Red-injective modules

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Abstract

Let $\text{Red}(M)$ be the sum of all reduced submodules of a module $M$. For modules over commutative rings, $\text{Soc}(M) \subseteq \text{Red}(M)$. By drawing motivation from how Soc-injective modules were defined by Amin et. al. in [1], we introduce Red-injective modules, study their properties and use them to characterize quasi-Frobenius rings and $V$-rings.

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1 Introduction

For a not necessarily commutative ring $R$, Lee and Zhou in [7] defined an $R$-module $M$ to be reduced if for all $r \in R$ and $m \in M$, $mr = 0$ implies that $Mr \cap mR = \{0\}$. This definition is equivalent to saying that for all $r \in R$ and $m \in M$, $mr^2 = 0$ implies that $mRr = \{0\}$, see [10] for the proof. However, for modules over commutative rings we get Definition 1 below.

Definition 1. An $R$-module $M$ is reduced if for all $r \in R$ and $m \in M$, $mr^2 = 0$ implies that $mr = 0$.

Except in Example 3.1, all rings are unital, commutative and associative. Modules are right unital defined over rings. A submodule is reduced if it is reduced as a module. A submodule of a reduced module is reduced but a factor module of a reduced module need not be reduced. The $\mathbb{Z}$-module $\mathbb{Z}$ is reduced but its factor module $\mathbb{Z}/n\mathbb{Z}$ is not reduced for a non-square free integer $n$. The socle of an

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$R$-module $M$, denoted by $\text{Soc}(M)$ is the sum of simple submodules of $M$. Let $\text{Red}(M)$ denote the sum of reduced submodules of $M$, i.e.,

$$\text{Red}(M) := \sum_{i \in I} \{N_i \mid N_i \text{ is a reduced submodule of } M \}.$$ 

**Definition 2.** An $R$-module $M$ is **semi-reduced** if $\text{Red}(M) = M$.

**Proposition 1.** For any $R$-module $M$, the following implications hold:

$$\text{simple} \Rightarrow \text{semi-simple} \Rightarrow \text{reduced} \Rightarrow \text{semi-reduced}.$$ 

**Proof:** We prove that a semi-simple module is reduced. The other implications follow from the definition of semi-simple and semi-reduced modules respectively. Since a simple module is prime\(^1\) and every prime module is reduced, a simple module is reduced. Suppose that $M$ is a semi-simple module and $mr^2 = 0$ where $m \in M$ and $r \in R$. Then, $(m_1, m_2, \ldots, m_i, \cdots) r^2 = 0$ where $(m_1, m_2, \ldots, m_i, \cdots) = m \in M = \bigoplus_{i \in I} M_i$ for some simple modules $M_i$. Since every simple module is reduced, $m_i r^2 = 0 \Rightarrow m_i r = 0 \forall \ i \in I$. Hence, $mr = 0$, and $M$ is reduced. \hfill \Box

**Corollary 1.** For any $R$-module $M$, $\text{Soc}(M) \subseteq \text{Red}(M)$.

**Proof:** The proof follows from the fact that a semi-simple module is semi-reduced which is proved in Proposition 1. \hfill \Box

Note that for semi-simple modules and for modules without nonzero reduced submodules, $\text{Soc}(M) = \text{Red}(M)$.

**Example 1.1.** A reduced module need not be semi-simple. $\mathbb{Z}$ and $\mathbb{Q}$ are reduced $\mathbb{Z}$-modules but they are not semi-simple.

### 1.1 Other basic definitions

**Definition 3.** \([1, \text{Definition 1.1}]\) Let $M$ and $N$ be $R$-modules. $M$ is **socle-$N$-injective** (Soc-$N$-injective) if any $R$-homomorphism $f : \text{Soc}(N) \rightarrow M$ extends to $N$. Equivalently, for any semi-simple submodule $K$ of $N$, any $R$-homomorphism $f : K \rightarrow M$ extends to $N$. An $R$-module $M$ is **Soc-quasi-injective** if $M$ is Soc-$M$-injective. $M$ is **Soc-injective** if $M$ is Soc-$R$-injective. $R$ is right (self-) **Soc-injective**, if the module $R_R$ is Soc-injective (equivalently, if $R_R$ is Soc-quasi-injective).

**Definition 4.** \([1, \text{Definition 1.2}]\) An $R$-module $M$ is called **strongly Soc-injective**, if $M$ is Soc-$N$-injective for all $R$-modules $N$. A ring $R$ is called **strongly Soc-injective**, if the module $R_R$ is strongly Soc-injective.

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\(^1\)An $R$-module $M$ for which $RM \neq \{0\}$ is **prime** if for all $a \in R$ and every $m \in M$, $am = 0$ implies that $m = 0$ or $aM = \{0\}$. 

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2
Definitions 3 and 4 together with Corollary 1 motivate us to have Definitions 5 and 6 respectively.

**Definition 5.** An $R$-module $M$ is called **Red-$N$-injective** if any $R$-homomorphism $f: K \to M$ extends to $N$ for any semi-reduced submodule $K$ of $N$. $M$ is called **Red-quasi-injective** if it is Red-$M$-injective. $M$ is called **Red-injective** if it is Red-$R$-injective.

**Definition 6.** An $R$-module $M$ is called **strongly-Red-injective**, if $M$ is Red-$N$-injective for all $R$-modules $N$.

In Definition 7, we recall different generalizations of injective modules that we later use in the sequel. As with Soc-injective and Red-injective modules defined above, these generalizations of injective modules were defined by relaxing conditions on the lifting property of homomorphisms.

**Definition 7.** If $M$ and $N$ are $R$-modules, then

1. $M$ is **$N$-injective** if every $R$-homomorphism from a submodule of $N$ into $M$ can be extended to an $R$-homomorphism from $N$ into $M$.

2. $M$ is **quasi-injective** if it is $M$-injective.

3. $M$ is **$N$-simple-injective** if for any submodule $L$ of $N$, any homomorphism $\theta: L \to M$ with $\theta(L)$ simple, can be extended to a homomorphism $\beta: N \to M$.

4. $M$ is **simple-injective** if it is simple $R$-injective.

5. $M$ is **strongly simple-injective**, if $M$ is simple-$N$-injective for all right $R$-modules $N$.

6. $M$ is **$min$-$N$-injective** if, for every simple submodule $L$ of $N$, every homomorphism $\gamma: L \to M$ extends to $N$.

7. $M$ is **min-injective** if it is min-$R$-injective.

8. $M$ is **strongly min-injective**, if it is min-$N$-injective for all $R$-modules $N$.

9. $M$ is **pseudo-injective** if any monomorphism from a submodule of $M$ to $M$ extends to an endomorphism of $M$.

### 1.2 Notation

Throughout this paper, $N \subseteq^e M$, $N \oplus M$, $N \subseteq^\oplus M$, and $N \leq M$, mean that $N$ is an essential submodule of $M$, a direct sum of $N$ and $M$, $N$ is a direct summand of $M$, $N$ is a submodule of $M$ respectively.
1.3 Paper roadmap

In Section 1, we have given the introduction, defined key terms, given the notation used and the roadmap for the paper.

Section 2 is devoted to obtaining properties of Red-injective modules and their generalizations. An equivalent definition of a Red-injective module is obtained. It is shown that any injective module is strongly Red-injective and a Red-injective module is Soc-injective. Other implications with known generalizations of injective modules are given. The class of (strongly) Red-injective $R$-modules is closed under isomorphisms, direct products and summands. If $M$ is a Noetherian module, then a direct sum of Red-$M$-injective is Red-$M$-injective. For a family of $R$-modules $\{M_i : i \in I\}$, an $R$-module $N$ is Red-$\left(\oplus_{i \in I} M_i\right)$-injective if and only if it is Red-$M_i$-injective for each $i$. For a projective $R$-module $M$, every quotient of a Red-$M$-injective $R$-module is Red-$M$-injective if and only if Red$(M)$ is projective if and only if every quotient of an injective $R$-module is Red-$M$-injective. Over a principal ideal domain a free module is Red-injective if each of its submodule is Red-injective. Red$(N)$-lifting modules are introduced. It is shown that if a module $N$ is Red$(N)$-lifting, then any $R$-module $K$ is Red-$N$-lifting if and only if $K$ is $N$-injective. It is shown that Red-quasi-injective modules inherit a weaker version of C2-condition and C3-condition.

In Section 3, we characterize quasi-Frobenius rings and right $V$-rings in terms of strongly Red-injective modules. A ring $R$ is quasi-Frobenius if and only if every strongly Red-injective $R$-module is projective. A ring $R$ is a right $V$-ring if and only if every simple $R$-module is strongly Red-injective. A question is raised as to whether Red-quasi-injective modules and Soc-quasi-injective modules are clean and or satisfy the exchange property.

2 Red-injective modules

Proposition 2. For $R$-modules $K$, $M$ and $N$, the following statements are equivalent:

1. Any $R$-homomorphism $f : K \to M$ extends to $N$ for any semi-reduced submodule $K$ of $N$.
2. Any $R$-homomorphism $f : \text{Red}(N) \to M$ extends to $N$.

Proof:

1 $\Rightarrow$ 2 since $\text{Red}(N)$ is semi-reduced.

2 $\Rightarrow$ 1. Suppose $f : K \to M$ is an $R$-homomorphism and $K$ is a semi-reduced submodule of $N$. Since $K \leq \text{Red}(N)$, then $f$ extends to $N$.

$\square$
**Proposition 3.** If $N$ is an $R$-module, then

1. any injective module is strongly Red-injective,
2. a Red($N$)-injective module is Soc($N$)-injective.

**Proof:**

1. Let $M$ be an injective module. Then $M$ is $N$-injective for every $R$-module $N$. For every submodule $K$ of $N$, any $R$-homomorphism $f : K \to M$ extends to $N$. For every module $N$, any $R$-homomorphism $f : \text{Red}(N) \to M$ extends to $N$. Hence, $M$ is strongly Red-injective.

2. Suppose $f : \text{Soc}(N) \to M$ is an $R$-homomorphism and $M$ is Red($N$)-injective. By Proposition 1, Soc($N$) is a semi-reduced submodule of $N$. Hence by Definition 5, $f$ extends to $N$. Thus, $M$ is Soc-$N$-injective.

\[\blacksquare\]

Every projective module over a right Noetherian right self-injective ring is strongly Red-injective. Let $R$ be a ring for which each module $M$ has Red($M$) = \{0\}. Then, $M$ is strongly Red-injective.

A Red-injective module need not be injective. The module $\mathbb{Z}$ is Red-injective but not injective.

**Theorem 1.** Let $\{M_i : i \in I\}$ be a family of $R$-modules and $N$, $M$, $A$, $C$, $S$ and $K$ be $R$-modules. Then the following conditions hold:

1. A direct product $\prod_{i \in I} M_i$ is Red-$N$-injective if and only if each $M_i$ is Red-$N$-injective.
2. For $S \leq N$, if $M$ is Red-$N$-injective, then $M$ is Red-$S$-injective.
3. For $M \cong N$; $M$ is Red-$S$-injective if and only if $N$ is Red-$S$-injective.
4. For $A \cong B$; $C$ is Red-$A$-injective if and only if it is Red-$B$-injective.
5. For $N \subseteq \bigoplus M$, if $M$ is Red-$K$-injective, then $N$ is Red-$K$-injective.

**Proof:**

1. We prove only for $M = M_i \times M_j$ where $i$, $j \in I$. The proof for the general case is analogous. Let $M_i$ and $M_j$ be Red-$N$-injective $R$-modules, $h : \text{Red}(N) \to N$ and $f : \text{Red}(N) \to M_i \times M_j$ be any $R$-homomorphisms.

Define

\[f_{M_i} : \text{Red}(N) \to M_i\] such that $\pi_{M_i} \circ f = f_{M_i}$

and

\[f_{M_j} : \text{Red}(N) \to M_j\] such that $\pi_{M_j} \circ f = f_{M_j}$.
where \( \pi_{M_i} : M_i \times M_j \rightarrow M_i \) and \( \pi_{M_j} : M_i \times M_j \rightarrow M_j \) are \( R \)-homomorphisms. Since \( M_i \) and \( M_j \) are \( R \)-injective there exists \( f'_{M_i} : N \rightarrow M_i \) and \( f'_{M_j} : N \rightarrow M_j \) such that
\[
f_{M_i} = f'_{M_i} \circ h \quad \text{and} \quad f_{M_j} = f'_{M_j} \circ h.
\]
By the uniqueness part of the universal property of direct product there exists an \( R \)-homomorphism \( f' : N \rightarrow M_i \times M_j \) such that \( f = f' \circ h \). It follows that \( \pi_{M_i} \circ (f' \circ h) = f_{M_i} \) and \( \pi_{M_j} \circ (f' \circ h) = f_{M_j} \).

By the uniqueness of the universal property we conclude that \( f = f' \circ h \). Hence, \( f : \text{Red}(N) \rightarrow M_i \times M_j \) extends to \( N \). Thus \( M_i \times M_j \) is \( R \)-injective. Conversely, assume that \( M_i \times M_j \) is \( R \)-injective. Let \( h : \text{Red}(N) \rightarrow N \) and \( f_{M_i} : \text{Red}(N) \rightarrow M_i \) be any \( R \)-homomorphisms. Choose \( f_{M_j} : \text{Red}(N) \rightarrow M_j \) to be the zero \( R \)-homomorphism. We obtain \( f' : N \rightarrow M_i \times M_j \) such that \( f = f' \circ h \). Finally we obtain \( f_{M_i} = \pi_{M_i} \circ f = (\pi_{M_i} \circ f') \circ h \). Hence \( \pi_{M_i} \circ f' : N \rightarrow M_i \) is an extension of \( f_{M_i} \). Thus, \( M_i \) is \( R \)-injective. Similarly, \( M_j \) is \( R \)-injective.

2. Consider the diagram in Figure 1, where \( M \) is \( R \)-injective.

\[
\begin{array}{ccc}
\text{Red}(S) & \xrightarrow{k} & \text{Red}(N) \\
& \downarrow{g} & \downarrow{h} \\
S & \xrightarrow{\iota} & N
\end{array}
\]

\[
f' \circ \iota : S \rightarrow M \quad \text{is an extension for any} \quad R \text{-homomorphism} \quad q : \text{Red}(S) \rightarrow M. \quad \text{Thus} \quad M \quad \text{is} \quad R \text{-injective.}
\]

3. Let \( N \cong M \) where \( \theta : N \rightarrow M \) is an \( R \)-isomorphism between them. Let \( f_N : \text{Red}(S) \rightarrow N \) be any \( R \)-homomorphism. Since \( M \) is \( R \)-injective, any \( R \)-homomorphism \( f_M : \text{Red}(S) \rightarrow M \) extends to \( f'_M : S \rightarrow M \). So for any \( R \)-homomorphism \( h : \text{Red}(S) \rightarrow S \), \( f_M = f'_M \circ h \).

Since \( M \) and \( N \) are isomorphic there exists an inverse homomorphism \( \theta^{-1} : M \rightarrow N \) such that \( \theta^{-1} \circ f'_M : S \rightarrow N \) is an \( R \)-homomorphism. Define \( f'_N = \theta^{-1} \circ f'_M : S \rightarrow N \). Then, \( f'_N \) is an extension of \( f_N \). Thus, \( N \) is \( R \)-injective. Similarly, if \( N \) is \( R \)-injective then \( M \) is \( R \)-injective.
4. Suppose that \( A \cong B \) and \( C \) is Red-\( A \)-injective. We show that \( C \) is Red-\( B \)-injective.

Consider the diagram in Figure 2, where \( f'_A : A \to C \) is the extension of \( f_A : \text{Red}(A) \to C \). Let also \( f_B : \text{Red}(B) \to C \) be an \( R \)-homomorphism. Define \( f'_B = f'_A \circ \theta : B \to C \). Then \( f'_B : B \to C \) is the extension of \( f_B \). Thus \( C \) is Red-\( B \)-injective. A similar argument works for the converse.

5. Let \( N \subseteq \oplus M \) and \( M \) be Red-\( K \)-injective. We show that \( N \) is Red-\( K \)-injective. Since \( N \subseteq \oplus M \), there exists an \( R \)-submodule \( N' \) of \( M \) such that \( N \oplus N' = M \). Let \( \pi_N : N \oplus N' \to N \) be the projection \( R \)-homomorphism. Since \( M \) is Red-\( K \)-injective, any \( R \)-homomorphism \( f_M : \text{Red}(K) \to M \) extends to \( f'_M : K \to M \). Suppose \( f_N = \pi_N \circ f_M : \text{Red}(K) \to N \). Define \( f'_N = \pi_N \circ f'_M : K \to N \). Then \( f'_N : K \to N \) is the extension of \( f_N \). Hence, \( N \) is Red-\( K \)-injective.

\( \square \)

**Corollary 2.** Let \( N \) be an \( R \)-module, then

1. a finite direct sum of Red-\( N \)-injective modules is again Red-\( N \)-injective. In particular, a finite direct sum of Red-injective (resp., strongly Red-injective) modules is again Red-injective (resp., strongly Red-injective);

2. a direct summand of Red-quasi-injective (resp., Red-injective, strongly Red-injective) module is again Red-quasi-injective (resp., Red-injective, strongly Red-injective) module.

**Proposition 4.** If \( M \) is a Noetherian \( R \)-module, then a direct sum of Red-\( M \)-injective modules is Red-\( M \)-injective.

**Proof:** For \( D = \bigoplus_{i \in I} D_i \), a direct sum of Red-\( M \)-injective modules, let \( f : K \to D \) be an \( R \)-homomorphism, where \( K \) is any semi-reduced submodule of \( M \). Since \( K \) is finitely generated, \( f(K) \leq \bigoplus_{i=1}^n D_i \) for some positive integer \( n \). Since \( \bigoplus_{i=1}^n D_i \) is Red-\( M \)-injective, then \( f \) can be extended to an \( R \)-homomorphism \( \hat{f} : M \to D \).

\( \square \)
Corollary 3. Let $R_R$ be a Noetherian module. Then, a direct sum of $\text{Red}$-injective modules is $\text{Red}$-injective.

Proposition 5. Let $\{M_i : i \in I\}$ be a family of $R$-modules and $N$ be an $R$-module. Then, $N$ is $\text{Red}(\bigoplus_{i \in I} M_i)$-injective if and only if it is $\text{Red}$-$M_i$-injective for each $i$.

Proof:

($\Rightarrow$). Suppose that $N$ is $\text{Red}(\bigoplus_{i \in I} M_i)$-injective. Let $f : \text{Red}(M_i) \to N$ be any $R$-homomorphism. By hypothesis, any $R$-homomorphism $g : \text{Red}(\bigoplus_{i \in I} M_i) \to N$ extends to $\bar{g} : \bigoplus_{i \in I} M_i \to N$. The required extension of $f$ is $\bar{g} \circ \iota$ where $\iota$ is the injection $\iota : M_i \to \bigoplus_{i \in I} M_i$.

($\Leftarrow$). Suppose that $N$ is $\text{Red}$-$M_i$-injective for each $i \in I$. Since $N$ is $\text{Red}$-$M_i$-injective for each $i \in I$; let $\theta_i : M_i \to N$ be the extension of $f_i : \text{Red}(M_i) \to N$ for each $i \in I$. Let also $g : \text{Red}(\bigoplus_{i \in I} M_i) \to N$ be any $R$-homomorphism. By the fundamental property of direct sum of modules, there exists an $R$-homomorphism $\theta = \langle \theta_i \rangle : \bigoplus_{i \in I} M_i \to N$ such that $\theta \circ \iota_i = \theta_i$ for all $i \in I$; where $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ is the injection $R$-homomorphism for each $i \in I$. Then $\theta$ is an extension of $g : \text{Red}(\bigoplus_{i \in I} M_i) \to N$. Hence, $N$ is $\text{Red}(\bigoplus_{i \in I} M_i)$-injective.

Corollary 4. If $A$, $B$, $C$, and $Q$ are $R$-modules and the short exact sequence $\{0\} \to A \xrightarrow{\mu} B \xrightarrow{\varepsilon} C \to \{0\}$ splits, then the following conditions hold:

1. $Q$ is $\text{Red}$-$C$-injective if and only if it is $\text{Red}$-$(B/\mu(A))$-injective.
2. $Q$ is $\text{Red}$-$B$-injective if and only if it is $\text{Red}$-$A$-injective and $\text{Red}$-$C$-injective.

Proof:

1. This follows from the fact that $B/\mu(A) \cong C$.
2. Follows from Proposition 4, Theorem 1 and the fact that $B \cong A \oplus C$.

Proposition 6. Let $N$ and $M$ be $R$-modules. Then the following conditions hold:

1. $M$ is injective $\Rightarrow$ $M$ is $N$-injective $\Rightarrow$ $M$ is $\text{Red}$-$N$-injective $\Rightarrow$ $M$ is $\text{Soc}$-$N$-injective $\Rightarrow$ $M$ is $\text{min}$-$N$-injective.
2. $M$ is injective $\Rightarrow$ $M$ is strongly $\text{Red}$-injective $\Rightarrow$ $M$ is strongly $\text{Soc}$-injective $\Rightarrow$ $M$ is strongly $\text{min}$-injective $\Leftrightarrow$ $M$ is strongly simple-injective.
Proposition 7. For an $R$-module $M$, if $\text{Red}(M)$ is a direct summand of $M$, then every $R$-module is $\text{Red-M}$-injective.

Proof: Suppose that $K$ is an $R$-module and $\text{Red}(M) \subseteq \oplus M$. We show that $K$ is $\text{Red-M}$-injective. Let $f : \text{Red}(M) \to K$ be any $R$-homomorphism. Since $\text{Red}(M)$ is a direct summand of $M$, there exists a proper $R$-submodule $P$ of $M$ such that $M = \text{Red}(M) \oplus P$. There exists an $R$-homomorphism $f' : M \to \text{Red}(M)$ such that $f'(n+p) = n$, for all $n \in \text{Red}(M)$ and $p \in P$. Then, the $R$-homomorphism $f \circ f' : M \to K$ is an extension of $f$ because $(f \circ f')(n+p) = f(f'(n+p)) = f(n)$ for all $n+p \in M$. Hence, $K$ is $\text{Red-M}$-injective. $\square$

Theorem 2. For a projective $R$-module $M$, the following conditions are equivalent:

1. Every quotient of a $\text{Red-M}$-injective $R$-module is $\text{Red-M}$-injective.
2. Every quotient of an injective $R$-module is $\text{Red-M}$-injective.
3. $\text{Red}(M)$ is a projective $R$-module.

Proof:

(1 $\Rightarrow$ 2). This is due to the fact that every injective $R$-module is $\text{Red-M}$-injective.

(2 $\Rightarrow$ 3). Consider the diagram in Figure 3 below:

$$
\begin{array}{ccccccc}
E & \xrightarrow{\varepsilon} & N & \xrightarrow{i} & \{0\} \\
\downarrow f & & \downarrow & & \\
\{0\} & \xrightarrow{f} & \text{Red}(M) & \xrightarrow{i} & M
\end{array}
$$

Figure 3

where $E$ and $N$ are $R$-modules, $\varepsilon$ an $R$-epimorphism, and $f$ an $R$-homomorphism. By [4, Proposition 5.1], assume that $E$ is injective. Since $N$ is $\text{Red-M}$-injective $f$ can be extended to an $R$-homomorphism $g : M \to N$. Since $M$ is projective, $g$ can be lifted to an $R$-homomorphism $\bar{g} : M \to E$ such that $\varepsilon \circ \bar{g} = g$. Define $\bar{f} : \text{Red}(M) \to E$ by $\bar{f} = \bar{g}|_{\text{Red}(M)}$. Then $\varepsilon \circ \bar{f} = \varepsilon \circ \bar{g}|_{\text{Red}(M)} = f$. Hence, $\text{Red}(M)$ is projective.

(3 $\Rightarrow$ 1). Let $N$ and $L$ be $R$-modules with $\varepsilon : N \to L$ an $R$-epimorphism and $N$ is $\text{Red-M}$-injective. Consider the diagram in Figure 4.
Since Red(M) is projective, \( f \) can be lifted to an \( R \)-homomorphism \( g : \text{Red}(M) \to N \) such that \( \varepsilon \circ g(m) = f(m) \), for all \( m \in \text{Red}(M) \). Since \( N \) is Red-\( M \)-injective, \( g \) can be extended to an \( R \)-homomorphism \( \tilde{g} : M \to N \). Hence, \( \varepsilon \circ \tilde{g} : M \to L \) extends \( f \).

\[ \varepsilon \]

\[ \text{Corollary 5.} \] The following conditions are equivalent for a reduced projective \( R \)-module:

1. Every quotient of a Red-injective \( R \)-module is Red-injective.
2. Every quotient of an injective \( R \)-module is Red-injective.
3. \( \text{Red}(R_R) \) is a projective module.

In addition, if every semi-reduced submodule of a projective \( R \)-module is projective, then \( \text{Red}(R_R) \) is a projective module.

**Proof:** 1 \( \iff \) 1 \( \iff \) 4 follows from Theorem 2. The additional case follows from the fact that \( \text{Red}(R_R) \) is a semi-reduced submodule of a projective module \( R_R \).

**Proposition 8.** Let \( R \) be a Principal Ideal Domain (PID) and \( N \) be an \( R \)-module. Then, the following statements hold:

1. If every free \( R \)-module is Red-\( N \)-injective then each of its submodules is Red-\( N \)-injective.
2. If every projective \( R \)-module is Red-\( N \)-injective then each of its submodules is Red-\( N \)-injective.
3. Every projective \( R \)-module is Red-\( N \)-injective if and only if every free \( R \)-module is Red-\( N \)-injective.

**Proof:**

1. Suppose that every free \( R \)-module \( M \) is Red-\( N \)-injective, and \( L \leq M \). Since over a PID a submodule of a free module is free, \( L \) is free. By hypothesis, \( L \) is Red-\( N \)-injective.
2. Suppose that every projective $R$-module $P$ is Red-$N$-injective, and $K \leq P$. Since over a PID a submodule of a projective $R$-module is projective, $K$ is projective. By hypothesis, $K$ is Red-$N$-injective.

3. Over a PID every projective module is free. The converse holds since any free module is projective.

Definition 8. Let $X$ be a submodule of a module $M$. We say that Red$(M)$ respects $X$ if there exists a direct summand $A$ of $M$ contained in $X$ such that $X = A \oplus B$ and $B \leq \text{Red}(M)$. $M$ is called Red$(M)$-lifting if Red$(M)$ respects every submodule of $M$.

Proposition 9. Let $N$ be an $R$-module. If $N$ is Red$(N)$-lifting, then any $R$-module $K$ is Red-$N$-injective if and only if $K$ is $N$-injective.

Proof:

$(\Rightarrow)$. Suppose that $K$ is Red-$N$-injective. Let $L$ be any submodule of $N$, $\iota : L \to N$ the inclusion map and $f : L \to K$ any $R$-homomorphism. Since Red$(N)$ respects $L$, $L$ has a decomposition $L = A \oplus B$ such that $A \subseteq N$ and $B \leq \text{Red}(N)$. $N = A \oplus A'$ for some submodule $A'$ of $N$. Then, $L = A \oplus (L \cap A')$ and $L \cap A'$ is semi-reduced. Let $i : L \cap A' \to L$ be the inclusion map and $f|_{L \cap A'} : L \cap A' \to K$. Since $K$ is Red-$N$-injective, there exists an $R$-homomorphism $g : N \to K$ such that $g \circ \iota \circ i = f|_{L \cap A'}$. Now, define $h : N \to K$ by $h(a + a') = f(a) + g(a')$ ($a \in A, a' \in A'$). Then $h \circ \iota = f$, and hence $K$ is $N$-injective.

$(\Leftarrow)$. Every $N$-injective module is Red-N-injective. This is due to the fact that for every $N$-injective module $K$, any $R$-homomorphism from any submodule of $N$ to $K$ extends to $N$. 

Proposition 9 shows that Red-quasi-injective modules inherit a weaker version of C2-condition and C3-conditions.
**Proposition 10.** Suppose that an $R$-module $N$ is Red-quasi-injective.

1. (Red-C2) If $P$ and $Q$ are semi-reduced submodules of $N$, $P \cong Q$ and $P \subseteq N$, then $Q \subseteq N$.

2. (Red-C3) Let $P$ and $Q$ be semi-reduced submodules of $N$ with $P \cap Q = \{0\}$. If $P \subseteq N$ and $Q \subseteq N$; then $P \oplus Q \subseteq N$.

Proof:

1. Since $P \cong Q$, and $P$ is Red-$N$-injective, being a direct summand of a Red-quasi-injective module $N$, $Q$ is Red-$N$-injective by Corollary 2(2). If $i : Q \rightarrow N$ is the inclusion map, the identity $id_Q : Q \rightarrow Q$ has an extension $\eta : N \rightarrow Q$ such that $\eta \circ i = id_Q$, and hence $Q \subseteq N$.

2. Since both $P$ and $Q$ are direct summands of $N$; then both $P$ and $Q$ are Red-$N$-injective. Then the semi-reduced module $P \oplus Q$ is Red-$N$-injective, and so a direct summand of $N$ by an argument similar to the one given in 1.

3 **Strongly Red-injective modules**

In this section, we characterize quasi-Frobenius rings and right $V$-rings in terms of strongly Red-injective modules. A ring $R$ is called right semi-Artinian if every non-zero $R$-module has nonzero socle. A submodule $S \leq M$ is small if, for any submodule $N \leq M$, $S + N = M$ implies that $N = M$. The projective cover of an $R$-module $M$ is a projective module $P$ for which there is an epimorphism $P \rightarrow M$ whose kernel is small. A ring $R$ is left perfect if every $R$-module has a projective cover.

**Proposition 11.** The following implications hold:

- $R$ is right semi-Artinian $\Rightarrow$ every strongly Red-injective $R$-module is injective $\Rightarrow$ every strongly Red-injective $R$-module is quasi-continuous.

In particular, over a left perfect ring $R$, every strongly Red-injective right $R$-module is injective.

**Proof:** For a right semi-Artinian ring $R$, suppose that a non-zero $R$-module $M$ is strongly Red-injective. Then, $\{0\} \neq \text{Soc}(M) \subseteq M$. Amin et al., in [1, Corollary 3.2] showed that a strongly Soc-injective module with essential socle is injective. Since $M$ has essential socle, it is injective. $M$ is quasi-continuous because every injective module is quasi-continuous see [8, p.18]. The last statement follows from the fact that every left perfect ring is right semi-Artinian, see [6, Theorem 11.6.3].

A ring $R$ is called quasi-Frobenius if $R$ is right (or left) Artinian and right (or left) self-injective. Equivalently, $R$ is quasi-Frobenius if and only if every injective $R$-module is projective if and only if
every projective \( R \)-module is injective. A ring whose all simple right modules are injective is called a right \( V \)-ring.

**Theorem 3.** A ring \( R \) is quasi-Frobenius if and only if every strongly Red-injective module is projective.

**Proof:** If \( R \) is quasi-Frobenius, then \( R \) is right semi-Artinian and so by Proposition 10 every strongly Red-injective module is injective, and hence projective since \( R \) is quasi-Frobenius. Conversely, if every strongly Red-injective module is projective, then in particular every injective module is projective, and so \( R \) is quasi-Frobenius. \( \square \)

**Theorem 4.** \( R \) is a right \( V \)-ring if and only if every simple \( R \)-module is strongly Red-injective.

**Proof:** Suppose that \( M \) is a simple \( R \)-module where \( R \) is a right \( V \)-ring. Then, by definition of a \( V \)-ring, \( M \) is injective. Hence, \( M \) is strongly Red-injective. Conversely, suppose that any simple module \( M \) is strongly Red-injective. Since \( M \) is simple, \( \text{Soc}(M) = M \) and hence \( \{0\} \neq \text{Soc}(M) \subseteq^e M \). Since \( M \) has essential socle, it is injective by [1, Corollary 3.2]. Hence, \( R \) is a right \( V \)-ring. \( \square \)

**Corollary 6.** Let \( M \) be an \( R \)-module with essential socle. The following statements are equivalent:

1. \( M \) is injective.
2. \( M \) is strongly Red-injective.
3. \( M \) is strongly Soc-injective.

**Proof:** By Proposition 5(2), \( 1 \Rightarrow 2 \Rightarrow 3 \). By [1, Corollary 3.2], \( 3 \Rightarrow 1 \) which completes the proof. \( \square \)

**Remark 1.** We make the following observations:

1. It is easy to check that any sort of injectivity that lies between injective and strongly Red-injective modules would lead to Theorems 3 and 4.
2. Red-injectivity is a less restricted notion than injectivity but carries most of the properties of injectivity.
3. Red-injectivity is much closer to injectivity than Soc-injectivity.
4. When the ring is not commutative, a semi-simple module need not be semi-reduced, see Example 3.1 below:

**Example 3.1.** Let the ring \( R \) be the collection of all \( 2 \times 2 \) matrices over the field of real numbers. The module \( M = R_R \) is semi-simple but not reduced. For if \( m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M \) and \( r = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in R \), then \( mr \neq 0 \) but \( mr^2 = 0 \). Since a direct sum of reduced modules is reduced, \( M \) is a direct sum of simple modules which is not reduced. A simple module over a not necessarily commutative ring need not be reduced. \( M \) is not semi-reduced.
The following implications hold:

Injective ⇒ quasi-injective ⇒ pseudo-injective ⇒ Red-quasi-injective.

For the first two implications, see [9]. The last implication is trivial, it follows directly from the definitions.

**Example 3.2.** Let $R$ be the ring of all eventually constant sequences $(x_n)_{n \in \mathbb{N}}$ of elements in $\mathbb{F}_2$, the field of two elements. Then, $E(R_R) = \prod_{n \in \mathbb{N}} \mathbb{F}_2$, which has only one automorphism, namely the identity automorphism. By [5, Example 9], $R_R$ is pseudo-injective but it is not quasi-injective. It therefore follows that $R_R$ is Red-quasi-injective but not injective.

An $R$-module $M$ is said to satisfy the **exchange property** if for every $R$-module $A$ and any two direct sum decomposition $A = M' \bigoplus N = \bigoplus_{i \in I} A_i$ with $M' = M$, there exists a submodule $B_i$ of $A_i$ such that $A = M' \bigoplus (\bigoplus_{i \in I} B_i)$. An $R$-module is called **clean** if its endomorphism ring, $\text{End}_R(M)$, is clean, i.e., for all $f \in \text{End}_R(M)$, $f = e + u$ with $e$ idempotent and $u$ a unit. Pseudo-injective modules (and hence quasi-injective and injective modules) are clean and also satisfy the exchange property, see [2] and [3]. Note that pseudo-injective modules are equivalent to automorphism-invariant modules as they are being referred to in [2] and [3]. The equivalence was proved in [5]. We now ask:

**Question 1.** Are the Red-quasi-injective modules and Soc-quasi-injective modules clean? Do they satisfy the exchange property?

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