Conservation laws of some lattice equations

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Abstract

We derive infinitely many conservation laws for some multi-dimensionally consistent lattice equations from their Lax pairs. These lattice equations are the Nijhoff-Quispel-Capel equation, lattice Boussinesq equation, lattice nonlinear Schrödinger equation, modified lattice Boussinesq equation, Hietarinta’s Boussinesq-type equations, Schwarzian lattice Boussinesq equation and Toda-modified lattice Boussinesq equation.

Keywords: conservation laws, Lax pairs, multi-dimensionally consistent lattice equations, discrete integrable systems

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1 Introduction

In recent years the study of integrable partial difference equations has progressed rapidly. The property of multi-dimensional consistency acts as an important role in the research of discrete integrable systems. By this property as a criteria and through searching approaches many multi-dimensionally consistent lattice equations are found. For such equations one can easily write out their Bäcklund transformations and Lax pairs, which have been used to derive solutions and conservation laws.

Possessing infinitely many conservation laws is one of the important characters of integrable systems. For discrete integrable systems, many methods have been developed to find infinitely many conservation laws. Recently, we proposed an approach to derive infinitely many conservation laws for the Adler-Bobenko-Suris (ABS) lattice equations from their Lax pairs. In this paper we will apply the same method to some multi-component multi-dimensionally consistent lattice equations. We will first in the next section, taking the Nijhoff-Quispel-Capel (NQC) equation and discrete Boussinesq (DBS) equation as examples, describe our approach. Then in Sec.3 we derive conservation laws for lattice nonlinear Schrödinger equation, modified lattice Boussinesq equation, Hietarinta’s Boussinesq-type equations, Schwarzian lattice Boussinesq equation and Toda-modified lattice Boussinesq equation. We use Lax pairs collected in Ref.
2 Conservation laws of the NQC equation and DBSQ equation

Let us take the NQC equation and DBSQ equation as examples to review the approach that we used in [16] for deriving conservation laws. Conservation laws of these two equations have also been considered in [11] and [15] through direct approach and symmetry approach.

2.1 The NQC equation

Consider a quadrilateral lattice equation

\[ Q(u, \tilde{u}, \hat{u}, p, q) = 0, \quad (2.1) \]

where

\[ u = u(n, m), \quad \tilde{u} = E_n u = u(n + 1, m), \quad \hat{u} = E_m u = u(n, m + 1), \quad \tilde{u} = u(n + 1, m + 1), \]

\( E_n \) and \( E_m \) respectively serve as shift operators in direction \( n \) and \( m \), \( p \) and \( q \) are spacing parameters of direction \( n \) and \( m \), respectively. A conservation law of equation (2.1) is defined by

\[ \Delta_m F(u) = \Delta_n J(u), \quad (2.2) \]

where \( \Delta_m = E_m - 1, \Delta_n = E_n - 1 \), and \( u \) is a generic solution to (2.1).

The NQC equation is \[ 18 \parallel 19 \]

\[ [(p - \alpha)u - (p + \beta)\tilde{u}] [(p - \beta)\tilde{u} - (p + \alpha)\hat{u}] - [(q - \alpha)u - (q + \beta)\hat{u}] [(q - \beta)\hat{u} - (q + \alpha)\tilde{u}] = 0, \quad (2.3) \]

where \( \alpha, \beta \) are constants, and its Lax pair reads (cf. [17])

\[ \begin{align*}
\tilde{\phi} &= \gamma_1 \left( \begin{array}{c}
(p - \alpha)(p - \beta)u - (p^2 - r^2)\tilde{u} \\
(r + \alpha)(r + \beta)
\end{array} \right) \phi,
\end{align*} \quad (2.4a)\]

\[ \begin{align*}
\hat{\phi} &= \gamma_2 \left( \begin{array}{c}
(q - \alpha)(q - \beta)u - (q^2 - r^2)\hat{u} \\
(r + \alpha)(r + \beta)
\end{array} \right) \phi,
\end{align*} \quad (2.4b)\]

where \( \phi = (\phi_1, \phi_2)^T \), \( \gamma_1 \) is either

\[ \gamma_1 = \frac{1}{\sqrt{[(\beta - p)u + (\alpha + p)\tilde{u}] [(\alpha - p)u + (\beta + p)\hat{u}]}} \quad \text{or} \quad \gamma_1 = \frac{1}{(\alpha - p)u + (\beta + p)\tilde{u}} \quad \text{or} \quad \gamma_1 = \frac{1}{(\beta - p)u + (\alpha + p)\hat{u}}, \quad (2.5a)\]

and \( \gamma_2 \) follows from the above \( \gamma_1 \)'s by replacing \( (p, \tilde{u}) \) by \( (q, \hat{u}) \).

Eliminating \( \phi_1 \) from (2.4a) one finds

\[ A\tilde{\phi}_2 + B\hat{\phi}_2 + \varepsilon C\phi_2 = 0, \quad (2.6a)\]

where \( \varepsilon = p^2 - r^2 \),

\[ A = \frac{1}{\gamma_1}, \quad B = (p + \alpha)(p + \beta)\tilde{u} - (p - \alpha)(p - \beta)u, \quad (2.6b)\]

\[ C = \gamma_1 [(\alpha^2 + \beta^2 - 2p^2)u\tilde{u} + (p - \alpha)(p - \beta)u^2 + (p + \alpha)(p + \beta)\tilde{u}^2]. \quad (2.6c)\]
yields a discrete Riccati equation

\[ A\tilde{\theta} + B\theta + \varepsilon C = 0, \quad (2.7) \]

with \( \theta = \tilde{\phi}_2/\phi_2 \), which is then solved by

\[ \theta = \rho \varepsilon (1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^j), \quad (2.8a) \]

with

\[ \rho = -\frac{C}{B}, \quad \theta_{j+1} = -\frac{A\tilde{\rho}}{B} \sum_{i=0}^{j} \tilde{\theta}_i \theta_{j-i}, \quad (\theta_0 = 1), \quad j = 0, 1, 2, \ldots \quad (2.8b) \]

Next, going back to the Lax pair (2.4) we can easily find

\[ \theta = \frac{\phi_2}{\phi_2} = \gamma_1 [(r + \alpha)(r + \beta)\frac{\phi_1}{\phi_2} - (p + \alpha)(p + \beta)\tilde{u} + (p^2 - r^2)u], \quad (2.9a) \]
\[ \eta = \frac{\phi_2}{\phi_2} = \gamma_2 [(r + \alpha)(r + \beta)\frac{\phi_1}{\phi_2} - (q + \alpha)(q + \beta)\tilde{u} + (q^2 - r^2)u], \quad (2.9b) \]

from which eliminating \( \phi_1/\phi_2 \) we reach to the relation

\[ \eta = \omega (1 + \sigma \theta), \quad (2.10a) \]

with

\[ \omega = \gamma_2 [(p + \alpha)(p + \beta)\tilde{u} - (q + \alpha)(q + \beta)\tilde{u} + (q^2 - p^2)u], \quad (2.10b) \]
\[ \sigma = \frac{1}{\gamma_1 [(p + \alpha)(p + \beta)\tilde{u} - (q + \alpha)(q + \beta)\tilde{u} + (q^2 - p^2)u]}. \quad (2.10c) \]

Meanwhile, due to \( \theta = \tilde{\phi}_2/\phi_2, \quad \eta = \tilde{\phi}_2/\phi_2 \), we get

\[ \Delta_m \ln \theta = \Delta_n \ln \eta, \quad (2.11) \]

which provides a formal conservation law for the NQC equation. Finally, what we need is to insert the explicit form (2.8) of \( \theta \) into (2.11) and then expand it in terms of \( \varepsilon \). The coefficient of each power of \( \varepsilon \) provides a conservation law for the NQC equation, which is expressed through (cf. [16])

\[ \Delta_m \ln \rho = \Delta_n \ln \omega, \quad (2.12a) \]
\[ \Delta_m h_s(\theta) = \Delta_n h_s(\sigma \rho \theta), \quad (s = 1, 2, 3, \cdots), \quad (2.12b) \]

where

\[ \theta = (\theta_1, \theta_2, \cdots), \quad \underline{\theta} = (1, \theta_1, \theta_2, \cdots), \quad \text{and} \quad \sigma \rho \underline{\theta} = (\sigma \rho, \sigma \rho \theta_1, \sigma \rho \theta_2, \cdots), \quad (2.12c) \]

with \( \rho, \omega, \sigma \) and \( \{\theta_j\} \) given by (2.8b), (2.10b), (2.10c) and (2.8b). \( \{h_s(t)\} \) are polynomials defined as the following [16].
Proposition 1. The following expansion holds,

$$\ln \left(1 + \sum_{i=1}^{\infty} t_i k^i\right) = \sum_{j=1}^{\infty} h_j(t) k^j, \quad (2.13a)$$

where

$$h_j(t) = \sum_{|\alpha| = j} (-1)^{|\alpha| - 1}(|\alpha| - 1)! \frac{t^\alpha}{\alpha!}, \quad (2.13b)$$

and

$$t = (t_1, t_2, \cdots), \quad \alpha = (\alpha_1, \alpha_2, \cdots), \quad \alpha_i \in \{0, 1, 2, \cdots\}, \quad (2.13c)$$

$$t^\alpha = \prod_{i=1}^{\infty} t_i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^{\infty} (\alpha_i!), \quad |\alpha| = \sum_{i=1}^{\infty} \alpha_i, \quad ||\alpha|| = \sum_{i=1}^{\infty} i\alpha_i. \quad (2.13d)$$

The first few of \{h_j(t)\} are

$$h_1(t) = t_1, \quad (2.14a)$$

$$h_2(t) = -\frac{1}{2} t_1^2 + t_2, \quad (2.14b)$$

$$h_3(t) = \frac{1}{3} t_1^3 - t_1 t_2 + t_3, \quad (2.14c)$$

$$h_4(t) = -\frac{1}{4} t_1^4 + t_1^2 t_2 - t_1 t_3 - \frac{1}{2} t_2^2 + t_4. \quad (2.14d)$$

2.2 The DBSQ equation

Now let us look at the DBSQ equation [20]

$$\tilde{z} - x \tilde{x} + y = 0, \quad \tilde{z} - x \tilde{x} + y = 0, \quad (\tilde{x} - x) (z - x \tilde{x} + \tilde{y}) - p + q = 0. \quad (2.15)$$

Its Lax pair reads

$$\tilde{\phi} = \begin{pmatrix} -\tilde{x} & 1 & 0 \\ -\tilde{y} & 0 & 1 \\ p - r - x \tilde{y} + \tilde{x}z & -z & x \end{pmatrix} \phi, \quad (2.16a)$$

$$\hat{\phi} = \begin{pmatrix} -\hat{x} & 1 & 0 \\ -\hat{y} & 0 & 1 \\ q - r - x \hat{y} + \hat{x}z & -z & x \end{pmatrix} \phi, \quad (2.16b)$$

where \(\phi = (\phi_1, \phi_2, \phi_3)^T\). From (2.16a) we can eliminate \(\phi_2, \phi_3\) and get

$$\tilde{\phi}_1 + (\tilde{x} - x) \tilde{\phi}_1 + (\tilde{y} + z - x \tilde{x}) \tilde{\phi}_1 + \varepsilon \phi_1 = 0,$$

where \(\varepsilon = r - p\), and then a discrete Riccati equation

$$\tilde{\theta} \theta + (\tilde{x} - x) \tilde{\theta} + (\tilde{y} + z - x \tilde{x}) \theta + \varepsilon = 0, \quad (2.17)$$
with \( \theta = \tilde{\phi}_1 / \phi_1 \). This is a third-order equation and solved by

\[
\theta = \rho \varepsilon \left( 1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^j \right),
\]

(2.18a)

with

\[
\rho = \frac{1}{\bar{y} + z - \bar{x}},
\]

(2.18b)

\[
\theta_1 = -\frac{\tilde{\rho}(\bar{x} - x)}{\bar{y} + z - \bar{x}},
\]

(2.18c)

\[
\theta_{j+2} = -\frac{\tilde{\rho}}{\bar{y} + z - \bar{x}} \left[ \tilde{\rho} \sum_{i=0}^{j} \tilde{\theta}_i \tilde{\theta}_{j-i} + (\bar{x} - x) \sum_{i=0}^{j+1} \tilde{\theta}_i \tilde{\theta}_{j+1-i} \right], \quad (\theta_0 = 1),
\]

(2.18d)

for \( j = 0, 1, 2, \ldots \). Meanwhile, from the Lax pair (2.16) we have

\[
\theta = \frac{\tilde{\phi}_1}{\phi_1} = -\bar{x} + \frac{\phi_2}{\phi_1},
\]

\[
\eta = \frac{\hat{\phi}_1}{\phi_1} = -\bar{x} + \frac{\phi_2}{\phi_1},
\]

which yields

\[
\eta = \omega (1 + \sigma \theta),
\]

(2.19a)

with

\[
\omega = \bar{x} - \bar{x},
\]

(2.19b)

\[
\sigma = \frac{1}{x - \bar{x}}.
\]

(2.19c)

Next, from the formal conservation law \( \Delta_m \ln \theta = \Delta_n \ln \eta \), we get infinitely many conservation laws

\[
\Delta_m \ln \rho = \Delta_n \ln \omega,
\]

(2.20a)

\[
\Delta_m h_s(\theta) = \Delta_n h_s(\sigma \rho \theta), \quad (s = 1, 2, 3, \ldots),
\]

(2.20b)

where

\[
\theta = (\theta_1, \theta_2, \ldots), \quad \underline{\theta} = (1, \theta_1, \theta_2, \ldots), \quad \text{and} \quad \sigma \rho \underline{\theta} = (\rho, \sigma \rho \theta_1, \sigma \rho \theta_2, \ldots),
\]

(2.20c)

with \( \rho, \omega, \sigma \) and \( \{\theta_j\} \) given by (2.18b), (2.19b), (2.19c), (2.18c) and (2.18d). \( \{h_s(t)\} \) are polynomials defined in Proposition 1.
3 Conservation laws of some multi-component lattice equations

3.1 Generic description

We first list all multi-component lattice equations involved in this part.

\[
\begin{align*}
\bar{y} - 
\hat{\phi} = N_1 \phi, \\
\hat{\phi} = N_2 \phi,
\end{align*}
\]

where \(N_1\) and \(N_2\) are \(N \times N\) matrices and \(\phi = (\phi_1, \phi_2, \ldots, \phi_N)^T\). There is some certain \(\phi_{i0}\) such that one can from (3.1a) eliminate other \(\phi_j\)'s and get a scalar form spectral problem in
terms of $\phi_{i_0}$, say, the following
\[ A \tilde{\phi}_{i_0} + B \tilde{\phi}_{i_0} + (\varepsilon C + D)\tilde{\phi}_{i_0} + \varepsilon G\phi_{i_0} = 0, \]
(3.2)
where $A, B, C, D, G$ are functions of $(E^j_n u, p)$, and $\varepsilon$ is a constant related to $p$ and $r$. From this we reach to a discrete Riccati equation
\[ A\tilde{\theta}\tilde{\theta} + B\tilde{\theta} + (\varepsilon C + D)\theta + \varepsilon G = 0, \]
(3.3)
with
\[ \theta = \frac{\tilde{\phi}_{i_0}}{\phi_{i_0}} \]
(3.4)
As for solutions to (3.3) we have

**Proposition 2.** The discrete Riccati equation (3.3) is solved by
\[ \theta = \rho \varepsilon \left( 1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^j \right), \]
(3.5a)
with
\[ \rho = -\frac{G}{D}, \]
(3.5b)
\[ \theta_1 = -\frac{1}{D}(B\rho + C), \]
(3.5c)
\[ \theta_{j+2} = -\frac{1}{D} \left( A\rho \sum_{i=0}^{j} \sum_{k=0}^{j-i} \tilde{\theta}_i \tilde{\theta}_k \theta_{j-i-k} + B\rho \sum_{i=0}^{j+1} \tilde{\theta}_i \theta_{j+1-i} + C\theta_{j+1} \right), \quad (\theta_0 = 1), \]
(3.5d)
for $j = 0, 1, 2, \cdots$.

Next, the following relation is also available (recalling (2.19a)),
\[ \eta = \frac{\tilde{\phi}_{i_0}}{\phi_{i_0}} = \omega(1 + \sigma \theta), \]
(3.6)
where $\omega$ and $\sigma$ are functions of $(u, \tilde{u}, \hat{u}, p, q)$ related to considered equations and they satisfy
\[ \omega(u, \tilde{u}, \hat{u}, p, q) = -\frac{1}{\sigma(u, \tilde{u}, \hat{u}, q, p)}. \]
(3.7)
Then, the infinitely many conservation laws can be described as following (cf. [16]).

**Proposition 3.** From the formal conservation law
\[ \Delta_m \ln \theta = \Delta_n \ln \eta, \]
(3.8)
one has
\[ \Delta_m \ln \rho = \Delta_n \ln \omega, \]
(3.9a)
\[ \Delta_m h_s(\theta) = \Delta_n h_s(\sigma \rho \theta), \quad (s = 1, 2, 3, \cdots), \]
(3.9b)
where
\[ \theta = (\theta_1, \theta_2, \cdots), \quad \theta = (1, \theta_1, \theta_2, \cdots), \] (3.9c)
with \( \rho, \{\theta_i\}, \omega \) and \( \sigma \) given by \([3.5]\) and \([3.6]\) and \( h_n(t) \) defined in Proposition 7. The first few conservation laws are
\[ \Delta_m \ln \left( -\frac{G}{D} \right) = \Delta_n \ln \omega, \] (3.10a)
\[ \Delta_m \frac{C\tilde{D} - B\tilde{G}}{DD} = \Delta_n \frac{G\sigma}{D}, \] (3.10b)
\[ \Delta_m \left[ \frac{(B\tilde{G} - C\tilde{D})^2}{2D^2D^2} + \frac{B\tilde{G}(B\tilde{G} - C\tilde{D})}{DD\tilde{D}} - \frac{A\tilde{G}\tilde{G}}{DD\tilde{D}} \right] = \Delta_n \frac{G\sigma}{2D^2D} \left[ 2(C\tilde{D} - B\tilde{G}) - \tilde{D} G\sigma \right]. \] (3.10c)

3.2 Main results

We find each multi-component lattice system we list out in our paper falls in the above frame and therefore they can share those formulae of conservation laws with concrete \( \{A, B, C, D, G, \omega, \sigma\} \) where in some cases \( A \) can also be scaled to 1. In the following we skip details and list out \( A, B, C, D, G, \omega \) and \( \sigma \) for each equation.

**Proposition 4.** For lNLS equation, \( i_0 = 1 \),

\[ A = 0, \quad B = \frac{1}{\tilde{x}}, \quad C = \frac{1}{\tilde{x}}, \quad D = \frac{1 + \tilde{y}}{\tilde{x}}, \quad G = \frac{1}{\tilde{x}}, \quad \omega = \frac{\hat{x} - \tilde{x}}{\tilde{x}}, \quad \sigma = \frac{\hat{x} - \tilde{x}}{\hat{x} - \tilde{x}}. \] (3.11a)

For mDBSQ equation, \( i_0 = 3 \),

\[ A = \frac{1}{\tilde{y}1\tilde{y}1\tilde{y}}, \quad B = -\frac{\tilde{p}[\tilde{y}(\tilde{xy} + \tilde{xy}) + \tilde{xy}]}{\tilde{y}1\tilde{x}\tilde{xy}y}, \quad C = 0, \quad G = \gamma_1, \] (3.12a)
\[ \omega = \frac{\gamma_2(y(q\tilde{xy} - p\tilde{xy}))}{x\tilde{y}}, \quad \sigma = \frac{\gamma_1y(q\tilde{xy} - p\tilde{xy})}{\gamma_1y(q\tilde{xy} - p\tilde{xy})}. \] (3.12b)

For (C-2.1) equation, \( i_0 = 3 \),

\[ A = \frac{1}{\tilde{y}1\tilde{y}1\tilde{y}}, \quad B = \frac{1}{\tilde{y}1\tilde{y}1\tilde{y}}[\tilde{z} + z\tilde{p} + \tilde{p} + \tilde{z}], \quad C = 1, \] (3.13a)
\[ \omega = \gamma_1(z - \tilde{z}), \quad \sigma = \frac{1}{\gamma_1(z - \tilde{z})}. \] (3.13b)

For (C-2.2) equation, \( i_0 = 3 \),

\[ A = \frac{\tilde{x}}{\tilde{y}1\tilde{y}1\tilde{y}}, \quad B = \frac{1}{\tilde{y}1\tilde{y}1\tilde{y}}[\tilde{z} + \tilde{z} + \tilde{p} + \tilde{z}], \quad C = \tilde{x}, \] (3.14a)
\[ \omega = \gamma_2(\tilde{z} - \tilde{z}), \quad \sigma = \frac{1}{\gamma_1(\tilde{z} - \tilde{z})}. \] (3.14b)
For (A-2) equation, $i_0 = 3$,

\[
A = \frac{\bar{x}}{\gamma_1 \gamma_1 \bar{z}}, \quad B = \frac{1}{\gamma_1}(-x + \bar{x} \bar{z} - \bar{y}), \quad C = 0, \quad G = \gamma_1 \bar{x} \bar{z}, \quad (3.15a)
\]

\[
D = \bar{z}(\bar{p}x - x \bar{z} + y), \quad \omega = \gamma_2 z(\bar{z} - \bar{x}), \quad \sigma = \frac{1}{\gamma_1 \bar{z}(\bar{z} - \bar{x})}. \quad (3.15b)
\]

For (B-2) equation, $i_0 = 3$,

\[
A = \frac{1}{\gamma_1 \gamma_1 \bar{x} \bar{z}}, \quad B = \frac{1}{\gamma_1 \bar{x}}(x - \bar{x} d), \quad C = 0, \quad G = \gamma_1 x, \quad (3.16a)
\]

\[
D = d(x - \bar{x}) - x \bar{x} + y + \bar{z}, \quad \omega = \gamma_2 x(\bar{x} - \bar{x}), \quad \sigma = \frac{1}{\gamma_1 x(\bar{x} - \bar{x})}. \quad (3.16b)
\]

For (C-3) equation, $i_0 = 3$,

\[
A = \frac{\bar{y}}{\gamma_1 \gamma_1 \bar{z} z}, \quad B = \frac{1}{\gamma_1 \bar{z} z}(y \bar{z} + \bar{y} \bar{z} + p \bar{y} \bar{z} - d_2 \bar{x} - d_1), \quad C = 0, \quad (3.17a)
\]

\[
D = \frac{1}{\bar{z}}[y \bar{z} + p(\bar{y} \bar{z} + \bar{y} \bar{z}) - d_2 \bar{x} - d_1], \quad G = \gamma_1 (\bar{x} - x), \quad \omega = \gamma_2 (\bar{z} - \bar{x}), \quad \sigma = \frac{1}{\gamma_1 (\bar{z} - \bar{x})}. \quad (3.17b)
\]

For (C-4) equation, $i_0 = 2$,

\[
A = \frac{1}{\gamma_1 \gamma_1 \bar{x} \bar{y}}, \quad B = \frac{\bar{z}}{\gamma_1 \bar{x} \bar{y}} - \frac{-x \bar{x} + p \bar{y} \bar{z} + y \bar{z} - d}{\gamma_1 \bar{y} (\bar{x} - \bar{x})}, \quad C = 0, \quad (3.18a)
\]

\[
D = \frac{\bar{z}(-x \bar{y} + x \bar{y} + p \bar{y} \bar{z} + d \bar{z})}{\gamma_1 \bar{y} (\bar{x} - \bar{x})}, \quad G = \gamma_1 z \bar{z} \bar{z} \bar{y}, \quad \omega = \gamma_2 z (\bar{x} - \bar{x}), \quad \sigma = \frac{\bar{x} - x}{\gamma_1 z (\bar{x} - \bar{x})}. \quad (3.18b)
\]

For SDBSQ equation, $i_0 = 3$,

\[
A = \frac{\bar{x}}{\gamma_1 \gamma_1 \bar{y}}, \quad B = -\frac{1}{\gamma_1 \bar{y}}(x \bar{y} + \bar{x} \bar{y} + p \bar{x} \bar{y}), \quad C = 0, \quad (3.19a)
\]

\[
D = p(x \bar{y} + \bar{x} \bar{y}) + \bar{x} \bar{y}, \quad G = \gamma_1 \bar{x} \bar{y} \bar{y}, \quad \omega_2 = \gamma_2 (\bar{y} - \bar{y}), \quad \sigma = \frac{1}{\gamma_1 (\bar{y} - \bar{y})}. \quad (3.19b)
\]

For Toda-mDBSQ equation, $i_0 = 2$,

\[
A = \frac{1}{\gamma_1 \gamma_1 \gamma_1 \bar{z}}, \quad B = \frac{1 - p}{\gamma_1 \gamma_1 \gamma_1 \bar{y}} + \frac{\bar{z} + z - 2p}{\gamma_1 \gamma_1 \gamma_1 \bar{y}}, \quad C = 0, \quad (3.20a)
\]

\[
D = \frac{p - 1}{\gamma_1 \gamma_1 \gamma_1 \bar{y}} (-\bar{x} - z + 2p) + \frac{p^2 + p + 1}{y}, \quad (3.20b)
\]

\[
G = \frac{\gamma_1}{y}, \quad \omega = \frac{\gamma_2[(q - 1) \bar{y} - (p - 1) - \bar{y}]}{\gamma_1 [(q - 1) \bar{y} - (p - 1) \bar{y}]}, \quad \sigma = \frac{\bar{y}}{\gamma_1 [(q - 1) \bar{y} - (p - 1) \bar{y}]} \quad (3.20a)
\]

For each equation, the function $\gamma_j$ is defined in Appendix A.

For each equation, from Proposition 2 and Proposition 4 we can find that $\rho$ is related to $\gamma_1$ and $\omega$ is related to $\gamma_2$ while $\{\theta_j\}$ and $\sigma \rho$ are independent of $\gamma_1$ and $\gamma_2$, thus by Proposition 3 all conservation laws except the first one (3.9a) are independent of $\gamma_1$ and $\gamma_2$. 

9
4 Conclusion

We have shown some examples of deriving infinitely many conservation laws from Lax pairs for some lattice equations, particularly for multi-component discrete systems. These systems are all integrable in the sense of multi-dimensional consistency. Such integrability is used to construct Lax pairs. In [11] three-point conservation laws were found via direct approach. Here the simplest nontrivial conservation law of the NQC equation is a four-point one (see Appendix B). However, the approach using Lax pairs looks quite natural and can provide infinitely many conservation laws. And more important, it works for most of known multi-dimensionally consistent systems, including one-component and multi-component discrete systems. We also note that if we conduct the same procedure starting from \((q, \bar{q})\) part of Lax pairs, we only need to switch \((p, \bar{p})\) and \((q, \bar{q})\) in the present results and this is guaranteed by the symmetric property (3.7).

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A Lax pairs of lattice equations listed in Sec.3 (cf. [17])

For each equation we only list out the matrix \(N_1\) in the Lax pair. Matrix \(N_2\) follows from \(N_1\) by switching \((1, p, \bar{p}) \rightarrow (2, q, \bar{q})\).

For the INLS equation

\[
N_1 = \gamma_1 \begin{pmatrix}
-1 & \bar{x} \\
y & r - p - y\bar{x}
\end{pmatrix}, \text{ with } \gamma_1 = 1.
\]

For the mDBSQ equation

\[
N_1 = \gamma_1 \begin{pmatrix}
p\bar{y} & 0 & -r \\
-r\bar{y} & py & 0 \\
0 & -r\bar{y} & \bar{p}x
\end{pmatrix}, \text{ with } \gamma_1 = \sqrt{x/y^2}, \text{ or } \gamma_1 = \frac{1}{y}, \text{ or } \gamma_1 = \frac{1}{\bar{y}}.
\]

For (C-2.1) equation

\[
N_1 = \gamma_1 \begin{pmatrix}
-\bar{z} & \bar{x} \\
z\bar{z} & -z(p + \bar{x}) & r\bar{z}\bar{z} \\
0 & 1 & -\bar{z}
\end{pmatrix}, \text{ with } \gamma_1 = \frac{1}{\sqrt{z\bar{z}}}, \text{ or } \gamma_1 = \frac{1}{z}, \text{ or } \frac{1}{\bar{z}}.
\]

For (C-2.2) equation

\[
N_1 = \gamma_1 \begin{pmatrix}
-\bar{z} & \bar{x} \\
\bar{z} & \bar{z}(d + p\bar{x}) & 0 \\
0 & 1 & -\bar{z}
\end{pmatrix}, \text{ with } \gamma_1 = \sqrt{x/z\bar{z}}, \text{ or } \gamma_1 = \frac{1}{z}, \text{ or } \frac{1}{\bar{z}}.
\]
For (A-2) equation

\[
N_1 = \gamma_1 \left( \begin{array}{ccc}
\frac{y}{z} & \frac{x}{z} & \frac{rx-pzx-uz}{z} \\
0 & \frac{x}{z} & -\frac{zx}{z} \\
0 & -1 & -\frac{z}{z}
\end{array} \right)
\]

, with \( \gamma_1 = \sqrt[3]{\frac{x}{x^2z}} \), or \( \gamma_1 = \frac{1}{z} \), or \( \gamma_1 = \frac{1}{z} \).

For (B-2) equation

\[
N_1 = \gamma_1 \left( \begin{array}{ccc}
-(dx + x^2) & dx + y & k_1 \\
-x & \frac{z}{z} & xz - x \\
0 & -1 & -\frac{z}{z}
\end{array} \right)
\]

, where \( k_1 = (z - x\bar{x})(dx + y) + \bar{z}(d + x^2) + x(p - r) \),

with \( \gamma_1 = \frac{1}{\sqrt{ax}} \), or \( \gamma_1 = \frac{1}{x} \), or \( \gamma_1 = \frac{1}{x} \).

For (C-3) equation

\[
N_1 = \gamma_1 \left( \begin{array}{ccc}
d_1 & d_2 & \frac{pz}{y} \\
0 & -z & \frac{xz - x}{y} \\
1 & 0 & -\frac{z}{z}
\end{array} \right)
\]

, with \( \gamma_1 = \sqrt[3]{\frac{y}{yz^2z}} \), or \( \gamma_1 = \frac{1}{z} \), or \( \gamma_1 = \frac{1}{z} \).

For (C-4) equation

\[
N_1 = \gamma_1 \left( \begin{array}{ccc}
d & \frac{(r-x)z}{y} & \frac{(d+x^2)z}{y} \\
0 & -z & \frac{xz - x}{y} \\
1 & 0 & -\frac{z}{y}
\end{array} \right)
\]

, with \( \gamma_1 = \sqrt[3]{\frac{y}{yz^2z}} \), or \( \gamma_1 = \frac{1}{z} \), or \( \gamma_1 = \frac{1}{z} \).

For the SDBSQ equation

\[
N_1 = \gamma_1 \left( \begin{array}{ccc}
p & \frac{r}{z} & \frac{r}{y} \\
\frac{x}{y} & 0 & 0 \\
-1 & 0 & \frac{y}{y}
\end{array} \right)
\]

, with \( \gamma_1 = \sqrt[3]{\frac{x}{y^2(z - x)}} \), or \( \gamma_1 = \frac{1}{y} \), or \( \gamma_1 = \frac{1}{y} \).

For the Toda-mDBSQ equation

\[
N_1 = \gamma_1 \left( \begin{array}{ccc}
r + p - z & \frac{1+r+z^2}{y} & k_1 \\
0 & p - 1 & (1 - r)\tilde{y} \\
1 & 0 & p - r - \frac{z}{z}
\end{array} \right)
\]

, where \( k_1 = (p^2 - r^2) - \tilde{x}(p + r) \)

\[+\frac{z(r - p + \tilde{x}) - \tilde{y}(p^2 + p + 1)}{y}, \]

with \( \gamma_1 = \sqrt[3]{\frac{y}{y}} \), or \( \gamma_1 = 1 \).

B First few conservation laws of some lattice equations

For the NQC equation, the first two conservation laws are

\[
\Delta_m \ln \gamma_1 \left[ (\alpha^2 + \beta^2 - 2p^2)u\tilde{u} + P_{-u}^2 + P_{+u}^2 \right] = \Delta_n \ln \gamma_2 \left[ P_{+\tilde{u}} - Q_{+\tilde{u}} + (q^2 - p^2)u \right],
\]

(B.1a)

\[
\Delta_m \left( \frac{\alpha^2 + \beta^2 - 2p^2)u\tilde{u} + P_{-u}^2 + P_{+u}^2}{P_{-u} - P_{+\tilde{u}}(P_{-\tilde{u}} - P_{+\tilde{u}})} \right) = \Delta_n \left( \frac{(\alpha^2 + \beta^2 - 2p^2)u\tilde{u} + P_{-u}^2 + P_{+\tilde{u}}^2}{[P_{+\tilde{u}} - Q_{+\tilde{u}} + (q^2 - p^2)u][P_{-u} - P_{+\tilde{u}}]} \right),
\]

(B.1b)
where $P_+ = (p + \alpha)(p + \beta)$, $P_- = (p - \alpha)(p - \beta)$, $Q_+ = (q + \alpha)(q + \beta)$. For the DBSQ equation, the first two conservation laws are

$$\Delta_m \ln \frac{1}{x \tilde{x} - y - z} = \Delta_n \ln (\tilde{x} - \tilde{\tilde{x}}),$$  \hspace{1cm} (B.2a)

$$\Delta_m \frac{-x + \tilde{y}}{(x \tilde{x} - y - z)(\tilde{x} \tilde{y} - \tilde{z})} = \Delta_n \frac{1}{(x - \tilde{x})(\tilde{x} - \tilde{\tilde{x}})}.$$

(B.2b)

For the lNLS equation, the first two conservation laws are

$$\Delta_m \ln \frac{-\tilde{x}}{x(1 + \tilde{x} y)} = \Delta_n \ln \frac{\tilde{x} - \tilde{\tilde{x}}}{\tilde{x}},$$ \hspace{1cm} (B.3a)

$$\Delta_m \frac{\tilde{x} \tilde{y} (1 + \tilde{x} y)}{x \tilde{x} y (1 + \tilde{x} y)} = \Delta_n \frac{\tilde{x} \tilde{y}}{x(1 + \tilde{x} y)(\tilde{x} - \tilde{\tilde{x}})}.$$

(B.3b)

For the SDBSQ equation, the first two conservation laws are

$$\Delta_m \ln \frac{-\gamma_1 \tilde{x} \tilde{y} y}{p(\tilde{x} y + \tilde{\tilde{x}} y) + x \tilde{y}} = \Delta_n \ln \gamma_2 (\tilde{y} - \tilde{\tilde{y}}),$$ \hspace{1cm} (B.4a)

$$\Delta_m \frac{\tilde{x} \tilde{y} (\tilde{x} \tilde{y} + \tilde{\tilde{x}} \tilde{y} + px \tilde{y})}{[p(\tilde{x} y + \tilde{\tilde{x}} y) + x \tilde{y}] [p(\tilde{x} y + \tilde{\tilde{x}} y) + \tilde{\tilde{x}} \tilde{y}]} = \Delta_n \frac{\tilde{x} \tilde{y} y}{(\tilde{y} - \tilde{\tilde{y}})[p(\tilde{x} y + \tilde{\tilde{x}} y) + x \tilde{y}]}. \hspace{1cm} (B.4b)$$

References

[1] F.W. Nijhoff, A.J. Walker, The discrete and continuous Painlevé VI hierarchy and the Garnier systems, Glasg. Math. J., 43A (2001) 109-23.

[2] A.I. Bobenko, Yu.B. Suris, Integrable systems on quad-graphs, Intl. Math. Res. Notices, No.11 (2002) 573-611.

[3] V.E. Adler, A.I. Bobenko, Yu.B. Suris, Classification of integrable equations on quad-graphs. The consistency approach, Commun. Math. Phys., 233 (2003) 513-43.

[4] J. Hietarinta, Searching for CAC-maps, J. Nonlinear Math. Phys., 12 (2005) Suppl.2, 223-30.

[5] J. Hietarinta, Boussinesq-like multi-component lattice equations and multi-dimensional consistency, J. Phys. A: Math. Theor., 44 (2011) 165204 (25pp).

[6] J. Atkinson, J. Hietarinta, F.W. Nijhoff, Seed and soliton solutions for Adler’s lattice equation, J. Phys. A: Math. Theor., 40 (2007) F1-F8.

[7] J. Atkinson, J. Hietarinta, F.W. Nijhoff, Soliton solutions for Q3, J. Phys. A: Math. Theor., 41 (2008) 142001 (11pp).

[8] J. Hietarinta, D.J. Zhang, Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization, J. Phys. A: Math. Theor., 42 (2009) 404006 (30pp).

[9] J. Hietarinta, D.J. Zhang, Multisoliton solutions to the lattice Boussinesq equation, J. Math. Phys., 51 (2010) 033505 (12pp).
[10] J. Hietarinta, D.J. Zhang, Soliton taxonomy for a modification of the lattice Boussinesq equation, SIGMA, 7 (2011) 061 (14pp).

[11] O.G. Rasin, P.E. Hydon, Conservation laws for NQC-type difference equations, J. Phys. A: Math. Gen., 39 (2006) 14055-66.

[12] A.G. Rasin, J. Schiff, Infinitely many conservation laws for the discrete KdV equation, J. Phys. A: Math. Theor., 42 (2009) 175205 (16pp).

[13] A.G. Rasin, Infinitely many symmetries and conservation laws for quad-graph equations via the Gardner method, J. Phys. A: Math. Theor., 43 (2010) 235201 (11pp).

[14] P. Xenitidis, Symmetries and conservation laws of the ABS equations and corresponding differential-difference equations of Volterra type, J. Phys. A: Math. Theor., 44 (2011) 435201 (22pp).

[15] P. Xenitidis, F.W. Nijhoff, Symmetries and conservation laws of lattice Bossinesq equations, Phys. Lett. A, 376 (2012) 2394-401.

[16] D.J. Zhang, J.W. Cheng, Y.Y. Sun, Deriving conservation laws for ABS lattice equations from Lax pairs, arXiv:1210.3454v1.

[17] T. Bridgman, W. Hereman, G.R.W. Quispel, P.H. Van der Kamp, Symbolic computation of Lax pairs of partial difference equations using consistency around the cube, Found. Comput. Math., doi 10.1007/s10208-012-9133-9.

[18] F.W. Nijhoff, G.R.W. Quispel, H.W. Capel, Direct linearization of nonlinear difference-difference equations, Phys. Lett. A, 97 (1983) 125-8.

[19] F.W. Nijhoff, J. Atkinson, J. Hietarinta, Soliton solutions for ABS lattice equations: I. Cauchy matrix approach, J. Phys. A: Math. Theor., 42 (2009) 404005 (34pp).

[20] A. Tongas, F.W. Nijhoff, The Boussinesq integrable system: compatible lattice and continuum structures, Glasg. Math. J., 47 (2005) 205-19.

[21] A.V. Mikhailov, From automorphic Lie algebras to discrete integrable systems, in Programme on Discrete Integrable Systems (Isaac Newton Institute for Mathematical Sciences, Cambridge, 2009). http://www.newton.ac.uk/programmes/DIS/seminars/061714001.html.

[22] A. Walker, Similarity reductions and integrable lattice equations, PhD Thesis, University of Leeds, UK, (2001).

[23] F.W. Nijhoff, On some “Schwarzian” equations and their discrete analogues, in Algebraic Aspects of Integrable Systems, In memory of Irene Dorfman, ed. by A.S. Fokas, I.M. Gel’fand (Birkhäuser, New York, 1996), pp. 237-60.

[24] F.W. Nijhoff, V.G. Papageorgiou, H.W. Capel, G.R.W. Quispel, The lattice Gel’fand-Dikii hierarchy, Inverse Probl., 8 (1992) 597-621.