Phenomenological signatures of gauge invariant theories of gravity with vectorial and gradient nonmetricity

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In this paper we discuss on the phenomenological footprints of theories where the gravitational effects are due not only to spacetime curvature, but also to nonmetricity. These theories are characterized by gauge invariance. Due to their simplicity, here we focus in theories with vectorial nonmetricity. We make special emphasis in gradient nonmetricity theories which are based in Weyl integrable geometry (WIG) spaces. While arbitrary and vectorial nonmetricities may have played a role in the quantum epoch, gradient nonmetricity can be important for the description of gravitational phenomena in our classical world instead. This would entail that gauge symmetry may be an actual symmetry of our past, present and future universe, without conflict with the standard model of particles (SMP). We show that, in a gauge invariant world modeled by WIG spacetime, the vacuum energy density is a dynamical quantity, so that the cosmological constant problem (CCP) may be avoided. Besides, due to gauge invariance, and to the fact that photons and radiation do not interact with nonmetricity, the accelerated pace of cosmic expansion can be explained without the need for the dark energy. We also discuss on the “many-worlds” interpretation of the resulting gauge invariant framework, where general relativity (GR) is just a specific gauge of the theory. The unavoidable discrepancy between the present value of the Hubble parameter computed on the GR basis and its value according to the gauge invariant theory, may explain the Hubble tension issue. It will be shown also that, due to gauge freedom, inflation is not required in order to explain the flatness, horizon and relict particles abundance problems within the present framework.

I. INTRODUCTION

Weyl geometry [1], the theoretical framework where gauge symmetry was introduced for the first time, played an important role in the early search for alternatives of unification of the fundamental interactions [2–10]. It represented an interesting generalization of Riemann geometry where vectorial nonmetricity – see Eq. (6) below – was assumed. Nonetheless, discussions on the occurrence of the so called “second clock effect” (SCE) [11–22], ruled it out as phenomenologically nonviable.

Recently generalized nonmetricity theories, where the covariant derivative of the metric does not vanish [23],

\[ \nabla_{\alpha}g_{\mu\nu} = -Q_{\alpha\mu\nu}, \]

with \( Q_{\alpha\mu\nu} \) – the nonmetricity tensor, have played an interesting role in the search for alternative explanations to fundamental questions of current interest. The recent resurrection of nonmetricity theories is mainly focused in the so called teleparallel [24–32] and, specially, the symmetric teleparallel theories [33–42] and their cosmological applications [43–49]. However, in the bulk of these papers, gauge invariance is ignored. Only in Ref. [43], where the role of conformal symmetry within symmetric teleparallel framework is discussed, and in Ref. [24], nonmetricity is investigated from the point of view of gauge symmetry.\(^1\) This is one of the most important properties of nonmetricity geometry due to covariance of \( g_{\mu\nu} \) under the following Weyl gauge transformations [23, 45]:

\[ g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \]
\[ Q^\alpha_{\;\mu\nu} \rightarrow Q^\alpha_{\;\mu\nu} - 2\partial_\alpha \ln \Omega g_{\mu\nu}, \]
\[ Q_{\alpha\mu\nu} \rightarrow \Omega^2 \left( Q_{\alpha\mu\nu} - 2\partial_\alpha \ln \Omega g_{\mu\nu} \right), \]
\[ Q_\alpha \rightarrow Q_\alpha - 2\partial_\alpha \ln \Omega, \]

where the conformal transformation of the metric does not represent a diffeomorphism or, properly, a conformal isometry, i. e., the spacetime coincidences or events (as well as the spacetime coordinates that label the events,) are not modified by the conformal transformations. In Eq. (2) the positive smooth function \( \Omega = \Omega(x) \) is the conformal factor and we used the following definition \( Q_\alpha := Q^\lambda_{\alpha\lambda} = Q^\lambda_{\alpha\lambda} \).

Gauge freedom, an immediate consequence of gauge invariance, represents a challenge from the point of view of its physical and geometrical implications within the gravitational context. In order to understand the reach of the challenge let us make a comparison with \( U(1) \) gauge freedom in electrodynamics. In this case gauge fixing the vector potential amounts to a mathematical procedure allowing simplification of subsequent computations. However, within the framework of a gauge invariant theory of gravitation, since we deal with a whole class of conformal equivalent theories, gauge fixing amounts to choosing a specific gravitational theory. Hence, gauge fixing in this case is not a trivial question. There should be a way in which we could determine the gauge where we (and the rest of the matter fields in the universe) live in. As we shall show, this would be the one which better describes the existing amount of observational and

\(^1\) Gauge symmetry and its breaking within the framework of Weyl geometry has been recently investigated in [50–52] in connection with model building beyond the SMP and inflation.
experimental evidence.

Another aspect of the challenge deals with the compatibility of Weyl gauge symmetry with the SMP. It is a well-known fact that photons and radiation in general do not interact with nonmetricity \[ \mathbb{R} \] or, in other words; photons and radiation are blind to nonmetricity so that these “see” only the pseudo-Riemann structure of spacetime. This is the reason why the Weyl gauge vector cannot be identified with the electromagnetic potential. Hence, before electroweak (EW) symmetry breaking the massless particles of the SMP react only to the curvature of pseudo-Riemann space so that nonmetricity may be ignored. After \( SU(2) \times U(1) \) symmetry breaking the SMP particles acquire masses but, then, as widely accepted the existence of masses breaks the gauge symmetry associated with nonmetricity. It seems that there is no room for nonmetricity in the SMP.

Can anyway EW symmetry breaking and Weyl gauge symmetry coexist together? \( SU(2) \times U(1) \) symmetry breaking may be associated with the Higgs Lagrangian density,

\[
\mathcal{L}_H = - \frac{1}{2} |D_\mu H|^2 - \frac{\lambda}{2} (|H|^2 - v_0^2)^2, \tag{3}
\]

where \( v_0 \) is the EW mass parameter, \( H \) is the Higgs doublet, and we use the following notation: \( |H|^2 \equiv H^\dagger H \), \( |D_\mu H|^2 \equiv g^\mu\nu (D^\mu_H H) (D^\nu_H H) \),

\[
D^\mu_H H = \left( \partial_\mu + \frac{i}{2} g W^k_\mu \sigma^k + \frac{i}{2} g B_\mu \right) H, \tag{4}
\]

with \( W^k_\mu \) – the \( SU(2) \) bosons, \( B_\mu \) – the \( U(1) \) boson, \( (g, g') \) – gauge couplings and \( \sigma^k \) are the Pauli matrices. Under the conformal transformation in Eq. \( \{2\} \), \( (H, H^\dagger) \rightarrow \Omega^{-1} (H, H^\dagger) \), so that the Higgs action,

\[
S_H = \int d^4x \sqrt{-g} \mathcal{L}_H, \tag{5}
\]

is not invariant under the (Weyl) gauge transformations \( \{2\} \). Hence, if we expect gauge symmetry to survive EW symmetry breaking, the Lagrangian density \( \{3\} \) has to be modified. The required modification amounts to lifting the mass parameter \( v_0 \) to a point dependent field \( \{53, 54\} \): \( v_0 \rightarrow v(x) \), such that under \( \{2\} \), \( v^2 \rightarrow \Omega^{-2} v^2 \).

Besides, the EW gauge covariant derivative in Eq. \( \{1\} \) is to be replaced as well: \( D^\mu_H \rightarrow D^\mu_H - Q_\mu/2 \), so that, under the gauge transformations \( \{2\} \), \( D^\mu_H \rightarrow \Omega^{-1} D^\mu_H \Rightarrow |D_\mu H|^2 \rightarrow \Omega^{-4} |D_\mu H|^2 \). Lifting of the mass parameter to a point dependent field \( v(x) \) leads to the masses acquired by the particles of the SMP after EW symmetry breaking, being point dependent quantities as well:\(^2\)

\[ m = m(x) \]. Under \( \{2\} \) the mass \( m \) of given particle transforms like \( m \rightarrow \Omega^{-4} m \). Hence, Weyl gauge symmetry may survive after \( SU(2) \times U(1) \) symmetry breaking and thus it may play a role in the past, present and future of the cosmic evolution of our universe. Notice that our approach is very different from the one undertaken in the bulk of papers on gauge theories of gravity, where the Weyl gauge symmetry breaks down either through the Higgs procedure or through other alternative mechanisms \( \{54, 55\} \).

Separate comment deserves Weyl integrable geometry. It is a particular case of Weyl geometry which is characterized by gradient nonmetricity (see below). WIG is usually detracted by incorrectly identifying it with pseudo-Riemann geometry and the corresponding theory of gravity is identified with general relativity.\(^3\) This is incorrect! WIG is a class of geometries while the corresponding gauge invariant theory of gravity is a whole class of theories. GR is just a specific gauge in this class. Different gauges represent different theories. As we shall see these can be differentiated through the check of the observational and experimental evidence.

The main goal of the present paper is to show how gauge freedom explains several outstanding puzzles in the forefront of current science. As we shall see gauge freedom may offer a completely new alternative explanation of several fundamental problems in cosmology, such as: i) the cosmological constant problem \( \{61–65\} \), ii) the accelerated pace of the cosmic expansion \( \{62, 66–74\} \), iii) the Hubble constant tension \( \{75\} \) and other puzzles. We shall focus in spaces with vectorial nonmetricity \( Q_{\alpha\mu\nu} = Q_{\alpha g_{\mu\nu}} \), since arbitrary nonmetricity brings with it several issues of fundamental character such as ambiguity in the definition of the gauge covariant derivative operators \( \{20\} \) and in the determination of the actual role geodesics and autoparallels play within the geometrical structure of generalized Weyl spaces \( \{40, 41\} \) (in spaces with arbitrary nonmetricity the autoparallels and the geodesics do not coincide.) We shall show that while vectorial (and arbitrary) nonmetricity can play a role in the quantum gravitational laws, only gradient nonmetricity \( (\partial_\alpha \varphi g_{\mu\nu}) \) may play in important role in the classical gravitational description of our past, present and future world.

This paper can be divided into three parts. The first part, consisting of Secs. \( \{IV\} \) is dedicated mainly to expose the required mathematical formalism. In the second part (Secs. \( \{VI-VIII\} \)) we discuss on the geometrical and physical aspects of a gauge invariant theory of gravity with vectorial nonmetricity and we conclude that such a theory can have impact in the description of quantum gravitational phenomena but not in our clas-

\(^2\) Point dependent masses which transform under the conformal transformation of the metric as \( m \rightarrow \Omega^{-4} m \), are considered by Dicke in his paper \( \{53\} \).

\(^3\) There are, however, several works \( \{53, 54, 55\} \) where the role played by WIG in the description of the gravitational phenomena is investigated.
sical world. On the contrary, a gauge invariant theory based in WIG spaces with gradient nonmetricity, may play an important role in the classical description of the gravitational phenomena. The third part of the paper, composed of Secs. [X] to [XVII] is dedicated to the investigation of the phenomenological consequences of such a theory. We shall show that gauge freedom, an inevitable consequence of gauge invariance, can take account of the CCP (Sec. [XIII]), of the accelerated pace of the expansion (Sec. [XIII]) and of the flatness, horizon and relict particles abundance problems (Sec. [XVI]), at once. Other current cosmological issues such as the Hubble tension problem are explained in this framework as well (see Sec. [XVII]). A required discussion of the main results of this research, as well as brief conclusions are given in section [XVII].

Unless otherwise stated, here we use the units \( h = c = 1 \) and the following signature of the metric is chosen: \((- + + +)\). Greek indices run over spacetime \( \alpha, \beta, ..., \mu, ... = 0, 1, 2, 3 \), while latin indices \( i, j, k, ... = 1, 2, 3 \) run over three-dimensional space. Some times the spatial components of a vector \( v^i \) will be represented as three-dimensional vectors \( \vec{v} \). Bold-type letters \( \mathbf{v} \) will represent four-dimensional (spacetime) vectors instead. Hence, for instance, \( \mathbf{v} = \{v^0, \vec{v}\} \).

II. BACKGROUND AND CONVENTIONS

Weyl geometry space, denoted here by \( \tilde{W}_4 \), is defined as the class of four-dimensional (torsionless) manifolds \( \mathcal{M}_4 \) that are paracompact, Hausdorff, connected \( C^\infty \), endowed with a locally Lorentzian metric \( g \) that obeys the vectorial nonmetricity condition:

\[
\nabla_\alpha g_{\mu\nu} = -Q_\alpha g_{\mu\nu},
\]

where \( Q_\alpha \) is the Weyl gauge vector and the covariant derivative \( \nabla_\mu \) is defined with respect to the torsion-free affine connection of the manifold:

\[
\Gamma^\alpha_{\mu\nu} = \left\{^\alpha_{\mu\nu}\right\} + L^\alpha_{\mu\nu},
\]

where

\[
\left\{^\alpha_{\mu\nu}\right\} := \frac{1}{2} g^{\alpha\lambda} \left( \partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda} \right),
\]

is the Levi-Civita (LC) connection, while

\[
L^\alpha_{\mu\nu} := \frac{1}{2} \left( Q_\nu \delta^\alpha_\mu + Q_\mu \delta^\alpha_\nu - Q^\alpha g_{\mu\nu} \right),
\]

is the disformation tensor. The Weyl gauge vector \( Q_\alpha \) measures how much the length of given vector varies during parallel transport.

In this paper we call as “generalized curvature tensor” of \( \tilde{W}_4 \) spacetime, the curvature of the connection, symbolically \( R(\Gamma) \), whose coordinate components are,

\[
R^\alpha_{\sigma\mu\nu} := \partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\sigma},
\]

or, if take into account the decomposition (7):

\[
R^\alpha_{\sigma\mu\nu} = \hat{R}^\alpha_{\sigma\mu\nu} + \nabla_\mu L^\alpha_{\nu\sigma} - \nabla_\nu L^\alpha_{\mu\sigma}
\]

+ \hat{L}^\lambda_{\mu\nu} L^\alpha_{\nu\lambda} - L^\alpha_{\nu\lambda} \hat{L}^\lambda_{\mu\sigma},
\]

where \( \hat{R}^\alpha_{\sigma\mu\nu} \) is the Riemann-Christoffel or LC curvature tensor,

\[
\hat{R}^\alpha_{\sigma\mu\nu} := \partial_\mu \left\{^\alpha_{\nu\sigma}\right\} - \partial_\nu \left\{^\alpha_{\mu\sigma}\right\}
\]

+ \left\{^\alpha_{\nu\lambda}\right\} \left\{^\lambda_{\mu\sigma}\right\} - \left\{^\alpha_{\nu\sigma}\right\} \left\{^\lambda_{\mu\lambda}\right\},
\]

and \( \hat{\nabla}_\alpha \) is the LC covariant derivative. Besides, the LC Ricci tensor \( R_{\mu\nu} = \hat{R}^\alpha_{\mu\alpha\nu} \), and LC curvature scalar read:

\[
\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu},
\]

respectively. We call \( R^\alpha_{\sigma\mu\nu} \) as generalized curvature tensor because it is contributed both by LC curvature \( \hat{R}^\alpha_{\sigma\mu\nu} \), and by nonmetricity through disformation \( L^\alpha_{\mu\nu} \). We have that,

\[
R_{\mu\nu} = \hat{R}_{\mu\nu} + \hat{\nabla}_\lambda L^\lambda_{\mu\nu} - \hat{\nabla}_\nu L^\lambda_{\lambda\mu}
\]

+ \hat{L}^\kappa_{\lambda\mu} L^\lambda_{\nu\kappa} - L^\lambda_{\nu\kappa} \hat{L}^\kappa_{\lambda\mu},
\]

\[
R = \hat{R} + Q + \partial Q,
\]

where the nonmetricity scalar \( Q \) and the boundary term \( \partial Q \) are defined as it follows:

\[
Q := L^\tau_{\tau\lambda} L^\lambda_{\kappa\kappa} - L^\tau_{\tau\kappa} L^\lambda_{\lambda\kappa} = -\frac{3}{2} Q_\mu Q^\mu,
\]

\[
\partial Q := \hat{\nabla}_\lambda \left( L^\lambda_{\kappa\kappa} - L^\kappa_{\kappa\lambda} \right) = -3 \hat{\nabla}_\mu Q^\mu.
\]

The generalized curvature tensor \( R^\alpha_{\sigma\mu\nu} \) has various contractions. In order to show these contractions let us write Eq. (14) in the following form:

\[
R_{\alpha\sigma\mu\nu} = \hat{R}_{\alpha\sigma\mu\nu} + \nabla_\mu L^\alpha_{\sigma\nu} - \nabla_\nu L^\alpha_{\mu\sigma}
\]

+ \hat{L}^\lambda_{\alpha\lambda} L^\lambda_{\nu\sigma} - \hat{L}^\lambda_{\alpha\lambda} \hat{L}^\lambda_{\mu\sigma}.
\]

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4 When the generalized nonmetricity condition (1) is satisfied, the resulting space is denoted by \( \tilde{W}_4 \) and it is called as generalized Weyl space.
The various linearly independent contractions of the generalized curvature tensor are,

\[ R_{\mu\nu} := g^{\lambda\kappa} R_{\lambda\mu\kappa\nu}, \quad \hat{R}_{\mu\nu} := g^{\lambda\kappa} R_{\mu\lambda\kappa\nu}, \]

\[ R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \hat{R}_{\mu\nu}. \quad (18) \]

The first two of these amount to,

\[ R_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} \left( Q_\lambda Q^\lambda + \hat{\nabla}_\lambda Q^\lambda \right) g_{\mu\nu} + \frac{1}{2} Q_\mu Q_{\nu} - \hat{\nabla}_\nu Q_\mu + \frac{1}{2} \left( \hat{\nabla}_\nu Q_\mu - \hat{\nabla}_\mu Q_\nu \right), \quad (19) \]

and to,

\[ \hat{R}_{\mu\nu} = \hat{\nabla}_\mu Q_{\nu} - \hat{\nabla}_\nu Q_\mu. \quad (21) \]

There are two more contractions of the generalized curvature tensor in \( \mathcal{W}_4 \) space differ from those in Riemann space \( V_4 \). For instance:

\[ R^{\alpha}_{\sigma\mu\nu} = -R^{\alpha}_{\sigma\nu\mu}, \quad (29) \]

\[ R_{\alpha\sigma\mu\nu} = -R_{\alpha\sigma\nu\mu} - (\partial_{[\mu} Q_{\nu]} - \partial_{[\nu} Q_{\mu]}) g_{\alpha\sigma}. \quad (30) \]

The last equation, known as the third Bianchi identity, in compact form can be written in the following way:

\[ \nabla_\mu R^{\sigma}_{\alpha\lambda\nu\sigma} + \nabla_\nu R^{\alpha}_{\sigma\lambda\mu\sigma} + \nabla_\sigma R^{\alpha}_{\lambda\mu\nu\sigma} = 0, \quad (27) \]

where we have defined \( \nabla_\mu := \nabla_\mu + Q_\mu \) (see below in Eq. \( (11) \) of Sect. \( IV \) the definition of the gauge covariant derivative \( \nabla_\mu^g \)). Hence, \( \nabla_\alpha^g g_{\mu\nu} = 0 \).

A. Properties and identities of the curvature in \( \mathcal{W}_4 \)

The torsionless connection \( \nabla \) of \( \mathcal{W}_4 \) space satisfies the first (cyclic) identity \( \nabla_\mu R^{\lambda}_{\nu\kappa\lambda} + \nabla_\kappa R^{\lambda}_{\nu\mu\lambda} + \nabla_\lambda R^{\lambda}_{\nu\mu\kappa} = 0 \), as well as the second Bianchi identity,

\[ \nabla_\mu R^{\alpha}_{\nu\lambda\kappa} + \nabla_\nu R^{\alpha}_{\lambda\mu\kappa} + \nabla_\kappa R^{\alpha}_{\nu\lambda\mu} = 0. \]

In general the symmetries of the generalized curvature tensor are, relations:

\[ R^{\alpha}_{\mu\nu\sigma} = R^{\alpha}_{\nu\mu\sigma}, \quad (26) \]

\[ R^{\alpha}_{\sigma\mu\nu} = R^{\alpha}_{\sigma\nu\mu}, \quad (29) \]

\[ R_{\alpha\sigma\mu\nu} = -R_{\alpha\sigma\nu\mu} - (\partial_{[\mu} Q_{\nu]} - \partial_{[\nu} Q_{\mu]}) g_{\alpha\sigma}. \]

The identity can be written, alternatively, as it follows:

\[ \partial_{[\mu} Q_{\nu]} g_{\alpha\sigma} = R_{(\alpha\sigma)[\mu\nu]}, \quad (31) \]

respectively. We shall call \( R_{\mu\nu} \) as first Ricci tensor while \( \hat{R}_{\mu\nu} \) we shall call as second Ricci tensor. Notice that only the second Ricci tensor is symmetric in its indices: \( \hat{R}_{\mu\nu} = \hat{R}_{\nu\mu} \). Both \( R_{\mu\nu} \) and \( \hat{R}_{\mu\nu} \) are related by the following relationships:

\[ R_{\mu\nu} - \hat{R}_{\mu\nu} = \hat{\nabla}_\mu Q_{\nu} - \hat{\nabla}_\nu Q_\mu, \quad (21) \]

and

\[ R_{\mu\nu} + \hat{R}_{\mu\nu} = 2 \hat{\nabla}_\mu Q_{\nu} - \left( Q_\lambda Q^\lambda + \hat{\nabla}_\lambda Q^\lambda \right) g_{\mu\nu} + Q_\mu Q_{\nu} - 2 \hat{\nabla}_\nu Q_\mu, \quad (22) \]

respectively.

\[ R_{\alpha\mu\nu} = \hat{R}_{\alpha\mu\nu}, \quad (23) \]

\[ R_{[\mu\nu]} = \hat{\nabla}_\mu Q_{\nu} - \hat{\nabla}_\nu Q_\mu. \quad (24) \]

Besides, for the Einstein's tensor \( G_{\mu\nu} := R_{\mu\nu} - g_{\mu\nu} R/2 \) we obtain that,

\[ G_{(\mu\nu)} = \hat{G}_{(\mu\nu)} = \hat{R}_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R. \quad (25) \]

\[ 5 \] In Eq. 30 we have taken into account the symmetry of the connection in its second and third indices:

\[ \nabla_\mu Q_{\nu} - \nabla_\nu Q_\mu = \nabla_\mu Q_{\nu} - \nabla_\nu Q_\mu = \partial_{[\mu} Q_{\nu]} - \partial_{[\nu} Q_{\mu]}. \]
\[ \nabla^\lambda R_{\lambda\mu} = \nabla^\lambda \tilde{R}_{\lambda\mu} + \nabla^\lambda (\nabla_\lambda Q_\mu - \nabla_\mu Q_\lambda), \]

then,

\[ \nabla^\lambda \tilde{G}_{\lambda\mu} = -\frac{1}{2} \nabla^\lambda (\nabla_\lambda Q_\mu - \nabla_\mu Q_\lambda). \] \hspace{1cm} (33)

On the other hand, if take into account Eq. (24), then

\[ \nabla^\lambda G_{\mu\lambda} = -\frac{2}{3} \nabla^\lambda (\nabla_\lambda Q_\mu - \nabla_\mu Q_\lambda) \]

or

\[ \nabla^\lambda G_{(\lambda\mu)} = \nabla^\lambda \tilde{G}_{\lambda\mu} = -\frac{1}{2} \nabla^\lambda (\nabla_\lambda Q_\mu - \nabla_\mu Q_\lambda). \] \hspace{1cm} (34)

III. WEAy GAUGE SYMMEtRY

Weyl gauge symmetry (WGS) or invariance under local changes of scale, is a manifest symmetry of \( W_4 \) spaces.\(^6\) The geometric laws that define \( W_4 \), among which is the nonmetricity condition (6), are invariant under generalized (local) Weyl rescalings or, also, Weyl gauge transformations. These represent a particular case of (2) and amount to simultaneous conformal transformations of the metric and gauge transformations of the vector \( Q_\alpha \):

\[ g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \Omega^{-2} g^{\mu\nu}, \]

\[ Q_\alpha \rightarrow Q_\alpha - 2\partial_\alpha \ln \Omega, \] \hspace{1cm} (35)

respectively. Notice that, while the gauge vector \( Q_\alpha \) transforms like in (35), its variation \( \delta Q_\mu = Q'_\mu - Q_\mu \), where \( Q_\mu \) and \( Q'_\mu \) are different vector functions (although they are very close to each other), is not transformed by the Weyl gauge transformations. In other words, the variation vector \( \delta Q_\alpha \) has vanishing weight \( w(\delta Q_\alpha) = 0 \).

In what follows we shall call the transformations (35) either as Weyl gauge transformations or, simply, as gauge transformations. Under (35):

\[ \{^\alpha_{\mu\nu}\} \rightarrow \{^\alpha_{\mu\nu}\} + (\delta^\alpha_\mu \partial_\nu + \delta^\alpha_\nu \partial_\mu - g_{\mu\nu} \partial^\alpha) \ln \Omega, \]

\[ L^\alpha_{\mu\nu} \rightarrow L^\alpha_{\mu\nu} - (\delta^\alpha_\mu \partial_\nu + \delta^\alpha_\nu \partial_\mu - g_{\mu\nu} \partial^\alpha) \ln \Omega, \] \hspace{1cm} (36)

so that the generalized affine connection \( \Gamma \) is unchanged by the Weyl rescalings:

\[ \Gamma^\alpha_{\mu\nu} \rightarrow \Gamma^\alpha_{\mu\nu}. \] \hspace{1cm} (37)

This means that the generalized curvature tensor \( R^a_{\sigma\mu\nu} \) in (10) and the generalized Ricci tensor, \( R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \), are unchanged as well, \( R^a_{\mu\sigma\nu} \rightarrow R^a_{\mu\sigma\nu}, \quad R_{\mu\nu} \rightarrow R_{\mu\nu} \), while the generalized curvature scalar transforms as,

\[ R \rightarrow \Omega^{-2} R. \] \hspace{1cm} (38)

It can be straightforwardly demonstrated, as well, that the third Bianchi identity \( \Box \) is a gauge invariant expression.

Another important quantity through this paper is the traceless second-rank tensor with coordinate components,

\[ Q_{\mu\nu} := 2\nabla_{[\mu} Q_{\nu]} = \nabla_\mu Q_\nu - \nabla_\nu Q_\mu = \partial_\mu Q_\nu - \partial_\nu Q_\mu. \] \hspace{1cm} (39)

Under the gauge transformations (35) it is not transformed, so that it has vanishing weight \( w(\delta Q_{\mu\nu}) = 0 \). The quantity \( \Box \) represents that part of the curvature which is due to nonmetricity of \( W_4 \) space. We shall call it as gauge tensor. It is not difficult to demonstrate that,

\[ Q^{\mu\nu} = g^{\mu\lambda} g^{\nu\kappa} Q_{\lambda\kappa} = \nabla^{\mu} Q^{\nu} - \nabla^{\nu} Q^{\mu}, \] \hspace{1cm} (40)

so that \( w(Q^{\mu\nu}) = -4 \).

IV. GAUGE SYMMETRY AND PARALLEL TRANSPORT

Parallel transport consistent with gauge symmetry is required to define gauge covariant differentiation of vectors and tensors in generalized Weyl spaces. Below we shall expose the theory of gauge invariant parallel transport, which is cornerstone to discuss, among others, on the second clock effect. Although our exposition contains new elements not previously considered, to a great extent it is based in the work of references [3, 4].

A. Gauge derivative operators

In order to make the gauge symmetry compatible with well-known derivation rules and with the inclusion of fields into \( W_4 \), it is necessary to introduce the Weyl gauge derivative operators in a way that is equivalent to the one appearing in [3, 4, 6]. Let \( T \) be a \((p, q)\)-tensor in \( W_4 \), with coordinate components \( T^\alpha_{\beta_1\beta_2...\beta_q} \) and with conformal weight \( w(T) = w \), so that under (35): \( T \rightarrow \Omega^w T \). Then, the Weyl gauge differential of the tensor, its Weyl
gauge derivative and Weyl gauge covariant derivative, respectively, are defined as it follows:

\[ d^* T := dT + \frac{w}{2} Q_\lambda dx^\lambda T, \]
\[ \partial^*_\alpha T := \partial_\alpha T + \frac{w}{2} Q_\alpha T, \]
\[ \nabla^*_\alpha := \nabla_\alpha + \frac{w}{2} Q_\alpha. \]  (41)

where

\[ d^* T = dx^\mu \partial^*_\mu T. \]  (42)

In general, for any geometrical object \( O \) with weight \( w \), the gauge covariant derivative reads,

\[ \nabla^*_\alpha O = \nabla_\alpha O + \frac{w}{2} Q_\alpha O. \]  (43)

The above definitions warrant that the gauge differential, the gauge derivative and the gauge covariant derivative, transform like the geometrical object itself, i.e., under (35):

\[ d^* T \rightarrow \Omega^* d^* T, \]
\[ \partial^*_\alpha T \rightarrow \Omega^* \partial^*_\alpha T, \]
\[ \nabla^*_\alpha T \rightarrow \Omega^* \nabla^*_\alpha T, \]
\[ \nabla^*_\alpha O \rightarrow \Omega^* \nabla^*_\alpha O. \]  (44)

As an illustration, the gauge covariant derivative of the metric tensor (the conformal weight of the metric \( w(g) = 2 \)), vanishes:

\[ \nabla^*_\alpha g_{\mu \nu} = 0, \]
\[ \nabla^*_\alpha \sqrt{-g} = 0 \Rightarrow \nabla_\alpha \sqrt{-g} = -2 \sqrt{-g} Q_\alpha, \]  (45)

while for the vector density: \( O = \sqrt{-g} V^\mu \), where the weights \( w(\sqrt{-g}) = 4 \) and \( w(V^\mu) = w_V \), we have that,

\[ \nabla^*_\alpha (\sqrt{-g} V^\mu) = \nabla_\alpha (\sqrt{-g} V^\mu) \]
\[ + \left( \frac{4 + w_V}{2} \right) \sqrt{-g} Q_\alpha V^\mu. \]  (46)

From this equation it follows, in particular, that for a vector density \( \sqrt{-g} V^\mu \) with vanishing weight \( w(\sqrt{-g} V^\mu) = 0 \) (i.e., \( w_V = -4 \)),

\[ \nabla^*_\mu (\sqrt{-g} V^\mu) = \nabla_\mu (\sqrt{-g} V^\mu) \]
\[ = \sqrt{-g} \nabla_\mu V^\mu = \partial_\mu (\sqrt{-g} V^\mu). \]  (47)

Another useful expression involving the gauge covariant derivative is the following one,

\[ \nabla^*_\mu Q_\nu = \nabla_\mu Q_\nu = \hat{\nabla}_\mu Q_\nu = \partial_\mu Q_\nu. \]  (48)

### B. Parallel transport in \( \tilde{W}_4 \) space

Let \( C \) be a curve in \( \tilde{W}_4 \) that is parametrized by the affine parameter \( \xi \). I.e., \( C \) has coordinates \( x^\mu(\xi) \). We can define the gauge covariant derivative along the path \( x^\mu(\xi) \) to be given by the following operator:

\[ \frac{D^*}{d\xi} := \frac{dx^\mu}{d\xi} \nabla^*_\mu, \]  (49)

where the gauge covariant derivative \( \nabla^*_\mu \) is given by (41). Then, the parallel transport of given tensor \( T \) with coordinate components \( T^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} \), along the path \( x^\mu(\xi) \), is defined by the following requirement (this definition coincides with the one in [3, 6]):

\[ \frac{D^* T}{d\xi} := \frac{dx^\mu}{d\xi} \nabla^*_\mu T = 0 \Leftrightarrow \frac{D^* T}{d\xi} T^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} = 0. \]  (50)

The above parallel transport law is satisfied by any tangent vector of weight \( w = -1 \), as well. Let us consider, for instance, a space-like unit vector \( t \) with coordinate components \( t^\mu \), such that:

\[ t^2 := (t, t) = g_{\mu \nu} t^\mu t^\nu = 1. \]  (51)

Since \( w(t^2) = w(1) = 0 \) and \( w(g_{\mu \nu}) = 2 \), then the weight of the tangent unit vector is \( w(t^\mu) = -1 \). Applying (49) to both sides of (51) and taking into account (49), we get that:

\[ \frac{D^* t^\mu}{d\xi} = 0, \]  (52)

as it should be.

### V. GAUGE SYMMETRY, AUTOPARALLELS AND GEODESICS

In general autoparallels – “straightest curves” of the geometry – do not coincide with the geodesics, which are the “shortest curves” [40, 41, 76]. There goes a disussion on whether autoparallels or geodesics describe the motion of test particles in spaces \( W_4 \) with generalized nonmetricity \( Q_{\alpha \mu \nu} \) [20, 40, 41]. However, in Weyl space \( \tilde{W}_4 \), autoparallels and geodesics coincide as in GR. Anyway, geodesics and autoparallels can be associated exclusively with the motion of spinless point particles. Spinor fields like the fermions obey the Dirac equation in curved background, while extended spinning test bodies obey the Mathisson-Papapetrou-Dixon equations [77, 80].

#### A. Auto-parallels

In Weyl space \( \tilde{W}_4 \) the “timelike” autoparallels are those curves along which the gauge covariant derivative
of the tangent four-velocity vector $u$, vanishes. Here $u^\mu = dx^\mu/d\tau$ are the coordinate components of $u$ and, as long as this does not cause loss of generality, we chose the proper time $\tau$ to be the affine parameter along the autoparallel curve. The conformal weight of the four-velocity vector $u^\mu\nabla^*_\mu u = 0$, or, in explicit form, in terms of the arc-length $d\tau \to ds$:

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\mu \nu}^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \frac{1}{2} Q_\mu h^\mu_\alpha = 0 \Leftrightarrow \frac{d^2 x^\alpha}{ds^2} + \frac{(\alpha_{\mu \nu})}{2} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \frac{1}{2} Q_\mu h^\mu_\alpha = 0, \quad (54)$$

where

$$h^\mu_\alpha := g^\mu_\alpha + u^\mu u^\alpha = g^\mu_\alpha - \frac{dx^\mu}{ds} \frac{dx^\alpha}{ds}, \quad (55)$$

is the orthogonal projection tensor, which projects any vector or tensor onto the hypersurface orthogonal to the four-velocity vector $u^\mu = dx^\mu/d\tau$.

The same parallel transport law is obeyed by the four-momentum vector $p = mu$, where $m$ is the mass of the point particle. Since under (35) the point mass transforms like $d\tau \to \Omega^{-1} m$, it has a conformal weight $w(m) = -1$. Consequently, the weight of the four-momentum $w(p) = -2$. Hence, from the law of parallel transport of the four-momentum

$$\frac{D^* p}{d\tau} = u^\mu \nabla^*_\mu p = u^\mu (\nabla^*_\mu m) u + mu^\mu \nabla^*_\mu u = 0, \quad (56)$$

it follows that,

$$u^\mu \nabla^*_\mu m = 0 \Rightarrow \delta m - \frac{m}{2} Q_\mu dx^\mu = 0. \quad (57)$$

Here we use variation instead of differentiation to underline that, in general, $\delta m$ is not a perfect differential. Integration of equation (57) along given path $C$, joining the origin $x = \{0\}$ with the point with coordinates $x = \{\tau\}$ and parametrized by some affine parameter $\tau$, yields

$$m(x, C) = m_0 \exp \left( \frac{1}{2} \int_C Q_\mu dx^\mu \right) = m_0 \exp \left( \frac{1}{2} \int_C Q_\mu u^\mu d\tau \right), \quad (58)$$

where $m_0$ is an integration constant that we can identify with the value of the parameter $m$ evaluated at the origin $m_0 = m(0)$. Eq. (58) is the basis of the second clock effect. It says that the mass $m$ of a point particle at some point $x^\mu$ depends not only on the point but also on the path joining this point with the origin.

In the same fashion, in $\mathcal{W}_4$ the “null” autoparallels are those curves along which the gauge covariant derivative of the wave vector $k$ with components $k^\mu := dx^\mu/d\lambda$ ($\lambda$ is a parameter along the null autoparallel), vanishes:

$$\frac{D^* k^\alpha}{d\lambda} = k^\mu \nabla^*_\mu k^\alpha = 0 \Rightarrow \frac{dk^\alpha}{d\lambda} + \Gamma_{\mu \nu}^\alpha k^\mu k^\nu - Q_\mu k^\mu k^\alpha = 0, \quad (59)$$

where we have taken into account that the conformal weight of the wave vector $w(k) = -2$, i. e., it coincides with the weight of the four-momentum since, in the quantum limit both should be related by $p = \hbar k$. Then, the autoparallel null curves satisfy the following equations:

$$\frac{dk^\alpha}{d\lambda} + \Gamma_{\mu \nu}^\alpha k^\mu k^\nu - Q_\mu k^\mu k^\alpha = 0 \Leftrightarrow \frac{dk^\alpha}{d\lambda} + \left( \alpha_{\mu \nu} \right) k^\mu k^\nu = 0, \quad (59)$$

where, in the last line, we took into account that $\left( k, \kappa \right) = g_{\mu \nu} k^\mu k^\nu = 0$. In other words: photons and radiation in general do not interact with the gauge vector $Q_\alpha$.

It is seen that both, time-like and null autoparallels Eqs. (57) and (59), respectively, are invariant under the gauge transformations Eq. (35). The same is true of the corresponding geodesic equations which, as we shall see below, coincide with the autoparallel equations. While gauge invariance is manifest for the time-like geodesics (autoparallels), for the null geodesics this is less clear. In order to show that Eq. (59) is indeed a gauge covariant equation, it is enough to realize that, under (35) the wave vector $k^\alpha \to \Omega^{-2} k^\alpha \Rightarrow d\lambda \to \Omega^2 d\lambda$, and that

$$\frac{dk^\alpha}{d\lambda} \to \Omega^{-4} \left[ \frac{dk^\alpha}{d\lambda} - 2k^\alpha \frac{d\ln \Omega}{d\lambda} \right], \quad \left( \alpha_{\mu \nu} \right) k^\mu k^\nu \to \Omega^{-4} \left[ \alpha_{\mu \nu} k^\mu k^\nu + 2k^\alpha \frac{d\ln \Omega}{d\lambda} \right].$$

B. Geodesic equations

The geodesic equations are equations of motion in the sense that these are the result of applying the variational

---

Here it is implicitly assumed that the universal constant $\hbar$ is not transformed by the Weyl gauge transformations.
principle of least action. Time-like and null particles follow geodesics. When these are compared with the corresponding auto-parallels one can measure how much the motion paths depart from the straightest curves of the geometry.

In the GR context the action of timelike particles reads \( S = m \int ds \), where \( m \) is the constant mass parameter. In Weyl space \( \tilde{W}_4 \), the mass, being the squared length of the four-momentum of the particle, varies in spacetime, then \( m \) can not be taken out of the action integral. The action integral in \( \tilde{W}_4 \) reads:

\[
S = \int \frac{m}{ds} ds.
\]

From this action the following equations of motion – geodesic equations – can be derived:

\[
\frac{d^2 x^\alpha}{ds^2} + \left(\alpha_{\mu\nu}\right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{1}{2} \frac{\delta m}{m \delta x^\nu} h^{\alpha\nu} = 0,
\]

(60)

where the non-Riemannian term \( \frac{\delta m}{m \delta x^\mu} \) accounts for the variation of mass during parallel transport. Hence, from Eq. (57) it follows that,

\[
\frac{1}{m} \frac{\delta m}{\delta x^\nu} = \frac{1}{2} Q_\mu.
\]

(61)

This shows that time-like autoparallels and time-like geodesics coincide in \( \tilde{W}_4 \) space.

In a similar way the null geodesic equations can be derived from the following action:

\[
S_{\text{null}} = \frac{1}{2} \int g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\xi,
\]

(62)

where the dot accounts for derivative with respect to the parameter \( \lambda \) of the path \( x^\mu(\lambda) \) followed by photons (by radiation in general). From (62) the GR null geodesic equations are obtained. These coincide with the null autoparallels Eq. (57). Hence, the null geodesic equations do not depend on \( Q_{\alpha\mu} \). This means that photons and radiation probe the Riemann affine structure of spacetime. In other words, photons and radiation interact only with the metric field, i.e., with the LC curvature of spacetime. These do not interact with the gauge vector \( Q_\alpha \).

VI. GAUGE INVARIANT THEORY OF GRAVITY

Let us consider the simplest Einstein-Hilbert (EH) gravitational action in \( \tilde{W}_4 \) space:

\[
S_{\text{EH}} = \frac{1}{2} \int_{\mathcal{M}_4 \in \tilde{W}_4} d^4 x \sqrt{-g} m_{\text{pl}}^2 R,
\]

(63)

where \( m_{\text{pl}} \) is the (reduced) Planck mass and \( R = g^{\mu\nu} R_{\mu\nu} \) is the generalized curvature scalar of \( \tilde{W}_4 \), while the generalized Ricci tensor \( R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \), with the generalized curvature tensor of \( \tilde{W}_4 \) given by Eq. (10).

In general the gravitational action (63) is not gauge invariant. Actually, under the gauge transformations (35), \( \sqrt{-g} \rightarrow \Omega^4 \sqrt{-g} \) and \( R \rightarrow \Omega^{-2} R \), so that \( \sqrt{-g}R \rightarrow \Omega^2 \sqrt{-g}R \). The only way in which (63) can be transformed into a gauge invariant action is by lifting the Plank mass to a point-dependent quantity, as any other mass parameter, so that, under (35): \( m_{\text{pl}}^2 \rightarrow \Omega^{-2} m_{\text{pl}}^2 \). This is, precisely, the approach we shall follow here.

Other pieces have to be included in the action in order to get rid of higher energy gravitational effects. Higher order curvature terms are required by renormalization of (a would be) quantum gravity \( [81, 82] \). Besides, quadratic (and also cubic) corrections arise as counterterms at one loop in gravity coupled to matter \( [83] \) and at two loops in pure gravity \( [84] \) and also in the effective string gravitational action \( [85, 87] \). Here, in addition to the Einstein-Hilbert action Eq. (63), we shall consider quadratic contributions to the curvature:

\[
S_{\text{gauge}} = \frac{\alpha^2}{c} \int_{\mathcal{M}_4 \in \tilde{W}_4} d^4 x \sqrt{-g} \left\{ a R_{[\mu\nu]} R^{[\mu\nu]} + b R_{(\lambda\kappa)[\mu\nu]} R^{(\lambda\kappa)[\mu\nu]} \right\},
\]

(64)

where \( a, b \) and \( c \) are real constants such that \( a + b = c \), \( \alpha^2 \) is a dimensionless coupling constant and the gauge tensor \( Q_{\mu\nu} \) is defined in Eq. (49). According to the third Bianchi identity in the form of Eq. (52) we have that,

\[
R_{(\lambda\kappa)[\mu\nu]} R^{(\lambda\kappa)[\mu\nu]} = R_{[\mu\nu]} R^{[\mu\nu]} = Q_{\mu\nu} Q^{\mu\nu}.
\]

(65)

Hence the action (64) can be written in the following way:

\[
S_{\text{gauge}} = \alpha^2 \int_{\mathcal{M}_4 \in \tilde{W}_4} d^4 x \sqrt{-g} Q_{\mu\nu} Q^{\mu\nu}.
\]

(66)

Of course, this piece of action is gauge invariant as well. Equation (66) shows that the precise form of the quadratic terms in the action Eq. (64) leads to second-order equations of motion, hence, to the absence of Ostrogradski ghosts.

VII. VARIATIONAL PRINCIPLE AND THE EQUATIONS OF MOTION

During the variational process, whenever it is possible, covariant derivatives in \( \tilde{W}_4 \) will be replaced by gauge covariant derivatives \( [11] \). This warrants that gauge symmetry is manifest at any stage of the variational process.

We shall vary separately the actions (63) and (66). For simplicity we omit explicit writing of the integration domain.
A. Variation of the EH action

Let us vary the EH action in \( \mathcal{W}_4 \) space, given by Eq. \( (63) \), with respect to the metric:

\[
\delta g_{\text{EH}} = \frac{1}{2} \int d^4x m_{\text{pl}}^2 \delta g (\sqrt{-g} R)
\]

\[
= \frac{1}{2} \int d^4x \sqrt{-g} m_{\text{pl}}^2 \left( \delta g_{\mu\nu} \frac{1}{2} g_{\mu\nu} R \delta g_{\mu\nu} \right). \quad (67)
\]

According to Eq. \( (18) \):

\[
\delta R = \delta (g^{\mu\nu} R_{\mu\nu}) = \delta \left( g^{\mu\nu} \tilde{R}_{\mu\nu} \right)
\]

\[
= g^{\mu\nu} \delta \tilde{R}_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}, \quad (68)
\]

where in the last step we took into account Eq. \( (70) \), which leads to,

\[
g^{\mu\nu} \delta \tilde{R}_{\mu\nu} = g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \delta R_{\mu\nu}.
\]

Hence,

\[
\delta g_{\text{EH}} = \frac{1}{2} \int d^4x \sqrt{-g} m_{\text{pl}}^2 g^{\mu\nu} \delta R_{\mu\nu}
\]

\[
+ \frac{1}{2} \int d^4x \sqrt{-g} m_{\text{pl}}^2 \delta g^{\mu\nu} \left( \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right). \quad (69)
\]

We have that,

\[
\delta R_{\mu\nu} = \nabla^\lambda \Gamma_{\nu\mu}^\lambda - \nabla_{\nu} \Gamma_{\lambda\mu}^\lambda. \quad (70)
\]

But since the conformal weight of the generalized connection vanishes (the generalized connection is not transformed by the gauge transformations), then we may safely replace the covariant derivatives in \( (70) \) by the corresponding gauge covariant derivative: \( \nabla_{\mu} \rightarrow \hat{\nabla}_{\mu} \). We get that,

\[
g^{\mu\nu} \delta \tilde{R}_{\mu\nu} = g^{\mu\nu} \left( \nabla^\lambda \delta \Gamma_{\nu\mu}^\lambda - \nabla_{\nu} \delta \Gamma_{\lambda\mu}^\lambda \right) = \nabla^\mu \delta_{\mu}, \quad (71)
\]

where we have defined the vector \( \Gamma^\mu := g^{\mu\nu} \delta \Gamma_{\nu\lambda}^\mu - g^{\mu\nu} \delta \Gamma_{\lambda\nu}^\mu \). Hence, for the first integral in the right-hand side (RHS) of Eq. \( (69) \) we get,

\[
\int d^4x \sqrt{-g} m_{\text{pl}}^2 g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} m_{\text{pl}}^2 \nabla^\mu \delta_{\mu}
\]

\[
= \int d^4x \tilde{\nabla}_{\mu} \left( \sqrt{-g} m_{\text{pl}}^2 \Gamma^\mu \right) = \int d^4x \partial_{\mu} \left( \sqrt{-g} m_{\text{pl}}^2 \Gamma^\mu \right). \quad (72)
\]

This integral amounts to a total divergence and so it may be safely ignored in Eq. \( (69) \). While deriving the above equation we took into account Eqs. \( (57) \) and \( (57) \), where in the former equation we made the following identification: \( V^\mu \equiv m_{\text{pl}}^2 \delta Q^\mu \).

I. Variation with respect to the gauge vector

If take into account Eq. \( (15) \), the Einstein-Hilbert action in \( \mathcal{W}_4 \) space \( (63) \) can be written in the following way:

\[
S_{\text{EH}} = \int d^4x \sqrt{-g} m_{\text{pl}}^2 \tilde{R}
\]

\[
= \frac{1}{2} \int d^4x \sqrt{-g} m_{\text{pl}}^2 \left( \tilde{R} - \frac{3}{2} Q_\mu Q^\mu - 3 \nabla Q^\mu \right) \quad (73)
\]

Hence, variation of the Einstein-Hilbert action with respect to the gauge vector \( Q_\mu \) is given by the following expression:

\[
\delta Q S_{\text{EH}} = - \frac{3}{2} \int d^4x \sqrt{-g} m_{\text{pl}}^2 \left( Q_\mu \delta Q^\mu + \tilde{\nabla}_{\mu} \delta Q^\mu \right). \quad (74)
\]

The vector \( \delta Q^\mu = g^{\mu\nu} \delta Q_\nu \) has weight \( w(\delta Q^\mu) = -2 \). Hence,

\[
\tilde{\nabla}_{\mu} \delta Q^\mu = \nabla_{\mu} \delta Q^\mu - Q_\mu \delta Q^\mu
\]

\[
= \nabla_{\mu} \delta Q^\mu + Q_\mu \delta Q^\mu. \quad (75)
\]

In consequence, Eq. \( (74) \) can be written in the following way:

\[
\delta Q S_{\text{EH}} = - \frac{3}{2} \int d^4x \sqrt{-g} m_{\text{pl}}^2 \nabla_{\mu} \delta Q^\mu
\]

\[
= - \frac{3}{2} \int d^4x \nabla_{\mu} \left( \sqrt{-g} m_{\text{pl}}^2 \delta Q^\mu \right)
\]

\[
= - \frac{3}{2} \int d^4x \partial_{\mu} \left( \sqrt{-g} m_{\text{pl}}^2 \delta Q^\mu \right), \quad (76)
\]

so that it is a boundary term that may be omitted since variation of the gauge field \( \delta Q^\mu \) vanishes on the boundary.

---

\*8 While deriving Eq. \( (69) \) we took into account Eqs. \( (17) \) and \( (57) \), where in the former equation we made the following identification: \( V^\mu \equiv m_{\text{pl}}^2 \delta Q^\mu \).
B. Variation of the gauge field action

Let us now vary the piece of action \( \delta g S_{\text{gauge}} \) first with respect to the metric and then with respect to the gauge field. Variation of \( S_{\text{gauge}} \) with respect to the metric yields,

\[
\delta g S_{\text{gauge}} = -\frac{1}{2} \int d^4 x \sqrt{-g} \delta g^\mu\nu T^{(Q)}_{\mu\nu}, \tag{77}
\]

where

\[
T^{(Q)}_{\mu\nu} := -4\alpha^2 \left( Q^\lambda_{\mu} Q_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} Q_{\lambda\kappa} Q^{\lambda\kappa} \right), \tag{78}
\]

is the effective stress-energy tensor of the gauge field.

Variation of the action \( S_{\text{gauge}} \) with respect to the \( Q_{\mu} \)-field yields,

\[
\delta Q S_{\text{gauge}} = 2\alpha^2 \int d^4 x \sqrt{-g} Q^{\mu\nu} \delta Q_{\nu},
\]

\[
= 4\alpha^2 \int d^4 x \sqrt{-g} Q^{\mu\nu} \nabla^\mu \delta Q_{\nu}, \tag{79}
\]

where we have taken into account the following expressions:

\[
\begin{align*}
Q_{\mu\nu} &= \nabla_{\mu} Q_{\nu} - \nabla_{\nu} Q_{\mu} = \nabla^\mu Q_{\nu} - \nabla^\nu Q_{\mu}, \\
\Rightarrow \delta Q_{\mu\nu} &= \nabla^\mu \delta Q_{\nu} - \nabla^\nu \delta Q_{\mu}, \\
Q^{\mu\nu} \delta Q_{\mu\nu} &= Q^{\mu\nu} \nabla^\mu \delta Q_{\nu} - Q^{\mu\nu} \nabla^\nu \delta Q_{\mu} \\
&= 2Q^{\mu\nu} \nabla^\mu \delta Q_{\nu}.
\end{align*}
\]

Besides, according to Eq. \( \text{(47)} \),

\[
\nabla^\mu \left( \sqrt{-g} Q^{\mu\nu} \delta Q_{\nu} \right) = \partial_{\mu} \left( \sqrt{-g} Q^{\mu\nu} \delta Q_{\nu} \right),
\]

where we have replaced \( V^\mu \) by \( Q^{\mu\nu} \delta Q_{\nu} \). We have also,

\[
\sqrt{-g} Q^{\mu\nu} \nabla^\mu \delta Q_{\nu} = \partial_{\mu} \left( \sqrt{-g} Q^{\mu\nu} \delta Q_{\nu} \right) - \sqrt{-g} \nabla^\nu Q^{\mu\nu} \delta Q_{\nu}. \tag{80}
\]

Substituting this last equation into Eq. \( \text{(79)} \) we finally get,

\[
\delta Q S_{\text{gauge}} = -4\alpha^2 \int d^4 x \sqrt{-g} \nabla^\mu Q^{\mu\nu} \delta Q_{\nu}. \tag{81}
\]

C. Variation of the matter Lagrangian

Let us consider a matter piece of action,

\[
S_m = \int d^4 x \sqrt{-g} L_m(\psi, \nabla^\mu \psi, g), \tag{82}
\]

where \( \psi \) denotes the matter fields. As usual, variation of the above action with respect to the matter leads to,

\[
\delta g S_m = -\frac{1}{2} \int d^4 x \sqrt{-g} g^{\mu\nu} T^{(m)}_{\mu\nu}, \tag{83}
\]

where

\[
T^{(m)}_{\mu\nu} := -2 \frac{\partial}{\partial g^{\mu\nu}} \left( \sqrt{-g} L_m \right), \tag{84}
\]

is the stress-energy tensor of the matter fields.

D. Equations of motion

Let us consider the overall action,

\[
S_{\text{tot}} = S_{\text{EH}} + S_{\text{gauge}} + S_m. \tag{85}
\]

Variation of \( S_{\text{tot}} \) with respect to the metric leads to the following equations of motion (EOM):

\[
\eta^2 \bar{G}_{\mu\nu} = T^{(Q)}_{\mu\nu} + T^{(m)}_{\mu\nu}, \tag{86}
\]

while variations with respect to the gauge and matter fields, lead to,

\[
\nabla^\nu Q_{\mu\nu} = 0 \Leftrightarrow \nabla^\nu \left( \nabla_{\mu} Q_{\nu} - \nabla_{\nu} Q_{\mu} \right) = 0, \tag{87}
\]

and to the formal Euler equation for the matter fields:

\[
\nabla^\mu \left( \frac{\partial L_m}{\partial (\nabla^\mu \psi)} \right) - \frac{\partial L_m}{\partial \psi} = 0, \tag{88}
\]

respectively.

If take into account the noncommuting property of the generalized covariant derivatives of given vector \( V_{\alpha} \):

\[
\nabla_{\mu} \nabla_{\nu} V_{\alpha} - \nabla_{\nu} \nabla_{\mu} V_{\alpha} = R^\lambda_{\mu\nu\alpha} V_{\lambda}, \tag{89}
\]

the equation of motion Eq. \( \text{(87)} \) can be written in the following equivalent way:

\[
\nabla^\mu \nabla_{\nu} Q_{\mu\nu} - (\nabla_{\nu} - Q_{\nu}) \nabla^\nu Q_{\mu} = R^\mu_{\nu\mu} Q_{\mu}. \tag{90}
\]

VIII. OBSERVATIONAL SIGNATURES OF GAUGE INVARIANT THEORY WITH VECTORIAL NONMETRICITY

In general relativity when two identical clocks, initially synchronized, are parallel transported along different paths, a certain loss of synchronization arises that
is called as the “first clock effect.” In Weyl spacetimes $W_4$, where the vectorial nonmetricity condition (5) is satisfied, an additional effect arises: the two clocks not only have lost their initial synchronization, but, they go at different rates. It is known as the “second clock effect” \[ \text{[16, 13, 22]} \]. The SCE causes a serious observational issue: an unobserved broadening of spectral lines. This issue was enough to reject the original Weyl’s gauge invariant gravitational theory and its related geometrical framework \[ \text{[1, 10]} \]. In spite of this, several recent papers have put into discussion the occurrence of the second clock effect \[ \text{[16, 13, 21, 22]} \].

The mathematical basis for the SCE is given by Eq. \[ \text{[58]} \],

\[
m(x, C) = m_0 \exp \left( \frac{1}{2} \int_C Q_\mu u^\mu d\tau \right), \tag{91}
\]

where \( C \) is the path joining the origin \( x_0 = \{0, 0\} \) with an arbitrary point with coordinates \( x^i = \{t, x^i\} \), \( u^\mu = dx^\mu/d\tau \) is the four-velocity, \( \tau \) is an affine parameter along the path and the integration constant \( m_0 = m(0) \).

Let us base the physical analysis of the SCE on the functioning of an atomic clock which, for definiteness and simplicity, we shall assume is made of hydrogen atoms. The principle of operation of such an atomic clock is based on atomic physics: it measures the electromagnetic signal that electrons in the hydrogen atoms emit when they change energy levels. For instance, the energy of each energy level in the hydrogen atom, labeled by \( n \), is given by: \( E_n \approx -ma^2/2n^2 \), where \( m \) is the mass of the electron and \( a \approx 1/137 \) is the fine-structure constant.

Any changes in the mass \( m \) over spacetime will cause changes in the energy levels and, consequently, in the energy of the atomic transitions

\[
\nu_{ij} = |E_{nj} - E_{ni}| = \frac{ma^2}{2} \left( \frac{1}{n_j^2} - \frac{1}{n_i^2} \right).
\]

Hence, the functioning of atomic clocks will be affected by the variation of masses over spacetime, according to equation (91).

Let us consider a collection of identical hydrogen atoms that are parallel transported along neighboring paths from the origin \( x_0 = \{0, 0\} \) to a given point \( x \). Let us take the larger difference arising between the masses of any two atoms in the collection at \( x \): \( \Delta m = m(x) - \bar{m}(x) \). Then, according to (91) one gets the following gauge invariant ratio:

\[
\frac{\Delta \nu_{ij}}{\nu_{ij}} = \frac{\Delta m}{m} = 1 - \exp \left( \frac{1}{2} \left( \int_C Q_\mu dx^\mu - \int_C Q_\mu dx^\mu \right) \right), \tag{92}
\]

where \( \Delta \nu_{ij} \) quantifies the broadening of a given spectral line. The spectral line is sharp: \( \Delta \nu_{ij} = 0 \), only if either \( \bar{C} = C \), or if \( Q_\mu = \partial_\mu \phi \), where \( \phi \) is the Weyl gauge scalar. In this last case \( \int_C \phi dx^\mu = \phi(x) - \phi(0) \), independent of the path joining the starting and final points.

WIG is the resulting geometric structure. Hence, only for gradient nonmetricity the second clock effect does not arise.\(^9\) The SCE alone is sufficient to reject theories with vectorial nonmetricity as phenomenologically nonviable descriptions of our classical world.

It has been stated \[ \text{[5]} \] that vanishing of the quantity \( Q_{\mu\nu} := 2\partial_\mu Q_{\nu}\phi = 0 \), is the necessary and sufficient condition for the given vector (four-momentum in particular) to return to itself after parallel transport in a closed trajectory in \( W_4 \) space. Actually, due to the Stokes theorem:

\[
\oint_C Q_\mu dx^\mu = \frac{1}{2} \int_S Q_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{93}
\]

where \( Q_{\mu\nu} \) is defined by Eq. (39) and \( C \) is the boundary of the oriented surface \( S \), if \( Q_{\mu\nu} = 0 \), then \( \oint C Q_\mu dx^\mu = 0 \). Nevertheless, the demonstration that the SCE does not arise based in the argument that Eq. (39) vanishes when \( \partial_\mu Q_{\nu} = 0 \) \[ \text{[16, 21]} \], would be incorrect in general \[ \text{[20]} \]. As a matter of fact, the above conclusion would require the given body – in our case an hydrogen atomic clock – to be submitted to parallel transport in a closed spacetime trajectory \( C \). In general, closed paths in spacetime carry causality issues and timelike worldlines of observers with clocks – aimed at the check of the SCE – are not the exception.

In this regard we should differentiate the timelike worldlines \( C_{\text{open}} \) with coordinates \( x^\epsilon(\xi) \), where \( \epsilon \) is an affine parameter along the worldline, which start and end up at a same spatial point (different values of the time coordinate):

\[
x^\epsilon(\xi_{\text{start}}) \neq x^\epsilon(\xi_{\text{end}}), \quad x^\epsilon(\xi_{\text{start}}) = x^\epsilon(\xi_{\text{end}}) \Rightarrow x^\mu(\xi_{\text{start}}) \neq x^\mu(\xi_{\text{end}}), \tag{94}
\]

from those worldlines \( C_{\text{closed}} \), which start and end up at the same spacetime point:

\[
x^\mu(\xi_{\text{start}}) = x^\mu(\xi_{\text{end}}) \Rightarrow x^\epsilon(\xi_{\text{start}}) = x^\epsilon(\xi_{\text{end}}), \tag{95}
\]

While timelike worldlines of type \( C_{\text{open}} \) can be associated with real classical motions, timelike worldlines of type \( C_{\text{closed}} \) are usually called as closed timelike curves (CTCs) and are plagued by causality issues as long as a CTC represents time travel \[ \text{[58, 93]} \]. Closed curves of this latter type are the ones that are considered in Eq. (93).

\(^9\) Worthy of mention is the fact that, due to the third Bianchi identity \[ \text{[31]} \], for the so called teleparallel theories which satisfy the “flatness” requirement, \( R^\mu_{\nu\rho\sigma} = 0 \), the vectorial nonmetricity \( Q_{\alpha\mu\nu} = Q_{\alpha} g_{\mu\nu} \) can be exclusively of gradient type, \( Q_\mu = \partial_\mu \phi \), so that the SCE does not arise in this case.
A. Quantum considerations

The above causality argument that is based in the existence of CTC-s, may not be correct in the quantum domain. Hence, the impact of vector nonmetricity in quantum gravitational phenomena may not be underestimated. Following a reasoning line similar to that of the pioneering work [2], one can make the replacement \( Q_{\mu} \to 2q_{\mu} \) in Eq. (93), where \( i \) is the imaginary unit. Then one gets that the following quantization requirement,

\[
\oint_C q_\mu dx^\mu = \frac{1}{2} \int_S f_{\mu\nu} dx^\mu \wedge dx^\nu = 2n\pi,
\]  

where \( f_{\mu\nu} \equiv \partial_\mu q_\nu - \partial_\nu q_\mu \) is the field strength associated with the quantum nonmetricity vector \( q_\mu \) and \( n \) is an integer, avoids the occurrence of the SCE. Hence, the phenomenological viability of vectorial nonmetricity in the quantum world amounts to quantization of the flux of the vector \( q_\mu \) through the surface \( S \) in spacetime. This is similar to the phenomenon of magnetic flux quantization.

Similar argument may be applied to arbitrary nonmetricity \( Q_{\alpha\mu\nu} \),

\[
Q_{\alpha\mu\nu} \to iq_{\alpha\mu\nu},
\]

where the above field strength \( f_{\mu\nu} \) is to be replaced as it follows:

\[
f_{\mu\nu} \to (\nabla_\mu q_\alpha\beta - \nabla_\nu q_\alpha\beta) u^\alpha u^\beta.
\]

Hence, the impact of arbitrary nonmetricity in quantum gravitational phenomena may not be excluded.

IX. THEORY WITH GRADIENT NONMETRICITY

In this and the remaining part of this paper we shall seek for the phenomenological signatures of a particular case of the above gauge invariant theory, which is free of the SCE and so it could take account of classical gravitational phenomena. We shall investigate Weyl geometry space \( \tilde{W}_4^{\text{int}} \) with gradient nonmetricity,

\[
Q_{\mu} = \partial_{\mu}\varphi.
\]  

In this equation \( \varphi \) is the gauge scalar. An immediate consequence of Eq. (97) is that,

\[
Q_{\mu\nu} = 0 \Rightarrow T_{\mu\nu}^{\text{Q}} = 0,
\]

\[
\tilde{G}_{\mu\nu} = G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.
\]

Besides, according to Eq. (54), the second Bianchi identity in this case leads to:

\[
\nabla^\lambda G_{\lambda\mu} = 0.
\]  

In the bibliography \( \tilde{W}_4^{\text{int}} \) spaces are better known as WIG spaces. Obviously \( \tilde{W}_4^{\text{int}} \subset \tilde{W}_4 \). This type of theory is interesting because the SCE does not arise so that it is not classically forbidden from start. WIG and related gauge invariant theory of gravity are usually detracted by incorrectly identifying them with pseudo-Riemann geometry and with general relativity, respectively. As we shall show below, while a given gauge invariant theory of gravity based on WIG space is in fact a class of theories, GR is just a gauge. Each gauge represents a whole theory which can be differentiated from any other theory in the class through the check of the experimental and observational evidence.

A. Equations of motion

Taking into account Eq. (97) the equation of motion Eq. (86) simplifies to:

\[
m_{pl}^2 G_{\mu\nu} = T_{\mu\nu}^{(m)},
\]  

while the EOM (87) becomes an identity. According to Eq. (57), the effective (reduced) Planck mass in this equation satisfies,

\[
m_{pl}^2 = m_{pl,0}^2 \exp [-\varphi(x)] \exp (\varphi(x)) = m_{pl,0}^2 \exp [\varphi(x)],
\]  

where \( m_{pl,0}^2 \equiv m_{pl,0}^2 (\varphi(x)) = m_{pl,0}^2 (0) \exp [-\varphi(0)] \) and the constant \( m_{pl,0}^2 (0) \) is the magnitude of the Planck mass squared evaluated at the origin. For convenience we may define the gauge invariant stress-energy tensor of matter,

\[
T_{\mu\nu}^{(m,\ast)} := e^{-\varphi(x)} T_{\mu\nu}^{(m)},
\]  

so that the corresponding Einstein’s equation Eq. (99) in \( \tilde{W}_4^{\text{int}} \) can be rewritten in the following equivalent way:

\[
G_{\mu\nu} = \frac{1}{m_{pl,0}^2} T_{\mu\nu}^{(m,\ast)}.
\]  

This equation is manifestly gauge invariant since both, the Einstein’s tensor \( G_{\mu\nu} \) and the gauge invariant stress-energy tensor \( T_{\mu\nu}^{(m,\ast)} \) have vanishing conformal weight.\(^{10}\)

\(^{10}\) We recall that \( w(T^{(m)}) = -2 \) while, according to Eq. (103), \( w(e^{-\varphi}) = 2 \), so that \( w(T^{(m,\ast)}) = 0 \). We are using the following property of weights: \( w(a \cdot b) = w(a) + w(b) \).
Hence, neither is transformed by the Weyl gauge transformations Eq. (103) which, in the present case, amount to:

\[
g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \Omega^{-2} g^{\mu\nu}, \quad \sqrt{-g} \rightarrow \Omega^4 \sqrt{-g}, \quad \varphi \rightarrow \varphi - 2 \ln \Omega \Rightarrow e^\varphi \rightarrow \Omega^{-2} e^\varphi. \tag{103}
\]

Here we are assuming that integration constants and/or their combination, such as \( m_0^2 \), are not transformed by the above gauge transformations.

Notice that, due to the identity Eq. (108), the following conservation equation takes place:

\[
\nabla^\lambda T^{(m,*)}_{\lambda\mu} = 0. \tag{104}
\]

### B. Motion of test particles

In the present theory, depicted by the EOMs (102), (104), point-like test particles follow the following geodesic curves in spacetime:

\[
d^2 x^\alpha / ds^2 + \left\{ \alpha \mu \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \frac{1}{2} h^{\alpha\mu} \partial_\mu \varphi = 0, \tag{105}
\]

for time-like test particles and

\[
dk^\alpha / d\lambda + \left\{ \alpha \mu \right\} k^\mu k^\nu = 0, \tag{106}
\]

for null particles. In Eq. (105) we have taken into account that the mass of a given test particle obeys,

\[
\frac{1}{m} \frac{\delta m}{\delta x^\mu} = \frac{1}{2} \partial_\mu \varphi \Rightarrow m(x) = m_0 \exp[\varphi(x)/2]. \tag{107}
\]

where the constant \( m_0 \equiv m(0) \exp[-\varphi(0)/2] \) is not transformed by the gauge transformations Eq. (103). As shown in Sec. IX, both Eq. (105) and (106) are gauge covariant equations. In particular, the demonstration that (106) – which coincides with the null geodesic in general relativity – is gauge invariant, can be found at the end of subsection IX.A.

As in the more general situation with vectorial nonmetricity, in the present case photons and radiation do not interact with the gauge scalar \( \varphi \) (see Eq. (103)). Only time-like point particles interact with the gauge scalar as seen in Eq. (105).

The remaining part of this paper will be dedicated to investigate the geometrical and phenomenological consequences, as well as the observational signatures, of our theory: the gauge invariant theory of gravity over \( \tilde{W}_4^{\text{int}} \) space with gradient nonmetricity, which is mathematically described by the equations of motion (102) and (104).

### X. GAUGE FREEDOM

In general there is not an independent equation for the gauge scalar \( \varphi \) since, in addition to the four degrees of freedom to make diffeomorphisms, we have an additional degree of freedom to make gauge transformations (103). Different choices of the function \( \varphi(x) \) lead to different gauges of the theory (102), (104). Otherwise, one may fix one of the independent components of the metric, leaving \( \varphi \) as an independent degree of freedom. In this case one have to solve a differential equation on \( \varphi \) that is obtained by taking the trace of Einstein’s equation Eq. (102):

\[
-\tilde{R} = T^{(m,*)}/m_0^2, \text{ or, in the Riemannian decomposition Eq. (114):}
\]

\[
\tilde{\nabla}^2 \varphi + \frac{1}{2} (\partial \varphi)^2 \frac{1}{3} \tilde{R} = \frac{1}{3m_0^2} \tilde{T}^{(m,*)}, \tag{108}
\]

where we have used the following notation: \( T^{(m,*)} \equiv g^{\mu\nu} T_{\mu\nu}^{(m,*)} \), \( \tilde{\nabla}^2 \equiv g^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \) and \( (\partial \varphi)^2 \equiv g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \).

Different choices of either the gauge scalar \( \varphi \) or one of the metric functions \( g_{\mu\nu} \), with the remaining degrees of freedom completely determined by the equations of motion, lead to physically equivalent gauges in the sense that the same laws of gravity (102), (104) are satisfied in any gauge. However, once a given gauge is picked up, manifest gauge symmetry is broken down. Each gauge represents a different gravitational theory. Yet, these theories are related by the gauge transformations (103).

### A. General relativity gauge

Let us discuss about one of the most outstanding gauges in the present theory: the general relativity gauge. The GR gauge is defined by the following choice of the gauge scalar:

\[
\varphi = \varphi_0 = \text{const.} \tag{109}
\]

Given that in the present case the nonmetricity vanishes \( Q_\mu = \partial_\mu \varphi_0 = 0 \), the \( \tilde{W}_4^{\text{int}} \) space is replaced by Riemann space \( V_4: \tilde{W}_4^{\text{int}} \rightarrow V_4 \). The gauge is characterized by constant mass of time-like point particles: \( m = m_0 \exp(\varphi_0/2) \).

The resulting theory is Einstein’s \( \text{GR} \) in \( V_4 \). Actually, under the choice (109) the gravitational equations of motion Eq. (99) or, equivalently, Eq. (102) read

\[
\tilde{G}_{\mu\nu} = \frac{1}{m_0^2 \exp(\varphi_0)} T^{(m)}_{\mu\nu}, \tag{110}
\]
where, we recall, geometric objects and operators with a hat are defined with respect to the LC connection \( \tilde{\nabla} \). From the second Bianchy identity and Eq. (110) it follows the standard continuity equation,

\[
\tilde{\nabla}^\lambda T^{(\mu)}_{\lambda \mu} = 0. \quad (111)
\]

Although the manifest symmetry of our gauge invariant theory, that is represented by Eqs. (102) and (104), is lost once a given gauge is fixed, gauge symmetry is implicit in the specific transformations (103) that link any gauge with every other one. In this regard the GR gauge is not an exception: it can be linked with any other — in principle arbitrary — gauge through the Weyl rescalings (103). Let \((\mathcal{M}_4, g_{\mu \nu}, \varphi) \in \tilde{W}_4^{\text{int}}\) be a Weyl integrable space. Under the gauge transformation (103),

\[
g_{\mu \nu} \rightarrow \Omega^2 g_{\mu \nu}, \quad e^\varphi \rightarrow \Omega^{-2} e^{\varphi_0},
\]
or, equivalently, under

\[
g_{\mu \nu} \rightarrow e^{\varphi_0 - \varphi} g_{\mu \nu}^{\text{GR}}. \quad (112)
\]

one can map any \( \tilde{W}_4^{\text{int}} \) space into Riemannian manifold:

\[
[(\mathcal{M}_4, g_{\mu \nu}, \varphi) \in \tilde{W}_4^{\text{int}}] \rightarrow [(\mathcal{M}_4, g_{\mu \nu}^{\text{GR}}, \varphi_0) \in V_4]. \quad (113)
\]

The converse is also true: under the inverse gauge transformation

\[
g_{\mu \nu}^{\text{GR}} \rightarrow e^{-\varphi - \varphi_0} g_{\mu \nu}, \quad (114)
\]
one can map any Riemannian GR \( V_4 \) space into a Weyl integrable \( \tilde{W}_4^{\text{int}} \) one,

\[
[(\mathcal{M}_4, g_{\mu \nu}^{\text{GR}}, \varphi_0) \in V_4] \rightarrow [(\mathcal{M}_4, g_{\mu \nu}^{\text{GR}}, \varphi) \in \tilde{W}_4^{\text{int}}].
\]

Hence, GR belongs in the class of conformal equivalence of our gauge invariant theory. In other words, GR theory is just one of the feasible gauges in the infinite equivalence class. Although the different gauges are physically equivalent in the sense that the same gravitational laws (102), (104) are satisfied in each one of them, each gauge yields a different geometrical (and physical) description of the world.

We want to underline that different values \( \varphi_{i0} \) \((\varphi_0 \in \mathbb{R}, i \in \mathbb{N})\) of the constant \( \varphi_0 \) lead to different physical pictures in the GR gauge, since each one has a different value of the measured Planck mass squared \( M_{\text{pl},i}^2 \equiv m_{\text{pl},i0}^2 \exp(\varphi_{i0}) \), and also different value of the mass parameter \( \nu_i = \nu_0 \exp(\varphi_{i0}/2) \) in the standard model of particles (SMP), so that a given particle of the SMP acquires different mass in the different pictures: \( m_i = m_0 \exp(\varphi_{i0}/2) \). Hence, the GR gauge is in fact a (in principle infinite) set of \( i \) copies of general relativity with different values of several universal constants \( \{M_{\text{pl},i}, \nu_i, \ldots\} \). Other constants such as the Planck constant \( \hbar \), the speed of light \( c \), the electron charge \( e \), etc., are the same in all \( i \) copies of GR theory. The resulting overall picture could be the theoretical basis of a restricted type of anthropic principle [4,103] in a universe where gravity is governed by the GR laws.

### XI. OUTSTANDING SOLUTIONS

Due to their simplicity and importance here we shall discuss on a pair of relevant solutions of the EOMs (102) and (104) which are special in some sense.

#### A. Minkowski space

In this case we fix the metric functions without invoking the EOM (102), while the gauge scalar can be found with the help of the equation (108). Minkowski space is one of the simplest solutions of any theory of gravity. In the present case its simplicity allows us to qualitatively discuss on certain phenomenological consequences of the theory.

According to this solution the gravitational effects are described in flat Minkowski space with metric \( \eta_{\mu \nu} = \text{diag}(−1,1,1,1) \) by the gauge scalar exclusively. The equations of motion (108) and (104), simplify to

\[
\partial^2 \phi = \frac{T^{(m)}}{6m_{\text{pl},0}^2 \phi}, \quad (115)
\]

and to

\[
\partial^\lambda T^{(m)}_{\mu \lambda} = 2 \frac{\partial_{\mu} \phi}{\phi} T^{(m)}, \quad (116)
\]

respectively, where we have rescaled the gauge scalar \( \phi = \exp(\varphi/2) \) and we adopted the following notation: \( \partial^2 ≡ \partial^\mu \partial_{\mu} \).

While in Riemannian spacetimes the Minkowski metric corresponds to vacuum solution, in \( \tilde{W}_4^{\text{int}} \) space it is not associated with vacuum as it can be seen from equations Eqs. (115) and (116). These equations are fully equivalent to the equations of Nordström’s theory of gravity (106). This theory is ruled out by Solar system’s experiments (107). In particular, as seen from the null geodesic equation (106) which, in the present case amounts to \( dk^\mu/d\lambda = 0 \), there is no light-bending in this theory. The redshift of frequencies may be explained as due to point-dependent property of the mass expressed in Eq. (107).
B. de Sitter space and the cosmological constant problem

In the present theoretical framework the de Sitter space is defined by,

\[ R_{\mu\nu} = \frac{R}{4} g_{\mu\nu} = \Lambda_{\text{eff}} g_{\mu\nu} \Rightarrow G_{\mu\nu} = -\Lambda_{\text{eff}} g_{\mu\nu}, \quad (117) \]

where the effective cosmological constant \( \Lambda_{\text{eff}} \) is dynamical. Actually, since the product \( \Lambda_{\text{eff}} g_{\mu\nu} \) must be gauge invariant and, given that under \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \), then \( \Lambda_{\text{eff}} \rightarrow \Omega^2 \Lambda_{\text{eff}} \). This means that we can write \( \Lambda_{\text{eff}} = \Lambda e^\varphi \), where \( \Lambda \) is a free constant with dimensions of mass squared. In consequence, the motion equation \( [69] \) transforms into,

\[ G_{\mu\nu} = \frac{1}{m_{\text{pl}}^2} T_{\mu\nu}^{\text{vac}} = -\Lambda e^\varphi g_{\mu\nu}, \quad (118) \]

where the stress-energy tensor of vacuum is given by: \( [12] \)

\[ T_{\mu\nu}^{\text{vac}} = -\Lambda m_{\text{pl}}^2 e^\varphi g_{\mu\nu}, \quad (119) \]

with \( m_{\text{pl}}^2 \) given by Eq. \( [100] \). The gauge invariant continuity equation \( [101] \) reads, \( [13] \)

\[ \nabla^\nu T_{\nu\mu}^{\text{vac}} = \nabla^\nu \varphi T_{\nu\mu}^{\text{vac}}. \quad (120) \]

Meanwhile, the Ricci curvature of de Sitter-\( \tilde{W}_4 \) space satisfies, \( R = 4 \Lambda e^\varphi \).

The effective "cosmological constant",

\[ \Lambda_{\text{eff}} = \Lambda e^\varphi, \quad (121) \]

as well as the energy density of vacuum,

\[ \rho_{\text{vac}} = \Lambda m_{\text{pl}}^2 e^\varphi, \quad (122) \]

both depend on the spacetime point. As we shall see this dynamical behavior could explain the cosmological constant problem (CCP) \( [61-65] \).

Let us consider a FRW spacetime with flat spatial sections which is given by the line element,

\[ ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j, \quad i, j = 1, 2, 3, \quad (123) \]

where \( a(t) \) is the dimensionless scale factor and \( t \) is the cosmic time. According to Eq. \( [122] \) the energy density of vacuum is a function of the cosmic time \( t \),

\[ \rho_{\text{vac}}(t) = \Lambda m_{\text{pl}}^2(t) e^{\varphi(t)}. \quad (124) \]

Here we shall use the following convenient time normalization:

\[ m_{\text{pl}}^2(t) = M_{\text{pl}}^2 e^{2\varphi(t)-\varphi(t_{\text{pl}})}, \quad (125) \]

where \( t_{\text{pl}} \) represents the Planck time elapsed after the bigbang at \( t = 0 \), while the constant \( M_{\text{pl}}^2 \) is the value of the Planck mass squared measured at Planck time: \( m_{\text{pl}}^2(t_{\text{pl}}) = M_{\text{pl}}^2 \). By substituting Eq. \( [123] \) into Eq. \( [124] \) one gets that,

\[ \rho_{\text{vac}}(t) = \Lambda M_{\text{pl}}^2 e^{2\varphi(t)-\varphi(t_{\text{pl}})} \quad (126) \]

We assume next that \( \varphi(t) \) is a monotonically decreasing function: \( [14] \)

\[ \varphi(t) - \varphi(t_{\text{pl}}) < 0 \quad \text{for any} \ t > t_{\text{pl}}. \]

From Eq. \( [126] \) it follows that the energy density of vacuum at Planck time was,

\[ \rho_{\text{vac}}(t_{\text{pl}}) = \Lambda M_{\text{pl}}^2 e^{\varphi(t_{\text{pl}})}, \quad (127) \]

while at present time \( t = t_0 \) it is

\[ \rho_{\text{vac}}(t_0) = \Lambda M_{\text{pl}}^2 e^{2\varphi(t_0)-\varphi(t_{\text{pl}})}, \quad (128) \]

or if take into account Eq. \( [127] \),

\[ \rho_{\text{vac}}(t_0) = \rho_{\text{vac}}(t_{\text{pl}}) e^{2[\varphi(t_0)-\varphi(t_{\text{pl}})]}. \quad (129) \]

According to the cosmological data the present measured value of the vacuum energy density \( \rho_{\text{vac}}(t_0) \) is about 120 orders of magnitude smaller than its Planck’s value:

\[ \rho_{\text{vac}}(t_0) \approx 10^{-120} \rho_{\text{vac}}(t_{\text{pl}}). \quad (129) \]

Then it follows that,

\[ \rho_{\text{vac}}(t_0) \approx 10^{-120} \rho_{\text{vac}}(t_{\text{pl}}). \quad (129) \]

---

12 Notice that, under the gauge transformation \( [103] \),

\[ T_{\mu\nu}^{\text{vac}} \rightarrow \Omega^{-2} T_{\mu\nu}^{\text{vac}}. \]

13 In a cosmological setting– spatially flat Friedmann-Robertson-Walker (FRW) spacetime –, the continuity equation \( [120] \) simplifies to the following equation:

\[ \dot{\varphi} = \Lambda_{\text{eff}} \dot{\varphi}, \]

which can be found as well by deriving equation \( [121] \) with respect to the cosmic time.

14 Since \( \varphi \) is a gauge field (we can replace it by any function of the cosmic time) this assumption does not affect the generality of our analysis.
\[ \Delta \varphi = \varphi(t_{pl}) - \varphi(t_0) \sim 60 \ln 10 \approx 138. \]  

(130)

This means that a reasonable difference of about \( \Delta \varphi \sim 138 \) between the value of the dimensionless gauge field at Planck era and its present value, resolves the apparent discrepancy between the measured value of the vacuum energy density and the energy density the quantum vacuum should have according to well-motivated estimates \(^6\).

**XII. REDSHIFT OF FREQUENCY**

In this section we shall show that the overall redshift effect in the gravitational theory described by the EOMs \(^{102}\) and \(^{104}\), which is built on \( W_4^{\text{int}} \) space, is the result of the combined effect of the cosmological redshift of frequencies – due to photon’s propagation in a background spacetime with nonvanishing curvature – and of an additional redshift which results from spacetime variation of masses.

We shall consider the FRW metric Eq. \(^{123}\). In the GR gauge the FRW line element reads,\(^15\)

\[ ds^2 = -d\hat{t}^2 + \hat{a}^2(t)\delta_{ij}dx^idx^j, \]  

(131)

where, according to Eq. \(^{112}\),

\[ d\hat{t} = \Omega dt, \quad \hat{a} = a, \quad \Omega = e^{(\varphi - \varphi_0)/2}. \]

A. Redshift due to photon propagation in a curved spacetime

According to the null geodesic Eq. \(^{116}\), photons and radiation do not interact with the gauge scalar \( \varphi \). This means that null particles are not able to “see” the WIG structure of \( W_4^{\text{int}} \) space. Instead, these see the Riemannian structure of \( V_4 \) space, just as in general relativity. For the null geodesic equation Eq. \(^{116}\), in FRW space we have that,

\[ g_{\mu\nu}k^\mu k^\nu = 0 \Rightarrow -\omega^2 + a^2(t)\delta_{ij}k^i k^j = 0. \]

(133)

If substitute this equation back into \(^{132}\) we get that,

\[ \frac{d\omega}{d\xi} + \frac{\dot{a}}{a} \omega^2 = 0, \]

(134)

or, if recall that \( \omega = 2\pi \nu \) (\( \nu \) is the photon’s frequency) and that \( \omega = dt/d\xi \Rightarrow \dot{\omega} = da/d\xi \), then the null geodesic equation can be written in the following way:

\[ \frac{d\nu}{d\xi} + \frac{1}{a} \frac{da}{d\xi} \nu = 0. \]

(135)

Straightforward integration of this equation leads to the redshift of photon’s frequency:\(^16\)

\[ \nu = \frac{\nu_0}{a}, \quad \nu_0 \equiv C, \]

(136)

where \( C \) is an integration constant. This is the GR redshift effect that is due to the propagation of photons (and radiation in general) in a curved spacetime background.

Let us assume that a photon with frequency \( \nu_{em} = \nu(x) \) is emitted at some spacetime point \( x \). After propagating in a given FRW background, the photon will have a frequency \( \nu_{abs} = \nu(0) \) at the coordinate origin where it is absorbed. Hence, we may define the (relative) cosmological redshift of frequency which is originated by the effect of the curvature of spacetime on the photon during its propagation, or simply GR redshift:

\[ z_{GR} \equiv \frac{\nu_{em} - \nu_{abs}}{\nu_{abs}} = \frac{\nu_{em}}{\nu_{abs}} - 1 = \frac{\dot{a}(0)}{\dot{a}(t)} - 1, \]

(137)

where we have written the hat over FRW metric coefficients in Riemannian space (in particular over the scale factor) in order to underline that this is a GR effect that arises primarily in Riemann space \( V_4 \) (basically in the GR gauge).

Following the above analysis, in general, in a FRW-\( W_4^{\text{int}} \) spacetime, the redshift effect due to the curvature of spacetime is given by the following expression:

\[ z_{\text{curv}} = \frac{a(0)}{a(t)} - 1, \]

(138)

where \( a \) is the scale factor in a FRW-\( W_4^{\text{int}} \) space with metric \(^{123}\).

\(^{15}\) In order to keep our notation in this section consistent with notation in previous sections, quantities with an over-hat will amount to GR/Riemannian quantities.

\(^{16}\) Since \( \nu \lambda = 1 \), where \( \lambda \) is the photon’s wavelength, we may call this effect as wavelength redshift as well.
B. Redshift due to mass variation

In addition to the above redshift effect due to the propagation of light in curved spacetime, in $\tilde{W}_4$ space with gradient nonmetricity, an additional redshift arises as we shall see. We shall base our discussion on the role of the mass parameter given that photons are emitted and absorbed by atoms. Let us consider, for instance, the hydrogen atom. In the hydrogen atom the energy of each energy level, which is labeled by $n$, is given by:

$$E_n = -\frac{m_e\alpha^2}{2n^2},$$

(139)

where $m_e$ is the mass of the electron and $\alpha \approx 1/137$ is the fine-structure constant. Any changes in the mass $m_e$ over spacetime will cause changes in the energy levels and, consequently, in the energy of the atomic transitions between states $n_i$ and $n_f$,

$$\nu_{f} = |E_{n_f} - E_{n_i}| = \frac{m_e\alpha^2}{2} \left| \frac{1}{n_f^2} - \frac{1}{n_i^2} \right|. \quad (140)$$

Hence, the frequency of either emitted or absorbed photons will be affected by the variation of the electron mass over spacetime, which is given by Eq. (107):

$$m_e(x) = m_{e0} \exp[\varphi(x)/2],$$

(141)

where $m_{e0}$ is the measured mass of the electron at the origin. Let us imagine that an hydrogen atom, which is placed at point $x \equiv \{x^\mu\}$, emits a photon with frequency:

$$\nu_{f}(x) = \frac{m_{e0}\alpha^2}{2} \left| \frac{1}{n_f^2} - \frac{1}{n_i^2} \right| e^{\varphi(x)/2},$$

(142)

while a second hydrogen atom, which is placed at the coordinate origin $x = 0$, absorbs the emitted photon. In order for the photon to be absorbed by the atom at the origin, its frequency must equal,

$$\nu_{f}(0) = \frac{m_{e0}\alpha^2}{2} \left| \frac{1}{n_f^2} - \frac{1}{n_i^2} \right| e^{\varphi(0)/2}. \quad (143)$$

This causes an additional cosmological redshift of frequency,

$$z_{\text{add}} = \frac{\nu_{f}(x)}{\nu_{f}(0)} - 1 = e^{\varphi(x)/2} - e^{\varphi(0)/2} - 1,$$

(144)

which does not arise in general relativity. The overall redshift of frequency in $\tilde{W}_4$ space equals, the GR FRW scale factor transforms in the following way:

$$\hat{a} \to e^{\varphi/2}a,$$

(146)

where $a(t)$ is the time dependent scale factor in any other gauge of our present theory while, on the other hand, according to (137) we have that

$$\frac{1}{a(t)} = z_{\text{GR}} + 1 \Rightarrow \hat{a}(t) = e^{-\varphi/2}a(t) = \frac{1}{z_{\text{GR}} + 1}. \quad (147)$$

Hence, it follows that,

$$\frac{1}{a(t)e^{\varphi/2}} = z_{\text{GR}} + 1 \Rightarrow a(t) = e^{-\varphi/2}z_{\text{GR}} + 1. \quad (148)$$

Taking into account equations (138), (144) and (148), we finally get that,

$$z_{\text{tot}} = e^{\varphi/2}z_{\text{GR}} + 2 e^{\varphi/2} - 1. \quad (149)$$

This means that if our theory were correct, the redshift $z_{\text{GR}}$ in astrophysical data sets and in tables is to be replaced by the overall redshift in Eq. (149): $z_{\text{tot}} \to z_{\text{GR}}$. Let us assume also that $\varphi$ is a small quantity, so that Eq. (149) can be written in the following way,
FIG. 1: Plots of the apparent magnitude Eq. (155) vs redshift for two choices of the cosmological parameters $\Omega_0^m \equiv \rho_m(0)/3M_\text{pl}^2H_0^2$—present value of the dimensionless energy density of the dark matter and $\Omega_0^\Lambda \equiv \Lambda/3M_\text{pl}^2H_0^2$—present value of the dimensionless energy density of the cosmological constant. We have not included the error bars since the plots are for illustrative purposes. The dash-dot curve corresponds to the choice $\Omega_0^m = 0.3$, $\Omega_0^\Lambda = 0.7$, while the solid curve is for $\Omega_0^m = 1$, $\Omega_0^\Lambda = 0$. The crosses and the circles represent observational points corresponding to small-redshift data and to high-redshift data, respectively. Both sets of data are taken from TAB. 1 (high-redshift data) and from TAB. 2 (small-redshift data) in Ref. [67]. We have not included all data points that appear in the mentioned reference but just a representative number of them. In the left figure we have used the values of the (Riemannian) GR-based redshift $z_{\text{GR}}$, which is the one that appears in TABs. 1 and 2 of Ref. [67], while in the right figure we have used the redshift $z_{\text{tot}}$ in Eq. (151), that arises in our theory under the assumption of small $\varphi(z_{\text{GR}}) = \epsilon z_{\text{GR}}$ (we have chosen $\epsilon = 0.15$.) It is seen that, while in (pseudo)Riemannian GR the $\Lambda$CDM model is favored by the high-redshift data, in our gauge invariant setup, under the assumed approximation, the CDM-dominated universe $3H^2 = M_\text{pl}^{-2}\rho_m$ with vanishing cosmological constant is favored instead.

$$z_{\text{tot}} \approx \left[1 + \frac{\varphi(z_{\text{GR}})}{2}\right] z_{\text{GR}} + \varphi(z_{\text{GR}}),$$

where, for definiteness, we choose the simplest possible function $\varphi = \epsilon z_{\text{GR}}$ ($\epsilon$ is a small parameter). Hence,

$$z_{\text{tot}} = (1 + \epsilon) z_{\text{GR}} + \frac{\epsilon^2}{2} z_{\text{GR}}^2.$$  \hfill (151)

This will be our master equation to determine the overall (observed) redshift in our theory. Our approximation is not bad for redshifts $z_{\text{GR}} < 1$, so that in the present case it may serve our illustrative purpose.

A. Accelerating or decelerating expansion?

Measuring distances to distant stars is not an easy task. Several quantities such as the apparent magnitude $m$, the absolute magnitude $M$ and the luminosity distance $d_L$ are involved. The latter is related with the energy flux $\Phi$ measured by the observer at $z = 0$, which comes from a distant source with actual luminosity $L$, in the following way,

$$d_L = \sqrt{\frac{L}{4\pi\Phi}}.$$  \hfill (152)

Meanwhile, the absolute and apparent magnitudes $M$ and $m$, are logarithmic measures of luminosity and flux, respectively.

The luminosity distance $d_L$ is, in general, a model dependent quantity. Actually, it can be related to theoretical quantities as it follows:

$$d_L = \chi(1 + z),$$  \hfill (152)

where the comoving distance $\chi$ is given by

$$\chi = \int_0^z \frac{dt'}{a(t')} = \int_0^z \frac{dz'}{H(z')}.$$  \hfill (153)
In this equation the expression for the Hubble parameter as a function of the redshift parameter \( z \) can be found from the Friedmann equation of the model. For instance, for the \( \Lambda \)CDM model we have that,\(^{17}\)

\[
H_0d_L = 1 + z \sqrt{\frac{\Omega_0^m}{\rho_m}} \int_0^z \frac{dz'}{(1 + z')^3 + \Omega_0^\Lambda / \Omega_0^m},
\]

where \( \Omega_0^m = \Omega_m(0) = \rho_m(0)/3M_p^2H_0^2 \) is the present value of the dimensionless (normalized) energy density of the cold dark matter (CDM), while \( \Omega_0^\Lambda = \Lambda/3M_p^2H_0^2 \) is the present value of the dimensionless energy density of the cosmological constant \( \Lambda \).

According to Eq. (31) of reference \(^{63}\), the redshift-dependent apparent magnitude \( m \) of distant type IA supernovae is given by\(^{18}\)

\[
m(z) = M - 5 \log_{10} h + 42.38 + 5 \log_{10} [H_0d_L(z)].
\]

Here we take \( M = -19.09 \) and \( h = 0.7 \) so that,

\[
m(z) = 5 \log_{10}(H_0d_L) + 24.06,
\]

which is the master equation to determine the theoretical values of the apparent magnitude.

Let us notice that, since the quantities \( d_L, M \) and \( m \) are related with luminosity and the flux of photons, i.e., with the propagation of photons in spacetime, neither is modified by the nonmetricity (recall that photons do not interact with nonmetricity but only with the curvature of space.) This means that these are gauge invariant quantities (null geodesics satisfy gauge covariant equations) which equal their GR value in any gauge. This is why, in what follows – without loss of generality – we compute \( m(z) \) in the GR gauge exclusively. What changes from gauge to gauge is the magnitude of the overall redshift factor \( z_{\text{tot}} \).

Now we are in position to explain the way in which the accelerated expansion/dark energy issue can be avoided in our theory. As illustration we shall consider one of the first data sets on high redshift supernovae measurements \(^{67}\), which provided early evidence for accelerated expansion of the universe.

On the one hand we shall explore the \( \Lambda \)CDM model, which arises in the GR gauge (\( \varphi = 0 \)) when nonvanishing cosmological constant is considered, for two arrangements of the present values of the dimensionless energy densities \( \Omega_0^m \) and \( \Omega_0^\Lambda \); \( (\Omega_0^m, \Omega_0^\Lambda) = (0.3, 0.7) \) and \( (\Omega_0^m, \Omega_0^\Lambda) = (1, 0) \). It is well-known that most of the existing data-sets point to a relationship \( \Omega_0^m/\Omega_0^\Lambda \approx 0.43 \), so that, for instance, the choice where \( \Omega_0^m = 1 \) and \( \Omega_0^\Lambda = 0 \), does not fit well the observational evidence. On the other hand we shall consider the same two choices of the mentioned cosmological parameters but in our theory \(^{102}, \, ^{104}\) in FRW-\( W^4 \)-spacetime, in a gauge where \( \varphi(z_{\text{GR}}) = \varepsilon z_{\text{GR}} \) (\( \varepsilon \) is a small number which, for our illustrative purpose, we choose to be \( \varepsilon = 0.15 \)). Since the values of the apparent magnitude – which coincide with their GR values – are the same in all of gauges, the only thing we have to modify in the chosen data set are the values of the redshift parameter which is \( z_{\text{GR}} \) in the GR gauge, but in the alternative gauge of our theory (\( \varphi = \varepsilon z_{\text{GR}} \)) it should be replaced by the overall redshift \( z_{\text{tot}} \) in Eq. (151).

The results are shown in FIG. 1 where the dash-dot curve corresponds to theoretical predictions of the apparent magnitude \( m(z) \) for the \( \Lambda \)CDM model with \( \Omega_0^m = 0.3 \) and \( \Omega_0^\Lambda = 0.7 \), while the solid curve corresponds to the choice \( \Omega_0^m = 1, \Omega_0^\Lambda = 0 \). This last choice amounts to matter-dominated (decelerated) Friedmann expansion: \( 3H^2 = M_p^{-2} \rho_m \). A plot of the apparent magnitude \( m \) vs curvature redshift \( z_{\text{GR}} \) is shown in the left figure, while in the right figure in FIG. 1 a plot of the apparent magnitude vs the overall redshift \( z_{\text{tot}} \) Eq. (151) is shown. In other words, in the left figure we have a fit of the \( \Lambda \)CDM model in the GR gauge of our gauge invariant setup with \( \varphi = 0 \) – formally it is just standard general relativity – to observational data, for two different arrangements of the cosmological parameters \( \Omega_0^m \) and \( \Omega_0^\Lambda \).

Meanwhile in the right figure what we have is a fit of the same model – with the same arrangements of the cosmological parameters – but in another gauge (\( \varphi = \varepsilon z_{\text{GR}} \)). While in the GR gauge, where the spacetime background has (pseudo)Riemannian geometrical structure, the redshift \( z_{\text{GR}} \) is due exclusively to propagation of photons in a curved spacetime, in any other gauge with \( \varphi = \varphi(t) \), where the background space belongs in \( W^4 \) (spaces with gradient nonmetricity), the overall redshift \( z_{\text{tot}} \) gets contributions both from propagation of photons in a curved background and from variation of masses of the atoms (and of SM particles in general) from point to point in spacetime.

It is seen in the left figure in FIG. 1 that the observational data favors the (Riemannian) GR-based \( \Lambda \)CDM model with \( \Omega_0^m = 0.3 \) and \( \Omega_0^\Lambda = 0.7 \). Meanwhile in the right figure the same observational data, interpreted in a specific gauge of our gauge invariant setup over \( W^4 \) spacetime (namely, \( \varphi = \varepsilon z_{\text{GR}} \)), seems to favor cosmological evolution dominated by CDM (vanishing cosmological constant). In this last case the supernova data brings no evidence for accelerating expansion, so that the dark energy plays no role in the cosmological dynamics and may be safely ignored.

We want to underline that the present analysis has been mostly illustrative and it did not involve other data

\(^{17}\) Although for simplicity here we do not use quantities with the hat, it is implicit that the \( \Lambda \)CDM model is based on \(^{110}\) which is satisfied only in the GR gauge (see also Eq. \(^{113}\)). In the present case in Eq. \(^{110}\) one have to make the replacement, \( T^{(m)}_{\mu\nu} \rightarrow T^{(m)}_{\mu\nu} + T^{(v)}_{\mu\nu} \).

\(^{18}\) Since we consider the dimensionless combination \( H_0d_L(z) \) rather than \( d_L(z) \), this means that in the equation to determine the apparent magnitude a term \( 5 \log_{10}(H_0) \) is to be taken away.
sets than one of the first high-redshift measurements reported in [67]. Other evidences for accelerating expansion within the framework of Riemannian GR-based ΛCDM model, such as those related with cosmic microwave background (CMB) temperature anisotropies, baryon acoustic oscillations (BAO), etc., should be carefully analyzed in our present theory before we may come to a definitive (solid) conclusion on the possible abandonment of the dark energy idea.

XIV. "MANY-WORLDS" INTERPRETATION OF THE GAUGE INVARIANT THEORY OF GRAVITY

The gauge invariant theory exposed in Sect. [X] shows a distinctive feature: due to invariance of the EOMs [102] and [114] under the gauge transformations [113], in addition to the four degrees of freedom to make diffeomorphisms, there is an additional degree of freedom to make gauge transformations. This is reflected in the fact that there is not an additional (independent) equation of motion for the gauge scalar \( \varphi \). Hence, any choice of \( \varphi = \varphi(x) \), satisfying the equations of motion Eqs. [102] and [104]. Different choices of the gauge scalar lead to different gauges of the present gauge invariant theory (see the discussion in Sec. [X]).

Although the gravitational laws [112], [104] are the same in all of possible gauges, each gauge

\[
\mathcal{G}_i = \{ M_4, g_{\mu \nu}^{(i)}(x), \varphi_i(x) \}, \quad i = 1, \ldots, N, \tag{156}
\]

where \( N \to \infty \) and each \( \varphi_i \) belongs in the set of real-valued, smooth continuous functions, provides a potentially different description of the universe. The different gauges belong in the same class of conformal equivalence since appropriate conformal transformations relate one given frame with any other one in the class. Actually, any two gauges \( \mathcal{G}_i \) and \( \mathcal{G}_j \) are joined by a gauge transformation,

\[
g_{\mu \nu}^{(i)} \rightarrow \Omega_{ij} g_{\mu \nu}^{(j)}, \quad \varphi_i \rightarrow \varphi_j - 2 \ln \Omega_{ij},
\]
or

\[
g_{\mu \nu}^{(i)} \rightarrow e^{\varphi_j - \varphi_i} g_{\mu \nu}^{(j)}.
\]

This means that all gauges in our theory belong in a class of conformal equivalence. Despite obvious differences, the resulting geometrical picture reminds us the “many-worlds” interpretation of quantum physics [108–117] since different gauges represent different theories, yielding different complete descriptions of the gravitational laws. Below we shall illustrate this in the cosmological context.

A. Many worlds and cosmology

In order to illustrate the resulting geometrical picture, for definiteness and also for simplicity, let us consider the cosmological setup where the background spacetime is FRW with flat spatial sections. In this case the different gauges can be defined as it follows,

\[
\mathcal{G}_i = \{ M_4, a_i^2(\tau), \varphi_i(\tau) \}, \quad i = 1, \ldots, N, \tag{157}
\]

where \( N \to \infty \), \( a_i(\tau) \) is the scale factor in the \( i \)-th gauge and \( \tau \) is the conformal time, which is related with the cosmic time \( t \) through, \( \tau = \int a^{-1} dt \).

A outstanding gauge is the one generated by the choice \( \varphi = \text{const} \). It is called as GR gauge and to all purposes it is no more than standard general relativity in Riemannian \( V_4 \) space (see Sect. [X A]). It is usually argued that GR is not a gauge invariant theory (it is not). Notwithstanding, in the present theoretical framework general relativity is no more than a specific gauge of the overall gauge invariant theory [102], [104], so that manifest gauge invariance is broken down by the gauge choice. Consequently, although gauge symmetries are not a manifest symmetry of general relativity, gauge invariance of the overall theory is implicitly shared by GR as well. The elements of the GR gauge can be expressed as,

\[
\mathcal{G}_{0k} = \{ M_4, a_{0k}^2(\tau), \varphi_{0k} \}, \quad k \in \mathbb{N}. \tag{158}
\]

The different constants \( \varphi_{0k} \in \mathbb{R} \) generate different sets of physical constants: \( \{ M_{pl,k}^2, \nu_k, \ldots \} \), where \( M_{pl,k}^2 = m_{pl,0}^2 \exp(\varphi_{0k}) \) is the Planck mass squared in the \( k \)-th element of the GR gauge, \( \nu_k = \nu_0 \exp(\varphi_{0k}/2) \) is the corresponding mass parameter of the SMP and the ellipsis stand for other possible physical constants which are transformed by the conformal transformations of the metric. Constants such as \( \hbar \), the electron charge \( e \), the speed of light in vacuum \( c \) and the fine structure constant \( \alpha \), which are not affected by the gauge transformations [113], are the same in every GR gauge \( \mathcal{G}_{0k} \).

In order to have a more clear idea of the many-worlds picture associated with the existence of infinitely many different gauges in our present theory, let us imagine that, at some initial post-Planckian time

\[
\tau_0 > \tau_{pl}, \quad \tau_{pl} \approx \frac{t_{pl}}{a(t_{pl})},
\]

where the quantum gravitational effects are subdominant,\(^{19}\) a large number \( N \) (\( N \to \infty \)) of identical copies

\(^{19}\) Our present theory with EOMs [102] and [104] driving the dy-
of a FRW-$\tilde{W}^4_{\text{int}}$ universe that is governed by Eqs. (102) and (104) (see Eqs. (159) and (160) below), have been prepared with the same initial conditions. These copies share same particle content, same non-gravitational laws, etc. They differ only in one function: $\varphi = \varphi(\tau)$ – the gauge scalar, so that each copy is actually a gauge. Once the $N$ copies are prepared in this initial state, each one of them evolves according to the laws Eqs. (102), (104) with the specified functional form of the gauge scalar. After the conformal time $\Delta \tau = \tau - \tau_0$ has elapsed, each one of the copies, say the $i$-th copy, distinguishes from each other. Hence, what we have is a set of $N$ different universes evolving according to the same laws of gravity (102) and (104) but with different functions $\varphi_i(t)$ ($i = 1, ..., N$). The different gauges represent physically equivalent descriptions of the gravitational laws since these are the same in any gauge.

Since the number of gauges $N$ can be very large ($N \rightarrow \infty$), in principle any possible pattern of cosmic evolution can be reproduced by given gauges in the set. Hence, the question is: which would be the predictive power of such a theory? In the next section we shall answer this question. As we shall see different gauges fit differently the same observational evidence. The right question then turns out to be which copy – in the very large number $N$ of copies of the universe – is the one that better fits the existing amount of observational evidence? The winning copy will be the one where we belong in.

**XV. GAUGES AND OBSERVATIONAL EVIDENCE**

Here we shall look for specific evolution in conformal time of the unknowns $a(\tau)$ and $\varphi(\tau)$, in order to differentiate given gauges. Then we shall seek for observational data fitting of these gauges. As before we shall consider only the high redshift SN-Ia data from Ref. [67]. Besides, for simplicity, we shall assume vanishing vacuum energy density: $T^\nu_\mu = 0$ in Eq. (119), i.e., we shall assume vanishing cosmological constant $\Lambda = 0$. This would entail, in particular, that the GR gauge (basically standard general relativity) can not fit the observational evidence on high redshift SN-Ia.

Let us write the (00)-component of the equations of motion Eq. (102) in terms of the FRW metric Eq. (123),

$$3 \left( H + \frac{\dot{\varphi}}{2} \right)^2 = \frac{e^{-\varphi}}{M^2_p} \rho_m, \quad (159)$$

as well as the null-component of the conservation equation Eq. (103),

$$\dot{\rho}_m + 3H(\rho_m + p_m) - \frac{\dot{\varphi}}{2}(\rho_m - 3p_m) = 0, \quad (160)$$

where the dot accounts for derivative with respect to the cosmic time $t$ (below we shall come back to the conformal time.) In the above equations we have considered the time normalization Eq. (125) with $\varphi(t_{\text{pl}}) = 0$, and the stress-energy tensor of a prefect fluid,

$$T^m_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu},$$

where $\rho_m$ and $p_m$ are the energy density and the pressure of the perfect fluid, respectively, while $u_\mu = \delta^0_\mu$ is the four-velocity vector of the fluid in the co-moving frame. Let us further, for simplicity, consider pressureless dust ($p_m = 0$). In this case the conservation equation Eq. (160) simplifies to,

$$\dot{\rho}_m + \left( 3H - \frac{\dot{\varphi}}{2} \right) \rho_m = 0.$$

Integration of this equation leads to,

$$\rho_m = \frac{M^4 e^{\varphi/2}}{a^3}, \quad (161)$$

where $M^4$ is an integration constant (do not confound with the absolute magnitude $M$ of distant type IA supernovae.) If we substitute $\rho_m$ from Eq. (161) back into Eq. (159) and we replace the cosmic time $t$ by the conformal time $\tau = \int a^{-1} dt$, then the latter equation can be written in the following way:

$$e^{\chi/2} d\chi = \sqrt{\frac{M^4}{3M^2_p}} d\tau, \quad (162)$$

where we have also introduced the new gauge invariant variable $\chi \equiv \ln a + \varphi/2$. Straightforward integration of Eq. (162) yields,

$$ae^{\varphi/2} = \frac{M^4}{12M^2_p} (\tau - \tau_0)^2, \quad (163)$$

where $\tau_0$ is another integration constant. This solution, as any other solution of the equations of motion (102), (104), shows that we can determine, at most, the gauge invariant product $a \exp(\varphi/2)$. As already mentioned, this is a direct consequence of gauge invariance: we can choose any $\varphi(\tau)$ we want or, in its place, we can choose any $a = a(\tau)$. The different choices define different gauges. Each gauge amounts to a possible (complete) history of the universe. The universe we observe is appropriately described by one of the possible gauges.

Let us consider, for illustrative purposes, three gauges among the infinite set of them:
1. **GR gauge.** In this case \( \varphi = \text{const} \). For definiteness we choose \( \varphi = 0 \). This is one of the elements in the infinite set of elements in the GR gauge (recall that other values of the constant \( \varphi = \varphi_0 \) also define possible elements of the GR gauge.) In this case Eq. (163) simplifies to,

\[
a_{GR}(\tau) = \frac{M^4}{12M_{pl}^2}(\tau - \tau_0)^2, \tag{164}
\]

where \( \tau_0 \) marks the starting point of the cosmic expansion and we shall use the following conformal time normalization: at present time \( \tau = \tau_* \) we have that \( a_{GR}(\tau_*) = 1 \). Hence, from Eq. (164) it follows that,

\[
\tau_* - \tau_0 = \sqrt{\frac{12M_{pl}^2}{M^4}}. \tag{165}
\]

Besides (recall that \( dt = a d\tau \)),

\[
H_{GR} = \frac{\dot{a}_{GR}}{a_{GR}} = \frac{a'_{GR}}{a_{GR}} = \frac{24M_{pl}^2}{M^4} \left( \frac{\tau}{\tau_0} - 1 \right)^{-3}, \tag{166}
\]

where the comma denotes derivative with respect to the conformal time \( \tau \). We have also,

\[
\dot{H}_{GR} = \frac{H'_{GR}}{a_{GR}} = -\frac{864M_{pl}^4}{M^8} \left( \frac{\tau}{\tau_0} - 1 \right)^{-6}. \tag{167}
\]

Hence, the deceleration parameter,

\[
q \equiv -\left( 1 + \frac{\dot{H}_{GR}}{H_{GR}^2} \right) = \frac{1}{2}, \tag{168}
\]

so that the expansion is decelerated.

We can write the conformal time in terms of the redshift \( \chi_{GR} \),

\[
\tau - \tau_0 = \sqrt{\frac{12M_{pl}^2/M^4}{\sqrt{1 + \chi_{GR}}}}. \tag{169}
\]

For the comoving distance we have that,

\[
\chi = \int \frac{da}{a^2 H(a)} = \int d\tau = \tau - \tau_0, \tag{170}
\]

i.e., modulo a constant, the comoving distance coincides with the conformal time. By deriving (169) we get that,

\[
d\tau = -\sqrt{\frac{3M_{pl}^2}{M^4}} \frac{dz_{GR}}{(1 + \chi_{GR})^{3/2}},
\]

which, after integration, yields

\[
\chi = \tau = \sqrt{\frac{3M_{pl}^2}{M^4}} \int_{\tau_0}^{\tau} \frac{dz'_{GR}}{(1 + \chi_{GR})^{3/2}}. \tag{171}
\]

Comparing this equation with Eq. (154) with \( \Omega_0^m = 1 \) and \( \Omega_0^\Lambda = 0 \), and comparing with Eq. (169), one gets that the integration constant,

\[
M^4 = 3H_0^2M_{pl}^2, \tag{172}
\]

where \( H_0 \) is the present value of the GR Hubble parameter. Actually, substituting Eq. (172) into Eq. (169) yields that Eq. (169) can be written in the following way:

\[
H_{GR}(\chi_{GR}) = H_0 (1 + \chi_{GR})^{3/2}, \tag{173}
\]

so that \( H_{GR}(0) = H_0 \).

The plot of the apparent magnitude vs redshift \( \chi_{GR} \) for this gauge – general relativity with a vanishing cosmological constant – corresponds to the solid curve in the left panel of FIG. 1. It is seen that, in what regards to the high-redshift type IA supernovae data, this gauge can not explain the resulting accelerated pace of the cosmic expansion.

2. **Flat gauge.** Since the curvature of spacetime vanishes, then \( \Phi = 0 \). This is the only gauge – general relativity with a vanishing cosmological constant – corresponds to the solid curve in the left panel of FIG. 1. It is seen that, in what regards to the high-redshift type IA supernovae data, this gauge can not explain the resulting accelerated pace of the cosmic expansion.

\[
e^{\varphi/2} = \frac{M^4}{12M_{pl}^2} (\tau - \tau_0)^2. \tag{174}
\]

In the present case the comoving distance obeys Eq. (170), and the only contribution to the redshift comes from variation of masses due to gradient nonmetricity. Hence,

\[
e^{\varphi/2} = \frac{1}{1 + \chi_{GR}}. \tag{175}
\]

Combining Eqs. (173), (175) and (170), for the comoving distance one gets the same expression (171) that we obtained in the former gauge. What this means is that, although in the present gauge the universe is not expanding \( H = 0 \), the fit of the model to the observational data on high-redshift
type IA supernovae is the same as in the GR gauge (solid curve in the left figure in FIG. 1). This means that the model is ruled out by the cosmological observations. Only if add the problematic cosmological constant term in the above and in the present gauges we obtain a good fit to the data on high-redshift measurements (dash-dot curve in the left panel of FIG. 1).

In addition, the flat gauge is phenomenologically ruled out because the light rays follow straight lines and do not suffer gravitational bending. Actually, since, on the one hand, photons do not interact with nonmetricity, in flat background space the null geodesics Eq. (146), read

\[ \frac{dk^a}{d\lambda} = 0. \]

On the other hand, in the flat gauge gravity is a manifestation of nonmetricity exclusively. Therefore, photons do not suffer neither curvature redshift of frequency nor gravitational bending. Means that this model does not pass the Solar system test on light bending.

3. Third gauge. Let us consider a third (“intermediate") possibility where neither the scale factor \( a \), nor the gauge scalar \( \varphi \) are constants. Since the product \( a \exp(\varphi/2) \) is gauge invariant, this means that \( a \exp(\varphi/2) = a_{GR} = 1/(1 + z_{GR}) \), where \( a_{GR} \) is the scale factor in the GR gauge. Hence, according to Eq. (163), the following relationship between the conformal time \( \tau \) and the redshift \( z \), can be established:

\[ 1 + z_{GR} = \frac{4}{H_0^2 (\tau - \tau_0)^2}, \]

where we have considered Eq. (172). In this case the redshift of frequency is contributed both by the curvature of space and by nonmetricity. The overall redshift \( z_{tot} \) is given by Eq. (145), which can be written as in Eq. (149). In Sect. XIII we have shown that an acceptable fit to the high-redshift data set of Ref. [67] is obtained if assume that \( \varphi = \varepsilon z_{GR} \) is a small quantity, so that the overall redshift can be written in terms of the GR redshift through Eq. (161). Here we shall assume this is the case, so that the “third gauge” is consistent with the observations. Of course the above is only an approximate expression for the gauge scalar \( \varphi \), which is valid for redshifts \( z_{GR} < 1 \). This means that the correct gauge should contain this (or a similar) approximation as a particular limit. Yet, since our discussion is mostly illustrative (qualitative), it suffices to consider the above approximation as a gauge covering the whole cosmic history from the distant past \( z_{GR} \to \infty \) to the asymptotic future \( z_{GR} \to -1 \).

In this gauge, from Eq. (163) it follows that,

\[ a(\tau) = \exp(\frac{1}{2}(1 - \xi)/\xi), \]  

(177)

where we have defined the following function of the conformal time \( \tau \):

\[ \xi = \xi(\tau) \equiv \frac{4}{H_0^2 (\tau - \tau_0)^2}. \]

Besides, the Hubble parameter reads

\[ H(\tau) = H_0 \sqrt{2} \exp(\frac{1}{4}(1 + \xi)/\xi) \]

(179)

Given our normalization in Eqs. (163) and (172), it follows that at present time \( \tau = \tau_0, \tau_0 - \tau_0 = 2H_0^{-1} \), so that \( \xi(\tau_0) = 1 \). In consequence in this gauge \( a(\tau_0) = 1 \), while the present value of the Hubble parameter,

\[ H(\tau_0) = H_0 \left(1 + \frac{c}{2}\right) \]

(180)

slightly differs from its GR value \( H_{GR}(\tau_0) = H_0 \). In general, for arbitrary function \( \varphi = \varphi(z_{GR}) \), we have that,

\[ H = H_0 \sqrt{2} \left(1 + \frac{z_{GR}}{2} \frac{d\varphi}{dz_{GR}}\right) \]

(181)

while the overall redshift and the GR redshift are related by:

\[ z_{tot} + 2 = e^{\varphi/2} (z_{GR} + 2). \]

(182)

The question under scrutiny is, which would be the predictive power of a theory that admits almost any possible evolution pattern? In order to answer this question let us first summarize our above results.

Among the infinite number of different gauges, above we have chosen three of them: i) the GR gauge, ii) the flat gauge and iii) the third gauge. The different gauges provide different but equivalent descriptions of the cosmological evolution. While in the GR gauge gravity is due to the curvature of FRW-V$_4$ (Riemann) space, in the flat space gauge it is completely due to gradient nonmetricity in Minkowski space. In the third gauge the gravitational phenomena are associated both with curvature of FRW-V$_4$ space and with gradient nonmetricity.
The different gauges are physically equivalent since they satisfy the same laws of gravity Eqs. (102), (104). Gauge equivalence is due to invariance of the gauge-invariant quantities such as, for instance \( a \exp(\varphi/2) \), under the transformations (103).

The critical argument in order to answer the question on the predictive power of our theory is the following. Although all three gauges above: GR, flat and third gauges, yield different but equivalent descriptions of the same gravitational laws, only one of them: the third gauge, fits well the observational evidence from high-redshift type IA supernovae (see the solid curve in the right panel of FIG. 1) Neither the GR gauge nor the flat one fit well enough the observational SN-Ia data (recall that for sake of simplicity here we consider vanishing cosmological constant.) In addition, the flat gauge is not compatible with light bending in a gravitational field so that Solar system tests rule out this gauge. In consequence, the GR and flat gauges are ruled out by experiments/observations.

Experiment and observations in general, play a crucial role in determining which one, in the large number \( N \to \infty \) of possible gauges, is the one that better describes our causally accessible universe. In a sense experiment allows us to determine the gauge “which we live in,” which in what follows we shall call as “world-gauge.” Once this gauge is fully determined, which means that we fully determined the gauge scalar \( \varphi = \varphi(\tau) \), one can make predictions with the help of the gravitational laws (159) and (160), which are valid in our world-gauge.

A. The \( H_0 \) tension issue

In the above case observations and experiments favor the third gauge which we identify with our causally accessible universe. One of the predictions we can make on the basis of the laws governing our world-gauge, is that the present value of the Hubble parameter \( H(\tau_0) \) in Eq. (180) differs from the one computed on the basis of the GR equations \( H_0 \). This discrepancy is unavoidable, unless \( \varphi = \text{const} \). This latter case is the trivial one and corresponds to the GR gauge, i.e., to general relativity framework. Actually, from Eq. (181) it follows that at present \( (z_{GR} = 0) \),

\[
H(0) = H_0 \left(1 + \frac{1}{2} \frac{d\varphi}{dz_{GR}}|_{z_{GR}=0}\right),
\]

where, without loss of generality, we are using the normalization in which \( \varphi(0) = 0 \).

The discrepancy arising in our gauge invariant framework may explain the disagreement between the present value of the Hubble parameter locally measured (no specific model assumed) of about \( H_0 \approx 73.2 \) km s\(^{-1}\) Mpc\(^{-1}\) at 68\% confidence level and the one inferred from GR (plus the cosmological constant) of about \( H_0 \approx 67.3 \) km s\(^{-1}\) Mpc\(^{-1}\) by evaluating early times physics \([75]\). As a matter of fact, under the assumption that at redshifts \( z < 1 \) our approximation Eq. (154) is not bad, in equation (180) one may set \( \epsilon \approx 0.15 \). Hence, assuming that the present value of the Hubble parameter computed in our world-gauge coincides with the one measured in local experiments \( H(\tau_0) \approx 73.2 \) km s\(^{-1}\) Mpc\(^{-1}\), from Eq. (180) one gets that the corresponding GR value \( H_{GR} = H_0 \approx 68.1 \) km s\(^{-1}\) Mpc\(^{-1}\). These estimates can be improved by improving our knowledge of the curvature of the spatial sections and \( \varphi(\tau)/\varphi(z_{GR}) \). Recall that we are using a very rough approximation \( \varphi \approx \epsilon z_{GR} \), which is valid only as long as \( \epsilon z_{GR} \ll 1 \).

According to above explanation the \( H_0 \) tension issue \([75]\) is due to the assumption of an incorrect theory – general relativity, in particular the ΛCDM model – in order to compute the present value of the Hubble constant by evaluating early time physics. In our gauge invariant framework we do not need neither GR nor the cosmological constant in order to explain the accelerated expansion of the cosmos (see Sec. XIII). Hence, if assume that the theory (159), (160) correctly describes the classical laws of gravity, including the early times stage, there will be no discrepancy between the present value of the Hubble constant computed by extrapolating early times physics to the present, with the one inferred from local measurements (late time physics).

XVI. FLATNESS, HORIZON AND RELICT PARTICLES ABUNDANCE PROBLEMS

Any theoretical framework that pretends to explain the past, present and future of the cosmic expansion, should be able to explain the flatness, horizon and relict particles abundance problems. Cosmic inflation \([118–131]\) has been developed, precisely, with the aim to explain these puzzles. Here we shall show that the neither the flatness nor the horizon and relict particle abundance problems arise in the present gauge invariant theory, so that no inflationary mechanism is required.

A. Flatness problem

Let us briefly explain what is the flatness problem about. For this purpose let us to write the GR equations of motion Eqs. (110), (111) in FRW background with the following line element in comoving spherical coordinates \( t, r, \theta, \phi \):

\[
ds^2_{GR} = -dt^2 + \frac{a_{GR}^2(t)}{1 - kr^2} dr^2 + r^2 a_{GR}^2(t) d\Omega^2, \quad (183)
\]

where \( a_{GR}(t) \) is the scale factor, \( k = \pm 1, 0 \) account for the curvature of the spatial sections and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). The FRW GR-EOMs read:
\[
3H_{GR}^2 + \frac{3k}{a_{GR}^2} = \frac{1}{M_{pl}^2} \rho_m, \quad (184)
\]
\[
\rho_m + 3\gamma_m H_{GR} \rho_m = 0, \quad (185)
\]
where \(H_{GR} = \dot{a}_{GR}/a_{GR}\) is the GR Hubble rate, \(M_{pl}\) is the Planck mass and \(\rho_m, p_m = (\gamma_m - 1) \rho_m\) are the energy density and the barotropic pressure of the background matter fluid, respectively (\(\gamma_m\) is the barotropic index of the matter fluid.) In terms of the dimensionless (normalized) energy density: \(\Omega_m = \rho_m/3M_{pl}^2H_{GR}^2\), the Friedmann equation Eq. (184) can be written in the following alternative way:

\[
\xi_{GR} := \Omega_m - 1 = \frac{k}{a_{GR}^2 H_{GR}^2}, \quad (186)
\]
where the quantity \(\xi_{GR}\) measures the departure from spatial flatness. Integration of Eq. (185) yields \(\rho_m = M^4 a^{-3\gamma_m} (M\text{ is an integration constant with mass units,})\) so that Eq. (186) can be written in the following way:

\[
|\xi_{GR}| = \frac{|k|}{H_0^2 a^{2-3\gamma_m} - k}, \quad (187)
\]
where we took into account Eq. (172). From this equation it follows that for \(\gamma_m > 2/3\), a requirement that is satisfied by dust (\(\gamma_m = 1 > 2/3\)) and also by radiation (\(\gamma_m = 4/3 > 2/3\)), the cosmic expansion leads to \(|\xi_{GR}| \rightarrow 1\) at late times.

The only way in which the departure from spatial flatness decreases with the curse of the cosmic expansion is for matter with \(\gamma_m < 2/3\). In this case, for \(a_{GR} \gg (3kM_{pl}^2/M^4)^{1/(2-3\gamma_m)}\),

\[
|\xi_{GR}| \approx \frac{3kM_{pl}^2}{M^4 a_{GR}^2 - k},
\]
so that, at late times \(|\xi_{GR}| \rightarrow 0\). The inflaton \(\phi\) is a kind of dynamical self-interacting scalar field which can lead to fulfillment of the condition \(\gamma_{\phi} < 2/3\). The latter requirement is obviously satisfied by vacuum fluid (\(\gamma_m = 0\)) as well.

In order to show that this problem does not arise in our gauge invariant theory, let us write the EOM Eqs. (102) and (104) in terms of the metric (183):

\[
3 \left( H + \frac{\dot{\phi}}{2} \right)^2 + \frac{3k}{a^2} = \frac{e^{-\phi}}{M_{pl}^2} \rho_m, \quad (188)
\]
\[
\dot{\rho}_m + \left( 3H - \frac{\dot{\phi}}{2} \right) \rho_m = 0, \quad (189)
\]
where, for simplicity, we have chosen dust fluid (\(p_m = 0\)). If we integrate Eq. (189) and substitute into Eq. (188), the latter equation can be written in the following way,

\[
H^2 + \frac{k}{a^2} = \frac{H_{GR}^2}{a^2}, \quad (190)
\]
where we have introduced the gauge invariant variable \(\chi = ae^{\phi/2}\) and a new Hubble parameter \(H_{\chi} = \dot{\chi}/\chi\). Besides, we took into account Eq. (172). The parameter which measures departure from spatial flatness can be defined as,

\[
|\xi| = \frac{|k|}{a^2 H_{\chi}^2 - k}, \quad (191)
\]
Gauge freedom can explain flatness. Actually, what we need is that during the curse of the cosmic evolution \(\chi \rightarrow 0\). Given the freedom in choosing the function \(\varphi\), this can be easily achieved. Let us set, as illustration, \(\chi^{-1} = a t^n\), where \(a \geq 0\) and \(n \geq 0\) are non-negative reals.\(^\text{20}\) Hence,

\[
|\xi| = \frac{|k|}{a H_0^2 t^n - k},
\]
so that, at large \(t \gg (k/a H_0^2)^{1/n}\),

\[
|\xi| \approx \frac{k}{a H_0^2} t^{-n}, \quad \lim_{t \to \infty} |\xi| = 0.
\]

If substitute the above choice \(\chi = a^{-1} t^{-n}\) into Eqs. (188) and (189), we get that,

\[
a(t) = \frac{t}{n} \sqrt{\alpha H_0^2 t^n - k},
\]
\[
\varphi(t) = \varphi_0 - 2(n + 1) \ln t - \ln (\alpha H_0^2 t^n - k),
\]
where \(\varphi_0 = 2 \ln(n/\alpha)\). There are many other possible choices of \(\chi\) that lead to avoidance of the flatness problem (see Ref. [54]), however, the above choice is enough in order to illustrate how gauge freedom allows to avoid this problem.

**B. Horizon problem**

In order to simplify the analysis in this subsection we shall consider FRW spacetime with flat spatial sections (\(k = 0\).) For definiteness let us consider background radiation \(p_r = p_r/3\). The cosmological equations read,
FIG. 2: Plots of the scale factor and of the distance to the causal horizon vs cosmic time $t$ (solid and dash-dot curves, respectively). We arbitrarily chose the following values of the constants: $H_0 = 0.7$, $\mu = 1.5$. The left figure is for GR with background radiation where the horizon problem is evident (equations (194) for the scale factor and (195) for the distance to the causal horizon), while the middle figure depicts GR-de Sitter expansion where the horizon puzzle is settled (equations (196) and (197)). In the right figure the plots are for an arbitrarily chosen gauge of the present gauge invariant theory. This gauge is specified by equations (198) for the scale factor and (199) for the distance to the causal horizon. In this last case the horizon issue does not arise.

The horizon problem arises because scales that originated outside of the causal horizon will eventually enter our past light cone and hence these will become part of our observable Universe [122]. In consequence anisotropies are expected to be observed on large scales [132, 133]. Within the GR framework length scales grow as the scale factor:

$$a_{GR}(t) = \sqrt{2H_0 t^{1/2}}.$$  

(194)

Meanwhile, the causal horizon [133], which amounts to the maximal physical distance light can travel from the co-moving position of an observer at some initial time to time $t$ [122]:

$$d_H(t) = a_{GR} \int_0^t \frac{dt'}{a_{GR}(t')} = 2t.$$  

(195)

Hence, the distance to the causal horizon grows faster than co-moving separations i.e., than the scale factor.

The horizon problem is illustrated in the left figure of FIG. 2 where the plots of $a_{GR}$ (dash-dots) and of $d_H$ (solid curve) vs the cosmic time $t$, are shown. It is seen that at early times, very close to the bigbang, $t_{bb}$ and up to the “equality time” $t_{eq}$, time at which the dash-dot and the solid curves meet again, the curve representing $d_H(t)$ lies below the curve for $a_{GR}(t)$. During this time interval, scales that at certain $t_{bb} \leq t \leq t_{eq}$ were located above the solid curve and below of the dash-dots, are not in causal contact with the co-moving position, while scales that are located below of the solid curve are causally connected with co-moving observer instead. After the equality time $t_{eq}$ those scales that were out of causal contact since the bigbang $t_{bb}$ and up to $t_{eq}$, enter the causal horizon so these can be seen by a co-moving observer.

Inflation can take account of the horizon problem since during the de Sitter expansion period:

$$a_{GR}(t) = \exp(H_0 t),$$  

(196)

while

$$d_H(t) = \frac{1}{H_0} (e^{H_0 t} - 1).$$  

(197)

The plots of $a_{GR}$ Eq. (196) and of $d_H$ Eq. (197) for this case are shown in the middle figure of FIG. 2. It is seen that scales that in the past were out of causal contact keep causally disconnected for all time.

As it was for the flatness problem, within the present gauge invariant framework, due to gauge freedom, inflation is not required in order to explain the horizon issue. Since $\chi = ae^{\varphi/2}$ is a gauge invariant quantity, for the present case – spatially flat FRW metric with background radiation – we have that,
\[ ae^{\varphi/2} = a_{GR} = \sqrt{2H_0} \sqrt{t} \Rightarrow a(t) = \sqrt{2H_0} \sqrt{t} e^{-\varphi/2}, \]

where Eq. (194) has been taken into account. Let us choose the gauge with \( \varphi(t) = -2\mu \sqrt{t} \), where \( \mu \) is a constant parameter with the dimensions of the square of mass. For the scale factor and the distance to the causal horizon we get that,

\[
a(t) = \sqrt{2H_0} \sqrt{t} e^{\mu \sqrt{t}}, \quad (198)
\]
\[
d_H(t) = \frac{2\sqrt{t}}{\mu} \left( e^{\mu \sqrt{t}} - 1 \right), \quad (199)
\]

respectively. In this gauge those scales which were out of causal contact in the past, will be causally disconnected for all future times, as it was for inflation within the GR framework. This is illustrated in the right figure in FIG. 2 where the dash-dot curve represents the evolution of the scale factor (198) while the solid curve represents the evolution of the distance to the causal horizon \( d_H \) in cosmic time Eq. (199).

C. Abundance of relict particles

In a similar fashion gauge freedom may explain the problem with the abundance of relict particles such as magnetic monopoles, gravitinos, moduli fields, etc. The question in this case is why the universe is not dominated by these heavy relict particles at present? In order to explain why this is not so within GR-based cosmology, it is again required the inflationary stage, so that any initially existing amount of relict particles will be very quickly diluted

\[
\rho_{rel}^{GR} \propto \exp(-3\gamma H_0 t).
\]

This means, in turn, that within the GR framework an additional inflaton field is required.

In our gauge invariant theory inflation is not necessary to explain the problem with the abundance of relict particles, as we shall see. In this case the gravitational laws are given by Eqs. (159) and (160), respectively. As before, for simplicity we shall consider spatially flat FRW background space. We also assume that the background fluid is a perfect fluid of relict particles with energy density and pressure \( \rho_{rel}, p_{rel} \), which satisfy

\[
\rho_{rel} = \frac{M^4 e^{\frac{4-3\gamma}{3\gamma}}} {a^3 \gamma} = \frac{M^4 e^{2\varphi}} {\chi^{3\gamma}} = \frac{M^4 e^{2\varphi}} {a_{GR}^{3\gamma}},
\]

or

\[
\rho_{rel} = e^{2\varphi} \rho_{rel}^{GR},
\]

where \( \rho_{rel}^{GR} \) is the solution of the GR conservation equation \( \rho_{rel}^{GR} + 3\gamma H \rho_{rel}^{GR} = 0 \). We can find a gauge where the density of the fluid of relict particles very quickly decays with the course of the cosmic expansion. In particular, the gauge \( \varphi(t) = -2\mu \sqrt{t} \) which solved the horizon problem above, can take account of the abundance of relict particles as well. In this gauge the Friedmann equation can be written as,

\[
\left( \frac{\dot{\chi}}{\chi} \right)^2 = \frac{H_0^2 e^{-2\mu \sqrt{t}}}{\chi^{3\gamma}}.
\]

By straightforward integration of this equation one finds,

\[
\chi(t) = \left( \frac{3\gamma H_0}{\mu} \right)^{\frac{2}{3}} e^{-\frac{2\mu \sqrt{t}}{3\gamma}} \left( e^{\mu \sqrt{t}} - 1 - \mu \sqrt{t} \right)^{\frac{2}{3}},
\]

so that

\[
\rho_{rel}^{GR} = \frac{M^4}{\chi^{3\gamma}} = \frac{\mu^2 M_{pl}^2 e^{2\mu \sqrt{t}}}{3\gamma^2 \left( e^{\mu \sqrt{t}} - 1 - \mu \sqrt{t} \right)^2}, \quad (202)
\]
\[
\rho_{rel} = \frac{\mu^2 M_{pl}^2}{3\gamma^2 e^{2\mu \sqrt{t}} \left( e^{\mu \sqrt{t}} - 1 - \mu \sqrt{t} \right)^2}. \quad (203)
\]

The ratio of the energy densities of the relict particles according to our gauge invariant theory \( \rho_{rel} \) and according to GR \( \rho_{rel}^{GR} \),

\[
\frac{\rho_{rel}}{\rho_{rel}^{GR}} = e^{-4\mu \sqrt{t}}, \quad (204)
\]

very quickly goes to zero. This result is independent of the behavior of the GR energy density of relict particles.

What we have shown is that the abundance of relict particles in our theory does not represent a problem. These results are illustrated in FIG. 3.

XVII. DISCUSSION AND CONCLUSION

Nonmetricity theories have received renewed interest within the gravitational community due to the cosmological applications of the symmetric teleparallel theories of
FIG. 3: In the left figure the plots of the energy densities $\rho_{GR}$ in Eq. (202) – dash-dots – and $\rho_{rel}$ in Eq. (203) – solid curve – vs cosmic time $t$ are shown. We arbitrarily chose the following values of the constants: $H_0 = 0.7$, $\mu = 1.5$, $M_{pl}^2 = 1$ and $\gamma = 1$ (dust of relict particles). The ratio $\rho_{rel}/\rho_{GR}$ in Eq. (204) vs cosmic time $t$ is drawn in the right figure. It is seen that, independent of the behavior of $\rho_{GR}(t)$, the energy density of relict particles in the present gauge of our gauge invariant theory, very quickly dilutes with the expansion.

However, due to the SCE this theory is phenomenologically ruled out as a classical theory of gravity. Notwithstanding, it may have led its footprints in the quantum era. When in the above theory we replace vector by gradient nonmetricity, the quadratic term vanishes and, besides, the SCE does not arise. Hence, the resulting gauge invariant theoretical framework, which is given by the EOMs (102) and (104), may serve to search for the classical phenomenological and observational consequences of gauge symmetry.

Gauge freedom can be associated with a physical (also geometrical) picture resembling the many-worlds interpretation of quantum physics [108–117]. Given that the gauge scalar $\varphi$ may be fixed at will, in equations (102), (104) we may choose any function $\varphi(x)$ we want. The result will be a specific theory associated with this choice or a gauge. Hence, each gauge represents a whole theory of gravity over WIG ($\tilde{W}_4^{\text{int}}$) space, which is characterized by a specific behavior in spacetime of several fundamental “constants,” the mass of the SMP particles, etc. An outstanding gauge in this theoretical framework is the so called GR gauge, which is a set of GR theories specified by the choice $\varphi = \varphi_0 i$ ($i = 1, 2, ..., N$), where the $\varphi_0 i$ are different constants. In this gauge the gravitational laws look (and are) exactly the same, so that each member in the GR gauge differs from any other in the values of the fundamental constant $M_{pl,i}^2$ and of the EW mass parameter $v_{\text{ew}}^2$, among others. Hence, the constant mass of given gravity [29–31, 33–42]. However, one of the most important properties of nonmetricity geometry: gauge invariance, has not been investigated with the same interest. It has been known since the first theory of this kind was published by Weyl [1, 10], that gauge symmetry is the distinctive feature of nonmetricity. This is not modified by considering the teleparallel condition that $R^\alpha_{\mu\nu\sigma} = 0$. The argument many authors make to justify not considering gauge symmetry is that one has the freedom to choose the action of the theory, which is the one that defines the underlying symmetries. Others base their work on the dynamical equivalence existing between GR and (pure nonmetricity) symmetric teleparallel equivalent of GR (STEGR) [42]. Then, since GR is not a Weyl gauge invariant theory, the STEGR should not possess gauge symmetry. The question is, why a theory where gravity has geometric nature should not respect the underlying symmetry of the geometrical background? Weyl avoided this question by choosing an action for his theory which satisfied gauge symmetry.

In this paper we have followed a similar approach by looking for a (quadratic) gravitational theory whose action [39] possesses the same symmetries of the $\tilde{W}_4$ geometric background. This opened up the possibility to search for the phenomenological and observational consequences of gauge symmetry. A distinctive feature of this quadratic theory is that it is free of the Ostrogradski instability since the equations of motion are second order. However, due to the SCE this theory is phenomenologically ruled out as a classical theory of gravity. Notwithstanding, it may have led its footprints in the quantum era. When in the above theory we replace vector by gradient nonmetricity, the quadratic term vanishes and, besides, the SCE does not arise. Hence, the resulting gauge invariant theoretical framework, which is given by the EOMs (102) and (104), may serve to search for the classical phenomenological and observational consequences of gauge symmetry.
SMP particle is different in each member of the gauge. In this gauge invariant framework general relativity is just a subclass of a bigger theory. Manifest gauge symmetry is broken down once a specific gauge has been chosen. This is why GR seems to evade this symmetry. Yet, it is a residual symmetry since any gauge of \( \text{(102)}, \text{(104)} \) is related with any other gauge through the transformations Eq. \( \text{(105)} \) (see the related discussion in Sec. \( \text{XIV} \)).

The main objection against our theory could be associated with its predictive power, given our freedom to choose \( \varphi(x) \). Notwithstanding, on the basis of equations \( \text{(102)}, \text{(104)} \) we can make predictions as in any other theory of gravity. The only thing we have to achieve is to fully determine the gauge where we “live” in or our world-gauge. I. e., we need to fully determine the gauge scalar \( \varphi(x) \) which is consistent with the existing amount of experimental and observational evidence. This is when experiments/observations make their magic. It happens that gauge freedom can be experimentally tested in the sense that astrophysical and cosmological data sets as well as other experimental results are able to pick up, among the infinite number of equivalent gauges, our world-gauge. One example is provided by the high-redshift SN-Ia data sets \( \text{[60, 68, 74]} \), as explained in Sec. \( \text{XIII} \). In this case one looks for the dependence of the apparent magnitude of supernovae type Ia on the redshift.

The lucky circumstance here is that, on the one hand, the apparent magnitude \( m \) (do not confound with the mass parameter) is a gauge invariant quantity since it has to be with the propagation of light in spacetime: light does not interact with nonmetricity so that its propagation may be affected by spacetime curvature exclusively. On the other hand the source of light (atoms) is point dependent: the energy of atomic transitions varies from point to point in spacetime. This entails that the overall redshift is contributed both by the propagation of photons in a curved space and by the nonmetricity through spacetime variation of the atoms masses: different amounts of nonmetricity lead to different values of the overall redshift. Hence, data sets of \( m(z) \) allow us to pick up a gauge which the best fit.

In the present paper we have been able just to qualitatively illustrate the possibility to determine the gauge function \( \varphi(x) \) by means of the check of observational data sets. It is necessary to go further and to look for new and more encompassing checks which may allow us to determine our world-gauge with more accuracy. This will be possible once we develop the theory of the cosmological perturbations that is adequate for the present gauge invariant framework. Then we will be able to make new predictions on the basis of the present theory which is based on equations \( \text{(102)}, \text{(104)} \). An area of opportunity we may easily identify is the one related with the correlation between measurements of cosmic distances and redshift. Let us consider the following simplified situation. Suppose that an observer \( O \) is correlating the measurements of distance to a distant star \( S \) with the corresponding redshift value. Suppose further that the distance to \( S \) measured by some specific method is \( d_S \).

Take two hydrogen atoms: one atom \( H_{\text{int}} \) which is located in the deep interstellar space at a distance \( d_{H_{\text{int}}} = d_S \) from \( O \), and the other one \( H_S \), which located on the surface of the distant star. We assume that the distance to the star is much bigger than its size so that we may consider that both atoms are at the same distance from the observer \( O \). Since \( H_{\text{int}} \) and \( H_S \) are not only at different values of the gravitational field but also of the gauge scalar \( \varphi \), then the overall redshift will be contributed by: i) the gravitational redshift due to variations of the gravitational potential, ii) the cosmological redshift due to propagation of photons in an expanding FRW spacetime and iii) the gravitational redshift due to nonmetricity. The overall redshift for the atom \( H_{\text{int}} \) will be different from the one for \( H_S \) despite that both are at the same distance from the observer. This means that, if we (also the observer, the star, the hydrogen atoms and the remaining matter in the universe) live in a gauge different from the GR gauge, measurements of the redshift must be corrected on the basis of gravitational and nonmetricity contributions and not only of gravitational contributions as it should be in the framework of general relativity.

If our theoretical framework is the one that correctly describes the gravitational phenomena in our universe, then we have been looking for answers to the wrong questions: i) why the cosmological constant so tiny and why its associated energy density is of the order of the present value of the dark matter energy density precisely at present?, ii) which is the nature of the dark energy?, iii) what inflates the expansion of the universe at early stages of the cosmic evolution? among others. Instead we should be wondering which is our world-gauge among the infinity of possible gauges of the gauge invariant theory of gravity.

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