A Most General Edge Elimination Polynomial — Thickening of Edges

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Abstract

We consider a graph polynomial \( \xi(G; x, y, z) \) introduced by Averbouch, Godlin, and Makowsky (2007). This graph polynomial simultaneously generalizes the Tutte polynomial as well as a bivariate chromatic polynomial defined by Dohmen, Pönitz and Tittmann (2003). We derive an identity which relates the graph polynomial of a thicked graph (i.e. a graph with each edge replaced by \( k \) copies of it) to the graph polynomial of the original graph. As a consequence, we observe that at every point \((x, y, z)\), except for points lying within some set of dimension 2, evaluating \( \xi \) is \#P-hard.

1 Introduction

We consider the following three-variable graph polynomial which has been introduced by I. Averbouch, B. Godlin, and J. A. Makowsky \cite{AGM07}:

\[
\xi(G; x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \cup B) - k_{\text{cov}}(B)} \cdot y^{|A| + |B| - k_{\text{cov}}(B)} \cdot z^{k_{\text{cov}}(B)},
\]

where \( G = (V, E) \) is a graph with multiple edges and self loops allowed, \( A \sqcup B \) denotes a vertex-disjoint union of edge sets \( A \) and \( B \), \( k(A \cup B) \) is the number of components of \((V, A \cup B)\), and \( k_{\text{cov}}(B) \) is the number of components of \((V(B), B)\).

The polynomial \( \xi \) simultaneously generalizes two interesting graph polynomials: the Tutte polynomial and a bivariate chromatic polynomial \( P(G; x, y) \) defined by K. Dohmen, A. Pönitz, and P. Tittmann \cite{DPT03}.

It is known that the Tutte polynomial of a graph with “thicked” edges evaluated at some point equals the Tutte polynomial of the original graph evaluated at another...
where in this section we apply Sokal’s approach to $\xi$ as on the Tutte polynomial (Theorem 3). In Section 3 we conclude that for every point $(x, y, z) \in \mathbb{Q}^3$, except on a set of dimension at most 2, it is $\#P$-hard to compute $\xi(G; x, y, z)$ from $G$ (Theorem 5). This supports a difficult point conjecture for graph polynomials [Mak07 Conjecture 1], [AGM07 Question 1].

2 A point-to-point reduction from thickening

In this section we apply Sokal’s approach to $\xi$ and obtain Lemma 2 (cf. [Sok05, Section 4.4]), the main technical contribution of this note.

We define the following auxiliary polynomial, which has a different $y$-variable for each edge of the graph, $\tilde{y} = (y_e)_{e \in E(G)}$.

$$\psi(G; x, \tilde{y}, z) = \sum_{(A \cup B) \subseteq E(G)} w(G; x, \tilde{y}, z; A, B),$$

where

$$w(G; x, \tilde{y}, z; A, B) = x^{k(A \cup B)} \left( \prod_{e \in (A \cup B)} y_e \right)^z.$$ We write $\psi(G; x, y, z)$ for $\psi(G; x, \tilde{y}, z)$ if for each $e \in E(G)$ we have $y_e = y$.

**Lemma 1.** We have the polynomial identities $\psi(G; x, y, zx^{-1}y^{-1}) = \xi(G; x, y, z)$ and $\xi(G; x, y, zxy) = \psi(G; x, y, z)$. □

Let $G$ be a graph and $e \in E(G)$ an edge. Let $E' := E \setminus \{e\}$ and $G_{ee}$ be the graph $G$ with $e$ doubled, i.e. $G_{ee} = (V(G), E' \cup \{e_1, e_2\})$ with $e_1, e_2$ being new edges.

**Lemma 2.** $\psi(G_{ee}; x, \tilde{y}, z) = \psi(G; x, \tilde{y}, z)$ with $Y_e = (1 + y_{e_1})(1 + y_{e_2}) - 1$ and $Y_{\tilde{e}} = y_{\tilde{e}}$ for all $\tilde{e} \in E'$.

**Proof.** Let $M(G) = \{(A, B) \mid A \cup B \subseteq E(G)\}$ and $M(G_{ee}) = \{(\tilde{A}, \tilde{B}) \mid \tilde{A} \cup \tilde{B} \subseteq E(G_{ee})\}$. We define a map $\tau : M(G) \rightarrow 2^{M(G_{ee})}$ in the following way. Consider $(A, B) \in M(G)$. If $e \not\in A \cup B$, we set $\tau(A, B) = \{(A, B)\}$. If $e \in A$, we let $A' := A \setminus \{e\}$ and define $\tau(A, B) = \{(A' \cup \{e_1\}, B), (A' \cup \{e_2\}, B), (A' \cup \{e_1, e_2\}, B)\}$. (Note that in this case $e \not\in B$, as $A$ and $B$ are vertex-disjoint.) If $e \in B$, we let $B' := B \setminus \{e\}$ and define $\tau(A, B) = \{(A, B' \cup \{e_1\}), (A, B' \cup \{e_2\}), (A, B' \cup \{e_1, e_2\})\}$. Observe that

$$M(G_{ee}) = \bigcup_{(A, B) \in M(G)} \tau(A, B),$$

(3)
and that this union is a union of pairwise disjoint sets.

Calculation yields

$$w(G; x, \bar{Y}, z; A, B) = \sum_{(\tilde{A}, \tilde{B}) \in \tau(A,B)} w(G_{ee}; x, \bar{y}, z; \tilde{A}, \tilde{B})$$

for every $(A, B) \in M(G)$. Thus,

$$\psi(G_{ee}; x, \bar{y}, z) = \sum_{(\tilde{A}, \tilde{B}) \in M(G_{ee})} w(G_{ee}; x, \bar{y}, z; \tilde{A}, \tilde{B})$$

by (3)

$$= \sum_{(A,B) \in M(G)} \sum_{(\tilde{A}, \tilde{B}) \in \tau(A,B)} w(G_{ee}; x, \bar{y}, z; \tilde{A}, \tilde{B})$$

by (4)

$$= \sum_{(A,B) \in M(G)} w(G; x, \bar{Y}, z; A, B)$$

by (1)

$$= \psi(G; x, \bar{Y}, z).$$

\[\square\]

Applying Lemma 2 repeatedly and Lemma 1 to convert between $\psi$ and $\xi$ we obtain

**Theorem 3.** Let $G_k$ be the $k$-thickening of $G$ (i.e. the graph obtained out of $G$ by replacing each edge by $k$ copies of it). Then

$$\psi(G_k; x, y, z) = \psi(G; x, (1 + y)^k - 1, z),$$

(5)

$$\xi(G_k; x, y, z) = \xi(G; x, (1 + y)^k - 1, z\frac{(1+y)^k-1}{y}).$$

(6)

**3 Hardness**

The following theorem has been proven independently by I. Averbouch (J. A. Makowsky, personal communication, October 2007).

**Theorem 4.** Let $P$ denote the bivariate chromatic polynomial defined by K. Dohmen, A. Pönnitz, and P. Tittmann [DPT03]. For every $(x, y) \in \mathbb{Q}$, $y \neq 0$, $(x, y) \notin \{(1, 1), (2, 2)\}$, it is \#P-hard to compute $P(G; x, y)$ from $G$. 

3
Proof (Sketch). Given a graph $G = (V, E)$ let $\tilde{G}$ denote the graph obtained out of $G$ by inserting a new vertex $\tilde{v}$ and connecting $\tilde{v}$ to all vertices in $V$. Let $P(G; y)$ denote the chromatic polynomial \[\text{[Rea68]}\]. It is well known that

$$P(\tilde{G}; y) = yP(G; y - 1).$$

(7)

From this and \[\text{[DPT03, Theorem 1]}\] we can derive

$$P(\tilde{G}; x, y) = yP(G; x - 1, y - 1) + (x - y)P(G; x, y).$$

(8)

The proof of the theorem now works in the same fashion as a proof that $P(G; y)$ is \#P-hard to evaluate almost everywhere using (7) would work: using (8) we reduce along the lines $x = y + d$, which eventually enables us to evaluate $P$ at $(1 + d, 1)$ (if $y$ is a positive integer, we reach $(1 + d, 1)$ directly; otherwise we obtain arbitrary many points on the line $x = y + d$, which enables us to interpolate the polynomial on this line). On the line $y = 1$ the polynomial $P$ equals the independent set polynomial \[\text{[DPT03, Corollary 2]}\], which is \#P-hard to evaluate almost everywhere \[\text{[AM07, BH07]}\].

Theorem 5. For every $(x, y, z) \in \mathbb{Q}, x \neq 0, z \neq -xy, (x, z) \not\in \{(1, 0), (2, 0)\}, y \not\in \{-2, -1, 0\}$, the following statement holds true: It is \#P-hard to compute $\xi(G; x, y, z)$ from $G$.

Proof (Sketch). For $x, y \in \mathbb{Q}, x, y \neq 0$ and $(x, y) \not\in \{(1, 1), (2, 2)\}$ the following problem is \#P-hard by Theorem 4: Given $G$, compute

$$P(G; x, y) = \xi(G; x, -1, x - y) = \psi\left(G; x, -1, \frac{y - x}{x}\right),$$

where the first equality is by \[\text{[AGM07, Proposition 18]}\] and the second by Lemma 1. We will argue that, for any fixed $\tilde{y} \in \mathbb{Q} \setminus \{-2, -1, 0\}$, this reduces to compute $\psi(G; x, \tilde{y}, \frac{y - x}{x})$ from $G$. We have

$$\psi\left(G; x, \tilde{y}, \frac{y - x}{x}\right) = \xi(G; x, \tilde{y}, (y - x)\tilde{y})$$

by Lemma 1. An easy calculation converts the conditions on $x, \tilde{y}, y$ into conditions on $x, y, z$ and yields the statement of the theorem.

Now assume that we are able to evaluate $\psi$ at some fixed $(x, y, z) \in \mathbb{Q}^3$, i.e. given $G$ we can compute $\psi(G; x, y, z)$. Then Theorem 3 allows us to evaluate $\psi$ at $(x, y', z)$ for infinitely many different $y' = (1 + y)^k - 1$ provided that $|1 + y| \neq 0$ and $|1 + y| \neq 1$. As $\psi$ is a polynomial, this enables interpolation in $y$ and eventually gives us the ability to evaluate $\psi$ at $(x, y', z)$ for any $y' \in \mathbb{Q}$. In particular, being able to evaluate $\psi$ at $(x, \tilde{y}, \frac{y - x}{x})$, $\tilde{y} \in \mathbb{Q} \setminus \{-2, -1, 0\}$, implies the ability to evaluate it at $(x, -1, \frac{y - x}{x})$. \[\square\]
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