Supplementary Information for

Synthetically Non-Hermitian Nonlinear Wave-like Behavior in a Topological Mechanical Metamaterial

H.X., I.F., H.L., K.Q., S.S., B.C.M., Z.C., N.B., X.M.

Zi Chen.
E-mail: zchen33@bwh.harvard.edu

Nicholas Boechler.
E-mail: nboechler@eng.ucsd.edu

Xiaoming Mao.
E-mail: maox@umich.edu

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**Supporting Information Text**

**Note 1: Floppy Modes of Maxwell Lattice and Control Variables**

For a general mechanical structural frame (a Maxwell lattice herein) in a d-dimensional domain that has point masses hinge-connected by central-force bonds, one can apply the Calladine index theorem (1–3)* to count the number of zero energy lattice modes (ZMs). The ZMs are given by

\[ dN - N_b = N_0 - N_s, \]

where \( N_0 \) is the number of ZMs, \( N_s \) is the number of states of self-stress (SSSs), \( N \) is the number of nodes, and \( N_b \) is the number of bonds in the lattice. A periodic Maxwell lattice with no boundaries always yields \( dN - N_b = 0 \), resulting in no ZMs in the lattice unless pairs of SSS and ZM arise due to the geometric singularity, which is the case at topological transitions. However, the generation of ZMs can be achieved by selecting a finite piece from an infinite lattice. In that case, \( dN - N_b > 0 \), and the number of floppy modes are related to the length of the boundary. Considering a two-dimensional (2D) finite Maxwell lattice consisting of \( N_y \) columns by \( N_y \) rows of unit cells (the dashed box area in Fig. S1 represents a unit cell), the total number of nodes and bonds are \( N = 3N_xN_0 + N_s + N_b \) and \( N_b = 6N_xN_y \), respectively, and consequently, the number of ZMs is \( N_0 = 2N_x + 2N_y + N_s \). Removing the number of rigid body planar degrees of freedom (DOFs), the remaining floppy modes (FMs) of the finite lattice is \( 2N_x + 2N_y + N_s - 3 \) (note in the main text FMs and ZMs are used almost interchangeable since for a large lattice they are approximately equal to each other and only differ by three rigid body rotations). In other words, by controlling these \( 2N_x + 2N_y - 3 \) floppy modes (FMs) of a Maxwell lattice with \( N_x \times N_y \) unit cells (here \( N_s = 0 \)), the configuration of the lattice is fully determined.

A deformed kagome lattice (the type of Maxwell lattice considered herein) is configured by the internal angles of \( \theta, \alpha, \) and \( \gamma \) between triangles, each one of which is formed by three bonds. The angle, \( \theta \), is the one between the red and blue triangles in the unit cell. The notation of \( \alpha \) represents the angle between triangles from different unit cells connected in adjacent rows, while \( \gamma \) shows the angle between triangles from different unit cells connected in adjacent columns. The Maxwell-Calladine counting rule, for a deformed kagome lattice with periodic left-right boundaries has \( 2(N_x + N_y) - 3 \) floppy modes originally, then periodic boundary conditions give additional constraints on each row (except the bottom row) removing the \( 2N_y - 2 \) modes. Additionally connecting the right-most node to the left-most node in the bottom row removes \( \theta_{1,+} \) as a free variable leaving \( 2N_x - 2 \) floppy modes which can be fully defined by choosing \( \theta \) and \( \gamma \) angles along the bottom. Each \( \alpha_{i,+} = \alpha_{N_x} \) angle is numerically determined such that periodicity is met (see Fig. S1). For example, a lattice consisting of 3 \( \times \) 3 unit cells can be determined by 4 variables including \( \theta_{1,1}, \theta_{1,2}, \gamma_{1,1}, \) and \( \gamma_{1,2} \). Here it is assumed \( \theta_{1,1} = \theta_{1,1} \) from periodic boundary conditions. All other angles of \( \theta_{1,1}, \alpha_{1,1}, \) and \( \gamma_{1,1} \) inside the lattice can be solved by these variables, and the remaining boundary variables \( \theta_{2,1}, \theta_{1,2}, \alpha_{2,1} \), and \( \alpha_{3,1} \) can be numerically solved using the periodic boundary conditions as constraints. The non-linear configuration of the Maxwell lattice in the following sections will be solved using these defined independent variables.

**Note 2: Nonlinear Exact Solution of Zero Energy Configurations**

The internal geometry formed by unit cells in the lattice is investigated and the corresponding structure of the lattice is then studied and presented. As can be seen in Fig. S1, a hexagon is generated by the adjacent four unit cells, including two entire unit cells of \( \theta_{i,j} \) and \( \theta_{i+1,j+1} \), a blue triangle from the unit cell of \( \theta_{i+1,j} \), and a red triangle from the one of \( \theta_{i,j+1} \), where the first subscript represents the row of the lattice while the second subscript denotes the column. The nonlinear configuration of the Maxwell lattice can be determined based on all angles in the lattice.

**Geometry of a General Hexagon.** Consider a hexagon defined in a Cartesian coordinate system (Fig. S2(a)) with the origin at site \( A \), which is located at the angle of \( \theta_{1,j} \). The \( x_{ij} \)-axis is aligned with side \( c_o \) of the blue triangle of the bottom row. Sites \( B \) to \( F \) correspond to the angles of \( \gamma_{i,j+1}, \alpha_{i+1,j+1}, \theta_{i+1,j+1}, \gamma_{i+1,j+1}, \alpha_{i+1,j+1}, \) and \( \alpha_{i+1,j+1} \), respectively. If initial conditions are given for the left and bottom edges of the lattice, all other angles can be obtained by solving hexagons from the bottom left corner to the top right corner in the lattice. Starting from a single hexagon, using known angles, \( \theta_{i,j}, \gamma_{i,j+1} \) and \( \alpha_{i+1,j} \) and related sides of triangles, the other unknown angles \( \theta_{i+1,j+1}, \gamma_{i+1,j+1} \) and \( \alpha_{i+1,j+1} \) can be determined by calculating the intersection coordinate of site \( D \). The coordinates of the remaining vertexes are given:

\[
\begin{align*}
x_A &= 0, & y_A &= 0, & x_B &= c_o, & y_B &= 0, \\
x_C &= c_o - c_o \cos(\gamma_{i,j+1} + \psi_{ab} + \psi_{ar}), & y_C &= -c_o \sin(\gamma_{i,j+1} + \psi_{ab} + \psi_{ar}), \\
x_F &= b_c \cos(\theta_{i,j} + \psi_{ab} + \psi_{cr}), & y_F &= -b_c \sin(\theta_{i,j} + \psi_{ab} + \psi_{cr}), \\
x_E &= b_c \cos(\theta_{i,j} + \psi_{ab} + \psi_{cr}) - b_c \cos(\theta_{i,j} + \alpha_{i+1,j} + \psi_{ab} + \psi_{cr} + \psi_{ab} + \psi_{ar}), & y_E &= -b_c \sin(\theta_{i,j} + \psi_{ab} + \psi_{cr}) + b_c \sin(\theta_{i,j} + \alpha_{i+1,j} + \psi_{ab} + \psi_{cr} + \psi_{ab} + \psi_{ar}),
\end{align*}
\]

where both the sides of the triangles \( c_o, b_c, \) and \( \alpha_c \), as well as the internal angles of the triangles \( \psi_{ab}, \psi_{ab}, \psi_{ab}, \psi_{ar}, \psi_{br} \) and \( \psi_{cr} \) are given in Fig. S2. Based on the geometry of the hexagon, the last site \( D \) should satisfy the following constraints:

\[
\begin{align*}
(x_D - x_E)^2 + (y_D - y_E)^2 &= a_r^2, \\
(x_D - x_C)^2 + (y_D - y_C)^2 &= a_r^2.
\end{align*}
\]

*The numbers of References in the SI are independent of those in the main text.
Equations (S.3) result in two solutions, one of which \((x_{D_2}, y_{D_2})\) forms \(\theta_{i+1,j+1} > \pi\) (site \(D_2\) in Fig. S2(a), which we call the concave solution), and the other of which \((x_{D_1}, y_{D_1})\) corresponds to \(\theta_{i+1,j+1} \leq \pi\) (site \(D_1\) in Fig. S2(a), which we call the convex solution). A hexagon is considered concave or convex based on the \(\theta_{i+1,j+1}\) angle, or in coordinate basis the angle \(EDC\). There exists a particular situation where two solutions overlap and hence \(\theta_{i+1,j+1} = \pi\). The concave solution \(D_1\) is chosen if the previous \(\theta_{i,j} < \pi\), and the convex solution is chosen if \(\theta_{i,j} > \pi\).

The coordinate of site \(D\) can be presented in two ways using known angles defined above and written as

\[
X_D = b_x \cos(\theta_{i,j} + \psi_{bb} + \psi_{cr}) - b_y \cos(\theta_{i,j} + \psi_{bb} + \psi_{cr} + \psi_{bc} + \psi_{ar}) \\
+ a_x \cos(\gamma_{i+1,j+1} - \theta_{i,j} - \alpha_{i+1,j} - \psi_{bb} - \psi_{cr} - \psi_{bc} - \psi_{ar}),
\]

\[
Y_D = -b_x \sin(\theta_{i,j} + \psi_{bb} + \psi_{cr}) + b_y \sin(\theta_{i,j} + \psi_{bb} + \psi_{cr} + \psi_{bc} + \psi_{ar}) \\
+ a_x \sin(\gamma_{i+1,j+1} - \theta_{i,j} - \alpha_{i+1,j} - \psi_{bb} - \psi_{cr} - \psi_{bc} - \psi_{ar}).
\]

By solving Eqs. (S.4) and (S.5), angles of \(\gamma_{i+1,j+1}\) and \(\alpha_{i+1,j+1}\) can be achieved. The last angle \(\theta_{i+1,j+1}\) is then obtained by the sum of the inner angles of the hexagon subtracting the other five angles. Figure S2(c) illustrates possible three known angles, which can generate hexagons using the configuration and parameters given from Fig. S2(b). Constraints, such as the strain free condition, are shown by blue, red, and green dots. The multicolored line (grey for \(R_T = 0\), light green for \(R_T = a_2\), and light yellow for \(R_T = a_3\)) represents the family of angles generating homogeneous Maxwell lattices. Processing along this line represents a uniform twisting synonymous with the Guest-Hutchinson modes in the system. The choice of solutions \(D_1\) and \(D_2\) to calculate the value of \(\theta_{i+1,j+1}\) for the current hexagon is determined by the value of \(\theta_{i,j}\). For the topological Maxwell lattice, we assume the more favorable solution for a lattice configuration is one with less discontinuous jumps in perturbation, thus we choose either the concave or convex \(\theta_{i+1,j+1}\) based on the previous result \(\theta_{i,j}\). If \(\theta_{i,j} > \pi\), \(\theta_{i+1,j+1}\) will still be greater than \(\pi\), and vice versa. However, we note that for all cases shown herein, the hexagons do not transition between different convexities.

Note that if a lattice is perturbed at the hard edge, we solve all unknown angles of hexagons \((\theta_{i+1,j+1}, \gamma_{i+1,j+1}\) and \(\alpha_{i+1,j+1}\) using known angles of \(\theta_{i,j}, \gamma_{i,j+1}\) and \(\alpha_{i+1,j}\), shown in Fig. S2(a). If a lattice is perturbed at the soft edge, we start solving angles of \(\theta_{i,j}, \gamma_{i,j+1}\), and \(\alpha_{i+1,j}\), which are complementary angles of \(\gamma_{i,j}, \gamma_{i,j+1}\), and \(\alpha_{i+1,j}\) subtracting two neighboring inner angles of triangles, using \(\theta_{i+1,j+1}, \gamma_{i+1,j+1}\), and \(\alpha_{i+1,j+1}\).

### Transformation of Coordinates for Unit Cells

The positions of each unit cell can be solved once all necessary angles in the lattice are obtained. Here, local coordinate systems from each unit cell associated with \(\theta_{i,j}\) must be rotated to align within a global coordinate system, set by the original unit cell and the value of \(\theta_{i,j}\).

To simplify the notation of the lattice, where all nodes except those on boundaries are shared with two surrounding unit cells, only three nodes are counted for one unit cell, noted by \(\hat{a}, \hat{b},\) and \(\hat{c}\) (symbols in the parentheses in Fig. S2(a) from red triangles). The coordinates of nodes \(\hat{a}_{n,m}, \hat{b}_{n,m}\) and \(\hat{c}_{n,m}\) in the unit cell at \(n\)th row and \(m\)th column (the lattice consists of \(N_x \times N_y\) unit cells) are shown below:

**Unit cell of \(\theta_{n,m}\) (\(n \leq m\)):**

\[
\begin{pmatrix}
\hat{x} \\
\hat{y}
\end{pmatrix}_{\hat{a}_{n,m}} = \sum_{p=1}^{m-n} \left( \prod_{i=1}^{p} R(\varphi_{i1}) \right) \begin{pmatrix}
x_{\hat{a}_{n,m},p} \\
y_{\hat{a}_{n,m},p}
\end{pmatrix}_{\varphi_{1}} + \sum_{p=1}^{n-1} \left( \prod_{i=1}^{p} R(\varphi_{i1}) \prod_{i=p+1}^{n-m} R(\varphi_{i,n-m}) \right) \begin{pmatrix}
x_{\hat{a}_{n,m},p} \\
y_{\hat{a}_{n,m},p}
\end{pmatrix}_{\varphi_{n-m}}
\]

**Unit cell of \(\theta_{n,m}\) (\(n > m\)):**

\[
\begin{pmatrix}
\hat{x} \\
\hat{y}
\end{pmatrix}_{\hat{a}_{n,m}} = \sum_{p=1}^{n-m} \left( \prod_{i=1}^{p} R(\varphi_{i1}) \right) \begin{pmatrix}
x_{\hat{a}_{n,m},p} \\
y_{\hat{a}_{n,m},p}
\end{pmatrix}_{\varphi_{1}} + \sum_{p=1}^{m-1} \left( \prod_{i=1}^{p} R(\varphi_{i1}) \prod_{i=p+1}^{n-m} R(\varphi_{i,n-m}) \right) \begin{pmatrix}
x_{\hat{a}_{n,m},p} \\
y_{\hat{a}_{n,m},p}
\end{pmatrix}_{\varphi_{n-m}}
\]

where \(\sum\) and \(\prod\) are cumulative sum and product operators. The coordinate transformation matrix, \(R(\varphi_{i,j})\), is given by

\[
R(\varphi_{i,j}) = \begin{bmatrix}
\cos \varphi_{i,j} & -\sin \varphi_{i,j} \\
\sin \varphi_{i,j} & \cos \varphi_{i,j}
\end{bmatrix}
\]

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where $\varphi_{i,j}$ is the deflection angle between $x_{i,j}$-axis and the previous $x$-axis ($\varphi_{11} = 0$), and:

\[
\begin{align*}
\varphi_{1,j} &= \gamma_{1,j} + \theta_{1,j-1} + \psi_{ab} + \psi_{bb} - 2\pi, \\
\varphi_{i,1} &= 2\pi - \alpha_{i,1} - \theta_{i-1,1} - \psi_{ar} - \psi_{cr}, \\
\varphi_{i,j} &= \gamma_{i-1,j} - \alpha_{i,j} + \psi_{bb} + \psi_{ab} + \psi_{br} - \pi \quad (i, j > 1).
\end{align*}
\]  

[S.9]

The cumulative coordinates with subscripts $\theta_{1,j} \rightarrow \theta_{1,j+1}$ (in Fig. S3(a)), $\theta_{i,1} \rightarrow \theta_{i+1,1}$ (in Fig. S3(b)), and $\theta_{i,j} \rightarrow \theta_{i+1,j+1}$ (in Fig. S2(a)) are those converted from $(1, j+1)^{th}$ to $(1, j)^{th}$ unit cell, from $(i+1, 1)^{th}$ to $(i, 1)^{th}$ unit cell and from $(i+1, j+1)^{th}$ to $(i, j)^{th}$ unit cell, respectively:

\[
\begin{align*}
x_{\theta_{1,j} \rightarrow \theta_{1,j+1}} &= c_b - a_x \cos(\psi_{bb} + \gamma_{1,j+1}), \\
y_{\theta_{1,j} \rightarrow \theta_{1,j+1}} &= -a_x \sin(\psi_{bb} + \gamma_{1,j+1}), \\
x_{\theta_{i,1} \rightarrow \theta_{i+1,1}} &= b_x \cos(\theta_{i,1} + \psi_{bb} + \psi_{cr}) - a_x \cos(\alpha_{i+1,1} + \theta_{i+1,1} + \psi_{ar} + \psi_{cr} + \psi_{bb}), \\
y_{\theta_{i,1} \rightarrow \theta_{i+1,1}} &= -b_x \sin(\theta_{i,1} + \psi_{bb} + \psi_{cr}) + a_x \sin(\alpha_{i+1,1} + \theta_{i+1,1} + \psi_{ar} + \psi_{cr} + \psi_{bb}), \\
x_{\theta_{i,j} \rightarrow \theta_{i+1,j+1}} &= c_b - c_r \cos(\gamma_{i,j+1} + \psi_{ab} + \psi_{br}) + a_x \cos(\gamma_{i,j+1} + \alpha_{i+1,j+1} + \psi_{ab} + \psi_{br}), \\
y_{\theta_{i,j} \rightarrow \theta_{i+1,j+1}} &= -c_r \sin(\gamma_{i,j+1} + \psi_{ab} + \psi_{br}) + a_x \sin(\gamma_{i,j+1} - \alpha_{i+1,j+1} + \psi_{ab} + \psi_{br}).
\end{align*}
\]

[S.10]

Equations (S.6) and (S.7) solve for the number of nodes of $3N_x N_y$ related to all red triangles, while the remaining $(N_x + N_y)$ nodes come from the first row $\theta_{1,m}$ and last column $\theta_{n,N_x}$, which are respectively given by:

\[
\begin{align*}
(x)_{\theta_{1,m}} &= (x)_{\theta_{1,m}} + \sum_{i=1}^{m} R(\varphi_{11}) \left( a_b \cos(\psi_{bb}) - a_b \sin(\psi_{bb}) \right), \\
(y)_{\theta_{1,m}} &= \left( y \right)_{\theta_{1,m}} + \sum_{i=1}^{m} R(\varphi_{11}) \left( c_b \right), \\
(x)_{\theta_{n,N_x}} &= \left( x \right)_{\theta_{n,N_x}} + \sum_{i=1}^{N_x} R(\varphi_{i1}) \left( c_b \right), \\
(y)_{\theta_{n,N_x}} &= \left( y \right)_{\theta_{n,N_x}} + \sum_{i=1}^{N_x} R(\varphi_{i1}) \left( c_b \right).
\end{align*}
\]  

[S.11]

**Periodic Boundary Conditions.** Newton’s method, an algorithm for finding roots of a function $f(x)$ such that $f(x) = 0$, is used in order to achieve periodic boundary conditions. Herein, the goal is to find periodic boundary conditions such that $f(x)$ is a function of $\alpha$, and periodic boundary conditions are met when $f(\alpha_{i+1}) = \alpha_n - \alpha_{i+1} = 0$, for the $i$-th row of an $M \times N$ lattice. The general Newton’s method algorithm is given below, where:

\[
\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}.
\]  

[S.13]

The input variable $\alpha_n$ is the initial guess for $\alpha_{i,1}$. The derivative $f'(\alpha_n)$ is calculated numerically using a central difference method, such that:

\[
f'(\alpha_n) = \frac{f(\alpha + h/2) - f(\alpha - h/2)}{h},
\]  

[S.14]

where $h/2$ is an angular increment $\delta \alpha$ that is small in comparison to $\alpha$. In this paper, a value of $\delta \alpha = 0.1 \mu$rad was used. For each value of $f(\alpha)$, $f(\alpha + \delta \alpha)$, and $f(\alpha - \delta \alpha)$, the corresponding configuration of the row must be calculated. The algorithm continues to update the initial guess of $\alpha$, until the residual is less than a specified tolerance value. In this paper, we set the tolerance value as $0.01 \mu$rad.

**Linear Mode Analysis.** In the small amplitude perturbation limit, where $\varepsilon$ is small (on the order of $10^{-6}$ rad, for our simulations $10^{-3}$ rad was considered 'large perturbations'), the numerically solved wave profile in the static limit can be approximated by a linear mode analysis. In our case, of a finite boundary in $y$ and a periodic boundary in $x$, one can prescribe a real $k_x$ and solve for a complex $k_y$ that satisfies the condition det $C(k) = 0$ to find the ZMs (2). The resulting complex $k_y = k_y'(k_x) + ik_y''(k_x)$ has $k_y' = \Re(k_y')$ being the real spatial wave profile in the lattice and $k_y'' = \Im(k_y')$ being the decay rate of the ZMs. The relation of $k_x$ and $k_y$ for the given homogeneous lattice ($\alpha_0 = 1.3144$ rad) is shown in Fig. S4 and Fig. 2 of the main text. The numerical results shown in Fig. 2(a) are in close agreement with the linear ZM analysis.

**Note 3: Space-Time Mapping in Maxwell Lattices**

In this section, we discuss the general elasticity theory of 2D materials with a soft strain (or “mechanism”), and how their static zero energy configurations map to 1D dynamical systems in linear and weakly nonlinear cases.
From soft strain to partial differential equations of ZMs in linear theory. We start by considering the general case of elasticity in 2D, where, to leading order in strain and spatial gradient of deformation, the elastic energy density can be written as:

\[ f_{el} = \frac{1}{2} \varepsilon_{ij} K_{ab} \delta_{ij}, \]  

[S.15]

where \( \varepsilon = \{ \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \} \) are the strains in Voigt notation, forming a 3-dimensional vector, and \( K \) is the (real symmetric) elastic constant matrix. This elastic energy density can be generally applied to any 2D elastic structure. For lattices considered in this paper, this continuum elastic energy density can be obtained starting from the full dynamical matrix \( D \) of the system and properly taking the long-wavelength limit by integrating out high frequency eigenmodes as previously discussed in \((6, 7)\). The expression in Eq. S.15 generally applies for any elastic system. For the lattice structure with springs and point masses considered here, the total energy of the system to the harmonic order of the displacement vector \( u \) can be written as \( F = \frac{1}{2} u^T D u \), where \( D \) is the dynamical matrix of the lattice system. Starting from the dynamical matrix, the continuum elastic energy density in Eq. S.15 can be obtained by taking a long wavelength limit as is shown in Refs. \((6, 7)\). Furthermore, the Guest Hutchinson mode, or the “soft strain” (the eigenvector corresponding to the zero eigenvalue of the matrix \( K \)), is completely determined by the geometry of the structure \((6, 8)\).

Now, let’s consider the case when this material has a soft strain, which means the \( K \) matrix has an eigenvalue that is zero or much smaller than the other two eigenvalues. This can arise from microscopic mechanisms of the material, e.g. auxetic foams or pentamode metamaterials \((9, 10)\). In this paper, we are particularly interested in the case of Maxwell lattices, where this soft mode is guaranteed to arise, taking the form of Guest-Hutchinson modes \((6, 8)\).

This soft strain has far-reaching effects on the elasticity of these lattices. To see this, we diagonalize the \( K \) matrix, \( K = V^T \Lambda V \), where \( V \) is a 3 \( \times \) 3 orthogonal matrix, the row vectors of which formed by eigenvectors of \( K \), and \( \Lambda \) is a diagonal matrix with two positive eigenvalues:

\[ V = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]  

[S.16]

where the last strain vector \( \epsilon_0 \) is the soft deformation, which comes with the zero eigenvalue in \( \Lambda \).

To see the consequence of this elastic energy density on spatially varying ZMs in the material, we rewrite Eq. (S.15) in terms of displacement field \( u = \{ u_x, u_y \} \):

\[ f_{el} = \frac{1}{2} \varepsilon_{ij} K_{ab} \delta_{ij} = \frac{1}{2} u_x S_{ia} K_{ab} \delta_{ij} u_j \equiv \frac{1}{2} u_i K_{ij} u_j, \]  

[S.17]

where

\[ S \equiv \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix}, \]  

[S.18]

with \( k_x, k_y \) representing \( x \) and \( y \) directional wave vectors and the overhead arrow denoting that these wave vectors are differential operators (in real space) acting on the wave to the right. Similarly, the left overhead arrow on \( S_{ia}^T \) acts on \( u_i \) to its left.

With the diagonalization in Eq. (S.16), the stiffness matrix \( K \) is given by:

\[ K = S^T S = S^T V^T \Lambda V S. \]  

[S.19]

Because the last eigenvalue of \( \Lambda \) vanishes, we can define \( \hat{V} \) to be the 2 \( \times \) 3 matrix with the last row removed from \( V \), and \( \hat{\Lambda} \) to the \( 2 \times 2 \) matrix with the last row removed in \( \Lambda \). Thus, we have a 2 \( \times \) 2 matrix \( \hat{C} \) that maps displacement fields to the “nonsoft” strains:

\[ \hat{C} = \hat{V} S, \]  

[S.20]

and correspondingly

\[ \hat{K} = \hat{C}^T \hat{\Lambda} \hat{C}. \]  

[S.21]

Therefore, zero modes in this media are waves that satisfy:

\[ \hat{C} u = 0. \]  

[S.22]

We can eliminate \( u_y \) and write this as a differential equation of \( u_x \) only, which becomes:

\[ \det \hat{C} u_x = 0. \]  

[S.23]

This \( \hat{C} \) matrix is the same as the final \( 2 \times 2 \) matrix in the continuum theory (after integrating out high frequency modes and expanding to the linear order in wavenumbers \( k_x \) and \( k_y \)) for topological ZMs in Maxwell lattices \((7)\).

Using the fact that \( \{ \epsilon_1, \epsilon_2, \epsilon_0 \} \) form an orthogonal triplet, we have:

\[ \det \hat{C} = -\frac{1}{2} \left( \epsilon_{02} k_x^2 - 2 \epsilon_{03} k_x k_y + \epsilon_{01} k_y^2 \right). \]  

[S.24]
It is perhaps more transparent if we turn $q$ into differential operators, where Eq. (S.22) becomes:

$$\left(\epsilon_{02}\partial_{x}^{2} - 2\epsilon_{03}\partial_{x}\partial_{y} + \epsilon_{01}\partial_{y}^{2}\right)u_{x} = 0,$$ \hfill [S.25]

where $\{\epsilon_{01}, \epsilon_{02}, \epsilon_{03}\}$ are the 3 components of the soft strain $\epsilon_{0}$ (using the Voigt notation). When $\det \epsilon_{0} > 0$, the material is dilation dominated (auxetic), and this equation is elliptic; when $\det \epsilon_{0} < 0$, the material is shear dominated (non-auxetic), and this equation is hyperbolic. The relationship between the real and imaginary components of the $x$ and $y$ component wavenumbers for the first Brillouin zone of an az polarized lattice are shown in Fig. S4. Furthermore, following (7), we also evaluated the parameters in Eq. (S.25) for the system considered in the main text with $\alpha = 1.3144$ to obtain the following equation:

$$\left(-0.08524k_{x}^{2} - 0.1276k_{x}k_{y} - 0.02223k_{y}^{2}\right)u_{x} = 0,$$ \hfill [S.26]

where $k_{x}$ and $k_{y}$ are written in units of rads/unit cell. Solving for $k_{y}$ as a function of $k_{x}$ we obtain two solutions for the zero modes: $k_{y} = -4.9661k_{x}$ and $k_{y} = -0.7720k_{x}$ which we plot in dashed lines in Figs. 2(b) and (e) as well as Fig. S4(a). They match very well with exact numerical curves for small $k = |k|$.

**Interpreting Eq.** Eq. (S.25) as a wave equation when $\det \epsilon_{0} < 0$. Since for $\det \epsilon_{0} < 0$, Eq. (S.25) is hyperbolic, it can be thought of as a wave equation in a 1D medium if we interpret spatial direction $y$ as time $t$. With this interpretation, Eq. (S.25) can be rewritten as:

$$\left(\epsilon_{01}\partial_{x}^{2} - 2\epsilon_{03}\partial_{x}\partial_{t} + \epsilon_{02}\partial_{t}^{2}\right)u_{x} = 0.$$ \hfill [S.27]

The presence of the second term means that the equation is not time reversal invariant since under the transformation $t \to -t, \partial_{x} \to -\partial_{x}$. This manifests itself in the nonsymmetric dispersion relation once we plug in the plane wave ansatz $u_{x} \sim e^{i(kx - \omega t)}$:

$$\omega = -\frac{\epsilon_{03}}{\epsilon_{01}}k \pm k \sqrt{\frac{\epsilon_{03}}{\epsilon_{01}}} \frac{-\epsilon_{02}}{\epsilon_{01}} = -\frac{\epsilon_{03}}{\epsilon_{01}}k \pm k \sqrt{-\det \epsilon_{0}} \left|\frac{1}{\epsilon_{01}}\right|,$$ \hfill [S.28]

i.e., $\omega(k) \neq \omega(-k)$. Note that $\omega$ is real since $\det \epsilon_{0} < 0$. And since the equations of motion is not time reversal symmetric, it is non-reciprocal.

Even though Eq. (S.27) is not time reversal symmetric, it is energy conserving. This can be seen from the following argument. This wave equation can be written as the Euler-Lagrange equation of the action:

$$S = \int dx \int dt \left[\frac{1}{2}\epsilon_{01}(\partial_{t}u_{x})^{2} + \frac{1}{2}\epsilon_{02}(\partial_{x}u_{x})^{2} - \epsilon_{03}(\partial_{x}u_{x})(\partial_{t}u_{x})\right],$$ \hfill [S.29]

where $\mathcal{L}$ is the Lagrangian density. The Euler-Lagrange equation of this action is:

$$\partial_{t} \frac{\partial \mathcal{L}}{\partial(\partial_{t}u_{x})} + \partial_{x} \frac{\partial \mathcal{L}}{\partial(\partial_{x}u_{x})} = \frac{\partial \mathcal{L}}{\partial u_{x}} \Rightarrow (\epsilon_{01}\partial_{t}^{2} - 2\epsilon_{03}\partial_{t}\partial_{x} + \epsilon_{02}\partial_{x}^{2})u_{x} = 0,$$ \hfill [S.30]

which confirms the validity of the action. Since this Lagrangian is invariant under time translation $t \to t + t_{0}$, according to Noether’s theorem, the energy $E$, defined below, is conserved:

$$E = \int dx \left[\frac{\partial \mathcal{L}}{\partial(\partial_{t}u_{x})}(\partial_{t}u_{x}) - \mathcal{L}\right] = \int dx \frac{1}{2}\left[\epsilon_{01}(\partial_{t}u_{x})^{2} - \epsilon_{02}(\partial_{x}u_{x})^{2}\right].$$ \hfill [S.31]

The fact that energy $E$ is conserved can be seen directly in the following way:

$$\frac{dE}{dt} = \int dx \frac{1}{2}\partial_{t}\left[\epsilon_{01}(\partial_{t}u_{x})^{2} - \epsilon_{02}(\partial_{x}u_{x})^{2}\right]$$

$$= \int dx \left[\epsilon_{01}\partial_{x}\partial_{t}^{2}u_{x} \right. - \epsilon_{02}\partial_{x}u_{x}\partial_{t}\partial_{x}u_{x} \right]$$

$$= \int dx \left[\partial_{x}u_{x}(2\epsilon_{03}\partial_{t}\partial_{x}u_{x} - \epsilon_{02}\partial_{x}u_{x}\partial_{t}\partial_{x}u_{x}) \right. - \epsilon_{02}\partial_{x}u_{x}\partial_{t}\partial_{x}u_{x} \right]$$

$$= \int dx \partial_{x}\left[\epsilon_{03}(\partial_{t}u_{x})^{2} - \epsilon_{02}(\partial_{x}u_{x})(\partial_{t}u_{x})\right]$$

$$= 0,$$

where we used Eq. (S.27) from second to third equality. The last equality is due to periodic boundary conditions.
Higher order corrections to Eq. (S.22), topological polarization, and interpretation of the resulting equation as time evolution of non-Hermitian 1D system. So far we have kept to linear order terms in \( k_x \) and \( k_y \) in the effective 2 \( \times \) 2 compatibility matrix \( \tilde{C} \). This corresponds to a classical elastic energy in terms of the strain tensor (without strain gradient terms). In this case, the ZMs are delocalized bulk modes. This can be seen from the fact that \( \det \tilde{C} = -\frac{1}{2} \left( \epsilon_{02} k_x^2 - 2 \epsilon_{01} k_x k_y + \epsilon_{01} k_y^2 \right) \) results in \( k_y = \frac{\epsilon_{03} k_x \pm k_x \sqrt{-\det \epsilon_{01}}}{\epsilon_{01}} \equiv \tilde{k}_y \), which are real—meaning that the ZMs are plane waves in the bulk.

The localization of the ZMs to edges are captured when terms up to the second order in \( k_x \) and \( k_y \) in the effective 2 \( \times \) 2 compatibility matrix \( \tilde{C} \) are kept (6, 7). To see this, we follow the steps from Eq. (S.22) to Eq. (S.24) keeping terms up to quadratic order in \( \tilde{C} \) to get:

\[
\delta \tilde{C} = -\frac{1}{2} \left( \epsilon_{02} k_x^2 - 2 \epsilon_{01} k_x k_y + \epsilon_{01} k_y^2 \right) - \frac{i}{2} (C_1 k_x^2 + C_2 k_x k_y + C_3 k_y^2 + C_4) + \mathcal{O}(k^4),
\]

where the parameters \( C_i, i = 1, 2, 3, 4 \) are real (this is due to time reversal symmetry of the Maxwell lattice) and are evaluated numerically using the scheme described in Ref. (7). The cubic terms in \( k \) in \( \delta \tilde{C} \) gives correction of order \( k_x^2 \) to solutions \( k_y = \tilde{k}_y \) mentioned above. The real and imaginary components of \( k_x \) and \( k_y \) are shown in Fig. S4 for an \( a_2 \) polarized hyperbolic PDE. To get the corrections, we plug in the ansatz \( k_y = k_y \mp i \delta \), \( k_x \) in Eq. (S.33), and keep the terms up to the cubic order in \( k_x \) to get:

\[
\left( \epsilon_{02} k_x^2 - 2 \epsilon_{01} k_x k_y + \epsilon_{01} k_y^2 \right) + i(C_1 k_x^2 + C_2 k_x k_y + C_3 k_y^2 + C_4) = 0 \Rightarrow \delta_y = \frac{C_1 + C_2 k_x + C_3 k_y + C_4}{2 \epsilon_{01} - 2 \epsilon_{02} k_x},
\]

where we defined \( \tilde{k}_y \equiv k_y/k_x = \frac{\epsilon_{03}}{\epsilon_{01}} \pm \frac{\sqrt{-\det \epsilon_{01}}}{\epsilon_{01}} \). Note that the solutions \( k_y = \tilde{k}_y k_x \mp i \delta \) to the order \( k_x^2 \) are complex numbers, meaning these zero modes are actually localized at the edges of the system. In the case of topologically polarized Maxwell lattices, \( \delta_x \) and \( \delta_y \) have the signs implying that both zero modes are at the same edge, hence the polarization. For the system considered in the main text with \( \alpha = 1.3144 \), following the procedure described in (7), we obtain:

\[
\delta \tilde{C} = \left( -0.08524 k_x^2 - 0.1276 k_x k_y - 0.02232 k_y^2 \right) + i \left( +0.000886403 k_x^2 + 0.0873246 k_x k_y + 0.0887591 k_x k_y^2 + 0.00878779 k_y^2 \right),
\]

and solving for the \( \delta_x \) using the procedure described above, we get for the two zero modes the dispersion relation to be:

\[
k_y = -4.9661 k_x - 17.6796 i k_x^2 \quad \text{and} \quad k_y = -0.7720 k_x - 0.15049 i k_x^2.
\]

The real and imaginary parts of \( k_y \) as functions of \( k_x \) have been plotted in Figs. S4(a-b), they are in good agreement with the full numerical calculation for small \( |k_x| \).

Now, we can turn to the corresponding wave equation in a 1-dimensional dynamical system by replacing \( k_i \) with \( -i \partial_x \) and interpreting direction \( y \) as time \( t \):

\[
- \left[ \left( \epsilon_{01} \partial_t^2 - 2 \epsilon_{03} \partial_x \partial_t + \epsilon_{02} \partial_x^2 \right) + (C_1 \partial_t^2 + C_2 \partial_x \partial_t + C_3 \partial_x^2 + C_4 \partial_t^2) \right] u_x = 0.
\]

We showed in the previous section that the terms with second order derivatives can be obtained from a time-invariant Lagrangian and hence energy conserving. However, the third order derivative terms cannot be obtained from a Lagrangian. The reason is the following. If there were a Lagrangian from which these terms with derivatives could be obtained, then that Lagrangian would have to have three derivatives and have to be of the order \( u_x^3 \); hence the most generic form of the Lagrangian would be \( u_x \partial_x \partial_t \partial_x \partial_t u_x \) \( (i, j, l \in \{t, x \}) \). By taking variation of the action, we would obtain the following:

\[
\delta S = S[u_x + \delta u_x] - S[u_x] = \int dt dx \left[ u_x \partial_x \partial_t \partial_x \partial_t u_x + \delta u_x \partial_x \partial_t \partial_x \partial_t u_x \right] = \int dt dx \left[ u_x \partial_x \partial_t \partial_x \partial_t u_x + \delta u_x \partial_x \partial_t \partial_x \partial_t u_x \right] = 0,
\]

where we used integration by parts from the second to the third equality and assumed that the variations at the boundary of the integration domain are zero. This shows that the Euler-Lagrange equation of terms \( u_x \partial_x \partial_t \partial_x \partial_t u_x \) in the Lagrangian are zero; hence terms with third order derivatives in an equation of motion cannot be obtained from a Lagrangian. As a result the argument of energy conservation in the previous subsection does not hold anymore as we include the terms with third order derivatives in the equation of motion. Furthermore, plugging the plane wave ansatz \( u_x = e^{-i(k_x x - \omega t)} \) along with \( \omega = -\frac{\epsilon_{04} k_x \pm k_x \sqrt{-\det \epsilon_{01}}}{\epsilon_{01}} + i \delta_x k^2 \) in Eq. (S.36), and carrying out a calculation similar to Eq. (S.34), we would get complex \( \omega \). Depending on the sign of the imaginary part of \( \omega \), the wave would grow/decay exponentially with time. Hence, we can interpret the system as active/dissipative. Interestingly, the distinction between active and dissipative in this 1D problem is determined by the topological polarization of the 2D lattice.
**Note 4: Simulated Soft Edge Sinusoidal Perturbation of an \( a_2 - a_1 \) Polarized Lattice**

In addition to the simulations of the soft edge sinusoidal perturbation of \( a_2 \) lattices in the main text (Figs. 1 and 2), here we show simulations with similar perturbation, but of \( a_2 - a_1 \) polarized lattices. Figure S5 and Fig. S6 shows the simulated deformation field for the \( a_2 - a_1 \) polarized lattices at low and high perturbation magnitudes, respectively. The key difference that can be observed between cases shown here and the \( a_2 \) polarized lattices shown in the main text, is that the two excited modes propagate in opposite directions, rather than the same direction, as was the case for the \( a_2 \) polarized lattices.

**Note 5: Additional Wave Amplification Simulations**

In Fig. S7, we show additional computational examples of wave amplification for 60-column-wide lattices with periodic boundary conditions over an array of \( a_0 \) values. We observe polarization domain switching in Fig. S7(a,b,d,e,f) and high frequencies generation (compared to the input wavelength) in Fig. S7(e,f).

**Note 6: Solitary-Wave-Like Behavior of the Deformed Kagome Lattice**

In addition to the example of the 10000 × 3000 \( a_2 \) polarized lattice in the main text, here we show a 10000 × 600 \( a_2 \) polarized lattice with a point perturbation applied at the top (floppy) edge. The computational results are shown in Figs. S8 and S9.

In Fig. S10, we further analyze the simulated results from Fig. S8. Fig. S10(a) [(b)] is obtained by tracking the peak magnitude of \( d^2 \alpha/dr^2 \) (where derivatives with respect to \( r \) are found via a finite difference formulation subtracting over columns) in each row for slower [faster] moving solitary wave using the findpeaks command in MATLAB. Figure S10(c) shows the rate of change of average \( \alpha \), where \( \alpha_{avg} = \sum \alpha/c = \alpha/600 \), with respect to rows, as a function of point perturbation value \( \varepsilon \). This is tracked by taking \( \alpha_{avg} \) at each row and performing a linear fitting \( \alpha_{avg} = m/r + b \) ranging the entire lattice (rows \( 1 - 10000 \)), using the polyfit command in MATLAB. \( d\alpha_{avg}/dr \) is thus given by the fitting parameter \( m \).

Figure S10(d) shows the fitted decay rate \( \sigma \) of the slower moving solitary wave, as a function of point perturbation value \( \varepsilon \), obtained by finding \( \alpha_{max} \) and \( \alpha_{min} \) using the max and \( min \) commands in MATLAB, and then calculating the peak-to-peak amplitude \( \alpha_{max} - \alpha_{min} \) at each row. The behavior of the peak to peak amplitude is oscillatory and decaying, in order to fit the decay rate, we first find the peak values of \( \alpha_{max} - \alpha_{min} \) in the plot of \( \alpha_{max} - \alpha_{min} \) versus \( r \) using the findpeaks command in MATLAB, then fit these peak values to an exponential function \( Ae^{-\sigma r} \). Figure S10(e) shows the fitted decay rate \( \sigma \) for both slower and faster moving solitary waves, as a function of point perturbation value \( \varepsilon \), based on \( d^2 \alpha/dr^2 \) at each row, instead of \( \alpha_{max} - \alpha_{min} \). Here, for each solitary wave, \( d^2 \alpha/dr^2_{min} \) represents the local minimum and \( d^2 \alpha/dr^2_{max} \) at each row. For the results shown in Fig. S10(e), rows 50 – 300 are the chosen range for fitting and \( d^2 \alpha/dr^2_{max} - d^2 \alpha/dr^2_{min} = Ae^{-\sigma r} \) since the solitary waves separate from each other and have not collided again.

The “speed” of the solitary waves for the 600 column wide lattice is shown in Fig. S10(f). Differing algorithms were used to find the speed for the 3000 (main text) and 600 (here) column wide lattices, since the number of interactions in the 600 column wide lattice makes it difficult to locate the individual solitary wave peaks in each row. For the 3000-column-wide lattice shown in the main text, the speed \( d(col)/d(row) \rightarrow dc/dr \) is obtained by tracking the location of peaks for \( d^2 \alpha/dr^2 \) at each row, which can correspond with slower (bigger peak) and faster solitary waves (smaller peak). Again, the findpeaks command in MATLAB is used here. During collision no data is collected. Once peaks of \( d^2 \alpha/dr^2 \) are found at each row, a linear fitting is performed on rows 50 – 400. The row range is chosen for the same reason as the previous fitting in Fig. S10(e). For each solitary wave, columns can be related to rows in the linear form \( r = n/m + \epsilon \) the slope \( n/m \) gives wave-speed \( dc/dr \). For the 600 column wide lattice, the local maximum is found using the same peak detection algorithm, however this time, the data analyzed comes from one column for a prescribed set of rows, such as in Fig S9(c). The peaks denote when the solitary wave passes the column of interest, the distance between peaks \( \Delta r \) is calculated. The columns traversed is the width of the lattice, thus the velocity is given by \( dc/dr = \Delta r/\Delta t = 600/\Delta r \). In order to avoid the interaction between the two solitary waves at the beginning rows of the lattice, the slower moving solitary wave is investigated in the range of \( r/3 - 4r/5 \). The final speed in main text Fig. 5 is calculated as the average of \( 600/\Delta r \) for all \( \Delta r \) in the specified range. The error bars are calculated by taking the maximum and minimum solitary wave velocities for a particular \( \epsilon \) value. The amplitude of the faster moving solitary wave is found by looking at the far right column (column 600) and the first 500 rows of the acceleration term \( d^2 \alpha/dr^2 \). The peaks and their corresponding row locations of the fast solitary wave are obtained using the findpeaks command, which gives \( \Delta r \), the faster solitary wave’s velocity is then found in the same manner as before \( dc/dr = 600/(\Delta r) \). The error is found similar to the slower solitary wave case by taking the difference between the maximum and the minimum velocities.

Additional computational results of a 10000 × 3000 \( a_2 \) polarized lattice are shown in Fig. S11.

**Note 7: Experimental Setup and Image Processing**

The dimensions of the laser cut unit cells used in Fig. 7(a-d) and Fig. 7(e-h) of the main text are shown in Fig. S12 and Fig. S13 respectively. Photographs of the assembled lattices were taken and the hinges of each triangle are found using image processing in MATLAB R2021a, an example of which can be seen in Fig. S14. First, the image is converted to a grey scale data structure, then is converted to a binary data structure by assigning a threshold. The nodes of each triangle are then found by applying the function *imfindcircles* to the binary structure. Repeated circles are removed by averaging circles together within a 50 px distance of each other. Additional thresholds based on the size of the triangle nodes helps the algorithm to locate the circles.
For the images used in this project, a lower threshold of 28 px and a higher threshold of 40 px were used. A sensitivity for the circle finding algorithm is set to 0.9 out of 1.0. From the coordinates of each unit cell $\alpha, \theta$, and $\gamma$ values can then be solved for.

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Fig. S1. Schematic of a deformed kagome lattice with $3 \times 3$ unit cells. Each unit cell consists of a rigid red and blue triangle connected by hinges. The configuration of the lattice is represented by the angles between the triangles, $\theta$, $\alpha$, and $\gamma$. 
**Fig. S2.** (a) Schematic of a general hexagon in the lattice. The hexagon is generated by sides from three red triangles and three blue triangles. Site $A$, $B$, and $F$ correspond to three known angles, $\theta_{i,j}$, $\gamma_{i,j+1}$, and $\alpha_{i+1,j}$, respectively, while site $C$, $D$, and $E$ are related to the other three angles ($\alpha_{i+1,j+1}$, $\theta_{i+1,j+1}$, and $\gamma_{i+1,j+1}$) to be solved. (b) Dimensions of a unit cell from (a). (c) Possible given angles that can generate hexagons shown in (a). Pink dots represent all possible given angles, which can form a hexagon without constraints (i.e. triangles can overlap indicating not a strain free condition or ZM), while blue, red, and green dots indicate all feasible angles under a strain free constraint. Blue and red dots show the convex ($D_1$) and concave ($D_2$) cases, respectively, and green dots imply that the given angles can create both convex and concave solutions.
Fig. S3. Schematic of two unit cells from (a) the bottom edge and (b) the left edge of the lattice.
Fig. S4. Comparison between results obtained from the PDE in Eq. 1 of the main text and full numerical calculation for an $\alpha_{2}$ polarized lattice. (a) and (b) show the “dispersion relations” $k'_{y} = \Re(k_{y})$ vs $k_{x}$ and $k''_{y} = \Im(k_{y})$ vs $k_{x}$ of the two edge ZMs, respectively. In each of these subplots, the black and red curves are obtained from the numerical linear mode analysis of the lattice, whereas the blue and green curves are obtained from the long wavelength PDE approximation in Eq. (S.35). (c) Deformation field of PDE response of a lattice with $\alpha_0 = 1.3144$ rad due to a sinusoidal perturbation with $k_x = 0.0524$ rad/unit cell and $\varepsilon = 1 \mu$rad.
Fig. S5. (a) Perturbation from homogeneous configuration $\alpha - \alpha_0$ for a 600 x 600 $a_2 - a_1$ polarized lattice ($\alpha_0 = 1.9144$) with periodic boundary conditions given a sinusoidal wave perturbation on the top floppy edge with $k_x = 0.0523$ (rad/unit cell) and $\varepsilon = 1 \mu$rad. (b) 2D Fourier transform of (a). (c) Wave shapes of select rows (top, middle, and bottom) from the perturbed lattice for (a).
Fig. S6. (a) Perturbation from homogeneous configuration $\alpha - \alpha_0$ for a $600 \times 600 \ a_2 - a_1$ polarized lattice ($\alpha_0 = 1.9144$) with periodic boundary conditions given a sinusoidal wave perturbation on the top floppy edge with $k_x = 0.0523$ (rad/unit cell) and $\varepsilon = 1$ mrad. (b) 2D Fourier transform of (a). (c) Wave shapes of select rows (top, middle, and bottom) from the perturbed lattice for (a).
Fig. S7. Computational results ($\alpha$) of wave amplification in 60-column-wide lattices with periodic boundary conditions due to sinusoidal perturbation ($k_x = 0.314$) for an array of $\alpha_0$ values. (a,d) $\alpha_0$ originally in $a_2$ polarization, near the boundary with $R_T = 0$. (b,e) $\alpha_0$ in $R_T = 0$ polarization near the boundary with $a_2$. (c,f) $\alpha_0$ further in $R_T = 0$ than before. (a-c) Perturbation amplitude $\varepsilon = 1$ mrad. (a) Bottom edges after perturbing at the top (soft) edges. (b,c) Bottom edges after perturbing at the top edge of the unpolarized lattices. (d-f) Perturbation amplitude $\varepsilon = 1$ $\mu$rad. (d) Top edges after perturbing at the bottom (hard) edges. (e,f) Top edges after perturbing the bottom of the unpolarized lattices. In (a), 60 rows is chosen; In (b-f), the displayed row is the last row before the lattice breaks.
Fig. S8. Computational results of a 10000 × 600 α2-polarized lattice with periodic left-right boundary conditions. The lattice has \( \alpha_0 = 1.3144 \) rad, and the point perturbation is applied on the top row at column 50. (a) \( \alpha - \alpha_0 \), (b) \( d\alpha / dr \), and (c) \( d^2\alpha / dr^2 \). Logarithmic scale is used in the colorbars of (b) and (c). Note, (b) [c], \( d\alpha / dr \) [\( d^2\alpha / dr^2 \)] is saturated at the lower limit of \( 10^{-6} \) rad \( [10^{-8} \) rad].
Fig. S9. Selected computational results for the two solitary waves observed in the lattice in Fig. S8. (a-c) Slower moving solitary wave: (a) $\alpha - \alpha_0$, (b) $d\alpha/dr$, and (c) $d^2\alpha/dr^2$ at column 300 (middle column). Rows 3700 – 4800 are chosen to minimize the interaction with the faster moving solitary wave. (d-f) Faster moving solitary wave: (d) $\alpha - \alpha_0$, (e) $d\alpha/dr$, and (f) $d^2\alpha/dr^2$ at columns 500 (red) and 600 (blue), respectively. Rows 70 – 300 are chosen to avoid the effect of substantial amplitude decay. Note that we avoid very early rows 1 – 70 to give the solitary waves adequate time to separate.
Fig. S10. (a,b) Peak magnitude of $d^2\alpha/dr^2$ as a function of rows for (a) slower and (b) faster moving solitary waves. The fluctuation of magnitude is due to the interaction of two solitary waves. As a function of point perturbation magnitude $\epsilon$ the: (c) rate of change of $\alpha_{avg}$ with respect to rows, (d) decay rate $s$ based on the amplitude $\alpha_{max} - \alpha_{min}$ at each row of the slower moving solitary wave, (e) decay rate $\sigma$ based on the peak-to-peak magnitude of $d^2\alpha/dr^2$ at each row for both solitary waves, and (f) wave-speeds of slower and faster moving solitary waves versus the point perturbation value $\epsilon$. 
Fig. S11. Additional computational results of a 10000 × 3000 a2p polarized lattice (α0 = 1.3144 rad) with periodic left-right boundary conditions and the point perturbation applied on the top row at column 50. As a function of point perturbation magnitude ε, the: (a) rate of change of α_{avg} with respect to rows, and (b) decay rate s based on the amplitude α_{max} − α_{min} at each row of the slower moving solitary wave.
Fig. S12. Dimensions (in the unit of mm) of laser cut acrylic triangles in the unit cell used in Fig. 7(a) of the main text.
Fig. S13. Dimensions (in the unit of mm) of laser cut acrylic triangles in the unit cell used in Fig. 7(d) of the main text.
Fig. S14. Example of image processing algorithm. Located circles are colored in green, the magenta lines separate each row of the lattice. For each hexagon color is determined by the difference between the back calculated $\alpha$ value and the homogeneous $\alpha_0$ value.