ON COHERENT HOPF 2-ALGEBRAS

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Abstract. We construct a coherent Hopf 2-algebra as a quantization of coherent 2-group, which consists of two Hopf coquasigroups and a coassociator. We also study quasi coassociative Hopf coquasigroups, which are shown to be coherent Hopf 2-algebras with nontrivial coassociators. As an example, we study functions on a Cayley algebra basis.

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1. Introduction

In this paper, we are going to study coherent Hopf 2-algebras (or noncoassociative quantum 2-groups) as a generalization of strict Hopf 2-algebras [5]. Since Hopf coquasigroups [6] are noncoassociative quantum groups that offer many interesting examples, we are motivated to construct coherent Hopf 2-algebras by Hopf coquasigroups. Moreover, the coassociators of Hopf coquasigroups will play an important role in the coherence condition of coherent Hopf 2-algebras.

Recall that, for a classical coherent 2-group, all the 1-arrows and 2-arrows are weakly invertible. Moreover, for the set of 2-arrows, there are two products: the ‘horizontal’ and the ‘vertical’ products, which form a nonassociative group and groupoid respectively. Therefore, by applying the idea of ‘2-arrow’ quantization, a coherent Hopf 2-algebra could consist of two Hopf coquasigroups, which correspond to the ‘quantum 1-arrows’ and ‘quantum 2-arrows’. Moreover, for the ‘quantum 2-arrows’, it is on the one hand, a Hopf coquasigroup corresponds to the ‘horizontal’ coproduct; on the other hand, a Hopf algebroid corresponds to the ‘vertical’ coproduct. These two coproducts also satisfy the interchange law. As a coherent Hopf 2-algebra, the coherence condition will be described by a coassociator, which satisfies the ‘3-cocycle’ condition.

We also study crossed comodules of Hopf coquasigroups as a generalization of crossed comodules of Hopf algebras [4]. We show that if a crossed comodule of Hopf coquasigroup is quasi coassociative, one can construct a coherent Hopf 2-algebra with a non-trivial coassociator. As an example, we study the Hopf coquasigroup which consists of functions
on Cayley algebra basis. We show that this Hopf coquasigroup is quasi coassociative and its coassociator is controlled by a 3-coboundary.

The paper is organised as follows: In §2 and §3, we will give a brief introduction of coherent 2-groups, Hopf coquasigroups and Hopf algebroids. In §4, we will define coherent Hopf 2-algebras and study their properties. In §5, we will introduce crossed comodules of Hopf coquasigroups and quasi coassociative Hopf coquasigroups, which are shown to be coherent Hopf 2-algebras under some conditions. In §6, we study a special coherent Hopf 2-algebras, namely, a Hopf coquasigroup which consists of functions on a Cayley algebra basis.

2. Coherent 2-groups

In this section we will give an introduction to quasigroups and coherent 2-groups [1][10].

2.1. Quasigroups. By [6], we have the definition of quasigroup:

Definition 2.1. A quasigroup is a set $G$ with a product and identity, for each element $g$ there is an inverse $g^{-1} \in G$, such that $g^{-1}(gh) = h$ and $(hg^{-1})g = h$ for any $h \in G$.

For a quasigroup, the multiplicative associator $\beta : G^3 \rightarrow G$ is defined by

$$g(hk) = \beta(g, h, k)(gh)k,$$

(2.1)

for any $g, h, k \in G$. The group of associative elements $N(G)$ is given by

$$N(G) = \{a \in G|(ag)h = a(gh), \ g(ah) = (ga)h, \ (gh)a = g(ha), \ \forall g, h \in G\},$$

which sometimes called the ‘nucleus’. A quasigroup is called quasiassociative, if $\beta$ has its image in $N(G)$ and $uN(G)u^{-1} \subseteq N(G)$ for any $u \in G$. It is clear that any element in $N(G)$ can ‘pass through’ the brackets of a product. For example, $(g(hx))k = (gh)(xk)$ for any $g, h, k \in G$ and $x \in N(G)$. By [6], we also have the following Lemma:

Lemma 2.2. Let $G$ be a quasiassociative quasigroup, then we have the following 3-cocycle condition:

$$(g\beta(h, k, l)g^{-1})\beta(g, hk, l)\beta(g, h, k) = \beta(g, h, kl)\beta(gh, k, l)$$

(2.2)

for any $g, h, k, l \in G$.

Lemma 2.2 will be useful in the proof of Theorem 2.3

2.2. Coherent 2-groups. By [10], we know that a coherent 2-group is a monoidal category, in which every object is weakly invertible and every morphism is invertible. More precisely, we have the following definition:

Definition 2.3. A coherent 2-group is a monoidal category $(G, \otimes, I, \alpha, r, l)$, with the unit $I$ and three natural isomorphisms, namely, the associator, with components $\alpha_{g,h,k} : (g \otimes h) \otimes k \rightarrow g \otimes (h \otimes k)$, the right and left unitor, with components $r_g : g \otimes I \rightarrow g$ and $l_g : I \otimes g \rightarrow g$, such that the following diagrams commute:

(1) The pentagon diagram:
The inverse corresponding to this product is given by
\[ g^{-1} := g^{-1}g. \]

If \( g \) is well defined for any \( g \in G \), we can see that the adjoint action \( \text{Ad} : G \to \text{Aut}(G) \) given by \( \text{Ad}_g(m) := gm^{-1}g \) is well defined for any \( g \in G \) and \( m \in N(G) \), since for any \( g \in G \) and \( m, n \in N(G) \), we have
\[
(g^{-1}m^{-1})(g^{-1}n^{-1}) = ((gm)^{-1})(gn)^{-1} = (gm)(g^{-1}(gn^{-1})) = (gm)(mg^{-1})^{-1} = g(mg^{-1})^{-1} = g(mn)^{-1}. \]
For any \( g, h \in G \) and \( m \in N(G) \), we have
\[
((gh)m(gh)^{-1})g = (g(hm(gh)^{-1})g) = (gh)(h^{-1}(hm^{-1})).
\]
On the other hand,
\[
(g(hm(gh)^{-1})g^{-1}) = (gm)h^{-1} = (gh)(h^{-1}(hm^{-1})).
\]
Therefore, by multiplying \( g^{-1} \) on the right side of \((gh)m(gh)^{-1})g = (g(hm(gh)^{-1})g^{-1})\), we have \((gh)m(gh)^{-1} = g(hm(gh)^{-1})g^{-1}. \) So
\[
\text{Ad}_{gh}(m) = (gh)m(gh)^{-1} = g(hm(gh)^{-1})g^{-1} = \text{Ad}_g \circ \text{Ad}_h(m).
\]
Now we can see $H$ is a quasigroup:
\[
((n, g) \otimes (m, h)) \otimes (m, h)^{-1} = (n(gm^{-1}), gh) \otimes (h^{-1}m^{-1}h, h^{-1})
\]
\[
= ((n(gm^{-1}))(Ad_{gh}(Ad_{h^{-1}}(m^{-1}))), (gh)h^{-1}) = ((n(gm^{-1}))(gm^{-1}g^{-1}), g)
\]
\[
=(n, g),
\]
for any $(n, g), (m, h) \in H$. Similarly, we have $(n, g)^{-1} \otimes ((n, g) \otimes (m, h)) = (m, h)$.

Then we can construct the set of objects by $G$ and the tensor product $\otimes : G \times G \rightarrow G$ is defined to be the multiplication of $G$. The source and target maps $s, t : H \rightarrow G$ are given by
\[
s(n, g) := g, \quad t(n, g) := ng.
\]
The identity morphism $id : G \rightarrow H$ is given by
\[
id(g) := (1, g).
\]
The composition $\circ : H \times_t H \rightarrow H$ is given by
\[
(n, mg) \circ (m, g) := (nm, g).
\]
The composition inverse is given by
\[
(n, g)^* := (n^{-1}, ng).
\]
Thus we get a groupoid with the associative composition, inverse, source and target maps. Moreover, the maps $s, t$ and $id$ preserve the tensor product. Indeed, by using the fact that $G$ is quasiasociative we have
\[
t((n, g) \otimes (m, h)) = t(n(gm^{-1}), gh) = (n(gm^{-1}))gh = n((gm^{-1}))(gh)
\]
\[
=n(((gm^{-1})g)h) = n((gm)h) = n(g(mh)) = (ng)(mh) = t(n, g) \otimes t(m, h).
\]
The interchange law of products can be checked as follows: On the one hand, we have
\[
((n_1, m_1g) \circ (m_1, g)) \otimes ((n_2, m_2h) \circ (m_2, h)) = (n_1m_1g) \otimes (n_2m_2, h)
\]
\[
=(n_1m_1Ad_g(n_2m_2), gh).
\]
On the other hand,
\[
((n_1, m_1g) \otimes (n_2, m_2h)) \circ ((m_1, g) \otimes (m_2, h))
\]
\[
=(n_1Ad_{m_1g}(n_2), (m_1g)(m_2h)) \circ (m_1Ad_g(m_2), gh)
\]
\[
=(n_1Ad_{m_1g}(n_2)m_1Ad_g(m_2), gh)
\]
\[
=(n_1m_1Ad_g(n_2)m_1^{-1}m_1Ad_g(m_2), gh)
\]
\[
=(n_1m_1Ad_g(n_2)Ad_g(m_2), gh)
\]
\[
=(n_1m_1Ad_g(n_2m_2), gh)
\]
where the second step is well defined since
\[
m_1Ad_g(m_2)(gh) = m_1(((gm_2^{-1})(gh)) = m_1(((gm_2^{-1})g)h) = m_1(g(m_2h))
\]
\[
=(m_1g)(m_2h).
\]
The associator $\alpha$ is given by
\[
\alpha_{g,h,k} := (\beta(g, h, k), (gh)k),
\]
this is well defined since the image of $\beta$ belongs to $N(G)$, and the source of $\alpha_{g,h,k}$ is $(gh)k$ and the target of $\alpha_{g,h,k}$ is $g(hk)$. Because $G$ is a quasigroup, we have $\alpha_{g,g^{-1},h} = \alpha_{h,g^{-1},g} = \cdots$
We get \( g, h, k, l \) and on the other hand (\( l, g, (m, h), (n, k) \in H \), we can see on the one hand
\[
\alpha_{lg, mh, nk} \circ (((l, g) \otimes (m, h)) \otimes (n, k)) \\
= (\beta(lg, mh, nk), ((lg)(mh))(nk)) \circ ((l(Ad_g(m)))(Ad_{gh}(n)), (gh)k) \\
= (\beta(lg, mh, nk)((l(Ad_g(m)))(Ad_{gh}(n))),(gh)k),
\]
and on the other hand
\[
((l, g) \otimes ((m, h) \otimes (n, k))) \circ \alpha_{g, h, k} \\
= (l(Ad_g(mAd_h(n)), g(hk)) \circ (\beta(g, h, k), (gh)k) \\
= (l(Ad_g(mAd_h(n)))\beta(g, h, k),(gh)k).
\]
Since
\[
\beta(lg, mh, nk)((l(Ad_g(m)))(Ad_{gh}(n)))(gh)k = \beta(lg, mh, nk)(l(Ad_g(m)))(Ad_{gh}(n))(gh)k \\
= \beta(lg, mh, nk)(l(Ad_g(m)))(gh)nk) \\
= \beta(lg, mh, nk)(l(Ad_g(m))(gh))(nk) \\
= \beta(lg, mh, nk)(l(Ad_g(m))(gh))(nk) \\
= (lg)((mh)(nk)) \\
= (lAd_g(mAd_h(n)))\beta(g, h, k),(gh)k,
\]
we get
\[
\beta(lg, mh, nk)((l(Ad_g(m)))(Ad_{gh}(n))) = (lAd_g(mAd_h(n)))\beta(g, h, k),
\]
therefore,
\[
\alpha_{lg, mh, nk} \circ (((l, g) \otimes (m, h)) \otimes (n, k)) = ((l, g) \otimes ((m, h) \otimes (n, k))) \circ \alpha_{g, h, k}.
\]
The pentagon diagram can be proved by Lemma 2.2. Indeed, let \( g, h, k, l \in G \), on the one hand we have
\[
\alpha_{g, h, kl} \circ \alpha_{gh, k, l} = (\beta(g, h, kl), (gh)(kl)) \circ (\beta(gh, k, l), ((gh)k)l) \\
= (\beta(g, h, kl)\beta(g, h, k, l), ((gh)k)l)
\]
On the other hand,
\[
(id_g \otimes \alpha_{h, k, l}) \circ \alpha_{g, h, k, l} \circ (\alpha_{g, h, k} \otimes id_l) \\
= (g(\beta(h, k, l)g^{-1}, g((hk)l)) \circ (\beta(g, hk, l), (g(hk))l) \circ (\beta(g, h, k), ((gh)k)l) \\
= (g(\beta(h, k, l)g^{-1})\beta(g, hk, l)\beta(g, h, k), ((gh)k)l),
\]
by using Lemma 2.2 we get the pentagon. For the natural transformations \( l, r, e, i \), we can see \( l_g = r_g = (1, g) \), and \( i_g = e_g = (1, 1) \) for any \( g \in G \), which satisfy all the axioms of a coherent 2-group. \( \square \)
3. HOPF COQUASIGROUPS AND HOPF ALGEBROIDS

In this section, we will recall some material about Hopf algebras, Hopf coquasigroups and their modules and comodules. We will also review Hopf algebroids over commutative rings.

**Definition 3.1.** A bialgebra is an algebra \( H \) with two algebra maps \( \Delta_H : H \to H \otimes H \) (called the coproduct) and \( \epsilon : H \to \mathbb{C} \) (called the counit), such that

\[
(id_H \otimes \Delta_H) \circ \Delta_H = (\Delta_H \otimes id_H) \circ \Delta, \quad (id_H \otimes \epsilon_H) \circ \Delta_H = id_H = (\epsilon_H \otimes id_H) \circ \Delta_H,
\]

where the algebra multiplication on \( H \otimes H \) is given by \((h \otimes g) \cdot (h' \otimes g') := (hh' \otimes gg')\) for any \( h \otimes g, h' \otimes g' \in H \otimes H \). Moreover, if there is a linear map \( S : H \to H \otimes H \) (called the antipode), such that

\[
m_H \circ (S \otimes id_H) \circ \Delta_H = 1_H \epsilon_H = m_H \circ (id_H \otimes S) \circ \Delta_H,
\]

where \( m_H \) is the product on \( H \), then \( H \) is called a Hopf algebra.

For the coproduct of a bialgebra, we use the sumless Sweedler notation \( \Delta_H(h) = h_{(1)} \otimes h_{(2)}, \) and its iterations: \( \Delta^n = (id_H \otimes \Delta_H) \circ \Delta_H^{n-1} : h \mapsto h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n+1)}. \) In order to define a quantization of a quasigroup, we will relax the condition that the coproduct is coassociative. In other words, the coproduct of a bialgebra will no longer satisfy the first equation of (3.1). By [6], we have a quantization of a quasigroup:

**Definition 3.2.** A Hopf coquasigroup \( H \) is an unital associative algebra, equipped with counital algebra homomorphisms \( \Delta : H \to H \otimes H, \epsilon : H \to \mathbb{C} \) (called the coproduct and counit), and a linear map \( S_H : H \to H \) (called the antipode) such that

\[
(m_H \otimes id_H)(S_H \otimes id_H \otimes id_H)(id_H \otimes \Delta_H) = 1 \otimes id_H = (m_H \otimes id_H)(id_H \otimes S_H \otimes id_H)(id_H \otimes \Delta_H),
\]

\[
(id_H \otimes m_H)(id_H \otimes S_H \otimes id_H)(\Delta \otimes id_H) = id_H \otimes 1 = (id_H \otimes m_H)(id_H \otimes id_H \otimes S_H)(\Delta \otimes id_H).
\]

A morphism between two Hopf coquasigroups is an algebra map \( f : H \to G \), such that for any \( h \in H \), \( f(h_{(1)} \otimes f(h_{(2)}) = f(h_{(1)}) \otimes f(h_{(2)}) \) and \( \epsilon_G(f(h)) = \epsilon_H(h) \).

**Remark 3.3.** Since the coproduct of a Hopf coquasigroup is not necessarily coassociative, we cannot use the Sweedler index notion for the iterated coproduct \( \Delta^n \) (but we still use \( h_{(1)} \otimes h_{(2)} \) as the image of the coproduct \( \Delta \)). For example, since \((\Delta \otimes id) \circ \Delta \neq (id \otimes \Delta) \circ \Delta\) we won’t have: \( h_{(1)_1} \otimes h_{(1)_2} \otimes h_{(2)} = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = h_{(1)} \otimes h_{(2)_1} \otimes h_{(2)_2} \). It is given in [6] that the antipode \( S_H \) of a Hopf coquasigroup also satisfies:

- \( h_{(1)}S_H(h_{(2)}) = \epsilon(h) = S_H(h_{(1)})h_{(2)}, \)
- \( S_H(hh') = S_H(h')S_H(h), \)
- \( S_H(h_{(1)} \otimes S_H(h_{(2)}) = S_H(h_{(2)}) \otimes S_H(h_{(1)}), \)

for any \( h, h' \in H \).

Given a Hopf coquasigroup \( H \), we can define a linear map \( \beta : H \to H \otimes H \otimes H \) (called the comultiplicative coassociator) by

\[
\beta(h) := h_{(1)(1)}S_H(h_{(2)_1}) \otimes h_{(1)(2)}S_H(h_{(2)_2}) \otimes h_{(1)(2)_1}S_H(h_{(2)_2})
\]

for any \( h \in H \). We can see that

\[
\beta \ast ((\Delta \otimes id_H) \circ \Delta) = (id_H \otimes \Delta) \circ \Delta,
\]
where * is the convolution product in the vector space $H' := \text{Hom}(H, H \otimes H \otimes H)$, namely $(f * g)(h) := f(h_1)g(h_2)$ for any $f, g \in H'$. More precisely, a Hopf coquasigroup $H$ is a Hopf algebra if and only if $\beta(h) = \epsilon(h)1_H \otimes 1_H \otimes 1_H$.

Given a Hopf coquasigroup $H$, a left $H$-coaction is a vector space $V$ carrying a left $H$-coaction, namely a linear map $\delta^V : V \to H \otimes V$ such that
\[
(id_H \otimes \delta^V) \circ \delta^V = (\Delta \otimes \text{id}_V) \circ \delta^V, \quad (\epsilon \otimes \text{id}_V) \circ \delta^V = \text{id}_V. \tag{3.7}
\]
In the sumless Sweedler notation, $\delta^V : v \mapsto v^{(-1)} \otimes v^{(0)}$, and the left $H$-comodule properties read
\[
v^{(-1)}(1) \otimes v^{(-1)}(2) \otimes v^{(0)} = v^{(-1)} \otimes v^{(0)(-1)} \otimes v^{(0)(0)} , \quad \epsilon(v^{(-1)}) v^{(0)} = v,
\]
for all $v \in V$.

In particular, a left $H$-comodule algebra is an algebra $A$, such that the coaction $\delta : A \to H \otimes A$ is an algebra map. A left $H$-comodule coalgebra is a coalgebra $C$, which is a left $H$-comodule and such that the coproduct and the counit of $C$ are morphisms of $H$-comodules. Explicitly, this means that, for each $c \in C$,
\[
c^{(-1)} \otimes c^{(0)(1)} \otimes c^{(0)(2)} = c^{(-1)} c^{(2)(-1)} \otimes c^{(1)(0)} \otimes c^{(2)(0)} , \quad \epsilon_C(c) = \epsilon_C(c^{(0)}).
\]

**Definition 3.4.** A coassociative pair $(A, B, \phi)$ consists of a Hopf coquasigroup $B$ and a Hopf algebra $A$, together with a Hopf coquasigroup morphism $\phi : B \to A$, such that
\[
\begin{align*}
\phi(b_{(1)(1)}) \otimes b_{(1)(2)} \otimes b_{(2)} &= \phi(b_{(1)}) \otimes b_{(2)(1)} \otimes b_{(2)(2)} \\
b_{(1)(1)} \otimes \phi(b_{(1)(2)}) \otimes b_{(2)} &= b_{(1)} \otimes \phi(b_{(2)(1)}) \otimes b_{(2)(2)} \\
b_{(1)(1)} \otimes b_{(1)(2)} \otimes \phi(b_{(2)}) &= b_{(1)} \otimes b_{(2)(1)} \otimes \phi(b_{(2)(2)}). \tag{3.8}
\end{align*}
\]

**Remark 3.5.** A coassociative pair can be viewed as the dual case of a group and a quasigroup, such that there is a quasigroup morphism which maps the group into the associative elements of the quasigroup. More precisely, let $H$ be a group, $G$ be a quasigroup, and $\phi : H \to G$ be a morphism of quasigroup, such that the $\phi(H) \subseteq N(G)$. Then we have
\[
\begin{align*}
(\phi(h)g)g' &= \phi(h)(gg') \\
(g\phi(h))g' &= g(\phi(h)g') \\
g(g'\phi(h)) &= (gg')\phi(h),
\end{align*}
\]
for any $h \in H$ and $g, g' \in G$.

For a Hopf coquasigroup, the $n$-th iterated coproducts $\Delta^n_I$ (for $n \geq 2$) are not always equal, where we use index $I$ to distinguish different kinds of iterated product. We have:

**Lemma 3.6.** Let $(A, B, \phi)$ be a coassociative pair of a Hopf algebra $A$ and a Hopf coquasigroup $B$, and let $\Delta^n_I, \Delta^n_J$ be $n$-th iterated coproducts of $B$ with $\Delta^n_I(b) = b_{1} \otimes b_{2} \otimes \cdots \otimes b_{I_{n+1}}$ and $\Delta^n_J(b) = b_{J_{1}} \otimes b_{J_{2}} \otimes \cdots \otimes b_{J_{n+1}}$. If
\[
b_{1} \otimes b_{2} \otimes \cdots \otimes \epsilon_B(b_{I_{m_1}}) \otimes \cdots \otimes \epsilon_B(b_{I_{m_k}}) \otimes \cdots \otimes b_{I_{n+1}} = b_{J_{1}} \otimes b_{J_{2}} \otimes \cdots \otimes \epsilon_B(b_{J_{m_1}}) \otimes \cdots \otimes \epsilon_B(b_{J_{m_k}}) \otimes \cdots \otimes b_{J_{n+1}}
\]
then
for $1 \leq m_1 < m_2 < \cdots < m_k \leq n + 1$ (and $m_{i+1} - m_i \leq 1$ for any $1 \leq i \leq k - 1$), then

$$b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes \phi(b_{I_{m_1}}) \otimes \cdots \otimes \phi(b_{I_{m_k}}) \otimes \cdots \otimes b_{I_{n+1}} = b_{I_{m_1}} \otimes b_{I_{m_2}} \otimes \cdots \otimes \phi(b_{I_{m_1}}) \otimes \cdots \otimes \phi(b_{I_{m_k}}) \otimes \cdots \otimes b_{I_{n+1}}.$$ 

**Proof.** First, we show that if

$$b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \epsilon_B(b_{I_{m_1}}) \otimes \cdots \otimes b_{I_{m+n-1}} = b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \epsilon_B(b_{I_{m_1}}) \otimes \cdots \otimes b_{I_{m+n-1}} \quad (3.9)$$

for $1 \leq m \leq n + 1$, then

$$b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \phi(b_{I_{m_1}}) \otimes \cdots \otimes b_{I_{m+n-1}} = b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \phi(b_{I_{m_1}}) \otimes \cdots \otimes b_{I_{m+n-1}}.$$

We can prove this inductively. For $n = 2$, this is obvious by the definition of a coassociative pair. Now consider the case for $n \geq 3$. We can see both sides of equation (3.9) are equal to the image of an $(n - 1)$-th iterated coproduct $\Delta^{n-1}_K$, which can be written as $\Delta^{n-1}_K = (\Delta^{p}_K \otimes \Delta^{q}_K) \circ \Delta$ for some iterated coproducts $\Delta^{p}_K$, $\Delta^{q}_K$ with $p + q = n - 2$. Assume this proposition is correct for $n = N - 1$. We have two cases for the index of $I_m$ and $J_m$:

The first case is that the first index of $b_{I_m}$ and $b_{J_m}$ are the same (where the first index means the first Sweedler index on the left, for example, the first index of $b_{(2)(1)}$ is 2). When the first indices of $b_{I_m}$ and $b_{J_m}$ are 1. In this case, we can see $\Delta^{1}_I = (\Delta^{p+1}_I \otimes \Delta^{q+1}_K) \circ \Delta$ and $\Delta^{1}_J = (\Delta^{p+1}_I \otimes \Delta^{q+1}_K) \circ \Delta$, for some $(p + 1)$-th iterated coproducts $\Delta^{p+1}_I$ and $\Delta^{p+1}_J$. By applying the hypotheses for $\Delta^{p+1}_I$ and $\Delta^{p+1}_J$, we get the result. When the first indices of $b_{I_m}$ and $b_{J_m}$ are 2, the situation is similar.

The second case is that the first indices of $b_{I_m}$ and $b_{J_m}$ are different. Assume the first index of $b_{I_m}$ is 1 and $b_{J_m}$ is 2. In this case $m$ has to be equal to $p + 2$, and $\Delta^m_I = (\Delta^{p+1}_E \otimes \Delta^{q+1}_K) \circ \Delta$ and $\Delta^m_J = (\Delta^{p+1}_K \otimes \Delta^{q+1}_E) \circ \Delta$ for some iterated $(p + 1)$-th coproduct $\Delta^{p+1}_E$ with $(\epsilon_B \otimes \epsilon_B) \circ \Delta^{p+1}_E = \Delta^{p+1}_K$ and iterated $(q + 1)$-th coproduct $\Delta^{q+1}_E$ with $(\epsilon_B \otimes \epsilon_B) \circ \Delta^{q+1}_E = \Delta^{q+1}_K$. Define $\Delta^{p+1}_G := (\Delta^{p+1}_K \otimes \epsilon_B) \circ \Delta$ and $\Delta^{q+1}_H := (\epsilon_B \otimes \Delta^{q+1}_K) \circ \Delta$ (notice that $\Delta^{p+1}_E$ is not necessarily equal to $\Delta^{p+1}_G$, and $\Delta^{q+1}_E$ is not necessarily equal to $\Delta^{q+1}_H$), then we can see

$$b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \phi(b_{I_{m_1}}) \otimes b_{I_{m+1}} \otimes \cdots \otimes b_{I_{n+1}}$$

where $b_{I_{m_1}}$ and $b_{I_{m+1}}$ are 1, and first index of $b_{I_{m_2}}$ and $b_{I_{m+2}}$ are 2, then we go back to the case for $k = 1$ by considering only the terms with the first index equal to 1 or 2. If the first index of $b_{I_{m_1}}$, $b_{I_{m_1}}$ and $b_{I_{m_2}}$ are 1 and the first
index of $b_{I_{m_2}}$ is 2, then $b_{I_{m_2}}$ is the last term whose first index are 1 and $b_{I_{m_2}}$ is the first term whose first index are 2. Therefore, we can use the same method above for the $k = 1$ case, where the first index of $b_{I_m}$ is 1 and $b_{I_m}$ is 2. If the first index of $b_{I_{m_1}}$, $b_{I_{m_2}}$ are 1 and the first index of $b_{I_{m_1}}$, $b_{I_{m_2}}$ are 2, then $b_{I_{m_1}}$, $b_{I_{m_2}}$ are the last two terms whose first index are 1 and $b_{I_{m_1}}$, $b_{I_{m_2}}$ are the first two terms whose first index are 2, we can also use the same method as above, the only different is to move two terms from the ‘left-hand’ side to the ‘right-hand’ side, we will omit the detail here.

There is a dual version of the Hopf coquasigroup \[6\]:

**Definition 3.7.** A Hopf quasigroup $A$ is a coassociative coalgebra with a coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to k$, together with a unital and possibly nonassociative algebra structure, such that the coproduct and counity are algebra maps. Moreover, there is a linear map (the antipode) $S_A : A \to A$ such that:

\[
m(id_A \otimes m)(S_A \otimes id_A \otimes id_A)(\Delta \otimes id_A) = \epsilon \otimes id_A = m(id_A \otimes m)(id_A \otimes S_A \otimes id_A)(\Delta \otimes id_A)
\]

(3.10)

\[
m(m \otimes id_A)(id_A \otimes S_A \otimes id_A)(id_A \otimes \Delta) = id_A \otimes \epsilon = m(m \otimes id_A)(id_A \otimes id_A \otimes S_A)(id_A \otimes \Delta).
\]

(3.11)

A Hopf quasigroup is a Hopf algebra if and only if it is associative. In the following, we will introduce central Hopf algebroids, which is a Hopf algebroid \[3\] with images of the source and target maps belonging to the center.

**Definition 3.8.** Let $B$ be a commutative algebra, a central Hopf algebroid over $B$ is an algebra $H$ with two algebra maps (called source and target map) $s : B \to H$ and $t : B \to H$. In addition, there is an antialgebra map (called Hopf algebroid antipode) $S : H \to H$, such that:

1. The image of $s$ and $t$ belongs to the center of $H$.
2. $(H, \Delta, \epsilon)$ is a $B$-coring with the $B$-bimodule structure given by $bc \cdot h \cdot b' = s(b)ht(b')$, for any $h \in H$ and $b, b' \in B$. Namely, there are two $B$-bimodule maps $\Delta : H \to H \otimes_B H$ and $\epsilon : H \to B$, such that

\[
(\Delta \otimes_B id) \circ \Delta = (id \otimes_B \Delta) \circ \Delta, \quad (\epsilon \otimes_B id) \circ \Delta = id = (id \otimes_B \epsilon) \circ \Delta,
\]

where the balanced tensor product $\otimes_B$ is induced by the $B$-bimodule structure of $H$, i.e. $g \otimes_B s(b)h = t(b)g \otimes_B h$ for any $g, h \in H$ and $b \in B$.
3. $\Delta$ and $\epsilon$ are algebra maps.
4. For any $h \in H$ and $b, b' \in B$,

\[
S(t(b)hs(b')) = t(b')S(h)s(b).
\]

(3.12)

(5) $m \circ (S \otimes_B id_H) \circ \Delta = t \circ \epsilon$ and $m \circ (id_H \otimes_B S) \circ \Delta = s \circ \epsilon$.

In (5), we can see the product $m$ factors through the balanced tensor product $\otimes_B$ because of \[3.12\].

Let $H$ be a central Hopf algebroid over $B$. We define the convolution product $\ast : _BHom_B(H, B) \times _BHom_B(H, B) \to _BHom_B(H, B)$ by $(f \ast g)(h) := f(h^{(1)})g(h^{(2)})$ for any $f, g \in _BHom_B(H, B)$ and $h \in H$, where $_BHom_B(H, B)$ is the vector space of $B$-bimodule maps and we use the upper Sweeder index notation for the coproduct Hopf algebroids, i.e. $\Delta(h) = h^{(1)} \otimes h^{(2)}$. By \[2\] we know $_BHom_B(H, B)$ is an algebra with unit $\epsilon : H \to B$. 

4. COHERENT HOPF-2-ALGEBRAS

In [4], quantum 2-groups can be given by crossed modules and crossed comodules of Hopf algebras. In this paper, we will define quantum 2-groups in a different way, which is based on the idea of 2-arrows quantization.

Definition 4.1. A coherent Hopf 2-algebra consists of a commutative Hopf coquasigroup \((B, m_B, 1_B, \Delta_B, \epsilon_B, S_B)\) and a Hopf coquasigroup \((H, m, 1_H, \Delta, \epsilon_S, S)\), and a central Hopf algebroid \((H, m, 1_H, \Delta, \epsilon, S)\) over \(B\). Moreover, there is an algebra map (called coassociator) \(\alpha : H \to B \otimes B \otimes B\), such that all the structures above satisfy the following axioms:

(i) The underlying algebra of the Hopf coquasigroup \((H, m, 1_H, \Delta, \epsilon_S, S)\) and the Hopf algebroid \((H, m, 1_H, \Delta, \epsilon, S)\) are the same.

(ii) \(\epsilon : H \to B\) and \(s, t : B \to H\) are morphisms of Hopf coquasigroups.

(iii) The two coproducts \(\Delta\) and \(\Delta_B\) satisfies the following cocommutation relation:

\[
(\Delta \otimes 1_H) \circ \Delta = (\text{id}_H \otimes \text{flip} \otimes \text{id}_H) \circ (\Delta \otimes B \Delta) \circ \Delta,
\]

where \(\text{id}_H \otimes \text{flip} \otimes \text{id}_H : H \otimes H \otimes B \otimes B H \otimes H \to H \otimes B H \otimes H \otimes B H\) is given by \(\text{id}_H \otimes \text{flip} \otimes \text{id}_H : X \otimes Y \otimes B \otimes B Z \otimes W \mapsto (X \otimes B Z) \otimes (Y \otimes B W)\).

(iv) For the coassociator,

\[
\alpha \circ t = (\Delta_B \otimes \text{id}_B) \circ \Delta_B, \quad \alpha \circ s = (\text{id}_B \otimes \Delta_B) \circ \Delta_B.
\]

(v) Let \(\star\) be the convolution product corresponding to the Hopf algebroid coproduct, we have

\[
((s \otimes s \otimes s) \circ \alpha) \star ((\Delta \otimes \text{id}_H) \circ \Delta) = ((\text{id}_H \otimes \Delta) \circ \Delta) \star ((t \otimes t \otimes t) \circ \alpha)
\]

(vi) The 3-cocycle condition:

\[
((\epsilon \otimes \alpha) \circ \Delta) \star ((\text{id}_B \otimes \Delta_B \otimes \text{id}_B) \circ \alpha) \star ((\alpha \otimes \epsilon) \circ \Delta) = ((\text{id}_B \otimes \text{id}_B \otimes \Delta_B) \circ \alpha) \star ((\Delta_B \otimes \text{id}_B \otimes \text{id}_B) \circ \alpha).
\]

A coherent Hopf 2-algebra is a strict Hopf 2-algebra, if \(H\) and \(B\) are coassociative \((H\) and \(B\) are Hopf algebras), and \(\alpha = (\epsilon \otimes \epsilon \otimes \epsilon) \circ (\Delta \otimes \text{id}_H) \circ \Delta\).

Remark 4.2. In general, for any Hopf algebroid over an algebra \(B\), the base algebra \(B\) is not necessarily commutative. However, in order to give a correct definition of coherent Hopf 2-algebras, we need the maps \(\epsilon, s, t\) to be Hopf algebra maps, since only in that case condition (v) and (vi) make sense. As a result, we assume that the Hopf algebroid \(H\) is a central Hopf algebroid, and the base algebra \(B\) is a commutative algebra. In the following, we denote the image of \(\alpha\) by \(\alpha(h) := h^1 \otimes h^2 \otimes h^3\) for any \(h \in H\), and use different sumless Sweedler notations for the coproducts of Hopf coquasigroup and Hopf algebroid structure on \(H\), namely, \(\Delta(h) := h_{(1)} \otimes h_{(2)}\).

1. In condition (iii), \(\Delta \otimes_B \Delta : H \otimes_B H \to (H \otimes H) \otimes_B (H \otimes H)\) is well defined since \(H \otimes H\) has \(B \otimes B\)-bimodule structure: \((b \otimes b') \circ (h \otimes h') = s(b)h \otimes s(b')h'\) and \((h \otimes h') \circ (b \otimes b') = t(b)h \otimes t(b')h'\), for any \(b, b' \in B \otimes B\) and \(h, h' \in H \otimes H\).
Indeed, for any \( b \in B \) and \( h, h' \in H \) we have
\[
(\bullet \otimes_B \bullet)(h \otimes_B b \triangleright h') = (\bullet \otimes_B \bullet)(h \otimes_B s(b)h')
\]
\[
= (h_{(1)} \otimes h_{(2)}) \otimes_{B \otimes B} (s(b)_{(1)}h'_{(1)} \otimes s(b)_{(2)}h'_{(2)})
\]
\[
= (t(b)_{(1)}h_{(1)} \otimes t(b)_{(2)}h_{(2)}) \otimes_{B \otimes B} (h'_{(1)} \otimes h'_{(2)})
\]
\[
= (t(b)_{(1)}h_{(1)} \otimes t(b)_{(2)}h_{(2)}) \otimes_{B \otimes B} (h'_{(1)} \otimes h'_{(2)})
\]
where in the 2nd step we use the fact that \( \bullet \) is an algebra map, and in the 3rd step we use the fact that \( s \) is a coalgebra map. Clearly, the map \( id_H \otimes \text{flip} \otimes id_H : h \otimes h' \otimes_{B \otimes B} g \otimes g' \mapsto (h \otimes_B g) \otimes (h' \otimes_B g') \) is also well defined for any \( h \otimes h', g \otimes g' \in H \otimes H \). More precisely, (iv) can be written as
\[
\begin{align*}
& h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} = h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(1)}^{(1)} \otimes h_{(2)}^{(2)},
\end{align*}
\]
(4.5)
for any \( h \in H \).

(3) By using condition (iv) and the fact that \( s, t \) are bialgebra maps, we can see (4.3) is well defined since \((s \otimes s \otimes s) \circ \alpha \circ t = (\bullet \otimes id_H) \circ \bullet \circ s \circ (t \otimes t \otimes t) \circ \alpha \circ s = (id_H \otimes \bullet) \circ \bullet \circ t \).

(4) The left hand side of (4.4) is well defined since:
\[
((\epsilon \otimes \alpha) \circ \bullet)(t(b)) = (s(\epsilon(h_{(1)}^{(1)})))h_{(2)}^{(2)} \otimes s(\epsilon(h_{(2)}^{(2)}))h_{(3)}^{(3)} \otimes s(\epsilon(h_{(3)}^{(3)}))h_{(4)}^{(4)}
\]
\[
= ((\bullet \otimes id_H) \circ \bullet)(h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(3)}^{(3)} \otimes h_{(4)}^{(4)})
\]
\[
= ((s \otimes s \otimes s) \circ \alpha) \ast ((\bullet \otimes id_H) \circ \bullet)(h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(3)}^{(3)} \otimes h_{(4)}^{(4)})
\]
For a strict Hopf 2-algebra, we can see the morphisms \( s, t \) and \( \epsilon \) are morphisms of Hopf algebras, and (iv), (v), (vi) are automatically satisfied. For (4.3), we have
\[
((s \otimes s \otimes s) \circ \alpha) \ast ((\bullet \otimes id_H) \circ \bullet)(h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(3)}^{(3)} \otimes h_{(4)}^{(4)})
\]
\[
= (\epsilon(h_{(1)}^{(1)}))h_{(2)}^{(2)} \otimes s(\epsilon(h_{(3)}^{(3)}))h_{(4)}^{(4)}
\]
\[
= ((id_H \otimes \bullet) \circ \bullet \ast ((t \otimes t \otimes t) \circ \alpha)(h_{(1)}^{(1)} \otimes h_{(2)}^{(2)} \otimes h_{(3)}^{(3)} \otimes h_{(4)}^{(4)}))
\]
For (4.4) we can see the left hand side is
\[
\epsilon(h_{(1)}^{(1)}h_{(2)}^{(2)})h_{(3)}^{(3)}h_{(4)}^{(4)} = \epsilon(h_{(1)}^{(1)}h_{(2)}^{(2)})h_{(3)}^{(3)}h_{(4)}^{(4)}
\]
while the right hand side is
\[
\epsilon(h_{(1)}^{(1)}h_{(2)}^{(2)}h_{(3)}^{(3)})h_{(4)}^{(4)} = \epsilon(h_{(1)}^{(1)}h_{(2)}^{(2)}h_{(3)}^{(3)})h_{(4)}^{(4)}
\]
using the fact that \( \epsilon(h^{(1)})\epsilon(h^{(2)}) = \epsilon(s(\epsilon(h^{(1)})))\epsilon(h^{(2)}) = \epsilon(s(\epsilon(h^{(1)}))h^{(2)}) = \epsilon(h) \), we get that the left and right hand side of (ix) are equal.

Now let’s explain why Definition 4.1 is a quantisation of a coherent 2-group, whose objects form a quasigroup. First, the morphisms and their composition form a groupoid, which corresponds to a Hopf algebroid. Second, the tensor products of objects and morphisms form two quasigroups, which corresponds to two Hopf coquasigroups.

By the definition of monoidal category, we can see that axiom (ii) is natural, since the source and target maps from objects to morphisms preserve the tensor product, and the identity map from objects to morphisms also preserves the tensor product. The interchange law corresponds to condition (iii). The source and target of the morphism \( \alpha_{g,h,k} \) is \((gh)k\) and \(g(hk)\), which corresponds to condition (iv). The naturality of \( \alpha \) corresponds to (v). The pentagon diagram corresponds to condition (vi). Here we will still call Definition 4.1 coherent Hopf 2-algebra even though it only corresponds to a special case of ‘quantum’ coherent 2-group whose set of objects and set of morphisms are quasigroups.

**Proposition 4.3.** Given a coherent Hopf 2-algebra as in Definition 4.1, the antipodes satisfy the following property:

(i) \( \Delta \circ S_H = (S_H \otimes B S_H) \circ \Delta \).

(ii) \( S \) is a coalgebra map on \((H, \triangle, \epsilon_H)\). In other words, \( \Delta \circ S = (S \otimes S) \circ \Delta \) and \( \epsilon_H \circ S = \epsilon_H \).

(iii) If \( H \) is commutative, \( S \circ S_H = S_H \circ S \).

**Proof.** For (i), \( \forall h \in H \) we can see \( S_H(h^{(1)}) \otimes B S_H(h^{(2)}) \) is well defined. Since the image of source and target maps belongs to the center of \( H \), we have

\[
S_H(t(b)h^{(1)}) \otimes B S_H(h^{(2)}) = S_H(t(b))S_H(h^{(1)}) \otimes B S_H(h^{(2)}) = t(S_B(b))S_H(h^{(1)}) \otimes B S_H(h^{(2)}) = S_H(h^{(1)}) \otimes B s(S_B(b))S_H(h^{(2)}) = S_H(h^{(1)}) \otimes B S_H(s(b)h^{(2)}).
\]

We can see

\[
S_H(h^{(1)}) \otimes B S_H(h^{(2)})
\]

\[
= (S_H(h^{(1)})) \otimes B S_H((h^{(2)}))((\Delta(h^{(2)}))S_H(h^{(2)}))
\]

\[
= (S_H(h^{(1)})) \otimes B S_H((h^{(2)}))((h^{(1)}(2) \otimes B (h^{(2)}(2))((S_H(h^{(2)}))((h^{(1)}(2) \otimes B (S_H(h^{(2)})(2)))
\]

\[
= (\epsilon_H(h^{(1)})) \otimes B (\epsilon_H(h^{(2)}))((S_H(h^{(2)}))) ((S_H(h^{(2)}))((S_H(h^{(2)}))((h^{(1)}(2) \otimes B (S_H(h^{(2)})(2)))
\]

\[
= (\epsilon_B(\epsilon(h^{(1)}))\epsilon(h^{(2)}))((S_H(h^{(2)})) ((S_H(h^{(2)}))((h^{(1)}(2) \otimes B (S_H(h^{(2)})(2)))
\]

\[
= (\epsilon_H(h^{(1)}))((S_H(h^{(2)}))((h^{(1)}(2) \otimes B (S_H(h^{(2)})(2)))
\]

\[
= (S_H(h^{(1)})) \otimes B (S_H(h^{(2)})).
\]

For (ii), we can first observe that for any \( h \in H \), \( (S(h^{(1)})) \otimes B (S(h^{(2)})) )((h^{(1)}(2) \otimes h^{(2)}(2))) \) is well defined. Indeed,

\[
(S((t(b)h^{(1)})(1) \otimes S((t(b)h^{(1)})(2)))(h^{(2)}(1) \otimes h^{(2)}(2))
\]

\[
= (S(t(b^{(1)}))S(t(b^{(2)})))h^{(2)}(1) \otimes h^{(2)}(2)
\]

\[
= (S(h^{(1)})) \otimes B (S(h^{(2)})) )((s(b^{(1)})) \otimes s(b^{(2)}))(h^{(2)}(1) \otimes h^{(2)}(2))
\]

\[
= (S(h^{(1)})) \otimes B (S(h^{(2)})) )((s(b)) \otimes s(b)(2))(h^{(2)}(1) \otimes h^{(2)}(2))
\]

\[
= (S(h^{(1)})) \otimes B (S(h^{(2)})) )((s(b)(2))(h^{(2)}(1) \otimes h^{(2)}(2)).
\]
Similarly, since $H$ is a central Hopf algebroid, \((h^{(1)}(1) \otimes h^{(2)}(2))(S(h^{(2)}(1) \otimes S(h^{(2)}(2))))\) is defined as well. So on the one hand

\[
(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))(h^{(2)}(1) \otimes h^{(2)}(2))(S(h^{(2)}(1)) \otimes S(h^{(2)}(2)))
=\(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))(h^{(2)}(1) \otimes h^{(2)}(2))(S(h^{(2)}(1)) \otimes S(h^{(2)}(2)))
=\(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))((h^{(2)}(1)S(h^{(2)}(2))))
=\(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))((s(\epsilon(h^{(2)})))_{(1)} \otimes s(\epsilon(h^{(2)})))_{(2)}
=\(S(h^{(1)}(1)) \otimes S(h^{(1)}(2))(s(\epsilon(h^{(2)}))) \otimes s(\epsilon(h^{(2)})))
=\(S(h^{(1)}(1) t(\epsilon(h^{(2)}))) \otimes S(h^{(2)}(1) t(\epsilon(h^{(2)}))))
=\(S(h^{(1)}) \otimes S(h^{(2)}),\)

on the other hand

\[
(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))(h^{(2)}(1) \otimes h^{(2)}(2))(S(h^{(2)}(1)) \otimes S(h^{(2)}(2)))
=\(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))(h^{(2)}(1) \otimes h^{(2)}(2))(S(h^{(2)}(1)) \otimes S(h^{(2)}(2)))
=\(S(h^{(1)}(1)) \otimes S(h^{(1)}(2)))(t(\epsilon(h^{(1)}))) \otimes t(\epsilon(h^{(1)})))S(h^{(2)}(1) \otimes S(h^{(2)}(2)))
=\(\epsilon(\epsilon(h^{(1)}))S(h^{(2)}(1) \otimes t(\epsilon(h^{(1)})))S(h^{(2)}(2)))
=\(\epsilon(\epsilon(h^{(1)}))S(h^{(2)}))
=\(\epsilon(\epsilon(h))\).
\]

so \(\epsilon(\epsilon(h)) = S(h^{(1)}) \otimes S(h^{(2)}).\) We also have

\[
\epsilon_{H}(S(h)) = \epsilon_{H}(S(s \circ \epsilon(h^{(2)}))) = \epsilon_{H}(S(h^{(1)})t \circ \epsilon(h^{(2)})) = \epsilon_{H}(S(h^{(1)})t \circ \epsilon(h^{(2)})) = \epsilon_{H}(S(h^{(1)}))\epsilon_{H}(t \circ \epsilon(h^{(2)})) = \epsilon_{H}(S(h^{(1)}))\epsilon_{H}(h^{(2)})) = \epsilon_{H}(S(h^{(1)}))\epsilon_{H}(h^{(2)})) = \epsilon_{H}(t(\epsilon(h)))
=\epsilon_{B} \circ \epsilon(h) = \epsilon_{H}(h).
\]

For (iii), we can first observe that \(S(S_{H}(h^{(1)}))S_{H}(h^{(2)})\) is well defined. Indeed,

\[
S(S_{H}(t(b)h^{(1)}))S_{H}(h^{(2)}) = S(S_{H}(h^{(1)}))t \circ S_{B}(b))S_{H}(h^{(2)}) = S(S_{H}(h^{(1)}))s \circ S_{B}(b)S_{H}(h^{(2)})
= S(S_{H}(h^{(1)}))S_{H}(s(b)h^{(2)}).
\]

Similarly, \(S_{H}(h^{(1)}))S_{H}(S(h^{(2)}))\) is also well defined. Indeed,

\[
S_{H}(t(b)h^{(1)}))S_{H}(S(h^{(2)})) = S_{H}(h^{(1)}))S_{H}(t(b))S_{H}(S(h^{(2)})) = S_{H}(h^{(1)}))S_{H}(t(b))S_{H}(S(h^{(2)}))
= S_{H}(h^{(1)}))S_{H}(S(s(b)h^{(2)})).
\]

So on the one hand

\[
S(S_{H}(h^{(1)}))S_{H}(h^{(2)})S_{H}(S(h^{(3)})) = S(S_{H}(h^{(1)}))S_{H}(h^{(2)})S(h^{(3)}))
= S(S_{H}(h^{(1)}))S_{H}(s(\epsilon(h^{(2)}))) = S(S_{H}(h^{(1)}))S_{H}(S_{B}(\epsilon(h^{(2)})))
= S(S_{H}(h^{(1)}))t(S_{B}(\epsilon(h^{(2)}))) = S(S_{H}(h^{(1)}))S_{H}(t(\epsilon(h^{(2)})))
= S(S_{H}(h^{(1)}))t(\epsilon(h^{(2)}))).
\]
where in the first step we use the fact that $H$ is commutative. On the other hand
\[
S(S_H(h^{(1)}))S_H(h^{(2)})S_H(S(h^{(3)})) = S(S_H(h^{(1)})^{(1)})S_H(h^{(1)})^{(2)}S_H(S(h^{(2)}))
\]
\[
= t(e(S_H(h^{(1)})))S_H(S(h^{(2)})) = S_H(t(e(h^{(1)})))S_H(S(h^{(2)}))
\]
\[
= S_H(S(s(e(h^{(1)})))h^{(2)})) = S_H(S(h)),
\]
where the first step uses (i) of this Proposition.

\[
\square
\]

5. Crossed comodule of Hopf coquasigroups

In this section, we will study crossed comodules of Hopf coquasigroups. Moreover, we will use crossed comodules of Hopf coquasigroups to construct coherent Hopf 2-algebras with non-trivial coassociators.

**Definition 5.1.** A crossed comodule of Hopf coquasigroup $(A, B, \phi, \delta)$ is a coassociative pair $(A, B, \phi)$, such that

1. $A$ is a left $B$ comodule coalgebra and left $B$ comodule algebra with coaction $\delta$;
2. For any $b \in B$,
   \[
   \phi(b)^{-1} \otimes \phi(b)^{(0)} = b_{(1)(1)}S_B(b_{(2)}) \otimes \phi(b_{(1)(2)}) = b_{(1)}S_B(b_{(2)(2)}) \otimes \phi(b_{(2)(1)});
   \]  
   \[
   \phi(a^{-1}) \otimes a^{(0)} = a_{(1)}S_A(a_{(3)}) \otimes a_{(2)}. \tag{5.2}
   \]

3. For any $a \in A$,

If $B$ is coassociative the crossed comodule of Hopf coquasigroup is a crossed comodule of Hopf algebra [4]. In the following, we will present a useful Lemma. We will still write down the proof as it is slightly different with Proposition 5.14 of [6].

**Lemma 5.2.** Let $(A, B, \phi, \delta)$ be a crossed comodule of Hopf coquasigroup, if $B$ is commutative, then the tensor product $H := A \otimes B$ is a Hopf coquasigroup, with factorwise multiplication, and unit $1_A \otimes 1_B$. The coproduct is given by $\Delta(a \otimes b) := a_{(1)} \otimes a_{(0)}^{-1}b_{(1)} \otimes a_{(0)}^{(0)} \otimes b_{(2)}$, for any $a \otimes b \in A \otimes B$, the counit is given by $\epsilon_H(a \otimes b) := \epsilon_A(a)\epsilon_B(b)$. The antipode is given by $S_H(a \otimes b) := S_A(a^{(0)}) \otimes S_B(a^{(-1)}b)$. Moreover, if $B$ is coassociative, then $H$ is a Hopf algebra.

**Proof.** $A \otimes B$ is clearly an unital algebra and it is clearly counital. We first show $H$ is also a bialgebra:

\[
\Delta(a \otimes b) = a_{(1)}a'_{(1)} \otimes a_{(2)}^{-1}a'_{(2)}b'_{(1)} \otimes a_{(2)}^{(0)}b_{(2)} \otimes b_{(2)}^{(0)}b_{(2)}^{(0)}
\]
\[
= a_{(1)}a'_{(1)} \otimes a_{(2)}^{-1}b_{(1)}^{(0)}a'_{(2)}^{(0)}b_{(2)}^{(0)}b_{(2)}^{(0)}b_{(2)}^{(0)}b_{(2)}^{(0)}
\]
\[
= \Delta(a \otimes b) \Delta(a' \otimes b'),
\]
here we use the fact that $B$ is a commutative algebra in the 2nd step. Thus $H$ is a bialgebra. Now check the antipode $S_H$ for $h = a \otimes b$,

\[
h_{(1)(1)}S_H(h_{(1)(2)})h_{(2)}
\]
\[
= a_{(1)(1)} \otimes a_{(1)(2)}^{-1}a_{(2)}^{(0)}a_{(2)}^{(0)}a_{(2)}^{(0)}a_{(2)}^{(0)}b_{(1)(2)}b_{(2)}
\]
\[
= a_{(1)(1)} \otimes a_{(1)(2)}^{-1}a_{(1)(2)}^{-1}b_{(1)(1)} \otimes S_A(a_{(1)(2)}^{(0)}a_{(2)}^{(0)}a_{(2)}^{(0)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}
\]
\[
= a_{(1)(1)} \otimes S_A(a_{(1)(2)}^{(0)}a_{(2)}^{(0)}a_{(2)}^{(0)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}b_{(2)}
\]
\[
= a \otimes b_{(1)(1)} \otimes 1 \otimes S_B(b_{(1)(2)})b_{(2)}
\]
\[
= a \otimes b \otimes 1_A \otimes 1_B
\]
where the third step uses the fact that \( a^{(-1)} \otimes S_A(a^{(0)}) = S_A(a)^{(-1)} \otimes S_A(a)^{(0)} \). Indeed,

\[
a^{(-1)} \otimes S_A(a^{(0)}) = a_{(1)(1)}^{(-1)} a_{(1)(2)}^{(-1)} S_A(a_{(2)})^{(-1)} \otimes S_A(a_{(1)(1)}^{(0)}) a_{(1)(2)}^{(0)} S_A(a_{(2)})^{(0)} = a_{(1)}^{(-1)} S_A(a_{(2)})^{(-1)} \otimes S_A(a_{(1)(1)}^{(0)}) a_{(1)(2)}^{(0)} S_A(a_{(2)})^{(0)}
\]

\[
= a_{(1)}^{(-1)} S_A(a_{(2)})^{(-1)} \otimes \epsilon_A(a_{(1)}^{(0)}) S_A(a_{(2)})^{(0)} = S_A(a)^{(-1)} \otimes S_A(a)^{(0)},
\]

where in the second and third steps we use the comodule coalgebra property. The rest axioms of Hopf coquasigroups are similar. Thus \( H \) is a Hopf coquasigroup.

When \( B \) is coassociative, for any \( a \otimes b \in A \otimes B \), we also have

\[
((\text{id}_H \otimes \triangle) \circ \triangle)(a \otimes b) = (\text{id}_H \otimes \triangle)(a \otimes (a \otimes b)^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes b_{(2)})
\]

\[
= a_{(1)}^{(-1)} a_{(2)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes a_{(2)}^{(0)} b_{(2)} \otimes a_{(2)}^{(0)} b_{(3)} = a_{(1)}^{(-1)} a_{(2)}^{(-1)} a_{(2)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes a_{(2)}^{(0)} b_{(2)} \otimes a_{(2)}^{(0)} b_{(3)}
\]

\[
= (\triangle \otimes \text{id}_H)(a \otimes a_{(2)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes b_{(2)}) = ((\triangle \otimes \text{id}_H) \circ \triangle)(a \otimes b),
\]

where in the 3rd step we use the fact that \( A \) is a comodule coalgebra and in the 4th step we use the fact that \( A \) is a left \( B \) comodule. So \( (H, \triangle, \epsilon_H) \) is coassociative.

From the proof above we can also see that even if \( A \) is a Hopf coquasigroup, we can also get a Hopf coquasigroup \( A \otimes B \), with the same coproduct, count, and antipode as above.

**Lemma 5.3.** Let \((A, B, \phi, \delta)\) be a crossed comodule of Hopf coquasigroup. If \( B \) is commutative and the image of \( \phi \) belongs to the center of \( A \), then \( H = A \otimes B \) is a central Hopf algebroid over \( B \) with the source, target and counit (of the bialgebroid structure) being bialgebra maps. Moreover,

\[
(\Delta \otimes \Delta) \circ \triangle = (\text{id}_H \otimes \text{flip} \otimes \text{id}_H) \circ (\triangle \otimes_B \triangle) \circ \Delta.
\]

**Proof.** The source and target maps \( s, t : B \to H \) are given by \( s(b) := \phi(b_{(1)}) \otimes b_{(2)} \), and \( t(b) := 1_A \otimes b \), for any \( b \in B \). The counit map \( \epsilon : H \to B \) is defined to be \( \epsilon(a \otimes b) := \epsilon_A(a) b \), and the central Hopf algebroid coproduct is defined to be \( \Delta(a \otimes b) := (a_{(1)} \otimes 1_B) \otimes_B (a_{(2)} \otimes b) \). The antipode of the Hopf algebroid is given by \( S(a \otimes b) := S_A(a) \phi(b_{(1)}) \otimes b_{(2)} \). Now we show all the structures above form a central Hopf algebroid structure on \( H \). First, we can see that \( s, t \) are algebra maps. Now we show \( H \) is a \( B \)-coring. Here the \( B \)-bimodule structure on \( H \) is given by \( b' \triangleright (a \otimes b) \triangleleft b'' = s(b') t(b'')(a \otimes b) \) for \( a \otimes b \in H, b', b'' \in B \). So we have

\[
\epsilon(b' \triangleright (a \otimes b) \triangleleft b'') = \epsilon(s(b') t(b'')(a \otimes b)) = \epsilon_A(\phi(b'_{(1)}) a) b'_{(2)} b'' b
\]

\[
= \epsilon_B(b'_{(1)}) \epsilon_A(a) b'_{(2)} b'' b = b' \epsilon(a \otimes b) b'',
\]

\[
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\]
where we use the fact that $\phi$ is a bialgebra map in the 3rd step. Clearly, $\epsilon$ is an algebra map from $A \otimes B$ to $B$. We also have

$$
(\epsilon \otimes \epsilon)(\mathbf{\Delta}(a \otimes b)) = (\epsilon \otimes \epsilon)(a_{(1)} \otimes a_{(2)} (-1)b_{(1)} \otimes a_{(2)}^{(0)} \otimes b_{(2)})
$$

$$
= \epsilon_A(a_{(1)})a_{(2)}(-1)b_{(1)} \otimes \epsilon_A(a_{(2)}^{(0)})b_{(2)}
$$

$$
= a(-1)b_{(1)} \otimes \epsilon_A(a^{(0)})b_{(2)}
$$

$$
= \epsilon_A(a)b_{(1)} \otimes b_{(2)}
$$

$$
= \Delta_B(\epsilon(a \otimes b)),
$$

where for the 3rd step we use the fact that $A$ is a comodule algebra. So we can see that $\epsilon$ is a bialgebra map from $A \otimes B$ to $B$. We can also show $s$ and $t$ are also coalgebra maps

$$
\mathbf{\Delta}(s(b)) = \mathbf{\Delta}(s(b_{(1)}) \otimes b_{(2)})
$$

$$
= \phi(b_{(1)(1)}) \otimes \phi(b_{(1)(2)}) \otimes (b_{(2)(1)} \otimes \phi(b_{(2)(2)})) \otimes b_{(2)(2)}
$$

$$
= \phi(b_{(1)(1)}) \otimes \phi(b_{(1)(2)}) \otimes (b_{(2)(1)} \otimes \phi(b_{(2)(2)})) \otimes b_{(2)(2)}
$$

$$
= \phi(b_{(1)(1)}) \otimes b_{(1)(2)} \otimes \phi(b_{(2)(1)}) \otimes b_{(2)(2)}
$$

$$
= (s \otimes s)(\Delta_B(b)),
$$

where in the 4th and 7th steps we use Lemma 3.6 and in the 5th step we use (5.1). We also have

$$
\mathbf{\Delta}(t(b)) = \mathbf{\Delta}(1 \otimes b) = 1 \otimes b_{(1)} \otimes 1 \otimes b_{(2)} = (t \otimes t)(\Delta_B(b)),
$$

for any $b \in B$. We can also show $\Delta$ is a $B$-bimodule map:

$$
\Delta(b' \triangleright (a \otimes b)) = \Delta(\phi(b'_{(1)})a \otimes b'_{(2)}b)
$$

$$
= (\phi(b'_{(1)}a_{(1)} \otimes 1) \otimes_B (\phi(b'_{(1)}a_{(2)} \otimes b'_{(2)}b))
$$

$$
= (\phi(b'_{(1)}a_{(1)} \otimes 1) \otimes_B (\phi(b'_{(1)}a_{(2)} \otimes b'_{(2)}b))
$$

$$
= (\phi(b'_{(1)}a_{(1)} \otimes 1) \otimes_B s(b'_{(2)})(a_{(2)} \otimes b)
$$

$$
= t(b'_{(1)})(\phi(b'_{(1)}a_{(1)} \otimes 1) \otimes_B (a_{(2)} \otimes b)
$$

$$
= (\phi(b'_{(1)}a_{(1)} \otimes b'_{(2)}) \otimes_B (a_{(2)} \otimes b)
$$

$$
= b' \triangleright \Delta(a \otimes b),
$$

where in the fourth step we use Lemma 3.6. We also have

$$
\Delta((a \otimes b) \triangleleft b') = \Delta(a \otimes bb') = (a_{(1)} \otimes 1) \otimes_B (a_{(2)} \otimes bb') = \Delta(a \otimes b) \triangleleft b'
$$

for any $a \otimes b \in A \otimes B$ and $b' \in B$. $\Delta$ is clearly coassociative and counital by straightforward computation. Up to now we have already shown that $H$ is a $B$-coring. Clearly, $\Delta$ is also an algebra map from $H$ to $H \otimes_B H$. Since the image of $\phi$ belongs to the center of $A$, we can check that $S(t(b')(a \otimes b)s(b'')) = t(b'')(S(a \otimes b)s(b')$ for any $a \otimes b \in H$ and $b', b'' \in B$. 

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Indeed,
\[
S(t(b') (a \otimes b) s(b'')) = S(a\phi(b'_0) \otimes b'bb'')
\]
\[
= S_A(a\phi(b'_{(1)}) \phi(b'_{(2)})) sentence \otimes b''_0 b''_{(2)} b''_{(2)}
\]
\[
= S_A(a\phi(b') \phi(b'_0) \otimes b'' \otimes b'')
\]
\[
=t(b'') S(a \otimes b) s(b'),
\]
where the 3rd step uses the fact that \(B\) is commutative and its image of \(\phi\) belongs to the center of \(A\). Now we can see that
\[
S(a_{(1)} \otimes 1)(a_{(2)} \otimes b) = S_A(a_{(1)}) a_{(2)} \otimes b = (t \circ \epsilon)(a \otimes b),
\]
and
\[
(a_{(1)} \otimes 1)S(a_{(2)} \otimes b) = a_{(1)} S_A(a_{(2)}) \phi(b_{(1)}) \otimes b_{(2)} = (s \circ \epsilon)(a \otimes b)
\]
So \(H\) is a central Hopf algebroid.

Let \(h = a \otimes b \in H\), on the one hand
\[
(\Delta \otimes \Delta) \circ (\phi(h)) = a_{(1)} \otimes 1 \otimes_B a_{(2)} \otimes a_{(3)}^{(-1)} b_{(1)} \otimes a_{(3)}^{(0)} (1) \otimes 1 \otimes_B a_{(3)}^{(0)} (2) \otimes b_{(2)},
\]
on the other hand
\[
(H \otimes \text{flip} \otimes H) \circ (\phi(h)) = a_{(3)} \otimes a_{(2)}^{(-1)} \otimes_B a_{(3)} \otimes a_{(4)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes 1 \otimes_B a_{(4)}^{(0)} \otimes b_{(2)}
\]
\[
= a_{(1)} \otimes 1 \otimes_B \phi(a_{(2)}^{(-1)} a_{(3)} \otimes a_{(4)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes 1 \otimes_B a_{(4)}^{(0)} \otimes b_{(2)}
\]
\[
= a_{(1)} \otimes 1 \otimes_B a_{(2)}^{(1)} S_A(a_{(3)} a_{(4)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes 1 \otimes_B a_{(4)}^{(0)} \otimes b_{(2)}
\]
\[
= a_{(1)} \otimes 1 \otimes_B a_{(2)}^{(2)} \otimes a_{(3)} \otimes a_{(4)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes 1 \otimes_B a_{(4)}^{(0)} \otimes b_{(2)}
\]
\[
= a_{(1)} \otimes 1 \otimes_B a_{(2)}^{(3)} \otimes a_{(3)} \otimes a_{(4)}^{(-1)} b_{(1)} \otimes a_{(2)}^{(0)} \otimes 1 \otimes_B a_{(4)}^{(0)} \otimes b_{(2)}
\]
where the second step uses the balanced tensor product over \(B\), the fourth step uses (5.2), and the last step uses the fact that \(A\) is a comodule coalgebra of \(B\).

### 5.1. Quasi coassociative Hopf coquasigroup

In this section we will construct a coherent Hopf 2-algebra in terms of a crossed comodule of Hopf coquasigroup. First we define quasi coassociative Hopf coquasigroups, which can be viewed as a quantization of quasiasociative quasigroups.

**Definition 5.4.** A coassociative pair \((C, B, \phi)\) is called **quasi coassociative** if:

- \(\phi : B \to C\) is a surjective morphism of Hopf coquasigroups.

- For any \(i \in I_B := \ker(\phi)\),
  \[
  \begin{cases}
  i_{(1)(1)} S_B(i_{(2)}) \otimes i_{(1)(2)} & \in B \otimes I_B, \\
  i_{(1)} S_B(i_{(2)(2)}) \otimes i_{(1)(1)} & \in B \otimes I_B.
  \end{cases}
  \tag{5.3}
  \]

- \(I_B \subseteq \ker(\beta), \) where \(\beta : B \to B \otimes B \otimes B\) is the comultiplicative coassociator (3.3).

Since \(\phi\) is surjective, any element in \(C\) can be given by \([c] := \phi(c)\) for some element \(c \in B\). If \(B\) is a quasi coassociative, by (5.3) there is a linear map \(\text{Ad} : C \to B \otimes C, \)
\[
\text{Ad}([c]) := c_{(1)} S_B(c_{(2)}) \otimes S_B(c_{(3)}) : c_{(3)(1)} S_B(c_{(2)}) \otimes S_B(c_{(3)}) = c_{(1)} S_B(c_{(2)(2)}) \otimes S_B(c_{(3)(1)}) \text{ for any } [c] \in C
\]
the last equality hold because of Lemma 3.6. For any \(b \in B\), there is an important result of Lemma 3.6
\[
b_{(1)(1)} S_B(b_{(1)(2)}) b_{(2)} \otimes b_{(1)(2)} = b_{(1)(1)} S_B(b_{(1)(2)}) b_{(2)} \otimes b_{(1)(1)(2)} = b_{(1)} \otimes [b_{(2)}].
\tag{5.4}
\]
Similarly,
\[ S_B(b_{(1)})b_{(2)(1)}S_B(b_{(2)(2)}) \otimes [b_{(2)(2)}] = S_B(b_{(2)}) \otimes [b_{(1)}]. \] (5.5)

Since \( I_B \subseteq \ker(\beta) \), the comultiplicative coassociator \( \beta \) factors through \( C \), namely, there is a linear map \( \tilde{\beta} : C \to B \otimes B \otimes B \) given by \( \tilde{\beta}([c]) := \beta(c) = c^1 \otimes c^2 \otimes c^3 \).

**Lemma 5.5.** Let \((C, B, \phi)\) be quasi coassociative. If \( B \) is commutative, then \((C, B, \phi, \text{Ad})\) is a crossed comodule of Hopf coquasigroup.

**Proof.** Since
\[ c(\epsilon(1))S_B(c(2)) \otimes c(\epsilon(2))S_B(c(2)) = c(\epsilon(1))S_B(c(2)) \otimes c(\epsilon(2))S_B(c(2)), \]
for any \( c \in B \), by Lemma 3.6 we can prove \( \text{Ad} \) is a coaction. Indeed,
\[ c(\epsilon(1))S_B(c(2)) \otimes c(\epsilon(2))S_B(c(2)) \subseteq [c(2)] \]
We can see that \( \text{Ad} \) is an algebra map, since \( B \) is commutative. Now let’s show that \( \text{Ad} \) is a comodule coalgebra map. On the one hand
\[ [c]^{-1} \otimes [c]^{00} \otimes [c]^{00} = c(\epsilon(1))S_B(c(2)) \otimes [c(2)](1) \otimes [c(2)](2). \]
On the other hand
\[ [c]^{-1}(1) \otimes [c]^{00}(1) \otimes [c]^{00}(2) = c(\epsilon(1))S_B(c(2)) \otimes [c(2)](1) \otimes [c(2)](2). \]
The last step uses Lemma 3.6. And
\[ \epsilon_C([c]) = \epsilon_B(c) = c(\epsilon(1))S_B(c(2)) \epsilon_C([c]) = [c]^{-1}\epsilon_C([c]^{00}). \]

We can see (5.1) and (5.2) are given by the definition of \( \text{Ad} \).

We just considered above. In the following, we always assume \( B \) is commutative. Compare to Definition 4.1, the first Hopf coquasigroup is \( H := C \otimes B \). By Lemma 5.2, as a Hopf coquasigroup the coproduct, counit and antipode are given by the following:

\[ \Delta([c] \otimes b) := [c](1) \otimes [c](2)^{-1} \otimes [c]^{00}(1) \otimes b_{(2)}, \] (5.6)
\[ \epsilon_H([c] \otimes b) := \epsilon_B(c)\epsilon_B(b), \] (5.7)
\[ S_H([c] \otimes b) := [S_B(c(1))] \otimes S_B(c(2))S_B(b) = S_C([c]^{00}) \otimes S_B([c]^{-1}b). \] (5.8)

By Lemma 5.3 with the source and target maps \( s, t : B \to H \) are given by
\[ s(b) := [b_{(1)}] \otimes b_{(2)}, \quad t(b) := 1_C \otimes b, \] (5.9)
for any \( b \in B \). The Hopf algebroid coproduct is given by
\[ \Delta([c] \otimes b) := ([c(1)] \otimes 1_B) \otimes_B ([c(2)] \otimes b), \] (5.10)
and the counit is given by
\[ \epsilon([c] \otimes b) := \epsilon_B(c)b. \] (5.11)

The antipode is
\[ S([c] \otimes b) := [S_B(c)b_{(1)}] \otimes b_{(2)}. \] (5.12)
We define the coassociator \( \alpha : H \to B \otimes B \otimes B \) by
\[
\alpha([c] \otimes b) := \beta(c)(b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}) = c^1 b_{(1)(1)} \otimes c^2 b_{(1)(2)} \otimes c^3 b_{(2)}. \quad (5.13)
\]
By using \((3.6)\) we can check condition (iv) of Definition 4.1.
\[
\alpha(t(b)) = b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}
\]
and
\[
\alpha(s(b)) = (b_{(1)})^1 b_{(2)(1)(1)} \otimes (b_{(1)})^2 b_{(2)(1)(2)} \otimes (b_{(1)})^3 b_{(2)(2)}
\]
\[= b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)},
\]
where the second step uses \((3.6)\). Now let’s check (v) and (vi) of Definition 4.1.

**Lemma 5.6.** The coassociator \( \alpha : H \to B \otimes B \otimes B \) satisfies:
\[
((s \otimes s \otimes s) \circ \alpha) \ast ((\triangle \otimes \text{id}_H) \circ \triangle) = ((\text{id}_H \otimes \triangle) \circ \triangle) \ast ((t \otimes t \otimes t) \circ \alpha).
\]

More precisely, for any \( h \in H \), we have
\[
s(h^{(1)1})h^{(2)}_{(1)(1)} \otimes s(h^{(1)2})h^{(2)}_{(1)(2)} \otimes s(h^{(1)3})h^{(2)}_{(2)}
\]
\[= h^{(1)}_{(1)} t(h^{(2)1}) \otimes h^{(1)}_{(2)(1)} t(h^{(2)2}) \otimes h^{(1)}_{(3)(2)} t(h^{(2)3}).
\]

**Proof.** Let \( h = [c] \otimes b \). The left hand side of the equation above is:
\[
s((c_{(1)})^1)([c_{(2)(1)(1)}] \otimes [c_{(2)(1)(2)}]^{−1})[c_{(2)(2)}]^{−1} b_{(1)(1)} \otimes s((c_{(1)})^2)([c_{(2)(1)(2)}]^{00} \otimes [c_{(2)(2)}]^{−1}) b_{(1)(2)}
\]
\[\otimes s((c_{(1)})^3)([c_{(2)(2)}]^{00} \otimes b_{(2)}),
\]
while the right hand side of the equation is
\[
[c_{(1)(1)}] \otimes [c_{(1)(2)}]^{−1} (c_{(2)})^1 b_{(1)(1)} \otimes [c_{(1)(2)}]^{00} \otimes [c_{(1)(2)}]^{−1} (c_{(2)})^2 b_{(1)(2)}
\]
\[\otimes [c_{(1)(2)}]^{00} \otimes (c_{(2)})^3 b_{(2)}.
\]

So it is sufficient to show
\[
s((c_{(1)})^1)([c_{(2)(1)(1)}] \otimes [c_{(2)(1)(2)}]^{−1})[c_{(2)(2)}]^{−1} (c_{(2)})^1 \otimes s((c_{(1)})^2)([c_{(2)(1)(2)}]^{00} \otimes [c_{(2)(2)}]^{−1}) (c_{(2)})^2 \otimes [c_{(1)(2)}]^{00} \otimes (c_{(2)})^3
\]
\[= [c_{(1)(1)}] \otimes [c_{(1)(2)}]^{−1} (c_{(2)})^1 \otimes [c_{(1)(2)}]^{00} \otimes [c_{(1)(2)}]^{−1} (c_{(2)})^2 \otimes [c_{(1)(2)}]^{00} \otimes (c_{(2)})^3.
\]

By the definition of Hopf coquasigroup, this is equivalent to
\[
s((c_{(1)(1)(2)})^1)([c_{(1)(1)(2)(1)(1)}] \otimes [c_{(1)(1)(2)(1)(2)}]^{−1})[c_{(1)(1)(2)(2)}]^{−1} c_{(1)(2)(1)(1)} S_B(c_{(2)(1)(1)})
\]
\[\otimes s((c_{(1)(1)(2)})^2)([c_{(1)(1)(2)(2)}]^{00} \otimes [c_{(1)(2)(2)}]^{−1}) c_{(1)(2)(1)(2)} S_B(c_{(2)(1)(2)})
\]
\[\otimes s((c_{(1)(1)(2)})^3)([c_{(1)(2)(2)}]^{00} \otimes c_{(1)(2)(2)} S_B(c_{(2)}))
\]
\[= [c_{(1)(1)(1)}] \otimes [c_{(1)(2)(2)}]^{−1} (c_{(1)(2)})^1 c_{(1)(2)(1)(1)} S_B(c_{(2)}) S_B(c_{(2)}),
\]
\[\otimes [c_{(1)(1)(1)}]^{00} \otimes [c_{(1)(2)(2)}]^{−1} (c_{(1)(2)})^2 c_{(1)(2)(1)(2)} S_B(c_{(2)}) S_B(c_{(2)}),
\]
\[\otimes [c_{(1)(1)(2)}]^{00} \otimes (c_{(1)(1)(2)})^3 S_B(c_{(2)}).
\]
Thus it is sufficient to show
\[s((c_{(1)(1)}^1)((c_{(1)(2)2(1)(2)}) \otimes [c_{(1)(2)2(1)(2)}]^{-1})[c_{(1)(2)(2)}]^{-1}(1)c_{(2)(1)(1)})
\]
\[\otimes s((c_{(1)(1)}^2)((c_{(1)(2)(2)}) \otimes \lfloor c_{(1)(2)(2)} \lfloor^{-1}(2) c_{(2)(1)(2)}) \otimes s((c_{(1)(1)}^3)((c_{(1)(2)(2)}) \otimes c_{(2)(2)})
\]
\[=\lfloor c_{(1)(1)(1)} \otimes [c_{(1)(2)(2)}]^{-1}(c_{(2)(1)(1)}^1) c_{(2)(1)(1)}
\]
\[\otimes [c_{(1)(2)(1)(2)}]^{-1}(c_{(1)(2)(2)}^1) c_{(1)(2)(2)} \otimes [c_{(1)(2)(1)(2)}]^{-1}(c_{(1)(2)(2)}^2) c_{(2)(1)(2)} \otimes [c_{(1)(2)(1)(2)}]^{-1}(c_{(1)(2)(2)}^3) c_{(2)(1)(2)}
\]

The left hand side is
\[s((c_{(1)(1)}^1)((c_{(1)(2)(1)(2)} \otimes [c_{(1)(2)(1)(2)}]^{-1})[c_{(1)(2)(2)}]^{-1}(1)c_{(2)(1)(1)})
\]
\[\otimes s((c_{(1)(1)}^2)(c_{(1)(2)(2)}) \otimes \lfloor c_{(1)(2)(2)} \lfloor^{-1}(2) c_{(2)(1)(2)}) \otimes s((c_{(1)(1)}^3)((c_{(1)(2)(2)}) \otimes c_{(2)(2)})
\]
\[=s((c_{(1)(1)}^1)(c_{(2)(1)(1)(1)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(1)c_{(2)(1)(2)(2)})
\]
\[\otimes s((c_{(1)(1)}^2)(c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(2) c_{(2)(1)(2)(2)}) \otimes s((c_{(1)(1)}^3)((c_{(2)(1)(2)(2)}) \otimes c_{(2)(2)(2)})
\]
\[=s((c_{(1)(1)}^1)(c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(1)c_{(2)(1)(2)(2)})
\]
\[\otimes s((c_{(1)(1)}^2)(c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(2) c_{(2)(1)(2)(2)}) \otimes s((c_{(1)(1)}^3)((c_{(2)(1)(2)(2)}) \otimes c_{(2)(2)(2)})
\]
\[=[(c_{(1)(1)}^1) \otimes c_{(2)(1)(2)(2)} \otimes ((c_{(1)(1)}^2) \otimes c_{(2)(1)(2)(2)} \otimes ((c_{(1)(1)}^3) \otimes c_{(2)(1)(2)(2)}
\]
\[=[c_{(1)(1)}] \otimes c_{(2)(1)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(1)c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(2) c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(3) c_{(2)(1)(2)(2)}
\]
\[=\lfloor c_{(1)(1)(1)} \otimes [c_{(1)(2)(1)(2)}]^{-1}(1)c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(2) c_{(2)(1)(2)(2)} \otimes [c_{(2)(1)(2)(2)}]^{-1}(3) c_{(2)(1)(2)(2)}
\]

where for the 1st, 2nd, 3rd, 5th step we use Lemma 3.6 and the fact that \( \beta \) factors through \( C \), in the 2nd, 4th step we use (3, 4), and the last step uses (3, 0). The right hand side is:
where in the 1st, 3rd, 5th and last step we use Lemma 3.6 and the fact that $\beta$ factors through $C$, the 2nd step uses (3.4), and the 4th and 6th step use (5.3).

Lemma 5.7. The coassociator $\alpha : H \to B \otimes B \otimes B$ satisfies the 3-cocycle condition:

$((\epsilon \otimes \alpha) \circ \Delta) \ast ((\text{id}_B \otimes \Delta_B \otimes \text{id}_B) \circ \alpha) = ((\text{id}_B \otimes \text{id}_B \otimes \Delta_B) \circ \alpha) \ast ((\Delta_B \otimes \text{id}_B \otimes \text{id}_B) \circ \alpha)$.

More precisely, for any $h \in H$,

$h^{(1)} h^{(2)} 1^{(1)} \otimes h^{(1)} 2^{(2)} h^{(2)} 1^{(3)} \otimes h^{(2)} 2^{(3)} h^{(3)} 1^{(3)} \otimes h^{(3)} 2^{(3)}

= \epsilon(h^{(1)}) h^{(2)} 1^{(1)} \otimes h^{(1)} 2^{(2)} h^{(1)} 2^{(1)} \otimes h^{(2)} 2^{(2)} h^{(3)} 3 \otimes h^{(3)} 2^{(3)} \epsilon(h^{(3)} 2)$.

Proof. Let $h = [c] \otimes b$, the left hand side is

$(c^{(1)})^1 (c^{(2)})^1 (b^{(1,1,1)} \otimes (c^{(1)})^2 (c^{(2)})^2 (b^{(1,1,2)} \otimes (c^{(1)})^3 (c^{(2)})^3 (b^{(2)}),$

the right hand side is

$c^{(1)}(c^{(2)})^1 (c^{(2)})^1 (b^{(1,1,1)} \otimes (c^{(1)})^2 (c^{(2)})^2 (b^{(1,1,2)} \otimes (c^{(1)})^3 (c^{(2)})^3 (b^{(2)}),$

We first observe that

$(c^{(1)})^1 (c^{(2)})^1 (c^{(2)})^2 (c^{(2)})^2 (c^{(2)})^3 (c^{(2)})^3

= (c^{(1)})^1 (c^{(2)})^1 (c^{(1)})^2 (c^{(2)})^2 (c^{(1)})^3 (c^{(2)})^3$

and

$c^{(1)}(c^{(2)})^1 (c^{(2)})^1 (c^{(2)})^1 (c^{(2)})^2 (c^{(2)})^2 (c^{(2)})^3 (c^{(2)})^3

= (c^{(1)})^1 (c^{(2)})^1 (c^{(1)})^2 (c^{(2)})^2 (c^{(1)})^3 (c^{(2)})^3$

Thus to show this lemma it is sufficient to show:

$(c^{(1)})^1 (c^{(2)})^1 (c^{(2)})^1 (c^{(2)})^2 (c^{(2)})^1 (c^{(2)})^1 (c^{(2)})^2 (c^{(2)})^2

\otimes (c^{(1)})^3 (c^{(2)})^3 (c^{(2)})^3 (c^{(2)})^3 (c^{(2)})^3 (c^{(2)})^3 (c^{(2)})^3 (c^{(2)})^3$
Since \( \ker(\phi) \subseteq \ker(\beta) \) the left hand side of the above equation becomes

\[
(c_{(1)})^1(c_{(2)(1)})^1(c_{(2)(2)(1)(1)})^1 \otimes (c_{(1)})^2(c_{(2)(1)})^2(c_{(2)(2)(1)(1)})^2 \\
\otimes (c_{(1)})^3(c_{(2)(1)})^3(c_{(2)(2)(1)(2)})^2 \otimes (c_{(1)})^4(c_{(2)(1)})^4(c_{(2)(2)(2)}),
\]

\[
= (c_{(1)})^1(c_{(2)(1)})^1 \otimes (c_{(1)})^2(c_{(2)(1)})^2 \otimes (c_{(1)})^3(c_{(2)(1)})^3 \otimes (c_{(1)})^4(c_{(2)(1)})^4.
\]

where the first step uses (3.6). By Lemma 3.6 and the fact that \( \beta \) factors through \( C \), the right hand side becomes

\[
c_{(1)(1)}S_B(c_{(1)(2)})(c_{(2)(1)})^1(c_{(2)(2)(1)(1)})^1 \otimes (c_{(1)(1)(2)})^1(c_{(2)(1)})^2(c_{(2)(2)(1)(2)})^2 \\
\otimes (c_{(1)(1)(2)})^3(c_{(2)(1)})^3(c_{(2)(2)(2)(2)})^2 \otimes (c_{(1)(1)(2)})^4(c_{(2)(1)})^4(c_{(2)(2)(2)}),
\]

\[
= c_{(1)(1)}S_B(c_{(1)(2)})(c_{(2)(1)})^1(c_{(2)(2)(1)(1)})^1 \otimes (c_{(1)(1)(2)})^1(c_{(2)(1)})^2(c_{(2)(2)(1)(2)})^2 \\
\otimes (c_{(1)(1)(2)})^3(c_{(2)(1)})^3(c_{(2)(2)(2)(2)})^2 \otimes (c_{(1)(1)(2)})^4(c_{(2)(1)})^4(c_{(2)(2)(2)}),
\]

\[
= c_{(1)(1)}S_B(c_{(1)(2)})(c_{(2)(1)})^1(c_{(2)(2)(1)(1)})^1 \otimes (c_{(1)(1)(2)})^1(c_{(2)(1)})^2(c_{(2)(2)(1)(2)})^2 \\
\otimes (c_{(1)(1)(2)})^3(c_{(2)(1)})^3(c_{(2)(2)(2)(2)})^2 \otimes (c_{(1)(1)(2)})^4(c_{(2)(1)})^4(c_{(2)(2)(2)}),
\]

\[
= c_{(1)}(c_{(2)(1)})^1(c_{(2)(2)(1)(1)})^1 \otimes (c_{(1)(1)(2)})^1(c_{(2)(1)})^2(c_{(2)(2)(1)(2)})^2 \\
\otimes (c_{(1)(1)(2)})^3(c_{(2)(1)})^3(c_{(2)(2)(2)(2)})^2 \otimes (c_{(1)(1)(2)})^4(c_{(2)(1)})^4(c_{(2)(2)(2)}),
\]

where in the 1st, 4th and 6th step we use Lemma 3.6 and the fact that \( \beta \) factors through \( C \), for the 2nd, 3rd and last step we use (3.6), and the 5th step uses (5.4) (since \( \ker(\phi) \subseteq \ker(\beta) \)).

As a result of Lemma 5.2, 5.3, 5.5, 5.6 and 5.7 we have

**Theorem 5.8.** If \( (C, B, \phi) \) is quasi coassociative and \( B \) is commutative, then \( H = C \otimes B \) is a coherent Hopf 2-algebra with the structure maps given by:

\[
([c] \otimes b)([c'] \otimes b') = [cc'] \otimes bb',
\]

\[
\Delta([c] \otimes b) = [c]_{(1)} \otimes [c]_{(2)}^{-1}b_{(1)} \otimes [c]_{(a)}^{(0)} \otimes b_{(2)},
\]

\[
\epsilon_H([c] \otimes b) = \epsilon_C([c]) \epsilon_B(b),
\]

\[
S_H([c] \otimes b) = S_C([c]^{(0)}) \otimes S_B([c]^{-1}b),
\]

\[
s(b) = \phi(b_{(1)}) \otimes b_{(2)},
\]

\[
t(b) = 1 \otimes b,
\]

\[
\Delta([c] \otimes b) = [c]_{(1)} \otimes 1 \otimes B [c]_{(2)} \otimes b,
\]

\[
\epsilon([c] \otimes b) = \epsilon_C([c]) b,
\]

\[
S([c] \otimes b) = S_C([c]) \phi(b_{(1)}) \otimes b_{(2)},
\]

\[
\alpha([c] \otimes b) = \beta(c)(b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}) = c^1b_{(1)(1)} \otimes c^2b_{(1)(2)} \otimes c^3b_{(2)}.
\]

Since for strict Hopf 2-algebra the coassociator is trivial, we can conclude:
Corollary 5.9. Let \((A, B, \phi, \delta)\) be a crossed comodule of Hopf algebra, if \(B\) is commutative and the image of \(\phi\) belongs to the center of \(B\), then \(H = A \otimes B\) is a strict Hopf 2-algebra.

Here are some examples of strict Hopf 2-algebras:

Example 5.10. Let \(\phi : B \to A\) be a surjective morphism of Hopf algebras, and \(B\) is commutative, such that for any \(i \in I := \ker(\phi)\), \(i(\phi)S_B(i(\phi)) \otimes i(\phi) \in B \otimes I\). Since the left co-adjoint coaction factors through \(A\) and \(\phi\) is surjective, we can define \(\delta : A \to B \otimes A\) by \(\delta([a]) := a_1B(a_2) \otimes [a_{(2)}]\), where \([a] := \phi(a)\). We can see that \(A\) is a comodule coalgebra and comodule algebra of \(B\), since \(A\) is commutative. Moreover, (5.1) and (5.2) are also satisfied. Therefore \((A, B, \phi, \delta)\) forms a strict Hopf 2-algebra.

Example 5.11. Let \(G \to H \to E\) be a short exact sequence of Hopf algebras with injection \(i : G \to H\), surjection \(\pi : H \to E\) and \(G\) commutative, such that \(h_{(1)} \otimes \pi(h_{(2)}) = h_{(2)} \otimes \pi(h_{(1)})\) for any \(h \in H\). For any \(k \in H\), we can see \(k_{(1)}S_H(k_{(2)}) \otimes k_{(2)} \in i(G) \otimes H\), since \(k_{(1)}S_H(k_{(1)}) \otimes k_{(2)} \in \ker(\pi) \otimes H\). Therefore, we can define a coaction \(\delta : H \to G \otimes H\) by \(\delta(h) := h_{(1)}S_H(h_{(3)}) \otimes h_{(2)}\) (here we identify \(G\) with \(i(G) \subseteq H\)). We can see that \(H\) is a \(G\)-comodule algebra and comodule coalgebra. Moreover, (5.1) and (5.2) are also satisfied. Thus \((H, G, i, \delta)\) is a crossed comodule of Hopf algebra. Moreover, if the image of \(i\) belongs to the centre of \(H\), then \((H, G, i, \delta)\) forms a strict Hopf 2-algebra.

6. Finite dimensional coherent Hopf 2-algebras and examples

Now we will give an explicit example of a coherent Hopf 2-algebra based on the Cayley algebra. By [6] the unital basis of Cayley algebras \(G_n := \{\pm e_a \mid a \in \mathbb{Z}_2^n\}\) is a quasigroup, with the product controlled by a 2-cochain \(F : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to k^*\), more precisely, \(e_a e_b := F(a, b)e_{a+b}\). From now on we define \(e^0_a := e_a\) and \(e^1_a := -e_a\), i.e. \(G_n = \{e^i_a \mid a \in \mathbb{Z}_2^n, i \in \mathbb{Z}_2\}\), so we have \(e^i_a e^j_b = F(a, b)e^{i+j}_{a+b}\). We define \(kG_n\) as the linear extension of \(G_n\), which is a Hopf quasigroup with the coalgebra structure given by \(\Delta(u) = u \otimes u\), \(\epsilon(u) = 1\), and \(S(u) := u^{-1}\) on the basis elements. As we already know from [6] that

\[
kG_n \simeq \begin{cases} 
\mathbb{C} & \text{if } n = 1 \\
\mathbb{H} & \text{if } n = 2 \\
\mathbb{O} & \text{if } n = 3.
\end{cases}
\]

We also have

\[
N_kG_n \simeq \begin{cases} 
\mathbb{C} & \text{if } n = 1 \\
\mathbb{H} & \text{if } n = 2 \\
\mathbb{R} & \text{if } n = 3.
\end{cases}
\]

By [6, Prop 3.6], we know \(G_n\) is quasiassociative and \(B := k[G_n]\) given by functions on \(G_n\) is a Hopf coquasigroup. Let \(f^i_a \in k[G_n]\) be the delta function on each element of \(G_n\), i.e. \(f^i_a(e^j_b) = \delta_{a,b}\delta_{i,j}\). We can see \(k[G_n]\) is an algebra with generators \(\{f^i_a \mid a \in \mathbb{Z}_2^n, i \in \mathbb{Z}_2\}\) subject to the relations:

\[
f'^{i}af^{i'}_{a'} = \begin{cases} 
f^{i'}_a & \text{if } a = a' \text{ and } i = i' \\
0 & \text{otherwise}.
\end{cases}
\]
The unit of \( k[G_n] \) is \( \sum_{a \in \mathbb{Z}_2^*, i \in \mathbb{Z}_2} f_a^i \). The coproduct, counit and antipode are given by

\[
\Delta_B(f_a^i) := \sum_{b+c=a, j+k=i} F(b, c) f_b^j \otimes f_c^k, \quad (6.1)
\]
\[
\epsilon_B(f_a^i) := \delta_{a, 0} \delta_{i, 0}, \quad (6.2)
\]
\[
S_B(f_a^i) := F(a, a) f_a^0. \quad (6.3)
\]

The previous structures make \( k[G_n] \) a Hopf coquasigroup. Now we will show \( (k[G_0], k[G_n], \pi) \) is quasi coassociative, where \( \pi : k[G_n] \rightarrow k[G_0] \) is the canonical projection map given by the pull back of the inclusion \( \{-e_0, e_0\} \subseteq G_n \). First, since \( F(a, 0) = 1 \), we can see \( (\pi \otimes \text{id}) \circ \Delta(f_a^i) = \sum_{j+k=i} F(0, a) f_0^j \otimes f_a^k = \sum_{j+k=i} f_0^j \otimes f_a^k \), and \( (k[G_0], k[G_n], \pi) \) is a coassociative pair. Second, we have

\[
x_{(1)(1)} S_B(x_{(2)(1)}) \otimes x_{(1)(2)} \in B \otimes I_B
\]
\[
x_{(1)} S_B(x_{(2)(2)}) \otimes x_{(2)(1)} \in B \otimes I_B,
\]

for any \( x \in I_B \). Indeed, let \( x \in I_B = \ker(\pi) \), then \( x \) is a linear combination of \( f_a^i \) with \( a \neq 0 \). Without losing generality (since every map below is linear), assuming \( x = f_a^i \), we can see

\[
x_{(1)} S_B(x_{(2)(2)}) \otimes x_{(2)(1)} = \sum_{b+c+d=a, j+k+l=i} F(b, c + d) F(c, d) F(d, d) f_b^j f_c^k \otimes f_d^l, \quad (6.4)
\]

the right hand side of the equality is not zero only if \( b = d \). As a result \( c \) is equal to \( a \), and \( x_{(1)} S_B(x_{(2)(2)}) \otimes x_{(2)(1)} \in B \otimes I_B \). Similarly, we also have \( x_{(1)(1)} S_B(x_{(2)}) \otimes x_{(1)(2)} \in B \otimes I_B \). Moreover, we can see the left adjoint coaction on \( C \) is trivial, namely, \( \text{Ad}([f_0]) = 1 \otimes [f_0] \).

Finally, recall the linear map \( \beta : B \rightarrow B \otimes B \otimes B \)

\[
\beta(b) = b_{(1)(1)} S_B(b_{(2)(1)}) \otimes b_{(1)(2)} S_B(b_{(2)(2)}) \otimes b_{(1)(2)} S_B(b_{(2)(2)})
\]

for any \( b \in B \). We can see \( I_B \subseteq \ker(\beta) \). Indeed, without losing generality, let \( x = f_a^i \) with \( a \neq 0 \), we have:

\[
\beta(x) = \beta(f_a^i) = \sum_{j+k+l+m+n+p=i, b+c+d+e+f+g=a} F(b+c+d, e+f+g) F(b, c+d) F(c, d) F(e, f+g) F(f, g)
\]
\[
F(e, e) F(f, f) F(g, g) f_b^j f_c^k \otimes f_e^m f_f^l \otimes f_d^m f_e^m.
\]

Since \( a \neq 0 \), we can see the right hand side of the above equation is zero (by using \( b + c + d + e + f + g = a \)). So \( I_B \) belongs to the kernel of \( \beta \). By Definition 5.4 we can see \( (k[G_0], k[G_n], \pi) \) is quasi coassociative. Thus by Theorem 5.3 there is a coherent Hopf 2-algebra structure, with \( C = k[G_0] \), and \( H = k[G_0] \otimes k[G_n] \). More precisely, the Hopf
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where α
Recall From the formula of β
Cheng, Adam Magee and Veronica Fantini for their proof reading. Chenchang Zhu and Daan Van De Weem for many useful discussions, and also Dr. Song
C
Since the left adjoint coaction of C is trivial, the Hopf coquasigroup structure of H is given by:
\[ \Delta(f_0^i \otimes f_a^j) = \sum_{m+n=l \atop b+c+d=a} F(b, c) f_0^i \otimes f_b^m \otimes f_c^n \otimes f_a^j; \]
\[ \epsilon(f_0^i \otimes f_a^j) = \delta_{i,0} f_a^j; \]
\[ S(f_0^i \otimes f_a^j) = f_0^i \otimes f_{a^{-1}}^j; \]
\[ s(f_a^j) = \sum_{m+n=l} f_0^m \otimes f_n^l; \]
\[ t(f_a^j) = 1 \otimes f_a^j. \]

Since the left adjoint coaction of C is trivial, the Hopf coquasigroup structure of H is given by:
\[ \Delta(f_0^i \otimes f_a^j) = \sum_{m+n=l \atop b+c+d=a} F(b, c) f_0^i \otimes f_b^m \otimes f_c^n \otimes f_a^j; \]
\[ \epsilon_H(f_0^i \otimes f_a^j) = \epsilon_B(f_0^i f_a^j) = \delta_{i,0} \delta_{a,0}; \]
\[ S_H(f_0^i \otimes f_a^j) = F(a, a) f_0^i \otimes f_a^j. \]

Recall α : H → B ⊗ B ⊗ B in [5,13], we have
\[ \alpha(f_0^i \otimes f_a^j) = \sum_{k+m+n=l \atop b+c+d=a} \beta(f_0^m)(F(b + c + d) F(b, c) f_b^k \otimes f_c^n \otimes f_a^j). \]

From the formula of β, we can see that α is controlled by a 3-coboundary ∂F. In fact,
\[ \beta(f_0^i) = \sum_{j, k, t \in \mathbb{Z}^2 \atop b, c, d \in \mathbb{Z}_n^3} F(b + c + d, b + c + d) F(b, c + d) F(c, d) F(d, c + b) F(c, b) \]
\[ \cdot F(d, d) F(c, c) F(b, b) f_b^j \otimes f_c^k \otimes f_d^l \]
\[ = \sum_{j, k, t \in \mathbb{Z}^2 \atop b, c, d \in \mathbb{Z}_n^3} F(b + c + d, b + c + d) \partial F(b, c, d) F(d, d) F(c, c) F(b, b) f_b^j \otimes f_c^k \otimes f_d^l, \]
where \( \partial F \) is the 3-coboundary given by [9]:
\[ \partial F(b, c, d) = \frac{F(b, c + d) F(c, d)}{F(d, c + b) F(c, b)} = F(b, c + d) F(c, d) F(d, c + b) F(c, b). \]

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