Gauge Invariant Effective Potentials and Higgs Mass Bounds

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\textbf{Abstract}

The problem of defining a gauge invariant effective potential with a strict energetic interpretation is examined in the context of spontaneously broken gauge theories. It is shown that such a potential can be defined in terms of a composite gauge invariant order parameter in physical gauges. This effective potential is computed through one loop order in a model with scalars and fermions coupled to an abelian gauge theory, which serves as a simple model of the situation in electroweak theory, where vacuum stability arguments based on the scalar effective potential have been used to place lower bounds on the Higgs mass.

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1 Introduction

Effective potential calculations in electroweak theory have been the subject of sustained interest as a consequence of the observation that phenomenologically interesting lower bounds on the Standard Model Higgs mass can be obtained from vacuum stability considerations (see [1] for a comprehensive review of work through 1989, [2] for a review of more recent work). Although the gauge interactions do not play a qualitatively significant role in these estimates, it is important to include them if one wishes to obtain as precise a bound as possible. Unfortunately, the conventional effective potential is an inescapably gauge-variant object, as the scalar field must transform nontrivially under the gauge group in order to break the symmetry spontaneously in the first place. While general theorems [3] ensure that the value of the effective potential at local extrema is a gauge-invariant quantity, the location of the extrema and the behavior of the potential between extrema can vary widely from one gauge to another. This makes it difficult to reach unambiguous conclusions on the basis of vacuum stability arguments [4].

There is a long history of attempts to formulate spontaneous symmetry breaking (SSB) in terms of gauge-invariant order parameters [5, 6, 7]. In the case of dynamical symmetry breaking, where the spontaneous breakdown arises from the appearance of a vacuum expectation value for a gauge-invariant operator, such order parameters have necessarily been taken to be composite operators [8, 9, 10]. Thus one is led to the introduction of an effective action for composite gauge-invariant fields, defined in the usual way as the Legendre transform of the connected generating functional for n-point insertions of the composite operator [11]. Such approaches can further be divided into those based on a local composite operator, and those where the order parameter involves a bilocal operator, i.e. a product of fields at separated space-time points. In the former case it is well-known [5, 12] that the additional subtractions needed to render the n-point functions of the local composite operator finite vitiate the energetic interpretation of the effective potential (obtained as the translationally invariant limit of the effective action). This is obviously fatal for any attempt to study vacuum stability using such potentials. Other authors have employed bilocal operators [5] with gauge-invariantizing factors, but in non-physical covariant gauges where a rigorous energy interpretation is again lacking.

In this paper we present a complete calculation to the one-loop level of a composite effective potential based on a gauge-invariant order parameter in a physical gauge (Coulomb) where the energetic interpretation of the potential is preserved.
The bilocal operators used involve a smearing function which is largely arbitrary. The smearing dependence of the results can however be understood in a completely physical way, and is found to be numerically insignificant in the regime of interest. The formalism needed is illustrated first in a simple scalar model (Section 2), and then extended to the Higgs-abelian model coupled to fermions (Section 3). The effective potential computed in Section 3 is renormalized and the final explicit finite result for arbitrary smearing functions given in Section 4. Section 5 contains an explicit analytic evaluation for the special case of a smearing function with a sharp cutoff in momentum space. In Section 6 we apply the composite operator formalism to the issue of electroweak vacuum stability bounds. Renormalization group improvement of the composite effective potential is also discussed here. Some explicit numerical results are presented in Section 7. The issue of extension of the formalism to non-abelian gauge groups leads naturally to the subject of composite effective potentials in axial gauge. The choice of an appropriate bilocal operator in this case involves a number of subtleties at both the perturbative and nonperturbative level. Some of these issues are discussed in Section 8, although a full computation at the one loop level is deferred to a future publication [13].

2 One-loop effective potential for a composite operator: a simple example

In order to establish notation, and to remind the reader of some features of the loop expansion of effective potentials which arise when the order parameter is a bilocal composite field, we first consider a simple model of a single real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m_0^2 \phi^2 - P(\phi)$$

(1)

Conventionally one studies symmetry breaking in this model by constructing an effective potential defined [14] as the Legendre transform of $W(j)$, the generating functional of connected graphs:

$$W(j) \equiv -i \log Z(j) = -i \log \int \mathcal{D}\phi e^{i \int (\mathcal{L} - j \phi) d^4x}$$

(2)

The corresponding Legendre transform, $\Gamma(\phi)$ (where now $\phi$ represents a classical field) has a straightforward interpretation in terms of one particle irreducible (1PI) graphs and is easily evaluated graphically at one-loop order. If we wish instead to add a source for a composite bilocal operator, the graphical interpretation is more complicated, and it turns out to be easier to construct the one-loop potential by a direct
semi-classical expansion. The calculation is also readily performed for a composite
operator in which the field point splitting is smeared spatially (in all directions) with
a function $K(\vec{r})$, which is essentially arbitrary, except for (a) being nonsingular at
the origin $\vec{r} = 0$ (in particular, there must be no delta function singularities there),
and (b) having a positive Fourier transform $\tilde{K}(\vec{p})$. The reason for the first condi-
tion is well-known- $n$-point functions of local composite fields will require additional
subtractions which correspond to nonlinear source terms which ruin the energy
interpretation of the effective potential. The positivity requirement on $\tilde{K}(\vec{p})$ ensures
that if we define our composite field as

$$
\chi(x) \equiv \int \phi(x + \vec{r}/2)K(\vec{r})\phi(x - \vec{r}/2)d^3\vec{r}
$$

(3)

then expectation values of $\chi$ are finite (once the field $\phi$ is renormalized) and positive. The latter property is easily seen by inserting a complete set of states between the two
$\phi$ fields in $<0|\chi(x)|0>$ and using translational invariance. This is in distinction to the
case of a local composite field, where the subtractions needed to define the composite
operator would ruin the positivity, as well as introducing the need for counterterms nonlinear in the source which ruin the energy interpretation of the effective potential. Here we are assured that the appropriate domain for the effective potential $V(\chi)$ is
$\chi > 0$. Finally, it will be convenient to normalize the smearing function $K(\vec{r})$ by
$\int d\vec{r}K(\vec{r}) = 1$. This ensures that at the classical level, $\chi$ reduces simply to $\phi^2$ in the
translationally invariant case $\phi(x) = \text{const}$. In momentum space, this means that we
take $\tilde{K}(\vec{p} = 0) = 1$. A convenient choice is the Gaussian $\tilde{K}(\vec{p}) = e^{-\vec{p}^2/(4\rho^2)}$ where
the parameter $\rho$ is roughly the inverse smearing separation of the fields in coordinate
space. However, it is not essential that $\tilde{K}$ be a smooth function- one may also use a
sharp cutoff in momentum space, with $\tilde{K}(\vec{p}) = \theta(\rho - |\vec{p}|)$, which has the advantage
that the one-loop integrals for the composite effective potential can be performed
analytically. As we shall see below, the qualitative results are similar in both cases
when $\rho$ takes a physically sensible value.

The field $\chi$ is not an order parameter for symmetry breaking in the conventional
sense, vanishing exactly in the symmetric phase and giving a non-zero expectation
value in the broken phase. Evidently, $\chi \neq 0$ even in the symmetric phase. However,
we shall see below that if the smearing scale $\frac{1}{\rho}$ is taken large compared to the inverse
mass gap in the theory, the value of $\chi$ at the minimum of the potential does go
to zero in the symmetric phase, while approaching the square of the conventional
vacuum expectation value of $\phi$ in the broken phase. Consequently, the scale (if any)
at which symmetry breaking occurs can just as well be studied by looking for global
minima of the potential $V(\chi)$, provided that such a potential is defined in a manner consistent with an energy interpretation. This means that composite operators must be point-split (to avoid counterterms nonlinear in the source) and the point-splitting must be spatial only, thereby maintaining the Schrödinger picture derivation of the energetics of the system perturbed by a source $j \chi$.

The calculation of the Legendre transform $\Gamma(\chi)$ of $W(j)$

$$W(j) \equiv -i \log Z(j) = -i \log \int \mathcal{D}\phi e^{\frac{i}{\hbar}\int (\mathcal{L}-j\chi)d^4x}$$

through one loop is most easily performed by saddle-point techniques. As we are interested in the result only through $O(\bar{\hbar})$, for the calculation of $W(j)$ we need only keep Gaussian fluctuations around the saddle-point $\phi_0$ of the source-augmented action:

$$(\Box + m_0^2)\phi_0(x) + P'(\phi_0) + 2 \int d\bar{r}j(x - \frac{\bar{r}}{2})K(\bar{r})\phi_0(x - \bar{r}) = 0$$

(5)

Passing to the Legendre transform will require the elimination of the source $j(x)$ in favor of the transform variable $\chi$. Taking the Gaussian smearing $K(\bar{p}) = e^{-\bar{p}^2/(4\rho^2)}$ to be specific, one may write formally (by Taylor expanding the fields $\phi(x \pm \frac{1}{2}\bar{r})$ in the definition (3))

$$\chi(x) = \phi(x)^2 - \frac{1}{8\rho^2}(|\nabla\phi(x)|^2 + ...)$$

(6)

Evidently, as asserted previously, in the translationally invariant limit we recover simply $\chi = \phi^2$. Taking $P(\phi) = \lambda_0 \phi^4/4$, the saddle-point equation (5) then implies

$$j_0 = -\frac{1}{2}(m_0^2 + \lambda_0 \phi_0^2) = -\frac{1}{2}(m_0^2 + \lambda_0 \chi_0)$$

(7)

in the translationally invariant limit. Here, the subscript “0” on the fields $\phi, \chi$ refers to the leading order in the $\hbar$ loop expansion, while the subscripts on $m, \lambda$ remind us that these parameters are bare ones, with $m_0 = m_R + \hbar \cdot (1 - \text{loop counterterms})$.

The leading contribution to the effective potential (defined as the effective action, or Legendre transform of $W(j)$ per unit volume in the translationally invariant limit where $\chi(x) = \text{constant}$), is obtained by evaluating the classical action at the saddle point $\phi_0$ and eliminating $\phi_0$ in favor of $\chi = \phi_0^2 + O(\hbar)$. As the saddle-point is by definition an extremum, the $O(\hbar)$ correction in $\chi$ only affects the result at $O(\hbar^2)$ and so may be neglected to 1-loop order. Not surprisingly, we recover at order $\hbar^0$ the expected tree potential, this time written as a function of the classical composite field $\chi$

$$V_{\text{tree}}(\chi) = \frac{1}{2} m_0^2 \chi + \frac{\lambda_0}{4} \chi^2$$

(8)
The one-loop contribution is obtained by integrating out the quadratic fluctuations around the saddle-point field $\phi_0$. Unlike the usual case where the source is coupled to the elementary scalar, so that the source term does not contribute to the quadratic part, here the source term acts as a momentum-dependent mass term at the one-loop level. Defining $P_{\text{tot}}(\phi) \equiv P(\phi) + j\chi$ the one-loop integral gives a contribution to the effective potential

$$V_{1-\text{loop}} = -\frac{i}{2} \text{Tr} \ln (\Box + m_0^2 + P''_{\text{tot}}(\phi)|_{\phi_0})$$

For translationally invariant saddle points, the determinant in (9) reduces to (we are still in Minkowski space)

$$\text{Tr} \ln (\Box + m_0^2 + P''_{\text{tot}}) = \int \frac{d^4 p}{(2\pi)^4} \ln (p^2 - m_0^2 - P''(\phi_0) - 2j_0\tilde{K}(\vec{p}))$$

where $j_0$ is to be written in terms of $\chi$ using (7), and we may replace $\chi_0$ by $\chi$ everywhere in (9) as we are already at $O(\bar{\hbar})$. After a Wick rotation to Euclidean space, the 1-loop contribution may now be written

$$V_{1-\text{loop}} = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} (\ln (p^2 + m_0^2(1 - \tilde{K}(\vec{p}))) + 3\lambda_0\chi - \chi_0\chi\tilde{K}(\vec{p})) - \ln (p^2))$$

The second term subtracts off a field-independent divergence which is physically insignificant. Although we continue to write the 1-loop contribution in terms of the bare parameters $m_0, \lambda_0$, these may be replaced with renormalized parameters in the 1-loop term, which is already of order $\bar{\hbar}$. It will be convenient to perform the renormalization of the theory using the $\overline{\text{MS}}$ scheme, as we shall shortly be moving on to gauge theories. In this scheme the relation between bare and renormalized quantities is (to order $\bar{\hbar}$) in $4-\epsilon$ dimensions

$$\lambda_0 = \lambda_R + \bar{\hbar} \frac{9}{16\pi^2} \lambda_R^2 \Delta_\epsilon$$

$$m_0^2 = m_R^2 \left(1 + \bar{\hbar} \frac{3}{16\pi^2} \lambda_R \Delta_\epsilon \right)$$

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi$$

where $\gamma$ is the Euler constant and the factor of $\bar{\hbar}$ makes the 1-loop order at which we are working explicit.

As the smearing function only involves the spatial components of the four vector $p$, we first integrate out the energy component explicitly, leaving a $3-\epsilon$ dimensionally regulated integrand:

$$V_{1-\text{loop}} = \frac{1}{2} \bar{\hbar} \int \frac{d^{3-\epsilon} p}{(2\pi)^{3-\epsilon}} (\sqrt{\vec{p}^2 + 2\lambda_R \chi + (m_R^2 + \lambda_R \chi)(1 - \tilde{K}(\vec{p}))} - |\vec{p}|)$$
where we can replace bare by renormalized quantities as we are already at 1-loop order. The divergent parts of this expression may be isolated by writing

\[ V_{1\text{-loop}} = V\text{fin} + V\text{sub} \]

\[ V\text{fin} = \frac{1}{2} \hbar \int \frac{d^3p}{(2\pi)^3} \left( \sqrt{\vec{p}^2 + 2\lambda_R \chi + (m_R^2 + \lambda_R \chi)(1 - \tilde{K}(\vec{p}))} - |\vec{p}| \right) \]

\[ V\text{sub} = -\frac{1}{16} \hbar \int \frac{d^3p}{(2\pi)^3} \frac{(m_R^2 + 3\lambda_R \chi)^2}{(|\vec{p}|^2 + \eta^2)^{3/2}} \]

Here \( \eta \) is an infrared cutoff introduced to ensure that \( V\text{fin} \) remains both infrared and ultraviolet finite when \( \epsilon \) is taken to zero. The first subtraction term vanishes in dimensional regularization. The subtraction term is easily computed in \( 3 - \epsilon \) dimensions:

\[ V\text{sub} = -\frac{1}{64\pi^2} \hbar (m_R^2 + 3\lambda_R \chi)^2 (\Delta_\epsilon - \ln \frac{\eta^2}{\mu^2}) \]

with \( \mu \) the usual renormalization scale which is introduced by the dimensional continuation. Adding together the tree contribution \( \mathbb{N} \) (with the bare quantities replaced by the renormalized ones using \( \mathbb{D} \)) one finds the renormalized composite field potential through 1-loop, up to an irrelevant additive constant, to be

\[ V_{\text{ren}}(\chi) = \frac{1}{2} m_R^2 \chi + \frac{1}{4} \lambda_R \chi^2 + V\text{fin}(\chi) + \frac{1}{64\pi^2} (m_R^2 + 3\lambda_R \chi)^2 \ln \frac{\eta^2}{\mu^2} \]

The logarithmic dependence on \( \eta \) in the last term is cancelled by a similar dependence in \( V\text{fin} \), so we can take \( \eta \) to zero after the integrals are performed.

The local limit for the composite field corresponds to taking \( \rho \) in \( \tilde{K}(\vec{p}) = e^{-\vec{p}^2/(4\rho^2)} \) to infinity, i.e. \( \tilde{K} \to 1 \). If we do this, a quadratic divergence reappears in \( V\text{fin} \), as the \( \chi \)-dependence at large momentum no longer matches that of the counterterms needed to renormalize the theory. This is a well-known difficulty, much discussed in the literature. Alternately, if we take \( \rho \) small, corresponding to separating the field points in the bilocal composite by large distances, the potential simply goes over smoothly to the effective potential for the elementary field \( \phi \), with the replacement \( \chi \to \phi^2 \). From \( \mathbb{D} \) the leading correction for small \( \rho \) is evidently

\[ \delta_K V_{\text{eff}} = -\frac{1}{4} (m_R^2 + \lambda_R \chi) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2 + m_R^2 + 3\lambda_R \chi}} \tilde{K}(\vec{p}) \]
which is clearly of order $\hbar^2$ at the extremum (where $m^2_R + \lambda_R \chi \sim O(\hbar)$), showing that
the extremal energy is independent of the choice of smearing function to this order. The actual value of the composite
field in the ground state is shifted by the smearing. The amount of the shift (extremizing $V$ to leading order in $\hbar$)
is easily seen to be

$$\delta_K \chi = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2 + m^2_{ph}}} \tilde{K}(\vec{p})$$

(22)

where $m_{ph} = m^2_R + 3\lambda_R \chi_{ex}$ is the physical scalar mass ($\chi_{ex}$ is the extremum value of the composite field). To understand the origin of this shift we can write

$$<0| \int d\vec{r} \phi(x + \vec{r}/2) K(\vec{r}) \phi(x - \vec{r}/2)|0> = <\phi >^2 + \int d\vec{r} K(\vec{r}) \Delta F(\vec{r})$$

(23)

The second term is just the connected contribution to the expectation value of the composite bilocal operator, i.e the propagator for the physical mode smeared with $K$. Going to momentum space and integrating out the energy component, we find

$$\int d\vec{r} K(\vec{r}) \Delta F(\vec{r}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \tilde{K}(\vec{p}) \frac{1}{\sqrt{\vec{p}^2 + m^2_{ph}}}$$

(24)

Thus we see that the shift in $<\chi>$ at the minimum $\chi = -\frac{m^2_R}{3\lambda_R}$ induced by the smearing is exactly equal to (22). In general, the composite field can be interpreted as equivalent to the square of the usual elementary order parameter as long as the smearing is carried out over scales much larger than the inverse mass gap of the physical modes. The above argument also makes it clear that the domain of definition of the effective potential, for a fixed finite smearing scale $\rho$, actually begins at $\chi = \chi_{\min} > 0$ where $\chi_{\min}$ is, from the Lehmann representation, at least as large as

$$z \int \frac{d^3p}{(2\pi)^3} \tilde{K}(\vec{p}) \frac{1}{\sqrt{\vec{p}^2 + m^2_{ph}}},$$

$z$ is the (finite) residue of the renormalized $\phi$ propagator at the physical pole $\vec{p}^2 = m^2_{ph}$. For $\rho$ small, this end-point can be made as close as we wish to zero.

The upshot of the preceding discussion is simply this: the smearing dependence of the effective potential defined with a smeared composite field will be exponentially small as long as the scale of smearing is kept large compared to the Compton wavelength of the physical scalar. Once the field points are sufficiently far apart, the expectation of the composite field reverts by clustering to the square of the $vev$ of the elementary field, with exponentially small corrections arising from the connected part of the bilocal expectation value. This means that ambiguities arising from various choices of the bilocal smearing are (a) physically well understood, and (b) reducible to an arbitrarily small value.
3 Composite Effective Potential for the Higgs-Abelian model- Coulomb Gauge

In this section we shall repeat the calculation of the 1-loop composite effective potential in a physically more interesting case, that of a spontaneously broken gauge symmetry. As we wish to work with gauge-invariant quantities while retaining a strict energy interpretation for the potential, we shall use a gauge-invariant bilocal operator as our probe of symmetry breaking, but work in a physical gauge where a positive Hamiltonian and positive metric Hilbert space obtain. In such a situation the standard argument \cite{16} shows that the value of the effective potential at any field value is the minimum energy compatible with that value in a physical state (with the usual proviso that the effective potential is the convex hull of the perturbatively computed Legendre transform). Either Coulomb or axial gauge would be suitable for this purpose. The calculation is somewhat simpler in Coulomb gauge, where we can maintain the smooth smearing procedure used in Section 2 for the discrete symmetry case, so we shall use this gauge henceforth.

The Lagrangian is a Higgs-abelian model, with an additional fermion (the top quark, in the electroweak context) coupled chirally as in the sigma-model:

$$
\mathcal{L} = \left| (\partial_{\mu} + ieA_{\mu}) \phi \right|^2 - \frac{1}{4} F^{\mu\nu} - m^2 \phi \phi - P(\phi^* \phi) + \bar{\psi}i(\gamma - ie \gamma_5 A)\psi - g_y \bar{\psi}(\phi_1 + i\gamma_5 \phi_2)\psi
$$

(25)

where the scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ is now complex. The bilocal operator used to define a potential will now be taken as

$$
\chi(t, \vec{x}) \equiv \int d\vec{r} \phi^*(t, \vec{x} + \vec{r}/2) K(\vec{r}) \int d^3 y \vec{J}(\vec{x}, \vec{y}, \vec{r}) \cdot \vec{A}(t, \vec{y}) \phi(t, \vec{x} - \vec{r}/2)
$$

(26)

where the c-number current $\vec{J}(\vec{x}, \vec{y}, \vec{r})$ satisfies

$$
\nabla_{\vec{y}} \cdot \vec{J}(\vec{x}, \vec{y}, \vec{r}) = \delta(\vec{x} - \vec{y} - \vec{r}/2) - \delta(\vec{x} - \vec{y} + \vec{r}/2)
$$

(27)

$$
\vec{J}(\vec{x}, \vec{y}, \vec{r}) = \nabla_{\vec{y}} \sigma
$$

(28)

Here $\sigma(\vec{x})$ is a c-number scalar field which is formally identical to the electrostatic potential in a dipole field (where $\vec{J}$ corresponds to an electric dipole field). The choice of $\vec{J}$ ensures that $\chi$ is a gauge-invariant field. Moreover, the gauge-invariantizing phase factor vanishes in Coulomb gauge, by a spatial integration by parts in the exponent. Also, the composite field $\chi$ contains by explicit construction only fields on a single time-slice, so the Schrödinger picture essential for the energy interpretation of the potential is maintained. Rigorous arguments show \cite{17} that the bilocal operator
with a smeared gauge-invariantizing string of this type is a valid order parameter
for SSB in this theory, as the expectation value of this bilocal operator remains
nonzero even when the field points are taken infinitely far apart. Finally, insertion of
a complete set of states (in Coulomb gauge we have a positive metric Hilbert space)
leads to positivity of \( < 0 | \chi | 0 > \) as for the simple theory of Section 2 provided the
Fourier transform of the smearing function \( K(\vec{r}) \) is positive. In a slight change of
notation from the preceding section, henceforth bare masses and couplings will be
unsubscripted, and we no longer display explicitly powers of \( h \).

As we are dealing with an abelian theory here, the functional integral over the
gauge degrees of freedom is gaussian, and it will be convenient to perform this integra-
tion (in Coulomb gauge) immediately. The integration over the fermion field may
likewise be performed at the outset. The fermion determinant gives a contribution
to the effective action which is explicitly of order \( h \), and which is a functional of the
scalar and gauge fields. The gauge field dependence of this functional only contributes
at order \( h^2 \), and may therefore be neglected when the gauge field integrations are per-
formed. (The scalar field dependence of the fermionic determinant will be computed
and included below.) The remaining functional integral over the scalar field will then
be evaluated using a saddle-point expansion as in Section 2. In doing the gauge inte-
grations, it is best to separate the transverse and Coulomb modes, which give rise to
physically distinct contributions. Thus, defining

\[
J^\mu \equiv ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)
\]

\[
D \equiv \Box + 2e^2 \phi^* \phi
\]

we find for the integral over the transverse modes

\[
-i \ln (Z_{\text{tr}}) \equiv -i \ln \int D\vec{A} \delta(\vec{\nabla} \cdot \vec{A}) e^{i \int (\frac{1}{2} \vec{A}D\vec{A} + \vec{J} \cdot \vec{A})} d^4 x
\]

\[
= \frac{1}{2} \int d^4 x J_i D^{-1/2}(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}) D^{-1/2} J_j + i \text{tr} \ln D
\]

while the integral over the Coulomb mode yields

\[
-i \ln (Z_{\text{c}}) \equiv -i \ln \int DA_0 e^{i \int (\frac{1}{2} A_0 (\Delta + 2e^2 \phi^* \phi) A_0 - J_0 A_0) d^4 x}
\]

\[
= -\frac{1}{2} \int d^4 x J_0 \frac{1}{-\Delta + 2e^2 \phi^* \phi} J_0 + \frac{i}{2} \text{tr} \ln (-\Delta + 2e^2 \phi^* \phi)
\]

Note that terms quadratic in \( J \) are effectively tree-level in powers of Planck’s constant,
and must therefore be included in the effective scalar action when we perform the
saddle-point evaluation of the effective potential.
After integrating out the gauge degrees of freedom, the effective tree-level scalar Lagrangian, augmented by a source term for the bilocal $\chi$ field, becomes

$$\mathcal{L}_{\text{eff}}^\phi = \partial_\mu \phi^* \partial^\mu \phi - \phi^* \bar{m}^2 \phi - P(\phi^* \phi) + \frac{1}{2} J_i(\phi) D^{-1/2}(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}) D^{-1/2} J_j(\phi)$$

$$- \frac{1}{2} J_0(\phi) \frac{1}{-\Delta + 2e^2 \phi^* \phi} J_0(\phi)$$

(35)

where $\bar{m}^2 \equiv m^2 + jK$ incorporates the source bilinear in the field together with the bare mass term into a single nonlocal kernel. Now we expand $\phi = \phi_0 + \hat{\phi}$, where $\frac{\partial L^{\phi^* \phi}}{\partial \phi}(\phi = \phi_0) = 0$ and take the source $j(x)$ and consequently the response $\phi_0(x) = \phi_0$ constant. One finds

$$\mathcal{L}_{\text{eff}}^\phi(\hat{\phi}) \simeq \mathcal{L}_{\text{eff}}^\phi(\phi_0) + \frac{1}{2}(\hat{\phi}^* \hat{\phi}) \mathcal{M}_\phi \left( \frac{\hat{\phi}^*}{\hat{\phi}} \right)$$

(36)

Note that only the Coulomb part of (35) contributes in the translationally invariant limit to the quadratic form $\mathcal{M}_\phi$, as in this limit $J_i \to ie(\phi_0^i \partial_i \hat{\phi} - \phi_0 \partial_i \hat{\phi}^*)$ and $(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}) \partial_j \hat{\phi} = 0$. Explicitly,

$$\mathcal{M}_\phi =$$

$$\left( \begin{array}{cc}
-\frac{\partial^2 P}{\partial \phi^2} |_{\phi_0} - e^2 \phi_0^2 \frac{\partial^2}{\Delta + 2e^2 |\phi_0|^2} & -(\Box + \bar{m}^2) - \frac{\partial^2 P}{\partial \phi^2} |_{\phi_0} + e^2 |\phi_0|^2 \frac{\partial^2}{\Delta + 2e^2 |\phi_0|^2} \\
-(\Box + \bar{m}^2) - \frac{\partial^2 P}{\partial \phi^2} |_{\phi_0} + e^2 |\phi_0|^2 \frac{\partial^2}{\Delta + 2e^2 |\phi_0|^2} & -\frac{\partial^2 P}{\partial \phi^2} |_{\phi_0} - e^2 \phi_0^2 \frac{\partial^2}{\Delta + 2e^2 |\phi_0|^2}
\end{array} \right)$$

Specializing to the case $P(\phi) = \lambda(\phi^* \phi)^2$, and noting that in the translationally invariant limit $\chi = \int d^4x d\vec{r} \phi^*(x - \vec{r}/2)K(\vec{r})\phi(x + \vec{r}/2) = |\phi_0|^2$, this simplifies to

$$\mathcal{M}_\phi =$$

$$\left( \begin{array}{cc}
-(2\lambda + e^2 \frac{\partial^2}{\Delta + 2e^2 \chi}) \phi_0^2 & -(\Box + \bar{m}^2 + 6\lambda \chi) + (2\lambda + e^2 \frac{\partial^2}{\Delta + 2e^2 \chi}) \chi \\
-(\Box + \bar{m}^2 + 6\lambda \chi) + (2\lambda + e^2 \frac{\partial^2}{\Delta + 2e^2 \chi}) \chi & -(2\lambda + e^2 \frac{\partial^2}{\Delta + 2e^2 \chi}) \phi_0^2
\end{array} \right)$$

whence the integral over the fluctuation fields $\hat{\phi}$ yields directly

$$\ln \det(\mathcal{M}_\phi) = \ln \det \{(\Box + \bar{m}^2 + 6\lambda \chi)^2 - (\Box + \bar{m}^2 + 6\lambda \chi)(4\lambda \chi + 2e^2 \chi \frac{\partial^2}{\Delta + 2e^2 \chi})\}$$

$$= \ln \det(\Box + \bar{m}^2 + 6\lambda \chi) + \ln \det \{(\Box + \bar{m}^2 + 2\lambda \chi - 2e^2 \chi \frac{\partial^2}{\Delta + 2e^2 \chi}) - \ln \det(-\Delta + 2e^2 \chi)$$

$$+ \ln \det \left(1 + \frac{-\Delta + 2e^2 \chi}{-\Delta} \right) (\Box + 2e^2 \chi \bar{m}^2 + 2\lambda \chi) \right) + \text{constant}$$

(37)
Adding in the one-loop contributions from the gauge integrations we find

\[-V^{1\text{-loop}}_{\text{gauge+scalar}} = \frac{3}{2} \text{tr} \ln (\Box + 2e^2 \chi) + \frac{1}{2} \text{tr} \ln (\Box + \tilde{m}^2 + 6\lambda \chi) + \frac{1}{2} \text{tr} \ln \{1 + \left(\frac{-\Delta + 2e^2 \chi}{-\Delta}\right) \frac{1}{\Box + 2e^2 \chi} (\tilde{m}^2 + 2\lambda \chi)\}\] (38)

The unpleasant term \(\ln \det(-\Delta + 2e^2 \chi)\), which contains a UV divergence of the form \(\int \text{d}p_0\) not present in the counterterms, has cancelled, and the remaining terms are easily seen to correspond in the broken symmetry phase to three massive gauge vector modes and a single massive scalar mode. The complicated and peculiar last term is a remnant of the long-range Coulomb interaction, and contains divergences which will be removed by the counterterms of the theory, as we shall see shortly. In these one-loop contributions, the source augmented mass \(\tilde{m}\) contains the source \(j_0\) of the composite field \(\chi\), determined at tree level from \(\frac{\partial L^\phi}{\partial \phi} \bigg|_{\phi_0} = 0\) for \(\phi_0\) constant, so

\[
\begin{align*}
  j_0 & = (-m^2 + V'(\chi)) = -(m^2 + 2\lambda \chi) \\
  \tilde{m}^2 & = m^2 + j_0 K = m^2 (1 - K) - 2\lambda \chi K
\end{align*}
\] (39) (40)

Integrating out the fermion field gives a similar contribution (except for the characteristic change of sign)

\[-V^{1\text{-loop}}_{\text{fermion}} = -i \text{tr} \ln (i\slashed{\partial} - g_y (\phi_1 + i\gamma_5 \phi_2))\] (41)

The full effective potential through 1-loop is thus given by

\[V(\chi) = m^2 \chi + \lambda \chi^2 + V_{\text{gauge+scalar}}^{1\text{-loop}} + V_{\text{fermion}}^{1\text{-loop}}\] (42)

This expression is as yet unrenormalized- we must rewrite the bare parameters \(m, \lambda, g_y\) in terms of renormalized parameters and rescale the fields appropriately, at which point the divergences in the one-loop contributions will be seen to cancel completely. This calculation will be performed in the next section.

4 Renormalized Composite Effective Potential for the Higgs-Abelian Model

To renormalize the result (42) obtained above, one may integrate out the energy component \(p_0\) in momentum space and then dimensionally regularize the resulting
purely spatial integrals (which are then carried out in 3-\(\epsilon\) dimensions). For example the contribution from the massive gauge vector loop was found to be

\[
V_A = -\frac{3}{2} \text{tr} \ln (\Box + 2e^2 \chi)
\]

\[
= -\frac{3}{2} \int \frac{d^4 p}{(2\pi)^4} \ln (-p^2 + 2e^2 \chi) \quad \text{Minkowski}
\]

\[
= \text{const} + \frac{3}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left(1 + \frac{2e^2 \chi}{p^2 + \tilde{p}^2}\right) \quad \text{Euclidean}
\]

\[
\rightarrow \frac{3 - \epsilon}{2} \int \frac{d^{3-\epsilon} p}{(2\pi)^{3-\epsilon}} \ln \left(\sqrt{p^2 + 2e^2 \chi} - |\tilde{p}|\right)
\]

where the final spatial integral, together with the prefactor \((g^\mu_\mu - 1)\) counting spatial gauge modes, has been dimensionally continued. The divergent part of \(V_A\) can be separated off as follows:

\[
V_A = \frac{3 - \epsilon}{2} \int \frac{d^{3-\epsilon} p}{(2\pi)^{3-\epsilon}} \ln \left(\sqrt{p^2 + 2e^2 \chi} - |\tilde{p}|\right) - \frac{e^2 \chi}{|\tilde{p}|} + \frac{e^4 \chi^2}{2 (p^2 + \eta^2)^{3/2}} + V_{A,\text{div}}
\]

\[
V_{A,\text{div}} \equiv -\frac{3 - \epsilon}{4} \int \frac{d^{3-\epsilon} p}{(2\pi)^{3-\epsilon}} \frac{e^4 \chi^2}{(p^2 + \eta^2)^{3/2}}
\]

\[
= -\frac{3}{16\pi^2} e^4 \chi^2 (\Delta_\epsilon - \ln \left(\frac{\eta^2}{\mu^2}\right) - \frac{2}{3})
\]

Here the divergent piece \(\Delta_\epsilon\) is defined in (12), and \(\eta\) is an infrared cutoff which will be set to zero at the end. The first term in (14) has all the subtractions needed to ensure that it remains finite as \(\epsilon \to 0\) so we may define a MS-renormalized gauge boson contribution (obtained as usual by subtracting the part proportional to \(\Delta_\epsilon\) in \(V_{A,\text{div}}\)) as

\[
V_{A,\text{ren}} = \frac{3}{2} \int \frac{d^3 p}{(2\pi)^3} \ln \left(\sqrt{p^2 + 2e^2 \chi} - |\tilde{p}|\right) - \frac{e^2 \chi}{|\tilde{p}|} + \frac{e^4 \chi^2}{2 (p^2 + \eta^2)^{3/2}} + \frac{3}{16\pi^2} e^4 \chi^2 (\ln \left(\frac{\eta^2}{\mu^2}\right) + \frac{2}{3})
\]

Similar manipulations can be used to extract the singular parts of the massive Higgs, residual Coulomb, and fermion contributions to the 1-loop composite effective potential. One obtains for the massive scalar loop contribution

\[
V_B = \frac{1}{2} \int \frac{d^{3-\epsilon} p}{(2\pi)^{3-\epsilon}} \ln \left(\sqrt{p^2 + 4\lambda \chi} + (m^2 + 2\lambda \chi)(1 - K(\tilde{p})) - |\tilde{p}|\right)
\]

\[
= V_{B,\text{ren}} - \frac{1}{64\pi^2} (m^2 + 6\lambda \chi)^2 \Delta_\epsilon
\]
with

\[
V_{B,\text{ren}} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left( \sqrt{\vec{p}^2} + 4\lambda \chi + (m^2 + 2\lambda\chi)(1 - \tilde{K}(\vec{p})) - |\vec{p}| - \frac{1}{2|\vec{p}|}(m^2 + 6\lambda\chi) \right) + \frac{1}{8} \frac{(m^2 + 6\lambda\chi)^2}{(\vec{p}^2 + \eta^2)^{3/2}} + \frac{1}{64\pi^2} (m^2 + 6\lambda\chi)^2 \ln \left( \frac{\eta^2}{\mu^2} \right)
\] (48)

The residual Coulomb contribution also yields a divergent contribution:

\[
V_C = \frac{1}{2} \int \frac{d^3\epsilon p}{(2\pi)^3} \sqrt{\vec{p}^2} + 2e^2\chi(\sqrt{1 + (m^2 + 2\lambda\chi)\frac{1 - \tilde{K}(\vec{p})}{\vec{p}^2} - 1}) (49)
\]

\[
= V_{C,\text{ren}} + \frac{1}{16\pi^2} e^2\chi(m^2 + 2\lambda\chi) \Delta\epsilon - \frac{1}{64\pi^2} (m^2 + 2\lambda\chi)^2 \Delta\epsilon
\] (50)

This term couples the Higgs and gauge sectors and the renormalized contribution is rather complicated. One finds

\[
V_{C,\text{ren}} = \frac{1}{2} \int \frac{d^3\epsilon p}{(2\pi)^3} \left\{ \frac{(\sqrt{\vec{p}^2} + 2e^2\chi - |\vec{p}| - e^2\chi |\vec{p}|)}{|\vec{p}|} \left( \sqrt{1 + (m^2 + 2\lambda\chi)\frac{1 - \tilde{K}(\vec{p})}{\vec{p}^2} - 1} \right) \right. \\
\left. + |\vec{p}| \left( \sqrt{1 + (m^2 + 2\lambda\chi)\frac{1 - \tilde{K}(\vec{p})}{\vec{p}^2} - 1} - \frac{m^2 + 2\lambda\chi}{2\vec{p}^2} + \frac{(m^2 + 2\lambda\chi)^2}{8(\vec{p}^2 + \eta^2)^2} \right) \right. \\
\left. + \frac{e^2\chi |\vec{p}|}{|\vec{p}|} \left( \sqrt{1 + (m^2 + 2\lambda\chi)\frac{1 - \tilde{K}(\vec{p})}{\vec{p}^2} - 1} - \frac{m^2 + 2\lambda\chi}{2(\vec{p}^2 + \eta^2)} \right) \right\} \\
- \frac{1}{16\pi^2} e^2\chi(m^2 + 2\lambda\chi)(\ln \left( \frac{\eta^2}{\mu^2} \right) - 2 + 2 \ln 2) \\
+ \frac{1}{64\pi^2} (m^2 + 2\lambda\chi)^2 (\ln \left( \frac{\eta^2}{\mu^2} \right) - 1 + 2 \ln 2)
\] (51)

Finally, the 1 loop fermion contribution gives

\[
V_D = -2 \int \frac{d^3\epsilon p}{(2\pi)^3} \left( \sqrt{\vec{p}^2} + 2g_y^2\chi - |\vec{p}| \right) (52)
\]

\[
= V_{D,\text{ren}} + \frac{1}{4\pi^2} g_y^4\chi^2 \Delta\epsilon
\] (53)

where

\[
V_{D,\text{ren}} = -2 \int \frac{d^3\epsilon p}{(2\pi)^3} \left( \sqrt{\vec{p}^2} + 2g_y^2\chi - |\vec{p}| - \frac{g_y^2\chi}{2|\vec{p}|} + \frac{1}{8} \frac{g_y^4\chi^2}{(\vec{p}^2 + \eta^2)^{3/2}} \right) \\
- \frac{1}{4\pi^2} g_y^4\chi^2 \ln \left( \frac{\eta^2}{\mu^2} \right)
\] (54)
The 1-loop renormalization of the theory (mass, coupling and field rescaling) implies the following cutoff dependence for the coefficients of $\chi$ and $\chi^2$ in the tree contribution to the potential (individual counterterm contributions are listed in Appendix A):

\begin{align}
  m^2 &\rightarrow m_R^2 (1 + \frac{\Delta \epsilon}{16\pi^2} (-e^2_R + 4\lambda_R)) \\
  \lambda &\rightarrow \lambda_R + \frac{\Delta \epsilon}{16\pi^2} (-2e^2_R\lambda_R + 10\lambda^2_R + 3e^4_R - 4g_s^4) \quad (55) \quad (56)
\end{align}

When these replacements are made in the unrenormalized potential (42), one finds that all divergences cancel and we are left with an explicitly finite result for the composite potential (now in terms of $\overline{\text{MS}}$-renormalized fields, masses and couplings):

$$V_{\text{ren}}(\chi) = m^2_R \chi + \lambda_R \chi^2 + V_{A,\text{ren}}(\chi) + V_{B,\text{ren}}(\chi) + V_{C,\text{ren}}(\chi) + V_{D,\text{ren}}(\chi) \quad (57)$$

with the renormalized potentials given explicitly in (45, 48, 51, 54) with the mass and couplings therein interpreted as renormalized ones.

## 5 Analytic Formulae for Sharp Momentum-space Smearing

As mentioned previously, any choice of smearing function $K(\vec{r})$ which adequately suppresses short-distance contributions to $\langle \chi(x) \rangle$ could be used in defining the composite operator. To the extent that different $K(\vec{r})$ isolate only the longest-distance modes relevant for SSB they should give similar values for $\langle \chi(x) \rangle$ and thus express the same physics. In particular, they should unambiguously signal the location and depth of symmetry-breaking extrema of the effective potential. One simple choice which permits closed-form expressions for the dimensionally-regulated momentum integrals is the step function, $\tilde{K}(\vec{p}) = \theta(\rho - |\vec{p}|)$. With this choice, the $K$-dependent integrals in $V_B$ and $V_C$ can be written generically (to $\mathcal{O}(\epsilon^0)$) as

$$\int d^{3-\epsilon} p \ f (\tilde{K}(|\vec{p}|)) = \int_{|\vec{p}| > \rho} d^{3-\epsilon} p \ f (\tilde{K} = 0) + \int_{|\vec{p}| < \rho} d^{3} p \ f (\tilde{K} = 1) = \int d^{3-\epsilon} p \ f (0) + \int_{|\vec{p}| < \rho} d^{3} p \ (f (1) - f (0)) \quad (58)$$

The first integral contains the UV divergence and can be evaluated in the usual manner. It is independent of the smearing radius $\rho$. The second integral is manifestly UV finite, contains all of the dependence on $\rho$, and is also straightforward to evaluate.
Thus we see that at one loop the effective potential for the bilocal composite operator naturally decomposes into a piece which is the elementary field effective potential (written in terms of $\chi = \phi^* \phi$) and a piece dependent on the smearing scale which introduces no new UV divergences, unlike the local composite operator $\phi^2(x)$.

The explicit expressions for terms of the effective potential are:

\[
V_A(\chi) = \frac{3m_V^2}{64\pi^2} \left[ -\Delta_\epsilon + \ln \frac{m_V^2}{\mu^2} - \frac{5}{6} \right] \tag{59}
\]

\[
V_B(\chi) = V_{B,1}(\chi) + V_{B,2}(\chi) \tag{60}
\]

\[
V_{B,1}(\chi) = \frac{m_h^2}{64\pi^2} \left[ -\Delta_\epsilon + \ln \frac{m_h^2}{\mu^2} - \frac{3}{2} \right] \tag{61}
\]

\[
V_{B,2}(\chi) = \frac{1}{64\pi^2} \left\{ 2\rho \sqrt{\rho^2 + 4\lambda \chi} \left( 2\rho^2 + 4\lambda \chi \right) - (4\lambda \chi)^2 \ln \frac{\rho + \sqrt{\rho^2 + 4\lambda \chi}}{4\lambda \chi} \right\} -
\]

\[
\frac{1}{64\pi^2} \left\{ 2\rho \sqrt{\rho^2 + m_h^2} \left( 2\rho^2 + m_h^2 \right) - m_h^4 \ln \left( \frac{\rho + \sqrt{\rho^2 + m_h^2}}{m_h^2} \right) \right\} \tag{62}
\]

\[
V_C(\chi) = V_{C,1}(\chi) + V_{C,2}(\chi) \tag{63}
\]

\[
V_{C,1}(\chi) = \frac{1}{64\pi^2} \left\{ -(m_g^2 - m_V^2)^2 \left( \Delta_\epsilon + \frac{3}{2} \right) - 2m_g m_V (m_g - m_V)^2 +
\right.
\]

\[
(m_g^2 - m_V^2)^2 \ln \left( \frac{m_g + m_V}{\mu^2} \right) \left\} - \frac{m_V^4}{64\pi^2} \left\{ - \left( \Delta_\epsilon + \frac{3}{2} \right) + \ln \frac{m_V^2}{\mu^2} \right\} \right\} \tag{64}
\]

\[
V_{C,2}(\chi) = \frac{2}{64\pi^2} \left\{ \rho \sqrt{m_g^2 + \rho^2} \left( m_g^2 + 2\rho^2 \right) - m_g^4 \ln \frac{\rho + \sqrt{\rho^2 + m_g^2}}{m_g} \right\} -
\]

\[
\frac{2}{64\pi^2} \left\{ \sqrt{m_g^2 + \rho^2} m_V^2 + \rho^2 \left( m_g^2 + m_V^2 + 2\rho^2 \right) - m_g m_V (m_g^2 + m_V^2) -
\right.
\]

\[
(m_g^2 - m_V^2)^2 \ln \frac{\sqrt{\rho^2 + m_g^2} + \sqrt{\rho^2 + m_V^2}}{m_g + m_V} \right\} \tag{65}
\]

\[
V_D(\chi) = -\frac{4}{64\pi^2} m_f^4 \left[ -\Delta_\epsilon + \ln \frac{m_f^2}{\mu^2} - \frac{3}{2} \right] \tag{66}
\]

where $m_g^2 = m^2 + 2\lambda \chi$, $m_V^2 = 2e^2 \chi$, $m_h^2 = m^2 + 6\lambda \chi$ and $m_f^2 = 2g_f^2 \chi$.

As noted previously, for $\rho \to 0$ the $\rho$-dependent terms go smoothly to zero. For $\rho \to \infty$ new quadratic divergences arise as the point-split composite operator approaches the local composite operator.
6 Renormalization Group Improvement and Vacuum Stability Bounds

The composite operator effective potential formalism developed above provides a gauge-invariant framework for obtaining the information about symmetry breaking which is partially obscured by gauge dependence and UV problems in other treatments. While the value of the elementary-field effective potential is gauge invariant at any of its local extrema, the field value at which the extremum occurs (i.e. the expectation value of the elementary scalar field in the corresponding phase of the theory) is not. Thus, associating the gauge dependent expectation value of the elementary field at which some feature of the effective potential occurs with gauge invariant physical quantities is a practice of questionable validity. The expectation value of the composite operator is by construction free of these gauge ambiguities. Because the effective potential of the composite operator has an energy interpretation, the value of the effective potential at any given value of the composite order parameter corresponds to the minimum physical energy density compatible with that value. Consequently, the global minimum of this effective potential should correspond to the true vacuum of the theory and the value of $\chi$ at which it occurs should also have physical meaning.

One context in which such a treatment is useful is in the formulation of a lower bound on the Standard Model Higgs mass from vacuum stability considerations. If the electroweak vacuum is assumed to be stable (rather than merely metastable) then it must be the global minimum of the effective potential (in both the case of the composite operator effective potential and of the elementary field effective potential). In practice, however, we expect that the Standard Model is the low-energy effective theory of a more fundamental high-scale theory and so will only be an accurate description of physics up to some energy scale (the ‘new physics’ scale, perhaps characterized by the masses of new heavy particles). Thus, it is only consistent to demand that the electroweak vacuum is the global minimum up to the scale at which the effective theory breaks down, beyond which the model is simply no longer accurate.

For the elementary field effective potential, the statement that the value of the effective potential is larger than the value at the electroweak minimum up to some scale $\Lambda$ is a gauge-dependent statement [4]. As such it is an unsatisfactory criterion for defining a ‘new physics’ scale. For the composite operator effective potential defined and evaluated in this paper there is no gauge dependence and we may interpret the composite operator expectation value as a physically meaningful energy scale.
Since the Standard Model may be valid up to very high energies it will be necessary to study the effective potential for field expectation values much larger than the electroweak scale. However, it is well-known that in order to study the effective potential at large field values, the usual perturbative loop expansion for the effective potential may be inadequate. For the elementary field effective potential, the loop expansion generates terms generically of form \( \frac{g^4 \phi^4}{(4\pi)^2} \ln \frac{\mu^2}{g^2 \phi^2} \) (where \( g \) is any of the couplings). If the couplings \( \hat{g} \) and the renormalization scale \( \mu \) are given values characteristic of the electroweak scale (e.g. \( \mu = M_Z \) and the couplings given their values at the Z-scale), considering \( \phi \gg \mu \) will invalidate the perturbative expansion. The range of validity of the approximation may be improved by utilizing the renormalization group (RG) improved effective potential [18, 19]. Similar considerations apply to the composite operator effective potential.

The full (all-orders) effective potential is independent of \( \mu \). This independence can be expressed as a first-order differential equation

\[
\mu \frac{d}{d\mu} V_{\text{eff}}(\chi, m, e, \lambda, g_y, \mu) = 0 \tag{67}
\]

This equation can be solved and, combined with dimensional analysis, yields the RG-improved effective potential [4, 19]

\[
V_{\text{eff}}(s^2 \chi_i, \hat{g}_i, m_i, \mu) = [\zeta(s)]^4 V_{\text{eff}}(\chi_i, \hat{g}(s, \hat{g}_i), m(s, m_i), \mu) \tag{68}
\]

where

\[
\gamma_{\phi} = \frac{1}{2} \frac{\mu}{Z_{\chi}} \frac{dZ_{\chi}}{d\mu} \tag{69}
\]

Note that \( \gamma_{\phi} \) in (69) is really one half of the anomalous dimension of the gauge-invariant composite operator \( \chi \). In Coulomb gauge (and only in Coulomb gauge) the gauge-invariantizing string factor in \( \chi \) is equal to unity and the operator therefore renormalizes as the square of the elementary field in this gauge. Accordingly \( \gamma_{\phi} \) is just the usual anomalous dimension of the elementary scalar in Coulomb gauge. In other gauges, there will be a divergent contribution from a nontrivial string factor which compensates for the gauge-variance of \( \gamma_{\phi} \). The scale factor in (68) is just

\[
\zeta(s) = \exp \left[ \int_0^{\log s} \frac{1}{\gamma_{\phi}(x) + 1} dx \right] \tag{70}
\]

Here \( s^2 = \frac{1}{\chi_i} \) and \( \hat{g} \) represents the set of couplings \( g_y, \lambda, \) and \( e \). The running couplings \( \hat{g} \) are solutions to the equations

\[
\frac{d\lambda(s)}{ds} = \beta_\lambda(\hat{g}(s)) = \frac{\beta_\lambda}{1 + \gamma_\phi} , \quad \lambda(0) = \lambda_i \tag{71}
\]
and analogous for the other couplings. The \( \hat{g}_i \) are the values for the couplings at the initial scale \( \chi_i \), which we will take to be around the electroweak scale. The \( \beta \) functions can be computed (in a loop expansion) from the \( \overline{MS} \) counterterms of the theory and are listed in Appendix A. Whereas the unimproved effective potential was trustworthy only for \( \chi \) such that \( \frac{g_i^2}{(4\pi)^2} \ln g_i^2 s^2 \ll 1 \), the RG improved effective potential will be reliable as long as the running couplings remain small.

It has been demonstrated (in the context of the elementary field effective potential) that the \( n \)-loop effective potential improved using \( n + 1 \) loop \( \beta \) and \( \gamma \) functions resums the \( n \)th-to-leading logs \([20, 21]\). Since we are demonstrating the utility of a calculational tool rather than pursuing a precise numerical result, we will be satisfied to sum the leading logs only. It is thus sufficient to consider the one-loop effective potential with the tree-level piece run with one-loop \( \beta \) functions. In the large-field \((\chi \gg m^2)\) limit the RG improved effective potential can be written \([22, 23, 24]\)

\[
V_{\text{eff}}(s^2 \chi_i, \hat{g}_i, \mu) = \lambda_{\text{eff}}(\chi_i, \hat{g}(s, \hat{g}_i), \mu) \left[ \chi_i \zeta^2(s) \right]^2
\]

\[
\lambda_{\text{eff}}(s) = \lambda(s) + \Delta \lambda(\hat{g}_i, \chi_i, \mu)
\]

We will demonstrate the calculation of the vacuum stability bound using the RG-improved composite operator effective potential in the context of a toy model, the abelian Higgs model coupled to a fermion. The issue of vacuum stability arises in qualitatively the same way in the abelian Higgs+fermion model as in the Standard Model, as a result of the large negative contribution of top quark loops to \( \beta_\lambda \). For some electroweak-scale boundary condition on \( \lambda(s) \), \( \lambda(s_{\text{EW}}) = \lambda_i \), the top quark term in \( \beta_\lambda \) drives \( \lambda_{\text{eff}}(s) \) negative for sufficiently large \( s \). For large \( s \) the tree-level RG-improved effective potential \( V_{\text{eff}}(\chi_i, s) \approx \lambda_{\text{eff}}(s) \chi_i^2 \zeta(s)^4 \) then falls rapidly below \( V_{\text{eff}}(s_{\text{EW}}) \) and the electroweak vacuum is unstable. We define \( s_{VI} \) as the value of \( s \) at which at which the RG improved effective potential equals zero. Since \( \zeta(s) > 0 \), \( V_{\text{eff}}(s_{VI}) = 0 \) is equivalent to \( \lambda_{\text{eff}}(s_{VI}) = 0 \). For the electroweak vacuum to remain the global minimum of the theory, the ‘new physics’ must enter before \( s_{VI} \) (however, see \([25]\) for qualification). For a fixed \( s_{VI} \) there corresponds a minimum \( \lambda_i \) below which the electroweak vacuum is destabilized too early. This then translates to a lower bound on the Higgs pole mass.

### 7 Numerical Results

We present here some numerical results to demonstrate the qualitative features of the composite operator effective potential and its use in the study of vacuum stability.
bounds. A set of initial values of parameters has been chosen ($\epsilon_i^2 = 0.15, g_y^2 = 0.5, v = 246 \text{ GeV}, \mu = v$) to resemble the Standard Model, but these plots should not be construed as a serious attempt to calculate Standard Model quantities. Figures 1 and 2 illustrate the gauge dependence of the elementary field effective potential (prior to RG improvement) for $\lambda_i = 0.2$. Fig. 1 shows the effective potential in the $R_\xi$ gauge for several values of the gauge parameter and explicitly indicates the shift in the location of the ‘electroweak’ minimum as a function of the gauge parameter $\xi e^2$. In Fig. 2 the one-loop corrections are isolated to highlight the gauge dependence. In Fig. 3 we plot the one-loop composite operator effective potential with $p$-space smearing function $\tilde{K}(\vec{p}) = \theta(\rho - |\vec{p}|)$ for several values of $\rho$. Since the mass parameter $m$, which introduces a typical energy scale for the theory, has been taken to be 110 GeV, we expect that for $\rho$ much smaller than this the composite operator will be insensitive to all but SSB effects, which are purely infrared. For $\rho$ much larger than this the composite operator will detect the shorter-wavelength fluctuations not associated with SSB. As is shown in the figure, curves with $\rho$ of order the mass parameter are very close to the $\rho = 0$ curve. Only for $\rho \sim 1000 \text{ GeV}$ does the
Figure 2: One-loop corrections to the effective potentials in the $R_\xi$ gauge for several values of the gauge parameter.
shape of the composite operator effective potential deviate significantly from the \( \rho = 0 \) curve, so we are confident in the ability of the composite operator to provide a reliable separation of the physical scales of the problem and to isolate only the SSB effects. Fig. 4 shows the results of a vacuum stability bound calculation using the composite operator effective potential with \( \rho = 0 \). For a choice of \( s_{VI} \) apply the boundary condition \( \lambda_{\text{eff}}(s_{VI}) = \lambda(s_{VI}) + \Delta \lambda = 0 \) and use the RG equation to run \( \lambda(s) \) down to \( s = 0(\chi = \chi_i) \) (we neglect the mass parameter in our expression for \( \Delta \lambda \)). The resulting \( \lambda_i \) is the minimum value for which the RG-improved effective potential remains positive up to scale \( s_{VI} \). In a complete treatment this lower bound on \( \lambda_i \) could be converted to a lower bound on the Higgs pole mass. We have performed a similar calculation for the elementary field effective potential in Landau gauge and find the results virtually indistinguishable. For the case studied here, with weak gauge couplings, this is simply a reflection of the small quantitative contribution from the gauge sector, and not a statement of identity of Landau and Coulomb effective potentials. This suggests that in the Standard Model as well the Landau gauge
results may be similar to those obtained using a gauge-invariant method. However, due to numerical differences in the $\beta$ functions, the effect of QCD in the running of $g_y^2$, and of course the different gauge group of the Standard Model, no immediate conclusions can be drawn in that context. In Fig. 5 we plot the one-loop correction to the elementary-field effective potential in Landau gauge and the composite operator effective potential with ($\rho = 0$). The fact that the curves are so close in the region in which we choose initial scale $\chi_i$ explains why the corresponding $\Delta \lambda$ and thus the $\lambda_{i,\text{min}}$ curves are so similar for the two effective potentials.

8 Composite Operator Effective Potentials for Non-abelian Gauge Theories

In an abelian gauge theory, the use of Coulomb gauge was facilitated by the observation that a smeared string corresponding to a dipole field (cf. (25-27)) yields a
Figure 5: One-loop corrections to the elementary field effective potential in Landau gauge and composite operator effective potential (for cutoff momentum zero), for $g^2 = 0.15$, $g_s^2 = 0.5$, $\lambda = 0.2$, $\mu = v$. Both curves are without RG improvement.
gauge-invariantizing factor for a bilocal operator which automatically reduces to unity in Coulomb gauge. The need for path ordering of such factors in the nonabelian case means that this procedure fails and we are forced to seek a more convenient physical gauge. At first sight, axial gauge would seem to fit the bill- in this gauge, there is a positive definite state space and a well defined Hamiltonian, and the ordinary straight line string factor can be used to define a gauge-invariant composite field in the usual way

\[
\chi_{1D}(t, x) = \int dl \phi^* \left( t, \vec{x} + \frac{l}{2} \hat{z} \right) K(l) \mathcal{P} e^{i \int dt' \hat{A}(t') \cdot \hat{z} \phi^* \left( t, \vec{x} - \frac{l}{2} \hat{z} \right)} \tag{74}
\]

Here the scalar fields have been fixed to lie along an arbitrarily chosen line \( \hat{z} \). The path ordering along the one-dimensional (spacelike) path is unambiguous. If the calculation is then performed in the \( A \cdot \hat{z} = 0 \) spacelike axial gauge the string operator again reduces to unity.

The use of axial gauge in this fashion leads however to a number of interesting delicate points, which divide into two categories- perturbative ultraviolet problems in defining the effective potential, and nonperturbative infrared problems associated with the loss of long range order \[26\] when such string operators are employed as order parameters in a broken gauge theory.

First, we discuss the problems arising with the definition of a finite composite potential in perturbation theory. These problems are not specific to gauge theory - arising simply from the UV singularities of one-dimensionally smeared bilocal operators- and can be illustrated in the simple scalar model of Section 2. We found there that the one loop composite effective potential yields a term

\[
\int \frac{d^3 p}{(2\pi)^3} \left( \sqrt{\vec{p}^2 + 2\lambda_R \chi} + (m_R^2 + \lambda_R \chi)(1 - \tilde{K}(\vec{p})) - |\vec{p}| \right) \tag{75}
\]

where \( \tilde{K}(\vec{p}) \) is the Fourier transform of the coordinate space bilocal smearing function \( K(\vec{r}) \). It is crucial that the terms proportional to powers of \( \tilde{K} \) in the expansion of this expression in powers of \( \chi \) not introduce additional ultraviolet divergences, as there are no counterterms available in the Lagrangian to absorb them. If we simply point split the fields (along the \( z \) direction, say) by taking \( K(\vec{r}) = \frac{1}{2} \delta(x) \delta(y)(\delta(x - 1/\rho) + \delta(x + 1/\rho)) \), corresponding to \( \tilde{K} = \cos \frac{2\pi}{\rho} \), we find that the one-loop potential expanded in \( \chi \) is UV finite at \( O(\chi) \) but logarithmically divergent at \( O(\chi^2) \). Namely:

\[
\int d^3 p \frac{\cos(2\pi \rho / \rho)}{\sqrt{\vec{p}^2 + m^2 + 3\lambda R \chi}} < \infty \tag{76}
\]

\[
\int d^3 p \frac{\cos^2(2\pi \rho / \rho)}{(\vec{p}^2 + m^2 + 3\lambda R \chi)^{3/2}} \sim \frac{1}{2} \int d^3 p \frac{1}{(\vec{p}^2 + m^2 + 3\lambda R \chi)^{3/2}} + \text{finite} \tag{77}
\]

25
The origin of the divergence is not hard to find— at order $\chi^2$, the momentum space double insertion of the point split operator $\phi(\vec{r} + \frac{1}{\rho} \hat{z}) \phi(\vec{r} - \frac{1}{\rho} \hat{z})$ leads to a UV divergence when the coordinate space integration defining the Fourier transform brings both pairs of field operators simultaneously together. Clearly, holding the two fields at a fixed separation is a prescription for trouble. On the other hand, taking a smooth smearing function such as $\tilde{K} = e^{-p^2/4\rho^2}$ (in analogy to the 3 dimensional Gaussian smearing of Sections 2,3), leads to an uncompensated divergence at $O(\chi)$, with order $\chi^2$ and higher terms finite. The linear divergence in $\chi \int d^3 p \frac{e^{-p^2/4\rho^2}}{\sqrt{p^2 + m^2 + 3\lambda \chi}}$ is also readily understood in operator terms. Here the one-dimensional smearing integral leads directly to a linear divergence due to the quadratic short-distance divergence of a product of two scalar field operators approaching the same point.

The solution to these ultraviolet problems is actually quite simple— one need only choose a smearing function which vanishes at the origin in coordinate space. This removes the divergence from the one dimensional integral in the region where the two field points approach one another in a single insertion of the composite field $\chi$, while removing the logarithmic divergence at order $\chi^2$ by smearing out the split field locations. For example, a suitably normalized smearing function is

$$K(\vec{r}) = \frac{2}{\sqrt{\pi}} \rho^2 \delta(x) \delta(y) z^2 e^{-\rho^2 z^2}$$

(79)

As for the three dimensional smearings of Sections 2-4, one should take the smearing parameter $\rho$ small, corresponding to widely separated peaks in $K(\vec{r})$. In the opposite limit of $\rho$ large, the UV divergences will reappear, distorting the shape of the effective potential. Of course, the requirement that $K(0) = 0$ now means that the Fourier transform $\tilde{K}$ is not positive for all $\vec{p}$, so there is no rigorous argument that the natural domain of $\chi$ is restricted to the positive real axis. Still, for small $\rho$ in the symmetry-broken phase, we expect that the minimum of the effective potential will be found at a safely positive location, so this property is perhaps not terribly important.

A further difficulty which one encounters in employing $\chi_{1D}$ as an order parameter is specific to gauge theories with nontrivial topological structure. The decorrelating effects induced by instantons on bilocal operators with a gauge-invariantizing string was first pointed out by Fröhlich et. al. [26]. The existence of Gribov copies [27] can also lead to a destruction of long range order, again as a result of large nonperturbative field configurations. All such effects are presumably of order $e^{-c/\bar{\hbar}}$ and therefore not visible in a standard perturbative loop expansion. From a practical point of
view, in the weakly coupled electroweak case instanton effects (unenhanced by large combinatoric prefactors as is potentially the case in multiparticle production at high energy) are extremely small, so these effects are clearly ignorable. In strongly coupled theories—when studying dynamical symmetry breaking, for example—the problem returns and caution will be required when taking the limit $\rho \to 0$ in an axial type gauge. We should remind the reader that for the Coulomb gauge smeared bilocal operator, Kennedy and King have shown [17] that this limit leads to a nonvanishing order parameter in the symmetry broken phase.

9 Conclusions

The computation of lower bounds on the Higgs mass from vacuum stability constraints using the elementary field effective potential results results in unphysical gauge dependence in such bounds. We have formulated an effective potential in terms of gauge invariant composite operators which avoids the ultraviolet problems of local composite operators while retaining an energy interpretation. We have shown how this could be used to calculate the vacuum stability bound on the Higgs mass in the context of a toy model, the abelian Higgs model coupled to a fermion, and for the set of parameters chosen (corresponding to a weakly coupled gauge sector) find that the results are quantitatively close to those obtained from the elementary field effective potential in Landau gauge. While the extension to a nonabelian gauge theory using a related composite operator in axial gauge introduces additional subtleties, we see no serious obstacles to the complete calculation. Thus, the tool might be extended to models of phenomenological interest, such as the Standard Model or its extensions. Composite effective potentials are essential in the study of dynamical symmetry breaking, so the issues discussed in this paper should prove of value in that context as well.
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Appendix A: $\overline{MS}$ renormalization

$\overline{MS}$ counterterms

\[
\begin{align*}
\delta Z_\phi &= (-2e^2 + 2g_y^2) \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right] \\
\delta Z_\lambda &= \left( -10\lambda + 6e^2 - 3\frac{e^4}{\lambda} - 4g_y^2 + 4\frac{g_y^4}{\lambda} \right) \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right] \\
\delta Z_{g_y} &= \left( \frac{3}{4}e^2 - 2g_y^2 \right) \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right] \\
\delta Z_{m^2} &= \left( -4\lambda + 3e^2 - 2g_y^2 \right) \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right] \\
\delta Z_\epsilon &= -\frac{1}{3}e^2 \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right] \\
\delta Z_A &= \frac{2}{3}e^2 \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right] \\
\delta Z_t &= g_y^2 \left[ \frac{-\Delta_\epsilon}{(4\pi)^2} \right]
\end{align*}
\]  

(80)

$\beta$ functions

The relevant one-loop $\overline{MS}$ $\beta$ and $\gamma$ functions for the theory are:

\[
\begin{align*}
\beta_\lambda &= \frac{1}{16\pi^2} \left( 20\lambda^2 + 6e^4 - 8g_y^4 - 12e^2\lambda + 8\lambda g_y^2 \right) \\
\beta_\epsilon &= \frac{1}{16\pi^2} \left( \frac{2}{3}e^3 \right) \\
\beta_{g_y} &= \frac{1}{16\pi^2} g_y \left( -\frac{3}{2}e^2 + 4g_y^2 \right) \\
\gamma_\phi &= \frac{1}{16\pi^2} \left( -e^2 + g_y^2 \right)
\end{align*}
\]  

(81, 82, 83, 84)
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