Representations of Clifford Algebras with Hyperbolic Numbers

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Abstract. The representations of Clifford algebras and their involutions and anti-involutions are fully investigated since decades. However, these representations do sometimes not comply with usual conventions within physics. A few simple examples are presented, which point out that the hyperbolic numbers can close this gap.

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1. Introduction

The Clifford algebra approach to physics has become more and more popular over the last decades. The main idea is to replace the basis vectors of a real linear space by matrices, or even to rely purely on the algebraic properties of the basis matrices in a form which is independent of an explicit representation. The application of Clifford algebras to physics has been strongly promoted by Hestenes [1, 2, 3, 4] and has been investigated beside others also by Gull, Doran, and Lasenby [5, 6], and Rodrigues et al. [7].

The explicit matrix representations of Clifford algebras are well investigated. For a complete summary it is referred to the textbooks of Porteous [8, 9]. Though not mandatory needed, these representations give additional interesting insights. All representations of Clifford algebras can be constructed based on the real numbers, complex numbers, quaternions, and the double fields over these number systems. Equivalently, and more familiar to physicists, the real numbers, complex numbers, hyperbolic numbers, and the elements of the Pauli algebra can be chosen as basic building blocks (see also Keller [10]).

In the same way as complex numbers can parametrize a circle in two dimensions, hyperbolic numbers and the double field parametrize a hyperbola in a plane [11], where the two number systems refer to the two possible mathematical representations of a hyperbola. The hyperbolic numbers are also known as split-complex, paracomplex or double numbers [12]. It is clear that the hyperbolic numbers offer an elegant way to parametrize the hyperbolic spacetime metric of relativistic physics. But even more important is the fact that they give the possibility to consider the general and special linear groups as unitary groups [9, 13, 14, 15]. With this property the hyperbolic numbers are able to relate concepts from different areas in physics, namely general relativity and quantum physics. Furthermore, it has been shown by Hucks [16] that the hyperbolic complex spinor combines the so-called dotted and undotted spinor representations of relativistic physics in a single expression, which opens the way to a new representation of the Dirac theory.

Recently, the hyperbolic numbers have been used by da Rocha and Vaz [17] in order to study chirality in the context of extended Grassmann and Clifford algebras. Khrennikov et al. disproved the von Neumann uniqueness theorem in hyperbolic quantum mechanics [19]. Boccaletti et al. [18] investigated the twin-paradox with the help of hyperbolic numbers. More references on hyperbolic numbers can be found in [20].

Most applications of Clifford algebras to physics so far rely on real Clifford algebras. Complex Clifford algebras have been recently used in the investigation of the CPT-group by Varlamov [21, 22], and in the context of the Dirac theory by Sabadini et al. [23], Miralles et al. [24], Marchuk [25], and by Avramidi [26]. Polyvector Super-Poincaré algebras have been investigated by Alekseevsky et al. [27]. In the context of String theory complex Clifford algebras have been used by Asakawa et al. [28]. They complexify with the hyperbolic unit, as well as Moffat in his Noncommutative Quantum Gravity [29].

It is one advantage of the explicit representations of Clifford algebras, based on the building blocks mentioned above, that the complexification of a real algebra can be performed in a clear and straightforward way. One example for such a complexification is based on the $\mathbb{R}_{3,0}$ paravector algebra, which has been considered by Sobczyk and Baylis [30, 31] for the representation of relativistic vectors. Baylis has shown that the theory of electrodynamics can be fully expressed in terms of this algebra. In his textbook [32] a wide range of explicit physical applications of the $\mathbb{R}_{3,0}$ algebra can be found.

The algebra $\mathbb{R}_{3,0}$ can be represented with the help of the hyperbolic unit. In this case the algebra will be denoted as hyperbolic algebra, which has been used recently also by Wei and Xuegang [33] for the investigation of the Lorentz transformation and related special relativistic physics. The algebra can be complexified further to provide the complex Clifford algebra $\mathbb{C}_{3,0}$. It has been proposed in [34] to use this algebra to represent physical operators, like the mass operator, in their most general form. This idea is reconsidered in this work, which leads to an investigation of the Clifford algebra $\mathbb{R}_{0,5}$ and its complexification.