IDENTITIES INVOLVING NARAYANA POLYNOMIALS AND CATALAN NUMBERS

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Abstract. We first establish the result that the Narayana polynomials can be represented as the integrals of the Legendre polynomials. Then we represent the Catalan numbers in terms of the Narayana polynomials by three different identities. We give three different proofs for these identities, namely, two algebraic proofs and one combinatorial proof. Some applications are also given which lead to many known and new identities.

Keywords: Narayana polynomials, Legendre polynomials, Inverse relations, Catalan numbers

2000 Mathematics Subject Classification: Primary 05A05, 05A15

1. Introduction

The Catalan numbers [23, Sequence A000108] are defined by $C_n = \frac{1}{n+1} \binom{2n}{n}$, for all $n \geq 0$. The Narayana polynomials $\mathcal{N}_n(q)$ and the associated Narayana polynomials $\mathcal{N}_n(q)$ [2] are defined by

$$\mathcal{N}_n(q) = \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^k \quad \text{and} \quad \mathcal{N}_n(q) = q^n \mathcal{N}_n(q^{-1}) = \mathcal{N}_n(q)/q,$$

for $n \geq 1$, with the initial values $\mathcal{N}_0(q) = \mathcal{N}_0(q) = 1$. The coefficients $N_{n,k} = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}$ with $N_{0,0} = 1$ are called the Narayana numbers, and it is well known that the sequence $\{\mathcal{N}_n(1)\}_{n \geq 0}$ is the sequence of the Catalan numbers, while the sequence $\{\mathcal{N}_n(2)\}_{n \geq 0}$ is the sequence of the large Schröder numbers [23, Sequence A006318]. The Narayana polynomials and associated Narayana polynomials have been considered by several authors, see [2, 8, 19, 21, 20, 23]. For instance, Bonin, Shapiro and Simion [2] showed that the polynomial $\mathcal{N}_n(1 + q) = q$-analog of the $n$-th large Schröder numbers. Coker [8] provided several different expressions:

$$\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^{k-1} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1}{2k} C_k q^{k} (1 + q)^{n-2k-1}, \tag{1.1}$$

$$\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^{2(k-1)(1 + q)^{2(n-k)}} = \sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1} q^{k} (1 + q)^{k}. \tag{1.2}$$

Identity (1.1) was studied by Simion and Ullman [21] and proved combinatorially by Chen, Deng and Du [4]. Later, Chen, Yan and Yang [6] proved (1.1) and (1.2) combinatorially in...
terms of weighted 2-Motzkin paths. Recently, the authors \[14\] presented a new expression for Narayana polynomials

\[
N_n(q) = \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^k = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \binom{2n-k}{n} (q-1)^k,
\]

whose generalized version is appeared in \[15\] Example 2.13. Chen and Pang \[5\] also independently deduced (1.3) combinatorially.

The main result of this paper can be formulated as follows.

**Theorem 1.1.** For any integer \(n \geq 0\), there hold

\[
C_n = \sum_{k=0}^{n} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} N_k(q)(1-q)^{n-k},
\]

\[
q^{n+1} C_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} N_{k+1}(q)(1+q)^{n-k},
\]

\[
q^{n+2} C_{n+1} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} N_{k+1}(q^2)(1-q)^{2(n-k)},
\]

where \(C_n\) is zero if \(n\) is odd.

In this paper, we first establish the result that the Narayana polynomials can be represented as the integrals of the Legendre polynomials \[9\] in Sections 2. Then we give three different proofs for Theorem 1.1, see Sections 3-5, including two algebraical proofs and one combinatorial proof. Some applications are also given which lead to many known and new identities.

### 2. Narayana polynomials

Recall that the Legendre polynomials \(P_n(x)\) \[9\ \[18\], which are most familiar in the form

\[
P_n(x) = 2^{-n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \binom{2n-2k}{n-k} x^{n-2k},
\]

have an alternate expression, namely,

\[
P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left( \frac{x-1}{2} \right)^k = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} \left( \frac{x-1}{2} \right)^k,
\]

so that

\[
P_n(2x-1) = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} (x-1)^k.
\]

Note that an equivalent form of (1.3) is

\[
\sum_{k=0}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} q^k = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} (q-1)^{n-k}.
\]

Then (2.1) and (2.2) generate the following result.

**Theorem 2.1.** For any integer \(n \geq 1\), there holds

\[
N_n(q) = (q-1)^{n+1} \int_0^{1/q} P_n(2x-1) \, dx
\]

\[
= q(q-1)^n \int_0^{2q/(q-1)} P_n(P_n(2x-1) \, dx).
\]
Proof. It is clear that \( N_n(0) = 0 \) for all \( n \geq 1 \). Then we have

\[
q(q - 1)^n \int_0^1 P_n(\frac{2q}{q-1}x - 1)dx
\]

\[= (q - 1)^{n+1} \int_0^q P_n(2x - 1)dx\]

\[= (q - 1)^{n+1} \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} \int_0^{q-1} (x - 1)^k dx\]

\[= \sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k}(q - 1)^{n-k}\]

\[+ (q - 1)^{n+1} \sum_{k=0}^n \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k}(-1)^k\]

\[= N_n(q) - N_n(0)(1 - q)^{n+1} = N_n(q),\]

which completes the proof. \( \square \)

Theorem 2.1 signifies that many classical sequences such as Catalan numbers and Schröder numbers can be represented as the integrals of Legendre polynomials.

Example 2.2. (i) Let \( q = -1 \). Using the parity identity [8, 26]

\[ N_n(-1) = \begin{cases} 0 & \text{if } n = 2r, \\ (-1)^{r+1}C_r & \text{if } n = 2r + 1, \end{cases} \]

we have for \( n \geq 0 \),

\[ C_n = 2^{2n+1} \int_0^1 P_{2n+1}(x - 1)dx. \]

(ii) Let \( q = 2 \), we have that the large Schröder numbers \( N_n(2) \) satisfy

\[ N_n(2) = \int_0^2 P_n(2x - 1)dx. \]

Remark 2.3. Simons [22] established the following curious (in fact, it is not curious) identity

\[ \sum_{k=0}^n \frac{(-1)^{n-k}(n+k)!(1+x)^k}{(n-k)!k!^2} = \sum_{k=0}^n \frac{(n+k)!x^k}{(n-k)!k!^2}, \]

or equivalently

\[ \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} (1+x)^k = \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} x^k, \]

which was proved by Chapman [3], Prodinger [17], Wang and Sun [27]. It has been pointed out by Hirschhorn [13] that (2.4) is a special case of the Pfaff identity [12]. Recently, Munarini [16] gave a generalization of (2.4).

Obviously, (2.1) and (2.4) generate that

\[ (-1)^nP_n(-2x - 1) = P_n(2x + 1), \]

which can be easily derived by the generating function of Legendre polynomials [9],

\[ \sum_{n\geq0} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}. \]
3. Proof of Theorem 1.1 and inverse relations

In this section, using three well known inverses relations, we present our first proof for Theorem 1.1. The Legendre inverse relation [18]

\[
A_n = \sum_{k=0}^{n} \binom{n+k}{n-k} B_k \iff B_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} A_k,
\]

and the left-inversion formula [10] reads

\[
A_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n+p}{sk+p} B_k \implies B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{sn+p}{k+p} A_k,
\]

which, in the case \( s = 1, p = 0 \), implies the binomial inverse relation

\[
A_n = \sum_{k=0}^{n} \binom{n}{k} B_k \iff B_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k.
\]

Now we are ready to present the proof of Theorem 1.1.

3.1. Proof of (1.4). Rewriting (2.2), we have

\[
\mathcal{N}_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} C_k (q-1)^{-k},
\]

and using (3.1), we obtain two expressions for the Catalan numbers,

\[
C_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} \mathcal{N}_k(q)(q-1)^{n-k},
\]

which completes the proof of (1.4).

3.2. Proof of (1.5). Rewriting (1.1) in another form after replacing \( n \) by \( n+1 \),

\[
\mathcal{N}_{n+1}(q) = \sum_{k=0}^{n} \binom{n+k}{2k} C_k q^{k+1}(1+q)^{-2k},
\]

and using (3.2) in the case \( s = 2, p = 0 \), we deduce another expression for Catalan numbers,

\[
q^{n+1} C_n = \sum_{k=0}^{2n} (-1)^{k} \binom{2n+1}{k} \mathcal{N}_{k+1}(q)(1+q)^{2n-k},
\]

which motivates us to consider the following related summation

\[
f_n(q) = \sum_{i=1}^{2n+2} f_i q^i = \sum_{k=0}^{2n+1} (-1)^{k} \binom{2n+1}{k} \mathcal{N}_{k+1}(q)(1+q)^{2n+1-k}.
\]

Lemma 3.1. For all \( n \geq 0 \), \( f_n(q) = 0 \).

Proof. Comparing the coefficients of two sides in (3.5), we have

\[
f_m = \sum_{j=0}^{m} \sum_{k=0}^{2n+1} (-1)^{k} \binom{2n+1}{k} N_{k+1,j} \binom{2n+1-k}{m-j}.
\]
Noting that $N_{k+1,j}(\frac{2n+1-k}{m-j})$ is a polynomial on $k$ with degree $m+j-2$, which does not exceed $2n$ when $1 \leq m \leq n+1$. According to the well-known difference formula
\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (x-k)^{r} = \begin{cases} 
0 & \text{if } 0 \leq r < n, \\
n! & \text{if } r = n,
\end{cases}
\]
we can derive that each inner sum is zero in $f_{m}$ for $1 \leq m \leq n+1$. Note that $f_{n}(q) = q^{2n+3}f_{n}(q^{-1})$ by $q^{n+1}N_{n}(q^{-1}) = \mathcal{N}_{n}(q)$, which implies that $f_{m} = 0$ for $n+2 \leq m \leq 2n+2$. Hence, $f_{n}(q) = 0$ for $n \geq 0$, as claimed.

By combining (3.4) and (3.5), using Lemma 3.1, we obtain (1.5).

3.3. Proof of (1.6). Rewriting (1.2) in another form after replacing $n$ by $n+1$,
\[
\mathcal{N}_{n+1} \left( \frac{q^{2}}{(1+q)^{2}} \right) (1+q)^{2n+2} = \sum_{k=0}^{n} \binom{n}{k} C_{k+1} q^{k+2} (1+q)^{k},
\]
and using (3.3), we deduce that
\[
q^{n+2} (1+q)^{n} C_{n+1} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathcal{N}_{k+1} \left( \frac{q^{2}}{(1+q)^{2}} \right) (1+q)^{2k+2}.
\]
Replacing $q$ by $\frac{q}{1-q}$, after simplification, we get (1.6).

3.4. Applications. Theorem 1.1 can produce numerical known or new identities. For instance,

- The case $q = -1$ in (1.4) together with (2.3), lead to a new identity
\[
(2^{n} - 1) C_{n} = \sum_{r=0}^{\frac{n-1}{2}} (-1)^{r} \frac{4r + 3}{2n+1} \binom{2n+1}{n-2r-1} 2^{n-2r-1} C_{r},
\]
which implies that $C_{2k} \equiv 0 \mod 2$ and $C_{2k-1} \equiv C_{k-1} \mod 2$ for $k \geq 1$, from which one can easily derive that $C_{n}$ is odd if and only if $n = 2^{k} - 1$ for some $k \geq 0$.

- Taking the coefficient of $q^{n}$ in both sides of (1.4), we get another parity identity
\[
\sum_{k=0}^{n} (-1)^{k} 2^{k+1} \binom{2n+1}{n-k} = 0, \quad (n \geq 1),
\]
which has been proved by Chen, Li and Shapiro [7].

- The case $q = 1$ in (1.5) leads to a new identity
\[
C_{n} = \sum_{k=0}^{2n} (-1)^{k} \binom{2n}{k} C_{k+1} 2^{2n-k},
\]

- The case $q = -1$ in (1.6) leads to a known identity [1, 8, 18]
\[
C_{n+1} = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} C_{k+1} 4^{n-k},
\]
and the case $q = \sqrt{-1}$ in (1.6) leads to the Touchard identity [1, 8]
\[
C_{n+1} = \sum_{k=0}^{n} \binom{n}{2k} C_{k} 2^{n-2k}.
\]
• Let \( q = \sqrt{2} \) in (1.6), by the relation \((1 - \sqrt{2})^n = (P_n + P_{n-1}) - P_n\sqrt{2}\), where \( P_n \) is the \(n\)-th Pell number (defined by the recurrence relation \( P_{n+1} = 2P_n + P_{n-1} \) with \( P_{-1} = 1, P_0 = 0 \)), we have new identities involving Catalan numbers, large Schröder numbers, and Pell numbers

\[
2^{n+1}C_{2n+1} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \mathcal{N}_{k+1}(2) P_{4n-2k-1},
\]

\[
2^{n+1}C_{2n+2} = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \mathcal{N}_{k+1}(2) P_{4n-2k+2}.
\]

• Let \( q = \sqrt{5} \) in (1.6), by the relation \((1 - \sqrt{5})^n = (L_n - F_n)\sqrt{5}\), where \( L_n \) and \( F_n \) are respectively the \(n\)-th Lucas number and the \(n\)-th Fibonacci number (defined by the same recurrence relation \( G_{n+1} = G_n + G_{n-1} \) with \( G_{-1} = 2, G_0 = 1 \) for \( L_n \) and \( G_{-1} = 0, G_0 = 1 \) for \( F_n \)), we have new identities involving Catalan numbers, Lucas numbers, and Fibonacci numbers

\[
5^{n+1}C_{2n+1} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \mathcal{N}_{k+1}(5) L_{4n-2k-1} 2^{4n-2k-1},
\]

\[
5^{n+1}C_{2n+2} = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \mathcal{N}_{k+1}(5) F_{4n-2k+1} 2^{4n-2k+1}.
\]

4. PROOF OF THEOREM 1.1 AND GENERATING FUNCTIONS

In this section we present our second proof for Theorem 1.1 which is based on generating function techniques.

Recall that \( C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \) is the generating function for the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \), which satisfies the relation \( C(x) = 1 + xC(x)^2 = \frac{1}{1 - xC(x)} \). By Lagrange inversion formula [28], one can deduce that

\[
[x^{n-k}]C(x)^{2k+1} = \frac{2k+1}{2n+1} \binom{2n+1}{n-k}.
\]

Define \( \Omega(q, x) = \sum_{n \geq 0} \mathcal{N}_n(q)x^n \), then \( \Omega(q, x) \) has the explicit expression [11]

\[
\Omega(q, x) = \frac{1 + x - qx - \sqrt{1 - 2x + x^2 - 2qx - 2qx^2 + q^2x^2}}{2x},
\]

which can be rewritten as

\[
\Omega(q, x) = \frac{1}{1 + x - qx} C \left( \frac{x}{(1 + x - qx)^2} \right) = 1 + \frac{qx}{1 - x - qx} C \left( \frac{q^2x^2}{(1 - x - qx)^2} \right).
\]
By the Cauchy Residue Theorem, we have

\[
\sum_{k=0}^{n} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} \mathfrak{N}_k(q)(1-q)^{n-k}
= \sum_{k=0}^{n} \text{Res}_x C((1-q)x)^{2k+1} \frac{\Omega(q,y)}{x^{n-k+1}} \text{Res}_y \frac{\Omega(q,y)}{y^{k+1}}
= \text{Res}_x C((1-q)x)^{\frac{1}{x^{n+1}}} \sum_{k=0}^{n} x^k C((1-q)x)^{2k} \text{Res}_y \frac{\Omega(q,y)}{y^{k+1}}
= \text{Res}_x C((1-q)x) \frac{x \Omega(q,x)}{1+(1-q)x} C \left( \frac{xC((1-q)x)^2}{1+(1-q)xC((1-q)x)^2} \right) x^{-n-1}
= \text{Res}_x C(x) x^{-n-1} = C_n,
\]

and

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathfrak{N}_{k+1}(q)(1+q)^{n-k}
= \text{Res}_x \frac{(1-(1+q)x)^n \Omega(q,x) - 1}{x^{n+2}}
= \text{Res}_z \frac{\Omega(q, \frac{x}{1+(1+q)x}) - 1}{z^{n+2}}, \quad \text{(by setting } x = \frac{z}{1+(1+q)z})
= \text{Res}_z \left\{ \frac{q x}{1-(1+q)x} C \left( \frac{q x^2}{(1-(1+q)x)^2} \right) \right\} x=\frac{z}{1+(1+q)z}
= \text{Res}_z \frac{q C(q^2) \Omega(q^2,x)}{z^{n+1}} = q^{\frac{n+1}{2}} C_n
\]

and

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathfrak{N}_{k+1}(q^2)(1-q)^{2(n-k)}
= \text{Res}_x \frac{(1-(1-q)^2x)^n \Omega(q^2,x) - 1}{x^{n+2}}
= \text{Res}_z \frac{\Omega(q^2, \frac{x}{1+(1-q)^2x}) - 1}{z^{n+2}}, \quad \text{(by setting } x = \frac{z}{1+(1-q)^2z})
= \text{Res}_z \left\{ \frac{q^2 x}{1-(1+q^2)x} C \left( \frac{q^2 x^2}{(1-(1+q^2)x)^2} \right) \right\} x=\frac{z}{1+(1-q)^2z}
= \text{Res}_z \frac{q^2 z}{1-2qz} C \left( \frac{q^2 z^2}{(1-2qz)^2} \right) z^{-n-1}
= \text{Res}_w \frac{q z^{n+2}(1+2w)^n C(w^2)}{w^{n+1}}, \quad \text{(by setting } z = \frac{w}{q(1+2w)})
= q^{n+2} \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = q^{n+2} C_{n+1},
\]
which give (1.4), (1.5) and (1.6), respectively. Note that the last equation follows by the well-known Touchard’s identity, which can also be derived by setting \( q = 1 \) in (1.1) after replacing \( n \) by \( n + 1 \).

5. Combinatorial Proof of Theorem 1.1

In order to give the combinatorial proof of (1.4), we need the following definitions. A Dyck path of length \( 2n \) is a lattice path from \((0,0)\) to \((2n,0)\) in the first quadrant of \(xoy\)-plane, consisting of up-steps \( u = (1,1) \) and down-steps \( d = (1,-1) \), which never passes below the \(x\)-axis. We will refer to \( n \) as the semilength of the path. It is well known that the set of Dyck paths of semilength \( n \) is counted by the Catalan number \( C_n = \frac{1}{2n+1} \binom{2n+1}{n} \). A peak in a Dyck path is an occurrence of \( ud \). By the height of a step we mean the ordinate of its endpoint. By a return step we mean a down-step ending at height zero. Dyck paths that have exactly one return step are said to be primitive. If \( D_1 \) and \( D_2 \) are Dyck paths, we define \( D_1 D_2 \) to be the concatenation of \( D_1 \) and \( D_2 \).

A weighted Dyck path is a Dyck path \( D \) for which every up-step is endowed with a weight. The weight of a Dyck path \( D \) is the product of the weights of its up-steps, the weight of a set \( S \) of Dyck paths means the sum of the weights of \( D \) in \( S \).

Let \( \mathcal{P}_{n,k} \) denote the set of weighted Dyck paths of length \( 2n \) consisting of an ordered sequence of \( 2k + 2 \) Dyck paths \( D^{(0)}, D^{(1)}, \ldots, D^{(2k+1)} \) such that

- \( D^{(0)} \) is a Dyck path of length \( 2k \) for \( 0 \leq k \leq n \) with an up-step in each peak weighted by \( q \) and other up-steps weighted by 1;
- There are totally \( n - k \) up-steps in the rest \( 2k + 1 \) Dyck paths, and all up-steps of each \( D^{(i)} \) are weighted by 1 or \(-q\) for \( 1 \leq i \leq 2k + 1 \), i.e., such up-steps can be regarded to be weighted by \((1 - q)\);
- Each \( D^{(i)} \) is inserted into the \( i \)-th endpoint of \( D^{(0)} \) including the beginning point for \( 1 \leq i \leq 2k + 1 \).

Let \( \mathcal{P}_n = \bigcup_{k=0}^n \mathcal{P}_{n,k} \). For any \( D \in \mathcal{P}_n \), denote by \( w(D) \) the weight of \( D \). Let \( \tilde{\mathcal{P}}_n \) denote the subset of Dyck paths \( D \in \mathcal{P}_n \) such that all up-steps in \( D \) are weighted by 1, such Dyck paths only appear in \( \mathcal{P}_{n,0} \).

**Theorem 5.1.** There exists a sign reversing involution \( \varphi \) on the set \( \mathcal{P}_n \setminus \tilde{\mathcal{P}}_n \).

**Proof.** For any \( D \in \mathcal{P}_n \setminus \tilde{\mathcal{P}}_n \), it can be uniquely written as \( D = D_1 D_2 \cdots D_m \) for some \( 1 \leq m \leq n \), where \( D_i \)'s are weighted primitive Dyck paths. Obviously, there exist at least a \( D_i \) such that \( D_i \) has an up-step weighted by \( q \) or \(-q\). Now we can recursively construct the involution \( \varphi \) as follows. First find the maximum \( i \) for \( 1 \leq i \leq m \) such that \( D_i \) has an up-step weighted by \( q \) or \(-q\), then define \( \varphi(D) = D_1 \cdots D_{i-1} \varphi(D_i) D_{i+1} \cdots D_m \). Note that \( D_i = u D^*_i d \), where \( D^*_i \in \mathcal{P}_j \) for some \( 0 \leq j \leq n - 1 \).

- If the first up-step in \( D_i = u D^*_i d \) has weight \( q \) or \(-q\), then \( D^*_i \) has no up-steps with weight \( q \), otherwise it will contradict with the definition of \( D \in \mathcal{P}_n \). Let \( D'_i \) denote the weighted Dyck path obtained from \( D_i \) by changing the sign of the weight of the first up-step. Then define \( \varphi(D_i) = D'_i \);
- If the first up-step in \( D_i = u D^*_i d \) has weight 1, then \( D^*_i \in \mathcal{P}_j \setminus \tilde{\mathcal{P}}_j \) for some \( 1 \leq j \leq n - 1 \), define \( \varphi(D_i) = u \varphi(D^*_i) d \).

It is clear that the \( \varphi \) is a sign reversing involution on the set \( \mathcal{P}_n \setminus \tilde{\mathcal{P}}_n \). See Figure 4 for an illustration. \( \square \)
Let $T$ be a plane tree, means the sum of the weights of its vertices, the weight of a plane tree $T$ is defined recursively (see for example [24]) as a finite set of vertices $\phi$ such that a distinguished vertex $u$, and the remaining vertices are put into an ordered partition $(T_1, T_2, \ldots, T_m)$ of $m \geq 0$ disjoint non-empty sets, each of which is a plane tree called the subtree of $T$ of vertex $u$. The root $u_i$ of $T_i$ is called the child of $u$, and $u$ is called the father of $u_i$. The out-degree of a vertex of $T$ is the number of its subtrees. An internal vertex of $T$ is a vertex of out-degree at least one. A vertex of out-degree zero is called a leaf of $T$. A complete binary tree is a plane tree such that each internal vertex has out-degree two.

A weighted plane tree is a plane tree for which every vertex is endowed with a weight. The weight of a plane tree $T$ is the product of the weights of its vertices, the weight of a set $S$ of plane trees means the sum of the weights of $T$ in $S$.

Let $\mathcal{P}_{n,k}$ denote the set of weighted plane trees of $n + 2$ vertices such that

- The leaves have weight $q$;
- There exist $n - k$ vertices of out-degree one, except for the root, with weight $-1$ or $-q$; in other words, such vertices can be regarded to be weighted by $-(1 + q)$;
- All other internal vertices have weight 1, there may exist vertices of out-degree one with weight 1.

Proof of (1.4). It is clear that the weight of $\mathcal{D}_n$ is the $n$-th Catalan number $C_n$. According to the definition of $D \in \mathcal{D}_{n,k}$, it is easy to derive the weight of $\mathcal{D}_{n,k}$. On one hand, it is well known that the total weight for $D^{(0)}$ is the Narayana polynomial $N_k(q)$. On the other hand, the total product of the weights of $D^{(1)}, D^{(2)}, \ldots, D^{(2k+1)}$ is just the coefficient of $x^{n-k}$ in $(1 - q)^{n-k}C(x)^{2k+1}$. Then by (1.1) we have

$$w(\mathcal{D}_{n,k}) = N_k(q)(1 - q)^{n-k}[x^{n-k}]C(x)^{2k+1} = \frac{2k + 1}{2n + 1} \binom{2n + 1}{n - k} N_k(q)(1 - q)^{n-k}.$$ 

By Theorem 5.1 we have $w(\mathcal{D}_n) = \sum_{k=0}^{n} w(\mathcal{D}_{n,k})$, which completes the proof. \hfill $\Box$

Remark 5.2. Specially, let $\mathcal{D}_n$ denote the subset of Dyck paths $D \in \mathcal{D}_n$ such that all up-steps in $D$ are weighted by $q$ or $-q$, such Dyck paths can only appear in $\mathcal{D}_{n,k}$ for $0 \leq k \leq n$ satisfying that (a) $D^{(0)} = (ud)^k$ with up-steps weighted by $q$, and (b) all up-steps in $D^{(i)}$ are weighted by $-q$ for $1 \leq i \leq k$. Let $\varphi_{\mathcal{D}_n}$ be the $\varphi$ restricted to $\mathcal{D}_n$, it is clear that $\varphi_{\mathcal{D}_n}$ is a sign reversing involution on $\mathcal{D}_n$. Then (1.2) is followed immediately by $\varphi_{\mathcal{D}_n}$ and (1.1). In order to give the combinatorial proof of (1.5) and (1.6), we need the following definitions. A plane tree $T$ can be defined recursively (see for example [24]) as a finite set of vertices such that a distinguished vertex $u$ is called the root of $T$, and the remaining vertices are put into an ordered partition $(T_1, T_2, \ldots, T_m)$ of $m \geq 0$ disjoint non-empty sets, each of which is a plane tree called the subtree of $u$. The root $u_i$ of $T_i$ is called the child of $u$, and $u$ is called the father of $u_i$. The out-degree of a vertex of $T$ is the number of its subtrees. An internal vertex of $T$ is a vertex of out-degree at least one. A vertex of out-degree zero is called a leaf of $T$. A complete binary tree is a plane tree such that each internal vertex has out-degree two.

A weighted plane tree is a plane tree for which every vertex is endowed with a weight. The weight of a plane tree $T$ is the product of the weights of its vertices, the weight of a set $S$ of plane trees means the sum of the weights of $T$ in $S$.

Let $\mathcal{P}_{n,k}$ denote the set of weighted plane trees of $n + 2$ vertices such that

- The leaves have weight $q$;
- There exist $n - k$ vertices of out-degree one, except for the root, with weight $-1$ or $-q$; in other words, such vertices can be regarded to be weighted by $-(1 + q)$;
- All other internal vertices have weight 1, there may exist vertices of out-degree one with weight 1.
Let \( \mathcal{P}_n = \bigcup_{k=0}^{n} \mathcal{P}_{n,k} \). For any \( T \in \mathcal{P}_n \), denote by \( w(T) \) the weight of \( T \). Let \( \mathcal{P}_n^* \) denote the subset of \( \mathcal{P}_n \) such that there are at least one vertex, except for the root, of out-degree one weighted by 1 or \(-1\), let \( \tilde{\mathcal{P}}_n \) denote the subset of \( \mathcal{P}_n \) such that the root has out-degree one, and all other internal vertices have out-degree two. It is clear that \( \mathcal{P}_n^* \cap \tilde{\mathcal{P}}_n = \emptyset \).

**Theorem 5.3.** There exists a sign reversing involution \( \psi \) on the set \( \mathcal{P}_n \setminus \tilde{\mathcal{P}}_n \).

**Proof.** Note that a tree \( T \in \mathcal{P}_n \) is in \( \mathcal{P}_n^* \) if and only if it contains a non-rooted vertex of out-degree one with weight either 1 or \(-1\). Consider the first occurrence of such vertex, denoted by \( v \), when traversing the weighted plane tree \( T \) in pre-order, (i.e., visiting the root first, then traversing its subtrees from left to right). Then replace the weight of \( v \) in \( T \) by \(-w(v)\), we obtain another weighted plane tree \( T^* \) in \( \mathcal{P}_n^* \), and then define \( \psi(T) = T^* \). See Figure 2 for example.

![Figure 2](image-url)  
**Figure 2.** The involution on \( \mathcal{P}_n^* \)

For any \( T \in \mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n) \), we can construct recursively the involution \( \psi \) as follows. First we should consider the following two cases:

- The root of \( T \) has out-degree one, in this case, the unique subtree of the root has either a vertex of out-degree greater than two or a vertex of out-degree one with weight \(-q\).

- The root of \( T \) has out-degree not less than two.

If the root \( u \) of \( T \) has \( i (\geq 2) \) number of subtrees, denoted by \( T_1, T_2, \ldots, T_i \), when \( T_1 \) is a complete binary tree, then delete the subtree \( T_2 \) of the root \( u \), regard \( T_2 \) as the subtree of the right-most leaf \( v \) of \( T_1 \), and replace the weight \( q \) of \( v \) by \(-q\), hence we obtain a new weighted plane tree \( T^* \) in \( \mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n) \), and define \( \psi(T) = T^* \); when \( T_1 \) is not a complete binary tree, consider the left-most component of \( T \), that is the tree \( T_1' \) with the root \( u \) and \( T_1 \) as its unique subtree, then \( \psi(T) \) is obtained by adding the subtrees \( T_2, \ldots, T_i \) right to the root \( u \) of \( \psi(T_1') \) step by step. See Figure 3 for example. If the root \( u \) of \( T \) has a unique subtree, denoted by \( T' \), let \( u' \) be the root of \( T' \) which is the only child of \( u \).

(i) If the out-degree of \( u' \) is greater than two, then \( \psi(T) \) is defined to be the tree \( T^* \) in \( \mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n) \) which has the root \( u \) with a unique subtree \( \psi(T') \); See Figure 4 for example.

(ii) If the out-degree of \( u' \) is one or two, find the right-most leaf \( v' \),

- if there exist vertices of weight \(-q\) in the path \( u'v' \), then choose the vertex \( v \) which is first occurring in the path \( u'v' \), denoted by \( T'' \) as the subtree of \( v \), if deleting \( T'' \) in \( T' \), the resulting tree is a complete binary tree. Then deleting the subtree \( T'' \) in \( T \), annexing it to the right of \( u \), and changing the weight \(-q \) of \( v \) to \( q \), we obtain a new weighted plane tree \( T^* \) in \( \mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n) \), and define \( \psi(T) = T^* \); See Figure 5 for example.
• if deleting $T''$ in $T'$, the resulting tree is not a complete binary tree or if there is no vertex of weight $-q$ in the path $u'v'$, then $u'$ must have out-degree two. Let $T_1, T_2$ be the left and right subtrees of $u'$ and $T'_1, T'_2$ be the left and right components of $u'$ respectively. If $T_1$ is not a complete binary tree, then $\psi(T)$ is defined to be the tree $T^*$ in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$ by replacing $T'_1$ in $T$ by $\psi(T'_1)$; If $T_1$ is a complete binary tree, so $T_2$ must not be a complete binary tree, then $\psi(T)$ is defined to be the tree $T^*$ in $\mathcal{P}_n \setminus (\mathcal{P}_n^* \cup \tilde{\mathcal{P}}_n)$ by replacing $T'_2$ in $T$ by $\psi(T'_2)$. See Figure 6 for example.

Clearly, the $\psi$ as defined is indeed a sign reversing involution on the set $\mathcal{P}_n \setminus \tilde{\mathcal{P}}_n$. \qed
Let $Q_{n,k}$ denote the set of weighted plane trees of $n + 2$ vertices such that
- The leaves have weight $q^2$;
- There exist $n - k$ vertices of out-degree one, except for the root, with weight $-1$, $2q$ or $-q^2$, in other words, such vertices can be regarded to be weighted by $-(1 - q)^2$;
- All other internal vertices have weight 1, there may exist vertices of out-degree one with weight 1.

Let $Q_n = \bigcup_{k=0}^{n} Q_{n,k}$, and $Q_n^*$ denote the subset of $Q_n$ such that there are at least one vertex, except for the root, of out-degree one weighted by 1 or $-1$, let $Q_n^\circ$ denote the subset of $Q_n$ such that the root has out-degree one, and all other internal vertices have either out-degree two or out-degree one with weight $2q$. It is clear that $Q_n^* \cap Q_n^\circ = \emptyset$. Similar to the proof of Theorem 5.3, a sign reversing involution can be constructed on $Q_n \setminus Q_n^\circ$, the detail leaves to the interested readers. Hence we have

**Theorem 5.4.** There exists a sign reversing involution on the set $Q_n \setminus Q_n^\circ$.

Now we can give the combinatorial proof of identities (1.5) and (1.6).

**Proof of (1.5) and (1.6).** For any $T \in \mathcal{P}_{k,k}$, namely, $T$ is a weighted plane tree of $k + 2$ vertices with leaves weighted by $q$ and all other internal vertices weighted by 1, inserting $n - k$ vertices of weight $-(1 + q)$ into the $k + 1$ edges of $T$ (repetition allowed), we can obtain \( \binom{n}{k} \) number of weighted plane trees in $\mathcal{P}_{n,k}$. It is well-known that the weight of $\mathcal{P}_{k,k}$ is $\bar{N}_{k+1}(q)$, then $\mathcal{P}_{n,k}$ has the weight \( \binom{n}{k} \bar{N}_{k+1}(q)(-1 - q)^{n - k} \). Similarly, $\mathcal{P}_{n,k}$ has the weight \( (-1)^{n-k} \binom{n}{k} \bar{N}_{k+1}(q^2)(1 - q)^{2(n-k)} \).

On the other hand, for any $T \in \mathcal{P}_n$, we know that the root of $T$ has out-degree one and has only one subtree $T'$ which is a weighted complete binary tree with $n + 1$ vertices, it is well known that the number of complete binary trees with $n + 1$ vertices is counted by Catalan number $C_n$, where $C_n = 0$ if $n$ is odd. So the weight of $\mathcal{P}_n$ is $q^{n+1}C_n$.

For any $T \in \mathcal{D}_n$, let $\mathcal{D}_{n,k}$ denote the subset of $\mathcal{D}_n$ such that $T$ has $n - 2k$ vertices, except for the root, of out-degree one with weight $2q$. For any $T \in \mathcal{D}_{2k,k}$, we know that the root of $T$ has out-degree one and has only one subtree $T'$ which is a weighted complete binary tree with $2k + 1$ vertices, inserting $n - 2k$ vertices of weight $2q$ into the $2k + 1$ edges of $T$ (repetition allowed), we can obtain \( \binom{n}{2k} \) number of weighted plane trees in $\mathcal{D}_{n,k}$. It is clear that
\(\mathcal{D}_{2k,k}\) is counted by Catalan numbers \(C_k\) and has weight \(q^{2k+2}C_k\), then \(\mathcal{D}_{n,k}\) has weight 
\[(2q)^{n-2k}\binom{n}{2k}q^{2k+2}C_k = q^{n+2}\binom{n}{2k}C_k2^{n-2k}\.\] Hence \(\mathcal{D}_n\) has weight \(q^{n+2}\sum_{k=0}^{n}\binom{n}{2k}C_k2^{n-2k}\), which is \(q^nC_{n+1}\) by Touchard identity.

Using Theorem 5.3 and 5.4, one can easily obtain that the weight of \(\mathcal{P}_n\) (resp. \(\mathcal{D}_n\)) equals that of \(\mathcal{P}_n\) (resp. \(\mathcal{D}_n\)), which completes the proof. \(\square\)

Acknowledgements

The second author was supported by The National Science Foundation of China.

References

[1] M. Aigner, Catalan-like Numbers and Determinants, *J. Combin. Theory, Series A* 87 (1999) 33–51.
[2] J. Bonin, L. Shapiro and R. Simion, Some \(q\)-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, *J. Statist. Plann. Inference* 34 (1993) 35–55.
[3] R. Chapman, A curious identity, revisited, *The Math. Gazette* 87 (2003) 139–141.
[4] W.Y.C. Chen, E.Y.P. Deng and R.R.X. Du, Reduction of \(m\)-regular noncrossing partitions, *European J. Combin.* 26 (2005) 237–243.
[5] W.Y.C. Chen and S. X. M. Pang, On the combinatorics of the Pfaff identity, submitted.
[6] W.Y.C. Chen, S.H.F. Yan and L.L.M. Yang, Identities from weighted 2-Motzkin paths, *Adv. Appl. Math.*, to appear.
[7] C. Coker, Enumerating a class of lattice paths, *Disc. Math.* 271 (2003) 13–28.
[8] L. Comtet, Advanced Combinatorics, D.Reidel, Dordrecht-Holland, 1970.
[9] C. Corsani, D. Merlini and R. Sprugnoli, Left-inversion of combinatorial sums, *Disc. Math.* 180:1-3 (1998) 107–122.
[10] E. Deutsch, Dyck path enumeration, *Disc. Math.* 204 (1999) 167–202.
[11] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2nd Edition, Encyclopedia of Mathematics And Its Applications, vol. 96, Cambridge University Press, 2004.
[12] M. D. Hirschhorn, Comment on a “curious” identity, *The Math. Gazette* 87 (2003) 528–530.
[13] T. Mansour and Y. Sun, Dyck Paths and partial Bell polynomials, *Australasian J. Combinatorics.*, to appear.
[14] T. Mansour and Y. Sun, Bell polynomials and \(k\)-generalized Dyck Paths, *Disc. Appl. Math.*, to appear.
[15] T. Mansour and Y. Sun, Dyck Paths and partial Bell polynomials, *Integers: Elect. J. Combin. Number Theory* 5 (2005) #A15.
[16] H. Prodinger, A curious identity proved by Cauchy’s integral formula, *The Math. Gazette* 89 (2005) 266–267.
[17] J. Riordan, Combinatorial Identities, New York: John Wiley & Sons, Inc. 1968
[18] D. G. Rogers, Rhyming schemes: crossings and coverings, *Disc. Math.* 33 (1981) 67–77.
[19] D. G. Rogers, L.W. Shapiro, Deques, trees and lattice paths, *Lect. Notes Math.* 884 (1981) 293–303.
[20] R. Simion and D. Ullman, On the structure of the lattice of noncrossing partitions, *Disc. Math.* 98 (1991) 193–206.
[21] S. Simons, A curious identity, *The Math. Gazette* 85 (2001) 296–298.
[22] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, www.research.att.com/jas/sequences.
[23] R. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, Cambridge, 1997.
[24] R. Sulanke, Counting lattice paths by Narayana polynomials, *Electron. J. Comb.*, 2 (2000) #R40.
[25] Y. Sun, The statistic “number of \(udu’s\)” in Dyck paths, *Disc. Math.* 287:1-3 (2004) 177-186.
[26] X. Wang and Y. Sun, A new proof of a curious identity, *The Math. Gazette* 91 (502) (2007) 105–106.
[27] H. Wilf, Generatingfunctionology, Academic Press, New York, 1990.