Note on the Riemann solutions to the Euler equations of gas dynamics in the vanishing pressure limit

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Abstract

The behaviour of the solutions of the Riemann problem for the isentropic Euler equations in vanishing pressure limit is analyzed. It is shown that any solution composed of a 1-shock wave combined with a 2-rarefaction wave tends to a two-shock waves when the pressure coefficient gets smaller than a fixed value determined by the Riemann data. In contrast, any solution composed of a 1-rarefaction wave combined with a 2-shock wave tends to a two-rarefaction waves when the pressure coefficient gets smaller than a fixed value determined by the Riemann data. The two situations are illustrated with a numerical test.

Keywords: Euler equations, vanishing pressure limit, δ-shocks, vacuum states, isentropic fluids, pressureless fluids

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1. Introduction

The one-dimensional isentropic Euler equations of gas dynamics writes as

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} &= 0, \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + p)}{\partial x} &= 0,
\end{align*}
\]

(1.1)

where \(\rho > 0\), \(v\) and \(p\) are the density, velocity and pressure of the gas, respectively. The pressure is a function of the density and is determined from the constitutive thermodynamic relations of the gas under consideration. We restrict ourselves to polytropic perfect gases for which the state equation for the pressure is given by

\[ p = p(\rho) = \kappa \rho^\gamma, \quad \kappa > 0, \gamma > 1. \]

(1.2)

Formally, the limit system of the isentropic Euler equations, when the pressure vanishes, forms the pressureless Euler equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} &= 0, \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2)}{\partial x} &= 0.
\end{align*}
\]

(1.3)

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This system can be used to model the motion of free particles which stick under collision [1]. It has been extensively analyzed, see for instance [2, 1, 3, 4, 5, 6, 7]. Bouchut [2] first established the existence of measure solutions of the Riemann problem. The 1-D and 2-D Riemann problems were solved by Sheng and Zhang [3] with the characteristic and vanishing viscosity methods. It is proven in [3] that δ-shock waves and vacuum states do occur in the solutions of the Riemann problem.

The system (1.1) describes the flow of isentropic compressible fluids. There has been a great interest in the analysis of the isentropic Euler equations for gas dynamics [8, 9, 10, 11, 12, 13, 14]. In 2003, Chen and Liu [13] identified and analyzed the phenomena of concentration and cavitation and the formation of δ-shock waves and vacuum states in solutions of the system (1.1)-(1.2) as the pressure vanishes. They rigorously proved that: any two-shock Riemann solution of the Euler equations for isentropic fluids tends to a δ-shock solution of the Euler equations for pressureless fluids, and the intermediate density between the two shocks tends to a weighted δ-measure that forms a δ-shock; by contrast, any two-rarefaction Riemann solution of the Euler equations for isentropic fluids tends to a two-contact-discontinuity solution of the Euler equations for pressureless fluids, whose intermediate state between the two contact discontinuities is a vacuum state even when the initial data stays away from the vacuum. These results were extended for nonisentropic flows [15], for the relativistic Euler equations for polytropic gases [16], and recently for the modified Chaplygin gas pressure law [17]. These papers only cover solutions that contain two shocks or two rarefactions waves.

In this paper we are concerned with the behaviour of the two others possible configurations for the Riemann problem, namely a 1-shock wave combined with a 2-rarefaction wave and a 1-rarefaction wave combined with a 2-shock wave. It is shown that any Riemann solution composed of a 1-shock wave and 2-rarefaction wave converts to a two-shock waves solution when the pressure coefficient κ gets smaller than a fixed value κ_{sr} determined by the Riemann data. In contrast, any Riemann solution composed of a 1-rarefaction wave and 2-shock wave converts to a two-rarefaction waves solution when the coefficient κ gets smaller than a fixed value κ_{rs} determined by the initial data. As far as we know, the proof and complete analysis of these two cases were left over in the literature on the degeneracy of the Euler equations.

The rest of this paper is organized as follows. In section 2, we recall the relations for shock and rarefaction waves for the Riemann problem of the system (1.1)-(1.2). In section 3, we analyze the behaviour of a solution composed of a 1-shock wave combined with a 2-rarefaction wave and a solution composed of a 1-rarefaction wave combined with a 2-shock wave, when the pressure coefficient vanishes. Numerical illustrations are carried out in section 4.

2. Rarefaction and shock curves

The Jacobian matrix of the system (1.1)-(1.2) has two eigenvalues
$$
\lambda_1 = v - c, \quad \lambda_2 = v + c,
$$
where
$$
c = c(\rho) = \sqrt{p'(\rho)} = \sqrt{\kappa \gamma \rho^{\gamma-1}}.
$$
is called the sound speed. The j-Riemann invariants are
$$
\phi_1 = v + \frac{2}{\gamma - 1} c, \quad \phi_2 = v - \frac{2}{\gamma - 1} c.
$$
The Riemann problem for the isentropic Euler equation for gas dynamics consists in the solutions of the system (1.1)-(1.2) with initial data

\[ (\rho, v)(x, 0) = \begin{cases} 
(\rho_-, v_-), & x < 0, \\
(\rho_+, v_+), & x > 0,
\end{cases} \]

(2.4)

where \( \rho_-, \rho_+ \in \mathbb{R}^+ \) and \( v_-, v_+ \in \mathbb{R} \), are given constants. The solution of this problem is established in [8, 9]. The solutions of the Riemann problem consist of rarefaction and shock waves. We briefly recall the relations for these waves for completeness.

2.1. Rarefaction curves

Given a state \((\rho_-, v_-)\), we look for the set of states \((\rho^\kappa, v^\kappa)\) that can be connected from the left to the state \((\rho_-, v_-)\) by a 1-rarefaction wave. The states \((\rho^\kappa, v^\kappa)\) depend on the pressure \(p\) which is parametrized by the parameter \(\kappa\). Therefore, in the following, these states are denoted by \((\rho^\kappa_\kappa, v^\kappa_\kappa)\).

Using the fact that in a \(j\)-rarefaction wave, a \(j\)-Riemann invariant is constant (see [10], theorem 3.2) and \(\lambda_1\) increases from the left to the right, one establishes that the state \((\rho^\kappa_\kappa, v^\kappa_\kappa)\) satisfies

\[ v^\kappa_\kappa = v_- + \frac{2}{\gamma - 1} (c_- - c^\kappa_\kappa) = v_- + \frac{2\sqrt{\kappa \gamma}}{\gamma - 1} \left( \rho^-_{\frac{3\gamma - 2}{\gamma - 1}} - (\rho^\kappa_\kappa)^{\frac{3\gamma - 2}{\gamma - 1}} \right), \quad \rho^\kappa_\kappa < \rho_- . \]  

(2.5)

In a same way, one establishes that a state \((\rho^\kappa_\kappa, v^\kappa_\kappa)\) that can be connected from the right to a given state \((\rho_+, v_+)\) by a 2-rarefaction wave, satisfies

\[ v^\kappa_\kappa = v_+ - \frac{2\sqrt{\kappa \gamma}}{\gamma - 1} \left( \rho^+_{\frac{3\gamma - 2}{\gamma - 1}} - (\rho^\kappa_\kappa)^{\frac{3\gamma - 2}{\gamma - 1}} \right), \quad \rho^\kappa_\kappa < \rho_+ . \]  

(2.6)

2.2. Shock curves

Given a state \((\rho_-, v_-)\), we look for the set of states \((\rho^\kappa_\kappa, v^\kappa_\kappa)\) that can be connected from the left to the state \((\rho_-, v_-)\) by a 1-shock wave. Combining the Rankine-Hugoniot conditions and the Lax entropy conditions, one establishes that the state \((\rho^\kappa_\kappa, v^\kappa_\kappa)\) satisfies

\[ v^\kappa_\kappa = v_- - \sqrt{\frac{\kappa (R_\kappa)^{\gamma} - \rho_{-\gamma}}{\rho^\kappa_\kappa (\rho^\kappa_\kappa - \rho_-)}} (\rho^\kappa_\kappa - \rho_-), \quad \rho^\kappa_\kappa > \rho_- . \]  

(2.7)

In a analogous way, one establishes that a state \((\rho^\kappa_\kappa, v^\kappa_\kappa)\) that can be connected from the right to a given state \((\rho_+, v_+)\) by a 2-shock, satisfies

\[ v^\kappa_\kappa = v_+ + \sqrt{\frac{\kappa (\rho^\gamma_\kappa - \rho^\kappa_\kappa)^{\gamma}}{\rho^\kappa_\kappa (\rho^\kappa_\kappa - \rho_+)}} (\rho^\kappa_\kappa - \rho_+), \quad \rho^\kappa_\kappa > \rho_+ . \]  

(2.8)

3. Behaviour of the solutions of the Riemann problem in the pressure vanishing limit

The solution of the Riemann problem for the pressureless gas system (1.3) with Riemann initial data (2.4) is established in [3]. In the case \(v_- < v_+\), the solution of the Riemann problem consists
of two contact discontinuities and a vacuum state. That is
\[
(\rho, v)(x, t) = \begin{cases} 
(\rho_-, v_-), & -\infty < \frac{x}{t} \leq v_-, \\
(0, v(\frac{x}{t})), & v_- \leq \frac{x}{t} \leq v_+, \\
(\rho_+, v_+), & v_+ \leq \frac{x}{t} < \infty,
\end{cases}
\tag{3.1}
\]
where \(v(\frac{x}{t})\) is any smooth function satisfying \(v(v_\pm) = v_\pm\). In the case \(v_- > v_+\), the solution of the Riemann problem is given by a \(\delta\)-shock solution, that is
\[
(\rho, v)(x, t) = \begin{cases} 
(\rho_-, v_-), & -\infty < x < \sigma t, \\
(\omega(t)\delta(x - \sigma t), \sigma), & x = \sigma t, \\
(\rho_+, v_+), & \sigma t < x < \infty,
\end{cases}
\tag{3.2}
\]
where
\[
\sigma = \frac{\sqrt{\rho_- v_+} + \sqrt{\rho_+ v_-}}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \quad \omega(t) = \frac{t}{\sqrt{1 + \sigma^2}} \left((\rho_+ v_+ - \rho_- v_-) - \sigma(\rho_+ - \rho_-)\right)
\]
and \(\delta\) is the Dirac delta distribution centered at the origin. The function \(\omega\) is called the weight of the \(\delta\)-shock. For more details, see \[3\].

Chen and Liu\[13\] have identified and analyzed the phenomena of concentration and cavitation, and the formation of \(\delta\)-shocks and vacuum states in the Riemann solutions to the isentropic Euler equations \((1.1)+(1.2)\) in the vanishing pressure limit. They have established the two results:

**Theorem 3.1.** Let \(v_- > v_+\) and \(\rho_\pm > 0\). For any \(\kappa > 0\), assume that \((\rho^\kappa, \rho^\kappa v^\kappa)\) is a 1-shock and 2-shock solution of \((1.1)-(1.2)\) with the Riemann data \((2.4)\). Then, when \(\kappa \to 0\), \(\rho^\kappa\) and \(\rho^\kappa v^\kappa\) converge in the sense of distributions, and the limit functions \(\rho\) and \(\rho v\) are sums of a step function and a \(\delta\)-measure with weights
\[
\frac{t}{1 + \sigma^2} \left((\rho_+ v_+ - \rho_- v_-) - \sigma(\rho_+ - \rho_-)\right)
\]
respectively, which form the \(\delta\)-shock solution \((3.2)\) of the pressureless gas system \((1.3)\) with the same Riemann data.

**Proof.** See the proof of theorem 3.1 in \[13\].

**Theorem 3.2.** Let \(v_- < v_+\) and \(\rho_\pm > 0\). For any \(\kappa > 0\), assume that \((\rho^\kappa, \rho^\kappa v^\kappa)\) is a 1-rarefaction and 2-rarefaction solution of \((1.1)-(1.2)\) with the Riemann data \((2.4)\). Then, when \(\kappa \to 0\), the solution \((\rho^\kappa, \rho^\kappa v^\kappa)\) tends to the two-contact-discontinuity solution \((3.1)\) of the pressureless gas system \((1.3)\) with the same Riemann data.

**Proof.** See section 4, in \[13\].

Chen and Liu\[13\] also mentioned without any proof that the behaviour of a solution of the Riemann problem in the two remaining cases, namely the 1-shock wave combined with a 2-rarefaction wave and the 1-rarefaction wave combined with a 2-shock wave, can be deduced from the two above results. Since these two others cases must be dealt in a non-trivial manner, we will provide a complete proof.
3.1. Behaviour of a 1-shock and 2-rarefaction solution as the pressure vanishes

Let \( v_- > v_+ \) and \( \rho_\pm > 0 \). For \( \kappa > 0 \), let \( (\rho^s_\pm, \rho^s v_\pm) \) be the intermediate state of a solution \((\rho^s, \rho^s v^\kappa)\) of the system (1.1)+(1.2) with Riemann data (2.4), in the sense that \( v_- \) and \( v_+^s \) are connected by a 1-shock wave, and \( v_+^s \) and \( u_+ \) are connected by a 2-rarefaction wave. Then, this intermediate state is determined by (2.6) and (2.7), from which we immediately deduce that \( \rho_+ > \rho_- \). The following results also hold:

**Lemma 3.3.**

\[
v_- > v_+^s > v_+^\kappa, \quad \forall \kappa \in (0, \kappa_{sr}), \quad \text{with} \quad v_+^s = v_+ \leftrightarrow \kappa = \kappa_{sr} := \frac{\rho_- \rho_+(v_- - v_+)^2}{(\rho_+^\gamma - \rho_-^\gamma)(\rho_+ - \rho_-)}. \quad (3.3)
\]

**Proof.** Let \( \kappa > 0 \). We first prove the equivalence in (3.3). Assume that \( v_+^s = v_+ \). From (2.6), we get \( \rho_+^s = \rho_+ \). Using the equalities \( v_+^s = v_+ \) and \( \rho_+^s = \rho_+ \), we obtain

\[
v_+ - v_- = \sqrt{\frac{\kappa(\rho_+^\gamma - \rho_-^\gamma)}{\rho_+\rho_-(\rho_+ - \rho_-)}}(\rho_+ - \rho_-) = -\sqrt{\frac{\kappa(\rho_+^\gamma - \rho_-^\gamma)(\rho_+ - \rho_-)}{\rho_+\rho_-}},
\]

which implies, by taking the square in both side, that

\[
(v_+ - v_-)^2 = \frac{\kappa(\rho_+^\gamma - \rho_-^\gamma)(\rho_+ - \rho_-)}{\rho_+\rho_-}.
\]

This last relation gives rise to \( \kappa = \kappa_{sr} \). Inversely, suppose that \( \kappa = \kappa_{sr} \). Then

\[
v_+ - v_+ = \sqrt{\frac{\kappa_{sr}(\rho_+^\gamma - \rho_-^\gamma)}{\rho_+\rho_-(\rho_+ - \rho_-)}}(\rho_+ - \rho_-). \quad (3.4)
\]

We claim that \( \rho_{sr}^s = \rho_+ \). In fact, assume that \( \rho_{sr}^s < \rho_+ \). By combining (2.6) and (2.7), we obtain

\[
v_+ - v_+ = \sqrt{\frac{\kappa_{sr}((\rho_{sr}^s)^\gamma - \rho_-^\gamma)}{\rho_{sr}^s\rho_-(\rho_{sr}^s - \rho_-)}}(\rho_{sr}^s - \rho_-) - \frac{2}{\gamma - 1}(\rho_+ - (\rho_{sr}^s)^{\frac{\gamma - 1}{\gamma}}).
\]

with \( \rho_- < \rho_{sr}^s < \rho_+ \). Consider the function \( h_1 : \rho \rightarrow (\rho^\gamma - \rho_-^\gamma)(1 - \frac{\rho_-}{\rho}) \). From the monotonic increasing property of \( h_1 \), we get

\[
\sqrt{\frac{\kappa_{sr}(\rho_+^\gamma - \rho_-^\gamma)}{\rho_+\rho_-(\rho_+ - \rho_-)}}(\rho_+ - \rho_-) > \sqrt{\frac{\kappa_{sr}((\rho_{sr}^s)^\gamma - \rho_-^\gamma)}{\rho_{sr}^s\rho_-(\rho_{sr}^s - \rho_-)}}(\rho_{sr}^s - \rho_-) - \frac{2}{\gamma - 1}(\rho_+ - (\rho_{sr}^s)^{\frac{\gamma - 1}{\gamma}}).
\]

This last inequality implies that (3.4) and (3.5) cannot both be true. Hence, \( \rho_{sr}^s = \rho_+ \). Using this equality in (2.7), we obtain

\[
v_+ - v_{sr}^s = \sqrt{\frac{\kappa_{sr}(\rho_+^\gamma - \rho_-^\gamma)}{\rho_+\rho_-(\rho_+ - \rho_-)}}(\rho_+ - \rho_-), \quad (3.6)
\]
which, combined with (3.4), implies that \( v^{\kappa^s}_s = v^{\kappa}_s = v_+ \). So, the equivalence in (3.3) holds. From (2.7), we get \( v^{\kappa}_s < v_- \). It remains to prove that \( v^{\kappa}_s > v_+ \) for all \( \kappa \in (0, \kappa_{sr}) \). We proceed by contradiction. Suppose there exists \( \kappa_1 \in (0, \kappa_{sr}) \) such that \( v^{\kappa_1}_s \) and \( v_+ \) are connected by a 2-rarefaction wave. On the one hand, using the inequality \( \kappa_1 < \kappa_{sr} \) in the definition of \( \kappa_{sr} \) in (3.3), one gets

\[
v_- - v_+ > \sqrt{\frac{\kappa_1 (\rho^+_s - \rho^-_s)}{\rho^+_s \rho^-_s (\rho^+_s - \rho^-_s)}} (\rho^+_s - \rho^-_s).
\]

On the other hand, as the intermediate state \((\rho^{\kappa_1}_s, v^{\kappa_1}_s)\) satisfies both (2.6) and (2.7) then

\[
v_- - v_+ = \sqrt{\frac{\kappa_1 ((\rho^{\kappa_1}_s)^\gamma - \rho^-_s\gamma)}{\rho^{\kappa_1}_s \rho^-_s (\rho^{\kappa_1}_s - \rho^-_s)}} (\rho^{\kappa_1}_s - \rho^-_s) - \frac{2\sqrt{\gamma \kappa_1 \gamma}}{\gamma - 1} (\rho^+_s - (\rho^{\kappa_1}_s)^\gamma) - (\rho^+_s)^\gamma),
\]

with \( \rho^- < \rho^{\kappa_1}_s < \rho^+ \). Again from the monotonic increasing property of \( h_1 \), we get

\[
\sqrt{\frac{\kappa_1 (\rho^+_s - \rho^-_s)}{\rho^+_s \rho^-_s (\rho^+_s - \rho^-_s)}} (\rho^+_s - \rho^-_s) > \sqrt{\frac{\kappa_1 ((\rho^{\kappa_1}_s)^\gamma - \rho^-_s\gamma)}{\rho^{\kappa_1}_s \rho^-_s (\rho^{\kappa_1}_s - \rho^-_s)}} (\rho^{\kappa_1}_s - \rho^-_s) - \frac{2\sqrt{\gamma \kappa_1 \gamma}}{\gamma - 1} (\rho^+_s - (\rho^{\kappa_1}_s)^\gamma) - (\rho^+_s)^\gamma).
\]

This last inequality implies that (3.7) and (3.8) cannot both be true. Hence, \( v^{\kappa}_s \) and \( v_+ \) are connected by a 2-shock wave for all \( \kappa \in (0, \kappa_{sr}) \). This completes the proof.

**Theorem 3.4.** Let \( v_- > v_+ \) and \( \rho_+ > 0 \). For some \( \kappa > 0 \), assume that \((\rho^\kappa, \rho^\kappa v^\kappa)\) is a 1-shock wave and 2-rarefaction wave solution of (1.1)-(1.2) with the Riemann data (2.4). Then, when \( \kappa \to 0 \), the solution \((\rho^\kappa, \rho^\kappa v^\kappa)\) tends to the \( \delta \)-shock solution (3.2) of the pressureless gas system (1.3) with the same Riemann data.

**Proof.** Lemma 3.3 says that the solution \((\rho^\kappa, \rho^\kappa v^\kappa)\) converts to a two-shock waves solution of the system (1.1)+(1.2) when \( \kappa \) gets smaller than \( \kappa_{sr} \). From theorem 3.1 we conclude that, when \( \kappa \to 0 \), the solution \((\rho^\kappa, \rho^\kappa v^\kappa)\) tends to the \( \delta \)-shock solution of the system (1.3) with the same initial data.

### 3.2. Behaviour of a 1-rarefaction and 2-shock solution as the pressure vanishes

Let \( v_- < v_+ \) and \( \rho_+ > 0 \). For \( \kappa > 0 \), let \((\rho^\kappa_s, \rho^\kappa v^\kappa_s)\) be the intermediate state of a solution \((\rho^\kappa, \rho^\kappa v^\kappa)\) of the system (1.1)-(1.2) with Riemann data (2.4), in the sense that \( v_- \) and \( v^\kappa_s \) are connected by a 1-rarefaction wave, and \( v^\kappa_s \) and \( u_+ \) are connected by a 2-shock wave. Then, this intermediate state is determined by (2.5) and (2.8), which imply that \( \rho^- > \rho^+_s \). The following results also hold:

**Lemma 3.5.**

\[
v_- < v^\kappa_s < v_+, \quad \forall \kappa \in (0, \kappa_{rs}), \text{ with } v^\kappa_s = v_+ \iff \kappa = \kappa_{rs} := \left( \frac{(v_+ - v_-)(\gamma - 1)}{2\sqrt{\gamma (\rho^-_s - \rho^+_s)^2}} \right)^2.
\]

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Proof. Let $\kappa > 0$. We first prove the equivalence in (3.9). Assume that $v_+^\kappa = v_+$. Using this equality in (2.8), we get $\rho^\kappa_+ = \rho_+$. Now, taking $v_+^\kappa = v_+$ and $\rho^\kappa_+ = \rho_+$ in (2.5), we obtain

$$v_+ = v_- + \frac{2\sqrt{K\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_+^{\gamma - 1})$$

(3.10)

which implies that $\kappa = \kappa_{rs}$. Inversely, suppose that $\kappa = \kappa_{rs}$. Then

$$v_+ - v_- = \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_+^{\gamma - 1}).$$

(3.11)

We claim that $\rho_{rs}^\kappa = \rho_+$. In fact, assume that $\rho_+ < \rho_{rs}^\kappa$. Combining (2.5) and (2.8), we obtain

$$v_+ - v_- = \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_{rs}^{\kappa^{\gamma - 1}}) - \sqrt{\frac{K_{rs}}{\rho_{rs}^\kappa \rho_+ (\rho_{rs}^\kappa - \rho_+)}} (\rho_{rs}^\kappa - \rho_+)$$

(3.12)

with $\rho_+ < \rho_{rs}^\kappa < \rho_-$. Consider the function $h_2 : \rho \to \rho_+^{\gamma - 1} - \rho_+^{\gamma - 1}$. Since $h_2$ is strictly increasing, we get

$$\frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_+^{\gamma - 1}) > \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - (\rho_{rs}^\kappa)^{\gamma - 1}) - \sqrt{\frac{K_{rs}}{\rho_{rs}^\kappa \rho_+ (\rho_{rs}^\kappa - \rho_+)}} (\rho_{rs}^\kappa - \rho_+).$$

This last inequality implies that (3.11) and (3.12) cannot both be true. Hence, $\rho_{rs}^\kappa = \rho_+$. Taking this equality in (2.5), we obtain

$$v_{rs}^\kappa - v_- = \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_+^{\gamma - 1}),$$

(3.13)

which, combined with (3.11), implies that $v_+^\kappa = v_{rs}^\kappa = v_+$. So, the equivalence in (3.9) holds. From (2.5), we get $v_+^\kappa > v_-$. It remains to prove that $v_+^\kappa < v_+$ for all $\kappa \in (0, \kappa_{rs})$. We proceed by contradiction. Suppose that there exists $\kappa_2 \in (0, \kappa_{rs})$ such that $v_{rs}^\kappa$ and $v_+$ are connected by a 2-shock wave. Then, using the inequality $\kappa_2 < \kappa_{rs}$ in the definition of $\kappa_{rs}$ in (3.9), we get

$$v_+ - v_- > \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_+^{\gamma - 1}).$$

(3.14)

As the intermediate state $(\rho_{rs}^\kappa, v_{rs}^\kappa)$ satisfies both (2.5) and (2.8) then

$$v_+ - v_- = \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - (\rho_{rs}^\kappa)^{\gamma - 1}) - \sqrt{\frac{K_{rs}}{\rho_{rs}^\kappa \rho_+ (\rho_{rs}^\kappa - \rho_+)}} (\rho_{rs}^\kappa - \rho_+)$$

(3.15)

with $\rho_+ < \rho_{rs}^\kappa < \rho_-$. Again, using the function $h_2$, we get

$$\frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - \rho_+^{\gamma - 1}) > \frac{2\sqrt{K_{rs}\gamma}}{\gamma - 1} (\rho_+^{\gamma - 1} - (\rho_{rs}^\kappa)^{\gamma - 1}) - \sqrt{\frac{K_{rs}}{\rho_{rs}^\kappa \rho_+ (\rho_{rs}^\kappa - \rho_+)}} (\rho_{rs}^\kappa - \rho_+).$$

This last inequality implies that (3.14) and (3.15) cannot both be true. Hence, $v_+^\kappa$ and $v_+$ are connected by a 2-rarefaction wave for all $\kappa \in (0, \kappa_{rs})$. This completes the proof. \(\square\)
Theorem 3.6. Let \( v_- < v_+ \) and \( \rho_\pm > 0 \). For some \( \kappa > 0 \), assume that \((\rho^*, \rho^* v^*)\) is a 1-rarefaction wave and 2-shock wave solution of (1.1)-(1.2) with the Riemann initial data (2.4). Then, when \( \kappa \to 0 \), the solution \((\rho^*, \rho^* v^*)\) tends to the two-contact-discontinuity solution (3.1) of the pressureless gas system (1.3) with the same Riemann data.

Proof. Lemma 3.5 says that the solution \((\rho^*, \rho^* v^*)\) converts to a two-rarefaction waves solution of the system (1.1)+(1.2) when \( \kappa \) gets smaller than \( \kappa_{rs} \). From theorem 3.2, we conclude that, when \( \kappa \to 0 \), the solution \((\rho^*, \rho^* v^*)\) tends to the two-contact-discontinuity solution of the system (1.3) with the same initial data. \(\square\)

4. Numerical illustrations

This section is devoted to the numerical illustration of the theoretical results. We use the modified Lax Friedrich scheme from [18] to discretize the isentropic Euler equations (1.1)-(1.2) and we take \( \gamma = 1.4 \).

We first illustrate the behaviour of a solution composed of a 1-rarefaction wave combined with a 2-shock wave when the pressure coefficient \( \kappa \) vanishes. The Riemann data are

\[
(r, v)(x, 0) = \begin{cases} 
(1.0, 0.8), & \text{for } x < 0, \\
(0.5, 1.0), & \text{for } x > 0,
\end{cases}
\]  

(4.1)

from which, we calculate by using (3.9), the coefficient \( \kappa_{rs} \approx 0.07 \). Numerical results, when the pressure coefficient is decreased, are represented on Figure 1. When \( \kappa = \kappa_{rs} \), we observe that the 2-shock wave disappears. When \( \kappa \) gets smaller than \( \kappa_{rs} \), the solution converts to a two-rarefaction waves. When \( \kappa \) tends to zero, the two-rarefaction waves tends to a two-contact-discontinuity, whose intermediate state between the two contact discontinuities tends to a vacuum state.

The behaviour of a solution composed of a 1-shock wave combined with a 2-rarefaction wave, is illustrated using the Riemann data

\[
(r, v)(x, 0) = \begin{cases} 
(0.2, 1.5), & \text{for } x < 0, \\
(0.7, 1.0), & \text{for } x > 0,
\end{cases}
\]

(4.2)

from which, we calculate by using (3.3), the coefficient \( \kappa_{sr} \approx 0.14 \). Numerical solutions, when the pressure coefficient \( \kappa \) is decreased, are shown on Figure 2. When \( \kappa = \kappa_{sr} \), we notice that the 2-rarefaction wave disappears. When \( \kappa \) gets smaller than \( \kappa_{sr} \), the solution converts to a two-shock waves that tends to a delta shock wave as the pressure coefficient \( \kappa \) progressively vanishes.

5. Conclusion

The results in this paper show how one can deduce, in the vanishing pressure limit, the behaviour of a solution composed of a 1-shock wave combined with a 2-rarefaction wave or a 1-rarefaction wave combined with a 2-shock wave, from the work of Chen and Liu[13]. As for the two-shock or two-rarefaction cases, the phenomena of concentration and cavitation are the leading comes behind the formation of delta shock waves and vacuum states in solutions of the isentropic Euler equations for gas dynamics when the pressure vanishes.
Figure 1: Behaviour of a 1-rarefaction wave and 2-shock wave solution when the pressure coefficient $\kappa$ is decreased. $\gamma = 1.4$, $t = 0.63$, $\Delta x = 10^{-4}$ and $\Delta t = 2 \times 10^{-5}$.

Figure 2: Behaviour of a 1-shock wave and 2-rarefaction wave solution when the pressure coefficient $\kappa$ is decreased. $\gamma = 1.4$, $t = 0.63$, $\Delta x = 10^{-4}$ and $\Delta t = 2 \times 10^{-5}$. 
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