Relative $p$-adic Hodge theory, I: Foundations

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Abstract

We initiate a new approach to relative $p$-adic Hodge theory based on systematic use of Witt vector constructions and nonarchimedean analytic geometry in the style of Berkovich. In this paper, we give a thorough development of $\varphi$-modules over a relative Robba ring associated to a perfect Banach ring of characteristic $p$, including the relationship between these objects and étale $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems on the algebraic and analytic spaces associated to the base ring, and the relationship between étale cohomology and $\varphi$-cohomology. We also make a critical link to mixed characteristic by exhibiting an equivalence of tensor categories between the finite étale algebras over an arbitrary perfect Banach algebra over a nontrivially normed complete field of characteristic $p$ and the finite étale algebras over a corresponding Banach $\mathbb{Q}_p$-algebra. This recovers the homeomorphism between the absolute Galois groups of $\mathbb{F}_p((\pi))$ and $\mathbb{Q}_p(\mu_{p^\infty})$ given by the field of norms construction of Fontaine and Wintenberger, as well as generalizations considered by Andreatta, Brinon, Faltings, Gabber, Ramero, Scholl, and most recently Scholze. Applications to the description of étale local systems on nonarchimedean analytic spaces over $p$-adic fields will be described in subsequent papers.

Contents

0 Introduction
  0.1 Artin-Schreier theory and $(\varphi, \Gamma)$-modules
  0.2 Arithmetic vs. geometric
  0.3 Goals of geometric relative $p$-adic Hodge theory
  0.4 Analytic spaces associated to Banach algebras
  0.5 Comparison of étale topologies
  0.6 Robba rings and slope theory
  0.7 $\varphi$-modules and local systems

1 Algebro-geometric preliminaries
  1.1 Definitions and conventions
  1.2 Comparing étale algebras
0  Introduction

After its formalization by Deligne [30], the subject of Hodge theory may be viewed as the study of the interrelationship among different cohomology theories associated to algebraic varieties over \( \mathbb{C} \), most notably singular (Betti) cohomology and the cohomology of differential forms (de Rham cohomology). From work of Fontaine and others, there emerged a parallel subject of \( p \)-adic Hodge theory concerning the interrelationship among different cohomology theories associated to algebraic varieties over finite extensions of \( \mathbb{Q}_p \), most notably étale cohomology with coefficients in \( \mathbb{Q}_p \) and algebraic de Rham cohomology.

In ordinary Hodge theory, the relationship between Betti and de Rham cohomologies is forged using the Riemann-Hilbert correspondence, which relates topological data (local systems) to analytic data (integrable connections). In \( p \)-adic Hodge theory, one needs a similar correspondence relating de Rham data to étale \( \mathbb{Q}_p \)-local systems, which arise from the étale cohomology functor on schemes of finite type over \( K \). However, in this case the local systems turn out to be far more plentiful, so it is helpful to first build a correspondence relating them to some sort of intermediate algebraic objects. This is achieved by the theory of \((\varphi, \Gamma)\)-modules, which gives some Morita-type dualities relating étale \( \mathbb{Q}_p \)-local systems over \( K \) (i.e., continuous representations of the absolute Galois group \( G_K \) on finite-dimensional \( \mathbb{Q}_p \)-vector spaces) with modules over certain mildly noncommutative topological algebras. The latter appear as topological monoid algebras over certain commutative period rings for certain continuous operators (the eponymous \( \varphi \) and \( \Gamma \)).

One of the key features of Hodge theory is that it provides information not just about individual varieties, but also about families of varieties through the mechanism of variations of Hodge structures. Only recently has much progress been made in developing any analogous constructions in \( p \)-adic Hodge theory; part of the difficulty is that there are two very different directions in which relative \( p \)-adic Hodge theory can be developed. In the remainder of this introduction, we first give a bit more background about \((\varphi, \Gamma)\)-modules, and contrast the arithmetic and geometric forms of relative \( p \)-adic Hodge theory. We then describe our plans in geometric relative \( p \)-adic Hodge theory for this series of papers, and indicate how much is achieved in this particular paper.

0.1  Artin-Schreier theory and \((\varphi, \Gamma)\)-modules

For \( K \) a perfect field of characteristic \( p \), the discrete representations of the absolute Galois group \( G_K \) of \( K \) on finite dimensional \( \mathbb{F}_p \)-vector spaces form a category equivalent to
the category of $\varphi$-modules over $K$, i.e., finite-dimensional $K$-vector spaces equipped with isomorphisms with their $\varphi$-pullbacks. This amounts to a nonabelian generalization of the Artin-Schreier description of $(\mathbb{Z}/p\mathbb{Z})$-extensions of fields of characteristic $p$ [32, Expos´e XXII, Proposition 1.1].

A related result is that the continuous representations of $G_K$ on finite free $\mathbb{Z}_p$-modules form a category equivalent to the category of finite free $W(K)$-modules equipped with isomorphisms with their $\varphi$-pullbacks. Here $W(K)$ denotes the ring of Witt vectors over $K$, and $\varphi$ denotes the unique lift to $W(K)$ of the absolute Frobenius on $K$. One can further globalize this result to arbitrary smooth schemes over $K$, in which the corresponding category becomes a category of unit-root $F$-crystals; see [24, Theorem 2.2] or [66, Proposition 4.1.1].

Fontaine’s theory of $(\varphi, \Gamma)$-modules [40] provides a way to extend such results to mixed-characteristic local fields. The observation underpinning the theory is that a sufficiently wildly ramified extension of a mixed-characteristic local field behaves Galois-theoretically just like a local field of positive characteristic. For example, for $K_0$ a finite unramified extension of $\mathbb{Q}_p$ (or the completion of an infinite algebraic unramified extension of $\mathbb{Q}_p$) with residue field $k_0$, the fields $K_0(\mu_{p^{\infty}})$ and $k_0((T))$ have homeomorphic Galois groups. One can describe representations of the absolute Galois group of a local field by restricting to a suitably deeply ramified extension, applying an Artin-Schreier construction, then adding appropriate descent data to get back to the original group. This assertion is formalized in the theory of fields of norms introduced by Fontaine and Wintenberger [41] [100]; some of the analysis depends on Sen’s calculation of ramification numbers in $p$-adic Lie extensions [93]. See [18, Part 4] for a detailed but readable exposition.

0.2 Arithmetic vs. geometric

As noted earlier, there are two different directions in which one can develop relative forms of $p$-adic Hodge theory. We distinguish these as arithmetic and geometric.

In arithmetic relative $p$-adic Hodge theory, one still treats continuous representations of the absolute Galois group of a finite extension of $\mathbb{Q}_p$. However, instead of taking representations simply on vector spaces, one allows finite locally free modules over affinoid algebras over $\mathbb{Q}_p$. Interest in the arithmetic theory arose originally from the consideration of $p$-adic analytic families of automorphic forms and their associated families of Galois representations; such families include Hida’s $p$-adic interpolation of ordinary cusp forms [62], the eigencurve of Coleman-Mazur [21], and further generalizations. Additional interest has come from the prospect of a $p$-adic local Langlands correspondence which would be compatible with formation of analytic families on both the Galois and automorphic sides. For the group $GL_2(\mathbb{Q}_p)$, such a correspondence has recently emerged from the work of Breuil, Colmez, Emerton, et al. (see for instance [23]) and has led to important advances concerning modularity of Galois representations, in the direction of the Fontaine-Mazur conjecture [76], [36].

In the arithmetic setting, there is a functor from Galois representations to families of $(\varphi, \Gamma)$-modules, constructed by Berger and Colmez [11]. However, this functor is not an equivalence of categories; rather, it can only be inverted locally [28], [74]. It seems that in this setting, one is forced to study families of Galois representations in the context of the
larger category of families of \((\varphi, \Gamma)\)-modules. For instance, one sees this distinction in the relative study of Colmez’s trianguline Galois representations [22], as in the work of Bellaïche [7] and Pottharst [86].

By contrast, in geometric relative \(p\)-adic Hodge theory, one continues to consider continuous representations acting on finite-dimensional \(\mathbb{Q}_p\)-vector spaces. However, instead of the absolute Galois group of a finite extension of \(\mathbb{Q}_p\), one allows étale fundamental groups of affinoid spaces over finite extensions of \(\mathbb{Q}_p\). The possibility of developing an analogue of \((\varphi, \Gamma)\)-module theory in this setting emerged from the work of Faltings, particularly his almost purity theorem [37, 38], and has been carried out most thoroughly to date by Andreatta and Brinon [1, 2]. A similar construction was described by Scholl [90]. (See also the exposition by Olsson [84].)

0.3 Goals of geometric relative \(p\)-adic Hodge theory

We now describe the target results for this series of papers. For a description of what appears in this particular paper, see the remainder of this introduction.

A principal goal of this series of papers is to construct an equivalence of categories from the category of étale \(\mathbb{Q}_p\)-local systems on a fairly general nonarchimedean analytic space in the sense of Berkovich [12, 13] to a category of objects resembling \((\varphi, \Gamma)\)-modules. This requires a generalization of the work of Andreatta and Brinon with better functoriality properties and without any smoothness conditions. Using methods of Fargues and Fontaine (first introduced in [39] and developed in subsequent work), one can replace the category of \((\varphi, \Gamma)\)-modules by the category of equivariant vector bundles on a suitable scheme carrying an action of a \(p\)-adic Lie group, subject to a stability condition. This creates a \(p\)-adic picture with strong resemblance to the correspondence between stable vector bundles on compact Riemann surfaces and irreducible unitary fundamental group representations, as constructed by Narasimhan and Seshadri [83].

Another goal is to construct certain “tautological” local systems on period spaces of \(p\)-adic Hodge structures (filtered \((\varphi, N)\)-modules). Such period spaces arise, for instance, in the work of Rapoport and Zink on period mappings on deformation spaces of \(p\)-divisible groups [87]. The construction we have in mind is outlined in [71].

0.4 Analytic spaces associated to Banach algebras

We now turn to the topics addressed by this particular paper. The first substantial chunk of the paper concerns the study of Gel’fand spectra of nonarchimedean (commutative) Banach rings, in the sense of Berkovich. Most of the basic theory is well-understood for the spectra of affinoid algebras over fields, which are used to build Berkovich’s analogues of Tate’s rigid analytic spaces. However, in our work, we will be forced to deal with Banach rings which are not affinoid algebras, so it is important to see how far the basic theory can be developed without restricting to the affinoid case. To avoid difficulties with nilpotents (which are in any case irrelevant to the study of étale local systems), we work primarily with uniform Banach rings, for which the norm is not just submultiplicative but power-multiplicative. For
uniform Banach rings, one has a usable notion of rational subspaces; one key result, proved by an approximation argument using affinoid algebras, is that finite étale morphisms glue over finite coverings by rational subspaces (Theorem 2.6.10).

We consider Berkovich spaces rather than rigid analytic spaces because Berkovich’s framework provides a convenient formalism for comparing the geometry of analytic spaces in different characteristics. However, while Berkovich’s framework is sufficient for our present purposes, it may be necessary in the future to switch to Huber’s broader theory of adic spaces [64]. For instance, adic spaces appear to be the narrowest category suitable for the work of Hellmann [60, 61] on the moduli spaces for Breuil-Kisin modules described by Pappas and Rapoport [85]. The study of these spaces seems to include features of both arithmetic and geometric relative $p$-adic Hodge theory, which appears to render both Tate and Berkovich spaces insufficient. (Adic spaces are also the preferred language of Scholze [91, 92].)

0.5 Comparison of étale topologies

As noted earlier, one of the main techniques of $p$-adic Hodge theory is the relationship between the absolute Galois groups of certain fields of mixed and positive characteristic, such as $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((T))$. For relative $p$-adic Hodge theory, it is necessary to extend this correspondence somewhat further. However, instead of an approach dependent on ramification theory, we use a construction based on analysis of Witt vectors.

Suppose first that $L$ is a perfect field of characteristic $p$ complete for a multiplicative norm, with valuation ring $\mathfrak{o}_L$. Let $W(\mathfrak{o}_L)$ be the ring of $p$-typical Witt vectors over $\mathfrak{o}_L$. One can generate certain fields of characteristic 0 by quotienting $W(\mathfrak{o}_L)$ by certain principal ideals and then inverting $p$. For instance, for $L$ the completed perfect closure of $\mathbb{F}_p((\pi))$ and $z = \sum_{i=0}^{p-1} \frac{1}{i} \in W(\mathfrak{o}_L)$, we may identify $W(\mathfrak{o}_L)/(z)$ with the completion of $\mathbb{Q}_p(\mu_{p^\infty})$. The relevant condition (that of being primitive of degree 1 in the sense of Fargues and Fontaine [39]) is a Witt vector analogue of the property of an element of $\mathbb{Z}_p[T]$ being associated to a monic linear polynomial whose constant term is not invertible in $\mathbb{Z}_p$ (which allows use of the division algorithm to identify the quotient by this element with $\mathbb{Z}_p$). See Definition 3.3.4.

Using this construction, we obtain a correspondence between certain complete fields of mixed characteristic (which we call perfectoid fields, following [91]) and perfect fields of characteristic $p$ together with appropriate principal ideals in the ring of Witt vectors over the valuation ring (see Theorem 3.5.3). It is not immediate from the construction that the former category is closed under formation of finite extensions, but this turns out to be true and not too difficult to check (see Theorem 3.5.6). In particular, we recover the field of norms correspondence.

To extend this correspondence to more general Banach algebras, we exploit a relationship developed in [72] between the Berkovich space of a perfect uniform Banach ring $R$ of characteristic $p$ and the Berkovich space of the ring $W(R)$. (For instance, any subspace of $\mathcal{M}(R)$
has the same homotopy type as its inverse image under $\mu$ [72, Corollary 7.9]. This leads to a correspondence between uniform Banach algebras over a perfectoid field and uniform Banach algebras over the corresponding field of characteristic $p$, which is compatible with formation of both rational subspaces and finite étale covers (see Theorem 3.6.5 and Theorem 3.6.20). One also obtains a result in the style of Faltings’s *almost purity theorem* [37, 38] (see also [44, 45]), which underlies the aforementioned generalization of $(\varphi, \Gamma)$-modules introduced by Andreatta and Brinon [1, 2]. This will lead us to a variation on the work of Andreatta and Brinon, to be discussed in a subsequent paper.

After preparing the initial version of this paper, we discovered that similar results had been obtained by Scholze [91]. Subsequently, discussions with Scholze led to some refinements on both sides; in particular, the term *perfectoid* is taken from [91], and the formulation of Theorem 3.6.14 and Proposition 3.6.9 owe a great debt to these discussions. (The effect in the opposite direction is described in the introduction to [91].) It is worth noting that Scholze takes the perfectoid correspondence somewhat further than we do, introducing a theory of *perfectoid spaces* in which the previous discussion is the local situation. (Note that these spaces are adic spaces rather than Berkovich spaces, although the distinction is not so serious in this context.) This leads Scholze to an unexpected and spectacular application: the resolution of some new cases of the weight-monodromy conjecture in $\ell$-adic étale cohomology.

### 0.6 Robba rings and slope theory

Another important technical device in usual $p$-adic Hodge theory is the classification of Frobenius-semilinear transformation on modules over certain power series by *slopes*, in rough analogy with the classification of vector bundles on curves. This originated in work of the first author [67]; we extend this work here to the relative setting. (The relevance of such results to $p$-adic Hodge theory largely factors through the work of Berger [8, 9], to which we will return in a later paper.)

In [67], one starts with the *Robba ring* of germs of analytic functions on open annuli of outer radius 1 over a $p$-adic field, and then passes to a certain “algebraic closure” thereof. The latter can be constructed from the ring of Witt vectors over the completed algebraic closure of a power series field. One is thus led naturally to consider similar constructions starting from the ring of Witt vectors over a general analytic field; the analogues of the results of [67] were worked out by the first author in [68].

Using the previously described work largely as a black box, we are able to introduce analogues of Robba rings starting from the ring of Witt vectors of a perfect uniform $F_p$-algebra (and obtain some weak analogues of the theorems of Tate and Kiehl), and to study slopes of Frobenius modules thereof. We obtain semicontinuity of the slope polygon as a function on the spectrum of the base ring (Theorem 7.4.5) as well as a slope filtration theorem when this polygon is constant (Theorem 7.4.8). (Similar results in the arithmetic relative setting have recently been obtained by the second author [80].)

We also obtain a description of Frobenius modules over relative Robba rings in the style of Fargues and Fontaine, using vector bundles over a certain scheme (Theorem 6.3.12). When the base ring is an analytic field, the scheme in question is connected, regular, separated,
and noetherian of dimension 1; it might thus be considered to be a complete absolute curve. (The adjective absolute means that the curve cannot be seen as having relative dimension 1 over a point; it is a scheme over $\mathbb{Q}_p$, but not of finite type.)

0.7 $\varphi$-modules and local systems

Putting everything together, we obtain a link between étale local systems and $\varphi$-modules, in what amounts to a broad nonabelian generalization of Artin-Schreier theory. We obtain some equivalences of tensor categories between $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems on certain $\mathbb{Q}_p$-algebras and Frobenius modules ($\varphi$-modules) over some rings derived from Witt vectors (Theorems 8.1.2, 8.1.4, 8.1.9). We are also able to describe étale cohomology of local systems in terms of cohomology of $\varphi$-modules (Theorems 8.2.1, 8.2.3, 8.3.6) and coherent sheaf cohomology (Theorem 8.3.3).

In subsequent papers, these results will be used to construct and analyze relative $(\varphi, \Gamma)$-modules. Some links will also be made with the work of Scholze on relative étale-de Rham comparison isomorphisms [92].

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1 Algebro-geometric preliminaries

Before proceeding to analytic geometry, we start with some background facts from algebraic geometry.

Hypothesis 1.0.1. Throughout this paper, fix a prime number $p$.

Convention 1.0.2. When we refer to a tensor category, we will always assume it is equipped not just with the usual monoidal category structure, but also with a rank function into some abelian group. Equivalences of tensor categories (or for short tensor equivalences) will be assumed to respect rank.

1.1 Definitions and conventions

Convention 1.1.1. Throughout this paper, all rings are assumed to be commutative and unital unless otherwise indicated.
Definition 1.1.2. Let $M$ be a module over a ring $R$. We say that $M$ is pointwise free if $M \otimes_R R_p$ is a free module over $R_p$ for each maximal ideal $p$ of $R$ (and hence for each prime ideal). The term locally free is sometimes used for this, but it is better to make this term match its usual meaning in sheaf theory by saying that $M$ is locally free if there exist $f_1, \ldots, f_n \in R$ generating the unit ideal such that $M \otimes_R R[f_i^{-1}]$ is a free module over $R[f_i^{-1}]$ for $i = 1, \ldots, n$.

We say that $M$ is projective if it is a direct summand of a free module. The following conditions are equivalent [16 §II.5.2, Théorème 1].

(a) $M$ is finitely generated and projective.

(b) $M$ is a direct summand of a finite free module.

(c) $M$ is finitely presented and pointwise free.

(d) $M$ is finitely generated and pointwise free of locally constant rank. (The rank of $M$ at $p \in \text{Spec}(R)$ is defined as $\dim_{R_p/pR_p}(M_p/pM_p)$.)

(e) $M$ is finitely generated and locally free.

For $R$ reduced, it is enough to check that $M$ is finitely generated of locally constant rank; see [34, Exercise 20.13]. For an analogous argument for Banach rings, see Lemma 2.3.12(b) and Lemma 2.6.4.

Definition 1.1.3. Let $M$ be a module over a ring $R$. We say $M$ is faithfully flat if $M$ is flat and $M \otimes_R M \neq 0$ for every nonzero $R$-module $N$. Since tensor products commute with direct limits, $M$ is faithfully flat if and only if it satisfies the following conditions.

(a) For any injective homomorphism $N \to P$ of finite $R$-modules, $M \otimes_R N \to M \otimes_R P$ is injective.

(b) For any nonzero finite $R$-module $N$, $M \otimes_R N$ is nonzero.

For other characterizations, see [16 §I.3.1, Proposition 1].

Lemma 1.1.4. A flat ring homomorphism $R \to S$ is faithfully flat if and only if every maximal ideal of $R$ is the contraction of a maximal ideal of $S$.

Proof. See [16 §I.3.5, Proposition 9].

1.2 Comparing étale algebras

We will expend a great deal of effort comparing finite étale algebras over different rings. A key case is given by base change from a ring to a quotient ring, in which the henselian property plays a key role. (A rather good explanation of this material can be obtained from [44] by specializing from almost ring theory to ordinary ring theory.)
Definition 1.2.1. As in [52 Définition 17.3.1], we say a morphism of schemes is étale if it is locally of finite presentation and formally étale. (The latter condition is essentially a unique infinitesimal lifting property; see [52 Définition 17.1.1].) A morphism of rings is étale if the corresponding morphism of affine schemes is étale.

For any ring $R$, let $\mathbf{FÉt}(R)$ denote the tensor category of finite étale algebras over the ring $R$, with morphisms being arbitrary morphisms of $R$-algebras. Such morphisms are themselves finite and étale (e.g., by [52 Proposition 17.3.4]). Any $S \in \mathbf{FÉt}(R)$ is finite as an $R$-module and finitely presented as an $R$-algebra, and hence finitely presented as an $R$-module by [51 Proposition 1.4.7]. Also, $S$ is flat over $R$ by [52 Théorème 17.6.1]. By the criteria described in Definition 1.2.2, $S$ is finite projective as an $R$-module. Conversely, an $R$-algebra $S$ which is finite projective as an $R$-module is finite étale if and only if the $R$-module homomorphism $S \rightarrow \text{Hom}_R(S, R)$ taking $x$ to $y \mapsto \text{Trace}_{S/R}(xy)$ is an isomorphism. (Namely, since étaleness is an open condition [52 Remarques 17.3.2(iii)], this reduces to the case where $R$ is a field, which is straightforward.)

For short, we will describe a finite étale $R$-algebra which is faithfully flat over $R$ (or equivalently of positive rank everywhere over $R$, by Lemma 1.1.4 and the going-up theorem) as a faithfully finite étale $R$-algebra.

Definition 1.2.2. Let $R$ be a ring and let $U$ be an element of $\mathbf{FÉt}(R)$. Then there exists an idempotent element $e_{U/R} \in U \otimes_R U$ mapping to $1$ via the multiplication map $\mu : U \otimes_R U \rightarrow U$ and killing the kernel of $\mu$; see for instance [44 Proposition 3.1.4]. For $V$ another $R$-algebra and $e \in U \otimes_R V$ an idempotent element, let $\Gamma(e) : V \rightarrow e(U \otimes_R V)$ be the morphism of $R$-algebras sending $x \in V$ to $e(1 \otimes x)$. In case $\Gamma(e)$ is an isomorphism, we obtain a morphism $\psi_e : U \rightarrow V$ of $R$-algebras by applying the natural map $\Delta(e) : U \rightarrow e(U \otimes_R V)$ sending $x$ to $e(x \otimes 1)$ followed by $\Gamma(e)^{-1}$. Conversely, for $\psi : U \rightarrow V$ a morphism of $R$-algebras, the idempotent $e_\psi = (1 \otimes \psi)(e_{U/R}) \in U \otimes_R V$ has the property that $\Gamma(e_\psi)$ is an isomorphism (see [44 Proposition 5.2.19]).

Lemma 1.2.3. For $R$ a ring, $U \in \mathbf{FÉt}(R)$, and $V$ an $R$-algebra, the function $\psi \mapsto e_\psi$ defines a bijection from the set of $R$-algebra morphisms from $U$ to $V$ to the set of idempotent elements $e \in U \otimes_R V$ for which $\Gamma(e)$ is an isomorphism. The inverse map is $e \mapsto \psi_e$.

Proof. See [44 Lemma 5.2.20].

Lemma 1.2.4. Let $R$ be a ring and let $I$ be an ideal contained in the Jacobson radical of $R$.

(a) No two distinct idempotents of $R$ are congruent modulo $I$.

(b) For any integral extension $S$ of $R$, the ideal $IS$ is contained in the Jacobson radical of $S$.

Proof. For (a), let $e, e' \in R$ be idempotents with $e - e' \in I$. Modulo each maximal ideal of $R$, $e$ and $e'$ either both reduce to $1$ or both reduce to $0$; it follows that $(e + e' - 1)^2 \in 1 + I$, so $e + e' - 1$ is a unit in $R$. Since $(e - e')(e + e' - 1) = 0$, we have $e = e'$.

For (b), note that the set of $y \in S$ which are roots of monic polynomials over $R$ whose nonleading coefficients belong to $I$ is an ideal. Consequently, if $x \in 1 + IS$, it is a root of a
monic polynomial \( P \in R[T] \) for which \( P(T - 1) \) has all nonleading coefficients in \( I \). It follows that the constant coefficient \( P_0 \) of \( P \) belongs to \( \pm 1 + I \) and so is a unit; since \( P_0 = P_0 - P(x) \) is divisible by \( x \) (being the evaluation at \( x \) of the polynomial \( P_0 - P \)), \( x \) is also a unit. \( \square \)

**Proposition 1.2.5.** (a) Let \( R \rightarrow R' \) be a homomorphism of rings such that every element of \( R \) whose image in \( R' \) is a unit is itself a unit. Suppose that for each \( S \in \text{FÉt}(R) \), every idempotent element of \( S \otimes_R R' \) is the image of a unique idempotent element of \( S \). Then the base change functor \( \text{FÉt}(R) \rightarrow \text{FÉt}(R') \) is rank-preserving and fully faithful.

(b) Let \( R \) be a ring, and let \( I \) be an ideal contained in the Jacobson radical of \( R \). Suppose that for each \( S \in \text{FÉt}(R) \), every idempotent element of \( S/IS \) lifts to \( S \). Then the base change functor \( \text{FÉt}(R) \rightarrow \text{FÉt}(R/I) \) is rank-preserving and fully faithful.

Note that in case (a), if \( R \rightarrow R' \) is injective, then \( S \) injects into \( S \otimes_R R' \) because \( S \) is locally free as an \( R \)-module (see Definition 1.2.1), so the condition simply becomes that every idempotent element of \( S \otimes_R R' \) belongs to \( S \).

**Proof.** The rank-preserving property is evident in both cases. To check full faithfulness in case (a), note that the hypothesis on the homomorphism \( R \rightarrow R' \) implies that a map between finite projective \( R \)-modules is an isomorphism if and only if its base extension to \( R' \) is an isomorphism (because the isomorphism condition amounts to invertibility of the determinant). Consequently, for \( U, V \in \text{FÉt}(R) \) and \( e \in U \otimes_R V \) an idempotent element, the map \( \Gamma(e) : V \rightarrow e(U \otimes_R V) \) is an isomorphism if and only if its base extension to \( R' \) is an isomorphism. Lemma 1.2.3 then implies that every morphism \( U \otimes_R R' \rightarrow V \otimes_R R' \) of \( R' \)-algebras descends to a morphism \( U \rightarrow V \) of \( R \)-algebras, as desired.

To check full faithfulness in case (b), note that for \( U, V \in \text{FÉt}(R) \) and \( \overline{\tau} \in (U/\overline{I}U) \otimes_{R/I R} (V/\overline{I}V) \) an idempotent element, by Lemma 1.2.4 there is at most one idempotent \( e \in U \otimes_R V \) lifting \( \overline{\tau} \). Moreover, because \( I \) is contained in the Jacobson radical of \( R \), an element of \( R \) is a unit if and only if its image in \( R/I \) is a unit. We may thus apply (a) to deduce the claim. \( \square \)

**Definition 1.2.6.** A pair \((R, I)\) consisting of a ring \( R \) and an ideal \( I \subseteq R \) is said to be **henselian** if the following conditions hold.

(a) The ideal \( I \) is contained in the Jacobson radical of \( R \).

(b) For any monic \( f \in R[T] \), any factorization \( \overline{f} = \overline{g}\overline{h} \) in \((R/I)[T]\) with \( \overline{g}, \overline{h} \) monic and coprime lifts to a factorization \( f = gh \) in \( R[T] \).

For example, if \( R \) is \( I \)-adically complete, then \((R, I)\) is henselian by the usual proof of Hensel’s lemma. A local ring \( R \) with maximal ideal \( m \) is **henselian** if the pair \((R, m)\) is henselian.

**Remark 1.2.7.** There are a number of equivalent formulations of the definition of a henselian pair. For instance, let \((R, I)\) be a pair consisting of a ring \( R \) and an ideal \( I \) contained in the Jacobson radical of \( R \). By [46, Theorem 5.11], \((R, I)\) is henselian if and only if every monic polynomial \( f = \sum_j f_j T^j \in R[T] \) with \( f_0 \in I, f_j \in R^x \) has a root in \( I \). (In other words, it suffices to check the lifting condition for \( \overline{g} = T \).) See [88, Exposé XI, §2] for some other formulations.
Theorem 1.2.8. Let \((R, I)\) be a henselian pair. Then the base change functor \(\text{FÉt}(R) \to \text{FÉt}(R/I)\) is a tensor equivalence.

Proof. See [77, Satz 4.4.7], [78, Satz 4.5.1], or [55]. See also [44, Theorem 5.5.7] for a more general assertion in the context of almost ring theory.

Remark 1.2.9. Let \(\{R_i\}_{i \in I}\) be a direct system in the category of rings. We may then define the categorical direct limit \(\varinjlim R_i\) as follows. Start with the category of pairs \(\{(i, S_i) : i \in I, S_i \in \text{FÉt}(R_i)\}\), in which morphisms \((i, S_i) \to (j, S_j)\) exist only when \(j \geq i\), in which case they correspond to morphisms \(S_i \otimes_{R_i} R_j \to S_j\) in \(\text{FÉt}(R_j)\). Then formally invert each morphism \((i, S_i) \to (j, S_i \otimes_{R_i} R_j)\) corresponding to the identity map on \(S_i \otimes_{R_i} R_j\).

For \(R = \varinjlim R_i\), there is a natural functor \(\varinjlim \text{FÉt}(R_i) \to \text{FÉt}(R)\) given by base extension to \(R\). This functor is fully faithful by [52, Proposition 17.7.8(ii)] (since affine schemes are quasicompact). To see that it is essentially surjective, start with \(S \in \text{FÉt}(R)\). Since \(S\) is finitely presented as an \(R\)-algebra, by [51, Lemme 1.8.4.2] it has the form \(S_i \otimes_{R_i} R\) for some \(i \in I\) and some finitely presented \(R_i\)-algebra \(S_i\). By [52, Proposition 17.7.8(ii)] again, there exists \(j \geq i\) such that \(S_j = S_i \otimes_{R_i} R_j\) is finite étale over \(R_j\). We conclude that \(\varinjlim \text{FÉt}(R_i) \to \text{FÉt}(R)\) is an equivalence of tensor categories.

1.3 Descent formalism

We will make frequent use of faithfully flat descent for modules, as well as variations thereof (e.g., for Banach algebras). It is convenient to frame this sort of argument in standard abstract descent formalism, since this language can also be used to discuss glueing of modules (see Example [1.3.3]). We set up in terms of cofibred categories rather than fibred categories, as appropriate for studying modules over rings rather than sheaves over schemes.

Definition 1.3.1. Let \(F : \mathcal{F} \to \mathcal{C}\) be a covariant functor between categories. For \(X\) an object (resp. \(f\) a morphism) in \(\mathcal{C}\), let \(\text{F}^{-1}(X)\) (resp. \(\text{F}^{-1}(f)\)) denote the class of objects (resp. morphisms) in \(\mathcal{C}\) carried to \(X\) (resp. \(f\)) via \(F\).

For \(f : X \to Y\) a morphism in \(\mathcal{C}\) and \(E \in \text{F}^{-1}(X)\), a pushforward of \(E\) along \(f\) is a morphism \(\tilde{f} : E \to f_*E \in \text{F}^{-1}(f)\) such that any \(g \in \text{F}^{-1}(f)\) with source \(E\) factors uniquely through \(\tilde{f}\). (We sometimes call the target \(f_*E\) a pushforward of \(E\) as well, understanding that it comes equipped with a fixed morphism from \(E\).) We say \(\mathcal{F}\) is a cofibred category over \(\mathcal{C}\), or that \(F : \mathcal{F} \to \mathcal{C}\) defines a cofibred category, if pushforwards always exist and the composition of two pushforwards (when defined) is always a pushforward.

Definition 1.3.2. Let \(\mathcal{C}\) be a category in which pushouts exist. Let \(F : \mathcal{F} \to \mathcal{C}\) be a functor defining a cofibred category. Let \(f : X \to Y\) be a morphism in \(\mathcal{C}\). Let \(\pi_1, \pi_2 : Y \to Y \sqcup_X Y\) be the coprojection maps. Let \(\pi_{12}, \pi_{23}, \pi_{23} : Y \sqcup_X Y \to Y \sqcup_X Y \sqcup_X Y\) be the coprojections such that \(\pi_{ij}\) carries the first and second factors of the source into the \(i\)-th and \(j\)-th factors in the triple coproduct (in that order). A descent datum in \(F\) along \(f\) consists of an object \(M \in \text{F}^{-1}(Y)\) and an isomorphism \(\iota : \pi_{1*}M \to \pi_{2*}M\) between some choices of pushforwards, satisfying the following cocycle condition. Let \(M_1, M_2, M_3\) be some pushforwards of \(M\) along
the three coprojections $Y \to Y \sqcup_X Y \sqcup_Y Y$. Then $\iota$ induces a map $\iota_{ij} : M_i \to M_j$ via $\pi_{ij}$; the condition is that $\iota_{23} \circ \iota_{12} = \iota_{13}$.

For example, any object $N \in F^{-1}(X)$ induces a descent datum by taking $M$ to be a pushforward of $N$ along $f$ and taking $\psi$ to be the map identifying $\pi_{1*}M$ and $\pi_{2*}M$ with a single pushforward of $M$ along $X \to Y \sqcup_X Y$. Any such descent datum is said to be effective. We say that $f$ is an effective descent morphism for $F$ if the following conditions hold.

(a) Every descent datum along $f$ is effective.

(b) For any $M, N \in \mathcal{F}$ with $F(M) = F(N)$, the morphisms $M \to N$ in $\mathcal{F}$ lifting the identity morphism are in bijection with morphisms between the corresponding descent data. (We leave the definition of the latter to the reader.)

Example 1.3.3. Let $\mathcal{C}$ be the category of rings. Let $\mathcal{F}$ be the category of modules over rings, with morphisms defined as follows: for $R_1, R_2 \in \mathcal{C}$ and $M_i \in \mathcal{F}$ a module over $R_i$, morphisms $M_1 \to M_2$ consist of pairs $(f, g)$ with $f : R_1 \to R_2$ a morphism in $\mathcal{C}$ and $g : f^*M_1 \to M_2$ a morphism of modules over $R_2$. Let $F : \mathcal{F} \to \mathcal{C}$ be the functor taking each module to its underlying ring; this functor defines a cofibred category with pushforwards defined as expected.

Let $R \to R_1, \ldots, R \to R_n$ be ring homomorphisms corresponding to open immersions of schemes which cover $\text{Spec } R$, and put $S = R_1 \oplus \cdots \oplus R_n$. Then $R \to S$ is an effective descent morphism for $\mathcal{F}$; this is another way of stating the standard fact that any quasicoherent sheaf on an affine scheme is represented uniquely by a module over the ring of global sections [48 Théorème 1.4.1]. This fact is generalized by Theorem 1.3.4.

Theorem 1.3.4. Any faithfully flat morphism of rings is an effective descent morphism for the category of modules over rings (Example 1.3.3).

Proof. See [54 Exposé VIII, Théorème 1.1].

Theorem 1.3.5. For $f : R \to S$ a faithfully flat morphism of rings, an $R$-module $U$ is finite (resp. finite projective) if and only if $f^*U = U \otimes_R S$ is a finite (resp. finite projective) $S$-module. A $R$-algebra $U$ is finite étale if and only if $f^*U$ is a finite étale $S$-algebra.

Proof. For the first assertion, see [54 Exposé VIII, Proposition 1.10]. For the second assertion, see [54 Exposé IX, Proposition 4.1].

For a morphism of rings which is faithful but not flat (e.g., a typical adic completion of a nonnoetherian ring), it is difficult to carry out descent except for modules which are themselves flat. Here is a useful example due to Beauville and Laszlo [6]. (Note that even the noetherian case of this result is not an immediate corollary of faithfully flat descent; see [4 §2].)

Proposition 1.3.6. Let $R$ be a ring. Suppose that $t \in R$ is not a zero divisor and that $R$ is $t$-adically separated. Let $\widehat{R}$ be the $t$-adic completion of $R$.
(a) For any flat $R$-module $M$, the sequence

$$0 \to M \to (M \otimes_R R[t^{-1}]) \oplus (M \otimes_R \hat{R}) \to M \otimes_R \hat{R}[t^{-1}] \to 0,$$

in which the last nontrivial arrow is the difference between the two base extension maps, is exact.

(b) The morphism $R \to R[t^{-1}] \oplus \hat{R}$ is an effective descent morphism for the category of finite projective modules over rings.

In order to carry out analogous arguments in other contexts (as in Proposition 2.7.5), it is helpful to introduce some formalism. (One can use these arguments to recover Proposition 1.3.6; we leave this as an exercise for the reader.)

**Definition 1.3.7.** Let

$$
\begin{array}{ccc}
R & \longrightarrow & R_1 \\
\downarrow & & \downarrow \\
R_2 & \longrightarrow & R_{12}
\end{array}
$$

be a commuting diagram of ring homomorphisms such that the sequence

$$0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0$$

of $R$-modules, in which the last nontrivial arrow is the difference between the given homomorphisms, is exact. By a **glueing datum** over this diagram, we will mean a datum consisting of modules $M_1, M_2, M_{12}$ over $R_1, R_2, R_{12}$, respectively, equipped with isomorphisms $\psi_1 : M_1 \otimes_{R_1} R_{12} \cong M_{12}$, $\psi_2 : M_2 \otimes_{R_2} R_{12} \cong M_{12}$. We say such a glueing datum is finite or finite projective if the modules are finite or finite projective over their corresponding rings.

When considering a glueing datum, it is natural to consider the kernel $M$ of the map $\psi_1 - \psi_2 : M_1 \oplus M_2 \to M_{12}$. There are natural maps $M \to M_1$, $M \to M_2$ of $R$-modules, which by adjunction correspond to maps $M \otimes_R R_1 \to M_1$, $M \otimes_R R_2 \to M_2$.

**Lemma 1.3.8.** Consider a diagram as in Definition 1.3.7. Suppose that for any finite glueing datum, there exists a finitely generated $R$-submodule $M_0$ of $M$ such that $M_0 \otimes_R R_1 \to M_1$ is surjective. Then for any such datum, we have the following.

(a) The map $\psi_1 - \psi_2 : M_1 \oplus M_2 \to M_{12}$ is surjective.

(b) The submodule $M_0$ can be chosen so that $M_0 \otimes_R R_2 \to M_2$ is also surjective.

**Proof.** The surjection $M_0 \otimes_R R_1 \to M_1$ induces a surjection $M_0 \otimes_R R_{12} \to M_{12}$, and hence to a surjection $M_0 \otimes_R (R_1 \oplus R_2) \to M_{12}$. Since this map factors through $\psi_1 - \psi_2$, the latter is surjective. This yields (a).

Let $v_1, \ldots, v_n$ be generators of $M_2$ as an $R_2$-module. For $j = 1, \ldots, n$, $\psi_2(v_j)$ lifts to $M_0 \otimes_R (R_1 \oplus R_2)$; we can thus find $w_{i,j}$ in the image of $M_0 \otimes_R R_1$ such that $\psi_1(w_{1,j}) - \psi_2(w_{2,j}) = \psi_2(v_j)$. Put $v'_j = (w_{1,j}, v_j + w_{2,j}) \in M_1 \oplus M_2$; note that $v'_j \in M$ by construction. For $M'_0$ the submodule of $M$ generated by $M_0$ together with $v'_1, \ldots, v'_n$, $M'_0 \otimes_R R_1 \to M_1$ is clearly still surjective. On the other hand, the image of $M'_0 \otimes_R R_2 \to M_2$ contains $w_{2,j}$ and $v_j + w_{2,j}$ for each $j$, so it contains $v_1, \ldots, v_n$. Hence $M'_0 \otimes_R R_2 \to M_2$ is surjective, yielding (b).
Lemma 1.3.9. Suppose that the hypotheses of Lemma 1.3.8 are satisfied. Then for any finite projective glueing datum, $M$ is a finitely presented $R$-module and $M \otimes_R R_1 \to M_1$, $M \otimes_R R_2 \to M_2$ are bijective.

Proof. Choose $M_0$ as in Lemma 1.3.8(b). Choose a surjection $F \to M_0$ of $R$-modules with $F$ finite free, and put $F_1 = F \otimes_R R_1$, $F_2 = F \otimes_R R_2$, $F_{12} = F \otimes_R R_{12}$, $N = \ker(F \to M)$, $N_1 = \ker(F_1 \to M_1)$, $N_2 = \ker(F_2 \to M_2)$, $N_{12} = \ker(F_{12} \to M_{12})$. From Lemma 1.3.8 we have a commutative diagram

$$
\begin{array}{ccccccccc}
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0 & \to & N & \to & N_1 \oplus N_2 & \to & N_{12} & \to & 0 \\
0 & \to & F & \to & F_1 \oplus F_2 & \to & F_{12} & \to & 0 \\
0 & \to & M & \to & M_1 \oplus M_2 & \to & M_{12} & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
$$

(1.3.9.1)

with exact rows and columns, excluding the dashed arrows. Since $M_i$ is projective, the exact sequence

$$
0 \to N_i \to F_i \to M_i \to 0
$$

splits, so

$$
0 \to N_i \otimes_{R_i} R_{12} \to F_{12} \to M_{12} \to 0
$$

is again exact. Thus $N_i$ is finite projective over $F_i$ and admits an isomorphism $N_i \otimes_{R_i} F_{12} \cong N_{12}$. By Lemma 1.3.8 again, the dashed horizontal arrow in (1.3.9.1) is surjective. By diagram chasing, the dashed vertical arrow in (1.3.9.1) is also surjective; that is, we may add the dashed arrows to (1.3.9.1) while preserving exactness of the rows and columns. In particular, $M$ is a finitely generated $R$-module; we may repeat the argument with $M$ replaced by $N$ to deduce that $M$ is finitely presented.

For $i = 1, 2$, we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
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N \otimes_{R_i} R_i & \to & F_i & \to & M \otimes_{R_i} R_i & \to & 0 \\
0 & \to & N_i & \to & F_i & \to & M_i & \to & 0 \\
\end{array}
$$

with exact rows: the first row is derived from the left column of (1.3.9.1) by tensoring over $R$ with $R_i$, while the second row is derived from the middle column of (1.3.9.1). Since the left vertical arrow is surjective (Lemma 1.3.8 once more), by the five lemma, the right vertical arrow is injective. We thus conclude that the map $M \otimes_{R_i} R_i \to M_i$, which was previously shown (Lemma 1.3.8) to be surjective, is in fact a bijection. □
Corollary 1.3.10. Suppose that the hypotheses of Lemma 1.3.8 are satisfied. Then the natural functor

$$\mathbf{F\acute{e}t}(R) \to \mathbf{F\acute{e}t}(R_1) \times_{\mathbf{F\acute{e}t}(R_{12})} \mathbf{F\acute{e}t}(R_2)$$

is an equivalence of categories. (For example, with notation as in Proposition 1.3.6, this holds for $R_1 = R[t^{-1}]$, $R_2 = \hat{R}$, $R_{12} = \hat{R}[t^{-1}]$.)

Proof. Choose $A_1 \in \mathbf{F\acute{e}t}(R_1)$, $A_2 \in \mathbf{F\acute{e}t}(R_2)$, $A_{12} \in \mathbf{F\acute{e}t}(R_{12})$ equipped with isomorphisms $A_1 \otimes R_1 \cong A_2 \otimes R_2 \cong A_{12}$, and view this package as a finite projective glueing datum. By Lemma 1.3.9, the kernel $A$ of $A_1 \oplus A_2 \to A_{12}$ is a finite projective $R$-module and the natural maps $A \otimes R 1 \to A_1$, $A \otimes R 2 \to A_2$ are isomorphisms. Using the exact sequence

$$0 \to A \to A_1 \oplus A_2 \to A_{12} \to 0, \quad (1.3.10.1)$$

the multiplication maps on $A_1, A_2, A_{12}$ define a multiplication map on $A$, making it a flat $R$-algebra. By Lemma 1.3.9 again, we also have an exact sequence

$$0 \to \text{Hom}_R(A, R) \to \text{Hom}_{R_1}(A_1, R_1) \oplus \text{Hom}_{R_2}(A_2, R_2) \to \text{Hom}_{R_{12}}(A_{12}, R_{12}) \to 0. \quad (1.3.10.2)$$

Using (1.3.10.1), (1.3.10.2), and the snake lemma, we see that the trace pairing on $A$ defines an isomorphism $A \to \text{Hom}_R(A, R)$. This proves the claim. \hfill \Box

1.4 Étale local systems

Definition 1.4.1. For $X$ a scheme, the small étale site $X_{\acute{e}t}$ of $X$ is the category of étale $X$-schemes and étale morphisms, equipped with the Grothendieck topology generated by set-theoretically surjective families of morphisms.

A sheaf $T$ of torsion $\mathbb{Z}_p$-modules on $X_{\acute{e}t}$ is lisse if it is represented by a finite étale $X$-scheme. A lisse $\mathbb{Z}_p$-sheaf on $X_{\acute{e}t}$ is an inverse system $T = \{ \cdots \to T_1 \to T_0 \}$ in which each $T_n$ is a lisse sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules and each arrow $T_{n+1} \to T_n$ identifies $T_n$ with the cokernel of multiplication by $p^n$ on $T_{n+1}$. A lisse $\mathbb{Z}_p$-sheaf on $X_{\acute{e}t}$ is also called an (étale) $\mathbb{Z}_p$-local system on $X$. Such objects may be constructed using faithfully flat descent (Theorem 1.3.3 and Theorem 1.3.5).

A lisse $\mathbb{Q}_p$-sheaf on $X_{\acute{e}t}$ is an element of the isogeny category of lisse $\mathbb{Z}_p$-sheaves (in which multiplication by $p$ on morphisms is formally inverted). A lisse $\mathbb{Q}_p$-sheaf on $X_{\acute{e}t}$ is also called an (étale) $\mathbb{Q}_p$-local system on $X$. Let $\mathbb{Z}_p$-$\text{Loc}(X)$ and $\mathbb{Q}_p$-$\text{Loc}(X)$ denote the categories of $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems, respectively, on $X$.

Remark 1.4.2. Let $\mathbf{F\acute{e}t}(X)$ be the subcategory of $X_{\acute{e}t}$ in which all internal morphisms and all structure morphisms are finite étale. This acquires the structure of a site (the small finite étale site) by restriction. The category $\mathbb{Z}_p$-$\text{Loc}(X)$ is then equivalent to the category of sheaves of finite free $\mathbb{Z}_p$-modules on $\mathbf{F\acute{e}t}(X)$ which are locally constant modulo $p^n$ for each positive integer $n$. (See also [53, Exposé VI, §1.2.4] in other words, one can describe étale $\mathbb{Z}_p$-local systems just in terms of their sections over finite étale $X$-schemes, without reference to general étale $X$-schemes.)
Remark 1.4.3. One consequence of Remark 1.4.2 is that for $X$ connected, we may identify étale $\mathbb{Z}_p$-local systems (resp. $\mathbb{Q}_p$-local systems) on $X$ with continuous representations of the étale fundamental group $\pi_1^{\text{ét}}(X, \mathfrak{r})$ of $X$ in finite free $\mathbb{Z}_p$-modules (resp. finite-dimensional $\mathbb{Q}_p$-vector spaces), for any choice of the geometric base point $\mathfrak{r}$. For example, if $X = \text{Spec}(K)$, we get precisely the continuous representations of the absolute Galois group $G_K$.

Remark 1.4.4. Another consequence of Remark 1.4.2 is that for any rings $A, B$, any tensor equivalence $\text{FÉt}(A) \cong \text{FÉt}(B)$ induces tensor equivalences $\mathbb{Z}_p\text{-Loc}(\text{Spec } A) \cong \mathbb{Z}_p\text{-Loc}(\text{Spec } B)$, $\mathbb{Q}_p\text{-Loc}(\text{Spec } A) \cong \mathbb{Q}_p\text{-Loc}(\text{Spec } B)$.

It is difficult to handle $\mathbb{Q}_p$-local systems unless we assume that the base scheme $X$ is noetherian. In that case, we may bring to bear the theory of constructible sheaves (whose development depends heavily on noetherian induction). See [31, §1.1] for a fuller synopsis than given here, and [42, Chapter I] for a more detailed development.

Definition 1.4.5. Let $X$ be a noetherian scheme. A sheaf $T$ of torsion $\mathbb{Z}_p$-modules on $X_{\text{ét}}$ is constructible if there exists a finite partition of $X$ into locally closed subschemes $X_i$ on each of which $T$ is locally constant with finitely generated fibres. This is equivalent to requiring that $T$ be represented by an étale algebraic space $\tilde{T}$ over $X$ [42, Proposition I.4.6].

Consider an inverse system $T = \{\cdots \to T_1 \to T_0\}$ of torsion $\mathbb{Z}_p$-modules on $X_{\text{ét}}$. For $r$ a nonnegative integer, let $T[r]$ denote the shifted system $\{\cdots \to T_{r+1} \to T_r\}$. We say $T$ is a null system if there exists $r$ for which the natural map $T[r] \to T$ is zero. By the AR category (or Artin-Rees category) of $X_{\text{ét}}$, we will mean the quotient category of inverse systems as above by the Serre subcategory of null systems. In other words, the objects of this category are again inverse systems of torsion $\mathbb{Z}_p$-modules on $X_{\text{ét}}$, but with morphisms now given by

$$\text{Hom}_{\text{AR}}(T, T') = \lim_{r \geq 0} \text{Hom}(T[r], T').$$

A constructible $p$-adic system on $X_{\text{ét}}$ is an inverse system $T = \{\cdots \to T_1 \to T_0\}$ in which each $T_n$ is a constructible sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules and each arrow $T_{n+1} \to T_n$ identifies $T_n$ with the cokernel of multiplication by $p^n$ on $T_{n+1}$. A constructible $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ is an element of the AR category of $X_{\text{ét}}$ isomorphic to a constructible $p$-adic system.

A constructible $\mathbb{Q}_p$-sheaf on $X_{\text{ét}}$ is an object $\mathcal{F}$ in the isogeny category of constructible $\mathbb{Z}_p$-sheaves on $X_{\text{ét}}$ (again obtained by inverting multiplication by $p$ on morphisms). We formally write $\mathcal{F} = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to indicate that $T$ is a constructible $\mathbb{Z}_p$-sheaf giving rise to $\mathcal{F}$.

Remark 1.4.6. Let $X$ be a noetherian scheme. Let $T = \{\cdots \to T_1 \to T_0\}$ be an inverse system in which each $T_n$ is a constructible sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules on $X_{\text{ét}}$. Then $T$ is a constructible $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ if and only if the following conditions hold.

(a) There exists a nonnegative integer $r$ such that $T[t]$ and $T[r]$ have the same image in $T$ for all $t \geq r$.

(b) Let $\tilde{T}_n$ be the image of $T_{n+t}$ in $T_n$. Then there exists a nonnegative integer $s$ such that $\tilde{T}_{n+t}/p^n\tilde{T}_{n+t} \cong \tilde{T}_{n+s}/p^n\tilde{T}_{n+s}$ for all $t \geq s$. 

17
More precisely, under these conditions, the sheaves $\tilde{T}_{n+s}/p^n\tilde{T}_{n+s}$ form the unique constructible $p$-adic system on $X_{\acute{e}t}$ isomorphic to $T$ in the AR category.

An immediate consequence is that the property of an inverse system being a constructible $\mathbb{Z}_p$-sheaf is local for the étale topology. With some more work, it can be shown that constructible $\mathbb{Z}_p$-sheaves form an abelian category \cite[Proposition I.12.11]{[2]}, as then do constructible $\mathbb{Q}_p$-sheaves. Note that this last fact is a rather deep fact of algebraic geometry which fails completely for general topological spaces; see Remark 1.4.9 below.

Remark 1.4.7. Remark 1.4.6 can be used to glue constructible $\mathbb{Q}_p$-sheaves for the étale topology, as follows. Let $U_1, \ldots, U_n$ be an étale covering of a noetherian scheme $X$, and put $U_{ik} = U_i \times_X U_k$. Consider a descent datum for this covering in the category of constructible $\mathbb{Q}_p$-sheaves over noetherian schemes; this consists of a constructible $\mathbb{Q}_p$-sheaf $F_i$ on each $U_i$, a constructible $\mathbb{Q}_p$-sheaf $F_{ik}$ on each $U_{ik}$, and isomorphisms $F_i|_{U_{ik}} \cong F_{ik}$ satisfying a cocycle condition. Let $E_i$ and $E_{ik}$ be the direct images of $F_i$ and $F_{ik}$, respectively, under the étale morphisms from $U_i$ and $U_{ik}$ to $X$. These are again constructible $\mathbb{Q}_p$-sheaves (since constructibility for $\mathbb{Z}/p^n\mathbb{Z}$-sheaves is stable under direct image along étale morphisms, e.g., by the characterization using algebraic spaces), and restriction from $U_i$ to $U_{ik}$ induces a morphism $E_i \to E_{ik}$. Taking the difference of the two restriction maps gives a map $E_i \oplus E_k \to E_{ik}$. Define a morphism

$$\bigoplus \limits_i E_i \to \bigoplus \limits_{i,k} E_{ik}$$

as the composition of the diagonal mapping of $\oplus \limits_i E_i$ into $\oplus \limits_{i,k} (E_i \oplus E_k)$ followed by the direct sum of the maps $E_i \oplus E_k \to E_{ik}$ described above. Let $F$ be the kernel of this morphism; it is again a constructible $\mathbb{Q}_p$-sheaf because such objects form an abelian category by Remark 1.4.6. This sheaf restricts to $F_i$ on each $U_i$ compatibly with the descent datum, as desired.

Remark 1.4.8. Let $X$ be a noetherian scheme. Then any lisse $\mathbb{Z}_p$-sheaf defines a constructible $p$-adic system, so every lisse $\mathbb{Q}_p$-sheaf gives rise to a constructible $\mathbb{Q}_p$-sheaf. Moreover, a constructible $\mathbb{Q}_p$-sheaf arises from a lisse $\mathbb{Q}_p$-sheaf if and only if it does so locally for the étale topology, by Remark 1.4.6. Consequently, using Remark 1.4.7 we may define lisse $\mathbb{Q}_p$-sheaves using descent data for the étale topology.

Remark 1.4.9. Remark 1.4.8 can be used to illustrate a difference between étale local systems on schemes and on topological spaces. On a connected topological space, topological $\mathbb{Q}_p$-local systems correspond to representations of the ordinary fundamental group on $\mathbb{Q}_p$-vector spaces, whereas étale local systems correspond only to those representations which extend continuously to the profinite completion of the fundamental group. In particular, such representations must map into a compact subgroup of $\text{GL}_n(\mathbb{Q}_p)$. Thus one cannot hope to glue étale local systems for topological coverings, as otherwise they would include all topological local systems on spaces admitting a good covering. For instance, on a circle, one obtains a topological local system of rank 1 that does not descend to an étale local system by requiring the monodromy transformation to act as multiplication by $p$. (See [2,§2.8] for a recurrence of this point.)
This suggests that the special topological nature of schemes is needed in order to make the theory of constructible local systems work. This nature is normally harnessed by making a noetherian hypothesis and then freely employing noetherian induction; it is not obvious whether there is a method available to extend any of the framework of constructible sheaves to more general schemes.

In some cases, one can even construct local systems using descent data in the fpqc topology. Here is a concrete example in the style of Proposition 1.3.6.

**Lemma 1.4.10.** Let \( R \) be a normal noetherian integral domain. Pick \( f \in R \) and let \( S \) be the \( f \)-adic completion of \( R \). Then the morphism \( R \to R[f^{-1}] \oplus S \) is an effective descent morphism for lisse \( \mathbb{Q}_p \)-sheaves over noetherian rings.

**Proof.** Choose a separable closure of \( K = \text{Frac}(R) \); this fixes identifications of the étale fundamental groups of \( \text{Spec}(R) \) and \( \text{Spec}(R[f^{-1}]) \) with quotients of the absolute Galois group \( G_K \) of \( K \). Let \( \mathcal{F} \) be a \( \mathbb{Q}_p \)-local system over \( R[f^{-1}] \), identified with a continuous representation of the étale fundamental group of \( \text{Spec}(R[f^{-1}]) \) on a finite-dimensional \( \mathbb{Q}_p \)-vector space. This representation factors through the étale fundamental group of \( \text{Spec}(R) \) if and only if it kills the inertia subgroup of \( G_K \) associated to each irreducible component of the zero locus of \( f \). However, this is ensured by the existence of an extension to \( \text{Spec}(S) \) of the restriction of \( \mathcal{F} \) to \( \text{Spec}(S[f^{-1}]) \).

**Remark 1.4.11.** For \( F \) a finite extension of \( \mathbb{Q}_p \), one can define étale \( F \)-local systems and étale \( F \)-local systems on a scheme \( X \) by analogy with the case \( F = \mathbb{Q}_p \). We will only need this observation in the case where \( F = \mathbb{Q}_p^d \) is a finite unramified extension of \( \mathbb{Q}_p \) of degree \( d \), in which case we label the resulting categories \( \mathbb{Z}_p^d \)-Loc\((X) \) and \( \mathbb{Q}_p^d \)-Loc\((X) \).

### 1.5 Semilinear actions

**Convention 1.5.1.** For \( S \) a ring equipped with an endomorphism \( \varphi \) and \( M \) an \( S \)-module, a *semilinear \( \varphi \)-action* on \( M \) will always mean an isomorphism (not just an endomorphism) \( \varphi^* M \to M \) of \( S \)-modules. We will commonly interpret such an action as a \( \varphi \)-semilinear map \( M \to M \).

Although we have not found a precise reference, we believe that the following is a standard lemma in algebraic \( K \)-theory, specifically from the study of polynomial extensions (as in [5, Chapter XII]).

**Lemma 1.5.2.** Let \( S \) be a ring equipped with an endomorphism \( \varphi \). Let \( M \) be a finitely generated \( S \)-module equipped with a semilinear \( \varphi \)-action. Then there exists a finite free \( S \)-module \( F \) equipped with a semilinear \( \varphi \)-action and a \( \varphi \)-equivariant surjection \( F \to M \).

**Proof.** Choose generators \( v_1, \ldots, v_n \) of \( M \), and use them to define a surjection \( E = S^n \to M \) of \( S \)-modules. Let \( T : \varphi^* M \to M \) be the given isomorphism. Choose \( A_{ij}, B_{ij} \in S \) so that...
\[ T(v_j \otimes 1) = \sum_i A_{ij} v_i, \quad T^{-1}(v_j) = \sum_i B_{ij} (v_i \otimes 1); \text{ by writing} \]
\[ v_k = T(T^{-1}(v_k)) = T \left( \sum_j B_{jk} (v_j \otimes 1) \right) = \sum_{i,j} A_{ij} B_{jk} v_i, \]
we see that the columns of the matrix \( C = AB - 1 \) are elements of \( N = \ker(E \to M) \).

Let \( D \) be the block matrix \( \begin{pmatrix} A & C \\ 1 & B \end{pmatrix} \). By using row operations to clear the bottom left block, we find that \( \det(D) = \det(AB - C) = 1 \). Consequently, \( D \) is invertible over \( S \), so we may use it to define an isomorphism \( \varphi^* F \to F \) for \( F = E \oplus E \). This isomorphism carries \( \varphi^* (N \oplus E) \) into \( N \oplus E \), so we obtain a \( \varphi \)-equivariant surjection \( F \to M \) as desired. \( \square \)

**Definition 1.5.3.** Let \( S \) be a ring equipped with an endomorphism \( \varphi \). Let \( M \) be a module over \( S \) equipped with a semilinear \( \varphi \)-action. We then write
\[ H^0_{\varphi}(M) = \ker(\varphi - 1, M), \quad H^1_{\varphi}(M) = \coker(\varphi - 1, M), \]
and \( H^i_{\varphi}(M) = 0 \) for \( i \geq 2 \). The groups \( H^i_{\varphi}(M) \) may be interpreted as the Yoneda extension groups \( \text{Ext}^i(S, M) \) in the category of left modules over the twisted polynomial ring \( S\{\varphi\} \), by tensoring \( M \) over \( S \) with the free resolution
\[ 0 \to S\{\varphi\} \xrightarrow{\varphi^{-1}} S\{\varphi\} \to S \to 0 \]
of \( S \). (For a detailed development of Yoneda extension groups, see for instance [63, §IV.9].)

**Remark 1.5.4.** In Definition 1.5.3, if \( M \) is a module over \( S \) equipped with a semilinear \( \varphi^d \)-action for some positive integer \( d \), we may identify \( H^i_{\varphi^d}(M) \) with \( H^i_{\varphi}(N) \) for \( N = M \oplus \varphi^* M \oplus \cdots \oplus (\varphi^{d-1})^* M \).

## 2 Spectra of nonarchimedean Banach rings

We set notation and terminology concerning spectra of nonarchimedean (commutative) Banach rings and particularly affinoid algebras, following Berkovich [12, 13]. Although this setup is preliminary to a construction of analytic spaces, we will not refer to such spaces in this paper, so we postpone their introduction to a subsequent paper.

**Convention 2.0.1.** For \( M \) a matrix over a ring equipped with a submultiplicative seminorm \( \alpha \), we write \( \alpha(M) \) for \( \sup_{i,j} \{ \alpha(M_{ij}) \} \).

### 2.1 Seminorms on groups and rings

**Definition 2.1.1.** Consider the following conditions on an abelian group \( G \) and a function \( \alpha : G \to [0, +\infty) \).
(a) For all $g, h \in G$, we have $\alpha(g - h) \leq \max\{\alpha(g), \alpha(h)\}$.

(b) We have $\alpha(0) = 0$.

(b') For all $g \in G$, we have $\alpha(g) = 0$ if and only if $g = 0$.

We say $\alpha$ is a (nonarchimedean) seminorm if it satisfies (a) and (b), and a (nonarchimedean) norm if it satisfies (a) and (b'). Any seminorm $\alpha$ induces a norm on $G/\ker(\alpha)$.

If $\alpha, \alpha'$ are two seminorms on the same abelian group $G$, we say $\alpha$ dominates $\alpha'$, and write $\alpha \geq \alpha'$ or $\alpha' \leq \alpha$, if there exists $c \in (0, +\infty)$ for which $\alpha'(g) \leq c\alpha(g)$ for all $g \in G$. If $\alpha$ and $\alpha'$ dominate each other, we say they are equivalent; in this case, $\alpha$ is a norm if and only if $\alpha'$ is.

**Definition 2.1.2.** Let $G, H$ be two abelian groups equipped with nonarchimedean seminorms $\alpha, \beta$, and let $\varphi : G \to H$ be a homomorphism. We say $\varphi$ is bounded if $\alpha \geq \beta \circ \varphi$, and isometric if $\alpha = \beta \circ \varphi$. (An intermediate condition is that $\varphi$ is submetric, meaning that $\alpha(g) \geq \beta(\varphi(g))$ for all $g \in G$.)

The quotient seminorm induced by $\alpha$ is the seminorm $\overline{\alpha}$ on $\operatorname{image}(\varphi)$ defined by

$$\overline{\alpha}(h) = \inf\{\alpha(g) : g \in G, \varphi(g) = h\}.$$ 

If $H$ is also equipped with a seminorm $\beta$, we say $\varphi$ is strict if the two seminorms $\overline{\alpha}$ and $\beta$ on $\operatorname{image}(\varphi)$ are equivalent; this implies in particular that $\varphi$ is bounded. We say $\varphi$ is almost optimal if $\overline{\alpha}$ and $\beta$ coincide. We say $\varphi$ is optimal if every $h \in \operatorname{image}(\varphi)$ admits a lift $g \in G$ with $\alpha(g) = \overline{\alpha}(h)$. Any optimal homomorphism is almost optimal, and any optimal homomorphism is strict, but not conversely.

**Remark 2.1.3.** Berkovich uses the term admissible in place of strict, but the latter is well-established in the context of topological groups, as in [13, §III.2.8]. However, there is no perfect choice of terminology; our convention will create some uncomfortable linguistic proximity during the discussions of strictly affinoid algebras and strict p-rings.

**Definition 2.1.4.** For $G$ an abelian group equipped with a nonarchimedean seminorm $\alpha$, equip the group of Cauchy sequences in $G$ with the seminorm whose value on the sequence $g_0, g_1, \ldots$ is $\lim_{i \to \infty} \alpha(g_i)$. The quotient by the kernel of this seminorm is the separated completion $\widehat{G}$ of $G$ under $\alpha$. For the unique continuous extension of $\alpha$ to $\widehat{G}$, the homomorphism $G \to \widehat{G}$ is isometric, and injective if and only if $\alpha$ is a norm (in which case we call $\widehat{G}$ simply the completion of $G$).

**Definition 2.1.5.** Let $A$ be a ring. Consider the following conditions on a (semi)norm $\alpha$ on the additive group of $A$.

(c) For all $g, h \in A$, we have $\alpha(gh) \leq \alpha(g)\alpha(h)$.

(c') We have (c), and for all $g \in A$, we have $\alpha(g^2) = \alpha(g)^2$. (Equivalently, $\alpha(g^n) = \alpha(g)^n$ for all $g \in A$ and all positive integers $n$. In particular, $\alpha(1) \in \{0, 1\}$.)
We say \( \alpha \) is submultiplicative if it satisfies (c), power-multiplicative if it satisfies (c'), and multiplicative if it satisfies (c''). Note that if \( \alpha \) is a submultiplicative seminorm and \( \alpha' \) is a power-multiplicative seminorm, then \( \alpha \) dominates \( \alpha' \) if and only if \( \alpha(a) \geq \alpha'(a) \) for all \( a \in A \).

**Example 2.1.6.** For any abelian group \( G \), the trivial norm on \( G \) sends 0 to 0 and any nonzero \( g \in G \) to 1. For any ring \( A \), the trivial norm on \( A \) is submultiplicative in all cases, power-multiplicative if and only if \( A \) is reduced, and multiplicative if and only if \( A \) is an integral domain. (As usual, the zero ring is not considered to be a domain.)

**Definition 2.1.7.** For a ring equipped with a submultiplicative norm \( \alpha \), define

\[
\mathfrak{a}_A = \{ a \in A : \alpha(a) \leq 1 \}
\]

\[
\mathfrak{m}_A = \{ a \in A : \alpha(a) < 1 \}
\]

\( \kappa_A = \mathfrak{a}_A / \mathfrak{m}_A. \)

If \( \alpha(1) \leq 1 \), then \( \mathfrak{a}_A \) is a ring and \( \mathfrak{m}_A \) is an ideal of \( \mathfrak{a}_A \). If \( A \) is a field equipped with a multiplicative norm, then \( \mathfrak{a}_A \) is a valuation ring with maximal ideal \( \mathfrak{m}_A \) and residue field \( \kappa_A \).

**Example 2.1.8.** Let \( I \) be an arbitrary index set. For each \( i \in I \), specify a ring \( A_i \) and a power-multiplicative seminorm \( \alpha_i \) on \( A_i \). Put \( A = \prod_{i \in I} A_i \), and define the function \( \alpha : A \to [0, +\infty] \) by setting \( \alpha((a_i)_{i \in I}) = \sup_{i} \{ \alpha_i(a_i) \} \). Then the subset \( A_0 \) of \( A \) on which \( \alpha \) takes finite values is a subring on which \( \alpha \) restricts to a power-multiplicative seminorm. This example is in some sense universal; see Theorem 2.3.10.

**Definition 2.1.9.** Let \( A \) be a ring equipped with a submultiplicative seminorm \( \alpha \). The spectral seminorm on \( A \) is the power-multiplicative seminorm \( \alpha_{sp} \) defined by \( \alpha_{sp}(a) = \lim_{s \to \infty} \alpha(a^s)^{1/s} \). (The existence of the limit is an exercise in real analysis known as Fekete’s lemma.) Note that equivalent choices of \( \alpha \) define the same spectral seminorm.

**Definition 2.1.10.** Let \( A, B, C \) be rings equipped with submultiplicative seminorms \( \alpha, \beta, \gamma \), and let \( A \to B, A \to C \) be bounded homomorphisms. The product seminorm on the ring \( B \otimes_A C \) is defined by taking \( f \in B \otimes_A C \) to the infimum of \( \max_i \{ \beta(b_i) \gamma(c_i) \} \) over all presentations \( f = \sum_i b_i \otimes c_i \). The separated completion of \( B \otimes_A C \) for the product seminorm is denoted \( B \hat{\otimes}_A C \) and called the completed tensor product of \( B \) and \( C \) over \( A \).

**Remark 2.1.11.** The definition of a submultiplicative seminorm \( \alpha \) on \( A \) does not include the condition that \( \alpha(1) \leq 1 \). However, if we define the operator seminorm \( \alpha' \) by the formula

\[
\alpha'(a) = \inf \{ c \geq 0 : \alpha(ab) \leq ca(b) \text{ for all } b \in A \},
\]

then \( \alpha' \) is a submultiplicative norm, \( \alpha'(1) \leq 1 \), and \( \alpha'(a) \leq \alpha(a) \) for all \( a \in A \). Moreover, if \( \alpha(1) > 0 \), then we may take \( b = 1 \) in (2.1.11.1) to deduce that \( \alpha'(a) \geq \alpha(1)^{-1} \alpha(a) \). Consequently, in all cases \( \alpha' \) is equivalent to \( \alpha \) (this being trivially true if \( \alpha(1) = 0 \)).
2.2 Banach rings and modules

**Definition 2.2.1.** By a *Banach ring*, we will mean a commutative ring equipped with a submultiplicative norm under which it is complete. Note that the zero ring with the trivial norm is permitted; this convention is required for the completed tensor product to define an operation on the category of Banach rings. Note also that what we call Banach rings would more commonly be called *commutative Banach rings*, but we will not use noncommutative Banach rings in this paper.

A Banach ring is *uniform* if its norm is power-multiplicative; this forces the ring to be reduced. For example, for any Banach ring $A$, the separated completion of $A$ for the spectral seminorm associated to the original norm is a uniform Banach ring, called the *uniformization* of $A$. (Beware that the class of uniform Banach rings is not closed under completed tensor product; one should instead take the uniformization of the tensor product. See however Remark 3.1.5.)

A Banach ring whose underlying ring is a field and whose underlying norm is multiplicative will also be called an *analytic field*. For $K$ an analytic field, any finite extension of $K$ admits a unique structure of an analytic field extending $K$ [19, Theorem 3.2.3/2].

**Lemma 2.2.2.** Let $I$ be a nontrivial ideal in a Banach ring $A$. Then the closure of $I$ is also a nontrivial ideal. In particular, any maximal ideal in $A$ is closed.

**Proof.** If the closure were trivial, then $I$ would contain an element $x$ for which $1 - x \in m_A$. But then the series $\sum_{i=0}^{\infty} (1 - x)^i$ would converge in $A$ to an inverse of $x$, contradicting the assumption that $I$ is a nontrivial ideal.

For $A$ a Banach ring, it is easy to check (using Remark 1.2.7) that the pair $(o_A, m_A)$ is henselian. The following refinement of this observation will also prove to be useful. See also Proposition 2.6.9.

**Lemma 2.2.3.** Let $\{(A_i, \alpha_i)\}_{i \in I}$ be a direct system in the category of Banach rings and submetric homomorphisms. Equip the direct limit $A$ of the $A_i$ in the category of rings with the infimum $\alpha$ of the quotient seminorms induced by the $\alpha_i$.

(a) The pair $(A, \ker(\alpha))$ is henselian.

(b) The pair $(o_A, m_A)$ is henselian.

**Proof.** In both cases, we check the criterion of Remark 1.2.7. To check (a), note first that $\ker(\alpha)$ is contained in the Jacobson radical of $A$: if $a - 1 \in \ker(\alpha)$, then there exists some index $i$ for which $a - 1$ is an element of $A_i$ of norm less than 1. Since $A_i$ is complete, this forces $a$ to be invertible. With that in mind, let $f = \sum_i f_i T^i \in A[T]$ be a monic polynomial with $f_0 \in \ker(\alpha), f_1 \in A^\times$. We construct a root of $f$ using the Newton-Raphson iteration as follows. Put $x_0 = 0$. Given $x_l \in A$ for some nonnegative integer $l$ such that $x_l \in \ker(\alpha), f'(x_l)$ is invertible modulo $\ker(\alpha)$ and hence is a unit. We may thus define $x_{l+1} = x_l - f(x_l)/f'(x_l)$ and note that $x_{l+1} \in \ker(\alpha)$. For any sufficiently large $i \in I$, the sequence $\{x_i\}$ is Cauchy in $A_i$, and so has a limit which is a root of $f$.
To check (b), let \( f = \sum_i f_i T^i \in \mathfrak{o}_A[T] \) be a monic polynomial with \( f_0 \in \mathfrak{m}_A, f_1 \in \mathfrak{o}_A^* \); then \( f \) admits a root \( r \) in \( \widehat{\mathfrak{m}}_A \). Choose any \( s \in A \) with \( \alpha(r - s) < 1 \), and put \( x_0 = s, \) \( x_{i+1} = x_i - f(x_i)/f'(x_i) \). For any sufficiently large \( i \in I \), the sequence \( \{x_i\} \) is Cauchy in \( A_i \), and so has a limit which is a root of \( f \).

**Lemma 2.2.4.** Retain notation as in Lemma 2.2.3.

(a) The base change functor \( \text{FÉt}(A) \to \text{FÉt}(\widehat{A}) \) is rank-preserving and fully faithful.

(b) Suppose that \( \alpha \) is a multiplicative seminorm and \( K = A/\ker(\alpha) \) is a field. Then the base change functor \( \text{FÉt}(A) \to \text{FÉt}(\overline{K}) \) is an equivalence of categories.

**Proof.** To check (a), we first observe that by Lemma 2.2.3(a), \( \ker(\alpha) \) is contained in the Jacobson radical of \( A \). Also, for any \( x \in A \) which becomes a unit in \( \widehat{A} \), we can find \( y \in A \) for which \( \alpha(xy - 1) < 1 \), so \( xy \) is a unit in some \( A_i \) and so \( x \) is a unit in \( A \). Finally, for \( S \in \text{FÉt}(A) \), note that any idempotent \( S \otimes_A \widehat{A} \) can have at most one preimage in \( S \otimes_A A/\ker(\alpha) \) (since \( A/\ker(\alpha) \) injects into \( \widehat{A} \) and \( S \) is a projective \( A \)-module) and hence at most one preimage in \( S \) (by Lemma 1.2.4). We conclude by Proposition 1.2.5 that to check (a), it suffices to verify that for each \( S \in \text{FÉt}(A) \); every idempotent of \( S \otimes_A \widehat{A} \) arises from some idempotent of \( S \).

Since \( S \in \text{FÉt}(A) \), by Remark 1.2.9, we can choose an index \( i \in I \) for which \( S = S_i \otimes_{A_i} A \) for some \( S_i \in \text{FÉt}(A_i) \). Since \( S_i \) is a finitely locally free \( A_i \)-module (see Definition 1.2.1), we can choose a finite free \( A_i \)-module \( F_i \) admitting a direct sum decomposition \( F_i \cong S_i \oplus T_i \). Choose a basis \( x_1, \ldots, x_n \) of \( F_i \) and let \( y_1, \ldots, y_n \) be the projections of \( x_1, \ldots, x_n \) onto \( S_i \). For \( h, k \in \{1, \ldots, n\} \), write the image of \( y_h y_k \) in \( F_i \) as \( \sum_l c_{hkl} x_l \) with \( c_{hkl} \in A_i \), so that in \( S_i \) we have \( y_h y_k = \sum_l c_{hkl} y_l \). Put \( c = \max\{1, \sup_{h, k, l} \{a_i(c_{hkl})\}\} \).

For each \( j \in I \) with \( i \leq j \), let \( \beta_j \) be the restriction to \( S_j = S_i \otimes_{A_i} A_j \) of the supremum norm on \( F_j = F_i \otimes_{A_i} A_j \) defined by the basis \( x_1, \ldots, x_n \). Note that \( \beta_j(xy) \leq c \beta_j(x) \beta_j(y) \) for all \( j \) and all \( x, y \in S_j \). Similarly, let \( \beta \) be the supremum seminorm on \( S \otimes_A \widehat{A} \) defined by the basis \( x_1, \ldots, x_n \), so that \( \beta(xy) \leq c \beta(x) \beta(y) \) for all \( x, y \in S \otimes A \widehat{A} \). In particular, any nonzero idempotent element \( e \in S \otimes_A \widehat{A} \) satisfies \( \beta(e) \geq c^{-1} \).

Let \( e \in S \otimes_A \widehat{A} \) be an idempotent element. Choose \( \epsilon > 0 \) with \( \epsilon \max\{\beta(e), 1\} < 1 \). Since \( e^2 = e \in S \otimes_A \widehat{A} \), we can choose \( j \in I \) and \( x \in S_j \) with \( \beta(x - e) < c^{-1} \) and \( \beta_j(x^2 - x) \leq c^{-1} \epsilon \). Define the sequence \( x_0, x_1, \ldots \) by \( x_0 = x \) and \( x_{i+1} = 3x_i^2 - 2x_i^3 \). We then have \( \beta_j(x_i^2 - x_i) \leq c^{-1} \epsilon^{i+1} \) by induction on \( i \), by writing

\[
x_{i+1}^2 - x_{i+1} = 4(x_i^2 - x_i)^3 - 3(x_i^2 - x_i)^2.
\]

Also, \( x_{i+1} - x_i = (x_i^2 - x_i)(1 - 2x_i) \), so by induction on \( i \), \( \beta_j(x_{i+1}) \leq \max\{\beta_j(x_i), 1\} \) and \( \beta(x_i) \leq \max\{\beta(x), 1\} \). Using the equation \( x_{i+1} - x_i = (x_i^2 - x_i)(1 - 2x_i) \) again, we see that the \( x_i \) form a Cauchy sequence, whose limit \( y \) in \( S_j \) must satisfy \( y^2 = y \). In addition, \( \beta(y - x) \leq c^{-1} \epsilon \max\{\beta(x), 1\} \), so \( \beta(y - e) < c^{-1} \). Since \( y - e \) is an idempotent element of \( S \otimes_A \widehat{A} \), this is only possible if \( y = e \). This completes the proof of (a).
To check (b), note that the hypotheses ensure that the completion $\hat{K}$ of $K$ is an analytic field. It suffices to show that an arbitrary finite separable field extension $\hat{L}$ of $\hat{K}$ occurs in the essential image of the base change functor. By the primitive element theorem, we can write $\hat{L} \cong \hat{K}[T]/(P)$ for some monic separable polynomial $P \in \hat{K}[T]$. By Hensel’s lemma (or more precisely Krasner’s lemma), we also have $\hat{L} \cong \hat{K}[T]/(Q)$ for any monic polynomial $Q \in \hat{K}[T]$ whose coefficients are sufficiently close to those of $P$. In particular, we may choose $Q \in K[T]$, in which case we may write $\hat{L} = L \otimes_K \hat{K}$ for $L = K[T]/(Q) \in \text{F\text{\textregistered}Et}(K)$. Since $\text{F\text{\textregistered}Et}(A) \to \text{F\text{\textregistered}Et}(K)$ is essentially surjective by Lemma 2.2.3 plus Theorem 1.2.8 this proves the claim.

Lemma 2.2.5. Let $K$ be an analytic field with norm $\alpha$, and let $L$ be a finite extension of $K$. Then the unique multiplicative extension of $\alpha$ to $L$ (Definition 2.2.1) is also the unique power-multiplicative extension of $\alpha$ to $L$.

Proof. Let $\beta$ be the multiplicative extension of $\alpha$ to $L$, and let $\gamma$ be a power-multiplicative extension of $\alpha$ to $L$. Note that for $x \in K^\times, y \in L$, we have

$$\gamma(xy) \leq \gamma(x)\gamma(y) \leq \gamma(x^{-1})^{-1}\gamma(y) \leq \gamma(xy),$$

so $\gamma(xy) = \gamma(x)\gamma(y)$.

Given $x \in L^\times$, let $P \in K[T] | x$ be the minimal polynomial of $x$ over $K$; since $K$ is complete, the Newton polygon of $P$ consists of a single segment. In other words, if we write $P(T) = \sum_{i=0}^n P_i T^i$ with $P_n = 1$, then $|P_{n-i}|^{1/i} \leq |P_0|^{1/n} = \beta(x)$ for $i = 1, \ldots, n$. (See [70, §2.1] for more discussion of Newton polygons.)

If $\gamma(x) > |P_0|^{1/n}$, then under $\gamma$ the sum $0 = \sum_{i=0}^n P_i x^i$ would be dominated by the term $P_n x^n$, a contradiction. Hence $\gamma(x) \leq |P_0|^{1/n} = \beta(x)$ and similarly $\gamma(x^{-1}) \leq \beta(x^{-1})$; by writing

$$1 = \gamma(x \cdot x^{-1}) \leq \gamma(x)\gamma(x^{-1}) \leq \beta(x)\beta(x^{-1}) = 1,$$

we see that $\gamma(x) = \beta(x)$ as desired. \hfill \square

Before moving on to Banach modules, we make one observation about modules over a Banach ring.

Lemma 2.2.6. Let $R$ be a Banach ring.

(a) For any finite $R$-module $M$, the quotient seminorm defined by a surjection $\pi: R^n \to M$ of $R$-modules does not depend, up to equivalence, on the choice of the surjection.

(b) Let $R \to S$ be a bounded homomorphism of Banach rings. Let $M$ be a finite $R$-module, let $N$ be a finite $S$-module, and let $M \to N$ be an additive semilinear map (i.e., a map induced by a homomorphism $M \otimes_R S \to N$ of $S$-modules). Then this map becomes bounded if we equip $M$ and $N$ with seminorms as described in (a).
Proof. To prove (a), let \( \pi' : R^m \to M \) be a second surjection, and combine \( \pi \) and \( \pi' \) to obtain a third surjection \( \pi'' : R^{n+m} \to M \). It is enough to check that the quotient seminorms \( |\cdot|, |\cdot|'' \) induced by \( \pi, \pi'' \) are equivalent, as then the same argument will apply with \( \pi \) and \( \pi' \) interchanged.

Let \( e_1, \ldots, e_{n+m} \) be the standard basis of \( R^{n+m} \). On one hand, we clearly have \( |\cdot|'' \leq |\cdot| \) because lifting an element of \( M \) to \( R^n \) also gives a lift to \( R^{n+m} \). On the other hand, for \( j = n + 1, \ldots, n + m \), we can write \( \pi(e_j) = \sum_{i=1}^{n+m} A_{ij} \pi(e_i) \) for some \( A_{ij} \in R \). If an element of \( M \) lifts to \( \sum_{i=1}^{n+m} c_i e_i \in R^{n+m} \), it also lifts to \( \sum_{i=1}^{n} c_i + \sum_{j=n+1}^{n+m} A_{ij} c_j \). Consequently, we have \( |\cdot| \leq \max \{1, |A|\} |\cdot|'' \). This yields (a).

To prove (b), choose surjections \( R^m \to M, S^n \to N \) of \( R \)-modules. We may then lift the composition \( R^m \to M \to N \) to a homomorphism \( R^m \to S^n \) which is evidently bounded. This proves the claim. \( \square \)

**Definition 2.2.7.** Let \( R \) be a Banach ring. A Banach module over \( R \) is an \( R \)-module \( M \) whose additive group is complete for a norm \( |\cdot|_M \) for which \( |rv|_M \leq |r||v|_M \) for all \( r \in R, v \in M \). A Banach module over \( R \) which is itself a Banach ring is called a Banach algebra over \( R \).

One has an analogue of the Banach-Schauder open mapping theorem in the nonarchimedean setting.

**Theorem 2.2.8.** Let \( \varphi : V \to W \) be a bounded surjective homomorphism of Banach modules over a nontrivially normed analytic field. Then \( \varphi \) is open and strict.

Proof. See [17, §I.3.3, Théorème 1] or [89, Proposition 8.6]. \( \square \)

**Definition 2.2.9.** Let \( R \) be a Banach ring. A finite Banach module/algebra over \( R \) is a Banach module/algebra \( M \) over \( R \) admitting a strict surjection \( R^n \to M \) of Banach modules over \( R \) for some nonnegative integer \( n \) (for the supremum norm on \( R^n \) defined by the canonical basis). By Lemma 2.2.6, the equivalence class of the norm on \( M \) is determined by the underlying \( R \)-module.

Beware that in general, a Banach module whose underlying \( R \)-module is finite need not be a finite Banach module; see the remark following [12, Proposition 2.1.9]. However, if \( R \) is a Banach algebra over a nontrivially normed analytic field, any Banach module structure on a finite \( R \)-module defines a finite Banach module, by Theorem 2.2.8.

**Lemma 2.2.10.** Let \( V, W, X \) be Banach modules over an analytic field \( K \).

(a) The map \( V \otimes_K W \to V \hat{\otimes}_K W \) is injective.

(b) Let \( f : V \to W \) be a bounded homomorphism and let \( f_X : V \hat{\otimes}_K X \to W \hat{\otimes}_K X \) be the induced map. Then the natural map \( \ker(f) \hat{\otimes}_K X \to \ker(f_X) \) is a bijection.
Proof. Both parts reduce immediately to the case where \( V, W, X \) contain dense \( K \)-vector subspaces of at most countable dimension (i.e., they are separable Banach modules). In this setting, (a) follows from the existence of Schauder bases for \( V \) and \( W \); see for instance [70, Lemma 1.3.11]. Similarly, (b) follows from the existence of a Schauder basis for \( X \). \( \square \)

Remark 2.2.11. In general, a finite module \( M \) over a Banach ring \( R \) does not receive a finite Banach module structure from Lemma 2.2.6 because \( M \) need not be separated or complete under the quotient seminorm induced by a presentation. In fact, if \( R \) is a Banach algebra over a nontrivially normed analytic field, then the following conditions on \( R \) are equivalent [19, Propositions 3.7.3/2, 3.7.3/3].

(a) The ring \( R \) is noetherian.

(b) Every ideal of \( R \) is closed.

(c) The forgetful functor from finite Banach \( R \)-modules to finite \( R \)-modules is an equivalence of categories.

For Banach rings which are not noetherian, as noted in Remark 2.2.11, we cannot equip arbitrary finite modules over \( R \) with natural Banach module structures. However, we can do so for finite projective \( R \)-modules.

Lemma 2.2.12. Let \( R \) be a Banach ring. Let \( P \) be a finite projective \( R \)-module. Choose a finite projective \( R \)-module \( Q \) and an isomorphism \( P \oplus Q \cong R^n \) of \( R \)-modules, for \( n \) a suitable nonnegative integer. Equip \( R^n \) with the supremum norm defined by the canonical basis.

(a) The subspace norm on \( P \) for the inclusion into \( R^n \) is equivalent to the quotient norm for the projection from \( R^n \), and gives \( P \) the structure of a finite Banach module over \( R \).

(b) The equivalence class of the norms described in (a) is independent of the choice of \( Q \) and of the presentation \( P \oplus Q \cong R^n \).

(c) The above construction defines a fully faithful functor from finite projective \( R \)-modules to finite Banach modules over \( R \) whose underlying \( R \)-modules are projective, which is a section of the forgetful functor.

Proof. Let \( P', Q' \) be copies of \( P, Q \), respectively. Note that the supremum norms \( | \cdot |_1, | \cdot |_2 \) on \( P \oplus P' \oplus Q \oplus Q' \) defined by the presentations

\[
(P \oplus Q) \oplus (P' \oplus Q') \cong R^n \oplus R^n, \quad (P \oplus Q') \oplus (P' \oplus Q) \cong R^n \oplus R^n
\]

are equivalent by Lemma 2.2.6.

It is clear that the subspace and quotient norms on \( P \oplus Q \) induced by \( | \cdot |_1 \) are identical, and that \( P \oplus Q \) is complete under these norms. Consequently, the subspace and quotient norms on \( P \oplus Q \) induced by \( | \cdot |_2 \) are equivalent, and \( P \oplus Q \) is complete under these norms. Restricting to \( P \) yields the subspace and quotient norms induced by the original presentation, so these two are equivalent. Moreover, \( P \) is the intersection of the closed subspaces \( P \oplus Q \) and \( P \oplus Q' \) of \( P \oplus P' \oplus Q \oplus Q' \). This proves (a). Parts (b) and (c) follow from Lemma 2.2.6. \( \square \)
Lemma 2.2.13. Let $P$ be a finite projective module over a Banach ring $R$, and choose a norm on $P$ as in Lemma 2.2.12. Let $e_1, \ldots, e_n$ be a finite set of generators of $P$ as an $R$-module. Then there exists $c > 0$ such that any $e_1', \ldots, e_n' \in P$ with $|e_i' - e_i| < c$ for $i = 1, \ldots, n$ also form a set of generators of $P$ as an $R$-module.

Proof. The conclusion does not depend on the choice of the norm (only the constant $c$ does), so we may use the restriction the supremum norm on $R^n$ along the homomorphism $P \to R^n$ defined by $e_1, \ldots, e_n$. In this case, the claim is evident with $c = 1$, as then the matrix $A$ defined by $e_j' = \sum_i A_{ij} e_i$ satisfies $|A - 1| < 1$ and hence is invertible.

2.3 The spectrum of a Banach ring

We now introduce the topological space corresponding to a Banach ring.

Hypothesis 2.3.1. For the remainder of §2, let $A$ be a Banach ring with norm denoted by $|\cdot|$.

Definition 2.3.2. The spectrum (or Gel’fand spectrum) $\mathcal{M}(A)$ of $A$ is the set of multiplicative seminorms $\alpha$ on $A$ bounded above by $|\cdot|$ (or equivalently, dominated by $|\cdot|$). We topologize $\mathcal{M}(A)$ as a closed subspace of the product $\prod_{a \in A} [0, |a|]$; hence $\mathcal{M}(A)$ is compact by Tikhonov’s theorem [15, §1.9.5, Théorème 3] (see also [12, Theorem 1.2.1]). A subbasis of the topology on $\mathcal{M}(A)$ is given by the sets $\{\alpha \in \mathcal{M}(A) : \alpha(f) \in I\}$ for each $f \in A$ and each open interval $I \subseteq \mathbb{R}$. Any bounded homomorphism $\varphi : A \to B$ between Banach rings induces a continuous map $\varphi^* : \mathcal{M}(B) \to \mathcal{M}(A)$ by restriction.

Remark 2.3.3. One can use Definition 2.3.2 to define the spectrum $\mathcal{M}(A)$ more generally for any ring $A$ equipped with a submultiplicative seminorm. However, this will provide no useful additional generality, because the map $A \to \widehat{A}$ always induces a homeomorphism $\mathcal{M}(\widehat{A}) \to \mathcal{M}(A)$.

Berkovich’s first main theorem about the spectrum is the following.

Theorem 2.3.4. For $A$ nonzero, $\mathcal{M}(A) \neq \emptyset$.

Proof. See [12, Theorem 1.2.1].

Corollary 2.3.5. For any nontrivial ideal $I$ of $A$, there exists $\alpha \in \mathcal{M}(A)$ such that $\alpha(f) = 0$ for all $f \in I$.

Proof. Let $J$ be the closure of $I$. By Lemma 2.2.2, $A/J$ is nonzero, so $\mathcal{M}(A/J) \neq \emptyset$ by Theorem 2.3.4. Any element of $\mathcal{M}(A/J)$ restricts to an element $\alpha \in \mathcal{M}(A)$ of the desired form. (Compare [12, Corollary 1.2.4].)

Corollary 2.3.6. A finite set $f_1, \ldots, f_n$ of elements of $A$ generates the unit ideal if and only if for each $\alpha \in \mathcal{M}(A)$, there exists an index $i \in \{1, \ldots, n\}$ for which $\alpha(f_i) > 0$. 28
Proof. If there exist \( u_1, \ldots, u_n \in A \) for which \( u_1 f_1 + \cdots + u_n f_n = 1 \), then for each \( \alpha \in \mathcal{M}(A) \), we have \( \max_i \{ \alpha(u_i) \alpha(f_i) \} \geq 1 \) and so \( \alpha(f_i) > 0 \) for some index \( i \). Conversely, suppose that \( f_1, \ldots, f_n \) generate a nontrivial ideal \( I \); then by Corollary 2.3.5, we can choose \( \alpha \in \mathcal{M}(A) \) such that \( \alpha(f) = 0 \) for all \( f \in I \). \[\square\]

**Corollary 2.3.7.** An element \( f \in A \) is a unit if and only if \( \alpha(f) > 0 \) for all \( \alpha \in \mathcal{M}(A) \).

**Definition 2.3.8.** For \( \alpha \in \mathcal{M}(A) \), define the prime ideal \( p_\alpha = \alpha^{-1}(0) \); then \( \alpha \in \mathcal{M}(A) \) induces a multiplicative norm on \( A/p_\alpha \). The completion of \( \text{Frac}(A/p_\alpha) \) for the unique multiplicative extension of this norm is called the residue field of \( \alpha \), and denoted \( H(\alpha) \). The image of the map \( \mathcal{M}(A) \to \text{Spec}(A) \) taking \( \alpha \) to \( p_\alpha \) contains all maximal ideals, by Corollary 2.3.5; see Lemma 2.3.12 for some consequences of this observation.

The *Gel'fand transform* of \( A \) is the map \( A \to \prod_{\alpha \in \mathcal{M}(A)} H(\alpha) \); it is bounded for the supremum norm on the product (or more precisely, on the subring of the product on which the supremum is finite).

We say that \( A \) is free of trivial spectrum if there exists no \( \alpha \in \mathcal{M}(A) \) such that the norm on \( H(\alpha) \) is trivial. For instance, this occurs if \( A \) is a Banach algebra over a nontrivially normed analytic field.

**Lemma 2.3.9.** The Banach ring \( A \) is free of trivial spectrum if and only if the ideal \( I \) generated by \( m_A \) is trivial.

Proof. If \( I \) is trivial, then for each \( \alpha \in \mathcal{M}(A) \), there must exist \( x \in m_A \) which does not belong to \( p_\alpha \). We then have \( 0 < \alpha(x) \leq |x| < 1 \), so the norm on \( H(\alpha) \) is not trivial.

Conversely, suppose that \( I \) is nontrivial. By Corollary 2.3.5, there exists \( \beta \in \mathcal{M}(A) \) such that \( \beta(x) = 0 \) for all \( x \in I \). Let \( \alpha \) be the seminorm on \( A \) obtained by restricting the trivial norm on \( A/p_\beta \); then for \( x \in A \setminus p_\beta \), we have \( \beta(x) > 0 \), so \( x \notin I \) and so \( |x| \geq 1 \). It follows that \( \alpha \in \mathcal{M}(A) \) and the norm on \( H(\alpha) \) is trivial. \[\square\]

Berkovich’s second main theorem about the spectrum is the following result.

**Theorem 2.3.10.** The restriction of the supremum norm on \( \prod_{\alpha \in \mathcal{M}(A)} H(\alpha) \) along the Gel’fand transform is the spectral seminorm on \( A \). In particular, this equals the norm on \( A \) if \( A \) is uniform.

Proof. See [12, Corollary 1.3.2]. \[\square\]

**Remark 2.3.11.** We collect several remarks about Theorem 2.3.10.

(a) Theorem 2.3.10 implies Theorem 2.3.4 if \( A \) is nonzero, the spectral seminorm of \( 1 \in A \) equals \( 1 \).

(b) The supremum norm in Theorem 2.3.10 is always achieved if \( A \) is nonzero: for each \( f \in A \), the map \( f \mapsto \alpha(f) \) is continuous on the compact space \( \mathcal{M}(A) \), and so achieves its maximum. Consequently, Theorem 2.3.10 may be viewed as a form of the *maximum modulus principle* in nonarchimedean analytic geometry. For an analogous result in rigid analytic geometry, see [19, Proposition 6.2.1/4].
(c) It is not generally true that $A$ is complete under its spectral seminorm when the latter is a norm. This is true for affinoid algebras over an analytic field, though; see Corollary 2.5.9

**Lemma 2.3.12.** (a) A homomorphism $M \to N$ of $A$-modules, with $N$ a finite $A$-module, is surjective if and only if $M \otimes_A \mathcal{H}(\alpha) \to N \otimes_A \mathcal{H}(\alpha)$ is surjective for all $\alpha \in \mathcal{M}(A)$.

(b) Suppose that $A$ is uniform. If $M$ is a finitely generated $A$-module and $\dim_{\mathcal{H}(\alpha)}(M \otimes_A \mathcal{H}(\alpha))$ is the same for all $\alpha \in \mathcal{M}(A)$, then $M$ is locally free. (See Lemma 2.6.4 for a refinement of this result.)

**Proof.** To prove (a), suppose that $M \otimes_A \mathcal{H}(\alpha) \to N \otimes_A \mathcal{H}(\alpha)$ is surjective for all $\alpha \in \mathcal{M}(A)$. For each maximal ideal $p$ of $A$, choose $\alpha \in \mathcal{M}(A)$ with $p_\alpha = p$. Then $A/p \to \mathcal{H}(\alpha)$ is an extension of fields, so surjectivity of $M \otimes_A \mathcal{H}(\alpha) \to N \otimes_A \mathcal{H}(\alpha)$ implies surjectivity of $M \otimes_A A/p \to N \otimes_A A/p$. This in turn implies surjectivity of $M \otimes_A A_\alpha \to N \otimes_A A_\alpha$ by Nakayama’s lemma, and hence surjectivity of $M \to N$.

To prove (b), let $n$ be the common value of $\dim_{\mathcal{H}(\alpha)} M \otimes_A \mathcal{H}(\alpha)$. For each maximal ideal $p$ of $A$, we may choose $\alpha \in \mathcal{M}(A)$ with $p_\alpha = p$ (see Definition 2.3.8). Choose elements $v_1, \ldots, v_n$ of $M$ whose images in $M \otimes_A \mathcal{H}(\alpha)$ are linearly independent. Then $v_1, \ldots, v_n$ form a basis of $M \otimes_A A/p$, so they also generate $M \otimes_A A_\alpha$ by Nakayama’s lemma. We may then choose $f \in A \setminus p$ so that $v_1, \ldots, v_n$ generate $M \otimes_A A[f^{-1}]$. Suppose that $v_1, \ldots, v_n$ fail to form a basis of $M \otimes_A A[f^{-1}]$; then there must exist $a_1, \ldots, a_n \in A$ not all mapping to zero in $A[f^{-1}]$ and a nonnegative integer $m$ such that $f^m a_1 v_1 + \cdots + f^m a_n v_n = 0$. For each $\beta \in \mathcal{M}(A)$, if $\beta(f) = 0$, then obviously $\beta(f^m a_i) = 0$ for $i = 1, \ldots, n$. Otherwise, $p_\beta \in \text{Spec}(A[f^{-1}])$ and so $v_1, \ldots, v_n$ generate $M \otimes_A A/p_\beta$, again without relations because $\dim_{\mathcal{H}(\beta)}(M \otimes_A \mathcal{H}(\beta)) = n$. Hence $\beta(f^m a_i) = 0$ for $i = 1, \ldots, n$ again. By Theorem 2.3.10 we deduce that $f^m a_i = 0$ for $i = 1, \ldots, n$, a contradiction. We conclude that $M \otimes_A A[f^{-1}]$ is a free $A[f^{-1}]$-module; in other words, $M$ is free over a distinguished open subset of $\text{Spec}(A)$ containing the original maximal ideal $p$ as well as all other prime ideals contained in $p$. We may thus cover $\text{Spec}(A)$ by such open subsets, so $M$ is locally free. (As noted in Definition 1.1.2 a closely related statement is [34 Exercise 20.13].)

**Lemma 2.3.13.** For $A \to B$, $A \to C$ homomorphisms of Banach rings, the map $\mathcal{M}(B \widehat{\otimes}_A C) \to \mathcal{M}(B) \times_{\mathcal{M}(A)} \mathcal{M}(C)$ is surjective.

**Proof.** This reduces to the case where $A, B, C$ are all analytic fields, for which we may apply Lemma 2.2.10(a) and Theorem 2.3.4. See also [72, Lemma 1.20].

**Remark 2.3.14.** Let $R$ be a uniform Banach ring with norm $\alpha$. For $S \in \text{FÉt}(R)$, we may see that there is a maximal power-multiplicative seminorm $\beta$ on $S$ for which the homomorphism $R \to S$ is bounded, as follows.

- When $R$ and $S$ are both analytic fields, this is clear: since the homomorphism $R \to S$ is nonzero, it is bounded if and only if it is isometric, and the only power-multiplicative extension of $\alpha$ to $S$ is the multiplicative extension (Lemma 2.2.5).
• When \( R \) is an analytic field and \( S \) is arbitrary, we may split \( S \) as a direct sum \( S_1 \oplus \cdots \oplus S_n \) of finite separable field extensions of \( R \). The maximal power-multiplicative seminorm in this case is the supremum of the maximal seminorms on the \( S_i \).

• For general \( R \), we may write \( \alpha = \sup \{ \gamma : \gamma \in \mathcal{M}(R) \} \) by Theorem 2.3.10. We then take \( \beta \) to be the supremum of the restrictions to \( S \) of the maximal power-multiplicative seminorm on the rings \( S \otimes_R \mathcal{H}(\alpha) \); this is maximal by Theorem 2.3.10 again.

From the construction, \( \beta \) is a norm on \( S \), but it is not guaranteed in general that \( S \) is complete under \( \beta \); that is, it is not guaranteed that \( S \) is a finite Banach module over \( R \). For a special case where these properties do hold, see Remark 3.1.8.

From the construction of the norm on \( S \), it follows that the integral closure of \( o_R \) in \( S \) is contained in \( o_S \), but this containment may be strict. However, if \( S \) is a finite Banach module over \( R \), we can at least check that \( m_S \) is contained in the integral closure of \( m_R \) in \( S \); this containment is maximal by Theorem 2.3.10 again.

**Remark 2.3.15.** When studying spectra, it is helpful to use general facts about compact topological spaces. Here are a few that we will need.

(a) The image of a quasicompact topological space under a continuous map is quasicompact [15, §I.9.4, Théorème 2]. Consequently, any continuous map \( f : Y \to X \) from a quasicompact topological space to a Hausdorff topological space is closed [15, §I.9.4, Corollaire 2].

(b) With notation as in (a), if \( V \) is open in \( Y \), then \( W = X \setminus f(Y \setminus V) \) is open. One consequence is that if \( Z \) is closed in \( X \) and \( V \) is an open neighborhood of \( f^{-1}(Z) \), then \( W \) is an open neighborhood of \( Z \) and \( f^{-1}(Z) \subseteq V \). Another consequence is that the quotient and subspace topologies on \( \text{image}(f) \) coincide: if \( U \subseteq \text{image}(f) \) and \( V = f^{-1}(U) \) is open in \( Y \), then \( U = \text{image}(f) \cap W \) is open in \( \text{image}(f) \). That is, any continuous surjection (resp. bijection) from a quasicompact space to a Hausdorff space is a quotient map (resp. a homeomorphism).

(c) If \( X \) is the inverse limit of an inverse system \( \{X_i\}_{i \in I} \) of nonempty compact spaces, then \( X \) is nonempty and compact. This follows from Tikhonov’s theorem, or see [15, §I.9.6, Proposition 8]. As a corollary, for \( i \in I \) and \( Z \) a closed subset of \( X_i \), \( Z \) has empty inverse image in \( X \) if and only if there exists an index \( j \geq i \) for which \( Z \) has empty inverse image in \( X_j \).

(d) With notation as in (c), for any \( i \in I \) and any open subsets \( V_{1,i}, \ldots, V_{n,i} \) of \( X_i \) whose inverse images in \( X \) form a covering, there exists an index \( j \geq i \) for which the inverse images \( V_{1,j}, \ldots, V_{n,j} \) of \( V_{1,i}, \ldots, V_{n,i} \) in \( X_j \) form a covering of \( X_j \) itself: apply (c) to the
closed set \( X_i \setminus (V_{1,i} \cup \cdots \cup V_{n,i}) \). As a corollary, any finite open covering of \( X \) is refined by the pullback of a finite open covering of some \( X_i \).

(e) With notation as in (c), any disconnection of \( X \) (i.e., any partition of \( X \) into two disjoint closed-open subsets \( U_1, U_2 \)) is the inverse image of a disconnection of some \( X_j \), by the following argument. Choose any \( i \in I \). By (a), the images \( V_{1,i}, V_{2,i} \) of \( U_1, U_2 \) in \( X_i \) are closed and disjoint; they may thus be covered by disjoint open neighborhoods \( W_{1,i}, W_{2,i} \). By (d), we can find an index \( j \geq i \) such that \( X_j \) is covered by the inverse images \( W_{1,j}, W_{2,j} \) of \( W_{1,i}, W_{2,i} \) in \( X_j \). Since \( W_{1,j}, W_{2,j} \) are open and disjoint, they form a disconnection of \( X_j \) which pulls back to the given disconnection of \( X \).

### 2.4 Affinoid and rational subdomains

We next introduce affinoid algebras, then use them to define some important subspaces of the spectrum.

**Definition 2.4.1.** For \( r_1, \ldots, r_n > 0 \), define the *Tate algebra* over the Banach ring \( A \) with radii \( r_1, \ldots, r_n \) to be the ring

\[
A\{T_1/r_1, \ldots, T_n/r_n\} = \left\{ f = \sum_I a_I T^I : a_I \in A, \lim_{I \to \infty} |a_I| r^I = 0 \right\},
\]

where \( I = (i_1, \ldots, i_n) \) runs over \( n \)-tuples of nonnegative integers, \( T^I = T_1^{i_1} \cdots T_n^{i_n} \), and \( r^I = r_1^{i_1} \cdots r_n^{i_n} \). (That is, the series in question converge on the closed polydisc defined by the conditions \( |T_i| \leq r_i \) for \( i = 1, \ldots, n \).) The set \( A\{T_1/r_1, \ldots, T_n/r_n\} \) is a subring of \( A\llbracket T_1, \ldots, T_n \rrbracket \) complete for the *Gauss norm*

\[
\sum_I |a_I T^I|_r = \sup_I \{|a_I| r^I\},
\]

which is easily seen to be submultiplicative (resp. power-multiplicative, multiplicative) if the seminorm on \( A \) is; see [72, Lemma 1.7].

A bounded homomorphism \( A \to B \) of Banach rings is *affinoid* if it factors as \( A \to A\{T_1/r_1, \ldots, T_n/r_n\} \to B \) for some positive integer \( n \), some \( r_1, \ldots, r_n > 0 \), and some strict surjection \( A\{T_1/r_1, \ldots, T_n/r_n\} \to B \). We also say that \( B \) is an *affinoid algebra* over \( A \). In case \( A \) is a Banach algebra over a nontrivially normed analytic field and we have such a factorization with \( r_1 = \cdots = r_n = 1 \), we say that the homomorphism \( A \to B \) is *strictly affinoid*, or that \( B \) is a *strictly affinoid algebra* over \( A \).

**Convention 2.4.2.** We will encounter a number of statements which are true either when stated in terms of strictly affinoid algebras, or when stated *mutatis mutandis* in terms of affinoid algebras. To assert these statements compactly, we will phrase them in terms of the adjective *(strictly)* affinoid. This adjective can either be read as affinoid or strictly affinoid, provided that the same interpretation is taken consistently within the statement in question.
A similar convention applies to other adjectives to which the adverb \textit{strictly} applies, as in Definition 2.4.7; again, this only applies when \( A \) is a Banach algebra over a nontrivially normed analytic field.

**Definition 2.4.3.** Assume that \( A \) is uniform, and let \( U \) be a closed subset of \( \mathcal{M}(A) \). We say that \( U \) is a (\textit{strictly}) affinoid subdomain of \( \mathcal{M}(A) \) if there exists a (\textit{strictly}) affinoid homomorphism \( \varphi : A \to B \) which is initial among bounded homomorphisms \( \psi : A \to C \) of uniform Banach rings for which \( \psi^{\ast}(\mathcal{M}(C)) \subseteq U \). This implies that \( \varphi \) is an epimorphism in the category of uniform Banach rings; we say that \( \varphi \) represents the (\textit{strictly}) affinoid subdomain \( U \). We will also say that \( \varphi \) is a (\textit{strictly}) affinoid localization.

**Lemma 2.4.4.** Let \( \varphi : A \to B \) be a bounded homomorphism of uniform Banach rings representing the affinoid subdomain \( U \) of \( \mathcal{M}(A) \). Then \( \varphi^{\ast} \) induces a homeomorphism \( \mathcal{M}(B) \cong U \).

**Proof.** The map \( \varphi^{\ast} \) by definition gives a continuous map from \( \mathcal{M}(B) \) to \( U \). To see that the map is bijective, choose any \( \alpha \in U \). The map \( A \to \mathcal{H}(\alpha) \) factors uniquely through a bounded homomorphism \( B \to \mathcal{H}(\alpha) \); the restriction of the norm on \( \mathcal{H}(\alpha) \) to \( B \) defines the unique point of \( \mathcal{M}(B) \) mapping to \( \alpha \) under \( \varphi^{\ast} \).

Since \( U \) is closed in \( \mathcal{M}(A) \), it is compact. Since \( \varphi^{\ast} : \mathcal{M}(B) \to U \) is a continuous bijection of compact sets, it is a homeomorphism by Remark 2.3.15(b).

**Remark 2.4.5.** Note that the intersection of (\textit{strictly}) affinoid subdomains \( U_1, U_2 \) represented by homomorphisms \( A \to B_1, A \to B_2 \) is again a (\textit{strictly}) affinoid subdomain, represented by \( A \to C \) for \( C \) the uniformization of \( B_1 \mathcal{H} B_2 \). Also, if \( A \to B \) and \( B \to C \) are (\textit{strictly}) affinoid localizations, then so is the composition \( A \to C \); that is, a (\textit{strictly}) affinoid subdomain of a (\textit{strictly}) affinoid subdomain is again a (\textit{strictly}) affinoid subdomain.

**Definition 2.4.6.** A covering family of (\textit{strictly}) affinoid localizations for \( A \) is a finite set \( A \to B_1, \ldots, A \to B_n \) of homomorphisms representing (\textit{strictly}) affinoid subdomains, such that the images of \( \mathcal{M}(B_1), \ldots, \mathcal{M}(B_n) \) in \( \mathcal{M}(A) \) cover \( \mathcal{M}(A) \). We will also refer to the subsets \( \mathcal{M}(B_1), \ldots, \mathcal{M}(B_n) \) as a covering family of (\textit{strictly}) affinoid subdomains on \( \mathcal{M}(A) \). If \( \mathcal{M}(A) \) is also covered by the relative interiors of \( \mathcal{M}(B_1), \ldots, \mathcal{M}(B_n) \), we say that the covering family is strong.

The general definition of an affinoid subdomain is too fragile to be of much use in our work. We instead restrict to the following subclasses which can be described more explicitly.

**Definition 2.4.7.** A \textit{Weierstrass subdomain} of \( \mathcal{M}(A) \) is a closed subspace of the form

\[
U = \{ \alpha \in \mathcal{M}(A) : \alpha(f_i) \leq p_i \quad (i = 1, \ldots, m) \}
\]

for some \( f_1, \ldots, f_m \in A \) and some \( p_1, \ldots, p_m > 0 \). More generally, a \textit{Laurent subdomain} of \( \mathcal{M}(A) \) is a closed subspace of the form

\[
U = \{ \alpha \in \mathcal{M}(A) : \alpha(f_i) \leq p_i, \alpha(g_j) \geq q_j \quad (i = 1, \ldots, m; j = 1, \ldots, n) \}
\]

33
for some \( f_1, \ldots, f_m, g_1, \ldots, g_n \in A \) and some \( p_1, \ldots, p_m, q_1, \ldots, q_n > 0 \). Note that \( \mathcal{M}(A) \) admits a neighborhood basis consisting of Laurent subdomains. If the \( p_i \) and \( q_j \) are all equal to 1, we get the definition of a strictly Weierstrass or strictly Laurent subdomain of \( \mathcal{M}(A) \).

Even more generally, a rational subdomain of \( \mathcal{M}(A) \) is a closed subspace of the form

\[
U = \{ \alpha \in \mathcal{M}(A) : \alpha(f_i) \leq p_i \alpha(g) \quad (i = 1, \ldots, n) \}
\]

(2.4.7.3)

for some \( f_1, \ldots, f_n, g \in A \) which generate the unit ideal and some \( p_1, \ldots, p_n > 0 \). If the \( p_i \) are all equal to 1, we get the definition of a strictly rational subdomain of \( \mathcal{M}(A) \).

The intersection of two (strictly) Weierstrass (resp. (strictly) Laurent, (strictly) rational) subdomains is again (strictly) Weierstrass (resp. (strictly) Laurent, (strictly) rational) [19, Proposition 7.2.3/7], and a (strictly) Weierstrass (resp. rational) subdomain of a (strictly) Weierstrass (resp. rational) subdomain is (strictly) Weierstrass (resp. rational) [19, Proposition 7.2.4/1]. However, the Laurent property is not transitive; see the counterexample following [19, Corollary 7.2.4/3].

We will show below (Lemma 2.4.9) that if \( A \) is uniform, then any (strictly) rational subdomain is a (strictly) affinoid subdomain. A homomorphism \( \varphi : A \to B \) representing a (strictly) Weierstrass (resp. (strictly) Laurent, (strictly) rational) subdomain \( U \) of \( \mathcal{M}(A) \) will be described as a (strictly) Weierstrass localization (resp. (strictly) Laurent localization, (strictly) rational localization). For such a homomorphism, if \( U \) contains a neighborhood of \( \alpha \in \mathcal{M}(A) \), we will say that \( \varphi \) encircles \( \alpha \), or that \( U \) encircles \( \alpha \).

**Remark 2.4.8.** Given a rational subdomain \( U \) of \( \mathcal{M}(A) \) as in (2.4.7.3), one has \( \alpha(g) > 0 \) for all \( \alpha \in U \), so by compactness, \( \inf \{ \alpha(g) : \alpha \in U \} > 0 \). Consequently, any (strictly) rational subdomain may be viewed as a (strictly) Weierstrass subdomain of a (strictly) Laurent subdomain.

From this, it is easy to exhibit \( \epsilon > 0 \) such that any \( f'_1, \ldots, f'_n, g' \in A \) satisfying \( |f'_i - f_i| < \epsilon, |g' - g| < \epsilon \) generate the unit ideal and satisfy

\[
U = \{ \alpha \in \mathcal{M}(A) : \alpha(f'_i) \leq p_i \alpha(g') \quad (i = 1, \ldots, n) \}.
\]

See for instance [72, Remark 1.15]; the proof of [19, Proposition 7.2.4/1] is similar.

**Lemma 2.4.9.** If \( A \) is uniform, then any (strictly) rational subdomain of \( \mathcal{M}(A) \) is a (strictly) affinoid subdomain. More precisely, for \( U \) as in (2.4.7.2), \( U \) is represented by the homomorphism \( A \to B \) for \( B \) the uniformization of

\[
A\{T_1/p_1, \ldots, T_m/p_m, U_1/q_1^{-1}, \ldots, U_n/q_n^{-1}\}/(T_1 - f_1, \ldots, T_m - f_m, U_1g_1 - 1, \ldots, U_ng_n - 1)
\]

(2.4.9.1)

for the quotient norm associated to the Gauss norm. Similarly, for \( U \) as in (2.4.7.3), \( U \) is represented by the homomorphism \( A \to B \) for \( B \) the uniformization of

\[
A\{T_1/p_1, \ldots, T_n/p_n\}/(gT_1 - f_1, \ldots, gT_n - f_n)
\]

(2.4.9.2)

for the quotient norm associated to the Gauss norm.
Proof. We discuss only the rational case, the Laurent case being analogous. Let \( \psi : A \to C \) be a bounded homomorphism of uniform Banach rings for which \( \psi^*(\mathcal{M}(C)) \subseteq U \). In particular, \( \alpha(\psi(g)) > 0 \) for all \( \alpha \in \mathcal{M}(C) \), so \( \psi(g) \) is a unit in \( C \) by Corollary 2.3.7. The map \( \psi \) thus factors uniquely through the ring homomorphism \( A \to A[T_1, \ldots, T_n]/(gT_i - f_i, \ldots, gT_n - f_n) \) by mapping \( T_i \) to \( \psi(f_i)\psi(g)^{-1} \). By Theorem 2.3.10, \( \psi(f_i)\psi(g)^{-1} \) has norm at most \( p_i \). It now follows that \( \psi \) factors through the algebra described in (2.4.9.2), and through its uniformization. (Note that the uniformization is needed for \( B \) to be eligible for the universal property.)

Remark 2.4.10. Note that \( \mathcal{M}(A) \) admits a neighborhood basis consisting of open sets of the form

\[
\{ \alpha \in \mathcal{M}(A) : \alpha(f_i) < p_i, \alpha(g_j) > q_j \quad (i = 1, \ldots, m; j = 1, \ldots, n) \}. \tag{2.4.10.1}
\]

A set of this form is contained in the relative interior of the Laurent subspace described in (2.4.7.2), but this containment need not be an equality. Nonetheless, any strong covering family of rational subdomains can be refined to a covering by open sets as in (2.4.10.1). This means in that most arguments involving a strong covering family of rational subdomains, we may assume that replacing the nonstrict inequalities defining the rational subspaces with strict inequalities still gives a covering of \( \mathcal{M}(A) \).

Definition 2.4.11. For \( A \) uniform and \( \alpha \in \mathcal{M}(A) \), the localization of \( A \) at \( \alpha \), denoted \( A_\alpha \), is defined as the direct limit over all Laurent localizations (or equivalently all rational localizations) encircling \( \alpha \).

Lemma 2.4.12. For \( A \) uniform and \( \alpha \in A \), the localization \( A_\alpha \) is a henselian local ring whose residue field is dense in \( H(\alpha) \).

Proof. The ring \( A_\alpha \) is local by Corollary 2.3.7. The henselian property follows from Lemma 2.2.3(a).

Remark 2.4.13. While Lemma 2.4.12 and Lemma 2.2.4 together imply that any finite étale algebra over \( H(\alpha) \) lifts to a finite étale algebra over \( B \) for some rational localization \( A \to B \) encircling \( \alpha \), they do not imply the corresponding statement for \( \mathbb{Z}_p \)-local systems or \( \mathbb{Q}_p \)-local systems.

The following argument of Tate allows us to reduce certain questions about coverings to those of a very simple form. See also [19, §8.2.2].

Definition 2.4.14. Choose \( f_1, \ldots, f_n \in A \) and \( p_1, \ldots, p_n > 0 \). For \( i = 1, \ldots, n \), define the Laurent subdomains

\[
S_{i,-} = \{ \alpha \in \mathcal{M}(A) : \alpha(f_i) \leq p_i \}, \quad S_{i,+} = \{ \alpha \in \mathcal{M}(A) : \alpha(f_i) \geq p_i \}.
\]

For each \( e = (e_1, \ldots, e_n) \in \{-, +\}^n \), put

\[
S_e = \bigcap_{i=1}^n S_{i,e_i}. \tag{2.4.14.1}
\]
This forms a covering of \( \mathcal{M}(A) \) by Laurent subspaces. Any such covering is called a *standard Laurent covering*; in case \( n = 1 \), we will call it a *simple Laurent covering*. In case \( p_1 = \cdots = p_n = 1 \), we will refer to *standard strictly Laurent coverings* and *simple strictly Laurent coverings*.

**Proposition 2.4.15.** Let \( \mathcal{C} \) be a subcategory of the category of uniform Banach algebras over analytic fields which is closed under formation of (strictly) rational localizations. Let \( \mathcal{P} \) be a property of covering families of (strictly) rational localizations within \( \mathcal{C} \). Suppose that the following conditions hold.

(a) The property \( \mathcal{P} \) is local: if it holds for a refinement of a given covering, it also holds for the original covering.

(b) The property \( \mathcal{P} \) is transitive: if it holds for a covering \( R \rightarrow R_1, \ldots, R \rightarrow R_n \) and for coverings \( R_i \rightarrow R_{ij} \), then it holds for the composite covering \( R \rightarrow R_{ij} \).

(c) The property \( \mathcal{P} \) holds for any simple (strictly) Laurent covering.

Then the property \( \mathcal{P} \) holds for any covering in \( \mathcal{C} \).

**Proof.** The claim follows from the following observations.

(i) Since any (strictly) rational localization is the composition of two (strictly) Laurent localizations (Remark 2.4.8), any covering family of (strictly) rational localizations may be refined by a composition of covering families of (strictly) Laurent localizations.

(ii) Any covering family of (strictly) Laurent localizations may be refined by a standard (strictly) Laurent covering.

(iii) Any standard (strictly) Laurent covering is a composition of simple (strictly) Laurent coverings.

**Remark 2.4.16.** Suppose that \( A \) is an affinoid algebra over an analytic field, but is not necessarily uniform. If one modifies the universal property defining affinoid subdomains to allow for maps into arbitrary affinoid algebras (over arbitrary analytic fields), one can then hope to exhibit affinoid subdomains of \( \mathcal{M}(A) \). For instance, every (strictly) rational subdomain is a (strictly) affinoid subdomain, with representing algebra given by \( (2.4.9.2) \) without further uniformization; see [19, Proposition 7.2.3/4] for the strictly affinoid case and [12, Corollary 2.1.5] for the general case. (This does not contradict Lemma 2.4.9 because in case \( A \) is a uniform affinoid algebra over an analytic field, the algebra in \( (2.4.9.2) \) will be reduced, and so its norm will be equivalent to its spectral seminorm by Corollary 2.5.9.) This more general fact will be referenced implicitly in some statements in 2.5 but will otherwise not be used in this paper.

It would be nice to have a theory of affinoid subdomains that handles both affinoid algebras and uniform Banach algebras in a consistent way. One natural approach (suggested
to us by Vladimir Berkovich) is the following. Say that a Banach ring $A$ is quasiaffinoid if for each $f \in A$, there exists a constant $c > 0$ such that $|f^n| \leq c|f|^n$ for all sufficiently large positive integers $n$. This includes both uniform Banach rings and arbitrary affinoid algebras over analytic fields. However, it is not clear how to reformulate Lemma 2.4.9 if we expand the universal mapping property for affinoid subdomains to include maps with quasiaffinoid targets: this would seem to require a left adjoint to the forgetful functor from quasiaffinoid Banach rings to affinoid algebras over quasiaffinoid Banach rings.

2.5 Coherent sheaves on affinoid spaces

We now restrict to the setting of affinoid spaces over an analytic field, where a good theory of coherent sheaves is available.

**Hypothesis 2.5.1.** Throughout §2.5 assume that $A$ is an affinoid algebra over an analytic field $K$. For the purposes of this paper, it is sufficient to assume that $A$ is reduced (which will imply that it is isomorphic to a uniform Banach algebra; see Corollary 2.5.9), but all statements in §2.5 hold without this condition unless otherwise specified. Note that we cannot consider strictly affinoid algebras unless $K$ has nontrivial norm.

**Lemma 2.5.2.** The ring $A$ is noetherian, so (by Remark 2.2.11) any ideal of $A$ is closed. Moreover, any finite $A$-module may be viewed as a finite Banach $A$-module in a canonical way, under which any $A$-linear homomorphism of finite $A$-modules is continuous and strict.

**Proof.** For the first assertion, see [19, Proposition 6.1.1/3] for the strictly affinoid case and [12, Proposition 2.1.3] for the general case. For the other assertions (which apply to any noetherian Banach ring), see [19, §3.7.3].

**Remark 2.5.3.** A refinement of Lemma 2.5.2 is that any $K$-affinoid algebra is excellent in the sense of Grothendieck [33, Théorème 2.6], and hence catenary.

**Theorem 2.5.4.** Let $A \rightarrow B$ be an epimorphism in the category of (strictly) affinoid algebras over $K$. Then there exists a covering family $A \rightarrow B_1, \ldots, A \rightarrow B_n$ of (strictly) rational localizations such that for each $i$, the composition $B_i \rightarrow B_i \hat{\otimes}_A B$ can be written as a Weierstrass localization followed by a strict surjection. In particular, any affinoid subdomain of $\mathcal{M}(A)$ is a finite union of rational subdomains (but not conversely).

**Proof.** This statement is a generalization due to Temkin [98, Theorem 3.1] of the Gerritzen-Grauert theorem in rigid analytic geometry [19, Theorem 7.3.5/1].

**Lemma 2.5.5.** For any analytic field $L$ containing $K$, the map $A \rightarrow A_L = A \hat{\otimes}_K L$ is strict and faithfully flat, and the corresponding restriction map $\mathcal{M}(A_L) \rightarrow \mathcal{M}(A)$ is surjective.

**Proof.** Strictness of $A \rightarrow A_L$ follows from the existence of a Schauder basis for $A$ over $K$, as in the proof of Lemma 2.2.10 while faithful flatness holds by [13, Lemma 2.1.2]. For the surjectivity of $\mathcal{M}(A_L) \rightarrow \mathcal{M}(A)$, observe that for $\alpha \in \mathcal{M}(A)$, $\mathcal{H}(\alpha) \hat{\otimes}_K L$ is nonzero by Lemma 2.2.10(a), and so by Theorem 2.3.4 admits a bounded multiplicative seminorm. The latter restricts to a seminorm on $A_L$ lifting $\alpha$. 

37
Remark 2.5.6. Note that if $A \to B$ is a (strictly) Weierstrass, (strictly) Laurent, or (strictly) rational localization, then so is $A_L \to B_L$. Moreover, if $A \to B_1, \ldots, A \to B_n$ is a covering family of affinoid localizations, then so is $A_L \to B_{L,1}, \ldots, A_L \to B_{L,n}$ for $B_{L,i} = B_i \hat{\otimes}_K L$, by the following reasoning. For $\beta \in \mathcal{M}(A_L)$, let $\alpha$ be the restriction of $\beta$ to $A$, and choose an index $i \in \{1, \ldots, n\}$ for which $\alpha$ extends to $\alpha_i \in \mathcal{M}(B_i)$. Since $\mathcal{H}(\alpha) \cong \mathcal{H}(\alpha_i)$, $\beta$ defines a multiplicative seminorm on $H(\alpha) \hat{\otimes} H(\alpha_i)$. The latter ring receives a map from $A_L \hat{\otimes} A B_i \cong B_{L,i}$, along which we restrict to obtain an element of $\mathcal{M}(B_{L,i})$ which restricts to $\beta \in \mathcal{M}(A_L)$.

As noted by Berkovich [12, §2.1], for any affinoid algebra $A$ over $K$, we can construct an analytic field $L$ containing $K$ so that $A_L$ is a strictly affinoid algebra over $L$. For instance, choose a strict surjection $K\{T_1/r_1, \ldots, T_n/r_n\} \to A$. Put $L_0 = K$. For $i = 1, \ldots, n$, if $r_i$ is in the divisible closure of $|L_i^x|$, then put $L_{i+1} = L_i$; otherwise, put $L_{i+1} = L_i\{T/r_i, r_i/T\}$, which is an analytic field [12, Proposition 2.1.2]. The field $L = L_n$ has the desired property. This construction is used repeatedly in [12, §2] to extend many properties of strictly affinoid algebras established in [19] to more general affinoid algebras.

Lemma 2.5.7 (Noether normalization). For any nonzero strictly $K$-affinoid algebra $A$, there exists a finite strict monomorphism $K\{T_1, \ldots, T_n\} \to A$ for some $n \geq 0$.

Proof. See [19, Corollary 6.1.2/2].

Corollary 2.5.8. Let $A$ be a strictly $K$-affinoid algebra.

(a) Every maximal ideal of $A$ has residue field finite over $K$.

(b) The formula $\alpha \mapsto p_\alpha$ defines a bijection from the points of $\mathcal{M}(A)$ with residue field finite over $K$ to the maximal ideals of $A$.

(c) If $A$ is nonzero, then the sets in (b) are nonempty.

Proof. Assertion (a) follows from Lemma 2.5.7. For (b), note that on one hand, if $\mathcal{H}(\alpha)$ is finite over $K$, then $A \to \mathcal{H}(\alpha)$ is surjective because its image generates a dense subfield containing $K$. Consequently, $p_\alpha$ is a maximal ideal of $A$. Conversely, if $m$ is a maximal ideal of $A$, then $A/m$ is a finite extension of $K$, and so is complete for the unique multiplicative extension of the norm on $K$. It thus may be identified with $\mathcal{H}(\alpha)$ for some $\alpha \in \mathcal{M}(A)$. This proves (b); since any nonzero ring has a maximal ideal, (b) implies (c).

Corollary 2.5.9. For any reduced $K$-affinoid algebra $A$, the spectral seminorm on $A$ is a norm equivalent to the given norm.

Proof. See [12, Proposition 2.1.4].

Lemma 2.5.10. Any homomorphism $A \to B$ representing an affinoid subdomain of $\mathcal{M}(A)$ is flat.

Proof. See [12, Proposition 2.2.4].
Proposition 2.5.11. Let $A \to B_1, \ldots, A \to B_n$ be a covering family of affinoid localizations. Then the ring homomorphism $A \to B_1 \oplus \cdots \oplus B_n$ is faithfully flat.

Proof. The homomorphism is flat by Lemma 2.5.10. To check faithful flatness, we set notation as in Remark 2.5.6. By Corollary 2.5.8 and Lemma 1.1.4, the map $A_L \to B_{L,1} \oplus \cdots \oplus B_{L,n}$ is faithfully flat. By Lemma 2.5.5 $A \to A_L$ is faithfully flat, so the composition $A \to A_L \to B_{L,1} \oplus \cdots \oplus B_{L,n}$ is also. This composition also factors as $A \to B_1 \oplus \cdots \oplus B_n \to B_{L,1} \oplus \cdots \oplus B_{L,n}$, so $A \to B_1 \oplus \cdots \oplus B_n$ is forced to be faithfully flat as desired. □

Corollary 2.5.12. Let $A \to B_1, \ldots, A \to B_n$ be a covering family of affinoid localizations.

(a) A finite $A$-module $M$ is locally free if and only if $M \otimes_A B_i$ is a locally free $B_i$-module for $i = 1, \ldots, n$.

(b) A finite $A$-algebra $R$ is étale if and only if $R \otimes_A B_i$ is an étale $B_i$-algebra for $i = 1, \ldots, n$.

Proof. This follows from Proposition 2.5.11 plus Theorem 1.3.5 □

Theorem 2.5.13. Let $A \to B_1, \ldots, A \to B_n$ be a covering family of affinoid localizations. Put $B_{ij} = B_i \hat{\otimes}_A B_j$, so that $A \to B_{ij}$ is again an affinoid localization [19, Proposition 7.2.2/4].

(a) For any finite $A$-module $M$, the augmented Čech complex

$$0 \to M \to M \otimes_A \left( \bigoplus_{i=1}^n B_i \right) \to M \otimes_A \left( \bigoplus_{i,j=1}^n B_{ij} \right) \to \cdots$$

is exact. (For example, the last map shown is induced by the map $\bigoplus_{i=1}^n B_i \to \bigoplus_{i,j=1}^n B_{ij}$ taking $(b_i)_{i=1}^n$ to $(b_i - b_j)_{i,j=1}^n$.) Moreover, the sequence becomes strict exact if we view $M$ as a finite Banach module using Lemma 2.5.3.

(b) The morphism $A \to \bigoplus_{i=1}^n B_i$ is an effective descent morphism for the category of finite Banach modules over Banach rings.

Proof. Part (a) was established in the strictly affinoid case by Tate [19, Corollary 8.2.1/5] and extended to the general case by Berkovich [12, Proposition 2.2.5]. Part (b) was established in the strictly affinoid case by Kiehl [19, Theorem 9.4.3/3]. It is noted in [13, §1.2] that this implies the general case, but perhaps some further details are warranted.

A descent datum for the morphism $A \to \bigoplus_{i=1}^n B_i$ consists of some finite modules $M_i$ over $B_i$ and some isomorphisms $M_i \hat{\otimes}_B B_j \cong M_j \hat{\otimes}_B B_j$, satisfying the cocycle condition. Put $M_{ij} = M_i \hat{\otimes}_B B_{ij}$, and let $g : \bigoplus_{i=1}^n M_i \to \bigoplus_{i,j=1}^n M_{ij}$ be the map taking $(m_i)_{i=1}^n$ to $(m_i - m_j)_{i,j=1}^n$.

Use the recipe of Remark 2.5.6 to produce an analytic field $L$ containing $K$ for which $A_L = A \hat{\otimes}_K L$ and $B_{L,i} = B_i \hat{\otimes}_K L$ are strictly affinoid algebras over $L$. Put $M_{L,i} = M_i \hat{\otimes}_K L$, $M_{L,ij} = M_{ij} \hat{\otimes}_K L$, and let $g_L : \bigoplus_{i=1}^n M_{L,i} \to \bigoplus_{i,j=1}^n M_{L,ij}$ be the map taking $(m_i)_{i=1}^n$ to $(m_i - m_j)_{i,j=1}^n$. Put $M = \ker(g)$ and $M_L = \ker(g_L)$; the natural map $\hat{\otimes}_K L \to M_L$ is an isomorphism by
Lemma 2.2.10(b). By Kiehl’s theorem, $M_L$ is a finite Banach module over $A_L$, and for $j = 1, \ldots, n$, the composition $M_L \to \bigoplus_{i=1}^n M_{L,i} \to M_{L,j}$ induces an isomorphism $M_L \otimes_A B_{L,j} \to M_{L,j}$. (We may omit the completion of this last tensor product thanks to Remark 2.2.11.)

By [12, Proposition 2.1.11], $M$ is a finite Banach module over $A$. The map $M \otimes_A B_j \to M_j$ induced by the composition $M \to \bigoplus_{i=1}^n M_i \to M_j$ becomes an isomorphism upon tensoring (without completion) over $B_j$ with $B_{L,j}$; since $B_j \to B_{L,j}$ is faithfully flat by Lemma 2.5.5 it follows that $M \otimes_A B_j \to M_j$ is an isomorphism. We conclude that $M$ gives rise to the original descent datum (necessarily uniquely by (a)), so $f$ is an effective descent morphism as desired.

**Corollary 2.5.14.** Let $U$ be a closed and open subset of $\mathcal{M}(A)$. Then there exists a unique idempotent element $e \in A$ whose image in $\mathcal{H}(\alpha)$ is 1 if $\alpha \in U$ and 0 if $\alpha \notin U$. In particular, the projection $A \to eA$ taking $x \in A$ to $ex$ induces a homeomorphism $\mathcal{M}(eA) \cong U$.

**Proof.** Since $U$ is open, it can be covered with rational subdomains of $\mathcal{M}(A)$; since $U$ is closed in $\mathcal{M}(A)$ and hence compact, only finitely many rational subdomains $U_1, \ldots, U_m$ are needed. Similarly, $\mathcal{M}(A) \setminus U$ can be covered with finitely many rational subdomains $V_1, \ldots, V_n$ of $\mathcal{M}(A)$. Let $B_1, \ldots, B_m, C_1, \ldots, C_n$ be the affinoid algebras over $A$ representing $U_1, \ldots, U_m, V_1, \ldots, V_n$, respectively. By Theorem 2.5.13(a) applied with $M = A$, the element $((1, \ldots, 1), (0, \ldots, 0))$ of $(B_1 \oplus \cdots \oplus B_m) \oplus (C_1 \oplus \cdots \oplus C_n)$ determines an idempotent element $e \in A$ with the desired property.

To verify uniqueness, let $e' \in A$ be another idempotent of the desired form. Then $1 - e - e'$ maps to 1 or $-1$ in $\mathcal{H}(\alpha)$ for each $\alpha \in \mathcal{M}(A)$, and so is a unit in $A$ by Corollary 2.3.7. Now $(e - e')(1 - e - e') = 0$, so $e - e' = 0$ as desired. □

**Remark 2.5.15.** Theorem 2.5.13(a) implies that the structure presheaf for the affinoid G-topology on $\mathcal{M}(A)$ (the set-theoretic Grothendieck topology in which the admissible open sets are affinoid subdomains and the admissible coverings are the finite coverings) is a sheaf, as is the presheaf induced by any finite $A$-module. Theorem 2.5.13(b) implies that any coherent sheaf on the resulting locally G-ringed space is represented by a finite $A$-module.

**2.6 Affinoid systems**

We will find it useful to consider more general Banach algebras over analytic fields. To study these, we use an analogue of the observation that every ring is a direct limit of noetherian subrings (namely its finitely generated $\mathbb{Z}$-subalgebras).

**Definition 2.6.1.** By an **affinoid system**, we will mean a directed system $\{(A_i, \alpha_i)\}_{i \in I}$ in the category of Banach rings and submetric homomorphisms, in which each $A_i$ is an affinoid algebra over an analytic field. (It is not required that the $A_i$ are all affinoid algebras over the same field, though this extra condition will be satisfied in our applications.) We will mostly consider only **uniform** affinoid systems, in which $A_i$ is uniform (i.e., reduced and equipped with the spectral norm). Given such an affinoid system, equip the direct limit $A$ of the $A_i$ in the category of rings with the submultiplicative seminorm $\alpha$ given by taking the
Lemma 2.6.2. Let $K$ be an analytic field, and let $R$ be a Banach algebra over $K$.

(a) There exists an affinoid system $\{(A_i, \alpha_i)\}_{i \in I}$ with completed direct limit $R$.

(b) If $R$ is uniform, then there exists a uniform affinoid system with completed direct limit $R$.

Proof. Let $\alpha$ denote the norm on $R$. Let $I$ be the set of finite subsets of $R$. For each $i \in I$, let $A_i$ be the quotient of $K\{s/\alpha(s) : s \in i\}$ by the kernel of the map to $R$ taking $s$ (as a generator of the ring) to $s$ (as an element of $R$); this is an affinoid algebra over $K$. Equip $A_i$ with the quotient norm in case (a), and the spectral norm in case (b). This gives the desired affinoid system.

The previous observation has some strong consequences for uniform Banach algebras.

Remark 2.6.3. Let $\{(A_i, \alpha_i)\}_{i \in I}$ be an affinoid system. The inverse limit $\lim_{\leftarrow} \mathcal{M}(A_i)$ is compact by Remark 2.3.15(c). For $(A, \alpha)$ the direct limit of the $A_i$ and $R$ the separated completion of $A$, the restriction map $\mathcal{M}(R) \to \lim_{\leftarrow} \mathcal{M}(A_i)$ is continuous, and also bijective because specifying a compatible system of multiplicative seminorms on each $A_i$ bounded by $\alpha_i$ is equivalent to specifying a multiplicative seminorm on $A$ bounded by $\alpha$. We thus obtain a homeomorphism $\mathcal{M}(R) \cong \lim_{\leftarrow} \mathcal{M}(A_i)$ by Remark 2.3.15(b). By Remark 2.3.15(e), any disconnection of $\mathcal{M}(R)$ is induced by a disconnection of some $\mathcal{M}(A_i)$, and hence by some idempotent element of some $A_i$ by Corollary 2.5.14, and hence by some idempotent element of $R$. By Lemma 2.6.2, this last conclusion is valid for any Banach algebra over an analytic field. (See [12, Theorem 7.4.1] for another proof.)

We may now obtain the following refinement of Lemma 2.3.12(b). (We note again the parallel to [34, Exercise 20.13].)

Lemma 2.6.4. Let $R$ be a uniform Banach algebra over an analytic field, and let $M$ be a finitely generated $R$-module. Then the following conditions are equivalent.

(a) The module $M$ is projective.

(b) The rank function $\beta \mapsto \dim_{\mathcal{H}(\beta)}(M \otimes_R \mathcal{H}(\beta))$ on $\mathcal{M}(R)$ is continuous.

(c) There exists a bounded homomorphism $R \to S$ of uniform Banach algebras such that $\mathcal{M}(S) \to \mathcal{M}(R)$ is surjective and $M \otimes_R S$ is a projective $S$-module.

Proof. If (a) holds, then the function $p \mapsto \dim_{\mathcal{H}(p)}(M \otimes_R (R/p))$ on $\mathrm{Spec}(R)$ is continuous. By restricting along the map $\mathcal{M}(R) \to \mathrm{Spec}(R)$, we obtain (b). Conversely, if (b) holds, then the function $\beta \mapsto \dim_{\mathcal{H}(\beta)}(M \otimes_R \mathcal{H}(\beta))$ is constant on each set in some finite disconnection of $\mathcal{M}(R)$. By Remark 2.6.3, this disconnection descends to $\mathrm{Spec}(R)$, so we may reduce to
the case where \( \dim_{\mathcal{H}(\beta)}(M \otimes_R \mathcal{H}(\beta)) \) is constant. We may then apply Lemma \( 2.3.12(b) \) to deduce (a).

If (a) holds, then (c) is evident. Conversely, if (c) holds, then the function \( \gamma \mapsto \dim_{\mathcal{H}(\gamma)}(M \otimes_R \mathcal{H}(\gamma)) \) on \( \mathcal{M}(S) \) is continuous by the previous paragraph. This function factors through the function \( \beta \mapsto \dim_{\mathcal{H}(\beta)}(M \otimes_R \mathcal{H}(\beta)) \) on \( \mathcal{M}(R) \); the latter is forced to be continuous because \( \mathcal{M}(S) \to \mathcal{M}(R) \) is a surjective continuous map of compact spaces and hence a quotient map (Remark \( 2.3.15(b) \)). We thus deduce (b). Hence all three conditions are equivalent. \( \square \)

We next relate rational localizations of the completed direct limit of an affinoid system with the corresponding objects defined on individual terms of the system.

**Lemma 2.6.5.** Let \( \{(A_i, \alpha_i)\}_{i \in I} \) be a uniform affinoid system with direct limit \( (A, \alpha) \). Let 
\( R \) be the separated completion of \( A \).

(a) For any rational localization \( R \to S \), there exist an index \( i \in I \) and a rational localization \( A_i \to B_i \) such that \( S \) is the uniformization of \( B_i \otimes_{A_i} R \). The same is then true for each \( j \geq i \) for \( B_j = B_i \otimes_{A_i} A_j \); in fact, the \( B_j \) (when equipped with their spectral norms) form another uniform affinoid system with completed direct limit \( S \).

(b) With notation as in (a), for any \( \beta \in \mathcal{M}(R) \) restricting to \( \beta_i \in \mathcal{M}(A_i) \), \( \beta \) belongs to \( \mathcal{M}(S) \) if and only if \( \beta_i \) belongs to \( \mathcal{M}(B_i) \).

(c) With notation as in (a), for any \( \beta \in \mathcal{M}(R) \) restricting to \( \beta_i \in \mathcal{M}(A_i) \), \( R \to S \) encircles \( \beta \) if and only if there exists an index \( j \geq i \) for which \( A_j \to B_j \) encircles \( \beta_j \).

**Proof.** To prove (a), invoke Remark \( 2.4.8 \) to write
\[
\mathcal{M}(S) = \{ \beta \in \mathcal{M}(R) : \beta(f_j) \leq p_j \beta(g) \quad (j = 1, \ldots, n) \}\quad (2.6.5.1)
\]
with \( f_1, \ldots, f_n, g \in A_i \) for some \( i \in I \) and \( p_1, \ldots, p_n > 0 \). For \( j \geq i \), equip the reduced affinoid algebra \( B_j = A_j \langle T_1/p_1, \ldots, T_n/p_n \rangle / (gT_1 - f_1, \ldots, gT_n - f_n) \) with its spectral norm; by Lemma \( 2.4.9 \), \( S \) is the uniformization of \( B_i \otimes_{A_i} R \). We may now view \( \{B_j\}_{j \geq i} \) as a uniform affinoid system, and use the universal property of a rational localization to identify the completed direct limit with \( S \).

To prove (b), use \( 2.6.5.1 \) to identify \( \mathcal{M}(S) \) with the preimage of \( \mathcal{M}(B_i) \) under the restriction map \( \mathcal{M}(R) \to \mathcal{M}(A_i) \).

To prove (c), since \( \mathcal{M}(R) \) admits a neighborhood basis consisting of Laurent subdomains, we may shrink \( \mathcal{M}(S) \) (and possibly increase the \( f_j \) and \( g \)) to arrive at the case where \( \beta(f_j) < p_j \beta(g) \) for \( j = 1, \ldots, n \). In this case, it is clear from (b) that \( A_i \to B_i \) encircles \( \beta_i \). \( \square \)

**Corollary 2.6.6.** Let \( \{(A_i, \alpha_i)\}_{i \in I} \) be a uniform affinoid system with direct limit \( (A, \alpha) \). Let 
\( R \) be the completion of \( A \). For \( \beta \in \mathcal{M}(R) \) and \( i \in I \), let \( A_{i, \beta} \) denote the localization of \( A_i \) at the restriction of \( \beta \). Then \( \lim_{i \in I} A_{i, \beta} \) is a local ring whose residue field is dense in \( \mathcal{H}(\beta) \).
Remark 2.6.7. With notation as in Lemma 2.6.5, note that any simple Laurent covering on $R$ is defined over some $A_i$: if the family is defined by the element $f \in R$ and the number $q > 0$, then we get the same family after replacing $f$ by any $f' \in A$ satisfying $a(f - f') < q$.

Remark 2.6.8. It is not immediate that one can extend Remark 2.6.7 to show that any covering family of rational localizations of $R$ can be defined over some $A_i$. However, this can be shown for strong covering families using Remark 2.3.15(d). As we will not use this result, we omit further details.

We have the following extension of Lemma 2.2.4.

Proposition 2.6.9. Let $\{(A_i, \alpha_i)\}_{i \in I}$ be a uniform affinoid system with direct limit $(A, \alpha)$. Put $I = \ker(\alpha)$, $\overline{A} = A/I$, and $R = \hat{A}$. Then the base change functors $\text{FÉt}(A) \to \text{FÉt}(\overline{A}) \to \text{FÉt}(R)$ are tensor equivalences.

Proof. The base change functor $\text{FÉt}(A) \to \text{FÉt}(\overline{A})$ is a tensor equivalence by Lemma 2.2.3(a) and Theorem 1.2.8. The functor $\text{FÉt}(\overline{A}) \to \text{FÉt}(R)$ is rank-preserving and fully faithful by Lemma 2.2.4(a). It is thus enough to check that $\text{FÉt}(A) \to \text{FÉt}(R)$ is essentially surjective.

Choose any $V \in \text{FÉt}(R)$. For each $\beta \in \mathcal{M}(R)$, for $A_{i,\beta}$ the localization of $A_i$ at the restriction of $\beta$, the functor $\text{FÉt}(\lim_{i \in I} A_{i,\beta}) \to \text{FÉt}(R_{\beta})$ is an equivalence by Corollary 2.6.6 (to see that both $\lim_{i \in I} A_{i,\beta}$ and $R_{\beta}$ have dense images in $\mathcal{H}(\beta)$) and Lemma 2.2.4(b). We can thus choose an index $i \in I$ and a Laurent localization $A_i \to B_i$ encircling $\beta$ such that for $S$ the uniformization of $R \otimes_{A_i} B_i$, the object $V \otimes_R S$ in $\text{FÉt}(S)$ descends to an object in $\text{FÉt}(B_i)$. By the compactness of $\mathcal{M}(R)$, we can find an index $i \in I$ and a covering family $A_i \to B_{i,1}, \ldots, A_i \to B_{i,n}$ of Laurent localizations such that for $S_j$ the uniformization of $R \otimes_{A_i} B_{i,j}$, the object $V \otimes_R S_j$ in $\text{FÉt}(S_j)$ descends to an object $U_{i,j}$ in $\text{FÉt}(B_{i,j})$. If we write $B_{i,j}$ and $S_j$ for the uniformizations of $B_{i,j} \otimes_{A_i} B_{i,1}$ and $S_j \otimes_R S_i$, the functor $\text{FÉt}(\lim_{i \in I} B_{i,j}) \to \text{FÉt}(S_{jl})$ is fully faithful; we thus obtain (after suitably increasing $i$) isomorphisms among the $U_{i,j}$ on overlaps satisfying the cocycle condition. By Theorem 2.5.13(b) and Corollary 2.5.12, the $U_{i,j}$ glue to an object in $\text{FÉt}(A_i)$, and hence in $\text{FÉt}(A)$. This proves the claim. 

Using Lemma 2.6.5, we obtain a weak extension of Theorem 2.5.13 to arbitrary Banach algebras over a field. A better result would be to glue finite projective modules, but this is more difficult; see 2.7.

Theorem 2.6.10. Let $R$ be a uniform Banach algebra over an analytic field. Let $R \to R_1, \ldots, R \to R_n$ be a covering family of rational localizations. Then the homomorphism $R \to R_1 \oplus \cdots \oplus R_n$ is an effective descent morphism for finite étale algebras over uniform Banach rings.

Proof. By Proposition 2.4.13, it is enough to check the claim for a simple Laurent covering. By Lemma 2.6.2(a), we can construct a uniform affinoid system $\{(A_i, \alpha_i)\}_{i \in I}$ with completed direct limit $R$. By Remark 2.6.7, the chosen simple Laurent covering is defined by some
For some index $i$ and some $q > 0$. By Proposition 2.6.9, the claim reduces to the fact that for each $j \geq i$, the simple Laurent covering of $A_j$ defined by $f$ and $q$ is an effective descent morphism for finite étale algebras over Banach rings. This holds by Theorem 2.5.13(b) (to uniquely glue the underlying finite algebras) and Corollary 2.5.12 (to show that the resulting algebra is finite étale).

Corollary 2.6.11. With notation as in Theorem 2.6.10, the morphism $R \rightarrow R_1 \oplus \cdots \oplus R_n$ is an effective descent morphism for étale $\mathbb{Z}_p$-local systems over Banach rings.

Proof. This follows from Theorem 2.6.10 and Remark 2.8.2. \qed

2.7 Glueing of finite modules

We now turn to the problem of gluing finite modules over uniform Banach algebras, using the formalism of §1.3 as a starting point.

Remark 2.7.1. For $R$ a uniform Banach algebra, it is not clear in general whether any rational localization $R \rightarrow S$ is flat, as is the case when $R$ is a (not necessarily uniform) affinoid algebra by Lemma 2.5.10. If such a statement turns out to be true, it should be possible to verify it using affinoid systems; however, we have not succeeded in doing so. Consequently, we limit our glueing ambitions to cases where the modules being glued are themselves flat.

In a certain sense, the only obstructions to extending Theorem 2.5.13 at least to finite projective modules over uniform Banach rings occur in acyclicity (Tate’s theorem) and not in glueing (Kiehl’s theorem). We now make this observation precise.

Lemma 2.7.2. Let $R_1 \rightarrow S$, $R_2 \rightarrow S$ be bounded homomorphisms of Banach rings such that the sum homomorphism $\psi : R_1 \oplus R_2 \rightarrow S$ of groups is strict. Then there exists a constant $c > 0$ such that for every positive integer $n$, every matrix $U \in \text{GL}_n(S)$ with $|U - 1| < c$ can be written in the form $\psi(U_1)\psi(U_2)$ with $U_i \in \text{GL}_n(R_i)$.

Proof. By hypothesis, there exists a constant $d \geq 1$ such that every $x \in S$ lifts to some pair $(y_1, y_2) \in R_1 \oplus R_2$ with $|y_1|, |y_2| \leq d|x|$. We may prove the claim for $c = d^{-2}$ as follows. Given $U \in \text{GL}_n(S)$ with $|U - 1| < c$, put $V = U - 1$, and lift each entry $V_{ij}$ to a pair $(X_{ij}, Y_{ij}) \in R_1 \oplus R_2$ with $|X_{ij}|, |Y_{ij}| \leq d|V_{ij}|$. Then the matrix $U' = \psi(1 - X)U\psi(1 - Y)$ satisfies $|U' - 1| \leq d|U - 1|^2$. If $|U - 1| \leq d^{-l}$ for some integer $l \geq 2$, then $|U' - 1| \leq d^{-l-1}$, so we may construct the desired matrices by iterating the construction. (See [43, Lemma 4.5.3] for a similar argument or [70, Theorem 2.2.2] for a more general result.) \qed

Definition 2.7.3. Let

```
\begin{array}{ccc}
R & \longrightarrow & R_1 \\
\downarrow & & \downarrow \\
R_2 & \longrightarrow & R_{12}
\end{array}
```

be a commutative diagram of bounded homomorphisms of Banach rings. We call this diagram a glueing square if the following conditions hold.
(a) The sequence
\[ 0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0 \]
of \(R\)-modules, in which the last nontrivial arrow takes \((s_1, s_2)\) to \(s_1 - s_2\), is strict exact.

(b) The map \(R_2 \to R_{12}\) has dense image.

(c) The map \(\mathcal{M}(R_1 \oplus R_2) \to \mathcal{M}(R)\) is surjective.

We define glueing data on a glueing square as in Definition 1.3.7.

The following argument is a variant of Lemma 2.2.13.

**Lemma 2.7.4.** Consider a glueing square as in Definition 2.7.3, and let \(M_1, M_2, M_{12}\) be a finite glueing datum. Let \(M\) be the kernel of the map \(M_1 \oplus M_2 \to M_{12}\) taking \((m_1, m_2)\) to \(\psi_1(m_1) - \psi_2(m_2)\).

(a) There exists a finitely generated \(R\)-submodule \(M_0\) of \(M\) such that for \(i = 1, 2\), the natural map \(M_0 \otimes_R R_i \to M_i\) is surjective.

(b) The map \(M_1 \oplus M_2 \to M_{12}\) is surjective.

**Proof.** We follow [43, Lemmas 4.5.4 and 4.5.5]. Choose generating sets \(v_1, \ldots, v_n\) and \(w_1, \ldots, w_n\) of \(M_1\) and \(M_2\), respectively, of the same cardinality. We may then choose \(n \times n\) matrices \(A, B\) over \(R_{12}\) such that \(\psi_2(w_j) = \sum_i A_{ij} \psi_1(v_i)\) and \(\psi_1(v_j) = \sum_i B_{ij} \psi_2(w_i)\).

By hypothesis, the map \(R_2 \to R_{12}\) has dense image. We may thus choose an \(n \times n\) matrix \(B'\) over \(R_2\) so that \(A(B' - B)\) has norm less than the constant \(c\) of Lemma 2.7.2. We may then write \(1 + A(B' - B) = C_1C_2^{-1}\) with \(C_i \in \text{GL}_n(R_i)\).

We now may define elements \(x_j \in M_1 \oplus M_2\) by the formula
\[ x_j = (x_{j,1}, x_{j,2}) = \left( \sum_i (C_1)_{ij} v_i, \sum_i (B'C_2)_{ij} w_i \right) \quad (j = 1, \ldots, n). \]

Then
\[ \psi_1(x_{j,1}) - \psi_2(x_{j,2}) = \sum_i (C_1 - AB'C_2)_{ij} \psi_1(v_i) = \sum_i ((1 - AB)C_2)_{ij} \psi_1(v_i) = 0, \]
so \(x_j \in M\). Let \(M'_0\) be the \(R\)-submodule of \(M\) generated by the \(x_i\). Since \(C_1 \in \text{GL}_n(R_1)\), the \(x_{i,1}\) generate \(M_1\) over \(R_1\), so the map \(M'_0 \otimes_R R_1 \to M_1\) is surjective. We may now apply Lemma 1.3.8 to deduce (a) and (b).

**Proposition 2.7.5.** Consider a glueing square as in Definition 2.7.3 in which \(R\) is a uniform Banach algebra over an analytic field, and let \(M_1, M_2, M_{12}\) be a finite projective glueing datum. Let \(M\) be the kernel of the map \(M_1 \oplus M_2 \to M_{12}\) taking \((m_1, m_2)\) to \(\psi_1(m_1) - \psi_2(m_2)\). Then \(M\) is a finite projective \(R\)-module, and the natural maps \(M \otimes_R R_i \to M_i\) are isomorphisms.
Proof. It follows from Lemma 1.3.9 that $M$ is a finitely generated (and even finitely presented) $R$-module and that $M \otimes_R R_i \rightarrow M_i$ is a bijection for $i = 1, 2$. Since $M \otimes_R (R_1 \oplus R_2) \cong M_1 \oplus M_2$ is a finite projective module over $R_1 \oplus R_2$, we may apply Lemma 2.6.4 to deduce that $M$ is a finite projective $R$-module, as desired.

Definition 2.7.6. We say that a uniform Banach ring $R$ satisfies the (strictly) Tate sheaf property if for any covering family $R \rightarrow S_1, \ldots, R \rightarrow S_n$ of (strictly) rational localizations, the augmented Čech complex

$$0 \rightarrow R \rightarrow \bigoplus_{i=1}^n S_i \rightarrow \bigoplus_{i,j=1}^n S_i \hat{\otimes} S_j \rightarrow \cdots$$

is strict exact. This implies immediately that for any finite projective $R$-module $M$, the augmented Čech complex

$$0 \rightarrow M \rightarrow M \otimes_R \left( \bigoplus_{i=1}^n S_i \right) \rightarrow M \otimes_R \left( \bigoplus_{i,j=1}^n S_i \hat{\otimes} S_j \right) \rightarrow \cdots$$

is exact (where the tensor products are taken to be uniformized as well as completed). We may also view $M$ as a finite Banach module as in Lemma 2.2.12, then replace the ordinary tensor products with completed tensor products without changing the underlying groups, to obtain a strict exact sequence.

Let $F$ be the category of finite projective modules over uniform Banach rings, viewed as a cofibred category over the category $C$ of uniform Banach rings as in Example 1.3.3. We say that $R$ satisfies the (strictly) Kiehl glueing property if for any covering family $R \rightarrow S_1, \ldots, R \rightarrow S_n$ of (strictly) rational localizations, the morphism $R \rightarrow \oplus_i S_i$ in $C$ is an effective descent morphism for $F$.

Proposition 2.7.7. If $R$ is a uniform Banach ring satisfying the (strictly) Tate sheaf property, then $R$ also satisfies the (strictly) Kiehl glueing property.

Proof. This follows from Proposition 2.4.15 and Proposition 2.7.5.

Remark 2.7.8. By Theorem 2.5.13 any reduced (strictly) affinoid algebra over an analytic field satisfies the (strictly) Tate sheaf and (strictly) Kiehl glueing properties. Some additional cases in which we will establish the Tate sheaf property and Kiehl glueing properties are perfect uniform Banach $\mathbb{F}_p$-algebras (Theorem 3.1.12) and perfectoid algebras (Theorem 3.6.15).

2.8 Étale local systems on spectra

We will need to make a distinction between étale local systems on a Banach ring and étale local systems on the Gel’fand spectrum.
Definition 2.8.1. Let $R$ be a uniform Banach ring. For each strong covering family $R \rightarrow R_1, \ldots, R \rightarrow R_n$ of rational localizations, construct the categories of descent data for étale $\mathbb{Z}_p$-local systems and $\mathbb{Q}_p$-local systems for the homomorphism $R \rightarrow R_1 \oplus \cdots \oplus R_n$ of Banach rings. (That is, specify a local system over each $R_i$, then define isomorphisms over the uniformization of each $R_i \hat{\otimes}_{R} R_j$.) Then form the categorical direct limit (as in Remark 1.2.9) over all strong covering families; we call the resulting categories the categories of étale $\mathbb{Z}_p$-local systems over $\mathcal{M}(R)$ and étale $\mathbb{Q}_p$-local systems over $\mathcal{M}(R)$. When $R$ is an affinoid algebra over an analytic field, these categories are equivalent to the analogous categories defined by de Jong [29] in terms of étale covering spaces.

Remark 2.8.2. If $R$ is a uniform Banach algebra over an analytic field, then the natural functor from étale $\mathbb{Z}_p$-local systems over Spec$(R)$ to étale $\mathbb{Z}_p$-local systems over $\mathcal{M}(R)$ is an equivalence of categories, since $\mathbb{Z}_p$-local systems are determined by finite étale algebras (Remark 1.4.2) and these glue over strong covering families of rational localizations (Theorem 2.6.10). However, for étale $\mathbb{Q}_p$-local systems, de Jong has observed that the natural functor is fully faithful but not essentially surjective. For instance, when $R$ is a connected affinoid algebra over an analytic field, étale $\mathbb{Q}_p$-local systems over $\mathcal{M}(R)$ correspond to continuous representations of the étale fundamental group of $\mathcal{M}(R)$ (as defined in [29, §2.6]) on finite-dimensional $\mathbb{Q}_p$-vector spaces, and such representations can fail to have compact image. Typical examples arise from instances of $p$-adic uniformization, such as the Tate uniformization of an elliptic curve of split multiplicative reduction; see Example 8.1.14. More examples of this sort arise from Rapoport-Zink period morphisms, which we will encounter later in this series of papers; see [29, §7].

Remark 2.8.3. A related observation to Remark 2.8.2 is that while étale $\mathbb{Z}_p$-local systems over $\mathcal{M}(R)$ admit glueing for covering families which are not strong (again by Remark 1.4.2 and Theorem 2.6.10), it is unclear whether the same holds in general for étale $\mathbb{Q}_p$-local systems. This is related to the fact that the Seifert-van Kampen theorem for glueing topological fundamental groups only applies to coverings by open subsets. For a special case where such glueing works, see Corollary 8.1.12.

One case where the aforementioned subtleties do not occur is the following.

Proposition 2.8.4. Let $R$ be a reduced normal noetherian ring equipped with the trivial norm, viewed as a uniform Banach ring. Then the functor from étale $\mathbb{Q}_p$-local systems on Spec$(R)$ to étale $\mathbb{Q}_p$-local systems on $\mathcal{M}(R)$ is an equivalence of categories.

Proof. We may assume throughout that $R$ is connected, and hence irreducible. We first show that any simple Laurent covering defines an effective descent morphism for étale $\mathbb{Q}_p$-local systems over affine schemes. Choose a simple Laurent covering

$$S_- = \{\alpha \in \mathcal{M}(R) : \alpha(f) \leq q\}, \quad S_+ = \{\alpha \in \mathcal{M}(R) : \alpha(f) \geq q\}$$

for some $f \in R$ and some $q > 0$. We may assume $q < 1$, as otherwise $S_- = \mathcal{M}(R)$ and there is nothing to check. The representing algebra of $S_-$ is the $f$-adic completion $\hat{R}$ of $R$,
while the representing algebra of $S_+$ is the localization $R[f^{-1}]$. The representing algebra of $S_\cap S_+$ is the completed tensor product $\widehat{R} \otimes_R R[f^{-1}]$, which in this case happens to coincide with the ordinary tensor product $\widehat{R} \otimes_R R[f^{-1}] = \widehat{R}[f^{-1}]$. We thus have a descent datum for the covering $\text{Spec}(R[f^{-1}] \oplus \widehat{R}) \to \text{Spec}(R)$ in the flat (fpqc) topology; Lemma 1.4.10 then produces the desired $\mathbb{Q}_p$-local system on $\text{Spec}(R)$.

We next check that any standard Laurent covering defines an effective descent morphism for étale $\mathbb{Q}_p$-local systems over affine schemes. With notation as in Definition 2.4.14, we proceed by induction on $n$. Suppose we are given a descent datum for this covering. By the induction hypothesis, we obtain local systems on $\text{Spec}(R[f_n^{-1}])$ and on $\text{Spec}(R/(f_n))$. We may promote the latter to $\text{Spec}(\widehat{R})$ for $\widehat{R}$ equal to the $f_n$-adic completion of $R$. The original descent datum now gives a descent datum for the covering of $\text{Spec}(R)$ in the flat topology by $\text{Spec}(R[f_n^{-1}])$ and $\text{Spec}(\widehat{R})$, which as in the previous paragraph induces a local system on $\text{Spec}(R)$.

To conclude, note that any strong covering family can be refined to a standard Laurent covering because $\mathcal{M}(R)$ admits a neighborhood basis of Laurent subdomains. \hfill \Box

Remark 2.8.5. Proposition 2.8.4 fails without the normality hypothesis. For instance, let $k$ be a field, and let $R$ be the coordinate ring of a nodal cubic curve in $\text{Spec} k[x, y]$. Then $\mathcal{M}(R)$ has the homotopy type of a circle, and one can construct a $\mathbb{Q}_p$-local system on $\mathcal{M}(R)$ that does not descend to $\text{Spec}(R)$ by arguing as in Remark 1.4.9.

Remark 2.8.6. Note that the proof of Proposition 2.8.4 involves glueing algebraic $\mathbb{Q}_p$-local systems over certain special covering families which are not strong, but the general glueing property is only achieved for strong covering families. Proposition 2.4.15 does not apply because triviality of the norm is not stable under rational localizations.

Remark 2.8.7. For any uniform Banach ring $R$ and any étale $\mathbb{Q}_p$-local systems $V_1, V_2$ on $\text{Spec}(R)$, any extension $0 \to V_1 \to V \to V_2 \to 0$ in the category of étale $\mathbb{Q}_p$-local systems on $\mathcal{M}(R)$ descends to an extension of étale $\mathbb{Q}_p$-local systems on $\text{Spec}(R)$. To see this, start with identifications $V_i = T_i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for some étale $\mathbb{Z}_p$-local systems on $\text{Spec}(R)$. The extension $0 \to V_1 \to V \to V_2 \to 0$ then corresponds locally on $\mathcal{M}(R)$ to a class in $\text{Ext}(T_2, T_1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; since $\mathcal{M}(R)$ is compact, we can find a strong covering family $R \to S_1, \ldots, R \to S_n$ of rational localizations such that the extension corresponds to a class $x_i$ in $\text{Ext}(T_2, T_1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ over $\text{Spec}(S_i)$. Let $S_{ij}$ be the uniformization of $S_i \otimes_R S_j$; then $x_i - x_j$ vanishes as an element of $\text{Ext}(T_2, T_1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ over $\text{Spec}(S_{ij})$. We may rescale $T_1$ by a power of $p$ first to force each $x_i$ into $\text{Ext}(T_2, T_1)$ over $\text{Spec}(S_i)$, then to force $x_i - x_j$ to vanish in $\text{Ext}(T_2, T_1)$ over $\text{Spec}(S_{ij})$. The extensions then glue to define an extension of étale $\mathbb{Z}_p$-local systems on $\mathcal{M}(R)$. We may now invoke Remark 2.8.2 to conclude.

3 Perfect rings and strict $p$-rings

Recall that there is a natural way to lift perfect rings of characteristic $p$ to strict $p$-rings of characteristic 0 (see Definition 3.2.1), and that these can be used to describe étale lo-
cal systems on perfect rings of characteristic $p$ using a nonabelian generalization of Artin-Schreier-Witt theory (see Proposition 3.2.7). We take a first step towards exploiting this description in $p$-adic Hodge theory by setting up a correspondence between certain highly ramified analytic fields of mixed characteristics and perfect analytic fields of characteristic $p$. This correspondence, which has also been described recently by Scholze [91], generalizes the field of norms construction of Fontaine-Wintenberger [41] but with a rather different proof. We then extend the correspondence to Banach algebras, in the direction of generalizing Faltings’s almost purity theorem; however, this will not be completed until we introduce extended Robba rings (see §5.5).

3.1 Perfect $\mathbb{F}_p$-algebras

We begin with some observations about perfect rings of characteristic $p$, which we may more briefly characterize as perfect $\mathbb{F}_p$-algebras.

**Definition 3.1.1.** For $R$ an $\mathbb{F}_p$-algebra, let $\varphi : R \to R$ denote the $p$-th power map, i.e., the Frobenius endomorphism. We say $R$ is perfect if $\varphi$ is a bijection; this forces $R$ to be reduced. Note that any localization of a perfect $\mathbb{F}_p$-algebra is also perfect.

By a perfect uniform Banach $\mathbb{F}_p$-algebra, we will mean a uniform Banach algebra $R$ over $\mathbb{F}_p$ (for the trivial norm on the latter) whose underlying $\mathbb{F}_p$-algebra is perfect. For any rational localization $R \to R_1$, $R_1$ is again perfect.

**Lemma 3.1.2.** Let $R$ be a perfect $\mathbb{F}_p$-algebra. If $e \in R$ satisfies $e^p = e$, then

$$e_i = \prod_{j \in \mathbb{F}_p \setminus \{ i \}} \frac{e - j}{i - j} \quad (i \in \mathbb{F}_p)$$

is an idempotent in $R$ and $\sum_{i \in \mathbb{F}_p} ie_i = e$.

**Proof.** Note that $e_i(e - i)$ is divisible by $e^p - e$ and thus equals 0. Hence

$$e_i^2 = e_i \prod_{j \in \mathbb{F}_p \setminus \{ i \}} \frac{e - j}{i - j} = e_i \prod_{j \in \mathbb{F}_p \setminus \{ i \}} \frac{i - j}{i - j} = e_i.$$

The identity $\sum_{i \in \mathbb{F}_p} ie_i = e$ arises from Lagrange interpolation of the polynomial $T \in \mathbb{F}_p[T]$ at the points of $\mathbb{F}_p$.

**Corollary 3.1.3.** The ring $R^\varphi$ is the $\mathbb{F}_p$-algebra generated by the idempotents of $R$. In particular, this equals $\mathbb{F}_p$ if and only if $R$ is connected.

**Lemma 3.1.4.** For $R$ a perfect $\mathbb{F}_p$-algebra, any $S \in \mathcal{F} \text{Ét}(R)$ is also perfect.

**Proof.** Since $S$ is étale over the reduced ring $R$, it is also reduced [52, Proposition 17.5.7]; hence $\varphi : S \to S$ is injective. Since $\Omega_{S/R} = 0$ by [52, Proposition 17.2.1], $S$ is generated over $R$ by $S^p$ by [51, Proposition 0.21.1.7]. Combining this with the surjectivity of $\varphi : R \to R$ yields surjectivity of $\varphi : S \to S$. Hence $S$ is perfect, as desired.
The study of perfect uniform Banach \( \mathbb{F}_p \)-algebras is greatly simplified by the following observations. For some related results in characteristic 0, see \[3.6\]

**Remark 3.1.5.** Let \( R, S, T \) be perfect uniform Banach \( \mathbb{F}_p \)-algebras.

(a) Any strict homomorphism \( f : R \to S \) is almost optimal (but not necessarily optimal).

(b) If \( f_1, f_2 : R \to S \) are homomorphisms such that \( f_1 - f_2 \) is strict, then \( f_1 - f_2 \) is almost optimal.

(c) For any bounded homomorphisms \( T \to R, T \to S \), the completed tensor product \( R \hat{\otimes}_T S \) is again a perfect uniform Banach \( \mathbb{F}_p \)-algebra. (By contrast, the completed tensor product of uniform Banach rings is not guaranteed to be uniform in general.)

The proofs are all similar, so we only describe (a) in detail. Choose \( c > 0 \) such that every \( x \in \text{image}(f) \) lifts to some \( y \in R \) with \( |y| \leq c|x| \). For any positive integer \( n \), \( x^{p^n} \) then lifts to some \( y_n \in R \) with \( |y_n| \leq c|x|^{p^n} \); we may then lift \( x \) to \( y_n^{p^{-n}} \) and note that

\[
|y_n^{p^{-n}}| = |y_n|^{p^{-n}} \leq c^{p^{-n}}|x|^{p^n} = c^{p^{-n}}|x|.
\]

Since \( c^{p^{-n}} \) can be made arbitrarily close to 1, this proves the claim.

For perfect uniform Banach \( \mathbb{F}_p \)-algebras, we have the following refinement of Lemma \[2.4.9\]

**Remark 3.1.6.** Let \( R \) be a perfect uniform Banach \( \mathbb{F}_p \)-algebra, choose \( f_1, \ldots, f_n, g \in R \) generating the unit ideal and \( p_1, \ldots, p_n > 0 \), and consider the rational subdomain

\[
\{ \beta \in \mathcal{M}(R) : \beta(f_i) \leq p_i \beta(g) \quad (i = 1, \ldots, n) \}
\]

of \( \mathcal{M}(R) \). Let \( S \) be the quotient of the completed perfected closure of \( R\{T_1/p_1, \ldots, T_n/p_n\} \) by the closed ideal generated by \( (gT_i - f_i)^{p^{-h}} \) for \( i = 1, \ldots, n \) and \( h = 0, 1, \ldots \). By arguing as in Remark \[3.1.5\], we see that \( S \) is perfect uniform. We may then verify that the map \( R \to S \) is a rational localization by checking the universal property.

One can also make a somewhat more refined construction. Choose \( h_1, \ldots, h_n, k \in R \) for which \( h_1 f_1 + \cdots + h_n f_n + k g = 1 \). Suppose \( x \in S \) lifts to \( y = \sum_{i_1, \ldots, i_n} y_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n} \) in the completed perfect closure of \( R\{T_1/p_1, \ldots, T_n/p_n\} \), where \( i_1, \ldots, i_n \) each run over the nonnegative elements of \( \mathbb{Z}[p^{-1}] \). Then \( x \) also lifts to

\[
z = (k + h_1 T_1 + \cdots + h_n T_n)^n \sum_{i_1, \ldots, i_n} y_{i_1, \ldots, i_n} f_1^{i_1-[i_1]} \cdots f_n^{i_n-[i_n]} g^{n-(i_1-[i_1]) - \cdots i_n-[i_n])} T_1^{[i_1]} \cdots T_n^{[i_n]},
\]

which is an element of \( R\{T_1/p_1, \ldots, T_n/p_n\} \) of norm bounded by a constant times the norm of \( y \). Consequently, the homomorphism \( R\{T_1/p_1, \ldots, T_n/p_n\} \to S \) is a strict (but not almost optimal) surjection; that is, the natural homomorphism \( R\{T_1/p_1, \ldots, T_n/p_n\}/(gT_1 - f_1, \ldots, gT_n - f_n) \to S \) is an isomorphism of Banach algebras.
Lemma 3.1.7. Let $R$ be a perfect uniform Banach $\mathbb{F}_p$-algebra with norm $\alpha$. Let $S$ be a finite perfect $R$-algebra admitting the structure of a finite Banach module over $R$ for some norm $\beta$. (Such $\beta$ exists when $S$ is projective as an $R$-module by Lemma 2.2.12 and hence when $S \in \mathcal{F}\mathcal{E}\mathcal{t}(R)$.) Then there exists a unique norm $\gamma$ equivalent to $\beta$ under which $S$ is a perfect uniform Banach algebra.

Proof. Equip $S \otimes_R S$ with the product seminorm induced by $\beta$. By Lemma 2.2.6 the multiplication map $\mu : S \otimes_R S \to S$ is bounded. Consequently, there exists $c > 0$ such that

$$\beta(xy) \leq c\beta(x)\beta(y) \quad (x, y \in S). \quad (3.1.7.1)$$

Rewrite (3.1.7.1) as $c\beta(xy) \leq (\beta(x))(\beta(y))$, then apply Fekete's lemma to deduce that the limit $\gamma(x) = \lim_{n \to \infty}(c\beta(x^n))^{1/n}$ exists. From (3.1.7.1) again, we see that $\gamma$ is a power-multiplicative seminorm on $S$ and that $\gamma(x) \leq c\beta(x)$.

Let $R'$ be a copy of $R$ equipped with the norm $\alpha^p$; the homomorphism $\overline{\gamma}^{-1} : R \to R'$ is isometric because $R$ is uniform. Let $S'$ be a copy of $S$ equipped with the norm $\beta^p$; then $S'$ is a finite Banach module over $R'$ and the map $\overline{\gamma}^{-1} : S \to S'$ is semilinear with respect to $\overline{\gamma}^{-1} : R \to R'$. By Lemma 2.2.6 again, $\overline{\gamma}^{-1} : S \to S'$ is bounded; that is, there exists $d > 0$ such that for all $x \in S$, $\beta(x^{1/p})^p \leq d\beta(x)$. Equivalently, for all $x \in S$, $\beta(x^p) \geq d^{-1}\beta(x)^p$. By induction on the positive integer $n$, we have $c\beta(x^n) \geq cd^{-1-p-\ldots-p^{n-1}}\beta(x)^p$; by taking $p^n$-th roots and then taking the limit as $n \to \infty$, we deduce that $\gamma(x) \geq d^{-1/(p-1)}\beta(x)$.

From the preceding paragraphs, $\gamma$ is equivalent to $\beta$; it is thus a norm on $S$ under which $S$ is a perfect uniform Banach algebra. The uniqueness of $\gamma$ follows from the observation that any two equivalent power-multiplicative seminorms on the same ring are identical. \[\square\]

Remark 3.1.8. Let $R$ be a perfect uniform Banach $\mathbb{F}_p$-algebra. For $S \in \mathcal{F}\mathcal{E}\mathcal{t}(R)$, the norm $\beta$ described in Lemma 3.1.7 is power-multiplicative, so it is dominated by the norm constructed in Remark 3.1.14. In fact, the reverse inequality also holds: this is straightforward when $R$ is an analytic field, and the general case follows because $\beta$ dominates the corresponding norm on $S \otimes_R \mathcal{H}(\alpha)$ for each $\alpha \in \mathcal{M}(R)$. Consequently, the norm of Lemma 3.1.7 coincides with the norm of Remark 2.3.14, providing an instance where the latter is a Banach norm.

Lemma 3.1.9. Let $R$ be a perfect uniform Banach $\mathbb{F}_p$-algebra, and let $\gamma$ be an isometric automorphism of $R$ extending to an automorphism of $S \in \mathcal{F}\mathcal{E}\mathcal{t}(R)$. Then $\gamma$ is also isometric on $S$ for the norm provided by Lemma 3.1.7.

Proof. Suppose first that $R = L$ is an analytic field. Given $y \in S$, let $P = \sum_i P_i T^i \in L[T]$ be the minimal polynomial of $y$. As in the proof of Lemma 2.2.5 we have $|y| = |P_0|^{1/d}$ for $d = \text{deg}(P)$ and $|\gamma(y)| = |\gamma(P_0)|^{1/d} = |y|$ as desired.

We reduce the general case to the case of an analytic field using Theorem 2.3.10. More precisely, for each $\beta \in \mathcal{M}(R)$, we may use $\gamma$ to identify $\mathcal{H}(\beta)$ with $\mathcal{H}(\gamma^*(\beta))$, then use the extended action of $\gamma$ to define an automorphism of $S \otimes_R \mathcal{H}(\beta)$. Since this automorphism is isometric by the previous paragraph, we may apply Theorem 2.3.10 and Remark 3.1.8 to deduce that the action of $\gamma$ on $S$ is isometric. \[\square\]
Remark 3.1.10. Suppose that $R$ is a perfect uniform $\mathbb{F}_p$-algebra over a nontrivially normed analytic field $L$. For $S \in \text{FÉt}(R)$, $\mathfrak{o}_S$ is perfect because $S$ is, so $\Omega_{\mathfrak{o}_S/\mathfrak{o}_R} = 0$. This does not imply that $\mathfrak{o}_S$ is finite étale over $\mathfrak{o}_R$, because $\mathfrak{o}_S$ need not be a finite $\mathfrak{o}_R$-module. However, we can say that the quotient of $\mathfrak{o}_S$ by the sum of its finitely generated $\mathfrak{o}_R$-submodules is killed by $\mathfrak{m}_L$: we can find a single submodule the quotient by which is killed by a nonzero element $\tau \in \mathfrak{m}_L$, then use perfectness to replace $\tau$ by $\tau^{p^{-n}}$ for any nonnegative integer $n$. A related statement in the language of almost ring theory is that $\mathfrak{o}_S$ is almost finite étale over $\mathfrak{o}_R$; see Theorem 5.5.9 for a similar statement and derivation.

As a consequence of these observations, we obtain analogues of the Tate-Kiehl theorems for perfect uniform Banach algebras. For extensions of these results, see Theorems 5.3.4 and 5.3.6; for an analogue for perfectoid algebras, see Theorem 3.6.15.

Lemma 3.1.11. Let $R$ be a perfect uniform Banach $\mathbb{F}_p$-algebra. Choose $f \in R$ and $q > 0$, and let $R \to R_1, R \to R_2, R \to R_{12}$ be the Laurent localizations representing the subdomains

$$\{ \beta \in \mathcal{M}(R) : \beta(f) \leq q \}, \quad \{ \beta \in \mathcal{M}(R) : \beta(f) \geq q \}, \quad \{ \beta \in \mathcal{M}(R) : \beta(f) = q \}$$

of $\mathcal{M}(R)$.

(a) Let $(A_i, \alpha_i)_{i \in I}$ be a uniform affinoid system with completed direct limit $R$ such that $f \in A_i$ for some $i \in I$. For each $j \geq i$, put $B_{j,1} = B_j\{T/q\}/(T - f)$, $B_{j,2} = B_j\{q/T\}/(f/T - 1)$, and $B_{j,12} = B_j\{q/T, T/q\}/(T - f)$. Then the completed direct limits of $B_{j,1}^\text{perf}, B_{j,2}^\text{perf}, B_{j,12}^\text{perf}$ are isomorphic to $R, R_1, R_2, R_{12}$, respectively.

(b) The sequence

$$0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0 \quad (3.1.11.1)$$

is almost optimal exact (i.e., exact with each morphism being almost optimal).

Proof. Part (a) follows from Remark 3.1.6. Part (b) may be obtained in two ways. One way is to combine Remark 3.1.6 with the usual proof of Tate’s theorem [19, §8.2.3] to obtain strict exactness of (3.1.11.1), then invoke Remark 3.1.5 to pass from strict to almost optimal. The other way is to apply Lemma 2.6.2 and Remark 2.6.7 to construct an affinoid system as in (a). For each $j \geq i$,

$$0 \to A_j \to B_{j,1} \oplus B_{j,2} \to B_{j,12} \to 0$$

is strict exact by Theorem 2.5.13(a). Since taking direct limits is exact, the sequence

$$0 \to A_j^\text{perf} \to B_{j,1}^\text{perf} \oplus B_{j,2}^\text{perf} \to B_{j,12}^\text{perf} \to 0$$

is also exact. However, if we equip each term of the resulting sequence with the spectral seminorm, then by the argument of Remark 3.1.3, each map is not just strict but almost optimal. We thus may complete to obtain an almost optimal exact sequence of perfect uniform Banach algebras (which would not have been possible for strict maps without some uniformity on the implied constants). By taking the direct limit over $j$, completing again, and invoking (a), we obtain (3.1.11.1).
Theorem 3.1.12. Any perfect uniform Banach $\mathbb{F}_p$-algebra satisfies the Tate sheaf and Kiehl glueing properties (see Definition 2.7.6).

Proof. By Remark 3.1.5, a completed tensor product of uniform Banach $\mathbb{F}_p$-algebras is again uniform. Consequently, Lemma 3.1.11 implies the Tate sheaf property using Proposition 2.4.15. The Kiehl glueing property then follows from Proposition 2.7.7. 

We will sometimes have need to pass from an $\mathbb{F}_p$-algebra to an associated perfect $\mathbb{F}_p$-algebra.

Definition 3.1.13. Let $R$ be an $\mathbb{F}_p$-algebra. The perfect closure of $R$ is the limit $R^{\text{perf}}$ of the direct system

$$R \xrightarrow{x_n} R \xrightarrow{x_m} \cdots,$$

viewed as an $R$-algebra via the map to the first factor. (The map $R \to R^{\text{perf}}$ induces an injection of the reduced quotient of $R$ into $R^{\text{perf}}$.) Any power-multiplicative seminorm on $R$ extends uniquely to a power-multiplicative seminorm on $R^{\text{perf}}$; in particular, given a power-multiplicative norm on $R$, we may extend it to $R^{\text{perf}}$ and then obtain a homeomorphism $\mathcal{M}(R^{\text{perf}}) \to \mathcal{M}(R)$. We will also call $R^{\text{perf}}$ the direct perfection of $R$, to distinguish it from the inverse perfection in which one takes the arrows in the opposite direction; we will have more use for the latter construction in §3.4.

Example 3.1.14. For $X$ an arbitrary (possibly infinite) set and $R$ a ring, let $R[X]$ denote the free commutative $R$-algebra generated by $X$, and write $R[X^{p^{-\infty}}]$ for $\bigcup_{n=1}^{\infty} R[X^{p^{-n}}]$. Then for any $\mathbb{F}_p$-algebra $R$, we have a natural (in $R$) identification $R[X]^{\text{perf}} \cong R^{\text{perf}}[X^{p^{-\infty}}].$

The operation of forming the perfect closure, or the completed perfect closure in case we have a power-multiplicative norm, does not change the étale fundamental group.

Theorem 3.1.15. Let $R$ be a perfect $\mathbb{F}_p$-algebra.

(a) The base change functor $\mathbf{F\acute{e}t}(R) \to \mathbf{F\acute{e}t}(R^{\text{perf}})$ is a tensor equivalence.

(b) Suppose that $R$ is complete for a power-multiplicative norm, and let $S$ be the completion of $R^{\text{perf}}$. Then the base change functor $\mathbf{F\acute{e}t}(R^{\text{perf}}) \to \mathbf{F\acute{e}t}(S)$ is a tensor equivalence.

Proof. The morphism $\text{Spec}(R^{\text{perf}}) \to \text{Spec}(R)$ is surjective, integral, and radicial, so by [52 Corollaire 18.12.11], it is a universal homeomorphism. In particular, it is universally submersive [54 Exposé IX, Définition 2.1], so $\mathbf{F\acute{e}t}(R) \to \mathbf{F\acute{e}t}(R^{\text{perf}})$ is fully faithful by [54 Exposé IX, Corollaire 3.3]. On the other hand, by Remark 1.2.9, $\mathbf{F\acute{e}t}(R^{\text{perf}})$ is the direct limit of $\mathbf{F\acute{e}t}(T)$ as $T$ runs over all $R$-subalgebras of $R^{\text{perf}}$ of the form $\mathbb{R}[x_1^{1/p^m}, \ldots, x_n^{1/p^m}]$ for some nonnegative integer $m$ and some $x_1, \ldots, x_n \in R$. For each such $T$, the morphism $\text{Spec}(T) \to \text{Spec}(R)$ is finite, radicial, surjective, and of finite presentation, so $\mathbf{F\acute{e}t}(R) \to \mathbf{F\acute{e}t}(T)$ is essentially surjective by [54 Exposé IX, Théorème 4.10]. We deduce that $\mathbf{F\acute{e}t}(R) \to \mathbf{F\acute{e}t}(R^{\text{perf}})$ is also essentially surjective. This proves (a).
If $R$ is complete for a power-multiplicative norm, by Lemma 2.6.2 we can write $R$ as the completion of the direct limit $A$ of some uniform affinoid system $\{A_i\}_{i \in I}$. If we enlarge this affinoid system by adding $\varphi^{-j}(A_i)$ for all nonnegative integers $j$, the direct limit becomes $A^{perf}$, whose completion is $S$. Applying $\text{FÉt}$ to the commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & R \\
\downarrow & & \downarrow \\
A^{perf} & \longrightarrow & S
\end{array}
\]

yields tensor equivalences along the horizontal arrows by Proposition 2.6.9 and along the left vertical arrow by (a). Hence the right vertical arrow also becomes a tensor equivalence; this proves (b).

\[\square\]

3.2 Strict $p$-rings

Perfect $\mathbb{F}_p$-algebras lift naturally to characteristic zero, as follows. (Our derivations follow [95, §5]; see [65, §1] for a discussion more explicitly in terms of Witt vectors.)

Definition 3.2.1. A strict $p$-ring is a $p$-torsion-free, $p$-adically complete ring $S$ for which $S/(p)$ is perfect. Given such a ring, for any $p$-adically complete ring $U$ and any ring homomorphism $\bar{t} : S/(p) \to U/(p)$, $\bar{t}$ lifts uniquely to a multiplicative map $t : S/(p) \to U$; more precisely, for any $\overline{x} \in S/(p)$ and any $y \in U$ lifting $\bar{t}(x^{p^{-n}})$, we have $t(\overline{x}) \equiv y^{p^n} \pmod{p^{n+1}}$. In particular, the projection $S \to S/(p)$ admits a multiplicative section $[\cdot] : S/(p) \to S$, called the Teichmüller map; each $x \in S$ can be written uniquely as $\sum_{n=0}^{\infty} p^n \overline{x_n}$ with $\overline{x_n} \in S/(p)$.

Lemma 3.2.2. Let $S$ be a strict $p$-ring, let $U$ be a $p$-adically complete ring, and let $\pi : U \to U/(p)$ be the natural projection. Let $\bar{t} : S/(p) \to U/(p)$ be a ring homomorphism, and lift $\bar{t}$ to a multiplicative map $t : S/(p) \to U$ as in Definition 3.2.1. Then the formula

\[
T \left( \sum_{n=0}^{\infty} p^n \overline{x_n} \right) = \sum_{n=0}^{\infty} p^n t(\overline{x_n}) \quad (\overline{x_0, x_1, \cdots} \in S/(p))
\]

(3.2.2.1)

defines a (necessarily unique) homomorphism $T : S \to U$ such that $T \circ [\cdot] = t$.

Proof. We check by induction that for each positive integer $n$, $T$ induces an additive map $S/(p^n) \to U/(p^n)$. This holds for $n = 1$ because $\pi \circ t$ is a homomorphism. Suppose the claim holds for some $n \geq 1$. For $x = [\overline{x}] + px_1, y = [\overline{y}] + py_1, z = [\overline{z}] + pz_1 \in S$ with $x + y = z$,

\[
\begin{align*}
[\overline{x}] & \equiv ([\overline{x}]^{p^{-n}} + [\overline{y}]^{p^{-n}})^{p^n} \pmod{p^{n+1}} \\
t(\overline{x}) & \equiv (t([\overline{x}]^{p^{-n}}) + t([\overline{y}]^{p^{-n}}))^{p^n} \pmod{p^{n+1}}
\end{align*}
\]

as in Definition 3.2.1. In particular,

\[
T([\overline{x}]) - T([\overline{y}]) - T([\overline{y}]) \equiv \sum_{i=1}^{p^n-1} \binom{p^n}{i} T([\overline{x}]^{p^{-n-1}} [\overline{y}]) \pmod{p^{n+1}}.
\]

(3.2.2.2)

54
On the other hand, since $\frac{1}{p} (p^n) \in \mathbb{Z}$ for $i = 1, \ldots, p^n - 1$, we may write

$$z_1 - x_1 - y_1 = \frac{[\bar{x}] + [\bar{y}] - [\bar{z}]}{p} \equiv - \sum_{i=1}^{p^n-1} \frac{1}{p} \binom{p^n}{i} [x^{ip^{-n}}y^{1-ip^{-n}}] \quad (\text{mod } p^n),$$

apply $T$, invoke the induction hypothesis on both sides, and multiply by $p$ to obtain

$$pT(z_1) - pT(x_1) - pT(y_1) \equiv - \sum_{i=1}^{p^n-1} \left( \binom{p^n}{i} T([x^{ip^{-n}}y^{1-ip^{-n}}]) \right) \quad (\text{mod } p^{n+1}). \quad (3.2.3)$$

Since $T(x) = T([\bar{x}]) + pT(x_1)$ and so on, we may add $(3.2.2)$ and $(3.2.3)$ to deduce that $T(z) - T(x) - T(y) \equiv 0 \mod p^{n+1}$, completing the induction. Hence $T$ is additive; multiplicativity of $t$ forces $T$ to also be multiplicative, as desired.

\[\square\]

**Remark 3.2.3.** For $X$ an arbitrary set, the $p$-adic completion $S$ of $\mathbb{Z}[X^{p^{-\infty}}]$ is a strict $p$-ring with $S/(p) \cong \mathbb{F}_p[X^{p^{-\infty}}]$. If we take $X = \{\bar{x}, \bar{y}\}$, then

$$[\bar{x}] + [\bar{y}] = \sum_{n=0}^{\infty} p^n [P_n(\bar{x}, \bar{y})] \quad (3.2.3.1)$$

for some $P_n(\bar{x}, \bar{y})$ in the ideal $(\bar{x}^{p^{-\infty}}, \bar{y}^{p^{-\infty}}) \subset \mathbb{F}_p[\bar{x}^{p^{-\infty}}, \bar{y}^{p^{-\infty}}]$ and homogeneous of degree 1. For instance, $P_0(\bar{x}, \bar{y}) = \bar{x} + \bar{y}$ and $P_1(\bar{x}, \bar{y}) = - \sum_{i=1}^{p-1} \frac{1}{i} \binom{p}{i} \bar{x}^{i/p} \bar{y}^{1-i/p}$. By Lemma 3.2.2 $(3.2.3.1)$ is also valid for any strict $p$-ring $S$ and any $\bar{x}, \bar{y} \in S/(p)$. One can similarly derive formulas for arithmetic in a strict $p$-ring in terms of Teichmüller coordinates; these can also be obtained using Witt vectors (Definition 3.2.5).

**Theorem 3.2.4.** The functor $S \rightsquigarrow S/(p)$ from strict $p$-rings to perfect $\mathbb{F}_p$-algebras is an equivalence of categories.

**Proof.** Full faithfulness follows from Lemma 3.2.2. To prove essential surjectivity, let $R$ be a perfect $\mathbb{F}_p$-algebra, choose a surjection $\psi : \mathbb{F}_p[X^{p^{-\infty}}] \to R$ for some set $X$, and put $\overline{T} = \ker(\psi)$. Let $S_0$ be the $p$-adic completion of $\mathbb{Z}[X^{p^{-\infty}}]$; this is a strict $p$-ring with $S_0/(p) \cong \mathbb{F}_p[X^{p^{-\infty}}]$. Put $I = \{ \sum_{n=0}^{\infty} p^n [\bar{x}_n] \in S_0 : \bar{x}_0, \bar{x}_1, \ldots \in \overline{T} \}$; this forms an ideal in $S_0$ by Remark 3.2.3. Then $S = S_0/I$ is a strict $p$-ring with $S/(p) \cong R$. \[\square\]

**Definition 3.2.5.** For $R$ a perfect $\mathbb{F}_p$-algebra, let $W(R)$ denote the strict $p$-ring with $W(R)/(p) \cong R$; this object is unique up to unique isomorphism by Theorem 3.2.4. More concretely, we may identify $W(R)$ with the set of infinite sequences over $R$ so that the sequence $(\bar{x}_0, \bar{x}_1, \ldots)$ corresponds to the ring element $\sum_{n=0}^{\infty} p^n [\bar{x}_n]$. This is a special case of the construction of the ring of $p$-typical Witt vectors associated to a ring $R$, hence the notation. The construction of $W(R)$ is functorial in $R$, so for instance $\bar{\varphi}$ lifts functorially to an endomorphism $\varphi$ of $W(R)$. It is common shorthand to write $W_n(R)$ for $W(R)/(p^n)$. 55
One of the key roles that strict $p$-rings play in our work is in the classification of local systems over rings of positive characteristic. The central point is a nonabelian version of Artin-Schreier-Witt theory, for which we follow [60 Proposition 4.1.1] (see also [24 Theorem 2.2]).

**Lemma 3.2.6.** Let $R$ be a perfect $\mathbb{F}_p$-algebra, and let $n$ be a positive integer. Let $M$ be a finite projective $W_n(R)$-module of everywhere positive rank, equipped with a semilinear $\varphi^a$-action for some positive integer $a$. Then there exists a faithfully finite étale $R$-algebra $S$ such that $M \otimes_{W_n(R)} W_n(S)$ admits a basis fixed by $\varphi^a$. More precisely, if $m < n$ is another positive integer and $M \otimes_{W_n(R)} W_m(R)$ admits a $\varphi^a$-fixed basis, then $S$ can be chosen so that this basis lifts to a $\varphi^a$-fixed basis of $M \otimes_{W_n(R)} W_n(S)$.

**Proof.** We treat the case $n = 1$ first. Suppose first that $M$ is free; choose a basis $e_1,\ldots,e_m$ of $M$ on which $\varphi^a$ acts via a matrix $A$ over $W_1(R) \cong R$. Let $X$ be the closed subscheme of $\text{Spec}(R[U_{ij} : i,j = 1,\ldots,m])$ defined by the matrix equation $\varphi^a(U) = A^{-1}U$. The morphism $X \to \text{Spec}(R)$ is finite (evidently) and étale (by the Jacobian criterion), so $X = \text{Spec}(S)$ for some finite étale $R$-algebra $S$. The elements $v_1,\ldots,v_m$ of $M \otimes_R S$ defined by $v_j = \sum_i U_{ij} e_i$ form a basis fixed by $\varphi^a$. Since the construction is naturally independent of the choice of the original basis, for general $M$ we can glue to obtain a finite étale $R$-algebra $S$ and a fixed basis of $M \otimes_R S$.

What is left to check is that $S$ has positive rank everywhere as an $R$-module. This can be checked pointwise on $R$, and also may be checked after faithfully flat descent, so we may reduce to the case where $R$ is an algebraically closed field. It is enough to check that the map $U \mapsto U^{-1}\varphi^a(U)$ on $\text{GL}_m(R)$ is surjective; this observation is due to Lang and is proved as follows (following [96 §VI.1, Proposition 4], [32 Exposé XXII, Proposition 1.1]). For each $A \in \text{GL}_m(R)$, the map $L_A : U \mapsto U^{-1}A\varphi^a(U)$ induces a bijective map from the tangent space at the identity matrix $I$, so the image of $L_A$ contains a nonempty Zariski open subset $V_A$ of $\text{GL}_m(R)$. Since $\text{GL}_m$ is a connected group scheme, the open sets $V_A$ and $V_I$ must intersect in some matrix $B$, for which

$$B = U_1^{-1}\varphi^a(U_1) = U_2^{-1}A\varphi^a(U_2)$$

for some $U_1, U_2 \in \text{GL}_m(R)$. We then have $A = U^{-1}\varphi^a(U)$ for $U = U_1U_2^{-1}$.

The case $n = 1$ is now complete; we treat the case $n > 1$ by induction on $n$. We may assume that there exists a basis $e_1,\ldots,e_m$ of $M$ on which $\varphi^a$ acts via a matrix $A$ congruent to $I$ modulo $p^{n-1}$. We may then take $\text{Spec}(S)$ to be the closed subscheme of $\text{Spec}(R[U_{ij} : i,j = 1,\ldots,m])$ defined by the matrix equation $\varphi^a(U) - U + p^{1-n}(A - I) = 0$: this subscheme is again finite étale (and hence affine) over $R$, and the elements $v_1,\ldots,v_m$ of $M \otimes_{W(R)} W(S)$ defined by $v_j = e_j + \sum_i p^{n-1}U_{ij} e_i$ form a basis fixed by $\varphi^a$ modulo $p^n$. \qed

**Proposition 3.2.7.** For $R$ a perfect $\mathbb{F}_p$-algebra, for each positive integer $n$, there is a natural (in $R$ and $n$) tensor equivalence between étale $\mathbb{Z}/p^n\mathbb{Z}$-local systems on $R$ and finite projective modules over $W_n(R)$ equipped with semilinear $\varphi$-actions.
Proof. Let $T$ be an étale $(\mathbb{Z}/p^n\mathbb{Z})$-local system on $R$, represented by the scheme Spec$(R_n)$. In case $T$ is of constant rank $d$, $R_n$ carries an action of the group $G = GL_d(\mathbb{Z}/p^n\mathbb{Z})$, so we may define $M(T) = W_n(R_n)^G$; by faithfully flat descent (Theorems 1.3.4 and 1.3.5), $M(T)$ is projective of constant rank $d$ over $W_n(R)$. The construction extends naturally to general $T$.

Let $M$ be a finite projective module over $W_n(R)$ equipped with a semilinear $\varphi$-action. The assignment
\[ S \mapsto (M \otimes_{W_n(R)} W_n(S))^\varphi \]
defines an étale sheaf $T(M)$ on Spec$(R)$ thanks to Lemma 3.2.6. It is easy to check that the functors $T \sim M(T)$ and $M \sim T(M)$ form an equivalence.

Remark 3.2.8. One might like to assert Proposition 3.2.7 with $GL_d$ replaced by other group schemes. The main difficulty is that the analogue of Hilbert’s Theorem 90 is not always valid; this is related to the classification of special groups by Serre [94] and Grothendieck [47]. One tractable special case is that of a unipotent group scheme; see Proposition 3.2.9.

Proposition 3.2.9. Let $d, m, n$ be integers with $d, m \geq 1$ and $n \geq 2$ (we may also take $n = 1$ in case $p > 2$). Let $g$ be an algebraic Lie subalgebra of the Lie algebra of $d \times d$ matrices over $\mathbb{Q}_p$. Let $g_n$ be the intersection of $g$ with the Lie algebra of $d \times d$ matrices over $p^n\mathbb{Z}_p$. Let $G_{n,m}$ be the unipotent group scheme defined by the Lie algebra $g_n \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/(p^m)$, viewed over $\mathbb{F}_p$ by Greenberg realization (i.e., identifying $\mathbb{Z}_p$ with the Witt vectors of $\mathbb{F}_p$). For $R$ a perfect $\mathbb{F}_p$-algebra, define the equivalence relation $\sim$ on $G_{n,m}(R)$ by declaring that $g_1 \sim g_2$ if there exists $h \in G_{n,m}(R)$ for which $h^{-1}g_1 \varphi(h) = g_2$. Then there is a natural (in $G, R, n, m$) bijection
\[ G_{n,m}(R)/\sim \to H^1_{\text{ét}}(R, G_{n,m}(\mathbb{F}_p)). \]

Proof. For $g \in G_{n,m}(R)$, as in Lemma 3.2.6, we may construct a faithfully finite étale $R$-algebra $S$ such that $g = h^{-1}\varphi(h)$ for some $h \in G_{n,m}(S)$. The choice of $h$ then defines an element of $H^1_{\text{ét}}(R, G_{n,m}(\mathbb{F}_p))$ which depends only on $g$ up to equivalence. This gives the claimed map; its injectivity is straightforward. Surjectivity comes down to the fact that $H^1_{\text{ét}}(R, G_{n,m})$ is trivial as a pointed set, which holds because $G_{n,m}$ is unipotent.

The operation of direct perfection can be extended to certain $p$-torsion-free rings in order to generate strict $p$-rings.

Definition 3.2.10. Let $A$ be a $p$-torsion-free ring with $A/pA$ reduced, equipped with an endomorphism $\varphi_A : A \to A$ inducing some power of the $p$-power Frobenius map on $A/pA$ and with an identification $(A/pA)_{\text{perf}} \cong R$. Then $\varphi$ induces a map $s_\varphi : A \to W(R)$ satisfying $\varphi \circ s_\varphi = s_\varphi \circ \varphi_A$; this may be seen by using the uniqueness property of $W(R)$ to identify it with the $p$-adic completion of the limit of the direct system
\[ A \xrightarrow{s_\varphi} A \xrightarrow{s_\varphi} \cdots. \]
(For more details, we follow [65 (1.3.16)] in suggesting the reference [79 VII, §4].) We describe $W(R)$ as the direct perfection of $A$ with respect to $\varphi$. 57
Example 3.2.11. Put $A = \mathbb{Z}[T], R = \mathbb{F}_p[T]_{\text{perf}}$, and identify $(A/pA)_{\text{perf}}$ with $R$ by mapping the class of $T$ to $[T]$. For the endomorphism $\varphi_A: A \to A$ defined by $\varphi_A(T) = T^p$, the map $s_\varphi$ takes $T$ to $[T]$. However, note for instance that $s_\varphi(T + 1) \neq [T + 1]$.

3.3 Norms on strict $p$-rings

We now take a more metric look at strict $p$-rings.

Hypothesis 3.3.1. Throughout §3.3 let $R$ be a perfect $\mathbb{F}_p$-algebra.

We introduce some operations relating the spectra of $R$ and $W(R)$. For variants that do not require the norm on $R$ to be trivial, see Proposition 5.1.2.

Definition 3.3.2. For $\alpha$ a submultiplicative (resp. power-multiplicative, multiplicative) seminorm on $R$ bounded by the trivial norm,

$$\lambda(\alpha) \left( \sum_{i=0}^{\infty} p^i [\pi_i] \right) = \sup_i \{ p^{-i} \alpha(\pi_i) \}$$

(3.3.2.1)

is a submultiplicative (resp. power-multiplicative, multiplicative) seminorm on $W(R)$ bounded by the $p$-adic norm [72, Lemma 4.1]. For $\beta$ a submultiplicative (resp. power-multiplicative, multiplicative) seminorm on $W(R)$ bounded by the $p$-adic norm,

$$\mu(\beta)(\pi) = \beta([\pi])$$

(3.3.2.2)

is a submultiplicative (resp. power-multiplicative, multiplicative) seminorm on $R$ bounded by the trivial norm [72, Lemma 4.2].

Lemma 3.3.3. Equip $R$ with the trivial norm and $W(R)$ with the $p$-adic norm. Then the functions $\lambda: \mathcal{M}(R) \to \mathcal{M}(W(R))$ and $\mu: \mathcal{M}(W(R)) \to \mathcal{M}(R)$ are continuous, and satisfy $(\mu \circ \lambda)(\alpha) = \alpha$ and $(\lambda \circ \mu)(\beta) \geq \beta$.

Proof. See [72, Theorem 4.5].

Definition 3.3.4. Suppose that $R$ is equipped with a power-multiplicative norm $\alpha$. An element $z = \sum_{i=0}^{\infty} p^i [\pi_i] \in W(\mathfrak{o}_R)$ is primitive of degree 1 if $\pi_0 \in R^\times$, $\alpha(\pi_0) = p^{-1}$, $\alpha(\pi_0^{-1}) = p$, and $\pi_1 \in \mathfrak{o}_R^\times$. For example, any $z$ of the form $p - [u]$ for $u \in R^\times$ with $\alpha(u) = p^{-1}$ and $\alpha(u^{-1}) = p$ is primitive of degree 1. Note that the conditions on $\pi_0$ imply the limited multiplicativity property

$$\alpha(xz_0) = \alpha(x)\alpha(z_0) \quad (x \in R),$$

(3.3.4.1)

by writing

$$\alpha(xz_0) \leq \alpha(x)\alpha(z_0) = \alpha(x\alpha(z_0^{-1})) \leq \alpha(xz_0).$$

Conversely, let $S$ be a ring equipped with a power-multiplicative norm $\alpha$ bounded above by the trivial norm. Let $z_0 \in S$ be an element for which $\alpha(xz_0) = p^{-1}\alpha(x)$ for all $x \in S$, and for which every element of $S$ of norm at most $p^{-1}$ is divisible by $z_0$. We can then extend $\alpha$ to
$R = S[\overline{z}_0]^{-1}$ so as to maintain (3.3.4.1); moreover, we may then identify $S$ with $o_R$. We say that $z = \sum_{i=0}^\infty p^i[\overline{z}_i] \in W(S)$ is primitive of degree 1 if the same holds when $z$ is viewed as an element of $W(o_R)$ (i.e., if $\overline{z}_i \in o_R^\times$). Note that from (3.3.4.1), it follows that the principal ideal $(z)$ in $W(o_R)$ is closed (see [72, Theorem 5.11]).

The terminology is modeled on that of [39], in which a result similar to our Theorem 3.3.7 can be found; the wording is meant to evoke an analogy with the theory of Weierstrass preparation for analytic power series. (Note however that when $R = L$ is an analytic field, our definition is more restrictive than that used in [39], in which the condition $\alpha(\overline{z}_0) = p^{-1}$ is relaxed to $\alpha(\overline{z}_0) < 1$.)

A key example of the previous definition is the following.

**Example 3.3.5.** Choose $\overline{\pi} \in R^\times$ with $\alpha(\overline{\pi}) = p^{-p/(p-1)}$ and $\alpha(\overline{\pi}^{-1}) = p^{p/(p-1)}$, and put

$$z = \sum_{i=0}^{p-1} [\overline{\pi} + 1]^{i/p} = \sum_{i=0}^\infty p^i[\overline{z}_i].$$

Then $\overline{z}_0 = \overline{\pi}^{(p-1)/p}$, so $\alpha(\overline{z}_0) = p^{-1}$ and $\alpha(\overline{z}_0^{-1}) = p$. We may check that $\overline{z}_i \in o_R^\times$ by noting that under the map $W(F_p[\overline{\pi}]^\text{perf}) \to W(F_p)$ induced by reduction modulo $\overline{\pi}$, the image of $\sum_{i=0}^{p-1}[\overline{\pi} + 1]^{i/p}$ is $\sum_{i=0}^{p-1}1 = p$. Hence $z$ is primitive of degree 1.

**Lemma 3.3.6.** Suppose that $R$ is complete with respect to a power-multiplicative norm $\alpha$ and that $z \in W(o_R)$ is primitive of degree 1. Then any $x \in W(o_R)$ is congruent modulo $z$ to some $y = \sum_{i=0}^\infty p^i[\overline{y}_i] \in W(o_R)$ with $\alpha(\overline{y}_0) \geq \alpha(\overline{y}_i)$ for all $i > 0$.

**Proof.** See [72, Lemma 5.5].

**Theorem 3.3.7.** Take $R, z$ as in Lemma 3.3.6.

(a) For each submultiplicative (resp. power-multiplicative, multiplicative) seminorm $\gamma$ on $o_R$ bounded by the trivial norm, the quotient seminorm $\sigma(\gamma)$ on $W(o_R)/(z)$ induced by $\lambda(\gamma)$ is submultiplicative (resp. power-multiplicative, multiplicative) and satisfies $\mu(\sigma(\gamma)) = \gamma$.

(b) Equip $W(o_R)$ with the power-multiplicative norm $\lambda(\alpha)$. Then the map $\sigma : \mathcal{M}(o_R) \to \mathcal{M}(W(o_R))$ indicated by (a) is a continuous section of $\mu$, which induces a homeomorphism of $\mathcal{M}(o_R)$ with $\mathcal{M}(W(o_R)/(z))$. Under this homeomorphism, a subspace of $\mathcal{M}(o_R)$ is Weierstrass (resp. Laurent, rational) if and only if the corresponding subspace of $\mathcal{M}(W(o_R)/(z))$ is Weierstrass (resp. Laurent, rational).

(c) The homeomorphism of (b) induces a homeomorphism of $\mathcal{M}(R)$ with $\mathcal{M}(W(o_R)[[\overline{z}]]^{-1}/(z))$ under which Weierstrass, Laurent, and rational subspaces again correspond.

For more on the relationship between $\mathcal{M}(R)$ and $\mathcal{M}(W(o_R)[[\overline{z}]]^{-1}/(z))$, see [3.6] and [3.4].
Proof. For (a), see [72, Theorem 5.11(a)] (which is itself an easy corollary of Lemma 3.3.6). For (b), see [72, Corollary 7.2]. Note that for these results, \( z \) need not be primitive of degree 1; it is enough to assume that \( \alpha(z_0) \leq p^{-1} \) and \( z_1 \in \mathcal{O}_R^* \).

By assuming that \( z \) is primitive of degree 1, however, we ensure that the quotient norm \( \beta \) on \( M(W(\mathcal{O}_R)/(z)) \) has the property that \( \beta(px) = p^{-1} \beta(x) \), so that we may extend \( \beta \) after inverting \( p \). We may then identify

\[
M(R) = \left\{ \gamma \in M(\mathcal{O}_R) : \gamma(z_0) \geq p^{-1} \right\} \\
M(W(\mathcal{O}_R)[[z]]^{-1})/(z)) = \left\{ \gamma \in M(W(\mathcal{O}_R)/(z)) : \gamma(z_0) \geq p^{-1} \right\}.
\]

Since these are rational subspaces, we may deduce (c). \( \square \)

**Example 3.3.8.** Let \( L \) be a perfect analytic field of characteristic \( p \), let \( \alpha \) be the norm on \( L \), and choose \( z \in W(\mathcal{O}_L) \) which is primitive of degree 1 (this forces \( \alpha \) to be nontrivial). By Theorem 3.3.7(a), the quotient norm on \( W(\mathcal{O}_L)/(z) \) induced by \( \lambda(\alpha) \) is multiplicative. Moreover, by Lemma 3.3.6, every nonzero element of \( W(\mathcal{O}_L)/(z) \) can be lifted to an element of \( W(\mathcal{O}_L) \) which becomes invertible in \( W(\mathcal{O}_L)[[z]]^{-1} \). It follows that \( W(\mathcal{O}_L)/(z) \) is the valuation subring of an analytic field \( F = W(\mathcal{O}_L)[[\pi]]^{-1} \), whose residue field is the same as that of \( L \). (In terms of the rings \( \mathcal{R}_{\mathcal{O},r} \) to be introduced in Definition 4.2.2 below, we can also realize \( F \) as \( \mathcal{R}_{\mathcal{O},r}^{\text{int}}/(z) \) for any \( r \geq 1 \). See Lemma 5.5.5.) For instance, for \( L \) the completed perfection of \( \mathbb{F}_p((\pi)) \) and \( z \) as in Example 3.3.5, \( F \) is the completion of \( \mathbb{Q}_p(\mu_{p^n}) \) for the \( p \)-adic norm.

Note that

\[
\mathcal{O}_L[[z]] \equiv W(\mathcal{O}_L)/(p, [z]) = W(\mathcal{O}_L)/(p, z) = \mathcal{O}_F/(p).
\]

This implies that \( \mathcal{O} \) is surjective on \( \mathcal{O}_F/(p) \) and that \( \mathcal{O}_F \) is not discretely valued. These conditions turn out to characterize the fields \( F \) which arise in this manner; see Lemma 3.3.2.

The following refinement of [72, Lemma 5.16] is useful for some calculations.

**Lemma 3.3.9.** Take \( R, z \) as in Lemma 3.3.6. Then for any \( \epsilon > 0 \) and any nonnegative integer \( m \), every \( x \in W(\mathcal{O}_R)[[z]]^{-1} \) is congruent modulo \( z \) to some \( y = \sum_{n=0}^{\infty} p^{\alpha(\gamma_n)} \gamma_n \in W(\mathcal{O}_R)[[z]]^{-1} \) such that for each \( \alpha \in M(R) \),

\[
\alpha(\gamma_1) \leq \max\{p^{-\epsilon}, \alpha(\gamma_0), \epsilon\} \\
\alpha(\gamma_n) \leq \max\{\alpha(\gamma_0), \epsilon\} \quad (n > 1).
\]

**Proof.** Define the sequence \( x = x_0, x_1, \ldots \) as in the proof of [72, Lemma 5.5]. That is, let \( w \) be the inverse of \( p^{-1}(z - \pi) \) in \( W(\mathcal{O}_R) \), then write \( x_i = \sum_{j=0}^{\infty} p^j \gamma_{ij} \] with \( \gamma_{ij} \in R \) and put

\[
x_{i+1} = x_i - p^{-1} w(x_i - \gamma_{ij}) z = [\gamma_{ij}] - p^{-1} w(x_i - \gamma_{ij})[z].
\]

The proof of [72, Lemma 5.16] shows that there exists \( i_0 \) such that for each \( \alpha \in M(R) \),

\[
\alpha(\gamma_{ij}) \leq \max\{\alpha(\gamma_{i0}), \epsilon\} \quad (i \geq i_0, j > 0).
\]
If we take \( y = x_{i_0+k} \) for some nonnegative integer \( k \), then (3.3.9.2) is satisfied. If \( \alpha(x_{i_0}) \leq \epsilon \), then (3.3.9.1) is also satisfied.

Suppose instead that \( \alpha(x_{i_0}) > \epsilon \); in this case, it will complete the proof to show that (3.3.9.1) is satisfied whenever \( k \geq m \). It will suffice to check that for each nonnegative integer \( k \),

\[
\alpha(x_{i_0+k}) \leq \max\{p^{-1}\cdots p^{-k}|x_{i_0+k}|, \epsilon\}. \tag{3.3.9.3}
\]

We have this for \( k = 0 \), so we may proceed by induction on \( k \). Given (3.3.9.3) for some \( k \), write

\[
x_{i_0+k+1} \equiv \frac{x_{i_0+k+1}}{w_{i_0+k+1}} = w_{i_0+k+1}z + pw_{i_0+k+1}z \pmod{p^3}
\]

and then deduce that

\[
\frac{x_{i_0+k+1}}{w_{i_0+k+1}} = \frac{\frac{x_{i_0+1}}{w_{i_0+1}}}{\frac{w_{i_0+1}}{w_{i_0+k+1}z}} = P\left(\left(\frac{x_{i_0+k+1}z}{w_{i_0+k+1}}\right)^{1/p}, \left(\frac{w_{i_0+k+1}}{w_{i_0+1}}\right)^{1/p}\right)
\]

for \( P(x, y) = p^{-1}(x^p - y^p - (x - y)^p) \in \mathbb{Z}[x, y] \). From this we deduce

\[
\alpha(x_{i_0+k+1}) = \alpha(x_{i_0+k}) \leq \max\{\alpha(w_{i_0+k+1}z), \alpha(\frac{x_{i_0+k+1}z}{w_{i_0+k+1}})^{1/p}\}.
\]

Since \( \alpha(z) = p^{-1} \), this yields the analogue of (3.3.9.3) with \( k \) replaced by \( k+1 \).

\[\square\]

### 3.4 Inverse perfection

We have already introduced one method for passing from an \( \mathbb{F}_p \)-algebra to a perfect \( \mathbb{F}_p \)-algebra, that of direct perfection. We now consider the dual operation of inverse perfection, which has the advantage of capturing useful information from characteristic 0.

**Definition 3.4.1.** For any ring \( A \), define the inverse perfection \( A^{\text{frep}} \) of \( A \) as the inverse limit of the system

\[
\cdots \xrightarrow{\phi} A/pA \xrightarrow{\varphi} A/pA.
\]

This evidently gives a perfect \( \mathbb{F}_p \)-algebra. There is a natural projection \( A^{\text{frep}} \to A/pA \) by projection onto the last factor; this is surjective as long as \( \varphi : A/pA \to A/pA \) is surjective.

**Lemma 3.4.2.** Let \( A \) be a ring. For any ideal \( I \) of \( A \) satisfying \( I^m \subseteq (p) \subseteq I \) for some positive integer \( m \), the natural map \( A^{\text{frep}} \to (A/I)^{\text{frep}} \) is an isomorphism.

**Proof.** We may assume \( m = p^k \) for some positive integer \( k \). Let \( y = (\ldots, y_1, y_0) \) be an element of \( A^{\text{frep}} \) whose image in \( (A/I)^{\text{frep}} \) is zero. For each nonnegative integer \( n \), we then have \( y_{n+k} \equiv 0 \pmod{I} \), and so

\[
y_n \equiv y_{n+k} \equiv 0 \pmod{(p + I^k)}.
\]

Hence \( y_n \equiv 0 \pmod{p} \), and so \( y = 0 \) in \( A^{\text{frep}} \).

Given \( z = (\ldots, z_1, z_0) \in (A/I)^{\text{frep}} \), choose any lifts \( \tilde{z}_n \in A \) of \( z_n \). Put \( y_n = z_n^{p^k} \); then the congruence \( z_n^{p^k} \equiv \tilde{z} \equiv 0 \pmod{I} \) implies \( y_n \equiv y_{n+1} \equiv y_n \pmod{(p + I^k)} \). Hence \( y = (\ldots, y_1, y_0) \) forms an element of \( A^{\text{frep}} \) lifting \( z \). \[\square\]
**Definition 3.4.3.** Let $A$ be a ring, and let $\hat{A}$ denote the $p$-adic completion of $A$. From the projection $A^{\text{frep}} \to A/pA$, we obtain first a multiplicative map $A^{\text{frep}} \to \hat{A}$ and then by Lemma 3.2.2 a homomorphism $\theta : W(A^{\text{frep}}) \to \hat{A}$. Note that $\theta$ is surjective if and only if $\overline{\varphi} : A/pA \to A/pA$ is surjective.

**Remark 3.4.4.** Let $A$ be a $p$-adically separated ring, let $\beta$ be any submultiplicative (resp. power-multiplicative, multiplicative) seminorm on $A$ bounded by the $p$-adic norm, and extend $\beta$ to $\hat{A}$ by continuity. Then $\alpha = \mu(\theta^*(\beta))$ is a submultiplicative (resp. power-multiplicative, multiplicative) seminorm on $A^{\text{frep}}$ bounded by the trivial norm. In particular, the map $\theta$ is bounded for the seminorm $\lambda(\alpha)$ on $W(A^{\text{frep}})$ and the seminorm $\beta$ on $\hat{A}$.

**Lemma 3.4.5.** In Remark 3.4.4 suppose that $\beta$ is power-multiplicative (resp. multiplicative) norm and that $A$ is complete under $\beta$. Then $\alpha = \mu(\theta^*(\beta))$ is a power-multiplicative (resp. multiplicative) norm under which $A^{\text{frep}}$ is complete.

**Proof.** For $x = (\ldots, x_1, x_0) \in A^{\text{frep}}$ and any lifts $x_i \in A$ of $\overline{x}_i$, we have

$$\max\{\alpha(x), \beta(p)i\} = \max\{\beta(x_i)p^i, \beta(p)^i\}$$

for all $i$. In particular, if $\alpha(x) \neq 0$, then $x_i \neq 0$ for all sufficiently large $i$, so $x \neq 0$. Hence $\alpha$ is a norm.

Let $y_0, y_1, \ldots$ be a sequence in $A^{\text{frep}}$ which is Cauchy with respect to $\alpha$. For each nonnegative integer $i$, the sequence $y_0^{p^{-i}}, y_1^{p^{-i}}, \ldots$ is also Cauchy with respect to $\alpha$, so $[y_0^{p^{-i}}], [y_1^{p^{-i}}], \ldots$ is Cauchy with respect to $\lambda(\alpha)$. The images of $[y_0^{p^{-i}}], [y_1^{p^{-i}}], \ldots$ in $A$ then form a Cauchy sequence with respect to $\beta$, which by hypothesis has a limit $\tilde{z}_i \in A$. Let $z_i \in A/pA$ be the image of $\tilde{z}_i$; then $(\ldots, z_1, z_0)$ forms an element of $A^{\text{frep}}$ which is the limit of the $y_n$. Hence $A^{\text{frep}}$ is complete. 

**Remark 3.4.6.** For $R$ a perfect $\mathbb{F}_p$-algebra, the natural map $W(R)^{\text{frep}} \cong R$ is an isomorphism, as then is the map $\theta : W(W(R)^{\text{frep}}) \to W(R)$.

**Lemma 3.4.7.** Let $A_1$ be a $p$-adically separated ring written as a union $\bigcup_{i \in I} A_{1,i}$ such that for each $i \in I$, $\overline{\varphi}$ is surjective on $A_{1,i}/pA_{1,i}$. Let $A_2$ be a $p$-adically separated ring equipped with a norm $\beta_2$ bounded by the $p$-adic norm. Let $\psi : A_1 \to A_2$ be a homomorphism. Put $\alpha_2 = \mu(\theta^*(\beta_2))$.

(a) Suppose that $\psi$ has dense image. Then the induced map $\psi^{\text{frep}} : \bigcup_{i \in I} A_{1,i}^{\text{frep}} \to A_2^{\text{frep}}$ has dense image.

(b) Suppose that the image of $\psi$ has dense intersection with $m_{A_2}$. Then the image of $\psi^{\text{frep}}$ has dense intersection with $m_{A_2}^{\text{frep}}$.

**Proof.** Given $w = (\ldots, w_1, w_0) \in A_2^{\text{frep}}$, if we choose a nonnegative integer $n$, we can choose $i \in I$ and $\tilde{w}_n \in A_{1,i}$ so that $\alpha_2(\psi(\tilde{w}_n) - w_n) \leq p^{-1}$. Since $\overline{\varphi}$ is surjective on $A_{1,i}/pA_{1,i}$, we can find $x = (\ldots, x_1, x_0) \in A_{1,i}^{\text{frep}}$ with $x_n$ equal to the image of $\tilde{w}_n$ in $A_{1,i}/pA_{1,i}$. Let
\[ y = (\ldots, y_i, y_\ell) \in A^{\text{frep}}_0 \text{ be the image of } x \text{ under } \psi^{\text{frep}}; \text{ then } \alpha_2(y_n-w_n) \leq p^{-1}, \text{ so } \alpha_2(y-w) \leq p^{-p^n}. \text{ Since this holds for any } n \text{ (for some } i, y \text{ depending on } n), \text{ it follows that } \cup_{i \in I} A^{\text{frep}}_{i,1} \text{ has dense image in } A^{\text{frep}}_2. \text{ This proves (a); the proof of (b) is similar.} \]

**Lemma 3.4.8.** Let \( A \) be a \( p \)-adically separated \( p \)-torsion-free ring complete under a power-multiplicative norm \( \beta \) bounded by the \( p \)-adic norm, and put \( \alpha = \mu(\theta^*(\beta)) \). Suppose that there exists \( z \in W(A^{\text{frep}}) \) primitive of degree \( 1 \) with \( \theta(z) = 0 \). Extend \( \theta \) to a map \( W(A^{\text{frep}})[[\overline{z}]] \to A[\theta([\overline{z}])]^{-1} \).

\((a)\) The ideal \( \ker(\theta) \subset W(A^{\text{frep}})[[\overline{z}]]^{-1} \) is generated by \( z \).

\((b)\) The extended map \( \theta \) is optimal.

\((c)\) The map \( \overline{\varphi} : A/(p) \to A/(p) \) is surjective if and only if \( \theta \) has dense image in \( A[\theta([\overline{z}])]^{-1} \).

**Proof.**

Given \( x \in W(A^{\text{frep}}) \) not divisible by \( z \), choose \( y = \sum_{i=0}^{\infty} p^i[y_i] \) as in Lemma 3.3.6. Then \( \theta(x) = \theta(y) = \theta(\overline{y_0}) + \theta(y - \overline{y_0}) \) and

\[ \beta(\theta(\overline{y}_0)) = \alpha(\overline{y}_0) > \lambda(\alpha)(y - \overline{y}_0) \geq \beta(\theta(y - \overline{y}_0)). \]

Consequently, \( \beta(\theta(x)) = \alpha(\overline{y}_0) > 0 \). This implies (a) and (b). To check (c), note that strictness of \( \theta \) (from (b)) implies that \( \theta \) is surjective if and only if it has dense image. \( \square \)

**Remark 3.4.9.** If \( \overline{\varphi} : A/pA \to A/pA \) is surjective, then so is \( \overline{\varphi} : A/I \to A/I \) for any ideal \( I \) for which \( I^m \subseteq (p) \subseteq I \) for some positive integer \( m \). The converse is not true: e.g., take \( A = \mathbb{Z}_p[\sqrt{p}] \) and \( I = (\sqrt{p}) \).

One correct partial converse is that if there exist \( x, y \in A \) such that \( \overline{\varphi} : A/(x, p) \to A/(x, p) \) is surjective, \( x^m \in (p) \) for some positive integer \( m \), and \( y^p \equiv x \pmod{(x^2, p)} \), then \( \overline{\varphi} : A/pA \to A/pA \) is surjective. To see this, we prove by induction that \( \overline{\varphi} : A/(x^i, p) \to A/(x^i, p) \) is surjective for \( i = 1, \ldots, m \), the case \( i = 1 \) being given and the case \( i = m \) being the desired result. Given the claim for some \( i \), note first that \( (y^p, p) = (x, p) \) and that \( y^{p^a} \equiv x^{i} \pmod{(x^{i+1}, p)} \). For any \( z \in A, \) we can find \( w_0, z_1 \in A \) with \( z - w_0^p - x^i z_1 \in pA \). We can then find \( w_1 \in A \) with \( z_1 - w_1^p \in (x^i, p) \); then \( w = w_0 + y^i w_1 \) satisfies \( w^p \equiv w_0^p + y^{p^a} w_1^p \equiv w_0^p + y^{p^a} z_1 \equiv w_0^p + x^i z_1 \equiv z \pmod{(x^{i+1}, p)} \).

### 3.5 The perfectoid correspondence for analytic fields

In order to bring nonabelian Artin-Schreier-Witt theory to bear upon \( p \)-adic Hodge theory, one needs a link between étale covers of spaces of characteristic 0 and characteristic \( p \). We first make this link at the level of analytic fields; this extends the field of norms correspondence introduced by Fontaine and Wintenberger [41], upon which usual \( p \)-adic Hodge theory is based. Similar results have been obtained by Scholze [91] using a slightly different method; see Remark 3.5.13. (See also [73] for a self-contained presentation of the correspondence following the approach taken here.) See [31.6] for an extension to more general Banach algebras.

63
Definition 3.5.1. An analytic field \( F \) is perfectoid if \( F \) is of characteristic 0, \( \kappa_F \) is of characteristic \( p \), \( F \) is not discretely valued, and \( \overline{\varphi} \) is surjective on \( \mathfrak{o}_F/(p) \). For example, any field \( F \) appearing in Example 3.3.8 is perfectoid; the converse is also true by Lemma 3.5.2 below.

Lemma 3.5.2. Let \( F \) be a perfectoid analytic field with norm \( \beta \). Put \( R = (\mathfrak{o}_F/(p)^{\text{frep}}) \), let \( \theta : W(R) \to \mathfrak{o}_F \) be the surjective homomorphism from Definition 3.4.3, and define the multiplicative norm \( \alpha = \mu(\theta^*(\beta)) \) on \( R \) as in Remark 3.4.4.

(a) The ring \( K = \text{Frac}(R) \) is an analytic field under \( \alpha \) which is perfect of characteristic \( p \), and \( R = \mathfrak{o}_K \).

(b) We have \( \beta(F^\times) = \alpha(K^\times) \).

(c) For any \( z \in K \) with \( \alpha(z) = p^{-1} \) (which exists by (b)), there is a natural (in \( F \)) isomorphism \( \mathfrak{o}_F/(p) \cong \mathfrak{o}_K/(z) \). In particular, we obtain a natural isomorphism \( \kappa_F \cong \kappa_K \).

(d) There exists \( z \in W(\mathfrak{o}_K) \) which is primitive of degree 1. Consequently (by Lemma 3.4.8), the kernel of \( \theta : W(\mathfrak{o}_K)[[\overline{\varphi}^{-1}]] \to F \) is generated by \( z \).

Proof. Note that \( R \) is already complete under \( \alpha \) by Lemma 3.4.5. Hence to prove (a), it suffices to check that for any nonzero \( x,y \in R \) with \( \alpha(x) \leq \alpha(y) \), \( x \) is divisible by \( y \) in \( R \). Write \( x = (\ldots, x_0), y = (\ldots, y_0) \) and lift \( x_n, y_n \) to \( x_n, y_n \in \mathfrak{o}_F \). Choose \( n_0 \geq 0 \) so that \( \alpha(x), \alpha(y) > p^{-n_0} \). For \( n \geq n_0 \), as in the proof of Lemma 3.4.5, we have \( \beta(x_n) = \alpha(x)p^{-n}, \beta(y_n) = \alpha(y)p^{-n} \), so \( \beta(x_n) \leq \beta(y_n) \). Since \( \mathfrak{o}_F \) is a valuation ring, \( z_n = x_n/y_n \) belongs to \( \mathfrak{o}_F \). Since \( \alpha(x_n^p - x_n), \alpha(y_n^p - y_n) \leq p^{-1} \), we have \( \alpha(z_n^p - z_n) \leq p^{-1}/\alpha(y_n) \). This last quantity is bounded away from 1 for \( n \geq n_0 \), so by Lemma 3.4.2, the \( z_n \) define an element \( z \in R \) for which \( x = yz \).

To establish (b), note that the group \( \alpha(K^\times) \) is \( p \)-divisible and that \( \alpha(K^\times) \cap (p^{-1},1) = \beta(F^\times) \cap (p^{-1},1) \) by (a). Since \( F \) is not discretely valued, we can choose \( r \in \beta(F^\times) \cap (1,p^{1/p}) \). For any such \( r \), we have \( r^{-1}, p^{-1}r \in \beta(F^\times) \cap (p^{-1},1) \subseteq \alpha(K^\times) \), so \( p^{-1} \in \alpha(K^\times) \).

To establish (c), note that from the definition of the inverse perfection, we obtain a homomorphism \( \mathfrak{o}_K \to \mathfrak{o}_F/(p) \). By comparing norms, we see that the kernel of this map is generated by \( \overline{\varphi} \).

To establish (d), keep notation as in (c). Note that \( \theta([\overline{\varphi}]) \) is divisible by \( p \) in \( \mathfrak{o}_F \) and that \( \theta : W(\mathfrak{o}_K) \to \mathfrak{o}_F \) is surjective. We can thus find \( z_1 \in W(\mathfrak{o}_K)^\times \) with \( \theta(z_1) = \theta([\overline{\varphi}])/p \); we then take \( z = [\overline{\varphi}] - pz_1 \).

Theorem 3.5.3 (Perfectoid correspondence). The constructions

\[
F \leadsto (\text{Frac}(\mathfrak{o}_F/(p)^{\text{frep}}), \ker(\theta)), \quad (L, I) \leadsto \text{Frac}(W(\mathfrak{o}_L)/I)
\]

define an equivalence of categories between perfectoid analytic fields \( F \) and pairs \((L, I)\) in which \( L \) is a perfect analytic field of characteristic \( p \) and \( I \) is a principal ideal of \( W(\mathfrak{o}_L) \) admitting a generator which is primitive of degree 1.
Proof. This follows immediately from Example 3.3.8 and Lemma 3.5.2.

We next study the compatibility of this correspondence with finite extensions of fields.

**Lemma 3.5.4.** Fix $F$ and $(L, I)$ corresponding as in Theorem 3.5.3. Then for any finite extension $M$ of $L$, the pair $(M, IW(\mathfrak{o}_M))$ corresponds via Theorem 3.5.3 to a finite extension $E$ of $F$ with $[E : F] = [M : L]$.

**Proof.** Suppose first that $M$ is Galois over $L$, and put $G = \text{Gal}(M/L)$. Since $I$ is a principal ideal, averaging over $G$ induces a projection

$$E = \frac{W(\mathfrak{o}_M)[p^{-1}]}{W(\mathfrak{o}_L)[p^{-1}] I W(\mathfrak{o}_M)[p^{-1}]} = \frac{W(\mathfrak{o}_L)[p^{-1}]}{W(\mathfrak{o}_L)[p^{-1}] \cap I W(\mathfrak{o}_M)[p^{-1}]} = F.$$  

Consequently, $E^G = F$, so by Artin’s lemma, $E$ is a finite Galois extension of $F$ and $[E : F] = \#G = [M : L]$. This proves the claim when $M$ is Galois; the general case follows by Artin’s lemma again.

For the reverse direction, the crucial case is when the characteristic $p$ field is algebraically closed.

**Lemma 3.5.5.** Fix $F$ and $(L, I)$ corresponding as in Theorem 3.5.3. If $L$ is algebraically closed, then so is $F$.

**Proof.** Let $\beta$ denote the norm on $F$. Let $P(T) \in \mathfrak{o}_F[T]$ be an arbitrary monic polynomial of degree $d \geq 1$; it suffices to check that $P(T)$ has a root in $\mathfrak{o}_F$. We will achieve this by exhibiting a sequence $x_0, x_1, \ldots$ of elements of $\mathfrak{o}_F$ such that for all $n \geq 0$, $\beta(P(x_n)) \leq p^{-n}$ and $\beta(x_{n+1} - x_n) \leq p^{-n/d}$. This sequence will then have a limit $x \in \mathfrak{o}_F$ which is a root of $P$.

To begin, take $x_0 = 0$. Given $x_n \in \mathfrak{o}_F$ with $\beta(P(x_n)) \leq p^{-n}$, write $P(T + x_n) = \sum_i Q_i T^i$. If $Q_0 = 0$, we may take $x_{n+1} = x_n$, so assume hereafter that $Q_0 \neq 0$. Put

$$c = \min \{ \beta(Q_0/Q_j)^{1/j} : j > 0, Q_j \neq 0 \};$$

by taking $j = d$, we see that $c \leq \beta(Q_0)^{1/d}$. Also, $\beta(F^x) = \alpha(L^x)$ by Lemma 3.5.2 and the latter group is divisible because $L$ is algebraically closed; we thus have $c = \beta(u)$ for some $u \in \mathfrak{o}_F$.

Apply Lemma 3.5.2 to construct $\overline{\mathfrak{r}} \in \mathfrak{o}_L$ with $\alpha(\overline{\mathfrak{r}}) = p^{-1}$. For each $i$, choose $\overline{R}_i \in \mathfrak{o}_L$ whose image in $\mathfrak{o}_L/(\overline{\mathfrak{r}}) \cong \mathfrak{o}_F/(p)$ is the same as that of $Q_i u^i / Q_0$. Define the polynomial $\overline{R}(T) = \sum_i \overline{R}_i T^i \in \mathfrak{o}_L[T]$. By construction, the largest slope in the Newton polygon of $\overline{R}$ is 0; by this observation plus the fact that $L$ is algebraically closed, it follows that $\overline{R}(T)$ has a root $y' \in \mathfrak{o}_L^\times$. Choose $y \in \mathfrak{o}_F^\times$ whose image in $\mathfrak{o}_F/(p) \cong \mathfrak{o}_L/(\overline{\mathfrak{r}})$ is the same as that of $y'$, and take $x_{n+1} = x_n + uy$. Then $\sum_i Q_i u^i y^i / Q_0 \equiv 0 \pmod{p}$, so $\beta(P(x_{n+1})) \leq p^{-1} \beta(Q_0) \leq p^{-n-1}$ and $\beta(x_{n+1} - x_n) = \beta(u) \leq \beta(Q_0)^{1/d} \leq p^{-n/d}$. We thus obtain the desired sequence, proving the claim.

\[ \end{proof} \]
Theorem 3.5.6. For $F$ and $(L, I)$ corresponding as in Theorem 3.5.3, the correspondence described in Lemma 3.5.4 induces a tensor equivalence $\mathbf{F\acute{e}t}(F) \cong \mathbf{F\acute{e}t}(L)$. In particular, every finite extension of $F$ is perfectoid, and the absolute Galois groups of $F$ and $L$ are homeomorphic.

Proof. Let $M$ be the completion of an algebraic closure of $L$. Via Theorem 3.5.3, $(M, IW(o_M))$ corresponds to a perfectoid analytic field $E$, which by Lemma 3.5.5 is algebraically closed.

By Lemma 3.5.4 each finite Galois extension of $L$ within $M$ corresponds to a finite Galois extension of $F$ within $E$ which is perfectoid. The union of the latter is an algebraic extension of $F$ whose closure is the algebraically closed field $E$; the union is thus forced to be separably closed by Krasner’s lemma. Since $F$ is of characteristic 0 and hence perfect, every finite extension of $F$ is thus forced to lie within a finite Galois extension which is perfectoid; the rest follows from Theorem 3.5.3.

Remark 3.5.7. Using Theorem 3.5.6, it is not difficult to show that the functor $F \rightsquigarrow L$ induced by Theorem 3.5.3 by forgetting the ideal $I$ is not fully faithful. For instance, as $F$ varies over finite totally ramified extensions of the completion of $\mathbb{Q}_p(\mu_{p^\infty})$ of a fixed degree, the fields $L$ are all isomorphic.

Theorem 3.5.6 implies that the perfectoid property moves up along finite extensions of analytic fields. It also moves in the opposite direction.

Lemma 3.5.8. Let $K$ be a perfect analytic field of characteristic $p$, and let $G$ be a finite group that acts faithfully on $K$ by isometric automorphisms. Then $H^1(G, 1 + m_K) = 0$.

Proof. We start with an observation concerning additive Galois cohomology. Since $K$ is an acyclic $K^G[G]$-module by the normal basis theorem, the complex

$$K \to K^G \to K^{G^2} \to \cdots$$

computing Galois cohomology is exact. Using the inverse of Frobenius as in Remark 3.1.5 we see that this complex is in fact almost optimal exact for the supremum norm on each factor.

Now let $f : G \to 1 + m_K$ be a 1-cocycle, and put $\delta = \max\{|f(g) - 1| : g \in G\} < 1$. If we view $f$ as an element of $K^G$, its image in $K^{G^2}$ has supremum norm at most $\delta^2$. By the previous paragraph, we can modify $f$ by an element of $1 + m_K$ of norm at most $\delta^{1/2}$ to get a new multiplicative cocycle $f'$ such that $\max\{|f'(g) - 1| : g \in G\} \leq \delta^{3/2}$. By iterating the construction, we obtain the desired conclusion.

Proposition 3.5.9. Let $E/F$ be a finite extension of analytic fields such that $E$ is perfectoid. Then $F$ is also perfectoid.

Proof. By Theorem 3.5.6 we are free to enlarge $E$, so we may assume $E/F$ is Galois with group $G$. Let $(L, I)$ be the pair corresponding to $E$ via Theorem 3.5.3 then $G$ acts on both $L$ and $I$. 

66
We first check that $I$ admits a $G$-invariant generator (this is immediate if $F$ is already known to contain a perfectoid field, but not otherwise). Let $z \in I$ be any generator. Write $z$ as $[\overline{z}] + pz_1$ with $z_1 \in W(\mathfrak{o}_E)^\times$; then $z_1^{-1}z$ is also a generator. Define the function $f : G \to W(\mathfrak{o}_E)^\times$ taking $g \in G$ to $f(z_1^{-1}z)/(z_1^{-1}z)$. The composition $G \to W(\mathfrak{o}_E)^\times \to W(\mathfrak{k}_E)^\times$ is identically 1, so we may apply Lemma 3.5.8 to trivialize the 1-cocycle; that is, there exists $y \in W(\mathfrak{o}_E)^\times$ with $f(g) = g(y)/y$ for all $g \in G$. Then $(yz_1)^{-1}z$ is a $G$-invariant generator of $I$.

Put $K = L^G$; by Artin’s lemma, $L$ is Galois over $K$ of degree $#G = [E : F]$. Since $I$ admits a generator contained in $W(\mathfrak{o}_K)^\times$ (which is then primitive of degree 1), we may apply Theorem 3.5.6 to the pair $(K, I \cap W(\mathfrak{o}_K))$ to obtain a perfectoid field $F'$. By Lemma 3.5.4 $E$ is Galois over $F'$ of degree $[L : K] = [E : F]$ with Galois group $G$; consequently, $F' = E^G = F$. This proves the claim.

**Definition 3.5.10.** An analytic field $K$ is *deeply ramified* if for any finite extension $L$ of $K$, $\mathfrak{O}_{\mathfrak{O}_K}^L = 0$; that is, the morphism $\text{Spec}(\mathfrak{O}_L) \to \text{Spec}(\mathfrak{O}_K)$ is formally unramified. (Beware that this morphism is usually not of finite type if $K$ is not discretely valued.)

**Theorem 3.5.11.** Any perfectoid analytic field is deeply ramified. (The converse is also true; see [44, Proposition 6.6.6].)

**Proof.** Let $F$ be a perfectoid field, and let $E$ be a finite extension of $F$. Since $E/F$ is separable, $\Omega_{E/F} = 0$; it follows easily that $\Omega_{\mathfrak{O}_E/\mathfrak{O}_F}$ is killed by some nonzero element of $\mathfrak{O}_F$. On the other hand, since $E$ is perfectoid by Theorem 3.5.6 for any $x \in \mathfrak{O}_E$, we can find $y \in \mathfrak{O}_E$ for which $x \equiv y^p \pmod{p}$. Hence $\Omega_{\mathfrak{O}_E/\mathfrak{O}_F} = p\Omega_{\mathfrak{O}_E/\mathfrak{O}_F}$; it now follows that $\Omega_{\mathfrak{O}_E/\mathfrak{O}_F} = 0$. 

**Remark 3.5.12.** Many cases of Theorem 3.5.6 in which $F$ is the completion of an algebraic extension of $\mathbb{Q}_p$ arise from the *field of norms* construction of Fontaine and Wintenberger [44] [100]. For instance, one may take any *arithmetically profinite* extension of $\mathbb{Q}_p$ thanks to Sen’s theory of ramification in $p$-adic Lie extensions [93]. The approach to Theorem 3.5.6 instead requires only checking the perfectoid condition for a single analytic field, as then it is transmitted along finite extensions. For example, for $F$ the completion of $\mathbb{Q}_p(\mu_{p^\infty})$, the perfectoid condition is trivial to check.

**Remark 3.5.13.** Theorems 3.5.6 and 3.5.11 have also been obtained by Scholze [91] using an analysis of valuation rings made by Gabber and Ramero [44, Chapter 6] in the language of *almost ring theory*. This generalizes the alternate proof of the Fontaine-Wintenberger theorem introduced by Faltings; see [13, Exercise 13.7.4]. Scholze uses the term *tilting* to refer to the relationship between $F$ and $L$, as well as to the corresponding relationship between Banach algebras introduced in Theorem 3.6.5. (The term *perfectoid* is also due to Scholze.)

### 3.6 The perfectoid correspondence for Banach algebras

Once one fixes a pair of analytic fields corresponding as in Theorem 3.5.3 one can describe a corresponding correspondence of Banach algebras. This correspondence will ultimately
be compatible with the étale topology, but we must postpone the discussion of finite étale covers until we have introduced relative Robba rings (see [55]). A parallel but more thorough development, including a construction of perfectoid spaces, has been given by Scholze [91].

The form of the following definition is taken from [25], where the perfectoid condition is studied from a purely ring-theoretic point of view.

**Definition 3.6.1.** A uniform Banach \( \mathbb{Q}_p \)-algebra \( A \) is perfectoid if \( \overline{\varphi} : \mathfrak{o}_A / (p) \rightarrow \mathfrak{o}_A / (p) \) is surjective and there exists \( x \in \mathfrak{o}_A \) with \( x^p \equiv p \pmod{p^2\mathfrak{o}_A} \).

It is worth pointing out some equivalent formulations of the perfectoid property.

**Proposition 3.6.2.** Let \( F \) be an analytic field of mixed characteristics and let \( A \) be a uniform Banach \( F \)-algebra.

(a) If \( F \) is not discretely valued, then \( A \) is perfectoid if and only if \( \overline{\varphi} : \mathfrak{o}_A / (p) \rightarrow \mathfrak{o}_A / (p) \) is surjective.

(b) The field \( F \) is perfectoid as a uniform Banach \( \mathbb{Q}_p \)-algebra as per Definition 3.6.1 if and only if it is perfectoid as an analytic field as per Definition 3.5.1.

(c) If \( F \) is not discretely valued, then \( A \) is perfectoid if and only if there exists \( c \in (0, 1) \) such that for every \( x \in A \), there exists \( y \in A \) with \( |x - y^p| \leq c|x| \). Moreover, if this holds for some \( c \), it holds for any \( c \in (p^{-1}, 1) \).

**Proof.** To check (a), assume that \( \overline{\varphi} : \mathfrak{o}_A / (p) \rightarrow \mathfrak{o}_A / (p) \) is surjective. Since \( F \) is not discretely valued, there exists \( a \in F \) with \( p^{-1} < |a| < 1 \). By hypothesis, there exist \( b, c \in \mathfrak{o}_A \) with \( b^p \equiv a \pmod{p\mathfrak{o}_A} \), \( c^p \equiv (p/a) \pmod{p\mathfrak{o}_A} \). In particular, \( b^p/a \) and \( ac^p/p \) are elements of \( \mathfrak{o}_A \) and so are units. We can thus find \( d \in \mathfrak{o}_A \) with \( dp^p \equiv (b^p/a)^{-1}(ac^p/p)^{-1} \pmod{p\mathfrak{o}_A} \), and then \( x = bcd \) has the property that \( x^p \equiv p \pmod{p^2\mathfrak{o}_A} \). Hence \( A \) is perfectoid. This yields (a), from which (b) follows by taking \( A = F \).

To check (c), note that the given condition (for any given \( c \in (0, 1) \)) implies that \( \overline{\varphi} : A/pA \rightarrow A/pA \) is surjective using Remark 3.4.9 and the hypothesis that \( F \) is not discretely valued. (Namely, choose \( x \in F \) with \( c < |x| < 1 \), so that there exists \( y \in A \) with \( |x - y^p| \leq c|x| \) and so \( y^p \equiv x \pmod{x^2} \).) By (a), this implies that \( A \) is perfectoid. Conversely, if \( A \) is perfectoid, then for every nonnegative integer \( m \), we can find \( x_m \in A \) with \( x_m^m/p \in \mathfrak{o}_A^\times \). Given \( x \in A \) nonzero and \( c \in (p^{-1}, 1) \), choose a nonnegative integer \( m \) and an integer \( t \) such that \( p^{-1}/c < |x/x^t| \leq 1 \). Since \( A \) is perfectoid, we can find \( w \in \mathfrak{o}_A \) with \( w^p \equiv (x/x^t)^{m} \pmod{p} \); then \( y = x^t_m w \) satisfies \( |x - y^p| \leq c|x| \).

**Lemma 3.6.3.** Let \( A \) be a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra. Then the homomorphism \( \theta : W(\mathfrak{o}_A^{\text{rep}}) \rightarrow \mathfrak{o}_A \) is surjective, with kernel generated by an element \( z \) which is primitive of degree 1.

**Proof.** The map \( \theta \) is surjective because \( \overline{\varphi} \) is surjective on \( \mathfrak{o}_A / (p) \). To produce a primitive element, start with \( x \in \mathfrak{o}_A \) with \( x^p \equiv p \pmod{p^2} \). Choose \( \overline{\varphi} = (\ldots, \overline{z}_1, \overline{z}_0) \in \mathfrak{o}_A^{\text{rep}} \) with \( \overline{z}_1 \)
equal to the reduction of \( x \) modulo \( p \); then \( \theta([x]) \equiv p \pmod{p^2} \). For \( n \geq 1 \), choose \( z_n \in o_A \)
lifting \( z_n \); then \( z_n^p \equiv p \pmod{p^2} \), so \( z_n \in A^{\times} \) and \( |z_n y| = p^{-n} |y| \) for all \( y \in A \).

Let \( \overline{y} = (\ldots, \overline{y}_1, \overline{y}_0) \) be an arbitrary element of \( o_A^{\text{rep}} \). For \( n \geq 1 \), choose \( y_n \in o_A \) lifting \( \overline{y}_n \). By the previous paragraph, \( |y_n z_n| = |y_n| |z_n| \), so \( \overline{y} \overline{z} = [\overline{y}] [\overline{z}] \). On the other hand, if \( |\overline{y}| \leq p^{-1} \), then the elements \( y_n/z_n \in o_A \) define an element of \( \varprojlim o_A / (y) \), and hence of \( o_A^{\text{rep}} \) by Lemma 3.4.4. Hence \( \overline{y} \) is divisible by \( \overline{z} \); we may thus argue as in the proof of Lemma 3.5.2(d) to produce an element which is primitive of degree 1. \( \Box \)

**Definition 3.6.4.** Let \( A \) be a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra. By Lemma 3.6.3, the kernel of \( \theta : W(o_A^{\text{rep}}) \to o_A \) is generated by some element \( z \) which is primitive of degree 1. Let \( \overline{z} \in o_A^{\text{rep}} \) be the reduction of \( A \), and define \( R(A) = o_A^{\text{rep}}[\overline{z}^{-1}] \). By Lemma 3.4.5 this is a perfect uniform Banach \( \mathbb{F}_p \)-algebra. Note that this construction does not depend on the choice of \( z \): if \( z' \) is another element of \( \ker(\theta) \) which is primitive of degree 1, with reduction \( \overline{z}' \in o_A^{\text{rep}} \), then \( \overline{z}' \) is divisible by \( \overline{z} \) since \( |\overline{z}'| \leq p^{-1} \), and vice versa. Also write \( I(A) = \ker(\theta) = zW(o_A^{\text{rep}}) \).

For \( R \) a perfect uniform Banach \( \mathbb{F}_p \)-algebra and \( I \) an ideal of \( W(o_R) \) generated by an element \( z \) which is primitive of degree 1, write \( A(R, I) = (W(o_R)/I)[\overline{z}^{-1}] \). By Lemma 3.3.6 the surjective map \( W(o_R)[[\overline{z}^{-1}]] \to A(R, I) \) is optimal, so \( A(R, I) \) is a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra. We may write \( A(R) \) if the choice of \( I \) is to be understood.

**Theorem 3.6.5.** The functors

\[
A \mapsto (R(A), I(A)), \quad (R, I) \mapsto A(R, I)
\]

define an equivalence of categories between perfectoid uniform Banach \( \mathbb{Q}_p \)-algebras \( A \) and pairs \((R, I)\) in which \( R \) is a perfect uniform Banach \( \mathbb{F}_p \)-algebra and \( I \) is a principal ideal of \( W(o_R) \) generated by an element which is primitive of degree 1.

**Proof.** Given \( A \), the surjectivity of \( \theta : W(o_A^{\text{rep}}) \to o_A \) provides a natural isomorphism \( A(R(A), I(A)) \cong A \). Conversely, given \((R, I)\), note that \( o_{A(R,I)} = W(o_R)/I \), so \( o_{A(R,I)}/(p) = W(o_R)/(p, I) = o_R/\overline{z} \) for any \( z \in I \) which is primitive of degree 1. This yields a natural isomorphism \( o_{A(R,I)}^{\text{rep}} \cong o_R \), under which \( I(A(R, I)) \subset W(o_{A(R,I)}^{\text{rep}}) \) corresponds to \( I \subset W(o_R) \). \( \Box \)

**Remark 3.6.6.** For \( A \) and \((R, I)\) corresponding as in Theorem 3.6.5, \( \text{Spec}(R) \) and \( \text{Spec}(A) \) have the same closed-open subsets thanks to Remark 2.6.3. However, there is no reason to expect \( \text{Spec}(R) \) and \( \text{Spec}(A) \) to have the same irreducible components.

For some applications, it will be useful to have the following refinement of the criterion from Proposition 3.6.2(c).

**Corollary 3.6.7.** Let \( A \) be a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra with norm \( |\cdot| \), and let \( R \) be the perfect uniform Banach \( \mathbb{F}_p \)-algebra corresponding to \( A \) via Theorem 3.6.5. Then for any \( \epsilon > 0 \), any nonnegative integer \( m \), and any \( x \in A \), there exists \( y \in A \) of the form \( \theta([\overline{y}]) \) for some \( \overline{y} \in R \), such that

\[
\beta(x - y^p) \leq \max\{p^{-1-p^{-1}\cdots-p^{-m}}\beta(x), \epsilon\} \quad (\beta \in M(A)). \tag{3.6.7.1}
\]
In particular, we may choose \( y \) such that
\[
|x - y^p| \leq p^{-(p-1) - p^{-1} - \cdots - p^{-m}} |x|.
\] (3.6.7.2)

Proof. Lift \( x \) along \( \theta \) to \( \tilde{x} \in W(\mathfrak{o}_R)[[z]^{-1}] \) and then apply Lemma 3.3.9 with \( \tilde{x} \) playing the role of \( x \). Let \( \sum_{n=0}^{\infty} p^n [z_n] \) be the resulting element of \( W(\mathfrak{o}_R)[[z]^{-1}] \); then \( y = \theta([z_0]_p) \) satisfies (3.6.7.1). To obtain (3.6.7.2), take \( y = 0 \) if \( x = 0 \), and otherwise apply (3.6.7.1) with \( \epsilon \) equal to the right side of (3.6.7.2).

Remark 3.6.8. The constant in (3.6.7.2) cannot be improved to \( p^{-p/(p-1)} \). See [25, Example 6.8].

Using Theorem 3.6.5, we may replicate the conclusions of Remark 3.1.5 with perfect uniform Banach \( \mathbb{F}_p \)-algebras replaced by perfectoid algebras, in the process obtaining compatibility of the correspondence described in Theorem 3.6.5 with various natural operations on Banach rings. We begin with a correspondence between strict maps that includes an analogue of Remark 3.1.5(a) in characteristic 0.

Proposition 3.6.9. Keep notation as in Theorem 3.6.5.

(a) Let \( \overline{\psi} : R \to S \) be a strict (and hence almost optimal, by Remark 3.1.5) homomorphism of perfect uniform Banach \( \mathbb{F}_p \)-algebras, and apply the functor \( A \) to obtain \( \psi : A \to B \). Then \( \psi \) is almost optimal (and surjective if \( \psi \) is).

(b) Let \( \overline{\psi}_1, \overline{\psi}_2 : R \to S \) be homomorphisms of perfect uniform Banach \( \mathbb{F}_p \)-algebras, and apply the functor \( A \) to obtain \( \psi_1, \psi_2 : A \to B \). If \( \overline{\psi}_1 - \overline{\psi}_2 \) is strict surjective (and hence almost optimal, by Remark 3.1.5), then \( \psi_1 - \psi_2 \) is almost optimal and surjective.

(c) Let \( \psi : A \to B \) be a strict homomorphism of perfectoid uniform Banach \( \mathbb{Q}_p \)-algebras. Then \( \psi \) is almost optimal.

(d) With notation as in (c), apply the functor \( R \) to obtain \( \overline{\psi} : R \to S \). Then \( \overline{\psi} \) is also almost optimal (and surjective if \( \psi \) is).

Proof. We first check (a) in case \( \overline{\psi} \) is strict surjective. By Remark 3.1.5, \( \overline{\psi} \) is almost optimal; in particular, every element of \( S \) of norm strictly less than 1 lifts to an element of \( R \) of norm strictly less than 1. By Lemma 3.3.8, every element of \( B \) of norm strictly less than 1 lifts to an element of \( A \) of norm strictly less than 1. Consequently, \( \psi \) is almost optimal and surjective. A similar argument yields (b).

We now check (a) in the general case. We may factor \( \psi \) as a composition \( R \to S_0 \to S \) with \( R \to S_0 \) strict surjective and \( S_0 \to S \) an isometric injection (since \( R \) and \( S \) are uniform). For \( z \) a generator of \( I \), we have \( zW(\mathfrak{o}_S) \cap W(\mathfrak{o}_{S_0}) = zW(\mathfrak{o}_{S_0}) \), so the map \( S_0 \to S \) corresponds to an isometric injection \( B_0 \to B \). We may thus deduce (a) from the previous paragraph.

To check (c), let \( \alpha, \beta \) be the norms on \( A, B \). Choose a constant \( c \geq 1 \) such that every \( b \in \text{image}(\psi) \) admits a lift \( a \in A \) with \( \alpha(a) \leq c \beta(b) \). We will then prove that the same conclusion holds with \( c \) replaced by \( c^{1/p} \); this is enough to imply the desired result.

70
Suppose that \( b_1 \in \text{image}(\psi) \) for some nonnegative integer \( l \). Lift \( b_1^p \) to \( a_1 \in A \) with \( \alpha(a_1) \leq c\beta(b_1^p) = c\beta(b_1)p^p \). Apply Lemma 3.3.9 (or Lemma 5.16) to find \( \bar{x} \in R \) such that
\[
\gamma(a_1 - \theta([x])) \leq p^{-1} \max\{\gamma(a_1), \beta(b_1)^p\} \quad (\gamma \in \mathcal{M}(A)).
\] (3.6.9.1)
In particular,
\[
\gamma(\theta([x])) \leq \max\{\gamma(a_1), p^{-1}\gamma(a_1), p^{-1}\beta(b_1)^p\} \leq c\beta(b_1)^p.
\]
Put \( u_l = \theta([x]) \), \( v_l = \psi(u_l) \), and \( b_{l+1} = b_l - v_l \); note that \( \alpha(u_l) \leq c^{l/p}\beta(b_l) \).

For each \( \gamma \in \mathcal{M}(B) \), by applying (3.6.9.1) to the restriction of \( \gamma \) to \( A \), we find that
\[
\gamma(b_l^p - v_l^p) \leq p^{-1} \max\{\gamma(b_l)^p, \beta(b_l)^p\} = p^{-1}\beta(b_l)^p.
\]
We now consider three cases.

(i) If \( \gamma(b_l^p - v_l^p) > \gamma(b_l)^p \), then \( \gamma(b_l^p - v_l^p) = \gamma(v_l)^p \), so \( \gamma(b_{l+1}) = \gamma(v_l) > \gamma(b_l) \). It follows that \( \gamma(b_{l+1}) = \gamma(b_l^p - v_l^p)^{1/p} \leq p^{-1}\beta(b_l)^p \).

(ii) If \( p^{-1}\gamma(b_l)^p \leq \gamma(b_l^p - v_l^p) \leq \gamma(b_l)^p \), we may apply Lemma 10.2.2 to deduce that
\[
\gamma(b_{l+1}) \leq \gamma(b_l^p - v_l^p)^{1/p} \leq p^{-1}\beta(b_l)^p.
\]
(iii) If \( \gamma(b_l^p - v_l^p) \leq p^{-1}\gamma(b_l)^p \), then by Lemma 10.2.2 again, \( \gamma(b_{l+1}) \leq p^{-1}\gamma(b_l) \leq p^{-1}\beta(b_l)^p \).

It follows that \( \gamma(b_{l+1}) \leq p^{-1}\beta(b_l) \) for all \( \gamma \in \mathcal{M}(B) \), and so \( \beta(b_{l+1}) \leq p^{-1}\beta(b_l) \).

If we now start with \( b = b_0 \in \text{image}(\psi) \) and recursively define \( b_l, u_l, v_l \) as above, the \( b_l \) converge to 0, so the series \( \sum_{l=0}^{\infty} v_l \) converges to \( b \). Meanwhile, the series \( \sum_{l=0}^{\infty} u_l \) converges to a limit \( a \in A \) satisfying \( \psi(a) = b \) and \( \alpha(a) \leq c^{1/p}\beta(b) \). This proves (c).

To obtain (d), by noting that a strict injection of uniform Banach rings is isometric, we may reduce to the case where \( \psi \) is strict surjective. Let \( \bar{x}, \bar{y} \) be the norms on \( R, S \). Given \( \gamma \in S, \) by (c) we may lift \( \theta(\gamma) \in B \) to some \( a \in A \) with \( \alpha(a) \leq p^{1/2}\beta(\gamma) \). By Lemma 3.3.9 again, we may find \( \bar{x} \in R \) such that \( \gamma(a - \theta([x])) \leq p^{-1}\max\{\gamma(a), \beta(\gamma)\} \) for all \( \gamma \in \mathcal{M}(A) \); in particular,
\[
\gamma(\theta([x])) \leq p^{-1}\max\{\gamma(a), \beta(\gamma)\} \leq p^{1/2}\beta(\gamma).
\]
Let \( \bar{x} \in S \) be the image of \( \bar{x} \). For all \( \gamma \in \mathcal{M}(B) \), we have \( \gamma(\theta([x])) \leq p^{1/2}\beta(\gamma) \) and
\[
\gamma(\theta([y] - [z])) \leq p^{-1}\max\{\gamma(\theta([y])), \beta(\gamma)\} = p^{-1}\beta(\gamma).
\]
Put \( \gamma = \mu(\theta(\gamma)) \in \mathcal{M}(S) \); then \( \gamma(z) \leq p^{1/2}\beta(\gamma) \). If we expand \( \gamma(z) - [z] = \sum_{i=0}^{\infty} p^i \omega_i \), for \( i > 0 \) we have \( \gamma(\theta(p^i\omega_i)) \leq p^{-1}\max\{\gamma(\omega_i), \gamma(\omega_i^*)\} \leq p^{-1}\beta(\gamma) \). Since \( \omega_0 = \gamma - \gamma(z) \), it follows that \( \gamma(\gamma - \gamma(z)) = \gamma(\theta(\gamma - \gamma(z))) \leq p^{-1}\beta(\gamma) \). By iterating the construction as in the proof of (b), we see that every \( \bar{y} \in S \) admits a lift \( \bar{x} \in R \) for which \( \bar{x}(\bar{y}) \leq p^{1/2}\beta(\gamma) \). Hence \( \psi \) is strict, and hence almost optimal by Remark 3.1.5.

We thus may deduce (d) except for the fact that if \( \psi \) is surjective, then so is \( \bar{\psi} \). This follows from (c) plus Lemma 3.4.7 (b).

Remark 3.6.10. Note that in Proposition 3.6.9 (a), strict surjectivity of \( \bar{\psi} \) does not imply that \( \sigma_R \) surjects onto \( \sigma_S \). Similarly, in part (d), strict surjectivity of \( \psi \) does not imply that \( \sigma_A \) surjects onto \( \sigma_B \).
We next establish compatibility of the correspondence with completed tensor products, and obtain an analogue of Remark 3.1.5(c).

Proposition 3.6.11. Let \( A \to B, A \to C \) be morphisms of perfectoid uniform Banach \( \mathbb{Q}_p \)-algebras. Let \( (R,I) \) be the pair corresponding to \( A \) via Theorem 3.6.7, and put \( S = R(B), T = R(C) \). Then the completed tensor product \( B \hat{\otimes}_A C \) with the tensor product norm is the perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra corresponding to \( S \hat{\otimes}_R T \).

Proof. Put \( U = S \hat{\otimes}_R T \), which is a perfect uniform Banach \( \mathbb{F}_p \)-algebra by Remark 3.1.5(c). Put \( D = A(U) \); then \( D \) is perfectoid. It remains to check that the natural map \( B \hat{\otimes}_A C \to D \) is an isometric isomorphism of Banach \( \mathbb{Q}_p \)-algebras. To see this, let \( \alpha, \beta, \gamma, \delta, \xi, \beta, \gamma, \delta \) denote the norms on \( A, B, C, D, R, S, T, U \), respectively. Choose any \( x \in D \) and any \( \epsilon > 1 \), and apply Lemma 3.3.6 to find \( y = \sum_{n=0}^{\infty} p^n [y_n] \in W(a_U)[[\mathfrak{a}]^{-1}] \) with \( \theta(y) = x \) and \( \delta(y_0) \geq \delta(y_n) \) for all \( n > 0 \). Then write each \( y_n \) as a convergent sum \( \sum_{i=0}^{\infty} \alpha_{ni} \otimes \gamma_{ni} \) with \( \alpha_{ni} \in S, \gamma_{ni} \in T \) and \( \beta(\alpha_{ni}), \gamma(\gamma_{ni}) < (c\delta(y_n))^{1/2} \). (More precisely, by Remark 3.1.5(c) we can ensure that \( \beta(\alpha_{ni}) \alpha(\gamma_{ni}) < c\delta(y_n) \), but then we can enforce the desired inequality by transferring a suitable power of \( \xi \) between the two terms.) We can then write \( [y_n] \) as a convergent sum for the \((p,z)\)-adic topology, each term of which is a power of \( p \) times the Teichmüller lift of an element of \( S \) times the Teichmüller lift of an element of \( T \); moreover, each of these terms has norm at most \( c\delta(y_n) \). It follows that \( x \) is the image of an element of \( B \hat{\otimes}_A C \) of norm at most \( c\delta(y) \); since \( \epsilon > 1 \) was arbitrary, this yields the desired result.

Remark 3.6.12. At this point, we have analogues for perfectoid algebras of parts (a) and (c) of Remark 3.1.5. It would be useful to also have an analogue of part (b); that is, if \( \psi_1, \psi_2 : A \to B \) are two homomorphisms of perfectoid uniform Banach \( \mathbb{Q}_p \)-algebras such that \( \psi = \psi_1 - \psi_2 \) is strict, one would expect that \( \psi \) is almost optimal. Unfortunately, the technique of proof of Proposition 3.6.9(c) does not suffice to establish this, due to the fact that the image of \( \psi \) is not closed under taking \( p \)-th powers.

We next establish compatibility with rational localizations and an analogue of Remark 3.1.6, stating the latter first.

Remark 3.6.13. Let \( A \) be a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra, and put \( R = R(A) \). Let \( A \to B \) be the rational localization representing a rational subdomain of \( \mathcal{M}(A) \) of the form \( U = \{ \beta \in \mathcal{M}(A) : \beta(f_1) \leq p_1 \beta(g), \ldots, \beta(f_n) \leq p_n \beta(g) \} \) for some \( f_1, \ldots, f_n, g \in A \) and \( p_1, \ldots, p_n > 0 \) such that \( g \) is invertible in \( A \). By Theorem 3.6.14, \( A \to B \) corresponds to a rational localization \( R \to S \). More precisely, this localization represents the rational subdomain of \( \mathcal{M}(R) \) given by \( V = \{ \beta \in \mathcal{M}(R) : \beta(f_1) \leq p_1 \beta(g), \ldots, \beta(f_n) \leq p_n \beta(g) \} \) for any \( f_1, \ldots, f_n, g \in R \) such that \( \theta([f_1]), \ldots, \theta([f_n]), \theta([g]) \) are sufficiently close to \( f_1, \ldots, f_n, g \), respectively.

Using the first part of Remark 3.1.6, we may write \( B \) as an almost optimal quotient of the completion of \( A\{U_1/p_1, \ldots, U_n/p_n\} [U_1^{-\infty}, \ldots, U_n^{-\infty}] \) by the ideal generated by \( U_i^{p_i-h} \theta([f_i]^{-h}) - \theta([g]^{-h}) \) for \( i = 1, \ldots, n \) and \( h = 0, 1, \ldots \). This implies that \( B \) is also perfectoid.

By imitating the second part of Remark 3.1.6, we may rewrite \( B \) as a strict quotient of \( A\{U_1/p_1, \ldots, U_n/p_n\} \) by the ideal generated by \( U_i \theta([f_i]) - \theta([g]) \) for \( i = 1, \ldots, n \). This
quotient is in turn isomorphic to the quotient of $A\{T_1/p_1, ..., T_n/p_n\}$ by the ideal generated by $T_if_i - g$ for $i = 1, ..., n$. That is, the natural homomorphism $A\{T_1/p_1, ..., T_n/p_n\} \to B$ taking $T_i$ to $f_i/g$ is a strict surjection with kernel generated by $f_1T_1 - g_1, ..., f_nT_n - g_n$.

**Theorem 3.6.14.** Suppose that $A$ and $(R, I)$ correspond as in Theorem 3.6.3. Choose a (strictly) rational subdomain $U$ of $\mathcal{M}(R)$, and let $V$ be the (strictly) rational subdomain of $\mathcal{M}(A)$ corresponding to $U$ via the homeomorphism $\mathcal{M}(R) \cong \mathcal{M}(A)$ of Theorem 3.3.7. Let $R \to S$, $A \to B$ be the rational localizations representing $U$ and $V$, respectively. Then $B$ is again perfectoid, and there are natural identifications $S \cong R(B)$, $B \cong A(S, IW(\mathfrak{o}_S))$.

**Proof.** In what follows, we omit mention of the ideal parameter in the functor $A$, since it will always be the extension of the originally specified ideal $I$. Note that $B$ is perfectoid by Remark 3.6.13. Applying the functors $R$ and then $A$ to the natural map $A \to B$ gives a map $A \cong A(R(A)) \to A(R(B))$; composing this map with $\theta : A(R(B)) \to B$ gives back the original map. By Theorem 3.3.7 the image of $\mathcal{M}(A(R(B)))$ in $\mathcal{M}(A)$ is the same as the image of $\mathcal{M}(B)$, so we may invoke the universal property of the rational localization $A \to B$ to obtain a factorization $A \to B \to A(R(B))$. The resulting composition $A \to B \to A(R(B)) \to B$ is again the given map, so by the uniqueness aspect of the universal property, $B \to A(R(B)) \to B$ must be the identity map. It follows that $\theta : A(R(B)) \to B$ is surjective.

On the other hand, it is a bounded homomorphism of uniform Banach algebras inducing a homeomorphism on spectra, so it is isometric by Theorem 2.3.10 and hence injective. We thus conclude that $A(R(B)) \cong B$, so in particular, $\varphi$ is surjective on $\mathfrak{o}_B/(p)$. Consequently, $B$ appears in the correspondence defined by Theorem 3.6.5, as does $S$ since it is perfect uniform. It thus suffices to match them up on one of the two sides of the correspondence.

By the universal property of $B$, the map $A = A(R) \to A(S)$ factors through $B$. By applying $R$, we obtain a factorization $R = R(A) \to R(B) \to R(A(S)) = S$. The map $R(B) \to S$ is a bounded homomorphism of uniform Banach algebras inducing a homeomorphism on spectra, so it is again injective.

On the other hand, by the universal property of $S$, we may factor $R \to R(B)$ uniquely as $R \to S \to R(B)$. Inserting this into the previous factorization yields a sequence of maps $R \to S \to R(B) \to S$ whose composition is the original map $R \to S$. By the universal property of $S$ again, $S \to R(B) \to S$ must be the identity map on $S$, so $R(B) \to S$ is also surjective. We thus identify $R(B)$ with $S$, completing the proof.

As a corollary, we obtain the Tate and Kiehl properties for perfectoid algebras.

**Theorem 3.6.15.** Any perfectoid Banach algebra satisfies the Tate sheaf and Kiehl glueing properties (Definition 2.7.6).

**Proof.** To prove the Tate sheaf property, we may use Proposition 2.4.15 and Proposition 3.6.11 to reduce to the case of a simple Laurent covering of $\mathcal{M}(R)$. Let $R \to R_1, R \to R_2$ be rational localizations defining such a covering and put $R_{12} = R_1 \hat{\otimes}_R R_2$. By Theorem 3.6.14 we may apply the functor $A$ to $R \to R_1, R \to R_2, R \to R_{12}$ to obtain rational localizations $A \to A_1, A \to A_2, A \to A_{12}$. As in the proof of Proposition 3.6.9(a,b), the strict exactness of the sequence $0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0$ implies strict exactness of the sequence...
0 \rightarrow A \rightarrow A_1 \oplus A_2 \rightarrow A_{12} \rightarrow 0$. This proves the Tate sheaf property; the Kiehl glueing property now follows by Proposition 2.7.4.

We next establish compatibility with formation of quotients. It will be useful later to consider not just surjective homomorphisms, but also homomorphisms with dense image.

**Theorem 3.6.16.** Suppose that $A$ and $(R, I)$ correspond as in Theorem 3.6.5.

(a) Let $\overline{\psi} : R \rightarrow S$ be a bounded homomorphism of uniform Banach $F_p$-algebras with dense image. Then $S$ is also perfect, and the corresponding homomorphism $\psi : A \rightarrow B$ of perfectoid uniform Banach $F_p$-algebras also has dense image.

(b) Let $B$ be a uniform Banach $\mathbb{Q}_p$-algebra admitting a bounded homomorphism $\psi : A \rightarrow B$ with dense image. Then $B$ is also perfectoid, and the corresponding homomorphism $\overline{\psi} : R \rightarrow S$ of perfect uniform Banach $F_p$-algebras also has dense image.

(c) In (a) and (b), $\overline{\psi}$ is surjective if and only if $\psi$ is.

**Proof.** In the setting of (a), the ring $S$ is reduced and admits the dense perfect $F_p$-subalgebra $\overline{\psi}(R)$, so $S$ is also perfect. Let $\alpha, \beta, \overline{\alpha}, \overline{\beta}$ denote the norms on $A, B, R, S$, respectively. By Lemma 3.3.6, for any $x \in B$ and $\epsilon > 0$, we can find a finite sum $\sum_{i=0}^{n} p^i [\overline{x}_i] \in W(S)$ such that $\beta(x - \theta(\sum_{i=0}^{n} p^i [\overline{x}_i])) < \epsilon$. For $i = 0, \ldots, n$, if $\overline{x}_i = 0$, then $\overline{y}_i = 0$, otherwise choose $\overline{y}_i \in R$ with

$$
\overline{\beta}(\overline{x}_i - \overline{\psi}(\overline{y}_i)) < \inf \{ e^{p^i (i+j)p^j} \beta(\overline{x}_i) : j = 0, 1, \ldots \}.
$$

(Note that the sequence whose infimum is sought tends to $+\infty$ as $j \rightarrow \infty$, since it is dominated by $p^{jp^i}$, so the infimum is positive.) Put $y = \sum_{i=0}^{n} p^i \theta([\overline{y}_i]) \in A$; then

$$
\beta \left( \sum_{i=0}^{n} p^i \theta([\overline{x}_i] - \overline{\psi}(\overline{y}_i)) \right) \leq \max \{ p^{-i} \beta(\theta([\overline{x}_i] - \overline{\psi}(\overline{y}_i))) : i = 0, \ldots, n \}
$$

$$
\leq \max \{ p^{-i-j} \beta(\overline{x}_i) : i = 0, \ldots, n; j = 0, 1, \ldots \} < \epsilon.
$$

It follows that $\beta(x - \psi(\sum_{i=0}^{n} p^i [\overline{y}_i])) < \epsilon$, yielding (a).

In the setting of (b), let $\alpha, \beta$ be the norms on $A, B$. Given $x \in \psi(A) \cap \mathfrak{o}_B$, choose $w \in \psi^{-1}(x)$. By Lemma 3.3.9, we can find $\overline{w} \in R$ such that

$$
\gamma(w - \theta([\overline{w}])) \leq p^{-1} \max \{ \gamma(w), \beta(x) \} \quad (\gamma \in \mathcal{M}(A)).
$$

Put $y = \psi(\theta([\overline{w}^1/p^1]));$ then $\gamma(x - y^p) \leq p^{-1} \beta(x)$ for all $\gamma \in \mathcal{M}(B)$, so $\beta(x - y^p) \leq p^{-1} \beta(x)$. Since $\psi(A)$ is dense in $B$, $B$ is perfectoid. Given $\overline{x} \in S$, choose $w \in A$ with $\beta(\psi(w) - \theta([\overline{x}])) \leq p^{-1} \beta(x)$, then apply Lemma 3.3.9 again to choose $\overline{w} \in R$ such that

$$
\gamma(w - \theta([\overline{w}])) \leq p^{-1} \max \{ \gamma(w), \overline{\beta}(\overline{x}) \} \quad (\gamma \in \mathcal{M}(A)).
$$

Put $\overline{y} = \overline{\psi}(\overline{w});$ then $\gamma(\theta([\overline{x}] - [\overline{y}])) \leq p^{-1} \overline{\beta}(\overline{x})$ for $\gamma \in \mathcal{M}(B)$, so $\gamma(\overline{x} - \overline{y}) \leq p^{-1} \overline{\beta}(\overline{x})$. This yields (b).

In the setting of (c), if either $\overline{\psi}$ or $\psi$ is surjective, then it is strict by the open mapping theorem (Theorem 2.2.8). Consequently, (c) follows from Proposition 3.6.9(a,d).  \[\Box\]
Corollary 3.6.17. Let \( A \to B, B \to C, B \to D \) be bounded homomorphisms of uniform Banach \( \mathbb{Q}_p \)-algebras, and let \( E \) denote the uniform completion of \( C \otimes_B D \). If \( A, C, D \) are perfectoid, then so is \( E \).

Proof. By Proposition 3.6.11 \( C \hat{\otimes} A D \) is perfectoid. Since there is a natural map \( C \hat{\otimes} A D \to E \) having dense image, the claim then follows from Theorem 3.6.16(b).

A related observation is that the perfectoid property is preserved under completions.

Proposition 3.6.18. Let \( A \) be a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra. Let \( J \) be a finitely generated ideal of \( \mathcal{O}_A \) which contains \( p \). Equip each quotient \( \mathcal{O}_A / J_n \) with the quotient norm, equip the inverse limit \( R \) with the supremum norm, and put \( B = R[p^{-1}] \). Then \( B \) is again a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra.

Proof. Choose generators \( x_1, \ldots, x_m \) of \( J \); then \( R \) can also be written as the inverse limit of the quotients \( \mathcal{O}_A / (p^n, x_1^{p^n}, \ldots, x_m^{p^n}) \). Consequently, each element \( y \) of \( R \) can be written as an infinite series

\[
\sum_{n=0}^{\infty} (a_{n0} p^n + a_{n1} x_1^{p^n} + \cdots + a_{nm} x_m^{p^n})
\]

with all of the \( a_{ni} \) in \( \mathcal{O}_A \). Choose \( b_{ni} \in \mathcal{O}_A \) with \( b_{ni} \equiv a_{ni} \pmod{p} \); then the series

\[
b_{n0} + \sum_{n=0}^{\infty} (b_{n1} p^n x_1^n + \cdots + b_{nm} p^n x_m^n)
\]

converges to an element \( z \) of \( R \) satisfying \( z^p \equiv y \pmod{pR} \). From this, the claim follows at once.

We finally establish compatibility with finite étale covers. As in the case of analytic fields, the first step is to lift from characteristic \( p \). One can give an alternate proof of this result using relative Robba rings; see Remark 5.5.7.

Lemma 3.6.19. For \( A \) and \((R, I)\) corresponding as in Theorem 3.6.5 and \( S \in \text{FÉt}(R) \), the perfectoid Banach algebra \( B \) over \( \mathbb{Q}_p \) corresponding to \((S, IW(\mathcal{O}_S))\) via Theorem 3.6.7 belongs to \( \text{FÉt}(A) \) and is a finite Banach module over \( A \).

Proof. We may assume that \( S \) is of constant rank \( d > 0 \) as an \( R \)-module. Let \( z \) be a generator of \( I \) which is primitive of degree 1. Since \( S \) is a finite \( R \)-module and \( \overline{\varphi} \) is bijective on \( S \), we can find \( \overline{\tau}_1, \ldots, \overline{\tau}_n \in \mathcal{O}_S \) such that \( \mathcal{O}_S / (\overline{\tau}_1 \mathcal{O}_R + \cdots + \overline{\tau}_n \mathcal{O}_R) \) is killed by \( \overline{\tau}_1 \). Using Remark 3.2.3 it follows that \( W(\mathcal{O}_S) / ([\overline{\tau}_1] W(\mathcal{O}_R) + \cdots + [\overline{\tau}_n] W(\mathcal{O}_R)) \) is killed by \( [\overline{\tau}_1] \). Quotienting by \( z \) and then inverting \( p \), we find that \( B \) is a finite \( A \)-module. By Lemma 2.3.12(b) and Lemma 3.5.4 \( B \) is locally free of constant rank \( d \) as an \( A \)-module.

We now know that \( B \) is a finite projective \( A \)-module. To check that \( B \in \text{FÉt}(A) \), it remains to check that the map \( B \to \text{Hom}_A(B, A) \) taking \( x \) to \( y \mapsto \text{Trace}_{B/A}(xy) \) is surjective. By Lemma 2.3.12(a), it suffices to check this pointwise; we may thus apply Lemma 3.5.4 to conclude.
Finally, note that since $B$ is a Banach ring and a finite $A$-module, and $A$ is a Banach module over a nontrivially normed analytic field, $B$ is a finite Banach $A$-module by the open mapping theorem (Theorem 2.2.8).

**Theorem 3.6.20.** For $A$ and $(R, I)$ corresponding as in Theorem 3.6.5, for each $B \in \mathbf{F\acute{e}t}(A)$, the Banach norm on $B$ provided by Lemma 2.2.12 is equivalent to a power-multiplicative norm, under which $B$ is a perfectoid uniform Banach $F$-algebra. Consequently, the correspondence of Lemma 3.6.19 induces a tensor equivalence $\mathbf{F\acute{e}t}(A) \cong \mathbf{F\acute{e}t}(R)$. (As in Theorem 3.5.11, it follows that $\Omega_{oB/oA} = 0$.)

**Proof.** It suffices to check that any $B \in \mathbf{F\acute{e}t}(A)$ arises as in Lemma 3.6.19. Recall that by Theorem 3.3.7 there is a homeomorphism $\mathcal{M}(A) \cong \mathcal{M}(R)$ matching up rational subdomains on both sides. For each $\delta \in \mathcal{M}(A)$, let $\gamma \in \mathcal{M}(R)$ be the corresponding point. Using Theorem 3.5.6 we may transfer $B \otimes_A \mathcal{H}(\delta) \in \mathbf{F\acute{e}t}(\mathcal{H}(\delta))$ to some $S(\gamma) \in \mathbf{F\acute{e}t}(\mathcal{H}(\gamma))$. By Lemma 2.2.3(a) (and Theorem 1.2.8), there exists a rational localization $R \to R_1$ encircling $\gamma$ such that $S(\gamma)$ extends to $S_1 \in \mathbf{F\acute{e}t}(R_1)$. Let $A \to A_1$ be the rational localization corresponding to $R \to R_1$ via Theorem 3.6.14. Applying Lemma 3.6.19, we may lift $S_1$ to $B_1 \in \mathbf{F\acute{e}t}(A_1)$. By Lemma 2.2.3(a) again, by replacing $A \to A_1$ by another rational localization encircling $\delta$, we can ensure that $B_1 \cong B \otimes_A A_1$. In particular, $B \otimes_A A_1$ is perfectoid.

By compactness, we obtain a covering family $A \to A_1, \ldots, A \to A_n$ of rational localizations corresponding to a covering family $R \to R_1, \ldots, R \to R_n$ as in Theorem 3.6.14 such that for each $i$, $B \otimes_A A_i$ corresponds to some $S_i \in \mathbf{F\acute{e}t}(R_i)$ as in Theorem 3.6.5. Put $A_{ij} = A_i \otimes_A A_j$, so that $A \to A_{ij}$ is the rational localization corresponding to $\mathcal{M}(A_i) \cap \mathcal{M}(A_j)$ by Proposition 3.6.11. Let $R \to R_{ij}$ be the corresponding rational localization. Using Theorem 3.6.14 we may transfer the isomorphisms $(B \otimes_A A_i) \otimes_A A_{ij} \cong (B \otimes_A A_j) \otimes_A A_{ij}$ to obtain isomorphisms $S_i \otimes_{R_i} R_{ij} \cong S_j \otimes_{R_j} R_{ij}$ satisfying the cocycle condition. By Theorem 2.6.10 we may glue the $S_i$ to obtain $S \in \mathbf{F\acute{e}t}(R)$.\n
Apply Lemma 3.6.19 to lift $S$ to $C \in \mathbf{F\acute{e}t}(A)$. By Theorem 3.6.5 we have isomorphisms $B \otimes_A A_i \cong C \otimes_A A_i$ which again satisfy the cocycle condition. They thus glue to an isomorphism $B \cong C$ by Theorem 2.6.10 again, so $B$ is perfectoid as desired.

**Remark 3.6.21.** By analogy with Proposition 3.5.9 one might hope that if $A$ and $B$ are uniform Banach $\mathbb{Q}_p$-algebras, $B \in \mathbf{F\acute{e}t}(A)$, and $B$ is perfectoid, then $A$ is also perfectoid. We do not know whether or not this holds; it is probably necessary to assume also that $B$ is a finite Banach $A$-module, as this is not immediate from the other hypotheses. It may also be necessary to assume that $A$ is a Banach algebra over a perfectoid analytic field.

**Remark 3.6.22.** Scholze observes [91] that for $A$ and $(R, I)$ corresponding as in Theorem 3.6.5, Theorem 3.6.14 and Theorem 3.6.20 imply that the small étale sites of the adic spaces associated to $R$ and $A$ are naturally equivalent. This observation is the point of departure of his theory of perfectoid spaces, which casts the aforementioned results in more geometric terms. Besides the expected consequences for relative $p$-adic Hodge theory, as in the study of relative comparison isomorphisms between étale and de Rham cohomology [92],

76
there are some unexpected consequences; for instance, Scholze derives some new cases of the
weight-monodromy conjecture in étale cohomology \[91\]. There may also be consequences
in the direction of Hochster’s direct summand conjecture in commutative algebra, as in the
work of Bhatt \[14\] (see also the discussion in \[45\]).

Remark 3.6.23. While the correspondence described above is sufficient for some applica-
tions (e.g., Remark 3.6.22), relative \(p\)-adic Hodge theory tends to requires somewhat more
refined information. The most common approach to getting this extra information is through
variants of the almost purity theorem of Faltings \[37, 38\]. We will instead take an alternative
approach based on relative Robba rings, as introduced in \[34\]. For the relationship between
the two points of view, see \[5, 5\].

Remark 3.6.24. For \(A\) and \((R, I)\) corresponding as in Theorem 3.6.5 it is straightforward
to check that the map \(\mathfrak{F} \mapsto \theta([\mathfrak{F}])\) induces an isomorphism between \(R\) and the inverse limit
of \(A\) under the \(p\)-th power map (by rescaling to reduce to the corresponding statement for
\(\sigma R\) and \(\sigma A\)). This is the characterization of the perfectoid correspondence used in \[91\].

4 Robba rings and \(\varphi\)-modules

An important feature of the approach to \(p\)-adic Hodge theory used in this series of papers
(and in the work of Berger and others) is the theory of slopes of Frobenius actions on modules
over certain rings. This bears some resemblance to the theory of slopes for vector bundles
on Riemann surfaces, including the relationship of those slopes to unitary representations of
fundamental groups. (A more explicit link to vector bundles appears in the work of Fargues
and Fontaine \[39\]; see \[6.3\]) We review here some of the principal results of the first author
which are pertinent to \(p\)-adic Hodge theory, mostly omitting proofs. Besides serving as a
model, some of these results provide key inputs into our work on relative \(p\)-adic Hodge theory.

4.1 Slope theory over the Robba ring

We begin by introducing several key rings used in \(p\)-adic Hodge theory, and the basic theory
of slopes of Frobenius actions. Our description here is rather minimal; see \[69\] for a more
detailed discussion.

Hypothesis 4.1.1. Throughout \[4.1\] put \(K = \text{Frac}(W(k))\) for some perfect field \(k\) of
characteristic \(p\). Equip \(K\) with the \(p\)-adic norm and the Frobenius lift \(\varphi_K\) induced by Witt
vector functoriality. Fix also a choice of \(\omega \in (0, 1)\).

Definition 4.1.2. For \(r > 0\), put

\[ \mathcal{R}_K^r = \left\{ \sum_{i \in \mathbb{Z}} c_i T^i : c_i \in K, \lim_{i \to \pm \infty} |c_i| \rho^i = 0 \quad (\rho \in [\omega^r, 1)) \right\}. \]

In other words, \(\mathcal{R}_K^r\) consists of formal sums \(\sum_{i \in \mathbb{Z}} c_i T^i\) in the indeterminate \(T\) with coefficients
in \(K\) which converge on the annulus \(\omega^r \leq |T| < 1\). The set \(\mathcal{R}_K^r\) forms a ring under formal
series addition and multiplication; let \( R_{K}^{\text{int},r} \) be the subring of \( R_{K}^{r} \) consisting of series whose coefficients have norm at most 1, and put \( R_{K}^{\text{bd},r} = R_{K}^{\text{int},r}[p^{-1}] \). Put

\[
R_{K}^{\text{int}} = \bigcup_{r > 0} R_{K}^{\text{int},r}, \quad R_{K}^{\text{bd}} = \bigcup_{r > 0} R_{K}^{\text{bd},r}, \quad R_{K} = \bigcup_{r > 0} R_{K}^{r}.
\]

The ring \( R_{K}^{\text{int}} \) is a local ring with residue field \( k((T)) \) which is not complete but is henselian. (See for instance [67, Lemma 3.9]. For a similar argument, see Proposition 5.5.3.) The completion of the field \( R_{K}^{\text{bd}} \) is the field

\[
E_{K} = \left\{ \sum_{i \in \mathbb{Z}} c_{i} T^{i} : c_{i} \in K, \sup_{i} |c_{i}| < +\infty, \lim_{i \to -\infty} |c_{i}| = 0 \right\}.
\]

The units of \( R_{K} \) are precisely the nonzero elements of \( R_{K}^{\text{bd}} \), as may be seen by considering Newton polygons [67, Corollary 3.23].

**Definition 4.1.3.** We will need to consider several different topologies on the rings described above.

(a) Those rings contained in \( E_{K} \) carry both a \( p \)-adic topology (the metric topology defined by the Gauss norm) and a weak topology (in which a sequence converges if it is bounded for the Gauss norm and converges \( T \)-adically modulo any fixed power of \( p \)). For both topologies, \( E_{K} \) is complete.

(b) Those rings contained in \( R_{K}^{r} \) carry a Fréchet topology, in which a sequence converges if and only if it converges under the \( \omega^{r} \)-Gauss norm for all \( s \in (0, r] \). For this topology, \( R_{K}^{r} \) is complete.

(c) Those rings contained in \( R_{K} \) carry a limit-of-Fréchet topology, or LF topology. This topology is defined on \( R_{K} \) by taking the locally convex direct limit (in the sense of [17, §II.4]) of the \( R_{K}^{r} \) (each equipped with the Fréchet topology). In particular, a sequence converges in \( R_{K} \) if and only if it is a convergent sequence in \( R_{K}^{r} \) for some \( r > 0 \).

**Remark 4.1.4.** The convergence of the formal expression \( x = \sum_{i} c_{i} T^{i} \) for various of the topologies described in Definition 4.1.3 is useful for defining operations such as Frobenius lifts (see Definition 4.1.6 below). In \( E_{K} \), the formal sum converges for the weak topology but not the \( p \)-adic topology. In \( R_{K}^{r} \), the sum converges for the Fréchet topology. In \( R_{K} \), the sum converges for the LF topology.

**Remark 4.1.5.** Note that a sequence of elements of \( R_{K}^{\text{bd},r} \) which is \( p \)-adically bounded and convergent under the \( \omega^{r} \)-Gauss norm also converges in the weak topology.

**Definition 4.1.6.** A Frobenius lift \( \varphi \) on \( R_{K} \) is an endomorphism defined by the formula

\[
\varphi \left( \sum_{i \in \mathbb{Z}} c_{i} T^{i} \right) = \sum_{i \in \mathbb{Z}} \varphi_{K}(c_{i}) u^{i}
\]
for some \( u \in \mathcal{R}_{K}^{\text{int}} \) with \( |u - T^p| < 1 \), where the right side may be interpreted as a convergent sum using Remark 4.1.4. Such an endomorphism also acts on \( \mathcal{R}_{K}^{\text{int}}, \mathcal{R}_{K}^{\text{bd}}, \mathcal{E}_{K} \), but not on \( \mathcal{R}_{K}^{r} \) for any individual \( r > 0 \); rather, for \( r > 0 \) sufficiently small, \( \varphi \) carries \( \mathcal{R}_{K}^{r} \) into \( \mathcal{R}_{K}^{r/p} \). The action of \( \varphi \) is continuous for each of the topologies described in Definition 4.1.3.

Choose a Frobenius lift \( \varphi \) on \( \mathcal{R}_{K} \). For \( R \in \{ \mathcal{R}_{K}^{\text{int}}, \mathcal{R}_{K}^{\text{bd}}, \mathcal{R}_{K}, \mathcal{O}_{E_{K}}, \mathcal{E}_{K} \} \), a \( \varphi \)-module over \( R \) is a finite free \( R \)-module \( M \) equipped with a semilinear \( \varphi \)-action (i.e., an \( R \)-linear isomorphism \( \varphi^{*}M \to M \)). Since the action of \( \varphi \) takes any basis of \( M \) to another basis, the \( p \)-adic valuation of the matrix via which \( \varphi \) acts on a basis of \( M \) is both finite and independent of the choice of the basis. We call the negative of this quantity the degree of \( M \), denoted \( \deg(M) \). For \( M \) nonzero, we define the slope of \( M \) to be \( \mu(M) = \deg(M)/\text{rank}(M) \).

For \( s \in \mathbb{Q} \), we say \( M \) is pure of slope \( s \) if for some (hence any) \( c, d \in \mathbb{Z} \) with \( d > 0 \) and \( c/d = s \), \( p^{r}\varphi^{d} \) acts on \( M \) via a matrix \( U \) such that the entries of \( U \) and \( U^{-1} \) all have Gauss norm at most 1. This evidently implies \( \mu(M) = s \). If \( s = 0 \), we also say \( M \) is étale. (Note that our definitions force any nonzero \( \varphi \)-module over \( \mathcal{R}_{K}^{\text{int}} \) or \( \mathcal{E}_{K} \) to be étale.)

**Remark 4.1.7.** For \( M \) a \( \varphi \)-module over a ring \( R \), there is a natural way to equip the dual module \( M^{\vee} = \text{Hom}_{R}(M, R) \) with a \( \varphi \)-module structure so that the pairing map \( M \otimes_{R} M^{\vee} \to R \) is \( \varphi \)-equivariant (i.e., is a morphism of \( \varphi \)-modules). If \( M \) is pure of slope \( s \), then \( M^{\vee} \) is pure of slope \( -s \).

**Proposition 4.1.8.** For any \( s \in \mathbb{Q} \), we have the following.

(a) The functor \( M \mapsto M \otimes_{\mathcal{R}_{K}^{\text{bd}}} \mathcal{E}_{K} \) gives a fully faithful functor from \( \varphi \)-modules over \( \mathcal{R}_{K}^{\text{bd}} \) which are pure of slope \( s \) to \( \varphi \)-modules over \( \mathcal{E}_{K} \) which are pure of slope \( s \).

(b) The functor \( M \mapsto M \otimes_{\mathcal{R}_{K}^{\text{bd}}} \mathcal{R}_{K} \) gives an equivalence of categories between \( \varphi \)-modules over \( \mathcal{R}_{K}^{\text{bd}} \) which are pure of slope \( s \) and \( \varphi \)-modules over \( \mathcal{R}_{K} \) which are pure of slope \( s \).

**Proof.** See [69] Proposition 1.2.7 for (a) and [69] Theorem 1.6.5 for (b). For (b), see also Remark 4.3.4.

The main theorem about slopes of \( \varphi \)-modules over \( \mathcal{R}_{K} \) can be formulated in several ways. This formulation asserts that \( M \) is pure if and only if \( M \) is semistable with respect to slope in the sense of geometric invariant theory [82].

**Theorem 4.1.9.** Let \( M \) be a nonzero \( \varphi \)-module over \( \mathcal{R}_{K} \) with \( \mu(M) = s \). Then \( M \) is pure of slope \( s \) if and only if there exists no nonzero proper \( \varphi \)-submodule of \( M \) of slope greater than \( s \).

**Proof.** See [69] Theorem 1.7.1. An essentially equivalent formulation, incorporating an analogue of the Harder-Narasimhan filtration for vector bundles, is the following.

**Theorem 4.1.10.** Let \( M \) be a nonzero \( \varphi \)-module over \( \mathcal{R}_{K} \). Then there exists a unique filtration \( 0 = M_{0} \subset \cdots \subset M_{l} = M \) by saturated \( \varphi \)-submodules, such that \( M_{1}/M_{0}, \ldots, M_{l}/M_{l-1} \) are pure \( \varphi \)-modules and \( \mu(M_{1}/M_{0}) > \cdots > \mu(M_{l}/M_{l-1}) \).
Proof. See again [69, Theorem 1.7.1].

Remark 4.1.11. Beware that Proposition 4.1.8 does not imply anything about maps between \( \varphi \)-modules which are pure of different slopes. For instance, it is common in \( p \)-adic Hodge theory (in the study of trianguline representations; see for instance [22]) to encounter short exact sequences of the form \( 0 \to M_1 \to M \to M_2 \to 0 \) of \( \varphi \)-modules over \( \mathcal{R}_K \) in which for some positive integer \( m \), \( M_1 \) has rank 1 and slope \(-m\), \( M_2 \) has rank 1 and slope \( m \), and \( M \) is étale. While each term individually descends uniquely to \( \mathcal{R}_K^{\text{bd}} \) by Proposition 4.1.8(b), the sequence cannot descend because there are no nonzero maps between pure \( \varphi \)-modules over \( \mathcal{R}_K^{\text{bd}} \) of different slopes. (Note that \( \mu(M_1) < \mu(M) \), so there is no contradiction with Theorem 4.1.9.)

Remark 4.1.12. The sign convention used here for degrees of \( \varphi \)-modules is opposite to that used in previous work of the first author [67, 68, 69, 70]. We have changed it in order to match the sign convention used in geometric invariant theory [82], in which the ample line bundle \( \mathcal{O}(1) \) on any projective space has degree +1. This choice of sign also creates agreement with the work of Hartl and Pink [59] and of Fargues and Fontaine [39].

Convention 4.1.13. It will be convenient at several points to speak also of \( \varphi^d \)-modules for \( d \) an arbitrary positive integer. We adopt the convention that the degree of a \( \varphi^d \)-module is defined by computing the \( p \)-adic valuation of the determinant of the matrix on which \( \varphi^d \) acts on some basis, then dividing by \( d \). This has the advantage that the degree is preserved upon restriction of a \( \varphi^d \)-module to a \( \varphi^d \)-module. (One can even replace \( \varphi \) with a map on \( \tilde{\mathcal{R}}_K \) lifting the \( p^d \)-power absolute Frobenius on \( k((T)) \), not necessarily given by raising a \( p \)-power Frobenius lift to the \( d \)-th power, by modifying Definition 4.1.6 in the obvious way. We will not need this extra generality.)

4.2 Slope theory and Witt vectors

One can generalize the slope theory for Frobenius modules over the Robba ring by first making explicit the role of the \( T \)-adic norm on the residue field of \( \mathcal{R}_K^{\text{bd}} \), then replacing this norm with something more general. This second step turns out to be a bit subtle unless we first perfect the residue field of \( \mathcal{R}_K^{\text{bd}} \) and pass to Witt vectors (as in Definition 3.2.10); this gives a slope theory introduced in [68] and reviewed here. In fact, this study is integral to the slope theory over the Robba ring itself; see Remark 4.3.5. We take up the relative version of this story starting in §5.

Hypothesis 4.2.1. Throughout §4.2 let \( L \) be a perfect analytic field of characteristic \( p \) with norm \( \alpha \).

Definition 4.2.2. For \( r > 0 \), let \( \tilde{\mathcal{R}}_L^{\text{int}, r} \) be the set of \( x = \sum_{i=0}^{\infty} p^i [\pi_i] \in W(L) \) for which \( \lim_{i \to \infty} p^{-i} \alpha(\pi_i)^r = 0 \). Thanks to the homogeneity property of Witt vector addition (Remark 3.2.3), this set forms a ring on which the formula
\[
\lambda(\alpha^s)(x) = \max\{p^{-i}\alpha(\pi_i)^s\}
\]
defines a multiplicative norm \( \lambda(\alpha^s) \) on \( \tilde{R}^{\text{int},r}_L \) for each \( s \in [0, r] \). (For an explicit argument, see Proposition 5.1.2)

Put \( \tilde{R}^{\text{bd},r}_L = \tilde{R}^{\text{int},r}_L[p^{-1}] \). Let \( \tilde{R}^r_L \) be the Fréchet completion of \( \tilde{R}^{\text{bd},r}_L \) under the norms \( \lambda(\alpha^s) \) for \( s \in (0, r] \). Put

\[
\tilde{R}^{\text{int}}_L = \bigcup_{r > 0}\tilde{R}^{\text{int},r}_L, \quad \tilde{R}^{\text{bd}}_L = \bigcup_{r > 0}\tilde{R}^{\text{bd},r}_L, \quad \tilde{R}_L = \bigcup_{r > 0}\tilde{R}^r_L.
\]

Again, \( \tilde{R}^{\text{int}}_L \) is an incomplete but henselian (see [68, Lemma 2.1.12] or Proposition 5.5.3) local ring with residue field \( L \); the completion of the field \( \tilde{R}^{\text{bd}}_L \) is simply \( \tilde{E}_L = W(L)[p^{-1}] \).

One can again identify the units of \( \tilde{R}_L \) as the nonzero elements of \( \tilde{R}^{\text{bd}}_L \); see Corollary 4.2.5.

We call \( \tilde{R}_L \) the extended Robba ring with residue field \( L \). This terminology is not used in [68], but is suggested in [69].

**Lemma 4.2.3.** For \( x \in \tilde{R}^r_L \), the function \( s \mapsto \log \lambda(\alpha^s)(x) \) is convex on \( (0, r] \).

**Proof.** The function is affine in case \( x = p^i[\overline{x}] \) for some \( i \in \mathbb{Z}, \overline{x} \in L \). In case \( x \) is a finite sum of such terms, the function is the maximum of finitely many affine functions, and hence is convex. Since such finite sums are dense in \( \tilde{R}^r_L \), the general case follows. \( \square \)

**Lemma 4.2.4.** For \( x \in \tilde{R}_L \), we have \( x \in \tilde{R}^{\text{bd}}_L \) if and only if for some \( r > 0 \), \( \lambda(\alpha^s)(x) \) is bounded for \( s \in (0, r] \).

**Proof.** If \( x \in \tilde{R}^{\text{bd}}_L \), then as \( s \) tends to 0, \( \lambda(\alpha^s)(x) \) tends to the \( p \)-adic norm of \( x \). Hence for some \( r > 0 \), \( \lambda(\alpha^s)(x) \) is bounded for \( s \in (0, r] \). Conversely, suppose that for some \( r > 0 \), \( \lambda(\alpha^s)(x) \) is bounded for \( s \in (0, r] \). To prove that \( x \in \tilde{R}^{\text{bd}}_L \), we may first multiply by a power of \( p \); we may thus ensure that for some \( r > 0 \), \( x \in \tilde{R}^r_L \) and \( \lambda(\alpha^s)(x) \leq 1 \) for \( s \in (0, r] \). We will show in this case that \( x \in \tilde{R}^{\text{int},r}_L \).

Write \( x \) as the limit of a sequence \( x_0, x_1, \ldots \) with \( x_i \in \tilde{R}^{\text{bd},r}_L \). For each positive integer \( j \), we can find \( N_j > 0 \) such that

\[
\lambda(\alpha^s)(x_i - x) \leq p^{-j} \quad (i \geq N_j, \ s \in [p^{-j}r, r]).
\]

Write \( x_i = \sum_{l=m(i)}^\infty p^l[\overline{x}_l] \), and put \( y_i = \sum_{l=0}^\infty p^l[\overline{x}_l] \in \tilde{R}^{\text{int},r}_L \). For \( i \geq N_j \), we have \( \lambda(\alpha^{p^{-j}r})(x_i) \leq 1 \) and so

\[
\alpha(\overline{x}_l) \leq p^{lp^j/l} \quad (i \geq N_j, \ l < 0).
\]

Since \( p^{-l}p^l \leq p^{1-p^l} \) for \( l \leq -1 \), we deduce that \( \lambda(\alpha^r)(x_i - y_i) \leq p^{1-p^l} \).

Consequently, the sequence \( y_0, y_1, \ldots \) converges to \( x \) under \( \lambda(\alpha^r) \), and hence under \( \lambda(\alpha^s) \) for \( s \in (0, r] \) by Lemma 4.2.3; it follows that \( x \in \tilde{R}^{\text{int},r}_L \) as desired. (See also [68, Corollary 2.5.6] for a slightly different argument.) \( \square \)

**Corollary 4.2.5.** The units in \( \tilde{R}_L \) are precisely the nonzero elements of \( \tilde{R}^{\text{bd}}_L \).

**Proof.** Suppose \( x \in \tilde{R}_L \) is a unit with inverse \( y \). Choose \( r > 0 \) so that \( x, y \in \tilde{R}^r_L \). Then the functions \( \log \lambda(\alpha^s)(x), \log \lambda(\alpha^s)(y) \) are convex by Lemma 4.2.3 but their sum is the constant function 0. Hence both functions are affine in \( s \); in particular, \( \lambda(\alpha^s)(x) \) is bounded for \( s \in (0, r] \). By Lemma 4.2.4 this forces \( x \in \tilde{R}^{\text{bd}}_L \). (Compare [68, Lemma 2.4.7].) \( \square \)
**Lemma 4.2.6.** The rings $\tilde{R}_L$ and $\hat{R}_L$ are Bézout domains, i.e., integral domains in which every finitely generated ideal is principal.

**Proof.** See [68, Theorem 2.9.6].

**Remark 4.2.7.** A fact closely related to Lemma 4.2.6 is that for $0 < s \leq r$, the completion of $\tilde{R}_L$ with respect to the norm $\max\{\lambda(\alpha^r), \lambda(\alpha^s)\}$ is a principal ideal domain, and even a Euclidean domain. See [68, Proposition 2.6.8]. (This ring will later be denoted $\tilde{R}_L^{[s,r]}$; see Definition 5.1.1.)

**Definition 4.2.8.** To each nonzero $x \in \tilde{R}_L$ is associated its Newton polygon, i.e., the convex dual of the function $f_x(t) = \log \lambda(\alpha^t)(x)$ (which is convex by Lemma 4.2.3). The slopes of the Newton polygon of $x$ are the values $t$ where $f_x$ changes slope; for short, we call these the slopes of $x$. The multiplicity of a slope $s$ is the width of the corresponding segment of the Newton polygon. Note that $r$ fails to receive a multiplicity under this definition; to correct this, choose any $s > r$ and any $y \in \tilde{R}_L$ with $\lambda(\alpha^r)(x - y) < \lambda(\alpha^s)(x)$, then define the multiplicity of $r$ as a slope of $x$ to be its multiplicity as a slope of $y$. This does not depend on the choice of $y$. (See [68, Definition 2.4.4] for an alternate definition.)

For our present purposes, the most important properties of slopes are the following.

(a) If $x = yz$, then the slopes of $x$ are precisely the slopes of $y$ and $z$. More precisely, the multiplicity of any $s \in (0, r]$ as a slope of $x$ equals the sum of its multiplicities as a slope of $y$ and of $z$.

(b) If $x \in \tilde{R}_L$ is a unit, then by (a) it has no slopes. The converse is also true, by the following argument. If $x$ has no slopes, then $x \in \tilde{R}_L^{\text{bd}}$ by Lemma 4.2.4. Write $x = p^m y$ with $m \in \mathbb{Z}$ and $y \in \tilde{R}_L^{\text{int},r}$ not divisible by $p$; we must then have $\lambda(\alpha^r)(y - [\mathfrak{p}]) < \lambda(\alpha^s)(y)$ by definition of the multiplicity of $r$. It follows that $x$ is a unit in $\tilde{R}_L$.

**Definition 4.2.9.** The Frobenius lift $\varphi$ on $W(L)$ acts on $\tilde{R}_L^{\text{int}}$, and extends by continuity to $\tilde{R}_L^{\text{bd}}, \hat{R}_L, \hat{E}_L$. Note the following useful identity:

$$\lambda(\alpha^s)(\varphi(x)) = \lambda(\alpha^{bs})(x) \quad (x \in \hat{R}_L, s > 0).$$  \hfill (4.2.9.1)

We define $\varphi$-modules over these rings, degrees, slopes, and the pure and étale conditions as in Definition 4.1.6. (We also consider $\varphi^d$-modules for $d$ a positive integer, keeping in mind Convention 4.1.13)

**Lemma 4.2.10.** We have $\hat{E}_L^\varphi = \mathbb{Q}_p$ and $\tilde{R}_L^{\varphi} = \mathbb{Q}_p$. More generally, for any positive integer $a$, the elements of $\hat{E}_L$ and $\hat{R}_L$ fixed by $\varphi^a$ constitute the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^a}$.

**Proof.** The first equality holds because $\hat{E}_L = W(L)[p^{-1}]$, so $\tilde{E}_L^\varphi = W(L^\varphi)[p^{-1}] = \mathbb{Q}_p$. For the second equality, suppose $x \in \tilde{R}_L^{\varphi}$ is nonzero; then by (4.2.9.1), $\lambda(\alpha^s)(x) = \lambda(\alpha^{bs})(x)$ for all $s > 0$. It follows that $\lambda(\alpha^r)(x)$ is bounded over all $r > 0$, and so by Lemma 4.2.4 $x \in (\tilde{R}_L^{\text{bd}})^r \subseteq \tilde{E}_L^\varphi = \mathbb{Q}_p$. The final assertion is proved similarly. \hfill $\square$

82
We have the following analogue of Proposition 4.1.8. One key difference is that the functor in part (a) can be shown to be essentially surjective; see Theorem 8.1.2.

**Proposition 4.2.11.** For any \( s \in \mathbb{Q} \), we have the following.

(a) The functor \( M \mapsto M \otimes_{\tilde{\mathcal{R}}_{bd} L} \tilde{\mathcal{E}}_{L} \) gives a fully faithful functor from \( \varphi \)-modules over \( \tilde{\mathcal{R}}_{bd} L \) which are pure of slope \( s \) to \( \varphi \)-modules over \( \tilde{\mathcal{E}}_{L} \) which are pure of slope \( s \).

(b) The functor \( M \mapsto M \otimes_{\tilde{\mathcal{R}}_{bd} L} \tilde{\mathcal{R}}_{bd} L \tilde{\mathcal{E}}_{L} \) gives an equivalence of categories between \( \varphi \)-modules over \( \tilde{\mathcal{R}}_{bd} L \) which are pure of slope \( s \) and \( \varphi \)-modules over \( \tilde{\mathcal{R}}_{bd} L \) which are pure of slope \( s \).

**Proof.** See [68, Theorem 6.3.3(a,b)] or Remark 4.3.4.

We also have analogues of Theorem 4.1.9 and Theorem 4.1.10.

**Theorem 4.2.12.** Let \( M \) be a nonzero \( \varphi \)-module over \( \tilde{\mathcal{R}}_{bd} L \) with \( \mu(M) = s \). Then \( M \) is pure of slope \( s \) if and only if there exists no nonzero proper \( \varphi \)-submodule of \( M \) of slope greater than \( s \).

**Proof.** See [68, Proposition 6.3.5, Corollary 6.4.3].

**Theorem 4.2.13.** Let \( M \) be a nonzero \( \varphi \)-module over \( \tilde{\mathcal{R}}_{bd} L \). Then there exists a unique filtration \( 0 = M_0 \subset \cdots \subset M_l = M \) by saturated \( \varphi \)-submodules, such that \( M_1/M_0, \ldots, M_l/M_{l-1} \) are pure and \( \mu(M_1/M_0) > \cdots > \mu(M_l/M_{l-1}) \).

**Proof.** See [68, Theorem 6.4.1].

**Corollary 4.2.14.** Let \( M \) be a nonzero \( \varphi \)-module over \( \tilde{\mathcal{R}}_{bd} L \) with \( \mu(M) = s \). Let \( L' \) be a perfect analytic field containing \( L \) with compatible norms. Then \( M \) is pure of slope \( s \) if and only if \( M \otimes_{\tilde{\mathcal{R}}_{bd} L} \tilde{\mathcal{R}}_{bd} L' \) is pure of slope \( s \).

**Proof.** From the definition of purity, it is clear that if \( M \) is pure of slope \( s \), then so is \( M \otimes_{\tilde{\mathcal{R}}_{bd} L} \tilde{\mathcal{R}}_{bd} L' \). On the other hand, from the alternate criterion for purity given by Theorem 4.2.12 it is also clear that if \( M \) fails to be pure of slope \( s \), then so does \( M \otimes_{\tilde{\mathcal{R}}_{bd} L} \tilde{\mathcal{R}}_{bd} L' \).

In addition, in case \( L \) is algebraically closed, we get an analogue of Manin’s classification of rational Dieudonné modules.

**Proposition 4.2.15.** Suppose that \( L \) is algebraically closed. Let \( M \) be a \( \varphi \)-module over \( \tilde{\mathcal{E}}_{L} \) (resp. \( \tilde{\mathcal{R}}_{bd} L \)) which is pure of slope \( s \). Then for any \( c, d \in \mathbb{Z} \) with \( d > 0 \) and \( c/d = s \), and any basis of \( M \) on which \( p^c \varphi^d \) acts via an invertible matrix over \( W(L) \) (resp. \( \tilde{\mathcal{R}}_{bd} L' \)), the \( W(L) \)-span (resp. \( \tilde{\mathcal{R}}_{bd} L' \)-span) of this basis admits another basis fixed by \( p^c \varphi^d \).

**Proof.** Both assertions reduce easily to the case \( s = 0 \) provided that we allow \( \varphi \) to be replaced by a power (which does not affect the argument). The assertion about \( \tilde{\mathcal{E}}_{L} \) is fairly standard; see for instance [10, Proposition A1.2.6]. The assertion about \( \tilde{\mathcal{R}}_{bd} L \) follows from the assertion about \( \tilde{\mathcal{E}}_{L} \) as in [69, Proposition 2.5.8]. See also Proposition 7.3.6 for a stronger statement.
One has an analogue of Proposition 4.2.15 for $\varphi$-modules over $\hat{\mathcal{R}}_L$ which need not be pure.

**Proposition 4.2.16.** Suppose that $L$ is algebraically closed. Let $M$ be a $\varphi$-module over $\hat{\mathcal{R}}_L$. Then for some positive integer $d$, there exists a basis of $M$ on which $\varphi^d$ acts via a diagonal matrix with diagonal entries in $p\mathbb{Z}$.

**Proof.** Using Theorem 4.2.13 and Proposition 4.2.15, this reduces to the assertion that for any positive integers $c,d$, the map $x \mapsto x^{\varphi^d} - p^c x$ on $\hat{\mathcal{R}}_L$ is surjective. For this, see [68, Proposition 3.3.7(c)]. See also [68, Proposition 4.5.3] for a detailed proof of the original statement.

**Remark 4.2.17.** The use of growth conditions to cut out subrings of the rings of Witt vectors also appears in the work of Fargues and Fontaine [39] with which we make contact later (§6.3), as well as in the approach to constructing $p$-adic cohomology via the overconvergent de Rham-Witt complex of Davis, Langer, and Zink [26, 27].

### 4.3 Comparison of slope theories

The slope theories for $\varphi$-modules over $\mathcal{R}_K$ and $\hat{\mathcal{R}}_L$ can be related as follows. Throughout §4.3, retain Hypothesis 4.1.1.

**Definition 4.3.1.** Equip the field $k((T))$ with the $T$-adic norm $\alpha$ for the normalization $\alpha(T) = \omega$. Let $L$ be the completed perfection of $k((T))$ for the unique multiplicative extension of $\alpha$. Proceeding as in Definition 3.2.10, we obtain a map $s_\varphi : \mathcal{E}_K \to \hat{\mathcal{E}}_L$; more precisely, $\hat{\mathcal{E}}_L$ is the completion of the direct perfection of $\mathcal{E}_K$ for the weak topology. For $r > 0$ small enough that the $\omega^r/p$-Gauss norm of $\varphi(T)/T^p - 1$ is less than 1, $s_\varphi$ takes $\mathcal{R}_K^{\text{int},r}$ into $\mathcal{R}_L^{\text{int},r}$. In fact, this map is isometric for the $\omega^r$-Gauss norm on the source and the norm $\lambda(\alpha^r)$ on the target [68, Lemma 2.3.5]. We thus obtain a $\varphi$-equivariant homomorphism $\mathcal{R}_K \to \hat{\mathcal{R}}_L$.

**Example 4.3.2.** For the Frobenius lift $\varphi(T) = (T+1)^p - 1$ and $\omega = p^{-p/(p-1)}$, $s_\varphi$ is isometric for the $\omega^r$-Gauss norm for $r \in (0, 1)$.

We can use the extended rings to trivialize $\varphi$-modules over the smaller rings, as follows.

**Proposition 4.3.3.** Identify the maximal unramified extension $\mathcal{E}_K^{\text{unr}}$ of $\mathcal{E}_K$ with a subring of $\hat{\mathcal{E}}_L$.

(a) Let $M$ be an étale $\varphi$-module over $\mathfrak{o}_{\mathcal{E}_K}$. Then the $\mathbb{Z}_p$-module

$$V = (M \otimes_{\mathfrak{o}_{\mathcal{E}_K}} \mathfrak{o}_{\mathcal{E}_K^{\text{unr}}})^p$$

has the property that the natural map

$$V \otimes_{\mathbb{Z}_p} \mathfrak{o}_{\mathcal{E}_K^{\text{unr}}} \to M \otimes_{\mathfrak{o}_{\mathcal{E}_K}} \mathfrak{o}_{\mathcal{E}_K^{\text{unr}}}$$

is an isomorphism.
(b) Let $L'$ be the completed direct perfection of $\mathcal{O}_{E_k}/(p)$ (which is algebraically closed). Let $M$ be an étale $\varphi$-module over $\mathcal{R}_K^{\text{int}}$. Then the $\mathbb{Z}_p$-module

$$V = (M \otimes_{\mathcal{R}_K^{\text{int}}} (\mathcal{O}_{E_k^{unr}} \cap \mathcal{R}_L'))^\varphi$$

has the property that the natural map

$$V \otimes_{\mathbb{Z}_p} (\mathcal{O}_{E_k^{unr}} \cap \mathcal{R}_L') \to M \otimes_{\mathcal{R}_K^{\text{int}}} (\mathcal{O}_{E_k^{unr}} \cap \mathcal{R}_L')$$

is an isomorphism. Moreover,

$$V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = (M \otimes_{\mathcal{R}_K^{\text{int}}} \mathcal{R}_L')^\varphi = (M \otimes_{\mathcal{R}_K^{\text{int}}} \mathcal{E}_L')^\varphi.$$

**Proof.** Both parts follow from Proposition 4.2.15 plus Lemma 4.2.10.

**Remark 4.3.4.** For any $\varphi$-modules $M_1, M_2$ over a ring $R$, there is a natural identification

$$\text{Hom}_R(M_1, M_2) = M_1^\vee \otimes_R M_2.$$

If $M_1, M_2$ are both pure of slope $s$, then $M_1^\vee \otimes_R M_2$ is étale. By this reasoning, Proposition 4.1.8 reduces to Proposition 4.3.3 while Proposition 4.2.11 reduces to a similar consequence of Proposition 4.2.15 (derived using Lemma 4.2.10).

**Remark 4.3.5.** Note that any pure $\varphi$-module over $\mathcal{R}_K$ remains pure upon base extension to $\mathcal{R}_L$, while any $\varphi$-module over $\mathcal{R}_K$ whose base extension to $\mathcal{R}_L$ is semistable, i.e., which does not have any nonzero proper $\varphi$-submodule of larger slope, is also itself semistable. The reverse implications also hold, and in fact form part of the proof of Theorem 4.1.9 (in the form of a reduction to the somewhat more tractable Theorem 4.2.12). One approach to the reverse implications is to make somewhat careful calculations, as in [68]; a simpler approach is to use faithfully flat descent, as in [69, §3].

**Remark 4.3.6.** On the topic of descent, we record some minor inaccuracies in the statement and proof of [69, Proposition 3.3.2].

(a) The module $M$ should not be assumed to be a $\varphi$-module over $R$, but only an $R$-module equipped with an isomorphism $\varphi^*M \cong M$. That is, we should not assume $M$ is finite free over $R$. That is because in the proof of [69, Theorem 3.1.3], we need to take $R = \mathcal{R}_L^{\text{bd}}$ and $S = \mathcal{R}_L^{\text{bd}}$, and to take $M$ to be the restriction of scalars of a $\varphi$-module over $\mathcal{R}$.

(b) The conclusion should not state that $N$ is a $\varphi$-module over $R$, only a finite locally free $R$-module equipped with an isomorphism $\varphi^*N \cong N$. The proof of [69, Proposition 3.2.2] invokes [54, Exposé VIII, Corollaire 1.3], which only guarantees the existence and uniqueness of the module $N$. It should instead invoke [54, Exposé XIII, Théorème 1.1] (i.e., Theorem 1.3.3(a)) to recover both $N$ and the isomorphism $\varphi^*N \cong N$, plus [54, Exposé VIII, Proposition 1.10] (i.e., Theorem 1.3.5) to deduce that $N$ is finite locally free over $R$.

Note that the modified statement suffices for the applications to [69, Theorems 3.1.2 and 3.1.3] because in those cases $R$ is a Bézout domain (Lemma 4.2.6), over which any finite locally free module is free [69, Remark 1.1.2].
5 Relative Robba rings

We now begin in earnest to consider the relative setting. Although we will ultimately need to consider analogues of the Robba ring itself, these are not so straightforward to construct, and we leave them to a subsequent paper. Here, we treat only the analogue of the extended Robba ring in which the field of positive characteristic (over which we define Witt vectors) is replaced by a more general ring. As noted in the introduction, this pertains to a “geometric” relativization of slope theory, which is rather different from an “arithmetic” relativization in which one works with power series over a more general ring. (See Remark 7.4.10 for some discussion of the latter.)

Hypothesis 5.0.1. For the remainder of the paper, let $R$ be a perfect uniform Banach algebra over $\mathbb{F}_p$, with norm $\alpha$. When $R$ has been assumed to be an analytic field, we conventionally change its name from $R$ to $L$, but this change is pointed out explicitly in each instance.

Remark 5.0.2. For $x = \sum_{i=m}^{\infty} p^i [\alpha_i] \in W(R)[p^{-1}]$, for each $h \in \mathbb{Z}$, the set
\[ \{ \beta \in \mathcal{M}(R) : \beta([\alpha_i]) = 0 \text{ for all } i \leq h \} \]
is closed in $\mathcal{M}(R)$. Consequently, the $p$-adic absolute value of the image of $x$ in $W(\mathcal{H}(\beta))[p^{-1}]$ is a lower semicontinuous function of $\beta \in \mathcal{M}(R)$. When $x$ is a unit, this function is seen to be continuous by applying the same argument to $x^{-1}$.

5.1 Relative extended Robba rings

We start by generalizing the definition of the extended Robba rings and their subrings.

Definition 5.1.1. Define the rings $\mathcal{E}_R$, $\mathcal{R}^{\text{int},r}_R$, $\mathcal{R}^{\text{int}}_R$, $\mathcal{R}^{\text{bd},r}_R$, $\mathcal{R}^{\text{bd}}_R$, $\mathcal{R}_R$, $\mathcal{R}^{\text{bd},r}_R$ by changing $L$ to $R$ in Definition 4.2.2. That is, for $r > 0$, put $\mathcal{E}_R = W(R)[p^{-1}]$, let $\mathcal{R}^{\text{int},r}_R$ be the ring of $x = \sum_{i=0}^{\infty} p^i [\alpha_i] \in W(R)$ for which $\lim_{i \to \infty} p^{-i} \alpha([\alpha_i]) = 0$, and extend $\lambda(\alpha^s)$ to a power-multiplicative norm on $\mathcal{R}^{\text{int},r}_R$ for $s \in (0, r]$ by putting
\[ \lambda(\alpha^s) \left( \sum_{i=0}^{\infty} p^i [\alpha_i] \right) = \max_i \{ p^{-i} \alpha([\alpha_i]) \} \).

(See Proposition 5.1.2(a) for more details.) Put $\mathcal{R}^{\text{bd},r}_R = \mathcal{R}^{\text{int},r}_R[p^{-1}]$, let $\mathcal{R}^{\text{bd}}_R$ be the Fréchet completion of $\mathcal{R}^{\text{bd},r}_R$ under $\lambda(\alpha^s)$ for $s \in (0, r]$, and drop $r$ from the superscript to indicate the union over all $r > 0$.

For $0 < s \leq r$, let $\mathcal{R}^{[s,r]}_R$ be the Fréchet completion of $\mathcal{R}^{\text{bd},r}_R$ under the norms $\lambda(\alpha^t)$ for $t \in [s, r]$; it will follow from Lemma 5.2.1 below that $\mathcal{R}^{[s,r]}_R$ is also complete under $\max \{ \lambda(\alpha^s), \lambda(\alpha^r) \}$, and so is a Banach ring. Put $\mathcal{R}^{\text{bd},+}_R = W(\mathcal{O}_R)$, $\mathcal{R}^{\text{bd},+_r}_R = W(\mathcal{O}_R)[p^{-1}]$, and let $\mathcal{R}^{\text{bd},+}_R$ be the Fréchet completion of $\mathcal{R}^{\text{bd},+_r}_R$ under $\lambda(\alpha^s)$ for all $s > 0$. Note that ring is in general properly contained in $\mathcal{R}^{\infty}_R = \cap_{r>0} \mathcal{R}^{r}_R$.
We need the following mild extension of the basic constructions of [72, §4]. For more discussion of topological aspects (e.g., continuity of $\lambda$ and $\mu$), see [72, §5.4].

**Proposition 5.1.2.** Choose $0 < s \leq r$.

(a) The set $\tilde{\mathcal{R}}^{int,r}_R$ is a ring on which $\lambda(\alpha)$ is a power-multiplicative norm. Moreover, $\lambda(\alpha)$ is multiplicative in case $\alpha$ is.

(b) For $\beta$ a submultiplicative (resp. power-multiplicative, multiplicative) (semi)norm on $R$ dominated by $\max\{\alpha^s, \alpha^r\}$, the formula

$$
\lambda(\beta) \left( \sum_{i=0}^{\infty} p^i [\pi_i] \right) = \max_i \{ p^{-i} \beta(\pi_i) \}
$$

defines a submultiplicative (resp. power-multiplicative, multiplicative) (semi)norm on $\tilde{\mathcal{R}}^{int,r}_R$ dominated by $\max\{\lambda(\alpha)^s, \lambda(\alpha)^r\}$.

(c) In (b), if $\beta$ is power-multiplicative (resp. multiplicative), then $\lambda(\beta)$ extends to a power-multiplicative (resp. multiplicative) (semi)norm on $\tilde{\mathcal{R}}^{bd,r}_R$, and then extends further by continuity to $\tilde{\mathcal{R}}^{[s,r]}_R$.

(d) For $\gamma$ a power-multiplicative (resp. multiplicative) (semi)norm on $\tilde{\mathcal{R}}^{int,r}_R$ dominated by $\max\{\lambda(\alpha)^s, \lambda(\alpha)^r\}$, the formula

$$
\mu(\gamma)(\pi) = \gamma([\pi])
$$

defines a power-multiplicative (resp. multiplicative) (semi)norm on $R$ dominated by $\max\{\alpha^s, \alpha^r\}$. Moreover, $\gamma$ is dominated by $\lambda(\mu(\gamma))$.

**Proof.** To check (a), we follow the argument of [72, Lemma 4.1], omitting those details which remain unchanged. Closure of $\tilde{\mathcal{R}}^{int,r}_R$ under addition and the inequality $\lambda(\alpha)(x+y) \leq \max\{\lambda(\alpha)(x), \lambda(\alpha)(y)\}$ follow from the homogeneity of the Witt vector addition formula [72, Remark 3.7] (see also Remark 3.2.3). This easily implies that $\tilde{\mathcal{R}}^{int,r}_R$ is closed under multiplication and that $\lambda(\alpha)$ is a submultiplicative norm, as in [72, Lemma 4.1]. To check that $\lambda(\alpha)$ is multiplicative whenever $\alpha$ is, it is enough to check that $\lambda(\alpha)(xy) \geq \lambda(\alpha)(x)\lambda(\alpha)(y)$ in case the right side of this inequality is positive. Write $x = \sum_{i=0}^{\infty} p^i [\pi_i], y = \sum_{i=0}^{\infty} p^i [\eta_i]$. Let $j, k$ be the largest indices maximizing $p^{-j}\alpha(\pi_j), p^{-k}\alpha(\eta_k)$. As in [72, Lemma 4.1], we use the fact that $\lambda(\alpha)$ is a submultiplicative norm to reduce to the case where $\pi_i = 0$ for $i < j$ and $\eta_i = 0$ for $i < k$. Then $xy = \sum_{i=j+k}^{\infty} p^i [\zeta_i]$ with $\zeta_{j+k} = \pi_j \eta_k$, proving the desired inequality. To check that $\lambda(\alpha)$ is power-multiplicative whenever $\alpha$ is, one makes the same argument with $y = x$. This yields (a); we may check (b) by imitating the proof of (a), and (c) is clear.

To check (d), we introduce an alternate proof of [72, Lemma 4.4]. Again from [72, Remark 3.7], we deduce that for $\pi, \eta \in R$, $\gamma([\pi + \eta]) \leq \max\{\gamma([\pi]), \gamma([\eta])\}$. (Note that this requires at least power-multiplicativity, not just submultiplicativity.) By rewriting this
inequality as \( \mu(\gamma)(\overline{x} + \overline{y}) \leq \max\{\mu(\gamma)(\overline{x}), \mu(\gamma)(\overline{y})\} \), we see that \( \mu(\gamma) \) is a (semi)norm. The power-multiplicativity or multiplicativity of \( \mu(\gamma) \) follows from the corresponding property of \( \gamma \). The fact that \( \gamma \) is dominated by \( \lambda(\mu(\gamma)) \) follows as in [72, Theorem 4.5].

**Definition 5.1.3.** As in Definition 4.1.3, we impose topologies on the aforementioned rings as follows.

(a) Those rings contained in \( \tilde{\mathcal{E}}_R \) carry both a \textit{p-adic topology} (the metric topology defined by the Gauss norm) and a \textit{weak topology} (in which a sequence converges if it is bounded for the Gauss norm and converges under \( \lambda(\alpha) \) modulo any fixed power of \( p \)). For both topologies, \( \tilde{\mathcal{E}}_R \) is complete.

(b) Those rings contained in \( \tilde{\mathcal{R}}^r \) carry a \textit{Fréchet topology}, in which a sequence converges if and only if it converges under \( \lambda(\alpha^s) \) for all \( s \in (0, r] \). For this topology, \( \tilde{\mathcal{R}}^r \) is complete.

(c) Those rings contained in \( \tilde{\mathcal{R}}_R \) carry a \textit{limit-of-Fréchet topology}, or \textit{LF topology}. This topology is defined by taking the locally convex direct limit of the \( \tilde{\mathcal{R}}^r \) (each equipped with the Fréchet topology).

The analogue of Remark 4.1.5 is true: a sequence in \( \mathcal{R}^{bd,r} \) which is \( p \)-adically bounded and convergent under \( \lambda(\alpha^r) \) also converges in the weak topology.

**Remark 5.1.4.** If \( \alpha \) is the trivial norm, then \( \tilde{\mathcal{E}}_R, \tilde{\mathcal{R}}^{bd}, \tilde{\mathcal{R}}_R \) all coincide. This makes it possible to abbreviate some arguments. However, one must be careful when using this trick, as the extension of \( \alpha \) along a rational localization will usually no longer be trivial.

**Remark 5.1.5.** Recall that \( \overline{\gamma} \) acts as the identity map on \( R \) if and only if \( R \) is generated over \( \mathbb{F}_p \) by idempotent elements (Lemma 3.1.2). In this case, the power-multiplicative norm \( \alpha \) on \( R \) must be trivial, so by Remark 5.1.4 all of the topologies in Definition 5.1.3 coincide with the \( p \)-adic topology.

**Remark 5.1.6.** All of the constructions in Definition 5.1.1 are functorial with respect to bounded homomorphisms \( \psi : R \to S \) in which \( S \) is another perfect uniform Banach \( \mathbb{F}_p \)-algebra with norm \( \beta \). If \( \psi \) is strict injective, then \( \psi \) is isometric by Remark 3.1.5 and it is evident that the functoriality maps induced by \( \psi \) are strict injective, for all of the topologies named in Definition 5.1.3 (The case when \( \psi \) is injective but not strict is more subtle; we do not treat it here.)

Similarly, if \( \psi \) is strict surjective, then the functoriality maps induced by \( \psi \) are again strict surjective, by the following argument. Choose \( c > 0 \) such that any \( \overline{y} \in S \) admits a lift \( \overline{x} \in R \) with \( \alpha(\overline{x}) \leq c \beta(\overline{y}) \). (In fact any \( c > 1 \) has this property by Remark 3.1.5 but we do not need this here.) By lifting each Teichmüller element separately, we can lift each \( y \in \mathcal{R}^{int,r}_S \) to some \( x \in \mathcal{R}^{int,r}_S \) for which \( \lambda(\alpha^s)(x) \leq c^s \lambda(\beta^s)(y) \) for all \( s \in (0, r] \). From this, the claim follows. (See Lemma 5.5.2 for a similar argument.)
5.2 Reality checks

The operation of Fréchet completion in Definition 5.1.1 leaves the structure of the resulting rings a bit mysterious. To clarify these, we make some calculations akin to the reality checks of [68, §2.5].

Lemma 5.2.1. For each \( x \in \tilde{\mathcal{R}}_{[s,r]} \), the function \( t \mapsto \log \lambda(\alpha^t)(x) \) is convex. In particular, \( \max\{\lambda(\alpha^s), \lambda(\alpha^r)\} = \sup\{\lambda(\alpha^t) : t \in [s, r]\} \).

Proof. As in Lemma 4.2.3.

Lemma 5.2.2. For \( x \in \tilde{\mathcal{R}}_R \), we have \( x \in \tilde{\mathcal{R}}_{bd} \) if and only if for some \( r > 0 \), \( \lambda(\alpha^s)(x) \) is bounded for \( s \in (0, r) \).

Proof. As in Lemma 4.2.4.

Using this criterion, we obtain a generalization of Corollary 4.2.5.

Corollary 5.2.3. Any unit in \( \tilde{\mathcal{R}}_R \) is also a unit in \( \tilde{\mathcal{R}}_{bd} \).

Proof. Suppose \( x \in \tilde{\mathcal{R}}_R \) is a unit with inverse \( y \). Choose \( r > 0 \) so that \( x, y \in \tilde{\mathcal{R}}_{[r,r']} \). For each \( \beta \in \mathcal{M}(R) \), the function \( \log \lambda(\beta^s)(x) \) is affine in \( s \), as in the proof of Corollary 4.2.5. We thus have

\[
\lim_{s \to 0^+} \log \lambda(\beta^s)(x) = 2 \log \lambda(\beta^{r/2})(x) - \log \lambda(\beta^r)(x) = 2 \log \lambda(\beta^{r/2})(x) + \log \lambda(\beta^r)(y).
\]

Taking suprema over \( \mathcal{M}(R) \) and invoking Theorem 2.3.10 yields

\[
\limsup_{s \to 0^+} \log \lambda(\alpha^s)(x) \leq 2 \log \lambda(\alpha^{r/2})(x) + \log \lambda(\alpha^r)(y) < +\infty.
\]

Consequently, \( x \in \tilde{\mathcal{R}}_{bd} \) by Lemma 5.2.2. Similarly, \( y \in \tilde{\mathcal{R}}_{bd} \), so \( x \) is a unit in \( \tilde{\mathcal{R}}_{bd} \) as desired.

Corollary 5.2.4. We have \( \tilde{\mathcal{E}}_R^\varphi = \tilde{\mathcal{R}}_R^\varphi = W(\mathcal{R}^\varphi)[p^{-1}] \). In particular, by Corollary 3.1.3, \( W(\mathcal{R}^\varphi)[p^{-1}] = \mathbb{Q}_p \) if and only if \( R \) is connected.

Proof. It is clear that \( \tilde{\mathcal{E}}_R^\varphi = W(\mathcal{R}^\varphi)[p^{-1}] \subseteq \tilde{\mathcal{R}}_R^\varphi \). We have \( \tilde{\mathcal{R}}_R^\varphi = (\tilde{\mathcal{R}}_{bd})^\varphi \subseteq \mathcal{E}_R^\varphi \) by Lemma 5.2.2 as in the proof of Lemma 4.2.10.

Lemma 5.2.5. For \( 0 < r \leq r' \), inside \( \tilde{\mathcal{R}}_{[r,r']}^\varphi \) we have

\[
\tilde{\mathcal{R}}_{[r,r']}^\varphi \cap \tilde{\mathcal{R}}_{[r,r']}^{\text{int}} = \tilde{\mathcal{R}}_{[r,r']}^{\text{int}}.
\]

Proof. The case \( r' = r \) is trivial, so assume that \( r' > r \). Take \( x \in \tilde{\mathcal{R}}_{[r,r']}^{\text{int}} \cap \tilde{\mathcal{R}}_{[r,r']}^{[r,r']}, \) and write \( x \) as the limit in \( \tilde{\mathcal{R}}_{[r,r']}^{[r,r']} \) of a sequence \( x_0, x_1, \ldots \) with \( x_i \in \tilde{\mathcal{R}}_{[r,r']}^{[r,r']} \). For each positive integer \( j \), we can find \( N_j > 0 \) such that

\[
\lambda(\alpha^s)(x_i - x) \leq p^{-j} \quad (i \geq N_j, s \in [r, r']).
\]
Write $x_i = \sum_{l=m(i)}^{\infty} p^l \overline{x}_i$ and put $y_i = \sum_{l=0}^{\infty} p^l \overline{x}_i \in \hat{R}_R^{\text{int}, r'}$. For $i \geq N_j$, having $x \in \hat{R}_R^{\text{int}, r}$ and $\lambda(\alpha^r)(x_i - y_i) \leq p^{-j}$ implies that $\lambda(\alpha^r)(p^l \overline{x}_i) \leq p^{-j}$ for $l < 0$. That is,

$$\alpha(\overline{x}_i) \leq p^{(l-j)/r} \quad (i \geq N_j, l < 0).$$

Since $p^{-l} p^{(l-j)/r} \leq p^{1+(l-j)/r}$ for $l \leq -1$, we deduce that $\lambda(\alpha^r)(x_i - y_i) \leq p^{1+(l-j)/r}$ for $i \geq N_j$. Consequently, the sequence $y_0, y_1, \ldots$ converges to $x$ under $\lambda(\alpha^r)$; it follows that $x \in \hat{R}_R^{\text{int}, r'}$. This proves the claim.

\[\square\]

**Remark 5.2.6.** In both Lemma 4.2.4 and Lemma 5.2.5, the key step was to split an element of $\hat{R}_R^{\text{bd}}$ into what one might call an integral part and a fractional part. One cannot directly imitate the construction for elements of $\hat{R}_R^{s, r}$ because they cannot be expressed as sums of Teichmüller elements. One can give presentations of a slightly less restrictive form with which one can make similar arguments (the seminunit presentations of [68, §2]); the stable presentations of [72, §5] are similar. We will instead rely on Lemma 5.2.7 (see below) to simulate splittings into integral and fractional parts.

As noted in Remark 5.2.6, the following lemma extends to $\hat{R}_R^{s, r}$ the splitting argument for elements of $\hat{R}_R^{\text{bd}}$ used previously. Its formulation is modeled on [68, Lemma 2.5.11].

**Lemma 5.2.7.** For $0 < s \leq r$ and $n \in \mathbb{Z}$, any $x \in \hat{R}_R^{s, r}$ can be written as $y + z$ with $y \in p^n \hat{R}_R^{\text{int}, r}$, $z \in \cap_{r' \geq r} \hat{R}_R^{s, r'}$, and

$$\lambda(\alpha^t)(z) \leq p^{(1-n)(1-t/r)} \lambda(\alpha^r)(x)^{t/r} \quad (t \geq r). \quad \text{(5.2.7.1)}$$

**Proof.** In case $x \in \hat{R}_R^{\text{bd}}$, write $x = \sum_{i=m(x)}^{\infty} p^i \overline{x}_i$, and put $y = \sum_{i=n}^{\infty} p^i \overline{x}_i$ and $z = y - x$. This works because for $i \leq n - 1$ and $t \geq r$,

$$\lambda(\alpha^t)(p^i \overline{x}_i) = p^{-i} \alpha(\overline{x}_i)^t = p^{-i(1-t/r)} \lambda(\alpha^r)(p^i \overline{x}_i)^{t/r} \leq p^{(1-n)(1-t/r)} \lambda(\alpha^r)(p^i \overline{x}_i)^{t/r}. \quad \text{(5.2.7.2)}$$

To handle the general case, choose $x_0, x_1, \ldots \in \hat{R}_R^{\text{bd}, r}$ so that

$$\lambda(\alpha^t)(x - x_0 - \cdots - x_i) \leq p^{-i-1} \lambda(\alpha^t)(x) \quad (i = 0, 1, \ldots; t \in [s, r]).$$

The series $\sum_{i=0}^{\infty} x_i$ converges to $x$ under $\lambda(\alpha^t)$ for $t \in [s, r]$, and $\lambda(\alpha^t)(x_i) \leq p^{-i} \lambda(\alpha^t)(x)$ for $i = 0, 1, \ldots$ and $t \in [s, r]$. Split each $x_i$ as $y_i + z_i$ as above. Since the sum $\sum_{i=0}^{\infty} y_i$ converges under $\lambda(\alpha^r)$ and consists of elements of $\hat{R}_R^{\text{int}, r}$, it converges under $\lambda(\alpha^t)$ for all $t \in (0, r]$ and defines an element $y$ of $\hat{R}_R^{\text{int}, r}$. Put $z = y - x$; then the series $\sum_{i=0}^{\infty} z_i$ converges to $z$ under $\lambda(\alpha^t)$ for $t \in [s, r]$. On the other hand, for $t > r$, by (5.2.7.2) we have

$$\lambda(\alpha^t)(z_i) \leq p^{(1-n)(1-t/r)} \lambda(\alpha^r)(x_i)^{t/r} \leq p^{(1-n)(1-t/r)} p^{-i(t/r)} \lambda(\alpha^r)(x)^{t/r}.$$  

Consequently, $\sum_{i=0}^{\infty} z_i$ also converges to $z$ under $\lambda(\alpha^t)$, and (5.2.7.1) holds.

\[\square\]

We will also have use for the following variant, where we separate in terms of $\alpha$ rather than the $p$-adic norm.
Lemma 5.2.8. Fix a quantity $c < 1$ and a positive integer $a$, and put $q = p^a$. Then for $0 \leq s < r$, each $x \in \tilde{R}_R^{[s,r]}$ can be written as $y + z$ with $y \in \tilde{R}_R^{[s,q,r]}$ and $z \in \tilde{R}_R^{[a,rq]}$, such that for all $t \in [s, r]$,

$$
\lambda(\alpha^t)(y), \lambda(\alpha^t)(z) \leq \lambda(\alpha^t)(x), \\
\lambda(\alpha^t)(\varphi^{-a}(y)) \leq c^{-(q-1)\ell/q} \lambda(\alpha^t)(x), \\
\lambda(\alpha^t)(\varphi^a(z)) \leq c^{(q-1)\ell} \lambda(\alpha^t)(x).
$$

Proof. By approximating $x$ as in the proof of Lemma 5.2.7, it is enough to consider the case $x \in \tilde{R}_R^{bd,r}$. In this case, write $x = \sum_{i=m}^{\infty} p^i [\pi_i]$, let $y$ be the sum of $p^i [\pi_i]$ over all indices $i$ for which $\alpha(\pi_i) \geq c$, and put $z = x - y$.

As an immediate application of Lemma 5.2.7, we extend Lemma 5.2.5 as follows.

Lemma 5.2.9. For $0 < s \leq s' \leq r$, inside $\tilde{R}_R^{[s,r]}$ we have

$$
\tilde{R}_R^{[s,r]} \cap \tilde{R}_R^{[s',r]} = \tilde{R}_R^{[s',r]}.
$$

Proof. Given $x$ in the intersection, apply Lemma 5.2.7 to write $x = y + z$ with $y \in \tilde{R}_R^{int,r}$, $z \in \tilde{R}_R^{[s',r]}$. By Lemma 5.2.5,

$$
y = z - x \in \tilde{R}_R^{int,r} \cap \tilde{R}_R^{[s',r]} = \tilde{R}_R^{int,r'} \subseteq \tilde{R}_R^{[s',r]},
$$

so $x \in \tilde{R}_R^{[s,r]}$ as desired.

Lemma 5.2.10. (a) An element $x \in \tilde{R}_R^\infty$ belongs to $\tilde{R}_R^+$ if and only if

$$
\lim_{r \to +\infty} \sup_{r} \lambda(\alpha^r)(x)^{1/r} \leq 1.
$$

(b) For $r > 0$ and $x \in \tilde{R}_R^r$, we have $x = y + z$ for some $y \in \tilde{R}_R^{bd,r}$ and $z \in \tilde{R}_R^+$.

(c) We have $\tilde{R}_R^{int,r} \cap \tilde{R}_R^+ = \tilde{R}_R^{int,+}$.

Proof. We first prove (a) for $x = \sum_{i=m}^{\infty} p^i [\pi_i] \in \tilde{R}_R^{bd} \cap \tilde{R}_R^\infty$, by observing that

$$
\lim_{r \to +\infty} \sup_{r} \lambda(\alpha^r)(x)^{1/r} = \lim_{r \to +\infty} \sup_{r} \{p^{-i/r}\alpha(\pi_i)\}.
$$

If $x \in \tilde{R}_R^{bd,+}$, then for some $m$, we can bound the quantity $p^{-i/r}\alpha(\pi_i)$ from above by $p^{m/r}$, and so the limit superior in question is at most 1. Conversely, if $x \notin \tilde{R}_R^{bd,+}$, then there exists an index $i$ for which $\alpha(\pi_i) > 1$; we can then find $\epsilon > 0$ so that $p^{-i/r}\alpha(\pi_i) > 1 + \epsilon$ for $r$ large, so the limit superior is at least $1 + \epsilon$. This proves (a) for such $x$.

We next prove (b). For $r > 0$ and $x \in \tilde{R}_R^r$, choose $n \in \mathbb{Z}$ so that $n < 1$ and $p^{(n-1)/(2r)}\lambda(\alpha^r)(x)^{1/r} < 1$. Set notation as in the proof of Lemma 5.2.7 for $t \geq 2r$, we
have $\lambda(\alpha^r)(z_i)^{1/\mu} \leq p^{(1-n)(1/t-1/r)} \lambda(\alpha^r)(x)^{1/r} \leq 1$, so $z_i \in \tilde{R}_R^{bd,+}$ by the previous paragraph. Hence $z \in \tilde{R}_R^{bd}$ as needed.

We next return to (a). By the first paragraph, if $x \in \tilde{R}_R^+$, then $\limsup_{r \to +\infty} \lambda(\alpha^r)(x)^{1/r} \leq 1$. Conversely, if $x \in \tilde{R}_R^\infty$ and $\limsup_{r \to +\infty} \lambda(\alpha^r)(x)^{1/r} \leq 1$, apply (b) to write $x = y + z$ with $y \in \tilde{R}_R^{bd}$ and $z \in \tilde{R}_R^+$. We may then apply the first paragraph to $y$ to deduce that $x \in \tilde{R}_R^+$.

To deduce (c), use Lemma 5.2.5 to deduce that $\tilde{R}_R^{\text{int},r} \cap \tilde{R}_R^+ \subseteq \cap_{s>0} \tilde{R}_R^{\text{int},s}$, then argue as in the proof of (a).

**Corollary 5.2.11.** For $n$ a nonnegative integer, $d$ a positive integer, $q = p^d$, and $r > 0$, the inclusions

$$
\{x \in \tilde{R}_R^+: \varphi^d(x) = p^nx\} \subseteq \{x \in \tilde{R}_R: \varphi^d(x) = p^nx\},
$$

$$
\{x \in \tilde{R}_R^+: \varphi^d(x) = p^nx\} \subseteq \{x \in \tilde{R}_R^{[r/q,r]}: \varphi^d(x) = p^nx\}
$$

are bijective.

**Proof.** Suppose first that $x \in \tilde{R}_R^\infty$ and $\varphi^d(x) = p^nx$. For each $r > 0$,

$$
\lambda(\alpha^{rq})(x) = \lambda(\alpha^r)(\varphi^d(x)) = p^{-n}\lambda(\alpha^r)(x).
$$

It follows that for any fixed $s > 0$,

$$
\limsup_{r \to +\infty} \lambda(\alpha^r)(x)^{1/r} \leq \sup_{r \in [s,qs]} \{\limsup_{n \to +\infty} p^{-n/(rq^n)} \lambda(\alpha^r)(x)^{1/(rq^n)}\} \leq 1,
$$

so $x \in \tilde{R}_R^+$ by Lemma 5.2.10(a).

Given $x \in \tilde{R}_R^+$ for which $\varphi^d(x) = p^nx$, there exists $r > 0$ for which $x \in \tilde{R}_R^{r/q,r}$, but then $x = p^{-n} \varphi^{-d}(x) \in \tilde{R}_R^{r/q,r}$. Consequently, $x \in \tilde{R}_R^\infty$, so by the previous paragraph, $x \in \tilde{R}_R^+$.

Given $x \in \tilde{R}_R^{[r/q,r]}$ for which $\varphi^d(x) = p^nx$, for each positive integer $m$ we also have $x \in \tilde{R}_R^{[r/q^m,r]}$. Namely, this holds for $m = 1$, and given the statement for some $m$, we also have $x = p^{-n} \varphi^d(x) \in \tilde{R}_R^{[r/q^{m+1},r/q]}$, so $x \in \tilde{R}_R^{[r/q^{m+1},r]}$ by Lemma 5.2.9. It follows that $x \in \tilde{R}_R^+$, so by the previous paragraph, $x \in \tilde{R}_R^+$.

**Remark 5.2.12.** Let $R \subseteq S' \subseteq S$ and $R \subseteq S'' \subseteq S$ be inclusions of perfect uniform Banach $\mathbb{F}_p$-algebras, and suppose that within $S$ we have $S' \cap S'' = R$. Then

$$
*_{S'} \cap *_{S''} = *_R \quad * = \mathcal{E}, \tilde{R}_R^{\text{int},r}, \tilde{R}_R^{\text{int},+}, \tilde{R}_R^{\text{int},}, \tilde{R}_R^{\text{bd},r}, \tilde{R}_R^{\text{bd},+}, \tilde{R}_R^{\text{bd}}
$$

with the intersection taking place within $*_S$; namely, $W(S') \cap W(S'') = W(R)$ within $W(S)$.

Now suppose that all of the inclusions in question are strict (and hence isometric by Remark 3.1.5). Then for $r > 0$, it is not hard to check that within $\tilde{R}_R^{[r,r]}$ we have $\tilde{R}_S^ {[r,r]} \cap \tilde{R}_S^ {[r,r]} = \tilde{R}_R^{[r,r]}$. Using Lemma 5.2.7 and the previous paragraph, this implies

$$
*_{S'} \cap *_{S''} = *_R \quad * = \tilde{R}_S^{[s,r]}, \tilde{R}_R^{r}, \tilde{R}_R^{+}, \tilde{R}_R.
$$

If the inclusions are not strict, it is unclear whether such maps as $*_R \to *_{S'}$ are even injective.
5.3 Finite projective modules

We collect some auxiliary results concerning finite projective modules over the rings we have just introduced, including glueing results both with respect to the base ring \( R \) and with respect to the exponent of the Gauss norm. (As noted in Remark 2.7.1, we are forced to consider only projective modules because we do not know that rational localizations are flat.)

Lemma 5.3.1. Choose \( 0 < s \leq r \) and let \( \beta \) be a multiplicative seminorm on \( \tilde{R}^{[s,r]}_R \).

(a) The seminorm \( \beta \) is dominated by \( \max\{\lambda(\alpha^s), \lambda(\alpha^r)\} \) if and only if there exists \( t \in [s, r] \) such that \( \beta \) is dominated by \( \lambda(\alpha^t) \).

(b) Suppose that \( \beta \) is dominated by \( \lambda(\alpha^t) \) for some \( t \in [s, r] \), and put \( \gamma = \mu(\beta)^{1/t} \in \mathcal{M}(R) \). Then \( \beta \) extends uniquely to a multiplicative seminorm on \( \tilde{R}^{[s,r]}_{\mathcal{H}(\gamma)} \).

Proof. We first address (a). If \( \beta \) is dominated by \( \lambda(\alpha^t) \) for some \( t \in [s, r] \), then \( \beta \) is dominated by \( \max\{\lambda(\alpha^s), \lambda(\alpha^r)\} \) by Lemma 5.2.1. Conversely, suppose that \( \beta \) is dominated by \( \max\{\lambda(\alpha^s), \lambda(\alpha^r)\} \).

Suppose first that \( R \) is a Banach algebra over a nontrivially normed analytic field \( K \). Pick \( \pi \in K^\times \) with \( \alpha(\pi) < 1 \). Then \( \beta(\pi) \in [\alpha^s(\pi), \alpha^s(\pi)] \), so there exists \( t \in [s, r] \) such that \( \beta(\pi) = \alpha^t(\pi) \).

For \( \pi \in R \) such that \( \alpha(\pi) \leq 1 \), we also have \( \beta(\pi) \leq 1 \). If we take \( \pi = \pi^m \pi^{-n} \) for \( m \) a positive integer and \( n \) an arbitrary integer, we deduce that if \( \alpha(\pi)^m \alpha(\pi)^{-n} \leq 1 \), then \( \beta(\pi)^m \beta(\pi)^{-n} \leq 1 \). That is, if \( \alpha(\pi) \leq \alpha(\pi)^{n/m} \), then \( \beta(\pi) \leq \alpha(\pi)^{nt/m} \). Since \( n/m \) can be chosen to be any rational number, it follows that \( \beta(\pi) \leq \alpha(\pi)^t \).

We deduce the general case by reduction to the previous case. Choose any \( \rho < 1 \), equip \( \mathbb{F}_p((\pi)) \) with the \( \rho \)-Gauss norm, and let \( \tilde{K} \) be the completion. Note that \( \tilde{K} \) is a nontrivially normed analytic field. Equip \( \tilde{R}' = \tilde{R} \otimes_{\mathbb{F}_p} \mathbb{F}_p((\pi)) \) with the product norm \( \alpha' \), which is power-multiplicative by Remark 3.1.3. Since \( \mathbb{F}_p \to \tilde{K} \) is split (by the map extracting the constant coefficient of a power series), the homomorphism \( \tilde{R} \to \tilde{R}' \) is also split by [72, Lemma 1.17], and so \( \mathcal{M}(R') \to \mathcal{M}(R) \) is surjective by [72, Lemma 1.19]. In particular, \( \mu(\beta) \) (defined as in Proposition 5.1.2) extends to some \( \gamma \in \mathcal{M}(R') \). By the previous paragraph, there exists \( t \in [s, r] \) such that \( \gamma \) is dominated by \( (\alpha')^t \); then \( \mu(\beta) \) is dominated by \( \alpha'^t \), and so \( \beta \) is dominated by \( \lambda(\alpha^t) \) as desired. This completes the proof of (a).

To deduce (b), note first that \( \beta \) extends uniquely to the localization of \( \tilde{R}^{[s,r]}_R \) at the multiplicative set consisting of \( [\pi] \) for each \( x \in R \setminus p_\gamma \), and that this extension is dominated by \( \max\{\lambda(\gamma^s), \lambda(\gamma^r)\} \).

Observe that the separated completion under \( \max\{\lambda(\gamma^s), \lambda(\gamma^r)\} \) of this localization is precisely \( \tilde{R}^{[s,r]}_{\mathcal{H}(\gamma)} \).

Corollary 5.3.2. Let \( R \to R_1, \ldots, R \to R_n \) be a covering family of rational localizations. Let \( I \) be a closed subinterval of \( (0, +\infty) \) and let \( I_1, \ldots, I_m \) be a covering of \( I \) by closed subintervals.

(a) The morphisms \( \tilde{R}^{I_j}_R \to \tilde{R}^{I_j}_{R_i} \) form a covering family of rational localizations.
(b) Let $M$ be a finitely generated module over $\tilde{\mathcal{R}}_R$. Then a finite subset $S$ of $M$ generates $M$ as a module over $\tilde{\mathcal{R}}_R$ if and only if it generates $M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_{R_i}$ as a module over $\tilde{\mathcal{R}}_{R_i}$ for $i = 1, \ldots, n, j = 1, \ldots, m$.

Proof. It suffices to prove (a), as then (b) follows by Lemma 2.3.12(a). It is clear that the maps in question are rational localizations, so we need only check the covering property. Using Lemma 5.3.1(a), we may reduce to the case $m = 1$. Given $\beta \in \mathcal{M}(\tilde{\mathcal{R}}_R)$, set notation as in Lemma 5.3.1(a,b). We may then obtain $\beta$ by restriction from a particular element $\delta$ of $\mathcal{M}(\tilde{\mathcal{R}}_{R(\gamma)})$, choose $i$ so that $\gamma \in \mathcal{M}(R_i)$, and then restrict $\delta$ to $\mathcal{M}(\tilde{\mathcal{R}}_{R_i})$. We end up with an element of $\mathcal{M}(\tilde{\mathcal{R}}_{R_i})$ which restricts to $\beta$, proving the desired result.

Lemma 5.3.3. With notation as in Lemma 3.1.11, the sequence

$0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0$ (5.3.3.1)

remains exact, and the morphisms remain almost optimal, when $R_*$ is replaced by any of $W(R_*)$ or $\mathcal{E}_R$, (for the $p$-adic norm), $\tilde{\mathcal{R}}_{R_*}^{\int,r}$ (for the norm $\lambda(\alpha^r)$), $\tilde{\mathcal{R}}_{R_*}^{\bd,r}$ (for the maximum of $\lambda(\alpha^r)$ and the $p$-adic norm), or $\tilde{\mathcal{R}}_{R_*}^{[s,r]}$ (for the norm $\max\{\lambda(\alpha^r), \lambda(\alpha^s)\}$).

Proof. This is a straightforward consequence of Lemma 3.1.11.

Theorem 5.3.4. Let $R \to S_1, \ldots, R \to S_n$ be a covering family of rational localizations. Then for any finite projective $R$-module $M$, the augmented Čech complex

$0 \to M \to M \otimes_R \left( \bigoplus_{i=1}^n S_i \right) \to M \otimes_R \left( \bigoplus_{i,j=1}^n S_i \otimes S_j \right) \to \cdots$

is exact, and similarly with $R$ replaced by $\ast_R$ for $\ast = \mathcal{E}, \tilde{\mathcal{R}}_{R_*}^{\int,r}, \tilde{\mathcal{R}}_{R_*}^{\bd,r}, \tilde{\mathcal{R}}_{R_*}^{[s,r]}$. (Note that the completed tensor products are again uniform by Remark 3.1.3.) Also, we may view $M$ as a finite Banach module as in Lemma 2.2.12, then replace the ordinary tensor products with completed tensor products without changing the underlying groups, to obtain a strict exact sequence.

Proof. By Proposition 2.4.15 and Remark 2.6.7, the claim reduces to the case of a simple Laurent covering. The claim then follows from Lemma 5.3.3 by tensoring the exact sequence (5.3.3.1) with $M$ (it remains exact because $M$ is projective).

Lemma 5.3.5. With notation as in Lemma 5.3.3, the maps $R \to R_1, R \to R_2, R \to R_{12}$ form a glueing square in the sense of Definition 2.7.3. The same holds with $R_*$ replaced by $\tilde{\mathcal{R}}_{R_*}^{\int,r}$ or $\tilde{\mathcal{R}}_{R_*}^{[s,r]}$ for any $0 < s \leq r$.

Proof. Note that $R_2 \to R_{12}$ has dense image because $f$ is already invertible in $R_2$, and that by construction, $\mathcal{M}(R_1 \oplus R_2) \to \mathcal{M}(R)$ is surjective. Hence $R \to R_1, R \to R_2, R \to R_{12}$ form a glueing square by Lemma 5.3.3. The other assertions follow similarly, keeping in mind Lemma 5.3.1 in order to get surjectivity of $\mathcal{M}(\tilde{\mathcal{R}}_{R_1}^{\int,r} \oplus \tilde{\mathcal{R}}_{R_2}^{\int,r}) \to \mathcal{M}(\tilde{\mathcal{R}}_{R_1}^{\int,r})$ and $\mathcal{M}(\tilde{\mathcal{R}}_{R_1}^{[s,r]} \oplus \tilde{\mathcal{R}}_{R_2}^{[s,r]}) \to \mathcal{M}(\tilde{\mathcal{R}}_{R}^{[s,r]}).

94
Theorem 5.3.6. Let \( \mathcal{F} \) be the category of finite projective modules over perfect uniform Banach \( \mathbb{F}_p \)-algebras, viewed as a cofibred category over the category \( \mathcal{C} \) of perfect uniform Banach \( \mathbb{F}_p \)-algebras as in Example 7.3.3. Let \( R \to S_1, \ldots, R \to S_n \) be a covering family of rational localizations. Then the morphisms \( R \to \bigoplus_i S_i, \tilde{\mathcal{R}}_{R}^{[s,r]} \to \bigoplus_i \tilde{\mathcal{R}}_{S_i}^{[s,r]}, \) and \( \tilde{\mathcal{R}}_{R}^{[s,r]} \to \bigoplus_i \tilde{\mathcal{R}}_{S_i}^{[s,r]} \) in \( \mathcal{C} \) are effective descent morphisms for \( \mathcal{F} \).

Proof. By Proposition 2.4.15, this reduces to Lemma 5.3.5 and Proposition 2.7.3.

Remark 5.3.7. Note that Theorem 5.3.6 cannot be extended to \( \tilde{\mathcal{R}}_{R}^{\text{bd},r} \), even though the latter is complete with respect to the maximum of \( \lambda(\alpha^r) \) and the \( p \)-adic norm. Adapting the given proof fails because under this norm, \( \tilde{\mathcal{R}}_{R}^{\text{bd},r} \) is not dense in \( \tilde{\mathcal{R}}_{R}^{\text{bd},r} \). In fact, no such extension is possible; see Example 8.1.14. Similar arguments point against an analogue of Theorem 5.3.6 for \( \tilde{\mathcal{E}}_{R} \).

Lemma 5.3.8. For \( 0 < s \leq s' \leq r \), the homomorphisms \( \tilde{\mathcal{R}}_{R}^{[s,r]} \to \tilde{\mathcal{R}}_{R}^{[s,s']}, \tilde{\mathcal{R}}_{R}^{[s,s']} \to \tilde{\mathcal{R}}_{R}^{[s',r]}, \tilde{\mathcal{R}}_{R}^{[s,r]} \to \tilde{\mathcal{R}}_{R}^{[s',s']} \) form a glueing square as in Definition 2.7.3. Consequently, by Proposition 2.7.3, any finite projective glueing datum gives rise to a unique (up to unique isomorphism) finite projective module over \( \tilde{\mathcal{R}}_{R}^{[s,r]} \).

Proof. We first check strict exactness of the sequence
\[
0 \to \tilde{\mathcal{R}}_{R}^{[s,r]} \to \tilde{\mathcal{R}}_{R}^{[s,s']} \oplus \tilde{\mathcal{R}}_{R}^{[s',s']} \to \tilde{\mathcal{R}}_{R}^{[s',r]} \to 0. \tag{5.3.8.1}
\]

The first nontrivial arrow is isometric and hence strict and injective. Strict exactness in the middle follows from Lemma 5.2.9. To check strict surjectivity of the last nontrivial arrow, we show that the map is in fact optimal; that is, every \( x \in \tilde{\mathcal{R}}_{R}^{[s',s']} \) can be written as \( y - z \) with \( y \in \tilde{\mathcal{R}}_{R}^{[s,s']}, z \in \tilde{\mathcal{R}}_{R}^{[s',r]} \), and
\[
\lambda(\alpha^s(y)) \leq \lambda(\alpha^r(x)) \quad (t \in [s, s'])
\]
\[
\lambda(\alpha^r(z)) \leq \lambda(\alpha^s(x)) \quad (t \in [s', r]).
\]

As in the proof of Lemma 5.2.7, this claim reduces to the case \( x \in \tilde{\mathcal{R}}_{R}^{\text{bd},s'} \). In this case, we argue as in Lemma 5.2.8. Write \( x = \sum_i p_i x_i \in \tilde{\mathcal{R}}_{R}^{\text{bd},s'} \), let \( y \) be the sum of \( p_i x_i \) over those indices \( i \) for which \( \alpha(x_i) \geq 1 \), and put \( z = y - x \). This decomposition of \( x \) has the desired form, and thus completes the proof of strict exactness of (5.3.8.1).

We next observe that \( \tilde{\mathcal{R}}_{R}^{[s,r]} \) has dense image in \( \tilde{\mathcal{R}}_{R}^{[s',s']} \), so the intermediate rings also have dense images. Finally, the map \( \mathcal{M}(\tilde{\mathcal{R}}_{R}^{[s,s']} \oplus \tilde{\mathcal{R}}_{R}^{[s',r]}) \to \mathcal{M}(\tilde{\mathcal{R}}_{R}^{[s,r]}) \) is surjective by Lemma 5.3.1. We thus verify conditions (a), (b), (c) of Definition 2.7.3, proving the claim.

Lemma 5.3.9. For some \( 0 < s \leq r \), let \( M \) be a finite projective module over \( \tilde{\mathcal{R}}_{R}^{[s,r]} \). Choose \( \beta \in \mathcal{M}(R) \), and choose \( e_1, \ldots, e_n \in M \) to form a set of module generators of \( M \otimes \tilde{\mathcal{R}}_{R}^{[s,r]} \tilde{\mathcal{R}}_{R}^{[s,r]} \). Then there exists a rational localization \( R \to R' \) encircling \( \beta \) such that \( e_1, \ldots, e_n \) also form a set of module generators of \( M \otimes \tilde{\mathcal{R}}_{R'}^{[s,r]} \tilde{\mathcal{R}}_{R'}^{[s,r]} \).

95
Proof. For each $\gamma \in \mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]})$, by Nakayama’s lemma, $e_1, \ldots, e_n$ form a set of module generators of $M \otimes_{\mathcal{R}_{H(\beta)}^{[s,r]}} S_\gamma$ for some rational localization $\mathcal{R}_{H(\beta)}^{[s,r]} \to S_\gamma$ encircling $\gamma$. We may cover $\mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]})$ with finitely many of the $\mathcal{M}(S_\gamma)$; by Remark 2.3.15(b), these also cover $\mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]})$ for some rational localization $R \to R'$ encircling $\beta$. It follows that $e_1, \ldots, e_n$ also form a set of module generators of $M \otimes_{\mathcal{R}_{H(\beta)}^{[s,r]}} H(\gamma)$ for each $\gamma \in \mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]})$; this implies the claim using Lemma 2.3.12(a). \qed

5.4 Some geometric observations

We mention in passing some observations concerning the geometry of the spaces $\mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]})$, in the spirit of [72]. In the process, we extend some results from [72] by relaxing the restriction to the trivial norm; this extends the process begun by Proposition 5.1.2.

Theorem 5.4.1. Define $\lambda : \mathcal{M}(R) \to \mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]})$, $\mu : \mathcal{M}(\mathcal{R}_{H(\beta)}^{[s,r]}) \to \mathcal{M}(R)$ as in Proposition 5.1.2.

(a) The maps $\lambda$ and $\mu$ are continuous. Moreover, the inverse image under either map of a finite union of Weierstrass (resp. Laurent, rational) subdomains has the same form.

(b) For all $\beta \in \mathcal{M}(R)$, $(\mu \circ \lambda)(\beta) = \beta$.

(c) For all $\gamma \in \mathcal{M}(W(R))$, $(\lambda \circ \mu)(\gamma) \geq \gamma$.

Proof. The proof of [72] Theorem 4.5] carries over without change. \qed

Lemma 5.4.2. Let $R \to S$ be a bounded homomorphism of perfect uniform Banach $\mathbb{F}_p$-algebras, such that $\mathcal{M}(S) \to \mathcal{M}(R)$ is surjective. Then for any $r > 0$, the map $\mathcal{M}(\mathcal{R}_{H(\beta)}^{int,r}) \to \mathcal{M}(\mathcal{R}_{H(\beta)}^{int,r})$ is also surjective.

Proof. Equip $R[T]$ and $S[T]$ with the $p^{-1}$-Gauss norm, and let $R'$ and $S'$ be the completions of $R[T]^{perf}$ and $S[T]^{perf}$. We may then identify $S'$ with $S \otimes_R R'$; since $\mathcal{M}(S) \to \mathcal{M}(R)$ is surjective, so is $\mathcal{M}(S') \to \mathcal{M}(R')$ by [72] Lemma 1.20.

Given $\gamma \in \mathcal{M}(\mathcal{R}_{H(\beta)}^{int,r})$, put $\beta = \mu(\gamma)$ and $\mathfrak{o} = \mathfrak{o}_{H(\beta)}$, extend $\gamma$ to $\mathcal{R}_{H(\beta)}^{int,r}$ by continuity, then restrict to $W(\mathfrak{o})$. Let $\mathfrak{o}'$ be the completion of $\mathfrak{o}[T]$ for the $p^{-1}$-Gauss norm; as in [72] Definition 7.5], we may extend $\gamma$ from $W(\mathfrak{o})$ to a seminorm $\gamma'$ on $W(\mathfrak{o}')$ in such a way that $\gamma'(p - T) = 0$. By [72] Remark 5.14], this extension computes the quotient norm on $W(\mathfrak{o}')/(p - T)$ induced by $\lambda(\beta')$ for $\beta' = \mu(\gamma')$.

Choose $\beta \in \mathcal{M}(S)$ lifting $\beta$, put $\mathfrak{o} = \mathfrak{o}_{H(\beta)}$, and let $\mathfrak{o}'$ be the completion of $\mathfrak{o}[T]$ for the $p^{-1}$-Gauss norm. We may then identify $\mathfrak{o}'$ with $\mathfrak{o} \otimes_{\mathfrak{o}} \mathfrak{o}'$; by [72] Lemma 1.20], the map $\mathcal{M}(\mathfrak{o}') \to \mathcal{M}(\mathfrak{o}')$ is surjective.

We can thus lift $\beta'$ to a seminorm $\tilde{\beta}'$ on $\mathfrak{o}'$. Let $\tilde{\beta}'$ be the quotient norm on $W(\mathfrak{o}')/(p - T)$ induced by $\lambda(\tilde{\beta}')$, viewed as a seminorm on $W(\mathfrak{o}')$. We may then restrict $\tilde{\beta}'$ to a seminorm $\tilde{\beta}$ on $W(\mathfrak{o})$, extend multiplicatively to $\mathcal{R}_{H(\beta)}^{int,r}$, then restrict to $\mathcal{R}_{H(\beta)}^{int,r}$. This proves the claim. \qed
Definition 5.4.3. Choose \( r > 0 \) and \( \gamma \in \tilde{\mathcal{R}}^\text{int,}^r \). For \( \beta = \mu(\gamma)^{1/r} \), we may extend \( \gamma \) to \( \tilde{\mathcal{R}}^\text{int,}^r \) and then restrict to \( W(\mathfrak{o}_{H(\beta)}) \). We may then define the multiplicative seminorm \( H(\gamma, t) \) on \( W(\mathfrak{o}_{H(\beta)}) \) as in [72, Definition 7.5], extend multiplicatively to \( \tilde{\mathcal{R}}^\text{int,}^r \), then restrict back to \( \tilde{\mathcal{R}}^\text{int,}^r \). From [72, Theorem 7.8], the construction has the following properties.

(a) We have \( H(\gamma, 0) = \gamma \).
(b) We have \( H(\gamma, 1) = (\lambda \circ \mu)(\gamma) \).
(c) For \( t \in [0, 1] \), \( \mu(H(\gamma, t)) = \mu(\gamma) \).
(d) For \( t, u \in [0, 1] \), \( H(H(\gamma, t), u) = H(\gamma, \max\{t, u\}) \).

Theorem 5.4.4. For any \( r > 0 \), the map \( H : \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r) \times [0, 1] \to \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r) \) is continuous.

Proof. Recall that for the trivial norm on \( R \), the map \( H : \mathcal{M}(W(R)) \times [0, 1] \to \mathcal{M}(W(R)) \) is continuous by [72, Theorem 7.8]. In case \( \alpha \) is not trivial but is dominated by the trivial norm, if we let \( S \) be a copy of \( R \) equipped with the trivial norm, we may identify \( \mathcal{M}(W(R)) \) with a closed subspace of \( \mathcal{M}(W(S)) \) and deduce the claim from the previous assertion.

For general \( R \), equip the Laurent polynomial ring \( R[T^\pm] \) with the \( p^{-1} \)-Gauss norm, and let \( S \) be the completion of \( R[T^\pm] \) perf. Since every element of \( S \) equals an element of \( \mathfrak{o}_S \) times a unit, we may identify \( \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r_S) \) with a closed subspace of \( \mathcal{M}(W(\mathfrak{o}_S)) \). By the previous paragraph, the map \( H : \mathcal{M}(W(\mathfrak{o}_S)) \times [0, 1] \to \mathcal{M}(W(\mathfrak{o}_S)) \) is continuous; we thus obtain a continuous map \( \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r_S) \times [0, 1] \to \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r_S) \). We can now construct a diagram

\[
\begin{array}{ccc}
\mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r_S) \times [0, 1] & \to & \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r) \\
\downarrow & & \downarrow \\
\mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r_S) \times [0, 1] & \to & \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}^r)
\end{array}
\]

in which the diagonal arrow and the vertical arrow are continuous. Since the vertical arrow is a surjective (by Lemma 5.4.2) continuous map between compact spaces, it is a quotient map by Remark 2.3.15(b). Hence the horizontal arrow is continuous, as desired.

Remark 5.4.5. One can go further with analysis of this sort; for instance, one can show that the fibres of \( \mu \) bear a strong resemblance to the spectra of one-dimensional affinoid algebras over a nontrivially normed analytic field. See [72, §8].

Proposition 5.4.6. Suppose that \( R \) is free of trivial spectrum (as in Definition 2.3.8). Define the topological space

\[ T_R = \bigcup_{0 < s < r} \mathcal{M}(\tilde{\mathcal{R}}^\text{int,}[s, r]). \]

(a) For each \( \beta \in T_R \), there is a unique value \( t \in (0, +\infty) \) for which \( \alpha^t \) dominates \( \mu(\beta) \) (or equivalently, \( \lambda(\alpha^t) \) dominates \( \beta \)).
(b) Let \( t : T_R \to (0, +\infty) \) be the map described in (a). Then the formula \( \beta \mapsto (\mu(\beta)^{1/t(\beta)}, t(\beta)) \) defines a continuous map \( T_R \to \mathcal{M}(R) \times (0, +\infty) \). In particular, \( t \) is continuous.

(c) The group \((\varphi^*)^\mathbb{Z}\) acts properly discontinuously on \( T_R \) with compact quotient \( X_R \). (Note that the map in (b) induces a continuous map \( X_R \to \mathcal{M}(R) \times S^1 \).

(d) The map \( T_R \to \mathcal{M}(R) \times (0, +\infty) \) is a strong deformation retract, and induces a strong deformation retract \( X_R \to \mathcal{M}(R) \times S^1 \).

**Proof.** To check (a), note that the existence of \( t \) is guaranteed by Lemma 5.3.4 while the uniqueness of \( t \) is guaranteed because \( R \) is free of trivial spectrum.

To check (b), choose \( \beta_0 \in T_R \) and put \( t_0 = t(\beta_0) \). Let \( U \) be any open neighborhood of \( \mu(\beta_0) \) of the form \( \{ \gamma \in \mathcal{M}(R) : \gamma(f_1) \in I_1, \ldots, \gamma(f_n) \in I_n \} \) for some \( f_1, \ldots, f_n \in R \) and some open intervals \( I_1, \ldots, I_n \). Let \((a, b)\) be any subinterval of \((0, +\infty)\) containing \( t_0 \). Choose \( \overline{z} \in R \) for which \( 0 < \mu(\beta_0)(\overline{z}) < 1 \), and put \( z = |\overline{z}| \). Choose \( \delta > 0 \) such that \( a < t_0/\delta, t_0\delta < b \). For \( i = 1, \ldots, n \), choose an open neighborhood \( J_i \) of \( \beta_0(f_i) \) such that for all \( x \in J_i \) and all \( u \in [1/\delta, \delta], x^{1/(\mu(x))} \in J_i \). Put

\[
V = \{ \gamma \in T_R : \gamma(f_i) \in J_1, \ldots, \gamma(f_n) \in J_n, \gamma(z) \in (\beta_0(z)^{t_0/\delta}, \beta_0(z)^{t_0\delta}) \};
\]

this is an open subset of \( T_R \) with the property that for any \( \gamma \in V \), \( \mu(\gamma)^{1/t(\gamma)} \in U \) and \( t(\gamma) \in (a, b) \). This gives the desired continuity.

To check (c), first apply (b) after observing that for all \( \beta \in T_R \), \( t(\varphi^*(\beta)) = pt(\beta) \). We see from this that the action is properly discontinuous, so \( X_R \) is Hausdorff. Then note that for any \( r > 0 \), the projection \( \mathcal{M}(\check{R}_R^{[r/p, r]}) \to X_R \) is surjective. Since \( X_R \) receives a surjective continuous map from a compact space, it is quasicompact (Remark 2.3.15(a)) and hence compact.

To check (d), argue as in the proof of Theorem 5.4.4 to produce a continuous map \( H : \mathcal{M}(\check{R}_R^{\text{int}, r}) \times [0, 1] \to \mathcal{M}(\check{R}_R^{\text{int}, r}) \). Then observe by Lemma 5.3.4 that the image of \( \mathcal{M}(\check{R}_R^{\text{int}, r}) \times \{ 1 \} \) may be identified with \( \mathcal{M}(R) \times (0, r] \) by mapping \((\gamma, s)\) to \( \lambda(\gamma^s) \), and that the resulting map \( \mathcal{M}(\check{R}_R^{\text{int}, r}) \to \mathcal{M}(R) \times (0, r] \) is precisely \( T_R \).

**Remark 5.4.7.** When \( \mathcal{M}(R) \) is contractible, Proposition 5.4.6 asserts that \( X_R \) has the homotopy type of a circle, \( T_R \) is the universal covering space of \( X_R \), and \( \varphi^* \) acts on \( T_R \) as a deck transformation generating the fundamental group. For instance, this is the case when \( R = L \) is a nontrivially normed analytic field; in this case, the étale fundamental group of \( X_R \) is an extension of the topological fundamental group \( \mathbb{Z} \) by the Galois group of \( L \). When \( L \) is a finite extension of \( \mathbb{F}_p((\pi)) \), this suggests a relationship with the Weil group of the field \( \check{R}_L^{\text{int}, 1}/(z) \) for \( z = \sum_{i=0}^{p-1}[1 + \pi]^{i/p} \) (this field being a finite extension of \( \mathbb{Q}_p(\mu_{p^{\infty}}) \)). However, this relationship remains to be clarified.

5.5 Compatibility with finite étale extensions

We next establish a compatibility between the construction of extended Robba rings and formation of finite étale ring extensions. As promised earlier, this yields a variant of Faltings’s almost purity theorem, thus refining the perfectoid correspondence introduced in [3.6].

98
Convention 5.5.1. For $S \in \mathbf{F}{\text{Ét}}(R)$, we will always equip $S$ with the norm provided by Lemma 5.1.7. As noted in Remark 5.1.8, this makes $S$ both a perfect uniform Banach $\mathbf{F}_p$-algebra and a finite Banach $R$-module.

Lemma 5.5.2. Let $\psi : R \to S$ be a bounded homomorphism from $R$ to a perfect uniform $\mathbf{F}_p$-algebra $S$ with norm $\beta$. Use $\psi$ to view $S$ as an $R$-algebra. Let $x_1, \ldots, x_n$ be elements of $\mathcal{R}_S^\text{int}$ whose reductions $\overline{x}_1, \ldots, \overline{x}_n$ modulo $p$ define a strict surjection $R^n \to S$ of Banach modules over $R$. Then $x_1, \ldots, x_n$ generate $\mathcal{R}_S^\text{int}$ as a module over $\mathcal{R}_R^\text{int}$.

In case $R$ is a Banach algebra over a nontrivially normed analytic field, it is enough to assume that $\overline{x}_1, \ldots, \overline{x}_n$ generate $S$ as an $R$-module, by Theorem 2.2.8.

Proof. It is harmless to assume that $\overline{x}_1, \ldots, \overline{x}_n$ are all nonzero. Since the surjection $R^n \to S$ is strict, we can find $c \geq 1$ such that for each $\overline{x} \in S$, there exist $\overline{x}_1, \ldots, \overline{x}_n \in R$ for which $\overline{x} = \sum_{i=1}^n \overline{a}_i \overline{x}_i$ and $\alpha(\overline{a}_i) \beta(\overline{x}_i) \leq c \beta(\overline{x})$ for $i = 1, \ldots, n$.

Given $z \in \mathcal{R}_S^\text{int}$, for $l = 0, 1, \ldots$ we choose $z_l \in \mathcal{R}_S^\text{int}$ and $a_{l,1}, \ldots, a_{l,n} \in \mathcal{R}_R^\text{int}$ as follows. Put $z_0 = 0$. Given $z_l$, let $\overline{z}_l$ be its reduction modulo $p$, and invoke the previous paragraph to construct $\overline{a}_{l,1}, \ldots, \overline{a}_{l,n} \in R$ with $\alpha(\overline{a}_{l,i}) \beta(\overline{x}_i) \leq c \beta(\overline{x}_i)$ for $i = 1, \ldots, n$ such that $\overline{z}_l = \sum_{i=1}^n \overline{a}_{l,i} \overline{x}_i$. Then put $a_{l,i} = [\overline{a}_{l,i}]$ and $z_{l+1} = p^{-1}(z_l - \sum_{i=1}^n a_{l,i} x_i)$.

For all sufficiently small $r > 0$, we have $x_1, \ldots, x_n, z \in \mathcal{R}_S^\text{int}$ and $\lambda(\beta^r)(x_i - \overline{a}_{l,i}) < \beta(\overline{x}_i)^r$. For each such $r$, we have $\lambda(\beta^r)(z_{l+1}) \leq c^r p \lambda(\beta^r)(z_l)$, and so $\lambda(\beta^r)(z_l) \leq (c^r p^l \lambda(\beta^r))(z_0)$. In particular, $\alpha(\overline{a}_{l,i}) \leq c \beta(\overline{x}_i)^{-1} \lambda(\beta^r)(z_0)^{1/r}(cp^{1/r})^l$.

It follows that the series $\sum_{i=0}^{\infty} p^l a_{l,i}$ converges under $\lambda(\alpha^s)$ whenever $c^s p^{s/r-1} < 1$. Since this holds for any sufficiently small $s > 0$ (depending on $r$), $z$ belongs to the $\mathcal{R}_R^\text{int}$-span of $x_1, \ldots, x_n$. This proves the claim.

Proposition 5.5.3. Choose any $r > 0$.

(a) The base extension functors

$$\mathbf{F}{\text{Ét}}(\mathcal{R}_R^\text{int}, r) \to \mathbf{F}{\text{Ét}}(\mathcal{R}_R^\text{int}) \to \mathbf{F}{\text{Ét}}(W(R)) \to \mathbf{F}{\text{Ét}}(R)$$

are tensor equivalences.

(b) For $S \in \mathbf{F}{\text{Ét}}(R)$, the corresponding element of $\mathbf{F}{\text{Ét}}(\mathcal{R}_R^\text{int}, r)$ may be identified with $\mathcal{R}_S^\text{int}$.

Proof. For each $r > 0$, $\mathcal{R}_R^\text{int}$ is complete with respect to the maximum of $\lambda(\alpha^s)$ and the $p$-adic norm. For these norms, the maps $\mathcal{R}_R^\text{int} \to \mathcal{R}_R^\text{int,s}$ for $0 < s \leq r$ are submetric by Lemma 5.2.1. Consequently, Lemma 2.2.3(b) implies that the pair $(\mathcal{R}_R^\text{int}, (p))$ is henselian. By Theorem 1.2.8 the functors $\mathbf{F}{\text{Ét}}(\mathcal{R}_R^\text{int}) \to \mathbf{F}{\text{Ét}}(W(R)) \to \mathbf{F}{\text{Ét}}(R)$ are tensor equivalences.

Given $S \in \mathbf{F}{\text{Ét}}(R)$, by applying the previous paragraph to the isomorphism $\mathcal{S}(S) \cong S$, for some nonnegative integer $n$ we obtain $U \in \mathbf{F}{\text{Ét}}(\mathcal{R}_R^\text{int}, r/p^n)$ with $U/p \cong S$ admitting an isomorphism $\varphi^*(U) \cong U \otimes_{\mathcal{R}_R^\text{int}, r/p^n} \mathcal{R}_R^\text{int}$. By pulling back along $\varphi^*$, we obtain analogous
data with $n$ replaced by $n - 1$. Repeating, we eventually obtain an element of $\mathbf{F}^\text{ét}(\hat{R}_{R}^{\text{int}, r})$ lifting $S$. Consequently, the functor $\mathbf{F}^\text{ét}(\hat{R}_{R}^{\text{int}, r}) \to \mathbf{F}^\text{ét}(R)$ is essentially surjective. A similar argument gives full faithfulness; this yields (a).

To prove (b), take $S \in \mathbf{F}^\text{ét}(R)$; it is sufficient to check that $\hat{R}_{R}^{\text{int}} \in \mathbf{F}^\text{ét}(\hat{R}_{R}^{\text{int}})$, as one may then conclude by applying $\varphi^{-1}$ as in the proof of (a). Construct $U_S \in \mathbf{F}^\text{ét}(\hat{R}_{R}^{\text{int}})$ corresponding to $S$ via (a). We may identify $U_S/(p)$ and $\hat{R}_{S}^{\text{int}}/(p)$ with $S$; the $p$-adic completions of $U_S$ and $\hat{R}_{S}^{\text{int}}$ may then be identified with $W(S)$ by the uniqueness property of the latter.

Let $\pi_1, \pi_2 : S \to S \otimes_R S$ denote the structure morphisms. Put $V = U_S \otimes_{\hat{R}_{R}^{\text{int}}} \hat{R}_{R}^{\text{int}}$, and let $\tilde{\pi}_1 : U_S \to V$ and $\tilde{\pi}_2 : \hat{R}_{R}^{\text{int}} \to V$ denote the structure morphisms. Note that $\tilde{\pi}_2$ is the distinguished lift of $\pi_2$ from $\mathbf{F}^\text{ét}(S)$ to $\mathbf{F}^\text{ét}(\hat{R}_{S}^{\text{int}})$ provided by (a). Consequently, if we view the multiplication map $\mu : S \otimes_R S \to S$ as a map in $\mathbf{F}^\text{ét}(S)$ by equipping $S \otimes_R S$ with the structure morphism $\pi_2$, then (a) provides a lift $\tilde{\mu}$ of $\mu$ to $\mathbf{F}^\text{ét}(\hat{R}_{S}^{\text{int}})$. The composition $\psi = \tilde{\mu} \circ \tilde{\pi}_1 : U_S \to \hat{R}_{S}^{\text{int}}$ lifts the identity map modulo $p$. As noted above, the injection $U_S \to W(S)$ factors through $\psi$, so $\psi$ is injective; since $\psi$ is $\hat{R}_{R}^{\text{int}}$-linear, it is also surjective by Lemma 5.5.2 and Convention 5.5.1. This yields (b). □

**Proposition 5.5.4.** Let $S$ be a (faithfully) finite étale $R$-algebra. Then for

$$* \in \{ \tilde{\mathcal{E}}, \tilde{\mathcal{R}}^{\text{int}, r}, \tilde{\mathcal{R}}^{\text{int}}, \tilde{\mathcal{R}}^{\text{bd}}, \tilde{\mathcal{R}}^{\text{bd}, r}, \tilde{\mathcal{R}}^{[s, r]}, \tilde{\mathcal{R}}, \tilde{\mathcal{R}}, \}$

the natural homomorphism $*_R \to *_{S}$ is (faithfully) finite étale.

**Proof.** The cases $* = \tilde{\mathcal{E}}, \tilde{\mathcal{R}}^{\text{int}, r}, \tilde{\mathcal{R}}^{\text{int}}, \tilde{\mathcal{R}}^{\text{bd}}, \tilde{\mathcal{R}}^{\text{bd}, r}$ follow at once from Proposition 5.5.3. To handle the cases $* = \tilde{\mathcal{R}}^{[s, r]}, \tilde{\mathcal{R}}^r, \tilde{\mathcal{R}}^{r}$ it suffices to check that the natural map $\tilde{\mathcal{R}}_{R}^{r} \otimes_{\tilde{\mathcal{R}}_{R}^{\text{bd}, r}} \tilde{\mathcal{R}}_{S}^{\text{bd}, r} \to \tilde{\mathcal{R}}_{S}^{r}$ is an isometric isomorphism with respect to $\lambda(\alpha^r)$. We proceed by first noticing that $\tilde{\mathcal{R}}_{R}^{r} \otimes_{\tilde{\mathcal{R}}_{R}^{\text{bd}, r}} \tilde{\mathcal{R}}_{S}^{\text{bd}, r} = \tilde{\mathcal{R}}_{R}^{r} \otimes_{\tilde{\mathcal{R}}_{R}^{\text{bd}}, \tilde{\mathcal{R}}_{S}^{\text{bd}, r}} \tilde{\mathcal{R}}_{S}^{\text{bd}, r}$ because $\tilde{\mathcal{R}}_{S}^{\text{bd}, r}$ is a finite projective $\tilde{\mathcal{R}}_{R}^{\text{bd}, r}$-module (see Definition 1.2.1). We may then argue as in Lemma 5.5.2 that $\tilde{\mathcal{R}}_{R}^{r} \otimes_{\tilde{\mathcal{R}}_{R}^{\text{bd}, r}} \tilde{\mathcal{R}}_{S}^{\text{bd}, r} \to \tilde{\mathcal{R}}_{S}^{r}$ is a strict surjection for sufficiently small $r > 0$, and hence for all $r > 0$ by applying $\varphi^{-1}$ as needed. In particular, there exists $c > 0$ (depending on $r$) for which any element $z \in \tilde{\mathcal{R}}_{S}$ can be lifted to $\sum_i x_i \otimes y_i \in \tilde{\mathcal{R}}_{R}^{r} \otimes_{\tilde{\mathcal{R}}_{R}^{\text{bd}, r}} \tilde{\mathcal{R}}_{S}^{\text{bd}, r}$ with $\max\{\lambda(\alpha^r)(x_i y_i)\} \leq c \lambda(\alpha^r)(z)$. Finally, given a nonzero element $z \in \tilde{\mathcal{R}}_{S}$, choose $z_0 \in \tilde{\mathcal{R}}_{S}^{\text{bd}, r}$ with $\lambda(\alpha^r)(z - z_0) < c^{-1} \lambda(\alpha^r)(z)$, and lift $z - z_0$ to $\sum_i x_i \otimes y_i$ with $\max\{\lambda(\alpha^r)(x_i y_i)\} \leq c \lambda(\alpha^r)(z - z_0)$. The representation $1 \otimes z_0 + \sum_i x_i \otimes y_i$ of $z$ then shows that the map $\tilde{\mathcal{R}}_{R}^{r} \otimes_{\tilde{\mathcal{R}}_{R}^{\text{bd}, r}} \tilde{\mathcal{R}}_{S}^{\text{bd}, r} \to \tilde{\mathcal{R}}_{S}^{r}$, which is evidently submetric, is in fact isometric. In particular, it is injective, completing the argument. □

To link these results to almost purity, we use the following extension of Lemma 3.3.6.

**Lemma 5.5.5.** For any $z \in W(\alpha R)$ primitive of degree 1 and any $r \geq 1$, the map $W(\alpha R)[[\mathcal{Z}^{-1}]]/(z) \to \tilde{\mathcal{R}}_{R}^{\text{int}, r}/(z)$ is bijective. More precisely, any $x \in \tilde{\mathcal{R}}_{R}^{\text{int}, r}/(z)$ lifts to $y = \sum_{i=0}^{\infty} p^i [\overline{\mathcal{Y}}_i] \in W(R)$ with $\alpha(\overline{\mathcal{Y}}_i) \geq \alpha(\overline{\mathcal{Y}}_i)$ for all $i$.

**Proof.** Since $z$ is primitive of degree 1, we have $p = y(z - [\mathcal{Z}])$ for some unit $y \in W(\alpha R)$. Given $x = \sum_{i=0}^{\infty} p^i [\overline{\mathcal{Y}}_i] \in \tilde{\mathcal{R}}_{R}^{\text{int}, r}$, the series $\sum_{i=0}^{\infty} (-y)^i [\overline{\mathcal{Y}}, \overline{\mathcal{Z}}]$ converges to an element of $W(\alpha R)[[\mathcal{Z}^{-1}]]$
congruent to \( x \) modulo \( z \). This proves surjectivity of the map; the existence of lifts of the desired form follows by applying Lemma \( \ref{lem:lifting} \) to \( x[z]^n \) for a large positive integer \( n \).

To check injectivity, by Lemma \( \ref{lem:lifting} \) it is enough to check that for \( y = \sum_{i=0}^{\infty} p^i \bar{y}_i \in W(\mathfrak{o}_R) \) with \( \alpha(\bar{y}_0) \geq \alpha(\bar{y}_i) \) for all \( i \), \( y \) cannot be divisible by \( z \) in \( \tilde{R}_{\mathbb{A}(\beta)}^{\text{int},r} \) if it is nonzero. To see this, pick \( \beta \in \mathcal{M}(R) \) such that \( \beta(\bar{y}_0) > p^{-1} \alpha(\bar{y}_0) \); we then see that the image of \( y \) in \( \tilde{R}_{\mathbb{A}(\beta)}^{\text{int},r} \) cannot be divisible by \( z \) therein by considering Newton polygons (as in Definition \( \ref{def:newton} \)).

**Corollary 5.5.6.** For any \( z \in W(\mathfrak{o}_R) \) primitive of degree 1 and any \( r \geq 1 \), for \( A = W(\mathfrak{o}_R)[[\bar{z}]]/(z) = \tilde{R}_R^{\text{int},r}/(z) \) (by Lemma \( \ref{lem:lifting} \)), we have a 2-commuting diagram

\[
\begin{array}{ccc}
\text{FÉt}(\tilde{R}_R^{\text{int},r}) & \longrightarrow & \text{FÉt}(A) \\
\downarrow & & \downarrow \\
\text{FÉt}(R) & \longrightarrow & \\
\end{array}
\]  

(5.5.6.1)

in which the solid arrows are base extensions and the dashed arrow is the one provided by Theorem \( \ref{thm:main} \). In particular, each of these is a tensor equivalence.

**Proof.** The commutativity comes from Lemma \( \ref{lem:lifting} \). The dashed arrow is a tensor equivalence by Theorem \( \ref{thm:main} \) while the vertical arrow is an equivalence by Proposition \( \ref{prop:tensor} \). \( \square \)

**Remark 5.5.7.** Note that Lemma \( \ref{lem:lifting} \) provides an alternate proof of Lemma \( \ref{lem:almost} \) (and even Lemma \( \ref{lem:smooth} \)) by tracing along the solid arrows rather than the dashed arrow. This refinement makes it possible to study the effect of Frobenius on the perfectoid correspondence, leading to the almost purity theorem (Theorem \( \ref{thm:almost} \)).

In fact, it should be possible to give an alternate proof of Theorem \( \ref{thm:main} \) by directly establishing essential surjectivity of \( \text{FÉt}(\tilde{R}_R^{\text{int},r}) \rightarrow \text{FÉt}(A) \) using lifting arguments for smooth algebras, as in the work of Elkik \( \ref{ref:elkik} \) and Arabia \( \ref{ref:arabia} \). Thanks to Arthur Ogus for this suggestion.

To assert an almost purity theorem, we need a few definitions from almost ring theory; for these we follow \( \ref{ref:almost} \).

**Definition 5.5.8.** Fix an analytic field \( F \). An \( \mathfrak{o}_F \)-module is _almost zero_ if it is killed by \( \mathfrak{m}_F \). The category of _almost modules_ over \( \mathfrak{o}_F \) is the localization of the category of \( \mathfrak{o}_F \)-modules at the set of morphisms with almost zero kernel and cokernel.

Let \( A \) be an \( \mathfrak{o}_F \)-algebra. An \( A \)-module \( B \) is _almost finite projective_ if for each \( t \in \mathfrak{m}_F \), there exist a finite free \( A \)-module \( F \) and some morphisms \( B \rightarrow F \rightarrow B \) of \( A \)-modules whose composition is multiplication by \( t \). (See \( \ref{ref:almost} \) Lemma 2.4.15 for some equivalent formulations.)

For \( B \) an \( A \)-algebra whose underlying \( A \)-module is uniformly almost finite projective, there is a well-defined trace map \( \text{Trace} : B \rightarrow A \) in the category of almost modules over \( \mathfrak{o}_F \) \( \ref{ref:almost} \) §4.1.7; we say that \( B \) is _almost finite étale_ if the trace pairing induces an almost isomorphism \( B \rightarrow \text{Hom}_A(B, A) \). This is not the definition used in \( \ref{ref:almost} \), but is equivalent to it via \( \ref{ref:almost} \) Theorem 4.1.14.

101
We are now ready to fulfill the promise made in Remark 3.6.23. The key new ingredient provided by relative Robba rings is an action of Frobenius (or more precisely its inverse) in characteristic 0, which can be used in much the same way that Frobenius can be used to give a cheap proof of almost purity in positive characteristic (see Remark 3.1.10 and then [44, Chapter 3]).

**Theorem 5.5.9** (Almost purity). Let $F$ be a perfectoid analytic field, and let $A$ be a perfectoid uniform Banach $F$-algebra. Then for any $B \in \mathrm{F}\acute{e}t(A)$, $\mathfrak{o}_B$ is uniformly almost finite projective and almost étale over $\mathfrak{o}_A$.

**Proof.** Let $(L, I)$ be the pair corresponding to $F$ as in Theorem 3.5.3. By Theorem 3.6.20, $B$ is also perfectoid. Let $R, S$ correspond to $A, B$ via Theorem 3.6.5, so that $S \in \mathrm{F}\acute{e}t(A)$ by Theorem 3.6.20 again. Write $\mathfrak{o}_A$ as shorthand for $\mathfrak{o}_{\mathfrak{R}_{\mathrm{int}}}$. By Lemma 5.5.5, for each nonnegative integer $n$, we have isomorphisms $\mathfrak{o}_{R}^m/\mathfrak{o}_{R}^n \cong \mathfrak{o}_A$, $\mathfrak{o}_{S}^m/\mathfrak{o}_{S}^n \cong \mathfrak{o}_B$.

By Proposition 5.5.3, $\mathfrak{R}_{S}^{\mathrm{int}, 1}$ is the object of $\mathrm{F}\acute{e}t(\mathfrak{R}_{R}^{\mathrm{int}, 1})$ corresponding to $S$. In particular, $\mathfrak{R}_{S}^{\mathrm{int}, 1}$ is a finite projective $\mathfrak{R}_{R}^{\mathrm{int}, 1}$-module, so there exist morphisms $\mathfrak{R}_{S}^{\mathrm{int}, 1} \to (\mathfrak{R}_{R}^{\mathrm{int}, 1})^m \to \mathfrak{R}_{S}^{\mathrm{int}, 1}$ of $\mathfrak{R}_{R}^{\mathrm{int}, 1}$-modules whose composition is the identity. For a suitable $\mathfrak{p} \in \mathfrak{p}_1$, we may multiply through to obtain morphisms $\mathfrak{o}_S^m \to (\mathfrak{o}_R^m)^m \to \mathfrak{o}_S^m$ of $\mathfrak{o}_R^m$-modules whose composition is multiplication by $\varphi^{-n}$. By applying $\varphi^{-n}$ and then quotienting by $I$, we obtain morphisms $\mathfrak{o}_B \to \mathfrak{o}_B^m \to \mathfrak{o}_B$ of $\mathfrak{o}_A$-modules whose composition is multiplication by $\theta([\mathfrak{p}^{-n}])$. Since the norm of $\theta([\mathfrak{p}^{-n}])$ tends to 1 as $n \to \infty$, it follows that $\mathfrak{o}_B$ is almost finite projective over $\mathfrak{o}_A$. Similarly, starting from the perfection of the trace pairing on $\mathfrak{R}_{S}^{\mathrm{int}, 1}$ over $\mathfrak{R}_{R}^{\mathrm{int}, 1}$, then applying $\varphi^{-n}$, and finally quotienting by $I$, we deduce that the trace pairing defines an almost isomorphism $\mathfrak{o}_B \to \text{Hom}_{\mathfrak{o}_A}(\mathfrak{o}_B, \mathfrak{o}_A)$. \hfill \blacksquare

**Remark 5.5.10.** The original almost purity theorem of Faltings [37, 38] differs a bit in form from Theorem 5.5.9 in that it refers to a specific construction to pass from a suitable affinoid algebra over a complete discretely valued field of mixed characteristics to a perfectoid algebra. We will encounter this construction, which uses toric local coordinates, in a subsequent paper.

After Faltings introduced the concept of almost purity, and the broader context of almost ring theory, it emerged that his results could be generalized quite significantly. An abstract framework for such results has been introduced by Gabber and Ramero [44], and used by them to establish significant generalizations of Faltings’s theorem [45].

Theorem 5.5.9 appears to be of about the same strength as the main result of [45], but the proof is much simpler. In place of some complicated analysis in the style of Grothendieck’s proof of Zariski-Nagata purity (as in the original work of Faltings), the proof of Theorem 5.5.9 ultimately rests on the local nature of the perfectoid correspondence.

An independent derivation of Theorem 5.5.9 based on the same set of ideas, has been given by Scholze [91]. Scholze goes further, extending the perfectoid correspondence and the almost purity theorem to a certain class of adic analytic spaces; he then uses these to establish relative versions of the de Rham-étale comparison isomorphism in $p$-adic Hodge theory [92]. We will make contact with the latter results later in this series.

102
6 \( \varphi \)-modules

We now introduce \( \varphi \)-modules over the rings introduced in \( \S 5 \). In order to avoid some headaches later when working in the relative setting, we expend some energy here to relate \( \varphi \)-modules to more geometrically defined concepts, including a relative analogue of the vector bundles considered by Fargues and Fontaine \( [39] \).

**Hypothesis 6.0.1.** Throughout \( \S 6 \) continue to retain Hypothesis \( 5.0.1 \). In addition, let \( a \) denote a positive integer, and put \( q = p^a \).

### 6.1 \( \varphi \)-modules and \( \varphi \)-bundles

**Definition 6.1.1.** A \( \varphi^a \)-module over \( W(R) \) (resp. \( \bar{\mathcal{E}}_R, \bar{\mathcal{E}}_R^{\int}, \bar{\mathcal{E}}_R^{\bd}, \bar{\mathcal{E}}_R^+, \bar{\mathcal{E}}_R^\infty \)) is a finite locally free module \( M \) equipped with a semilinear \( \varphi^a \)-action. For example, one may take the direct sum of one or more copies of the base ring and use the action of \( \varphi^a \) on the ring; any such \( \varphi^a \)-module is said to be trivial.

For \( \ast \in \{ \bar{\mathcal{E}}_R^{\int}, \bar{\mathcal{E}}_R^{\bd}, \bar{\mathcal{E}}_R \} \) and \( r > 0 \), note that any \( \varphi^a \)-module over \( \ast_R \) descends uniquely to a finite locally free module \( M_r \) over \( \ast_R^r \) equipped with an isomorphism \( \varphi^a M_r \cong M_r \otimes \ast_R^r \ast_R^{\ast/r} \) of modules over \( \ast_R^r \). (The argument is as in the proof of Proposition \( 5.5.3(a) \).) We call \( M_r \) the model of \( M \) over \( \ast_R^r \).

Unfortunately, it is not straightforward to deal with \( \varphi^a \)-modules over \( \bar{\mathcal{E}}_R \), because of the complicated nature of the base ring. We are thus forced to introduce an auxiliary definition with a more geometric flavor.

**Definition 6.1.2.** For \( 0 < s \leq r/q \), a \( \varphi^a \)-module over \( \bar{\mathcal{E}}_R^{[s,r]} \) is a finite locally free module \( M \) equipped with an isomorphism \( \varphi^a M \otimes \bar{\mathcal{E}}_R^{[s/r,q]} \cong M \otimes \bar{\mathcal{E}}_R^{[s,r]} \) of modules over \( \bar{\mathcal{E}}_R^{[s,r]} \). A \( \varphi^a \)-bundle over \( \bar{\mathcal{E}}_R \), consists of a \( \varphi^a \)-module \( M_I \) over \( \bar{\mathcal{E}}_R^I \) for every interval \( I = [s,r] \) with \( 0 < s \leq r/q \), together with isomorphisms \( \psi_{I,I'} : M_I \otimes \bar{\mathcal{E}}_R^{I} \cong M_{I'} \) for every pair of intervals \( I, I' \) with \( I' \subseteq I \), satisfying the cocycle condition \( \psi_{I',I''} \circ \psi_{I,I'} = \psi_{I,I''} \). We refer to \( M_I \) as the model of the \( \varphi^a \)-bundle over \( \bar{\mathcal{E}}_R^I \); we may freely pass between \( \varphi^a \)-modules and models using Lemma \( 6.1.3 \) below. We define base extensions and exact sequences of \( \varphi^a \)-modules in terms of models.

For \( M = \{ M_I \} \) a \( \varphi^a \)-bundle over \( \bar{\mathcal{E}}_R \) and \( J \) a subinterval of \( (0, +\infty) \), a \( J \)-section of \( M \) consists of an element \( v_J \in M_I \) for each \( I \subseteq J \) such that \( \psi_{I,I'}(v_I) = v_{I'} \). When \( J = (0, +\infty) \), we also call such an object a global section of \( M \); note that \( \varphi \) acts on global sections.

**Remark 6.1.3.** The kernel of a surjective morphism of finite projective modules over a ring is itself a finite projective module. Consequently, the kernel of a surjective morphism of \( \varphi^a \)-modules or \( \varphi^a \)-bundles (where surjectivity in the latter case means that each map of models is surjective) is again a \( \varphi^a \)-module or \( \varphi^a \)-bundle, respectively.

The following lemma makes it unambiguous to say that a \( \varphi^a \)-bundle is generated by a given finite set of global sections. The proof is loosely modeled on that of \( [75 \text{ Satz 2.4}] \).
Lemma 6.1.4. Let $M = \{M_t\}$ be a $\varphi^a$-bundle over $\hat{\mathcal{R}}_R$. Suppose that $v_1, \ldots, v_n$ are global sections of $M$ which generate $M_t$ as a module over $\hat{\mathcal{R}}^I_R$ for every closed interval $I \subset (0, +\infty)$. Then $v_1, \ldots, v_n$ also generate the set of global sections of $M$ as a module over $\hat{\mathcal{R}}^\infty_R$.

Proof. For each nonnegative integer $l$, choose a morphism $\psi_t : M_{[p^{-l}, p^l]} \to (\hat{\mathcal{R}}_{R,p}^{[p^{-l}, p^l]})^n$ of $\hat{\mathcal{R}}_{R,p}^{[p^{-l}, p^l]}$-modules whose composition with the map $(\hat{\mathcal{R}}_{R,p}^{[p^{-l}, p^l]})^n \to M_{[p^{-l}, p^l]}$ defined by $v_1, \ldots, v_n$ is the identity. By Lemma 2.2.12 we can choose $c_l > 0$ such that the subspace norm on $M_{[p^{-l}, p^l]}$ defined by $\psi_{l+1}$ (or rather its base extension from $\hat{\mathcal{R}}_{R,p}^{[p^{-l+1}, p^{l+1}]}$ to $\hat{\mathcal{R}}_{R,p}^{[p^{-l}, p^l]}$) and the quotient norm on $M_{[p^{-l}, p^l]}$ differ by a multiplicative factor of at most $c_l$.

Given $w \in N$, we choose elements $a_{il} \in \hat{\mathcal{R}}_{R,p}^{[p^{-l}, p^l]}$, $b_{il} \in \hat{\mathcal{R}}_{R,p}^\infty$ for $i = 1, \ldots, n$, $l = 0, 1, \ldots$ as follows.

- Given the $b_{ij}$ for $j < l$, use $\psi_t$ to construct $a_{il}$ so that $w - \sum_i \sum_{j<l} b_{ij}v_i = \sum_i a_{il}v_i$.
- Given the $a_{il}$, choose the $b_{il}$ so that $\lambda(\alpha^t) (b_{il} - a_{il}) \leq p^{-1} c_l^{-1} \lambda(\alpha^t) (a_{il})$ for $i = 1, \ldots, n$ and $t \in [p^{-l}, p^l]$.

Note that for $t \in [p^{-l}, p^l]$, we have $\max_i \{\lambda(\alpha^t)(a_{i(t+1)})\} \leq p^{-1} \max_i \{\lambda(\alpha^t)(a_{it})\}$. Consequently, the series $\sum_{l} a_{il}$ converges to a limit $a_i \in \hat{\mathcal{R}}_{R,p}^\infty$ satisfying $w = \sum_i a_i v_i$; this proves the claim.

Lemma 6.1.5. For $0 < s \leq r/q$, the projection functor from $\varphi^a$-bundles over $\hat{\mathcal{R}}_R$ to $\varphi^a$-modules over $\hat{\mathcal{R}}_{R,s,r}^{[s,r]}$ is a tensor equivalence.

Proof. For each nonnegative integer $n$, we may uniquely lift a $\varphi^a$-module over $\hat{\mathcal{R}}_{R,s,r}^{[s,r]}$ to $\hat{\mathcal{R}}_{R,s,r}^{[sq^{-n}, sr^n]}$ by pulling back along positive and negative powers of $\varphi^a$, then glueing using Lemma 5.3.8. The claim follows at once.

Remark 6.1.6. There is a natural functor from $\varphi^a$-modules over $\hat{\mathcal{R}}_R$ to $\varphi^a$-bundles over $\hat{\mathcal{R}}_R$: given a $\varphi^a$-module over $\hat{\mathcal{R}}_R$, form its model $M_r$ over $\hat{\mathcal{R}}_R$ and then base extend to $\hat{\mathcal{R}}_{R,s,r}^{[s,r]}$ to obtain a $\varphi^a$-module over $\hat{\mathcal{R}}_{R,s,r}^{[s,r]}$. We may recover $M_r$ as the set of $(0, r]$-sections of the resulting $\varphi^a$-bundle, so this functor is fully faithful. If $R$ is free of trivial spectrum, we can say more; see Theorem 6.3.12.

6.2 Construction of $\varphi$-invariants

We next introduce some calculations that allow us to construct $\varphi^a$-invariants. These are relative analogues of results from [68, §4] which were used as part of the construction of slope filtrations.

Definition 6.2.1. For $M$ a $\varphi^a$-module or $\varphi^a$-bundle and $n \in \mathbb{Z}$, define the twist $M(n)$ of $M$ to be the same underlying module or bundle with the $\varphi^a$-action multiplied by $p^{-n}$.

Proposition 6.2.2. Let $M = \{M_t\}$ be a $\varphi^a$-bundle over $\hat{\mathcal{R}}_R$. Then there exists an integer $N$ such that for $n \geq N$ and $0 < s \leq r$, the map $\varphi^a - 1 : M_{[s, rq]}(n) \to M_{[s, r]}(n)$ is surjective. Moreover, if $M$ arises from a trivial $\varphi^a$-module, we may take $N = 1$. 

104
Proof. We first assume that \( r/s \leq q^{1/2} \); by applying a suitable power of \( \varphi^a \) as needed, we may also reduce to the case where \( r \in [1, q] \). Choose module generators \( v_1, \ldots, v_m \) of \( M_{[s/q, r]} \) and representations \( \varphi^{-a}(v_j) = \sum_i A_{ij} v_i, \varphi^a(v_j) = \sum_i B_{ij} v_i \) with \( A_{ij} \in \tilde{R}_{[s/r]}^t, B_{ij} \in \tilde{R}_R^t \). Put
\[
\begin{align*}
c_1 &= \sup\{\lambda(\alpha^t)(A) : t \in [q^{-1/2}, q]\}, \\
c_2 &= \sup\{\lambda(\alpha^t)(B) : t \in [q^{-1/2}, q]\}.
\end{align*}
\]
We take \( N \) large enough so that
\[
N \geq 0, \quad p^N c_2 > 1, \quad p^{N(1-q^{1/2})} \max\{c_1^{q^{1/2}}, 1\} c_2 < 1;
\]
note that we may take \( N = 1 \) if \( M \) arises from a trivial \( \varphi \)-module. The choice of \( N \) ensures that for \( n \geq N \), we can choose \( c \in (0, 1) \) so that
\[
\epsilon = \max\{p^{-n} c_1 e^{-(q-1)r/q}, p^{n} c_2 e^{(q-1)s}\} < 1. \tag{6.2.2.1}
\]
Given \( (x_1, \ldots, x_m) \in (\tilde{R}_R^{[s,r]})^m \), apply Lemma 5.2.8 to write \( x_i = y_i + z_i \) with \( y_i \in \tilde{R}_R^{[s/r]} \), \( z_i \in \tilde{R}_R^{[s,r]} \) such that for \( t \in [s, r] \),
\[
\begin{align*}
\lambda(\alpha^t)(y_i), \lambda(\alpha^t)(z_i) &\leq \lambda(\alpha^t)(x_i), \\
\lambda(\alpha^t)(\varphi^{-a}(y_i)) &\leq e^{-(q-1)t/q} \lambda(\alpha^t)(x_i), \\
\lambda(\alpha^t)(\varphi^a(z_i)) &\leq e^{(q-1)t} \lambda(\alpha^t)(x_i).
\end{align*}
\]
Put
\[
x'_i = p^n \sum_j A_{ij} \varphi^{-a}(y_j) + p^{-n} \sum_j B_{ij} \varphi^a(z_j),
\]
so that
\[
\sum_i x'_i v_i = p^n \varphi^{-a} \left( \sum_i y_i v_i \right) + p^{-n} \varphi^a \left( \sum_i z_i v_i \right)
\]
and
\[
\max_i \{\lambda(\alpha^t)(x'_i)\} \leq \epsilon \max_i \{\lambda(\alpha^t)(x_i)\} \quad (t \in [s, r]).
\]
Let us view \( y = (y_1, \ldots, y_m), z = (z_1, \ldots, z_m) \), and \( x' = (x'_1, \ldots, x'_m) \) as functions of \( x = (x_1, \ldots, x_m) \). Given \( x(0) \in (\tilde{R}_R^{[s,r]})^m \), define \( x_{(t+1)} = x'(x(t)) \), and put
\[
v = \sum_{i=0}^\infty \left( -p^n \varphi^{-a} \left( \sum_i y(x(t)_i) v_i \right) + \sum_i z(x(t)_i) v_i \right). \tag{6.2.2.2}
\]
This series converges to an element of \( M_{[s,r]} \) satisfying \( v - p^{-n} \varphi^a(v) = w \) for \( w = \sum_{i=0}^n x(0)_i v_i \in M_{[s,r]} \). More precisely, from the choice of the \( y_i \) and \( z_i \), \( \sum_i \varphi^{-a}(y(x(t)_i)) \) converges under \( \lambda(\alpha^t) \) in the ranges \( t \in [s, r] \) and \( t \in [sq, rq] \), hence for \( t \in [s, r] \) by Lemma 5.2.1 a similar argument applies to \( \sum_i z(x(t)_i) \).

For general \( r, s \), given \( w \in M_{[s,r]} \), the previous paragraph produces \( v \in M_{[t,r]} \) with \( t = \max\{rq^{-1/2}, s\} \) such that \( v - p^{-n} \varphi^a(v) = w \). By rewriting this equation as \( w + p^{-n} \varphi^a(v) = v \) and invoking Lemma 5.2.9 we find that \( v \in M_{[t', r]} \cap M_{[t,r]} \) for \( t' = \max\{t/q, s\} \). Repeating this argument, we eventually obtain \( v \in M_{[s,r]} \) as desired. \qed
Corollary 6.2.3. Let \(0 \to M_1 \to M \to M_2 \to 0\) be an exact sequence of \(\varphi^a\)-bundles over \(\hat{\mathcal{R}}_R\). Then there exists an integer \(N\) such that for \(n \geq N\), the sequence

\[
0 \to M_1(n)^{\varphi^a} \to M(n)^{\varphi^a} \to M_2(n)^{\varphi^a} \to 0
\]

is again exact.

**Proof.** Apply Proposition [6.2.2](#) and the snake lemma.

---

**Lemma 6.2.4.** Let \(M = \{M_l\}\) be a \(\varphi^a\)-bundle over \(\hat{\mathcal{R}}_R\). For \(N\) as in Proposition [6.2.2](#) and \(n \geq N\), for any \(\beta \in \mathcal{M}(R)\) which does not dominate the trivial norm on \(R/p_\beta\), there exist finitely many \(\varphi^a\)-invariant global sections of \(M(n)\) which generate \(M_\beta = M \otimes_{\hat{\mathcal{R}}_R} \hat{\mathcal{R}}_{M(\beta)}\). (If \(M\) is obtained from a trivial \(\varphi^a\)-module of rank \(m\), then we may take \(N = 1\) and use only \(2m\) global sections.)

**Proof.** Let \(M_{\beta,I}\) denote the model of \(M_\beta\) over \(\hat{\mathcal{R}}_{M(\beta)}^I\). Pick any \(r > 0\), and set notation as in the proof of Proposition [6.2.2](#) with \(s = rq^{-1/2}\). By our choice of \(\beta\), there exists \(\pi \in R\) with \(0 < \beta(\pi) < 1\). For any \(n \geq N\), we can find a positive rational number \(u \in \mathbb{Z}[p^{-1}]\) so that \(c = \beta(\pi^u)\) satisfies (6.2.2.1). For \(i = 1, \ldots, m\), define \(w_i\) to be the sum of a series as in (6.2.2.2) in which

\[
x(0) = 0, \quad (y(x(0))_j, z(x(0))_j) = \begin{cases} (-[\pi^u], [\pi^u]) & (j = i) \\ (0,0) & (j \neq i) \end{cases}
\]

this gives an element of \(M_{[rq^{-1/2},rq]}\) killed by \(\varphi^a - 1\), and hence a \(\varphi^a\)-invariant global section of \(M\). We can write \(w_j = [\pi^u]v_j + \sum_i X_{ij}v_i\) with \(\lambda(\beta^t)(X_{ij}) \leq \epsilon(\beta)\) for all \(i, j\) and all \(t \in [rq^{-1/2},r]\). It follows that the matrix \(1 + X\) is invertible over \(\hat{\mathcal{R}}_{M(\beta)}^{[rq^{-1/2},r]}\), so \(w_1, \ldots, w_m\) generate \(M_{\beta,[rq^{-1/2},r]}\).

By repeating the argument with \(r\) replaced by \(rq^{-1/2}\), we obtain \(\varphi^a\)-invariant global sections \(w'_1, \ldots, w'_m\) which generate \(M_{\beta,[rq^{-1},r]}\). By applying powers of \(\varphi^a\) and invoking Corollary [5.3.2](#) we see that \(w_1, \ldots, w_m, w'_1, \ldots, w'_m\) generate \(M_{\beta,I}\) for any \(I\).

**Remark 6.2.5.** One cannot hope to refine the calculation in Proposition [6.2.2](#) to cover the entire interval \([r/q, r]\) and thus prove Lemma [6.2.4](#) in one step. This approach is obstructed by the following observation: take \(R = L\) and obtain \(M\) from a trivial \(\varphi^a\)-module with \(\varphi^a\)-invariant basis \(v_1, \ldots, v_m\). If it were possible to refine the construction in Proposition [6.2.2](#) so that the elements \(w_1, \ldots, w_m\) produced in Lemma [6.2.4](#) were generators of \(M\), we would have produced two isomorphic \(\varphi\)-modules with different degrees, a contradiction.

**Proposition 6.2.6.** Suppose that \(R\) is free of trivial spectrum. Let \(M = \{M_l\}\) be a \(\varphi^a\)-bundle over \(\hat{\mathcal{R}}_R\). For \(N\) as in Proposition [6.2.2](#) and \(n \geq N\), there exist finitely many \(\varphi^a\)-invariant global sections of \(M(n)\) which generate \(M\).
Proof. For each \( \beta \in \mathcal{M}(R) \), the hypothesis of Lemma 6.2.4 is satisfied thanks to Lemma 2.3.9. We can thus find a finite set \( S_\beta \) of \( \varphi^a \)-invariant global sections of \( M(n) \) which generates \( M \otimes_{\tilde{R}_R} \tilde{R}_H(\beta) \). By Lemma 5.3.9 (applied to one model of \( M \)), the set \( S_\beta \) also generates \( M \otimes_{\tilde{R}_R} \tilde{R}_{R'} \) for some rational localization \( R \to R' \) encircling \( \beta \). By covering \( \mathcal{M}(R) \) with finitely many such rational localizations, we deduce the claim. \( \square \)

Remark 6.2.7. In case \( R \) is a Banach algebra over a nontrivially normed analytic field, the proof of Lemma 6.2.4 directly implies Proposition 6.2.6, with no need to work locally on \( \mathcal{M}(R) \).

6.3 Vector bundles à la Fargues-Fontaine

We now make contact with the new perspective on \( p \)-adic Hodge theory provided by the work of Fargues and Fontaine [39] (see Remark 6.3.18).

Definition 6.3.1. Define the reduced graded ring \( P = \bigoplus_{n=0}^\infty P_n \) by
\[
P_n = \{ x \in \tilde{R}_R : \varphi^a(x) = p^nx \} = \{ x \in \tilde{R}_R : \varphi^a(x) = p^n x \}
\]
(\( n = 0, 1, \ldots \)). The last equality holds by Corollary 5.2.11. (We will write \( P_R \) instead of \( P \) in case it becomes necessary to specify \( R \).) For \( d > 0 \) and \( f \in P_d \), let \( P[f^{-1}]_0 \) denote the degree zero subring of \( P[f^{-1}] \); the affine schemes \( D_+(f) = \text{Spec}(P[f^{-1}]_0) \) glue to define a reduced separated scheme \( \text{Proj}(P) \) as in [49, Proposition 2.4.2]. The points of \( \text{Proj}(P) \) may be identified with the homogeneous prime ideals of \( P \) not containing \( P_+ = \bigoplus_{n>0} P_n \), with \( D_+(f) \) consisting of those ideals not containing \( f \).

Definition 6.3.2. For \( f \in P_d \) for some \( d > 0 \), for \( M = \{ M_I \} \) a \( \varphi^a \)-bundle over \( \tilde{R}_R \), define
\[
M_f = \bigcup_{n \in \mathbb{Z}} f^{-n} M(dn)^{\varphi^a}
\]
(6.3.2.1) as a module over \( P[f^{-1}]_0 \). (In other words, \( M_f = M[f^{-1}]^{\varphi^a} \).) For any closed interval \( I \subset (0, +\infty) \), we have a natural map
\[
M_f \otimes_{P[f^{-1}]_0} \tilde{R}_R[f^{-1}] \to M_I \otimes_{\tilde{R}_H} \tilde{R}_R[f^{-1}].
\]
(6.3.2.2)

Lemma 6.3.3. Choose \( f \in P_d \) for some \( d > 0 \). Let \( 0 \to M_1 \to M \to M_2 \to 0 \) be a short exact sequence of \( \varphi^a \)-bundles over \( \tilde{R}_R \). Then the sequence
\[
0 \to M_1,f \to M_f \to M_2,f \to 0
\]
(6.3.3.1) is exact.

Proof. This follows from (6.3.2.1) and Corollary 6.2.3. \( \square \)

Corollary 6.3.4. For \( f \in P_d \) for some \( d > 0 \), \( M \) a \( \varphi^a \)-bundle over \( \tilde{R}_R \), and \( M_1, M_2 \) two \( \varphi^a \)-subbundles of \( M \) for which \( M_1 + M_2 \) is again a \( \varphi^a \)-subbundle, the natural inclusion \( M_1,f + M_2,f \to (M_1 + M_2)_f \) within \( M_f \) is an equality.
Proof. Apply Lemma 6.3.3 to the surjection \( M_1 \oplus M_2 \rightarrow M_1 + M_2 \) (after extending this to an exact sequence using Remark 6.1.3).

Definition 6.3.5. For \( A \) an integral domain, the following conditions are equivalent \([16, IV.2, Exercise 12]\).

(a) Every finitely generated ideal of \( A \) is projective.

(b) Every finitely generated torsion-free module over \( A \) is projective.

(c) For all ideals \( I_1, I_2, I_3 \) of \( A \), the inclusion \( I_1 \cap I_2 + I_1 \cap I_3 \rightarrow I_1 \cap (I_2 + I_3) \) is an equality.

Note that it is sufficient to test finitely generated ideals.

An integral domain satisfying any of these conditions is called a Prüfer domain. For example, any Bézout domain is a Prüfer domain (but not conversely). Note that a noetherian Prüfer domain is a principal ideal domain.

Lemma 6.3.6. Suppose that \( R = L \) is an analytic field. Then for \( f \in P_d \) for some \( d > 0 \), the ring \( P[f^{-1}]_0 = (\mathcal{R}_L[f^{-1}])^{\varphi^a} \) is a Prüfer domain.

Proof. We check criterion (c) of Definition 6.3.5. Let \( I_1, I_2, I_3 \) be three finitely generated ideals of \( P[f^{-1}]_0 \). For \( j = 1, 2, 3 \), \( I_j \otimes_{P[f^{-1}]_0} \mathcal{R}_L[f^{-1}] \) can be generated by a finite set of elements \( x_{j,1}, \ldots, x_{j,m} \in \mathcal{R}_L \) such that for each \( i \), \( \varphi^a(x_{j,i}) = p^h \) for some \( h \in \mathbb{Z} \). Let \( M_j \) be the ideal of \( \mathcal{R}_L \) generated by \( x_{j,1}, \ldots, x_{j,m} \); it is principal (because \( \mathcal{R}_L \) is a Bézout domain by Lemma 4.2.6) and \( \varphi^a \)-stable (because each \( x_{j,i} \) generates a \( \varphi^a \)-stable ideal), and hence a \( \varphi^a \)-module over \( \mathcal{R}_L \). Since \( \mathcal{R}_L \) is a Bézout domain and hence a Prüfer domain, by Definition 6.3.5 the inclusion

\[
M_1 \cap M_2 + M_1 \cap M_3 \rightarrow M_1 \cap (M_2 + M_3)
\]

is surjective. By Lemma 6.3.3 the map

\[
(M_1 \cap M_2 + M_1 \cap M_3)_f \rightarrow (M_1 \cap (M_2 + M_3))_f
\]

is also surjective. The operation \( M \mapsto M_f \) clearly distributes across intersections; it also distributes across sums thanks to Corollary 6.3.3 and the fact that the sum of two \( \varphi^a \)-submodules of \( \mathcal{R}_L \) is again a \( \varphi^a \)-submodule (by the Bézout property again). By identifying \( I_j \) with \( M_j,f \), we verify criterion (c) of Definition 6.3.5 so \( P[f^{-1}]_0 \) is a Prüfer domain as desired.

Lemma 6.3.7. Suppose that \( R \) is free of trivial spectrum.

(a) For any \( d > 0 \), \( P_d \) generates the unit ideal in \( \mathcal{R}_R^\infty \). In particular, we may choose \( f_1, \ldots, f_m \in P_d \) which generate the unit ideal in \( \mathcal{R}_R^\infty \).
(b) For any such elements, the ideal in $P$ generated by $f_1, \ldots, f_m$ is saturated (i.e., its radical equals $P_+$). Consequently, the schemes $D_+(f_1), \ldots, D_+(f_m)$ cover $\text{Proj}(P)$, so $\text{Proj}(S)$ is quasicompact.

Proof. To prove (a), apply Proposition 6.2.6 and Lemma 6.1.4 to $\tilde{\text{invariant}}$ global sections of $M$. Define a surjection $\text{proof}. \text{Apply Proposition 6.2.6 to construct an integer } d > 0. \text{Let } P \text{ be any maximal ideal of } P[f^{-1}]_0, \text{and let } q \text{ be the corresponding homogeneous prime ideal of } P \text{ not containing } f.$

(a) The ideal in $\mathcal{R}_R^{[r/q,r]}$ generated by $q$ and $f$ is trivial.

(b) The ideal in $\mathcal{R}_R^{[r/q,r]}$ generated by $q$ is not trivial.

Proof. The homogeneous ideal in $P$ generated by $q$ and $f$ contains $P_{dn}$ for some $n > 0$. By Lemma 6.3.7, $q$ and $f$ generate the trivial ideal in $\mathcal{R}_R^\infty$ and hence also in $\mathcal{R}_R^{[r/q,r]}$. This yields (a).

Suppose now that $q$ contains elements $g_1, \ldots, g_m$ which generate the unit ideal in $\mathcal{R}_R^{[r/q,r]}$. We may as well assume $g_1, \ldots, g_m \in P_{dn}$ for some $n > 0$; then these elements define a map $(\mathcal{R}_R^\infty(-dn))^{\oplus m} \rightarrow \mathcal{R}_R^\infty$ of $\varphi$-modules. This map is surjective by Lemma 6.1.5 by Remark 6.1.3 and Lemma 6.3.3; we again get a surjective map upon inverting $f$ and taking $\varphi$-invariants. But this implies that $f \in q$, a contradiction. This yields (b).

Theorem 6.3.9. Suppose that $R$ is free of trivial spectrum. (For instance, this holds when $R$ is a Banach algebra over a nontrivially normed analytic field.) Choose $f \in P_d$ for some $d > 0$. Let $M = \{M_I\}$ be a $\varphi^a$-bundle over $\mathcal{R}_R$. Then $M_f$ is a finite projective module over $P[f^{-1}]_0$ and (6.3.2.2) is bijective for every interval $I$.

Proof. Apply Proposition 6.2.6 to construct an integer $n$ and a finite set $w_1, \ldots, w_m$ of $\varphi^a$-invariant global sections of $M(dn)$ which generate $M$. We may then view $f^{-n}w_1, \ldots, f^{-n}w_m$ as elements of $M_I$; this implies that (6.3.2.2) is surjective.

Let $M'$ be the $\varphi^a$-bundle associated to the $\varphi^a$-module $(\mathcal{R}_R^\infty(-dn))^{\oplus m}$, so that $w_1, \ldots, w_m$ define a surjection $M'(dn) \rightarrow M(dn)$ and hence a surjection $M' \rightarrow M$. By Remark 6.1.3 we obtain an exact sequence $0 \rightarrow M'' \rightarrow M' \rightarrow M \rightarrow 0$ of $\varphi^a$-bundles. By Lemma 6.3.3, the sequence

$$0 \rightarrow M''_0 \rightarrow M'_0 \rightarrow M_0 \rightarrow 0 \quad (6.3.9.1)$$

is exact. Consequently, $M_f$ is a finitely generated module over $P[f^{-1}]_0$. For any interval $I$, we obtain a commuting diagram

\[
\begin{array}{ccccccccc}
M''_f \otimes_{P[f^{-1}]_0} \mathcal{R}_R^I[f^{-1}] & \longrightarrow & M'_f \otimes_{P[f^{-1}]_0} \mathcal{R}_R^I[f^{-1}] & \longrightarrow & M_f \otimes_{P[f^{-1}]_0} \mathcal{R}_R^I[f^{-1}] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M''_I \otimes_{\mathcal{R}_R^I} \mathcal{R}_R^I[f^{-1}] & \longrightarrow & M'_I \otimes_{\mathcal{R}_R^I} \mathcal{R}_R^I[f^{-1}] & \longrightarrow & M_I \otimes_{\mathcal{R}_R^I} \mathcal{R}_R^I[f^{-1}] & \longrightarrow & 0
\end{array}
\]

109
with exact rows. (The left exactness in the last row follows from the exactness of localization.) Since the left vertical arrow is surjective (by the first part of the proof) and the middle vertical arrow is bijective, by the five lemma the right vertical arrow is injective. Hence (6.3.2.2) is a bijection. We may also repeat the arguments with $M$ replaced by $M''$ to deduce that $M_f$ is finitely presented.

The exact sequence $0 \to M'' \to M' \to M \to 0$ corresponds to an element of $H^1_{ϕ^a}(M' \otimes M'')$. By Proposition 6.2.2, for $m$ sufficiently large, we have $H^1_{ϕ^a}(M' \otimes M''(dm)) = 0$. That is, if we form the commutative diagram

$$
\begin{array}{ccc}
0 & \to & M'' \\
& \downarrow & \downarrow \\
0 & \to & f^{-m}M'' \\
& \downarrow & \downarrow \\
& N & \to M \\
& \downarrow & \downarrow \\
& \to 0 & \to 0
\end{array}
$$

by pushing out, then the exact sequence in the bottom row splits in the category of $ϕ^a$-bundles. By Lemma 6.3.3, we obtain a split exact sequence

$$0 \to (f^{-m}M'')_f \to N_f \to M_f \to 0$$

of modules over $P[f^{-1}]_0$; however, $M'[f^{-1}] \cong N[f^{-1}]$ and so $M_f' = M'[f^{-1}]^ϕ \cong N[f^{-1}]^ϕ = N_f$. Since the construction of $M'$ guarantees that $M'_f$ is a free module over $P[f^{-1}]_0$, the same is true of $N_f$; it follows that $M_f$ is a projective module over $P[f^{-1}]_0$, as desired.

Theorem 6.3.9 can be reformulated in terms of an equivalence of categories between $ϕ^a$-bundles and vector bundles on Proj$(P)$. 

**Definition 6.3.10.** Suppose that $R$ is free of trivial spectrum. Let $V$ be a quasicoherent finite locally free sheaf on Proj$(P)$. For each homogeneous $f \in P_+$, form the quasicoherent finite locally free sheaf on $D_+(f)$ corresponding to the module $Γ(D_+(f), V) \otimes_{P[f^{-1}]_0} \tilde{R}_R[f^{-1}]$. Since $P_+$ generates the unit ideal in $\tilde{R}_R^\infty$ by Lemma 6.3.7, we may glue to obtain a quasicoherent finite locally free sheaf on Spec$(\tilde{R}_R^\infty)$. Let $M(V)$ be the module of global sections of this sheaf; then $V \hookrightarrow M(V)$ defines an exact functor from quasicoherent finite locally free sheaves on Proj$(P)$ to $ϕ^a$-modules over $\tilde{R}_R^\infty$.

**Definition 6.3.11.** Suppose that $R$ is free of trivial spectrum. Let $M$ be a $ϕ^a$-bundle over $\tilde{R}_R$. By Theorem 6.3.9, for each $f \in P_+$, $M_f$ is a finite locally free module over $P[f^{-1}]_0$ and the natural map (6.3.2.2) is an isomorphism. In particular, the $M_f$ glue to define a quasicoherent finite locally free sheaf $V(M)$ on Proj$(P)$ (which one might call a *vector bundle* on Proj$(P)$).

**Theorem 6.3.12.** Suppose that $R$ is free of trivial spectrum. Then the following tensor categories are equivalent.

(a) The category of quasicoherent finite locally free sheaves on Proj$(P)$.

(b) The category of $ϕ^a$-modules over $\tilde{R}_R^\infty$.  

110
(c) The category of \( \varphi^a \)-modules over \( \tilde{R}_R \).

(d) The category of \( \varphi^a \)-bundles over \( \tilde{R}_R \).

More precisely, the functor from (a) to (b) is the functor \( V \mapsto M(V) \) given in Definition 6.3.10, the functor from (b) to (c) is base extension, the functor from (c) to (d) is the one indicated in Remark 6.1.6, and the functor from (d) to (a) is the functor \( M \mapsto V(M) \) given in Definition 6.3.11.

\textbf{Proof.} This is immediate from Theorem 6.3.9 \( \blacksquare \)

It is worth mentioning the following refinement of Theorem 6.3.12 in the case of an analytic field.

\textbf{Theorem 6.3.13.} Suppose that \( R = L \) is a nontrivially normed analytic field. Then the functor of Definition 6.3.10 defines an equivalence of categories between the category of coherent sheaves on \( \text{Proj}(P) \) and the category of finitely generated \( \tilde{R}_L \)-modules equipped with semilinear \( \varphi^a \)-actions.

\textbf{Proof.} Since \( \tilde{R}_L \) is a Bézout domain by Lemma 4.2.6 and hence a Prüfer domain, any finitely generated module over \( \tilde{R}_L \) is automatically finitely presented. The claim thus follows from Theorem 6.3.12 \( \blacksquare \)

\textbf{Remark 6.3.14.} The obstruction to generalizing Theorem 6.3.13 is that it is unclear whether the rings \( P[f^{-1}]_0 \) have the property that every finitely generated ideal is finitely presented (i.e., whether these rings are \textit{coherent}). This is most likely not true in general; however, we do not know what to expect if \( R \) is restricted to being the completed perfection of an affinoid algebra over a nontrivially normed analytic field.

\textbf{Remark 6.3.15.} One can improve the formal analogy between \( \text{Proj}(P) \) and the projective line over a field by defining \( \mathcal{O}(n) \) for \( n \in \mathbb{Z} \) as the invertible sheaf on \( \text{Proj}(P) \) corresponding via Theorem 6.3.12 to the \( \varphi^a \)-module over \( \tilde{R}_R \) free on one generator \( v \) satisfying \( \varphi^a(v) = p^{-n}v \). For \( V \) a quasicoherent sheaf on \( \text{Proj}(P) \), write \( V(n) \) for \( V \otimes \mathcal{O}(n) \); we may then naturally identify \( M(V(n)) \) with \( M(V)(n) \). For \( V \) a quasicoherent finite locally free sheaf on \( \text{Proj}(P) \), \( V(n) \) is generated by finitely many global sections for \( n \) large (by Theorem 6.3.12 and Proposition 6.2.6). For a vanishing theorem for \( H^1 \) in the same vein, see Corollary 8.3.4.

However, this analogy has its limits. For instance, one cannot in general obtain the \( \mathcal{O}(n) \) in the usual manner of viewing \( P \) as a graded module over itself after shifting by \( n \) and then passing to the associated quasicoherent sheaf on \( \text{Proj}(P) \). For this to work, one must assume that \( P \) is generated by \( P_1 \) as a \( P_0 \)-algebra [49, Proposition 3.2.5], which is not always the case. For instance, if \( R = L \) is a nontrivially normed analytic field, then \( P \) is generated by \( P_1 \) as a \( P_0 \)-algebra if and only if \( L \) is algebraically closed; see [39] for further discussion of this case.

It will be useful for subsequent developments to explain how to add topologies to both types of objects appearing in Theorem 6.3.12.
Lemma 6.3.16. Let $M = \{M_t\}$ be a $\varphi^a$-bundle over $\breve{R}_R$. Choose $r > 0$, and induce from $\lambda(\alpha^r)$ a norm on $M_{[r/q,r]}$ as in Lemma 2.2.13. Then for each $n \in \mathbb{Z}$, the equivalence class of the restriction of this norm to $\{v \in M_{[r/q,r]} : \varphi^n(v) = p^{-a}v\} \cong M(n)\varphi^a$ is independent of $r$ and of the choice of the norm on $M$. (However, the construction is not uniform in $n$.)

Proof. It is clear that the choice of the norm on $M$ makes no difference up to equivalence, so we need only check the dependence on $r$. For any $s \in (0, r/q]$, by fixing a set of generators for $M_{[s,r]}$, we obtain norms $|\cdot|_t$ induced by $\lambda(\alpha^t)$ for all $t \in [s, r]$. We can choose $c_1, c_2 > 0$ so that these norms satisfy $c_1|v|_t \leq |\varphi^a(v)|_t/q \leq c_2|v|_t$ for all $v \in M_{[s,r]}$ and all $t \in [sq, r]$.

For $v \in M_{[r/q,r]}$ with $\varphi^a(v) = p^{-a}v$, we have $|\varphi^a(v)|_{r/q} = p^a|v|_{r/q}$. Consequently, $|\cdot|_r$ and $|\cdot|_{r/q}$ have equivalent restrictions. By induction, we see that $r$ and $rq^m$ give equivalent norms for any $m \in \mathbb{Z}$. To complete the proof, it is enough to observe that for $t$ in the interval between $r$ and $rq^m$ (inclusive), we have $|\cdot|_t \leq \max\{|\cdot|_r, |\cdot|_{rq^m}\}$ by Lemma 5.2.1; this then implies that the norm induced by $r$ dominates the norm induced by $s$, and vice versa by symmetry.

Definition 6.3.17. Let $G$ be a profinite group acting continuously on $\breve{R}_R^r$ for each $r > 0$ and commuting with $\varphi^a$; then $G$ also acts on $P$ (but not continuously). Let $M$ be a $\varphi^a$-module over $\breve{R}_R$, and apply Theorem 6.3.12 to construct a corresponding quasicoherent finite locally free sheaf $V$ on $\text{Proj}(P)$. Then the following conditions on an action of $G$ on $V$ (or equivalently on $M$) are equivalent.

(a) The action of $G$ on $M$ is continuous for the LF topology.

(b) For each $n \in \mathbb{Z}$, the action of $G$ on $M(n)\varphi^a = \Gamma(\text{Proj}(P), V(n))$ is continuous for any norm as in Lemma 6.3.16. (This implies (a) by Proposition 6.2.6)

If these equivalent conditions are satisfied, we say the action is continuous.

Remark 6.3.18. Theorem 6.3.13 is essentially due to Fargues and Fontaine [39], who have further studied the structure of the scheme $\text{Proj}(P)$ when $R = L$ is a nontrivially normed analytic field. They show that it is a complete absolute curve in the sense of being noetherian, connected, separated, and regular of dimension 1, with each closed point having a well-defined degree and the total degree of any principal divisor being 0. (If $L$ is algebraically closed, then the degrees are all equal to 1.) One corollary is that the rings $P[f^{-1}]_0$ for $f \in P_+$ homogeneous are not just Prüfer domains but Dedekind domains.

Many of the basic notions in $p$-adic Hodge theory can be interpreted in terms of the theory of vector bundles on $\text{Proj}(P_L)$; this is the viewpoint developed in [39], which we find appealing and suggestive. For instance, the slope polygon of a $\varphi^a$-module over $\breve{R}_L$ can be interpreted as the Harder-Narasimhan polygon of the corresponding vector bundle, with étale $\varphi^a$-modules corresponding to semistable vector bundles of degree 0 via the slope filtration theorem over $\breve{R}_L$ (Theorem 4.2.12). The correspondence between étale $\varphi^a$-modules and étale local systems then bears a remarkable formal similarity to the correspondence between stable vector bundles on compact Riemann surfaces and irreducible unitary representations of the fundamental group, due to Narasimhan and Seshadri [83]. (A materially equivalent
construction was given by Berger [10] in the somewhat less geometric language of B-pairs. See Remark 8.3.8 for a reinterpretation of Berger’s point of view in the language used here.)

For general $R$, we expect the scheme $\text{Proj}(P)$ to exhibit much less favorable behavior (see for instance Remark 6.3.15). However, it may still be profitable to view the relationship between étale local systems and $\varphi$-modules through the optic of vector bundles over $\text{Proj}(P)$. One possibly surprising aspect is that étale $\mathbb{Q}_p$-local systems on $\mathcal{M}(R)$, which need not descend to $\text{Spec}(R)$ (as in Example 8.1.14), nonetheless give rise to algebraic vector bundles on $\text{Proj}(P)$ via Theorem 6.3.12 and Theorem 8.1.9.

6.4 Towards finite generation of $\varphi$-bundles

We now discuss the following proposed extension of Theorem 6.3.12, in which we relax the hypothesis that $R$ be free of trivial spectrum.

**Conjecture 6.4.1.** The following tensor categories are equivalent.

(a) The category of $\varphi^a$-modules over $\tilde{\mathcal{R}}_R^\infty$.

(b) The category of $\varphi^a$-modules over $\tilde{\mathcal{R}}_R$.

(c) The category of $\varphi^a$-bundles over $\mathcal{R}_R$.

More precisely, the functor from (a) to (b) is base extension, while the functor from (b) to (c) is as described in Remark 6.4.1.

**Remark 6.4.2.** One important consequence of Conjecture 6.4.1 would be that the categories of $\varphi^a$-modules over $\tilde{\mathcal{R}}_R^\infty$ and $\tilde{\mathcal{R}}_R$ admit glueing for covering families of rational localizations on $R$, by virtue of the corresponding fact about $\varphi^a$-bundles (see Theorem 5.3.6).

Another important consequence of Conjecture 6.4.1 is the following statement at the level of cohomology.

**Proposition 6.4.3.** Suppose that Conjecture 6.4.1 holds for $R$. Let $M$ be a $\varphi^a$-module over $\tilde{\mathcal{R}}_R^\infty$.

(a) For $r > 0$, put $M_r = M \otimes_{\tilde{\mathcal{R}}_R^\infty} \tilde{\mathcal{R}}_R$. Then the vertical arrows in the diagram

\[
\begin{array}{ccccccccc}
0 & \to & M & \xrightarrow{\varphi^a-1} & M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_r & \xrightarrow{\varphi^a-1} & M_{r/q} & \to & 0
\end{array}
\]

induce an isomorphism on the cohomology of the horizontal complexes. In particular, the lower complex computes $H^i_{\varphi^a}(M)$.

(b) The map $M \to M \otimes_{\tilde{\mathcal{R}}_R^\infty} \tilde{\mathcal{R}}_R$ induces an isomorphism on cohomology.
(c) For \( r, s \) with \( 0 < s \leq r/q \), put \( M_{[s,r]} = M \otimes_{\hat{R}_R} \hat{R}_R^{[s,r]} \). Then the vertical arrows in the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & M \\
\downarrow & & \downarrow \phi^a-1 \\
0 & \longrightarrow & M_{[s,r]} \\
\end{array}
\]

induce an isomorphism on the cohomology of the horizontal complexes. In particular, the lower complex computes \( H^1_{\phi^a}(M) \).

**Proof.** Note that (b) follows from (a) by taking direct limits, so we need only treat (a) and (c). Write \( H^i_{\phi^a}(M_r) \) and \( H^i_{\phi^a}(M_{[s,r]}) \) as shorthand for the kernel and cokernel of the second row in (a) and (c), respectively. Since \( M \) is finite projective over \( \hat{R}_R^\infty \), the maps \( M \to M_r, M \to M_{[s,r]} \) are injective; consequently, the maps \( H^0_{\phi^a}(M) \to H^0_{\phi^a}(M_r), H^0_{\phi^a}(M) \to H^0_{\phi^a}(M_{[s,r]}) \) are injective. Conversely, for \( v \in H^0_{\phi^a}(M_r) \), we also have \( v = \phi^{-a}(v) \in M_q \). By induction, we have \( v \in M_{qr} \) for all \( n \) and so \( v \in M \); that is, \( H^0_{\phi^a}(M) \to H^0_{\phi^a}(M_r) \) is surjective. Similarly, for \( v \in H^0_{\phi^a}(M_{[s,r]}) \), we may apply powers of \( \phi^a \) and invoke Lemma 5.2.9 to deduce that \( v \in M \); that is, \( H^0_{\phi^a}(M) \to H^0_{\phi^a}(M_{[s,r]}) \) is surjective.

By similar reasoning, if \( v \in M_r \) (resp. \( v \in M_{[s,r]} \)) is such that \( (\phi^a - 1)(v) \in M_{r/q} \), then \( v \in M \). Consequently, the maps \( H^1_{\phi^a}(M) \to H^1_{\phi^a}(M_r), H^1_{\phi^a}(M) \to H^1_{\phi^a}(M_{[s,r]}) \) are injective. To see that \( H^1_{\phi^a}(M) \to H^1_{\phi^a}(M_r) \) is surjective, note that any class in the target defines an extension of \( \phi^a \)-modules over \( \hat{R}_R \), which lifts to an extension of \( \phi^a \)-modules over \( \hat{R}_R^\infty \) by the assumption of Conjecture 6.4.1. The argument for \( H^1_{\phi^a}(M) \to H^1_{\phi^a}(M_{[s,r]}) \) is similar. \( \square \)

**Remark 6.4.4.** Some evidence for Conjecture 6.4.1 is that it holds in two diametrically opposite cases: when the norm on \( R \) dominates the trivial norm, in which case the assertion holds for trivial reasons (namely that the rings \( \hat{R}_R \) all coincide), and when \( R \) is free of trivial spectrum, in which case the assertion is contained in Theorem 6.3.12. In particular, the equivalence holds when \( R = L \) is an analytic field; one can also treat this case without using the action of \( \phi \) (see [68, Theorem 2.8.4]).

For general \( R \), the main difficulty seems to be to show that the module of global sections of a \( \phi^a \)-bundle over \( \hat{R}_R \) is finitely generated, or equivalently (by Lemma 6.4.4), that there exist finitely many global sections generating the \( \phi^a \)-bundle. Given this, one can apply Lemma 6.4.6 below to recover a \( \phi^a \)-module over \( \hat{R}_R^\infty \).

We offer the following partial results towards Conjecture 6.4.1. The following construction of global sections of an arbitrary \( \phi^a \)-bundle is modeled on that of [75, Satz 2.4.1]. Unfortunately, it is unclear how to make the construction uniform in the interval \( I \).

**Lemma 6.4.5.** For any \( \phi^a \)-bundle \( M = \{M_I\} \) over \( \hat{R}_R \), for each interval \( I \), there exist finitely many global sections of \( M \) which generate \( M_I \).

**Proof.** Choose \( r > 0 \) and choose generators \( v_1, \ldots, v_n \) of \( M_I \). For \( l = 0, 1, \ldots \), put \( I_l = \bigcup_{i=-l} I_l^i \) and \( I_l = \bigcup_{i=-l} I_l^i \); we construct elements \( w_{l,1}, \ldots, w_{l,n} \) of \( M_I \) as follows. For \( l = 0 \), take \( w_{l,i} = v_i \). Given \( w_{l,1}, \ldots, w_{l,n} \) for some \( l \), by arguing as in the proof of Lemma 2.7.3
to glue the elements $w_{i,1}, \ldots, w_{i,n}$ of $M_I$ with the generators $\varphi^{-l-1}(v_1), \ldots, \varphi^{-l-1}(v_n)$ of $M_{q+1,1}$, we can construct an $n \times n$ matrix $A$ over $\mathcal{R}_R$ such that $\lambda(\alpha^s)(A - 1) \leq p^{-l}$ for $s \in I_l$ and the elements $w_{l,i} = \sum_i A_{ij} w_{l,j}$ are elements of $M_{I_l}$ which generate $M_{I_l}$. Repeating the process to glue these generators with the generators $\varphi^{l+1}(v_1), \ldots, \varphi^{l+1}(v_n)$ of $M_{q+1,1}$, we produce another $n \times n$ matrix $B$ over $\mathcal{R}_R$ such that $\lambda(\alpha^s)(B - 1) \leq p^{-l}$ for $s \in I_l$ and the elements $\tilde{w}_{l+1,j} = \sum_i B_{ij} \tilde{w}_{l+1,i}$ are elements of $M_{I_{l+1}}$ which generate $M_{I_{l+1}}$. This completes the construction; in the limit as $l \to \infty$, the $w_{l,i}$ tend to global sections $w_1, \ldots, w_n$ which generate $M_I$.

The following base extension argument has been used already in Lemma 5.3.3.

**Lemma 6.4.6.** Let $M = \{M_I\}$ be a $\varphi^a$-bundle over $\mathcal{R}_R$ such that the module $M_{\infty}$ of global sections is finitely generated over $\mathcal{R}_R^\infty$. Then $M_{\infty}$ is a finite projective module over $\mathcal{R}_R^\infty$, and hence a $\varphi^a$-module.

**Proof.** Let $K$ be the completed perfect closure of $\mathbb{F}_p((\pi))$; then $R' = R \hat{\otimes}_{\mathbb{F}_p} K$ is free of trivial spectrum. Identify $W(R')$ with the $p$-adic completion of $W(R)((\pi^{p^{-\infty}}))$ so that $[\pi]$ corresponds to $\pi$. We may then view each element of $\mathcal{R}_R^\infty$ as a convergent series in which each term is an element of $\mathcal{R}_R^\infty$ times a fractional power of $\pi$. We obtain a $\mathcal{R}_R^\infty$-linear section of the map $\mathcal{R}_R^\infty \to \mathcal{R}_R^\infty$ by writing an element of $\mathcal{R}_R^\infty$ in terms of fractional powers of $\pi$, then retaining only the integral powers.

Choose a finite free module $F$ over $\mathcal{R}_R^\infty$ and a surjection $F \to M_{\infty}$. Put $M' = M \hat{\otimes}_{\mathcal{R}_R} \mathcal{R}_R^\infty$ and $F' = F \hat{\otimes}_{\mathcal{R}_R} \mathcal{R}_R^\infty$, and let $M_{\infty}'$ be the set of global sections of $M'$. By tensoring the section $\mathcal{R}_R^\infty \to \mathcal{R}_R^\infty$ with $F$ and $M$ over $\mathcal{R}_R^\infty$, we obtain $\mathcal{R}_R^\infty$-linear sections $F' \to F$, $M_{\infty}' \to M_{\infty}$ of the natural maps $F \to F'$, $M_{\infty} \to M_{\infty}'$.

By Theorem 6.3.12 $M_{\infty}'$ is a projective module over $\mathcal{R}_R^\infty$, so we obtain a map $M_{\infty}' \to F'$ such that the composition $M_{\infty}' \to F' \to M_{\infty}'$ is the identity. Then the map $M_{\infty} \to F$ defined as the composition $M_{\infty} \to M_{\infty}' \to F' \to F$ has the property that the composition $M_{\infty} \to F \to M_{\infty}$ is also the identity (because it can be refactored as $M_{\infty} \to M_{\infty}' \to F' \to M_{\infty}' \to M_{\infty}$). Therefore $M_{\infty}$ is a finite projective module over $\mathcal{R}_R^\infty$, as desired. \qed

7  Slopes in families

When one considers $\varphi$-modules over relative Robba rings, one has not one slope polygon but a whole family of polygons indexed by the base analytic space. We now study the variation of the slope polygon in such families. Throughout §7, continue to retain Hypothesis 6.0.1.

**Remark 7.0.1.** Many of the statements we make below about $\varphi^a$-modules over $\mathcal{R}_R$ also hold for $\varphi^a$-bundles, but we will not make these statements explicit because one never encounters the difference between these categories in practice (see Remark 6.4.4).
7.1 An approximation argument

Much of our analysis of slopes in families depends on the following argument for spreading out certain bases of \( \varphi \)-modules, modeled on \[68\] Lemma 6.1.1 and Proposition 6.2.2.

**Lemma 7.1.1.** Let \( M \) be a \( \varphi \)-module over \( \tilde{R}_R \). For \( r > 0 \), let \( M_r \) be the model of \( M \) over \( \tilde{R}^r_R \). Let \( \{ M_l \} \) be the \( \varphi \)-bundle associated to \( M \). Suppose that there exists a basis \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) of \( M_{[r/q,r]} \) on which \( \varphi \) acts via an invertible matrix \( F \) over \( \tilde{R}^r_R \). Then \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) is a basis of \( M_r \).

**Proof.** As in Lemma \[6.1.4\] it suffices to prove that \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) is a basis of \( M_{[r/q^l,r]} \) for each nonnegative integer \( l \). As the case \( l = 0 \) is given, we may proceed by induction on \( l \).

Suppose that \( l > 0 \) and that the claim is known for \( l - 1 \), so that \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) form a basis of \( M_{[r/q^l,r]} \). Then \( \varphi^a(\mathbf{v}_1), \ldots, \varphi^a(\mathbf{v}_n) \) is a basis of \( M_{[r/q^{l+1},r/q]} \). By hypothesis, \( \mathbf{v}_j \) can be written as a \( \tilde{R}^r_R \)-linear combination of the \( \varphi^a(\mathbf{v}_i) \) and vice versa, so the \( \mathbf{v}_j \) also form a basis of \( M_{[r/q^{l+1},r/q]} \). By Corollary \[5.3.2\] \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) form a basis of \( M_{[r/q^{l+1},r]} \) as desired. \( \square \)

**Lemma 7.1.2.** Let \( M \) be a \( \varphi \)-module over \( \tilde{R}_R \). For \( r > 0 \), let \( M_r \) be the model of \( M \) over \( \tilde{R}^r_R \). Let \( \{ M_l \} \) be the associated \( \varphi \)-bundle. Suppose that there exist a nonnegative integer \( h \), a diagonal matrix \( D \) with diagonal entries \( p^{d_1}, \ldots, p^{d_n} \) for some \( d_1, \ldots, d_n \in \mathbb{Z} \) no two of which differ by more than \( h \), and a basis \( \mathbf{e}_1, \ldots, \mathbf{e}_n \) of \( M_{[r/q,r]} \) on which \( \varphi^a \) acts via a matrix \( F \) over \( \tilde{R}^{[r/q,r]}_R \) such that \( \lambda(\alpha^{r/q})(FD - 1) < p^{-h} \). Then there exists a basis \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) of \( M_r \) on which \( \varphi^a \) acts via a matrix \( F' \) over \( \tilde{R}^{[r/q,r]}_R \) such that \( F'D - 1 \) has entries in \( p\tilde{R}^{[r/q,r]}_R \), and for which the invertible matrix \( U \) over \( \tilde{R}^{[r/q,r]}_R \) defined by \( \mathbf{v}_j = \sum_i U_{ij} \mathbf{e}_i \) satisfies \( \lambda(\alpha^{r/q})(U - 1), \lambda(\alpha^r)(D^{-1}UD - 1) < 1 \).

**Proof.** Put \( c_0 = p^h\lambda(\alpha^{r/q})(FD - 1) < 1 \). We construct a sequence of invertible \( n \times n \) matrices \( U_0, U_1, \ldots \) over \( \tilde{R}^{[r/q,r]}_R \) such that the following conditions hold for \( l = 0, 1, \ldots \).

\[\begin{align*}
\text{(a)} & \quad \text{We have } \lambda(\alpha^{r/q})(U_l - 1), \lambda(\alpha^r)(D^{-1}U_l D - 1) \leq c_0p^{-h}. \\
\text{(b)} & \quad \text{For } F_l = U_l^{-1}F\varphi^a(U_l) \text{ (which has entries in } \tilde{R}^{[r/q,r]}_R \text{ and satisfies } \lambda(\alpha^{r/q})(F_l D - 1) \leq c_0p^{-h}), \text{ there exists a matrix } X_l \text{ over } \tilde{R}^{[r/q,r]}_R \text{ such that } F_l D - X_l - 1 \text{ has entries in } p\tilde{R}^{[r/q,r]}_R \text{ and } \lambda(\alpha^{r/q})(X_l) \leq c_0^{l+1}p^{-h}. \\
\end{align*}\]

For \( l = 0 \), we may take \( U_0 = 1 \) and \( X_0 = F_0D - 1 \). Given \( U_l \) for some \( l > 0 \), by applying Lemma \[5.2.7\] to the entries of \( p^{-1}X_l \), we construct a matrix \( Y_l \) over \( \tilde{R}^{[r/q,r]}_R \) such that \( F_l D - Y_l - 1 \) has entries in \( p\tilde{R}^{[r/q,r]}_R \), \( \lambda(\alpha^{r/q})(Y_l) \leq c_0^{l+1}p^{-h} \), and \( \lambda(\alpha^r)(Y_l) \leq c_0^{l+1}p^{-h} \). Since

\[\lambda(\alpha^{r/q})(D^{-1}\varphi^a(Y_l)D) \leq p^h\lambda(\alpha^r)(Y_l) \leq p^h c_0^{l+1}p^{-h} \leq c_0^{l+2}p^{-h}, \tag{7.1.2.1}\]

\[1 + Y_l \text{ and } 1 + D^{-1}\varphi^a(Y_l)D \text{ are both invertible over } \tilde{R}^{[r/q,r]}_R. \]

We may thus put

\[\begin{align*}
U_{l+1} &= U_l(1 + Y_l) \\
F_{l+1} &= U_l^{-1}F\varphi^a(U_l) = (1 + Y_l)^{-1}F_l(1 + \varphi^a(Y_l)) \\
X_{l+1} &= F_{l+1}D - 1 - (F_l D - Y_l - 1),
\end{align*}\]
so that $F_{t+1}D - X_{t+1} - 1 = F_tD - Y_t - 1$ has entries in $\mathcal{R}_R^{int,r/q}$. By writing

$$X_{t+1} = (1 + Y_t)^{-1}(F_tD)(D^{-1}\varphi^a(Y_t)D) + Y_t^2(1 + Y_t)^{-1}F_tD + Y_t(1 - F_tD),$$

we see that $\lambda(\alpha^{r/q})(X_{t+1}) \leq \beta^l + 2p^{-h}$. Hence $U_{t+1}$ has the desired properties.

The matrices $U_l$ converge to an invertible matrix $U$ over $\mathcal{R}_R^{r,q,r}$ for which the matrix $G = U^{-1}F\varphi^a(U)D$ has entries in $\mathcal{R}_R^{int,r/q}$ and $G - 1$ has entries in $p\mathcal{R}_R^{int,r/q}$ (because $G$ is the limit of the Cauchy sequence $F_lD$ with respect to $\lambda(\alpha^{r/q})$). If we put $v_j = \sum_i U_{ij}e_i$, we obtain a basis of $M_{[r,q,r]}$ on which $\varphi^a$ acts via $GD^{-1}$. By Lemma 7.1.1, $v_1, \ldots, v_n$ form a basis of $M_r$, proving the desired result. \qed

\section{Rank, degree, and slope}

\textbf{Definition 7.2.1.} Define the \textit{rank} and \textit{degree} of a $\varphi^a$-module $M$ over $W(R)$ (resp. $\mathcal{E}_R$, $\mathcal{R}_R^{int}$, $\mathcal{R}_R^{bd}$, $\mathcal{R}_R$) as the functions $\text{rank}(M, \cdot) : \mathcal{M}(R) \to \mathbb{Z}$ and $\text{deg}(M, \cdot) : \mathcal{M}(R) \to \mathbb{Z}$ whose values at $\beta \in \mathcal{M}(R)$ are the rank and degree, respectively, of the $\varphi^a$-module obtained by base extension from $M$ by passing from $R$ to $\mathcal{H}(\beta)$ (recalling Convention 4.1.13 in case $a > 1$). Note that by our definitions, the degree of a $\varphi^a$-module over $W(R)$ or $\mathcal{R}_R^{int}$ is identically zero.

\textbf{Lemma 7.2.2.} The rank and degree of a $\varphi^a$-module over any of the rings allowed in Definition 6.1.1 are continuous on $\mathcal{M}(R)$. In other words, the set of all points at which the rank or degree takes any given value is closed and open in $\mathcal{M}(R)$.

\textit{Proof.} The rank function is continuous for the Zariski topology on Spec$(R)$, and hence also for the topology on $\mathcal{M}(R)$. Continuity of the degree follows from Corollary 5.2.3 and Remark 5.0.2. \qed

\textbf{Definition 7.2.3.} Let $M$ be a $\varphi^a$-module over one of the rings allowed in Definition 6.1.1 of nowhere zero rank (that is, $\text{rank}(M, \cdot)$ never takes the value 0). The \textit{slope} of $M$ is then defined as the function $\mu(M, \cdot) : \mathcal{M}(R) \to \mathbb{Q}$ given by $\mu(M, \beta) = \text{deg}(M, \beta)/\text{rank}(M, \beta)$. By Lemma 7.2.2, $\mu(M, \cdot)$ is continuous for the discrete topology on $\mathbb{Q}$.

The \textit{pure locus} (resp. \textit{étale locus}) of $M$ is the set of $\beta \in \mathcal{M}(R)$ for which $M$ becomes pure (resp. étale) upon passing from $R$ to $\mathcal{H}(\beta)$. If this locus is all of $\mathcal{M}(R)$, we say that $M$ is \textit{pointwise pure} (resp. \textit{pointwise étale}). These conditions have a more global interpretation; see Corollary 7.3.9.

\textbf{Convention 7.2.4.} At a point where a $\varphi^a$-module has rank 0, it is considered to be pointwise pure of every slope. This is the correct convention for defining the categories of pure and étale $\varphi$-modules, so that they admit kernels for surjective morphisms (by Remark 6.1.3).

\section{Pure models}

\textbf{Definition 7.3.1.} Fix integers $a, c, d$ with $d$ a positive multiple of $a$. Let $M$ be a $\varphi^a$-module over $\mathcal{E}_R$ (resp. $\mathcal{R}_R^{bd}$, $\mathcal{R}_R$). A $(c,d)$-pure model of $M$ is a finite $W(R)$-submodule (resp. $\mathcal{R}_R^{int}$-submodule, $\mathcal{R}_R^{int}$-submodule) $M_0$ of $M$ with $M_0 \otimes_{W(R)} \mathcal{E}_R \cong M$ (resp. with $M_0 \otimes_{\mathcal{R}_R^{int}} \mathcal{R}_R^{bd} \cong M$, $M_0 \otimes_{\mathcal{R}_R^{int}} \mathcal{R}_R \cong M$, $M_0 \otimes_{\mathcal{R}_R^{int}} \mathcal{R}_R^{ent} \cong M$).
Let $M$ be a pure model at $\beta$ if and only if it admits a free local $(c,d)$-pure model at $\beta$. The existence of such a model implies that $M$ is pointwise pure of constant slope $c/d$. A pure model is locally free or free if its underlying module is locally free or free.

For $\beta \in \mathcal{M}(R)$, a (locally free, free) local $(c,d)$-pure model of $M$ at $\beta$ consists of a rational localization $R \to R'$ encircling $\beta$ and a (locally free, free) $(c,d)$-pure model of $M \otimes_{\tilde{E}_R} \tilde{E}_{R'}$ (resp. $M \otimes_{\tilde{R}_R^{bd}} \tilde{R}_R^{bd}$).

A $(0,1)$-pure model will also be called an étale model, and likewise with the modifiers local, locally free, or free in place.

**Remark 7.3.2.** Note that if $M$ is a $\varphi^a$-module over $\tilde{R}_R$ and $M_0$ is a $(c,d)$-pure model, it is only assumed that $M_0[p^{-1}]$ is stable under the action of $\varphi^d$, rather than $\varphi^a$. However, stability under $\varphi^a$ will follow later from Theorem [8.1.4] which will imply that the isomorphism $(\varphi^a)^* M \cong M$ descends to $M_0[p^{-1}]$. Until then, we will not assume this stability.

**Lemma 7.3.3.** Keep notation as in Definition [7.3.1].

(a) The $\varphi^a$-module $M$ admits a locally free local $(c,d)$-pure model at $\beta$ if and only if it admits a free local $(c,d)$-pure model at $\beta$.

(b) If $M$ is a $\varphi^a$-module over $\tilde{R}_R^{bd}$, then $M$ admits a free local $(c,d)$-pure model at $\beta$ if and only if $M \otimes_{\tilde{R}_R^{bd}} \tilde{E}_R$ does.

**Proof.** Part (a) follows at once from the fact that the direct limit of $W(S)$ (resp. $\tilde{R}_R^{int}$) over all rational localizations $R \to S$ encircling $\beta$ is a local ring; namely, it contains $p$ in its Jacobson radical, and its quotient by the ideal $(p)$ is the local ring $R_\beta$ (see Lemma [2.4.12]).

To deduce (b), we may assume that $M' = M \otimes_{\tilde{R}_R^{bd}} \tilde{E}_R$ admits a free $(c,d)$-pure model $M_0'$. Let $e_1, \ldots, e_n$ be a basis of $M_0'$, let $v_1, \ldots, v_m$ be generators of $M$, and write $e_j = \sum_i B_{ij} v_i$, $v_j = \sum_i C_{ij} e_i$ with $B_{ij}, C_{ij} \in \tilde{E}_R$. Note that $CB = 1$ and that $\sum_i (1 - BC)_{ij} v_i = 0$.

Since $\tilde{R}_R^{bd}$ is dense in $\tilde{E}_R$ for the $p$-adic topology, we can find elements $e'_1, \ldots, e'_n$ of $M$ by the formula $e'_j = \sum_i B'_{ij} v_i$. Then $e'_j = \sum_i (CB')_{ij} e_i = \sum_i X_{ij} e_i$, so $e'_1, \ldots, e'_n$ form another basis of $M_0'$ and hence of $M'$. Note that for each maximal ideal $m$ of $\tilde{R}_R^{bd}$, we can find a maximal ideal $m'$ of $\tilde{E}_R$ containing $m$. (Otherwise, $m$ would generate the unit ideal in $\tilde{E}_R$, and so would contain an element of $\tilde{R}_R^{int}$ congruent to 1 modulo $p$. But the latter would be a unit, contradiction.) Since $v_1, \ldots, v_m$ generate $M$, there exists an $n$-element subset $J$ of $\{1, \ldots, m\}$ such that the $v_j$ for $j \in J$ form a basis of $M/mM$. Since $e'_1, \ldots, e'_n$ form a basis of $M'$, the maximal minor of $B'$ corresponding to $J$ is nonzero in $\tilde{E}_R/m'$ and hence also in $\tilde{R}_R^{bd}/m$. By Nakayama’s lemma, $e'_1, \ldots, e'_n$ generate the localization of $M$ at $m$; since this holds for all $m$, $e'_1, \ldots, e'_n$ generate $M$. This proves the desired result. □

**Definition 7.3.4.** Let $M$ be a $\varphi^a$-module over one of $\tilde{E}_R, \tilde{R}_R^{bd}, \tilde{R}_R$. For $\beta \in \mathcal{M}(R)$ and $s \in \mathbb{Q}$, we say $M$ is pure (of slope $s$) at $\beta$ if $M$ admits a locally free local $(c,d)$-pure model at $\beta$ for some pair $(c,d)$ of integers with $d$ a positive multiple of $a$ and $c/d = s$. This forces
s = \mu(M) \text{ if } \text{rank}(M, \beta) > 0; \text{ if } \text{rank}(M, \beta) = 0, \text{ then } M \text{ is pure of every slope at } \beta \text{ (see Convention 7.2.4). We say } M \text{ is pure if it is pure at each } \beta \in \mathcal{M}(R); \text{ in this case, we can cover all } \beta \in \mathcal{M}(R) \text{ using finitely many local pure models thanks to the compactness of } \mathcal{M}(R). \text{ In these definitions, we regard étale as a synonym for pure of slope 0.}

In case we need to be precise about the choice of } c \text{ and } d, \text{ we will say that a module is } (c,d)-\text{pure rather than pure of slope } s. \text{ This will ultimately be rendered unnecessary by the observation that purity for one pair } (c,d) \text{ with } d > 0 \text{ and } c/d = s \text{ implies the same for any other slope (Corollary 8.1.10).}

We also say that } M \text{ is globally pure/étale if it admits a locally free pure/étale model. For more on this condition, see Remark 7.3.5.}

**Remark 7.3.5.** Consider the following conditions on a } \varphi^a-\text{module } M.

(a) The } \varphi^a-\text{module } M \text{ is globally pure (i.e., admits a locally free pure model).

(b) The } \varphi^a-\text{module } M \text{ admits a pure model.

(c) The } \varphi^a-\text{module } M \text{ is pure (i.e., admits locally free local pure models).

(d) The } \varphi^a-\text{module } M \text{ admits local pure models.

In all cases, (a) implies (b) and (c), which in turn each imply (d).

Over } \tilde{E}_R \text{ or } \tilde{R}_R^{bd}, \text{ (d) implies (b): given some local pure models on a finite covering of } \mathcal{M}(R), \text{ the elements of } M \text{ which restrict into each pure model form a global pure model. Also, conditions (a) and (b) are equivalent in some cases; see Proposition 8.1.13. In such cases, (a), (b), (c), (d) are all equivalent. (In general, it is unclear whether (a) and (c) are equivalent; see Remark 5.3.7.)

Over } \tilde{R}_R, \text{ conditions (c) and (d) are equivalent (see Corollary 7.3.10), while conditions (a) and (b) are equivalent in the same cases as over } \tilde{R}_R^{bd}. \text{ However, the two pairs of conditions are definitely not equivalent to each other; see Example 8.1.14.}

**Proposition 7.3.6.** Let } M \text{ be a } \varphi^a-\text{module over } \tilde{E}_R \text{ (resp. } \tilde{R}_R^{bd}, \tilde{R}_R) \text{ admitting a free } (c,d)-\text{pure model } M_0 \text{ for some } c,d \in \mathbb{Z} \text{ with } d \text{ a positive multiple of } a. \text{ Then there exists an } R\text{-algebra } S \text{ which is the completed direct limit of some faithfully finite étale } R\text{-subalgebras, such that } M_0 \otimes_{W(R)} W(S) \text{ (resp. } M_0 \otimes_{\tilde{R}_R^{int}} \tilde{R}_S^{int}, M_0 \otimes_{\tilde{R}_R} \tilde{R}_S^{int}) \text{ admits a basis fixed by } p^c\varphi^d.

**Proof.** The assertion over } \tilde{E}_R \text{ follows from Lemma 3.2.6. To handle the other cases, invoke the case } n = 1 \text{ of Lemma 3.2.6 to reduce to the case where } M_0 \text{ admits a basis } e_1, \ldots, e_n \text{ on which } p^c\varphi^d \text{ acts via a matrix } A \text{ over } \tilde{R}_R^{int} \text{ congruent to 1 modulo } p. \text{ Choose } S \text{ so that } M_0 \otimes_{\tilde{R}_R} \tilde{E}_S \text{ contains a basis } v_1, \ldots, v_n \text{ fixed by } p^c\varphi^d \text{ with } v_i \equiv e_i \pmod{p}. \text{ Since } A - 1 \text{ has } p\text{-adic absolute value less than 1, we have } \lambda(\alpha^r)(A - 1) < 1 \text{ for all sufficiently small } r > 0. \text{ For any such } r, \text{ the matrix } U \text{ over } W(S) \text{ defined by } v_j = \sum_i U_{ij} e_i \text{ is congruent to 1 modulo } p \text{ and satisfies } A\varphi^d(U) = U.
We now argue as in [69] Proposition 2.5.8. We may assume \(d = a\), so \(q = p^d\). Put 
\[
C = \max\{p^{-1}, \lambda(\alpha^r)(A - 1)\} < 1
\]
We prove by induction that for each positive integer \(m\), \(U\) is congruent modulo \(p^m\) to an invertible matrix \(V_m\) over \(\mathcal{R}_S^{\int, rq}\) with 
\[
\lambda(\alpha^r)(V_m - 1), \lambda(\alpha^q)(V_m - 1) \leq C.
\]
This is obvious for \(m = 1\) by taking \(V_m = 1\). Given the claim for some \(m\), \(U\) is congruent modulo \(p^{m+1}\) to a matrix \(V_m + p^m X\) in which each entry \(X_{ij}\) is the Teichmüller lift of some \(\lambda(\alpha^r)X_{ij} \in S\). We have 
\[
\varphi^d(X) - X \equiv p^{-m}(V_m - \varphi^d(V_m) - (A - 1)\varphi^d(V_m)) \pmod{p},
\]
from which it follows that 
\[
\alpha(X_{ij})^r \leq \max\{1, (p^m C)^q^{-1}\} = (p^m C)^q^{-1}.
\]
If we put \(V_{m+1} = V_m + p^m X\), then 
\[
\lambda(\alpha^r)(V_{m+1} - 1) \leq \max\{C, p^{-m}(p^m C)^q^{-1}\} \leq C
\]
\[
\lambda(\alpha^q)(V_{m+1} - 1) \leq \max\{C, p^{-m}(p^m C)\} = C
\]
as desired.

From the previous induction, we conclude that \(U\) has entries in \(\mathcal{R}_S^{\int, rq}\) and satisfies 
\[
\lambda(\alpha^r)(U - 1), \lambda(\alpha^q)(U - 1) < 1.
\]
It is thus invertible over \(\mathcal{R}_S^{\int}\), so \(v_1, \ldots, v_n\) form a basis of 
\(M_0 \otimes_{\mathcal{R}_R} \mathcal{R}_H(\beta)\) extends to a free local \((c, d)\)-pure model of \(M\) at \(\beta\).

**Theorem 7.3.7.** Let \(M\) be a \(\varphi^a\)-module over \(\mathcal{R}_R\) of nowhere zero rank. Choose \(\beta \in \mathcal{M}(R)\) and choose \(c, d \in \mathbb{Z}\) with \(d\) a positive multiple of \(a\) and \(c/d = \mu(M, \beta)\). Suppose that \(\beta\) belongs to the pure locus of \(M\). Then any \((c, d)\)-pure model of \(M \otimes \mathcal{R}_R \mathcal{R}_H(\beta)\) extends to a free local \((c, d)\)-pure model of \(M\) at \(\beta\).

**Proof.** We may assume \(d = a\), so \(q = p^d\). Choose a \((c, d)\)-pure model \(M_{0, \beta}\) of \(M \otimes \mathcal{R}_R \mathcal{R}_H(\beta)\). By Proposition 4.2.15 for \(L\) a completed algebraic closure of \(\mathcal{H}(\beta)\), for some choice of \(r\), there exists a basis \(e_1, \ldots, e_n\) of \(M_r \otimes_{\mathcal{R}_R} \mathcal{R}_L^r\) on which \(p^r \varphi^d\) acts via the identity matrix. We may also ensure that \(e_1, \ldots, e_n\) also form a basis of \(M_{0, \beta} \otimes_{\mathcal{R}_R^{\int, H(\beta)}} \mathcal{R}_L^{\int}\).

By Lemma 2.2.13 any elements \(e'_1, \ldots, e'_n\) of \(M_r \otimes_{\mathcal{R}_R} \mathcal{R}_L^{[r/q, r]}\) which are sufficiently close to \(e_1, \ldots, e_n\) also generate \(M_r \otimes_{\mathcal{R}_R} \mathcal{R}_L^r\). Since the separable closure of \(\mathcal{H}(\beta)\) in \(L\) is dense, we can take \(e'_1, \ldots, e'_n\) to be generators of \(M_r \otimes_{\mathcal{R}_R} \mathcal{R}_E^{[r/q, r]}\) for some finite Galois extension \(E\) of \(\mathcal{H}(\beta)\).

Since \(R_\beta\) is a henselian local ring (by Lemma 2.4.12), by Theorem 1.2.8 we can find a rational localization \(R \to R'\) encircling \(\beta\) and a faithfully finite étale \(R'\)-algebra \(S\) such that \(S\) is Galois over \(R'\) (i.e., \(R'\) is the quotient of \(S\) by a finite group action), \(S\) admits a unique extension \(\gamma\) of \(\beta\), and said extension has residue field \(E\). By Lemma 5.3.9 for a suitable
choice of $R'$, we can take $e'_1, \ldots, e'_n$ to be a basis of $M_r \otimes_{\mathcal{R}_R} \mathcal{R}_S^{[r/q,r]}$ on which the action of $p^c \varphi^d$ is via a matrix $F$ for which $\lambda(\alpha^{r/q})(F - 1) < 1$.

In this setting, Lemma 7.3.2 produces a basis $e''_1, \ldots, e''_n$ of $M \otimes_{\mathcal{R}_R} \mathcal{R}_S$ on which $p^c \varphi^d$ acts via an invertible matrix over $\mathcal{R}_S^{[r/q,r]}$ congruent to 1 modulo $p$. More precisely, we have $e''_i = \sum_i U_{ij} e'_i$ for some invertible matrix $U$ over $\mathcal{R}_S^{[r/q,r]}$ for which $\lambda(\alpha^{r/q})(U - 1), \lambda(\alpha^r)(U - 1) < 1$. Starting from this new basis and then applying Proposition 7.3.6, we obtain, for some $S'$-algebra $S'$ which is the completed direct limit of faithfully finite étale subalgebras (and which maps to $L$), a basis $e'''_1, \ldots, e'''_n$ of $M \otimes_{\mathcal{R}_R} \mathcal{R}_S'$ fixed by $p^c \varphi^d$. More precisely, we have $e'''_i = \sum_i V_{ij} e''_i$ for some invertible matrix $V$ over $\mathcal{R}_S^{[r/q,r]}$ for which $\lambda(\alpha^r)(V - 1) < 1$ (this bound following from the proof of Proposition 7.3.6). We may also choose $S'$ to have an automorphism lifting each element of $G = \text{Gal}(E/\mathcal{H}(\beta))$.

We thus have $e'''_i = \sum_i (C U V)_{ij} e'_i$, and the matrix $C U V$ over $\mathcal{R}_L^{[r/q,r]}$ satisfies $\lambda(\beta^{r/q})(C U V - 1), \lambda(\beta^r)(C U V - 1) < 1$. However, both $e_i$ and $e''_i$ are fixed by $p^c \varphi^d$, so the entries of $C U V$ must be fixed by $\varphi^d$. By Lemma 4.2.10, $C U V$ has entries in the field $W(\mathbb{F}_q)[p^{-1}]$. Since $\lambda(\beta^{r/q})(C U V - 1) < 1$, we conclude that $C U V$ has entries in $pW(\mathbb{F}_q)$. In particular, $e''_i, \ldots, e'''_n$ form a basis of $M_{0, \beta} \otimes_{\mathcal{R}_S^{\text{int}(\beta)}} \mathcal{R}_L^{\text{int}}$ and hence also a basis of $M_{0, \beta} \otimes_{\mathcal{R}_S^{\text{int}(\beta)}} \mathcal{R}_E^{\text{int}}$.

This last result implies that the action of any element $\tau \in G$ on $S$ induces an automorphism of the $\mathcal{R}_E^{\text{int}}$-span of $e''_i, \ldots, e'''_n$. For each $\tau$, choose a lift of $\tau$ to $S'$ and define the matrix $T_\tau$ over $\mathcal{R}_S'$ by $\tau(e'''_i) = \sum_i (T_\tau)_{ij} e''_i$. Again, since $e'''_i$ and $\tau(e'''_i)$ are fixed by $p^c \varphi^d$, the entries of $T_\tau$ are forced to belong to the $\varphi^d$-fixed subring of $\mathcal{R}_S'$, which Corollary 5.2.4 identifies as $W((S')^{\varphi^d})[p^{-1}]$. By construction, the images of these entries in $\mathcal{R}_L$ belong to $\mathcal{R}_L^{\text{int}}$, and hence to $a_L^{\varphi^d}$. By Remark 5.1.5 and Remark 5.0.2, the condition of an element of $W((S')^{\varphi^d})[p^{-1}]$ belonging to $W(S')$ is closed and open on $\mathcal{M}(R')$; consequently, by shrinking $\mathcal{M}(R')$ again (in a manner dependent on $\tau$), we can force the entries of $T_\tau$ into $W((S')^{\varphi^d})$. (Note that this last step fails if we try to argue directly with the $e''_i$ rather than with the $e'''_i$).

After the resulting shrinking of $\mathcal{M}(R')$, the $\mathcal{R}_S^{\text{int}}$-span of $e'_1, \ldots, e'_n$ admits an action of $G$, so we can apply faithfully flat descent (Theorem 1.3.4) to descend it to a local $(c, d)$-pure model of $M$ at $\beta$. (We are here using Proposition 5.5.3(b) to deduce that $\mathcal{R}_E^{\text{int}}$ is faithfully finite étale over $\mathcal{R}_R^{\text{int}}$.) This local model is locally free by Theorem 1.3.5 we obtain a free local model at $\beta$ by applying Lemma 7.3.3.

**Corollary 7.3.8.** For any $\varphi^a$-module $M$ over $\mathcal{R}_R$, the pure locus and étale locus of $M$ are open.

**Proof.** The openness of the pure locus is immediate from Theorem 7.3.7. The étale locus is open because it is the intersection of the pure locus with the closed and open (by Lemma 7.2.2) subset of $\mathcal{M}(R)$ on which the degree of $M$ is zero.

**Corollary 7.3.9.** For any $\varphi^a$-module $M$ over $\mathcal{R}_R$, $M$ is étale (resp. pure) if and only if $M$ is pointwise étale (resp. pointwise pure).

**Corollary 7.3.10.** For any $\varphi^a$-module $M$ over $\mathcal{R}_R$ and any $\beta \in \mathcal{M}(R)$, $M$ is pure at $\beta$ if and only if $M$ admits a (not necessarily locally free) local pure model at $\beta$.
Proof. If $M$ admits a local pure model $M_0$ at $\beta$, then $M_0$ also generates a pure model of $M \otimes_{\tilde{R}_L} \tilde{R}_{H(\beta)}$, which is necessarily free because $\tilde{R}_{H(\beta)}^{\text{int}}$ is a principal ideal domain. Hence $\beta$ belongs to the pure locus of $M$, which by Theorem 7.3.7 implies that $M$ is pure at $\beta$. \hfill \Box

Remark 7.3.11. Let $M$ be a $\varphi^a$-module over $\tilde{R}_{R}^\text{bd}$. Whereas Lemma 7.3.3 implies that $M$ is pure if and only if $M \otimes_{\tilde{R}_{R}^\text{bd}} \tilde{E}_R$ is pure, it is not the case that purity of $M$ can be deduced from purity of $M \otimes_{\tilde{R}_{R}^\text{bd}} \tilde{R}_R$.

7.4 Slope filtrations in geometric families

At this point, it is natural to discuss generalizations to the relative case of the existence of slope filtrations for Frobenius modules over the Robba ring (Theorem 4.1.10) or the extended Robba ring (Theorem 4.2.13). We do not have in mind an explicit use for these in $p$-adic Hodge theory, but we expect them to become relevant in the same way that slope theory over the Robba ring appears in the work of Colmez on trianguline representations [22].

We first give a brief review of the formalism of slope filtrations and slope polygons. See [68, §3.5] for a more thorough discussion, keeping in mind the change in sign convention. (To compensate for that change, we have also swapped the order of slopes in the slope polygon in order to preserve the convex shape of the polygon.)

Definition 7.4.1. Suppose that $R = L$ is an analytic field, and let $M$ be a $\varphi^a$-module over $\tilde{R}_L$. Let $0 = M_0 \subset \cdots \subset M_l = M$ be the filtration provided by Theorem 4.2.13 in which $M_1/M_0, \ldots, M_l/M_{l-1}$ are pure and $\mu(M_1/M_0) > \cdots > \mu(M_l/M_{l-1})$. Define the slope polygon of $M$ to be the polygonal line starting at $(0,0)$ and consisting of, for $i = l, \ldots, 1$ in order, a segment of horizontal width $\text{rank}(M_i/M_{i-1})$ and slope $\mu(M_i/M_{i-1})$. Note that the right endpoint of the polygon is $(\text{rank}(M), \text{deg}(M))$.

For $M$ a $\varphi^a$-module over $\tilde{R}_L^\text{bd}$, there are two natural ways to associate a slope polygon to $M$. One is to first extend scalars to $\tilde{R}_L$ and use the definition given in the previous paragraph; this gives the special slope polygon of $M$. The other is to first extend scalars to $\tilde{E}_L$, identify the latter with $\tilde{R}_L$ for the trivial norm on $L$, then invoke the previous paragraph. This gives the generic slope polygon of $M$. (It is equivalent to define the generic slope polygon using the usual Dieudonné-Manin definition of slopes; see for instance [70, Chapter 14].)

Remark 7.4.2. For $R = L$ an analytic field and $L'$ a complete extension of $L$, passing from $L$ to $L'$ does not change slope polygons, by virtue of Corollary 4.2.14 and the uniqueness of the slope filtration in Theorem 4.2.13.

Proposition 7.4.3. Let $R = L$ be an analytic field, and let $M$ be a $\varphi^a$-module over $\tilde{R}_L^\text{bd}$.

(a) The generic slope polygon lies on or above the special slope polygon, with the same endpoints.

(b) If the two polygons coincide, then the slope filtration of $M \otimes_{\tilde{R}_L^\text{bd}} \tilde{R}_L$ (Theorem 4.2.13) descends to $M$. 

122
Proof. For (a), see [68, Proposition 5.5.1]. For (b), see [68, Theorem 5.5.2].

Lemma 7.4.4. Let $M$ be a $\varphi^a$-module over $\mathcal{R}_R^{\text{bd}}$ admitting a basis on which $\varphi^a$ acts via a matrix of the form $AD$, where $D$ is a diagonal matrix with diagonal entries in $p^{\mathbb{Z}}$, and $A$ is a square matrix such that $A - 1$ has entries in $p\mathcal{R}_R^{\text{int}}$. Then there exist an $R$-algebra $S$ which is the completion of faithfully finite étale $R$-subalgebras and an invertible matrix $U$ over $W(S)$ congruent to $1$ modulo $p$, such that $U^{-1}AD\varphi^a(U) = D$. In particular, for each $\beta \in \mathcal{M}(R)$, the generic slopes of $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_H^{\text{bd}}(\beta)$ are the negatives of the $p$-adic valuations of the entries of $D$.

Proof. As in [67, Lemma 5.9].

Theorem 7.4.5. For any $\varphi^a$-module $M$ over $\mathcal{R}_R$, the function mapping $\beta \in \mathcal{M}(R)$ to the slope polygon of $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_H(\beta)$ is lower semicontinuous. In other words, if $\text{rank}(M)$ is constant (which is true locally by Lemma [7.2.2], for any $x \in [0, \text{rank}(M)]$, the $y$-coordinate of the point of the slope polygon of $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_H(\beta)$ is a lower semicontinuous function of $\beta$. (Note that this function is locally constant for $x = \text{rank}(M)$, by Lemma [7.2.2] again.)

Proof. Choose $\beta \in \mathcal{M}(R)$. Let $L$ be a completed algebraic closure of $\mathcal{H}(\beta)$. By Proposition [4.2.16] for some positive multiple $d$ of $a$, $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_L$ admits a basis on which $\varphi^a$ acts via a diagonal matrix $D$ with entries in $p^{\mathbb{Z}}$. We now proceed as in Theorem [7.3.7] by applying Lemma [7.4.2] to a suitably good approximation of this basis. As a result, we obtain a rational localization $R \to R'$ encircling $\beta$, a faithfully finite étale $R'$-algebra $S$, and a basis $v_1, \ldots, v_n$ of $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_S$ on which $\varphi$ acts via an invertible matrix over $\mathcal{R}_S$ of the form $FD$, where $F - 1$ has entries in $p\mathcal{R}_S^{\text{int}}$. Let $N$ be the $\varphi^a$-module over $\mathcal{R}_S^{\text{bd}}$ spanned by $v_1, \ldots, v_n$.

By Remark [7.4.2] the slope polygon of $M$ at a given point of $\mathcal{M}(R')$ is the same as at any point of $\mathcal{M}(S)$ restricting to the given point. For one, this means that the negatives of the $p$-adic valuations of the diagonal entries of $D$ give the slopes in the slope polygon of $M$ at $\beta$. By Lemma [7.4.4] this polygon also computes the generic slope polygon of $N$ at each $\gamma \in \mathcal{M}(S)$. By Proposition [7.4.3] (a), we conclude that the special slope polygon of $N$ at each $\gamma \in \mathcal{M}(S)$, or in other words the slope polygon of $M$ at $\gamma$, lies on or above the slope polygon of $M$ at $\beta$. By Remark [7.4.2] again, this implies that the slope polygon is lower semicontinuous as a function on $\mathcal{M}(R)$, as desired.

In addition to semicontinuity, we have the following boundedness property for the slope polygon.

Proposition 7.4.6. For any $\varphi^a$-module $M$ over $\mathcal{R}_R$, the function mapping $\beta \in \mathcal{M}(R)$ to the slope polygon of $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_H(\beta)$ is bounded above and below.

Proof. Since slope polygons are invariant under base extension (Remark [7.4.2]), we may use the method of the proof of Lemma [6.4.6] to reduce to the case where $R$ is free of trivial spectrum. Choose $N$ as in Proposition [6.2.6] we then obtain a surjection $\mathcal{R}_R^{\text{bd}}(-N) \to M$ of $\varphi^a$-modules for some nonnegative integer $m$. For each $\beta \in \mathcal{M}(R)$, for $s$ the smallest slope in the slope polygon of $M$ at $\beta$, $M \otimes_{\mathcal{R}_R^{\text{bd}}} \mathcal{R}_H(\beta)$ surjects onto a nonzero $\varphi^a$-module over
Theorem 7.4.5; using faithfully flat descent, we may reduce to the case where $M$ is of constant rank, on which we induct. Set notation as in the proof of Theorem 7.4.4, plus faithfully flat descent, we may split $M$ uniquely as a direct sum of globally pure $\varphi$-submodules. We will show that this splitting descends to $N$; for this, we may by the proof of Proposition 7.4.5 (plus a descent argument) reduce to the case where $N$ admits a basis $e_1, \ldots, e_n$ on which $\varphi$ acts via a matrix of the form $UD$, where $U$ is an upper triangular unipotent matrix congruent to 1 modulo $p$, and $D_{ii} = p^{a_i}$ with $c_1 \geq \cdots \geq c_n$.

The argument now amounts to solving a series of equations of the form $y - p^c \varphi^d(y) = x$ for some $x \in \mathcal{H}(\beta)$, some nonnegative integer $c$, and some positive integer $d$, and showing that in each case the unique solution $y \in \mathcal{E}(\beta)$ belongs to $\mathcal{H}(\beta)$. This is easy to see for $c = 0$, so we may assume $c > 0$. We may also assume that $x \in \mathcal{H}(\beta)$, in which case $y \in \mathcal{W}(\beta)$. Fix $r$ for which $x \in \mathcal{H}(\beta)$. Then note that if $y \in \mathcal{H}(\beta)$, then we must have $y \in \mathcal{H}(\beta + r)$ and hence

$$\lambda(\alpha^r)(y) \leq \lambda(\alpha^{rp^d})(y)^{p^{-d}} = \lambda(\alpha^r)(\varphi^d(y))^{p^{-d}}.$$
Consequently, if $\lambda(\alpha^r)(\varphi^d(y)) > p^{c/(1-p^{-d})}$, then $\lambda(\alpha^r)(y) < p^{-c}\lambda(\alpha^r)(\varphi^d(y))$, so the equation $y-p^{c\varphi^d(y)} = x$ forces $\lambda(\alpha^r)(\varphi^d(y)) = \lambda(\alpha^r)(x)$. That is, $\lambda(\alpha^r p^d)(y) \leq \max\{p^{c/(1-p^{-d})}, \lambda(\alpha^r)(x)\}$.

In other words, if the solution $y$ is in $\mathcal{R}_R^{\text{int}}$, then we have a uniform bound on its size determined by $x$. It follows that we may check for this property pointwise on $\mathcal{M}(R)$. But for $R = L$ an analytic field, the presence of $y$ in $\mathcal{R}_R^{\text{int}}$ is precisely what is guaranteed by [68, Theorem 5.5.2]. This completes the proof.

**Corollary 7.4.9.** Let $M$ be a $\varphi$-module over $\mathcal{R}_R$. Then there exists an integer $N$ such that for $n \geq N$, $H^0_{\varphi}(M(-n)) = 0$, $H^0_{\varphi}(M \otimes_{\mathcal{R}_R} \mathcal{R}_{H(\beta)}(-n)) = 0$ for all $\beta \in \mathcal{M}(R)$, and the map

$$H^1_{\varphi}(M(-n)) \to \prod_{\beta \in \mathcal{M}(R)} H^1_{\varphi}(M(-n) \otimes_{\mathcal{R}_R} \mathcal{R}_{H(\beta)})$$

is injective.

**Proof.** By extending scalars as in Lemma [6.4.6] we may reduce to the case where $R$ is free of trivial spectrum. We first treat the case $M = \mathcal{R}_R$, in which case we prove the claim for $N = 1$. For $n \geq 1$, by Theorem [4.2.12] $H^0_{\varphi}(\mathcal{R}_{H(\beta)}(-n)) = 0$ for all $\beta \in \mathcal{M}(R)$; since the map

$$H^0_{\varphi}(\mathcal{R}_R(-n)) \to \prod_{\beta \in \mathcal{M}(R)} H^0_{\varphi}(\mathcal{R}_{H(\beta)}(-n))$$

is evidently injective, it follows that $H^0_{\varphi}(\mathcal{R}_R(-n)) = 0$. Next, suppose that $x \in H^1_{\varphi}(\mathcal{R}_R(-n))$ has zero image in $H^1_{\varphi}(\mathcal{R}_{H(\beta)}(-n))$ for each $\beta \in \mathcal{M}(R)$. Then $x$ defines an extension

$$0 \to \mathcal{R}_R(-n) \to P \to \mathcal{R}_R \to 0$$

whose base extension to $\mathcal{R}_{H(\beta)}$ splits for each $\beta \in \mathcal{M}(R)$. This implies that the slope polygon of $M$ is the constant polygon with slopes $-n, 0$, so the extension splits by Theorem [7.4.8]. Hence $x = 0$ as desired.

In the general case, by Proposition [6.2.6] for any sufficiently large integer $n_0$, there exists a surjection $\mathcal{R}_R^m(-n_0) \to M'$ of $\varphi$-modules for some positive integer $m$ (depending on $n_0$). The kernel of this map is again a $\varphi$-module (by Remark [6.1.3]), so we may write it as $P'$ and then transpose to obtain a short exact sequence

$$0 \to M \to \mathcal{R}_R^m(n_0) \to P \to 0.$$ 

For $n$ sufficiently large, $H^0_{\varphi}(\mathcal{R}_R^m(n_0 - n)) = 0$ and $H^0_{\varphi}(\mathcal{R}_{H(\beta)}^m(n_0 - n)) = 0$ for all $\beta \in \mathcal{M}(R)$. For such $n$, $H^0_{\varphi}(M(-n)) = 0$ and $H^0_{\varphi}(M \otimes_{\mathcal{R}_R} \mathcal{R}_{H(\beta)}(-n)) = 0$ for all $\beta \in \mathcal{M}(R)$. By the same reasoning, for $n$ sufficiently large, $H^0_{\varphi}(P'(-n)) = 0$. It follows that in the commutative diagram

$$\begin{array}{ccc}
H^1_{\varphi}(M(-n)) & \longrightarrow & H^1_{\varphi}(\mathcal{R}_R^m(n_0 - n)) \\
\downarrow & & \downarrow \\
\prod_{\beta \in \mathcal{M}(R)} H^1_{\varphi}(M(-n) \otimes_{\mathcal{R}_R} \mathcal{R}_{H(\beta)}) & \longrightarrow & \prod_{\beta \in \mathcal{M}(R)} H^1_{\varphi}(\mathcal{R}_{H(\beta)}^m(n_0 - n)),
\end{array}$$
for \( n \) sufficiently large, the top horizontal arrow (by the snake lemma) and the right vertical arrow (by the previous paragraph) are injective. Hence the left vertical arrow is injective, as desired.

**Remark 7.4.10.** As noted earlier, there is a generalization of slope theory for \( \varphi \)-modules orthogonal to the one given here, where one continues to work with rings of power series but with coefficients in more general rings (such as affinoid algebras over \( \mathbb{Q}_p \)). These are called *arithmetic families* in [74], where they are distinguished from the *geometric families* arising here. Unfortunately, it seems difficult to achieve any results in the context of arithmetic families as complete as those given here, in no small part because such results would most likely require a heretofore nonexistent slope theory for Frobenius modules over a Robba ring consisting of Laurent series over a *nondiscretely* valued field. One does however get some important information by working in neighborhoods of rigid analytic points; for instance, one can construct global slope filtrations in such neighborhoods [80], which is relevant for applications to \( p \)-adic automorphic forms via the study of eigenvarieties.

# 8 \( \varphi \)-modules and local systems

We conclude by relating \( \varphi \)-modules to étale local systems. This relationship forms the foundation of our study of relative \( p \)-adic Hodge theory, to be continued in subsequent papers.

**Hypothesis 8.0.1.** Throughout \( \S 8 \) continue to let \( R \) be a perfect uniform Banach algebra over \( \mathbb{F}_p \). However, fix also a realization of \( R \) as the completed perfection of the direct limit \((A, \alpha)\) of a uniform affinoid system \( \{(A_i, \alpha_i)\}_{i \in I} \); such a realization always exists thanks to Lemma 2.6.2(b). (All constructions will be independent of this choice except when the affinoid system explicitly appears.)

## 8.1 \( \varphi \)-modules and local systems

We now relate pure and étale \( \varphi \)-modules to étale local systems, starting with the case of \( \mathbb{Z}_p \)-local systems.

**Lemma 8.1.1.** For any \( c > 1 \), any positive integer \( d \), and any \( x \in R \), there exists \( y \in R \) with \( \alpha(x - y + y^{pd}) < c \).

**Proof.** If \( \alpha(x) \leq 1 \), we may take \( y = 0 \). Otherwise, we may take \( y = -(x^{p^{-d}} + \cdots + x^{p^{-md}}) \) for any positive integer \( m \) which is large enough that \( \alpha(x)^{p^{-md}} < c \), as then \( x - y + y^{pd} = x^{p^{-md}} \).

**Theorem 8.1.2.** For \( d \) a positive integer such that \( \mathbb{F}_p^d \subseteq A \), let \( \mathbb{Z}_p^d \) denote the valuation subring of the finite unramified extension of \( \mathbb{Q}_p \) of degree \( d \). Then the following categories are equivalent.

(a) The category of étale \( \mathbb{Z}_p^d \)-local systems over \( \text{Spec}(R) \).
(b) The category of étale $\mathbb{Z}_p$-local systems over $\text{Spec}(A)$.

(c) The category of étale $\mathbb{Z}_p$-local systems over $\text{Spec}(\mathcal{R}_R^{\text{int},1}/(z))$ for any $z \in W(\mathfrak{o}_R)$ which is primitive of degree 1.

(d) The category of $\varphi^d$-modules over $W(R)$.

(e) The category of $\varphi^d$-modules over $\mathcal{R}_R^{\text{int}}$.

More precisely, the functor from (e) to (d) is base extension.

Proof. The equivalences between (a) and (b) and between (a) and (c) follow from Remark 1.3.3 combined with Theorem 3.1.5 and Theorem 3.6.20, respectively. The equivalence between (a) and (d) follows immediately from Proposition 3.2.7.

We next check that the base extension functor from (e) to (d) is fully faithful. As in Remark 1.3.4, it suffices to check that for $M$ a $\varphi^d$-module over $\mathcal{R}_R^{\text{int}}$, any $\varphi^d$-stable element $v \in M \otimes_{\mathcal{R}_R^{\text{int}}} W(R)$ belongs to $M$. This claim may be checked locally on $\mathcal{M}(R)$, so we may assume that $M$ is free over $\mathcal{R}_R^{\text{int}}$. By Proposition 1.3.6 we can choose an $R$-algebra $S$ which is a completed direct limit of faithfully finite étale $R$-subalgebras, in such a way that $M \otimes_{\mathcal{R}_R^{\text{int}}} \mathcal{R}_S$ admits a $\varphi^d$-invariant basis $e_1, \ldots, e_n$. If we write $v = \sum_i x_i e_i$ with $x_i \in W(S)$, then $x_i \in W(S)^{\varphi^d} = W(S^{\varphi^d})$, and the latter ring is contained in $\mathcal{R}_S$ by Remark 3.1.3. Since $W(R) \cap \mathcal{R}_R^{\text{int}} = \mathcal{R}_R^{\text{int}}$, it follows that $v \in M$ as desired.

We finally check that the functor from (e) to (d) is essentially surjective. Let $M$ be a $\varphi^d$-module over $W(R)$. By faithfully flat descent (Theorem 1.3.4 and Theorem 1.3.5), to check that $M$ arises by base extension from $\mathcal{R}_R^{\text{int}}$, it suffices to do so after replacing $R$ with a faithfully finite étale extension. Since (a) and (d) are equivalent, we may reduce to the case where $M$ admits a basis $e_1, \ldots, e_n$ on which $\varphi^d$ acts via a matrix $F$ for which $F - 1$ has entries in $pW(R)$.

We define matrices $F_n, G_n$ for each positive integer $n$ such that $F_1 = F, G_1 = 1, F_n - 1$ has entries in $pW(R), G_n$ has entries in $\mathcal{R}_R^{\text{int},1}, \lambda(\alpha)(G_n - 1) < 1$, and $X_n = p^{-n}(F_n - G_n)$ has entries in $W(R)$. Namely, given $F_n$ and $G_n$, apply Lemma 8.1.1 to construct a matrix $\overline{Y}_n$ over $R$ so that $\alpha(\overline{X}_n - \overline{Y}_n + \varphi^d(\overline{Y}_n)) < p^{n/2}$. Then put $U_n = 1 + p^n[\overline{Y}_n]$ (the entries of $[\overline{Y}_n]$ are Teichmüller liftings of corresponding entries of $\overline{Y}_n$), $F_{n+1} = U_n^{-1}F_n\varphi^d(U_n), G_{n+1} = G_n + p^n[\overline{X}_n - \overline{Y}_n + \varphi^d(\overline{Y}_n)]$. The product $U_1U_2 \cdots$ converges to a matrix $U$ so that $U^{-1}F\varphi^d(U)$ is equal to the $p$-adic limit of the $G_n$, which is invertible over $\mathcal{R}_R^{\text{int},r}$ for any $r \in (0, 1)$. Consequently, the $\mathcal{R}_R^{\text{int}}$-span of the vectors $v_1, \ldots, v_n$ defined by $v_j = \sum_i U_{ij} e_i$ gives a $\varphi^d$-module $N$ over $\mathcal{R}_R^{\text{int}}$ for which $N \otimes_{\mathcal{R}_R^{\text{int}}} W(R) \cong M$. \hfill \Box

Remark 8.1.3. In Theorem 8.1.2 the equivalence between étale $\mathbb{Z}_p$-local systems over $R$ and $\varphi$-modules over $W(R)$ can also be interpreted as a form of nonabelian Artin-Schreier theory, using Lang torsors. For example, see [S1, Proposition 4.12] for a derivation of ordinary Artin-Schreier theory in this framework.

We next consider $\mathbb{Q}_p$-local systems, for which the situation is complicated by the distinction between $\text{Spec}(R)$ and $\mathcal{M}(R)$ (Remark 2.8.2). We start by considering the former.
Theorem 8.1.4. For $d$ a positive integer such that $\mathbb{F}_{p^d} \subseteq A$, let $\mathbb{Q}_{p^d}$ denote the finite unramified extension of $\mathbb{Q}_p$ of degree $d$. Then the following categories are equivalent.

(a) The category of étale $\mathbb{Q}_{p^d}$-local systems over $\text{Spec}(R)$.

(b) The category of étale $\mathbb{Q}_{p^d}$-local systems over $\text{Spec}(A)$.

(c) The category of étale $\mathbb{Q}_{p^d}$-local systems over $\text{Spec}(\mathcal{R}_R^{\text{int},1}/(z))$ for any $z \in W(\mathfrak{o}_R)$ which is primitive of degree 1.

(d) The category of globally étale $\varphi^d$-modules over $\mathcal{E}_R$.

(e) The category of globally étale $\varphi^d$-modules over $\mathcal{R}_R^{\text{bd}}$.

(f) The category of globally étale $\varphi^d$-modules over $\mathcal{R}_R$.

More precisely, the functors from (e) to (d) and (f) are base extensions.

Proof. The equivalences between (a) and (b) and between (a) and (c) again follow from Remark 1.4.4 combined with Theorem 3.1.15 and Theorem 3.6.20, respectively.

The functor from (a) to (e) is constructed as follows. Let $V$ be an étale $\mathbb{Q}_{p^d}$-local system over $\text{Spec}(R)$; we may write $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for some $\mathbb{Z}_p$-local system $T$ on $\text{Spec}(R)$. The latter corresponds to a $\varphi^d$-module $M_0$ over $\mathcal{R}_R^{\text{int}}$ by Theorem 8.1.2. The assignment $V \mapsto M_0$ then defines a fully faithful functor by Corollary 5.2.4; by the same reasoning, the resulting functors from (a) to (d) and from (a) to (f) are fully faithful.

To construct the functor from (e) back to (a), given a $\varphi^d$-module $M$ over $\mathcal{R}_R^{\text{bd}}$ admitting a locally free étale model $M_0$, apply Theorem 8.1.2 to convert $M_0$ into a $\mathbb{Z}_p$-local system $T$ on $\text{Spec}(R)$. The assignment $M \mapsto T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ defines a quasi-inverse to the functor from (a) to (e). By similar reasoning, the functors from (a) to (d) and from (a) to (f) are equivalences of categories.

Definition 8.1.5. For $c,d \in \mathbb{Z}$ with $d > 0$, define an étale $(c,d)$-$\mathbb{Q}_p$-local system on a scheme $X$ to be a $\mathbb{Q}_{p^d}$-local system $V$ on $X$ equipped with a semilinear action of the Frobenius automorphism $\tau$ of $\mathbb{Q}_{p^d}$ on sections, such that $p^c \tau^d$ acts as the identity. For $c',d' \in \mathbb{Z}$ with $d' > 0$ and $c'/d' = c/d$, the categories of étale $(c,d)$-$\mathbb{Q}_p$-local systems and étale $(c',d')$-$\mathbb{Q}_p$-local systems on any scheme are naturally equivalent: this reduces to the case where $c' = ce, d' = de$ for some positive integer $e$, in which case the claim is an easy exercise using Hilbert’s Theorem 90. (See [10, Théorème 3.2.3] for a similar construction.)

Theorem 8.1.6. For $d$ a positive integer such that $\mathbb{F}_{p^d} \subseteq A$ and any integer $c$, the following categories are equivalent.

(a) The category of étale $(c,d)$-$\mathbb{Q}_p$-local systems over $\text{Spec}(R)$.

(b) The category of étale $(c,d)$-$\mathbb{Q}_p$-local systems over $\text{Spec}(A)$.
(c) The category of étale \((c,d)\)-\(\Q_p\)-local systems over \(\text{Spec}(\hat{R}_R^{\text{int},1}/(z))\) for any \(z \in W(\mathfrak{o}_R)\) which is primitive of degree 1.

(d) The category of globally \((c,d)\)-pure \(\varphi\)-modules over \(\hat{E}_R\).

(e) The category of globally \((c,d)\)-pure \(\varphi\)-modules over \(\hat{R}_R^{\text{bd}}\).

(f) The category of globally \((c,d)\)-pure \(\varphi\)-modules over \(\hat{R}_R\).

More precisely, the functors from (e) to (d) and (f) are base extensions.

**Proof.** This is immediate from Theorem 8.1.4.

**Definition 8.1.7.** For \(d\) a positive integer, for each strong covering family \(R \to R_1, \ldots, R \to R_n\) of rational localizations, construct the categories of descent data for \(\varphi^d\)-modules over \(\hat{E}_r, \hat{R}_R^{\text{bd}}, \hat{R}_R\) for the homomorphism \(R \to R_1 \oplus \cdots \oplus R_n\) of perfect uniform Banach algebras over \(\mathbb{F}_p\). Then form the categorical direct limit (as in Remark 1.2.9) over all strong covering families; we call the resulting categories the categories of local \(\varphi^d\)-modules over \(\hat{E}_R, \hat{R}_R^{\text{bd}}, \hat{R}_R\). Since the pure and étale conditions are local on \(\mathcal{M}(R)\), we immediately obtain categories of pure and étale local \(\varphi^d\)-modules.

**Remark 8.1.8.** There is a natural functor from \(\varphi^d\)-modules to local \(\varphi^d\)-modules over each of \(\hat{E}_R, \hat{R}_R^{\text{bd}}, \hat{R}_R\). These functors are fully faithful by Theorem 5.3.4. The functors for \(\hat{E}_R\) and \(\hat{R}_R^{\text{bd}}\) are not equivalences of categories (see Example 8.1.14); but the functor for \(\hat{R}_R\) is an equivalence of categories when \(R\) is free of trivial spectrum (see Remark 6.4.2); in fact, \(\varphi^d\)-modules over \(\hat{R}_R\) admit glueing even for covering families which are not strong. Thanks to the local nature of the pure and étale conditions, one sees easily that a \(\varphi^d\)-module is pure or étale if and only if the corresponding local \(\varphi^d\)-module has this property.

**Theorem 8.1.9.** For \(d\) a positive integer such that \(\mathbb{F}_{p^d} \subseteq A\) and any integer \(c\), the following categories are equivalent.

(a) The category of étale \((c,d)\)-\(\Q_p\)-local systems over \(\mathcal{M}(R)\).

(b) The category of étale \((c,d)\)-\(\Q_p\)-local systems over \(\mathcal{M}(A)\).

(c) The category of étale \((c,d)\)-\(\Q_p\)-local systems over \(\mathcal{M}(\hat{R}_R^{\text{int},1}/(z))\) for any \(z \in W(\mathfrak{o}_R)\) which is primitive of degree 1.

(d) The category of \((c,d)\)-pure local \(\varphi\)-modules over \(\hat{E}_R\).

(e) The category of \((c,d)\)-pure local \(\varphi\)-modules over \(\hat{R}_R^{\text{bd}}\).

(f) The category of \((c,d)\)-pure local \(\varphi\)-modules over \(\hat{R}_R\).

More precisely, the functors from (e) to (d) and (f) are base extensions.

**Proof.** This is immediate from Theorem 8.1.4 and Theorem 3.3.7(c).
Corollary 8.1.10. For a a positive integer, a \( \varphi^a \)-module over \( \mathcal{E}_R \), \( \mathcal{R}_R^\bd \), \( \mathcal{R}_R \) is pure of slope \( s \) at some \( \beta \in M(R) \) if and only if it is \( (c, d) \)-pure at \( \beta \) for every (not just one) pair \( c, d \) of integers for which \( d \) is a positive multiple of \( a \) and \( c/d = s \).

Proof. It is enough to check that for any positive integer \( e \), a \( \varphi^a \)-module \( M \) is \( (c, d) \)-pure at \( \beta \) if and only if it is \( (ce, de) \)-pure at \( \beta \). Using faithfully flat descent, this reduces to the case where \( \mathbb{F}_{p^e} \subseteq A \). (This requires a bit of care to produce the descent data on local models; see the end of the proof of Theorem 7.3.7.) In this case, the claim follows from Theorem 8.1.9 plus the corresponding equivalence on the side of local systems (Definition 8.1.3).

Corollary 8.1.11. Let \( R \to S \) be a bounded homomorphism for which \( M(S) \to M(R) \) is surjective. For a a positive integer, let \( M \) be a \( \varphi^a \)-module over \( \mathcal{R}_R^\bd \) (resp. \( \mathcal{R}_R \)). Then \( M \) is pure if and only if \( M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S^\bd \) (resp. \( M \otimes_{\mathcal{R}_R} \mathcal{R}_S \)) is pure. In particular, \( M \) is étale if and only if \( M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S^\bd \) (resp. \( M \otimes_{\mathcal{R}_R} \mathcal{R}_S \)) is étale.

Proof. By Remark 8.1.8, it is sufficient to prove the same assertion for local \( \varphi^a \)-modules. Suppose first that \( M \) is a local \( \varphi^a \)-module over \( \mathcal{R}_R \). If \( M \) is pure, then so is \( M \otimes_{\mathcal{R}_R} \mathcal{R}_S \). Conversely, if \( M \otimes_{\mathcal{R}_R} \mathcal{R}_S \) is pure, then it is pointwise pure. By Corollary 4.12.14, \( M \) is pointwise pure, so by Corollary 7.3.9, \( M \) is pure.

Suppose next that \( M \) is a local \( \varphi^a \)-module over \( \mathcal{R}_R^\bd \). If \( M \) is pure, then so is \( M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S^\bd \). Conversely, if \( M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S^\bd \) is pure, then by the first paragraph, \( M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_R \) is pure. By Theorem 8.1.9 (and a descent argument as in Corollary 8.1.10), there exists a unique pure \( \varphi^a \)-module \( N \) over \( \mathcal{R}_R^\bd \) such that \( N \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_R \cong M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_R \). Moreover, the resulting isomorphism \( N \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S \cong M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S \) descends to an isomorphism \( N \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S \cong M \otimes_{\mathcal{R}_R^\bd} \mathcal{R}_S \). Since for any \( x \in \mathcal{R}_R \), if the image of \( x \) in \( \mathcal{R}_S \) belongs to \( \mathcal{R}_S^\bd \), then \( x \) belongs to \( \mathcal{R}_R^\bd \) (by Lemma 5.2.2), and \( M \) and \( N \) are locally free over \( \mathcal{R}_R^\bd \) by definition, we deduce that \( N \cong M \), so \( M \) is pure.

As a corollary, we obtain the following glueing result. Remember that the definition of an étale local system only incorporates glueing for strong covering families, so this is not tautologous (see Remark 2.8.3).

Corollary 8.1.12. For any covering family \( R \to R_1, \ldots, R \to R_n \) of rational localizations, the homomorphism \( R \to R_1 \oplus \cdots \oplus R_n \) is an effective descent morphism for the opposite category of étale \((c, d)\)-\( \mathbb{Q}_p \)-local systems on \( M(\bullet) \) for \( \bullet \) a perfect uniform Banach \( \mathbb{F}_p \)-algebra.

Proof. This follows from Theorem 8.1.9 and Theorem 5.3.6.

Proposition 8.1.13. Suppose that the ring \( A \) is noetherian and normal. Then for any positive integer \( a \), any \( \varphi^a \)-module over \( \mathcal{E}_R \) or \( \mathcal{R}_R^\bd \) admitting a pure model admits a locally free pure model.

Proof. We first prove the claim over \( \mathcal{E}_R \). The hypothesis and conclusion do not refer to the space \( M(R) \) or the norm on \( R \), so we may equip \( R \) with the trivial norm, and then identify
\( \mathcal{E}_R \) with \( \tilde{\mathcal{R}}_R \). The resulting \( \varphi^a \)-module over \( \tilde{\mathcal{R}}_R \) is pure by Corollary 7.3.10, Theorem 8.1.9 then produces an étale \( \mathbb{Q}_p \)-local system over \( \mathcal{M}(A) \), which by Proposition 2.8.4 arises from a local system over \( \operatorname{Spec}(A) \). By Theorem 8.1.4, the original \( \varphi^a \)-module is forced to admit a locally free pure model.

Suppose now that \( M \) is a \( \varphi^a \)-module over \( \tilde{\mathcal{R}}_R^{\text{bd}} \) admitting a pure model. By Theorem 8.1.9 \( M \) corresponds to an étale \( \mathbb{Q}_p \)-local system over \( \mathcal{M}(R) \). By applying the previous paragraph to \( M \otimes_{\tilde{\mathcal{R}}_R^{\text{bd}}} \mathcal{E}_R \), we see that the local system over \( \mathcal{M}(R) \) arises from a local system over \( \operatorname{Spec}(R) \). By Theorem 8.1.4 again, \( M \) is forced to admit a locally free pure model. \( \square \)

Here is an example to illustrate the difference between \( \varphi \)-modules and local \( \varphi \)-modules.

**Example 8.1.14.** Put \( K = \mathbb{F}_p((q)) \) for an arbitrary normalization \( |q| = \omega < 1 \) of the \( q \)-adic norm, and define the strictly affinoid algebras

\[
B = K\{\omega^2/T,T,U\}/(U(T-q)-1), \quad B_1 = K\{\omega^2/T,T/\omega^2\}, \quad B_2 = K\{1/T,T\}.
\]

over \( K \). In words, \( \mathcal{M}(B) \) is the annulus \( \omega^2 \leq |T| \leq 1 \) minus the circle \( |T-q| < \omega^2 \), and \( \mathcal{M}(B_1) \) and \( \mathcal{M}(B_2) \) are the boundary circles \( |T| = \omega^2 \) and \( |T| = 1 \), respectively, within \( \mathcal{M}(B) \). Let \( \sigma_q : B_2 \to B_1 \) be the substitution \( T \mapsto q^2T \). If we quotient \( \mathcal{M}(B) \) by the identification \( \mathcal{M}(B_2) \cong \sigma^*_q \mathcal{M}(B_1) \), we obtain a strictly affinoid subspace \( \mathcal{M}(A) \) of the Tate curve over \( X \) for the parameter \( q^2 \). The latter is the analytification of a smooth projective curve over \( K \) of genus 1; see for instance [97, Theorem V.3.1] for explicit equations.

We may construct an étale \( \mathbb{Q}_p \)-local system \( V \) on \( \mathcal{M}(A) \) as follows. Let \( \tilde{V} \) be the trivial \( \mathbb{Q}_p \)-local system on \( \mathcal{M}(B) \), equipped with the distinguished generator 1. Let \( \tilde{V}_1, \tilde{V}_2 \) be the restrictions of \( \tilde{V} \) to \( \mathcal{M}(B_1), \mathcal{M}(B_2) \), respectively. To specify \( V \), it suffices to specify an isomorphism \( \tilde{V}_1 \cong \sigma_q^* \tilde{V}_2 \); we choose the isomorphism matching 1 in \( \tilde{V}_1 \) with \( p \) in \( \sigma_q^* \tilde{V}_2 \).

Let \( R, S, S_1, S_2 \) be the completed perfections of \( A, B, B_1, B_2 \), respectively. We claim that \( V \) cannot correspond to an étale \( \varphi \)-module \( M \) over \( \mathcal{E}_R \) (or over the subring \( \tilde{\mathcal{R}}_R^{\text{bd}} \) thereof). To check this, suppose the contrary, and choose any nonzero element \( \mathbf{v} \in M \). The pullback of \( M \) to \( \mathcal{E}_S \) can be identified with the trivial \( \varphi \)-module \( \mathcal{E}_S \) itself, and \( \mathbf{v} \) must correspond to an element \( x \in \mathcal{E}_S \). Let \( x_1, x_2 \) be the images of \( x \) in \( \mathcal{E}_{S_1}, \mathcal{E}_{S_2} \), respectively. We must then have

\[
x_2 = p\sigma_q(x_1) \in \mathcal{E}_{S_2} = W(S_2)[p^{-1}].
\]

However, this is impossible: the maps \( S \to S_1, S_2 \) are injective, so an element of \( W(S)[p^{-1}] \) which maps to \( W(S_1) \) or \( W(S_2) \) must itself belong to \( W(S) \). Thus if \( x \in p^m W(S) \) for some \( m \in \mathbb{Z} \), then also \( x \in p^{m+1} W(S) \), which cannot hold for all \( m \) if \( x \neq 0 \).

By contrast, by Theorem 8.1.9 \( V \) does correspond to an étale \( \varphi \)-module over \( \tilde{\mathcal{R}}_R \). It would be interesting to describe this \( \varphi \)-module explicitly.

**Remark 8.1.15.** If \( A \) is a connected affinoid algebra over an analytic field \( K \), then an étale fundamental group of \( \mathcal{M}(A) \) has been defined by de Jong [29]; its continuous representations on finite-dimensional \( \mathbb{Q}_p \)-vector spaces correspond precisely to étale \( \mathbb{Q}_p \)-local systems on \( \mathcal{M}(A) \) in our sense [29, Lemma 2.6]. It should be possible to show using Theorem 3.3.7(c) and Theorem 8.3.6.20 that \( \mathcal{M}(A) \) and \( \mathcal{M}(\tilde{\mathcal{R}}_R^{\text{int,1}}/(z)) \) have the same étale fundamental group;
the only serious issue is that $\hat{R}^\text{int,1}_R((z))$ need not be an affinoid algebra over an analytic field, so some work is needed to define the étale fundamental group and check some basic properties.

### 8.2 Comparison of cohomology

We conclude by relating the cohomology of étale local systems, $\varphi$-modules, and sheaves on $\text{Proj}(P)$. Again, this comes down to nonabelian Artin-Schreier-Witt theory.

**Theorem 8.2.1.** Let $d$ be a positive integer such that $F_{p^d} \subseteq A$. Let $X$ be one of $\text{Spec}(A)$, $\text{Spec}(R)$, or $\text{Spec}(\hat{R}^\text{int,1}_R((z)))$ for any $z \in W(o_R)$ which is primitive of degree 1. Let $T$ be an étale $\mathbb{Z}_{p^d}$-local system on $X$. Let $M$ be the $\varphi^d$-module over $\hat{R}^\text{int}_R$ or $W(R)$ corresponding to $T$ via Theorem 8.1.2. Then there are natural (in $\varphi^d$-modules) bijections $H^i_\text{ét}(X,T) \cong H^i_\varphi(M)$ for all $i \geq 0$.

**Proof.** By Theorem 8.1.2, we may assume without loss of generality that $X = \text{Spec}(R)$. Suppose first that $M$ is defined over $W(R)$. For each positive integer $n$, view $T/p^nT$ as a locally constant étale sheaf on $\text{Spec}(R)$, and let $\tilde{M}_n$ be the étale sheaf on $\text{Spec}(R)$ corresponding to the quasicoherent sheaf on $\text{Spec}(W(R)/(p^n))$ with global sections $M/p^nM$. We then have an exact sequence

$$0 \to T/p^nT \to \tilde{M}_n \xrightarrow{\varphi^d-1} \tilde{M}_n \to 0,$$

where exactness at the right is given by Theorem 8.1.2 (or more directly by Proposition 7.3.6). We see by induction on $n$ that $\tilde{M}_n$ is acyclic: it is enough to check that $\tilde{M}_n/\tilde{M}_{n-1}$ is acyclic, which is because it arises from a quasicoherent sheaf on an affine scheme. Taking the long exact sequence in cohomology thus yields the desired result.

Suppose next that $M$ is defined over $\hat{R}^\text{int}_R$; by the previous paragraph, we need only check the cases $i = 0, 1$. For these, interpret $H^i_\varphi(M)$ as an extension group as in Definition 1.5.3, then note that any extension of two $\varphi^d$-modules is again a $\varphi^d$-module. By Theorem 8.1.2, the 0th and 1st extension groups do not change upon base extension to $W(R)$; we may thus deduce the claim from the previous paragraph.

**Lemma 8.2.2.** Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of $\varphi$-modules over $\hat{R}_R$. If any two of $M, M_1, M_2$ are $(c,d)$-pure, then so is the third.

**Proof.** By Corollary 7.3.9, it suffices to treat the case that $R = L$ is an analytic field. In this case, the lemma follows immediately from Theorem 4.2.12.

We have a similar result for $\mathbb{Q}_p$-local systems. Note that this result can also be formulated in terms of $\varphi$-bundles using Proposition 6.4.3.

**Theorem 8.2.3.** Let $d$ be a positive integer such that $F_{p^d} \subseteq A$. Let $X$ be one of $\text{Spec}(A)$, $\text{Spec}(R)$, or $\text{Spec}(\hat{R}^\text{int,1}_R((z)))$ for any $z \in W(o_R)$ which is primitive of degree 1. Let $E$ be an étale $\mathbb{Q}_p$-local system on $X$. Let $M$ be the globally étale $\varphi^d$-module over $\hat{R}^\text{int,bl}_R$, $\hat{E}_R$, or $\hat{R}_R$ corresponding to $E$ via Theorem 8.1.4. Then there are natural (in $E$ and $A$) bijections $H^i_\text{ét}(X,E) \cong H^i_\varphi(M)$ for all $i \geq 0$. 

132
**Proof.** We first treat the case over \( \tilde{R}_{R}^{d} \), the case over \( \mathcal{E}_{R} \) being similar. Let \( T \) be an étale \( \mathbb{Z}_{p^{d}} \)-local system on \( X \) for which \( E = T \otimes \mathbb{Z}_{p} \mathbb{Q}_{p} \). Let \( M_{0} \) be the \( \varphi^{d} \)-module over \( \tilde{R}_{R}^{d} \) corresponding to \( T \). By Theorem 8.2.1, we have natural (in \( T \) and \( A \)) bijections \( H_{\text{ét}}^{i}(X, T) \cong H_{\varphi^{d}}^{i}(M_{0}) \) for all \( i \geq 0 \). By definition, we may identify \( H_{\text{ét}}^{i}(X, E) \) with \( H_{\text{ét}}^{i}(X, T) \otimes \mathbb{Z}_{p} \mathbb{Q}_{p} \); in particular, it is zero for \( i > 1 \). On the other hand, for \( i = 0, 1 \), we may identify \( H_{\varphi^{d}}^{i}(M) \) with \( H_{\varphi^{d}}^{i}(M_{0}) \otimes \mathbb{Z}_{p} \mathbb{Q}_{p} \) by identifying \( M \) with \( M_{0} \otimes \mathbb{Z}_{p} \mathbb{Q}_{p} = \bigcup_{n=0}^{\infty} p^{-n} M_{0} \) and noting that the computation of \( H_{\varphi^{d}}^{i} \) commutes with direct limits. This proves the claim in this case.

We next treat the case over \( \tilde{R}_{R} \). Put \( M_{1} = M_{0} \otimes \mathbb{Z}_{p} \mathbb{Q}_{p} \), so that we may identify \( M \) with \( M_{1} \otimes \mathcal{R}_{R}^{d} \tilde{R}_{R} \). It follows from Theorem 8.1.4 that the natural map \( H_{\varphi^{d}}^{0}(M_{1}) \to H_{\varphi^{d}}^{0}(M) \) is bijective; hence \( H_{\varphi^{d}}^{0}(M) \) is naturally isomorphic to \( H_{\text{ét}}^{0}(X, E) \). Recall that by Remark 2.8.7, the extension of two étale \( \mathbb{Q}_{p^{d}} \)-local systems on \( \text{Spec}(R) \) in the category of étale \( \mathbb{Q}_{p^{d}} \)-local systems on \( \mathcal{M}(R) \) descends to an extension of étale \( \mathbb{Q}_{p^{d}} \)-local systems on \( \text{Spec}(R) \). That is, we may compute \( H_{\text{ét}}^{1}(X, E) \) as an extension group in the category of étale local systems over \( \mathcal{M}(R) \). We then obtain an isomorphism between this group and \( H_{\varphi^{d}}^{1}(M) \) by applying Theorem 6.2.9 and noting that any extension of étale \( \varphi^{d} \)-modules over \( \mathcal{R}_{R} \) is again étale by Lemma 8.2.2. \( \square \)

**Remark 8.2.4.** One can formulate an analogue of Theorem 8.2.3 for étale \((c, d)\)-local systems comparing a suitably modified étale cohomology to \( H_{\varphi^{c}}^{0}(d) \) of the corresponding \( \varphi \)-module. As this statement is a formal consequence of the one given, and we have no particular use for it, we omit further details.

There is also an analogous result for étale \( \mathbb{Q}_{p} \)-local systems on \( \mathcal{M}(A) \), but it is somewhat subtler (especially in the vanishing aspect for \( i > 1 \)); it is easiest to obtain it by going through the Fargues-Fontaine vector bundles interpretation. See Theorem 8.3.6.

### 8.3 More comparison of cohomology

We can get some further results concerning cohomology of \( \varphi \)-modules by restricting the ring \( R \) to be a perfect uniform Banach algebra over a nonarchimedean field. This is harmless for our intended applications, since this is a necessary condition for the existence of elements of \( W(\mathfrak{o}_{R}) \) which are primitive of degree 1.

**Hypothesis 8.3.1.** Throughout 8.3 assume that \( R \) is a perfect uniform Banach algebra over a nontrivially normed analytic field \( L \), and let \( z \in W(\mathfrak{o}_{R}) \) be an element which is primitive of degree 1. Also fix a positive integer \( a \), as used in Definition 6.3.1.

**Remark 8.3.2.** By Lemma 6.2.4, there exist nonzero homogeneous elements \( f_{1}, f_{2} \in P_{L,+} \) which generate the unit ideal in \( \mathcal{R}_{L} \), and hence also in \( \mathcal{R}_{R} \). This has the following further consequences.

(a) For any positive integers \( m, n \) such that \( f_{1}^{m} \) and \( f_{2}^{n} \) have the same degree \( d \), if we write \( P_{(d)} = \bigoplus_{i=0}^{\infty} P_{i} \) then the scheme \( \text{Proj}(P/(f_{1}^{m})) \) is isomorphic to \( \text{Spec}(P_{(d)}[f_{2}^{-n}] / (f_{1}^{m} f_{2}^{-n})) \), and hence is affine.
(b) By Lemma 6.3.7, \( \text{Proj}(P) \) is covered by the two open affine subsets \( D_+(f_1), D_+(f_2) \). Since \( \text{Proj}(P) \) is separated, we may use Čech cohomology for this covering to compute sheaf cohomology for quasicoherent sheaves [50, Proposition 1.4.1], so the cohomology of any quasicoherent sheaf on \( \text{Proj}(P) \) vanishes in degree greater than 1.

See Remark 8.3.8 for a related observation.

**Theorem 8.3.3.** For any quasicoherent finite locally free sheaf \( V \) on \( \text{Proj}(P) \), there are natural (in \( V \) and \( R \)) bijections \( H^i(\text{Proj}(P), V) \to H^i_{\varphi^a}(M(V)) \) for all \( i \geq 0 \).

**Proof.** For \( X \) a scheme which is quasicompact and semiseparated (i.e., \( X \) is covered by finitely many open affine subschemes, any two of which have affine intersection), the category of quasicoherent sheaves on \( X \) has enough injectives, and the resulting derived functors agree with sheaf cohomology as defined using the full category of sheaves of abelian groups [99, Proposition B.8]. These results apply to \( X = \text{Proj}(P) \) since the latter is quasicompact (by Lemma 6.3.7) and separated.

By the previous paragraph plus [63, Theorem IV.9.1], we may identify the sheaf cohomology groups \( H^i(\text{Proj}(P), V) \) with the Yoneda extension groups \( \text{Ext}^i(\mathcal{O}, V) \) in the category of quasicoherent sheaves. For \( i = 0, 1 \), this computation involves only quasicoherent finite locally free sheaves, so we may apply Theorem 6.3.12 to obtain the desired isomorphisms. For \( i \geq 2 \), \( H^i(\text{Proj}(P), V) = 0 \) by Remark 8.3.2(b) while \( H^i_{\varphi^a}(M(V)) = 0 \) by definition, so we again obtain an isomorphism. \( \square \)

**Corollary 8.3.4.** Let \( V \) be a quasicoherent finite locally free sheaf on \( \text{Proj}(P) \). Then there exists \( N \in \mathbb{Z} \) such that for all \( n \geq N \), \( H^1(\text{Proj}(P), V(n)) = 0 \).

**Proof.** This follows from Theorem 8.3.3 plus Proposition 6.2.2. \( \square \)

**Proposition 8.3.5.** Let \( M \) be a \( \varphi^a \)-module over \( \tilde{R}_R \). Then the Yoneda extension groups \( \text{Ext}^i(\tilde{R}_R, M) \) are the same whether computed in the category of \( \tilde{R}_R \)-modules equipped with a semilinear \( \varphi \)-action or the sheafification of this category for strong covering families of rational localizations.

**Proof.** To clarify notation, we write \( \text{Ext}^i(\tilde{R}_R, M) \) for the extension groups in the original category and \( \textbf{Ext}^i(\tilde{R}_R, M) \) for the extension groups in the sheafified category; there is then a natural map \( \text{Ex}^i(\tilde{R}_R, M) \to \text{Ext}^i(\tilde{R}_R, M) \). We prove that this is a bijection for \( i \geq 0 \) by induction on \( i \), the case \( i = 0 \) being evident. Given the claim for \( i - 1 \) for some \( i > 0 \) (for all \( M \)), by Proposition 6.2.6, for any sufficiently large integer \( n \), there exists a \( \varphi \)-equivariant surjection \( \tilde{R}_R^\oplus(-n) \to M^\vee \) for some nonnegative integer \( m \). In particular, we may also require \( n \geq 1 \). Since \( M \) is finite projective over \( \tilde{R}_R \), the kernel of this surjection can be written as \( P^\vee \) for some \( \varphi \)-module \( P \) over \( \tilde{R}_R \); we may thus transpose to obtain an exact sequence

\[
0 \to M \to \tilde{R}_R^\oplus(n) \to P \to 0.
\]

For any rational localization \( R \to S \), we have \( H^1_{\varphi}(\tilde{R}_S(n)) = 0 \) by Proposition 6.2.2; consequently, \( \textbf{Ext}^i(\tilde{R}_R, \tilde{R}_R(n)) \) is computed by the Čech complex associated to the functor taking
each rational localization $R 	o S$ to $\tilde{R}_S^{\equiv \mathfrak{p}^n}$. This Čech complex is acyclic because $\tilde{R}_R$ satisfies an analogue of Tate’s theorem (Theorem 5.3.4), so $\text{Ext}^i(\tilde{R}_R, \tilde{R}_R(n)) = 0$. We thus have a commuting diagram

$$
\begin{array}{c}
\text{Ext}^{i-1}(\tilde{R}_R, \tilde{R}_R^\oplus_m(n)) \to \text{Ext}^{i-1}(\tilde{R}_R, P) \to \text{Ext}^i(\tilde{R}_R, M) \to \text{Ext}^i(\tilde{R}_R, \tilde{R}_R^\oplus_m(n)) = 0
\end{array}
$$

with exact rows; by the five lemma, we obtain the desired isomorphism.

We may finally extend Theorem 8.2.3 to étale local systems on $\mathcal{M}(A)$. (Again, this result can be formulated in terms of $\varphi$-bundles using Proposition 6.4.3)

**Theorem 8.3.6.** Let $d$ be a positive integer such that $\mathbb{F}_{p^d} \subseteq A$. Let $X$ be one of $\mathcal{M}(A)$, $\mathcal{M}(R)$, or $\mathcal{M}(\tilde{R}_{R}^{\text{int,1}}/(z))$ for any $z$. Let $E$ be an étale $\mathbb{Q}_{p^d}$-local system on $X$. Let $M$ be the étale $\varphi^d$-module over $\tilde{R}_R$ corresponding to $E$ via Theorem 8.1.3. Then there are natural (in $E$ and $A$) bijections $H^i_{\text{ét}}(X, E) \cong H^i_{\varphi^d}(M)$ for all $i \geq 0$.

**Proof.** If $E$ arises from a local system on $\text{Spec}(R)$ (or equivalently by Theorem 8.1.4 if $M$ is globally étale), then the étale cohomology of $E$ is the same whether computed on $\text{Spec}(R)$ or $\mathcal{M}(R)$, so the claim holds by Theorem 8.2.3. In the general case, using the spectral sequence for Čech cohomology, it follows that the $H^i_{\text{ét}}(X, E)$ compute Yoneda extension groups in the sheafification of the category of left $\tilde{R}_{R^{\{\varphi^d\}}}$-modules with respect to strong covering families of rational localizations on $R$. By Proposition 8.3.5 these are the same as the extension groups without sheafification; this proves the claim.

**Remark 8.3.7.** Let $E$ be an étale $\mathbb{Q}_p$-local system on $\mathcal{M}(R)$. Via Theorem 8.1.9 $E$ corresponds to an étale $\varphi$-module $M$ over $\tilde{R}_R$. Via Theorem 6.3.12 $M$ in turn corresponds to a quasicoherent finite locally free sheaf $V$ on $\text{Proj}(P)$. By combining Theorem 8.3.6 with Theorem 8.3.3 we obtain identifications $H^i_{\text{ét}}(\mathcal{M}(R), E) \cong H^i(\text{Proj}(P), V)$; in other words, we can represent étale cohomology on $\mathcal{M}(R)$ with coefficients in a $\mathbb{Q}_p$-local system in terms of coherent cohomology on $\text{Proj}(P)$.

**Remark 8.3.8.** Suppose $t \in P_{L,1}$ is nonzero. From the proof of Lemma 6.2.4 it follows that there exists $t' \in P_{L,1}$ such that $t, t'$ generate the unit ideal in $\tilde{R}_R$. As in Remark 8.3.2 $\text{Proj}(P)$ is covered by the affine open subschemes $D_+(t) = \text{Spec}(P[t^{-1}]_{0})$ and $D_+(t') = \text{Spec}(P[(t')^{-1}]_{0})$.

Put $R_1 = P[t^{-1}]_0$, let $R_2$ be the $(t/t')$-adic completion of $P[(t')^{-1}]_0$, and put $R_3 = R_2[t'/t]$. In case $R = L$, the morphism $\text{Spec}(R_1 \oplus R_2) \to \text{Proj}(P)$ is faithfully flat, so it can be used to define quasicoherent sheaves and compute their cohomology; this is related to Berger’s construction of $B$-pairs [10]. In general, one might expect the same to hold, but one cannot quite prove it using faithfully flat descent because it is unclear whether $\text{Spec}(R_2) \to \text{Proj}(P)$ is a flat morphism. Nonetheless, we can salvage something; see Proposition 8.3.10
Lemma 8.3.9. Set notation as in Remark 8.3.8 and put \( S = P[(t')^{-1}]_0 \) and \( R'_1 = S[t'/t] = P[(tt')^{-1}]_0 \).

(a) For any \( S \)-module \( M \), the complex
\[
0 \to M \to (M \otimes_S R'_1) \oplus (M \otimes_S R_2) \to M \otimes_S R_3 \to 0,
\]
where the right nontrivial arrow is given by the difference between the two natural restriction maps, is exact.

(b) The morphism \( S \to R'_1 \oplus R_2 \) is an effective descent morphism for the category of finite projective modules over rings.

Proof. Note first that \( (t/t') \) is not a zero divisor in \( S \): otherwise, \( t \) would be a zero divisor in \( \tilde{R}_R \) and hence in \( \tilde{R}_{H(\beta)} \) for some \( \beta \in \mathcal{M}(R) \), but this last ring is a domain. Note next that \( S \) is \( (t/t') \)-adically separated: otherwise, this would fail over some analytic field, but in that case the ring \( S \) is noetherian (see Remark 6.3.18). By virtue of these conditions, we may apply Proposition 1.3.6 to deduce the claims.

Proposition 8.3.10. Set notation as in Remark 8.3.8.

(a) For any quasicoherent sheaf \( V \) on \( \text{Proj}(P) \), the cohomology of the complex
\[
0 \to \Gamma(V, \text{Spec}(R_1)) \oplus \Gamma(V, \text{Spec}(R_2)) \to \Gamma(V, \text{Spec}(R_3)) \to 0
\]
where the arrow is given by the difference between the two natural restriction maps, may be naturally identified with \( H^i(\text{Proj}(P), V) \).

(b) The morphism \( \text{Spec}(R_1 \oplus R_2) \to \text{Proj}(P) \) is an effective descent morphism for the category of quasicoherent finite locally free sheaves over schemes (after reversing all arrows).

Proof. This follows from Remark 8.3.8 (for the covering of \( \text{Proj}(P) \) by \( D_+(t) \) and \( D_+(t') \)) and Lemma 8.3.9.

Remark 8.3.11. Let \( V \) be an arbitrary quasicoherent sheaf on \( \text{Proj}(P) \), not necessarily finite or locally free. Again as in the proof of Theorem 8.3.3, we have a natural identification \( H^i(\text{Proj}(P), V) \cong \text{Ext}^i(O, V) \). By the construction of Definition 6.3.10, we obtain a natural map \( \text{Ext}^i(O, V) \to H^i_{\varphi_a}(M(V)) \).

Proposition 8.3.12. Choose \( t \in P_{L,d} \) nonzero for some positive integer \( d \), and put \( S = (\tilde{R}_R/(t))^{\varphi_a} \). Let \( M \) be a \( \varphi_a \)-module over \( \tilde{R}_R \). Then \( H^i_{\varphi_a}(M/tM) = 0 \), \( H^0_{\varphi_a}(M/tM) \) is a finite projective \( S \)-module, and the natural map \( H^0_{\varphi_a}(M/tM) \otimes_S \tilde{R}_R/(t) \to M/tM \) is a bijection.

Proof. Let \( V, W \) be the sheaves on \( \text{Proj}(P) \) corresponding to \( M, tM \) via Theorem 6.3.12. By Theorem 8.3.3, we have natural isomorphisms \( H^i(\text{Proj}(P), V) \cong H^i_{\varphi_a}(M) \), \( H^i(\text{Proj}(P), W) \cong H^i_{\varphi_a}(tM) \). Using Remark 8.3.11, we then obtain isomorphisms \( H^i(\text{Proj}(P), V/W) \cong H^i_{\varphi_a}(M/tM) \). Since \( V/W \) is supported on the closed subscheme \( \text{Proj}(P/t) \) which is affine (Remark 8.3.2), this yields the desired results. (We also obtain an identification of \( S \) with the coordinate ring of \( \text{Proj}(P/t) \).)
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