The configuration space of general relativity is *superspace* - the space of all Riemannian 3-metrics modulo diffeomorphisms. However, it has been argued that the configuration space for gravity should be *conformal superspace* - the space of all Riemannian 3-metrics modulo diffeomorphisms and conformal transformations. Recently a manifestly 3-dimensional theory was constructed with conformal superspace as the configuration space. Here a fully 4-dimensional action is constructed so as to be invariant under conformal transformations of the 4-metric using general relativity as a guide. This action is then decomposed to a (3 + 1)-dimensional form and from this to its Jacobi form. The surprising thing is that the new theory turns out to be *precisely* the original 3-dimensional theory. The physical data is identified and used to find the physical representation of the theory. In this representation the theory is extremely similar to general relativity. The clarity of the 4-dimensional picture should prove very useful for comparing the theory with those aspects of general relativity which are usually treated in the 4-dimensional framework.

1 Introduction

As formulated by Einstein, the natural arena for gravity as represented by general relativity is spacetime. We have a purely 4-dimensional structure and the 4-geometry reigns. The reformulation of the theory in canonical dynamical form by Dirac [1] and Arnowitt, Deser and Misner (ADM) [2] led away from the 4-dimensional picture and placed the emphasis more on the 3-geometry. The configuration space is superspace and general relativity describes the evolution of the 3-geometry in time (geometrodynamics). York [3] went further and the identified the conformal 3-geometry with the dynamical degrees of freedom of the gravitational field. The correct configuration space for gravity should not be superspace but rather *conformal superspace* - superspace modulo conformal transformations.

Alas, general relativity and conformal superspace are not entirely compatible. Barbour and Ó Murchadha [4] constructed a theory with conformal superspace at the very core. They took the Jacobi action for general relativity of Baerlein, Sharp and Wheeler (BSW) [5] and constructed an action which is invariant under conformal transformations of the 3-metric. Conformal superspace arises naturally in this theory.

In this paper we construct a *four*-dimensional action based on conformal transformations of the *four*-metric. We then decompose this to a (3 + 1)-dimensional form and from this we find the Jacobi action of the theory. Incredibly, it turns out to be the same as that of Barbour and Ó Murchadha. We will begin with a review of some general relativity before considering the new theory.
2 The Action

The Einstein-Hilbert action of general relativity is well known. It has the form

\[ S = \int \sqrt{-g} \, (4) R \, d^4x \]  \hspace{1cm} (1)

where \( g_{\alpha\beta} \) is the 4-metric and \( (4) R \) is the four-dimensional Ricci scalar. The action is varied with respect to \( g_{\alpha\beta} \) and the resulting equations are the (vacuum) Einstein equations

\[ G^{\alpha\beta} = \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0 \]  \hspace{1cm} (2)

We would like to construct an action which is invariant under conformal transformations of the metric

\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \]  \hspace{1cm} (3)

where \( \Omega \) is a strictly positive function using the Einstein-Hilbert action as a guide. First we need to develop some machinery for dealing with conformal transformations.

2.1 Dimensional Properties of Conformal Transformations

A supposed problem with conformal transformations and different numbers of dimensions is that various coefficients change when the number of dimensions changes. This turns out not to be a problem in this analysis as will be shown.

Let us consider conformal transformations and the scalar curvature. If we make a conformal transformation of the metric of the form in equation (3) above then the Ricci tensor transforms as

\[ (n) R_{\alpha\beta} \rightarrow (n) R_{\alpha\beta} + 2(n-2) \left( \nabla_\alpha \Omega \right) \nabla_\beta \Omega \Omega^2 - (n-2) \frac{\nabla_\alpha \nabla_\beta \Omega}{\Omega} \]  \hspace{1cm} (4)

Then we can contract to find that the scalar curvature transforms as

\[ (n) R \rightarrow \Omega^{-2} \left( (n) R - 2(n-1) \right. \frac{\nabla_\alpha \nabla_\beta \Omega}{\Omega} + (n-1)(4-n) \frac{\nabla_\alpha \nabla_\gamma \Omega}{\Omega^2} \right) \]  \hspace{1cm} (5)

where \( n \) is the number of dimensions. A consequence is that the combination

\[ \phi^2/s \left( (n) R - \frac{4(n-1)}{(n-2)} \frac{\nabla_\alpha \nabla_\beta \Omega}{\Omega} \right) \]  \hspace{1cm} (6)

is conformally invariant for any scalar function \( \phi \) under the combined transformation

\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \; , \; \phi \rightarrow \Omega^s \phi \]  \hspace{1cm} (7)

where \( s = 1 - \frac{n}{2} \). While this is true in any number of dimensions we are of course most concerned with the 3-dimensional and 4-dimensional cases. In 3 dimensions we have \( s = -\frac{1}{2} \). Thus we get that

\[ \phi^{-4} \left( (3) R - \frac{8 \nabla^2 \phi}{\phi} \right) \]  \hspace{1cm} (8)
is conformally invariant under the transformation
\[ g_{ab} \rightarrow \Omega^2 g_{ab}, \quad \phi \rightarrow \frac{\phi}{\sqrt{\Omega}} \] (9)

In four dimensions \( s = -1 \) and the combination
\[ \phi^{-2} \left( (4)R - \frac{6\Box \phi}{\phi} \right) \] (10)
is conformally invariant under the transformation
\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta}, \quad \phi \rightarrow \frac{\phi}{\Omega} \] (11)

Then the combination
\[ \sqrt{-(4)g} \phi^2 \left( (4)R - \frac{6\Box \phi}{\phi} \right) \] (12)
is also conformally invariant. This will be our Lagrangian density \( L \). Thus our action is
\[ S = \int L \, d^4x \] (13)

Before we decompose this to a \((3 + 1)\)-dimensional form let us consider the 4-dimensional structure and see what emerges.

2.2 Varying with respect to \( g_{\alpha\beta} \)

The variation with respect to \( g_{\alpha\beta} \) is quite straightforward. The resulting equations of motion are
\[ -\phi^2 \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) + 4 \nabla^\alpha \phi \nabla^\beta \phi - g^{\alpha\beta} \nabla^\gamma \phi \nabla^\gamma \phi - 2 \phi \nabla^\alpha \nabla^\beta \phi + 2 g^{\alpha\beta} \phi \Box \phi = 0 \] (14)

This looks quite complicated but it is actually just
\[ \widetilde{G}^{\alpha\beta} = 0 \] (15)

where \( \widetilde{G}^{\alpha\beta} \) is just the Einstein tensor conformally transformed with conformal factor \( \phi \). Equivalently, this is the Einstein tensor for the metric \( \phi^2 g_{\alpha\beta} \). This interpretation will prove useful later.

2.3 Varying with respect to \( \phi \)

Again, this variation is fairly straightforward. We get
\[ (4)R - \frac{6\Box \phi}{\phi} = 0 \] (16)

This is actually the trace of \( \Box \phi \)¹ and so, as such, is redundant. This can be viewed as a result of \( \phi \) being pure gauge. Work by Barbour on the variation of gauge variables [7] shows that for a pure gauge variable \( \psi \) we may vary the action with respect to both \( \psi \) and its time derivative \( \dot{\psi} \) independently. We are permitted to perform so-called free-end point variations. Because \( \phi \) is pure gauge here we may vary the action with respect to \( \phi \) and \( \dot{\phi} \) independently. This will be crucial in the theory.. We shall return to this.

¹We use the signature \{−, +, +, +\} which results in the minus sign here.
2.4 A note on the action

The form of the action as it stands is not conventional as it contains second time derivatives of the metric. However, the combination

\[ (4) R + 2A^\alpha_{;\alpha} \]  

(17)

where \( A^\alpha = (n^\alpha tr K + a^\alpha) \), \( n^\alpha \) is the unit timelike normal and \( a^\alpha \) is the four-acceleration of an observer travelling along \( n \), contains no second time derivatives. (The coordinates \( \alpha \) are general.) We write our Lagrangian as

\[ L = \sqrt{-(4)g} \left[ (4) R + 2A^\alpha_{;\alpha} - 2A^\alpha_{;\alpha} - \frac{6\Box \phi}{\phi} \right] \]  

(18)

which then becomes

\[ L = \sqrt{-(4)g} \left[ \phi^2 \left( (4) R + 2A^\alpha_{;\alpha} \right) + 4\phi \phi_{,\alpha} A^\alpha + 6g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] \]  

(19)

after some integration by parts.

This Lagrangian contains no second time derivatives of the metric. Varying this with respect to \( \phi \) and \( \dot{\phi} \) gives two conditions which combine to give equation (16). Although we may do these variations here in a general coordinate form it will be more instructive to do a \((3 + 1)\)-dimensional decomposition and get the corresponding equations there.

3 (3+1)-Decomposition

Before we consider the new theory it will be instructive to recall the ADM treatment of general relativity as much of this will carry straight over to the new theory.

The idea in the ADM treatment is that a thin-sandwich 4-geometry is constructed from two 3-geometries separated by the proper time \( d\tau \). The 4-metric found from the ADM construction is

\[
\begin{pmatrix}
(4)g_{00} & (4)g_{0k} \\
(4)g_{0k} & (4)g_{kk}
\end{pmatrix}
= 
\begin{pmatrix}
N^s N_s - N^2 & N_k \\
N_i & g_{ik}
\end{pmatrix}
\]

(20)

\( N = N(t, x, y, z) \) is the lapse function given by

\[ d\tau = N(t, x, y, z) dt \]

(21)

and \( N^i = N^i(t, x, y, z) \) are the shift functions given by

\[ x^i_2(x^m) = x^i_1 - N^i(t, x, y, z) dt \]

(22)

where \( x^i_2 \) is the position on the “later” hypersurface corresponding to the position \( x^i_1 \) on the “earlier” hypersurface. The indices in the shift are raised and lowered by the 3-metric \( g_{ij} \).
The reciprocal 4-metric is

\[
\begin{pmatrix}
(4)g^{00} & (4)g^{0k} \\
(4)g^{i0} & (4)g^{ik}
\end{pmatrix} =
\begin{pmatrix}
-1/N^2 & N^k/N^2 \\
N^i/N^2 & g^{ik} - N^iN^k/N^2
\end{pmatrix}
\] (23)

The volume element has the form

\[
\sqrt{(4)g} d^4x = N \sqrt{g} dt dx^3
\] (24)

This construction of the 4-metric also automatically determines the components of the unit timelike normal vector \( n \). We get

\[
n_\beta = (-N, 0, 0, 0)
\] (25)

and raising the indices using \((4)g^{\alpha\beta}\) gives us

\[
n^\alpha = (1/N, -N^m/N)
\] (26)

Consider now the Einstein-Hilbert action

\[
S = \int \sqrt{-g} (4)R d^4x
\] (27)

Using the Gauss-Codazzi relations we get

\[
(4)R = R - (trK)^2 + K_{ab}K_{ab} - 2A^\alpha_{;\alpha}
\] (28)

where \( A^\alpha \) is given by (as earlier)

\[
A^\alpha = (n^\alpha trK + a^\alpha)
\] (29)

\( n^\alpha \) is the unit timelike normal and

\[
a^\alpha = n^\alpha_{;\beta} n^\beta
\] (30)

is the four-acceleration of an observer travelling along \( n \). It is easily verified that \( a^0 = 0 \) and that \( a^i = \nabla^i N/N \). Substituting into the action gives

\[
S = \int N \sqrt{g} (R - (trK)^2 + K_{ab}K_{ab}) dt dx^3
\] (31)

where the total divergence \( A^\alpha_{;\alpha} \) has been discarded. \( K \) is the extrinsic curvature given by

\[
K = -\frac{1}{2} \mathcal{L}_n g
\] (32)

the Lie derivative of the 3-metric \( g \) along \( n \). In the coordinates we are using here the extrinsic curvature takes the form

\[
K_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - N_{a;b} - N_{b;a} \right)
\] (33)

The action is varied with respect to \( \frac{\partial g_{ab}}{\partial t} \) to get the canonical momentum

\[
\pi^{ab} = \sqrt{g} (g^{ab} trK - K^{ab})
\] (34)
and varied with respect to $N$ and $N_a$ to give the initial value equations

$$\mathcal{H} = 0 \quad \text{and} \quad \mathcal{H}^a = 0$$

(35)

respectively, where

$$\mathcal{H} = \sqrt{g} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} (\text{tr}\pi)^2 \right) - \sqrt{g} R$$

(36)

and

$$\mathcal{H}^a = -2 \pi^{ab} ;_b$$

(37)

We are now ready to consider the new action. This is

$$S = \int \sqrt{-g} \phi^2 \left( R - \frac{6}{\phi} \bigg( R - \frac{6}{\phi} \bigg) \bigg) d^4x$$

(38)

The 4-dimensional scalar curvature decomposes as earlier. The action becomes

$$S = \int \sqrt{-g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} - 2A^\alpha_{;\alpha} - \frac{6}{\phi} \bigg) d^4x$$

(39)

Let’s separate this into two terms $S_1$ and $S_2$ where,

$$S_1 = \int \sqrt{-g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} - 2A^\alpha_{;\alpha} \right) d^4x$$

(40)

and

$$S_2 = -\int 6 \sqrt{-g} \phi \square \phi d^4x$$

(41)

Consider the first term. In the ADM theory $A^\alpha_{;\alpha}$ leads to a total divergence which is discarded. However, the presence of the $\phi^2$ here changes this. Integrating by parts we get

$$-2\phi^2 A^\alpha_{;\alpha} \rightarrow 2(\phi^2)_{;\alpha} A^\alpha$$

(42)

discarding the total divergence again. Decomposing this gives

$$2 \left( \phi^2 \left( n^0 \text{tr}K + a^0 \right) + \phi^2 \left( n^i \text{tr}K + a^i \right) \right)$$

$$= 4\phi \left( \phi n^0 \text{tr}K + \phi_i n^i \text{tr}K \right) + 4\phi \phi_i a^i$$

(43)

using equation (26). Then,

$$S_1 = \int N \sqrt{g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} \right) dt \ dt^3x$$

$$+ \int 4\sqrt{g} \phi \left[ \left( \phi - \phi \phi_i N^i \right) \text{tr}K + N \phi_i a^i \right] dt \ dt^3x$$

(44)

We must now deal with $S_2$. After a little integration by parts this is

$$S_2 = \int 6 \sqrt{-g} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi d^4x$$

(45)
Decomposing this gives

\[ S_2 = \int 6N\sqrt{g} \left( -\frac{1}{N^2} \dot{\phi}^2 + \frac{2N^i}{N^2} \phi_{,i} + \left( g^{ij} - \frac{N^i N^j}{N^2} \right) \phi_{,i} \phi_{,j} \right) dt \, d^3x \]  

(46)

The full action is now

\[ S = \int N\sqrt{g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} \right) dt \, d^3x + \int 4\sqrt{g} \phi \left( \dot{\phi} - \phi_{,i} N^i \right) trK + N \phi_{,i} a^i \right) dt \, d^3x + \int 6N\sqrt{g} \left( -\frac{1}{N^2} \dot{\phi}^2 + \frac{2N^i}{N^2} \phi_{,i} + \left( g^{ij} - \frac{N^i N^j}{N^2} \right) \phi_{,i} \phi_{,j} \right) dt \, d^3x \]  

(47)

This looks like a much more complicated object than we began with. There will, however, be much simplification. First, let’s write it as

\[ S = \int N\sqrt{g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} \right) dt \, d^3x + \int 4\sqrt{g} \phi \left( \dot{\phi} - \phi_{,i} N^i \right) trK + N \phi_{,i} a^i \right) dt \, d^3x \]  

(48)

where we have used \( a^i = \frac{\nabla^i N}{N} \). If we set \( \theta = -\frac{2}{\phi} \left( \dot{\phi} - \phi_{,i} N^i \right) \) then we get

\[ S = \int N\sqrt{g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} \right) dt \, d^3x - \int 2\sqrt{g} \theta \phi^2 trK dt \, d^3x - \int 3\frac{\sqrt{g} \theta^2 \phi^2}{N} dt \, d^3x + \int 4\sqrt{g} \phi \left( \dot{\phi} - \phi_{,i} N^i \right) N \phi_{,i} \phi_{,i} \phi dt \, d^3x \]  

(49)

This becomes

\[ S = \int N\sqrt{g} \phi^2 \left( R - (\text{tr}K)^2 + K^{ab} K_{ab} \right) dt \, d^3x - \int 2\sqrt{g} \theta \phi^2 trK dt \, d^3x - \int 3\frac{\sqrt{g} \theta^2 \phi^2}{N} dt \, d^3x + \int 2N\sqrt{g} \nabla_i \phi \nabla^i \phi dt \, d^3x - \int 3N\sqrt{g} \nabla^2 \phi dt \, d^3x \]  

(50)

after some integration by parts. We notice that there might be a possibility of “completing some squares” with terms involving \( K \) and those involving \( \theta \). We have,

\[ -N(\text{tr}K)^2 + NK^{ab} K_{ab} - 2\theta trK - \frac{3}{2} \frac{\theta^2}{N} \]  

(51)

Let’s try the combination,

\[ -N \left( \text{tr}K + A \frac{\theta}{N} \right)^2 + N \left( K_{ab} + B \frac{\theta g_{ab}}{N} \right) \left( K^{ab} + B \frac{\theta g^{ab}}{N} \right) \]  

(52)

This gives us

\[ -N(\text{tr}K)^2 - 2A\theta trK - A^2 \frac{\theta^2}{N} + NK^{ab} K_{ab} + 2B\theta trK + 3B^2 \frac{\theta^2}{N} \]  

(53)
Comparing coefficients with equation (51) gives us,

\[-2A + 2B = -2 \text{ and } -A^2 + 3B^2 = -\frac{3}{2}\]  

(54)

Solving here gives \(A = \frac{3}{2}\) and \(B = \frac{1}{2}\) and so we have,

\[-N\left(trK + \frac{3}{2N}\right)^2 + N\left(K_{ab} + \frac{1}{2N}g_{ab}\right)\left(K^{ab} + \frac{1}{2N}g^{ab}\right)\]  

(55)

Finally, let us set

\[B_{ab} = \left(K_{ab} + \frac{\theta}{2N}g_{ab}\right)\]  

(56)

Thus we get,

\[-N\left(trB\right)^2 + NB^{ab}B_{ab}\]  

(57)

overall. Our full action is now,

\[S = \int N\sqrt{g}\phi^4\left(R - (trB)^2 + B^{ab}B_{ab}\right) dt d^3x\]  

\[+ \int 2N\sqrt{g}\nabla_i \phi \nabla^i \phi dt d^3x\]  

\[-\int 4N\sqrt{g}\phi \nabla^2 \phi dt d^3x\]  

(58)

We are now in a \((3 + 1)\)-dimensional form and so we would like to use the power of \(\phi\) which is appropriate in 3 dimensions. From the earlier discussion of conformal invariance in different numbers of dimensions we find that we should use \(\psi = \phi^{1/2}\) (or \(\psi^2 = \phi\)). This is no more than a relabelling to make things look neater and there is no real change to the theory in this relabelling. We get,

\[S = \int N\sqrt{g}\psi^4\left(R - 8\nabla^2 \psi \psi - (trB)^2 + B^{ab}B_{ab}\right) dt d^3x\]  

(59)

Thus the action is

\[S = \int N\sqrt{g}\psi^4\left(R - 8\nabla^2 \psi \psi - (trB)^2 + B^{ab}B_{ab}\right) dt d^3x\]  

(60)

This looks much better! We notice too that \(\left(R - 8\nabla^2 \psi \psi\right)\) is the conformally uniform 3-D version of \(\left(R - \frac{4(n-1)}{(n-2)^2} \Box \psi\right)\). In fact, if we start with the ADM \((3+1)\) action, equation (31) and perform a conformal transformation

\[g_{\alpha \beta} \rightarrow \psi^4 g_{\alpha \beta}\]

\[N \rightarrow \psi^2 N\]

\[N_i \rightarrow \psi^4 N_i\]

(61)

(which is simply \(g_{\alpha \beta} \rightarrow \phi^2 g_{\alpha \beta}\) ; that is, a conformal transformation of the 4-metric of the form equation (3)) we get precisely the action of equation (51).
Note: We had $\theta$ in terms of $\phi$: $\theta = -\frac{2}{\psi}(\dot{\phi} - \dot{\phi}_iN^i)$. We may, of course, write it in terms of $\psi$: $\theta = -\frac{4}{\psi}(\dot{\psi} - \dot{\psi}_iN^i)$. We can also find a coordinate independent form for $B$. This is

$$B = -\frac{1}{2}\psi^{-4}f_B(\psi^4 g)$$

(62)

This is analogous to the expression

$$K = -\frac{1}{2}f_B(g)$$

(63)

for the extrinsic curvature $K$ in general relativity.

4 Jacobi Action

Baerlein, Sharp and Wheeler [5] constructed a Jacobi Action for general relativity. Their action was,

$$S = \pm \int d\lambda \int \sqrt{g} \sqrt{R} \sqrt{T_{GR}} d^3x$$

(64)

where

$$T_{GR} = \left( g^{ae}g^{bd} - g^{ab}g^{cd} \right) \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} \right) \left( \frac{\partial g_{cd}}{\partial t} - (KN)_{cd} \right)$$

(65)

Variation with respect to $\frac{\partial g_{ab}}{\partial t}$ gives

$$\pi^{ab} = \sqrt{gT} \left( g^{ae}g^{bd} - g^{ab}g^{cd} \right) \left( \frac{\partial g_{cd}}{\partial t} - (KN)_{cd} \right)$$

(66)

This expression is squared to give the Hamiltonian constraint. The variation with respect to $N_a$ gives the momentum constraint. The evolution equations are found in the usual way. The equations found with the Jacobi action are those of general relativity if we identify $2N$ and $\sqrt{\frac{T}{R}}$. We want to construct the analogous case in this theory. Let us return to our (3+1) action, equation (60),

$$L = N \sqrt{g} \psi^4 \left[ R - 8\nabla^2 \frac{\psi}{\psi'} - (trB)^2 + B^{ab}B_{ab} \right]$$

(67)

We can write the Lagrangian as

$$L = \sqrt{g} \psi^4 \left[ N \left( R - 8\nabla^2 \frac{\psi}{\psi'} \right) + \frac{1}{4N} \left( \beta^{ab} \beta_{ab} - (tr\beta)^2 \right) \right]$$

(68)

where $\beta_{ab} = -2NB_{ab} = \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} - \theta g_{ab} \right)$. We now extremise with respect to $N$. This gives us,

$$N = \pm \frac{1}{2} \left( \beta^{ab} \beta_{ab} - (tr\beta)^2 \right)^{\frac{1}{2}} \left( R - 8\nabla^2 \frac{\psi}{\psi'} \right)^{-\frac{1}{2}}$$

(69)

Substituting this back into the action gives us

$$S = \pm \int d\lambda \int \sqrt{g} \psi^4 \sqrt{R - 8\nabla^2 \frac{\psi}{\psi'} \sqrt{T} d^3x}$$

(70)

where $T = \left( \beta^{ab} \beta_{ab} - (tr\beta)^2 \right)$. This is the conformal gravity version of the BSW action [61].
Amazingly, this is precisely the action that Barbour and Ó Murchadha found by starting with the BSW action and conformalising it under conformal transformations of the 3-metric

$$g_{ab} \rightarrow \psi^4 g_{ab}$$

(71)

The Jacobi action is manifestly 3-dimensional and its configuration space is naturally conformal superspace - the space of all 3-D Riemannian metrics modulo diffeomorphisms and conformal rescalings. However, we found this action starting with a fully four-dimensional theory!

Omitting the details (which are to be found in [8]) the constraints are found to be

$$\pi^{ab} \pi_{ab} - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0$$

(72)

$$\nabla_b \pi^{ab} = 0$$

(73)

$$tr \pi = 0$$

(74)

$$N \psi^3 \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \nabla^2 \left( N \psi^3 \right) = 0$$

(75)

The evolution equations are

$$\frac{\partial g_{ab}}{\partial t} = N g^{-\frac{3}{2}} \psi^{-4} \pi_{ab} + (KN)_{ab} + \theta g_{ab}$$

(76)

and

$$\frac{\partial \pi^{ab}}{\partial t} = - \sqrt{g} N \psi^4 \left( R_{ab} - g^{ab} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) - 2N g^{-\frac{3}{2}} \psi^{-4} \pi^{ac} \pi^{bc}$$

$$+ \nabla^a \nabla^b \left( \sqrt{g} N \psi^4 \right) - \sqrt{g} g^{ab} \nabla^2 \left( N \psi^3 \right) + 4g^{ac} N \sqrt{g} \psi^3 \nabla^2 \psi$$

$$- 8 \sqrt{g} \nabla^a \left( N \psi^3 \right) \nabla^b \psi + 4g^{ab} \sqrt{g} \nabla_c \left( N \psi^3 \right) \nabla^c \psi$$

$$- \nabla_c N^{ac} \pi^{bc} - \nabla_c N^{bc} \pi^{ac} + \nabla_c \left( N^{ab} \pi^{bc} \right)$$

(77)

5 Conformally Related Solutions

In conformal superspace conformally related metrics are equivalent. Thus, conformally related solutions of the theory must be physically equivalent and so it is crucial that we have a natural way to relate such solutions. Suppose we have one set of initial data \((g_{ab}, \pi^{ab})\). These must satisfy the constraints (73) and (74). We solve the Hamiltonian constraint (72) for our “conformal field” \(\psi\). Suppose now we start with a different pair \((h_{ab}, \rho^{ab})\) where \(h_{ab} = \alpha^4 g_{ab}\) and \(\rho^{ab} = \alpha^{-4} \pi^{ab}\). Our new initial data is conformally related to the original set of initial data. This is allowed as “transverse-traceless”-ness is conformally invariant and so our initial data constraints are satisfied. All we must do is solve the new Hamiltonian constraint for our new conformal field \(\chi\) say. This constraint is now

$$\rho^{ab} \rho_{ab} = h \chi^8 \left( R_h - 8 \frac{\nabla_h^2 \chi}{\chi} \right)$$

(78)

The subscript \(h\) on \(R\) and \(\nabla\) is because we are now dealing with the new metric \(h_{ab}\). We now solve this for \(\chi\). It can be shown that we must have \(\chi = \psi^4\). That is, \(\psi\) is automatically transformed when our
initial data is transformed.

Now,
\[ \chi^4 h_{ab} = \frac{\psi^4}{\alpha^4} g_{ab} = \psi^4 g_{ab} \]  
\( (79) \)

and
\[ \chi^{-4} \rho^{ab} = \psi^{-4} \pi^{ab} \]  
\( (80) \)

If we label these as \( \tilde{g}_{ab} = \psi^4 g_{ab} \) and \( \tilde{\pi}^{ab} = \psi^{-4} \pi^{ab} \) then we can write our constraints as
\[ \tilde{\pi}^{ab} \tilde{\pi}_{ab} - \tilde{g} R = 0 \]  
\( (81) \)

\[ \nabla_b \tilde{\pi}^{ab} = 0 \]  
\( (82) \)

\[ t_r \pi = 0 \]  
\( (83) \)

\[ \tilde{N} R - \nabla^2 \tilde{N} = 0 \]  
\( (84) \)

All conformally related solutions are identical in this form (which is also the representation which most closely resembles general relativity). We shall call this the physical representation as it is the combination \( \psi^4 g_{ab} \) which is the physical quantity. The momentum constraint is identical in the two theories.

This is the distinguished representation of \( \mathbb{S} \). However, it was not shown there why or how this representation is the physical one. This is explicitly shown here.

The Hamiltonian constraint of general relativity on a maximal slice is identical to that here. The slicing equation in the physical representation looks just like the maximal slicing equation of general relativity. In this representation the evolution equations are
\[ \frac{\partial g_{ab}}{\partial t} = N g^{-\frac{1}{4}} \pi_{ab} + (KN)_{ab} \]  
\( (85) \)

and
\[ \frac{\partial \pi^{ab}}{\partial t} = - \sqrt{g} \left( R^{ab} - g^{ab} R \right) - 2N g^{-\frac{1}{4}} \pi^{ac} \pi_{bc} \]
\[ + \nabla^a \nabla^b \left( \sqrt{g} N \right) - \sqrt{g} g^{ab} \nabla^2 N \]
\[ - \nabla_c N^a \pi^{bc} - \nabla_c N^b \pi^{ac} + \nabla_c \left( N^c \pi^{ab} \right) \]  
\( (86) \)

These are exactly those of general relativity on a maximal slice. Thus, solutions of general relativity in maximal slicing gauge are also solutions here. There are of course solutions of general relativity which do not have a maximal slicing and these are not solutions of the conformal theory.

Consider again the full four-dimensional form of the theory. Suppose we have a solution of the equations \( g_{\alpha \beta} \) and \( \phi \). If we perform a conformal transformation on this metric with conformal factor \( \alpha \), say, the new metric \( h_{\alpha \beta} = \alpha^2 g_{\alpha \beta} \) must still be a solution. We find that the conformal factor this time is \( \eta = \frac{\phi}{\alpha} \)
and so we have $\phi^2 g_{\alpha\beta} = \eta^2 h_{\alpha\beta}$ which is yet another demonstration of the identification of conformally related solutions. In the physical representation the 4-dimensional equations take the form of the Einstein equations in vacuum

$$G^{\alpha\beta} = 0$$  \hspace{1cm} (87)

However, these are supplemented with the slicing conditions $\mathbf{33}$ and $\mathbf{34}$ thus giving the theory a distinguished slicing which sets it apart from general relativity.

### 6 Topological Considerations

So far we have not considered any implications which the topology of the manifold may have. In $\mathbf{8}$ it was shown that if the manifold was compact without boundary then there would be frozen dynamics. That is, the only solution for the lapse would be $N \equiv 0$. The problem was resolved by introducing the volume into the action explicitly. With the necessary changes the constraints were found to be

$$\pi^{ab}\pi_{ab} - \frac{g\psi^8}{V(\psi)^{\frac{3}{2}}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0$$  \hspace{1cm} (88)

$$\nabla_b \pi^{ab} = 0$$  \hspace{1cm} (89)

$$tr\pi = 0$$  \hspace{1cm} (90)

$$N\psi^3 \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \nabla^2 \left( N\psi^3 \right) = C\psi^5$$  \hspace{1cm} (91)

Of course, with the introduction of the volume term we have a change in the original four-dimensional action also. This becomes,

$$S = \int \frac{\sqrt{-(4)g} \phi^2 \left( (4) R - \frac{6\psi \phi}{\phi} \right)}{V(\phi)^{\frac{3}{2}}} \, d^4 x$$  \hspace{1cm} (92)

We have an implicit $(3+1)$ split here because $V$ is a purely three-dimensional quantity. We vary with respect to $(4)g_{0a}$ and $(4)g_{ij}$ separately. (We vary with respect to the lower index case as $(4)g_{ij} = g_{ij}$ and so both the numerator and the denominator may be varied with respect to the spatial part of the metric.) The variations give

$$G^{\mu\nu} = 0$$  \hspace{1cm} (93)

and

$$N\sqrt{g}\phi^2 G_{ij} + \frac{2}{3} y^{ij} \sqrt{g} C \phi^3 = 0$$  \hspace{1cm} (94)

where

$$C = \int \frac{N\sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right)}{V(\psi)} \, d^3 x$$  \hspace{1cm} (95)

arises, as usual, due to variation of the volume. As earlier, $G^{\alpha\beta}$ is the Einstein tensor of the metric $\phi^2 g_{\alpha\beta}$ and $\psi^2 = \phi$. We have used the Hamiltonian constraint to simplify $C$. 

12
We can combine the equations to get
\[ G^\alpha\beta + \frac{2}{3N} h^{\alpha\beta} C \phi = 0 \] (96)

where \( h^{\alpha\beta} \) is the induced 3-metric. This has the form
\[
\begin{bmatrix}
  h_{00} & h_{0k} \\
  h_{i0} & h_{ik}
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 \\
  0 & g_{ik}
\end{bmatrix}
\] (97)

We may lower the indices using \( g_{\alpha\beta} \) to get
\[
\begin{bmatrix}
  h_{00} & h_{0k} \\
  h_{i0} & h_{ik}
\end{bmatrix}
= \begin{bmatrix}
  N^s N_s & N_k \\
  N_i & g_{ik}
\end{bmatrix}
\] (98)

In the physical representation equation (96) becomes
\[ G^\alpha\beta + \frac{2}{3N} h^{\alpha\beta} C = 0 \] (99)

where now \( C = \langle NR \rangle \).

Of these ten equations, the four \( 0 \alpha \) equations are identical to those in general relativity while the remaining six differ by the new term which arose due to the variation of the volume. This new term is both time dependent and position dependent and so behaves like a “non-constant cosmological constant.” It will undoubtedly lead to new features, particularly in cosmology. Some of these are discussed in [8].

We must also do the variations with respect to \( \phi \) and \( \dot{\phi} \). The volume is independent of \( \dot{\phi} \) and so this variation gives us exactly the same result as earlier, namely
\[ trB = 0 \] (100)

However the volume is not independent of \( \phi \) and so we will have a slight change. Varying with respect to \( \phi \) gives us exactly what we found when we did the variation on the Jacobi form of the action (of course)
\[ N\psi^3 \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \psi^2 \left( N\psi^3 \right) = C \psi^5 \] (101)

where
\[ C = \int \frac{N\sqrt{g}\psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) d^3x}{V(\psi)} \] (102)

This becomes
\[ NR - \nabla^2 N = \langle NR \rangle \] (103)
Let us consider equation (99) again. Taking the trace gives us

$$-(4)R + 2\left\langle NR/N \right\rangle = 0$$

(104)

or

$$N^{(4)}R = 2\left\langle NR \right\rangle$$

(105)

If we average both sides of this equation we get

$$\frac{\int N\sqrt{g}(4)Rd^3x}{\int \sqrt{g}d^3x} = 2\left\langle NR \right\rangle$$

(106)

It turns out that this equation is equivalent to the equation

$$\frac{\partial tr\pi}{\partial t} = 0$$

(107)

Of course, this is already known from the propagation of the $tr\pi$ constraint. Thus we have demonstrated that there is no inconsistency in the equations.

7 Discussion

The initial idea was to construct a theory with conformal superspace as its configuration space. These are 3-dimensional ideas and it was not expected that such a clear 4-dimensional picture would emerge. The clarity of the 4-dimensional picture should allow easy comparison with aspects of general relativity which have traditionally been treated in the 4-dimensional framework. Furthermore, this formulation may be useful in a path integral approach to quantisation.

The field equations of the theory are almost identical to Einstein’s equations. The differences are entirely due to the emergence of a preferred frame and it is this which breaks the explicit 4-covariance of the theory. Of course, there will be a quite different cosmology not least due to the fact that since the volume does not change expansion is automatically ruled out along with anything explained by expansion (most notably the redshift). These are discussed in [8] and so we will not delve any further into this here.

There has been much work on other aspects of this theory. Among these are the Hamiltonian formulation including the constraint algebra and some Hamilton-Jacobi theory [9], and coupling of the theory to matter [8]. Cosmological considerations and quantisation are further issues which are currently being investigated. Furthermore, a number of related theories are also being actively investigated and will be the topics of future articles.

Whether or not conformal gravity proves to be a viable theory of gravity remains to be seen. Nonetheless, the quantisation of the theory may teach some valuable lessons with regard to a full quantum theory of gravity. This is in itself a worthwhile pursuit.
8 Acknowledgements

I wish to thank Niall Ó Murchadha for many discussions. These were invaluable from both technical and non-technical viewpoints. I would also like to thank Ed Anderson, Julian Barbour and Brendan Foster for discussions. This work was partially supported by Enterprise Ireland.

References

[1] P.A.M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, NY, 1964)

[2] R. Arnowitt, S. Deser and C.W. Misner, in Gravitation: An Introduction to Current Research ed L. Witten (Wiley, New York 1962)

[3] J.W. York, Phys. Rev. Lett. 26 1656 (1971); Phys. Rev. Lett. 28 1082 (1972).

[4] J. Barbour and N. Ó Murchadha, Preprint gr-qc/9911071

[5] R.F. Baerlein, D. Sharp and J.A. Wheeler, Phys. Rev. 126 1864 (1962)

[6] R.M. Wald, General Relativity (The University of Chicago Press, 1984)

[7] J. Barbour, private communication

[8] E. Anderson, J. Barbour, B.Z. Foster, N. Ó Murchadha, Class. Quantum Grav. 20 1571 (2003) (Preprint gr-qc/0211022)

[9] B. Kelleher, The Hamiltonian Formulation of Conformal Gravity (MSc Thesis, UCC, Cork, Ireland, 1999)