Classification of integrable boundary equations
for integrable quad-graph systems

Pengyu Sun, Cheng Zhang†

Department of Mathematics
Shanghai University
Shanghai, 200444, China

Abstract

In the context of integrable systems on quad-graphs, the boundary consistency around a
half of a rhombic dodecahedron, as a companion notion to the three-dimensional consistency
around a cube, was introduced as a criterion for defining integrable boundary conditions for
quad-graph systems with a boundary. In this paper, we formalize the notions of boundary
equations as boundary conditions for quad-graph systems, and provide a systematic method
for solving the boundary consistency, which results in a classification of integrable boundary
equations for quad-graph equations in the Adler-Bobenko-Suris classification. This relies on
factorizing, first the quad-graph equations into pairs of dual boundary equations, and then the
consistency on a rhombic dodecahedron into two equivalent boundary consistencies. General-
izations of the method to rhombic-symmetric equations are also considered.

1 Introduction

In the context of integrable partial difference equations, also known as fully discrete integrable
equations in contrast to some well-known continuous integrable equations in the soliton theory,
the notion of three-dimensional consistency has emerged as the key feature of defining integrability
for fully discrete systems on quad-graphs [31, 12]. A major result was the classification of scalar
equations on quad-graphs led by Adler, Bobenko and Suris [2, 4], also known as the ABS classifica-
tion. One of the particular aspects of fully discrete integrable equations is that classes of equations
are formulated as polynomial or rational functions, the classification techniques are naturally con-
nected to languages and methods in some classical aspects of geometry and invariant theories. For
instance, quad-graph equations in the ABS classification are expressed in terms of affine-linear
polynomials of four variables, and the classification was based on analysis of canonical forms of
multivariate polynomials and their invariants with respect to common Möbius transformations.
Also in the classification of Yang-Baxter maps [3, 34], the maps are birational functions, and the
Yang-Baxter properties were described as consistency conditions among classical geometric objects
such as lines and pencils of quadratic curves. This is in an apparent contrast to classifications of
continuous integrable systems, where notions in differential geometry and related group/algebraic
structures are of prominent importance.

This paper aims to classify one particular type of discrete equations, known as integrable bound-
dary equations, which, together with the 3D-consistent equations, satisfy the boundary consistency condition. The latter condition was introduced as the integrability criterion for integrable bound-
dary conditions for quad-graph systems with boundary [16]. In this introductory section, necessary
backgrounds on quad-graph systems with boundary are provided.

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†Correspondence: ch.zhang.maths@gmail.com
1.1 Quad-graph systems with a boundary

The notion of integrable systems on quad-graph was introduced by Bobenko and Suris [12] (partly inspired by earlier work of Mercat [30] on discretizing Riemann surfaces). A quad-graph means a cellular decomposition of a surface with all faces being quadrilateral. It can be obtained from an arbitrary planar graph: first, set up the dual graph of an arbitrary cellular decomposition of a surface called original graph, then connect the adjacent vertices of original graph and its dual graph.

A quad-graph system is determined by equations of elementary type that are quad equations in the form (possibly depending on the lattice parameters, see Figure 1)

\[ Q(x, u, v, y; \alpha, \beta) = 0. \]  

(1.1)

Quad-graphs were constructed for original graphs without boundary. When taking an original graph with boundary, it was realized in [16] that the boundary should be represented by triangles as illustrated by Figure 2. Then, a quad-graph system with boundary is determined by two types of elementary equations: the quad equations representing the bulk dynamics, and the triangle equations, called boundary equations in this context, representing the boundary conditions.

A boundary equation with dependence on the lattice parameters is expressed as (see Figure 1)

\[ q(x, y, z; \alpha, \beta) = 0. \]  

(1.2)

By convention, the first and third arguments in \( q \), i.e. \( x, z \) in (1.2), are the values at the boundary vertices, and \( \alpha, \beta \) are the lattice parameters attached to edges connecting the value \( y \) to the boundary values \( x \) and \( z \), respectively.

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Figure 1: Elementary configurations for a quad-graph with boundary: a quadrilateral supporting a bulk quad equation (left); and a triangle supporting a boundary equation (right) where the thick line represents the boundary connected by boundary values.

Cauchy problems for generic (integrable or nonintegrable) quad-graph systems without boundary were investigated in [7] with the help of characteristic lines connecting the lattice parameters. A constructive approach was provided in [27]. In a broader geometric setting, the quad-graph systems appeared as a natural approach to discretizing Riemann surfaces [30, 10] (see also the monograph [13]). The question of connecting quad-graph systems with concrete examples of discrete Riemann surfaces and discrete complex analysis has its own significance that is well beyond the original motivations of solving initial value problems for quad-graph systems, c.f. [11].

Similar questions could be asked for quad-graph systems with boundary. Some basic examples of well-posed initial-boundary value problems were given in [16, 17]. However, a general criterion for the well-posedness of initial-boundary problems for quad-graphs with boundary still remains to be investigated. And the connections of quad-graphs with boundary to discrete Riemann surfaces and discrete complex analysis are completely open. The present work is partly motivated by these tantalizing perspectives.

1.2 Consistency as integrability criterion

We take the integrability of quad equations as synonymous with the 3D consistency. By 3D consistency, it means that a quad equation \( Q = 0 \) can be consistently imposed on a cube [31, 12]. When taking \( Q = 0 \) as a two-dimensional lattice equation depending on the lattice parameters, the 3D consistency implies that a collective shift, or covariance, of \( Q = 0 \) in a multi-dimensional...
lattice is an auto-Bäcklund transformation (see Figure 3). For a scalar quad equation, this interpretation, together with some extra assumptions of $Q$ such as irreducibility and affine-linearity, allows us to express $Q = 0$ in terms of the discrete zero curvature conditions. Namely, the equation $Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) = 0$ is a result of the constraint
\begin{align}
g_{x_1}(x_{12}; \alpha_1)g_{x_2}(\alpha_3) = g_{x_2}(x_{12}; \alpha_1)g_{x_1}(\alpha_3),
\end{align}
where $g$ is certain transition matrix in forms of Möbius transformation acting on the multi-dimensional lattice. For instance, $x_{13} = g(x, x_1; \alpha_1)[x_3]$, where $g(x, x_1; \alpha_1)$ is attached to the edge $(x, x_1; \alpha_1)$ with $\alpha_3$ being the Bäcklund (spectral) parameter.

Based on the 3D consistency condition, the ABS classification \cite{2}, up to few extra assumptions such as the $D_4$-symmetry, affine-linearity and tetrahedron property (these properties are also discussed in Section 2.1), exhausted all scalar integrable quad equations. The classification contains nine equations (their canonical forms are provided in Appendix A) called respectively $Q_4$, $Q_3(\delta)$, $Q_2$, $Q_1(\delta)$ as $Q$-type equations, $A_2$, $A_1(\delta)$ as $A$-type equations, and $H_3(\delta)$, $H_2$, $H_1$ as $H$-type equations. These equations are discrete analogs of the (continuous) Korteweg-de Vries (KdV)-type equations. For instance, the $Q_4$ equation \cite{1, 6} as the “master” equation in the classification is the lattice version of the famous Krichever-Novikov equation \cite{29}; $Q_1(0)$, $H_3(0)$ and $H_1$ are in connection to the Schwarzian KdV, modified KdV, and KdV equations respectively \cite{33}. A follow-up
study of the classification, by relaxing the $D_4$-symmetry and tetrahedron property, was taken following a more in-depth analysis of multi-affine polynomials and their invariants [4]. In particular, an extra list of integrable quad equations was provided, and named $\text{H'-type}$ (see Section 4). The list contains three rhombic-symmetric equations, and can be seen as an extension of the $\text{H-type}$ list. These integrable quad equations are the main objects of study of the paper.

As to quad-graph systems with boundary, the integrability criterion for boundary equations, known as the boundary consistency condition, was introduced in [16]. It emerged from the notions of set-theoretical reflection equation and its solutions as reflection maps [19, 37, 18] as companions to the set-theoretical Yang–Baxter equation [20] and its solutions as Yang-Baxter maps [38]. In a similar way to connecting integrable quad equations with Yang-Baxter maps [35], integrable boundary equations were connected with reflection maps, and the integrability condition inherits from that of the set-theoretical reflection equation, which can be imposed on a half of a rhombic dodecahedron (see Figure 4) [16]. Here, we adapt the definition given in [16] to our context.

Figure 4: Boundary consistency around a half of a rhombic dodecahedron (left) and its planar projection (right), where $Q = 0$ is imposed on four quadrilaterals and $q = 0$ is imposed on four triangles. The (closed) characteristic lines (the dotted lines) are reflected at the boundary edges.

**Definition 1** A nondegenerate boundary equation $q = 0$ is boundary consistent with an integrable quad equation $Q = 0$ if there is an involution relation $\sigma$ between the parameters such that the initial value problem on the half rhombic dodecahedron in Figure 4 with $\beta = \sigma(\alpha)$ and $\eta = \sigma(\lambda)$ is well-posed, i.e., the three ways of computing $t$ from initial values $x, y, u,$ and $\alpha, \lambda$ yield the same result. A boundary equation which is boundary consistent is called integrable.

The nuance between the above definition and that given in [16] is the requirement that $q = 0$ is nondegenerate, which will clarified in Definition 5 below. Therefore, the above definition of boundary consistency based on the involution relation $\sigma$ is reserved to nondegenerate boundary equations. For nondegenerate $q = 0$, the involution relation $\sigma$, as consistency between the parameters (see Figure 4), is taken as part of an integrable boundary equation [16]. Comparing to (1.2), an integrable boundary equation is simply put as

$$q(x, y, z; \alpha) = 0,$$

with $\beta = \sigma(\alpha)$. In the picture of classical soliton dynamics, $\sigma$ can be understood as the change of velocities when solitons interacting with an integrable boundary. We refer readers to [17] for the origin and significance of $\sigma$ in this respect. Despite of its importance, $\sigma$ obtained in some known examples was largely based on guesswork [16]. In this paper, a systematic derivation of $\sigma$ is provided.
As to degenerate \( q = 0 \), a class of boundary equations, known as degenerate integrable boundary equations, is shown to be boundary consistent with \( Q = 0 \) in this paper as well. However, it turns out that the involution relation \( \sigma \) is no longer needed for degenerate integrable boundary equations (see Section 3.3).

1.3 Preliminary results

Various aspects of integrable boundary equations were obtained in \([16, 17]\). These include the notions of dual boundary equations and discrete boundary zero curvature conditions. A partial list of integrable boundary equations was also given for quad equations in ABS classification. We illustrate these results by taking \( Q(0) \) as our primary example.

\( Q(0) \), also known as the lattice Schwarzian KdV, or cross-ratio equation, reads

\[
Q := Q(x, u, v; \alpha, \beta) = \alpha(x-v)(u-y) - \beta(x-u)(v-y).
\]

(1.5)

It has a pair of integrable boundary equations \( p = 0 \) and \( q = 0 \) with \( \sigma(\alpha) = \mu^2/\alpha \)

\[
q(x, y, z; \alpha) = \mu(x-y) + \alpha(y-z), \quad p(x, y, z; \alpha) = \alpha(y-z) + \mu(y-z),
\]

(1.6)

where \( \mu \neq 0 \) is an arbitrary constant. They have the following properties.

Duality property: \( q = 0 \) and \( p = 0 \) are dual in the following sense: taking the compatibility of

\[
Q(x, u, v, y; \alpha, \sigma(\alpha)) = 0, \quad q(x, u, y; \alpha) = 0.
\]

(1.7a)

Here, the compatibility means to put \( Q = 0 \) and \( q = 0 \) on the same quadrilateral. By eliminating, for instance, \( y \), the remaining equation is proportional to \( p(x, v, x; \alpha) = 0 \). Similarly, taking the compatibility of

\[
Q(x, u, v, y; \alpha, \sigma(\alpha)) = 0, \quad p(u, x, v; \alpha) = 0,
\]

(1.7b)

by eliminating \( v \), the remaining equation is proportional to \( q(x, u, y; \alpha) = 0 \). For a given integrable quad equation \( Q = 0 \) and an integrable boundary equation \( q = 0 \), one could obtain a dual \( p = 0 \) which is also an integrable boundary equation, and vice versa.

Discrete boundary zero curvature conditions: consider the pair of dual integrable boundary equations \( p = 0 \) and \( q = 0 \) given in \([17]\). Taking, for instance, \( p(x, y, z; \alpha) = 0 \), one can express \( z \) as

\[
z = k_\alpha(\alpha)[x],
\]

(1.8)

where \( k_\alpha(\alpha) \) is called boundary matrix that is attached to the value \( y \) with \( \alpha \) being a parameter. Clearly, it can be set up as a Möbius transformation. One requires that the action of a boundary matrix is accompanied by a change of parameter following \( \alpha \to \sigma(\alpha) \). Then, the integrable boundary equation \( q = 0 \), dual to \( p = 0 \), can be cast into

\[
g_{(u, y, \sigma(\alpha))}(\sigma(\lambda)) g_{(x, u, \alpha)}(\sigma(\lambda)) k_x(\lambda) = k_y(\lambda) g_{(u, y, \sigma(\alpha))}(\lambda) g_{(x, u, \alpha)}(\lambda),
\]

(1.9)

which holds precisely when \( q(x, u, y; \alpha) = 0 \). The above equation is referred to as the discrete boundary zero curvature condition for \( q = 0 \), where the boundary matrix is obtained using its dual boundary equation \( p = 0 \).

In \([17]\), the notion of duality was formulated in a more general setting, and the discrete boundary zero curvature condition was constructed following a dual boundary consistency. Moreover, a special type of initial-boundary value problems for quad-graph systems on a trip with two parallel boundaries was investigated, which led to the open boundary reduction technique, as an alternative to the periodic reduction. Some nontrivial examples of integrable mappings \([17, 26]\) were accordingly obtained.

In this paper, based on a systematic characterization of boundary equations in terms of multivariate polynomials, the duality property is formulated as the factorization property of quad equations. The role of the dual boundary equations played in the boundary consistency is also clarified.
1.4 Plan of the paper

The aim of the paper is to classify integrable boundary equations for quad equations in the ABS classification. In Section 2, after giving necessary ingredients to the ABS classification (detailed expressions are given in Appendix A), we provide the notions of boundary polynomials and boundary equations, and formulate the duality property as factorization of quad equations. This leads to pairs of boundary equations dual to each. We classify all such pairs for quad equations in the ABS classification. The results are given in Appendix B and C. In Section 3, for a given quad equation, we provide a criterion of selecting the integrable ones among the list of dual boundary equations. This relies on the factorization of the consistent system around a rhombic dodecahedron into two equivalent halves. The construction gives rise to the involution relation $\sigma$ needed in Definition 1. The degenerate boundary equations are also considered, and contribute to degenerate integrable boundary equations as solutions to the boundary consistency. In Section 4, we derive integrable boundary equations for the $H_1$ equation. Since $H_1$ is rhombic-symmetric only, one can obtain two sets of boundary consistency conditions depending on the patterns of the quadrilaterals and triangles. This allows us to design some interesting initial-boundary value problems on quad-graphs on a strip with two parallel boundaries that consist of different patterns of triangles. Section 5 contains concluding remarks.

2 Factorization of quad equations

In this section, we provide the notions of boundary polynomials and boundary equations that are the natural objects to characterize discrete boundary conditions for quad-graph systems with boundary. In the context of quad equations in the ABS classification, we provide systematic methods for factorizing the quad equations, which leads to pairs of boundary equations dual to each other. Factorization of quad equations plays a crucial role in the derivation of integrable boundary equations. The exhausted list of boundary equations satisfying the factorization property is obtained and provided in Appendix B (for Q-type and A-type equations) and C (for H-type equations).

2.1 ABS classification

Let us first collect some useful ingredients of the ABS classification. Their canonical forms (with respect to simultaneous Möbius transformations) of types Q, H and A are given in Appendix A. Details of the classification results can be found in the original papers [2, 4].

A quad equation $Q = 0$ in the ABS classification is in the form (1.1) (see Figure 1). The arguments $x, u, v, y$ are understood as discrete fields, and $\alpha, \beta$ are the lattice parameters. Each argument of $Q$ is living in the complex projective space $\mathbb{CP}^1$, and $Q$ is an irreducible polynomial affine-linear with respect to each of its arguments. In our context of quad-graph systems with boundary, we refer to such $Q$ as bulk polynomial and to $Q = 0$ as the associated bulk equation. Due to the irreducibility of $Q$, one requires $\alpha \neq \beta$. Here, we briefly present a series of properties of $Q$ that is of crucial importance in the rest of the paper.

$D_4$-symmetry: a bulk polynomial $Q$ obeys

\begin{align}
Q(x, u, v; y; \alpha, \beta) &= Q(y, v, x; u; \alpha, \beta), \\
Q(x, u, v; y; \alpha, \beta) &= -Q(x, v, y; \beta, \alpha).
\end{align}

Biquadratic polynomials: let a bulk polynomial $Q := Q(x, u, v, y; \alpha, \beta)$ be given, one can associate the edges and diagonals of the underlying quadrilateral with biquadratic polynomials that are obtained using a discriminant-type operator $\delta_{v, y}$, which eliminates two of the four arguments. For instance, one has (with the subscripts denoting partial derivatives)

$\delta_{v, y}Q := Q_{vy} - Q_y Q_v$.

One can check by direct computations that all bulk polynomials listed in Appendix A can be reduced to $Q|_{\alpha=\beta} \propto (u-v)(x-y)$. 

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as a biquadratic polynomial in $x, u$ attached to the edge connected by $x$ and $u$, and

$$\delta_{u,v}Q := QQ_{u} - Q_{u}Q_{v},$$

(2.3)
as a biquadratic polynomial in $x, y$ attached to the diagonal connected by $x$ and $y$. The biquadratic polynomials attached to other edges (resp. to the other diagonal) can be similarly derived, and have similar forms as (2.2) (resp. as (2.3)) due to the $D_4$-symmetry.

**Singular solutions:** a solution $(x, u, v, y)$ of (1.1) is said to be singular with respect to an argument, say $x$, if $Q = 0$ holds independently of $x$, c.f. [4].

The vanishing biquadratic polynomial leads to singular solutions of the bulk equation $Q = 0$.

For instance, if $(x, u) = (X, U)$ such that $\delta_{v,y}Q = 0$, then $Q$ becomes reducible and can be factorized as a product of two polynomials

$$Q|_{\delta_{v,y}Q=0} = F(X, U, v; \alpha, \beta)G(X, U, y; \alpha, \beta).$$  

(2.4)

In this case, $G(X, U, y; \alpha, \beta) = 0$ provides a singular solution of $Q = 0$ with respect to $v$. Similarly, $F(X, U, v; \alpha, \beta) = 0$ provides a solution singular with respect to $y$.

**Tetrahedron property:** for a given bulk equation $Q = 0$, it follows from the 3D consistency property that one can consistently assign values to all vertices on a cube using, for instance, $x, x_1, x_2, x_3$ as well as the parameters $\alpha_1, \alpha_2, \alpha_3$ as initial data (see Figure 5). By tetrahedron property, we mean that the values on the vertices connected by the diagonals of the faces of the cube (which form a tetrahedron) satisfy certain bulk equation of $Q$-type\footnote{2}. We use $Q^\top = 0$ to denote such equation that is called the tetrahedron equation of $Q = 0$.

For examples, let $Q$ be $H_1$, then $x_1, x_2, x_3, x_{123}$ that are the black dots in Figure 5 satisfy

$$Q^\top(x_{123}, x_1, x_2, x_3; \alpha_3 - \alpha_2, \alpha_3 - \alpha_1) = 0,$$

(2.5a)

where $Q^\top$ is the $Q_1(0)$ polynomial. Alternatively, the white dots $x, x_{12}, x_{13}, x_{23}$ satisfy

$$Q^\top(x, x_{12}, x_{13}, x_{23}; \alpha_1 - \alpha_2, \alpha_1 - \alpha_3) = 0.$$  

(2.5b)

In the case of $Q$-type equations, $Q^\top$ coincides with $Q$ itself.

**Symmetry of $Q$-type equations:** recall a M"obius transformation $m : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$,

$$m(x) = \frac{m_1x + m_2}{m_3x + m_4}, \quad m_1m_4 - m_2m_3 \neq 0.$$  

(2.6)

\footnote{2} Apparently, the diagonal biquadratic polynomials of all bulk equations coincide with the edge biquadratic polynomials of certain $Q$-type equation. Since a bulk equation can be identified with its edge biquadratic polynomials, the equation $Q^\top = 0$ relating the values on the tetrahedron (connected by the black or white vertices on the cube in Figure 5) is a $Q$-type equation.  

\[7\]
A Möbius transformation can be decomposed as a sequence of elementary transformations such as the translation $x \mapsto x + c$, the scaling transformation $x \mapsto cx$ and the inversion $x \mapsto 1/x$. By a symmetry of a bulk equation $Q = 0$, we mean that, under simultaneous Möbius transformations on all arguments of $Q$, one has

$$Q(m(x), m(u), m(v), m(y); \alpha, \beta) = 0,$$

when $Q(x, u, v, y; \alpha, \beta) = 0$ holds. Note that this notion of symmetry coincides with the Lie-point symmetry [35, 36, 39].

Symmetries of Q-type equations are listed in Table 1. They are useful devices for the derivation of integrable boundary equations. The Q1(0) equation has the generic Möbius transformation as its symmetry, while Q2 only has the trivial symmetry, i.e. $x \mapsto x$.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
& Q1(0) & Q1(\delta \neq 0) & Q2 & Q3(0) & Q3(\delta \neq 0) & Q4 \\
\hline
& cx, x + c, 1/x & \pm x, x + c & x & cx, 1/x & \pm x & \pm x, \pm 1/(kx) \\
\hline
\end{array}
$$

Table 1: Symmetries of the bulk equations. For Q4, $k$ denotes the elliptic modulus.

Moreover, each Q-type bulk polynomial also satisfies

$$Q(x, u, v, y; \alpha, \beta) = -Q(x, u, v, y; -\alpha, -\beta).$$

Equivalence class: each bulk polynomial $Q$ in its canonical form is a representative of an equivalence class $[Q]$ with respect to simultaneous Möbius transformations on all arguments of $Q$. We take the Möbius transformations (2.6), the action amounts to

$$Q_m := \Lambda(x)\Lambda(u)\Lambda(v)\Lambda(y)Q(m(x), m(u), m(v), m(y); \alpha, \beta),$$

where $\Lambda(\ast) := m_3 \ast + m_4$ is understood. Then, $\{Q \in [Q] : Q \sim Q_m\}$. A remarkable consequence of the ABS classification is that $Q_m = 0$ is also 3D-consistent. The related properties of $Q_m$ can be derived from those of $Q$.

In the rest of paper, we will closely work with the canonical forms of $Q$ in view of solving the boundary consistency. Having the canonical forms of bulk equations, the above properties can be directly checked by computations. For completeness, we provide the biquadratic polynomials as well as the tetrahedron equation $Q^\top = 0$ for all bulk equations in Appendix A. Their importance in deriving integrable boundary equations are becoming apparent.

2.2 Boundary polynomials and boundary equations

We proceed to formalizing the notions of boundary polynomials and boundary equations. Similar to the way of defining bulk equations, we use multivariate polynomials to characterize boundary equations. With some extra properties, these polynomials provide a natural way to define boundary equations considered in this paper.

**Definition 2** A multivariate polynomial $q$ depending on three arguments and two parameters in the form (see Figure 1)

$$q(x, y, z; \alpha, \beta)$$

is called a **boundary polynomial**, if 1) it is irreducible in $x, y, z$ and affine-linear with respect to $x$ and $z$, 2) it is $\mathbb{Z}_2$-symmetric meaning that there exists a parity parameter $\gamma = \pm 1$ such that

$$q(x, y, z; \alpha, \beta) = \gamma q(z, y, x; \beta, \alpha).$$

We call $q = 0$ the associated **boundary equation**.

Let us comment on some consequences of the above definition.
1. Consider a boundary equation in the form (see Figure 1)

\[ q(x, y, z; \alpha, \beta) = 0. \]

(2.12)

It is understood that the first argument \( x \) and the third argument \( z \) in \( q \) represent the values at the boundary vertices, and the parameters \( \alpha, \beta \) are the lattice parameters attached to the edges connecting the value \( y \) to the boundary values \( x \) and \( z \) respectively.

2. The \( \mathbb{Z}_2 \)-symmetry puts the two lattice directions associated with the parameters \( \alpha \) and \( \beta \) at the same footing. This is in line with the \( D_4 \)-symmetry of the bulk polynomials. In particular, a new ingredient \( \gamma \) called parity, enters into the definition of boundary polynomials. A boundary polynomial has parity as its extra characteristic. Note that the notion of parity for boundary conditions was also mentioned in [10] in the context of integrable PDEs.

3. The affine-linearity as well as the \( \mathbb{Z}_2 \)-symmetry allows us to express \( q \) as

\[ q(x, y, z; \alpha, \beta) = r_1(y; \alpha, \beta)xz + r_2(y; \alpha, \beta)x + r_3(y; \alpha, \beta)z + r_4(y; \alpha, \beta), \]

where \( r_j(y; \alpha, \beta), j = 1, \ldots, 4 \), are polynomials in \( y \) satisfying

\[ r_{1\Delta}(y; \alpha, \beta) = \gamma r_{1\Delta}(y; \beta, \alpha), \quad r_{2\Delta}(y; \alpha, \beta) = \gamma r_{3\Delta}(y; \beta, \alpha). \]

(2.14)

We do not exclude that the case that some or all \( r_j \) are independent of \( y \).

4. A boundary polynomial is required to be irreducible. This excludes the case that the quantity (we take \( q \) as shown in (2.13) by omitting the inputs)

\[ \delta_{x,z} q := q_{xz} - q_x q_z = r_1 - r_2 r_3, \]

(2.15)

vanishes identically for generic input \( y \). The irreducibility of \( q \) also excludes the case that \( q \) is decomposed as \( q = f(y)q^* \) with \( q^* \) being a boundary polynomial.

Remark 3 Taking account of the parity, a boundary polynomial \( q \) in the form (2.13) can always be reduced to one of the following two forms with opposing parities

\[ \gamma = 1, \quad q(x, y, z; \alpha, \beta) = r_1(y; \alpha, \beta)xz + r_2(y; \alpha, \beta)x + r_3(y; \alpha, \beta)z + r_4(y; \alpha, \beta), \]

(2.16a)

\[ \gamma = -1, \quad q(x, y, z; \alpha, \beta) = r_2(y; \alpha, \beta)x - r_3(y; \beta, \alpha)z, \]

(2.16b)

where \( r_{1\Delta}(y; \alpha, \beta) = r_{1\Delta}(y; \beta, \alpha) \) is understood. However, there is still a remaining freedom for the parity. For a given \( q \), an equivalence class can be defined using \( q \sim q^1 \), if \( q^1 = g(\alpha, \beta)q \) for some function \( g(\alpha, \beta) \) depending on the parameters only. Clearly, an anti-symmetric \( g(\alpha, \beta) \), i.e. \( g(\alpha, \beta) = -g(\beta, \alpha) \), leads to \( q^1 \) with opposing parity to \( q \). We will see examples later where the parity is fixed in order to have an appropriate form for \( q \).

Remark 4 A boundary equation in the form (2.13) can be used to define an initial value problem on a single triangle by using, for instance, the values \( x, y \) as well as the parameters \( \alpha, \beta \) as initial values to uniquely express \( z \). This can be generalized to the study of Cauchy problems for generic quad-graph systems with boundary similar to how this is done for quad-graph systems without boundary [7, 27]. Note that some well-posed initial-boundary value problems on quad-graphs with quad-graph systems with boundary similar to how this is done for quad-graph systems without boundary [16, 17].

As explained in Section 1.3, let a bulk equation \( Q = 0 \) be given, the key step to derive a boundary zero curvature condition for an integrable boundary equation \( q = 0 \) is based on the existence of a dual boundary equation \( p = 0 \) as a result of the compatibility of \( Q = 0 \) and \( q = 0 \). Here, we provide some simple consequences by taking a bulk polynomial \( Q \) and a boundary polynomial \( q \) defined on the same quadrilateral.

Let \( Q \) and \( q \) be in the forms

\[ Q(x, u, v; \alpha, \beta) = S(x, u, v; \alpha, \beta) + T(x, u, v; \alpha, \beta)y, \]

(2.17a)

\[ q(x, u, y; \alpha, \beta) = s(x, u; \alpha, \beta) + t(x, u; \alpha, \beta)y, \]

(2.17b)

where \( S, T, s, t \) are polynomials in their respective arguments. By irreducibility of \( Q \), we require that \( \alpha \neq \beta \). Moreover, we use the following notion of degeneracy for \( q = 0 \).
Definition 5 Let $Q$ and $q$ be defined on the same quadrilateral as given in (2.17). Then, the boundary equation $q = 0$ is said to be nondegenerate, if it does not lead to any singular solution to $Q = 0$.

By assuming $q = 0$ to be nondegenerate, we exclude the following two cases:

- let $q = 0$ be a boundary equation that leads to $\delta_{u,v}Q = 0$, then $Q$ becomes reducible, and $Q = 0$ has singular solutions with respect to either $u$ or $v$. This could happen for H-type bulk polynomials, and for $Q1(\delta)$, $Q3(0)$ and A-type bulk polynomials, where $\delta_{u,v}Q$ can be factorized as a product of two boundary polynomials (see Appendix A for details).

- one can express $Q$ in (2.17) as $Q = S(u, x; y; \alpha, \beta) + T(u, x; y; \alpha, \beta)v$ thanks to the symmetry (2.1a). Let $q$ be a linear combination of $S(u, x; y; \alpha, \beta)$ and $T(u, x; y; \alpha, \beta)$, namely,

$$q(x, u; y; \alpha, \beta) = c_1S(u, x; y; \alpha, \beta) + c_0T(u, x; y; \alpha, \beta).$$

Due to the symmetry (2.1b), both $S(u, x; y; \alpha, \beta)$ and $T(u, x; y; \alpha, \beta)$ are affine-linear with respect to $x, y$ and $\mathbb{Z}_2$-symmetric by interchanging $x, \alpha$ and $y, \beta$. Such $q$ is a boundary polynomial with parity $\gamma = -1$. This form is excluded, since $q = 0$, together with $Q = 0$, implies that either $S(u, x; y; \alpha, \beta) = 0$ and $T(u, x; y; \alpha, \beta) = 0$ should hold simultaneously, or a particular value $v = c_0/c_1$ is a solution of $Q = 0$. Both cases lead to singular solutions to $Q = 0$.

Remark 6 In this section, we do not consider degenerate boundary equations discussed in the above two cases, since they do not lead to any dual boundary equations (see next subsection for details). However, some of them satisfy the boundary consistency, and contribute to the classification of integrable boundary equations in the degenerate case (see Section 4.3).

By eliminating $y$ in (2.17), the resulting polynomial, denoted by $H$, can be expressed as

$$H := Q q_y - Q_y q = S t - T s.$$  

(2.19)

It depends on $x, u, v$ as well as on the parameters $\alpha, \beta$. $H$ will be the central object of interest.

Since the polynomials $S, T$ are of degree 1 in their respective arguments, and $s, t$ are of degree 1 in $x$, one has $\deg_x H \leq 2$ and $\deg_x H \leq 1$, and $H$ can be generically expressed as

$$H = H_0v + H_1,$$

(2.20)

where $H_0$ and $H_1$ are polynomials possibly depending on $x, u$. Under the assumption that $q = 0$ is nondegenerate, we can show $H$ is affine-linear with respect to $v$, and the equation $H = 0$ yields $v = -H_1/H_0$, which is, in general, a rational expression of $x, u$. Note that due to the $\mathcal{D}_4$-symmetry of $Q$ and $\mathbb{Z}_2$-symmetry of $q$, one can derive similar results for (2.17) by eliminating $x$.

Lemma 7 Consider the polynomial $H$ defined in (2.19). Then, $H_0$ and $H_1$ in (2.20) are nonvanishing polynomials for generic $x, u$, and $H_0$ is not proportional to $H_1$. Moreover, eliminating $v$ in $Q$ and $H$ yields

$$Q H_v - Q_v H = - (\delta_{v,y}Q) q(x, u; y; a, b),$$

where $\delta_{v,y}Q$ is the biquadratic polynomial defined in (2.21)

Proof: Taking $Q$ in the form (by omitting the inputs)

$$Q = R_1v_y + R_2v + R_3y + R_4,$$

(2.22)

where $R_j, j = 1, \ldots, 4$, are polynomials of $x, u$, $H_0, H_1$ in (2.20) can be expressed as

$$H_0 := (R_2t - R_1s), \quad H_1 := (R_4t - R_3s).$$

(2.23)

Let $H = 0$ as a result of $Q = 0$ and $q = 0$. For generic $v$, if $H_{0,1} = 0$, then $H_{1,0} = 0$. This implies that $R_2t - R_1s = 0$ and $R_4t - R_3s = 0$ should hold simultaneously for generic $x, u$, which means $\delta_{v,y}Q = R_1R_4 - R_2R_3 = 0$. This violates our assumption for $q$. 

10
For non vanishing $H_0$, $H_1$. If $\delta_{v,y}Q = R_1 R_4 - R_2 R_3 \neq 0$, assume there is a particular value $v = c_0/c_1$ such that $H = 0$, then $H_0$ is proportional to $H_1$, namely, $c_0 H_0 + c_1 H_1 = 0$. This implies, for instance, $H_0 = -c_1 h$ and $H_1 = c_0 h$ for $h$ being a non vanishing function depending on $x,u$. One could set $h = R_1 R_4 - R_2 R_3$, this allows us to express $s,t$ as
\begin{equation}
\begin{aligned}
s &= c_0 R_2 + c_1 R_3, \\
t &= c_0 R_1 + c_1 R_3.
\end{aligned}
\end{equation}

and has $q$ in the form
\begin{equation}
q = c_0 (R_2 + R_1 y) + c_1 (R_4 + R_3 y),
\end{equation}
which is precisely (2.18), since $S, T$ in (2.18) can be read from (2.22) as $S = R_1 + R_3 y$ and $T = R_2 + R_1 y$. This case is also excluded.

Knowing that $H$ is affine-linear with respect to $v$, one has
\begin{equation}
Q H_v - Q_v H = Q (Q_v q_y - Q_y q) - Q_v (Q q_y - Q_y q) = -(Q Q_v y - Q_v Q_y) q,
\end{equation}
which corresponds to (2.21).

2.3 Factorization of $Q$ and duality of boundary equations

Consider the polynomials $Q$ and $q$ given in (2.17). Assume $q = 0$ to be nondegenerate. The existence of a dual boundary equation implies that one can extract a boundary polynomial in the sense of Definition 2 from the polynomial $H$. This amounts to the following factorized form of $H$.

**Definition 8** Consider the polynomial $H$ defined in (2.19) as a result of eliminating $y$ in (2.17). If $H$ can be factorized as
\begin{equation}
H := Q q_y - Q_y q = \chi(x,u;\alpha,\beta)p(u,x,v;\alpha,\beta),
\end{equation}
where $p$ is also a boundary polynomial and $\chi$ is certain polynomial depending on $x,u$, then we say that the boundary equation $q = 0$ factorizes the bulk equation $Q = 0$, and $p = 0$ is the dual boundary equation of $q = 0$.

![Figure 6: Factorization of $Q$](image)

A significant consequence of the factorized form (2.27) is that the resulting boundary polynomial $p$ plays exactly the same role as $q$. For simplicity, we use $\Gamma$ to denote the biquadratic polynomial $\delta_{v,y}Q$, namely,
\begin{equation}
\Gamma(x,u;\alpha,\beta) := \delta_{v,y}Q(x,u,v;\alpha,\beta).
\end{equation}
Lemma 9 Assume the polynomial $H$ defined in (2.19) admits the factorized form (2.27). Then,

$$Q p_v - Q_v p = \zeta(x, u; \alpha, \beta)q(x, u; y; \alpha, \beta),$$  \hfill (2.29)

where $\zeta$ is certain polynomial depending on $x, u$. Moreover, $\chi$ and $\zeta$ are connected to $\Gamma$ as

$$\chi(x, u; \alpha, \beta)\zeta(x, u; \alpha, \beta) = -\Gamma(x, u; \alpha, \beta).$$  \hfill (2.30)

Proof: It follows from the factorized form (2.27) that (2.21) can be written as

$$Q H_v - Q_v H = \chi(Q p_v - Q_v p) = -\Gamma q.$$  \hfill (2.31)

The polynomials $\chi$ and $\Gamma$ are independent of $y$, while $Q p_v - Q_v p$ is affine-linear with respect to $y$ as a result of Lemma 7. Then, $Q p_v - Q_v p$ must be proportional to $q$, which amounts to (2.29). The last formula (2.30) is a consequence of (2.21) and (2.29). □

The boundary equation $p = 0$ also factorizes the bulk equation $Q = 0$ as shown in (2.29). This factorization process is illustrated in Figure 6. In this sense, we say that a bulk polynomial $Q$ (resp. a bulk equation $Q = 0$) admits a pair of boundary polynomials $p$ and $q$ (resp. a pair of boundary equations $q = 0$ and $p = 0$) dual to each other.

Another important consequence of (2.27) is that the set of equations $Q = 0$, $p = 0$ and $q = 0$ form a compatible system on a single quadrilateral with the values on the opposite triangles of the quadrilateral governed by the same boundary equation (see Figure 7). Consider, for instance, $x, u$ as well as the parameters $\alpha, \beta$ as initial values on a quadrilateral, which are subject to the equations $Q(x, u, v; y; \alpha, \beta) = 0$ and $q(x, u, y; \alpha, \beta) = 0$. Due to the $\mathbb{D}_4$-symmetry of $Q$ and $\mathbb{Z}_2$-symmetry of $q$, one can obtain the dual boundary equation $p = 0$ on the two opposite triangles, i.e. $p(v, y; u; \alpha, \beta) = 0$ and $p(u, x; v; \alpha, \beta) = 0$, sharing the same boundary that is the diagonal connected by the values $u, v$. Similarly, the $\mathbb{D}_4$-symmetry of $Q$ and $\mathbb{Z}_2$-symmetry of $p$ implies that one can obtain the equation $q$ on the two opposite triangles sharing the same boundary connected by the values $x, y$.

$$q(y, v, x; \alpha, \beta) = 0 \quad \iff \quad Q(x, u, v; y; \alpha, \beta) = 0$$

$$p(v, y, u; \alpha, \beta) = 0 \quad \iff \quad p(u, x; v; \alpha, \beta) = 0$$

Figure 7: Factorization of $Q$ along the two diagonals (thick lines).

There is also a duality between the factorized forms (2.27) and (2.29), and the polynomials $\chi$ and $\zeta$ dual to each other are connected by the formula (2.30). This allows us to determine the degrees of the middle argument of $q$ and $p$. When $Q$ is a $Q$-type or $A$-type bulk polynomial, $\Gamma$ is of bidegree $(2, 2)$ in $x, u$. It follows from (2.30) that $\Gamma$ can be factorized as a product of $\chi$ and $\zeta$, therefore $\chi$ can be either of bidegree $(0, 0)$ in $x, u$ ($\zeta$ of bidegree $(2, 2)$ in $x, u$), or of bidegree $(1, 1)$ in $x, u$ ($\zeta$ of bidegree $(1, 1)$ in $x, u$). The former case is dual to the case where $\chi$ is of bidegree $(2, 2)$ ($\zeta$ of bidegree $(0, 0)$), and the latter case only happens when $Q$ is $Q1(\delta)$, $Q3(\delta = 0)$, or $A$-type polynomials where $\Gamma$ can be factorized as a product of two polynomials of bidegree $(1, 1)$.

Lemma 10 Let $Q$ be a $Q$-type or $A$-type bulk polynomial admitting a pair of dual boundary polynomials $q, p$ as given in (2.21) and (2.24). Let $c, d$ denote respectively the degree of the middle argument of $q, p$, and $e, e'$ denote respectively the degree of $\chi, \zeta$ in one of its arguments, namely,

$$c = \deg_u q, \quad d = \deg_x p, \quad e = \deg_x \chi = \deg_u \chi, \quad e' = \deg_x \zeta = \deg_u \zeta.$$  \hfill (2.32)
Then,
\[ c = e, \quad d = e', \quad c + d = e + e' = 2. \] (2.33)

**Proof:** One has the explicit degrees of the polynomials
\[ Q(x, u, v, y), \quad q(x, y), \quad p(x, v), \quad \chi(x, u), \quad \zeta(x, u), \quad \Gamma(x, u), \] (2.34)
where \( e + e' = 2 \) due to (2.30). Counting the degrees of (2.27) and (2.29) in \( x, u \) leads to
\[ 1 + e \leq 1 + c, \quad e + d \leq 2, \quad 1 + e' \leq 1 + d, \quad e' + e \leq 2. \] (2.35)
Arranging the above inequalities leads to the results (2.33).

**Remark 11** Similar to Lemma 10 for H-type polynomials, one has
\[ c + d \leq 2, \] (2.36)
where \( c, d \) denote respectively the degree of the middle argument of \( q, p \). The proof is less straightforward. This is because H-type polynomials are degenerate in the ABS classification in the sense that \( \Gamma \) in the canonical form is of bidegree \((0, 0)\) (for H1) or \((1, 1)\) (for H2, H3), c.f. [4]. In these cases, some zeros of \( \Gamma \) are located at \( \infty \) for one or both of its arguments, and there are some extra freedoms for determining \( c, d \) (see comments below). Derivations of \( q, p \) for H-type polynomials require the action of simultaneous M"{o}bius transformations on the factorized forms (2.27) and (2.29). The proof of (2.30) requires some explicit computations that are provided in Appendix C.

Let us comment on the action of simultaneous M"{o}bius transformations on the factorized forms (2.27) and (2.29). Take the M"{o}bius transformation (2.30). Similar to (2.31), one has
\[ q_m := \Lambda(x)\Lambda^c(u)\Lambda(y)q(m(x), m(u), m(y); \alpha, \beta), \quad \chi_m := \Lambda^c(x)\Lambda^c(u)\chi(m(x), m(u); \alpha, \beta), \] (2.37a)
\[ p_m := \Lambda(u)\Lambda^d(x)\Lambda(v)p(m(u), m(x), m(v); \alpha, \beta), \quad \zeta_m := \Lambda^d(x)\Lambda^d(u)\zeta(m(x), m(u); \alpha, \beta), \] (2.37b)
where the degrees of the factors \( \Lambda \) follow from (2.32). Clearly, \( q_m, p_m \) are boundary polynomials in the sense of Definition 2 if so are \( q, p \). For \( Q_m \) defined in (2.31), one also has
\[ \delta_{v,y}Q_m = \Delta^2_m \Gamma_m, \quad \Gamma_m := \Lambda^2(x)\Lambda^2(u)\Gamma(m(x), m(u); \alpha, \beta), \] (2.38)
where \( \Delta_m = m_1m_4 - m_2m_3. \) Moreover, it follows from (2.27) and (2.29) that
\[ Q_{m,y}q_{m,y} - Q_{m,y}q_m = \Delta_m \Lambda^{2-c-d}(x)\Lambda^{c-e}(u)\chi_mp_m, \] (2.39a)
\[ Q_{m,v}p_{m,v} - Q_{m,v}p_m = \Delta_m \Lambda^{2-e'-(c)}(x)\Lambda^{d-e}(u)\zeta_mp_m, \] (2.39b)
and from (2.30) that
\[ \Lambda^{2-c-e'}(x)\Lambda^{2-c-e'}(u)\chi_m\zeta_m = -\Gamma_m. \] (2.40)
If \( Q \) is a Q-type or A-type bulk polynomial, \( e + e' = 2, \) \( e = c, \) \( d = e' \), the degrees of the factorized forms (2.30) are completely fixed for a given \( \chi \) (or \( \zeta \)). However, for H-type polynomials, one has \( e + e' \leq 1 \), the degrees of the factorized forms (2.30) could have extra freedoms. For instance, for H1, if \( e = e' = 0 \), both \( c, d \) could take values in \( 0, 1, 2 \), and it might happen that \( c = d = 2 \). However, this contradicts (2.36) and is indeed excluded following explicit computations (see Appendix C).

### 2.4 Classification of boundary polynomials dual to each other

For a bulk polynomial \( Q \) in the ABS classification, we provide two systematic approaches to deriving boundary polynomials that lead to the factorized form (2.27). The boundary polynomials obtained following these two approaches are respectively called M"{o}bius case (having \( \mathcal{M}_\pm \) cases and dual \( \mathcal{M}_\pm \) cases as subcases) and singular case. As an illustration, explicit computations are provided for the
Q1(δ) polynomial. We conclude this section by arguing that these two cases exhaust all possible pairs of dual boundary polynomials for the ABS classification.

Recall the forms of Q, q, p, χ, ζ, Γ presented in the previous subsection

\[ Q := Q(x, u, v; y; \alpha, \beta), \quad q := q(x, u, v; \alpha, \beta), \quad p := p(u, x, v; \alpha, \beta), \]  
\[
\chi := \chi(x, u; \alpha, \beta), \quad \zeta := \zeta(x, u; \alpha, \beta), \quad \Gamma := \Gamma(x, u; \alpha, \beta). \tag{2.41a, 2.41b}
\]

**Möbius case:** let q be a boundary polynomial in one of the following two forms called respectively \( M_+ \) and \( M_- \) cases according to the parity

\[
\begin{align*}
M_+ & : \quad \gamma = 1, \quad q = r_1 xy + r_2 x + r_2 y + r_4, \tag{2.42a} \\
M_- & : \quad \gamma = -1, \quad q = x - y, \tag{2.42b}
\end{align*}
\]

where \( r_j, j = 1, 2, 4, \) in the \( M_+ \) case is independent of \( u \) but possibly depends on the parameters \( \alpha, \beta \) (in the parameter-dependent case, \( r_j \) is a symmetric function, c.f. Remark 3), and \( r_1 r_4 - r_2^2 \neq 0 \). Now, \( q \) has \( x \) and \( y \) connected by a “constant” Möbius transformation independent of \( u \). By eliminating \( y \) in \( Q \) and \( q \), one automatically has the factorized form

\[ Q = S + Ty \] as given in (2.17a), one has

\[
\begin{align*}
\text{dual } M_+ & : \quad H = r_1 S x + r_2 (S - T)x - r_4 T = p, \tag{2.43a} \\
\text{dual } M_- & : \quad H = -S - T x = \kappa p, \tag{2.43b}
\end{align*}
\]

where the factor \( \kappa \) in the dual \( M_- \) case is introduced as a function of \( \alpha, \beta \) satisfying \( \kappa(\alpha, \beta) = -\kappa(\beta, \alpha) \). The second equality in both equations serves as a definition for \( \kappa \). Since the polynomials \( S, T \) are affine-linear with respect to \( u, v \), and \( \mathbb{Z}_2 \)-symmetric with parity \(-1\) by interchanging \( u, \alpha \) and \( v, \beta \) (due to the symmetry (2.1b)), \( p \) must be affine-linear with respect to \( u, v \), and \( \mathbb{Z}_2 \)-symmetric. Moreover, one can show that \( p \) is irreducible. Therefore, \( p \) is a boundary polynomial in the sense of Definition 2.

For the \( M_+ \) and dual \( M_+ \) cases, \( q \) and \( p \) are, in general, three-parameter family of polynomials having \( r_1, r_2, r_4 \) as parameters. For the dual \( M_- \) case, one has \( p := (S - T x)/\kappa \). Since \( \kappa \) is an anti-symmetric function, \( q \) has parity \( \gamma = 1 \), c.f. Remark 3. Related properties of the boundary polynomials \( p, q \) in the \( M_\pm \) and dual \( M_\pm \) cases are listed in Table 2.

| \( q \) | \( M_+ \) and dual \( M_+ \) | \( M_- \) and dual \( M_- \) |
|---|---|---|
| \( \gamma = 1, \ c = 0, \ \chi = 1 \) | \( \gamma = 1, \ c = 0, \ \chi = \kappa \) | \( \gamma = 1, \ d \leq 2, \ \zeta = -1/\kappa \) |
| \( \gamma = -1, \ d \leq 2, \ \zeta = -1 \) | \( \gamma = -1, \ c = 0, \ \chi = \kappa \) | \( \gamma = 1, \ d \leq 2, \ \zeta = -1/\kappa \) |

Table 2: Möbius case: pairs of dual boundary polynomials \( q, p \) constructed using a “constant” Möbius transformation. The degrees of the middle argument are denoted by \( c, d \).

**Singular case:** this case only applies to Q1(δ), Q3(0), A-type and H-type bulk polynomials. Since H-type polynomials are treated in details in Appendix C, we only take \( Q \) to be Q1(δ), Q3(0), or A-type polynomials. In this case, \( \Gamma \) is of bidegree \((2, 2)\) but can be factorized as a product of two polynomials of bidegree \((1, 1)\). Let \( \chi \) be one of the polynomials, due to (2.30), if \( \chi = 0 \), then \( \Gamma = 0 \) and \( H \) in (2.27) vanishes identically for generic \( v \). This implies \( Q \) and \( q \) possess a common factor as

\[ Q|_{\chi=0} = F(x, U(x), v; \alpha, \beta) G(x, U(x), y; \alpha, \beta), \quad q|_{\chi=0} \propto G(x, U(x), y; \alpha, \beta), \tag{2.44} \]

where \((x, u) = (x, U(x))\) such that \( \chi = 0 \). Since \( \chi \) is of bidegree \((1, 1)\) in \( x, u, U(x) \) is of degree 1 in \( x \). Due to Lemma 10 \( q \) is of degree 1 in its middle argument, and can be generically expressed as

\[ q = (r_{10} + r_{11} u) x y + (r_{20} + r_{21} u) x + (r_{30} + r_{31} u) y + (r_{40} + r_{41} u), \tag{2.45} \]

with \( r_{ij} := r_{ij}(\alpha, \beta) \) to be determined. Such \( q \) can be reduced to

\[ q|_{\chi=0} = (c_0 x + c_1) G(x, U(x), y; \alpha, \beta), \tag{2.46} \]
where \( c_0, c_1 \) are assumed to be some constants. By comparing the coefficients of \([2.10]\) evaluated at \((x, u) = (x, U(x))\) with those of \([2.16]\), one sets up six equations allowing us to express six out of eight parameters in \([2.16]\) in terms of \(c_0, c_1\). The remaining freedoms can be fixed by taking account of the \(Z_2\)-symmetries of \(q, p\).

As a result, one can always obtain a pair of two-parameter family of boundary polynomials dual to each other in the forms

\[
q = c_0 q_0 + c_1 q_1, \quad p = c_0 p_0 + c_1 p_1, \tag{2.47}
\]

where \(q_j\) is dual to \(p_j, j = 1, 2\), and \(c_0, c_1\) do not vanish simultaneously. The case \(c_0 = 0\) or \(c_1 = 0\) represents a subcase.

**Q1(\(\delta\)) as an example:** the bulk polynomial \(Q\) reads

\[
\alpha(x - v)(u - y) - \beta(x - u)(v - y) + \delta^2 \alpha \beta (\alpha - \beta), \tag{2.48}
\]

with

\[
\Gamma = \delta_\nu \nu Q = -\beta(\alpha - \beta)(x - u + \delta \alpha)(x - u - \delta \alpha), \tag{2.49}
\]

which is a product of two polynomials of bidegree \((1, 1)\). The \(M_\pm\) and dual \(M_\pm\) cases can be directly read from \([2.42]\) and \([2.43]\). For \(M_-\) case, one sets \(\chi := \kappa = \alpha - \beta\) which yields

\[
p = uv - x(u + v) + x^2 - \delta \alpha \beta, \quad \zeta = \beta ((x - u)^2 - \delta^2 \alpha^2), \tag{2.50}
\]

for \(\delta \neq 0\). The \(M_-\) case is excluded for Q1(0), since \(x - y\) leads to \(\delta_{uv} Q = 0\).

For the singular case, let \(\{\alpha, \beta\} \rightarrow \{\alpha^2, \beta^2\}\) to avoid square roots in the parameters. Without loss of generality, take \(\chi \propto x - u + \delta \alpha^2\), then \(\zeta \propto x - u - \delta \alpha^2\). Following \([2.35]\) and \([2.46]\), one gets explicitly \(r_{11} = 0, r_{21} = -c_0, r_{31} = c_0 - r_{10}, r_{20} = -c_1 - r_{41} + \delta^2 c_0, r_{30} = c_1 - \delta^2 (c_0 - r_{10})\) and \(r_{40} = \delta c_1 (\beta^2 - \alpha^2) - \alpha^2 r_{41} \delta\).

It remains to fix \(r_{10}, r_{41}\). Due to the \(Z_2\)-symmetries of \(q, p\), one can assume

\[
q(x, u, y; \alpha, \beta) = \gamma(\alpha, \beta) q(y, u, x; \beta, \alpha), \quad p(u, x, v; \alpha, \beta) = \eta(\alpha, \beta) p(v, x, u; \beta, \alpha), \tag{2.51}
\]

with \(\gamma(\alpha, \beta) \gamma(\beta, \alpha) = \eta(\alpha, \beta) \eta(\beta, \alpha) = 1\). This further leads to \(\gamma(\alpha, \beta) = \pm \beta / \alpha, \eta(\alpha, \beta) = -\beta^2 / \alpha^2, r_{10} = c_0 (\alpha \pm \beta) / \alpha\) and \(r_{41} = -c_1 (\alpha \pm \beta) / \alpha\). Without loss of generality, one can absorb the sign \(\pm\) into \(\beta\), since \(Q\) is unchanged under \(\beta \rightarrow -\beta\). Thus, the forms of \(q, p\) are completely fixed. Lastly, taking \(q \rightarrow \alpha q\) which normalizes \(\gamma(\alpha, \beta) \rightarrow 1\), one can express \(q, p\) in the forms of \([2.47]\).

Precisely,

\[
q_0 = (\alpha + \beta) xy - (\alpha u - \delta \alpha \beta^2) x - (\beta u - \delta \alpha^2 \beta) y, \tag{2.52a}
\]

\[
q_1 = \beta x + \alpha y + (\alpha + \beta) (\delta \alpha \beta - u), \tag{2.52b}
\]

\[
p_0 = - (\alpha - \beta) uv + \alpha xv - \beta x v + \delta \alpha \beta (\alpha - \beta) (\delta \alpha \beta + x), \tag{2.52c}
\]

\[
p_1 = \beta u - \alpha v + (\alpha - \beta) (\delta \alpha \beta + x), \tag{2.52d}
\]

where \(q\) has parity 1 and \(p\) has parity \(-1\). The duality also holds under \(\delta \rightarrow -\delta\).

**Classification of boundary polynomials dual to each other:** it turns out that the two approaches presented here allow us to list all possible dual boundary polynomials \(q, p\) for a given \(Q\) in the ABS classification. This relies on exhausting \(\chi\) (or \(\zeta\)). The statement can be clearly justified for Q-type and A-type polynomials, since the degrees of \(q, p\) can be fixed thanks to Lemma \([10]\). The complete list of \(q, p\) for Q-type and A-type polynomials is provided in Appendix \([13]\). The situation for H-type polynomials is less straightforward, and explicit computations are provided in Appendix \([10]\).

- **Q-type polynomials:** it follows Lemma \([10]\) that for a given \(\chi\) (or \(\zeta\)), the degrees of the middle argument of \(q, p\) are fixed. Then, it suffices to list all possible cases for \(\chi\).

  Due to \([2.39]\), one has two possibilities:
1. Möbius case: $\chi$ is of bidegree $(0,0)$ in $x,u$ and $\zeta$ is of bidegree $(2,2)$ in $x,u$. Then, deg$_u q = 0$, deg$_u p = 2$, the deg$_u q = 0$ polynomials are exhausted by the $M_+$ cases. The case $\chi$ is of bidegree $(2,2)$ is dual.

2. singular case: if $Q$ is $Q1(\delta)$ or $Q3(0)$, it is possible that both $\chi$ and $\zeta$ are of bidegree $(1,1)$ in $x,u$ since $\Gamma$ can be factorized as a product of two polynomials of bidegree $(1,1)$. This case can also be exhausted in the singular case approach.

- **A-type polynomials:** the arguments for Q-type polynomials hold here, since $A1(\delta)$, $A2$ can be obtained respectively from $Q1(\delta)$ and $Q3(0)$ (see Appendix A.3).

- **H-type polynomials:** for $H1$ in its canonical form, $\Gamma$ is of bidegree $(0,0)$ in $x,u$. Under the simultaneous Möbius transformations in sense of (2.29) and (2.37), one transforms $\Gamma$ to $\Gamma_m$ which is of bidegree $(2,2)$ in $x,u$. Then, all possible $q,p$ can be derived following the singular case approach by exhausting all possible $\chi$ (or $\zeta$) following (2.40). By explicit computations, it turns out that all possible $q,p$ of $H1$ belong to the $M_+$ and dual $M_+$ cases.

For $H2$, $H3(\delta)$, $\Gamma$ is of bidegree $(1,1)$ in $x,u$. Similarly to $H1$, under suitable simultaneous Möbius transformations, all possible $q,p$ can be obtained following the singular case approach by exhausting all possible $\chi$ (or $\zeta$). Besides the Möbius case, it turns out that both $H2$ and $H3$ possess a pair of boundary polynomials in the forms (2.47) that can only be obtained from the singular case.

Let us summarize the above results.

**Theorem 12** The boundary polynomials $q,p$ dual to each other satisfying the factorization property (2.27) for $Q$ in the ABS classification can be exhausted into the following three categories.

1. $M_+$ and dual $M_+$: this applies to all bulk equations. In general, $q,p$ are three-parameter families of boundary polynomials in the forms

   $$
   M_+: q = r_1 xy + r_2 x + r_2 y + r_4, \quad \text{dual } M_+: p = r_1 p_1 + r_2 p_2 + r_4 p_4.
   $$

   where $r_j, j = 1, 2, 4,$ is a symmetric function of $\alpha, \beta,$ and $r_1 r_4 - r_2^2 \neq 0$.

2. $M_-$ and dual $M_-$: this applies to all bulk equations except for $Q1(0), A1(0), H1$ due to the nondegeneracy requirement. $q,p$ are in the forms

   $$
   M_- : q = x - y, \quad \text{dual } M_- : p = p.
   $$

3. Singular case: for $Q3(0), Q1(\delta), A2, A1(\delta), H3(\delta \neq 0)$ and $H2$, exactly one pair of $q,p$, not obtainable as a subcase of (2.53), exists. The pair $q,p$ are, in general, two-parameter families of boundary polynomials in the forms

   $$
   q = c_0 q_0 + c_1 q_1, \quad p = c_0 p_0 + c_1 p_1,
   $$

   where $q_j$ is dual to $p_j, j = 1, 2,$ and $c_0, c_1$ are constants that do not vanish simultaneously.

**Remark 13** There are still some subtleties here. For $H2$ and $H3(\delta \neq 0)$ as degenerate polynomials in the ABS classification, it happens that the pair $q_1, p_1$ in the singular case coincides with the $M_-$ and dual $M_-$ cases (see Appendix A). For $Q3(0), Q1(\delta), q,p$ in the singular case were obtained following certain changes of the lattice parameters (see Appendix B) to avoid square roots in the expressions, and certain $q_j$ either coincides with, or is equivalent to (in the sense of equivalence given in Remark 1) $p_j, j = 0, 1$: for $Q1(0)$, $q_0/p_1$ coincides with $p_0/p_1$ under irrelevant change $\beta \rightarrow -\beta$, so that $q$ coincides with $p$ in the singular case; for $Q1(\delta \neq 0)$, $q_1$ coincides with $p_1$ under $\beta \rightarrow -\beta$; for $Q3(0)$, $q_0/p_1$ is equivalent to $p_0/p_1$ under irrelevant change $\beta \rightarrow -\beta$, but $q,p$ are different from each other.
3 Factorization of consistency on rhombic dodecahedron and classification of integrable boundary equations

Based on the factorization of $Q = 0$ by pairs of boundary equations $p = 0$ and $q = 0$ dual to each other, we provide a criterion to select admissible boundary equations that are boundary consistent with $Q = 0$. This relies on the factorization of a consistent system around a rhombic dodecahedron into two equivalent halves. In particular, the construction allows us to identify the involution relation $\sigma$ needed in Definition 1 for nondegenerate boundary consistency. The degenerate boundary equations $q = 0$ (see Definition 5) are also considered. Some of them contribute to degenerate integrable boundary equations.

3.1 Boundary consistency as a factorized rhombic dodecahedron

Any bulk equation $Q = 0$ in the ABS classification is consistent on a hypercube that is a four-dimensional cube (see Figure 8). This implies that $Q = 0$ is consistent around a rhombic dodecahedron.

**Theorem 14** Let a $Q$ in the ABS classification be given. Consider an initial value problem for $Q = 0$ on a rhombic-dodecahedron with initial values $x, x_1, x_2, x_3, x_4$ as well as the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (see Figure 9). Then, $Q = 0$ is consistent around a rhombic-dodecahedron which means the four ways to compute $x_{1234}$ give the same result.

**Proof:** A rhombic-dodecahedron can be constructed from two possible “shadow projections” of a hypercube cube (see Figure 9). Then, the consistency for $Q = 0$ around a rhombic-dodecahedron follows from the consistency around a hypercube.

Note that the consistency around a rhombic dodecahedron for $Q = 0$ was also used in [24] as an important devise to reduce the bulk equation to certain discrete Painlevé equations. In our context, the consistency around a rhombic dodecahedron is needed since the boundary consistency is derived by factorizing the consistent system into two equivalent ones around the two halves of the rhombic dodecahedron (see Figure 10). In this process, the notion of factorization of $Q$ provided in Section 2.3 is the key ingredient.

**Theorem 15** Let a $Q$ in the ABS classification be given. Let $q = 0$ and $p = 0$ be a pair of boundary equations dual to each other of $Q = 0$. Consider an initial value problem for $Q = 0$ on a rhombic-dodecahedron (see Figure 10) with $x, x_1, x_2$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as the initial data. Moreover, assume...
the values $x_3, x_4$ are obtained using one of the boundary equations, say $p = 0$, by solving

\[ p(x_1, x, x_4; \alpha_1, \alpha_4) = 0, \]  
\[ p(x_2, x, x_3; \alpha_2, \alpha_3) = 0. \]  

We call $p = 0$ the companion boundary equation for the boundary consistency. If the values of $x_{124}, x_{14}, x_{134}$ satisfy again the boundary equation $p = 0$, namely,

\[ p(x_{124}, x_{14}, x_{134}; \alpha_2, \alpha_3) = 0, \]  

which is accompanied by certain constraints on the lattice parameters, then the boundary equation $q = 0$ dual to $p = 0$ is boundary consistent with $Q = 0$.

**Proof:** The sets of values $x, x_1, x_4, x_{14}$ and $x, x_2, x_3, x_{23}$ satisfy the bulk equations $Q = 0$

\[ Q(x, x_1, x_4, x_{14}; \alpha_1, \alpha_4) = 0, \]  
\[ Q(x, x_2, x_3, x_{23}; \alpha_2, \alpha_3) = 0. \]  

Since the set of equations $Q = 0, p = 0$ and $q = 0$ form a compatible system on a single quadrilateral (see Figure 7), by imposing (3.1), $x, x_1, x_{14}$ and $x, x_2, x_{23}$ obey the boundary equations $q = 0$

\[ q(x, x_1, x_{14}; \alpha_1, \alpha_4) = 0, \]  
\[ q(x, x_2, x_{23}; \alpha_2, \alpha_3) = 0. \]  

If the condition (3.2) is satisfied, using symmetry arguments by interchanging the indices $1, 2$ and $3, 4$, one can show $x_{123}, x_{23}, x_{234}$ also satisfy $p = 0$, namely,

\[ p(x_{123}, x_{23}, x_{234}; \alpha_1, \alpha_4) = 0. \]  

Based on the duality between $q = 0$ and $p = 0$, the values on the rhombic dodecahedron can be uniquely determined with the initial data $x, x_1, x_2$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ using $Q = 0$ and $q = 0$, since the “front”, “back”, “top” and “bottom” equations of the rhombic dodecahedron (see Figure 10) admit the factorization property as shown in Figure 7 thanks to the equations (3.1), (3.2) and (3.3). This implies the sets of values $x_{14}, x_{124}, x_{1234}$ and $x_{23}, x_{123}, x_{1234}$ also satisfy $q = 0$, namely,

\[ q(x_{14}, x_{124}, x_{1234}; \alpha_2, \alpha_3) = 0, \]  
\[ q(x_{23}, x_{123}, x_{1234}; \alpha_1, \alpha_4) = 0. \]
Figure 10: Factorization of a rhombic dodecahedron into two halves: the factorization of the “front” equation involving \( x, x_2, x_3, x_{23} \) (resp. of the “bottom” equation involving \( x, x_1, x_4, x_{14} \)) is transformed to the “back” equation involving \( x_{14}, x_{124}, x_{134}, x_{1234} \) (resp. to the “top” equation involving \( x_{23}, x_{123}, x_{234}, x_{1234} \)).

Finally, the consistency around the rhombic dodecahedron for \( Q = 0 \) implies the consistency around a half of the rhombic dodecahedron for \( Q = 0 \) and \( q = 0 \) modulo certain constraints on the lattice parameters.

We have just shown that (3.2) provides a sufficient condition that enables us to select integrable ones among the list of dual boundary equations of a given \( Q = 0 \). In the following, the companion boundary equation for the boundary consistency is denoted by \( p = 0 \), so that its dual boundary equation \( q = 0 \) is the “potentially” integrable one satisfying the boundary consistency.

Remark 16 One could interpret the condition (3.2) as a Bäcklund transformation of (3.1b). By the tetrahedron property, the sets of values \( x_{124}, x_1, x_4 \) and \( x_{134}, x_1, x_3 \) satisfy the tetrahedron equations

\[
Q^T(x_{124}, x_1, x_4; \alpha_{12}, \alpha_{14}) = 0, \quad Q^T(x_{134}, x_1, x_3; \alpha_{13}, \alpha_{14}) = 0,
\]

where \( \alpha_{ij} = \alpha_i - \alpha_j \) and \( Q^T = 0 \) is given in (2.5a). Then, \( x_{124}, x_{134} \) can be expressed as

\[
x_{124} = g^T(x_1, x_4; \alpha_{14})[x_2], \quad x_{134} = g^T(x_1, x_4; \alpha_{13})[x_3],
\]

where \( g^T \) is the transition matrix associated to \( Q^T = 0 \) (see (1.3)). Using the boundary equations (3.4a) and (3.1a), one can express \( x_{14} \) and \( x_4 \) as

\[
x_{14} = h_{x_1}(\alpha_1, \alpha_4)[x], \quad x_4 = k_{x_1}(\alpha_1, \alpha_4)[x_1],
\]

where \( h, k \) are the same boundary matrices similar to (1.8). If the condition (3.2) holds, then

\[
p(g^T(x_1, x_4; \alpha_{14})(\alpha_{12})[x_2], h_{x_1}(\alpha_1, \alpha_4)[x], g^T(x_1, x_4; \alpha_{13})(\alpha_{13})[x_3]; \alpha_2, \alpha_3) = 0,
\]

provided that \( x_4 = k_{x_1}(\alpha_1, \alpha_4)[x_1] \), which is a Bäcklund transformation of (3.1b). In other words, using the set of transformations (3.8), one transforms the factorization property of \( Q \) from the “front” equation to the “back” equation, and similarly from the “bottom” equation to the “top” equation.

The condition (3.2), or alternatively (3.9), amounts to certain constraints on the parameters \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \). This constraint gives rise to the involution relation \( \sigma \) as the consistency condition needed in Definition 1 among the parameters.
3.2 Deriving integrable boundary equations and classification

To derive integrable boundary equations, one could readily check the condition (3.2), or alternatively (3.3), by “brute force” using the list of dual boundary equations \( q = 0 \) and \( p = 0 \) derived in Section 2. This will done for \( q, p \) in the singular case. For the boundary equations derived from the Möbius case, i.e. \( q, p \) in the \( M_\pm \) and dual \( M_\pm \) cases, we provide an efficient method for determining whether they are integrable. The full list of integrable boundary equations obtained following Theorem 15 is given in Table 3 and 4.

Möbius case I: \( q \) in the \( M_\pm \) case, \( p \) in the dual \( M_\pm \) case. Assuming the sets of values \( x, x_1, x_{14} \) and \( x, x_2, x_{23} \) satisfying \( q = 0 \) (see Figure 9) with \( q \) being in the \( M_\pm \) cases (2.42). We make a further restriction that the parameters \( r_1, r_2, r_4 \) appearing in the \( M_+ \) case are independent of \( \alpha, \beta \). Then, \( x_{14}, x_{23} \) are Möbius transformations of \( x \) as

\[
M_+: \ x_{14} = x_{23} = m_0^+(x) = -\frac{r_2 x + r_4}{r_1 x + r_2}, \quad M_-: \ x_{14} = x_{23} = m_0^-(x) = x. \tag{3.10}
\]

It follows from the tetrahedron property (2.5a) that

\[
Q^T(x, x_{12}, x_{23}, x_{13}; \alpha_{21}, \alpha_{23}) = -Q^T(x, x_{12}, m_0^+(x), x_{13}; \alpha_{12}, \alpha_{32}) = 0, \tag{3.11a}
\]

\[
Q^T(x_{1234}, x_{12}, x_{14}, x_{13}; \alpha_{34}, \alpha_{32}) = Q^T(x_{1234}, m_0^+(x), x_{13}; \alpha_{34}, \alpha_{32}) = 0, \tag{3.11b}
\]

where \( \alpha_{ij} = \alpha_i - \alpha_j \), and the second equality in (3.11a) is a result of (2.8). Comparing the above two equations, one has \( x_{1234} = x \), if \( \alpha_{12} = \alpha_{34} \) holds. This leads to

\[
\alpha_3 - \alpha_4 = \alpha_1 - \alpha_2 \quad \Rightarrow \quad \alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 = \mu, \tag{3.12}
\]

where \( \mu \) is an arbitrary constant. This allows us to set

\[
\sigma(\alpha) = -\alpha + \mu, \tag{3.13}
\]

as an involution relation on the parameters, so that \( \alpha_4 = \sigma(\alpha_1) \) and \( \alpha_3 = \sigma(\alpha_2) \). Then, \( x_{1234} \) is connected to \( x_{14} \) as \( x_{1234} = m_0^+(x_{14}) \), and the condition (3.2) holds.

Möbius case II: \( q \) in the dual \( M_\pm \) case, \( p \) in the \( M_\pm \) case. The companion boundary equation \( p = 0 \) is in the \( M_\pm \) cases. Let \( r_1, r_2, r_4 \) be independent of \( \alpha, \beta \). Then, (see Figure 9)

\[
x_{4/1} = m_0^+(x_{1/4}), \quad x_{3/2} = m_0^+(x_{2/3}). \tag{3.14}
\]

It also follows from the tetrahedron property that

\[
Q^T(x_{124}, x_4, x_2, x_1, \alpha_{12}, \alpha_{14}) = 0, \quad Q^T(x_{134}, x_1, x_3, x_4, \alpha_{34}, \alpha_{14}) = 0. \tag{3.15}
\]

Assume \( m_0^+ \) is a Möbius transformation making \( Q^T = 0 \) invariant. Then, \( 0 = Q^T(x_{124}, x_4, x_2, x_1, \alpha_{12}, \alpha_{14}) \propto Q^T(m_0^+(x_{124}), m_0^+(x_4), m_0^+(x_2), m_0^+(x_1); \alpha_{12}, \alpha_{14}) = Q^T(m_0^+(x_{124}), x_1, x_3, x_4; \alpha_{12}, \alpha_{14}), \tag{3.16}
\]

Comparing this with the second equation in (3.15), the condition (3.2) holds as

\[
m_0^+(x_{124}) = x_{134}, \tag{3.17}
\]

provided that \( \alpha_{12} = \alpha_{34} \), which imposes a \( \sigma \) in the form of (3.13).

Collecting the above arguments, the integrable boundary equations among the boundary equations dual to each other in the \( M_\pm \) and dual \( M_\pm \) cases can be stated as follows.

Theorem 17 For a given \( Q \) in the ABS classification, let \( q, p \) be a pair of dual boundary polynomials in the \( M_\pm \) and dual \( M_\pm \) cases as classified in Theorem 15. Assume \( r_1, r_2, r_4 \) in the \( M_+ \) case are independent of the lattice parameters \( \alpha, \beta \) and obey \( r_1 r_4 - r_2^2 \neq 0 \). Then, \( q = 0 \) is boundary consistent with \( Q = 0 \) in the sense of Definition 7 with \( \sigma(\alpha) = -\alpha + \mu \), where \( \mu \) is an arbitrary parameter, in the following two cases.
Table 3: Möbius case II: with $\sigma(\alpha) = -\alpha + \mu$, $q = 0$ is integrable. We take $q = r_1 q_1 + r_2 q_2 + r_4 q_4$ for $q$ in the dual $\mathcal{M}_+$ case, and $q = r$ for $q$ in the dual $\mathcal{M}_-$ case. Their explicit forms are given in Appendix B and C (represented by $p$). For $Q4$, $k$ denotes the elliptic modulus.

1. Möbius case I: $q$ is in the $\mathcal{M}_\pm$ case;

2. Möbius case II: $p$ is in the $\mathcal{M}_\pm$ case such that $m_\pm^\delta$ is a symmetry of the tetrahedron equation $Q^T = 0$.

The proof follows from the arguments given above. For $q = 0$ being a nondegenerate boundary equation either in the $\mathcal{M}_\pm$ case, or in the dual $\mathcal{M}_-$ case (the companion equation $p = 0$ is in the $\mathcal{M}_-$ case which is a trivial symmetry for any $Q^T = 0$), the boundary consistency is always satisfied provided that $\sigma(\alpha) = -\alpha + \mu$. For $q = 0$ in the dual $\mathcal{M}_+$ case, the requirement that $m_\pm^\delta$ is a symmetry for $Q^T = 0$ (which is a $Q$-type bulk equation) imposes certain restriction on the parameters $r_1, r_2, r_4$. Comparing to the symmetries of the $Q$-type bulk equations shown in Table 1, this restriction can be easily obtained. The complete list of integrable $q = 0$ with $q$ being in the dual $\mathcal{M}_\pm$ case (then, $p$ is in the $\mathcal{M}_\pm$ case with $p = 0$ being the companion equation) for $Q$-type and $H$-type bulk polynomials is given in Table 3. The $A2$ (resp. $A1(\delta)$) equation has the same results as $Q3(0)$ (resp. $Q1(\delta)$).

Table 4: Singular case: integrable boundary equations for $q,p$ as a pair of dual boundary polynomials in the forms (2.55a). The parameter $\mu$ can be taken freely ($\mu \neq 0$ expect for $H3(\delta \neq 0)$).

Now turn to $q,p$ as a pair of dual boundary polynomials in the singular case given in Theorem 3.2. The condition (3.2), or alternatively (5.9), can be checked by direct computations respectively for $q_j$ and $p_j$, $j = 0, 1$. The integrable ones give rise to certain constraint on the parameters that can be cast into an involution relation $\sigma$ similar to (3.12). For instance, for $Q1(0)$, the condition (3.2) implies that both $q_0 = 0$ and $q_1 = 0$ are integrable provided that

$$\alpha_1 \alpha_4 = \alpha_2 \alpha_3 \Rightarrow \alpha_3/4 = \sigma(\alpha_{2/1}) = \frac{\mu}{\alpha_{2/1}}, \quad (3.18)$$

with $\mu \neq 0$ being an arbitrary constant. It turns out that $q = 0$ with $q = c_0 q_0 + c_1 q_1$ as a linear combination of $q_0$ and $q_1$ is also integrable with $\sigma(\alpha) = -\alpha + \mu$.

The complete list of integrable boundary equations for $Q3(0)$, $Q1(\delta)$, $H3(\delta \neq 0)$ is provided in Table 4. It turns out that there is also a free parameter $\mu$ in $\sigma$ in all cases. For $Q3(0)$ and $H3(\delta \neq 0)$, the branches of the parameters should, a priori, be fixed in order to have $\sigma$ as an involution. $H2$ and $H3(\delta \neq 0)$ have their pairs $q_1 = 0, p_1 = 0$ satisfying the boundary consistency with $\sigma(\alpha) = -\alpha + \mu$. 21
However, they coincide with the $\mathcal{M}_-$ and dual $\mathcal{M}_+$ cases. For $Q1(0)$, $Q1(\delta \neq 0)$ and $Q3(0)$, the integrable boundary equations coincide with their dual boundary equations (see Remark 13), it suffices to list one of them.

Following Theorem 13 the results listed in Table 3 and 4 together with the boundary equations in the Möbius case I, exhaust all integrable boundary equations in the sense of Definition 1 for Q-type and H-type equations in ABS classification. The results of $A1(\delta)$ (or $A2$) follow trivially from those of $Q1(\delta)$ (or $Q3(0)$), and are omitted. Therefore, for bulk equations in the ABS classification, we claim a classification of integrable boundary equations based on the factorization approaches: first the quad-graph equations are factorized into pairs of dual boundary equations, and then the boundary consistency is obtained by factorizing the consistency around a rhombic dodecahedron into two equivalent halves. Comparing to known results obtained in [16], equations listed in Table 3 and 4 are, in general new, and have the partial list of integrable boundary equations in [16] as subcases. The involution relation $\sigma$, a result of the boundary consistency, enters systematically into the definition integrable boundary equation. Namely, for given $\sigma$ and $q$, the integrable boundary equation is defined as

$$q(x, u, y; \alpha, \sigma(\alpha)) = 0, \quad \text{or} \quad q(x, u, y; \alpha) = 0. \quad (3.19)$$

To each integrable $q = 0$, one can associate a discrete boundary zero curvature condition as given in (3.19) where the boundary matrix is obtained using the companion (dual) equation

$$m_p = 0.$$

Remark 18 Let us also comment on the action of simultaneous Möbius transformations on the boundary consistency. Recall the equivalence class $[Q]$ having 3D-consistent $Q_m = 0$. As a result of Theorem 14, $Q_m$ is also consistent around a rhombic dodecahedron. Similarly, it follows from Theorem 15 that the boundary consistency is preserved under simultaneous Möbius transformations in the sense of (2.38) and (2.37) on both the bulk equation $Q = 0$ and its integrable boundary equation $q = 0$. Therefore, with a given $\sigma$, $Q_m = 0$ has $q_m = 0$ as its integrable boundary equation.

| $Q$          | $q = r_1 q_1 + r_2 q_2 + r_4 q_4$ | $q = r_1 xy + r_2 x + y + r_4$ |
|--------------|----------------------------------|----------------------------------|
| $Q1(\delta)$ | $q_2$                            | $x + y$                          |
| $Q3(0)$      | $q_1 + q_4, q_2$                 | $xy + 1, x + y$                  |
| $Q3(\delta \neq 0)$ | $q_2$               | $x + y$                          |
| $Q4$         | $k q_1 + 1, q_2$                 | $kxy + 1, x + y$                 |
| $H1$         | $q_1 + c q_2$                    | $xy + c, x + y$                  |
| $H2$         | $q_2 + c q_4$                    | $x + y + c$                      |
| $H3(0)$      | $q_1 + q_4, q_2$                 | $xy + 1, x + y$                  |
| $H3(\delta \neq 0)$ | $q_1 + c q_2$ | $xy + c, x + y$                  |

Table 5: Canonical forms of integrable $q = 0$ in the $\mathcal{M}_+$ and dual $\mathcal{M}_+$ cases. For $H2, c$ is an arbitrary parameter; for $H1$ and $H3(\delta \neq 0), c \neq 0$. For $Q4, k$ denotes the elliptic modulus.

Remark 19 One can further reduce the integrable boundary equations $q = 0$ appearing the $\mathcal{M}_+$ and dual $\mathcal{M}_+$ cases to their canonical forms by factoring out the action of simultaneous Möbius transformations that are also symmetries of $Q = 0$. For instance, for $Q1(0)$, the generic Möbius transformation is a symmetry. One can reduce the integrable boundary equation $q = r_1 xy + r_2 (x + y) + r_4$ in the $\mathcal{M}_+$ case to $q = x + y$ that is a parameter-free boundary equation without changing the form of the bulk equation $Q = 0$. This is also accompanied by reducing the parameters appearing in the dual $\mathcal{M}_+$ case. For a bulk-equation in its canonical form, its integrable boundary equations with least number of parameters are referred to as canonical forms (they are not unique). The canonical forms are listed in Table 6. For the derivation of canonical forms of $H$-type equations, their symmetries are needed (they are $\pm x, x + c$ for $H1$; $x$ for $H2$; $cx, 1/x$ for $H3(\delta = 0)$; $\pm x$ for $H3(\delta \neq 0)$). Moreover, for $Q1(0)$, the canonical form of the integrable boundary equation $q = 0$ in the singular case given in Table 4 can be either $q_0 = 0$ or $q_1 = 0$ (they can be transformed from one to the other using $m(x) = 1/x$).
3.3 Degenerate integrable boundary equations

The above derivations exclude the degenerate boundary equations, since they do not lead to any
dual boundary equations through (2.13). Here, we provide a criterion to select integrable
degenerate boundary equations. It turns out that the boundary consistency in this case can be understood as
certain “degeneration” of the consistent system around the rhombic dodecahedron. In particular,
the boundary consistency does not require any involution relation $\sigma$, this is in contrast to the
nondegenerate boundary consistency given in Definition 1.

There are two possibilities for $q = 0$ to be degenerate (see discussions right below Definition
5). They are respectively called degenerate case I and II.

Degenerate case I: let $q = 0$ be a boundary equation leading to $\delta_{a,e}Q = 0$. This applies to
the H-type equations, $Q_1(\delta)$, $Q_3(0)$ and A-type equations. Take $Q_1(\delta)$, $Q_3(0)$ as examples.
Arguments for H-type equations and A-type equations follow similarly.

Let $Q$ to be $Q_1(\delta)$ or $Q_3(0)$, one has

$$Q_1(\delta): \quad q = x - y (\alpha - \beta) \delta, \quad Q_3(0): \quad q = e^\alpha x - e^\beta y, \text{ or } q = e^\beta x - e^\alpha y. \quad (3.20)$$

It can be checked by direct computations that $q = 0$ is consistent with the associated $Q = 0$
aver around a half of a rhombic dodecahedron without any restriction on the lattice parameters.

This can be understood as a “degeneration” of the consistency around a rhombic dodecahedron:
following Figure 9 by imposing the above $q$ to the “bottom” (resp. “front”) bulk equation $Q = 0$
with $q(x, x_1, x_14; \alpha_1, \alpha_4) = 0$ (resp. $q(x, x_2, x_23; \alpha_2, \alpha_3) = 0$), then the “bottom”
(resp. “front”) boundary equations have a singular solution with respect to $x_1$ (resp. $x_2$),
which allows us to assign generic value to $x_1$ (resp. $x_2$). The tetrahedron equation
$Q^T(x, x_12, x_14, x_24; \alpha_{12}, \alpha_{14}) = 0$, where $\alpha_{ij} = \alpha_i - \alpha_j$, also has a singular solution with
respect to $x_{12}$, and the values $x_{14}, x_{23}, x_{24}$ are fixed and depend on $x$ only. Moreover, the
tetrahedron equation $Q^T(x_{1234}, x_{12}, x_{23}, x_{24}; \alpha_{34}, \alpha_{41}) = 0$ with $x_{23}, x_{14}$ obtained above is also
singular with respect to $x_{12}$, and the values $x_{1234}, x_{23}$ can be connected using $q = 0$. The
values $x_{1234}, x_{14}$ can also be connected using $q = 0$, since $Q^T(x_{1234}, x_{14}, x_{24}, x_{34}; \alpha_{32}, \alpha_{31}) = 0$
is singular with respect to $x_{34}$. The boundary consistency around a half of the rhombic
dodecahedron follows from the consistency around the whole rhombic dodecahedron.

| $Q_1(0)$ | $Q_1(\delta \neq 0)$ | $Q_2$ | $Q_3(0)$ | $Q_3(\delta \neq 0)$ | $Q_4$ |
|---------|-------------------|-------|-----------|-------------------|-------|
| (c_0, c_1) ≠ (0, 0) | c_1 = 0 | c_1 = 0 | c_0 = 0 or c_1 = 0 | c_1 = 0 | N/A |

Table 6: Possibilities of $c_0, c_1$ not vanishing simultaneously to make a $Q$-type bulk equation

holds for generic $y$.

Degenerate case II: take $q$ in the form (2.13). In this case, the consistency between $Q = 0$
and $q = 0$ around a half of the rhombic dodecahedron can also be understood as certain
“degeneration” of the consistency around the whole rhombic dodecahedron.

Following Figure 9, one imposes $q = 0$ to the “bottom” and “front” equations, and uses
$x, x_1, x_2$ as well as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as initial data. This leads to $x_4 = x_3 = c_0/c_1$ (see discussions
after (2.13)), and the “bottom” and “front” bulk equations reduce to $q = 0$, and are
independent of $x_4$ and $x_3$ respectively. If the “top” and “back” equations also hold independ-
ently of $x_{234}$ and $x_{134}$ respectively with $x_{234} = x_{134} = c_0/c_1$, then they reduce to $q = 0$, the
consistency around the left half of the rhombic dodecahedron can be fulfilled without any
restriction on the lattice parameters.

Having $x_4 = x_3 = c_0/c_1$, the condition $x_{234} = x_{134} = c_0/c_1$ can be checked using the
tetrahedron equations

$$Q^T(x_{134}, x_1, x_4, x_3; \alpha_{43}, \alpha_{41}) = 0, \quad Q^T(x_{234}, x_2, x_3, x_4; \alpha_{42}, \alpha_{43}) = 0. \quad (3.21)$$

In other words, one needs the first equation holds with $x_4 = x_3 = x_{134} = c_0/c_1$ for generic
input $x_1$, and similarly to the second equation with $x_4 = x_3 = x_{234} = c_0/c_1$ for generic input
$x_2$. This amounts to examining if the tetrahedron equation

$$Q^T(c_0/c_1, c_0/c_1, c_0/c_1, y; \alpha, \beta) = 0. \quad (3.22)$$
as a Q-type bulk equation, holds for generic $y$. This is related to properties of the Q-type bulk equations, and possible values of $c_0, c_1$ are listed in Table 5.

Let us summarize the above results.

| $Q$ | degenerate case I | degenerate case II (in the form (2.18)) |
|-----|------------------|----------------------------------|
| $Q_1(0), H_1$ | $x - y$ | $c_1 S + c_0 T$ |
| $Q_1(\delta), H_2$ | $x - y \pm (\alpha - \beta) \delta$ | $T$ |
| $Q_2$ | not needed (n.n.) | $T$ |
| $Q_3(0), H_3(\delta)$ | $e^\alpha x - e^\beta y, e^{\alpha x} - e^{\alpha y}$ | $S, T$ |
| $Q_3(\delta)$ | not needed | $T$ |

Table 7: Degenerate integrable boundary equations: the boundary consistency holds without any restriction on the parameters. For $H_2$ in the degenerate case I, $\delta = 1$. Degenerate case II is derived using Table 6 and explicit forms of $S, T$ are given in Table 8.

| $Q$ | $S$ | $T$ |
|-----|-----|-----|
| $Q_1(0)$ | $\alpha x (u - y) - \beta y (x - u)$ | $\alpha (y - u) - \beta (x - u) u$ |
| $Q_1(\delta) \neq 0$ | not needed (n.n.) | $\alpha (y - u) - \beta (x - u) u$ |
| $Q_2$ | n.n. | $\alpha y - \beta x + (\alpha - \beta)(\alpha \beta - u)$ |
| $Q_3(0)$ | $\sinh(\alpha) u x - \sinh(\beta) u y - \sinh(\alpha - \beta) x y$ | $\sinh(\alpha) y - \sinh(\beta) x - u \sinh(\alpha - \beta)$ |
| $Q_3(\delta) \neq 0$ | n.n. | $\sinh(\alpha) y - \sinh(\beta) x - u \sinh(\alpha - \beta)$ |
| $H_1$ | $u (x - y) - \alpha + \beta$ | $y - x$ |
| $H_2$ | n.n. | $y - x - \alpha + \beta$ |
| $H_3(\delta)$ | $(e^\alpha x - e^\beta y) u + \delta (e^{2\alpha} - e^{2\beta})$ | $e^\alpha y - e^\beta x$ |

Table 8: Explicit forms of $S, T$ needed in Table 7.

**Theorem 20** Let $q = 0$ be a degenerate boundary equation, and $Q = 0$ a bulk equation in the ABS classification. We say $q = 0$ is boundary consistent with $Q = 0$, if the initial value problem on the half rhombic dodecahedron, c.f. left picture in Figure 4, is well-posed. The list of such $q = 0$ belonging to degenerate case I and II as discussed above is given in Table 7.

**Remark 21** In [8, 9], the so-called “singular boundary” reductions were considered for Q-type bulk equations in the ABS classification, where the initial data of an initial value problem on a $\mathbb{Z}^2$-lattice were taken as singular solutions to $Q = 0$. Some of their results are in connection to the integrable degenerate boundary equations of case II. For instance, for $Q_1(0)$, one has $q = 0$ with $q$ in the form (2.18) which provides a singular solution to $Q_1(0)$. Then, an initial-boundary value problem on a $\mathbb{Z}^2$-lattice with a boundary with $q = 0$ being the boundary conditions corresponds to an initial value problem with a “singular boundary” considered in [8, 9]. This is also the cases for other Q-type equations except for $Q_4$, where the “singular boundary” solutions provided in [8, 9] can not be written as a boundary equation formulated in this paper, since $Q_4$ does not have any integrable degenerate boundary equations.

### 4 Integrable boundary equations for $H_1^c$

In this section, we consider integrable boundary equations for the $H_1^c$ equation which belongs to the $H^c$-type (also called $H^4$-type in [11]) equations as an extension of H-type equations in ABS classification [5]. The derivation of integrable $q = 0$ is based on the successive factorization approaches presented in this paper. Here, we do not intend to provide the complete list of integrable $q = 0$, since it requires a substantial amount of computations but following the exactly same techniques as presented above.

One particularity of the $H_1^c$ and other $H^c$-type equations is that their bulk polynomials are rhombic-symmetric only, which implies that one should also take the patterns of the quadrilateral
Quad-graph systems with boundary for the $H^\varepsilon$-type equations could have two types of boundary conditions depending on the patterns of the triangles assigned to the boundary, and the boundary consistency conditions should also be adapted to the patterns of the quadrilaterals and triangles. Here, we provide examples of integrable boundary equations for $H^1_\varepsilon$, and also give quad-graph systems on a strip with two boundaries that are of different pattern types (see Figure 13).

The bulk polynomial $Q$ of $H^1_\varepsilon$ reads

$$Q := Q(x, u, v, y; \alpha, \beta) = (x - y)(u - v) + (\beta - \alpha)(1 + \epsilon uv), \quad (4.1)$$

which is affine-linear with respect to each of its arguments. When $\epsilon = 0$, it coincides with $H_1$. In contrast to $H_1$, or any $Q$ in the ABS classification, it is rhombic-symmetric

$$Q(x, u, v, y, \alpha, \beta) = -Q(x, v, u, y, \beta, \alpha), \quad Q(x, u, v, y, \alpha, \beta) = -Q(y, u, v, x, \beta, \alpha), \quad (4.2)$$

meaning that it is irrelevant to interchange the black and white vertices (one gets a different bulk equation by doing so). We set the convention following Figure 11: the first and fourth arguments of $Q$ are assigned to the white vertices, while the second and third arguments are assigned to the black vertices. The biquadratic polynomials of $H^1_\varepsilon$ are

$$\Gamma := \delta_{v,y}Q = -(\alpha - \beta)(1 + \epsilon u^2), \quad \delta_{u,v}Q = (x - y)^2 + \epsilon(\alpha - \beta)^2, \quad \delta_{x,y}Q = (u - v)^2. \quad (4.3)$$

Due to the symmetry (4.2), the biquadratic polynomials attached to the edges of the underlying quadrilateral take similar forms, but the two biquadratic polynomials attached to the two crossing diagonals are of different forms.

The $H^1_\varepsilon$ equation is 3D-consistent following the assignments of white and black vertices as shown in Figure 13. It also processes the tetrahedron properties with two different tetrahedron equations $Q^T_w = 0$ and $Q^T_b = 0$ that are satisfied respectively by the values on the white and black vertices of the cube. Following Figure 13, one has

$$Q^T_w(x, x_{12}, x_{13}, x_{23}; \alpha_{12}, \alpha_{13}) = 0, \quad Q^T_b(x_{123}, x_1, x_2, x_3; \alpha_{32}, \alpha_{31}) = 0, \quad (4.4)$$

where $\alpha_{ij} = \alpha_i - \alpha_j$, and $Q^T_w = 0$ is $Q_1(\sqrt{-\epsilon})$ and $Q^T_b = 0$ is $Q_1(0)$.

Now, we turn to the factorization of $H^1_\varepsilon$ to derive its pairs of dual boundary equations. We follow the definition of boundary equations given in section 2.2. In addition to consider the boundary polynomial (2.17), whose boundary vertices are white dots, one could also take a boundary polynomial whose boundary vertices are black dots. These correspond to two patterns of triangles (see Figure 11), and the associated boundary polynomials are respectively denoted by $q_w$ and $q_b$ as

$$q_w(x, u, v; \alpha, \beta), \quad q_b(u, x, v; \alpha, \beta). \quad (4.5)$$

Both $q_w$ and $q_b$ should have dual boundary polynomials, and they are not necessarily dual to each other. Let us illustrate this property by taking $q_w$ and $q_b$ in the $\mathcal{M}_+$ case.

One can express $Q$ as

$$Q = S + Ty = M + Nv, \quad (4.6)$$
where \( S, T, \) and \( M, N \) are polynomials affine-linear with respect to its arguments. They are in the forms

\[
S = (\beta + \alpha)(1 + \epsilon uv) + (u - v)x, \quad T = v - u, \quad (4.7a)
\]

\[
M = \beta - \alpha + u(x - y), \quad N = \epsilon(\beta - \alpha)u - x + y. \quad (4.7b)
\]

If \( \epsilon = 0 \), then \( S \) (resp. \( N \)) coincides with \( M \) (resp. \( T \)). Let \( q_w \) be in the \( \mathcal{M}_+ \) case, i.e. \( q_w = r_1xy + r_2x + r_2y + r_4 \), then it has a dual boundary polynomial \( p_b \) as

\[
\text{dual \( \mathcal{M}_+ \) case: } p_b = r_1 S x + r_2(S - T x) - r_4 T. \quad (4.8)
\]

Alternatively, let \( q_b \) be in the \( \mathcal{M}_+ \) case, i.e. \( q_b = \rho_1uv + \rho_2u + \rho_2v + \rho_4 \), it has

\[
\text{dual \( \mathcal{M}_+ \) case: } p_w = \rho_1 M u + \rho_2(M - N u) - \rho_4 N. \quad (4.9)
\]

as its dual boundary polynomial. For simplicity, we assume \( r_j \) and \( \rho_j, j = 1, 2, 4 \), are independent of the lattice parameters. One obtains two pairs of dual boundary polynomials \( \{q_w, p_b\} \) and \( \{q_b, p_w\} \), and clearly, \( p_b \) is different from \( p_w \).

The criterion for them to be boundary consistent with \( H_1 \) is explained in Section 3.2. One needs to respectively consider the symmetries of \( Q_w^{\perp} = 0 \) and \( Q_b^{\perp} = 0 \) which are different equations here. One also need to interchange the patterns of white and black vertices appearing in the rhombic dodecahedron of Figure 9 which results in two boundary consistency conditions with different patterns of quadrilaterals and triangles as shown in Figure 12. Based on arguments given in Möbius cases I and II in Section 3.2 it is straightforward to conclude that both \( q_b = 0 \) and \( q_w = 0 \) are integrable with \( \sigma(\alpha) = -\alpha + \mu \). With the same involution relation \( \sigma(\alpha) = -\alpha + \mu \), \( p_w = 0 \) is integrable provided that \( \rho_1 = 0 \) and \( \rho_2 \neq 0 \) since \( Q_w^{\perp} = 0 \) is \( Q_1(\sqrt{-\epsilon}) \), and \( p_b = 0 \) is integrable with \( r_1r_4 - r_2^2 \neq 0 \) since \( Q_b^{\perp} = 0 \) is \( Q_1(0) \). Interestingly, the integrable boundary equations also depend on the patterns of the underlying triangles. In Figure 13 we provide two examples of quad-graph systems on a trip with two parallel boundaries, where the patterns of the boundary equations are of different types.

5 Conclusion

In this paper, we provide a classification of boundary equations that are boundary-consistent with quad equations in the ABS classification.

First, we provide the notions of boundary polynomials and boundary equations in Section 2.2 as the natural objects to characterize boundary conditions for quad-graph systems with boundary. The classification is based on factorization approaches that are introduced along the paper. The
factorization of quad equation is formulated in Section 2.3, which, in turn, leads to the duality property of its boundary equations. The exhausted list of nondegenerate boundary equations dual to each other is provided in Section 2.4. This list is taken in Section 3 to factorize the consistent system around a rhombic dodecahedron into two equivalent halves, which amounts to the boundary consistency and the associated integrable boundary equations. In particular, the involution relation \( \sigma \) appearing in the definition of boundary consistency is systematically derived.

A criterion for degenerate boundary equations that are boundary consistent with a given quad equation is also provided, and the degenerate boundary consistency does not require any involution \( \sigma \). We extend our method to derive integrable boundary equations for the \( H^1 \) equation, that is rhombic-symmetric only. In this case, the patterns of boundary equations should be taken into account, and the boundary consistency conditions are also adapted according to the patterns. Examples of quad-graph systems on a strip with different type of patterns are also provided.

The results obtained in this paper lays foundations for what we believe could be some new areas of research in discrete integrable systems. Among many open questions we can think of, we would mention some that are of importance: an inverse scattering transform for discrete integrable equations on a “half-plane” can be readily developed, notably for some Q-type equations including Q1(0) and Q4 equations. This is in analogy to a half-line problem for soliton equations, and the technique could follow the half-line inverse scattering transform recently introduced by one of the authors [10]. We would expect that discrete soliton solutions as given in [32] [23] reflect at the boundary with soliton parameters changing according to \( \sigma \). More generic quad-graph systems with boundary could also be investigated as have been done in [17]. The quad-graph systems without boundary are in conjunction with examples of discrete Riemann surface and discrete complex analysis [30] [11] [12]. Similar aspects for quad-graph systems with boundary are completely open. Moreover, the factorization approach developed in this paper could also be adapted to systems of 3D consistent equations such as the list of discrete Boussinesq-type equations obtained in [21]. Lastly, we would like to mention that classification problems for discrete integrable systems represent a very active research areas in connection to many aspects of mathematical physics. Recent results in various contexts can be found, for instance, in [5] [25] [26].

Figure 13: Well-posed quad-graph systems on trips with two parallel boundaries with odd (left) and even (right) initial data (denoted by dashed lines and the parameters are omitted). The odd case has the same patterns of triangles as boundaries, while the even case has boundaries with different patterns.
A ABS classification

The canonical forms of the bulk polynomial $Q := Q(x, v, y; \alpha, \beta)$, the associated biquadratic polynomials, and the tetrahedron equation $Q^T = 0$ are listed below.

A.1 Q-type bulk polynomials

Q1(δ): the bulk polynomial $Q$ reads

$$
\alpha(x - v)(u - y) - \beta(x - u)(v - y) + \delta^2 \alpha \beta (\alpha - \beta) .
$$

It has Q1(δ) as $Q^T = 0$. The factorized forms of the biquadratic polynomials are

$$
\delta_{v,y} Q = -\beta(\alpha - \beta)((x - u) + \delta \alpha)((x - u) - \delta \alpha) ,
$$

$$
\delta_{u,v} Q = \alpha \beta((x - y) + \delta(\alpha - \beta))((x - y) - \delta(\alpha - \beta)) .
$$

Q2: the bulk polynomial $Q$ reads

$$
\alpha(x - v)(u - y) - \beta(x - u)(v - y) + \alpha \beta(\alpha - \beta)(x + u + v + y) - \alpha \beta(\alpha - \beta)(\alpha^2 - \alpha \beta + \beta^2) .
$$

It has Q2 as $Q^T = 0$. The biquadratic polynomials are

$$
\delta_{v,y} Q = -(\alpha - \beta)((x - u)^2 - 2(x + u)\alpha^2 + \alpha^4) ,
$$

$$
\delta_{u,v} Q = \alpha \beta((x - y)^2 - 2(x + y)(\alpha - \beta)^2 + (\alpha - \beta)^4) .
$$

Q3(δ): the bulk polynomial $Q$ reads

$$
\sinh(\alpha)(x u + vy) - \sinh(\beta)(x v + uy) - \sinh(\alpha - \beta)(x y + uv) + \delta \sinh(\alpha - \beta) \sinh(\alpha) \sinh(\beta)
$$

It has Q3(δ) as $Q^T = 0$. The biquadratic polynomials are

$$
\delta_{v,y} Q = -\sinh(\alpha - \beta) \sinh(\beta)(x^2 + u^2 - 2xu \cosh(\alpha) - \delta \sinh^2(\alpha)) ,
$$

$$
\delta_{u,v} Q = \sinh(\alpha) \sinh(\beta)(x^2 + y^2 - 2xy \cosh(\alpha - \beta) - \delta \sinh^2(\alpha - \beta)) .
$$

When δ = 0, the factorized forms of the biquadratic polynomials are

$$
\delta_{v,y} Q = -\sinh(\alpha - \beta) \sinh(\beta)(u - e^{-\alpha}x)(u - e^\alpha x) ,
$$

$$
\delta_{u,v} Q = \sinh(\alpha) \sinh(\beta)(x - e^{-\alpha+\beta}y)(x - e^{\alpha-\beta}y) .
$$

Q4: the bulk polynomial $Q$ reads

$$
\text{sn}(\alpha)(x u + vy) - \text{sn}(\beta)(x v + uy) - \text{sn}(\alpha - \beta)(x y + uv) + \text{sn}(\alpha - \beta)\text{sn}(\alpha)\text{sn}(\beta)(1 + k^2 x u v y) ,
$$

where $\text{sn}$ is the Jacobi elliptic functions of modulus $k$, $k \neq 0, \pm 1$. It has Q4 as $Q^T = 0$, and

$$
\delta_{v,y} Q = -\text{sn}(\alpha - \beta)\text{sn}(\beta)(x^2 + u^2 - k^2 \text{sn}^2(\alpha)x^2 u^2 - 2\text{cn}(\alpha)\text{dn}(\alpha)x u - \text{sn}^2(\alpha)) ,
$$

$$
\delta_{u,v} Q = \text{sn}(\alpha)\text{sn}(\beta)(u^2 + v^2 - k^2 \text{sn}^2(\alpha - \beta)u^2 v^2 - 2\text{cn}(\alpha - \beta)\text{dn}(\alpha - \beta)uv - \text{sn}^2(\alpha - \beta)) .
$$

A.2 H-type bulk polynomials

H1: the bulk polynomial $Q$ reads

$$
(x - y)(u - v) + \beta - \alpha .
$$

It has Q1(0) as $Q^T = 0$. The biquadratic polynomials are

$$
\delta_{v,y} Q = -\alpha + \beta , \quad \delta_{u,v} Q = (x - y)^2 .
$$
H2: the bulk polynomial \( Q \) reads
\[
(x - y)(u - v) + (\beta - \alpha)(x + u + v + y) + \beta^2 - \alpha^2.
\]
It has \( Q_1(1) \) as \( Q^T = 0 \). The biquadratic polynomials are
\[
\delta_{v,y} Q = 2(\alpha - \beta)(\alpha + u + x), \quad \delta_{u,v} Q = (x - y + \alpha - \beta)(x - y - \alpha + \beta).
\]

H3(\( \delta \)): the bulk polynomial \( Q \) reads
\[
e^\alpha(xu + vy) - e^\beta(xv + uy) + \delta(e^{2\alpha} - e^{2\beta}).
\]
It has \( Q_3(0) \) as \( Q^T = 0 \). The biquadratic polynomials are
\[
\delta_{v,y} Q = (e^{2\alpha} - e^{2\beta})(ux + e^\alpha \beta), \quad \delta_{u,v} Q = (e^\alpha x - e^\beta y)(e^\beta x - e^\alpha y).
\]

A.3 A-type bulk polynomials

A1(\( \delta \)): the bulk polynomial \( Q \) reads
\[
\alpha(x + v)(u + y) - \beta(x + u)(v + y) - \delta^2 \alpha \beta(\alpha - \beta).
\]
It can be obtained from \( Q_1(\delta) \) using \( u \to -u, v \to -v \). It has \( Q_1(\delta) \) as \( Q^T = 0 \), and the biquadratic polynomials follow from those of \( Q_1(\delta) \).

A2: the bulk polynomial \( Q \) reads
\[
\sinh(\alpha)(xv + vy) - \sinh(\beta)(xu + vy) - \sinh(\alpha - \beta)(1 + xuvy).
\]
It can be obtained from \( Q_3(0) \) using \( u \to 1/u, v \to 1/v \). It has \( Q_3(0) \) as \( Q^T = 0 \), and the biquadratic polynomials follow from those of \( Q_3(0) \).

B Boundary polynomials for Q- and A-type polynomials

We follow the forms of \( Q, q, p, \chi, \zeta, \Gamma \) given in [241].

B.1 Q-type polynomials

Q1(\( \delta \)): \( M_+ \) case:
\[
\begin{align*}
p_1 &= (\alpha - \beta)uvx - x^2(\alpha u - \beta v) - \delta^2 (\alpha - \beta)\alpha \beta x, \\
p_2 &= (\alpha - \beta)uv - (\alpha + \beta)x(u - v) - (\alpha - \beta)(x^2 + \alpha \beta \delta^2), \\
p_4 &= - \beta u + \alpha v - (\alpha - \beta)x.
\end{align*}
\]

\( M_- \) case: set \( \kappa = \alpha - \beta \), then
\[
p = uv - x(u + v) + x^2 - \delta^2 \alpha \beta.
\]

When \( \delta = 0 \), the \( M_- \) boundary polynomial is excluded since \( q = 0 \) leads to \( Q_{u,v} = 0 \).

Singular case: let \( \{ \alpha, \beta \} \to \{ \alpha^2, \beta^2 \} \), then
\[
\begin{align*}
q_0 &= (\alpha + \beta)xy - (\alpha u - \delta \alpha \beta^2)x - (\beta u - \delta \alpha^2 \beta)y, \\
q_1 &= \beta x + \alpha y + (\alpha + \beta)(\delta \alpha \beta - u), \\
p_0 &= -(\alpha - \beta)uv + \alpha ux - \beta xv + \delta \alpha \beta(\alpha - \beta)(\delta \alpha \beta + x), \\
p_1 &= \beta u - \alpha v + (\alpha - \beta)(\delta \alpha \beta + x).
\end{align*}
\]
Q2: $M_+$ case:
\[ p_1 = x(- (\alpha - \beta)uv + \alpha \beta (\alpha - \beta)(u + v) + \alpha xu - \beta xv - \alpha \beta (\alpha - \beta)(\alpha^2 - \alpha \beta + \beta^2 - x)) , \]
\[ p_2 = - (\alpha - \beta)uv + (\alpha + \beta)x(u - v) + \alpha \beta (\alpha - \beta)(u + v) - (\alpha - \beta)(\alpha \beta(\alpha^2 - \alpha \beta + \beta^2) - x^2) , \]
\[ p_4 = \beta u - \alpha v - (\alpha - \beta)(\alpha \beta - x) . \]

$M_-$ case: set $\kappa = \alpha - \beta$, then
\[ p = uv - (\alpha \beta + x)(u + v) + \alpha \beta(\alpha^2 - \alpha \beta + \beta^2) - 2\alpha \beta x + x^2 . \]

Q3(\delta): $M_+$ case:
\[ p_1 = x(- \sinh(\alpha - \beta)uv + \sinh(\alpha)xu - \sinh(\beta)xv + \delta \sinh(\alpha) \sinh(\beta) \sinh(\alpha - \beta)) , \]
\[ p_2 = - \sinh(\alpha - \beta)uv + x(\sinh(\alpha) + \sinh(\beta))(u - v) + \sinh(\alpha - \beta)(x^2 + \delta \sinh(\alpha) \sinh(\beta)) , \]
\[ p_4 = \sinh(\beta)u - \sinh(\alpha)v + x \sinh(\alpha - \beta) . \]

$M_-$ case: set $\kappa = \sinh(\alpha - \beta)$, then
\[ p = uv - \cosh(\alpha - \beta)(\sinh(\alpha) - \sinh(\beta))x(u + v) - \delta \sinh(\alpha) \sinh(\beta) + x^2 . \]

Singular case: when $\delta = 0$, let $\{e^\alpha, e^\beta\} \to \{\cosh(\alpha), \cosh(\beta)\}$, then
\[ q_0 = (\sinh(\alpha) + \sinh(\beta))xy - u \sinh(\alpha) \cosh(\beta)x - u \cosh(\alpha) \sinh(\beta)y , \]
\[ q_1 = \sinh(\beta)x + \sinh(\alpha)y - u \sinh(\alpha + \beta) , \]
\[ p_0 = (\sinh(\alpha) + \sinh(\beta))(-(\sinh(\alpha) - \sinh(\beta))uv + x \sinh(\alpha) \cosh(\beta)u - x \cosh(\alpha) \sinh(\beta)v) , \]
\[ p_1 = \sinh(\alpha + \beta)(\sinh(\beta)u - \sinh(\alpha)v + x \sinh(\alpha - \beta)) . \]

Q4: $M_+$ case:
\[ p_1 = x(- \sinh(\alpha - \beta)uv + (\sinh(\alpha) - \sinh(\beta))x(u - v) + \sinh(\alpha) \sinh(\beta) \sinh(\alpha - \beta)) , \]
\[ p_2 = - \sinh(\alpha - \beta)(1 + K^2 \sinh(\alpha) \sinh(\beta)x^2)uv + (\sinh(\alpha) + \sinh(\beta))x(u - v) + \sinh(\alpha - \beta)(\sinh(\alpha) \sinh(\beta) + x^2) , \]
\[ p_4 = - K^2 \sinh(\alpha) \sinh(\beta)(\sinh(\alpha - \beta)uv + \sinh(\beta)u - \sinh(\alpha)v + \sinh(\alpha - \beta)x) . \]

$M_-$ case: set $\kappa = \sinh(\alpha - \beta)$, then
\[ p = -(K^2 \sinh(\alpha) \sinh(\beta)x^2 - 1)uv - \sinh(\alpha - \beta)(\sinh(\alpha) - \sinh(\beta))x(u + v) - \sinh(\alpha) \sinh(\beta) + x^2 . \]

B.2 A-type polynomials

A1(\delta) (or A2) can be derived from Q1(\delta) (or Q3(\delta)) using $\{u, v\} \to \{\phi(u), \phi(v)\}$ (\phi is a Möbius transformation, c.f. [A3]). Let $Q$ denote Q1(\delta) (or Q3(\delta)), and $\tilde{Q}$ denote A1(\delta) (or A2), then
\[ \tilde{Q}(x, u, v; y; \alpha, \beta) = \Lambda(u)\Lambda(v)Q(x, \phi(u), \phi(v), y; \alpha, \beta) , \]
with $\Lambda(*)$ defined in [2.9]. Let $q, p$ be a pair of dual boundary polynomials of $Q$, it follows from some simple considerations that $\tilde{q}, \tilde{p}$ in the forms
\[ \tilde{q}(x, u, y; \alpha, \beta) = \Lambda^\delta(u)q(x, \phi(u), y; \alpha, \beta) , \]
\[ \tilde{p}(u, x, v; \alpha, \beta) = \Lambda(u)\Lambda(v)p(\phi(u), x, \phi(v); \alpha, \beta) , \]
are boundary polynomials dual to each other for $\tilde{Q}$. Precisely, one gets

A1(\delta): $M_+$ case:
\[ p_1 = x(- (\alpha - \beta)uv - x(\alpha u - \alpha v) + \delta^2(\alpha - \beta)\alpha \beta) , \]
\[ p_2 = - (\alpha - \beta)uv - (\alpha + \beta)x(u - v) + (\alpha - \beta)(x^2 + \delta^2 \alpha \beta) , \]
\[ p_4 = - \beta u + \alpha v + (\alpha - \beta)x . \]
\( \mathcal{M}_- \) case: set \( \kappa = \alpha - \beta \), then
\[
p = uv + x(u + v) + x^2 - \delta^2 \alpha \beta.
\]
When \( \delta = 0 \), the \( \mathcal{M}_- \) boundary polynomial is excluded since \( q = 0 \) leads to \( Q_{u,v} = 0 \).

**Singular case:** let \( \{\alpha, \beta\} \to \{\alpha^2, \beta^2\} \), then
\[
q_0 = (\alpha + \beta)xy + \alpha(u + \delta \beta^2)x + \beta(u + \delta \alpha^2)y, \\
p_0 = (\alpha - \beta)uv + \alpha x - \beta y - \delta \alpha \beta (\alpha - \beta)(x + \delta \alpha \beta), \\
p_1 = \beta u - \beta v - (\alpha - \beta)(x + \delta \alpha \beta).
\]

**A2:** \( \mathcal{M}_+ \) case:
\[
p_1 = x(-\sinh(\beta)xu + \sinh(\alpha)xy - \sinh(\alpha - \beta)), \\
p_2 = \sinh(\alpha - \beta)x^2uv - (\sinh(\alpha) + \sinh(\beta))x(u - v) - \sinh(\alpha - \beta), \\
p_4 = \sinh(\alpha - \beta)xuv - \sinh(\alpha)u + \sinh(\beta)v.
\]

**\( \mathcal{M}_- \) case:** one set \( \kappa = \sinh(\alpha - \beta) \), then
\[
p = x^2uv - \cosh(\alpha - \beta)(\sinh(\alpha) - \sinh(\beta))x(u + v) + 1.
\]

**Singular case:** let \( \{\cosh, \beta\} \to \{\cosh(\alpha), \sinh(\beta)\}, \) then
\[
q_0 = (\sinh(\alpha) + \sinh(\beta))uxy - \sinh(\alpha) \cosh(\beta)x - \cosh(\alpha) \sinh(\beta)y, \\
p_0 = \sinh(\beta)ux + \sinh(\alpha)uy - \sinh(\alpha + \beta), \\
p_1 = \sinh(\alpha + \beta)(\sinh(\alpha - \beta)xuv - \sinh(\alpha)u + \sinh(\beta)v).
\]

\section*{C Boundary polynomials for H-type polynomials}

We follow the forms of \( Q, q, p, \chi, \zeta, \Gamma \) given in (\ref{boundary-poly}).

\subsection*{C.1 Results according to Theorem \ref{thm:boundary}}

**H1:** \( \mathcal{M}_+ \) case:
\[
p_1 = x(x(u - v) - \alpha + \beta), \\
p_2 = 2x(u - v) - \alpha + \beta, \\
p_4 = u - v.
\]

**H2:** \( \mathcal{M}_+ \) case:
\[
p_1 = x(-(\alpha - \beta)(u + v) + x(u - v) - (\alpha - \beta)(\alpha + \beta + x)), \\
p_2 = -(\alpha - \beta)(u + v) + 2x(u - v) - \alpha^2 + \beta^2, \\
p_4 = u - v + \alpha - \beta.
\]

**\( \mathcal{M}_- \) case:** one set \( \kappa = \alpha - \beta \), then
\[
p = u + v + \alpha + \beta + 2x.
\]

**Singular case:**
\[
q_0 = 2xy + u(x + y) + \beta x + \alpha y, \\
q_1 = x + y + \alpha + \beta + 2u, \\
p_0 = -(\alpha - \beta)(u + v) + 2x(u - v) - (\alpha - \beta)(\alpha + \beta + 2x), \\
p_1 = 2(u - v).
\]

Here, \( p_1, q_1 \) coincide with the \( \mathcal{M}_- \) and dual \( \mathcal{M}_- \) cases.
H3(δ): $M_+$ case:

\[ p_1 = x(e^\alpha xu - e^\beta xv + \delta(2\alpha - e^\beta)), \]
\[ p_2 = (e^\alpha + e^\beta)(u(u - v) + \delta(e^\alpha - e^\beta)), \]
\[ p_4 = e^\beta u - e^\alpha v. \]

$M_-$ case: one set $\kappa = e^\alpha - e^\beta$ then

\[ p = -x(u + v) - \delta(e^\alpha + e^\beta). \]

Singular case: when $\delta \neq 0$,

\[ q_0 = 2uxy + \delta(e^\beta x + e^\alpha y), \]
\[ q_1 = y(x + y) + \delta(e^\alpha + e^\beta), \]
\[ p_0 = 2x(e^\alpha u - e^\beta v) + \delta(2\alpha - e^\beta), \]
\[ p_1 = (e^\alpha + e^\beta)(u - v). \]

Here, $p_1, q_1$ coincide with the $M_-$ and dual $M_-$ cases.

C.2 Proof

It is stated in Theorem 12 that H-type bulk polynomials only have the above list of $q, p$. Here, we provide a direct proof.

H1: $\Gamma$ is $\beta - \alpha$ that is of bidegree $(0, 0)$ in $x, u$. Transform $Q$ to $Q_m$ using $m(x) = 1/x$, then

\[ Q_m = (x - y)(u - v) + (\beta - \alpha)xyuv, \quad \Gamma_m = (\beta - \alpha)x^2u^2. \] (C.1)

We aim to classify all possible $q, p$ for (C.1), which, through the equivalence class with respect to simultaneous Möbius transformations, provides all $q, p$ for $Q$ in the canonical form. For simplicity, the subscript $m$ is omitted here. Take a generic boundary polynomial $q$ with 12 parameters

\[ q = \sum_{j=0}^{2} r_{1j}u^jxy + \sum_{j=0}^{2} r_{2j}u^jx + \sum_{j=0}^{2} r_{3j}u^jy + \sum_{j=0}^{2} r_{4j}u^j, \] (C.2)

with $r_{ij}$ to be determined. The idea is to combine the above forms of $Q$ and $q$ following the singular case approach provided in Section 2.4. To classify all possible $q, p$, it suffices to classify all possible $\chi, \zeta$ dual to each other, which, as a result of (2.43), obey

\[ \chi \zeta = -(\beta - \alpha)x^2u^2, \] (C.3)

Due to the duality between $\chi$ and $\zeta$, one only needs to take six possible cases of $\chi$, namely, $\chi \propto x^2u^2, \chi \propto x^2u, \chi \propto x^2, \chi \propto u^2, \chi \propto xu^2, \chi \propto xu$.

Let $\deg_x \chi$ be 2, which contains the first three cases. Assumming the existence of $p$, by the analysis of the degrees of the middle argument of $q, p$ provided in Section 2.3, the polynomial $Qq_y - Qyq$ defined in (2.39) should be of degree 2 in $x$, which implies that $\deg_x p = 0$. This means $p$ belongs to the $M_+$ case. This amounts to the only possible $q, p$, and the results are listed above.

One can show that the remaining three cases are not possible. For instance, let $\chi \propto u^2$, then $Qq_y - Qyq$ has a factor $x^2$. This leads to restrictions on the parameters of $q$ as $r_{10} = 0, r_{40} = 0, r_{30} = -r_{20}, r_{41} = 0, r_{31} = -r_{21}$ and $r_{11} = (\alpha - \beta)r_{20}$. Since $q, p$ are irreducible, we have $q|_{u=0} = r_{20}(x - y) \neq 0$ and $p|_{x=0} = r_{42}(u - v) \neq 0$. This imposes

\[ r_{20} \neq 0, \quad r_{42} \neq 0. \] (C.4)
On the other hand, due to the $\mathbb{Z}_2$-symmetry of $q, p$, assume

$$q(x, u, y; \alpha, \beta) = \gamma(x, \beta)q(y, u, x; \beta, \alpha), \quad p(u, x, v; \alpha, \beta) = \eta(x, \beta)p(v, x, u; \beta, \alpha),$$

with $\gamma(x, \beta)\gamma(\beta, \alpha) = \eta(x, \beta)\eta(\beta, \alpha) = 1$. Together with the above results, this leads to

$$r_{20}(\alpha, \beta) = -\gamma(\alpha, \beta)r_{20}(\beta, \alpha) = -\eta(\alpha, \beta)r_{20}(\beta, \alpha), \quad \text{(C.6a)}$$

$$r_{42}(\alpha, \beta) = \gamma(\alpha, \beta)r_{42}(\beta, \alpha) = -\eta(\alpha, \beta)r_{42}(\beta, \alpha), \quad \text{C.6b}$$

which require that $\gamma(\alpha, \beta) = \eta(\alpha, \beta), \gamma(\alpha, \beta) = -\eta(\alpha, \beta)$. One has $\gamma(\alpha, \beta) = \eta(\alpha, \beta) = 0$, which is impossible.

**H2, H3**: they can be treated similarly, since $\Gamma$ in both cases is of bidegree $(1, 1)$ in $x, u$. Take H2 as an example. Using transformation $m(x) = 1/x$, one gets

$$Q_m = (x-y)(u-v) + (\beta-\alpha)(xuv + xvy + xuy + uvy) + (\beta^2 - \alpha^2)xuvy,$$

with

$$\Gamma_m = -2(\alpha-\beta)xu(x + u + \alpha xu).$$

One aims to classify all possible $q, p$ for $Q_m$. Drop the subscript $m$ for simplicity. Due to the duality between $\chi$ and $\zeta$, one needs to take four possibilities for $\chi$, namely, $\chi \propto xu(x + u + \alpha xu), \chi \propto x(x + u + \alpha xu), \chi \propto u(x + u + \alpha xu), \chi \propto x + u + \alpha xu.$ The first two cases have $\operatorname{deg}_{\chi} \chi = 2$, which implies the dual boundary polynomials belong to the $M_+$ or $M_-$ case. In the case $\chi \propto x + u + \alpha xu$, $\chi$ and $\zeta$ are of bidegree $(1, 1)$ in $x, u$, and all possible $q, p$ can be exhausted following the singular case approach. Lastly, the case $\chi \propto u(x + u + \alpha xu)$ is impossible. This follows the same arguments presented above for H1.

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