Modelling and control of a shell structure based on a finite dimensional variational formulation

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A mathematical model of a controlled shell structure based on Hamilton’s principle and the generalized Ritz–Galerkin method is proposed in this paper. The problem of minimizing the stress energy is solved explicitly for a static version of this model. For the dynamical system under consideration, a procedure for estimating external disturbances and the state vector is derived. We also propose an observer design scheme and solve the stabilization problem for an arbitrary dimension of the linearized model. This approach allows us to perform control design for double-curved shells of complex geometry by combining analytical computation of the controller parameters with numerical data that represent the reference configuration and modal displacements of the shell. As an example, the parameters of our model are validated by results of a finite element analysis for the Stuttgart SmartShell structure.

Keywords: elastic shell; Hamilton’s principle; Ritz–Galerkin method; stabilization; observability; modal solution

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1. Introduction

The paper is devoted to the development of analytical methods for the modelling and vibration damping of controlled elastic shells. Such shells are of particular interest in civil engineering as hybrid intelligent construction elements (HICE). These innovative construction elements may be used, for example in a lightweight roof system with adaptive support. A practical problem of significant importance is to optimize the stress distribution and to reduce the vibrations of such adaptive roof under external actions caused by wind, snow, etc., by applying controls at the support surface. These static and dynamic phenomena can be experimentally tested on the SmartShell structure constructed at the University of Stuttgart (see Figure 1). This shell structure has already been the subject of several investigations. In the paper [1], the dynamics of the shell is represented by a linearized system of ordinary differential equations (ODEs). The coefficients of that system are computed by using finite element modelling in ANSYS. The authors of Reference [1] reduce the original finite element model with 4762 degrees of freedom to a system with 18 modes. It is pointed out that the damping matrix cannot be computed directly as the corresponding material data are not available. Thus, the damping matrix is assumed to be

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a linear combination of the mass and the stiffness matrices with coefficients defined by an identification procedure. For this model, several load scenarios are considered in order to illustrate the influence of static loads on the shell dynamics. These scenarios correspond to snow loads as they have the highest impact on the system behaviour. The paper [1] treats the optimization of the sensor locations as a two objective problem with objectives being the number of sensors and an observability measure represented in terms of the observability Gramian. This optimization task is analysed by estimating the Pareto front. The problem of fault diagnosis of the Stuttgart SmartShell structure is considered in the papers [2,3].

Although finite element models, considered in the above publications, are of significant importance from the engineering viewpoint, coefficients of these models cannot be easily interpreted in terms of the original mechanical parameters. In contrast to the pure finite element approach, the present paper aims to derive a reduced control system analytically in terms of available representations of displacement modes. It should be emphasized that, despite the extremely rich literature on the mechanics of shells, analytical solutions are available only in exceptional cases for shells of simple forms (see, e.g. [4–10]). We also refer to the paper [11], where exact solutions are presented for simply supported, doubly curved, cross-ply laminated shells under sinusoidal, uniformly distributed and concentrated point load at the centre. In the paper [12], theoretical values of the shear correction factors are computed for the general six-field geometrically nonlinear theory of elastic shells. A survey of recent developments in the nonlinear theory of vibrations of elastic shells is presented in Reference [13].

In the paper [14], a feedback control on the boundary of a spherical shell is constructed. It is shown that the resulting closed-loop system generates a semigroup of contractions in a suitable function space. Then the exponential decay of the semigroup solutions is established. That paper also contains a result on the exact controllability by using boundary controls. Continuous observability estimates are established in Reference [15] for a Koiter shell model with the Dirichlet boundary conditions. These estimates are also used to describe controllability conditions. In the paper [16], modal equations of motion of a shell structure with discretely distributed piezoelectric sensors and actuator
patches are studied. The authors of that paper use quasi-modal sensors and quasi-modal actuators in order to perform independent modal control of smart shells.

This paper is organized as follows. Basic geometric characteristics of a controlled shell are introduced in Section 2. Then Hamilton’s principle is applied to derive the variational formulation of the dynamical equations in a neighbourhood of the equilibrium position in Section 3. We give explicit formulas for the elastic potential in orthotropic and isotropic cases. The generalized Ritz–Galerkin method is applied for deriving a finite dimensional model in Section 4. For this linearized model, we solve problems related to the minimization of the stress, identification of disturbances, observer design and stabilization in Section 5. Section 6 contains some simulation results for the Stuttgart SmartShell structure.

2. Modelling of a shell
Consider an elastic shell of thickness $h$ and denote its median surface by $\Omega$. Let the position of a point $M$ on $\Omega$ at time $t$ be defined by the radius vector

$$
r = r(\xi^1, \xi^2, t) = \sum_{i=1}^{3} r'(\xi^1, \xi^2, t)v_i, \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2.
$$

Here $(v_1, v_2, v_3)$ is a Cartesian basis in $\mathbb{R}^3$. We treat $(\xi^1, \xi^2)$ as the Lagrangian coordinates of $M$ and assume that $D \subset \mathbb{R}^2$ is a closed bounded domain with piecewise smooth boundary. Let $\hat{\Omega}$ be the reference configuration of the median surface given by

$$
r = \hat{r}(\xi^1, \xi^2) = \sum_{i=1}^{3} \hat{r}^i(\xi^1, \xi^2)v_i, \quad (\xi^1, \xi^2) \in D.
$$

We assume that $\hat{\Omega}$ corresponds to an equilibrium position of the shell with constant external loads and constant controls. The coordinate functions $r'(\xi^1, \xi^2, t)$ and $\hat{r}^i(\xi^1, \xi^2)$ are assumed to be of class $C^2$ in their domains of definition. In typical cases, the median surface $\hat{\Omega}$ is parameterized as

$$
r^1 = \xi^1, \quad r^2 = \xi^2, \quad r^3 = \hat{r}^3(\xi^1, \xi^2),
$$

so that we can identify the Lagrangian coordinates $(\xi^1, \xi^2)$ with $(r^1, r^2)$ at the reference configuration $\hat{\Omega}$ and treat $D$ as the projection of $\hat{\Omega}$ on the $(r^1, r^2)$-plane.

To describe the strain of the shell, we introduce the first and the second quadratic forms of $\hat{\Omega}$ and $\Omega$ at a point $M$ with the Lagrangian coordinates $(\xi^1, \xi^2)$:

$$
\left(\frac{dr}{d\xi}\right)^2 = \sum_{i,j=1}^{2} g_{ij} d\xi^i d\xi^j,
$$

$$
(dr)^2 = \sum_{i,j=1}^{2} g_{ij} d\xi^i d\xi^j,
$$
\[
(n, d^2 r) = \sum_{i,j=1}^{2} q_{ij} \, d\xi^i \, d\xi^j, \quad (n, d^2) = \sum_{i,j=1}^{2} q_{ij} \, d\xi^i \, d\xi^j.
\]

The coefficients of these quadratic forms are
\[
g_{ij} = \left( \frac{\partial r}{\partial \xi^1}, \frac{\partial r}{\partial \xi^2} \right), \quad q_{ij} = \left( \frac{\partial^2 r}{\partial \xi^1 \partial \xi^2} \right),
\]
where
\[
\hat{n} = \left| \frac{\partial r}{\partial \xi^1} \right|^{-1/2} \left[ \frac{\partial r}{\partial \xi^1}, \frac{\partial r}{\partial \xi^2} \right], \quad n = \left| G \right|^{-1/2} \left[ \frac{\partial r}{\partial \xi^1}, \frac{\partial r}{\partial \xi^2} \right]
\]
are normal vectors to \( \hat{\Omega} \) and \( \Omega \), respectively. We use notations \((\cdot, \cdot)\) for the scalar product and \([\cdot, \cdot]\) for the vector product in \( \mathbb{R}^3 \). The matrices of quadratic forms are denoted by \( \hat{G} = (\hat{g}_{ij}) \), \( G = (g_{ij}) \), \( \hat{Q} = (\hat{q}_{ij}) \) and \( Q = (q_{ij}) \),
\[
\left| \hat{G} \right| = \hat{g}_{11} \hat{g}_{22} - \hat{g}_{12}^2, \quad \left| G \right| = g_{11} g_{22} - g_{12}^2.
\]

To measure the strain and bending, we introduce tensors \( \varepsilon = (\varepsilon_{ij}) \) and \( \kappa = (\kappa_{ij}) \) as follows:
\[
\varepsilon_{ij} = \frac{1}{2} (g_{ij} - \hat{g}_{ij}), \quad \kappa_{ij} = q_{ij} - \hat{q}_{ij}.
\]

Note that the above \( \varepsilon_{ij} \) are components of the strain tensor at the median surface considered as a two-dimensional body (cf. \[17\]).

Following the classic shell theory \[18, Chapter IV\], we write the kinetic energy \( K \) and the potential energy \( U \) for the case of small vibrations of the shell around its reference configuration as follows:
\[
K = \frac{1}{2} \int_{D} \rho \left( \frac{\partial r}{\partial t} \right)^2 \left| \hat{G} \right|^{-1/2} d\xi^1 d\xi^2, \quad (2)
\]
\[
U = \int_{D} \{ \Psi(\varepsilon, \kappa) + \rho(\gamma, r) \} \left| \hat{G} \right|^{-1/2} d\xi^1 d\xi^2, \quad (3)
\]
where \( \rho \) is the surface density (mass per unit area of the shell), function \( \Psi \) is the measure of strain and bending and \( \gamma \) is the vector pointing opposite the gravity field such that \( |\gamma| \) is the standard acceleration due to gravity.

3. Hamilton’s principle

We assume that external disturbances act as the force \( W \) distributed on the surface of the shell, and that the control force \( F \) and control torque \( R \) act on the boundary \( \Gamma \) of \( \Omega \).
Let \( \delta r = \delta r(\xi^1, \xi^2, t) \) be a virtual displacement for \( r = r(\xi^1, \xi^2, t) \). Then the work of \( W, F \) and \( R \) along this displacement is

\[
\delta A = \int_D (W, \delta r) |G|^{1/2} d\xi^1 d\xi^2 + \int_\Gamma \{ (F, \delta r) + R(n', \delta n) \} ds,
\]

where \( \delta n \) is the variation of the normal vector to \( \Omega \) at \( \Gamma \), and \( n' \) is the outer normal for \( \Gamma \) which belongs to the tangent space of \( \Omega \).

If \( r = r(\xi^1, \xi^2, t) \) satisfies the geometric boundary conditions and defines the motion of the shell on a time segment \( t \in [t_0, t_1] \), then the dynamics is governed by Hamilton’s principle [18]:

\[
\int_{t_0}^{t_1} \{ \delta (K - U) + \delta A \} \, dt = 0,
\]

for any admissible variation \( \delta r \) such that \( \delta r|_{t=t_0} = \delta r|_{t=t_1} = 0 \). Here \( K, U \) and \( A \) are defined by formulas (2), (3) and (4), respectively. We assume that the class of admissible variations satisfies all the geometric boundary conditions. In particular, if the shell is supported along a part of its boundary, we assume that \( r(\xi^1, \xi^2, t) \) is fixed and \( \delta r(\xi^1, \xi^2, t) = 0 \) for all \( (\xi^1, \xi^2) \in \tilde{\Gamma} \subset \Gamma \).

As we assume that \( \Omega \) corresponds to an equilibrium of the shell, then \( r = \bar{r}(\xi^1, \xi^2) \) satisfies relation (5) for some disturbance \( W = W_0(\xi^1, \xi^2) \) and controls \( F = F_0(\xi^1, \xi^2) \), \( R = R_0(\xi^1, \xi^2) \). From the physical viewpoint, such controls compensate the action of gravity, internal stresses and external disturbances. In order to describe the perturbed dynamics of the shell, we introduce the displacement vector \( \tilde{r} \) such that

\[
r(\xi^1, \xi^2, t) = \bar{r}(\xi^1, \xi^2) + \tilde{r}(\xi^1, \xi^2, t)
\]

and consider the following perturbed quantities:

\[
W = W_0(\xi^1, \xi^2) + \tilde{W}(\xi^1, \xi^2, t),
\]

\[
F = F_0(\xi^1, \xi^2) + \tilde{F}(\xi^1, \xi^2, t), \quad R = R_0(\xi^1, \xi^2) + \tilde{R}(\xi^1, \xi^2, t).
\]

By substituting these expressions into the variational formulation (5) and performing the integration by parts with respect to \( t \), we get:

\[
\int_D \left\{ \rho \left( \frac{\partial^2 \tilde{r}}{\partial t^2}, \delta r \right) + \sum_{j=1}^{n} \left\{ \frac{1}{2} \frac{\partial \Phi}{\partial e_{ij}} \delta g_{ij} + \frac{\partial \Phi}{\partial \xi_{ij}} \delta q_{ij} \right\} - \left( \bar{W}, \delta r \right) \right\} |G|^{1/2} d\xi^1 d\xi^2
\]

\[- \int_\Gamma \{ (\tilde{F}, \delta r) + R(n', \delta n) \} ds = 0, \quad t \in [t_0, t_1],
\]

for each admissible variation \( \delta r \) such that \( \delta r|_{t=t_0} = \delta r|_{t=t_1} = 0 \). The variations of components of tensors
\[
\delta g_{ij} = \left( \frac{\partial \delta r}{\partial \xi^i}, \frac{\partial \delta r}{\partial \xi^j} \right) + \left( \frac{\partial \delta r}{\partial \xi^i}, \frac{\partial r}{\partial \xi^j} \right),
\]
\[
\delta q_{ij} = \frac{2g_{12}\delta g_{12} - g_{22}\delta g_{11} - g_{11}\delta g_{22}}{2|G|^{3/2}} \left( \frac{\partial r}{\partial \xi^1}, \frac{\partial r}{\partial \xi^2}, \frac{\partial^2 r}{\partial \xi^1 \partial \xi^2} \right) + \frac{1}{|G|^{1/2}} \times \left\{ \frac{\partial \delta r}{\partial \xi^1}, \frac{\partial r}{\partial \xi^2}, \frac{\partial^2 r}{\partial \xi^1 \partial \xi^2} \right\}
\]

are evaluated at \( r = r(\xi^1, \xi^2, t) \). Here \( (a, b, c) = (a, [b, c]) = ([a, b], c) \) stands for the scalar product of vectors in \( \mathbb{R}^3 \).

The function \( \Phi = \Phi(\epsilon, \chi) \) measures the energy density in Equation (6). In this work, we will use one of two formulas for \( \Phi(\epsilon, \chi) \). First we consider the case of an isotropic shell (cf. [18, Chapter IV]):

\[
\Phi(\epsilon, \chi) = \mu h \left\{ \sigma v^2 \left( \epsilon G^{-1} \right) + v \left( \epsilon G^{-1} \right)^2 \right\} + \frac{\mu h^3}{12} \left\{ \sigma v^2 \left( \chi G^{-1} \right) + v \left( \chi G^{-1} \right)^2 \right\}, \quad \sigma = \frac{\lambda}{\lambda + 2\mu} = \frac{v}{1-v},
\]

where \( \lambda = \frac{\nu E}{1 + \nu(1-\nu)} \) and \( \mu = \frac{E}{2(1+\nu)} \) are the Lamé parameters, \( h \) is the thickness of the shell, \( E \) is the Young’s modulus and \( \nu \) is the Poisson’s ratio. Expression (7) may be written in component form as follows:

\[
\Phi(\epsilon, \chi) = \frac{\mu h}{|G|^2} \left\{ \sigma \left( \epsilon_{11} \epsilon_{22} - 2 \epsilon_{12} \epsilon_{12} + \epsilon_{22} \epsilon_{11} \right)^2 + 2 \epsilon_{12}^2 \left[ \epsilon_{11} \epsilon_{22}^2 + \epsilon_{22} \epsilon_{11}^2 \right] + \epsilon_{11}^2 \epsilon_{22}^2 + \epsilon_{22}^2 \epsilon_{11}^2 + 2 \epsilon_{12} \epsilon_{12} \epsilon_{11} \epsilon_{22} - 4 \epsilon_{11} \epsilon_{12} \epsilon_{12} \epsilon_{22} - 4 \epsilon_{12} \epsilon_{22} \epsilon_{11} \epsilon_{12} \right\} + \frac{\mu h^3}{12 |G|^2} \left\{ \sigma \left( \chi_{11} \chi_{22} - 2 \chi_{12} \chi_{12} + \chi_{22} \chi_{11} \right)^2 + 2 \chi_{12}^2 \left[ \chi_{11} \chi_{22}^2 + \chi_{22} \chi_{11}^2 \right] + \chi_{11}^2 \chi_{22}^2 + \chi_{22}^2 \chi_{11}^2 + 2 \chi_{11} \chi_{22} \chi_{12} \chi_{12} - 4 \chi_{11} \chi_{12} \chi_{12} \chi_{22} - 4 \chi_{12} \chi_{22} \chi_{11} \chi_{12} \right\},
\]

where \( |G| = \epsilon_{11} \epsilon_{22} - \epsilon_{12}^2 > 0 \) for the metric tensor.

For the second case, we consider an orthotropic shell and assume that its axes of symmetry of mechanical properties correspond to curvilinear coordinates \( \xi^1 \) and \( \xi^2 \). Thus, we take the following energy density (cf. [19]):

\[
2\Phi(\epsilon, \chi) = h \left\{ E_{11}\chi_{11}^2 + 2E_{12}\chi_{11}\chi_{22} + E_{22}\chi_{12}^2 + 4G_{12}\chi_{12}^2 \right\} + \frac{\mu h^3}{12} \left\{ E_{11}\chi_{11}^2 + 2E_{12}\chi_{11}\chi_{22} + E_{22}\chi_{12}^2 + 4G_{12}\chi_{12}^2 \right\},
\]

\[
E_{11} = \frac{E_1}{1 - v_1 v_2}, \quad E_{22} = \frac{E_2}{1 - v_1 v_2}, \quad E_{12} = \frac{v_1 E_2}{1 - v_1 v_2} = \frac{v_2 E_1}{1 - v_1 v_2},
\]

where \( E_i \) is the Young’s modulus along the \( i \)th direction (the axis corresponding to \( \xi^i \)), \( v_i \) is the Poisson’s ratio that corresponds to a contraction in direction \( j \neq i \) when an extension
is applied in direction \(i\) and \(G_{ij}\) is the shear modulus in direction \(j\) on the plane whose normal is in direction \(i\).

**Remark 1.** Note that for the isotropic shell with
\[
E_1 = E_2 = E, \quad v_1 = v_2 = v, \quad G_{12} = \frac{E}{2(1 + v)}
\]
and with the unit metric tensor \(G = I\), both formulas (7) and (8) give the same expression:
\[
\Phi(e, \varkappa) = \frac{Eh}{2(1 - v^2)} \left\{ e_{11}^2 + e_{22}^2 + 2v e_{11} e_{22} + 2(1 - v)e_{12}^2 \right\}
\]
\[
+ \frac{Eh^3}{24(1 - v^2)} \left\{ \varkappa_{11}^2 + \varkappa_{22}^2 + 2v \varkappa_{11} \varkappa_{22} + 2(1 - v)\varkappa_{12}^2 \right\}.
\]

**Remark 2.** Assume that the reference configuration of the shell corresponds to a plate with the median surface \(\xi^3 = 0\) in the \((\xi^1, \xi^2, \xi^3)\) space, that is \(r(\xi^1, \xi^2) = (\xi^1, \xi^2, 0) \in \mathbb{R}^3\), \((\xi^1, \xi^2) \in D \subset \mathbb{R}^2\). If we take \(r(\xi^1, \xi^2, t) = (\xi^1, \xi^2, w(\xi^1, \xi^2, t))\) and \(\delta r = (0, 0, \delta w(\xi^1, \xi^2, t))\), then the variational formulation (6) with \(\tilde{W} = 0\), \(\tilde{F} = 0\), \(\tilde{R} = 0\) and \(\Phi\) given by Equation (9) yields the following differential equation:
\[
\rho \frac{\partial^2 w(\xi^1, \xi^2, t)}{\partial t^2} + \frac{Eh^3}{12(1 - v^2)} \Delta^2 w(\xi^1, \xi^2, t) = 0
\]
and appropriate boundary conditions, where \(\Delta = \frac{\partial^2}{\partial (\xi^1)^2} + \frac{\partial^2}{\partial (\xi^2)^2}\). Equation (10) describes the Kirchhoff plate model (see, e.g., [10,20,21, Chapter 6]).

### 4. Modal analysis

To derive an approximate model, we fix a number \(N\) and assume that
\[
\tilde{r} = a_1(t) r_1(\xi^1, \xi^2), \quad \tilde{r} = a_2(t) r_2(\xi^1, \xi^2), \quad \ldots, \quad \tilde{r} = a_N(t) r_N(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D,
\]
satisfy formula (6) with
\[
\tilde{W} = 0, \quad \tilde{F} = 0, \quad \tilde{R} = 0,
\]
for any admissible variation \(\delta r\). Moreover, we assume that \((r_1(\xi^1, \xi^2), \ldots, r_N(\xi^1, \xi^2))\) are linearly independent in \(D\), and we refer to \(r_i(\xi^1, \xi^2)\) as to eigenfunctions or modal forms of the shell.

Then we apply the generalized Ritz–Galerkin method (see, e.g., [22]) to approximate the dynamics by the following linear combination of modes:
\[
\tilde{r}(\xi^1, \xi^2, t) = \sum_{j=1}^{N} q_j(t) r_j(\xi^1, \xi^2). \tag{11}
\]
By substituting representation (11) into Equation (6) and assuming that Equation (6) holds for each variation \( \delta r = a(t) r_l(\xi^1, \xi^2) \) with \( a \in C^2[t_0, t_1], \quad a(t_0) = a(t_1) = 0, \quad l = 1, 2, \ldots, N \), we obtain a system of \( N \) second-order ODEs with respect to \( q_l(t) \). The linear part of this control system can be written in the following form:

\[
M \ddot{q}(t) + K q(t) = \bar{B} u + d(t), \quad q(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \tag{12}
\]

where \( u \in \mathbb{R}^m \) is the control and \( d(t) \in \mathbb{R}^N \) is the disturbance vector. The components of matrices \( M \) and \( K \) in Equation (12) are computed by using integrals from Equation (6) with \( \Phi \) given by formula (8):

\[
M = (M_{lk})_{l,k=1}^N, \quad K = (K_{lk})_{l,k=1}^N;
\]

\[
M_{lk} = \int_D \rho(r_l(\xi^1, \xi^2), r_k(\xi^1, \xi^2)) \left| \hat{G}(\xi^1, \xi^2) \right|^{1/2} d\xi^1 d\xi^2, \tag{13}
\]

\[
K_{lk} = \int_D \left( K^*_l(\xi^1, \xi^2) + K^*_k(\xi^1, \xi^2) \right) \left| \hat{G}(\xi^1, \xi^2) \right|^{1/2} d\xi^1 d\xi^2.
\]

For the isotropic shell with the energy density \( \Phi \) given by formula (7), the integrands in expressions (13) are as follows:

\[
K^*_l(\xi^1, \xi^2) = \frac{Eh}{2(1-\nu^2)|\hat{G}|^2} \left\{ (\nu - 1)|\hat{G}|(\varepsilon^l_{22}\dot{g}^l_{q_{11}} - 2\varepsilon^l_{12}\dot{g}^l_{q_{12}} + \varepsilon^l_{11}\dot{g}^l_{q_{22}}) \\
+ \left( \hat{g}^l_{22}\varepsilon^l_{11} - 2\hat{g}^l_{12}\varepsilon^l_{12} + \hat{g}^l_{11}\varepsilon^l_{22} \right) \left( \dot{g}^l_{22}\delta g^l_{q_{11}} - 2\dot{g}^l_{12}\delta g^l_{q_{12}} + \dot{g}^l_{11}\delta g^l_{q_{22}} \right) \right\},
\]

\[
K^*_k(\xi^1, \xi^2) = \frac{Eh^3}{12(1-\nu^2)|\hat{G}|^2} \left\{ (\nu - 1)|\hat{G}|(\varepsilon^k_{22}\dot{q}^k_{11} - 2\varepsilon^k_{12}\dot{q}^k_{12} + \varepsilon^k_{11}\dot{q}^k_{22}) \\
+ \left( \hat{g}^k_{22}\varepsilon^k_{11} - 2\hat{g}^k_{12}\varepsilon^k_{12} + \hat{g}^k_{11}\varepsilon^k_{22} \right) \left( \dot{g}^k_{22}\delta q^k_{11} - 2\dot{g}^k_{12}\delta q^k_{12} + \dot{g}^k_{11}\delta q^k_{22} \right) \right\},
\]

where

\[
2\varepsilon^l_{ij}(\xi^1, \xi^2) = \delta g^l_{ij}(\xi^1, \xi^2) = \left( \frac{\partial \hat{r}}{\partial \xi^l}, \frac{\partial r_k}{\partial \xi^l}, \frac{\partial r_k}{\partial \xi^l} \right) + \left( \frac{\partial \hat{r}}{\partial \xi^l}, \frac{\partial r_k}{\partial \xi^l}, \frac{\partial r_k}{\partial \xi^l} \right), \tag{14}
\]
\[
\chi_j^k(\xi^1,\xi^2) = \delta q_j^k(\xi^1,\xi^2) = \left\{ \begin{array}{l}
\left( \frac{\partial r_{ik}}{\partial \xi^1}, \frac{\partial^2 \bar{r}}{\partial \xi^1 \partial \xi^1}, \frac{\partial^2 \bar{r}}{\partial \xi^2 \partial \xi^1} \right) - 
\left( \frac{\partial r_{ik}}{\partial \xi^2}, \frac{\partial^2 \bar{r}}{\partial \xi^2 \partial \xi^1}, \frac{\partial^2 \bar{r}}{\partial \xi^2 \partial \xi^2} \right)
\end{array} \right.
+ \left( \frac{\partial^2 r_{ik}}{\partial \xi^1 \partial \xi^1}, \frac{\partial r_{ik}}{\partial \xi^1}, \frac{\partial r_{ik}}{\partial \xi^2} \right) \left\lvert \hat{\sigma} \right\rvert^{-1/2} + \left\{ \hat{\sigma} \right\rvert^{-3/2}.
\]

(15)

For the orthotropic case (formula (8)), the functions \(K_{il}^\nu(\xi^1,\xi^2)\) and \(K_{il}^\nu(\xi^1,\xi^2)\) in expressions (13) take the following form:

\[
K_{il}^\nu = \frac{E_1 h}{2(1 - v_1 v_2)} \left[ (\epsilon_{11}^k + 2\epsilon_{22}^k)\delta l_{11} + (\epsilon_{22}^k + \frac{E_2}{E_1}\epsilon_{22}^k)\delta l_{22} + 2\gamma_{12}\epsilon_{12}^k\delta l_{12} \right],
\]

\[
K_{il}^\nu = \frac{E_1 h^3}{12(1 - v_1 v_2)} \left[ (\epsilon_{11}^k + 2\epsilon_{22}^k)\delta l_{11} + (\epsilon_{22}^k + \frac{E_2}{E_1}\epsilon_{22}^k)\delta l_{22} + 2\gamma_{12}\epsilon_{12}^k\delta l_{12} \right],
\]

where

\[
\gamma_{12} = \frac{2G_{12}(1 - v_1 v_2)}{E_1}.
\]

The above computations can be performed numerically if the eigenfunctions \(r_{ik}(\xi^1,\xi^2)\) are defined in terms of nodal displacements.

We assume that controls \(u \in \mathbb{R}^m\) correspond to the action of \(m\) actuators, and the \(i\)th actuator applies the force \(u_l \in \mathbb{R}\) along the unit vector \(f_l \in \mathbb{R}^3\) at a point with the Lagrangian coordinates \(\xi_l = (\xi_l^1,\xi_l^2) \in \Gamma, i = 1, 2, \ldots, m\). Thus, we replace the integral term over \(\Gamma\) in Equation (6) with the sum of \((\delta r_l f_l)u_l\) in the sequel. The value \(u_l\) is treated as the perturbed force, that is the difference between the real force applied and the force needed to compensate the equilibrium state \(\bar{r}\). The work of such controls along a virtual displacement \(\delta r = a(t)r_l(\xi^1,\xi^2)\) is

\[
\delta A_u = \sum_{i=1}^{m} a(t)(r_l(\xi_l^1,\xi_l^2),f_l)u_l.
\]

Thus, the matrix \(\bar{B}\) in Equation (12) has the following structure:

\[
\bar{B} = (\bar{B}_{il}), \quad \bar{B}_{il} = (r_l(\xi_l^1,\xi_l^2),f_l), \quad l = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, m,
\]

and the components of the disturbance vector \(d(t) = (d_1(t), d_2(t), \ldots, d_N(t))^T\) are

\[
d_l(t) = \int_D \left( \bar{W}(\xi^1,\xi^2,t), r_l(\xi^1,\xi^2) \right) \left\lvert \hat{\sigma} \right\rvert^{1/2} d\xi^1 d\xi^2, \quad l = 1, 2, \ldots, N.
\]
In order to measure the state of the system considered, we assume that there are \( p \) strain gauges located at points with the coordinates \( \xi_l = (\xi^1_l, \xi^2_l) \in D, \ l = 1, 2, \ldots, p \). Assume, moreover, that the \( h \)th strain gauge measures the strain along the direction

\[
e_l = e^1_l \frac{\partial r}{\partial \xi^1} + e^2_l \frac{\partial r}{\partial \xi^2},
\]

so that \((e^1_l, e^2_l)\) are local coordinates of the vector \( e_l \neq 0 \). Let \( |\hat{e}_l| \) be the length of \( e_l \) in the reference configuration \( \hat{\Omega} \). Thus, the strain along \( e_l \) is

\[
e_l = \frac{|e_l| - |\hat{e}_l|}{|\hat{e}_l|} \approx \frac{\sum_{i,j=1}^{2} \overline{e}_{ij} e^i_l e^j_l}{\sum_{i,j=1}^{2} \overline{g}_{ij} e^i_l e^j_l},
\]

and the terms of order \( O\left(\left(|e_l| - |\hat{e}_l|\right)^2\right) \) have been truncated in the right-hand side of this formula. Here the tensors \( \overline{g}_{ij} \) and \( \overline{e}_{ij} \) correspond to the inner or outer surface of the shell (where the sensor is located), that is

\[
\overline{e}_{ij} = e_{ij} - \frac{h}{2} \chi_{ij}, \quad \overline{g}_{ij} = g_{ij} - h q_{ij}, \quad 1 \leq i, j \leq 2,
\]

where \((e_{ij}, \chi_{ij})\) and \((g_{ij}, q_{ij})\) are evaluated at the median surface of the shell for \( r = r(\xi^1_l, \xi^2_l, t) \) and \( \hat{r} = \hat{r}(\xi^1_l, \xi^2_l) \), respectively. Formulas (17) are obtained by exploiting the Kirchhoff–Love hypothesis and neglecting the terms of order \( O(h^2) \), where \( h \) is the thickness of the shell.

Formulas (17) correspond to the sensor located at the direction \( nh/2 \) from the median surface\(^3\); if the sensor is located at the direction \( -nh/2 \), then one should change the sign of \( h \) in formulas (17). By using representations (16) and (17), we conclude that the following output signals are available for system (12):

\[
y_l(t) = \sum_{k=1}^{N} C_{lk} q_k(t), \quad l = 1, 2, \ldots, p,
\]

where

\[
C_{lk} = \frac{\sum_{i,j=1}^{2} \left( c^i_{ij}(\xi^1_l, \xi^2_l) - \frac{h}{2} \chi^i_{ij}(\xi^1_l, \xi^2_l) \right) e^i_l e^j_l}{\sum_{i,j=1}^{2} \left( g^i_{ij}(\xi^1_l, \xi^2_l) - h q^i_{ij}(\xi^1_l, \xi^2_l) \right) e^i_l e^j_l},
\]

and \( c^i_{ij}(\xi^1, \xi^2), \chi^i_{ij}(\xi^1, \xi^2) \) are computed by formulas (14) and (15).
We assume that the eigenfunctions \( r_l(\xi^1, \xi^2) \) are chosen in an appropriate way, so that the matrix \( M \) is nondegenerate in system (12). Then by introducing new notations

\[ \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0_{N\times N} & I \\ -M^{-1}K & 0_{N\times N} \end{pmatrix}, \quad B = \begin{pmatrix} 0_{N\times m} \\ M^{-1}B \end{pmatrix}, \quad P = \begin{pmatrix} 0_{N\times N} \end{pmatrix}, \quad C = \begin{pmatrix} \tilde{C}, 0_{p\times N} \end{pmatrix}, \]

we rewrite system (12) in its normal form as follows:

\[ \dot{x}(t) = Ax(t) + Bu + Pd(t), \quad y(t) = Cx(t), \]

where \( x(t) \in \mathbb{R}^{2N} \) is the state, \( u \in \mathbb{R}^m \) is the control, \( d(t) \in \mathbb{R}^N \) is the disturbance and \( y(t) \in \mathbb{R}^p \) is the output.

Control system (19) and (20) is considered as a finite dimensional representation of the shell dynamics with modal coordinates around its equilibrium position \( \Omega \).

5. Control problems

This section addresses problems related to the optimization and control of system (19). The list of these problems is summarized as follows:

1. For a given disturbance \( d \), find a control \( u \) that minimizes the total stress in the shell,
2. identification of the disturbance vector \( d \),
3. observer design problem for system (19) with the output (20),
4. stabilization of system (19) by a state feedback law and with an observer-based controller.

We will propose solutions to these problems below. Note that an algorithm for active vibration damping has been proposed in Reference [23] for a finite-element model of the Stuttgart SmartShell structure.

5.1. Static minimization of the stress energy

First we consider the problem of minimizing the stress in the shell by considering constant disturbances and control. Let us assume that the elastic potential matrix \( K \) is positive definite. Thus, for each constant value of \( d \in \mathbb{R}^N \) and \( u \in \mathbb{R}^m \), there is an equilibrium state \( x_e = \begin{pmatrix} q_e \\ 0 \end{pmatrix} \in \mathbb{R}^{2N} \) of system (19) with

\[ q_e(d, u) = K^{-1}Bu + K^{-1}d. \]
For further analysis, we take the energy of stress as an integral characteristics of the stress in system (19) at a state $x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \in \mathbb{R}^{2N}$:

$$J(q) = \sum_{i,j=1}^{N} K_{ij} q_i q_j = q^T K q.$$  \hspace{1cm} (22)

Then we consider the problem of minimizing $J(q_e(d, u))$ in the class of all controls $u \in \mathbb{R}^m$. The basic result in this area is as follows.

**Lemma 5.1.** Let $d \in \mathbb{R}^N$ be given, and let $K$ be a positive definite $N \times N$ matrix. Then there exists a unique solution $u = \bar{u} \in \mathbb{R}^m$ of the optimization problem

$$J(q_e(d, u)) \rightarrow \min, \quad u \in \mathbb{R}^m$$

where $q_e(d, u)$ is defined by formula (21). The solution to this optimization problem can be written as

$$\bar{u} = -\left(\bar{B}^T K^{-1} \bar{B}^T\right)^{-1} \bar{B}^T K^{-1} d.$$  \hspace{1cm} (23)

**Proof.** By substituting the value of $q = q_e$ given by formula (21) into the cost function (22), we formulate the problem under consideration as the minimization of

$$f(u) = J(q_e) = \left(u^T \bar{B}^T K^{-1} + d^T K^{-1}\right) K^{-1} B u + K^{-1} d).$$

Then we obtain expression (23) from the condition $\nabla f(u) = 0$, where $\nabla f(u)$ is the gradient of $f(u)$. The global character of the minimum at $u = \bar{u}$ follows from the fact that $J(q)$ is a positive definite quadratic form under the assumptions of this lemma.

### 5.2. Identification of disturbances and observer design

In Lemma 5.1, we have assumed that the disturbance $d(t)$ is known and constant. If it is not the case, the values of $d(t)$ should be reconstructed according to the output signal $y(t)$ and the input $u(t)$ of system (19). This problem can be solved in a straightforward way for $d(t) = \text{const}$ by using an extended output vector. For this purpose we compute successive derivatives of the output signal $y(t)$ of system (19) by assuming that $u(t)$ is sufficiently smooth and $d$ is constant:

$$\frac{d^j}{dt^j} y(t) = CA^j x(t) + CA^j P d + C \sum_{i=0}^{j-1} A^{j-i-1} B \frac{d^i}{dt^i} u(t), \quad j = 1, 2, \ldots.$$  \hspace{1cm} (24)

Based on this representation, we introduce the vector function $Y(t) \in \mathbb{R}^{lp}$ for some $l \geq 1$ such that the components of $Y(t)$ can be evaluated in terms of $y(t)$, $u(t)$ and their derivatives:
\[
Y(t) = \begin{pmatrix}
Y_1(t) \\
Y_2(t) \\
\vdots \\
Y_l(t)
\end{pmatrix},
\quad Y_1(t) = y(t),
\quad Y_j(t) = \frac{d^{j-1}}{dt^{j-1}}y(t) - C \sum_{i=0}^{j-2} A^{j-i-2} B \frac{d^i}{dt^i} u(t),
\quad j \geq 2.
\]

(25)

The following result allows us to represent \(d\) and \(x(t)\) in terms of \(Y(t)\).

**Lemma 5.2.** Assume that \(\text{rank } S = 3N\) for some \(k \geq 1\), where

\[
S = \begin{pmatrix}
S_0 \\
S_1 \\
\vdots \\
S_k
\end{pmatrix},
\quad S_0 = (\bar{C} \quad 0 \quad 0),
\quad S_j = (-1)^{j-1} \begin{pmatrix}
0_{p \times N} \\
\bar{CA}_1^{j-1} \\
0 \\
\bar{CA}_1^{j-1} \bar{M}^{-1}
\end{pmatrix},
\quad j \geq 1,
\]

\[
A_1 = M^{-1} K.
\]

(26)

If \(x(t)\) is a solution of system (19) with a control \(u \in C^{2k}[t_0, t_1)\) \((t_0 < t_1 \leq +\infty)\) and a disturbance \(d = \text{const} \in \mathbb{R}^p\), then the components of \(x(t)\) and \(d\) are uniquely expressed in terms of \(y(t)\), \(u(t)\) and their derivatives:

\[
\begin{pmatrix}
x(t) \\
d
\end{pmatrix} = S^+ Y(t),
\quad \text{for all } t \in [t_0, t_1),
\]

(27)

where \(S^+\) is the pseudoinverse of \(S\),

\[
Y(t) = \begin{pmatrix}
y(t) \\
y(t) \\
Y_3(t) \\
\vdots \\
Y_{2k+1}(t)
\end{pmatrix},
\quad Y_j(t) = \frac{d^{j-1}}{dt^{j-1}}y(t) + \sum_{i=0}^{j-2} (-1)^{j+i-1} \bar{CA}_1^i \bar{M}^{-1} B \frac{d^{j-2r-3}}{dt^{j-2r-3}} u(t).
\]

Proof. Representations (24) and (25) with \(l = 2k + 1\) allow us to write the following system of linear algebraic equations with respect to \(x(t)\) and \(d\):

\[
S \begin{pmatrix}
x(t) \\
d
\end{pmatrix} = Y(t),
\]

(28)

where the blocks of \(S\) are obtained from formulas (26) by collecting the coefficients with \(x\) and \(d\) in expressions (24). We also exploit the block structure of \(A, B, C\) and \(P\) here. Note that system (28) is consistent as \(x(t)\) is a solution corresponding to \(u(t)\) and \(d\). The uniqueness of its solution (formula (27)) follows from the assumption that the matrix \(S\) is of full rank.

The procedure described in Lemma 5.2 requires the computation of time-derivatives of the input and output signals, which may not be efficient from the numerical viewpoint. To
avoid possible difficulties concerning numerical differentiation, we present another approach for estimating $x(t)$ by means of the Luenberger observer (cf. [24, Chapter 3], [25]) below.

**Theorem 5.3.** Let $M$ and $K$ be positive definite matrices, and let the rank condition

$$\text{rank}(\tilde{C}^T, A_0\tilde{C}^T, \ldots, A_0^{N-1}\tilde{C}^T) = N, \quad A_0 = KM^{-1}$$

be satisfied. Then, for any initial data $x(0) = x^0 \in \mathbb{R}^{2N}$ and admissible control $u \in L^\infty([0, +\infty); \mathbb{R}^m)$, the corresponding solution $x(t)$ of system (19) satisfies the property

$$\|x(t) - z(t)\| = O(e^{-\lambda t}) \to 0 \text{ as } t \to +\infty,$$

for some $\lambda > 0$ and any solution $z(t)$ of the observer system

$$\dot{z}(t) = (A - HC)z(t) + Bu(t) + Pd(t) + Hy(t).$$

Here $y(t)$ is the output signal of system (19) and (20),

$$H = \begin{pmatrix} \beta_0 K^{-1}\tilde{C}^T \\ 0_{N \times p} \end{pmatrix},$$

and $\beta_0$ is an arbitrary positive constant.

**Proof.** Let us introduce the observation error $\theta(t) = z(t) - x(t)$ and write the error dynamics from differential equations (19) and (31) as follows:

$$\dot{\theta}(t) = (A - HC)\theta(t).$$

In order to prove the exponential stability of the equilibrium $\theta = 0$, we introduce a quadratic Lyapunov function

$$2V(\theta) = (K\theta_1, \theta_1) + (M\theta_2, \theta_2), \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \theta_1 \in \mathbb{R}^N, \quad \theta_2 \in \mathbb{R}^N.$$

Then the time-derivative of $V$ along the trajectories of system (33) is

$$\dot{V}(\theta) = (K\theta_1, \theta_2 - \beta_0 K^{-1}\tilde{C}^T\tilde{C}\theta_1) + (M\theta_2, -M^{-1}K\theta_1).$$

As $M$ and $K$ are symmetric, the above expression takes the form

$$\dot{V}(\theta) = -\beta_0(\theta_1, \tilde{C}^T\tilde{C}\theta_1) = -\beta_0\|\tilde{C}\theta_1\|^2 \leq 0.$$

Under conditions of Theorem 5.3, the function $V$ is positive definite and its time-derivative is nonpositive. Thus, the trivial solution of system (33) is stable in the sense of Lyapunov. To prove its asymptotic stability, we apply the Barbashin–Krasovskii theorem.
[26] (or LaSalle’s invariance principle). For this purpose we show that the only invariant subset of

\[ S_0 = \left\{ \theta \in \mathbb{R}^{2N} | \dot{V}(\theta) = -\beta_0 \| \bar{C} \theta_1 \|^2 = 0 \right\} \]  

(34)
is the equilibrium point \( \theta = 0 \). Indeed, let \( \theta(t) = \left( \theta_1(t), \theta_2(t) \right) \) be a solution of system (33) such that \( \theta(t) \in S_0 \) for all \( t \geq 0 \). Then formula (34) implies that \( \bar{C} \theta_1(t) = 0 \) for all \( t \geq 0 \). By differentiating this identity with respect to \( t \), we obtain:

\[ \bar{C} \theta_1(t) = \bar{C} \dot{\theta}_1(t) = \bar{C} \ddot{\theta}_1(t) = \ldots = \bar{C} \frac{d^{2N-1}}{dt^{2N-1}} \theta_1(t) = 0. \]

Then we represent the derivatives of \( \theta_1(t) \) in terms of \( \theta_1(t) \) and \( \theta_2(t) \) from system (33) as follows:

\[
\begin{align*}
\bar{C} \theta_1 &= 0, \\
\bar{C} \dot{\theta}_1 &= \bar{C} \theta_2 = 0, \\
\bar{C} \ddot{\theta}_1 &= -\bar{C} M^{-1} K \theta_1 = 0, \\
\bar{C} \frac{d^{3} \theta_1}{dt^{3}} &= -\bar{C} M^{-1} K \theta_2 = 0, \\
\vdots \quad \bar{C} \frac{d^{2N-2} \theta_1}{dt^{2N-2}} &= \bar{C} (-M^{-1} K)^{N-1} \theta_1 = 0, \\
\bar{C} \frac{d^{2N-1} \theta_1}{dt^{2N-1}} &= \bar{C} (-M^{-1} K)^{N-1} \theta_2 = 0.
\end{align*}
\]

(35)
The rank condition (29) implies that equalities (35) may be satisfied only with \( \theta_1 = \theta_2 = 0 \). It means that the only solution \( \theta(t) \in S_0, t \geq 0 \) of system (33) is \( \theta(t) = 0 \). Hence, the trivial solution of system (33) is asymptotically stable by the Barbashin–Krasovskii theorem. This proves the assertion of Theorem 5.3 as the properties of asymptotic and exponential stability are equivalent for linear system (33).

5.3. Stabilization problem

For the rest of this section we assume that the disturbance \( d(t) \) can be compensated by an appropriate control action \( u \) in system (19), that is \( d(t) \in \mathcal{R}(\bar{B}) \) for all \( t \geq 0 \). Under this assumption we introduce the function \( d^0(t) = \bar{B}^+ d(t) \), where \( \mathcal{R}(\bar{B}) \) stands for the range of the operator \( \bar{B} : \mathbb{R}^m \rightarrow \mathbb{R}^N \), and \( \bar{B}^+ \) is the pseudoinverse of \( \bar{B} \).

Let us first consider the stabilization problem with a state feedback law.

**Theorem 5.4.** Let \( M \) and \( K \) be positive definite matrices, \( d(t) \in \mathcal{R}(\bar{B}) \) for all \( t \geq 0 \), and let the rank condition

\[
\operatorname{rank}(\bar{B}, A_0 \bar{B}, \ldots, A_0^{N-1} \bar{B}) = N, \quad A_0 = KM^{-1},
\]

(36)
be satisfied. Then the trivial solution of the closed-loop system (19) with

\[ u = -Fx(t) - \bar{B}^+ d(t), \quad F = (0_{m \times N}, \beta_1 \bar{B}^T) \]

(37)
is exponentially stable. Here $\beta_1$ is an arbitrary positive constant.

**Proof.** The closed-loop system (19) with (37) the following form in matrix notations:

$$\dot{x}(t) = (A - BF)x(t). \quad (38)$$

To prove the exponential stability of the solution $x = 0$ of this system, we consider the following Lyapunov function

$$2V(x) = (Kx_1, x_1) + (Mx_2, x_2), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_1 \in \mathbb{R}^N, \ x_2 \in \mathbb{R}^N.$$

The time-derivative of this function along the trajectories of system (38) is

$$\dot{V}(x) = (Kx_1, x_2) + (Mx_2, -M^{-1}Kx_1 + M^{-1}Bu + M^{-1}d) = -\beta_1 \left\| \bar{B}^T x_2 \right\|^2 \leq 0.$$

We see that $V(x)$ is positive definite and $\dot{V}(x) \leq 0$ under the conditions of this theorem. To complete the proof, we check the largest invariant subset of

$$Z_0 = \{ x \in \mathbb{R}^{2N} | \dot{V}(x) = 0 \} = \{ x \in \mathbb{R}^{2N} | \bar{B}^T x_2 = 0 \}.$$

As in Theorem 5.3, we conclude that the only solution $x(t) \in Z_0 \ (t \geq 0)$ of system (38) is $x(t) \equiv 0$. Thus, the trivial solution of this closed-loop system is asymptotically stable by the Barbashin–Krasovskii theorem. $\square$

By combining Theorems 5.3 and 5.4, we obtain the following result on the observer-based stabilization.

**Theorem 5.5.** Let $M$ and $K$ be positive definite matrices, $d(t) \in \mathcal{R}(\bar{B})$ for all $t \geq 0$, and let the rank conditions (29) and (36) be satisfied.

Then the solution $x = z = 0$ of the extended system (19), (20), (31) with

$$u = -Fz(t) - \bar{B}^T d(t) \quad (39)$$

is exponentially stable, where the matrices $H$ and $F$ are defined in formulas (32) and (37), $\beta_0$ and $\beta_1$ are arbitrary positive constant.

**Proof.** Let us rewrite the extended system (19), (20), (31) and (39) with respect to functions $x(t)$ and $\theta(t) = z(t) - x(t)$ in the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = X \begin{pmatrix} x \\ \theta \end{pmatrix}, \quad X = \begin{pmatrix} A - BF & -BF \\ 0_{2N \times 2N} & A - HC \end{pmatrix}.$$

(40)

The spectrum of $X$ is the union of the spectra of its blocks $A - BF$ and $A - HC$. Thus, exponential stability results established in Theorem 5.3 for system (33) and in Theorem 5.4 for system (38) imply that the matrix $X$ is Hurwitz. Therefore, the trivial solution of system (40) is exponentially stable, which proves the assertion of Theorem 5.5.
6. Simulation results

In this section, we present some numerical results concerning the dynamics of the Stuttgart SmartShell structure in a neighbourhood of its equilibrium configuration. We assume that the median surface \( \Omega \) of the shell is parameterized as \( z = r^3(x,y) \) for this equilibrium. Here \( Oxyz \) is a Cartesian frame in \( \mathbb{R}^3 \), and the domain \( D \) is chosen as the projection of \( \Omega \) to the horizontal plane \( Oxy \). Thus, we identify the Lagrangian coordinates \((\xi^1, \xi^2) \in D \) with \((x,y) \) at the reference equilibrium: \( r_j(\xi^1, \xi^2) = \xi^j, j = 1,2 \). The shell is supported at its corner parts \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) on the boundary of \( D \) (Figure 2).

We treat the shell as isotropic and use the following mechanical parameters:

\[
E = 10^{10} \text{Pa}, \quad \nu = 0.065, \quad \rho = 16.92 \text{ kg/m}^2, \quad h = 0.04 \text{ m}.
\]

In order to compute the coefficients of control system (19), we use numerical data concerning the reference configuration \( r(\xi^1, \xi^2) \) and the first two modes \( r_k(\xi^1, \xi^2), k = 1,2 \). Such information is obtained from the finite element model of the Stuttgart SmartShell structure constructed in ANSYS and described in the paper [1]. Namely, we use nodal displacements of that model with 4762 nodes as approximate values of \( r(\xi^1, \xi^2) \) and \( r_k(\xi^1, \xi^2), k = 1,2 \). The nodal displacements, represented as arrays of \( xyz \)-coordinates, allow us to compute the integrals in formulas (13) numerically. For this computation, we generate a triangulation of \( D \) and replace the integrals over \( D \) with finite sums over the triangles in formulas (13). The triangulation with 371 triangles whose vertices are located at nodes of the finite element model is used in our simulation. The spatial derivatives of \( r(\xi^1, \xi^2) \) and \( r_k(\xi^1, \xi^2) \) are approximated by finite differences with six-point stencils at the nodes.

Our algorithm for computing the integrals in formulas (13), implemented in C++, produces the following matrices of systems (12) and (19):

![Figure 2. The domain D.](image-url)
\[
M = \begin{pmatrix}
0.1348 & 0.0017 \\
0.0017 & 0.1408
\end{pmatrix}, \quad
K = \begin{pmatrix}
33.4242 & 1.3555 \\
1.3555 & 32.7192
\end{pmatrix},
\]

\[
\bar{B} = \begin{pmatrix}
2.53 & 8.81 \\
33.9 & 17.5
\end{pmatrix} \cdot 10^{-3}, \quad
d = \begin{pmatrix}
0.00705 \\
0.00726
\end{pmatrix} \phi_x,
\]

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-247.78 & -7.09 & 0 & 0 \\
-6.6 & -232.33 & 0 & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0.016 & 0.064 \\
0.24 & 0.124
\end{pmatrix}, \quad
Pd = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0.0516 \\
0.051
\end{pmatrix} \phi_x.
\]

(41)

The matrix \( B \) corresponds to controls \( u_1 \) and \( u_2 \) which act as \( z \)- and \( x \)-components, respectively, of the force applied at a point close to \( \Gamma_1 \). The term \( Pd \) stands for the disturbance force \( \phi_x \) applied along the \( x \)-direction at the top of the shell (i.e. at the point with \( \xi^1 = \xi^2 = 0 \)).

Note that we do not use any information concerning the modal frequencies or other parameters of the finite element model for the derivation of Galerkin’s system (19).

The matrix \( A \) in formulas (41) has two pairs of purely imaginary eigenvalues \( \pm i \omega_1 \) and \( \pm i \omega_2 \). These eigenvalues allow us to compute the first two modal frequencies \( \nu_k = \frac{\omega_k}{2\pi} (k = 1, 2) \) for system (19), as shown in Table 1. We see that the corresponding modal frequencies of both methods are very close to each other.

To illustrate the response of the shell under different transient loadings, we compute solutions \( x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T \) of system (19) with coefficients given by formulas (41) under several choices of controls \( u_1 \) and \( u_2 \). We present the displacement \( \Delta(t) \) at the point \( \xi^1 = \xi^2 = 0 \) as a characteristic of the shell vibrations in all time plots of this section:

\[
\Delta(t) = \| x_1(t)r^1(0, 0) + x_2(t)r^2(0, 0) \|.
\]

The response under the load \( \phi_x = 5000 \text{ N} \) is illustrated in Figure 3 for the initial data \( x(0) = 0 \) and parameters \( u_1 = u_2 = 0 \). We observe that the upper point of the shell performs undamped oscillations with the amplitude \( \Delta_{\text{max}} \approx 3 \text{ cm} \). In order to reduce these vibrations, we will apply feedback controls of form (37).

Let us first consider the problem of stabilizing the shell by applying a force in the \( z \)-direction only, that is we put \( u_2 = 0 \) and define \( u_1(x) \) by formula (37) as in the single-input case. This yields the following controller:

Table 1. Modal frequencies for the finite element model (FEM) and system (19).

| Frequency | \( \nu_1 \) (Hz) | \( \nu_2 \) (Hz) |
|-----------|----------------|----------------|
| FEM       | 2.4112         | 2.4763         |
| System (19)| 2.4123        | 2.5184         |
where $\bar{B}_1$ is the first column of $\bar{B}$ from formulas (41), and the tuning parameter is chosen as $\beta_1 = 150$. The response of the closed-loop system (19) and (42) is shown in Figure 4 for $\phi_x = 5000$ N and $x(0) = 0$. The overshoot for $\Delta(t)$ is about 2 cm, and $\Delta(t)$ is close to 1 cm after the time $t = 30$ s. We see that the control force $u_1$, acting in the z-direction close to $\Gamma_1$, is able to reduce vibrations of the shell; however, the relaxation of these vibrations is rather slow. We also note that $\Delta(t)$ does not tend to zero for large $t$ because the condition $d(t) \in R(\bar{B}_1)$ of Theorem 5.4 is not satisfied in this case. Thus, only partial compensation of the disturbance $\phi_x$ by means of the control $u_1$ is possible.

In addition to the previous case, let us now consider two independent controls $u_1$ and $u_2$, acting in the z- and x-directions, respectively. We get the following feedback law from formula (37):

$$u_1(x) = -\left(0_{1 \times 2}, \beta_1 \bar{B}_1^T\right)x(t) - \bar{B}_1^+ d(t) = \beta_1 (0.00253x_3 + 0.0339x_4) - 0.2284\phi_x,$$

$$u_2(x) = \beta_1 (0.00881x_3 + 0.0175x_4) - 0.867\phi_x,$$

(43)

Figure 5 illustrates the behaviour of the closed-loop system (19) and (43) for the initial disturbance $\Delta(0) \approx 2$ cm and parameters $\phi_x = 5000$ N, $\beta_1 = 150$.

One can see that the application of two independent controls (Figure 5) is able to improve the transient behaviour of our model significantly in comparison with the single-input case (Figure 4).
Figure 4. Closed-loop response: single input.

Figure 5. Closed-loop response: two inputs.
7. Conclusions

In this paper, Hamilton’s principle is used to derive the variational problem describing the dynamics of a controlled shell. For this variational formulation, a family of finite-dimensional approximations is constructed by exploiting the generalized Ritz–Galerkin method. The main advantage of our approach relies on an analytic representation of the coefficients of control system (19) and explicit control design in Section 5. As Table 1 shows, our approach produces approximately the same modal frequencies as the finite element model. Although exponential stability results in Theorems 5.3–5.5 are valid for any degrees of freedom of the model, the computational efficiency of our control design remains to be an issue for future work.

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Notes

1. We have also used the fundamental lemma of the calculus of variations to derive formula (6).
2. Nonlinear terms are omitted in differential equation (10).
3. The normal vector \( n \) to the median surface is represented in Equation (1).
4. Here, and in the sequel, we denote by \( 0_{i \times j} \) the block of zeroes with \( i \) rows and \( j \) columns.

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