Domain Walls
and Dimensional Reduction

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Abstract

We study some properties of a dimensional reduction mechanism for fermions in an odd number $D + 1$ of spacetime dimensions. A fermionic field is equipped with a mass term with domain wall like defects along one of the spacelike dimensions, which is moreover compactified. We show that there is a regime such that the only relevant degrees of freedom are massless fermionic fields in $D$ dimensions. For any fixed gauge field configuration, the extra modes may be decoupled, since they can be made arbitrarily heavy. This decoupling combines the usual Kaluza-Klein one, due to the compactification, with a mass enhancement for the non-zero modes provided by the domain wall mechanism. We obtain quantitative results on the contribution of the massive modes in the cases $D = 2$ and $D = 4$.

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1 Introduction.

Domain wall fermions have been a very active subject of theoretical research because of their many interesting properties and applications. They have recently attracted increased attention in their application to Kaplan’s domain wall approach to the problem of putting chiral fermions on a lattice. This proposal has evolved into another, more abstract construction, the so-called ‘overlap’ formalism [2, 3, 4], that was designed in order to bypass Nielsen-Ninomiya’s no-go theorem [5]. It gives a procedure to define a chiral determinant in $D = 2k$ dimensions in terms of the ‘overlap’ (that is, scalar product) between two different Dirac vacua. These vacua correspond to two Dirac Hamiltonians in $D = 2k + 1$ spacetime dimensions, in the presence of the same static external gauge field, but with opposite signs for their (Dirac) mass terms. This idea has been extensively applied to different models, testing its predictions and consequences for both its continuum and lattice versions. Particular interest has been directed to the issue of chiral anomalies. The original idea has been also extended to the odd dimensional case [6], and to the bosonic case [7]. Recently, fermions of domain-wall like defects have also been studied in the context of condensed matter systems [8, 9].

Up to know, all the tests performed so far suggest that, when properly regularized, domain wall fermions may indeed provide a satisfactory definition of a chiral fermionic determinant. This definition, either in its overlap or domain-wall version, demands the introduction of a parameter $\Lambda$, with the dimensions of a mass, which has to go to infinity at some point of the construction. A finite-$\Lambda$ definition, on the other hand, provides an effective definition of the chiral determinant, valid when low-momentum external gauge-field configurations are considered. Chiral $2k$ dimensional modes are ‘confined’ to a strip of width $\sim \frac{1}{\Lambda}$ around a $2k$-dimensional hypersurface defined by the position of the domain wall, say, $x_{2k+1} = 0$.

In this paper we study domain wall fermions from the dimensional reduction point of view. That is to say, we assume the extra dimension $x_{2k+1}$ to be compactified, and we want to understand the combined effects of the domain walls and the compact dimension on the effective dimensionally reduced theory. The effective low-energy theory arising in this kind of system involves fields of both chiralities, since the use of a compactified extra dimension automatically produces a domain anti-domain wall pair and their companion fermionic modes of opposite chiralities. We obtain the conditions that have to be satisfied by the parameters of the theory, in order to have a
regime where the massive modes are decoupled. In other words, we find the conditions for the theory to be effectively dimensionally reduced to a model of massless Dirac fermions in $2k$ dimensions.

This paper is organized as follows: In section 2 we define and discuss, in the functional integral approach, the dynamics corresponding to the fermionic modes in an odd dimensional theory with a compactified dimension, regarding the gauge field as external (non-dynamical). We first deal with a restricted class of external gauge field configurations, which are in fact similar to the static ones of the overlap formalism, and afterwards extend the study to more general gauge field configurations. We then show how to extract the effective dynamics of the reduced theory, and to assure the decoupling of the higher modes. As explicit examples, the decoupling for the Abelian case in $4 + 1$ and $2 + 1$ dimensions is discussed in the framework of perturbation theory.

2 Dimensional reduction

The Euclidean functional integral corresponding to a fermionic field in the presence of an external gauge field $A$, in $D + 1$ dimensions, and with a mass term depending on the extra coordinate $x_{D+1} \equiv s$, is defined by

$$Z(A) = \int [d\Psi][d\bar{\Psi}] \exp\left(-\int d^{D+1}x \, \bar{\Psi} D \Psi\right)$$  \hspace{1cm} (1)

where the Euclidean Dirac operator $D$ appearing in the fermionic action is

$$D = \gamma_s D_s + m(s) + \Phi(A).$$  \hspace{1cm} (2)

We shall adopt the conventions that Greek indices run from 1 to $D$; $x = (x_{\mu}), \mu = 1 \cdots D$ will denote the $D$ uncompactified ‘physical’ dimensions, while $s$ stands for the compactified one ($x_{D+1}$). Also, Dirac’s $\gamma$ matrices are Hermitian, obeying the relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, \gamma_s\} = 0, \quad \gamma_s^2 = 1. \hspace{1cm} (3)$$

2.1 Restricted gauge field configurations

We want to express the Euclidean functional integral (1) for $Z(A)$ in a form that renders the existence and properties of the domain wall fermions more
transparent. To this effect we shall introduce an expansion of the fermionic fields in terms of a suitable Hermitian operator. As $D$ is not Hermitian, we introduce the explicitly Hermitian, positive definite operator $H$, defined in terms of $D$ as follows:

$$H = D^\dagger D.$$  \hfill (4)

We first assume that $A_{D+1} = 0$, and $A_\mu = A_\mu(x)$. This restriction (to be relaxed in subsection [22]) is imposed in order to have a clean factorization into the dynamics along the $x$-coordinates and along $s$, as then $H$ is the sum of two commuting pieces

$$H = h - \mathcal{D}^2$$  \hfill (5)

where $h$ acts on the extra coordinate $s$ only

$$h = -\partial_s^2 + m^2(s) - \gamma_s m'(s),$$  \hfill (6)

with $m'(s) = \frac{d}{ds} m(s)$. We also define projectors corresponding to the two eigenvalues $\pm 1$ of $\gamma_s$, named $\mathcal{P}_{L,R}$, respectively, and two operators $a, a^\dagger$, acting on the Hilbert space of functions depending on the coordinate $s$

$$a = \partial_s + m(s), \quad a^\dagger = -\partial_s + m(s).$$  \hfill (7)

The boundary conditions for the functions upon which these operators act are always going to be such that they become relatively adjoint. That will hold true for both the compact and non-compact cases.

After these definitions, $H$ may be written as

$$H = a^\dagger a \mathcal{P}_L + aa^\dagger \mathcal{P}_R - \mathcal{D}^2$$  \hfill (8)

or

$$H = (a^\dagger a - \mathcal{D}^2) \mathcal{P}_L + (aa^\dagger - \mathcal{D}^2) \mathcal{P}_R.$$  \hfill (9)

We define normalized eigenfunctions corresponding to the Hermitian and positive operators $a^\dagger a$ and $aa^\dagger$,

$$a^\dagger a f_n^{(+)}(s) = \mathcal{E}_n^2 f_n^{(+)}(s)$$

$$aa^\dagger f_n^{(-)}(s) = \mathcal{E}_n^2 f_n^{(-)}(s),$$  \hfill (10)

and note that, as the extra dimension is compactified, both operators shall have zero modes, namely, eigenfunctions $f_0^{(+)}(s)$ and $f_0^{(-)}(s)$ with $\mathcal{E}_0 = 0$. They shall correspond to the equations

$$a f_0^{(+)}(s) = 0, \quad a^\dagger f_0^{(-)}(s) = 0.$$  \hfill (11)
respectively. Had the extra dimension been of infinite extension, one of the zero modes \( f^+(+) \), \( f^-(+) \) would have disappeared, because it must have had an infinite norm. We shall here assume that the extra dimension is finite: \( s \in [-L, L] \), and that \( m(s) \) is periodic, so that we always have both zero modes. Moreover, they can be explicitly written as

\[
f_0^{(\pm)}(s) = N_0^{(\pm)} \exp[\mp \int_0^s dt m(t)]
\]

where

\[
N^{(\pm)} = \left\{ \int_{-L}^{+L} ds \exp[\mp 2 \int_0^s dt m(t)] \right\}^{-\frac{1}{2}}.
\]

It is convenient at this point to be more specific about the functional form of \( m(s) \). The simplest possibility within the compactified case is to give \( m(s) \) a positive step at \( s = 0 \)

\[
m(s) = \Lambda \text{sign}(s),
\]

which, because of periodicity, does also have a negative jump at \( s = +L \equiv -L \). This yields for the zero modes the explicit expression

\[
f_0^{(+)}(s) = \sqrt{\frac{\Lambda}{1 - e^{-2\Lambda L}}} e^{-\Lambda|s|}
\]

\[
f_0^{(-)}(s) = \sqrt{\frac{\Lambda}{e^{2\Lambda L} - 1}} e^{\Lambda|s|},
\]

which shall correspond to modes localized around \( s = 0 \) and \( s = L \), respectively. In the uncompactified case, on the other hand, as \( L \to \infty \), only the mode \( f_0^{(+)} \) survives. It is evident how to extend the previous zero mode solutions to the general case of a stepwise mass profile \( m(s) \) such that \( m^2(s) = \Lambda^2 \).

Keeping (8) in mind, we see that the eigenfunctions of \( \mathcal{H} \) shall be products of eigenfunctions of \( a \dagger a \mathcal{P}_L \) or \( aa \dagger \mathcal{P}_R \), times eigenfunctions of \( -\mathcal{D}^2 \). We then expand the fermionic fields \( \Psi(x, s) \) and \( \bar{\Psi}(x, s) \) in terms of these eigenfunctions. Of course, each term will also depend on an arbitrary \( D \)-dimensional spinor \( \psi^{(n)}(x) \):

\[
\Psi(x, s) = \sum_n \left[ f_n^{(\pm)}(s) \mathcal{P}_L \psi^{(n)}(x) + f_n^{(-\dagger)}(s) \mathcal{P}_R \psi^{(n)}(x) \right]
\]

\[
\bar{\Psi}(x, s) = \sum_n \left[ f_n^{(\dagger \pm)}(s) \bar{\psi}^{(n)}(x) \mathcal{P}_R + f_n^{(-\dagger \dagger)}(s) \bar{\psi}^{(n)}(x) \mathcal{P}_L \right],
\]

5
or

\[
\Psi(x, s) = \sum_n \left[ f^{(+)}_n(s) \psi^{(n)}_L(x) + f^{(-)}_n(s) \psi^{(n)}_R(x) \right]
\]

\[
\bar{\Psi}(x, s) = \sum_n \left[ \bar{\psi}^{(n)}_L(x) f^{(+)*}_n(s) + \bar{\psi}^{(n)}_R(x) f^{(-)*}_n(s) \right],
\]

(17)

with \( \psi^{(n)}_{L,R}(x) = \mathcal{P}_{L,R} \psi^{(n)}(x) \), and \( \bar{\psi}^{(n)}_{L,R}(x) = \bar{\psi}^{(n)}(x) \mathcal{P}_{R,L} \).

We could, of course, expand each \( \psi^{(n)} \) in terms of the eigenfunctions of \(- \mathcal{D}^2\), but that shall not be necessary. Note that no approximation has been invoked in order to obtain the expansions (16) and (17), since we are only applying the property that \( \mathcal{H} \) can be written as the sum of two commuting pieces, and we can always expand any state in such a way as to make one of the pieces diagonal.

Using expansion (17), we have a decomposition of the fermionic integration measure,

\[
[d\Psi][d\bar{\Psi}] = \prod_n \left( [d\psi^{(n)}_L][d\bar{\psi}^{(n)}_L][d\psi^{(n)}_R][d\bar{\psi}^{(n)}_R] \right),
\]

(18)

and a series for the fermionic action. It is convenient, in this series, to separate what corresponds to the zero modes, from the contributions due to the higher \( n \)'s, since they shall have quite different properties. Explicitly, it reads

\[
S = \int d^Dx \left[ \bar{\psi}^{(0)}_L(x) \mathcal{D} \psi^{(0)}_L(x) + \bar{\psi}^{(0)}_R(x) \mathcal{D} \psi^{(0)}_R(x) \right]
\]

\[
\sum_{n \neq 0} \int d^Dx \left\{ \bar{\psi}^{(n)}_L(x) \mathcal{D} \psi^{(n)}_L(x) + \bar{\psi}^{(n)}_R(x) \mathcal{D} \psi^{(n)}_R(x) + \mathcal{E}_n \left[ \bar{\psi}^{(n)}_R(x) \psi^{(n)}_L(x) + \bar{\psi}^{(n)}_L(x) \psi^{(n)}_R(x) \right] \right\}.
\]

(19)

This shows that there appear massless \( D \)-dimensional fermionic fields localized around the domain wall defects, and that there is an infinite tower of massive states. These states can be decoupled of the massless ones, since their masses \( \mathcal{E}_n \) will satisfy, as shown below, the inequality

\[
\mathcal{E}_n \geq \sqrt{|\Lambda|^2 + \left( \frac{\pi}{L} \right)^2}.
\]

(20)

Thus the masses of these higher modes can be made arbitrarily large by proper choices of the parameters \( \Lambda, L \).
Let us prove inequality (20) for the eigenvalues of $h$ corresponding to the modes with $n \neq 0$. We first use the property that, for stepwise profiles $m(s)$, such that $m^2(s) = \Lambda^2$, the only localized states are the zero modes. This can be proved just by taking into account that localized states are combinations of real exponentials, and those combinations are completely determined once the matching conditions due to the $\delta$ functions are imposed. For the non-zero modes, the states are combinations of exponentials of $\pm ik \cdot x$. The periodicity of the extra coordinate fixes the minimum $k$ to be $\pi L$, and the $\delta$-function term only fixes boundary conditions, thus for any eigenstate $f^{(\pm)}_n$, 

$$\langle f^{(\pm)}_n|h|f^{(\pm)}_n\rangle = \mathcal{E}_n^2 \geq \left(\frac{\pi}{L}\right)^2 + \Lambda^2. \quad (21)$$

Let us consider the issue of decoupling. Equations (18) and (19) already indicate that

$$\mathcal{Z} = \mathcal{Z}^{(0)} \times \prod_{n \neq 0} \mathcal{Z}^{(n)} \quad (22)$$

where $\mathcal{Z}^{(0)}$ contains the massless modes of the dimensionally reduced theory

$$\mathcal{Z}^{(0)} = \int [d\bar{\psi}_L^{(0)}][d\psi_L^{(0)}][d\bar{\psi}_R^{(0)}][d\psi_R^{(0)}]$$

$$\times \exp \left[ - \int d^Dx (\bar{\psi}_L^{(0)} \slashed{D}\psi_L^{(0)} + \bar{\psi}_R^{(0)} \slashed{D}\psi_R^{(0)}) \right], \quad (23)$$

while

$$\mathcal{Z}^{(n)} = \int [d\bar{\psi}^{(n)}][d\psi^{(n)}] \exp \left[ - \int d^Dx (\bar{\psi}^{(n)} \slashed{D}\psi^{(n)} + \mathcal{E}_n \bar{\psi}^{(n)} \psi^{(n)}) \right]$$

$$= \det(\mathcal{P} + \mathcal{E}_n) \quad (24)$$

are the massive, decoupling modes. Of course they will only decouple if some conditions are imposed on the external gauge fields, namely, their momentum dependence cannot be arbitrary. In order to achieve decoupling, we have to assume that the external momenta $p_\mu$, corresponding to the Fourier components of the gauge fields, are small in comparison to $\mathcal{E}_n$. This guarantees the existence of a momentum expansion for the massive determinants of (24). The terms of such an expansion are suppressed by powers of $\frac{p^2}{\mathcal{E}_n^2}$, since the Feynman diagrams are analytic below the threshold.
2.2 General gauge field configurations

In the preceding subsections we have addressed the issue of decoupling in the presence of external gauge fields satisfying the conditions,

$$\partial_s A_\mu = 0 , \quad A_s = 0 .$$  \hspace{1cm} (25)

As mentioned in subsection 2.1, under conditions (25), the eigenfunctions of $D^\dagger D$ can be factored into eigenfunctions of $h$ times eigenfunctions of $(\not D)^2$. In the present subsection we present a perturbative proof of the fact that, even for the general case, namely, relaxing (25), we still have a non-vanishing gap of $O(\Lambda)$ in the spectrum of $D^\dagger D$. The spirit of the proof is to use the fact that, by performing a perturbative expansion around the restricted case (25), the correction to the gap is negligible when the gauge field is smooth.

We split $D^\dagger D$ into free ($\mathcal{H}$) and perturbation ($V$) terms

$$D^\dagger D = \mathcal{H} + V$$  \hspace{1cm} (26)

where $\mathcal{H}$ is the one discussed in 2.1, i.e.,

$$\mathcal{H} = m(s)^2 - \gamma_s (s) - \partial_s^2 - (\not D)^2 (A_\mu (x, s = 0))$$  \hspace{1cm} (27)

and $V$ is defined by

$$V = -\partial_s A_s - A_s^2 - \gamma_\mu \gamma_s F_{\mu s} + \not D^2 (A_\mu (x, s = 0)) - \not D^2 (A_\mu) .$$  \hspace{1cm} (28)

The free term is $O(\Lambda^2)$, while for the perturbation we shall assume that the gauge field is smooth, in the same sense as in the restricted case, namely, their derivatives are small when compared with $\Lambda$. Thus the perturbation is $O(\Lambda^0)$. It is also useful to note that the perturbation is a periodic function of $s$.

In order to compute the corrections to the spectrum of (27) we should consider matrix elements of the following kind:

$$< n|V|m > = \int_{-L}^L ds \ f_n^\pm (s) \left\{ \int d^d x \bar{\psi}_{LR}^n (x) V (x, s) \psi_{LR}^n (x) \right\} f_m^\pm (s)$$  \hspace{1cm} (29)

where $V(x, s)$ may be written as a Fourier series,

$$V(x, s) = \sum_n \left[ V_n^e (x) \cos \left( \frac{n \pi s}{L} \right) + V_n^o (x) \sin \left( \frac{n \pi s}{L} \right) \right]$$  \hspace{1cm} (30)
and the functions $f_n^\pm(s)$ appearing in (29) are the excited states eigenfunctions of $h$. They are odd or even ($h$ is invariant under $s \leftrightarrow -s$) and are explicitly given by,

$$f_n^{+,o}(s) = \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi s}{L}\right) = f_n^-(s),$$
$$f_n^{+,e}(s) = \frac{1}{\sqrt{L(1 + (\frac{n\Lambda}{L})^2)}} \left(\cos\left(\frac{n\pi s}{L}\right) \pm \frac{a\Lambda}{n\pi} \sin\left(\frac{n\pi s}{L}\right)\right)$$

with eigenvalues,

$$\epsilon_n^2 = \left(\frac{n\pi}{L}\right)^2 + \Lambda^2$$

in both cases. In order to estimate the perturbative correction, we see that the $\Lambda$ dependence of the matrix elements (29) is determined by the $s$-integration, since the parameter $\Lambda$ only appears in the $f_n^\pm$ functions. From formulae (30) and (31) it is evident that this matrix elements are at most of $O(\Lambda^0)$. Therefore, when the Fourier components of the gauge fields are small compared with $\Lambda$, the corrections to the unperturbed energy levels (32) do not eliminate the gap of $O(\Lambda)$ of the unperturbed theory. Indeed, the corrections to the eigenvalues are not capable of modifying a gap of order $\Lambda$, since they always involve $O(\Lambda^0)$ terms.

### 2.3 The fermionic propagator

The results of 2.1 were important in order to understand the mechanism on decoupling, from the point of view of the gauge field action. Namely, the contribution to the effective gauge field action due to the fermion loops is shown to be given mainly from the contribution of the massless, domain wall modes. However, for physical processes involving external fermions, we need to consider a different case, since we have to include fermionic sources $\bar{\eta}(x, s) \eta(x, s)$ into the functional integral (1):

$$Z(\bar{\eta}, \eta; A) = \left[Z(A)^{-1}\int [d\bar{\Psi}][d\Psi] \exp \{-S\}ight.\left. + \int ds d^D x [\bar{\eta}(x, s) \Psi(x, s) + \bar{\Psi}(x, s) \eta(x, s)]\right].$$

Expanding the measure and the action, according to (13) and (19), respectively, we have, after integrating out the fermions

$$Z(\bar{\eta}, \eta; A) = Z^{(0)}(\bar{\eta}, \eta; A) \times \prod_{n \neq 0} Z^{(n)}(\bar{\eta}, \eta; A).$$
$Z^{(0)}(\bar{\eta}, \eta; A)$ is the generating functional corresponding to the domain wall fermions:

$$Z^{(0)}(\bar{\eta}, \eta; A) = \exp \left\{ - \int ds d^D x \int ds' d^D x' \bar{\eta}(x, s) [f_0^{(+)}(s) \mathcal{P}_R + f_0^{(-)}(s) \mathcal{P}_L] \mathcal{D}^{-1}(x, x') [f_0^{(+)}(s') \mathcal{P}_L + f_0^{(-)}(s') \mathcal{P}_R] \eta(x', s') \right\},$$  \hspace{1cm} (35)

while $Z^{(n)}(\bar{\eta}, \eta; A)$ contains the infinite tower of massive fermionic fields

$$Z^{(n)}(\bar{\eta}, \eta; A) = \exp \left\{ - \int ds d^D x \int ds' d^D x' \bar{\eta}(x, s) [f_0^{(+)}(s) \mathcal{P}_R + f_0^{(-)}(s) \mathcal{P}_L] (\mathcal{D} + \mathcal{E}_n)^{-1}(x, x') [f_0^{(+)}(s') \mathcal{P}_L + f_0^{(-)}(s') \mathcal{P}_R] \eta(x', s') \right\}. \hspace{1cm} (36)$$

Again, decoupling is achieved when the masses ($\mathcal{E}_n$) in the propagators derived from (36) are large compared with the external momenta (now the momenta of the fermions are also relevant). The domain wall piece (35) allows us to write down their contribution to the fermionic propagator

$$\langle \Psi(x, s) \bar{\Psi}(x', s') \rangle = \frac{\Lambda}{1 - e^{-2\Lambda L}} e^{-\Lambda(|s| + |s'|)} \mathcal{D}^{-1}(x, x') \mathcal{P}_L + \frac{\Lambda}{e^{2\Lambda L} - 1} e^{\Lambda(|s| + |s'|)} \mathcal{D}^{-1}(x, x') \mathcal{P}_R. \hspace{1cm} (37)$$

It is straightforward to obtain the expression for $j_\mu(x, s)$, the vacuum current contribution due to this part of the fermion propagator. With the usual definitions, we see that

$$j_\mu(x, s) = \langle \bar{\Psi}(x, s) \gamma_\mu \Psi(x, s) \rangle - \text{tr} \left[ \gamma_\mu (\bar{\Psi}(x, s) \bar{\Psi}(x, s)) \right]$$

$$\frac{\Lambda}{1 - e^{-2\Lambda L}} e^{-2\Lambda |s|} j^L_\mu(x) + \frac{\Lambda}{e^{2\Lambda L} - 1} e^{2\Lambda |s|} j^R_\mu(x) \hspace{1cm} (38)$$

where $j^L_\mu(x)$ and $j^R_\mu(x)$ denote the corresponding vacuum currents in $D$ dimensions:

$$j^L_\mu(x) = \langle \bar{\psi}_L(x) \gamma_\mu \psi_L(x) \rangle$$

$$j^R_\mu(x) = \langle \bar{\psi}_R(x) \gamma_\mu \psi_R(x) \rangle. \hspace{1cm} (39)$$
It becomes clear from the above that the contributions of the chiral zero modes to the current are localized around each domain wall, with a localization length $\sim \frac{1}{\Lambda}$. In order to be able to resolve the two currents, the localization length should be smaller than the compactification length, what amounts to the inequality
\[ \Lambda \times L > 1. \] (40)

Regardless of whether the two chiral currents have a large overlap or not, the integral of the $D + 1$ dimensional current $j_\mu(x, s)$ along the $s$ coordinate always produces the result corresponding to the current of a Dirac fermions in $D$ dimensions
\[ \int_{-L}^{L} ds \, j_\mu(x, s) = j_\mu(x) = \langle \bar{\psi}(x) \gamma_\mu \psi(x) \rangle, \] (41)

where $\psi = \psi_L + \psi_R$.

We shall also present, for the sake of completeness, a derivation of the fermionic propagator obtained by directly inverting the operator $D$ of (3), in $4 + 1$ dimensions. We consider the free ($A = 0$) case for the sake of clarity, since, for this calculation, the necessary changes for the non-free case are easy to introduce. Moreover, this free propagator in $4 + 1$ is used in the next subsection, in the perturbation theory example. This system is described by the free Euclidean action:
\[ S = \int ds dx L_F, \quad L_F = \bar{\Psi}(\hat{\partial} + m(s))\Psi \] (42)

The free Dirac operator $\hat{\partial}$ acts on the five co-ordinates:
\[ \hat{\partial} \equiv \gamma_\mu \partial_\mu = \gamma_s \partial_s + \gamma_\mu \partial_\mu. \] (43)

Dirac’s matrices are chosen to be in the representation:
\[ \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu^\dagger \\ \sigma_\mu & 0 \end{pmatrix}, \quad \gamma_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_\mu = (\bar{\sigma}, i1) \] (44)

where $\bar{\sigma}$ denotes the three familiar Pauli’s matrices. Finally, the domain wall mass term $m(s)$ is assumed to be of the form:
\[ m(s) = \Lambda \text{sign}(s). \] (45)
namely, it contains a domain wall like effect localized at \( s = 0 \), and an anti domain wall companion at \( s = L \). From its definition, it follows that the fermionic propagator \( S \) satisfies

\[
[\partial_{x,s} + m(s)] S(x, s; x', s') = \delta^{(4)}(x - x') \delta(s - s')
\]  

(46)

where the subscripts \( x, s \) mean that derivatives act with respect to the \( x, s \)-coordinates. Because of translation invariance in \( x_\mu \), a Fourier transformation in these co-ordinates suggests itself:

\[
S(x, s; x', s') = \int \frac{d^4 k}{(2\pi)^4} e^{ik_\mu(x - x')} \tilde{S}_k(s, s').
\]  

(47)

Hence equation (46) implies for \( \tilde{S}_k \)

\[
[\gamma_s \partial_s + i\gamma_\mu k_\mu] \tilde{S}_k(s, s') = \delta(s - s').
\]  

(48)

To solve this equation, it is convenient to define an auxiliary function \( G_k \), determined from \( S_k \) by the relation

\[
\tilde{S}_k(s, s') = [-\gamma_s \partial_s - i\gamma_\mu k_\mu + m(s)]G_k(s, s')
\]  

(49)

which substituted into (46) yields for \( G_k \) the equation

\[
[\partial_s^2 + 2\Lambda \gamma_s (\delta(s) - \delta(s - L)) + k^2 + \Lambda^2]G_k(s, s') = \delta(s, s')
\]  

(50)

where \( k^2 = k_\mu k_\mu \). Then, \( G_k \) is the inverse of a Hamiltonian operator \( \mathcal{H}_k \) which contains a delta-like potential in a one-dimensional quantum mechanical system

\[
\mathcal{H}_k = -\frac{d^2}{ds^2} + 2\Lambda \gamma_s (\delta(s) - \delta(s - L)) + \omega_k^2
\]  

(51)

where \( \omega_k^2 = k^2 + \Lambda^2 \). In order to invert this operator, it is convenient to diagonalize it, and then we are back into the issue of knowing the spectrum of \( \hbar \), the operator considered in the general derivation of the previous subsection. Again, the domain wall modes correspond to the bound states of the Hamiltonian. Keeping only the contribution coming from these states, we get for \( G_k \)

\[
G_k(s, s') = \frac{\Lambda}{k^2} \left[ (e^{2\Lambda L} - 1)^{-1} e^{\Lambda(s' + |s|)} \right] \mathcal{P}_L
\]
\[+(1 - e^{-2\Lambda})^{-1} e^{-\Lambda(|s| + |s'|)} \mathcal{P}_R\]  
\[(52)\]

which, when introduced in \[(49)\] yields,

\[\tilde{S}_k(s, s') = \frac{\Lambda}{1 - e^{-2\Lambda} e^{-\Lambda(|s| + |s'|)}} (i \not{k})^{-1} \mathcal{P}_L\]

\[\frac{\Lambda}{e^{2\Lambda} - 1} e^{\Lambda(|s| + |s'|)} (i \not{k})^{-1} \mathcal{P}_R. \]  
\[(53)\]

### 2.4 Decoupling of the massive modes

To render the general derivations of the previous paragraphs more concrete, we present here their realization for two particular examples. We consider the cases \(D = 4\) and \(D = 2\), with Abelian external gauge fields. Making the redefinitions \(A \rightarrow i eA\), and recalling equations \((23)\) and \((24)\), we may write for the full functional \(Z(A)\),

\[Z(A) = \exp [-\Gamma(A)] \]  
\[(54)\]

where the effective action \(\Gamma(A)\) is a sum

\[\Gamma(A) = \Gamma^{(0)}(A) + \sum_{n \neq 0} \Gamma^{(n)}(A) \]  
\[(55)\]

with the domain wall effective action

\[\Gamma^{(0)} = -\text{Tr} \ln[\not{\partial} + i A] \]  
\[(56)\]

and the contribution of the massive modes

\[\Gamma^{(n)} = -\text{Tr} \ln[\not{\partial} + i A + \mathcal{E}_n]. \]  
\[(57)\]

Note that the issue of the convergence of the series over \(n\) is not clear at all, unless one obtain a more explicit expression, in terms of \(A\), for the contribution of the massive modes. To check this in the \(D = 4\) case, we may use the known results \[10\] for the derivative expansions of the Abelian fermionic determinants in 4 dimensions, obtaining for the leading terms

\[\Gamma^{(n)}_{D=4} = \int d^4x \left\{-\frac{e^2}{120\pi^2 \mathcal{E}_n^2} F_{\mu\nu} \partial^2 e^2 F_{\mu\nu} + \frac{e^4}{1440\pi^2 \mathcal{E}_n^4} [(F^2)^2 + \frac{7}{16} (F\tilde{F})^2] \right\}, \]  
\[(58)\]
while in $D = 2$ a similar calculation yields
\[
\Gamma_{D=2}^{(n)} = \int d^2 x \left\{ \frac{e^2}{24\pi E_n^2} F_{\mu\nu} F_{\mu\nu} + \frac{e^2}{360\pi E_n^4} F_{\mu\nu} \partial^2 F_{\mu\nu} \right\},
\]  
where we are of course neglecting higher dimensional terms. Thus, in order to assure the decoupling of these unwanted contributions to $\Gamma$, either in $D = 4$ or $D = 2$, we must be able to render
\[
\sum_{n \neq 0} \frac{1}{E_n^2}, \quad \sum_{n \neq 0} \frac{1}{E_n^4},
\]
(and all the series with higher even powers of $E_n$ in the denominator) arbitrarily small by tuning $\Lambda$ and $L$. But that can always be done, because for large $n$, the eigenvalues of $E_n$ will obviously grow at least like $n$, what makes the series above absolutely convergent. Then we adjust $\Lambda$ and $L$ to make those sums arbitrarily small (note that we could interchange the order of the limit $\Lambda \to \infty$ or $L \to 0$ and the summation of the series). It goes without saying that higher dimensional terms carry higher powers of $\frac{1}{E_n}$, what makes them more convergent.

This concludes our particular examples of decoupling. It is easy to verify that the convergence arguments remain the same for the case of external non-Abelian fields, since the power of $E_n$ that appears is 1 fixed by dimensional arguments, identical for the Abelian and non-Abelian cases.

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