DO COSMIC STRINGS
GIVE RISE TO VACUUM FLUCTUATIONS?

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Abstract
The role of the identification of the vacuum and non-vacuum space-times in the computation of vacuum fluctuations in the presence of a cosmic string is discussed and an alternative interpretation of the renormalization is proposed. This procedure does not give rise to vacuum fluctuations.

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1. Introduction.

Whether vacuum fluctuations exist in the presence of cosmic strings was an important question in recent years. This problem has been studied by various authors and it was answered affirmatively [1-5]. The method for the computation of vacuum fluctuations is to obtain the Greens function in the space-time with a cosmic string and to renormalize this function by subtracting the vacuum contribution, i.e. the Greens function for the Minkowski space. This renormalization process involves the difference of two functions defined on different manifolds, hence we need to define a map to identify the points on the space-time with a cosmic string and the Minkowski space-time. In this paper we discuss the role of this identification of in the renormalization process.

In the literature the computation and the renormalization of the Greens function for the space-time with a cosmic string is adopted from [6] where the Greens function for a flat space with a wedge cut out is obtained and renormalized. We claim that the identification scheme underlying this renormalization is not appropriate for the identification of a space-time with a cosmic string and the Minkowski space, and we propose a different identification which leads to a renormalization that do not remove the singularity. We then, calculate the energy difference treating the energy shift, as if this is due to a perturbation in Minkowski space, and show that the difference in energies is proportional to the energy itself. Hence we conclude that subtracting the vacuum can not tame the singularity.

We recall that an infinitely long straight cosmic string may be described by a locally flat space-time with a “topological defect”. That is in cylindrical coordinates, the range of the polar angle is \((0, 2\pi \beta)\) with \(\beta < 1\) instead of being \((0, 2\pi)\), but otherwise the metric is flat. This description of the cosmic string provides an analogy with the computation of the Greens function for a flat space with a wedge cut out [6]. In fact the expression of the Greens function for the string in the coordinate system above can be obtained directly from the Greens function of the flat space with a wedge, by imposing appropriate boundary conditions. However there is a crucial difference in the renormalization as discussed in Section 3.
2. Preliminaries.

Let \( M \) be the spacetime manifold with a metric \( g \), and \((U, h)\), where \( h \) is a homeomorphism from \( U \) to \( M \) be a chart domain. In the computations below we emphasize the distinction between the quantities on \( M \) and their local coordinate expressions. For example, \( g \) is the metric on the manifold, and \( g \circ h \) is its expression in local coordinates. Similarly the Greens function \( G(p, p') \), \( p, p' \in M \) is a function on the manifold, but \((G \circ h)(x, x')\), \( x, x' \in U \) is a function on \( U \). In practice we obtain \( G \circ h \) by solving the Klein-Gordon equation in \( U \) with boundary conditions that reflect the physical situation on the spacetime \( M \).

We note that, here (and elsewhere in the literature except [7], where a \( C^\infty \) metric is used) the space-time with a cosmic string has a time independent metric, hence this metric does not represent the “formation” of the cosmic string. The information related to the formation of the cosmic string is coded in the subtraction of the vacuum contribution.

We list below the Greens function used in the literature, with an emphasis on the local coordinates used in each case. The computation methods are standard and are given in [8]. We use the metric signature \((+,-,-,-)\).

**Minkowski space with cylindrical coordinates:** Let \( M_o \) be \( \mathbb{R}^4 \) with flat metric \( g_o \). The usual cylindrical coordinates provide a (global) coordinate chart. The chart domain \( U \) is

\[
U = \{(t, r, z, \phi) \in \mathbb{R}^4 : -\infty < t < \infty, 0 < r < \infty, -\infty < z < \infty, 0 < \phi < 2\pi\} \quad (2.1)
\]

and \( g \circ h \) leads to the line element

\[
ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\phi^2. \quad (2.2)
\]

In these coordinates the Greens function \( G_o \circ h \) can be obtained as the solution of the Klein-Gordon equation with the boundary condition \((G \circ h) \mid_{\phi=0} = (G \circ h) \mid_{\phi=2\pi} \). If we identify all the coordinates except \( \phi \) and \( \phi' \), \( G \circ h \) reduces to

\[
(G \circ h)(\phi, \phi') = \frac{1}{16\pi^2 r^2 \sin^2 \left(\frac{\phi-\phi'}{2}\right)} \quad (2.3)
\]

This is the standard result for vacuum.
Minkowski space with a wedge cut out: Let $N$ be the subset of the Minkowski space with a wedge cut out. The metric $g'$ is the restriction of Minkowski metric to this subset. We use the same (global) coordinate chart as above with the restriction of the domain. Namely

$$V = \{(t,r,z,\theta) \in \mathbb{R}^4 : -\infty < t < \infty, r_0 < r < \infty, -\infty < z < \infty, 0 < \theta < 2\pi\beta, \beta < 1\}$$

(2.4)

and $h' = h|_V$. In this case the Klein-Gordon equation is same as above but boundary conditions for the Greens are different. As a special case, if we impose periodic boundary conditions, i.e. $(G \circ h')|_{\theta=0} = (C \circ h')|_{\theta=2\pi\beta}$, and take the coincidence limit for all other coordinates, we obtain

$$(G \circ h')(\theta, \theta) = \frac{1}{16r^2\beta^2\pi^2 \sin^2 \frac{\theta - \theta'}{2\beta}}.$$  

(2.5)

This is the result in [6].

Space-time with cosmic string: We recall that a spacetime $M$ with an infinitely long cosmic string along the $z$ axis is locally a flat spacetime with a conical defect. There are two standard local coordinate descriptions for this space-time: One can either “straighten” the conical surface to a plane by cutting out a wedge, or “stretch” the conical spacetime to map the surface of the cone to a plane. In both cases cylindrical provide a (global) chart.

(i) “Straightened” coordinates $(V,k)$: $V$ is the open subset of $\mathbb{R}^4$ described above in (2.4) and $k$ is a homeomorphism of $V$ into $M$. The expression of the metric in these coordinates, i.e. $g \circ k$ leads to the line element

$$ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\theta^2$$

(2.6)

In this coordinate system, the local expression of the Klein-Gordon operator is the same as in the Minkowski space, but the boundary conditions are $(G \circ k)|_{\theta=0} = (C \circ k)|_{\theta=2\pi\beta}$. Hence the expression of the Greens function in these coordinates, i.e. $G \circ k$ is given by (2.5), which is the result given in [1].

(ii) “Stretched” coordinates $(U,l)$: $U$ is the open subset of $\mathbb{R}^4$ described in (2.1), and $l$ is a homeomorphism of $U$ into $M$. The chart transformation between these two coordinates
is given by \( \{t, z, r, \theta\} \to \{t, r, z, \phi\} \) where \( \phi = \theta / \beta \). The expression of the metric in these coordinates, i.e. \( g \circ l \) leads to the line element

\[
ds^2 = dt^2 - dr^2 - dz^2 - \beta^2 r^2 d\phi^2, \quad \beta < 1.
\] (2.7)

The expression of the Greens function in these coordinates after the coincidence limit is then

\[
(G \circ l)(\phi, \phi') = \frac{1}{16\pi^2 r^2 \beta^2 \sin^2 \left(\frac{\phi - \phi'}{2}\right)}. \tag{2.8}
\]

This result coincides with that of [1] if we set \( \phi = \theta / \beta \).

Up to this point there is no problem in working with different coordinate systems, since the Greens functions are related via coordinate transformations. The problems arise at the stage of normalization where we need to identify the points on the spacetime \( M \) with the vacuum spacetime, and at this stage we need to be careful in using local coordinates.

3. Renormalization. The description of the vacuum contribution is the subtle part of the problem. The Minkowski space and the space-time with a cosmic string are two distinct manifolds, and \( G \) and \( G_o \) are functions on these. In order to subtract the vacuum contribution, we need to define a map identifying the points on \( M \) and \( M_o \). Let \( \varphi : M \to M_o \) be this identifying map (see Figure 1). Then the renormalized Green function \( G_r \) can be defined as a function on the manifold \( M \), as

\[
G_r(p, p') = G(p, p') - G_o(\varphi(p), \varphi(p')). \tag{3.1}
\]

We claim that a point \( p \) in \( M \) with “stretched” coordinates \( \{t, r, z, \phi\} \) should be identified with the point \( p_o \) in the Minkowski space \( M_o \) with coordinates \( \{t, r, z, \phi\} \). Thus the identifying map is

\[
\varphi : M \to M_o, \quad \varphi = h \circ l^{-1} \tag{3.2}
\]

where \( h : U \to M_o \) and \( l : U \to M \) are the coordinate functions described above. We can express the renormalized Green function on \( U \): if \( p = l(x), \ x \in U \), and \( \varphi(p) = h(x) \), we have

\[
G_r \circ l = G \circ l - G_o \circ h \tag{3.3}
\]
The expression of $G \circ l$ and $G_o \circ h$ are given respectively in (2.8) and (2.3), hence the renormalization leads to the result

$$(G_r \circ l)(\phi, \phi') = \frac{1}{16\pi^2 r^2 \sin^2 \left(\frac{\phi - \phi'}{2}\right)} \left(\frac{1}{\beta^2} - 1\right).$$

(3.4)

This result shows that the renormalization does not eliminate the singularity.

In the wedge calculation, one should identify a point $p$ in the flat space with a wedge cut out, with coordinates $\{t, r, z, \theta\}$ with the point in the Minkowski space with same coordinates, in other words, the identification map $N \rightarrow M_\alpha$ is the inclusion. Then the renormalized Green function is

$$(G_r \circ h') = (G \circ h') - (G_o \circ h \circ i)$$

where $i$ is the inclusion map from $V$ to $U$. Then the renormalization gives a finite result as obtained in [6].

4. The perturbation approach. An alternative computation of this phenomena may be instructive at this stage to justify our claim even more strongly. Essentially we have a scalar field $\Phi$ in a flat space with a defect. We can rewrite the problem as a scalar particle subject to an interaction where the Hamiltonian density is given as

$$H = H_0 + V$$

(4.1)

$$H_0 = \partial_\mu \Phi \partial^\mu \Phi$$

(4.2)

$$V = \left(\frac{1}{\beta^2} - 1\right) \left(\frac{\partial \Phi}{\partial \phi}\right)^2$$

(4.3)

This is an exactly solvable model. Assuming exponential behaviour for time dependence, we can calculate the energy eigenvalue $\omega$ in both the perturbed and unperturbed cases. If we calculate the total energy, we see that

$$\int d^3 p \omega_\beta \propto \Lambda^4$$

(4.4)

$$\int d^3 p \omega \propto \Lambda^4$$

(4.5)
\[ \int d^3 p \, (\omega_\beta - \omega) \propto \Lambda^4 \quad (4.6) \]

Here \( \omega_\beta \) and \( \omega \) are the energy eigenvalues with and without the interaction and \( \Lambda \) is the cut-off. If the subtraction of the vacuum resulted in a regularization, the difference of the energies would have a milder divergence.

One consistent way to make this contribution finite is to use a counterterm that cancels it completely. Since the interaction is only bilinear, we cannot regenerate it in the next order, as in the case with trilinear or higher couplings. We get zero for the energy with this regularization.

5. Conclusion.

Here we tried to point to the difference in identifying the vacuum in the case of a wedge calculation in electromagnetic theory and in a spacetime with cosmic string. These two situations can be described by with the same metric hence the local coordinate formulation of the Greens function is the same. However the renormalization process, i.e. the quantum part of the problem involves the identification of the the points in different manifolds. Our claim is that the physically meaningful identification corresponds to subtracting (2.3) from (2.8) which results in (3.4). Since we cannot tame the singularity by subtracting the vacuum, we suggest that this whole term should be subtracted, thus yielding no vacuum fluctuations in the presence of a cosmic string.

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