On The Babenko-Bechner-Type Inequality associated with the Weinstein Operator

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Abstract

In this paper, we study the Babenko-Bechner-type inequality for the Fourier Weinstein transform $\mathcal{F}_{\Delta_{w,d}}$ associated with the Weinstein operator $\Delta_{w,d}$. We use this inequality to establish a new version of Young’s type inequality.

keywords. Weinstein transform, Weinstein operator, Babenko inequality, Young’s type inequality

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1 Introduction

Inequalities are a basic tool in the study of the classical Fourier analysis. The result relating $L^p$ estimates for a function and its Fourier transform is the Hausdorff-Young inequality (see[1]) it given for $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\|\hat{f}\|_q \leq \|f\|_p$ where $\hat{f}$ is the classical Fourier transform defined by:

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} e^{-2\pi \langle \lambda, x \rangle} f(x) dx. \quad (1.1)$$

This sharp form of the Hausdorff-Young inequality was extended by W.Bechner [2] in the form:

$$\|\hat{f}\|_q \leq \left( \frac{\frac{1}{p} \|f\|_p}{q^{-\frac{1}{q}}} \right)^{\frac{1}{p}} \|f\|_p. \quad (1.2)$$

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Next, using the method as W. Bechner, A. Fitouhi [3] in 1975 proved that the Fourier-Bessel transform $F(f)$ of $f \in L^p(\mathbb{R}_+, \frac{x^{2\alpha+1}}{2^{\alpha+1}} dx)$ satisfies the following inequality:

$$\|F(f)\|_{q, \alpha} \leq \left( \frac{p^b}{q^a} \right)^\alpha \|f\|_{p, \alpha}, \quad \alpha > -\frac{1}{2},$$

where $F$ is given by:

$$F(f)(\lambda) = \frac{1}{2^\alpha \Gamma(\alpha + 1)} \int_0^{+\infty} f(x) j_\alpha(\lambda x) x^{2\alpha+1} dx,$$  \hspace{1cm} (1.4)

and $j_\alpha$ is the normalized Bessel function of index $\alpha$, defined by:

$$j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left(\frac{z}{2}\right)^{2n}, \quad \forall z \in \mathbb{C}.$$ \hspace{1cm} (1.5)

Moreover, F. Bouzeffour [8] in 2014 adapted these ideas and he applied them to obtain the Babenko-Bechner-type inequality for the Dunkl transform in real line. In [6, 7], Ben Salem, N and Ben Nahia, Z was defined the Weinstein operator $\Delta_{w,d}^{\alpha}$ on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times [0, +\infty]$ by

$$\Delta_{w,d}^{\alpha} = \Delta_d + L_{\alpha}, \quad \alpha > -\frac{1}{2},$$ \hspace{1cm} (1.6)

where $\Delta_d$ is the ordinary Laplacian on $\mathbb{R}^d$ and $L_{\alpha}$ is the Bessel operator for the last variable $x_{d+1}$.

The Weinstein operator $\Delta_{w,d}^{\alpha}$ has several applications in pure and applied mathematics, especially in fluid mechanics [5]. The Weinstein transform generalizing the usual Fourier transform, is given for $f \in L_1^{\alpha}(\mathbb{R}_+^{d+1})$ and $\lambda \in \mathbb{C}_+^{d+1}$, by

$$F_{w,d}^{\alpha}(f)(\lambda) = \int_{\mathbb{R}_+^{d+1}} f(x) \Psi_{\alpha,d}(x, \lambda) d\nu_{\alpha,d}(x),$$

where $d\nu_{\alpha,d}(x)$ is the measure on $\mathbb{R}_+^{d+1}$ and $\Psi_{\alpha,d}$ is the Weinstein kernel and given respectively later by (2.1) and (2.2). The purpose of the present paper is to establish the Babenko-Bechner-type inequality for the Weinstein transform associated to the Weinstein operator. The paper is organized as follows: In the first section we recall some basic Harmonic results related with the differential operator $\Delta_{w,d}^{\alpha}$ given by (1.6). In the second section we establish a Babenko-Bechner-type inequality for the Weinstein transform. As application, new Young’s type inequality is given.

## 2 Preliminaries

In this section, we collect some notations and results related to the Weinstein operator $\Delta_{w,d}^{\alpha}$ (see [6, 7]).

In the following we denote by
• \( \mathbb{R}^{d+1} = \mathbb{R}^d \times [0, +\infty[ \)

• \( x = (x_1, \ldots, x_d, x_{d+1}); \quad x' = (x_1, \ldots, x_d) \)

• \( \|x\| = \sqrt{x_1^2 + \cdots + x_{d+1}^2} \)

• \( C_*(\mathbb{R}^{d+1}), \) the space of continuous functions on \( \mathbb{R}^{d+1}, \) even with respect to the last variable.

• \( S_*(\mathbb{R}^{d+1}), \) the space of the \( C^\infty \) functions, even with respect to the last variable and rapidly decreasing together with their derivatives.

• \( D_*(\mathbb{R}^{d+1}), \) the space of the \( C^\infty \) functions on \( \mathbb{R}^{d+1} \) which are of compact support, even with respect to the last variable.

• \( L^p_\alpha(\mathbb{R}^{d+1}), \) \( 1 \leq p \leq +\infty, \) the space of measurable functions \( f \) on \( \mathbb{R}^{d+1}_+ \) such that:

\[
\|f\|_{\alpha,p} = \left( \int_{\mathbb{R}^{d+1}_+} |f(x)|^p d\nu_{\alpha,d}(x) \right)^{\frac{1}{p}} < +\infty, \quad \text{if } p \in [1, +\infty[,
\]

\[
\|f\|_{\alpha,\infty} = \text{ess sup}_{x \in \mathbb{R}^{d+1}_+} |f(x)| < +\infty,
\]

where \( d\nu_{\alpha,d} \) is the measure on \( \mathbb{R}^{d+1}_+ \) given by:

\[
d\nu_{\alpha,d}(x) = \frac{x_{d+1}^{2\alpha+1}}{(2\pi)^{\frac{d+1}{2}} 2^\alpha \Gamma(\alpha + 1)} dx. \quad (2.1)
\]

The Weinstein Kernel \( \Psi_{\alpha,d}(x,.) \) is given by:

\[
\Psi_{\alpha,d}(x, z) = e^{-i<x';z'>} j_\alpha(x_{d+1}z_{d+1}) \quad \forall (x, z) \in \mathbb{R}^{d+1} \times \mathbb{C}^{d+1}. \quad (2.2)
\]

The function \( (x, z) \mapsto \Psi_{\alpha,d}(x, z) \) has a unique extention to \( \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} \) and has the following integral operator:

\[
\Psi_{\alpha,d}(x, z) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} e^{-i<x';z'>} \int_{-1}^{1} (1 - u^2)^{\alpha - \frac{1}{2}} e^{-ix_{d+1}z_{d+1}u} du. \quad (2.3)
\]

The Weinstein Kernel satisfies the following properties:

• For all \( (x, z) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} \) we have:

\[
\Psi_{\alpha,d}(x, z) = \Psi_{\alpha,d}(z, x); \quad \Psi_{\alpha,d}(x, 0) = 1.
\]

• For all \( (x, z) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} \) and \( \lambda \in \mathbb{C} \) we have:

\[
\Psi_{\alpha,d}(\lambda x, z) = \Psi_{\alpha,d}(x, \lambda z).
\]
For all \(v \in \mathbb{N}^{d+1}, x \in \mathbb{R}^{d+1}\) and \(z \in \mathbb{C}^{d+1}\):
\[
|D^v_z \Psi_{\alpha,d}(x,z)| \leq \|x\|^{|v|} \exp \left( \|x\| \|I mg(z)\| \right),
\]  
(2.4)

where
\[
D^v_z = \frac{\partial^v}{\partial z_1^{v_1} \ldots \partial z_{d+1}^{v_{d+1}}},
\]
and
\[
|v| = v_1 + v_2 + \ldots + v_{d+1}.
\]

For all \(x, y \in \mathbb{R}^{d+1}\), we have:
\[
|\Psi_{\alpha,d}(x,y)| \leq 1.
\]  
(2.5)

**Definition 2.1** The Weinstein transform is given for \(f \in L^1_{\alpha}(\mathbb{R}^{d+1}, d\nu_{\alpha,d}(x))\) by:
\[
\mathcal{F}^{\alpha,d}_{\omega}(f)(\lambda) = \int_{\mathbb{R}^{d+1}} f(x) \Psi_{\alpha,d}(x,\lambda) d\nu_{\alpha,d}(x), \quad \forall \lambda \in \mathbb{R}^{d+1}.
\]  
(2.6)

We list some known basic properties of the Weinstein transform:

- For \(f \in L^1_{\alpha}(\mathbb{R}^{d+1}, d\nu_{\alpha,d}(x))\), we have:
\[
\|\mathcal{F}^{\alpha,d}_{\omega}(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}.
\]  
(2.7)

- Let \(f \in L^1_{\alpha}(\mathbb{R}^{d+1}, d\nu_{\alpha,d}(x))\) such that the function \(\mathcal{F}^{\alpha,d}_{\omega}(f) \in L^1_{\alpha}(\mathbb{R}^{d+1}, d\nu_{\alpha,d}(x))\), we have the following inversion formula:
\[
f(x) = \int_{\mathbb{R}^{d+1}} \mathcal{F}^{\alpha,d}_{\omega}(f)(y) \Psi_{\alpha,d}(-x,y) d\nu_{\alpha,d}(y).
\]  
(2.8)

- For all \(f, g \in S_{\ast}(\mathbb{R}^{d+1})\), we have the Parseval formula:
\[
\int_{\mathbb{R}^{d+1}} f(x)\overline{g(x)} d\nu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}} \mathcal{F}^{\alpha,d}_{\omega}(f)(\lambda) \overline{\mathcal{F}^{\alpha,d}_{\omega}(g)(\lambda)} d\nu_{\alpha,d}(\lambda).
\]  
(2.9)

- For every \(f \in L^2_{\alpha}(\mathbb{R}^{d+1}, d\nu_{\alpha,d}(x))\), we have the Plancherel formula:
\[
\int_{\mathbb{R}^{d+1}} |f(x)|^2 d\nu_{\alpha,d}(x) = \int_{\mathbb{R}^{d+1}} |\mathcal{F}^{\alpha,d}_{\omega}(f)(\lambda)|^2 d\nu_{\alpha,d}(\lambda).
\]  
(2.10)

**Definition 2.2** The generalized translation operator \(T^{\alpha,d}_{\omega} x, x \in \mathbb{R}^{d+1}\) associated with the Weinstein operator \(\Delta^{\alpha,d}_{\omega}\) is defined for \(y \in \mathbb{R}^{d+1}\) and \(f \in C_{\ast}(\mathbb{R}^{d+1})\) by:
\[
T^{\alpha,d}_{\omega} f(y) = \frac{a_\alpha}{2} \int_0^\pi f \left( x' + y', \sqrt{x'^2_{d+1} + y'^2_{d+1} + 2x_{d+1}y_{d+1} \cos \theta} \right) (\sin \theta)^{2\alpha} d\theta,
\]  
(2.11)

where \(a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi^{\alpha+\frac{1}{2}}}}\).
Definition 2.3 The convolution product of \( f, g \in L^1_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \) is defined for every \( x \in \mathbb{R}^{d+1}_+ \) by:

\[
f \ast_w g(x) = \int_{\mathbb{R}^{d+1}_+} T_x^{\alpha,d} f(y) g(y) d\nu_{\alpha,d}(y).
\] (2.12)

The convolution product satisfies the following properties:

- For all \( f, g \in L^1_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \), \( f \ast_w g \in L^1_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \), and we have:
  \[
  \mathcal{F}^{\alpha,d}_w(f \ast_w g) = \mathcal{F}^{\alpha,d}_w(f) \mathcal{F}^{\alpha,d}_w(g).
  \] (2.13)

- The function \( \Psi_{\alpha,d}(x, \lambda) \) satisfies the following product formula:
  \[
  \forall y \in \mathbb{R}^{d+1}_+, \Psi_{\alpha,d}(x, \lambda) \Psi_{\alpha,d}(y, \lambda) = T_x^{\alpha,d} [\Psi_{\alpha,d}(\cdot, \lambda)](y).
  \] (2.14)

- For every \( f \in L^1_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \),
  \[
  \mathcal{F}^{\alpha,d}_w(T_x^{\alpha,d} f) = \Psi_{\alpha,d}(x, y) \mathcal{F}^{\alpha,d}_w(f)(y).
  \] (2.15)

- Let \( p, q, r \in [1, +\infty) \) such that \( \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \), if \( f \in L^p_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \)
  \( g \in L^q_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \), then \( f \ast_w g \in L^r_\alpha(\mathbb{R}^{d+1}_+; d\nu_{\alpha,d}) \) and we have:
  \[
  \|f \ast_w g\|_{r,\nu_{\alpha,d}} \leq \|f\|_{p,\nu_{\alpha,d}} \|g\|_{q,\nu_{\alpha,d}}.
  \] (2.16)

3 Babenko-type inequality for the Weinstein transform

The following result is a consequence of the central limit theorem \([4]\) to obtain the gaussian measure \( d\mu_{\alpha,d}(x) \), for this we define the Weinstein convolution for two positives and bounded measures \( \gamma_1 \) and \( \gamma_2 \) in \( \mathbb{R}^{d+1}_+ \) and for \( f \in D_*(\mathbb{R}^{d+1}) \) by:

\[
\gamma_1 \ast_w \gamma_2(f) = \int_{\mathbb{R}^{d+1}_+} T_x^{\alpha,d} f(y) d\gamma_1(x) d\gamma_2(y).
\] (3.1)

Inspired by the work of \([2, 3]\), we consider the following measures \( \beta \) and \( \beta_n(x) \) given by:

\[
\beta = \bigotimes_{j=1}^{d+1} \beta_j,
\] (3.2)

with

\[
\begin{cases}
  \beta_j = \frac{1}{2} (\delta_{j-1} + \delta_1) & 1 \leq j \leq d, \\
  \beta_{d+1} = \frac{1}{2} \left( \delta_0 + \delta_{\sqrt{4\alpha+1}} \right),
\end{cases}
\]

and for \( x = (x_1, \ldots, x_d, x_{d+1}) \in \mathbb{R}^{d+1}_+ \); \( n = (n_1, \ldots, n_{d+1}) \in \mathbb{N}^{d+1} \),
\[ \beta_n(x) = \beta(\sqrt{n_1}x_1, \ldots, \sqrt{n_d}x_d) = \otimes_{j=1}^{d+1} \beta_j(\sqrt{n_j}x_j), \quad n_j \in \mathbb{N} \setminus \{0\}. \] (3.3)

The n-fold convolution of \( \gamma_n \) is given by:

\[ \gamma_n = \beta_n \ast_w \cdots \ast_w \beta_n, \quad n = (n_1, \ldots, n_{d+1}) \in \mathbb{N}^{d+1}. \] (3.4)

Then the sequence of measure \{\( \gamma_n \)\} defined in (3.4) converge weakly to\[ d\mu_{\alpha,d}(x) = \frac{x_{d+1}^{2\alpha+1}}{(2\pi)^{d+2} \Gamma(\alpha + 1)} e^{-\frac{\|x\|^2}{2}} dx. \] (3.5)

In what follows we use the standard multi-index notation.\[ m! = m_1! \cdots m_{d+1}!, \quad \text{for} \quad m = (m_1, \cdots, m_{d+1}) \in \mathbb{N}^{d+1}, \]

and\[ x^n = x_1^{n_1} \cdots x_{d+1}^{n_{d+1}}, \quad \text{for} \quad x = (x_1, \cdots, x_{d+1}). \]

### 3.1 Multidimensional Hermite polynomials

We consider the system of multi-dimensional Hermite polynomial (see [13, 14]):\[ H_m^\alpha(x) = \left( \prod_{k=1}^{d} H_{m_k}(x_k) \right) \times \ell_{m_{d+1}}^\alpha(x_{d+1}), \] (3.6)

where \( m = (m_1, \cdots, m_{d+1}) \in \mathbb{N}^{d+1}, \quad x = (x_1, x_2 \cdots x_{d+1}) \in \mathbb{R}^{d+1}, \quad H_{m_k}(x_k) \) are the one-dimensional Hermite polynomial, \{\( k = 1 \cdots d \)\} and \( \ell_{m_{d+1}}^\alpha \) is given by:

\[ \ell_{m_{d+1}}^\alpha(x_{d+1}) = 2^{m_{d+1}} m_{d+1}! L_{m_{d+1}}^\alpha(\frac{x_{d+1}^2}{2}), \]

where \( L_{m_{d+1}}^\alpha(x_{d+1}) \) represents the Laguerre polynomial of index \( \alpha \geq -\frac{1}{2} \).

The polynomials \( H_m^\alpha \) satisfy the orthogonality relation:

\[ \int_{\mathbb{R}^{d+1}} H_n^\alpha(x) H_m^\alpha(x) d\mu_{\alpha,d}(x) = \gamma_m^{-1} \prod_{i=1}^{d+1} \delta_{n_i, m_i}, \] (3.7)

where\[ \gamma_m = \prod_{k=1}^{d} \frac{\gamma_{m_k}}{m_k!}, \] (3.8)

with\[ \gamma_{m_{d+1}} = \frac{(-1)^{m_{d+1}} \Gamma(\alpha + 1)}{m_{d+1}! 2^{m_{d+1}} \Gamma(m_{d+1} + \alpha + 1)}, \] (3.9)

and \( \delta_{n_i, m_i} \) is the Kronecker delta.
Proposition 3.1 Let \( t = (t_1, \cdots t_{d+1}) \in \mathbb{R}^{d+1}_+ \) and \( m = (m_1, \cdots, m_{d+1}) \in \mathbb{N}^{d+1} \), the generating function of \( \{H_m^{\alpha,d}(x)\} \) is given by:

\[
e^{-\frac{\|t\|^2}{2}} \Psi_{\alpha,d}(ix, t) = \sum_{m_1, \cdots, m_{d+1}}^{0,\infty} \gamma_m H_m^{\alpha,d}(x) t^{m_1 \cdots m_{d+1}}, \tag{3.10}
\]

where \( \gamma_m \) is defined in (3.3).

Proof. For \( t \in \mathbb{R}^{d+1}_+ \), we consider the function:

\[
f(t) = e^{-\frac{\|t\|^2}{2}} \Psi_{\alpha,d}(ix, t) = \left[ \prod_{k=0}^{d} e^{-\frac{t_k^2}{2}} e^{x_k t_k} \right] \left[ e^{-\frac{(t_{d+1})^2}{2}} f_{\alpha}(ix_{d+1} t_{d+1}) \right].
\]

Since \( f \) is the product of the \( d+1 \) functions which have developable in integer series then

\[
f(t) = \sum_{m_1, \cdots, m_{d+1}=0}^{\infty} \left[ \frac{\partial^{m_1}}{\partial t_1^m} \cdots \frac{\partial^{m_{d+1}}}{\partial t_{d+1}^m} e^{-\frac{\|t\|^2}{2}} \Psi_{\alpha,d}(ix, t) \right]_{|t|=0} \frac{t_1^{m_1}}{m_1!} \cdots \frac{t_{d+1}^{m_{d+1}}}{m_{d+1}!}.
\]

Now, from the generating functions of \( H_m (x_k) \) and the generating functions of \( \ell_{m_{d+1}} (x_{d+1}) \), we get:

\[
= \prod_{k=0}^{d} \left( \frac{\partial^{m_k}}{\partial t_k^{m_k}} e^{-\frac{t_k^2}{2}} e^{x_k t_k} \right)_{t_k=0} \times \left( \frac{\partial^{m_{d+1}}}{\partial t_{d+1}^{m_{d+1}}}(e^{-\frac{(t_{d+1})^2}{2}} f_{\alpha}(ix_{d+1} t_{d+1})) \right)_{t_{d+1}=0}
\]

\[
= \frac{(-1)^{m_{d+1}} \Gamma(\alpha + 1)}{2^{m_{d+1}} \Gamma(m_{d+1} + \alpha + 1)} \left( \prod_{k=0}^{d} H_m (x_k) \right) \times \ell_{m_{d+1}} (x_{d+1}).
\]

Definition 3.1 For \( |w| < 1 \), we define the integral operator \( K_w(f) \) by:

\[
K_w(f)(y) = \int_{\mathbb{R}^{d+1}_+} K_w(x,y) f(x) d\mu_{\alpha,d}(x), \tag{3.11}
\]

where \( K_w(x,y) \) is the Mehler Kernel-type given by:

\[
K_w(x,y) = \frac{1}{(1 - u^2)^{\alpha+1/2}} e^{\exp \left[ \frac{-w^2 (\|x\|^2 + \|y\|^2)}{2(1 - u^2)} \right]} \Psi_{\alpha,d} \left( \frac{-i w}{1 - u^2}, x, y \right). \tag{3.12}
\]

Proposition 3.2 Let \( n = (n_1, \cdots, n_{d+1}) \in \mathbb{N}^{d+1} \), for \( |w| < 1 \), we have:

\[
K_w(H_n^{\alpha,d})(y) = w^{|n|+n_{d+1}} H_n^{\alpha,d}(y). \tag{3.13}
\]
Proof. From the expression of the Mehler Kernel-type $K_w(x, y)$ defined in (3.12) and the polynomials $H_{m}^{\alpha, d}(x)$ defined in (3.6), we can easily find the result.

**Lemma 3.1** Let $n, m \in \mathbb{N}$, $x = (x_1, x_2, \ldots, x_n)$, $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, the Hermite polynomial satisfies the following relation:

$$H_m(x_1u_1 + \cdots + x_nu_n) = \sum_{s=0}^{m} \frac{2^m m!}{(m-s)!} h_s(x_1(u_1 - 1), \ldots, x_n(u_n - 1)) H_{m-s}(x_1 + \cdots + x_n),$$

where the homogenous symmetric polynomial $h_s(y)$ is given by

$$h_0(y) = 1 \quad \text{and} \quad h_s(y) = \sum_{|\mathbf{k}| = s} \frac{y^k}{k!}; \quad y \in \mathbb{R}^n.$$

**Proof.** For the proof, we refer [8].

**Lemma 3.2** If $x_i = (x_{i1}, \ldots, x_{id+1}) \in \mathbb{R}^{d+1}$ for $1 \leq i \leq n$, satisfies the following conditions:

$$\begin{cases} x_{i1}^2 = x_{i2}^2 = \cdots = x_{i_n}^2 = \frac{1}{n}, & 1 \leq j \leq d, \\ x_{id+1} = \cdots = x_{nd+1} = \frac{4\alpha+4}{nd+1} \text{ or } 0, \end{cases}$$

then, for every $m = (m_1, \ldots, m_{d+1}) \in \mathbb{N}^{d+1}$, such that $d+1 \leq |m| \leq |n|$ we have:

$$\gamma_{m_{d+1}} T_{x_1}^{\alpha, d} \circ T_{x_2}^{\alpha, d} \circ \cdots \circ T_{x_{n-1}}^{\alpha, d} H_m(x_n) = \phi_{m, n}^{\alpha, d}(x_1, \ldots, x_n) + \mathcal{P}_{m, n}^{\alpha, d}(x_1, \ldots, x_n), \quad (3.14)$$

where $\gamma_{m_{d+1}}$ is defined in (3.13), $\mathcal{P}_{m, n}^{\alpha, d}(x_1, \ldots, x_n)$ is a polynomial of degree less than $|m| - 1$ such that

$$\mathcal{P}_{m, n}^{\alpha, d}(x_1, \ldots, x_n) \to 0, \quad n_j \to \infty, \quad \forall \ 1 \leq j \leq d+1,$$

and $\phi_{m, n}^{\alpha, d}(x_1, \ldots, x_n)$ is a homogeneous symmetric polynomial of degree $|m|$ which is defined as follow

$$\phi_{m, n}^{\alpha, d}(x_1, \ldots, x_n) = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \frac{\prod_{k=1}^{d} m_k! \sigma_{n_k, m_k}(x_{1k}, \ldots, x_{nk})}{\prod_{s=0}^{m_{d+1}} \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \frac{2^{m_{d+1}} h_s(x_{1d+1}, \ldots, x_{nd+1}) \sigma_{n_{d+1} m_{d+1} - s}(x_{1d+1}, \ldots, x_{nd+1})}{k!}}.$$

where

$$\sigma_{n_i, l}(x_{i1}, \ldots, x_{in_i}) = \sum_{1 \leq j_1 < \cdots < j_l \leq n_i} x_{j_1} \cdots x_{j_l} \quad 1 \leq i \leq d+1,$$

and

$$h_s(x_{1d+1}, \ldots, x_{nd+1}) = (-1)^s \sum_{|\mathbf{k}| = s} \frac{1}{k!} C_{k_1 \cdots k_{d+1}} \frac{k_1 d+1 \cdots k_{nd+1}}{x_{1d+1} \cdots x_{nd+1}}.$$

with

$$C_{k_1 \cdots k_{d+1}} = 2^{2(n_{d+1})\alpha + |k|} \prod_{i=1}^{n_{d+1}} \frac{\Gamma(\alpha + k_{i(d+1)} + \frac{1}{2})\Gamma(\alpha + 1)}{\pi^{\frac{1}{2}} \Gamma(2\alpha + k_{i(d+1)} + 1)}.$$
Proof. On the one hand, from the product formula for the Weinstein function defined in (2.14) and the relation (3.10), we have:
\[
e^{-\\frac{|t|^2}{2}} \Psi_{\alpha,d}(ix_1, t) \cdots \Psi_{\alpha,d}(ix_n, t) = \sum_{m_1, \ldots, m_{d+1}=0}^{\infty} \gamma_m T_{x_1}^{\alpha,d} \circ T_{x_2}^{\alpha,d} \circ \cdots \circ T_{x_{n-1}}^{\alpha,d} H_m^{\alpha,d}(x_n)^{t_{m_1}} \cdots t_{d+1}^{2m_{d+1}}.
\]
(3.16)

On the other hand, from the relations (2.2) and (2.3) we get:
\[
e^{-\\frac{|t|^2}{2}} \Psi_{\alpha,d}(ix_1, t) \cdots \Psi_{\alpha,d}(ix_n, t) = e^{-\\frac{|t|^2}{2}} e^{<x'_{1,t'}, j_0(i x_{d+1} t_{d+1})} \cdots e^{<x'_{n,t'}, j_0(i x_{n_d+1} t_{d+1})},
\]

then
\[
e^{-\\frac{|t|^2}{2}} \Psi_{\alpha,d}(ix_1, t) \cdots \Psi_{\alpha,d}(ix_n, t)
\]
\[
= a_{n_d+1}(\alpha) \left( \prod_{k=1}^{d} e^{\frac{|t|^2}{2} e^{l_{k} \sum_{j=1}^{n_k} x_{j_k}}} \right) \int_{[-1,1]^{n_{d+1}}} e^{-\frac{t_{d+1}^2}{2} + t_{d+1}(x_{d+1} u_{d+1} + \cdots + x_{n_{d+1}} u_{n_{d+1}})} w_{\alpha}(u) du,
\]
(3.17)

where
\[
w_{\alpha}(u) = \prod_{k=1}^{n_{d+1}} (1 - u_{k_d+1}^2)^{\alpha - \frac{1}{2}} \quad \text{and} \quad a_{n_d+1}(\alpha) = \frac{(\Gamma(\alpha + 1))^{n_{d+1}}}{\pi^{-\frac{n_{d+1}}{2}} (\Gamma(\alpha + \frac{1}{2}))^{n_{d+1}}}.
\]

The generating function for the Hermite polynomials gives:
\[
e^{t(y_{u_1} + \cdots + y_{u_n}) - \frac{t^2}{2}} = \sum_{k=0}^{\infty} H_k(y_{u_1} + \cdots + y_{u_n}) \frac{t^k}{k!}.
\]
(3.18)

From the identities (3.17) and (3.18), we have:
\[
e^{-\\frac{|t|^2}{2}} \Psi_{\alpha,d}(ix_1, t) \cdots \Psi_{\alpha,d}(ix_n, t) = \prod_{k=1}^{d} \left( \sum_{l_k=0}^{\infty} H_{l_k}(x_{1_k} + \cdots + x_{n_k}) \frac{t_{l_k}}{l_k!} \right)
\]
\[
\times \left( a_{n_d+1}(\alpha) \int_{[-1,1]^{n_{d+1}}} \left[ \sum_{l_{d+1}=0}^{\infty} H_{l_{d+1}}(x_{d+1} u_{d+1} + \cdots + x_{n_{d+1}} u_{n_{d+1}}) \frac{t_{l_{d+1}}}{l_{d+1}!} \right] w_{\alpha}(u) du \right).
\]

Equating the coefficients of the power of $t$ in the last relation and (3.16), we obtain:
\[
\gamma_m T_{x_1}^{\alpha,d} \circ T_{x_2}^{\alpha,d} \circ \cdots \circ T_{x_{n-1}}^{\alpha,d} H_m^{\alpha,d}(x_n) = \prod_{k=1}^{d} H_{m_k}(x_{1_k} + \cdots + x_{n_k}) \frac{1}{m_k!}
\]
\[
\times \left[ a_{n_d+1}(\alpha) \int_{[-1,1]^{n_{d+1}}} \frac{1}{2m_{d+1}!} H_{2m_{d+1}}(x_{1_d+1} u_{1_d+1} + \cdots + x_{n_{d+1}} u_{n_{d+1}}) w_{\alpha}(u) du \right].
\]

From lemma 3.1 we get:
\[
a_{n_d+1}(\alpha) \int_{[-1,1]^{n_{d+1}}} \frac{1}{2m_{d+1}!} H_{2m_{d+1}}(x_{1_d+1} u_{1_d+1} + \cdots + x_{n_{d+1}} u_{n_{d+1}}) w_{\alpha}(u) du
\]
\[
\sum_{s=0}^{2m_{d+1}} \frac{2^{2m_{d+1}}}{(2m_{d+1} - s)!} \tilde{h}_s(x_{1,d+1}, \ldots, x_{n_{d+1}}) H_{2m_{d+1}-s}(x_{1,d+1} + \cdots + x_{n_{d+1}}),
\]

where

\[
\tilde{h}_s(x_{1,d+1}, \ldots, x_{n_{d+1}}) = a_{n_{d+1}}(\alpha) \int_{[-1,1]^{n_{d+1}}} h_s(x_{1,d+1}(u_{1,d+1} - 1), \ldots, x_{n_{d+1}}(u_{n_{d+1}} - 1)) w_\alpha(u) du,
\]

\[
= (-1)^s \sum_{|k|=s} \frac{1}{k!} C_{k_{1,d+1}, \ldots, k_{n_{d+1}}} x_{1,d+1}^{k_{1,d+1}} \cdots x_{n_{d+1}}^{k_{n_{d+1}}},
\]

with

\[
C_{k_{1,d+1}, \ldots, k_{n_{d+1}}} = 2^{2(n_{d+1})\alpha+|k|} \prod_{i=1}^{n_{d+1}} \frac{\Gamma(\alpha + k_{i,d+1} + \frac{1}{2})\Gamma(\alpha + 1)}{\pi^{\frac{1}{2}}\Gamma(2\alpha + k_{i,d+1} + 1)}.
\]

We can prove for

\[
\left\{ \begin{array}{l}
x_{1,j}^2 = x_{2,j}^2 = \cdots = x_{n,j}^2 = \frac{1}{n_j}, \quad 1 \leq j \leq d, \\
x_{1,d+1}^2 = \cdots = x_{n,d+1}^2 = \frac{d+1}{n_{d+1}} \text{ or } 0,
\end{array} \right.
\]

that the Hermite polynomial \( H_l(x_{1,i} + \cdots + x_{n_i}) \) for \( 1 \leq i \leq d + 1 \) can be approximated by the elementary symmetric functions \( \sigma_{n_i}(x_{1,i}, \ldots, x_{n_i}), 1 \leq l \leq n_i \), where

\[
\sigma_{n_i}(x_{1,i}, \ldots, x_{n_i}) = \sum_{1 \leq j_1 < \ldots < j_l \leq n_i} x_{j_1} \cdots x_{j_l},
\]

as follows:

\[
H_l(x_{1,i} + \cdots + x_{n_i}) = l! \sigma_{n_i}(x_{1,i}, \ldots, x_{n_i}) + \frac{1}{n_i} \sum_{r=1}^{[n_i]} b_{l,r}(n_i) H_{l-2r}(x_{1,i} + \cdots + x_{n_i}), \quad (3.19)
\]

where \( b_{l,r}(n_i) \) are bounded with respect to \( n_i \) for a fixed \( l \), (see [2, 8]).

Thus we get

\[
\prod_{k=1}^{d} H_{m_k}(x_{1,k} + \cdots + x_{n_k}) = \quad (3.20)
\]

\[
\prod_{k=1}^{d} \left( m_k! \sigma_{n_k,m_k}(x_{1,k}, \ldots, x_{n_k}) + \frac{1}{n_k} \sum_{r=1}^{[m_k]} b_{m_k,r}(n_k) H_{m_k-2r}(x_{1,k} + \cdots + x_{n_k}) \right),
\]

and

\[
\sum_{s=0}^{2m_{d+1}} \frac{2^{2m_{d+1}}}{(2m_{d+1} - s)!} \tilde{h}_s(x_{1,d+1}, \ldots, x_{n_{d+1}}) H_{2m_{d+1}-s}(x_{1,d+1} + \cdots + x_{n_{d+1}})
\]
The restriction  

\[ \phi_{m+1}(x_{1d+1}, \ldots, x_{nd+1}) = \frac{1}{n_{d+1}} P_{m+1}(x_{1d+1}, \ldots, x_{nd+1}), \quad (3.21) \]

where

\[
\phi_{m+1}(x_{1d+1}, \ldots, x_{nd+1}) = \sum_{s=0}^{2m_{d+1}} 2^{2m_{d+1}-s} \overline{h}_s(x_{1d+1}, \ldots, x_{nd+1}) \sigma_{n_{d+1}} 2m_{d+1}-s(x_{1d+1}, \ldots, x_{nd+1}),
\]

and

\[
P_{m+1}(x_{1d+1}, \ldots, x_{nd+1}) = \sum_{s=0}^{2m_{d+1}} \sum_{r=1}^{2m_{d+1}-s} b_{2m_{d+1}-s,r}(n_{d+1}) H_{2m_{d+1}-s-2r}(x_{1d+1} + \ldots + x_{nd+1}).
\]

The result follows from relations (3.20) and (3.21).

### 3.2 Babenko Inequality

This subsection is devoted to establish the Babenko-Bechner-type inequality for the Weinstein transform.

The product measures \( \beta_n(x_1) \ldots \beta_n(x_n) \) defined in (3.3) are discrete, all functions over these measures spaces can be identified as polynomials of the form \( \prod_{k=1}^n P(x_k) \) where

\[ P(x_k) = \prod_{j=1}^d (a_{kj} + b_{kj} x_{kj}) \left(a_{kd+1} + b_{1d+1} x_{kd+1}^2\right); \forall x_k = (x_{k1}, \ldots, x_{kd+1}) \in \mathbb{R}_{+}^{d+1} \]

We define an analogue \( C \) of the multiplier \( K_w \) on the measure space over \( \beta \) defined in (3.2).

\[
K_w : aH_0^\alpha, d + bH_1^\alpha, d \mapsto aH_0^\alpha, d + bw^{d+2} H_1^\alpha, d
\]

\[
C : a + b \left[ \prod_{j=1}^d x_j \{ (2\alpha + 2) - x_{d+1}^2 \} \right] \mapsto a + bw^{d+2} \left[ \prod_{j=1}^d x_j \{ (2\alpha + 2) - x_{d+1}^2 \} \right].
\]

The operator \( C : L^p(\beta) \rightarrow L^q(\beta) \) is bounded, (see [2]).

For \( 1 \leq k \leq n \), define operators:

\[
C_{n,k} = a + b \left[ \prod_{j=1}^d x_{j,k} \{ (2\alpha + 2) - x_{d+1,k}^2 \} \right] \mapsto a + bw^{d+2} \left[ \prod_{j=1}^d x_{j,k} \{ (2\alpha + 2) - x_{d+1,k}^2 \} \right]
\]

where \( a \) and \( b \) are functions of the remaining \( n-1 \) vectors.

and defines

\[
D_{w,n} = C_{n,1} C_{n,2} \ldots C_{n,n}. \quad (3.22)
\]

\( D_{w,n} \) is a bounded linear operator in \( L^p[\beta_n(x_1) \ldots \beta_n(x_n)] \) to \( L^q[\beta_n(x_1) \ldots \beta_n(x_n)] \), (see [2] and [3]).

The restriction \( \overline{D_{w,n}} \) of the operator \( D_{w,n} \) will also be a linear operator. We denote this function space of symmetric functions over the product measures \( \beta_n(x_1) \ldots \beta_n(x_n) \) by \( X_n \).

**Theorem 3.1** Let \( 1 < p \leq 2 \), with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( w = i\sqrt{p-1} \). Then the operator \( K_w \) satisfies the following inequality:

\[
\| K_w(f) \|_{q, \mu, a, d} \leq \| f \|_{p, \mu, a, d}. \quad (3.23)
\]
Proof. Using the relation expressed in (3.14), the polynomials \( T_{x_{k+1}} \circ T_{x_{k+2}} \circ \ldots \circ T_{x_{n-1}} H^\alpha_{m,n}(x_n) \) are approximated by the polynomials \( \phi^\alpha_{m,n}(x_1, \ldots, x_n) \). Since, the operator \( D_{w,n} \) defined in (3.22) acts on \( \phi^\alpha_{m,n}(x_1, \ldots, x_n) \) as:

\[
D_{w,n} \phi^\alpha_{k,n}(x_1, \ldots, x_n) = \phi^\alpha_{k,n}(w x_1, \ldots, w x_n) = w^{|k|+k+1} \phi^\alpha_{k,n}(x_1, \ldots, x_n),
\]

(3.24)

then the natural replacement for \( K_w \) is the operator \( D_{w,n} \).

Let \( g(x) \) be a polynomial, then there exist a vectors \( v_1, \ldots, v_M \) such that:

\[
g(x) = \sum_{l=0}^M v_l H^\alpha_l(x); \quad v_l = (v_{l0}, \ldots, v_{ld+1}).
\]

Put

\[
G_n(x_1, \ldots, x_n) = \sum_{l=0}^M v_l \phi^\alpha_{l,n}(x_1, \ldots, x_n).
\]

By lemma 3.1 we have:

\[
T_{x_1}^\alpha \circ T_{x_2}^\alpha \circ \ldots \circ T_{x_{n-1}}^\alpha (g)(x_n) = \sum_{l=0}^M v_l \phi^\alpha_{l,n}(x_1, \ldots, x_n),
\]

(3.25)

and

\[
T_{x_1}^\alpha \circ T_{x_2}^\alpha \circ \ldots \circ T_{x_{n-1}}^\alpha K_w(g)(x_n) = \sum_{l=0}^M v_l w^{|l|+l+1} \phi^\alpha_{l,n}(x_1, \ldots, x_n). \tag{3.26}
\]

Thus

\[
\lim_{n_i \to \infty} \left( \int_{(\mathbb{R}^d)^{n_i}} \left| T_{x_1}^\alpha \circ T_{x_2}^\alpha \circ \ldots \circ T_{x_{n-1}}^\alpha (g)(x_n) \right|^p d\beta(x_1) \ldots d\beta(x_n) \right)^{\frac{1}{p}} \quad 1 \leq i \leq d + 1
\]

\[
= \lim_{n_i \to \infty} \left( \int_{(\mathbb{R}^d)^{n_i}} \left| G_n(x_1, \ldots, x_n) \right|^p d\beta(x_1) \ldots d\beta(x_n) \right)^{\frac{1}{p}},
\]

and

\[
\lim_{n_i \to \infty} \left( \int_{(\mathbb{R}^d)^{n_i}} \left| T_{x_1}^\alpha \circ T_{x_2}^\alpha \circ \ldots \circ T_{x_{n-1}}^\alpha K_w(g)(x_n) \right|^q d\beta(x_1) \ldots d\beta(x_n) \right)^{\frac{1}{q}}
\]

\[
= \lim_{n_i \to \infty} \left( \int_{(\mathbb{R}^d)^{n_i}} \left| D_{w,n} G_n(x_1, \ldots, x_n) \right|^q d\beta(x_1) \ldots d\beta(x_n) \right)^{\frac{1}{q}},
\]

where \((\mathbb{R}^d)^{n_i} = \prod_{k=1}^{d} \mathbb{R}^{n_i} \times \mathbb{R}^{d+i} \).
The identities (3.4) and (3.5) lead to:

\[
\lim_{n_i \to \infty} \left( \int_{\mathbb{R}^{d+1}} |T_{x_1}^{\alpha,d} \circ T_{x_2}^{\alpha,d} \circ \cdots \circ T_{x_{n-1}}^{\alpha,d} (g)(x_n)|^p \, d\beta_n(x_1) \cdots d\beta_n(x_n) \right)^{\frac{1}{p}} = \lim_{n_i \to \infty} \left( \int_{\mathbb{R}^{d+1}} |g(x)|^p \, d\gamma_n \right)^{\frac{1}{p}},
\]

then

\[
\lim_{n_i \to \infty} \left( \int_{\mathbb{R}^{d+1}} |T_{x_1}^{\alpha,d} \circ T_{x_2}^{\alpha,d} \circ \cdots \circ T_{x_{n-1}}^{\alpha,d} (g)(x_n)|^p \, d\beta_n(x_1) \cdots d\beta_n(x_n) \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^{d+1}} |g(x)|^p \, d\mu_{\alpha,d}(x) \right)^{\frac{1}{p}}. \tag{3.27}
\]

By a similar argument

\[
\lim_{n_i \to \infty} \left( \int_{\mathbb{R}^{d+1}} |T_{x_1}^{\alpha,d} \circ T_{x_2}^{\alpha,d} \circ \cdots \circ T_{x_{n-1}}^{\alpha,d} (K_w g)(x_n)|^q \, d\beta_n(x_1) \cdots d\beta_n(x_n) \right)^{\frac{1}{q}} = \left( \int_{\mathbb{R}^{d+1}} |K_w g(x)|^q \, d\mu_{\alpha,d}(x) \right)^{\frac{1}{q}}. \tag{3.28}
\]

By the relations (3.27) and (3.28), we obtain:

\[
\|K_w g\|_{q,\mu_{\alpha,d}} \leq \|g\|_{p,\mu_{\alpha,d}}.
\]

By density, the result is deduced (see [2]).

**Theorem 3.2 (Babenko Inequality)**

Let \(1 \leq p \leq 2\), \(q = \frac{p}{p-1}\) and let \(A_p = \frac{p}{q} \Gamma(\frac{p}{2})\). If \(f \in L^p(\mathbb{R}^{d+1}; d\nu_{\alpha,d})\), then the Weinstein transform \(F_{\alpha,d}^w(f) \in L^q(\mathbb{R}^{d+1}; d\nu_{\alpha,d})\) and we have:

\[
\|F_{\alpha,d}^w(f)\|_{q,\nu_{\alpha,d}} \leq A_p^{\frac{d}{2} + 1} \|f\|_{p,\nu_{\alpha,d}}. \tag{3.29}
\]

**Proof.** Let \(w = i\sqrt{p-1}\), \(x = \sqrt{pu}\), \(y = \sqrt{qv}\) and \(q = \frac{p}{p-1}\), then the kernel \(k_w(x,y)\) given by the relation (3.12) becomes:

\[
K_w(x,y) = \frac{1}{p^{\alpha+1} + q^{\alpha+1}} \left( \frac{u}{2\pi} \right)^{\frac{d-1}{2} \alpha + 1} \cdot \Psi_{\alpha,d}(u,v).
\]

Using the relation (3.3), we have:

\[
d\mu_{\alpha,d}(x) = e^{-\frac{|u|^2}{2 \sigma^2}} \frac{p^{\alpha+1} + q^{\alpha+1}}{(2\pi)^{\frac{d-1}{2} \alpha + 1} \Gamma(\alpha + 1)} \, du,
\]

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and
\[ d\mu_{\alpha,d}(y) = e^{-\frac{\|y\|^2}{2}} \frac{q^{\alpha+1+\frac{d}{2}} v_d^{2\alpha+1}}{(2\pi)^{\frac{d}{2}2^\alpha \Gamma(\alpha + 1)}} dv. \] (3.32)

Hence, by relations (2.1) and (3.30), we have:
\[
\int_{\mathbb{R}^d} K_w(x, y) g(x) d\mu_{\alpha,d}(x)
\]
\[ = \int_{\mathbb{R}^d} \frac{1}{p^{\alpha+1+\frac{d}{2}}} e^{\frac{(p-1)\|u\|^2}{2}} e^{\frac{\|y\|^2}{2}} \Psi_{\alpha,d}(u, v) g(\sqrt{p}u) e^{-\frac{\|u\|^2}{2}} \frac{p^{\alpha+1+\frac{d}{2}} v_d^{2\alpha+1}}{(2\pi)^{\frac{d}{2}2^\alpha \Gamma(\alpha + 1)}} du
\]
\[ = e^{-\frac{\|y\|^2}{2}} \int_{\mathbb{R}^d} g(\sqrt{p}u) e^{-\frac{\|y\|^2}{2}} \Psi_{\alpha,d}(u, v) d\nu_{\alpha,d}(u). \]

If we put
\[ f(u) = g(\sqrt{p}u) e^{-\frac{\|u\|^2}{2}}, \] (3.33)
then, from the Weinstein transform defined in (2.6), we get
\[
\left| \int_{\mathbb{R}^d} K_w(x, y) g(x) d\mu_{\alpha,d}(x) \right|^q = e^{\frac{q\|y\|^2}{2}} \left| \mathcal{F}_w^{\alpha,d}(f)(v) \right|^q. \] (3.34)

So, by the relations (3.3), (3.32) and (3.34), we obtain:
\[
\|K_w(g)\|_{q,\mu_{\alpha,d}} = \left( \int_{\mathbb{R}^d} \left| K_w(g)(y) \right|^q d\mu_{\alpha,d}(y) \right)^{\frac{1}{q}}
\]
\[ = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_w(x, y) g(x) d\mu_{\alpha,d}(x) \right)^q d\mu_{\alpha,d}(y) \]
\[ = \left( \int_{\mathbb{R}^d} e^{\frac{q\|y\|^2}{2}} \left| \mathcal{F}_w^{\alpha,d}(f)(v) \right|^q d\mu_{\alpha,d}(y) \right)^{\frac{1}{q}}
\]
\[ = \left( \int_{\mathbb{R}^d} e^{\frac{q\|y\|^2}{2}} \left| \mathcal{F}_w^{\alpha,d}(f)(v) \right|^q e^{-\frac{\|y\|^2}{2}} \frac{q^{\alpha+1+\frac{d}{2}} v_d^{2\alpha+1}}{(2\pi)^{\frac{d}{2}2^\alpha \Gamma(\alpha + 1)}} dv \right)^{\frac{1}{q}}
\]
\[ = \left( \int_{\mathbb{R}^d} q^{\alpha+1+\frac{d}{2}} \left| \mathcal{F}_w^{\alpha,d}(f)(v) \right|^q d\nu_{\alpha,d}(v) \right)^{\frac{1}{q}}
\]
we deduce that:
\[
\|K_w(g)\|_{q,\mu_{\alpha,d}} = q^{\frac{\alpha+1+\frac{d}{2}}{q}} \|\mathcal{F}_w^{\alpha,d}(f)\|_{q,\nu_{\alpha,d}}. \] (3.35)
On the other hand, the identities \((3.31), (3.33)\) lead to:

\[
\|g\|_{p,\mu,\alpha,d} = \left( \int_{\mathbb{R}^{d+1}} |g(x)|^p d\mu(\alpha,d(x)) \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^{d+1}} |g(\sqrt{p}u)|^p e^{-\frac{|u|^2}{2}} \frac{p^{\alpha+\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}2^{\alpha}(\alpha + 1)} u^2_{d+1} du \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^{d+1}} p^{\alpha+\frac{d}{2}} |f(u)|^p d\nu_{\alpha,d}(u) \right)^{\frac{1}{p}}.
\]

So, we obtain:

\[
\|g\|_{p,\mu,\alpha,d} = p^{\frac{\alpha+\frac{d}{2}+1}{p}} \|f\|_{p,v,\alpha,d}.
\] (3.36)

Using the relations \((3.23), (3.35)\) and \((3.36)\), we get:

\[
\|F_{\alpha,d}^w(f)\|_{q,\nu,\alpha,d} \leq A_p^{\frac{d}{p}+\alpha+1} \|f\|_{p,\nu,\alpha,d}.
\]

4 Application

Proposition 4.1 (Young’s-type inequality)

Let \(p, q\) and \(r\) three real numbers such that \(1 \leq p, q, r \leq 2\) and \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1\). Then for \(f \in L^p(\mathbb{R}^{d+1}; d\nu_{\alpha,d})\) and \(g \in L^q(\mathbb{R}^{d+1}; d\nu_{\alpha,d})\), we have \(f \ast_w g \in L^r(\mathbb{R}^{d+1}; d\nu_{\alpha,d})\) and

\[
\|f \ast_w g\|_{r,\nu,\alpha,d} \leq \left( \frac{r^\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}}{\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}} \right)^{\alpha+\frac{d}{p}+1} \|f\|_{p,\nu,\alpha,d} \|g\|_{q,\nu,\alpha,d}.
\] (4.1)

where \(p_1 = \frac{p^\gamma}{p-1}, q_1 = \frac{q}{q-1}\) and \(r_1 = \frac{r}{r-1}\).

Proof. By the identities \((2.13), (2.16)\) and \((3.29)\), we have:

\[
\|f \ast_w g\|_{r,\nu,\alpha,d} \leq \left( \frac{r_1^\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}}{\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}} \right)^{\alpha+\frac{d}{p}+1} \|F_{\alpha,d}^w(f \ast_w g)\|_{r_1,\nu,\alpha,d}
\]

\[
\leq \left( \frac{r_1^\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}}{\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}} \right)^{\alpha+\frac{d}{p}+1} \|F_{\alpha,d}^w(f)F_{\alpha,d}^w(g)\|_{r_1,\nu,\alpha,d}
\]

\[
\leq \left( \frac{r_1^\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}}{\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}} \right)^{\alpha+\frac{d}{p}+1} \|F_{\alpha,d}^w(f)\|_{p_1,\nu,\alpha,d} \|F_{\alpha,d}^w(g)\|_{q_1,\nu,\alpha,d}
\]

\[
\leq \left( \frac{r_1^\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}}{\frac{1}{p}p^\frac{1}{p}q^\frac{1}{q}} \right)^{\alpha+\frac{d}{p}+1} \|f\|_{p,\nu,\alpha,d} \|g\|_{q,\nu,\alpha,d}.
\]
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