Causal feature of central singularity and gravitational mass

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Abstract

Mass of singularity is defined, and its relation to whether the singularity is spacelike, timelike or null is discussed for spherically symmetric spacetimes. It is shown that if the mass of singularity is positive (negative) the singularity is non-timelike (non-spacelike). The connection between the sign of the mass and the force on a particle is also discussed.

I. INTRODUCTION

In discussing cosmic censorship [1] of singular spacetimes in general relativity it is important whether the singularity is “spacelike”, “timelike”, or “null”, which can be defined by conformal embedding. A spacelike singularity immediately implies existence of an event horizon and save the cosmic censorship. On the contrary, a timelike singularity is naked, hence violates the cosmic censorship.
It is important to know what physical quantity determines the causal feature of a singularity. This quantity should be defined locally since the causal feature is a local concept.

The concept of quasi-local mass has been proposed by many authors [2–8]. Though asymptotic mass is well-defined such as the ADM [9] or the Bondi [10] mass, there is no satisfactory definitions of quasi-local mass. When we discuss an appropriate definition of quasi-local mass, its physical implication is no less important than mathematical property to be considered as energy. If we regard the quasi-local mass as the generalization of the Schwarzschild mass, we may expect in a general spacetime that the quasi-local mass play similar roles as the Schwarzschild mass.

Let us consider the singularity $r = 0$ of Schwarzschild spacetime,

$$ds^2 = -(1 - 2mr^{-1})dt^2 + (1 - 2mr^{-1})^{-1}dr^2 + r^2(d\theta^2 + \sin \theta d\phi^2).$$

(1.1)

The usual definition of quasi-local mass $M(t, r)$ in spherical symmetric spacetimes gives $M(t, r) = m$. The mass of the singularity can be defined to be the limit of $M(t, r)$ when $r \to 0$, which is $m$ in this case. The singularity is spacelike in the case $m > 0$ and is timelike in the case $m < 0$.

In this paper we define mass of singularity and discuss the relation between the mass of singularity and whether it is spacelike, timelike or null. We concentrated on spherically symmetric spacetimes because they are essentially two-dimensional and have a conformal embedding to the flat two-dimensional flat space with infinity and singularity being its boundary. We investigate the statement in general spherically symmetric: if the mass of the singularity is positive, the singularity is spacelike, and if it is negative, the singularity is timelike. Further if we see that the negative mass implies the existence of naked singularities, it may be possible to prove the cosmic censorship theorem by discussing the inadequateness of negative mass.

In §2, we define the terms in the above statement precisely. Whether it holds or not is investigated in §3. The §4 reveals the motion of a particle near the negative mass singularity. In negative mass Schwarzschild spacetime no particle hits the singularity. The final section
II. MASS OF SINGULARITY IN SPHERICALLY SYMMETRIC SPACETIMES

We assume that the spacetime manifold \((M, g_{ab})\) is an \(S^2\)-bundle over a two-dimensional manifold \((M_1, (g_1)_{ab})\) so that the metric is given as

\[
(g)_{ab} = (g_1)_{ab} + (g_2)_{ab},
\]

(2.1)

where \((g_2)_{ab}\) is the metric of a sphere characterized by the area \(A = 4\pi R^2\) with \(R\) being a (positive) function on \(M_1\).

Two-dimensional space \(M_1\), or at least a part of it under consideration, is conformally embedded into two-dimensional flat space \((\tilde{M}_1, (\tilde{g}_1)_{ab}) = (\tilde{M}_1, \eta_{ab})\) as

\[
\tilde{M}_1 = M_1 \cup \partial M,
\]

(2.2)

\[
(\tilde{g}_1)_{ab} = \eta_{ab} = \Omega^2 (g_1)_{ab}.
\]

(2.3)

Let us call the boundary on which \(R = 0\) central. We consider point \(p\) of the central boundary and the neighborhood \(U\) of \(p\) in \(\tilde{M}_1\). The function \(R\) is continuous on \(M\) and the central boundary. We assume that the function \(R\) and either \(\Omega\) or \(\Omega^{-1}\) can be continuously extended to \(U \cap \partial M_1\).

The line element is written in coordinates as

\[
ds^2 = ds_1^2 + ds_2^2,
\]

\[
ds_1^2 = -2\Omega^{-2}(u, v)du dv = -a^2(t, r)dt^2 + b^2(t, r)dr^2,
\]

\[
ds_2^2 = R^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]

(2.4)

Dual-null tetrad can be chosen as

\[
l^a = \Omega(\partial_u)^a = \frac{1}{\sqrt{2} ab}(b(\partial_t)^a + a(\partial_r)^a),
\]

\[
n^a = \Omega(\partial_v)^a = \frac{1}{\sqrt{2} ab}(b(\partial_t)^a - a(\partial_r)^a),
\]
\[(e_1)^a = R^{-1}(\partial_\theta)^a,\]
\[(e_2)^a = R^{-1}(\sin \theta)^{-1}(\partial_\phi)^a;\]

(2.5)

where \(l^a\) and \(n^a\) are the out-going and in-going null vectors, respectively.

In spherically symmetric spacetimes the known quasi-local masses such as the Penrose mass and the Hawking mass fall into the standard one \(R^3(\Phi_{11} + \Lambda - \Phi_2)\). The Hawking’s expression for \(S^2(q)\), the round sphere determined by \(q \in M_1\) is given by

\[
M(q) = \frac{1}{32\pi^{3/2}}A^{1/2} \int_S \mu(R + \theta_+ \theta_-) = \frac{R^3}{4}(R + \theta_+ \theta_-) \tag{2.6}
\]

where \(\mu\) is the volume form on \(M_2\). The scalar curvature \(R\) of \(S^2(q)\) and the expansions \(\theta_+\) and \(\theta_-\) of out-going and in-going null rays, respectively, are given in our coordinates as

\[
R = 2R^{-2},
\]
\[
\theta_+ = 2\Omega R^{-1} R_v = (1/\sqrt{2})(Rab)^{-1}(bR_t - aR_r),
\]
\[
\theta_- = 2\Omega R^{-1} R_u = (1/\sqrt{2})(Rab)^{-1}(bR_t + aR_r). \tag{2.7}
\]

Eq. (2.6) yields

\[
M(q) = R(\Omega^2 R_u R_v + \frac{1}{2})
\]
\[\]
\[
= \frac{R}{2} (a^{-2} R_t^2 - b^{-2} R_r^2 + 1). \tag{2.8}
\]

The mass \(M_S\) of a point \(p\) of the central boundary is defined to be the limit of \(M(q)\) for \(q \rightarrow p\), where \(q\) is a point in \(U \cap M\).

From (2.8) we have

\[
M_S(p) = \lim_{q \rightarrow p} \Omega^2 R R_u R_v. \tag{2.9}
\]

We present some examples of spherically symmetric spacetimes with central singularity. The mass is calculated by the formula (2.8). The conformal diagram of the first two examples are well known [11]. The conformal diagram of the Tolman-Bondi spacetime [12,13] is found by Eardley and Smarr [14]. The statement given in the bottom of §4 is true for all of them.
### TABLES

| Spacetime                        | Causal feature | Mass of central boundary |
|----------------------------------|----------------|--------------------------|
| The Schwarzschild spacetime      |                |                          |
| \(a^2 = b^{-2} = 1 - 2mr^{-1}, R = r\) | (positive-mass) | spacelike | \(m(> 0)\) |
|                                  | (negative-mass) | timelike | \(m(< 0)\) |
| The Reissner-Nordström spacetime |                |                          |
| \(a^2 = b^{-2} = 1 - 2mr^{-1} + e^2r^{-2}, R = r\) | timelike | \(-\infty\) |
| The Friedmann-Robertson-Walker spacetime |                |                          |
| \(a = 1, b = b(t), R = bc,\)   |                |                          |
| \(b^2 = \frac{8}{3}\pi b^2 \rho(t) - k,\) | (dust-filled) | spacelike | \((4\pi/3)R^3\rho(=\text{constant}>0)\) |
| \(c = \sin r, r \text{ or sinh } r \text{ for } k = 0, 1, -1,\) | (radiation-filled) | spacelike | \(+\infty\) |
| \(\rho(t): \text{ density}\) |                |                          |
| The Tolman-Bondi spacetime       |                |                          |
| \(a = 1, R_r = W(r)b,\)        |                |                          |
| \(\frac{1}{2}R^2_r - m_0(r)R^{-1} = \frac{1}{2}(W^2(r) - 1),\) | spacelike | \(m_0(r)(>0)\) |
| \(m_0(r) = 4\pi \int_0^r \rho_0 r^2 dr,\) |                |                          |
| \(W(r): \text{ any function}\) | null           | 0                        |

**TABLE I.**
III. CAUSAL FEATURE

If the boundary point $R = 0$ is not singular the mass must be zero. So if the central mass is nonzero the boundary point at which $R = 0$ must be a singularity. In this section we investigate causal feature of central singularities.

A. Trappedness

The mass $M_S$ of the singularity $p$ immediately gives the informations of whether the points near the singularity are trapped or not. From (2.6) and (2.7) one has

$$\theta_+ - \theta_- = \frac{2}{R^2} \left( \frac{2M(q)}{R} - 1 \right).$$

(3.1)

If $M_S(p) > 0$ then $\theta_+ - \theta_-$ becomes positive because $M(q) \to M_S(p) > 0$ and $R \to 0$ as $q$ approaches $p$. This means that the points near the singularity are trapped. If $M_S(p) < 0$ then $\theta_+ - \theta_-$ becomes negative because $M(q) \to M_S(p) < 0$. This means that the points near the singularity are not trapped. Therefore, one cannot see positive-mass singularities.

B. Spacelikeness and timelikeness of singularity

The argument in the previous section suggests the following. In the positive-mass case both of the expansions $\theta_+ - \theta_-$ are negative (positive), which implies that both of the future (past) directed out-going and in-going null geodesics approaches the $R = 0$ singularity. In the negative-mass case one of them approaches and the other goes away from the singularity. One may intuitively think that positive-mass singularities are spacelike and negative-mass singularities are timelike. We discuss this more precisely.

We define the boundary $R = 0$ to be spacelike, timelike, or null by the conformally related metric $(\tilde{g}_1)_{ab} = \eta_{ab}$. There must be a strictly increasing smooth function $F$ of $R$ such that $(dF)_a$ can be extended smoothly to the central boundary and does not vanish there.
Let us say that the central boundary is spacelike, timelike, or null if \((\tilde{g}_1)_{ab}(dR)_a(dR)_b\) is positive, negative, or zero, respectively. The norm is given by

\[
(\tilde{g}_1)_{ab}(dF)_a(dF)_b = -2F_R R_u R_v. \tag{3.2}
\]

By (2.9), if \(R\Omega^2 F_R^{-2}\) is bounded, positive mass implies that the norm (3.2) is negative infinity so that the singularities must be spacelike. If \(R\Omega^2 F_R^{-2}\) is not bounded the singular boundary is timelike or null. An example of the positive mass with a null singular boundary is given by \(R = -uv, \Omega = m^{1/2} R^{-1}(m > 0)\), where the mass on a boundary point is given by \(M_S = m\). Negative central mass implies that the singularity is timelike or null by the same argument.

The zero-mass boundary point has the following possibilities: it is a spacelike singularity, it is a timelike singularity, it is a null singularity, or it is not a singularity.

\section*{IV. Dynamics of a Particle Near Singularity}

In the negative-mass Schwarzschild spacetime any inertial observer cannot fall into the singularity, though any observer can see it. We expect that the situation is the same in general cases of negative-mass singularity.

In the Schwarzschild case it is verified by writing down the geodesic equations and obtain the first integral of them. The equation can be treated as a problem of a particle in a potential which is infinite at \(R = 0\). Since general spherically symmetric spacetimes do not admit calculating such first integrals explicitly we investigate whether the negative-mass singularity has infinite repulsive force by calculating the acceleration of an observer.

If the singularity is timelike, we can choose the coordinate \(r\) to be the area radius \(R\) in the metric (2.4):

\[
ds^2 = -a^2(t, R) dt^2 + b^2(t, R) dR^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{4.1}
\]

The formula (2.8) implies that \(b\) is expressed by the quasi-local mass as
\[ b^2 = \left( 1 - \frac{2M(t,R)}{R} \right)^{-1}. \]  

(4.2)

Let \( u^a \) be the 4-velocity of an observer with constant \( R \) and \( s^a \) be a radial unit vector normal to \( u^a \) i.e.

\[ u^a = a^{-1}(\partial_t)^a, \quad s^a = b^{-1}(\partial_R)^a. \]  

(4.3)

The acceleration of this observer is

\[ f = s_au^b \nabla_b u^a = \frac{a_{,R}}{a} \left( 1 - \frac{2M(t,R)}{R} \right)^{\frac{1}{2}} \]  

(4.4)

where \( \nabla_a \) is the covariant derivative of \( g_{ab} \). The (tt) and (RR)-components of Einstein’s equation reduce to

\[ M_{,R} = 4\pi R^2 \rho, \]  

(4.5)

\[ f = \frac{1}{R^2} \left( M(t,R) + 4\pi R^3 p_1 \right) \left( 1 - \frac{2M(t,R)}{R} \right)^{-\frac{1}{2}}, \]  

(4.6)

where \( \rho = T_{ab}u^au^b \) and \( p_1 = T_{ab}s^as^b \) with \( T_{ab} \) being the energy-momentum tensor. Equation (4.3) corresponds to the Gauss law

\[ M(t,R) = MS + 4\pi \int_0^R R^2 \rho dR. \]  

(4.7)

Equation (4.3) is the expression of the acceleration of a particle caused by gravitational potential. The second term in the first parenthesis is a general relativistic effect. Using this equation, the acceleration near the central singularity is given by

\[ f \to \lim_{q \to p} \frac{|M_S|^{1/2}}{R^{3/2}} \left( -1 + \frac{4\pi R^3 p_1}{|M_S|} \right). \]  

(4.8)

The observer needs an infinite amount of inward acceleration if \( p_1 \) behaves as \( \mathcal{O}(R^{-3+\epsilon}) \) (\( \epsilon > 0 \)) or it is nonpositive to stay on the constant \( R \) world-line. This means that the observer feels infinite repulsive force near the singularity.

Of course, (4.6) is valid when the surface does not contain singularities. Since the repulsive force disperses positive-mass particles the negative-mass region causes instability. Large amount of positive pressure could prevent such instabilities formalism of naked singularities.
V. CONCLUSIONS AND DISCUSSIONS

We showed that in spherically symmetric spacetimes positive-mass singularities cannot be naked. If the mass of the central singularity is positive the singularity is spacelike or null, and if negative it is timelike or null. Negative quasi-local mass causes as repulsive force on a point particle in the same way as Newtonian gravity. Whereas positive pressure weaken the repulsive force as a general relativistic effect. Sufficient positive pressure keeps from negative-mass singularities from appearing in the spacetime.

There was no null singularities with mass $M = 0$ among well-known exact solutions we have presented in §2. We might expect that a certain condition forbid them.

Since the sign of singularity mass determines the causal feature of the singularity, we may translate the cosmic censorship conjecture into a problem of singularity mass. If conditions imposed on the matter restricts the sign of singularity mass to be positive spacetimes with naked singularity are ruled out.

In the case of spacetimes without spherical symmetry, it is difficult to investigate causal feature of singularities, since general spacetimes do not admit a conformal embedding into a compact manifold with boundary. In such a spacetime the various definitions of quasi-local mass give different values. Our arguments suggest that the Hawking mass is preferred to the other quasi-local masses for the definition of mass of singularity.

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REFERENCES

[1] R. Penrose, *Riv. Nuovo Cim.* 1 (1969) 252.

[2] A. Komar, *Phys. Rev.* 113 (1959) 934.

[3] S. W. Hawking, *J. Math. Phys.* 9 (1968) 598.

[4] R. Penrose, *Proc. Roy. Soc.* A381 (1982) 53.

[5] M. Ludvigsen and J. A. G. Vickers, *J. Phys. A: Math. Gen.* 15 (1982) L67.

[6] R. Kulkarni, V. Chellathurai, and N. Dadhich, *Class. Quantum Grav.* 5 (1988) 1443.

[7] A. Dougan and L. J. Mason, *Twistor Newslett.* 30 (1990) 6.

[8] S. A. Hayward, Max-Planck-Institut für Astrophysik preprint (1993).

[9] R. Arnowitt, S. Deser, and C. W. Misner, *Phys. Rev.* 117 (1960) 1595.

[10] H. Bondi, M. G. J. Van der Burg, and A. W. K. Metzner, *Proc. Roy. Soc. London* A269 (1962) 21.

[11] S. W. Hawing and G. F. R. Ellis, *Large scale structure of the spacetime*, Cambridge, 1973.

[12] R. C. Tolman, *Proc. Natl. Acad. Sci. USA.* 20 (1934) 164.

[13] H. Bondi, *Mon. Not. R. Astron. Soc.* 107 (1947) 410.

[14] D. M. Eardley and L. Smarr, *Phys. Rev.* 19 (1979) 2239.