More on the anti-automorphism of the Steenrod algebra

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The relations of Barratt and Miller are shown to include all relations among the elements $P_i P^{n-i}$ in the mod $p$ Steenrod algebra, and a minimal set of relations is given.

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1 Introduction

Milnor [4] observed that the mod 2 Steenrod algebra $A$ forms a Hopf algebra with commutative diagonal determined by

$$\Delta Sq^n = \sum_i Sq^i \otimes Sq^{n-i}.$$  (1)

This allowed him to interpret the Cartan formula as the assertion that the cohomology of a space forms a module-algebra over $A$. The anti-automorphism $\chi$ in the Hopf algebra structure, defined inductively by

$$\chi Sq^0 = Sq^0, \quad \sum_i Sq^i \chi Sq^{n-i} = 0 \quad \text{for } n > 0,$$  (2)

has a topological interpretation too: If $K$ is a finite complex then the homology of the Spanier–Whitehead dual $DK_+$ of $K_+$ is canonically isomorphic to the cohomology of $K$. Under this isomorphism the left action by $\theta \in A$ on $H^*(K)$ corresponds to the right action of $\chi \theta \in A$ on $H_*(DK_+)$.  

In 1974 Davis [3] proved that sometimes much more efficient ways exist to compute $\chi Sq^n$; for example

$$\chi Sq^{2^r-1} = Sq^{2^r-1} \chi Sq^{2^r-1},$$  (3)

$$\chi Sq^{2^r-r-1} = Sq^{2^r-1} \chi Sq^{2^r-1-r} + Sq^{2^r-1} \chi Sq^{2^r-1-r-1}.$$  (4)

Similarly, Straffin [6] proved that if $r \geq 0$ and $b \geq 2$ then

$$\sum_i Sq^{2^r i} \chi Sq^{2^r (b-i)} = 0.$$  (5)
Both authors give analogous identities among reduced powers and their images under $\chi$ at an odd prime as well. Further relations among the Steenrod squares and their conjugates appear in these articles and elsewhere (eg Silverman [5]).

Barratt and Miller [1] found a general family of identities which includes (3), (4) and (5), and their odd-prime analogues, as special cases. We state it for the general prime. When $p = 2$, $P^n$ denotes $\text{Sq}^n$. Let $\alpha(n)$ denote the sum of the $p$–adic digits of $n$.

**Theorem 1.1** [1; 2] For any integer $k$ and any integer $l \geq 0$ such that $pl - \alpha(l) < (p-1)n$,

\[ \sum_i \binom{k-i}{l} p^i \chi P^{n-i} = 0. \]

The relations defining $\chi$ occur with $l = 0$. Davis’ formulas (for $p = 2$) are the cases in which $(n, l, k) = (2^r - 1, 2^r-1 - 1, 2^r - 1)$ or $(n, l, k) = (2^r - r - 1, 2^{r-1} - 2, 2^r - 2)$.

Straffin’s identities (for $p = 2$) occur as $(n, l, k) = (2^r b, 2^r - 1, -1)$.

Since $\binom{k+1-i}{l} - \binom{k-i}{l-1} = \binom{k-i}{l-1}$, the cases $(l, k + 1)$ and $(l, k)$ of (6) imply it for $(l - 1, k)$. Thus the relations for $l = \phi(n) - 1$, where

\[ \phi(n) = 1 + \max\{ j : pj - \alpha(j) < (p-1)n \}, \]

imply all the rest. Here we have adopted the notation $\phi(n)$ used in [2]; we note that it is not the Euler function $\varphi(n)$.

When $p = 2$, $\phi(2^r - 1) = 2^{r-1}$ and $\phi(2^r - r - 1) = 2^{r-1} - 1$, so Davis’s relations are among these basic relations.

Two questions now arise. To express them uniformly in the prime, let $\mathcal{P}$ denote the algebra of Steenrod reduced powers (which is the full Steenrod algebra when $p = 2$), but assign $P^n$ degree $n$. Write

\[ V_n = \text{Span}\{ P^i \chi P^{n-i} : 0 \leq i \leq n \} \subseteq \mathcal{P}^n. \]

It is natural to ask:

- Are there yet other linear relations among the $n + 1$ elements $P^i \chi P^{n-i}$ in $\mathcal{P}^n$?
- What is a basis for $V_n$?

We answer these questions in Theorem 1.4 below.

Write $e_i, 0 \leq i \leq n$, for the $i$–th standard basis vector in $\mathbb{F}_p^{n+1}$.

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Proposition 1.2  For any integers $l, m, n$, with $0 \leq l \leq n$,

\[ \left\{ \sum_i \binom{k-i}{l} e_i : m \leq k \leq m + l \right\} \]

is linear independent in $\mathbb{F}_p^{n+1}$.

Proposition 1.3  The set

\[ \left\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \right\} \]

is linearly independent in $\mathcal{P}^n$.

Define a linear map

\[ \mu: \mathbb{F}_p^{n+1} \to \mathcal{P}^n, \quad \mu e_i = P^i \chi P^{n-i}. \]

Theorem 1.1 implies that if $l = \phi(n) - 1$ the elements in (8) lie in ker $\mu$, so Propositions 1.2 and 1.3 imply that (8) with $l = \phi(n) - 1$ is a basis for ker $\mu$ and that (9) is a basis for $V_n \subseteq \mathcal{P}^n$. Thus:

Theorem 1.4  Any $\phi(n)$ consecutive relations from the set (6) with $l = \phi(n) - 1$ form a basis of relations among the elements of $\{ P^i \chi P^{n-i} : 0 \leq i \leq n \}$. The set $\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \}$ is a basis for $V_n$.

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2  Independence of the relations

We wish to show that (8) is a linearly independent set. Regard elements of $\mathbb{F}_p^{n+1}$ as column vectors, and arrange the $l + 1$ vectors in (8) as columns in a matrix, which we claim is of rank $l + 1$. The top square portion is the mod $p$ reduction of the $(l + 1) \times (l + 1)$ integral Toeplitz matrix $A_I(m)$ with $(i, j)$–th entry

\[ \binom{m + j - i}{l}, \quad 0 \leq i, j \leq l. \]

Lemma 2.1  $\det A_I(m) = 1$. 

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Proof. By induction on $m$. Since $\binom{-1}{i} = (-1)^i$ and $\binom{-1+j}{i} = 0$ for $0 < j \leq l$, $A_l(-1)$ is lower triangular with determinant $((-1)^l)^{l+1} = 1$. Now we note the identity

$$BA_l(m) = A_l(m + 1)$$

where

$$B = \begin{bmatrix}
\binom{l+1}{1} & -\binom{l+1}{2} & \cdots & (-1)^{l-1}\binom{l+1}{l} & (-1)^l\binom{l+1}{l+1} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}.$$

The matrix identity is an expression of the binomial identity

$$(11) \quad \sum_k (-1)^k \binom{l+1}{k} \binom{n-k}{l} = 0$$

(taking $n = m + 1 - j$ and $k = j + 1$). Since $\det B = 1$, the result follows for all $m \in \mathbb{Z}$.

For completeness, we note that (11) is the case $m = l + 1$ of the equation

$$(12) \quad \sum_k (-1)^k \binom{m}{k} \binom{n-k}{l} = \binom{n-m}{l-m}.$$  

To prove this formula, note that the defining identity for binomial coefficients implies the case $m = 1$, and also that both sides satisfy the recursion $C(l, m, n) - C(l, m, n - 1) = C(l, m + 1, n)$.

3 Independence of the operations

We will prove Proposition 1.3 by studying how $P^i \chi P^{n-i}$ pairs against elements in $P_*$, the dual of the Hopf algebra of Steenrod reduced powers. According to Milnor [4], with our grading conventions

$$P_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots], \quad |\xi_j| = \frac{p^j - 1}{p - 1},$$

$$\Delta \xi_k = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j.$$
For a finitely nonzero sequence of nonnegative integers \( R = (r_1, r_2, \ldots) \) write \( \xi^R = \xi_r^1 \xi_r^2 \cdots \) and let \( \|R\| = r_1 + pr_2 + p^2r_3 + \cdots \) and
\[
|R| = |\xi^R| = r_1 + \left( \frac{p^2 - 1}{p - 1} \right) r_2 + \left( \frac{p^3 - 1}{p - 1} \right) r_3 + \cdots.
\]
The following clearly implies Proposition 1.3.

**Proposition 3.1** For any integer \( n > 0 \) there exist sequences \( R_{n,j}, 0 \leq j \leq n - \phi(n) \), such that \( |R_{n,j}| = n \) and
\[
\langle P^i \chi P^{n-i}, \xi_{R_{n,j}} \rangle = \begin{cases} 
+1 & \text{for } i = n - j \\
0 & \text{for } i > n - j
\end{cases}.
\]
The starting point in proving this is the following result of Milnor.

**Lemma 3.2** \([4, \text{Corollary 6}]\) \( \langle \chi P^n, \xi^R \rangle = \pm 1 \) for all sequences \( R \) with \( |R| = n \).

In the basis of \( \mathcal{P} \) dual to the monomial basis of \( \mathcal{P}_* \), the element corresponding to \( \xi^1 \) is \( P^1 \). Since the diagonal in \( \mathcal{P}_* \) is dual to the product in \( \mathcal{P} \), it follows from (13) and Lemma 3.2 that
\[
\langle P^i \chi P^{n-i}, \xi^R \rangle = \begin{cases} 
+1 & \text{for } i = \|R\| \\
0 & \text{for } i > \|R\|
\end{cases}.
\]
So we wish to construct sequences \( R_{n,j} \), for \( \phi(n) \leq j \leq n \), such that \( |R_{n,j}| = n \) and \( \|R_{n,j}\| = j \). We deal first with the case \( j = \phi(n) \).

**Proposition 3.3** For any \( n \geq 0 \) there is a sequence \( M = (m_1, m_2, \ldots) \) such that
\begin{enumerate}
  \item \( |M| = n \),
  \item \( 0 \leq m_i \leq p \) for all \( i \),
  \item if \( m_j = p \) then \( m_i = 0 \) for all \( i < j \).
\end{enumerate}

For any such sequence, \( \|M\| = \phi(n) \).

**Proof** Give the set of sequences of dimension \( n \) the right-lexicographic order. We claim that the maximal sequence satisfies the hypotheses.

Suppose that \( R = (r_1, r_2, \ldots) \) does not satisfy the hypotheses. If \( r_1 > p \) then the sequence \( (r_1 - (p + 1), r_2 + 1, r_3, \ldots) \) is larger. If \( r_j > p \), with \( j > 1 \), then the sequence \( (r_1, \ldots, r_{j-2}, r_{j-1} + p, r_j - (p + 1), r_{j+1} + 1, r_{k+2}, \ldots) \) is larger. This proves (2). To prove (3), suppose that \( r_j = p \) with \( j > 1 \), and suppose that some earlier entry is nonzero. Let \( i = \min\{k : r_k > 0\} \). If \( i = 1 \), then the sequence
(r_1 - 1, r_2, \ldots, r_{j-1}, 0, r_{j+1} + 1, r_{j+2}, \ldots) is larger. If i > 1, then S with s_k = 0 for \( k < i - 1 \) and \( i \leq k \leq j \), \( s_{i-1} = p \), \( s_{j+1} = r_{j+1} + 1 \), and \( s_k = r_k \) for \( k > j + 1 \), is larger. Let \( M \) be a sequence satisfying (1)–(3), and write \( l = \|M\| - 1 \). To see that \( l = \phi(n) - 1 \) we must show that

(14) \[ p(l + 1) - \alpha(l + 1) \geq (p - 1)n, \]

(15) \[ pl - \alpha(l) < (p - 1)n. \]

The excess \( e(R) \) is the sum of the entries in \( R \), so that \( p\|R\| - e(R) = (p - 1)|R| \). The \( p \)-adic representation of a number minimizes excess, so for any sequence \( R \) we have \( e(R) \geq \alpha(\|R\|) \) and hence \( p\|R\| - \alpha(\|R\|) \geq (p - 1)|R| \); so (14) holds for any sequence.

To see that (15) holds for \( M \), let \( j = \min\{i : m_i > 0\} \), so that \( (p - 1)n = (p^j - 1)m_j + (p^{j+1} - 1)m_{j+1} + \cdots \) and \( l + 1 = p^{j-1}m_j + p^jm_{j+1} + \cdots \). The hypotheses imply that \( l \) has \( p \)-adic expansion

\[ (1 + \cdots + p^{j-2})(p - 1) + p^{j-1}(m_j - 1) + p^jm_{j+1} + \cdots, \]

so

\[ \alpha(l) = (j - 1)(p - 1) + (m_j - 1) + m_{j+1} + \cdots, \]

from which we deduce

\[ pl - \alpha(l) = (p - 1)(n - j) < (p - 1)n. \]

This completes the proof of Proposition 3.3.

\[ \square \]

**Corollary 3.4** The function \( \phi(n) \) is weakly increasing.

**Proof** Let \( M \) be a sequence satisfying the conditions of Proposition 3.3, and note that the sequence \( R = (1, 0, 0, \ldots) + M \) has \( |R| = n + 1 \) and \( \|R\| = \|M\| + 1 = \phi(n) + 1 \). If \( p \) does not occur in \( M \), then \( R \) satisfies the hypotheses of the proposition (in degree \( n + 1 \)) and hence \( \phi(n) \leq \phi(n + 1) \). If \( p \) does occur in \( M \), then the moves described above will lead to a sequence \( M' \) satisfying the hypotheses. None of the moves decrease \( \|\cdot\| \), so \( \phi(n) \leq \phi(n + 1) \).

\[ \square \]

**Remark 3.5** Properties (1)–(3) of Proposition 3.3 in fact determine \( M \) uniquely.

**Proof of Proposition 3.1** Define \( R_{n, \phi(n)} \) to be a sequence \( M \) as in Proposition 3.3. Then inductively define

\[ R_{n, j} = (1, 0, 0, \ldots) + R_{n-1, j-1} \quad \text{for } \phi(n) < j \leq n. \]

This makes sense by monotonicity of \( \phi(n) \), and the elements clearly satisfy \( |R_{n, j}| = n \) and \( \|R_{n, j}\| = j \). This completes the proof.

\[ \square \]
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