Comparing the Dirac Hamiltonians for a neutron subjected to either a Schwartzchild gravitational field or a uniform acceleration, we observe that the difference between the two is precisely the sort that might be eliminated by the introduction of a new quantum number. The origin of this quantum number lies in the noncommutation of an acceleration with the quark operators that constitute the neutron. We show that the term containing the new quantum number only acts on very long length scales. Furthermore, the symmetries of an acceleration prevent the effects of this term from being periodic.

PACS numbers: 03.65.-w, 04.62.+v
I. INTRODUCTION

Any laboratory on the Earth simultaneously experiences both acceleration and gravity. Einstein’s equivalence principle states that, neglecting curvature effects, the acceleration and gravitation are indistinguishable. The success of General Relativity is a testament to the accuracy of this statement in the classical regime. The equivalence principle has also been tested in quantum mechanical phenomena. First, Colella et al. [1] performed their celebrated neutron interferometry experiment, observing a gravitationally-induced phase shift. Using similar apparatus, Bonse and Wroblewski [2] found the same phase shift when their interferometer was uniformly accelerated. Together, these experiments provided some confirmation of the equivalence principle in quantum mechanics.

These two neutron interferometry experiments did not measure any spin-dependent effects, leaving open the question of whether spin may be affected differently by acceleration and uniform gravitation. Varjú and Ryder [3] have suggested a particular spin effect that may be able to distinguish between the two. They compared two Dirac (spin 1/2) Hamiltonians. One was the Hamiltonian for a particle in a Schwartzchild gravitational field, drawing on earlier calculations by Fischbach, et al. [4]. The other Hamiltonian represented a Dirac particle subjected to a uniform acceleration, as found by M. S. Altschul [5] and later by Hehl and Ni [6]. An apparent difference between these Hamiltonians calls into question whether the equivalence principle holds for spin effects.

We will show that an additional accelerational effect could eliminate the difference between the two Hamiltonians, restoring the equivalence principle. However, it is natural that this effect has not yet been observed, since it acts on very long length scales.

II. COMPARISON OF HAMILTONIANS

In comparing uniform gravitation and acceleration, we will evaluate the Hamiltonians to first order. For the acceleration, that means to first order in \( \mathbf{a} \), the acceleration vector. For the gravitational Hamiltonian, we will work to first order in \( \mathbf{g} \), where \( g_\mu = -\frac{\partial \Phi}{\partial x^\mu}c^2 \). Here, \( \Phi = \frac{GM}{c^2r} \), so \( \mathbf{g} = \frac{-4\pi^2}{c} \mathbf{x} \) is just the normal gravity vector. Each of these Hamiltonians may be found by using three successive Foldy-Wouthuysen transformations [7].

The gravitational Hamiltonian found by Varjú and Ryder takes the form

\[
H = \beta mc^2 + \beta m g \cdot x + \frac{\beta}{2m} p^2 - \frac{\beta}{2mc^2} \mathbf{p} \cdot (\mathbf{g} \times \mathbf{p}) + \frac{\hbar \beta}{4mc^2} \sigma \cdot (\mathbf{g} \times \mathbf{p}).
\]

(1)

The accelerational Hamiltonian has been found in two different (and both flawed) ways. The result was

\[
H = \beta mc^2 - \beta mg \cdot x + \frac{\beta}{2m} p^2 - \frac{\beta}{2mc^2} \mathbf{p} \cdot (\mathbf{a} \cdot \mathbf{x}) \mathbf{p} + \frac{\hbar \beta}{4mc^2} \sigma \cdot (\mathbf{a} \times \mathbf{p}).
\]

(2)

The equivalence principle indicates that these two Hamiltonians should be identical when we set \( \mathbf{a} = -\mathbf{g} \), except for tidal terms in (1). On initial inspection, this does not appear to be the case. There are two differences. The last term of (1) is entirely absent from (2). Although only first order in \( \mathbf{g} \), this term is actually a tidal term, as Varjú and Ryder demonstrated, so its absence from (2) is expected.

The other difference between the Hamiltonians less easily dismissed. While (1) includes the term

\[
\frac{\beta \hbar}{2mc^2} \sigma \cdot (\mathbf{g} \times \mathbf{p}),
\]

(3)

(2) contains

\[
\frac{\beta \hbar}{4mc^2} \sigma \cdot (\mathbf{a} \times \mathbf{p}).
\]

(4)

When we equate \( \mathbf{a} = -\mathbf{g} \), these two terms differ by a factor of \(-2\). It appears that this difference could allow an observer to distinguish between the acceleration and the gravitation. However, our closer analysis indicates that this may not be the case.

III. ANOMALOUS ACCELERATION MOMENT

The Hamiltonian (2) was first derived by purely electromagnetic means, for a charged particle in a uniform electric field. Then (3) enters as a charge-dependent term, vanishing for a neutral particle. The behavior of (1) in this regard is the same as that of the spin magnetic moment term in a uniform magnetic field \( \mathbf{B} \). However, we know that the spin magnetic moment is actually an additional parameter of the Dirac Hamiltonian; it is not zero for neutral particles with internal structure. Since \( \mathbf{a} \) and \( \mathbf{B} \) have the same three-dimensional vector group structure, there should be an analogous parameter in the accelerational Hamiltonian.

In fact, when we looks at the infinitesimal generator for an acceleration, we may identify this new parameter. As shown in (3), the generator has the form

\[
\chi = \chi_0 + \frac{\beta \hbar}{4c^2} \sigma \cdot \nabla \mathbf{p} + f_\chi.
\]

(5)

For small values of \( \mathbf{a} \) this generates the Hamiltonian (2) from the free particle Hamiltonian \( H_0 \) by \( H = H_0 + [\mathbf{a} \cdot \chi, H_0] \). Under this prescription, the first term of \( \chi \) generates the “classical” potential energy \( \beta ma \cdot x \), and
$f_\chi$ commutes with $H_0$. The second term, containing $\sigma$, is the one of interest. We will call this term $\chi'$. It is $\chi'$ that generates the problematic term \( \Box \).

From the form of \( \Box \), it is clear that there is a free parameter. The coefficient of $\chi'$ may be freely adjusted without affecting any lower-order terms in the Hamiltonian. An “anomalous acceleration moment,” $\mu_a$ may be introduced. The presence of $\mu_a$ changes the generator according to

$$
\chi' = (1 + \mu_a) \frac{\beta \hbar}{4c^2} \sigma \times \nabla p.
$$

(6)

The variation of $\mu_a$ creates a one-parameter family of representations of the acceleration group.

It remains to consider for what particles this anomalous contribution may be nonzero and significant. For charged particles, the contribution is certainly not significant. Attempts to accelerate such particles uniformly are prevented by the emission of electromagnetic radiation and the induced radiative reaction force \( \Box \). For neutral particles, this problem does not appear, so we shall consider neutrons.

From the pure Dirac standpoint, the neutron should have no magnetic moment. However, there is a neutron magnetic moment caused by QCD. Similarly, there may be an nonzero anomalous acceleration moment for the neutron. In fact, it is expected to be nonzero, for the following reason. The value $\mu_a = 0$ is required by the conditions of Lorentz invariance and charge conservation \( \Box \). However, these conditions do not hold under acceleration. Since a charge density causes acceleration, the charge and acceleration operators do not commute. More specifically, the quark operators $\psi_q^\dagger$ that compose a neutron do not commute with acceleration, so the quark content of an accelerated neutron is not well-defined. We may estimate the effective commutator from the fact that a free quark would induce a force $F \approx 10^{22}$ GeV/cm \( \Box \) on every differently-colored quark in the universe. This yields an order of magnitude of

$$
|\langle a, \psi_q^\dagger \rangle| \sim \frac{2}{3} \frac{F}{m_q} \approx 10^{36} \text{ cm } \cdot \text{s}^{-2},
$$

(7)

since the quark mass $m_q$ is a few MeV $\cdot$ c$^{-2}$ for the $u$ and $d$ quarks. The lowest-order approximation for terms stemming from this commutator to enter the Hamiltonian is through $\mu_a$.

This noncommutation is what interferes with Hehl and Ni’s derivation of the accelerational Hamiltonian, in which they simply replace the partial derivatives in the Dirac equation with covariant derivatives. Although Varjú and Ryder use the same substitution to find the gravitational Hamiltonian, the noncommutation of $g$ with the quark operators lies in the realm of quantum gravity, and so should be negligible. So the existence of $\mu_a$ allows for the restoration of the equivalence principle. In fact, the equivalence principle provides a specific prediction of its value, $\mu_a = -3$.

IV. LENGTH SCALE FOR THE ACCELERATION MOMENT

It is important to investigate the dynamical effects caused by $\mu_a$. We shall begin this process by examining the length scale on which $\chi'$ operates. The order of magnitude of $\mu_a$ will probably not deviate too much from unity, since it is a dimensionless parameter of nuclear structure. This is certainly true if $\mu_a$ takes the value indicated by the equivalence principle.

To get a length scale from $\chi'$, we must insert the quantum-mechanical prescription that $i \hbar \nabla_p = x$, to get

$$
\chi' = (1 + \mu_a) \frac{\beta}{4c^2} \sigma \times x.
$$

(8)

The $i$ disappears when we take $\exp(ia \cdot \chi) \approx 1 + ia \cdot \chi$. The terms multiplying $x$ in $\chi'$ give us a length scale for the action of this effect in the Hamiltonian. Although the generator is only correct for infinitesimal accelerations, it should give the right dependence for the length scale. We drop $\beta$ and $\sigma$, since they only take the values $\pm 1$. This leaves

$$
\frac{|1 + \mu_a|}{4c^2} x.
$$

Since the original dimensionless quantity for generating the Hamiltonian was $a \cdot \chi$, the length scale must depend on $a = |a|$. The final expression for determining the length scale is then

$$
a \cdot \chi' \sim \frac{|1 + \mu_a|}{4c^2} |x||a|.
$$

(9)

Therefore, the characteristic length scale for the accelerated neutron is

$$
x_a = \frac{4c^2}{|1 + \mu_a||a|}.
$$

(10)

In cgs units, this has the numerical value of

$$
x_a = \frac{3.6 \times 10^{21}}{|1 + \mu_a||a|} \text{ cm}.
$$

(11)

For reasonable values of $a$ and $|1 + \mu_a|$, the scale of $x_a$ is astrophysical. A natural place to look for this effect would be the solar wind, since the solar wind contains many neutrons and it undergoes continuous acceleration. The gravitational acceleration at the surface of the sun is $a_\odot = GM_\odot/R_\odot^2 = 2.7 \times 10^6 \text{ cm } \cdot \text{s}^{-2}$. If $|1 + \mu_a| = 2$, this indicates a length of $x_a = 6.6 \times 10^{16} \text{ cm}$ or 0.036 LY. This is a substantial distance. Moreover, this is not the
flight distance of the neutrons from the sun. Instead, it represents their retardation by the sun’s gravitational attraction. The actual flight distance would be even larger, and the sun’s gravitational acceleration is certainly not uniform over the distance. Thus, it is almost certainly not possible to observe any effect on the solar wind neutrons.

Since the length scale due to the sun’s gravitational acceleration is too large, it is natural to consider situations in which $a$ is substantially greater. In particular, we will look at the acceleration at the surface of a neutron star, with mass $1.5M_\odot$ and radius $10^6$ cm. For this body, $a_{NS} = 2.0 \times 10^{14}$ cm·s$^{-2}$. In this case, the value of $x_a$ is $9.1 \times 10^6$ cm, substantially smaller. However, this is still too large for the gravitational field to be nearly uniform. In any case, it is unlikely that particles at a neutron star’s surface would be free from far stronger interactions, so the effect is almost certainly not significant in this case, either.

These calculated length scales indicate that the term of the Hamiltonian containing $\mu_a$ does not have a sizeable effect except at very long distances or very high resolutions. Thus, it is natural that $\mu_a$ has not yet been observed in any experiment. Moreover, this term of the Hamiltonian probably has no direct astrophysical implications.

V. ACCELERATION MOMENT EFFECTS ALLOWED BY SYMMETRY

At this point, it is not at all clear what sort of effects will occur on the length scale $x_a$. Fischbach et al. identify the term $\mathcal{H}_a$ as a standard spin-orbit coupling term, since

$$\sigma \cdot (g \times p) = -\frac{\Phi r^2}{r^2} \sigma \cdot (x \times p) = -\frac{2\Phi r^2}{\hbar r^2} L \cdot S. \quad (12)$$

This is correct in general, but it is not useful in the approximation of a uniform gravitational field $a = -g$, since the meaningful behavior of $L = x \times p$ relies on $x$ not remaining constant. We shall therefore work only with $\mathcal{H}_a$, which is clearly not a simple $L \cdot S$ interaction. In fact, we shall find that the simple precession characteristic of $L \cdot S$ interactions does not occur in this problem. Our arguments will be qualitative, based on the symmetries of the situation, so the numerical difference between $\mathcal{H}_a$ and $\mathcal{H}_b$ is not significant.

As noted in $\mathcal{B}$, $\chi'$ does not affect the space-time trajectory of a neutron. However, $\chi'$ may affect the spin vector, similar to the effect of a magnetic field, $\mathbf{B}$. If we express the direction of the spin vector as $(\theta, \phi)$ and apply the magnetic field $\mathbf{B}$ along the $z$-axis, $\phi$ precesses. However, the acceleration vector $\mathbf{a}$ possesses a higher symmetry than $\mathbf{B}$, making the situation more complex.

The symmetry point group of $\mathbf{B}$ is $SO(2)$, since the physics are unchanged by rotations about the $z$-axis. The acceleration $\mathbf{a}$ exhibits the same properties. However, the magnetic field is an axial vector, changing sign under reflection. Acceleration is a vector, so its symmetry point group is $O(2)$.

We will consider the action of a known acceleration $\mathbf{a}$, suitably parameterized. The operation of the acceleration on the spin vector $(\theta, \phi)$ must commute with the elements of $O(2)$. This means that $\phi$ must remain constant, since no rotation about the $z$-axis commutes with the reflections in $O(2)$. So the only rotations allowed are in the polar angle $\theta$. These rotations may not take $\theta$ to 0 or $\pi$, since $\phi$ is indeterminate at these poles, so the transformation would not be invertible. If $\theta$ is the only coordinate changing, it can not oscillate, since it is restricted to the interval $(0, \pi)$, and the transformation induces a rotation of definite direction at each point of this interval.

Changes in $\theta$ must rotate the spin vector either toward or away from the $z$-axis, the direction of acceleration. Although the rate of this rotation may vary, it must satisfy

$$\dot{\theta}|_{\theta=\theta_0} = \dot{\theta}|_{\theta=\pi-\theta_0} \quad (13)$$

in order to be consistent with the action of the inverse transformation. The inverse, which would correspond to acceleration along the $-z$-axis, inverts $\theta$. Symmetry dictates that if, at the angle $\theta_0$, the regular rotation was toward the $z$-axis, the inverse rotation at the angle $\pi - \theta_0$ must be toward the $-z$-axis at the same rate. Since the inverse transformation just reverses the rotation direction, the regular rotation at $\pi - \theta_0$ is the same as it is at $\theta_0$.

Although we have not demonstrated this fact, it seems likely that the range of rotation of $\theta$ is the entire range $(0, \pi)$, with the sign of $\dot{\theta}$ the same everywhere. This would result in the spin vector continuously rotating towards or away from the direction of the acceleration, with the angular frequency symmetric about $\theta = \frac{\pi}{2}$ and gradually decreasing as $\theta$ approaches the pole.

VI. CONCLUSIONS

The existence of $\mu_a$ does not ensure the validity of the equivalence principle. $\chi'$ itself gives us no indication of the magnitude of the “anomalous acceleration moment.” However, $\mu_a$ does provide a convenient method for restoring the equivalence principle, if $\mu_a = -3$. This specific prediction gives us another avenue for analysis of both $\mu_a$ and the equivalence principle. The particular value of $\mu_a$ is a property of the quark operators under acceleration. By examining the problem from the QCD standpoint, it may be possible to determine $\mu_a$ directly, either confirming or rejecting the equivalence principle.

If $\mu_a$ exists for the neutron, it almost certainly exists for the other neutral baryons. Whether it has an
analogue for other neutral particles is not as clear. For higher-spin mesons, such as the $\rho(770)^0$, with spin 1, there may well be other “anomalous acceleration moments.” To see how such moments would enter the Hamiltonian, it is necessary to examine the acceleration generators for these particles.

ACKNOWLEDGMENTS

The author wishes to thank Martin S. Altschul for his extensive and useful comments.

[1] R. Colella, A. W. Overhauser, S. A. Warner, Phys. Rev. Lett. 34 1472 (1975).
[2] U. Bonse, T. Wroblewski, Phys. Rev. Lett. 51 1401 (1983).
[3] K. Varjú, L. H. Ryder, Phys. Lett. A 250 263 (1998).
[4] E. Fischbach, B. S. Freeman, W.-K. Cheng, Phys. Rev. D 23 2157 (1981).
[5] M. S. Altschul, Found. Phys. 8 69 (1978).
[6] F. W. Hehl, W. T. Ni, Phys. Rev. D 42 2046 (1990).
[7] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78 29 (1950).
[8] J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1999), p. 748.
[9] R. L. Jaffe and A. Manohar, Nuc. Phys. B 337 509 (1990).
[10] W. B. Rolnick, The Fundamental Particles and Their Interactions (Addison-Wesley, Reading, Mass., 1994) p. 148.