N-body Efimov states of trapped bosons

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Abstract – We demonstrate the possibility of existence of meta-stable N-body Efimov states in trapped Bose systems with large scattering length. We calculate spectra of trapped systems of \( N = 3, 4, 5, 6, \) and 7 bosons using a stochastic variational method with a restricted correlated Gaussian basis. For each system the calculations reveal a series of Efimov states where the energy and the r.m.s. radius exhibit the characteristic exponential dependence upon the state number. We also estimate the contribution of these states to the recombination rate of Bose-Einstein condensates.

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Introduction. – The Efimov effect [1] appears in quantum three-body systems when attractive interactions between at least two pairs of particles are such that the scattering length is much larger than the range of the interaction; in other words two of the three two-body subsystems are close to the threshold of binding. Under these conditions a characteristic series of weakly bound and spatially extended states, called Efimov states [2], appears in the system. These states appear due to specific long-range two-body correlations between particles caused by the large scattering length.

The effect is easiest to see in the hyper-spherical adiabatic approximation where the slow adiabatic variable is the hyper-radius \( \rho \) (root square radius of the system).

It has been shown that close to the two-body threshold the effective adiabatic potential \( W(\rho) \) is attractive and asymptotically proportional to the inverse square of the hyper-radius,

\[
W(\rho) = -\frac{\hbar^2}{2m} \frac{\xi^2 - \frac{1}{4}}{\rho^2},
\]

where \( m \) is the mass scale, and \( \xi \) is a constant depending on masses of the particles [2–4].

A sufficiently large positive \( \xi^2 \) leads to a geometric series of bound states with exceedingly small energies, \( E_n \propto e^{-\xi n} \), and exceedingly large root mean square radii, \( R_n \propto e^{\frac{1}{2}\xi n} \), where \( n \) is the state number and \( \xi = \frac{2\pi}{\zeta} \).

There is a general theoretical consensus [4–6] that the first excited state of the helium trimer \( ^4\text{He}_3 \) is an Efimov state, although the experimental observation so far proved elusive [7].

The three-body Efimov effect has recently got some support [8] from an experiment with trapped Bose gases where the recombination rate was measured as a function of the scattering length. The latter was varied using the Feschbach resonance technique by applying an external magnetic field. A sharp peak in the recombination rate was detected and interpreted as a three-body Efimov resonance in qualitative agreement with theoretical predictions [9].

It has been shown that in an \( N \)-body system with \( N > 3 \) the Efimov effect does not exist at the \( N - 1 \) threshold [10]. At the two-body threshold the \( N \)-body Efimov states with \( N > 3 \) cannot exist either as the clusters with 3 and higher number of particles are generally deeply bound at this point. The Efimov states would then be unstable due to lower lying thresholds and would decay into deeply bound cluster states.

However it has been suggested in [11] that a sequence of meta-stable \( N \)-body states with the characteristic exponential energy dependence can yet show up at the two-body threshold. Using the \( N \)-body hyper-spheric method it has been shown that an \( N \)-body system at the two-body threshold has a hyper-spheric adiabatic potential with inverse-square dependence. This peculiar adiabatic potential appears due to the same mechanism as for three particles and thus gives rise to \( N \)-body Efimov states with a structure similar to that of three-body Efimov states: an (otherwise) uncorrelated system with very specific two-body correlations caused by the large scattering lengths.
This specific hyper-spheric adiabatic potential is not the lowest one as different bound clusters with lower thresholds create lower-lying adiabatic potentials. However, although not truly bound, these $N$-body Efimov states might still exist as meta-stable states slowly decaying into clusters, much like the Bose-Einstein condensate states. The structure of the Efimov states is determined by the long-range two-body correlations and is thus quite dissimilar to the structure of clustered states with short-range many-body correlations. Therefore the overlap between Efimov states and clustered states should be small and consequently the lifetime should be large.

The conclusions about meta-stable $N$-body Efimov states were obtained in [11] in an extreme hyper-spheric adiabatic approximation where the couplings to all other channels were neglected. In this letter we report on a more realistic calculation of $N$-body Efimov states with a different method, namely the restricted correlated Gaussian stochastic variational method where no adiabatic approximation is assumed.

**The system and the method.** – We consider a system of $N$ identical bosons with mass $m$ and coordinates $r_i$ in a spherical harmonic trap with frequency $\omega$, where the scattering length is assumed to be tuned to a large value, facilitating the Efimov effect.

The Hamiltonian of the system is given by

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial r_i^2} + \sum_{i<j} V(|r_i - r_j|) + \frac{m\omega^2}{2} \sum_{i=1}^{N} r_i^2,$$

where the system parameters are $m = 86.909$ u, and the trap length $b_t = \sqrt{\hbar/(m\omega)} = 46189$ au.

The two-body potential is an attractive Gaussian with the range $b = 11.65$ au and depth $1.24825$ au corresponding to the scattering length of $a = -1.4 \times 10^{6}$ au $\gg b_t$.

The wave function of the system is represented as a linear combination of $K$ basis-functions taken in the form of symmetrized correlated Gaussians,

$$\Psi = \hat{S} \sum_{k=1}^{K} C_k \exp \left( -\frac{1}{2} \sum_{i<j}^{N} \alpha_{ij}(k)(r_i - r_j)^2 \right),$$

where the total angular momentum is zero, $\hat{S}$ is the symmetrization operator, and $C_k$ and $\alpha_{ij}(k)$ are variational parameters. The linear parameters $C_k$ are determined by diagonalization of the Hamiltonian while the non-linear parameters $\alpha_{ij}(k)$ are optimized stochastically [12,13] by random sampling from a region that covers the spatial distances from $b$ to $b_t$.

During calculations of a given system the number of Gaussians in the basis is increased and the stochastic optimization is carried out until all energy levels of interest are converged.

With uncorrelated Gaussians the method is equivalent [13] to a mean-field approximation which is unable to describe the Efimov effect and which has the validity condition $na^3 \ll 1$. Introducing successive correlations beyond the mean-field improves this validity condition and allows calculations of the Efimov states.

The structure of the $N$-body Efimov states is analogous to that of the three-body Efimov states: the spatial extension of these states is much larger than the range of the potential $b$, hence the density of the system is small, $nb^3 \ll 1$; and there are no cluster substructures. For low-density systems, $nb^3 \ll 1$, only two-body correlations are of importance.

Allowing only two-body correlations, the variational wave function can be simplified as

$$\Psi_{2b} = \hat{S} \sum_{k=1}^{K} C_k \times \exp \left( -\frac{1}{2} \alpha^{(k)}(r_i - r_j)^2 \right),$$

where $\rho^2 = \sum_{i<j}(r_i - r_j)^2$ is the hyper-radius and $\alpha^{(k)}$ and $\beta^{(k)}$ are the nonlinear parameters. The symmetrization of this function can be done analytically [13] which greatly simplifies the numerical calculations.

This form of the variational space proved very successful in describing the Bose-Einstein condensate states [14] which have similar structure. The clustered states, uninteresting in the present context, are explicitly excluded while the crucial two-body correlations responsible for the existence of the Efimov states are retained. Otherwise it would be impossible to calculate the Efimov states due to the enormous number of cluster thresholds and clustered states in an $N$-body system. Omitting the clustered states, into which the Efimov states would decay, effectively disregards the widths of the Efimov states. However, the clustered states differ significantly in spatial structure and therefore should overlap very little with the computed Efimov states. Indeed an estimate [15] of the lifetimes of the molecular three-body Efimov states showed that their widths are extremely small and that they can be considered bound for all practical purposes.

**Results.** – The energies of Efimov states in a trap should be, on the one hand, much larger than the typical energy scale of a cluster state, $\hbar^2/(2m\omega)$, and, on the other hand, smaller than the oscillator energy $\hbar\omega = \frac{\hbar^2}{m\omega^2}$.

We have calculated the spectrum of trapped boson systems with the Hamiltonian equation (2) with $N = 3, 4, 5, 6,$ and $7$. The calculated energies are shown on fig. 1. Indeed in the indicated energy region for each of the $N$-body systems there is a series of states with exponential dependence upon the state number, $E_n \propto e^{-\zeta_n n}$.

The exponential fits give the numbers $\zeta_3 = 6.33 \pm 0.032$, $\zeta_4 = 3.40 \pm 0.14$, $\zeta_5 = 1.79 \pm 0.034$, $\zeta_6 = 1.31 \pm 0.020$, and $\zeta_7 = 1.01 \pm 0.007$. The value for $N = 3$ agrees within 3 sigma with the known analytical result of 6.244 (see [3] and
The spatial extension of Efimov states in trapped systems must be much larger than the interaction range $b$ and much smaller than the trap length $b_t$. In fig. 3 are shown the calculated r.m.s. radii $R_n$ as a function of state number $n$. The radii of the Efimov states identified in fig. 1 are reproduced well with the exponentials $R_n \propto e^{\frac{1}{2} \zeta_N n}$, where the parameters $\zeta_N$ are taken from the fits in fig. 1. Apparently all these states fall within the correct boundaries and the values of the radii follow the correct exponential trend.

There are several states in the $N=6$ and $7$ systems with radii much smaller than those of the typical states in the series with similar energies. Clearly these states are not Efimov states but rather relatively compact states with a different structure.

Recombination reactions into shallow $N$-body states. – The $N$-body Efimov states could in principle be identified by their contribution to the recombination rate of a cold gas as a function of scattering length similar to the three-body case [8]. As the scattering length is increased using the Feschbach technique, the $N$-body Efimov states crossing the threshold should produce peaks in the recombination rate. Since the $n$-th Efimov states appears when the scattering length is about $a \approx a_0 e^{\frac{1}{2} \zeta_n}$ (where $a_0$ is the scattering length corresponding to the lowest Efimov state) the peaks will appear as an exponential sequence as a function of the scattering length [17].

The scale of the rate can be estimated as follows. Let us first consider a three-body reaction. The rate of the loss of particles from a cold gas due to the three-body recombination reaction into a shallow dimer with the
energy $\frac{\hbar^2}{m a^2}$ is given by Fermi’s golden rule,

$$\frac{dN_0}{dt} = \frac{3N_0^3}{6} \frac{2\pi}{\hbar} |T_{fi}|^2 \frac{d\nu_f}{dE_f}, \quad (6)$$

where the factor 3 is there since each recombination reaction removes three particles from the cold gas, $N_0^3/6$ is the number of triples in the gas of $N_0$ particles, $d\nu_f$ is the number of final states (dimer plus the third particle) with relative momentum $q_f = \frac{2}{\sqrt{3}} a^{-1}$ and the relative kinetic energy  $E_f = \frac{\hbar^2 q_f^2}{2(\frac{\nu}{m})} = \frac{\hbar^2}{m a^2}$,

$$d\nu_f = \frac{V V_m}{d\psi R} \frac{2}{3\pi^2} \frac{V_m}{\hbar^2 a^2} dE_f, \quad (7)$$

and $T_{fi}$ is the transition matrix element from the initial three-body to the final dimer+particle state.

In the typical experimental regime, where the scattering length is still much smaller than the size of the trap, $a \ll b$, the transition matrix element for the non-resonant three-body recombination rate from a cold gas state into a shallow dimer state can be estimated perturbatively substituting the asymptotic expressions for the initial and final wave functions,

$$T_{fi} = \int d^3rd^3R \left[ \psi_0(r) e^{i q R} \frac{d}{\sqrt{V}} \right] \times \left( U \left( R - \frac{1}{2} r \right) + U \left( R + \frac{1}{2} r \right) \right) \left[ e^{i q R} \frac{d}{\sqrt{V}} \right], \quad (8)$$

where $r$ is the distance between two particles, $R$ is the distance between their center of mass and the third particle, $V \propto b^3$ is the normalization volume, $k$ and $q$ ($k \sim q \propto b^{-1} \ll a^{-1}$) are the initial momenta of the cold gas particles, $\psi_0(r)$ is the $s$-wave function of the shallow dimer with the binding energy $\frac{\hbar^2}{m a^2}$. Using the zero-range approximation for the transition interaction, $U(r) = \frac{\delta(r)}{m}$, the matrix element (8) in the limit $k \sim q \ll a^{-1}$ is estimated as (cf. [18])

$$T_{fi} \propto \frac{\hbar^2 a^{5/2}}{V^{1/2} m}, \quad (9)$$

Finally, the non-resonant factor of the three-body recombination rate becomes (cf. [3,18])

$$\frac{dN_0}{dt} \bigg|_{\text{3-body}} \propto n^3 \frac{\hbar a^4}{m}, \quad (10)$$

where $n = N_0/V$ is the density.

For an $N$-body recombination reaction into a shallow $(N - 1)$-body Efimov state with the binding energy of the order of $\frac{\hbar^2}{m a^2}$ the modifications to the expression for the rate (6) include the extra $3(N - 3)$ spatial dimensions in the integral in eq. (8), which gives an extra factor $(a^{3/2} V^{-1/2})^{2(N-3)}$, and also the factor $N_0^3/6$ is substituted by $N_0^N/N!$. Thus for the $N$-body recombination rate we have

$$\frac{dN_0}{dt} \bigg|_{N}\propto n^3 \frac{\hbar a^4}{m} (na^3)^{N-3}. \quad (11)$$

We emphasize that these simple estimates only refer to the non-resonant contributions and they also do not include other types of decays where the final-state structure might be substantially different from the shallow $N-1$ cluster plus one particle.

Although in the regime $na^3 \ll 1$ the $N$-body recombination has an additional small factor $(na^3)^{N-3}$ it might still be possible to observe the $N$-body Efimov states as a sequence of resonant peaks in the recombination rate as function of scattering length. The ratio of scattering lengths corresponding to the adjacent peaks, say number $n + 1$ and number $n$, is given as $\frac{2(a^{3(n+1)})}{a^{3n}} = e^{2\delta N}$.

**Conclusion.** We have calculated the spectrum of trapped $N$-boson systems with $N = 3, 4, 5, 6,$ and 7 using the stochastic variation method with a restricted correlated Gaussian basis. Only two-body correlations were allowed in the variational space. Thus the cluster states were a priori mostly excluded from the variation space, which made the calculations technically possible.

For each system a series of states is found with specific exponential dependences of the energies and r.m.s radii on the state number, which is a characteristic feature of Efimov states. For the $N = 3$ system the exponent obtained agrees well with the known analytical value and the trend also agrees well with the existing asymptotic estimate.

Inclusion of the cluster states would turn the Efimov states into meta-stable states. However, the lifetime of these states should be comparable with the Bose-Einstein condensate states which have a similar structure and similar decay modes.

It might be possible to observe the 4-body Efimov states as peaks in the recombination rate of a condensate as function of the scattering length, similar to the 3-body case. However, for a dilute gas the 4-body recombination rate is a factor $na^4$ smaller than the 3-body rate, therefore the accuracy constraints on the experiment would be higher.

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