Optimal Feedback Controls of Stochastic Linear Quadratic Control Problems in Infinite Dimensions with Random Coefficients

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Abstract

It is a longstanding unsolved problem to characterize the optimal feedback controls for general linear quadratic optimal control problem of stochastic evolution equation with random coefficients. A solution to this problem is given in [21] under some assumptions which can be verified for interesting concrete models, such as controlled stochastic wave equations, controlled stochastic Schrödinger equations, etc. More precisely, the authors establish the equivalence between the existence of optimal feedback operator and the solvability of the corresponding operator-valued, backward stochastic Riccati equations. However, their result cannot cover some important stochastic partial differential equations, such as stochastic heat equations, stochastic stokes equations, etc. A key contribution of the current work is to relax the $C_0$-group assumption of unbounded linear operator $A$ in [21] and using contraction semigroup assumption instead. Therefore, our result can be well applicable in the linear quadratic problem of stochastic parabolic partial differential equations. To this end, we introduce a suitable notion to the aforementioned Riccati equation, and some delicate techniques which are even new in the finite dimensional case.

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1 Introduction

In Control Theory, one of the fundamental issues is to find optimal feedback controls, which are particularly important in practical applications since the main advantage of feedback controls is to keep the corresponding control to be robust with respect to (small) perturbation/ disturbance, which is usually unavoidable in real world. Unfortunately, it is actually very difficult to find optimal feedback controls for many optimal control problems. So far, the most successful attempt in this respect is that for linear quadratic control problems (LQ problems for short) which are extensively studied in Control Theory. The study of LQ problems dates back at least to [3] in which the system is governed by a linear ordinary differential equation. Later, in the seminal work of [4], the matrix-valued Riccati equations were brought into LQ problems to construct (linear) feedback

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controls. It is well-known that, Kalman’s theory for LQ problems is among the three milestones in modern (finite dimensional) optimal control theory. Due to the elegant and fruitful mathematical structure, LQ problems were investigated extensively in the literature for a variety of deterministic control systems (e.g., [2, 4]).

LQ problems for controlled stochastic differential equations (SDEs) was first studied in [33]. More precisely, given proper \((s, \eta)\), the state equation is described as

\[
\begin{aligned}
    &dX(t) = \left[A X(t) + B u(t)\right] dt + \left[C X(t) + D u(t)\right] dW(t), \quad t \in (s, T], \\
    &X(s) = \eta,
\end{aligned}
\]

and the cost functional is defined as

\[
J(s, \eta; u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_s^T \left( \langle QX(t), X(t) \rangle_{\mathbb{R}^n} + \langle Ru(t), u(t) \rangle_{\mathbb{R}^m} \right) dt + \langle GX(T), X(T) \rangle_{\mathbb{R}^n} \right].
\]

In the above, \(W(\cdot)\) is a standard Brownian motion defined on complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) with \(\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}\) generated by \(W(\cdot)\) augmented by all \(\mathbb{P}\)-null sets. The coefficients \(A, B, C, D\), the control variable \(u(\cdot)\), and the state variable \(X(\cdot)\) are stochastic processes with suitable measurability and integrability such that both (1.1) and (1.2) are well-defined. The optimal control problem is to find (if possible) a \(\bar{u}(\cdot)\) to minimize the cost functional. There are a huge amount of works addressing the LQ problems for controlled SDEs (see [5, 8, 19, 28, 29, 30, 33] and the rich references therein). To simplify the notations, here and henceforth the sample point \(\omega(\in \Omega)\) and/or the time variable \(t(\in [0, T])\) in the coefficients are often suppressed, in the case that no confusion would occur.

Similar as the deterministic setting, to find the optimal feedback controls, people introduce the following Riccati equation:

\[
\begin{aligned}
    &dP = -\left[PA + A^\top P + \Lambda C + C^\top \Lambda + C^\top PC + Q - L^\top K^\top L\right] dt + \Lambda dW(t) \quad \text{in } [0, T], \\
    &P(T) = G,
\end{aligned}
\]

where \(K \equiv R + D^\top P D, L = B^\top P + D^\top (PC + \Lambda)\), and \(K^\dagger\) is the Moore-Penrose pseudo-inverse of \(K\), is introduced as fundamental tools to represent the linear feedback controls. We point out that (1.3) is actually a backward stochastic differential equation (BSDE), whose solvability is highly challenging and nontrivial. Even though the well-posedness is guaranteed, as pointed in [19, Remark 1.2], the solution is not fully qualified to serve as the design of optimal feedback controls. Naturally, one may ask a question:

**Is it possible to link the existence of optimal feedback control for LQ problems with the solvability of (1.3)?**

Along this line, an affirmative answer was given in [19].

Now let us move to the linear quadratic for stochastic evolution equations (SEEs), which is the main concern of the current paper. Let \(H\) and \(U\) be separable Hilbert spaces, and \(A\) be an unbounded linear operator (with domain \(D(A)\) on \(H\)), which generates a \(C_0\)-semigroup \(\{e^{At}\}_{t \geq 0}\) on \(H\). Denote by \(A^*\) the adjoint operator of \(A\), which generates the adjoint \(C_0\)-semigroup \(\{e^{A^*t}\}_{t \geq 0}\) on \(H\). Given proper \((s, \eta)\), consider the following controlled linear SEE:

\[
\begin{aligned}
    &dx(t) = \left[(A + A_1)x + Bu\right] dt + (Cx + Du) dW(t) \quad \text{in } (s, T], \\
    &x(s) = \eta,
\end{aligned}
\]

with the quadratic cost functional

\[
J(s, \eta; u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_s^T \left( \langle Qx(t), x(t) \rangle_H + \langle Ru(t), u(t) \rangle_U \right) dt + \langle Gx(T), x(T) \rangle_H \right].
\]
Here $u(\cdot)$ is the control variable, $x(\cdot)$ is the state variable, and the coefficients (say, $A_1$, $B$, $C$ and $D$) are stochastic processes. Under proper conditions (See Section 2), (1.4) admits a unique solution and (1.5) becomes well-defined. The optimal control problem is to find (if possible) a control $\bar{u}(\cdot)$ to minimize $J$ in (1.5).

SEEs are used to describe a lot of random phenomena appearing in physics, chemistry, biology, and so on. In many situations SEEs are more realistic mathematical models than the deterministic ones (e.g. [7, 15, 23]). Consequently, there are many works addressing the optimal control problems for SEEs. In particular, we refer the readers to [1, 11, 17, 21, 23] and the rich references therein for LQ problems for controlled SEEs.

To design optimal feedback controls for linear quadratic problem of SEEs, similar to (1.3), we introduce the following operator-valued Riccati equation:

\[
\begin{align*}
    dP &= -\left[P(A + A_1) + (A + A_1)^*P + \Lambda C + C^*\Lambda \right. \\
    &\quad \quad + C^*PC + Q - L^*K^{-1}L \big]dt + \Lambda dW(t) \quad \text{in } [0, T), \\
    P(T) &= G,
\end{align*}
\]

where $K \equiv R + D^*PD$, $L = B^*P + D^*(PC + \Lambda)$. Besides the difficulties for the case of finite dimensions, there exists several new essentially ones in the study of (1.6) when $\dim H = \infty$:

- There is no stochastic integration/evolution equation theory in general Banach spaces that can be employed to handle the (stochastic integral) term “$\Lambda dW(t)$” in (1.6) effectively. Due to this difficulty, there exist only a quite limited number of works, see [1, 9, 10] and the papers therein. We refer to [21] for the detailed introduction.

- The appearance of $D$, and the resulting term $L^*K^{-1}L$ lead to another essential difficulty in the study of (1.6). Notice that the second term is already quite subtle even in the finite dimensional case. Usually, people assume that $D = 0$ to avoid that difficulty (e.g., [9, 10]). We study the case that $D \neq 0$ according to the following reasons:

  On one hand, only when the controls/decisions could or would influence the scale of uncertainty do the stochastic problems differ from the deterministic ones. On the other hand, once one put a control in the drift term, it will influence the diffusion term.

  Generally speaking, to study the difficult operator-valued stochastic differential equations, people need to introduce suitable new concept of solution. For example, in [20, 22], the authors introduced the concept of transposition solution to operator-valued, backward stochastic (linear) Lyapunov equations to study the maximum principle for optimal control problems of SEEs. In [21], the authors defined a type of transposition solution to treat (1.6), and obtained equivalence between the existence of optimal feedback operator and the solvability of BSRE (1.6) in the sense of transposition solution. We point out that such a definition of transposition solution can bypass the previous two difficulties. However, due to the methodology limitation in obtaining BSREs by the existence of optimal feedback operators, the authors made a common yet technical assumption of $A$, i.e., $A$ generates a $C_0$-group on $H$. In fact, they followed the forward-backward stochastic system ideas (see e.g. [19, 27] for the finite dimensional case), and therefore used the inverse of solution to certain forward stochastic differential equation to construct the solution of Riccati equation. In the finite dimensional case, the inverse of the solution to the forward stochastic system is easy to see. However, in the infinite dimensions, to achieve such a goal, it requires $-A^*$ generates a $C_0$-semigroup as well, which means that $A^*$ generates a $C_0$-group. As a result, their equivalence ruled out some important stochastic partial differential equations, say stochastic parabolic equation and stochastic stokes equation for example.
In this paper, by supposing $A$ generating a contraction semigroup, we establish the equivalence between optimal feedback controls and backward stochastic Riccati equations in infinite dimensions. As a result, we not only have to deal with the two difficulties mentioned in the last paragraph, but also need to overcome the methodology limitation in [17, 21, 23]. Notice that all these obstacles can be well-treated in the finite dimensional framework. To figure out these issues, let us look at the following observation which holds in the finite dimensional framework: the BSRE (1.3) can be rewritten as a linear BSDE with a parameter $\Theta$:

$$\begin{align*}
    &\frac{dP}{dt} = -[P(A + B\Theta) + (C + D\Theta)^\top P(C + D\Theta) + (C + D\Theta)^\top \Lambda + \Lambda(C + D\Theta) + Q + \Theta^\top R\Theta] dt + \Lambda dW(t) \\
    &P(T) = G,
\end{align*}$$

where the parameter $\Theta$ satisfies the following:

$$D^\top PC + B^\top P + D^\top \Lambda + (R + D^\top PD)\Theta = 0. \tag{1.8}$$

This important fact was used in several existing papers under Markovian framework, see e.g. [16, Theorem 3.3] and [22, Theorem 2.3] for the mean-field stochastic LQ problems for SDEs, and [18, Lemma 2.8] for the mean-field stochastic LQ problem for SEEs. Clearly, one advantage of (1.7) lies in its linearity in contrast with (1.3). As a tradeoff, one has to explore the additional relation (1.8).

In this article, we successfully apply this idea into our infinite dimensional framework with random coefficients and derive the new equivalence between optimal feedback operators and BSREs in the following three steps:

1. We establish the well-posedness of an infinite dimensional version of (1.7), i.e., operator-valued backward stochastic Lyapunov equation (BSLE), in the sense of $H_\lambda$-transposition solution (inspired by [20]) (see Subsection 4.1).

2. We provide a relationship between the optimal feedback operator and the operator-valued BSLE (see Subsection 4.2). More precisely, with subtle choice of $v$, small $\varepsilon > 0$ such that $t + \varepsilon \leq T$, we introduce control variables $\Theta_\varepsilon X$ and $\Theta_\varepsilon X v,\varepsilon + v I_{t,t+\varepsilon}$ associated with optimal feedback operator $\Theta$. At this moment, we point out one novel technical contribution. We refer to Remark 4.1 for more relevant details.

3. Using the result established in step 2, we prove that the aforementioned $H_\lambda$-transposition solution of BSLE is just the transposition solution of BSREs.

At this very moment, let us summarize our contributions of the current article:

- From the viewpoint of methodology, we develop some useful procedures to overcome the essential obstacles and limitation in the current relevant literature. Interestingly, some technologies happen to be new even in finite dimensional scenario.

- From the viewpoint of conclusions, we considerably relax the $C_0$-group assumption of operator $A$ and impose the $C_0$-semigroup assumption instead. This allows us to cover LQ problem for stochastic parabolic PDEs, which is one of the most important class of SPDEs.

It is worth noting that, besides the study of optimal feedback controls for LQs, operator-valued Riccati equations have other applications (e.g., [26]). We believe that both the results and the methods in this paper have other applications, such as the study of open quantum systems with continuation observation. Nevertheless, a detailed study for it is beyond the scope of this paper and will be done in our future works.
The rest of this paper is organized as follows. In Section 2, we present some necessary notations, important remarks and a complete formulation of our optimal control problems. In Section 3, we state our main result, i.e., the characterization of optimal feedback operators by means of well-posedness of operator-valued backward stochastic Riccati equation in the suitable transposition sense. In Section 4, we first establish the well-posedness of operator-valued backward stochastic Lyapunov equation (BSLE for short) in Subsection 4.1, then give a characterization of optimal feedback operator via the operator-valued BSLE in Subsection 4.2 and then demonstrate the proof of the main result. In Section 5, we present an illustrative example for LQ problem of stochastic parabolic PDEs.

2 Preliminary notations

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete filtered probability space (satisfying the usual conditions), on which a 1-dimensional standard Brownian motion $\{W(t)\}_{t \in [0, T]}$ is defined. Here $\mathbb{F}^{\Delta} = \{\mathcal{F}_t\}_{t \in [0, T]}$ is the natural filtration generated by $W(\cdot)$. Denote by $\mathbb{F}$ the progressive $\sigma$-field (in $[0, T] \times \Omega$) with respect to $\mathbf{F}$.

Let $\mathcal{X}$ be a Banach space. For any $t \in [0, T]$ and $p \in [1, \infty)$, denote by $L^p_{\mathcal{F}_t}(\Omega; \mathcal{X})$ the Banach space of all $\mathcal{F}_t$-measurable random variables $\xi : \Omega \to \mathcal{X}$ such that $\mathbb{E}[|\xi|^p_{\mathcal{X}}] < \infty$, with the canonical norm. Denote by $L^p_{\mathcal{F}}(\Omega; C([t, T]; \mathcal{X}))$ the Banach space of all $\mathcal{X}$-valued $\mathbf{F}$-adapted continuous processes $\phi(\cdot)$, with the norm

$$|\phi(\cdot)|_{L^p_{\mathcal{F}}(\Omega; C([t, T]; \mathcal{X}))} \triangleq \left[ \mathbb{E} \sup_{\tau \in [t, T]} |\phi(\tau)|^p_{\mathcal{X}} \right]^{1/p}.$$

Similarly, one can define $L^p_{\mathcal{F}}(\Omega; C([\tau_1, \tau_2]; \mathcal{X}))$ for two stopping times $\tau_1$ and $\tau_2$ with $\tau_1 \leq \tau_2$, $\mathbb{P}$-a.s. Also, denote by $C_\mathcal{F}([t, T]; L^p(\Omega; \mathcal{X}))$ the Banach space of all $\mathcal{X}$-valued $\mathbf{F}$-adapted processes $\phi(\cdot)$ such that $\phi(\cdot) : [t, T] \to L^p_{\mathcal{F}}(\Omega; \mathcal{X})$ is continuous, with the norm

$$|\phi(\cdot)|_{C_\mathcal{F}([t, T]; L^p(\Omega; \mathcal{X}))} \triangleq \sup_{\tau \in [t, T]} \left[ \mathbb{E}[|\phi(\tau)|^p_{\mathcal{X}}] \right]^{1/p}.$$

Fix any $p_1, p_2, p_3, p_4 \in [1, \infty]$. Put

$$L^p_{\mathcal{F}}(\Omega; L^{p_2}(t, T; \mathcal{X})) = \left\{ \varphi : (t, T) \times \Omega \to \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbf{F}-\text{adapted and } \mathbb{E} \left( \int_t^T |\varphi(\tau)|^{p_2}_{\mathcal{X}} d\tau \right)^{\frac{p_4}{p_2}} < \infty \right\},$$

$$L^p_{\mathcal{F}}(t, T; L^{p_1}(\Omega; \mathcal{X})) = \left\{ \varphi : (t, T) \times \Omega \to \mathcal{X} \mid \varphi(\cdot) \text{ is } \mathbf{F}-\text{adapted and } \int_t^T \left( \mathbb{E}[|\varphi(\tau)|^{p_1}_{\mathcal{X}}] \right)^{\frac{p_4}{p_1}} d\tau < \infty \right\}.$$

(When any one of $p_j$ ($j = 1, 2, 3, 4$) is equal to $\infty$, it is needed to make the usual modifications in the above definitions of $L^p_{\mathcal{F}}(\Omega; L^{p_2}(t, T; \mathcal{X}))$ and $L^p_{\mathcal{F}}(0, T; L^{p_1}(\Omega; \mathcal{X}))$. If $\mathcal{X} = \mathbb{R}$, then we may omit the term $\mathcal{X}$ in the above spaces. Clearly, both $L^p_{\mathcal{F}}(\Omega; L^{p_2}(t, T; \mathcal{X}))$ and $L^p_{\mathcal{F}}(t, T; L^{p_1}(\Omega; \mathcal{X}))$ are Banach spaces with the canonical norms. If $p_1 = p_2$, we simply write the above spaces as $L^p_{\mathcal{F}}(t, T; \mathcal{X})$. For $r \in [0, T]$ and $f \in L^1_{\mathcal{F}_r}(\Omega; \mathcal{X})$, denote by $\mathbb{E}_r f$ the conditional expectation of $f$ with respect to $\mathcal{F}_r$ and by $\mathbb{E} f$ the mathematical expectation of $f$.

Let $\mathcal{Y}$ be another Banach space. Denote by $\mathcal{L}(\mathcal{X}; \mathcal{Y})$ the Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, with the usual operator norm (When $\mathcal{Y} = \mathcal{X}$, we simply write $\mathcal{L}(\mathcal{X})$ instead of $\mathcal{L}(\mathcal{X}; \mathcal{X})$). Suppose $\mathcal{X}_j$ and $\mathcal{Y}_j$ ($j = 1, 2$) are Banach spaces satisfying $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_2$ and $\mathcal{Y}_1 \subset \mathcal{Y} \subset \mathcal{Y}_2$. If $M \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ can be extended as an operator $\tilde{M} \in \mathcal{L}(\mathcal{X}_2; \mathcal{Y}_2)$, then, to simplify
the notations, (formally) we also write \( M \in \mathcal{L}(\mathcal{X}_2; \mathcal{Y}_2) \). Similarly, if \( M|_{\mathcal{X}_1} \in \mathcal{L}(\mathcal{X}_1; \mathcal{Y}_1) \), then, we write \( M \in \mathcal{L}(\mathcal{X}_1; \mathcal{Y}_1) \).

Throughout this paper, for any operator-valued process/random variable \( M \), we write \( M^* \) for its pointwise dual. For example, if \( M \in L^p_2(0, T; L^2(\Omega; \mathcal{L}(H))) \), then \( M^* \in L^q_2(0, T; L^{q'}(\Omega; \mathcal{L}(H))) \), and \( |M|_{L^p_2(0, T; L^2(\Omega; \mathcal{L}(H)))} = |M^*|_{L^q_2(0, T; L^{q'}(\Omega; \mathcal{L}(H)))} \). When \( \mathcal{X} \) is a Hilbert space, denote by \( \mathcal{S}(\mathcal{X}) \) the set of all self-adjoint operators on \( \mathcal{X} \). Further, fix any \( p_3, p_4 \in [1, \infty] \), we put

\[
\mathbf{L}\left(L^p_2(\Omega; L^p_2(t_1, t_2; \mathcal{X})) \right); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right)
\]

\[
\triangleq \{ L \in \mathcal{L}\left(L^p_2(\Omega; L^p_2(t_1, t_2; \mathcal{X})) \right); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right) \}
\]

(2.1) for a.e. \((t, \omega) \in (t_1, t_2) \times \Omega\), there exists \( L(t, \omega) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) verifying

\[
\text{that } (L_f(\cdot))(t, \omega) = L(t, \omega)f(t, \omega), \forall f(\cdot) \in L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{X})) \}
\]

and

\[
\mathbf{L}(\mathcal{X}; L^p_2(t_1, t_2; L^p_4(\Omega; \mathcal{Y})))
\]

\[
\triangleq \{ L \in \mathcal{L}(\mathcal{X}; L^p_2(t_1, t_2; L^p_4(\Omega; \mathcal{Y}))) \} \}
\]

(2.2) for a.e. \((t, \omega) \in (t_1, t_2) \times \Omega\), there exists \( L(t, \omega) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) verifying that \( (L_x)(t, \omega) = L(t, \omega)x, \forall x \in \mathcal{X} \}. \]

To simplify the notations, in what follows we shall identify the above \( \mathbf{L} \) with \( L(\cdot, \cdot) \). Similarly, one can define the spaces \( \mathbf{L}\left(L^p_2(\Omega; \mathcal{X}_1); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right) \) and \( \mathbf{L}\left(L^p_2(\Omega; \mathcal{X}_2); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right) \), etc. In the sequel, we shall call an element in the sets \( \mathbf{L}\left(L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{X}_1)); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right) \)

\( \mathbf{L}\left(L^p_2(\Omega; \mathcal{X}_2); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right) \), \( \mathbf{L}\left(L^p_2(\Omega; \mathcal{X}_2); L^p_2(\Omega; L^p_3(t_1, t_2; \mathcal{Y})) \right) \), \( \mathbf{L}(\mathcal{X}; L^p_2(t_1, t_2; L^p_4(\Omega; \mathcal{Y}))) \), \( \mathbf{L}(\mathcal{X}; L^p_2(t_1, t_2; L^p_4(\Omega; \mathcal{Y}))) \), and so on on a pointwise defined operator.

Put

\[
\mathbf{Y}_p(\mathcal{X}; \mathcal{Y}) \triangleq \left\{ L(\cdot, \cdot) \in \mathbf{L}(L^2_\infty(\Omega; L^p(0, T; \mathcal{X}))); L^2_\infty(\Omega; L^p(0, T; \mathcal{Y})) \right\}
\]

(2.3) \( |L(\cdot, \cdot)|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} \in L^\infty_\infty(\Omega; L^p(0, T; \mathcal{Y})) \).

We shall simply denote \( \mathbf{Y}_p(\mathcal{X}; \mathcal{Y}) \) by \( \mathbf{Y}_p(\mathcal{X}) \).

**Lemma 2.1** Assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are separable Hilbert spaces. For any \( L \in \mathbf{Y}_p(\mathcal{X}; \mathcal{Y}) \), there exist a sequence \( \{L_n\}_{n=1}^\infty \subset L^\infty(\Omega; L^p(0, T; \mathcal{L}(\mathcal{X}; \mathcal{Y}))) \) such that

\[
\lim_{n \to \infty} L_n v = L v \quad \text{in } L^2_\infty(\Omega; L^p(0, T; \mathcal{Y})), \quad \forall v \in L^2_\infty(\Omega; L^\infty(0, T; \mathcal{X})).
\]

(2.4)

**Proof.** Since \( \mathcal{X} \) (resp. \( \mathcal{Y} \)) is a separable Hilbert space, there is an orthonormal basis \( \{e_k\}_{k=1}^\infty \) (resp. \( \{\tilde{e}_k\}_{k=1}^\infty \) of \( \mathcal{X} \) (resp. \( \mathcal{Y} \)). Denote by \( \mathcal{X}_n \) (resp. \( \mathcal{Y}_n \)) the subspace spanned by \( \{e_k\}_{k=1}^n \) (resp. \( \{\tilde{e}_k\}_{k=1}^n \) and by \( \Gamma_n \) (resp. \( \tilde{\Gamma}_n \)) the orthogonal projection operator from \( \mathcal{X} \) (resp. \( \mathcal{Y} \)) to \( \mathcal{X}_n \) (resp. \( \mathcal{Y}_n \)).

Let \( L_n = \tilde{\Gamma}_n L \Gamma_n \). We first prove that \( \{L_n\}_{n=1}^\infty \subset L^\infty(\Omega; L^p(0, T; \mathcal{L}(\mathcal{X}; \mathcal{Y}))) \).

Since \( L_n(t, \omega) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) for a.e. \((t, \omega) \in [0, T] \times \Omega\), we have \( L_n(t, \omega) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \) for a.e. \((t, \omega) \in [0, T] \times \Omega\). Moreover, from the definition of \( L_n \), we know that for a.e. \((t, \omega) \in [0, T] \times \Omega\), \( L_n(t, \omega) \) can be regarded as an element in \( \mathcal{L}(\mathcal{X}_n; \mathcal{Y}_n) \).

Since \( L(\cdot, \cdot) \in \mathbf{L}(L^2_\infty(\Omega; L^\infty(0, T; \mathcal{X}))); L^2_\infty(\Omega; L^p(0, T; \mathcal{Y}))) \), for any \( x \in \mathcal{X} \), we have that \( L_n(\cdot, \cdot) x \in L^2_\infty(\Omega; L^p(0, T; \mathcal{Y}_n)) \). Noting that \( \mathcal{L}(\mathcal{X}_n; \mathcal{Y}_n) \) is isomorphic to \( \mathbb{R}^{n \times n} \), we get that \( L_n(\cdot, \cdot) \in L^2_\infty(\Omega; L^p(0, T; \mathcal{L}(\mathcal{X}_n; \mathcal{Y}_n))) \). Further, noting that

\[
|L_n(\cdot, \cdot)|_{\mathcal{L}(\mathcal{X}_n; \mathcal{Y}_n)} = |L_n(\cdot, \cdot)|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} \leq |L(\cdot, \cdot)|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})};
\]
it follows from $|L(\cdot, \cdot)|_{L^2(\mathcal{X}; Y)} \in L^\infty_F(\Omega; L^p(0, T))$ that $|L_n(\cdot, \cdot)|_{L^2(\mathcal{X}; Y)} \in L^\infty_F(\Omega; L^p(0, T))$. This, together with $L_n(\cdot, \cdot) \in L^\infty_F(\Omega; L^p(0, T; \mathcal{L}(X_n, \mathcal{Y}_n)))$, implies that $L_n(\cdot, \cdot) \in L^\infty_F(\Omega; L^p(0, T; \mathcal{L}(X_n, \mathcal{Y}_n)))$. Hence, $L_n(\cdot, \cdot) \in L^2_F(\Omega; L^p(0, T; \mathcal{L}(X, \mathcal{Y})))$.

For any $v \in L^2_F(\Omega; L^\infty(0, T; \mathcal{X}))$, we have

$$
|L_n v - L v|_{L^2_F(\Omega; L^p(0, T; Y))} = |\Gamma_n L v - L v|_{L^2_F(\Omega; L^p(0, T; Y))} \\
\leq |\Gamma_n L v - \tilde{\Gamma}_n L v|_{L^2_F(\Omega; L^p(0, T; Y))} + |\tilde{\Gamma}_n L v - L v|_{L^2_F(\Omega; L^p(0, T; Y))} \\
\leq |\Gamma_n L v - L v|_{L^2_F(\Omega; L^p(0, T; Y))} + |\tilde{\Gamma}_n L v - L v|_{L^2_F(\Omega; L^p(0, T; Y))}.
$$

(2.5)

Since

$$
\lim_{n \to \infty} |\Gamma_n L v - L v|_Y = 0
$$

and

$$
|\Gamma_n L v - L v|_Y \leq 2|L|_{L^2(\mathcal{X}; Y)}|v|_Y,
$$

by Lebesgue’s dominated convergence theorem, we get that

$$
\lim_{n \to \infty} |\Gamma_n L v - L v|_{L^2_F(\Omega; L^p(0, T; Y))} = 0. \quad (2.6)
$$

Similarly, we can prove that

$$
\lim_{n \to \infty} |\tilde{\Gamma}_n L v - L v|_{L^2_F(\Omega; L^p(0, T; Y))} = 0. \quad (2.7)
$$

Combining (2.5), (2.7), we obtain (2.4).

**Remark 2.1** One can show that $L^\infty_F(\Omega; L^p(0, T; \mathcal{L}(X, \mathcal{Y}))) \subset \mathcal{Y}_p(\mathcal{X}; \mathcal{Y})$. For any $L(\cdot, \cdot) \in \mathcal{Y}_p(\mathcal{X}; \mathcal{Y})$, one does not need to have $L(\cdot, \cdot) \in L^\infty_F(\Omega; L^p(0, T; \mathcal{L}(X, \mathcal{Y})))$. The reason for introducing such set is the lack of measurability for operator-valued functions and processes appeared in concrete problems. For example, generally speaking, a $C_0$-semigroup is not measurable with respect to the time variable (e.g., Subsection 1.1.c). In such case, one has to relax the condition on the measurability of such functions and processes. Nevertheless, as we shall see later, in some sense $\mathcal{Y}_p(\mathcal{X}; \mathcal{Y})$ is a nice “replacement” of the space $L^\infty_F(\Omega; L^p(0, T; \mathcal{L}(X, \mathcal{Y})))$ in the study of stochastic LQ problems.

Now we put the following assumptions on the operator $A$ in the control system (1.4).

**(AS0)** The $C_0$-semigroup $\{e^{At}\}_{t \geq 0}$ generated by $A$ is contractive in the sense that $|S(t)|_{L(H)} \leq e^{kt}$ for some $k \in \mathbb{R}$ and all $t \in [0, T]$.

**Remark 2.2** (AS0) is not restrictive in the sense that, as far as we know, all the linear SPDEs satisfy it.

**(AS1)** The eigenvectors $\{e_j\}_{j=1}^\infty$ of $A$ such that $|e_j|_H = 1$ for all $j \in \mathbb{N}$ constitute an orthonormal basis of $H$.

**Remark 2.3** (AS1) is not restrictive in the sense that, many important SPDEs, such as stochastic wave equations, stochastic heat equations and stochastic Schrödinger equations, involved on bounded domains fulfill it.
The following assumption is put on the coefficients of the control system (1.4) and the cost functional (1.5).

**Assumption 2.2** The coefficients satisfy that

\[
\begin{align*}
A_l(\cdot) &\in L^1_T(0, T; L^\infty(\phi; \mathcal{L}(H))), & B_l(\cdot) &\in L^\infty_T(0, T; \mathcal{L}(U; L^2(H))), \\
C_l(\cdot) &\in L^1_T(0, T; L^\infty(\phi; \mathcal{L}(H))), & D_l(\cdot) &\in L^\infty_T(0, T; \mathcal{L}(U; H)), \\
Q_l(\cdot) &\in L^\infty_T(0, T; \mathcal{L}(S(H))), & R_l(\cdot) &\in L^\infty_T(0, T; \mathcal{L}(U)), & G &\in L^\infty_T(\phi; \mathcal{L}(S(H))), \\
G &\geq 0, & R &> 0, & Q &\geq 0 \text{ for a.e. } (t, \omega) \in [0, T] \times \Omega.
\end{align*}
\]

Under (AS1), the control system (1.4) admits a unique solution \(x(\cdot) \in C_p([s, T]; L^2(\phi; H))\), and the cost functional (1.5) becomes well-defined. If in addition (AS0) is true, then \(x(\cdot) \in L^2_T(\phi; C([s, T]; H))\). Let us consider the following optimal control problem:

**Problem (SLQ):** For each \((s, \eta) \in [0, T) \times L^2_{\mathcal{F}_x}(\phi; H), \) find (if possible) a control \(\bar{u}(\cdot) \in L^2_T(s, T; U)\) such that

\[
\mathcal{J}(s, \eta; \bar{u}(\cdot)) = \inf_{u(\cdot) \in L^2_T(s, T; U)} \mathcal{J}(s, \eta; u(\cdot)). \tag{2.8}
\]

If the above is possible, then **Problem (SLQ)** is called **solvable**. Any \(\bar{u}(\cdot)\) satisfying (2.8) is called an **optimal control**. If the \(\bar{u}(\cdot)\) which fulfills (2.8) is unique, then **Problem (SLQ)** is called **uniquely solvable**. The corresponding state \(\bar{x}(\cdot)\) is called an **optimal state**, and \((\bar{x}(\cdot), \bar{u}(\cdot))\) is called an **optimal pair**. Under (AS2), we know that **Problem (SLQ)** admits at most one optimal control (see [21, Proposition 2.1]).

Set

\[
\ell^2_+ \triangleq \left\{ \{\lambda_j\}_{j=1}^\infty \in \ell^2 \mid \lambda_j \in (0, +\infty) \text{ for each } j \in \mathbb{N} \right\}.
\]

For any given \(\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^2_+\), define a norm \(| \cdot |_{H^I_\lambda}\) on \(H\) as follows (Recall that \(| \cdot |_{D(A)}\) stands for the graph norm of the operator \(A\)):

\[
|h|_{H^I_\lambda} = \sum_{j=1}^\infty \lambda_j^2 |e_j|_{D(A)}^2 |h_j|^2, \quad \forall h = \sum_{j=1}^\infty h_j e_j \in H.
\]

Denote by \(H^I_\lambda\) the completion of \(H\) with respect to the norm \(| \cdot |_{H^I_\lambda}\). Then, \(H^I_\lambda\) is a Hilbert space, \(H \subset H^I_\lambda\) and \(\{\lambda_j^{-1} |e_j|_{D(A)} e_j\}_{j=1}^\infty\) is an orthonormal basis of \(H^I_\lambda\). Write \(H_{\lambda}\) for the dual space of \(H^I_\lambda\) with respect to the pivot space \(H \equiv H^I\). Hence, \(H_{\lambda} \subset H \equiv H^I \subset H^I_\lambda\).

For any \(\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^2_+\), from the definition of \(H^I_{\lambda}\), we see that \(\{e_j\}_{j=1}^\infty \subset H_{\lambda}\) and the norm on \(H_{\lambda}\) is given as follows:

\[
|h|_{H_{\lambda}} = \sum_{j=1}^\infty |h_j|_{D(A)}^2 \lambda_j^{-2}, \quad \forall h \in H_{\lambda},
\]

where \(\lambda_j = \langle \xi, e_j \rangle_{H}\) for \(j \in \mathbb{N}\). Furthermore, \(\{\lambda_j |e_j|_{D(A)}^{-1} e_j\}_{j=1}^\infty\) is an orthonormal basis of \(H_{\lambda}\).

**Remark 2.4** Note the set \(H^I_\lambda\) depends on the parameter \(\lambda\). Once a \(\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^2_+\) is given, we get an \(H^I_\lambda\), namely, \(H^I_\lambda\) depends on the choice of \(\lambda\). Generally speaking, for two different \(\lambda, \hat{\lambda} \in \ell^2_+\), the corresponding spaces \(H^I_\lambda\) and \(H^I_{\hat{\lambda}}\) are different.
Denote by $\mathcal{L}_2(H; H'_\lambda)$ the set of all Hilbert-Schmidt operators from $H$ to $H'_\lambda$. Namely,

$$\mathcal{L}_2(H; H'_\lambda) \triangleq \left\{ F \in \mathcal{L}(H; H'_\lambda) \mid \sum_{j=1}^{\infty} |F e_j|_{H'_\lambda}^2 < \infty \right\}.$$ 

It is well-known that $\mathcal{L}_2(H; H'_\lambda)$ is a Hilbert space itself with the inner product

$$\langle F_1, F_2 \rangle_{\mathcal{L}_2(H; H'_\lambda)} \triangleq \sum_{j=1}^{\infty} \langle F_1 e_j, F_2 e_j \rangle_{H'_\lambda}.$$ 

Denote by $\mathcal{L}_2(H_\lambda; H)$ the set of all Hilbert-Schmidt operators from $H_\lambda$ to $H$. Similarly, $\mathcal{L}_2(H_\lambda; H)$ is also a Hilbert space itself. We refer to [25] for more details on Hilbert-Schmidt operators.

We also need the following technical assumption:

(AS3) There exists $\lambda = \{ \lambda_j \}_{j=1}^{\infty} \in \ell^2_+$ such that

$$\begin{align*}
A_1 \in L^1_F(0, T; L^\infty(\Omega; \mathcal{L}(H'_\lambda))), & \quad C \in L^2_F(0, T; L^\infty(\Omega; \mathcal{L}(H'_\lambda))), \\
Q \in L^2_F(\Omega; L^2(0, T; \mathcal{L}(H'_\lambda))), & \quad G \in L^2_F(\Omega; \mathcal{L}(H'_\lambda)), \\
A_1 \in L^1_F(0, T; L^\infty(\Omega; \mathcal{L}(H_\lambda))), & \quad C \in L^2_F(0, T; L^\infty(\Omega; \mathcal{L}(H_\lambda))), \\
Q \in L^2_F(\Omega; L^2(0, T; \mathcal{L}(H_\lambda))), & \quad G \in L^2_F(\Omega; \mathcal{L}(H_\lambda)).
\end{align*} \tag{2.9}$$

Remark 2.5 In Assumption (AS3), we assume the existence of one $\lambda \in \ell^2_+$ so that (2.9) holds. Nevertheless, there may exist another $\hat{\lambda} \in \ell^2_+$ such that (2.9) does not hold when $\lambda$ is replaced by $\hat{\lambda}$. Fortunately, this will not influence the main results of this paper. In other words, we only need the existence of one $\lambda \in \ell^2_+$ such that (2.9) holds to prove our main results.

Remark 2.6 For an LQ problem of a given SPDE, Assumption (AS3) means that some coefficients in the equation and the cost functional are smooth in some sense (See Section 5 for example). This is not very restrictive for many controlled SPDEs.

Following [21], we also introduce the notion of optimal feedback operator.

Definition 2.1 An operator $\Theta(\cdot) \in \mathcal{T}_2(H; U)$ is called an optimal feedback operator for Problem (SLQ) if

$$\mathcal{J}(s, \eta; \Theta(\cdot)\bar{x}(\cdot)) \leq \mathcal{J}(s, \eta; u(\cdot)), \quad \forall (s, \eta) \in [0, T) \times L^2_F(\Omega; H), \quad u(\cdot) \in L^2_F(s, T; U),$$

where $\bar{x}(\cdot) = \bar{x}(\cdot; s, \eta, \Theta(\cdot)\bar{x}(\cdot))$ solves the following equation:

$$\begin{align*}
d\bar{x}(t) & = [(A + A_1)\bar{x}(t) + B\Theta\bar{x}(t)]dt + (C\bar{x}(t) + D\Theta\bar{x}(t))dW(t) \quad \text{in } (s, T], \\
\bar{x}(s) & = \eta.
\end{align*} \tag{2.11}$$

In Definition 2.1, $\Theta(\cdot)$ is independent of $\eta \in H$. For a fixed $\eta \in H$, the inequality (2.10) implies that the control $\tilde{u}(\cdot) \equiv \Theta(\cdot)\bar{x}(\cdot) \in L^2_F(0, T; U)$ is optimal for Problem (SLQ). Therefore, for Problem (SLQ), the existence of an optimal feedback operator on $[0, T]$ implies the existence of an optimal control of any $\eta \in H$. 

9
3 The statement of the main results

To characterize the optimal feedback operator, we need the following operator-valued, backward stochastic Riccati equation:

\[
\begin{cases}
    dP = -[P(A + A_1) + (A + A_1)^* P + \Lambda C + C^* \Lambda P] + C^* P C + Q - L^* K^{-1} L \, dt + \Lambda dW(t) & \text{in } [0, T), \\
    P(T) = G,
\end{cases}
\]  

(3.1)

where

\[ K \equiv R + D^* P D, \quad L = B^* P + D^* (PC + \Lambda). \]  

(3.2)

Before presenting the main result, we need to recall the definition of transposition solution \((P, \Lambda)\) of (3.1). To this end, let us consider the following two (forward) SEEs:

\[
\begin{cases}
    dx_1 = [(A + A_1)x_1 + u_1] \, d\tau + (Cx_1 + v_1) \, dW(\tau) & \text{in } (t, T], \\
    x_1(t) = \xi_1
\end{cases}
\]  

(3.3)

and

\[
\begin{cases}
    dx_2 = [(A + A_1)x_2 + u_2] \, d\tau + (Cx_2 + v_2) \, dW(\tau) & \text{in } (t, T], \\
    x_2(t) = \xi_2.
\end{cases}
\]  

(3.4)

Here \( t \in [0, T) \), \( \xi_1, \xi_2 \) are suitable random variables and \( u_1, u_2, v_1, v_2 \) are suitable stochastic processes.

Put

\[
C_{F,w}([0, T]; L^\infty(\Omega; \mathcal{L}(H))) \triangleq \left\{ P \in C([0, T] \times \Omega \rightarrow \mathcal{L}(H)) \mid P|_{\mathcal{L}(H)} \in L^\infty_F(0, T), \chi_{[t, T]} P(\cdot) \xi \in C_{\mathcal{F}}([t, T]; L^2(\Omega; H)), \forall \xi \in L^2_F(\Omega; H) \right\},
\]

(3.5)

\[
C_{F,w}([0, T]; L^\infty(\Omega; \mathcal{S}(H))) \triangleq \left\{ P \in C_{F,w}([0, T]; L^\infty(\Omega; \mathcal{L}(H))) \mid P(t, \omega) \in \mathcal{S}(H) \text{ for a.e. } (t, \omega) \in [0, T] \times \Omega \right\},
\]

and

\[
L^2_{F,D,w}(0, T; \mathcal{S}(H)) \triangleq \left\{ \Lambda : [0, T] \times \Omega \rightarrow \mathcal{S}_2(H; H'_\Lambda) \mid |\Lambda|_{\mathcal{L}(H; H'_\Lambda)} \in L^2_F(0, T), \\
D^* \Lambda |_{\mathcal{L}(H; U)} \in L^\infty_F(\Omega; L^2(0, T)) \right\},
\]

where

\[
\mathcal{S}_2(H; H'_\Lambda) \triangleq \left\{ F \in \mathcal{L}_2(H; H'_\Lambda) \mid F|_{H_\Lambda}, \text{ the restriction of } F \text{ on } H_\Lambda, \text{ is a Hilbert-Schmidt operator from } H_\Lambda \text{ to } H, \text{ and } (F|_{H_\Lambda})^* = F \right\},
\]

is a Hilbert space with the inner product inherited from \( \mathcal{L}_2(H; H'_\Lambda) \).

These spaces are Banach spaces with the norms

\[
|P|_{C_{F,w}([0, T]; L^\infty(\Omega; \mathcal{S}(H)))} \triangleq \left( |P|_{\mathcal{L}(H)} \right)^{L^\infty_F(0, T)}, \quad \forall P \in C_{F,w}([0, T]; L^\infty(\Omega; \mathcal{S}(H)))
\]

and

\[
|\Lambda|_{L^2_{F,D,w}(0, T; \mathcal{S}(H))} \triangleq \left( |\Lambda|_{\mathcal{L}_2(H; H'_\Lambda)} \right)^{L^2_F(0, T)} + \left( |D^* \Lambda|_{\mathcal{L}(H; U)} \right)^{L^\infty_F(\Omega; L^2(0, T))}, \quad \forall \Lambda \in L^2_{F,D,w}(0, T; \mathcal{S}(H)),
\]

respectively.
**Definition 3.1** We call \((P(\cdot), \Lambda(\cdot)) \in C_{F,w}([0,T]; L^\infty(\Omega; S(H))) \times L^2_{F,D,w}(0,T; S(H))\) a transposition solution to (3.1) if the following conditions hold:

1) \(K(t,\omega)(t,\omega) + D(t,\omega)^* P(t,\omega) D(t,\omega) > 0\) and its left inverse \(K(t,\omega)^{-1}\) is a densely defined closed operator for a.e. \((t,\omega) \in [0,T] \times \Omega;\)

2) For any \(t \in [0,T], \xi_1, \xi_2 \in L^2_F(\Omega; H), u_1(\cdot), u_2(\cdot) \in L^4_F(\Omega; L^2(t,T; H))\) and \(v_1(\cdot), v_2(\cdot) \in L^4_F(\Omega; L^2(t,T; H))\), with \(L \triangleq D^* P C + B^* P + D^* \Lambda\), it holds that

\[
\begin{align*}
\mathbb{E}\langle G x_1(T), x_2(T) \rangle_H + \mathbb{E} \int_t^T \langle Q(\tau)x_1(\tau), x_2(\tau) \rangle_H d\tau - \mathbb{E} \int_t^T \langle K(\tau)^{-1} L(\tau) x_1(\tau), L(\tau) x_2(\tau) \rangle_H d\tau \\
= \mathbb{E}\langle P(t)\xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(\tau) u_1(\tau), x_2(\tau) \rangle_H d\tau + \mathbb{E} \int_t^T \langle P(\tau) u_2(\tau), v_2(\tau) \rangle_H d\tau \\
+ \mathbb{E} \int_t^T \langle P(\tau) C(\tau) x_1(\tau), v_2(\tau) \rangle_H d\tau + \mathbb{E} \int_t^T \langle P(\tau) v_1(\tau), C(\tau) x_2(\tau) + v_2(\tau) \rangle_H d\tau \\
+ \mathbb{E} \int_t^T \langle v_1(\tau), \Lambda(\tau) x_2(\tau) \rangle_{H_\alpha, H'_\alpha} d\tau + \mathbb{E} \int_t^T \langle \Lambda(\tau) x_1(\tau), v_2(\tau) \rangle_{H'_\alpha, H'_\alpha} d\tau,
\end{align*}
\]

where \(x_1(\cdot)\) and \(x_2(\cdot)\) solve (3.3) and (3.4), respectively.

Before stating the main result, which reveals the relationship between the existence of optimal feedback operator for Problem (SLQ) and the well-posedness of (3.1) in the sense of transposition solution, we also need the following technical condition:

(AS4) There is a dense subspace \(\bar{U}\) of \(U\) such that \(R \in L^\infty_F(0,T; L(\bar{U}))\) and \(B, D \in L^\infty_F(0,T; L(\bar{U}; H))\), where \(H_\alpha\) is given in Assumption (AS3).

**Remark 3.1** Similar to Assumption (AS3), for an LQ problem of a given SPDE, Assumption (AS4) means that some coefficients in the equation and the cost functional are smooth in some sense (See Section 2 for example). This is not very restrictive for many controlled SPDEs.

Put

\[
\begin{align*}
\Upsilon_2(H;U) \cap \Upsilon_2(H;\tilde{U}) \triangleq \{ L \in \Upsilon_2(H;U) | \text{For any } f \in L^2_F(\Omega; L^\infty(0,T; H)), Lf \in L^2_F(0,T; \tilde{U}), |L(\cdot, \cdot)|_{L(H_\alpha; \tilde{U})} \in L^\infty_F(\Omega; L^P(0,T)) \}\,
\end{align*}
\]

**Theorem 3.1** Let (AS0) – (AS4) hold. Then, Problem (SLQ) admits a unique optimal feedback operator \(\Theta(\cdot) \in \Upsilon_2(H;U) \cap \Upsilon_2(H;\tilde{U})\) if and only if the Riccati equation (3.1) admits a unique transposition solution \((P(\cdot), \Lambda(\cdot)) \in C_{F,w}([0,T]; L^\infty(\Omega; S(H))) \times L^2_{F,D,w}(0,T; S(H))\) such that

\[
K(\cdot)^{-1}[B(\cdot)^* P(\cdot) + D(\cdot)^* P(\cdot) C(\cdot) + D(\cdot)^* \Lambda(\cdot)] \in \Upsilon_2(H;U) \cap \Upsilon_2(H;\tilde{U}).
\]

In this case, the optimal feedback operator \(\Theta(\cdot)\) is given by

\[
\Theta(\cdot) = -K(\cdot)^{-1}[B(\cdot)^* P(\cdot) + D(\cdot)^* P(\cdot) C(\cdot) + D(\cdot)^* \Lambda(\cdot)].
\]

Furthermore,

\[
\inf_{u \in L^2_F(s,T;\tilde{U})} J(s, \eta; u) = \frac{1}{2} \mathbb{E}\langle P(s) \eta, \eta \rangle_H.
\]
Remark 3.2 If $H, U$ are finite dimensional, then (AS1), (AS3) and (AS4) hold true. The transposition solution becomes equivalent with the classical adapted solution of (3.4), and our conclusion is reduced to [19, Theorem 2.1].

Remark 3.3 It is more natural to require the optimal feedback operator $\Theta(\cdot) \in \Upsilon_2(H; U)$ other than $\Theta(\cdot) \in \Upsilon_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})$. Nevertheless, we do not know how to prove Theorem 3.1 with such condition now.

In the above, we require $X \in L^2_\Sigma(\Omega; C([0, T]; H))$ and $\Theta \in \Upsilon_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})$ to guarantee that $\Theta X \in L^2_\Sigma(0, T; U)$. By relaxing the condition on $X$ and strengthening that on $\Theta$ as $X \in C_{\Sigma}([0, T]; L^2(\Omega; H))$ and $\Theta \in \Upsilon_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})$, where

$$\tilde{\Upsilon}_2(H; U) \overset{\Delta}{=} \{ L \in L(L^\infty([0, T]; L^2(\Omega; H)); H^2(0, T; U)) || L|_{G(H; U)} \in L^2_\Sigma(0, T; L^\infty(\Omega)) \}$$

and

$$\tilde{\Upsilon}_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U}) \overset{\Delta}{=} \{ L \in \tilde{\Upsilon}_2(H; U) | \text{For any } f \in L^\infty_\Sigma(0, T; L^2(\Omega; H_\lambda)), Lf \in L^2_\Sigma(0, T; \tilde{U}), \}$$

$$|L(\cdot, \cdot)|_{G(H_\lambda; \tilde{U})} \in L^2_\Sigma(0, T; L^\infty(\Omega)) \}.$$}

one can also obtain $\Theta X \in L^2_\Sigma(0, T; U)$. In this case, we can relax (AS0) as $A$ generates a $C_0$-semigroup on $H$. We state the analogue result of Theorem 3.1 and omit the proof, which is similar to the one for Theorem 3.1.

Theorem 3.2 Let (AS1)–(AS4) hold and $A$ generates a $C_0$-semigroup on $H$. Then, Problem (SLQ) admits a unique optimal feedback operator $\Theta(\cdot) \in \tilde{\Upsilon}_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})$ if and only if the Riccati equation (3.1) has a unique transposition solution $(P(\cdot), \Lambda(\cdot)) \in C_{\Sigma, w}([0, T]; L^\infty(\Omega; SS(H))) \times L^2_{\Sigma, D; w}(0, T; S(H))$ such that

$$K(\cdot)^{-1}[B(\cdot)*P(\cdot) + D(\cdot)*P(\cdot)C(\cdot) + D(\cdot)*\Lambda(\cdot)] \in \tilde{\Upsilon}_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U}).$$

In this case, the optimal feedback operator $\Theta(\cdot)$ is given by

$$\Theta(\cdot) = -K(\cdot)^{-1}[B(\cdot)*P(\cdot) + D(\cdot)*P(\cdot)C(\cdot) + D(\cdot)*\Lambda(\cdot)].$$

Furthermore,

$$\inf_{u \in L^2_\Sigma([0, T]; U)} \mathcal{J}(s, \eta; u) = \frac{1}{2} \mathbb{E}(P(s)\eta, \eta)_H.$$

Remark 3.4 At a first glance, it seems that Theorem 3.2 is more general than Theorem 3.1. However, we believe that Theorem 3.1 is more interesting due to the following reasons: first, as we have explained in Remark 2.2 (AS0) is always fulfilled by controlled SPDEs; second, the space $\tilde{\Upsilon}_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})$ is larger than the space $\tilde{\Upsilon}_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})$.

4 Proof of the main result

This section is devoted to the proof of Theorem 3.1. Inspired by [16, Theorem 3.3], we transform the nonlinear BSRE (3.4) into a proper linear BSEE with the solution satisfying an appropriate equality condition. To this end, we need two preliminary results, i.e., Lemma 4.1 and Lemma 4.7 which are obtained in the following two subsections respectively.
4.1 A new transposition solution for backward stochastic Lyapunov equations in infinite dimensions

Consider the following BSEE:

\[
\begin{aligned}
dP &= -\left[P(A + A_\Theta) + (A + A_\Theta)^*P + C_\Theta^*PC_\Theta + C_\Theta\Lambda + \Lambda C_\Theta + Q + \Theta^*R\Theta\right]ds \\
&\quad + \lambda dW(s) \quad \text{in } [0, T], \\
P(T) &= G,
\end{aligned}
\]

(4.1)

where \( A_\Theta \triangleq A_1 + B\Theta \) and \( C_\Theta \triangleq C + D\Theta \) for some \( \Theta \in \mathcal{Y}_2(H; U) \cap \mathcal{Y}_2(H_\lambda; U) \). Since \( A_\Theta \) and \( C_\Theta \) may not belong to \( L_\infty^\infty(0, T; \mathcal{L}(H)) \), we cannot employ the existing result to obtain the well-posedness of (4.1) (see [20] for example).

Inspired by [20], we first define the \( H_\lambda \)-transposition solution of (4.1). To this end, we consider the following two (forward) SEEs:

\[
\begin{aligned}
d\tilde{x}_1 &= [(A + A_\Theta)\tilde{x}_1 + \tilde{u}_1]dr + (C_\Theta\tilde{x}_1 + \tilde{v}_1)dw(r) \quad \text{in } (t, T], \\
\tilde{x}_1(t) &= \xi_1,
\end{aligned}
\]

(4.2)

\[
\begin{aligned}
d\tilde{x}_2 &= [(A + A_\Theta)\tilde{x}_2 + \tilde{u}_2]dr + (C_\Theta\tilde{x}_2 + \tilde{v}_2)dw(r) \quad \text{in } (t, T], \\
\tilde{x}_2(t) &= \xi_2,
\end{aligned}
\]

(4.3)

where \( t \in [0, T) \).

Let \( p > 1 \). If \( \xi_1, \xi_2 \in L^p_\mathcal{F}_t(\Omega; H_\lambda) \) and \( \tilde{u}_1(\cdot), \tilde{u}_2(\cdot), \tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \in L^p_\mathcal{F}(\Omega; L^2(t, T; H_\lambda)) \), by [23] Theorem 3.20, the equation (4.2) (resp. (4.3)) admits a unique solution \( \tilde{x}_1 = \tilde{x}_1(\cdot; t, \xi_1, \tilde{u}_1, \tilde{v}_1) \) (resp. \( \tilde{x}_2 = \tilde{x}_2(\cdot; t, \xi_2, \tilde{u}_2, \tilde{v}_2) \)) in \( L^p_\mathcal{F}(\Omega; C([t, T]; H_\lambda)) \subset L^p_\mathcal{F}(\Omega; C([t, T]; H)) \). Further, for \( j = 1, 2 \), we have

\[
|\tilde{x}_j|_{L^p_\mathcal{F}(\Omega; C([t, T]; H_\lambda))} \leq \mathcal{C}(|\xi_j|_{L^p_\mathcal{F}_t(\Omega; H_\lambda)} + |\tilde{u}_j|_{L^p_\mathcal{F}(\Omega; L^2(t, T; H_{\lambda}))} + |\tilde{v}_j|_{L^p_\mathcal{F}(\Omega; L^2(t, T; H_{\lambda}))}).
\]

(4.4)

Here and in what follows, we denote by \( \mathcal{C} \) a generic positive constant, which may be different from one place to another.

If \( \xi_1, \xi_2 \in L^2_\mathcal{F}_t(\Omega; H) \), \( \tilde{u}_1(\cdot), \tilde{u}_2(\cdot) \in L^2_\mathcal{F}((\Omega; L^2(t, T; H))) \) and \( \tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \in L^2_\mathcal{F}(\Omega; L^2(t, T; H)) \), by [23] Theorem 3.20 again, the equation (4.2) (resp. (4.3)) admits a unique solution \( \tilde{x}_1 \) (resp. \( \tilde{x}_2 \)) in \( L^2_\mathcal{F}(\Omega; C([t, T]; H)) \) and for \( j = 1, 2 \), we have

\[
|\tilde{x}_j|_{L^2_\mathcal{F}(\Omega; C([t, T]; H))} \leq \mathcal{C}(|\xi_j|_{L^2_\mathcal{F}_t(\Omega; H)} + |\tilde{u}_j|_{L^2_\mathcal{F}(\Omega; L^2(t, T; H))} + |\tilde{v}_j|_{L^2_\mathcal{F}(\Omega; L^2(t, T; H))}).
\]

(4.5)

**Definition 4.1** We call \( (P(\cdot), \Lambda(\cdot)) \in C_{F,w}([0, T]; \mathcal{L}(\Omega; \mathcal{L}(H))) \times L^2(0, T; L^2(\Omega; \mathcal{S}_2(H; H^*_\lambda))) \) an \( H_\lambda \)-transposition solution to the equation (4.1) if for any \( t \in [0, T], \xi_1, \xi_2 \in L^2_\mathcal{F}_t(\Omega; H), \tilde{u}_1(\cdot), \tilde{u}_2(\cdot) \in L^2_\mathcal{F}(\Omega; L^2(t, T; H)) \) and \( \tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \in L^2_\mathcal{F}(\Omega; L^2(t, T; H)) \), it holds that

\[
\begin{aligned}
\mathbb{E}\langle G\tilde{x}_1(T), \tilde{x}_2(T)\rangle_H + \mathbb{E}\int_t^T \langle [Q(r) + \Theta(r)^*R(r)\Theta(r)]\tilde{x}_1(r), \tilde{x}_2(r)\rangle_H dr \\
= \mathbb{E}\langle P(t)\xi_1, \xi_2\rangle_H + \mathbb{E}\int_t^T \langle P(r)\tilde{u}_1(r), \tilde{x}_2(r)\rangle_H dr + \mathbb{E}\int_t^T \langle P(r)\tilde{x}_1(r), \tilde{u}_2(r)\rangle_H dr \\
+ \mathbb{E}\int_t^T \langle P(r)C_\Theta(r)\tilde{x}_1(r), \tilde{v}_2(r)\rangle_H dr + \mathbb{E}\int_t^T \langle P(r)\tilde{v}_1(r), C_\Theta(r)\tilde{x}_2(r) + \tilde{v}_2(r)\rangle_H dr \\
+ \mathbb{E}\int_t^T \langle \tilde{v}_1(r), \Lambda^*(r)\tilde{x}_2(r)\rangle_{H_\lambda, H^*_\lambda} dr + \mathbb{E}\int_t^T \langle \tilde{v}_2(r), \Lambda(r)\tilde{x}_1(r)\rangle_{H_\lambda, H^*_\lambda} dr.
\end{aligned}
\]

(4.6)

Here \( \tilde{x}_1(\cdot) \) and \( \tilde{x}_2(\cdot) \) solve (4.2) and (4.3), respectively.
Lemma 4.1 The equation (4.1) admits a unique $H_\lambda$-transposition solution $(P(\cdot), \Lambda(\cdot))$. Moreover,
\begin{align*}
|P(\cdot), \Lambda(\cdot)|_{C_{\text{loc}}([0,T]; L^\infty(\Omega; S(H)))} &
\lesssim C(|Q + \Theta^* R\Theta|_{L^p(\Omega; L^1(0,T))} + |G|_{L^p(\Omega; \mathcal{M}(H))}).
\end{align*}

Lemma 4.1 is a slight modification of [20, Theorem 1.2]. Here we relax the assumption on the measurability of the nonhomogeneous terms and coefficients, due to the appearance of the feedback operator $\Theta$ in the nonhomogeneous terms and coefficients of the equation (4.1). The proof of Lemma 4.1 is similar as the one for [20, Theorem 1.2]. Hence, we provide details for the different part and give a sketch of the proof.

We first present the following preliminary results.

Lemma 4.2 [21, Lemma 3.5] If $\{S(t)\}_{t \geq 0}$ is a contraction semigroup on $\mathcal{H}$, then it is also a contraction semigroup on $\mathcal{H}_\lambda$.

Lemma 4.3 For each $t \in [0,T]$, if $u_2 = v_2 = 0$ in the equation (4.3), then there exists an operator $\Phi(\cdot,t) \in \mathcal{L}(L^2_\lambda(\Omega; \mathcal{H}), L^2_\lambda(\Omega; C([t,T]; \mathcal{H})))$ such that the solution to (4.3) can be represented as $\hat{x}_2(\cdot) = \Phi(\cdot,t)\hat{x}_2$. Further, for any $t \in [0,T)$, $\xi \in L^2_\lambda(\Omega; \mathcal{H})$ and $\varepsilon > 0$, there is a $\delta \in (0, T-t)$ such that for any $s \in [t, t+\delta]$, it holds that
\begin{align*}
|\Phi(\cdot,t)\xi - \Phi(\cdot,s)\xi|_{L^2_\lambda(\Omega; C([s,T]; \mathcal{H}))} < \varepsilon.
\end{align*}

The proof of Lemma 4.3 is very similar to that of [22, Lemma 2.6]. Hence we omit it here.

Lemma 4.4 The set
\begin{align*}
\{\hat{x}_2(\cdot) \mid \hat{x}_2(\cdot) \text{ solves (4.3) with } t = 0, \hat{x}_2 = 0, \tilde{v}_2 = 0 \text{ and } \tilde{u}_2 \in L^2_\lambda(\Omega; L^2(0,T; \mathcal{H}))\}
\end{align*}
is dense in $L^2_\lambda(0,T; \mathcal{H})$.

The proof of Lemma 4.4 is almost the same as the one for [21, Lemma 3.9]. We omit it.

Lemma 4.5 [22, Lemma 2.5] Assume that $p \in (1, \infty)$, $q = \begin{cases} \frac{p-1}{p} & \text{if } p \in (1, \infty), \\ 1 & \text{if } p = \infty, \end{cases}$ \(f_1 \in L^p_\lambda(0,T; L^2(\Omega; \mathcal{H}))\) and $f_2 \in L^2_\lambda(0,T; L^2(\Omega; \mathcal{H}))$. Then there exists a monotonic sequence $\{h_n\}_{n=1}^\infty$ of positive numbers such that $\lim_{n \to \infty} h_n = 0$, and
\begin{align*}
\lim_{n \to \infty} \frac{1}{h_n} \int_t^{t+h_n} \mathbb{E}(f_1(t), f_2(\tau))_H d\tau = \mathbb{E}(f_1(t), f_2(t))_H, \quad \text{a.e. } t \in [0,T].
\end{align*}

Lemma 4.6 Let $L \in \mathcal{L}(\mathcal{H})$. Then $L \in \mathcal{L}_\lambda(\mathcal{H}; \mathcal{H}_\lambda^\prime)$.

Proof. For $j \in \mathbb{N}$, let $\ell_{jk} = \langle L e_j, e_k \rangle_H$ for $k \in \mathbb{N}$. Then $L e_j = \sum_{k=1}^\infty \ell_{jk} e_j$ and
\begin{align*}
\sum_{k=1}^\infty |\ell_{jk}|^2 = |L e_j|^2_H \leq |L|^2_{\mathcal{L}(\mathcal{H})}.
\end{align*}

Noting that $\ell_{jk} = \langle L e_j, e_k \rangle_H = \langle e_j, L^* e_k \rangle_H$, we have $L^* e_k = \sum_{k=1}^\infty \ell_{jk} e_j$ and
By (4.9), we have that
\[
\sum_{j=1}^{\infty} |f_j|_H^2 = |L^* e_k|_H^2 \leq |L^*|_{\mathcal{L}(H)}^2 = |L|_{\mathcal{L}(H)}^2.
\] (4.9)

By Lemma 2.1, there is \( \{\Theta_n\}_{n=1}^{\infty} \subset L_{\infty}^\infty(\Omega; L^2(0, T; \mathcal{L}(H; U))) \) such that
\[
\lim_{n \to \infty} \Theta_n v = \Theta v \quad \text{in } L_{\infty}^2(0, T; U), \quad \forall \, v \in L_{\infty}^2(\Omega; L^\infty(0, T; H)).
\] (4.10)

Now we are in a position to prove Lemma 4.1.

**Proof of Lemma 4.1.** We divide the proof into six steps.

**Step 1.** By Lemma 2.1, there is \( \{\Theta_n\}_{n=1}^{\infty} \subset L_{\infty}^\infty(\Omega; L^2(0, T; \mathcal{L}(H; U))) \) such that
\[
\lim_{n \to \infty} \Theta_n v = \Theta v \quad \text{in } L_{\infty}^2(0, T; U), \quad \forall \, v \in L_{\infty}^2(\Omega; L^\infty(0, T; H)).
\] (4.10)

Define a family of operators \( \{\mathcal{T}(t)\}_{t \geq 0} \) on \( L^2(\mathcal{H}; H') \) as follows:
\[
\mathcal{T}(t)\mathcal{O} = e^{At}e^{A^* t}, \quad \forall \, \mathcal{O} \in L^2(\mathcal{H}; H').
\]

Similar to Step 1 in the proof of [20] Theorem 1.2, we can show that \( \{\mathcal{T}(t)\}_{t \geq 0} \) is a contraction semigroup on \( L^2(\mathcal{H}; H') \), and its restriction on \( L^2(\mathcal{H}_L; H) \) is a contraction semigroup on \( L^2(\mathcal{H}_L; H) \).

Denote by \( \mathcal{A} \) the infinitesimal generater of \( \{\mathcal{T}(t)\}_{t \geq 0} \). Consider the following \( L^2(\mathcal{H}; H') \)-valued BSEE:
\[
\begin{align*}
\left \{ \begin{array}{l}
dP_n &= −A^* P_n dt + f_n(t, P_n, \Lambda_n) dt + \Lambda_n dW(t) \quad \text{in } [0, T), \\
P(T) &= G,
\end{array} \right.
\]
(4.11)

where
\[
f_n(t, P_n, \Lambda_n) = −(P_n A\Theta_n + A^* \Theta_n P_n + C^* \Theta_n P_n C\Theta_n + C^* \Theta_n \Lambda_n + \Lambda_n C\Theta_n + Q + \Theta_n^* R\Theta_n) \quad \text{with } A\Theta_n \triangleq A_1 + B\Theta_n \quad \text{and } C\Theta_n \triangleq C + D\Theta_n.
\]

Since \( G \in L_{T}^2(\Omega; \mathcal{L}(H)) \), by Lemma 1.6, we get that
\[
|G|_{L_{T}^2(\Omega; \mathcal{L}(H,H'))} = ||G|_{\mathcal{L}(H,H')}|_{L_{T}^2(\Omega)} \leq |C||G|_{\mathcal{L}(H)}|_{L_{T}^2(\Omega)}.
\] (4.13)

Similarly, we have that
\[
|Q|_{L_{T}^\infty(\Omega; L^2(0,T;\mathcal{L}(H,H')))} \leq |C||Q|_{L_{T}^\infty(\Omega; L^2(0,T;\mathcal{L}(H)))}
\] (4.14)

and
\[
|\Theta^* R\Theta_n|_{L_{T}^\infty(\Omega; L^1(0,T;\mathcal{L}(H,H')))} \leq |C||\Theta^* R\Theta_n|_{\mathcal{L}(H)}|L_{T}^\infty(\Omega; L^1(0,T))|.
\] (4.15)

Since \( |\Theta_n|_{\mathcal{L}(H,U)} \leq |\Theta|_{\mathcal{L}(H,U)} \), we get from (4.15) that
\[
|\Theta^* R\Theta_n|_{L_{T}^\infty(\Omega; L^1(0,T;\mathcal{L}(H,H')))} \leq |C||\Theta^*|_{\mathcal{L}(H,U)}|R|_{\mathcal{L}(U)}|\Theta_n|_{\mathcal{L}(H,U)}|L_{T}^\infty(\Omega; L^1(0,T))|.
\] (4.16)
For any \( \eta_j \in \mathcal{L}_2(H; H') \ (j = 1, 2, 3, 4) \), we have

\[
|f(t, \eta_1, \eta_2) - f(t, \eta_3, \eta_4)|_{\mathcal{L}_2(H; H')}
\leq \left( |A_1|_{\mathcal{L}(H)} + |A_1'|_{\mathcal{L}(H')} + |C|_{\mathcal{L}(H)} |C'|_{\mathcal{L}(H')} + |B|_{\mathcal{L}(U; H)} |\Theta_n|_{\mathcal{L}(H; U)} + |B^*|_{\mathcal{L}(H', \tilde{U'})} |\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')}
\right.
\]

\[
+ |C'|_{\mathcal{L}(H')} |D|_{\mathcal{L}(U; H)} |\Theta_n|_{\mathcal{L}(H; U)} + |C|_{\mathcal{L}(H)} |D^*|_{\mathcal{L}(H', \tilde{U'})} |\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')}
\]

\[
+ |D|_{\mathcal{L}(U; H)} |\Theta_n|_{\mathcal{L}(H; U)} |D^*|_{\mathcal{L}(U'; \tilde{U'})} |\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')}
\]

\[
+ \left( |C|_{\mathcal{L}(H)} + |C'|_{\mathcal{L}(H')} + |D|_{\mathcal{L}(U; H)} |\Theta_n|_{\mathcal{L}(H; U)} + |D^*|_{\mathcal{L}(H', \tilde{U'})} |\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')}ight) |\eta_3 - \eta_4|_{\mathcal{L}_2(H; H')},
\]

From (AS2), (AS3), (23) and noting that

\[
|\Theta_n|_{\mathcal{L}(H; U)} \leq |\Theta|_{\mathcal{L}(H; U)}, \quad |\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')} \leq |\Theta^*|_{\mathcal{L}(\tilde{U'}; H'')},
\]

we see that there exist \( L_1 \in L^1(0, T) \) (independent of \( n \)) and \( L_2 \in L^2(0, T) \) such that

\[
|f(t, \eta_1, \eta_2) - f(t, \eta_3, \eta_4)|_{\mathcal{L}_2(H; H')} \leq L_1(t) |\eta_1 - \eta_2|_{\mathcal{L}_2(H; H')} + L_2(t) |\eta_3 - \eta_4|_{\mathcal{L}_2(H; H')}, \quad \mathbb{P}\text{-a.s.}
\]

Noting (4.13), (4.14), (4.16) and (4.18), by [23, Theorem 4.10], there exists a unique mild solution \((P_n, A_n) \in L^2_\mathcal{D}(\Omega; C([0, T]; \mathcal{L}_2(H; H'))) \times L^2_\mathcal{D}(0, T; \mathcal{L}_2(H; H')) \) of (4.11) such that

\[
\begin{align*}
|\Theta_n|_{\mathcal{L}(H; U)} & \leq |\Theta|_{\mathcal{L}(H; U)}, \\
|\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')} & \leq |\Theta^*|_{\mathcal{L}(\tilde{U'}; H'')},
\end{align*}
\]

\[
\left( |P_n|_{L^2_\mathcal{D}(\Omega; C([0, T]; \mathcal{L}_2(H; H')))} \times L^2_\mathcal{D}(0, T; \mathcal{L}_2(H; H')) \right)
\]

\[
\leq C(L_1, L_2) \left( |G|_{L^2_\mathcal{D}(\Omega; \mathcal{L}(H; H'))} + |Q + \Theta_n^* R \Theta_n|_{L^2_\mathcal{D}(\Omega; L^1(0, T; \mathcal{L}_2(H; H')))} \right).
\]

Hence,

\[
\begin{align*}
|\Theta_n|_{\mathcal{L}(H; U)} & \leq |\Theta|_{\mathcal{L}(H; U)}, \\
|\Theta_n^*|_{\mathcal{L}(\tilde{U'}; H'')} & \leq |\Theta^*|_{\mathcal{L}(\tilde{U'}; H'')},
\end{align*}
\]

\[
\begin{align*}
|P_n|_{L^2_\mathcal{D}(\Omega; C([0, T]; \mathcal{L}_2(H; H')))} & \leq C \left( |G|_{L^2_\mathcal{D}(\Omega; \mathcal{L}(H))} + |Q|_{L^2_\mathcal{D}(\Omega; L^1(0, T; \mathcal{L}(H)))} + |\Theta_n^* R \Theta_n|_{\mathcal{L}(H; U)} \right) \\
& \leq C \left( |G|_{L^2_\mathcal{D}(\Omega; \mathcal{L}(H))} + |Q|_{L^2_\mathcal{D}(0, T; \mathcal{L}(H))} + |\Theta_n|_{\mathcal{L}(H; U)} |R|_{\mathcal{L}(H)} |\Theta_n|_{\mathcal{L}(H; U)} \right) \\
& \leq C \left( |G|_{L^2_\mathcal{D}(\Omega; \mathcal{L}(H))} + |Q|_{L^2_\mathcal{D}(0, T; \mathcal{L}(H))} + |\Theta_n|_{\mathcal{L}(H; U)} |R|_{\mathcal{L}(H)} |\Theta_n|_{\mathcal{L}(H; U)} \right),
\end{align*}
\]

where the constant \( C \) is independent of \( n \in \mathbb{N} \).

**Step 2.** For \( t \in [0, T) \), consider the following two (forward) SEEs:

\[
\begin{align*}
d\tilde{x}_{1,n}(t) &= [(A + A \Theta_n) \tilde{x}_{1,n} + \tilde{u}_1] d\tau + \left( C \Theta_n \tilde{x}_{1,n} + \tilde{v}_1 \right) dW(\tau) \quad \text{in } (t, T], \\
\tilde{x}_{1,n}(t) &= \tilde{x}_1,
\end{align*}
\]

and

\[
\begin{align*}
d\tilde{x}_{2,n}(t) &= [(A + A \Theta_n) \tilde{x}_{2,n} + \tilde{u}_2] d\tau + \left( C \Theta_n \tilde{x}_{2,n} + \tilde{v}_2 \right) dW(\tau) \quad \text{in } (t, T], \\
\tilde{x}_{2,n}(t) &= \tilde{x}_2.
\end{align*}
\]

From (4.17), we know there exist \( L_3 \in L^1(0, T) \) and \( L_4 \in L^2(0, T) \) such that for all \( n \in \mathbb{N} \),

\[
|A \Theta_n|_{\mathcal{L}(H)} + |A \Theta_n|_{\mathcal{L}(H')} \leq L_3(t), \quad \mathbb{P}\text{-a.s.},
\]

and

\[
|C \Theta_n|_{\mathcal{L}(H)} + |C \Theta_n|_{\mathcal{L}(H')} \leq L_4(t), \quad \mathbb{P}\text{-a.s.}
\]
Let \( p > 1 \). If \( \xi_1, \xi_2 \in L^p_{\mathcal{F}}(\Omega; H_\lambda) \) and \( \hat{u}_1(\cdot), \hat{u}_2(\cdot), \hat{v}_1(\cdot), \hat{v}_2(\cdot) \in L^p_{\mathcal{F}}(\Omega; L^2(t, T; H_\lambda)) \), by [23] Theorem 3.20, the equation (4.20) (resp. (4.21)) admits a unique solution \( \check{x}_{1,n} = \check{x}_{1,n}(\cdot; t, \xi_1, \hat{u}_1, \hat{v}_1) \) (resp. \( \check{x}_{2,n} = \check{x}_{2,n}(\cdot; t, \xi_2, \hat{u}_2, \hat{v}_2) \)) in \( L^p_{\mathcal{F}}(\Omega; C([t, T]; H_\lambda)) \) and for \( j = 1, 2, \)

\[
|\check{x}_{j,n}|_{L^p_{\mathcal{F}}(\Omega; C([t, T]; H_\lambda))} \leq C(L_3, L_4)(|\hat{\xi}_j|_{L^p_{\mathcal{F}}(\Omega; H_\lambda)} + |\tilde{u}_j|_{L^p_{\mathcal{F}}(\Omega; L^2(t, T; H_\lambda))} + |\tilde{v}_j|_{L^p_{\mathcal{F}}(\Omega; L^2(t, T; H_\lambda))}), \tag{4.22}
\]

where the constant \( C(L_3, L_4) \) is independent of \( n \in \mathbb{N} \).

If \( \xi_1, \xi_2 \in L^p_{\mathcal{F}}(\Omega; H) \), \( \hat{u}_1(\cdot), \hat{u}_2(\cdot) \in L^p_{\mathcal{F}}(\Omega; L^2(t, T; H)) \) and \( \hat{v}_1(\cdot), \hat{v}_2(\cdot) \in L^p_{\mathcal{F}}(\Omega; L^2(t, T; H)) \), by [23] Theorem 3.20 again, the equation (4.20) (resp. (4.21)) admits a unique solution \( \check{x}_{1,n} \) (resp. \( \check{x}_{2,n} \)) in \( L^p_{\mathcal{F}}(\Omega; C([t, T]; H)) \) and for \( j = 1, 2, \)

\[
|\check{x}_{j,n}|_{L^p_{\mathcal{F}}(\Omega; C([t, T]; H))} \leq C(L_3, L_4)(|\hat{\xi}_j|_{L^p_{\mathcal{F}}(\Omega; H)} + |\tilde{u}_j|_{L^p_{\mathcal{F}}(\Omega; L^2(t, T; H))} + |\tilde{v}_j|_{L^p_{\mathcal{F}}(\Omega; L^2(t, T; H))}), \tag{4.23}
\]

where the constant \( C(L_3, L_4) \) is independent of \( n \in \mathbb{N} \).

We claim that for \( j = 1, 2 \) and \( p = 4, \)

\[
\lim_{n \to \infty} |\check{x}_{j,n} - \check{x}_j|_{L^4_{\mathcal{F}}(\Omega; C([t, T]; H))} = 0. \tag{4.24}
\]

Denote by \([a]\) the integer part of a number \( a \in \mathbb{R} \). Write

\[
N = \left[ \frac{1}{\varepsilon} \left( \int_T (|A_1|_{L^{4}(\Omega; L^4(t,T))} + |B|_{L^4(U;H)}|\Theta|_{L^4(U;H)}|\Theta(t,\omega)|_{L^4(U;H)}dr \right)^4 \right] + 1,
\]

where \( \varepsilon > 0 \) is a constant to be determined later. Define a sequence of stopping times \( \{\tau_{j,\varepsilon}\}_{j=1}^{N} \) as follows:

\[
\tau_{1,\varepsilon}(\omega) = \inf \left\{ \tau \in [t, T] \left| \int_t^\tau (|A_1(r,\omega)|_{L^4(H)} + |B(r,\omega)|_{L^4(U;H)}|\Theta(r,\omega)|_{L^4(U;H)}) dr \right|^2 = \varepsilon \right\},
\]

\[
\tau_{k,\varepsilon}(\omega) = \inf \left\{ \tau \in [\tau_{k-1,\varepsilon}(\omega), T] \left| \int_{\tau_{k-1,\varepsilon}}^\tau (|A_1(r,\omega)|_{L^4(H)} + |B(r,\omega)|_{L^4(U;H)}|\Theta(r,\omega)|_{L^4(U;H)}) dr \right|^2 = \varepsilon \right\},
\]

\[
k = 2, \ldots, N.
\]

Here, we agree that \( \inf \emptyset = T \).

For any \( s \in [t, \tau_{1,\varepsilon}] \), by Burkholder-Davis-Gundy’s inequality, we get that

\[
\mathbb{E} \sup_{\tau \leq t, \tau_{1,\varepsilon}} |\check{x}_{j,n}(\tau) - \check{x}_j(\tau)|^4_H \leq \mathbb{E} \sup_{\tau \leq t, \tau_{1,\varepsilon}} \left| \int_t^\tau e^{A_{\Theta}(\tau,\cdot)} (A_{\Theta} \check{x}_j - A_{\Theta} \check{x}_{j,n}) d\tau + \int_t^\tau e^{A_{\Theta}(\tau,\cdot)} (C_{\Theta} x_j - C_{\Theta} \check{x}_{j,n}) dW(\tau) \right|^4_H \leq CE \sup_{\tau \leq t, \tau_{1,\varepsilon}} \left( \int_t^\tau e^{A_{\Theta}(\tau,\cdot)} |A_{\Theta} (\check{x}_j - \check{x}_{j,n})|^4_H d\tau \right)^4_H \leq \int_t^\tau e^{A_{\Theta}(\tau,\cdot)} |A_{\Theta} (\check{x}_j - \check{x}_{j,n})|^4_H d\tau \right|^4_H
\]

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where the constant $C$ is independent of $\tau_{1,\varepsilon}$.

Let us choose $\varepsilon = \frac{1}{2}$. We find that

$$
\mathbb{E} \sup\limits_{\tau \in [t, \tau_{1,\varepsilon}]} |\tilde{x}_{j,n}(\tau) - \tilde{x}_j(\tau)|_H^4
\leq C \mathbb{E} \left[ \left( \int_{t}^{\tau_{1,\varepsilon}} |B|_{L(U;H)} |(\Theta - \Theta_n)\tilde{x}_j|_U d\tau \right)^4 + \left( \int_{t}^{\tau_{1,\varepsilon}} |D|_{L^2(U;H)}^2 |(\Theta - \Theta_n)\tilde{x}_j|_U^2 d\tau \right)^2 \right].
$$

By (4.25) and (4.26), we see that

$$
\lim_{n \to \infty} \mathbb{E} \sup\limits_{\tau \in [t, \tau_{1,\varepsilon}]} |\tilde{x}_{j,n}(\tau) - \tilde{x}_j(\tau)|_H^4 = 0.
$$

If $\tau_{1,\varepsilon} = T$, $\mathbb{P}$-a.s., then we get (4.24). Otherwise, by repeating the above argument $N$ times, we can prove (4.22).

**Step 3.** By (4.19), we know that $\{(P_n, \Lambda_n)\}_{n=1}^\infty$ is a bounded sequence in $L^2_P(0, T; L^2(H; \dot{H}^1_\lambda)) \times L^2_P(0, T; L^2(H; \dot{H}^1_\lambda))$. Hence, there is a subsequence of $\{(P_n, \Lambda_n)\}_{n=1}^\infty$, denoted by $\{(P_{n_k}, \Lambda_{n_k})\}_{k=1}^\infty$, such that

$$
\{(P_{n_k}, \Lambda_{n_k})\}_{n=1}^\infty \text{ converges weakly to some } (P, \Lambda)
\text{ in } L^2_P(0, T; L^2(H; \dot{H}^1_\lambda)) \times L^2_P(0, T; L^2(H; \dot{H}^1_\lambda)) \text{ as } k \to \infty.
$$

By (4.19) again, we know that for any $t \in [0, T], \{P_{n_k}(t)\}_{n=1}^\infty$ is a bounded sequence in $L^2_{\mathcal{F}_T}(\Omega; L^2(H; \dot{H}^1_\lambda))$. Hence, for every $t \in [0, T)$, there is a subsequence of $\{P_{n_k}(t)\}_{n=1}^\infty$, denoted by $\{P_{n_k,t}(t)\}_{n=1}^\infty$, such that

$$
\{P_{n_k,t}\}_{n=1}^\infty \text{ converges weakly to some } P^t \text{ in } L^2_{\mathcal{F}_T}(\Omega; L^2(H; \dot{H}^1_\lambda)) \text{ as } k \to \infty.
$$
Similar to Steps 2-3 of the proof of \[20\] Theorem 1.2, we have that
\[
\begin{align*}
\mathbb{E}\langle G\tilde{x}_{1,n}(T),\tilde{x}_{2,n}(T)\rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta_n(r)^* R(r) \Theta_n(r)]\tilde{x}_{1,n}(r),\tilde{x}_{2,n}(r)\rangle_H dr \\
= \mathbb{E}\langle P_n(t)^\dagger \xi_1, \xi_2\rangle_{H^*_\lambda H^*_{\lambda}} + \mathbb{E} \int_t^T \langle P_n(r)^\dagger \tilde{u}_1(r), \tilde{x}_{2,n}(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle P_n(r)\tilde{x}_{1,n}(r),\tilde{u}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr + \mathbb{E} \int_t^T \langle P_n(r)\Theta_n(r)\tilde{x}_{1,n}(r),\tilde{v}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle P_n(r)\tilde{v}_1(r), C\Theta_n(r)\tilde{x}_{2,n}(r) + \tilde{v}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle \tilde{v}_1(r), \Lambda_n^*(r)\tilde{x}_{2,n}(r)\rangle_{H_{\lambda} H_{\lambda}} dr + \mathbb{E} \int_t^T \langle \tilde{v}_2(r), \Lambda_n(r)\tilde{x}_{1,n}(r)\rangle_{H_{\lambda} H_{\lambda}} dr.
\end{align*}
\]

Hence, for any \( t \in [0, T) \),
\[
\begin{align*}
\mathbb{E}\langle G\tilde{x}_{1,n,k,t}(T),\tilde{x}_{2,n,k,t}(T)\rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta_n(k,t)^* R(r) \Theta_n(k,t)]\tilde{x}_{1,n}(r),\tilde{x}_{2,n}(r)\rangle_H dr \\
= \mathbb{E}\langle P_{n,k}(t)^\dagger \xi_1, \xi_2\rangle_{H^*_\lambda H^*_{\lambda}} + \mathbb{E} \int_t^T \langle P_{n,k}(r)^\dagger \tilde{u}_1(r), \tilde{x}_{2,n,k,t}(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle P_{n,k}(r)\tilde{x}_{1,n}(r),\tilde{u}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr + \mathbb{E} \int_t^T \langle P_{n,k}(r)\Theta_n(k,t)\tilde{x}_{1,n,k,t}(r),\tilde{v}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle P_{n,k}(r)\tilde{v}_1(r), C\Theta_n(k,t)\tilde{x}_{2,n,k,t}(r) + \tilde{v}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle \tilde{v}_1(r), \Lambda_n^*(k,t)\tilde{x}_{2,n,k,t}(r)\rangle_{H_{\lambda} H_{\lambda}} dr + \mathbb{E} \int_t^T \langle \tilde{v}_2(r), \Lambda_n(k,t)\tilde{x}_{1,n,k,t}(r)\rangle_{H_{\lambda} H_{\lambda}} dr.
\end{align*}
\]

Letting \( k \to \infty \) in (4.31), by (4.19), (4.27) and (4.28), we obtain that
\[
\begin{align*}
\mathbb{E}\langle G\tilde{x}_1(T),\tilde{x}_2(T)\rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta(r)^* R(r) \Theta(r)]\tilde{x}_1(r),\tilde{x}_2(r)\rangle_H dr \\
= \mathbb{E}\langle P^\dagger \xi_1, \xi_2\rangle_{H^*_\lambda H^*_{\lambda}} + \mathbb{E} \int_t^T \langle P(r)^\dagger \tilde{u}_1(r), \tilde{x}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle P(r)\tilde{x}_1(r),\tilde{u}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr + \mathbb{E} \int_t^T \langle P(r)\Theta\tilde{x}_1(r),\tilde{v}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle P(r)\tilde{v}_1(r), C\Theta\tilde{x}_2(r) + \tilde{v}_2(r)\rangle_{H^*_\lambda H^*_{\lambda}} dr \\
\quad + \mathbb{E} \int_t^T \langle \tilde{v}_1(r), \Lambda^*(r)\tilde{x}_2(r)\rangle_{H_{\lambda} H_{\lambda}} dr + \mathbb{E} \int_t^T \langle \tilde{v}_2(r), \Lambda(r)\tilde{x}_1(r)\rangle_{H_{\lambda} H_{\lambda}} dr.
\end{align*}
\]

**Step 4.** In this step, we prove that \( P \in C\mathbb{F}_\omega([0, T]; L^\infty(\Omega; L(H))) \).

We first show that for any \( t \in [0, T] \), \( |P^\dagger|_{L(H)} \in L^\infty_{\mathbb{F}_t}(\Omega) \) by a contradiction argument. Assume that this was untrue for some \( t \in [0, T] \). Noting that \( H \) is separable, by (6) Corollary 2.3, we can find two sequences \( \{\eta_n\}_{n=1}^\infty \), \( \{\tilde{\eta}_n\}_{n=1}^\infty \) with \( \eta_n|_{L^2_{\mathbb{F}_t}(\Omega; H)} = \tilde{\eta}_n|_{L^2_{\mathbb{F}_t}(\Omega; H)} = 1 \) for all \( n \in \mathbb{N} \) such that
\[
\mathbb{E}\langle P^\dagger \eta_n, \tilde{\eta}_n\rangle_{H^*_\lambda H^*_{\lambda}} \geq n \text{ for all } n \in \mathbb{N}.
\]
Thanks to (4.32), we get that for any $n \in \mathbb{N}$,

$$\mathbb{E}\langle P^t \eta_n, \tilde{\eta}_n \rangle_{H_{\lambda}} = \mathbb{E}\langle G_1(T; t, \eta_n, 0, 0), \tilde{x}_2(T; t, \tilde{\eta}_n, 0, 0) \rangle_{H}$$

$$- \mathbb{E} \int_t^T \langle (Q + \Theta^* R \Theta) \tilde{x}_1(s; t, \eta_n, 0, 0), \tilde{x}_2(s; t, \tilde{\eta}_n, 0, 0) \rangle_{H} ds. \tag{4.34}$$

By (4.34) and (4.23), for any $n \in \mathbb{N}$, it holds that

$$|\mathbb{E}\langle P^t \eta_n, \tilde{\eta}_n \rangle_{H_{\lambda}}|$$

$$\leq |G|_{L_{\infty}^\infty(\mathcal{L}(H))} |x_1(T; t, \eta_n, 0, 0)|_{L_2^2(\mathcal{L}(H))} |x_2(T; t, \tilde{\eta}_n, 0, 0)|_{L_2^2(\mathcal{L}(H))}$$

$$+ \left| \langle (Q + \Theta^* R \Theta) \eta_n \rangle_{L_2^2(\mathcal{L}(H))} \right| |x_1(T; t, \eta_n, 0, 0)|_{L_2^2(\mathcal{L}(H))}$$

$$+ \left| \langle (Q + \Theta^* R \Theta) \tilde{\eta}_n \rangle_{L_2^2(\mathcal{L}(H))} \right| |x_2(T; t, \tilde{\eta}_n, 0, 0)|_{L_2^2(\mathcal{L}(H))}$$

$$\leq C \left( |G|_{L_{\infty}^\infty(\mathcal{L}(H))} + \left| \langle (Q + \Theta^* R \Theta) \eta_n \rangle_{L_2^2(\mathcal{L}(H))} \right| + \left| \langle (Q + \Theta^* R \Theta) \tilde{\eta}_n \rangle_{L_2^2(\mathcal{L}(H))} \right| \right)$$

$$\leq C \left( |G|_{L_{\infty}^\infty(\mathcal{L}(H))} + \left| \langle (Q + \Theta^* R \Theta) \eta_n \rangle_{L_2^2(\mathcal{L}(H))} \right| + \left| \langle (Q + \Theta^* R \Theta) \tilde{\eta}_n \rangle_{L_2^2(\mathcal{L}(H))} \right| \right) \tag{4.35}.$$
It follows from (4.40) and (4.41) that for any $\xi \in L^2_{F^t}(\Omega; H),$

\[
\begin{align*}
\mathbb{E}|P^\tau \xi - P^t \xi|^2_H \\
\leq C \left\{ \mathbb{E} \left[ \Phi(T, \tau)^*G\Phi(T, \tau)\xi - \int_\tau^T \Phi(r, \tau)^*(Q + \Theta^* R\Theta)\Phi(r, \tau)\xi dr \right] \right. \\
- \mathbb{E} \left[ \Phi(T, t)^*G\Phi(T, t)\xi - \int_t^T \Phi(r, t)^*(Q + \Theta^* R\Theta)\Phi(r, t)\xi dr \right] \bigg|_H^2 \\
+ \mathbb{E} \left[ \Phi(T, t)^*G\Phi(T, t)\xi - \int_t^T \Phi(r, t)^*(Q + \Theta^* R\Theta)\Phi(r, t)\xi dr \right] \bigg|_H^2 \right\}.
\end{align*}
\]

(4.42)

Since $F$ is the natural filtration of $W(\cdot)$, by the Martingale representation theorem, we know that any martingale on $(\Omega, F, P)$ is continuous. Thus,

\[
\lim_{\tau \to t^+} \mathbb{E} \left[ \Phi(T, \tau)^*G\Phi(T, \tau)\xi - \int_\tau^T \Phi(r, \tau)^*(Q + \Theta^* R\Theta)\Phi(r, \tau)\xi dr \right] \\
- \mathbb{E} \left[ \Phi(T, t)^*G\Phi(T, t)\xi - \int_t^T \Phi(r, t)^*(Q + \Theta^* R\Theta)\Phi(r, t)\xi dr \right] \bigg|_H^2 = 0.
\]

(4.43)

On the other hand,

\[
\begin{align*}
\mathbb{E} \left[ \Phi(T, \tau)^*G\Phi(T, \tau)\xi - \int_\tau^T \Phi(r, \tau)^*(Q + \Theta^* R\Theta)\Phi(r, \tau)\xi dr \right] \\
- \mathbb{E} \left[ \Phi(T, t)^*G\Phi(T, t)\xi - \int_t^T \Phi(r, t)^*(Q + \Theta^* R\Theta)\Phi(r, t)\xi dr \right] \bigg|_H^2 \\
\leq C \left\{ \mathbb{E} \left[ \Phi(T, \tau)^*G\Phi(T, \tau)\xi - \Phi(T, t)^*G\Phi(T, t)\xi \right]^2_H \right. \\
+ \mathbb{E} \left[ \int_\tau^T \left( \Phi(r, \tau)^*(Q + \Theta^* R\Theta)\Phi(r, \tau)\xi - \Phi(r, t)^*(Q + \Theta^* R\Theta)\Phi(r, t)\xi \right) dr \right]^2_H \\
+ \mathbb{E} \left. \left[ \int_t^T \Phi(r, \tau)^*(Q + \Theta^* R\Theta)\Phi(r, \tau)\xi dr \right]^2_H \right\}.
\end{align*}
\]

(4.44)

By Lemma 4.3, we know that the three terms in the right hand side of (4.44) tend to zero as $\tau \to t^+$. Consequently, we obtain that

\[
\lim_{\tau \to t^+} \mathbb{E} \left[ \Phi(T, \tau)^*G\Phi(T, \tau)\xi - \int_\tau^T \Phi(r, \tau)^*(Q + \Theta^* R\Theta)\Phi(r, \tau)\xi dr \right] \\
- \mathbb{E} \left[ \Phi(T, t)^*G\Phi(T, t)\xi - \int_t^T \Phi(r, t)^*(Q + \Theta^* R\Theta)\Phi(r, t)\xi dr \right] \bigg|_H^2 = 0.
\]

(4.45)

From (4.42), (4.43) and (4.45), we see that for any $\xi \in L^2_{F^t}(\Omega; H),$

\[P^\tau \xi \in C_{\mathcal{F}^t}(0, T; L^2(\Omega; H))\]

and

\[|P^\tau \xi|_{C_{\mathcal{F}^t}(0, T; L^2(\Omega; H))} \leq C \left( |Q + \Theta^* R\Theta|_{L^\infty(\Omega; L^1(0, T))} + |G|_{L^\infty(\Omega; L^1(0, T))} \right) |\xi|_{L^2_{F^t}(\Omega; H)}.\]

Hence, we get that $P^\tau \in C_{\mathcal{F}^t,w}(0, T; L^\infty(\Omega; L(H))).$

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Step 5. In this step, we prove that

\[ P(t, \omega) = P^t(\omega), \text{ a.e. } (t, \omega) \in [0, T] \times \Omega. \]  

(4.46)

To show this, for any \( 0 \leq t_1 < t_2 < T \), we choose \( \tilde{x}_1(t_1) = \tilde{x}_1 \in L^2_{F_{t_1}}(\Omega; H) \) and \( \tilde{u}_1 = \tilde{v}_1 = 0 \) in the equation (4.2), and \( \tilde{x}_2(t_1) = 0, \tilde{u}_2(\cdot) = \chi_{[t_2-t_1]} \tilde{\xi}_2 \) with \( \tilde{\xi}_2 \in L^2_{F_{t_1}}(\Omega; H) \) and \( \tilde{v}_2 = 0 \) in the equation (4.3). By (4.32) and recalling the definition of the evolution operator \( \Phi(\cdot, \cdot) \) (see Lemma 4.3), we see that

\[
\frac{1}{t_2 - t_1} \mathbb{E} \int_{t_1}^{t_2} \left\langle P(r)\Phi(r, t_1)\tilde{\xi}_1, \tilde{x}_2(\tau) \right\rangle_H d\tau
\]

\[
= \mathbb{E} \left( G\Phi(T, t_1)\tilde{x}_1, \tilde{x}_2(T; t_2, 0, \tilde{u}_2, 0) \right)_H - \mathbb{E} \int_{t_1}^{T} \left\langle (Q + \Theta^* R \Theta)\Phi(r, t_1)\tilde{x}_1, \tilde{x}_2(r; t_1, 0, \tilde{u}_2, 0) \right\rangle_H d\tau.
\]

(4.47)

It is clear that

\[
\tilde{x}_2(r; t_1, 0, \tilde{u}_2, 0) = \begin{cases}
\displaystyle \int_{t_1}^{r} e^{A^{(r-\tau)}} A_{\Theta}(\tau)\tilde{x}_2(\tau) d\tau + \int_{t_1}^{r} e^{A^{(r-\tau)}} C_{\Theta}(\tau)\tilde{x}_2(\tau) dW(\tau) \\
\phantom{=} + \frac{1}{t_2 - t_1} \int_{t_1}^{s} e^{A^{(s-\tau)}} \tilde{\xi}_2 d\tau, & r \in [t_1, t_2], \\
\Phi(r, t_2)\tilde{x}_2(t_2), & r \in [t_2, T].
\end{cases}
\]

(4.48)

Hence, we have

\[
\mathbb{E} \left| \tilde{x}_2(t_2) - \tilde{\xi}_2 \right|_H^4 \\
\leq C \left\{ \left( \int_{t_1}^{t_2} ||A_{\Theta}(\tau)||_{L^\infty(\Omega)} d\tau \right)^4 + \left( \int_{t_1}^{t_2} ||C_{\Theta}(\tau)||_{L^\infty(\Omega)} d\tau \right)^2 \right\} \mathbb{E} \sup_{s \in [t_1, t_2]} |\tilde{x}_2(s)|_H^4 \\
+ \mathbb{E} \left| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^{A^{(t_2-\tau)}} \tilde{\xi}_2 d\tau - \tilde{\xi}_2 \right|_H^4.
\]

This, together with (4.5), implies that

\[
\lim_{t_2 \to t_1 + 0} \mathbb{E} \left| \tilde{x}_2(t_2) - \tilde{\xi}_2 \right|_H^4 \\
\leq C \lim_{t_2 \to t_1 + 0} \mathbb{E} \left| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^{A^{(t_2-\tau)}} \tilde{\xi}_2 d\tau - \tilde{\xi}_2 \right|_H^4 = 0.
\]

Therefore, for any \( r \in [t_2, T] \), by Lemma 4.3 we get that

\[
\lim_{t_2 \to t_1 + 0} \mathbb{E} \left| \Phi(r, t_2)\tilde{x}_2(t_2) - \Phi(r, t_1)\tilde{\xi}_2 \right|_H^4
\]

\[
\leq C \lim_{t_2 \to t_1 + 0} \left( \mathbb{E} \left| \Phi(r, t_2)\tilde{x}_2(t_2) - \Phi(r, t_2)\tilde{\xi}_2 \right|_H^4 + \mathbb{E} \left| \Phi(r, t_2)\tilde{\xi}_2 - \Phi(r, t_1)\tilde{\xi}_2 \right|_H^4 \right)
\]

\[
\leq C \lim_{t_2 \to t_1 + 0} \left( \mathbb{E} \left| \tilde{x}_2(t_2) - \tilde{\xi}_2 \right|_H^4 + \mathbb{E} \left| \Phi(r, t_2)\tilde{\xi}_2 - \Phi(r, t_1)\tilde{\xi}_2 \right|_H^4 \right) = 0.
\]

Hence, we obtain that

\[
\lim_{t_2 \to t_1 + 0} \tilde{x}_2(s) = \Phi(s, t_1)\tilde{\xi}_2 \text{ in } L^4_{F_s}(\Omega; H), \quad \forall s \in [t_2, T].
\]

(4.49)

By (4.5) and (4.39), we conclude that

\[
\lim_{t_2 \to t_1 + 0} \left[ \mathbb{E} \left( G\Phi(T, t_1)\tilde{\xi}_1, \tilde{x}_2(T) \right)_H - \mathbb{E} \int_{t_1}^{T} \left\langle (Q + \Theta^* R \Theta)\Phi(s, t_1)\tilde{x}_1, \tilde{x}_2(s) \right\rangle_H ds \right]
\]

\[
= \mathbb{E} \left( G\Phi(T, t_1)\tilde{\xi}_1, \Phi(T, t_1)\tilde{\xi}_2 \right)_H - \mathbb{E} \int_{t_1}^{T} \left\langle (Q + \Theta^* R \Theta)\Phi(s, t_1)\tilde{x}_1, \Phi(s, t_1)\tilde{\xi}_2 \right\rangle_H ds.
\]

(4.50)
By choosing $\tilde{x}_1(t_1) = \tilde{\zeta}_1$ and $\tilde{u}_1 = \tilde{v}_1 = 0$ in (4.12), and $\tilde{x}_2(t_1) = \tilde{\zeta}_2$ and $\tilde{u}_2 = \tilde{v}_2 = 0$ in (4.13), from (4.32), we find that
\[
\mathbb{E}\langle P^{t_1}\tilde{x}_1, \tilde{x}_2 \rangle_H
= \mathbb{E}\langle G\Phi(T, t_1)\tilde{\xi}_1, \Phi(T, t_1)\tilde{\xi}_2 \rangle_H - \mathbb{E}\int_{t_1}^{T} \langle (Q + \Theta^*R\Theta)\Phi(s, t_1)\tilde{\xi}_1, \Phi(s, t_1)\tilde{\xi}_2 \rangle_H ds. \tag{4.51}
\]
Combining (4.47), (4.50) and (4.51), we obtain that
\[
\lim_{t_2 \to t_1 + 0} \frac{1}{t_2 - t_1} \mathbb{E}\int_{t_1}^{t_2} \langle P(s)\Phi(s, t_1)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H ds = \mathbb{E}\langle P^{t_1}\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H, \quad \text{a.e. } t_1 \in [0, T). \tag{4.52}
\]
By Lemma 4.5 we see that there is a monotonically decreasing sequence $\{t_2^{(k)}\}_{k=1}^{\infty}$ with $t_2^{(k)} > t_1$ for every $k \in \mathbb{N}$, such that
\[
\lim_{t_2^{(k)} \to t_1 + 0} \frac{1}{t_2^{(k)} - t_1} \mathbb{E}\int_{t_1}^{t_2^{(k)}} \langle P(s)\Phi(s, t_1)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H ds = \mathbb{E}\langle P^{t_1}\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H, \quad \text{for a.e. } t_1 \in [0, T). \tag{4.53}
\]
This, together with (4.52), implies that
\[
\mathbb{E}\langle P(t_1)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H = \mathbb{E}\langle P^{t_1}\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H, \quad \text{for a.e. } t_1 \in [0, T).
\]
Since $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are arbitrary elements in $L^4_{\mathcal{F}_t}(\Omega; H)$, we conclude (4.46). By modifying $P(\cdot, \cdot)$ on a null measure subset of $[0, T] \times \Omega$, we get that $P(\cdot, \cdot) = P^{t_1}(\cdot)$ for all $t \in [0, T]$. Hence, $P \in C_{F,w}([0, T]; L^{\infty}(\Omega; L(H)))$. Further, we get from (4.32) that
\[
\mathbb{E}\langle G\tilde{x}_1(T), \tilde{x}_2(T) \rangle_H + \mathbb{E}\int_{t}^{T} \langle [Q(r) + \Theta(r)^*R(r)\Theta(r)]\tilde{x}_1(r), \tilde{x}_2(r) \rangle_H dr
= \mathbb{E}\langle P(t)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_{H^*_X,H_X} + \mathbb{E}\int_{t}^{T} \langle P(r)\tilde{u}_1(r), \tilde{x}_2(r) \rangle_{H^*_X,H_X} dr
+ \mathbb{E}\int_{t}^{T} \langle P(r)\tilde{u}_2(r), H^*_X,H_X \rangle dr
+ \mathbb{E}\int_{t}^{T} \langle P(r)\tilde{v}_1(r), C\Theta(r)\tilde{x}_1(r), \tilde{v}_2(r) \rangle_{H^*_X,H_X} dr
+ \mathbb{E}\int_{t}^{T} \langle \tilde{v}_1(r), \Lambda^*\tilde{x}_2(r) \rangle_{H^*_X,H_X} dr + \mathbb{E}\int_{t}^{T} \langle \tilde{v}_2(r), \Lambda\tilde{x}_1(r) \rangle_{H^*_X,H_X} dr. \tag{4.54}
\]

**Step 6.** In this step, we complete the proof. For any given $\tilde{\xi}_1, \tilde{\xi}_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, let $\{\tilde{\xi}_1,k\}_{k=1}^{\infty}$, $\{\tilde{\xi}_2,k\}_{k=1}^{\infty} \subset L^4_{\mathcal{F}_t}(\Omega; H)$ such that for $j = 1, 2,$
\[
\lim_{k \to \infty} \tilde{\xi}_{j,k} = \tilde{\xi}_j \quad \text{in } L^4_{\mathcal{F}_t}(\Omega; H). \tag{4.55}
\]
For any given $\tilde{u}_1, \tilde{u}_2 \in L^4_{\mathcal{F}_t}(\Omega; L^2(0, T; H))$, let $\{\tilde{u}_1,k\}_{k=1}^{\infty}$, $\{\tilde{u}_2,k\}_{k=1}^{\infty} \subset L^4_{\mathcal{F}_t}(\Omega; L^2(0, T; H))$ such that for $j = 1, 2,$
\[
\lim_{k \to \infty} \tilde{u}_{j,k} = \tilde{u}_j \quad \text{in } L^4_{\mathcal{F}_t}(\Omega; L^2(0, T; H)). \tag{4.56}
\]
From \((4.53)\), we have that

\[
\mathbb{E}\langle G\tilde{x}_{1,k}(T), \tilde{x}_{2,k}(T) \rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta(r)^* R(r) \Theta(r)] \tilde{x}_{1,k}(r), \tilde{x}_{2,k}(r) \rangle_H dr
\]

\[
= \mathbb{E}\langle P(t)\tilde{\xi}_{1,k}, \tilde{\xi}_{2,k} \rangle_{H^\lambda} + \mathbb{E} \int_t^T \langle P(r)\tilde{u}_{1,k}(r), \tilde{x}_{2,k}(r) \rangle_{H^\lambda} dr
\]

\[
+ \mathbb{E} \int_t^T \langle P(r)\tilde{x}_{1,k}(r), \tilde{u}_{2,k}(r) \rangle_{H^\lambda} dr + \mathbb{E} \int_t^T \langle P(r)C_\Theta(r)\tilde{x}_{1,k}(r), \tilde{v}_2(r) \rangle_{H^\lambda} dr
\]

\[
+ \mathbb{E} \int_t^T \langle \tilde{v}_1(r), \Lambda^*(r)\tilde{x}_{2,k}(r) \rangle_{H^\lambda} dr + \mathbb{E} \int_t^T \langle \tilde{v}_2(r), \Lambda(r)\tilde{x}_{1,k}(r) \rangle_{H^\lambda} dr
\]

\[
(4.56)
\]

where \(\tilde{x}_{1,k}\) (resp. \(\tilde{x}_{2,k}\)) solves \((4.2)\) (resp. \((4.3)\)) with \(\tilde{\eta}_1\) (resp. \(\tilde{\eta}_2\)) and \(\tilde{u}_1\) (resp. \(\tilde{u}_2\)) replaced by \(\tilde{\eta}_{1,k}\) (resp. \(\tilde{\eta}_{2,k}\)) and \(\tilde{u}_{1,k}\) (resp. \(\tilde{u}_{2,k}\)).

Similar but simpler to the proof of \((4.24)\), for \(j = 1, 2\), we can get that

\[
\lim_{k \to \infty} \tilde{x}_{j,k} = \tilde{x}_j \quad \text{in} \quad L^2_{F_t}(\Omega; C([t, T]; H)).
\]

Noting that \(P \in C_{F,w}([0,T]; L^2(\Omega; L^2(H)))\), from \((4.54)\), \((4.55)\) and \((4.57)\), by taking \(k \to \infty\) in \((4.56)\), we obtain that

\[
\mathbb{E}\langle G\tilde{x}_1(T), \tilde{x}_2(T) \rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta(r)^* R(r) \Theta(r)] \tilde{x}_1(r), \tilde{x}_2(r) \rangle_H dr
\]

\[
= \mathbb{E}\langle P(t)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H + \mathbb{E} \int_t^T \langle P(r)\tilde{u}_1(r), \tilde{x}_2(r) \rangle_H dr
\]

\[
+ \mathbb{E} \int_t^T \langle P(r)\tilde{x}_1(r), \tilde{u}_2(r) \rangle_H dr + \mathbb{E} \int_t^T \langle P(r)C_\Theta(r)\tilde{x}_1(r), \tilde{v}_2(r) \rangle_H dr
\]

\[
+ \mathbb{E} \int_t^T \langle \tilde{v}_1(r), \Lambda^*(r)\tilde{x}_2(r) \rangle_{H^\lambda} dr + \mathbb{E} \int_t^T \langle \tilde{v}_2(r), \Lambda(r)\tilde{x}_1(r) \rangle_{H^\lambda} dr
\]

\[
(4.58)
\]

Assume that

\[
(P_1(\cdot), \Lambda_1(\cdot)), (P_2(\cdot), \Lambda_2(\cdot)) \in C_{F,w}([0,T]; L^\infty(\Omega; L^2(H))) \times L^2_T(0, T; \mathcal{L}_2(H)).
\]

satisfy \((4.58)\).

By choosing \(\tilde{u}_1 = \tilde{u}_2 = 0\) and \(\tilde{v}_1 = \tilde{v}_2 = 0\), from \((4.58)\), we have that for any \(t \in [0,T)\) and \(\tilde{\xi}_1, \tilde{\xi}_2 \in L^2_{F_t}(\Omega; H),\)

\[
\mathbb{E}\langle P_1(s)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H = \mathbb{E}\langle P_2(s)\tilde{\xi}_1, \tilde{\xi}_2 \rangle_H.
\]

\[
(4.59)
\]

This implies

\[
P_1(\cdot) = P_2(\cdot).
\]

\[
(4.60)
\]

Let \(\tilde{v}_2 = 0\) in \((4.59)\). By \((4.60)\), we see that for any \(\tilde{\xi}_1 \in L^4_{F_0}(\Omega; H), \tilde{u}_1 \in L^4_{F}(\Omega; L^2(0, T; H)),\)

\(\tilde{v}_1 \in L^4_{F}(\Omega; L^2(0, T; H^\lambda)),\)

\[
0 = \mathbb{E} \int_0^T \langle \tilde{v}_1(r), (\Lambda_1(r) - \Lambda_2(r)) \tilde{x}_2(r) \rangle_{H^\lambda} dr.
\]

\[
(4.61)
\]
Consequently,
\[(\Lambda_1 - \Lambda_2)\bar{x}_2 = 0 \quad \text{in} \quad L^4_{\bar{\nu}}(\Omega; L^2(0, T; H'))\].
This, together with Lemma 4.4, implies that
\[\Lambda_1 = \Lambda_2 \quad \text{in} \quad L^2_{\bar{\nu}}(H; H')\].

Hence, the desired uniqueness follows, which also implies the uniqueness of the $H_\lambda$-transposition solution to (4.1).

From (4.58), we see that if $(P, \Lambda)$ satisfies (4.58), then $(P^*, \Lambda^*)$ also satisfies (4.58). Hence, $(P, \Lambda) = (P^*, \Lambda^*)$. This concludes that $(P(\cdot), \Lambda(\cdot)) \in C_{F,w}([0, T]; L^\infty(\Omega; L^2(H))) \times L^2_{\bar{\nu}}(0, T; L^2(\Omega; S_2(H; H'))).

### 4.2 A novel characterization of optimal feedback operator via $H_\lambda$-transposition solution

The following result, which plays indispensable role in proving Theorem 3.1, reveals another characterization of the optimal feedback operator $\Theta$.

**Lemma 4.7** Let (AS0)-(AS4) hold and \(\Theta \in \Upsilon_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U})\), and $(\bar{P}, \bar{\Lambda})$ be the $H_\lambda$-transposition solution of (4.1) corresponding to $\Theta$. Let
\[
\bar{K} = R + D^*\bar{P}D, \quad \bar{L} = D^*\bar{P}C + B^*\bar{P} + D^*\bar{\Lambda}.
\]
Then $\Theta$ is an optimal feedback operator for Problem (SLQ) if and only if
\[
\bar{K}(t, \omega) \geq 0, \quad \text{for a.e. } (t, \omega) \in (0, T) \times \Omega,
\]
\[
\bar{L} \in \Upsilon_2(H; U) \cap \Upsilon_2(H_\lambda; \tilde{U}),
\]
and
\[
\bar{K}(t, \omega)\Theta(t, \omega) + \bar{L}(t, \omega) = 0 \quad \text{in} \quad L(H; U) \text{ for a.e. } (t, \omega) \in (0, T) \times \Omega.
\]

Before proving Lemma 4.7, we need another preliminary result. The proof is almost the same as the one for [12, Lemma 3.4], where a very similar result is proved when $H = \mathbb{R}^l$. Here we provide it for the convenience of readers.

**Lemma 4.8** Suppose $Y(\cdot) \in L^2_\bar{\nu}(0, T; H)$ is a given process satisfying that for a.e. $t \in [0, T]$, there is a sequence of decreasing positive numbers \(\{\varepsilon_n\}_{n=1}^\infty\) such that
\[
\lim_{\varepsilon_n \to 0} \int_t^{t+\varepsilon_n} \frac{E_t Y(r)}{\varepsilon_n} \, dr = 0 \quad \text{in} \quad H, \quad \mathbb{P}\text{-a.s.}
\]
Then we have $Y = 0$, for a.e. $(t, \omega) \in [0, T) \times \Omega$.

**Proof.** Since $L^2_\bar{\nu}(\Omega; H)$ is separable, it follows from the (deterministic) Lebesgue differentiation theorem that there is a countable dense subset $\mathcal{D} \subset L^2_\bar{\nu}(\Omega; H)$ of $L^2_\bar{\nu}(\Omega; H)$ and a full measure set $\Gamma_1 \subset [0, T)$ (depending on $Y(\cdot)$) such that for all $t \in \Gamma_1$, we have
\[
\lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} \frac{E\langle Y(r), \eta \rangle_H}{\varepsilon} \, dr = E\langle Y(t), \eta \rangle_H, \quad \forall \eta \in \mathcal{D},
\]
\[
\lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} \frac{\mathbb{E}[Y(r)]^2}{\varepsilon} dr = \mathbb{E}[Y(t)]^2. \tag{4.68}
\]

For any \( \eta \in D \), put
\[
\tilde{\eta}(r) \triangleq \mathbb{E}_r \eta \text{ for } r \in [0, T]. \tag{4.69}
\]

Since \( \eta \) is essentially bounded, so is \( \tilde{\eta}(\cdot) \). Further, since \( \tilde{\eta}(\cdot) \) is a martingale on the natural filtration of the Brownian motion \( \mathbb{W} \), it is continuous.

From (4.69), we have that
\[
\mathbb{E}\langle Y(r), \tilde{\eta} \rangle_H = \mathbb{E}\langle Y(r), \eta(r) \rangle_H \text{ for } r \in [0, T],
\]
and for all \( t \in \Gamma_1 \),
\[
\left| \lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} \frac{\mathbb{E}\langle Y(r), \tilde{\eta}(r) - \tilde{\eta}(t) \rangle_H}{\varepsilon} dr \right|
\leq \lim_{\varepsilon \to 0} \left[ \int_t^{t+\varepsilon} \frac{\mathbb{E}[\tilde{\eta}(r) - \tilde{\eta}(t)]^2}{\varepsilon} dr \right]^{\frac{1}{2}} \left[ \int_t^{t+\varepsilon} \frac{\mathbb{E}[\tilde{\eta}(r) - \tilde{\eta}(t)]^2}{\varepsilon} dr \right]^{\frac{1}{2}}
\leq 2 \lim_{\varepsilon \to 0} \left[ \int_t^{t+\varepsilon} \frac{\mathbb{E}[\tilde{\eta}(r)]^2}{\varepsilon} dr \right]^{\frac{1}{2}} \sup_{s \in [t, t+\varepsilon]} \mathbb{E}[\tilde{\eta}(r) - \tilde{\eta}(t)]^2_H
\leq 2 \mathbb{E}[\tilde{\eta}(t)]^2_H \lim_{\varepsilon \to 0} \sup_{s \in [t, t+\varepsilon]} \mathbb{E}[\tilde{\eta}(r) - \tilde{\eta}(t)]^2_H = 0,
\]
where the last inequality is due to the continuity of \( \tilde{\eta}(\cdot) \). Hence, for every \( \eta \in D \), we have
\[
\mathbb{E}\left[ \langle Y(t), \tilde{\eta}(t) \rangle_H \right] = \lim_{\varepsilon \to 0} \int_t^{t+\varepsilon} \frac{\mathbb{E}\langle Y(r), \tilde{\eta}(t) \rangle_H}{\varepsilon} dr
= \lim_{\varepsilon \to 0} \mathbb{E} \left\langle \int_t^{t+\varepsilon} \frac{\mathbb{E}Y(r)}{\varepsilon} dr, \tilde{\eta}(t) \right\rangle_H.
\]

For any \( t \in [0, T] \) such that \( t + \varepsilon \leq T \),
\[
\mathbb{E}\left[ \int_t^{t+\varepsilon} \frac{\mathbb{E}Y(r)}{\varepsilon} dr \right]^2_H \leq \left( \int_t^{t+\varepsilon} \frac{1}{\varepsilon} dr \right) \mathbb{E} \int_t^{t+\varepsilon} \frac{\mathbb{E}Y(r)^2}{\varepsilon} dr \leq \frac{1}{\varepsilon} \mathbb{E} \int_t^{t+\varepsilon} \mathbb{E}Y(r)^2_H dr. \tag{4.70}
\]

From (4.68) and (4.70), we know that for any \( t \in \Gamma_1 \), there exists a constant \( \delta(t) > 0 \) such that
\[
\mathbb{E}\left[ \int_t^{t+\varepsilon} \frac{\mathbb{E}Y(r)}{\varepsilon} dr \right]^2_H < \mathbb{E}[Y(t)]^2_H + 1, \quad \forall \varepsilon \in (0, \delta(t)).
\]

This implies that
\[
\mathcal{Y}(\varepsilon; t) \triangleq \int_t^{t+\varepsilon} \frac{\mathbb{E}Y(r)}{\varepsilon} dr
\]
is uniformly integrable in \( \varepsilon_n \in (0, \delta(t)) \). Hence, by (4.67), for a.e. \( t \in \Gamma_1 \), there is a sequence \( \{\varepsilon_n\}_{n=1}^\infty \) of decreasing positive numbers such that
\[
\lim_{\varepsilon_n \to 0} \mathbb{E}\left[ \int_t^{t+\varepsilon_n} \frac{\mathbb{E}Y(r)}{\varepsilon_n} dr \right]_H = \mathbb{E}\lim_{\varepsilon_n \to 0} \left| \int_t^{t+\varepsilon_n} \frac{\mathbb{E}Y(r)}{\varepsilon_n} dr \right|_H = \mathbb{E}\lim_{\varepsilon_n \to 0} \left| \int_t^{t+\varepsilon_n} \frac{\mathbb{E}Y(r)}{\varepsilon_n} dr \right|_H = 0.
\]
It follows from the essential boundedness of $\tilde{\eta}(\cdot)$ that there exists a constant $C > 0$ such that
\[
\lim_{\varepsilon_n \to 0} \left| \mathbb{E}\left( \int_t^{t+\varepsilon_n} \frac{E_t Y(r)}{\varepsilon_n} dr, \tilde{\eta}(t) \right)_H \right| \leq C \lim_{\varepsilon_n \to 0} \mathbb{E}\left( \int_t^{t+\varepsilon_n} \frac{E_t Y(r)}{\varepsilon_n} dr \right)_H = 0.
\]

Consequently, for any $\eta \in \mathcal{D}$,
\[
\mathbb{E}\langle Y(t), \eta \rangle_H = 0 \quad \text{for a.e. } t \in \Gamma_1.
\]

This concludes that
\[
Y(t) = 0 \quad \text{for a.e. } (t, \omega) \in [0, T] \times \Omega.
\]

**Proof of Lemma** [4.7] The “if” part. Let us divide the proof into several steps.

**Step 1.** For any $\tilde{v} \in \tilde{U}$ and given $t \in [0, T)$, let
\[
\xi \triangleq \mathbb{E}_t \int_t^{t+\varepsilon} \left[ K(r)\Theta(r) + L(r) \right] \tilde{v} dr \in L_{\mathbb{F}}^2(\Omega; H).
\]

Given $\varphi \in L_{\mathbb{F}}^2(0, T; \tilde{U})$, consider the following equation:
\[
\begin{align*}
&\{ 
&\quad dX = \left[ (A + A\Theta) X + B\varphi \right] dr + (C\Theta X + D\varphi) dW(r) \quad \text{in } (t, T], \\
&\quad X(t) = \xi.
\end{align*}
\]

From (AS4), we have that $D\varphi \in L_{\mathbb{F}}^2(0, T; H)$. Thanks to (AS0), the solution $X \in L_{\mathbb{F}}^2(\Omega; C([t, T]; H))$. By the definition of the $H_\lambda$-transposition solution to (4.11) and Lemma [4.11] we have
\[
\begin{align*}
&\mathbb{E}\langle GX(T), X(T) \rangle_H + \mathbb{E}\int_t^T \langle [Q(r) + \Theta(r)^* R(r)\Theta(r)] X(r), X(r) \rangle_H dr \\
= &\mathbb{E}\langle P(t)\xi, \xi_H \rangle_H + \mathbb{E}\int_t^T \langle P(r)B(r)\varphi(r), X(r) \rangle_H dr + \mathbb{E}\int_t^T \langle P(r)X(r), B(r)\varphi(r) \rangle_H dr \\
&+ \mathbb{E}\int_t^T \langle P(r)C\Theta(r)X(r), D(r)\varphi(r) \rangle_H dr + \mathbb{E}\int_t^T \langle P(r)D(r)\varphi(r), C\Theta(r)X(r) + D(r)\varphi(r) \rangle_H dr \\
&+ \mathbb{E}\int_t^T \langle D(r)\varphi(r), \bar{X}(r)X(r) \rangle_{H_\lambda, H_\lambda'} dr + \mathbb{E}\int_t^T \langle D(r)\varphi(r), \bar{X}(r)X(r) \rangle_{H_\lambda, H_\lambda'} dr \\
&= \mathbb{E}\langle P(t)\xi, \xi_H \rangle_H + \mathbb{E}\int_t^T \langle P(r)B(r)\varphi(r), X(r) \rangle_H dr + \mathbb{E}\int_t^T \langle P(r)X(r), B(r)\varphi(r) \rangle_H dr \\
&+ 2\mathbb{E}\int_t^T \langle P(r)C\Theta(r)X(r), D(r)\varphi(r) \rangle_H dr + \mathbb{E}\int_t^T \langle P(r)D(r)\varphi(r), D(r)\varphi(r) \rangle_H dr \\
&+ 2\mathbb{E}\int_t^T \langle D(r)\varphi(r), \bar{X}(r)X(r) \rangle_{H_\lambda, H_\lambda'} dr.
\end{align*}
\]

Denote by $\bar{x}(\cdot) \in L_{\mathbb{F}}^2(\Omega; C([t, T]; H))$ the solution to (4.72) with $x$ given by (4.71) and $\varphi = 0$. Let $\varepsilon > 0$ be such that $t + \varepsilon \leq T$. Write $x^{v, \varepsilon}(\cdot)$ for the solution to (4.72) with $x$ given by (4.71) and $\varphi(\cdot) = \chi_{[t, t+\varepsilon]}(\cdot)v$ for some $v \in \tilde{U}$. It follows from (4.72) and the choice of $\varphi$ that $x^{v, \varepsilon}(t) = \bar{x}(t)$. 27
Let \( t_\varepsilon \in [t, t + \varepsilon] \) be a stopping time to be fixed later. For \( r \in [t, t_\varepsilon] \),

\[
\mathbb{E} \sup_{r \in [t, t_\varepsilon]} |x^{v, \varepsilon}(r) - \mathbf{\overline{r}}(r)|_H^2
\]

\[
= \mathbb{E} \sup_{r \in [t, t_\varepsilon]} \left[ \int_t^r e^{A(\tau-t)} A_\Theta(\tau)(x^{v, \varepsilon}(\tau) - \mathbf{\overline{r}}(\tau)) d\tau + \int_t^r e^{A(\tau-t)} C_\Theta(\tau)(x^{v, \varepsilon}(\tau) - \mathbf{\overline{r}}(\tau)) dW(\tau) + \int_t^r e^{A(\tau-t)} B(\tau) \varphi(\tau) d\tau + \int_t^r e^{A(\tau-t)} D(\tau) \varphi(\tau) dW(\tau) \right]^2_H
\]

\[
\leq C \mathbb{E} \sup_{r \in [t, t_\varepsilon]} \left[ \int_t^r e^{A(\tau-t)} A_\Theta(\tau)(x^{v, \varepsilon}(\tau) - \mathbf{\overline{r}}(\tau)) d\tau + \int_t^r e^{A(\tau-t)} C_\Theta(\tau)(x^{v, \varepsilon}(\tau) - \mathbf{\overline{r}}(\tau)) dW(\tau) + \int_t^r e^{A(\tau-t)} B(\tau) \varphi(\tau) d\tau + \int_t^r e^{A(\tau-t)} D(\tau) \varphi(\tau) dW(\tau) \right]^2_H
\]

\[
\leq C \left( \int_t^{t_\varepsilon} |A_\Theta(\tau)|_{\mathcal{L}(H)} d\tau \right)^2 + \int_t^{t_\varepsilon} |C_\Theta(\tau)|_{\mathcal{L}(H)} d\tau + \mathbb{E} \sup_{r \in [t, t_\varepsilon]} |x^{v, \varepsilon}(r) - \mathbf{\overline{r}}(r)|_H^2 d\tau + C(t_\varepsilon - t),
\]

where the constant \( C \) is independent of the stopping time \( t_\varepsilon \). Let us choose \( t_\varepsilon \) as follows:

\[
t_\varepsilon = \begin{cases} 
\inf \left\{ r \in [t, t + \varepsilon] \mid \left( \int_t^r |A_\Theta(\tau)|_{\mathcal{L}(H)} d\tau \right)^2 + \int_t^r |C_\Theta(\tau)|_{\mathcal{L}(H)}^2 d\tau = \frac{1}{2C} \right\}, \\
\text{if } \left\{ r \in [t, t + \varepsilon] \mid \left( \int_t^r |A_\Theta(\tau)|_{\mathcal{L}(H)} d\tau \right)^2 + \int_t^r |C_\Theta(\tau)|_{\mathcal{L}(H)}^2 d\tau = \frac{1}{2C} \right\} \neq \emptyset, \\
t + \varepsilon, \quad \text{otherwise.}
\end{cases}
\]

From (4.74), we have that

\[
\mathbb{E} \sup_{r \in [t, t_\varepsilon]} |x^{v, \varepsilon}(r) - \mathbf{\overline{r}}(r)|_H^2 \leq C(t_\varepsilon - t), \quad \forall r \in [t, t_\varepsilon].
\]

If \( t_\varepsilon = \varepsilon \), then we get

\[
\mathbb{E} \sup_{r \in [t, \varepsilon]} |x^{v, \varepsilon}(r) - \mathbf{\overline{r}}(r)|_H^2 \leq C \varepsilon, \quad \forall r \in [t, t + \varepsilon]. \tag{4.75}
\]

Otherwise, noting that \( |A_\Theta|_{\mathcal{L}(H)} \in L^\infty(\Omega; L^1(0, T)) \) and \( |C_\Theta|_{\mathcal{L}(H)} \in L^\infty(\Omega; L^2(0, T)) \), we can repeat the above argument in finite steps to get (4.75).

**Step 2.** From (4.73), we have that

\[
2 \mathcal{J}(t, \xi; \Theta \mathbf{\overline{r}})
\]

\[
= \mathbb{E}(G \mathbf{\overline{r}}(T), \mathbf{\overline{r}}(T))_H + \mathbb{E} \int_t^T \left[ \langle Q(r) \mathbf{\overline{r}}(r), \mathbf{\overline{r}}(r) \rangle_H + \langle R(r) \Theta(r) \mathbf{\overline{r}}(r), \Theta(r) \mathbf{\overline{r}}(r) \rangle_H \right] dr
\]

\[
= \mathbb{E}(G \mathbf{\overline{r}}(T), \mathbf{\overline{r}}(T))_H + \mathbb{E} \int_t^T \langle Q(r) + \Theta(r)^* R(r) \Theta(r) \rangle \mathbf{\overline{r}}(r), \mathbf{\overline{r}}(r) \rangle_H dr
\]

\[
= \mathbb{E}(\mathbf{\overline{r}}(t) \xi, \xi)_H
\]

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Since $Θ$ is an optimal feedback operator for Problem (SLQ), for any $t \in [0, T)$, we have

$$2 \mathcal{J}(t, ξ; Θx^{v, ε} + φ) - 2 \mathcal{J}(t, ξ; Θx^{v, ε} + φ) = \mathbb{E} \int_t^{t+ε} \left[ \left\langle φ(r), K(r)φ(r) \right\rangle_U + 2\left[ K(r)Θ(r) + L(r) \right] x^{v, ε}(r), φ(r) \right] dr \geq 0.$$ (4.79)
Since $\varphi(\cdot)$ is a bounded process, we have that
\[
\begin{align*}
\left| E \int_t^{t+\varepsilon} \frac{\left( [K(r)\Theta(r) + \mathcal{L}(r)] \left[ x^{v,\varepsilon}(r) - \varpi(r) \right], \varphi(r) \right)_{\tilde{U}',\tilde{U}} \, dr}{\varepsilon} \right| \\
\leq C E \int_t^{t+\varepsilon} \frac{\left( [K(r)\Theta(r)]_{L(H;\tilde{U}')} + [\mathcal{L}(r)]_{L(H;\tilde{U}')} \right) |x^{v,\varepsilon}(r) - \varpi(r)|_H \, dr}{\varepsilon} \\
\leq \frac{C}{\varepsilon} \left[ \int_t^{t+\varepsilon} E\left( [K(r)\Theta(r)]_{L(H;\tilde{U}')} + [\mathcal{L}(r)]_{L(H;\tilde{U}')} \right)^2 dr \right]^{\frac{1}{2}} \left[ \int_t^{t+\varepsilon} E|x^{v,\varepsilon}(r) - \varpi(r)|_H^2 \, dr \right]^{\frac{1}{2}} \\
\leq C \left[ \int_t^{t+\varepsilon} E\left( [K(r)\Theta(r)]_{L(H;\tilde{U}')} + [\mathcal{L}(r)]_{L(H;\tilde{U}')} \right)^2 dr \right]^{\frac{1}{2}} \left[ \frac{1}{\varepsilon^2} \int_t^{t+\varepsilon} E|x^{v,\varepsilon}(r) - \varpi(r)|_H^2 \, dr \right]^{\frac{1}{2}} \\
\leq C \left[ \int_t^{t+\varepsilon} E\left( [K(r)\Theta(r)]_{L(H;\tilde{U}')} + [\mathcal{L}(r)]_{L(H;\tilde{U}')} \right)^2 dr \right]^{\frac{1}{2}} \left[ \frac{1}{\varepsilon} \sup \right. \left. \left\{ \int_{t\in[t,t+\varepsilon]} E|x^{v,\varepsilon}(r) - \varpi(r)|_H^2 \, dr \right\} \right]^{\frac{1}{2}}.
\end{align*}
\] 

This, together with (4.75), implies that
\[
\lim_{\varepsilon \to 0} \left| E \int_t^{t+\varepsilon} \frac{\left( [K(r)\Theta(r) + \mathcal{L}(r)] \left[ x^{v,\varepsilon}(r) - \varpi(r) \right], \varphi(r) \right)_{\tilde{U}',\tilde{U}} \, dr}{\varepsilon} \right| = 0. \tag{4.81}
\]

Therefore, we get that for any $v \in \tilde{U}$,
\[
f(t, v) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E \int_t^{t+\varepsilon} \langle v, K(r)v \rangle_{L} \, dr + \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} E \int_t^{t+\varepsilon} \langle [K(r)\Theta(r) + \mathcal{L}(r)] \xi, v \rangle_{\tilde{U}',\tilde{U}} \, dr \geq 0. \tag{4.82}
\]

By (4.82), we see that for all $n \in \mathbb{N}$ and $v \in \tilde{U}$,
\[
nf\left( t, \frac{v}{n} \right) \geq 0, \quad \text{for a.e. } t \in [0, T]. \tag{4.83}
\]

Letting $n \to \infty$ in (4.83), we have that
\[
\lim_{n \to \infty} \frac{1}{n} E \int_t^{t+\varepsilon} \left[ [K(r)\Theta(r) + \mathcal{L}(r)] \xi \right]_{\tilde{U}',\tilde{U}} \, dr \geq 0, \quad \forall v \in \tilde{U}. \tag{4.84}
\]

By the arbitrariness of $v \in \tilde{U}$, one has
\[
\lim_{\varepsilon \to 0} E \int_t^{t+\varepsilon} \left[ [K(r)\Theta(r) + \mathcal{L}(r)] \xi \right]_{\tilde{U}',\tilde{U}} \, dr = 0 \text{ in } \tilde{U}'. \tag{4.85}
\]

By the definition of $\xi$ (see (4.71)), we know that for any $\tilde{v} \in \tilde{U}$,
\[
\lim_{\varepsilon \to 0} E \left[ E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \, dr \left( E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \tilde{v} \, dr \right) \right] = 0.
\]

Consequently,
\[
\begin{align*}
\left\langle \lim_{\varepsilon \to 0} E \left[ E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \, dr \left( E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \tilde{v} \, dr \right) \right] \right\rangle_{\tilde{U}',\tilde{U}} \\
= \lim_{\varepsilon \to 0} E \left[ E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \, dr \left( E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \tilde{v} \, dr \right) \right]_{\tilde{U}',\tilde{U}} \\
= \lim_{\varepsilon \to 0} E \left[ E_t \int_t^{t+\varepsilon} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]}{\varepsilon} \, dr \right]^2 \bigg|_H = 0.
\end{align*}
\]
Hence there exists a sequence \( \{ \varepsilon_n \}_{n=1}^{\infty} \) of positive numbers (depending on \( \tilde{v} \) and \( t \)) such that
\[
\lim_{\varepsilon_n \to 0^+} \mathbb{E}_{t} \int_{t}^{t+\varepsilon_n} \frac{[K(r)\Theta(r) + \mathcal{L}(r)]^* \tilde{v}}{\varepsilon_n} dr = 0.
\]
This, together with Lemma 4.8, implies that
\[
[K(t)\Theta(t) + \mathcal{L}(t)]^* \tilde{v} = 0 \text{ for a.e. } t \in [0, T], \quad \mathbb{P}\text{-a.s.}
\]
By separability of \( \tilde{U} \), we can find a density countable subset \( \tilde{U}_1 \) of \( \tilde{U} \) and a full measurable subset \( \Gamma_2 \) of \([0, T]\) such that such that for all \( \tilde{v} \in \tilde{U}_1 \),
\[
[K(t)\Theta(t) + \mathcal{L}(t)]^* \tilde{v} = 0 \text{ for all } t \in \Gamma_2, \quad \mathbb{P}\text{-a.s.}
\]
This implies that for all \( \tilde{v} \in \tilde{U} \),
\[
[K(t)\Theta(t) + \mathcal{L}(t)]^* \tilde{v} = 0 \text{ for all } t \in \Gamma_2, \quad \mathbb{P}\text{-a.s.}
\]
Consequently,
\[
[K(t)\Theta(t) + \mathcal{L}(t)]^* = 0 \text{ for a.e. } t \in [0, T], \quad \mathbb{P}\text{-a.s.}
\]
Thus,
\[
K(t)\Theta(t) + \mathcal{L}(t) = 0 \text{ in } \mathcal{L}(H; \tilde{U}') \text{ for a.e. } t \in [0, T], \quad \mathbb{P}\text{-a.s.}
\]
This, together with (4.69), implies that
\[
D(t)^*K(t) = -D(t)^*\mathcal{P}(t)C(t) - B(t)^*\mathcal{P}(t) - K(t)\Theta(t). \tag{4.86}
\]
Noting that the right hand side of (4.86) belongs to \( \mathcal{Y}_2(H; U) \cap \mathcal{Y}_2(H_\lambda; \tilde{U}) \), we have
\[
D^*K \in \mathcal{Y}_2(H; U) \cap \mathcal{Y}_2(H_\lambda; \tilde{U}). \tag{4.87}
\]
Consequently, (4.65) and (4.66) hold.

**Step 3.** In this step, we prove (4.64).

For any \( t \in [0, T) \), \( \xi \in L^2_{\mathcal{F}_t}(\Omega; H) \) and \( u \in L^2_{\mathbb{P}}(t, T; U) \), consider the following equation:
\[
\begin{aligned}
  \begin{cases}
    dy = \left[ (A + A_1)y + Bu \right] dr + (Cy + Du) dW(r) \quad \text{in } (t, T], \\
    y(t) = \xi,
  \end{cases}
\end{aligned} \tag{4.88}
\]
which admits a unique solution \( y(\cdot) \in L^2_{\mathbb{P}}(\Omega; C([t, T]; H)) \) under (AS0) and (AS2). Let \( \varphi \overset{\Delta}{=} u - \Theta y \).
We can rewrite (4.88) as
\[
\begin{aligned}
  \begin{cases}
    dy = \left[ (A + A_\Theta)y + B\varphi \right] dr + [C_\Theta y + D\varphi] dW(r) \quad \text{in } (t, T], \\
    y(t) = \xi.
  \end{cases}
\end{aligned} \tag{4.89}
\]
It follows from (1.5) that
\[
\begin{aligned}
  2\mathcal{J}(t, \xi; u(\cdot)) &= \mathbb{E}\langle Gy(T), y(T) \rangle_H + \mathbb{E} \int_{t}^{T} \left[ \langle Q(r)y(r), y(r) \rangle_H + \langle R(r)u(r), u(r) \rangle_U \right] dr \\
  &= \mathbb{E} \langle Gy(T), y(T) \rangle_H \\
  &\quad + \mathbb{E} \int_{t}^{T} \left[ \langle Q(r)y(r), y(r) \rangle_H + \langle R(r)(\varphi(r) + \Theta y(r)), (\varphi(r) + \Theta y(r)) \rangle_H \right] dr \\
  &= \mathbb{E} \langle Gy(T), y(T) \rangle_H + \mathbb{E} \int_{t}^{T} \langle [Q(r) + \Theta y(r)]R(r)(\varphi(r) + \Theta y(r)), (\varphi(r) + \Theta y(r)) \rangle_H \rangle_H dr \\
  &\quad + \mathbb{E} \int_{t}^{T} \left[ \langle R(r)\varphi(r), \varphi(r) \rangle_U + 2\langle R(r)\Theta y(r), \varphi(r) \rangle_U \right] dr.
\end{aligned} \tag{4.90}
\]
Since \( \varphi \in L^2_F(0, T; U) \), and \( \tilde{U} \) is a dense subspace of \( U \), there exists a sequence of \( \{ \varphi_n \}_{n=1}^\infty \subset L^2_F(0, T; \tilde{U}) \) such that
\[
\lim_{n \to \infty} \varphi_n = \varphi \quad \text{in} \quad L^2_F(0, T; U). \tag{4.91}
\]
Let \( y_n \) be the solution of (4.89) corresponding with \( \varphi_n \). Similar to the proof of (4.24), one can show that
\[
\lim_{n \to \infty} y_n = y \quad \text{in} \quad L^2_F(\Omega; C([0, T]; H)).
\]
Hence,
\[
2J(t, \xi; u(\cdot)) = \lim_{n \to \infty} \left\{ \mathbb{E}\langle Gy_n(T), y_n(T) \rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta(r)^* R(r) \Theta(r)] y_n(r), y_n(r) \rangle_H \, dr \right. \tag{4.92}
\]
\[
\left. + \mathbb{E} \int_t^T \left( \langle R(r) \varphi_n(r), \varphi_n(r) \rangle_U + 2 \langle R(r) \Theta(r) y_n(r), \varphi_n(r) \rangle_U \right) \, dr \right\}.
\]
By (4.73) and (4.86), we have that
\[
\mathbb{E}\langle Gy_n(T), y_n(T) \rangle_H + \mathbb{E} \int_t^T \langle [Q(r) + \Theta(r)^* R(r) \Theta(r)] y_n(r), y_n(r) \rangle_H \, dr 
\]
\[
= \mathbb{E}\langle \mathcal{P}(t) \xi, \xi \rangle_H + \mathbb{E} \int_t^T \langle \mathcal{P}(r) B(r) \varphi_n(r), y_n(r) \rangle_H \, dr + \mathbb{E} \int_t^T \langle \mathcal{P}(r) y_n(r), B(r) \varphi_n(r) \rangle_H \, dr 
\]
\[
+ 2\mathbb{E} \int_t^T \langle \mathcal{P}(r) (C(r) + D(r) \Theta(r)) y_n(r), D(r) \varphi_n(r) \rangle_H \, dr 
\]
\[
+ \mathbb{E} \int_t^T \langle \mathcal{P}(r) D(r) \varphi_n(r), D(r) \varphi_n(r) \rangle_H \, dr + 2\mathbb{E} \int_t^T \langle D(r) \varphi_n(r), \mathcal{P}(r) y_n(r) \rangle_{H, H'} \, dr 
\]
\[
= \mathbb{E}\langle \mathcal{P}(t) \xi, \xi \rangle_H + \mathbb{E} \int_t^T \langle \mathcal{P}(r) D(r) \varphi_n(r), D(r) \varphi_n(r) \rangle_H \, dr - 2\mathbb{E} \int_t^T \langle R(r) \Theta(r) y_n(r), \varphi_n(r) \rangle_H \, dr.
\]
Combining (4.92) and (4.93), and recalling (4.91), we obtain that
\[
2J(t, \xi; u(\cdot)) = \lim_{n \to \infty} \left\{ \mathbb{E}\langle \mathcal{P}(t) \xi, \xi \rangle_H + \mathbb{E} \int_t^T \langle \mathcal{P}(r) D(r) \varphi_n(r), D(r) \varphi_n(r) \rangle_H \, dr + \mathbb{E} \int_t^T \langle R(r) \varphi_n(r), \varphi_n(r) \rangle_U \, dr \right\} 
\]
\[
= \mathbb{E}\langle \mathcal{P}(t) \xi, \xi \rangle_H + \mathbb{E} \int_t^T \langle \mathcal{K}(r) \varphi(r), \varphi(r) \rangle_U \, dr \tag{4.94}
\]
\[
= \mathbb{E}\langle \mathcal{P}(t) \xi, \xi \rangle_H + \mathbb{E} \int_t^T \langle \mathcal{K}(u(r) - \Theta(r)y(r)), u(r) - \Theta(r)y(r) \rangle_U \, dr.
\]
Noting that \( \Theta \) is an optimal feedback operator, we get that
\[
\frac{1}{2} \mathbb{E}\langle \mathcal{P}(t) \xi, \xi \rangle_H = J(t, \xi; \Theta(\cdot) \mathcal{P}(\cdot)) \leq J(t, \xi; u(\cdot)), \quad \forall u(\cdot) \in L^2_F(t, T; U). \tag{4.95}
\]
Combining (4.94) and (4.95), we get (4.64).

The “only if” part. Suppose \( \Theta \in \mathcal{Y}_2(H; U) \) is an operator satisfying (4.64) and (4.66). Following the exact same procedures as the above Step 3, we can show that \( \Theta \) is optimal.

□
Remark 4.1 We point out one essential difference from the Markovian study in [32, Theorem 3.3], [18, Lemma 2.8] due to the appearance of random coefficients. To this end, let us take a closer look at (4.85). In the Markovian setting, the expectation $\mathbb{E}$ can be omitted. Then by the separable property of Hilbert space $H$, $U$ or the Euclidean space, and the classical Lebesgue’s differential theorem, one can obtain the pointwise limit of $\{4.85\}$. This method does not work in our setting. To overcome this obstacle, we introduce a proper form of $\xi$ in (4.71), and suitable Lebesgue’s differential theorem (Lemma 4.8). To the best of our knowledge, these two technologies are even new in the counterpart finite dimensional situation.

4.3 Proof of Theorem 3.1

Based on the preliminary results in the previous subsections, we are ready to prove the main conclusion of the current paper, i.e. Theorem 3.1. To begin with, let us recall the following result.

Lemma 4.9 [21, Lemma 3.9] The set

$$\{ x_2(\cdot) \mid x_2(\cdot) \text{ solves } (3.2) \text{ with } t = 0, \xi_2 = 0, v_2 = 0 \text{ and } u_2 \in L^2_r(\Omega; L^2(0,T;H)) \}$$

is dense in $L^2_r(0,T;H)$.

Proof of Theorem 3.1: We first prove the “only if” part. The basic idea is to prove that the $H_\lambda$-transposition solution $(\overline{P}, \overline{A})$ obtained in Theorem 4.7 is also the unique transposition solution in terms of Definition 3.1 satisfying (3.7). We divide the proof of the “only if” part into two steps.

Step 1. In this step, we prove the existence of a transposition solution. From (AS2) and (4.64), we know that

$$\overline{K} > 0 \text{ for a.e. } (t, \omega) \in [0,T] \times \Omega.$$  \hfill (4.96)

By the same argument in Step 7 of [21, Theorem 2.2], we can show that the domain of $\overline{K}(t,\omega)^{-1}$ is dense in $U$ for a.e. $(t,\omega) \in [0,T] \times \Omega$ and $\overline{K}(t,\omega)$ is a closed operator for a.e. $(t,\omega) \in (0,T) \times \Omega$.

Let $\xi_1 \in L^4_{\overline{F}'}(\Omega; H_\lambda)$ (resp. $\xi_2 \in L^4_{\overline{F}'}(\Omega; H_\lambda)$) and $u_1, v_1 \in L^4_{\overline{F}'}(\Omega; L^2(t,T;H_\lambda))$ (resp. $u_2, v_2 \in L^4_{\overline{F}'}(\Omega; L^2(t,T;H_\lambda))$) in (3.3) (resp. (3.4)). Let $\tilde{\xi}_1 = \xi_1$ (resp. $\tilde{\xi}_2 = \xi_2$), $\tilde{u}_1 = u_1 - B\Theta \tilde{x}_1$ (resp. $\tilde{u}_2 = u_2 - B\Theta \tilde{x}_2$) and $\tilde{v}_1 = v_1 - D\Theta \tilde{x}_1$ (resp. $\tilde{v}_2 = v_2 - D\Theta \tilde{x}_2$) in (4.2) (resp. (4.3)). Then $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = x_2$, where $x_1$ and $x_2$ are solutions to (3.3) and (3.4), respectively. From (4.6), we have that

$$\mathbb{E}\langle gx_1(T), x_2(T) \rangle_H + \mathbb{E} \int_t^T \langle (Q(r) + \Theta(r)^* R(r) \Theta(r)) x_1(r), x_2(r) \rangle_H dr$$

$$= \mathbb{E}\langle \overline{P}(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle \overline{P}(r)(u_1(r) - B(r) \Theta(r) x_1(r)), x_2(r) \rangle_H dr$$

$$+ \mathbb{E} \int_t^T \langle \overline{P}(r)(v_1(r) - D(r) \Theta(r) x_1(r)), (C(r) + D(r) \Theta(r)) x_2(r) - D(r) \Theta(r) x_2(r) \rangle_H dr$$

$$+ \mathbb{E} \int_t^T \langle \overline{P}(r)(v_2(r) - D(r) \Theta(r) x_2(r)), (C(r) + D(r) \Theta(r)) x_1(r) + v_2(r) - D(r) \Theta(r) x_2(r) \rangle_H dr$$

$$+ \mathbb{E} \int_t^T \langle v_1(r) - D(r) \Theta(r) x_1(r), \overline{A}(r)(x_2(r) - x_2(r)) \rangle_{H_\lambda, H_\lambda} dr$$

$$+ \mathbb{E} \int_t^T \langle v_2(r) - D(r) \Theta(r) x_2(r), \overline{A}(r)(x_1(r) - x_1(r)) \rangle_{H_\lambda, H_\lambda} dr$$ \hfill (4.97)

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Noting
\[ \mathbb{E} \int_t^T \langle \Theta(r)x_1(r), (B(r)^* \mathbf{P}(r) + D(r)^* \mathbf{P}(r)C(r) + D(r)^* \mathbf{X}(r))x_2(r), \Theta(r)x_1(r) \rangle_H dr \]
and
\[ \mathbb{E} \int_t^T \langle \Theta(r)^* R(r) \Theta(r)x_1(r), x_2(r) \rangle_H dr + \mathbb{E} \int_t^T \langle \mathbf{P}(r)D(r) \Theta(r)x_1(r), D(r)\Theta(r)x_2(r) \rangle_H dr \]
we get from (4.97) that
\[ \mathbb{E}\langle Gx_1(T), x_2(T) \rangle_H + \mathbb{E} \int_t^T \langle Q(r)x_1(r), x_2(r) \rangle_H dr \]
\[ -\mathbb{E} \int_t^T \langle \mathbf{K}(r)^{-1} \mathbf{L}(r)x_1(r), \mathbf{L}(r)x_2(r) \rangle_H dr \]
\[ = \mathbb{E} \langle \mathbf{P}(t)\xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle \mathbf{P}(r)u_1(r), x_2(r) \rangle_H dr + \mathbb{E} \int_t^T \langle \mathbf{P}(r)x_1(r), u_2(r) \rangle_H dr \]
\[ + \mathbb{E} \int_t^T \langle \mathbf{P}(r)C(r)x_1(r), v_2(r) \rangle_H dr + \mathbb{E} \int_t^T \langle \mathbf{P}(r)v_1(r), C(r)x_2(r) + v_2(r) \rangle_H dr \]
\[ + \mathbb{E} \int_t^T \langle v_1(r), \mathbf{X}(r)x_2(r) \rangle_{H_\lambda,H_\lambda} dr + \mathbb{E} \int_t^T \langle \mathbf{X}(r)x_1(r), v_2(r) \rangle_{H_\lambda,H_\lambda} dr. \]

Next we look at the case that \( \xi_j \in L^4_{\mathcal{F}_T}(\Omega; H), u_j \in L^2_{\mathcal{F}}(\Omega; L^2(t, T; H)) \) and \( v_j \in L^4_{\mathcal{F}}(\Omega; L^2(t, T; H_\lambda)) \) \((j = 1, 2)\). Recall that \( H_\lambda \) is dense in \( H \), for \( j = 1, 2 \), we can find \( \{\xi_{j,n}\}_{n=1}^\infty \subset L^4_{\mathcal{F}_1}(\Omega; H_\lambda) \) and \( \{u_{j,n}\}_{n=1}^\infty \subset L^2_{\mathcal{F}}(\Omega; L^2(0, T; H_\lambda)) \) such that
\[ \lim_{n \to \infty} \|\xi_j - \xi_{j,n}\|_{L^4_{\mathcal{F}_1}(\Omega; H)} = 0, \]
\[ \lim_{n \to \infty} \|u_j - u_{j,n}\|_{L^4(\Omega; L^2(0, T; H_\lambda))} = 0. \]
Let \(x_{1,n}\) (resp. \(x_{2,n}\)) be the solution to (3.3), (resp. (3.4)) with \(\xi_1\) (resp. \(\xi_2\)) and \(u_1\) (resp. \(u_2\)) replaced by \(\xi_{1,n}\) (resp. \(\xi_{2,n}\)) and \(u_{1,n}\) (resp. \(u_{2,n}\)) and \(v_1\) (resp. \(v_2\)). Similar to the proof of (4.24), for \(j = 1, 2\), we can get that
\[
\lim_{n \to \infty} x_{j,n} = x_j \quad \text{in } L^2_{H}(\Omega; C(t, T; H)).
\]

Then similar to the proof of (4.53), we can get that (4.98) holds with \(\xi_j \in L^1_{\mathcal{J}_1}(\Omega; H), u_j \in L^4_{\mathcal{P}}(\Omega; L^2(t, T; H))\) and \(v_j \in L^4_{\mathcal{P}}(\Omega; L^2(t, T; H_\lambda))\) \((j = 1, 2)\). Hence, \((\mathcal{T}, \mathcal{X})\) is a transposition solution to (3.1).

**Step 2.** In this step, we prove the uniqueness of the transposition solution to (3.1).

Assume that
\[
(P_1(\cdot), \Lambda_1(\cdot)), (P_2(\cdot), \Lambda_2(\cdot)) \in C_{\mathcal{F},w}([0, T]; L^\infty(\Omega; \mathcal{L}(H))) \times L^2_{\mathcal{F},D,w}(0, T; \mathcal{L}(H))
\]
are two transposition solutions to (3.1).

From (4.95), we have that for any \(s \in [0, T]\) and \(\eta \in L^2_{\mathcal{F}_s}(\Omega; H),\)
\[
\frac{1}{2} \mathbb{E}(P_1(s)\eta, \eta)_H = \mathcal{J}(s, \eta; \Theta(\cdot)\eta(\cdot)) = \frac{1}{2} \mathbb{E}(P_2(s)\eta, \eta)_H. \tag{4.99}
\]
Thus, for any \(\xi, \eta \in L^2_{\mathcal{F}_s}(\Omega; H)\), we have that
\[
\mathbb{E}(P_1(s)(\eta + \xi), \eta + \xi)_H = \mathbb{E}(P_2(s)(\eta + \xi), \eta + \xi)_H,
\]
and
\[
\mathbb{E}(P_1(s)(\eta - \xi), \eta - \xi)_H = \mathbb{E}(P_2(s)(\eta - \xi), \eta - \xi)_H.
\]
These, together with \(P_1(\cdot) = P_1(\cdot)^*\) and \(P_2(\cdot) = P_2(\cdot)^*\), imply that
\[
\mathbb{E}(P_1(s)\eta, \xi)_H = \mathbb{E}(P_2(s)\eta, \xi)_H, \quad \forall \xi, \eta \in L^2_{\mathcal{F}_s}(\Omega; H). \tag{4.100}
\]
Hence,
\[
P_1(s) = P_2(s) \quad \text{for any } s \in [0, T], \mathbb{P}\text{-a.s.}
\]

Let \(v_2 = 0\) in (3.4). By (3.4) and noting
\[
\langle K(\cdot)^{-1}L(\cdot)x_1(\cdot), L(\cdot)x_2(\cdot) \rangle_H = -\langle \Theta(\cdot)x_1(\cdot), K(\cdot)\Theta(\cdot)x_2(\cdot) \rangle_H, \tag{H.9}
\]
we see that for any \(\xi_1 \in L^2_{\mathcal{F}_0}(\Omega; H), u_1 \in L^4_{\mathcal{P}}(\Omega; L^2(0, T; H))\) and \(v_1 \in L^4_{\mathcal{P}}(\Omega; L^2(0, T; H_\lambda))\),
\[
0 = \mathbb{E} \int_0^T \langle v_1(t), (\Lambda_1(t) - \Lambda_2(t))x_2(t) \rangle_{H_\lambda, H_\lambda'} dt.
\]
Consequently,
\[
(\Lambda_1 - \Lambda_2)x_2 = 0 \quad \text{in } L^4_{\mathcal{P}}(\Omega; L^2(0, T; H_\lambda')).
\]
This, together with Lemma 4.9 implies that
\[
\Lambda_1 = \Lambda_2 \quad \text{in } L^2_{\mathcal{P}}(0, T; \mathcal{L}(H; H_\lambda')).
\]
Hence, the desired uniqueness follows.
Next, we prove the “if” part. The proof is similar to **Step 3** in the proof for Lemma 4.7 (see also [21, Theorem 2.1]). For readers’ convenience, we give a sketch.

For any \( u \in L^2_F(0, T; U) \), since \( \bar{U} \) is a dense subspace of \( U \), there exists a sequence \( \{u_n\}_{n=1}^\infty \subset L^2_F(0, T; \bar{U}) \) such that
\[
\lim_{n \to \infty} u_n = u \quad \text{in } L^2_F(0, T; U). \tag{4.101}
\]
Let \( y_n \) be the solution to (4.88) corresponding with \( u_n \) and a \( \xi \in L^2_F(\Omega; H) \). Then, similar to the proof of (4.24), we can obtain that
\[
\lim_{n \to \infty} y_n = y \quad \text{in } L^2_F(\Omega; C([t, T]; H)). \tag{4.102}
\]
Hence
\[
2J(t, \xi; u(\cdot)) = \lim_{n \to \infty} E\langle Gy_n(T), y_n(T)\rangle_H + E\int_t^T \left[ \langle Q(r)y_n(r), y_n(r)\rangle_H + \langle R(r)u_n(r), u_n(r)\rangle_U \right] dr. \tag{4.103}
\]
By the definition of transposition solution in Definition 3.1, we have
\[
E\langle Gy_n(T), y_n(T)\rangle_H + E\int_t^T \left[ \langle Q(r)y_n(r), y_n(r)\rangle_H + \langle R(r)u_n(r), u_n(r)\rangle_U \right] dr
= E\langle P(t)\xi, \xi\rangle_H + 2E\int_t^T \langle u_n(r), L(r)y_n(r)\rangle_U dr + E\int_t^T \langle K(r)u_n(r), u_n(r)\rangle_U dr
+ E\int_t^T \langle K^{-1}(r)L(r)u_n(r), L(r)y_n(r)\rangle_H dr
= E\langle P(t)\xi, \xi\rangle_H + E\int_t^T \langle K(r)[u_n(r) + K^{-1}(r)L(r)y_n(r)], u_n(r) + K^{-1}(r)L(r)y_n(r)\rangle_U dr. \tag{4.104}
\]
From (4.101) and (4.102), we have that
\[
\lim_{n \to \infty} E\int_t^T \langle K(r)[u_n(r) + K^{-1}(r)L(r)y_n(r)], u_n(r) + K^{-1}(r)L(r)y_n(r)\rangle_U dr
= E\int_t^T \langle K(r)[u(r) + K^{-1}(r)L(r)y(r)], u(r) + K^{-1}(r)L(r)y(r)\rangle_U dr. \tag{4.105}
\]
Combining (4.103)–(4.105), we get that
\[
2J(t, \xi; u(\cdot)) = E\int_t^T \langle K(r)[u(r) + K^{-1}(r)L(r)y(r)], u(r) + K^{-1}(r)L(r)y(r)\rangle_U dr.
\]
This implies that
\[
2J(t, \xi; u(\cdot)) \geq 2J(t, \xi; \bar{u}(\cdot)), \quad \forall u \in L^2_F(t, T; U),
\]
with \( \bar{u} \equiv -K^{-1}Ly \). Hence \( -K^{-1}L \) is an optimal feedback operator.

At last, we prove the uniqueness of the optimal feedback operator. Suppose that \( \Theta' \) is another optimal feedback operator. According to the proof of “only if” part of Theorem 3.1, Riccati equation (3.1) admits a transposition solution, denoted by \( \left(P'(\cdot), \Lambda'(\cdot)\right) \). By the uniqueness of the Riccati equation (3.1), we have \( \left(P'(\cdot), \Lambda'(\cdot)\right) = \left(P(\cdot), \Lambda(\cdot)\right) \), which leads to \( \Theta = \Theta' \). This completes the proof of Theorem 3.1. \( \square \)
5 SLQs for stochastic parabolic equations

Stochastic parabolic equations are widely used to describe diffusion processes under the perturbations of random noises (e.g., [15]). In this section, we consider a linear quadratic control problem for such equation.

Let \( m \in \mathbb{N} \) and \( \mathcal{O} \subset \mathbb{R}^m \) be a bounded domain with a \( C^\infty \) boundary \( \partial \mathcal{O} \). Consider the following control system:

\[
\begin{aligned}
    dy - \Delta y dt &= (a_1 y + b_1 u) dt + (a_2 y + b_2 u) dW(t) \quad \text{in } \mathcal{O} \times (0, T), \\
    y &= 0 \quad \text{on } \partial \mathcal{O} \times (0, T), \\
    y(0) &= y_0 \quad \text{in } \mathcal{O},
\end{aligned}
\]

with the following cost functional

\[
J(0, y_0; u) \triangleq \mathbb{E} \int_0^T \int_\mathcal{O} (q|y|^2 + r|u|^2) dx dt + \mathbb{E} \int_\mathcal{O} g|y(T)|^2 dx.
\]

Here \( y_0 \in L^2(\mathcal{O}) \) and \( u \in L^2_F(0, T; L^2(\mathcal{O})) \) is the control variable. The conditions on the coefficients will be given below.

Our optimal control problem is as follows:

**Problem (sSLQ).** For each \( y_0 \in L^2(\mathcal{O}) \), find a \( \tilde{u}(\cdot) \in L^2_F(0, T; L^2(\mathcal{O})) \) such that

\[
J(0, y_0; \tilde{u}(\cdot)) = \inf_{u(\cdot) \in L^2_F(0, T; L^2(\mathcal{O}))} J(0, y_0; u(\cdot)).
\]

Problem (sSLQ) is a concrete example of Problem (SLQ) with the following setting:

- \( H = U = L^2(\mathcal{O}); \)
- The operator \( A \) is defined as follows:
  \[
  \begin{aligned}
  D(A) &= H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}), \\
  A\varphi &= \Delta \varphi, \quad \forall \varphi \in D(A);
  \end{aligned}
  \]
- \( A_1 y = a_1 y, \ B_u = b_1 u, \ C_y = a_2 y \) and \( Du = b_2 u; \)
- The operators \( Q, R \) and \( G \) are given by
  \[
  \begin{aligned}
  \langle Q(t)y, y \rangle_H &= \int_\mathcal{O} q(t)|y|^2 dx, \quad \langle R(t)u, u \rangle_H = \int_\mathcal{O} r(t)|u|^2 dx, \ \text{a.e.} \ t \in [0, T], \\
  \langle G y(t), y(T) \rangle_H &= \int_\mathcal{O} g|y(T)|^2 dx.
  \end{aligned}
  \]

From (5.3), we get (AS0). To guarantee that (AS1)–(AS4) hold, we assume the coefficients fulfill the following conditions:

\[
\begin{aligned}
  a_1 &\in L^\infty_F(0, T; L^\infty(\Omega; C^{2m}(\bar{\mathcal{O}}))), &\quad b_1 &\in L^\infty_F(0, T; L^2(\Omega; C^{2m}(\bar{\mathcal{O}}))), \\
  a_2, q &\in L^\infty_F(0, T; L^\infty(\Omega; C^{2m}(\bar{\mathcal{O}}))), &\quad b_2, r &\in L^\infty_F(0, T; C^{2m}(\bar{\mathcal{O}})), \\
  g &\in L^\infty_F(\Omega; C^{2m}(\bar{\mathcal{O}})), &\quad q &\geq 0, \quad r > 0, \quad g \geq 0 \ \text{for a.e.} \ (t, x, \omega) \in [0, T] \times \mathcal{O} \times \Omega.
\end{aligned}
\]
From (5.4), one can check that (AS1) holds.

Write \( \{\mu_j\}_{j=1}^{\infty} \) for the eigenvalues of \( A \) and \( \{\epsilon_j\}_{j=1}^{\infty} \) the corresponding eigenvectors satisfying \( |\epsilon_j|_{L^2(O)} = 1 \) for all \( j \in \mathbb{N} \). It is well known that \( \{\epsilon_j\}_{j=1}^{\infty} \) constitutes an orthonormal basis of \( L^2(O) \).

Hence, (AS2) holds.

Let \( \lambda_j = |\mu_j|^{-1} \) for \( j \in \mathbb{N} \) and \( \lambda = \{\lambda_j\}_{j=1}^{\infty} \). Then \( H'_\lambda \) is the completion of the Hilbert space \( L^2(O) \) with respect to the norm

\[
|f|_{H'_\lambda} = \sqrt{\sum_{j=1}^{\infty} |\epsilon_j|^{-m}(1 + |\mu_j|^{-2}) |f_j|^2}, \quad \text{for } f = \sum_{j=1}^{\infty} f_j \epsilon_j \in L^2(O).
\]

By the asymptotic distribution of eigenvalues of \( A \) (e.g., [24] Chapter 1, Theorem 1.2.1 and Remark 1.2.2), we have \( \mu_j = C_1 j^{\frac{2}{m}} + O(1) \) for some constant \( C_1 > 0 \). Therefore,

\[
|f|_{L^2(H;H'_\lambda)}^2 = \sum_{j=1}^{\infty} \langle \epsilon_j, \epsilon_j \rangle_{H'_\lambda} \leq 4 \sum_{j=1}^{\infty} |\mu_j|^{-n} < \infty.
\]

Hence, the embedding from \( L^2(O) \) to \( H'_\lambda \) is Hilbert-Schmidt. From the definition of \( H'_\lambda \), it follows that \( H_\lambda = D(A^\frac{m}{2}+1) \).

Let \( \alpha \in C^{2m}(\bar{O}) \). For any \( f \in H_\lambda \), one has

\[
|\alpha f|_{H_\lambda} = |\alpha f|_{D(A^\frac{m}{2}+1)} = \sqrt{|\alpha f|_{L^2(O)}^2 + |A^\frac{m}{2}+1(\alpha f)|_{L^2(O)}^2} \leq C|\alpha|_{C^{2m}(\bar{O})}|f|_{D(A^\frac{m}{2}+1)} = C|\alpha|_{C^{2m}(\bar{O})}|f|_{H_\lambda}.
\]

From this, and recalling (5.4), we conclude that \( A_1 \in L^2_{\bar{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H_\lambda))) \), \( C \in L^2_{\bar{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H_\lambda))) \), \( G \in L^\infty_{\bar{F}}(\Omega;\mathcal{L}(H_\lambda)) \) and \( Q \in L^\infty_{\bar{F}}(\Omega;L^2(O;\mathcal{L}(H_\lambda))) \). Similarly, we can prove that \( A_1 \in L^1_{\bar{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H'_\lambda))) \), \( C \in L^1_{\bar{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H'_\lambda))) \), \( G \in L^\infty_{\bar{F}}(\Omega;\mathcal{L}(H'_\lambda)) \) and \( Q \in L^\infty_{\bar{F}}(\Omega;L^2(O;\mathcal{L}(H'_\lambda))) \). Thus, (AS3) holds.

Let \( \tilde{U} = D(A^\frac{m}{2}+1) \). Then \( \tilde{U} \) is dense in \( L^2(O) \). Let \( \alpha \in C^{2m}(\bar{O}) \). For any \( f \in \tilde{U} \), one has

\[
|\alpha f|_{H_\lambda} = |\alpha f|_{D(A^\frac{m}{2}+1)} = |A^\frac{m}{2}+1(\alpha f)|_{L^2(O)} \leq |\alpha|_{C^{2m}(\bar{O})}|f|_{D(A^\frac{m}{2}+1)} = |\alpha|_{C^{2m}(\bar{O})}|f|_{\tilde{U}}.
\]

From this, and using (5.4), we find that \( B \in L^\infty_{\bar{F}}(\Omega;L^2(\tilde{U};\mathcal{L}(\tilde{U};H_\lambda))) \), \( D \in L^\infty_{\bar{F}}(0,T;\mathcal{L}(\tilde{U};H_\lambda)) \) and \( R \in L^\infty_{\bar{F}}(0,T;\mathcal{L}(\tilde{U};H_\lambda)) \). Similarly, we can prove that \( B \in L^\infty_{\bar{F}}(\Omega;L^2(O;\mathcal{L}(U';H'_\lambda))) \), \( D \in L^\infty_{\bar{F}}(0,T;\mathcal{L}(U';H'_\lambda)) \) and \( R \in L^\infty_{\bar{F}}(0,T;\mathcal{L}(U';H'_\lambda)) \). Therefore, (AS4) holds.

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