Resonant Excitation of Disk Oscillations in Deformed Disks III: 
Revision of Mathematical Treatment

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Abstract

In previous studies, we have examined a resonant excitation of disk oscillations in deformed disks. In these studies, however, the mathematical treatment around the resonant points was not rigorous. In this paper the inadequate point is corrected, but with no essential changes in the final results. For this excitation process to work, disks must be general relativistic. That is, a non-monotonic radial distribution of the epicyclic frequency in relativistic disks is essential for the presence of the resonance, and for trapping oscillations. In this paper, the growth rate of resonant oscillations is expressed in a form more suitable for numerical calculations.

Key words: accretion, accretion disks — black holes — high-frequency quasi-periodic neutron stars — relativity — oscillations — stability — X-rays: stars

1. Introduction

In previous papers (Kato 2004, 2008a), we proposed a resonant excitation mechanism of disk oscillations in deformed disks. The purpose here is to propose an excitation mechanism of quasi-periodic oscillations (QPOs) observed in neutron-star and black-hole X-ray binaries (e.g., Kato & Fukue 2007, Kato 2008b). In this resonant excitation model of disk oscillations, a deformation of disks from an axially symmetric equilibrium state is essential. The deformation to be considered is a warp or an eccentric deformation of disks in the equatorial plane.

An outline of the model is as follows. A non-linear coupling between a disk oscillation (hereafter we call it the original oscillation) and a deformed part of the disks (warp or eccentric deformation) brings about some forced disk oscillations (we call them intermediate oscillations). The intermediate oscillations make a resonant coupling with the unperturbed disk at particular radii of the disk. After this resonant coupling, the intermediate oscillations feed back to the original oscillation by a nonlinear coupling with the deformed part of the disk. Since this nonlinear feedback process involves a resonance, the original oscillation is amplified or dampened. Kato (2004, 2008a) examined this nonlinear feedback processes, and derived an excitation criterion and growth (damping) rate of the resonant oscillations. It is of importance to note that in Keplerian disks this resonant excitation process works only when the disks are general relativistic. That is, a non-monotonic radial distribution of epicyclic frequency is necessary for the appearance of resonance, and for the trapping of oscillations.

In these studies the intermediate disk oscillations were assumed to be local near to the resonant radius in the sense that their radial wavelength is shorter than the characteristic radial length of the disks. That is, the radial derivative, $\partial/\partial r$, operated to the wave quantities was taken to be $ik$. Further, $k$ was assumed to be constant. This treatment of $k = \text{const.}$, however, was inadequate around the resonant point. If this inadequate treatment is properly corrected, we find that, unlike in the previous papers, the resonant point is not the place where a local dispersion relation of the intermediate oscillations is satisfied with a constant $k$, but the radii of Lindblad resonance for intermediate oscillations (when resonance occurs by horizontal motions) or the radii where the intermediate oscillations are trapped in the vertical direction (when resonance occurs by vertical motions). In this paper, we modify the analyses in our previous papers so that the above-mentioned inadequate treatment is corrected. We apply mathematical techniques used by Meyer-Vernet and Sicardy (1987) in their study of resonant disk-satellite interaction. The results show that the stability criterion derived in the previous papers is unchanged. The expression for the growth rate obtained in the previous papers is found to still be applicable, but in this paper the expression is changed to a form more suitable for numerical calculations.

We first summarize in section 2 the basic equations and relations necessary to the study the present resonant excitation problem, although they are given in the previous papers. In section 3, the stability criterion and growth rate are derived for the case of pressure-less disks, since it is instructive, and for the case of disks with pressure in section 4. The final section is devoted to a brief discussion on the meaning of the instability criterion.

2. Brief Summary of Basic Equations

We briefly summarize here the basic equations and relations [see Kato (2008a) for details] that are necessary in the subsequent sections.

2.1. Nonlinear Hydrodynamical Equations

In the present problem, general relativity is essential. For simplicity, however, the formulation in this paper is done within the framework of a pseudo-Newtonian using the gravitational potential introduced by Paczyński and Wiita (1980), assuming that the central object has no rotation. We
adopt a Lagrangian formulation proposed by Lynden-Bell and Ostriker (1967).

The unperturbed disk is in a steady state with a steady flow, \( u_0 \). By using a displacement vector, \( \xi \), a weakly nonlinear hydrodynamical equation describing adiabatic, non-self-gravitating perturbations is written as, after generalizing the linear equation derived by Lynden-Bell and Ostriker (1967),

\[
\rho_0 \frac{\partial^2 \xi}{\partial t^2} + 2 \rho_0 (u_0 \cdot \nabla) \frac{\partial \xi}{\partial t} + L(\xi) = \rho_0 C(\xi, \xi),
\]

where \( L(\xi) \) is a linear Hermitian operator with respect to \( \xi \) (Lynden-Bell & Ostriker 1967) and is

\[
L(\xi) = \rho_0 (u_0 \cdot \nabla)(u_0 \cdot \nabla)\xi - \rho_0 (\xi \cdot \nabla)(\nabla \psi_0) + \nabla \left( (1 - \Gamma) p_0 \div \xi \right) - p_0 \nabla \div (\xi),
\]

and \( \rho_0(r) \) and \( p_0(r) \) are the density and pressure in the unperturbed state; \( \Gamma \) is the barotropic index specifying the linear part of the relation between the pressure, \( \rho \), and the specific internal energy, \( u \) [see equation (6)]. Here, \( \Gamma \) is assumed to have frequency \( \omega \) and azimuthal wavenumber \( m \). Separating the time and azimuthal dependences from \( \xi(r, t) \), we introduce \( \hat{\xi} \) as

\[
\xi(r, t) = \exp[i(\omega t - m\phi)]\hat{\xi}(r, \phi).
\]

As for the deformation, a warp will be the most probable, but it is not the only candidate of possible deformations. A plane-symmetric one-armed spiral deformation is one other possible candidate that can excite disk oscillations. In both cases we have \( m_W = 1 \).

Our purpose here is to examine how the behavior of disk oscillations is affected by the disk deformation. As shown below, some of the oscillation modes are resonantly excited on deformed disks through nonlinear coupling with disk deformation. The nonlinear coupling processes are schematically shown in figure 1 of Kato (2004).

The displacement vector associated with a disk oscillation, \( \xi \), is assumed to have frequency \( \omega \) and azimuthal wavenumber \( m \). Separating the time and azimuthal dependences from \( \xi(r, t) \), we introduce \( \hat{\xi} \) as

\[
\xi(r, t) = i(\omega t - m\phi)\hat{\xi}(r, \phi).
\]

The first step of the nonlinear interaction between the disk oscillation characterized by \( (\omega, m) \) and the deformation characterized by \( (1, 0) \) introduces two kinds of intermediate oscillations with the azimuthal wavenumber being \( m + 1 \) and \( m - 1 \). Let us denote the displacement vector associated with these intermediate oscillations by

\[
\hat{\xi}^\text{int}(r, t) = \exp[i(\omega t - m\phi)]\hat{\xi}^\text{int}(r, \phi).
\]

Next, the second stage of the nonlinear coupling is considered, which is a feedback process returning to the original oscillation, \( \hat{\xi} \), by the intermediate oscillations, \( \hat{\xi}^\text{int} \), interacting with \( \hat{\xi}^W \). In the case where the coupling occurs through \( \hat{\xi}^+ \), the feedback is described by

\[
-\omega^2 \rho_0 \hat{\xi}^+ + 2 \omega \rho_0 (u_0 \cdot \nabla)\hat{\xi}^+ + L(\hat{\xi}^+). \tag{10}
\]

On the other hand, in the case where the feedback occurs through \( \hat{\xi}^- \), the equation corresponding to equation (10) is

\[
\hat{\xi}^W(r, t) = \exp(-im\omega t)\hat{\xi}^W(r, t). \tag{5}
\]
\[-\omega^2 \rho_0 \ddot{\xi} + 2i \omega \rho_0 (u_0 \cdot \nabla) \dot{\xi} + L(\dot{\xi}) = \frac{1}{2} [\rho_0 C(\dot{\xi}^{\text{int}}, \dot{\xi}^W) + \rho_0 C(\dot{\xi}^W, \dot{\xi}^{\text{int}})] . \tag{11}\]

An important point to be noted here is that as a result of this feedback process, the original disk oscillation is amplified or dampened, i.e., the frequency, \( \omega_t \), can no longer be real, since in the feedback processes a resonance is involved. How much is the imaginary part of \( \omega_t \)? This can be examined from equations (10) and (11) using the fact that the operators \( i \rho_0 (u_0 \cdot \nabla) \) and \( L \) are Hermitian (Lynden-Bell & Ostriker 1967). After some calculations, the imaginary part of the frequency, \( \omega_t \), can be formally expressed as (e.g., Kato 2008a)

\[-\omega_t = \frac{W_\pm}{2E}, \tag{12}\]

where \( \pm \) denotes the cases of coupling through \( \dot{\xi}_+ \) and \( \dot{\xi}_- \), respectively, and

\[
W_+ = \frac{\omega_0}{2} \int \frac{1}{2} \rho_0 \dot{\xi}_+ [C(\dot{\xi}_+^\text{int}, \dot{\xi}^W) + C(\dot{\xi}^W, \dot{\xi}_+^\text{int})] dV ,
\]

\[
W_- = \frac{\omega_0}{2} \int \frac{1}{2} \rho_0 \dot{\xi}_- [C(\dot{\xi}_-^\text{int}, \dot{\xi}^W) + C(\dot{\xi}^W, \dot{\xi}_-^\text{int})] dV ,
\]

and

\[
E = \frac{1}{2} \omega_0 \int \rho_0 \dot{\xi}_+ [\omega - i (u \cdot \nabla)] \dot{\xi} dV . \tag{15}\]

where \( \omega_0 \) is the frequency of the original oscillation before the mode couplings. The above expressions for \( W_\pm \) and \( E \) have physical meanings, such that \( W_\pm \) is the rate at which work is done on the original oscillations by the nonlinear resonant processes, and \( E \) is the wave energy of the original oscillations [see equation (93) by Kato (2001)].

It is important to change the above expressions for \( W_\pm \) by using the commutative relation (4) as

\[
W_+ = \frac{\omega_0}{2} \int \frac{1}{2} \rho_0 \dot{\xi}_+^\text{int} [C(\dot{\xi}_+^\text{int}, \dot{\xi}^W) + C(\dot{\xi}^W, \dot{\xi}_+^\text{int})] dV ,
\]

\[
W_- = \frac{\omega_0}{2} \int \frac{1}{2} \rho_0 \dot{\xi}_-^\text{int} [C(\dot{\xi}_-^\text{int}, \dot{\xi}^W) + C(\dot{\xi}^W, \dot{\xi}_-^\text{int})] dV .
\]

2.3. Oscillations in Isothermal Disks

Hereafter, we restrict our attention to oscillations in geometrically thin disks. The steady unperturbed disks are axially-symmetric, and have no motion, except for rotation. Here, cylindrical coordinates \((r, \varphi, z)\) are employed, in which the \( z \)-axis is perpendicular to the disk plane and the origin of the coordinates is at the disk center. The unperturbed flow is then described as \( u_0 = [0, r \Omega(r), 0] \), where \( \Omega(r) \) is the angular velocity of disk rotation. We further assume, for simplicity, that the disk is isothermal in the vertical direction. In vertically isothermal disks, the unperturbed density in disks, \( \rho_0(r, z) \), is stratified as (e.g., Kato et al. 1998)

\[
\rho_0(r, z) = \rho_{00}(r) \exp \left[ -\frac{z^2}{2H^2(r)} \right] , \tag{18}\]

where \( \rho_{00} \) is the density on the equatorial plane, and \( H \) is the half-thickness of the disk, and is related to the vertical epicyclic frequency, \( \Omega_z \), by

\[
\Omega_z^2 H^2 = \frac{\rho_0}{\rho_{00}} = c_s^2(r) , \tag{19}\]

where \( c_s \) is the isothermal acoustic speed. The vertical epicyclic frequency, \( \Omega_z \), is equal to the angular velocity of the Keplerian rotation, \( \Omega_K \), in the case of the central object being non-rotating, and practically equal to \( \Omega_z \) since the disk is assumed to be geometrically thin. Hereafter, however, we use \( \Omega_z \) without using \( \Omega_K \) or \( \Omega_z \), so that we can trace back the effects of \( \Omega_z \) on the final results.

Concerning oscillations superposed on such disks, we consider those whose radial wavelength is moderately short, so that the radial variations of the physical quantities in the unperturbed state [including the radial variation of \( H(r) \)] are neglected compared with the radial variation of the wave quantities. Furthermore, we assume that the oscillations also occur isothermally. Then, we can neglect the term \( \nabla[(1 - \Gamma_1) \rho_0 \dot{\xi}] \) in operator \( L \) in equation (2).

If the above simplifications and approximations are adopted, the \( r \)- and \( z \)-dependences of \( \dot{\xi}(r, z) \) are approximately separated. That is, the \( z \)-dependence of \( \dot{\xi}(r, z) \) is obtained by solving an eigen-value problem, and the eigen-functions are found to be an orthogonal set of Hermite polynomials (Okazaki et al. 1987). Restricting our attention to one of such oscillations, we write \( \dot{\xi}(r, z) \) as

\[
\dot{\xi}_r(r, z) = \tilde{\xi}_{r, n}(r) \mathcal{H}_n(z/H) , \tag{20}\]

\[
\dot{\xi}_{\varphi}(r, z) = \tilde{\xi}_{\varphi, n}(r) \mathcal{H}_n(z/H) , \tag{21}\]

\[
\dot{\xi}_z(r, z) = \tilde{\xi}_{z, n}(r) \mathcal{H}_{n-1}(z/H) , \tag{22}\]

where \( \mathcal{H}_n \) is an Hermite polynomial of argument \( z/H \), and \( n(=0, 1, 2, \ldots) \) characterizes the number of node(s) of oscillations in the vertical direction. It is noted that the number of node(s) of \( \dot{\xi}_z \) in the vertical direction is smaller than those of \( \dot{\xi}_r \) and \( \dot{\xi}_\varphi \) by one, as shown in equation (22). However, the subscript \( n \) (not \( n - 1 \)) is attached to \( \dot{\xi}_{z, n} \), in order to emphasize that \( \dot{\xi}_{r, n}, \dot{\xi}_{\varphi, n} \) and \( \dot{\xi}_{z, n} \) are a set of solutions.

Under these approximations, we express the \( r \)-, \( \varphi \)-, and \( z \)-components of the homogeneous parts of wave equation (1) as

\[
- \frac{\varphi - m \Omega}{2} + k^2 - 4 \Omega^2 - c_s^2 \frac{d^2}{dr^2} \tilde{\xi}_{r, n} = 0 , \tag{23}\]

\[
-i 2 \Omega (\omega - m \Omega) \tilde{\xi}_{\varphi, n} + \Omega_z H \frac{d^2 \tilde{\xi}_{z, n}}{dr^2} = 0 , \tag{24}\]

\[
- (\omega - m \Omega)^2 \frac{d^2 \tilde{\xi}_{z, n}}{dr^2} + \frac{d^2 \tilde{\xi}_{r, n}}{dr^2} = 0 . \tag{25}\]

Nonlinear coupling between \( \xi \) and \( \xi^W \) introduces intermediate oscillations, \( \xi^\text{int} \). The azimuthal wavenumber of the intermediate oscillations are \( m + 1 \) or \( m - 1 \). Their \( z \)-dependence are characterized by \( n + 1 \) or \( n - 1 \) when \( n_W = 1 \), since \( \mathcal{H}_1 \mathcal{H}_n = \mathcal{H}_{n+1} + n \mathcal{H}_{n+1} \) and by \( n \) when \( n_W = 0 \). To consider these various coupling cases separately, we write \( \dot{\xi}_\pm \) in the forms of
where the subscript $\tilde{n}$ represents $n + 1$ or $n - 1$ or $n$.

Nonlinear coupling terms are also separated into terms proportional to $\exp[-i(m \pm 1)\varphi]$ and $\mathcal{H}_\delta(r/z)$. That is, in the case of coupling through $\xi^{\text{int}}_{r,\pm} = 0$, we write the coupling terms as

$$
\frac{1}{2} \rho_0 \mathcal{H}(\xi, \xi^{W}) + C(\xi, \xi^{W}, \xi),
$$

$$
\rho_0 \sum_{r, \pm} A_{r, \pm}(r) \exp[i(\omega t - m \varphi)] \mathcal{H}_\delta(z/H) + \cdots,
$$

$$
\frac{1}{2} \rho_0 \mathcal{H}(\xi, \xi^{W}) + C(\xi, \xi^{W}, \xi),
$$

$$
\rho_0 \sum_{r, \pm} A_{r, \pm}(r) \exp[i(\omega t - m \varphi)] \mathcal{H}_\delta(z/H) + \cdots,
$$

$$
\frac{1}{2} \rho_0 \mathcal{H}(\xi, \xi^{W}) + C(\xi, \xi^{W}, \xi),
$$

$$
\rho_0 \sum_{r, \pm} A_{r, \pm}(r) \exp[i(\omega t - m \varphi)] \mathcal{H}_\delta(z/H) + \cdots,
$$

where $+ \cdots$ denotes terms orthogonal both to $\mathcal{H}_\delta$ and $\mathcal{H}_\delta^{-1}$, and the subscript $+$ is added to the $\dot{A}$'s in order to emphasize that they are related to the $\varphi$-dependence of $\exp[-i(m + 1)\varphi]$. In a similar way, in the case of coupling through the intermediate oscillations of $\xi^{\text{int}}_{r}$, $\xi^{\text{int}}_{\psi}$, $\xi^{\text{int}}_{r, \pm}$, $\xi^{\text{int}}_{\psi, \pm}$, $\xi^{\text{int}}_{r, \pm, \delta}$, and $\xi^{\text{int}}_{\psi, \pm, \delta}$ can be expressed in forms similar to equations (29)–(31), introducing $A_{r, \pm}, A_{\psi, \pm, \delta}, A_{r, \pm, \delta}$.

The equations describing intermediate oscillations are then written as

$$
\left( - (\omega - m \Omega)^2 + \kappa^2 - 4 \Omega^2 - c_s^2 \frac{d^2}{dr^2} \right) \xi^{\text{int}}_{r, \pm, \delta} = 0,
$$

$$
-i 2 \Omega (\omega - m \Omega) \xi^{\text{int}}_{r, \pm, \delta} + \Omega^2 H \frac{d \xi^{\text{int}}_{r, \pm, \delta}}{dr} = \dot{A}_{r, \pm, \delta},
$$

$$
- (\omega - m \Omega)^2 \xi^{\text{int}}_{\psi, \pm, \delta} + i 2 \Omega (\omega - m \Omega) \xi^{\text{int}}_{\psi, \pm, \delta} = \dot{A}_{\psi, \pm, \delta},
$$

$$
- (\omega - m \Omega)^2 + \tilde{n} \Omega^2 \xi^{\text{int}}_{z, \pm, \delta} - \tilde{n} \Omega^2 H \frac{d \xi^{\text{int}}_{z, \pm, \delta}}{dr} = \dot{A}_{z, \pm, \delta},
$$

where $\tilde{m}$ represents $m + 1$ or $m - 1$, and $\tilde{n}$ does $n + 1$ or $n - 1$ when $n_W = 1$ and $\tilde{n} = n$ when $n_W = 0$. From equations (32) and (33) we can eliminate $\xi^{\text{int}}_{\psi, \pm, \delta}$ to give

$$
\left[ - (\omega - m \Omega)^2 + \kappa^2 \right] \xi^{\text{int}}_{r, \pm, \delta} - c_s^2 \frac{d^2 \xi^{\text{int}}_{r, \pm, \delta}}{dr^2} + \Omega^2 H \frac{d \xi^{\text{int}}_{r, \pm, \delta}}{dr} = \dot{A}_{r, \pm, \delta}.
$$

Hereafter, we use equations (32)–(35) as the basic equations describing the intermediate oscillations.

Finally, we should notice that in the case of the isothermal disks described above, $W_+$ and $W_-$ given by equations (16) and (17) are written in the forms of

$$
W_+ = \frac{\alpha_0}{2} \int \rho_0(r) [ (2\pi)^{3/2} n^2 \Omega H ]
$$

$$
\cdot \left[ (\xi_{r, \pm} + F_{r, \pm}) \right] d r,
$$

and

$$
W_- = \frac{\alpha_0}{2} \int \rho_0(r) [ (2\pi)^{3/2} n^2 \Omega H ]
$$

$$
\cdot \left[ \xi_{r, \pm} + F_{r, \pm} \right] d r.
$$

where $[(2\pi)^{3/2} n^2 \Omega H]$ comes from the part of volume integration in the $\varphi$- and $z$-directions. Furthermore, the wave energy, $E$, given by equation (15) can be expressed as

$$
E = \frac{(2\pi)^{3/2}}{2} \omega_0^2 H \rho_0 \epsilon E_n,
$$

where $E_n$ is a dimensionless quantity, given by

$$
E_n = \int \frac{r \rho_0}{(r \rho_0 c)^2} \omega - m \Omega \omega \left[ n^2 \left( \frac{\xi_{r, \pm}^2}{r_c^2} + (n - 1)! \left( \frac{\xi_{r, \pm}^2}{r_c^2} \right)^2 \right) \right] d r,
$$

and the subscript $c$ denotes the values at the resonant radius, which is defined by $J_1 = 0$ or $J_2 = 0$ (see below). It is noted that in the case of $n = 0$, $(n - 1)!$ is zero.

3. Resonant Excitation of Oscillations in Pressure-Less Disks

Before examining general cases of disks with pressure, we consider here the limiting case of pressure-less disks, i.e., $c_s = 0$ and $H = 0$. Equations (35) and (36) show that $\xi^{\text{int}}_{r}$ and $\xi^{\text{int}}_{\psi}$ respond resonantly, respectively, at radii where $-(\omega - m \Omega)^2 + \kappa^2$ is zero and at radii where $-(\omega - m \Omega)^2 + \tilde{n} \Omega^2 = 0$ holds. We define here $J_1(r)$ and $J_2(r)$ by

$$
J_1(r) = -(\omega - m \Omega)^2 + \kappa^2
$$

and

$$
J_2(r) = -(\omega - m \Omega)^2 + \tilde{n} \Omega^2.
$$

We call the resonance at $J_1 = 0$ horizontal resonances (Lindblad resonances) and at $J_2 = 0$ vertical resonances. We consider these two resonances separately.

As a typical example in infinitesimally thin disks, we consider here the case where all motions including those associated with disk deformations are in the equatorial plane, i.e., $n = 0$, $n_W = 0$, and $\tilde{n} = 0$. At the radii of $J_1 = 0$, $\xi^{\text{int}}_{r}$ and $\xi^{\text{int}}_{\psi}$ resonantly respond to $\dot{A}$'s as [see equations (35) and (33)]

$$
\xi^{\text{int}}_{r, \pm, \delta} = \frac{1}{J_1} \left( \dot{A}_{r, \pm, \delta} - i \frac{2 \Omega}{\omega - m \Omega} \dot{A}_{r, \pm, \psi} \right),
$$

and

$$
\xi^{\text{int}}_{\psi, \pm, \delta} = \frac{1}{J_1} \left( i \frac{2 \Omega}{\omega - m \Omega} \dot{A}_{r, \pm, \psi} - \dot{A}_{r, \pm, \delta} \right).
$$

This expression for $E_n$ is different from that given in the previous paper (Kato 2008a), since the previous one is not suitable to represent the sign of wave energy.

3 In the following formulations, however, we retain $n$, $n_W$, and $\tilde{n}$ in general without specifying to particular values.
The quantity $\dot{q}^\text{int}_z$ can be taken to be zero, since it does not respond resonantly and has no contribution to the final results. To examine the resonant interaction at the radii of $J_1 = 0$, we introduce a small imaginary part of $\omega$, i.e., $\omega_i$, as $\omega = \omega_r + i\omega_i$, where the imaginary part, $\omega_i$, is tentatively assumed to be negative so that causality is satisfied. Later, the results obtained by letting $\omega_i < 0$ are extended analytically to the whole region of $\omega_i$, as usually done in stability analyses. Hereafter, the real part, $\omega_r$, is written as $\omega$ or $\omega_i$, without confusion.

Near to the resonance, $J_1$ is expanded as

$$J_1 = J'_1 \left[ (r - r_c) - i \frac{2(\omega - \tilde{\omega}_i)}{J'_1} \omega_i \right],$$

where the subscript $c$ represents the values at the resonant radii, and $J'_1$ is

$$J'_1 = 2 \left[ \tilde{m}(\omega - \tilde{\omega}_i) \Omega^2 \frac{d\ln \Omega}{dr} + k^2 \frac{d\ln k}{dr} \right].$$

Let us now consider the integration of $f(r)/J_1$, where $f$ is an arbitrary smooth real function, along the radial direction including the resonant region. The imaginary part of the integration comes from the path near to the pole. Thus, we have

$$\Im \int \frac{f(r)}{J_1(r)} dr = \left\{ \begin{array}{ll} -\pi f_c/J'_1 & \text{when } (\omega - \tilde{\omega}_i)/J'_1 > 0 \\ \pi f_c/J'_1 & \text{when } (\omega - \tilde{\omega}_i)/J'_1 < 0. \end{array} \right.$$ (45)

This can be summarized as

$$\Im \int \frac{f(r)}{J_1(r)} dr = -\pi \frac{f}{J'_1} \text{sign}(\omega - \tilde{\omega}_i)c,$$

where $\text{sign}(\omega - \tilde{\omega}_i)c$ is the sign of $\omega - \tilde{\omega}_i$ at the resonant radii.

Let us now estimate the rate of work done by the resonance on the original oscillations by using equation (46). In the case of coupling through $\dot{q}^\text{int}_z$, the work is given by equations (36) and (37). Since $\ddot{X}_{\pm H}$ and $\dot{\dot{X}}^\text{int}_{\pm H}$ are given by equations (41) and (42), respectively, we have from equations (36) and (37)

$$W_\pm = -\frac{\omega_0}{2} \pi (2\pi)^{3/2} \frac{n}{\Omega R_{\text{pool}}} \left( \frac{\Omega}{J'_1} \right) \text{sign}(\omega - \tilde{\omega}_i)c \times \frac{1}{\omega - \tilde{\omega}_i} \frac{\dot{X}_{\pm H} - i \frac{2\Omega}{\omega - \tilde{\omega}_i} \dot{X}_{\mp H}}{c},$$ (47)

where we have used $\int_{-\infty}^\infty \exp(-z^2/2H^2)H_z^2(z/H)dz = (2\pi)^{1/2}n!H$. Hence, the growth rate, $-\omega_1$, see equation (12), can be expressed as

$$-\omega_1,H,\pm,H = -\frac{\pi n!}{2r_c^3 |J'_1| |\omega_0 E_n|} \frac{\text{sign}(\omega - \tilde{\omega}_i)c}{\omega - \tilde{\omega}_i} \times \frac{1}{\omega - \tilde{\omega}_i} \frac{\dot{X}_{\pm H} - i \frac{2\Omega}{\omega - \tilde{\omega}_i} \dot{X}_{\mp H}}{c},$$ (48)

where subscript $H$ is attached to $\omega_0$ in order to emphasize that this is a case of horizontal resonances. It is noted that the above expression for the growth rate is valid even when $\tilde{n} \neq 0$. Equations (47)–(48) formally hold even in the case with pressure, as shown in the next section.

The vertical resonance might be unrealistic in pressure-less disks, but we briefly summarize the formal results here, since the expression for the growth rate is applicable even in the case of disks with pressure, as shown in the next section. We now expand $J_2$ around $J_1 = 0$ as

$$J_2 = J'_2 \left[ (r - r_c) - i \frac{2(\omega - \tilde{\omega}_i)}{J'_2} \omega_i \right],$$ (49)

where

$$J'_2 = 2 \left[ \tilde{m}(\omega - \tilde{\omega}_i) \Omega^2 \frac{d\ln \Omega}{dr} + \tilde{\omega}_i^2 \frac{d\ln k}{dr} \right].$$ (50)

and $r_c$ is the resonant radius of $J_2 = 0$. In the above two cases of the horizontal and vertical resonances, the resonant radii are different each other. However, the same notation, $r_c$, is used here and hereafter without confusion.

After this preparation, we calculate the rate of work done on oscillations by the same procedures as those in the case of horizontal resonance. The results show that the growth rate is given by

$$-\omega_1,V,\pm,H = \frac{\pi (\tilde{n} - 1)!}{2r_c^3 |J'_2| |\omega_0 E_n|} \text{sign}(\omega - \tilde{\omega}_i)c \frac{\dot{X}_{\pm H}^\text{int} - i \frac{2\Omega}{\omega - \tilde{\omega}_i} \dot{X}_{\mp H}^\text{int}}{c}.$$ (51)

4. Resonances in Disks with Pressure

Equations describing the intermediate oscillations [equations (32)–(34) or equations (35) and (34)] are now solved without assuming $c_0 = 0$ and $H = 0$. Since we are interested in the behaviors of $\ddot{X}^\text{int}_z$ in a region close to the resonant point of $J_1 = 0$ or $J_2 = 0$, of all of the coefficients of $\ddot{X}^\text{int}_z$ in equations (32)–(34) are assumed to be constant, except for $J_1$ or $J_2$. That is, in the case of the horizontal resonance (Lindblad resonance), $J_2 = \text{const.}$, and $J_1$ is taken as [see equations (43) and (44)]

$$J_1 = J_1^c x - 2i(\omega - \tilde{\omega}_i)c \omega_1,$$ (52)

where $x$ is the radial distance from the resonant radius, $r_c$, i.e., $x = r - r_c$. In the case of vertical resonance, on the other hand, $J_1 = \text{const.}$, and $J_2$ is taken as [see equations (49) and (50)]

$$J_2 = J_2^c x - 2i(\omega - \tilde{\omega}_i)c \omega_1,$$ (53)

where $x = r - r_c$. The resonant radii in the horizontal and vertical resonances are different from each other, but the same notation, $r_c$, has been adopted here without confusion.

In disks with pressure, different from the case of pressure-less disks, the resonant region is not narrow. That is, the resonant region is widened by the effects of pressure (see below), and thus the coupling terms, $\dot{X}_{\pm H}$, $\dot{X}_{\pm v}$, and $\dot{X}_z$, can be no longer regarded to be spatially constant, in general. Considering these situations, we introduce Fourier transforms of $\ddot{X}^\text{int}_z$ and $\dot{X}$ as

$$\ddot{X}^\text{int}_z(k) = \int_{-\infty}^{\infty} \exp(-ikx)\ddot{X}^\text{int}_z(x)dx,$$ (54)

$$\dot{X}(k) = \int_{-\infty}^{\infty} \exp(-ikx)\dot{X}(x)dx.$$ (55)

After these preparations, to solve the set of equations (32)–(34), we apply a method of Fourier transform adopted by Meyer-Vernet and Sicardy (1987) in their study of a resonant disk-satellite interaction. Here and hereafter, such subscripts as $\pm$ and $\tilde{n}$ to be attached to $\ddot{X}^\text{int}_z$, $\ddot{X}^\text{int}_v$, $\dot{X}$, and $\dot{X}$ are sometimes
omitted in order to avoid unnecessary complications, unless they are explicitly necessary.

4.1. Horizontal Resonances (Lindblad Resonances)

In the present case, the Fourier transform of equations (35) and (34) give, respectively,

\[ iJ_1 \frac{d \xi_{r,\text{int}}(k)}{dk} + i \xi_{s,\text{int}} + c_2^2 k^2 \xi_{s,\text{int}} + i k \Omega z H \xi_{z,\text{int}} = \hat{A}_s - i \frac{2 \Omega}{\omega - m \Omega} \hat{A}_\psi, \]

where

\[ J_2 \xi_{s,\text{int}} - i k \Omega z H \xi_{z,\text{int}} = \hat{A}_z, \]

(56)

and

where \( \epsilon \equiv -2(\omega - m \Omega) \omega_\Omega \).

Eliminating \( \xi_{s,\text{int}} \) from equations (56) and (57), we have an inhomogeneous differential equation of \( \xi_{r,\text{int}}(k) \) with respect to \( k \):

\[ \frac{d \xi_{r,\text{int}}(k)}{dk} + \frac{\epsilon}{J_1} \xi_{r,\text{int}}(k) + i \frac{c_2^2 k^2}{J_1} + \frac{(\omega - m \Omega)^2}{J_2} \xi_{s,\text{int}} = -i \frac{1}{J_1} \hat{A}(k), \]

(58)

where \( \hat{A}(k) \) is defined by

\[ \hat{A}(k) = \frac{2 \Omega}{\omega - m \Omega} \Omega(z) - i \frac{k \Omega z H}{J_2} \Omega(z). \]

(59)

Let us first consider the case of \( \epsilon / J_1 > 0 \). Equation (58) is then solved to lead

\[ \xi_{r,\text{int}}(k) = D_{(+)}(k) \exp \left[ - \frac{\epsilon k^2 (\omega - m \Omega)^2}{3 J_1 J_2} k^3 \right], \]

(60)

where

\[ D_{(+)}(k) = -i \frac{1}{J_1} \int_{-\infty}^{k} \hat{A}(k') \exp \left[ \frac{\epsilon}{J_1} k' + i \frac{c_2^2 (\omega - m \Omega)^2}{3 J_1 J_2} k'^3 \right] dk'. \]

(61)

In determining the integration range of equation (61), we have used a boundary condition allowing \( \xi_{r,\text{int}} \to 0 \) for \( k \to -\infty \). This boundary condition is required, since \( \xi_{s,\text{int}} \) is the Fourier transform of \( \xi_{r,\text{int}}(x) \).

By using the inverse Fourier transform,

\[ \xi_{r,\text{int}}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi_{r,\text{int}}(k) \exp(ikx) dk, \]

(62)

we can express \( \xi_{r,\text{int}}(x) \) in an integration form as

\[ \xi_{r,\text{int}}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} D_{(+)}(k) \exp \left[ - \frac{\epsilon}{J_1} k + i \left( k x - \frac{1}{3} \alpha_y k^3 \right) \right] dk, \]

(63)

where \( \alpha_y \) is defined by

\[ \alpha_y^3(x) = \frac{c_2^2 (\omega - m \Omega)^2}{J_1 J_2}. \]

Since \( \xi_{q,\text{int}}(x) \) and \( \xi_{z,\text{int}}(x) \) are related to \( \xi_{r,\text{int}} \) by [see equations (33) and (34)]

\[ \xi_{q,\text{int}}(x) = i \frac{2 \Omega}{\omega - m \Omega} \xi_{r,\text{int}} - \frac{\hat{A}_q}{(\omega - m \Omega)^2}, \]

\[ \xi_{z,\text{int}}(x) = \frac{1}{J_1} \left( \Omega z H \frac{d \xi_{s,\text{int}}}{dr} + \hat{A}_z \right). \]

(65)

the work integral, \( W_{H+} \), in the case of coupling through \( \xi_{r,\text{int}} \), is written as, using equation (36),

\[ W_{H+} = \frac{\rho_0}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \frac{1}{\rho_0 \Omega(z)} \left[ (2\pi)^{3/2} \Omega z H \right] \]

\[ \times \left[ D_{(+)}(k) \exp \left[ - \frac{\epsilon}{J_1} k + i \left( k x - \frac{1}{3} \alpha_y^3 k^3 \right) \right] \right] \]

\[ \times \left[ \hat{A}_{r,\text{int}}(x) + i \frac{2 \Omega}{\omega - m \Omega} \hat{A}_{s,\text{int}}(x) \right]^{\text{even}} + \frac{1}{J_2} \left( \Omega z H \hat{A}_{r,\text{int}}(x) \right). \]

(66)

where the real parts on the right-hand side of equation (66) have been omitted.

In the case of \( \epsilon / J_1 < 0 \), the expression for \( \xi_{r,\text{int}}(x) \) is slightly changed from equation (60) as

\[ \xi_{r,\text{int}}(k) = D_{(-)}(k) \exp \left[ - \frac{\epsilon}{J_1} k - i \frac{c_2^2 (\omega - m \Omega)^2}{3 J_1 J_2} k^3 \right], \]

(67)

where \( D_{(-)}(k) \) is given by

\[ D_{(-)}(k) = -i \frac{1}{J_1} \int_{-\infty}^{k} \hat{A}(k') \exp \left[ \frac{\epsilon}{J_1} k' + i \frac{c_2^2 (\omega - m \Omega)^2}{3 J_1 J_2} k'^3 \right] dk'. \]

(68)

where \( \hat{A}(k) \) is defined by equation (59). The integration range in equation (68) is changed from equation (61) so that the boundary condition \( \xi_{r,\text{int}} = 0 \) is satisfied at \( k = \infty \). By performing the same procedures as those in the case of \( \epsilon / J_1 > 0 \), we can derive an expression for \( W_{H+} \). The expression for \( W_{H+} \) in this case is the same as equation (66), except that \( D_{(+)} \) is now changed to \( D_{(-)}, \) i.e.,

\[ W_{H+} = \frac{\rho_0}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk \frac{1}{\rho_0 \Omega(z)} \left[ (2\pi)^{3/2} \Omega z H \right] \]

\[ \times \left[ D_{(-)}(k) \exp \left[ - \frac{\epsilon}{J_1} k + i \left( k x - \frac{1}{3} \alpha_y^3 k^3 \right) \right] \right] \]

\[ \times \left[ \hat{A}_{r,\text{int}}(x) + i \frac{2 \Omega}{\omega - m \Omega} \hat{A}_{s,\text{int}}(x) \right]^{\text{even}} + \frac{1}{J_2} \left( \Omega z H \hat{A}_{r,\text{int}}(x) \right). \]

(69)

Expressions of the work integral in the case of the coupling through \( \xi_{z,\text{int}} \) can be derived by similar procedures as those in the case of coupling through \( \xi_{r,\text{int}} \). In this case, the subscript + attached to the A’s in equations (66) and (69) is changed to − and \( \bar{m} = m - 1 \).
4.1.1. The case of constant coupling terms

First, we consider the case where the coupling terms, \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \), can be regarded to be spatially constant in the resonant region. (This is really realized in a particular case, as mentioned in the next section.) Another purpose of examining the case of constant coupling terms is to have a rough image on the width of the resonant region in disks with pressure.

If \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \) are spatially constant, their Fourier transforms give

\[
\tilde{A}_r(k) = 2\pi \tilde{A}_r \delta(k), \quad \tilde{A}_\varphi = 2\pi \tilde{A}_\varphi \delta(k), \quad \tilde{A}_z(k) = 2\pi \tilde{A}_z \delta(k).
\]

(70)

In this case, \( D_{(+)}(k) \) given by equation (61) is reduced simply to

\[
D_{(+)}(k) = -\frac{2i}{\pi} J_1(k) \left( \tilde{A}_r - i \frac{2\Omega}{\omega - m\Omega} \tilde{A}_\varphi \right),
\]

(71)

where \( H_+(k) \) is the unit-step function, i.e., \( H_+(k) = 0 \) for \( k < 0 \), while \( H(k) = 1 \) for \( k > 0 \). Since \( D_{(+)}(k) \) does not involve \( x \), the integration with respect to \( x \) in equation (66) leads to \( 2\pi \delta(k) \). Hence, performing the integration with respect to \( k \) in equation (66), we have

\[
W_{H, +} = -\frac{\omega_0}{2} \pi (2\pi)^{3/2} \bar{n} \frac{(r H \rho_0 \Omega c)}{|J_1|} \left| \tilde{A}_r - i \frac{2\Omega}{\omega - m\Omega} \tilde{A}_\varphi \right|^2,
\]

(72)

in the case of \( \epsilon / J_1 > 0 \), where \( \bar{m} = m + 1 \) and \( \bar{n} = n - 1 \), or \( n \), or \( n + 1 \), depending on the type of couplings and the form of deformation.

In the case of \( \epsilon / J_1 < 0 \), \( \tilde{\xi}_{\text{int}}(k) \) is given by equation (67). By performing the same procedures as those in the case of \( \epsilon / J_1 > 0 \), we can derive an expression for \( W_{H, +} \). The final expression for \( W_{H, +} \) in this case has the opposite sign from equation (72).

The inequality \( \epsilon / J_1 > 0 \) means that \( (\omega - m\Omega) / J_1 > 0 \), since \( -\omega_0 \) should be taken tentatively to be negative as mentioned before, while the inequality \( \epsilon / J_1 < 0 \) means \( (\omega - m\Omega) / J_1 > 0 \). Considering this difference, we can summarize \( W_{H, +} \) in the above two cases of \( \epsilon / J_1 > 0 \) and \( \epsilon / J_1 < 0 \) to be

\[
W_{H, \pm, \pm} = -\frac{\omega_0}{2} \pi (2\pi)^{3/2} \bar{n} \frac{(r H \rho_0 \Omega c)}{|J_1|} \left| \tilde{A}_r - i \frac{2\Omega}{\omega - m\Omega} \tilde{A}_\varphi \right|^2,
\]

(73)

where \( \bar{m} = m + 1 \).

In the case of coupling through \( \tilde{\xi}_{\text{int}} \), we have, after similar processes as the above,

\[
W_{H, \pm, \mp} = -\frac{\omega_0}{2} \pi (2\pi)^{3/2} \bar{n} \frac{(r H \rho_0 \Omega c)}{|J_1|} \left| \tilde{A}_r - i \frac{2\Omega}{\omega - m\Omega} \tilde{A}_\varphi \right|^2,
\]

(74)

where \( \bar{m} = m - 1 \).

Finally, from \( W_{H, \pm, \pm} \) given above and the wave energy, \( E \), given by equation (15), we find that the growth rate, equation (38), is written as

\[
-\omega_0 H, \pm, \pm = -\frac{\pi \bar{n}}{2r c^2 |J_1| |\omega_0| E_{n}} \text{sign}(\omega - m\Omega)c \\
\times \left| \tilde{A}_r, \pm, \pm - i \frac{2\Omega}{\omega - m\Omega} \tilde{A}_\varphi, \pm, \pm \right|^2,
\]

(75)

which is identical to equation (48), and also equal to the growth rate derived in the previous paper (Kato 2008a), as will be mentioned at the end of this section.

4.1.2. The case of radially changing coupling terms

The arguments in subsection 4.1.1 show that in the case where \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \) are radially constant, the radial structure of resonant region is determined by [see equation (66), and we notice that \( D_{(+)}(k) \) has the step-function]

\[
\int_0^\infty dk \exp \left[ -\frac{\epsilon}{J_1} k + i \left( k x - \frac{1}{3} \alpha_3^3 k^3 \right) \right],
\]

(76)

where the term with \( \epsilon \) can be practically neglected. The \( x \)-dependence of this integral has been examined by Meyer-Vernet and Sicardy (1987) [see equation (41) and figure 5 in their paper]. They show that the half width of the resonant region is \( \sim \alpha_3 \), which is on the order of \( (r H^2)^{1/3} \), i.e., \( r (H / r)^{2/3} \), in the present case.

In general, \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \) radially oscillate with the wavelength determined by the product of \( \bar{\xi} \) and \( \bar{\xi}^W \). Hence, in some cases the scale-length of the variation of \( \tilde{A}_r \)'s will be shorter than \( (r (H / r)^{2/3}) \), although it depends on where the resonance occurs (see the next section). To examine general cases, numerical calculations are needed. However, in order to have a rough image of the effects of the spatial variation of \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \) on the work integral, \( W_{H, \pm, \pm} \), we consider here a limiting case where \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \) oscillate periodically in the radial direction. That is, we assume

\[
\tilde{A}_r(x) = \tilde{A}_r^{(0)} \exp(ik_0 x), \quad \tilde{A}_\varphi(x) = \tilde{A}_\varphi^{(0)} \exp(ik_0 x), \quad \tilde{A}_z(x) = \tilde{A}_z^{(0)} \exp(ik_0 x),
\]

(77)

where \( \tilde{A}_r^{(0)} \), \( \tilde{A}_\varphi^{(0)} \), and \( k_0 \) are the amplitudes and wavenumber of the spatial variations. Then, the Fourier transform, defined by equation (55), shows that

\[
\tilde{A}_r(k) = 2\pi \tilde{A}_r^{(0)} \delta(k - k_0), \quad \tilde{A}_\varphi(k) = 2\pi \tilde{A}_\varphi^{(0)} \delta(k - k_0), \quad \tilde{A}_z(k) = 2\pi \tilde{A}_z^{(0)} \delta(k - k_0).
\]

(78)

Let us consider the case of \( \epsilon / J_1 > 0 \). In this case, \( D_{(+)}(k) \) given by equation (61) is reduced to [cf. equation (71)]

\[
D_{(+)}(k) = -\frac{2i}{\pi} J_1 H_+(k - k_0) \times \left[ \tilde{A}_r^{(0)} - i \frac{2\Omega}{\omega - m\Omega} \tilde{A}_\varphi^{(0)} \right] \exp \left( \frac{\epsilon k_0^2}{J_1} + i \frac{1}{3} \alpha_3^3 k_0^3 \right).
\]

(79)

Since \( \tilde{A}_r, \tilde{A}_\varphi, \) and \( \tilde{A}_z \) are given by equation (77), the performance of integration with respect to \( x \) in equation (66) gives \( 2\pi \delta(k - k_0) \). Next, we perform the integration with respect to \( k \). Since \( D_{(+)}(k) \) is now given by equation (79), we have
In deriving equation (85), in order to avoid unnecessary complications, we have neglected \(k^2c^2_z\) compared with \(J_1\), since the magnitude of \(kH\) contributing to the present formulations is smaller than unity, i.e., \(kH \ll 1\).

In the case of \(\epsilon/J_1^2 > 0\), equation (85) is solved as

\[
\tilde{\xi}^\text{int}_z(k) = D_{V,(+)}(k)\exp\left(-\frac{\epsilon}{J_2} - i\frac{1}{3}\alpha_{p,V}^2k^3\right),
\]

where the coefficient \(D_{V,(+)}(k)\) is given by

\[
D_{V,(+)}(k) = -i\frac{1}{J_2} \int_{-\infty}^{k} \left[i\tilde{k}\bar{\Omega}_\xi^2 \frac{H}{J_1} \times \left(\tilde{A}_r - i\frac{2\Omega}{\omega - m\Omega}\bar{A}_\psi\right) + \tilde{A}_z\right] \times \exp\left(\frac{\epsilon}{J_2}k' + i\frac{1}{3}\alpha_{p,V}^2k^3\right)dk'.
\]

By using \(\tilde{\xi}^\text{int}_z(k)\) given above, we can write \(\tilde{\xi}^\text{int}_r\) and \(\tilde{\xi}^\text{int}_\psi\) as [see equations (83) and (33)]

\[
\tilde{\xi}^\text{int}_r = -i\frac{k\bar{\Omega}_r^2 \tilde{H} \tilde{\xi}^\text{int}_z + 1}{J_1} \left(\tilde{A}_r - i\frac{2\Omega}{\omega - m\Omega}\bar{A}_\psi\right),
\]

and

\[
\tilde{\xi}^\text{int}_\psi = \frac{2\Omega}{\omega - m\Omega}\tilde{H} \tilde{\xi}^\text{int}_z + \frac{1}{J_1} \left(\tilde{A}_r - i\frac{2\Omega}{\omega - m\Omega}\bar{A}_\psi\right) - \frac{1}{(\omega - m\Omega)^2}\bar{A}_\psi.
\]

Substitution of these relations into the work integral (36) gives

\[
W_{V,+} = \frac{\omega_0}{2} \int dx \int_{-\infty}^{\infty} dk \frac{1}{2\pi} \rho_0(2\pi)^{3/2}\tilde{n}!rH
\]

\[
\times \left[D_{V,(+)}(k)\exp\left(-\frac{\epsilon}{J_2} - i\frac{1}{3}\alpha_{p,V}^2k^3\right)\right]
\]

\[
\times \left[D_{V,(-)}(k)\exp\left(-\frac{\epsilon}{J_2} - i\frac{1}{3}\alpha_{p,V}^2k^3\right)\right]
\]

\[
\times \int_{-\infty}^{k} \left[i\tilde{k}\bar{\Omega}_\xi^2 \frac{H}{J_1} \times \left(\tilde{A}_r - i\frac{2\Omega}{\omega - m\Omega}\bar{A}_\psi\right) + \tilde{A}_z\right] \times \exp\left(\frac{\epsilon}{J_2}k' + i\frac{1}{3}\alpha_{p,V}^2k^3\right)dk'.
\]

In the case of \(\epsilon/J_1^2 < 0\), \(W_{V,+}\) has a similar form, but we should use, instead of \(D_{V,(+)}(k)\), \(D_{V,(-)}(k)\), which is given by

\[
D_{V,(-)}(k) = -i\frac{1}{J_2} \int_{-\infty}^{k} \left[i\tilde{k}\bar{\Omega}_\xi^2 \frac{H}{J_1} \times \left(\tilde{A}_r - i\frac{2\Omega}{\omega - m\Omega}\bar{A}_\psi\right) + \tilde{A}_z\right] \times \exp\left(\frac{\epsilon}{J_2}k' + i\frac{1}{3}\alpha_{p,V}^2k^3\right)dk'.
\]

As in subsubsection 4.1.1, we consider two cases where i) \(\tilde{A}_r\) is constant in the resonant region and ii) \(\tilde{A}_z\) is spatially periodic as \(\tilde{A}_z^{(0)}(\text{exp}(ik_0x)). In the former case, summing the two cases of \(\epsilon/J_1^2 > 0\) and \(\epsilon/J_2\), we have

\[
W_{V,+} = -\frac{\omega_0}{2}(2\pi)^{3/2}(\tilde{n} - 1)! \frac{(rH\rho_0)c}{|J_2|^2} \times \text{sign}(\omega - m\Omega)\tilde{A}_{\text{int}+}^2,
\]

and in the latter case we have

\[
W_{V,+} = -\frac{\omega_0}{2}(2\pi)^{3/2}(\tilde{n} - 1)! \frac{(rH\rho_0)c}{|J_2|^2} \times \text{sign}(\omega - m\Omega)\tilde{A}_{\text{int}+}^2\tilde{A}_{\text{int}+}^2.
\]
In the coupling through $\xi_r$, the work integral, $W_{\psi,-\psi}$, has the same expression as equations (93) and (94), except that the subscript $+$ is now changed to $-\psi$.

After the above considerations, we summarize the growth rate, $\omega_{h,\psi,\pm,\psi}$, in the former case of $\Delta_1$ being constant as

$$-\omega_{h,\psi,\pm,\psi} = -\frac{\pi(\tilde{u} - 1)!}{2r^2|J|_2|0_0|E_n} \text{sign}(\omega - \tilde{m}\Omega) \big| \tilde{A}_{\psi,\pm,\psi} \big|^2. \quad (95)$$

In the case where $\Delta_1$ changes as $\Delta_1 = \Delta_{01}^2 \exp(i k_{01} x)$, we have

$$-\omega_{h,\psi,\pm,\psi} = -\frac{\pi(\tilde{u} - 1)!}{2r^2|J|_2|0_0|E_n} \text{sign}(\omega - \tilde{m}\Omega) \bigg( \tilde{\Delta}_{\psi,\pm,\psi} - i \frac{2\Omega}{\omega - \tilde{m}\Omega} \bigg) \bigg( \tilde{A}_{\psi,\pm,\psi} \bigg)^2. \quad (96)$$

4.3. Comparison with Previous Results

Finally, we note here that the growth rates obtained in previous papers are the same as those obtained in subsection 4.1.1, although the final expression in the previous papers are different from those in subsection 4.1.1. In the case of horizontal resonance, for example, $G_{H,\psi}$ in Kato (2008a) is equal to $r_{cJ}/2$ in the present paper. Furthermore, if $\hat{\xi}_{\psi}^{\text{int}}$ and $\hat{\xi}_{\psi}^{\text{ext}}$ are eliminated from equations (46)–(49) (Kato 2008a), we have

$$\hat{\xi}_{\psi}^{\text{int}} = \frac{J_2}{D} \bigg( \tilde{A}_{\psi} - i \frac{2\Omega}{\omega - \tilde{m}\Omega} \bigg). \quad (97)$$

where $D$ is $J_1 J_2 - c_k^2 \kappa^2 (\omega - \tilde{m}\Omega)^2$. This expression for $\hat{\xi}_{\psi}^{\text{int}}$ shows that $\hat{\xi}_{\psi}$ defined by equation (52) in Kato (2008a) can be written as

$$\hat{\xi}_{\psi} = J_2 \bigg( \tilde{A}_{\psi} - i \frac{2\Omega}{\omega - \tilde{m}\Omega} \bigg). \quad (98)$$

Then, substituting $G_{H} = r_{cJ}/2$ and equation (98) with $(J_2/c_k = (\kappa^2 + \tilde{m}\Omega)^2)$ into equation (70) in Kato (2008a), we obtain an expression for growth rate in the case of horizontal resonances, which is identical to equation (75) in subsection 4.1.1.

Similar arguments easily show that the growth rate given by Kato (2008a) for vertical resonances, i.e., equation (71) in Kato (2008a), is identical to that in subsection 4.2, i.e., equation (94).

5. Discussion

In this paper, we have derived a stability criterion on the resonant excitation of disk oscillations in a deformed disk. The deformation of disks that can lead to the excitation of disk oscillations are a warp and an eccentric deformation of the disk plane, i.e., $m_W = 1$, although this is not discussed in this paper (see Kato 2004, 2008a). In previous papers (Kato 2004, 2008a), the above resonant excitation problem was examined and a condition for the excitation of disk oscillations was derived. From a mathematical point of view, however, the treatment of disk oscillations around a resonant point was not suitable in the previous papers. In this paper, we corrected the inadequate points. The results of the correction show that the stability condition and the growth rate derived in the previous papers remain qualitatively unchanged. In this paper, however, the growth rates of oscillations is written in a form applicable more easily to numerical calculations of the growth rate.

In disks with pressure, the resonant region is widened by pressure effects. Based on arguments by Meyer-Vernet and Sicardy (1987), we show in section 4 that the half-width of the resonant region is $\sim r(H/r)^{1/3}$ [or $\sim H(r/H)^{1/3}$], which is shorter than $r$, but longer than $H$. The coupling terms come from nonlinear products of the original oscillation, $\xi$, and disk deformation, $\xi_W$. If the disk deformation is assumed to be global, the scale of the spatial variation of the coupling terms is mainly governed by the spatial behavior of the original oscillation, $\xi$. Hence, in general, the width of the resonant region is comparable to the wavelength of the original oscillations, and may not be narrow enough to regard the coupling terms $\Delta_1$ as being constant in the region. Hence, in section 4, we examine two limiting cases where i) the coupling terms are constant in the resonant region and ii) they vary sinusoidally in the region. Analyses in section 4 show that in these two limiting cases there are no essential differences in the growth rate, especially no difference in the criterion for stability. To examine what happens in intermediate cases of the above two limiting ones, however, numerical calculations will be needed.

It is noted that the case where the coupling terms, $\Delta_1$, are roughly constant in the resonant region is realized when resonance occurs near to the boundary between the propagation and evanescent regions of the original oscillations. This is because near to the boundary, the radial wavelength of the oscillations is long. Applications of the present disk oscillation model to high-frequency QPOs by Kato and Fukue (2006) and Kato (2008a) are of concern in such cases. Another limiting case where the coupling terms vary in the radial direction is realized when resonance occurs in the propagation region far apart from the boundary between the propagation and evanescent regions.

There are two kinds of resonances, i.e., horizontal and vertical. In both resonances, the condition of excitation of oscillations is that the wave energy of the original oscillation, $E_n$, and $\omega - \tilde{m}\Omega$ at the resonant radius have opposite signs, i.e., $E_n \cdot \text{sign}(\omega - \tilde{m}\Omega)_c < 0$. For the resonance to occur efficiently, the resonant radius must be within the region where both the original and intermediate oscillations exist predominantly. Hence, it will be reasonable to suppose that the sign of $(\omega - \tilde{m}\Omega)_c$ roughly represents the sign of the energy of the intermediate oscillation. [Notice that the sign of the wave energy of the intermediate oscillation is determined by the sign of $\omega - \tilde{m}\Omega$ in the region where the wave exists predominantly; see equations (38) and (39)]. In other words, the condition $E_n \cdot \text{sign}(\omega - \tilde{m}\Omega)_c < 0$ is roughly equal to the condition that the original and intermediate oscillations have energies of opposite signs. This instability criterion can thus be understood as being a result of energy exchange between the original and intermediate oscillations at the resonant point (Ferreira & Ogilvie 2008).

The resonant instability condition, i.e., $E_n \cdot \text{sign}(\omega - \tilde{m}\Omega)_c < 0$, however, is not exactly equal to the condition of both
oscillations (original and intermediate one) having opposite signs. This suggests that the energy exchange at the resonant point is not only between the above two oscillations, but that disk rotation also plays a part, and gains or losses its energy. In the case of \( E_n \cdot \text{sign}(\omega - \dot{m}\Omega) > 0 \), the original oscillation is damped. Roughly speaking, in this case, both oscillations (original and intermediate ones) have the same signs for their wave energy. Hence, in this case, the main energy flow will be between disk rotation and the oscillations, so that the oscillations are damped. It is noticed that in vertical resonance we always have \( E_n \cdot \text{sign}(\omega - \dot{m}\Omega) > 0 \) in the case where a resonant radius can appear in the disk plane, and the oscillations are always damped (see Kato 2004, 2008a).

Finally, it is noted that the present resonant excitation mechanism has been applied to quasi-periodic oscillations observed in black-hole and neutron-star X-ray binaries, for example, by Kato and Fukue (2006), Kato (2008a) and Kato et al. (2008).

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