On coefficient ideals

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Abstract
Let \((A, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension \(d \geq 2\) with infinite residue field and let \(I\) be an \(\mathfrak{m}\)-primary ideal. Let \(0 \leq i \leq d\) let \(I_i\) be the \(i\)th-coefficient ideal of \(I\). Also let \(I = I_d\) denote the Ratliff-Rush closure of \(A\). Let \(G = G_i(A)\) be the associated graded ring of \(I\). We show that if \(\dim H^j_{G_i}(G) \leq j - 1\) for \(1 \leq j \leq i \leq d - 1\) then \((I^n)_{d-i} = \bar{I}^n\) for all \(n \geq 1\). In particular if \(G\) is generalized Cohen-Macaulay then \((I^n)_1 = \bar{I}^n\) for all \(n \geq 1\).

1. Introduction

Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \(d \geq 1\) and let \(I\) be an \(\mathfrak{m}\)-primary ideal. If \(M\) is an \(A\)-module let \(\lambda(M)\) denote its length. Let \(P_I(z) \in \mathbb{Q}[z]\) be the Hilbert-Samuel polynomial of \(I\); i.e., \(P_I(n) = \lambda(A/I^{n+1})\) for all \(n \gg 0\). Write

\[
P_I(z) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{z + d - i}{d - i}.
\]

The integer \(e_i(I)\) is called the \(i\)th-Hilbert coefficient of \(I\). The number \(e_0(I)\) is called the multiplicity of \(A\) with respect to \(I\). Let \(\bar{I}\) denote the integral closure of \(I\). For \(0 \leq i \leq d\) set

\[
E_i(I) = \{ J \mid J \supseteq I \text{ and } e_j(I) = e_j(J) \text{ for } 0 \leq j \leq i \}.
\]

Now assume \(A\) is quasi-unmixed with infinite residue field. By a work of Rees [9], \(E_0(I)\) has a unique maximal element \(\bar{I}\). Shah [10] proved that each \(E_i\) has a unique maximal element \(I_i\) which is called the \(i\)th coefficient ideal of \(I\). So we have a chain of ideals

\[
I \subseteq I_d \subseteq I_{d-1} \subseteq I_{d-2} \subseteq \cdots \subseteq I_1 \subseteq I_0 = \bar{I}.
\]

If depth \(A > 0\) then \(I_d = \bar{I}\) the Ratliff-Rush closure of \(I\). Recall

\[
\bar{I} = \bigcup_{n \geq 1} (I^n + I^{n+1}).
\]

So \(\bar{I} = (I^{r+1} : I^r)\) for all \(r \gg 0\) (see [8]).

Let \(G(I) = \bigoplus_{n>0} I^n/I^{n+1}\) be the associated graded ring of \(I\). Fix an integer \(r\) with \(1 \leq r \leq d\). Then Shah proved that if depth \(G(I) \geq r\) then \(I^s = (I^r)\), for \(d + 1 - r \leq j \leq d\), and for all \(s \geq 1\); see [10, Theorem 5]. In particular if \(G\) is Cohen-Macaulay then \(I^s = (I^r)\) for all \(s \geq 1\).
To state our results we need to introduce some concepts. Let \( R = \bigoplus_{n \geq 0} R_n \) be a standard graded algebra over a Artin local ring \((R_0, m_0)\). Let \( M \) be a finitely generated graded \( R \)-module. Let \( H^i(M) \) denote the \( i \)-th local cohomology module of \( M \) with respect to \( R_+ \). It is well-known that \( H^i(M) \) are \(*\)-Artinian \( R \)-module (i.e., every descending chain of graded submodules stabilize). It follows that its Matlis-dual \( H^i(M)^\vee \) is a finitely generated \( R \)-module. If \( M \) is non-zero and of dimension \( r \) then it is known that \( \dim H^i(M)^\vee \leq i \) for \( 0 \leq i \leq r - 1 \) and \( \dim H^r(M)^\vee = r \); see \([1, 17.1.9 \text{ and } 17.1.10]\). We set dimension of the zero module to be \(-1\). In this paper we prove

**Theorem 1.1.** Let \((A, m)\) be a Cohen-Macaulay local ring of dimension \( d \geq 2 \) and with infinite residue field. Let \( I \) be an \( m \)-primary ideal of \( A \). Fix an integer \( r \) with \( 1 \leq r \leq d - 1 \). If \( \dim H^i(\mathcal{G}(I))^\vee \leq i - 1 \) for \( 1 \leq i \leq r \) then \((I^n)_d, r = \tilde{I}^n \) for all \( n \geq 1 \).

As an easy consequence we obtain

**Corollary 1.2.** (with hypotheses as in 1.1) If \( G(I) \) is generalized Cohen-Macaulay then \((I^n)_1 = \tilde{I}^n \) for all \( n \geq 1 \).

In [3] the first coefficient ideal \( I \) is related to the \( S_2 \)-ification of the Rees algebra. From his results it follows that if \( A \) is an analytically unramified Cohen-Macaulay of dimension \( d \geq 2 \) then the \( S_2 \)-ification of the Rees algebra \( \mathcal{R}(I) = A[It] \) is \( \bigoplus_{n \geq 0} (I^n)_1 \). As a consequence we obtain

**Corollary 1.3.** (with hypotheses as in 1.2) Further assume \( A \) is analytically unramified domain. If \( G(I) \) is generalized Cohen-Macaulay then the \( S_2 \)-ification of the Rees algebra \( \mathcal{R}(I) \) is \( \bigoplus_{n \geq 0} \tilde{I}^n \). In particular \( \mathcal{R}(I^n) \) is \( S_2 \) for all \( n \gg 0 \).

**Remark 1.4.** In dimension one a similar result to Corollary 1.3 was proved by Noh and Vasconcelos [4, 2,13, 2.14].

We now describe in brief the contents of this paper. In section two we introduce some notation and discuss some preliminary results that we need. In section three we recall the \( \mathcal{R}(I) \)-module \( L^I(M) = \bigoplus_{n \geq 0} M/I^{n+1}M \) and discuss some of its properties that we need. In section four we discuss our results on dimensions of duals of certain local cohomology modules. In section five we prove Theorem 1.1.

### 2. Notation and preliminaries

In this section we introduce some notation and discuss a few preliminaries which will be used in this paper. In this paper all rings are commutative Noetherian and all modules (unless stated otherwise) are assumed finitely generated. We use terminology from [2]. Let \((A, m)\) be a local ring of dimension \( d \) with residue field \( k = A/m \). Let \( M \) be Cohen-Macaulay \( A \)-module of dimension \( r \). Throughout \( I \) is an \( m \)-primary ideal.

#### 2.1.

Set \( G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1} \) to be the associated graded ring of \( A \) with respect to \( I \). If \( M \) is an \( A \)-module then set \( G_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1}M \) to be the associated graded module of \( M \) with respect to \( I \).

#### 2.2.

For definition and basic properties of superficial sequences see [5, p. 86–87].
2.3.

**Base change:** Let \( \phi: (A, m) \to (A', m') \) be a flat local ring homomorphism with \( mA' = m' \). Set \( I' = IA' \) and if \( N \) is an \( A \)-module set \( N' = N \otimes A' \). In these cases it can be seen that

1. \( \lambda_A(N) = \lambda_{A'}(N') \).
2. \( H^i(M, n) = H^i(M', n) \) for all \( n \geq 0 \).
3. \( \dim M = \dim M' \) and \( \text{grade}(K, M) = \text{grade}(KA', M') \) for any ideal \( K \) of \( A \).
4. \( \text{depth} G_i(M) = \text{depth} G_i(M') \).

We need the following result. It is definitely known to the experts. However we are unable to find a reference. We sketch a proof.

**Proposition 2.1.** Let \( R = \bigoplus_{n \geq 0} R_n \) be a standard graded algebra over a complete Noetherian local ring \( (R_0, m_0) \). Let \( M \) be a non-zero finitely generated graded \( R \)-module of dimension \( r \) with \( \lambda(M_n) < \infty \) for all \( n \in \mathbb{Z} \). Set \( r = \dim M \). If \( E \) is a graded \( R \)-module, set \( E^\vee \) to be the Matlis dual of \( E \) with respect to \( R \). Let \( m = m_0 \oplus R_+ \) be the unique maximal homogeneous ideal of \( R \). Then \( \dim H^i_m(M)^\vee \leq i \) for \( i = 0, \ldots, r - 1 \) and \( \dim H^i_m(M)^\vee = r \).

**Sketch.** The result is known if \( R_0 \) is Artin local [1, 17.1.9 and 17.1.10]. Set \( S = R/\text{ann}_R M \) and let \( n \) be the maximal homogeneous ideal of \( S \). Then note that \( S_0 \) is Artin local and \( H^n_S(M) = H^n_m(M) \) by graded independence theorem of local cohomology, [1, 13.1.6]. We claim that the Matlis dual of \( W = H^n_m(M) \) as an \( R \)-module is isomorphic to the Matlis dual of \( W \) as an \( S \)-module. Notice

\[
W^\vee \cong \text{Hom}_R(W, E_R(k)) \cong \text{Hom}_R(W \otimes S, E_R(k)) \cong \text{Hom}_S(W, \text{Hom}_R(S, E_R(k))) \cong \text{Hom}_S(W, E_S(k)).
\]

The result follows from the Artin local case.

3. Some properties of \( L^i(M) \)

In this section we collect some of the properties of \( L^i(M) = \bigoplus_{n \geq 0} M/I^{n+1}M \) which we proved in [6]. Throughout this section \( (A, m) \) is a local ring with infinite residue field, \( M \) is a Cohen-Macaulay module of dimension \( r \geq 1 \) and \( I \) an \( m \)-primary ideal.

3.1.

Set \( \mathcal{R}(I) = A[I] \); the Rees Algebra of \( I \). In [6, 4.2] we proved that \( L^i(M) \) is a \( \mathcal{R}(I) \)-module. Note that \( L^i(M) \) is not finitely generated \( \mathcal{R}(I) \)-module.

3.2.

Let \( \mathfrak{M} = (m, \mathcal{R}(I)_+) \) be the maximal graded ideal of \( \mathcal{R}(I) \). Let \( H^i(\mathfrak{M}, \cdot) = H^i_{\mathfrak{M}_+}(\cdot) \) denote the \( i \)-th local cohomology functor with respect to \( \mathfrak{M} \). Recall a graded \( \mathcal{R}(I) \)-module \( V \) is said to be \( * \)-Artinian if every descending chain of graded submodules of \( V \) terminates. For example if \( E \) is a finitely generated \( \mathcal{R}(I) \)-module then \( H^i(E) \) is \( * \)-Artinian for all \( i \geq 0 \).
3.3. In [6, 4.7] we proved that

\[ H^0(L^1(M)) = \bigoplus_{n \geq 0} \frac{I^{n+1}M}{I^nM} \cdot \]

Here \( \widetilde{KM} \) denotes the Ratliff-Rush closure of \( M \) with respect to an ideal \( K \). Recall

\[ \widetilde{KM} = \bigcup_{i \geq 1} K^{i+1}M : K^i. \]

3.4. For \( L^1(M) \) we proved that for \( 0 \leq i \leq r - 1 \)

1. \( H^i(L^1(M)) \) are \( * \)-Artinian; see [6, 4.4].
2. \( H^i(L^1(M))_n = 0 \) for all \( n \gg 0 \); see [6, 1.10].
3. \( H^i(L^1(M))_n \) has finite length for all \( n \in \mathbb{Z} \); see [6, 6.4].
4. For \( 0 \leq i \leq r - 1 \) there exists a polynomial \( q_i(z) \in \mathbb{Q}[z] \) such that \( q_i(n) = \ell(H^i(L^1(M))_n) \) for all \( n \ll 0 \).

3.5. Let \( x \) be \( M \)-superficial with respect to \( I \), i.e., \( (I^{n+1}M : x) = I^nM \) for all \( n \gg 0 \). Set \( N = M/xM \) and \( u = xt \in \mathcal{R}(I)_1 \). Notice \( L^1(M)/uL^1(M) = L^1(N) \). For each \( n \geq 1 \) we have the following exact sequence of \( A \)-modules:

\[ 0 \longrightarrow \frac{I^{n+1}M}{I^nM} : x \longrightarrow M \xrightarrow{\psi_n} \frac{M}{I^nM} \longrightarrow \frac{N}{I^{n+1}N} \longrightarrow 0, \]

where \( \psi_n(m + I^nM) = xm + I^{n+1}M \).

This sequence induces the following exact sequence of \( \mathcal{R} \)-modules:

\[ 0 \longrightarrow B^1(x, M) \longrightarrow L^1(M)(-1) \xrightarrow{\psi_u} L^1(M) \xrightarrow{\delta^x} L^1(N) \longrightarrow 0, \tag{1} \]

where \( \psi_u \) is left multiplication by \( u \) and

\[ B^1(x, M) = \bigoplus_{n \geq 0} \frac{(I^{n+1}M : Mx)}{I^nM}. \]

We call (1) the second fundamental exact sequence.

3.6. Notice \( \lambda(B^1(x, M)) < \infty \). A standard trick yields the following long exact sequence connecting the local cohomology of \( L^1(M) \) and \( L^1(N) \):

\[ 0 \longrightarrow B^1(x, M) \longrightarrow H^0(L^1(M))(−1) \longrightarrow H^0(L^1(M)) \longrightarrow H^0(L^1(N)) \]

\[ \quad \longrightarrow H^1(L^1(M))(−1) \longrightarrow H^1(L^1(M)) \longrightarrow H^1(L^1(N)) \tag{2} \]

\[ \ldots \]
3.7.
One huge advantage of considering \( L^t(M) \) is that it behaves well with respect to the Veronese functor. Notice

\[
(L^t(M)(-1))^{<t>} = L^t(M)(-1) \quad \text{for all } t \geq 1.
\]

Also note that \( \mathcal{R}(I)^{<t>} = \mathcal{R}(I^t) \) and that \( (\mathcal{M}_{\mathcal{R}(I)})^{<t>} = \mathcal{M}_{\mathcal{R}(I^t)} \). It follows that for all \( i \geq 0 \)

\[
(H_{\mathcal{M}_{\mathcal{R}(I)}}(L^t(M)(-1)))^{<t>} \cong H_{\mathcal{M}_{\mathcal{R}(I^t)}}(L^t(M)(-1)).
\]

4. Results on dimensions of duals of local cohomology modules
Throughout this section \( (A, m) \) is a complete Noetherian local ring with infinite residue field and \( M \) is a Cohen-Macaulay \( A \)-module of dimension \( r \geq 1 \). Furthermore we will assume that \( I \) is an \( m \)-primary ideal. In this section we prove some results regarding dimensions of duals of graded local cohomology modules of \( G_t(M) \) and \( L^t(M) \). Throughout we compute local cohomology with respect to the graded maximal ideal of the Rees algebra \( \mathcal{R}(I) = A[It] \). If \( E \) is a graded \( \mathcal{R} \)-module then we denote its Matlis dual by \( E^\vee \).

4.1.
Let \( \text{mod}^f(\mathcal{R}(I)) \) be the category of finitely generated graded \( \mathcal{R}(I) \)-modules \( M \) with \( \lambda(M_n) < \infty \) for all \( n \in \mathbb{Z} \). Let \( \text{A}^f(\mathcal{R}(I)) \) be the category of \(*\)-Artinian graded \( \mathcal{R}(I) \)-modules \( L \) with \( \lambda(L_n) < \infty \) for all \( n \in \mathbb{Z} \). The usual Matlis duality between finitely generated graded \( \mathcal{R}(I) \)-modules and \(*\)-Artinian \( \mathcal{R}(I) \)-modules (see [2, 3.6.17]) restricts to a duality between \( \text{mod}^f(\mathcal{R}(I)) \) and \( \text{A}^f(\mathcal{R}(I)) \).

Lemma 4.1. Let \( M \in \text{mod}^f(\mathcal{R}(I)) \). Then there exists a non-empty Zariski open subset \( U \) of \( \mathfrak{m}I \) such that if \( x \in I \) such that \( \overline{x} \in U \) then

1. \( \ker(M(-1) \xrightarrow{x} M) \) has finite length.
2. \( \coker(M^\vee(-1) \xrightarrow{x} M^\vee) \) has finite length.

Proof. By Matlis duality we get (2) from (1).

(1) If \( \dim M = 0 \) then any \( x \in I \) will do the job.

Assume \( \dim M > 0 \). Set \( N = M/H^0_{\mathcal{M}}(M) \). Then note that \( N \in \text{mod}^f(\mathcal{R}(I)) \) and if \( xt \) is \( N \)-regular then (by using snake lemma) it can be shown that \( \ker(M(-1) \xrightarrow{x} M) \) has finite length. Thus it suffices to assume depth \( M > 0 \).

Claim: Assume depth \( M > 0 \). If \( P \in \text{Ass}_{\mathcal{R}(I)} M \) then \( P \not\supseteq \mathcal{R}(I)_+ \).

Suppose if possible \( P \supseteq \mathcal{R}(I)_+ \) for some \( P \in \text{Ass}_{\mathcal{R}(I)} M \). Note that as \( M \in \text{mod}^f(\mathcal{R}(I)) \) it follows that \( \mathfrak{m}P \subseteq P \). Therefore \( P = \mathfrak{m}P \). So depth \( M = 0 \), a contradiction.

Set \( V = I/\mathfrak{m}I \). For \( P \in \text{Ass}_{\mathcal{R}(I)} M \) set

\[
V_P = \frac{((It \cap P) + \mathfrak{m}I)}{\mathfrak{m}I}.
\]

Note \( V_P \) is a \( k \)-subspace of \( V \). Furthermore \( V_P \neq V \) for otherwise \( P \supseteq \mathcal{R}(I)_+ \) which is not possible by our earlier claim. We note that as \( \text{Ass}_{\mathcal{R}(I)} M \) is a finite set, the set

\[
U = V \setminus \bigcup_{P \in \text{Ass}_{\mathcal{R}(I)} M} V_P \quad \text{is a non-empty Zariski open in } V.
\]

If \( x \in I \) such that \( \overline{x} \in U \) then clearly \( x \) is \( M \)-regular. The result follows.
4.2.

Let \( E \in A^i(\mathcal{R}(I)) \). Then by Matlis duality it follows that there exists a polynomial \( q_E(z) \in \mathbb{Q}[z] \) such that \( q_E(n) = \lambda(E_n) \) for all \( n \ll 0 \). Furthermore if \( E \neq 0 \) then \( \deg q_E(z) = \dim E^\vee - 1 \). We call \( q_E(z) \) as the dual Hilbert-polynomial of \( E \).

Our first result is

**Proposition 4.2.** Let \( M \) be a Cohen-Macaulay \( A \)-module of dimension \( r \geq 1 \). Then \( \dim H^i(L^1(M))^\vee \leq i \) for \( 0 \leq i \leq r - 1 \).

**Proof.** We prove the result by induction on \( r = \dim M \). If \( r = 1 \) then by 3.3 we get that \( H^0(L^1(M)) \) has finite length. So we have nothing to show. Now assume \( r \geq 2 \) and the result has been shown for Cohen-Macaulay modules of dimension \( r - 1 \). By (4.1) we may choose \( x \in I \) which is \( M \)-superficial and \( \text{coker } H^i(L^1(M))(-1) \xrightarrow{x} H^i(L^1(M)) \) has finite length for \( i = 1, \ldots, r - 1 \). Let \( q_i^M(z) \) be the dual Hilbert-polynomial of \( H^i(L^1(M)) \). By (3.3) we get that \( H^0(L^1(M)) \) has finite length. Set \( N = M/xM \). Now assume that \( 1 \leq i \leq r - 1 \). Then by our construction and (3.6) there exist finite length modules \( U_i, V_i \) such that we have an exact sequence

\[
0 \to U_i \to H^{i-1}(L^1(N)) \to H^i(L^1(M))(-1) \xrightarrow{x} H^i(L^1(M)) \to V_i \to 0.
\]

By our induction hypothesis it follows that \( \deg(q_i^M(z) - q_i^M(z - 1)) \leq i - 2 \). So \( \deg q_i^M(z) \leq i - 1 \). So \( \dim H^i(L^1(M))^\vee \leq i \). The result follows.

**Proposition 4.3.** Let \( M \) be a Cohen-Macaulay \( A \)-module of dimension \( r \geq 1 \). Assume that there is \( s \) with \( 2 \leq s \leq r - 1 \) such that for \( 1 \leq i \leq s \) we have

\[
\dim H^i(G_t(M))^\vee \leq i - 1.
\]

Then there exists a non-empty Zariski-open subset \( W \) of \( I/mI \) such that if \( x \in I \) and \( \overline{x} \in W \) we have \( \dim H^i(G_t(M/xM))^\vee \leq i - 1 \) for \( 1 \leq i \leq s - 1 \).

**Proof.** By (4.1) it follows that we may choose a non-empty Zariski-open subset \( W \) of \( I/mI \) such that if \( x \in I \) and \( \overline{x} \in W \) then \( x \) which is \( M \)-superficial and \( \text{coker } H^i(G_t(M))(-1) \xrightarrow{x} H^i(G_t(M)) \) has finite length for \( i = 1, \ldots, r - 1 \). Set \( N = M/xM \). Then by our construction there exist finite length modules \( U_i, V_i \) such that we have an exact sequence for \( 2 \leq i \leq r - 1 \).

\[
U_i \to H^{i-1}(G_t(M)/xtG_t(M)) \to H^i(G_t(M))(-1) \xrightarrow{x} H^i(G_t(M)) \to V_i.
\]

Using dual Hilbert polynomials it follows that for \( \dim H^i(G_t(M)/xtG_t(M))^\vee \leq i - 1 \) for \( 1 \leq i \leq s - 1 \). The natural map \( G_t(M)/xtG_t(M) \to G_t(N) \) is surjective and has finite length kernel. Therefore

\[
H^i(G_t(N)) \cong H^i(G_t(M)/xtG_t(M)) \quad \text{for } i \geq 1.
\]

**Proposition 4.4.** Let \( M \) be a Cohen-Macaulay \( A \)-module of dimension \( r \geq 1 \). Assume that there is \( s \) with \( 1 \leq s \leq r - 1 \) such that for \( 1 \leq i \leq s \) we have

\[
\dim H^i(G_t(M))^\vee \leq i - 1.
\]

Then \( \dim H^i(L^1(M))^\vee \leq i - 1 \) for \( 1 \leq i \leq s \).

**Proof.** We do by induction on \( s \). For \( s = 1 \) the result follows from [7, 5.2]. Assume \( s \geq 2 \). Using (4.1) and (4.3) it follows that there exists a non-empty Zariski-open subset \( W \) of \( I/mI \) such that if \( x \in I \) and \( \overline{x} \in W \) we have

1. \( x \) is \( M \)-superficial with respect to \( I \).
2. \( \dim H^i(G_t(M/xM))^\vee \leq i - 1 \) for \( 1 \leq i \leq s - 1 \).
3. \( \text{coker } H^i(L^1(M))(-1) \xrightarrow{x} H^i(L^1(M)) \) has finite length for \( i = 1, \ldots, r - 1 \).
Let \( q_i^M(z) \) be the dual Hilbert-polynomial of \( H^i(I^j(M)) \). Set \( N = M/xM \). By induction hypothesis we have \( \dim H^i(L^j(N)) \leq i - 1 \) for \( 1 \leq i \leq s - 1 \). We prove \( \dim H^i(L^j(M)) \leq i - 1 \) for \( 1 \leq i \leq s \). For \( i = 1 \) the result follows from [7, 5.2]. Now let \( 2 \leq i \leq s \). By our construction and (3.6) there exist finite length modules \( U_i, V_i \) such that we have an exact sequence

\[
0 \to U_i \to H^{i-1}(L^j(N)) \to H^i(L^j(M))(-1) \to H^i(L^j(M)) \to V_i \to 0.
\]

By our induction hypothesis it follows that \( \deg(q_i^M(z) - q_i^M(z - 1)) \leq i - 3 \). So \( \deg q_i^M(z) \leq i - 2 \). So \( \dim H^i(L^j(M)) \leq i - 1 \). The result follows. \( \square \)

5. **Proof of Theorem 1.1**

In this section we prove our main result.

5.1.

For our arguments we have to go to the completion. The Ratliff-Rush closure and integral closure of an \( m \)-primary ideal behave well with respect to completion. However to the best of the authors knowledge it is not known whether other coefficient ideals behave well with respect to completion. Set

\[ E'_i(I) = \{ J \mid J \supseteq \overline{I} \text{ and } e_j(I) = e_j(I) \text{ for } 0 \leq j \leq i \}. \]

It is clear that if \( J \in E'_i(I) \) then \( J\overline{A} \in E'_i(J\overline{A}) \).

5.2.

Fix \( i \) with \( 1 \leq i \leq d \). Let \( J \in E'_i(I) \) then note \( J \subseteq \overline{I} \). So \( \mathcal{R}(J) \) is a finite \( \mathcal{R}(I) \)-module. Set \( W(J) = \mathcal{R}(J)/\mathcal{R}(I) \). Note \( W(J)_n \) has finite length for all \( n \). We show

**Proposition 5.1.** *(with setup as in 5.2)* \( \dim W(J) \leq d - i \).

**Proof.** Set \( W = W(J) \). From the short exact sequence

\[
0 \to W(+1) \to \bigoplus_{n \geq 0} A/I^{n+1} \to \bigoplus_{n \geq 0} A/I^{n+1}_i \to 0
\]

we have

\[
\lambda(W_{n+1}) = \lambda(A/I^{n+1}_i) - \lambda(A/I^{n+1})
\]

\[
= \{ e_0(I) \left( \begin{array}{c} n+d \\ d \end{array} \right) + \cdots + (-1)^i e_i(I) \left( \begin{array}{c} n+d-j \\ d-j \end{array} \right) + \cdots \}
\]

\[
- \{ e_0(J) \left( \begin{array}{c} n+d \\ d \end{array} \right) + \cdots + (-1)^i e_i(J) \left( \begin{array}{c} n+d-j \\ d-j \end{array} \right) + \cdots \}
\]

for all \( n \gg 0 \). As \( e_j(I) = e_j(J) \) for all \( 0 \leq j \leq i \) so we get

\[
\lambda(W_{n+1}) = (-1)^{i+1} (e_{i+1}(I) - e_{i+1}(J)) \left( \begin{array}{c} n+d-i-1 \\ d-i-1 \end{array} \right) + \text{lower terms}.
\]

Thus \( n \mapsto \lambda(W_{n+1}) \) is a polynomial of degree at most \( d - i - 1 \). Hence \( \dim W \leq d - i \). \( \square \)

**Theorem 1.1** is an easy consequence of the following result.

**Theorem 5.2.** Let \( (A, m) \) be a complete Cohen-Macaulay local ring of dimension \( d \geq 2 \) and with infinite residue field. Let \( I \) be an \( m \)-primary ideal of \( A \). Fix an integer \( r \) with \( 1 \leq r \leq d - 1 \). Assume \( \dim H^i(L^j(A))^\vee \leq i - 1 \) for \( 1 \leq i \leq r \). Then \( E_{d-r}^\vee(I^n) = \{ \overline{P} \} \) for all \( n \geq 1 \).
Proof. We will first prove the result when \( n = 1 \). For the convenience of the reader we first prove the case when \( r = 1 \). It suffices to show that \( E'_{d-1}(I) = \{ \widetilde{I} \} \). As discussed earlier we may assume that \( A \) is complete. Let \( J \in E'_{d-1}(I) \). Set \( W = W(J) = \mathcal{R}(J)/\mathcal{R}(I) \). By (5.1) we get \( \dim W \leq 1 \). We have an exact sequence
\[
0 \to W(+1) \to L^I(A) \to L^I(A) \to 0.
\]
This induces a long exact sequence in cohomology
\[
\cdots \to H^0(L^I(A)) \to H^1(W(+1)) \to H^1(L^I(A)) \to \cdots.
\]
Taking Matlis duals we obtain an exact sequence
\[
\cdots \to H^1(L^I(A))^\vee \to H^1(W(+1))^\vee \to H^0(L^I(A))^\vee \to \cdots.
\]
By our assumption and (3.3) it follows that \( \dim H^1(W(+1)) \leq 0 \), as \( \dim W \leq 1 \) it follows from 2.1 that \( W \) is zero-dimensional. Thus \( J^m = I^m \) for all \( m \gg 0 \). Therefore \( J \subseteq \widetilde{I} \). But by definition of \( E'_{d-1}(I) \) we have \( J \supseteq \widetilde{I} \). Thus \( E'_{d-1}(I) = \{ \widetilde{I} \} \).

Now assume \( r \geq 2 \). Let \( J \in E'_{d-r}(I) \). Set \( W = W(J) = \mathcal{R}(J)/\mathcal{R}(I) \). By (5.1) we get \( \dim W \leq r \). We assert \( \dim W = 0 \). Suppose if possible \( \dim W = c > 0 \). Note \( c \leq r \). We have an exact sequence
\[
0 \to W(+1) \to L^I(A) \to L^I(A) \to 0.
\]
This induces a long exact sequence in cohomology
\[
\cdots \to H^{c-1}(L^I(A)) \to H^c(W(+1)) \to H^c(L^I(A)) \to \cdots.
\]
Taking Matlis duals we obtain an exact sequence
\[
\cdots \to H^c(L^I(A))^\vee \to H^c(W(+1))^\vee \to H^{c-1}(L^I(A))^\vee \to \cdots.
\]
By our assumption and (4.2) it follows that \( \dim H^c(W(+1)) \leq c - 1 \). This contradicts (2.1). So \( \dim W = 0 \). Thus \( J^m = I^m \) for all \( m \gg 0 \). Therefore \( J \subseteq \widetilde{I} \). But by definition of \( E'_{d-r}(I) \) we have \( J \supseteq \widetilde{I} \). Thus \( E'_{d-r}(I) = \{ \widetilde{I} \} \). Thus we have proved the result when \( n = 1 \).

Now assume that \( n > 1 \). We note that \( L^m(A)(-1) \) is the \( n \)-th Veronese module of \( L^I(A)(-1) \); see (3.7). As local cohomology module commutes with Veronese it follows that \( \dim H^i(L^m(A)) \leq i - 1 \) for \( 1 \leq i \leq r \). The result now follows from the \( n = 1 \) case. \( \square \)

We now give

Proof of Theorem 1.1. It suffices to prove that \( E'_{d-r}(I^n) = \{ \widetilde{I^n} \} \) for all \( n \geq 1 \). We note that \( G(I) = G(I\hat{A}) \). By (5.1) we may assume that \( A \) is complete. By (4.4) it follows that \( \dim H^i(L^I(A))^\vee \leq i - 1 \) for \( 1 \leq i \leq r \). The result now follows from Theorem 5.2. \( \square \)

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References
[1] Brodmann, M. P., Sharp, R. Y. (1998). Local Cohomology: An Algebraic Introduction with Geometric Applications, Vol. 60, Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press.
[2] Bruns, W., Herzog, J. (1993). Cohen-Macaulay Rings, Vol. 39, Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press.
[3] Ciupercă, C. (2001). First coefficient ideals and the \( S_2 \)-ification of a Rees algebra. J. Algebra 242:782–794.
[4] Noh, S., Vasconcelos, W. (1993). The \( S_2 \)-closure of a Rees algebra. Results Math. 23(1–2):149–162.
[5] Puthenpurakal, T. J. (2003). Hilbert coefficients of a Cohen-Macaulay module. J. Algebra 264:82–97.
[6] Puthenpurakal, T.J. (2007). Ratliff-Rush filtration, regularity and depth of higher associated graded modules: Part I. *J. Pure Appl. Algebra* 208(1):159–176.

[7] Puthenpurakal, T.J. (2017). Ratliff-Rush filtration, regularity and depth of higher associated graded modules: Part II. *J. Pure Appl. Algebra* 221(3):611–631.

[8] Ratliff, L. J., Rush, D. (1978). Two notes on reductions of ideals. *Indiana Univ. Math. J.* 27:929–934.

[9] Rees, D. (1961). a-Transform of local rings and a theorem on multiplicities of ideals. *Proc. Cambridge Philos. Soc.* 57:8–17.

[10] Shah, K. (1991). Coefficient ideals. *Trans. Amer. Math. Soc.* 327(1):373–384.