A scheme for amplification and discrimination of photons

A R Usha Devi\(^1,2\), R Prabhu\(^1,3\) and A K Rajagopal\(^2,4\)

\(^1\) Department of Physics, Bangalore University, Bangalore-560 056, India
\(^2\) Inspire Institute Inc., McLean, VA 22101, USA
\(^3\) Department of Physics, Kuvempu University, Shankaraghatta, Shimoga-577 451, India
\(^4\) Center for Quantum Studies, George Mason University, Fairfax, VA 22030, USA

E-mail: arutth@rediffmail.com

Received 31 July 2008, in final form 24 September 2008
Published 19 November 2008
Online at stacks.iop.org/JPhysB/41/235501

Abstract

A scheme for exploring photon number amplification and discrimination is presented based on the interaction of a large number of two-level atoms with a single-mode radiation field. The fact that the total number of photons and atoms in the excited states is a constant under time evolution in the Dicke model is exploited to rearrange the atom–photon numbers. Three significant predictions emerge from our study: threshold time for initial exposure to photons, time of perception (time of maximum detection probability) and discrimination of first few photon states.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A scheme for investigating amplification and discrimination of input photons is presented based on the interaction of a large number of identical two-level atoms and low-intensity single-mode radiation. This analysis provides a deeper understanding of the cumulative response of millions of atoms to few input photons within the basic model [1] of atom–field interaction.

Historically, collective behaviour of \(N\) two-level atoms with single-mode radiation, investigated by Dicke [1] in 1954, has led to a vast array of interesting physical phenomena in quantum optics and recently in artificial condensed matter systems [2]. The present investigation is an entirely novel application of the Dicke model. The fact that the sum of the number of photons and the number of atoms in the excited states is a constant under time evolution has a natural consequence of rearranging the atom–photon numbers. A projective measurement on the temporally evolving combined multiatom–radiation system is shown to lead to both these features.

2. The physical model

The Hamiltonian characterizing the interaction of \(N\) two-level atoms with a single-mode radiation is given by (\(\hbar = 1\))

\[
H = \omega(a^\dagger a) + \omega_0 S_z + \frac{g}{\sqrt{N}}(S_+ a + S_- a^\dagger),
\]

(1)

where \(g\) denotes the atom–photon coupling parameter, \(\omega_0\), the atomic splitting and \(\omega\) the field frequency; \(a^\dagger(a)\) are creation (annihilation) operators of the field satisfying the bosonic commutation relations \([a, a^\dagger] = 1\). The collective (pseudo) spin operators of the two-state atoms,

\[
S_\pm = S_x \pm iS_y = \frac{1}{2} \sum_{\alpha=1}^{N} \sigma_{\alpha \pm}, \quad S_z = \frac{1}{2} \sum_{\alpha=1}^{N} \sigma_{\alpha z},
\]

(2)

obey the commutation relations

\[
[S_+, S_-] = 2S_z, \quad [S_z, S_\pm] = \pm S_\pm.
\]

(3)

It is well known [3] that the excitation operator

\[
N_{ex} = a^\dagger a + S_z + \frac{N}{2}
\]

(4)

remains constant as the system evolves under the Hamiltonian (1). The degenerate atom–photon states \(|S = \frac{N}{2}, M = \pm \frac{N}{2}\rangle\)
in terms of bosonic operators and photon amplification in this model. The symmetric atomic Dicke states \( |S = \frac{N}{2}, M = n_e - \frac{N}{2}\rangle \) (except for \( n_e = 0 \) and \( N \)) are well known for their entanglement properties [4–6]. These are essential in the ensuing discussion. The collective ground state of the atomic system is denoted by \(|0\rangle = \left| \frac{N}{2}, -\frac{N}{2}\right\rangle\). A given initial state of the combined atom–radiation system, \( \rho_{AR}(0) \), evolves to

\[
\rho_{AR}(t) = e^{-iHt} \rho_{AR}(0) e^{iHt}. \tag{5}
\]

Consequently, a projection operator \( \Pi_{A0} \otimes I_R \), with \( \Pi_{A0} = |0\rangle \langle 0 | \) and \( I_R \) the unit operator in the radiation space, gives us the conditional density matrix of the collective atom–photon system (where all the atoms are projected to their collective ground state), subjected to the constraint (4). This projective measurement has the implication that a maximum number of photons are emitted consequently. 3. Photon amplification with different initial states

3.1. Pure atom–photon state \( |n_e; n\rangle \)

Let us consider an initial atom–photon state,

\[
|S = \frac{N}{2}, M = n_e - \frac{N}{2}\rangle \otimes |n\rangle \equiv |n_e; n\rangle, \tag{6}
\]

with \( n_e \ll N \), and the number of atoms \( N \) sufficiently large so that the Holstein–Primakoff mapping [7],

\[
S_+ = b^\dagger \sqrt{N - b b^\dagger}, \quad S_- = \sqrt{N - b^\dagger b}, \quad S_z = b^\dagger b - \frac{N}{2}, \tag{7}
\]

in terms of bosonic operators \( b, b^\dagger \) satisfying

\[
[b, b^\dagger] = 1, \tag{8}
\]

reduces the Hamiltonian (1) into a two-mode bosonic interaction structure [2] (up to the order \( O(1/N) \)):

\[
H \sim \omega N u_x = -\frac{N}{2} + g(b^\dagger a + b). \tag{9}
\]

Temporal evolution in this approximation thus assumes the form

\[
U(\tau) = e^{-iH\tau} = e^{-i(t^2/2N, -\frac{N}{2})} e^{-i(b^\dagger a + ba^\dagger)/2}. \tag{10}
\]

(Here we have denoted \( g(t) = \tau \). It may be noted that \( U(\tau) \) acts as a passive unitary beam splitter on the two-mode bosonic Fock states \( |n_e; n\rangle \). Thus, initial states of the form (6) get confined within the space spanned by the \((n_e + 1)\) basis states \(|n_e'; n'\rangle \), with \( n_e', n' = 0, 1, 2, \ldots \) such that \( n_e' + n' = n_e + n \) under time evolution. It is interesting to note that at scaled time \( \tau = \pi/2 \) the unitary operator \( U(\tau) \) swaps [8] the atom–photon numbers i.e., \( |n_e; n\rangle \rightarrow |n; n_e\rangle \) with unit probability.

In general, we have

\[
U(\tau)|n_e; n\rangle = e^{-i(t^2/2N, -\frac{N}{2})} \sum_{n_e', n'} |n_e', n'\rangle |\psi_{n_e, n}^{n_e', n'}\rangle, \tag{11}
\]

where

\[
d_{m,m'}^{n_e,n}(\tau) = \sum_k (j+m)!/(j-m)! (j+m')!(j-m')! (-1)^{j-m+m'} 
\times (\cos^2\tau)^{j-2k+m-m'} (\sin^2\tau)^{2k-m+m'} = d_{m,m'}^{n_e,n}(\tau), \tag{12}
\]

with the sum over \( k \) taken such that none of the arguments of the factorials in the denominator are negative.

A measurement \( \Pi_{A0} \otimes I_R \) at an instant \( t \) projects all the atoms to the ground state:

\[
\Pi_{A0} \otimes I_R U(\tau)|n_e; n\rangle = 0, \quad n_e' \neq n_e, \quad n' \neq n, \quad \text{and for photon amplification} \quad n \rightarrow n + n_e, \quad \text{is given by}
\]

\[
\mathcal{P}(n_e, n, \tau) = \left( \frac{n + n_e}{n_e} \right) \cos^{2n}(\tau) \sin^{2n}(\tau). \tag{14}
\]

In figures 1(a)–(c), we have displayed the probability of maximum photon emission for three choices of \( n_e = 1, 10 \) and 25, each with different values for the number of photons \( n = 0, 1, 5, 10 \), as a function of scaled time \( \tau \). For \( n = 0 \) i.e., the dark photon input state, the probability has a single peak\(^7\) in the range \( 0 \leq \tau \leq \pi \) with a maximum value \( \mathcal{P}(n_e, 0, \tau = \pi/2) = 1 \), which corresponds to swapping

\(^5\) The Fock states \(|n\rangle, n = 0, 1, 2, \ldots \) denote eigenstates of the photon number operator \( a^\dagger a \) with eigenvalue \( n \), and the Dicke states \(|S = \frac{N}{2}, M\rangle\) are joint eigenstates of \( S^+ = S_+^2 + S_0^2 + S_z^2 \) and \( S_+^0 \) with eigenvalues \( \frac{N}{2}(2\ell + 1), M \), respectively.

\(^6\) Dicke states with a specific permutation symmetry are chosen here because of the greatest simplicity offered by them in their theoretical analysis and also due to their current experimental relevance [4,5].

\(^7\) The probability \( \mathcal{P}(n_e, n, \tau) \) exhibits a repetitive structure with a period \( \pi \), as is evident from (14).
of atom–photon numbers $|n_c; 0⟩ → |0; n_c⟩$. However, when $n ≠ 0$ the projective measurement $\Pi_{A0} ⊗ I_B$ leads to $|n_c; n⟩ → |0; n + n_c⟩$ corresponding to maximum photon emission; this is not a complete swap action and occurs with the probability $0 ≤ \mathcal{P}(n_c, n ≠ 0, τ) < 1$. Clearly, the probability $\mathcal{P}(n_c, n ≠ 0, τ)$ vanishes at scaled time $τ = π/2$, in order to give way to complete swap action $|n_c; n⟩ → |n; n_c⟩$. The probability profile for maximum photon emission $n → n + n_c$ therefore reveals two peaks around $τ = π/2$ for $n ≠ 0$. The maxima of probabilities corresponding to different input photon numbers are well separated, and they appear periodically at $τ_n = $ arccos $\sqrt{n/n_c}$, allowing for photon number discrimination.

We define the time of perception $τ_p(n_c, n)$, for a given $n_c$ and $n$, as the time at which the probability $\mathcal{P}(n_c, n, τ)$ attains its maximum. This determines the efficient detection of the photons. An examination of these figures reveals: (i) for a given $n_c$, the time of perception $τ_p(n_c, n_c)$ reduces as more and more photons are detected—with less and less efficiency. Moreover the widths in the probabilities $\mathcal{P}(n_c, n, τ)$ reduce correspondingly. (ii) As $n_c$ increases, there is a threshold time for the detection of photons, before which the probabilities are zero. In particular, for $n_c = 1$, there is an instant response to photons (for all $n$) as seen in figure 1(a), whereas for higher values of $n_c$ there is a delay in such a response (see figures 1(b) and (c)). (iii) For a given photon number $n$, the profile of probability as a function of time sharpens as $n_c$ increases; however, the maximum value of the probability drops with this, as is evident from figure 1.

### 3.2. Pure state of atoms with low-intensity coherent radiation

The above discussion was confined to pure photon number states. We now show that an enhanced photon amplification behaviour is realized, when initially a low-intensity coherent state of radiation,

$$|α⟩ = e^{-|α|^2/2} \sum_{n=0}^{∞} \frac{α^n}{\sqrt{n!}} |n⟩,$$

$$|α|^2 < 1,$$  \hspace{1cm} (15)

is considered as input. The probability of finding the atoms in the ground state is obtained by following the procedures given in (11) and (13):

$$\mathcal{P}(n_c, α, τ) = e^{-|α|^2} \sum_{n=0}^{∞} \mathcal{P}(n_c, n, τ) \frac{|α|^{2n}}{n!}.$$

The probability profile (16)—with coherent radiation as input—is a series involving individual atom–photon Fock state probabilities $\mathcal{P}(n_c, n, τ)$, dark photon probability $\mathcal{P}(n_c, 0, τ)$ being the leading term for the low value of intensity $|α|^2$. Thus, the probability response to low-intensity coherent light has a similar structure as that of dark photon probability $\mathcal{P}(n_c, 0, τ)$ (see figure 1) with small contribution from higher-order terms $|α|^{2n}, n = 1, 2, \ldots$. It may be seen from figure 2 that the probability $\mathcal{P}(n_c, α, τ)$ is found to be nearly zero in the beginning and raises to a maximum at $τ = π/2$ (peak value being $\mathcal{P}(n_c, α, π/2) \sim e^{-|α|^2}$). An increase in the initial intensity of coherent radiation has the effect of reducing the probability of finding the atoms in their ground state. Amplification of the intensity of radiation,
Figure 2. Probability $P(n_e, \alpha, \tau)$ of finding the atoms in the ground state as a function of scaled time $\tau$ for different values of initial intensity $|\alpha|^2$ and for different choices of $n_e$, the initial number of atoms in the excited state.

Figure 3. Probability of finding the atoms in the ground state under time evolution governed by the Hamiltonian (9), when the initial atomic state is chosen as (i) a mixed state

$$\rho_{atom}=\frac{1}{n_e+1}\sum_{m=0}^{n_e}|m\rangle\langle m|, \quad n_e \ll N,$$

along with low-intensity coherent radiation. The probability of finding the atoms in the ground state under time evolution in this model is readily found to be

$$P(\rho_{atom}, \alpha, \tau) = \frac{e^{-|\alpha|^2}}{n_e+1} \sum_{m=0}^{n_e} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} P(m, n, \tau).$$

The probability $P(\rho_{atom}, \alpha, \tau)$ as well as $P(n_e, \alpha, \tau)$ (see equation (16)) associated with a pure atomic state $|n_e\rangle$ is plotted, as a function of scaled time $\tau$, in figures 3(a) and (b) for two different choices of the initial intensity of radiation for a fixed $n_e = 25$. We observe that while the maximum probability remains the same for a given initial intensity of radiation $|\alpha|^2$, the mixed states $\rho_{atom}$ result in a wider spread around the maximum value (at $\tau = \pi/2$) compared to their pure state counterparts. Also, the probability $P(\rho_{atom}, \alpha, \tau)$ builds above the background value $\frac{1}{n_e+1}$. Moreover, as the intensity of
the radiation $|\alpha|^2$ increases, the maximum probability drops down. It may be noted that the photon amplification factor $I(t)/I(0)$ approaches the value $\frac{n_e}{2|\alpha|^2}$ as $\tau \to \pi/2$ for mixed atomic states $\rho_{\text{atom}}$, in contrast to its corresponding value $\frac{n_e}{|\alpha|^2}$ in the case of pure states $|n_e\rangle$.

4. Summary

We have presented a scheme for photon amplification and discrimination based on the Dicke model interaction between single-mode radiation and the $N$-atom system. This involves the Holstein–Primakoff large-$N$ approximation [7], where the number $n_e$ of atoms in excited states is assumed to be much smaller than the total number $N$ of atoms. Further, this approximation enforces that the number $n$ of input photons is small compared to $N$. This leads to a beam-splitter feature for time evolution under the interaction Hamiltonian (9), thus ensuring entanglement between the atoms in collective excited states and the photons [10]. Emission of maximum number of photons corresponds to a completely uncorrelated atom–radiation system, with all the atoms in the ground state, and this is achieved via a projective measurement at a suitable interval of time. Concepts such as threshold time and time of perception for exposure to low-intensity light emerge as characteristic features of our investigation. Discrimination of a small number of photons is also realized in this scheme. This provides motivation for a quantum mechanical analysis of the collective response of millions of rods in the eye, which act as nearly perfect photon detectors, initiating the associated process of night vision [11–13].

Acknowledgments

We thank the referees for their insightful comments, which improved the presentation of our work in this revised version. We also thank Dr C Ramakrishna for introducing us to the biology of vision and for several useful discussions on our model.

References

[1] Dicke R H 1954 Phys. Rev. 93 99
[2] Brandes T 2005 Phys. Rep. 408 315
[3] Tavis M and Cummings F W 1967 Phys. Rev. 170 379
[4] Thiel C, von Zanthier J, Bastin T, Solano E and Agarwal G S 2007 Phys. Rev. Lett. 99 193602
[5] Bastin T, Thiel C, von Zanthier J, Lamata L, Solano E and Agarwal G S 2007 arXiv:0710.3720
[6] Usha Devi A R, Prabhu R and Rajagopal A K 2007 Phys. Rev. A 76 012322
[7] Holstein T and Primakoff H 1940 Phys. Rev. 58 1098
[8] Wang X 2001 J. Phys. A: Math. Gen. 34 9577
[9] Sakurai J J 1999 Modern Quantum Mechanics (New York: Addison-Wesley) p 223
[10] Wang X B 2002 Phys. Rev. A 66 024303
[11] Kandel E R, Schwartz J H and Jassel T M 2000 Principles of Neural Science (New York: McGraw-Hill)
[12] Ramakrishna C, Rajagopal A K and Usha Devi A R 2008 Proc. Indo-US Workshop on Science and Technology at the Nano-Bio Interface (19–22 February 2008, Bhubaneswar, India) (arXiv:0804.0190)
[13] Rieke F and Baylor D A 1998 Rev. Mod. Phys. 70 1027