Comparison principles and input-to-state stability for stochastic impulsive systems with delays

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ABSTRACT
In this paper, the input-to-state stability (ISS), the $p$th moment exponential ISS and the stochastic input-to-state stability issues are investigated for the stochastic impulsive systems with time delays via the comparison principle method. Firstly, several general comparison principles in vector-version are proposed guaranteeing the existence, uniqueness and magnitude for solutions of the addressed system. Then based on these established comparison principles, the ISS-related properties are investigated for the stochastic impulsive delayed model. Finally, one example is given to illustrate effectiveness of the obtained results.

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1. Introduction
In recent years, the input-to-state stability (ISS) and extensions of the ISS on different systems have attracted widespread attentions in the literature (Damak, 2021; Khalil, 2002; Sontag, 2004; Sontag & Wang, 1996) due to their extensive usage in characterizing the effects of external inputs (such as sensor noise (Liu et al., 2012), actuator disturbances (Liao et al., 2014), parameter perturbations, or measurement errors) on the considered systems. The notion of ISS, firstly proposed in Sontag (1989) for continuous-time nonlinear systems, is formed to investigate how the external disturbance affects the system stability. Roughly speaking, the property of ISS means that no matter what the magnitude of the initial state is, the system state will ultimately approach to a neighbourhood of the origin whose magnitude is proportional to the size of the input. Recently, various extensions of the ISS, such as integral ISS (iISS), stochastic ISS (SIS), and finite-time ISS, have been proposed successfully to different kinds of dynamical systems, for instance, impulsive systems (Liu & Hill, 2010; Wu et al., 2016; Yang et al., 2013), switched systems (Ren & Xiong, 2016; Wakaiki & Yamamoto, 2017), discrete-time dynamical networks (Jiang et al., 2004; Teel, 1998; Zhao et al., 2015), and nonlinear systems with delays (Li et al., 2016; Liu, 2017; Wang et al., 2018).

Impulsive systems, as an important portion of the hybrid systems, combine the continuous dynamical systems with the discrete-time instantaneous state jumps or resets (also be referred to as impulses). Impulsive control models have attracted considerable concern in both the theoretical and the practical fields during the past few years, since they occur naturally from a wide range of areas such as biomechanical systems (Karafyllis et al., 2008), economic systems (An et al., 2019), and the like. Meanwhile, they can be extended as the impulsive time delay plants (Liu et al., 2011; Ren & Xiong, 2019) if there exists time delay in the continuous-time or discrete-time dynamical impulsive systems. On the other hand, great efforts have been devoted to the stability analysis for stochastic systems (Hu et al., 2020; Liu, 2008; Mao, 2007; Song et al., 2020) because lots of natural processes and real-world systems are disturbed unavoidably by stochastic perturbation. For example, the networked systems are ubiquitously subject to random noises such as data dropouts or congestions (Antunes et al., 2013). By resorting to multiple-Lyapunov functions, the stability has been analysed for the deterministic and stochastic models (Chatterjee & Liberzon, 2006). With the method of matrix measure, the problem of reliable dissipative control is discussed in Zhang et al. (2008) for a class of stochastic systems. Based on the average dwell-time laws, global asymptotic stability in probability (GASiP) and SIS are investigated in Zhao, Feng et al. (2012) for the switched stochastic nonlinear systems. It should be noted that it is important and necessary to further develop the ISS-related results for the stochastic impulsive time delay systems (Niu et al., 2012; Zhao, Kang et al., 2012). However, up to now, only scattered efforts have been carried out for a comprehensive analysis on the ISS-related
properties (including ISS, IS$E_B$, $p$th moment exponential ISS, and SISS) for the addressed stochastic impulsive time delay plants (Wu et al., 2016), which motivates the present study in this paper.

Lyapunov method, one of the main techniques contributed to analysing the stability of dynamical systems (Hu et al., 2021; Kundu & Anghel, 2017; Li et al., 2020; Ren & Xiong, 2017; Xu et al., 2016), has also been employed extensively to investigate the ISS property for various systems (Chen & Zheng, 2009; Ren & Xiong, 2018). As an example, SISS has been tackled in Ren and Xiong (2019) for the stochastic impulsive systems by resorting to the Lyapunov function. The Lyapunov conditions are deduced to be input-to-state stable in mean, the comparison principles in vector-version are established for the stochastic time delay systems by employing the Lyapunov method. Firstly, the comparison principle method, which comprised the other comparison system. To the best of the authors’ knowledge, there are only few (if not none) relevant literatures concerning about the ISS-related results on impulsive edge, there are only few (if not none) relevant literatures
deduced for the stochastic impulsive delayed system, which generalizes the scalar version of the classic comparison principle in Khalil (2002) and facilitates the examination of the ISS-related properties.

The existence and uniqueness of the solution are deduced for the stochastic impulsive delayed system with external inputs by comparing it with a lower-dimensional deterministic impulsive one which is assumed to have a global solution, where the linear growth constraints in Wu et al. (2016) are no longer necessary.

As for the stochastic impulsive delay-free (or time delay) systems with external inputs, it is the first time that the ISS-related properties are derived in terms of the vector-version comparison principles instead of the Lyapunov functions.

The rest of this paper is organized as follows. In Section 2, the problem to be addressed is formulated and some preliminaries are presented. In Section 3, we establish the general comparison principles in vector-version for the stochastic impulsive time delay system with external inputs, which guarantee the existence, uniqueness, and magnitude of the corresponding solutions. In Section 4, the ISS-related criteria are established for the considered stochastic impulsive model. In Section 5, one numerical example is provided to demonstrate the effectiveness of the proposed criteria. The conclusion is given in Section 6.

Notations: Standard notation is used in this paper. Let $\mathbb{R}_{\geq t_0} := [t_0, +\infty)$, $\mathbb{R}_{> t_0} := (t_0, +\infty)$, and $\mathbb{N}_+$ be the set of positive integers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the $n$-dimensional real space and the $n \times m$-dimensional real matrix space. For vector $x \in \mathbb{R}^n$, $\|x\|$ represents its norm with $\|x\| := \sum_{i=1}^{n} |x_i|$. $\mathcal{PC}([a,b];\mathbb{R}^n)$ stands for the class of piecewise continuous functions from $[a,b]$ to $\mathbb{R}^n$ and having finite right-hand continuous jumps on $[a,b]$. For any $\xi(\cdot) \in \mathcal{PC}([t_0 - \tau, t_0];\mathbb{R}^n)$, define $\|\xi\|_{\tau} = \sup_{t_0 - \tau \leq t \leq t_0} \{\|\xi(t)\|\}$. For any $u(\cdot) \in \mathcal{PC}([t_0, \infty);\mathbb{R}^n)$, we define $\|u\|_{[t_0, t]} = \sup_{t_0 \leq s \leq t} \{\|u(s)\|\}$ and $\|u\|_{\infty} = \sup_{s \geq t_0} \{\|u(s)\|\}$. Let $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for any $i \in \{1, 2, \ldots, n\}$. For any $g_1(\cdot)$, $g_2(\cdot) \in \mathcal{PC}([t_0, \infty);\mathbb{R}^n)$, we call $g_1(\cdot) \leq g_2(\cdot)$ if $g_1(s) \leq g_2(s)$ for all $s \geq t_0$. For a given matrix $A$, $A^T$ denotes its transpose. In addition, $E(\cdot)$ denotes the
mathematical expectation, and $P\{\cdot\}$ represents the probability measure.

2. Problem formulation

In this paper, we consider the following stochastic impulsive system with time delays:

$$
\begin{aligned}
\text{dx}(t) &= f(t, x_t, u(t)) \, dt + g(t, x_t, u(t)) \, dB(t), \\
t \in & \mathbb{R}_\geq 0 \setminus \mathcal{T}, \\
x(t) = (t, x(t), u(t)), & \ t \in \mathcal{T}, \\
x(t) = \xi(t), & \ t \in [t_0 - \tau, t_0]
\end{aligned}
$$

(1)

where $u(t) \in PC([t_0, \infty); \mathbb{R}^n_u)$ denotes the external input, $x(t) \in \mathbb{R}^n_x$ is the system state, $x_t := x(t - \tau(t))$, in which $\tau(t)$ is a continuous function representing the time delay which satisfies $0 \leq \tau(t) \leq \tau$ with $\tau$ being a constant, and $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^T$ stands for an $n_w$-dimensional $\mathcal{F}_t$-adapted Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, P)$ in which $\mathcal{F}$ is a $\sigma$-field and $\{\mathcal{F}_t\}_{t \geq t_0}$ is a filtration. The given impulsive time sequence $\tau := \{t_k : k \in \mathbb{N}_+\}$ is strictly increasing such that $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$. Denote $x(t^-) := \lim_{s \uparrow t} x(t+s)$. The initial function $\xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^n_x)$ is a bounded $\mathcal{F}_{t_0}$-adapted random process with $E[\|\xi\|^2_p] < \infty$. The functions $f(\cdot) : \mathbb{R}_\geq 0 \times \mathbb{R}^n_x \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g(\cdot) : \mathbb{R}_\geq 0 \times \mathbb{R}^n_x \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_x \times n_w}$, and $l(\cdot) : \mathbb{R}^n_x \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous Borel measurable functions and $f(\cdot), g(\cdot)$ satisfy the Lipschitz condition. Moreover, suppose that $f(t, 0, 0) \equiv 0$, $g(t, 0, 0) \equiv 0$ for all $t \in \mathbb{R}_\geq 0$, and $l(0, 0) \equiv 0$, which infer that system (1) has a trivial solution $x(t) \equiv 0$ if the initial function $\xi(\cdot) \equiv 0$.

Remark 2.1: If there exists a positive integer $k^*$ such that $t_{k^*} \in \mathcal{T}, t_{k^*} < +\infty$ and $t_{k^*+1} = +\infty$, then system (1) reduces to the normal stochastic delayed system, of which the ISS and relevant issues have been extensively studied in the previous literature (Mao, 2007). Based on this fact, in this paper, the following assumption is proposed.

Assumption 2.1: Assume that the impulsive instances $\{t_k, k \in \mathbb{N}_+\}$ satisfy:

$$
\begin{aligned}
\Delta_{\sup} &:= \sup_{k \in \mathbb{N}_+} \{t_{k+1} - t_k\} < \infty, \\
\Delta_{\inf} &:= \inf_{k \in \mathbb{N}_+} \{t_{k+1} - t_k\} > 0.
\end{aligned}
$$

Definition 2.1 (Khalil, 2002): A function $\gamma(\cdot) : \mathbb{R}_\geq 0 \rightarrow \mathbb{R}_\geq 0$ is said to belong to class-$\mathcal{K}$ (i.e. $\gamma(\cdot) \in \mathcal{K}$) if it is a continuous and strictly increasing function with $\gamma(0) = 0$. It is said to belong to class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A vector function $l(t) = (l_1(t), l_2(t), \ldots, l_m(t))^T : \mathbb{R}_\geq 0 \rightarrow \mathbb{R}^m_\geq 0$ is of class-$\mathcal{K}_m^\infty$ if $l(\cdot) \in \mathcal{C}(\mathbb{R}_\geq 0; \mathbb{R}^m_\geq 0)$ and $l_i(\cdot) \in \mathcal{K}_\infty$ for all $1 \leq i \leq m$, where $\mathcal{C}(\mathbb{R}_\geq 0; \mathbb{R}^m_\geq 0)$ denotes the set of all continuous functions from $\mathbb{R}_\geq 0$ to $\mathbb{R}^m_\geq 0$ with $\mathbb{R}^m_\geq 0$ being the set

$$
\mathbb{R}_\geq 0 \times \mathbb{R}_\geq 0 \times \ldots \times \mathbb{R}_\geq 0.
$$

A continuous function $\beta(\cdot, \cdot) : \mathbb{R}_\geq 0 \times \mathbb{R}_\geq 0 \rightarrow \mathbb{R}_\geq 0$ is said to belong to class-$\mathcal{KL}$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to 0 for each $s \geq 0$. A continuous function $h(\cdot, \cdot) : \mathbb{R}_\geq 0 \times \mathbb{R}_\geq 0 \rightarrow \mathbb{R}_\geq 0$ is of class-$\mathcal{KB}$ if $h(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geq 0$ and $h(s, \cdot)$ is uniformly bounded for all $s \geq 0$.

Definition 2.2: Given an impulsive time sequence $\mathcal{T}$, system (1) is said to be $p$th $(p > 0)$ moment exponentially input-to-state stable in mean if there exist functions $\beta(\cdot, \cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}_\infty$ such that, for all $\xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^n_x), u(\cdot) \in PC([t_0, +\infty); \mathbb{R}^n_u)$ and $t \in \mathbb{R}_\geq 0$,

$$
E[\|x(t)\|^p] \leq \beta(E[\|\xi\|^p], t - t_0) + \gamma(\|u\|_{[t_0, t]}). 
$$

(2)

In particular, when $p = 1$, system (1) is said to be input-to-state stable in mean.

Definition 2.3: Given an impulsive time sequence $\mathcal{T}$, system (1) is said to be input-to-state $\mathcal{KB}$-stable in mean if there exist functions $\beta(\cdot, \cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}_\infty$ such that, for all $\xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^n_x), u(\cdot) \in PC([t_0, +\infty); \mathbb{R}^n_u)$ and $t \in \mathbb{R}_\geq 0$,

$$
E[\|x(t)\|] \leq \beta(E[\|\xi\|], t - t_0) + \gamma(\|u\|_{[t_0, t]}). 
$$

(3)

Remark 2.2: Constraint (2) with $p = 1$ (or (3) respectively) in Definition 2.2 (respectively, Definition 2.3) is equivalent to the following property: there exist functions $\beta(\cdot, \cdot) \in \mathcal{KL}$ (respectively, $\beta(\cdot, \cdot) \in \mathcal{KB}$) and $\gamma(\cdot) \in \mathcal{K}_\infty$ such that

$$
E[\|x(t)\|] \leq \max(\beta(E[\|\xi\|], t - t_0), \gamma(\|u\|_{[t_0, t]})),
$$

which is frequently used in the literature (Jiang et al., 2004; Teel, 1998) instead of (2) with $p = 1$ (or (3) respectively).

Definition 2.4: Given an impulsive time sequence $\mathcal{T}$, system (1) is said to be stochastic input-to-state stable, if for any given $\varepsilon \in (0, 1)$, there exist functions $\beta(\cdot, \cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}_\infty$ such that, for all $\xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^n_x)$, $u(\cdot) \in PC([t_0, +\infty); \mathbb{R}^n_u)$ and $t \in \mathbb{R}_\geq 0$,

$$
P\{\|x(t)\| \leq \beta(E[\|\xi\|], t - t_0) + \gamma(\|u\|_{[t_0, t]})\} \geq 1 - \varepsilon. 
$$

(4)
Remark 2.3: The above definition of ISS in mean is parallel to the one given in Wu et al. (2016) with minor modifications. When \( u(\cdot) \equiv 0 \), inequality (2) with \( p = 1 \) reduces to \( E\|x(t)\| \leq \beta(t)E\|x\|_{\tau}, t \leq t_0 \), which indicates the global asymptotic stability (GAS) in mean. As a result, the conclusion that system (1) is input-to-state stable in mean implies that the corresponding unforced system is globally asymptotically stable in mean. Furthermore, when \( u(\cdot) \equiv 0 \), inequality (4) reduces to \( P(\|x(t)\| \leq \beta(\|x\|_{\tau}, t - t_0)) \geq 1 - \epsilon \), which infers that the corresponding unforced system of (1) is globally asymptotically stable in probability. Similar definition has been given in Zhao et al. (2015) for the discrete-time stochastic nonlinear systems.

The main aim of this paper is to propose several new comparison principles, based on which the existence and uniqueness of solutions are discussed for the stochastic impulsive delayed system (1) with time delays. In addition, the ISS-related issues concerning model (1) are also analysed dexterously. To proceed, the following definitions and lemma are further introduced.

**Definition 2.5:** For a function \( W : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \), the upper right Dini derivative of \( W(\cdot) \) at time \( t \in \mathbb{R}_{\geq 0} \) is defined as

\[
D^+ W(t) = \lim_{s \to t^+} \sup \frac{W(t + s) - W(t)}{s}.
\]

**Definition 2.6 (Mao, 2007):** \( C^{1,2} \) represents the class of continuous functions \( V(t, x) : [t_0 - \tau, +\infty) \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_Z} \) \((n_z \leq n_x)\), where \( \partial V(t, x)/\partial t, \partial V(t, x)/\partial x, \partial^2 V(t, x)/\partial x^2 \) are continuous, and \( V(t, x) \) is concave in \( x \). Given function \( V(t, x) = (V_1(t, x), V_2(t, x), \ldots, V_{n_z}(t, x))^T \) in \( C^{1,2} \), for any \( j \in \{1, 2, \ldots, n_z\} \) and \( t \in (t_0, +\infty) \setminus T \), the infinitesimal operator of \( V_j(t, x(t)) \) for system (1) is defined as

\[
\mathcal{L}V_j(t, x(t)) = \frac{\partial V_j(t, x(t))}{\partial t} + \frac{\partial V_j(t, x(t))}{\partial x(t)} f(t, x(t), u(t)) + \frac{1}{2} \text{tr} \left[ g^T(t, x(t), u(t)) \frac{\partial^2 V_j(t, x(t))}{\partial x^2(t)} g(t, x(t), u(t)) \right].
\]

**Lemma 2.1:** Let constants \( p > 0 \) and \( a_i \geq 0 \) \((i = 1, 2, \ldots, m)\), where \( m \) is a positive integer, then

\[
\left( \sum_{i=1}^{m} a_i \right)^p \leq \max(2^{(m-1)(p-1)}, 1) \cdot \sum_{i=1}^{m} a_i^p.
\]

**Proof:** It is a deduction of the Holder inequality so that the details are omitted for space consideration.

**3. General comparison principle**

In this section, two new comparison principles are established for the stochastic impulsive delayed system (1) with external inputs.

For system (1), the following simple impulsive delayed model is considered as a comparison system:

\[
\begin{align*}
\begin{cases}
    dz(t) = h_1(t, z(t), v(t)) \, dt, & t \in \mathbb{R}_{\geq 0} \setminus T \\
    z(t) = \psi(z(t^-), v(t^-)), & t \in T \\
    z(t) = \zeta_1(t), & t \in [t_0 - \tau, t_0]
\end{cases}
\end{align*}
\]

where \( z(t) \in \mathbb{R}_{\geq 0}^{n_z} \) with \( n_z \leq n_x \) is the system state, the initial function \( \zeta_1(\cdot) \in \mathcal{PC}([t_0 - \tau, t_0]; \mathbb{R}_{\geq 0}^{n_z}) \) is bounded, \( v(t) \in \mathcal{PC}([t_0, \infty); \mathbb{R}_{\geq 0}^{n_v}) \) denotes the external input. Function \( h_1(t, z_t, v(t)) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}^{n_z} \times \mathbb{R}_{\geq 0}^{n_v} \rightarrow \mathbb{R}_{\geq 0}^{n_z} \) is a continuous Borel measurable function, which is concave and nondecreasing with respect to \( z_t \). Function \( \psi(z(t), v(t)) : \mathbb{R}_{\geq 0}^{n_z} \times \mathbb{R}_{\geq 0}^{n_v} \rightarrow \mathbb{R}_{\geq 0}^{n_z} \) satisfying \( \psi(0, 0) \equiv 0 \) is continuous, nondecreasing and concave in \( z(t) \).

In the practical engineering, the target system (1) might be large-scaled. To effectively reduce the dimension of the comparison system (5), the following one-dimensional system is further considered:

\[
\begin{align*}
\begin{cases}
    dy(t) = h_2(t, y(t), w(t)) \, dt, & t \in \mathbb{R}_{\geq 0} \setminus T \\
    y(t) = \phi(y(t^-), w(t^-)), & t \in T \\
    y(t) = \zeta_2(t), & t \in [t_0 - \tau, t_0]
\end{cases}
\end{align*}
\]

where \( y(t) \in \mathbb{R}_{\geq 0} \) is the system state, \( w(t) \in \mathbb{R}_{\geq 0} \) denotes the external input, \( h_2(\cdot, \cdot, \cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is a continuous Borel measurable function and satisfies the Lipschitz condition. Function \( \phi(y(t), w(t)) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) satisfying \( \phi(0, 0) = 0 \) is continuous, nondecreasing and concave in \( y(t) \). The initial function \( \zeta_2(\cdot) \in \mathcal{PC}([t_0 - \tau, t_0]; \mathbb{R}_{\geq 0}) \) is bounded. It follows from Luo and Shen (2006) that for any \( \zeta_2(t) \), system (6) has an unique solution. We further suppose that \( h_1(t, 0, 0) \equiv 0, h_2(t, 0, 0) \equiv 0 \) for all \( t \in \mathbb{R}_{\geq 0} \), which mean that systems (5) and (6) admit the trivial solution. Let the solutions of systems (5) and (6) be denoted by \( z(t) := z(t; t_0, \zeta_1, v) \) and \( y(t) := y(t; t_0, \zeta_2, w) \), respectively.

Given \( l(\cdot) = (l_1(\cdot), l_2(\cdot), \ldots, l_{n_z}(\cdot))^T \in \mathcal{K}_{\infty} \) and \( \zeta_2(\cdot) \in \mathcal{PC}([t_0 - \tau, t_0]; \mathbb{R}_{\geq 0}) \), let \( (l_1 \circ \zeta_2)(\cdot) \in \mathcal{PC}([t_0 - \tau, t_0]; \mathbb{R}_{\geq 0}) \) be defined as \( (l_1 \circ \zeta_2)(\cdot) := (l_1(\zeta_2(\cdot)), l_2(\zeta_2(\cdot)), \ldots, l_{n_z}(\zeta_2(\cdot)))^T \). For any given \( x_t \) and \( V(\cdot, \cdot) \in \mathcal{PC}_{V}^{1,2} \), let \( V_t \) be defined as \( V_t := V(t - \tau(t), x_t) \).

**Theorem 3.1 (Comparison principle for magnitude of solution):** Consider the stochastic impulsive delayed system (1) and the comparison systems (5) and (6). Assume that there exist functions \( V(\cdot, \cdot) \in C^{1,2} \) and \( l(\cdot) \in \mathcal{K}_{\infty} \) with
\(\frac{dl(y(t))}{dt} \geq l(dy(t)/dt)\) for \(t \in \mathbb{R}_{\geq 0} \setminus \mathcal{T}\), then

\[
EV(t, x(t)) \leq z(t) \leq l(y(t)), \quad t \geq t_0
\]  

(7)

holds provided that \(EV(t, \xi(t)) \leq \zeta(t) \leq (l \circ \zeta_2)(t)\) for all \(t \in [t_0 - \tau, t_0]\) and the following conditions are satisfied:

(i) let \( LC(t, x(t)) := (LC_1(t, x(t)), \ldots, LC_n(t, x(t)))^T\), functions \(h_1(\cdot, \cdot, \cdot, \cdot)\) and \(\psi(\cdot, \cdot, \cdot)\) in (5) are subject to

\[
\begin{align*}
LC(t, x(t)) &< h_1(t, V_t, V(t)), \quad t \in \mathbb{R}_{\geq 0} \setminus T \\
V(t, x(t)) &\leq \psi(V(t, x(t)), V(t)), \quad t \in T
\end{align*}
\]

(8)

(ii) for given constant \(r^* > 0\) (or \(r^* = +\infty\)), the constraints below are valid for all \(a \in [0, r^*] : \)

\[
\begin{align*}
&h_1(t, l \circ y, V(t)) < l(h_2(t, y, w(t))), \quad t \in \mathbb{R}_{\geq 0} \setminus T \\
&\psi(l(a), V(t)) \leq l(\psi(a), w(t)), \quad t \in T
\end{align*}
\]

(9)

(iii) whenever \(\zeta_2(\cdot) \in \mathbb{P}_{CL}([t_0 - \tau, t_0]; [0, r^*])\), it is known that

\[
y(t) \in [0, r^*], \quad t \in \mathbb{R}_{\geq 0}.
\]

(10)

**Proof:** Let \(z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T\), and we have known that \(EV(t, x(t)) \leq z(t) \leq l(y(t))\) holds for \(t \in [t_0 - \tau, t_0]\). Firstly, it will be proved via reductio ad absurdum that

\[
EV(t, x(t)) \leq z(t) \leq l(y(t)), \quad t \in [t_0, t_1].
\]

(11)

Suppose that \(EV(t, x(t)) \leq z(t)\) holds for all \(t \in [t_0, t_1]\). If not, there should exist some \(t \in (t_0, t_1)\) and certain subindex \(j \in \mathbb{N}_+\) such that component \(EV_j(t, x(t))\) satisfies \(EV_j(t, x(t)) > z_j(t)\). Set \(t^* := \inf\{t \in (t_0, t_1) : EV_j(t, x(t)) > z_j(t)\}\), and let \(j^*\) be the subindex corresponding to \(t^*\). Since \(EV(t, x(t))\) and \(z(t)\) are continuous on \([t_0, t_1]\), it implies that \(EV_{j^*}(t^*, x(t^*)) = z_{j^*}(t^*)\) and \(EV_{j^*}(t, x(t)) > z_{j^*}(t)\) for \(t \in (t^*, t^* + \varepsilon)\), where \(\varepsilon > 0\) is small enough and satisfies \(t^* + \varepsilon < t_1\). Therefore, for all \(t \in (t^*, t^* + \varepsilon)\), we have

\[
\frac{EV_j(t^*, x(t^*)) - EV_j(t^*, x(t^* + \varepsilon))}{t - t^*} > \frac{z_j(t^*) - z_{j^*}(t^*)}{t - t^*}
\]

i.e.,

\[
D^+ EV_j(t^*, x(t^*)) \geq \frac{d}{dt} z_j(t)|_{t=t^*}.
\]

(12)

On the other hand, noticing that \(h_1(t, z_j(t), V(t))\) is concave and nondecreasing with respect to \(z_j(t)\), it follows from the first inequality of (8) that

\[
\begin{align*}
D^+ EV(t^*, x(t^*)) &\leq E(LV(t^*, x(t^*))) \\
&< h_1(t^*, EV_{j^*}, V(t^*)) \\
&\leq h_1(t^*, z_{j^*}, V(t^*)) = \frac{d}{dt} z(t)|_{t=t^*}
\end{align*}
\]

(13)

which contradicts inequality (12). Therefore, it concludes that \(EV(t, x(t)) \leq z(t)\) holds for all \(t \in [t_0, t_1]\). Similarly, by using (9), we can conclude that \(z(t) \leq l(y(t))\) holds for all \(t \in [t_0, t_1]\). Thus, inequality (11) is proved to be valid.

Next, we prove (7) by using the method of mathematical induction. Suppose that, for \(r = 0, 1, \ldots, k - 1\) and \(k \in \mathbb{N}_+\),

\[
EV(t, x(t)) \leq z(t) \leq l(y(t)), \quad t \in [t_r, t_{r+1})
\]

then for a scalar \(\rho > 0\) which is small enough, we have

\[
EV(t, x(t)) \leq z(t) \leq l(y(t)), \quad t \in [t_k - \rho, t_k).
\]

(14)

Further recalling that \(\psi(z(t), V(t))\) is nondecreasing and concave in \(z(t)\), it follows from the second inequality of (8) that

\[
\begin{align*}
EV(t_k, x(t_k)) &\leq E\psi(V(t_k^-, x(t_k^-)), V(t_k^-)) \\
&\leq \psi(E(V(t_k^-, x(t_k^-)), V(t_k^-))) \\
&\leq \psi(z(t_k^-), V(t_k^-)) = z(t_k).
\end{align*}
\]

Similarly, by using the second inequality of (9) as well as condition (10), we could derive \(z(t_k) \leq l(y(t_k))\). Thus one gets

\[
EV(t, x(t)) \leq z(t) \leq l(y(t)), \quad t \in [t_k - \rho, t_k).
\]

(15)

If \(EV(t, x(t)) \leq z(t)\) is not valid for all \(t \in (t_k, t_{k+1})\), then there exists \(t^*\) such that \(t^* = \inf\{t \in (t_k, t_{k+1}) : \exists j \in \{1, 2, \ldots, n_2\} such that EV_j(t, x(t)) > z_j(t)\}\). By using the similar proof process of (11), we can get \(EV(t, x(t)) \leq z(t)\) for all \(t \in [t_k, t_{k+1})\). In the same way, \(z(t) \leq l(y(t))\) holds for all \(t \in [t_k, t_{k+1})\). Based on the mathematical induction, it follows that (7) holds for all \(t \geq t_0\).

**Remark 3.1:** Theorem 3.1 provides a new comparison principle in vector-version for solutions of the stochastic impulsive delayed system (1) with external inputs, which generalizes the scalar version of the classical comparison principle for nonlinear systems in Khalil (2002) that has been widely used in the literature (Chatterjee & Liberzon, 2006; Mao, 2007; Niu et al., 2012; Zhang et al., 2008; Zhao, Kang et al., 2012). Moreover, Theorem 3.1 contains the scalar results of Lemma 1 in Wu et al. (2016), where the conditions are proposed under linear assumption, which is relaxed in Theorem 3.1.

**Theorem 3.2 (Comparison principle for the existence and uniqueness of the solution):** Consider the stochastic impulsive delayed system (1) and the comparison system (5). Assume that there exist functions \(V(\cdot, \cdot) \in C^{1,2}\)
and \( \chi_i(\cdot) \in K_\infty \) \((1 \leq i \leq n_z)\) such that condition (i) of Theorem 3.1 and the following inequalities hold:

\[
\chi_i(\|x(t)\|) \leq V_i(t, x(t)), \quad t \geq t_0 - \tau \tag{16}
\]

then the existence of the global solution \( z(t) \) of system (5) implies that system (1) has an unique global solution \( x(t) \) for any given initial function \( \xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^{n_x}) \).

**Proof:** Due to the fact that functions \( f(t, x_t, u(t)) \) and \( g(t, x_t, u(t)) \) both satisfy the Lipschitz condition, it follows from Mao (2007) that the continuous dynamic system corresponding to model (1) has an unique solution on \((t_0 - \tau, \sigma_\infty)\) for any given \( \xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^{n_x}) \), where \( \sigma_\infty \leq +\infty \) and \((t_0 - \tau, \sigma_\infty)\) is the maximal existence interval of the solution. Then system (1) itself also has an unique solution \( x(t) \) on \((t_0 - \tau, \sigma_\infty)\) for the given initial condition \( \xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^{n_x}) \) by the approach of solution's continuation. Next, we only need to prove that \( \sigma_\infty = +\infty \).

For any \( k \in \mathbb{N}_+ \), let \( s_k = \{x(t) : \|x(t)\| \geq k\} \) and define the stopping time \( \rho(k) \) as

\[
\rho(k) = \inf\{t \in [t_0, \sigma_\infty) : x(t) \in s_k\} \tag{17}
\]

Obviously, the sequence \( \{\rho(k) : k \in \mathbb{N}_+\} \) is increasing with \( k \). Assume \( \sigma_\infty < +\infty \), then \( \lim_{k \to \infty} \rho(k) = \rho(\infty) \) exists, and \( \rho(k) \leq \rho(\infty) \leq \sigma_\infty \) for any \( k \in \mathbb{N}_+ \). It is known that there exists some integer \( k_0 \in \mathbb{N}_+ \) such that \( \sigma_\infty \in (t_{k_0-1}, t_{k_0}) \), and for any \( k \in \mathbb{N}_+ \), some \( i_k \) exists satisfying \( i_k \leq k_0 \) and \( \rho(k) \in (t_{i_k-1}, t_{i_k}) \). It follows from the first inequality of (8) and Itô’s formula that

\[
D^+EV(t, x(t)) < h_1(t, EV_t, v(t)), \quad t \in \mathbb{R}_{\geq t_0} \setminus T \tag{18}
\]

Let \( \tau = \min\{t, \rho(k)\} \) with \( t \in (t_{i_k-1}, t_{i_k}) \), based on (18) we have

\[
EV(\tilde{t}, x(\tilde{t})) - EV(t_{i_k-1}, x(t_{i_k-1})) \leq \int_{t_{i_k-1}}^{\tau} h_1(s, EV_s, v(s)) \, ds
\]

\[
\leq \int_{t_{i_k-1}}^{\tau} h_1(s, EV_s, v(s)) \, ds. \tag{19}
\]

For \( t \in (t_{i_k-1}, t_{i_k}) \), define \( m(t) = EV(t, x(t)) \) and \( m_t = EV_t \), then it is obvious that \( m(t) \) is continuous on \((t_{i_k-1}, t_{i_k})\).

For \( t \in (t_0, \sigma_\infty) \setminus (t_{i_k-1}, t_{i_k}) \), define \( m(t) = EV(t, x(t)) \) and \( m_t = EV_t \). Combine (8) with (19) and (19) follows that

\[
\begin{align*}
D^+m(t) &< h_1(t, m_t, v(t)), & t &\in (t_0, \sigma_\infty) \setminus (t_{i_k-1}, t_{i_k}) \setminus T \\
nm(t) &\leq \psi(m(t^-), v(t^-)), & t &\in \{t_k : k \leq k_0 - 1\} \setminus \mathbb{N}_+ \\
m(t) &= \zeta_1(t), & t &\in [t_0 - \tau, t_0].
\end{align*} \tag{20}
\]

Comparing system (5) with equation (20), we immediately get

\[
m(t) \leq z(t), \quad t \in (t_0, \sigma_\infty) \cup (t_{i_k-1}, t_i_k).
\]

Based on which we have that, for any \( t \in (t_{i_k-1}, t_{i_k}) \),

\[
EV(\tilde{t}, x(\tilde{t})) = m(t) \leq z(t). \tag{21}
\]

It then yields from inequalities (16) and (21) that, for any \( i = 1, 2, \ldots, n_z \),

\[
z_i(\tilde{t}) \geq EV_i(\tilde{t}, x(\tilde{t})) \geq E\chi_i(||x(\tilde{t})||).
\]

Let \( k \to \infty \) and \( t \to \min\{t, \sigma_\infty\} \) in the above expression, it achieves

\[
z(\rho(\infty)) \geq E\chi_i(||x(\rho(\infty))||)
\]

\[
= E\chi_i(||x(\sigma_\infty)||)
\]

\[
= E\chi_i(\infty) = +\infty.
\]

This is a contradiction with the existence condition of the global solution \( z(t) \) for system (5). Therefore we achieve \( \sigma_\infty = +\infty \), which means the system (1) has an unique global solution \( x(t) \) for any given initial function \( \xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^{n_x}) \).

**Remark 3.2:** Theorem 3.2 presents a new comparison principle in vector-version for the existence and uniqueness of solution for the stochastic impulsive delayed system (1) with external inputs. As is known to all, the linear growth assumption is needed in many literatures (Wu et al., 2016) to guarantee the existence and uniqueness of solution for the stochastic impulsive systems, or sometimes the existence/uniqueness of solutions are directly assumed without any proof (Ren & Xiong, 2019). By employing Theorem 3.2, the existence and uniqueness of solutions for model (1) can be deduced by comparing it with a lower-dimensional deterministic impulsive system (5) which is assumed to have a global solution, and the linear growth constraints on model (1) are no longer necessary.

### 4. ISS-related analysis

In this section, based on the comparison principles established in the previous section, criteria are to be derived for the stochastic impulsive system (1) concerning the properties of ISS in mean, SISS and the \( p \)th moment exponential ISS in mean.

**Theorem 4.1:** Assume that all conditions of Theorem 3.1 hold and there exist convex functions \( \alpha_1(\cdot) \in K_\infty \), concave functions \( \alpha_2(\cdot) \in K_\infty \) \((j = 1, 2, \ldots, n_z)\) and \( \varphi_1(\cdot), \varphi_2(\cdot) \in K_\infty \) such that

\[
\alpha_1(||x(t)||) \leq V_j(t, x(t)) \leq \alpha_2(||x(t)||), \quad t \geq t_0 - \tau. \tag{22}
\]
\[
\|w\|_{[t_0,t]} \leq \varphi_1(\|v\|_{[t_0,t]}) \leq \varphi_2(\|u\|_{[t_0,t]}), \quad t \geq t_0. \tag{23}
\]

Then existence of the global solution \( z(t) \) of system (5) implies that system (1) has an unique global solution \( x(t) \) for any given initial function \( \xi(\cdot) \in PC([t_0 - \tau, t_0]; \mathbb{R}^{n_x}) \). Moreover, the ISS property of system (5) (or system (6)) implies ISS in mean and ISS in of model (1). In particular, if condition (22) is replaced by

\[
c_1\|x(t)\|^p \leq V_j(t, x(t)) \leq c_2\|x(t)\|^p, \quad t \geq t_0 - \tau \tag{24}
\]

where \( c_1, c_2 \) and \( p \) are given positive scalars, then the ISS property of system (5) implies the \( p \)th moment exponential ISS in mean of model (1). In addition, if there exist constant \( q > 0 \), vectors \( \eta_1 = (\lambda_1, \lambda_2, \ldots, \lambda_{n_z})^T, \eta_2 = (\mu_1, \mu_2, \ldots, \mu_{n_z})^T \in \mathbb{R}_{\geq 0}^{n_z}\setminus\{0\} \) such that

\[
r^q\eta_1 \leq l(r) \leq r^q\eta_2, \quad r \in [0, r^*] \tag{25}
\]

then the ISS property of system (6) implies the \( p \)th moment exponential ISS in mean of model (1).

**Proof:** Firstly, it follows from Theorem 3.2 and condition (22) that system (1) has an unique global solution \( x(t) \) if system (5) has a global solution.

Secondly, we need to prove the ISS in mean of system (1) provided that model (5) is input-to-state stable. Another case is similar and will be omitted for space consideration. If system (5) is input-to-state stable, then there exist functions \( \beta_2(\cdot, \cdot) \in \mathcal{KL} \) and \( \gamma_2(\cdot) \in \mathcal{K}_\infty \) such that, for all \( \xi(\cdot) \in PC([0, t_0]; \mathbb{R}^{n_x}), \xi(t) \in PC([t_0, +\infty); \mathbb{R}^{n_z}) \) and \( t \in \mathbb{R}_{\geq 0} \),

\[
\|z(t)\| \leq \max\{\beta_2(\xi_1, t - t_0), \gamma_2(\|v(t_0, \xi(t))\|)\}. \tag{26}
\]

For any given initial function \( \xi(\cdot) \in PC([0, t_0], \mathbb{R}^{n_x}) \), let \( \bar{\xi}(\cdot) := EV_j(\xi(\cdot)) \), where \( V_j(\cdot) \) is defined by \( V_j(t) = V_j(t, \xi(t)) \) for \( t \in [0, t_0 - \tau] \). Then by Theorem 3.1, we obtain that \( EV_j(t, x(t)) \leq z(t) \) for all \( t \geq t_0 \), i.e.

\[
EV_j(t, x(t)) \leq z(t), \quad t \geq t_0, j = 1, 2, \ldots, n_z. \tag{27}
\]

From (22), we can get the following inequalities:

\[
\alpha_1(\mathbb{E}\|x(t)\|) \leq EV_j(t, x(t)), \quad j = 1, 2, \ldots, n_z \tag{28}
\]

where \( t \geq t_0 - \tau, \alpha_1(\cdot) := \min_{1 \leq j \leq n_z} \alpha_j(\cdot) \), and the convex property of function \( \alpha_j(\cdot) \) has been utilized. We claim that \( \alpha_1(\cdot) \in \mathcal{K}_\infty \). Obviously, \( \alpha_1(0) = \min_{1 \leq j \leq n_z} \alpha_j(0) = 0 \), and for any \( s > 0 \), \( \alpha_1(s) = \min_{1 \leq j \leq n_z} \alpha_j(s) > 0 \). For any \( 0 < s_1 < s_2 \), without loss of generality assume \( \alpha_1(s_1) = \alpha_1(s_2) = \min_{1 \leq j \leq n_z} \alpha_j(s_1) \), then for any \( j = 1, 2, \ldots, n_z \), it is obtained that \( \alpha_1(s_1) = \alpha_1(s_2) \leq \alpha_j(s_1) < \alpha_j(s_2) \), and hence \( \alpha_1(s_1) < \alpha_1(s_2) \), which means that \( \alpha_1(\cdot) \) is strictly increasing. In addition, it is obvious that \( \lim_{s \to \infty} \alpha_1(s) = +\infty \). At last, we claim that function \( \alpha_1(\cdot) \) is continuous. If this is not true, then there must exist one \( s > 0 \) such that \( \lim_{s \to 0} \alpha_1(s + \Delta s) \neq \alpha_1(s) = \alpha_1(\cdot) \). Since \( \alpha_{1, p}(\cdot) \in \mathcal{K}_\infty \), we have \( \lim_{s \to 0} \alpha_{1, p}(s + \Delta s) = \alpha_{1, p}(s) \). Assume \( \lim_{s \to 0} \alpha_1(s + \Delta s) = \lim_{s \to 0} \min_{1 \leq s \leq n_z} \{\alpha_1(s + \Delta s)\} = \alpha_{1, p}(s) \), it follows that \( \alpha_{1, p}(s) > \alpha_{1, p}(s) \), which means \( \min_{1 \leq s \leq n_z} \{\alpha_1(s + \Delta s)\} = \alpha_{1, p}(s + \Delta s) \) is small enough, which is a contradiction. Now, we have proved that \( \alpha_1(\cdot) \in \mathcal{K}_\infty \). Therefore, it is achieved from (26)–(28) that

\[
E\|x(t)\| \leq \alpha_1^{-1}(z_j(t)) \leq \alpha_1^{-1}(\|z(t)\|)
\]

\[
\leq \alpha_1^{-1}(\max\{\beta_2(\bar{\xi_1}, t - t_0), \gamma_2(\|v(t_0, \xi(t))\|)\})
\]

\[
\leq \alpha_1^{-1}\left(\beta_2(\|\xi_1\|, t - t_0) + \gamma_2(\|v(t_0, \xi(t))\|)\right).
\tag{29}
\]

On the other hand, from (22), we have

\[
\max_{1 \leq j \leq n_z} EV_j(t, x(t)) \leq \alpha_2(\mathbb{E}\|\xi(\cdot)\|), \quad t \in [t_0 - \tau, t_0] \tag{30}
\]

where \( \alpha_2(\cdot) := \max_{1 \leq j \leq n_z} \alpha_2(\cdot) \) and it can be proved similarly that \( \alpha_2(\cdot) \in \mathcal{K}_\infty \). Combining this with (23) and (29), one gets

\[
E\|x(t)\| \leq \alpha_1^{-1}\left(\bar{h}(\cdot)\right) + \alpha_1^{-1}(\gamma_2(\|v(t_0, \xi(t))\|))
\]

\[
\leq \alpha_1^{-1}\left(\beta_2(n_2 \sup_{t_0 - \tau \leq s \leq t_0} (\alpha_2(\mathbb{E}\|\xi(\cdot)\|)), t - t_0)\right)
\]

\[
+ \alpha_1^{-1}(\gamma_2(\|v(t_0, \xi(t))\|))
\]

\[
\leq \alpha_1^{-1}\left(\beta_2(n_2 \alpha_2(\mathbb{E}\|\xi(\cdot)\|)), t - t_0\right)
\]

\[
+ \alpha_1^{-1}(\gamma_2(\|v(t_0, \xi(t))\|))
\]

\[
= \beta(\mathbb{E}\|\xi(\cdot)\|, t - t_0) + \gamma(\|v(t_0, \xi(t))\|) \quad t \in \mathbb{R}_{\geq 0} \tag{31}
\]

where

\[
\bar{h} = \beta_2\left(\sup_{t_0 - \tau \leq s \leq t_0} (n_2 \max_{1 \leq j \leq n_z} EV_j(s, x(s))), t - t_0\right).
\]

\[
\beta(s, t) := \alpha_1^{-1}\left(\beta_2(n_2 \alpha_2(s, t))\right),
\]

\[
\gamma(s) := \alpha_1^{-1}(\gamma_2(\|v(s, \xi(\cdot))\|)).
\]

Because \( \alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty \), \( \beta_2(\cdot, \cdot) \in \mathcal{K}_{\infty} \), \( \beta_2(\cdot, \cdot) \in \mathcal{K}_{\infty} \), we achieve that \( \beta(\cdot, \cdot) \in \mathcal{K}_{\infty} \) and \( \gamma(\cdot) \in \mathcal{K}_{\infty} \). Thus, by inequality (31), it yields that system (1) is ISS in mean.

Thirdly, we shall prove that the ISS of system (5) would guarantee the ISS property of model (1). If system (5) is input-to-state stable, from the above proof process we know that there exist functions \( \beta(\cdot, \cdot) \in \mathcal{KL} \) and \( \gamma(\cdot) \in \mathcal{KL} \).
Let $\beta(s,t) := (1/\epsilon)\beta(s,t)$ and $\gamma(\cdot) = (1/\epsilon)\gamma(\cdot)$, by the Chebyshev inequality one then gets

$$P \{ \|x(t)\| \geq \beta_x(E\|x\|, t - t_0) + \gamma(\|u\|_{[t_0,t]}) \} \leq \frac{E\|x(t)\|^p}{\beta_x(E\|x\|, t - t_0) + \gamma(\|u\|_{[t_0,t]})} \leq \epsilon$$

which yields that model (1) is stochastic input-to-state stable by the following inequality which is equivalent to inequality (32):

$$P \{ \|x(t)\| \leq \beta_x(E\|x\|, t - t_0) + \gamma(\|u\|_{[t_0,t]}) \} \geq 1 - \epsilon.$$ 

Fourthly, assume that system (5) is input-to-state stable and inequality (24) is valid, we shall prove that model (1) is $p$th moment exponentially input-to-state stable in mean. Let $\xi_1(\cdot)$ be defined the same as above so that inequalities in (27) are valid. From condition (24) it follows that

$$E\|x(t)\|^p \leq \frac{1}{c_1} EV(t,x(t)), \quad t \geq t_0 - \tau, j = 1, 2, \ldots, n_z$$

which combining with inequalities (26)–(27) infers that

$$E\|x(t)\|^p \leq \frac{1}{c_1} z_j(t) \leq \frac{1}{c_1} \|z(t)\|^p \leq \frac{1}{c_1} \beta_2(\|\xi_1\|, t - t_0) + \frac{1}{c_1} \gamma(\|v\|_{[t_0,t]}).$$

By further considering condition (23), it follows that

$$E\|x(t)\|^p \leq \frac{1}{c_1} \beta_2(\|\xi_1\|, t - t_0) + \frac{1}{c_1} \gamma(\|v\|_{[t_0,t]})$$

where

$$\beta(s,t) := \frac{1}{c_1} \beta_2(nz_2s, t), \quad \gamma(s) := \frac{1}{c_1} \gamma(\|v\|^p_{[t_0,t]}).$$

Obviously, $\beta^{-1}(\cdot, \cdot) \in \mathcal{KL}$ and $\gamma^{-1}(\cdot) \in \mathcal{K}_{\infty}$, and equality (34) means that system (1) is $p$th moment exponentially input-to-state stable in mean.

Finally, suppose that system (6) is input-to-state stable, then there exist functions $\beta_y(\cdot, \cdot) \in \mathcal{KL}$ and $\gamma_y(\cdot) \in \mathcal{K}_{\infty}$ such that, for all $\xi_j(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $w(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $t \in \mathbb{R}_{\geq t_0}$,

$$\|y(t)\| \leq \beta_y(\|\xi_2\|, t - t_0) + \gamma_y(\|w\|_{[t_0,t]}).$$

We shall show that model (1) is $p$th moment exponentially input-to-state stable in mean. For any $\xi(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let $\tilde{\xi}_1(t) = EV(t)$, and let

$$\tilde{\xi}_2(t) = \max_{1 \leq i \leq n_z} \left\{ \frac{EV(t,\xi(t))}{\lambda_i} \right\}^{1/q}$$

with $t \in [t_0 - \tau, t_0]$. From condition (25) we have

$$EV(t,x(t)) \leq \sum_{j=1}^{n_z} \mu_j \cdot \max\{2^q - 1, 1\} \cdot \frac{\beta_2(\|\xi_2\|, t - t_0) + \gamma(\|w\|_{[t_0,t]}))}{\lambda_i}$$

According to Lemma 2.1, condition (25) and inequality (35), it follows that

$$\|y(t)\| \leq \sum_{j=1}^{n_z} \mu_j \cdot \max\{2^q - 1, 1\} \cdot \frac{(\|\xi_2\|^p_{[t_0,t]}))}{\lambda_i} + \gamma(\|w\|_{[t_0,t]}), \quad t \geq t_0.$$ (37)

For any $t \in [t_0 - \tau, t_0]$, we achieve from the definition of $\tilde{\xi}_2(\cdot)$ and condition (24) that

$$\tilde{\xi}_2^q(t) \leq \max_{1 \leq i \leq n_z} \left\{ \frac{EV(t,\xi(t))}{\lambda_i} \right\}^{1/q} \leq \frac{1}{\min_{1 \leq i \leq n_z}\{\lambda_i\}} \max_{1 \leq i \leq n_z} \{EV(t,\xi(t))\}^{1/q}$$

which obviously infers that

$$\|\tilde{\xi}_2\| \leq \left( \frac{C_2}{\min_{1 \leq i \leq n_z}\{\lambda_i\}} \right)^{1/q} \cdot (\|\xi\|^p_{[t_0,t]}{1/q}).$$

Combining (38) with (23), (24), (36) and (37), we can get

$$E\|x(t)\|^p \leq \frac{1}{c_1} \beta_2(nz_2s, t) + \frac{1}{c_1} \gamma(\|v\|^p_{[t_0,t]}).$$
\[
\leq \frac{1}{c_1} E V(t, x(t)) \leq \frac{1}{c_1} ||y(t)||
\]

\[
\leq \sum_{j=1}^{n_2} \mu_j \cdot \max\{2^{q-1}, 1\}
\]

\[
\times \beta^q \left( \frac{c_2}{\min_{1 \leq l \leq n_2} \lambda_l} \right)^{\frac{1}{q}} \cdot \left( E \|\xi\|^p, t - t_0 \right)
\]

\[
+ \sum_{j=1}^{n_2} \mu_j \cdot \max\{2^{q-1}, 1\}
\]

\[
\gamma^q \left( \varphi_2 (\|u\|_{[t_0, t]} ) \right)
\]

\[
= \beta (E \|\xi\|^p, t - t_0) + \gamma (\|u\|_{[t_0, t]}), \quad t \in \mathbb{R}_{\geq t_0}
\]

where

\[
\beta (s, t) := \sum_{j=1}^{n_2} \mu_j \cdot \max\{2^{q-1}, 1\}
\]

\[
\times \frac{c_2}{\min_{1 \leq l \leq n_2} \lambda_l} \left( \frac{c_2}{\min_{1 \leq l \leq n_2} \lambda_l} \right)^{\frac{1}{q}} \cdot s^{1/q}, t
\]

\[
\gamma (s) := \sum_{j=1}^{n_2} \mu_j \cdot \max\{2^{q-1}, 1\}
\]

\[
\gamma^q \left( \varphi_2 (s) \right)
\]

for any \( s \in \mathbb{R}_{\geq 0}. \) Obviously, \( \beta^q (\cdot, \cdot) \in K \mathcal{L} \) and \( \gamma (\cdot) \in K \mathcal{C} \), thus by (39) it yields that system (1) is pth moment exponentially input-to-state stable in mean.

**Remark 4.1:** Theorem 4.1 can be viewed as a comparison principle for the ISS-related properties of the stochastic impulsive delayed system (1). By using it, the ISS-related criteria concerning model (1) can be derived by the corresponding ISS-related properties of the deterministic impulsive systems. On another front, condition (23) in Theorem 4.1 should be relaxed to the case with \( \|u\|_{\infty} < \infty \), \( \|v\|_{\infty} < \infty \) and \( \|w\|_{\infty} < \infty \) when the term \( \|u\|_{[t_0, t]} \) in Definitions 2.2–2.4 is replaced by \( \|u\|_{\infty} \), provided that the input \( u (\cdot) \) is known to be globally bounded, that is, \( \|u\|_{\infty} < \infty \).

**Remark 4.2:** Based on the Lyapunov method, the ISS issue has been analysed for stochastic delay-free systems in Zhao, Kang et al. (2012) and then discussed in Yao et al. (2014) for stochastic impulsive models without delays. Soon afterwards, the ISS-type issues for stochastic impulsive systems with time delays are considered in Wu et al. (2016), where a scalar Lyapunov function is also resorted to under the assumption of the linear growth constraint which is no longer necessary here in Theorem 4.1. To be more specific, when investigating the ISS-related characteristics of the stochastic impulsive delayed model (1), influence of the time delay should be carefully/extendively taken into account, especially on the magnitude of the solutions of the target plant. This paper tries its first attempt to analyse these properties by introducing the lower-dimensional comparison systems which are also delay-dependent, and the comparison principles in vector version are established which are much more easier to be implemented in practice.

**Corollary 4.1:** Assume that conditions (22)–(23) of Theorem 4.1 hold with \( V(t, x) \in \mathbb{R}_{\geq 0}. \) Moreover, there exist positive constants \( \lambda_1, \lambda_2, \mu \in (0, 1) \), and \( \varphi (\cdot) \in K \mathcal{C} \) such that

\[
\left\{\begin{array}{l}
\mathcal{L} V(t, x(t)) < \lambda_1 V(t, x(t)) + \lambda_2 V_t + \varphi (\|v(t)\|),
\end{array}\right.
\]

\[
t \in \mathbb{R}_{\geq t_0} \setminus T
\]

\[
V(t, x(t)) \leq \mu V(t^-, x(t^-)) + \varphi (\|v(t^-)\|),
\]

\[
t \in T
\]

\[
T_a < -\frac{\ln \mu}{\lambda_1 + \mu - N_0 \lambda_2}
\]

where \( T_a \) is the average impulsive interval for the impulsive time sequence \( T \) and \( N_0 \) means the relating elasticity number. Then system (1) is input-to-state stable in mean and stochastic input-to-state stable. Particularly, if condition (22) in Theorem 4.1 is replaced by

\[
c_1 \|x(t)\|^p \leq V(t, x(t)) \leq c_2 \|x(t)\|^p,
\]

\[
t \geq t_0 - \tau
\]

where \( c_1, c_2, p \) are given positive constants, then system (1) is pth moment exponentially input-to-state stable in mean.

**Proof:** The conclusion can be derived by using Theorems 3.1 and 4.1.

**Remark 4.3:** Conditions (40)–(41) in Corollary 4.1 are almost the same as those in Theorem 2 of Wu et al. (2016), where condition (40) is proposed under the linear growth assumption and condition (41) is given to guarantee the ISS property of the one-dimensional comparison system defined in the following form:

\[
\left\{\begin{array}{l}
\dot{z}(t) = \lambda_1 z(t) + \lambda_2 z_t + \varphi (\|v(t)\|),
\end{array}\right.
\]

\[
t \in \mathbb{R}_{\geq t_0} \setminus T
\]

\[
\dot{z}(t) = \mu z(t^-) + \varphi (\|v(t^-)\|),
\]

\[
t \in T
\]

\[
\dot{z}(t) = E \xi(t),
\]

\[
t \in [t_0 - \tau, t_0]
\]

**Corollary 4.2:** Assume that all conditions of Theorem 3.1 and constraints (22)–(23) in Theorem 4.1 hold. Then the ISKB property of system (5) implies the ISKB property of system (6) in the mean of model (1), and the ISKB property of system (6) implies the ISKB property of system (6) in the mean of model (1) and model (5).

**Proof:** If system (5) is input-to-state KB-stable, then employ the similar method as that of Theorem 4.1, it will be directly proved that system (1) is input-to-state KB-stable in mean, where \( \beta_2 (\cdot) \in K \mathcal{B} \) in inequality (31).
5. Illustrative example

In this section, one example is given for illustration. Consider the following stochastic impulsive time delayed system:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} A & \tilde{B} \\ 0 & 6 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f_j(x(t)) + \begin{bmatrix} C \\ 0 \end{bmatrix} u(t), \quad t \in \mathbb{R}_{\geq 0} \setminus \mathcal{T} \\
\end{align*}
\]

where \( x(t) \in \mathbb{R}^2 \) is the system state, \( u(t) \in \mathbb{R}^2 \) is the input disturbance, \( f_j(x(t)) = (f_1(x_1(t)), f_2(x_2(t)))^T \) with \( f_j(x_j(t)) = \tanh(x_j(t)) \) for \( j \in \{1, 2\} \), and

\[
A = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -1 \\ -0.5 & -1.5 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 1.6 & 0.9 \\ 0.4 & 0.5 \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}.
\]

The comparison system for model (43) is as

\[
\begin{align*}
\dot{z}(t) &= \lambda_1 z(t) + \lambda_2 z_t + \varphi((V_i(t))), \quad t \in \mathbb{R}_{\geq 0} \setminus \mathcal{T} \\
\end{align*}
\]

where \( \lambda_1 = 10, \lambda_2 = 7, \mu = 0.64 \). Construct function \( V(t, x(t)) = (V_1(t, x(t)), V_2(t, x(t)))^T \) with \( V_j(t, x(t)) = x_j^2(t) \) for \( j \in \{1, 2\} \), then it is easy to get:

\[
\begin{align*}
\mathcal{L} V_1(t, x_1) &\leq 10x_1^2(t) + x_1(x_1 x_2 - 4x_1x_1(t - \tau_1) - 2x_1(t - \tau_2) + u_2(t), \\
\mathcal{L} V_2(t, x_1) &\leq 6x_2^2(t) + 0.3x_1 x_2 - 1.5x_2 x_1(t - \tau_1) - 3x_2 x_2 - 2u_2(t).
\end{align*}
\]

Let \( N_0 = 1, T_a < 0.1523 \). Then (43) is input-to-state stable in mean and stochastic input-to-state stable according to Theorem 4.1 as well as Corollary 4.1.

6. Conclusion

This article has addressed the ISS-related issues for stochastic impulsive systems with time-varying delay and external inputs. Firstly, a reduced-order deterministic impulsive delayed system and an one-dimensional impulsive system with time delays are selected as the comparison models, based on which the generalized vector-version comparison principles have been established assuring the existence and uniqueness of solutions for the addressed system as well as the magnitude of the solutions. Then the ISS in mean problem, the \( p \)th moment exponential ISS problem and SISS problem have been separately analysed for the considered stochastic impulsive system by employing the comparison principles. One example has been provided finally to demonstrate feasibility of the acquired theoretical results.

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References

An, C., Liu, C., & Bi, S. (2019, June 03–05). Stability in distribution and optimal control in an impulsive toxin input bioeconomic system with stochastic fluctuations and time delays. In The 31st Chinese control and decision conference, Nanchang, China (pp. 3638–3642).

Antunes, D., Hespanha, J., & Silvestre, C. (2013). Stability of networked control systems with asynchronous renewal links: An impulsive systems approach. *Automatica, 49*(2), 402–413. https://doi.org/10.1016/j.automatica.2012.11.033

Chatterjee, D., & Liberzon, D. (2006). Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions. *SIAM Journal on Control and Optimization, 45*(1), 174–206. https://doi.org/10.1137/040619429

Chen, W.-H., & W. X. Zheng (2009). Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica, 45*(6), 1481–1488. https://doi.org/10.1016/j.automatica.2009.02.005

Damak, H. (2021). Input-to-state stability and integral input-to-state stability of non-autonomous infinite-dimensional systems. *International Journal of Systems Science, 1–14*. doi:10.1080/00207721.2021.1879306

Hu, J., Cui, Y., Lv, C., Chen, D., & Zhang, H. (2020). Robust adaptive sliding mode control for discrete singular systems with randomly occurring mixed time-delays under uncertain occurrence probabilities. *International Journal of Systems Science, 51*(6), 987–1006. https://doi.org/10.1080/00207721.2020.1746439

Hu, J., Zhang, H., Liu, H., & Yu, X. (2021). A survey on sliding mode control for networked control systems. *International Journal of Systems Science, 1–19*. doi:10.1080/00207721.2021.1885082

Jiang, Z.-P., Lin, Y., & Wang, Y. (2004). Nonlinear small-gain theorems for discrete-time feedback systems and applications. *Automatica, 40*(12), 2129–2136. https://doi.org/10.1016/j.automatica.2004.08.002

Karafyllis, I., Kravaris, C., Syrou, L., & Lyberatos, G. (2008). A vector Lyapunov function characterization of input-to-state stability with application to robust global stabilization of the chemostat. *European Journal of Control, 14*(1), 47–61. https://doi.org/10.3166/ejc.14.47-61

Khalil, H. K. (2002). *Nonlinear Systems*. Prentice Hall.

Kundu, S., & Anghel, M. (2017). A multiple-comparison-systems method for distributed stability analysis of large-scale nonlinear systems. *Automatica, 78*(6), 25–33. https://doi.org/10.1016/j.automatica.2016.12.003
Li, Q., Liang, J., & Gong, W. (2020). Stabilization of piecewise-homogeneous Markovian switching CVNNs with mode-dependent delays and incomplete transition rates. *Systems Science & Control Engineering: An Open Access Journal, 8*(1), 206–221. https://doi.org/10.1080/21642583.2020.1737847

Li, Z., Liu, L., & Zhu, Q. (2016). Mean-square exponential input-to-state stability of delayed Cohen–Grossberg neural networks with Markovian switching based on vector Lyapunov functions. *Neural Networks, 84*(4), 39–46. https://doi.org/10.1016/j.neunet.2016.08.001

Liao, F., Yang, K., & Ji, H. (2014, August 8–10). Adaptive integrated guidance and control with actuator failures based on backstepping and input-to-state stability. In *IEEE Chinese guidance, navigation and control conference*, Yantai, China (pp. 49–54).

Liu, B. (2008). Stability of solutions for stochastic impulsive systems via comparison approach. *IEEE Transactions on Automatic Control, 53*(9), 2128–2133. https://doi.org/10.1109/TAC.2008.930185

Liu, L. (2017). New criteria on exponential stability for stochastic delay differential systems based on vector Lyapunov function. *IEEE Transactions on Systems, Man, and Cybernetics: Systems, 47*(11), 2985–2993. https://doi.org/10.1109/TSMC.2016.2558047

Liu, B., & Hill, D. J. (2010). Uniform stability and ISS of discrete-time impulsive hybrid systems. *Nonlinear Analysis: Hybrid Systems, 4*(2), 319–333. https://doi.org/10.1016/j.nahs.2009.05.002

Liu, T., Jiang, Z.-P., & Hill, D. J. (2012). Decentralized output-feedback control of large-scale nonlinear systems with sensor noise. *Automatica, 48*(10), 2560–2568. https://doi.org/10.1016/j.automatica.2012.06.054

Liu, J., Liu, X., & Xie, W.-C. (2011). Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica, 47*(5), 899–908. https://doi.org/10.1016/j.automatica.2011.01.061

Liu, B., Xu, C., & Liu, D. (2013). Input-to-state-stability-type comparison principles and input-to-state stability for discrete-time dynamical networks with time delays. *International Journal of Robust and Nonlinear Control, 23*(4), 450–472. https://doi.org/10.1002/rnc.1846

Luo, J. (2006). Comparison principle and stability of Ito stochastic differential delay equations with Poisson jump and Markovian switching. *Nonlinear Analysis: Theory, Methods & Applications, 64*(2), 253–262. https://doi.org/10.1016/j.na.2005.06.048

Luo, Z., & Shen, J. (2006). Global existence results for impulsive functional differential equations. *Journal of Mathematical Analysis and Applications, 323*(1), 644–653. https://doi.org/10.1016/j.jmaa.2005.10.066

Mao, X. (2007). *Stochastic Differential Equations and Applications*. Horwood Publishing Limited.

Niu, Y., Liao, D., & Wang, P. (2012). Stochastic asymptotical stability for stochastic impulsive differential equations and it is application to chaos synchronization. *Communications in Nonlinear Science and Numerical Simulation, 17*(2), 505–512. https://doi.org/10.1016/j.cnsns.2011.07.011

Ren, W., & Xiong, J. (2016). Stability and stabilization of switched stochastic systems under asynchronous switching. *Systems & Control Letters, 97*(9), 184–192. https://doi.org/10.1016/j.sysconle.2016.09.005

Ren, W., & Xiong, J. (2017, December 17–20). “Exponential stability of stochastic impulsive switched delayed systems based on vector Lyapunov functions,” In *The 11th Asian Control Conference*, Gold Coast, Australia (pp. 1888–1893).

Ren, W., & Xiong, J. (2018). Lyapunov conditions for stability of stochastic impulsive switched systems. *IEEE Transactions on Circuits and Systems-I: Regular Papers, 65*(6), 1994–2004. https://doi.org/10.1109/TCSI.2017.2771432

Ren, W., & Xiong, J. (2019). Vector-Lyapunov-function-based input-to-state stability of stochastic impulsive switched time-delay systems. *IEEE Transactions on Automatic Control, 64*(2), 654–669. https://doi.org/10.1109/TAC.2018.2836191

Rüffer, B. S., Kellett, C. M., & Weller, S. R. (2009, December 16–18). Integral input-to-state stability of interconnected iISS systems by means of a lower-dimensional comparison system. In *Joint 48th IEEE conference on decision and control/28th Chinese control conference*, Shanghai, China (pp. 638–643).

Song, J., Ding, D., Liu, H., & Wang, X. (2020). Non-fragile distributed state estimation over sensor networks subject to DoS attacks: The almost sure stability. *International Journal of Systems Science, 51*(6), 1119–1132. https://doi.org/10.1080/00207721.2020.1752843

Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control, 34*(4), 435–443. https://doi.org/10.1109/9.28018

Sontag, E. D. (2004, June 19–29). Input to state stability: Basic concepts and results. In *CIME Summer School on Nonlinear and Optimal Control Theory*, Cetraro, Italy.

Sontag, E. D., & Wang, Y. (1996). New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control, 41*(9), 1283–1294. https://doi.org/10.1109/9.536498

Teel, A. R. (1998). Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control, 43*(7), 960–964. https://doi.org/10.1109/9.701099

Wakai, M., & Yamamoto, Y. (2017). Stabilization of switched linear systems with quantized output and switching delays. *IEEE Transactions on Automatic Control, 62*(6), 2958–2964. https://doi.org/10.1109/TAC.2016.2604924

Wang, W., Gong, S., & Chen, W. (2018). New result on the mean-square exponential input-to-state stability of stochastic delayed recurrent neural networks. *Systems Science & Control Engineering: An Open Access Journal, 6*(1), 501–509. https://doi.org/10.1080/21642583.2018.1544512

Wu, X., Tang, Y., & Zhang, W. (2016). Input-to-state stability of impulsive stochastic delayed systems under linear assumptions. *Automatica, 66*(7), 195–204. https://doi.org/10.1016/j.automatica.2016.01.002

Xu, D., Wang, X., Hong, Y., Jiang, Z.-P., & Xu, S. (2016). Output feedback stabilization and estimation of the region of attraction for nonlinear systems: A vector control Lyapunov function perspective. *IEEE Transactions on Automatic Control, 61*(12), 4034–4040. https://doi.org/10.1109/TAC.2016.2543198

Yang, S., Shi, B., & Hao, S. (2014). Impulsive stabilization of discrete-time nonlinear impulsive systems with delays. *International Journal of Robust and Nonlinear Control, 24*(4), 400–418. https://doi.org/10.1002/rnc.v23.4

Yao, F., Qiu, L., & Shen, H. (2014). On input-to-state stability of impulsive stochastic systems. *Journal of the Franklin Institute, 351*(9), 4636–4651. https://doi.org/10.1016/j.jfranklin.2014.06.011

Zhang, H., Guan, Z.-H., & Feng, G. (2008). Reliable dissipative control for stochastic impulsive systems. *Automatica,*
Zhao, P., Feng, W., & Kang, Y. (2012). Stochastic input-to-state stability of switched stochastic nonlinear systems. *Automatica, 48*(10), 2569–2576. https://doi.org/10.1016/j.automatica.2012.06.058

Zhao, P., Kang, Y., & Zhai, D. (2012). On input-to-state stability of stochastic nonlinear systems with Markovian jumping parameters. *International Journal of Control, 85*(4), 343–349. https://doi.org/10.1080/00207179.2011.651749

Zhao, P., Zhao, Y., & Guo, R. (2015, July 28–30). Input-to-state stability for discrete-time stochastic nonlinear systems. In *The 34th Chinese control conference*, Hangzhou, China (pp. 1799–1803).