Limitations of the Hyperplane Separation Technique for Bounding the Extension Complexity of Polytopes

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Abstract

This note illustrates the limitations of the hyperplane separation bound, a non-combinatorial lower bound on the extension complexity of a polytope. Most notably, this bounding technique is used by Rothvoß (J ACM 64.6:41, 2017) to establish an exponential lower bound for the perfect matching polytope. We point out that the technique is sensitive to the particular choice of slack matrix. For the canonical slack matrices of the spanning tree polytope and the completion time polytope, we show that the lower bounds produced by the hyperplane separation method are trivial.

1 Introduction

The extension complexity of a polytope $P$, denoted by $xc(P)$, is the minimum number of facets of any polytope $Q$ that affinely projects onto $P$. A linear description of such a polytope $Q$ (together with the corresponding projection) is an extended formulation of $P$. If we define the size of an extended formulation as the number of its inequalities, the minimum size of any extended formulation of $P$ equals $xc(P)$.

Building on Yannakakis’ seminal work [31], there has recently been a renewed interest in the study of extended formulations (see, e.g., [1,8,11–13,18,19,24,25]). For many polytopes associated with NP-hard combinatorial optimization problems, we now know that their extension complexity cannot be bounded by a polynomial in their dimension; among them are TSP polytopes, cut and correlation polytopes, and stable set polytopes [12,19]. An exponential lower bound also holds for the extension complexity of the (perfect) matching polytope [25] (even though one can optimize over it in polynomial time). Well-known polytopes that do admit nontrivial polynomial-size extended formulations include, among many others, parity polytopes [2,31], independence polytopes of regular matroids [1], and two families of polytopes considered here, spanning tree polytopes and completion time polytopes. We refer to the surveys by Conforti et al. [4] and Kaibel [16] for an overview and more examples.

The spanning tree polytope of a connected graph $G = (V,E)$ is the convex hull of the incidence vectors of the spanning trees in $G$,

$$P_{st}(G) := \text{conv} \{\chi(T) \in \{0,1\}^E : T \subseteq E \text{ is a spanning tree in } G\},$$  \hspace{1cm} (1)

where $\chi(T)$ denotes the incidence vector of $T$. Although $P_{st}(G)$ has exponentially many facets in general, there are extended formulations of size $O(|V| \cdot |E|)$ due to Wong [30] and Martin [21] (see also [5,31]). Special classes of graphs admit even smaller extended formulations: For

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Supported by the Alexander von Humboldt Foundation with funds from the German Federal Ministry of Education and Research (BMBF).
instance, Williams [28] gives a formulation of size $O(|V|)$ for planar graphs. Some progress has also been made for graphs of bounded genus, more generally, by Fiorini et al. [10].

On the other hand, it is known that the extension complexity of a polytope is at least its dimension [11]. Thus, if $K_n$ is the complete graph on $n$ vertices, $\Omega(n^2)$ is a trivial lower bound on the extension complexity of $P_{st}(K_n)$. The question whether this bound can be improved is open [27]. Khoskhah and Theis [20] show that every combinatorial lower bound (that is, one that depends only on the vertex-facet incidence structure of the polytope and, thus, is unable to distinguish between combinatorially equivalent polytopes [11]), achieves at most $O(n^2 \log n)$. In [20], the authors ask whether using non-combinatorial techniques instead may lead to stronger lower bounds.

One candidate is the hyperplane separation bound proposed by Fiorini [9] and applied by Rothvoß [25] in his proof of the exponential lower bound for the matching polytope. It is a lower bound on the extension complexity of a polytope $P$ that essentially depends on the coefficients in a given linear description of $P$. We show that, for Edmonds’ [7] canonical description of $P_{st}(K_n)$, the hyperplane separation technique fails to produce a lower bound stronger than $\Omega(n^2)$. In this sense, the trivial dimension bound is already at least as strong. Our proof in Section 3 relies on a dual interpretation of the method, which will be explained in Section 2.

At the same time, we stress that our result does not rule out the possibility of obtaining meaningful bounds for a description of $P_{st}(K_n)$ by a different system of linear inequalities. To the best of the author’s knowledge, this issue has not been addressed explicitly in the study of the hyperplane separation technique. In particular, we consider a description of $P_{st}(K_n)$ obtained from the canonical one by suitably scaling the inequalities. While scaling in this particular way does improve on the hyperplane separation bound, we are able to prove that it can only do so by a factor of at most $O(n \log n)$.

The limitations of the hyperplane separation method can be observed in another family of well-understood polytopes as well. Consider $n$ jobs with processing times $p = (p_1, \ldots, p_n) \in \mathbb{R}_{\geq 0}^n$ to be scheduled on a single machine. Every permutation $\pi \in S_n$ (the symmetric group on $[n]$) defines a feasible schedule without idle time where job $j$ is completed at time $C_{\pi}^j := \sum_{i=1}^{n} p_{\pi^{-1}(i)}$ for $j = 1, \ldots, n$. The completion time polytope $P_{ct}(p)$ is defined as

$$P_{ct}(p) := \text{conv}\{(C_{\pi}^1, \ldots, C_{\pi}^n) \in \mathbb{R}^n : \pi \in S_n\}.$$

Wolsey observed (see remark in [17]) that $P_{ct}(p)$ is a zonotope, the affine linear image of a hypercube with $n(n-1)$ facets. In fact, no smaller extended formulation of $xc(P_{ct}(p))$ is known to date. In case that $p_j = 1$ for all $j = 1, \ldots, n$, $P_{ct}(p)$ is known as the $n$th permutohedron and equals $\text{conv}\{(\pi(1), \ldots, \pi(n)) : \pi \in S_n\}$. For this polytope, Goemans [15] gives an asymptotically minimal extended formulation of size $\Theta(n \log n)$. The lower bound in [15] is established via a purely combinatorial argument. Since any two completion time polytopes on $n$ jobs are combinatorially equivalent for strictly positive processing times, $\Omega(n \log n)$ is therefore best possible for any combinatorial lower bound on $xc(P_{ct}(p))$, for any $p \in \mathbb{R}_{\geq 0}^n$.

In Section 4, we show that, regardless of $p$, the hyperplane separation bound for the canonical linear description of $P_{ct}(p)$ due to Wolsey [29] and Queyranne [23] is at most a constant. In fact, we obtain our result in the more general setting of graphic zonotopes, a natural generalization of completion time polytopes inspired by Wolsey’s observation.
2 Slack matrices and the hyperplane separation bound

Given a nonnegative matrix $S \in \mathbb{R}^{m \times n}_{\geq 0}$, the nonnegative rank of $S$, denoted by $\text{rk}_+(S)$, is defined as the minimum $r \in \mathbb{N}$ such that $S = UV$ for two nonnegative matrices $U \in \mathbb{R}^{m \times r}_{\geq 0}, V \in \mathbb{R}^{r \times n}_{\geq 0}$. Equivalently, it is the minimum $r \in \mathbb{N}$ such that $S$ can be written as the sum of $r$ nonnegative matrices of rank one [3].

Consider a polytope $P = \text{conv}(X) = \{x \in \mathbb{R}^n : Ax \leq b, x \in \text{aff}(P)\}$ for some finite set $X = \{x^1, \ldots, x^p\} \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{f \times n}, b \in \mathbb{R}^f$ such that every inequality in $Ax \leq b$ defines a nonempty face of $P$. The $f \times v$ matrix whose $j$th column equals $b - Ax^j$ is a slack matrix of $P$. If $X$ is the set of vertices of $P$, we refer to the corresponding slack matrix as the slack matrix of $P$ with respect to the linear description above. In particular, any slack matrix of a polytope is a nonnegative matrix whose nonnegative rank satisfies the following property due to Yannakakis [31].

**Proposition 1.** Let $S$ be a slack matrix of a polytope $P$. Then $\text{xc}(P) = \text{rk}_+(S)$.

This result is the key to many techniques for bounding the extension complexity of $P$. This paper is concerned with one such technique. For two matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}$, we denote by $\langle A, B \rangle := \sum_{i=1}^m \sum_{j=1}^n a_{ij}b_{ij}$ their Frobenius inner product and let $\|A\|_\infty := \max_{i,j} |a_{ij}|$.

**Proposition 2** (Hyperplane separation bound [25]). Let $S \in \mathbb{R}^{m \times n}_{\geq 0}$ with at least one positive entry, and let $\mathcal{R}$ denote the set of rank-one matrices in $\{0,1\}^{m\times n}$. We further let

$$\text{hsb}(S) := \sup \left\{ \frac{\langle S, X \rangle}{\|S\|_\infty \rho(X)} : X \in \mathbb{R}^{m \times n} \right\},$$

(2)

where $\rho(X) := \max \{\langle X, R \rangle : R \in \mathcal{R}\}$ for every $X \in \mathbb{R}^{m \times n}$. Then $\text{rk}_+(S) \geq \text{hsb}(S)$.

Normalizing $X$ such that $\rho(X) = 1$ in the definition of hsb$(S)$, we may rewrite (2) as follows:

$$\|S\|_\infty \text{hsb}(S) = \sup \left\{ \langle S, X \rangle : X \in \mathbb{R}^{m \times n}, \rho(X) = 1 \right\}$$

$$= \sup \left\{ \langle S, X \rangle : X \in \mathbb{R}^{m \times n}, \rho(X) \leq 1 \right\}$$

$$= \max \left\{ \langle S, X \rangle : X \in \mathbb{R}^{m \times n}, \langle X, R \rangle \leq 1 \forall R \in \mathcal{R} \right\}. \quad (3)$$

In the last step, we used the fact that the supremum of $\langle S, \cdot \rangle$ is finite: Any $X \in \mathbb{R}^{m \times n}$ with $\rho(X) \leq 1$ satisfies $\langle X, R \rangle \leq 1$ for all $R$ with singleton support, that is, every entry of $X$ is at most one. As $S$ is nonnegative, the sum of its entries is an upper bound on $\langle S, X \rangle$.

Note that (3) is a linear program (LP). From strong LP duality, we obtain the following dual characterization of the hyperplane separation bound, which already appears in [26], although derived differently.

**Proposition 3.** In the situation of Proposition 2 we have that

$$\text{hsb}(S) = \min \left\{ \|S\|_\infty^{-1} \sum_{R \in \mathcal{R}} y_R : y \in \mathbb{R}_{\geq 0}^{\mathcal{R}}, \sum_{R \in \mathcal{R}} y_R R = S \right\}. \quad (4)$$

The feasible region of the LP in (4) corresponds to a particular type of nonnegative factorization of $S$, namely the decomposition of $S$ into the weighted sum of 0/1 matrices of rank one. Not only will this observation be the key ingredient of our proofs in Sections 3 and 4, it also motivates an alternate proof of Proposition 2 which is slightly simpler than the original one in [26].
Proof of Proposition 3. Without loss of generality, we may assume that \( \|S\|_\infty = 1 \). Let 
\[ S = \sum_{k=1}^{r} A_k \] 
for \( r \in \mathbb{N} \) and rank-one matrices \( A_k \in [0,1]^{m \times n}, k = 1, \ldots, r \). We claim that for every \( k = 1, \ldots, r \), we have that \( A_k \in \text{conv}(R) \), i.e., 
\[ A_k = \sum_{R \in R} y_R R \] 
for some coefficients \( y_R \geq 0, \sum_{R \in R} y_R = 1 \). Then \( y \in \mathbb{R}^R \) defined by 
\[ y_R = \sum_{k=1}^{r} y_R, \quad R \in R, \] 
is a feasible solution of the LP in (4), and \( \text{hsb}(S) \leq \sum_{R \in R} y_R = r \) by Proposition 3.

It remains to prove the claim. Let \( A \in [0,1]^{m \times n} \) be of rank one. Then \( A = vw^T \) for some \( v \in \mathbb{R}^m, w \in \mathbb{R}^n \), which, by scaling, can be assumed to be \([0,1]\)-valued vectors. Then \( v \in [0,1]^m \) can be written as a convex combination \( v = \sum_{i=1}^{p} \lambda_i v^i \) for some \( v^i \in \{0,1\}^m, \lambda_i \geq 0 \) and \( \sum_{i=1}^{p} \lambda_i = 1 \). Similarly, \( w = \sum_{j=1}^{q} \mu_j w^j \) for some \( w^j \in \{0,1\}^n, \mu_j \geq 0 \) and \( \sum_{j=1}^{q} \mu_j = 1 \). Then

\[ A = vw^T = \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j \cdot v^i(w^j)^T \] 
and \( \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_i \mu_j = 1 \).

Note that \( \text{hsb}(S) \) is invariant under multiplying \( S \) by positive scalars, under transposition, and under permutations of rows and columns of \( S \), respectively. It further satisfies the following two useful properties on submatrices, both of which are immediate consequences of Proposition 3.

Lemma 1. Let \( S = (A \ B) \) for nonnegative matrices \( A \) and \( B \). Then

(i) \( \text{hsb}(S) \leq \text{hsb}(A) + \text{hsb}(B) \),

(ii) \( \|S\|_\infty \text{hsb}(S) \geq \|A\|_\infty \text{hsb}(A) \).

Recall that any two slack matrices of a given polytope have identical nonnegative rank. (This is a consequence of Proposition 4) In this sense, the nonnegative rank is well-defined for polytopes. The situation for the hyperplane separation bound, however, is fundamentally different. Let us highlight this difference with two examples.

Consider the standard hypercube \( C_n = [0,1]^n \) and let \( S_n \) denote its slack matrix w.r.t. the (minimal) description \( C_n = \{ x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \ i = 1, \ldots, n \} \). The inequality \( \sum_{i=1}^{n} x_i \geq 0 \) is valid for \( C_n \) (defining the vertex \( 0 \in \mathbb{R}^n \)). Adding this inequality to the minimal description of \( C_n \) adds one row to \( S_n \), which equals the sum of the rows corresponding to the facets defined by \( x_i \geq 0 \) for \( i = 1, \ldots, n \). Let \( S'_n \) denote the slack matrix with this additional row. Then \( \|S'_n\|_\infty = n \) and it is not difficult to check that \( \|S_n\|_\infty \text{hsb}(S_n) = \|S'_n\|_\infty \text{hsb}(S'_n) \). Thus, \( \text{hsb}(S'_n) = \frac{n}{\lambda} \text{hsb}(S'_n) \).

Not even slack matrices w.r.t. minimal linear descriptions behave identically under the hyperplane separation bound: The \( n \)-simplex spanned by the canonical unit vectors in \( \mathbb{R}^n \) and the origin is the set of all \( x \in \mathbb{R}^n \) satisfying \( x_1 + \cdots + x_n \leq 1, \ x_i \geq 0 \) for \( i = 1, \ldots, n - 1 \), and \( \lambda x_n \geq 0 \) for any \( \lambda \geq 1 \). Every inequality defines a facet of the simplex. Modulo permutations of rows and columns, the associated slack matrix \( S_{n,\lambda} \) is obtained from the \( (n + 1) \times (n + 1) \) identity by multiplying the first row by \( \lambda \). One can show that \( \text{hsb}(S_{n,\lambda}) = \frac{n}{\lambda} + 1 \) while \( \text{rk}(S_{n,\lambda}) = n + 1 \).

Motivated by the latter example, let us consider the effect of normalizing the rows of a slack matrix independently. Note that this leaves the nonnegative rank unchanged.

Lemma 2. Let \( S \in \mathbb{R}^{n \times n}_{\geq 0} \) with rows \( s^i \in \mathbb{R}^n \), \( i = 1, \ldots, m \), and suppose that every row contains at least one positive entry. Let \( S' \) denote the matrix obtained from \( S \) by dividing the \( i \)-th row by \( \|s^i\|_\infty \). Then

\[ 1 \leq \frac{\text{hsb}(S')}{\text{hsb}(S)} \leq \|S\|_\infty \sum_{\delta \in \Delta} \frac{1}{\delta}, \] 
where \( \Delta := \{ \|s^i\|_\infty : i = 1, \ldots, m \} \).
**Theorem 1.** Let $E$ where

(i) using parts $\setminus ij$ every spanning tree $T$ in (4). Our construction is inspired by Martin’s extended formulation [21]. For

| $T$ is a bipartite graph, both vertex classes in a bipartition are stable sets in $T$ |

| $3$ The spanning tree polytope |

Proof. We clearly have that $\|S’\|\infty = 1$. Let $X \in \mathbb{R}^{m \times n}$ be a feasible solution of the LP in (3) for $S$ (and, thus, for $S’$ too). Denoting the $i$th row of $X$ by $x^i$, we obtain

$$\langle S’, X \rangle = \sum_{i=1}^{m} \|s^i\|\infty^{-1}(s^i)^T x^i \geq \sum_{i=1}^{m} \|S\|\infty^{-1}(s^i)^T x^i = \|S\|\infty^{-1}\langle S, X \rangle.$$ |

Now observe that $\Delta$ defines a partition of $S$ into row submatrices $S^{ij}$ consisting of maximum norm $\delta \in \Delta$. The corresponding row submatrix of $S’$ is $\frac{1}{\delta} S^{ij}$. It follows that

$$\text{hsb}(S’) \leq \sum_{\delta \in \Delta} \text{hsb}\left(\frac{1}{\delta} S^{ij}\right) = \sum_{\delta \in \Delta} \text{hsb}(S^{ij}) \leq \|S\|\infty \text{hsb}(S) \sum_{\delta \in \Delta} \frac{1}{\delta},$$

using parts [1i] and [ii] of Lemma 1 in the first and second inequality, respectively. \(\square\)

By transposition, an analogous statement holds true for normalizing columns instead of rows.

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**3 The spanning tree polytope**

Let $G = (V, E)$ be a connected graph. The spanning tree polytope of $G$ given in [1] is completely described by the following system due to Edmonds [7]:

$$P_{st}(G) = \{x \in \mathbb{R}_\geq 0^E : x(E) = |V| - 1, x(E(U)) \leq |U| - 1 \quad \forall \emptyset \neq U \subseteq V\},$$

where $E(U)$ is the set of all edges with both endpoints in $U$. We will denote an edge $\{i, j\}$ by $ij$. Further, let $V(H)$ denote the set of vertices of a subgraph $H$ of $G$ and let $c(U, F)$ for $U \subseteq V$ and $F \subseteq E$ denote the number of connected components of the subgraph $(U, F \cap E(U))$.

**Theorem 1.** Let $G = (V, E)$ be a connected graph and let $S_G$ denote the slack matrix of $P_{st}(G)$ w.r.t. the description (5). Then

$$\text{hsb}(S_G) \leq O(|E|).$$

Proof. Since there are $|E|$ many nonnegativity constraints in (5), it suffices to consider the row submatrix of $S_G$ restricted to the set inequalities in (5) only, which will be denoted by $S_G$ again. The bound for the entire slack matrix then follows from Lemma 1[i].

Indexing the rows of $S_G$ by the nonempty subsets of $V$ and the columns by the spanning trees in $G$, the entry in row $U \subseteq V$ and column $T$ equals $c(U, T) - 1$. First, observe that

$$\|S_G\|\infty \geq \frac{1}{2} |V| - 1.$$ |

For, if $T$ is a spanning tree in $G$ and $U \subseteq V$ a stable set in $T$, then $c(U, T) = |U|$. Because $T$ is a bipartite graph, both vertex classes in a bipartition are stable sets in $T$. At least one of them is of size $|V|/2$.

We shall now construct a nonnegative factorization of $S_G$ which is feasible in the sense of the dual LP in (4). Our construction is inspired by Martin’s extended formulation [21]. For every spanning tree $T$ in $G$, let $\tau(T)$ be the set of all triples of vertices $(i, j, k) \in V^3$ such that $ij \in E$ is an edge of the unique $i$-$k$ path in $T$. From each nonempty subset $U \subseteq V$, we choose an arbitrary representative $k(U) \in U$. For every triple $(i, j, k) \in V^3$ where $ij \in E$, define the set

$$R(i, j, k) := \{U \subseteq V : i \in U, j \notin U, k = k(U)\} \times \{T \text{ spanning tree: } (i, j, k) \in \tau(T)\}.$$
For every such triple \((i,j,k)\), there is a unique 0/1 matrix indexed in the same way as \(S_G\) whose support equals \(R(i,j,k)\). We claim that these matrices, which clearly are of rank at most one, add up to \(S_G\). Indeed, let \(\emptyset \neq U \subseteq V\) and \(T\) be a spanning tree in \(G\). Letting \(c := c(U,T)\), it suffices to show that

\[
\left| \left\{ (i,j,k) \in V^3 : ij \in E, \ (U,T) \in R(i,j,k) \right\} \right| = c - 1.
\]

If \(c = 1\), the statement is clear. Let \(c \geq 2\), and let \(F_1, \ldots, F_c\) be the connected components of the subgraph \((U,T \cap E(U))\). Without loss of generality, we may assume that \(k(U) \in V(F_c)\). For \(l = 1, \ldots, c - 1\), we say that a path in \(T\ connects\ \(k(U)\) and \(F_1\) if its two endpoints are \(k(U)\) and some vertex in \(V(F_1)\) and no other vertex on the path belongs to \(V(F_l)\). For every \(l = 1, \ldots, c - 1\), there exists a unique path in \(T\ connecting\ \(k(U)\) and \(F_1\). Let \(i_l\) be its endpoint in \(V(F_l)\), and let \(j_l\) be the neighbour of \(i_l\) on the path. Then \(j_l \notin U\), and \((U,T) \in R(i_l,j_l,k(U))\) for every \(l = 1, \ldots, c - 1\).

On the other hand, if \((U,T) \in R(i,j,k)\) for some \((i,j,k) \in V^3\) with \(ij \in E\), then \(k = k(U)\) and \(i \notin V(F_c)\), say, \(i \in V(F_1)\). Since \(j \notin U\), the path connecting \(i\) and \(k(U)\) in \(T\) cannot visit any other vertex in \(V(F_1)\). Hence, it connects \(k(U)\) and \(F_1\) and we conclude that \((i,j) = (i_l,j_l)\).

This shows that the sets \(R(i,j,k)\) induce a decomposition of \(S_G\) into \(2|E| \cdot |V|\) summands which are 0/1 matrices. From \([6]\) and Proposition \([3]\) we conclude that

\[
\text{hsb}(S_G) \leq \frac{2|E| \cdot |V|}{|V|/2 - 1} = O(|E|).
\]

In the light of the previous section, let us briefly discuss how normalizing the rows of the slack matrix \(S_G\) defined above may strengthen the hyperplane separation bound. Note that the entries of \(S_G\) are in \([0,1,\ldots,|V| - 2]\). Normalizing \(S_G\) row by row, we obtain another slack matrix \(S_G'\) with \(\|S_G'\|_\infty = 1\). Lemma \([2]\) then implies that

\[
\frac{\text{hsb}(S_G')}{\text{hsb}(S_G)} \leq (|V| - 2) \sum_{k=1}^{\lfloor |V|/2 \rfloor} \frac{1}{k} = O(|V| \cdot \log |V|).
\]

It is easy to see that, if \(G\) is the complete graph, \(S_G'\) is the slack matrix of \(P_{\text{el}}(G)\) w.r.t. the description obtained from \([5]\) by dividing every set inequality for \(|U| \geq 2\) by \(|U| - 1\).

### 4 Graphic zonotopes

Given two sets \(X, Y \subseteq \mathbb{R}^n\), their **Minkowski sum** is \(X + Y := \{ x + y : x \in X, y \in Y \}\). A **zonotope** is the affine linear image of a hypercube. Equivalently, every zonotope is the Minkowski sum of a finite number of line segments, where a line segment in \(\mathbb{R}^n\) is a set \([x, y] := \text{conv}(\{x, y\})\) for some \(x, y \in \mathbb{R}^n\). Given a graph \(G = (V, E)\) on \(V = [n]\), a **graphic zonotope** of \(G\) is the Minkowski sum of line segments in the directions \(\{w^j - w^i\}_{ij \in E}\) (see \([22]\)), where \(w^i\) denotes the \(i\)th canonical unit vector in \(\mathbb{R}^n\). Let \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) be a symmetric nonnegative matrix. We associate with \(A\) a zonotope \(Z(A) \subseteq \mathbb{R}^n\) as follows:

\[
Z(A) := \sum_{1 \leq j \leq n} a_{jj} w^j + \sum_{1 \leq i < j \leq n} a_{ij} [w^i, w^j].
\]  

(7)

Up to translations, the graphic zonotopes of graphs on \(n\) vertices are exactly those of the above form for some symmetric and nonnegative matrix \(A\) (where \(a_{ij} > 0\) if and only if \(ij \in E\).
We will now derive a description of the facets of \( Z(A) \), generalizing remarks in [32] Example 7.15 and [14]. To this end, define the set function \( g_A : 2^{[n]} \to \mathbb{R} \) by

\[
[n] \ni S \mapsto g_A(S) := \sum_{i,j \in S: i \leq j} a_{ij}.
\]

Note that \( g_A \) is supermodular, and it is strictly supermodular if and only if \( A \) is positive. The supermodular base polytope (see, e.g., [14]) of a supermodular function \( g : 2^{[n]} \to \mathbb{R} \) with \( g(\emptyset) = 0 \) is defined as

\[
B(g) := \{ x \in \mathbb{R}^n : x([n]) = g([n]), x(S) \geq g(S) \forall S \subseteq [n] \}.
\]

**Lemma 3.** Let \( A \in \mathbb{R}^{n \times n} \) be symmetric and nonnegative. Then \( Z(A) = B(g_A) \).

**Proof.** It suffices to show that, for every linear functional \( w \in \mathbb{R}^n \), the minima of \( w \) over \( Z(A) \) and \( B(g_A) \) coincide. After a permutation of the coefficients of \( w \), we may assume that \( w_1 \geq \cdots \geq w_n \). The greedy rule (see [3]) then implies that a minimizer \( \pi \) over \( B(g_A) \) is given by

\[
\pi_j := g_A([j]) - g_A([j - 1]) = \sum_{i=1}^{j} a_{ij}, \quad j = 1, \ldots, n.
\]

Minimizing \( w \) over the zonotope \( Z(A) \) can be done over each summand in the Minkowski sum in (7) individually. For \( 1 \leq i < j \leq n \), it is easy to see that the minimum of \( w \) on the line segment \([a_{ij}w_i^j, a_{ij}w_i^j]\) is attained in the first endpoint since \( w_i \geq w_j \). Hence the minimum over \( Z(A) \) is attained in the point \( \pi \).

**Lemma 4.** For every symmetric positive matrix \( A \in \mathbb{R}^{n \times n} \), the \( n \)th permutahedron and \( Z(A) \) are combinatorially equivalent.

**Proof.** From the proof of Lemma 3, we conclude that the vertices of \( Z(A) \) and the permutations in \( \mathcal{S}_n \) correspond via the map

\[
\mathcal{S}_n \ni \pi \quad \mapsto \quad x^{\pi} \in \mathbb{R}^n; \quad x^{\pi}_j = \sum_{i=1}^{\pi(j)} a_{\pi^{-1}(i),j}, \quad j = 1, \ldots, n.
\]

Since \( A \) is positive, this is a bijection. Moreover, \( g_A \) is strictly supermodular and therefore, all inequalities in (7) for \( \emptyset \neq S \subseteq [n] \) define facets of \( Z(A) \) [23].

Now let \( \pi \in \mathcal{S}_n \) and \( \emptyset \neq S \subseteq [n] \). Let \( x^{\pi} \) denote the vertex of \( Z(A) \) induced by \( \pi \) via the bijection in (9). Then

\[
x^{\pi}(S) - g_A(S) = \sum_{i \in [n], j \in S: \pi(i) \leq \pi(j)} a_{ij} - \sum_{i \in [n], j \in S: \pi(i) \leq \pi(j)} a_{ij} = \sum_{i \notin S, j \in S: \pi(i) \leq \pi(j)} a_{ij},
\]

using symmetry of \( A \) in the first equation. Since \( A \) is positive, it follows that \( x^{\pi} \) belongs to the facet defined by \( S \) if and only if \( \pi(S) = [|[S]|] \). In other words, (9) is a bijection between the vertices of \( Z(A) \) and those of the \( n \)th permutahedron which preserves all vertex-facet incidences.

Given the structural insights above, it is not difficult to recognize that graphic zonotopes do indeed generalize completion time polytopes: If \( A = pp^T \) for some column vector \( p \in \mathbb{R}^n \), then \( Z(A) \) is the image of \( P_{ct}(p) \) under the linear transformation which sends a vector \( x \in \mathbb{R}^n \) to
to its componentwise product with $p$. Note that $A$ has rank one in this case. Conversely, every symmetric positive $A \in \mathbb{R}^{n \times n}$ of rank one can be written as an outer product of some positive vector with itself. Indeed, if $A = vw^T = A^T$ with $v, w \in \mathbb{R}^n_{> 0}$, then $w_1v = v_1w$ by symmetry. Letting $\rho := \sqrt{v_1/w_1} > 0$, we obtain $A = (pw)(pw)^T$. Thus, up to a coordinate transformation, the completion time polytopes for strictly positive processing times are exactly the zonotopes $Z(A)$ for symmetric positive rank-one matrices $A$.

Moreover, this generalization is compatible with both the upper and lower bounds on the extension complexity known for completion time polytopes. Recall that $Z(A)$, as defined in (7), can be written as the affine linear image of the hypercube $[0,1]^{n(n-1)/2}$ and, hence, $\text{xc}(Z(A)) \leq n(n-1)$. If $A$ is positive, $\Omega(n \log n)$ is a lower bound on $\text{xc}(Z(A))$ by combining Lemma 4 and the lower bound for the $m$th permutahedron in (15).

Let us now study the hyperplane separation bound in the case of $Z(A)$ for symmetric and nonnegative $A \in \mathbb{R}^{n \times n}$. To this end, let $M_A$ denote the slack matrix of $Z(A)$ w.r.t. its linear description (8). In what follows, we shall identify the rows of $M_A$ with the nontrivial subsets of $[n]$ and the columns with the permutations in $S_n$.

**Lemma 5.** Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive, and let $M_A$ be the slack matrix of $Z(A)$ w.r.t. (8). Then

$$
\|M_A\|_\infty = \max_{S \subseteq [n]} \sum_{i \notin S, j \in S} a_{ij}.
$$

**Proof.** From (10), we conclude that the entry in row $S \subseteq [n]$ and column $\pi \in S_n$ of $M_A$ equals

$$
x^\pi(S) - g_A(S) = \sum_{i \notin S, j \in S: \pi(i) \leq \pi(j)} a_{ij} \leq \sum_{i \notin S, j \in S} a_{ij}
$$

with equality if and only if $\pi([n] \setminus S) = [n - |S|]$. \hfill \Box

**Theorem 2.** Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive, and let $M_A$ be the slack matrix of $Z(A)$ w.r.t. (8). Then

$$
\text{hsb}(M_A) \leq \|M_A\|_\infty^{-1} \sum_{i \neq j} a_{ij} \leq 4.
$$

**Proof.** For every pair $i, j \in [n], i \neq j$, let

$$
R(i, j) := \{S \subseteq [n]: i \notin S, j \in S\} \times \{\pi \in S_n: \pi(i) \leq \pi(j)\}
$$

and let $\hat{R}(i, j)$ denote the unique 0/1 matrix indexed like $M_A$ whose support equals $R(i, j)$. Note that $\hat{R}(i, j)$ has rank one and, by (10),

$$
\sum_{i \neq j} a_{ij} \hat{R}(i, j) = M_A.
$$

The first inequality in the statement then follows from Proposition 3.

In order to show the second inequality, let $S^* \subseteq [n]$ be a subset attaining the maximum in the equation in Lemma 5. We claim that

$$
\sum_{i \in S^* \setminus \{j\}} a_{ij} \leq \sum_{i \notin S^*} a_{ij} \quad \text{for every } j \in S^*.
$$

Indeed, suppose that there were some $k \in S^*$ such that $\sum_{i \in S^* \setminus \{k\}} a_{ik} > \sum_{i \notin S^*} a_{ik}$. Letting $S' := S^* \setminus \{k\}$ and using the symmetry of $A$, we obtain

$$
\sum_{i \notin S^* \setminus S'} a_{ij} - \sum_{i \notin S^* \setminus S'} a_{ij} = \sum_{j \in S^* \setminus \{k\}} a_{kj} - \sum_{i \notin S^*} a_{ik} > 0,
$$

and thus

$$
\sum_{i \notin S^*} a_{ij} > \sum_{i \notin S^*} a_{ij},
$$

contradicting the maximality of $S^*$. \hfill \Box
contradicting the choice of \( S^* \). From (11), it follows that

\[
\sum_{i,j \in S^*: i \neq j} a_{ij} = \sum_{j \in S^*} \sum_{i \in S^* \setminus \{j\}} a_{ij} \leq \sum_{j \in S^*} \sum_{i \in S^* \setminus \{j\}} a_{ij} = \|M_A\|_\infty.
\]

The symmetric argument for \([n] \setminus S^*\) yields

\[
\sum_{i \neq j} a_{ij} = \sum_{i,j \in S^*: i \neq j} a_{ij} + \sum_{i,j \in S^*: i \neq j} a_{ij} + 2 \sum_{i \neq S^*: j \in S^*} a_{ij} \leq 4\|M_A\|_\infty.
\]

This completes the proof.

We conclude this section with a remark on the normalized slack matrix proposed in Lemma 2. More precisely, let us revisit the special case of the \( n \)th permutahedron. Recall that this is the zonotope \( Z(A) \) where \( A \) is the \( n \times n \) all-one matrix. From Lemma 5 and the proof thereof, we have that the slack matrix \( M_A \) as defined above satisfies \( \|M_A\|_\infty = \lfloor n/2 \lfloor n/2 \rfloor \) and the set \( \Delta \) of row maxima of \( M_A \) equals \( \Delta = \{ k(n-k): k = 1, \ldots, n-1 \} \). One can further show that

\[
\sum_{\delta \in \Delta} \frac{1}{\delta} = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{k(n-k)} = \Theta \left( \frac{\log n}{n} \right).
\]

From Lemma 2 we conclude that \( hsb(M'_A) \leq O(n \log n) \), where \( M'_A \) is the slack matrix obtained from \( M_A \) by normalizing rows independently. Recall that \( \text{rk}_+(M'_A) = \Theta(n \log n) \) [15].

5 Concluding remarks

For both families of polytopes studied in this note and their canonical linear descriptions, we have shown that the hyperplane separation technique is unable to improve on the currently best known lower bounds on their extension complexity. In contrast to the nonnegative rank, the hyperplane separation bound depends on the choice of slack matrix. By making a more careful choice, it is conceivable that the technique does indeed yield more meaningful bounds than the ones in Sections 3 and 4.

In particular, the rows and columns of a given nonnegative matrix can be scaled in such a way that the maximum entry in every row and column equals one. While preserving the nonnegative rank, this strengthens the hyperplane separation bound as argued in Section 2. In other words, the hyperplane separation method produces the strongest lower bounds for slack matrices which have been scaled in this way.

How much can one gain by this? Although Lemma 2 attempts to provide an answer to this question, it is not clear to the author whether the ratio in Lemma 2 attains the given upper bound when applied to the polytopes considered in this note and their canonical slack matrices. For instance, assuming this to hold true (up to a multiplicative constant) for the permutahedron, the hyperplane separation method would be capable of confirming Goemans’ lower bound [15] in a non-combinatorial way. In the case of the spanning tree polytope of \( K_n \), the improvement factor gained by scaling rows in the sense of Lemma 2 is at most \( O(n \log n) \). Of course, the hyperplane separation bound of any slack matrix of \( P_{st}(K_n) \) is at most \( O(n^3) \).

Acknowledgements. The author is grateful to Andreas S. Schulz and Stefan Weltge for helpful discussions and comments.
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