Cofiniteness with respect to extension of Serre subcategories

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Abstract

Let \( a \) be an ideal of a commutative noetherian ring \( R \), \( S \) a Serre subcategory of \( R \)-modules satisfying the condition \( C_a \) and \( N \) the subcategory of finitely generated \( R \)-modules. In this paper, we continue the study of \( N S-a \)-cofinite modules with respect to the extension subcategory \( N S \), show that some classical results of \( a \)-cofiniteness hold for \( N S-a \)-cofiniteness in the cases \( \dim R = d \) or \( \dim R/a = d - 1 \), where \( d \) is a positive integer. We also study \( N S-a \)-cofiniteness of local cohomology modules and the modules \( \text{Ext}^i_R(N, M) \) and \( \text{Tor}^i_R(N, M) \).

Key Words: Serre subcategory; \( N S-a \)-cofinite module

2020 Mathematics Subject Classification: 13E05; 13C15

Introduction and Preliminaries

Throughout this paper, \( R \) is a commutative noetherian ring with identity, \( a \) is a proper ideal of \( R \) and \( S \) is a Serre subcategory of \( R \)-modules, that is, \( S \) is closed under taking submodules, quotients and extensions. Alipour and Sazeedeh [3] introduced the cofiniteness with respect to \( S \) and \( a \). An \( R \)-module \( M \) is said to be \( S-a \)-cofinite if \( \text{Supp}_R M \subseteq \text{V}(a) \) and \( \text{Ext}^i_R(R/a, M) \in S \) for all \( i \geq 0 \).

Let \( N \) be the subcategory of finitely generated \( R \)-modules. The extension subcategory induced by \( N \) and \( S \) is denoted by \( NS \), consisting of those \( R \)-modules \( M \) for which there exist an exact sequence \( 0 \to N \to M \to S \to 0 \) such that \( N \in N \) and \( S \in S \). It follows from [25, Corollary 3.3] that \( NS \) is Serre. When \( S = 0 \), an \( NS-a \)-cofinite module was known as classical \( a \)-cofinite module, defined for the first time by Hartshorne [15], giving a negative answer to a question of [14, Expos XIII, Conjecture 1.1], studied by numerous authors [6, 7, 8, 20, 21, 23]. When \( S = A \) the subcategory of artinian modules, they are \( a \)-cominimax modules studies in [6, 26] and when \( S = F \) the subcategory of all modules of finite support, they are \( a \)-weakly cofinite modules studies in [3, 12]. Recall that \( S \) satisfies the condition \( C_a \) if for every \( R \)-module \( M \), the following implication holds.

\[ C_a: \text{If } M \in S, \text{ then } (0 :_M a) \text{ is in } S. \]

By [2, Lemma 2.2], the following Serre subcategories satisfy the condition \( C_a \). The class of zero modules; The class of artinian \( R \)-modules; The class of artinian \( a \)-cofinite \( R \)-modules;
The class of $R$-modules with finite support; The class of $R$-modules with finite Krull dimension. In this paper, we always assume that $S$ satisfies the condition $C_a$.

The support of the Serre subcategory $S$ is denoted by $\text{Supp}S$ which is

$$\text{Supp}S = \bigcup_{M \in S} \text{Supp}_R M = \{p \in \text{Spec}R \mid R/p \in S\}.$$ 

For an $R$-module $M$, we denote by $\text{Max}M$ the set of maximal ideal in $\text{Supp}_R M$. Assume that $S$ satisfies the condition $C_a$. Alipour and Sazeedeh [24] extended the fundamental results about $a$-cofinite modules at small dimensions to $\mathcal{N}S$-$a$-cofinite modules. They showed that if $M$ is an $\mathcal{NS}$-$a$-cofinite $R$-module of dimension $\leq 1$ with $\text{Max}M \subseteq \text{Supp}S$ and $N$ is a finitely generated $R$-module, then $\text{Ext}^i_R(N, M)$ is $\mathcal{NS}$-$a$-cofinite for each $i \geq 0$ (see [24, Theorem 2.7]); if $\dim R/a = 1$ and $\text{Max}M \subseteq \text{Supp}S$ then $M$ is $\mathcal{NS}$-$a$-cofinite if and only if $\text{Supp}_R M \subseteq V(a)$ and $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for $i = 0, 1$ (see [3, Theorem 3.2]); if $R$ is a local ring with $\dim R/a = 2$ and satisfies some further conditions, then an $R$-module $M$ is $\mathcal{NS}$-$a$-cofinite if and only if $\text{Supp}_R M \subseteq V(a)$ and $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for $i = 0, 1, 2$ (see [24, Corollary 2.11]). They also investigated $\mathcal{NS}$-$a$-cofiniteness of local cohomology modules (see [24, Theorem 2.13]).

The first aim of this paper is to improve Alipour and Sazeedeh’s results in [3], that is to say, eliminate the hypothesis $\text{Max}M \subseteq \text{Supp}S$ entirely. We show that

**Theorem 1.** Let $M$ be an $R$-module with $\dim R M \leq 1$. Then $M$ is $\mathcal{NS}$-$a$-cofinite if and only if $\text{Supp}_R M \subseteq V(a)$ and $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for $i = 0, 1$ (see Theorem 1.3).

The second aim of this paper is to extend the results about $a$-cofiniteness in the cases $\dim R = d \geq 1$ or $\dim R/a = d - 1$ to $\mathcal{NS}$-$a$-cofiniteness, and improve Sazeedeh’s some results in [24]. More precisely, we show that

**Theorem 2.** Let $a$ be an ideal of $R$ such that either $\dim R/a = d - 1$ or $\dim R = d$. Then an $a$-torsion $R$-module $M$ is $\mathcal{NS}$-$a$-cofinite if and only if $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for $i \leq d - 1$ (see Theorem 3.4 and Corollary 1.7).

**Theorem 3.** Let $M$ be an $\mathcal{NS}$-$a$-cofinite $R$-module and $N$ a finitely generated $R$-module with $\dim R N \leq 2$. Then the $R$-modules $\text{Ext}^i_R(N, M)$ and $\text{Tor}^i_R(N, M)$ are $\mathcal{NS}$-$a$-cofinite for all $i \geq 0$ (see Theorem 3.4).

As applications of these results, we show that if either $\dim R \leq 2$ or $\dim R/a \leq 1$ then the subcategory $\mathcal{NS}(R, a)_{cof} = \{M \in \text{Mod-}R \mid R M \in \mathcal{NS}$-$a$-cofinite $\}$ is abelian, and some results about $\mathcal{NS}$-$a$-cofiniteness of local cohomology modules are given.

Next we recall some notions which we will need later.

We write $\text{Spec}R$ for the set of prime ideals of $R$ and $\text{Max}R$ for the set of maximal ideals of $R$. For an ideal $a$ in $R$, we set

$$V(a) = \{p \in \text{Spec}R \mid a \subseteq p\}.$$
Let $M$ be an $R$-module. The **associated prime** of $M$, denoted by $\text{Ass}_RM$, is the set of prime ideals $p$ of $R$ such that there exists a cyclic submodule $N$ of $M$ with $p = \text{Ann}_RN$. The set of prime ideals $p$ such that there exists a cyclic submodule $N$ of $M$ with $p \supseteq \text{Ann}_RN$ is well-known to be the **support** of $M$, denoted by $\text{Supp}_RM$, which is equal to the set
\[ \{ p \in \text{Spec}R | M_p \neq 0 \}. \]

A prime ideal $p$ is said to be an **attached prime** of $M$ if $p = \text{Ann}_R(M/L)$ for some submodule $L$ of $M$. The set of attached primes of $M$ is denoted by $\text{Att}_RM$. If $M$ is artinian, then $M$ admits a minimal secondary representation $M = M_1 + \cdots + M_r$ so that $M_i$ is $p_i$-secondary for $i = 1, \cdots, r$. In this case, $\text{Att}_RM = \{ p_1, \cdots, p_r \}$.

The **arithmetic rank** of $a$, denoted by $\text{ara}(a)$, is the least number of elements of $R$ required to generate an ideal which has the same radical as $a$, i.e.,
\[ \text{ara}(a) = \min\{ n \geq 0 | \exists a_1, \cdots, a_n \in R \text{ with } \text{Rad}(a_1, \cdots, a_n) = \text{Rad}(a) \}. \]

For an $R$-module $M$, the arithmetic rank of $a$ with respect to $M$, denoted by $\text{ara}_M(a)$, is defined by the arithmetic rank of the ideal $a + \text{Ann}_RM/\text{Ann}_RM$ in the ring $R/\text{Ann}_RM$.

The $i$th **local cohomology** of an $R$-module $M$ with respect to $a$ is
\[ H^i_a(M) := \lim_{\to} \text{Ext}^i_R(R/a^n, M). \]

The reader can refer to [11] for more details about local cohomology. The module $M$ is called **$a$-torsion** if $\Gamma_a(M) := H^0_a(M) = M$, or equivalently, $\text{Supp}_RM \subseteq V(a)$.

For an arbitrary $R$-module $M$, set
\[ \text{cd}(a, M) = \sup\{ n \in \mathbb{Z} | H^n_a(M) \neq 0 \}. \]

The **cohomological dimension** of $a$ is
\[ \text{cd}(a, R) = \sup\{ \text{cd}(a, M) | M \text{ is an } R\text{-module} \}. \]

#### 1. Cofiniteness with respect to extension subcategories

Let $d$ be a positive integer such that either $\dim R/a = d - 1$ or $\dim R = d$. It is shown that an $R$-module $M$ is $\mathcal{N}S$-$a$-cofinite if and only if $\text{Supp}_RM \subseteq V(a)$ and $\text{Ext}^i_R(R/a, M) \in \mathcal{N}S$ for $i = 0, \cdots, d - 1$. Moreover, we show that the subcategory $\mathcal{N}S(R, a)_{\text{cof}}$ is abelian in the cases $\dim R \leq 2$ and $\dim R/a \leq 1$.

**Lemma 1.1.** Let $M$ be an $R$-module such that $(0 :_M a) \in \mathcal{N}S$. Then $(0 :_M a^n) \in \mathcal{N}S$ for all $n \geq 1$.

**Proof.** This follows from the proof of [24, Theorem 2.15]. \qed

The following lemma is used at several places of this paper.

**Lemma 1.2.** Let $M$ be an $R$-module of zero dimension. Then $M$ is $\mathcal{N}S$-$a$-cofinite if and only if $\text{Supp}_RM \subseteq V(a)$ and $\text{Hom}_R(R/a, M) \in \mathcal{N}S$. 

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Proof. ‘Only if’ part is trivial.

‘If’ part. By assumption, there exists a short exact sequence

\[ 0 \to N \to \text{Hom}_R(R/a, M) \to S \to 0 \]

with \( N \in \mathcal{N} \) and \( S \in \mathcal{S} \). If \( \text{Ass}_R M \subseteq \text{Supp}\mathcal{S} \), then \( \text{Ass}_R N \subseteq \text{Supp}\mathcal{S} \) and so \( N \in \mathcal{S} \) by a finite filtration of \( N \). Thus \( \text{Hom}_R(R/a, M) \in \mathcal{S} \). Since \( \mathcal{S} \) satisfies the condition \( C_a \), one has \( M \in \mathcal{S} \). Hence \( [4 \text{ Lemma } 2.1] \) implies that \( \text{Ext}^i_R(R/a, M) \in \mathcal{N}\mathcal{S} \) for all \( i \geq 0 \). Now assume that \( \text{Ass}_R M \not\subseteq \text{Supp}\mathcal{S} \), and let \( \Phi = \{ p | p \in \text{Ass}_R M \cap \text{Supp}\mathcal{S} \} \). By \( [10 \text{ Ch.IV, Section } 1.2, \text{ Proposition } 4] \), there is a submodule \( K \) of \( M \) such that \( \text{Ass}_R K = \text{Ass}_R M \setminus \Phi \) and \( \text{Ass}_R M/K = \Phi \subseteq \text{Supp}\mathcal{S} \). As \( \text{Supp}_R K \cap \text{Supp}\mathcal{S} = \emptyset \) and \( \text{Hom}_R(R/a, K) \in \mathcal{N}\mathcal{S} \), it follows that \( \text{Hom}_R(R/a, K) \) has finite length. So \( K \) is artinian \( a \)-cofinite by \( [20 \text{ Proposition } 4.1] \) and then \( \text{Ext}^i_R(R/a, K) \in \mathcal{N} \) for all \( i \geq 0 \). Hence the exact sequence \( 0 \to K \to M \to M/K \to 0 \) implies that \( \text{Hom}_R(R/a, M/K) \in \mathcal{N}\mathcal{S} \). Since \( \text{Ass}_R M/K \subseteq \text{Supp}\mathcal{S} \), by the preceding proof, \( M/K \in \mathcal{S} \). Hence the above sequence yields that \( \text{Ext}^i_R(R/a, M) \in \mathcal{N}\mathcal{S} \) for all \( i \geq 0 \). \( \square \)

We now present the first main theorem of this section, which eliminates the hypothesis \( \text{Max}M \subseteq \text{Supp}\mathcal{S} \) in \( [3 \text{ Theorem } 3.2] \).

**Theorem 1.3.** Let \( M \) be an \( R \)-module with \( \dim_R M \leq 1 \). Then \( M \) is \( \mathcal{N}\mathcal{S} \)-\( a \)-cofinite if and only if \( \text{Supp}_R M \subseteq V(a) \) and \( \text{Ext}^i_R(R/a, M) \in \mathcal{N}\mathcal{S} \) for \( i = 0, 1 \).

Proof. ‘Only if’ part is obvious.

‘If’ part. By Lemma \( [1 \text{ Lemma } 1.2] \) we may assume \( \dim_R M = 1 \), and let \( t = \text{ara}_M(a) \). If \( t = 0 \), then \( M = (0 :_M a^n) \) for some \( n \geq 1 \), and so the assertion follows by Lemma \( [1 \text{ Lemma } 2.1] \). Next assume that \( t > 0 \). Let \( \Phi = \{ p | p \in \text{Ass}_R M \cap \text{Supp}\mathcal{S} | \dim_R/p = 1 \} \). Then there is a submodule \( K \) of \( M \) so that \( \text{Ass}_R K = \Phi \) and \( \text{Ass}_R M/K = \text{Ass}_R M/\Phi \) by \( [10 \text{ Ch.IV, Section } 1.2, \text{ Proposition } 4] \). As \( \text{Hom}_R(R/a, K) \in \mathcal{N}\mathcal{S} \), there exists an exact sequence

\[ 0 \to N' \to \text{Hom}_R(R/a, K) \to S' \to 0 \]

with \( N' \in \mathcal{N} \) and \( S' \in \mathcal{S} \). Note that \( \text{Ass}_R K \subseteq \text{Supp}\mathcal{S} \), so a finite filtration of \( N' \) forces that \( N' \in \mathcal{S} \), and hence \( \text{Hom}_R(R/a, K) \in \mathcal{S} \). As \( \mathcal{S} \) satisfies the condition \( C_a \), one has \( K \in \mathcal{S} \) and then \( \text{Ext}^i_R(R/a, K) \in \mathcal{S} \) for all \( i \geq 0 \). Replacing \( M \) by \( M/K \) we may assume that every \( p \in \text{Ass}_R M \) with \( \dim_R/p = 1 \) is not in \( \text{Supp}\mathcal{S} \). Let \( \Phi = \{ p \in \text{Ass}_R M | \dim_R/p = 1 \} \). There is a submodule \( L \) of \( M \) so that \( \text{Ass}_R L = \text{Ass}_R M \setminus \Phi \) and \( \text{Ass}_R M/L = \Phi \) by \( [10 \text{ Ch.IV, Section } 1.2, \text{ Proposition } 4] \). Since \( \text{Hom}_R(R/a, L) \in \mathcal{N}\mathcal{S} \) and \( \dim_R L = 0 \), it follows from Lemma \( [1 \text{ Lemma } 1.2] \) that \( L \) is \( \mathcal{N}\mathcal{S} \)-\( a \)-cofinite and hence \( \text{Ext}^i_R(R/a, M/L) \in \mathcal{N}\mathcal{S} \) for \( i = 0, 1 \). Replacing \( M \) by \( M/L \) we may further assume that \( \text{Ass}_R M = \{ p \in \text{Supp}_R M | \dim_R/p = 1 \} \). Since \( \text{Hom}_R(R/a, M) \in \mathcal{N}\mathcal{S} \), there exists a short exact sequence

\[ 0 \to N \to \text{Hom}_R(R/a, M) \to S \to 0 \]
with $N \in \mathcal{N}$ and $S \in \mathcal{S}$, which implies that the set $\text{Ass}_R M$ is finite. Also for each $p \in \text{Ass}_R M$, the $R_p$-module $\text{Hom}_{R_p}(R_p/aR_p, M_p)$ is finitely generated and $M_p$ is $aR_p$-torsion with $\text{Supp}_{R_p} M_p \subseteq V(pR_p)$, it follows from [20 Proposition 4.1] that the $R_p$-module $M_p$ is artinian $aR_p$-cofinite. Let $\text{Ass}_R M = \{ p_1, \ldots, p_n \}$. It follows from [6 Lemma 2.5] that $V(aR_p) \cap \text{Att}_{R_p} M_p \subseteq V(p_j R_p)$ for $j = 1, \ldots, n$. Set

$$U = \bigcup_{j=1}^n \{ q \in \text{Spec} R | q R_{p_j} \in \text{Att}_{R_{p_j}} M_{p_j} \}. $$

Then $U \cap V(a) \subseteq \text{Ass}_R M$. On the other hand, for each $q \in U$ we have $q R_{p_j} \in \text{Att}_{R_{p_j}} M_{p_j}$, for some $1 \leq j \leq n$. Thus

$$(\text{Ann}_R M) R_{p_j} \subseteq \text{Ann}_{R_{p_j}} M_{p_j} \subseteq q R_{p_j},$$

and so $\text{Ann}_R M \subseteq q$. Since $t = \text{ara}_M(a) \geq 1$, there exist $y_1, \ldots, y_t \in a$ such that

$$\text{Rad}(a + \text{Ann}_R M) = \text{Rad}((y_1, \ldots, y_t) + \text{Ann}_R M).$$

As $a \notin (\bigcup_{q \in U \setminus V(a)} q) \cup (\bigcup_{p \in \text{Ass}_R M} p)$, we have $(y_1, \ldots, y_t) \notin (\bigcup_{q \in U \setminus V(a)} q) \cup (\bigcup_{p \in \text{Ass}_R M} p)$. Hence [19 Ex.16.8] provides an element $a_1 \in (y_2, \ldots, y_t)$ so that $y_1 + a_1 \notin (\bigcup_{q \in U \setminus V(a)} q) \cup (\bigcup_{p \in \text{Ass}_R M} p)$. Set $x = y_1 + a_1$. Then $x \in a$ and there is an exact sequence $0 \to M \xrightarrow{a} M \to M/xM \to 0$. By assumption, $\text{Hom}_R(R/a, M/xM) \in \mathcal{NS}$, it follows from Lemma 1.2 that $M/xM$ is $\mathcal{NS}$-$a$-cofinite since $\dim_R M/xM = 0$. Therefore, by [3 Lemma 2.2], one has $M$ is $\mathcal{NS}$-$a$-cofinite, as desired. □

The following corollary generalizes [21 Theorem 2.3] and [3 Theorem 3.2].

**Corollary 1.4.** If $\dim R/a \leq 1$, then an $a$-torsion $R$-module $M$ is $\mathcal{NS}$-$a$-cofinite if and only if $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for $i = 0, 1$.

An $R$-module $M$ is said to be weakly Laskerian if the set $\text{Ass}_R M/N$ is finite for each submodule $N$ of $M$.

**Corollary 1.5.** Let $M$ be a weakly Laskerian $R$-module such that $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for $i = 0, 1$. Then $M$ is $\mathcal{NS}$-$a$-cofinite.

**Proof.** As $M$ is weakly Laskerian, there is an exact sequence $0 \to N \to M \to F \to 0$ such that $N \in \mathcal{N}$ and $F \in \mathcal{F}$ by [21 Theorem 3.3]. Note that $\dim_R F \leq 1$ and $\text{Ext}^i_R(R/a, F) \in \mathcal{NS}$ for $i = 0, 1$ by assumption, it follows from Theorem 1.3 that $F$ is $\mathcal{NS}$-$a$-cofinite, and then $M$ is $\mathcal{NS}$-$a$-cofinite. □

The next is the second main theorem of this section, which is a nice generalization of [21 Theorem 2.3] and [3 Theorem 3.5].

**Theorem 1.6.** Assume that $\dim R/a = d \geq 1$. Then an $R$-module $M$ is $\mathcal{NS}$-$a$-cofinite if and only if $\text{Supp}_R M \subseteq V(a)$ and $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for all $i \leq d$. 

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Proof. ‘Only if’ part is trivial.
   ‘If’ part. We proceed by induction on $d$. If $d = 1$ then the assertion follows by Corollary 1.4. Suppose, inductively, $d > 1$ and the result has been proved for smaller values of $d$. If $a$ is nilpotent, say $a^n = 0$ for some integer $n$, then $M = (0:_R a^n) \in NS$ by Lemma 1.1 as $(0:_M a) \in NS$ and so $M$ is $NS$-a-cofinite. Now assume that $a$ is not nilpotent. We can choose a positive integer $n$ such that $(0:_R a^n) = \Gamma_a(R)$. Put $\overline{R} = R/\Gamma_a(R)$ and $\overline{M} = M/(0:_M a^n)$ which is an $\overline{R}$-module. Taking $\overline{a}$ as the image of $a$ in $\overline{R}$, we have $\Gamma_\pi(\overline{R}) = 0$. Thus $\overline{a}$ contains an $\overline{R}$-regular element so that $\dim R/a + \Gamma_a(R) \leq d - 1$. Note that $\text{Supp}_R(R/a + \Gamma_a(R)) \subseteq V(a)$, by the assumption and [3, Lemma 2.1], one has $\text{Ext}_R^i(R/a + \Gamma_a(R), M) \in NS$ for $i \leq d$. Also $(0:_M a^n) \in NS$, and thus $\text{Ext}_R^i(R/a + \Gamma_a(R), \overline{M}) \in NS$ for $i \leq d$. On the other hand, it is clear that $\text{Supp}_R\overline{M} \subseteq V(a + \Gamma_a(R))$. By the inductive hypothesis, the $R$-module $\overline{M}$ is $NS$-a + $\Gamma_a(R)$-cofinite, and then $\overline{M}$ is $NS$-a-cofinite by the proof of [24 Theorem 2.15]. Therefore, $(0:_M a^n) \in NS$ forces that $M$ is $NS$-a-cofinite.

The following corollary is a nice generalization of [24 Corollaries 2.16 and 2.18].

Corollary 1.7. If $\dim R = d \geq 1$, then an $a$-torsion $R$-module $M$ is $NS$-a-cofinite if and only if $\text{Ext}_R^i(R/a, M) \in NS$ for all $i \leq d - 1$.

Proof. Let $a$ be an ideal of $R$ with $\dim R/a \leq d - 1$. It follows from Theorem 1.6 that $M$ is $NS$-a-cofinite if and only if $\text{Ext}_R^i(R/a, M) \in NS$ for all $i \leq d - 1$. Hence [24 Theorem 2.15] yields the desired statement. □

The next result eliminates the hypothesis $\text{Max} M \subseteq \text{Supp} S$ in [3, Theorem 3.4].

Corollary 1.8. (1) Let $NS^1(R, a)_{cof}$ denote the category of $NS$-a-cofinite $R$-modules $M$ with $\dim_R M \leq 1$. Then $NS^1(R, a)_{cof}$ is abelian.
   (2) If either $\dim R \leq 2$ or $\dim R/a \leq 1$, then $NS(R, a)_{cof}$ is abelian.

Proof. We just prove (2) since the proof of (2) is similar.

Given an $R$-homomorphism $f : M \to N$ in $NS^1(R, a)_{cof}$, set $K = \ker f$, $I = \text{im} f$ and $C = \text{coker} f$. It is easy to obtain that $\text{Hom}_R(R/a, K), \text{Ext}_R^1(R/a, K) \in NS$ and hence the module $K \in NS^1(R, a)_{cof}$ by Theorem 1.4. This implies that $I \in NS^1(R, a)_{cof}$ and consequently $C \in NS^1(R, a)_{cof}$, as required. □

The following corollary is a generalization [3, Theorem 2.7].

Corollary 1.9. Let $M$ be an $NS$-a-cofinite $R$-module with $\dim_R M \leq 1$ and $N$ a finitely generated $R$-module. Then the $R$-modules $\text{Tor}_i^R(N, M)$ and $\text{Ext}_R^i(N, M)$ are $NS$-a-cofinite for all $i \geq 0$.

Proof. Since $N$ is finitely generated, $N$ has a free resolution

$$F^\bullet : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$
where all $F_i$ have finite ranks. Then $\text{Tor}^R_i(N, M) = H_i(F^\bullet \otimes_R M)$ and $\text{Ext}^i_R(N, M) = H^i(\text{Hom}_R(F^\bullet, M))$ are subquotients of a direct sum of finitely many copies of $M$. Now, the assertion follows from Corollary 1.8(1). □

The next result is a more general version of [23, Theorem 2.8].

**Corollary 1.10.** If either $\dim R = d \geq 3$ or $\dim R/a = d - 1$, then the subcategory $\mathcal{NS}(R, a)_{\text{cof}}$ is abelian if and only if for any homomorphism $f : M \to N$ in $\mathcal{NS}(R, a)_{\text{cof}}$ and $i \leq d - 2$, $\text{Ext}^i_R(R/a, \text{coker}f) \in \mathcal{NS}$.

**Proof.** ‘Only if’ part is trivial.

‘If’ part. Since $\text{Ext}^i_R(R/a, \text{coker}f) \in \mathcal{NS}$ for all $i \leq d - 2$, we have $\text{Ext}^i_R(R/a, \text{im}f) \in \mathcal{NS}$ for all $i \leq d - 1$, and hence $\text{im}f$ is $\mathcal{NS}$-$a$-cofinite by Theorem 1.6 and Corollary 1.7. This implies that $\ker f$ and therefore $\text{coker}f$ is $\mathcal{NS}$-$a$-cofinite, as desired. □

The following result is a generalization of [23, Proposition 2.10].

**Proposition 1.11.** Let $a$ and $b$ be two ideals of $R$ with $b \subseteq a$ and $M$ an $R$-module. If $n$ is a nonnegative integer such that $\text{Ext}^i_R(R/b, M)$ are $\mathcal{NS}$-$a$-cofinite for all $i \leq n$, then $\text{Ext}^i_R(R/a, M) \in \mathcal{NS}$ for all $i \leq n$.

**Proof.** Assume that $0 \to M \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$ is an injective resolution of $M$. We get the exact sequences $0 \to M^i \to E^i \to M^{i+1} \to 0$ and isomorphisms

$$\text{Ext}^{i+1}_R(R/a, M) \cong \text{Ext}^{i}_R(R/a, M^i), \text{ Ext}^{i+1}_R(R/b, M) \cong \text{Ext}^{i}_R(R/b, M^i),$$

where $M^i = \ker d^i$ for $i \geq 0$. Hence, for each $i \geq 0$, there is an exact sequence

$$0 \to (0 :)_{M^i} \to (0 :)_{E^i} \xrightarrow{f^i} (0 :)_{M^{i+1}} \to \text{Ext}^{i+1}_R(R/b, M) \to 0.$$  

We first show that $\text{Ext}^s_{R/b}(R/a, \text{Ext}^i_R(R/b, M)) \in \mathcal{NS}$ for all $s \geq 0$ and $0 \leq i \leq n$. Consider the Grothendieck spectral sequences

$$E^{p,q}_2 = \text{Ext}^p_{R/b}(\text{Tor}^R_q(R/b, R/a), \text{Ext}^i_R(R/b, M)) \Rightarrow \text{Ext}^{p+q}_R(R/a, \text{Ext}^i_R(R/b, M)).$$

For $s = 0$, we have $\text{Hom}_{R/b}(R/a, \text{Ext}^i_R(R/b, M)) \cong \text{Hom}_R(R/a, \text{Ext}^i_R(R/b, M)) \in \mathcal{NS}$ for $0 \leq i \leq n$. Now, assume that $s > 0$ and the result has been proved for all values smaller than $s$. Then $E^{p,0}_2 = \text{Ext}^p_{R/b}(R/a, \text{Ext}^i_R(R/b, M)) \in \mathcal{NS}$ for all $0 \leq p < s$. Since $\text{Supp}_{R/b} \text{Tor}^R_q(R/b, R/a) \subseteq \text{Supp}_{R/b} R/a$, it follows from [3, Lemma 2.4] that $E^{p,q}_2 \in \mathcal{NS}$ for all $0 \leq p < s$ and $q \geq 0$. There exists a finite filtration

$$0 = \Phi^{s+1}H^s \subset \cdots \subset \Phi^1H^s \subset \Phi^0H^s \subset H^s := \text{Ext}^i_R(R/a, \text{Ext}^i_R(R/b, M)),$$

such that $E^{s,0}_\infty \cong \Phi^sH^s/\Phi^{s+1}H^s = \Phi^sH^s$ is a submodule of $H^s \in \mathcal{NS}$, and so $E^{s,0}_\infty \in \mathcal{NS}$. For $r \geq 2$, consider the differential
Theorem 2.1. Let 
result generalizes [23, Theorem 3.3 and Proposition 3.4] and [3, Theorem 3.5].

For $0 \leq \dim_\mathbb{K} \mathcal{H}$, we have an exact sequence 

\[ E^s_{-r-1} \xrightarrow{d_r^s} E^s_0 \xrightarrow{d_r^0} E^s_{-r+1} = 0. \]

We have an exact sequence $E^s_{-r-1} \to E^s_0 \to E^s_{-r+1} \to 0$. As $E^s_0 \cong E^s_\infty \in \mathcal{NS}$ for $r \gg 0$, the sequence implies that $E_2^s = \text{Ext}^s_{R/\mathbb{K}}(R/\mathfrak{a}, \text{Ext}^i_R(R/\mathfrak{b}, M)) \in \mathcal{NS}$ for all $s \geq 0$ and $0 \leq i \leq n$. Next consider the Grothendieck spectral sequences

\[ E_2^{p,q} = \text{Ext}^p_{R/\mathbb{K}}(R/\mathfrak{a}, \text{Ext}^q_R(R/\mathfrak{b}, M)) \Rightarrow \text{Ext}^{p+q}_R(R/\mathfrak{a}, M). \]

For $0 \leq i \leq n$, there exists a finite filtration

\[ 0 = \Phi^{i+1}H^i \subseteq \Phi^iH^i \subseteq \cdots \subseteq \Phi^0H^i \subseteq \Phi^0H^i = H^i := \text{Ext}^i_R(R/\mathfrak{a}, M), \]

such that $\Phi^pH^i/\Phi^{p+1}H^i \cong E^{p,i-p}_r$ for $0 \leq p \leq i$. As $E^{p,i-p}_\infty$ is a subquotient of $E^{p,i-p}_2$, a successive use of the exact sequence

\[ 0 \to \Phi^{p+1}H^i \to \Phi^pH^i \to \Phi^pH^i/\Phi^{p+1}H^i \to 0 \]

implies that $\text{Ext}^i_R(R/\mathfrak{a}, M) \in \mathcal{NS}$ for all $i \leq n$. \[ \square \]

Corollary 1.12. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$ with $\mathfrak{b} \subseteq \mathfrak{a}$ and $M$ an $\mathfrak{a}$-torsion $R$-module.

1. If $\text{Ext}^i_R(R/\mathfrak{b}, M) \in \mathcal{NS}$-$\mathfrak{a}$-cofinite for each $i \geq 0$, then $M$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite.
2. For a non-negative integer $d$, if $\dim R/\mathfrak{a} = d$ and $\text{Ext}^i_R(R/\mathfrak{b}, M)$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite for $0 \leq i \leq d$, then $M$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite.

2. $\mathcal{NS}$-$\mathfrak{a}$-cofiniteness of local cohomology modules

This section, we study $\mathcal{NS}$-$\mathfrak{a}$-cofiniteness of local cohomology modules. The following result generalizes [23, Theorem 3.3 and Proposition 3.4] and [3, Theorem 3.5].

Theorem 2.1. Let $M$ be an $R$-module and $n$ a non-negative integer. If either $\dim R/\mathfrak{a} = 1$ or $\dim R/\mathfrak{a} = 2$, then $\text{Ext}^i_R(R/\mathfrak{a}, M) \in \mathcal{NS}$ for all $i \leq n+1$ if and only if $H^i_\mathfrak{a}(M)$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite for all $i \leq n$ and $\text{Hom}_R(R/\mathfrak{a}, H^{n+1}_\mathfrak{a}(M)) \in \mathcal{NS}$.

Proof. ‘If’ part follows from [9, Theorem 2.1].

‘Only if’ part. Set $s = 1$ in [9, Theorem 2.9], it is enough to show that $H^i_\mathfrak{a}(M)$ are $\mathcal{NS}$-$\mathfrak{a}$-cofinite for all $i \leq n$. We prove by induction on $n$. If $n = 0$ and $\text{Ext}^i_R(R/\mathfrak{a}, M) \in \mathcal{NS}$ for $i = 0$, then $\text{Hom}_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)), \text{Ext}^1_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)) \in \mathcal{NS}$, and so $\Gamma_\mathfrak{a}(M)$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite by Theorem [1,3] and Corollaries [1,4] and [1,7] and $\text{Hom}_R(R/\mathfrak{a}, H^1_\mathfrak{a}(M)) \in \mathcal{NS}$ by [9, Theorem 2.9]. Now, suppose that $n > 0$ and the result has been proved for smaller values of $n$. Then $H^i_\mathfrak{a}(X)$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite for $i \leq n - 1$ by the induction. Hence [9, Theorem 2.9] implies that $\text{Ext}^1_R(R/\mathfrak{a}, H^{n+1}_\mathfrak{a}(M)) \in \mathcal{NS}$ for $i = 0, 1$, and hence $H^i_\mathfrak{a}(M)$ is $\mathcal{NS}$-$\mathfrak{a}$-cofinite by Corollaries [1,4] and [1,7] and $\text{Hom}_R(R/\mathfrak{a}, H^{n+1}_\mathfrak{a}(M)) \in \mathcal{NS}$ by [9, Theorem 2.9]. \[ \square \]
Corollary 2.2. Let \( a, b \) be two ideals of \( R \) with \( b \subseteq a \), \( n \) a non-negative integer and \( M \) be an \( R \)-module such that \( H^i_b(M) \) is \( \mathcal{NS} \)-\( a \)-cofinite for \( i \leq n + 1 \). If either \( \dim R/a \leq 1 \) or \( \dim R \leq 2 \), then \( H^i_a(M) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \leq n \).

**Proof.** Consider the Grothendieck spectral sequence

\[
E_2^{p,q} = \text{Ext}_R^p(R/a, H^q_b(M)) \Rightarrow \text{Ext}_R^{p+q}(R/a, M).
\]

For \( 0 \leq i \leq n + 1 \), there exists a finite filtration

\[
0 = H^{i+1} \subseteq H^i \subseteq \cdots \subseteq H^1 \subseteq H^0 = H := \text{Ext}^i_R(R/a, M),
\]

such that \( \Phi^p H^i / \Phi^{p+1} H^i \cong E_2^{p,i-p} \) for \( 0 \leq p \leq i \). As \( E_2^{p,i-p} \) is a subquotient of \( E_2^{p,i-1} \), a successive use of the exact sequence

\[
0 \to \Phi^p H^i \to \Phi^p H^i / \Phi^{p+1} H^i \to 0
\]

implies that \( \text{Ext}_R^i(R/a, M) \in \mathcal{NS} \) for \( i \leq n + 1 \), and hence, by Theorem 2.1, \( H^i_a(M) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \leq n \). \( \square \)

Corollary 2.3. Let \( M \) be a weakly Laskerian \( R \)-module such that \( \text{Ext}^i_R(R/a, M) \in \mathcal{NS} \) for \( i = 0, 1 \). If either \( \dim R/a \leq 1 \) or \( \dim R \leq 2 \), then \( H^i_a(M) \) is \( \mathcal{NS} \)-\( a \)-cofinite for every \( i \geq 0 \).

**Proof.** As \( M \) is weakly Laskerian, there is an exact sequence \( 0 \to N \to M \to F \to 0 \) so that \( N \in \mathcal{N} \) and \( F \in \mathcal{F} \) by [5, Theorem 3.3]. Then \( F \) is \( \mathcal{NS} \)-\( a \)-cofinite by Theorem 2.1. Hence the above sequence and Theorem 2.1 yield the desired statement. \( \square \)

The next corollary is a more general version of [6, Theorem 2.15] and [16, Theorem 2.6].

Corollary 2.4. Let \( a, b \) be two ideals of \( R \) with \( b \subseteq a \), \( n \) a non-negative integer and \( M \) be an \( R \)-module such that \( \text{Ext}^i_R(R/b, M) \in \mathcal{NS} \) for \( i \leq n + 1 \). If \( \dim R/a = \dim R/b \leq 1 \), then \( H^i_a(H^j_b(M)) \) is \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \geq 0 \) and \( j \leq n \).

**Proof.** By Theorem 2.1, one has \( H^j_b(M) \) are \( \mathcal{NS} \)-\( b \)-cofinite for all \( j \leq n \), which implies that \( \text{Ext}^i_R(R/a, H^j_b(M)) \in \mathcal{NS} \) for all \( i \) and \( j \leq n \). Hence \( H^i_a(H^j_b(M)) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \geq 0 \) and \( j \leq n \) by Theorem 2.1 again. \( \square \)

The next corollary is a generalization of [20, Corollary 3.14 and Theorem 7.10] and [21, Corollary 2.12].

Corollary 2.5. If either \( \dim R/a \leq 1 \) or \( \dim R \leq 2 \) or \( \text{cd}(a, R) \leq 1 \), then \( H^i_a(M) \) is \( \mathcal{NS} \)-\( a \)-cofinite for any \( M \in \mathcal{NS} \) and every \( i \geq 0 \).

**Proof.** This follows from Theorem 2.1 and [9, Theorem 2.9]. \( \square \)

Corollary 2.6. Let \( M \neq 0 \) be in \( \mathcal{NS} \) such that \( \dim R/M/\mathfrak{a}M \leq 1 \). Then for each finitely generated \( R \)-module \( N \), the \( R \)-modules \( \text{Ext}^i_R(N, H^j_a(M)) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i, j \geq 0 \).
Proof. As $\text{Supp}_R H^d_a(M) \subseteq \text{Supp}_R M/aM$, it follows from Theorem 2.1 that $H^d_a(M)$ are $\mathcal{NS}$-$a$-cofinit for all $j \geq 0$. Now the assertion follows from Corollary 1.8. 

The following proposition is a more general version of [8] Theorem 3.7.

Proposition 2.7. Let $n$ be a non-negative integer such that $\text{Ext}_R^i(R/a, M) \in \mathcal{NS}$ for all $i \leq n + 1$. If either $\dim R = d \geq 3$ or $\dim R/a = d - 1$, then $H^i_a(M)$ is $\mathcal{NS}$-$a$-cofinite for $i < n$ if and only if $\text{Hom}_R(R/a, H^{i+d-3}_a(M)), \cdots, \text{Ext}_{R}^{d-3}(R/a, H^{i}_a(M)) \in \mathcal{NS}$ for $i \leq n$.

Proof. This follows from [9, Theorem 2.9] and Theorem 1.6 and Corollary 1.7.

The next result is a generalization of [20, Proposition 5.1].

Proposition 2.8. Let $M \in \mathcal{NS}$ be an $R$-module of dimension $d$. Then the top local cohomology module $H^d_a(M)$ is $\mathcal{NS}$-$a$-cofinite of zero dimension.

Proof. We use induction on $d$. This is clear if $d = 0$. So assume that $d > 0$ and replacing $M$ with $M/\Gamma_a(M)$, we may assume that $a$ contains an $M$-regular element $x$. By induction, $H^{d-1}_a(M/xM)$ is $\mathcal{NS}$-$a$-cofinite of zero dimension. Then the exact sequence

$$H^{d-1}_a(M/xM) \to H^d_a(M) \xrightarrow{\partial} H^d_a(M) \to 0$$

and Lemma 3.3 imply that $(0 : H^d_a(M) x)$ is $\mathcal{NS}$-$a$-cofinite of zero dimension. Thus, by [3, Lemma 2.2], $H^d_a(M)$ is $\mathcal{NS}$-$a$-cofinite of zero dimension.

An $R$-module $M$ is minimax if there is a finitely generated submodule $N$ of $M$, such that $M/N$ is artinian.

Corollary 2.9. Let $M$ be a minimax $R$-module of dimension $d$. Then $H^d_a(M)$ is artinian.

Proof. By Proposition 2.8, there is an exact sequence

$$0 \to N \to \text{Hom}_R(R/a, H^d_a(M)) \to A \to 0$$

with $N \in \mathcal{N}$ and $A \in \mathcal{A}$. But $\dim_R H^d_a(M) = 0$, it follows that $\text{Hom}_R(R/a, H^d_a(M))$ artinian, and so $H^d_a(M)$ is artinian.

The following proposition is a generalization of [11, Theorem 7.1.3].

Proposition 2.10. If $R/a \in \mathcal{S}$, then $H^i_a(M) \in \mathcal{S}$ for every $M \in \mathcal{NS}$ and all $i \geq 0$.

Proof. We use induction on $i$. First since $M \in \mathcal{NS}$, there is an exact sequence $0 \to N \to M \to S \to 0$ with $N \in \mathcal{N}$ and $S \in \mathcal{S}$. Then $H^0_a(N) = (0 : a^n)$ for some $n \geq 1$. Since $\text{Ass}_R(0 : a^n) \subseteq V(a)$, a finite filtration of $(0 : a^n)$ forces that $H^0_a(N) \in \mathcal{S}$. Also $H^0_a(S) \in \mathcal{S}$, so $H^0_a(M) \in \mathcal{S}$. Now assume, inductively, that $i > 0$ and that $H^{i-1}_a(M') \in \mathcal{S}$ for all finitely generated $R$-modules $M'$. Since $H^i_a(M) \cong H^i_a(M/\Gamma_a(M))$ for all $i > 0$, we may assume that $\Gamma_a(M) = 0$, and the ideal $a$ contains an $M$-regular element $x$. Then the exact sequence $0 \to M \xrightarrow{\partial} M \to M/xM \to 0$ induces the following exact sequence.
By induction, $H^i_a(M/xM) \rightarrow H^i_a(M) \rightarrow H^i_a(M)$. By Lemma 3.1, we may assume $\dim_a(S)$. This follows from [3, Lemma 2.3] and [13, Corollary 2.2.13].

**Corollary 2.11.** (1) Let $M$ be a minimax $R$-module. Then the $R$-module $H^i_a(M)$ is artinian for every $i \geq 0$ and $a \in \Max R$.

(2) Let $R$ be a local ring and $M$ a weakly Laskerian $R$-module. If $\dim R/a \leq 1$, then the set $\Supp_R H^i_a(M)$ is finite for every $i \geq 0$.

3. $\mathcal{N}S$-a-cofiniteness for extension and torsion functors

This section investigates $\mathcal{N}S$-a-cofiniteness of the $R$-modules $\Ext^i_R(N, M)$ and $\Tor^i_R(N, M)$. It is shown that $\Ext^i_R(N, M)$ and $\Tor^i_R(N, M)$ are $\mathcal{N}S$-a-cofinite for all $i \geq 0$ whenever $N$ is finitely generated with $\dim_R N \leq 2$ and $M$ is $\mathcal{N}S$-a-cofinite.

**Lemma 3.1.** Let $M$ be an $\mathcal{N}S$-a-cofinite $R$-module and $N$ a non-zero finite length $R$-module. Then $\Ext^i_R(N, M)$ and $\Tor^i_R(N, M)$ are in $\mathcal{N}S$ for all $i \geq 0$.

**Proof.** This follows from [3, Lemma 2.3] and [13, Corollary 2.2.13].

**Lemma 3.2.** Let $M$ be an $\mathcal{N}S$-a-cofinite $R$-module and $N$ a finitely generated $R$-module with $\dim_R N \leq 1$. Then $\Ext^i_R(N, M)$ and $\Tor^i_R(N, M)$ are $\mathcal{N}S$-a-cofinite of zero dimension for every $i \geq 0$.

**Proof.** By Lemma 3.1 we may assume $\dim_R N = 1$. It follows from [3, Lemma 2.1] that $\Ext^i_R(\Gamma_a(N), M) \in \mathcal{N}S$, and so $\Tor^i_R(\Gamma_a(N), M) \in \mathcal{N}S$ for all $i \geq 0$ by [13, Corollary 2.2.13]. The exact sequence $0 \to \Gamma_a(N) \to N \to \Gamma_a(N) \to 0$ induces the following two exact sequences

$$\Ext^{i-1}_R(\Gamma_a(N), M) \to \Ext^i_R(N/\Gamma_a(N), M) \to \Ext^i_R(N, M) \to \Ext^{i+1}_R(\Gamma_a(N), M),$$

$$\Tor^i_R(\Gamma_a(N), M) \to \Tor^i_R(N, M) \to \Tor^i_R(N/\Gamma_a(N), M) \to \Tor^{i+1}_R(\Gamma_a(N), M).$$

We may assume $\Gamma_a(N) = 0$. Then $a \not\subseteq \bigcup_{p\in \Ass_R N} p$ by [11, Lemma 2.1.1], and there exists an element $x \in a$ and an exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$, which induces the following exact sequence

$$\Ext^i_R(N/xN, M) \to \Ext^i_R(N, M) \xrightarrow{x} \Ext^i_R(N, M) \to \Ext^{i+1}_R(N/xN, M),$$

$$\Tor^i_R(N/xN, M) \to \Tor^i_R(N, M) \xrightarrow{x} \Tor^i_R(N, M) \to \Tor^{i+1}_R(N/xN, M).$$
for all \( i \geq 0 \). Hence we have an exact sequence 
\[
\text{Ext}_i^R(N/xN, M) \rightarrow (0 : \text{Ext}_i^R(N, M)) \rightarrow 0
\]
and 
\[
\text{Tor}_i^R(N/xN, M) \rightarrow (0 : \text{Tor}_i^R(N, M)) \rightarrow 0
\]
for \( i \geq 0 \). As the \( R \)-module \( N/xN \) is of finite length, 
\[
(0 : \text{Ext}_i^R(N, M)), (0 : \text{Tor}_i^R(N, M)) \in \mathcal{NS}
\]
by Lemma 3.1 and \( \dim_R(0 : \text{Ext}_i^R(N, M)) = 0 = \dim_R(0 : \text{Tor}_i^R(N, M)) \) for all \( i \geq 0 \). Thus 
\[
(0 : \text{Ext}_i^R(N, M)), (0 : \text{Tor}_i^R(N, M)) \in \mathcal{NS}
\]
dim \( (0 : \text{Ext}_i^R(N, M)) \) = \( 0 = \dim_R(0 : \text{Tor}_i^R(N, M)) \) for all \( i \geq 0 \). Note that \( \text{Supp}_R\text{Ext}_i^R(N, M) \), \( \text{Supp}_R\text{Tor}_i^R(N, M) \subseteq \text{V}(a) \), we have \( \dim_R\text{Ext}_i^R(N, M) = 0 = \dim_R\text{Tor}_i^R(N, M) \). Hence Lemma 3.2 implies that \( \text{Ext}_i^R(N, M) \) and \( \text{Tor}_i^R(N, M) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \geq 0 \). □

**Lemma 3.3.** The class of \( \mathcal{NS} \)-\( a \)-cofinite \( R \)-modules of zero dimension is closed under taking submodules and quotients.

*Proof.* Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be a short exact of \( R \)-modules with \( M \), \( \mathcal{NS} \)-\( a \)-cofinite of zero dimensions. Then \( \text{Hom}_R(R/a, L) \in \mathcal{NS} \). So \( L \) is \( \mathcal{NS} \)-\( a \)-cofinite by Lemma 1.2 and therefore \( N \) is \( \mathcal{NS} \)-\( a \)-cofinite. □

The next main theorem of this section generalizes [24, Theorems 2.8 and 2.10] and [11, Theorem 2.4] and [22, Theorem 2.4].

**Theorem 3.4.** Let \( M \) be an \( \mathcal{NS} \)-\( a \)-cofinite \( R \)-module and \( N \) a finitely generated \( R \)-module with \( \dim_R N \leq 2 \). Then \( \text{Ext}_i^R(N, M) \) and \( \text{Tor}_i^R(N, M) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \geq 0 \).

*Proof.* By analogy with the proof of Lemma 3.2 we may assume \( \Gamma_a(N) = 0 \) and \( \dim_R N = 2 \). Then there exists an element \( x \in a \) and an exact sequence \( 0 \rightarrow N_x \rightarrow N \rightarrow N/xN \rightarrow 0 \), which induces two exact sequences
\[
\text{Ext}_i^R(N/xN, M) \rightarrow \text{Ext}_i^R(N, M) \rightarrow \text{Ext}_i^{i+1}(N/xN, M),
\]
\[
\text{Tor}_i^{i+1}(N/xN, M) \rightarrow \text{Tor}_i^R(N, M) \rightarrow \text{Tor}_i^R(N/xN, M)
\]
for \( i \geq 0 \). Since \( \dim_R N/xN = 1 \), it follows from Lemma 3.2 that \( \text{Ext}_i^R(N/xN, M) \) and \( \text{Tor}_i^R(N/xN, M) \) are \( \mathcal{NS} \)-\( a \)-cofinite of zero dimension. Thus \( (0 : \text{Ext}_i^R(N, M)), (0 : \text{Tor}_i^R(N, M)) \) and \( \text{Ext}_i^R(N, M)/x\text{Ext}_i^R(N, M), \text{Tor}_i^R(N, M)/x\text{Tor}_i^R(N, M) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \geq 0 \) by Lemma 3.3. Consequently, by [3, Lemma 2.2], the \( R \)-modules \( \text{Ext}_i^R(N, M) \) and \( \text{Tor}_i^R(N, M) \) are \( \mathcal{NS} \)-\( a \)-cofinite for all \( i \geq 0 \). □

The following result is a generalization of [11, Theorems 2.5 and 2.10].

**Corollary 3.5.** Let \( (R, m) \) be local, and let \( M \) be an \( \mathcal{NF} \)-\( a \)-cofinite \( R \)-module and \( N \) a finitely generated \( R \)-module such that either \( \dim_R M = 2 \) or \( \dim_R N = 3 \). Then \( \text{Ext}_i^R(N, M) \) and \( \text{Tor}_i^R(N, M) \) are \( \mathcal{NF} \)-\( a \)-cofinite for all \( i \geq 0 \).

*Proof.* Denote \( \Phi \) the set of all modules \( \text{Ext}_i^R(R/a, \text{Ext}_i^R(N, M)) \) and \( \text{Ext}_i^R(R/a, \text{Tor}_i^R(N, M)) \) for \( i, j \geq 0 \). Let \( L \in \Phi \) and \( L' \) be a submodule of \( L \). It is enough to show that \( \text{Ass}_R L/L' \) is finite. To this end, according to [19, Exercise 7.7] and [17, Lemma 2.1] we may assume that
$R$ is complete. Suppose the contrary is true. Then there exists a countably infinite subset $\{p_k\}_{k=1}^\infty$ of $\text{Ass}_R L/L'$, such that none of which is not equal to $m$, and hence $m \not\subseteq \bigcup_{k=1}^\infty p_k$ by Lemma 3.2]. Let $S = R \setminus \bigcup_{k=1}^\infty p_k$. Then the $S^{-1}R$-module $S^{-1}M$ is $\mathcal{N}_S^{-1}a$-cofinite with $\dim_{S^{-1}R}S^{-1}M \leq 1$ or $\dim_{S^{-1}R}S^{-1}N \leq 2$, it follows from Corollary 1.9 and Theorem 3.4 that $S^{-1}L$ is a weakly Laskerian $S^{-1}R$-module and so $\text{Ass}_{S^{-1}R}(S^{-1}L/S^{-1}L')$ is a finite set. But $S^{-1}p_k \in \text{Ass}_{S^{-1}R}(S^{-1}L/S^{-1}L')$ for all $k = 1, 2, \cdots$, which is a contradiction. \[ \square \]

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