The method of separation of variables for the Frobenius-Perron operator associated to a class of two dimensional chaotic maps

José-Rubén Luévano
Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana, Unidad Azcapotzalco, CP 02200, México, D.F., México.
E-mail: jrle@correo.azc.uam.mx

Abstract. Analytical expressions for the invariant densities for a class of discrete two dimensional chaotic systems are given. The method of separation of variables for the associated Frobenius-Perron equation is introduced. These systems are related to nonlinear difference equations which are of the type \( x_{k+2} = T(x_k) \). The function \( T \) is a chaotic map of an interval whose chaotic behaviour is inherited to the two dimensional one. We work out in detail some examples, with \( T \) an expansive or intermittent map, in order to expose the method. Finally, we discuss how to generalize the method to higher dimensional maps.

1. Introduction
Explicit computations of invariant densities for higher dimensional maps are seldom found. In contrast, in one dimensional dynamics there are lots of exactly solvable chaotic maps, which can be generated e.g., using the conjugation property [1, 2, 3, 4, 5, 6, 7] or by means of the Schröder method, see [8, 9] and references therein. Recently the problem of continuous iteration has become an interesting topic which can be associated to our treatment [10].

From the statistical point of view, the knowledge of the invariant density of a map is of particular importance because that enables us to obtain with detail some of its statistical properties [6, 8, 11, 12, 13, 14, 15].

The invariant density of a chaotic map is an eigenfunction of the Frobenius Perron operator induced by it [6, 11, 12, 14, 16]. The associated eigenvalue problem is given by a functional equation but there are no known general methods to solve it. The methods used in the field of functional equations have not yet provided a general way to solve them, instead one can find a lot of operational methods adapted to each class of equations, see for example [17, 18]. Therefore, to find reliable methods to obtain these densities becomes a very important task.

In this work a methodology is developed to solve the Frobenius-Perron functional equation, which roughly consists in introducing the method of separation of variables for it. We use several examples to show how it works.

The paper is organized as follows: in section 2 we discuss the Lauwerier map; section 3 introduce the Frobenius-Perron operator and the notion of invariant measure; sections 4 and 5 develops the method of separation of variables for the Frobenius-Perron equation and presents some examples of it; section 6 extend that treatment to some more general discrete systems;
section 7 briefly discuss the relation with the theory of nonlinear difference equations, and finally we give some discussion and conclusions.

2. Lauwerier’s map
We start considering a map introduced by Lauwerier in [19]. For any \((x, y) \in \mathbb{R}^2\), the first order nonlinear system of difference equations:

\[
\begin{align*}
  x_{n+1} &= y_n, \\
  y_{n+1} &= 4x_n(1-x_n),
\end{align*}
\]

(1)
defines a discrete dynamical system for which the unit square \(\Omega = [0, 1] \times [0, 1]\) is invariant, and the dynamics in \(\Omega\) is chaotic. As remarked in that paper, the invariant density for this example can be exactly computed and is given by:

\[
\rho(x, y) = \frac{1}{\pi^2 \sqrt{xy(1-x)(1-y)}}.
\]

(2)

However, no comments about its computation are given. We point out that \(\rho(x, y) = A(x)A(y)\), where \(A(x) = 1/\pi \sqrt{x(1-x)}\). In other words, \(\rho\) is the product of two identical functions \(A\) of a single variable, each one depending on \(x\) and \(y\) independently. The invariant measure \(\mu(x, y) = \int_{(0, 0)} \rho(u, v) \, du \, dv\), is given by

\[
\mu(x, y) = \frac{1}{\pi^2} \arcsin(\sqrt{x}) \arcsin(\sqrt{y}).
\]

(3)

The main purpose of the present paper is to generalize this result to more general two dimensional maps. To this end we are implementing the method of separation of variables for the Frobenius-Perron operator associated to the corresponding transformation.

3. The Frobenius-Perron equation
Let us consider a higher dimensional nonlinear transformations \(F : U \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d\), and denote \(F(X) = (f_1(X), f_2(X), \ldots, f_d(X))\) for \(X = (x_1, x_2, \ldots, x_d) \in U \subseteq \mathbb{R}^d\). We are interested in the study of the discrete dynamical system \(X_{n+1} = F(X_n)\). We assume that \(F\) is, at least, a \(C^1\) function. Also, that there exists a set of functions \(\phi_j : U \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d\), such that \(F(\phi_j(X)) = X\), for \(j = 1, 2, \ldots, d\). In other words, \(\phi_j(X)\) is the \(j\)th branch of the inverse function of \(F\). The Frobenius Perron operator associated to \(F\) is defined by [6, 11, 12, 14, 16]:

\[
\mathcal{L}_F[\rho(X)] = \sum_{F(Y) = X} \frac{\rho(Y)}{|\det D_Y F|},
\]

(4)

where the Jacobian matrix \(D_Y F = [\partial F_i/\partial x_j]\) is evaluated at \(Y\). The scalar function supported on a given \(U \subseteq \mathbb{R}^d\), denoted by \(\rho\), which solves the functional equation

\[
\sum_{F(Y) = X} \frac{\rho(Y)}{|\det D_Y F|} = \rho(X),
\]

(5)
is called the invariant density for \(F\).

4. The method of separation of variables
The method presented in this section is called separation of variables, because of its resemblance with the traditional one used in differential equations. The method is exposed in detail in all the examples given in the following sections. For convenience, in the sections below, we use the standard representation of the Frobenius-Perron equation by means of the Dirac delta function.
4.1. One dimensional maps

We need a brief exposition of the one-dimensional case. Let us start considering a piecewise continuous transformation $T : I \to I$ where the interval $I = [a, b] \subset \mathbb{R}$. Moreover, for a partition of $I$, i.e. $I = \bigcup_j C_j = \bigcup_j [a_j, a_{j+1}]$, where $a = a_0, \ldots, a_k = b$, we assume the map $T$ is monotonous, continuous and onto each $C_j = [a_j, a_{j+1}]$, and also that $T$ is differentiable on each $(a_j, a_{j+1})$.

If $T$ has an invariant density $\rho(x)$, then it must satisfy the Frobenius-Perron equation:

$$L_T[\rho(x)] = \int_I dy \delta(x - T(y))\rho(y) = \rho(x),$$

$$= \sum_{j=1}^r \frac{\rho(T_j^{-1}(x))}{\left| T_j' \circ T_j^{-1}(x) \right|} = \rho(x).$$

In particular, we study some examples defined in terms of the Chebyshev polynomials because these functions considered as maps share nice ergodic properties, not only ergodicity but also stronger ones: mixing and exactness [6]. Notice moreover that these polynomials satisfy the desired properties mentioned above [2, 6, 8].

4.2. Two dimensional maps

For the sake of clarity we begin our exposition with a straightforward generalization of the equation (1). Now, for any $(x, y) \subset \mathbb{R}^2$ we consider the system:

$$x_{n+1} = y_n,$$

$$y_{n+1} = T(x_n),$$

where $T : I \to I$ with $I \subseteq \mathbb{R}$, is a map satisfying conditions stated in the preceding section. Also, we assume that the set $\Omega = I \times I$ is invariant. Now, the Frobenius-Perron equation associated to equation (8) is given by

$$L[\rho(x, y)] = \sum_{j=1}^r \frac{\rho(T_j^{-1}(x, y))}{\left| T_j' \circ T_j^{-1}(y) \right|} = \rho(x, y).$$

We can see that this functional equation can be solved by the ansatz $\rho(x, y) = A(x)B(y)$, whereby

$$L[\rho(x, y)] = L_T[A(y)]B(x) = A(x)B(y).$$

The method of separation of variables for the equation (9) works as follows:

$$\frac{L_T[A(y)]}{B(y)} = \frac{A(x)}{B(x)} = c,$$

where we denote by $c$ the constant of separation, such that

$$L_T[A(y)] = cB(y),$$

$$A(x) = cB(x).$$

Note now that this is a system of two coupled functional equations. To proceed to decoupling them we make the substitution of the second equation in the first one. Hence, we obtain $L_T[A(y)] = A(y)$. In other words, if the map $T$ has an invariant density $A$, we must finally have

$$\rho(x, y) = \frac{1}{c}A(x)A(y).$$
The constant \( c \) is obtained by normalization:

\[
\int_{\Omega} \rho(x, y) dxdy = 1
\]

hence,

\[
c = \left[ \int_I A(x) dx \right]^2.
\] (14)

4.3. Analytical examples

In this section we provide three exactly solvable examples of systems like equation (8).

4.3.1. Chebyshev maps I

If the transformation \( T \) in equation (8) is given by the \( k \)-th Chebyshev polynomial, \( P_k(x) \), then as it is well known, the invariant density for this map is given by

\[
\rho(x) = \frac{1}{\pi \sqrt{1-x^2}}.
\]

Hence, by the equation (12) the invariant density for the corresponding two dimensional system is given by:

\[
\rho(x, y) = \frac{1}{\pi^2 \sqrt{(1-x^2)(1-y^2)}},
\] (15)

where the support for the invariant measure is the set \( \Omega = [-1, 1] \times [-1, 1] \).

4.3.2. Intermittency

An example of a map with a marginally stable fixed point is the cusp-shaped map \( T(x) = 1 - 2\sqrt{|x|} \). It is well known that this map has an invariant density \( \rho(x) = (1-x)/2, x \in [-1, 1] \), see [4]. Therefore, the corresponding system, given by equation (8) has an invariant density

\[
\rho(x, y) = (1-x)(1-y)/4.
\] (16)

for any \((x, y) \in \Omega = [-1, 1] \times [-1, 1]\).

4.3.3. Transformations of the plane onto itself

From Newton’s method to search the roots of the equation \( x^2 + 1 = 0 \), [20, 21, 22], a chaotic iteration of the real line is found \( T(x) = (x - 1/x)/2 \). Also, its invariant density is known to be \( \rho(x) = 1/(\pi(1 + x^2)) \). Therefore the corresponding system, equation (8), has an invariant density

\[
\rho(x, y) = \frac{1}{\pi^2 \sqrt{(1+x^2)(1+y^2)}},
\] (17)

where \((x, y) \in \Omega = \mathbb{R} \times \mathbb{R}\).

5. A further generalization

Now we consider two different transformations, \( S : J \to J \) and \( T : I \to I \), both of them satisfying conditions stated in section (4.1), but each one having \( r \) and \( s \) monotone pieces respectively. Therefore, \( T^{-1} \) and \( S^{-1} \) have \( r \) and \( s \) monotone branches on their respective intervals. Then, defining a discrete dynamical system on \((x, y) \in \Omega = I \times J\) by

\[
\begin{align*}
x_{n+1} &= S(y_n), \\
y_{n+1} &= T(x_n),
\end{align*}
\] (18)

the associated Frobenius Perron operator is given by

\[
\mathcal{L}[\rho(x, y)] = \int_{\Omega} dudv \delta(x - S(v))\delta(y - T(u))\rho(u, v).
\] (19)
Now, using the definition of a higher dimensional $\delta$ function \[\], we have
\[
L[\rho(x, y)] = \sum_{j=1}^{r} \sum_{l=1}^{s} \frac{\rho(T^{-1}_j(y), S^{-1}_l(x))}{\big| T' \circ T^{-1}_j(y) \big| \big| S' \circ S^{-1}_l(x) \big|}.
\]
(20)

By analogy with to the previous section we introduce the ansatz $\rho(x, y) = A(x)B(y)$, which allows us to have
\[
L[\rho(x, y)] = L_T[A(y)]L_S[B(x)] = A(x)B(y).
\]
(21)

This time the method of separation of variables gives us
\[
\frac{L_S[B(x)]}{A(x)}\frac{L_T[A(y)]}{B(y)} = 1,
\]
(22)
then, introducing a constant $c$, it provides the following system
\[
L_T[A(y)] = cB(y),
L_S[B(x)] = \frac{1}{c}A(x).
\]
(23)

Now, this system can be decoupled giving us $L_{SOT}[A(x)] = A(x)$ and $L_{TOS}[B(y)] = B(y)$. In other words, the system (23) is decoupled in two functional equations of a single variable. More important, this time the functions $A(x)$ and $B(x)$ are (if they exists) the invariant densities of $S \circ T$ and $T \circ S$ respectively. We finally obtain
\[
\rho(x, y) = A_{SOT}(x)B_{SOT}(y).
\]
(24)

5.1. Examples

The following two examples are related to the last discussed case.

5.1.1. Chebyshev maps II

As stated before, the Chebyshev maps are a very important examples among the class of integrable chaotic maps. Also, it is well known that all the Chebyshev maps share the same invariant density $A(x) = 1/(\pi\sqrt{1-x^2})$. Now we consider the following system
\[
x_{n+1} = P_k(y_n),
y_{n+1} = P_l(x_n),
\]
(25)
where $P_k$ and $P_l$ are the Chebyshev polynomials of $k$ and $l$ degrees respectively. From the semigroup property $P_l \circ P_k = P_k \circ P_l = P_{kl}$, we have that $L_{P_{kl}}[A(x)] = A(x)$. Therefore
\[
\rho(x, y) = \frac{1}{\pi^2\sqrt{(1-x^2)(1-y^2)}}.
\]
(26)
for any $(x, y) \in \Omega = [-1, 1] \times [-1, 1]$. We show a plot of this density in figure (1).

5.1.2. Cournot maps

Some mathematical models in economy are known as Cournot duopoly games [23, 24]. They are described by on a system consisting of two coupled logistic maps $F_r(x) = r x(1-x)$:
\[
x_{n+1} = r y_n(1-y_n),
y_{n+1} = s x_n(1-x_n),
\]
(27)
where $r, s > 0$ and $(x, y) \in [0, 1] \times [0, 1]$. We restrict our analysis to the case $r = s = 4$. It is well known that $F_4(x) = 4x(1 - x)$ is conjugated to $N_4(x) = (-1)^{[2x]}\{2x\}$. Here $[rx]$ denotes the integer part of $rx$ and $\{rx\}$ means $rx \mod 1 = rx - n$ where $n$ is the largest integer such that $rx - n \geq 0$ for any $r \in \mathbb{N}$. Hence, the map $F_4 \circ F_4(x) = F_4^{(2)}(x)$ is conjugated to $N_4(x) = (-1)^{[4x]}\{4x\}$, so that both maps share the same invariant density $A(x) = 1/\pi \sqrt{x(1-x)}$. To be more specific, $F_4(\sin^2(\theta)) = \sin^2(2\theta)$ then $F_4^{(2)}(\sin^2(\theta)) = \sin^2(4\theta)$. Because the invariant measure for $F_4$ is given by $\mu(x) = (2/\pi) \arcsin(\sqrt{x})$, therefore $\mathcal{L}_{F_4 \circ F_4}[A(y)] = \mathcal{L}_{F_4^{(2)}}[A(y)] = A(y)$, and finally

$$
\rho(x, y) = \frac{1}{\pi^2 \sqrt{xy(1-x)(1-y)}}. \tag{28}
$$

6. General maps

Our method is not restricted to the class of systems strictly given in the form of equations (8) and (18). Of course, it is possible to find new exactly solvable maps which are equivalent to these systems by means of a change of variables. In the following we give an example showing how this works. A nontrivial example which is studied in [2] is provided by

$$
x_{n+1} = (y_n^2 - x_n - 1)^2 - (y_n^2 + x_n - 1)^2,
$$
$$
y_{n+1} = \sqrt{(y_n^2 - x_n - 1)^2 + (y_n^2 + x_n - 1)^2}. \tag{29}
$$

In that reference the authors shows that this system is exactly solvable one. After making the change of variables (this one is different from that employed in [2]) $x_n = (\eta_n - \xi_n)/2$ and

![Figure 1. The invariant density for the map given by equation (25), $\rho(x, y) = \frac{1}{\pi^2 \sqrt{(1-x^2)(1-y^2)}}$.](image)
Figure 2. The invariant density for the map given by equation (29). It is noted the sharp singularities at $(\pm 1, \pm 1)$. \[ \rho(x, y) = \frac{4y}{\sqrt{(2-(y^2-x))(y^2-x)(2-(y^2+x))(y^2+x)}}. \]

After some algebra, we find that the domain of that density is given by the set $\Omega = \{(x, y) \in \mathbb{R}^2\}$ such that the following inequalities (also given in [2])

\[ y^2 - x < 2, \ y^2 - x > 0, \ y^2 + x < 2, \ y^2 + x > 0, \]  

are valid. Hence, this is the invariant set for the system. Since these inequalities are satisfied for $y > 0$, the absolute value in equation (32) becomes irrelevant. A plot of the associated invariant density is given in figure (2).
7. Delayed nonlinear difference equations
In recent times the study of nonlinear difference equations has become an intensive area of study from the point of view of dynamical systems [24, 25]. It is worth mentioning that the system (8) is equivalent to the nonlinear second-order difference equation of the form:

\[ x_{n+2} = T(x_n). \]  

(35)

We point out that the change of variables \( x_{n+1} = y_n \) gives us \( y_{n+1} = T(x_n) \), i.e. the two dimensional system given by equation (8). A straightforward generalization is given by the delayed difference equation:

\[ x_{n+k} = T(x_n) \]  

(36)

where \( k \in \mathbb{N} \). We can see that equation (36) becomes equivalent to a \( k \)-dimensional first order system which is a generalization of the equation (8).

8. Discussion and Conclusions
We have given a generalization of a discrete two dimensional chaotic system studied by Lauwerier some years ago. For these systems the corresponding Frobenius Perron equation is solved by a method of separation of variables. In this way we can transform a two dimensional system into 2 one-dimensional systems. Moreover, if the one-dimensional systems are chaotic, then this property is inherited to the two dimensional one. Several examples were worked out in detail to describe the method. The Chebyshev polynomials and logistic maps play an important role in our study, but others maps like the Jacobi sine function and rational functions could be considered, see [3, 5, 7].

Also, we related these systems to some class of nonlinear delayed difference equations. This relation can be exploited to build higher dimensional maps which can be solved by our method. Moreover, it can be interesting to apply our method to coupled maps lattices.

In this work we were only concerned with the computation of invariant densities. Some important topics in ergodic theory were not addressed, for example mixing and exactness properties. In particular, it will be interesting to explore the spectral theory of the Frobenius-Perron operator for the systems worked out here. Moreover, there is still the question about the possibility to give explicit expressions for the generalized spectra as studied in [29]. Some progress in such direction will be reported elsewhere.

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