Deformation Quantisation of Gravity

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Abstract

We study the deformation (Moyal) quantisation of gravity in both the ADM and the Ashtekar approach. It is shown, that both can be treated, but lead to anomalies. The anomaly in the case of Ashtekar variables, however, is merely a central extension of the constraint algebra, which can be “lifted”.

Finally we write down the equations defining physical states and comment on their physical content. This is done by defining a loop representation. We find a solution in terms of a Chern-Simons state, whose Wigner function then becomes related to BF-theory. This state exist even in the absence of a cosmological constant but only if certain extra conditions are imposed. Another solution is found where the Wigner function is a Gaussian in the momenta.

Some comments on “quantum gravity” in lower dimensions are also made. PACS: 04.60.-m, 04.60.Ds, 03.65.Ca, 11.15.Tk
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1 Introduction

This paper is based on [1], where it was shown in more detail how to perform a deformation quantisation of constrained systems. Among other things it was
shown how classical second class constraints could be turned into first class
quantum constraints. It was also shown how, in certain cases, simple kind
of anomalous contributions could be “lifted”, i.e., one could find quantum
constraints which did not have this anomalous contribution to their Moyal
algebra, but instead might have a singular naive classical limit, ℏ → 0.
Consequently, the correct classical limit is obtained as the principal part of
the ℏ → 0 limit.

We will give a short summary of the techniques here, and then expand on
the resulting treatment of gravitation sketched upon in [1].

Deformation quantisation, [2], consists essentially in replacing the classical
Poisson bracket, \{·, ·\}_PB by a new bracket known as the Moyal bracket

\[ [f, g]_M := i\hbar \{f, g\}_PB + O(\hbar^2) \]  

where \( f, g \) are functions on the classical phase-space \( \Gamma \). Hence, deformation
quantisation keeps the classical phase-space but endows it with a new, de-
formed bracket. For a flat phase-space, i.e., \( \Gamma \simeq \mathbb{R}^{2n} \), the Moyal bracket is
essentially unique, [3, 17]. It is given by

\[ [f, g]_M = 2i f \sin \left( \frac{1}{2} \hbar \Delta \right) g \]  

where \( \Delta \) is the bidifferential operator giving the Poisson bracket, i.e.,

\[ f \triangle g := \{f, g\}_PB \]  

Consequently,

\[ [f, g]_M = i \sum_{n=0}^{\infty} \frac{(-1)^n \hbar^{2n+1}}{4^n (2n+1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\partial^n f}{\partial q^{n-k} \partial p^k} \frac{\partial^n g}{\partial p^{n-k} \partial q^k} \]  

\[ := i \sum_{n=0}^{\infty} \hbar^{2n+1} \omega_{2n+1}(f, g) \]  

We will also introduce the anti-Moyal bracket

\[ [f, g]^+_M := 2f \cos \left( \frac{1}{2} \hbar \Delta \right) g \]  

Both of these can be written in terms of a twisted product \( * \) as

\[ [f, g]_M = f * g - g * f \quad [f, g]_M^+ = f * g + g * f \]  

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This twisted product is a deformation of the usual product of functions,

\[ f \ast g = fg + O(h) \]  \hspace{1cm} (8)

and comes from the *Weyl map* associating functions \( A_W(q,p) \) on the classical phase-space to operators \( \hat{A} \) acting on \( L^2(Q) \) where \( Q \) is the coordinate manifold. This map is given by, \[8, 7\]

\[ A_W(q, p) := \int e^{iuq - ivp} \text{Tr}(\Pi(u, v) \hat{A}) dudv \]  \hspace{1cm} (9)

where \( \Pi(u, v) \) is a translation operator on \( T^*Q \). When \( Q = \mathbb{R}^n \), it is given explicitly by

\[ \Pi(u, v) = e^{iu\hat{q} - iv\hat{p}} \]  \hspace{1cm} (10)

which form a ray representation of the Euclidean group. The inverse of this map is

\[ A^W := \int A_W(q, p) \Pi(u, v) e^{iuq - ivp} dudvdqdp \]  \hspace{1cm} (11)

and the twisted product is induced from the noncommutative product of operators

\[ (\hat{A} \hat{B})_W = A_W \ast B_W \]  \hspace{1cm} (12)

implying

\[(\hat{A}, \hat{B})_W = [A_W, B_W]_M \hspace{1cm} \{\hat{A}, \hat{B}\}_W = [A_W, B_W]_M^\dagger \]  \hspace{1cm} (13)

where \( \{\cdot, \cdot\} \) is the anticommutator. This relationship illuminates the standard Heisenberg-Dirac rule

\[ \{f, g\}_\text{PB} \to \frac{1}{i\hbar} [\hat{f}, \hat{g}] \]  \hspace{1cm} (14)

Furthermore, the Heisenberg-Dirac rule is known not always to work (the Groenwold-van Hove no-go theorem), \[4, 14\], whereas deformation quantisation always work, \[2, 18\].

For constrained systems, deformation quantisation was studied, probably for the first time, in \[1\]. Here I'll just give a brief outline.

Consider a flat phase-space \( \Gamma \) and a set of constraints \( \phi_a(q, p) \). Assume first of all that these are all first class, i.e.,

\[ \{\phi_a, \phi_b\}_\text{PB} = c_{ab}^c \phi_c \]  \hspace{1cm} (15)
and in involution with the Hamiltonian $h$, i.e.,

$$\{h, \phi_a\}_\text{PB} = V^b_a \phi_b \tag{16}$$

Then deformation quantisation consists in finding quantum constraints $\Phi_a$ and Hamiltonian $H$, such that

$$[\Phi_a, \Phi_b]_M = i\hbar c_{ab}^c \Phi_c \tag{17}$$
$$[H, \Phi_a]_M = i\hbar V^b_a \Phi_b \tag{18}$$

By assuming $\Phi_a, H$ to be analytic functions of the deformation parameter $\hbar$, i.e., that they can be Taylor expanded, the above Moyal brackets define a recursive scheme for finding $\Phi_a, H$. Furthermore, when $\phi_a, h$ are at most cubic in $p, q$ or has the form of a cubic polynomial plus a function of only one of the canonical variables, $q$, say, then we can pick $H = h, \Phi_a = \phi_a$.

If the constraints are second class, $\{\phi_a, \phi_b\}_\text{PB} = \chi_{ab} \neq c_{ab}^c \phi_c$, or not in involution with the Hamiltonian, $\{h, \phi_a\}_\text{PB} \neq V^b_a \phi_b$, then new quantum constraints and/or Hamiltonian, $\Phi_a, H$, can be found which are first class and in involution. The price one has to pay is the inclusion of negative powers of $\hbar$ in the formal power-series defining $H, \Phi_a$, it might also be necessary to allow the structure coefficients to receive quantum corrections. I.e., one can obtain

$$[\Phi_a, \Phi_b]_M = i\hbar \tilde{c}_{ab}^c \Phi_c \tag{19}$$
$$[H, \Phi_a]_M = i\hbar \tilde{V}^b_a \Phi_b \tag{20}$$

with

$$\Phi_a = h^{-1} \Phi_a^{(-1)} + \phi_a + h \Phi_a^{(1)} + ... \tag{21}$$
$$H = h + hH^{(1)} + ... \tag{22}$$
$$\tilde{c}_{ab}^c = c_{ab}^c + O(\hbar) \tag{23}$$
$$\tilde{V}^b_a = V^b_a + O(\hbar) \tag{24}$$

A similar trick can also take care of “anomalies” where $[\phi_a, \phi_b]_M = i\hbar \{\phi_a, \phi_b\}_\text{PB} + O(\hbar^3)$, at least when the quantum correction is a constant (i.e., the constraint algebra gets centrally extended when one naively uses the classical constraints in the Moyal brackets). See [1] for more details. It should be noticed that Hamachi independently has arrived at a similar “quantum smoothening” of anomalies, [15], albeit in somewhat simpler situations only applicable to toy
models, at least at the present state, but with a much higher level of mathematical rigour.

The classical condition picking out physical states is simply \( \phi_a(q, p) = 0, \forall a \).
In the standard Dirac quantisation picture, this would get replaced by \( \hat{\phi}_a|\psi\rangle = 0, \forall a \), where \( \hat{\phi}_a \) is some operator corresponding to \( \phi_a \), i.e., satisfying the same algebra but with Poisson brackets replaced by \((i\hbar)^{-1}\) times commutators.

In an improved version of the Dirac condition, the BRST-condition, one imposes instead \( \hat{\Omega}|\psi\rangle = 0 \) for a state \( |\psi\rangle \) and \([\hat{\Omega}, \hat{A}] = 0 \) for an observable \( \hat{A} \). Here \( \hat{\Omega} \) is the BRST-operator, \( \hat{\Omega} = \eta^a \hat{\phi}_a + \ldots \) where \( \eta^a \) are ghosts and where the commutator is understood to be graded appropriately, \([13, 1]\).

Deformation quantisation comes with another alternative. In \([1]\) it was proposed to use
\[
[\Phi_a, W]_M = 0, \quad \forall a
\] (25)
to pick out physical states \( W \) (Wigner functions), and similarly for other observables \( A \), \([\Phi_a, A]_M = 0, \forall a \). The Wigner-Weyl-Moyal formalism treats observables, states and transitions on an equal footing. The semi-classical limit of (25) is
\[
\phi_a W^{(0)} = 0
\] (26)
where \( \phi_a, W^{(0)} \) are the \( \hbar \to 0 \) limits of \( \Phi_a, W \) respectively. Thus
\[
W^{(0)} \propto \prod_a \delta(\phi_a)
\] (27)
i.e., \( W^{(0)} \) vanishes away from the constraint surface, at least in a distributional sense. We can also write this as
\[
\text{supp } W^{(0)} \subseteq \bigcap_a \ker \phi_a
\] (28)

The replacement of Poisson brackets by Moyal ones has been used by Strachan, Takasaki, Plebański and coworkers to study self-dual gravity and Yang-Mills theory, see e.g. \([16]\).

A few comments are in order. First, the replacement of Poisson brackets

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\(^1\)It ought to be emphasised that BRST is also subject to the Groenwald-van Hove no-go theorem, but it ameliorates the problem by using the underlying cohomological structure better that usual Dirac quantisation. The BRST symmetry is after all a classical symmetry, and the transition \( \Omega \to \hat{\Omega} \) is subject to more or less the same problems as the naive transition \( (q, p) \to (\hat{q}, \hat{p}) \). In both cases, the step of quantisation can probably be defined rigorously only through a deformation quantisation procedure in general.
by Moyal ones implies that the corresponding “gauge” transformations acquire quantum modifications. If the classical constraints are denoted by $\phi_a$ they generate (infinitesimal) “gauge” transformations $\delta_\omega f := \{\omega^a \phi_a, f\}_{PB}$, the corresponding quantum version is

$$\delta_\omega F := [\omega^a \Phi_a, F]_M = i\hbar \{\omega^a \phi_a, F\}_{PB} + \text{other terms}$$

which a priori differs from the classical expression. The discrepancy between the classical and the quantum “gauge” transformations show up in higher order derivatives, which seems to suggest that the quantum transformations are “larger”, i.e., slightly less local than their classical counterparts.

Another point to check is whether the space of physical quantities is invariant under such transformations. Consider thus an element $A$ satisfying $[\Phi_a, A]_M^+ = 0, \forall a$, when then wants to prove that a “gauge” transformation does not take us away from this subspace, i.e., $\delta_\omega [\Phi_a, A]_M^+ = 0, \forall a$. We get

$$\delta_\omega ([\Phi_a, A]_M^+) = [\delta_\omega \Phi_a, A]_M^+ + [\Phi_a, \delta_\omega A]_M^+ = [\omega^b c_{ba}^c \Phi_c, A]_M^+ + [\Phi_a, [\omega^b \Phi_b, A]_M^+]$$

the first term is zero provided $c_{ab}^c$ has vanishing Moyal brackets with $A$, e.g., if the structure coefficient is independent of the phase-space variables. The second term is also zero, as one can see by noting that $\Phi_a \ast A = -A \ast \Phi_a, \forall a$, since we can then rewrite the second term as

$$\delta_\omega ([\Phi_a, A]_M^+) = \omega^b ([\Phi_b, \Phi_a]_M, A]_M^+ = 0$$

Even if $c_{ab}^c$ depends upon the phase-space variables, as is the case, for instance, in gravity, then $\delta_\omega [\Phi_a, A]_M^+$ still vanishes since we have in general

$$\delta_\omega ([\Phi_a, A]_M^+) = \omega^b ([\Phi_b, \Phi_a]_M, A]_M^+ + ([\Phi_a, \Phi_b]_M, A]_M^+) = 0$$

Hence the condition $0 = [\Phi_a, A]_M^+$ is a consistent quantum analogue of $\phi_a = 0$, as we had anticipated.

It is argued in [?] that the approach put forward here is in fact compatible with the classical BRST-symmetry, and moreover, that the entire deformation quantisation procedure can be expressed in geometrical terms (through a sheaf over the real axis).

We will now apply this formalism to gravity.
2 ADM Variables

In the approach due to Arnowit, Desser and Misner, one splits up the metric as

\[ g_{\mu \nu} = \begin{pmatrix} N & N_i \\ N_j & g_{ij} \end{pmatrix} \] (33)

where \( N \) is known as the lapse function and \( N_i \) as the shift vector – these are Lagrange multipliers just like \( A_0 \) for the Yang-Mills case. The proper canonical variables are then the 3-metric \( g_{ij} \) (again, for Yang-Mills theory it is the 3-vector \( A_0 \)) and its conjugate momentum \( \pi^{ij} \). Hence the spacetime manifold has to be globally hyperbolic, \( M \simeq \Sigma \times \mathbb{R} \), where \( \Sigma \) is a spatial hypersurface. Consequently, the ADM-approach tells us that spacetime is to be considered not as a single four-dimensional entity but rather as a foliation by spatial hypersurfaces, i.e., \( M \) is the family \( \{ \Sigma_t \}_{t \in \mathbb{R}} \) where \( \Sigma_t \simeq \Sigma, \forall t \in \mathbb{R} \). The choice of “time” \( t \) is a gauge fixing, as are the choices of \( N, N_i \).

The action can be written as

\[ S = \int (\dot{g}_{ij}\pi^{ij} - N\mathcal{H}_\perp - N^i\mathcal{H}_i)dx \] (34)

where \( \mathcal{H}_\perp, \mathcal{H}_i \) are constraints depending only on \( g_{ij} \) and \( \pi^{ij} \) – the equations of motion of \( N, N_i \) gives the constraints \( \mathcal{H}_\perp = \mathcal{H}_i = 0 \).

\[ \mathcal{H}_\perp(x) = g^{-1/2}(\frac{1}{2}\pi^2 - \pi^i\pi^i) + \sqrt{g}R := G_{ijkl} \pi^{ij}\pi^{kl} + \sqrt{g}R \] (35)

\[ \mathcal{H}_i(x) = -2D_j \pi^j_i \] (36)

with \( g_{ij} \) the 3-metric, \( \pi^{ij} \) its conjugate momentum,

\[ \{g_{ij}(x), \pi^{kl}(x')\}_{PB} = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^k_j \delta^l_i)\delta(x, x') \] (37)

\( R \) the curvature scalar of \( g_{ij} \) (i.e., the three dimensional one) and \( g \) the determinant of the 3-metric. The first constraint is known as the Hamiltonian one, and the last, the \( \mathcal{H}_i \), as the diffeomorphism one. The algebra is

\[ \{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\}_{PB} = (g^{ij}(x)\mathcal{H}_j(x) + g^{ij}(x')\mathcal{H}_j(x'))\delta_{,i}(x, x') \] (38)

\[ \{\mathcal{H}_\perp(x), \mathcal{H}_i(x')\}_{PB} = \mathcal{H}_i(x)\delta_{,i}(x, x') \] (39)

\[ \{\mathcal{H}_i(x), \mathcal{H}_i(x')\}_{PB} = \mathcal{H}_i(x')\delta_{,i}(x, x') + \mathcal{H}_j(x)\delta_{,i}(x, x') \] (40)
where the subscript $\delta_i$ denotes the partial derivative with respect to $x^i$. The convention is the standard one in which $\delta(x, x')$ is a scalar in the first argument and a density in the second (the curved spacetime Dirac $\delta$ has a $g^{-1/2}$ in it).

The algebra of $H_i$ is $\text{diff}(\Sigma)$, the algebra of spatial diffeomorphism, i.e., the symmetry given by this first class constraint is the diffeomorphism symmetry of $\Sigma$. The Hamiltonian constraint generates “motion” away from one spatial slice $\Sigma_t$ to another $\Sigma_{t'}$, $t' \geq t$, i.e., the time evolution (with the given definition of time coordinate) of the three-manifold.

It is important to notice that the structure coefficients of the constraint algebra depend upon the phase-space variables (the 3-metric). Consequently, a naive canonical quantisation is very troublesome; when $g_{ij}, \pi^{ij}$ becomes operators the structure coefficients will no longer commute with the constraints, so in which order is one to write down the quantum constraint algebra, are the $g_{ij}$ to stand to the left or the right of the constraints? In order to ensure that time evolution does not take one away from the constraint surface, one has to demand that the constraints stand to the right of the structure coefficients on the right hand side of the constraint algebra, and it is this which makes a standard canonical quantisation of gravity so difficult.

Deformation quantisation does not care about such problems.

The algebra of the diffeomorphism constraint will not be deformed as their form is $H_i \sim \partial \pi + g^2 \pi$ and thus has $\omega_3 \equiv 0$. The Hamiltonian constraint, however, has as well a $g\pi^2$ as a $g^2 \pi$ term, and will consequently not have vanishing $\omega_3$. We should thus expect the algebraic relations involving $H_{\perp}$ to receive $\hbar^3$ corrections (but no higher order corrections since no higher powers of $\pi$ are present). This is precisely what we find. Moreover, the Christoffel symbols and the $\sqrt{g}$ contain, in a Taylor series, the metric to infinite order, whence we should expect infinite order equations to turn up.

In fact, gravity in the ADM approach with constraints $H_{\perp}, H_i$ is anomalous upon a deformation quantisation in the sense that

$$\begin{align*}
[&H_{\perp}(x), H_{\perp}(x')]_M = i\hbar\{H_{\perp}(x), H_{\perp}(x')\}_{\text{PB}} + i\hbar^3 k(x, x') \\
[&H_{\perp}(x), H_i(x')]_M = i\hbar\{H_{\perp}(x), H_i(x')\}_{\text{PB}} + i\hbar^3 k_i(x, x')
\end{align*}$$

whereas the spatial diffeomorphism subalgebra generated by the $H_i$ is non-anomalous.

A straightforward computation yields

$$k(x, x') = -\frac{1}{8} \left( (\varepsilon^{ijkl}_{\text{mnab}} \pi^{ab})(x) G^{mn}_{ijkl}(x') - (x \leftrightarrow x') \right)$$

(41)
with

\[ \Xi_{ijkl}^{mnab} = \frac{\delta^3 \mathcal{H}_\perp}{\delta g_{ij} \delta g_{kl} \delta \pi^{mn}} \]  
(42)

\[ = G_{mnab}(g^{ik} g^{jl} - \frac{1}{2} g^{ij} g^{kl}) - \frac{1}{2} g^{ij} \left( g_{mn} \delta^k_a \delta_l^b + g_{ma} \delta^k_n \delta_l^b + g_{mb} \delta^k_m \delta_l^a + g_{am} \delta^k_n \delta_l^b + g_{bm} \delta^k_m \delta_l^a + g_{an} \delta^k_n \delta_l^b \right) \]  
(43)

\[ G_{ijkl}^{mn} = \frac{\delta^3 \mathcal{H}_\perp}{\delta \pi^{ij} \delta \pi^{kl} \delta g_{mn}} \]  
(44)

\[ = -\frac{1}{2} g^{mn} G_{ijkl} + g^{-1/2} \left( g_{jl} (\delta^m_i \delta^n_k + \delta^m_k \delta^n_i) + g_{ik} (\delta_j^m \delta^n_l + \delta_j^n \delta^m_l) \right) - g_{kl} \delta^m_i \delta^n_j - g_{ij} \delta^m_k \delta^n_l \]  
(45)

One should note that \([\mathcal{H}_\perp, k]_M \neq 0\) hence we get an anomaly which is not a central extension of the original algebra. Explicitly

\[ [\mathcal{H}_\perp, k]_M = i \hbar \{\mathcal{H}_\perp, k\}_{PB} + i \hbar^3 \frac{3}{4} \Xi_{ijkl}^{mnab} \frac{\delta^3 k}{\delta \pi^{ij} \delta \pi^{kl} \delta g_{mn}} \neq 0 \]  
(46)

For the ADM constraints, the structure coefficients depend on the fields, consequently the anomaly too depends upon \((g, \pi)\).

Similarly, the relation mixing \(\mathcal{H}_\perp\) and \(\mathcal{H}_i\) receives a \(\hbar^3\) correction of the form

\[ k_i(x, x') \equiv -\frac{1}{8} \frac{\delta^3 \mathcal{H}_i(x)}{\delta \pi^{ij} \delta \pi^{kl} \delta g_{ab} \delta g_{jk} \delta g_{lm} \delta \pi^{ab}} \frac{\delta^3 \mathcal{H}_i(x')}{\delta \pi^{ij} \delta \pi^{kl} \delta g_{ab} \delta g_{jk} \delta g_{lm} \delta \pi^{ab}} \]  
(47)

which one easily finds to be

\[ k_i(x, x') = -\frac{1}{4} G_{jklm} \gamma_{ipq} \]  
(48)

with

\[ \frac{\delta^3 \mathcal{H}_i(x)}{\delta g_{jk}(x') \delta g_{lm}(x'') \delta \pi^{ab}(y)} = 2 \gamma_{jk}^{pq} (x, x', x'') \delta(x, y) \]  
(49)

\[ = \delta_{(a}^{\alpha} \delta_{b)}^{\beta} \delta(x, y) \left\{ \delta_{(a}^{\alpha} \delta_{b)}^{\beta} \delta(x, x'') (\delta_{c}^{\gamma} \delta_{d}^{\nu} \delta(x, x')) + \frac{1}{2} \delta_{(r}^{\gamma} \delta_{s}^{\nu} \delta_{b)} \delta_{c}^{\alpha} \delta_{d}^{\beta} \delta_{e}^{\mu} \delta_{f}^{\nu} \delta_{g}^{\mu} \delta_{h}^{\nu} \right\} \delta(x, x') + \delta_{(a}^{\alpha} \delta_{b)}^{\beta} \delta(x, x') \left( \delta_{c}^{\gamma} \delta_{d}^{\nu} \delta(x, x'') + \delta_{c}^{\alpha} \delta_{d}^{\beta} \delta_{e}^{\mu} \delta_{f}^{\nu} \delta_{g}^{\mu} \delta_{h}^{\nu} \right) \]  
(50)
\[
\frac{1}{2} g^{rs} \left( \delta^l_r \delta^m_s \partial_c + \delta^l_s \delta^m_r \partial_r - \delta^l_r \delta^m_s \partial_s \right) \delta(x, x') \left( x, x'' \right) - g_{ni} \delta(x, x'') \left[ g^{rl} g^{km} \Gamma^i_{rc} - g^{rk} g^{jm} \Gamma^i_{rc} \right] + \frac{1}{2} g^{rk} g^{js} \left( \delta^l_r \delta^m_s \partial_c + \delta^l_s \delta^m_r \partial_r - \delta^l_r \delta^m_s \partial_s \right) \delta(x, x') \right] \quad (c \rightarrow r, n \rightarrow c, r \rightarrow n) \quad (48)
\]

The spatial diffeomorphism subalgebra spanned by \( \mathcal{H}_i \) does not receive any quantum corrections since the constraints are only linear in the momentum.

We have relied on the following

\[
\begin{align*}
\frac{\delta g(x)}{\delta g_{ij}(x')} & = gg^{ij} \delta(x, x') \\
\frac{\delta g^{ij}(x)}{\delta \Gamma^i_{jk}(x')} & = -g^{ik} g^{jl} \delta(x, x') \\
\frac{\delta \Gamma^i_{jk}(x)}{\delta g_{mn}(x')} & = -g^{im} \Gamma^m_{jk} \delta(x, x') + \frac{1}{2} g^{kl} \left( \delta^m_i \delta^n_j \partial_l + \delta^m_k \delta^n_i \partial_j - \delta^m_i \delta^n_k \partial_l \right) \delta(x, x')
\end{align*}
\]

the first of which can be found in [11], and the last is a straightforward consequence of the first two relations.

### 2.1 Physical States

The set of physical states are defined as the functions \( W \) satisfying the two infinite order functional differential equations

\[
\begin{align*}
0 & = [\mathcal{H}_\bot, W]_M^+ \quad (49) \\
0 & = [\mathcal{H}_i, W]_M^+ \quad (50)
\end{align*}
\]

these are infinite order since the Christoffel symbols (and hence the covariant derivative and the curvature scalar) has an inverse metric in them, similarly the supermetric \( G_{ijkl} \) too has an inverse metric inside. Thus the constraints are not polynomial in the metric, but instead “meromorphic”.

Written out more explicitly, the physicality conditions read

\[
0 = 2\mathcal{H}_\bot W + \sum_{k=1}^{\infty} \left( -1 \right)^k 2^{-2k-1} \frac{h^{2k}}{(2k)!} \frac{\delta^{2k} \mathcal{H}_\bot}{\delta g_{ij_1 j_2} \cdots \delta g_{i_2 j_{2k-1} j_{2k-1}} \delta g_{mn}} \left( \delta^{2k} W \right) \frac{\delta^{2k} \mathcal{H}_\bot}{\delta \pi_{i_1 j_1} \cdots \delta \pi_{i_{2k-1} j_{2k-1}} \delta \pi_{mn}}
\]

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From this we get that the equations defining the physical state space become infinite order. If one assumes the Wigner function to be analytic in \( \hbar \), one can Taylor expand it \( W = \sum_{n=0}^{\infty} \hbar^n W_n \), and arrive at the following recursive formulas for the \( n \)'th order coefficients, \( W_n \):

\[
0 = 2\mathcal{H}_i W_0 = \mathcal{H}_{\perp} W_0
\]

\[
0 = 2\mathcal{H}_{\perp} W_N + \sum_{k=1}^{[N/2]} (-1)^k 2^{-2k-1} \frac{1}{(2k)!} \left( \frac{\delta^k \mathcal{H}_{\perp}}{\delta g_{i_1 j_1} \ldots \delta g_{i_{2k-1} j_{2k-1}} \delta \pi^{mn} \delta \pi^{i_1 j_1} \ldots \delta \pi^{i_{2k-1} j_{2k-1}}} \right) W_{N-2k}
\]

\[
0 = 2\mathcal{H}_i W_N + \sum_{k=1}^{[N/2]} (-1)^k 2^{-2k-1} \frac{1}{(2k)!} \left( \frac{\delta^k \mathcal{H}_i}{\delta g_{i_1 j_1} \ldots \delta g_{i_{2k-1} j_{2k-1}} \delta \pi^{mn} \delta \pi^{i_1 j_1} \ldots \delta \pi^{i_{2k-1} j_{2k-1}}} \right) W_{N-2k}
\]

where \( k, N \geq 1 \).

From this we get that the \( W_{2n} \) decouple from the \( W_{2n+1} \). Explicitly,

\[
W_0 \propto \delta(\mathcal{H}_1) \delta(\mathcal{H}_2) \delta(\mathcal{H}_3) \delta(\mathcal{H}_{\perp}) := \delta^4(\mathcal{H})
\]

\[
W_1 \propto \delta^4(\mathcal{H})
\]

\[
W_2 = \frac{1}{16\mathcal{H}_{\perp}} \left( \frac{\delta^2 \mathcal{H}_{\perp}}{\delta g \delta \pi} \frac{\delta^2 W_0}{\delta \pi^2 \delta g} - 2 \frac{\delta^2 \mathcal{H}_{\perp}}{\delta g^2} \frac{\delta^2 W_0}{\delta \pi^2} \right)
\]

\[
= \frac{1}{16\mathcal{H}_i} \frac{\delta^2 \mathcal{H}_i}{\delta g \delta \pi} \frac{\delta^2 W_0}{\delta \pi^2 \delta g}
\]
etc., where we have suppressed the indices on $g, \pi$. From the two expressions for $W_2$ we get another equation for $W_0$, namely

$$
\mathcal{H}_i \left( \frac{\delta^2 \mathcal{H}_\perp}{\delta g \delta \pi} \frac{\delta^2 W_0}{\delta \pi \delta g} - 2 \frac{\delta^2 \mathcal{H}_\perp}{\delta \pi^2} \frac{\delta^2 W_0}{\delta g^2} \right) = \mathcal{H}_\perp \frac{\delta^2 \mathcal{H}_i}{\delta g \delta \pi} \frac{\delta^2 W_0}{\delta \pi \delta g} \tag{60}
$$

This can be written as a condition on $W_0 = \frac{\delta W_0}{\delta g}$. We get

$$
\mathcal{F}_i(g, \pi) \frac{\delta W_0}{\delta \pi} - \mathcal{G}_i(g, \pi) \frac{\delta W_0}{\delta g} = 0 \tag{61}
$$

where

$$
\mathcal{F}_i = \mathcal{H}_i \frac{\delta^2 \mathcal{H}_\perp}{\delta g \delta \pi} - \mathcal{H}_\perp \frac{\delta^2 \mathcal{H}_i}{\delta \pi \delta g},
\mathcal{G}_i = 2 \mathcal{H}_i \frac{\delta^2 \mathcal{H}_\perp}{\delta \pi^2}
$$

Consequently, the equation (61) is integrable provided

$$
\frac{\delta \mathcal{F}_i}{\delta g} = -\frac{\delta \mathcal{G}_i}{\delta \pi} \tag{62}
$$

But this cannot be the case, since the right hand side is independent of $\pi$ ($\mathcal{H}_i$ is linear and $\mathcal{H}_\perp$ quadratic in $\pi$, since the latter is already differentiated twice with respect to $\pi$ the result follows) whereas the left hand side isn’t. Hence $W_0$ is not an exact form in $(g, \pi)$-space. The solutions of (61) and hence of the equation for $W_2$ are then parametrised by the non-trivial elements of the first cohomology class of $(g, \pi)$-space. Since this is an infinite dimensional space (being the cotangent bundle of Wheeler’s superspace of all 3-metrics) the computation of its cohomology is highly non-trivial, and we will not attempt it here.

The conclusion so far is then that in the ADM formalism gravity is anomalous when one attempts a deformation quantisation, and furthermore, that the physicality conditions are related to the cohomology classes of the cotangent bundle of superspace – an infinite dimensional manifold. I have so far not been able to lift the anomaly in this approach. Thus deformation quantisation of gravity in the ADM-variables is a highly non-trivial procedure, we will therefore turn to another description which shows more promise.
3 Ashtekar Variables

We saw that the anomalous nature of the quantum deformed algebra of the constraints in the ADM formalism were due to the constraints being non-polynomial. It is therefore interesting to consider another formulation, the Ashtekar variables [12], where the constraints are polynomials. It has also been suggested, [6], that Wigner functions are easier to define in the Ashtekar approach than in the ADM approach, thus hinting that the former are better suited for deformation quantisation purposes.

In four dimensions (and four dimensions only) with Lorentz signature (and not with Euclidean metric) we have an isomorphism between the Lorentz algebra (which is the local gauge algebra of gravitation) and $su_2 \otimes \mathbb{C}$,

$$so_{3,1} \simeq su_2 \otimes \mathbb{C}$$

Consequently, we can consider gravitation as a complexified $su_2$ gauge theory. In this formulation the canonical coordinates are a complex $su_2$-connection $A^a_i$ and its momentum (a densitized dreibein/triad) $E^i_a$ (i.e., $E^i_a E^j_b \delta^{ab} = gg^{ij}$), and the constraints are

$$H = F^a_{ij} E^i_b E^j_c \varepsilon^{bc}_a$$
$$G_a = D_i E^i_a$$
$$D_i = F^a_{ij} E^j_a$$

where $F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + \epsilon^{a}_{bc} A^b_i A^c_j$ is the field strength (or curvature) 2-form, and $D_i$ is the gauge covariant derivative, $F^a_{ij} \propto [D_i, D_j]$. As always, these quantities are three dimensional objects, i.e., $i, j = 1, 2, 3$ – once more a globally hyperbolic spacetime is assumed, $M \simeq \mathbb{R} \times \Sigma$, where $\Sigma$ is a Cauchy surface.

The first constraint, $H$, is referred to as the Hamiltonian, the $G_a$ as the Gauss and the $D_i$ as the diffeomorphism constraint. It will turn out that in these variables the anomaly is much simpler, namely merely a central extension. In fact, for gravity in the Ashtekar variables, the only anomalous bracket is

$$[\mathcal{H}(x), \mathcal{D}_i(x')]_M = i\hbar \{\mathcal{H}(x), \mathcal{D}_i(x')\}_{PB} - 9i\hbar^3 \delta_{i}(x, x')$$

Consequently the anomaly is a central extension and can be lifted.

This is seen as follows. Only the following two brackets can possibly receive
any quantum corrections, and then only to lowest order

\[ [\mathcal{H}(x), \mathcal{H}(x')]_M = \frac{3}{4} i \hbar^3 \left( \delta \mathcal{H}(x) \delta^3 \mathcal{H}(x') \right) \]

\[ (x \leftrightarrow x') \]

where we have suppressed the indices on the \( A, E \). An explicit and straightforward computation gives

\[ \omega_3(\mathcal{H}(x), \mathcal{H}(x')) = -12i \delta(x, x') \left( E^i_a(x) A^a_j(x') - E^i_a(x') A^a_j(x) \right) \]

\[ \omega_3(\mathcal{H}(x), D_m(x')) = 9i \delta_m(x, x') \]

We notice that the first of these vanish in the sense of distributions, hence the only quantum correction is the constant (w.r.t. the phase-space variables) \( 9i \delta_m(x, x') \). Consequently, the anomalous nature of gravity shows itself in the Ashtekar variables simply in a central extension of the constraint algebra (similar to a Schwinger term in current algebra).

\[ [\mathcal{H}(x), D_i(x')]_M = i \hbar \{ \mathcal{H}(x), D_i(x') \}_\text{PB} - \frac{3}{4} i \hbar^3 \delta^3 \mathcal{D}_i(x') \delta^3 \mathcal{D}_j(x') \delta^3 A_i \delta^3 A_j \delta^3 E \]

As mentioned earlier, such central extensions can be “lifted” by means of a redefinition of the quantum constraints.

Now, one can define Ashtekar variables also in \( d = 2 + 1 \) dimensions. Gravity in \( d \leq 3 \) is classically trivial, and what one gets in \( d = 2 + 1 \) in the Ashtekar approach is an \( SO(2, 1) \) Yang-Mills theory with flat gauge curvature, i.e., the constraints are

\[ \epsilon^{ij} D_i E^a_j = 0 \quad F^a_{ij} = 0 \]

Consequently, 2 + 1 dimensional Ashtekar gravity is not anomalous upon a deformation quantisation since no higher powers of the momenta occurs and hence \( \omega_3 \equiv 0 \). This shows that the anomaly is very much dependent on the dimensionality of spacetime.

### 3.1 Lifting the Anomaly

We will now attempt to find new quantum constraints \( H, D_i \) such that the anomaly vanishes (or rather, is absorbed into the redefinition of either the
constraints or the structure coefficients).

We will thus write

\[ H = \sum_{n \in \mathbb{Z}} h^n H_n \quad D_i = \sum_{n \in \mathbb{Z}} h^n D_i^{(n)} \quad (72) \]

with \( H_0 = \mathcal{H}, D_i^{(0)} = \mathcal{D}_i \).

The first Ansatz would naturally be to assume

\[ H = \sum_{n \in \mathbb{Z}} (\alpha_n h^n \mathcal{H} + h^n \beta_n) \quad D_i = \sum_{n \in \mathbb{Z}} a_n h^n \mathcal{D}_i \quad (73) \]

inspired by the lifting of the anomaly presented in [1]. A priori, one could have \( a_n \) be a matrix, but the recursion relations arising from this show that it will have to be proportional to the unit matrix.

Inserting (73) into

\[
\begin{align*}
[H(x), H(x')]_M &= i\hbar \left( g^{ij}(x) D_i(x) \delta_j(x, x') + (x \leftrightarrow x') \right) \\
[H(x), D_i(x')]_M &= i\hbar H(x) \delta_i(x, x')
\end{align*}
\quad (74)
\]

we get the following relations

\[
\begin{align*}
\alpha_k &= \sum_{n \in \mathbb{Z}} \alpha_n a_{k-n} \quad (76) \\
a_k &= \sum_{n \in \mathbb{Z}} \alpha_n \alpha_{k-n} \quad (77) \\
\beta_k &= -9 \sum_{n \in \mathbb{Z}} \alpha_n \alpha_{k-n-2} \quad (78)
\end{align*}
\]

subject to \( \alpha_0 = a_0 = 1, \beta_n = 0, n \leq 0 \). A solution is \( \beta_2 = -9, \alpha_0 = a_0 = 1 \) all other coefficients vanishing. This is the solution one would suspect looking at the anomalous bracket. Consequently

\[ H(x) = \mathcal{H}(x) - 9h^2 \quad (79) \]

and \( D_i(x) = \mathcal{D}_i(x) \). The anomaly represents, then, a zero-point energy or cosmological constant. Inflation can be interpreted as being related to negative zero-point energy, [19]. Thus the form of \( H(x) \) suggests that gravitational fluctuations can inflate and become macroscopic universes (a big bang scenario). Furthermore, the fact that the negative value is \( O(h^2) \) shows that is a very small quantity and will hence only be important in the very early universe, and/or at the Planck scale. At larger scales (corresponding to later times) it will be negligible.
3.2 Physical States

Since the constraints are polynomial in the phase-space variables the equations defining the physical state space, $\tilde{C}_{\text{phys}}$, become finite order differential equations. Explicitly, since the constraints are at most quartic in the phase-space variables we get

$$0 = [H, W]_{\tilde{\mathcal{M}}} + \frac{1}{2} \hbar^2 \left( E_b^k E_c^l \epsilon_{a}^{k} \epsilon_{f}^{l} \frac{\delta^2 W}{\delta E_k^a \delta E_l^f} + \epsilon_{a}^{bc} F_{ij}^{a} \frac{\delta^2 W}{\delta A_i^{a} \delta A_j^{b}} \right)$$

$$2 \epsilon_{a}^{bc} \left( - \delta_{i}^{a} (\delta_{j}^{k} \partial_j - \delta_{j}^{k} \partial_i) + \epsilon_{pq}^{a} (\delta_{i}^{p} \delta_{j}^{k} A_{q}^{p} + \delta_{i}^{q} \delta_{j}^{k} A_{p}^{q}) \right) \frac{\delta^2 W}{\delta E_k^a \delta A_i^{b}} \right)$$

$$0 = [D_{i}, W]_{\tilde{\mathcal{M}}} = 2 D_{i} W - \frac{1}{2} \hbar^2 \left( \epsilon_{ef}^{a} E_j^{f} \frac{\delta^2 W}{\delta E_i^{e} \delta E_j^{f}} - 2 \left( - \delta_{e}^{a} (\delta_{j}^{k} \partial_j - \delta_{j}^{k} \partial_i) + \epsilon_{mn}^{a} (\delta_{e}^{m} \delta_{i}^{k} A_{n}^{m} + \delta_{e}^{n} \delta_{i}^{k} A_{m}^{n}) \right) \frac{\delta^2 W}{\delta E_k^a \delta A_i^{b}} \right)$$

$$0 = [G_{a}, W]_{\tilde{\mathcal{M}}} = 2 G_{a} W + \frac{1}{4} i \hbar^2 \delta_{i}^{e} \epsilon_{ab}^{e} \frac{\delta^2 W}{\delta A_j^{a} \delta E_k^{b}}$$

These coupled equations constitute the equations for the Wigner function for Ashtekar gravity in vacuum. If we replace $H(x)$ by the quantum Hamiltonian $H(x)$, the only change in these equations will be the appearance of a $-18 \hbar^2 W$ term on the right hand side of (80).

3.3 Loop Formalism and Solutions

One of the most promising aspects of Ashtekar gravity is the loop transform of Rovelli and Smolin, [20]. It is in this formalism that the classical constraints can be solved seemingly opening up for a consistent quantisation. Furthermore, the loop states are closely related to Chern-Simons theory (to be expected considering Witten’s expression for knot-invariants) and to spin networks. Particularly important in the latter case is the appearance of a quantised spacetime structure in the sense of length, area and volume operators with discrete spectra. Consequently, a loop formulation of the above deformation quantisation is desirable.
We will first attempt a direct solution of the equations for a physical state and hence find $W$, we will then consider the only known solution, namely the Chern-Simons state $\psi[A]$ and then forms its Wigner function to see what conditions have to be imposed for this to be a solution.

By using the Gauss constraint equation we can rewrite the physicality equations as a pair of equations:

$$0 = (2\mathcal{H} + 4i\mathcal{G}^2)W - \frac{1}{2} \hbar^2 \left\{ i\epsilon^{bc} \epsilon^{gh} E_b^i E_c^j \frac{\delta^2 W}{\delta E_g^i \delta E_h^j} + \epsilon^{bc} F_{ij}^a \frac{\delta^2 W}{\delta A_i^a \delta A_j^c} \right\} +$$

$$\frac{5}{4} \hbar^4 \frac{\delta^4 W}{\delta A_i^a \delta A_j^b \delta E_a^i \delta E_b^j}$$  \hspace{1cm} (83)

$$0 = 2F_{ij}^a E_a^i W - \frac{1}{2} \hbar^2 \left( \frac{i\epsilon^{bc} F_{ij}^a}{\delta E_a^i \delta E_b^j} - 2D_i \frac{\delta^2 W}{\delta E_a^i \delta A_j^c} \right) \hspace{1cm} (84)$$

We will make the Ansatz

$$\frac{\delta^2 W}{\delta E_a^i \delta E_b^j} = \alpha_{ij}^{ab} W$$  \hspace{1cm} (85)

with $\alpha_{ij}^{ab}$ independent of $E$. Inserting this in the two constraint equations we arrive at

$$\alpha_{ij}^{ab} = 4\hbar^{-2} \left( \delta^{ab} D_i D_j + i\epsilon^{bc} F_{ij}^c \right)$$  \hspace{1cm} (86)

Thus\(^2\)

$$W[A, E] = \exp \left( 4\hbar^{-2} \int \delta^{ab}(D_i E_a^i(x))(D_j E_b^j(y)) + i\epsilon^{bc} \delta^{ij}(x, y) F_{ij}^c(x) E_a^i(x) E_b^j(y) dx dy + \ldots \right)$$  \hspace{1cm} (87)

where the remaining terms can be at most linear in $E$, hence we can write $W$ in symbolic notation as (invoking a generalised summation convention implying an integration over the continuous variables too)

$$W[A, E] = e^{\alpha_{ij}^{ab}[A] E_a^i E_b^j + \beta^a[A] E_a^i + \gamma[A]}$$  \hspace{1cm} (88)

Now, as was also found for $\alpha$, the coefficients $\beta, \gamma$ can only depend on $A$ and, moreover, must do so through covariant combinations. This implies (upto

\(^2\)For a Yang-Mills theory only the Gaussian constraints appear and the corresponding physicality condition can in fact be solved by a similar Gaussian Wigner function, namely $W[A, E] \sim \exp(-\alpha_a^{bc} F_{ij}^a E_i^a E_j^a)$ as shown in [?].
constant terms, which we’ll ignore)
\[ \beta_i^a = \beta \epsilon_i^{jk} F_{jk}^a \]
\[ \gamma = \gamma_1 \text{Tr} F^2 + \gamma_2 \text{Tr} \epsilon^{ijk} A_i F_{jk} \]
(89)
where \( \beta, \gamma_1, \gamma_2 \) are constants. Notice the appearance of the (three dimensional) Yang-Mills and Chern-Simons actions in \( \gamma \). Inserting this into the equations one can collect powers of \( E \) to find expressions for the coefficients, but as these are rather messy we will not do so here.

Notice, furthermore, that this solution gives us directly a Wigner function, whether this is the Weyl transform of a pure state or not, and if so, what this pure state is has not been answered. Secondly, we will start from a pure state, \( \psi[A] \), and then construct out of it a Wigner function. We will then investigate when this state gives rise to a physical solution.

Consider, then, a connection \( A \) and a function \( \psi[A] \). The formal Wigner function is
\[ W_{\psi}[A,E] := \int e^{i \int B \cdot E dx} \bar{\psi}[A + \frac{1}{2} B] \psi[a - \frac{1}{2} B] \text{D}B \]
(91)
We will also consider its Fourier transform \( E \rightarrow B \), which is just
\[ \tilde{W}_{\psi}[A,B] := \bar{\psi}[A + \frac{1}{2} B] \psi[a - \frac{1}{2} B] \]
(92)
The quantities \( A, B \) are both Lie algebra valued one forms, and we can perform a loop transform \( A \rightarrow \alpha, B \rightarrow \beta \), where \( \alpha, \beta \) are loops. Now, a general function \( \psi[A] \) of a connection can be mapped into a functional of a loop by means of the loop transform \( [20, 21, 22, 23] \)
\[ \psi[\alpha] := \int \psi[A] h[\alpha, A] \text{D}A \]
(93)
where \( h[\alpha, A] \) is the trace of the holonomy (the Wilson loop), i.e.,
\[ h[\alpha, A] := \text{Tr} Pe^{\int_{\alpha} A} \]
(94)
Doing this we can then define the following functions
\[ W_{\psi}[\alpha, E] := \int W_{\psi}[A, E] h[\alpha, A] \text{D}A \]
(95)
\[ \tilde{W}_{\psi}[\alpha, B] := \int \tilde{W}_{\psi}[A, B] h[\alpha, A] \text{D}A \]
(96)
\[ \tilde{W}_{\psi}[\alpha, \beta] := \int \tilde{W}_{\psi}[A, B] h[\alpha, A] h[\beta, B] \text{D}A \text{D}B \]
(97)
We will refer to any of these as a loop transform of the Wigner function $W_\psi$. Furthermore, we will usually omit the subscript $\psi$.

In terms of loop variables the classical Gauss constraint vanishes identically, $G_a = 0$. Hence, the quantum modified Gauss constraint equation, $[G_a, W]\hat{=}_M = 0$, reduces to

$$\epsilon^a_{bc} B^b_i \hat{T}^i_\alpha W[\alpha, B] = 0 \quad (98)$$

where we have used

$$\int \frac{\delta \tilde{W}[A, B]}{\delta A^a_i} h[\alpha, A] DA = - \int \tilde{W}[A, B] \frac{\delta}{\delta A^a_i} h[\alpha, A] DA$$

$$= - \int \tilde{W}[A, B] \int ds \delta(x, \alpha(s)) \dot{\alpha}^i(s) \hat{T}_a(s) h[\alpha, A] DA$$

$$:= - \hat{T}^i_\alpha \tilde{W}[\alpha, B] \quad (99)$$

which follows from the standard formula

$$\frac{\delta}{\delta A^a_i} h[\alpha, A] = \int ds \delta(x, \alpha(s)) \dot{\alpha}^i(s) \text{Tr} (e^{\int_x A^a_\tau})$$

$$:= \int ds \delta(x, \alpha(s)) \dot{\alpha}^i(s) \hat{T}_a(s) h[\alpha, A] \quad (100)$$

it being understood that the Lie algebra generator $\tau_a$ is inserted at the point $x = \alpha(s)$ along the loop. A possible solution to the quantum Gauss constraint equation is consequently quite simply

$$\dot{\alpha}^i \perp B^a_i \forall a$$

(102)

showing that $B^a_i$ provides a framing of the loop $\alpha$. It is very interesting that framing appears naturally in this formalism without the need of putting it in by hand. In the original relationship between Chern-Simons theory and knot invariants, $[24]$, the framing was needed in order to make the functional integrals convergent. This is a further suggestion that deformation quantisation automatically takes care of regularisation. It is already known that the twisted product can be seen as regularising the usual one, making products of $\delta$-functions possible. The possibility of using a Lie algebra-valued one form for framing the loop was already mentioned in $[20]$.

The next constraint equation to consider is the (spatial) diffeomorphism one,
Performing the Fourier transform $E \rightarrow B$ we get straightforwardly ($\hat{W} = \tilde{W}[A, B]$)

\[
0 = -2iF_{ij}^a \frac{\delta}{\delta B_j^a} \hat{W} + \frac{1}{2}i\hbar^2 \epsilon_{bc}B_i^bB_j^c \frac{\delta \hat{W}}{\delta B_j^a} + \frac{1}{2}i\hbar \left[ \epsilon^a(\delta_i^k \partial_j - \delta_j^k \partial_i) - \epsilon^a_{\ bc}(\delta_j^k \delta_i^b A_j^c + \delta_i^k \delta_j^b A_j^c) \right] \left( B_k^c \frac{\delta \hat{W}}{\delta A_j^a} \right) \tag{103}
\]

In order to be able to carry out the loop transform $A \rightarrow \alpha$, we must then compute the loop transform of $\hat{W}, F_i^a, \tilde{W}$ and $A \frac{\delta \hat{W}}{\delta A}$. These are found by brute force computations.

First

\[
\int A_i^a \hat{W}[A, B] h[\alpha, A] D A = \frac{1}{2} \frac{\delta}{\delta \hat{T}_a^i} \hat{T}_a^i \hat{W}[\alpha, B] \tag{104}
\]

From the definition of $F_{ij}^a$ as the covariant derivative of $A$ we get

\[
F_{ij}^a h[\alpha, A] = \left( \partial_i \frac{\delta}{\delta \hat{T}_j^a} \hat{T}_j^a - \partial_j \frac{\delta}{\delta \hat{T}_i^a} \hat{T}_i^a + \epsilon_{\ bc} \hat{T}_b^h \frac{\delta^2}{\delta \hat{T}_a^i \delta \hat{T}_j^b} \right) h \tag{105}
\]

This can also be written in terms of the area derivative, $\Delta_{ij}$, as [21, 22]

\[
F_{ij}^a = \Delta_{ij}(s) \hat{T}_a^i \tag{106}
\]

Putting all of this together we get the following expression for the quantum diffeomorphism constraint condition ($\hat{W} = \tilde{W}[\alpha, B]$)

\[
0 = 2i \frac{\delta}{\delta B_j^a} \Delta_{ij} \hat{T}_j^a \hat{W} + \frac{1}{2}i\hbar^2 \epsilon_{bc}B_i^bB_j^c \frac{\delta}{\delta B_j^a} \hat{W} - \frac{1}{2}i\hbar \left[ \epsilon^a(\delta_i^k \partial_j - \delta_j^k \partial_i) \left( B_k^c \frac{\delta \hat{W}}{\delta A_j^a} \right) + \epsilon^a_{\ bc} \frac{\delta^2}{\delta \hat{T}_a^i \delta \hat{T}_j^b} \hat{T}_b^h \hat{W} \right] \tag{107}
\]

Now, the first part of this is the usual Dirac version of the classical diffeomorphism constraint which is usually solved by demanding that $\tilde{W}[\alpha, B]$ be a knot invariant, such as the Chern-Simons state, $\psi[A] = \exp(-iS_{\text{Chern-Simons}}[A])$, [24, 20]. We see that we have slightly more freedom here. In fact we have two possibilities, either $\hat{W}$ is a knot invariant and this equation then gives a
condition for the \( B \)-dependence of \( \tilde{W} \), or \( \tilde{W} \) is allowed to be non-invariant, or rather a deformed knot-invariant, perhaps defined through the \( q \)-deformed Chern-Simons state of [27]. The Chern-Simons state leads to

\[
W[A,E] = \delta \left( E_i^a - \frac{k}{4\pi} \epsilon_j^i F_{jk}^a \right) \tag{108}
\]

\[
\tilde{W}[A,B] = e^{-i \frac{4\pi}{k} \int \mathrm{Tr} \epsilon^{ijk} B_i F_{jk} d^3 x} = e^{-i S_{BF}[A,B]} \tag{109}
\]

\[
\tilde{W}[\alpha,B] = \langle h[\alpha] \rangle_{BF} \tag{110}
\]

where \( S_{BF} \) is the action of the topological BF-theory, and \( \langle \cdot \rangle_{BF} \) denotes the corresponding expectation values. One should note that \( \tilde{W}[\alpha,B] \) is a knot-invariant, since it is diffeomorphism invariant (coming from a topological field theory) but it is not the usual Jones polynomial found by Witten, [24] (see also [10])

\[
\langle h[\alpha] \rangle_{CS} \propto c(k)^{-w(\alpha)} J_q(\alpha)
\]

with \( q = \exp\frac{2\pi i}{k+2} \) and \( w(\alpha) \) the writhe of the loop. That a relationship between BF-theory and knots exists was shown by Bimonte et al. in [26]. But the observables which Bimonte et al. show gives the reciprocal of the Alexander-Conway polynomials is not the Wilson loops which we consider here.

To get a feeling for the nature of these Wigner functions we can note that it is defined for any Lie algebra, and then consider the simplest possible example. For a \( U(1) \) theory we can actually arrive at an explicit result for \( \tilde{W}[\alpha,B] \) and thus see what it looks like, since

\[
\left. \tilde{W}[\alpha,B] \right|_{U(1)} = \int e^{-i \frac{k}{4\pi} \int \epsilon^{ijk} B_i F_{jk} d^3 x + i \int_0^1 \dot{\alpha}^i(s) A_i(\alpha(s)) ds} \mathcal{D} A = \int e^{-i \frac{k}{4\pi} \int \epsilon^{ijk} B_i F_{jk} d^3 x + i \int \Delta^i(\alpha,x) A_i(x) d^3 x} \mathcal{D} A = \delta(\Delta^i(\alpha,x)) + \frac{k}{4\pi} \epsilon^{ijk} \partial_j B_k
\]

Implying the “Maxwell equation” \( \nabla \times B = -\frac{4\pi}{k} \Delta \), i.e., the form factor \( \Delta^i \) acts like a source for the “magnetic” field. The “physical” interpretation of the Wigner functions in the Abelian case is the “Maxwell” equations \( E = \frac{k}{4\pi} \nabla \times A \) and \( \nabla \times B = -\frac{4\pi}{k} \Delta \). The first imply \( \nabla \cdot E := \rho = 0 \) whereas the latter imply conservation of the current \( j = \Delta, \nabla \cdot \Delta = 0 \). One would expect charges to
be located at the ends of curves, “field lines”, and since we only deal with loops \( \rho = 0 \). For \( SL_2(\mathbb{C}) \) we cannot find such a simple relationship.

For the Chern-Simons state, or, rather, the BF-state we can derive a few simple but useful identities. By direct computation one sees that

\[
\frac{\delta}{\delta A^a_i} \hat{W}[A, B] = -\frac{ik}{4\pi} \epsilon^{ijk} (D_k B_j)^a \hat{W}[A, B] \\
\frac{\delta}{\delta B^a_i} \hat{W}[A, B] = -\frac{ik}{4\pi} \epsilon^{ijk} F^a_{jk} \hat{W}[A, B]
\]

Inserting the BF-state into the Gauss constraint we get

\[
0 = -\frac{k}{2\pi} \epsilon^{ijk} (D_i F^a_{jk}) \hat{W} - \frac{k}{16\pi} \hbar^2 \epsilon^{ijk} \epsilon^a_{bc} D_j (B^b_i B^c_k) \hat{W}
\]

(111)

the first part, the classical contribution, vanishes by virtue of the Bianchi identity, thus confirming that the state is in fact (classically) gauge invariant. The second part then gives an equation the \( B \)-field has to satisfy, namely

\[
\epsilon^{ijk} \epsilon^a_{bc} D_i (B^b_i B^c_k) = 0
\]

(112)

One particular solution to this is \( \epsilon^a_{bc} B^b_i B^c_j \propto F^a_{ij} \). The Bianchi identity then ensures that \( \hat{W} \) satisfies the quantum Gauss constraint. In any case, we see that this condition is closely related to the cohomology of the space, since it says that \( B^b_i B^c_j \epsilon^a_{bc} \) is a Lie algebra valued two form, which is closed with respect to the covariant derivative. Consequently, if \( D^2 = 0 \) then a solution is for it to be \( D \)-exact. Now, since \( D^2 \omega = F \wedge \omega \) for any \( p \)-form \( \omega \), we see that only flat connections satisfy \( D^2 = 0 \). In that case, locally we have \( D \omega = 0 \Rightarrow \omega = D \chi \), whether this holds globally depends on the de Rham cohomology of the manifold and on the Lie algebra cohomology. Even if \( D^2 \neq 0 \), we can still find a solution by putting \( \epsilon^a_{bc} B^b_i B^c_j \propto F^a_{ij} \), since the Bianchi identity then ensures this to be a solution. The solution we will find which also satisfies the spatial diffeomorphism constraint equation will be precisely of this form.

For the spatial diffeomorphism constraint equation we similarly get (after the Fourier transform \( E \rightarrow B \), but before the loop transform \( A \rightarrow \alpha \)) by inserting the BF-state

\[
0 = \frac{k}{2\pi} \epsilon^{ijkl} \delta_{ab} F^a_{ij} F^b_{kl} + \frac{\hbar^2 k}{8\pi} \epsilon_{abc} \epsilon^{ijkl} B^b_\ell B^c_j F^a_{kl} + \frac{\hbar^2 k}{8\pi} \left[ \delta^a_(\delta^b_i \partial_j - \delta^b_j \partial_i) - \epsilon^a_{bc} (\delta^b_i \partial_i A^c_j + \delta^c_i \partial_j A^b_i) \right] (B^e_k \epsilon^{ijm} (D_m B_l)) (113)
\]
which can be rewritten using $\text{Tr} \tau_a \tau_b = 2 \delta_{ab}$ in the following way

$$0 = -\epsilon^{ijkl} \text{Tr} F_{ij} F_{kl} + \frac{1}{2} \hbar^2 \epsilon_{abc} \epsilon^{ijkl} B^b_i B^c_j F^a_{kl} +$$

$$\frac{1}{4} \hbar^2 \epsilon^{lmn} \text{Tr} \left[ (D_l B_j - D_j B_l)(D_m B_l) + B_i (D_j D_m B_l) - B_j (D_l D_m B_l) \right]$$

(114)

Using the Bianchi identity and making the Ansatz

$$B^a_i B^b_j \epsilon_{abc} = \alpha F^c_{ij}$$

(115)

this reduces to the following conditions

$$\alpha = 4 \hbar^{-2}$$

(116)

$$\text{Tr} \epsilon^{ijkl} (D_j (B_i D_m B_l) - D_i (B_j D_m B_l)) = 0$$

(117)

We will take the latter as a condition on $B$.

Moving on, and considering the Hamiltonian constraint we get, after a bit of algebraic manipulation, the following condition

$$0 = \frac{5 k^2}{8 \pi^2} \left( 1 + \alpha \hbar^2 \right) \epsilon_{abc} \epsilon^{ipq} \epsilon^{jrs} F^a_{ij} F^b_{pq} F^c_{rs} +$$

$$\frac{h^2 k^2}{32 \pi^2} \left( 1 + \frac{5}{2} \hbar^2 \alpha^{-1} \right) \epsilon_{abc} \epsilon^{ipq} \epsilon^{jrs} F^a_{ij} (D_p B_q) (D_r B_s)^c +$$

$$\frac{3}{2} \iota \hbar^2 k \delta_{abc} \epsilon^{ijk} B^a_i F^b_j + 27 \hbar^2$$

(118)

To find a solution to this we have to impose extra conditions of $B$ and $F$. The form of the Hamiltonian constraint equation suggests the following condition

$$(D_i B_j)^a = \beta F^a_{ij}$$

(119)

(that this is compatible with (117) follows by acting with $\epsilon^{ijk} D_k$ on (117))

and the equation then implies (where we have inserted the expression for $\alpha$)

$$\beta^2 = -20 \hbar^{-2} (1 + \frac{5}{8} \hbar^4)^{-1}$$

(120)

i.e., $\beta$ must be purely imaginary. The condition then also leads to (from the last line of (120))

$$\delta_{ab} \epsilon^{ijk} B^a_i F^b_j = \frac{21}{9} \hbar^{-1}$$

(121)
and thus the only states with a finite Wigner functions (i.e., the ones for which the BF-action is finite) are the ones of compact volume.

Now, the various conditions on $B$ and $A$ can be summarised in the following pair of equations

$$[B_i, B_j] = \frac{\alpha}{\beta} D_i B_j$$

$$\text{Tr} \, \epsilon^{ijk} B_i F_{jk} = \frac{2i}{9\beta k}$$

There is an interesting interpretation of this set of equations: The first states that $B_i$ is a kind of Maurer-Cartan form (on an associated bundle to the original $su_2$-principal bundle) and the second can be rewritten as $\text{Tr} \, \epsilon^{ijk} B_i [B_j, B_k] = \text{const.}$ which states that a certain cocycle (existing, by the way for any Lie algebra) is constant, corresponding to a notion of some kind of constant volume.

Thus the only spacetimes for which the Chern-Simons state is a solution to this set of quantum deformed physicality conditions is a space where the “dual” BF-theory is such that $B$ is an “associated Maurer-Cartan form”, which is the “squareroot of the field strength tensor” and where the volume of three-space is finite.

We will now turn to the general form of the constraints in the loop formalism. Performing the transformation $E \to B, A \to \alpha$, the Hamiltonian constraint equation can be written (for a general state, not just the Chern-Simons/BF-state)

$$0 = -2\epsilon_a^{bc} \frac{\delta^2}{\delta B_i^b \delta B_j^c} \Delta_{ij} \hat{T}^a \hat{W}[\alpha, B] -$$

$$\frac{1}{2} \hbar^2 \epsilon_a^{bc} \Delta_{ij} \hat{T}^a \hat{T}_i^j \hat{T}_c \hat{W}[\alpha, B] -$$

$$\frac{1}{2} \hbar^2 \epsilon_{abc} \epsilon^{aef} B_k^b B_l^c \delta^2 \delta B_k^e \delta B_l^f \hat{W}[\alpha, B] +$$

$$2\hbar^2 \epsilon_a^{bc} \partial_l \delta B_l^b \left( B_c^b \hat{T}_l^i \hat{W} \right) +$$

$$2\hbar^2 \epsilon_a^{bc} \epsilon_{bd} \epsilon^{ef} \delta \delta B_k^e \left( B_c^b \delta \delta \hat{T}^d \hat{T}_c^j \hat{W} \right) -$$

$$\frac{5}{4} \hbar^4 \epsilon_a^{bc} \epsilon_a^{ef} B_k^b B_l^c \hat{T}_e^k \hat{T}_l^j \hat{W}[\alpha, B]$$

(124)
where the last term actually vanishes upon imposing the Gauss condition. A physical, geometrical interpretation of this equation is difficult to give, and we will consequently move on.

Noting that $B$ is again a one-form, we can perform a second loop transform $B \rightarrow \beta$, thereby arriving at a Wigner function $\tilde{W}[\alpha, \beta]$. The Gauss condition then reads

$$e^{a}_{bc}\frac{\delta}{\delta \beta^i} \hat{S}^b \hat{T}^i_a \tilde{W}[\alpha, \beta] = 0 \quad (125)$$

where $\hat{S}^b$ is defined in the same way as $\hat{T}^b$ but with the loop $\beta$ replacing $\alpha$. Similarly the diffeomorphism constraint reads

$$0 = \frac{1}{8}h^2 e^{a}_{bc} \frac{\delta}{\delta \beta^k} \hat{S}^b \hat{T}^a \tilde{W} + \frac{1}{4}h^2 (\delta^i \partial_j - \delta^j \partial_i) \frac{\delta}{\delta \beta^k} \hat{S}^a \hat{T}^i_a \tilde{W}$$

$$(126)$$

while the Hamiltonian one becomes

$$0 = -2\hat{S}^j \Delta_{ij} \hat{T}^a \tilde{W} + \frac{1}{2}h^2 e^{a}_{bc} \frac{\delta}{\delta \beta^k} \hat{S}^b \hat{T}^c \hat{T}^i_a \tilde{W} - 15\hat{W} - h^2 \hat{S}^{k} \delta^{i} \partial_{j} \tilde{W}$$

$$(127)$$

These conditions are related to the entanglement of the loops (the equations do not separate, so $\tilde{W}[\alpha, \beta] = \tilde{W}_1[\alpha] \tilde{W}_2[\beta]$ is not a solution), but I haven’t been able to get any further with them. They are merely included for completeness. An interesting possibility presents itself though, namely of trying to define a kind of “regularised” composition of loops $\alpha \oplus \beta$ such that $\tilde{W}[\alpha, \beta] = \tilde{W}[\alpha \oplus \beta]$.

Presumably, one could gain some insight by using spin networks at this stage. In particular, there ought to be a close relationship between the formalism proposed here and the $q$-deformed spin networks introduced by Major and Smolin, [28], since the coupling constant in the Chern-Simons state is left
arbitrary in our approach, and it is this coupling constant which provides
the quantum deformation in the paper by Major and Smolin. But all of that
is beyond the scope of the present paper.

4 Ashtekar Gravity Lower Dimensions

In order to shed some light on the meaning of this formalism we will briefly
study the deformation quantisation of gravity in $d = 2 + 1$ and $d = 1 + 1$
dimensions. Now, these theories are classical trivial, the Einstein equations
amounting to Ricci flatness which in three dimensions imply flatness
(i.e., vanishing of the entire curvature tensor), and an empty set of equations
in two dimensions. Thus “gravity” in $d \leq 3$ is a theory without local physical
degrees of freedom. Most of the insights already gained are specific to the
proper physical dimensionality of four, but the topological aspects will turn
up much clearer in lower dimensions.

As already mentioned, the constraints in $d = 2 + 1$ are

\begin{align}
\epsilon^{ij} D_i E^a_j & = 0 \\
F^a_{ij} & = 0
\end{align}

which precisely state that the space is Ricci flat, and if the metric is non-
singular, the space is completely flat. Consequently, we will expect the metric
in a quantum theory to be trivial except in a number of isolated points
where the metric is singular. Note, furthermore, that the field $E^a_i$ is not a
zweibein/dyad but a $SO(2, 1)$-valued vector field ($i = 1, 2$ but $a = 1, 2, 3$).

The conditions for a state $W$ to be physical turn out to be

\begin{align}
0 & = [\epsilon^{ij} D_i E^a_j, W]_M = 2\epsilon^{ij} D_i E^a_j + \frac{2}{2} i\hbar^2 \epsilon^{ij} \epsilon_{bc} \frac{\delta^2 W}{\delta A^b \delta E^c_j} \\
0 & = [F^a_{ij}, W]_M = 2F^a_{ij} W - \frac{2}{2} i\hbar^2 g_{kj} g_{il} \epsilon^{abc} \frac{\delta^2 W}{\delta A^b_k \delta E^c_j} 
\end{align}

In a loop transformed formulation the first will again give the $B$-field Fourier-
dual to $E$ to be framing the loop. The only difference from $d = 3 + 1$ is that
the condition will now be $B^a_i \delta_j \epsilon^{ij} = 0$ and not $B^a \perp \dot{\alpha}$. In fact the loop plus
BF formulation (i.e., $A \rightarrow \alpha$, $E \rightarrow B$) of these constraints is

\begin{align}
0 & = \epsilon^{ij} \epsilon_{bc} B^b_k \tilde{T}^c_j \tilde{W} \\
0 & = 2\Delta_{ij} \tilde{T}^a \tilde{W} + \frac{1}{2} i\hbar^2 \epsilon^{abc} B^b_j \tilde{T}^c_i \tilde{W}
\end{align}
which can be reduced to simply (by contracting with $\epsilon^{ij}$)

$$\Delta_{ij} \hat{T}^a \bar{W} = 0 \quad (134)$$

stating that $\bar{W}$ is a diffeomorphism invariant of a framed loop.

Loops in two dimensions, i.e., on some surface of genus $g$, can only depend on the homotopy class of the loop, suggesting that $\bar{W}$ is a homotopy invariant. Consequently, $\bar{W}$ can only depend on a loop $\alpha$ through its winding numbers around the $g$ different holes in the surface, i.e., $\bar{W}$ depends on $g$ integers $n_1, ..., n_g$ which are the winding numbers of the loop $\alpha$. This is also what the classical analysis shows, [25], but this is hardly surprising since we have just seen that the quantum constraints reduce to their classical counterparts, plus a relationship between the field $B$ and the loop. The only possible extra quantum modification is $E^a_i$, and hence the metric $g_{ij}$, to be singular at a finite number of points. In that case $\bar{W}$ will also depend on the residues at these points. In a two-loop formalism (i.e., $B \rightarrow \beta$), the Wigner function can only depend on the intersection number of the two loops besides their homotopy class.

We cannot carry the analogy with $d = 3 + 1$ gravity any further since the Chern-Simons state does not exist in two dimensions. The BF-state does, but the $B$ field is then a zero-form and thus not related at all to the electric field $E$ on the 2d surface. In the full 2 + 1 dimensional spacetime manifold one can take $B \sim E$ to get $S_{BF} \sim S_{EH}$ where $S_{EH}$ is the Einstein-Hilbert (or rather Palatini) action, which is itself a BF-theory in $d = 2 + 1$ dimensions. For $d = 1 + 1$, the Lorentz algebra becomes the Abelian algebra $so(1,1)$, hence the constraints becomes simply

$$\partial E = 0 \quad F = \partial A = 0 \quad (135)$$

which implies that the quantum physicality conditions reduces to their classical part

$$W \propto \delta(\partial E)\delta(\partial A) \quad (136)$$

Thus, the theory is completely trivial classically as well as quantum theoretically and is not worth spending any more time on.

5 Conclusion

We have seen that a deformation quantisation of gravity is possible, although anomalies turn up in as well the ADM as the Ashtekar formulation. In the
latter, however, the anomaly is merely a central extension and hence liftable. In any case, the presence of an anomaly signals the breakdown of diffeomorphism invariance. This can either imply (1) the presence of a non-vanishing zero point energy, or (2) the appearance of a scale below which classical gravitation fails. In any case, it shows that the “time evolution” constraint, the Hamiltonian one, is no longer described by a scalar on the spatial hypersurface $\Sigma$. Hence, the quantum version of it must contain some information which the classical doesn’t – this could be a preferred direction, a scale or an origin. In the first case, we would expect the constraint to be vector-like, but that does not seem to be the case.

We showed that a solution could be found by assuming a Chern-Simons state $\psi[A]$ (which then gave rise to a BF-state) even for $\Lambda = 0$, but only if the $B$-field of the corresponding BF-theory was an “associated Maurer-Cartan form” and if the volume of three-space was finite.

In the general loop formalism, the field $B$ – the Fourier transformed of the dreibein – became related to the imbedding of the loop. In a two-loop formalism this were formulated as the necessity of entanglement of the two loops. One should also note that for the Chern-Simons state, the formal Wigner function becomes a knot invariant, closely related to the usual Jones polynomial. It is also worth noticing that framing of loops appeared naturally, was in fact imposed by the quantum modified Gauss constraint, in this formalism, and didn’t have to be introduced by hand in order to give well-defined expectation values.

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