A criterion for the existence of Killing vectors in 3D

Boris Kruglikov\textsuperscript{1,2} and Kentaro Tomoda\textsuperscript{3,}\textcopyright{}

\textsuperscript{1} Department of Mathematics and Statistics, Faculty of Science and Technology, UiT the Arctic University of Norway, Tromsø 90-37, Norway
\textsuperscript{2} Department of Mathematics and Natural Sciences, University of Stavanger, 40-36 Stavanger, Norway
\textsuperscript{3} Advanced Mathematical Institute, Osaka City University, Osaka 558-8585 Japan

E-mail: boris.kruglikov@uit.no and k-tomoda@sci.osaka-cu.ac.jp

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Abstract
A three-dimensional Riemannian manifold has locally 6, 4, 3, 2, 1 or no independent Killing vectors. We present an explicit algorithm for the computing dimension of infinitesimal isometry algebra. It branches according to the values of curvature invariants. These are relative differential invariants computed via curvature, but they are not scalar polynomial Weyl invariants. We compare our obstructions to the existence of Killing vectors with the known existence criteria due to Singer, Kerr and others.

Keywords: Killing vector, isometry group, curvature invariant, Cartan–Karlhede algorithm

1. Introduction

The problem of determining the number of Killing tensors (KTs) of a given Riemannian metric $g_{ab}$ on a manifold $M$ is classical. Such tensors give first integrals polynomial in momenta and allow integrating the geodesic flow of $g_{ab}$. The integrals linear in momenta are tantamount to Killing vectors (KVs) and they generate the isometry algebra of $g_{ab}$.

For surfaces Darboux found a criterion of local existence of KVs that we recall in figure 1 below (the scalar curvature $R$ of $g_{ab}$ can be equivalently changed to the Gaussian curvature), see [1] and also [2]. The corresponding problem for KT of order 2 is much more involved [3]. While a principal approach was sketched in [4], it was only relatively recently that the final solution was found in [5], including specification of the number of KT of order 2 depending on the curvature invariants of $g_{ab}$. Criteria for the existence of higher order KT in dimension 2 and KT of order 2 in general dimension are overly complicated; see [6] for further discussion.

In this paper, we devise an algorithm for computing the number of local KVs for Riemannian manifolds $(M, g_{ab})$ of dimension 3. The problem of finding dimension of the isometry algebra
of $g_{ab}$ is of fundamental importance in applications to general relativity. It was addressed before in arbitrary dimension, so let us recall the status of knowledge.

In [7] Singer characterised homogeneous spaces locally via the Riemann tensor $R^{abcd}$ and its covariant derivatives. The problem was later revisited by Nomizu [8], Prüfer–Tricerri–Vanhecke [9] and Console–Olmos [10]. In the latter two references the homogeneity was given via scalar Weyl invariants of $g_{ab}$.

Recall that scalar Weyl (or polynomial curvature) invariants $I$ are obtained from the covariant derivatives $\nabla_a \nabla^a R^{bcde}$ by tensor products and complete contractions; such invariants are said to have order $\sigma$. As proved by Weyl [11] these are the only invariants of $g_{ab}$ polynomial in derivatives of the metric components $g_{ij}$.

Using those, Kerr [12] found a criterion for the existence of KVs and expressed dimension of the isometry algebra in terms of the curvature. The criterion is as follows.

Let $I_s = \{ I : \text{ord}(I) \leq s \}$ be the set of scalar Weyl invariants of order $\leq s$. For a point $o \in M$ denote $O^o_s = \{ x \in M : I(x) = I(o), \forall I \in I_s \}$, $R^o_s = \{ A \in SO(T_o M, g) : A \text{ preserves } \nabla_a \cdots \nabla^a R^{bcde}(o), \sigma \leq s \}$, and let $m_s = \dim O^o_s$, $r_s = \dim R^o_s$. Then dimension of the algebra of isometries (or the group of motion) is

$$d = m_s + r_s, \quad (1)$$

for sufficiently large $s$.

It is possible to show that generally one should take $s = \binom{n+1}{2}$ in the above formula, for determining homogeneity it suffices to take $s = \binom{n}{2}$, see [7, 10]. Also one can modify tensor invariants in the definition of $R^o_s$ to the differentials of the scalar invariants from $I_s$. Let us count the number of scalar invariants involved in the formula (1).

To simplify the evaluation let us relax the condition of polynomial invariants to general differential invariants, i.e. arbitrary scalar (analytic) functions of the metric components $g_{ij}$ and their derivatives invariant under coordinate changes [13]. For Riemannian metrics the number of such invariants was computed using the method of Sophus Lie by Zorawski [14] for dimension $n = 2$ and Haskins [15] for higher dimensions $n > 2$.

Denote the number of functionally independent invariants of order $\sigma = 2 + s$ (also independent of invariants of order $< \sigma$) by $\delta_\sigma$. The result for $n = 2$ is this: $\delta_0 = 0$, $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_\sigma = \sigma - 1$ for $\sigma > 3$. Thus with $s = \binom{3}{2} = 3$ we get $\sigma = 5$ and the number of involved invariants of order $\leq 5$ in $g_{ab}$ is $\sum_{\sigma \leq 5} \delta_\sigma = 1 + 1 + 3 + 4 = 9$. This definitely exceeds the number 2 of invariants (of orders 4 and 5) in figure 1, so even in this simplest case Kerr’s criterion is less effective than that by Darboux.
For dimension \( n = 3 \) the situation is even more striking. The count of invariants is as follows: \( \delta_0 = 1 = 0, \delta_2 = 3 \) and \( \delta_\sigma = \frac{2}{(\sigma - 1)(\sigma + 2)} \) for \( \sigma > 2 \). Thus with \( s = \binom{4}{2} = 6 \) we get \( \sigma = 8 \) and the number of independent invariants of order \( \leq 8 \) in \( g_{ab} \) is \( \sum_{\sigma \leq 8} \delta_\sigma = 3 + 15 + 27 + 42 + 60 + 81 + 105 = 333 \). This is the lower bound for the number of scalar Weyl invariants involved in Kerr’s criterion. Those have never been written down even for the case \( n = 3, s \leq 6 \). Though in theory it is possible to find 333 functionally independent among all Weyl invariants, in practice it is rather non-trivial. One can construct a list of 960 scalar Weyl invariants using the Hamilton–Cayley theorem applied to the space of covariant derivatives of the curvature of order \( s \leq 6 \). Due to large size of those invariants, to extract 333 functionally independent of them is a demanding computational task. To use those further to check the number of KVs is a tremendous calculation.

We aim at a more effective criterion to decide the existence and number of KVs. To this end we devise an algorithm that brings the Killing equations to involution and branches depending on ranks of the equations in the prolonged system. The PDE system encoding the condition that a vector field \( K^a \) is a KV for \( g_{ab} \) is

\[
\mathcal{L}_K g_{ab} = 0 \iff \nabla_{(b}K_{a)} = 0.
\]  

Here we raise and lower indices with the help of \( g_{ab} \), \( \nabla_a \) is the Levi-Civita connection of \( g_{ab} \) and \( \mathcal{L}_K \) is the Lie derivative along \( K \). The first compatibility conditions of this overdetermined system are [6]:

\[
\mathcal{L}_K R_{abc}^d = 0.
\]  

In dimension 3 the Riemann curvature is expressed through the Ricci tensor \( R_{ab} \), and we deduce

\[
\mathcal{L}_K R^{(1)} = 0, \quad \mathcal{L}_K R^{(2)} = 0, \quad \mathcal{L}_K R^{(3)} = 0,
\]  

where \( R^{(1)} = R, \ R^{(2)} = R^{ab}R_{ab}, \ R^{(3)} = R^{a_bR^b_cR^c_a} \) are the principal traces of powers of the Ricci tensor. Thus the matrix equation

\[
R_a K^a = 0,
\]  

must be satisfied. Here we define an obstruction \( 3 \times 3 \) matrix \( R_a \) as

\[
R_a = \begin{pmatrix}
\nabla_a R^{(1)} \\
\nabla_a R^{(2)} \\
\nabla_a R^{(3)}
\end{pmatrix}.
\]  

The matrix equation (5) yields an immediate consequence: Any KV must be in the kernel of \( R_a \). Hence, the determinant of \( R_a \)

\[
\det R_a = dR^{(1)} \wedge dR^{(2)} \wedge dR^{(3)},
\]  

has to vanish. This is a relative differential invariant. Otherwise \( \ker R_a = 0 \) and consequently there is no KV.

It is clear that the orbit dimension of the motion group action is bounded as \( m = m_s \leq \dim \ker R_a = 3 - \text{rank} R_a \). Since \( r = r_s \leq \binom{m}{2} \), dimension of this group (= number of KVs) is \( d = m + r \leq \binom{m + 1}{2} \), and this is bounded via the rank of (6).
Our analysis proceeds as follows. We consider cases 2, 1, 0, called so according to rank \( R_a = 2, 1, 0 \) and, respectively, \( n = \dim \ker R_a = 1, 2, 3 \). In each case, a general solution to matrix equation (5) is given by

\[
K^a = \sum_{i=1}^{n} \omega_i E^a_i ,
\]

where \( E^a_i \)'s span the annihilator of the matrix \( R_a \) and scalars \( \omega_i \)'s are yet indeterminate. By substituting equation (8) into Killing equation (2), we obtain a PDE system of \( \omega_i \)'s. Moreover, since equation (5) is just a necessary condition for equation (2), the compatibility conditions of these PDEs reveal further algebraic constraints on \( \omega_i \)'s. This adds rows to the matrix \( R_a \) initially defined in (6), that results in decrease of \( n \). So we restart the analysis with a new choice of \( R_a \) in (5) and updated equation (8). The procedure stabilises when all compatibilities are met.

We will examine branching governing the number of KVs depending on the values of scalar invariants and specify compatibilities in all cases. Our result can be summarised so:

**Theorem.** Let \((M, g_{ab})\) be a 3-dimensional Riemannian manifold. The dimension of the isometry algebra can be computed via differential invariants of the algorithm outlined in figure 2. This algorithm includes further branching given in figures 3–5.

Notice that the differential invariants used in our theorem are rational in derivatives of the metric components \( g_{ij} \), and so they are not necessarily Weyl invariants. However since we use only differential relations, these can be assumed relative scalar polynomial invariants.

The rest of the paper is composed as follows. In sections 2–4, we correspondingly give the formulation and proof for the algorithm. In section 5 we present two examples of its application. In section 6 we close this paper with a comment on relations of our method to that of Cartan. Two appendices contain technical formulae.

### 2. Analysis of case 2

In this case, it follows from the rank-nullity theorem that \( \dim \ker R_a = 1 \). So if we have an annihilator of \( R_a \), any KV can be written by

\[
K^a = \omega U^a ,
\]

where \( U^a \) is determined by

\[
U^a = \frac{1}{U} \left( \nabla_b R(1) \nabla_c R(2) \right) ,
\]

so as to satisfy \( U^a U_a = 1 \). If \( U^a \) vanishes identically, two scalars \( (R(1), R(2)) \) in the definition must be replaced by \( (R(2), R(3)) \) or \( (R(3), R(1)) \).

Using the concrete form (9), we write out the components of the Killing equation (2). To this end, we introduce the projection tensor onto the hyperplanes orthogonal to \( U^a \) as

\[
q_{ab}(U) = g_{ab} - U_a U_b ,
\]

that is endowed with a projection property and an orthogonality

\[
q_{ac} q^c_b = q_{ab} , \quad q_{ab} U^b = 0 .
\]
The \( UU, Uq \) and \( qq \)-parts of the Killing equation \( (2) \) have respectively 1, 2 and 3 components as follows.

\[
0 = U^a U^b \nabla_{(a} K_{b)} = \mathcal{L}_U \omega, \tag{14a}
\]

\[
0 = U^a q^b \nabla_{(a} K_{b)} = \frac{1}{2} (\nabla_c \omega - (\mathcal{L}_U \omega) U_c + \Omega_c \omega), \tag{14b}
\]

\[
0 = q^c q^d \nabla_{(a} K_{b)} = \omega \kappa_{cd}, \tag{14c}
\]

where \( \mathcal{L}_U \) is the Lie derivative along \( U^a \), \( \Omega_a \) and \( \kappa_{ab} \) are defined as

\[
\Omega_a(U) \equiv U^b \nabla_b U_a, \quad \kappa_{ab}(U) \equiv q^c q^d \nabla_{(a} U_{b)} \nabla_{(d} U_{c)}. \tag{15}
\]

It can be concluded that the Killing equation \( (2) \) can be rewritten as

\[
\kappa_{ab} = 0, \quad \nabla_a \omega = -\Omega_a \omega. \tag{16}
\]

The integrability condition for equation \( (16) \) is given by

\[
\nabla_a [\Omega_b] = 0. \tag{17}
\]

If the annihilator \( U^a \) passes the two tests,

\[
\kappa_{ab} = 0, \quad \nabla_a [\Omega_b] = 0, \tag{18}
\]

then there are no extra conditions that must be satisfied, thereby allowing us to confirm that one KV exists. The results obtained here are summarised in figure 3.

Observe that our results \( (18) \) do not depend on the explicit form of \( U^a \), equation \( (10) \). Therefore if we can write KVs in the form of equation \( (9) \), our analysis here will be recyclable.

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\( ^4 \) Note that the second equation implies equation \( (14a) \).
3. Analysis of case 1

Again by the rank-nullity theorem, \( \text{dim ker } R_a = 2 \). Then KVs take the form

\[
K^a = \omega_N N^a + \omega_B B^a ,
\]

where \((N^a, B^a)\) are two annihilators of \( R_a \) and \((\omega_N, \omega_B)\) are two unknown functions. We assume that \((N^a, B^a)\) are unit vector fields satisfying an orthogonality \( N^a B_a = 0 \). In order to provide a versatile algorithm, we do not make any more assumptions.

By combining \((N^a, B^a)\) and a unit vector field \( T^a \) defined as

\[
T^a \equiv \frac{\nabla_a R(i)}{\sqrt{\langle \nabla_b R(i) \rangle \langle \nabla_b R(i) \rangle}} ,
\]

for the non-zero \( \nabla_a R(i) \) \((i = 1, 2 \text{ or } 3)\), we obtain a nonholonomic orthonormal basis \((T^a, N^a, B^a)\),

\[
\delta^a_b = T^a T_b + N^a N_b + B^a B_b .
\]

By using equations (19) and (21), the \( TT \)-part of the Killing equation can formally be written by

\[
0 = (\omega_N N^a + \omega_B B^a) T^b \nabla_b T_a ,
\]

which gives a test for \( R(i) \). If there are two linearly independent KVs, then equation (22) implies \( T^b \nabla_b T_a = 0 \). Depending on whether the gradient of \( R(i) \) satisfies the geodesic equation

\[
\langle \nabla^b R(i) \rangle \nabla_b \nabla_a R(i) \propto \nabla_a R(i) ,
\]

our analysis branches off.

3.1. Branch where \( \nabla_a R(i) \) is not a geodesic

In this branch \( T^a \) and its acceleration \( T^b \nabla_b T^a \) are linearly independent. It is therefore possible to define the Frenet–Serret frame as

\[
T^a \equiv \frac{\nabla^a R(i)}{\sqrt{\langle \nabla_b R(i) \rangle \langle \nabla_b R(i) \rangle}} , \quad N^a \equiv \frac{T^b \nabla_b T^a}{\sqrt{\langle T^c \nabla_c T^d \rangle \langle T^d \nabla_c T^d \rangle}} , \quad B^a \equiv \epsilon^{abc} T_b N_c .
\]

This frame obeys the so-called Frenet–Serret formulæ

\[
T^b \nabla_b \begin{pmatrix} T^a \\ N^a \\ B^a \end{pmatrix} = \begin{pmatrix} 0 & \kappa_T & 0 \\ -\kappa_T & 0 & \tau_T \\ 0 & -\tau_T & 0 \end{pmatrix} \begin{pmatrix} T^a \\ N^a \\ B^a \end{pmatrix} ,
\]

where

\[
\begin{align*}
\kappa_T &= \frac{\langle \nabla^a R(i) \rangle}{\sqrt{\langle \nabla_a R(i) \rangle \langle \nabla_b R(i) \rangle}} , \\
\tau_T &= \frac{\langle \nabla^b R(i) \rangle}{\sqrt{\langle \nabla_a R(i) \rangle \langle \nabla_b R(i) \rangle}} .
\end{align*}
\]
\( \kappa_T \equiv \nabla_b T_a \), \( \tau_T \equiv \nabla_b N_a \),
\begin{equation}
\text{(26)}
\end{equation}
are respectively the geodesic curvature and torsion of an integral curve of \( T^a \).

Now, the \( TT \)-part of the Killing equation (22) reads
\[ \kappa_T \omega_N = 0. \]
\begin{equation}
\text{(27)}
\end{equation}
Since \( \kappa_T = 0 \) contradicts \( T^a \nabla_b T^a \neq 0, \omega_N \) must be zero. As KVs take the same form as equation (9), \( K^a = \omega_B B^a \), our analysis reduces to that of case 2 with identification of \( B^a \rightarrow U^a \) and \( \omega_B \rightarrow \omega \). Hence, \( \Omega_B(B) \) and \( \kappa_{\omega_B}(B) \) defined in equation (15) give the tests for \( B^a \). Since in this branch the rank of \( R_a \) is at least 2, there is at most one KV.

3.2. Branch where \( \nabla_a R^{(i)} \) is a geodesic

Although the following analysis does not depend on the choice of the orthonormal basis \((T^a, N^a, B^a)\), we shall comment on it at any rate. For Riemannian metrics, there are two natural bases depending on the property of \( T^a \). If \( T^a \) is an eigenvector of the Ricci tensor \( R_{ab} \), we can take the orthonormal basis as the eigensystem of \( R_{ab} \). Otherwise \((N^a, B^a)\) can be taken to be
\[ N^a \equiv \epsilon^{abc} T^d R_{cd} N_b, \quad B^a \equiv \epsilon^{abc} T^d B_b, \]
\begin{equation}
\text{(28)}
\end{equation}
since \( T^a \) and \( R_{ab} T^b \) are linearly independent. Here \( N \) is the normalisation factor.

In this branch, the \( TT \)-part of the Killing equation (22) is identically satisfied. To write out the remaining parts, we introduce the Ricci rotation coefficients as
\begin{equation}
\text{(29a)}
\end{equation}
\begin{equation}
\text{(29b)}
\end{equation}
\begin{equation}
\text{(29c)}
\end{equation}
where
\[ \kappa_N \equiv T^a \nabla_b N_a, \quad \eta_N \equiv B^a \nabla_b N_a, \quad \tau_N \equiv B^a \nabla_b T_a, \]
\[ \kappa_B \equiv T^a B^b \nabla_b B_a, \quad \eta_B \equiv N^a B^b \nabla_b B_a, \quad \tau_B \equiv N^a B^b \nabla_b T_a. \]
\begin{equation}
\text{(30)}
\end{equation}
\( \kappa_N(\kappa_B), \eta_N(\eta_B) \) and \( \tau_N(\tau_B) \) are respectively the geodesic, normal curvature and relative torsion of an integral curve of \( N^a(\eta_B) \). Since the derivatives of the Ricci rotation coefficients are not independent, we collect their relations in appendix A.1.

Using the Ricci rotation coefficients, the remaining parts of the Killing equation read
\[ \mathcal{L}_T \omega_N = - \kappa_N \omega_N + (\tau_T + \tau_N) \omega_B, \]
\begin{equation}
\text{(31a)}
\end{equation}
\[ \mathcal{L}_N \omega_N = \eta_N \omega_B, \]
\begin{equation}
\text{(31b)}
\end{equation}
\[ \mathcal{L}_B \omega_N = - \eta_B \omega_B - \epsilon_N \omega_N - \bar{\omega}, \]
\begin{equation}
\text{(31c)}
\end{equation}
\[ L_\tau \omega_B = -(\tau_T - \tau_B)\omega_N - \kappa_B\omega_B , \quad (31d) \]
\[ L_N \omega_B = \tilde{\omega} , \quad (31e) \]
\[ L_B \omega_B = \eta_B \omega_N , \quad (31f) \]

where equation (31e) defines a new variable \( \tilde{\omega} \). Clearly, the above equations are not closed with respect to unknown scalars \( (\omega_N, \omega_B, \tilde{\omega}) \). We thus need the information of the second order derivatives of \( (\omega_N, \omega_B) \). From the identities \( \nabla_{[\mu} \nabla_{\nu]} \omega_N = \nabla_{[\mu} \nabla_{\nu]} \omega_B = 0 \), we obtain the constraint equations

\[ 0 = 2(\tau_B - \tau_N)\tilde{\omega} + \left[ 2\eta_N(\tau_B - \tau_N) - L_B(\tau_B - \tau_N) + L_N(\kappa_B + \kappa_N) \right]\omega_N \]
\[ + \left[ L_B(\kappa_B + \kappa_N) + L_N(\tau_B - \tau_N) \right]\omega_B . \quad (32a) \]
\[ 0 = 2(\tau_B + \tau_N)\tilde{\omega} + \left[ L_N(\kappa_B - \kappa_N) - L_B(\tau_B - \tau_N) \right]\omega_N \]
\[ + \left[ 2\eta_B(\tau_B + \tau_N) + L_B(\kappa_B - \kappa_N) - L_N(\tau_B - \tau_N) \right]\omega_B . \quad (32b) \]
\[ 0 = (\kappa_B - \kappa_N)\tilde{\omega} - (L_N\tau_B)\omega_N + \left[ \eta_B(\kappa_B - \kappa_N) - L_B\tau_N \right]\omega_B . \quad (32c) \]

Notice that the remaining parts of \( \nabla_{[\mu} \nabla_{\nu]} \omega_N = \nabla_{[\mu} \nabla_{\nu]} \omega_B = 0 \) constitute the equations of evolution of \( \tilde{\omega} \). However, it follows from equations (32a)–(32c) that \( \tilde{\omega} \) can be expressed in terms of \( (\omega_N, \omega_B) \) except when \( \kappa_B - \kappa_N = \tau_B = \tau_N = 0 \). Depending on the values of \( (\kappa_B - \kappa_N, \tau_B, \tau_N) \), our analysis falls into several sub-branches.

### 3.2.1 Sub-branch where \( \tau_B = \kappa_B - \kappa_N = 0 \)

In this sub-branch, constraints (32b) and (32c) are trivially satisfied, so we use the remaining parts of the identities \( \nabla_{[\mu} \nabla_{\nu]} \omega_N = \nabla_{[\mu} \nabla_{\nu]} \omega_B = 0 \) and obtain

\[ L_\tau \tilde{\omega} = (\eta_N \kappa_N + L_B \kappa_N - L_T \eta_N)\omega_N - (\eta_N \tau_T + L_N \kappa_N)\omega_B , \quad (33a) \]
\[ L_N \omega_B = - (L_N \eta_B)\omega_N + \eta_B(\kappa_B + \kappa_N)\omega_B , \quad (33b) \]
\[ L_B \omega_B = (L_N \eta_B - \eta_B^2)\omega_N + \eta_N \eta_B \omega_B + \eta_N \tilde{\omega} . \quad (33c) \]

Therefore, the equations of evolution of \( (\omega_N, \omega_B, \tilde{\omega}) \) take the form

\[ \nabla_{[\mu} \omega_{\nu]} = \Omega_\mu^{(1)} \omega_{\nu}], \quad \omega = \begin{pmatrix} \omega_N \\ \omega_B \\ \tilde{\omega} \end{pmatrix} . \quad (34) \]

where

\[ \Omega_\mu^{(1)} \equiv T_a \begin{pmatrix} -\kappa_N & \tau_T & 0 \\ -\tau_T & -\kappa_N & 0 \\ \eta_N \kappa_N + L_B \kappa_N - L_T \eta_N & -\eta_N \tau_T - L_N \kappa_N & 0 \end{pmatrix} \]
\[ + N_a \begin{pmatrix} 0 & \eta_N & 0 \\ 0 & 0 & 1 \\ -L_N \eta_N & \eta_B - L_B \eta_N - L_N \eta_B & 0 \end{pmatrix} + B_a \begin{pmatrix} -\eta_N & -\eta_B & 0 \\ \eta_B & 0 & 0 \\ \eta_B \eta_B & \eta_N \eta_B & \eta_N \end{pmatrix} . \quad (35) \]
The integrability condition for equation (34) reads
\[
(\nabla_a \Omega_b^{(1)} - \Omega_a^{(1)} \Omega_b^{(1)}) \omega = 0,
\] (36)
or in component form
\[
R_{\alpha \lambda}^{(1)} \omega = 0,
\] (37)
where
\[
R_{\alpha \lambda}^{(1)} = \begin{pmatrix}
\mathcal{L}_{\xi \xi} - \mathcal{L}_{\xi B} & \mathcal{L}_{\xi \lambda} - \mathcal{L}_{B \lambda} \\
\mathcal{L}_{\xi \lambda} - \mathcal{L}_{B \lambda} & 0
\end{pmatrix},
\] (38)
with \( \xi \equiv \mathcal{R}_{ab} \mathcal{N}^a \mathcal{N}^b \). Note that the first line correspond to equation (32a).

We call \( R_{\alpha \lambda}^{(1)} \) the first obstruction matrix of case 1 whose rank governs the number of KVs. If \( \text{rank} R_{\alpha \lambda}^{(1)} = 0 \), 3 KVs exist. If \( \text{rank} R_{\alpha \lambda}^{(1)} = 2 \), there is no KV. If \( \text{rank} R_{\alpha \lambda}^{(1)} = 1 \), our analysis reduces to that of case 2 with appropriate identifications of \( U^b \).

3.2.2. Sub-branch where \( \tau_N = \tau_B = 0 \) but \( \kappa_N \neq \kappa_B \). In this sub-branch, from equation (32c) the function \( \omega \) takes the form
\[
\omega = -\eta_B \omega_B.
\] (39)
Substituting this form into equation (31), we obtain
\[
\nabla \omega = \Omega_B^{(2)} \omega, \quad \omega \equiv \begin{pmatrix} \omega_N \\ \omega_B \end{pmatrix},
\] (40)
where
\[
\Omega_B^{(2)} \equiv T_a \begin{pmatrix} -\kappa_N & \tau_B \\ -\tau_B & -\kappa_B \end{pmatrix} + N_a \begin{pmatrix} 0 & \eta_N \\ 0 & -\eta_B \end{pmatrix} + B_a \begin{pmatrix} -\eta_N & 0 \\ \eta_B & 0 \end{pmatrix}.
\] (41)
Its integrability condition \( (\nabla_a \Omega_B^{(2)} - \Omega_a^{(2)} \Omega_B^{(2)}) \omega = 0 \) leads to \( R_{\alpha \lambda}^{(2)} \omega = 0 \) where
\[
R_{\alpha \lambda}^{(2)} = \begin{pmatrix}
\mathcal{L}_{\xi \xi} - \mathcal{L}_{\xi B} & \mathcal{L}_{\xi \lambda} - \mathcal{L}_{B \lambda} \\
\mathcal{L}_{\xi \lambda} - \mathcal{L}_{B \lambda} & 0
\end{pmatrix}.
\] (42)
We call \( R_{\alpha \lambda}^{(2)} \) the second obstruction matrix of case 1. In a way analogous to \( R_{\alpha \lambda}^{(1)} \), \( \text{rank} R_{\alpha \lambda}^{(2)} \) reveals the number of KVs. If \( \text{rank} R_{\alpha \lambda}^{(2)} = 0 \), two KVs exist. If \( \text{rank} R_{\alpha \lambda}^{(2)} = 2 \), there is no KV. Otherwise, when \( \text{rank} R_{\alpha \lambda}^{(2)} = 1 \), our analysis reduces to that of case 2 with an appropriate identification of \( U^b \).

3.2.3. Sub-branch where \( \tau_N = \tau_B \neq 0 \). In this sub-branch, it follows from equation (32b) that

\(^5\) For instance, if \( \mathcal{L}_{\xi \xi} \neq 0 \), from equation (37) we can write \( \omega_B = -\frac{\omega_B}{\mathcal{R}_{ab} \mathcal{N}^a \mathcal{N}^b} \) and then KVs take the same form as equation (9), \( \mathcal{K}^a = \omega_B \left( B^a - \frac{\mathcal{R}_{ab} \mathcal{N}^a}{\mathcal{R}_{ab} \mathcal{N}^b} \right) \). Thus, under identification of \( B^a \equiv \alpha \left( B^a - \frac{\mathcal{R}_{ab} \mathcal{N}^a}{\mathcal{R}_{ab} \mathcal{N}^b} \right) \rightarrow U^a \) and \( \omega_B / \alpha \rightarrow \omega \), where \( \alpha \) is chosen so as to satisfy the normalisation \( \mathcal{B}^a \mathcal{B}_a = 1 \), the computations made in section 2 are reusable: \( \Omega_c (\mathcal{B}) \) and \( \mathcal{R}_{ab} (\mathcal{B}) \) defined in equation (15) give the tests for \( B^a \).
\[ \bar{\omega} = -\eta_B \omega_B + \frac{1}{4\tau_N} \left[ \omega_N \mathcal{E}_N (\kappa_N - \kappa_B) + \omega_B \mathcal{E}_B (\kappa_N - \kappa_B) \right], \quad (43) \]

By using this, we rewrite equation (31) as

\[ \nabla_a \omega = \Omega_a^{(3)} \omega, \quad \omega \equiv \begin{pmatrix} \omega_N \\ \omega_B \end{pmatrix}, \quad (44) \]

where

\[ \Omega_a^{(3)} \equiv T_a \begin{pmatrix} -\kappa_N \tau_B + \tau_N - \kappa_B \\ -\tau_N + \tau_B - \kappa_B \end{pmatrix} + N_a \begin{pmatrix} 0 \\ \frac{1}{\tau_N} \mathcal{E}_N (\kappa_N - \kappa_B) + \eta_B + \frac{\tau_N}{\tau_B} \mathcal{E}_B (\kappa_N - \kappa_B) \end{pmatrix} \]

\[ + B_a \begin{pmatrix} -\eta_N - \frac{\tau_N}{\tau_B} \mathcal{E}_N (\kappa_B - \kappa_N) - \eta_B - \frac{\tau_N}{\tau_B} \mathcal{E}_B (\kappa_B - \kappa_N) \\ 0 \end{pmatrix}. \quad (45) \]

Using the shorthand notation for the Ricci rotation coefficients,

\[ \kappa_\Delta \equiv \kappa_B - \kappa_N, \quad \kappa_\Sigma \equiv \kappa_B + \kappa_N, \quad (46) \]

the integrability condition for equation (44) can be written as follows.

\[ 0 = (\mathcal{E}_N \kappa_\Sigma) \omega_N + (\mathcal{E}_B \kappa_\Sigma) \omega_B, \quad (47a) \]

\[ 0 = \left[ \mathcal{E}_N (\kappa_\Delta^2 + 4\tau_N^2) \right] \omega_N + \left[ \mathcal{E}_B (\kappa_\Delta^2 + 4\tau_N^2) \right] \omega_B, \quad (47b) \]

\[ 0 = \left[ \mathcal{E}_N \mathcal{E}_B \kappa_\Delta - 4\tau_N \mathcal{E}_N \tau_B - \frac{(\mathcal{E}_N \tau_N + \mathcal{E}_B \kappa_\Delta)}{\tau_N} \right] \omega_N \]

\[ + \left[ \mathcal{E}_B \mathcal{E}_B \kappa_\Delta - 4\tau_N \mathcal{E}_B \tau_B - \frac{(\mathcal{E}_B \tau_N + \mathcal{E}_B \kappa_\Delta)}{\tau_N} \right] \omega_B, \quad (47c) \]

\[ 0 = \left[ \mathcal{E}_N \mathcal{E}_B \kappa_\Delta - 4\tau_N \mathcal{E}_N \tau_B - \frac{(4\mathcal{E}_N \tau_N + \mathcal{E}_B \kappa_\Delta)}{4\tau_N} \right] \omega_N \]

\[ + \left[ \mathcal{E}_B \mathcal{E}_B \kappa_\Delta - 4\tau_N \mathcal{E}_B \tau_B - \frac{(4\mathcal{E}_B \tau_N + \mathcal{E}_B \kappa_\Delta)}{4\tau_N} \right] \omega_B, \quad (47d) \]

\[ 0 = \left[ \mathcal{E}_N \mathcal{E}_B \kappa_\Delta + 4\tau_N \mathcal{E}_N \tau_B - \frac{(4\mathcal{E}_N \tau_N - \mathcal{E}_B \kappa_\Delta)}{4\tau_N} \right] \omega_N \]

\[ + \left[ \mathcal{E}_B \mathcal{E}_B \kappa_\Delta + 4\tau_N \mathcal{E}_B \tau_B - \frac{(4\mathcal{E}_B \tau_N - \mathcal{E}_B \kappa_\Delta)}{4\tau_N} \right] \omega_B. \quad (47e) \]

Rewriting equation (47) as \( R^{(3)}_{\bar{u}} \omega = 0 \), the rank of the third obstruction matrix of case 1 \( R^{(3)}_{\bar{u}} \) governs the number of KVs in a way analogous to that of \( R^{(1)}_{\bar{u}} \) and \( R^{(2)}_{\bar{u}} \).

3.2.4. **Sub-branch where** \( \tau_N \neq \tau_B \). As similar to the previous sub-branch, we can put \( \bar{\omega} \) into \((\omega_N, \omega_B)\) by using equation (32a),

\[ \bar{\omega} = \frac{1}{2\tau_\Delta} \left[ \mathcal{E}_B \tau_\Delta + \tau_\Delta (\eta_\Delta - \eta_\Sigma) - \mathcal{E}_N \kappa_\Sigma \right] \omega_N - \frac{1}{2\tau_\Delta} \left[ \mathcal{E}_N \tau_\Delta + \mathcal{E}_B \kappa_\Sigma \right] \omega_B, \quad (48) \]

where we have used the shorthand notations (46) and
Thus, we rewrite equation (31) as

$$
\nabla_a \omega = \Omega_\mu^{(4)} \omega, \quad \omega \equiv \left( \frac{\omega_N}{\omega_B} \right),
$$

where

$$
\Omega_\mu^{(4)} = T_a \left( \begin{array}{cc}
\frac{\kappa_\mu - \kappa_N}{2 \xi^2 + \tau^2} & \frac{\tau}{2} - \frac{\kappa_\mu - \kappa_N}{2 \xi^2 + \tau^2} \\
\frac{\tau}{2} - \frac{\kappa_\mu - \kappa_N}{2 \xi^2 + \tau^2} & -\frac{\kappa_N}{2} + \frac{\kappa_\mu}{2}
\end{array} \right) + N_e \left( \begin{array}{cc}
0 & \frac{\eta_N - \eta_\mu}{2 \tau^2} \\
\frac{\eta_N - \eta_\mu}{2 \tau^2} & -\frac{\eta_\mu}{2} + \frac{\eta_N}{2}
\end{array} \right) + B_a \left( \begin{array}{cc}
\frac{\kappa_\mu - \kappa_N}{2 \xi^2 + \tau^2} & \frac{\tau}{2} + \frac{\kappa_\mu - \kappa_N}{2 \xi^2 + \tau^2} \\
\frac{\tau}{2} + \frac{\kappa_\mu - \kappa_N}{2 \xi^2 + \tau^2} & -\frac{\kappa_N}{2} + \frac{\kappa_\mu}{2}
\end{array} \right),
$$

Its integrability condition leads to tolerably complicated relations

$$
0 = \left[ \tau_\Delta \left( L_B \tau_\Delta - L_N \kappa_\Delta \right) + \tau_\Sigma \left( \tau_\Delta \left( \eta_\Delta - \eta_\Sigma \right) - L_B \tau_\Delta + L_N \kappa_\Sigma \right) \right] \omega_N \\
+ \left[ \tau_\Delta \left( L_N \tau_\Delta - L_B \kappa_\Delta \right) + \tau_\Sigma \left( -\tau_\Delta \left( \eta_\Delta + \eta_\Sigma \right) + L_B \kappa_\Sigma + L_N \tau_\Delta \right) \right] \omega_B,
$$

$$
0 = \left[ \kappa_\Delta \left( L_B \tau_\Delta - L_N \kappa_\Sigma \right) + \tau_\Sigma \left( \kappa_\Delta \left( \eta_\Delta - \eta_\Sigma \right) - L_N \left( \tau_\Delta + \tau_\Sigma \right) \right) \right] \omega_N \\
+ \left[ \tau_\Delta \left( \kappa_\Delta \left( \eta_\Delta + \eta_\Sigma \right) + L_B \left( \tau_\Delta - \tau_\Sigma \right) \right) - \kappa_\Delta \left( L_B \kappa_\Sigma + L_N \tau_\Delta \right) \right] \omega_B,
$$

$$
0 = \left[ \kappa_\Delta L_B \tau_\Delta - \tau_\Sigma L_N \tau_\Delta + \kappa_\Sigma L_B \kappa_\Sigma - L_N L_T \kappa_\Sigma \right] \\
+ \tau_\Delta \left( \kappa_\Delta \left( \eta_\Delta - \eta_\Sigma \right) + \tau_\Sigma \left( \eta_\Delta + \eta_\Sigma \right) + L_B \left( \kappa_\Delta - \kappa_\Sigma \right) - L_N \left( 2 \tau_\Delta - \tau_\Sigma \right) \right) \omega_N \\
+ \left[ \kappa_\Delta L_N \tau_\Delta + \tau_\Sigma L_B \tau_\Delta + \kappa_\Sigma L_B \kappa_\Sigma - L_B L_T \kappa_\Sigma \right] \\
+ \tau_\Delta \left( -\kappa_\Delta \left( \eta_\Delta + \eta_\Sigma \right) + \tau_\Sigma \left( \eta_\Delta - \eta_\Sigma \right) + L_N \left( \kappa_\Delta + \kappa_\Sigma \right) - L_B \left( 2 \tau_\Delta - \tau_\Sigma \right) \right) \omega_B,
$$

$$
0 = \left[ \left( \kappa_\Sigma - \kappa_\Delta \right) \tau_\Delta^3 + \left( \left( \eta_\Delta - \eta_\Sigma \right) L_B \kappa_\Sigma + \left( \eta_\Delta - \eta_\Sigma \right) L_N \tau_\Delta - 2 L_N L_B \tau_\Delta + 2 L_N L_B \kappa_\Sigma \right) \frac{\tau_\Delta}{2} \right] \omega_N \\
+ \frac{1}{2} \left( \left( L_B \tau_\Delta \right) \left( L_B \kappa_\Sigma \right) + 3 \left( L_B \tau_\Delta \right) \left( L_N \tau_\Delta \right) - \left( L_B \kappa_\Sigma \right) \left( L_N \kappa_\Sigma \right) - 3 \left( L_N \tau_\Delta \right) \left( L_N \kappa_\Sigma \right) \right) \omega_N \\
+ \left[ \left( \eta_\Delta - \eta_\Sigma \right) L_B \tau_\Delta - \left( \eta_\Delta - \eta_\Sigma \right) L_N \kappa_\Sigma + 2 L_N L_B \kappa_\Sigma + 2 L_N L_B \tau_\Delta \right] \frac{\tau_\Delta}{2} \\
- \frac{1}{2} \left( \left( L_B \kappa_\Sigma \right)^2 + 3 \left( L_B \tau_\Delta \right)^2 \right) \omega_B,
$$

(52d)
Rewriting equation (52) as $\omega = 0$, the rank of the fourth obstruction matrix of case 1 $R_{cs1}^{(4)}$ governs the number of KVs in a way analogous to that of $R_{cs1}^{(1)}$, $R_{cs1}^{(2)}$ and $R_{cs1}^{(2)}$.

We are at the end of this branch and summarise our results in figure 4.

4. Analysis of case 0

In this case, we have $dR^{(1)} = dR^{(2)} = dR^{(3)} = 0$. This implies that $R_a$ is a zero matrix, and all eigenvalues of the Ricci tensor are constant. Here we take advantage of this property and use the eigensystem of the Ricci tensor, $(E_a^1, E_a^2, E_a^3)$, as an orthonormal basis. As we will see below, our analysis depends on Segre types of the Ricci tensor.

Since an arbitrary vector can be the annihilator of $R_a$, any KV can be written as

$$K^a = \tau_1 E_1^a + \tau_2 E_2^a + \tau_3 E_3^a,$$

where $(\tau_1, \tau_2, \tau_3)$ are unknown functions. We also introduce the Ricci rotation coefficients for the eigensystem as

$$E_a^b \nabla_b \begin{pmatrix} E_1^a \\ E_2^a \\ E_3^a \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \eta_1 \\ -\kappa_1 & 0 & \tau_1 \\ -\eta_1 & -\tau_1 & 0 \end{pmatrix} \begin{pmatrix} E_1^a \\ E_2^a \\ E_3^a \end{pmatrix},$$
\[
E_a^b \nabla_b \pmatrix{E_1^a \\ E_2^a \\ E_3^a} = \pmatrix{0 & -\kappa_2 & \tau_2 \\ \kappa_2 & 0 & \eta_2 \\ -\tau_2 & -\eta_2 & 0} \pmatrix{E_1^a \\ E_2^a \\ E_3^a},
\]
\[
E_a^b \nabla_b \pmatrix{E_1^a \\ E_2^a \\ E_3^a} = \pmatrix{0 & \tau_3 & -\kappa_3 \\ -\tau_3 & 0 & -\eta_3 \\ -\kappa_3 & \eta_3 & 0} \pmatrix{E_1^a \\ E_2^a \\ E_3^a},
\]

where
\[
\kappa_1 \equiv E_1^b E_1^a \nabla_b (E_1)_a, \quad \eta_1 \equiv E_1^b E_1^a \nabla_b (E_1)_a, \quad \tau_1 \equiv E_3^b E_3^a \nabla_b (E_2)_a, \\
\kappa_2 \equiv E_1^b E_2^a \nabla_b (E_2)_a, \quad \eta_2 \equiv E_1^b E_2^a \nabla_b (E_2)_a, \quad \tau_2 \equiv E_3^b E_3^a \nabla_b (E_1)_a, \\
\kappa_3 \equiv E_1^b E_3^a \nabla_b (E_3)_a, \quad \eta_3 \equiv E_1^b E_3^a \nabla_b (E_3)_a, \quad \tau_3 \equiv E_3^b E_3^a \nabla_b (E_1)_a.
\]

Using equations (53) and (55), we can write out the equations of evolution of \((\pi_1, \pi_2, \pi_3)\). However, the equation will not be used and is somewhat lengthy. So we leave this to appendix B and just cite its integrability condition
\[
0 = (\lambda_1 - \lambda_2) \left[ \kappa_2 \pi_2 - (\tau_1 + \tau_3) \pi_3 + \pi_2 \right],
\]
\[
0 = (\lambda_2 - \lambda_3) \left[ (\tau_1 - \tau_2) \pi_1 - \eta_3 \pi_3 - \pi_3 \right],
\]
\[
0 = (\lambda_3 - \lambda_1) \left[ \eta_1 \pi_1 + (\tau_2 - \tau_3) \pi_2 + \pi_1 \right],
\]

where \(\lambda_i\) are the eigenvalues of the Ricci tensor,
\[
R^a_b E^b_i = \lambda_i E^b_i, (i = 1, 2, 3)
\]

and \(\bar{\pi}_i\) are new variables defined as
\[
\bar{\pi}_1 \equiv L_3 \pi_1, \quad \bar{\pi}_2 \equiv L_1 \pi_2, \quad \bar{\pi}_3 \equiv L_2 \pi_3.
\]

Here \(L_i\) denotes the Lie derivative along \(E^b_i\).

The integrability condition (56) is trivially satisfied if the Segre type is \([1111]\), \(\lambda_1 = \lambda_2 = \lambda_3\). In the remaining parts of this subsection, we discuss the Segre types \([1110]\) and \([1111]\).

**4.1. Branch where the Segre type is \([1110]\)**

In this branch, we can assume \(\lambda_1 \neq \lambda_2 = \lambda_3\) without loss of generality. From the constancy of the eigenvalues (A.5), we have
\[
\kappa_2 + \kappa_3 = 0, \quad \kappa_1 = 0, \quad \eta_1 = 0.
\]

Under equation (59), the integrability condition (56) reads
\[
\bar{\pi}_1 = - (\tau_2 - \tau_3) \pi_2, \quad \bar{\pi}_2 = - \kappa_2 \pi_2 + (\tau_1 + \tau_3) \pi_3.
\]

Substituting equation (60) into equation (B.1a), we obtain the equations for \((\tau_1, \tau_2, \tau_3, \pi_3)\).

We leave its explicit form to equation (B.2a) in appendix B. The integrability condition of equation (B.2a) reads

\[
\]
\[
\begin{align*}
0 &= (\tau_2 + \tau_3) \bar{\pi}_3 + \tau_2 (\tau_2 + \tau_3) \pi_1 - (\mathcal{L}_2 \kappa_2) \pi_2 - (2 \eta_2 \kappa_2 + \mathcal{L}_2 \tau_3) \pi_3, \\
0 &= 2 \kappa_2 \bar{\pi}_3 + 2 \kappa_2 \tau_2 \pi_1 + (\mathcal{L}_2 \tau_2) \pi_2 + (2 \eta_3 \kappa_2 + \mathcal{L}_3 \tau_3) \pi_3, \\
0 &= [\mathcal{L}_2 (\tau_2 - \tau_3)] \pi_2 + [\mathcal{L}_3 (\tau_2 - \tau_3)] \pi_3, \\
0 &= \left[ \frac{\tau_2 - \tau_3}{2} \mathcal{L}_2 \tau_2 - \tau_2 \mathcal{L}_2 \tau_3 - \kappa_2 (2 \eta_3 \kappa_2 + \mathcal{L}_3 \tau_2 - \eta_2 (\tau_2 + \tau_3)) \right] \pi_2 \\
&\quad + \left[ \frac{\tau_2 - \tau_3}{2} \mathcal{L}_3 \tau_2 - \tau_2 \mathcal{L}_3 \tau_3 - \kappa_2 (2 \eta_3 \kappa_2 - \mathcal{L}_3 \tau_3 + \eta_3 (\tau_2 + \tau_3)) \right] \pi_3. 
\end{align*}
\]

If there are four linearly independent KVs, then equation (61) implies \( \tau_2 + \tau_3 = \kappa_2 = 0 \). We examine the sub-branch depending on whether \( \tau_2 + \tau_3 \) and \( \kappa_2 \) are zero, or not.

### 4.1.1. Sub-branch where \( \tau_2 + \tau_3 = \kappa_2 = 0 \)

In this sub-branch, the integrability condition (61) gives a relation

\[
0 = (\mathcal{L}_2 \tau_2) \pi_2 + (\mathcal{L}_3 \tau_3) \pi_3. 
\]

On the other hand, the derivatives of the Ricci rotation coefficients (55) are not independent as it is for case 1. We collect their relations in appendix A. From equation (A.3a), it follows that \( \mathcal{L}_2 \tau_2 = \mathcal{L}_3 \tau_3 = 0 \) automatically. As there are no extra conditions, we conclude that four KVs exist in this sub-branch.

### 4.1.2. Sub-branch where \( \tau_2 + \tau_3 = 0 \) but \( \kappa_2 \neq 0 \)

In this sub-branch, we can solve equation (61) for \( \bar{\pi}_3 \) and obtain

\[
\bar{\pi}_3 = -\tau_2 \pi_1 - \eta_3 \pi_3. 
\]

By using (63), we rewrite equation (B.2a) as

\[
\nabla_a \pi = \Pi_a^{(1)} \pi, \quad \pi \equiv \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix},
\]

where

\[
\Pi_a^{(1)} \equiv (E_1)_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\kappa_2 & \tau_1 - \tau_2 \\ 0 & \tau_2 - \tau_1 & \kappa_2 \end{pmatrix} + (E_2)_a \begin{pmatrix} 0 & 0 & 2 \tau_2 \\ \kappa_2 & 0 & \eta_2 \\ -2 \tau_2 & 0 & -\eta_3 \end{pmatrix} + (E_3)_a \begin{pmatrix} 0 & -2 \tau_2 & 0 \\ \tau_2 & -\eta_2 & 0 \\ -\kappa_2 & \eta_3 & 0 \end{pmatrix}.
\]

Its integrability condition, \( \nabla_a \Pi_a^{(1)} - \Pi_a^{(1)} \Pi_b^{(1)} \pi = 0 \), can be written by

\[
R^{(1)}_{a b c d} \pi = 0,
\]

where

\[
R^{(1)}_{a b c d} \equiv \begin{pmatrix} 0 & \mathcal{L}_2 \tau_2 & \mathcal{L}_3 \tau_2 \\ 0 & \mathcal{L}_2 \kappa_2 & \mathcal{L}_3 \kappa_2 \\ \mathcal{L}_1 \kappa_2 & \mathcal{L}_2 \kappa_2 & \mathcal{L}_3 \kappa_2 \\ \mathcal{L}_1 \eta_2 & \mathcal{L}_2 \eta_2 & \mathcal{L}_3 \eta_2 \\ \mathcal{L}_1 \eta_3 & \mathcal{L}_2 \eta_3 & \mathcal{L}_3 \eta_3 \end{pmatrix}.
\]
is the first obstruction matrix of case 0 whose rank governs the number of KVs. If $\text{rank} R_{cs,0}^{(1)} = 0$, three KVs exist. If $\text{rank} R_{cs,0}^{(1)} = 3$, there is no KV. Otherwise, when $\text{rank} R_{cs,1}^{(1)} = 1$ or 2, our analysis reduces to that of case 1 or 2 with appropriate identifications$^6$.

4.1.3. Sub-branch where $\tau_2 + \tau_3 \neq 0$. As similar to the previous sub-branch, equation (61a) allows us to put $\pi_3$ into $(\pi_1, \pi_2, \pi_3)$

$$\pi_3 = -\tau_2 \pi_1 + \frac{L_2 \kappa_2}{\tau_1 + \tau_3} \pi_2 + \left(\frac{L_3 \kappa_2}{\tau_2} + \eta_3\right) \pi_3. \quad (68)$$

By combining equations (68) and (B.2a), we obtain

$$\nabla_a \pi = \Pi_a^{(2)} \pi, \quad \pi \equiv \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \quad (69)$$

where

$$\Pi_a^{(2)} \equiv (E_1)_{a} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\kappa_2 & \tau_1 + \tau_3 \\ 0 & \tau_2 - \tau_1 & \kappa_2 \end{pmatrix} + (E_2)_{a} \begin{pmatrix} 0 & 0 & \tau_2 - \tau_3 \\ \kappa_2 & 0 & \eta_2 \\ -\tau_2 - \eta_2 + 2\eta_2 \kappa_2 + L_2 \kappa_2 + L_2 \kappa_2 \frac{\eta_2}{\tau_2 + \tau_3} \end{pmatrix}$$

$$+ (E_3)_{a} \begin{pmatrix} -\tau_3 - \eta_2 - \frac{\eta_2 \kappa_2 + L_2 \kappa_2}{\tau_2 + \tau_3} \\ -\kappa_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 - \tau_3 \\ \eta_2 - \frac{\eta_2 \kappa_2 + L_2 \kappa_2}{\eta_2 / \tau_2 + \tau_3} \end{pmatrix}. \quad (70)$$

By using the shorthand notation for the Ricci rotation coefficients,

$$\tau_{\pm} \equiv \tau_3 \pm \tau_2, \quad (71)$$

its integrability condition, $$(\nabla_a \Pi^{(2)}_a - \Pi^{(2)}_a \Pi^{(2)}_a) \pi = 0$$, reads

$$0 = (L_2 \tau_-) \pi_2 + (L_3 \tau_-) \pi_3, \quad (72a)$$

and

$$0 = (4L_2 \kappa_2^2 + L_2 \tau_+^2) \pi_2 + (4L_3 \kappa_2^2 + L_3 \tau_+^2) \pi_3, \quad (72b)$$

and

$$0 = \frac{1}{2} \left[ 2\eta_2 \tau_2 + \eta_2 \tau_4 (\tau_+ - \tau_-) + L_2 \kappa_2^2 + (\tau_+ - \tau_-) L_3 \kappa_2 \right] \pi_1$$

$$+ \left[ L_2 \kappa_2 \tau_- + \eta_2 \kappa_2 + \tau_+ + \eta_2 \kappa_2 \frac{(L_2 \kappa_2) L_2 (\tau_+ - \tau_-)}{2\tau_+} - \eta_2 \kappa_2 \frac{(L_2 \kappa_2) (L_2 \kappa_2 + L_2 \kappa_2)}{2\tau_+} \right] \pi_2$$

$$+ \left[ L_2 \kappa_2 \tau_- - \eta_2 \kappa_2 + \tau_+ \tau_- \tau_+ + \eta_2 \kappa_2 + \tau_+ \eta_2 \kappa_2 \frac{(L_2 \kappa_2) L_2 (\tau_+ - \tau_-)}{2\tau_+} \right] \pi_3, \quad (72c)$$

$^6$For instance, if $L_3 \tau_2 \neq 0$, we can write $\pi_1$ as $\pi_1 = -\frac{2\tau_2}{\tau_1} \pi_2 - \frac{\tau_2}{\tau_1} \pi_3$, leading to the same form as equation (19), $K^\alpha = \pi_1 \left( E_2 - \frac{L_2 \kappa_2}{\kappa_2} E_1^1 \right) + \pi_2 \left( E_2 - \frac{L_2 \kappa_2}{\kappa_2} E_1^1 \right)$ Thus, under identification of $N^\alpha = \alpha_2 \left( E_2 - \frac{L_2 \kappa_2}{\kappa_2} E_1^1 \right)$ $\alpha_1 \rightarrow \omega_0$, $B^\alpha = \alpha_3 \left( E_2 - \frac{L_2 \kappa_2}{\kappa_2} E_1^1 \right) \rightarrow B^\alpha, \pi_2 \rightarrow \omega_0$ and $\pi_1 / \alpha_1 \rightarrow \omega_0$, with $\alpha_2, \alpha_3$ normalised via $N^\alpha N_\alpha = B^\alpha \dot{B}_\alpha = 1$, the computations of section 3 are reusable: The algorithm shown in figure 4 gives the tests for $N^\alpha$ and $B^\alpha$. 

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$$0 = \frac{1}{2} \left[ 2\eta_2 \kappa_2 \tau_+ - \eta_3 \tau_+ (\tau_+ + \tau_-) - \mathcal{L}_3 \kappa_3^2 (\tau_+ + \tau_-) \mathcal{L}_2 \kappa_2 \right] \pi_1$$

$$+ \left[ \mathcal{L}_3 \mathcal{L}_2 \kappa_2 - 3\eta_2 \mathcal{L}_2 \kappa_2 - \tau_1 \tau_- \tau_+ + \eta_3 \mathcal{L}_2 \kappa_2 - \tau_- \mathcal{L}_2 \eta_3 + \frac{(\mathcal{L}_2 \kappa_2) \mathcal{L}_3 (\tau_- + \tau_+)}{2 \tau_+} + \frac{\eta_3 \mathcal{L}_3 \kappa_3^2}{\tau_+} - \frac{2(\mathcal{L}_2 \kappa_2)^2}{\tau_+} \right] \pi_2$$

$$+ \left[ \mathcal{L}_3 \mathcal{L}_3 \kappa_2 - 2\eta_2 \mathcal{L}_3 \kappa_2 - \tau_+ \mathcal{L}_3 \eta_3 - \frac{(\mathcal{L}_2 \kappa_2) \mathcal{L}_3 (\tau_- + \tau_+)}{2 \tau_+} + \frac{\eta_3 \mathcal{L}_3 \kappa_3^2}{\tau_+} - \frac{2(\mathcal{L}_2 \kappa_2)(\mathcal{L}_3 \kappa_2)}{\tau_+} \right] \pi_3. \quad (72d)$$

Rewriting equation (72) as $R^{(2)}_{\text{ck}0} \pi = 0$, rank $R^{(2)}_{\text{ck}0}$ controls the number of KVs in a way analogous to that of $R^{(1)}_{\text{ck}0}$.

4.2. Branch where the Segre type is [1 1 1]

In this branch, three eigenvalues of the Ricci tensor, $(\lambda_1, \lambda_2, \lambda_3)$ differ from each other. The constancy of the eigenvalues (A.5) leads to

$$\varphi_1 \kappa_2 - \kappa_3 = 0, \quad \varphi_2 \eta_3 - \kappa_1 = 0, \quad \varphi_3 \eta_1 - \eta_2 = 0, \quad (73)$$

where

$$\varphi_1 \equiv \frac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_1}, \quad \varphi_2 \equiv -\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_2}, \quad \varphi_3 \equiv \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_3}. \quad (74)$$

Moreover, we solve equation (56) for $(\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$ and obtain

$$\bar{\pi}_1 = -\eta_1 \pi_1 - (\tau_2 - \tau_3) \pi_2, \quad \bar{\pi}_2 = -\kappa_2 \pi_2 + (\tau_1 + \tau_3) \pi_3, \quad \bar{\pi}_3 = (\tau_1 - \tau_2) \pi_2 - \eta_3 \pi_3. \quad (75)$$

By using equations (73) and (75), we rewrite equation (B.1a) as

$$\nabla_a \pi = \Pi^{(3)}_a \pi, \quad \pi \equiv \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \quad (76)$$

where

$$\Pi^{(3)}_a \equiv (E_1)_a \begin{pmatrix} 0 & \varphi_2 \eta_3 & \eta_1 \\ 0 & -\kappa_2 & \tau_1 + \tau_3 \\ -\tau_2 - \tau_1 & -\varphi_1 \kappa_2 \\ \end{pmatrix} + (E_2)_a \begin{pmatrix} -\varphi_2 \eta_3 & 0 & \tau_2 - \tau_3 \\ \kappa_2 & 0 & \varphi_3 \eta_1 \\ \tau_1 - \tau_2 & 0 & -\eta_3 \end{pmatrix}$$

$$+ (E_3)_a \begin{pmatrix} -\eta_1 & \tau_3 - \tau_2 & 0 \\ -\tau_1 - \tau_3 & -\varphi_3 \eta_1 & 0 \\ \varphi_1 \kappa_2 & \eta_3 & 0 \end{pmatrix}. \quad (77)$$

Its integrability condition, $(\nabla_a \Pi^{(3)}_b - \Pi^{(3)}_a \Pi^{(3)}_b) \pi = 0$, is typified by $R^{(3)}_{\text{ck}0} \pi = 0$ where

$$R^{(3)}_{\text{ck}0} \equiv \begin{pmatrix} \mathcal{L}_1 \kappa_2 & \mathcal{L}_2 \kappa_2 & \mathcal{L}_3 \kappa_2 \\ \mathcal{L}_1 \eta_1 & \mathcal{L}_2 \eta_1 & \mathcal{L}_3 \eta_1 \\ \mathcal{L}_1 \kappa_3 & \mathcal{L}_2 \kappa_3 & \mathcal{L}_3 \kappa_3 \\ \mathcal{L}_1 \eta_3 & \mathcal{L}_2 \eta_3 & \mathcal{L}_3 \eta_3 \\ \mathcal{L}_1 \tau_1 & \mathcal{L}_2 \tau_1 & \mathcal{L}_3 \tau_1 \\ \mathcal{L}_1 \tau_3 & \mathcal{L}_2 \tau_3 & \mathcal{L}_3 \tau_3 \end{pmatrix}. \quad (78)$$
We therefore are at the conclusion that the rank of the third obstruction matrix of case 0 controls the number of KVs in a way analogous to that of $R^{(1)}_{cs.0}$ and $R^{(2)}_{cs.0}$. We summarise our results in figure 5.

5. Examples

This section is devoted to some relatively simple applications of our algorithm.

5.1. Hamiltonian system

It is well known that a 2-dimensional natural Hamiltonian

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2),$$

(79)

can be lifted to the 3-dimensional geodesic Hamiltonian

$$H(p, q) = \frac{1}{2}g^{ab}_E(q) p_a p_b, \quad \text{where} \quad g^{ab}_E(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2V(q_1, q_2) \end{pmatrix}.$$\n
(80)

The metric $(g_E)_ab$ is called the Eisenhart metric. We examine this metric with

$$V(q_1, q_2) = \frac{1}{4}(q_1^4 + q_2^4) + \frac{\epsilon}{2}q_1^2q_2^2,$$

(81)

by our branching algorithm.

After simple algebra, we can see that the first test $R_{ab} \propto g_{ab}$ fails for any $\epsilon$. Since the geodesic flow of $(g_E)_{ab}$ has the KV $\partial_{q_1}$, the metric passes the test $dR^{(1)} \wedge dR^{(2)} \wedge dR^{(3)} = 0$. Moreover, we find that the value of $dR^{(a)} \wedge dR^{(b)}$ depends on the value of $\epsilon$: If $\epsilon \neq 1$, the metric arrives at the analysis of case 1, and it passes the last two tests $\kappa_{ab} = \nabla_{[a} \Omega_{b]} = 0$. So there is only one KV in this case.

In case $\epsilon = 1$, the metric passes to the analysis of case 1, and $\nabla_{\omega} R^{(1)}$ is to be a geodesic as well as an eigenvector of $R^{(1)}_{\Omega b}$ with the eigenvalue $-6/(q_1^2 + q_2^2)$. The metric has the properties $\tau_N = \tau_B$, $\tau_N = 0$ and $\kappa_N \neq \kappa_B$. We see that $\text{rank} R^{(2)}_{cs.1} = 2$, and therefore there are two KVs: $\partial_{q_1} - \partial_{q_2}, \partial_{q_2}$. This result was known in the literature.
5.2. Foliation of the Zipoy–Voorhees spacetime

Consider the Zipoy–Voorhees spacetime \( \delta \geq 0 \) \([16, 17]\)

\[
g_\delta = \left( \frac{x+1}{x-1} \right)^\delta \left[ (x^2 - y^2) \left( \frac{x^2 - 1}{x^2 - y^2} \right)^\delta \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2)dz^2 \right] - \left( \frac{x-1}{x+1} \right)^\delta dz^2. \tag{82}
\]

This has Lorentzian signature in the region \( |y| < 1 < |x| \), and \( \partial_t \) is a timelike KV with orthogonal Frobenius-integrable distribution. Restriction of the metric on a leaf of the corresponding foliation is Riemannian:

\[
g_\delta = g_\delta |_{z = \text{const.}} \tag{83}
\]

The original \( g_\delta \) is Ricci-flat, but \( \bar{g}_\delta \) is flat only for \( \delta = 0 \). We get: \( \text{rank} R_a = 0 \) for \( \delta = 0 \), \( \text{rank} R_a = 1 \) for \( \delta = 1 \), and \( \text{rank} R_a = 2 \) else.

Our algorithm implies the number \( d \) of KVs for \( g_\delta \): 6 KVs for \( \delta = 0 \) (this corresponds to the Minkowski metric \( g_\delta \)), 3 KVs for \( \delta = 1 \) (this corresponds to the Schwarzschild metric \( \bar{g}_\delta \)) and 1 KV for \( \delta \neq 0, 1 \). The number \( d \) of KVs for \( g_\delta \) is equal to \( d + 1 \) if \( \delta \neq 0 \).

To see this note that \( R^{(1)} = 0 \) for the metric \( \bar{g}_\delta \) but the higher traces \( R^{(2)}, R^{(3)} \) are non-zero. These functions are required to be preserved by KVs. For \( \delta = 1 \) both functions, as well as the coefficient of \( dz^2 \) in \( \bar{g}_\delta \), depend on \( x \) only, so the KVs belong to the span of \( \partial_x, \partial_z \) and they lift to 4D. Similarly, for \( 0 < \delta \neq 1 \), both functions depend on \( x, y \), the KV belongs to the span of \( \partial_t \) and it lifts to 4D. No new KV arise when passing to 4D.

Further computations yield that \( \bar{g}_\delta \) has no KT of order 2 that are not combinations of symmetric products of known KVs, see \([18]\).

6. Conclusion

In this paper we demonstrated an algorithm computing dimension \( d \) of the local group of motions of a 3-dimensional Riemannian manifold \((M, g_{ab})\) via differential invariants of \( g_{ab} \).

The following properties are known. The maximal value of \( d \) is 6, which is realised if and only if \( g_{ab} \) is an Einstein metric: \( R^{(1)} = \frac{1}{3}R \partial_b \). This is equivalently characterised by the equations \( R^{(1)} = R, R^{(2)} = \frac{1}{2}R^2, R^{(3)} = \frac{1}{6}R^3 \). The submaximal value of \( d \) is 4, and metrics \( g_{ab} \) with that many KVs were characterised by differential invariants in \([19]\). Our contribution is to go beyond this value and determine when \( d \) can be 3, 2, 1 or 0.

It would be worthwhile to comment relations of our method to that of Cartan. Cartan’s equivalence method \([20]\) provides a canonical frame on the principal \( SO(g)\)-bundle over \( M \), called the Cartan bundle, and this allows to determine the number of KVs in principle. The Cartan–Karlhede algorithm is designed, in particular, for such purposes. However, to our knowledge, this was not used to solve the problem we address in this paper. In addition, the invariants the algorithm produces are functions on the Cartan bundle, while our invariants are defined on its base \( M \). When \( d \leq 3 \) and \( r = 0 \) (see equation \( (1) \); \( r \) always vanishes for \( d < 3 \)) the frame fixing allows to pull down the invariants from the Cartan–Karlhede algorithm to the base, but our algorithm works universally for all cases.

The method we discuss in this paper works also for a large class of Lorentzian metrics in dimension 3, however there exist metrics with vanishing scalar invariants and no symmetries \([21]\). Such metrics have received an increasing attention in general relativity, see e.g. \([22, 23]\).
Technically, they fail our algorithm because of normalisation, like the factor $U$ in section 2. It is possible to include such cases into consideration via additional branching, but we leave these elaborations to a future work.

Moreover, by taking into account the Weyl tensor, our method is applicable for higher-dimensional spaces. For instance, in dimension 4 we get a $4 \times 9$ obstruction matrix

$$\left( R_a, W_a \right) = \left( \nabla_a R^{(1)}, \ldots, \nabla_a R^{(4)}, \nabla_a W^{(2)}, \ldots, \nabla_a W^{(6)} \right)^T,$$

with $4 \times 4$ minors giving obstructions, playing a role similar to (7). Here $W^{(i)}$ are principal traces of the $i$-th powers of the Weyl tensor, considered as an endomorphism of $\Lambda^2 TM$. A criterion on existence of KVs in 4D should be based on this obstruction matrix.

The algorithm we designed can be applied to determine exact number of KVs for 4-dimensional spacetimes possessing a timelike KV with integrable orthogonal distribution, as in the case of Zipoy–Voorhees metric considered in section 5.2. In such situation the quotient along trajectories of the given timelike KV determines a local submersion of the Lorentzian 4-manifold onto a Riemannian 3-manifold, and our analysis of KVs can be invoked to simplify computation of additional isometries. Further consideration of this shall be done elsewhere.

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Appendix A. Relations between the Ricci rotation coefficients and the commutation relations

Generally, the derivatives of the Ricci rotation coefficients are not independent. In this appendix, we record the relations among equations (30) and (55). We also write the commutation relations of the orthonormal basis used in sections 3 and 4.

A.1. For analysis of case 1

When $\nabla_a R$ is a geodesic, that is $\kappa_T = \eta_T = 0$, the following relations hold true.

A.1.1. The components of the Ricci tensor.

$$R_{ab}T^aT^b = \mathcal{L}_T (\kappa_N + \kappa_B) - \kappa_N^2 - \kappa_B^2 - 2\tau_B\tau_N,$$  \hspace{1cm} (A.1a)

$$R_{ab}N^aN^b = \mathcal{L}_B\eta_N + \mathcal{L}_T\kappa_T + \mathcal{L}_N\eta_B - \kappa_N(\kappa_N + \kappa_B) - \eta_N^2 - \eta_B^2 + 2\tau_T\tau_B,$$  \hspace{1cm} (A.1b)

$$R_{ab}R^aB^b = \mathcal{L}_B\eta_N + \mathcal{L}_T\kappa_B + \mathcal{L}_N\eta_B - \kappa_B(\kappa_N + \kappa_B) - \eta_N^2 - \eta_B^2 - 2\tau_T\tau_N,$$  \hspace{1cm} (A.1c)

$$R_{ab}T^aN^b = \mathcal{L}_B\tau_N + \mathcal{L}_N\kappa_B + \eta_B(\kappa_N - \kappa_B) - \eta_N(\tau_N + \tau_B),$$  \hspace{1cm} (A.1d)

$$= \mathcal{L}_B\tau_T + \mathcal{L}_T\eta_B - \kappa_B\eta_B - \eta_N(\tau_T + \tau_B),$$  \hspace{1cm} (A.1e)
$$R_{ab}T^aB^b = L_B\kappa_N + L_N\tau_B - \eta_N(\kappa_N - \kappa_B) - \eta_B(\tau_N + \tau_B) , \quad (A.1f)$$
$$= L_T\tau_N - L_N\tau_T - \eta_N\kappa_N + \eta_B(\tau_T - \tau_N) , \quad (A.1g)$$
$$R_{ab}N^aB^b = - L_T\tau_B + \tau_B(\kappa_N + \kappa_B) + \tau_T(\kappa_N - \kappa_B) , \quad (A.1h)$$
$$= - L_T\tau_N + \tau_T(\kappa_N - \kappa_B) + \tau_N(\kappa_N + \kappa_B) . \quad (A.1i)$$

A.1.2. The commutation relations.

$$[T, N_1^a] = \kappa_N N^a + (\tau_T - \tau_N)B^a , \quad (A.2a)$$
$$[T, B_1^a] = - (\tau_T + \tau_B)N^a + \kappa_B B^a , \quad (A.2b)$$
$$[N, B_1^a] = - (\tau_T - \tau_B)T^a - \eta_N N^a + \eta_B B^a . \quad (A.2c)$$

A.2. For analysis of case 0

In case 0, the following relations hold true.

A.2.1. The components of the Ricci tensor.

$$R_{ab}E_1^aE_1^b = L_1(\kappa_2 + \kappa_3) + L_2\kappa_1 + L_3\eta_1 - \kappa_1(\kappa_1 + \eta_1) - \kappa_2^2 - \kappa_3^2 - \eta_1\eta_2 - 2\tau_2\tau_3 , \quad (A.3a)$$
$$R_{ab}E_2^aE_2^b = L_1\kappa_2 + L_2(\kappa_1 + \eta_3) + L_3\eta_2 - \kappa_2^2 - \kappa_2(\kappa_2 + \kappa_3) - \eta_2(\eta_1 + \eta_2) - \eta_3^2 + 2\tau_1\tau_3 , \quad (A.3b)$$
$$R_{ab}E_3^aE_3^b = L_1\kappa_3 + L_2\eta_3 + L_3(\eta_1 + \eta_2) - \kappa_3(\kappa_2 + \kappa_3) - \eta_1^2 - \eta_2^2 - \eta_3(\kappa_1 + \eta_3) - 2\tau_1\tau_2 , \quad (A.3c)$$
$$R_{ab}E_1^aE_2^b = L_2\kappa_3 + L_3\tau_2 + \eta_3(\kappa_2 - \kappa_3) - \tau_2(\eta_1 + \eta_2) + \tau_3(\eta_1 - \eta_3) , \quad (A.3d)$$
$$= L_1\eta_3 + L_3\tau_1 + \kappa_3(\kappa_1 - \eta_3) - \tau_1(\eta_1 + \eta_2) + \tau_3(\eta_1 - \eta_2) , \quad (A.3e)$$
$$R_{ab}E_2^aE_3^b = L_2\tau_3 + L_3\kappa_2 - \eta_2(\kappa_2 - \kappa_3) - \tau_2(\eta_3 - \kappa_1) - \tau_3(\kappa_1 + \eta_3) , \quad (A.3f)$$
$$= L_1\eta_2 - L_2\tau_1 + \kappa_2(\kappa_1 - \eta_2) + \tau_1(\eta_3 + \kappa_1) + \tau_2(\kappa_1 - \eta_3) , \quad (A.3g)$$
$$R_{ab}E_3^aE_3^b = L_3\kappa_1 - L_1\tau_3 - \eta_1(\kappa_1 - \eta_3) + \tau_1(\kappa_2 - \kappa_3) + \tau_3(\kappa_2 + \kappa_3) , \quad (A.3h)$$
$$= L_2\eta_1 - L_1\tau_2 - \kappa_1(\eta_1 - \eta_2) + \tau_1(\kappa_2 - \kappa_3) + \tau_2(\kappa_2 + \kappa_3) . \quad (A.3i)$$

A.2.2. The commutation relations.

$$[E_1, E_2]^a = - \kappa_1 E_1^a + \kappa_2 E_2^a + (\tau_1 - \tau_2)E_3^a , \quad (A.4a)$$
$$[E_1, E_3]^a = - \eta_1 E_1^a - (\tau_1 + \tau_3)E_2^a + \kappa_3 E_3^a , \quad (A.4b)$$
$$[E_2, E_3]^a = - (\tau_2 - \tau_3)E_1^a - \eta_2 E_2^a + \eta_3 E_3^a . \quad (A.4c)$$
In case 0, all the eigenvalues of the Ricci tensor are constant. This fact leads to the additional relations from the Lie derivatives of equations (A.3a)–(A.3c):

\[
(\lambda_1 - \lambda_2)\kappa_2 - (\lambda_3 - \lambda_1)\kappa_3 = (\lambda_1 - \lambda_2)\kappa_1 - (\lambda_3 - \lambda_2)\eta_3 = (\lambda_3 - \lambda_1)\eta_1 - (\lambda_2 - \lambda_1)\eta_2 = 0,
\]

where \((\lambda_1, \lambda_2, \lambda_3)\) are the eigenvalues defined in equation (57).

**Appendix B. Supplements for case 0**

In this appendix, we make up for deficiencies in section 4.

We firstly write the equations of evolution of \((\pi_1, \pi_2, \pi_3)\). By using the concrete form of a KV (53), we can write out the Killing equation as follows.

\[
\begin{align*}
\mathcal{L}_1 \pi_1 & = \kappa_1 \pi_2 + \eta_1 \pi_3, \quad (B.1a) \\
\mathcal{L}_2 \pi_1 & = -\kappa_1 \pi_1 - \kappa_2 \pi_2 + (\tau_1 + \tau_2) \pi_3 - \pi_2, \quad (B.1b) \\
\mathcal{L}_3 \pi_1 & = \pi_1, \quad (B.1c) \\
\mathcal{L}_1 \pi_2 & = \pi_2, \quad (B.1d) \\
\mathcal{L}_2 \pi_2 & = \kappa_2 \pi_1 + \eta_2 \pi_3, \quad (B.1e) \\
\mathcal{L}_3 \pi_2 & = -\eta_2 \pi_1 - (\tau_2 + \tau_3) \pi_2 - \eta_3 \pi_3 - \pi_3, \quad (B.1f) \\
\mathcal{L}_1 \pi_3 & = -\eta_1 \pi_1 - (\tau_1 - \tau_3) \pi_2 - \kappa_3 \pi_3 - \pi_1, \quad (B.1g) \\
\mathcal{L}_2 \pi_3 & = \pi_3, \quad (B.1h) \\
\mathcal{L}_3 \pi_3 & = \kappa_3 \pi_1 + \eta_3 \pi_2 \quad (B.1i)
\end{align*}
\]

where \((\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)\) are defined in equation (58). The above equations are not closed with respect to unknown scalars \((\pi_1, \pi_2, \pi_3, \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)\). We need the information of the second order derivatives of \((\pi_1, \pi_2, \pi_3)\). From the identities \(\nabla \nabla \nabla \pi_1 = \nabla \nabla \nabla \bar{\pi}_2 = \nabla \nabla \nabla \bar{\pi}_3 = \), we obtain

\[
\begin{align*}
\mathcal{L}_1 \bar{\pi}_1 & = [\kappa_1 (\tau_1 - \tau_2) + \kappa_1 \eta_1] \pi_1 + [\mathcal{L}_3 \kappa_1 - \eta_1 (\kappa_1 - \eta_3) - \kappa_1 \eta_2 + \kappa_2 (\tau_1 + \tau_2)] \pi_2 \\
& + [\mathcal{L}_3 \eta_1 - \eta_2^2 - \kappa_1 \eta_3 - (\tau_1 + \tau_2)(\tau_1 + \tau_2)] \pi_3 + \kappa_3 \bar{\pi}_1 + (\tau_1 + \tau_3) \pi_2 - \kappa_1 \bar{\pi}_3, \quad (B.1j) \\
\mathcal{L}_2 \bar{\pi}_1 & = [\kappa_1 \eta_2 + \kappa_2 (\tau_2 + \tau_3)] \pi_1 + [\mathcal{L}_1 \kappa_2 - \eta_2 (2\kappa_2 - \kappa_3) - \eta_1 (\tau_2 + \tau_3) + \kappa_1 (\tau_2 - \tau_3)] \pi_2 \\
& - [\mathcal{L}_2 \kappa_3 + \eta_2 (\tau_1 - \tau_3)] \pi_3 + \eta_2 \bar{\pi}_2 + (\kappa_2 - \kappa_3) \bar{\pi}_3, \quad (B.1k) \\
\mathcal{L}_3 \bar{\pi}_1 & = -[\mathcal{L}_1 \kappa_3 + \mathcal{L}_3 \eta_1 - \eta_2^2 - (\tau_1 - \tau_3)(\tau_2 + \tau_3)] \pi_1 + [\mathcal{L}_3 \tau_3 - \kappa_3 \eta_3 - 2\eta_2 \eta_3] \pi_2 \\
& - [\mathcal{L}_3 \kappa_3 - \eta_3 (\tau_1 - \tau_3)] \pi_3 - \eta_3 \bar{\pi}_2 - 2\tau_3 \bar{\pi}_3, \quad (B.1l) \\
\mathcal{L}_1 \bar{\pi}_2 & = -[\mathcal{L}_1 \kappa_1 + \eta_1 (\tau_1 + \tau_2)] \pi_1 - [\mathcal{L}_1 \kappa_2 + \mathcal{L}_2 \kappa_1 - \kappa_1^2 + (\tau_1 - \tau_3)(\tau_1 + \tau_2)] \pi_2 \\
& - [\mathcal{L}_1 \tau_1 - \kappa_1 \eta_1 - 2\kappa_1 \tau_1] \pi_3 - 2\tau_1 \bar{\pi}_1 - \eta_1 \bar{\pi}_3, \quad (B.1m) \\
\mathcal{L}_2 \bar{\pi}_2 & = [\mathcal{L}_1 \kappa_2 - \eta_1 \eta_2 - \kappa_1^2 + (\tau_2 + \tau_3)(\tau_1 + \tau_2)] \pi_1 + [\kappa_1 \kappa_2 - \eta_2 (\tau_2 - \tau_3)] \pi_2 \\
& + [\mathcal{L}_1 \eta_2 - \kappa_3 \eta_2 + (\kappa_2 + \eta_1)(\tau_1 - \tau_3)] \pi_3 - \eta_2 \bar{\pi}_1 + \kappa_1 \bar{\pi}_2 + (\tau_1 - \tau_3) \bar{\pi}_3, \quad (B.1n)
\end{align*}
\]
\[ \mathcal{L}_3 \pi_3 = - \left[ \mathcal{L}_3 \kappa_1 - \kappa_3 (\tau_1 + \tau_3) \right] \pi_1 + \left[ \kappa_3 \eta_2 + \eta_1 (\tau_1 - \tau_3) \right] \pi_2 \\
- \left[ \mathcal{L}_1 \eta_3 + \kappa_3 (\kappa_2 - 2 \eta_1) - (\tau_1 - \tau_3) (\eta_1 + \eta_2) \right] \pi_3 - (\kappa_1 - \eta_3) \pi_1 + \kappa_3 \pi_3 , \quad (B.1o) \]

\[ \mathcal{L}_1 \pi_3 = - \left[ \mathcal{L}_2 \eta_1 - \kappa_1 (2 \eta_1 - \eta_2) + \tau_1 (\kappa_2 - \kappa_3) + \tau_2 (\kappa_2 + \kappa_3) \right] \pi_1 - \left[ \mathcal{L}_1 \eta_2 + \kappa_1 (\tau_2 + \tau_3) \right] \pi_2 \\
+ \left[ \kappa_1 \eta_3 - \eta_1 (\tau_1 + \tau_2) \right] \pi_3 + \kappa_1 \pi_3 + (\eta_1 - \eta_2) \pi_2 , \quad (B.1p) \]

\[ \mathcal{L}_2 \pi_3 = - \left[ \mathcal{L}_2 \tau_2 + \kappa_2 \eta_2 - 2 \kappa_1 \tau_2 \right] \pi_1 - \left[ \mathcal{L}_2 \eta_2 - \kappa_2 (\tau_2 + \tau_3) \right] \pi_2 \\
- \left[ \mathcal{L}_2 \eta_3 + \kappa_2 \eta_2 - \eta_3^2 + (\tau_2 + \tau_3) (\tau_1 + \tau_2) \right] \pi_3 - \kappa_2 \pi_3 + 2 \tau_2 \pi_2 , \quad (B.1q) \]

\[ \mathcal{L}_3 \pi_3 = \left[ \mathcal{L}_2 \kappa_3 + \eta_3 (\kappa_2 - \kappa_3) - \kappa_1 \kappa_3 - \eta_1 (\tau_2 - \tau_3) \right] \pi_1 + \left[ \mathcal{L}_2 \eta_3 - (\tau_1 - \tau_3) (\tau_2 - \tau_3) \right] \\
- \eta_3^2 - \kappa_2 \kappa_3 \right] \pi_2 - \left[ \eta_2 \eta_3 + \kappa_3 (\tau_1 + \tau_3) \right] \pi_3 - (\tau_2 - \tau_3) \pi_1 - \kappa_3 \pi_2 + \eta_2 \pi_3 . \quad (B.1r) \]

Its integrability condition is given by equation (56).

If the Ricci tensor has the Segre type \([I I I I]\), it follows from equations (59) and (60) that the closed system \((B.1o takes the form

\[ \mathcal{L}_1 \pi_1 = 0 , \quad (B.2a) \]

\[ \mathcal{L}_2 \pi_1 = (\tau_2 - \tau_3) \pi_3 , \quad (B.2b) \]

\[ \mathcal{L}_3 \pi_1 = -(\tau_2 - \tau_3) \pi_2 , \quad (B.2c) \]

\[ \mathcal{L}_1 \pi_2 = -\kappa_2 \pi_2 + (\tau_1 + \tau_3) \pi_3 , \quad (B.2d) \]

\[ \mathcal{L}_2 \pi_2 = \kappa_2 \pi_1 + \eta_2 \pi_3 , \quad (B.2e) \]

\[ \mathcal{L}_3 \pi_2 = -(\tau_2 + \tau_3) \pi_1 - \eta_2 \pi_2 - \eta_3 \pi_3 - \pi_3 , \quad (B.2f) \]

\[ \mathcal{L}_1 \pi_3 = -(\tau_1 + \tau_2) \pi_2 + \kappa_2 \pi_3 , \quad (B.2g) \]

\[ \mathcal{L}_2 \pi_3 = \pi_3 , \quad (B.2h) \]

\[ \mathcal{L}_3 \pi_3 = -\kappa_2 \pi_1 + \eta_3 \pi_2 , \quad (B.2i) \]

\[ \mathcal{L}_1 \pi_3 = -2 \kappa_2 \tau_1 \pi_1 - (\mathcal{L}_1 \eta_2 - \kappa_2 \eta_2) \pi_2 - \eta_2 (\tau_1 + \tau_3) \pi_3 , \quad (B.2j) \]

\[ \mathcal{L}_2 \pi_3 = -(\mathcal{L}_2 \tau_2 + \kappa_2 \eta_2) \pi_1 - (\mathcal{L}_2 \eta_2) \pi_2 - (\mathcal{L}_2 \eta_3 + \mathcal{L}_3 \eta_2 - \eta_3^2 + (\tau_1 - \tau_2) (\tau_1 - \tau_3) \pi_3 , \quad (B.2k) \]

\[ \mathcal{L}_3 \pi_3 = -(\mathcal{L}_2 \kappa_2 - 2 \kappa_2 \eta_3) \pi_1 + (\mathcal{L}_2 \eta_3 - \eta_3^2 + (\tau_3 - \tau_2) (\tau_3 - \tau_2) \pi_2 + \eta_2 \eta_3 \pi_3 + \eta_2 \pi_3 . \quad (B.2l) \]

**ORCID iDs**

Kentaro Tomoda 🐦 [https://orcid.org/0000-0002-3713-5211](https://orcid.org/0000-0002-3713-5211)

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