Invariants and reduced matrix elements associated with the Lie superalgebra $gl(m|n)$

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Abstract

We construct explicit formulae for the eigenvalues of certain invariants of the Lie superalgebra $gl(m|n)$ using characteristic identities. We discuss how such eigenvalues are related to reduced Wigner coefficients and the reduced matrix elements of generators, and thus provide a first step to a new algebraic derivation of matrix element formulae for all generators of the algebra.

1 Introduction

The theory of basic classical Lie superalgebras was extensively developed in the late 1970s by Kac [1,2] and Scheunert et al. [3-5], motivated not only from the mathematical viewpoint of having a wondrous generalisation of the well established theory of Lie algebras, but also by progress at the time in elementary particle physics and generalised Fermi-Bose statistics [6-12]. We direct the reader to an informative review of physical applications of Lie superalgebras that were known at the time [13]. In more recent times, Lie superalgebras continue to be of pure mathematical interest (see, for example, the book by Musson [14]), and lie at the heart of many applications – to give some key examples, they appear as symmetry algebras in $\mathcal{N} = 4$ super Yang-Mills theory [15,16] and other supersymmetric integrable models [17-21], underly logarithmic conformal field theories [22,23] and play a role in systems combining parafermions and parabosons [24,26]. Undeniably, Lie superalgebras have made their way into the mainstream of modern mathematical physics.

In most applications, it is important to have a well-developed representation theory of the symmetry algebras involved. One fundamental question of representation theory is how to provide explicit formulae for matrix elements of generators. For Lie algebras, such a construction was unknown until the work of Gelfand and Tsetlin [27,28] where formulae for matrix elements of simple generators for the general linear and orthogonal Lie algebras were obtained. They moreover introduced combinatorial presentations of the basis vectors, now commonly referred to as Gelfand-Tsetlin (GT) patterns. Partially motivated by a desire to understand the results of these two brief articles, Baird and Biedenharn [29] developed pattern calculus techniques in order to derive and extend the remarkable formulae of Gelfand and Tsetlin. Many works followed (for a very readable account, see the review article by Molev [30] and references therein), some of which are of particular interest to our present investigation, including presentations of matrix element formulae [25,26,31-34] and branching rules [35-38] for certain classes of representations of a variety of Lie superalgebras.
There is a body of literature from the 1970s and 1980s, that was devoted to determining characteristic (polynomial) identities satisfied by generators of Lie algebras [39–42]. Curiously, such characteristic identities were noticed by Dirac as early as 1936 [43], and their usefulness observed by Baird and Biedenharn [44]. Of particular note are the applications of these characteristic identities to the derivation of reduced matrix elements [45], raising and lowering generators [46] and matrix elements [47, 48], even in the context of infinite dimensional irreducible representations for semisimple Lie algebras [49, 50].

A great deal is already known about Casimir invariants of Lie superalgebras and their eigenvalues on irreducible representations [51–54]. In the current article, we seek to construct invariants related to reduced matrix elements and reduced Wigner coefficients, in a similar vein to the treatment of classical Lie algebras found in [45, 55, 56]. In order to determine eigenvalues of these invariants on the irreducible representations, one could attempt to express them directly in terms of the Casimir invariants (c.f. the work of Green [39] for classical Lie algebras). To our knowledge, such an approach has not been attempted for Lie superalgebras, possibly for good reason. An alternative strategy, as presented in the current article, makes use of characteristic identities, and an important family of elements known as tensor operators.

Tensor operators play an important role in our work, especially since they serve as intertwining operators (see Section 4 for a more comprehensive discussion). Many textbooks on quantum mechanics present a treatment of tensor operators in the context of su(2) and the Wigner-Eckart theorem (see, for example, the book by Hannabuss [57]). The fact that tensor operators constitute intertwining operators in more general cases such as Lie algebras other than su(2) [45, 48, 58, 59], quantum groups [60] and Hopf algebras [61, 62], allows many of the standard results for su(2) to be extended. Discussions of tensor operators associated with Lie superalgebras have been presented for some special cases in [63–65], and it may seem at first that the situation is not so straightforward in the general case. We seek to clarify this, and in doing so, explain how the eigenvalues of the invariants that we construct are a first step in obtaining matrix element formulae for the generators of gl(m|n).

Of particular relevance to the current article is the seminal work of Jarvis and Green [66] where characteristic identities were developed for the general linear, special linear and orthosymplectic Lie superalgebras. Other works along these lines include the development of characteristic identities associated to the so-called “strange” Lie superalgebras [67] and simple Lie superalgebras [68]. More recently, techniques involving characteristic identities have been used to study the representation theory of certain polynomial deformations of Lie superalgebras [69]. The current article is concerned with generalising the techniques employing characteristic identities satisfied by generators of the Lie superalgebra gl(m|n), specifically to determining eigenvalues of invariants associated with tensor operators. These invariants are of interest since their eigenvalues correspond to the squared reduced matrix elements of the generators.

The current article has two main goals:

1. To highlight the effectiveness of the characteristic identities and the shift vector formalism in determining eigenvalues of certain invariants related to reduced matrix elements and reduced Wigner coefficients, by generalising known methods to the case of the Lie superalgebra gl(m|n);
2. To make a first step in providing the details of the derivation of matrix element formulae for all \( gl(m|n) \) generators on irreducible representations.

This second goal is largely in the spirit of Baird and Biedenharn \[29\], ultimately aimed at understanding the derivation of the matrix element formula, the focus being on the means by which the formulae are derived, and will be the subject of future work.

It is worth noting that our results have been obtained for any irreducible representation of \( gl(m|n) \), without any reference to unitary irreducible representations or their classification \[20\], for all generators (i.e. not just the simple ones), and without dependence on the precise branching rules. Moreover, our approach unifies and consolidates previous independent work of Palev \[32,33\], Stoilova and Van der Jeugt \[26\] and Molev \[34\] (see also Tolstoy et al. \[31\]) into one unifying framework.

The article is organised as follows. Section 2 introduces the basic notations used in the paper, and provides some basic constructions for Section 3, which establishes the form of the characteristic identities used throughout. Section 4 discusses tensor operators in a graded context, paving the way for Section 5, which introduces some of the main objects of our study – vector operators and their shift components. Section 6 looks at the branching rules associated with the subalgebra inclusion \( gl(m|n+1) \supset gl(m|n) \) and establishes necessary conditions in the form of betweenness conditions. The key result of Section 6 is given in Theorem 2. It turns out that our approach does not require precise branching rules, and Section 7 is a culmination of this fact and other results from the preceding sections, where we construct certain invariants. We demonstrate the complexities inherent in adopting the naive approach to evaluating the eigenvalues of these invariants by attempting to express them in terms of the Casimir invariants of \( gl(m|n+1) \) and \( gl(m|n) \). We then follow up with the more elegant approach using characteristic identities and vector shift operators to determine eigenvalue expressions. The main results are presented in Theorems 3 and 4 and Corollary 5. Section 7 also provides some motivation, in the context of unitary representations, for investigating these particular invariants by considering reduced matrix elements and reduced Wigner coefficients.

## 2 Preliminaries

Throughout we adopt the graded index notation of Jarvis and Green \[66\], where Latin indices \( 1 \leq i, j, k, \ell, \ldots \leq m \) are always assumed to correspond to “even” labels and Greek indices \( 1 \leq \mu, \nu, \ldots \leq n \) are assumed “odd”. We associate with even and odd indices the parity factor

\[
(i) = 0, \quad (\mu) = 1.
\]

This in fact corresponds to the standard \( \mathbb{Z}_2 \)-gradation for the vector representation. Occasionally we find it convenient to introduce ungraded indices \( 1 \leq p, q, r, s, \ldots \leq m + n \) for the sake of uniformity of exposition.

The \( gl(m|n) \) generators \( E_{pq} \) (\( 1 \leq p, q \leq m + n \)) satisfy the graded commutation relations

\[
[E_{pq}, E_{rs}] = \delta_{qr} E_{ps} - (-1)^{(p)+(q)((r)+(s))}\delta_{ps} E_{rq}, \tag{1}
\]
where the graded commutator is given by

\[ [E_{pq}, E_{rs}] = E_{pq}E_{rs} - (-1)^{(p+q)(r+s)} E_{rs}E_{pq}. \]

Note in particular that this bracket satisfies graded antisymmetry, i.e.

\[ [E_{pq}, E_{rs}] = -(-1)^{(p+q)(r+s)} [E_{rs}, E_{pq}]. \]

A basis for the Cartan subalgebra \( H \) of \( gl(m|n) \) comprises the set of mutually commuting generators \( E_{pp} \) whose eigenvalues are employed to label the weights occurring in the representations. Following Kac [1], we may expand a weight in terms of the fundamental weights \( \varepsilon_i \) (1 \( \leq \) \( i \) \( \leq \) \( m \)) and \( \delta_\mu \) (1 \( \leq \) \( \mu \) \( \leq \) \( n \)), which provides a convenient basis for \( H^* \).

Indeed we may expand a weight \( \Lambda \in H^* \) as

\[ \Lambda = \sum_{i=1}^{m} \Lambda_i \varepsilon_i + \sum_{\mu=1}^{n} \Lambda_\mu \delta_\mu. \]

With this convention, the root system is given by the set of even roots

\[ \pm(\varepsilon_i - \varepsilon_j), \quad 1 \leq i < j \leq m, \]
\[ \pm(\delta_\mu - \delta_\nu), \quad 1 \leq \mu < \nu \leq n, \]

and the set of odd roots

\[ \pm(\varepsilon_i - \delta_\mu), \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n. \] (2)

A system of simple roots is given by the distinguished set

\[ \{ \varepsilon_i - \varepsilon_{i+1}, \varepsilon_{m} - \delta_1, \delta_\mu - \delta_{\mu+1} \mid 1 \leq i < m, \ 1 \leq \mu < n \}. \]

The sets of even and odd positive roots are then given, respectively, by

\[ \Phi_0^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq m \} \cup \{ \delta_\mu - \delta_\nu \mid 1 \leq \mu < \nu \leq n \}, \]
\[ \Phi_1^+ = \{ \varepsilon_i - \delta_\mu \mid 1 \leq i \leq m, \ 1 \leq \mu \leq n \}. \]

We set \( \rho \) to be the graded half-sum of positive roots, i.e.

\[ \rho = \frac{1}{2} \sum_{\alpha \in \Phi_0^+} \alpha - \frac{1}{2} \sum_{\beta \in \Phi_1^+} \beta \]
\[ = \frac{1}{2} \sum_{j=1}^{m} (m - n - 2j + 1) \varepsilon_j + \frac{1}{2} \sum_{\nu=1}^{n} (m + n - 2\nu + 1) \delta_\nu. \] (3)

Every finite dimensional irreducible representation of \( gl(m|n) \) is a \( \mathbb{Z}_2 \)-graded vector space

\[ V = V_0 \oplus V_1, \]

(so that \( v \in V_j \) implies the grading \( (v) = j \) for \( j = 0, 1 \)) and admits a highest weight vector, whose weight \( \Lambda \) uniquely characterises the representation. We denote the associated irreducible highest weight module by \( V(\Lambda) \) and the associated representation by \( \pi_\Lambda \).
Relative to the $\mathbb{Z}_2$-grading, it is assumed, unless stated otherwise, that the highest weight vector $v^{\Lambda}$ has an even grading, i.e. $v^{\Lambda} \in V(\Lambda)_0$. Components of the highest weight $\Lambda$ satisfy the lexicality conditions

$$\Lambda_i - \Lambda_j \in \mathbb{Z}_+ \ (1 \leq i < j \leq m), \quad \Lambda_\mu - \Lambda_\nu \in \mathbb{Z}_+ \ (1 \leq \mu < \nu \leq n),$$

but we note that $\Lambda_i + \Lambda_\mu$ may be any complex number. As a simple example, the fundamental vector representation is denoted $V(\varepsilon_1)$ using this notation.

The fundamental vector representation $\pi_{\varepsilon_1}$ of $gl(m|n)$ is $(m + n)$ dimensional with a basis $\{ |p\rangle \mid 1 \leq p \leq m + n \}$ on which the generators $E_{pq}$ have the following action:

$$E_{pq} |s\rangle = \delta_{qs} |p\rangle,$$

so that

$$\langle r | E_{pq} |s\rangle = \delta_{qs} \langle r |p\rangle = \delta_{qs} \delta_{pr}$$
or alternatively

$$\pi_{\varepsilon_1} (E_{pq})_{rs} = \delta_{qs} \delta_{pr}.$$  

This gives rise to a non-degenerate even invariant bilinear form on $gl(m|n)$ defined by

$$(x, y) = \text{str}(\pi_{\varepsilon_1}(xy)) = \sum_{i=1}^{m} \pi_{\varepsilon_1}(xy)_{ii} - \sum_{\mu=1}^{n} \pi_{\varepsilon_1}(xy)_{\mu\mu},$$

where str denotes the supertrace given in [2]. In particular we have (sum over repeated indices)

$$(E_{pq}, E_{rs}) = (-1)^{(s)} \delta_{qr} \delta_{ps}, \tag{4}$$

which leads to a bilinear form on the fundamental weights

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\varepsilon_i, \delta_\mu) = 0, \quad (\delta_\mu, \delta_\nu) = -\delta_{\mu\nu},$$

which in turn induces a non-degenerate bilinear form on our weights $\Lambda$ given by

$$(\Lambda, \Lambda') = \sum_{i=1}^{m} \Lambda_i \Lambda'_i - \sum_{\mu=1}^{n} \Lambda_\mu \Lambda'_\mu. \tag{5}$$

Note that the left dual basis of $\{E_{pq}\}$ under the form [4] is given by $\{(\varepsilon_1)^{(q)}E_{qp}\}$, i.e.

$$((\varepsilon_1)^{(q)}E_{rs}, E_{pq}) = \delta_{qr} \delta_{ps}.$$  

It follows that the second order universal Casimir invariant is given by

$$I_2 = (-1)^{(q)}E_{pq}E_{qp}, \tag{6}$$

(summation over repeated indices assumed) which is a well-defined element of the universal enveloping algebra $U$ of $gl(m|n)$. Indeed, it may be verified directly that $I_2$ is central, i.e.

$$I_2 E_{rs} - E_{rs} I_2 = 0.$$
We can expand $I_2$ as

$$I_2 = \sum_{i,j} E_{ji} E_{ij} + \sum_{i,\mu} E_{i\mu} E_{\mu i} - \sum_{i,\mu} E_{i\mu} E_{\mu i} - \sum_{\mu,\nu} E_{\mu \nu} E_{\nu \mu}.$$  

Since $I_2$ is central, it must take a constant value on any (standard cyclic) highest weight module. For a module with highest weight vector $v$ corresponding to highest weight $\Lambda$ we have

$$\pi_\Lambda(I_2)v = \sum_{j=1}^{m} \Lambda_i^2 v + \sum_{i=1}^{m} (m - 2i - 1) \Lambda_i v - n \sum_{i=1}^{m} \Lambda_i v$$

$$-m \sum_{\mu=1}^{n} \Lambda_\mu v - \sum_{\mu=1}^{n} (n - 2\mu - 1) \Lambda_\mu v - \sum_{\mu=1}^{n} \Lambda_\mu^2 v$$

so that the eigenvalue of $I_2$, denoted $\chi_\Lambda(I_2)$ is given by

$$\chi_\Lambda(I_2) = \sum_{i=1}^{m} \Lambda_i (\Lambda_i + m - n - 2i + 1) - \sum_{\mu=1}^{n} \Lambda_\mu (\Lambda_\mu + m + n - 2\mu + 1).$$

Making use of equations (3) and (5), this may be conveniently expressed

$$\chi_\Lambda(I_2) = (\Lambda, \Lambda + 2\rho).$$  

### 3 Characteristic identities

If $\pi_\theta$ denotes a finite dimensional irreducible representation of $gl(m|n)$ with highest weight $\theta$, we may construct the tensor matrix $A^\theta$ with algebraic entries

$$A^\theta_{\alpha\beta} = -\sum_{p,q} (-1)^{(q)+(p)+(q)(\beta)} \pi_\theta(E_{pq})_{\alpha\beta} E_{qp}$$

where $\{e_\alpha\}$ is a fixed homogeneous basis for the $gl(m|n)$ module $V(\theta)$. Acting on a finite dimensional irreducible $gl(m|n)$ module $V(\Lambda)$ the matrix $A^\theta$ may be expressed in the invariant form

$$A^\theta = -\frac{1}{2} \left[ (\pi_\theta \otimes \pi_\Lambda)(\Delta(I_2)) - \pi_\theta(I_2) \otimes I - I \otimes \pi_\Lambda(I_2) \right],$$

where $\Delta : U \rightarrow U \otimes U$ is the usual coproduct on the universal enveloping algebra $U$, and $I$ denotes the identity matrix on $V(\Lambda)$, $V(\theta)$ respectively. If $\theta_1, \theta_2, \ldots, \theta_k$ denote the distinct weights occurring in $V(\theta)$, it follows from previous work of Gould [68] that the tensor matrix $A^\theta$ satisfies the following polynomial identity on $V(\Lambda)$:

$$\prod_{i=1}^{k} (A^\theta - \alpha_i) = 0$$
where

\[ \alpha_i = -\frac{1}{2} \left[ \chi_{\Lambda + \delta_i}(I_2) - \chi_\delta(I_2) - \chi_\Lambda(I_2) \right] \]

\[ = \frac{1}{2} (\theta_i \theta + 2 \rho - \frac{1}{2} (\theta_i, \theta_i + 2(\Lambda + \rho))). \quad (8) \]

We are concerned here with the vector representation \( \pi_{\varepsilon_1} \) (with \( \theta = \varepsilon_1 \)) in which case the tensor matrix \( A_{\varepsilon_1} \) is given by

\[
A_{\varepsilon_1}^{pq} = -\sum_{r,s} (-1)^{(s)+(r)+(p)} \pi_{\varepsilon_1}(E_{rs})_{pq} E_{sr}
\]

\[ = -(-1)^{(p)(q)} E_{qp}, \]

where indices \( 1 \leq p, q \leq m + n \) are assumed ungraded. We thus obtain what we call the \( gl(m|n) \) adjoint matrix:

\[
\tilde{A}^{pq} \equiv A_{\varepsilon_1}^{pq} = -(-1)^{(p)(q)} E_{qp}. \quad (9)
\]

The weights in the representation \( \pi_{\varepsilon_1} \) are of the form \( \varepsilon_i \) (\( 1 \leq i \leq m \)), \( \delta_\mu \) (\( 1 \leq \mu \leq n \)) from which it follows that the adjoint matrix \( \tilde{A} \) satisfies the characteristic identity

\[
\prod_{i=1}^{m} (\tilde{A} - \tilde{\alpha}_i) \prod_{\mu=1}^{n} (\tilde{A} - \tilde{\alpha}_\mu) = 0 \quad (10)
\]

when acting on an irreducible \( gl(m|n) \) module \( V(\Lambda) \), where the adjoint roots \( \tilde{\alpha}_i, \tilde{\alpha}_\mu \) are given, in accordance with equation (8), by

\[ \tilde{\alpha}_r = -\frac{1}{2} \left( \chi_{\Lambda + \delta_r}(I_2) - \chi(I_2) - \chi_\Lambda(I_2) \right) \]

where \( \chi(I_2) = m - n \) is the eigenvalue of \( I_2 \) on the vector representation. Using equation (8) we thus obtain

\[ \tilde{\alpha}_i = -\frac{1}{2} \left[ \chi_{\Lambda + \varepsilon_i}(I_2) + n - m - \chi_\Lambda(I_2) \right] \]

\[ = -\frac{1}{2} \left[ (\varepsilon_i, \varepsilon_i) + 2(\varepsilon_i, \Lambda + \rho) + n - m \right] \]

\[ = i - 1 - \Lambda_i. \quad (11) \]

Similarly for the odd adjoint roots we obtain

\[ \tilde{\alpha}_\mu = -\frac{1}{2} \left[ \chi_{\Lambda + \delta_\mu}(I_2) + n - m - \chi_\Lambda(I_2) \right] \]

\[ = \Lambda_\mu + m + 1 - \mu. \quad (12) \]

To construct the \( gl(m|n) \) vector matrix we take \( \tilde{\pi} \) to be the triple dual of the vector representation (viz. \( \tilde{\pi} = \pi_{\varepsilon_1}^{**} \)) defined by

\[ \tilde{\pi}(E_{pq})_{rs} = -(-1)^{(p)(q)} \delta_{qr} \delta_{ps}. \]
Our previous construction for the matrix $A^\theta$ with $\pi_\theta$ replaced by $\tilde{\pi}$ yields the $gl(m|n)$ vector matrix

$$A^p_q = -\sum_{r,s} (-1)^{(r)+(s)+(q)} \tilde{\pi} (E_{rs})_{pq} E_{sr}$$

$$= (-1)^{(p)} E_{pq}. \quad (13)$$

The weights occurring in $\tilde{\pi}$ are the $-\varepsilon_i (1 \leq i \leq m), -\delta_\mu (1 \leq \mu \leq n)$ from which it follows that acting on the irreducible $gl(m|n)$ module $V(\Lambda)$, the matrix $A$ satisfies the characteristic identity

$$\prod_{i=1}^m (A - \alpha_i) \prod_{\mu=1}^n (A - \alpha_\mu) = 0 \quad (14)$$

where our characteristic roots are given by

$$\alpha_i = -\frac{1}{2} [\chi_\Lambda - \varepsilon_i (I_2) + n - m - \chi_\Lambda (I_2)]$$

$$= \Lambda_i + m - n - i, \quad 1 \leq i \leq m, \quad (15)$$

$$\alpha_\mu = -\frac{1}{2} [\chi_\Lambda - \delta_\mu (I_2) + n - m - \chi_\Lambda (I_2)]$$

$$= \mu - \Lambda_\mu - n, \quad 1 \leq \mu \leq n. \quad (16)$$

We remark that in the above we used the fact that the eigenvalue of $I_2$ in the representation $\tilde{\pi}$ is given by

$$\tilde{\chi} (I_2) = m - n,$$

which is the same as the eigenvalue in the vector representation.

**Remarks:**

1. Note that

$$\alpha_i - \alpha_\mu + 1 = (\Lambda + \rho, \varepsilon_i - \delta_\mu) = -(\bar{\alpha}_i - \bar{\alpha}_\mu + 1).$$

For *atypical* irreducible representations [2], we have

$$(\Lambda + \rho, \varepsilon_i - \delta_\mu) = 0, \quad (17)$$

so that the equation $\alpha_\mu = \alpha_i + 1$ (or $\bar{\alpha}_\mu = \bar{\alpha}_i + 1$) for some $i$ and $\mu$ serves as an atypicality condition in terms of the characteristic roots. Equation (17) also implies that

$$\alpha_i - \alpha_\mu = (\Lambda - \varepsilon_i + \rho, \varepsilon_i - \delta_\mu),$$

$$\bar{\alpha}_i - \bar{\alpha}_\mu = -(\Lambda + \varepsilon_i + \rho, \varepsilon_i - \delta_\mu).$$
2. In the case we take $\pi_\theta = \pi_\varepsilon^*$ (the dual of the vector representation) we obtain a new matrix, referred to as the double adjoint:

$$\tilde{A}_{rq} = (-1)^{(q)} E_{rq} = (-1)^{(r)+(q)} A_{rq}.$$  

This is the matrix appearing in the work of Jarvis and Green [66]. If $p(x)$ is any polynomial we have (e.g. by induction on the degree of $p(x)$)

$$p \left( \tilde{A} \right)_{rq} = (-1)^{(r)+(q)} p(A)_{rq}$$

so that $\tilde{A}$ and $A$ satisfy the same characteristic identity. Equations (15) and (16) agree with the roots obtained by Jarvis and Green using different methods, except that equation (16) correctly replaces $-\mu$ with $\mu$ in the formula of Jarvis and Green.

3. We also have a triple adjoint matrix defined by

$$\tilde{\tilde{A}}^q_p = (-1)^{(p)+(q)} \tilde{A}^q_p$$

which satisfies the same characteristic identity as $\tilde{A}$. Note that the matrix $\tilde{A}$ (resp. $\tilde{\tilde{A}}$) is simply related to $A$ (resp. $\tilde{A}$) by the $gl(m|n)$ grading automorphism.

4. Polynomials in $A$ and $\tilde{A}$ are defined recursively according to

$$(A^{k+1})^p_q = \sum_r (A^k)^p_r A^r_q = \sum_r A^p_r (A^k)^r_q$$

$$(\tilde{A}^{k+1})^p_q = \sum_r (\tilde{A}^k)^p_r \tilde{A}^r_q = \sum_r \tilde{A}^p_r (\tilde{A}^k)^r_q.$$  

### 4 Tensor operators

Let $V(\Lambda)$ be a finite dimensional irreducible $gl(m|n)$ module with highest weight $\Lambda$ and let

$$T : V(\Lambda) \otimes V \longrightarrow W$$

be a (surjective) intertwining operator of degree $(\tau)$ where $V$ and $W$ are $gl(m|n)$ modules:

$$\pi_W(E_{pq}) T = (-1)^{(p)+(q)+(\tau)} T (\pi_\Lambda \otimes \pi) \Delta(E_{pq})$$

where $\pi$ (resp. $\pi_W$) is the representation of $gl(m|n)$ afforded by $V$ (resp. $W$). Now let $\{e_\alpha\}$ be a homogeneous basis for $V(\Lambda)$. Then we may define a collection of operators $\{T_\alpha\}$, called a tensor operator, operating on $V$ according to

$$T_\alpha v = T(e_\alpha \otimes v), \forall v \in V$$

with $T$ as above. We have, for arbitrary $v \in V$, and homogeneous $x \in gl(m|n)$

$$x T_\alpha v \equiv \pi_W(x) T (e_\alpha \otimes v)$$

$$= (-1)^{(x)(\tau)} \pi_{\Lambda}(x) \otimes \pi(I) + \pi_I(I) \otimes \pi(x) \pi(\epsilon_\alpha \otimes v)$$

$$= (-1)^{(x)(\tau)} \pi_{\Lambda}(x)_{\beta \alpha} e_\beta \otimes v + (-1)^{(x)(\alpha)} \epsilon_\alpha \otimes \pi(x) v$$

$$\equiv \left( -1 \right)^{(x)(\tau)} \pi_{\Lambda}(x)_{\beta \alpha} T_\beta + (-1)^{(x)(((\tau)+(\alpha)))} T_\alpha x) v.$$
Note: We have utilised the summation convention over repeated indices. We will adopt this convention throughout the paper.

Thus, by abstraction, we define an irreducible tensor operator of rank \( \Lambda \) and degree \( (\tau) \) as a collection of components \( \{ T_\alpha \} \) transforming according to

\[
[x, T_\alpha] = (-1)^{(x)(\tau)} \pi_\Lambda (x)_{\beta\alpha} T_\beta
\]

where \( (x) \) is the degree of \( x \in gl(m|n) \) and the graded bracket on the left hand side is given by

\[
[x, T_\alpha] = x T_\alpha - (-1)^{(x)((\alpha)+(\tau))} T_\alpha x.
\]

In the special case where \( \pi_\Lambda = \pi_{\epsilon^1} \) is the vector representation we obtain the transformation law of vector operators \( \psi^r \) \( (1 \leq r \leq m + n) \):

\[
[E_{pq}, \psi^r] = (-1)^{(\psi)((p)+(q))} \pi_{\epsilon_1} (E_{pq})_{sr} \psi^s
= (-1)^{(\psi)((p)+(q))} \delta_q^r \psi^p
\]

If \( (\psi) = 0 \) (resp. 1) we call \( \psi \) an even (resp. odd) vector operator: the case \( (\psi) = 0 \) corresponds to the definition of vector operator given by Jarvis and Green [66]. In the case that \( \pi_\Lambda = \pi_{\epsilon^1}^* \) is the dual of the vector representation we obtain the transformation law of contragredient vector operators \( \phi^r \) \( (1 \leq r \leq m + n) \):

\[
[E_{pq}, \phi^r] = (-1)^{(\phi)((p)+(q))} \pi_{\epsilon_1}^* (E_{pq})_{sr} \phi^s
\]

where

\[
\pi_{\epsilon_1}^* (E_{pq})_{sr} \overset{\text{def.}}{=} -(-1)^{(\phi)((p)+(q))} \pi_{\epsilon_1} (E_{pq})_{rs}
= -(-1)^{(\phi)((p)+(q))} \delta_{pr} \delta_{qs}
\]

\[
\Rightarrow [E_{pq}, \phi^r] = -(-1)^{(\phi)((p)+(q))((\phi)+(q))} \delta_{pr} \phi_q
\]

If \( (\phi) = 0 \) (resp. 1) we say that \( \phi \) is an even (resp. odd) homogeneous contragredient vector operator. Our main concern here, is with \( gl(m|n) \) vector and contragredient vector operators, whose transformation laws are given above. We should remark that if we take \( \pi_\Lambda \) in equation (18) to be one of the tensor representations, then appropriate transformation laws for higher order tensor operators can be given.

Remarks:

1. If \( \psi^r \) is a vector operator then

\[
\tilde{\psi}^r = (-1)^{(r)} \psi^r
\]

also constitutes a tensor operator whose components \( \tilde{\psi}^r \) transform according to the double dual \( \pi_{\epsilon_1}^{**} \) of the vector representation \( \pi_{\epsilon_1} \). Similarly if \( \phi_r \) transforms as in equation (19) then

\[
\tilde{\phi}_r = (-1)^{(r)} \phi_r
\]

constitutes a tensor transforming as the triple dual \( \pi_{\epsilon_1}^{**} \) of the vector representation \( \pi_{\epsilon_1} \).

2. An odd vector operator is equivalent to an even vector operator but with a reversal of the \( \mathbb{Z}_2 \)-grading in the vector representation.

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5 Vector operator shift components

Since \( \psi^p \) transforms as the vector representation \( \pi_{e_1} \) it follows that the \( gl(m|n) \) adjoint matrix (9) acts naturally on the right of \( \psi \):

\[
(\psi \vec{A})^p = \psi^q \vec{A}_q^p. \tag{20}
\]

We may then proceed to resolve \( \psi \) into its shift components via the use of projections [39, 40]:

\[
\psi^p = \sum_{i=1}^{m} \psi[i]^p + \sum_{\mu=1}^{n} \psi[\mu]^p
\]

where the shift components \( \psi[r] \) are constructed as follows (single ungraded index notation in use):

\[
\psi[r]^p = \psi^g \tilde{P}[r]_q^p \quad 1 \leq r \leq m + n \tag{21}
\]

where

\[
\tilde{P}[r] = \prod_{k \neq r}^{m+n} \left( \frac{\vec{A} - \bar{\alpha}_k}{\alpha_r - \bar{\alpha}_k} \right)
\]

is the appropriate projection operator constructed using the characteristic identity (10).

Before proceeding we show that the shift components (21) of a vector operator indeed constitute a vector operator. Using a simple induction argument, since \( \tilde{P}[r] \) is a polynomial in \( \vec{A} \), it suffices to show that if \( \psi^p \) is a homogeneous vector operator, then so too is

\[
(\psi \vec{A})^p = \psi^q \vec{A}_q^p = -(-1)^{(p+q)} \psi^q E_{pq}.
\]

We have

\[
[E_{pq}, (\psi \vec{A})^r] = -(-1)^{(r)(s)} [E_{pq}, \psi^s E_{rs}]
\]

\[
= -(-1)^{(r)(s)} \left( [E_{pq}, \psi^s] E_{rs} + (-1)^{(p+q)(s+\psi)} \psi^s [E_{pq}, E_{rs}] \right)
\]

\[
= -(-1)^{(r)(s)} \left( \delta^s_q \psi^p (-1)^{(p+q)(\psi)} E_{rs} + \delta^s_q (-1)^{(p+q)(s+\psi)} \psi^s E_{ps} - \delta^s_q (1)^{(p)(s+\psi)} \psi^s E_{rq} \right)
\]

\[
= -(-1)^{(p+q)(\psi)} \psi^s E_{ps}
\]

\[
= (-1)^{(p+q)(\psi)} \delta^s_q \psi^p \tilde{A}_q^p
\]

\[
= (-1)^{(p+q)(\psi)} \delta^s_q (\psi \vec{A})^p.
\]

Thus \( \psi \vec{A} \) is also a homogeneous vector operator of degree \( \psi \). Alternatively note that \( \tilde{P}[r] \) determines an intertwining operator and hence so too does \( \psi[r] = \psi \tilde{P}[r] \). It follows that the components of \( \psi[r] \) must also determine a vector operator of the same degree.

Remark:
At this point we highlight the fact that for certain irreducible representations, the characteristic roots may coincide (consider $\bar{\alpha}_r = \bar{\alpha}_k$ in the above formula). This is related to the occurrence of atypical irreducible representations in the tensor product of $V(\varepsilon_1) \otimes V(\Lambda)$ or $V(\varepsilon_1)^* \otimes V(\Lambda)$. The set of $\Lambda$ for which this happens, however, is closed in the Zariski topology \cite{71} on $H^*$. Therefore the roots of the characteristic identity are distinct on an open and hence dense subset of $H^*$. Hence without loss of generality, we will make the assumption that the roots are distinct throughout the remainder of the paper unless otherwise indicated. In fact it can be shown that under this assumption, the tensor products $V(\varepsilon_1) \otimes V(\Lambda)$ and $V(\varepsilon_1)^* \otimes V(\Lambda)$ are completely reducible (see Appendix B for details). Furthermore, it is worth remarking that the invariants to be evaluated in this paper determine (rational) polynomial functions which are continuous in the Zariski topology. Note however, care needs to be taken when applying our formulae, by first cancelling terms in numerators and denominators where appropriate.

From the previous work of Gould \cite{47}, the above shift components \eqref{21} effect the following shifts in the representation labels $\Lambda$:

\[
\psi[i] : \Lambda_j \rightarrow \Lambda_j + \delta_{ij} \quad (1 \leq i, j \leq m),
\]

\[
\psi[\mu] : \Lambda_\nu \rightarrow \Lambda_\nu + \delta_{\nu\mu} \quad (1 \leq \mu, \nu \leq n),
\]

the remaining labels remaining unchanged.

In a similar way, if $\phi_\rho$ is a contragredient vector operator then the matrix $\bar{A}$ acts naturally on the right of $\phi$:

\[
(\phi_\rho \bar{A})_p = \phi_q (\bar{A})^q_p = (-1)^{(p)+(q)} \phi_q A^q_p.
\]

Thus we obtain the resolution

\[
\phi_\rho = \sum_{i=1}^{m} \phi[i]_p + \sum_{\mu=1}^{n} \phi[\mu]_p \quad \text{ (22)}
\]

where

\[
\phi[i]_p = (-1)^{(p)+(q)} \phi_q P[r]^q_p \quad \text{ (23)}
\]

where our $gl(m|n)$ vector projectors are given by

\[
P[r] = \prod_{k \neq r} \frac{A - \alpha_k}{\alpha_r - \alpha_k}
\]

In this case the shift components \eqref{23} effect the following shifts on the representation labels:

\[
\phi[i] : \Lambda_j \rightarrow \Lambda_j - \delta_{ij} \quad (1 \leq i, j \leq m)
\]

\[
\phi[\mu] : \Lambda_\nu \rightarrow \Lambda_\nu - \delta_{\nu\mu} \quad (1 \leq \mu, \nu \leq n),
\]
the other labels remaining unchanged.

We remark that the shift components \( (22) \) of a contragredient vector \( \phi_r \) indeed constitute a contragredient vector, in a similar way to the case of vector operators.

The results above all hold regardless of whether our vector operators are even or odd. However the matrices which act on the left of vectors and contragredient vectors will depend explicitly on their degree (odd or even). We shall be primarily concerned with odd vector and contragredient vector operators in this paper so we shall concentrate on them in the following (although an analogous formalism can be set up for the even case \( (\tau) = 0 \)).

It turns out that odd vector (and contragredient vector) operators appear naturally in discussing the Lie superalgebra embedding \( gl(m|n+1) \supset gl(m|n) \). Throughout the remainder of the paper we assume, unless otherwise stated, that \( \psi_r \) (resp. \( \phi_r \)) denotes a \( gl(m|n) \) odd vector (resp. odd contragredient vector) operator. That is, we assume the transformation laws

\[
[E_{pq}, \psi_r] = (-1)^{(p)+(q)} \delta_q^p \psi^p, \\
[E_{pq}, \phi_r] = -(-1)^{(p)(p)+(q)} \delta_{pr} \phi_q.
\]

We note that the graded brackets on the left hand side are given by

\[
[E_{pq}, \psi_r] = E_{pq} \psi_r - (-1)^{(p)+(q)((r)+1)} \psi_r E_{pq} = -(-1)^{(p)+(q)((r)+1)} [\psi_r, E_{pq}],
\]

and similarly

\[
[E_{pq}, \phi_r] = -(-1)^{(p)+(q)((r)+1)} [\phi_r, E_{pq}],
\]

since we are assuming that \( (\psi) = 1 = (\phi) \).

Since the matrix \( \mathcal{A} \) acts naturally on the right of \( \psi^p \) we have

\[
(\psi \mathcal{A})^p = \psi^q \mathcal{A}_q^p
= -(-1)^{(p)(q)} \psi^q E_{pq}
= (-1)^{(p)(q)} \left( \psi^q, E_{pq} \right) + (-1)^{(p)+(q)((q)+1)} E_{pq} \psi^q
= (-1)^{(p)} \left( [E_{pq}, \psi^q] - E_{pq} \psi^q \right)
= (-1)^{(q)} \delta_q^p \psi^p - (-1)^{(p)} E_{pq} \psi^q
= (m - n - \mathcal{A})^p q \psi^q
\]

It follows that the \( gl(m|n) \) matrix \( \mathcal{A} \) acts naturally on the left of odd vector operators (while the double adjoint \( \mathcal{A} \) acts naturally on the left of even vector operators).
Similarly, for contragredient vectors $\phi_p$ we have

$$
(\phi \bar{A})_p = \phi_q \bar{A}_q^n 
= (-1)^{(p)} \phi_q E_{qp} 
= (-1)^{(p)} ([\phi_q, E_{qp}] + (-1)^{(p)+(q)+(q)+1} E_{qp} \phi_q) 
= (-1)^{(p)+(q)} (E_{qp} \phi_q - [E_{qp}, \phi_q]) 
= (-1)^{(p)} E_{qp} \phi_q + (-1)^{(q)} \delta_q^p \phi_p 
= (m - n - \bar{A})_p^q \phi_q.
$$

It thus follows that the matrix $\bar{A}$ acts naturally on the left of odd contragredient vectors. Thus we may project out the shift components of $\psi$ and $\phi$ from the left according to

$$
\psi[r]^p = P[r]^p \psi^q, \quad \phi[r]^p = \bar{P}[r]^p \phi_q
$$

where $P[r], \bar{P}[r]$ are the projection operators previously constructed in terms of the matrices $A, \bar{A}$ respectively.

6 Branching conditions for $gl(m|n+1) \supset gl(m|n)$

We now seek to determine necessary conditions for the branching rule $gl(m|n+1) \downarrow gl(m|n)$. They turn out to be similar in appearance to the betweenness conditions of [26] for example, but here we give a detailed proof of the necessary condition (but not sufficient) in a more general setting.

We first establish some notation. We set

$$
\hat{L} = gl(m|n+1) = \hat{L}_- \oplus \hat{L}_0 \oplus \hat{L}_+ 
$$

where

$$
\hat{L}_0 = gl(m) \oplus gl(n+1),
$$

and similarly

$$
L = gl(m|n) \oplus gl(1) = L_+ \oplus L_0 \oplus L_-
$$

where

$$
L_0 = gl(m) \oplus gl(n) \oplus gl(1).
$$

We also introduce the $L_0$-modules

$$
K_+ = \text{span} \{E_{i,m+n+1}\}_{i=1}^m, \quad K_- = \text{span} \{E_{m+n+1,i}\}_{i=1}^m
$$

so that

$$
\hat{L}_\pm = L_\pm \oplus K_\pm.
$$
We now recall some facts about the representation theory of the algebra $\hat{L}$. First, every finite-dimensional irreducible $\hat{L}$-module $V(\tilde{\Lambda})$ with highest weight
\[
\tilde{\Lambda} = \sum_{i=1}^{m} \tilde{\Lambda}_i \epsilon_i + \sum_{\mu=1}^{n+1} \tilde{\Lambda}_\mu \delta_\mu
\]
admits a $\mathbb{Z}$-gradation
\[
V(\tilde{\Lambda}) = \bigoplus_{i=-d}^{0} V_i(\tilde{\Lambda})
\]
in which case we say that $V(\tilde{\Lambda})$ admits $d + 1$ levels. Here, $V_0(\tilde{\Lambda})$ is called the maximal $\mathbb{Z}$-graded component which constitutes an irreducible $\hat{L}_0$-module of the same highest weight. We also observe that the decomposition (25) is in fact an $\hat{L}_0$-module decomposition.

Next we note [2] that any such irreducible $\hat{L}_0$-module $V_0(\tilde{\Lambda})$ admits an invariant inner product $\langle \cdot , \cdot \rangle$ which extends in a unique way to an invariant non-degenerate sesquilinear form $\langle \cdot , \cdot \rangle$ on all of $V(\tilde{\Lambda})$ which satisfies the symmetry
\[
\langle v, w \rangle = \langle w, v \rangle, \forall v, w \in V(\tilde{\Lambda})
\]
and the invariance condition given by
\[
\langle av, w \rangle = \langle v, a^\dagger w \rangle, \forall a \in \hat{L}
\]
where $\dagger$ is the conjugation operation defined on $\hat{L}$ by
\[
(E_{pq})^\dagger = E_{qp}.
\]
Such an invariant sesquilinear form has all the properties of an inner product except it is not necessarily positive definite. When it is, we call $V(\tilde{\Lambda})$ unitary of type 1.

**Remarks:**

1. We may define matrix elements, Wigner coefficients etc. even for non-unitary irreps in the usual way – except we work with a non-degenerate sesquilinear form (26) rather than an inner product.

2. We also have another conjugation operation on $\hat{L}$ defined by
\[
(E_{pq})^\dagger = (-1)^{(p)+(q)} E_{qp}.
\]
Then $V(\tilde{\Lambda})$ also admits a unique invariant non-degenerate sesquilinear form (26) with respect to the conjugation operation (27). When this form is positive definite we call $V(\tilde{\Lambda})$ unitary of type 2. It can be shown [70] that the two types of unitary irreps are related by duality.
We observe a number of properties of the form $\langle \cdot, \cdot \rangle$ of equation (26). First the decomposition (25) is orthogonal with respect to the form. Next $V(\hat{\Lambda})$ decomposes into a direct sum of irreducible $L_0$ and $\hat{L}_0$ modules. Two irreducible $L_0$ (respectively $\hat{L}_0$) modules with different highest weights are necessarily orthogonal under the form.

Before we present the main result of this section, we first note from the PBW theorem that

\[
V(\hat{\Lambda}) = U(\hat{L}_-)V_0(\hat{\Lambda}) = U(L_-)U(K_-)V_0(\hat{\Lambda}) = U(L_-)W(\hat{\Lambda})
\]  

where

\[
W(\hat{\Lambda}) = U(K_-)V_0(\hat{\Lambda}).
\]

This leads to the following useful Lemma.

**Lemma 1** Let $0 \neq v_+ \in V(\hat{\Lambda})$ be an $L$-maximal weight vector. Then we have

\[
\langle v_+, W(\hat{\Lambda}) \rangle \neq (0).
\]

**Proof:** Otherwise we would have

\[
(0) = \langle U(L_+)v_+, W(\hat{\Lambda}) \rangle \overset{(26)}{=} \langle v_+, U(L_-)W(\hat{\Lambda}) \rangle \overset{(28)}{=} \langle v_+, V(\hat{\Lambda}) \rangle \Rightarrow v_+ = 0,
\]

a contradiction. ■

We are now in a position to prove our main result:

**Theorem 2** Let $v_+ \in V(\hat{\Lambda})$ be an $L$-maximal weight vector of weight

\[
\Lambda = \sum_{i=1}^{m} \Lambda_i \epsilon_i + \sum_{\mu=1}^{n} \Lambda_\mu \delta_\mu.
\]

Then:

(i) The components of $\Lambda$ must satisfy the betweenness conditions

\[
\hat{\Lambda}_\mu \geq \Lambda_\mu \geq \hat{\Lambda}_{\mu+1}, \quad 1 \leq \mu \leq n
\]

\[
\hat{\Lambda}_i \geq \Lambda_i \geq \hat{\Lambda}_{i-1}, \quad 1 \leq i \leq m.
\]

(ii) $v_+$ is the unique (up to scalar multiples) $L$-maximal weight vector in $V(\hat{\Lambda})$ of weight $\Lambda$. 

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Proof: To prove the theorem we note that $W(\tilde{\Lambda})$ of equation (28) gives rise to a $L_0$-module and decomposes into a direct sum of irreducible $L_0$ submodules

$$W(\tilde{\Lambda}) = \bigoplus_{\Lambda} \hat{V}_0(\Lambda)$$  \hspace{1cm} \text{(30)}$$

with highest weights $\Lambda$ which satisfy precisely condition (i) of the theorem.

If $v_+$ has weight $\nu$ then it cyclically generates an irreducible $L_0$-module $\hat{V}_0(\nu)$ of the same highest weight. It follows that $\nu$ must occur in the decomposition of (30) (that is, $\nu$ must satisfy the betweenness conditions (i) of the theorem) else $\hat{V}_0(\nu)$ would be orthogonal to $W(\tilde{\Lambda})$ in contradiction to Lemma 1. This proves part (i) of the theorem.

As to part (ii), let $v_+$ be as above and let $v_0^\nu \in W(\tilde{\Lambda})$ be the unique (up to scalar multiples) maximal weight vector in $W(\tilde{\Lambda})$ with highest weight $\nu$. Suppose $0 \neq w_+ \in V(\Lambda)$ is another $L$ highest weight vector of the same highest weight $\nu$. Then from Lemma 1 we must have

$$\langle w_+, v_0^\nu \rangle \neq 0, \quad \langle v_+, v_0^\nu \rangle \neq 0.$$  

Now consider the $L$-maximal vector

$$\tilde{v}_+ = \langle w_+, v_0^\nu \rangle v_+ - \langle v_+, v_0^\nu \rangle w_+.$$  

By construction we have

$$\langle \tilde{v}_+, v_0^\nu \rangle = 0$$  

$$\Rightarrow \langle \tilde{v}_+, \hat{V}_0(\nu) \rangle = \langle \tilde{v}_+, W(\tilde{\Lambda}) \rangle = (0)$$  

$$\Rightarrow \tilde{v}_+ = 0$$  

by Lemma 1. This proves that $w_+ = \kappa v_+$, with

$$\kappa = \frac{\langle w_+, v_0^\nu \rangle}{\langle v_+, v_0^\nu \rangle},$$

and hence we have proved part (ii) of the theorem. □

For the irreducible $gl(m|n)$-module $V(\Lambda)$ occurring in the irreducible $gl(m|n+1)$-module $V(\tilde{\Lambda})$, Theorem 2 states that the conditions (29) must be satisfied by the highest weights. Therefore the significance of Theorem 2 is that the conditions (29) are necessary, but not sufficient, so we refer to them as branching conditions. For an alternative perspective on the branching rule which we believe will give the reader some deeper insight, see the discussion in Appendix A.

7 Invariants and their eigenvalues

Throughout we let $\psi^p$ denote the odd $gl(m|n)$ vector operator $\psi^p = (-1)^{(p)}E_{p,m+n+1}$ ($1 \leq p \leq m+n$) and we let $\phi_p$ denote the odd $gl(m|n)$ contragredient vector operator
\( \phi_p = (-1)^{p}E_{m+n+1,p}. \) There are two natural \( gl(m|n) \) invariants we can construct from these vector operators, namely, for \( 1 \leq r \leq m+n, \)

\[
\gamma_r = (-1)^{(q)}\phi[r]q\psi[r]^q = (-1)^{(q)}\phi_q\psi[r]^q = E_{m+n+1,q}\psi[r]^q, \tag{31}
\]

\[
\bar{\gamma}_r = \psi[r]^q\phi[r]q = \psi^q\phi[r]q = (-1)^{(p)}E_{p,m+n+1}\phi[r]p, \tag{32}
\]

where, as usual, \( \psi[r]^q, \phi[r]q \) denote the shift components of \( \psi^q, \phi_q \) respectively. To see that these operators do indeed constitute invariants, consider the following, noting that we only consider \( 1 \leq p, q \leq m+n: \)

\[
[E_{pq}, \gamma_r] = [E_{pq}, E_{m+n+1,s}\psi[r]^s] = [E_{pq}, E_{m+n+1,s}\psi[r]^s] + (-1)^{(p)+(q)(1)+(s)}E_{m+n+1,s}[E_{pq}, \psi[r]^s] + (-1)^{(p)+(q)(1)+(s)}\delta^s_qE_{m+n+1,s}(-1)^{(p)+(q)}\psi[r]^p = 0.
\]

A similar calculation can be done in order to determine

\[
[E_{pq}, \bar{\gamma}_r] = 0.
\]

Our interest in these invariants stems from the fact that, by analogy with the normal Lie algebra situation (see e.g. [45]), their eigenvalues determine the squared reduced matrix elements of the \( gl(m+n+1) \) generators \( \psi^p \) and \( \phi_p \) respectively.

As motivation, consider the matrix elements of the \( \psi^p \), in a unitary (star) representation of \( gl(m|n+1) \) [70]. Using a notation reminiscent of [58], and keeping in mind (21) and (24), these matrix elements may be expressed

\[
\left\langle \left( \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda \\ \end{array} \right) | \psi[r]^p | \left( \begin{array}{c} \Lambda \\ \lambda' \\ \end{array} \right) \right\rangle = \langle \Lambda + \varepsilon_r | | \psi| | \Lambda \rangle \left\langle \left( \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda \\ \end{array} \right) | e_p \otimes \left( \begin{array}{c} \Lambda \\ \lambda' \\ \end{array} \right) \right\rangle,
\]

where \( \left( \begin{array}{c} \Lambda \\ \lambda' \\ \end{array} \right) \) denotes a suitable orthonormal basis for the \( gl(m|n) \) module \( V(\Lambda) \subseteq V(\tilde{\Lambda}) \) concerned, \( \{ e_p \} \) is a basis for the vector representation, and \( \langle \Lambda + \varepsilon_r | | \psi| | \Lambda \rangle \) is the reduced matrix element. Furthermore, let \( V(\tilde{\Lambda}) \) be a unitary representation such that

\[
\psi^1[r]^p = \phi[r]p,
\]

from which we obtain

\[
\left\langle \left( \begin{array}{c} \Lambda \\ \lambda' \\ \end{array} \right) | \phi[r]q\psi[r]^q | \left( \begin{array}{c} \Lambda \\ \lambda'' \\ \end{array} \right) \right\rangle = |\langle \Lambda + \varepsilon_r | | \psi| | \Lambda \rangle|^2 \times \sum_{\lambda} \left\langle \left( \begin{array}{c} \Lambda \\ \lambda' \\ \end{array} \right) \otimes e_q | \left( \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda \\ \end{array} \right) \right\rangle \left\langle \left( \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda \\ \end{array} \right) | e_p \otimes \left( \begin{array}{c} \Lambda \\ \lambda'' \\ \end{array} \right) \right\rangle = |\langle \Lambda + \varepsilon_r | | \psi| | \Lambda \rangle|^2 \left\langle \left( \begin{array}{c} \Lambda \\ \lambda' \\ \end{array} \right) | \tilde{P}[r]q^p | \left( \begin{array}{c} \Lambda \\ \lambda'' \\ \end{array} \right) \right\rangle. \tag{33}
\]
At this point we find it instructive to introduce the gl(m|n + 1) matrices
\[
\begin{align*}
\mathcal{B}_p &= (-1)^{(p)}E_{pq}, \\
\mathcal{B}_q &= (-1)^{(p)(q)}E_{pq},
\end{align*}
\tag{34}
\]
and the corresponding gl(m|n + 1) characteristic roots \( \bar{\beta}_r \) and \( \beta_r \) (1 ≤ \( r \) ≤ m + n + 1), determined by analogy with equations (11), (12), (15) and (16), and which evaluate on an irreducible gl(m|n + 1)-module \( V(\hat{\Lambda}) \) to
\[
\begin{align*}
\bar{\beta}_i &= i - 1 - \hat{\Lambda}_i, \\
\bar{\beta}_\mu &= \hat{\Lambda}_\mu + m + 1 - \mu, \\
\beta_i &= \hat{\Lambda}_i + m - n - 1 - \mu, \\
\beta_\mu &= \mu - \hat{\Lambda}_\mu - n - 1,
\end{align*}
\]
with 1 ≤ \( i \) ≤ \( m \) and 1 ≤ \( \mu \) ≤ \( n + 1 \). We then also have the associated projection operators
\[
Q[r] = \prod_{k \neq r}^{m+n+1} \left( \mathcal{B} - \beta_k \right), \quad \bar{Q}[r] = \prod_{k \neq r}^{m+n+1} \left( \mathcal{B} - \bar{\beta}_k \right).
\]

It is our aim here to evaluate the invariants \( \bar{c}_1 \) and \( \bar{c}_2 \) as rational polynomial functions in the gl(m|n) and gl(m|n + 1) characteristic roots.

Note first that the \((m + n + 1, m + n + 1)\) entries of the projection matrices,
\[
c_r = Q[r]^{m+n+1}_{m+n+1}, \quad \bar{c}_r = \bar{Q}[r]^{m+n+1}_{m+n+1}, \quad 1 ≤ r ≤ m + n + 1 \tag{35}
\]
clearly determine (even) invariants of gl(m|n) whose eigenvalues, by analogy with the Lie algebra situation \[47\], are given by certain gl(m|n + 1) ⊃ gl(m|n) reduced Wigner coefficients.

To demonstrate this point by way of example, note that in the context of \( \text{gl}(m|n + 1) \), using a basis of the unitary \( \text{gl}(m|n + 1) \) module \( V(\hat{\Lambda}) \) symmetry adapted to \( \text{gl}(m|n) \), we have
\[
\left\langle \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda \\
\lambda
\end{array} \right) \middle| \bar{Q}[r] \right| p \right\rangle \left\langle \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda' \\
\lambda'
\end{array} \right) \right\rangle = \sum_{\lambda',\lambda} \left\langle \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda \\
\lambda
\end{array} \right) \otimes e_p \left| \left( \begin{array}{c}
\hat{\Lambda} + \varepsilon_r \\
\Lambda' \\
\lambda
\end{array} \right) \right\rangle \left\langle \left( \begin{array}{c}
\hat{\Lambda} + \varepsilon_r \\
\Lambda' \\
\lambda
\end{array} \right) \middle| e_q \otimes \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda' \\
\lambda'
\end{array} \right) \right\rangle.
\]
In particular, setting \( p = q = m + n + 1 \) gives
\[
\left\langle \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda \\
\lambda
\end{array} \right) \middle| \bar{c}_r \right\rangle \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda' \\
\lambda'
\end{array} \right) = \delta_{\Lambda,\Lambda'} \delta_{\lambda,\lambda'} \left| \left\langle \left( \begin{array}{c}
\hat{\Lambda} + \varepsilon_r \\
\Lambda \\
\lambda
\end{array} \right) \middle| e_{m+n+1} \otimes \left( \begin{array}{c}
\hat{\Lambda} \\
\Lambda \\
\lambda
\end{array} \right) \right\rangle \right|^2,
\]
where we have used the fact that \( \bar{c}_r \) leaves the representation labels of \( \text{gl}(m|n) \) unchanged. Thus the invariants \( \bar{c}_r \) (and similarly \( c_r \)) determine squares of reduced Wigner coefficients. Note that this is the archetypical example where the reduced Wigner coefficient and Wigner coefficient (c.f. coupling coefficient or Clebsch-Gordan coefficient) coincide.
To appreciate the strength of our approach, we demonstrate some results for the types of invariants we are considering. Recall the recursive definition for powers of the matrices $\mathcal{A}$ and $\mathcal{B}$ respectively, namely

$$(\mathcal{A}^k)_q^p = \sum_{r=1}^{m+n} \mathcal{A}_r^p \left(\mathcal{A}^{k-1}\right)_q^r, \quad \left(\mathcal{A}^0\right)_q^p \equiv \delta_{pq}, \quad 1 \leq p, q \leq m + n. \quad (36)$$

and

$$(\mathcal{B}^k)_q^p = \sum_{r=1}^{m+n+1} \mathcal{B}_r^p \left(\mathcal{B}^{k-1}\right)_q^r, \quad \left(\mathcal{B}^0\right)_q^p \equiv \delta_{pq}, \quad 1 \leq p, q \leq m + n + 1. \quad (37)$$

We define the two $gl(m|n)$ invariants occurring in $gl(m|n+1)$:

$$\tau_k = (\mathcal{B}^k)_{m+n+1}^{m+n+1},$$

$$\sigma_k = \mathcal{B}_{m+n+1}^{m+n+1}(\mathcal{A}^k)_{m+n+1}^{m+n+1},$$

where $\mathcal{A}$ is the $(m+n) \times (m+n)$ submatrix of $\mathcal{B}$ that contains the $gl(m|n)$ generators. It is worth observing that the invariants we consider below can then be expressed in terms of the $\sigma_k$ and $\tau_k$.

Following Green \[39\] who studied similar objects associated with classical Lie algebras, we could adopt the approach of explicitly determining the eigenvalues of these invariants by attempting to express the invariants themselves in terms of the $gl(m|n+1)$ Casimir invariants $\hat{I}_k$ and the $gl(m|n)$ Casimir invariant $I_k$ contained in $gl(m|n+1)$, whose eigenvalues in turn can be expressed in terms of the highest weights labels (see for example equation (7)). The Casimir invariants are defined by

$$\hat{I}_k = (-1)^{(p)}(\mathcal{B}^k)_p^p = (-1)^{(p)}B_{q_1}B_{q_2}B_{q_3} \cdots B_{p_{k-1}}, \quad 1 \leq p, q_1, \ldots, q_{k-1} \leq m + n + 1, \quad (38)$$

$$I_k = (-1)^{(\hat{p})}(\mathcal{A}^k)_p^p = (-1)^{(\hat{p})}A_{q_1}A_{q_2}A_{q_3} \cdots A_{\hat{p}_{k-1}}, \quad 1 \leq \hat{p}, \hat{q}_1, \ldots, \hat{q}_{k-1} \leq m + n. \quad (39)$$

In Appendix C, we give recursion formulae that enable us to express the $\sigma$ invariants in terms of the $\tau$ invariants, and more importantly, we are also able to express the $\tau$ invariants in terms of lower order $\tau$ invariants and the Casimir invariants $I_k$ and $\hat{I}_k$. Some examples of the results of such calculations are as follows:

$$2\tau_2 = \left(I_2 - \hat{I}_2\right) + \left(I_1 - \hat{I}_1\right)^2 - I_1 + (m - n) \left(I_1 - \hat{I}_1\right),$$

$$3\tau_3 = \left(I_3 - \hat{I}_3\right) + \tau_1 \left(I_2 - \hat{I}_2\right) + \tau_2 \left(I_1 - \hat{I}_1\right) - 2(\tau_2 + \hat{I}_2) - (\tau_1 + \hat{I}_1) + 2(m - n)\tau_2 + (m - n)\tau_1 + \tau_2 + (\tau_1)^2,$$

$$4\tau_4 = \left(I_4 - \hat{I}_4\right) + \tau_1 \left(I_3 - \hat{I}_3\right) + \tau_2 \left(I_2 - \hat{I}_2\right) + \tau_3 \left(I_1 - \hat{I}_1\right) - 3(\hat{I}_3 - (m - n - 1)\tau_3) - \hat{I}_2\tau_1 + \hat{I}_1\tau_2 + 4\tau_1\tau_2 - (\tau_1)^3 - 3((m - n - 1)\tau_2 - \hat{I}_2 + \tau_3) - (\tau_1)^2 + (m - n - 1)(\tau_1)^2 - \tau_1\hat{I}_1 - (m - n - 1)\tau_1 + \hat{I}_1 - \tau_2.$$
See Appendix C for details.

It is clear from these calculations that taking the approach outlined above for determining the invariants, or more to the point their eigenvalues, leads to complicated recursion relations. The results of the present article, however, completely bypass such complexities, and we find that we are able to present elegant eigenvalue formulae using the characteristic identities.

**Remark:**

Let \( \hat{Z} \) be the centre of \( U(\hat{L}) \) and \( Z \) the centre of \( U(L) \) for \( \hat{L} \) and \( L \) Lie superalgebras such that \( L \subseteq \hat{L} \). In the spirit of Joseph’s second commutant theorems \([74]\), we conjecture that the embedding \( gl(m|n) \subset gl(m|n + 1) \) is canonical, i.e. that the double commutant of \( L = gl(m|n) \) in \( U\left(\hat{L} = gl(m|n + 1)\right) \) is precisely \( U(L)\hat{Z} \). Hence for this case the centraliser

\[
C(L) = \left\{ u \in U(\hat{L}) \mid ux - xu = 0, \forall x \in L \right\}
\]

of \( L \) in \( U(\hat{L}) \) is given explicitly by \( C(L) = \hat{Z}Z \).

From the \( gl(m|n + 1) \) characteristic identity (i.e. the \( gl(m|n + 1) \) analogue of \((14)\)), we have \((1 \leq p \leq m + n)\)

\[
\sum_{q=1}^{m+n+1} (\mathcal{B}^p_q - \beta_r \delta^p_q)Q[r]^q_r = 0,
\]

in particular

\[
\sum_{q=1}^{m+n} \mathcal{B}^p_q Q[r]^q_{m+n+1} + \mathcal{B}^p_{m+n+1}c_r = \beta_r Q[r]^p_r
\]

which may be rearranged to give

\[
\psi^p c_r = \sum_{q=1}^{m+n} (\beta_r - \mathcal{A})^p_q Q[r]^q_{m+n+1}
\]  \( (40) \)

where \( \mathcal{A} \) is the \( gl(m|n) \) matrix and we have employed the result

\[
\mathcal{B}^p_{m+n+1} = (-1)^{(p)}E_{p,m+n+1} = \psi^p, \quad 1 \leq p \leq m + n.
\]

We note that the \( gl(m|n + 1) \downarrow gl(m|n) \) branching rules imply that there may exist degeneracies between the even roots of \( gl(m|n + 1) \) and those of \( gl(m|n) \). Indeed, as we pointed out when the \( gl(m|n + 1) \) characteristic roots were introduced, the even \( gl(m|n + 1) \) roots are expressible in terms of the \( gl(m|n + 1) \) representation labels \( \hat{\Lambda}_i \) as

\[
\beta_i = \tilde{\Lambda}_i + m - n - 1 - i, \quad 1 \leq i \leq m.
\]
In view of the betweenness conditions \((29)\) we thus have
\[
\beta_i = \begin{cases} 
\alpha_i, & \tilde{\Lambda}_i = 1 + \Lambda_i \\
\alpha_i - 1, & \tilde{\Lambda}_i = \Lambda_i
\end{cases} \quad (41)
\]
This suggests that we introduce the even index sets
\[
I_0 = \{1 \leq i \leq m \mid \alpha_i = \beta_i\}, \quad \tilde{I}_0 = \{1 \leq i \leq m \mid \alpha_i = 1 + \beta_i\}
\]
and the full index sets
\[
I = I_0 \cup I_1, \quad \tilde{I} = I \cup \{m + n + 1\}
\]
where \(I_1\) denotes the set of odd indices \(\mu = 1, 2, \ldots, n\).

The importance of introducing the above index sets lies in the fact that if \(i \in \tilde{I}_0\) then the shift components \(\psi[i]^p\) must vanish on the representation of \(gl(m|n)\) concerned, since the label \(\Lambda_i, i \in \tilde{I}_0\), already takes its maximum value. In a similar way, the operator \(c_i, i \in \tilde{I}_0\), must vanish. On the other hand, for \(i \in I_0\), the operators \(\phi[i]^p\) and \(\tilde{c}_i\) must vanish on the representation concerned. We note also that the operators \(Q[i]^p_{m+n+1} (i \in \tilde{I}_0)\) and \(\tilde{Q}[i]^p_{m+n+1} (i \in I_0)\) must vanish on the representation of \(gl(m|n) \subset gl(m|n+1)\) concerned by an analogous argument.

Now inverting equation \((40)\) gives
\[
Q[r]_{m+n+1}^p = \sum_{q=1}^{m+n} ((\beta_r - \mathcal{A})^{-1})^p_q \psi^q \psi^r_c = \sum_{q=1}^{m+n} \sum_{r=1}^{m+n} (\beta_r - \alpha_r)^{-1} P[r]_q^p \psi^q \psi^r_c
\]
\[
= \sum_{r=1}^{m+n} (\beta_r - \alpha_r)^{-1} \psi[r]^p \psi^r_c = \sum_{s \in \tilde{I}} (\beta_r - \alpha_s)^{-1} \psi[s]^p \psi^r_c, \quad r \in \tilde{I}
\]
where we have used the fact that \(\psi[s]^p, Q[r]^p_{m+n+1}\) and \(c_r\) all vanish for \(r, s \in \tilde{I}_0\), and where \((\beta_r - \mathcal{A})^{-1}\) denotes the matrix
\[
(\beta_r - \mathcal{A})^{-1} = \sum_{r=1}^{m+n} (\beta_r - \alpha_r)^{-1} P[r].
\]
It is straightforward to establish the shift relation
\[
(\beta_r - \alpha_s)^{-1} \psi[s] = \psi[s](\beta_r - \alpha_s - (-1)^{(s)})^{-1},
\]
which then allows us to write
\[
Q[r]_{m+n+1}^p = \sum_{s \in \tilde{I}} \psi[s]^p (\beta_r - \alpha_s - (-1)^{(s)})^{-1} c_r, \quad r \in \tilde{I}. \quad (42)
\]
Summing this equation over \( r \) we thus obtain
\[
\sum_{s \in I} \psi[s]^p \sum_{r \in I} (\beta_r - \alpha_s - (-1)^{(s)})^{-1} c_r = 0
\]  
(43)
which is a direct consequence of the identity resolution
\[
\sum_{r=1}^{m+n+1} Q[r]^p m+n+1 = \sum_{r \in I} Q[r]^p m+n+1 = \delta^p_{m+n+1} = 0, \quad 1 \leq p \leq m+n.
\]
Resolving equation (43) into shift components we obtain, in view of the linear independence of the \( \psi[s]^p \), the following set of equations:
\[
\sum_{r \in I} (\beta_r - \alpha_s - (-1)^{(s)})^{-1} c_r = 0, \quad s \in I.
\]  
(44)
Equation (44) yields \(|I|\) relations in \(|\tilde{I}| = 1 + |I|\) unknowns \( c_r \). To uniquely determine the \( c_r \) we need the extra relation
\[
\sum_{r \in \tilde{I}} c_r = 1
\]  
(45)
which follows from the identity resolution
\[
\sum_{r=1}^{m+n+1} Q[r]^m+n+1 = \sum_{r \in \tilde{I}} Q[r]^m+n+1 = \delta^m+n+1_{m+n+1} = 1.
\]

Using straightforward techniques of linear algebra, equations (44) and (45) yield the unique solution
\[
c_s = \prod_{k \in \tilde{I}, k \neq s} (\beta_s - \beta_k)^{-1} \prod_{r \in I} (\beta_s - \alpha_r - (-1)^{(r)})^{-1}, \quad s \in \tilde{I}
\]  
(46)
which may be expressed in terms of odd and even indices according to
\[
c_i = 0, \quad i \in \tilde{I}_0, \\
\]
c_i = - \prod_{j \in I_0, j \neq i} \left( \frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \right)^{n+1} \prod_{\mu=1}^{n} (\beta_i - \beta_\mu)^{-1} \prod_{\mu=1}^{n} (\beta_i - \alpha_\mu + 1), \quad i \in I_0, \\
\]
c_\mu = \prod_{i \in I_0} \left( \frac{\beta_\mu - \beta_i - 1}{\beta_\mu - \beta_i} \right)^{n+1} \prod_{\nu \neq \mu} (\beta_\mu - \beta_\nu)^{-1} \prod_{\nu=1}^{n} (\beta_\mu - \alpha_\nu + 1), \quad 1 \leq \mu \leq n+1.
\]
We note that these formulae can be expressed independently of the index set notation as
\[
c_i = (\alpha_i - \beta_i - 1) \prod_{k \neq i}^m \left( \frac{\beta_i - \beta_k - 1}{\beta_i - \alpha_k} \right)^{n+1} \prod_{\nu=1}^{n} (\beta_i - \beta_\nu)^{-1} \prod_{\nu=1}^{n} (\beta_i - \alpha_\nu + 1), \quad 1 \leq i \leq m, \\
c_\mu = \prod_{k=1}^{m} \left( \frac{\beta_\mu - \beta_k - 1}{\beta_\mu - \alpha_k} \right)^{n+1} \prod_{\nu \neq \mu} (\beta_\mu - \beta_\nu)^{-1} \prod_{\nu=1}^{n} (\beta_\mu - \alpha_\nu + 1), \quad 1 \leq \mu \leq n+1.
\]
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Equation (46) enables a uniform evaluation of all $gl(m|n)$ invariants of the form
\[ p(B)^{m+n+1}_{m+n+1} \]
for arbitrary polynomials $p(x)$, according to the spectral resolution
\[ p(B)^{m+n+1}_{m+n+1} = \sum_{r \in I} p(\beta_r)c_r. \]

Using the $gl(m|n+1)$ adjoint identity we similarly have
\[ \tilde{A}_p q\tilde{Q}[r]_q^{m+n+1} + \tilde{B}_p^{m+n+1}c_r = \tilde{\beta}_r \tilde{Q}[r]_p^{m+n+1} \]
which may be rearranged to give
\[ -\phi_p \tilde{c}_r = (\tilde{\beta}_r - \tilde{A}) q\tilde{Q}[r]_q^{m+n+1}, \]
where we have used $\tilde{B}_p^{m+n+1} = -(-1)^{(p)}E_{m+n+1,p} = -\phi_p$. Inverting this equation and resolving $\phi_p$ into its shift components as before, noting in this case that
\[ \phi[i]_p = \tilde{Q}[i]_p^{m+n+1} = 0, \quad i \in I_0, \]
we obtain
\[ \tilde{Q}[r]_p^{m+n+1} = -\sum_{s \in I'} \phi[s]_p(\tilde{\beta}_s - \tilde{\alpha}_s - (-1)^{(s)}(r))^{-1} \tilde{c}_r, \quad r \in \tilde{I}' \tag{47} \]
where $\tilde{I}' = I' \cup \{m+n+1\}$ denotes the index set given by $I' = \tilde{I}_0 \cup I_1$ (note that $I' \cap I = I_1$ is the set of odd indices). In this case, we obtain by analogy with equation (46),
\[ \tilde{c}_s = \prod_{k \in I', k \neq s} (\tilde{\beta}_s - \tilde{\beta}_k)^{-1} \prod_{r \in I'} (\tilde{\beta}_s - \tilde{\alpha}_r - (-1)^{(r)}) , \quad s \in \tilde{I}' \tag{48} \]
and, of course, $\tilde{c}_s = 0$ for $s \in I_0$. As in the case of the $c_s$, we can express these formulae independently of the index set notation as
\[ \tilde{c}_i = (\beta_i - \alpha_i) \prod_{k \neq i}^{m} \left( \frac{\beta_k - \beta_i - 1}{\alpha_k - \beta_i - 1} \right) \prod_{\nu=1}^{n+1} (\beta_\nu - \beta_i - 2)^{-1} \prod_{\nu=1}^{n} (\alpha_\nu - \beta_i - 2), \quad 1 \leq i \leq m, \]
\[ \tilde{c}_\mu = \prod_{k=1}^{m} \left( \frac{\beta_k - \beta_\mu + 1}{\alpha_k - \beta_\mu + 1} \right) \prod_{\nu \neq \mu}^{n+1} (\beta_\nu - \beta_\mu)^{-1} \prod_{\nu=1}^{n} (\alpha_\nu - \beta_\mu), \quad 1 \leq \mu \leq n + 1. \]

We summarise the results in the following theorem.

**Theorem 3** The $gl(m|n)$ invariants $c_s$ and $\tilde{c}_s$, as given in (45), have eigenvalues on an irreducible $gl(m|n)$ module, with highest weight subject to the branching conditions (29), given respectively by equations (44) and (48).
To evaluate the invariants $\gamma_r$ of equation (31), we invert equation (42) to obtain $\psi[r]^p (r \in I)$ as a linear combination of the $Q[s]^p_{m+n+1} (s \in I)$. This leads us to look for the unique solution $\gamma_{rs} (s \in I, r \in I)$ to the set of equations

$$\sum_{s \in I} \gamma_{rs} \left( \beta_s - \alpha_q - (-1)^{(q)} \right) c_s = \delta_{rq}, \quad r, q \in I, \quad (49)$$

$$\sum_{s \in I} \gamma_{rs} c_s = 0. \quad (50)$$

Then for each $r \in I$, equations (49) and (50) yield $|\tilde{I}| = |I| + 1$ equations in $|\tilde{I}|$ unknowns $\gamma_{rs} (s \in I)$. These equations are easily solved using standard techniques of linear algebra and yield the unique solution

$$\gamma_{rs} = -\gamma_r \left( \beta_s - \alpha_r - (-1)^{(r)} \right)^{-1}, \quad r \in I, \quad s \in I,$$

where

$$\gamma_r = (-1)^{|\tilde{I}|} \prod_{q \in I, q \neq r} \left( \alpha_r - \alpha_q + (-1)^{(r)} - (-1)^{(q)} \right)^{-1} \prod_{p \in I} \left( \beta_p - \alpha_r - (-1)^{(r)} \right). \quad (51)$$

As the notation above suggests, formula (51) determines the eigenvalues of the invariants (31), as we shall now demonstrate. Multiplying equation (42) on the right by $\gamma_{qr}$, summing on $r \in I$ and making use of equation (49) we have

$$\sum_{r \in I} Q[r]^p_{m+n+1} \gamma_{qr} = \sum_{s \in I} \psi[s]^p \sum_{r \in I} \gamma_{qr} \left( \beta_r - \alpha_s - (-1)^{(s)} \right)^{-1} c_r$$

$$= \psi[q]^p.$$

Thus we obtain

$$(-1)^{(p)} \phi[q] \psi[q]^p = (-1)^{(p)} \phi_p \psi[q]^p$$

$$= \sum_{r \in I} (-1)^{(p)} \phi_p Q[r]^p_{m+n+1} \gamma_{qr}$$

$$= - \sum_{r \in I} \left( \beta_r - B^{m+n+1}_{m+n+1} \right) c_r \gamma_{qr}$$

$$= - \sum_{r \in I} \beta_r c_r \gamma_{qr}$$

where we have used the result $(-1)^{(p)} \phi_p = E_{m+n+1,p} = -B^{m+n+1}_{m+n+1}$ and employed equation
where the index sets $I$ the invariants $\bar{\psi}$ of equation (31) are given by the formula (51).

In graded index notation, we thus obtain the following formulae for the invariants $\bar{\psi}$ in the last step. Thus we may write

$$(-1)^p \phi[q] \psi[q]^p = - \sum_{r \in I} \beta_r c_r \gamma_{qr}$$

$$= \gamma_q \sum_{r \in I} \beta_r c_r \left( \beta_r - \alpha_q - (-1)^q \right)^{-1}$$

$$= \gamma_q \left( \sum_{r \in I} c_r + (\alpha_q + (-1)^q) \sum_{r \in I} c_r \left( \beta_r - \alpha_q - (-1)^q \right)^{-1} \right)$$

$$= \gamma_q,$$

where in the last step we employed equations (44) and (45). This shows the required result, that the eigenvalues of the $gl(m|n)$ invariants

$$\gamma_q = (-1)^p \phi[q] \psi[q]^p$$

of equation (31) are given by the formula (51).

In a similar way, by employing equation (47), we arrive at the following formula for the invariants $\bar{\gamma}_q = \psi[q] \psi[q]_\rho$ of equation (32):

$$\bar{\gamma}_q = (-1)^{|I'|} \prod_{r \in I', r \neq q} (\bar{\alpha}_q - \bar{\alpha}_r + (-1)^q - (-1)^{|r|})^{-1} \prod_{s \in I'} (\bar{\beta}_s - \bar{\alpha}_q - (-1)^q), \quad (52)$$

where the index sets $I', \bar{I}'$ are as in equation (48). Note that for $i \in I_0$, $\bar{\gamma}_i = 0$, while for $i \in \bar{I}_0$, $\gamma_i = 0$. The remaining non-zero cases are given by equations (51) and (52). Using the easily established relations

$$\alpha_s + \bar{\alpha}_s = m - n - (-1)^{(s)}, \quad \beta_s + \bar{\beta}_s = m - n - 1 - (-1)^{(s)},$$

we note that equation (52) may be alternatively expressed

$$\bar{\gamma}_q = (-1)^{|I'|} \prod_{r \in I', r \neq q} (\alpha_q - \alpha_r)^{-1} \prod_{s \in I'} (\beta_s - \alpha_q + (-1)^{(s)} + 1), \quad q \in I'. \quad (53)$$

In graded index notation, we thus obtain the following formulae for the invariants $\gamma_r, \bar{\gamma}_r$:

$$\gamma_i = 0, \quad i \in \bar{I}_0,$$

$$\gamma_i = \prod_{j \in I_0, j \neq i} \left( \frac{\alpha_i - \alpha_j + 1}{\alpha_i - \alpha_j} \right) \prod_{o = 1}^n (\alpha_i - \alpha_o + 2)^{-1} \prod_{\mu = 1}^{n+1} (\alpha_i - \beta_\mu + 1), \quad i \in I_0,$$

$$\gamma_\mu = \prod_{j \in I_0} \left( \frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \right) \prod_{\nu \neq \mu} (\alpha_\mu - \alpha_o)^{-1} \prod_{\nu = 1}^{n+1} (\alpha_\mu - \beta_\nu - 1), \quad 1 \leq \mu \leq n,$$

$$\bar{\gamma}_i = 0, \quad i \in I_0,$$

$$\bar{\gamma}_i = \prod_{j \in \bar{I}_0, j \neq i} \left( \frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \right) \prod_{o = 1}^n (\alpha_i - \alpha_o)^{-1} \prod_{\mu = 1}^{n+1} (\alpha_i - \beta_\mu), \quad i \in \bar{I}_0,$$

$$\bar{\gamma}_\mu = \prod_{j \in \bar{I}_0} \left( \frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \right) \prod_{\nu \neq \mu} (\alpha_\mu - \alpha_o)^{-1} \prod_{\nu = 1}^{n+1} (\alpha_\mu - \beta_\nu), \quad 1 \leq \mu \leq n.
Alternatively, we may express these formulae independently of the index sets:

\[
\gamma_i = (\beta_i - \alpha_i + 1) \prod_{k \neq i} \left( \frac{\alpha_k - \alpha_i - 1}{\beta_k - \alpha_i} \right) \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_i - 1) \prod_{\nu \neq \mu}^{n} (\alpha_\nu - \alpha_i - 2)^{-1}, \quad 1 \leq i \leq m,
\]

\[
\gamma_\mu = \prod_{k=1}^{m} \left( \frac{\alpha_k - \alpha_\mu + 1}{\beta_k - \alpha_\mu + 2} \right) \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_\mu + 1) \prod_{\nu \neq \mu}^{n} (\alpha_\nu - \alpha_\mu)^{-1}, \quad 1 \leq \mu \leq n,
\]

\[
\bar{\gamma}_i = (\alpha_i - \beta_i) \prod_{k \neq i} \left( \frac{\alpha_k - \alpha_i + 1}{\beta_k - \alpha_i + 1} \right) \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_i) \prod_{\nu \neq \mu}^{n} (\alpha_\nu - \alpha_i)^{-1}, \quad 1 \leq i \leq m,
\]

\[
\bar{\gamma}_\mu = - \prod_{k=1}^{m} \left( \frac{\alpha_k - \alpha_\mu + 1}{\beta_k - \alpha_\mu + 1} \right) \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_\mu) \prod_{\nu \neq \mu}^{n} (\alpha_\nu - \alpha_\mu)^{-1}, \quad 1 \leq \mu \leq n.
\]

We summarise the previous discussion in the following theorem, which is analogous to Theorem 3 but for \(\gamma_r\) and \(\bar{\gamma}_r\).

**Theorem 4** The \(gl(m|n)\) invariants \(\gamma_r\) and \(\bar{\gamma}_r\), as given by equations (54) and (55) respectively, have eigenvalues on an irreducible \(gl(m|n)\) module, with highest weight subject to the branching conditions (29), that are either zero or given respectively by equations (51) and (52).

We now note, as in the corresponding Lie algebra case [45], that

\[
(-1)^{(q)} \psi[r]^p (\gamma_r)^{-1} \phi[r]_q = P[r]^p_q, \quad r \in I',
\]

\[
\phi[r]^p (\bar{\gamma}_r)^{-1} \psi[r]_q = \bar{P}[r]^p_q, \quad r \in I.
\]

The above suggests that we define new invariants \(\delta_r, \bar{\delta}_r\) such that

\[
\delta_r \psi[r] = \psi[r] \gamma_r, \quad \bar{\delta}_r \phi[r] = \phi[r] \bar{\gamma}_r,
\]

in which case equations (54) and (55) may be rewritten

\[
(-1)^{(q)} \psi[r]^p \phi[r]_q = \delta_r P[r]^p_q, \quad r \in I',
\]

\[
\phi[r]^p \psi[r]_q = \bar{\delta}_r \bar{P}[r]^p_q, \quad r \in I.
\]

To justify the relevance of these equations to reduced matrix elements, compare equation (58) to equation (33) in the case of unitary representations. In such a case, this shows that the \(gl(m|n)\) invariants \(\delta_r\) determine the squared reduced matrix elements of \(\psi[r]^p\). We can apply an analogous argument to show the invariants \(\delta_r\) determine the squared reduced matrix elements of \(\phi[r]_p\). Even in the non-unitary case, we expect these invariants will still determine squared reduced matrix elements. We may therefore be inclined to refer to \(\delta_r, \bar{\delta}_r\) as **generalised squared reduced matrix elements**.

Using our formulae for \(\gamma_r, \bar{\gamma}_r\), we easily obtain the following corollary to Theorem 4.
Corollary 5 The \( gl(m|n) \) invariants \( \delta_r \) and \( \tilde{\delta}_r \), satisfying (57), have eigenvalues on an irreducible \( gl(m|n) \) module, with highest weight subject to the branching conditions (29), given by
\[
\delta_r = (-1)^{|I|} \prod_{s \in I, s \neq r} (\alpha_r - \alpha_s - (1)^{(s)})^{-1} \prod_{q \in I} (\beta_q - \alpha_r), \quad r \in I',
\]
(59)
\[
\tilde{\delta}_r = (-1)^{|I'|} \prod_{q \in I', q \neq r} (\alpha_r - \alpha_q + (1)^{(r)})^{-1} \prod_{s \in I'} (\beta_s - \alpha_r + (1)^{(s)} - (1)^{(r)} + 1), r \in I.
\]
(60)

We remark that the formulae (59) and (60) of Corollary 5 may be expressed independently of the index set notation as
\[
\delta_i = (\beta_i - \alpha_i) \prod_{k \neq i} (\frac{\alpha_k - \alpha_i}{\beta_k - \beta_i}) \prod_{\nu=1}^{n} (\alpha_\nu - \alpha_i - 1)^{-1} \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_i), \quad 1 \leq i \leq m,
\]
\[
\delta_\mu = - \prod_{k=1}^{m} (\frac{\alpha_k - \alpha_\mu}{\beta_k - \alpha_\mu}) \prod_{\nu=1}^{n} (\alpha_\nu - \alpha_\mu - 1)^{-1} \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_\mu), \quad 1 \leq \mu \leq n,
\]
\[
\tilde{\delta}_i = (\beta_i - \alpha_i + 1) \prod_{k \neq i} (\frac{\alpha_k - \alpha_i}{\beta_k - \beta_i}) \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_\mu - 1)^{-1} \prod_{\nu=1}^{n} (\alpha_\nu - \alpha_i - 1)^{-1}, \quad 1 \leq i \leq m,
\]
\[
\tilde{\delta}_\mu = - \prod_{k=1}^{m} (\frac{\alpha_k - \alpha_\mu + 2}{\beta_k - \alpha_\mu + 2}) \prod_{\nu=1}^{n+1} (\beta_\nu - \alpha_\mu + 1)^{-1} \prod_{\nu=1}^{n} (\alpha_\nu - \alpha_\mu + 1)^{-1}, \quad 1 \leq \mu \leq n.
\]

As a check on these equations, we note that taking the supertrace of equations (57) and (58) shows that the invariants discussed are related by
\[
\tilde{\gamma}_r = \delta_i \text{str}(P[r]),
\]
(61)
\[
\gamma_r = \tilde{\delta}_i \text{str}(\tilde{P}[r]),
\]
(62)
which can be regarded as defining \( \delta_r, \tilde{\delta}_r \). We note immediately that
\[
\delta_i = 0, \quad i \in I_0, \quad \tilde{\delta}_i = 0, \quad i \in \tilde{I}_0,
\]
so that equations (59) and (60) give all the non-zero eigenvalues of the invariants \( \delta_r, \tilde{\delta}_r \).

Our interest here is the fact that equations (61) and (62) enable us to evaluate the supertraces of the \( gl(m|n) \) projection operators and hence the eigenvalues of all \( gl(m|n) \) Casimir invariants, previously obtained by Jarvis and Green [66] using other methods. We have
\[
\text{str}(P[r]) = \tilde{\gamma}_r \delta_r^{-1}.
\]
Note that strictly speaking, this formula is only valid on the Zariski dense subset as mentioned earlier. For more on the general case, see [72][73]. In graded index notation,
we furthermore have
\[ \tilde{\gamma}_r = (-1)^{|I'|} \prod_{i \in \bar{I}_0, i \neq r} (\alpha_r - \alpha_i)^{-1} \prod_{i \in I_0} (\alpha_i + 1 - \alpha_r)^{-1} \prod_{\nu \neq r}^{\nu = 1} (\alpha_r - \alpha_\nu)^{-1} \prod_{\nu = 1}^{n+1} (\beta_\nu - \alpha_r) \]

\[ (\delta_r)^{-1} = (-1)^{|I'|} \prod_{i \in I_0} (\alpha_r - \alpha_i - 1) \prod_{i \in I_0} (\alpha_i + 1 - \alpha_r)^{-1} \prod_{\nu \neq r}^{\nu = 1} (\alpha_r - \alpha_\nu + 1) \prod_{\nu = 1}^{n+1} (\beta_\nu - \alpha_r)^{-1} \]

\[ \Rightarrow \ \text{str}(P[r]) = \prod_{q \neq r}^{m+n} \left( \frac{\alpha_r - \alpha_q - (-1)^{(q)}}{\alpha_r - \alpha_q} \right) \quad (63) \]

where in the above, we exploited the fact that \( \beta_i = \alpha_i \) (respectively \( \alpha_i - 1 \)) for \( i \in \bar{I}_0 \) (respectively \( I_0 \)). Equation (63) agrees with the supertrace formula, previously derived by Jarvis and Green [66], which provides a check on our formalism. Care is needed, however, in dealing with atypical representations.

It follows, in view of the spectral resolution, that the eigenvalues of the superinvariants

\[ I_k = \text{str}(A^k) = \sum_{p=1}^{m+n} (-1)^{(p)} (A^k)^p \]

are given by

\[ I_k = \sum_{r=1}^{m+n} (\alpha_r)^k \text{str}(P[r]). \]

In a similar way we have the adjoint invariants

\[ \bar{I}_k = \text{str}(\bar{A}^k) = \sum_{r=1}^{m+n} (\bar{\alpha}_r)^k \text{str}(\bar{P}[r]) \]

which may be evaluated with the help of the easily established formula

\[ \text{str}(\bar{P}[r]) = \gamma_r (\bar{\delta}_r)^{-1} = \prod_{q \neq r}^{m+n} \left( \frac{\alpha_r - \alpha_q + (-1)^{(r)}}{\alpha_r - \alpha_q + (-1)^{(r)} - (-1)^{(q)}} \right). \]

8 Concluding remarks

In this paper we have demonstrated how characteristic identities and projection techniques can be applied to determine certain \( gl(m|n) \) invariants in (an algebraic extension of) the enveloping algebra of \( gl(m|n+1) \). We have provided explicit derivations of the eigenvalues of such invariants on any irreducible representation, and our key formulae were presented in Theorems 3 and 4 and Corollary 5.

The next step will be the application of these results to unitary irreducible representations, and the explicit derivation of the matrix elements of generators. In fact, in Section 7 of the current article we mention the class of unitary representations as motivation for
the current work, although our results are not restricted to this class. Referring to the
classification of such irreducible representations given in [70], caution is needed since the
tensor product of a type 1 unitary with a type 2 unitary may contain indecomposables.
We note that the vector representation is type 1, and its dual is type 2, so based on the
presentation in this article, we would need to present separate treatments of the cases
where \( V(\Lambda) \) is type 1 or type 2, although the two are related via duality. Indeed, the
current article demonstrates how far one can go without making further restrictions on
the class of irreducible representation.

Appendix A

To assist in understanding the properties of odd vector and contragredient vector op-
erators, we now construct the appropriate odd vector representation. A realisation of odd
vector operators can be constructed by introducing a set of fermion operators
\[
\begin{align*}
    a^\dagger_i, a_i, & \quad 1 \leq i \leq m \\
    b^\dagger_\mu, b_\mu, & \quad 1 \leq \mu \leq n
\end{align*}
\]
and bosons
\[
\begin{align*}
    a^\dagger_i, a_i, & \quad 1 \leq i \leq m \\
    b^\dagger_\mu, b_\mu, & \quad 1 \leq \mu \leq n
\end{align*}
\]
according to which our \( gl(m|n) \) generators are realised by
\[
\begin{align*}
    E_{ij} = a^\dagger_i a_j, & \quad E_{i\mu} = a^\dagger_i b_\mu, & \quad E_{\mu i} = b^\dagger_\mu a_i, & \quad E_{\mu\nu} = b^\dagger_\mu b_\nu
\end{align*}
\]
(note the reversal of grading implicit with the description). Then the operators
\[
\begin{align*}
    \chi^i = a^\dagger_i, & \quad \chi^\mu = b^\dagger_\mu
\end{align*}
\]
transform as an odd vector operator. We have the rank two tensor
\[
T^{ij} = a^\dagger_i a^\dagger_j = -T^{ji}, T^{i\mu} = T^{\mu i} = a^\dagger_i b^\dagger_\mu, T^{\mu\nu} = T^{\nu\mu} = b^\dagger_\mu b^\dagger_\nu
\]
whose components transform as the representation \((1, 1, 0)\) of \( gl(m|n) \). The components
of the tensor \( T \) thus satisfy
\[
T^{pq} = (-1)^{(p+1)(q+1)}T^{qp}. \tag{64}
\]
The weights in the representation \((1, 1, 0)\) are of the form (in the notation of Kac [2])
\[
\epsilon_i + \epsilon_j \ (i \neq j), \quad \epsilon_i + \delta_\mu, \quad \delta_\mu + \delta_\nu.
\]
We wish to apply these results to investigate odd vector operators whose components commute (in the graded bracket sense).
Hence we now assume that \( \psi^p \) is an odd vector operator whose components are graded-commuting:
\[
[\psi^p, \psi^q] = 0
\]
where the graded bracket is given by
\[ [\psi^p, \psi^q] = \psi^p \psi^q - (-1)^{(p+1)(q+1)} \psi^q \psi^p. \]
In other words, we are assuming that the components of \( \psi \) satisfy the symmetry rule
\[ \psi^p \psi^q = (-1)^{(p+1)(q+1)} \psi^q \psi^p. \]
(65)
Equating even shift components of equation (65) we obtain
\[ \psi[i]^p \psi[i]^q = (-1)^{(p+1)(q+1)} \psi[i]^q \psi[i]^p, \quad 1 \leq i \leq m. \]
It follows, in view of equation (64), that
\[ T^{pq} = \psi[i]^p \psi[i]^q \]
constitutes a tensor operator of \( gl(m|n) \) transforming as the representation \( (1,1,0) \) and which increases the representation labels by the weight \( 2\varepsilon_i \). But this latter weight is not a weight of \( (1,1,0) \) so we must have
\[ \psi[i]^p \psi[i]^q = 0, \quad 1 \leq i \leq m. \]
More generally, by focusing on higher order tensors, we may establish the result
\[ \ldots \psi[i]^p \psi[i]^r \ldots \psi[u]^s \psi[i]^q \ldots = 0, \quad 1 \leq i \leq m. \]
That is, any product of shift components of \( \psi \) must vanish if there are two even shift indices which are equal. This result is a direct consequence of the oddness of \( \psi \) and the fact that its components are graded-commuting. Similarly, if \( \phi_p \) is an odd contragredient vector operator such that
\[ \phi_p \phi_q = (-1)^{(p+1)(q+1)} \phi_q \phi_p \]
then
\[ \phi[i]^p \phi[i]^q = 0, \quad 1 \leq i \leq m \]
and more generally
\[ \ldots \phi[i]^p \phi[i]^r \ldots \phi[u]^s \phi[i]^q \ldots = 0, \quad 1 \leq i \leq m. \]
It turns out that these results may be applied to provide useful information on the \( gl(m|n+1) \downarrow gl(m|n) \) branching conditions derived in Section 3.
Let \( V(\tilde{\Lambda}) \) be an irreducible finite dimensional \( gl(m|n+1) \) module with highest weight
\[ \tilde{\Lambda} = \sum_{i=1}^{m} \tilde{\Lambda}_i \varepsilon_i + \sum_{\mu=1}^{n+1} \tilde{\Lambda}_\mu \delta_\mu = \tilde{\Lambda}_0 + \tilde{\Lambda}_1. \]
Let
\[ V_0(\tilde{\Lambda}) = V(\tilde{\Lambda}_0) \otimes V(\tilde{\Lambda}_1) \]
be the irreducible $\hat{L}_0 = gl(m) \oplus gl(n+1)$ module which has a grading index 0 under the usual $\mathbb{Z}$-gradation induced on $V(\tilde{\Lambda})$ by the odd roots, i.e.
\[ E_{i\mu} V_0(\tilde{\Lambda}) = 0, \quad 1 \leq i \leq m, \quad 1 \leq \mu \leq n + 1. \]
Then this representation decomposes into irreducible representations of $gl(m) \oplus gl(n)$ according to
\[ V_0(\tilde{\Lambda}) = \bigoplus_{\Lambda} V_0(\Lambda), \quad V_0(\Lambda) = V(\tilde{\Lambda}_0) \otimes V(\Lambda_1) \quad (66) \]
where $V_0(\Lambda)$ is an irreducible representation of $gl(m) \oplus gl(n)$ and the sum is over all $gl(m|n)$ highest weights
\[ \Lambda = \sum_{i=1}^{m} \tilde{\Lambda}_i \epsilon_i + \sum_{\mu=1}^{n} \Lambda_\mu \delta_\mu = \tilde{\Lambda}_0 + \Lambda_1 \quad (67) \]
where the odd components $\Lambda_\mu$ are subject to the usual Gelfand-Tsetlin betweenness conditions [27]:
\[ \tilde{\Lambda}_\mu \geq \Lambda_\mu \geq \tilde{\Lambda}_{\mu+1}, \quad 1 \leq \mu \leq n \quad (68) \]
Applying the odd $gl(m|n)$ lowering operators $E_{i\mu}$ ($1 \leq i \leq m, \quad 1 \leq \mu \leq n$) to the left of equation (66) it follows that
\[ V(\tilde{\Lambda}) \supseteq W \quad (69) \]
where
\[ W = \bigoplus_{\Lambda} V(\Lambda) \]
where $V(\Lambda)$ is the irreducible module of $gl(m|n)$ with highest weight (67) and the sum is over all $\Lambda$ subject to the betweenness conditions (68).

We now note that the $gl(m|n)$ generator
\[ \phi_p = (-1)^{(p)} E_{m+n+1,p}, \quad 1 \leq p \leq m + n \]
transforms as an odd contragredient vector operator of $gl(m|n)$:
\[ [E_{pq}, \phi_r] = -(-1)^{(p)(q)+(p)} \delta_{pr} \phi_q, \]
whose components commute (in the usual graded bracket sense). Applying the $gl(m|n+1)$ generators $\phi_p$ ($1 \leq p \leq m + n$) to the left of equation (69) we obtain
\[ V(\tilde{\Lambda}) = W + \sum_{i=1}^{m} \phi_i W + \sum_{i \neq j} \phi_i \phi_j W + \ldots + \phi_1 \phi_2 \ldots \phi_m W, \]
where we have used the fact that the space $V_0(\tilde{\Lambda})$ (c.f. equation (66)) is stable under the action of the even generators $\phi_\mu$ ($1 \leq \mu \leq n$).

We note that only the even shift components of $\phi$ make a new contribution since for $V(\Lambda) \subseteq W$ we have

$$\phi[\mu]_i V(\Lambda) \subseteq V(\Lambda - \delta_\mu) \subseteq W.$$ 

Thus we have

$$V(\tilde{\Lambda}) = W \oplus \sum_{i,j} \phi[i]_j W \oplus \ldots \oplus \sum_{\pi \in S_m} \phi[1]_{\pi(1)} \ldots \phi[m]_{\pi(m)} W$$

where $S_m$ denotes the symmetric group on $m$ elements, and we have used the previously established result that no two even shift indices can occur in a product of shift components of an odd contragredient vector operator.

This produces an alternative perspective to understanding the $gl(m|n+1) \downarrow gl(m|n)$ branching conditions presented in Section 6. While the result of Theorem 2 is quite rigorous, the discussion in this Appendix merely provides insight into the general branching rule. Indeed, the outcome of this discussion is a rough sketch of the branching rule, where we have overlooked complications such as the possibility that some of the $gl(m|n)$ modules occurring may not be irreducible but only indecomposable (or possibly even zero).

**Appendix B**

Recall the characteristic roots

$$\bar{\alpha}_r = -\frac{1}{2} [\chi_{\Lambda+\varepsilon_r}(I_2) + n - m - \chi_{\Lambda}(I_2)]$$

of the adjoint identity. We make the basic assumption that these roots are distinct or equivalently the numbers

$$\chi_{\Lambda+\varepsilon_r}(I_2) \equiv \chi_{\Lambda+\varepsilon_1}(I_2), \chi_{\Lambda+\delta_\mu}(I_2)$$

are all distinct.

Throughout $V = V_0 \oplus V_1$ (usual $\mathbb{Z}$-grading) is the vector module (i.e $V = V(\varepsilon_1)$). It is our aim to prove the following theorem under the assumption (70).

**Theorem 6** $V \otimes V(\Lambda)$ is completely reducible and the allowed highest weights are of the form $\Lambda + \varepsilon_r$ ($1 \leq r \leq m + n$) each occurring at most once.

Throughout $V_0(\Lambda)$ denotes the maximal $\mathbb{Z}$-graded component of $V(\Lambda)$ which constitutes an irreducible $L_0$-module.

Here $L$ denotes the Lie superalgebra $gl(m|n)$ which admits the usual $\mathbb{Z}$-gradation

$$L = L_- \oplus L_0 \oplus L_+$$

with

$$L_0 = gl(m) \oplus gl(n).$$

We begin with the following easily established result:
Lemma 7 $V \otimes V(\Lambda)$ is cyclically generated as an $L$-module by the $L_0$-submodule $V \otimes V_0(\Lambda)$ and

$$V \otimes V(\Lambda) = U(L_-)V \otimes V_0(\Lambda).$$

Note that we have the following decomposition into irreducible $L_0$-modules:

$$V_0 \otimes V_0(\Lambda) = \bigoplus_i V_0(\Lambda + \varepsilon_i) \quad (71)$$

$$V_1 \otimes V_0(\Lambda) = \bigoplus_\mu V_0(\Lambda + \delta_\mu) \quad (72)$$

where each module on the RHS is understood to vanish identically if one of the corresponding $(\Lambda + \varepsilon_i)$ or $(\Lambda + \delta_\mu)$ is non-dominant. The above gives rise to the following decomposition into irreducible $L_0$-submodules for the cyclic module of Lemma 7:

$$V \otimes V_0(\Lambda) = \bigoplus_r V_0(\Lambda + \varepsilon_r). \quad (73)$$

Now let $\langle \ , \ \rangle$ be the (unique) non-degenerate sesquilinear form on $V(\Lambda)$ satisfying

$$\langle av, w \rangle = \langle v, a^\dagger w \rangle, \ \forall v, w \in V(\Lambda)$$

with $^\dagger$ the usual conjugation operation on $L$. We recall [2] that the corresponding form $\langle \ , \ \rangle$ on the vector module $V$ gives rise to an inner product. We let $\langle \ , \ \rangle$ denote the form induced on $V \otimes V(\Lambda)$.

The following Lemma will prove useful:

Lemma 8 Let $0 \neq v_+ \in V \otimes V(\Lambda)$ be a maximal weight vector. Then $\langle v_+, V \otimes V_0(\Lambda) \rangle \neq (0)$.

Proof: Otherwise we would have

$$0 = \langle U(L_+)v_+, V \otimes V_0(\Lambda) \rangle$$

$$= \langle v_+, U(L_-)V \otimes V_0(\Lambda) \rangle$$

$$= \langle v_+, V \otimes V(\Lambda) \rangle \quad \text{(from Lemma 7)}$$

$$\Rightarrow v_+ = 0$$

since the naturally induced form on $V \otimes V(\Lambda)$ is also non-degenerate. ■

We now aim to prove the following weaker version of the Theorem above, using the Lemma:

Proposition 9 Let $v_+ \in V \otimes V(\Lambda)$ be a maximal weight vector of weight $\nu$. Then

(i) $\nu \in \{\Lambda + \varepsilon_r | 1 \leq r \leq m + n\}$

(ii) $v_+$ is the unique (up to scalar multiples) maximal weight vector in $V \otimes V(\Lambda)$ of weight $\nu$. 

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Proof: From the decomposition (73) this is enough to prove part (i) of the proposition. As to the second part we proceed as in the proof of Theorem 2 and suppose $0 \neq w_+ \in V \otimes V(\Lambda)$ is also a maximal weight vector of the same weight $\nu$. We let $v_0^\nu$ be the $L_0$ maximal vector of the irreducible $L_0$-module $V_0(\nu) \subseteq V \otimes V_0(\Lambda)$. Then we observe that

$$\tilde{v}_+ \equiv \langle v_0^\nu, w_+ \rangle v_+ - \langle v_0^\nu, v_+ \rangle w_+$$

is also a maximal vector of weight $\nu$ satisfying

$$\langle \tilde{v}_+, V \otimes V_0(\Lambda) \rangle = (0) \Rightarrow \tilde{v}_+ = 0$$

from the Lemma, so

$$w_+ = \kappa v_+ , \quad \kappa = \frac{\langle v_0^\nu, w_+ \rangle}{\langle v_0^\nu, v_+ \rangle}$$

which proves the proposition. ■

Now observe from equation (71) that each irreducible $L_0$-module

$$V_0(\Lambda + \varepsilon_i) \subseteq V_0 \otimes V_0(\Lambda)$$

contains a maximal weight vector of weight $\Lambda + \varepsilon_i$, which is also a maximal weight vector of $L$ cyclically generating an indecomposable $L$-module

$$V(\Lambda + \varepsilon_i) = U(L-)V_0(\Lambda + \varepsilon_i). \quad (74)$$

If $V(\Lambda + \varepsilon_i)$ is not irreducible it must contain a maximal weight vector which, in view of Proposition 3, must have weight of the form $\Lambda + \varepsilon_r, r \neq i$. But then this would imply that

$$\chi_{\Lambda+\varepsilon_i}(I_2) = \chi_{\Lambda+\varepsilon_r}(I_2), \text{ for some } r \neq i$$

in contradiction to our basic assumption (70). Hence it follows that each $L$-module

$$V(\Lambda + \varepsilon_i) \subseteq U(L-)V_0 \otimes V_0(\Lambda) \equiv W$$

is irreducible. Thus we have a decomposition into irreducible $L$-modules

$$W = \bigoplus_{i=1}^{m} V(\Lambda + \varepsilon_i), \quad (75)$$

induced by the decomposition (71). This then gives an $L_0$-module decomposition

$$V \otimes V(\Lambda) = U(L-)V_1 \otimes V_0(\Lambda) + W: \quad (76)$$

We observe that the form $\langle , \rangle$ on $V \otimes V(\Lambda)$, restricted to each irreducible $L$-module $V(\Lambda + \varepsilon_i)$ is necessarily non-degenerate and that the decomposition (76) is orthogonal. In view of (76) it follows that the form $\langle , \rangle$ restricted to $W$ is non-degenerate also. We let $P_W$ be the (self-adjoint) projection onto the submodule (75) (which thus intertwines the action of $L$). This then gives an orthogonal decomposition of $L$-modules

$$V \otimes V(\Lambda) = W \oplus W^\perp \quad (77)$$
where

\[ W^\perp = (1 - P_W)V \otimes V(\Lambda) \]

\[ \overset{(76)}{=} (1 - P_W)U(L_-)V_1 \otimes V_0(\Lambda) \]

\[ = U(L_-)(1 - P_W)V_1 \otimes V_0(\Lambda). \]

We observe that

\[ L_+(1 - P_W)V_1 \otimes V_0(\Lambda) \]

\[ = (1 - P_W)L_+[V_1 \otimes V_0(\Lambda)] \]

\[ = (1 - P_W)V_0 \otimes V_0(\Lambda) \]

\[ \subseteq (1 - P_W)W = (0) \]

so that the decomposition of equation \((72)\) gives

\[ (1 - P_W)V_1 \otimes V_0(\Lambda) = \bigoplus_{\mu} (1 - P_W)V_0(\Lambda + \delta_\mu) \]

where \( L_+(1 - P_W)V_0(\Lambda + \delta_\mu) = (0) \).

It thus follows that if non-zero, \((1 - P_W)V_0(\Lambda + \delta_\mu)\) cyclically generates an indecomposable \(L\)-module with highest weight \(\Lambda + \delta_\mu\). As before, under our basic assumption \((70)\), this module is in fact irreducible giving a decomposition of irreducible \(L\)-modules

\[ W^\perp = \bigoplus_{\mu} V(\Lambda + \delta_\mu). \]

The proof of the theorem then follows from the decompositions \((75 \), \(77)\) and \((78)\).

It is worth remarking that if we assume that the roots \(\alpha_r, 1 \leq r \leq m + n\), of the characteristic identities are distinct, or equivalently that the numbers

\[ \chi_{\Lambda - \varepsilon_r}(I_2), 1 \leq r \leq m + n \]

are distinct, we may similarly prove that:

**Theorem 10** \(V^* \otimes V(\Lambda)\) is completely reducible with \(L\)-maximal weight vectors of the form \(\Lambda - \varepsilon_r\) \((1 \leq r \leq m + n)\), each occurring at most once.

**Appendix C**

Here we will define two \(gl(m|n)\) invariants that play a central role in determining reduced matrix elements and reduced Wigner coefficients. We will show that these invariants are indeed subalgebra invariants by expressing them solely in terms of the Casimir invariants.

Here we consider the characteristic matrices \(A\) and \(B\) defined in equations \((13)\) and \((34)\) respectively. Powers of these matrices are defined recursively as given in equations
Proposition 11

The set \( \{ \tau_1, \tau_2, \tau_3, \ldots \} \) is a set of commuting operators. That is,

\[ [\tau_\ell, \tau_k] = 0 \quad \forall \ell, k \quad (82) \]

**Proof:** We proceed by induction. For \( \ell = 1 \), equation (82) is seen to be valid immediately by applying the commutation relation given in equation (81).

\[
[\tau_1, \tau_k] = [B_1^+, (B^k)^+] = -\delta^+_q (B^k)^+ + \delta^+_q (B^k)^+
\]

\[ = 0 \quad \forall k. \quad (83) \]

For \( \ell > 1 \) we have

\[
[\tau_\ell, \tau_k] = [(B^{\ell-1})^+_r, (B^k)^+] \\
= (B^{\ell-1})^+_r [B_1^+, (B^k)^+] + (B^{\ell-2})^+_q [B_2^+, (B^k)^+] B_1^+ + (B^{\ell-3})^+_q [B_3^+, (B^k)^+] (B^2)_r \\
+ \ldots + [B_p^+, (B^k)^+] (B^{\ell-1})^+_p \\
= (B^{\ell-1})^+_r [B_1^+, (B^k)^+] + [B_2^+, (B^k)^+] (B^{\ell-1})^+_p + \sum_{j=1}^{\ell-2} (B^{\ell-j-1})^+_q [B_j^+, (B^k)^+] (B^j)_r.
\]

By use of equation (81) to evaluate the commutators, one can readily show that

\[
\sum_{j=1}^{\ell-2} (B^{\ell-j-1})^+_q [B_j^+, (B^k)^+] (B^j)_r = \sum_{j=1}^{\ell-2} [\tau_j, \tau_{k+j-j-1}]
\]

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and also
\[
(B^{l-1})_q^+ [B^q_+, (B^k)^+_p] + [B^l_+, (B^k)^+] (B^{l-1})_q^+ = (B^{l-1})_q^+ \left( -(B^k)^+_q + \delta^+_q (B^k)^+_p \right) \\
+ \left( -\delta^+_p (B^k)^+_q + (B^k)^+_{p} \right) (B^{l-1})_q^+ \\
= -\tau_{l+k-1} + \tau_{l-1} \tau_k - \tau_k \tau_{l-1} + \tau_{l+k-1} \\
= [\tau_{l-1}, \tau_k].
\]

We have shown that \([\tau_{l}, \tau_k]\) can be written in terms of commutators with lower order \(\tau\) invariants on the LHS. That is,
\[
[\tau_{l}, \tau_k] = \sum_{j=1}^{\ell-1} [\tau_j, \tau_{k+\ell-j-1}]
\]
so by induction we have \([\tau_{l}, \tau_k] = 0\) since \([\tau_1, \tau_k] = 0\) \(\forall k\).

The next three propositions will be required when expressing \(\tau_k\) in terms of the Casimir invariants. Recall the Casimir invariants \(\hat{I}_k\) and \(I_k\) are defined in equations (38) and (39) respectively. Using the commutation relations (84) the following result can be established:

**Proposition 12**
\[
\sum_{p=1}^{m+n+1} (-1)^{(p)} [(B^l)^+_p, (B^k)^+_{p}] = \sum_{i=0}^{\ell-1} \left( \hat{I}_i \tau_{l+k-1-i} - \hat{I}_{l+k-1-i-1} \right). \tag{84}
\]

**Definition.** The order of the term
\[
(-1)^{(p)} (B^k)^+_p \tau_{s_1} \tau_{s_2} \ldots \tau_{s_j} (B^l)^+_p
\]
is defined to be the sum of the powers of the \(B\)'s within the term which in the above case is
\[
k + \ell + \sum_{i=1}^{j} s_i.
\]

It is important to note that for sufficiently low orders, this term will degrade to two possible cases. The first case is when \(k = 0\) which gives
\[
\sum_{p} (-1)^{(p)} (B^0)^+_p \tau_{s_1} \tau_{s_2} \ldots \tau_{s_j} (B^l)^+_p = \sum_{p} (-1)^{(p)} \delta^+_p \tau_{s_1} \tau_{s_2} \ldots \tau_{s_j} (B^l)^+_p \\
= -\tau_{s_1} \tau_{s_2} \ldots \tau_{s_j} (B^l)^+_p \\
= -\tau_{s_1} \tau_{s_2} \ldots \tau_{s_j} \tau_{\ell}.
\]
The second case is \(s_i = 0\) \(\forall i\) where proposition [12] gives
\[
\sum_{p} (-1)^{(p)} (B^k)^+_p \tau_{0} \tau_{0} \ldots \tau_{0} (B^l)^+_p = \sum_{p} (-1)^{(p)} (B^k)^+_p (B^l)^+_p \\
= \sum_{p} (-1)^{(p)} [(B^k)^+_p, (B^l)^+_p] + \tau_{k+\ell} \\
= \sum_{i=0}^{k-1} \left( \hat{I}_i \tau_{k+\ell-i-1} - \hat{I}_{k+\ell-i-1} \right) + \tau_{k+\ell}.
\]

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By repeated use of the graded commutation relations we have:

**Proposition 13** A general term of the form

$$\sum_{p=1}^{m+n+1} (-1)^{(p)} (B^k)_p \tau_{s_1} \tau_{s_2} \ldots \tau_{s_j} (B^l)_p$$

which is of order

$$M = k + \ell + \sum_{i=1}^{j} s_i$$

can be written as a series of terms of the form

$$\tau_{s_1} \tau_{s_2} \ldots \tau_{s_j}$$

(and products of them) where the order of each term (and therefore the order of each \(\tau\) and \(I\) within a term) is strictly less than \(M\).

We are now in a position to state the two main theorems of this appendix.

**Theorem 14** The invariant \(\tau_k\) can be written as the sum of \(I_j - \hat{I}_j\) and a series of terms of the form

$$\tau_{s_1} \tau_{s_2} \ldots \tau_{s_j}$$

(and products of them) where the order of each term (and therefore the order of each \(\tau\) and \(I\) within a term) is strictly less than \(k\).

**Proof:** Firstly, we consider the difference of the Casimir invariants \(\hat{I}_k\) and \(I_k\). By definition, we have

$$\hat{I}_k = (-1)^{(p)} (B^k)_p = (-1)^{(p)} B^{q_1} B^{q_2} B^{q_3} \ldots B^{q_{k-1}}$$

and

$$I_k = (-1)^{(p)} (A^k)_p \quad 1 \leq p \leq m+n$$

$$= (-1)^{(p)} (1 - \delta^p_+) (B^k)_p \quad 1 \leq p \leq m+n+1$$

$$= (-1)^{(p)} (1 - \delta^p_+) (1 - \delta^q_1) (1 - \delta^q_2) \ldots (1 - \delta^q_{k-1}) B^{q_1} B^{q_2} B^{q_3} \ldots B^{q_{k-1}}$$

$$= \delta^p_+ (1 - \delta^p_+) (1 - \delta^q_1) (1 - \delta^q_2) \ldots (1 - \delta^q_{k-1}) B^{q_1} B^{q_2} B^{q_3} \ldots B^{q_{k-1}}$$

$$+ (-1)^{(p)} (1 - \delta^q_1) (1 - \delta^q_2) \ldots (1 - \delta^q_{k-1}) B^{q_1} B^{q_2} B^{q_3} \ldots B^{q_{k-1}}.$$

Now we note that

$$\delta^p_+ (B^{q_1} B^{q_2} B^{q_3} \ldots B^{q_{k-1}}) = \tau_k$$

$$\delta^p_+ \delta^q (B^{q_1} B^{q_2} B^{q_3} \ldots B^{q_{k-1}}) = B^{q_1} \ldots B^{q_{j-1}} B^{q_j+1} \ldots B^{q_{k-1}}$$

$$= \tau_{j-1} \tau_{k-j} \quad \text{for } i < j$$

$$\ldots$$
and observe that in each case the terms are of order $k$ (since the subscripts within each term sum to $k$). Similarly we have

\[
(-1)^{(p)} (B_{q_1}^{p} B_{q_2}^{q_1} B_{q_3}^{q_2} \cdots B_{q_{p-1}}^{q_{p-1}}) = \hat{I}_k
\]

\[
(-1)^{(p)} \delta^{(q_1,q_2)}_{\ell} (B_{q_1}^{p} B_{q_2}^{q_1} B_{q_3}^{q_2} \cdots B_{q_{p-1}}^{q_{p-1}}) = (-1)^{(p)} B_{q_1}^{p} \cdots B_{q_{p-1}}^{q_{p-1}} (B_{q_{p-1}}^{+} B_{q_{p-1}}^{+} \cdots B_{q_{p-1}}^{+})
\]

\[
(-1)^{(p)} \delta^{(q_1,q_2)}_{\ell+1} (B_{q_1}^{p} B_{q_2}^{q_1} B_{q_3}^{q_2} \cdots B_{q_{p-1}}^{q_{p-1}}) = (-1)^{(p)} (B_{q_1}^{p} \cdots B_{q_{p-1}}^{q_{p-1}} (B_{q_{p-1}}^{+} B_{q_{p-1}}^{+} \cdots B_{q_{p-1}}^{+})
\]

where each term is also of order $k$. By defining the summation symbol $\sum_{(k)}$ to be the sum of all terms such that the powers of $B$ within each term are positive and sum to $k$ (implying that each term is of order $k$) we can write $I_k$ as

\[
I_k = \tau_k - \sum_{(k)} \tau_a \tau_b + \sum_{(k)} \tau_a \tau_b \tau_c - \sum_{(k)} \tau_a \tau_b \tau_c \tau_d + \ldots
\]

\[
+ \hat{I}_k - (-1)^{(p)} \sum_{(k)} (B_{q_1}^{p} (B_{q_2}^{+} + (1)^{(p)} \sum_{(k)} (B_{q_2}^{+} \tau_j (B_{q_2}^{+}) + \ldots
\]

Rearranging gives

\[
\tau_k = I_k - \hat{I}_k + \sum_{(k)} \tau_a \tau_b - \sum_{(k)} \tau_a \tau_b \tau_c + \sum_{(k)} \tau_a \tau_b \tau_c \tau_d - \ldots
\]

\[
+ (-1)^{(p)} \sum_{(k)} (B_{q_1}^{p} (B_{q_2}^{+} + (1)^{(p)} \sum_{(k)} (B_{q_2}^{+} \tau_j (B_{q_2}^{+}) + \ldots
\]

which together with proposition 13 completes the proof. ■

**Theorem 15** The invariants $\tau_k, \sigma_{\ell}$ can be expressed in terms of products of Casimir invariants $I_j$ and $\hat{I}_j$ where $j \leq k$ and $j \leq \ell + 2$.

**Proof:** For $\tau_k$ the proof is obtained by applying Theorem 14 recursively and observing that $\tau_0 = 1$ and $\tau_1 = I_1 - \hat{I}_1$ (since $\tau_1 = B_{q_1}^{+} = -E_{q_1}^{+}$).

For the $\sigma_{\ell}$ we have

\[
\sigma_{\ell} = B_{q_1}^{+} (A_{j}^{l} B_{j}^{+}) (1 \leq \hat{p}, \hat{q} \leq m + n)
\]

\[
= (1 - \delta_{\hat{p}}^{q_1}) (1 - \delta_{\hat{q}_2}^{q_2}) \cdots (1 - \delta_{\hat{q}_{\ell+1}}^{q_{\ell+1}}) B_{q_1}^{+} B_{q_2}^{+} B_{q_3}^{+} \cdots B_{q_{\ell+1}}^{+} (1 \leq q_1 \leq m + n + 1)
\]

\[
= \left(1 - \sum_{i} \delta_{\hat{p}}^{q_i} + \sum_{i \neq j} \delta_{\hat{q}}^{q_i} \delta_{\hat{p}}^{q_j} - \sum_{i,j,k \neq i,j} \delta_{\hat{q}}^{q_i} \delta_{\hat{p}}^{q_j} \delta_{\hat{p}}^{q_k} + \ldots \right) B_{q_1}^{+} B_{q_2}^{+} B_{q_3}^{+} \cdots B_{q_{\ell+1}}^{+}
\]

\[
= \tau_{\ell+2} - \sum_{i=1}^{\ell+1} \tau_i \tau_{\ell+2-i} + \sum_{(\ell+2)} \tau_a \tau_b \tau_c - \sum_{(\ell+2)} \tau_a \tau_b \tau_c \tau_d + \ldots
\]

(85)

where again the summation symbol $\sum_{(\ell+2)}$ is defined to be the sum of all terms such that the powers of $B$ within each term are positive and sum to $\ell + 2$. ■

As an example, we give $\sigma_2$ in terms of $\tau$’s as

\[
\sigma_2 = \tau_4 - \tau_1 \tau_3 - \tau_2 \tau_2 - \tau_3 \tau_1 + \tau_1 \tau_1 \tau_2 + \tau_1 \tau_2 \tau_1 + \tau_2 \tau_1 \tau_1 - (\tau_1)^4.
\]

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References

[1] V.G. Kac, Adv. in Math. 26 (1977) 8.
[2] V.G. Kac, Lecture Notes in Math. 676, Springer, Berlin (1978) 597.
[3] W. Nahm and M. Scheunert, J. Math. Phys. 17 (1976) 868.
[4] M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. 17 (1976) 1626.
[5] M. Scheunert, Lecture Notes in Math. 716, Springer, Berlin (1979).
[6] P. Ramond, Phys. Rev. D 3 (1971) 2415.
[7] A. Neveu and J.H. Schwarz, Nucl. Phys. B 31 (1971) 86.
[8] D.V. Volkov and V.P. Akulov, Phys. Lett. B 46 (1973) 109.
[9] J. Wess and B. Zumino, Nucl. Phys. B 70 (1974) 39.
[10] A. Salam and J. Strathdee, Nucl. Phys. B 76 (1974) 477.
[11] J. Scherk, Rev. Mod. Phys. 47 (1975) 123.
[12] P. Fayat and S. Ferrara, Phys. Rep. 32 (1977) 249.
[13] L. Corwin, Y. Ne’eman and S. Sternberg, Rev. Mod. Phys. 47 (1975) 573.
[14] I.M. Musson, Graduate Studies in Math. 131, AMS (2012).
[15] N. Beisert and M. Staudacher, Nucl. Phys. B 670 (2003) 439.
[16] J.A. Minahan, Lett. Math. Phys. 99 (2012) 33.
[17] W. Galleas and M.J. Martins, Nucl. Phys. B 699 (2004) 455.
[18] F.H.L. Essler, H. Frahm and H. Saleur, Nucl. Phys. B 712 (2005) 513.
[19] S.Y. Zhao, W.L. Yang and Y.Z. Zhang, Commun. Math. Phys. 268 (2006) 505.
[20] E. Ragoucy and G. Satta, JHEP09 (2007) 001.
[21] H. Frahm and M.J. Martins, Nucl. Phys. B 847 [FS] (2011) 220.
[22] V. Schomerus and H. Saleur, Nucl. Phys. B 734 [FS] (2006) 221.
[23] D. Ridout, Nucl. Phys. B 810 [FS] (2009) 503.
[24] N.I. Stoilova and J. Van der Jeugt, J. Phys. A: Math. Theor. 41 (2008) 075202.
[25] S. Lievens, N.I. Stoilova and J. Van der Jeugt, Commun. Math. Phys. 281 (2008) 805.
[26] N.I. Stoilova and J. Van der Jeugt, J. Math. Phys. 51 (2010) 093523.

[27] I.M. Gelfand and M.L. Tsetlin, Dokl. Akad. Nauk., SSSR 71 (1950) 825 (Russian).
English translation in: I.M. Gelfand, “Collected Papers”, Vol II, Berlin: Springer-Verlag (1988) 653.

[28] I.M. Gelfand and M.L. Tsetlin, Dokl. Akad. Nauk., SSSR 71 (1950) 1017 (Russian).
English translation in: I.M. Gelfand, “Collected Papers”, Vol II, Berlin: Springer-Verlag (1988) 657.

[29] G.E. Baird and L.C. Biedenharn, J. Math. Phys. 4 (1963) 1449.

[30] A.I. Molev, Handbook of Algebra 4 (2006) 109.

[31] V.N. Tolstoy, I.F. Istomin and Yu.F. Smirnov, in “Group Theoretical Methods in Physics: Proceedings of the Third Yurmala Seminar”, Yurmala, USSR, 1985, Ed. M.A. Markov, V.I. Man’ko and V.V. Dodonov, VNU Science Press, Utrecht (1986) 337.

[32] T. D. Palev, Funct. Anal. Appl. 21 (1987) 245.

[33] T. D. Palev, Funct. Anal. Appl. 23 (1989) 141.

[34] A.I. Molev, Bull. Inst. Math. Acad. Sinica 6 (2011) 415.

[35] A.H. Kamupingene, N.A. Ky, T.D. Palev, J. Math. Phys. 30 (1989) 553.

[36] M.D. Gould, A.J. Bracken and J.W.B. Hughes, J. Phys. A: Math. Gen. 22 (1989) 2879.

[37] M.D. Gould, P.D. Jarvis and A.J. Bracken, J. Math. Phys. 31 (1990) 2803.

[38] T.D. Palev, N.I. Stoilova, J. Math. Phys. 31 (1990) 953.

[39] H.S. Green, J. Math. Phys. 12 (1971) 2106.

[40] A.J. Bracken and H.S. Green, J. Math. Phys. 12 (1971) 2099.

[41] D.M. O’Brien, A. Cant and A.L. Carey, Ann. Inst. Henri Poincaré, Section A: Physique théorique 26 (1977) 405.

[42] M.D. Gould, J. Austral. Math. Soc. Ser. B 26 (1985) 257.

[43] P.A.M. Dirac, Proc. R. Soc. Lond. A 155 (1936) 447.

[44] G.E. Baird and L.C. Biedenharn, J. Math. Phys. 5 (1964) 1723.

[45] M.D. Gould, J. Austral. Math. Soc. Ser. B 20 (1978) 401.

[46] M.D. Gould, J. Math. Phys. 21 (1980) 444.

[47] M.D. Gould, J. Math. Phys. 22 (1981) 15.
[48] M.D. Gould, J. Math. Phys. 22 (1981) 2376.
[49] M.D. Gould, J. Phys. A: Math. Gen. 17 (1984) 1.
[50] B. Kostant, J. Func. Anal 20 (1975) 257.
[51] A.M. Bincer, J. Math. Phys. 24 (1983) 2546.
[52] H.S. Green, P.D. Jarvis, J. Math. Phys. 24 (1983) 1681.
[53] M. Scheunert, J. Math. Phys. 24 (1983) 2681.
[54] J.R. Links and R.B. Zhang, J. Math. Phys. 34 (1993) 6016.
[55] M.D. Gould, J. Math. Phys. 27 (1986) 1944.
[56] M.D. Gould, J. Math. Phys. 27 (1986) 1964.
[57] K. Hannabuss, “An Introduction to Quantum Theory”, Oxford University Press, Oxford (1997).
[58] G.E. Baird and L.C. Biedenharn, J. Math. Phys. 5 (1964) 1730.
[59] J.D. Louck and L.C. Biedenharn, J. Math. Phys. 11 (1970) 2368.
[60] R.B. Zhang, M.D. Gould and A.J. Bracken, Nucl. Phys. B 354 (1991) 625.
[61] V. Rittenberg and M. Scheunert, J. Math. Phys. 33 (1992) 436.
[62] M. Mozrzymas, Int. J. Geom. Meth. Mod. Phys. 2 (2003) 393.
[63] A. Pais and V. Rittenberg, J. Math. Phys. 16 (1975) 2062.
[64] L. Mezincescu, J. Math. Phys. 18 (1977) 453.
[65] M. Mozrzymas, J. Phys. A: Math. Gen. 37 (2004) 9515.
[66] P.D. Jarvis and H.S. Green, J. Math. Phys. 20 (1979) 2115.
[67] P.D. Jarvis and M.K. Murray, J. Math. Phys. 24 (1983) 1705.
[68] M.D. Gould, J. Austral. Math. Soc. Ser. B 28 (1987) 310.
[69] P.D. Jarvis, G. Rudolph and L.A. Yates, J. Phys. A: Math. Theor. 44 (2011) 235205.
[70] M.D. Gould and R.B. Zhang, J. Math. Phys. 31 (1990) 2552.
[71] J.E. Humphreys, “Introduction to Lie algebras and representation theory”, Graduate Texts in Mathematics 9, New York: Springer (1972).
[72] J.R. Links and M.D. Gould, J. Math. Phys. 37 (1996) 484.
[73] M.D. Gould and J.R. Links, J. Phys. A: Math. Gen. 30 (1997) 1613.
[74] A. Joseph, Am. J. Math. 99 (1977) 1167.