THE PROOF OF GROMOLL-WALSCHAP CONJECTURE

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Abstract. It is conjectured that there are no Riemannian foliations on compact manifold of negative curvature by Gromoll-Walschap in [3]. In this note we give a positive answer.

1. Introduction

Let $M$ be an $n + m$ dimensional smooth connected compact manifold. An $n$-dimensional foliation $\mathcal{F}$ on $M$ is a partition of $M$ into $n$-dimensional submanifolds called leaves. $\mathcal{F}$ is called Riemannian if its leaves are locally equidistant. A Riemannian foliation always refers to a metric foliation in [3]. The leaves of a Riemannian foliation are locally given as fibers of a Riemannian submersion.

Riemannian submersion plays a key role in understanding the structure of Riemannian manifolds with nonnegative sectional curvature. The soul theorem proved by Perelman asserts that all complete noncompact Riemannian manifolds with nonnegative sectional curvature are Riemannian submersion over its soul ([7]). There are some known intriguing examples with respect to Riemannian foliations in nonnegative constant curvature spaces, such as one-dimensional foliation like Hopf fibration $U(1) \to S^{2n+1} \to \mathbb{CP}^n$, three dimensional Riemannian foliation like quaternionic Hopf fibration $Sp(1) \to S^{4n+3} \to \mathbb{PH}^n$, low-dimensional Riemannian foliations of Euclidean spheres ([2]), metric fibrations of arbitrary dimension on Euclidean space ([2]). Munteanu has considered one-dimensional Riemannian foliations on the Heisenberg group ([5]). For more examples about Riemannian foliation on compact manifolds with constant nonnegative curvature, the reader is referred to [3].

The nature question to be asked is whether there are Riemannian foliations on any compact negative sectional curvature manifold. Ranjan has shown that a compact manifold $M$ with negative Ricci curvature admits no one-dimensional metric foliations ([8]). In light of Ranjan’s result, Gromoll-Walschap in [3] (page 163, see the part before Theorem 4.5.2) conjectured that there are no metric foliations on compact manifolds of negative curvature.

In this note, we give a positive answer of Gromoll-Walschap conjecture. A key ingredient of the argument is to compute the divergence of mean curvature vector field $N$ of the leaves. The assumption of Riemannian foliation allows us to simplify greatly the structural equation derived by Ranjan in the framework of general foliations. That is, we obtain a formula $\text{div} \, N = S_{\text{mix}} - |A^*|^2$, where $S_{\text{mix}}$ represents mixed scalar curvature and $A^*$ the adjoint of the integrability tensor $A$. Furthermore, according to this formula and using Stokes theorem, we show that negative mixed scalar curvature is an obstruction to existence of Riemannian foliations in a compact manifold (see Theorem 1). Especially, Gromoll-Walschap conjecture is a direct result of our Theorem 1 (see Corollary 1). As mentioned above, Ranjan has obtained that a compact manifold $M$ with negative Ricci curvature admits no one-dimensional metric foliations ([8]). Beside his result, in this work, we also obtain that a compact manifold with

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negative Ricci curvature admits no codimension-one Riemannian foliations (see Corollary 2). In addition, when a compact manifold can be foliated by Riemannian foliations, we get a sufficient and necessary condition for flatness of the foliations (see Corollary 3).

2. The basic geometry on Riemannian foliation

In this section, we recall some basic facts about Riemannian foliation and Riemannian submersions, the reader is referred to [1], [3], [6] and [15] for more details.

Let $M, B$ be differential manifolds and $\pi : M \rightarrow B$ be a submersion, i.e. $\pi$ is a surjective differential maps of maximal rank. The vertical distribution $V$ of a submersion is defined to be the kernel of $\pi_\ast$. A vertical field is a section of the vertical subbundle $V \rightarrow M$. If $M$ is a Riemannian manifold, the orthogonal complement $H$ of $V$ is called the horizontal distribution of $\pi$. A horizontal field is a section of the horizontal subbundle $H \rightarrow M$. This means that the foliation $\mathcal{F}$ admits an orthogonal splitting $T M = H \bigoplus V$ of the tangent bundle of $M$.

For any $e \in T M$, the orthogonal splitting of the tangent bundle induces a decomposition $e = e^h + e^v \in H \bigoplus V$.

Let $M, B$ be Riemannian manifolds, a differential map $\pi : M \rightarrow B$ is called a Riemannian submersion if $\pi$ is a submersion and $\pi_\ast$ preserves the length of horizontal vectors, that is, $|\pi_\ast x| = |x|$ for all $x \in H$.

A horizontal vector field $X$ on $M$ is called basic if $\pi_\ast X = X \circ \pi$, where $X$ is a vector field on $B$. We always use $T, T_j$ to denote vertical fields, and $X, Y$ to denote basic fields on open set $U$ of $M$ related to $X, Y$ on $B$.

The integrability tensor $A$ of the horizontal bundle $H$ is the skew 2-form on $H$ with values in $V$, given by

$$A_X Y = \frac{1}{2} [X, Y]^v = \nabla^v_X Y$$

where $\nabla$ is the standard Riemannian connection of $M$. The foliation is flat if $A = 0$.

Denoting by $A_X : V \rightarrow H$ the adjoint of $A_X : H \rightarrow V$:

$$\langle A_X T, Y \rangle = \langle T, A_X Y \rangle, \quad X, Y \in H, \quad T \in V$$

If $X$ is basic, then for vertical $T$,

$$A_X T = -\nabla^h_X T = -\nabla^h_Y X,$$

The second fundamental tensor $S$ of $\mathcal{F}$ is the horizontal 1-form on $H$ with values in selfadjoint transformations of $V$,

$$S_X T = -\nabla^v_{T} X.$$

The mean curvature vector field of the Riemannian manifold $M$ is defined by

$$N = \sum_{j=1}^{n} \nabla^h_{T_j} T_j$$

where $\{T_1, \cdots, T_n\}$ is a local orthonormal basis of the vertical distribution.

The curvature tensor of $M$ always satisfies the relation (see 1.5.9 in [3])

$$R^h(X, T)Y = -(\nabla^v_Y A_X X - A_Y A_X X - (\nabla^v_X S)Y T + S_Y S_X T$$

It is remarked that $(\nabla^v_Y A_X X = -(\nabla^v_Y A_Y X)$. The sectional curvature $K(X, T) = \langle R(T, X)X, T \rangle$ is called mixed if $X \in H$ and $T \in V$. The mixed scalar curvature is an averaged mixed sectional...
curvature

\[ S_{\text{mix}} = \sum_{i=1}^{m} \sum_{j=1}^{n} K(X_i, T_j) \]

and is independent of the choice of a local orthonormal frame \( \{X_i, T_j\}_{1 \leq i \leq m, 1 \leq j \leq n} \) of \( TM \) adapted to \( H \) and \( V \). The mixed scalar curvature of a foliated Riemannian manifold has been considered by several geometers (see for examples \([9-13]\)).

3. Proof of Gromoll-Walschap conjecture

Lemma 1. (see lemma 1.4.1 in \([3]\)) Given a Riemannian submersion \( \pi : M \to B \), if \( X, Y \) are basic, then so is \( \nabla_X^h Y \). Moreover, \( \pi_* (\nabla_X^h Y) = (\nabla_X^B \iota) \circ \pi \), where \( \nabla^B \) is the Levi-Civita connection of \( B \).

Theorem 1. A compact manifold \( M \) with negative mixed scalar curvature admits no Riemannian foliations.

Proof. Let \( p \in M, \pi : U \to B \) a submersion defining \( \mathfrak{g} \) in a neighborhood \( U \) of \( p \), and \( \overline{X}_i \) local orthonormal fields on \( B \) with \( \nabla_{\overline{X}_i} \overline{X}_i (\pi(p)) = 0 \). Then the basic fields \( X_i \) on \( U \) that are \( \pi \)-related to \( \overline{X}_i \) satisfy \( \nabla_X^h X_j = 0 \) from lemma 1. Let \( T_1, \cdots, T_n \) be vertical orthonormal frame fields which span vertical bundle, the mean curvature vector field is given locally by \( N = \sum_{j=1}^{n} \nabla_{T_j}^h T_j \). The divergence of the mean curvature vector field reads

\[ \text{div } N = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle \nabla_X^h, \nabla_{T_j} T_j, X_i \rangle + \sum_{j,k=1}^{n} \langle \nabla_{T_k} \nabla_{T_j}^h T_j, T_k \rangle. \]

The second term on the right equals \(-|N|^2\). In order to simplify the first term, we firstly notice that

\[ \sum_{j=1}^{n} \langle S_{X_j} T_j, \nabla_{X_j} T_j \rangle = \sum_{j,k=1}^{n} \langle S_{X_j} T_j, T_k \rangle \langle \nabla_{X_j} T_j, T_k \rangle \]

\[ = - \sum_{j,k=1}^{n} \langle T_j, S_{X_j} T_k \rangle \langle T_j, \nabla_{X_j} T_k \rangle \]

\[ = - \sum_{j,k=1}^{n} \langle T_k, S_{X_j} T_j \rangle \langle T_k, \nabla_{X_j} T_j \rangle \]

\[ = - \sum_{j=1}^{n} \langle S_{X_j} T_j, \nabla_{X_j} T_j \rangle, \]
which means that $\sum_{j=1}^{n} \langle S_{X_j}, \nabla_{X_j} \rangle = 0$. As a result, it follows that

$$\sum_{j=1}^{n} \langle \nabla_X, \nabla_{T_j}^h T_j, X_i \rangle = \sum_{j=1}^{n} \langle X_i (\nabla_{T_j}^h T_j, X_i) - \langle \nabla_{T_j}^h T_j, \nabla_X, X_i \rangle \rangle$$

$$= \sum_{j=1}^{n} X_i \langle \nabla_{T_j}^h T_j, X_i \rangle = \sum_{j=1}^{n} X_i \langle S_{X_j}, T_j \rangle$$

$$= \sum_{j=1}^{n} \langle \nabla_{X_i} (S_{X_j}, T_j), T_j \rangle + \sum_{j=1}^{n} \langle S_{X_j}, \nabla_X, T_j \rangle$$

$$= \sum_{j=1}^{n} \langle (\nabla_{X_i}^h S) X_i, T_j \rangle$$

where we have used the fact that $\sum_{j=1}^{n} \langle S_{X_j}, \nabla_X, T_j \rangle = 0$ in the last equality. Using (1), we now obtain

$$\sum_{j=1}^{n} \langle \nabla_X, \nabla_{T_j}^h T_j, X_i \rangle = \sum_{j=1}^{n} \left( \langle R(T_j, X_i) X_i, T_j \rangle - |A_{X_j}^* T_j|^2 + |S_{X_j} T_j|^2 \right).$$

Observing that $|N|^2 = |\sum_{j=1}^{n} \nabla_{T_j}^h T_j|^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} |S_{X_j} T_j|^2$, we finally get

$$\text{div } N = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \langle R(T_j, X_i) X_i, T_j \rangle - |A_{X_j}^* T_j|^2 \right)$$

$$= S_{\text{mix}} - |A^*|^2$$

where $|A^*|$ denotes the Hilbert-Schmidt norm of the operator $X \rightarrow A_X^* Y$. The theorem follows, since the divergence of $N$ integrates to zero over $M$. \hfill \Box

It is obvious that the mix scalar curvature of a manifold $M$ with negative curvature is always negative, so we obtain the following Gromoll-Walschap conjecture.

**Corollary 1.** A compact manifold $M$ with negative curvature admits no Riemannian foliations.

In the case of one-dimensional and codimension one foliations, the mixed scalar curvature of a manifold $M$ becomes Ricci curvature. Consequently, we have

**Corollary 2.** A compact manifold $M$ with negative Ricci curvature admits no Riemannian one-dimensional and codimension-one foliations.

In addition, by a simple computation, we have $|A^*|^2 = |A|^2$, so it follows that

**Corollary 3.** Let $M$ be a compact manifold with Riemannian foliation. Then $\int_M S_{\text{mix}} \geq 0$ with equality iff the Riemannian foliation is flat.

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