On Bohmian trajectories in two-particle interference devices

Louis Marchildon

Département de physique, Université du Québec,
Trois-Rivières, Qc. Canada G9A 5H7
email: marchild@uqtr.uquebec.ca

Abstract

Claims have been made that, in two-particle interference experiments involving bosons, Bohmian trajectories may entail observable consequences incompatible with standard quantum mechanics. By general arguments and by an examination of specific instances, we show that this is not the case.

PACS No.: 03.65.Ta

1 Introduction

The question whether quantum mechanics can be derived from a deterministic theory has been raised for a long time [1, 2], and answered in the affirmative by Bohm [3]. In Bohmian mechanics [4, 5], each particle follows a trajectory that obeys equations of motion much like those of classical mechanics. To the total force acting on a particle, however, there is a contribution coming from a “quantum potential,” which is related to the amplitude of the Schrödinger wave function. Although a particle has, at any time \( t \), a well-defined position and momentum, these cannot be known exactly. One can only know the probability that, at time \( t \), the particle is in a given region of space, or has a momentum in a given range. The position probability is obtained from the absolute square of the wave function and, in this way, Bohmian mechanics reproduces exactly the statistical predictions of quantum mechanics.

Recent papers [6, 7, 8] have argued that although statistical predictions of Bohmian and quantum mechanics may coincide, the two will disagree as
far as certain individual events are concerned. This, it is claimed, will occur in two-particle interference experiments, where Bohmian mechanics would predict correlations in individual events that contradict quantum mechanics. The purpose of this note is firstly to give a general argument that this cannot be the case, and then to show how the specific experiments proposed fail to establish the intended conclusion.

2 Two-particle interference

The general situation can be developed in terms of a two-slit interferometer, as shown in Fig. 1. Two identical bosons are prepared in a state such that, at time $t = 0$, one is near the upper and the other one near the lower slit. The system is assumed to be symmetric with respect to the $yz$ plane.

![Two-slit interferometer](image)

Figure 1: Two-slit interferometer. Solid lines are schematic, and do not represent the exact shape of Bohmian trajectories.

The wave function can be written as

$$\Psi(r_1, r_2) = \psi_A(r_1)\psi_B(r_2) + \psi_A(r_2)\psi_B(r_1).$$  \hspace{1cm} (1)

As it should with bosons, it is symmetric with respect to the interchange of both particles. An alternative choice for the wave function is given by

$$\tilde{\Psi}(r_1, r_2) = [\tilde{\psi}_A(r_1) + \tilde{\psi}_B(r_1)][\tilde{\psi}_A(r_2) + \tilde{\psi}_B(r_2)].$$  \hspace{1cm} (2)

This allows both particles to go through the same slit. Note that (2) is a special case of (1) if we set $\psi_B = \psi_A$ and no longer require $\psi_A$ to be centered about a specific slit.
In quantum mechanics, the probability that one particle is detected in region $R_1$ and the other particle in region $R_2$ on the screen at time $t$ is given by

$$P(R_1, R_2; t) = \int_{R_1} \int_{R_2} |\Psi(r_1, r_2; t)|^2.$$

(3)

This, in general, will display interference patterns.

In Bohmian mechanics, each particle of a given pair has a well-defined trajectory associated with the following velocities:

$$v_1 = \frac{\hbar}{m} \text{Im} \frac{\nabla_1 \Psi}{\Psi} = \frac{1}{m} \nabla_1 S,$$

(4)

$$v_2 = \frac{\hbar}{m} \text{Im} \frac{\nabla_2 \Psi}{\Psi} = \frac{1}{m} \nabla_2 S.$$

(5)

Here $S(r_1, r_2; t)$ is the phase of the total wave function, in units of $\hbar$.

The initial wave functions of the two bosons are assumed to transform into each other under reflection. That is

$$\psi_A(r) = \psi_B(r'),$$

(6)

where $r'$ is obtained from $r$ by reflection in the plane of symmetry, specifically $x' = -x, \ y' = y, \ z' = z$. Since the experimental arrangement shares that symmetry, the wave functions will satisfy (6) at any time $t$. The Bohmian trajectories of a pair of bosons, however, will not in general transform into each other under reflection. This comes from the fact that the initial values of the position of each boson are unknowable in principle, and are each statistically distributed according to the absolute square of the wave function.

It is not difficult to show that

$$v_{1x}(r_1, r_2; t) = -v_{1x}(r_1', r_2'; t),$$

(7)

$$v_{2x}(r_1, r_2; t) = -v_{2x}(r_1', r_2'; t).$$

(8)

This implies that, if both particles are simultaneously on the plane of symmetry, both velocities vanish, and neither particle should cross the plane. If that were always so, Bohmian mechanics would predict that one particle would always be detected at positive, and the other one at negative values of $x$. However, the overwhelming majority of pairs are not simultaneously on the plane of symmetry. (7) and (8) therefore do not prevent them from crossing the plane.
A related kind of restriction to the motion of particles can be derived by considering the phase of the total wave function. Let us write it as \( S(\mathbf{r}, \mathbf{R}; t) \), where \( \mathbf{r} \) and \( \mathbf{R} \) are the relative and center-of-mass coordinates of the two particles. From (4) and (5), it is easy to see that

\[
v_1 + v_2 = \frac{1}{m} \nabla_1 S + \frac{1}{m} \nabla_2 S = \frac{1}{m} \nabla R S.
\]  

(9)

Thus, if the phase does not depend on the center-of-mass coordinate \( X \), we have \( v_{1x} + v_{2x} = 0 \), so that

\[
x_1 + x_2 = 2\bar{x},
\]  

(10)

where \( \bar{x} \) is a constant. This means that the motion of the particles along the \( x \) axis is symmetric with respect to the plane \( x = \bar{x} \). Suppose there is a limit to the value of \( |\bar{x}| \). Bohmian mechanics then predicts that there will be no pair of particles detected both above \( |\bar{x}| \) or both below \(-|\bar{x}|\). This has been taken to imply a contradiction with orthodox quantum mechanics where, it is claimed, there is a nonzero probability of finding both particles in one of these regions.

That there cannot be a contradiction of this kind can be seen as follows. Suppose for instance that Bohmian mechanics implies that two particles can never be simultaneously detected above the plane \( x = \bar{x} \). This is equivalent to the statement that the probability of both particles being above \( \bar{x} \) vanishes. In Bohmian mechanics, that probability is given by an expression like (3), where \( R_1 \) and \( R_2 \) are the regions \( x_1 > \bar{x} \) and \( x_2 > \bar{x} \), and \(|\Psi(\mathbf{r}_1, \mathbf{r}_2; t)|^2\) is the proportion of pairs whose true values of position at \( t \) are \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). Since the probability is computed in quantum mechanics with the same formula and with a \( \Psi \) which, albeit differently interpreted, has the same numerical value, it must vanish under exactly the same conditions as the Bohmian probability.

We shall now examine some specific cases and see how the agreement comes about.

### 3 Specific cases

Our first example is to take \( \psi_A \) and \( \psi_B \) to be given by plane waves. This is the case discussed by Ghose [6, 7]. At time \( t = 0 \), we set

\[
\psi_A(\mathbf{r}) = \exp\{i(k_x x + k_y y)\},
\]  

(11)
where for simplicity $z$ has been eliminated. From (6) we get
\[
\psi_B(r) = \exp\{i(-k_x x + k_y y)\}. \tag{12}
\]
Assuming free propagation with time, we find that the total wave function (1) is given by
\[
\Psi(r_1, r_2; t) = \psi_A(r_1; t)\psi_B(r_2; t) + \psi_A(r_2; t)\psi_B(r_1; t)
= 2 \cos\{k_x(x_1 - x_2)\} \exp\left\{i \left[k_y(y_1 + y_2) - \frac{\hbar}{m}(k_x^2 + k_y^2)t\right]\right\}. \tag{13}
\]
It is easy to show that not only does (10) hold, but $x_1$ and $x_2$ are separately constant. However, the marginal probability densities of $x_1$ and $x_2$ are both uniformly distributed. This means that for each pair of particles, the initial (and final) values of $x_1$ and $x_2$ are in no way constrained to lie on both sides of the plane $x = 0$, or any given plane $x = \text{constant}$ for that matter. The fact that both particles can be above, or below, the plane $x = 0$ also holds if plane waves are replaced with spherical waves [9, 10]. The agreement between Bohmian and quantum mechanics here comes from the fact that although for each pair there is a plane $x = \bar{x}$ about which both particles have symmetrical $x$ coordinates, the value of $\bar{x}$ changes from pair to pair, and is totally unknown for any given pair.

Of course, plane waves will not represent a realistic two-slit experiment, precisely because the $x$ coordinates have to be restricted at $t = 0$. But the example is instructive, and shows that with no restrictions on the initial $x$ positions of the particles, they will end up in all regions on the screen.

We will soon analyse a situation with waves initially restricted to the width of the slits. But let us first examine a one-dimensional case of restriction, using harmonic oscillator wave functions. We write
\[
\psi_A(x; t) \sim \exp\left\{-\frac{m\omega}{2\hbar}(x - a \cos \omega t)^2 \right.
- \frac{i}{2} \left[\omega t + \frac{m\omega}{2\hbar}(4ax \sin \omega t - a^2 \sin 2\omega t)\right]\right\}. \tag{14}
\]
This represents a wave packet of width $\sqrt{\hbar/2m\omega}$ whose center oscillates between $x = a$ and $x = -a$. Following (13), we set $\psi_B(x; t) = \psi_A(-x; t)$. For Bose-Einstein statistics, that is, for a total wave function given as in (1), Holland ([4], p. 300 ff.) shows that the phase does not depend on $x_1 + x_2$,
so that (10) holds. For each pair of oscillators, there is a plane \( x = \bar{x} \) that separates the two oscillators, so that their trajectories do not cross.

Let us assume that \( \sqrt{\hbar/2m\omega} \ll a \), that is, the width of the packet is much smaller than the amplitude of oscillation. Most values of \( |\bar{x}| \) are then smaller than or on the order of \( \sqrt{\hbar/2m\omega} \). We can see how the agreement between Bohmian and quantum mechanics comes about. For \( t \) such that \( |a \cos \omega t| \gg \sqrt{\hbar/2m\omega} \), members of a given Bohmian pair of oscillators are on different sides of the plane \( x = 0 \). But then the quantum wave packets are also widely separated. For \( t \) such that \( |a \cos \omega t| \approx \sqrt{\hbar/2m\omega} \), the wave packets overlap. But then two Bohmian oscillators on different sides of an \( x = \bar{x} \) plane can be on the same side of the \( x = 0 \) plane.

Let us now turn to a more realistic description of two-slit interference. A wave packet emerging from slit \( A \) can be modelled by a Gaussian wave function of the type

\[
\tilde{\psi}_A(r) = (2\pi\sigma_0^2)^{-1/4} \exp \left\{ -\frac{(x - a)^2}{4\sigma_0^2} + i[k_x(x - a) + k_y y] \right\},
\]

where \( \sigma_0 \) corresponds to the half-width of the slit and \( a \) is shown in Fig. 1.

We take \( \sigma_0 \ll a \). Assuming free propagation with time, we have

\[
\tilde{\psi}_A(r; t) = (2\pi\sigma_t^2)^{-1/4} \exp \left\{ -\frac{[x - a - (\hbar k_x/m)t]^2}{4\sigma_0\sigma_t} + i \left\{ k_x[x - a - (\hbar k_x/2m)t] + k_y y - (\hbar k_y^2/2m)t \right\} \right\},
\]

where

\[
\sigma_t = \sigma_0 \left( 1 + \frac{i\hbar t}{2m\sigma_0^2} \right).
\]

It is shown in [8] that, if (2) represents the total wave function and \( \tilde{\psi}_B(r; t) \) is given by (3), we have (in [8] \( x \) and \( y \) are interchanged and the term \( 2i k_x \) is in the end inadvertently omitted)

\[
v_{1x} + v_{2x} = \frac{(\hbar/2m\sigma_0^2)^2(x_1 + x_2)}{1 + (\hbar t/2m\sigma_0^2)^2} + \frac{\hbar}{m} \text{Im} \left\{ \frac{1}{\Psi} \left[ \frac{a + (\hbar k_x/m)t}{\sigma_0\sigma_t} + 2ik_x \right] \cdot \left[ \tilde{\psi}_A(r_1; t)\tilde{\psi}_A(r_2; t) - \tilde{\psi}_B(r_1; t)\tilde{\psi}_B(r_2; t) \right] \right\}.
\]
To establish a contradiction between Bohmian mechanics and quantum mechanics, Golshani and Akhavan consider separately the case where both particles go through different slits and the case where they go through the same slit. With the wave function (2), that distinction has meaning only in Bohmian mechanics, since in orthodox quantum mechanics we cannot assert that a particle has gone through a slit unless it has been measured to do so.

That the two particles go through different slits, in Bohmian mechanics, means the following: At \( t = 0 \),

\[
\begin{align*}
  a - \sigma_0 \lesssim x_1 \lesssim a + \sigma_0 \quad \text{and} \quad -a - \sigma_0 \lesssim x_2 \lesssim -a + \sigma_0,
\end{align*}
\]

or vice versa. Contrary to the claim made in [8], this does not imply that \( \psi_A(r_1) = \psi_B(r_2) \) or that \( \psi_A(r_2) = \psi_B(r_1) \).

The case where both particles go through different slits is best discussed in terms of wave function (1), where Bohmian mechanics definitely says so and quantum mechanics predicts with certainty (neglecting exponential tails of the wave function) that a measurement made at \( t = 0 \) finds both particles at different slits. There (18) holds with the last term absent. It is readily integrated as [8]

\[
x_1 + x_2 = 2\bar{x}\sqrt{1 + (\hbar t/2m\sigma_0^2)^2},
\]

where \( 2\bar{x} = x_1(0) + x_2(0) \). Note that \( -\sigma_0 \lesssim \bar{x} \lesssim \sigma_0 \), corresponding to the spread in values of \( x_1(0) \) and \( x_2(0) \). Does the constraint (20) imply observational consequences that contradict quantum mechanics?

That there are none can best be seen by examining the following limiting cases, where \( t_f \) is the time of arrival at the screen: (i) \( \hbar t_f/2m\sigma_0^2 \ll 1 \), and (ii) \( \hbar t_f/2m\sigma_0^2 \gg 1 \). For simplicity, we assume that \( k_x = 0 \). In (i), we have \( x_1(t_f) + x_2(t_f) \approx 2\bar{x} \). This suggests that \( x_1(t_f) \) and \( x_2(t_f) \) remain widely separated. But this is also what quantum mechanics predicts, since the wave functions \( \psi_A \) and \( \psi_B \) do not really spread beyond \( \sigma_0 \). In (ii), on the other hand, equation (16) shows that the spread of the wave functions at \( t_f \) is on the order of \( \hbar t_f/m\sigma_0 \). There is overlap if the spread is on the order of \( a \), and quantum mechanics no longer predicts that both particles are on different sides of the \( x = 0 \) plane. But then so does Bohmian mechanics, since \( x_1(t_f) + x_2(t_f) \approx \bar{x}\hbar t_f/m\sigma_0 \approx a \).
4 Conclusion

We have shown that there is no reason to expect discrepancies between Bohmian and quantum mechanics in the context of two-particle interference devices. Our general argument was illustrated with analysis of specific cases. Additional insight could be obtained by detailed numerical calculations of Bohmian trajectories associated with two-particle two-slit experiments.

It is a pleasure to thank Gianluca Introzzi for help in clarifying some of the issues discussed here.

References

[1] A. Einstein, B. Podolsky and N. Rosen. Phys. Rev. 47, 777 (1935).
[2] J. von Neumann. Mathematical foundations of quantum mechanics. Princeton University Press, Princeton. 1955.
[3] D. Bohm. Phys. Rev. 85, 166, 180 (1952).
[4] P.R. Holland. The quantum theory of motion. Cambridge University Press, Cambridge. 1993.
[5] D. Bohm and B.J. Hiley. The undivided universe. Routledge, London. 1993.
[6] P. Ghose. “Incompatibility of the de Broglie-Bohm theory with quantum mechanics.” quant-ph/0001024.
[7] P. Ghose. “An experiment to distinguish between de Broglie-Bohm and standard quantum mechanics.” quant-ph/0003037.
[8] M. Golshani and O. Akhavan. “A two-slit experiment which distinguishes between the standard and Bohmian quantum mechanics.” quant-ph/0009040.
[9] L. Marchildon. “No contradictions between Bohmian and quantum mechanics.” quant-ph/0007068.
[10] P. Ghose. “Reply to No contradictions between Bohmian and quantum mechanics.” quant-ph/0008007.