Qudit surface code and hypermap code

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Abstract

In this article, we define homological quantum codes in arbitrary qudit dimensions $D \geq 2$ by directly defining CSS operators on a 2-complex $\Sigma$. If the 2-complex is constructed from a surface, we obtain a qudit surface code. We then prove that the dimension of the code we define always equals the size of the first homology group of $\Sigma$. Additionally, we generalize the hypermap-homology quantum code proposed by Martin Leslie to the qudit case. For every such hypermap code, we construct an abstract 2-complex whose homological quantum code is equivalent to the hypermap code.

Keywords qudit, 2-complex, CSS code, surface code, hypermap

1 Introduction

Surface codes are an important class of error-correcting codes in fault-tolerant quantum computation. In the literature, rigorous constructions of surface codes are usually done for the case of $\mathbb{Z}_2$-vector spaces, which is reasonable because qubit quantum computation theories have been highly successful, and qubit quantum codes are still dominant in today’s research. However, higher-dimensional qudit quantum systems have been shown to have advantages in fault-tolerant schemes [1], and some numerical studies have been conducted using special qudit codes [2, 3, 4]. Therefore, a general discussion about the basic construction of qudit surface codes would be helpful.

The general theory of qudit stabilizer and surface codes is introduced in Bombin and Martin-Delgado [5], where the authors define them using symplectic codes. In this article, we provide a more direct construction of surface codes with arbitrary qudit dimension $D \geq 2$ in a way similar to the prevailing literature on qubit surface codes, such as Freedman and Meyer [6]. Following Gheorghiu [7], we define stabilizer codes simply as the subspace stabilized by a subgroup $S$ of the qudit Pauli group. Then, we use the usual CSS construction to obtain $S$ from an arbitrary 2-complex defined in Bombin and Martin-Delgado [5]. When the 2-complex is derived from a surface, we obtain the qudit surface code. In particular, even in arbitrary qudit dimensions, there is a size theorem that relates the “size” of a stabilizer code to the size of its stabilizer group [8]. In this article, we use this theorem to relate the size of the homology group...
of a 2-complex to that of its homological quantum code. This is more general than Theorem III.2 in Bombin and Martin-Delgado [5], whose proof relies on the dimension theory of vector spaces. In the general D-qudit case, we typically do not have a vector space but only \( \mathbb{Z}_D \)-modules.

As an application, we extend the hypermap-homology quantum code defined by Leslie[8] to the case of qudits. Unlike Leslie’s approach, which builds hypermap quantum codes from topological hypermaps, we define them directly from combinatorial hypermaps, making the statements more precise and rigorous at the cost of losing some geometric intuition. Furthermore, for a given hypermap quantum code, we construct an abstract 2-complex whose homological quantum code defined in this article is exactly equivalent to it. This approach was motivated by Sarvepalli’s work, which shows that any (canonical) hypermap quantum code is equivalent to a surface code that can be directly constructed on its underlying surface.[9]

2 Qudit Systems Of Dimension \( D^n \)

A qudit is a finite dimensional quantum system with dimension \( D \geq 2 \). As with the qubits’ case, two operators \( X \) and \( Z \) act on a single qudit, and is defined as:

\[
X = \sum_{j \in \mathbb{Z}_D} |j + 1\rangle\langle j| \\
Z = \sum_{j \in \mathbb{Z}_D} \omega_j |j\rangle\langle j|
\]

where \( \omega = e^{2\pi i/D} \) and \( \{ |j\rangle \mid j \in \mathbb{Z}_D \} \) is an orthonormal basis for the qudit Hilbert space \( \mathcal{H} \), also, the addition of integers in equation (1) is modulo \( D \). Notice that the definition of \( X \) there is the adjoint \( X^\dagger \) of that in Gheorghiu.[7] From the above equations, we have \( ZX = \omega XZ \), and \( X^D = Z^D = 1 \). As with the qubits’ Hadamard gate, there are so called Fourier gate which maps the \( \omega^k \)-eigenvector \( |k\rangle \) of \( Z \) to an \( \omega^k \)-eigenvector \( |H_k\rangle \) of \( X \), with

\[
|H_k\rangle = \frac{1}{\sqrt{D}} \sum_j \omega^{-jk} |j\rangle.
\]

For \( n \) qudits, the Hilbert space is denoted by \( \mathcal{H}_n \) and we have

\[
\mathcal{H}_n = \bigotimes_{i=1}^{n} \mathcal{H}
\]

with canonical basis the tensor products of \( |j\rangle \). Denote \( X_i \) and \( Z_i \) the corresponding \( X, Z \) operators acting on the \( i \)-th qudit, we call expressions of the form

\[
\omega^\lambda X^x Z^z = \omega^\lambda X_1^{x_1} Z_1^{z_1} \otimes X_2^{x_2} Z_2^{z_2} \otimes \cdots \otimes X_n^{x_n} Z_n^{z_n}
\]

the Pauli products,[7] where \( \lambda \) is an integer and the n-tuples \( x = (x_1, x_2, \cdots, x_n) \), \( z = (z_1, z_2, \cdots, z_n) \) belongs to \( \mathbb{Z}_D^n \). These Pauli products is closed under multiplication and form the Pauli group \( \mathcal{P}_n \).

Similar to the case with qubits, we can examine a subgroup \( S \) of \( \mathcal{P}_n \) and its stabilizer subspace \( \mathcal{C} := \{|\phi\rangle \in \mathcal{H}_n \mid s|\phi\rangle = |\phi\rangle, \forall s \in S \} \). Likewise, in order for \( (\mathcal{C}, S) \) to be called a stabilizer code, we require that \( \mathcal{C} \neq 0 \), and a necessary condition for this is
that \( S \) is commutative and does not contain any scalar products \( e^{i\theta}I \neq I \). Otherwise, we can always find a unit complex number \( e^{i\theta} 
eq 1 \) such that \( \forall \phi \in \mathbb{C}, e^{i\theta}\phi = |\phi| \), which implies \( C = 0 \). In fact, the condition of not containing scalar products \( e^{i\theta}I \neq I \) is both necessary and sufficient for \( C \neq 0 \), which is indicated by the following theorem:\[7\].

**Theorem 1.** Let \( C \) be the stabilizer subspace of a \( P_n \)'s subgroup \( S \) which does not contain any scalar multiplication other than identity. Then
\[
K \times |S| = D^n, \tag{6}
\]
where \( K \) is the dimension of \( C \), \(|S|\) is the size (cardinality) of the stabilizer group \( S \) and \( D \) is the dimension of the Hilbert space of one carrier qudit.

It should be noted that the first sentence of Theorem 1 is slightly more stringent than the original one presented in Gheorghiu’s work [3], as the proof actually relies on it. (See Appendix A.) Furthermore, while omitting nontrivial scalar multiplication results in the commutativity of \( S \), the converse does not hold.

## 3 2-Complexes And Qudit Surface Code

In contrast to the case of qubits, the orientation of the underlying 2-complex is relevant when constructing a qudit surface code. Therefore, we follow the definition of a 2-complex as introduced by Bombin and Martin-Delgado." An oriented graph is a graph with each edge assigned an orientation. From a combinatorial perspective, an oriented graph consists of a set of vertices \( V \), a set of edges \( E \), and two incidence functions \( I_v, I_t : E \to V \), which we refer to as the “source” and “target” functions. We say an edge \( e \) goes or points from \( I_v(e) \) to \( I_t(e) \). In addition, there is also the set of “inverse edges” \( E^{-1} = \{ e^{-1} \mid e \in E \} \). We define \( (e^{-1})^{-1} := e \) and \( I_v(e^{-1}) = I_t(e) \), \( I_t(e^{-1}) = I_v(e) \), which allows the inverse operation and the functions \( I_v, I_t \) to be extended to the whole set \( E = E \cup E^{-1} \). We can now define the concept of a closed walk. First, an \( n \)-tuple of extended edges is \((e_0, e_1, \ldots, e_{n-1})\) where \( e_i \in E \) with its index \( i \in \mathbb{Z}_n \), and satisfies \( I_v(e_i) = I_t(e_{i+1}) \). Then a “closed walk of length \( n \)” is an equivalence class of these \( n \)-tuples under the equivalence relation generated by cyclic permutations, i.e., \((e_0, e_1, \ldots, e_{n-1}) \sim (e_0, e_1, \ldots, e_{n-1}) \Leftrightarrow e_{i+k} = e_i \) for some \( k \in \mathbb{Z}_n \), and we denote the equivalence class of \((e_0, e_1, \ldots, e_{n-1})\) by
\[
\omega = [e_0, e_1, \ldots, e_{n-1}] \tag{7}
\]
which has a well-defined inverse
\[
\omega^{-1} := [e_0, e_1, \ldots, e_{n-1}] \tag{8}
\]
with \( e_i = e_{i+n-1}^{-1} \).

The 2-dimensional generalization of graphs is 2-complexes, combinatorially, an *oriented 2-complexes* is an oriented graph \( \Gamma = (V, E, I_v, I_t) \) with a set \( F \) called “faces” plus a function \( B : F \to W_{\Gamma} \), which comes from the gluing map of a 2-cell along its boundary in algebraic topology, where \( W_{\Gamma} \) is the set of all closed walks. Similarly, we expand \( F \) to \( \tilde{F} = F \cup F^{-1} \), together with the domain of the inverse operation and gluing map \( B \):
\[
B(f^{-1}) = B(f)^{-1}, \quad \forall f \in \tilde{F} \tag{9}
\]
Intuitively, a face \( f \) in \( F \) can be thought of as a closed circular shape or disk, with a normal vector field that determines its orientation. \((F^{-1}) \) represents now the same disk.
with the opposite orientation.) By convention, this orientation induces a counterclockwise orientation on the boundary circle that is relative to the direction of the normal vector field. When the face is attached to a graph, the orientation of the boundary circle determines the orientation of the closed walk. Although the combinatorial definition of 2-complexes does not allow gluing 2-cells into a single point, it is sufficient to define surface codes, as Bombin and Martin-Delgado have pointed out.\[4\]

Compact surfaces have finite cell divisions and can be combinatorially represented by 2-complexes. When a compact surface is also closed, a common representation consists of a vertex \( v \), \( g > 0 \) edges \( \{a_1, a_2, \cdots, a_g\} \) and a face \( f \) with

\[
B(f) = [a_1, a_1, \cdots, a_g, a_g]
\]

if the surface is non-orientable, and a vertex \( v \), \( 2g > 0 \) edges \( \{a_1, b_1, \cdots, a_g, b_g\} \) and a face \( f \) with

\[
B(f) = [a_1, b_1, a_1^{-1}, b_1^{-1}, \cdots, a_g, b_g, a_g^{-1}, b_g^{-1}]
\]

if the surface is orientable but not a sphere. In either case, the surface has genus \( g \), which is predetermined by its homeomorphism class. Notice that by equation (10), we understand that there exist an intrinsic index set \( \mathbb{Z}_{2g} \) such that \( (a_1, a_2, \cdots, a_g, a_g) = (e_1, e_2, \cdots, e_{2g}) \) and \( I_1(e_i) = I_1(e_{i+1}) \), thus \( [a_1, a_2, \cdots, a_g, a_g] = [e_1, e_2, \cdots, e_{2g}] \), the same holds for equation (11). For a sphere, its genus is defined to be \( g := 0 \), and has a 2-complex representation with two vertices \( v_0, v_1 \), an edge \( e \) pointing from \( v_0 \) to \( v_1 \) and a face \( f \) with \( B(f) = [e, e^{-1}] \). It is worth noting that a surface has many 2-complex representations other than those given above, which may be more useful in quantum error correction codes. However, not every 2-complex represent a surface, those do comes from a surface must satisfy the conditions of Surface 2-complex. For the purpose of this discussion, we will not provide a formal definition of a surface 2-complex, as it is not essential to our current topic. Interested readers can refer to Bombin and Martin-Delgado for more details.\[5\]

Given a 2-complex \( \Sigma = (V, E, I_1, I_2, F, B) \), we can define three \( \mathbb{Z}_D \) modules \( C_0(\Sigma) \), \( C_1(\Sigma) \), and \( C_2(\Sigma) \) as free modules generated by the sets \( V \), \( E \), and \( F \), respectively. For example, \( C_0(\Sigma) \) consists of all the formal sums \( r_1v_1 + r_2v_2 + \cdots + r_{|V|}v_{|V|} \), where \( r_i \in \mathbb{Z}_D, v_i \in V \). Then a boundary operator \( \partial_1 : C_1(\Sigma) \to C_0(\Sigma) \) is defined to be the unique homomorphism such that \( \partial_1(e) = I_1(e) - I_1(e) \) for each \( e \in E \). To define the boundary \( \partial_2 : C_2(\Sigma) \to C_1(\Sigma) \), first, for any closed walk \( \omega = [e_1^n, e_2^p, \cdots, e_n^h] \), \( e_i \in E \), \( \sigma_i = \pm 1 \), we define \( c_\omega := \sum_{i=1}^{n} \sigma_i e_i \), then \( \partial_2 \) is the unique homomorphism such that \( \partial_2(f) = c_{B(f)} \) for any \( f \in F \). Now, there is a simple but important equation

\[
\partial_1 \circ \partial_2 = 0.
\] (12)

We denote \( Z_1(\Sigma) := \ker \partial_1 \) whose elements are called cycles and \( B_1(\Sigma) := \text{im} \partial_2 \) whose elements are called boundaries. Equation (12) tells us that \( B_1(\Sigma) \subset Z_1(\Sigma) \), in particular, \( B_1(\Sigma) \) is a normal subgroup of \( Z_1(\Sigma) \), and we have the first homology group \( H_1(\Sigma) \) as the quotient group

\[
H_1(\Sigma) := Z_1(\Sigma)/B_1(\Sigma).
\] (13)

By writing out the matrix of \( \partial_1 \) under the natrual bases \( V, E, F \), a special kind of stabilizer codes called homological quantum codes can be constructed.\[8\] However, rather than using matrix arguments, we will introduce basic cohomology terms\[5\][8] that will provide a more concise and geometrically insightful approach. First, some
algebraic remarks. If $A$ is a module over a commutative ring $R$, then the set of all homomorphisms from $A$ to $R$ is an $R$-module called the dual modules of $A$ and is denoted by $A^*:=\text{Hom}_R(A,R)$. Now if $F$ is a free $R$-module with a finite basis $X$, for each $x \in X$, let $x^* : F \to R$ be the homomorphism given by $x^*(y) = \delta_{xy}$ (\forall y \in X), where $\delta_{xy}$ equals 0 in $R$ if $x \neq y$, and $1_R$ if $x = y$. Then a basic fact is that $F^*$ is a free $R$-module with basis $\{x^* \mid x \in X\}$. Denote $Z^i(\Sigma) := C^i(\Sigma)$, and also $(c^i, c_1) := c^i(c_1)$ for any $c^i \in C^i(\Sigma)$ and $c_1 \in C_1(\Sigma)$, the cobondary operator $\delta_{i+1} : C^i \to C^{i+1}, (i \in \{0,1\})$ is defined by

$$\delta_i(c^i, c_1) := (c^{i-1}, \delta_i(c_1)). \quad i = 1, 2. \quad (14)$$

Then, by equation (12), we have the cochain complex

$$\delta_2 \circ \delta_1 = 0. \quad (15)$$

along with so called first cohomology group $H^1(\Sigma) := Z^1(\Sigma)/B^1(\Sigma)$ where the cocycles is defined by $Z^1(\Sigma) := \ker \delta_2$, and coboundaries by $B^1(\Sigma) := \im \delta_1$. Now, let the star of a vertex $v \in V$ to be the set $\text{star}(v) := \{(e, \sigma) \in E \times \{-1,1\} \mid I_v(e^\sigma) = v\}$. Then we have a geometric explanation of $\delta_1$

$$\delta_1(v^*) = \sum_{(e, \sigma) \in \text{star}(v)} \sigma e^* \quad (16)$$

which is important in the construction of surface codes.

To construct a stabilizer code, we attach one qudit to each edge of a 2-complex, thus obtaining a $D^{[E]}$-dimensional Hilbert space $H_{[E]}$, what we need is to find a subgroup of $P_{[E]}$. To begin, we define two sets of operators.

- **Face operators**: For each face $f$, we have $\partial_2(f) = c_{B(f)} = \sum_{i=1}^{h} \sigma_i e_i$, where $\sigma \in \{-1,1\}$, an operator is defined by

$$B_f := \prod_{i=1}^{h} Z_i^\sigma \quad (17)$$

with $Z_i$ the $Z$ operator on $e_i$'s qudit.

- **Vertex operator**: For each vertex, we have equation (16), an operator is defined by

$$A_v := \prod_{(e, \sigma) \in \text{star}(v)} X_e^\sigma \quad (18)$$

with $X_e$ the $X$ operator on $e$'s qudit.

Notice that we can expand the index to all edges by forcing some exponential $\sigma$ equal to 0, i.e., we can write $B_f = \bigotimes_{i=1}^{E}[E] Z_i^\sigma_i$, and $A_v = \bigotimes_{i=1}^{E}[E] X_i^\sigma_i$, there thus is an $|E|$-tuple $\nu_f = (\sigma_1, \sigma_2, \cdots, \sigma_{|E|})$ for each face operator, and an $|E|$-tuple $\nu_v = (\sigma'_1, \sigma'_2, \cdots, \sigma'_{|E|})$ for each vertex operator. Unlike those in equation (17), there is possibility that for some $i$, $|\sigma_i| > 1$, because the closed walk may intersect with itself, in for example the case of a non-orientable surface. Now, multiplication of two face (vertex) operators $B_f, B_{f'} (A_v, A_{v'})$ correspond to addition of their $|E|$-tuples $\nu_f + \nu_{f'}$ ($u_v + u_{v'}$) in $Z_{[E]}^{|E|}$, which indicates that the subgroup $\mathcal{B}(A)$ of $P_{[E]}$ generated by all the face (vertex) operators $B_f (A_v)$ corresponds to a submodule of the free module $Z_{[E]}^{|E|}$ which are denoted by $r(\mathcal{B}) (r(A))$. Indeed, $r(\mathcal{B}) (r(A))$ is simply the set of coordinates of elements in $\im \partial_2 (\im \delta_1)$ under the basis $\{e \mid e \in E\}(\{e^* \mid e \in E\})$. 

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Lemma 2. The elements of $B$ commute with elements of $A$.

Proof. For any $f \in F$ and $v \in V$, we have $(\delta_1(v), \partial_2(f)) = \langle v, \partial_1 \circ \partial_2(f) \rangle = 0$ by equations (12) and (15), which implies that the inner product $v \cdot u = 0$ in $\mathbb{Z}_D^{|E|}$. If we denote $g^+ := \sum_{i \in I^+} \sigma_i \sigma_i'$ with $I^+ := \{ i \in \{1, 2, \ldots, |E|\} | \sigma_i \sigma_i' > 0 \}$, $g^- := \sum_{i \in I^-} |\sigma_i \sigma_i'|$ with $I^- := \{ i \in \{1, 2, \ldots, |E|\} | \sigma_i \sigma_i' < 0 \}$, we have $g^+ - g^- \equiv 0 \mod D$. Now, from the basic relation $ZX = \omega XZ$, we have $Z^{-1}X^{-1} =\omega^{-1}X^{-1}Z^{-1}$, $Z^{-1}X =\omega XZ^{-1}$ and $ZX^{-1} =\omega^{-1}X^{-1}Z$, which shows that if we interchange $B_f$ and $A_v$, there would have $\omega^{2\gamma}$ and $\omega^{-\gamma}$ generated, these together give 1.

Let $S$ be the subgroup generated by all $B_f$ and $A_v$, then by lemma 2 it is abelian and thus can be written as
\begin{equation}
    s = b \cdot a
\end{equation}
with $b \in B$ and $a \in A$, and thus cannot be some scalar multiplication other than identity. Then by theorem 1, the stabilizer code $C$ defined by $S$ has dimension $K = D^{|E|}/|S|$, and is called surface code when the 2-complex comes from a surface with or without boundary. In qubit’s case, it can be further showed that the number of logical qubits contained in $C$ equals the dimension of the first homology group, i.e., $\dim H_1(\Sigma)$. However, the arguments using the dimension property of vector spaces cannot be applied in general $D$-qubit’s case, for a $\mathbb{Z}_D$-module may not be vector space when $D$ is not a prime. Fortunately, the next theorem shows that even for arbitrary $D$, the size of $H_1(\Sigma)$ still gives a measurement of $K$.

Theorem 3. For any 2-complex $\Sigma$, let $\mathcal{S}$ be the subgroup generated by all face and vertex operators defined by equation (14) and (15), then the dimension $K$ of its stabilizer code $C$ equals the size of $H_1(\Sigma)$, i.e, we have
\begin{equation}
    K = |H_1(\Sigma)|.
\end{equation}

Proof. By equation (15) and the property that operators of the form $X^\alpha Z^\beta$ $(x, z \in \mathbb{Z}_D)$ is a basis of $L(H)$, we have $|S| = |B||A| = |r(B)||r(A)|$, so $K = D^{|E|}/|S| = |C_1(\Sigma)|/|\ker \partial_1|$, thus we only need to prove $|C_1(\Sigma)|/|\ker \partial_1| = |\ker \partial_1|$, i.e., $|\ker \partial_1| = D^{|E|}$. Notice that if $x \in \ker \partial_1$, then for every $\alpha \in \ker \partial_1$, there is a $\beta \in C^0(\Sigma)$ such that $\alpha = \partial_1 \beta$, and we have $(\alpha, x) = (\beta, \partial_1 x) = 0$. On the other hand, if $y \in C_1(\Sigma)$ such that for all $\alpha \in \ker \partial_1$, $(\alpha, y) = 0$, then for all $\beta \in C^0(\Sigma)$, we have $(\beta, \partial_1 y) = (\delta, \beta, y) = (\delta, \beta, y) = 0$, which means $\partial_1 y = 0$, i.e, $y \in \ker \partial_1$. These together shows that the set of coordinates of the elements in $\ker \partial_1$ is the submodule $r(A)^{\perp}$ of $\mathbb{Z}_{D^{|E|}}$. Now, by theorem 3.2 in Zhao et al. (See Appendix B.) there is $|r(A)||r(A)^{\perp}| = D^{|E|}$, which proves our result.

As an example, let us compute the dimension of the stabilizer subspace for a code on the projective plane $\mathbb{R}P^2$ and the torus $T^2$. For $\mathbb{R}P^2$, a standard 2-complex is composed of a point $p$, an edge $e$ with $I_\Sigma(e) = I_\Sigma(e) = v$, and a face $f$ with $B(f) = [e, e]$. Thus, $\partial_1(f) = e + e = 2e$, and we have $\text{im} \partial_1 \simeq \mathbb{Z}_D$. Furthermore, since $\partial_1(e) = v - v = 0$, we have $\ker \partial_1 = C_1(\Sigma) \simeq \mathbb{Z}_D$. Therefore, $H_1(\Sigma) \simeq \mathbb{Z}_D/2\mathbb{Z}_D$, which has two elements when $D$ is even, yielding $K = 2$, and has one element when $D$ is odd, yielding $K = 1$. Thus, when $D$ is an even integer greater than 2, $K$ cannot always be written in the form of $D^k, k \in \mathbb{Z}$, and saying that “the code contains $\log_D K$ logical qudits” is meaningless, unlike the case of qubits. For $T^2$, a standard 2-complex
is composed of a point \( v \), two edges \( e_1, e_2 \) with \( I_e(e_i) = I_e(e_i) = v \), and a face \( f \) with \( B(f) = [e_1, e_2, e_1^{-1}, e_2^{-1}] \). We have \( \partial_1(e_1) = 0 \) and \( \partial_2(f) = e_1 + e_2 - e_1 - e_2 = 0 \), which implies \( \text{im} \partial_2 = 0 \) and \( \ker \partial_1 = C_1(\Sigma) \). Therefore, \( H_1(\Sigma) \cong \mathbb{Z}_2 \), and the code has dimension \( D^2 \), implying the presence of two \( D \)-dimensional logical qudits. Note that \( H_1(\Sigma) \) only depends on the homotopy class of the surface, so these computations apply to other 2-complex representations of the above surfaces.

4 Qudit Hypermap Code

In a general 2-complex construction, even if the 2-complex comes from an oriented surface, there seems to have no canonical way of orienting the edges, i.e., defining the functions \( I_e, I_f \). In this section, we show that this arbitrariness can be avoided when the 2-complex comes in a certain way from a hypermap.

A hypermap, more precisely, a combinatorial hypermap, consists of a number set \( B_n = \{1, 2, \cdots, n\} \) with a pair of permutations \( (\alpha, \sigma) \in S_n \). For each element \( \gamma \in\alpha, \sigma \rangle \), define its orbits to be the equivalence classes of \( B_n \) under the relation \( i \sim j \Leftrightarrow \exists \gamma' \in \gamma >, \gamma'(i) = j \), then for each \( i \in B_n \), there is a positive integer \( r \) so that the \( \gamma \)-orbit it belongs to is \( \text{orb}_i(i) = \{i, \gamma(i), \cdots, \gamma^{-1}(i)\} \), with \( \gamma'(i) = i \). We call the orbits of \( \alpha \)-hyperedges, the orbits of \( \sigma \)-hypervertices, and the orbits of \( \alpha^{-1} \sigma \)-faces. For \( \alpha^{-1} \sigma \) here, we take the convention in Leslie, \cite{8} i.e., acting from left to right. In addition, we call the elements of \( B_n \) themselves *darts*, and denote \( v_{\beta i}, v_{\gamma i}, \) and \( f_{\beta i} \) the hyperedge, the hypervertex, and the face that dart \( i \) belongs to.

Let \( \mathcal{V}, \mathcal{E}, \mathcal{F} \) be the free \( \mathbb{Z}_D \)-modules generated by all hypervertices, hyperedges, and faces, also, \( \mathcal{W} \) be the free \( \mathbb{Z}_D \)-modules generated by all darts \( B_n \). We define a homomorphism \( d_2 : \mathcal{F} \rightarrow \mathcal{W} \) by \( d_2(f) = \sum_{i \in f} i \), and a homomorphism \( d_1 : \mathcal{W} \rightarrow \mathcal{V} \) by \( d_1(i) = v_{\beta^{-1} \alpha^{-1}(i)} - v_{\gamma i} \), then we have

**Lemma 4.** \( d_1 \circ d_2 = 0 \).

**Proof.** For an \( f \in \mathcal{F} \), we write its element as \( f = \{i_0, i_1, \cdots, i_{k-1}\} \) where the subscript \( s \in \mathbb{Z}_k \) with \( s_{i+1} = \alpha^{-1} \sigma(i_s) \), which implies \( v_{\beta^{-1} \alpha^{-1}(i_s)} = v_{\beta i_{s+1}} \), thus \( d_1 \circ d_2(f) = d_1 \sum_{s \in \mathbb{Z}_k} i_s = v_{\beta^{-1} \alpha^{-1}(i_0)} - v_{\beta i_1} + v_{\beta^{-1} \alpha^{-1}(i_1)} - v_{\beta i_2} + \cdots - v_{\beta^{-1} \alpha^{-1}(i_k-1)} - v_{\beta i_k} = 0 \). \( \square \)

Also, there is a homomorphism \( \iota : \mathcal{E} \rightarrow \mathcal{W} \) with \( \iota(e) = \sum_{i \in e} i \), which is very similar to \( d_2 \), and we have

**Lemma 5.** \( d_1 \circ \iota = 0 \).

Lemma 5 guarantees a well defined homomorphism \( \Delta_1 \) from the quotient module \( \mathcal{W}/\iota(\mathcal{E}) \) to \( \mathcal{V} \), with \( \Delta_1[\omega] = d_1 \omega \), where \([\omega]\) denotes the equivalence class of \( \omega \). Furthermore, if we define \( \Delta_2 : \mathcal{F} \rightarrow \mathcal{W}/\iota(\mathcal{E}) \) by \( \Delta_2 = \rho \circ d_2 \), where \( \rho \) is the natural projection from \( \mathcal{W} \) to \( \mathcal{W}/\iota(\mathcal{E}) \), we would have \( \Delta_1 \circ \Delta_2 = 0 \).

In the previous section, we discussed how to construct a homological quantum code in the qudit setting from a chain complex obtained from an abstract 2-complex. In fact, for any \( \mathbb{Z}_D \)-chain complex \( C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \), where \( C_i \) are all free modules with finite bases \( f_i, e_i, v_i \), if \( \partial_2(f_j) = \) \( \mathcal{F} \xrightarrow{d_2} \mathcal{W} \xrightarrow{d_1} \mathcal{V} \)

\[ \Delta_2 \]

\[ \mathcal{W}/\iota(\mathcal{E}) \]

\[ \Delta_1 \]

Figure 1: \( \Delta_1 \) are defined to make the diagram commute.

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which shows us to the situation in the previous section. In order to define $\Sigma$, we let $B$ be the set of all hypervertices, $V,E,I$ be the $2$-complex $\Sigma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$, such that $\sum_{i} e_i \delta_1(v_i) = \sum_{i} e_i c_i$, where $\sigma_i \in \mathbb{Z}_D$, we can construct generators $B_{f_i} = \bigotimes_j Z^{e_i j}$ and $A_{e_j} = \bigotimes_j X^{e_i j}$. The previous relevant proofs still apply, and we have a homological quantum code $(\mathcal{C}, \mathcal{S})$ that satisfies $\dim \mathcal{C} = |H_f|$. This means that if $W/\mathcal{I}(E)$ is a finitely generated free module, then we can directly construct a homological quantum code from the chain complex $F \xrightarrow{\Delta_2} W/\mathcal{I}(E) \xrightarrow{\Delta_1} \mathcal{V}$. This can be seen as a generalization of the hypermap quantum code in the qudit case. To do this, we choose one dart from each hyperedge and call it a special dart, and these special darts form a subset $S \subseteq B_n$. Now we have:

**Lemma 6.** $W/\mathcal{I}(E)$ is a free module with a basis $\{(i) | i \in B_n \setminus S\}$.

**Proof.** First, we show that this is a linear independent set. Suppose there are $k_i \in \mathbb{Z}_D$ such that $\sum_{i} k_i s_i = 0$, then we have $\sum_{i} k_i v_i = \sum_{i} k_i e_i$, $k_i \in \mathbb{Z}_D$. If we use $s_e$ to denote the special dart in the hyperedge $e$, then the right side of the equality sign becomes $\sum_{e} R_e s_e + \sum_{i} k_i h_i$ for some $h_i \in \mathbb{Z}_D$, thus $R_e = h_i - k_i = 0$ by linear independence of the set $B_n$ in $W$, which further indicates $k_i = 0$.

On the other hand, for every $\omega \in W$, we have some $R_e, h_i \in \mathbb{Z}_D$ such that $[\omega] = [\sum_{e} R_e s_e + \sum_{i} h_i]$.

However, there is an another approach, where we will instead construct an abstract $2$-complex $\Sigma = (V,E,I,\alpha,\beta,F)$ whose chain $C_2(\Sigma) \xrightarrow{\partial_2} C_1(\Sigma) \xrightarrow{\partial_1} C_0(\Sigma)$ is isomorphic to the chain of hypermap homology $F \xrightarrow{\Delta_2} W/\mathcal{I}(E) \xrightarrow{\Delta_1} \mathcal{V}$, which will lead us to the situation in the previous section. In order to define $\Sigma$, we let $V$ be the set of all hypervertices, $E$ be the set $B_n \setminus S$, and $F$ be the set of all faces, then apparently, we have $C_2(\Sigma) = F$, $C_1(\Sigma) \simeq W/\mathcal{I}(E)$, and $C_0(\Sigma) = V$. Further, for every $e \in E$, which is a non-special dart, i.e., $e = i \in B_n \setminus S$, define $I_i(e) = v_\alpha^{-1}(i)$, $I_i(e) = v_\beta$, then we have $\partial_1(e) = I_i(e) - I_j(e) = v_\alpha^{-1}(i) - v_\beta = di = \Delta_1[i]$, which means $\partial_1 \simeq \Delta_1$. The precise meaning of the symbol $\simeq$ is that the matrix of $\partial_1$ with respect to the basis $E, V$ is the same as the matrix of $\Delta_1$ with respect to the basis $[E], V$. To define $B$, notice that for every $f \in F$, there is a positive integer $r$ such that $f = \{i_0, i_1, \ldots, i_{r-1}\}$ with subscripts in $\mathbb{Z}_r$, and satisfy $i_{k+1} = \alpha^{-1}(i_k)$ for all $k$. Suppose that the subset which consists of all special darts in $f$ is $S_f = \{i_{k_1}, i_{k_2}, \ldots, i_{k_l}\}$, we have $[i_{k_l}] = [i_{k_l} - l(e_{\beta i_{k_l}})] = -\sum_{i=1}^{l(e_{\beta i_{k_l}})} [i_i]$ with $i_{l+1} = \alpha(i_l)$ for all $l \in \{1, 2, \ldots, l(e_{\beta i_{k_l}}) - 2\}$, plus $\alpha(i_{k_1}) = i_1^*$ and $\alpha(i_{l(e_{\beta i_{k_l}}) - 1}) = i_{k_l}$, where $i_l^* \in B_n \setminus S$. Thus we have

$$\Delta_2(f) = \sum_{i \in f \setminus S} [i] - \sum_{l=1}^{l(e_{\beta i_{k_l}}) - 1} \sum_{i=1}^{l(e_{\beta i_{k_l}})} [i_i]$$

and

$$\Delta_2(f) = \sum_{i \in f \setminus S} [i] - \sum_{l=1}^{l(e_{\beta i_{k_l}}) - 1} \sum_{i=1}^{l(e_{\beta i_{k_l}})} [i_i]$$

(21)
Lemma 7. In the r-tuple $(i_0, i_1, \cdots, i_{r-1})$ from $f$, if we replace each $i_k \in S_f$ by the tuple $p_i = (((i_{k_1})^{-1}, (i_{k_2})^{-1}, \cdots, (i_{|E|-1})^{-1}) \cup E^{-1}$, then we get a closed walk

$[i_0, i_1, \cdots, p_1, i_{k_1+1}, \cdots, p_s, i_{k_s+1}, \cdots, i_{r-1}]$.\[22]

Proof. If re-indexing the 'closed walk' by $e_i \in E \cup E^{-1}$ with $i \in Z_K$ where $K$ is the length, we only need to check that $I_s(e_{i+1}) = I_t(e_i)$. For example, $I_s((i_1)^{-1}) = v_{\gamma \alpha^{-1}(i_1)} = v_{\gamma \alpha^{-1}(i_1)} = v_{\gamma \alpha^{-1}(i_{k_1-1})} = v_{\gamma \alpha^{-1}(i_{k_1-1})} = I_t(i_{k_1-1})$, when $i_{k_1-1} \notin S_f$.\[22]

Now, if we define $B(f)$ to be the closed walk in lemma 7, then by equation (22) there is $\partial_2 \simeq \Delta_2$.

We have shown that every hypermap map code is the homological quantum code constructed from a 2-complex $\Sigma$. The most interesting observation about $\Sigma$ is that it should be a surface 2-complex. Actually, every hypermap $(\alpha, \sigma)$ has a geometrical representation $H = (M, \Gamma)$ called topological hypermap, where $M$ is an oriented surface, $\Gamma$ is an bipartite graph embedded in $M$ whose edges correspond to the darts in $B_n$, the normal vector field given by $M$'s orientation determines the maps $\alpha$ and $\sigma$. Then Sarvepalli showed that we can obtain by adding curves on $M$ an ordinary surface code which equals the original hypermap code constructed by Leslie.[9] In our language, Sarvepalli’s curves together with the vertices of $\Gamma$ they connected and $M$ itself from exactly the 2-complex $\Sigma$ we constructed, whose homological quantum code is exactly his surface code. A small difference is that in Sarvepalli, the curves are not oriented for they only deal with qubit quantum codes. However we do not try to prove directly that $\Sigma$ is surface 2-complex for the great possibility of a tedious argument, but show the simple fact that $\Sigma$ is orientable. If a 2-complex is from an orientable surface with a global normal vector field, moreover, the function $B : F \rightarrow E$ is determined by the global field because the restriction of the field in each face will induce an orientation of its closed walk, then we must have

$\sum_{f \in F} \partial_2(f) = 0$.

Let’s define orientable 2-complexes to be those satisfying equation (22), then we have

**Theorem 8.** The 2-complex $\Sigma$ constructed above is orientable.

Proof. We only have to show that $\sum_{f \in F} \Delta_2(f) = 0$, which is correct because $\sum_{f \in F} \partial_2(f) = \sum_{e \in B_n} i = \sum e \iota(e)$ where the last sum is done for all hyperedges.\[22]

**References**

[1] Earl T. Campbell, Hussain Anwar, and Dan E. Browne. Magic-state distillation in all prime dimensions using quantum reed-muller codes. *Phys. Rev. X*, 2:041021, Dec 2012.

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Appendices

A Proof of Theorem 1

We borrow the proof from [7], with emphasis on the need of excluding nontrivial scalar multiples.

Proof. Define

$$P := \frac{1}{|S|} \sum_{s \in S} s.$$  \hfill (23)

We first prove that $P$ is a projection operator on $C$.

By definition, we have

$$P = P^* = P^2,$$ \hfill (24)

where the two equalities come from the group properties of $S$. Equation (24) tells us that $P$ is a projection operator [12].

Let $|\psi\rangle$ be an arbitrary vector in $C$. Then, it follows from the definition in Equation (23) that $P|\psi\rangle = |\psi\rangle$. Therefore, if we let $W$ be the range of $P$, i.e., the space it projects onto, we have $C \subset W$. Now, let us choose an arbitrary $|\phi\rangle \in W$. Then, we have

$$|\phi\rangle = P|\phi\rangle = \frac{1}{|S|} \sum_{s \in S} s|\phi\rangle.$$ \hfill (25)

Multiplying both sides of Equation (25) by an arbitrary $t \in S$ and using the group properties of $S$, we obtain

$$t|\phi\rangle = tP|\phi\rangle = \frac{1}{|S|} \sum_{s \in S} ts|\phi\rangle = \frac{1}{|S|} \sum_{s \in S} s|\phi\rangle = P|\phi\rangle = |\phi\rangle.$$ \hfill (26)
Since $t$ is arbitrary, we have
\[ t|\phi\rangle = |\phi\rangle, \forall t \in \mathcal{S}. \]  
(27)
This means that $|\phi\rangle$ belongs to the stabilizer subspace $\mathcal{C}$, and hence $W \subset \mathcal{C}$. Therefore, the range of $P$ is $\mathcal{C}$, and its trace is the dimension of its range, i.e.,
\[ \text{Tr}(P) = K = \frac{1}{|\mathcal{S}|} D^n, \]  
(28)
where the second equality follows from the fact that the trace of any non-scalar element in $\mathcal{P}_0$ is zero, and the only scalar in $\mathcal{S}$ is $I$ whose trace is $D^n$. This completes the proof. \[ \square \]

B The proof of $|\!\!r(\mathcal{A})\!\!\rangle|_{|\!\!r(\mathcal{A})\!\!\rangle^\perp} = D^{\!\!|\mathcal{E}|}$ in Theorem 3.

We restate the property as the following lemma whose proof is adapted from [10], with some added and modified details.

Lemma 9. Let $D \geq 2$ be an integer and $n$ be a positive integer. If $E$ is a submodule of $\mathbb{Z}_D^n$, let $E^+ = \{ x \in \mathbb{Z}_D^n | x \cdot y = 0, \forall y \in E \}$, then $|E||E^+| = D^n$.

Proof. For a $D$-th root of unity $\omega = e^{\frac{2\pi i}{D}}$, we first prove that
\[ \sum_{x \in E} \omega^{\eta \cdot x} = \begin{cases} |E|, & \eta \in E^\perp \\ 0, & \eta \not\in E^\perp \end{cases} \]  
(29)
When $\eta \in E^\perp$, equation (29) is clearly true. Otherwise, we define a map $\eta : E \rightarrow \mathbb{Z}_D, x \mapsto \eta \cdot x$. Clearly, the map $\eta$ is a module homomorphism, and since $\eta \not\in E^\perp$, its image $\eta(E)$ is a nontrivial subgroup of $\mathbb{Z}_D$. We use the integers within 0 to $D - 1$ as representatives for the elements of $\eta(E)$ and label them along with $D$ on the number axis.

![Figure 2: Representatives of elements in $\eta(E)$.](image)

Now, all the points in the number axis are equidistant from each other, otherwise there exist three consecutive points $a_1, a_2, a_3$ such that $a_3 - a_2 > a_2 - a_1$ or $a_3 - a_2 < a_2 - a_1$. Without loss of generality, assume the latter is true. Then, since $a_3 - a_2 \in \eta(E)$, we have $a_1 + (a_3 - a_2) \in \eta(E)$, but this contradicts the fact that $a_1 < a_1 + (a_3 - a_2) < a_2$. We now let $m$ be the smallest positive integer labeled in the number axis and $|\eta(E)| = d$, then $D = md$ and $\eta(E) = \{0, m, 2m, \ldots (d - 1)m\}$, as shows in Figure 2. Therefore, we have
\[ \sum_{k \in \eta(E)} \omega^k = \sum_{l=0}^{d-1} \omega^{lm} = 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{d-1} = \frac{1 - \epsilon^d}{1 - \epsilon} = 0. \]  
(30)
Here, $\epsilon = \omega^m = e^{\frac{2\pi i}{d}} = e^{\frac{2\pi i}{d}}$ represents the $d$-th root of unity, and since $\eta(E)$ is non-zero, we have $d > 1$, implying $\epsilon \neq 1$. By the fundamental theorem of module homomorphisms, $\eta$ induces an isomorphism $\eta : E/\ker(\eta) \rightarrow \eta(E), x + \ker(\eta) \mapsto \eta(x)$. Therefore,

$$\sum_{x \in E} \omega^{\eta \cdot x} = \sum_{k \in \eta(E)} \sum_{x \in \eta^{-1}(k)} \omega^k = |\ker(\eta)| \sum_{k \in \eta(E)} \omega^k$$

(31)

$$= 0.$$  

According to equation (29) we just proved, we have:

$$|E||E^\perp| = \sum_{\eta \in E^\perp} \sum_{x \in E} \omega^{\eta \cdot x} = \sum_{\eta \in \mathbb{Z}_n^D} \sum_{x \in E} \omega^{\eta \cdot x}$$

$$= \sum_{x \in E} \sum_{\eta \in \mathbb{Z}_n^D} \omega^{\eta \cdot x}$$

(32)

Let us define the submodule $E' = \mathbb{Z}_n^D$, then $E'^\perp = 0$. Using equation (20) again, we have $\sum_{x \in E} \sum_{\eta \in \mathbb{Z}_n^D} \omega^{\eta \cdot x} = \sum_{\eta \in \mathbb{Z}_n^D} \omega^{\eta \cdot 0} = D^n$.  

$\square$