Composite bosons description of rapidly rotating Bose-Einstein condensates

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Abstract

Recently a novel phase transition has been observed \(^1\) in a rotating Bose-Einstein condensate when the rotating frequency \(\Omega\) reaches the transverse trap frequency \(\omega_\perp\) and eventually crosses it. We study certain aspects of this experiment in terms of the condensation of composite bosons and the corresponding vortex formation using a Chern-Simon Gross-Pitaevskii theory.

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I. INTRODUCTION

Following the observation of a regular array of vortices in a rotating Bose-Einstein condensates (BEC) [2], there have been a great amount of theoretical work [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and also some further experimental development [16, 17] to understand the behaviour of the condensate when it is rotated even faster. Apart from the fate of the vortex lattice submitted to a faster rotation, the interest stems also from the possibility of observing Laughlin-liquid like states in the condensate when the rotational frequency becomes almost equal to the transverse trap frequency ($\Omega \approx \omega_{\perp}$). Since the number of atoms in the rotating condensates is very large, it has been pointed out [7, 18, 19] that one can still use the Thomas-Fermi (TF) approximation where the kinetic energy is neglected compared to the interaction energy, even though the system is very close to the Landau level limit. Within the TF approximation it has been shown [7, 18] that the vortex lattice may survive when $\Omega$ is very close to $\omega_{\perp}$. Ho [8] reached the same conclusion using the Lowest-Landau level (LLL) approximation where the kinetic energy term is also frozen by the LLL constraint and thereby it is equivalent in this respect to the TF approximation. In a subsequent development Sinova et. al. [9] analyzed the Bogoliubov spectrum in the LLL limit and reached the conclusion that quantum fluctuations melt the vortex lattice. The same conclusion was reached by Wilkin et. al. [4] using exact particle diagonalization technique in a toroidal geometry. Fishcher et. al. [7] have addressed the same problem using an effective free energy for the interacting vortices in a rotating frame and provided a phase diagram consisting of regions of Laughlin liquid of bosons, Laughlin liquid of vortices and vortex lattice.

In a very recent experiment at ENS, Bretin et. al. [1] reached the Landau level (LL) limit of a BEC. They superimpose an additional quartic confining potential above the usual quadratic confining potential in the transverse plane, thereby eliminating the possibility of centrifugal instability in the limit $\Omega = \omega_{\perp}$. This experiment therefore reaches a regime where the TF approximation does not hold anymore. The vortex lattice is stable even at a very high value of the rotational frequency $\Omega$ (almost 98 percent of $\omega_{\perp}$). As $\Omega$ is further increased and eventually taken just beyond $\omega_{\perp}$, one obtains on both sides of $\Omega = \omega_{\perp}$ a blurred image of the vortex lattice indicating a phase transition. As $\Omega$ is further increased beyond $\omega_{\perp}$ a condensate like region reappears in a central region of the blurred image and again new
vortices start nucleating. At slightly higher $\Omega$, one recovers a collection of few such vortices again arranged in a regular array.

It should be pointed out that in most cases the rotational frequency $\Omega$ is equal to the stirring frequency $\Omega_{\text{stir}}$. However, in the ENS experiment $\Omega_{\text{stir}}$ is not always equal to $\Omega$. Though in most cases the two are almost equal, it has been pointed out that beyond $\Omega = \omega_\perp$ it is not correct anymore. Particularly at the highest reached value of $\Omega_{\text{stir}}$ at which a vortex lattice becomes clearly visible again, the frequency $\Omega$ is predicted to be much lower than $\Omega_{\text{stir}}$. However, it has been clearly demonstrated that when the rotational frequency of the condensate equals or exceeds $\omega_\perp$, the number of vortices in the condensate is much lower than expected at such a high $\Omega$.

In the present paper we argue that those features are a manifestation of boson-vortex duality which has been earlier predicted in the framework of Chern-Simon (CS) theory in order to explain certain aspects of the quantum Hall effect (QHE). That such a composite boson can be formed at $\Omega = \omega_\perp$ has been first pointed out by Wilkin et.al. In the quantum Hall system, however, this boson-vortex composites known as composite bosons remains a useful theoretical tool to understand the existence of the off-diagonal long-range order (ODLRO), a property of the one body density-matrix of the composite bosons rather than the one body density matrix of the electrons. However we shall argue that for a rapidly rotating BEC, composite bosons can actually be formed since the vortices are the zeroes of the condensate wave function and not a fictitious statistical flux. We propose to interpret the ENS experiment as a signature of the formation of composite-boson. We note that recently a CS field theory has been proposed to describe rotating bosons near a Feshbach resonance where we also need to consider the formation of molecules as a result of the strong interaction.

This paper is organized as follows. We shall start with the energy functional of the vortex lattice in the rotating frame as given by Fetter and show why it is energetically favourable to form at the frequency $\Omega = \omega_\perp$ a condensate of composite bosons of almost uniform density in a region where the quartic confinement can be considered to be smaller than the other relevant energy scales. We then write the free energy in terms of a Chern-Simon Gross-Pitaevskii (CSGP) theory which is very similar to the Chern-Simon Landau-Ginzburg (CSLG) theory of the quantum Hall effect. We shall show how this expression of the free energy can explain some of the new features observed in the ENS experiment.
We shall also point out that at the point $\Omega = \omega_{\perp}$ the model is self-dual. Then we make some suggestions on possible experiments to probe the nature of this new superfluid phase. Finally we summarize our observations.

II. GROSS-PITAEVSKII ENERGY FUNCTIONAL

The ENS experiment is performed with $N = 35.10^5$ $^{87}$Rb atoms in a cylindrical trap with $\omega_{\perp} = \omega_x = \omega_y = 2\pi 65.6 \text{Hz}$, $\omega_z = 2\pi 11 \text{Hz}$. In addition there is a confinement potential of the form $k r^4$ with $k = 6.54510^{-12} \text{S.I.}$ [1]. For convenience, we shall first omit the quartic term assuming that it does not play any other role than making the condensate stable. Nevertheless we must point out that this quartic potential on its own can add interesting features as pointed out by Lundh [6] (see also [21]) and besides it will influence the collective spectrum around the mean-field solution we shall provide here.

We start with the Gross-Pitaevskii free energy of the bosons of mass $M$ in a rotating frame. For the order parameter $\Psi$

$$F = \int dV \left[ \frac{\hbar^2}{2M} (\nabla \Psi)^2 + V_{\perp}(r_{\perp})|\Psi|^2 + \frac{1}{2}M\omega_z^2 z^2|\Psi|^2 + \frac{1}{2}g|\Psi|^4 - \Psi^* \Omega \times p \Psi \right]$$

(1)

It has been pointed out [18, 19] that the vortex lattice experiences a solid body rotation [20] with a solid-body velocity $v_{sb}$ given by

$$\nabla \times v_{sb} = 2\Omega$$

(2)

The existence of this solid body rotation then ensures that neither the phase $\hbar S$ or the superfluid velocity $v_s = \frac{\hbar}{M} \nabla S$ can be spatially periodic. The system thus behaves locally like a superfluid in each unit cell of the lattice, but globally as a rigid body provided the core size of each individual vortex is less than a lattice constant.

Under these assumption the Gross-Pitaevskii energy functional can be written as [18]

$$F = \int dV \left[ \frac{M}{2} (v_s - v_{sb})^2 |\Psi|^2 + \frac{\hbar^2}{2M} (\nabla \Psi)^2 + V_{\perp}(r_{\perp})|\Psi|^2 + V_{\text{cent}}|\Psi|^2 + \frac{1}{2}M\omega_z^2 z^2 |\Psi|^2 + \frac{1}{2}g|\Psi|^4 \right]$$

(3)
where \( V_{\text{cent}}(r_\perp) = -\frac{1}{2} M \Omega^2 r_\perp^2 \).

The Landau level limit is defined by the condition \( V_\perp(r_\perp) = -V_{\text{cent}}(r_\perp) \).

The minimum energy configuration, as we shall show using a singular gauge transformation leading to a composites of bosons and vortices, is achieved in the mean-field approximation by considering \( v_s = v_{sb} \) and \( |\Psi| = \text{constant} \), namely when the kinetic energy of the condensate in the presence of a vortex lattice is equal to zero. In this limit the solid body rotation becomes identical to the superfluid motion and we recover the superfluid order parameter throughout the bulk.

This new phase of composite of bosons and vortices shows superfluidity, namely off-diagonal long range order (ODLRO) \[23\] for the density matrix of the composite-bosons. It differs from the ODLRO of the bare atomic condensate with vortices obtained at low rotating frequencies. We assume that the energy to bind bosons to vortices results from the repulsive interacting energy of the original bosons, thus leading to a renormalization of \( g \). This behaviour is very naturally described through a Chern-Simon Gross-Pitaevskii approach which has the advantage to be independent of the explicit LLL constraint. For further discussion we also neglect the effect of confinement in the \( z \) direction, assuming that the profile of the condensate in this direction is the same as for the non-rotating case.

III. CHERN-SIMON GROSS-PITAEVSKII THEORY

Chern-Simon (CS) field theories have been originally proposed in order to describe the behaviour of charged planar matter interacting with photons whose dynamics is governed not only by the usual Maxwell density \(-\frac{1}{4} F_{\mu\nu} F_{\mu\nu}\) but also by the Chern Simon (CS) term \( \frac{\kappa}{4} \epsilon_{\mu\nu\sigma} F_{\mu\nu} a_{\sigma} \) which gives rise to topologically massive (2 + 1)-dimensional electrodynamics \[22\]. An important consequence of the CS term is that any charged excitation also carries a magnetic flux proportional to this charge. For the low-energy, long-wavelength physics in (2 + 1)-dimensional electrodynamics the CS Lagrangian is more important than the conventional Maxwell Lagrangian since it contains one less derivative. For a rapidly rotating BEC, the density of vortices is high and is assumed to form a vortex liquid, so that we can use the CS Lagrangian in order to couple it to the bosonic matter density \(|\Psi|^2\).

This coupling is achieved through the Chern-Simon transformation between the boson
order parameter $\Psi$ and the order parameter $\Phi_{cb}$ describing the composite bosons,

$$\Phi_{cb}(r_1, \ldots, r_N) = e^{-im\sum_{i<j} \theta_{ij}} \Psi(r_1, \ldots, r_N)$$  \hspace{1cm} (4)

where $\theta_{ij}$ is the angle describing the relative position of the $i$-th and the $j$-th bosons. Here $m$ is always an even integer. In two spatial dimensions, this singular gauge transformation attaches an integer number $m$ of angular momentum to each boson so that

$$\left[\left(\sum_i (-i\frac{\hbar}{M} \nabla_i + A(r_i)) \right)\right] \Psi = \left[\left(\sum_i (-i\frac{\hbar}{M} \nabla_i + (A(r_i) + a(r_i))) \right)\right] \Phi_{cb}$$  \hspace{1cm} (5)

where

$$a_i = \frac{\hbar m}{M} \sum_{i\neq j} \nabla_i \theta_{ij}$$  \hspace{1cm} (6)

and $A = \Omega \times r$. Here indices refer to bosons and hence repeated indices does not mean summation. In the CSLG description of QHE the flux of $a$ is known as the statistical flux. However here the vector potentials $a$ are associated to the vortices already present in the condensate, and in the limit where the number of bosons is equal to the $m$ times the number of the vortices the bosons and vortices make composites. One can physically view this mechanism as the re-entering of bosonic matter in the vortices which are zeroes of the condensate-wavefunction $\Psi$ and thereby making the condensate irrotational again. For $m = 0$ one recovers the original Gross-Pitaevskii description.

The Chern Simon transformation in the second quantized language is given by

$$\hat{\Phi}_{cb}(r) = e^{-\hat{J}(r)} \hat{\Psi}(r)$$  \hspace{1cm} (7)

where the operator $\hat{J}(r)$ is given by

$$\hat{J}(r) = im \int d\mathbf{r} \hat{\Psi}(\mathbf{r})^{+} Im \log(z - z') \hat{\Psi}(\mathbf{r}), z = re^{i\theta}$$  \hspace{1cm} (8)

This transformation at the mean-field level gives the phase of the Laughlin-wave function. Considering fluctuations around the meanfield and neglecting higher order terms one can produce only the modulus of the Laughlin wavefunction $\Psi$ [25, 31]. Using a modified form of the composite-boson field operator $\hat{\Phi}_{cb}$ [24] (whose auto-correlation function shows true long-range behaviour), Rajaraman and Sondhi [29] later generalized this transformation to produce the full Laughlin wavefunction at the mean-field level from a second quantized Chern-Simon description. The same procedure can be applied here also.
The Chern-Simon Gross-Pitaevskii Lagrangian of the rapidly rotating bosons is written as

\[ L_{csgp} = L_{gp} + L_{cs} \]  

with

\[ L_{gp} = \int \, dr \, - \, \hat{\Phi}_{cb}^*(i\hbar \frac{\partial \hat{\Phi}_{cb}}{\partial t}) + \frac{\hbar^2}{2M} \hat{\Phi}_{cb}^* \left( -i \nabla - \frac{M}{\hbar} (A + a) \right)^2 \hat{\Phi}_{cb} + g|\delta \hat{\rho}_{cb}|^2 \]  

where \( \delta \hat{\rho}_{cb} = \hat{\Phi}_{cb}^* \hat{\Phi}_{cb} - \rho_{cb} \) and

\[ L_{cs} = \frac{M}{2\pi\hbar} \int \, dr \, \varepsilon_{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma \]  

The term proportional to \( \delta \hat{\rho}_{cb} \) accounts for the attractive interaction between composite bosons. It is mediated by the gauge-fields \( a \) and thus is long ranged unlike the interaction between the bosons described by the order parameter \( \Psi \).

In the original problem of rotating bosons, the additional energy due to the applied rotation can be written as a sum of the self energy of each vortex and of an interaction term between the vortices which accounts for their short range and binary interacting potential [30]. The Chern Simon transformation maps this problem onto those of a liquid of interacting composite bosons where the dynamical gauge fields \( a \) play a role similar to the magnetic field in superconductors, by canceling the applied rotational velocity in the bulk of the system just like for the Meissner effect, leaving composite bosons whose interaction is described by the term proportional to \( \delta \hat{\rho}_{cb} \) in [10] where \( \bar{\rho}_{cb} \) accounts for the fact that in the absence of applied rotation, the Lagrangian \( L_{gp} \) vanishes at the mean field level. \( L_{cs} \) is a Lagrange’s multiplier which implements the constraint relating the rotational flux quanta to the original matter field density. Therefore, it does not contribute to the free energy which is identical to those given in [11] when transformed back. To obtain the the mean-field solution we first replace the operators \( \hat{\Phi}_{cb}, \hat{\Phi}_{cb}^*, a \) in the Lagrangian [9] with the corresponding functions \( \Phi_{cb}(r), \Phi_{cb}^*(r), a(r) \) by taking its expectation value in suitable field theoretical state. The corresponding Euler-Lagrange equations of motion are obtained through the functional derivatives with respect to \( \Phi_{cb}(r) \) and the dynamical gauge fields \( a(r) \) (For further description we shall omit the argument \( r \)). They are
\[\nabla \times a = -m \frac{\hbar}{M} \rho_{cb} \quad (12)\]

\[\epsilon_{\alpha\beta} \left( \frac{\partial}{\partial t} a_{\beta} - \frac{\partial}{\partial x_{\beta}} a_0 \right) = -m \frac{\hbar}{M} j_\alpha \quad (13)\]

\[\frac{1}{2M} \left[ -i\hbar \nabla - M (A + a) \right]^2 \Phi_{cb} + \Phi_{cb} \delta \rho_{cb} = 0 \quad (14)\]

where

\[j_\alpha = \frac{\hbar}{2m}\left[ \Phi_{cb}^* (\partial_\alpha \Phi_{cb}) - (\partial_\alpha \Phi_{cb}^*) \Phi_{cb} \right] - (A_\alpha + a_\alpha) \rho_{cb} \quad (15)\]

The first of these equations expresses the boson-vortex duality while the second is the Maxwell Ampere’s law. The last equation describes the motion of the composite bosons under the combined effect of an applied rotation and of the pseudo rotation generated by the vortices. The classical solution of these equations which minimizes the free energy is

\[\Phi_{cb}^{MF} = \sqrt{\rho_{cb}} \quad (16)\]

\[A + a = 0 \quad (17)\]

\[a_0 = 0 \quad (18)\]

This solution corresponds to a constant composite boson density and the complete cancellation of the external rotation by the pseudo-rotation generated by the vortex. This is equivalent to the Meissner effect in type II superconductors and the field \((\nabla \times a)\) plays here the role of the magnetic induction \(B\). Such a description is possible only when the number of vortices is high enough so that the discrete number of singular vortices can be replaced by a continuous function representing vortex density. At very low temperature the system should be well represented by this classical solution and small fluctuations around it. In a real system we expect to have density modulations and the corresponding gauge field fluctuations.

What will be the effect of an increasing rotation? When \(A\) is changed from its meanfield value \(-a\), the additional rotation gives a corresponding density modulation. We shall now see how that will produce a vortex. The vortex solution has a density profile which is identical to the mean field solution at long distance. Therefore one can write

\[\Phi_{cb}^{MF} = \sqrt{\rho_{cb}} e^{i\theta}, \quad r \to \infty \quad (19)\]
In order for this solution to satisfy asymptotically (14),

$$A + a = \frac{\hbar}{Mr} \theta$$  \hspace{1cm} (20)

must be satisfied at $r \to \infty$. In this way the vortex solution is accompanied by a change in the rotational velocity due to the gauge field $a$. Upon integration, the right hand side of (20) gives the extra flux $(\frac{\hbar}{M})$ associated with the deviation of $a$ from its mean-field value $-A$. This is equal to the circulation flux associated with a single vortex. Thus vortices in composite-bosons are like quasi-hole excitations over a Laughlin-liquid like ground state [5, 25]. The extra energy of a vortex can be evaluated by numerically solving (14) for a solution of the form $f(r)e^{i\theta}$ which satisfies (19) and (20). In the CSLG theory of QHE such solutions have been obtained by Tafelmayer, Curnoe and Weiss [33]. In the bosons-vortices composites, the energy required to create a vortex is going to be different from that in a bare atomic condensate. This is because of the renormalisation in the interaction strength of the composite bosons.

The above description does not include the effect of the confining quartic potential. It leads to a more pronounced effect at the edge of the system where the above description does not hold anymore. It will also affect the collective excitation spectrum around the mean-field solution. It is possible that the chiral Luttinger liquid description of QHE edge states may capture the phenomenology at the edge [34].

**IV. POSSIBLE PROBES FOR COMPOSITE BOSONS**

The Chern-Simon description provides a relation between the composite-boson density and the circulation flux quanta [26, 28]. This explains qualitatively why we should observe similar features in the rotating composite-boson condensate and the rotating (bare) Bose-Einstein condensate. However, it has been already pointed out that vortices in composite bosons are statistically different from their counterpart in the boson condensate. In the first quantized language they are equivalent to quasihole excitations in the bosonic Laughlin-liquid. A Berry phase measurement like the one suggested by Paredes et. al. [5] may reveal their statistics.

A composite-boson superfluid shows an ODLRO given in the mean field by

$$G(z-z') = \langle \Phi_{cb}^{MF}(z)\Phi_{cb}^{MF}(z')|\Phi_{cb}^{MF}\rangle$$  \hspace{1cm} (21)
which can be observed in the power law decay of the auto-correlation function of a single
composite-boson \[35\]. In the QHE this correlation function is difficult to measure since it is
built out of a composite made of real electrons and fictitious statistical fluxes. Here however,
it may be even possible to measure it directly as the composites are made of ordinary bosons
and real vortices.

There is another important difference between the ordinary BEC and the composite
BEC. We speculate that the interaction between the composite-bosons is much weaker than
between the original bosons. This conjecture may be verified by measuring the \(s\)-wave
scattering length when these composite bosons are formed. This may serve as an important
step to probe the composite boson superfluid. Alternatively, another possible way to probe
the composite boson superfluid is to look at the collective excitation spectrum \[36\] and
compare it with the Bogoliubov spectrum of the bare condensate. The measurement of
collective excitations may also reveal the renormalisation of the interaction strength between
the composite bosons.

For \(\Omega > \omega_\perp\) one can again use the Thomas-Fermi approximation for the composite boson
condensate, but with the renormalized interaction strength. By increasing \(\Omega\) further than
what has been achieved in the ENS experiment, one may be able to obtain some information
about this renormalized interaction strength from the evolution of the optical thickness of
the atomic cloud after a time of flight measurement and taking the \(s\)-wave scattering length
as a variational parameter in the resulting data \[1\]. In this limit, an increasing \(\Omega\) gives rise
to an effective confinement potential of the form \((-ar^2 + kr^4)\) with \(a > 0\) (Mexican hat)
and as a result the size of the condensate region must grow. The ENS experiment supports
this observation. In this region the composite-bosons are again in the lowest Landau level
and their kinetic energy can be neglected compared to the other terms in the expression of
the free energy. It will be interesting to study the collective excitation spectrum using the
Thomas-Fermi approximation in this regime and to compare it to the one obtained within
the CSGP framework \[36\].

At a frequency \(\Omega = \omega_\perp\), the ENS experiment \[1\] showed that the radial distribution of the
cloud developed a shallow local minima around the point \(r = 0\) where the mean-field CSGP
approximation is better justified. By reducing the strength \(k\) of the quartic confinement, this
region will be enlarged. In that case one shall recover superfluidity and vortex generation
for \(\Omega \geq \omega_\perp\) over a larger area compared to the present experiment.
V. SUMMARY AND OUTLOOK

We have proposed a Chern-Simon Gross-Pitaevskii theory to describe the behaviour of a rapidly rotating condensate at $\Omega \geq \omega_\perp$. The central idea in this formulation is the duality between bosons and vortices. We have studied certain features of a recent experiment [1] in terms of this theory. The description in terms of composite bosons is valid in the limit where the number of vortices in the condensate is of the same order as the number of atoms. There is no confirmation of this aspect at least at the present stage of the experiment. However this requires to probe the condensate in more details around $\Omega = \omega_\perp$ through an adiabatic changing of the rotational frequency. This should lead to a better comparison between the theory and the experiment. We also predict a renormalisation of the $s$-wave scattering length due to the formation of composite bosons. The statistical phase-measurement type experiments already suggested and the measurement of collective excitations around $\Omega = \omega_\perp$ may give more informations about the condensate of composite-bosons, namely the ODLRO and the nucleation of vortices.

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[35] The ODLRO defined here decays algebraically and hence shows quasi-long range order. A modified ODLRO can be defined \[24\] which shows true long-range order. The corresponding CSLG which involves a non-unitary transformation between composite bosons and bosons was developed in ref. \[29\].

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