Monstrous Moonshine
and the Classification of CFT

(16 lectures given in Istanbul, August 1998)

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Abstract

In these notes we give an introduction both to Monstrous Moonshine and to the classification of rational conformal field theories, using this as an excuse to explore several related structures and go on a little tour of modern math. We will discuss Lie algebras, modular functions, the finite simple group classification, vertex operator algebras, Fermat’s Last Theorem, category theory, (generalised) Kac-Moody algebras, denominator identities, the A-D-E meta-pattern, representations of affine algebras, Galois theory, etc. This work is informal and pedagogical, and aimed mostly at grad students in math or math phys, but I hope that many interested nonexperts will find something of value here — like any good Walt Disney movie I try not to completely ignore the ‘grown-ups’. My emphasis is on ideas and motivations, so these notes are intended to complement other papers and books where this material is presented with more technical detail. The level of difficulty varies significantly from topic to topic. The two parts — in fact any of the sections — can be read independently.
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Glossary

$24$ §1.6
$A_{\ell}$ $\text{sl}_{\ell+1}(\mathbb{C})$
affine algebra §1.4
algebra §1.2
category theory §1.1
central extension §§1.2, 1.4
CFT conformal field theory
character §§1.3, 1.4
chiral algebra §1.1
Fermat’s Last Theorem §2.3
field §1.2
finite simple group §2.2
genus §§1.1, 2.3
group §§1.3, 2.2
Hauptmodul §2.3
highest-weight §§1.3, 1.4
Kac-Moody algebra §2.7
lattice §1.6
Leech lattice $A_{24}$ §2.4
Lie algebra §1.2
Lie group §1.2
$M$ the Monster finite simple group
manifold §1.2
modular function §2.3
modular group $\text{PSL}_2(\mathbb{Z})$ or $\text{SL}_2(\mathbb{Z})$
$P_k^+$ §1.4
partition function §§1.1, 1.2, 1.5
Perron-Frobenius §1.7
physical invariant RCFT partition function, §1.5
QFT quantum field theory
RCFT rational conformal field theory
representation §1.3
Riemann surface §§1.1, 2.3
$S$, $T$ Kac-Peterson matrices, (1.4.2)
subfactor theory §1.9
topological §§1.1, 2.3
torus §2.3
$V^a$ the Moonshine module, §§2.4, 2.6
Verlinde’s formula (1.4.4)
Virasoro algebra (1.2.7)
VOA vertex operator algebra
Weyl group §1.3
WZW model §1.1
Part 1. The classification of conformal field theories

1.1. Informal motivation

In this section we will sketch a very informal and ‘hand-wavy’ motivation to what we shall call the classification problem for rational conformal field theory (RCFT). Much of this material is more carefully treated in e.g. [18].

A CFT is a quantum field theory (QFT), usually with a two-dimensional space-time, whose symmetries include the conformal transformations. There are different approaches to CFT — for one of these see [26,27]. Another formulation which has been deeply influential is due to Graeme Segal [52]. It is motivated by string theory and is phrased in an important mathematical language called category theory.

A category consists of two types of things. One are called objects, and the other are called arrows (or morphisms). An arrow, written \( f : A \to B \), has an initial and a final object (\( A \) and \( B \) respectively). Arrows \( f, g \) can be composed to yield a new arrow \( f \circ g \), if the final object of \( g \) equals the initial object of \( f \). Maps between categories are called functors if they take the objects (resp., arrows) of one to the objects (resp., arrows) of the other, and preserve composition.

The only difficulty people can have in understanding categories is in realising that there is no real content to them. It’s just a language, highly abstract like the more familiar set theory, but in many contexts (a great example is the theory of knot invariants [58]) one which is both natural and suggestive. It tries to deflect some of our instinctive infatuation with objects (nouns), to the mathematically more fruitful one with structure-preserving maps between objects (verbs). A gentle introduction to the mathematics and philosophy of categories is [43]; we’ll give a taste of this shortly.

The standard example of a category is called \( \text{Set} \), where the ‘objects’ are sets, and the arrows from \( A \) to \( B \) are functions \( A \to B \). A related example that Segal uses is \( \text{Vect} \), where the objects are complex vector spaces and the arrows are linear maps. A rather trivial example of a functor \( \mathcal{F} : \text{Vect} \to \text{Set} \) sends a vector space \( V \) to its underlying set, also called \( V \) — i.e. \( \mathcal{F} \) simply ‘forgets’ the vector space structure on \( V \) and ignores the fact that the arrows \( f \) in \( \text{Vect} \) are linear. The other category Segal uses he calls \( C \); its objects are disjoint unions of (parametrised) circles \( S^1 \), and the arrows are (conformal equivalence classes of) cobordisms, i.e. (Riemann) surfaces whose boundaries are those circles. Composition of arrows in \( C \) is accomplished by gluing the surfaces along the appropriate boundary circles.

Consider the usual definition of a one-to-one function: \( f(x) = f(y) \) only when \( x = y \). Category theory replaces this with the following. The arrow \( f : A \to B \) is called ‘monic’ if for any arrow \( g : C \to B \), there exists a unique arrow \( h : C \to A \) such that \( f \circ h = g \). So it’s a sort of factorisation property. You can easily verify that in \( \text{Set} \) the notions of ‘one-to-one’ and ‘monic’ coincide. What does this redefinition gain us? It certainly doesn’t seem any simpler. But it does change the focus from the argument of \( f \), to the global functional behaviour of \( f \), and a change of perspective can never be bad. And it allows us to transport the idea of one-to-one-ness to arbitrary categories. For instance, in the Riemann surface category \( C \), the ‘one-to-one functions’ are the genus-0 cobordisms.
Or consider the notion of product. In category theory, we say that the triple \((P, a, b)\) is a product of objects \(A, B\) if \(a : P \to A\) and \(b : P \to B\) are arrows, and if for any \(f : C \to A\), \(g : C \to B\), there is a unique arrow \(h : C \to P\) such that \(f = a \circ h\) and \(g = b \circ h\). This notion unifies several constructions (each of which is the ‘product’ in an appropriately chosen category): Cartesian product of sets; intersection of sets; multiplication of numbers; the logical operator ‘and’; direct product; infimum in a partially ordered set; etc. Sum can be defined similarly, unifying the constructions of disjoint union, ‘or’, addition, tensor product, direct sum, supremum, etc.

This generality of course comes with a price: it can wash away all of the endearing special features of a favourite theory or structure. There certainly are contexts where e.g. all human beings should be thought of as equal, but there are other contexts where the given human is none other than your mother and must be treated as such. It turns out that category theory provides a beautiful framework for understanding topological invariants such as the Jones-Reshetikhin-Turaev-Witten knot invariants (see e.g. [58]). And it seems to be a natural language for formulating CFT axiomatically, as we’ll now see.

According to Segal, a CFT is a functor \(T\) from \(C\) to \(\text{Vect}\), which obeys various properties. The picture comes from string theory: the fundamental object is a ‘vibrating’ loop; a state is given by a collection of these loops; each classical Feynman path from the initial to the final states is a world-sheet, i.e. a surface \(\Sigma\) whose boundary is all the loops in the initial and final states. QFT assigns a complex number \(I\) (the action) to each of these world-sheets, and the quantum amplitude, written \(\langle\text{final}|\text{initial}\rangle\), will then be the integral over all worldsheets of \(e^{iI/\hbar}\). (The quantum amplitude is how the theory makes contact with experiment, as it tells us the probability of the given process \(\langle\text{initial}\rangle \mapsto |\text{final}\rangle\) happening.) This is what Segal is trying to capture formally. The vector spaces in \(\text{Vect}\) come in in order to handle uniformly and simultaneously the various ‘vibrations’ of the strings. In particular there is one basic vector space \(H\) (a Hilbert space of quantum states), and the functor will take \(n\) copies of \(S^1\) to \(H^n = H \otimes \cdots \otimes H\).

The simplest interesting example here is the ‘tree-level creation of a string from the vacuum’. In this case the world-sheet looks like a bowl, i.e. \(\text{topologically}\) is a disk \(D\) (if we imagine the bowl to be made of rubber, we could grab its rim and stretch it down flat onto the table, so we say a bowl and a disk are topologically equivalent — see also §2.3). Segal’s functor gives us a linear map \(T(D) : \mathbb{C} \to H\) \((H^0\) is just \(\mathbb{C}\), which we can think of equivalently as the assignment of the vector \(T(D)(1)\) in \(H\) to \(D\). In the case of the standard unit disk (i.e. where the parametrisation of the boundary \(S^1\) is simply \(\theta \mapsto e^{2\pi i \theta}\)), this vector is called the \textit{vacuum state} \(\Omega = |0\rangle\).

For another example, consider the ‘vacuum-to-vacuum expectation value’. The initial and final states (objects) here are both the empty set, so the world-sheets (arrows) are closed Riemann surfaces. As usual in QFT, we can organise these by how many internal ‘loops’ are involved (this number is called the \textit{genus} of the surface): topologically, 0-loop (i.e. ‘tree-level’) world-sheets are spheres, 1-loop world-sheets are tori, etc. These closed Riemann surfaces are discussed in more detail in §2.3. The 0-loop contribution isn’t very interesting (there is only one conformal equivalence class of spheres), so let us look at the 1-loop contribution. It will be of the form \(\int Z(\text{torus}) d[\text{torus}]\), where \(Z\) is a complex-valued function called the partition function, and \([\text{torus}]\) is a conformal equivalence class.
of tori. In the Segal formalism we recover this in the following way: the functor takes \([\text{torus}]\) to a linear function from \(H^0 = \mathbb{C}\) to \(H^0 = \mathbb{C}\). Any such linear function is simply a \(1 \times 1\) matrix, i.e. a complex number, which we call \(Z([\text{torus}])\).

Now there is a nice parametrisation of conformal equivalence classes of tori, as we will see more explicitly in §2.3. Namely, a representative for each class can be chosen to be of the form \(\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)\) where \(\tau\) is in the upper half plane \(\mathcal{H}\). Thus we can write \(Z\) as a function of a complex variable \(\tau\). However, different \(\tau\) correspond to the same equivalence class of tori: the redundancy is exactly captured by the modular group \(\text{PSL}_2(\mathbb{Z})\). Namely, \(\tau\) and \(a\tau + b\) are equivalent, whenever \(a, b, c, d \in \mathbb{Z}\) and \(ad - bc = 1\). Thus \(Z(\tau) = Z(a\tau + b/c\tau + d)\).

In other words, the partition function is modular invariant!\(^1\)

There are two sectors in CFT, a holomorphic one and an antiholomorphic one, corresponding to the two directions (‘left-’ and ‘right-moving’) of motion on a string, or the two components of the group \(\text{Diff}(S^1)\) of diffeomorphisms of the circle. This means that many of the quantities (e.g. the partition function) factorise into parts depending holomorphically and anti-holomorphically on the modular parameters (e.g. \(\tau\) in genus 1). In a rational CFT there are finitely many ‘primary fields’ \(a \in \Phi\) — the precise meaning of this is not important here, but it says that the space of states for the theory decomposes into a finite sum\(^2\) \(H = \bigoplus_{a, b \in \Phi} M_{ab} H_a \otimes \overline{H}_b\), where \(M_{ab}\) are nonnegative integers which count the multiplicity of \(H_a \otimes \overline{H}_b\) in \(H\). The linear maps \(T(\Sigma) : H^m \to H^n\) in an RCFT will factorise similarly; this ‘chiral factorisation’ is captured by what Segal calls the ‘modular functor’ \(^5\). The partition function becomes

\[
Z(\tau) = \sum_{a, b \in \Phi} M_{ab} \chi_a(\tau) \chi_b(\tau)^* \tag{1.1.1}
\]

for certain holomorphic functions \(\chi_a\). One of the primary fields (we’ll denote it ‘0’) corresponds to the vacuum \(\Omega\), and uniqueness of the vacuum means that \(M_{00} = 1\).

\(H_0\) is called a chiral algebra; in the language of §2.6, \(H_0\) will be a vertex operator algebra (VOA). \(\Phi\) parametrises the irreducible \(H_0\)-modules and the \(\chi\)’s are their characters; in an RCFT we require this number to be finite. For example, for the Moonshine VOA \(V^{\natural}\) discussed in Part 2, \(\Phi\) consists of only one element.

The higher-genus behaviour of an RCFT is determined from the lower-genus behaviour, by composition of ‘arrows’ (i.e. the gluing together of surfaces) in \(\mathcal{C}\). See Figure 3 of [30] to find how a genus-2 surface is built up from genus-0 ones. In fact, it’s generally believed that an RCFT will be uniquely determined by: (i) the choice of chiral algebra; (ii) the partition function (which tells you the spectrum of the theory, i.e. how the two sectors link up); and (iii) the structure constants \(C_{ab}^c\), which in the Segal formalism correspond to the surfaces called ‘pairs-of-pants’, equivalently disks with two interior disks removed. Our approach will be to start with a chiral algebra, and find all possible partition functions. We will thus ignore the important question of existence and uniqueness of the structure

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\(^1\) In higher-dimensional string theories, a similar argument shows more generally that automorphic forms will appear naturally.

\(^2\) It seems though that ‘rational’ logarithmic CFT is trying to teach us the lesson that this familiar requirement can and should be weakened. See Gaberdiel-Kausch (1999).
constants, though at least for our chiral algebras, it seems to be generally believed that the structure constants will be unique.

Perhaps all chiral algebras come from standard constructions (e.g. orbifolds and the Goddard-Kent-Olive (GKO) coset construction — see e.g. [18]) involving lattices and affine Kac-Moody algebras. For instance a $\mathbb{Z}_2$-orbifold of the VOA of the Leech lattice gives us the Moonshine module $V^\natural$, and the so-called minimal models come from GKO cosets involving $A_1^{(1)}$. This is in line with the spirit of Tannaka-Krein duality (and its generalisations by Deligne and Doplicher-Roberts), which roughly says that if a bunch of things act like they’re the set of representations of a Lie group, then they are the set of representations of a Lie group.

In any case, one of the simplest, best understood, and important classes (called Wess-Zumino-Witten (WZW) models — see for instance [30,59] in this volume) of RCFTs correspond to affine Kac-Moody algebras at a positive integer level $k$. We will have much more to say later about these algebras, but for now let us remark that $\Phi$ here will be the (finite) set $P^+_k$ of integrable level $k$ highest weights $\lambda$. Their chiral algebras were constructed by Frenkel and Zhu. The following sections concern the attempt to classify the partition functions corresponding to Kac-Moody algebras — see especially §1.5. I will use this theme as an excuse to describe many other things, e.g. the A-D-E meta-pattern, Lie theory, Galois, fusion rings, ... I dedicate these notes to the profound friendship developing in recent years between mathematics and physics. As Victor Kac said in his 1996 Wigner medal acceptance speech, “Some of the best ideas come to my field from the physicists. And on top of this they award me a medal. One couldn’t hope for a better deal.”

1.2. Lie algebras

Lie algebras (and their nonlinear partners Lie groups) appear in numerous places throughout math and mathematical physics. A nice introduction is [9]; Lie theory is presented with more of a physics flavour in [24], as well as [59].

An algebra is a vector space with a way to multiply vectors which is compatible with the vector space structure (i.e. the vector-valued product is bilinear: $(a\vec{u}+a'\vec{u}') (b\vec{v}+b'\vec{v}') = ab \vec{u}\vec{v} + ab' \vec{u}\vec{v}' + a'b \vec{u}'\vec{v} + a'b' \vec{u}'\vec{v}')$. For example, the complex numbers $\mathbb{C}$ can be thought of as a 2-dimensional algebra over $\mathbb{R}$ (a basis is 1 and $i = \sqrt{-1}$; the scalars here are real numbers and the vectors are complex numbers). The quaternions are 4-dimensional over $\mathbb{R}$ and the octonions are 8-dimensional over $\mathbb{R}$. Incidentally, these are the only finite-dimensional algebras over $\mathbb{R}$ which obey the cancellation law: $\vec{u} \neq 0$ and $\vec{u}\vec{v} = 0$ implies $\vec{v} = 0$ (the reader should try to convince himself why the familiar vector product on $\mathbb{R}^3$ fails the cancellation law). This important little fact makes several unexpected appearances in math. For instance, it is trivially possible to ‘comb the hair’ on the circle $S^1$ without ‘cheating’ (i.e. needing a hair-part or exploiting a bald spot): just comb the hair clockwise for example. However it is not possible to comb the hair on the sphere $S^2$ (e.g. your own head) without cheating. The only other $k$-spheres $S^k$ which can be combed (i.e. for which there exist $k$ linearly independent continuous vector fields) are $k = 3$ and 7. This is intimately connected with the existence of $\mathbb{C}$, the quaternions, and octonions (namely,
$S^1, S^3, S^7$ can be thought of as the length 1 complex numbers, quaternions, and octonions, resp.).

In a Lie algebra $\mathfrak{g}$, the product is usually called a ‘bracket’ and is written $[xy]$. It is required to be ‘anti-commutative’ and ‘anti-associative’:

\[
[x] + [y] = 0 \tag{1.2.1a}
\]
\[
[x[yz]] + [y[zx]] + [z[xy]] = 0 \tag{1.2.1b}
\]

(like most other equalities in math, (1.2.1b) is usually called the Jacobi identity). Usually we will consider Lie algebras over $\mathbb{C}$, but sometimes over $\mathbb{R}$. Note that (1.2.1a) says $[xx] = 0$.

One important consequence of bilinearity is that it is enough to know the values of all the brackets $[x_ix_j]$ for $i < j$, for any basis $\{x_1, x_2, \ldots\}$ of $\mathfrak{g}$. (The reader should convince himself of this before proceeding.)

The simplest example of a Lie algebra is $\mathfrak{g} = \mathbb{C}$ (or $\mathfrak{g} = \mathbb{R}$), with the bracket $[xy]$ identically 0. In fact, this is the only 1-dimensional Lie algebra. It is a straightforward exercise for the reader to find all 2- and 3-dimensional Lie algebras (over $\mathbb{C}$) up to isomorphism (i.e. change of basis): there are precisely 2 and 6 of them, respectively (though one of the 6 depends on a complex parameter). Over $\mathbb{R}$, there are 2 and 9 (with 2 depending on real parameters). This exercise cannot be continued much further — e.g. not all 7-dimensional Lie algebras (over $\mathbb{C}$) are known. Nor is it obvious that this would be an interesting or valuable exercise. We should suspect that our definition of Lie algebra is probably a little too general for anything obeying it to be automatically an interesting structure. More often than not, a classification turns out to be a stale and useless list.

Two of the 3-dimensional Lie algebras are important in what follows. One of them is well-known to the reader: consider the vector-product (also called cross-product) in $\mathbb{C}^3$. Taking the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$, the bracket can be defined by the relations

\[
[e_1e_2] = e_3, \quad [e_1e_3] = -e_2, \quad [e_2e_3] = e_1. \tag{1.2.2a}
\]

This Lie algebra, denoted $A_1$ or $\text{sl}_2(\mathbb{C})$, can be called the ‘mother of all (semi-simple) Lie algebras’. A more familiar realisation of $A_1$ uses a basis $\{e, f, h\}$ with relations

\[
[ef] = h, \quad [he] = 2e, \quad [hf] = -2f. \tag{1.2.2b}
\]

The reader can find the change-of-basis (valid over $\mathbb{C}$ but not $\mathbb{R}$) showing that (1.2.2) define isomorphic complex (but not real) Lie algebras.

Another important 3-dimensional Lie algebra is called the Heisenberg algebra$^3$ and is the algebra of the canonical commutation relations in quantum mechanics: choosing a basis $x, p, h$, it is defined by

\[
[xp] = h, \quad [xp] = [ph] = 0. \tag{1.2.3}
\]

$^3$ Actually, ‘Heisenberg algebra’ refers to a family of Lie algebras, with (1.2.3) being the one of lowest dimension.
From our definition, it is far from clear that Lie algebras, as a class, should be natural and worth studying. After all, there are infinitely many possible axiomatic systems: why should the one defining a Lie algebra be anything special a priori? Perhaps this could have been anticipated by the following line of reasoning.

**Axiom.** Groups are important and interesting.

**Axiom.** Manifolds are important and interesting.

Manifolds are structures where calculus is possible; locally, a manifold looks like a piece of $\mathbb{R}^n$ (or $\mathbb{C}^n$), but these pieces can be bent and stitched together to create more interesting shapes. For instance a circle is a 1-dimensional manifold, while Einstein claimed space-time is a curved 4-dimensional one.

**Definition.** A Lie group is a manifold with a compatible group structure.

This means that ‘multiplication’ and ‘inverse’ are differentiable maps. $\mathbb{R}$ is a Lie group, under addition: obviously, $\mu : \mathbb{R}^2 \to \mathbb{R}$ and $\iota : \mathbb{R} \to \mathbb{R}$ defined by $\mu(a, b) = a + b$ and $\iota(a) = -a$ are both differentiable. (Why isn’t $\mathbb{R}$ a Lie group under multiplication?) A circle is also a Lie group: parametrise the points with the angle $\theta$ defined mod $2\pi$ (or mod 360 if you prefer); the ‘product’ of the point at angle $\theta_1$ with the point at angle $\theta_2$ will be the point at angle $\theta_1 + \theta_2$. Surprisingly, the only other $k$-sphere which is a Lie group is $S^3$ (the product can be defined using quaternions of unit length\(^4\), or using the matrix group $SU_2(\mathbb{C})$). Many but not all Lie groups can be expressed as matrix groups. Two other examples are $GL_n$ (invertible $n \times n$ matrices) and $SL_n$ (ones with determinant 1).

A consequence of the above axioms is then surely:

**Corollary.** Lie groups should be important and interesting.

Lie group structure theory can be thought of as a major generalisation of linear algebra. The basic constructions familiar to undergraduates have important analogues valid in many Lie groups. For instance, years ago we were taught to solve linear equations and invert matrices by using elementary row operations to reduce a matrix to row-echelon form. What this says is that any matrix $A \in GL_n(\mathbb{C})$ can be factorised $A = BPN$, where $N$ is uppertriangular with 1’s on the diagonal, $P$ is a permutation matrix, and $B$ is an uppertriangular matrix. This is essentially what is called the Bruhat decomposition of the Lie group $GL_n(\mathbb{C})$. More generally (where it applies to any ‘reductive’ Lie group $G$), $P$ will be an element of the so-called ‘Weyl group’ of $G$ (of which we’ll have much more to say later), and $B$ will be in a ‘Borel subgroup’.

Lie groups appear throughout physics. E.g. the orthogonal group $SO_3(\mathbb{R})$ is the configuration space of a rigid body centred at the origin, while $SU_2(\mathbb{C})$ is the set of states of an electron at rest. The gauge group of the Standard Model of particle physics is $SU_3(\mathbb{C}) \times SU_2(\mathbb{C}) \times U_1(\mathbb{C})$, while the Lorentz group of special relativity is $SO_{3,1}(\mathbb{R})$.

There is an important relation between Lie groups and Lie algebras.

**Fact.** The tangent space of a Lie group is a Lie algebra. Any (finite-dimensional real or complex) Lie algebra is the tangent space to some Lie group.

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\(^4\) Similarly, the 7-sphere inherits from the octonions a nonassociative (hence nongroup) product, compatible with its manifold structure.
More precisely, the tangent space at 1 (i.e. the set of all infinitesimal generators of the Lie group) can be given a natural Lie algebra structure. A Lie algebra, being a linearised Lie group, is much simpler and easier to handle. The Lie algebra preserves the local properties of the Lie group, though it loses global topological properties (like boundedness). A Lie group has a single Lie algebra, but a Lie algebra can correspond to many different Lie groups. The Lie algebra corresponding to both $\mathbb{R}$ and $S^1$ is $\mathfrak{g} = \mathbb{R}$ with trivial bracket. The Lie algebra corresponding to both $S^3 = SU_2(\mathbb{C})$ and $SO_3(\mathbb{R})$ is the cross-product algebra on $\mathbb{R}^3$ (usually called $so_3(\mathbb{R})$). Given the above fact, a safe guess would be:

**Conjecture.** Lie algebras are important and interesting.

From this line of reasoning, it should be expected that historically Lie groups arose first. Indeed that is the case: the Norwegian Sophus Lie introduced them in 1873 to try to develop a Galois theory for ordinary differential equations. As the reader may be aware, Galois theory is used for instance to show that not all 5th degree (or higher) polynomials can be explicitly ‘solved’ using radicals — we will meet Galois theory in §1.8. Lie wanted to study the explicit solvability (integrability) of differential equations, and this led him to develop what we now call Lie theory. The importance of Lie groups however have grown well beyond this initial motivation.

An important class of Lie algebras are the so-called finite-dimensional simple ones. Their definition and motivation will be studied in §2.7 below, but in a certain sense they serve as building blocks for all other finite-dimensional Lie algebras.

The classification of simple finite-dimensional Lie algebras over $\mathbb{C}$ is quite important and was accomplished at the turn of the century by Killing and Cartan. There are 4 infinite families $A_\ell (\ell \geq 1)$, $B_\ell (\ell \geq 3)$, $C_\ell (\ell \geq 2)$, and $D_\ell (\ell \geq 4)$, and 5 exceptionals $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$. $A_\ell$ can be thought of as $sl_{\ell+1}(\mathbb{C})$, the $(\ell + 1) \times (\ell + 1)$ matrices with trace 0. The orthogonal algebras $B_\ell$ and $D_\ell$ can be identified with $so_{2\ell+1}(\mathbb{C})$ and $so_{2\ell}(\mathbb{C})$, resp., where $so_n(\mathbb{C})$ is all $n \times n$ anti-symmetric matrices $A^t = -A$. The symplectic algebra $C_\ell$ is $sp_{2\ell}(\mathbb{C})$, i.e. all $2\ell \times 2\ell$ matrices $A$ obeying $A\Omega = -\Omega A^t$, where $\Omega = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$ and $I_\ell$ is the identity matrix. The exceptionals can be constructed e.g. using the octonions. In all these cases the bracket is given by the commutator

$$[AB] = [A, B] := AB - BA$$

(it is a good exercise for the reader to confirm that the commutator satisfies (1.2.1), and that e.g. $sl_n(\mathbb{C})$ is indeed closed under it). To see that (1.2.2b) truly is $sl_2(\mathbb{C})$, put

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ (1.2.5)

Incidentally the names $A$, $B$, $C$, $D$ have no significance: since the 4 series start at $\ell = 1, 2, 3, 4$, it seemed natural to call these $A, B, C, D$, resp. Unfortunately a bit of bad luck happened: $B_2$ and $C_2$ are isomorphic and so at random that algebra was placed in the orthogonal series; however affine Dynkin diagrams make it clear that it really is a

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5 Strictly speaking these are representations (see next section).
symplectic algebra which accidentally looks orthogonal; hence in hindsight the names of the B- and C-series really should have been switched.

This classification changes if the field — the choice of scalars=numbers — is changed. By a field, we mean we can add, subtract, multiply and divide, such that all the usual properties like commutativity and distributivity are obeyed. Fields will make a few different appearances in these notes. \( \mathbb{C}, \mathbb{R}, \) and \( \mathbb{Q} \) are fields, while \( \mathbb{Z} \) is not (you can’t always divide an integer by e.g. 3, and remain in \( \mathbb{Z} \)). The integers mod \( n \), which we will write \( \mathbb{Z}_n \), are a field iff \( n \) is prime (the reader can verify that in e.g. \( \mathbb{Z}_4 \), it is not possible to divide by the field element \( \{2\} \in \mathbb{Z}_4 \) even though \( \{2\} \neq \{0\} \) there). \( \mathbb{C} \) and \( \mathbb{R} \) are examples of fields of characteristic 0 — this means that 0 is the only integer \( k \) with the property that \( kx = 0 \) for all \( x \) in the field. \( \mathbb{Z}_p \) is the simplest example of a field with nonzero characteristic: in \( \mathbb{Z}_p \), multiplying by the integer \( p \) has the same effect as multiplying by 0, and so we say \( \mathbb{Z}_p \) has characteristic \( p \). Strange fields have important applications in e.g. coding theory and, ironically, in number theory itself — see e.g. §1.8.

As always, \( \mathbb{C} \) is better behaved than e.g. \( \mathbb{R} \) because every polynomial can be factorised over \( \mathbb{C} \) (we say \( \mathbb{C} \) is \textit{algebraically closed}) — this implies for example that every matrix has an eigenvector over \( \mathbb{C} \) but not necessarily over \( \mathbb{R} \). Over \( \mathbb{R} \), the difference in the simple Lie algebra classification is that each symbol \( X_\ell \in \{A_\ell, \ldots, G_2\} \) corresponds to a number of inequivalent algebras (over \( \mathbb{C} \), each algebra has its own symbol). For example, ‘\( A_1 \)’ corresponds to 3 different real simple Lie algebras, namely the matrix algebras \( \text{sl}_2(\mathbb{R}) \), \( \text{sl}_2(\mathbb{C}) \) (interpreted as a \textit{real} vector space), and \( \text{so}_3(\mathbb{R}) \). The simple Lie algebra classification has recently been done in any characteristic \( p > 7 \). It is surprising but very common that the smaller primes behave very poorly, and the classification for characteristic 2 is probably completely hopeless.

Associated with each simple algebra \( X_\ell \) is a Weyl group, and a (Coxeter-)Dynkin diagram. The Weyl group is a finite reflection group, e.g. for \( A_\ell \) it is the symmetric group \( S_{\ell+1} \). See Figure 7 of [59] for the Weyl group of \( A_2 \). The Dynkin diagram of \( X_\ell \) (see e.g. [24,36,38] or Figure 6 in [59]) is a graph with \( \ell \) nodes, and with possibly some double and triple edges. It says how to construct \( X_\ell \) abstractly using generators and relations — see §2.7. We will keep meeting both throughout these notes.

Another source of Lie algebras are the vector fields \( \text{Vect}(M) \) on a manifold \( M \). A vector field \( v \) is a choice (in a smooth way) of a tangent vector \( v(p) \in T_p M \) at each point of \( M \). It can be thought of as a (1st order) differential operator, acting on functions \( f : M \to \mathbb{R} \) (or \( f : M \to \mathbb{C} \)); at each point \( p \in M \) take the directional derivative of \( f \) in the direction \( v(p) \). For example the vector fields on the circle, \( \text{Vect}(S^1) \), can be thought of as anything of the form \( g(\theta) \frac{d}{d\theta} \) where \( g(\theta) \) can be any function with period 1. We can compose vector fields \( u \circ v \), but this will result in a 2nd order differential operator: e.g.

\[
(f(\theta) \frac{d}{d\theta}) \circ (g(\theta) \frac{d}{d\theta}) = f(\theta) g(\theta) \frac{d^2}{d\theta^2} + f(\theta) g'(\theta) \frac{d}{d\theta} .
\]

Instead, the natural ‘product’ of vector fields is given by their commutator \( [u, v] = u \circ v - v \circ u \), as it always results in a vector field: e.g.

\[
[f(\theta) \frac{d}{d\theta}, g(\theta) \frac{d}{d\theta}] = (f(\theta) g'(\theta) - f'(\theta) g(\theta)) \frac{d}{d\theta} .
\]
in \(	ext{Vect}(S^1)\). \(	ext{Vect}(M)\) with this bracket is an infinite-dimensional Lie algebra. In the case where \(M\) is a Lie group \(G\), the Lie algebra of \(G\) can be interpreted as a certain finite-dimensional subalgebra of \(	ext{Vect}(G)\) given by the ‘left-invariant vector fields’.

Simple algebras need not be finite-dimensional. An example of an infinite-dimensional one is the \textit{Witt algebra} \(W\), which can be defined (over \(\mathbb{C}\)) by the basis\(^6\) \(L_n, n \in \mathbb{Z}\), and the relations

\[
[L_m L_n] = (m - n) L_{m+n} .
\]  

Using the realisation \(L_n = -i e^{-in\theta} \frac{d}{d\theta}\), the Witt algebra can also be interpreted as the polynomial subalgebra of the complexification \(\mathbb{C} \otimes \text{Vect}(S^1)\) — i.e. change the scalar field of \(	ext{Vect}(S^1)\) from \(\mathbb{R}\) to \(\mathbb{C}\). Incidentally, infinite-dimensional Lie algebras need not have a corresponding Lie group: e.g. the real algebra \(	ext{Vect}(S^1)\) is the Lie algebra of the Lie group \(\text{Diff}^+(S^1)\) of orientation-preserving diffeomorphisms \(S^1 \to S^1\), but \(\mathbb{C} \otimes \text{Vect}(S^1)\) has no Lie group. \(\text{Diff}^+(S^1)\) plays a large role in CFT, by acting on the objects of Segal’s category \(\mathcal{C}\).

The Witt algebra appears naturally in CFT: e.g. using the realisation \(L_n = -z^{n+1} \frac{d}{dz}\) it is the polynomial subalgebra of the Lie algebra \(	ext{Vect}(\mathbb{C}/\{0\})\). Very carelessly, \(\text{Vect}(\mathbb{C}/\{0\})\) is often thought of as the infinitesimal conformal transformations on a suitable neighbourhood of 0 (yet clearly \(L_{-2}, L_{-3}, \ldots\) are singular at 0!). Indeed the CFT literature is very sloppy when discussing the conformal group in 2-dimensions. The unfortunate fact is that, contrary to claims, \textit{there is no infinite-dimensional conformal group} for \(\mathbb{C} \cong \mathbb{R}^2\). The best we can do is the 3-dimensional group \(\text{PSL}_2(\mathbb{C})\) of Möbius transformations \(z \mapsto \frac{az + b}{cz + d}\), which are orientation-preserving conformal transformations for the Riemann sphere \(\mathbb{C} \cup \{\infty\}\). There seem to be 2 ways out of this rather embarrassing predicament. One is to argue that we are really interested in ‘infinitesimal conformal invariance’ in some meromorphic sense, so the full Witt algebra can appear. The other way is to argue that it is the conformal group of ‘Minkowski space’ \(\mathbb{R}^{1,1}\) (or better, its compactification \(S^1 \times S^1\)) rather than \(\mathbb{R}^2 \cong \mathbb{C}\) (or its compactification \(S^2\)) which is relevant for CFT. That conformal group \textit{is} infinite-dimensional; for \(S^1 \times S^1\) it consists of 2 copies of \(\text{Diff}^+(S^1) \times \text{Diff}^+(S^1)\). For a more careful treatment of this point, see [51].

For reasons we will discuss in §1.4, we are more interested in the \textit{Virasoro algebra} \(\mathcal{V}\) rather than the Witt algebra \(W\). This is a ‘1-dimensional central extension’ of \(W\); as a vector space \(\mathcal{V} = \mathcal{W} \oplus \mathbb{C} C\) with relations given by

\[
[L_m L_n] = (m - n) L_{m+n} + \delta_{n,-m} \frac{m (m^2 - 1)}{12} C \quad (1.2.7a)
\]

\[
[L_mC] = 0 . \quad (1.2.7b)
\]

‘1-dimensional central extension’ means \(\mathcal{V}\) has one extra basis vector \(C\), which lies in the \textit{centre} of \(\mathcal{V}\) (i.e. \([xC] = 0\) for all \(x \in \mathcal{V}\)), and sending \(C \to 0\) recovers \(W\) (i.e. takes (1.2.7a) to (1.2.6)). A common mistake in the physics literature is to regard \(C\) as a number: it is in fact a vector, though in many (but not all) representations it is mapped to a scalar multiple of the identity.

\(^6\) In infinite dimensions, to avoid convergence complications, only finite linear combinations of basis vectors are generally permitted. Infinite linear combinations would involve taking some ‘completion’.
The reason for the strange-looking (1.2.7a) is that we have little choice: $V$ is the unique nontrivial 1-dimensional central extension of $W$. The factor $\frac{1}{12}$ there is conventional but standard, and has to do with ‘zeta-function regularisation’ in string theory — i.e. the divergent sum $\sum_{n=1}^{\infty} n$ is ‘reinterpreted’ as $\zeta(-1) = -\frac{1}{12}$, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Incidentally $\zeta(s)$ can be written as the product $\prod_p (1 - p^{-s})^{-1}$ over all primes $p = 2, 3, 5, \ldots$ (try to see why); hence $\zeta(s)$ has a lot to do with primes, in particular their distribution. In fact the most famous unsolved problem in math today is the Riemann conjecture, which says that $\zeta(s) \neq 0$ whenever $\text{Re}(s) \neq \frac{1}{2}$. One researcher recently described this conjecture as saying that the primes have music in them.

In CFT, $L_0$ is the energy operator. For example the partition function is given by $Z(\tau) = \text{Tr}_H(q^{L_0 - c/24} q^* L_0 - c/24)$ and the (normalised) character $\chi_a$ equals $\text{Tr}_{H_a} (q^{L_0 - c/24})$ for $q = e^{2\pi i \tau}$. $cI$ is the scalar multiple of the identity to which $C$ gets sent; it has a physical interpretation involving Casimir (vacuum) energy, which depends on space-time topology, and the strange shift by $c/24$ is due to an implicit mapping from the cylinder to the plane.

1.3. Representations of finite-dimensional simple Lie algebras

The representation theory of the simple Lie algebras can perhaps be regarded as an enormous generalisation of trigonometry. For instance the facts that

$$\frac{\sin(mx) \sin(nx)}{\sin(x)} = \sin((m + n)x) + \sin((m + n - 2)x) + \cdots + \sin((m - n)x)$$

for any $m, n \in \mathbb{Z}_{\geq}$, are both easy special cases of the theory.

The classic example of an algebraic structure are the numbers, and they prejudice us into thinking that commutativity and associativity are the ideal. We have learned over the past couple of centuries that commutativity can often be dropped without losing depth and usefulness, but most interesting structures seem to obey some form of associativity. Moreover, true associativity (as opposed to e.g. anti-associativity) really simplifies the arithmetic. Given the happy ‘accident’ that the commutator $[x, y] := xy - yx$ in any associative algebra obeys anti-associativity, it would seem to be both tempting and natural to study the ways (if any) in which associative algebras $\mathfrak{A}$ can ‘model’ or represent a given Lie algebra. Precisely, we are looking for a map $\rho : \mathfrak{g} \rightarrow \mathfrak{A}$ which preserves the linear structure (i.e. $\rho$ is a linear function), and which sends the bracket $[xy]$ in $\mathfrak{g}$ to the commutator $[\rho(x), \rho(y)]$ in $\mathfrak{A}$.

In practice groups (resp., algebras) often appear as symmetries (resp., infinitesimal generators of symmetries). These symmetries often act linearly. In other words, in practise the preferred associative algebras will usually be matrix algebras, and this is the usual

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7 See e.g. [25] for more details. Historically, representations of Lie algebras were considered even before representations of finite groups.
form for a representation and the only kind we will consider. The dimension of these representations is the size of the matrices.

Finding all possible representations, even for the simple Lie algebras, is probably hopeless. However, it is possible to find all finite-dimensional representations of the simple Lie algebras, and the answer is easy to describe. Given a simple Lie algebra $X_\ell$, there is a representation $L_\lambda$ for each $\ell$-tuple $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of nonnegative integers. $\lambda$ is called a highest-weight. Moreover, we can take direct sums $\oplus L_\lambda(\ell)$ of finitely many of these representations. The matrices in such a direct sum will be in block form. It turns out that, up to change-of-basis, this exhausts all finite-dimensional representations of $X_\ell$.

It is common to replace ‘representation $\rho$ of $g$’ with the equivalent notion of ‘$g$-module $M$’ — i.e. we think of the matrices $\rho(x)$ as linear maps $M \to M$. A $g$-module is a vector space on which $g$ acts (on the left). Instead of considering the matrix $\rho(x)$, we consider ‘products’ $xv$ (think of this as the matrix $\rho(x)$ times the column vector $v$) for $v \in M$. This product must be bilinear, and must obey $[xy]v = x(yv) - y(xv)$.

To get an idea of what $L_\lambda$ looks like, consider $A_1$. Recall its generators $e, f, h$ and relations (1.2.2b). Choose any $\lambda \in \mathbb{C}$. Define $x_0 \neq 0$ to formally obey $hx_0 = \lambda x_0$ and $ex_0 = 0$. Define inductively $x_{i+1} := fx_i$ for $i = 0, 1, \ldots$. Define $M_\lambda$ to be the span of all $x_i$ — we will see shortly that they are linearly independent (so $M_\lambda$ is infinite-dimensional). $M_\lambda$ is a module of $A_1$: the calculations $hx_{i+1} = hf x_i = ([hf] + fh)x_i = (-2f + fh)x_i$ and $ex_{i+1} = ef x_i = (e[f] + fe)x_i = (h + fe)x_i$ show inductively that $hx_i = (\lambda - m - 1)m x_i$ and $ex_m = (\lambda - m + 1)m x_{m-1}$. From these the reader can show that the $x_i$ are linearly independent. $M_\lambda$ is called a Verma module; $\lambda$ is called its highest-weight, and $x_0$ is called a highest-weight vector.

Now specialise to $\lambda = n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. Note that $ex_{n+1} = 0$ and $hx_{n+1} = (-n-2)x_{n+1}$. This means that, for these $n$, $M_n$ contains a submodule $M_{n+1}$ with highest-weight vector $x_{n+1}$, isomorphic to $M_{-n-2}$. $x_{n+1}$ is called a null vector. In other words, we could set $x_{n+1} := 0$ and still have an $A_1$-module. We would then get a finite-dimensional module which we’ll call $L_n := M_n/M_{-n-2}$ (not to be confused with the Virasoro generator in (1.2.7)). Its basis is $\{x_0, x_1, \ldots, x_n\}$ and so it has dimension $n + 1$.

For example, take $n = 1$. Note that what we get in terms of the basis $\{x_0, x_1\}$ is the familiar representation sl$_2(\mathbb{C})$ given in (1.2.5).

The situation for the other simple Lie algebras $X_\ell$ is similar.

It turns out to be hard to compare representations: $\rho$ and $\rho'$ could be equivalent (i.e. differ merely by a change-of-basis) but look very different. Or if we are given a representation, we may want to decompose it into the direct sum of some $L_\lambda(\ell)$. When working with representations, it is often very useful to avoid much of the extraneous basis-dependent detail present in the function $\rho$. Finite group theory suggests how to do this: we should use characters. The character of an $A_1$-module $M$ is given by Weyl: write $M$ as a direct sum of eigenspaces $M(m)$ of $h$; then define

$$\text{ch}_M(z) := \sum_m \dim M(m) \ e^{mz},$$

for any $z \in \mathbb{C}$. The $m$ are called weights and the $M(m)$ weight-spaces. For example, for
 Analogous formulas apply to any algebra $X_\ell$: the character will then be a function of an \( \ell \)-dimensional subspace \( \mathfrak{h} \) called the Cartan subalgebra, spanned by all the \( h_i \) (see §2.7), so can be thought of as a complex-valued function of \( \ell \) complex variables. The weights \( m \) will lie in the dual space to \( \mathfrak{h} \) — i.e. are linear maps \( \mathfrak{h} \to \mathbb{C} \) — so will have \( \ell \) components. See for instance Figure 8 in [59]. Incidentally, \( \ell \) is called the rank of \( X_\ell \).

Weyl’s definition works: two representations are equivalent iff their characters are identical, and \( M = \bigoplus_i L_{\lambda(i)} \) iff \( \text{ch}_M(z) = \sum_i \text{ch}_{\lambda(i)}(z) \). It also is enormously simpler: e.g. the smallest nontrivial representation of \( E_8 \) is a map from \( \mathbb{C}^{248} \) to the space of \( 248 \times 248 \) matrices, while its character is a function \( \mathbb{C}^8 \to \mathbb{C} \). But why is Weyl’s definition natural? How did he come up with it?

To answer that question, we must remind ourselves of the characters of finite groups\(^8\). A representation of a finite group \( G \) is a structure-preserving map \( \rho \) (i.e. a group homomorphism) from \( G \) to matrices. The group’s product becomes matrix product. In these notes we will be exclusively interested in group representations over \( \mathbb{C} \). Two representations \( \rho, \rho' \) are called equivalent if there exists a matrix (change-of-basis) \( U \) such that \( \rho'(g) = U \rho(g) U^{-1} \) for all \( g \). The character \( \text{ch}_\rho \) is the map \( G \to \mathbb{C} \) given by the trace:

\[
\text{ch}_\rho(g) = \text{tr}(\rho(g)).
\]

We see that equivalent representations will have the same character, because of the fundamental identity \( \text{tr}(AB) = \text{tr}(BA) \). This identity also tells us that the character is a ‘class function’, i.e. \( \text{ch}_\rho(hgh^{-1}) = \text{tr}(\rho(h) \rho(g) \rho(h)^{-1}) = \text{ch}_\rho(g) \) so \( \text{ch}_\rho \) is constant on each ‘conjugacy class’. Group characters are also enormously simpler than representations: e.g. the smallest nontrivial representation of the Monster \( \overline{\mathbb{M}} \) (see Part 2) consists of almost \( 10^{54} \) matrices, each of size \( 196883 \times 196883 \), while its character consists of 194 complex numbers. Incidentally, finite group representations behave analogously to the representations of \( X_\ell \): the role of the modules \( L_\lambda \) is played by the irreducible representations \( \rho_i \), and any finite-dimensional representation of \( G \) can be decomposed uniquely into a direct sum of various \( \rho_i \). The difference is that there are only finitely many \( \rho_i \) — their number equals the number of conjugacy classes of \( G \).

We can use this group intuition here. In particular, given any Lie algebra \( X_\ell \) and representation \( \rho \), we can think of the map \( e^x \mapsto e^{\rho(x)} \) as a representation of a Lie group \( G(X_\ell) \) corresponding to \( X_\ell \) (the exponential \( e^A \) of a matrix is defined by the usual power series; it will always converge). The trace of the matrix \( e^{\rho(x)} \) will be the group character value at \( e^x \in G(X_\ell) \), so we’ll define it to be the algebra character value at \( x \in X_\ell \). Again, it suffices to consider only representatives of each conjugacy class of \( G(X_\ell) \), because the character will be a class function. Now, almost every matrix is diagonalisable (since almost any \( n \times n \) matrix has \( n \) distinct eigenvalues), and so it would seem we aren’t losing much by restricting \( x \in X_\ell \) to diagonalisable matrices. Hence we may take our conjugacy class

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\(^8\) Surprisingly, what we now call the characters of group representations were invented almost a decade before group representations were.
representatives to be diagonal matrices \( x \in X_\ell \), i.e. (for \( X_\ell = A_1 \)) to \( x = zh \) for \( z \in \mathbb{C} \) \((h\) is diagonal in the \( x_i \) basis of \( L_\lambda \)). So the algebra character can be chosen to be a function of \( z \). Finally, the trace of \( e^{\rho(x)} = e^{zh} \) will be given by (1.3.1). This completes the motivation for Weyl’s character formula.

There is one other important observation we can make. Different diagonal matrices can belong to the same conjugacy class. For instance, 
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}^{-1}
= 
\begin{pmatrix}
b & 0 \\
0 & a
\end{pmatrix},
\]
so \( e^{zh} \) and \( e^{-zh} \) lie in the same \( G(A_1) = \text{SL}_2(\mathbb{C}) \) conjugacy class. Hence \( ch_M(z) = ch_M(-z) \). This symmetry \( z \mapsto -z \) belongs to the Weyl group for \( A_1 \). Each \( X_\ell \) has similar symmetries, and the Weyl group plays an important role in the whole theory, sort of analogous to the modular group for modular functions we’ll discuss in §2.3.

Weyl found a generalisation of the right-side of (1.3.2), valid for all \( X_\ell \). The character of \( L_\lambda \) can be written as a fraction (2.8.1): the numerator will be a alternating sum over the Weyl group, and the denominator will be a product over ‘positive roots’. This formula and its generalisations have profound consequences, as we’ll see in §2.8.

Incidentally, the trigonometric identities given at the beginning of this section are the tensor product formula of representations (interpreted as the product and sum of characters), and the fact that an arbitrary character can be written as a polynomial in the fundamental characters, both specialised to \( A_1 \) (see (1.3.2) for the \( A_1 \) characters).

1.4. Affine algebras and the Kac-Peterson matrices

The theory of nontwisted affine Kac-Moody algebras (usually called affine algebras or current algebras) is extremely analogous to that of the finite-dimensional simple Lie algebras. Nothing infinite-dimensional tries harder to be finite-dimensional than affine algebras. Standard references for the following material are [38,41,24].

Let \( X_\ell \) be any simple finite-dimensional Lie algebra. The affine algebra \( X_\ell^{(1)} \) is essentially the loop algebra \( \mathcal{L}(X_\ell) \), defined to be all possible ‘Laurent polynomials’ \( \sum_{n \in \mathbb{Z}} a_n t^n \) where each \( a_n \in X_\ell \) and all but finitely many \( a_n = 0 \). \( t \) here is an indeterminant. The bracket in \( \mathcal{L}(X_\ell) \) is the obvious one: e.g. \([a^n, b^m] = [ab]t^{n+m} \). Geometrically, \( \mathcal{L}(X_\ell) \) is the Lie algebra of polynomial maps \( S^1 \to X_\ell \) — hence the name (for that realisation, think of \( t = e^{2\pi i \theta} \)). Hence there are many generalisations of the loop algebra (e.g. any manifold in place of \( S^1 \) will do), closely related ones called toroidal algebras being the Lie algebra of maps \( S^1 \times \cdots \times S^1 \to X_\ell \). But the loop algebra is simplest and best understood, and the only one we’ll consider. Note that \( \mathcal{L}(X_\ell) \) is infinite-dimensional. Its Lie groups are the loop groups, consisting of all loops \( S^1 \to G(X_\ell) \) in a Lie group for \( X_\ell \).

We saw \( S^1 \) before, in the discussion of the Witt algebra. Thus the Virasoro and affine algebras should be related. In fact, the Virasoro algebra acts on the affine algebras as ‘derivations’, and this connection plays an important technical role in the theory.

\( X_\ell^{(1)} \) is in the same relation to the loop algebra, that the Virasoro \( \mathcal{V} \) is to the Witt \( \mathcal{W} \). Namely, it is its (unique nontrivial 1-dimensional) central extension — see e.g. (7.7.1)
of [38] for the analogue of (1.2.7a) here. In addition, for more technical reasons, a further (noncentral) 1-dimensional extension is usually made: the derivation $t\frac{d}{dt}$ is included (see footnote 33). $X^{(1)}_\ell$ is the simplest of the infinite-dimensional Kac-Moody algebras. The superscript ‘$(1)$’ denotes the fact that the loop algebra was twisted by an order-1 automorphism — i.e. that it is untwisted. It is called ‘affine’ because of its Weyl group, as we shall see.

Central extensions are a common theme in today’s infinite-dimensional Lie theory. Their raison d’être is always the same: a richer supply of representations. For example, $\mathcal{W}$ has several representations, but no nontrivial one is an ‘irreducible unitary positive-energy representation’ — the kind of greatest interest in math phys. On the other hand, its central extension $\mathcal{V}$ has a rich supply of those representations (e.g. there’s one for each choice of $c > 1, h > 0$, namely the Verma module $V_{c,h}$ corresponding to $L_0 x_0 = hx_0, Cx_0 = cx_0$). At the level of groups, central extensions allow projective representations (i.e. representations up to a scalar factor) to become true representations. Projective representations (hence central extensions) appear naturally in QFT because a quantum state vector $|v\rangle$ is physically indistinguishable from any nonzero scalar multiple $\alpha |v\rangle$.

All of the quantities associated to $X_\ell$ have an analogue here: Dynkin diagram, Weyl group, weights,... For instance, the affine Dynkin diagram is obtained from the Dynkin diagram for $X_\ell$ by adding one node. See for example Figure 9 of [59]. The extra node is always labelled by a ‘0’. The Cartan subalgebra $\mathfrak{h}$ here will be $\ell$-dimensional. Many of these details will be discussed in more detail in §2.7 below.

The construction of $X^{(1)}_\ell$ is so trivial that it seems surprising anything interesting and new can happen here. But a certain ‘miracle’ happens...

No interesting representation of $X^{(1)}_\ell$ is finite-dimensional. The analogue for $X^{(1)}_\ell$ of the finite-dimensional representations of $X_\ell$ are called the integrable highest-weight representations, and will be denoted $L_\lambda$. The highest-weight $\lambda$ here will be an $(\ell + 1)$-tuple $(\lambda_0, \lambda_1, \ldots, \lambda_\ell)$, $\lambda_i \in \mathbb{Z}_{\geq}$ (strictly speaking, it will be an $(\ell + 2)$-tuple, but the extra component is not important and is usually ignored). As for $X_\ell$, the highest-weights can be thought of as the assignment of a nonnegative integer to each node of the Dynkin diagram. The construction of $L_\lambda$ is as in the finite-dimensional case. They are called integrable because they are precisely those highest-weight representations which can be ‘integrated’ to a projective representation of the corresponding loop group, and hence a representation of a central extension of the loop group.

We define the character $\chi_\lambda$ as in (1.3.1), though now the weights $m$ will be $(\ell + 2)$-tuples, and there will be infinitely many of them. $\chi_\lambda$ will be a complex-valued function of $\ell + 2$ complex variables $(\vec{z}, \tau, u)$ (see (1.4.1a) below). It can be written as an alternating sum over the Weyl group $\mathcal{W}$, over a ‘nice’ denominator. The difference here is that $\mathcal{W}$ is now infinite.

Perhaps most of the interest in affine algebras can be traced to the ‘miracle’ that their Weyl groups are a semidirect product $Q^\vee \rtimes \mathcal{W}$ of translations in a lattice $\mathbb{Q}^\vee$ (the $\ell$-dimensional ‘co-root lattice’ of $X_\ell$ — see §1.6) with the (finite) Weyl group $\mathcal{W}$ of $X_\ell$.

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9 Incidentally the finite-dimensional simple Lie algebras do not have nontrivial central extensions.
See Figure 10 of [59] for the Weyl group of $A_2^{(1)}$. ‘Semidirect product’ means that any element of $W$ can be written uniquely as $(t, w)$ for some translation $t$ and some $w \in \overline{W}$, and $(t, w) \circ (t', w') = (\text{stuff}, w \circ w')$.

One thing this implies is that $\chi_\lambda$ will be of the form ‘theta function’/denominator. Theta functions are classically-studied modular forms (we will discuss these terms in §2.3), and thus the modular group $\text{SL}_2(\mathbb{Z})$ will make an appearance! To make this more precise, consider the highest-weight $\lambda = (\lambda_0, \lambda_1)$ of $A_1^{(1)}$, and write $k = \lambda_0 + \lambda_1$. Then

$$\chi_\lambda = \frac{\Theta^{(k+2)}_{\lambda_1+1} - \Theta^{(k+2)}_{-\lambda_1-1}}{\Theta^{(2)}_1 - \Theta^{(2)}_{-1}}$$

(1.4.1a)

where these functions all depend on 3 complex variables $z, \tau, u$, and

$$\Theta^{(n)}_m(z, \tau, u) := e^{-2\pi i nu} \sum_{\ell \in \mathbb{Z} + \frac{m}{2n}} \exp[\pi i n\ell^2 - 2\sqrt{2}\pi i \ell z].$$

(1.4.1b)

In (1.4.1a) we can see the alternating sum over the Weyl group of $A_1$ in the numerator (and denominator, since we’ve used the $A_1^{(1)}$ denominator identity in writing (1.4.1a)). For general $X_\ell^{(1)}$, the denominator will always be independent of $\lambda$, and the theta function (1.4.1b) will become a multidimensional one involving a sum over $Q'$ shifted by some weight and appropriately rescaled. The (co-)root lattice of $A_1$ is $\sqrt{2}\mathbb{Z}$. The key variable in (1.4.1a) is the modular one $\tau$, which will lie in the upper half complex plane $\mathcal{H}$ (in order to have convergence). In the applications to CFT, the other variables are often set to 0.

The number $k$ introduced in (1.4.1a) plays an important role in the general theory. In the representation $L_\lambda$, the central term $C$ will get sent to some multiple of the identity — the multiplier is labelled $k$ and is called the level of the representation. For any $X_\ell^{(1)}$ there is a simple formula expressing the level $k$ in terms of the highest-weight $\lambda$; e.g. for $A_\ell^{(1)}$ and $C_\ell^{(1)}$ it is given by $k = \lambda_0 + \lambda_1 + \cdots + \lambda_\ell$. Write $P^k_+$ for the (finite) set of level $k$ highest-weights (so the size of $P^k_+$ for $A_\ell^{(1)}$ is $\binom{k+\ell}{\ell}$). An important weight in $P^k_+$ is $(0, \ldots, 0)$. We will denote this ‘0’. In RCFT it corresponds to the vacuum.

The modular group $\text{SL}_2(\mathbb{Z})$ acts on the Cartan subalgebra $\mathfrak{h}$ of $X_\ell^{(1)}$ in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\bar{z}, \tau, u) = \left( \frac{\bar{z}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}; u - \frac{c\bar{z} \cdot \bar{z}}{2(c\tau + d)} \right).$$

Under this action, the characters $\chi_\lambda$ also transform nicely: in particular we find for any level $k$ weight $\lambda$

$$\chi_\lambda(\bar{z}, \frac{-1}{\tau}, u - \frac{c\bar{z} \cdot \bar{z}}{2\tau}) = \sum_{\mu \in P^k_+} S_{\lambda\mu} \chi_\mu(\bar{z}, \tau, u)$$

(1.4.2a)

$$\chi_\lambda(\bar{z}, \tau + 1, u) = \sum_{\mu \in P^k_+} T_{\lambda\mu} \chi_\mu(\bar{z}, \tau, u)$$

(1.4.2b)

---

10 This is also discussed briefly in section 2.2.
where $S$ and $T$ are complex matrices called the Kac-Peterson matrices. $S$ will always be symmetric and unitary, and has many remarkable properties as we shall see. Its entries are related to Lie group characters at elements of finite order (see (1.4.5) below). $T$ is diagonal and unitary; its entries are related to the eigenvalues of the quadratic Casimir.

For example, consider $A_1^{(1)}$ at level $k$. Then $S$ and $T$ will be $(k+1) \times (k+1)$ matrices given by

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \sin(\pi \frac{(\lambda_1 + 1)(\mu_1 + 1)}{k+2}), \quad T_{\lambda\mu} = \exp[\pi i \frac{(\lambda_1 + 1)^2}{2(k+2)} - \frac{\pi i}{4}] \delta_{\lambda,\mu}. \quad (1.4.3)$$

One important place $S$ appears is the famous Verlinde formula

$$N^\nu_{\lambda\mu} = \sum_{\kappa \in \Pi_k^+} S_{\lambda\kappa} S_{\mu\kappa} S^*_{0\kappa} \frac{S_{0\lambda}}{S_{0\kappa}}. \quad (1.4.4)$$

for the fusion coefficients $N^\nu_{\lambda\mu}$ of the corresponding RCFT. We will investigate some consequences of this formula in a later section. The fusion coefficients for the affine algebras are well-understood; see e.g. Section 4 of [59] for their interpretation (usually called the Kac-Walton formula) as ‘folded tensor product coefficients’.

We will see in §1.7 that symmetries of the extended Dynkin diagram have consequences for $S$ and $T$ (simple-currents, charge-conjugation). There is a ‘Galois action’ on $S$ which we will discuss in §1.8. There is a strange property of $S$ and $T$ called rank-level duality (see e.g. [45]): the matrices for $A_\ell^{(1)}$ at level $k$ are closely related to those of $A_{k-1}^{(1)}$ at level $\ell+1$, and similar statements hold for $B_\ell^{(1)}$, $C_\ell^{(1)}$ and $D_\ell^{(1)}$. Another reason $S$ is mathematically interesting is the formula

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \text{ch}_{\bar{\lambda}}(-2\pi i \frac{\mu + \rho}{k + h^\vee}). \quad (1.4.5)$$

The right-side is a character of $X_\ell$, and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$ means ignore the extended node. $\rho$ is the ‘Weyl vector’ $(1, 1, \ldots, 1)$ and $h^\vee$ is called the dual Coxeter number and is the level of $\rho$. For $A_\ell^{(1)}$, $h^\vee = \ell + 1$. Of course the right-side can also be regarded as a character for a Lie group associated to $X_\ell$, in which case the argument would have to be exponentiated and correspond to an element of finite order in the group. These numbers (1.4.5) have been studied by many people (most extensively by Pianzola) and have some nice properties. For instance Moody-Patera (1984) have argued that exploiting them leads to some quick algorithms for computing e.g. tensor product coefficients. Kac [37] found a curious application for them: a Lie theoretic proof of ‘quadratic reciprocity’!

Quadratic reciprocity is one of the gems of classical number theory. It tells us that the equations

$$x^2 \equiv a \pmod{b} \quad y^2 \equiv b \pmod{a}$$

are related; more precisely, for fixed $a$ and $b$ (for simplicity take them to both be primes $\neq 2$) the questions of whether there is a solution $x$ to the first equation and a solution $y$ to the
second, are related. They will both have the same yes or no answer, unless \( a \equiv b \equiv 3 \pmod{4} \), in which case they will have opposite answers. E.g. take \( a = 23 \) and \( b = 3 \), then we know the first equation does not have a solution (since \( a \equiv 2 \pmod{3} \) and \( x^2 \equiv 2 \) doesn’t have a solution mod 3), and hence the second equation must have a solution (indeed, \( y = 7 \) works). There are now many proofs for quadratic reciprocity, and Kac used Lie characters at elements of finite order to find another one.

What is interesting here is that Kac’s proof uses only certain special weights for \( A_\ell \). The natural question is: is it possible to find any generalisations of quadratic reciprocity using other weights and algebras? Many generalisations of quadratic reciprocity are known; will generalising Kac’s argument recover them, or will they perhaps yield new reciprocity laws? It seems no one knows.

The relation (1.4.5) is important because it connects finite-dimensional Lie data with infinite-dimensional Lie data. The ‘conceptual arrow’ can be exploited both ways: in the generalisations of the arguments of §1.9 to other algebras, (1.4.5) allows us to use our extensive knowledge of finite-dimensional algebras to squeeze out some information in the affine setting; but also it is possible to use the richer symmetries of the affine data to see ‘hidden’ symmetries in finite-dimensional data. For example it can be used (Gannon-Walton 1995) to find a sort of Galois symmetry of dominant weight multiplicities in \( X_\ell \), which would be difficult or impossible to anticipate without (1.4.5).

1.5. THE CLASSIFICATION OF PHYSICAL INVARIANTS

We are interested in the following classification problem. Choose any affine algebra \( X_\ell^{(1)} \) and level \( k \in \mathbb{Z}_\geq \). Find all matrices \( M = (M_{\lambda\mu})_{\lambda,\mu \in P_+^k} \) such that

\[ (P1) \quad MS = SM \text{ and } MT = TM, \]

\[ (P2) \quad \text{each entry } M_{\lambda\mu} \in \mathbb{Z}_\geq; \]

\[ (P3) \quad M_{00} = 1. \]

Any such \( M \), or equivalently the corresponding partition function \( Z = \sum_{\lambda,\mu} M_{\lambda\mu} \chi_{\lambda} \chi_{\mu}^\ast \), is called a physical invariant.

The first and most important classification of physical invariants was the Cappelli-Itzykson-Zuber A-D-E classification for \( A_1^{(1)} \) at all levels \( k \) [8]. We will give their result shortly. This implies for instance the minimal model RCFT classification, as well as the \( N = 1 \) super(symmetric)conformal minimal models. The other classifications of comparable magnitude are \( A_2^{(1)} \) for all \( k \); \( A_\ell^{(1)} \), \( B_\ell^{(1)} \) and \( D_\ell^{(1)} \) for all \( k \leq 3 \); \( (A_1 \oplus A_1)^{(1)} \) for all levels \((k_1, k_2)\); and \((u(1) \oplus \cdots \oplus u(1))^{(1)} \) for all (matrix-valued) levels \( k \). See e.g. [29] for references. The most difficult of these classifications is for \( A_2^{(1)} \), done by Gannon (1994).

In other words, very little in this direction has been accomplished in the 15 or so years this problem has existed. But this is not really a good measure of progress. The effort instead has been directed primarily towards the full classification; most of these partial results are merely easy spin-offs from that more serious and ambitious assault.

The proof in [8] was very complicated and followed the following lines. First, an explicit basis was found for the vector space (called the ‘commutant’) of all matrices
obeying (P1). Then (P2) and (P3) were imposed. Unfortunately their proof was long and formidable. Others tried to apply their approach to $A_2^{(1)}$, but without success. The eventual proof for $A_2^{(1)}$ was completely independent of the [8] argument, and exploited more of the structure implicit in the problem. As the $A_2^{(1)}$ argument became more refined, it became the model for the general assault. In §1.9 we sketch this new approach.

From this more general perspective, of these completed classifications only the level 2 $B_\ell^{(1)}$ and $D_\ell^{(1)}$ ones will have any lasting value (the orthogonal algebras at level 2 behave very peculiarly, possess large numbers of exceptional physical invariants, and must be treated separately). The others behave more generically and will fall out as special cases once the more general classifications are concluded. Other classifications which should be straightforward with our present understanding are $C_2^{(1)}$ at all $k$; $G_2^{(1)}$ at all $k$; and $B_\ell^{(1)}$ and $D_\ell^{(1)}$ at $k = 4$. The $C_2^{(1)}$ should be easiest and would imply the $C_\ell^{(1)}$ level 2, as well as the $B_\ell^{(1)}$ and $D_\ell^{(1)}$ level 5, classifications. A very safe conjecture is that the only exceptional physical invariants (we define this term in §1.7) for $C_2^{(1)}$ occur at $k = 3, 7, 8, 12$ — this is known to be true for all $k \leq 500$. $G_2^{(1)}$ would be more difficult but also much more valuable; its only known exceptionals occur at $k = 3, 4$, and these are the only exceptionals for $k \leq 500$, and a very safe conjecture is that there are no other $G_2^{(1)}$ exceptionals. $B_\ell^{(1)}$ and $D_\ell^{(1)}$ at level 4 will also be more difficult, but also would be valuable; less is understood about its physical invariants and there is a good chance new exceptionals exist there.

The most surprising thing about the known physical invariant classifications is that there so few surprises: almost every physical invariant is ‘generic’. We will see that the symmetries of the extended Dynkin diagram give rise to general families of physical invariants. We will call any physical invariants which do not arise in these generic ways (i.e. using what are called simple-currents or conjugations), exceptional. Many exceptionals have been found, and now we are almost at the point where we can safely conjecture the complete list of physical invariants for $X_\ell^{(1)}$ at any $k$, for $X_\ell$ a simple algebra.

Unfortunately the classification for semi-simple algebras $X_\ell_1 \oplus \cdots \oplus X_\ell_s$ does not reduce to the one for simple ones. In fact, any explicit classification of the physical invariants for $X^{(1)}$, for all semi-simple $X$, would easily be one of the greatest accomplishments in the history of math, for it would include as a small part such monumental things as an explicit classification of all positive-definite integral lattices. Thus we unfortunately cannot expect an explicit classification for the semi-simple algebras.

To make this discussion more concrete and explicit, consider $A_1^{(1)}$. For convenience drop $\lambda_0$, so $P^k_{\pm} = \{0, 1, \ldots, k\}$. Write $J$ for the permutation (called a simple-current) $Ja := k - a$. Then the complete list of physical invariants for $A_1^{(1)}$ is

$$A_{k+1} = \sum_{a=0}^{k} |\chi_a|^2, \quad \text{for all } k \geq 1$$

$$D_{\frac{k}{2}+2} = \sum_{a=0}^{k} \chi_a \chi_J^* a, \quad \text{whenever } \frac{k}{2} \text{ is odd}$$
\begin{align*}
\mathcal{D}_{k+2} &= |\chi_0 + \chi_{j0}|^2 + |\chi_2 + \chi_{j2}|^2 + \cdots + 2|\chi_{k}|^2, \quad \text{whenever } \frac{k}{2} \text{ is even} \\
\mathcal{E}_6 &= |\chi_0 + \chi_{6}|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2, \quad \text{for } k = 10 \\
\mathcal{E}_7 &= |\chi_0 + \chi_{10}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 \\
&\quad + \chi_8 (\chi_2 + \chi_{14})^* + (\chi_2 + \chi_{14}) \chi_8 + |\chi_8|^2, \quad \text{for } k = 16 \\
\mathcal{E}_8 &= |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2, \quad \text{for } k = 28.
\end{align*}

The physical invariants \( \mathcal{A}_n \) and \( \mathcal{D}_n \) are generic, corresponding respectively to the order 1 (i.e. identity) and order 2 (i.e. the simple-current \( J \)) Dynkin diagram symmetries, as we shall see in \( \S 1.7 \). Physically, they are the partition functions of WZW models on \( SU_2(\mathbb{C}) \) and \( SO_3(\mathbb{R}) \) group manifolds, resp. The exceptionals \( \mathcal{E}_6 \) and \( \mathcal{E}_8 \) are best interpreted as due to the \( C_{2,1} \supseteq A_{1,10} \) and \( G_{2,1} \supseteq A_{1,28} \) conformal embeddings (see \( \S 1.7 \); standard notation is to write ‘\( X_{\ell,k} \)’ for ‘\( X_{\ell}^{(1)} \)’ and level \( k \)’). The \( \mathcal{E}_7 \) exceptional is harder to interpret, but can be thought of as the first in an infinite series of exceptionals involving rank-level duality and \( D_4 \) triality.

Around Christmas 1985, Zuber wrote Kac about the \( A_{1}^{(1)} \) physical invariant problem, and mentioned the physical invariants he and Itzykson knew at that point (what we now call \( \mathcal{A}_* \) and \( \mathcal{D}_{\text{even}} \)). A few weeks later, Kac wrote back saying he found one more invariant, and jokingly pointed out that it must be indeed quite exceptional as the exponents of \( \mathcal{E}_6 \) appeared in it. “I must confess that I didn’t pay much attention to that last remark (I hardly knew what Coxeter exponents were, at the time!)” [63]. By spring 1986, Cappelli arrived in Paris and got things moving again; together Cappelli-Itzykson-Zuber found \( \mathcal{E}_7 \), \( \mathcal{D}_{\text{odd}} \), and then \( \mathcal{E}_8 \), and struggled to find more. “And it is only in August [1986], during a conversation with Pasquier, in which he was showing me his construction of lattice models based on Dynkin diagrams, that I suddenly remembered this cryptic but crucial! observation of Victor, rushed to the library to find a list of the exponents of the other algebras... and found with the delight that you can imagine that they were matching our list” [63]. Thus the A-D-E pattern to these physical invariants was discovered.

### 1.6. The A-D-E meta-pattern

Before we discuss *meta-patterns* in math, let’s introduce the notion of *lattice*\(^{11}\), a simple geometric structure we’ll keep returning to in these notes. The standard reference for lattice theory is [13].

Consider the real vector space \( \mathbb{R}^{m,n} \): its vectors look like \( \vec{x} = (\vec{x}_+; \vec{x}_-) \) where \( \vec{x}_+ \) and \( \vec{x}_- \) are \( m \)- and \( n \)-component vectors respectively, and dot products are given by \( \vec{x} \cdot \vec{y} = \vec{x}_+ \cdot \vec{y}_+ - \vec{x}_- \cdot \vec{y}_- \). The dot products \( \vec{x}_\pm \cdot \vec{y}_\pm \) are given by the usual product and sum of components. For example, the familiar Euclidean (positive-definite) space is \( \mathbb{R}^n = \mathbb{R}^{n,0} \), while Minkowski space is \( \mathbb{R}^{3,1} \).

\(^{11}\) There are many words in math which have several incompatible meanings. For example, there are *vector* fields and *number* fields, and modular *forms* and modular *representations*. ‘Lattice’ is another of these words. Aside from the geometric meaning we will use, it also refers to a ‘partially ordered set’.
Now choose any basis \( B = \{ \vec{x}_1, \ldots, \vec{x}_{m+n} \} \) in \( \mathbb{R}^{m,n} \). So \( \mathbb{R}^{m,n} = \mathbb{R}\vec{x}_1 + \cdots + \mathbb{R}\vec{x}_{m+n} \). Define the set \( \Lambda(B) := \mathbb{Z}\vec{x}_1 + \cdots + \mathbb{Z}\vec{x}_{m+n} \). This is a lattice, and all lattices can be formed in this way\(^{12}\). So a lattice is discrete and is closed under sums and integer multiples. For example, \( \mathbb{Z}^{m,n} \) is a lattice (take the standard basis in \( \mathbb{R}^{m,n} \)). A more interesting lattice is the hexagonal lattice (also called \( A_2 \)), given by the basis \( B = \{ (\sqrt{2}/2, \sqrt{3}/2), (\sqrt{2}, 0) \} \) of \( \mathbb{R}^2 \)—try to plot several points. If you wanted to slide a bunch of coins on a table together as tightly as possible, their centres would form this hexagonal lattice. Another important lattice is \( II_{1,1} \subset \mathbb{R}^{1,1} \), given by \( B = \{ (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \} \); equivalently it can be thought of as the set of all pairs \((a, b) \in \mathbb{Z}^2 \) with dot product

\[
(a, b) \cdot (c, d) = ad + bc .
\]

It is important to note that different choices of basis may or may not result in a different lattice. For a trivial example, consider \( B = \{1\} \) and \( B' = \{-1\} \) in \( \mathbb{R} = \mathbb{R}^{1,0} \): they both give the lattice \( \mathbb{Z} = \mathbb{Z}^{1,0} \). Two lattices are called equivalent if they only differ by a change-of-basis. E.g. \( B = \{ (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \} \) in \( \mathbb{R}^2 \) yields a lattice equivalent to \( \mathbb{Z}^2 \).

The dimension of the lattice is \( m + n \). The lattice is called positive-definite if it lies in some \( \mathbb{R}^m \) (i.e. \( n = 0 \)). The lattice is called integral if all dot products \( \vec{x} \cdot \vec{y} \) are integers, for \( \vec{x}, \vec{y} \in \Lambda \). A lattice \( \Lambda \) is called even if it is integral and in addition all norms \( \vec{x} \cdot \vec{x} \) are even integers. For example, \( \mathbb{Z}^{m,n} \) is integral but not even, while \( A_2 \) and \( II_{1,1} \) are even. The dual \( \Lambda^* \) of a lattice \( \Lambda \) consists of all vectors \( \vec{x} \in \mathbb{R}^{m,n} \) such that \( \vec{x} \cdot \Lambda \subset \mathbb{Z} \). So a lattice is integral iff \( \Lambda \subseteq \Lambda^* \). A lattice is called self-dual if \( \Lambda = \Lambda^* \). \( \mathbb{Z}^{m,n} \) and \( II_{1,1} \) are self-dual but \( A_2 \) is not.

There are lots of ‘meta-patterns’ in math, i.e. collections of seemingly different problems which have similar answers. Once one of these meta-patterns is identified it is always helpful to understand what is responsible for it. For example, while I was writing up my PhD thesis I noticed in several places the numbers 1, 2, 3, 4, and 6. For instance \( \cos(2\pi r) \in \mathbb{Q} \) for \( r \in \mathbb{Q} \) iff the denominator of \( r \) is 1, 2, 3, 4, or 6. This pattern was easy to explain: they are precisely those positive integers \( n \) with Euler totient \( \phi(n) \leq 2 \), i.e. there are at most 2 positive numbers less than \( n \) coprime\(^{13}\) to \( n \). The other incidences of these numbers can usually be reduced to this \( \phi(n) \leq 2 \) property (e.g. the dimension of the number field \( \mathbb{Q}[\cos(2\pi \phi)] \) (see §1.8) considered as a vector space over \( \mathbb{Q} \) will be \( \phi(b)/2 \).

A more interesting meta-pattern involves the number 24 and its divisors. One sees 24 wherever modular forms naturally appear. For instance, we see it in the critical dimensions in string theory: \( 24 + 2 \) and \( 8 + 2 \). Another example: the dimensions of even self-dual positive-definite lattices must be a multiple of 8 (e.g. the \( E_8 \) root lattice defined shortly has dimension 8, while the Leech lattice discussed in §2.4 has dimension 24). The meta-pattern 24 is also understood: the fundamental problem for which it is the answer is the following one. Fix \( n \), and consider the congruence \( x^2 \equiv 1 \pmod{n} \). Certainly in order to have a chance of satisfying this, \( x \) and \( n \) must be coprime. The extreme situation is when every

\(^{12}\) In most presentations a lattice is permitted to have smaller dimension than its ambient space, however that freedom gains no real generality.

\(^{13}\) We say \( m, n \) are coprime if any prime \( p \) which divides \( m \) does not divide \( n \), and vice versa.
number \(x\) coprime to \(n\) satisfies this congruence:
\[
gcd(x, n) = 1 \iff x^2 \equiv 1 \pmod{n} .
\] (1.6.2)

The reader can try to verify the following simple fact: \(n\) obeys this extreme situation (1.6.2) iff \(n\) divides 24.

What does this congruence property have to do with these other occurrences of 24? Let \(\Lambda\) be an even self-dual positive-definite lattice of dimension \(n\). Then an elementary argument shows that there will exist an \(n\)-tuple \(\vec{a} = (a_1, \ldots, a_n)\) of odd integers with the property that 8 must divide \(\vec{a} \cdot \vec{a} = \sum_i a_i^2\). But \(a_i^2 \equiv 1 \pmod{8}\), and so we get \(8|n\).

A much deeper and still not-completely-understood meta-pattern is called A-D-E (see [1] for a discussion and examples). The name comes from the so-called \textit{simply-laced algebras}, i.e. the simple finite-dimensional Lie algebras whose Dynkin diagrams — see Figure 6 in [59] — contain only single edges (i.e. no arrows). These are the \(A_n\) and \(D_n\)-series, along with the \(E_6\), \(E_7\) and \(E_8\) exceptional. The claim is that many other problems, which don’t seem to have anything directly in common with simple Lie algebras, have a solution which falls into this A-D-E pattern (for an object to be meaningfully labelled \(X_{\ell}\), some of the data associated to the algebra \(X_{\ell}\) should reappear in some form in that object). Let’s look at some examples.

Consider even positive-definite lattices \(\Lambda\). The smallest possible nonzero norm in \(\Lambda\) will be 2, and the vectors of norm 2 are special and are called \textit{roots}. The reason they are special is that reflecting through them will always be an automorphism of \(\Lambda\). That is, the reflection \(\vec{u} \mapsto \vec{u} - 2\vec{u} \cdot \vec{\alpha} / \vec{\alpha} \cdot \vec{\alpha}\) through \(\vec{\alpha} \neq \vec{0}\) won’t in general map \(\Lambda\) to itself, unless \(\vec{\alpha}\) is a root of \(\Lambda\). It is important in lattice theory to know the lattices which are spanned by their roots; it turns out these are precisely the orthogonal direct sums of lattices called \(A_n\), \(D_n\), and \(E_6\), \(E_7\) and \(E_8\). They carry those names for a number of reasons. For example, the lattice called \(X_n\) will have a basis \(\{\vec{\alpha}_1, \ldots, \vec{\alpha}_n\}\) with the property that the matrix \(A_{ij} := \vec{\alpha}_i \cdot \vec{\alpha}_j\) is the Cartan matrix (see §2.7) for the Lie algebra \(X_n\)! Also, the reflection group generated by reflections in the roots of the lattice \(X_n\) will be isomorphic to the Weyl group of the Lie algebra \(X_n\). Finally, to any simple Lie algebra there is canonically associated a lattice called the root lattice; for the simply-laced algebras, these will equal the corresponding lattice of the same name. Incidentally, the root lattices for the non-simply-laced simple algebras will (up to rescalings) be direct sums of the simply-laced root lattices.

We have already met the \(A_2\) lattice: it is the densest packing of circles in the plane. It has long been believed that the obvious pyramidal way to pack oranges is also the densest possible way — the centres of the oranges form the \(A_3\) root lattice. A controversial proof for this famous conjecture has been offered by W.-Y. Hsiang in 1991; in 1998 a new proof by Hale et al has been proposed. The densest known packings in dimensions 4,5,6,7,8 are \(D_4\), \(D_5\), \(E_6\), \(E_7\), \(E_8\), resp. \(E_8\) is the smallest even self-dual positive-definite lattice.

A famous A-D-E example is called the McKay\textsuperscript{14} correspondence. Consider any finite subgroup \(G\) of the Lie group SU\(_2\)(C) (i.e. the \(2 \times 2\) unitary matrices with determinant 1). For example, there is the cyclic group \(\mathbb{Z}_n\) of \(n\) elements generated by the matrix

\[
M_n = \begin{pmatrix}
\exp[2\pi i/n] & 0 \\
0 & \exp[-2\pi i/n]
\end{pmatrix}
\]

\textsuperscript{14} He is the same John McKay we will celebrate in section 2.1.
Let $R_i$ be the irreducible representations of $G$. For instance, for $\mathbb{Z}_n$, there are precisely $n$ of these, all 1-dimensional, given by sending the generator $M_n$ to $\exp[2k\pi i/n]$ for each $k = 1, 2, \ldots, n$. Now consider the tensor product $G \otimes R_i$, where we interpret $G \subset \text{SU}_2(\mathbb{C})$ here as a 2-dimensional representation. We can decompose that product into a direct sum $\oplus_j m_{ij} R_j$ of irreducibles (the $m_{ij}$ here are multiplicities). Now create a graph with one node for each $R_i$, and with the $i$th and $j$th nodes ($i \neq j$) connected with precisely $m_{ij}$ directed edges $i \to j$. If $m_{ij} = m_{ji}$, we agree to erase the double arrows from the $m_{ij}$ edges. Then McKay observed that the graph of any $G$ will be a distinct extended Dynkin diagram of A-D-E type! For instance, the cyclic group with $n$ elements corresponds to the extended graph of $A_{n-1}$.

How was McKay led to his remarkable correspondence? He knew that the sum of the ‘marks’ $a_i = 1, 2, 3, 4, 5, 6, 4, 2, 3$ associated to each node of the extended $E_8$ Dynkin diagram equaled 30, the Coxeter number of $E_8$. So what did their squares add to? 120, which he recognised as the cardinality of one of the exceptional finite subgroups of $\text{SU}_2(\mathbb{C})$, and that got him thinking...

Another famous example of A-D-E, due to Arnol’d, are the ‘simple critical points’ of smooth complex-valued functions $f$, on e.g. $\mathbb{C}^3$. For example, both $x^2 + y^2 + z^{n+1}$ and $x^2 + y^3 + z^5$ have singularities at $(0, 0, 0)$ (i.e. their first partial derivatives all vanish there), and they are assigned to $A_n$ and $E_8$, respectively. The $\text{SU}_2(\mathbb{C})$ subgroups can be related to singularities as follows. The group $\text{SU}_2(\mathbb{C})$ acts on $\mathbb{C}^2$ in the obvious way (matrix multiplication). If $G$ is a discrete subgroup, then consider the (ring of) polynomials in 2 variables $w_1, w_2$ invariant under $G$. It turns out it will have 3 generators $x(w_1, w_2), y(w_1, w_2), z(w_1, w_2)$, which are connected by 1 polynomial relation (syzygy). For instance, take $G$ to be the cyclic group $\mathbb{Z}_n$, then we’re interested in polynomials $p(w_1, w_2)$ invariant under $w_1 \mapsto \exp[2\pi i/n]w_1, w_2 \mapsto \exp[-2\pi i/n]w_2$. Any such invariant $p(w_1, w_2)$ is clearly generated by (i.e. can be written as a polynomial in) $w_1w_2, w_1^n$ and $w_2^n$. Choosing instead the generators $x = \frac{w_1^n - w_2^n}{2}, y = i\frac{w_1^n + w_2^n}{2}, z = w_1w_2$, we get the syzygy $z^n = -(x^2 + y^2)$. For any $G$, generators $x, y, z$ can always be found so that the syzygy will be one of the polynomials associated to a simple singularity, and in fact will give the equation of the algebraic surface $\mathbb{C}^2/G$ as a 2-dimensional complex surface in $\mathbb{C}^3$ (e.g. the complex surfaces $\mathbb{C}^2/\mathbb{Z}_n$ and $\{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^n = 0\}$ are equivalent).

Arguably the first A-D-E classification goes back to Theaetetus, around 400 B.C. He classified the regular solids. For instance the tetrahedron can be associated to $E_6$ while the cube is matched with $E_7$. This A-D-E is only partial, as there are no regular solids assigned to the A-series, and to get the D-series one must look at ‘degenerate regular solids’.

The closest thing to an explanation of the A-D-E meta-pattern would seem to be the notion of ‘additive assignments’ on graphs (which is a picturesque way of describing the corresponding eigenvalue problem). Consider any graph $\mathcal{G}$ with undirected edges, and none of the edges run from a node to itself. We can also assume without loss of generality that $\mathcal{G}$ is connected. Assign a positive number $a_i$ to each node. If this assignment has the property that for each $i$, $2a_i = \sum a_j$ where the sum is over all nodes $j$ adjacent to $i$ (counting multiplicities of edges), then we call it ‘additive’. For instance, for the graph $\circ \rightarrow \circ$, the assignment $a_1 = 1 = a_2$ is additive, but the assignment $a_1 = 1, a_2 = 2$ is not. The question is, which graphs have an additive assignment? The answer is: precisely the
extended Dynkin diagrams of A-D-E type! And their additive assignments are unique (up to constant proportionality) and are given by the marks $a_i$ of the algebra (see e.g. the Table on p.54 of [38]). For example the extended $A_n$ graph consists of $n + 1$ nodes arranged in a circle, and its marks $a_i$ all equal 1.

What do additive assignments have to do with the other A-D-E classifications? Consider a finite subgroup $G$ of SU$_2$(C). Take the dimension of the equation $G \otimes R_i = \oplus_j m_{ij} R_j$: we get $2d_i = \sum_j m_{ij} d_j$ where $d_j = \dim(R_j)$. Hence the dimensions of the irreducible representations define an additive assignment for each of McKay’s graphs, and hence those graphs must be of A-D-E type (provided we know $m_{ij} = m_{ji}$).

As Cappelli-Itzykson-Zuber observed, the physical invariants for $A^{(1)}_1$ also realise the A-D-E pattern, in the following sense. The Coxeter number $h$ of the name $X_\ell$ (i.e. the sum $\sum_i a_i$ of the marks) equals $k + 2$, and the exponents $m_i$ of $X_\ell$ equal those $a \in P^+_k$ for which $M_{aa} \neq 0$ (for the simply-laced algebras, the $m_i$ are defined by writing the eigenvalues of the corresponding Cartan matrix (see §2.7) as $4 \sin^2(\pi m_i / 2h)$ — the $m_i$ are integers and the smallest is always 1). Probably what first led Kac to his observation about the $E_6$ exponents was that $k + 2$ (this is how $k$ enters most formulas) for his exceptional equalled the Coxeter number 12 for $E_6$. More recently, the operator algebraists Ocneanu [48] and independently Böckenhauer-Evans [4] found an A-D-E interpretation for the off-diagonal entries $M_{ab}$ of the $A^{(1)}_1$ physical invariants, using subfactor theory.

We are not claiming that this $A^{(1)}_1$ classification is ‘equivalent’ to any other A-D-E one — that would miss the point of meta-patterns. What we really want to do is to identify some critical combinatorial part of an $A^{(1)}_1$ proof with critical parts in other A-D-E classifications — this is what we did with the other meta-patterns. A considerably simplified proof of the $A^{(1)}_1$ classification is now available [29], so hopefully this task will now be easier.

There has been some progress at understanding this $A^{(1)}_1$ A-D-E. Nahm [46] constructed the invariant $X_\ell$ in terms of the compact simply-connected Lie group of type $X_\ell$, and in this way could interpret the $k + 2 = h$ and $M_{m_i m_i} \neq 0$ coincidences. A very general explanation for A-D-E has been suggested by Ocneanu [48] using his theory of path algebras on graphs; although his work has never been published, others are now rediscovering (and publishing!) similar work (see e.g. [4]). Nevertheless, the A-D-E in CFT remains almost as mysterious now as it did a dozen years ago — for example it still isn’t clear how it directly relates to additive assignments.

There are 4 other claims for A-D-E classifications of families of RCFT physical invariants, and all of them inherit their (approximate) A-D-E pattern from the more fundamental $A^{(1)}_1$ one. One is the $c < 1$ minimal models, also proven in [8], and another is the $N = 1$ superconformal minimal models, proved by Cappelli (1987). In both cases the physical invariants are parametrised by pairs of A-D-E diagrams. The list of known $c = 1$ RCFTs also looks like A-D-E (two series parametrised by $\mathbb{Q}_+$, and three exceptionals), but the completeness of that list has never been rigourously established.

The fourth classification often quoted as A-D-E, is the $N = 2$ superconformal minimal models. Their classification was done by Gannon (1997). The connection here with A-D-E turns out to be rather weak: e.g. 20, 30, and 24 distinct invariants would have an equal
right to be called $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$ respectively. It would appear that the frequent claims that the $N = 2$ minimal models fall into an A-D-E pattern are rather dubious.

Hanany-He [35] suggest that the $A^{(1)}_1$ A-D-E pattern can be related to subgroups $G \subset SU_2(\mathbb{C})$ by orbifolding 4-dimensional $N = 4$ supersymmetric gauge theory by $G$, resulting in an $N = 2$ superCFT whose ‘matter matrix’ can be read off from the Dynkin diagram corresponding to $G$. The same game can be played with finite subgroups of $SU_3(\mathbb{C})$, resulting in $N = 1$ superCFTs whose matter matrices correspond to graphs very reminiscent of the ‘fusion graphs’ of Di Francesco-Petkova-Zuber (see e.g. [62]) corresponding to $A^{(1)}_2$ physical invariants. [35] use this to conjecture a McKay-type correspondence between singularities of type $\mathbb{C}^n/G$, for $G \subset SU_n(\mathbb{C})$, and the physical invariants of $A^{(1)}_{n-1}$. This in their view would be the form A-D-E takes for higher rank physical invariants. Their actual conjecture though is still somewhat too vague.

For a final example of meta-pattern, consider ‘modular function’ (see §2.3). After all, they appear in a surprising variety of places and disguises. Maybe we shouldn’t regard their ubiquity as fortuitous, instead perhaps there’s a deeper common ‘situation’ which is the source for that ubiquity. Just as ‘symmetry’ yields ‘group’, or ‘rain-followed-by-heat’ breeds mosquitos. Math is not above metaphysics; like any area it grows by asking questions, and changing your perspective — even to a metaphysical one — should suggest new questions.

1.7. SIMPLE-CURRENTS AND CHARGE-CONJUGATION

The key properties\footnote{A good exercise for the reader is to prove that if $S$ is unitary and symmetric, and obeys (1.7.1), then there will be at most finitely many physical invariants $M$ for that $S.T.$} of the matrix $S$ are that it’s unitary and symmetric (so $M$ in §1.5 equals $S M S^*$),

$$S_{0\mu} > 0 \text{ for all } \mu \in P^+_k,$$ (1.7.1)

and that the numbers $N_{\lambda \mu}$ defined by Verlinde’s formula (1.4.4) are nonnegative integers. These are obeyed by the matrix $S$ in any (unitary) RCFT. From these basic properties, we will obtain here some elementary consequences which have important applications.

But first, let’s make an observation which isn’t difficult to prove, but doesn’t appear to be generally known.

Verlinde’s formula looks strange, but it is quite generic,

and we can see it throughout math and mathematical physics. Consider the following.

Let $\mathfrak{A}$ be a commutative associative algebra, over $\mathbb{R}$ say. Suppose $\mathfrak{A}$ has a finite basis $\Phi$ (over $\mathbb{R}$) containing the unit 1. Define the ‘structure constants’ $N^{c}_{ab} \in \mathbb{R}$, for $a, b, c \in \Phi$, by $a b = \sum_{c \in \Phi} N^{c}_{ab} c$. Suppose there is an algebra homomorphism $*$ (so $*$ is linear, and $(xy)^* = x^*y^*$) which permutes the basis vectors (so $\Phi^* = \Phi$), and we have the relation $N^{1}_{ab} = \delta_{b,a^*}$. We call any such algebra $\mathfrak{A}$ a fusion algebra. Then any fusion algebra will necessarily have a unitary matrix $S$ with $S_{1a} > 0$ and with the structure constants given by Verlinde’s formula. Algebraically, the relation $S = S^t$ holds if $\mathfrak{A}$ is ‘self-dual’ in a certain natural sense.
Define the ‘fusion matrices’ $N_{\lambda}$ by $(N_{\lambda})_{\mu\nu} = N_{\lambda \mu}^{\nu}$. Then Verlinde’s formula says that the $\mu$th column $S_{\lambda \mu}^{*}$ of $S$ is an eigenvector of each fusion matrix $N_{\lambda}$, with eigenvalue $S_{\lambda \mu}^{0}$.

**Useful Fact.** If $v$ is a simultaneous eigenvector of each fusion matrix $N_{\lambda}$, then there exists a constant $c \in \mathbb{C}$ and a $\lambda \in \mathcal{P}_{k}^{+}$ such that $v = c S_{\lambda \mu}^{*}$.

For one consequence, take the complex conjugate of the eigenvector equation $N_{\lambda} S_{\lambda \mu}^{*} = S_{\lambda \mu}^{0} S_{\lambda \mu}^{*}$: we get that the vector $S_{\lambda \mu}^{*}$ is a simultaneous eigenvector of all $N_{\lambda}$, with eigenvalue $S_{\lambda \mu}^{0}$. The reader can verify that unitarity of $S$ forces $|c| = 1$, while (1.7.1) forces $c > 0$. Thus $c = 1$ and we obtain the formula

$$S_{\lambda \mu}^{*} = S_{\lambda \mu} C_{\mu} = S_{\lambda \mu} C_{\mu} C_{\mu} = S_{\lambda \mu} C_{\mu}.$$  

(1.7.2)

Also, unitarity and symmetry of $S$ forces $C = S^{2}$, while conjugating twice shows $C^{2} = id$. $C$ is an important matrix in RCFT, and is called charge-conjugation. When $C = id.$, then the matrix $S$ is real.

Note that (1.7.1) now implies $C 0 = 0$. Also $CT = TC$. Hence $M = C$ will always define a physical invariant, and if $M$ is any other physical invariant, the matrix product $MC = CM$ will define another physical invariant. Also, $N_{C \lambda, C \mu}^{C \nu} = N_{\lambda \mu}^{\nu}$ and $(N_{\lambda})^{t} = CN_{\lambda} = N_{\lambda} C = N_{C \lambda}$.

For the WZW (=affine) case, $C$ has a special meaning: $C \lambda$ is the highest-weight ‘contragredient’ to $\lambda$. $C$ corresponds to an order 2 (or 1) symmetry of the (unextended) Dynkin diagram. For example, for $A_{\ell}^{(1)}$, we have $C(\lambda_{0}, \lambda_{1}, \ldots, \lambda_{\ell-1}, \lambda_{\ell}) = (\lambda_{0}, \lambda_{\ell}, \lambda_{\ell-1}, \ldots, \lambda_{1})$. For $A_{1}^{(1)}$ then, $C = id.$, which can also be read off from (1.4.3).

The algebras $D_{even}^{(1)}$ all have at least one nontrivial symmetry of the (unextended) Dynkin diagram which isn’t the charge-conjugation. The most interesting example is $D_{4}^{(1)}$, which has 5 of these. By a conjugation, we will mean any symmetry of the unextended Dynkin diagram.

To go much further, we need a fascinating tool called Perron-Frobenius theory — a collection of results concerning the eigenvalues and eigenvectors of nonnegative matrices (i.e. matrices in which every entry is a nonnegative real number). Whenever you have such matrices in your problem, and it is natural to multiply them, then there is a good chance Perron-Frobenius theory will tell you something interesting. The basic result here is that if $A$ is a nonnegative matrix, then there will be a nonnegative eigenvector $x \geq 0$ with eigenvalue $\rho \geq 0$, such that if $\lambda$ is any other eigenvalue of $A$, then $|\lambda| \leq \rho$. There are lots of other results (see e.g. [44]), e.g. $\rho$ must be at least as large as any diagonal entry of $A$, and there must be a row-sum of $A$ no bigger than $\rho$, and another row-sum no smaller than $\rho$.

For instance, consider

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  

25
Perron-Frobenius eigenvectors for $A$ and $B$ are $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, with eigenvalues 3 and 2 resp. The other eigenvalue of $A$ is 0 (multiplicity 2), while those of $B$ are 0 and $-1$.

Fusion matrices $N_\lambda$ are nonnegative, and it is indeed natural to multiply them:

$$N_\lambda N_\mu = \sum_{\nu \in P_k^+} N_{\lambda\mu}^{\nu} N_{\nu}.$$

So we can expect Perron-Frobenius to tell us something interesting. This is the case, and we obtain the curious-looking inequalities

$$S_{\lambda\mu} S_{0\nu} \geq |S_{\lambda\mu}| S_{00}. \quad (1.7.3)$$

Squaring both sides, summing over $\mu$ and using unitarity, we get that $S_{\lambda\mu} \geq S_{00}$. In other words, the ratio $\frac{S_{\lambda\mu}}{S_{00}}$, called the quantum-dimension of $\lambda$, will necessarily be $\geq 1$.

The term ‘quantum-dimension’ comes from quantum groups, where $\frac{S_{\lambda\mu}}{S_{00}}$ is the quantum-dimension of the module labelled by $\lambda$ of the quantum group $U_q(X)$. The borderline case then is when a quantum-dimension equals 1. Any such weight is called a simple-current. The theory of simple-currents was developed most extensively by Schellekens and collaborators (see e.g. [50]). The simple-currents for the affine algebras were classified by J. Fuchs (1991), and the result is that (with one unimportant exception: $E_{8}^{(1)}$ at level 2) they all correspond to symmetries of the extended Dynkin diagrams. In particular, applying any such symmetry to the vacuum $0 = (k, 0, \ldots, 0)$ gives the list of simple-currents. For instance, the $\ell + 1$ weights of the form $(0, \ldots, k, \ldots, 0)$ ($k$ in the $i$th spot) are the simple-currents for $A_{\ell}^{(1)}$. There are 2 simple-currents for $B_{\ell}^{(1)}$, $C_{\ell}^{(1)}$ and $E_{7}^{(1)}$, 3 for $E_{6}^{(1)}$, and 4 for $D_{\ell}^{(1)}$. Simple-currents play a large role in RCFT, as we shall see.

Let $j$ be any simple-current. Then (1.7.3) becomes $S_{0\mu} \geq |S_{j\mu}|$ for all $\mu$, so unitarity forces $S_{0\mu} = |S_{j\mu}|$, that is

$$S_{j\mu} = \exp[2\pi i Q_j(\mu)] S_{0\mu} \quad \forall \mu \in P_k^+ \quad (1.7.4)$$

for some rational numbers $0 \leq Q_j(\mu) < 1$. Hence by diagonalising, we get $N_j N_{Cj} = I$. But the inverse of a nonnegative matrix $A$ is itself nonnegative, only if $A$ is a ‘generalised permutation matrix’, i.e. a permutation matrix except the 1’s can be replaced by any positive numbers. But $N_j$ and $N_{Cj}$ are also integral, and so they must in fact be permutation matrices. Write $(N_j)_{\lambda\mu} = \delta_{\mu, J\lambda}$ for some permutation $J$ of $P_k^+$. So $j = J0$. Then

$$\delta_{\lambda, \mu} = N_{j, \lambda}^{\mu} = \sum_{\nu} \exp[2\pi i Q_j(\nu)] S_{\lambda\nu} S_{j\mu, \nu}^\ast,$$

16 This seems to be a standard trick in math: when some sort of bound is established, look at the extremal cases which realise that bound. If your bound is a good one, it should be possible to say something about those extremal cases, and having something to say is always of paramount importance. This trick is used for instance in the definition of 24 last section, and the definition of normal subgroup in section 2.2.
so taking absolute values and using the triangle inequality and unitarity of $S$, we find that (1.7.4) generalises:

$$S_{J\lambda,\mu} = \exp[2\pi i Q_j(\mu)] S_{\lambda\mu} .$$

(1.7.5)

The simple-currents form a finite abelian group, corresponding to the composition of the permutations $J$. For any simple-currents $J,J'$, we get the symmetry $N^{J\nu}_{J',\mu} = N^{J'\nu}_{J\mu}$. The $\mathbb{Q}/\mathbb{Z}$-valued functions $Q_j$ define gradings on the fusion rings, and conversely any grading corresponds to a simple-current in this way.

For example, the simple-current $j = (0, k)$ of $A_1^{(1)}$ at level $k$ corresponds to $Q_j(\lambda) = \lambda_1/2$ and the permutation $J\lambda = (\lambda_1, \lambda_0)$. We can see this directly from (1.4.3). For $A_2^{(1)}$ level $k$, there are 2 nontrivial simple-currents, $(0, k, 0)$ and $(0, 0, k)$. The first of these corresponds to triality $Q(\lambda) = (\lambda_1 + 2\lambda_2)/3$ and $\lambda \mapsto (\lambda_2, \lambda_0, \lambda_1)$, while the second to $(2\lambda_1 + \lambda_2)/3$ and $\lambda \mapsto (\lambda_1, \lambda_2, \lambda_0)$. Similar statements hold for all affine algebras: e.g. for $B_n^{(1)}$ level $k$, the nontrivial simple-current has $Q_j(\lambda) = \lambda_k/2$ and $J\lambda = (\lambda_1, \lambda_0, \lambda_2, \ldots, \lambda_k)$.

One of the applications of simple-currents is that physical invariants can be built from them in generic ways. These physical invariants all obey the selection rule

$$M_{\lambda\mu} \neq 0 \implies \mu = J\lambda \text{ for some simple-current } J = J(\lambda, \mu) .$$

(1.7.6)

We will call any such physical invariant $M$ a \textit{simple-current invariant}. A special case is the $D_{\frac{3}{2}+2}$ physical invariant for $A_1^{(1)}$ at even level $k$. Up to a fairly mild assumption, all simple-current invariants have been classified for any RCFT by Schellekens and collaborators; given that assumption, they can all be constructed by generic methods. The basic construction is due to Bernard [3], though it has been generalised by others. In the WZW case, all simple-current invariants (except some for $D_{\ell+1}^{(1)}$) correspond to strings on nonsimply-connected Lie groups.

By a \textit{generic physical invariant} of $X_\ell^{(1)}$ we mean one of the form $M = C'M'C''$ where $C', C''$ are (charge-)conjugations, and $M'$ is a simple-current invariant. In other words, $M$ is constructed in generic ways from symmetries of the extended Dynkin diagram of $X_\ell$. Any other $M$ are called \textit{exceptional}.

All known results point to the validity of the following guess:

**Conjecture.** Choose a simple algebra $X_\ell$. Then for all sufficiently large $k$, all physical invariants of $X_\ell^{(1)}$ at level $k$ will be generic.

In other words, any given $X_\ell^{(1)}$ will have only finitely many exceptionals. For instance, for $A_1^{(1)}$ and $A_2^{(1)}$ at any $k > 28$ and $k > 21$ resp., all physical invariants are generic. For $C_2^{(1)}$ and $G_2^{(1)}$, $k > 12$ and $k > 4$ resp. should work.

The richest source of exceptionals are \textit{conformal embeddings}. In some cases the affine representations $L_\lambda$ for some algebra $X_\ell^{(1)}$ (necessarily at level 1) can be decomposed into \textit{finite} direct sums of representations of some affine subalgebra $Y_m^{(1)}$ (at some level $k$). In this case, a physical invariant for $X_\ell^{(1)}$ level 1 will yield a physical invariant for $Y_m^{(1)}$ level $k$, obtained by replacing every $X_\ell^{(1)}$ level 1 character $\chi_\lambda$ by the appropriate finite sum of
$Y_{m}^{(1)}$ level $k$ characters. An example will demonstrate this simple idea: $A_{1}^{(1)}$ level 28 is a conformal subalgebra of $G_{2}^{(1)}$ level 1, and we have the character decompositions

\[
\chi_{(1,0,0)} = \chi'_{(28,0)} + \chi'_{(18,10)} + \chi'_{(10,18)} + \chi'_{(0,28)}
\]

\[
\chi_{(0,0,1)} = \chi'_{(22,6)} + \chi'_{(16,12)} + \chi'_{(12,16)} + \chi'_{(6,22)}.
\]

Thus the unique level 1 $G_{2}^{(1)}$ physical invariant $|\chi_{0}|^{2} + |\chi_{(0,0,1)}|^{2}$ yields what we call the $E_{8}$ physical invariant of $A_{1}^{(1)}$. All level 1 physical invariants are known, as are all conformal embeddings and the corresponding character decompositions (branching rules).

1.8. Galois Theory

Evariste Galois was a brilliantly original French mathematician. Born shortly before Napoleon’s ill-fated invasion of Russia, he died shortly before the ill-fated 1832 uprising in Paris. His last words: “Don’t cry, I need all my courage to die at 20”.

Galois grew up in a time and place confused and excited by revolution. He was known to say “if I were only sure that a body would be enough to incite the people to revolt, I would offer mine”. On May 2 1832, after frustration over failure in love and failure to convince the Paris math establishment of the depth of his ideas, he made his decision. A duel was arranged with a friend, but only his friend’s gun would be loaded. Galois died the day after a bullet perforated his intestine. At his funeral it was discovered that a famous general had also just died, and the revolutionaries decided to use the general’s death rather than Galois’ as a pretext for an armed uprising. A few days later the streets of Paris were blocked by barricades, but not because of Galois’ sacrifice: his death had been pointless [56].

Galois theory in its most general form is the study of relations between objects defined implicitly by some conditions. For example, the objects could be the solutions to a given differential equation. In the incarnation of Galois we are interested in here, the objects are numbers, namely the zeros of certain polynomials. We will sketch this theory below, but see e.g. the article by Stark in [60] for more details.

Gauss seems to have been the first to show that ‘weird’ (complex) numbers could tell us about the integers. For instance, suppose we are interested in the equation $n = a^{2} + b^{2}$. Consider $5 = 2^{2} + 1^{2}$. We can write this as $5 = (2 + i)(2 - i)$, so we are led to consider complex numbers of the form $a + bi$, for $a, b \in \mathbb{Z}$. These are now called ‘Gaussian integers’.

Suppose we know the following theorem:

**Fact.** Let $p \in \mathbb{Z}$ be any prime number. Then $p$ factorises over the Gaussian integers iff $p = 2$ or $p \equiv 1 \pmod{4}$.

By ‘factorise’ there, we mean $p = zw$ where neither $z$ nor $w$ is a ‘unit’: $\pm 1, \pm i$.

Now suppose $p$ is a prime, $= 2$ or $\equiv 1 \pmod{4}$, and we write $p = (a + bi)(c + di)$. Then $p^{2} = (a^{2} + b^{2})(c^{2} + d^{2})$, so $a^{2} + b^{2} = c^{2} + d^{2} = p$. Conversely, suppose $p = a^{2} + b^{2}$, then $p = (a + bi)(a - bi)$. Thus:
Consequence. 17 Let \( p \in \mathbb{Z} \) be any prime number. Then \( p = a^2 + b^2 \) for \( a, b \in \mathbb{Z} \) iff \( p = 2 \) or \( p \equiv 1 \pmod{4} \).

Now we can answer the question: can a given \( n \) be written as a sum of 2 squares \( n = a^2 + b^2 \)? Write out the prime decomposition \( n = \prod p^{a_p} \). Then \( n = a^2 + b^2 \) has a solution iff \( a_p \) is even for every \( p \equiv 3 \pmod{4} \). For instance \( 60 = 2^2 \cdot 3^1 \cdot 5^1 \) cannot be written as the sum of 2 squares, but \( 90 = 2^1 \cdot 3^2 \cdot 5^1 \) can. We can also find (and count) all solutions: e.g. \( 90 = 2 \cdot 3^2 \cdot 5 = \{(1+i)3(1+2i)\}(1-i)3(1-2i)\}, giving \( 90 = (-3)^2 + 9^2 \).

This problem should give the reader a small appreciation for the power of using non-integers to study integers. Non-integers often lurk in the shadows, secretly watching their more arrogant brethren the integers strut. One of the consequences of their presence can be the existence of certain ‘Galois’ symmetries. Such happens in RCFT, as we will show below.

Look at complex conjugation: \( (wz)^* = w^*z^* \) and \( (w+z)^* = w^* + z^* \). Also, \( r^* = r \) for any \( r \in \mathbb{R} \). So we can say that * is a structure-preserving map \( \mathbb{C} \to \mathbb{C} \) (called an automorphism of \( \mathbb{C} \)) fixing the reals. We will write this \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{R}) \). ‘\( \text{Gal}(\mathbb{C}/\mathbb{R}) \)’ is the Galois group of \( \mathbb{C} \) over \( \mathbb{R} \); it turns out to contain only \( * \) and the identity.

A way of thinking about the automorphism * is that it says that, as far as the real numbers are concerned, \( i \) and \( -i \) are identical twins.

Let \( \mathbb{F} \) be any field containing \( \mathbb{Q} \) (we defined ‘field’ in §1.2). The Galois group \( \text{Gal}(\mathbb{F}/\mathbb{Q}) \) then will be the set of all automorphisms=symmetries of \( \mathbb{F} \) which fix all rationals.

For example, take \( \mathbb{F} \) to be the set of all numbers of the form \( a + b\sqrt{5} \), where \( a, b \in \mathbb{Q} \). Then \( \mathbb{F} \) will be a field, which is commonly denoted \( \mathbb{Q}[\sqrt{5}] \) because it is generated by \( \sqrt{5} \) and \( \sqrt{5} \). Let’s try to find its Galois group. Let \( \sigma \in \text{Gal}(\mathbb{F}/\mathbb{Q}) \). Then \( \sigma(a + b\sqrt{5}) = \sigma(a) + \sigma(b)\sigma(\sqrt{5}) = a + b\sigma(\sqrt{5}), \) so once we know what \( \sigma \) does to \( \sqrt{5} \), we know everything about \( \sigma \). But \( 5 = \sigma(5) = \sigma(\sqrt{5}^2) = (\sigma(\sqrt{5}))^2 \), so \( \sigma(\sqrt{5}) = \pm \sqrt{5} \) and there are precisely 2 possible Galois automorphisms here (one is the identity). As far as \( \mathbb{Q} \) is concerned, \( \pm \sqrt{5} \) are interchangeable: it cannot see the difference.

For a more important example, consider the cyclotomic field \( \mathbb{F} = \mathbb{Q}[\zeta_n] \), where \( \zeta_n := \exp[2\pi i/n] \) is an nth root of 1. So \( \mathbb{Q}[\zeta_n] \) consists of all complex numbers which can be expressed as polynomials \( a_n\zeta_n^m + a_{m-1}\zeta_n^{m-1} + \cdots + a_0 \) in \( \zeta_n \) with rational coefficients \( a_i \). Once again, to find the Galois group \( \text{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \), it is enough to see what an automorphism \( \sigma \) does to the generator \( \zeta_n \). Since \( \zeta_n^n = 1 \), we see that it must send it to another nth root of 1, \( \zeta_n^\ell \) say; in fact it is easy to see that \( \sigma(\zeta_n) \) must be another ‘primitive’ nth root of 1, i.e. \( \ell \) must be coprime to \( n \). So \( \text{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \) will be isomorphic to the multiplicative group \( \mathbb{Z}_n^* \) of numbers between 1 and \( n \) coprime to \( n \). The rationals can’t see any difference between the primitive nth roots of 1 — for instance \( \mathbb{Q} \) can’t tell that \( \zeta_n^{\pm 1} \) are ‘closer to 1’ than the other primitive roots. So any \( \sigma \in \text{Gal}(\mathbb{Q}[\zeta_n]/\mathbb{Q}) \) will correspond to some \( \ell \in \mathbb{Z}_n^* \), and to see what \( \sigma \) does to some \( z \in \mathbb{Q}[\zeta_n] \) what we do is write \( z \) as a polynomial in \( \zeta_n \) and then replace each occurrence of \( \zeta_n \) with \( \zeta_n^{\ell} \). For example,

\[
\sigma(\cos(2\pi a/n)) = \cos \left( \frac{\xi_n^a + \xi_n^{-a}}{2} \right) = \frac{\xi_n^{a\ell} + \xi_n^{-a\ell}}{2} = \cos(2\pi a\ell/n).
\]

\(^{17}\) This result was first stated by Fermat in one of his infamous margin notes (another is discussed in Section 2.3), and was finally proved a century later by Euler. A remarkable 1-line proof was found by Zagier [61].
So to summarise, Galois automorphisms are a massive generalisation of the idea of complex conjugation. If in your problem complex conjugation seems interesting, then there is a good chance more general Galois automorphisms will play an interesting role. This is what happens in RCFT, as we now show.

**Fact.** [14] Suppose $S$ is unitary and symmetric and each $S_{0a} > 0$.

(a) If in addition the numbers $N_{ab}^c$ given by Verlinde’s formula (1.4.4) are rational, then the entries $S_{ab}$ of $S$ must lie in a cyclotomic field.

(b) The numbers $N_{ab}^c$ will be rational iff for any $\sigma \in \text{Gal}(\mathbb{Q}[S]/\mathbb{Q})$, there is a permutation $a \mapsto a^\sigma$, and a choice of signs $\epsilon_\sigma(a) \in \{\pm 1\}$, such that

$$
\sigma(S_{ab}) = \epsilon_\sigma(a) S_{a^\sigma,b} = \epsilon_\sigma(b) S_{a,b^\sigma}.
$$

\(1.8.1\)

‘$\mathbb{Q}[S]$’ in part (b) denotes the field generated by $\mathbb{Q}$ and all matrix entries $S_{ab}$. The argument follows the one given for the charge-conjugation $C$ at the beginning of the last section. The kinds of complex numbers which lie in cyclotomic fields are $\sin(\pi r)$, $\cos(\pi r)$, $\sqrt{r}$ and $ri$ for any $r \in \mathbb{Q}$. Almost all complex numbers fail to lie in any cyclotomic field: e.g. generic cube roots, 4th roots, ..., of rationals, as well as transcendental numbers like $e$, $\pi$ and $e^\pi$.

Of course the affine algebras satisfy the conditions of the Fact, as does more generally the modular matrix $S$ for any unitary RCFT, and so these will possess the Galois action. For the affine algebras this action has a geometric interpretation in terms of multiplying weights by an integer $\ell$ and applying Weyl group elements — see [14] for a description.

This Fact is useful in both directions: as a way of testing whether a conjectured matrix $S$ has a chance of producing the integral fusions we want it to yield; and more importantly as a source of a symmetry of the RCFT which generalises charge-conjugation. Any statement about charge-conjugation seems to have an analogue for any of these Galois symmetries, although it is usually more complicated.

As an example, consider $A^{(1)}_1$: (1.4.3) shows explicitly that $S_{\lambda\mu}$ lies in the cyclotomic field $\mathbb{Q}[\zeta_{4(k+2)}]$. Write $\{x\}$ for the number congruent to $x$ mod $2(k+2)$ satisfying $0 \leq \{x\} < 2(k+2)$. Choose any Galois automorphism $\sigma$, and let $\ell \in \mathbb{Z}^\times_{4(k+2)}$ be the corresponding integer. Then if $\{\ell(a+1)\} < k+2$, we will have $a^\sigma = \{\ell(a+1)\} - 1$, while if $\{\ell(a+1)\} > k+2$, we’ll have $a^\sigma = 2(k+2) - \{\ell(a+1)\} - 1$. The sign $\epsilon_\sigma(a)$ will depend on a contribution from $\sqrt{2/k+2}$ (which for most purposes can be ignored), as well as the sign $+1$ or $-1$, resp., depending on whether or not $\{\ell(a+1)\} < k+2$.

Consider specifically $k = 10$, and the Galois automorphism $\sigma_5$ corresponding to $\ell = 5$. Then the permutation is $0 \leftrightarrow (6,4)$, $(9,1) \leftrightarrow (1,9)$, $(8,2) \leftrightarrow (2,8)$, $(4,6) \leftrightarrow (0,10)$, while $(7,3)$ and $(5,5)$ are fixed.

This Galois symmetry has been used to find certain exceptional physical invariants, but its greatest use so far is as a powerful selection rule we will describe next section.
In this final section we include some of the basic tools belonging to the ‘modern’ classifications of physical invariants, and we give a flavour of their proofs. We will state them for the $A_{1}^{(1)}$ level $k$ problem given above, but everything generalises without effort. See [29] and references therein for more details. Recall the matrices $S, T$ in (1.4.3).

First note that commutation of $M$ with $T$ implies the selection rule

$$M_{\lambda\mu} \neq 0 \implies (\lambda_1 + 1)^2 \equiv (\mu_1 + 1)^2 \pmod{4(k+2)}.
\tag{1.9.1}$$

It is much harder to squeeze information out of the commutation with $S$, but the resulting information turns out surprisingly to be much more useful. In fact, commutation with $S$ is almost incompatible with the constraint $M_{\lambda\mu} \in \mathbb{Z}_\geq$.

Note that the vacuum $0 \in P_+$ is both physically and mathematically special; our strategy will be to find all possible 0th rows and columns of $M$, and then for each of these possibilities to find the remaining entries of $M$.

The easiest result follows by evaluating $MS = SM$ at $(0, \lambda)$ for any $\lambda \in P_+$:

$$\sum_{\mu \in P_+} M_{0\mu} S_{\mu\lambda} \geq 0,
\tag{1.9.2}$$

with equality iff the $\lambda$th column of $M$ is identically 0. (1.9.2) has two uses: it severely constrains the values of $M_{0\mu}$ (similarly $M_{\mu0}$), and it says precisely which columns (and rows) are nonzero.

Next, let’s apply the triangle inequality to sums involving (1.7.5). Choose any $i, j \in \{0, 1\}$. Then

$$M_{J^i0, J^j0} = \sum_{\lambda, \mu} (-1)^{\lambda_1^i} S_{0\lambda} M_{\lambda\mu} (-1)^{\mu_1^j} S_{0\mu}.
\tag{1.9.3}$$

Taking absolute values, we obtain

$$M_{J^i0, J^j0} \leq \sum_{\lambda, \mu} S_{0\lambda} M_{\lambda\mu} S_{0\mu} = M_{00} = 1.
\tag{1.9.4}$$

Thus $M_{J^i0, J^j0}$ can equal only 0 or 1. If it equals 1, then we obtain the selection rule:

$$\lambda_1^i \equiv \mu_1^j \pmod{2}$$

whenever $M_{\lambda\mu} \neq 0$;

this implies the symmetry $M_{J^i\lambda, J^j\mu} = M_{\lambda\mu}$ for all $\lambda, \mu \in P_+$. We can see both of these in the list of physical invariants for $A_{1}^{(1)}$ level $k$. This explains a lot of the properties of those invariants. For instance, try to use this selection rule to explain why no $\chi_{\text{odd}}$ appears in the exceptional called $E_8$.

Our $M$ is nonnegative, and although multiplying $M$’s may not give us back a physical invariant, it will give us a matrix commuting with $S$ and $T$. In other words, the commutant
is much more than merely a vector space, it is in fact an algebra. Thus we should expect Perron-Frobenius to tell us something here. A first application is the following.

Suppose \( M_{\lambda 0} = \delta_{\lambda,0} \) — i.e. the 0th column of \( M \) is all zeros except for \( M_{00} = 1 \). Then Perron-Frobenius implies (with a little work) that \( M_{\lambda \mu} = \delta_{\mu,\pi \lambda} \), and \( S_{\pi \lambda,\pi \mu} = S_{\lambda \mu} \). This nice fact applies directly to the \( A \) and \( D_{\text{odd}} \) physical invariants of \( A^{(1)} \).

This is proved by studying the powers \((M^t M)^L\) as \( L \) goes to infinity: its diagonal entries will grow exponentially with \( L \), unless there is at most one nonzero entry on each row of \( M \), and that entry equals 1.

More careful reasoning along those lines tells us about the other generic situation here. Namely, suppose \( M_{\lambda 0} \neq 0 \) only for \( \lambda = 0 \) and \( \lambda = J0 \), and similarly for \( M_{0\lambda} \) — i.e. the 0th row and column of \( M \) are all zeros except for \( M_{J\lambda,J0} = 1 \). Then the \( \lambda \)th row (or column) of \( M \) will be identically 0 iff \( \lambda_1 \) is odd. Moreover, let \( \lambda, \mu \) be any non-fixed-points of \( J \), and suppose \( M_{\lambda \mu} \neq 0 \). Then

\[
M_{\lambda \nu} = \begin{cases} 1 & \text{if } \nu = \mu \text{ or } \nu = J\mu \\ 0 & \text{otherwise} \end{cases}
\]

with a similar formula for \( M_{\mu \nu} \). This applies to the \( D_{\text{even}} \) and \( E_7 \) invariants of \( A^{(1)} \).

Our final ingredient is the Galois symmetry (1.8.1) obeyed by \( S \). Choose any Galois automorphism \( \sigma \). It will correspond to some integer \( \ell \) coprime to \( 2(k+2) \). From (1.8.1) and \( M = SMS^* \) we get, for all \( \lambda, \mu \), the important relation

\[
M_{\lambda \mu} = \epsilon_\sigma(\lambda) \epsilon_\sigma(\mu) M_{\lambda \sigma^{-1} \mu} \quad . \quad (1.9.3)
\]

From (1.9.3) and the positivity of \( M \), we obtain the powerful Galois selection rule

\[
M_{\lambda \mu} \neq 0 \implies \epsilon_\sigma(\lambda) = \epsilon_\sigma(\mu) \quad . \quad (1.9.4)
\]

Next let us quickly sketch how these tools are used to obtain the \( A^{(1)} \) classification. For details the reader should consult [29].

The first step will be to find all possible values of \( \lambda \) such that \( M_{0\lambda} \neq 0 \) or \( M_{\lambda 0} \neq 0 \). These \( \lambda \) are severely constrained. We know two generic possibilities: \( \lambda = 0 \) (good for all \( k \)), and \( \lambda = J0 \) (good when \( \frac{k}{2} \) is even). We now ask the question, what other possibilities for \( \lambda \) are there? Our goal is to prove (1.9.7). Assume \( \lambda \neq 0, J0 \), and write \( a = \lambda_1 + 1 \) and \( n = k + 2 \).

There are only two constraints on \( \lambda \) which we will need. One is (1.9.1):

\[
(a - 1) (a + 1) \equiv 0 \pmod{4n} \quad . \quad (1.9.5)
\]

More useful is the Galois selection rule (1.9.4), which we can write as \( \sin(\pi \ell \frac{a}{n}) \sin(\pi \ell \frac{1}{n}) > 0 \), for all those \( \ell \). But a product of sines can be rewritten as a difference of cosines, so

\[
\cos(\pi \ell \frac{a - 1}{n}) > \cos(\pi \ell \frac{a + 1}{n}) \quad . \quad (1.9.6)
\]
(1.9.6) is strong and easy to solve; the reader should try to find her own argument.

What we get is that, provided \( n \neq 12, 30 \), \( M \) obeys the strong condition

\[
M_{\lambda_0} \neq 0 \text{ or } M_{0\lambda} \neq 0 \quad \implies \quad \lambda \in \{0, J0\} .
\]

Consider first case 1: \( M_{\lambda_0} = \delta_{\lambda, 0} \). From above, we know \( M_{\lambda_0} = \delta_{\mu, \pi \lambda} \) for some permutation \( \pi \) of \( P_+ \). From above, we know \( M_{\lambda_0} = \delta_{\mu, \pi \lambda} \) for some permutation \( \pi \) of \( P_+ \). We know \( \pi 0 = 0 \); put \( \mu := \pi(k - 1, 1) \). Then

\[
\sin(\frac{\pi 2}{n}) = \sin(\frac{4\mu + 1}{n}),
\]

and so we get either \( \mu = (k - 1, 1) \) or \( \mu = J(k - 1, 1) \). By \( T \)-invariance (1.9.1), the second possibility can only occur if \( 4 \equiv (n - 2)^2 \) (mod \( 4n \)), i.e. 4 divides \( n \). But for those \( n \), \( D_{n+1} \) is also a permutation matrix, so replacing \( M \) if necessary with the matrix product \( M D_{n+1} \), we can always require \( \mu = (k - 1, 1) \), i.e. \( \pi \) also fixes \( (k - 1, 1) \). It is now easy to show \( \pi \) must fix any \( \lambda \), i.e. that \( M \) is the identity matrix \( A_{n-1} \).

The other possibility, case 2, is that both \( M_{0, J0} \neq 0 \) and \( M_{J0, 0} \neq 0 \). (1.9.1) says \( 1 \equiv (n - 1)^2 \) (mod \( 4n \)), i.e. \( \frac{n}{2} \) is odd. The argument here is similar to that of case 1, but with \( (k - 2, 2) \) playing the role of \( (k - 1, 1) \). We can show that \( M_{(k-2,2),(k-2,2)} \neq 0 \), except possibly for \( k = 16 \), where we find the exceptional \( E_7 \). Otherwise we get \( M = D_{n+1} \).

For more general \( X_\ell \) level \( k \), the approach is

(i) to look at all the constraints on the \( \lambda \in P_+^k \) for which \( M_{\lambda_0} \neq 0 \) or \( M_{\lambda_0} \neq 0 \). Most important here are \( TM = MT \) (which will always be some sort of norm selection rule) and the Galois selection rule (1.9.4). Generically, what we will find is that such a \( \lambda \) must equal \( J0 \) for some simple-current \( J \), as in (1.9.7) for \( A_1^{(1)} \).

(ii) Solve this generic case (in the \( A_1^{(1)} \) classification, these were the physical invariants \( A_* \), \( D_* \), and \( E_7 \)).

(iii) Solve the nongeneric case. The worst of these are the orthogonal algebras at \( k = 2 \), as well as the places where conformal embeddings (see §1.7) occur.

(ii) has recently been completed for all simple \( X_\ell \), as has the \( k = 2 \) part of (iii). (i) is the main remaining task in the physical invariant classification for simple \( X_\ell \).

A natural question to ask is whether A-D-E has been observed in e.g. the \( A_2^{(1)} \) classification. The answer is no, although the fusion graph theory of Di Francesco-Petkova-Zuber [62] is an attempt to assign to these physical invariants graphs reminiscent of the A-D-E Dynkin diagrams. Also, there is related work trying to understand the \( A_2^{(1)} \) classification in terms of subgroups of \( SU_3(\mathbb{C}) \) (as opposed to \( SU_2(\mathbb{C}) \) for \( A_1^{(1)} \)) — see e.g. [35]. Finding the \( A_3^{(1)}, A_4^{(1)}, \ldots \) classifications would permit the clarification and testing of this vaguely conjectured relation between the \( A_n^{(1)} \) physical invariants, and singularities \( \mathbb{C}^{n+1}/G \) for \( G \) a finite subgroup of \( SU_{n+1}(\mathbb{C}) \).

However, a few years ago Philippe Ruelle was walking in a library in Dublin. He spotted a yellow book in the math section, called Complex Multiplication by Lang. A strange title for a book by Lang! After all, there can’t be all that much even Lang could really say about complex multiplication! Ruelle flipped it to a random page, which turned out to be p.26. On there he found what we would call the Galois selection rule for \( A_2^{(1)} \), analysed and solved for the cases where \( k + 3 \) is coprime to 6. Lang however didn’t know about physical invariants; he was reporting on work by Koblitz and Rohrlich on
decomposing the Jacobians of the Fermat curve $x^n + y^n = z^n$ into their prime pieces, called ‘simple factors’ in algebraic geometry. $n$ here corresponds to $k + 3$. Similarly, Itzykson discovered traces of the $A^{(1)}_2$ exceptionals — these occur when $k + 3 = 8, 12, 24$ — in the Jacobian of $x^{24} + y^{24} = z^{24}$. See [2] for further observations along these lines. These ‘coincidences’ are still far from understood. Nor is it known if, more generally, the $A^{(1)}_\ell$ level $k$ classification will somehow be related to the hypersurface $x^n_1 + \cdots + x^n_\ell = z^n$, for $n = k + \ell + 1$.

The $(u(1) \oplus \cdots \oplus u(1))^{(1)}$ classification has connections to rational points on Grassmannians. The Grassmannian is (essentially) the moduli space for the Narain compactifications of the (classical) lattice string. It would be very interesting to interpret other large families of physical invariants as special points on other moduli spaces.

These new connections relating various physical invariant classifications to other areas of math seem to indicate that although the physical invariant classifications are difficult, they could be well worth the effort and be of interest outside RCFT. Once the physical invariant lists are obtained, we will still have the fascinating task of explaining and developing all these mysterious connections. These thoughts keep me going!

Another motivation for completing these lists comes from their relation to subfactor theory in von Neumann algebras\textsuperscript{18}. These algebras (see e.g. [22]) can be thought of as symmetries of a (generally infinite) group. Their building blocks are called factors. Jones initiated the combinatorial study of subfactors $N$ of $M$ (i.e. inclusions $N \subseteq M$ where $M, N$ are factors), relating it to e.g. knots, and for this won a Fields medal in 1990. Jones assigned to each subfactor $N \subseteq M$ a numerical invariant called an ‘index’, a sort of (generally irrational) ratio of dimensions. Graphs (called principal and dual principal) are also associated to subfactors. A much more refined subfactor invariant, called a ‘paragroup’, has been introduced by Ocneanu. It is essentially equivalent to a (2+1)-dimensional topological field theory. Moreover, any RCFT can be assigned a paragroup, and any paragroup (via a process called asymptotic inclusion which is akin to Drinfeld’s quantum doubling of Hopf algebras) yields an RCFT. See [22] for details.

Böckenhauer-Evans [4] have recently developed this much further, and have clarified the fusion graph $\leftrightarrow$ physical invariant relation. The fusion graphs will correspond to subfactor principal graphs. In the work of Di Francesco-Petkova-Zuber, that relation seems to be only empirical (i.e. nonconceptual).

Subfactor theory together with singularity theory is our best hope at present for understanding and generalising the A-D-E meta-pattern.

\footnote{18 For reasons of necessity, in the following discussion I’ll take more liberties than usual in the presentation.}
Part 2. Monstrous Moonshine

2.1. Introduction

In 1978, John McKay made a very curious observation. One of the well-known functions of classical number theory is the $j$-function, given by

$$j(\tau) := \frac{(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} = \frac{\Theta_{E_8}(\tau)^3}{\eta(\tau)^{24}}$$

$$= q^{-1} + 744 + 196884 q + 21493760 q^2 + 86429970 q^3 + \cdots$$  

(2.1.1)

Here as elsewhere in this paper, $q = \exp[2\pi i \tau]$. Also, $\sigma_3(n) = \sum_{d|n} d^3$, $\Theta_{E_8}$ is the theta function of the $E_8$ root lattice, and $\eta$ is the Dedekind eta. What is important here are the values of the first few coefficients. What McKay noticed was that $196884 \approx 196883$. Closer inspection shows $21493760 \approx 21296876$, and $86429970 \approx 842609326$. In fact,

$$196884 = 196883 + 1$$  

(2.1.2a)

$$21493760 = 21296876 + 196883 + 1$$  

(2.1.2b)

$$86429970 = 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1$$  

(2.1.2c)

The numbers on the right-side are the dimensions of the smallest irreducible representations of the Monster finite simple group $\mathbb{M}$ (in 1978 it still wasn’t certain that $\mathbb{M}$ even existed so back then these numbers were merely conjectural). The same game could be played with other coefficients of the $j$-function. With numbers so large, it seemed to him doubtful that this numerology was merely a coincidence. On the other hand, it was hard to imagine any deep conceptual connection between the Monster and the $j$-function: they seem completely unrelated.

In November 1978 he mailed the ‘McKay equation’ (2.1.2a) to John Thompson. At first Thompson dismissed this as nonsense, but after checking the next few coefficients he became convinced. He then added a vital piece to the puzzle. It should be well-known that when one sees a nonnegative integer, it often helps to try to interpret it as the dimension of some vector space. Essentially, that is what McKay was proposing here. (2.1.2) are really hinting that there is a ‘graded’ representation $V$ of $\mathbb{M}$:

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots$$

where $V_{-1} = \rho_0$, $V_1 = \rho_1 \oplus \rho_0$, $V_2 = \rho_2 \oplus \rho_1 \oplus \rho_0$, $V_3 = \rho_3 \oplus \rho_2 \oplus \rho_1 \oplus \rho_0 \oplus \rho_0$, etc, where $\rho_i$ are the irreducible representations of $\mathbb{M}$ (ordered by dimension), and that

$$j(\tau) - 744 = \dim_q(V) := \dim(V_{-1}) q^{-1} + \sum_{i=1}^{\infty} \dim(V_i) q^i,$$  

(2.1.3)

19 ‘Well-known’ is math euphemism for ‘a basic result of which until recently we were utterly ignorant.’ As Conway later said, “the $j$-function was ‘well-known’ to other people, but not ‘well-known’ to me.”

20 This and other technical terms used in this introduction will be carefully explained in the following subsections. This section is merely offered as a quick overview.
the graded dimension of $V$.

Thompson suggested that we twist $\dim_q(V)$, i.e. that more generally we consider the series (now called the McKay-Thompson series)

$$T_g(\tau) := \text{ch}_{V,q}(g) = \text{ch}_{V,-1}(g) q^{-1} + \sum_{i=1}^{\infty} \text{ch}_{V_i}(g) q^i,$$

(2.1.4)

for each element $g \in \mathbb{M}$. The point is that, for any group representation $\rho$, the character value $\text{ch}_\rho(id.)$ equals the dimension of $\rho$, and so $T_{id.}(\tau) = j(\tau) - 744$ and we recover (2.1.2) as special cases. But there are many other possible choices of $g \in \mathbb{M}$. Thompson couldn’t guess what these functions $T_g$ would be, but he suggested that they too might be interesting. This is a nice thought: when we see a positive integer, we should try to interpret it as a dimension of a vector space; if there is a symmetry present, then it may act on the vector space — i.e. our vector space may carry a representation of that symmetry group — in which case we can apply the Thompson trick and see what if any significance the other character values have in our context.

Conway and Norton [12] did precisely what Thompson asked. Conway called it “one of the most exciting moments in my life” [11] when he opened Jacobi’s foundational (but 150 year old!) book on elliptic and modular functions and found that the first few terms of the McKay-Thompson series agreed perfectly with the first few terms of certain special functions, namely the Hauptmoduls of various genus 0 modular groups. Monstrous Moonshine was officially born.

The word ‘moonshine’ here is English slang for ‘unsubstantial or unreal’. It was chosen by Conway to convey as well the feeling that things here are dimly lit, and that Conway-Norton were ‘distilling information illegally’ from the Monster character table.

In fact the first incarnation of Moonshine goes back to Andrew Ogg in 1975. He was in France describing his result that the primes $p$ for which the group $\Gamma_0(p)^+$ has genus 0, are \{2, 3, 5, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}. $\Gamma_0(p)^+$ is the group generated by $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ and is the normaliser of $\Gamma_0(p)$ in $\text{SL}_2(\mathbb{R})$ (this sentence will make a little more sense after §2.3, but it isn’t important here to understand it). He also attended a lecture by Jacques Tits, who was describing a newly conjectured simple group. When Tits wrote down the prime decomposition of the order of that group (see (2.2.1) below), Ogg noticed its prime factors precisely equalled his list of primes. Presumably as a joke, he offered a bottle of Jack Daniels’ whisky to the first person to explain the coincidence.

The next step was accomplished by Griess in 1980, with the construction of the Monster\textsuperscript{21} $\mathbb{M}$, and with it the proof that the conjectured character table for $\mathbb{M}$ was correct. Griess did this by explicitly constructing the 196883-dimensional representation $\rho_1$; it turns out to have a (commutative nonassociative) algebra structure, now called the Griess algebra. Though this paper was clearly important, the construction was artificial and 100 pages long: since the Monster is presumably a natural mathematical object (see §2.2), an elegant construction for it should exist. This was ultimately accomplished in the mid 1980s with the construction by Frenkel-Lepowsky-Meurman [23] of the Moonshine module

\textsuperscript{21} Griess also came up with the symbol for the Monster; Conway came up with the name.
and its interpretation by Borcherds as a *vertex operator algebra*. The Griess algebra appears naturally in $V^2$, as we shall see. $V^2$ does indeed seem to be a ‘natural’ mathematical structure, and $\mathbb{M}$ is its automorphism group: in fact $V^2$ is the graded representation $V$ of $\mathbb{M}$ conjectured by McKay and Thompson.

Connections with physics (CFT) go back to Dixon-Ginsparg-Harvey [19] in 1988, in a paper titled “Beauty and the beast: Superconformal symmetry in a Monster module”. The Moonshine module $V^2$ can be interpreted as the string theory for a $\mathbb{Z}_2$-orbifold of free bosons compactified on the torus $\mathbb{R}^{24}/\Lambda_{24}$ ($\Lambda_{24}$ is the Leech lattice). Many aspects of Moonshine make complete sense within CFT, but some (e.g. the genus zero property) remain more obscure. (Though in 1987 Moore speculated that the 0-genus of $\Gamma_0(a)+$ could be related to the vanishing of the cosmological constant in certain string theories related to $\mathbb{M}$, and Tuite [57] related genus-zero with the conjectured uniqueness of $V^2$.) Nevertheless this helps make the words of Dyson ring prophetic: “I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the universe” [21].

Finally, in 1992 Borcherds [5] completed the proof of the Conway-Norton conjectures by showing $V^2$ is the desired representation $V$. The full conceptual relationship between the Monster and the Hauptmoduls (like $j$) seems to remain ‘dimly lit’, although much progress has been realised. This is a subject where it is much easier to conjecture than to prove, and we are still awash in unresolved conjectures.

McKay also noticed in 1978 that similar coincidences hold if $\mathbb{M}$ and $j(\tau)$ are replaced with the Lie group $E_8(\mathbb{C})$ and $(qj(q))^{\frac{1}{2}} = 1 + 248q + \cdots$. This turns out to be much easier to explain, and in 1980 both Kac and Lepowsky remarked that the unique level 1 highest-weight representation of the affine algebra $E_8(1)$ has graded dimension $(qj(q))^{\frac{1}{2}}$.

Moonshiners have a little chip on their shoulders. Modern math, they say, tends to be a little too infatuated with the pursuit of generalisations for generalisations’ sake. Surely a noble goal for math is to find interesting and fundamentally new theorems. It can be argued that both history and common-sense suggest that to this end it is most profitable to look simultaneously at both exceptional structures and generic structures, to understand the special features of the former in the context of the latter, and to be led in this way to a new generation of exceptional and generic structures. Moonshiners would sympathise with those biologists who study the duck-billed platypus and lungfish rather than hide them in the closet as monsters: BECAUSE they appear to be unique, those animals presumably have much to teach us about our general understanding of evolution, etc.

It often seems to people that Moonshine can’t be very deep: the Conway-Norton conjectures seem to be so finite and specialised. There only are 171 distinct McKay-Thompson series $T_g$ in Monstrous Moonshine, after all. The whole point though is to try to understand why the Monster and the Hauptmoduls are so related, and then to try to extend and apply this understanding to other contexts. Moonshine is still young, and our conjectures are finite (it is enough to check the first 1200 coefficients), and a slightly weaker form was quickly proved on a computer by Atkin, Fong and Smith [54]. However this sort of argument adds no light to Moonshine, and tells us nothing of $V$ except that it exists.

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\[ \text{[Footnote]} \]

Indeed the Moonshine conjectures are finite (it is enough to check the first 1200 coefficients), and a slightly weaker form was quickly proved on a computer by Atkin, Fong and Smith [54]. However this sort of argument adds no light to Moonshine, and tells us nothing of $V$ except that it exists.

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understanding remains incomplete. But already math has benefitted: e.g. we now have a natural definition of $\mathbb{M}$ (as the automorphism group of $V^\natural$), and Moonshine helped lead us to the rich structures of generalised Kac-Moody algebras and vertex operator algebras.

We will see that Moonshine involves the interplay between exceptional structures such as the number 24, the Leech lattice $\Lambda_{24}$, the Monster group $\mathbb{M}$, and the Moonshine module $V^\natural$, and generic structures such as modular functions, vertex operator algebras, generalised Kac-Moody algebras, and conformal field theories. The following sections will introduce the reader to many of these structures, as we use Moonshine as another happy excuse to take a second little tour through modern mathematics.

2.2. INGREDIENT #1: FINITE SIMPLE GROUPS AND THE MONSTER

A readable introduction to the basics of finite group representation theory is [25]. The finite simple groups are described in [33]; see also [11]. Group representations were introduced in §1.3.

A normal subgroup $H$ of a group is one obeying $gHg^{-1} = H$ for all $g \in G$. These are important because the set $G/H$ of ‘cosets’ $gH$ has a natural group structure precisely when $H$ is normal. Every group has two trivial normal subgroups: itself and $\{1\}$. If these are the only normal subgroups, the group is called simple. It is conventional to regard the trivial group $\{1\}$ as not simple (just as 1 is conventionally regarded as not prime). An alternate definition of a (finite) simple group $G$ is that if $\varphi : G \to H$ is any group homomorphism (i.e. structure-preserving map: $\varphi(gg') = \varphi(g)\varphi(g')$), then $\varphi$ is either constant (i.e. $\varphi(G) = \{1\}$), or $\varphi$ is one-to-one.

The importance of simple groups is provided by the Jordan-Hölder Theorem. By a ‘composition series’ for a group $G$, we mean a nested sequence

$$G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_k \supset H_{k+1} = \{1\}$$

of groups such that $H_i$ is normal in $H_{i-1}$, and $H_{i-1}/H_i$ (called a ‘composition factor’) is simple. Any finite group $G$ has at least one composition series. If $H'_0 \supset \cdots \supset H'_{k+1} = \{1\}$ is a second composition series for $G$, then Jordan-Hölder says that $k = \ell$ and, up to a reordering $\pi$, the simple groups $H_{i-1}/H_i$ and $H'_{\pi_j - 1}/H'_{\pi_j}$ are isomorphic.

For example, the cyclic group $\mathbb{Z}_n$ of order (=size) $n$ — you can think of it as the integers modulo $n$ under addition — is simple iff $n$ is prime. Consider the group $\mathbb{Z}_{12} = \langle 1 \rangle$. Two composition series are

$$\mathbb{Z}_{12} \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \langle 0 \rangle$$
$$\mathbb{Z}_{12} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \langle 0 \rangle$$

corresponding to composition factors $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbb{Z}_2$. Of course this is consistent with Jordan-Hölder. This is reminiscent of the fact that $2 \cdot 2 \cdot 3 = 3 \cdot 2 \cdot 2$ are both prime factorisations of 12.

There is some value to regarding finite groups as a massive generalisation of the notion of number. The number $n$ can be identified with the cyclic group $\mathbb{Z}_n$. The divisor of a
number corresponds to a normal subgroup, so a prime number corresponds to a simple group. The Jordan-Hölder Theorem generalises the uniqueness of prime factorisations. That you can build up any number by multiplying primes, is generalised to building up a group by semi-direct products (more generally, by group extensions): if $H$ is a normal subgroup of $G$, then $G$ will be an extension of $H$ by the quotient group $G/H$.

Note however that $\mathbb{Z}_6 \times \mathbb{Z}_2$ and $\mathfrak{S}_3 \times \mathbb{Z}_2$ — both different from $\mathbb{Z}_{12}$ — will also have $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3$ as composition factors: unlike for numbers, ‘multiplication’ here does not give a unique answer. The semidirect product $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ can equal either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, depending on how the product is taken. More precisely, the notation $G \rtimes G'$ means a group where every element can be written uniquely as a pair $(g, g')$, for $g \in G$ and $g' \in G'$, and where the group operation is $(g, g')(h, h') = (\text{stuff}, gh)$.

Thus simple groups have an importance for group theory approximating what primes have for number theory. One of the greatest accomplishments of twentieth century math is surely the classification of the finite simple groups. (On the other hand, group extensions turn out to be technically quite difficult and leads one into group cohomology.) This work, completed in the early 1980s (although gaps are continually being discovered and filled in the arguments), runs to approximately 15 000 journal pages, spread over 500 individual papers, and is the work of a whole generation of group theorists. A modern revision is currently underway (see e.g. [34]) to simplify the proof and find and fill all gaps, but the final proof is still expected to be around 4000 pages long. The resulting list is:

- the cyclic groups $\mathbb{Z}_p$ ($p$ a prime);
- the alternating groups $\mathfrak{A}_n$ for $n \geq 5$;
- 16 families of Lie type;
- 26 sporadics.

We’ve already met the cyclic groups. The alternating group $\mathfrak{A}_n$ consists of the even permutations in the symmetric group $\mathfrak{S}_n$, and so has order ($=\text{size}$) $\frac{1}{2} n!$. The groups of Lie type are essentially Lie groups defined over finite fields $\mathbb{F}_q$ (such as $\mathbb{Z}_p$), sometimes ‘twisted’ in certain senses. The simplest example is $\text{PSL}_n(\mathbb{F}_q)$, which consists of the $n \times n$ matrices with entries in $\mathbb{F}_q$, with determinant 1, quotiented out by the centre of $\text{SL}_n(\mathbb{F}_q)$ (namely the scalar matrices $\text{diag}(a, a, \ldots, a)$ for $a^n = 1$) (except for $\text{PSL}_2(\mathbb{Z}_2)$ and $\text{PSL}_2(\mathbb{Z}_3)$, which aren’t simple).

Note that the determinant $|\rho(g)|$ for any representation $\rho$ of any (noncyclic) simple group must be 1, otherwise we would violate the homomorphism definition of simple group (try to see why). Also, the centre of any (noncyclic) simple group must be trivial (why?). The smallest noncyclic simple group is $\mathfrak{A}_5$, with order 60.\textsuperscript{24} It is the same as (isomorphic to) $\text{PSL}_2(\mathbb{Z}_5)$ and $\text{PSL}_2(\mathbb{F}_4)$, and can also be expressed as the group of all rotations (reflections have determinant $-1$ and so cannot belong to any simple group) of $\mathbb{R}^3$ that bring a regular icosahedron back to itself.

\textsuperscript{23} There is a finite field with $q$ elements, iff $q$ is a power of a prime. For each such $q$, there is only 1 field of that size. The field with prime $p$ elements is the integers taken mod $p$.

\textsuperscript{24} This implies, incidentally, that if $G$ and $H$ are any two groups with the same order below 60, then they will have the same composition factors.
The smallest sporadic group is the Mathieu group $M_{11}$, order 7920, discovered in 1861\textsuperscript{25}. The largest is the Monster $\mathbb{M}$, conjectured by Fischer and Griess in 1973 and finally proved to exist by Griess in 1980. Its order is\textsuperscript{26}

$$||\mathbb{M}|| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \times 10^{53}.$$ (2.2.1)

20 of the 26 sporadics are involved in (i.e. are quotients of subgroups of) the Monster. Some relations among $\mathbb{M}$, the Leech lattice $\Lambda_{24}$ and the largest Mathieu group $M_{24}$ are given in Chapters 10 and 29 of [13].

Moonshine hints at a tantalising connection between the classification of finite simple groups, and the classification of RCFTs discussed in Part 1. Speculates [23] (page xli): “One can certainly hope for a uniform description of the finite simple groups as automorphism groups of certain vertex operator algebras — or conformal quantum field theories. If such a quantum field theory could somehow be attached a priori to a finite simple group, the classification of such theories, a problem of great current interest among string theorists, might some day be part of a new approach to the classification of the finite simple groups. On the other hand, can the known classification of the finite simple groups help in the classification of conformal field theories?”

2.3. Ingredient #2: Modular functions and Hauptmoduls

A readable introduction to some of the topics discussed in this section is [16,42,60].

We know from complex analysis that the group $\text{SL}_2(\mathbb{R})$ of $2 \times 2$ matrices with real entries and determinant 1, acts on the upper-half plane $\mathcal{H} = \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \}$ by fractional linear (or Möbius) transformations:

$$(a \ b \ c \ d) \cdot \tau := \frac{a \tau + b}{c \tau + d}.$$ (2.3.1)

For example the matrix $S:=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to the function $\tau \mapsto -1/\tau$, while the matrix $T:=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ corresponds to the translation $\tau \mapsto \tau + 1$. Since $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ correspond to the same Möbius transformation, strictly speaking our group here is $\text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{\pm I\}$.

The only reason this action (2.3.1) of the $2 \times 2$ matrices on complex numbers (or more precisely the Riemann sphere $\mathbb{C} \cup \{\infty\}$) might not look strange to us, is because familiarity breeds numbness. What we really have is a natural action of $n \times n$ matrices on $\mathbb{C}^n$, and this

\textsuperscript{25} Although his arguments apparently weren’t very convincing. In fact some people, including the Jordan of Jordan-Hölder fame, argued in later papers that the largest of Mathieu’s sporadic groups couldn’t exist.

\textsuperscript{26} The inquisitive reader, hungry for more ‘coincidences’, may have noticed that 196883 and 21296876 — see (2.1.2) — exactly divide the order of the Monster. Indeed this will hold for any finite group: the dimensions of the irreducible representations of a finite group will always divide its order.
induces their action on \( \mathbb{C}^{n-1} \) (together with a codimension-2 set of ‘points at infinity’) by interpreting \( \mathbb{C}^n \) as projective coordinates for \( \mathbb{C}^{n-1} \). Specialising to \( n = 2 \) gives us (2.3.1). In projective geometry, ‘parallel lines’ intersect at \( \infty \). Projective coordinates allow one to treat ‘finite’ and ‘infinite’ points on an equal footing.

Consider \( \Gamma := \text{SL}_2(\mathbb{Z}) \), the subgroup of \( \text{SL}_2(\mathbb{R}) \) consisting of the matrices with integer entries. It can be shown that it is generated by \( S \) and \( T \) (in other words, every matrix \( \alpha \in \Gamma \) can be expressed as a monomial in \( S \) and \( T \)). For reasons that will be clear shortly, consider the extended upper-half plane \( \mathbb{H} := \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q} \) — the extra points \( \{i\infty\} \cup \mathbb{Q} \) are called cusps. \( \Gamma \) acts on \( \mathbb{H} \) (e.g. \( S \) interchanges 0 and \( i\infty \)). By a modular function for \( \Gamma \), we mean a meromorphic function \( f : \mathbb{H} \to \mathbb{C} \), symmetric with respect to \( \Gamma \): i.e. \( f(\alpha(\tau)) = f(\tau) \) for all \( \alpha \in \Gamma \). Note that we require \( f \) to be meromorphic at the cusps (e.g. polynomials are meromorphic at \( i\infty \), but \( e^z \) is not).

It is not obvious why modular functions should be interesting, but in fact they are one of the most fundamental notions in modern number theory (see the last paragraph of §1.6).

For example, consider the question of writing numbers as sums of squares. We can write

\[ 5 = 1^2 + (-2)^2 = (-1)^2 + 1^2 + 1^2 + (-1)^2, \]

to give a couple of trivial examples. Let \( N_n(k) \) be the number of ways we can write the integer \( n \) as a sum of \( k \) squares, counting order and signs. For example \( N_5(1) = 0 \) (since 5 is not a perfect square), \( N_5(2) = 8 \) (since \( 5 = (\pm 1)^2 + (\pm 2)^2 = (\pm 2)^2 + (\pm 1)^2) \), \( N_5(3) = 24 \), etc. Their generating functions are:

\[
\sum_{n=0}^{\infty} N_n(k) q^n = (\theta_3(q))^k,
\]

where

\[ \theta_3(q) = 1 + 2q + 2q^4 + \cdots = \sum_{n \in \mathbb{Z}} q^{n^2} \]

is called a theta function. It turns out that \( \theta_3 \) transforms nicely with respect to \( \Gamma \), once we make the change-of-variables \( q = \exp[\pi i \tau] \). This takes work to show. For example, \( \theta_3 \) is clearly invariant under the action of \( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \), and a little work (from e.g. Poisson summation) shows that \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) takes \( \theta_3(\tau) \) to \( \sqrt{\frac{i}{\tau}} \theta_3(\tau) \). \( \theta_3 \) is not precisely a modular function (it is a ‘modular form of weight \( \frac{1}{2} \)’ for \( \Gamma_0(4) \)), but this simple example illustrates the point that \( \Gamma \) (and related groups) appear throughout number theory. More on this shortly.

That important change-of-variables \( q = \exp[\pi i \tau] \) was introduced by Jacobi early last century, in his analysis of ‘elliptic integrals’. The theory is beautiful and poorly remembered today, which is very disappointing considering how much of modern math was touched by it. I strongly recommend the book [10], written over a century ago; the style and motivation of math in our century is different from that in Jacobi’s, and we’ve lost a little in motivation what we’ve gained in power. I’ll briefly sketch Jacobi’s theory.
Just as we could develop a theory of ‘circular functions’ (i.e. sine etc.) starting from the integral \( s(a) = \int_0^a \frac{dx}{\sqrt{1-x^2}} \), so can we develop a theory of ‘elliptic functions’ starting from the ‘elliptic integral’ \( F(k, a) = \int_0^a \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \). Inverting \( s(a) \) gives a function both more useful and with nicer properties than \( s(a) \); we call it \( \sin(u) \). Similarly, for any \( k \) the elliptic function \( \text{sn}(k, u) \) is defined by \( u = F(k, \text{sn}(k, u)) \). Just as we can define a numerical constant \( \pi \) by \( \sin \left( \frac{1}{2} \pi \right) = 1 \) (i.e. \( \frac{1}{2} \pi = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \)), we get a function \( K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \). Just as \( \sin(u) \) has period \( 4(\frac{1}{2} \pi) \), so has its \( u \)-period \( 4K(k) \). \( \text{sn} \) also turns out to have \( u \)-period \( 4iK(k') \) where \( k' = \sqrt{1-k^2} \) — today we take this as the starting point and define an elliptic function to be doubly periodic (see [42] or Cohen in [60]).

The theta functions aren’t elliptic functions, but they are closely related, and e.g. \( \text{sn} \) can be written as a quotient of them. In Jacobi’s language, we have

\[
\theta_3 \left( \frac{iK(k')}{K(k)} \right) = \sqrt{\frac{2K(k)}{\pi}}.
\]

The ‘modular transformation’ \( \tau \mapsto \frac{1}{\tau} \) corresponds to interchanging the ‘modulus’ \( k \) with the ‘complementary modulus’ \( k' \), and thus is completely natural in Jacobi’s theory. The important formula \( \theta_3 \left( \frac{1}{\tau} \right) = \sqrt{\frac{1}{\tau}} \theta_3 (\tau) \) is trivial here.

A certain interpretation of modular functions also indicates their usefulness, and played an important role in Part 1. A torus is something that looks like the surface of a bagel, at least as far as its topology is concerned. For example, the Cartesian product \( S^1 \times S^1 \) of circles is a torus (think of one circle being the contact-circle of the bagel with the table on which it rests, then from each point on that horizontal circle imagine placing a vertical circle perpendicular to it, like a rib; together all these ribs fill out the bagel’s surface). A more sophisticated example of a torus is an elliptic curve (a complex curve of genus \( 1 \), of a bagel, at least as far as its topology is concerned. For example, the Cartesian product \( C \times \Lambda \) of the complex plane \( C \) with a 2-dimensional lattice \( \Lambda \) (we saw lattices in §1.6; \( \Lambda \) here will be a discrete doubly-periodic set of points in \( C \), containing \( 0 \). It turns out that certain equivalence classes of tori (e.g. with respect to conformal or complex-analytic equivalence) always contain a representative torus of the form \( C/\Lambda \), where \( \Lambda \) consists of all points \( \mathbb{Z} + \mathbb{Z} \tau \), for some \( \tau \in \mathcal{H} \). (Incidentally, the cusps correspond to degenerate tori.) In other words, these equivalence classes are parametrized by complex numbers \( \tau \) in \( \mathcal{H} \). So if we have a complex-valued function \( F \) on the set of all tori, which is e.g. conformally invariant (an example is the genus-one partition function \( Z \) in conformal field theories — see §1.1), then we can consider \( F \) as a well-defined function \( F : \mathcal{H} \to \mathbb{C} \). However, it turns out that different points \( \tau \) in \( \mathcal{H} \) correspond to the same equivalence class of tori: e.g. the lattice for \( \tau \) is the same as that for \( \tau + 1 \), and these are a rescaling of that for \(-1/\tau \). Thus \( F(\tau) = F(\tau + 1) = F(-1/\tau) \), because \( \tau, \tau + 1, -1/\tau \) all represent equivalent tori. Since \( \tau \mapsto \tau + 1 \) and \( \tau \mapsto -1/\tau \) generate \( \text{PSL}_2(\mathbb{Z}) \), what in fact we find is that \( F \) has \( \Gamma \) as its group of symmetries. One often says that \( \Gamma \) is the ‘modular group of the torus’, and that the orbit space \( \Gamma \backslash \mathcal{H} \) is the ‘moduli space’ of (conformal equivalence classes of) tori. \( \mathcal{H} \) is called its ‘Teichmüller space’ or ‘universal cover’. This is exactly
analogous to \( S^1 = \mathbb{R}/\mathbb{Z} \): \( \mathbb{R} \) is its universal cover and \( \mathbb{Z} \) is its ‘modular group’ (or ‘mapping class group’). Another example: the Teichmüller space for (conformal equivalence classes of) ‘pair-of-pants’, or equivalently a disc minus two open interior disks, is \( \mathbb{R}^3 \) (an ordered triple), while its modular group is the symmetric group \( S_3 \) and its moduli space consists of unordered triples. Incidentally, we write \( \Gamma \backslash \mathcal{H} \) instead of \( \mathcal{H}/\Gamma \) because the group \( \Gamma \) acts on \( \mathcal{H} \) ‘on the left’. A good introduction to the geometry here is [55].

In any case, a surprising number of innocent-looking questions in number theory can be dragged (usually with effort) into the richly developed realm of elliptic curves and modular functions, and it is there they are often solved. For instance, we all know the ancient Greeks were interested in Pythagorean triples: find all integer solutions \( a, b, c \) to \( a^2 + b^2 = c^2 \), i.e. find all integer (or if you prefer, rational) right-angle triangles. They solved this by elementary means: choose any integers (or rationals) \( x, y \) and put \( u = \frac{x^2-y^2}{x^2+y^2}, v = \frac{2xy}{x^2+y^2} \); then \( u^2 + v^2 = 1 \) and (multiplying by the denominator) this gives all Pythagorean triples.

There are two ways of extending this problem. One is to ask which \( n \in \mathbb{Z} \) can arise as areas of these rational right-angle triangles. It turns out \( n = 5 \) is the smallest one: \( a = \frac{3}{2}, b = \frac{20}{3}, c = \frac{41}{6} \) works (5 = \( \frac{1}{2} \) \( \left( \frac{3}{2} \right) \left( \frac{20}{3} \right) \) and \( \left( \frac{3}{2} \right)^2 + \left( \frac{20}{3} \right)^2 = \left( \frac{41}{6} \right)^2 \)). This is a hard problem — just try to show \( n = 1 \) cannot work. \( n = 157 \) turns out to work: the simplest triangle has \( a \) and \( b \) as quotients of integers of size around \( 10^{25} \), and \( c \) as the quotient of integers around \( 10^{47} \). Although this problem was studied by the ancient Greeks and also by the Arabs in the 10th century, it was finally cracked in the 1980s. It was solved by first translating it into the question of whether the elliptic curve \( y^2 = x^3 - n^2 x \) has infinitely-many rational points, and then applying all the rich 20th century machinery to answering that question.

The other continuation of the Pythagorean triples question is more famous: find all integer solutions to \( a^n + b^n = c^n \) (or equivalently all rational solutions to \( a^n + b^n = 1 \)). 350 years ago Fermat wrote in the margin of the book he was reading (the book was describing the Greek solution to Pythagorean triples) that he had found a “truly marvelous” proof that for \( n > 2 \) there are no nontrivial solutions, but that the margin was too narrow to contain it. This result came to be known as ‘Fermat’s Last Theorem’\(^{28} \) and despite considerable effort no one has succeeded in rediscovering his proof. Most people today believe that Fermat soon realised his ‘proof’ wasn’t valid, otherwise he would have alluded to it in later letters. In any case, a very long and complicated proof was finally achieved in the 1990s: the ‘Taniyama conjecture’ says that a certain function associated to any elliptic curve over \( \mathbb{Q} \) will be modular; if \( a^n + b^n = c^n \) for some \( n > 2 \), then the elliptic curve \( y^2 = x^3 + (a^n - b^n)x^2 - a^n b^n \) will violate the Taniyama conjecture; finally Wiles proved the Taniyama conjecture is true.

To most mathematicians, the ‘area-\( n \) problem’ and ‘Fermat’s Last Theorem’ are interesting only because they can be related to elliptic curves and modular forms — it’s easy to ask hard questions in math, but most questions tend to be stale. Number theory is infatuated with modular stuff because (in increasing order of significance) (a) it’s exceedingly rich, with lots of connections to other areas of math and math phys; (b) it’s

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\(^{28}\) It was called his ‘Last Theorem’ because it was the last of his 48 margin notes to be proved by other mathematicians — another one is discussed in Section 1.8. The story of Fermat’s Last Theorem is a fascinating one, but alas this footnote is too small to do it justice. See for instance the excellent book [53].
a battleground on which many innocent-looking but hard-to-crack problems can be slain; and (c) last generation’s number theorists also worked on modular stuff.

In any case, modular functions turn out to be important for math (and mathematical physics) even though they may at first glance look artificial. Poincaré explained how to study them. He said to look at the orbits of $\mathcal{H}$ with respect to $\Gamma$. For example, one orbit, hence one point in $\Gamma \backslash \mathcal{H}$, contains all cusps. We write this as $\Gamma \backslash \mathcal{H}$, and give it the natural topological structure (i.e. 2 points $[\tau], [\tau'] \in \Gamma \backslash \mathcal{H}$ are considered ‘close’ if the 2 sets $\Gamma \tau, \Gamma \tau'$ nearly overlap). Note first that by applying $T$ repeatedly, every point in $\mathcal{H}$ corresponds to a point in the vertical strip $-\frac{1}{2} \leq \text{Re}(\tau) \leq \frac{1}{2}$ — in fact to a unique point in that strip, if we avoid the two edges. $S$ is an inversion through the unit circle, so it permits us to restrict to those points in the vertical strip which are distance at least 1 from the origin. The resulting region $R$ is called a fundamental region for $\Gamma$. Apart from the boundary of $R$, every $\Gamma$-orbit will intersect $R$ in one and only one point.

What should we do about the boundary? Well, the edge $\text{Re}(\tau) = -\frac{1}{2}$ gets mapped by $T$ to the edge $\text{Re}(\tau) = \frac{1}{2}$, so we should identify (=glue together) these. The result is a cylinder running off to infinity, with a strange lip at the bottom. $S$ tells us how we should close that lip: identify $i e^{i \theta}$ and $i e^{-i \theta}$. This seals the bottom of the cylinder, so we get an infinitely tall cup with a strangely puckered base. In fact the top of this cup is also capped off, by the cusp $i \infty$. So what we have (topologically speaking) is a sphere. It does not look like a smooth sphere, but in fact it inherits the smoothness of $\mathcal{H}$.

Incidentally, topological manifolds of dimension $\leq 3$ always have a unique compatible smooth structure. ‘Topological structure’ means you can speak of continuity or closeness, ‘smooth structure’ means you can also do calculus. On the other hand, $\mathbb{R}^4$ has infinitely many smooth structures compatible with its topological structure; mysteriously, all other Euclidean spaces $\mathbb{R}^n$ have a unique smooth structure! Thus both mathematics and physics single out 4-space. Coincidence???

So anyways, what this construction of $\Gamma \backslash \mathcal{H}$ means is that a modular function can be reinterpreted as a meromorphic complex-valued function on this sphere. This is very useful, because our undergraduate complex variables class taught us all about meromorphic complex-valued functions $f$ on the Riemann sphere $\mathbb{C} \cup \infty$. There are many meromorphic functions on $\mathbb{C}$, but to also be meromorphic at $\infty$ forces $f$ to be rational, i.e. $f(w) = \frac{P(w)}{Q(w)}$, where $w$ is the complex parameter on the Riemann sphere. So our modular function $f(\tau)$ will simply be some rational function $P/Q$ evaluated at the change-of-variables function $w = c(\tau)$ which maps us from our sphere $\Gamma \backslash \mathcal{H}$ to the Riemann sphere. There are many different choices for this function $c(\tau)$, but the standard one is $c(\tau) = j(\tau)$, the $j$-function of (2.1.1)$^{29}$. Thus, any modular function can be written as a rational function $f(\tau) = P(j(\tau))/Q(j(\tau))$ in the $j$-function. Conversely, any such function will be modular.

This is analogous to saying that any function $g(x)$ periodic under $x \mapsto x + 1$ can be thought of as a function on the unit circle $S^1 \subset \mathbb{C}$ evaluated at the change-of-variables function $x \mapsto e^{2\pi i x}$, and hence has a Fourier expansion $\sum_n g_n \exp[2\pi inx]$.

---

$^{29}$ Historically, $j$ was the standard choice, but in Moonshine the preferred choice would be the function $J = j - 744$ with zero constant term.
We can generalise this argument. Consider a subgroup $G$ of $\text{SL}_2(\mathbb{R})$ which is both not too big, and not too small. ‘Not too big’ means it should be discrete, i.e. the matrices in $G$ can only get so close to the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. To make sure $G$ is ‘not too small’, it is enough to require that $G$ contains some subgroup of the form $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$, i.e. $G$ must contain all matrices in $\Gamma$ whose bottom-left entry is a multiple of $N$. So $G$ must contain $T$, for example. We will also be interested only in those $G$ which obey

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in G \Rightarrow t \in \mathbb{Z}$$

i.e. the only translations in $G$ are by integers. We will call a function $f : \mathcal{H} \to \mathbb{C}$ a modular function for $G$ if it is meromorphic (including at the cusps $\mathbb{Q} \cup \{i\infty\}$), and if also $f$ is symmetric with respect to $G$: $f \circ \alpha = f$ for all $\alpha \in G$. This implies we will be able to expand $f$ as a Laurent series in $q$. We analyse this as before: look at the orbit space $\Sigma = G \setminus \mathcal{H}$; because $G$ is not too big, $\Sigma$ will be a (Riemann) surface; because $G$ is not too small, $\Sigma$ will be compact.

The compact Riemann surfaces have been classified (up to homeomorphism — i.e. considering only topology as relevant), and are characterised by a number called the genus. Genus 0 is a sphere, genus 1 is a torus, genus 2 is like two tori resting side-by-side, etc. For example, the surface of a wine glass, or a fork, is topologically a sphere, while a coffee cup and a key will (usually) be tori. Eye glasses with the lenses popped out is a 2-torus, while a ladder with $n$ rungs on it has genus $n - 1$.

We will call $G$ ‘genus $g$’ if its surface $\Sigma$ has genus $g$. For example, $G = \Gamma_0(2)$ and $G = \Gamma_0(25)$ are both genus 0, while $\Gamma_0(50)$ is genus 2 and $\Gamma_0(24)$ is genus 3. Once again, we are interested here in the genus 0 case. As before, this means that there is a change-of-variables function we’ll denote $J_G$ which has the property that it’s a modular function for $G$, and all other modular functions for $G$ can be written as a rational function in it. Because of (2.3.2), we can choose $J_G$ to look like

$$J_G(\tau) = q^{-1} + a_1(G) q + a_2(G) q^2 + \cdots$$

So $J_G$ plays exactly the same role for $G$ that $J := j - 744$ plays for $\Gamma$. $J_G$ is called the Hauptmodul for $G$. (Incidentally for genus $> 0$, two generators, not one, are needed.)

For example, $\Gamma_0(2)$, $\Gamma_0(13)$ and $\Gamma_0(25)$ are all genus 0, with Hauptmoduls

$$J_2(\tau) = q^{-1} + 276 q - 2048 q^2 + 11202 q^3 - 49152 q^4 + 184024 q^5 + \cdots$$

$$J_{13}(\tau) = q^{-1} - q + 2 q^2 + q^3 + 2 q^4 - 2 q^5 - 2 q^7 - 2 q^8 + q^9 + \cdots$$

$$J_{25}(\tau) = q^{-1} - q + q^4 + q^6 - q^{11} - q^{14} + q^{21} + \cdots$$

The smaller the modular group, the smaller the coefficients of the Hauptmodul. In this sense, the $j$-function is optimally bad among the Hauptmoduls: e.g. for it $a_{23} \approx 10^{25}$. 

45
An obvious question is, how many genus 0 groups (equivalently, how many Hauptmoduls) are there? It turns out that \( \Gamma_0(p) \) is genus 0, for a prime \( p \), iff \( p - 1 \) divides 24. Thompson in 1980 proved that for any \( g \), there are only finitely many genus \( g \) groups obeying our two conditions (2.3.2). In particular this means there are only finitely many Hauptmoduls. Over 600 Hauptmoduls with integer coefficients \( a_i(G) \) are presently known.

2.4. THE MONSTROUS MOONSHINE CONJECTURES

We are now ready to make precise the main conjecture of Conway and Norton [12]. (We should emphasise though that there have been several other conjectures, some of which turned out to be partially wrong.)

They conjectured that for each element \( g \) of the Monster \( \mathbb{M} \), there is a Hauptmodul

\[
J_g(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n(g) q^n
\]  

(2.4.1)

for a genus 0 group \( G_g \) such that each coefficient \( a_n(g) \) is an integer, and for each \( n \) the map \( g \mapsto a_n(g) \) is a character of \( \mathbb{M} \). They also conjectured that \( G_g \) contains \( \Gamma_0(N) \) as a normal subgroup, for some \( N \) depending on the order of \( g \).

Another way of saying this is that there exists an infinite-dimensional graded representation \( V = V_{-1} \oplus \bigoplus_{n=1}^{\infty} V_n \) of \( \mathbb{M} \) such that the McKay-Thompson series \( T_g(\tau) \) in (2.1.4) is a Hauptmodul.

There are around \( 8 \times 10^{53} \) elements to the Monster, so naively we may expect around \( 8 \times 10^{53} \) different Hauptmoduls \( J_g = T_g \). However the character of a representation evaluated at \( g \) and at \( hgh^{-1} \) will always be the same, so \( J_g = J_{hgh^{-1}} \). Hence the relevant quantity is the number of conjugacy classes, which for \( \mathbb{M} \) is only 194. Moreover, a character evaluated at \( g^{-1} \) will always be the complex conjugate of its value at \( g \), but here all character values \( \chi_{V_n}(g) \) are integers (according to the conjecture). Thus \( J_g = J_{g^{-1}} \). The total number of distinct Hauptmoduls \( J_g \) arising in Monstrous Moonshine turns out to be only 171.

For example, if we choose \( g \) to be the identity, we recover \( T_{id} = J \). It turns out that there are precisely 2 different conjugacy classes of order 2 elements, one of them giving the Hauptmodul \( J_2 \) in (2.3.3). Similarly for 13, but \( J_{25} \) doesn’t correspond to any conjugacy class of \( \mathbb{M} \).

Moonshine provides an explanation for a forgotten mystery of classical mathematics: why are the coefficients of the \( j \)-function positive integers? On the other hand, that they are integers has long been important to number theory (complex multiplication, class field theory — see e.g. [16]).

There are lots of other less important conjectures. One which played a role in ultimately proving the main conjecture involves the replication formulae. Conway-Norton want to think of the Hauptmoduls \( J_g \) as being intimately connected with \( \mathbb{M} \); if so, then the group structure of \( \mathbb{M} \) should somehow directly relate different \( J_g \). In particular, consider the power map \( g \mapsto g^p \). Now, it was well-known that \( j(\tau) \) has the property that \( j(p\tau) + j\left(\frac{\tau}{p}\right) + j\left(\frac{\tau+1}{p}\right) + \cdots + j\left(\frac{\tau+p-1}{p}\right) \) equals a polynomial in \( j \), for any prime \( p \) (sketch
of proof: it’s a modular function for \( \Gamma \), and hence equals a rational function of \( j \); since its only poles will be at the cusps, the denominator polynomial must be trivial). Hence the same will hold for \( J \). Explicitly we get

\[
J(2\tau) + J\left(\frac{\tau}{2}\right) + J\left(\frac{\tau + 1}{2}\right) = J^2(\tau) - 2a_1 \\
J(3\tau) + J\left(\frac{\tau}{3}\right) + J\left(\frac{\tau + 1}{3}\right) + J\left(\frac{\tau + 2}{3}\right) = J^3(\tau) - 3a_1J(\tau) - 3a_2
\]

where \( J(\tau) = \sum_k a_kq^k \). Slightly more complicated formulas hold in fact for any composite \( n \). Conway and Norton conjectured that these formulas have an analogue for the Moonshine functions \( J_g \) in (2.4.1). In particular, (2.4.2) become for any \( g \in \mathbb{M} \)

\[
J_{g^2}(2\tau) + J_g\left(\frac{\tau}{2}\right) + J_g\left(\frac{\tau + 1}{2}\right) = J_{g^2}^2(\tau) - 2a_1(g) \tag{2.4.3a}
\\
J_{g^3}(3\tau) + J_g\left(\frac{\tau}{3}\right) + J_g\left(\frac{\tau + 1}{3}\right) + J_g\left(\frac{\tau + 2}{3}\right) = J_{g^3}^3(\tau) - 3a_1(g)J_g(\tau) - 3a_2(g) \tag{2.4.3b}
\]

These are examples of the replication formulae.

‘Replication’ concerns the power map \( g \mapsto g^n \) in \( \mathbb{M} \). Can Moonshine see more of the group structure of \( \mathbb{M} \)? A step in this direction was made by Norton [47], who associated a Hauptmodul to commuting elements \( g, h \) in \( \mathbb{M} \). Physically [19], this corresponds to orbifold traces, i.e. the \( V^g \) RCFT with boundary conditions twisted by \( g \) and \( h \) in the ‘time’ and ‘space’ directions. Still, we would like to see more of \( \mathbb{M} \) in Moonshine.

An important part of the Monstrous Moonshine conjectures came a few years after [12]. Frenkel-Lepowsky-Meurman [23] constructed a graded infinite-dimensional representation \( V^g \) of \( \mathbb{M} \) and conjectured (correctly) that it is the representation in (2.1.4). \( V^g \) has a very rich algebraic structure, which will be discussed in §2.6.

A major claim of [23] was that \( V^g \) is a ‘natural’ structure (hence their notation). To see what they mean by that, it’s best to view another simpler example of a natural construction: that of the Leech lattice \( \Lambda_{24} \). Recall the discussion of (root) lattices in §1.6. \( \Lambda_{24} \) is one of the most interesting lattices, and is related to Moonshine. It can be defined using ‘laminated lattices’. Start with the 0-dimensional lattice \( \Lambda_0 = \{0\} \), which consists of just a single point. Use it to construct a 1-dimensional lattice, with minimal (nonzero) norm 4, built out of infinitely many copies of \( \Lambda_0 \) laid side by side. The result of course is simply the even integers \( 2\mathbb{Z} \), which we will call here \( \Lambda_1 \). Now construct a 2-dimensional lattice, of minimal norm 4, built out of infinitely many copies of \( \Lambda_1 \) laid next to each other. There are lots of ways to do this, but choose the densest lattice possible. The result is unique: it is the hexagonal lattice \( A_2 \) scaled by a factor of \( \sqrt{2} \): call it \( \Lambda_2 \). Continue in this way: \( \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6, \Lambda_7, \) and \( \Lambda_8 \) will be the root lattices \( A_3, D_4, D_5, E_6, E_7 \) and \( E_8 \), respectively, all scaled by \( \sqrt{2} \). See [13] chapter 6 for a more complete treatment of laminated lattices.

The 24th repetition of this construction yields the Leech lattice. It is the unique 24-dimensional self-dual lattice with no norm-2 vectors, and provides among other things the densest known packing of 23-dimensional spheres in \( \mathbb{R}^{24} \). Many of its properties are discussed throughout [13]. So lamination provides us with a sort of no-input construction
of the Leech lattice, and a good example of the mathematical meaning of ‘natural’. After
dimension 24, it seems chaos results from the lamination procedure (there are 23 different
25-dimensional lattices that have an equal right to be called Λ25, and over 75 000 are
expected for Λ26).

It is natural to ask about Moonshine for other groups. There is a partial Moonshine
for the Mathieu groups M24 and M12 (which have about $2 \times 10^8$ and $10^5$ elements resp.),
the automorphism group .0 of Λ24 (which has about $8 \times 10^{18}$ elements), and a few others —
see e.g. [49]. These groups are either simple or almost simple (e.g. .0 is the direct product
of $\mathbb{Z}_2$ with the simple group .1). More generally, there will be some sort of Moonshine
for any group which is the automorphism group of a vertex operator algebra; the finite simple
groups of Lie type should be automorphism groups of VOAs closely related to the affine
algebras except defined over fields like $\mathbb{Z}_p$.

There is a geometric side to Moonshine, associated to names like Lian-Yau and
Hirzebruch. In particular, Hirzebruch’s ‘prize question’ asks for the construction of a
24-dimensional manifold on which $\mathbb{M}$ acts, whose twisted elliptic genus are the McKay-
Thompson series. This is still open.

It should be emphasised that Monstrous Moonshine is a completely unexpected con-
nection between finite groups and modular functions. Although there has been enormous
progress in our understanding of this connection (so much so, that Richard Borcherds
won the 1998 Fields medal for his work on this), there still is mystery at its heart. In
particular, that $\mathbb{M}$ is associated with modular functions can be explained mathematically
by it being the automorphism group of the Moonshine VOA $\mathcal{V}^\#$, and physically by the
associated RCFT, but what is so special about $\mathbb{M}$ that these modular functions should be
genus 0? We will come back to this in §2.9.

2.5. Formal Power Series

Vertex (operator) algebras (VOAs) are a mathematically precise formulation of the
notion of W-algebra or chiral algebra\textsuperscript{30} which is so central to conformal field theory (see
§1.1). VOAs were first defined by Borcherds, and their theory has since been developed
by a number of people (Frenkel, Lepowsky, Meurman, Zhu, Dong, Li, Mason, Huang, ...).
Because our primary motivation here is Moonshine, I will only focus on one aspect of their
theory (the connection with Lie algebras). Useful to consult while reading this review are
the notes [27] — they take a more analytic approach to many of the things we discuss, and
their approach (namely that of CFT) motivates beautifully much of VOA theory.

In quantum field theory the basic object is the quantum field, which roughly speaking
is a choice of operator $\hat{A}(x)$ at each space-time point $x$. ‘Operator’ means something
that ‘operates on’ functions or vectors. E.g. an indefinite integral is an operator, as is a
derivative. The operators in the QFT act on the space spanned by the states $|\star\rangle$, and
together form an infinite-dimensional vector space (e.g. a C$^*$ algebra) — this infinite-
dimensionality of QFT is a major source of its mathematical difficulties, and QFT still has
not been put on completely satisfactory mathematical grounds.

\textsuperscript{30} An alternate (and much more complicated) mathematical formulation of chiral algebra is due to Beilinson
and Drinfeld, and belongs to algebraic geometry. See [28] for a good — but still difficult — review.
But another difficulty is that the quantum field \( \hat{A} \) really isn’t an operator-valued function of space-time. ‘Function’ is too narrow a concept. For example, one of the most familiar ‘functions’ in quantum mechanics is the Dirac delta \( \delta(x) \). You see it for example in the canonical commutation relations: e.g. for a scalar field \( \hat{\varphi} \), we have \([\hat{\varphi}(\vec{x}, t), \frac{\partial}{\partial t} \hat{\varphi}(\vec{y}, t)] = i\hbar \delta^3(\vec{x} - \vec{y})\). \( \delta(x) \) has the property that for any other smooth function \( f \),

\[
\int_{-1}^{1} f(y) \delta(y) \, dy = f(0) , \quad \int_{-1}^{1} f(y) \delta'(y) \, dy = f'(0) ,
\]

etc. The problem is that \( \delta(x) \) isn’t a function — no function could possibly have those properties.

One way to make sense of ‘functions’ like the Dirac delta and its derivatives is distribution theory. Although it was first informally used in physics, it was rigourously developed around 1950 by Laurent Schwartz, and uses the idea of test functions. See e.g. [15].

What I will describe now is an alternate approach, algebraic as opposed to analytic. These two approaches are not equivalent: you can do some things in one approach which you can’t do in the other. But the algebraic approach is considerably simpler technically — no calculus or convergence to worry about — and it is remarkable how much can still be captured. This approach is the starting point for the VOA story described next section, and was first created around 1980 by Garland and Date-Kashiwara-Miwa. Keep in mind that what we are trying to capture is an operator-valued ‘function’ on space-time. Space-time in CFT is 2-dimensional, and so we can think of it (at least locally) as being on the complex plane \( \mathbb{C} \) (more precisely, we will usually associate the space-time point \((x, t)\) with the complex number \( z = e^{t+ix} \)). Good introductions to the material in this section are [23,39,31].

Let \( W \) be any vector space. We are most interested in it being an infinite-dimensional space of matrices (i.e. operators on an infinite-dimensional space), but forget that for now. Define \( W[[z, z^{-1}]] \) to be the set of all formal series \( \sum_{n=-\infty}^{\infty} w_n z^n \), where the coefficients \( w_n \) lie in our space \( W \). We don’t ask here whether a given series converges or diverges — \( z \) is merely a formal variable. We will also be interested in \( W[z, z^{-1}] \) (Laurent polynomials). We can add these formal series in the usual way, and multiply them by numbers (scalars) in the usual way.

Remember our ultimate aim here: we want to capture quantum fields. So we want our formal series to be operator-valued. The way to accomplish this is to choose \( W \) to be a vector space of operators, or matrices if you prefer. A fancy way to say this is ‘\( W = \text{End}(V) \)’, which means the things in \( W \) operate on vectors in \( V \). If we take \( V = \mathbb{C}^m \), then we can think of \( W \) as being the space of all \( m \times m \) complex matrices. We are ultimately interested in the case \( m = \infty \), but we won’t lose much now by taking \( m = 1 \), which would mean formal power series with numerical coefficients.

Because our coefficients \( w_n \) are operators, we can multiply our formal series. We define multiplication in the usual way. For example, consider \( W = V = \mathbb{C} \), and take \( c(z) = z^{21} - 5z^{100} \) and \( d(z) = \sum_{n=-\infty}^{\infty} z^n \). Then

\[
c(z) d(z) = \sum_{n=-\infty}^{\infty} z^{n+21} - 5 \sum_{n=-\infty}^{\infty} z^{n+100} = \sum_{n=-\infty}^{\infty} z^n - 5 \sum_{n=-\infty}^{\infty} z^n = -4d(z) .
\]
So far so good. Now try to compute the square $d(z)^2$. You get infinity. So the lesson is: you can’t always multiply in $W[[z, z^{-1}]]$. We’ll come back to this later.

But first, look again at that first product: $c(z) d(z) = -4d(z)$. One thing it tells us is that we can’t always divide (certainly $c(z)$ and $-4$ are two very different power series!). But there’s another lesson here: if you work out a few more multiplications of this kind, you’ll find that $f d(z) = f(1) d(z)$ for any $f$, at least for those $f$ for which $f(1)$ exists (e.g. any $f \in W[[z, z^{-1}]]$). Thus $d(z)$ is what we would call the Dirac delta $\delta(z - 1)!$

(You can think of it as the Fourier expansion of the Dirac delta, followed by a change of variables). Unfortunately, the standard notation here is to write it without the ‘$-1$’:

$$
\delta(z) := \sum_{n=-\infty}^{\infty} z^n
$$

and that is the notation we will also adopt. Similarly, $\delta(az)$ and $\delta'(z)$ etc (which are the formal series defined in the obvious way) act on $W[z, z^{-1}]$ in the way one would expect:

$$
f(z) \delta(az) = f(\frac{1}{a}) \delta(az) \quad \text{and} \quad f(z) \delta'(z) = f'(1) \delta'(z).$$

So of course it makes perfect sense that we couldn’t work out $d(z)^2$: we were trying to square the Dirac delta, which we know is impossible!

A similar theory can be developed for several variables $z_i$, with identities such as $f(z_1, z_2) \delta(z_1/z_2) = f(z_2, z_2) \delta(z_1/z_2) = f(z_1, z_1) \delta(z_1/z_2)$.

But we must not get too overconfident:

**Paradox 1.** Consider the following product:

$$
\delta(z) = \left[ \left( \sum_{n=0}^{\infty} z^n \right) (1 - z) \right] \delta(z) = \left( \sum_{n=0}^{\infty} z^n \right) \left[ (1 - z) \delta(z) \right] = \left( \sum_{n=0}^{\infty} z^n \right) [0 \delta(z)] = 0.
$$

When physicists are confronted with ‘paradoxes’ such as this, they tend to respond by keeping them in the back of their mind, by treading with care when they are involved in a calculation which reminds them of one of the paradoxes, and otherwise trusting their instincts. Mathematicians typically over-react: they kick themselves for getting overconfident and walking head-first into a ‘paradox’, and then they devise some rule which will absolutely guarantee that that paradox will always be safely avoided in the future. We will follow the mathematicians’ approach, and in the next few paragraphs will describe their rule for avoiding Paradox 1: to forbid certain innocent-looking products.

Remember that we are actually interested in the vector space $W = \text{End}(V)$. Suppose we have infinitely many matrices $w_i \in \text{End}(V)$. We will call them summable if for every column vector $v \in V$, only finitely many products $w_i(v) \in V$ are different from 0. In other words, only finitely many of the matrices $w_i$ have a nonzero first column, only finitely many have a nonzero second column, . . . .

We will certainly have a well-defined sum $\sum_i w_i(z)$ if for each fixed $n$, the set $\{w_i(n)\}$ (as $i$ varies) of matrices is summable. All other sums are forbidden. We will certainly have a well-defined product\(^{31}\) $\prod_{i=1}^{m} w_i(z)$ if for each $n$, the set $\{w_1(n_1) w_2(n_2) \cdots w_m(n_m)\}$

\(^{31}\) $m$ here will be finite: we permit infinite sums but only finite products.
(vary the \( n_i \) subject to \( \sum_i n_i = n \)) is summable. All other products are forbidden. This is reasonable because the sum of those matrix products \( w_1(n_1) \cdots w_m(n_m) \) will precisely equal the \( n \)th coefficient of the product \( \prod_{i=1}^{m} w_i(z) \).

Note that there are certainly more general ways to have a well-defined product (or sum). For example, according to our rule, we cannot even add \( \sum_n 2^{-n} \)! This way has the advantage of not touching the more complicated realm of convergence issues. We are doing algebra here, not analysis. The way out of Paradox 1 is that \( (\sum z^n)(1 - z) \) doesn’t equal 1 — rather, it’s a forbidden product.

An interesting consequence of the fact that we are doing algebra instead of analysis is that the product \( z^2 \delta(z) \) here does not and cannot equal \( 1^2 \delta(z) = \delta(z) \) — their formal power series are very different. In hindsight this ‘failing’ is understandable: algebraically, it seems artificial to prefer the positive root of 1 over the negative root.

**Paradox 2.** Expand \( \frac {1} {1 - z} \) in a formal power series in \( z \) to get \( \sum_{n \geq 0} z^n \). Next, expand \( \frac {1} {1 - z} = \frac {z^{-1}} {1 - z^{-1}} \) in a formal power series in \( z^{-1} \) to get \( -\sum_{n < 0} z^n \). Subtract these; we presumably should get 0, but we actually get \( \delta(z) \)!

The analytic explanation is that the first expression converges only for \( |z| < 1 \), and the second for \( |z| > 1 \), so it would be naive to expect their difference to be 0. We see from this ‘paradox’ that it really matters in which variable we expand rational functions. For instance, at first glance the identity

\[
z_0^{-1} \delta \left( \frac {z_1 - z_2} {z_0} \right) - z_0^{-1} \delta \left( \frac {z_2 - z_1} {-z_0} \right) = z_2^{-1} \delta \left( \frac {z_1 - z_0} {z_2} \right)
\]

is nonsense; it only holds if you expand the terms in positive powers of \( z_2, z_1, \) and \( z_0 \) respectively. The procedure of expanding a function in positive and negative powers of a variable and then subtracting the results, yields what are called expansions of zero; it is possible to show that expansions of zero will always be linear combinations of Dirac deltas \( \delta(az) \) and their various derivatives \( \delta^{(k)}(az) \), as we saw in Paradox 2.

### 2.6. Ingredient #3: Vertex Operator Algebras

We are now prepared to introduce the important new structure called vertex operator algebras (VOAs). They are essentially the chiral algebras of RCFTs — see [26,27] for excellent motivation of the 7 axioms below. A more detailed treatment of the basic theory of VOAs is provided by e.g. [23,39,31]. Although VOAs are natural from the CFT perspective and appear to be an important and rapidly developing area in math, their definition is not easy: Borcherds is known to have said that you either know what they are, or you don’t want to know.

A VOA is a (infinite-dimensional) graded vector space \( V = \oplus_{n \in \mathbb{Z}} V_n \) with infinitely many bilinear products \( u \ast_n v \) respecting the grading (in particular \( V_k \ast_n V_{\ell} \subseteq V_{k + \ell - n - 1} \)), which obey infinitely many constraints. ‘Bilinear’ means that for any \( a, a', b, b' \in \mathbb{C} \) and \( u, u', v, v' \in V \), \((au + a'u') \ast_n (bv + b'v') = ab u \ast_n v + a b' u \ast_n v' + a' b u' \ast_n v + a' b' u' \ast_n v' \) — i.e. that the products are compatible with the vector space structure of \( V \). The subspaces \( V_n \)
must all be finite-dimensional, and they must be trivial (i.e. \( V_n = \{0\} \)) for all sufficiently small \( n \) (i.e. for \( n \approx -\infty \)). Note that we can collect all these products into one generating function: a linear map \( Y : V \to (\text{End} V)[[z, z^{-1}]] \). That is, to each vector \( u \in V \) we associate the formal power series (called a vertex operator) \( Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1} \). For each \( u \), the coefficients \( u_n \) will be functions from \( V \) to \( V \). The idea is that the product \( u \star_n v \) will now be written \( u_n v := u_n(v) \). The bilinearity of \( \star_n \) translates into two things in this new language: that \( Y(\star, z) \) is linear, and that each function \( u_n \) is itself linear (i.e. they are endomorphisms).

The constraints are:

**VOA 1.** (regularity) \( u_n v = 0 \) for all \( n > N(u, v) \);

**VOA 2.** (vacuum) there is a vector \( 1 \in V \) such that \( Y(1, z) \) is the identity (i.e. \( 1_n v = \delta_{n,-1} v \));

**VOA 3.** (state-field correspondence) \( Y(u, 0) 1 = u \);

**VOA 4.** (conformal) there is a vector \( \omega \in V \), called the conformal vector, such that \( L_n := \omega_{n+1} \) gives us a representation of the Virasoro algebra \( V \), with central term \( C \mapsto cI \) for some \( c \in \mathbb{C} \);

**VOA 5.** (translation generator) \( Y(L_{-1} u, z) = \frac{d}{dz} Y(u, z) \);

**VOA 6.** (conformal weight) \( L_0 u = nu \) whenever \( u \in V_n \);

**VOA 7.** (locality) \( (z - w)^M [Y(u, z), Y(v, w)] = 0 \) for some integer \( M = M(u, v) \).

We saw the Virasoro algebra in Part 1 (see (1.2.7)). The number \( c \) in VOA 4 is called the central charge, and is an important invariant of \( V \). The peculiar-looking VOA 7 simply says that the commutator \( [Y(u, z), Y(v, w)] \) of two vertex operators will be a finite linear combination of derivatives of various orders of the Dirac delta centred at \( z - w \). A recommended exercise for the reader is to show that \( M = 4 \) works in VOA 7 for \( u = v = \omega \). Note that in a VOA, any \( Y(u, z)v \) will be a finite sum — i.e. the series \( Y(u, z) \) is summable (defined last section). It is a consequence of the axioms that \( 1 \in V_0 \) and \( \omega \in V_2 \): for instance, VOA 7 says all \( u_n 1 = 0 \) for any \( n \geq 0 \), so \( L_0 1 = \omega_1 1 = 0 \) and hence \( 1 \in V_0 \).

In RCFT, \( V \) would be the ‘Hilbert space of states’ (more carefully, \( V \) will be a dense subspace of it), and \( z = e^{i+ix} \) would be a local complex coordinate on a Riemann surface. \( L_0 \) generates time translations, and so its eigenvalues (the conformal weights) can be identified with energy. Physically, the requirement that \( V_n \to 0 \) for \( n \to -\infty \) corresponds to the energy of the RCFT being bounded from below. Also, \( z = 0 \) in VOA 3 corresponds to the time limit \( t \to -\infty \). For each state \( u \), the vertex operator \( Y(u, z) \) is a holomorphic (chiral) quantum field. The vector \( 1 \) is the vacuum \( |0\rangle \), and \( Y(\omega, z) \) is the stress-energy tensor \( T \). The most important axiom, VOA 7, says that vertex operators commute up to a possible pole at \( z = w \), and so are local quantum fields. It is equivalent to the duality axiom of many treatments of CFT. In the physics literature, there is a minor notational difference: for \( u \in V_k \), \( Y(u, z) = \sum u_n z^{-n-1} \) is written \( \sum u_{(n)} z^{-n-k} \). (Physicists prefer this because it cleans up some formulas a little; mathematicians abhor it because it artificially prefers the ‘homogeneous’ vectors \( u \in V_k \).)

In Segal’s language (see §1.1), \( Y(u, z) \) appears quite naturally. Consider the physical event of two strings combining to form a third. To first order (i.e. the tree-level Feynman diagram), this would correspond in Segal’s language to a ‘pair-of-pants’, or a sphere with three punctures, two of which are positively oriented (corresponding to the incoming
strings) and the other being negatively oriented. We can think of the sphere as the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \); put the punctures at \( \infty \) (outgoing) and \( z \) and 0 (the incomings). Segal’s functor \( T \) will associate to this a \( z \)-dependent homomorphism \( \varphi_z : V \times V \to V \). We write \( \varphi_z(u,v) \in V \) as \( Y(u, z)v \). Incidentally, the symbol ‘\( Y \)’ was chosen because of this ‘pair-of-pants’ picture (time flows from the top of the ‘\( Y \)’ to the bottom), as was the name ‘vertex operator’.

The original axioms by Borcherds were a little more complicated and general: he didn’t require \( \dim(V_n) < \infty \) nor the \( V_n \to 0 \) condition, and he only considered \( L_0 \) and \( L_{-1} \) rather than the full Virasoro algebra. The resulting generalisation is called a vertex algebra.

**VOA 7** can be rewritten in the form (usually called the Jacobi identity for the VOA)

\[
 z_0^{-1} \delta(\frac{z_1 - z_2}{z_0}) Y(u, z_1) Y(v, z_2) - z_0^{-1} \delta(\frac{z_2 - z_1}{z_0}) Y(v, z_2) Y(u, z_1) = z_2^{-1} \delta(\frac{z_1 - z_0}{z_2}) Y(Y(u, z_0)v, z_2),
\]

where the formal series are expanded in the appropriate way. This is the embodiment of commutativity and associativity in the VOA, as we will see. To bring it into a more useful form, hit it with \( t \in V \) and expand out into \( z_0^6 z_1^m z_2^n \); we obtain

\[
 \sum_{i \geq 0} (-1)^i \binom{\ell}{i} (u_{\ell + m - i} \circ v_{n + i} - (-1)^i v_{\ell + n - i} \circ u_{m + i}) = \sum_{i \geq 0} \binom{m}{i} (u_{\ell + i}v)_{m + n - i},
\]

where for any \( k \in \mathbb{Z}, j \in \mathbb{Z}_+ \), \( \binom{k}{j} := \frac{k!}{(k-j+1)j!} \). For instance, specialising (2.6.2) to \( \ell = 0 \) and \( m = 0 \), resp., gives us

\[
 [u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m + n - i},
\]

\[
 (u \ell v)_n = \sum_{i \geq 0} (-1)^i \binom{\ell}{i} (u_{\ell - i} \circ v_{n + i} - (-1)^i v_{\ell + n - i} \circ u_{i}) .
\]

Why is (2.6.1) called the Jacobi identity? Put \( \ell = m = n = 0 \) in (2.6.2): we get \( u_0(v_0 t) - v_0(u_0 t) = (u_0 v)_{0}t \). If we now formally write \( [x y] := x_0y \), then this becomes \([u[vt]] - [v[ut]] = [uv]t \), which is one of the forms of the Lie algebra Jacobi identity (1.2.1b). Even though \([x y] \neq -(y x) \) here, this formal little trick will turn out to be quite important next section.

The simplest examples of VOAs correspond to any even positive-definite lattice \( \Lambda \); for their construction see e.g. [27,39]. Physically, they correspond to a bosonic string compactified on the torus \( \mathbb{R}^n/\Lambda \cong S^1 \times \cdots \times S^1 \) (where \( n \) is the dimension of \( \Lambda \)); the central charge \( c = n \). Other important examples, first constructed by Frenkel-Zhu (again see e.g. [27,39]), correspond to affine Kac-Moody algebras \( X^{(1)}_k \) at level \( k \in \mathbb{Z}_+ \), and physically to WZW theories on simply-connected compact group manifolds. (We discussed affine algebras in §1.4.) These have central charge \( c = \frac{k \dim(X_k)}{k + h^\vee} \).

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In 1984 Frenkel-Lepowsky-Meurman [23] constructed the Moonshine module $V^\natural$. It is a VOA with $c = 24$, with $V^\natural = V_0^\natural \oplus V_1^\natural \oplus V_2^\natural \oplus \cdots$, where $V_0^\natural = \mathbb{C}1$ is 1-dimensional, $V_1^\natural = \{0\}$ is trivial, and $V_2^\natural = (\mathbb{C}\omega) \oplus$ (Griess algebra) is $(1 + 196883)$-dimensional. Its automorphism group (=symmetry group) is precisely the Monster $\mathbb{M}$. Each graded piece $V_n^\natural$ is a finite-dimensional representation of $\mathbb{M}$; Borcherds proved that in fact $V^\natural$ is the McKay-Thompson infinite-dimensional representation of $\mathbb{M}$. It can be regarded as the most natural representation of $\mathbb{M}$ — it is rather surprising that important aspects of a finite group need to be studied via an infinite-dimensional representation.

$V^\natural$ has an elegant physical interpretation. First construct the bosonic string on $\mathbb{R}^{24}/\Lambda_{24}$ (recall that $\Lambda_{24}$ is the Leech lattice). The resulting $c = 24$ VOA has partition function (=graded dimension) $J(\tau) + 24$, but although its graded pieces (at least for $n > 0$) have the right dimensions, they don’t carry a natural representation of $\mathbb{M}$ and so can’t qualify for the McKay-Thompson representation. To get $V^\natural$, orbifold this $\Lambda_{24}$ VOA by the order-2 automorphism of $\Lambda_{24}$ sending $\vec{x} \mapsto -\vec{x}$. $V^\natural$ thus corresponds to a holomorphic $c = 24$ RCFT, and Moonshine is related to physics. Most of Moonshine can be interpreted physically, except perhaps the genus 0 property of the McKay-Thompson series $T_g$.

There is a formal parallel between e.g. lattices and VOAs. For example, the Leech lattice $\Lambda_{24}$ and the Moonshine module $V^\natural$ play analogous roles: $\Lambda_{24}$ is the unique even lattice which (i) is self-dual, (ii) contains no norm 2 vectors, and (iii) has dimension 24; $V^\natural$ is believed to be the unique VOA which (i) possesses only one irreducible representation (namely itself), (ii) contains no conformal weight 1 elements, and (iii) has central charge $c = 24$. Analogies of these kinds are always useful as they suggest new directions to explore, and the history of math blooms with them. The battlecry ‘Why invent what can be profitably copied’ is not only heard in Hollywood.

We will end this section on a more speculative note. Witten (1986) said that to understand string theory conceptually, we need a new analogue of Riemannian geometry. Huang (1997) has pushed this thought a little further, saying that there is a more classical ‘particle-math’ and a more modern ‘string-math’. According to Huang we have the real numbers (particle physics) vrs the complex numbers (string theory); Lie algebras vrs VOAs; and the representation theory of Lie algebras vrs RCFT, etc. What are the stringy analogues of calculus, ordinary differential equations, Riemannian manifolds, the Atiyah-Singer Index theorem,...? At present these are all unknown. However, Huang suggests that just as we could imagine Moonshine as a mystery which is explained in some way by RCFT, perhaps the stringy version of calculus would similarly explain the mystery of 2-dimensional gravity, stringy ODEs would explain the mystery of infinite-dimensional integrable systems, stringy Riemannian manifolds would help explain the mystery of mirror symmetry, and the stringy index theorem would help explain the elliptic genus.

2.7. Ingredient #4: Generalised Kac-Moody algebras

In this section we investigate Lie algebras arising from VOAs. These Lie algebras are an interesting generalisation of Kac-Moody algebras. See e.g. [5,6,38 Chapter 11.13, 29].
Much of Lie theory (indeed much of algebra) is developed by analogy with simple properties of integers. In §2.2 I invited you to think of a finite group as a massive generalisation of the concept of whole number. Specifically, the number \( n \) can be identified with the cyclic group \( \mathbb{Z}_n \) with \( n \) elements. A divisor \( d \) of \( n \) generalises to a normal subgroup of a group. A prime number then corresponds to a simple group. Multiplying numbers corresponds to taking the semidirect product of groups (more generally, taking extensions of groups). Then we find that every group has a unique set of simple building blocks (although unlike numbers, different groups can have the same list of building blocks).

For a finite-dimensional Lie algebra, a divisor is called an ideal; a prime is called simple; and multiplying corresponds to semidirect sum. Lie algebras behave simpler than groups but not as simple as numbers, and the analogy sketched above is a reasonably satisfactory one. In particular, simple Lie algebras are important for similar reasons that simple groups are, and as mentioned in §1.2 can also be classified (with much less effort).

A good treatment of this important classification (over \( \mathbb{C} \)) is provided by [36]. The proof is now reaching the state of perfection of the formulation of classical mechanics. One unobvious discovery is that the best way to capture the structure of a simple Lie algebra is by an integer matrix, called the \textit{Cartan matrix}, or equivalently but more effectively (since most entries in the Cartan matrix are 0's) by using a graph called the \textit{(Coxeter-)Dynkin diagram}. For instance the Dynkin diagram for \( A_\ell \) consists of \( \ell \) nodes connected sequentially in a line. See Figure 6 in [59].

More precisely, define a \textit{symmetrised Cartan matrix} to be a symmetric real matrix

\[
A = (a_{ij})_{i,j \leq \ell} \text{ such that } a_{ij} \leq 0 \text{ if } i \neq j, \quad a_{ii} > 0, \quad \text{each } 2a_{ij}a_{ii} \in \mathbb{Z}, \quad \text{and } A \text{ is positive-definite.}
\]

Examples of \( 2 \times 2 \) symmetrised Cartan matrices are

\[
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}, \quad \begin{pmatrix}
1 & -1 \\
-1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
2 & -3 \\
-3 & 6
\end{pmatrix}
\]

The Dynkin diagram corresponding to \( A \) consists of \( \ell \) nodes; the \( i \)th and \( j \)th nodes are connected with \( 4a_{ij}^2/a_{ii}a_{jj} \) lines, and if \( a_{ii} \neq a_{jj} \), then we put an arrow over those lines pointing to \( i \) if \( a_{ii} < a_{jj} \). The Dynkin diagrams corresponding to those four Cartan matrices are respectively

\[
\circ\circ, \quad \circ \circ, \quad \circ \circ, \quad \circ \circ
\]

We may without loss of generality require \( A \) to be \textit{indecomposable}, or equivalently that the Dynkin diagram be connected. Of the 4 given above, only the second is decomposable.

To any \( \ell \times \ell \) symmetrisable Cartan matrix, we can construct the corresponding Lie algebra \( \mathfrak{g} \) in the following way. For each \( i \), create 3 generators \( e_i, f_i, h_i \) (so there are a total of \( 3\ell \) generators). The relations these generators obey are given by the following brackets:

\[
[e_i f_j] = \delta_{ij} h_i, \quad [h_i e_j] = a_{ij} e_j, \quad [h_i f_j] = -a_{ij} f_j, \quad \text{and for } i \neq j \quad \text{ad}(e_i)^n e_j = \text{ad}(f_i)^n f_j = 0
\]

where \( n = 1 - 2a_{ij}a_{ii} \). By \’ad\' here I mean the function \( \mathfrak{g} \to \mathfrak{g} \) defined by \( \text{ad}(e)f = [e f] \). So \( \text{ad}(e)^2 f = [e [e f]], \text{ad}(e)^3 f = [e [e [e f]]], \text{etc.} \)

\[32\] Note that our Cartan matrices differ from the usual definition, in which every diagonal entry equals 2.
To get a better feeling for these relations, consider a fixed $i$. The generators $e = \sqrt{\frac{2}{a_{ii}}} e_i$, $f = \sqrt{\frac{2}{a_{ii}}} f_i$, $h = \frac{2}{a_{ii}} h_i$ obey the relations (1.2.2b). In other words, every node in the Dynkin diagram corresponds to a copy of the $A_1$ Lie algebra. The lines connecting these nodes tells how these $\ell$ copies of $A_1$ intertwine.

For instance consider the first Cartan matrix given above. It corresponds to the Lie algebra $A_2$, or $\mathfrak{sl}_3(\mathbb{C})$. The two $A_1$ subalgebras which generate it (corresponding to the 2 nodes of the Dynkin diagram) can be chosen to be the trace-zero matrices of the form

\[
\begin{pmatrix}
\ast & \ast & 0 \\
\ast & \ast & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & \ast & \ast \\
0 & \ast & \ast
\end{pmatrix}.
\]

It can be shown that the Lie algebra corresponding to an indecomposable symmetrised Cartan matrix will be finite-dimensional and simple, and conversely that any finite-dimensional simple Lie algebra corresponds to an indecomposable symmetrisable Cartan matrix in this way.

A confusion sometimes arises between the terms ‘generators’ and ‘basis’. Both generators and basis vectors build up the whole algebra; the difference lies in which operations you are permitted to use. For a basis, you are only allowed to use linear combinations (i.e. addition of vectors and multiplication by numbers), while for generators you are also permitted multiplication of vectors (or the bracket, in the Lie case). ‘Dimension’ refers to basis, while ‘rank’ usually refers in some way to generators. For instance the (commutative associative) algebra of polynomials in one variable $x$ is infinite-dimensional — any basis needs infinitely many vectors. However, the single polynomial $x$ is enough to generate it (so we could say that its rank is 1). Although those Lie algebras have $3\ell$ generators, their dimensions in general will be greater.

From the point of view of generators and relations, the step from ‘finite-dimensional simple’ to ‘symmetrisable Kac-Moody’ is rather easy: the only difference is that we drop the ‘positive-definite’ condition (which was responsible for finite-dimensionality). Kac-Moody (KM) algebras are also generated by (finitely many) $A_1$ subalgebras, and their theory is quite parallel to that of the simple algebras. Compare Figures 6 and 9 in [59].

Now, it is easy to generalise something; the challenge is to generalise it in a rich and interesting direction. One natural and appealing strategy for generalisation was followed instinctively by a grad student named Robert Moody. Moody’s original motivation for developing the theory of Kac-Moody algebras was the Weyl group. If there were Lie algebras for finite Coxeter groups, he asked, why not also for the Euclidean (=affine) ones? For another example of this style of generalisation, consider the question: What is the analogue of calculus (or manifolds) over weird fields — fields (like $\mathbb{Z}_p$) for which the usual limit definitions make no sense? This question leads to the riches of algebraic geometry. Nevertheless this generalisation strategy, even in the hands of a master, will not always be successful. For instance, consider all the trouble the following metaphor has caused: my watch has a maker, so so should the Universe.

Recently Borcherds produced a further generalisation of finite-dimensional simple Lie algebras, which is rather less obvious than that of Kac-Moody algebras. It is easy to associate a Lie algebra to a matrix $A$, but which class of matrices will yield a deep theory?
Borcherds found such a class by holding in his hand a single algebra (the fake Monster Lie algebra, see §2.9) which acted a lot like a KM algebra, even though it had ‘imaginary simple roots’.

By a generalised symmetrised Cartan matrix $A = (a_{ij})$ we will mean a symmetric real matrix (possibly infinite), such that $a_{ij} \leq 0$ if $i \neq j$, and if $a_{ii} > 0$ then $2a_{ij}/a_{ii} \in \mathbb{Z}$ for all $j$. By a universal generalised Kac-Moody algebra (universal GKM) or universal Borcherds-Kac-Moody algebra $\mathfrak{g}$ we mean the algebra\(^{33}\) with generators $e_i, f_i, h_{ij}$, and with relations: $[e_i f_j] = h_{ij}; [h_{ij} e_k] = \delta_{ij} a_{ik} e_k; [h_{ij} f_k] = -\delta_{ij} a_{ik} f_k$; if $a_{ii} > 0$ and $i \neq j$ then $\text{ad}(e_i)^n e_j = \text{ad}(f_i)^n f_j = 0$ for $n = 1 - 2a_{ij}/a_{ii}$; and if $a_{ij} = 0$ then $[e_i e_j] = [f_i f_j] = 0$.

For example the Heisenberg algebra (1.2.3) corresponds to the choice $A = (0)$, while any other $1 \times 1 A = (a)$ corresponds to $A_1$. A universal GKM algebra differs from a KM algebra in that it is built up from Heisenberg algebras as well as $A_1$, and these subalgebras intertwine in more complicated ways. Nevertheless much of the theory for finite-dimensional simple Lie algebras continues to find an analogue in this much more general setting (e.g. root-space decomposition, Weyl group, character formula,...). This unexpected fact is the point of GKM algebras.

To get a feel for these algebras, let us prove a few simple results concerning the $h_{ij}$. Note first that, using the above relations together with the Jacobi identity, we obtain $[h_{ij}, h_{kl}] = \delta_{ij}(a_{jk} - a_{kj}) h_{kl}$. Comparing this with $[h_{kl} h_{ij}] = -[h_{ij} h_{kl}]$, we see that bracket must always equal 0. Hence all $h$’s pairwise commute, and $h_{ij} = 0$ unless the $i$th and $j$th columns of $A$ are identical. An easy exercise now is to show that when $i \neq j$, $h_{ij}$ will lie in the centre of the algebra (i.e. $h_{ij}$ will commute with all other generators).

Although the definition of universal GKM algebra is more natural, it turns out that an equivalent form can be more useful in practice. (It’s simpler to describe over $\mathbb{R}$, so in most expositions the reals are used, but alas it’s far too late for us to switch loyalties now.) By a generalised Kac-Moody algebra (or Borcherds-Kac-Moody algebra) $\mathfrak{g}$ we mean a (complex!) Lie algebra which is:

- $\mathbb{Z}$-graded, i.e. $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, where $[\mathfrak{g}_i \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$;
- $\mathfrak{g}_i$ is finite-dimensional for $i \neq 0$;
- $\mathfrak{g}$ has an antilinear involution $\omega$ (i.e. $\omega(kx + y) = k^* \omega(x) + \omega(y)$, $[\omega(x), \omega(y)] = \omega([x, y])$, and $\omega \circ \omega = id.$) which maps $\mathfrak{g}_i$ to $\mathfrak{g}_{-i}$ and acts as multiplication by $-1$ on some basis of $\mathfrak{g}_0$;
- $\mathfrak{g}$ has an invariant symmetric bilinear form $(\ast, \ast)$ (i.e. $(\{x, y\}, z) = (x, [y, z])$ and $(y, z) = (z, y) \in \mathbb{C}$), obeying $(\omega(x), \omega(y)) = (x, y)^\ast$, such that $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i \neq -j$;
- the Hermitian form defined by $(x, y) := -\omega(x, y)$ is positive-definite on $\mathfrak{g}_{i\neq0}$.

Note that for some basis $x_i$ of $\mathfrak{g}_0$, the third condition tells us $-x_i x_j = ([x_i, x_j] = [x_i, x_j]) = x_i x_j$, i.e. $\mathfrak{g}_0$ has a trivial bracket. It plays the role of the Cartan subalgebra $\mathfrak{h}$ in the theory.

For example, let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and recall (1.2.5). Then $\mathfrak{g}_1 = \mathbb{C} e, \mathfrak{g}_0 = \mathbb{C} h, \mathfrak{g}_{-1} = \mathbb{C} f$ is the root-space decomposition. $\omega(x) = -x^\dagger$ is the Cartan involution. $(x, y) = \text{tr}(xy)$ is the Killing form.

\(^{33}\) As with KM algebras, usually we want to extend it by some derivations; enough derivations are added so that the simple roots are linearly independent.
It turns out [5] that any universal GKM algebra is a GKM algebra, and any GKM algebra can be constructed from a unique universal GKM algebra (by quotienting out part of the centre and adding derivations), so in that sense the two structures are equivalent. This theorem is important, because it tells us that **GKM algebras are the ultimate generalisation of simple Lie algebras**, in the sense that any further generalisation will lose some basic structural ingredient.

We know simple Lie algebras (and groups) arise in both classical and quantum physics, and the affine KM algebras are important in CFT, as we saw in Part 1. GKM algebras have recently appeared in the physics literature (see Harvey-Moore) in the context of BPS states in string theory.

How do GKM algebras arise in VOAs? If we define \([xy] := x_0 y\), then as mentioned in §2.6 we get from the VOA Jacobi identity the equation \([x[yz]] - [y[xz]] = [[xy]z]\). Thus our bracket will be anti-associative if it is anti-commutative.

It can be shown

\[
u_n v = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{i+n+1}(L_{-1})^i(v_{n+i}u)
\]

so \(u_0 v \equiv -v_0 u\) if we look at things mod \(L_{-1} V\).

Since our bracket is clearly bilinear, we thus get a Lie algebra structure on \(V/L_{-1} V\). Similarly, we get a symmetric bilinear product on \(V/L_{-1} V\), given by \(\langle u, v \rangle := u_1 v\).

We would like \(\langle \ast, \ast \rangle\) to respect the Lie algebra structure, i.e. be \(\ast\)-invariant. We compute from (2.6.4)

\[
\langle [uv], t \rangle = -[v\langle u, t \rangle] + \langle u, [vt] \rangle.
\]

Since we would like to identify \(\langle \ast, \ast \rangle\) with the bilinear form in the GKM algebra definition, we also would like it to be number-valued (i.e. have 1-dimensional range).

There is a simple way to satisfy both of these. First, restrict attention to \(V_1\), i.e. the conformal weight 1 vectors: \(V_1 \cap (V/L_{-1} V) = V_1/L_{-1} V_0\). Then \(\langle u, v \rangle \in V_0\). Assume \(V_0\) is 1-dimensional: i.e. \(V_0 = \mathbb{C} 1\). Then \(\langle u, v \rangle\) will equal a number times 1, so call \(\langle u, v \rangle\) that number. Also, \([\langle u, t \rangle v] = (u, t)1_0 v = 0\), so \(\langle \ast, \ast \rangle\) and hence \(\langle \ast, \ast \rangle\), will be invariant. Of course, when \(V_0 = \mathbb{C} 1, L_{-1} V_0 = \{0\}\).

\(V_1/L_{-1} V_0\) is generally too large in practice to be useful; a subalgebra can be defined as follows. Let \(P_0\) be the ‘primary states with conformal weight \(n\)’, i.e. the \(u \in V_0\) killed by \(L_m\) for all \(m > 0\). Then \(g(V) := P_1/L_{-1} P_0\) will be a subalgebra of \(V_1/L_{-1} V_0\). Through the assignment \(u \mapsto u_0, g(V)\) acts on \(V\) and this action commutes with that of \(L_i\). This association of a Lie algebra to a VOA is due to Borcherds (1986).

Similar arguments show that when \(V_0\) is 1-dimensional and \(V_1\) is 0-dimensional, then \(V_2\) will necessarily be a commutative nonassociative algebra with product \(u \times v := u_1 v \in V_2\) and identity element \(\frac{1}{2} \omega\) (**proof**: \(\omega \times u = L_0 u = 2u\)). Now, those conditions on \(V_0, V_1\) are satisfied by the Moonshine module \(V^2\). We find that \(V^2\) is none other than the Griess algebra extended by an identity element.
2.8. Ingredient #5: Denominator identities

In §1.3, we discussed the representation theory of Lie algebras. An important invariant of a representation is its character. Simple Lie algebras possess a very useful formula for their characters, due to Weyl:

\[ \text{ch}_\lambda(z) := \sum_{\mu} \dim(L_\lambda(\mu)) e^{\mu \cdot z} = e^{-\rho \cdot z} \sum_{w \in W} \pm e^{w(\lambda + \rho) \cdot z} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha \cdot z}) , \]  

(2.8.1)

where \( W \) is the Weyl group, \( \Delta_+ \) the positive roots, and where \( \oplus \mu L_\lambda(\mu) \) is the weight-space decomposition of \( L_\lambda \) — i.e. the simultaneous eigenspaces of the \( h_i \). Here \( z \) belongs to the Cartan subalgebra \( h \); the character is complex-valued. Analogous statements hold for all GKM algebras.

It is rare indeed when a trivial special case of a theorem or formula produces something interesting. But that is what happens here. Consider the trivial representation: i.e. \( x \mapsto 0 \) for all \( x \in X_\ell \). Then the character is identically 1, by definition: \( \text{ch}_0 \equiv 1 \). Thus the character formula tells us that a certain alternating sum over a Weyl group, equals a certain product over positive roots. These formulas, called denominator formulas, are nontrivial even in the finite-dimensional cases.

Consider for instance the smallest simple algebra, \( A_1 \). Here the identity indeed is too trivial: it reads \( e^{z/2} - e^{-z/2} = e^{z/2}(1 - e^{-z}) \). For \( A_2 \) we get a sum of 6 terms equalling a product of 3 terms, and the complexity continues to rise from there.

Around 1970 Macdonald tried to generalise these finite denominator identities to infinite identities, corresponding to the extended Dynkin diagrams. These were later reinterpreted by Kac, Moody and others as denominator identities for affine nontwisted KM algebras. The simplest one was known classically as the Jacobi triple product identity:

\[ \sum_{n=-\infty}^{\infty} (-1)^n x^n y^n = \prod_{m=1}^{\infty} (1 - x^{2m})(1 - x^{2m-1}y)(1 - x^{2m-1}y^{-1}) \]

We now know it to be the denominator identity for the simplest infinite-dimensional KM algebra, \( A^{(1)}_1 \).

Freeman Dyson is famous for his work in quantum field theory, but he started as an undergraduate in number theory and still enjoys it as a hobby. Dyson [20] found a curious formula for the Ramanujan \( \tau \)-function, which can be defined by the generating function \( \sum_{n=1}^{\infty} \tau(n)x^n = \eta^{24}(x) := x \prod_{m=1}^{\infty} (1 - x^m)^{24} \). Dyson found the remarkable formula

\[ \tau(n) = \sum (a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e) \]

where the sum is over all 5-tuples \( (a,b,c,d,e) \equiv (1,2,3,4,5) \pmod{5} \) obeying \( a + b + c + d + e = 0 \) and \( a^2 + b^2 + c^2 + d^2 + e^2 = 10n \). Using this, an analogous formula can be found for \( \eta^{24} \). Dyson knew that similar-looking formulas were also known for \( \eta^d \) for the values \( d = 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots \).
What was ironic was that Dyson found that formula at the same time that Macdonald was finding the Macdonald identities. Both were at Princeton then, and would often chat a little when they bumped into each other after dropping off their daughters at school. But they never discussed work. Dyson didn’t realise that his strange list of numbers has a simple interpretation: they are precisely the dimensions of the simple Lie algebras! $3 = \dim(A_1)$, $8 = \dim(A_2)$, $10 = \dim(C_2)$, $14 = \dim(G_2)$, etc. In fact these formulas for $\eta^d$ are none other than (specialisations of) the Macdonald identities. For example, Dyson’s formula is the denominator formula for $A_4^{(1)}$ ($24 = \dim(A_4)$). If they had spoken, they would probably have anticipated the affine denominator identity interpretation.

One curiosity apparently still has no algebraic interpretation: No simple Lie algebra has dimension 26, so the formula for $\eta^{26}$ can’t correspond to any Macdonald identity.

Macdonald didn’t close the book on denominator identities. More recently Kac and Wakimoto [40] have used denominator identities for Lie superalgebras to obtain nice formulas for various generator functions involving sums of squares, sums of triangular numbers (triangular numbers are numbers of the form $\frac{1}{2}k(k+1)$), etc. For instance, the number of ways $n$ can be written as a sum of 16 triangular numbers is

$$\frac{1}{3 \cdot 4^2} \sum ab(a^2 - b^2)^2$$

where the sum is over all odd positive integers $a, b, r, s$ obeying $ar + bs = 2n + 4$ and $a > b$.

Another example of denominator identities is Borcherds’ use of them in proving the Moonshine conjectures. In particular this motivated his introduction of the GKM algebras. The denominator identities for other GKM algebras were used by Borcherds to obtain results on the moduli spaces of e.g. families of K3 surfaces. They are also often turned-around now and used for learning about the positive roots in a given GKM.

2.9. Proof of the Moonshine conjectures

The main Conway-Norton conjecture was proved almost immediately. Thompson showed that if $g \mapsto a_n(g)$ is a character for all $n \leq 1200$, then it will be for all $n$. He also showed that if certain congruence conditions hold for a certain number of $a_n(g)$ (all with $n \leq 100$), then all $g \mapsto a_n(g)$ will be virtual characters (i.e. a linear combination over $\mathbb{Z}$ of irreducible characters of $\mathbb{M}$; only if all coefficients are nonnegative will it be a true character). Atkin-Fong-Smith [54] used that to prove on a computer that indeed all were virtual characters. But their work didn’t say anything about the underlying (possibly virtual) representation $V$. The real challenge was to construct (preferably in a natural manner) the representation which works. Frenkel-Lepowsky-Meurman [23] constructed a candidate for it (the Moonshine module $V^\natural$); it was Borcherds who finally proved $V^\natural$ obeyed the Conway-Norton conjecture. A good overview of Borcherds’ work on Moonshine is provided in [32].

We want to show that the McKay-Thompson series $T_g(\tau) := q^{-1} \text{tr}_{V^\natural}(gq^{L_0})$ of (2.1.4) equals the Hauptmodul $J_g(\tau)$ in (2.4.1) (the ‘fudge factor’ $q^{-1} = q^{-c/24}$ is familiar to e.g. KM algebras and CFT and was discussed at the end of §1.2). Borcherds’ strategy
was to bring in Lie theory and to use the corresponding denominator identity to provide useful combinatorial data. The first guess for this ‘Monster Lie algebra’ was the Kac-Moody algebra whose Dynkin diagram is essentially the Leech lattice (i.e. a node for each vector in \( \Lambda_{24} \), and 2 nodes are connected by a number of edges depending on the value of a certain dot product). It was eventually discarded because some of the critical data (namely positive root multiplicities) needed in order to write down its denominator identity were too complicated. But looking at that failed attempt led Borcherds to a second candidate, now called the fake Monster Lie algebra \( g'_M \). In order to construct it, he developed the theory of VOAs; and in order to understand it, he developed the theory of GKM algebras. We will define it shortly. \( g'_M \) also turned out to be inadequate for proving the Moonshine conjectures; however it directly led him to the GKM algebra now called the true Monster Lie algebra \( g_M \). And that directly led to the proof of Moonshine.

**Step 1:** Construct \( g_M \) from \( V^2 = V_0^2 \oplus V_1^2 \oplus \cdots \). For later convenience, reparametrise these subspaces \( V^i := V_{i+1}^2 \). Recall the even indefinite lattice \( V_{1,1} \) defined in (1.6.1). Of course the direct choice \( g(V^2) \) is 0-dimensional because \( V_1^2 \) is trivial, so we must modify \( V^2 \) first.

The Monster Lie algebra is (essentially) defined to be the Lie algebra \( g(V^2 \otimes V_{1,1}) \) associated to the vertex algebra \( V^2 \otimes V_{1,1} \) (strictly speaking, more of \( g \) is quotiented away). By contrast, the fake Monster is the Lie algebra associated to the vertex algebra \( V_{24} \otimes V_{1,1} \cong V_{25,1} \). Both of these are vertex algebras as opposed to VOAs, because of the presence of the indefinite lattices, but this isn’t important here. \( g_M \) inherits a \( \Pi_{1,1} \)-grading from \( V_{1,1} \): the piece of grading \((m,n)\) is isomorphic (as a vector space) to \( V^{mn} \), if \((m,n) \neq (0,0)\); the \((0,0)\) piece is isomorphic to \( \mathbb{R}^2 \). Borcherds uses the No-Ghost Theorem of string theory to show that the homogeneous pieces of \( g_M \) are those of \( V^2 \).

Both \( g_M \) and \( g'_M \) are GKM algebras; for instance the \( \mathbb{Z} \)-grading of \( g_M \) is given by \( (g_M)_k = \oplus_{m+n=k} V^{mn} \) for \( k \neq 0 \), while the 0-part is \( V^{-1} \oplus V^{-1} \). Although \( g'_M \) is not used in the proof of the Monstrous Moonshine conjectures, it is related to some kind of Moonshine for the finite simple group .1, which is ‘half’ of the automorphism group .0 of the Leech lattice \( \Lambda_{24} \). \( g_M \) corresponds to the Cartan matrix

\[
\begin{pmatrix}
B_{-1,-1} & B_{-1,1} & B_{-1,2} & \cdots \\
B_{1,-1} & B_{1,1} & B_{1,2} & \cdots \\
B_{2,-1} & B_{2,1} & B_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where \( B_{i,j} \) for \( i, j \in \{-1, 1, 2, 3, \ldots\} \) is the \( a_i \times a_j \) block with fixed entry \(-i - j\) (the \( a_i \) as usual are the coefficients \( \sum_n a_n q^n \) of \( j = 744 \)).

**2nd step:** Compute the denominator identity of \( g_M \): we get

\[
p^{-1} \prod_{m > 0} \prod_{n \in \mathbb{Z}} (1 - p^{m} q^{n})^{a_{mn}} = j(z) - j(\tau)
\]  

(2.9.1)

where \( p = e^{2\pi iz} \). The result are various formulas involving the coefficients \( a_i \), for instance \( a_4 = a_3 + (a_1^2 - a_1)/2 \). It turns out to be possible to ‘twist’ (2.9.1) by each \( g \in \mathbb{M} \), obtaining

\[
p^{-1} \exp[- \sum_{k > 0} \sum_{m > 0} a_{mn}(g) \frac{p^{mk} q^{nk}}{k}] = T_g(z) - T_g(\tau).
\]

(2.9.2)
This looks a lot more complicated, but you can glimpse the Taylor expansion of \( \ln(1 - p^m q^n) \) there and in fact for \( g = \text{id} \) (2.9.2) reduces to (2.9.1). This formula gives more generally identities like \( a_4(g) = a_2(g) + (a_1(g)^2 - a_1(g^2))/2 \), where \( T_g(\tau) = \sum_i a_i(g)q^i \). These formulas involving the McKay-Thompson coefficients are equivalent to the replication formulae conjectured in §2.4.

3rd step: It was known earlier that all of the Hauptmoduls also obey those replication formula, and that anything obeying them will lie in a finite-dimensional manifold which we’ll call \( R \). In particular, if \( B(q) = q^{-1} + \sum_{n>0} b_n q^n \) and \( C(q) = q^{-1} + \sum_{n>0} c_n q^n \) both lie in \( R \), and \( b_n = c_n \) for \( n \leq 23 \), then \( B(q) = C(q) \). In fact, it turns out that if we verify for each conjugacy class \([g]\) of \( \mathbb{M} \) that the first, second, third, fourth and sixth coefficients of the McKay-Thompson series \( T_g \) and the corresponding Hauptmodul \( J_g \) agree, then \( T_g = J_g \), and we are done.

That is precisely what Borcherds then did: he compared finitely many coefficients, and as they all equalled what they should, this concluded the proof of Monstrous Moonshine!

However there was a disappointing side to his proof. While no one disputed its logical validity, it did seem to possess a disappointing conceptual gap. In particular, the Moonshine conjectures were made in the hope that proving them would help explain what the Monster had to do with the \( j \)-function and the other Hauptmoduls. A good proof says much more than ‘True’ or ‘False’. The case-by-case verification occurred at the critical point where the McKay-Thompson series were being compared directly to the Hauptmoduls. The proof showed that indeed the Moonshine module establishes some sort of relation between \( T_g \) and \( J_g \) (namely, they must lie in the same finite-dimensional space), but why couldn’t it be just a happy meaningless accident that they be equal? Of course we believe it’s more than merely an accident, so our proof should reflect this: we want a more conceptual explanation.

This conceptual gap has since been filled [17] — i.e. the case-by-case verification has been replaced with a general theorem. It turns out that something obeying the replicable formulas will also obey something called modular equations. A modular equation for a function \( f \) is a polynomial identity obeyed by \( f(x) \) and \( f(nx) \). The simplest examples come from the exponential and cosine functions: note that for any \( n > 0 \), \( \exp(nx) = (\exp(x))^n \) and \( \cos(nx) = T_n(\cos(x)) \) where \( T_n \) is a Tchebychev polynomial. A more interesting example of a modular equation is obeyed by \( J(\tau) = j(\tau) - 744 \): put \( X = J(\tau) \) and \( Y = J(2\tau) \), then

\[(X^2 - Y)(Y^2 - X) = 393768 (X^2 + Y^2) + 42987520 XY + 40491318744 (X + Y) - 120981708338256 .\]

Finding modular equations (for various elliptic functions) was a passion of the great mathematician Ramanujan. His notebooks are filled with them. See e.g. [7] for an application of Ramanujan’s modular equations to computing the first billion or so digits of \( \pi \). Many modular equations are also studied in [10]. For more of their applications, see e.g. [16]. It can be shown that the only functions \( f(\tau) = q^{-1} + a_1q + \cdots \) which obey modular equations for all \( n \), are \( J(\tau) \) and the ‘modular fictions’ \( q^{-1} \) and \( q^{-1} \pm q \) (which are essentially \( \exp \), \( \cos \), and \( \sin \)).
It was proved in [17] that, roughly speaking, a function \( B(\tau) = q^{-1} + \sum_{n>0} b_n q^n \)
which obeys enough modular equations, will either be of the form \( B(\tau) = q^{-1} + b_1 q \),
or will necessarily be a Hauptmodul for a modular group containing some \( \Gamma_0(N) \). The converse is also true: for instance, a modular equation for the Hauptmodul \( J_{25} \) of \( \Gamma_0(25) \) given in (2.3.5) is

\[
(X^2 - Y)(Y^2 - X) = -2(X^2 + Y^2) + 4(X + Y) - 4,
\]

where \( X = J_{25}(\tau) \) and \( Y = J_{25}(2\tau) \). To eliminate the conceptual gap, this result should then replace step 3. Steps 1 and 2 are still required, however.

This conceptual gap should not take away from what was a remarkable accomplishment by Borcherds: not only the proof of the Monstrous Moonshine conjectures, but also the definition of two new and important algebraic structures. I hope the preceding sections give the reader some indication of why Borcherds was awarded one of the 1998 Fields Medals.

Another approach to the Hauptmodul property is by Tuite [57], who related it to the (conjectured) uniqueness of \( V^\natural \). Norton has suggested that the reason \( \mathcal{M} \) is associated to genus-zero modular functions could be what he calls its ‘6-transposition’ property [47].

So has Moonshine been explained? According to Conway, McKay, and many others, it hasn’t. They consider VOAs in general, and \( V^\natural \) in particular, to be too complicated to be God-given. The progress, though impressive, has broadened not lessened the fundamental mystery, they would argue.

For what it’s worth, I don’t completely agree. Explaining away a mystery is a little like grasping a bar of soap in a bathtub, or quenching a child’s curiosity. Only extreme measures like pulling the plug, or enrollment in school, really work. True progress means displacing the mystery, usually from the particular to the general. Why is the sky blue? Because of how light scatters in gases. Why are Hauptmoduls attached to each \( g \in \mathcal{M} \)? Because of \( V^\natural \). Mystery exists wherever we can ask ‘why’ — like beauty it’s in the beholder’s eye.

Moonshine is now ‘leaving the nest’. We are entering a consolation phase, tidying up, generalising, simplifying, clarifying, working out more examples. Important and interesting discoveries will be made in the next few years, and yes there still is mystery, but no longer does a Moonshiner feel like an illicit distiller: Moonshine is now a day-job!

Acknowledgments. I warmly thank the Feza Gursey Institute in Istanbul, and in particular Teoman Turgut, for their invitation to the workshop and hospitality during my month-long stay. These notes are based on 16 lectures I gave there in Summer 1998. I’ve also benefitted from numerous conversations with Y. Billig, A. Coste, C. Cummins, M. Gaberdiel, J. McKay, M. Tuite, and M. Walton — Mark Walton in particular made a very careful reading of the manuscript (and hence must share partial blame for any errors still remaining). My appreciation as well goes to J.-B. Zuber and P. Ruelle for sharing with me their personal stories behind the discoveries of, respectively, A-D-E in \( A_1(1) \) and Fermat in \( A_2(1) \). The research was supported in part by NSERC.
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