Influence of Kappa Distributions on the Whistler Mode Instability

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Abstract  Kappa distributions possess an enhanced high-energy tail characterized by a spectral index $\kappa$. Here we consider how kappa distributions influence the whistler mode instability. In a relativistic regime, we analyze the effects of the bi-kappa and kappa loss-cone distributions on the linear and nonlinear growth of whistler mode waves. We find that for $\kappa = 2$, the linear growth rate corresponding to the bi-kappa distribution exceeds that for a bi-Maxwellian distribution ($\kappa = \infty$) for electron anisotropies less than a critical value. The threshold wave amplitude for nonlinear growth corresponding to a hot injected bi-kappa distribution can be sensitively dependent on the value of $\kappa$, but also depends crucially on the electron anisotropy, parallel hot electron temperature, and wave frequency. We plot time profiles of the wave magnetic field, frequency, total nonlinear growth rate and local nonlinear growth rate, and examine how these depend on $\kappa$. The sweep (chirp) rates of the whistler mode chorus waves are found to be in the range of observed values, for the adopted plasma parameters. For a given value of the loss-cone parameter $\sigma$, realistic whistler mode wave profiles may only exist for a restricted range of $\kappa$. For example, for $\sigma \geq 4$, realistic wave profiles require $\kappa \geq 4$ as well as a sufficiently large anisotropy. In summary, this new study finds that the influence of kappa distributions on the whistler mode instability is complex, not least because the instability depends on several system parameters as well as $\kappa$.

1. Introduction

Kappa distributions characteristically possess an enhanced high-energy tail. The simplest form of kappa distribution satisfies an inverse power-law relation at large energy, namely, $f(\xi) \propto (\xi)^{-1/(\kappa+1)}$, where $\kappa$ denotes the spectral index. The left panel of Figure 1 illustrates such a three-dimensional isotropic distribution (in the case of nonrelativistic momentum $p$) for $\kappa = 2, 4, 6, 8, 10, 16, 25$ and $\kappa = \infty$. This kappa distribution possesses the useful property that in the limit as $\kappa \to \infty$, the distribution approaches the standard Maxwellian distribution (Summers & Thorne, 1991). In the present study we employ, in a relativistic regime, the bi-kappa distribution (1) and the kappa loss-cone distribution (46). Other examples of kappa distributions are given, for instance, in Table 1 of Summers and Thorne (1991). Kappa distributions were first used in the 1960’s to model experimental particle data measured by spacecraft instruments (Binsack, 1966; Olbert, 1968; Vasyliunas, 1968). There are many applications of kappa distributions in space plasmas, including Earth’s radiation belts (e.g., Xiao et al., 2009), ring current (e.g., Pisarenko et al., 2002), ionosphere (e.g., Ogawara et al., 2017), plasma sheet (e.g., Kletzing et al., 2003), magnetosheath (e.g., Formisano et al., 1973), solar wind (e.g., Maksimovic et al., 1997), as well as the magnetospheres of the giant planets including Jupiter (e.g., Collier & Hamilton, 1995), Saturn (e.g., Livi et al., 2014), Uranus (e.g., Mauk et al., 1987), and Neptune (e.g., Krimigis et al., 1989). The monograph edited by Livadiotis (2017) contains an extensive list of further applications of kappa distributions. Part 1 of this monograph also provides a detailed account of the statistical origin of kappa distributions in the context of nonextensive statistical mechanics, including, in particular, discussion of the nonextensive Tsallis entropy (Tsallis, 1988) and its relation to the classical Boltzmann-Gibbs entropy. The thermodynamic origin of kappa distributions is addressed by Livadiotis (2018).

Many studies have been carried out on kinetic waves in plasmas described by kappa distributions, for example, see Vinas et al. (2017) and references therein. Probably the first authors to treat kinetic waves in plasmas with a non-Maxwellian high-energy tail were Abraham-Shrauner and Feldman (1977a, 1977b) who analyzed whistler mode and electromagnetic ion cyclotron waves in the solar wind. In the linear theory of waves in a hot plasma described by a Maxwellian distribution, the dispersion equation involves the Fried...
and Conte (1961) plasma dispersion function $Z$. As a natural analog to $Z$, a new special function $Z_{\kappa}$, called the modified plasma dispersion function (MPDF), was introduced by Summers and Thorne (1991) to describe waves in a hot plasma modeled by a kappa distribution (see also Summers & Thorne, 1992; Summers et al., 1994, 1996; Thorne & Summers, 1991). Various plasma microinstabilities have been analyzed using MPDF (or extensions, e.g., Mace & Hellberg, 1995) including the whistler (electron cyclotron) instability (e.g., Mace & Sydora, 2010; Summers & Thorne, 1992; Vinas et al., 2015), electromagnetic ion cyclotron instability (e.g., Cao et al., 2020; Chaston et al., 1997; Mace et al., 2011; Xue et al., 1993, 1996a, 1996b),

\[
\begin{align*}
\text{Left) Kappa distributions } & f_{\kappa}(p) = \frac{N_{\kappa}}{\pi^{3/2} \theta^3} \frac{\Gamma(\kappa + 1)}{\kappa^{3/2}} \frac{1 + \frac{p_{\perp}^2}{\kappa \theta_\perp^2}}{\kappa^{3/2} \Gamma(\kappa - 1/2)}^{-(\kappa + 1)} \\
\text{functions of the normalized momentum } & p/\theta, \text{ together with the Maxwellian distribution } f_{\text{Max}}(p) = \frac{N_{\kappa}}{\pi^{3/2} \theta^3} e^{-p^2/\theta^2}, \\
\text{corresponding to } & \kappa = \infty. \ (Right) Contours of constant phase space density for the bi-kappa distribution (1) for } \kappa = 2 \text{ (top) and the corresponding bi-Maxwellian distribution (bottom).}
\end{align*}
\]

![Figure 1](image.png)

**Figure 1.** (Left) Kappa distributions $f_{\kappa}(p) = \frac{N_{\kappa}}{\pi^{3/2} \theta^3} \frac{\Gamma(\kappa + 1)}{\kappa^{3/2} \Gamma(\kappa - 1/2)} \left(1 + \frac{p_{\perp}^2}{\kappa \theta_\perp^2}\right)^{-(\kappa + 1)}$ for the indicated $\kappa$-values, as functions of the normalized momentum $p/\theta$, together with the Maxwellian distribution $f_{\text{Max}}(p) = \frac{N_{\kappa}}{\pi^{3/2} \theta^3} e^{-p^2/\theta^2}$, corresponding to $\kappa = \infty$. (Right) Contours of constant phase space density for the bi-kappa distribution (1) for $\kappa = 2$ (top) and the corresponding bi-Maxwellian distribution (bottom).

| $f(p_\parallel, p_\perp)$ | $T_\parallel$ | $T_\perp$ | $A_T$ |
|--------------------------|-------------|-------------|------|
| 1. Bi-kappa | | | |
| $\frac{N_{\kappa}}{\pi^{3/2} \theta^3} \frac{\Gamma(\kappa + 1)}{\kappa^{3/2} \Gamma(\kappa - 1/2)} \left(1 + \frac{p_{\perp}^2}{\kappa \theta_\perp^2}\right)^{-(\kappa + 1)}$ | $\frac{\theta_\parallel^2}{2 \kappa - 3 \, m_e}$ | $\frac{\theta_\parallel^2}{2 \kappa - 3 \, m_e}$ | $\frac{\theta_\parallel^2}{\theta_\perp^2}$ |
| 2. Bi-Maxwellian | | | |
| $\frac{N_{\kappa}}{\pi^{3/2} \theta^3} \exp\left(-\frac{p_{\perp}^2}{\theta_\perp^2}\right)$ | $\frac{\theta_\parallel^2}{(2m_e)}$ | $\frac{\theta_\parallel^2}{(2m_e)}$ | $\frac{\theta_\parallel^2}{\theta_\perp^2}$ |
| 3. Kappa loss cone | | | |
| $\frac{N_{\kappa}}{\pi^{3/2} \theta^3} \left(\frac{p_{\perp}}{\theta_\perp}\right)^2 \frac{\Gamma(\kappa + \sigma + 1)}{\kappa^{3/2} \Gamma(\kappa - 1/2) \Gamma(\kappa + 1)} \left(1 + \frac{p_{\perp}^2}{\kappa \theta_\perp^2}\right)^{-(\kappa + \sigma + 1)}$ | $\frac{\theta_\parallel^2}{2 \kappa - 3 \, m_e}$ | $\frac{(\sigma + 1) \theta_\parallel^2}{2 \kappa - 3 \, m_e}$ | $\frac{\theta_\parallel^2}{\theta_\perp^2}$ |
| 4. Dory et al. (1965) loss cone | | | |
| $\frac{N_{\kappa}}{\pi^{3/2} \theta^3} \frac{\Gamma(\kappa + 1)}{\kappa^{3/2} \Gamma(\kappa + 1)} \left(\frac{p_{\perp}}{\theta_\perp}\right)^{2\sigma} \exp\left(-\frac{p_{\perp}^2}{\theta_\perp^2}ight)$ | $\frac{\theta_\parallel^2}{(2m_e)}$ | $\frac{(\sigma + 1) \theta_\parallel^2}{(2m_e)}$ | $\frac{(\sigma + 1) \theta_\parallel^2}{\theta_\perp^2}$ |
electrostatic ion cyclotron instability (e.g., Basu & Grossbard, 2011), electron acoustic instability (e.g., Balu-uku et al., 2011), and ion acoustic instability (e.g., Meng et al., 1992; Summers & Thorne, 1992). Sugiyama et al. (2015) analyzed electromagnetic ion cyclotron waves in a kappa-Maxwellian plasma and developed a full dispersion solver Kyoto University Plasma Dispersion Analysis Package using an extended version of MPDF (Hellberg & Mace, 2002). Astfalk et al. (2015) studied electromagnetic ion cyclotron waves propagating in a bi-kappa plasma using MPDF and created the dispersion solver Dispersion Solver for Homogeneous Plasmas with Anisotropic Kappa Distributions.

Whistler mode chorus is an electromagnetic emission typically comprising narrow-band rising-tone elements, in the frequency range 0.1–0.8|Ωe| where |Ωe| is the electron gyrofrequency (Anderson & Kurth, 1989; Aryan et al., 2016; Koons & Roeder, 1990; Li et al., 2016; Meredith et al., 2001, 2003; Santolik et al., 2003, 2004; Tsurutani & Smith, 1974, 1977). Chorus is often observed in two bands, 0.1–0.5|Ωe| and 0.5–0.8|Ωe|, with a gap at 0.5|Ωe| (Burtis & Helliwell, 1976; Meredith et al., 2001, 2003; Tsurutani & Smith, 1974). Chor- us waves are observed outside the plasmapause near the geomagnetic equator and generated by cyclotron resonance with anisotropic hot (10–100 keV) electrons injected from the plasmasheet (Anderson & Mæ - da, 1977; Kennel & Petschek, 1966; LeDocq et al., 1998; Li et al., 2008, 2009; Santolik et al., 2003).

In this study, we examine the effects of kappa distributions on the whistler mode instability. In a relativistic regime, we assume that whistler mode waves are generated by a small hot electron kappa distribution in the presence of a large cold background plasma. Specifically, we examine how the bi-kappa distribution (1) (introduced in Section 2) and kappa loss-cone distribution (46) (introduced in Section 5) influence both the linear and nonlinear phases of whistler mode wave growth. Our study is organized as follows. In Section 2, we calculate the linear growth rate of whistler mode waves generated by a bi-kappa distribution and analyze how the growth rate depends on κ. In Section 3, we briefly summarize the nonlinear wave growth theory used in this study. The theory was originally developed by Omura et al. (2008, 2009, 2012). Summers et al. (2012) extended the nonlinear theory to apply to a general distribution of hot electrons and used the bi-Maxwellian, bi-kappa, and Dory-Guest-Harris (DGH) (Dory et al., 1965) distributions as brief examples. The study by Summers et al. (2012) can be regarded as a useful background introduction to the present study. This present study is the first to analyze the effects of kappa distributions on the linear and nonlinear growth of whistler mode waves. In Section 3.1, we obtain the total nonlinear growth rate of whistler mode waves corresponding to an injected hot bi-kappa distribution. In Section 3.2, we present the “chorus equations” which model the nonlinear growth of rising-tone chorus elements. From the chorus equations, we obtain the threshold wave amplitude \( \tilde{B}_w \) required for sustained nonlinear wave growth and illustrate how \( \tilde{B}_w \) depends on κ. In Section 3.3, we introduce the local (temporal) nonlinear growth rate and present numerical solutions of the chorus equations. Furthermore, we investigate the values of κ and electron anisotropy for which the theoretical chorus sweep (chirp) rates lie in the expected range of observed values. In Section 4, we construct a smooth time profile for the whistler mode wave amplitude by matching linear and nonlinear profiles at a particular matching amplitude \( \tilde{B}_w \) which, for a bi-kappa distribution, is found to increase as κ decreases. In Section 5, we investigate how the kappa loss-cone distribution influences the linear and nonlinear growth rates, the solutions to the chorus equations and the chorus sweep rates. In Section 6, we present a brief discussion and summarize our results. Finally, in Section 7, we provide concluding remarks.

2. Linear Wave Growth

We consider whistler-mode waves generated by a small, hot (energetic) anisotropic electron population with number density \( N_h \) in the presence of a large cold background plasma with electron number density \( N_e \) and background neutralizing ions. We assume the waves propagate parallel to a uniform background magnetic field of strength \( B_0 \). We further assume that \( N_h/N_e \ll 1 \) and that the hot electrons are modeled by a (relativistic) bi-kappa momentum distribution given by

\[
f(p_x, p_z) = \frac{N_h}{\pi^{3/2} \theta_0^2 \Gamma(\kappa + 1)} \left(1 + \frac{p_x^2}{\kappa \theta_x^2} + \frac{p_z^2}{\kappa \theta_z^2}\right)^{-\kappa - 1/2}
\]  

(1)
where \( \kappa (>3/2) \) is the spectral index, \( \theta_i \) and \( \theta_\bot \) are the thermal momenta of the hot electrons parallel and perpendicular to the background magnetic field; \( p_i = \gamma m_e v_i \parallel \) and \( p_\bot = \gamma m_e v_i \bot \) are the component electron momenta, where \( v_i \parallel \) and \( v_i \bot \) are the velocity components parallel and perpendicular to the ambient magnetic field, \( v = \sqrt{v_i^2 + v_\bot^2} \) is the electron speed, and \( \Gamma \) is the gamma function. Distribution (1) has been normalized so that 
\[
\int f(p_\parallel, p_\bot) \, dp_\parallel dp_\bot = N_h
\]
where \( d^3p = 2\pi p_\bot dp_\bot dp_\parallel \).

It should be noted that in the limit as the spectral index \( \kappa \to \infty \), the bi-kappa distribution (1) tends to the bi-Maxwellian distribution, specifically distribution (2) listed in Table 1. In the top right panel of Figure 1, we show contours of constant phase space \( f = \text{const} \) for (1) with \( \kappa = 2 \); in the bottom right panel, we show corresponding contours for the bi-Maxwellian distribution. Comparison of these panels shows the enhanced tail in the bi-kappa distribution at larger momentum.

In this study, we use the standard cold-plasma dispersion relation for whistler-mode (electromagnetic R-mode) waves propagating parallel to the assumed uniform background magnetic field, namely,
\[
y^2 = x^2 + \frac{x}{a(1-x)}
\]
where
\[
x = \omega / |\Omega_e|, \quad y = ck / |\Omega_e|.
\]

and
\[
a = |\Omega_e|^2 / \omega_{pe}^2
\]
is a cold-plasma parameter; \( \omega \) is the (real) wave frequency, \( k \) is the (real) wave number, \( |\Omega_e| = eB_0/(m_e c) \) is the (nonrelativistic) electron gyrofrequency, \( \omega_{pe} = (4\pi N_e e^2 / m_e)^{1/2} \) is the electron plasma frequency, and \(-e\) is the electron charge.

Using the analysis of Xiao et al. (1998) and Summers et al. (2012), we find that the (temporal) linear growth rate \( \omega_i \) for field-aligned whistler-mode waves generated by the bi-kappa distribution (1) can be expressed in the form,
\[
\frac{\omega_i}{|\Omega_e|} = 2\sqrt{\pi} \frac{N_h}{N_0} \left[ \frac{m_e}{\theta_\parallel} \right] \frac{(1-x)^2}{1+2ax(1-x)^2} \frac{\Gamma(\kappa+1)}{\kappa^{3/2}\Gamma(\kappa-1/2)} \left\{ I_2(x,y) - x I_1(x,y) \right\},
\]
where
\[
I_1(x,y) = \left( \frac{\kappa+1}{\kappa} \right) \int_0^{x} \frac{z^3 dz}{\Delta_R} \left[ 1 + \frac{(1-\gamma_R x)^2}{\kappa y^2} \left( \frac{m_e}{\theta_\parallel} \right)^2 + \frac{z^2}{\kappa} \right]^{-(\kappa+2)}
\]
and
\[
I_2(x,y) = \left( \frac{\kappa+1}{\kappa} \right) \int_0^{\theta_\parallel} \frac{z^3 dz}{\Delta_R^2} \left[ 1 - \gamma_R x \right] \left[ 1 + \frac{(1-\gamma_R x)^2}{\kappa y^2} \left( \frac{m_e}{\theta_\parallel} \right)^2 + \frac{z^2}{\kappa} \right]^{-(\kappa+2)}
\]
In Equations 6 and 7, \( \gamma_R \) is the resonant value of the Lorentz factor given by
\[ \gamma_R = \frac{-x + y \left( \gamma^2 - x^2 \right) \left( 1 + z^2 \left( \frac{\theta}{m_c} \right)^2 \right) + 1}{\gamma^2 - x^2} \]  
\[ \Delta_R = 1 - \frac{x \gamma_R^2 - 1}{\gamma_R S^2} \]  

is a strictly positive resonant denominator.

Similarly, the linear growth rate for field-aligned whistler-mode waves corresponding to the bi-Maxwellian distribution (2) given in Table 1 can be readily found, and the result is given by Equations A1–A3 in the Appendix. As expected, it can be verified that the limiting form of the growth rate (5) for the bi-kappa distribution as \( \kappa \to \infty \) is precisely the growth rate (Equation A1) for the bi-Maxwellian distribution.

For a given relativistic momentum distribution \( F(p_\parallel, p_\perp) \), we may define the thermal anisotropy \( A_T \) as

\[ A_T = \frac{\left( p_\perp^2 \right)^{1/2}}{2 \left( p_\parallel^2 \right)^{1/2}} - 1 \]

where

\[ \left( p_\parallel^2 \right) = \int p_\parallel^2 F(p_\parallel, p_\perp) d^3p \]

and

\[ \left( p_\perp^2 \right) = \int p_\perp^2 F(p_\parallel, p_\perp) d^3p. \]

Hence, for the bi-kappa distribution (1), from Equations 10–12, we find that \( A_T = \theta / \theta^* - 1 \), as stated in Table 1. For the present study, we define the "effective" electron temperatures \( T_\parallel \) and \( T_\perp \) by the expressions

\[ T_\parallel = \left( p_\parallel^2 \right) / m_e, \quad T_\perp = \left( p_\perp^2 \right) / (2m_e) \]

(even though such definitions are only rigorously valid in the nonrelativistic regime), so that the thermal anisotropy then takes the familiar form \( A_T = T_\perp / T_\parallel - 1 \). In Table 1, we provide the results for \( T_\parallel \), \( T_\perp \), and \( A_T \) for four distributions, namely, the bi-kappa, bi-Maxwellian, kappa loss cone, and Dory et al. (1965) loss-cone distributions. We defer further reference to the loss-cone distributions until Section 5. The value \( \kappa = 2 \) is "iconic" in the literature on kappa distributions as it is the smallest integer satisfying \( \kappa > 3/2 \) which is the condition for the physical validity of temperature (see Table 1), and hence is the integral value of \( \kappa \) representative of the most pronounced high-energy tail.

In the derivation of the linear growth rates in this study, we apply the condition \( N_h / N_e \ll 1 \) since it is required that the hot electron density should make a small contribution to the total electron density. It is further required that the hot electron dynamic pressure should be much less than the background magnetic pressure, namely, the parallel electron beta parameter, defined by

\[ \beta = \frac{8\pi N_h T_h}{B_0^2}, \]
must be small ($\ll 1$). For the bi-kappa distribution (1), making use of the relation
$$\left(\frac{\delta \theta}{\ell_c}\right)^2 = \frac{2}{3} \left(\frac{m_e c}{T_m}\right)^2$$
from Table 1, and the cold-plasma parameter $a$ defined by Equation 4, we find that the condition $\beta \ll 1$
takes the form,
$$\left(\frac{\delta \theta}{\ell_c}\right)^2 = \frac{2}{3} \left(\frac{m_e c}{T_m}\right)^2 \ll 1.3$$
In all calculations in the present study, we set $a = 1/16$ which typifies conditions in Earth’s inner magnetosphere, with $N_0 = 44 \text{ cm}^{-3}$ at $L = 3.9$, assuming a dipole magnetic field. Tang et al. (2014) examined the effects of cold electron number density variation on whistler mode wave growth by varying the parameter $a$.

In the top panels of Figure 2, for each electron anisotropy $A_T = 0.5, 0.9, 2.0$, we show profiles of the linear growth rate (5) for whistler-mode waves generated by the bi-kappa distribution (1) with the hot electron parallel momentum $\theta_\parallel/(m_e c) = 0.248$, for $\kappa = 2, 4, 6, 10$. For comparison, we also show the corresponding linear growth rates Equation A1 in a bi-Maxwellian plasma.

Figure 2. Linear growth rate (5) for whistler mode waves generated by the bi-kappa distribution (1) with $N_0/N_h = 16 \times 10^{-4}$ for various $\kappa$-values and the indicated values of the parallel thermal momentum $\theta_\parallel/(m_e c)$ and anisotropy $A_T = \theta_\parallel^2 / \theta_\perp^2 - 1$. Also shown are the corresponding linear growth rates Equation A1 in a bi-Maxwellian plasma.

In the bottom panels of Figure 2, in which we take $A_T = 0.18, 0.24, 0.6$ and we set $\theta_\parallel/(m_e c) = 0.495$, further support this suggestion. We validify the hypothesis in Figure 3, in which we have constructed in the $(\theta_\parallel/(m_e c), A_T)$-parameter plane a “critical curve” below which the bi-kappa maximum linear growth rate (for $\kappa = 2$) exceeds the bi-Maxwellian maximum linear growth rate and above which the reverse holds. To construct the critical curve in Figure 3, we calculated the maximum linear growth rate corresponding to a large number of
pairs of \((\theta||/m_ec, A_T)\)-values which are indicated by the diamonds shown in the figure. Mace and Sydora (2010), using a hot dispersion relation in a nonrelativistic study, also showed that for a bi-kappa distribution the maximum linear growth rate for whistler-mode waves is enhanced for smaller temperature anisotropies and reduced at larger anisotropies.

In Figure 4, for the indicated values of the anisotropy \(A_T\), we plot the frequency \(\omega|| = \omega / |\Omega_e|\) at which the linear growth rate for whistler-mode waves maximizes, as a function of the spectral index \(\kappa\), for \(\theta||/m_ec = 0.248\) (top panel) and \(\theta||/m_ec = 0.495\) (bottom panel). We see that \(\omega||\) is a weakly increasing monotonic function of \(\kappa\), and, as expected, can significantly increase with increasing values of \(A_T\). As described below in Section 4, both \(\omega||\) and the maximum linear growth rate itself play an important role in the transition to nonlinear wave growth. See also the recent study by Tang and Summers (2019) who show that the generation of chorus waves is controlled by a critical bound on the maximum value of the linear growth rate.

3. Nonlinear Wave Growth

3.1. Total Nonlinear Growth Rate

The nonlinear wave growth theory of Omura et al. (2008, 2009, 2012) for magnetospheric chorus emissions was modified by Summers et al. (2012) to apply to an arbitrary distribution of hot electrons. Using the more general formulation of Summers et al. (2012), we find that for field-aligned whistler-mode waves of frequency \(\omega(t)\) and magnetic field amplitude \(B_\omega(t)\), the total nonlinear growth rate \(\Gamma_N\) can be expressed by the relation,

\[
\frac{dB_\omega}{dt} = \Gamma_N B_\omega, \tag{16}
\]

where

\[
\Gamma_N = \sqrt{\xi/\gamma^2} \frac{|\Omega_e|}{\Omega_e} \left[ \frac{B_\omega}{B_0} \right]^{1/2} \left( \frac{V_\omega}{c} \right)^{1/2} \left( \frac{|\Omega_e|}{\Omega} \right)^{1/2} \left( \frac{m_e^2 c^2 G}{aN_0} \right)^{1/2}, \tag{17}
\]

\[
\xi^2 = \frac{\omega E}{\omega_{pe}^2}, \tag{18}
\]

\[
\gamma^2 = \frac{1}{1 + \xi^2}, \tag{19}
\]

and

\[
\frac{V_\omega}{c} = \frac{\xi}{\gamma^2} \left[ \frac{\gamma^2}{2(\Omega_e - \omega)} \right]^{-1}, \tag{20}
\]

is the (normalized) wave group speed;
\[ \tilde{\gamma}_R = \left[ 1 - \left( \frac{\tilde{V}_R}{c} \right)^2 - \left( \frac{V_{\perp 0}}{c} \right)^2 \right]^{-1/2}, \]  

(21)

is the resonant Lorentz factor,

\[ \frac{\tilde{V}_R}{c} = \chi \left( 1 - \frac{|\Omega|}{\tilde{\gamma} R \theta} \right), \]  

(22)

is the (normalized) resonant parallel particle velocity, and \( V_{\perp 0} \) is the average perpendicular particle velocity (assumed constant). In Equation 17, the dimensionless parameter \( Q \) refers to the depth of the electron hole within which wave trapping takes place, \( a \) is the cold-plasma parameter defined by Equation 4, and the quantity \( \bar{G} \) represents the average value of the hot electron distribution trapped by the wave. Following Omura et al. (2009), we fix \( Q = 0.5 \).

Setting

\[ p_{\perp 0} = \gamma_0 m_e V_{\perp 0}, \quad \gamma_0 = \left( 1 - \frac{V_{\perp 0}^2}{c^2} \right)^{-1/2} \]  

(23)

and following the method used by Summers et al. (2012) to determine the trapped distribution we find

\[ p_{\perp 0} = \frac{\sqrt{\pi}}{2} \kappa^{1/2} \Gamma \left( \kappa - \frac{1}{2} \right) \theta_\perp. \]  

(24)

and

\[ \bar{G} = \frac{N_h}{\pi^2 \theta_\parallel \theta_\perp} \frac{\Gamma \left( \kappa + 1 \right) \Gamma \left( \kappa - \frac{1}{2} \right)}{\kappa \Gamma \left( \kappa - 1 \right) \Gamma \left( \kappa + \frac{1}{2} \right)} \left( 1 + \frac{\tilde{p}_R}{\kappa \theta_\parallel^2} \right)^{\kappa + 1}. \]  

(25)

where \( \tilde{p}_R = \tilde{\gamma}_R m_e \tilde{V}_R \).

It should be noted that Equations 23 and 24 imply a relationship between \( \theta_\parallel / (m_e c) \) and \( V_{\perp 0} / c \) which must be satisfied in the specification of these parameters.

The form of \( \bar{G} \) corresponding to the bi-Maxwellian distribution (2) in Table 1 can be obtained by letting \( \kappa \rightarrow \infty \) in Equation 25, the result being

\[ \bar{G} = \frac{N_h}{\pi^2 \theta_\parallel \theta_\perp} \exp \left( -\frac{\tilde{p}_R}{\kappa \theta_\parallel^2} \right). \]  

(26)

### 3.2. Chorus Equations

The nonlinear growth of a rising-tone chorus wave element can be described by a pair of nonlinear, differential “chorus equations,” originally developed by Omura et al. (2009). For a wave of normalized magnetic field amplitude \( \tilde{B}_w = B_w(t) / B_0 \) and normalized frequency \( \tilde{\omega} = \omega(t) / |\Omega| \), Summers et al. (2012, 2013) expressed the chorus equations in the form,

\[ \frac{\partial \tilde{B}_w}{\partial \tilde{t}} = \Gamma \tilde{B}_w - \frac{s_x}{s_0} \frac{V_w}{c} \tilde{a} \]  

(27)
\[
\frac{\partial \tilde{\omega}}{\partial \tilde{t}} = \frac{2\eta_0}{s_0} \frac{\partial \tilde{B}_n}{\partial \tilde{t}},
\]
(28)

where \( \tilde{t} = |\Omega|/\tau; \Gamma_N \) is the nonlinear growth rate given by Equation 17;

\[
s_0 = \frac{X}{\xi} \frac{V_{1,0}}{c},
\]
(29)

\[
s_1 = \tilde{\gamma} \left( 1 - \frac{\tilde{\gamma}}{\gamma} \right)^2,
\]
(30)

\[
s_2 = \frac{1}{2\xi \chi} \tilde{\gamma} \left( \frac{V_{1,0}}{c} \right)^2 - \left[ 2 + \chi \left( 1 - \tilde{\gamma} \right) \left( 1 - \tilde{\gamma} \right) \frac{V_{1,0}^{\gamma}}{c^2} \right],
\]
(31)

and

\[
\frac{V_{1,0}}{c} = \tilde{\chi} \xi
\]
(32)

is the (normalized) wave phase speed.

Equations 27 and 28 describe the generation of a chorus wave element at the equator of an assumed dipole-like magnetic field of the form,

\[
B = B_0 \left[ 1 + \tilde{a} \left( \frac{\Omega h}{c} \right)^2 \right],
\]
(33)

where \( h \) is the distance measured along the magnetic field line from the magnetic equator, and

\[
\tilde{a} = \frac{4.5}{LR_e} \left( \frac{c}{\Omega_e} \right)^2,
\]
(34)

is the dimensionless parameter also appearing in Equation 27; \( R_e \) is the Earth’s radius, and \( L \) denotes magnetic shell. We take \( \tilde{a} = 9.8 \times 10^{-7} \) which corresponds to \( L = 3.9 \) in a dipole field.

For chorus waves to undergo self-sustaining nonlinear growth and to propagate to higher latitudes, a threshold condition for the wave amplitude must be satisfied. The condition is

\[
\tilde{B}_n > \tilde{B}_{th}
\]

where \( \tilde{B}_{th} = B_{th} / B_0 \) is the normalized threshold amplitude, obtained by setting the right-hand-side of Equation 27 to zero. We find that

\[
\tilde{B}_{th} = \frac{25}{2} \frac{\xi \gamma}{\chi^2 \omega_0} \left( \frac{\tilde{a} \gamma}{Q} \right)^2 \left( \frac{c}{V_{1,0}} \right)^7 \frac{a^2 N_0^2}{\left( m_e c^2 G \right)^2}.
\]
(35)

The wave Equations 27 and 28 hold for lower band chorus for frequencies in the range, 0.1 ≤ \( \tilde{\omega} \) ≤ 0.5. We treat the solution of Equations 27 and 28 as an initial-value problem for \( \tilde{B}_n \tilde{t} \) and \( \tilde{\omega} \tilde{t} \). Corresponding to a chosen initial frequency \( \tilde{\omega_0}(0) = \tilde{\omega_0} \), we determine the value of the threshold wave amplitude \( \tilde{B}_{th} \) given by Equation 35. The initial wave amplitude \( \tilde{B}_n(0) = B_{th}(0) / B_0 \) must then be chosen to exceed \( \tilde{B}_{th} \). The solution of Equations 27 and 28 readily follows by routine numerical integration.

In the top four panels of Figure 5, we set \( \theta/(m_e c) = 0.248, N_e/N_0 = 16 \times 10^{-4} \), and plot the threshold wave amplitude \( \tilde{B}_{th} / B_0 = \tilde{B}_{th} \) given by Equation 35 as a function of frequency for the anisotropies \( A_T = 0.5, 1.5, \ldots \)
For each chosen anisotropy, we take the spectral indices $\kappa = 2, 4, 6, 10$ and we also show the bi-Maxwellian case ($\kappa = \infty$). The bottom four panels of Figure 5 have similar structure to the top panels except that we take $\theta_{\parallel}/(m_{e}c) = 0.495$. The influence of $\kappa$ on the threshold wave amplitude can range from minor to significant dependent on the wave frequency and also the values of the parameters $A_{T}$ and $\theta_{\parallel}/(m_{e}c)$. At lower electron temperature and lower anisotropy (panel [a], $\theta_{\parallel}/(m_{e}c) = 0.248, A_{T} = 0.5$), the threshold wave amplitude for $\kappa = 2$ is significantly lower than the corresponding bi-Maxwellian value, whereas at higher electron temperature and higher anisotropy (panel [h], $\theta_{\parallel}/(m_{e}c) = 0.495, A_{T} = 8.0$), the reverse is true over most of the frequency range. Detailed examination of panels (a–h) also reveals, for instance, that the threshold wave amplitude can be sensitive to changes in the electron temperature, for a given anisotropy, or to changes in the anisotropy, for a given electron temperature.

### 3.3. Local Nonlinear Growth Rate

The total nonlinear growth rate $\Gamma_N$ is defined by Equation 16 in which $d/dt = \partial/\partial t + V_{g}\partial/\partial h$ is the total derivative, where $V_{g}$ is the group speed and $h$ measures the distance along the magnetic field line from the magnetic equator. It is useful (Summers et al., 2012) to define the local nonlinear growth rate $\gamma_{N}$ by the relation

$$\frac{\partial B_{w}}{\partial t} = \gamma_{N} B_{w}. \tag{36}$$

Then by Equations 27 and 36, we find

$$\gamma_{N} = \Gamma_{N} - 5 \frac{\Omega_{e}}{B_{w}} \frac{s_{2}}{s_{0}} \frac{V_{g}}{c} \frac{\tilde{a}}{\tilde{\omega}} \tag{37}$$

The linear growth rate $\omega_{i}$ derived in Section 2 and given by Equation 5 satisfies...
Thus, comparing Equations 36 and 38, we observe that the local growth rate $\gamma_N$ in the nonlinear regime serves as an analog to the temporal growth rate $\omega_i$ in the linear regime. By making explicit use of these growth rates, in Section 4, we show how wave growth in the linear and nonlinear regimes can be smoothly linked. Prior to addressing this transition from the linear to the nonlinear regimes, we examine nonlinear wave growth as determined by solutions to the chorus Equations 27 and 28.

Figure 6. Corresponding to the bi-kappa distribution, solutions for the wave amplitude $B_w/B_0$ and frequency $\omega/|\Omega_e|$ as functions of time $|\Omega_e|t$, obtained from the chorus Equations 27 and 28, with $\theta_{\parallel}/(\eta_m c) = 0.248$ and $N_h/N_0 = 16 \times 10^{-4}$, for the indicated values of $\kappa$ and $A_T$. Also shown are the corresponding solutions for the total nonlinear growth rate $\Gamma_N/|\Omega_e|$ given by Equations 17 and 25, and the local nonlinear growth rate $\gamma_N/|\Omega_e|$ given by Equation 37. For comparison, in all panels solutions corresponding to the bi-Maxwellian distribution are also given.

\[ \frac{\partial \psi}{\partial t} = \omega_i \psi \]  

(38)
We solve Equations 27 and 28 subject to the initial conditions

\[ \omega(0) = \omega_m \text{ is the frequency at which the linear growth rate maximizes (for given values of } A_T, \theta_{\parallel} / (m_ec), \kappa), \text{ and } B_w(0) \text{ slightly exceeds the threshold wave amplitude } B_{th} \text{ at the frequency } \omega(0) = \omega_m. \]

In Figure 6, we show time profiles for the wave magnetic field \( B_w \) and frequency \( \omega \) obtained from Equations 27 and 28 in the case \( \theta_{\parallel} / (m_ec) = 0.248, N_0 / N_0 = 16 \times 10^{-4} \) for electron anisotropies \( A_T = 1.00, 2.00, 8.00 \). For each anisotropy, we set the spectral indices \( \kappa = 2, 3, 4, 6, 10 \), and also show the bi-Maxwellian case (\( \kappa = \infty \)) for comparison. Furthermore, we plot the corresponding time profiles for the total nonlinear growth rate \( \Gamma_N \) and the temporal nonlinear growth rate \( \gamma_N \). Figure 7 has identical format to Figure 6 except that we set \( \theta_{\parallel} / (m_ec) = 0.495 \).

**Figure 7.** As specified in Figure 6, except that here \( \theta_{\parallel} / (m_ec) = 0.495 \).
In both Figures 6 and 7, for all anisotropies, the threshold wave amplitude \( B_{th}(\sim B_{th}(0)) \) in the bi-Maxwellian case \((\kappa = \infty)\) is less than that for \( \kappa = 2 \). This follows because \( B_{th} \) is evaluated at \( \omega = \omega_m \). For, from Figure 4 we see that \( \omega_m \) increases with \( \kappa \), at any \( \Lambda_T \), as a result of which, from Figure 5, we find that at each anisotropy \( (B_{th})_{\kappa=2} > (B_{th})_{\kappa=\infty} \) since \( (\omega_m)_{\kappa=2} < (\omega_m)_{\kappa=\infty} \). Regarding the effects of an increase in anisotropy on \( B_{th} \) at a given \( \kappa \) value, we observe that in Figure 6 (in which \( \bar{\omega}_f / (m_c c) = 0.248 \) \( B_{th} \) monotonically decreases with an increase in \( \Lambda_T \), but that in Figure 7 (with \( \bar{\omega}_f / (m_c c) = 0.495 \) ), this is not the case. Again, this can be explained by combining the results in Figure 4 for \( \omega_m \) with the frequency profiles for \( B_{th} \), shown in Figure 5.

The time profiles for frequency \( \omega \) shown in Figures 6 and 7 characterize rising-tone forms typical of whistler mode chorus elements. We now examine the sweep rate (chirp rate) \( \partial \omega / \partial t \) for \( \kappa = 2 \) in comparison with that for the bi-Maxwellian \((\kappa = \infty)\). Sweep rates are of considerable interest in the literature on chorus waves (e.g., Cully et al., 2011; Macusova et al., 2010; Omura et al., 2009; Tao et al., 2012). In Figure 8, we set \( N_0 / N_0 = 30 \times 10^{-4} \) and plot \( \partial \omega / \partial t \) (kHz/sec), where \( f = \omega / 2\pi \), against wave amplitude \( B_w \) (pT) in the top panel for \( \bar{\omega}_f / (m_c c) = 0.248 \) and in the bottom panel for \( \bar{\omega}_f / (m_c c) = 0.248 \), for anisotropies \( \Lambda_T = 1, 2, 3, 4, 5, 6, 7, 8 \). Corresponding to each anisotropy, in Figure 8, we have plotted the results for \( \kappa = 2 \) and the bi-Maxwellian \((\kappa = \infty)\) (the exceptions being the solutions for \( \kappa = 2 \) which are outside the box). We have chosen the parameter box \( 0 < \partial \omega / \partial t < 10 \) kHz/sec, \( 0 < B_w < 100 \) pT as typical of observed chorus elements. Indeed, we find the theoretical sweep rate solutions to be comparable with observed sweep rate values (e.g., Cully et al., 2011; Macusova et al., 2010). However, we should point out that chorus sweep rates depend on many parameters, and hence direct comparison between theory and observations is challenging. We observe that in most cases, the sweep rates for \( \kappa = 2 \) slightly exceed the corresponding bi-Maxwellian rates. As well, the upper and lower panels imply that an increase in parallel electron temperature tends to favor chorus elements with higher wave amplitudes. We provide in Figure S1 in the Supporting Information, for \( \bar{\omega}_f / (m_c c) = 0.248 \) and the cases \( \kappa = 2 \) and \( \kappa = \infty \), the regions of \((N_0/N_0, \Lambda_T)\)-parameter space in which solutions of the chorus equations yield sweep rates in the typical observed range of \( \partial \omega / \partial t < 10 \) (kHz/s). This figure provides a guide to the parameter space over which the nonlinear theory presented in this study can be tested against observed sweep rates. Generally, we see that sufficiently large values of \( N_0 / N_0 \) are required for theoretical sweep rates to be in the range of observed values.

Since, from Equation 25, \( \bar{G} \propto N_0 \), then from Equation 35 it follows that \( \bar{B}_{th} \propto \left( N_0 / N_0 \right)^2 \). Thus, changes in the value of \( N_0 / N_0 \) can have significant influence on the nonlinear growth of whistler-mode waves. In Figure S2, in the Supporting Information, for the case \( \kappa = 2, \Lambda_T = 2.0 \), and \( \bar{\omega}_f / (m_c c) = 0.248 \), we plot solutions of Equations 27 and 28 for \( B_{th}, \alpha, \Gamma_N \), and \( \gamma_N \) for the normalized hot electron densities \( N_0 / N_0 = 5 \times 10^{-4}, 10 \times 10^{-4}, 15 \times 10^{-4}, 20 \times 10^{-4}, \) and \( 40 \times 10^{-4} \). The figure confirms the expected decrease in \( \bar{B}_{th} \sim \bar{B}_{th}(0) \) as \( N_0 / N_0 \) increases and also the decrease in the initial sweep rate \( \partial \omega / \partial t \) as \( N_0 / N_0 \) increases.

4. **Linear-Nonlinear Matching of Wave Amplitude Profiles**

As proposed by Summers et al. (2013), we assume that linear wave growth smoothly transitions to nonlinear wave growth when the local nonlinear growth rate equals the maximum value of the linear growth rate, that is,

\[ \gamma_N = \max \{ \omega_i \}, \]  

(39)
We suppose that Equation 39 holds at a specific “matching” wave amplitude $\tilde{B}_M = B_M / B_0$. Then using expression (Equation 37) (with Equation 17) for $\gamma_n$, and result (Equation 5) for $\omega$, we find from Equation 39 that $\tilde{B}_M$ satisfies the equation

$$k_1 \left( \frac{N_b}{N_0} \right) \tilde{B}_M^{1/2} - k_2 = k_3 \left( \frac{N_b}{N_0} \right) \tilde{B}_M,$$  \tag{40}$$

where

$$k_1 = \sqrt{2\pi} \chi^{3/2} \left[ \frac{2}{\kappa R} \right]^{1/2} \frac{1}{\omega_i^{1/2}} \left( \frac{V_e}{c} \right) \left( \frac{V_i}{c} \right)^{5/2} \frac{1}{\pi^2} \left( \frac{m_e}{\theta_i} \right) \left( \frac{m_i}{\theta_i} \right) \left[ \Gamma \left( \kappa + 1 \right) \Gamma \left( \kappa - \frac{1}{2} \right) \right] \left[ \frac{\Gamma \left( \kappa + 1 / 2 \right)}{\Gamma \left( \kappa - \frac{1}{2} \right)} \right] \frac{1}{\kappa \Gamma \left( \kappa + 1 / 2 \right)} \left( 1 + \frac{\rho \gamma_n}{\kappa \theta_i^2} \right)^{-1},$$  \tag{41}$$

and

$$k_2 = 5 \frac{s_2}{s_0} \left( \frac{V_e}{c} \right) \left( \frac{\tilde{a}}{\omega} \right),$$  \tag{42}$$

and

$$k_3 = 2 \sqrt{2\pi} \left( \frac{m_e}{\theta_i} \right)^{1/2} \left( \frac{\Gamma \left( \kappa + 1 \right)}{\Gamma \left( \kappa - \frac{1}{2} \right)} \right)^{1/2} \left( \frac{\Gamma \left( \kappa + 1 / 2 \right)}{\Gamma \left( \kappa - \frac{1}{2} \right)} \right) \left[ \frac{1}{1 + 2ax \left( 1 - x \right)^2} \right] \left[ \frac{I_2 \left( x, y \right)}{y} - x I_1 \left( x, y \right) \right].$$  \tag{43}$$

We evaluate the parameters $k_1$, $k_2$, and $k_3$ at $\tilde{\omega} = \tilde{\omega}_m = \omega_m / \left[ \Omega_i \right]$, where $\omega_m$ is the wave frequency at which the linear growth rate maximizes. Here, we regard $k_1$, $k_2$, $k_3$, and $\tilde{\omega}_m$ as functions of $\tilde{\omega} / (m_e c), A_T$, and the cold-plasma parameter $a$. Taking the lower root of Equation 40, we find that the matching wave amplitude $\tilde{B}_M$ can be expressed as

$$\tilde{B}_M = \frac{1}{4} \left( \frac{k_1}{k_3} \right)^2 \left[ 1 - \left[ 1 - \frac{4k_2k_3}{k_1^2} \left( \frac{N_b}{N_0} \right) \right]^{1/2} \right]^2.$$  \tag{44}$$

From Equation 44, we see that $\tilde{B}_M$ can be determined, so that linear-nonlinear matching is possible, only if

$$\frac{N_b}{N_0} > \frac{4k_2k_3}{k_1^2}.$$  \tag{45}$$

In practice, for a case in which values of the parameters $\tilde{\omega} / (m_e c)$ and $a$ are given, Equation 45 is readily satisfied by taking sufficiently large values of $N_b / N_0$. Condition (Equation 39) ensures that the matching wave amplitude $\tilde{B}_M$ is greater than the threshold wave amplitude $\tilde{B}_m$ given by Equation 35.

For given values of $\tilde{\omega} / (m_e c)$ and $a$, the linear-nonlinear matching region specified by Equation 45 is bounded by the curve $\frac{N_b}{N_0} = 4k_2k_3 / k_1^2$ in $(N_b / N_0, A_T)$-space. We set $a = 1/16$ and plot the boundary curves for $\kappa = 2, 4, 6, 10, \infty$ in each of the cases $\tilde{\omega} / (m_e c) = 0.248$ (Figure 9a) and $\tilde{\omega} / (m_e c) = 0.495$ (Figure 9b). From Figure 9, we see that the matching region, which is the region of $(N_b / N_0, A_T)$-space to the right of the boundary curve for a given $\kappa$-value, maximizes for $\kappa = \infty$ (the bi-Maxwellian case) and generally diminishes as $\kappa$ decreases from $\kappa = \infty$ toward $\kappa = 2$.

We carry out the procedure of linear-nonlinear matching of the wave amplitudes for the case $N_b / N_0 = 16 \times 10^{-4}, A_T = 2.5$, and $\tilde{\omega} / (m_e c) = 0.248$. The matching profiles are shown in Figure 10a for the spectral indices $\kappa = 2, 3, 4, 6, 10, \infty$ (bi-Maxwellian). The nonlinear (solid) profiles for $\tilde{B}_n$ are obtained by solving Equations 27 and 28 subject to the initial condition $\tilde{B}_n(0) = \tilde{B}_M, \tilde{\omega}(0) = \tilde{\omega}_m$; the matching linear (dashed) profiles are determined by solving Equation 38 with $\omega = \omega_m$. In panels (b–d) of Figure 10, we show the time profiles of $\omega, \Gamma_{\omega},$ and $\gamma_\kappa$ corresponding to the nonlinear solutions for $\tilde{B}_n$ shown in Figure 10a. In Figure S3...
in the Supporting Information, we show matching profiles in a similar format to Figure 10 except that there we set \( \frac{N_0}{N_0} = 16 \times 10^{-4} \), \( A_T = 1.25 \), and \( \frac{\theta_\parallel}{(m_e c)} = 0.495 \). We note that in both Figure 10 and Figure S3, the matching wave amplitude \( \tilde{B}_m \) increases as the spectral index \( \kappa \) decreases. In general, we expect that the variation of \( \tilde{B}_m \) with \( \kappa \) will depend subtly on the specification of the parameters \( \frac{N_0}{N_0}, A_T, \frac{\theta_\parallel}{(m_e c)} \), and \( a \).

5. Kappa Loss-Cone Distribution

So far in this study, we have examined whistler-mode waves generated by a hot electron distribution modeled by the bi-kappa distribution (1). We next consider the kappa loss-cone distribution (Summers & Thorne, 1991, 1995; J. F. Tang et al., 2012; Xiao et al., 2006, 2009) given by

\[
f(p_\parallel, p_\perp) = \frac{N_h}{\pi^{3/2} \theta_\perp^2} \left( \frac{p_\perp}{\theta_\perp} \right)^{2\sigma} \frac{\Gamma(\kappa + \sigma + 1)}{\kappa^{\sigma+3/2} \Gamma(\kappa - 1/2) \Gamma(\sigma + 1)} \left( 1 + \frac{p_\perp^2}{\kappa \theta_\perp^2} + \frac{p_\parallel^2}{\kappa \theta_\perp^2} \right)^{-(\kappa + \sigma + 1)}
\]  

(46)

where the parameter \( \sigma > 0 \) is the loss-cone index and \( \kappa > 3/2 \) is the spectral index. The parallel and perpendicular “temperatures” (from Equation 13) in terms of the thermal momenta \( \theta_\parallel \) and \( \theta_\perp \) are given in Table 1, and the anisotropy \( A_T \) defined by Equation 10 corresponding to Equation 46 is

\[
A_T = (\sigma + 1) \frac{\theta_\perp^2}{\theta_\parallel^2} - 1.
\]  

(47)

The kappa loss-cone distribution (46) has several useful features:

(i) When \( \sigma = 0 \), distribution (46) reduces to the bi-kappa distribution (1)

(ii) In the limit as \( \kappa \to \infty \), (46) tends to the well-known Dory et al. (1965) loss-cone distribution, given in Table 1

(iii) Distribution (46) contains the bi-Maxwellian distribution (also given in Table 1) as a special case, obtained by setting \( \sigma = 0 \) and \( \kappa = \infty \) (in either order)

(iv) When \( \theta_\parallel = \theta_\perp = \theta \) and \( p_\perp^2/\theta_\perp^2 \gg 1 \), distribution (46) takes the commonly used form \( f \propto \sin^2 \frac{\sigma}{p_\perp^2 (\kappa + 1)} \), originally introduced by Kennel and Petschek (1966), where \( \alpha = \arctan(p_\perp/\theta_\perp) \) is the particle pitch angle.

The kappa loss-cone distribution (46) provides two sources of free plasma energy, namely the loss-cone property \( (\sigma > 0) \) and the thermal anisotropy \( (\theta_\perp^2 > \theta_\parallel^2) \). Both free energy sources can excite wave growth, and both are represented in the anisotropy Equation 47. The loss-cone region determined by a positive perpendicular momentum gradient, \( \partial f/\partial p_\perp > 0 \), is given by

\[
\frac{p_\perp^2}{\theta_\perp^2} \left( 1 + \frac{p_\parallel^2}{\kappa \theta_\perp^2} \right)^{\frac{\kappa}{\kappa + 1}} \sigma < \frac{\kappa}{\kappa + 1}.
\]  

(48)

Both the loss-cone feature (paucity of particles in the loss cone) and pronounced high-energy tail of the kappa loss-cone distribution \( f \) (given by Equation 46 with \( \sigma = 4 \)) are clearly illustrated in the left and top right panels of Figure 11. The left panel shows perpendicular momentum profiles of \( f \) (for a given parallel momentum); the top right panel shows the
contours \( f(p_\parallel, p_\perp) = \text{constant} \) in momentum space. For comparison, the corresponding results for the Dory et al. (1965) distribution are shown in the left and bottom right panels. We use the value \( \sigma = 4 \) as a commonly adopted reference value for the loss-cone index.

Figure 10. (Left) (a) Matched linear and nonlinear time profiles of the wave amplitude for the bi-kappa distribution (1) for various \( \kappa \)-values and the given values of \( \Theta / (m_e c) \), \( N_h / N_0 \), and \( \Lambda_f \). Also shown are the corresponding profiles for the bi-Maxwellian distribution. (Right) (b)-(d) Time profiles of \( \omega / |\Omega_e| \), \( \Gamma N / |\Omega_e| \), and \( \gamma N / |\Omega_e| \) in the nonlinear regime.

Figure 11. (Left) Kappa loss-cone distributions \( f_{KL} = \left( \frac{\sigma}{\sigma + 1} \right)^{2} \) for \( \sigma = 4 \) and the indicated \( \kappa \)-values, as functions of the normalized perpendicular momentum \( p_\perp / \Theta \), for fixed parallel momentum \( p_\parallel / \Theta = 2 \). Also shown is the Dory et al. (1965) distribution \( f_{DGH} = \left( \frac{\sigma}{\sigma + 1} \right)^{2} \exp \left( -\frac{p_\parallel^2}{\kappa \Theta^2} \right) \) corresponding to \( \kappa = \infty \). (Right) Contours of constant phase space density for the kappa loss-cone distribution (46) with \( \sigma = 4 \), for \( \kappa = 2 \) (top) and the corresponding Dory et al. (1965) distribution (bottom).
By again using the formulation of Xiao et al. (1998) and Summers et al. (2012), we determine the linear growth rate $\omega_i$ for field-aligned whistler-mode waves corresponding to the kappa loss-cone distribution (46) and we provide the result (Equations A4–A6) in the appendix. We note that when the loss-cone index $\sigma = 0$, Equations A4–A6 reduce to results (Equations 5–7) corresponding to the kappa distribution. In the limit as $\kappa \to \infty$, Equations A4–A6 yield the linear growth rate corresponding to the Dory et al. (1965) loss-cone distribution. The latter growth rate is given by Equations A7–A9 in the appendix.

In Figure 12 (left panels) for the kappa loss-cone distribution, we show the variation of the frequency $\omega_m/|\Omega_e|$ at which the linear growth rate maximizes as a function of $\kappa$, for $\sigma = 4$ and the indicated values of anisotropy $A_T$. Left (top) and (bottom) panels correspond to $\sigma = 4$ and 10, respectively. In the top left panel, we set $\sigma = 4$ and in the bottom left panel $\sigma = 10$. Since in the present study nonlinear wave growth is initiated at the threshold wave amplitude $B_{th}$, then clearly $\omega_m$ is a crucial parameter. As we discuss below, the left panels of Figure 12 together with Figures S4 and S5 in the Supporting Information therefore aid in the understanding of Figures 13 and 14. In the limit as $\sigma \to \infty$, Equations A4–A6 yield the linear growth rate corresponding to the Dory et al. (1965) loss-cone distribution. The latter growth rate is given by Equations A7–A9 in the appendix.

In Figure 12 (right panels) for the kappa loss-cone distribution, we show the variation of the frequency $\omega_m/|\Omega_e|$ at which the linear growth rate maximizes as a function of $\sigma$, for anisotropies $A_T = 0.5, 1.0, 2.0, 4.0,$ and 8.0; in the top right panel, we set $\sigma = 4$ and in the bottom right panel $\sigma = 10$. Since in the present study nonlinear wave growth is initiated at the threshold wave amplitude $B_{th}$, then clearly $\omega_m$ is a crucial parameter. We note that the left panels of Figure 12 together with Figures S4 and S5 in the Supporting Information therefore aid in the understanding of Figures 13 and 14. In Figure 12 (right panels), we illustrate how the maximum value of the linear growth rate, $\max(\omega_m/|\Omega_e|)$, for the kappa loss-cone distribution varies with $\sigma$ and $\kappa$ in the case $A_T = 2.0, \theta_{\parallel}/(m_e c) = 0.248$. Broadly, we deduce that the maximum growth rate is a monotonically increasing function of $\kappa$ (for fixed $\sigma$) and is also a monotonically increasing function of $\sigma$ (for fixed $\kappa$). However, for fixed values of $\sigma$, the maximum growth rate does not increase significantly for $\kappa \geq 10$ and for fixed values of $\kappa$, the maximum growth rate does not increase significantly for $\sigma \geq 10$. 

Figure 12. Left (top) For the kappa loss-cone distribution, frequency at which the linear growth rate (Equation A4) of whistler mode waves maximizes as a function of $\kappa$, for $\sigma = 4$ and the indicated values of anisotropy $A_T$. Left (bottom) As left (top) but for $\sigma = 10$. Right (top) Maximum value of the linear growth rate (Equation A4) for whistler mode waves corresponding to the kappa loss-cone distribution as a function of $\kappa$, for $A_T = 2.0$ and the indicated values of $\sigma$. Right (bottom) As right (top) but the maximum value of the linear growth rate as a function of $\sigma$, for the indicated values of $\kappa$. 

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Following a similar method to that used in Section 3.1 to calculate the trapped electron distribution corresponding to the hot bi-kappa distribution (1), we obtain corresponding to the kappa loss-cone distribution the results,

\[
  \lambda_\parallel = \frac{\sigma + \frac{3}{2}}{\Gamma(\sigma + 1)} \Gamma(\frac{\kappa}{2} - 1) \theta_\perp
\]

(49)

and

\[
  \tilde{G} = \frac{N_h}{2\pi^{3/2}\theta_\perp} \Gamma(\sigma + 1) \Gamma(\kappa + \sigma + 1) \Gamma(\frac{\kappa}{2} - 1) \left(1 + \frac{\rho_R^2}{\kappa\theta_\parallel^2}\right)^{\frac{1}{2}}
\]

(50)

For the kappa loss-cone distribution as the chosen hot distribution, it follows that the nonlinear growth rate \( \Gamma_N \), the chorus equations, and the normalized threshold wave amplitude \( \gamma_{th} \) are respectively given by Equations 17, 27, 28 and 35 where \( \tilde{G} \) is given by Equation 50. For the choice of the parameters \( \theta_∥/(m,e) \), \( \theta_⊥/(m,e) \), \( \gamma \), and \( \nu_0 \) for the nonlinear calculations associated with the kappa loss-cone distribution, these parameters must be consistent with Equations 23 and 49 (in the same way that specification of the parameters \( \theta_∥/(m,e) \), \( \theta_⊥/(m,e) \), \( \gamma \), and \( \nu_0/c \) associated with the bi-kappa distribution must be consistent with Equations 23 and 24). Results (Equations 49 and 50) reduce to the respective results (Equations 24 and 25) for the bi-kappa distribution by setting \( \sigma = 0 \).

By letting \( \kappa \to \infty \) in Equations 49 and 50, we obtain the following results for the Dory et al. (1965) loss-cone distribution:

\[
  \lambda_\parallel = \frac{\sigma + \frac{3}{2}}{\Gamma(\sigma + 1)} \theta_\perp
\]

(51)

and

\[
  \tilde{G} = \frac{N_h}{2\pi^{3/2}\theta_\perp} \Gamma(\sigma + 1) \Gamma(\kappa + \sigma + 1) \Gamma(\frac{\kappa}{2} - 1) \exp \left(\frac{-\theta_\perp^2}{\kappa\theta_∥^2}\right).
\]

(52)

In Figure 13 for \( \theta_∥/(m,e) = 0.248 \), \( N_h/N_0 = 16 \times 10^{-4} \), and the case \( \sigma = 4 \), we show solutions for \( B_{\perp}/B_0 \), \( \omega/|\Omega_e| \), \( \Gamma_N/|\Omega_e| \), and \( \gamma_{th}/|\Omega_e| \) obtained from the chorus Equations 27 and 28. We choose \( \kappa = 2, 3, 4, 6, 10 \) and \( \kappa = \infty \) (the DGH case). In the left panels, we set \( A_T = 2.00 \) and in the right panels \( A_T = 8.00 \). It is instructive to compare Figure 13 (for \( \sigma = 4 \)) with the center and right panels of Figure 6 (for \( \sigma = 3 \)). While the corresponding panels of Figures 6 and 13 are similar qualitatively, there are important quantitative differences arising from the introduction of the loss-cone feature. The main differences result from the significant increase in the threshold wave amplitude (when \( \sigma = 4 \)) for the kappa loss-cone profiles (especially for \( \kappa = \infty \)) as compared to the bi-kappa case (for \( \sigma = 0 \)); note that the threshold wave amplitudes are identical to the \( B_{\perp}(0)/B_0 \)-values that can be read off from the top panels of Figure 13. Furthermore, comparing the corresponding \( \omega(t)/|\Omega_e| \) profiles in Figures 6 and 13. We observe the significant increase in sweep...
rates for the kappa loss-cone case in comparison to the bi-kappa case. In fact, for the kappa loss-cone case with $\kappa = 2$, as shown in Figure 13, the sweep rates are unrealistically high.

In Figures S4 and S5 in the Supporting Information, we plot respectively the linear growth rates and threshold wave amplitudes for the kappa loss-cone distribution, for $\kappa = 2, 4, 6, 10 \rightarrow \infty$ (the DGH case), with $A_t = 0.5, 2.0, 4.0, 8.0$ in each of the cases $\sigma = 4$ and $\sigma = 10$. These figures, together with the left panels of Figure 12, provide useful information on the choice of parameters likely to yield realistic whistler-mode wave growth profiles (and in particular realistic sweep rates) for $\sigma = 4, 10$, and the given range of anisotropies. For instance, parameters that imply a particular frequency $\omega_m/|\Omega|$ which subsequently yields a threshold wave amplitude $B_0$ that is unrealistically large, can be discounted. Broadly, for the kappa loss-cone distribution with loss-cone index $\kappa \geq 4$, we find that $\kappa$-values in the range $\kappa \geq 4$ are required for realistic wave profiles, and, even then, larger anisotropies may also be required, for example, see Figure S6 in the Supporting Information.

Typical realistic sweep rate solutions are shown in Figure 14 in which for the case $\theta/(m_e c) = 0.248$, we set $\sigma = 4$, $\kappa = 4$ in the top panel and $\sigma = 10$, $\kappa = 6$ in the bottom panel. As in Figure 8, we choose anisotropy values $A_t = 1, 2, 3, 4, 5, 6, 7, 8$, electron hot-to-cold number density ratio $N_h/N_c$, and $A_t$. Solutions corresponding to the Dory et al. (1965) distribution are shown for comparison.

The results shown in Figures 8 and 14 demonstrate that the nonlinear theory utilized in this study yields theoretical sweep rate solutions that lie in the typically observed range for certain sets of parameter values. We leave as a future project comparison of the theoretical sweep rates with observed sweep rates from actual experimental test cases. It should be noted that to carry out such a detailed comparison, experimental data are required on the parameters $N_h/N_c$, $A_t$, $\theta/(m_e c)$, and $a \approx (\Omega_e \Omega_i^2 / \omega_p^2)$, as well as $\kappa$ and $\sigma$.

6. Discussion and Summary

Whistler mode waves are a key ingredient of magnetospheric physics. Cyclotron resonant electron acceleration by whistler mode chorus can generate relativistic (MeV) electrons in Earth's outer radiation belt (Meredith et al., 2002, 2003; Miyoshi et al., 2003; Jaynes et al., 2015; Katoh et al., 2008; Omura et al., 2015; Su et al., 2014; Summers & Omura, 2007; Summers et al., 1998, 2002, 2007a, 2007b; Thorne et al., 2013). Resonant scattering by whistler mode waves can also induce losses of outer belt electrons via precipitation into the atmosphere (Harid et al., 2014; Hikishima et al., 2010; Lorentzen et al., 2001; O'Brien et al., 2004; Summers et al., 2007a, 2007b). Earth’s dayside diffuse auroral precipitation has been attributed to chorus wave scattering (Ni et al., 2014). Whistler wave activity can limit trapped electron fluxes in planetary magnetospheres (Mauk & Fox, 2010; Schulz & Davidson, 1988; Summers & Shi, 2014; Summers et al., 2009, 2011; R. Tang and Summers, 2012). Since space plasmas are typically described by kappa distributions, our present study is in principle relevant to all of the above applications of the whistler mode instability in magnetospheric physics (and, in fact, potentially beyond to heliophysics and astrophysics). Our conclusions relate broadly to the whistler mode instability as influenced by the kappa index that characterizes the high-energy tail of a typical space plasma distribution. The degree of influence of the kappa distribution...
will depend on the problem under consideration, but in general terms, the kappa index can be expected to influence the rate of electron acceleration (in the “killer electron” problem, for instance), the rate of electron pitch-angle scattering in the diffusive scattering problems, and, of course, the generation process of the whistler mode waves themselves. Observations of whistler mode waves at Jupiter and Saturn (e.g., Hospodarsky et al., 2012; Kurth et al., 2018; Menietti et al., 2014, 2016) provide potential future avenues of research on the influence of kappa distributions on wave-particle interactions at these planets.

In this study, we have examined the influence of kappa distributions on the growth of whistler mode waves in a relativistic regime. Specifically, we analyzed how the bi-kappa distribution (1) and kappa loss-cone distribution (46) influence both the linear and nonlinear growth processes. In fact, this is the first study with the primary aim of analyzing how the kappa index affects the linear and nonlinear growth of whistler mode waves. We summarize our results as follows:

1. The maximum linear growth rate corresponding to the bi-kappa distribution, for spectral index $\kappa = 2$, exceeds that for the bi-Maxwellian distribution for electron anisotropy $A_T$ values less than a critical value. For anisotropy values greater than the critical value, the reverse is true.

2. The threshold wave amplitude for nonlinear growth can depend sensitively on the value of $\kappa$, but also depends crucially on the anisotropy $A_T$, the parallel electron thermal speed $\theta_\parallel/(m_e c)$ and the wave frequency. For instance, for $\theta_\parallel/(m_e c) = 0.248$ and lower anisotropy (say $A_T = 0.5$), the threshold wave amplitudes can be much less than those for the bi-Maxwellian distribution, whereas for $\theta_\parallel/(m_e c) = 0.495$ and higher anisotropy (say $A_T = 8.0$), the threshold amplitude can be much greater than that for the bi-Maxwellian.

3. We solved the nonlinear chorus wave Equations 27 and 28 corresponding to the bi-kappa distribution for $\kappa = 2, 3, 4, 6, 10$ and $\kappa = \infty$ (the bi-Maxwellian limit) in both the cases $\theta_\parallel/(m_e c) = 0.248$ and $\theta_\parallel/(m_e c) = 0.495$. Time profiles were plotted of the corresponding solutions for the wave amplitude $B_\parallel/B_0$, wave frequency $\omega/|\Omega_\parallel|$, total nonlinear growth rate $\Gamma/|\Omega_\parallel|$, and local nonlinear growth rate $\gamma_\parallel/|\Omega_\parallel|$. The solutions typify the characteristic rising-tone nature of whistler mode chorus elements.

4. From solutions to the chorus equations, we plotted the sweep (chirp) rates $\partial f/\partial t$ (kHz/s) against the wave amplitude $B_\parallel$ (pT). For the adopted parameters, we found that generally the sweep rates were within the range of observed values, and that the sweep rates for $\kappa = 2$ slightly exceed the corresponding bi-Maxwellian rates. Since chorus sweep rates depend on many system parameters, however, detailed testing of the theoretical sweep rates against observed experimental data remains a topic of future study.

5. By assuming that the transition from linear wave growth to nonlinear growth occurs when the local nonlinear growth rate equals the maximum value of the linear growth rate, we were able to construct linear-nonlinear matching time profiles of the wave amplitude. In the cases considered, the matching wave amplitude was found to decrease as $\kappa$ decreases, though the matching process in general is expected to depend subtly on the system parameters as well as $\kappa$.

6. Compared to the results for the bi-kappa distribution, consideration of how the kappa loss-cone distribution influences the growth of whistler mode waves is further complicated by the inclusion of the additional parameter $\sigma$, the loss-cone index. We find it useful in this case to examine together the profiles of the linear growth rates and threshold wave amplitudes so as to select a range of parameter values that yield threshold wave amplitudes $B_\parallel/B_0 = B_\parallel(0)/B_0$ that are realistic (in particular not too large). For instance, we find that for the loss-cone index $\sigma \geq 4$, we require $\kappa \geq 4$ to yield realistic wave profiles, and even then, larger anisotropy values may be required. As an illustration, comparing the sweep rates corresponding to the kappa loss-cone and the DGH ($\kappa = \infty$) distributions in both the cases ($\sigma = 4, \kappa = 4$) and ($\sigma = 10, \kappa = 6$), we find that the kappa loss-cone solutions exist only for larger values of anisotropy and smaller values of wave amplitude, while the DGH ($\kappa = \infty$) solutions exist for wider ranges of both anisotropy and wave amplitude.

Additional figures associated with whistler mode waves generated by kappa loss-cone distributions are provided in the Supporting Information.

The work carried out in this study and the results obtained have been summarized in this section and the Abstract. However, to succinctly capture highlights of the study, we briefly review a particular subset of the figures, namely Figures 3, 5, 6, 10 and 13. Figure 3 shows how the relative magnitudes of the linear growth rates of whistler mode waves corresponding to the bi-kappa and bi-Maxwellian distributions depend on the anisotropy $A_T$ and the parallel thermal speed $\theta_\parallel/(m_e c)$. In fact, throughout the study, the degree of in-
fluence of kappa distributions on the growth of whistler mode waves is seen to depend on the values of \( A_T \) and \( \theta_\parallel / (m_e c) \). In Figure 5, we show that the threshold wave amplitude for nonlinear growth corresponding to the bi-kappa distribution can be much less than or greater than that for a bi-Maxwellian distribution, dependent on \( A_T \) and \( \theta_\parallel / (m_e c) \) (compare the top left and bottom right panels of Figure 5). The characteristic rising-tone elements of chorus waves are modeled in Figure 6 (top panels), and notably the threshold wave amplitude is seen to be much less for the bi-Maxwellian distribution than for the bi-kappa distribution with \( \kappa = 2 \). Correspondingly, the matched linear-nonlinear amplitude profiles illustrated in Figure 10 are more readily generated for the bi-Maxwellian than for the bi-kappa distribution. The influence of the spectral index \( \kappa \) on the growth of whistler mode waves in a kappa loss-cone distribution can be strongly influenced by the value of the loss-cone index \( \sigma \). Figure 13 illustrates the growth of chorus wave elements for a kappa loss-cone distribution and shows that the elements are more likely to be generated (namely with a lower threshold wave amplitude) for \( \kappa = \infty \), the DGH distribution, than for a distribution with a pronounced high-energy tail (\( \kappa = 2 \)).

7. Concluding Remarks

The total number of publications related to kappa (or Lorentzian) distributions has increased exponentially over the last 20 years (Livadiotis, 2017). Nevertheless, the importance of employing kappa distributions in problems in magnetospheric physics, for instance, has not yet been fully recognized. We find that analyzing the influence of kappa distributions on the standard whistler mode instability introduces nuance and complexity. For example, as we describe above, the linear growth rates do not simply increase with an increasingly enhanced high-energy tail. Furthermore, the threshold wave amplitude for nonlinear growth and consequently the growth of chorus elements themselves depend subtly on the spectral index \( \kappa \). The bi-kappa and kappa loss-cone distributions yield a rich array of solutions which are dependent on the parameters \( N_i / N_0 \), \( A_T \), \( \theta_\parallel / (m_e c) \), and \( a (= \Omega_x c |\Omega|^{-1}/e p_c^2) \), as well as \( \kappa \) and the loss-cone index \( \sigma \). Detailed comparison of the theory presented in this study against experimental particle and wave data presents an interesting future challenge.

There are many possibilities for extending the present study, including (i) carrying out a wider exploitation of parameter space, (ii) including hot-plasma effects, (iii) replacing the assumption of parallel propagation by considering oblique wave propagation, and (iv) examining how kappa distributions affect the linear and nonlinear growth of whistler mode hiss, magneto sonic waves, and EMIC waves.

Appendix A: Linear Growth Rates for Whistler-Mode Waves Generated by the Bi-Maxwellian, Kappa Loss-Cone, and Dory et al. (1965) Distributions

Corresponding to the bi-Maxwellian distribution (2) in Table 1, the linear growth rate for field-aligned whistler-mode waves is

\[
\frac{\omega_l}{\Omega_x} = 2 \sqrt{\pi} \frac{N_i}{N_0} \left( \frac{m_e c}{\theta_\parallel} \right) \left( 1 - x \right) \left( 1 + 2 a x (1 - x)^2 \right) \left\{ I_4(x,y) - \frac{x}{y} I_3(x,y) \right\},
\]

where

\[
I_3(x,y) = \int_0^\infty \frac{z^3 dz}{\Delta_R} \exp \left[ -z^2 - \left( \frac{1 - \gamma_R x}{y} \right)^2 \left( \frac{m_e c}{\theta_\parallel} \right)^2 \right],
\]

\[
I_4(x,y) = \left( \frac{\theta_\parallel^2}{\theta_\parallel^2 - 1} \right) \int_0^\infty \frac{z^3 dz}{\Delta_R} \exp \left[ -z^2 - \left( \frac{1 - \gamma_R x}{y} \right)^2 \left( \frac{m_e c}{\theta_\parallel} \right)^2 \right].
\]
Corresponding to the Dory et al. (1965) distribution (4) in Table 1, the linear growth rate is

\[
\frac{\omega_l}{\Omega_i} = 2\sqrt{\frac{N_b}{N_0}} \left( \frac{m_e}{e} \right) \frac{(1-x)^2}{1 + 2ax(1-x)} \Gamma\left( \kappa + \sigma + 1 \right) \frac{1}{1 + 2ax(1-x)} \frac{\Gamma\left( \kappa - 1/2 \right) \Gamma\left( \sigma + 1 \right)}{\Gamma\left( \kappa - 1/2 \right) \Gamma\left( \sigma + 1 \right)} \left[ I_6(x,y) - \frac{x}{y} I_5(x,y) \right],
\]

(A4)

where

\[
I_5(x,y) = \int_0^\infty \frac{dz}{\Delta_R} \left( \frac{1 - \gamma_R x}{y} \right) \left( \frac{\theta_1^2}{\theta_1^2 - 1} \right) \left( \frac{1 + \gamma_R x}{\kappa} \right) \left( \frac{1 + \gamma_R x}{\kappa} \right) z^{2\sigma+3} \left[ 1 + \left( 1 - \gamma_R x \right)^2 \left( \frac{m_c}{\theta_1} \right)^2 + \frac{z^2}{\kappa} \right]^{(\kappa+\sigma+2)}.
\]

(A5)

\[
I_6(x,y) = \int_0^\infty \frac{dz}{\Delta_R y_R} \left( 1 - \gamma_R x \right) \left( 1 - \gamma_R x \right) \left( \frac{\theta_1^2}{\theta_1^2 - 1} \right) \left( \frac{1 + \gamma_R x}{\kappa} \right) \left( 1 + \gamma_R x \right) z^{2\sigma+3} \left[ 1 + \left( 1 - \gamma_R x \right)^2 \left( \frac{m_c}{\theta_1} \right)^2 + \frac{z^2}{\kappa} \right]^{(\kappa+\sigma+2)}.
\]

(A6)

Corresponding to the Dory et al. (1965) distribution (4) in Table 1, the linear growth rate is

\[
\frac{\omega_l}{\Omega_i} = 2\sqrt{\frac{N_b}{N_0}} \left( \frac{m_e}{e} \right) \frac{(1-x)^2}{1 + 2ax(1-x)} \Gamma\left( \kappa + \sigma + 1 \right) \frac{1}{1 + 2ax(1-x)} \frac{\Gamma\left( \kappa - 1/2 \right) \Gamma\left( \sigma + 1 \right)}{\Gamma\left( \kappa - 1/2 \right) \Gamma\left( \sigma + 1 \right)} \left[ I_6(x,y) - \frac{x}{y} I_5(x,y) \right],
\]

(A7)

where

\[
I_5(x,y) = \int_0^\infty \frac{dz}{\Delta_R} \left( \frac{1 - \gamma_R x}{y} \right) \left( \frac{\theta_1^2}{\theta_1^2 - 1} \right) \left( \frac{1 + \gamma_R x}{\kappa} \right) \left( \frac{1 + \gamma_R x}{\kappa} \right) z^{2\sigma+3} \left[ 1 + \left( 1 - \gamma_R x \right)^2 \left( \frac{m_c}{\theta_1} \right)^2 + \frac{z^2}{\kappa} \right]^{(\kappa+\sigma+2)}.
\]

(A8)

\[
I_6(x,y) = \int_0^\infty \frac{dz}{\Delta_R y_R} \left( 1 - \gamma_R x \right) \left( 1 - \gamma_R x \right) \left( \frac{\theta_1^2}{\theta_1^2 - 1} \right) \left( \frac{1 + \gamma_R x}{\kappa} \right) \left( 1 + \gamma_R x \right) z^{2\sigma+3} \left[ 1 + \left( 1 - \gamma_R x \right)^2 \left( \frac{m_c}{\theta_1} \right)^2 + \frac{z^2}{\kappa} \right]^{(\kappa+\sigma+2)}.
\]

(A9)

The quantities \(\gamma_R, \Delta_R\) occurring in formulae (Equations A2, A3, A5, A6, A8 and A9) are defined in Equations 8 and 9. The remaining symbols are defined in Section 2.

**Data Availability Statement**

The computer code and data used for the numerical work in this study are available from the website (http://isst.ncu.edu.cn/Download/sjxz/jgrcode.html).
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