz,
\]
\[
T_{k, j} f(y, c) \overset{\text{def}}{=} \frac{1}{(u(y) - c)^j} \int_{y_c}^{y} f(z, c)(u(z) - c)^k \, dz.
\]

**Lemma 4.6** There exists a constant \( C \) independent of \( A \) so that

\[
\|T f\|_{Y_0} \leq \frac{C}{A^2} \|f\|_{Y_0}, \quad \|T f\|_{Y} \leq \frac{C}{A^2} \|f\|_{Y},
\]
\[
\|T f\|_{Y_l} \leq \frac{C}{A^2} \|f\|_{Y_l}, \quad \|T f\|_{Y_r} \leq \frac{C}{A^2} \|f\|_{Y_r}.
\]

**Proof** Let us prove the first inequality. A direct calculation shows that for \( k \geq 0 \),

\[
\|T_0 f\|_{X_0^k} = \sup_{(y, c) \in [0, 1] \times D_0} \left| \frac{1}{\cosh(A(y - y_c))} \int_{y_c}^{y} \frac{u'(y_c)^k f(z, c)}{\cosh(A(z - y_c))} \cosh(A(z - y_c)) \, dz \right|
\]
\[
\leq \sup_{(y, c) \in [0, 1] \times D_0} \left| \frac{1}{\cosh(A(y - y_c))} \int_{y_c}^{y} \cosh(A(z - y_c)) \, dz \right| \|f\|_{X_0^k}
\]
\[
\leq \frac{1}{A} \|f\|_{X_0^k}.
\]

Using the fact that for any \( 0 < y < y' < y_c \) or \( y_c < y' < y < 1 \)

\[
|u(y') - c| \leq |u(y) - c|,
\]
we deduce that

\[
\|T_{2,2}f\|_{\mathcal{X}^k_0} \leq \sup_{(y,c) \in [0,1] \times D_0} \left| \frac{y - y_c}{\cosh A(y - y_c)} \right| \int_0^1 \cosh tA(y - y_c)dt \|f\|_{\mathcal{X}^k_0}
\]

\[
\leq \frac{C}{A} \|f\|_{\mathcal{X}^k_0}.
\]

(4.7)

from which and (4.5), we deduce that for \(k \geq 0\),

\[
\|Tf\|_{\mathcal{X}^k_0} \leq \frac{C}{A^2} \|f\|_{\mathcal{X}^k_0}.
\]

(4.8)

By Lemma 4.1, we have

\[
0 \leq \frac{(u'(y) + u'(y_c))(y - y_c)}{u(y) - u(y_c)} \leq C,
\]

which along with (4.6) implies that

\[
\|T_{k,k+1}f\|_{\mathcal{X}^{k+1}_0} + \|u'(y)T_{k,k+1}f\|_{\mathcal{X}^k_0}
\]

\[
\leq C \sup_{(y,c) \in [0,1] \times D_0} \left| \frac{1}{\cosh A(y - y_c)} \right| \int_0^1 \cosh tA(y - y_c)dt \|f\|_{\mathcal{X}^k_0}
\]

\[
\leq C \|f\|_{\mathcal{X}^k_0}.
\]

(4.9)

It is easy to see that

\[
\partial_y Tf(y, c) = T_{2,2}f(y, c),
\]

\[
\partial_c Tf(y, c) = 2T_0 \circ T_{2,3}f(y, c) - 2T_0 \circ T_{1,2}f(y, c) + T\partial_c f(y, c),
\]

\[
\partial_y^2 Tf(y, c) = -2u'(y)T_{2,3}f(y, c) + f(y, c).
\]

Thus, it follows from (4.5), (4.7) and (4.9) that

\[
\|\partial_y Tf\|_{\mathcal{X}^k_0} + \frac{1}{A} \|\partial_y^2 Tf\|_{\mathcal{X}^k_0} + \|\partial_c Tf\|_{\mathcal{X}^{k+1}_0} \leq \frac{C}{A} \|f\|_{\mathcal{X}^k_0} + \frac{C}{A^2} \|\partial_c f\|_{\mathcal{X}^k_0}. \tag{4.10}
\]

For \(c \in D_0\), let

\[
u_1(y, c) = \frac{u(y) - c}{y - y_c} = \int_0^1 u'(y_c + t(y - y_c))dt.
\]

Then we have

\[
T_{k,k+1}f(y, c) = \int_0^1 f(y_c + t(y - y_c), c) \frac{t^ku_1(y_c + t(y - y_c), c)^k}{u_1(y, c)^{k+1}}dt.
\]
By Lemma 4.1, \( u_1(y, c) \sim y + y_c \). Then we can deduce that

\[
\left| \frac{u'(y_c) j \partial_c T_k, k+1 f(y, c)}{\cosh A(y - y_c)} \right| \leq C \left( \frac{\| f \|_{X_0^1}^{-1}}{u_1(y, c)} + \frac{\| \partial_y f \|_{X_0^1}^{-1} + \| \partial_c f \|_{X_0^1}}{u_1(y, c)} \right). \tag{4.11}
\]

Direct calculation gives

\[
\begin{align*}
\partial_y \partial_c T f(y, c) &= 2T_{2,3} f(y, c) - 2T_{1,2} f(y, c) + T_{2,2} \partial_c f(y, c), \\
\partial_y \partial_c^2 T f(y, c) &= 2\partial_c T_{2,3} f(y, c) - 2\partial_c T_{1,2} f(y, c) \\
&+ 2T_{2,3} \partial_c f(y, c) - 2T_{1,2} \partial_c f(y, c) + T_{2,2} \partial_c^2 f(y, c),
\end{align*}
\]

and

\[
\begin{align*}
\partial_c^2 T f(y, c) &= 2\partial_c T_0 T_{2,3} f(y, c) - 2\partial_c T_0 T_{1,2} f(y, c) + \partial_c T \partial_c f(y, c) \\
&= 2T_0 \partial_c T_{2,3} f(y, c) - 2T_0 \partial_c T_{1,2} f(y, c) + \frac{f(y_c, c)}{3u'(y_c)^2} \\
&+ 2T_0 T_{2,3} \partial_c f(y, c) - 2T_0 T_{1,2} \partial_c f(y, c) + T \partial_c^2 f(y, c).
\end{align*}
\]

Then it follows from (4.9) and (4.7) that

\[
\| \partial_y \partial_c T f \|_{X_0^1} \leq C \| f \|_{X_0^0} + \frac{C}{A} \| \partial_c f \|_{X_0^1}. \tag{4.12}
\]

By (4.11), we have

\[
\begin{align*}
\left| \frac{u'(y_c)^2 T_0 \partial_c T_k, k+1 f(y, c)}{\cosh A(y - y_c)} \right| &\leq \frac{u'(y_c)}{\cosh A(y - y_c)} \times \int_{y_c}^{y} \cosh A(y' - y_c) \left| \frac{u'(y_c) \partial_c T_k, k+1 f(y', c)}{\cosh A(y' - y_c)} \right| dy' \\
&\leq C u'(y_c) \left| \int_{y_c}^{y} \frac{1}{u_1(y', c)^2} dy' \right| \| f \|_{X_0^0} \\
&+ \frac{C}{A} \left( \| \partial_y f \|_{X_0^0} + \| \partial_c f \|_{X_0^1} \right) \\
&\leq C \| f \|_{X_0^0} + \frac{C}{A} \left( \| \partial_y f \|_{X_0^0} + \| \partial_c f \|_{X_0^1} \right),
\end{align*}
\]

which along with (4.5) and (4.9) gives

\[
\| \partial_c^2 T f \|_{X_0^2} \leq C \| f \|_{X_0^0} + \frac{C}{A} \| \partial_y f \|_{X_0^0} + \frac{C}{A} \| \partial_c f \|_{X_0^1} + \frac{C}{A^2} \| \partial_c^2 f \|_{X_0^2}. \tag{4.13}
\]
We infer from (4.9) and (4.11) that

$$\left| \frac{u'(y_c) \partial_y \partial_z T f(y, c)}{\cosh A(y - y_c)} \right| \leq \frac{C \|f\|_{X_0^1}}{u_1(y, c)^2} + \frac{C \|\partial_y f\|_{X_0^1}}{u'(y_c)} + \frac{C \|\partial_z f\|_{X_0^1}}{u'(y_c)} ,$$

which implies that

$$\|\partial_y \partial_z T f\|_{X_0^3} \leq C A \|f\|_{X_0^1} + C \|\partial_y f\|_{X_0^1} + C \|\partial_z f\|_{X_0^1} + \frac{C}{A} \|\partial_z^2 f\|_{X_0^1}. \quad (4.14)$$

Putting (4.8), (4.10) and (4.12)–(4.14) together, we conclude the first inequality.

The other inequalities of the lemma can be deduced in a similar derivation as in (4.10) except the Hölder estimates \(\|T f\|_{C^{1/2}}\) and \(\|T f\|_{C^{1/2}}\). For \(c_k = u(y_c) + i \epsilon_k\) with \(0 \leq \epsilon_2 < \epsilon_1\) and \(k = 1, 2\), we have

$$|T f(y, c_1) - T f(y, c_2)| = \int_{y_c}^{y} \int_{y_c}^{y'} \left( \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} - \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \right) f(z, c_1) dz dy' + \int_{y_c}^{y} \int_{y_c}^{y'} \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} (f(z, c_1) - f(z, c_2)) dz dy'.$$

Using the facts that for \(y_c \leq z \leq y'\) or \(y' \leq z \leq y_c\),

$$\left| \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} - \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \right| \leq C \left| \frac{(u(z) - c_1)}{(u(y') - c_1)} - \frac{(u(z) - c_2)}{(u(y') - c_2)} \right|$$

$$\leq C \frac{|\epsilon_1 - \epsilon_2| |u(z) - u(y')|}{|u(y') - u(y_c) + i \epsilon_1| |u(y') - u(y_c) + i \epsilon_2|}$$

$$\leq C \frac{|u(y') - u(y_c) + i \epsilon_1| |u(y') - u(y_c) + i \epsilon_2|}{|u_1(y', c)(y' - y_c)| + |\epsilon_1|} ,$$

and \(u_1(y', c) \geq C^{-1} (y' + y_c)\) and \(\epsilon_1 - \epsilon_2 \leq \epsilon_1\), we deduce that

$$\frac{1}{\cosh A(y - y_c)} \left| \int_{y_c}^{y} \int_{y_c}^{y'} \left( \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} - \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \right) f(z, c_1) dz dy_1 \right|$$

$$\leq C \frac{1}{\cosh A(y - y_c)} \left| \int_{y_c}^{y} \frac{(\epsilon_1 - \epsilon_2)|y' - y_c|}{|u_1(y', c)(y' - y_c)| + |\epsilon_1|} \cosh A(y' - y_c) dy' \right| \|f\|_{X_0}$$

$$\leq \frac{|\epsilon_1 - \epsilon_2|^2}{\cosh A(y - y_c)} \left| \int_{y_c}^{y} \cosh A(y' - y_c) dy' \right| \|f\|_{X_0}$$

$$\leq \frac{C}{A} |\epsilon_1 - \epsilon_2|^2 \|f\|_{X_0} .$$
For the second term, we have
\[
\left| \int_{y_c}^{y} \int_{y_c}^{y'} \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} (f(z, c_1) - f(z, c_2)) \, dz \, dy' \right|
\leq C|\epsilon_1 - \epsilon_2|^2 \left| \int_{y_c}^{y} \int_{y_c}^{y'} \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \cosh A(z - y_c) \, dz \, dy' \right| \|f\|_{C^{1/2}_{\partial \Omega}}
\leq \frac{C \cosh A(y - y_c)}{A^2} |\epsilon_1 - \epsilon_2|^2 \|f\|_{C^{1/2}_{\partial \Omega}}.
\]

This shows that
\[
\|Tf\|_{C^{1/2}_{\partial \Omega}} \leq \frac{C}{A} \|f\|_{X^0} + \frac{C}{A^2} \|f\|_{C^{1/2}_{\partial \Omega}} \leq \frac{C}{A} \|f\|_{X^0}.
\]

Similarly, for \( c_k = u(0) + \epsilon_k e^{i\theta} \) with \( k = 1, 2 \), we have
\[
|Tf(y, c_1) - Tf(y, c_2)| = \int_{0}^{y} \int_{0}^{y'} \left( \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} - \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \right) f(z, c_1) \, dz \, dy' + \int_{0}^{y} \int_{0}^{y'} \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} (f(z, c_1) - f(z, c_2)) \, dz \, dy'.
\]

Using the fact that for \( 0 \leq z \leq y' \),
\[
\left| \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} - \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \right| \leq C \left| \frac{(u(z) - c_1)}{(u(y') - c_1)} - \frac{(u(z) - c_2)}{(u(y') - c_2)} \right| \leq C \frac{|c_1 - c_2| |u(z) - u(y')|}{|u(y') - u(0)| + \epsilon_1 e^{i\theta} |u(y') - u(0)| + \epsilon_2 e^{i\theta} |u(y') - u(0)|}
\leq C \frac{\epsilon_1 - \epsilon_2}{y'^2 + |\epsilon_1|},
\]
we deduce that
\[
\frac{1}{\cos Ay} \left| \int_{0}^{y} \int_{0}^{y'} \left( \frac{(u(z) - c_1)^2}{(u(y') - c_1)^2} - \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \right) f(z, c_1) \, dz \, dy' \right|
\leq \frac{C}{\cosh Ay} \left| \int_{0}^{y} \frac{(\epsilon_1 - \epsilon_2)y'}{y'^2 + |\epsilon_1|} \cosh Ay' \, dy' \right| \|f\|_{X^0}
\leq \frac{C}{A} |\epsilon_1 - \epsilon_2|^2 \|f\|_{X^0},
\]
and

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\[
\left| \int_0^y \int_0^{y'} \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} (f(z, c_1) - f(z, c_2)) dz dy' \right|
\leq C|\epsilon_1 - \epsilon_2|^{1/2} \int_0^y \int_0^{y'} \frac{(u(z) - c_2)^2}{(u(y') - c_2)^2} \cosh A z dy' \parallel f \parallel_{C_{1/2}^1}
\leq \frac{C \cosh A y}{A^2} |\epsilon_1 - \epsilon_2|^{1/2} \parallel f \parallel_{C_{1/2}^1}.
\]

This shows that
\[
\parallel T f \parallel_{C_{1/2}^1} \leq \frac{C}{A} \parallel f \parallel_{Y_l}.
\]

The proof of the lemma is completed. \(\Box\)

4.3 Existence of the Solution

In this subsection, the constant \(C\) may depend on \(\alpha\).

Proposition 4.7 There exists a solution \(\phi(y, c) \in C^1([0, 1] \times \Omega_{e_0} \setminus D_0) \cap C([0, 1] \times \Omega_{e_0})\) of the Rayleigh equation (4.1). Let
\[
\phi_1(y, c) = \frac{\phi(y, c)}{u(y) - c}. \tag{4.15}
\]

There exists \(\epsilon_1 > 0\) such that for any \(\epsilon_0 \in [0, \epsilon_1)\) and \((y, c) \in [0, 1] \times \Omega_{e_0},\)
\[
|\phi_1(y, c)| \geq \frac{1}{2}, \quad |\phi_1(y, c) - 1| \leq C|y - y_c|^2.
\]

Lemma 4.8 Let \(c \in D_{e_0}\). Then there exists a solution \(\phi(x, c) \in Y\) to the Rayleigh equation
\[
\left\{ \begin{array}{l}
\phi'' - \alpha^2 \phi - \frac{u''}{u-c} \phi = 0, \\
\frac{\phi(y_c, c)}{u(y_c) - c} = 1, \quad \left( \frac{\phi(y, c)}{u(y) - c} \right)' \big|_{y=y_c} = 0.
\end{array} \right.
\]

Moreover, there holds
\[
\parallel \phi_1 \parallel_{Y} + \parallel \phi \parallel_{Y} \leq C.
\]

Lemma 4.9 Let \(c \in B_{e_0}^1\). Then there exists a solution \(\phi(x, c) \in Y_l\) to the Rayleigh equation
\[
\left\{ \begin{array}{l}
\phi'' - \alpha^2 \phi - \frac{u''}{u-c} \phi = 0, \\
\frac{\phi(0, c)}{u(0) - c} = 1, \quad \left( \frac{\phi(y, c)}{u(y) - c} \right)' \big|_{y=0} = 0.
\end{array} \right.
\]
Moreover, there holds
\[ \|\phi_1\|_{Y_l} + \|\phi\|_{Y_l} \leq C. \]

**Lemma 4.10** Let \( c \in B_{\epsilon_0}^r \). Then there exists a solution \( \phi(x, c) \in Y_r \) to the Rayleigh equation
\[
\begin{align*}
\phi'' - \alpha^2 \phi - \frac{u''}{u-c} \phi &= 0, \\
\frac{\phi(1,c)}{u(1)-c} &= 1, \\
\left( \frac{\phi(y, c)}{u(y)-c} \right)' \bigg|_{y=1} &= 0.
\end{align*}
\]
Moreover, there holds
\[ \|\phi_1\|_{Y_r} + \|\phi\|_{Y_r} \leq C. \]

**Lemma 4.11** Let \( c \in D_0 \). Then there exists a solution \( \phi(y, c) \in Y_0 \) to the Rayleigh equation
\[
\begin{align*}
\phi'' - \alpha^2 \phi - \frac{u''}{u-c} \phi &= 0, \\
\phi(y_c, c) &= 0, \\
\phi'(y_c, c) &= u'(y_c).
\end{align*}
\]
Moreover, there holds
\[ \|\phi_1\|_{Y_0} + \|\phi\|_{Y_0} \leq C. \]

With Lemma 4.6 and the formula
\[
\phi_1(y, c) = 1 + \int_{y_c}^{y} \frac{\alpha^2}{(u(y') - c)^2} \int_{y_c}^{y'} \phi_1(z, c)(u(z) - c)^2 \, dz \, dy' = 1 + \alpha^2 T \phi_1(y, c), \tag{4.16}
\]
the above lemmas can be proved as in Lemma 4.6–Lemma 4.9 in [31]. So, we omit the details.

As in the proof of Proposition 4.5 in [31], we define
\[
\phi(y, c) \overset{\text{def}}{=} \begin{cases} 
\phi^0(y, c) & \text{for } c \in D_0, \\
\phi^\pm(y, c) & \text{for } c \in D_{\epsilon_0}, \\
\phi^l(y, c) & \text{for } c \in B_{\epsilon_0}^l, \\
\phi^r(y, c) & \text{for } c \in B_{\epsilon_0}^r,
\end{cases}
\]
where \( \phi^\pm, \phi^l, \phi^r, \phi^0 \) are given by Lemma 4.8, Lemma 4.9, Lemma 4.10 and Lemma 4.11 respectively. Then \( \phi(y, c) \) is our desired solution.

We need the following further properties of \( \phi_1(y, c) \).
Lemma 4.12 For \( c \in D_0 \), it holds that 

1. \( \phi_1(y, c) \geq 1 \); 
2. \( \partial_y \phi_1(y, c) > 0 \) for \( y \in (y_c, 1) \) and \( \partial_y \phi_1(y, c) < 0 \) for \( y \in [0, y_c) \); 
3. for any given \( M_0 > 0 \), there exists a constant \( C \) only depending on \( M_0 \) such that for \( \beta + \gamma \leq 2 \), \( |y - y_c| \leq M_0/|\alpha| \)

\[
|u'(y_c)^Y \partial_y^\beta \partial_c^\gamma \phi_1(y, c)| \leq C|\alpha|^\beta + \gamma, \\
|u'(y_c)^2 \partial_y \partial_c^2 \phi_1(y, c)| \leq C(|\alpha|^3 + \alpha^2 y_c/(y + y_c)^2).
\]

**Proof** The first two properties are a direct consequence of \((4.16)\). The third property follows from Lemma 4.11 and the definition of \( Y_0 \) norm. \( \square \)

Lemma 4.13 For \( c_e \in D_e \cup D_0 \) and \( c_e \in B_{e}^r \), \( \phi_1'(0, c_e) = \partial_y \phi_1(0, c_e) \) is continuous in \( e \). For \( c_e \in B_{e}^r \), \( \phi_1'(0, c_e) = 0 \).

**Proof** Let \( c_{e1} = u(y_c) + i \epsilon_1 \in D_e \cup D_0 \) and \( c_{e2} = u(y_c) + i \epsilon_2 \in D_e \cup D_0 \) with \( 0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_0 \). By \((4.16)\), we have

\[
\phi_1'(0, c_{e1}) = \frac{\alpha^2}{(u(0) - c_{e1})^2} \int_{y_c}^0 (u(y') - c_{e1})^2 \phi_1(y', c_{e1})dy'.
\]

Therefore,

\[
|\phi_1'(0, c_{e1}) - \phi_1'(0, c_{e2})| \\
\leq C \left| \int_{y_c}^0 \frac{(u(y') - c_{e1})^2}{(u(0) - c_{e1})^2} (\phi_1(y', c_{e1}) - \phi_1(y', c_{e2}))dy' \right| \\
+ C \left| \int_{0}^{y_c} \left( \frac{(u(y') - c_{e1})^2}{(u(0) - c_{e1})^2} - \frac{(u(y') - c_{e2})^2}{(u(0) - c_{e2})^2} \right) \phi_1(y', c_{e2})dy' \right|.
\]

Using the fact that \( |\partial_y \phi_1| \leq \frac{C}{u(y_c)} \) and \( \frac{(u(y') - c_{e1})^2}{(u(0) - c_{e1})^2} \leq C \) for \( y' \leq y_c \), the first term is bounded by \( C|\epsilon_1 - \epsilon_2| \). For the second term, we have

\[
\left| \int_{0}^{y_c} \left( \frac{(u(y') - c_{e1})^2}{(u(0) - c_{e1})^2} - \frac{(u(y') - c_{e2})^2}{(u(0) - c_{e2})^2} \right) \phi_1(y', c_{e2})dy' \right| \\
\leq C \left| \int_{0}^{y_c} \frac{(u(y') - c_{e1})}{(u(0) - c_{e1})} - \frac{(u(y') - c_{e2})}{(u(0) - c_{e2})} \right| dy' \\
\leq C \left| \int_{0}^{y_c} \frac{(u(z) - u(0))(c_{e1} - c_{e2})}{(u(0) - c_{e1})(u(0) - c_{e2})} \right| dz \\
\leq C \frac{y_c|\epsilon_1 - \epsilon_2|}{y_c^2 + \epsilon_1} \leq C \sqrt{\epsilon_1 - \epsilon_2}.
\]

This shows that

\[
|\phi_1'(0, c_{e1}) - \phi_1'(0, c_{e2})| \leq C \sqrt{\epsilon_1 - \epsilon_2}.
\]
Similarly, we can prove that for $c_{\varepsilon_1}, c_{\varepsilon_2} \in B^r_{\varepsilon}$,

$$|\phi'_1(0, c_{\varepsilon_1}) - \phi'_1(0, c_{\varepsilon_2})| \leq C|\varepsilon_1 - \varepsilon_2|.$$ 

For $c_\varepsilon \in B^r_{\varepsilon}$, we have $y_\varepsilon = 0$, thus $\phi'_1(0, c_\varepsilon) = 0$. $\square$

### 4.4 Uniform Estimates of the Solution

In this subsection, we establish some uniform estimates in wave number $\alpha$ for the solution $\phi(y, c)$ for $c \in D_0$ of the Rayleigh equation given by Lemma 4.11. Without loss of generality, we always assume $\alpha \geq 1$ in the sequel.

Recall that $\phi_1(y, c) = \frac{\phi(y, c)}{u(y) - c}$ and $c = u(y_\varepsilon)$ for $c \in D_0$. We introduce

\begin{align}
\mathcal{F}(y, c) &= \frac{\partial_y \phi_1(y, c)}{\phi_1(y, c)}, \\
\mathcal{G}(y, c) &= \frac{\partial_c \phi_1(y, c)}{\phi_1(y, c)}, \\
\mathcal{G}_1(y, c) &= \frac{\mathcal{F}(y, c)}{u'(y_\varepsilon)} + \mathcal{G}(y, c) = \frac{1}{\phi_1} \left( \frac{\partial_y}{u'(y_\varepsilon)} + \partial_c \right) \phi_1(y, c).
\end{align}

(4.17)

(4.18)

It is easy to see that

$$\mathcal{F}(y_\varepsilon, c) = \mathcal{G}(y_\varepsilon, c) = \mathcal{G}_1(y_\varepsilon, c) = 0.$$

**Proposition 4.14** There exists a constant $C$ independent of $\alpha$ such that

$$\phi_1(y, c) - 1 \leq C \min\{\alpha^2|y - y_\varepsilon|^2, 1\} \phi_1(y, c),$$

$$C^{-1} e^{C^{-1}a|y - y_\varepsilon|} \leq \phi_1(y, c) \leq e^{C\alpha|y - y_\varepsilon|},$$

$$C^{-1} \alpha \min\{\alpha|y - y_\varepsilon|, 1\} \leq |\mathcal{F}(y, c)| \leq C\alpha \min\{\alpha|y - y_\varepsilon|, 1\},$$

and for $\beta + \gamma \leq 2$,

$$|u'(y_\varepsilon)^\gamma \partial_y^{\beta} \partial_c^\gamma \phi_1(y, c)| \leq C\alpha^{\beta + \gamma} \phi_1(y, c),$$

$$|\partial_c \phi_1(y, c)| \leq C \left( \frac{\alpha^2|y - y_\varepsilon|}{u'(y_\varepsilon)} + \frac{\alpha^3|y - y_\varepsilon|^2}{u'(y_\varepsilon)^2} \right) \phi_1(y, c).$$

Moreover, there holds

$$\left| \left( \frac{\partial_y}{u'(y_\varepsilon)} + \partial_c \right) \phi_1(y, c) \right| \leq C \frac{\min\{1, \alpha^2|y - y_\varepsilon|^2\} \phi_1}{u'(y_\varepsilon)^2},$$

$$\left| \left( \frac{\partial_y}{u'(y_\varepsilon)} + \partial_c \right)^2 \phi_1(y, c) \right| \leq C \frac{\min\{1, \alpha^2|y - y_\varepsilon|^2\} \phi_1}{u'(y_\varepsilon)^4}.$$

**Proof** Step 1. Estimates of $\mathcal{F}$ and $\partial_y^k \phi_1$ for $k = 0, 1, 2$.
Recall that $φ_1 = 1 + α^2 T(φ_1)$, which implies that

$$φ_1(y, c) - 1 ≤ C \min \{α^2 |y - y_c|^2, 1\} φ_1(y, c). \quad (4.19)$$

It is easy to check that $F$ satisfies

$$F' + F^2 + \frac{2u'}{u - c} F = α^2, \quad F(y_c, c) = 0. \quad (4.20)$$

Notice that

$$\lim_{y \to y_c} \frac{u' F(y, c)}{u - c} = \partial_y F(y_c, c).$$

So, $\partial_y F(y_c, c) = \frac{α^2}{3} > 0$. Thanks to $F(y, c) ≥ 0$ for $y ≥ y_c$ and $F(y, c) ≤ 0$ for $y ≤ y_c$, we also have $\frac{2u'}{u - c} F ≥ 0$. Then (4.20) implies that $|F(y, c)| ≤ α$, which in particular gives

$$e^{-α|y - y'|} ≤ \frac{φ_1(y', c)}{φ_1(y, c)} ≤ e^{α|y - y'|}. \quad (4.21)$$

Using the equation

$$\partial_y φ_1(y, c) = \frac{α^2}{(u(y) - c)^2} \int_{y_c}^y φ_1(y', c)(u(y') - c)^2 \, dy',$$

and the fact that $(u(y') - c)^2 ≤ (u(y) - c)^2$ and $φ_1(y', c) ≤ φ_1(y, c)$ for $y_c ≤ y' ≤ y$ or $y ≤ y' ≤ y_c$ (by Lemma 4.12), we infer that

$$|F(y, c)| ≤ Cα^2 |y - y_c|. \quad (4.22)$$

So, we get

$$|F(y, c)| ≤ Cα \min \{α |y - y_c|, 1\}. \quad (4.22)$$

Using Lemma 4.1 and (4.21), we deduce that for $0 ≤ y_c ≤ y ≤ \frac{1}{α} + y_c ≤ 1$,

$$\left\{ \begin{array}{ll} F(y, c) = \frac{α^2}{(u(y) - c)^2} \int_{y_c}^y φ_1(y', c) \frac{φ_1(y', c)}{φ_1(y, c)} (u(y') - c)^2 \, dy' \\ ≥ \frac{α^2}{(u(y) - c)^2} \int_{y_c}^y e^{-α|y - y'|}(u(y') - c)^2 \, dy' \\ ≥ C^{-1} \frac{α^2}{(y + y_c)^2 (y - y_c)^2} \end{array} \right.$$
\[
\int_{y_c+y}^{y} (y' + y_c)^2 (y' - y_c)^2 \, dy' \\
\geq C^{-1} \alpha^2 |y - y_c|,
\]

and for \(0 \leq y_c - \frac{1}{\alpha} \leq y \leq y_c \leq 1\),

\[
-
\mathcal{F}(y, c) \geq \frac{\alpha^2}{(u(y) - c)^2} \int_{y_c}^{y} e^{-\alpha |y - y'|} (u(y') - c)^2 \, dy' \\
\geq C^{-1} \frac{\alpha^2}{(y + y_c)^2 (y - y_c)^2} \int_{y}^{\frac{x+y}{2}} (y' + y_c)^2 (y' - y_c)^2 \, dy' \\
\geq C^{-1} \alpha^2 |y - y_c|.
\]

For \(0 \leq \frac{1}{\alpha} + y_c \leq y \leq 1\), we have

\[
\mathcal{F}(y, c) = \frac{\alpha^2}{(u(y) - c)^2} \int_{y_c}^{y} \phi_1(y', c) \phi_1(y, c) \, dy' \\
\geq \frac{\alpha^2}{(u(y) - c)^2} \int_{y_c}^{y} e^{-\alpha |y - y'|} (u(y') - c)^2 \, dy' \\
\geq C^{-1} \frac{\alpha^2}{(y + y_c)^2 (y - y_c)^2} \\
\int_{\max \left\{ \frac{x+y}{2}, y - \frac{1}{\alpha} \right\}}^{y} e^{-\alpha |y - y'|} (y' + y_c)^2 (y' - y_c)^2 \, dy' \\
\geq C^{-1} \alpha.
\]

For \(0 \leq y \leq y_c - \frac{1}{\alpha} \leq 1\), we have

\[
-\mathcal{F}(y, c) = \frac{\alpha^2}{(u(y) - c)^2} \int_{y_c}^{y} \phi_1(y', c) \phi_1(y, c) \, dy' \\
\geq C^{-1} \frac{\alpha^2}{(u(y) - c)^2} \int_{y}^{y_c} e^{-\alpha |y - y'|} (u(y') - c)^2 \, dy' \\
\geq C^{-1} \frac{\alpha^2}{(y + y_c)^2 (y - y_c)^2} \int_{y}^{\min \left\{ \frac{x+y}{2}, y + \frac{1}{\alpha} \right\}} e^{-\alpha |y - y'|} (y' + y_c)^2 (y' - y_c)^2 \, dy' \\
\geq C^{-1} \alpha.
\]

This shows that

\[
C^{-1} \alpha \min \{ \alpha |y - y_c|, 1 \} \leq |\mathcal{F}(y, c)| \leq C \alpha \min \{ \alpha |y - y_c|, 1 \}, \quad (4.23)
\]
which along with \( \phi_1(y_c, c) = 1 \) implies that
\[
C^{-1} e^{C^{-1} |\alpha y - y_c|} \leq \phi_1(y, c) \leq e^{C |\alpha y - y_c|}, \quad (4.24)
\]
\[|\partial_y \phi_1(y, c)| \leq C \alpha \min\{\alpha |y - y_c|, 1\} \phi_1(y, c). \quad (4.25)\]

Using \( \phi_1'' + \frac{2u'}{u-c} \phi_1' = \alpha^2 \phi_1 \) and \( (u'(y) + u'(y_c)) |y - y_c| \leq C |u - c| \), we obtain
\[|\partial_y^2 \phi_1(y, c)| \leq C \alpha^2 \phi_1(y, c).\]

**Step 2.** Estimates of \( \partial_c \phi_1, \partial_y \partial_c \phi_1 \) and \( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \phi_1 \).

It is easy to check that
\[
\partial_c F = \partial_y G, \quad \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) F = \partial_y G_1, \quad (4.24)
\]
\[
(\phi^2 \partial_c F)' + 2\phi^2 u' F = 0, \quad (\phi^2 \partial_y F)' + 2\phi^2 (u''(u - c) - u'^2) F = 0. \quad (4.25)
\]

Then we deduce that
\[
\partial_c F(y, c) = \frac{-2}{\phi(y, c)^2} \int_{y_c}^{y} \phi_1(y', c)^2 u'(y') F(y', c) dy',
\]
\[
= \frac{-2}{\phi(y, c)^2} T_0 (\phi_1^2 u' F), \quad (4.26)
\]
\[
\partial_y F(y, c) = \frac{-2}{\phi(y, c)^2} T_0 \left( \phi_1^2 (u''(u - c) - u'^2) F \right), \quad (4.27)
\]
\[
\left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) F = \partial_y G_1 = \frac{-2}{\phi(y, c)^2} T_0 \left( \phi_1^2 \frac{a_1}{u'(y_c)} F \right), \quad (4.28)
\]

where \( a_1(y, c) = u'(y)u'(y_c) + u''(y)(u(y) - c) - u'(y)^2 \).

It is easy to see that
\[
\partial_y a_1(y, c) = u''(y)(u'(y_c) - u'(y)) + u'''(y)(u(y) - c),
\]
\[
\partial_c a_1(y, c) = u'(y) \frac{u''(y_c)}{u'(y_c)} - u''(y), \quad a_1(y, c, c) = 0,
\]

which imply that
\[
|\partial_y a_1(y, c)| \leq C |y - y_c|, \quad |a_1(y, c)| \leq C |y - y_c|^2. \quad (4.29)
\]

Let
\[
a_2(y, c) = u'(y)(u''(y_c) - u''(y)) + u'''(y)(u(y) - c).
\]
Then we find that

\[
\left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) a_1(y, c) = \frac{a_2(y, c)}{u'(y_c)},
\]

\[
\partial_y a_2(y, c) = u''(y)(u''(y_c) - u''(y)) + u'''(y)(u(y) - c),
\]

and \(a_2(y, c) = 0\), therefore,

\[
|\partial_y a_2(y, c)| \leq C|y - y_c|, \quad |a_2(y, c)| \leq C|y - y_c|^2.
\] (4.30)

By Lemma 4.12, \(\partial_y \phi_1(y, c)\) has the same sign as \(y - y_c\). For \(y_c \leq y' \leq y\) or \(y \leq y' \leq y_c\), \(u'(y') \leq C(u'(y) + u'(y_c))\). So, we get by (4.26) and (4.19) that

\[
|\partial_y G(y, c)| = |\partial_c F(y, c)|
\]

\[
\leq C\frac{u'(y) + u'(y_c))}{\phi(y, c)^2} \int_{y_c}^{y} \phi_1(y', c) \partial_y \phi_1(y', c) dy'
\]

\[
\leq C \frac{\phi_1(y, c)^2 - 1}{\phi(y, c)^2} (u'(y) + u'(y_c))
\]

\[
\leq C \frac{\phi_1(y, c)^2 - 1}{\phi_1(y, c)} \frac{1}{|y - y_c|^2(u'(y) + u'(y_c))}
\]

\[
\leq C \min\{1, a^2|y - y_c|^2\} \frac{|y - y_c|^2}{|y - y_c|^2(u'(y) + u'(y_c))},
\] (4.31)

and by (4.28) and (4.29),

\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) F(y, c) \right| = |\partial_y G_1(y, c)|
\]

\[
\leq C\frac{|y - y_c|^2}{u'(y_c)\phi(y, c)^2} \int_{y_c}^{y} \phi_1(y', c) \partial_y \phi_1(y', c) dy'
\]

\[
\leq C \frac{\phi_1(y, c)^2 - 1}{u'(y_c)\phi(y, c)^2} |y - y_c|^2
\]

\[
\leq C \min\{1, a^2|y - y_c|^2\} \frac{|y - y_c|^2}{u'(y_c) (u(y) - c)^2}.
\] (4.32)

from which and \(G(y_c, c) = G_1(y_c, c) = 0\), it follows that

\[
|G(y, c)| \leq C \int_{y_c}^{y} \frac{\min\{1, a^2|y' - y_c|^2\}}{|y' - y_c|^2u'(y_c)} dy' \leq \frac{C a \min\{1, a|y - y_c|\}}{u'(y_c)},
\] (4.33)

and

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\[ |G_1(y, c)| \leq C \frac{\min\{1, \alpha^2|y - y_c|^2\}}{u'(y_c)} \left| \int_{y_c}^y \frac{|y' - y_c|^2}{(u(y') - c)^2} dy' \right| \leq C \frac{\min\{1, \alpha^2|y - y_c|^2\}}{u'(y_c)^2}. \]

Thus, we deduce that

\[ |\partial_c \phi_1(y, c)| \leq \frac{C\alpha \phi_1(y, c)}{u'(y_c)}, \quad |\partial_y \partial_c \phi_1(y, c)| \leq \frac{C\alpha^2 \phi_1(y, c)}{u'(y_c)}, \quad (4.34) \]

and

\[ \left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \phi_1(y, c) \right| \leq C \frac{\min\{1, \alpha^2|y - y_c|^2\} \phi_1}{u'(y_c)^2}. \quad (4.35) \]

On the other hand, we have

\[ \partial_c \phi_1 = 2\alpha^2 T_0 \circ T_{2.3}(\phi_1) - 2\alpha^2 T_0 \circ T_{1.2}(\phi_1) + \alpha^2 T(\partial_c \phi_1), \]

from which and (4.34), we infer that

\[ |\partial_c \phi_1(y, c)| \leq C\alpha^2 \left| \int_{y_c}^y \frac{\phi_1(y', c) - \alpha^2 \phi_1(y', c)}{u'(y_c)} dy' \right| + C\alpha^3 \left| \int_{y_c}^y \frac{\phi_1(y', c)}{u'(y_c)} dy' \right| \leq C\phi_1(y, c) \left( \frac{\alpha^2|y - y_c|}{u'(y_c)} + \frac{\alpha^3|y - y_c|^2}{u'(y_c)} \right). \]

Here we again used the fact that \( \phi_1(y', c) \leq \phi_1(y, c) \) for \( y_c \leq y' \leq y \) or \( y \leq y' \leq y_c \).

**Step 3.** Estimates of \( \partial_c^2 \phi_1 \) and \( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right)^2 \phi_1 \).

By Lemma 4.12, we get for \( |y - y_c| \leq \frac{M_0}{\alpha} \)

\[ |\partial_c^2 \mathcal{F}(y, c)| \leq \frac{C|\partial_c^2 \partial_y \phi_1|}{\phi_1} + \frac{C|\partial_c \partial_y \phi_1 \partial_c \phi_1|}{\phi_1^2} + \frac{C|\partial_c^2 \phi_1 \partial_y \phi_1|}{\phi_1^3} + \frac{C|\partial_c \phi_1 |^2 |\partial_y \phi_1|}{\phi_1^3} \leq \frac{C\alpha^3}{u'(y_c)^2} + \frac{C\alpha^2}{y_c(y + y_c)^2}. \]

It follows from (4.26) that

\[ \partial_c^2 \mathcal{F}(y, c) = \partial_y \partial_c \mathcal{G}(y, c) = -2 \int_{y_c}^y \partial_c \left( \frac{\phi_1(z, c)^2}{\phi(y, c)^2} \mathcal{F}(z, c) \right) u'(z) \, dz \]
\[
\begin{align*}
&= -2 \int_{y_c}^{y} 2 \frac{u'(z)\phi_1(z, c)}{\phi(y, c)} \left( \frac{\partial_c \phi_1(z, c)}{\phi(y, c)} - \frac{\phi_1(z, c)\partial_c \phi(y, c)}{\phi(y, c)^2} \right) \frac{\partial_c \phi_1(z, c)}{\phi_1(z, c)} dz \\
&\quad - 2 \int_{y_c}^{y} \frac{\phi_1(z, c)^2}{\phi(y, c)^2} \partial_c F(z, c)u'(z) dz,
\end{align*}
\]

which along with the fact \( \int_{y_c}^{y} 2 \phi_1(z, c)\partial_c \phi_1(z, c) dz = \phi_1(y, c)^2 - 1 \), (4.19), (4.23) and (4.31) implies that for \( |y - y_c| \geq \frac{M_0}{a} \),

\[
|\partial_c^2 F(y, c)| \leq \frac{C\alpha}{|y - y_c|^2 u'(y_c)^2}.
\]

Thanks to \( \partial_c G(y_c, c) = \partial_c^2 \phi_1(y_c, c) = \alpha^2 T \phi_1 = \frac{\alpha^2}{3u'(y_c)^2} \), we get

\[
|\partial_c G(y, c)| \leq \frac{C\alpha^2}{u'(y_c)^2}, \quad \text{thus, } |\partial_c^2 \phi_1| \leq \frac{C\alpha^2}{u'(y_c)^2}.
\]

Using the formula

\[
\left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \left( T_0(a) \right) = \frac{a(y, c)}{u'(y_c)} - \frac{a(y_c, c)}{u'(y_c)} + T_0(\partial_c a)
\]

\[
= T_0 \left( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) a \right),
\]

we obtain

\[
\left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \partial_y G_1(y, c) = \left( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \frac{-2}{\phi(y, c)^2} \right) T_0 \left( \phi_1^2 \frac{a_1}{u'(y_c)} F \right)
\]

\[
+ \frac{-2}{\phi(y, c)^2} T_0 \left( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \phi_1^2 \frac{a_1}{u'(y_c)} F \right).
\]

(4.36)

Using (4.35) and the fact that

\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) (u(y) - c) \right| = \left| \frac{u'(y)}{u'(y_c)} - 1 \right| \leq C \frac{|y - y_c|}{u'(y_c)} \leq C \frac{|u(y) - c|}{u'(y_c)^2},
\]

we infer that

\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \frac{-2}{\phi(y, c)^2} \right| \leq \frac{C}{\phi(y, c)^2 u'(y_c)^2}.
\]

(4.37)

Using the formula
\[
\left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \left( \phi_1^2 \frac{a_1(y, c) F}{u'(y_c)} \right)
= 2 \phi_1 \frac{a_1(y, c)}{u'(y_c)} F \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \phi_1 + \phi_1^2 \frac{a_2(y, c)}{u'(y_c)^2} F
- \phi_1^2 \frac{a_1(y, c)}{u'(y_c)^3} u''(y_c) F + \phi_1^2 \frac{a_1(y, c)}{u'(y_c)} \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) F,
\]
we get by (4.29), (4.30), (4.35) and (4.32) that
\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \left( \phi_1^2 \frac{a_1(y, c)}{u'(y_c)} F \right) \right| \\
\leq C \phi_1^2 \frac{|y - y_c|^2}{u'(y_c)^3} |F| + C \phi_1^2 \frac{|y - y_c|^2}{u'(y_c)} \frac{\phi_1(y, c)^2 - 1}{u'(y_c) \phi_1(y, c)^2} \frac{|y - y_c|^2}{(u(y) - c)^2}
\leq C \phi_1^2 \frac{|y - y_c|^2}{u'(y_c)^3} |F| + C |y - y_c| \frac{\phi_1(y, c)^2 - 1}{u'(y_c)^3}.
\]

which gives
\[
\left| T_0 \left( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \left( \phi_1^2 \frac{a_1(y, c)}{u'(y_c)} F \right) \right) \right| \\
\leq C \frac{|y - y_c|^2}{u'(y_c)^3} |T_0(\phi_1^2 |F|)| + C \frac{|y - y_c| \phi_1(y, c)^2 - 1}{u'(y_c)^3}
\leq C |y - y_c|^2 \frac{\phi_1(y, c)^2 - 1}{u'(y_c)^3}.
\]

Here we used
\[
|T_0(\phi_1^2 |F|)| \leq \int_{y_c}^y \phi_1(y', c) \partial_y \phi_1(y', c) dy' \leq \phi_1(y, c)^2 - 1.
\]

It follows from (4.36), (4.37), (4.28), (4.32) and (4.38) that
\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \partial_y G_1(y, c) \right| \\
\leq C \frac{\partial_y G_1(y, c)}{u'(y_c)^2} + C \frac{|y - y_c|^2 \phi_1(y, c)^2 - 1}{\phi(y, c)^2} \frac{|y - y_c|^2}{u'(y_c)^3} \phi_1(y, c)^2 \frac{u'(y_c)}{(u(y) - c)^2}.
\]

This together with \( \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) G_1(y_c, c) = \partial_c G_1(y_c, c) = 0 \) gives
\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) G_1(y, c) \right| \leq C \frac{\phi_1(y, c)^2 - 1}{u'(y_c)^3 \phi_1(y, c)^2} \\
\int_{y_c}^y \frac{|y' - y_c|^2}{(u(y') - c)^2} dy' \leq C \frac{(\phi_1(y, c)^2 - 1)}{u'(y_c)^4 \phi_1(y, c)^2},
\]

from which and (4.35), we infer that

\[
\left| \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right)^2 G_1(y, c) \right| \leq C \frac{\min\{1, \alpha^2 |y - y_c|^2\} \phi_1}{u'(y_c)^4}.
\]

This completes the proof of the proposition. \(\square\)

It is easy to see from the proof of Proposition 4.14 that

Lemma 4.15  It holds that

\[
C^{-1} \alpha \min\{\alpha y_c, 1\} \leq |F(0, c)| \leq C \alpha \min\{\alpha y_c, 1\},
\]

\[
|\partial_c F(0, c)| \leq \frac{C|F(0, c)|^2}{\alpha^2 y_c^3}, \quad |\partial_c^2 F(0, c)| \leq \frac{C \alpha \min\{\alpha y_c, 1\}}{y_c^4},
\]

\[
|G(0, c)| \leq \frac{C \alpha \min\{1, \alpha y_c\}}{y_c^4}, \quad |\partial_c G(0, c)| \leq \frac{C \alpha^2}{y_c^2}.
\]

5 Spectral Analysis of the Linearized Operator

In this section, we study the spectrum of the linearized operator \(R_\alpha\) defined by

\[
R_\alpha \hat{\psi} = -\left( \partial_y^2 - \alpha^2 \right)^{-1} (u''(y) - u(\partial_y^2 - \alpha^2)) \hat{\psi}.
\]

5.1 Spectrum and Embedding Eigenvalues

We consider the shear flows in \(\mathcal{K}\). Let us first recall some classical facts about the spectrum \(\sigma(R_\alpha)\) of the operator \(R_\alpha\) in \(L^2(-1, 1)\) (see [27,30,31] for more details):

1. The spectrum \(\sigma(R_\alpha)\) is compact;
2. \(\sigma_d(\mathcal{L}) = \bigcup_{\alpha} \sigma_d(i\alpha R_\alpha)\);
3. The continuous spectrum \(\sigma_c(R_\alpha)\) is contained in the range \(\text{Ran } u\) of \(u(y)\);
4. The eigenvalues of \(R_\alpha\) can not cluster except possibly along on \(\text{Ran } u\).

Usually, \(c \in \text{Ran } u\) is called an embedding eigenvalue of \(R_\alpha\) if there exists \(0 \neq \psi \in H_0^1(-1, 1)\) so that

\[
R_\alpha \psi = c \psi \quad \text{or} \quad (u - c)(-\partial_y^2 + \alpha^2) \psi + u'' \psi = 0. \quad (5.1)
\]

However, this definition is too general so that one can construct a nontrivial solution \(\psi \in H_0^1(-1, 1)\) to (5.1) for all \(c \in \text{Ran } u\). Thus, we introduce the following definition.
**Definition 5.1** We say that \( c \in \text{Ran} \ u \) is an embedding eigenvalue of \( \mathcal{R}_\alpha \) if there exists \( 0 \neq \psi \in H^1_0(-1,1) \) such that for all \( \varphi \in H^1_0(-1,1) \),

\[
\int_{-1}^{1} (\psi' \varphi' + \alpha^2 \psi \varphi) \, dy + \text{p.v.} \int_{-1}^{1} \frac{u'' \psi \varphi}{u - c} \, dy + i \pi \times \sum_{y \in u^{-1}[c], u'(y) \neq 0} \frac{(u'' \psi \varphi)(y)}{|u'(y)|} = 0. \tag{5.2}
\]

Let us first explain that the integration \( \text{p.v.} \int_{-1}^{1} \frac{u'' \psi \varphi}{u - c} \, dy \) is well defined. It suffices to consider the point where \( u'(y_0) = 0 \) and \( u(y_0) = c \). Without loss of generality, we assume \( u'(y) > 0 \) in \( (y_0, y_0 + \delta) \). Then in \( (y_0, y_0 + \delta) \), \( \psi(y) \) satisfies (5.1), thus has the form

\[
\psi(y) = C_1 \phi(y) + C_2 \phi(y) \int_{y_0 + \delta}^{y} \frac{1}{\phi(y')^2} \, dy',
\]

for two constants \( C_1, C_2 \), where \( \phi(y) = (u(y) - u(y_0)) \phi_1(y) \) with \( \phi_1(y) = 1 + \alpha^2 T(\phi_1) \) as in Sect. 4.3. Thus,

\[
\phi(y) \int_{y_0 + \delta}^{y} \frac{1}{\phi(y')^2} \, dy' \sim \frac{1}{y - y_0},
\]

which is not in \( L^2_{loc} \), thus \( C_2 = 0 \). Then \( \frac{\psi}{u - c} \) is bounded and the integral is well-defined. It is easy to see from (5.2) that

\[
-\psi'' + \alpha^2 \psi + \frac{u'' \psi}{u - c} = 0 \quad \text{in} \quad [-1,1] \setminus u^{-1}[c]. \tag{5.3}
\]

Moreover, for \( u(y) \) satisfying the condition \( (F) \): \( u''(y_1)u''(y_2) \geq 0 \) if \( u(y_1) = u(y_2) \), we can deduce from (5.2) that \( \psi \in H^2(-1,1) \) is a classical solution to (5.3) see [16]. Obviously, the condition \( (S) \) implies \( (F) \).

**Lemma 5.2** Let \( u(y) \) satisfy the condition \( (F) \). If \( c \) is an embedding eigenvalue of \( \mathcal{R}_\alpha \), then \( u''(y) = 0 \) for some \( y \in u^{-1}[c] \setminus \{\pm 1\} \).

**Proof** If \( u''(y) \neq 0 \) for all \( y \in u^{-1}[c] \setminus \{\pm 1\} \), taking \( \varphi = \overline{\psi} \) in (5.2), we deduce that \( \psi(y) = 0 \) for \( u(y) = c \) and

\[
\int_{-1}^{1} \left( |\psi|^2 + \alpha^2 |\psi|^2 + \frac{u'' |\psi|^2}{u - c} \right) \, dy = 0.
\]

Thus, we can use integration by parts to obtain

\[
\int_{-1}^{1} \left| \psi' - u' \frac{\psi}{u - c} \right|^2 \, dy + \alpha^2 \int_{-1}^{1} |\psi|^2 \, dy = 0,
\]

which implies \( \psi \equiv 0 \). \( \Box \)
5.2 A Criterion on Embedding Eigenvalues

Here we give an equivalent characterization of embedding eigenvalues for symmetric flow $u \in C^4([-1, 1])$ satisfying (S), which is key to solve the inhomogeneous Rayleigh equation.

For $c \in D_0$, let $\phi(y, c)$ be the solution of the homogeneous Rayleigh equation given by Lemma 4.11 and $\phi_1(y, c) = \frac{\phi(y, c)}{u(y) - c}$. We introduce

$$A(c) = A_1(c) + u'(y_c)\rho(c) \Pi_3(c), \quad B(c) = \pi \rho(c) \frac{u''(y_c)}{u'(y_c)^2}, \quad (5.4)$$

$$A_2(c) = (u(0) - c)A(c) + J(c), \quad B_2(c) = (u(0) - c)B(c), \quad (5.5)$$

where

$$\rho(c) = (c - u(0))(u(1) - c), \quad J(c) = \frac{u'(y_c)(u(1) - c)}{\phi_1(0, c)\phi'_1(0, c)}, \quad (5.6)$$

$$A_1(c) = \rho u'(y_c) \partial_c \left( p.v. \int_0^1 \frac{dy}{u(y) - c} \right), \quad (5.7)$$

$$\Pi_3(c) = \int_0^1 \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) dy. \quad (5.8)$$

By Lemma 4.12 and Proposition 4.7, $J(c)$ and $\Pi_3(c)$ are well-defined.

**Proposition 5.3** $c \in D_0$ is an embedding eigenvalue of $\mathcal{R}_\alpha$ if and only if

$$A(c)^2 + B(c)^2 = 0 \quad \text{or} \quad A_2(c)^2 + B_2(c)^2 = 0.$$ 

**Proof** Let us first assume that $c \in D_0$ is an embedding eigenvalue of $\mathcal{R}_\alpha$. Thus, $u''(y_c) = 0$ (thus, $B = B_2 = 0$) and there exists a nontrivial solution $\varphi$ of

$$\begin{cases} 
\varphi'' - \alpha^2 \varphi - \frac{u''}{u-c} \varphi = 0, \\
\varphi(-1) = \varphi(1) = 0.
\end{cases} \quad (5.9)$$

Let

$$\phi_o(y, c) = \frac{\varphi(y, c) - \varphi(-y, c)}{2}, \quad \phi_e(y, c) = \frac{\varphi(y, c) + \varphi(-y, c)}{2}.$$ 

Then $\phi_o$ and $\phi_e$ also satisfies the homogeneous Rayleigh equation in $[0, 1]$ with

$$\phi_o(0, c) = \phi_o(1, c) = 0, \quad \partial_y \phi_e(0, c) = \phi_e(1, c) = 0.$$ 

If $\phi_o$ is a nontrivial solution, then $\partial_y \phi_o(0, c) \neq 0$. Otherwise, it is trivial by the fact that $u'(y) > 0$ for $y \in (0, 1)$ and $u''(y_c) = 0$. Without loss of generality, we assume $\partial_y \phi_o(0, c) = 1$. Thus, $\phi_o(y, c)$ has the following representation formula

\[ 
\]
\[ \phi_o(y, c) = \phi_1(0, c)\varphi(y, c), \]

where \( \varphi(y, c) \) is given by
\[
\varphi(y, c) = \frac{\phi_1(y, c)}{u'(y_c)}(u(y) - u(0))
+ \frac{\phi_1(y, c)}{u'(y_c)}(u(y) - c)(u(0) - c)
\int_0^y \frac{u'(y_c) - u'(z)}{(u(z) - c)^2} dz
+ \phi_1(y, c)(u(y) - c)(u(0) - c)
\int_0^y \frac{1}{(u(z) - c)^2}\left(\frac{1}{\phi_1(z, c)^2} - 1\right) dz.
\]

In fact, the solution \( \phi_o(y, c) \) can be formally written as
\[
\phi_o(y, c) = \phi(0, c)\phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} dy'.
\]

Then we find that \( \phi_o(1, c) = 0 \) is equivalent to
\[
0 = \phi_o(1, c) = \phi_1(0, c)\varphi(1, c)
= \frac{\phi_1(0, c)}{u'(y_c)}(u(1) - u(0))
- \frac{\phi_1(0, c)\phi_1(1, c)}{u'(y_c)}\rho(c)
\int_0^1 \frac{u'(y_c) - u'(y)}{(u(y) - c)^2} dy
- \frac{\phi_1(0, c)\phi_1(1, c)\rho(c)}{u'(y_c)}
\int_0^1 \frac{1}{(u(y) - c)^2}\left(\frac{1}{\phi_1(y, c)^2} - 1\right) dy
= -\frac{\phi_1(0, c)\phi_1(1, c)}{u'(y_c)}A(c).
\]

Thus, \( A(c) = 0 \). Here we used
\[
A_1(c) = u(0) - u(1) - \rho(c)\Pi_2(c),
\]
\[
\Pi_2(c) = \text{p.v.} \int_0^1 \frac{u'(y) - u'(y_c)}{(u(y) - c)^2} dy.
\]  \( 5.10 \)

If \( \phi_e \) is a nontrivial solution, we may assume that \( \partial_y\phi_e(1, c) = 1 \). Thus, \( \phi_e(y, c) \) has the following representation formula
\[
\phi_e(y, c) = \phi(1, c)\tilde{\varphi}(y, c),
\]

where \( \tilde{\varphi}(y, c) \) is given by
\[
\tilde{\varphi}(y, c) = \frac{\phi_1(y, c)}{u'(y_c)}(u(y) - u(1))
+ \frac{\phi_1(y, c)}{u'(y_c)}(u(y) - c)(u(1) - c)
\int_1^y \frac{u'(y_c) - u'(z)}{(u(z) - c)^2} dz
+ \phi_1(y, c)(u(y) - c)(u(1) - c)
\int_1^y \frac{1}{(u(z) - c)^2}\left(\frac{1}{\phi_1(z, c)^2} - 1\right) dz.
\]
Then we find that
\begin{equation}
0 = \partial_y \tilde{\psi}(0, c) = \frac{\partial_y \phi_1(0, c)}{u'(y_c)} (u(0) - u(1)) \\
+ \frac{\partial_y \phi_1(0, c)}{u'(y_c)} (u(0) - c)(u(1) - c) \int_1^0 \frac{u'(y_c) - u'(z)}{(u(z) - c)^2} \, dz \\
+ (u(1) - c) \frac{1}{u(0) - c} \phi_1(0, c) \\
+ \partial_y \phi_1(0, c)(u(0) - c)(u(1) - c) \int_1^0 \frac{1}{(u(z) - c)^2} \left( \frac{1}{\phi(z, c)^2} - 1 \right) \, dz \\
= \frac{\partial_y \phi_1(0, c)}{(u(0) - c)u'(y_c)} A_2(c).
\end{equation}

Thus, $A_2(c) = 0$.

It remains to show that if $A(c)^2 + B(c)^2 = 0$ or $A_2(c)^2 + B_2(c)^2 = 0$, then $c \in D_0$ must be an embedding eigenvalue of $\mathcal{R}_\sigma$. In this case, we have $u''(y_c) = 0$. Thus, we can construct a nontrivial solution of the homogeneous Rayleigh equation with $\varphi(0, c) = 0$, $\partial_y \varphi(0, c) = 1$ (or $\varphi(1, c) = 0$, $\partial_y \varphi(1, c) = 1$). The above argument shows that $\varphi(1, c) = 0$ (or $\varphi(0, c) = 0$). This means that $c$ is an embedding eigenvalue of $\mathcal{R}_\sigma$ with the eigenfunction $\varphi(y, c)$. 

\section{The Limiting Absorption Principle}

In this section, we establish the limiting absorption principle for the inhomogeneous Rayleigh equation when $u \in K$:

\begin{equation}
(u - c)(\Phi'' - \alpha^2 \Phi) - u'' \Phi = \omega, \quad \Phi(-1) = \Phi(1) = 0,
\end{equation}

where $c \in \Omega_{\varepsilon_0} \setminus \text{Ran}(u)$ and $\Omega_{\varepsilon_0} = \{ c \in \mathbb{C} \mid \exists c_0 \in \text{Ran}(u) \text{ so that } |c - c_0| < \varepsilon_0 \}$.

The key ingredient is the following uniform estimate of the solution.

\textbf{Proposition 6.1} If $\mathcal{R}_\sigma$ has no embedding eigenvalues, then there exists $\varepsilon_0$ such that for $c \in \Omega_{\varepsilon_0} \setminus D_0$, the solution to (6.1) has the following uniform bound

$$\| \Phi \|_{H^1(-1, 1)} \leq C \| \omega \|_{H^1(-1, 1)}.$$ 

Here $C$ is a constant independent of $\varepsilon_0$.

The proof is based on blow-up analysis and compactness argument.

\textbf{Lemma 6.2} Let $\psi_n$, $\omega_n \in H^1(a, b)$, $u_n \in H^3(a, b)$ be a sequence, which satisfies

$$\psi_n \to \psi, \omega_n \to \omega \text{ in } H^1(a, b),$$

$$u_n \to u_0 \text{ in } H^3(a, b),$$

$$u_n(\psi''_n - \alpha^2_n \psi_n) - u''_n \psi_n = \omega_n.$$
and $\alpha_n \to \alpha$, $\Im u_n < 0$. Moreover, $\Im u_0 = 0$ in $[a, b]$, $u(y_0) = 0$, $y_0 \in [a, b]$, $u'(y)u'(y_0) > 0$ in $[a, b]$. Then we have $\psi_n \to \psi$ in $H^1(a, b)$, and for all $\varphi \in H_0^1(a, b)$,

$$\int_a^b (\psi' \varphi' + \alpha^2 \psi \varphi) dy + p.v. \int_a^b \frac{(u_0' \psi + \omega) \varphi}{u_0} dy + i\pi \frac{(u_0'' \psi + \omega) \varphi(y_0)}{|u'(y_0)|} = 0. \tag{6.2}$$

**Proof** Without loss of generality, we may assume that $u_0'(y) > 0$ for $y \in [a, b]$. Otherwise, we can consider $\overline{\psi_n} - \overline{\varphi_n} - \overline{u_n}$.

As $\psi_n \to \psi$, $\omega_n \to \omega$ in $H^1(a, b)$ and $u_n \to u_0$ in $H^3(a, b)$, $\psi_n, \omega_n, u_n''$ are uniformly bounded in $H^1(a, b)$. Let $g_n = u''n \psi + \omega_n$. Then $g_n$ is uniformly bounded in $H^1(a, b)$ and $\psi_n'' - \alpha_n^2 \psi_n = g_n / u_n$. Choose $N$ large enough so that $\Re u_n'(y) \geq c_0 > 0$ for $n \geq N$ and $y \in [a, b]$.

It is easy to see that

$$\left( \psi'_n - \frac{g_n}{u_n'} \ln u_n \right)' = \alpha_n^2 \psi_n - \left( \frac{g_n}{u_n'} \right)' \ln u_n.$$

Here $\ln(r e^{i\theta}) = \ln r + i\theta$ for $r > 0$, $\theta \in [-\pi, \pi]$. We know that $\left( \frac{g_n}{u_n'} \right)'$ is uniformly bounded in $L^2(a, b)$, and $\frac{g_n}{u_n'}$ is uniformly bounded in $L^\infty(a, b)$. Due to $\Re u_n'(y) \geq c_0 > 0$ for $y \in [a, b]$, $\ln u_n$ is uniformly bounded in $L^p(a, b)$, $p > 2$. Therefore, $\psi_n' - \frac{g_n}{u_n'} \ln u_n$ is uniformly bounded in $L^2(a, b)$ and $W^{1,1}(a, b)$, thus bounded in $L^\infty(a, b)$. Thus, we conclude

$$\lim_{b \to 0} \sup_{n} \| \psi_n' \|_{L^2((y_0 - \delta, y_0 + \delta) \cap (a, b))} = 0. \tag{6.3}$$

On the other hand, $\psi_n''$ is bounded in $L^\infty((a, b) \setminus (y_0 - \delta, y_0 + \delta))$. So, $\psi_n \to \psi$ in $H^1((a, b) \setminus (y_0 - \delta, y_0 + \delta))$, which together with (6.3) shows that $\psi_n \to \psi$ in $H^1(a, b)$ and $g_n \to g_0 = u_0'' \psi + \omega$ in $H^1(a, b)$.

Now we prove (6.2). As $\psi_n'' - \alpha_n^2 \psi_n = g_n / u_n$, we have

$$\int_a^b (\psi_n' \varphi' + \alpha_n^2 \psi_n \varphi) dy + \int_a^b \frac{g_n \varphi}{u_n} dy = 0$$

for any $\varphi \in H_0^1(a, b)$. Obviously,

$$\lim_{n \to \infty} \int_a^b (\psi_n' \varphi' + \alpha_n^2 \psi_n \varphi) dy = \int_a^b (\psi' \varphi' + \alpha^2 \psi \varphi) dy.$$

It remains to show that

$$\lim_{n \to \infty} \int_a^b \frac{g_n \varphi}{u_n} dy = p.v. \int_a^b \frac{g_0 \varphi}{u_0} dy + i\pi \frac{(g_0 \varphi)(y_0)}{|u_0'(y_0)|}.$$
Using the facts that
\[
\int_a^b \frac{g_n \varphi}{u_n} \, dy = - \int_a^b \left( \frac{g_n \varphi}{u_n} \right)' \ln u_n \, dy, \quad \left( \frac{g_n \varphi}{u_n} \right)' \rightarrow \left( \frac{g \varphi}{u_0} \right)',
\]
we infer that
\[
\int_a^b \frac{g_n \varphi}{u_n} \, dy \rightarrow - \int_a^b \left( \frac{g \varphi}{u_0} \right)' \ln |u_0| - i \pi \chi_{(a, b)} \quad \text{in} \quad L^2(a, b),
\]
then for
\[
\frac{g \varphi}{u_0} \rightarrow \frac{g \varphi}{u_0} \quad \text{in} \quad L^2(a, b),
\]
we infer that
\[
\int_a^b \frac{g_n \varphi}{u_n} \, dy \rightarrow - \int_a^b \left( \frac{g \varphi}{u_0} \right)' \ln |u_0| + i \pi \int_a^{y_0} \left( \frac{g \varphi}{u_0} \right)' \, dy
\]
\[
= \text{p.v.} \int_a^b \frac{g \varphi}{u_0} \, dy + i \pi \frac{(g \varphi)(y_0)}{|u_0(y_0)|}.
\]
This proves (6.2). \qed

Lemma 6.3 Assume that \( \psi, \omega \in H^1(a, b), u \in H^3(a, b) \) satisfy
\[
(u - c)(\psi'' - \alpha^2 \psi) - u''\psi = \omega,
\]
where \( c \notin \mathbb{R} \), and \( u \) is real-valued and \( u'(y_0) = 0 \), \( y_0 = \frac{a+b}{2} \in (a, b) \), \( u''(y)u''(y_0) > 0 \) in \([a, b] \). Then we have
\[
|\psi'' - \alpha^2 \psi y_0))| \leq C \min(|u(y_0) - c|^{-2}, |u(y_0) - c|^{-1})(\|\psi\|_{H^1(a, b)} + \|\omega\|_{H^1(a, b)}),
\]
where the constant \( C \) depends only on \( u, \alpha \) and \( \delta = y_0 - a \).

Proof We may assume that \( |u''| > c_0 > 0 \) in \([a, b] \). Then we have
\[
|u'(y)| \leq C|y - y_0|, \quad |u(y) - u(y_0)| \leq C|y - y_0|^2,
\]
\[
|u'(y) - u'(y_1)| \geq c_0|y - y_1| \quad \text{for} \quad y, y_1 \in [a, b].
\]
It is easy to see that
\[
|\psi'' - \alpha^2 \psi(y_0)| = \left| \frac{u''\psi - \omega}{u - c} \right| (y_0) \leq C \frac{\|\psi\|_{H^1(a, b)} + \|\omega\|_{H^1(a, b)}}{|u(y_0) - c|}.
\]
Thus, it suffices to consider the case \( |u(y_0) - c| < \delta^2 \). Let \( \delta_1 = |u(y_0) - c|^{\frac{1}{2}} \). So, \( \delta_1 < \delta \).

First of all, we consider the case of \( \omega(y_0) = 0 \). We normalize \( \|\psi\|_{H^1(a, b)} + \|\omega\|_{H^1(a, b)} = 1 \). Then for \( |y - y_0| < \delta_1 \),
\[
|\omega(y)| \leq C\delta_1^{\frac{1}{2}}, \quad |u(y) - c| \leq |u(y) - u(y_0)| + |u(y_0) - c| \leq C\delta_1^{\frac{1}{2}}.
\]
Let \( g = ((u - c)\psi' - u'\psi)' = \alpha^2 \psi (u - c) + \omega. \) Then for \(|y - y_0| < \delta_1\),

\[
|g(y)| \leq C|u(y) - c| + |\omega(y)| \leq C \delta_1^{\frac{1}{2}},
\]

which gives

\[
\left| \left. ((u - c)\psi' - u'\psi) \right|_{y_1}^{y_2} \right| = \left| \int_{y_1}^{y_2} g(y)dy \right| \leq C \delta_1^{\frac{3}{2}}.
\]

Choose \( \delta_2 \in (\delta_1/2, \delta_1) \), \( y_1 = y_0 - \delta_2 \), \( y_2 = y_0 + \delta_2 \) so that

\[
|\psi'(y_1)|^2 + |\psi'(y_2)|^2 \leq 2\delta_1^{-1} \|\psi'\|^2_{L^2(a,b)} \leq 2\delta_1^{-1},
\]

which gives

\[
\left| \left. (u - c)\psi' \right|_{y_1}^{y_2} \right| \leq ((|\psi'(y_1)| + |\psi'(y_2)|) \|u - c\|_{L^\infty([y_1,y_2])} \leq C \delta_1^{-\frac{1}{2}} \delta_1^2 = C \delta_1^{\frac{3}{2}}.
\]

This shows that \( u'\psi \left|_{y_1}^{y_2} \right. \leq C \delta_1^{\frac{3}{2}}. \) Notice that

\[
u'\psi \left|_{y_1}^{y_2} = \psi(y_0)u' \left|_{y_1}^{y_2} + u'(y_1)\psi \left|_{y_1}^{y_0} + u'(y_2)\psi \left|_{y_1}^{y_2}.\right.\right.\]

Thus, we have

\[
c_0\delta_1 |\psi(y_0)| \leq \left| \left. \psi u' \right|_{y_1}^{y_2} \right| + C \|u'\|_{L^\infty([y_1,y_2])} \delta_2^\frac{1}{2} \|\psi'\|_{L^2(a,b)} \leq C \delta_1^{\frac{3}{2}} + C \delta_1 \delta_1^{\frac{1}{2}} = C \delta_1^\frac{3}{2},
\]

which gives

\[
|\psi(y_0)| \leq C \delta_1^{\frac{1}{2}} \leq C|u(y_0) - c|^{\frac{1}{2}},
\]

and thus,

\[
|(\psi'' - \alpha^2 \psi)(y_0)| = \left| \frac{u''\psi(y_0)}{u(y_0) - c} \right| \leq C \delta_1^{-2} |\psi(y_0)| \leq C \delta_1^{-\frac{3}{2}}.
\]

For the general case, we introduce

\[
\Psi_s(y) = \psi(y) + \frac{\omega(y_0)}{u''(y_0)} \cosh \alpha (y - y_0),
\]

\[
\omega_s(y) = \omega(y) - u''(y) \frac{\omega(y_0)}{u''(y_0)} \cosh \alpha (y - y_0).
\]

We find that \( \Psi_s, \omega_s \in H^1(a, b), \omega_s(y_0) = 0, \) and

\[
(u - c)(\psi''_s - \alpha^2 \psi_s) - u'' \psi_s = \omega_s.
\]
Then the above argument shows that
\[
|\psi'' - \alpha^2 \psi(y_0)| = |\psi''_* - \alpha^2 \psi_*(y_0)|
\leq C \delta_1 \frac{1}{2} \left( \|\psi_*\|_{H^1(a,b)} + \|\omega_*\|_{H^1(a,b)} \right)
\leq C \delta_1 \frac{1}{2} \left( \|\psi\|_{H^1(a,b)} + \|\omega\|_{H^1(a,b)} + |\omega(y_0)| \right)
\leq C \delta_1 \frac{1}{2} \left( \|\psi\|_{H^1(a,b)} + \|\omega\|_{H^1(a,b)} \right).
\]

This gives our result by recalling \( \delta_1 = |u(y_0) - c|^{\frac{1}{2}} \). \( \square \)

**Lemma 6.4** Let \( \psi_n, \omega_n \in H^1(a, b), u \in H^3(a, b) \) be a sequence, which satisfies
\[
\psi_n \to 0, \quad \omega_n \to 0 \quad \text{in} \quad H^1(a, b),
\]
\[
(u - c_n)(\psi_n'' - \alpha^2 \psi_n) - u'' \psi_n = \omega_n,
\]
and \( \text{Im} \ c_n > 0, \ c_n \to u(y_0), \ u'(y_0) = 0, \ y_0 = \frac{a+b}{2} \in (a, b), \ \delta = y_0 - a \in (0, 1), \ u''(y)u''(y_0) > 0 \) in \( [a, b] \). Then we have \( \psi_n \to 0 \) in \( H^1(a, b) \).

**Proof** Without loss of generality, we may assume that \( y_0 = 0, \ u''(y_0) = 2, \ u(y_0) = 0 \).
Then \( [a, b] = [-\delta, \delta] \).
Let \( c_n = r_n e^{2 \delta \rho_n}, \ r_n > 0, \ 0 < \theta_n < \pi/2 \). So, \( r_n \to 0 \).
First of all, we consider the case of \( \omega_n(0) = 0 \). By Lemma 6.3 and (6.4), we have
\[
|\psi''_n - \alpha^2 \psi_n(0)| \leq C r_n^{\frac{1}{2}}, \quad |\psi_n(0)| \leq C r_n^{\frac{1}{2}}, \quad |u'' \psi_n + \omega_n(0)| \leq C r_n^{\frac{1}{2}}.
\]
We introduce
\[
\tilde{\psi}_n(y) = r_n^{-\frac{1}{2}} \psi_n(r_n y), \quad \tilde{\omega}_n(y) = r_n^{-\frac{1}{2}} \omega_n(r_n y), \quad u_n(y) = r_n^{-2}(u(r_n y) - u(0)).
\]
It holds that
\[
\left( u_n - e^{2i\theta_n} \right) \left( \tilde{\psi}_n'' - (\alpha r_n)^2 \tilde{\psi}_n \right) - u_n'' \tilde{\psi}_n = \tilde{\omega}_n,
\]
and
\[
|\tilde{\psi}_n(0)| = |r_n^{-\frac{1}{2}} \psi_n(0)| \leq C, \quad \|\tilde{\psi}_n'\|_{L^2(a/r_n, b/r_n)} = \|\psi_n'\|_{L^2(a, b)} \leq C,
\]
\[
|\tilde{\omega}_n(0)| = 0, \quad \|\tilde{\omega}_n'\|_{L^2(a/r_n, b/r_n)} = \|\omega_n'\|_{L^2(a, b)} \to 0.
\]
Thus, \( \tilde{\psi}_n \) is bounded in \( H^1_{loc}(\mathbb{R}) \), and \( \tilde{\omega}_n \to 0 \) in \( H^1_{loc}(\mathbb{R}) \). Up to a subsequence, we may assume that \( \tilde{\psi}_n \to \tilde{\psi}_0 \) in \( H^1_{loc}(\mathbb{R}) \), \( \theta_n \to \theta_0 \) with \( \psi_0' \in L^2(\mathbb{R}), \ \theta_0 \in [0, \pi/2] \).
Using the facts that
\[
u_n(y) = y^2 \int_0^1 \int_0^1 tu''(r_n y) ds dt, \quad u_n'(y) = y \int_0^1 u''(r_n y) dt,
\]
and \(u''(y) = u''(r_n y), u'''(y) = r_n u'''(r_n y)\), we can deduce that \(u_n \to y^2\) in \(H^3_{loc}(\mathbb{R})\).

Now, if \(\theta_0 \neq 0\), then \(\tilde{\psi}_n''\) is bounded in \(L^\infty_{loc}(\mathbb{R})\) and \(\tilde{\psi}_n \to \tilde{\psi}_0\) in \(C^1_{loc}(\mathbb{R})\), and

\[
\left( y^2 - e^{2i\theta_0} \right) \tilde{\psi}_0'' = 2\tilde{\psi}_0.
\]

If \(\theta_0 = 0\), then \(\tilde{\psi}_n''\) is bounded in \(L^\infty_{loc}(\mathbb{R}\setminus\{\pm 1\})\) and \(\tilde{\psi}_n \to \tilde{\psi}_0\) in \(C^1_{loc}(\mathbb{R}\setminus\{\pm 1\})\), and

\[
\left( y^2 - 1 \right) \tilde{\psi}_0'' = 2\tilde{\psi}_0 \quad \text{in} \quad \mathbb{R}\setminus\{\pm 1\}.
\]

Lemma 6.2 also ensures that \(\tilde{\psi}_n \to \tilde{\psi}_0\) in \(H^1(1 - \delta, 1 + \delta) \cap H^1(-1 - \delta, -1 + \delta)\), thus,

\[
\tilde{\psi}_n \to \tilde{\psi}_0 \quad \text{in} \quad H^1_{loc}(\mathbb{R}) \cap C^1_{loc}(\mathbb{R}\setminus\{\pm 1\})
\]

Thanks to \((y^2 - e^{2i\theta_0})\tilde{\psi}_0'' = 2\tilde{\psi}_0\), we deduce that for \(y > 1\),

\[
\tilde{\psi}_0(y) = (y^2 - e^{2i\theta_0}) \left( C_1 + C_2 \int_{y}^{+\infty} \frac{dz}{(z^2 - e^{2i\theta_0})^2} \right)
\]

where \(C_1, C_2\) are constants. As \(\tilde{\psi}_0' \in L^2(\mathbb{R})\), we have \(C_1 = 0\), thus \(\tilde{\psi}_0' \in L^2(2, +\infty)\).

Similarly, \(\tilde{\psi}_0 \in L^2(-\infty, -2)\). Hence, \(\tilde{\psi}_0' \in L^2(\mathbb{R})\), \(\tilde{\psi}_0 \in H^1(\mathbb{R})\).

If \(\theta_0 \neq 0\), then

\[
\int_{\mathbb{R}} |\tilde{\psi}_0'|^2 dy = -\int_{\mathbb{R}} \tilde{\psi}_0'' \tilde{\psi}_0' dy = -\int_{\mathbb{R}} \frac{2|\tilde{\psi}_0|^2}{y^2 - e^{2i\theta_0}} dy.
\]

Multiplying \(e^{i\theta_0}\) on both sides, and then taking the imaginary part, we obtain

\[
\sin \theta_0 \int_{\mathbb{R}} |\tilde{\psi}_0'|^2 dy = -\sin \theta_0 \int_{\mathbb{R}} \frac{2(y^2 + 1)|\tilde{\psi}_0|^2}{|y^2 - e^{2i\theta_0}|^2} dy,
\]

which implies \(\tilde{\psi}_0' = 0\).

If \(\theta_0 = 0\), we first claim that for any \(\varphi \in H^1(\mathbb{R})\),

\[
\int_{\mathbb{R}} \tilde{\psi}_0' \varphi' dy + \text{p.v.} \int_{\mathbb{R}} \frac{2\tilde{\psi}_0 \varphi}{y^2 - 1} dy + i\pi \sum_{y = \pm 1} \tilde{\psi}_0 \varphi = 0. \tag{6.5}
\]

Indeed, it holds that for \(y \in \mathbb{R}\setminus\{\pm 1\}\), \((y^2 - 1)\tilde{\psi}_0'' = 2\tilde{\psi}_0\), thus \((6.5)\) holds for \(\varphi \in H^1(\mathbb{R})\), \(\text{supp} \varphi \subset \mathbb{R}\setminus\{\pm 1\}\). Lemma 6.2 ensures that \((6.5)\) holds for \(\varphi \in H^1(\mathbb{R})\), \(\text{supp} \varphi \subset [1 - \delta, 1 + \delta]\) or \([-1 - \delta, -1 + \delta]\). Therefore, \((6.5)\) holds for any \(\varphi \in H^1(\mathbb{R})\).

Taking \(\varphi = \overline{\tilde{\psi}_0}\) in \((6.5)\) and taking the imaginary part, we deduce that \(\tilde{\psi}_0(\pm 1) = 0\), which implies \(\tilde{\psi}_0 \in H^2(\mathbb{R})\). As \(((y^2 - 1)\tilde{\psi}_0' - 2y\tilde{\psi}_0)' = 0\) and \(((y^2 - 1)\tilde{\psi}_0' - 2y\tilde{\psi}_0)|_{\pm 1} = 0\), we infer that

\[
\left( y^2 - 1 \right) \tilde{\psi}_0' - 2y\tilde{\psi}_0 = 0, \quad ((y^2 - 1)^{-1}\tilde{\psi}_0)' = 0 \quad \text{in} \quad \mathbb{R}\setminus\{\pm 1\},
\]
which mean that there exist constants $C_3, C_4, C_5$ so that $\tilde{\psi}_0 = C_3(y^2 - 1)$ in $(-\infty, -1)$, $\tilde{\psi}_0 = C_4(y^2 - 1)$ in $(-1, 1)$, $\tilde{\psi}_0 = C_5(y^2 - 1)$ in $(1, +\infty)$. However, $\tilde{\psi}_0 \in H^2(\mathbb{R})$, thus $C_3 = C_4 = C_5 = 0$. Then $\tilde{\psi}_0 = 0$.

Up to now, we proved that $\tilde{\psi}_n \to 0$ in $H^1_{loc}(\mathbb{R}) \cap C^1_{loc}(\mathbb{R} \setminus \{1\})$, thus $\|\psi_n\|_{L^2(2r_n, b)} = \|\tilde{\psi}_n\|_{L^2(-2, 2)} = 0$. Since $\psi_n''$ is bounded in $L^\infty_{loc}((a, b) \setminus \{0\})$ and $\psi_n \to 0$ in $H^1(a, b)$, we have $\psi_n \to 0$ in $C^1_{loc}((a, b) \setminus \{0\})$. Integration by parts gives

$$\int_{2r_n}^b (|\psi_n'|^2 + \alpha^2 |\psi_n|^2 + \frac{u''|\psi_n|^2}{u - c_n}) dy = -\int_{2r_n}^b \frac{\omega_n}{u - c_n} dy + \psi_n^2 \psi_n|_{2r_n}. \quad (6.6)$$

Notice that for $n$ sufficiently large and $y \in [2r_n, b]$, we have $u(y) \geq u(2r_n) > 2r_n^2 = 2|c_n|$. Then we have

$$\Re \frac{1}{u(y) - c_n} \geq \frac{1}{2(u(y) + |c_n|)} \geq \frac{1}{4u(y)} \geq \frac{1}{Cy^2},$$

$$\alpha^2 \frac{1}{u(y) - c_n} \leq \frac{1}{u(y) - |c_n|} \leq \frac{2}{u(y)} \leq \frac{C}{y^2}.$$ 

Taking the real part of $(6.6)$, we get

$$\int_{2r_n}^b \left(|\psi_n'|^2 + \alpha^2 |\psi_n|^2 + \frac{u''|\psi_n|^2}{u - c_n}\right) dy + C^{-1} \frac{\psi_n^2}{y^2} \leq C \psi_n \psi_n|_{2r_n} \leq \frac{\omega_n^2}{y^2} + \left|\psi_n^2 \psi_n|_{2r_n}\right|,$$

which implies that

$$\|\psi_n''\|_{L^2(2r_n, b)} \leq C \|\omega_n y\|_{L^2(2r_n, b)} \psi_n^2 \psi_n|_{2r_n}.$$ 

Notice that $\psi_n^2 \psi_n|_{2r_n} = \psi_n^2 \psi_n(b) - \tilde{\psi}_n^2 \psi_n(2) \to 0$, and by Hardy’s inequality, $\|\omega_n y\|_{L^2(2r_n, b)} \leq C \|\omega_n y\|_{L^1(a, b)} \to 0$. Thus, $\|\psi_n''\|_{L^2(2r_n, b)} \to 0$. Similarly, we have $\|\psi_n''\|_{L^2(-2, 2r_n)} \to 0$. This shows that $\|\psi_n''\|_{L^2(a, b)} \to 0$. As $\psi_n \to 0$ in $H^1(a, b)$, we have $\|\psi_n\|_{H^1(a, b)} \to 0$ and then $\|\psi_n\|_{H^1(a, b)} \to 0$.

For general case, we consider

$$\psi_{n*}(y) = \psi_n(y) + \frac{\omega_n(y_0)}{u''(y_0)} \cosh \alpha(y - y_0),$$

$$\omega_{n*}(y) = \omega_n(y) - \frac{\omega_n(y_0)}{u''(y_0)} \cosh \alpha(y - y_0).$$

Then $\psi_{n*}, \omega_{n*} \in H^1(a, b), \omega_{n*}(y_0) = 0$ and

$$(u - c_n)(\psi_{n*}'' - \alpha^2 \psi_{n*}) - u'' \psi_{n*} = \omega_{n*}.$$
Notice that
\[
\|\omega_n\|_{H^1(a,b)} + C|\omega_n(y_0)| \leq C\|\omega_n\|_{H^1(a,b)} \to 0,
\]
\[
\|\psi_n - \psi_n^{\ast}\|_{H^1(a,b)} \leq C|\omega_n(y_0)| \leq C\|\omega_n\|_{H^1(a,b)} \to 0,
\]
thus, \(\psi_n^{\ast} \to 0\) in \(H^1(a,b)\). This is reduced to the case of \(\omega(y_0) = 0\). Thus, \(\|\psi_n^\ast\|_{H^1(a,b)} \to 0\), hence \(\|\psi_n\|_{H^1(a,b)} \to 0\).

Now we are in a position to prove Proposition 6.1.

**Proof** We only consider \(\text{Im} c > 0\), and the case of \(\text{Im} c < 0\) can be proved by taking conjugation. We use the contradiction argument.

Assume that there exists \(\psi_n \in H_0^1(-1,1), \omega_n \in H^1(-1,1)\), and \(c_n\) with \(\text{Im} c_n > 0\), such that \(\|\psi_n\|_{H^1(-1,1)} = 1\), \(\|\omega_n\|_{H^1(-1,1)} \to 0\), \(c_n \to c \in \text{Ran} u\), and
\[
(u - c_n)\left(\psi''_n - \alpha^2 \psi_n\right) - u'' \psi_n = \omega_n.
\]
Then there exists a subsequence of \(\{\psi_n\}\) (still denoted by \(\{\psi_n\}\)) and \(\psi \in H_0^1(-1,1)\) such that \(\psi_n \to \psi\) weakly in \(H^1(-1,1)\). Since \(\psi''_n\) is bounded in \(L^\infty_{loc}((-1,1)\setminus u^{-1}\{c\})\), we have \(\psi_n \to \psi\) in \(H^1_{loc}((-1,1)\setminus u^{-1}\{c\})\) and \(\psi\) satisfies (5.3) in \([-1,1]\setminus u^{-1}\{c\}\) and (5.2) holds for \(\varphi \in H^1_0(-1,1)\), \(\sup \varphi \subset [-1,1]u^{-1}\{c\}\).

For \(y_0 \in u^{-1}\{c\}\), if \(u'(y_0) = 0\), then \(y_0 \not= \pm 1\), \(u''(y_0) \not= 0\), and there exists \(0 < \delta < 1 - |y_0|\) so that \(u''(y)u''(y_0) > 0\) for \(|y - y_0| \leq \delta\). By the argument in Sect. 5.1, we know that \(\psi/(u - c)\) is bounded in \([y_0 - \delta, y_0 + \delta]\), and then (5.2) holds for \(\varphi \in H^1_0(-1,1)\), \(\sup \varphi \subset (y_0 - \delta, y_0 + \delta)\). If \(u'(y_0) \not= 0\), there exists \(0 < \delta < 1\) so that \(u'(y)u'(y_0) > 0\) for \(|y - y_0(1 - \delta)| \leq \delta\). Then Lemma 6.2 ensures that (5.2) holds for \(\varphi \in H^1_0(-1,1)\), \(\sup \varphi \subset [y_0(1 - \delta) - \delta, y_0(1 - \delta) + \delta]\). Thus, (5.2) holds for all \(\varphi \in H^1_0(-1,1)\). Since \(R_\alpha\) has no embedding eigenvalues, \(\psi = 0\). Thus, \(\psi_n \to 0\) in \(H^1_{loc}((-1,1)\setminus u^{-1}\{c\})\). Furthermore, if \(u'(y_0) = 0\), Lemma 6.4 gives \(\psi_n \to 0\) in \(H^1(y_0 - \delta, y_0 + \delta)\); if \(u'(y_0) \not= 0\), Lemma 6.2 gives \(\psi_n \to 0\) in \(H^1(y_0(1 - \delta) - \delta, y_0(1 - \delta) + \delta)\). Thus, \(\psi_n \to 0\) in \(H^1(-1,1)\), which leads to a contradiction.

Now we establish the limiting absorption principle.

**Proposition 6.5** Assume that \(R_\alpha\) has no embedding eigenvalues. Then there exists \(\Phi_{\pm}(\alpha, \cdot, c) \in H^1_0((-1,1))\) for \(c \in \text{Ran} u\), such that \(\Phi(\alpha, \cdot, c \pm i\epsilon) \to \Phi_{\pm}(\alpha, \cdot, c)\) in \(C([-1,1])\) as \(\epsilon \to 0^+\) and
\[
\|\Phi_{\pm}(\alpha, \cdot, c)\|_{H^1(-1,1)} \leq C\|\omega\|_{H^1(-1,1)}.
\]

**Proof** We view \(\Phi\) as the map
\[
\Phi(\alpha, \cdot, \cdot) : c \in \Omega_{\epsilon_0} \setminus \mathbb{R} \to \Phi(\alpha, \cdot, c) \in C([-1,1]).
\]
Let us claim that the map \(\Phi(\alpha, \cdot, \cdot)\) is uniformly continuous in \(\Omega_+ = \{c + i\epsilon|c \in \text{Ran} u, 0 < \epsilon \leq \epsilon_0/2\}\) and \(\Omega_- = \{c - i\epsilon|c \in \text{Ran} u, 0 < \epsilon \leq \epsilon_0/2\}\). We first consider the domain \(\Omega_+\).
If it is not true, then there exists $c_n^1$, $c_n^2 \in \Omega_+$ and $\delta > 0$ so that

$$
\|\Phi(\alpha, \cdot, c_n^1) - \Phi(\alpha, \cdot, c_n^2)\|_{C([-1, 1])} > \delta, \quad |c_n^1 - c_n^2| \to 0.
$$

By Proposition 6.1, $\Phi(\alpha, \cdot, c_n^j) \ (j = 1, 2, n > 0)$ is bounded in $H^1(-1, 1)$. Up to a subsequence, we can assume $\Phi(\alpha, \cdot, c_n^j) \rightarrow \Phi_j$ weakly in $H^1(-1, 1)$ for some $\Phi_j \in H^1(-1, 1)$, and $c_n^j \rightarrow c_0$ for $j = 1, 2$. Moreover, $\Phi(\alpha, \cdot, c_n^j) \rightarrow \Phi_j, \ j = 1, 2$ in $C([-1, 1])$. Thus, $\|\Phi_1 - \Phi_2\|_{C([-1, 1])} \geq \delta$.

If $c_0 \notin \mathbb{R}$, then $\Phi_1$, $\Phi_2$ are solutions of (6.1) with $c = c_0$. Thus, $0 \neq \Phi_1 - \Phi_2$ is a solution of (6.1) with $\omega(y) = 0$, which is impossible by Proposition 6.1.

If $c_0 \in \mathbb{R}$, then $c_0 \in \text{Ran} \ u$. Following the arguments in Proposition 6.1, we have

$$
\int_{-1}^{1} (\Phi_j^\prime \varphi + c^2 \Phi_j \varphi) dy + p.v. \int_{-1}^{1} \frac{(u'' \Phi_j + \omega) \varphi}{u - c_0} dy + i\pi \times \sum_{y \in u^{-1}(c_0), u'(y) \neq 0} \frac{(u'' \Phi_j + \omega) \varphi(y)}{|u'(y)|} = 0
$$

for $j = 1, 2, \varphi \in H^1_{0}(-1, 1)$, $\text{supp } \varphi \subset [-1, 1] \setminus \{y : u(y) = c, u'(y) = 0\}$. Moreover, $\psi = \Phi_1 - \Phi_2$ satisfies (5.2) with $c = c_0$ for any $\varphi \in H^1_{0}(-1, 1)$, since $\psi/(u - c)$ is bounded near $y_0$ where $u(y_0) = c, u'(y_0) = 0$ (see Sect. 5.1). Hence, $c_0$ is an embedding eigenvalue of $\mathcal{R}_\alpha$, which leads to a contradiction.

By taking conjugation, the map $\Phi(\alpha, \cdot, \cdot)$ is uniformly continuous in $\Omega_-$.

Then the result follows easily from Proposition 6.1. \hfill \Box

### 7 Linear Damping and Vorticity Depletion

In this section, we prove the linear damping and the vorticity depletion phenomena for shear flows in $\mathcal{K}$.

In terms of the stream function $\hat{\psi}(t, \alpha, y)$, the linearized Euler equations take as follows

$$
\frac{1}{i\alpha} \partial_t \hat{\psi} = \left(\partial_x^2 - \alpha^2\right)^{-1} \left(u''(y) - u(y) \left(\partial_y^2 - \alpha^2\right)\right) \hat{\psi} = -\mathcal{R}_\alpha \hat{\psi}
$$

with the initial data

$$
\psi(0, \alpha, y) = (\alpha^2 - \partial_y^2)^{-1} \omega_0(\alpha, y).
$$

Let $\Omega$ be a simply connected domain including the spectrum $\sigma(\mathcal{R}_\alpha)$ of $\mathcal{R}_\alpha$. Then we have

$$
\hat{\psi}(t, \alpha, y) = \frac{1}{2\pi i} \int_{\partial \Omega} e^{-i\alpha t c} (c - \mathcal{R}_\alpha)^{-1} \hat{\psi}(0, \alpha, y) dc.
$$
Due to $P_{\mathcal{R}} \tilde{\psi}(0, \alpha, y) = 0$, we have

$$\tilde{\psi}(t, \alpha, y) = \frac{1}{2\pi i} \int_{\partial \Omega_\epsilon} e^{-ia\mathcal{R}c} (c - \mathcal{R}_\alpha)^{-1} \tilde{\psi}(0, \alpha, y) dc,$$

where $\Omega_\epsilon$ is defined by (4.2) with $\epsilon$ sufficiently small. Let $\Phi(\alpha, y, c)$ be the solution of (6.1) with $\omega = \tilde{\omega}_0(\alpha, y)/(i\alpha)$. It is easy to see that

$$(c - \mathcal{R}_\alpha)^{-1} \tilde{\psi}(0, \alpha, y) = i\alpha \Phi(\alpha, y, c).$$

This shows that

$$\tilde{\psi}(t, \alpha, y) = \frac{1}{2\pi} \int_{\partial \Omega_\epsilon} \alpha \Phi(\alpha, y, c)e^{-i\alpha ct} dc.$$  \hspace{1cm} (7.1)

Now we prove the linear damping.

**Proof of Theorem 1.1** By Proposition 6.1, $|\Phi(\alpha, y, c)| \leq C\|\tilde{\omega}_0(\alpha, \cdot)/\alpha\|_{H^1}$, which along with Proposition 6.5 ensures that

$$\tilde{\psi}(t, \alpha, y) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\partial \Omega_\epsilon} e^{-ia\mathcal{R}c} i\alpha \Phi(\alpha, y, c) dc$$

$$= \frac{1}{2\pi} \int_{\text{Ran } u} e^{-ia\mathcal{R}c} i\alpha (\Phi_-(\alpha, y, c) - \Phi_+ (\alpha, y, c)) dc$$

$$= \frac{1}{2\pi} \int_{\text{Ran } u} e^{-ia\mathcal{R}c} \tilde{\Phi}(\alpha, y, c) dc,$$

where $\tilde{\Phi}(\alpha, y, c) = \alpha (\Phi_-(\alpha, y, c) - \Phi_+ (\alpha, y, c))$.

For $k, l = 0, 1$, we have

$$\partial_t^k \partial_y^l \tilde{\psi}(t, \alpha, y) = \frac{1}{2\pi} \int_{\text{Ran } u} (-i\alpha c)^k e^{-ia\mathcal{R}c} \partial_y^l \tilde{\Phi}(\alpha, y, c) dc,$$

from which and Plancherel’s formula, we infer that

$$\|\tilde{V}(t, \alpha, y)\|_{L^2_y H^1_x}^2 = \int_{\mathbb{R}} (\|\tilde{V}(t, \alpha, \cdot)\|_{L^2_y}^2 + \|\partial_t \tilde{V}(t, \alpha, \cdot)\|_{L^2_y}^2) dt$$

$$\leq \int_{-1}^1 \int_{\mathbb{R}} (\alpha^2 |\tilde{\psi}(t, \alpha, y)|^2 + |\partial_y \tilde{\psi}(t, \alpha, y)|^2$$

$$+ \alpha^2 |\partial_t \tilde{\psi}(t, \alpha, y)|^2 + |\partial_t \partial_y \tilde{\psi}(t, \alpha, y)|^2) dt dy$$

$$= \frac{1}{2\pi |\alpha|} \int_{\text{Ran } u} (1 + |\alpha c|^2)(\alpha^2 |\tilde{\Phi}(\alpha, y, c)|^2 + |\partial_y \tilde{\Phi}(\alpha, y, c)|^2) dcdy$$

$$\leq C \int_{\text{Ran } u} \|\tilde{\Phi}(\alpha, \cdot, c)\|_{H^1_y}^2 dc \leq C \int_{\text{Ran } u} \|\tilde{\omega}_0(\alpha, \cdot)\|_{H^1_y}^2 dc = C \|\tilde{\omega}_0(\alpha, \cdot)\|_{H^1_y}^2.$$

\[\square\]
By Sobolev embedding, we deduce that as \( t \to \infty \),
\[
\| \hat{V}(t, \alpha, \cdot) \|_{L^2_y}^2 \leq C \int_{t-1}^{t+1} \left( \| \hat{V}(s, \alpha, \cdot) \|_{L^2_y}^2 + \| \partial_s \hat{V}(s, \alpha, \cdot) \|_{L^2_y}^2 \right) ds \to 0.
\]
This shows the linear damping. 

Next, we prove the vorticity depletion phenomena.

**Proof of Theorem 2.1** Let \( W(\alpha, y, c) = i\alpha(\alpha^2 - \partial_y^2) \Phi(\alpha, y, c) \). It follows from (7.1) that
\[
\hat{\omega}(t, \alpha, y) = \frac{1}{2\pi i} \int_{\partial\Omega} e^{-i\alpha t c} W(\alpha, y, c) dc.
\]
By Lemma 6.3 and Proposition 6.1, we have for \( c \in \Omega_{\epsilon_0} \setminus \mathbb{R} \),
\[
|W(\alpha, y_0, c)| \leq C|u(y_0) - c|^{-\frac{3}{2}} \left( \| \Phi(\alpha, \cdot, c) \|_{H^1_y} + \| \hat{\omega}_0(\alpha, \cdot) \|_{H^1_y} \right)
\leq C|u(y_0)| - c|^{-\frac{3}{2}} \| \hat{\omega}_0(\alpha, \cdot) \|_{H^1_y},
\]
which gives
\[
|W(\alpha, y_0, c)| \leq C e^{-\frac{3}{2} \| \hat{\omega}_0(\alpha, \cdot) \|_{H^1_y}} \text{ for } c \in \partial\Omega_{\epsilon},
\]
\[
|W(\alpha, y_0, c \pm i\epsilon)| \leq C|u(y_0)| - c|^{-\frac{3}{2}} \| \hat{\omega}_0(\alpha, \cdot) \|_{H^1_y} \text{ for } c \in \text{Ran} u.
\]
Thus, \( W(\alpha, y_0, \cdot \pm i\epsilon) \) is bounded in \( L^p_c(\text{Ran} u)(1 < p < 4/3) \) for \( 0 < \epsilon < \epsilon_0 \). There exists a subsequence \( \epsilon_n \to 0^+ \) and \( W_{\pm}(c) \in L^p_c(\text{Ran} u)(1 < p < 4/3) \) so that \( W(\alpha, y_0, \cdot \pm i\epsilon_n) \to W_{\pm} \) weakly in \( L^p_c(\text{Ran} u) \). Then we deduce that
\[
\hat{\omega}(t, \alpha, y_0) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\partial\Omega} e^{-i\alpha t c} W(\alpha, y_0, c) dc
= \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\text{Ran} u} \left( e^{-i\alpha (c - i\epsilon_n)} W(\alpha, y_0, c - i\epsilon_n)
- e^{-i\alpha (c + i\epsilon_n)} W(\alpha, y_0, c + i\epsilon_n) \right) dc
= \frac{1}{2\pi i} \int_{\text{Ran} u} e^{-i\alpha t c} (W_-(c) - W_+(c)) dc \to 0 \text{ as } t \to \infty,
\]
where in the last step we used Riemann–Lebesgue lemma. 

\( \square \)
8 The Inhomogeneous Rayleigh Equation for Symmetric Flow

In this section, we solve the inhomogeneous Rayleigh equation when $u \in C^4([-1, 1])$ satisfies (S):

$$\begin{aligned}
\Phi'' - \alpha^2 \Phi - \frac{u''}{u-c} \Phi &= f, \\
\Phi(-1) &= \Phi(1) = 0,
\end{aligned} \quad (8.1)$$

where $c \in \Omega_{\epsilon_0} \setminus D_0$. In what follows, we will suppress the variable $\alpha$ for simplicity.

8.1 Representation Formula of the Solution

We decompose $f$ into the odd part and even part, i.e.,

$$f = f_o + f_e,$$

where

$$f_o(y, c) = \frac{f(y, c) - f(-y, c)}{2}, \quad f_e(y, c) = \frac{f(y, c) + f(-y, c)}{2}.$$

Let $\Phi_0$ and $\Phi_e$ be the solution of the following Rayleigh equations:

$$\begin{aligned}
\Phi_o'' - \alpha^2 \Phi_o - \frac{u''}{u-c} \Phi_o &= f_o, \\
\Phi_o(-1) &= \Phi_o(1) = 0,
\end{aligned} \quad (8.2)$$

and

$$\begin{aligned}
\Phi_e'' - \alpha^2 \Phi_e - \frac{u''}{u-c} \Phi_e &= f_e, \\
\Phi_e(-1) &= \Phi_e(1) = 0.
\end{aligned} \quad (8.3)$$

Thus, $\Phi = \Phi_o + \Phi_e$ is the solution of the inhomogeneous Rayleigh equation (8.1) with $\Phi_o$ being an odd function and $\Phi_e$ being an even function. So, it suffices to solve the following Rayleigh equation in $[0, 1]$:

$$\begin{aligned}
\Phi_o'' - \alpha^2 \Phi_o - \frac{u''}{u-c} \Phi_o &= f_o, \\
\Phi_o(0) &= \Phi_o(1) = 0,
\end{aligned} \quad (8.2)$$

and

$$\begin{aligned}
\Phi_e'' - \alpha^2 \Phi_e - \frac{u''}{u-c} \Phi_e &= f_e, \\
\Phi_e'(0) &= \Phi_e(1) = 0.
\end{aligned} \quad (8.3)$$
Let $\varphi(y)$ be a solution of the homogenous Rayleigh equation

$$\varphi'' - \alpha^2 \varphi - \frac{u''}{u - c} \varphi = 0.$$ 

Then the inhomogeneous Rayleigh equations (8.2) and (8.3) are equivalent to

$$\begin{aligned}
\begin{cases}
\left( \varphi^2 \left( \frac{\Phi_o}{\varphi} \right) \right)' = f_o \varphi, \\
\Phi_o(0) = \Phi_o(1) = 0,
\end{cases}
\end{aligned} \quad (8.4)$$

and

$$\begin{aligned}
\begin{cases}
\left( \varphi^2 \left( \frac{\Phi_e}{\varphi} \right) \right)' = f_e \varphi, \\
\Phi_e(0) = \Phi_e(1) = 0.
\end{cases}
\end{aligned} \quad (8.5)$$

In particular, let $\varphi = \phi$ be the solution of the homogeneous Rayleigh equation constructed in Proposition 4.7. We deduce by integrating (8.4) and (8.5) twice and matching the boundary conditions that for $y \in [0, 1]$,

$$\begin{aligned}
\Phi_o(y, c) = & \phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_o \phi(y'', c) dy'' dy' \\
& + \mu_o(c) \phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} dy'
= & \phi(y, c) \int_1^y \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_o \phi(y'', c) dy'' dy' \\
& + \mu_o(c) \phi(y, c) \int_1^y \frac{1}{\phi(y', c)^2} dy',
\end{aligned} \quad (8.6)$$

where

$$\mu_o(c) = - \frac{\int_1^y \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_o \phi(y'', c) dy'' dy'}{\int_0^1 \frac{1}{\phi(y', c)^2} dy'},$$

and for $y \in [0, 1]$,

$$\begin{aligned}
\Phi_e(y, c) = & \phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_e \phi(y'', c) dy'' dy' \\
& + \mu_e(c) \phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} dy' + \nu_e(c) \phi(y, c)
\end{aligned}$$
\[\phi(y, c) \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_e \phi(y'', c) dy'' dy' + \mu^e(c) \phi(y, c) \frac{1}{\phi(y', c)^2} dy'\]

where \(\mu^e\) and \(\nu^e\) are determined by solving

\[
\int_0^1 \phi(y', c)^2 \int_{y_c}^{y'} f_e \phi(y'', c) dy'' dy' + \mu^e(c) \int_0^1 \phi(y', c)^2 dy' + \nu^e(c) = 0,
\]

\[
\nu^e(c) \phi(0, c) \phi'(0, c) + \int_{y_c}^0 f_e \phi(y'', c) dy'' + \mu^e(c) = 0.
\]

That is,

\[
\nu^e(c) = \frac{\int_0^1 \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_e \phi(y'', c) dy'' dy' - \left( \int_{y_c}^0 f_e \phi(y'', c) dy'' \right) \left( \int_0^1 \frac{1}{\phi(y', c)^2} dy' \right)}{\phi(0, c) \phi'(0, c) \int_0^1 \frac{1}{\phi(y', c)^2} dy' - 1},
\]

\[
\mu^e(c) = \frac{- \phi(0, c) \phi'(0, c) \int_0^1 \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f_e \phi(y'', c) dy'' dy' + \int_{y_c}^0 f_e \phi(y'', c) dy''}{\phi(0, c) \phi'(0, c) \int_0^1 \frac{1}{\phi(y', c)^2} dy' - 1}.
\]

Thus, the solution \(\Phi(y, c)\) of (8.1) can be written as

\[
\Phi(y, c) = \Phi_o(y, c) + \Phi_e(y, c)
\]

with \(\Phi_o\) and \(\Phi_e\) given by (8.6) and (8.7).

### 8.2 Representation Formula of the Limiting Solution

In this subsection, we give a precise representation formula of the limiting solution \(\Phi_{\pm}(y, c)\) obtained in Proposition 6.5 for the symmetric flow.

The following fact is classical [12].

**Lemma 8.1** For \(g \in H^2(a, b)\), the function

\[
F(c) = \int_a^b \frac{g(z)}{z-c} dz \quad \text{for} \quad \text{Im} \, c > 0
\]

can be \(C^1\) extended to the interval \((a, b)\) with

\[
F(c) = -H(g \chi_{[a, b]})(c) + i \pi g(c) \quad \text{for} \quad c \in (a, b).
\]

Here \(H(g)(c) = p.v. \int \frac{g(z)}{z-c} dz\) is the Hilbert transform of \(g\).
Remark 8.2 If \( g \in H^2((-b, b) \setminus \{0\}) \cap C([-b, b]) \), then \( F(z) \) can be \( C^1 \) extended to the interval \((-b, b) \setminus \{0\}\) with

\[
F(c) = -H(g \chi_{[-b, b]})(c) + i \pi g(c) \text{ for } c \in (-b, b) \setminus \{0\}.
\]

Let \( v(y) \) be as in (4.4) and \( \tilde{c} = v(y_c) \). We define

\[
\text{Int}(\varphi)(y) = \int_0^y \varphi(y') dy' \text{ for } y \in [0, 1], \quad \text{Int}(\varphi)(y) = \text{Int}(\varphi)(-y) \text{ for } y \in [-1, 0].
\]

We introduce

\[
\Pi_{1,1}(\varphi)(c) = p.v. \int_0^1 \frac{\text{Int}(\varphi)(y) - \text{Int}(\varphi)(y_c)}{(u(y) - u(y_c))^2} dy
\]

\[
= \partial_c \left( \int_0^1 \frac{\text{Int}(\varphi)(y) - \text{Int}(\varphi)(y_c)}{u(y) - c} dy \right)
\]

\[
+ \varphi(y_c) \left( p.v. \int_0^1 \frac{1}{u(y) - c} dy \right)
\]

\[
= \partial_c \left( p.v. \int_0^1 \frac{\text{Int}(\varphi)(y)}{u(y) - c} dy \right)
\]

\[
- \text{Int}(\varphi)(y_c) \partial_c \left( p.v. \int_0^1 \frac{dy}{u(y) - c} \right)
\]

\[
= \frac{1}{2\tilde{c}} \partial_c \left( \frac{1}{2\tilde{c}} p.v. \int_{-v(1)}^{v(1)} \frac{\text{Int}(\varphi)(v^{-1}(z))(v^{-1}'(z))}{z - \tilde{c}} dz \right)
\]

\[
- \frac{1}{2\tilde{c}} \partial_c \left( \frac{1}{2\tilde{c}} p.v. \int_{-v(1)}^{v(1)} \frac{(v^{-1}'(z))}{z - \tilde{c}} dz \right). \tag{8.9}
\]

Let us point out that \( \text{Int}(\varphi)(y) \notin H^2(-1, 1) \) if \( \varphi(0) \neq 0 \). However, \( \text{Int}(\varphi) \circ v^{-1} \in H^2((-v(1), v(1)) \setminus \{0\}) \cap C([-v(1), v(1)) \setminus \{0\}) \).

Lemma 8.3 Let \( c_\varepsilon = c + i \varepsilon \in D_{e_0} \). Then for any \( \varphi \in H^1(0, 1) \),

\[
\lim_{\varepsilon \to 0^+} \rho(c_\varepsilon) \int_0^1 \frac{\int_{y_\varepsilon} y \varphi(y') dy'}{(u(y) - c_\varepsilon)^2} dy = \rho(c) \Pi_{1,1}(\varphi)(c) + i \pi \frac{\varphi(y_c)}{u'(y_c)^2} \rho(c),
\]

\[
\lim_{\varepsilon \to 0^-} \rho(c_\varepsilon) \int_0^1 \frac{\int_{y_\varepsilon} y \varphi(y') dy'}{(u(y) - c_\varepsilon)^2} dy = \rho(c) \Pi_{1,1}(\varphi)(c) - i \pi \frac{\varphi(y_c)}{u'(y_c)^2} \rho(c),
\]

and

\[
\lim_{\varepsilon \to 0^+} \rho(c_\varepsilon) \tilde{c}_\varepsilon \int_0^1 \frac{dy}{(u(y) - c_\varepsilon)^2} = \frac{A_1(c) - i B(c)}{2 v'(y_c)},
\]

\[
\lim_{\varepsilon \to 0^-} \rho(c_\varepsilon) \tilde{c}_\varepsilon \int_0^1 \frac{dy}{(u(y) - c_\varepsilon)^2} = \frac{A_1(c) + i B(c)}{2 v'(y_c)}.
\]
Here \( \tilde{\omega}_e \) is a unique solution of \( c_e - u(0) = \tilde{c}_e^2 \) with \( \text{Im} \tilde{\omega}_e > 0 \).

**Proof** Thanks to \( u(y) - c_e = v(y)^2 - \tilde{c}_e^2 \), we have

\[
\int_0^1 \frac{\varphi'(y')}{(u(y) - c_e)^2} dy = p.v. \int_0^1 \frac{\text{Int}(\varphi)(y) - \text{Int}(\varphi)(y_c)}{(u(y) - c_e)^2} dy \\
= \partial_c \left( \int_0^1 \frac{\text{Int}(\varphi)(y) - \text{Int}(\varphi)(y_c)}{u(y) - c_e} dy \right) \\
+ \frac{\varphi(y_c)}{u'(y_c)} \int_0^1 \frac{1}{u(y) - c_e} dy \\
= \partial_c \left( \int_0^1 \frac{\text{Int}(\varphi)(y)}{u(y) - c_e} dy \right) \\
- \text{Int}(\varphi)(y_c) \partial_c \left( p.v. \int_0^1 \frac{dy}{u(y) - c_e} \right) \\
= \frac{1}{2\tilde{c}_e} \partial_{\tilde{c}_e} \left( \frac{1}{2\tilde{c}_e} p.v. \int_{-u(1)}^{u(1)} \frac{\text{Int}(\varphi)(v^{-1}(z))(v^{-1})'(z)}{z - \tilde{c}_e} dz \right) \\
- \frac{\text{Int}(\varphi)(y_c)}{2\tilde{c}_e} \partial_{\tilde{c}_e} \left( \frac{1}{2\tilde{c}_e} p.v. \int_{-u(1)}^{u(1)} \frac{(v^{-1})'(z)}{z - \tilde{c}_e} dz \right),
\]

from which and Remark 8.2, we deduce the first limit. Similarly, we have

\[
\rho(c_e)\tilde{\omega}_e \int_0^1 \frac{dy}{(u(y) - c_e)^2} = \rho(c_e)\tilde{\omega}_e \partial_c \left( p.v. \int_0^1 \frac{dy}{u(y) - c_e} \right) \\
= \frac{\rho(c_e)}{2} \partial_{\tilde{c}_e} \left( \frac{1}{2\tilde{c}_e} p.v. \int_{-u(1)}^{u(1)} \frac{(v^{-1})'(z)}{z - \tilde{c}_e} dz \right) \\
\to \frac{\rho(c)}{2} \partial_{\tilde{c}_e} \left( \frac{1}{2\tilde{c}_e} p.v. \int_{-u(1)}^{u(1)} \frac{(v^{-1})'(z)}{z - \tilde{c}_e} dz + \frac{\pi i(v^{-1})'(\tilde{c})}{2\tilde{c}_e} \right) = \frac{A_1(c) - iB(c)}{2v'(y_c)},
\]

as \( c_e \to c = u(y_c) \in D_0, \text{Im} c_e > 0 \).

The case of \( \text{Im} c_e < 0 \) is similar. We omit the details. \( \square \)

We decompose

\[
\hat{\omega}_0(y) = \hat{\omega}_o(y) + \hat{\omega}_e(y),
\]

where \( \hat{\omega}_o(y) = \frac{\hat{\omega}_0(y) - \hat{\omega}_0(-y)}{2} \) and \( \hat{\omega}_e(y) = \frac{\hat{\omega}_0(y) + \hat{\omega}_0(-y)}{2} \). We take \( f \) in (8.1) as follows

\[
f = \frac{\hat{\omega}_o(y)}{i\alpha(u(y) - c)} + \frac{\hat{\omega}_e(y)}{i\alpha(u(y) - c)} = f_o + f_e.
\]
We denote

\[ C_e(c) = \rho(c) \frac{\bar{\omega}_e(y_c)}{u'(y_c)}, \quad D_e(c) = u'(y_c)\rho(c)I_1(\bar{\omega}_e)(c), \]

\[ C_o(c) = \rho(c) \frac{\bar{\omega}_o(y_c)}{u'(y_c)}, \quad D_o(c) = u'(y_c)\rho(c)I_1(\bar{\omega}_o)(c), \]

\[ E_e(c) = E(\bar{\omega}_e)(c) \triangleq \int_{y_c}^{0} \bar{\omega}_e\phi_1(y, c) \, dy, \]

\[ \Pi_1(\varphi)(c) = \Pi_{1,1}(\varphi)(c) + \Pi_{1,2}(\varphi)(c), \]

where

\[ \Pi_{1,2}(\varphi)(c) = \int_{0}^{1} \int_{y_c}^{z} \varphi(y) \left( \frac{1}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) \, dy \, dz. \quad (8.10) \]

For \( c \in D_0 \), we introduce

\[ \Phi^o_{\pm}(y, c) \triangleq \begin{cases} \phi \int_{0}^{y} \frac{1}{\phi(z, c)^2} \int_{y_c}^{z} \phi f_o(y', c) \, dy' \, dz + \mu^o_{\pm}(c)\phi \int_{0}^{y} \frac{1}{\phi(y', c)^2} \, dy' & 0 \leq y \leq y_c, \\ \phi \int_{1}^{y} \frac{1}{\phi(z, c)^2} \int_{y_c}^{z} \phi f_o(y', c) \, dy' \, dz + \mu^o_{\pm}(c)\phi \int_{1}^{y} \frac{1}{\phi(y', c)^2} \, dy' & y_c \leq y \leq 1, \end{cases} \]

where

\[ \mu^o_{\pm}(c) = \frac{1}{\alpha} \left( \frac{iu'(y_c)\rho(c)I_1(\bar{\omega}_o)(c) - \rho(c)\frac{\bar{\omega}_o(y_c)}{u'(y_c)}}{A_1(c) - i\pi \rho(c)\frac{u''(y_c)}{u'(y_c)^2} + u'(y_c)\rho(c)I_3(c)} \right) \]

\[ = \frac{1}{\alpha} \left( \frac{C_o(c) + iD_o(c)}{A(c) - iB(c)} \right), \]

\[ \mu^o_{\pm}(c) = \frac{1}{\alpha} \left( \frac{iu'(y_c)\rho(c)I_1(\bar{\omega}_o)(c) + \rho(c)\frac{\bar{\omega}_o(y_c)}{u'(y_c)}}{A_1(c) + i\pi \rho(c)\frac{u''(y_c)}{u'(y_c)^2} + u'(y_c)\rho(c)I_3(c)} \right) \]

\[ = \frac{1}{\alpha} \left( \frac{C_o(c) + iD_o(c)}{A(c) + iB(c)} \right). \]

We also introduce

\[ \Phi^e_{\pm}(y, c) \triangleq \begin{cases} \phi \int_{0}^{y} \frac{1}{\phi(y', c)^2} \, dy' + \mu^e_{\pm}(c)\phi \int_{0}^{y} \frac{1}{\phi(y', c)^2} \, dy' \\ + \nu^e_{\pm}(c)\phi(y, c) & 0 \leq y \leq y_c, \\ \phi \int_{1}^{y} \frac{1}{\phi(y', c)^2} \, dy' + \mu^e_{\pm}(c)\phi \int_{1}^{y} \frac{1}{\phi(y', c)^2} \, dy' & y_c \leq y \leq 1, \end{cases} \]
where

$$
\begin{align*}
\mu^e_+(c) &= \frac{1}{\alpha} \frac{\phi(0, c)\phi'(0, c)(iD_x - C_x)(c) - iE_x(c)u'(y_c)\rho(c)}{\phi(0, c)\phi'(0, c)(A - iB)(c) - \rho(c)u'(y_c)}, \\
\mu^e_-(c) &= \frac{1}{\alpha} \frac{\phi(0, c)\phi'(0, c)(iD_x + C_x)(c) - iE_x(c)u'(y_c)\rho(c)}{\phi(0, c)\phi'(0, c)(A + iB)(c) - \rho(c)u'(y_c)}, \\
\nu^e_+(c) &= -\frac{1}{\alpha} \frac{iD_x(c) - C_x(c) - iE_x(c)(A - iB)(c)}{\phi(0, c)\phi'(0, c)(A - iB)(c) - \rho(c)u'(y_c)}, \\
\nu^e_-(c) &= -\frac{1}{\alpha} \frac{iD_x(c) + C_x(c) - iE_x(c)(A + iB)(c)}{\phi(0, c)\phi'(0, c)(A + iB)(c) - \rho(c)u'(y_c)}.
\end{align*}
$$

It is easy to see that

$$
\phi(0, c)\phi'(0, c)(A + iB)(c) - u'(y_c)\rho(c) = \phi_1(0, c)\phi_1'(0, c)(u(0) - c)(A_2 + iB_2)(c).
$$

Thus, if $\mathcal{R}_\alpha$ has no embedding eigenvalues, $\Phi^\alpha_\pm$ and $\Phi^e_\pm$ are well-defined by Proposition 5.3.

For $y \in [-1, 0]$, we let

$$
\Phi^\alpha_\pm(y, c) = -\Phi^\alpha_\pm(-y, c), \quad \Phi^e_\pm(y, c) = \Phi^e_\pm(-y, c),
$$

and for $y \in [-1, 1]$,

$$
\Phi_\pm(y, c) = \Phi^\alpha_\pm(y, c) + \Phi^e_\pm(y, c).
$$

**Proposition 8.4** Let $\Phi(y, c)$ be a solution of (8.1) with $f = \hat{\omega}/(\alpha(u-c))$ for $\hat{\omega} \in L^2(-1, 1)$ given by the formula (8.8). If $\mathcal{R}_\alpha$ has no embedding eigenvalues, then $\Phi(y, c)$ is well-defined for $c \in \Omega_{\epsilon_0}$ small enough. Moreover, it holds that for any $(y, c) \in [0, 1] \times D_0$ and $y \neq y_c$,

$$
\lim_{\epsilon \to 0^+} \Phi(y, c_\epsilon) = \Phi_+(y, c), \quad \lim_{\epsilon \to 0^-} \Phi(y, c_\epsilon) = \Phi_-(y, c),
$$

where $c_\epsilon = c + i\epsilon \in D_{\epsilon_0}$.

**Proof** By Proposition 4.7, $\phi_1(y, c)$ is continuous for $(y, c) \in [0, 1] \times \Omega_{\epsilon_0}$ and for $\epsilon_0$ small enough,

$$
|\phi_1(y, c)| > \frac{1}{2}, \quad |\phi_1(y, c) - 1| \leq C|y - y_c|^2.
$$

Thus, for $(y, c) \in [0, 1] \times \Omega_{\epsilon_0}$, there exists a constant $C$ so that

$$
\left| \frac{\rho(c)}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right| \leq C.
$$
and for \((y, c) \in [0, 1] \times \Omega_\epsilon_0\) with \(|z - y_c| \leq |y - y_c|\),

\[
\left| \frac{\rho(c)}{(u(y) - c)^2} \left( \frac{1}{\phi_1(z, c)} - 1 \right) \right| \leq C.
\]

This implies that as \(c_\epsilon \to u(y_c) = c\),

\[
\rho(c_\epsilon) \int_0^1 \frac{1}{(u(y) - c_\epsilon)^2} \left( \frac{1}{\phi_1(y, c_\epsilon)} - 1 \right) dy \to \rho(c) I_3(c),
\]

\[
\rho(c_\epsilon) \int_0^1 \int_{y_c}^y \frac{\varphi(z)}{(u(y) - c_\epsilon)^2} \left( \frac{\phi_1(z, c_\epsilon)}{\phi_1(y, c_\epsilon)} - 1 \right) dz dy \to \rho(c) I_{1,2}(\varphi)(c),
\]

from which and Lemma 8.3, it follows that as \(\epsilon \to 0\),

\[
u'(y_c) \rho(c_\epsilon) \int_0^1 \frac{1}{\phi(y', c_\epsilon)^2} dy' \to \nu'(y_c) \rho(c) I_3(c) + A_1(c) \mp iB(c),
\]

and

\[
i\alpha \nu'(y_c) \rho(c_\epsilon) \int_0^1 \frac{1}{\phi(y', c_\epsilon)^2} \int_{y_c}^y f_\alpha \phi(y'', c_\epsilon) dy'' dy' \to \rho(c) I_{1,2}((\tilde{\omega})'(c)) + \rho(c) I_{1,1}((\tilde{\omega})''(c)) \pm i\pi \frac{\tilde{\omega}(y_c)}{\nu'(y_c)} \rho(c).
\]

Thus, we get

\[
\lim_{\epsilon \to 0^\pm} \mu^\rho(c_\epsilon) = \mu^\rho_\pm(c).
\]

Similarly, we have

\[
\lim_{\epsilon \to 0^\pm} \mu^e(c_\epsilon) = \mu^e_\pm(c), \quad \lim_{\epsilon \to 0^\pm} \nu^e(c_\epsilon) = \nu^e_\pm(c).
\]

The following convergence is obvious: for \(0 \leq y < y_c\), as \(c_\epsilon \to u(y_c)\),

\[
\phi(y, c_\epsilon) \int_0^y \frac{1}{\phi(y', c_\epsilon)^2} \int_{y_c}^y f \phi(y'', c_\epsilon) dy'' dy' \to \phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} \int_{y_c}^y f \phi(y'', c) dy'' dy',
\]

\[
\phi(y, c_\epsilon) \int_0^y \frac{1}{\phi(y', c_\epsilon)^2} dy' \to \phi(y, c) \int_0^y \frac{1}{\phi(y', c)^2} dy',
\]
and for \( y_c < y \leq 1 \), as \( c_\epsilon \to u(y_c) \),

\[
\phi(y, c_\epsilon) \int_1^y \frac{1}{\phi(y', c_\epsilon)^2} \int_{y_c}^{y'} f \phi(y'', c_\epsilon) dy'' dy' \\
\to \phi(y, c) \int_1^y \frac{1}{\phi(y', c)^2} \int_{y_c}^{y'} f \phi(y'', c) dy'' dy',
\]

\[
\phi(y, c_\epsilon) \int_1^y \frac{1}{\phi(y', c_\epsilon)^2} dy' \to \phi(y, c) \int_1^y \frac{1}{\phi(y', c)^2} dy'.
\]

With the above information, the proposition follows easily. \( \square \)

## 9 Decay Estimates of the Velocity for Symmetric Flow

In this section, we establish the decay estimates of the velocity for symmetric flow by using the dual method.

We introduce that for \( y \in [0, 1] \),

\[
\Phi(y, c) = \Phi_o(y, c) + \Phi_\epsilon(y, c)
\]

\[
= \begin{cases} 
(\mu_o^-(c) - \mu_o^+(c))\phi(y, c) \int_0^y \frac{1}{\phi(z, c)^2} dz \\
(\mu_o^+(c) - \mu_o^-(c))\phi(y, c) \int_1^y \frac{1}{\phi(z, c)^2} dz \\
(\mu_\epsilon^-(c) - \mu_\epsilon^+(c))\phi(y, c) \int_0^y \frac{1}{\phi(z, c)^2} dz + (\nu_\epsilon^-(c) - \nu_\epsilon^+(c))\phi(y, c) \quad 0 \leq y < y_c, \\
(\mu_\epsilon^+(c) - \mu_\epsilon^-(c))\phi(y, c) \int_1^y \frac{1}{\phi(z, c)^2} dz \quad y_c < y \leq 1.
\end{cases}
\]

For \( y \in [-1, 0] \), we define

\[
\Phi(y, c) = -\Phi_o(-y, c) + \Phi_\epsilon(-y, c).
\]

Furthermore, we have

\[
\mu_o^-(c) - \mu_o^+(c) = \frac{2}{\alpha} \frac{AC_o + BD_o}{A^2 + B^2} \overset{\text{def}}{=} \frac{2}{\alpha} \rho(c) \mu_1(c),
\]

\[
\nu_\epsilon^-(c) - \nu_\epsilon^+(c) = -\frac{2}{\alpha} \frac{\phi(0, c)\phi'(0, c)(AC_\epsilon + BD_\epsilon) - u'(y_c)\rho(c)(BE_\epsilon + C_\epsilon)}{\phi(0, c)\phi'(0, c)A - u'(y_c)\rho(c) A^2 + B^2} \overset{\text{def}}{=} \frac{2}{\alpha} \nu_1(c),
\]

\[
\mu_\epsilon^-(c) - \mu_\epsilon^+(c) = -\frac{2}{\alpha} \frac{\phi(0, c)\phi'(0, c)v_1(c)}{\phi(0, c)\phi'(0, c)} \overset{\text{def}}{=} \frac{2}{\alpha} \mu_2(c).
\]

It follows from Proposition 6.1 and Proposition 8.4 that
\[
\hat{\psi}(t, \alpha, y) = \frac{1}{2\pi} \int_{u(0)}^{u(1)} \alpha \tilde{\Phi}(y, c) e^{-i\alpha ct} dc
\]
\[
= \frac{1}{2\pi} \int_{u(0)}^{u(1)} \alpha \tilde{\Phi}_o(y, c) e^{-i\alpha ct} dc
\]
\[
+ \frac{1}{2\pi} \int_{u(0)}^{u(1)} \alpha \tilde{\Phi}_e(y, c) e^{-i\alpha ct} dc
\]

\[\defeq \hat{\psi}_o(t, \alpha, y) + \hat{\psi}_e(t, \alpha, y).\]

### 9.1 The Odd Part

For \( f = g'' - \alpha^2 g \) with \( g \in H^2(0, 1) \cap H^1_0(0, 1) \), we have

\[
\int_0^1 \hat{\psi}_o(t, \alpha, y) f(y) dy
\]
\[
= \frac{1}{\pi} \int_0^1 f(y) \int_{u(0)}^{u(y)} \rho(c) \mu_1(c) \phi(y, c) \int_1^y \frac{1}{\phi(z, c)} dz e^{-i\alpha ct} dc dy
\]
\[
+ \frac{1}{\pi} \int_0^1 f(y) \int_{u(y)}^{u(1)} \rho(c) \mu_1(c) \phi(y, c) \int_0^y \frac{1}{\phi(z, c)} dz e^{-i\alpha ct} dc dy
\]
\[
= -\frac{1}{\pi} \int_{u(0)}^{u(1)} \rho(c) \mu_1(c) e^{-i\alpha ct} \int_0^{\hat{z}_c} f(y) \phi(y, c) dy \frac{1}{\phi(z, c)^2} dz dc.
\]

First of all, we have

\[
\rho(c) \mu_1(c) = \pi \frac{A(c) \rho(c) \tilde{\Phi}_o(\hat{y}_c) + \rho(c)^2 u''(\hat{y}_c) u'(\hat{y}_c) \Pi_1(\tilde{\Phi}_o)(c)}{A(c)^2 + B(c)^2}.
\]

We write

\[
\int_0^{\hat{z}_c} f(y) \phi(y, c) dy \frac{1}{\phi(z, c)^2} dz
\]
\[
= \int_0^{\hat{z}_c} f(y)(g''(y) - \alpha^2 g(y)) \phi(y, c) dy \frac{1}{\phi(z, c)^2} dz
\]
\[
= \int_0^{\hat{z}_c} f(y) \phi''(y) \phi(y, c) dy + g'(z) \phi(z, c) - g(z) \phi'(z, c) + g(\hat{y}_c) \phi'(\hat{y}_c, c)
\]
\[
= \Pi_1(g u'')(c) + p.v. \int_0^{\hat{z}_c} \left[ \frac{g(z)}{\phi(z, c)} \phi'(z, c) + \frac{g(\hat{y}_c) u'(\hat{y}_c)}{\phi(z, c)^2} \right] dz.
\]
For $c \in (u(0), u(1))$, we have
\[
\frac{g(z)}{\phi(z, c)} - \frac{g(y_c)}{u(z) - c} = \frac{g(z)(1 - \phi_1(z, c))}{(u(z) - c)\phi_1(z, c)} + \frac{g(z) - g(y_c)}{u(z) - c} \in C([0, 1]),
\]
which along with $g(0) = g(1) = 0$ gives
\[
p.v. \int_0^1 \left[ \left( \frac{g(z)}{\phi(z, c)} \right)' + \frac{g(y_c)u'(y_c)}{\phi(z, c)^2} \right] dz = \left( \frac{g(z)}{\phi(z, c)} - \frac{g(y_c)}{u(z) - c} \right) \bigg|_0^1 + p.v.
\]
\[
\int_0^1 \left[ \left( \frac{g(y_c)}{u(z) - c} \right)' + \frac{g(y_c)u'(y_c)}{\phi(z, c)^2} \right] dz = \frac{g(y_c)}{u(0) - c} - \frac{g(y_c)}{u(1) - c} + p.v.
\]
\[
\int_0^1 \left[ - \frac{g(y_c)(u'(z) - u'(y_c))}{(u(z) - c)^2} + \frac{g(y_c)u'(y_c)}{(u(z) - c)^2} \left( \frac{1}{\phi(z, c)^2} - 1 \right) \right] dz,
\]
\[
= \frac{g(y_c)(u(0) - u(1))}{\rho(c)} - g(y_c)\Pi_2(c) + g(y_c)u'(y_c)\Pi_3(c) = \frac{g(y_c)}{\rho(c)} A(c).
\]
Thus, we deduce that
\[
\int_0^1 \widehat{\psi}_o(t, \alpha, y) f(y) dy = - \int_{u(0)}^{u(1)} K_o(c, \alpha) e^{-i\alpha t} dc, \tag{9.1}
\]
where
\[
K_o(c, \alpha) = \frac{(A(c)\widehat{\omega}_o(y_c) + \rho(c)u''(y_c)\Pi_1(\widehat{\omega}_o)(c))\left(\Pi_1(gu'')(c)\rho(c) + g(y_c)A(c)\right)}{(A(c)^2 + B(c)^2)u'(y_c)}.
\]
We introduce
\[
\Lambda_1(\varphi)(c) = \Lambda_{1,1}(\varphi)(c) + \Lambda_{1,2}(\varphi)(c),
\]
\[
\Lambda_2(\varphi)(c) = \Lambda_{2,1}(\varphi)(c) + \Lambda_{2,2}(\varphi)(c),
\]
where
\[
\Lambda_{1,1}(\varphi)(c) = A_1(c)\varphi(y_c) + \rho(c)u''(y_c)\Pi_{1,1}(\varphi)(c),
\]
\[
\Lambda_{1,2}(\varphi)(c) = \rho(c)u''(y_c)\Pi_{1,2}(\varphi) + u'(y_c)\rho(c)\Pi_3(c)\varphi(y_c),
\]
\[
\Lambda_{2,1}(\varphi)(c) = A_1(c)\varphi(y_c) + \rho(c)\Pi_{1,1}(u''\varphi)(c),
\]
\[
\Lambda_{2,2}(\varphi)(c) = \rho(c)\Pi_{1,2}(u''\varphi)(c) + u'(y_c)\rho(c)\Pi_3(c)\varphi(y_c).
\]
Then we have

\[ K_\alpha(c, \alpha) = \frac{\Lambda_1(\tilde{\omega}_\alpha)(c)\Lambda_2(g)(c)}{(A(c)^2 + B(c)^2)u'(y_c)}. \tag{9.2} \]

### 9.2 The Even Part

For \( f = g'' - \alpha^2 g \) with \( g \in H^2(0, 1) \) and \( g'(0) = g(1) = 0 \), we have

\[
\int_0^1 \hat{\psi}_e(t, \alpha, y) f(y) dy \\
= \frac{1}{\pi} \int_0^1 f(y) \int_{u(0)}^{u(y)} \mu_2(c) \phi(y, c) \int_1^y \frac{1}{\phi(z, c)^2} dze^{-i\alpha c} dc dy \\
+ \frac{1}{\pi} \int_0^1 f(y) \int_{u(y)}^{u(1)} \mu_2(c) \phi(y, c) \int_0^y \frac{1}{\phi(z, c)^2} dze^{-i\alpha c} dc dy \\
+ \frac{1}{\pi} \int_0^1 f(y) \int_{u(y)}^{u(1)} \psi_1(c) \phi(y, c) e^{-i\alpha c} dc dy \\
= -\frac{1}{\pi} \int_{u(0)}^{u(1)} \mu_2(c) e^{-i\alpha c} \int_0^z f(y) \phi(y, c) dy dc \\
+ \frac{1}{\pi} \int_{u(0)}^{u(1)} \psi_1(c) e^{-i\alpha c} \int_0^{y_c} f(y) \phi(y, c) dy dc \\
= -\frac{1}{\pi} \int_{u(0)}^{u(1)} \mu_2(c) e^{-i\alpha c} \left( \int_0^1 \frac{f(y)}{\phi(z, c)^2} dy + \frac{f(y_c)}{\phi(0, c)\phi'(0, c)} \right) dc.
\]

Using the facts that

\[
\int_0^{y_c} \frac{f(y) \phi(y, c) dy}{\phi(z, c)^2} dz = \Pi_1(gu''(c)) + p.v. \int_0^1 \left[ \frac{g(z)}{\phi(z, c)} \right]' \frac{g(y_c)u'(y_c)}{\phi(z, c)^2} dz, \\
p.v. \int_0^1 \left[ \frac{g(z)}{\phi(z, c)} \right]' \frac{g(y_c)u'(y_c)}{\phi(z, c)^2} dz = \frac{g(y_c)}{\rho(c)} A(c) - \frac{g(0)}{\phi(0, c)}, \\
\int_0^{y_c} f(y) \phi(y, c) dy = -E(gu'')(c) - g(y_c)u'(y_c) + g(0)\phi'(0, c),
\]

we deduce that

\[
\int_0^1 \hat{\psi}_e(t, \alpha, y) f(y) dy \\
= -\frac{1}{\pi} \int_{u(0)}^{u(1)} \mu_2(c) \left( \frac{g(y_c)A(c)}{\rho(c)} \right)
\]

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\[ + \Pi_1(gu''(c)) + v_1(c)\left(\mathbb{E}(gu''(c)) + g(y_c)u'(y_c)\right) \right] e^{-i\alpha ct} dc \\
= - \int_{u(0)}^{u(1)} K_e(c, \alpha) e^{-i\alpha ct} dc, \quad (9.3) \]

where

\[ \pi K_e(c, \alpha) = \mu_2(c) \left( \frac{g(y_c)A(c)}{\rho(c)} + \Pi_1(gu''(c)) + v_1(c)\left(\mathbb{E}(gu''(c)) + g(y_c)u'(y_c)\right) \right) \]

Recall that

\[ v_1(c) = - \frac{\phi(0, c)\phi'(0, c)(AC_e + BD_e) - u'(y_c)\rho(c)(BE_e + C_e)}{(\phi(0, c)\phi'(0, c)A - u'(y_c)\rho(c))^2 + \phi(0, c)^2\phi'(0, c)^2(B(c)^2)} \]

\[ = -\rho_1(AC_e + BD_e) + J(BE_e + C_e) \]

\[ = -\pi \frac{-\rho_1(A\rho\frac{\omega_e(y_c)}{u'(y_c)} + \rho\frac{u''(y_c)}{u'(y_c)}u'(y_c)\rho\Pi_1(\omega_e) + J(\rho\frac{u''(y_c)}{u'(y_c)}E_e + \rho\frac{\omega_e(y_c)}{u'(y_c)}))}{(\rho_1J + \rho^2_1B^2)[\phi(0, c)\phi'(0, c)]} \]

\[ \mu_2(c) = -\phi(0, c)\phi'(0, c)v_1(c), \]

with \( \rho_1 = c - u(0) \) and

\[ J(c) = \frac{-u'(y_c)\rho(c)}{\phi(0, c)\phi'(0, c)} = \frac{u'(y_c)(u(1) - c)}{\phi(0, c)\phi'(0, c)}. \]

Let

\[ \Lambda_3(\omega_e)(c) = -\rho_1 \left( A(c)\omega_e(y_c) + u''(y_c)\rho\Pi_1(\omega_e) \right) + J \left( \frac{u''(y_c)}{u'(y_c)}E_e(c) + \omega_e(y_c) \right) \]

\[ = -\rho_1(c)\Lambda_1(\omega_e) + \Lambda_{3,1}(\omega_e), \]

\[ \Lambda_4(g)(c) = -\rho_1 \left( Ag(y_c) + \rho\Pi_1(gu'') \right) + J \left( \frac{E(gu'')}{u'(y_c)} + g(y_c) \right) \]

\[ = -\rho_1(c)\Lambda_2(g) + \Lambda_{4,1}(g), \]

with

\[ \Lambda_{3,1}(\omega_e)(c) = J(c) \left( \frac{u''(y_c)}{u'(y_c)}E(\omega_e(c) + \omega_e(y_c)) \right), \]

\[ \Lambda_{4,1}(g)(c) = J(c) \left( \frac{E(gu'')(c)}{u'(y_c)} + g(y_c) \right). \]
Thus, we get
\[ K_e(c, \alpha) = \frac{\Lambda_3(\hat{\omega}_c(c)\Lambda_4(g)(c))}{u'(y_c)((-\rho_1 A + J)^2 + \rho_1^2 B^2)(c)} = \frac{\Lambda_3(\hat{\omega}_c(c)\Lambda_4(g)(c))}{u'(y_c)(A_2^2 + B_2^2)(c)}. \] (9.4)

### 9.3 Decay Estimates and Scattering

The decay estimates are based on the following regularity estimates of the kernel.

**Proposition 9.1** Assume that \( f = g'' - \alpha^2 g \) with \( g \in H^2(0, 1) \cap H_0^1(0, 1) \) and \( \hat{\omega}_o(\alpha, y) = \frac{1}{2}(\hat{\omega}_0(\alpha, y) - \hat{\omega}_0(\alpha, -y)) \in H^2(0, 1) \). Then it holds that
\[ K_o(u(0), \alpha) = K_o(u(1), \alpha) = 0, \]
and there exists a constant \( C \) independent of \( \alpha \) so that
\[
\begin{align*}
\| K_o(\cdot, \alpha) \|_{L_c^1} & \leq C \| \hat{\omega}_o(\alpha, \cdot) \|_{L_2^1} \| g \|_{L_2^2}, \\
\| (\partial_c K_o)(\cdot, \alpha) \|_{L_c^1} & \leq C \| \hat{\omega}_o(\alpha, \cdot) \|_{H_1^1} \| g \|_{H^1}, \\
\| (\partial_c^2 K_o)(\cdot, \alpha) \|_{L_c^1} & \leq C \alpha^2 \| \hat{\omega}_o(\alpha, \cdot) \|_{H_2^2} \| f \|_{L_2^2}.
\end{align*}
\]

**Proposition 9.2** Assume that \( f = g'' - \alpha^2 g \) with \( g \in H^2(0, 1) \) and \( g'(0) = g(1) = 0 \), and \( \hat{\omega}_e(\alpha, y) = \frac{1}{2}(\hat{\omega}_0(\alpha, y) + \hat{\omega}_0(\alpha, -y)) \in H^2(0, 1) \). Then we have
\[ K_e(u(0), \alpha) = K_e(u(1), \alpha) = 0, \]
and there exists a constant \( C \) independent of \( \alpha \) such that
\[
\begin{align*}
\| K_e(\cdot, \alpha) \|_{L_c^1} & \leq C \| \hat{\omega}_e(\alpha, \cdot) \|_{L_2^1} \| g \|_{L_2^2}, \\
\| (\partial_c K_e)(\cdot, \alpha) \|_{L_c^1} & \leq C \alpha^2 \| \hat{\omega}_e(\alpha, \cdot) \|_{H_2^2} (\| g' \|_{L_2^2} + \alpha \| g \|_{L_2^2}), \\
\| (\partial_c^2 K_e)(\cdot, \alpha) \|_{L_c^1} & \leq C \alpha^3 \| \hat{\omega}_e(\alpha, \cdot) \|_{H_3^2} \| f \|_{L_2^2}.
\end{align*}
\]

With Propositions 9.1 and 9.2, we are in a position to prove Theorem 1.2. Using Propositions 9.1 and 9.2, we get by integration by parts that
\[
\begin{align*}
\| \hat{\psi}_o(t, \alpha, \cdot) \|_{L_2^2} & = 2 \sup_{\| f \|_{L_2^1} = 1} \left| \int_0^1 \hat{\psi}_o(t, \alpha, y) f(y) dy \right| \\
& = 2 \sup_{\| f \|_{L_2^1} = 1} \left| \int_{u(0)}^{u(1)} K_o(c, \alpha) e^{-i\alpha ct} dc \right| \\
& \leq C \alpha^{\frac{1}{2}} \| \hat{\omega}_o(\alpha, \cdot) \|_{H_2^2},
\end{align*}
\]
and

\[ \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y} = 2 \sup_{\Vert f \Vert_{L^2_y} = 1} \left| \int_0^1 \hat{\psi}_e(t, \alpha, y) f(y) \, dy \right| \]

\[ = 2 \sup_{\Vert f \Vert_{L^2_y} = 1} \left| \int_{u(0)}^{u(1)} K_e(c, \alpha) e^{-iact} \, dc \right| \]

\[ \leq C \frac{1}{\alpha^2 t^2} \Vert \hat{\omega}_o(\alpha, \cdot) \Vert_{H^2_y}. \]

Similarly, we have

\[ \alpha^2 \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}^2 + \Vert \partial_y \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}^2 \]

\[ = -2 \int_0^1 \hat{\psi}_e(t, \alpha, y) (\psi_e'' - \alpha^2 \psi_e)(t, \alpha, y) \, dy \]

\[ \leq \begin{cases} 
C \Vert \hat{\omega}_o(\alpha, \cdot) \Vert_{L^2_y} \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}, \\
C \frac{1}{\alpha t} \Vert \hat{\omega}_o(\alpha, \cdot) \Vert_{H^1_y} \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{H^1_y},
\end{cases} \]

and

\[ \alpha^2 \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}^2 + \Vert \partial_y \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}^2 \]

\[ = -2 \int_0^{\pi} \hat{\psi}_e(t, \alpha, y) (\psi_e'' - \alpha^2 \psi_e)(t, \alpha, y) \, dy \]

\[ \leq \begin{cases} 
C \Vert \hat{\omega}_e(\alpha, \cdot) \Vert_{L^2_y} \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}, \\
C \frac{1}{|\alpha|^2 |t|} \Vert \hat{\omega}_e(\alpha, \cdot) \Vert_{H^1_y} (|\alpha| \Vert \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y} + \Vert \partial_y \hat{\psi}_e(t, \alpha, \cdot) \Vert_{L^2_y}).
\end{cases} \]

Let

\[ V_o = \nabla^\perp \psi_o, \quad V_e = \nabla^\perp \psi_e. \]

Thus, we deduce that for \( t \leq 1, \)

\[ \| \hat{V}_o(t, \alpha, \cdot) \|_{L^2_y} \leq \alpha \| \hat{\psi}_o(t, \alpha, \cdot) \|_{L^2_y} + \| \partial_y \hat{\psi}_o(t, \alpha, \cdot) \|_{L^2_y} \]

\[ \leq C \alpha^{-1} \| \hat{\omega}_o(\alpha, \cdot) \|_{L^2_y}, \]

\[ \| \hat{V}_e(t, \alpha, \cdot) \|_{L^2_y} \leq \alpha \| \hat{\psi}_e(t, \alpha, \cdot) \|_{L^2_y} + \| \partial_y \hat{\psi}_e(t, \alpha, \cdot) \|_{L^2_y} \]

\[ \leq C \alpha^{-1} \| \hat{\omega}_e(\alpha, \cdot) \|_{L^2_y}, \]

and for \( t \geq 1, \)
\[
\| \hat{V}_o(t, \alpha, \cdot) \|_{L^2_y} \leq \alpha \| \hat{\psi}_o(t, \alpha, \cdot) \|_{L^2_y} + \| \partial_y \hat{\psi}_o(t, \alpha, \cdot) \|_{L^2_y} \\
\leq C \frac{1}{\alpha t} \| \hat{\omega}_o(\alpha, \cdot) \|_{H^1_y}, \\
\| \hat{V}_e(t, \alpha, \cdot) \|_{L^2_y} \leq \alpha \| \hat{\psi}_e(t, \alpha, \cdot) \|_{L^2_y} + \| \partial_y \hat{\psi}_e(t, \alpha, \cdot) \|_{L^2_y} \\
\leq C \frac{1}{\alpha \varepsilon t} \| \hat{\omega}_e(\alpha, \cdot) \|_{H^1_y}.
\]

Thanks to \( V = V_o + V_e \), we get
\[
\| V(t) \|_{L^2_x,y} \leq \frac{C}{\langle t \rangle} \| \omega_0 \|_{H^{-\frac{1}{2}}_x H^1_y}.
\]

Using \( \| \hat{V}^2 \|_{L^2(I_0)} \leq C \alpha \| \hat{\psi} \|_{L^2(I_0)} \leq C \frac{1}{\varepsilon^2} \| \hat{\omega}_0 \|_{H^2(I_0)} \), we get
\[
\| V^2(t) \|_{L^2_x,y} \leq \frac{C}{\langle t \rangle^2} \| \omega_0 \|_{H^1_x H^2_y}.
\]

The scattering part is the same as the monotonic case in [31]. Thus, we omit the details.

The remaining sections of this paper will be devoted to the proof of Propositions 9.1 and 9.2.

10 Estimates of Some Key Quantities

In this section, we present some estimates for some key quantities like \( A, B \) etc. appeared in \( K_o \) and \( K_e \).

In the sequel, we denote by \( \phi(y, c) \) the solution of the homogeneous Rayleigh equation given by Proposition 4.7 and \( \phi_1(y, c) = \frac{\phi(y, c)}{u(y)-c} \).

Let \( I_v = (-v(1), v(1)) \) with \( v \) given by (4.4) and \( \rho_0(c) = \rho(c)/u'(y_v) \sim y_v(1-y_v) \). We denote by \( \| \cdot \|_{L^p} \) the norm of \( L^p(I_v, d\tilde{c}) \), and \( C \) a constant independent of \( \alpha \), which may be different from line to line.

10.1 Estimate of II_3

Recall that
\[
II_3(c) = \int_0^1 \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) dy \leq 0.
\]

We have following estimates for \( II_3 \).
Lemma 10.1 It holds that

\[
C^{-1} \min \left\{ \frac{\alpha^2}{u'(y_c)}, \frac{\alpha}{u'(y_c)^2} \right\} \leq -\Pi_3(c) \leq C \min \left\{ \frac{\alpha^2}{u'(y_c)}, \frac{\alpha}{u'(y_c)^2} \right\},
\]

\[
|\rho(c)^k \partial^k_c \Pi_3(c)| \leq C \min \left\{ \frac{\alpha^2}{u'(y_c)}, \frac{\alpha}{u'(y_c)^2} \right\}, \quad k = 1, 2.
\]

**Proof** By Proposition 4.14, we have

\[
\phi_1(y, c) \geq 1, \quad \phi_1(y, c) - 1 \leq C \min\{\alpha^2|y - y_c|^2, 1\} \phi_1(y, c),
\]

which gives

\[
-\Pi_3(c) = \int_0^1 \frac{1}{(u(y) - c)^2} \left(1 - \frac{1}{\phi_1(y, c)^2}\right) dy
\]

\[
\geq C^{-1} \int_0^1 \frac{\alpha^2|y - y_c|^2}{(u(y) - c)^2} \chi_{\{|y - y_c| \leq \frac{1}{\alpha}\}} dy
\]

\[
\geq \begin{cases} 
C^{-1} \int_0^{\frac{1}{\alpha}} \frac{\alpha^2|y - y_c|^2}{u'(y_c)^2|y - y_c|^2} dy \geq \frac{\alpha^2}{Cu'(y_c)^2}, & 0 < y_c \leq \frac{1}{\alpha}, \\
C^{-1} \int_{\frac{1}{\alpha} - y_c}^{\frac{1}{\alpha}} \frac{\alpha^2|y - y_c|^2}{u'(y_c)^2|y - y_c|^2} dy \geq \frac{\alpha}{Cu'(y_c)^2}, & \frac{1}{\alpha} \leq y_c < 1,
\end{cases}
\]

and

\[
-\Pi_3(c) = \int_0^1 \frac{1}{(u(y) - c)^2} \left(1 - \frac{1}{\phi_1(y, c)^2}\right) dy
\]

\[
\leq C \int_0^1 \frac{\alpha^2|y - y_c|^2}{(u(y) - c)^2} dy \leq C \alpha^2 \int_0^1 \frac{1}{(y + y_c)^2} dy \leq C \frac{\alpha^2}{u'(y_c)^2},
\]

or

\[
-\Pi_3(c) = \int_0^1 \frac{1}{(u(y) - c)^2} \left(1 - \frac{1}{\phi_1(y, c)^2}\right) dy
\]

\[
\leq C \int_0^1 \min\{\alpha^2|y - y_c|^2, 1\} \frac{1}{u'(y_c)^2|y - y_c|^2} dy \leq C \frac{\alpha}{u'(y_c)^2}.
\]

This proves the first point of the lemma.

Direct calculation gives

\[
\partial_c \Pi_3(c) = \partial_c \int_0^1 \frac{1}{(u(y) - c)^2} \left(\frac{1}{\phi_1(y, c)^2} - 1\right) dy
\]

\[
= \int_0^1 \left( \frac{\partial_y}{u'(y_c)} + \partial_c \right) \left(\frac{1}{(u(y) - c)^2} \left(\frac{1}{\phi_1(y, c)^2} - 1\right)\right) dy
\]

\[
- \left. \frac{1}{u'(y_c)(u(y) - c)^2} \left(\frac{1}{\phi_1(y, c)^2} - 1\right) \right|_{y = 0}
\]
and
\[
\partial^2_c \Pi_3(c) = \int_0^1 \left( \frac{\partial y}{u'(y_c)} + \frac{\partial c}{c} \right)^2 \left( \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right) dy
- \frac{1}{u'(y_c)} \left( \frac{\partial y}{u'(y_c)} + \frac{\partial c}{c} \right) \left( \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right) \bigg|_{y=0}^1,
- \partial_c \left( \frac{1}{u'(y_c)(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right) \bigg|_{y=0}^1.
\]

By Proposition 4.14 and Lemma 4.1, we have
\[
\left| \left( \frac{\partial y}{u'(y_c)} + \frac{\partial c}{c} \right) \left( \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right) \right| \leq C \frac{\min\{\alpha^2|y - y_c|^2, 1\}}{u'(y_c)(u(y) - c)^2},
\]
\[
\left| \left( \frac{\partial y}{u'(y_c)} + \frac{\partial c}{c} \right)^2 \left( \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right) \right| \leq C \frac{\min\{\alpha^2, \alpha/u'(y_c)\}}{u'(y_c)|u(y) - c|}.
\]

On the other hand, we have for \( y = 0, 1, \)
\[
\left| \frac{1}{u'(y_c)(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right| \leq C \frac{\min\{\alpha^2|y - y_c|^2, 1\}}{u'(y_c)(u(y) - c)^2}
\]
\[
\leq C \frac{\min\{\alpha^2, \alpha/u'(y_c)\}}{u'(y_c)|u(y) - c|},
\]
and for \( y = 0, 1, \)
\[
\left| \frac{1}{u'(y_c)(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) \right| \leq C \left( \frac{1}{u'(y_c)^3(u(y) - c)^2 + u'(y_c)|u(y) - c|^3} \right) \left( \frac{1}{\phi_1(y, c)^2} - 1 \right)
\]
\[
+ \frac{2}{u'(y_c)(u(y) - c)^2} \frac{G(y, c)}{\phi_1(y, c)^2}
\]
\[
\leq C \left( \frac{\min\{\alpha^2|y - y_c|^2, 1\}}{u'(y_c)^3(u(y) - c)^2 + u'(y_c)|u(y) - c|^3} + \frac{\alpha \min\{\alpha|y - y_c|, 1\}}{u'(y_c)^2(u(y) - c)^2} \right)
\]
\[
\leq C \left( \frac{\min\{\alpha^2|y - y_c|^2, 1\}}{u'(y_c)|u(y) - c|\rho(c)^2} + \frac{\alpha \min\{\alpha|y - y_c|, 1\}}{u'(y_c)^2(u(y) - c)^2} \right)
\]
\[
\leq C \frac{\min\{\alpha^2, \alpha/u'(y_c)\}}{\rho(c)^2} \min \left\{ \frac{\alpha^2}{u'(y_c)}, \frac{\alpha}{u'(y_c)^2} \right\},
\]
here \( G(y, c) = \frac{\partial_c \phi_1(y, c)}{\phi_1(y, c)} \) and we used (4.33). Then the second inequality follows easily. \( \square \)

### 10.2 Estimate of \( A_1 \)

Recall that

\[
A_1(c) = \rho(c)u'(y_c)\partial_c \left( p.v. \int_0^1 \frac{dy}{u(y) - c} \right) = -\rho(c)u'(y_c)\partial_c \left( \frac{1}{2c} H \left( (v^{-1})' \chi_I(c) \right) \right).
\]

Here we made the change of variable \( \tilde{c} = v(y_c) \in I_v \).

**Lemma 10.2** It holds that for any \( p \in (1, +\infty) \),

\[
|A_1(c)| \leq C\tilde{c}^2, \quad |\partial_c A_1(c)| \leq C(\ln(u(1) - c)| + 1), \quad \|\rho\partial_c^2 A_1\|_{L_p^p} \leq C.
\]

**Proof** Let \( f_2(c) = \left((v^{-1})'(\tilde{c}) - \sqrt{\frac{2}{u''(0)}}\chi_I(c)\right) \). Here \( (v^{-1})'(0) = \frac{1}{v'(0)} \geq \sqrt{\frac{2}{u''(0)}} > 0 \). By Lemma 4.1, we know that \( f_2 \in C^3([-v(1), 0) \cup (0, v(1)]) \cap C^2(I_v) \) and \( zf_2(z) \in C^3(I_v) \). Then we have

\[
A_1 = -\rho(c)u'(y_c)\partial_c \left( \frac{1}{2\tilde{c}} H(f_2)(\tilde{c}) + \frac{1}{\tilde{c}} \sqrt{u''(0)} \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \right)
\]

\[= I_1 + I_2,
\]

where

\[
I_1 = -\rho(c)u'(y_c)\partial_c \left( \frac{1}{2\tilde{c}} H(f_2)(\tilde{c}) \right),
\]

\[
I_2 = \sqrt{\frac{1}{8u''(0)}} \tilde{c}(u(1) - c)u'(y_c)\partial_c \left( \frac{1}{\tilde{c}} \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \right).
\]

Let \( \eta(\tilde{c}) \geq 0 \) be a smooth even function, such that \( \eta(\tilde{c}) = 1 \) for \( |\tilde{c}| \leq \frac{v(1)}{4} \) and \( \eta(\tilde{c}) = 0 \) for \( |\tilde{c}| \geq \frac{3v(1)}{4} \). Let \( \eta_1(\tilde{c}) = \chi_I(\tilde{c}) - \eta(\tilde{c}) \). Using the fact that if \( f(\tilde{c}) \) is odd, then \( \partial_c(f(\tilde{c})) = \partial_c \left( \int_0^1 f'(t\tilde{c})dt \right) = f''(t\tilde{c})dt \) is odd, and \( u'(y_c) = 2\tilde{c}v'(y_c) \), we infer that

\[
|\eta(\tilde{c})I_2| \leq C\tilde{c}^3, \quad |\eta(\tilde{c})\partial_c I_2| + |\tilde{c}\eta(\tilde{c})\partial_c^2 I_2| \leq C. \quad (10.1)
\]

On the other hand, we have
Recall that\[ \eta_1(\tilde{c})(u(1) - c) \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \leq C \eta_1(\tilde{c}), \]
\[ \eta_1(\tilde{c})(u(1) - c) \left| \partial \tilde{c} \left( \frac{1}{\tilde{c}} \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \right) \right| \leq C \eta_1(\tilde{c}), \]
which along with (10.1) give\[ |I_2| \leq |\eta(\tilde{c})I_2| + |\eta_1(\tilde{c})I_2| \leq C|\tilde{c}|^3. \]
Using (10.1) and the facts that
\[ \tilde{c}(u(1) - c)u'(y_c)\partial \tilde{c} \left( \frac{1}{\tilde{c}} \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \right) \]
\[ = -2(u(1) - c)v'(y_c)\ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| - 2u'(y_c)v(1), \]
and
\[ \left| \eta_1(\tilde{c})v'(y_c) \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \right| \leq C|\ln(u(1) - c)| + C, \]
\[ \left| \eta_1(\tilde{c}) \frac{u(1) - c}{u'(y_c)} \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| \right| \leq C, \]
we infer that\[ |\partial_c I_2| \leq C(|\ln(u(1) - c)| + 1), \quad |\tilde{c}(u(1) - c)\partial_c^2 I_2| \leq C. \]
The estimates of $I_1$ are a direct consequence of Proposition 13.3. \hfill \Box

10.3 Estimate of $A^2 + B^2$

Recall that
\[ A(c) = A_1(c) + u'(y_c)\rho(c)\Pi_3(c), \quad B(c) = \pi\rho(c) \frac{u''(y_c)}{u'(y_c)^2}. \]

Lemma 10.3 It holds that
\[ C^{-1} (1 + \alpha\rho_0(c))^2 \leq A(c)^2 + B(c)^2 \leq C (1 + \alpha\rho_0(c))^2, \]
and
\[ \left| \partial_c \left( \frac{1}{(A^2 + B^2)u'(y_c)} \right) \right| \leq \frac{C (1 + \alpha u'(y_c))}{(1 + \alpha\rho_0)^3 u'(y_c)^3} \quad + \quad \frac{C (1 + |\ln(1 - y_c)|)}{(1 + \alpha\rho_0)^3 u'(y_c)}, \]
\[ \left| \partial_c^2 \left( \frac{1}{(A^2 + B^2)u'(y_c)} \right) \right| \leq \frac{C (1 + \alpha^2 u'(y_c)^2)}{(1 + \alpha\rho_0)^4 u'(y_c)^5} \quad + \quad \frac{C (1 + |\ln(1 - y_c)|)^2}{(1 + \alpha\rho_0)^4 u'(y_c)^5}. \]
Proof By Lemma 10.1 and Lemma 10.2, we get

\[ |A(c)| \leq C\gamma_c^2 + C(u(1) - c)\alpha y_c, \quad |B(c)| \leq C(u(1) - c), \]

which imply that

\[ |A(c)| + |B(c)| \leq C(1 + \alpha \rho_0(c)). \]

Now we prove the lower bound of \( A^2 + B^2 \). Proposition 5.3 ensures that for any fixed \( M > 0 \) and \( \alpha \leq M \), \( A(c)^2 + B(c)^2 \geq C^{-1} \). Thus, we may assume \( \alpha \gg 1 \). If \( \alpha y_c \leq 1 \), then \( |B(c)| \geq C^{-1} \). Since \( A_1(u(1)) = u(0) - u(1) < 0 \) (by 5.10) and \( A_1(c) \) is continuous, \( A_1(c) < \frac{u(0) - u(1)}{2} \) for \( y_c \in [1 - C/\alpha, 1] \). Thus, by \( \Pi_3 < 0 \), for \( y_c \in [1 - C/\alpha, 1] \),

\[ |A(c)| = -A_1(c) - u'(y_c)\rho\Pi_3 \geq \frac{u(1) - u(0)}{2}. \]

For \( y_c \in [0, 1 - C/\alpha] \) and \( \alpha y_c \geq 1 \), we get by Lemma 10.1 and Lemma 10.2 that

\[ |A(c)| \geq -u'(y_c)\rho\Pi_3 - |A_1| \geq C^{-1}_1\alpha(1 - y_c)y_c - C^{-1}_2\gamma_c^2 \geq \frac{1}{2}C^{-1}_1\alpha\rho_0(c). \]

Thus, we deduce that \( |A(c)| + |B(c)| > C^{-1}(1 + \alpha \rho_0(c)) \).

Notice that

\[ u'(y_c) = 2v(y_c)v'(y_c), \quad u''(y_c) = 2v'(y_c)^2 + 2v(y_c)v'(y_c), \]

which gives

\[ \rho(c) \frac{u''(y_c)}{u'(y_c)^2} = \frac{1}{2} \left( u(1) - c + \frac{(v(1)^2 - \tilde{c}^2)v''(y_c)}{v'(y_c)^2} \right). \]

(10.3)

Using the facts that \( \frac{\partial}{\partial c} \left( \frac{1}{v'(y_c)^2} \right) = -\frac{v''(y_c)}{v'(y_c)^3}, |v''(y_c)| \leq C|y_c|, \) and

\[ |\partial_c((v(1)^2 - \tilde{c}^2)v''(y_c))| \leq |(v(1)^2 - \tilde{c}^2)v''(y_c)| + 2\tilde{c}^2v''(y_c)| + |(v(1)^2 - \tilde{c}^2)v''(y_c)/v'(y_c)| \leq C|\tilde{c}|, \]

we infer that \( |\partial_c B(c)| \leq C \). Similarly, \( |\partial_c B| \leq C \).

A direct calculation gives
\[
\left| \frac{1}{(A^2 + B^2)u'(y_c)} \right| \\
\leq C \frac{|A\partial_c A + B\partial_c B|}{(A^2 + B^2)^2 u'(y_c)} + \frac{C}{(A^2 + B^2)u'(y_c)^3} \\
\leq C(|\partial_c A_1| + |\partial_c B| + |\partial_c (u'(y_c)\rho\Pi_3)|) \leq \frac{C}{(A^2 + B^2)^{3/2}u'(y_c)} + \frac{C}{(A^2 + B^2)u'(y_c)^3},
\]

and

\[
\left| \frac{1}{(A^2 + B^2)u'(y_c)} \right| \\
\leq \frac{C}{(A^2 + B^2)u'(y_c)^5} + \frac{C(|A\partial_c A| + |B\partial_c B|)}{(A^2 + B^2)^2 u'(y_c)^3} + \frac{C(|\partial_c A|^2 + |\partial_c B|^2)}{(A^2 + B^2)^2 u'(y_c)} + \frac{C(|\partial_c A|^2 + |B\partial_c B|^2)}{(A^2 + B^2)^2 u'(y_c)^5} \\
\leq \frac{C}{(A^2 + B^2)u'(y_c)^5} + \frac{C(|\partial_c A|^2 + |\partial_c B|^2)}{(A^2 + B^2)^2 u'(y_c)^3} + \frac{C(|\partial_c A|^2 + |B\partial_c B|^2)}{(A^2 + B^2)^2 u'(y_c)^5} \\
\leq \frac{C}{(A^2 + B^2)u'(y_c)^5} + \frac{C|\partial_c A|^2}{(A^2 + B^2)^2 u'(y_c)} + \frac{C|\partial_c A|^2}{(A^2 + B^2)^2 u'(y_c)^5}.
\]

Thus, by Lemmas 10.1 and 10.2, we obtain

\[
\left| \frac{1}{(A^2 + B^2)u'(y_c)} \right| \\
\leq \frac{C((A^2 + B^2)^{1/2} + u'(y_c) + \alpha u'(y_c))}{(A^2 + B^2)^{3/2}u'(y_c)^3} + \frac{C|\partial_c A_1|}{(A^2 + B^2)^{3/2}u'(y_c)} \\
\leq \frac{C(1 + \alpha u'(y_c))}{(1 + \alpha \rho_0)^3 u'(y_c)^3} + \frac{C(1 + |\ln(u(1) - c)|)}{(1 + \alpha \rho_0)^3 u'(y_c)}.
\]

and using the facts that

\[
|\partial_c A|^2 \leq C(|\partial_c A_1|^2 + |\partial_c (u'(y_c)\rho\Pi_3)|^2),
\]

\[
|\partial_c A|^2 \leq C(|\partial_c A_1|^2 + |\partial_c (u'(y_c)\rho\Pi_3)|),
\]

we deduce that

\[
\partial_c^2 \left( \frac{1}{(A^2 + B^2)u'(y_c)} \right) \leq \frac{C(1 + \alpha^2 u'(y_c)^2)}{(1 + \alpha \rho_0)^4 u'(y_c)^5} + \frac{C\alpha}{(1 + \alpha \rho_0)^3 \rho u'(y_c)} + \frac{C(1 + |\ln(u(1) - c)|)^2}{(1 + \alpha \rho_0)^4 u'(y_c)} + \frac{C|\rho \partial_c^2 A_1|}{(1 + \alpha \rho_0)^3 \rho u'(y_c)}.
\]

This completes the proof of the lemma. \qed
10.4 Estimate of $A_2^2 + B_2^2$ and $J$

Recall that

$$A_2(c) = (u(0) - c)A(c) + J(c), \quad B_2(c) = (u(0) - c)B(c),$$

$$J(c) = \frac{u'(y_c)(u(1) - c)}{\phi_1(0, c)\phi'_1(0, c)}.$$

**Lemma 10.4** It holds that

$$|J(c)| \leq \frac{C(1 - y_c)}{\phi_1(0, c)\alpha^2}, \quad |\partial_c J(c)| \leq \frac{C}{\phi_1(0, c)\alpha^2 y_c^2} \text{ and } |\partial_c^2 J(c)| \leq \frac{C}{\phi_1(0, c)\alpha^2 y_c^4}.$$

**Proof** We get by Lemma 4.15 that

$$|J(c)| \leq \frac{C y_c(1 - y_c)}{\phi_1(0, c)\alpha^2 \min\{\alpha y_c, 1\} \phi_1(0, c)\alpha^2} \leq \frac{C(1 - y_c)}{\phi_1(0, c)\alpha^2}.$$

Recall that $F = \frac{\partial y}{\phi_1}$ and $G = \frac{\partial_c \phi_1}{\phi_1}$ defined by 4.17. A direct calculation gives

$$\frac{1}{\phi_1 \partial_y \phi_1}(0, c) = \frac{1}{\phi_1(0, c)^2 F(0, c)},$$

$$\partial_c \left( \frac{1}{\phi_1 \partial_y \phi_1} \right)(0, c) = -\frac{\partial_c F}{\phi_1^2 F^2}(0, c) - 2\frac{G}{\phi_1^2 F}(0, c),$$

$$\partial_c^2 \left( \frac{1}{\phi_1 \partial_y \phi_1} \right)(0, c) = \frac{-2\partial_c G + 4G^2}{\phi_1^2 F}(0, c) + \frac{4G\partial_c F}{\phi_1 F^2}(0, c) - \frac{\partial_c^2 F}{\phi_1^3 F^2}(0, c) + \frac{2[\partial_c F]^2}{\phi_1^3 F^3}(0, c),$$

from which and Lemma 4.15, we infer that

$$\left| \frac{1}{\phi_1(0, c)^2 F(0, c)} \right| \sim \frac{1 + \alpha y_c}{\alpha^2 \phi_1(0, c)^2 y_c},$$

$$\left| \partial_c \left( \frac{1}{\phi_1 \phi'_1} \right)(0, c) \right| \leq \frac{C(1 + \alpha y_c)^2}{\alpha^2 \phi_1(0, c)^2 y_c^3},$$

$$\left| \partial_c^2 \left( \frac{1}{\phi_1 \phi'_1} \right)(0, c) \right| \leq \frac{C(1 + \alpha y_c)^3}{\alpha^2 \phi_1(0, c)^2 y_c^5}.$$

Therefore, we obtain
\[
\frac{\partial_c J(c)}{\phi_1\phi_1'(0,c)|u'(y_c)|} + \frac{1}{\phi_1\phi_1'(0,c)} \frac{\partial_c}{\partial_c} \leq \frac{C}{\phi_1(0,c) \alpha^2 y_c^2},
\]

\[
|\partial_c^2 J(c)| \leq C \sum_{k=0}^{2} u'(y_c)^{2k-3} \left| \frac{\partial_c^k}{\phi_1\phi_1'(0,c)} \right| \leq \frac{C}{\phi_1(0,c) \alpha^2 y_c^4}.
\]

This proves the lemma. \qed

**Lemma 10.5** It holds that

\[
C^{-1} \frac{(1 + \alpha \rho_0)^2(1 + \alpha y_c)^4}{\alpha^4} \leq A_2(c)^2 + B_2(c)^2 \leq C \frac{(1 + \alpha \rho_0)^2(1 + \alpha y_c)^4}{\alpha^4},
\]

and

\[
\left| \frac{1}{(A_2^2 + B_2^2) u'(y_c)} \right| \leq \frac{C\alpha^4}{(1 + \alpha \rho_0)^3(1 + \alpha y_c)^3 u'(y_c)^3} + \frac{C\alpha^4(1 + |\ln(1 - y_c)|)}{(1 + \alpha \rho_0)^3(1 + \alpha y_c)^4 u'(y_c)},
\]

\[
\left| \frac{1}{(A_2^2 + B_2^2) u'(y_c)} \right| \leq \frac{C\alpha^4}{(1 + \alpha \rho_0)^4(1 + \alpha y_c)^2 u'(y_c)^5} + \frac{C\alpha^4(|\ln(1 - y_c)|^2 + |\rho \partial_c^2 A_1| + \alpha^2)}{(1 + \alpha \rho_0)^3(1 + \alpha y_c)^4 \rho u'(y_c)}.
\]

**Proof** We get by Lemma 10.4 and (10.2) that

\[
|J(c)| \leq \frac{C}{\alpha^2}, \quad |(u(0) - c) A(c)| \leq C y_c^2 (1 + \alpha \rho_0), \quad |(u(0) - c) B(c)| \leq C y_c^2,
\]

which gives

\[
|A_2(c)| + |B_2(c)| \leq \frac{C}{\alpha^2} + C y_c^2 (1 + \alpha \rho_0) \leq \frac{C(1 + \alpha \rho_0)(1 + \alpha y_c)^2}{\alpha^2}.
\]

Proposition 5.3 ensures that for some fixed \(M\), \(|A_2(c)| + |B_2(c)| \geq C^{-1}\) for \(\alpha \leq M\).

Thus, we may assume that \(\alpha \gg 1\). Let \(\delta_1\) be a small constant. For \(0 \leq y_c \leq \frac{\delta_1}{\alpha}\), we have

\[
|J(c)| \geq \frac{y_c}{C\alpha \min\{\alpha y_c, 1\}} \geq \frac{1}{C\alpha^2},
\]

which gives for \(0 \leq y_c \leq \frac{\delta_1}{\alpha}\).
We known from the proof of Lemma 10.3 that
\[ \delta \]
For \( \alpha \) we have \( |B(c)| \geq 1/C \), since \( \lim_{c \to u(0)+} B(c) = \frac{\pi}{2}(u(1) - u(0)) > 0 \) by (10.3), hence,
\[ |B_2(c)| \geq \frac{|u(0) - c|}{C} \geq \frac{y_c^2}{C} \geq \frac{(1 + \alpha \rho_0)(1 + \alpha y_c)^2}{C \alpha^2}. \]

For \( \frac{1}{\delta_1 \alpha} \leq y_c \leq 1 \), we have
\[ |A_2(c)| \geq |(u(0) - c)A(c)| - |J(c)| \]
\[ \geq \frac{y_c^2(y_c + \alpha \rho_0)}{C} - \frac{C}{\alpha^2} \geq \frac{y_c^2(y_c + \alpha \rho_0)}{C} \geq \frac{(1 + \alpha \rho_0)(1 + \alpha y_c)^2}{C \alpha^2}. \]

Here we used the fact that \( |A(c)| \geq C^{-1}(1 + \alpha \rho_0(c)) \) for \( \alpha y_c \geq 1 \) (see the proof of Lemma 10.3). Summing up, we conclude the lower bound of \( |A_2(c)| + |B_2(c)| \).

Using Lemma 10.2, Lemma 10.1 and Lemma 10.4, we deduce that
\[ |\partial_c A_2(c)| \leq |\partial_c((u(0) - c)A_1)| + |\partial_c((u(0) - c)u'(y_c))| + |\partial_c J| \]
\[ \leq C y_c^2(1 + |\ln(1 - y_c)|) + C \alpha y_c + C(\alpha^2 y_c^2)^{-1} \]
\[ \leq C y_c^2(1 + |\ln(1 - y_c)|) + C(1 + \alpha^3 y_c^3)(\alpha^2 y_c^2)^{-1}, \]
\[ |\partial_c^2 A_2(c)| \leq |\partial_c((u(0) - c)A_1)| + |\partial_c^2((u(0) - c)u'(y_c))| + |\partial_c^2 J| \]
\[ \leq C(1 - y_c)^{-1} + C \alpha^2 y_c + C \alpha \rho_0^{-1} + C(\alpha^2 y_c^4)^{-1} \]
\[ \leq C(1 - y_c)^{-1} + C \alpha^2 y_c + C \alpha \rho_0^{-1} + C(\alpha^2 y_c^4)^{-1}. \]

We known from the proof of Lemma 10.3 that
\[ |B_2(c)| \leq Cy_c^2, \quad |\partial_c B_2(c)| \leq C, \quad |\partial_c^2 B_2(c)| \leq C. \]

With these estimates, we can deduce that
\[ \left| \partial_c \left( \frac{1}{(A_2^2 + B_2^2)u'(y_c)} \right) \right| \leq \frac{C \alpha^4}{(1 + \alpha \rho_0)^3(1 + \alpha y_c)^3 u'(y_c)^3} \]
\[ + \frac{C \alpha^4(1 + |\ln(1 - y_c)|)}{(1 + \alpha \rho_0)^3(1 + \alpha y_c^4) u'(y_c)^3}. \]
and

$$
\left| \partial^2_c \left( \frac{1}{(A_2^2 + B_2^2)u'(y_c)} \right) \right| \leq \frac{C}{(A_2^2 + B_2^2)u'(y_c)^5} + \frac{C|\partial_c A_2|^2}{(A_2^2 + B_2^2)^2u'(y_c)} + \frac{C|A_2 \partial^2_c A_2|}{(A_2^2 + B_2^2)^2u'(y_c)}
$$

$$
\leq \frac{Ca^4}{(1 + \alpha \rho_0)^2(1 + \alpha \gamma_c)^6u'(y_c)^5} + \frac{Ca^6(1 + |\ln(1 - y_c)|)^2 + (1 + \alpha \gamma_c)^6/(\alpha \gamma_c)^4}{(1 + \alpha \rho_0)^4(1 + \alpha \gamma_c)^8u'(y_c)}
$$

$$
+ \frac{Ca^6(|\partial_c^2 A_1|/(1 - y_c) + \alpha/\rho_0 + 1/(\alpha \gamma_c)^4))}{(1 + \alpha \rho_0)^3(1 + \alpha \gamma_c)^6u'(y_c)}
$$

$$
\leq \frac{Ca^4}{(1 + \alpha \rho_0)^2(1 + \alpha \gamma_c)^2u'(y_c)^5} + \frac{Ca^4(1 + |\ln(1 - y_c)|)^2 + |\partial_c^2 A_1| + \alpha^2}{(1 + \alpha \rho_0)^3(1 + \alpha \gamma_c)^4 \rho u'(y_c)}.
$$

This proves the lemma. \[\square\]

### 11 Estimates of Basic Integral Operators

In this section, we present some estimates for some basic integral operators appeared in $K_o$ and $K_e$.

Let $I_0 = (0, 1)$ and $I = (-1, 1).$ Let $I_v = (-v(1), v(1))$ with $v$ given by (4.4) and $ho_0(v) = \rho(v)/u'(y_c) \sim y_c(1 - y_c).$ We denote by $\| \cdot \|_{L^p_x}$ the norm of $L^p(I_v, d\tilde{v})$, and $C$ a constant independent of $\alpha$, which may be different from line to line.

#### 11.1 Estimates of the Operator $A_{j, 1}$ for $j = 1, 2$

Recall that

$$
II_{1, 1}(\phi)(c) = p.v. \int_0^1 \frac{\text{Int}(\phi)(y) - \text{Int}(\phi)(y_c)}{(u(y) - u(y_c))^2} dy
$$

$$
= \frac{1}{2\tilde{c}} \partial_c \left( \frac{1}{2\tilde{c}} p.v. \int_{-1}^1 \frac{\text{Int}(\phi)(y)}{v(y) - \tilde{v}} dy \right)
$$

$$
- \frac{\text{Int}(\phi)(v^{-1}(-\tilde{c}))}{2\tilde{c}} \partial_c \left( \frac{1}{2\tilde{c}} p.v. \int_{-v(1)}^{v(1)} \frac{(v^{-1})(z)}{z - \tilde{v}} dz \right).
$$

Here $\tilde{v} = v(y_c)$ and

$$
\text{Int}(\phi)(y) = \int_0^y \phi(y') dy' \quad \text{for} \quad y \in [0, 1],
$$

$$
\text{Int}(\phi)(y) = \text{Int}(\phi)(-y) \quad \text{for} \quad y \in [-1, 0].
$$
Thus, \( \text{Int}(\varphi)(y) \) is an even function with \( \text{Int}(\varphi)(0) = 0 \) and satisfies
\[
\| \text{Int}(\varphi) \|_{W^{1,p}(I)} \leq C \| \varphi \|_{L^p(I_0)}, \quad \left\| \frac{1}{y} \text{Int}(\varphi)(y) \right\|_{L^p(I)} \leq C \| \varphi \|_{L^p(I_0)}.
\] (11.1)

Let us also recall that
\[
\Lambda_{1,1}(\varphi)(c) = A_1(c)\varphi(y_c) + \rho(c)u''(y_c)\Pi_{1,1}(\varphi)(c), \\
\Lambda_{2,1}(\varphi)(c) = A_1(c)\varphi(y_c) + \rho(c)\Pi_{1,1}(u''\varphi)(c).
\]

**Lemma 11.1** For \( k = 0, 1, 2 \) and \( j = 1, 2 \), we have
\[
\left\| \partial^k_\zeta \Lambda_{j,1}(\varphi) \right\|_{L^2_\zeta} \leq C \| \varphi \|_{H^k(I_0)}, \\
\left\| \partial^k_\zeta (\Lambda_{j,1}(\varphi)) \right\|_{L^2_\zeta} \leq C \| \varphi \|_{H^k(I_0)}.
\]

If \( \varphi(0) = 0 \), then we have
\[
\left\| \partial^{-1}_\zeta \Lambda_{j,1}(\varphi) \right\|_{L^2_\zeta} \leq C \| \varphi \|_{H^1(I_0)}, \\
\left\| \partial^{-2}_\zeta \Lambda_{j,1}(\varphi) \right\|_{L^2_\zeta} + \left\| \partial_\zeta (\Lambda_{j,1}(\varphi)) \right\|_{L^2_\zeta} \leq C \| \varphi \|_{H^2(I_0)}.
\]

If \( \varphi(1) = 0 \), then we have
\[
\left\| (u(1) - c)^{-1} \Lambda_{j,1}(\varphi) \right\|_{L^2_\zeta} \leq C \| \varphi \|_{H^1(I_0)}.
\]

**Remark 11.2** Using Gagliardo-Nirenberg inequality and Lemma 11.1, we can deduce that
\[
\left\| \partial^2_\zeta (\Lambda_{j,1}(\varphi)) \right\|_{L^2_\zeta} \leq C \left( \left\| \partial^2_\zeta (\Lambda_{j,1}(\varphi)) \right\|_{L^2_\zeta} + \left\| \partial_\zeta (\Lambda_{j,1}(\varphi)) \right\|_{L^2_\zeta} \right) \left\| \partial_\zeta (\Lambda_{j,1}(\varphi)) \right\|_{L^2_\zeta}
\]
\[
\leq C \| \varphi \|_{H^2(I_0)} \| \varphi \|_{H^1(I_0)},
\]
and if \( \varphi(0) = 0 \),
\[
\left\| (u'(y_c))^{-1} \Lambda_{j,1}(\varphi) \right\|_{L^\infty_\zeta} \leq C \| \varphi \|_{H^2(I_0)}.
\]
If \( \varphi(1) = 0 \), due to \( \Lambda_{j,1}(\varphi)(1) = 0 \), we have
\[
\left\| \frac{\Lambda_{j,1}(\varphi)}{1 - y_c} \right\|_{L^\infty} \leq C \left\| \Lambda_{j,1}(\varphi) \right\|_{W^{1,\infty}} \leq C \left\| \Lambda_{j,1}(\varphi) \right\|_{H^1}^{\frac{1}{2}} \left\| \Lambda_{j,1}(\varphi) \right\|_{H^2}^{\frac{1}{2}} \leq C \left\| \varphi \right\|_{H^1}^{\frac{1}{2}} \left\| \varphi \right\|_{H^2}^{\frac{1}{2}}.
\]

**Proof** Let us recall that for \( y_c \in [0, 1] \),
\[
A_1 = -\rho u'(y_c) \partial_c \left( \frac{1}{2c} H((v^{-1})' \chi_{[v^{-1}(1), v(1)]})(\tilde{c}) \right) = -\rho u'(y_c) \partial_c \left( \frac{1}{2c} H(f_2(\tilde{c})) \right),
\]
\[
\Pi_{1,1}(\varphi) = -\partial_c \left( \frac{1}{2c} H(f_1 f_2) \right) + f_1 \partial_c \left( \frac{1}{2c} H(f_2) \right),
\]
and by the proof of Proposition 13.3, we find that
\[
\Lambda_{1,1}(\varphi)(c) = A_1 \varphi(y_c) + \rho(c)u''(y_c) \Pi_{1,1}(\varphi)(c)
\]
\[
= \rho(-u'(y_c)\varphi(y_c) + u''(y_c) f_1(\tilde{c})) \partial_c \left( \frac{1}{2c} H(f_2(\tilde{c})) \right) - \rho(c)u''(y_c) \partial_c \left( \frac{1}{2c} H(f_1 f_2) \right)
\]
\[
= (-u'(y_c)\varphi(y_c) + u''(y_c) f_1(\tilde{c})) \left( \frac{u(1) - c}{4c} H(Z(f_1 f_2)) \right) + \frac{f_2(v(1))v(1)}{2c} H(zZ(f_2))
\]
\[
+ \frac{u(1) - c}{4c} \left( \frac{u''(y_c) H(Z(f_1 f_2))}{\tilde{c}} f_1 f_2(v(1)) v(1) \right)
\]
\[
= \Lambda_{1,1,1}(\varphi) + \Lambda_{1,1,2}(\varphi),
\]
where \( f_1(\tilde{c}) = (\text{Int}(\varphi) \circ v^{-1}) \chi_{[-v(1), v(1)]}(\tilde{c}) \) and \( f_2(\tilde{c}) = (v^{-1})' \chi_{[-v(1), v(1)]}(\tilde{c}) \). Let
\[
\varphi_1(\tilde{c}) = Z(f_1 f_2)(\tilde{c}), \quad \varphi_2(\tilde{c}) = \tilde{c} Z(f_2)(\tilde{c}),
\]
\[
\varphi_3(\tilde{c}) = \frac{u''(y_c) f_1(\tilde{c}) - \varphi(y_c) u'(y_c)}{\tilde{c}}, \quad \varphi_4(\tilde{c}) = -u''(y_c).
\]

Then \( \varphi_1, \varphi_3 \) are odd in \( \tilde{c} \), and \( \varphi_2, \varphi_4 \) are even and in \( H^2(\tilde{c}; I_v) \), thus \( \Lambda_{1,1,2}(\varphi) \) is even. If \( \varphi \in H^k(I_v), k = 1, 2 \), then by Lemma 13.2, \( \varphi_1 \in H^k(I_v) \), and \( \varphi_3 \in H^k(I_v \setminus \{0\}) \).

As
\[
\partial_{\tilde{c}} \left( u''(y_c) f_1(\tilde{c}) - \varphi(y_c) u'(y_c) \right) = u'''(y_c) f_1(\tilde{c}) - \varphi'(y_c) u'(y_c),
\]
we find that
\[
\partial_{\tilde{c}} \left( u''(y_c) f_1(\tilde{c}) - \varphi(y_c) u'(y_c) \right) \bigg|_{\tilde{c}=0} = u''(y_c) f_1(\tilde{c}) - \varphi(y_c) u'(y_c) \bigg|_{\tilde{c}=0} = 0
\]
and
\[
\varphi_3|_{\tilde{c}=0} = 0, \quad \text{thus} \quad \varphi_3 \in H^k(I_v).
\]

Let us verify that
\[
\varphi_1(\tilde{c}) \varphi_4(\tilde{c}) + \varphi_3(\tilde{c}) \varphi_2(\tilde{c}) \bigg|_{\tilde{c} = \pm v(1)} = 0.
\]

We only need to check the case of \( \tilde{c} = v(1) \). Using the facts that \((v^{-1})'(v(1)) = \frac{1}{v'(1)}\),
\((v^{-1})''(v(1)) = -\frac{u''(1)}{v'(1)^2}, u'(1) = 2v(1)v'(1) \) and \( u''(1) = 2v'(1)^2 + 2v(1)v''(1) \), we deduce that
\[\square\]
\[ \varphi_1(v(1)) = (f_1 f_2)' - \frac{f_1 f_2}{v(1)} \]
\[ = \frac{\varphi(1)}{v'(1)^2} - f_1 \left( \frac{v''(1)}{v'(1)^3} + \frac{1}{v(1)v'(1)} \right) \]
\[ = \varphi(1) \frac{v''(1)}{v'(1)^3} - f_1 \frac{u''(1)}{v'(1)^2u'(1)} = -\frac{v(1)\varphi_3(v(1))}{v'(1)^2u'(1)} \]
\[ \varphi_2(v(1)) = -v(1) \frac{v''(1)}{v'(1)^3} - \frac{1}{v'(1)} \]
\[ = -\frac{u''(1)}{2v'(1)^3} = \frac{\varphi_4(v(1))}{2v'(1)^3} = \frac{v(1)\varphi_4(v(1))}{v'(1)^2u'(1)}. \]

This implies (11.2). By (13.2), we have \((u(1) - c)H(z \bar{Z}(f_2)) \in L^\infty\), and by Lemma 4.1, we have \(f_2 \in C^2(I_v)\) and \(zf_2(z) \in C^3(I_v)\), and then by Lemma 13.5 and Lemma 13.2, we have for \(k = 0, 1, 2, \)
\[ \left\| \partial_c^k \Lambda_{1,2}(\varphi) \right\|_{L^2_c} \leq C \left\| Z(f_1 f_2) \right\|_{H^k(I_v)} + C \left\| \varphi \right\|_{H^k(I_0)} \leq C \left\| \varphi \right\|_{H^k(I_0)}. \]

This shows that
\[ \left\| \partial_c^k (\Lambda_{1,1}(\varphi)) \right\|_{L^2_c} \leq C \left\| \varphi \right\|_{H^k(I_0)}, \]
\[ \left\| \tilde{c}^k \partial_c^k (\Lambda_{1,1}(\varphi)) \right\|_{L^2_c} \leq C \left\| \varphi \right\|_{H^k(I_0)}, \]

here we used \(\partial_c (\Lambda_{1,1}(\varphi)) |_{\tilde{c}=0} = 0\) for \(\varphi \in H^2(I_0)\), which follows from the facts that \(\Lambda_{1,1,2}(\varphi)\) is an even function in \(H^2_c(\varphi)\), \(\Lambda_{1,1}(\varphi)\) and \(\tilde{c}\) \(\partial_c (\Lambda_{1,1}(\varphi))\) \(\left\| \tilde{c}^k \partial_c^k (\Lambda_{1,1}(\varphi)) \right\|_{L^2_c} \leq C \left\| \varphi \right\|_{H^k(I_0)}\).

Next, we consider \(\Lambda_{2,1}(\varphi)\). Let \(\tilde{f}_1(\tilde{c}) = (\text{Int}(u''\varphi) \circ v^{-1}) \chi_{[-v(1), v(1)]}(\tilde{c}).\) Then we find that
\[ \Lambda_{2,1}(\varphi)(y_c) = \Lambda_1 \varphi(y_c) + \rho(c)\Pi_{1,1}(u''\varphi)(c) \]
\[ = \rho(-u'(y_c)\varphi(y_c) + \tilde{f}_1(\tilde{c}))\partial_c \left( \frac{1}{2\tilde{c}} H(f_2)(\tilde{c}) \right) - \rho(c)\partial_c \left( \frac{1}{2\tilde{c}} H(f_2)(\tilde{c}) \right) \]
\[ = \frac{v(1)}{2v'(1)} \int_1^{y_c} u''(z)\varphi(z)dz - \frac{v(1)}{2v'(1)}u'(y_c)\varphi(y_c) \]
\[ + \frac{u(1) - c}{4} \left( -H(H(\tilde{f}_2)(\tilde{c})) + \frac{\tilde{f}_1(\tilde{c}) - \varphi(y_c)u'(y_c)}{\tilde{c}} H(z \bar{Z}(f_2)) \right). \]
Let \( \varphi_1(\tilde{c}) = Z(f_1 f_2)(\tilde{c}) \), \( \varphi_2(\tilde{c}) = \tilde{Z}(f_2)(\tilde{c}) \), \( \varphi_3(\tilde{c}) = \frac{\tilde{f}_1(\tilde{c}) - \varphi(y_\infty) u'(y_\infty)}{\epsilon} \) and \( \varphi_4(\tilde{c}) = -\chi_{[-v(1), v(1)]}(\tilde{c}) \). We also have

\[
\varphi_1(\tilde{c}) \varphi_4(\tilde{c}) + \varphi_3(\tilde{c}) \varphi_2(\tilde{c}) = 0.
\]

Then as above, we can deduce from Lemma 13.5 that

\[
\| \partial_c^k (\Lambda_{2,1}(\varphi)) \|_{L^2_{\tilde{c}}} \leq C \| \varphi \|_{H^k(I_0)},
\]

\[
\| \tilde{c}^k \partial_c^k (\Lambda_{2,1}(\varphi)) \|_{L^2_{\tilde{c}}} \leq C \| \varphi \|_{H^k(I_0)},
\]

and \( \partial_c (\Lambda_{2,1}(\varphi)) \big|_{\tilde{c}=0} = 0 \) for \( \varphi \in H^2(I_0) \).

If \( \varphi(0) = 0 \) and \( \varphi \in H^1(I_0) \), we get by Remark 13.4 and Hardy’s inequality that

\[
\Lambda_{j,1}(\varphi) \big|_{\tilde{c}=0} = 0, \quad \| \tilde{c}^{-1} \Lambda_{j,1}(\varphi) \|_{L^2_{\tilde{c}}} \leq C \| \partial_c \Lambda_{j,1}(\varphi) \|_{L^2_{\tilde{c}}} \leq C \| \varphi \|_{H^1(I_0)}.\tag{11.3}
\]

If \( \varphi(1) = 0 \), due to \( \Lambda_{j,1}(\varphi) \big|_{\tilde{c}=1} = 0 \), we get by Hardy’s inequality and (11.3) that

\[
\| (u(1) - c)^{-1} \Lambda_{j,1}(\varphi) \|_{L^2_{\tilde{c}}} \leq C \| \partial_c \Lambda_{j,1}(\varphi) \|_{L^2_{\tilde{c}}} \leq C \| \varphi \|_{H^1(I_0)}.
\]

If \( \varphi(0) = 0 \) and \( \varphi \in H^2(I_0) \), then \( \Lambda_{j,1}(\varphi) \big|_{\tilde{c}=0} = \partial_c (\Lambda_{j,1}(\varphi)) \big|_{\tilde{c}=0} = 0 \), thus Hardy’s inequality gives

\[
\| \tilde{c}^{-2} \Lambda_{j,1}(\varphi) \|_{L^2_{\tilde{c}}} + \| \partial_c (\Lambda_{j,1}(\varphi)) \|_{L^2_{\tilde{c}}} \leq C \left\| \partial_c^2 \Lambda_{j,1}(\varphi) \right\|_{L^2_{\tilde{c}}} \leq C \| \varphi \|_{H^2(I_0)}.
\]

This gives our result by noting \( \| \cdot \|_{L^2_{\tilde{c}}} \sim \| \cdot \|_{L^2_{\tilde{c}}} \) due to \( v'(y) \geq c_1 \). \qed

### 11.2 Estimate of the Operator \( \Pi_{1,2} \)

Recall that

\[
\Pi_{1,2}(\varphi)(c) = \int_{0}^{1} \int_{y_c}^{c} \varphi(y) \left( \frac{1}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) dydz.
\]

We introduce for \( k = 0, 1, 2, \)

\[
\mathcal{L}_k(\varphi)(c) = \int_{0}^{1} \int_{y_c}^{c} \varphi(y) \left( \frac{\partial_z + \partial_y}{u'(y_c)} + \partial_c \right)^k \left( \frac{1}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) dydz.
\]
and for \( k, j = 0, 1, \)
\[
I_{k,j}(\varphi)(c) = \int_{y_c}^{z} \varphi(y) \left( \frac{\partial_z + \partial_y}{u'(y_c)} + \partial_c \right)^k \left( \frac{1}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) dy \bigg|_{z=j}.
\]

It is easy to see that \( \Pi_{1,2}(\varphi) = L_0(\varphi) \) and for \( k = 0, 1, \)
\[
\partial_c L_k(\varphi) = \frac{1}{u'(y_c)} \left( L_k(\varphi') - I_{k,1}(\varphi) + I_{k,0}(\varphi) \right) + L_{k+1}(\varphi). \tag{11.4}
\]

By Proposition 4.14, we deduce that for \( k = 0, 1, 2, \) and \( y \in [z, y_c] \) or \( y \in [y_c, z], \)
\[
\left| \left( \frac{\partial_z + \partial_y}{u'(y_c)} + \partial_c \right)^k \left( \frac{1}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) \right| \leq C \min \{ \alpha^2 |z - y_c|^2, 1 \} \frac{\alpha^2 |z - y_c|^{1-\frac{1}{p}}}{(z + y_c)^2} \int_{z-y_c}^{z+y_c} \frac{|z-y_c|^{1-\frac{1}{p}}}{(z+y_c)^2} \, dz + C \int_{|z-y_c| > \frac{y_c}{2}} \frac{|z-y_c|^{1-\frac{1}{p}}}{(z+y_c)^2} \, dz
\]
which implies that for \( k = 0, 1, 2, \) and \( p \in [1, +\infty), \)
\[
|u'(y_c)^2 L_k(\varphi)(c)| \leq C \frac{1-y_c}{|u(z) - c|^2} \min \{ \alpha^2 |z - y_c|^2, 1 \} \| \varphi \|_{L^p(I_0)}
\]
\[
|u'(y_c)^2 L_k(\varphi)(c)| \leq C \min \left\{ \frac{\alpha^{1+\frac{1}{p}}}{u'(y_c)^2}, \frac{\alpha^{\frac{1}{p}}}{u'(y_c)^2} \right\} \| \varphi \|_{L^p(I_0)}. \tag{11.5}
\]

Similarly, we have for \( k = 0, 1, \)
\[
|I_{k,1}(\varphi)(c)| \leq C \frac{1-y_c}{(u(1) - c)^2} u'(y_c)^2 \| \varphi \|_{L^{\infty}(I_0)}, \tag{11.6}
\]
\[
|I_{k,0}(\varphi)(c)| \leq C \frac{y_c}{(c-u(0))^2} u'(y_c)^2 \| \varphi \|_{L^{\infty}(I_0)}. \tag{11.7}
\]
Lemma 11.3 It holds that for $p \in [1, \infty)$,

$$
\|u'(y_c)^2 I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha^{\frac{1}{p}} \|\varphi\|_{L^p(I_0)},
$$

and for $p \in [1, \infty]$,

$$
\|u'(y_c) I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha^{1 + \frac{1}{p}} \|\varphi\|_{L^p(I_0)}.
$$

Proof The lemma follows from (11.5) except the case of $p = \infty$. We have

$$
\int_0^1 \frac{|z - y_c| \min\{\alpha^2|z - y_c|^2, 1\}}{|u(z) - c|^2} dz 
\leq C \int_{|z-y_c| \leq \frac{1}{\alpha}, 0 \leq z \leq 1} \frac{\alpha^2|z - y_c|}{(z + y_c)^2} dz 
\quad + C \int_{|z-y_c| > \frac{1}{\alpha}, 0 \leq z \leq 1} \frac{|z - y_c|^{-1}}{(z + y_c)^2} dz 
\leq C \int_0^1 \frac{|\alpha|}{(z + y_c)^2} dz \leq \frac{C \alpha}{u'(y_c)},
$$

which gives

$$
\|u'(y_c)^{2k+1} L_k(\varphi)\|_{L^\infty(I_0)} \leq C |\alpha| \|\varphi\|_{L^\infty(I_0)}. \quad (11.8)
$$

Thus, the second inequality holds for $p = + \infty$.

Lemma 11.4 It holds that

$$
\|u'(y_c)^3 \partial_c I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi\|_{L^\infty(I_0)} + C \alpha^{\frac{1}{2}} \|\varphi'\|_{L^2(I_0)},
$$

$$
\|u'(y_c)^3 \partial_c I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi\|_{L^\infty(I_0)} + C \alpha u'(y_c) \|\varphi'\|_{L^\infty(I_0)}.
$$

If $\varphi(0) = 0$, then for $p \in (1, \infty)$,

$$
\|u'(y_c) I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha^{\frac{1}{p}} \|\varphi\|_{W^{1,p}(I_0)},
$$

$$
\|I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi\|_{W^{1,\infty}(I_0)},
$$

$$
\|u'(y_c)^2 \rho \partial_c I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha^{\frac{1}{2}} \|\varphi\|_{H^1(I_0)},
$$

$$
\|u'(y_c)^2 \partial_c I_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi'\|_{L^\infty(I_0)}.
$$

Proof By (11.4), we have

$$
\partial_c I_{1,2}(\varphi) = \frac{1}{u'(y_c)} \left( L_0(\varphi') - I_{0,1}(\varphi) + I_{0,0}(\varphi) \right) + L_1(\varphi).
$$
It follows from (11.5) and (11.8) that
\[
\|u'(y_c)^2 \mathcal{L}_0(\varphi')\|_{L^\infty(I_0)} \leq C \alpha^{\frac{1}{2}} \|\varphi'\|_{L^2(I_0)},
\]
\[
\|u'(y_c)^3 \mathcal{L}_1(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi\|_{L^\infty(I_0)},
\]
which along with (11.7) and (11.6) gives
\[
\|u'(y_c)^3 \partial_c \Pi_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi\|_{L^\infty(I_0)} + C \alpha^{\frac{1}{2}} \|\varphi'\|_{L^2(I_0)}.
\]

On the other hand, we have by (11.8) that
\[
\|u'(y_c)^2 \mathcal{L}_0(\varphi')\|_{L^\infty(I_0)} \leq C \alpha \|\varphi'\|_{L^\infty(I_0)},
\]
which gives
\[
\|u'(y_c)^3 \partial_c \Pi_{1,2}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi\|_{L^\infty(I_0)} + C \alpha u'(y_c) \|\varphi'\|_{L^\infty(I_0)}.
\]
If \(\varphi(0) = 0\), then we have
\[
|\mathcal{L}_k(\varphi)(c)| \leq C \int_0^1 \frac{\min\{\alpha^2 |z - y_c|^2, 1\}}{u'(y_c)^{2k+1} |z - y_c|^{1+\frac{1}{p}}} \frac{\|\varphi\|}{y^{1/p}} \frac{dz}{L^p(I_0)}
\]
\[
\leq C \frac{\alpha^{\frac{1}{p}}}{u'(y_c)^{2k+1}} \|\varphi'\|_{L^p(I_0)}. \tag{11.9}
\]

Similar to (11.8), we have
\[
\|u'(y_c)^{2k} \mathcal{L}_k(\varphi)\|_{L^\infty(I_0)} \leq C \int_0^1 \frac{\min\{\alpha^2 |z - y_c|^2, 1\}}{|u(z) - c|} \frac{\|\varphi\|}{y^{1/p}} \frac{dz}{L^\infty(I_0)}
\]
\[
\leq C \int_0^1 \frac{\min\{\alpha^2 |z - y_c|^2, 1\}}{(z - y_c)^2} \frac{dz}{L^\infty(I_0)} \frac{\|\varphi\|}{y^{1/p}} \frac{dz}{L^\infty(I_0)}
\]
\[
\leq C \alpha \frac{\|\varphi\|}{y} \|_{L^\infty(I_0)} \leq C \alpha \|\varphi'\|_{L^\infty(I_0)}. \tag{11.10}
\]

Similarly, we can deduce that for \(k = 0, 1\) and \(j = 0, 1\),
\[
\|u'(y_c)^{2k} \rho I_{k,j}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha^{\frac{1}{p}} \|\varphi\|_{W^{1,p}(I_0)}, \tag{11.11}
\]
\[
\|u'(y_c)^{2k+1} I_{k,j}(\varphi)\|_{L^\infty(I_0)} \leq C \alpha \|\varphi'\|_{L^\infty(I_0)}. \tag{11.12}
\]

Then the last two inequalities follow from (11.4), (11.5) and the above estimates. \(\Box\)

**Lemma 11.5** It holds that
\[
|u'(y_c)^3 \rho^2 \partial_c^2 \Pi_{1,2}(\varphi)| \leq C \alpha^{\frac{1}{2}} u'(y_c) \|\varphi''\|_{L^2(I_0)} + C \alpha \|\varphi\|_{L^\infty(I_0)}
\]
\[
+ C \alpha u'(y_c) \|\varphi'\|_{L^\infty(I_0)} + C \alpha^{\frac{1}{2}} \|\varphi'\|_{L^2(I_0)}.
\]
If \( \varphi(0) = 0 \), then we have

\[
\| u'(y_c) \varphi \|_{L^\infty(I_0)} \leq C \alpha \left\| \varphi' \right\|_{L^\infty(I_0)} + C \alpha^\frac{1}{2} \left\| \varphi'' \right\|_{L^2(I_0)}.
\]

**Proof** First of all, we have

\[
\partial_c I_{0,j}(\varphi)(c) = -\frac{\varphi(y_c)}{u'(y_c)} \cdot \left( \frac{1}{(u(z) - c)^2} \right) \left( \frac{1}{\phi_1(z, c)^2} - 1 \right) \bigg|_{z=j} + \int_{y_c}^{j} \partial_c \left( \frac{\varphi(y)}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) dy \bigg|_{z=j}.
\]

By Proposition 4.14, we have for \( y \in [z, y_c] \) or \( y \in [y_c, z] \),

\[
\left| \partial_c \left( \frac{1}{(u(z) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) \right| \leq C \left| \left( \frac{1}{(u(z) - c)^3} \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} - 1 \right) \right) \right| + C \left| \frac{1}{|u(z) - c|^2} \partial_c \left( \frac{\phi_1(y, c)}{\phi_1(z, c)^2} \right) \right|
\]

\[
\leq C \min \{ \alpha^2 |z - y_c|^2, 1 \} \left\| \varphi \right\|_{L^\infty(I_0)} + C \frac{\min \{ \alpha^2 |z - y_c|, \alpha \} \varphi}{u'(y_c)|u(z) - c|^2}.
\]

Thus, we have

\[
|\partial_c I_{0,1}(\varphi)(c)| \leq C \left\| \varphi \right\|_{L^\infty(I_0)} \frac{\min \{ \alpha^2 (1 - y_c)^2, 1 \}}{(1 - y_c)^2 u'(y_c)^3}
\]

\[
+ C \left( \sup_{y \in [y_c, 1]} \left| \partial_c \left( \frac{1}{(u(1) - c)^2} \left( \frac{\phi_1(y, c)}{\phi_1(1, c)^2} - 1 \right) \right) \right| \right) \left| 1 - y_c \right| \left\| \varphi \right\|_{L^\infty(I_0)}
\]

\[
\leq C \left\| \varphi \right\|_{L^\infty(I_0)} \frac{\min \{ \alpha^2 (1 - y_c)^2, 1 \}}{(1 - y_c)^2 u'(y_c)^3}
\]

\[
+ C \left\| \varphi \right\|_{L^\infty(I_0)} \frac{\min \{ \alpha^2 (1 - y_c), \alpha \} \varphi}{|1 - y_c| u'(y_c)^3},
\]

and

\[
|\partial_c I_{0,0}(\varphi)(c)| \leq C \left\| \varphi \right\|_{L^\infty(I_0)} \left( \frac{\min \{ \alpha^2 y_c^2, 1 \}}{y_c^2 u'(y_c)^3} + \frac{\min \{ \alpha^2 y_c^2, \alpha \} \varphi}{y_c u'(y_c)^3} \right).
\]

This implies that

\[
\| \partial_c I_{0,1}(\varphi) \|_{L^\infty(I_0)} + \| \partial_c I_{0,0}(\varphi) \|_{L^\infty(I_0)} \leq C \left\| \varphi \right\|_{L^\infty(I_0)} \frac{\min \{ \alpha^2 \rho_0^2, \alpha \rho_0 \}}{u'(y_c)^3 \rho_0^2}.
\]

(11.13)
A direct calculation with (11.4) gives
\[
\partial_c^2 \Pi_{1,2}(\varphi) = \partial_c \left( \frac{1}{u'(y_c)} \left( \mathcal{L}_0(\varphi') - I_{0,1}(\varphi) + I_{0,0}(\varphi) \right) + \mathcal{L}_1(\varphi) \right)
\]
\[
= \frac{1}{u'(y_c)^2} \left( \mathcal{L}_0(\varphi'') - I_{0,1}(\varphi') + I_{0,0}(\varphi') \right) + \frac{1}{u'(y_c)} \mathcal{L}_1(\varphi')
\]
\[
- \frac{u''(y_c)}{u'(y_c)^3} \left( \mathcal{L}_0(\varphi') - I_{0,1}(\varphi) + I_{0,0}(\varphi) \right)
\]
\[
+ \frac{1}{u'(y_c)} \left( \mathcal{L}_1(\varphi') - I_{1,1}(\varphi) + I_{1,0}(\varphi) \right) + \mathcal{L}_2(\varphi),
\]
which along with (11.5), (11.8), (11.7), (11.6) and (11.13) gives the first inequality of
the lemma.

If \( \varphi(0) = 0 \), then we have
\[
|\partial_c I_{0,0}(\varphi)(c)| \leq C \left\| \frac{\varphi(y)}{y} \right\|_{L^\infty(I_0)} \left( \frac{\min\{\alpha^2 y_c^2, 1\}}{y_c^2 u'(y_c)^2} + \frac{\min\{\alpha^2 y_c, \alpha\}}{y_c u'(y_c)^2} \right),
\]
which gives
\[
\left\| u'(y_c)^2 y_c \partial_c I_{0,0}(\varphi) \right\|_{L^\infty(I_0)} \leq C \alpha \left\| \varphi' \right\|_{L^\infty(I_0)}.
\]
Similarly, we have
\[
\left\| u'(y_c)^2 (1 - y_c) \partial_c I_{0,1}(\varphi) \right\|_{L^\infty(I_0)} \leq C \alpha \left\| \varphi' \right\|_{L^\infty(I_0)}.
\]
Then the second inequality follows by combining the above inequalities with (11.5),
(11.10) and (11.12).

\[\square\]

12 \( W^{2,1} \) Estimate of the Kernel

For the sake of simplicity, we introduce the following notations:
- We denote by \( \mathcal{L}^p \) a function \( f \) which satisfies \( \| f \|_{L^p_{y_c}} \leq C \).
- We denote by \( a \mathcal{L}^p \cap b \mathcal{L}^q \) a function \( f \) which satisfies \( \| f/a \|_{L^p_{y_c}} + \| f/b \|_{L^q_{y_c}} \leq C \).

Here the constant \( C \) is independent of \( \alpha \), and may be different from line to line.

12.1 \( W^{2,1} \) Estimate of \( K_\alpha(c, \alpha) \)

Recall that
\[
K_\alpha(c, \alpha) = \frac{\Lambda_1(\hat{\omega}_o)(c) \Lambda_2(g)(c)}{(A(c)^2 + B(c)^2) u'(y_c)^2}.
\]
Proof of Proposition 9.1

Step 1. \(L^1\) estimate. We normalize \(\|\hat{\omega}_o\|_{L^2} \leq 1, \|g\|_{L^2} \leq 1\). By Lemmas 11.3 and 10.1, we get

\[
\Lambda_{1.2}(\hat{\omega}_o) = \alpha \rho_0 L^2 + \alpha^\frac{1}{2} L^\infty, \\
\Lambda_{2.2}(g) = \alpha \rho_0 L^2 + \alpha^\frac{1}{2} L^\infty,
\]

here \(\rho_0 = \frac{\rho}{u'(y_c)} \sim y_c(1 - y_c)\). Then we get by Lemma 11.1 that

\[
\Lambda_1(\hat{\omega}_o) = (1 + \alpha \rho_0)L^2 + \alpha^\frac{1}{2} L^\infty, \\
\Lambda_2(g) = (1 + \alpha \rho_0)L^2 + \alpha^\frac{1}{2} L^\infty,
\]

which gives

\[
\Lambda_1(\hat{\omega}_o)\Lambda_2(g) = (1 + \alpha \rho_0)^2 L^1 + (1 + \alpha \rho_0)\alpha^\frac{1}{2} L^2 + \alpha L^\infty.
\]

By Lemma 10.3, we have \(\frac{1}{A^2 + B^2} = \frac{C L^\infty}{(1 + \alpha \rho_0)^2}\). Thus, we obtain

\[
\frac{\Lambda_1(\hat{\omega}_o)(c)\Lambda_2(g)(c)}{(A(c)^2 + B(c)^2)} = L^1 + \frac{\alpha^\frac{1}{2} L^2}{1 + \alpha \rho_0} = L^1,
\]

which gives

\[
\|K_o(c, \alpha)\|_{L^1} = \|u'(y_c)K_o(c, \alpha)\|_{L^1_c} \leq C.
\]

Step 2. \(W^{1,1}\) estimate. We normalize \(\|\hat{\omega}_o\|_{H^1} \leq 1\) and \(\|g\|_{H^1} \leq 1\). Thanks to \(\hat{\omega}_o(\alpha, 0) = g(0) = g(1) = 0\), we get by Hardy’s inequality that

\[
\|\hat{\omega}_o/y\|_{L^2} + \|g/\rho_0\|_{L^2} \leq C. \tag{12.1}
\]

By Lemma 11.4, Lemma 10.1 and (12.1), we have

\[
\Lambda_{1.2}(\hat{\omega}_o) = \alpha \rho_0 L^\infty \cap \alpha \rho L^2 + \alpha^\frac{1}{2} \rho_0 L^\infty, \\
\Lambda_{2.2}(g) = \alpha^\frac{1}{2} \rho_0 L^\infty + \alpha^\frac{3}{2} L^2, \\
\partial_c \Lambda_{1.2}(\hat{\omega}_o) = \alpha^\frac{1}{2} L^\infty / u' + \alpha L^2, \\
\partial_c \Lambda_{2.2}(g) = \frac{(\alpha^\frac{1}{2} L^\infty + \alpha \rho_0 L^2)}{u'(y_c)}.
\]

from which and Lemma 11.1, we infer that

\[
\Lambda_1(\hat{\omega}_o) = \alpha \rho_0 L^\infty \cap \alpha \rho L^2 + \alpha^\frac{1}{2} \rho_0 L^\infty + y_c L^2 = u'(y_c)(1 + \alpha \rho_0) L^2, \\
\Lambda_2(g) = \alpha^\frac{1}{2} \rho_0 L^\infty + \alpha^\frac{3}{2} L^2 + \rho_0 L^2 = \rho_0(1 + \alpha \rho_0) L^2.
\]
\[ \partial_c \Lambda_1(\hat{\omega}_o) = \frac{\alpha^\frac{1}{2} L^\infty + (1 + \alpha u'(y_c))L^2}{u'(y_c)} = \frac{(1 + \alpha u'(y_c))L^2}{u'(y_c)}, \]
\[ \partial_c \Lambda_2(g) = \frac{\alpha^\frac{1}{2} L^\infty + (1 + \alpha \rho_0)L^2}{u'(y_c)} = \frac{(1 + \alpha \rho_0)L^2}{u'(y_c)}. \]

Then we obtain
\[ \Lambda_1(\hat{\omega}_o)\Lambda_2(g) = u'(y_c)(1 + \alpha \rho_0)L^2 \rho_0(1 + \alpha \rho_0)L^2 = \rho(1 + \alpha \rho_0)^2L^1, \]
and
\[ \partial_c(\Lambda_1(\hat{\omega}_o)\Lambda_2(g)) = \Lambda_2(g)\partial_c \Lambda_1(\hat{\omega}_o) + \Lambda_1(\hat{\omega}_o)\partial_c \Lambda_2(g) \]
\[ = \rho_0(1 + \alpha \rho_0)L^2 \frac{(1 + \alpha u'(y_c))L^2}{u'(y_c)} \]
\[ + u'(y_c)(1 + \alpha \rho_0)L^2 \frac{(1 + \alpha \rho_0)L^2}{u'(y_c)} \]
\[ = (1 - y_c)(1 + \alpha \rho_0)(1 + \alpha u'(y_c))L^1 \]
\[ + (1 + \alpha \rho_0)^2L^1 = (1 + \alpha \rho_0)^2L^1. \]

By Lemma 10.3, we have
\[ \partial_c \left( \frac{1}{(A^2 + B^2)u'(y_c)} \right) = \frac{L^\infty(1 + \alpha u'(y_c))}{(1 + \alpha \rho_0)^3u'(y_c)^3} + \frac{(L^2 \cap (L^\infty / \rho))}{(1 + \alpha \rho_0)^3u'(y_c)}. \]

Summing up, we deduce that
\[ u'(y_c)\partial_c K_\alpha(\alpha, c) \]
\[ = u'(y_c)\Lambda_1(\hat{\omega}_o)\Lambda_2(g)\partial_c \left( \frac{1}{(A^2 + B^2)u'(y_c)} \right) + \frac{\partial_c(\Lambda_1(\hat{\omega}_o)\Lambda_2(g))}{(A^2 + B^2)} \]
\[ = \rho(1 + \alpha \rho_0)^2L^1 \left( \frac{L^\infty(1 + \alpha u'(y_c))}{(1 + \alpha \rho_0)^3u'(y_c)^2} + \frac{(L^2 \cap (L^\infty / \rho))}{(1 + \alpha \rho_0)^3} \right) + \frac{(1 + \alpha \rho_0)^2L^1}{(1 + \alpha \rho_0)^2} \]
\[ = \frac{(1 - y_c)(1 + \alpha u'(y_c))L^1}{(1 + \alpha \rho_0)} + L^1 + L^1 = L^1, \]

which gives
\[ \| \partial_c K(c, \alpha) \|_{L^1} = \| u'(y_c)\partial_c K_\alpha(c, \alpha) \|_{L^1} \leq C. \]

**Step 3.** $W^{2,1}$ estimate. We normalize $\| \hat{\omega}_o \|_{H^2} \leq 1$ and $\| g'' \|_{L^2} + \alpha^2 \| g \|_{L^2} \leq 1$. Then we get
\[ \| g \|_{L^2} \leq \frac{C}{\alpha^2}, \quad \| g' \|_{L^2} \leq \frac{C}{\alpha}, \quad \| g'' \|_{L^2} \leq C \| g' \|_{L^2} \| g'' \|_{L^2} \leq \frac{C}{\sqrt{\alpha}}. \]
Thanks to $\hat{\omega}_o(\alpha, 0) = g(0) = g(1) = 0$, we have
\[
\|g/\rho_0\|_{L^\infty} \leq C\|g'\|_{L^\infty} \leq \frac{C}{\sqrt{\alpha}}, \quad \|\hat{\omega}_o/y\|_{L^\infty} \leq C. \quad (12.3)
\]
By Lemma 11.4, Lemma 10.1, Lemma 11.5, (12.2) and (12.3), we have
\[
\Lambda_{1,2}(\hat{\omega}_o) = \alpha \rho \mathcal{L}^\infty, \quad \Lambda_{2,2}(g) = \alpha^\frac{1}{2} \rho \mathcal{L}^\infty, \\
\partial_c \Lambda_{1,2}(\hat{\omega}_o) = \alpha \mathcal{L}^\infty, \quad \partial_c \Lambda_{2,2}(g) = \alpha^\frac{1}{2} \mathcal{L}^\infty, \\
\partial_c^2 \Lambda_{1,2}(\hat{\omega}_o) = \rho^{-1} \alpha \mathcal{L}^\infty + (u'(y_c))^{-1} \alpha \mathcal{L}^2, \\
\partial_c^2 \Lambda_{2,2}(g) = u'(y_c)^{-2}(\alpha^\frac{1}{2} \mathcal{L}^\infty + \alpha \rho_0 \mathcal{L}^2),
\]
from which and Lemma 11.1, we infer that
\[
\Lambda_1(\hat{\omega}_o) = (\rho \mathcal{L}^2 \cap u'(y_c) \mathcal{L}^\infty) + (u'(y_c)^2 + \alpha \rho) \mathcal{L}^\infty, \\
\partial_c (\Lambda_1(\hat{\omega}_o)) = (\mathcal{L}^2 + \alpha \mathcal{L}^\infty), \\
\partial_c^2 (\Lambda_1(\hat{\omega}_o)) = \rho^{-1}((1 + \alpha \rho_0) \mathcal{L}^2 + \alpha \mathcal{L}^\infty),
\]
and
\[
\Lambda_2(g) = \rho \mathcal{L}^2 + \alpha^\frac{1}{2} \rho \mathcal{L}^\infty = \rho (1 + \alpha \rho_0) \mathcal{L}^2, \\
\partial_c (\Lambda_2(g)) = (\mathcal{L}^2 + \alpha^\frac{1}{2} \mathcal{L}^\infty) = (1 + \alpha \rho_0) \mathcal{L}^2, \\
\partial_c^2 (\Lambda_2(g)) = u'(y_c)^{-2}((1 + \alpha \rho_0) \mathcal{L}^2 + \alpha^\frac{1}{2} \mathcal{L}^\infty) = u'(y_c)^{-2}(1 + \alpha \rho_0) \mathcal{L}^2.
\]
By Lemma 10.3, we have
\[
\partial_c^2 \left( \frac{1}{(A^2 + B^2)u'(y_c)} \right) = \frac{\mathcal{L}^\infty(1 + \alpha \rho_0)^2 u'(y_c)^2}{(1 + \alpha \rho_0) (1 + \alpha \rho_0)^2 u'(y_c)^5} + \frac{(\mathcal{L}^2 + \alpha \mathcal{L}^\infty)}{(1 + \alpha \rho_0)^3 \rho u'(y_c)}.
\]
Thus, we can deduce that
\[
\Lambda_1(\hat{\omega}_o) \Lambda_2(g) = \left( \rho \mathcal{L}^2 \cap u'(y_c) \mathcal{L}^\infty + (u'(y_c)^2 + \alpha \rho) \mathcal{L}^\infty \right) (1 + \alpha \rho_0) \mathcal{L}^2 \\
= \rho (1 + \alpha \rho_0) \left( \rho \mathcal{L}^1 \cap u' \mathcal{L}^2 + (u'(y_c)^2 + \alpha \rho) \mathcal{L}^2 \right),
\]
and
\[
\partial_c (\Lambda_1(\hat{\omega}_o) \Lambda_2(g)) \\
= \Lambda_2(g) \partial_c \Lambda_1(\hat{\omega}_o) + \Lambda_1(\hat{\omega}_o) \partial_c \Lambda_2(g) \\
= \rho (1 + \alpha \rho_0) \mathcal{L}^2 (\mathcal{L}^2 + \alpha \mathcal{L}^\infty) \\
+ \left( \rho \mathcal{L}^2 \cap u'(y_c) \mathcal{L}^\infty + (u'(y_c)^2 + \alpha \rho) \mathcal{L}^\infty \right) (1 + \alpha \rho_0) \mathcal{L}^2.
\]
Let us first present the following estimates of $\frac{\hat{\omega}}{\Lambda_1}$ which along with the fact $K_0$ gives as follows

\[ W_1 \geq 12.2 \]

With the above estimates, it is easy to deduce that

\[ \frac{\hat{\omega}}{\Lambda_1} \]

This completes the proof of the proposition. \( \Box \)

### 12.2 $W^{2,1}$ Estimate of $K_0(c, \alpha)$

Let us first present the following estimates of $\Lambda_{3,1}(\widehat{\omega}_e)$ and $\Lambda_{4,1}(g)$, which are defined as follows

\[ \Lambda_{3,1}(\widehat{\omega}_e)(c) = J(c) \left( \frac{u''(\gamma_c)}{u'(\gamma_c)} \mathcal{E}(\widehat{\omega}_e)(c) + \widehat{\omega}_e(\gamma_c) \right), \]

\[ \Lambda_{4,1}(g)(c) = J(c) \left( \frac{\mathcal{E}(gu'')(c)}{u'(\gamma_c)} + g(\gamma_c) \right), \]
where $E(\varphi)(c) = \int_{y_c}^{0} \varphi \phi_1(y, c) \, dy$.

**Lemma 12.1** 1. If $\|\hat{\omega}_e\|_{L^2} \leq 1$ and $\|g\|_{L^2} \leq 1$, then

$$\Lambda_{3,1}(\hat{\omega}_e) = \alpha^{-2} \mathcal{L}^2, \quad \Lambda_{4,1}(g) = \alpha^{-2} \mathcal{L}^2.$$  

2. If $\|\hat{\omega}_e\|_{H^1} \leq 1$ and $|\alpha||g|_{L^2} + \|g'\|_{L^2} \leq 1$, then

$$\Lambda_{3,1}(\hat{\omega}_e) = \frac{\rho_0 (\mathcal{L}^2 + \alpha \mathcal{L}^\infty)}{\alpha^2}, \quad \Lambda_{4,1}(g) = \frac{\rho_0 (\mathcal{L}^2 + \alpha^{1/2} \mathcal{L}^\infty)}{\alpha^2},$$

$$\partial_c(\Lambda_{3,1}(\hat{\omega}_e)) = \frac{\mathcal{L}^2 + \alpha \mathcal{L}^\infty}{\alpha^2 y_c}, \quad \partial_c(\Lambda_{4,1}(g)) = \frac{\mathcal{L}^2 + \alpha^{1/2} \mathcal{L}^\infty}{\alpha^2 y_c}.$$  

3. If $\|\hat{\omega}_e\|_{H^2} \leq 1$ and $\|\varphi'' - \alpha^2 \varphi\|_{L^2} \leq 1$ and $\hat{\omega}_e'(\alpha, 0) = g'(0) = 0$, then

$$\Lambda_{3,1}(\hat{\omega}_e) = \rho \mathcal{L}^\infty + C \alpha^{-2} (\rho_0 \mathcal{L}^\infty \cap \rho \mathcal{L}^2),$$

$$\partial_c \Lambda_{3,1}(\hat{\omega}_e) = \mathcal{L}^\infty + \alpha^{-2} \mathcal{L}^2,$$

$$\partial_c^2 \Lambda_{3,1}(\hat{\omega}_e) = \rho^{-1} (\mathcal{L}^\infty + \alpha^{-2} \mathcal{L}^2),$$

and

$$\Lambda_{4,1}(g) = \rho \alpha^{-3/2} \mathcal{L}^\infty + C \alpha^{-2} \rho \mathcal{L}^2,$$

$$\partial_c \Lambda_{4,1}(g) = \alpha^{-3/2} \mathcal{L}^\infty + C \alpha^{-2} \mathcal{L}^2,$$

$$\partial_c^2 \Lambda_{4,1}(g) = \gamma_c^{-2} (\alpha^{-3/2} \mathcal{L}^\infty + \alpha^{-2} \mathcal{L}^2).$$

**Proof** 

**Case 1.** $\|\hat{\omega}_e\|_{L^2} \leq 1$ and $\|g\|_{L^2} \leq 1$. Due to $\phi_1(y, c) \leq \phi_1(0, c)$, we have

$$\left\| \frac{E(\varphi)}{\phi_1(0, c) y_c} \right\|_{L^2} \leq C \|\varphi\|_{L^2},$$

which along with Lemma 10.4 implies that

$$\Lambda_{3,1}(\hat{\omega}_e) = \alpha^{-2} \mathcal{L}^2, \quad \Lambda_{4,1}(g) = \alpha^{-2} \mathcal{L}^2.$$  

**Case 2.** $\|\hat{\omega}_e\|_{H^1} \leq 1$ and $|\alpha||g|_{L^2} + \|g'\|_{L^2} \leq 1$. Thus, by Sobolev embedding, $\|\hat{\omega}_e\|_{L^\infty} \leq 1$ and $\|g\|_{L^\infty} \leq C \alpha^{-1/2}$.

We write

$$u''(y_c)E(\hat{\omega}_e) + u'(y_c)\hat{\omega}_e(y_c)$$

$$= \int_{y_c}^{0} (u''(y_c)\phi_1(y, c)\hat{\omega}_e(y) - u''(y)\hat{\omega}_e(y_c)\phi_1(y_c, c)) \, dy$$
\[
\begin{align*}
&= \int_{y_c}^{0} (u''(y_c) - u''(y)) \phi_1(y, c) \hat{\omega}_e(y) \, dy \\
&\quad + \int_{y_c}^{0} (\phi_1(y, c) - 1) u''(y) \hat{\omega}_e(y) \, dy + \int_{y_c}^{0} u''(y) (\hat{\omega}_e(y) - \hat{\omega}_e(y_c)) \, dy \\
&= I_1 + I_2 + I_3.
\end{align*}
\]

Using the facts that \(\phi_1(y, c) - 1 \leq C \min\{|\alpha|^2|y - y_c|^2, 1\} \phi_1(y, c)\) and \(|u''(y) - u''(y_c)| \leq C|y - y_c|y_c\) for \(y \leq y_c\) (since \(u\) is even), it is easy to get

\[
|I_1| \leq C|y_c|^3 \phi_1(0, c) \|\hat{\omega}_e\|_{L^\infty},
\|
\begin{align*}
|I_2| &\leq C|\alpha| |y_c|^2 \min \{1, |\alpha||y_c|\} \phi_1(0, c) \|\hat{\omega}_e\|_{L^\infty}, \\
\|I_3/y_c^2\|_{L^2} &\leq C\|\hat{\omega}_e'\|_{L^2}.
\end{align*}
\]

This shows that

\[
u''(y_c) E(\hat{\omega}_e) + u'(y_c) \hat{\omega}_e(y_c) = \phi_1(0, c) y_c^2 \alpha L^\infty + y_c^2 L^2,
\]

which along with Lemma 10.4 gives

\[
\Lambda_{3,1}(\hat{\omega}_e) = \frac{\rho_0(L^2 + \alpha L^\infty)}{\alpha^2}. 
\tag{12.4}
\]

We write

\[
\begin{align*}
E(u'' g) + u'(y_c) g(y_c) \\ = \int_{y_c}^{0} (u''(y) \phi_1(y, c) g(y) - u''(y) g(y_c) \phi_1(y_c, c)) \, dy \\
= \int_{y_c}^{0} (\phi_1(y, c) - 1) u''(y) g(y) \, dy + \int_{y_c}^{0} u''(y) (g(y) - g(y_c)) \, dy \\
= I_1 + I_2.
\end{align*}
\]

It is easy to see that

\[
|I_1| \leq C|\alpha| y_c^2 \min \{1, |\alpha||y_c|\} \phi_1(0, c),
\|
\begin{align*}
\|I_2/y_c^2\|_{L^2} &\leq C\|g'\|_{L^2}.
\end{align*}
\]

This shows that

\[
E(gu'') + u'(y_c) g(y_c) = \phi_1(0, c) y_c^2 \alpha^{1/2} L^\infty + y_c^2 L^2,
\]
which along with Lemma 10.4 gives

$$\Lambda_{4,1}(g) = \frac{\rho_0(\mathcal{L}^2 + \alpha \frac{1}{2} \mathcal{L}^\infty)}{a^2}. \quad (12.5)$$

A direct calculation gives

$$\partial_c \left( u''(y_c)E(\hat{\omega}_e) + u'(y_c)\hat{\omega}_e(y_c) \right)$$

$$= \frac{u''(y_c)}{u'(y_c)}E(\hat{\omega}_e) - \frac{u''(y_c)\hat{\omega}_e(y_c)}{u'(y_c)} + u''(y_c)$$

$$\int_{y_c}^{0} \hat{\omega}_e \partial_c \phi_1(y, c) dy + \frac{(u'\hat{\omega}_e)'(y_c)}{u'(y_c)}$$

$$= \frac{u''(y_c)}{u'(y_c)}E(\hat{\omega}_e) + \hat{\omega}_e'(y_c) + u''(y_c) \int_{y_c}^{0} \hat{\omega}_e \partial_c \phi_1(y, c) dy,$$

and

$$\partial_c \left( E(gu'') + u'(y_c)g(y_c) \right) = -\frac{gu''(y_c)}{u'(y_c)} + \int_{y_c}^{0} gu'' \partial_c \phi_1(y, c) dy + \frac{(u'g)'(y_c)}{u'(y_c)}$$

$$= g'(y_c) + \int_{y_c}^{0} gu'' \partial_c \phi_1(y, c) dy.$$

By Proposition 4.14, we know that for $0 \leq y \leq y_c$,

$$\left| \frac{\partial_c \phi_1(y, c)}{\phi_1(0, c)} \right| \leq \frac{C}{\alpha} \min\{\alpha, y_c \},$$

Thus, we can deduce that

$$\partial_c \left( u''(y_c)E(\hat{\omega}_e) + u'(y_c)\hat{\omega}_e(y_c) \right) = \phi_1(0, c)\alpha\mathcal{L}^\infty + \mathcal{L}^2,$$

$$\partial_c \left( E(gu'') + u'(y_c)g(y_c) \right) = \phi_1(0, c)\alpha\frac{1}{2}\mathcal{L}^\infty + \mathcal{L}^2,$$

from which and Lemma 10.4, we infer that

$$\partial_c \left( \Lambda_{3,1}(\hat{\omega}_e) \right) = \frac{\mathcal{L}^2 + \alpha\mathcal{L}^\infty}{\alpha^2 y_c}, \quad \partial_c \left( \Lambda_{4,1}(g) \right) = \frac{\mathcal{L}^2 + \alpha\frac{1}{2}\mathcal{L}^\infty}{\alpha^2 y_c}. \quad (12.6)$$

**Case 3.** $||\hat{\omega}_e||_{H^2} \leq 1$ and $||g'' - \alpha^2g||_{L^2} \leq 1$ and $\hat{\omega}'(\alpha, 0) = g'(0) = 0$. Thus, we have $||\hat{\omega}_e||_{L^\infty} + ||\hat{\omega}_e'||_{L^\infty} \leq C$ and $||g||_{L^\infty} \leq C\alpha^{-\frac{1}{2}}, \ ||g'||_{L^\infty} \leq C\alpha^{-\frac{1}{2}}$.

First of all, following the proof of (12.4), (12.5) and (12.6), we can deduce that

$$u''(y_c)E(\hat{\omega}_e) + u'(y_c)\hat{\omega}_e(y_c) = \phi_1(0, c)y_c^3\alpha^2\mathcal{L}^\infty + (y_c^3\mathcal{L}^2 \cap y_c^2\mathcal{L}^\infty),$$

$$\partial_c \left( u''(y_c)E(\hat{\omega}_e) + u'(y_c)\hat{\omega}_e(y_c) \right) = \phi_1(0, c)\alpha^2 y_c\mathcal{L}^\infty + y_c\mathcal{L}^2,$$
E(gu'') + u'(y_c)g(y_c) = \phi_1(0, c)\alpha^2 y_c^3 \mathcal{L}\infty + y_c^2 \mathcal{L}^2,
\partial_c \left( E(gu'') + u'(y_c)g(y_c) \right) = \phi_1(0, c)\alpha^2 y_c^3 \mathcal{L}\infty + y_c^2 \mathcal{L}^2,

from which and Lemma 10.4, we infer that

\Lambda_{3,1}(\hat{\omega}_c) = \rho \mathcal{L}\infty + C\alpha^{-2}(\rho_0 \mathcal{L}\infty \cap \rho \mathcal{L}^2), \quad \partial_c \Lambda_{3,1}(\hat{\omega}_c) = \mathcal{L}\infty + \alpha^{-2} \mathcal{L}^2,
\Lambda_{4,1}(g) = \rho\alpha^{-\frac{3}{2}} \mathcal{L}\infty + C\alpha^{-2} \mathcal{L}^2, \quad \partial_c \Lambda_{4,1}(g) = \alpha^{-\frac{3}{2}} \mathcal{L}\infty + C\alpha^{-2} \mathcal{L}^2.

We have

\partial_c^2 \left( E(gu'') + u'(y_c)g(y_c) \right) = \frac{g''(y_c)}{u'(y_c)} + \int_{y_c}^0 gu'' \partial_c^2 \phi_1(1, c)\,dy,

and

\partial_c^2 \left( u''(y_c)E(\hat{\omega}_c) + u'(y_c)\hat{\omega}_c(y_c) \right)
= \left( \frac{u'''(y_c)}{u'(y_c)^2} - \frac{(u''u')(y_c)}{u'(y_c)^3} \right) E(\hat{\omega}_c) + \hat{\omega}_c \partial_c \left( \frac{u''(y_c)}{u'(y_c)} \right) + 2 \frac{u'''(y_c)}{u'(y_c)} \int_{y_c}^0 \hat{\omega}_c \partial_c \phi_1(1, c)\,dy
+ u''(y_c) \int_{y_c}^0 \hat{\omega}_c \partial_c^2 \phi_1(1, c)\,dy - \frac{u''(y_c)}{u'(y_c)^2} \hat{\omega}_c(y_c).

By Proposition 4.14, we know that for 0 < c < y_c,

\left| \frac{\partial_c^2 \phi_1(1, c)}{\phi_1(1, c)} \right| \lesssim \frac{C\alpha^2}{u'(y_c)^2}.

Thus, we can obtain

y_c \partial_c^2 \left( u''(y_c)E(\hat{\omega}_c) + u'(y_c)\hat{\omega}_c(y_c) \right) = \phi_1(0, c)\alpha^2 \mathcal{L}\infty + \mathcal{L}^2,
y_c \partial_c^2 \left( E(gu'') + u'(y_c)g(y_c) \right) = \phi_1(0, c)\alpha^2 y_c^3 \mathcal{L}\infty + \mathcal{L}^2.

from which and Lemma 10.4, we infer that

\partial_c^2 \Lambda_{3,1}(\hat{\omega}_c) = \rho^{-1}(\mathcal{L}\infty + \alpha^{-2} \mathcal{L}^2), \quad \partial_c^2 \Lambda_{4,1}(g) = y_c^{-2}(\alpha^{-\frac{3}{2}} \mathcal{L}\infty + \alpha^{-2} \mathcal{L}^2).

The proof is completed. \square

Now we are in position to prove Proposition 9.2. Let us recall

\[
K_e(c, \alpha) = \frac{\Lambda_3(\hat{\omega}_c)(c) \Lambda_4(g)(c)}{u'(y_c)(A_2^2 + B_2^3)(c)}.
\]
Proof Step 1. $L^1$ estimate. We normalize $\|\hat{\varphi}_c\|_{L^2} \leq 1$ and $\|g\|_{L^2} \leq 1$. Similar to Step 1 in Proposition 9.1, we have

$$\Lambda_1(\hat{\varphi}_c) = (1 + \alpha \rho_0) \mathcal{L}^2 + \alpha \frac{1}{2} \mathcal{L}^\infty = (1 + \alpha \rho_0) \mathcal{L}^2,$$
$$\Lambda_2(g) = (1 + \alpha \rho_0) \mathcal{L}^2 + \alpha \frac{1}{2} \mathcal{L}^\infty = (1 + \alpha \rho_0) \mathcal{L}^2,$$

from which and Lemma 12.1, we infer that

$$\Lambda_3(\hat{\varphi}_c) = (\alpha^{-2} + y_c^2)(1 + \alpha \rho_0) \mathcal{L}^2,$$
$$\Lambda_4(g) = (\alpha^{-2} + y_c^2)(1 + \alpha \rho_0) \mathcal{L}^2.$$

By Lemma 10.5, we have $\frac{1}{A_2^2 + B_2^2} = \frac{\alpha^4 \mathcal{L}^\infty}{(1 + \alpha \rho_0)^2 (1 + \alpha y_c)^4}$. Thus,

$$u'(y_c) K_c(c, \alpha) = \frac{\Lambda_3(\hat{\varphi}_c) \Lambda_4(g)}{A_2^2 + B_2^2}$$
$$= \frac{\alpha^4 (\alpha^{-2} + y_c^2)(1 + \alpha \rho_0) \mathcal{L}^2 (\alpha^{-2} + y_c^2)(1 + \alpha \rho_0) \mathcal{L}^2}{(1 + \alpha \rho_0)^2 (1 + \alpha y_c)^4} = \mathcal{L}^1.$$ 

which gives

$$\left\| K_c(c, \alpha) \right\|_{L^1} = \left\| u'(y_c) K_c(c, \alpha) \right\|_{L^1_{y_c}} \leq C.$$

Step 2. $W^{1,1}$ estimate. We normalize $\|\hat{\varphi}_c\|_{H^1} \leq 1$ and $\|g\|_{L^2} + \|g\|_{L^2} \leq 1$. Then we have

$$\|g\|_{L^\infty} \leq C \|g\|_{L^2}^{1/2} \|g\|_{H^1}^{1/2} \leq C \alpha^{-\frac{1}{2}},$$
$$\|g/(1 - y_c)\|_{L^2} \leq C \|g\|_{L^2} \leq C.$$

By Lemma 11.3, Lemma 11.4 and Lemma 10.1, we have

$$\Lambda_{1,2}(\hat{\varphi}_c) = \alpha \rho_0 \mathcal{L}^\infty, \quad \Lambda_{2,2}(g) = \alpha \frac{1}{2} \rho_0 \mathcal{L}^\infty,$$
$$\partial_c \Lambda_{1,2}(\hat{\varphi}_c) = y_c^{-1} \alpha \mathcal{L}^\infty + C \alpha \mathcal{L}^2,$$
$$\partial_c \Lambda_{2,2}(g) = y_c^{-1} \alpha \frac{1}{2} \mathcal{L}^\infty + \alpha (1 - y_c) \mathcal{L}^2,$$

from which, Lemma 11.1 and Remark 11.2, we infer that

$$\Lambda_1(\hat{\varphi}_c) = \alpha \rho_0 \mathcal{L}^\infty + \mathcal{L}^\infty,$$
$$\Lambda_2(g) = \alpha \frac{1}{2} \rho_0 \mathcal{L}^\infty + \rho_0 \mathcal{L}^2,$$
$$\partial_c \Lambda_1(\hat{\varphi}_c) = y_c^{-1} \alpha \mathcal{L}^\infty + \alpha \mathcal{L}^2 + y_c^{-1} \mathcal{L}^2,$$
$$\partial_c \Lambda_2(g) = y_c^{-1} \alpha \frac{1}{2} \mathcal{L}^\infty + \alpha (1 - y_c) \mathcal{L}^2 + y_c^{-1} \mathcal{L}^2.$$
Then by Lemma 12.1, we obtain
\[
\Lambda_3(\tilde{\omega}_e) = \alpha^{-2} y_c (1 + \alpha y_c)^2 (L^2 + \alpha L^\infty), \\
\Lambda_4(g) = \alpha^{-2} \rho_0 (1 + \alpha y_c)^2 (L^2 + \alpha^\frac{1}{2} L^\infty), \\
\partial_c \Lambda_3(\tilde{\omega}_e) = \alpha^{-2} y_c^{-1} (1 + \alpha y_c)^2 (\alpha L^\infty + (1 + \alpha y_c)L^2), \\
\partial_c \Lambda_4(g) = \alpha^{-2} y_c^{-1} (1 + \alpha y_c)^2 ((1 + \alpha \rho_0) L^2 + \alpha^\frac{1}{2} L^\infty).
\]

By Lemma 10.5, we have
\[
\partial_c \left( \frac{1}{(A_2^2 + B_2^2)} u'(y_c) \right) = \frac{\alpha^4 L^\infty}{(1 + \alpha \rho_0)^3 (1 + \alpha y_c)^3 u'(y_c)^3} + \frac{\alpha^4 (L^2 \cap (L^\infty / \rho_0))}{(1 + \alpha \rho_0)^3 (1 + \alpha y_c)^4 u'(y_c)}.
\]

With these estimates, we deduce that
\[
\Lambda_3(\tilde{\omega}_e) \Lambda_4(g) = \alpha^{-4} \rho (1 + \alpha y_c)^4 (L^2 + \alpha L^\infty)(L^2 + \alpha^\frac{1}{2} L^\infty) \\
= \alpha^{-4} \rho (1 + \alpha y_c)^4 (L^1 + \alpha L^2 + \alpha^\frac{3}{2} L^\infty),
\]
and
\[
\partial_c \left( \Lambda_3(\tilde{\omega}_e) \Lambda_4(g) \right) = \Lambda_3(\tilde{\omega}_e) \partial_c \Lambda_4(g) + \Lambda_4(g) \partial_c \Lambda_3(\tilde{\omega}_e) \\
= \alpha^{-2} y_c (1 + \alpha y_c)^2 (L^2 + \alpha L^\infty) \alpha^{-2} y_c^{-1} (1 + \alpha y_c)^2 ((1 + \alpha \rho_0) L^2 + \alpha^\frac{1}{2} L^\infty) \\
+ C \alpha^{-2} \rho_0 (1 + \alpha y_c)^2 (L^2 + \alpha^\frac{1}{2} L^\infty) \left( \alpha^{-2} y_c^{-1} (1 + \alpha y_c)^2 (\alpha L^\infty + (1 + \alpha y_c)L^2) \right) \\
= \alpha^{-4} (1 + \alpha y_c)^4 ((1 + \alpha \rho_0) L^1 + \alpha (1 + \alpha \rho_0) L^2 + \alpha^\frac{3}{2} L^\infty).
\]

Thus, we obtain
\[
u'(y_c) \partial_c K_e(c, \alpha) = \frac{\partial_c \left( \Lambda_3(\tilde{\omega}_e) \Lambda_4(g) \right)}{A_2^2 + B_2^2} + u'(y_c) \Lambda_3(\tilde{\omega}_e) \Lambda_4(g) \partial_c \left( \frac{1}{(A_2^2 + B_2^2)} u'(y_c) \right) \\
= \frac{((1 + \alpha \rho_0) L^1 + \alpha (1 + \alpha \rho_0) L^2 + \alpha^\frac{3}{2} L^\infty)}{(1 + \alpha \rho_0)^2} \\
+ \rho (1 + \alpha y_c)(L^1 + \alpha L^2 + \alpha^\frac{3}{2} L^\infty) \left( \frac{L^\infty}{(1 + \alpha \rho_0)^3 u'(y_c)^2} + \frac{(L^2 \cap (L^\infty / \rho_0))}{(1 + \alpha \rho_0)^3 (1 + \alpha y_c)} \right) \\
= \alpha^\frac{1}{2} L^1.
\]

which gives
\[
\| \partial_c K_e(c, \alpha) \|_{L^1_c} = \| u'(y_c) \partial_c K_e(c, \alpha) \|_{L^1_{yc}} \leq C \alpha^\frac{1}{2}.
\]
Step 3. \( W^{2,1} \) estimate. We normalize \( \| \tilde{\omega}_e \|_{H^2} \leq 1 \) and \( \| g'' - \alpha^2 g \|_{L^2} \leq 1 \). That is, \( \| g'' \|_{L^2}^2 + \alpha^2 \| g' \|_{L^2}^2 + \alpha^4 \| g \|_{L^2}^2 \leq 1 \). Thus, we have

\[
\| g \|_{L^\infty} \leq C \| g \|_{L^2}^{\frac{1}{2}} \| g \|_{H^1}^{\frac{1}{2}} \leq C \alpha^{-\frac{3}{4}},
\]

\[
\| g' \|_{L^\infty} \leq C \| g' \|_{L^2}^{\frac{1}{2}} \| g' \|_{H^1}^{\frac{1}{2}} \leq \frac{C}{\sqrt{\alpha}}.
\]

Thanks to \( \tilde{\omega}_e(0) = 0 \) and \( g'(0) = g(1) = 0 \), we get by Hardy’s inequality that

\[
\| \tilde{\omega}_e' \|_{L^2} \leq C \| \tilde{\omega}_e \|_{L^2} \leq C,
\]

\[
\| g/(1 - y_c) \|_{L^\infty} \leq C \| g' \|_{L^\infty} \leq C \alpha^{-\frac{1}{4}},
\]

\[
\| g' \|_{L^2} \leq C \| g'' \|_{L^2} \leq C.
\]

By Lemma 11.3–Lemma 11.5 and Lemma 10.1, we have

\[
\Lambda_{1,2}(\tilde{\omega}_e) = \alpha \rho_0 \mathcal{L}^\infty, \quad \Lambda_{2,2}(g) = \alpha^{-\frac{1}{2}} \rho_0 \mathcal{L}^\infty,
\]

\[
\partial_c \Lambda_{1,2}(\tilde{\omega}_e) = y_c^{-1} \alpha \mathcal{L}^\infty + \alpha y_c \mathcal{L}^2 \cap \alpha \mathcal{L}^\infty, \quad \partial_c \Lambda_{2,2}(g) = (y_c^{-1} \alpha^{-\frac{1}{2}} + \alpha^\frac{1}{2}) \mathcal{L}^\infty,
\]

\[
\partial_c^2 \Lambda_{1,2}(\tilde{\omega}_e) = y_c^{-1} \rho^{-1} \alpha \mathcal{L}^\infty + \alpha y_c^{-1} \mathcal{L}^2, \quad \partial_c^2 \Lambda_{2,2}(g) = y_c^{-3} (\alpha^\frac{1}{2} y_c + \alpha^{-\frac{1}{2}}) \mathcal{L}^\infty + \alpha y_c^{-1} (1 - y_c) \mathcal{L}^2,
\]

from which, Lemma 11.1 and Remark 11.2, we infer that

\[
\Lambda_1(\tilde{\omega}_e) = \alpha \rho_0 \mathcal{L}^\infty + \mathcal{L}^\infty,
\]

\[
\Lambda_2(g) = \alpha^{-\frac{1}{2}} \rho_0 \mathcal{L}^\infty + (\alpha^{-\frac{1}{2}} (1 - y_c) \mathcal{L}^\infty \cap \alpha^{-\frac{3}{2}} \mathcal{L}^\infty) = \alpha^{-\frac{3}{4}} (1 + \alpha y_c)(1 - y_c) \mathcal{L}^\infty,
\]

\[
\partial_c \Lambda_1(\tilde{\omega}_e) = y_c^{-1} \alpha \mathcal{L}^\infty + \alpha y_c \mathcal{L}^2 \cap \alpha \mathcal{L}^\infty + y_c^{-1} \mathcal{L}^\infty, \quad \partial_c \Lambda_2(g) = (y_c^{-1} \alpha^{-\frac{1}{2}} + \alpha^\frac{1}{2}) \mathcal{L}^\infty,
\]

\[
\partial_c^2 \Lambda_1(\tilde{\omega}_e) = y_c^{-1} \rho^{-1} \alpha \mathcal{L}^\infty + \alpha y_c^{-1} \mathcal{L}^2 + y_c^{-2} \mathcal{L}^2, \quad \partial_c^2 \Lambda_2(g) = y_c^{-3} (\alpha^\frac{1}{2} y_c + \alpha^{-\frac{1}{2}}) \mathcal{L}^\infty + \alpha y_c^{-1} (1 - y_c) \mathcal{L}^2 + y_c^{-2} \mathcal{L}^2.
\]

Then we infer from Lemma 12.1 that

\[
\Lambda_3(\tilde{\omega}_e) = (\alpha^{-2} + y_c^2)(\rho \mathcal{L}^2 \cap \mathcal{L}^\infty) + y_c (y_c + \alpha \rho) \mathcal{L}^\infty,
\]

\[
\Lambda_4(g) = \alpha^{-2} (1 + \alpha y_c) \mathcal{L}^2 \cap (1 + \alpha y_c) \mathcal{L}^2 = \alpha^{-2} (1 + \alpha y_c) \mathcal{L}^2,(1 + \alpha y_c) \mathcal{L}^2,
\]

\[
\partial_c \Lambda_3(\tilde{\omega}_e) = (1 + \alpha y_c) \mathcal{L}^\infty + (\alpha^{-2} + y_c^2) \mathcal{L}^2, \quad \partial_c \Lambda_4(g) = \alpha^{-2} (1 + \alpha y_c)^2 \mathcal{L}^2 + \alpha^\frac{1}{2} \mathcal{L}^\infty = \alpha^{-2} (1 + \alpha y_c)^2 (1 + \alpha y_c) \mathcal{L}^2,
\]

\[
\partial_c^2 \Lambda_3(\tilde{\omega}_e) = \rho^{-1} ((1 + \alpha y_c) \mathcal{L}^\infty + (\alpha^{-2} + y_c^2)(1 + \alpha y_c) \mathcal{L}^2), \quad \partial_c^2 \Lambda_4(g) = (1 + \alpha^{-2} y_c^2)(1 + \alpha y_c) \mathcal{L}^2 + \alpha^\frac{1}{2} \mathcal{L}^\infty = (1 + \alpha^{-2} y_c^2) (1 + \alpha y_c) \mathcal{L}^2.
\]
By Lemma 10.5, we have

\[
\partial_c^2 \left( \frac{1}{(A^2 + B^2)} u'(y_c) \right) = \frac{\alpha^4 L^\infty}{(1 + \alpha \rho_0)^4 (1 + \alpha y_c^2) u'(y_c)^5} + \frac{\alpha^4 (L^2 + \alpha^2 L^\infty)}{(1 + \alpha \rho_0)^3 (1 + \alpha y_c^3) \rho u'(y_c)}. \]

With these estimates, we can deduce that

\[
\Lambda_3(\hat{\omega}_e) \Lambda_4(g) = ((\alpha^{-2} + y_c^2)(\rho L^2 \cap L^\infty) + y_c(y_c + \alpha \rho)L^\infty)\alpha^{-2} \\
\rho (1 + \alpha y_c)(1 + \alpha \rho_0)L^2 \\
= \alpha^{-4} \rho (1 + \alpha y_c)^3 (1 + \alpha \rho_0)(\rho L^1 \cap L^2) \\
+ \alpha^{-2} y_c^2 \rho (1 + \alpha y_c)(1 + \alpha \rho_0)^2 L^2, \]

and

\[
\partial_c \left( \Lambda_3(\hat{\omega}_e) \Lambda_4(g) \right) = \Lambda_4(g) \partial_c \Lambda_3(\hat{\omega}_e) + \Lambda_3(\hat{\omega}_e) \partial_c \Lambda_4(g) \\
= \alpha^{-2} \rho (1 + \alpha y_c)(1 + \alpha \rho_0)L^2 \left( ((1 + \alpha y_c)L^\infty + (\alpha^{-2} + y_c^2)L^2) \\
+ ((\alpha^{-2} + y_c^2)(\rho L^2 \cap L^\infty) + y_c(y_c + \alpha \rho)L^\infty)\alpha^{-2}(1 + \alpha y_c)^2 (1 + \alpha \rho_0)L^2 \right) \\
= \alpha^{-4} \rho (1 + \alpha y_c)^3 (1 + \alpha \rho_0)L^1 + \alpha^{-2} \rho (1 + \alpha y_c)^2 (1 + \alpha \rho_0)L^2 \\
+ \alpha^{-4} (1 + \alpha y_c)^4 (1 + \alpha \rho_0)(\rho L^1 \cap L^2) + \alpha^{-2} y_c^2 (1 + \alpha y_c)^2 (1 + \alpha \rho_0)^2 L^2 \\
= \alpha^{-4} \rho (1 + \alpha y_c)^4 (1 + \alpha \rho_0)L^1 + \alpha^{-2} y_c^2 (1 + \alpha y_c)^2 (1 + \alpha \rho_0)^2 L^2, \]

and

\[
\partial_c^2 \left( \Lambda_3(\hat{\omega}_e) \Lambda_4(g) \right) = \Lambda_4(g) \partial_c^2 \Lambda_3(\hat{\omega}_e) + \Lambda_3(\hat{\omega}_e) \partial_c^2 \Lambda_4(g) + 2 \partial_c \Lambda_4(g) \partial_c \Lambda_3(\hat{\omega}_e) \\
= \alpha^{-2} (1 + \alpha y_c)^2 (1 + \alpha \rho_0)(L^2 + \alpha^{-2}(1 + \alpha y_c)(1 + \alpha \rho_0)L^1) \\
+ ((\alpha^{-2} + y_c^2)(\rho L^2 \cap L^\infty) + y_c(y_c + \alpha \rho)L^\infty) \alpha^{-2}(1 + \alpha y_c)^2 (1 + \alpha \rho_0)L^2 \\
+ ((1 + \alpha y_c)L^\infty + (\alpha^{-2} + y_c^2)L^2) \alpha^{-2}(1 + \alpha y_c)^2 (1 + \alpha \rho_0)L^2 \\
= \alpha^{-2} (1 + \alpha y_c)^2 (1 + \alpha \rho_0)(L^2 + \alpha^{-2}(1 + \alpha y_c)(1 + \alpha \rho_0)L^1) \\
+ \alpha^{-2} (1 + \alpha y_c)^2 (1 + \alpha \rho_0)(\alpha^{-2}(1 + \alpha y_c)(1 + \alpha \rho_0)L^1 + (1 + \alpha \rho_0)L^2) \\
+ \alpha^{-2} (1 + \alpha y_c)^3 (1 + \alpha \rho_0)L^2 + \alpha^{-4} (1 + \alpha y_c)^4 (1 + \alpha \rho_0)L^1 \\
= \alpha^{-2} (1 + \alpha y_c)^3 (1 + \alpha \rho_0)L^2 + \alpha^{-4} (1 + \alpha y_c)^4 (1 + \alpha \rho_0)L^1, \]

from which, we can deduce that

\[
u'(y_c) \partial_c^2 K_e(c, \alpha) = \alpha^2 L^1. \]
which gives
\[
\|\partial_c^2 K_e(c, \alpha)\|_{L^1_\gamma} = \|u'(y_c)\partial_c^2 K_e(c, \alpha)\|_{L^1_\gamma} \leq C\alpha^{\frac{3}{2}}.
\]

Moreover, we have
\[
K_e(c, \alpha) = C_\alpha \rho_0 L^2,
\]
which along with the fact $K_e$ is continuous implies that $K_e(u(0), \alpha) = K_e(u(1), \alpha) = 0$. \hfill \Box

13 Appendix

In this “Appendix”, we present various estimates for some singular integral operators. Recall that the Hilbert transform $Hf(x)$ is defined by
\[
H(f)(x) = p.v. \int \frac{f(y)}{x-y} dy.
\]
In what follows, we assume that $u$ satisfies (S), and let $v$ be defined by (4.4) and $\tilde{c} = v(y_c)$. Let $I_v = (-v(1), v(1))$, $I_v^+ = (0, v(1))$, $I_v^- = (-v(1), 0)$.

For a function $f$ defined in $I_v$, $H(f)(\tilde{c})$ denotes the Hilbert transform of $\tilde{f}(\tilde{c}) = f \chi_{I_v}(\tilde{c})$. We denote by $\|\cdot\|_{L^p}$ the norm of $L^p(I_v)$, and $\|\cdot\|_{W^k,p}$ or $\|\cdot\|_{H^k}$ the Sobolev norm of $W^k,p(I_v)$ or $H^k(I_v)$.

**Lemma 13.1** Let $g \in W^{1,p}(I_v)$ for $p \in (1, +\infty)$. If $g$ is even, then we have
\[
\frac{d}{d\tilde{c}} H(g)(\tilde{c}) = H(g')(\tilde{c}) + \frac{2g(v(1))v(1)}{v(1)^2 - \tilde{c}^2}.
\]
If $g$ is odd, then we have
\[
\frac{d}{d\tilde{c}} H(g)(\tilde{c}) = H(g')(\tilde{c}) + \frac{2g(v(1))\tilde{c}}{v(1)^2 - \tilde{c}^2},
\]
\[
\tilde{c}H(g')(\tilde{c}) = H(zg')(\tilde{c}) + 2g(v(1)),
\]
\[
\tilde{c}H(g)(\tilde{c}) = H(zg)(\tilde{c}).
\]

**Proof** It is easy to check that
\[
\frac{d}{d\tilde{c}} H(g)(\tilde{c}) = H(g')(\tilde{c}) - \frac{g(v(1))}{\tilde{c} - v(1)} + \frac{g(-v(1))}{\tilde{c} + v(1)}, \tag{13.1}
\]
and for $g(z) = -g(-z)$,
\[
\tilde{c}H(g)(\tilde{c}) - H(zg)(\tilde{c}) = \int_{-v(1)}^{v(1)} \frac{\tilde{c}g(z) - zg(z)}{\tilde{c} - z} dz = 0,
\]
\[\Box\]
which give the lemma. □

We need the following point-wise estimate
\[
|H(g)(\tilde{c})| = \left| \text{p.v.} \int_{-v(1)}^{v(1)} \frac{g(z)}{z - \tilde{c}} \, dz \right|
\leq \left| g(z) \ln |z - \tilde{c}| \right|_{z=v(1)}^{v(1)} + \left| \int_{-v(1)}^{v(1)} g'(z) \ln |z - \tilde{c}| \, dz \right|
\leq C\|g\|_{L^\infty} (|\ln(u(1) - c)| + 1) + C\|g'\|_{L^2}
\leq C\|g\|_{H^1} (|\ln(u(1) - c)| + 1). \tag{13.2}
\]

We introduce the average operator \(A^{(z)}(g)(\tilde{c})\) defined by
\[
A^{(z)}(g)(\tilde{c}) = \frac{1}{z - \tilde{c}} \int_{\tilde{c}}^{z} g(z') \, dz'.
\]
Notice that for any \(k \in \mathbb{N}\),
\[
\frac{d^k}{d\tilde{c}^k} A^{(z)}(g)(\tilde{c}) = \frac{1}{z - \tilde{c}} \int_{\tilde{c}}^{z} g^{(k)}(z') \left( \frac{z' - \tilde{c}}{z - \tilde{c}} \right)^k \, dz'.
\]
We infer that for any \(p \in (1, +\infty]\)
\[
\|A^{(z)}(g)\|_{W^{k,p}} \leq C\|g\|_{W^{k,p}}. \tag{13.3}
\]

Lemma 13.2 We define
\[
Z(g)(\tilde{c}) = g'(\tilde{c}) - \frac{g(\tilde{c})}{\tilde{c}}.
\]
For any even function \(g \in W^{k,p}(I_v^\pm) \cap C(I_v)\) for \(p \in (1, +\infty),\) \(k = 2, 3\) and \(g(0) = 0,\) we have \(Z(g) \in W^{k-1,p}(I_v)\) and
\[
\|Z(g)\|_{W^{k-1,p}} \leq C\|g\|_{W^{k,p}(I_v^\pm)}.
\]

Proof Since \(g\) is even, \(Z(g)\) is odd. Thanks to \(Z(g)(z) = g'(z) - A^{(0)}(g') (z)\) by (13.3) we have \(Z(g) \in W^{k-1,p}(I_v \setminus \{0\})\) and \(\|Z(g)\|_{W^{k-1,p}} \leq C\|g\|_{W^{k,p}(I_v^\pm)}\). The fact that \(\lim_{z \to 0^\pm} Z(g)(z) = g'(0 \pm) - g'(0 \pm) = 0\) and that \(\partial_z Z(g)(z)\) is an even function imply \(Z(g) \in W^{k-1,p}(I_v)\), this completes the proof. □

Proposition 13.3 Let \(g \in C^2(I_v) \cap C^3(I_v^\pm)\) be an even function with \(g(0) = 0,\) \(\tilde{c} g(\tilde{c}) \in C^3(I_v)\). Then for any \(p \in (1, +\infty),\)
\[
\left\| \tilde{c} \rho(c) \partial_c^2 \left( \rho(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right) \right\|_{L^p} \leq C \|g\|_{C^2(I_v)} + C \|zg\|_{C^3(I_v)},
\]
\[
\left| \tilde{c} \partial_c \left( \rho(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right) \right| \leq C \left( \|g\|_{C^2(I_v)} + \|zg\|_{C^3(I_v)} \right) \ln(u(1) - c),
\]
\[
\left| \rho(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right| \leq C |\tilde{c}| \|g\|_{C^2(I_v)} + C |\tilde{c}| \|zg\|_{C^3(I_v)}.
\]

**Proof** If \( g \) is even, it follows from Lemma 13.1 that
\[
(v(1)^2 - \tilde{c}^2)(\tilde{c}(Hg)'(\tilde{c}) - Hg(\tilde{c})) = (v(1)^2 - \tilde{c}^2)(\tilde{c}H(g')(\tilde{c}) - Hg(\tilde{c})) + 2\tilde{c}g(v(1))v(1)
\]
\[
= (v(1)^2 - \tilde{c}^2)H(zg' - g)(\tilde{c}) + 2\tilde{c}g(v(1))v(1),
\]
which gives
\[
\rho(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) = \frac{\tilde{c}}{2}(v(1)^2 - \tilde{c}^2)\partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right)
\]
\[
= \frac{v(1)^2 - \tilde{c}^2}{4\tilde{c}}(\tilde{c}(Hg)'(\tilde{c}) - Hg(\tilde{c}))
\]
\[
= \frac{u(1) - c}{4} \frac{H(Z(g))}{\tilde{c}} + \frac{g(v(1))v(1)}{2}.
\]
Suppose also that \( g(0) = 0 \) then \( Z(g) \) is odd, and
\[
\rho(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) = \frac{u(1) - c}{4} H(Z(g))\tilde{c} + \frac{g(v(1))v(1)}{2}. \tag{13.4}
\]
We get by Lemma 13.1 again that
\[
\partial_c \left( \rho(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right) = -\frac{1}{4} H(Z(g))(\tilde{c})
\]
\[
+ \frac{v(1)^2 - \tilde{c}^2}{8\tilde{c}} (HZ(g))'(\tilde{c})
\]
\[
= -\frac{1}{4} H(Z(g)) + \frac{v(1)^2 - \tilde{c}^2}{8\tilde{c}} H(Z(g))'(\tilde{c}) + \frac{1}{4} Z(g)(v(1))
\]
\[
= -\frac{1}{4} H(Z(g)) + \frac{v(1)^2 - \tilde{c}^2}{8\tilde{c}^2} (H(zZ(g))'(\tilde{c})
\]
\[
+ 2Z(g)(v(1))) + \frac{1}{4} Z(g)(v(1)), \tag{13.5}
\]
and that

\( \tilde{c} \) Springer
\[
\partial_c^2 \left( \rho(c) \partial_c \left( \frac{1}{2c} H(g)(\tilde{c}) \right) \right) \\
= -\frac{1}{8c} \left( H(Z(g)')(\tilde{c}) + \frac{2Z(g)(v(1))\tilde{c}}{v(1)^2 - \tilde{c}^2} \right) \\
- \frac{v(1)^2}{8\tilde{c}^4} \left( H(zZ(g)')(\tilde{c}) + 2Z(g)(v(1)) \right) \\
+ \frac{v(1)^2 - \tilde{c}^2}{16c^3} \left( H((zZ(g)')(\tilde{c}) + \frac{2Z(g)(v(1))v(1)\tilde{c}}{v(1)^2 - \tilde{c}^2} \right) \\
= -\frac{v(1)^2 + \tilde{c}^2}{8c^3} H(Z(g)')(\tilde{c}) + \frac{v(1)^2 - \tilde{c}^2}{16c^3} H((zZ(g)')(\tilde{c}) \\
- \frac{Z(g)(v(1))}{(v(1)^2 - \tilde{c}^2) + \frac{Z(g)(v(1))v(1)}{8c^2}}.
\]

from which and Lemma 13.2, we infer that

\[
\left\| \tilde{c}^2 \rho(c) \partial_c \left( \rho(c) \partial_c \left( \frac{1}{2c} H(g)(\tilde{c}) \right) \right) \right\|_{L^p} \\
\leq C \left\| H(Z(g)') \right\|_{L^p} + C \| H((zZ(g)')(\tilde{c}) \|_{L^p} + C \| Z(g)\|_{W^{1,\infty}} \\
\leq C \left\| Z(g)' \right\|_{L^p} + C \| zZ(g)'(\tilde{c}) \|_{L^p} + C \| Z(g)\|_{W^{1,\infty}} \\
\leq C \| Z(g)\|_{W^{1,p}} + C \| zZ(g)\|_{W^{2,p}} + C \| Z(g)\|_{W^{1,\infty}} \\
\leq C \| \tilde{c}g \|_{C^3(I_v)} + C \| g \|_{C^2(I_v)}.
\]

As \( Z(g)' \) is even, we have

\[
(v(1)^2 - \tilde{c}^2) H(Z(g)')(\tilde{c}) \\
= (v(1)^2 - \tilde{c}^2) Z(g)'(v(1)) \ln \left| \frac{v(1) - \tilde{c}}{v(1) + \tilde{c}} \right| - (v(1)^2 - \tilde{c}^2) \\
\times \int_{-v(1)}^{v(1)} Z(g)''(z) \ln |\tilde{c} - z| dz, \\
\times \int_{-v(1)}^{v(1)} Z(g)''(z) \ln |\tilde{c} - z| dz = \frac{1}{2} \int_{-v(1)}^{v(1)} zZ(g)''(z) - \frac{1}{z} \ln \left| \frac{\tilde{c} - z}{\tilde{c} + z} \right| dz, \\
\left\| \frac{1}{z} \ln \left| \frac{\tilde{c} - z}{\tilde{c} + z} \right| \right\|_{L^1(I_v)} \leq C, \\
\| zZ(g)''(z) \|_{L^\infty(I_v)} \leq \| zZ(g)(z) \|_{C^2(I_v)} + 2 \| Z(g)(z) \|_{C^1(I_v)} \leq C \| g \|_{C^2(I_v)}.
\]

which implies that for any \( p \in (1, \infty), \)

\[
\left\| (v(1)^2 - \tilde{c}^2) H(Z(g)') \right\|_{L^p} \leq C \| \tilde{c}g \|_{C^3(I_v)} + C \| g \|_{C^2(I_v)}. \tag{13.6}
\]
Thus, we obtain
\[
\left\| \tilde{c} p(c) \partial_c^2 \left( p(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right) \right\|_{L^p} \leq C \| g \|_{C^2(I_v)} + C \| \tilde{c}g \|_{C^3(I_v)}.
\]

By (13.5), we have
\[
\tilde{c} \partial_c \left( p(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right) = -\frac{\tilde{c}}{4} H(Z(g))(\tilde{c}) + \frac{v(1)^2 - \tilde{c}^2}{8} H(Z(g)'(\tilde{c})) + \frac{\tilde{c}}{4} Z(g)(v(1)),
\]
which along with (13.2) and (13.6) gives
\[
\left| \tilde{c} \partial_c \left( p(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right) \right| \leq C \| H(Z(g)) \| + C \| \tilde{c}g \|_{C^3(I_v)} + C \| g \|_{C^2(I_v)}
\]
\[
\leq C \| g \|_{C^2(I_v)} \left( | \ln(u(1) - c) \right) + 1 \right) + C \| \tilde{c}g \|_{C^3(I_v)} + C \| g \|_{C^2(I_v)}.
\]
This along with the fact that \( p(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \bigg|_{\tilde{c}=0} = 0 \) (see remark 13.4) gives
\[
\left| p(c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) \right| \leq C \| \tilde{c} \| \| g \|_{C^2(I_v)} + C \| \tilde{c} \| \| \tilde{c}g \|_{C^3(I_v)}.
\]

This completes the proof of the proposition. \( \square \)

**Remark 13.4** Using the formula
\[
p.v. \int_{v(1)}^{u(1)} \frac{g(z)}{\tilde{c} - y} dy = p.v. \int_{v(1)}^{u(1)} \frac{g(z) - g(0)}{\tilde{c} - z} dz - g(0) \ln \left| \frac{\tilde{c} - v(1)}{\tilde{c} + v(1)} \right|
\]
and the fact that \( \xi(\tilde{c}) = \frac{1}{\tilde{c}} (g(\tilde{c}) - g(0)) \) is odd, we get by Lemma 13.1 that
\[
p.v. \int_{v(1)}^{u(1)} \frac{g(z)}{\tilde{c} - z} dz = \tilde{c} p.v. \int_{v(1)}^{u(1)} \frac{\xi(z)}{\tilde{c} - z} dz - g(0) \ln \left| \frac{\tilde{c} - v(1)}{\tilde{c} + v(1)} \right|
\]
Therefore,
\[
\frac{1}{2} \tilde{c} (u(1) - c) \partial_c \left( \frac{1}{2\tilde{c}} H(g)(\tilde{c}) \right) = \frac{1}{2} \tilde{c} (u(1) - c) \partial_c \left( \frac{1}{2} \int_{v(1)}^{u(1)} \frac{\xi(z)}{\tilde{c} - z} dz - \frac{g(0)}{2\tilde{c}} \ln \left| \frac{\tilde{c} - v(1)}{\tilde{c} + v(1)} \right| \right)
\]
\[= \frac{1}{2} \tilde{c} (u(1) - c) \left( \frac{1}{2} \int_{v(1)}^{u(1)} \frac{\xi(z)}{\tilde{c} - z} dz + \frac{\xi(0)\tilde{c}}{u(1) - c} \right) - \partial_c \left( \frac{g(0)}{2\tilde{c}} \ln \left| \frac{\tilde{c} - v(1)}{\tilde{c} + v(1)} \right| \right).
\]
which implies that $\rho(c)\partial_\epsilon\left(\frac{1}{\epsilon}H(g)(\tilde{c})\right)\bigg|_{\tilde{c}=0} = 0.$

**Lemma 13.5** Assume that $\varphi_k \in W^{1,p}(I_v)$, $p \in (1, +\infty)$ for $k = 1, 2, 3, 4$ satisfies

$$\varphi_1(\tilde{c})\varphi_4(\tilde{c}) + \varphi_3(\tilde{c})\varphi_2(\tilde{c})\bigg|_{\tilde{c}=\pm v(1)} = 0. \quad (13.7)$$

Let $G(\tilde{c}) = \varphi_4(\tilde{c})H(\varphi_1)(\tilde{c}) + \varphi_3(\tilde{c})H(\varphi_2)(\tilde{c})$. Then it holds that

$$\|\partial_\tilde{c}((u(1) - c)G)\|_{L^p} \leq C\|\varphi_1\|_{W^{1,p}}\|\varphi_4\|_{W^{1,p}} + C\|\varphi_2\|_{W^{1,p}}\|\varphi_3\|_{W^{1,p}},$$

$$\|\partial_\tilde{c}^2((u(1) - c)G)\|_{L^2} \leq C\|\varphi_1\|_{H^2}\|\varphi_4\|_{H^2} + C\|\varphi_2\|_{H^2}\|\varphi_3\|_{H^2}.$$ 

**Proof** By (13.1) and (13.7), it is easy to deduce that

$$\frac{d}{d\tilde{c}}(\varphi_4(\tilde{c})H(\varphi_1)(\tilde{c}) + \varphi_3(\tilde{c})H(\varphi_2)(\tilde{c}))$$

$$= \varphi'_4(\tilde{c})H(\varphi_1)(\tilde{c}) + \varphi'_3(\tilde{c})H(\varphi_2)(\tilde{c})$$

$$+ \varphi_4(\tilde{c})\frac{d}{d\tilde{c}}(H(\varphi_1)(\tilde{c}) + \varphi_3(\tilde{c})\frac{d}{d\tilde{c}}H(\varphi_2)(\tilde{c}))$$

$$= \varphi'_4(\tilde{c})H(\varphi_1)(\tilde{c}) + \varphi'_3(\tilde{c})H(\varphi_2)(\tilde{c})$$

$$+ \varphi_4(\tilde{c})\left(H(\varphi'_1)(\tilde{c}) - \varphi_1(\tilde{v}(1)) + \frac{\varphi_1(-\tilde{v}(1))}{\tilde{c} - \tilde{v}(1)} + \frac{\varphi_1(\tilde{v}(1))}{\tilde{c} + \tilde{v}(1)}\right)$$

$$+ \varphi_3(\tilde{c})\left(H(\varphi'_2)(\tilde{c}) - \varphi_2(\tilde{v}(1)) + \frac{\varphi_2(-\tilde{v}(1))}{\tilde{c} - \tilde{v}(1)} + \frac{\varphi_2(\tilde{v}(1))}{\tilde{c} + \tilde{v}(1)}\right)$$

$$= \varphi'_4(\tilde{c})H(\varphi_1)(\tilde{c}) + \varphi'_3(\tilde{c})H(\varphi_2)(\tilde{c}) + \varphi_4(\tilde{c})H(\varphi'_1)(\tilde{c}) + \varphi_3(\tilde{c})H(\varphi'_2)(\tilde{c})$$

$$- \varphi_1(\tilde{v}(1))A^{(\tilde{v}(1))}(\varphi'_4)(\tilde{c}) + \varphi_1(-\tilde{v}(1))A^{(-\tilde{v}(1))}(\varphi'_4)(\tilde{c})$$

$$- \varphi_2(\tilde{v}(1))A^{(\tilde{v}(1))}(\varphi'_3)(\tilde{c}) + \varphi_2(-\tilde{v}(1))A^{(-\tilde{v}(1))}(\varphi'_3)(\tilde{c}),$$

which along with (13.3) and (13.2) gives the first inequality of the lemma. The proof of the second inequality is similar.

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