Research Article

Some Identities of the Degenerate Multi-Poly-Bernoulli Polynomials of Complex Variable

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In this paper, we introduce degenerate multi-poly-Bernoulli polynomials and derive some identities of these polynomials. We give some relationship between degenerate multi-poly-Bernoulli polynomials degenerate Whitney numbers and Stirling numbers of the first kind. Moreover, we define degenerate multi-poly-Bernoulli polynomials of complex variables, and then, we derive several properties and relations.

1. Introduction

For any \( \lambda \in \mathbb{R} \setminus \{0\} \) (or \( \mathbb{C} \setminus \{0\} \)), degenerate version of the exponential function \( e^x \) is defined as follows (see [1–15])

\[
e^x := (1 + \lambda t)^x = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!},
\]

where \((x)_{n,\lambda} = x(x-\lambda) \cdots (x-(n-1)\lambda) \) for \( n \geq 1 \), (cf. [1–15]). It follows from (1) is \( \lim_{\lambda \to 0} e^x = e^t \). Note that \( e^x(t) = e^t(t) \).

Carlitz [1] introduced the degenerate Bernoulli polynomials as follows:

\[
e^x(t) = (1 + \lambda t)^x = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}.
\]

Note that

\[
\lim_{\lambda \to 0} \beta_n(x; \lambda) = B_n(x),
\]

where \( B_n(x) \) are the familiar Bernoulli polynomials (cf. [1, 3, 4, 6, 8, 11, 12, 14, 16–22])

\[
\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi).
\]

For \( k \in \mathbb{Z} \), the polyexponential function \( E_i^k(x) \) is defined by (see [21])

\[
E_i^k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \quad (k \in \mathbb{Z}).
\]

Setting \( k = 1 \) in (5), we have \( E_i^1(x) = e^x - 1 \).
The degenerate modified polyexponential function [12] is defined, for \( k \in \mathbb{Z} \) and \( |x| < 1 \), by
\[
E_{i,k\lambda}(x) = \sum_{n=0}^{\infty} \frac{(1)_{n,k}}{(n-1)!} t^n x^n.
\] (6)

Note that \( E_{i,k\lambda}(x) = e_\lambda(x) - 1 \).

Let \( k \in \mathbb{Z} \) and \( \lambda \in \mathbb{R} \). The degenerate poly-Bernoulli polynomials, cf. [12], are defined by
\[
E_{i,k\lambda}(\log(1+t)) \quad \text{and} \quad e_\lambda(t) = \sum_{n=0}^{\infty} B_{n,k\lambda}(x) \frac{t^n}{n!},
\] (7)

where \( \log(1+t) \) are the degenerate version of the logarithmic function (cf. [8, 12]), which is also the inverse function of the degenerate exponential function \( e_\lambda(t) \) as shown below (cf. [8])
\[
e_\lambda(\log(1+t)) = \log\left(e_\lambda(1+t)\right) = 1 + t. \] (9)

Letting \( x = 0 \) in (7), \( B_{n,k\lambda}(0) = e_{n,k\lambda}^{(k)} \) are called the type 2 degenerate poly-Bernoulli numbers.

The degenerate Stirling numbers of the first kind (cf. [8, 13]) and second kind (cf. [4–6, 9, 17]) are defined, respectively, by
\[
\frac{1}{k!} \left(\log(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,k\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0) , \] (10)
and (cf. [1–27])
\[
\frac{1}{k!} (e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,k\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0).
\] (11)

Note that \( \lim_{\lambda \to 0} \) in (10) and (1.8), we have (cf. [8, 13])
\[
\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad (k \geq 0),
\] (12)
and (cf. [4–6, 9, 17, 24])
\[
\frac{(e^t - 1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}, \quad (k \geq 0),
\] (13)

where \( S_1(n,k) \) and \( S_2(n,k) \) are called the Stirling numbers of the first kind and second kind.

The following paper is as follows. In Section 2, we define the degenerate multi-poly-Bernoulli polynomials and numbers by using the degenerate multiple polyexponential functions and derive some properties and relations of these polynomials. In Section 3, we consider the degenerate multi-poly-Bernoulli polynomials of a complex variable and then we derive several properties and relations. Also, we examine the results derived in this study [28, 29].

### 2. Degenerate Multi-Poly-Bernoulli Polynomials and Numbers

Let \( k_1, k_2, \ldots, k_r \in \mathbb{Z} \). The degenerate multiple polyexponential function \( E_{i,k_1\lambda,\ldots,k_r\lambda}(x) \) is defined (cf. [15]) by
\[
E_{i,k_1\lambda,\ldots,k_r\lambda}(x) = \sum_{0<n_1<\cdots<n_r} \frac{(1)_{n_1,k_1\lambda} \cdots (1)_{n_r,k_r\lambda} x^{n_r}}{(n_1-1)! \cdots (n_r-1)! n_1! \cdots n_r!},
\] (14)

where the sum is over all integers \( n_1, n_2, \ldots, n_r \) satisfying \( 0 < n_1 < n_2 < \cdots < n_r \). Utilizing this function, Kim et al. [15] introduced and studied the degenerate multi-poly-Genocchi polynomials given by
\[
2^r E_{i,k_1\lambda,\ldots,k_r\lambda}(\log(1+t)) \quad \text{and} \quad e_{\lambda}(t) = \sum_{n=0}^{\infty} g_{n,k_1\lambda,\ldots,k_r\lambda}(x) \frac{t^n}{n!}.
\] (15)

Inspired by the definition of degenerate multi-poly-Genocchi polynomials, using the degenerate multiple polyexponential function (14), we give the following definition.

**Definition 1.** Let \( k_1, k_2, \ldots, k_r \in \mathbb{Z} \) and \( \lambda \in \mathbb{R} \), we consider the degenerate multi-poly-Bernoulli polynomials given by
\[
r! E_{i,k_1\lambda,\ldots,k_r\lambda}(\log(1+t)) \quad \text{and} \quad e_{\lambda}(t) = \sum_{n=0}^{\infty} g_{n,k_1\lambda,\ldots,k_r\lambda}(x) \frac{t^n}{n!}.
\] (16)

Upon setting \( x = 0 \) in (16), the degenerate multi-poly-Bernoulli polynomials reduce to the corresponding numbers, namely, the type 2 degenerate multi-poly-Bernoulli numbers \( g_{n,k_1\lambda,\ldots,k_r\lambda}(0) = g_{n,k_1\lambda,\ldots,k_r\lambda} \).

**Remark 2.** As \( \lambda \to 0 \), the degenerate multi-poly-Bernoulli polynomials reduce to the multi-poly-Bernoulli polynomials given by
\[
r! E_{i,k_1\lambda,\ldots,k_r\lambda}(\log(1+t)) \quad \text{and} \quad e_{\lambda}(t) = \sum_{n=0}^{\infty} g_{n,k_1\lambda,\ldots,k_r\lambda}(x) \frac{t^n}{n!}.
\] (17)

**Remark 3.** Upon setting \( r = 1 \) in (16), the degenerate multi-poly-Bernoulli polynomials reduce to the degenerate poly-Bernoulli polynomials in (7).

Before going to investigate the properties of the degenerate multi-poly-Bernoulli polynomials, we first give the following result.
Proposition 4 (Derivative Property). For \( k_1, k_2, \ldots, k_r \in \mathbb{Z} \) and \( \lambda \in \mathbb{R} \), we have

\[
\frac{d}{dx} E_{k_1, k_2, \ldots, k_r, \lambda}(x) = \frac{1}{x} E_{k_1, k_2, \ldots, k_r, \lambda-1}(x).
\]

Proof. By (14), we see that

\[
\frac{d}{dx} E_{k_1, k_2, \ldots, k_r, \lambda}(x) = \frac{1}{x} \sum_{m=0}^{\infty} \sum_{0 \leq c_1 < c_2 < \cdots < c_n \leq \lambda} \frac{(1)_{n, \lambda} \cdots (1)_{n, \lambda} x^{n_r}}{(n_1 - 1)!(n_2 - 1)!n_1^{k_1} \cdots n_r^{k_r}}
\]

which gives the asserted result (18).

Theorem 5. The following relationship

\[
\mathcal{G}_{n, \lambda}^{(k_1, k_2, \ldots, k_r)}(x) = \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) \mathcal{G}_{m, j, \lambda}^{(k_1, k_2, \ldots, k_r)}(x)_{j, \lambda},
\]

holds for \( n \geq 0 \).

Proof. Recall Definition 1 that

\[
\sum_{m=0}^{\infty} \mathcal{G}_{n, \lambda}^{(k_1, k_2, \ldots, k_r)}(x) \frac{t^n}{n!} = \frac{r! E_{k_1, k_2, \ldots, k_r, \lambda}(log(1 + t))}{(e(t) - 1)^r} \mathcal{C}_r(t)
\]

which means the claimed result (23).

Theorem 6. For \( n \geq r \). Then

\[
\mathcal{G}_{n, \lambda}^{(k_1, k_2, \ldots, k_r)}(x) = \sum_{m=0}^{\infty} \sum_{0 \leq c_1 < c_2 < \cdots < c_n \leq m} \frac{(1)_{n, \lambda} \cdots (1)_{n, \lambda} x^{n_r}}{(n_1 - 1)!(n_2 - 1)!n_1^{k_1} \cdots n_r^{k_r}} \cdot \frac{n! n_r!}{(n+r)![(n_1 - 1)!(n_2 - 1)!n_1^{k_1} \cdots n_r^{k_r}]}.
\]

Proof. Recall from Definition 1 and (10) that

\[
\sum_{m=0}^{\infty} \mathcal{G}_{n, \lambda}^{(k_1, k_2, \ldots, k_r)}(x) \frac{t^n}{n!} = \frac{r! E_{k_1, k_2, \ldots, k_r, \lambda}(log(1 + t))}{(e(t) - 1)^r} \mathcal{C}_r(t)
\]

which gives the asserted result (20).
Proof. In view of Definition 1, we see that

\[
\sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(x+y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(x) \frac{t^n}{n!} \frac{t^m}{m!}
\]

which implies the desired result (25).

Theorem 8. The following relation

\[
\frac{d}{dx} \mathcal{B}_{n,k_2,\cdots,k_r}(x) = \sum_{l=1}^{n} \binom{n}{l} \mathcal{B}_{n-l,k_2,\cdots,k_r}(x)(-\lambda)^{l-1}(1-l)!
\]

is valid for \(k_1, k_2, \cdots, k_r \in \mathbb{Z} \) and \(n \geq 0\).

Proof. To investigate the derivative property of \(\mathcal{B}_{n,k_2,\cdots,k_r}(x)\) that

\[
\sum_{n=0}^{\infty} \frac{d}{dx} \mathcal{B}_{n,k_2,\cdots,k_r}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(x) \frac{t^n}{n!} \frac{d}{dx} \mathcal{B}_{n,k_2,\cdots,k_r}(x)
\]

which provides the asserted result (27).

We here give a relation including the degenerate multi-poly-Bernoulli polynomials with numbers and the degenerate Stirling numbers of the second kind.

Theorem 9. The following correlation

\[
\mathcal{B}_{n,k_2,\cdots,k_r}(x) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \binom{n}{m} S_{2,l}(m,l) \mathcal{B}_{n-m,k_2,\cdots,k_r},
\]

is valid for \(k_1, k_2, \cdots, k_r \in \mathbb{Z} \) and \(n \geq 0\).

Proof. By means of Definition 1, we attain that

\[
\sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(x) \frac{t^n}{n!} \frac{d}{dx} \mathcal{B}_{n,k_2,\cdots,k_r}(x)
\]

where the notation \((x)_l\) is falling factorial that is defined by \((x)_0 = 1\) and \((x)_n = x(x-1) \cdots (x-(n-1))\) for \(n \geq 1\), (cf. [1, 2, 5–14, 21, 23, 24]). So, the proof is completed.

Kim [5] introduced the degenerate Whitney numbers are given by

\[
\frac{(\epsilon_\lambda^m - 1)^k}{m!} = \sum_{n=0}^{\infty} W_{m,n}(k;\lambda) \frac{t^n}{n!} \quad (k \geq 0).
\]

Kim also provided several correlations including the degenerate Stirling numbers of the second kind and the degenerate Whitney numbers (see [5]).

We now give a correlation as follows.

Theorem 10. For \(k_1, k_2, \cdots, k_r \in \mathbb{Z} \) and \(n \geq 0\), we have

\[
\mathcal{B}_{n,k_2,\cdots,k_r}(xu + \alpha) = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} u^l(x) S_{2,l}(m,l) \mathcal{B}_{n-m,k_2,\cdots,k_r}(xu + \alpha).
\]

Proof. Using (31) and Definition 1, we acquire that

\[
\sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(xu + \alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{B}_{n,k_2,\cdots,k_r}(x) \frac{t^n}{n!} \frac{d}{dx} \mathcal{B}_{n,k_2,\cdots,k_r}(xu + \alpha)
\]
multi-poly-Bernoulli polynomials of complex variable as
gate some of their properties. Motivated and inspired by
and Bernoulli polynomials of complex variable and investi-


\begin{equation}
\sum_{n=0}^{\infty} \binom{n}{m} \frac{u'(x)}{x} W_{m,n}(m,1) \mathbb{B}_{n-m,m}(x)
\end{equation}

which implies the asserted result (32).

### 3. Degenerate Multi-Poly-Bernoulli Polynomials of Complex Variable

In [25], Kim et al. defined the degenerate sine sin_\(t\) and cosine cos_\(t\) functions by

\begin{equation}
\sin^{(x)}(t) = \frac{e^{ix}_t(t) - e^{-ix}_t(t)}{2i} \quad \text{and} \quad \cos^{(x)}(t) = \frac{e^{ix}_t(t) + e^{-ix}_t(t)}{2}.
\end{equation}

where \(i = \sqrt{-1}\). Note that \(\lim_{t \to 0} \sin^{(x)}(t) = \sin x \) and \(\lim_{t \to 0} \cos^{(x)}(t) = \cos x\). From (34), it readily that

\begin{equation}
\sin^{(x)}(t) = \cos^{(x)}(t) + i \sin^{(x)}(t).
\end{equation}

By these functions in (34), the degenerate sine-polynomials \(S_{k,1}(x, y)\) and degenerate cosine-polynomials \(C_{k,1}(x, y)\) are introduced (cf. [25]) by

\begin{equation}
\sum_{n=0}^{\infty} S_{k,1}(x, y) \frac{t^n}{n!} = e_t^{x}(t) \sin^{(y)}(t),
\end{equation}

\begin{equation}
\sum_{n=0}^{\infty} C_{k,1}(x, y) \frac{t^n}{n!} = e_t^{x}(t) \cos^{(y)}(t).
\end{equation}

Several properties of these polynomials in (36) and (37) are studied and investigated in [25]. Also, by means of these functions, Kim et al. [25] introduced the degenerate Euler and Bernoulli polynomials of complex variable and investigate some of their properties. Motivated and inspired by these considerations above, we define type 2 degenerate multi-poly-Bernoulli polynomials of complex variable as follows.

**Definition 11.** Let \(k_1, k_2, \ldots, k_s \in \mathbb{Z}\). We define a new form of the degenerate multi-poly-Bernoulli polynomials of complex variable by the following generating function:

\begin{equation}
\frac{r!E_{k_1, k_2, \ldots, k_s}(\log_1(1 + t))}{(e_1(t) - 1)^r} e_t^{ix}(t) = \sum_{n=0}^{\infty} B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x + iy) \frac{t^n}{n!}.
\end{equation}

By (34) and (38), we observe that

\begin{equation}
\sum_{n=0}^{\infty} \binom{n}{m} \frac{u'(x)}{x} W_{m,n}(m,1) \mathbb{B}_{n-m,m}(x)
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} \binom{n}{m} \frac{u'(x)}{x} W_{m,n}(m,1) \mathbb{B}_{n-m,m}(x)
\end{equation}

In view of (39) and (40), we consider the degenerate multi-poly-sine-Bernoulli polynomials \(B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y)\) with two parameters and the degenerate multi-poly-cosine-Bernoulli polynomials \(B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y)\) with two parameters as follows:

\begin{equation}
\sum_{n=0}^{\infty} B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y) \frac{t^n}{n!} = \frac{r!E_{k_1, k_2, \ldots, k_s}(\log_1(1 + t))}{(e_1(t) - 1)^r} e_t^{x}(t) \sin^{(y)}(t),
\end{equation}

\begin{equation}
\sum_{n=0}^{\infty} B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y) \frac{t^n}{n!} = \frac{r!E_{k_1, k_2, \ldots, k_s}(\log_1(1 + t))}{(e_1(t) - 1)^r} e_t^{x}(t) \cos^{(y)}(t).
\end{equation}

Note that

\begin{equation}
\lim_{\lambda \to 0} B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y) = B_{n}^{(k_1, k_2, \ldots, k_s)}(x, y) \quad \text{and} \quad \lim_{\lambda \to 0} B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y) = B_{n}^{(k_1, k_2, \ldots, k_s)}(x, y),
\end{equation}

which are multi-poly-sine-Bernoulli polynomials \(B_{n}^{(k_1, k_2, \ldots, k_s)}(x, y)\) and multi-poly-cosine-Bernoulli polynomials \(B_{n}^{(k_1, k_2, \ldots, k_s)}(x, y)\) with two parameters.

By (39)-(42), we see that

\begin{equation}
B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y) = \frac{B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x + iy) - B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x - iy)}{2i},
\end{equation}

\begin{equation}
B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x, y) = \frac{B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x + iy) + B_{n, \lambda}^{(k_1, k_2, \ldots, k_s)}(x - iy)}{2}.
\end{equation}

We now give the two summation formulae by the following theorem.
Theorem 12. For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have
\[
B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)}(x + iy) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\lambda}^{(k_1, k_2, \ldots, k_r)}(x)(iy)_m. 
\] (45)
\[
B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)}(x + iy) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\lambda}^{(k_1, k_2, \ldots, k_r)}(x + iy)_m. 
\] (46)

Proof. The proofs of this theorem can be done by using the same proof methods used in Theorems 5 and 7. So, we omit the proofs.

We here provide the two derivative formule by the following theorem.

Theorem 13. For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have
\[
dt \sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)}(x + iy) = n n_{x}^{(k_1, k_2, \ldots, k_r)}(x + iy), \]
\[
dt \sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)}(x + iy) = n n_{x}^{(k_1, k_2, \ldots, k_r)}(x + iy)(-\lambda)^{-l}(l - 1)! \]
\] (47)

Proof. The proofs of this theorem can be done by using the same proof methods used in Theorem 8. So, we omit the proofs.

We give the following theorem.

Theorem 14. For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have
\[
B_{n,\lambda}^{(k_1, k_2, \ldots, k_r, S)}(x,y) = \sum_{l=1}^{n} \binom{n}{l} B_{n-l,\lambda}^{(k_1, k_2, \ldots, k_r)} S_{l,\lambda}(x,y), 
\]
\[
B_{n,\lambda}^{(k_1, k_2, \ldots, k_r, C)}(x,y) = \sum_{l=1}^{n} \binom{n}{l} B_{n-l,\lambda}^{(k_1, k_2, \ldots, k_r)} C_{l,\lambda}(x,y). 
\] (48)

Proof. From (36), (37), and (38), we get
\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r, S)}(x,y) t^n n! 
= \sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)} (\log_\lambda (1 + t)) \sin(t) \lambda^{-l}(l - 1)! 
\]
\[
= \sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)} t^n \sum_{n=0}^{\infty} S_{l,\lambda}(x,y) t^n n! 
\]
\[
= \sum_{n=0}^{\infty} \left( \binom{n}{l} B_{n-l,\lambda}^{(k_1, k_2, \ldots, k_r)} S_{l,\lambda}(x,y) t^n n! \right) 
\] (49)

which complete the proof of the theorem.

We note that (cf. [25])
\[
\sin(t) = \sum_{n=0}^{\infty} \binom{n}{2} \lambda^n 2^{-l+1}(1) t^{2l+1} S_{l,\lambda}(n, 2k + 1) \frac{t^n}{n!}, 
\]
\[
\cos(t) = \sum_{n=0}^{\infty} \binom{n}{2} \lambda^n 2^{-l+1}(1) t^{2l+1} S_{l,\lambda}(n, 2k + 1) \frac{t^n}{n!}. 
\] (50)

We give the following theorem.

Theorem 15. For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$ and $n \geq 0$, we have
\[
B_{n,\lambda}^{(k_1, k_2, \ldots, k_r, S)}(x,y) = \sum_{l=1}^{n} \binom{n}{l} B_{n-l,\lambda}^{(k_1, k_2, \ldots, k_r)} S_{l,\lambda}(x,y), 
\]
\[
B_{n,\lambda}^{(k_1, k_2, \ldots, k_r, C)}(x,y) = \sum_{l=1}^{n} \binom{n}{l} B_{n-l,\lambda}^{(k_1, k_2, \ldots, k_r)} S_{l,\lambda}(x,y). 
\] (52)

where the notation $[\cdot]$ is Gauss’ notation and represents the maximum integer which does not exceed a number in the square bracket.

Proof. By (41)–(51), we observe that
\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r, S)}(x,y) t^n n! 
= \sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \ldots, k_r)} (x) t^n n! \sum_{l=1}^{\infty} \lambda^{2l+1}(1) t^{2l+1} S_{l,\lambda}(n, 2k + 1) \frac{t^n}{n!} 
\]
\[
= \sum_{n=0}^{\infty} \left( \binom{n}{l} B_{n-l,\lambda}^{(k_1, k_2, \ldots, k_r)} S_{l,\lambda}(x,y) t^n n! \right) 
\] (53)
So, the proof is completed.

We give the following proposition.

**Proposition 16.** The following relations

\[
B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(x + u, y) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l, \lambda}^{(k_1, k_2, \cdots, k_s)}(x, y)(u)_l, \\
B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(x + u, y) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l, \lambda}^{(k_1, k_2, \cdots, k_s, C)}(x, y)(u)_l.
\]

hold for \(k_1, k_2, \cdots, k_r \in \mathbb{Z}, u \in \mathbb{C}\) and \(n \geq 0\).

**Proof.** The proofs of this proposition can be done by utilizing the same proof methods used in Theorem 7. So, we omit the proofs.

Upon setting \(x = 0\) in (41) and (42), we consider the degenerate multi-poly-sine-Bernoulli polynomials \(B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(y)\) and the degenerate multi-poly-cosine-Bernoulli polynomials \(B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(y)\) as follows

\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(y) \frac{t^n}{n!} = \frac{r!E_{k_1, k_2, \cdots, k_s, \lambda}(\log(1 + t))}{(e_{\lambda}(t) - 1)^r} \sin_{\lambda}^{(y)}(t),
\]

\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(y) \frac{t^n}{n!} = \frac{r!E_{k_1, k_2, \cdots, k_s, \lambda}(\log(1 + t))}{(e_{\lambda}(t) - 1)^r} \cos_{\lambda}^{(y)}(t).
\]

We now provide the following theorem.

**Theorem 17.** For \(k_1, k_2, \cdots, k_r \in \mathbb{Z}\) and \(n > 0\), we have

\[
B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(x, y) = \sum_{m=0}^{k_s} \sum_{l=0}^{n} \binom{n}{m} S_{2,\lambda}(m, l) B_{n-m,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(x)(y)\text{ with } B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(x, y) = 0.
\]

**Proof.** The proofs of this theorem can be done by utilizing the same proof methods in Theorem 9.

Let \(\alpha\) be any fixed real (or complex) number. The Bernoulli polynomials of order \(\alpha\) is defined by (cf. [25])

\[
\left( \frac{t}{e^{\alpha t} - 1} \right)^{\alpha} e^{\alpha t} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi).
\]

When \(x = 0\), the Bernoulli polynomials of order \(\alpha\) reduce to the Bernoulli numbers of order \(\alpha\), denoted by \(B_{n,\alpha}^{(\alpha)}\). We give the following relation.

**Theorem 18.** For \(k_1, k_2, \cdots, k_r \in \mathbb{Z}\) and \(n \geq 0\), we have

\[
B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(1, y) - B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(y) = n \sum_{l=0}^{n-1} \binom{n-1}{l} B_{n-l,\lambda}^{(k_1, k_2, \cdots, k_s)}(y) B_{l}^{(-1)},
\]

\[
B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(1, y) - B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(y) = n \sum_{l=0}^{n-1} \binom{n-1}{l} B_{n-l,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(y) B_{l}^{(-1)}.
\]

**Proof.** By (55) and (56), we acquire

\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(1, y) \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(y) \frac{t^n}{n!} = r!E_{k_1, k_2, \cdots, k_s, \lambda}(\log(1 + t)) \frac{\sin_{\lambda}^{(y)}(t)}{(e_{\lambda}(t) - 1)^r},
\]

\[
\sum_{n=0}^{\infty} B_{n,\lambda}^{(k_1, k_2, \cdots, k_s, C)}(y) \frac{t^n}{n!} = r!E_{k_1, k_2, \cdots, k_s, \lambda}(\log(1 + t)) \frac{\cos_{\lambda}^{(y)}(t)}{(e_{\lambda}(t) - 1)^r}.
\]

Thus, (59) is proved. We prove (60) in the same way.

Here is a special case of Theorem 18.

**Corollary 19.** For \(k_1, k_2, \cdots, k_r \in \mathbb{Z}\) and \(n \geq 0\), we have

\[
B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(1) - B_{n,\lambda}^{(k_1, k_2, \cdots, k_s)}(y) = n \sum_{l=0}^{n-1} \binom{n-1}{l} B_{n-l,\lambda}^{(k_1, k_2, \cdots, k_s)} B_{l}^{(-1)},
\]

which is a relation including the degenerate multi-poly-
Bernoulli polynomials.

4. Conclusions

In this paper, we defined the degenerate multi-poly-Bernoulli polynomials by employing the degenerate multiple
polyexponential functions. We have established some identities and relations between degenerate Whitney numbers and degenerate Stirling numbers of the first kind. Also, we have established addition formulas and derivative formulas of degenerate multi-poly-Bernoulli polynomials. In the last section, we have defined degenerate multi-poly-Bernoulli polynomials of complex variables and then we have derived several properties and relations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References

[1] L. Carlitz, “Degenerate Stirling,” Bernoulli and Eulerian numbers. Utilitas Math., vol. 15, pp. 51–88, 1979.
[2] L. Carlitz, “A degenerate Staudt-Clausen theorem,” Archiv der Mathematik, vol. 7, no. 1, pp. 28–33, 1956.
[3] H. Haroon and W. A. Khan, “Degenerate Bernoulli numbers and polynomials associated with degenerate Hermite polynomials,” Communication of the Korean Mathematical Society, vol. 33, no. 2, pp. 651–669, 2018.
[4] W. A. Khan, M. Acikgoz, and U. Duran, “Note on the type 2 degenerate multi-poly-Euler polynomials,” Symmetry, vol. 12, no. 1091, pp. 1–10, 2020.
[5] T. Kim, “A note on degenerate Stirling polynomials of the second kind,” Proc. Jangjeon Math. Soc., vol. 20, no. 3, pp. 319–331, 2017.
[6] T. Kim, L.-C. Jang, D. S. Kim, and H.-Y. Kim, “Some identities on type 2 degenerate Bernoulli polynomials of the second kind,” Symmetry, vol. 12, no. 4, p. 510, 2020.
[7] T. Kim and D. S. Kim, “Degenerate polyexponential functions and degenerate Bell polynomials,” Journal of Mathematical Analysis and Applications, vol. 487, no. 2, p. 124017, 2020.
[8] D. S. Kim and T. Kim, “A note on a new type of degenerate Bernoulli numbers,” Russian Journal of Mathematical Physics, vol. 27, no. 2, pp. 227–235, 2020.
[9] T. Kim, D. S. Kim, H. Y. Kim, and J. Kwon, “Degenerate Stirling polynomials of the second kind and some applications,” Symmetry, vol. 11, no. 8, p. 1046, 2019.
[10] T. Kim and D. S. Kim, “Degenerate Laplace transform and degenerate gamma function,” Russian Journal of Mathematical Physics, vol. 24, no. 2, pp. 241–248, 2017.
[11] T. Kim, W. A. Khan, S. K. Sharma, and M. Ghayasuddin, “A note on parametric kinds of the degenerate poly-Bernoulli and poly-Genocchi polynomials,” Symmetry, vol. 12, no. 4, p. 614, 2020.
[12] T. Kim, D. S. Kim, J. Kwon, and H. Lee, “Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials,” Advances in Difference Equations, vol. 2020, no. 1, 2020.
[13] T. Kim, D. S. Kim, J. Kwon, and H. Y. Kim, “A note on degenerate Genocchi and poly-Genocchi numbers and polynomials,” Journal of Inequalities and Applications, vol. 2020, no. 1, 2020.
[14] T. Kim and D. S. Kim, “A note on degenerate multi-poly-Bernoulli numbers and polynomials,” 2020, http://arxiv.org/abs/2005.07319.
[15] T. Kim, D. S. Kim, H.-Y. Kim, and J. Kwon, “A note on degenerate multi-poly-Genocchi polynomials,” Advanced Studies in Contemporary Mathematics, vol. 30, no. 3, pp. 447–454, 2020.
[16] M. M. Jamei, M. R. Beyki, and W. Koepf, “An extension of the Euler-Maclaurin quadrature formula using a parametric type of Bernoulli polynomials,” Bulletin des Sciences Mathématiques, vol. 156, p. 102798, 2019.
[17] W. A. Khan, “Degenerate Hermite-Bernoulli polynomials and Bernoulli polynomials of the second kind,” Prespacetime Journal, vol. 7, no. 9, pp. 1297–1305, 2016.
[18] W. A. Khan, M. Ghayasuddin, and M. Shadab, “Multiple poly-Bernoulli polynomials of the second kind associated with Hermite polynomials,” Fasciculi Mathematici, vol. 58, no. 1, pp. 97–112, 2017.
[19] W. A. Khan, M. Ghayasuddin, N. Khan, and J. Choi, “The 3-variable Hermite poly-Bernoulli polynomials of the second kind,” Nonlinear Functional Analysis and Applications, vol. 22, no. 5, pp. 1013–1027, 2017.
[20] D. S. Kim and T. Kim, “Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation,” Bulletin of the Korean Mathematical Society, vol. 52, no. 6, pp. 2001–2010, 2015.
[21] D. S. Kim and T. Kim, “A note on polyexponential and unipoly functions,” Russian Journal of Mathematical Physics, vol. 26, no. 1, pp. 40–49, 2019.
[22] T. Kim, H.-I. Kwon, S.-H. Lee, and J. I. Seo, “A note on poly-Bernoulli numbers and polynomials of the second kind,” Advances in Difference Equations, vol. 2014, no. 1, 2014.
[23] D. V. Dolgy and L. C. Jang, “Some identities on the poly-Genocchi polynomials and numbers,” Symmetry, vol. 12, no. 6, p. 1007, 2020.
[24] U. Duran, M. Acikgoz, and S. Araci, “Construction of the type 2 pol-Frobenius-Genocchi polynomials with their certain applications,” Advances in Differences Equations, vol. 2020, no. 432, pp. 1–14, 2020.
[25] D. S. Kim, T. Kim, and H. Lee, “A note on degenerate Euler and Bernoulli polynomials of complex variable,” Symmetry, vol. 11, no. 9, p. 1168, 2019.
[26] F. Qi, D. S. Kim, and T. Kim, “Multiple poly-Bernoulli polynomials of the second kind,” Advanced Studies in Contemporary Mathematics, vol. 25, pp. 1–7, 2015.
[27] S. Roman, The Umbral Calculus, Academic Press, Inc., New York, NY, USA, 1984.
[28] G. Muhiuddin, W. A. Khan, U. Duran, and D. Al-Kadi, “A new class of higher-order hypergeometric bernoulli polynomials associated with lagrange-hermite polynomials,” Symmetry, vol. 13, no. 4, p. 648, 2021.
[29] G. Muhiuddin, W. A. Khan, and U. Duran, “Two-variable type 2 poly-fubini polynomials,” Mathematics, vol. 9, no. 3, p. 281, 2021.