Prewhitened Long-Run Variance Estimation Robust to Nonstationarity

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Abstract

We introduce a nonparametric nonlinear VAR prewhitened long-run variance (LRV) estimator for the construction of standard errors robust to autocorrelation and heteroskedasticity that can be used for hypothesis testing in a variety of contexts including the linear regression model. Existing methods either are theoretically valid only under stationarity and have poor finite-sample properties under nonstationarity (i.e., fixed-b methods), or are theoretically valid under the null hypothesis but lead to tests that are not consistent under nonstationary alternative hypothesis (i.e., both fixed-b and traditional HAC estimators). The proposed estimator accounts explicitly for nonstationarity, unlike previous prewhitened procedures which are known to be unreliable, and leads to tests with accurate null rejection rates and good monotonic power. We also establish MSE bounds for LRV estimation that are sharper than previously established and use them to determine the data-dependent bandwidths.

Keywords: Asymptotic Minimax MSE, Data-dependent bandwidths, HAC, HAR, Long-run variance, Nonstationarity, Prewhitening, Spectral density.

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1 Introduction

Heteroskedasticity and autocorrelation robust (HAR) inference requires estimation of the relevant asymptotic variance or simply the long-run variance (LRV). A large literature has considered this problem. In econometrics, Andrews (1991) and Newey and West (1987; 1994) extended the scope of kernel-based autocorrelation and heteroskedastic consistent (HAC) estimators of the LRV [see also de Jong and Davidson (2000) and Hansen (1992)]. Test statistics normalized by HAC estimators follow standard asymptotic distributions under the null hypothesis under mild conditions.

It was early noted that classical HAC estimators lead to test statistics that do not correctly control the rejection rates under the null hypothesis when there is strong serial dependence in the data. A vast literature has considered this issue. Kiefer, Vogelsang, and Bunzel (2000) and Kiefer and Vogelsang (2002; 2005) introduced the fixed-\(b\) LRV estimators for stationary sequences which are characterized by using a fixed bandwidth [e.g., the Newey-West/Bartlett estimator including all lags]. The crucial difference relative to HAC estimators is that the LRV estimator is not consistent under fixed-\(b\) asymptotics and inference is nonstandard. Test statistics under the null hypotheses asymptotically follow nonstandard distributions whose critical values are obtained numerically. This has limited the use of fixed-\(b\) in practice. The advantage of the fixed-\(b\) framework is that it yields test statistics with more accurate null rejection rates when there is strong dependence.\(^1\)

There is widespread evidence that the processes governing economic data are nonstationary. By nonstationary we mean non-constant moments. As in the literature, we consider processes whose sum of absolute autocovariances is finite. That is, we rule out processes with unbounded second moments (e.g., unit root). The latter can be handled by taking first-differences or applying some de-trending technique. Nonstationarity can occur for several reasons: changes in the moments induced by changes in the model parameters that govern the data (e.g., the Great Moderation with the decline in variance for many macroeconomic variables or the effects of the COVID-19 pandemic); smooth changes in the distributions of the processes that arise from transitory dynamics; and so on. HAR inference requires the estimation of the LRV of some relevant process, \(V_t\) say.\(^2\) We first analyze the case with \(\mathbb{E}(V_t) = 0\) for all \(t\), since it is the leading case that applies under the null hypothesis. This will allow us to derive useful properties to construct bandwidths (and so on) to have tests with the correct null rejection rates. Thus, under the null hypothesis, nonstationary occurs through time-varying autocovariances \(\mathbb{E}(V_t V_{t-k})\). We recognize that in some cases, the null hypothesis may involve a non-constant mean (e.g., when the model is misspecified).

\(^1\)See Jansson (2004) and Sun, Phillips, and Jin (2008) for theoretical results based on asymptotic expansions.

\(^2\)For example, in the linear regression model \(V_t = x_t e_t\) where \(x_t\) is a vector of regressors and \(e_t\) is a disturbance.
As in the literature, we do not address this case since the results can only be obtained on a case by case basis. Under the alternative hypothesis, \( E(V_t) \neq 0 \) and \( E(V_t) \) as well as \( E(V_t V_{t-k}) \) can be time-varying. In most HAR inference problems the leading case is with a non-zero mean. The literature has so far not properly addressed this leading case. Our aim is to devise a method for this leading case that delivers useful estimates such that the tests have good power. Hence, we shall also consider the properties of our estimate when the mean of \( V_t \) is non-zero and show that it leads to test having good monotonic power, unlike what is available in the literature.

The objective of this paper is to propose an estimator of the LRV that has the following properties: (i) it can be used for any hypothesis testing problem both within and outside the linear regression model and is valid under both stationarity and nonstationarity; (ii) it can be used without the need to develop further asymptotic analyses to determine the null limiting distribution of the test statistics; (iii) it leads to tests that have accurate null rejection rates even with strong dependence; such tests are consistent in any hypothesis testing problem, and in particular, in testing problems characterized by a nonstationary alternative hypothesis. None of the existing procedures satisfies all three properties. Fixed-\( b \) methods rely on nonstandard limit theory and require one to derive the null limiting distribution on a case-by-case basis. Casini (2021a) showed that existing fixed-\( b \) methods are not theoretically valid under nonstationarity since the null limiting distribution of the test statistics is then not pivotal. Traditional HAC estimators satisfy (i)-(ii) since they are consistent for the LRV so that a test statistic studentized by an HAC estimator follows asymptotically a standard distribution. A long-lasting problem with HAC estimators is that they lead to HAR tests that can be oversized when there is strong dependence. To address this issue, Andrews and Monahan (1992) proposed the prewhitened HAC estimators which substantially reduce the oversized problem under stationarity with HAR tests having null rejection rates similar to (and sometime better than) those of recent methods based on fixed-\( b \) [e.g., the EWP and EWC methods of Lazarus, Lewis, and Stock (2020) and Lazarus et al. (2018), respectively]. However, we show theoretically that existing prewhitened and non-prewhitened LRV estimators lead to HAR tests that are not consistent in contexts characterized by nonstationary alternative hypotheses. This has been a recurrent problem in the time series econometrics literature. It occurs, for

\[ ^3 \text{By nonstationary alternative hypothesis we mean alternative hypothesis such that } E(V_t) \text{ is time-varying.} \]

\[ ^4 \text{Lazarus, Lewis, and Stock (2020) pointed out the usefulness for empirical work of having test statistics that follow asymptotically standard distributions rather than nonstandard distributions whose critical value has to be obtained by simulations.} \]

\[ ^5 \text{Simulation evidence of serious (e.g., non-monotonic) power was documented by Altissimo and Corradi (2003), Casini (2018), Casini and Perron (2019, 2020a, 2020b), Chan (2020), Chang and Perron (2018), Crainiceanu and Vogelsang (2007), Demetrescu and Salish (2020), Deng and Perron (2006), Juhl and Xiao (2009), Kim and Perron (2009), Martins and Perron (2016), Otto and Breitung (2021), Perron and Yamamoto (2021), Shao and Zhang} \]
instance, when using tests involving structural breaks based on estimating the model under the null hypothesis; e.g., tests for forecast evaluation [e.g., Diebold and Mariano (1995), Giacomini and White (2006) and West (1996)], tests for forecast instability [cf. Casini (2018), Giacomini and Rossi (2009, 2010) and Perron and Yamamoto (2021)], CUSUM tests for structural change [see, e.g., Brown, Durbin, and Evans (1975) and Ploberger and Krämer (1992)] tests and inference in time-varying parameters models [e.g., Cai (2007) and Chen and Hong (2012)], tests and inference for regime switching models [e.g., Hamilton (1989) and Qu and Zhuo (2020)].

To improve the power properties of HAR tests based on HAC estimators, Casini (2021b) proposed to modify the HAC estimators by adding a second kernel which applies smoothing over time. Such double kernel HAC estimators (DK-HAC) satisfy properties (i)-(iii) except that they can be oversized when there is high serial correlation. We introduce a novel nonparametric non-linear VAR prewhitening step to apply prior to constructing the DK-HAC estimators. It is robust to nonstationarity unlike previous prewhitened procedures [e.g., Andrews and Monahan (1992), Preinerstorfer (2017), Rho and Shao (2013) and Xiao and Linton (2002)]. The latter are sensitive to estimation errors in the whitening step when there is nonstationarity in the autoregressive dynamics. For example, with AR(1) prewhitening the resulting LRV estimator is given by

$$\hat{J}_{HAC,pw} = \frac{\hat{J}_{HAC,V^*}}{(1-\hat{a}_1)^2}$$

where $\hat{a}_1$ is the estimated parameter in the regression $V_t = a_1 V_{t-1} + V^*_t$ involving the process of interest $\{V_t\}$ and $\hat{J}_{HAC,V^*}$ is a classical HAC estimator applied to the prewhitened residuals $\{V^*_t\}$. Under nonstationarity in $\{V_t\}$, $\hat{a}_1$ is biased toward one, [cf. Perron (1989)]. This makes the recoloring step unstable as $(1-\hat{a}_1)^2$ approaches zero and more so as the extent of the nonstationarity increases.

The consistency, rate of convergence and MSE of the new prewhitening step are established under nonstationarity using the segmented locally stationary framework. The prewhitened DK-HAC estimators lead to HAR tests with null rejection rates close to the nominal even with strong dependence. Furthermore, we show formally that prewhitened DK-HAC estimators lead to HAR tests that are consistent even under nonstationary alternative hypotheses. Finite-sample simulations demonstrate that the latter tests have good monotonic power in those situations.

We also consider the theoretical properties of DK-HAC estimators under general nonstationarity (i.e., unconditionally heteroskedastic processes). We show consistency and derive asymptotic MSE bounds that are sharper than the ones in Andrews (1991); they apply to a wider class of processes and are more informative because the bounds change with the nature of the nonstationarity. To save space, these results are presented in the supplement [cf. Casini and Perron (2021b)].

Andrews and Monahan (1992), Preinerstorfer (2017), Rho and Shao (2013) and Xiao and Linton (2002) among others.
The paper is organized as follows. Section 2 introduces the nonlinear VAR prewhitening procedure and its asymptotic results are established in Section 3. Section 4 presents some theoretical results about the power of HAR tests under nonstationary alternative hypotheses. Section 5 presents the simulation results. Section 6 concludes. The supplemental materials [cf. Casini and Perron (2021b) and an additional supplement not for publication] contain all mathematical proofs.

2 The Statistical Environment

HAR inference requires the estimation of asymptotic variances of the form $J \triangleq \lim_{T \to \infty} J_T$ where

$$J_T = T^{-1} \sum_{s=1}^{T} \sum_{t=1}^{T} \mathbb{E}(V_s(\beta_0)V_t(\beta_0)'),$$

with $V_t(\beta)$ a random $p$-vector for each $\beta \in \Theta \subset \mathbb{R}^{p\beta}$ and $\mathbb{E}(V_t(\beta_0)) = 0$ for all $t$ under the null hypothesis provided that the underlying model is correctly specified. We allow for $\mathbb{E}(V_t) \neq 0$ in Section 4 when we analyze the theoretical properties of the power of the tests. For the linear regression model $y_t = x_t' \beta_0 + e_t$, we have $V_t(\beta_0) = x_t e_t$. More generally, in nonlinear dynamic models, we have under mild conditions,

$$(B_T J_T B_T)^{-1/2} \sqrt{T}(\hat{\beta} - \beta_0) \overset{d}{\to} N(0, I_{p\beta}),$$

where $B_T$ is a nonrandom $p_{\beta} \times p$ matrix. Often it is easy to construct estimators $\hat{B}_T$ such that $\hat{B}_T - B_T \overset{p}{\to} 0$. Thus, one needs a consistent estimate of $J = \lim_{T \to \infty} J_T$ to construct a consistent estimate of $\lim_{T \to \infty} B_T J_T B_T'$. Our goal is to consider the estimation of $J$ under nonstationarity.

Let $\Gamma_u(k) \triangleq \mathbb{E}(V_{Tu} V_{Tu-k})$ where $u = t/T$ for some lag $k \in \mathbb{Z}$. From Casini (2021b), we have $\Gamma_u(k) = c(u, k) + O(T^{-1})$ where

$$c(u, k) \triangleq \int_{-\pi}^{\pi} e^{iuk} f(u, \omega) \, d\omega,$$

(2.1)

$i = \sqrt{-1}$ and $f(u, \omega)$ is the local spectral density of $\{V_t\}$ at the rescaled time $u \in [0, 1]$ and frequency $\omega$. Eq. (2.1) relates the local autocovariance of $\{V_t\}$ at rescaled time $u$ and lag $k$ to its local spectral density at $u$. Thus, the nonstationary properties of $\{V_t\}$, which are reflected in the time-varying behavior of the autocovariance function $\Gamma_u(k)$, depend on the smoothness properties of $f(u, \omega)$ in $u$. For example, if $\{V_t\}$ is stationary, then $\Gamma_u(k) = \Gamma(k)$ for all $u$, $c(u, k)$ is constant.
in $u$, $f(u, \omega) = f(\omega)$ and (2.1) reduces to $\Gamma(k) = \int_{-\pi}^{\pi} e^{i\omega k} f(\omega) \, d\omega$. These coincide with textbook definitions used under stationarity [see, e.g., Brillinger (1975)]. If $f(u, \omega)$ is continuous in $u$ then \{V_t\} is locally stationary [cf. Dahlhaus (1997)]. For example, consider a time-varying AR(1) \[ V_t = a(t/T) V_{t-1} + u_t \] where $u_t \sim$ i.i.d. $(0, 1)$ and $a(\cdot)$ is continuous with $a(t/T) \in (-1, 1)$ for all $t$. Then $V_t$ is a locally stationary AR(1) with a local spectral density $f(u, \omega)$ that is continuous in $u$.

**Assumption 2.1.** (i) \{V_t\} is zero-mean with local spectral density $f(u, \omega)$ that is piecewise Lipschitz continuous with $m_0$ discontinuity points; (ii) $f(u, \omega)$ is twice continuously differentiable in $u$ at all continuity points with uniformly bounded derivatives $(\partial / \partial u) f(u, \cdot)$ and $(\partial^2 / \partial u^2) f(u, \cdot)$, and Lipschitz continuous in the second component; (iii) $(\partial^2 / \partial u^2) f(u, \cdot)$ is Lipschitz continuous at all continuity points; (iv) $f(u, \omega)$ is twice left-differentiable at all discontinuity points with uniformly bounded derivatives $(\partial / \partial u) f(u, \cdot)$ and $(\partial^2 / \partial u^2) f(u, \cdot)$ and has piecewise Lipschitz continuous derivative $(\partial^2 / \partial u^2) f(u, \cdot)$.

Assumption 2.1 implies that \{V_t\} is segmented locally stationary (SLS). The class of SLS processes allows for relevant features such as structural change, regime switching-type and threshold model and includes general time-varying parameter processes, locally stationary processes and stationary processes. The zero-mean assumption holds under the null hypothesis. To focus on the main intuition, we consider the case of SLS processes and extend the results to general nonstationarity processes in the supplement. The latter require more technical notations and assumptions while the results of direct importance for applied work (i.e., consistency and rate of convergence) remain the same. In Section 2.1 we present the prewhitening DK-HAC estimator while in Section 2.2 we discuss its data-dependent bandwidths.

### 2.1 Prewhitening DK-HAC Estimator

Under Assumption 2.1, the results in Casini (2021b) suggest that \[ J = 2\pi \int_0^1 f(u, 0) \, du. \] The right-hand side can be seen as a function, say $\tilde{f}(\omega)$, evaluated at the zero frequency $\omega = 0$. The intuition behind prewhitening is simple, though the mechanics under nonstationarity are quite different.
Suppose one is estimating \( \tilde{f}(0) \) nonparametrically by averaging asymptotically unbiased estimates of \( \tilde{f}(\omega) \) at a number of points \( \omega \) in a neighborhood of 0. The flatter is the function \( \tilde{f}(\omega) \) around 0, the smaller the estimation bias. The idea is to transform the data such that the function of the transformed data, say \( \tilde{f}^*(\omega) \), is flatter in the neighborhood of \( \omega = 0 \). Then, using the transformed data one can estimate \( \tilde{f}^*(0) \) by averaging asymptotically unbiased estimates of \( \tilde{f}^*(\omega) \) at points \( \omega \) in the neighborhood of 0. The resulting bias should be less than that incurred by estimating \( \tilde{f}(0) \) since \( \tilde{f}^*(\omega) \) is flatter than \( \tilde{f}(\omega) \). Finally, one can apply the inverse of the transformation from \( \tilde{f}(\omega) \) to \( \tilde{f}^*(\omega) \) to obtain an estimator of \( \tilde{f}(\omega) \) from the estimator of \( \tilde{f}^*(\omega) \). This is how it works under stationarity. However, under nonstationarity one applies both the transformation and the inverse transformation locally in time, otherwise the prewhitening procedure may be unreliable.

The proposed prewhitening is based on the following three steps. Suppose \( \hat{\beta} \) is a \( \sqrt{T} \)-consistent estimator of \( \beta_0 \). Divide the sample in \( \lfloor T/n_T \rfloor \) blocks, each with \( n_T \) observations. For each block \( r = 0, \ldots, \lfloor T/n_T \rfloor \), one estimates a VAR(\( p_A \)) model for \( V_t(\hat{\beta}) \):

\[
V_t(\hat{\beta}) = \sum_{j=1}^{p_A} \hat{A}_{r,j}V_{t-j}(\hat{\beta}) + V^*_t(\hat{\beta}) \quad \text{for} \quad t = rn_T + 1, \ldots, (r+1)n_T,
\]

(for the last block, \( t = [T/n_T]n_T + 1, \ldots, T \)) where \( \hat{A}_{r,j} \) for \( j = 1, \ldots, p_A \) are \( p \times p \) least-squares estimates and \( \{V^*_t(\hat{\beta})\} \) is the corresponding residual vector. The order of the VAR, \( p_A \), can potentially change across blocks but, for notational ease, we assume it is the same for each \( r \). The choices of \( n_T \) and how to optimally split the sample depend on the property of the spectrum of \( \{V_t(\hat{\beta})\} \). A test for breaks versus smooth changes in the spectrum of \( \{V_t(\hat{\beta})\} \) is introduced in Casini and Perron (2021a). The latter can be employed here to efficiently determine the sample-splitting. This results in the sample being split in blocks with the property that within each block \( \{V_t(\hat{\beta})\} \) is locally stationary. Thus, least-squares estimation within blocks yields consistent estimates \( \hat{A}_{r,j} \). The fitted VAR need not be the true model; it is used only as a tool to “soak up” some of the serial dependence in \( \{V_t(\hat{\beta})\} \) and to leave one with residuals \( \{V^*_t(\hat{\beta})\} \) that are closer to white noise. This step is referred to as the whitening transformation.\(^9\)

The prewhitened DK-HAC estimator \( \hat{J}_{T,pw} \) is constructed by applying the DK-HAC estimator
to an inverse transformation of the VAR residuals \( \{ V_t^*(\hat{\beta}) \} \) (i.e., the recoloring step). Let

\[
\hat{J}_{pw,T} (\hat{b}_{1,T}^*, \hat{b}_{2,T}^*) = \frac{T}{T - p} \sum_{k=T+1}^{T-1} K_1 (\hat{b}_{1,T}^* k) \hat{\Gamma}_D^* (k),
\]

with \( K_1 (\cdot) \) a real-valued kernel in the class \( K_1 \) defined below, \( \hat{b}_{1,T}^* \) is a data-dependent bandwidth sequence to be discussed, \( n_T \to \infty \), and

\[
\hat{c}_{D,T}^* (rn_T/T, k) = \begin{cases} 
\left( T\hat{b}_{2,T}^* \right)^{-1} \sum_{s=k+1}^{T} K_2^* \left( \frac{((r+1)n_T-(s-k)/2)}{\hat{b}_{2,T}^*} \right) \hat{V}_{D,s}^* \hat{V}_{D,s-k}^*, & k \geq 0 \\
\left( T\hat{b}_{2,T}^* \right)^{-1} \sum_{s=-k+1}^{T} K_2^* \left( \frac{((r+1)n_T-(s+k)/2)}{\hat{b}_{2,T}^*} \right) \hat{V}_{D,s+k}^* \hat{V}_{D,s}^*, & k < 0 
\end{cases}
\]

\( \hat{V}_{D,s}^* = \hat{D}_s \hat{V}_s^* \), \( \hat{V}_s^* = V_s^*(\hat{\beta}) \), \( K_2^* \) being a kernel, \( \hat{b}_{2,T}^* \) a data-dependent bandwidth sequence to be defined below, and \( \hat{D}_s = \left( I_p - \sum_{j=1}^p \hat{A}_{D,s,j} \right)^{-1} \) with \( \hat{A}_{D,s,j} = \hat{A}_{r,j} \) for \( s = rn_T + 1, \ldots, (r+1)n_T \).

In order to guarantee positive semi-definiteness, one needs to use a data taper or, e.g., for \( k \geq 0 \),

\[
K_2^* \left( \frac{(r+1)n_T-(s-k)/2}{T\hat{b}_{2,T}^*} \right) = \left( K_2 \left( \frac{(r+1)n_T-s}{T\hat{b}_{2,T}^*} \right) K_2 \left( \frac{(r+1)n_T-(s-k)}{T\hat{b}_{2,T}^*} \right) \right)^{1/2},
\]

see Casini (2021b) for details. Below we assume that \( \hat{A}_{r,j} \xrightarrow{p} A_{r,j} \) for some \( A_{r,j} \in \mathbb{R}^{p \times p} \) for all \( r \) and \( j \) which follow from standard arguments. For \( K_1 \) we suggest using the Quadratic Spectral (QS) kernel

\[
K_1^{QS} (x) = \left( 25/ \left( 12\pi^2 x^2 \right) \right) \left[ \sin \left( \frac{6\pi x / 5}{6\pi x / 5} \right) - \cos \left( \frac{6\pi x / 5}{6\pi x / 5} \right) \right],
\]

and for \( K_2 \) a quadratic-type kernel [cf. Epanechnikov (1969)] given by \( K_2 (x) = 6x (1-x), 0 \leq x \leq 1 \). These kernels are optimal under an MSE criterion [see Casini (2021b)].

### 2.2 Data-Dependent Bandwidths

For data-dependent bandwidths, we use plug-in estimates of the optimal value that minimizes some MSE criterion, see Section S.B and Casini (2021b). Let \( \Gamma_{D,u} (k) = \text{Cov}(V_{D,Tu}^*, V_{D,Tu-k}^*) \) and \( C_{pp} = \sum_{j=1}^p \sum_{i=1}^p t_i t_j' \otimes t_i t_j' \), where \( t_i \) is the \( i \)-th elementary \( p \)-vector. The notation \( W \) and \( \hat{W} \) are
used for some \( p^2 \times p^2 \) weight matrices. Let \( F(K_2) \triangleq \int_0^1 K_2^2 (x) \, dx \), \( H(K_2) \triangleq (\int_0^1 x^2 K_2 (x) \, dx)^2 \),
\[
D_1 (u) \triangleq \text{vec} \left ( \partial^2 c_D^* (u, k) / \partial u^2 \right )' \tilde{W} \text{vec} \left ( \partial^2 c_D^* (u, k) / \partial u^2 \right ),
\[
D_2 (u) \triangleq \text{tr} [\tilde{W} (I_p^2 + C_{pp}) \sum_{l=-\infty}^{\infty} c_D^* (u, l) \otimes [2c_D^* (u, l)]],
\]
where \( c_D^* (u, l) = \text{Cov}(V_{D,Tu}^*, V_{D,Tu-l}^*), V_{D,t}^* = D_t V_t^* \),
\[
V_t^* = V_t - \sum_{j=1}^{p_A} A_{r,j} V_{t-j} \quad \text{for } t = rn_T + 1, \ldots, (r + 1) n_T,
\]
\[
D_t = (I_p - \sum_{j=1}^{p_A} A_{D,t,j})^{-1}, A_{D,t,j} = A_{r,j} \quad \text{for } t = rn_T + 1, \ldots, (r + 1) n_T.
\]
The optimal \( b_{2,T} \) is given by [see Casini (2021b)]
\[
b_{2,T}^{\text{opt},*} (u) = [H (K_2) D_1 (u)]^{-1/5} (F (K_2) (D_2 (u)))^{1/5} T^{-1/5}.
\]
Let \( K_{1,q} \triangleq \lim_{t \to 0} (1 - K_1 (x)) / |x|^q \) for \( q \in [0, \infty) \); \( K_{1,q} < \infty \) if and only if \( K_1 (x) \) is \( q \) times differentiable at zero. Let \( f_D^* (u, \omega) = \sum_{k=-\infty}^{\infty} c_D^* (u, k) e^{-i\omega k} \) and define the index of smoothness of \( f_D^* (u, \omega) \) at \( \omega = 0 \) by \( f_D^{*(q)} (u, 0) \triangleq (2\pi)^{-1} \sum_{k=-\infty}^{\infty} |k|^q c_D^* (u, k) \). Let
\[
\phi_D (q) = \frac{\text{vec} \left ( \int_0^1 f_D^{*(q)} (u, 0) \, du \right )' \tilde{W} \text{vec} \left ( \int_0^1 f_D^{*(q)} (u, 0) \, du \right )}{\text{tr} W (I_p^2 + C_{pp}) \left ( \int_0^1 f_D^{*(q)} (u, 0) \, du \right ) \otimes \left ( \int_0^1 f_D^{*(q)} (v, 0) \, dv \right )}.
\]
The optimal \( b_{1,T} \) given the optimal value \( b_{2,T}^{\text{opt},*} \) is given by [see Casini (2021b)],
\[
b_{1,T}^{\text{opt},*} = (2qK_{1,q}^{2} \phi_D (q) T \tilde{b}_{2,T}^{\text{opt}} / (\int K_1^2 (y) \, dy \int K_2^2 (x) \, dx))^{-1/(2q+1)},
\]
with \( \tilde{b}_{2,T}^{\text{opt}} = \int_0^1 b_{2,T}^{\text{opt},*} (u) \, du \). For the QS kernel, \( q = 2 \), \( K_{1,2} = 1.421223 \), and \( \int K_1^2 (x) \, dx = 1 \). For the optimal \( K_2 \) we have \( H(K_2^{\text{opt}}) = 0.09 \) and \( F(K_2^{\text{opt}}) = 1.2 \).

The bandwidths \( (b_{1,T}^{\text{opt},*}, \tilde{b}_{2,T}^{\text{opt},*}) \) are optimal under a sequential MSE criterion that determines the optimal \( b_1 \) as a function of the optimal \( b_2 (u) \). Thus, the latter influences the former but not vice-versa. However, this has the advantage that the optimal \( b_2 (u) \) is allowed to change over time. Belotti et al. (2021) proposed an alternative criterion that determines the optimal \( b_1 \) and \( b_2 \) that jointly minimize the global MSE. The latter yields an optimal \( b_2 \) that does not depend on \( u \) and so it does not perform as well as the sequential method when the data is far from stationary.
In order to construct a data-dependent bandwidth for $b_{2,T}(u)$, we need consistent estimates of $D_{1,D}(u)$ and $D_{2,D}(u)$. These are discussed in Casini (2021b). We set $\tilde{W}^{(r,r)} = p^{-1}$ for all $r$ which corresponds to the normalization used below for $W$. The estimate of $D_{1,D}(u)$ is then,

$$\hat{D}_{1,D}(u) \triangleq [S_\omega]^{-1} \sum_{s \in S_\omega} \left[ (3/\pi)(1 + 0.8 (\cos 1.5 + \cos 4\pi u) \exp (-i\omega_s))^{-4} (0.8 (-4\pi \sin (4\pi u))) \exp (-i\omega_s) \right. $$

$$- \pi^{-1} |1 + 0.8 (\cos 1.5 + \cos 4\pi u) \exp (-i\omega_s)|^{-3} \left( 0.8 (-16\pi^2 \cos (4\pi u)) \right) \exp (-i\omega_s) \right],$$

where $[S_\omega]$ is the cardinality of $S_\omega$ and $\omega_{s+1} > \omega_s$ with $\omega_1 = -\pi$, $\omega_{|S_\omega|} = \pi$. The estimate of $D_{2,D}(u)$ is given by

$$\hat{D}_{2,D}(u_0) \triangleq 2p^{-1} \sum_{r=1}^{p} \sum_{l=-[T^{4/25}]}^{[T^{4/25}]} (c_{D,T}^{(r,r)}(u_0, l))^2,$$

where the number of summands grows at the same rate as the inverse of the optimal bandwidth $b_{1,T}^{\text{opt,*}}$. Hence, the estimate of the optimal bandwidth $b_{2,T}^{\text{opt,*}}$ is given by

$$\hat{b}_{2,T}^* = (n_T/T) \sum_{r=1}^{[T^{1/5}]} \hat{b}_{2,T}^*(u_r),$$

where $$\hat{b}_{2,T}^*(u_r) = 1.7781 (\hat{D}_{1,D}(u_r))^{-1/5} (\hat{D}_{2,D}(u_r))^{1/5} T^{-1/5}, \quad u_r = rn_T/T. \quad (2.5)$$

The data-dependent bandwidth parameter $\hat{b}_{1,T}$ is then defined as follows. First, one specifies $p$ univariate approximating parametric models given by $\{V_{D,t}^{(r)}\}$ for $r = 1, \ldots, p$. Second, one estimates the parameters of the approximating parametric model by least-squares. Third, one substitutes these estimates into $\hat{b}_{1,T}$ with the estimate denoted by $\hat{\phi}_D(q)$. This yields the data-dependent bandwidth parameter

$$\hat{b}_{1,T}^* = \left[ 2qK_1^2 \hat{\phi}_D(q) T \hat{b}_{2,T}^* / \int K_1^2(x) dx \right]^{1/(2q+1)}.$$  \quad (2.6)$$

For the QS kernel, we have $\hat{b}_{1,T}^* = 0.6828(\hat{\phi}_D(2) T \hat{b}_{2,T}^*)^{-1/5}$. The suggested approximating parametric models are locally stationary first order autoregressive (AR(1)) models given by $V_{D,t}^{(r)} = a_1^{(r)}(t/T) V_{D,t-1}^{(r)} + u_t^{(r)}, r = 1, \ldots, p$. Let $\hat{a}_1^{(r)}(u)$ and $(\hat{\sigma}^{(r)}(u))^2$ be the least-squares estimates of
the autoregressive and innovation variance parameters computed using data close to \( u = t/T \):

\[
\hat{a}_1^{(r)}(u) = \frac{\sum_{j=t-n_2,T+1}^t \hat{V}_{D,j}^{(r)} \hat{V}_{D,j-1}^{(r)}}{\sum_{j=t-n_2,T+1}^t \left( \hat{V}_{D,j}^{(r)} \right)^2}, \quad \hat{\sigma}^{(r)}(u) = \left( \sum_{j=t-n_2,T+1}^t \left( \hat{V}_{D,j}^{(r)} - \hat{a}_1^{(r)}(u) \hat{V}_{D,j-1}^{(r)} \right)^2 \right)^{1/2},
\]

where \( n_{2,T} \to \infty \). These are simply least-squares estimates based on rolling windows. Then, for \( q = 2 \), we have

\[
\hat{\phi}_D(2) = \sum_{r=1}^p W^{(r,r)} \left( 18 \left( \frac{n_{3,T}}{T} \right)^{T/n_{3,T}} \sum_{j=0}^{T/n_{3,T}-1} \left( \frac{\hat{\sigma}^{(r)} \left( (jn_{3,T} + 1)/T \right)}{1 - \hat{a}_1^{(r)} \left( (jn_{3,T} + 1)/T \right)^2} \right)^2 \right) / \sum_{r=1}^p W^{(r,r)} \left( \frac{n_{3,T}}{T} \right)^{T/n_{3,T}} \sum_{j=0}^{T/n_{3,T}-1} \left( \frac{\hat{\sigma}^{(r)} \left( (jn_{3,T} + 1)/T \right)}{1 - \hat{a}_1^{(r)} \left( (jn_{3,T} + 1)/T \right)^2} \right)^2,
\]

where \( W^{(r,r)} \), \( r = 1, \ldots, p \) are pre-specified weights and \( n_{3,T} \to \infty \). The usual choice for \( W^{(r,r)} \) is one for all \( r \) except that which corresponds to an intercept in which case it is zero.

### 3 Large-Sample Results When \( \mathbb{E}(V_t) = 0 \)

In this section, we analyze the asymptotic properties of \( \hat{J}_{pw,T} \) for the case with \( \mathbb{E}(V_t) = 0 \) for all \( t \), which is relevant under the null hypothesis provided that the model is correctly specified. Let

\[
K_3 = \left\{ K_3(\cdot) \in K_1 : (i) \ |K_1(x)| \leq C_1 |x|^{-b} \right\}
\]

with \( b > \max(1 + 1/q, 4) \) for \( |x| \in [\pi_L, D_T h_T \pi_U] \), \( T^{-1/2} h_T \to \infty \), \( D_T > 0 \), \( \pi_L, \pi_U \in \mathbb{R} \), \( 1 \leq \pi_L < \pi_U \), and with \( b > 1 + 1/q \) for \( |x| \notin [\pi_L, D_T h_T \pi_U] \), and some \( C_1 < \infty \), where \( q \in (0, \infty) \) is such that \( K_{1,q} \in (0, \infty) \), (ii) \( |K_1(x) - K_1(y)| \leq C_2 |x - y| \ \forall x, y \in \mathbb{R} \) for some constant \( C_2 < \infty \), and (iii) \( q < 34/4 \).

\( K_3 \) contains commonly used kernels, e.g., QS, Bartlett, Parzen, and Tukey-Hanning, with the exception of the truncated kernel. For the QS, Parzen, and Tukey-Hanning kernels, \( q = 2 \). For the

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\(^{10}\)See, for example, Dahlhaus and Giraitis (1998) for a discussion about nonparametric local parameter estimates in the context of locally stationary time series.
Bartlett kernel, $q = 1$. We define

$$\text{MSE}(Tb_{1,T}b_{2,T}, \tilde{J}_T, J_T, W) = Tb_{1,T}b_{2,T}\mathbb{E}[\text{vec}(\tilde{J}_T - J_T)'W\text{vec}(\tilde{J}_T - J_T)].$$

We need to impose conditions on the temporal dependence of $\{V_t\}$. Let

$$\kappa_{V,t}^{(a_1,a_2,a_3,a_4)}(u, v, w) \triangleq \kappa^{(a_1,a_2,a_3,a_4)}(t, t + u, t + v, t + w) - \kappa_{V,t}^{(a_1,a_2,a_3,a_4)}(t, t + u, t + v, t + w)$$

$$\triangleq \mathbb{E}(V_t^{(a_1)}V_{t+u}^{(a_2)}V_{t+v}^{(a_3)}V_{t+w}^{(a_4)}) - \mathbb{E}V_{t+u}^{(a_2)}V_{t+v}^{(a_3)}V_{t+w}^{(a_4)},$$

where $\{V_{t+u}\}$ is a Gaussian sequence with the same mean and covariance structure as $\{V_t\}$. $\kappa_{V,t}^{(a_1,a_2,a_3,a_4)}(u, v, w)$ is the time-$t$ fourth-order cumulant of $(V_t^{(a_1)}V_{t+u}^{(a_2)}V_{t+v}^{(a_3)}V_{t+w}^{(a_4)})$ while $\kappa_{V,t}^{(a_1,a_2,a_3,a_4)}(t, t + u, t + v, t + w)$ is the time-$t$ centered fourth moment of $V_t$ if $V_t$ were Gaussian. Let $\lambda_{\text{max}}(A)$ denote the largest eigenvalue of the matrix $A$.

**Assumption 3.1.** (i) $\sum_{k=-\infty}^{\infty} \sup_{a \in [0,1]} \|c(u,k)\| < \infty$ and $\sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sup_{a} |\kappa_{V,t}^{(a_1,a_2,a_3,a_4)}(k, j, l)| < \infty$ for all $a_1, a_2, a_3, a_4 \leq p$. (ii) For all $a_1, a_2, a_3, a_4 \leq p$ there exists a function $\tilde{\kappa}_{a_1,a_2,a_3,a_4} : [0, 1] \times \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ that is piecewise continuous in the first argument such that

$$\sup_{u \in [0,1]} |\kappa_{V,t}^{(a_1,a_2,a_3,a_4)}(k, s, l)| \leq KT^{-1}$$

for some $K < \infty$; $\tilde{\kappa}_{a_1,a_2,a_3,a_4}(u,k,s,l)$ is twice differentiable in $u$ at all continuity points with uniformly bounded derivatives $(\partial/\partial u)$ $\tilde{\kappa}_{a_1,a_2,a_3,a_4}(u,k,s,l)$ and $(\partial^2/\partial u^2)\tilde{\kappa}_{a_1,a_2,a_3,a_4}(u,k,s,l)$, and twice left-differentiable in $u$ at all discontiuity points with uniformly bounded derivatives $(\partial/\partial_- u)\tilde{\kappa}_{a_1,a_2,a_3,a_4}(u,k,s,l)$ and $(\partial^2/\partial_- u^2)\tilde{\kappa}_{a_1,a_2,a_3,a_4}(u,k,s,l)$, and piecewise Lipschitz continuous derivative $(\partial^2/\partial_- u^2)\tilde{\kappa}_{a_1,a_2,a_3,a_4}(u,k,s,l)$.

If $\{V_{t+u}\}$ is stationary then the cumulant condition of Assumption 3.1-(i) reduces to the standard one used in the time series literature [see, e.g., Assumption A in Andrews (1991)]. We do not require fourth-order stationarity but only that the time-$t = Tu$ fourth order cumulant is locally constant in a neighborhood of a continuity point $u$. As explained in Casini (2021b), using an argument similar to that used in Lemma 1 in Andrews (1991), one can show that $\alpha$-mixing and moment conditions imply that the cumulant condition of Assumption 3.1-(i) holds.

**Assumption 3.2.** (i) $\sqrt{T}(\hat{\beta} - \beta_0) = O_p(1)$; (ii) $\sup_{u \in [0,1]} \mathbb{E}[|V_{Tu}|]^2 < \infty$; (iii) $\sup_{u \in [0,1]} \mathbb{E} \sup_{\beta \in \Theta} \|\partial/\partial^\beta V_{Tu}(\beta)\|^2 < \infty$; (iv) $\int_{-\infty}^{\infty} |K_1(y)| dy$, $\int_0^1 |K_2(x)| dx < \infty$.

Assumption 3.2-(i,iii) is the same as Assumption B in Andrews (1991). Part (iv) of the assumption is satisfied by most commonly used kernels. In order to obtain rate of convergence results we shall need to replace Assumption 3.1 with the following assumption.
Assumption 3.3. (i) Assumption 3.1-(i) holds with $V_t$ replaced by
\[
\left( V'_{[Tu]}, \text{vec}\left( \frac{\partial}{\partial \beta'} V_{[Tu]}(\beta_0) \right) - \mathbb{E}\left( \frac{\partial}{\partial \beta'} V_{[Tu]}(\beta_0) \right) \right)'.
\]
(ii) $\sup_{u \in [0,1]} \mathbb{E} (\sup_{\beta \in \Theta} \| (\partial^2 / \partial \beta \partial \beta') V_{[Tu]}(\beta) \|)^2 < \infty$ for all $r = 1, \ldots, p$.

Assumption 3.4. Let $W_T$ denote a $p^2 \times p^2$ weight matrix such that $W_T \xrightarrow{P} W$.

Assumption 3.5. (i) $\hat{\phi}_D(q) = O_p(1)$ and $1 / \hat{\phi}_D(q) = O_p(1)$; (ii) $\inf \{ T/n_{3,T}, \sqrt{n_{2,T}} \} (\hat{\phi}_D(q) - \phi_{\theta^*}) = O_p(1)$ for some $\phi_{\theta^*} \in (0, \infty)$ where $n_{2,T}/T + n_{3,T}/T \to 0$, $n_{2,T}/T \to [c_2, \infty)$, $n_{3,T}/T \to [c_3, \infty)$ with $0 < c_2, c_3 < \infty$; (iii) $\sup_{u \in [0,1]} \lambda_{\max}(\Gamma_{D,u}(k)) \leq C_3 k^{-l}$ for all $k \geq 0$ for some $C_3 < \infty$ and some $l > \max \{ 2, (4q + 2) / (2 + q), (11 + 6q) / (11 + 4q), (23 + 34q) / (23 + 10q) \}$, where $q$ is as in $K_3$; (iv) uniformly in $u \in [0,1]$, $\hat{D}_1(u)$, $\hat{D}_2(u)$, $1/\hat{D}_1(u)$ and $1/\hat{D}_2(u)$ are $O_p(1)$; (v) $\omega_{s+1} - \omega_s \to 0$, $[S_{u}]^{-1} \to \infty$ at rate $O(T^{-1})$ and $O(T)$, respectively; (vi) $\sqrt{T b_{2,T}(u)} \hat{D}_2(u) - D_2(u) = O_p(1)$ for all $u \in [0,1]$; (vii) $K_2$ includes kernels that satisfy $|K_2(x) - K_2(y)| \leq C_4 |x - y|$ for all $x, y \in \mathbb{R}$ and some constant $C_4 < \infty$.

Parts (i)-(ii) of Assumption 3.5 are the nonparametric analogue to Assumption E and F, respectively, in Andrews (1991). Part (iii) is satisfied if $\{V_t\}$ is strong mixing with mixing numbers that are less stringent than those sufficient for the cumulant condition in Assumption 3.1-(i). Part (iv) and (vi) extend (i)-(ii) to $\hat{D}_1$ and $\hat{D}_2$. Part (v) is needed to apply the convergence of Riemann sums. Part (vii) requires $K_2$ to satisfy Lipschitz continuity.

Assumption 3.6. $\sqrt{n_T} (\hat{A}_{r,j} - A_{r,j}) = O_p(1)$ for some $A_{r,j} \in \mathbb{R}^{p \times p}$ for all $j = 1, \ldots, p_A$ and all $r = 0, \ldots, [T/n_T]$.

Given the restrictions below on $n_T$, Assumption 3.6 is satisfied by standard nonparametric estimators. For the consistency of $\hat{J}_{T,\text{pw}}$, Assumption 2.1, 3.1-3.2, 3.5-(i,iv,vii) and 3.6 are sufficient. For the rate of convergence and asymptotic MSE results additional conditions are needed. Let
\[
b_{\theta_0,T} = \left( 2q K_{1,q}^2 \phi_{\theta^*} T b_{\theta_0,T} / \int K_1^2(y) \, dy \int_0^1 K_2^2(x) \, dx \right)^{-1/(2q+1)},
\]
where $b_{\theta_0,T} \triangleq \int_0^1 [H(K_2) D_1(u)]^{-1/5} (F(K_2) D_2(u))^{1/5} T^{-1/5} du$. Recall that the bandwidths $\tilde{b}_{2,T}$, $\tilde{b}_{2,T}$ and $\tilde{b}_{1,T}$ are defined by (2.4), (2.5) and (2.6), respectively.

Theorem 3.1. Suppose $K_1(\cdot) \in K_3$, $q$ is as in $K_3$, $\| \int_0^1 f_D^{(a)}(u, 0) \| < \infty$. Then, we have:
(i) If Assumption 2.1, 3.1-3.2, 3.5-(i,iv,vii) and 3.6 hold, \( \sqrt{n_T} \hat{b}_{1,T} \to \infty \), and \( q > 1/2 \), then
\[
\bar{J}_{pw}(\hat{b}_{1,T}, \hat{b}_{2,T}) - J_T \overset{p}{\to} 0.
\]

(ii) If Assumption 2.1, 3.1-(ii), 3.2-3.3, 3.5-(ii,iii,v,vi,vi) and 3.6 hold, and \( n_T/T \hat{b}_{1,T} \to 0 \), \( n_T/T(\hat{b}_{1,T})^q \to 0 \), \( T \hat{b}_{2,T}/(n_T^2 \hat{b}_{1,T}) \to 0 \), \( T \hat{b}_{2,T} \hat{b}_{1,T}/n_T \to 0 \), then
\[
\sqrt{T} \hat{b}_{1,T} \hat{b}_{2,T} (\hat{J}_{pw,T}(\hat{b}_{1,T}, \hat{b}_{2,T}) - J_T) = O_p(1).
\]

(iii) Let \( \gamma_{K,q} = 2qK_1^2 q\phi_{\theta_\ast}/(\int K_1^2(y)dy \int_0^1 K_2^2(x)dx) \). If Assumption 3.2-3.4 and 3.5-(ii,iii,v,vi, vii) hold, then
\[
\lim_{T \to \infty} \text{MSE}(T^{4q/10(2q+1)}, \hat{J}_{pw,T}(\hat{b}_{1,T}, \hat{b}_{2,T}), J_T, W_T)
= 4\pi^2 \left[ \gamma_{K,q} K_1^2 \mathbb{E} \left( \int_0^1 f_D^*(u, 0) du \right)^2 \right] W \mathbb{E} \left( \int_0^1 f_D(u, 0) du \right) + \int K_1^2(y)dy \int K_2^2(x)dx \text{tr}(W)(I_p - C_{pp}) \left( \int_0^1 f_D^*(u, 0) du \right) \otimes \left( \int_0^1 f_D(u, 0) du \right).
\]

A result corresponding to Theorem 3.1 for non-prewhitened DK-HAC estimators is established in Theorem 5.1 in Casini (2021b) under the same assumptions with the exception of Assumption 3.6. Note that for \( u \) a continuity point, \( f_D(u, \omega) = D(u, \omega) f^*(u, \omega) D(u, \omega)' \), where \( D(u, \omega) = (I_p - \sum_{j=1}^p A_{D,j}(u) e^{-ij\omega})^{-1} \) with \( A_{D,j}(u) = A_{D,Tu,j} + O(T^{-1}) \) and \( f^*(u, \omega) \) is the local spectral density of \( \{V_t\} \). Since \( D(u-k/T, \omega) = D(u, \omega) + O(T^{-1}) \) by local stationarity, we have
\[
f_D^*(u, 0) = (-)^{q/2} \frac{d^q}{d\omega^q} \left[ D(u, \omega)^{-1} f(u, \omega) \left( D(u, \omega)' \right)^{-1} \right]_{\omega=0} + O(T^{-1}), \quad q \text{ even.}
\]

A meaningful comparison between prewhitened and non-prewhitened DK-HAC estimators \( \hat{J}_T \) can be made only if reasonable choices of the bandwidths \( b_{1,T} \) and \( b_{2,T} \) are made. When the optimal bandwidths for \( \hat{J}_{pw,T} \) and \( \hat{J}_T \) are used we find that \( \hat{J}_{pw} \) has smaller asymptotic MSE than \( \hat{J}_T \) if and only if (assuming \( p = 1 \), i.e., the scalar case, with \( w_{1,1} = 1 \))
\[
\int_0^1 f_D^*(u, 0) du \left( \int_0^1 f_D(u, 0) du \right)^{2q} < \int_0^1 f(u, 0) du \left( \int_0^1 f(u, 0) du \right)^{2q}.
\]

A numerical comparison would be tedious since the condition depends on the true data-generating process of \( \{V_t\} \) and the VAR approximation for \( \{\tilde{V}_t\} \). Under stationarity, Grenander and Rosenblatt (1957) and Andrews and Monahan (1992) considered a few examples. We can make a few observations on the difference between the condition (3.1) and an analogous condition for the case
with \( \{V_t\} \) second-order stationary and \( D_s = D = (1 - \sum_{j=1}^{p_s} A_j)^{-1} \) for all \( s \) [cf. Andrews and Monahan (1992)]. The condition in Andrews and Monahan (1992) is then

\[ |f^{*(q)}(0)| D^2 < |f^q(0)|, \]

where the quantities \( f^q(0) \) and \( f^{*(q)}(0) \) do not depend on \( u \) by stationarity. The main difference between the two conditions (3.1)-(3.2) is that the part involving the asymptotic variance is missing in (3.2). The quantities \( |f^{*(q)}(0)| D^2 \) and \( |f^q(0)| \) are from the asymptotic squared bias. This is a consequence of the fact that prewhitened and non-prewhitened HAC estimators have the same asymptotic variance under stationarity when the optimal bandwidths are used. This property does not hold when \( \{V_t\} \) is nonstationary. The condition (3.1) suggests instead that, in general, both the asymptotic squared bias and asymptotic variance of prewhitened and non-prewhitened HAC estimators can be different. Simulations in Andrews and Monahan (1992) showed that this is indeed the case even under stationarity: the variance of the prewhitened HAC estimators is larger than that of the non-prewhitened HAC estimators—this feature is consistent with our theoretical results but not with theirs.

4 Theoretical Results About the Power of HAR Tests Under General \( \mathbb{E}(V_t) \)

A long-lasting problem in time series econometrics is the non-monotonic power of HAR inference tests under nonstationary alternative hypotheses. We provide theoretical results that are general enough to encompass most cases discussed in the literature. The problem involves HAR tests outside the linear regression model that can be characterized by an alternative hypothesis involving \( \mathbb{E}(V_t) = \mu_t \) with \( \mu_t \neq 0 \) for at least one \( t \). The process \( \mu_t \) can be any piecewise continuous function of \( t \). For example, tests for structural breaks, tests for regime switching and tests for time-varying parameters can be framed in this way. To see this, consider the following linear regression model,

\[ y_t = x_t' \beta_t + e_t, \quad t = 1, \ldots, T. \] (4.1)

The null hypothesis of no break in the regression coefficient of \( x_t \) is written as \( H_{\beta,0} : \beta_t = \beta_0 \) for all \( t \) for some \( \beta_0 \in \mathbb{R}^p \) [see, e.g., Andrews (1993)]. The alternative hypothesis may be of several forms. Let \( H_{\beta,1} : \beta_t = \beta(t/T) \) \( (t = 1, \ldots, T) \) for some piecewise continuous function \( \beta(\cdot) \). Estimating (4.1) by least-squares yields \( y_t = x_t' \widehat{\beta} + \widehat{e_t} \) for all \( t \) where \( \widehat{\beta} \) is the least-squares estimate and \( \{\widehat{e_t}\} \)
are the least-squares residuals. Letting $V_t = x_t \hat{\epsilon}_t$, the null hypothesis $H_{\beta,0}$ can be rewritten as $H_0 : \mathbb{E}(V_t) = 0$ for all $t$ while the alternative hypothesis $H_{\beta,1}$ can be rewritten as $H_1 : \mathbb{E}(V_t) = \mu_t$ where $\mu_t \neq 0$ for at least one $t$. Structural break tests are based on an estimate of the LRV of $V_t = x_t \hat{\epsilon}_t$. While under $H_0$ $V_t$ is zero-mean, under $H_1$ the mean of $V_t$ is time-varying. Under the alternative hypothesis, it is required for consistency of the test that the LRV estimator converges to some positive semidefinite matrix since the numerator of the test statistics diverges to infinity. However, time-variation in the mean of $V_t$ severely biases upward traditional LRV estimators which then lead to tests with non-monotonic power. Casini, Deng, and Perron (2021) established analytical results for this phenomenon, which they referred to as low frequency contamination. We show that the proposed nonlinear prewhitened DK-HAC estimator accounts for nonstationarity also under the alternative hypothesis and leads to consistent tests with good monotonic power.

We present theoretical results about the power of a popular forecast evaluation test, namely the Diebold-Mariano test [cf. Diebold and Mariano (1995)], which can be also framed as above. We focus on the Diebold-Mariano test for ease of the exposition. Similar results hold for the other HAR inference tests that can be framed as above, though the proofs change slightly depending on the specific test statistic. Suppose the goal is to forecast some variable $y_t$. Two forecast models are used: $y_t = \beta^{(1)}(1) + \beta^{(2)}(2)x_{t-1}^{(i)} + e_t$ where $x_{t-1}^{(i)}$ is some predictor and $i = 1, 2$. That is, each forecast model uses an intercept and a predictor. The parameters $\beta^{(1)}$ and $\beta^{(2)}$ are estimated using least-squares in the in-sample $t = 1, \ldots, T$ with a fixed forecasting scheme. Each forecast model generates a sequence of $\tau (=1)$-step ahead out-of-sample losses $L_t^{(i)}$ for $t = T + 1, \ldots, T - \tau$. Then $d_t \triangleq L_t^{(2)} - L_t^{(1)}$ denotes the loss differential at time $t$. Let $\overline{d}_L$ denote the average of the loss differentials. The Diebold-Mariano test statistic is defined as $t_{DM} \triangleq T^{1/2}\overline{d}_L / \hat{\mathbb{J}}_{dL,T}^{1/2}$, where $\hat{\mathbb{J}}_{dL,T}$ is an estimate of the LRV of the loss differentials and $T_n$ is the number of observations in the out-of-sample. Throughout, we use the quadratic loss. The true model is $y_t = \beta^{(1)}_0 + \beta^{(2)}_0 x_{t-1}^{(0)} + e_t$ where $x_{t-1}^{(0)}$ is a predictor and $e_t$ is a zero-mean error. We assume that the conditions for consistency and asymptotic normality of the least-squares estimates of $\beta^{(1)}_0$ and $\beta^{(2)}_0$ are satisfied.

Let $\hat{V}_t = d_t$. The hypothesis testing problem is given by

$$H_0 : \mathbb{E} \left( \hat{V}_t \right) = 0, \quad \text{for all } t,$$

$$H_1 : \mathbb{E} \left( \hat{V}_t \right) = \mu_t, \quad \text{with } \mu_t \neq 0 \text{ for at least one } t. \quad (4.2)$$

$H_0$ corresponds to equal predictive ability between the two forecast models while $H_1$ corresponds to the two forecast models performing differently.

Since we want to study the power of $t_{DM}$, we need to work under the alternative hypothesis.
The two competing forecast models are as follows: the first model uses the actual true predictor (i.e., \( x_{t-1}^{(1)} = x_{t-1}^{(0)} \) for all \( t \)) while the second model differs in that in place of \( x_{t-1}^{(0)} \) it uses \( x_{t-1}^{(2)} = x_{t-1}^{(0)} + u_{X,t} \) for \( t \leq T_b \) and \( x_{t-1}^{(2)} = \delta + x_{t-1}^{(0)} + u_{X,t} \) for \( t > T_b \) with \( T_b > T_m \), and \( u_{X,t} \) is a zero-mean error term. Evidently, the null hypotheses of equal predictive ability should be rejected whenever \( \delta > 0 \). We consider \( t_{DM} \) normalized by different LRV estimators. The HAC estimator is defined as

\[
\hat{J}_{d_L,HAC,T} \triangleq \sum_{k=-T+1}^{T-1} K_1(b_T k) \hat{\Gamma}(k), \quad \hat{\Gamma}(k) = T^{-1} \sum_{t=|k|+1}^{T} \hat{V}_t \hat{V}_{t-k},
\]

where \( K_1(\cdot) \) is a kernel (e.g., the Bartlett and QS) and \( b_T \) a bandwidth parameter. Kiefer, Vogelsang, and Bunzel (2000) proposed to use a LRV estimator that keeps the bandwidth at a fixed fraction of the sample size, i.e., \( \hat{J}_{KVB,T} \triangleq T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} (1 - |t-s|/T) \hat{V}_t \hat{V}_s \) which is equivalent to the Newey-West estimator with \( b_T = T^{-1} \).

We present theoretical results about the power of \( t_{DM} \). Let \( t_{DM,i} = T_n^{2/2} \sqrt{P_i} / \sqrt{\hat{J}_{d_L,i,T}} \) denote the DM test statistic where \( i = \text{DK}, \text{pwDK}, \text{KVB}, \text{EWC}, \text{A91}, \text{pwA91}, \text{NW87} \) and \text{pwNW87}. \( \hat{J}_{d_L,A91,T} \) and \( \hat{J}_{d_L,NW87,T} \) are \( \hat{J}_{d_L,HAC,T} \) where \( K_1(\cdot) \) is the Bartlett and QS kernel, respectively.\(^{11}\) \( \hat{J}_{d_L,pwA91,T} \) and \( \hat{J}_{d_L,pwNW87,T} \) are the prewhitened HAC estimators using the QS and Bartlett kernel, respectively, and the prewhitening procedure of Andrews and Monahan (1992). “DK” refers to the DK-HAC estimator from Casini (2021b) with the MSE-optimal kernels and bandwidths whereas “pwDK” refers to the prewhitened DK-HAC estimator \( \hat{J}_{pw,T} \) in (2.3). Define the power of \( t_{DM,i} \) as \( \mathbb{P}_\delta(t_{DM,i} > z_\alpha) \) where \( z_\alpha \) is the two-sided standard normal critical value and \( \alpha \in (0,1) \) is the significance level. To avoid repetitions we present the results only for \( i = \text{DK}, \text{pwDK}, \text{KVB}, \text{NW87} \) and \text{pwNW87}. The results concerning the EWC estimator are the same as those for the KVB’s fixed-b estimator. The results pertaining to Andrews’ (1991) HAC estimator (with and without prewhitening) are the same as those corresponding to Newey and West’s (1987) estimator (with and without prewhitening, respectively). For the HAC and DK-HAC estimators we report the results for the MSE-optimal bandwidth [see Andrews (1991), Casini (2021b) and Whilelm (2015)].\(^{12}\) We set \( n_T = n_{2,T} = n_{3,T} = T^{2/3} \) which satisfy the growth rate bounds [see Casini (2021b) for details]. Let \( n_\delta = T - T_b - 2 \) denote the length of the regime in which \( x_t^{(2)} \) exhibits a shift \( \delta \) in the mean. The alternative hypothesis depends on the shift magnitude \( \delta \) and on how long the shift lasts for. Here the latter is \( n_\delta \). More generally, this is the set of time points such that \( \mathbb{E}(\hat{V}_t) = \mu_t \neq 0 \) holds.

\(^{11}\)Since \( \{\hat{V}_t\} \) is only observed in the out-of-sample, the LRV estimators use a sample of \( T_n \) observations.

\(^{12}\)For the HAC estimators we also report the result for any bandwidth choice \( b_T \rightarrow 0 \) such that \( T b_T \rightarrow \infty \), which is sufficient for the consistency of the estimator.
**Theorem 4.1.** Let \( \{d_t - \mathbb{E}(d_t)\} \) be a SLS process satisfying Assumption 2.1 and 3.1, and \( n_\delta = O(T_n^{1/2+\zeta}) \) where \( \zeta \in (0, 1/2) \) such that \( T_n b_T^{1/2} \to 0 \) and \( T_n (\hat{b}_{1,T})^{1/2} \to 0 \). Then, we have:

(i) If \( b_T \to 0 \), then \( \mathbb{P}_\delta(|t_{DM,NW87}| > z_\alpha) \to 0 \). If \( b_T = O(T^{-1/3}) \), then \( |t_{DM,NW87}| = O_p(T_n^{-1/6}) \) and \( \mathbb{P}_\delta(|t_{DM,NW87}| > z_\alpha) \to 0 \).

(ii) If \( b_T \to 0 \), then \( \mathbb{P}_\delta(|t_{DM,pwNW87}| > z_\alpha) \to 0 \). If \( b_T = O(T^{-1/3}) \), then \( |t_{DM,pwNW87}| = O_p(T_n^{-1/6}) \) and \( \mathbb{P}_\delta(|t_{DM,pwNW87}| > z_\alpha) \to 0 \).

(iii) If \( b_T = T^{-1} \), then \( |t_{DM,KVB}| = O_p(T_n^{\zeta-1/2}) \) and \( \mathbb{P}_\delta(|t_{DM,KVB}| > z_\alpha) \to 0 \).

(iv) Under Assumption 3.2-(i-iii), \( |t_{DM,DK}| = \delta^2 O_p(T_n^{\zeta}) \) and \( \mathbb{P}_\delta(|t_{DM,DK}| > z_\alpha) \to 1 \).

(v) Under Assumption 3.2-(i-iii), 3.5-(i, iv) and 3.6, \( |t_{DM,pwDK}| = \delta^2 O_p(T_n^{\zeta}) \) and \( \mathbb{P}_\delta(|t_{DM,pwDK}| > z_\alpha) \to 1 \).

Note that \( b_T = O(T^{-1/3}) \) in parts (i)-(ii) refers to the MSE-optimal bandwidth for the Newey and West’s (1987) estimator. The conditions \( T_n b_T^{1/2} \to 0 \) and \( T_n (\hat{b}_{1,T})^{1/2} \to 0 \) mean that the length of the regime in which \( x_t^{(2)} \) exhibits a shift \( \delta \) in the mean increases to infinity at a slower rate than \( T \). Theorem 4.1 implies that when the (prewhitened or non-prewhitened) HAC estimators or the fixed-\( b \) LRV estimators are used, the DM test is not consistent and its power converges to zero. The theorem suggests that prewhitened and non-prewhitened HAC estimators suffer from this problem in a similar way. The theorem also implies that the power functions corresponding to tests based on HAC estimators lie above the power functions corresponding to those based on fixed-\( b \)/EWC LRV estimators. An additional feature is that \( |t_{DM,NW87}|, |t_{DM,pwNW87}| \) and \( |t_{DM,KVB}| \) do not increase in magnitude with \( \delta \) because \( \delta \) appears in both the numerator and denominator.

The results concerning the DK-HAC estimator and the prewhitened DK-HAC estimator \( \hat{J}_{pw,T} \) show that these issues do not occur when these estimators are used. In fact, the test is consistent and its power increases with \( \delta \) and with the sample size. We provide finite-sample evidence in support of these theoretical results in Section 5.

## 5 Small-Sample Evaluations

We now show that the prewhitened DK-HAC estimators lead to HAR inference tests that have accurate null rejection rates when there is strong dependence and have superior power properties relative to those based on traditional LRV estimators. We consider HAR tests in the linear regression model as well as applied to the forecast evaluation literature, namely the Diebold-Mariano test and the forecast breakdown test of Giacomini and Rossi (2009).

The linear regression models have an intercept and a stochastic regressor. We focus on
the $t$-statistics $t_r = \sqrt{T} (\hat{\beta}(r) - \beta_0^{(r)})/\sqrt{\hat{J}_{X,T}^{(r)}}$ where $\hat{J}_{X,T}$ is a consistent estimate of the limit of $\text{Var}(\sqrt{T}(\hat{\beta} - \beta_0))$ and $r = 1, 2$. $t_1$ is the $t$-statistic for the parameter associated to the intercept while $t_2$ is associated to the stochastic regressor. Two regression models are considered. We run a $t$-test on the intercept in model M1 whereas a $t$-test on the coefficient of the stochastic regressor is run in model M2. The models are,

$$y_t = \beta_0^{(1)} + \delta + \beta_0^{(2)} x_t + e_t, \quad t = 1, \ldots, T,$$

(5.1)

for the $t$-test on the intercept and

$$y_t = \beta_0^{(1)} + (\beta_0^{(2)} + \delta) x_t + e_t, \quad t = 1, \ldots, T,$$

(5.2)

for the $t$-test on $\beta_0^{(2)}$ where $\delta = 0$ under the null hypotheses. In model M1 we set $\beta_0^{(1)} = 0$, $\beta_0^{(2)} = 1$, $x_t \sim \text{i.i.d. } \mathcal{N}(1, 1)$ and $e_t = \rho e_{t-1} + u_t$, $\rho = 0.4, 0.9$, $u_t \sim \text{i.i.d. } \mathcal{N}(0, 0.7)$. Model M2 involves segmented locally stationary errors: $\beta_0^{(1)} = \beta_0^{(2)} = 0$, $x_t = 0.6 + 0.8 x_{t-1} + u_{x,t}$, $u_{x,t} \sim \text{i.i.d. } \mathcal{N}(0, 1)$ and $e_t = \rho e_{t-1} + u_t$, $\rho_t = \max\{0, 0.8 (\cos (1.5 - \cos (5t/T)))\}$ for $t < 4T/5$ and $e_t = 0.5 e_{t-1} + u_t$, $u_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$ for $t \geq 4T/5$. Note that $\rho_t$ varies smoothly between 0 and 0.7021. Then, $\hat{J}_{X,T} = (X'X/T)^{-1} \hat{J}_T (X'X/T)^{-1}$ where $X = [X_1, \ldots, X_T]'$ and $X_t = [1, x_t]'$.

Next, we move to the forecast evaluation tests. The Diebold-Mariano test statistic is defined as in Section 4, $t_{DM} \triangleq T_n^{1/2} \bar{d}_L / \hat{d}_{L,T}^{1/2}$. In model M3 we consider an out-of-sample forecasting exercise with a fixed scheme where, given a sample of $T$ observations, $0.5T$ observations are used for the in-sample and the remaining half is used for prediction. To evaluate the empirical size, we specify the following data-generating process and the two forecasting models that have equal predictive ability. The true model for $y_t$ is given by $y_t = \beta_0^{(1)} + \beta_0^{(2)} x_{t-1}^{(0)} + e_t$ where $x_{t-1}^{(0)} \sim \text{i.i.d. } \mathcal{N}(1, 1)$, $e_t = 0.8 e_{t-1} + u_t$ with $u_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$ and we set $\beta_0^{(1)} = 0$, $\beta_0^{(2)} = 1$. The two competing models differ on the predictor used in place of $x_{t-1}^{(0)}$. The first forecast model uses $x_t^{(1)}$ while the second uses $x_t^{(2)}$ where $x_t^{(1)}$ and $x_t^{(2)}$ are i.i.d. $\mathcal{N}(1, 1)$ sequences, both independent from $x_{t-1}^{(0)}$. Each forecast model generates a sequence of $\tau (= 1)$-step ahead out-of-sample losses $L_t^{(i)}$ ($i = 1, 2$) for $t = T/2 + 1, \ldots, T - \tau$. Then $d_t \triangleq L_t^{(2)} - L_t^{(1)}$ denotes the loss differential at time $t$. The test rejects the null of equal predictive ability when (after normalization) $\bar{d}_L$ is sufficiently far from zero.

Next, we specify the alternative hypotheses for the Diebold-Mariano test. The two competing forecast models are as follows: the first model uses the actual true data-generating process while the second model differs in that in place of $x_{t-1}^{(0)}$ it uses $x_t^{(2)} = x_{t-1}^{(0)} + u_{X,t}$ for $t \leq 3T/4$ and $x_t^{(2)} = \delta + x_{t-1}^{(0)} + u_{X,t}$ for $t > 3T/4$, with $u_{X,t} \sim \text{i.i.d. } \mathcal{N}(0, 1)$. The null hypotheses of equal
predictive ability should be rejected whenever $\delta > 0$.

Finally, we consider model M4 which we use for investigating the performance the $t$-test for forecast breakdown of Giacomini and Rossi (2009). Suppose we want to forecast a variable $y_t$ which follows $y_t = \beta_0^{(1)} + \beta_0^{(2)} x_{t-1} + \delta x_{t-1} 1\{t > T_0^0\} + e_t$ where $x_t \sim$ i.i.d. $\mathcal{N}(1.5, 1.5)$ and $e_t = 0.3e_{t-1} + u_t$ with $u_t \sim$ i.i.d. $\mathcal{N}(0, 0.7)$, $\beta_0^{(1)} = \beta_0^{(2)} = 1$ and $T_0^0 = T\lambda_1^0$ with $\lambda_1^0 = 0.85$. The test detects a forecast breakdown when the average of the out-of-sample losses differs significantly from the average of the in-sample losses. The in-sample is used to obtain estimates of $\hat{\beta}_0^{(1)}$ and $\hat{\beta}_0^{(2)}$ which are in turn used to construct out-of-sample forecasts $\hat{y}_t = \hat{\beta}_0^{(1)} + \hat{\beta}_0^{(2)} x_{t-1}$. The test is defined as $t^{GR} \triangleq T_n^{1/2} S_{\hat{J}^1/2}$ where $S_{\hat{J}} \triangleq T_n^{-1} \sum_{t=T_{n+1}}^{T} SL_{t+\tau}$, $SL_{t+\tau}$ is the surprise loss at time $t + \tau$, i.e., the difference between the time $t + \tau$ out-of-sample loss and in-sample-average loss, $SL_{t+\tau} = L_{t+\tau} - \overline{L}_{t+\tau}$. Here $T_n$ is the sample size in the out-of-sample, $T_m$ is the sample size in the in-sample and $\hat{J}_{SL}$ is a LRV estimator. We consider a fixed forecasting scheme and $\tau = 1$.

We consider the following DK-HAC estimators: $\hat{J}_{T,pw,SLS} = \hat{J}_{T,pw}$ as discussed in Section 2, $\hat{J}_{T,pw,1}$ which uses prewhitening with a single block [$n_T = T$ in (2.2)] (i.e., stationary prewhitening), $\hat{J}_{T,pw,SLS,\mu}$ which uses prewhitening involving a VAR(1) with time-varying intercept [i.e., with $\hat{\mu}_t$ in (2.2)]. The asymptotic properties of $\hat{J}_{T,pw,SLS,\mu}$ are the same as those of $\hat{J}_{T,pw,SLS}$ since $\hat{\mu}_t$ plays no role in the theory given the zero-mean assumption on $\{V_t\}$. However, it leads to power enhancement under nonstationary alternative hypotheses. The asymptotic properties of $\hat{J}_{T,pw,1}$ follows as a special case from the properties of $\hat{J}_{T,pw,SLS}$. We set $n_T = n_{2,T} = n_{3,T} = T^{2/3}$. For the test of Giacomini and Rossi (2009) we do not report the results for $\hat{J}_{T,pw,1}$ because the stationarity assumption is clearly violated under the alternative. We compare tests using these estimators to those using the following estimates: Andrews’ (1991) HAC estimator with automatic bandwidth; Andrews’ (1991) HAC estimator with automatic bandwidth and the prewhitening procedure of Andrews and Monahan (1992); Newey and West’s (1987) HAC estimator with the automatic bandwidth as proposed in Newey and West (1994); Newey and West’s (1987) HAC estimator with the automatic bandwidth as proposed in Newey and West (1994) and the prewhitening procedure; Newey-West with the fixed-$b$ method of Kiefer, Vogelsang, and Bunzel (2000); the Empirical Weighted Cosine (EWC) of Lazarus, Lewis, Stock, and Watson (2018). We consider the following sample sizes: $T = 200, 400$ for model M1-M2 and $T = 400, 800$ for model M3-M4. We set $T_m = 200, 400$ for M3 and $T_m = 240, 480$ for M4. The nominal size is $\alpha = 0.05$ throughout.

Table 1-2 report the rejection rates under the null hypothesis for model M1-M4. We begin with model M1 with medium dependence ($\rho = 0.4$). The prewhitened DK-HAC estimators lead to tests with accurate rejection rates that are slightly better than those obtained with Newey-
West with fixed-$b$ and to EWC. In contrast, the classical HAC estimators of Andrews (1991) and Newey and West (1987) are less accurate with rejection rates higher than the nominal level. The prewhitening of Andrews and Monahan (1992) helps to reduce the size distortions but they still persist for the Newey-West estimator even for $T = 400$. For higher dependence (i.e., $\rho = 0.9$), using EWC and $\hat{J}_{T,pw,SLS,\mu}$ yield oversized tests, though by a small margin. The best size control is achieved using the Newey-West with fixed-$b$ (KVB), $\hat{J}_{T,pw,1}$ and $\hat{J}_{T,pw,SLS}$.

For model M2, Newey-West with fixed-$b$ and the prewhitened DK-HAC ($\hat{J}_{T,pw,1}, \hat{J}_{T,pw,SLS}, \hat{J}_{T,pw,\mu}$) allow accurate rejection rates. In some cases, tests based on the prewhitened DK-HAC are superior to those based on fixed-$b$ (KVB). The tests with EWC are slightly oversized when $T = 200$ but close to the nominal level when $T = 400$. The classical HAC of Andrews (1991) and Newey and West (1987), either prewhitened or not, imply oversized tests with $T = 200$.

Turning to the HAR tests for forecast evaluation, Table 2 report some striking results. First, tests based on the Newey-West with fixed-$b$ (KVB) have size essentially equal to zero, while those based on the EWC and prewhitened or non-prewhitened classical HAC estimators are oversized. The prewhitened DK-HAC allows more accurate tests. For model M4, many of the tests have size equal to or close to zero. This occurs using the classical HAC, either prewhitened or not and EWC. The prewhitened DK-HAC estimators and Newey-West with fixed-$b$ (KVB) allow controlling the size reasonably well. Overall, Table 1-2 in part confirm previous evidence and in part suggest new facts. Newey-West with fixed-$b$ (KVB) leads to better size control than using the classical HAC estimators of Andrews (1991) and Newey and West (1987) even when the latter are used in conjunction with the prewhitening device of Andrews and Monahan (1992). The new result is that several of the LRV estimators proposed in the literature can lead to tests having null rejection rates equal to or close to zero. This occurs because the null hypotheses involves nonstationary data generating mechanisms. These LRV estimators are inflated and the associated test statistics are undersized. This is expected to have negative consequences for the power of the tests, as we will see below. The estimators proposed in this paper perform well in leading to tests that control the null rejection rates for all cases. They are in general competitive with using the Newey-West with fixed-$b$ (KVB) when the latter does not fail and in some cases can also outperform it.

Table 3-4 report the empirical power of the tests for model M1-M4. For model M1 with $\rho = 0.9$ and M2 we see that all tests have good and monotonic power. It is fair to compare tests based on the DK-HAC estimators relative to using Newey-West with fixed-$b$ (KVB) since they have similar well-controlled null rejection rates. Tests based on the Newey-West with fixed-$b$ (KVB) sacrifices power more than using the DK-HAC estimators and the difference is substantial. The
classical HAC estimators have higher power but it is unfair to compare them since they are often oversized. A similar argument applies to using the EWC.

We now move to the forecast evaluation tests. For both models M3 and M4 we observe several features of interests. Essentially all tests proposed previously experience severe power issues. The power is either non-monotonic, very low or equal zero. This holds when using the classical HAC estimators of Andrews (1991) as well as Newey and West (1987) irrespective of whether prewhitening is used, with the EWC and the Newey-West with fixed-\( b \) (KVB). The only exceptions are tests based on the Newey and West’s (1987) and Andrews’ (1991) HAC estimator with prewhitening in model M4 that display some power but much lower compared to using the prewhitened DK-HAC estimators. The latter have excellent power. The reason for the severe power problems for the previous LRV-based tests is that models M3 and M4 involve nonstationary alternative hypotheses. The sample autocovariances become inflated and overestimate the true autocovariances. The theoretical results about the power in Theorem 4.1 suggest that this issue becomes more severe as \( \delta \) increases, which explains the non-monotonic power for some of the tests, with tests based on fixed-\( b \) methods that include many lags suffering most. The double smoothing in the DK-HAC estimators allows to avoid this problem because it flexibly accounts for nonstationarity. The key idea is not to mix observations belonging to different regimes. Simulation results for additional data-generating processes involving ARMA, ARCH and heteroskedastic errors are not discussed here because the results are qualitatively equivalent.

6 Conclusions

We introduce a nonparametric nonlinear VAR prewhitened long-run variance (LRV) estimator for the construction of standard errors robust to autocorrelation and heteroskedasticity that can be used for hypothesis testing both within and outside the linear regression model. HAR tests normalized by the proposed estimator exhibit accurate null rejection rates even when there is strong dependence. We show theoretically that existing estimators lead to HAR tests that have non-monotonic power under nonstationary alternative hypotheses while the proposed estimator has good monotonic power thereby addressing a long-lasting problem in time series econometrics. The proposed method is theoretically valid under general nonstationary random variables. We also establish mean-squared error bounds for LRV estimation that are sharper than previously established and use them to determine the data-dependent bandwidths.
Supplemental Materials

The supplement for online publication [cf. Casini and Perron (2021b)] establishes the theoretical properties of DK-HAC estimators under general nonstationarity (i.e., unconditionally heteroskedastic random variables). We show consistency and derive asymptotic MSE bounds that are sharper than previously established. We also present the proofs of the results of Section 3. An additional supplement, not for publication, includes the proofs of the results of Section 4 and S.B.
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A Appendix

A.1 Tables

Table 1: Empirical small-sample size of t-test for model M1-M2

|                  | M1, \( \rho = 0.4 \) | M1, \( \rho = 0.9 \) | M2            |
|------------------|-----------------------|-----------------------|---------------|
| \( \alpha = 0.05 \) T = 200 | 0.054 | 0.045 | 0.085 | 0.065 | 0.061 | 0.053 |
| \( \hat{J}_T \), QS, prew | 0.052 | 0.043 | 0.086 | 0.051 | 0.065 | 0.054 |
| \( \hat{J}_T \), QS, prew, SLS | 0.049 | 0.048 | 0.103 | 0.092 | 0.063 | 0.054 |
| \( \hat{J}_T \), QS, prew, SLS, \( \mu \) | 0.082 | 0.065 | 0.162 | 0.118 | 0.095 | 0.050 |
| Andrews | 0.063 | 0.057 | 0.104 | 0.083 | 0.077 | 0.048 |
| Newey-West | 0.114 | 0.090 | 0.351 | 0.272 | 0.138 | 0.057 |
| Newey-West, prew | 0.075 | 0.064 | 0.110 | 0.077 | 0.090 | 0.059 |
| Newey-West, fixed-\( b \) (KVB) | 0.058 | 0.056 | 0.091 | 0.066 | 0.069 | 0.052 |
| EWC | 0.058 | 0.055 | 0.149 | 0.113 | 0.071 | 0.048 |

Table 2: Empirical small-sample size for model M3-M4

|                  | M3 | M4 |
|------------------|----|----|
| \( \alpha = 0.05 \) T = 400 | 0.065 | 0.060 |
| \( \hat{J}_T \), QS, prew, SLS | 0.065 | 0.061 |
| Andrews | 0.082 | 0.073 |
| Andrews, prew | 0.080 | 0.074 |
| Newey-West | 0.080 | 0.074 |
| Newey-West, prew | 0.078 | 0.073 |
| Newey-West, fixed-\( b \) (KVB) | 0.002 | 0.002 |
| EWC | 0.080 | 0.074 |
Table 3: Empirical small-sample power of $t$-test for model M1-M2

|            | M1             | M2             |            |            |            |            |
|------------|----------------|----------------|------------|------------|------------|------------|
| $\alpha = 0.05, T = 400$ | $\delta = 0.5$ | $\delta = 1$ | $\delta = 0.1$ | $\delta = 0.2$ | $\delta = 0.4$ |
| $J_T$, QS, prew | 0.344 | 0.807 | 1.000 | 0.387 | 0.889 | 1.000 |
| $J_T$, QS, prew, SLS | 0.378 | 0.787 | 1.000 | 0.330 | 0.813 | 1.000 |
| $J_T$, QS, prew, SLS, $\mu$ | 0.463 | 0.849 | 1.000 | 0.347 | 0.833 | 1.000 |
| Andrews | 0.430 | 0.864 | 1.000 | 0.450 | 0.922 | 1.000 |
| Andrews, prew | 0.360 | 0.812 | 1.000 | 0.433 | 0.911 | 1.000 |
| Newey-West | 0.630 | 0.958 | 1.000 | 0.511 | 0.938 | 1.000 |
| Newey-West, prew | 0.363 | 0.811 | 1.000 | 0.443 | 0.911 | 1.000 |
| Newey-West, fixed-b (KVB) | 0.274 | 0.655 | 0.980 | 0.329 | 0.758 | 0.990 |
| EWC | 0.436 | 0.886 | 1.000 | 0.392 | 0.890 | 1.000 |

Table 4: Empirical small-sample power for model M3-M4

|            | M3             | M4             |            |            |            |            |
|------------|----------------|----------------|------------|------------|------------|------------|
| $\alpha = 0.05, T = 400$ | $\delta = 0.5$ | $\delta = 2$ | $\delta = 6$ | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ |
| $J_T$, QS, prew, SLS | 0.495 | 0.920 | 1.000 | 0.613 | 0.923 | 1.000 |
| $J_T$, QS, prew, SLS, $\mu$ | 0.498 | 0.940 | 1.000 | 0.663 | 0.957 | 1.000 |
| Andrews | 0.158 | 0.014 | 0.000 | 0.000 | 0.043 | 0.073 |
| Andrews, prew | 0.224 | 0.056 | 0.000 | 0.351 | 0.942 | 0.952 |
| Newey-West | 0.179 | 0.302 | 0.587 | 0.019 | 0.821 | 1.000 |
| Newey-West, prew | 0.137 | 0.014 | 0.000 | 0.003 | 0.278 | 0.722 |
| Newey-West, fixed-b (KVB) | 0.059 | 0.008 | 0.000 | 0.000 | 0.000 | 0.000 |
| EWC | 0.087 | 0.018 | 0.000 | 0.062 | 0.000 | 0.000 |
Supplemental Material to

Prewhitened Long-Run Variance Estimation Robust to Nonstationarity

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Abstract

This supplemental material is for online publication and is structured as follows. Section S.A presents some preliminary notions. Section S.B establishes the theoretical properties of DK-HAC estimators under general nonstationarity. Section S.C presents the proofs of the results of Section 3.
S.A Preliminaries

In this section we present a formal definition of SLS processes which is implied by Assumption 2.1 on $f(u, \omega)$. Let $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{m_0} < \lambda_{m_0+1} = 1$ where $m_0$ may be fixed or grow to infinity. A function $G(u, \omega) : [0, 1] \times \mathbb{R} \to \mathbb{C}$ is said to be piecewise (Lipschitz) continuous in $u$ with $m_0 + 1$ segments if for each segment $j = 1, \ldots, m_0 + 1$ it satisfies $\sup_{u \neq v} |G(u, \omega) - G(v, \omega)| \leq K |u - v|$ for any $\omega \in \mathbb{R}$ with $\lambda_{j-1} < u, v \leq \lambda_j$ for some $K < \infty$. We define $G_j(u, \omega) = G(u, \omega)$ for $\lambda_{j-1} < u \leq \lambda_j$. A function $G(\cdot, \cdot) : [0, 1] \times \mathbb{R} \to \mathbb{C}$ is said to be left-differentiable at $u_0$ if $\lim_{u \to u_0^-} (G(u_0, \omega) - G(u, \omega))/ (u_0 - u)$ exists for any $\omega \in \mathbb{R}$.

**Definition S.A.1.** A sequence of stochastic processes $V_{t,T} (t = 1, \ldots, T)$ is called segmented locally stationary (SLS) with $m_0 + 1$ regimes, transfer function $A^0$ and trend $\mu$, if there exists a representation,

$$V_{t,T} = \mu_j (t/T) + \int_{-\pi}^{\pi} \exp (i \omega t) A^0_{j,t,T} (\omega) d\xi (\omega), \quad \left(t = T_{j-1}^0 + 1, \ldots, T_j^0\right),$$

for $j = 1, \ldots, m_0 + 1$, where by convention $T_0 = 0$ and $T_{m_0+1} = T$ and the following holds:

(i) $\xi (\lambda)$ is a stochastic process on $[-\pi, \pi]$ with $\xi (\omega) = \xi (-\omega)$ and

$$\text{cum} \{d\xi (\omega_1), \ldots, d\xi (\omega_r)\} = \varphi \left(\sum_{j=1}^{r} \omega_j \right) g_r (\omega_1, \ldots, \omega_{r-1}) d\omega_1 \ldots d\omega_r,$$

where $\text{cum} \{\cdot\}$ denotes the cumulant spectra of $r$-th order, $g_1 = 0$, $g_2 (\omega) = 1$, $|g_r (\omega_1, \ldots, \omega_{r-1})| \leq M_r$ for all $r$ with $M_r$ being a constant that may depend on $r$, and $\varphi (\omega) = \sum_{j=-\infty}^{\infty} \delta (\omega + 2\pi j)$ is the period $2\pi$ extension of the Dirac delta function $\delta (\cdot)$.

(ii) There exists a constant $K$ and a piecewise continuous function $A : [0, 1] \times \mathbb{R} \to \mathbb{C}$ such that, for each $j = 1, \ldots, m_0 + 1$, there exists a $2\pi$-periodic function $A_j : (\lambda_{j-1}^0, \lambda_j^0] \times \mathbb{R} \to \mathbb{C}$ with $A_j (u, -\omega) = A_j (u, \omega)$, $\lambda_j^0 \triangleq T_j^0/T$ and for all $T$,

$$A (u, \omega) = A_j (u, \omega) \text{ for } \lambda_{j-1}^0 < u \leq \lambda_j^0, \quad \text{(S.2)}$$

$$\sup_{1 \leq j \leq m_0+1} \sup_{0 < t < T_{j-1}^0} \left|A^0_{j,t,T} (\omega) - A_j (t/T, \omega)\right| \leq KT^{-1}. \quad \text{(S.3)}$$

(iii) $\mu_j (t/T)$ is piecewise continuous.

In the context of HAR inference $\mathbb{E} (V_t) = 0$ and so $\mu (t/T) = 0$ for all $t$ in Definition S.A.1. In view of Definition S.A.1, Assumption 2.1 also holds with $f(u, \omega)$ replaced by $A(u, \omega)$ and this property is used in some parts of the proofs. In Assumption 3.1-(ii), the continuity points are those $u \in [0, 1]$ such that $u \neq \lambda_j^0$ ($j = 1, \ldots, m_0 + 1$) whereas the discontinuity points are those $u \in [0, 1]$ such that $u = \lambda_j^0$ ($j = 1, \ldots, m_0 + 1$).

S.B Extension to General Nonstationary Random Variables

In this section we discuss the case where $\{V_t\}$ is unconditionally heteroskedastic and establish new MSE bounds which we compare to existing ones. To focus on the main intuition and for comparison purposes,
we consider the non-prewhitened DK-IIAC estimator
\[
\hat{J}_T(b_1,T, b_2,T) = \sum_{k=-T+1}^{T-1} K_1(b_1,Tk) \hat{\Gamma}(k),
\]
where \( \hat{\Gamma}(k) \) is defined analogously to \( \hat{\Gamma}_D(k) \) but with \( \hat{V}_t \) in place of \( \hat{V}_{D,t}^* \). Corresponding results for the prewhitened estimator \( \hat{J}_{pw,T} \) can be obtained by using the results of the Section 3, though the proofs are more lengthy with no special gain in intuition. We provide theoretical results under the assumption that \( \{V_t\} \) is generated by some distribution \( \mathcal{P} \). \( \mathbb{E}_\mathcal{P} \) denotes the expectation taken under \( \mathcal{P} \). We establish lower and upper bounds on the MSE under \( \mathcal{P} \) and use a minimax MSE criterion for optimality. Define the sample size dependent spectral density of \( \{V_t\} \) as
\[
f_{\mathcal{P},T}(\omega) \triangleq (2\pi)^{-1} \sum_{k=-T+1}^{T-1} \Gamma_{\mathcal{P},T}(k) \exp(-i\omega k), \text{ for } \omega \in [-\pi, \pi],
\]
where
\[
\Gamma_{\mathcal{P},T}(k) = \begin{cases} 
T^{-1} \sum_{t=k+1}^{T} \mathbb{E}_\mathcal{P}(V_tV_{t-k}'), & \text{for } k \geq 0 \\
T^{-1} \sum_{t=-k+1}^{T} \mathbb{E}_\mathcal{P}(V_{t+k}V_t'), & \text{for } k < 0.
\end{cases}
\]

The estimand is then given by \( J_{\mathcal{P},T} \triangleq \sum_{k=-T+1}^{T-1} \Gamma_{\mathcal{P},T}(k) \).

The theoretical bounds below are derived in terms of two distributions \( \mathcal{P}_w, w = L, U \), under which \( \{V_t\} \) is zero-mean SLS with \( m_0 + 1 \) regimes and satisfies Assumption 2.1 and 3.1 with autocovariance function \( \{\Gamma_{\mathcal{P},T}(k)\} \). Then, \( \{a'V_t\} \) has spectral density \( f_{\mathcal{P},a}(\omega) \triangleq \int_0^1 f_{\mathcal{P},a}(u, \omega) \, du \), where
\[
f_{\mathcal{P},a}(u, \omega) \triangleq (2\pi)^{-1} \sum_{k=\infty}^{\infty} a'\Gamma_{\mathcal{P},a}(k) \, a \exp(-i\omega k), \text{ for all } a \in \mathbb{R}^p.
\]

Let \( \kappa_{\mathcal{P},aV,t}(k, j, m) \) denote the time-\( t \) fourth-order cumulant of \( \{a'V_t, a'V_{t+k}, a'V_{t+j}, a'V_{t+m}\} \) under \( \mathcal{P} \). Define
\[
P_U \triangleq \left\{ \mathcal{P} : -\Gamma_{\mathcal{P},T}(k) \leq \Gamma_{\mathcal{P},T}(k) \leq \Gamma_{\mathcal{P},T}(k), \text{ and } |\kappa_{\mathcal{P},aV,t}(k, j, m)| \leq |\kappa^*_t(k, j, m)| \right\},
\]

\[\forall t \geq 1, k, j, m \geq -t + 1, a \in \mathbb{R}^p \text{ that satisfies } \sum_{k=\infty}^{\infty} \sum_{j=\infty}^{\infty} \sum_{m=\infty}^{\infty} \sup_{t} \kappa^*_t(k, j, m) < \infty \},
\]

and
\[
P_L \triangleq \left\{ \mathcal{P} : 0 \leq \Gamma_{\mathcal{P},T}(k) \leq \Gamma_{\mathcal{P},T}(k), \forall t \geq 1, k \geq -t + 1 \text{ and } \kappa_{\mathcal{P},aV,t}(k, j, m) \right\}
\]
satisfies the same condition as in \( P_U \).

To derive the MSE bounds for a given class of general nonstationary processes one needs to impose restrictions on the autocovariance function of the processes in the class relative to the autocovariance function of some process whose second-order properties are known. \( P_U \) includes all distributions such
that the autocovariances of \( \{V_t\} \) are bounded above by those of some SLS process with distribution \( \mathcal{P}_U \), thereby allowing considerable variability of \( \Gamma_{\mathcal{P}_t/T} (k) \) for given \( t \) and \( k \). The set \( \mathcal{P}_L \) requires the autocovariances of \( \{V_t\} \) to be bounded below by positive semidefinite autocovariances of some SLS process with distribution \( \mathcal{P}_L \). Let \( c_{\mathcal{P}_w} (u, k) = \int e^{i \omega k} \Gamma_{\mathcal{P}_w,u} (k) \, d\omega \) denote the local autocovariance associated to the distribution \( \mathcal{P}_w \), \( w = L, U \). Let

\[
K_1 = \left\{ K_1 (\cdot) : \mathbb{R} \to [-1, 1] : K_1 (0) = 1, K_1 (x) = K_1 (-x), \forall x \in \mathbb{R} \right\}
\]

\[
\int_{-\infty}^{\infty} K_1^2 (x) \, dx < \infty, K_1 (\cdot) \text{ is continuous at 0 and at all but finite numbers of points}\).
\]

Note that \( K_3 \subset K_1 \). In particular, \( K_1 \) includes also the truncated kernel.

### S.B.1 Consistency, Rate of Convergence and MSE Bounds

Consider the following generalization of Assumption 2.1 and 3.1:

**Assumption S.B.1.** \( \{V_t\} \) is a mean-zero sequence and satisfies \( \sum_{k=0}^{\infty} \sup_{l \geq 1} ||\mathbb{E}_\mathcal{P} (V_l V_{l+k}^\prime)|| < \infty \) and for all \( a_1, a_2, a_3, a_4 \leq p, \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sup_{l \geq 1} |\kappa_{\mathcal{P}_l V, \mathcal{P}_l V} (a_1, a_2, a_3, a_4) (k, j, m)| < \infty \).

Let \( \text{MSE}_{\mathcal{P}} (\cdot) \) denote the MSE of \( \cdot \) under \( \mathcal{P} \) and let \( K_{1,+} = \{ K_1 (\cdot) \in K_1 : K_1 (x) \geq 0, \forall x \} \). \( K_{1,+} \) is a subset of \( K_1 \) that contains all kernels that are non-negative and is used for some results below. The QS kernel is not in \( K_{1,+} \). The smoothness of \( f_{\mathcal{P}_{w,a}} (u, \omega) \) at \( \omega = 0 \) is indexed by

\[
f_{\mathcal{P}_{w,a}} (q) (u, 0) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} |k|^q a' \Gamma_{\mathcal{P}_{w,u}} (k) a, \quad \text{for } q \in [0, \infty), w = L, U.
\]

We first consider the MSE bounds for the estimator \( \tilde{J}_T \) is constructed using \( \hat{V}_t (\beta_0) \) rather than \( \hat{V}_t \). Let

\[
K_2 = \left\{ K_2 (\cdot) : \mathbb{R} \to [0, \infty] : K_2 (x) = K_2 (1-x), \int K_2 (x) \, dx = 1, K_2 (x) = 0, \text{ for } x \notin [0, 1], K_2 (\cdot) \text{ is continuous} \right\}.
\]

**Theorem S.B.1.** Suppose Assumption S.B.1 holds, \( K_2 (\cdot) \in K_2 \), \( b_1, b_2, a_1, a_2, a_3, a_4 \leq p \), \( \gamma = \infty, \gamma_1 = 0 \), \( b_1, b_2, b_3, b_4 \rightarrow 0 \) and \( 1/Tb_1 b_2 \rightarrow 0 \). If \( n_T/Tb_{1,T}^q \rightarrow 0 \), \( b_{1,T}^q b_{1,T}^q \rightarrow 0 \) and \( T b_{1,T}^{2q+1} b_{2,T} \rightarrow \gamma \in (0, \infty) \) for some \( q \in [0, \infty) \) for which \( K_{1,q} \), \( f_{\mathcal{P}_{w,a}} (u, 0) \) \( \text{MSE}_{\mathcal{P}} (a' \tilde{J}_T a) = 4\pi^2 \left[ \gamma K_{1,q}^2 \left( \int_0^1 f_{\mathcal{P}_{w,a}} (u, 0) \, du \right)^2 + 2 \int K_1^2 (y) \, dy \int_0^1 K_2^2 (x) \, dx \left( \int_0^1 f_{\mathcal{P}_{w,a}} (u, 0) \, du \right)^2 \right] \),

\( \forall K_1 (\cdot) \in K_1 \),

(i) for all \( K_1 (\cdot) \in K_1 \),

\[
\lim_{T \rightarrow \infty} T b_{1,T} b_{2,T} \sup_{\mathcal{P} \in \mathcal{P}_U} \text{MSE}_{\mathcal{P}} (a' \tilde{J}_T a) = 4\pi^2 \left[ \gamma K_{1,q}^2 \left( \int_0^1 f_{\mathcal{P}_{w,a}} (u, 0) \, du \right)^2 + 2 \int K_1^2 (y) \, dy \int_0^1 K_2^2 (x) \, dx \left( \int_0^1 f_{\mathcal{P}_{w,a}} (u, 0) \, du \right)^2 \right].
\]
(ii) for all $K_1 (\cdot) \in K_{1,+}$,

$$
\lim_{T \to \infty} T b_{1,T} b_{2,T} \inf_{P_L, \mathcal{P}} \text{MSE}_P \left( a' \tilde{J} \right) = 4\pi^2 \left[ \gamma K_{1,q}^2 \left( \int_0^1 f_{\mathcal{P}_L,a} (u, 0) \, du \right)^2 \right] + 2 \int K_1^2 (y) \, dy \int_0^1 K_2^2 (x) \, dx \left[ \int_0^1 f_{\mathcal{P}_L,a} (u, 0) \, du \right]^2).
$$

The theoretical bounds in Theorem S.B.1 are sharper than the ones in Andrews (1988; 1991) which are based on stationarity (i.e., the autocovariances that dominate the autocovariances of any $\mathcal{P} \in P_U$ are assumed by Andrews (1991)) to be those of a stationary process).\(^{13}\) Given that stationarity is a special case of SLS, our bounds apply to a wide class of processes. Furthermore, they are more informative because they change with the specific type of nonstationarity unlike Andrews’ (1991) bounds that depend on the spectral density of a stationary process.

The theorem is derived under the assumption $b_{2,T}^2 / b_{1,T}^q \to 0$. When instead $b_{2,T}^2 / b_{1,T}^q \to \nu \in (0, \infty)$, there is an additional term in the bound. For example, in part (i) this term is

$$
(2^{-1} \nu \int_0^1 x^2 K_2 (x) \sum_{k = -\infty}^\infty \int_0^1 (\partial^2 / \partial u^2) c_{\mathcal{P}_U} (u, k) \, du)^2.
$$

Some of the results of this paper are extended to the case $b_{2,T}^2 / b_{1,T}^q \to \nu \in (0, \infty)$ in Belotti et al. (2021). Thus, our bounds show how nonstationarity influences the bias-variance trade-off. They also highlight how it is affected by the smoothing over the time direction versus the autocovariance lags direction. These are important elements in order to understand the properties of HAR tests normalized by LRV estimators. We now extend the results in Theorem S.B.1 to the estimator $\tilde{J}$ that uses $V_t (\beta)$. The following assumptions extend Assumption 3.2-3.3 to the distribution $\mathcal{P}$.

**Assumption S.B.2.** Assumption 3.2 holds with $\mathbb{E}$ replaced by $\mathbb{E}_{\mathcal{P}}$.

**Assumption S.B.3.** (i) Assumption S.B.1 holds with $V_t$ replaced by $(V_{[u]}', \text{vec}((\partial / \partial \beta') V_{[u]} (\beta_0)) - \mathbb{E}_{\mathcal{P}_U} (\partial / \partial \beta') V_{[u]} (\beta_0))^\prime); (ii) sup_{u \in [0,1]} \mathbb{E}_{\mathcal{P}_U} (sup_{\beta \in \Theta} \| \partial^2 / \partial \beta \partial \beta' \| V_{[u]} (\beta))^2 \| < \infty (r = 1, \ldots, p)$.

To show the asymptotic equivalence of the MSE of $a' \tilde{J} a$ to that of $a' \tilde{J} a$ we need an additional assumption which was also used by Andrews (1991). Define

$$
H_{1,T} \triangleq b_{1,T} \sum_{k = -T+1}^{T-1} \left| K_1 (b_{1,T} k) \right| \times \frac{n_T}{T} \sum_{r=0}^{T/n_T} (T b_{2,T})^{-1/2} \sum_{s=k+1}^{T} K_2^2 \left( \frac{(r + 1) n_T - (s - k / 2)) / T}{b_{2,T}} \right) \frac{\partial}{\partial \beta} \partial d^2 \left( \beta_0 \right) d^2 \left( \beta_0 \right),
$$

$$
H_{2,T} \triangleq b_{1,T} \sum_{k = -T+1}^{T-1} \left| K_1 (b_{1,T} k) \sup_{\beta \in \Theta} \frac{n_T}{T} \sum_{r=0}^{T/n_T} (T b_{2,T})^{-1} \sum_{s=k+1}^{T} T
$$

\(^{13}\)There are a couple of technical issues in Section 8 in Andrews (1991). In particular, the MSE bound is not correct. See Casini (2021a) for details.
$\times K_s^2 \left( \frac{(r + 1) n_T - (s - k/2))}{b_{2,T}} \right) \frac{\partial^2}{\partial \beta \partial \beta'} d' V_s(\beta) a' V_{s-k}(\beta) \right|.$

Let $H^{(r)}_{1,T}, \beta^{(r)}$ and $\beta_0^{(r)}$ denote the $r$-th elements of $H_{1,T}, \beta$ and $\beta_0$, respectively, for $r = 1, \ldots, p$.

**Assumption S.B.4.** For all $r = 1, \ldots, p$, $\limsup_{T \to \infty} \sup_{\mathcal{F} \in \mathcal{P}_U} \mathbb{E}_{\mathcal{F}} (H^{(r)}_{1,T} \sqrt{T} (\beta^{(r)} - \beta_0^{(r)}))^2 < \infty$ and

$$\limsup_{T \to \infty} \sup_{\mathcal{F} \in \mathcal{P}_U} \mathbb{E}_{\mathcal{F}} (\sqrt{T} (\beta - \beta_0)^2) < \infty.$$

**Theorem S.B.2.** Suppose $K_1(\cdot) \in \mathcal{K}_1, K_2(\cdot) \in \mathcal{K}_2, b_{1,T}, b_{2,T} \to 0, n_T \to \infty, n_T / T \to 0$ and $1 / T b_{1,T} b_{2,T} \to 0$. We have:

(i) If Assumption S.B.1-S.B.2 hold, $\sqrt{T} b_{1,T} \to \infty$, then $\hat{J}_T - J_T \xrightarrow{P} 0$ and $\hat{J}_T - J_T \xrightarrow{} 0$.

(ii) If Assumption S.B.1-S.B.3 hold, $n_T / T b_{1,T} \to 0$, $n_T / T b_{2,T}^q \to 0$ and $T b_{1,T} b_{2,T} \to \gamma \in (0, \infty)$ for some $q \in [0, \infty)$ for which $K_{1,q}, \int_0^1 f_{\beta_{u,a}}(u, 0) \, du \in [0, \infty), w = U, L, a \in \mathbb{R}^p$, then $\sqrt{T} b_{1,T} b_{2,T} (\hat{J}_T - J_T) = O_{\mathcal{F}}(1)$ and $\sqrt{T} b_{1,T} b_{2,T} (\hat{J}_T - J_T) = O_{\mathcal{F}}(1)$.

(iii) Under the assumptions of part (ii) and Assumption S.B.4,

$$\lim_{T \to \infty} \sup_{\mathcal{F} \in \mathcal{P}_U} T b_{1,T} b_{2,T} |\text{MSE}_{\mathcal{F}}(a' \hat{J}_T a) - \text{MSE}_{\mathcal{F}}(a' J_T a)| = 0$$

for all $a \in \mathbb{R}^p$ such that $|\int_0^1 f_{\beta_{u,a}}(u, 0) \, du| < \infty$.

Theorem S.B.2 extends the consistency, rate of convergence, MSE results of Theorem 3.2 in Casini (2021b). The asymptotic equivalence of the MSE implies that the bounds in Theorem S.B.1 apply to $\hat{J}_T$ as well. The MSE equivalence is used to show that the optimal kernels and bandwidths results below apply to $\hat{J}_T$ as well as to $\hat{J}_T$. Similar results can be shown for the prewhitened estimator $\hat{J}_{T,pw}$. For this case, the sets $\mathcal{P}_U$ and $\mathcal{P}_L$ would need to be defined in terms of the autocovariance function of $V_{D,t}^* = D_t V_t^*$. The distributions $\mathcal{P}_U$ and $\mathcal{P}_L$ that form an envelope for the autocovariances of $V_{D,t}^*$ may depend on different prewhitening models.

### S.B.2 Optimal Bandwidths and Kernels

We use the sequential MSE procedure that first determines the optimal $b_{2,T}(u)$ and then determines the optimal $b_{1,T}$ as function of the integrated optimal $\tilde{b}_{2,T}$, see Casini (2021b). This contrasts with the MSE criterion used by Belotti et al. (2021) which determines the optimal $b_{1,T}$ and $b_{2,T}$ that jointly minimize the maximum asymptotic MSE bound. An advantage of the sequential criterion is that the optimal $b_{2,T}(u)$ is determined for each $u$. Thus, it accounts more accurately for nonstationarity. The results for the global MSE criterion can easily be extended using similar arguments as those used in this section. We consider distributions $\mathcal{F} \in \mathcal{P}_{U,2}$ where $\mathcal{P}_{U,2} \subseteq \mathcal{P}_U$ is defined below. We need to restrict attention to a subset $\mathcal{P}_{U,2}$ of $\mathcal{P}_U$ for technical reasons related to the derivation of the optimal bandwidth $b_{2,T}^{opt}$. The distributions in $\mathcal{P}_{U,2}$ restrict the degree of nonstationarity by requiring some smoothness of the local autocovariance. This is intuitive since the optimality of $b_{2,T}^{opt}$ is justified under smoothness locally in time. We remark however, that, the optimality of $b_{1,T}$ and $\tilde{K}_1$ determined below holds over all distributions $\mathcal{F} \in \mathcal{P}_U$. We show that the resulting optimal kernels are $K_1^{opt}(\cdot)$ and $K_2^{opt}(\cdot)$ from Section 3. For any $a \in \mathbb{R}^p$ consider...
the following inequality,
\[
\left| a' \left( \frac{\partial^2}{\partial^2 u} c_{\mathcal{F}}(u_0, k) \right) a \right| \leq \left| a' \left( \frac{\partial^2}{\partial^2 u} c_{\mathcal{F}}(u_0, k) \right) a \right|.
\]

We consider the following class of distributions,
\[
P_{U,2} \triangleq \{ \mathcal{F} : \mathcal{F} \in P_U, m_0 = 0, \text{ and (S.1) holds} \forall k \in \mathbb{R} \text{ and } \forall u_0 \in (0, 1) \}.
\]

Let
\[
D_{1,U,a}(u_0) \triangleq \left( a' \left( \frac{\partial^2 c_{\mathcal{F}}(u_0, k)}{\partial u^2} \right) a \right)^2,
\]
and
\[
D_{2,U,a}(u_0) \triangleq \sum_{l=-\infty}^{\infty} a'(c_{\mathcal{F}}(u_0, l)[c_{\mathcal{F}}(u_0, l) + c_{\mathcal{F}}(u_0, l + 2k)]')a.
\]

**Proposition S.B.1.** Suppose Assumption S.B.1-S.B.4 hold. For any sequence of bandwidth parameters \(\{b_{2,T}\} \) such that \(b_{2,T} \to 0\), we have
\[
\sup_{\mathcal{F} \in P_U} \text{MSE}_{\mathcal{F}}(a' \hat{c}_{T}(u_0, k) a) = \mathbb{E}_{\mathcal{F}}(a' \hat{c}_{T}(u_0, k) a - a' c_{\mathcal{F}}(u_0, k) a)^2 \leq \frac{1}{4} b_{2,T}^4 \left( \int_0^1 x K_2(x) dx \right)^2 \left( \frac{\partial^2}{\partial u^2} a' c_{\mathcal{F}}(u_0, k) a \right)^2
\]
\[
+ \frac{1}{T b_{2,T}^2} \int_0^1 K_2^2(x) dx \sum_{l=-\infty}^{\infty} a'(c_{\mathcal{F}}(u_0, l)[c_{\mathcal{F}}(u_0, l) + c_{\mathcal{F}}(u_0, l + 2k)]')a
\]
\[
+ \frac{1}{T b_{2,T}^2} \int_0^1 K_2^2(x) dx \sum_{h_1=-\infty}^{\infty} \sum_{h_2=-\infty}^{\infty} \kappa_{\mathcal{F},aV,T_{u_0}}(h_1, 0, h_2) + o \left( b_{2,T}^4 \right) + O \left( 1 / (b_{2,T}^2) \right),
\]
which is minimized for
\[
b_{2,T}^{\text{opt}}(u_0) = \left[ H \left( K_2^{\text{opt}} \right) D_{1,U,a}(u_0) \right]^{-1/5} \left( F \left( K_2^{\text{opt}} \right) (D_{2,U,a}(u_0) + D_{3,U}(u_0)) \right)^{1/5} T^{-1/5},
\]
where
\[
D_{3,U}(u_0) = \sum_{h_1=-\infty}^{\infty} \sum_{h_2=-\infty}^{\infty} \kappa_{\mathcal{F},aV,T_{u_0}}(h_1, 0, h_2),
\]
and \(K_2^{\text{opt}}(x) = 6x(1-x)\), \(0 \leq x \leq 1\). In addition, if \(\{V_t\}\) is Gaussian, then \(D_{3,U}(u_0) = 0\) for all \(u_0 \in (0, 1)\).

We now obtain the optimal \(K_1(\cdot)\) and \(b_{1,T}\) as a function of \(b_{2,T}^{\text{opt}}(\cdot)\) and \(K_2^{\text{opt}}(\cdot)\). For some results below, we consider a subset of \(K_1\) defined by \(\tilde{K}_1 = \{K_1(\cdot) \in K_1 | \tilde{K}(\omega) \geq 0 \forall \omega \in \mathbb{R}\} \) where \(\tilde{K}(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} K_1(x) e^{-ix\omega} dx\). The function \(\tilde{K}(\omega)\) is referred to as the spectral window generator.
corresponding to the kernel $K_1(\cdot)$. The set $\overline{K}_1$ contains all kernels $K_1$ that generate positive semidefinite estimators in finite samples. $\overline{K}_1$ contains the Bartlett, Parzen, and QS kernels, but not the truncated or Tukey-Hanning kernels. We adopt the notation $\hat{J}_T(b_1, T) = \hat{J}_T(b_1, T, K_{2,0})$ for the estimator $\hat{J}_T$ that uses $K_{2,0}(\cdot) \in K_2, b_1, T$ and $b_2, T = \frac{b_{\text{opt}}}{b_{2, T}} + o(T^{-1/5})$ where $b_{2, T} = \int_0^1 b_{\text{opt}}(u) \, du$. Let $\hat{J}_T^{\text{QS}}(b_1, T)$ denote the estimator based on the QS kernel $K_1^{\text{QS}}(\cdot)$. We then compare two kernels $K_1$ using comparable bandwidths $b_1, T$ which are defined as follows. Given $K_1(\cdot) \in \overline{K}_1$, the QS kernel $K_1^{\text{QS}}(\cdot)$, and a bandwidth sequence $\{b_1, T\}$ to be used with the QS kernel, define a comparable bandwidth sequence $\{b_1, T, K_1\}$ for use with $K_1(\cdot)$ such that both kernel/bandwidth combinations have the same maximum asymptotic variance over $\mathcal{P} \in P_U$ when scaled by the same factor $Tb_1, Tb_2, T$. This means that $b_1, T, K_1$ is such that

$$
\lim_{T \to \infty} \sup_{\mathcal{P} \in P_U} T b_1, T b_2, T \text{MSE}_{\mathcal{P}}(a' (\hat{J}_T^{\text{QS}}(b_1, T) - \mathbb{E}(\hat{J}_T^{\text{QS}}(b_1, T))) + J_T)a = \lim_{T \to \infty} \sup_{\mathcal{P} \in P_U} T b_1, T b_2, T \text{MSE}_{\mathcal{P}}(a' (\hat{J}^{\text{QS}}_T(b_1, T) - \mathbb{E}(\hat{J}_T(b_1, T))) + J_T)a.
$$

This definition yields $b_1, T, K_1 = b_1, T/(\int K_1^2(x) \, dx)$. Note that for the QS kernel, $K_1^{\text{QS}}(x)$, we have $b_1, T, \text{QS} = b_1, T$ since $\int K_1^{\text{QS}}(x)^2 \, dx = 1$.

**Theorem S.B.3.** Suppose Assumption S.B.1-S.B.4 hold, $\int_0^1 |f_{U,a}(u, 0)| \, du < \infty$, and $b_2, T \to 0$, $b_{2, T}^2 T \to \eta \in (0, \infty)$. For any bandwidth sequence $\{b_1, T\}$ such that $b_2, T / b_1, T \to 0$, $n_T / T b_1^2, T \to 0$ and $T b_{1, T}^2 b_2, T \to \gamma \in (0, \infty)$, and for any kernel $K_1(\cdot) \in \overline{K}_1$ used to construct $\hat{J}_T$, the QS kernel is preferred to $K_1(\cdot)$ in the sense that for all $a \in \mathbb{R}^p$,

$$
\liminf_{T \to \infty} Tb_1, T b_2, T \left( \sup_{\mathcal{P} \in P_U} \text{MSE}_{\mathcal{P}} \left( a' \hat{J}_T(b_1, T, K_1) a \right) - \sup_{\mathcal{P} \in P_U} \text{MSE}_{\mathcal{P}} \left( a' \hat{J}_T^{\text{QS}}(b_1, T) a \right) \right) = 4 \gamma \pi^2 \left( \int_0^1 f_{U,a}^{(2)}(u, 0) \, du \right)^2 \int_0^1 (K_{2, 0}(x))^2 \, dx \times \left[ K_{1, 2}^2 \left( \int K_{2}^2(y) \, dy \right)^4 - (K_{1, 2}^{\text{QS}})^2 \right] \geq 0.
$$

The inequality is strict if $K_1(x) \neq K_1^{\text{QS}}(x)$ with positive Lebesgue measure.

We now consider the asymptotically optimal choice of $b_1, T$ for a given kernel $K_1(\cdot)$ for which $K_1, q \in (0, \infty)$ for some $q$, and given $K_{2, \text{opt}}^{\text{QS}}$ and $b_{2, T}$. We continue to use a minimax optimality criterion. However, unlike the results of Proposition S.B.1 and Theorem S.B.3, in which an optimal kernel was found that was the same for any dominating distribution $P_{U,2}$ and $P_U$, respectively, the optimal bandwidth $b_1, T$ depends on a scalar parameter $\phi(q)$ that is a function of $\mathcal{P}_U$ in addition to $q$. Let $w_r, r = 1, \ldots, p$ be a set of non-negative weights summing to one. We consider a weighted squared error loss function

$$
L(\hat{J}_T, J_{\mathcal{P}, T}) = \sum_{r=1}^p w_r (\hat{J}_T^{(r,r)}(b_1, T) - J_{\mathcal{P}, T}^{(r,r)})^2.
$$

A common choice is $w_r = 1/p$ for $r = 1, \ldots, p$. For a given dominating distribution $\mathcal{P}_U$, define

$$
\phi(q) = \sum_{r=1}^p w_r \left( \int_0^1 f_{U,a(r)}(u, 0) \, du \right)^2 / \sum_{r=1}^p w_r \left( \int_0^1 f_{U,a(r)}(u, 0) \, du \right)^2,
$$

(S.3)
where \(a^{(r)}\) is a \(p\)-vector with the \(r\)-th element one and all other elements zero. For any given \(\phi(q) \in (0, \infty)\), let \(P_U(\phi)\) denote some set \(P_U\) whose dominating distribution \(P_U\) satisfies (S.3).

**Theorem S.B.4.** Suppose Assumption S.B.1-S.B.4 hold. For any given \(K_1(\cdot) \in K_1\) such that \(0 < K_{1,q} < \infty\) for some \(q \in (0, \infty)\), and any given sequence \(\{b_{1,T}\}\) such that \(b_{2,T}/b_{1,T} \to 0, Tb_{1,T}^{2q+1}b_{2,T} \to \gamma \in (0, \infty)\), the bandwidth defined by

\[
b_{1,T}^{\text{opt}} = \left(2qK_1^{2}\phi(q)Tb_{2,T}^{\text{opt}}\left(\int K_1^2(y)\,dy\int K_2^2(x)\,dx\right)\right)^{-1/(2q+1)},
\]

is optimal in the sense that,

\[
\lim \inf_{T \to \infty} T^{8q/5(2q+1)}\left(\sup_{\mathcal{P} \in P_U(\phi)} \mathbb{E}_{\mathcal{P}} L\left(\hat{J}_T(b_{1,T}), J_{\mathcal{P},T}\right) - \sup_{\mathcal{P} \in P_U(\phi)} \mathbb{E}_{\mathcal{P}} L\left(\hat{J}_T(b_{1,T}^\text{opt}), J_{\mathcal{P},T}\right)\right) \geq 0,
\]

provided \(f_{U,a^{(r)}}(q) > 0 \) and \(f^{(q)}_{U,a^{(r)}} > 0\) for some \(r\) for which \(w_r > 0\). The inequality is strict unless \(b_{1,T} = b_{1,T}^{\text{opt}} + o(T^{-4/5(2q+1)})\).

**S.B.3 Data-dependent DK-HAC Estimation**

We now show that the DK-HAC estimators based on data-dependent bandwidths with similar form as \(\hat{b}_{1,T} \) and \(\hat{b}_{2,T}\) (cf. Section 2) have the same asymptotic MSE properties as the estimators based on optimal fixed bandwidth sequences \(b_{1,T}^{\text{opt}}\) and \(b_{2,T}^{\text{opt}}\) that depend on the unknown distribution \(P\).

We choose a parametric model for \(\{a^{(r)}V_t\}, r = 1, \ldots, p\). We use the same locally stationary AR(1) models as in Section 3, i.e.,

\[
V_t^{(r)} = a_{1}^{(r)} (t/T) V_{t-1}^{(r)} + u_t^{(r)},
\]

with estimated parameters \(\hat{a}_{1}^{(r)}(\cdot)\) and \(\hat{\sigma}^{(r)}(\cdot)\). Let

\[
\hat{\Theta} = \left(\int_0^1 \hat{a}_{1}^{(1)}(u)\,du, \int_0^1 \left(\hat{\sigma}^{(1)}(u)\right)^2\,du, \ldots, \int_0^1 \hat{a}_{1}^{(p)}(u)\,du, \int_0^1 \left(\hat{\sigma}^{(p)}(u)\right)^2\,du\right)',
\]

and \(\theta_{\hat{\Theta}}\) denote the probability limit of \(\hat{\Theta}\). We only consider distributions \(P\) for which \(\theta_{\hat{\Theta}}\) exists. We construct \(\hat{\phi}(q) = \hat{\phi}_{D}(q)\) as in Section 2 but using the estimate \(\hat{\theta}\). The probability limit of \(\hat{\phi}(q)\) is denoted by \(\phi_{\hat{\Theta}}(q)\). Let \(\phi_{\hat{\Theta}}(\cdot)\) be the value of \(\phi(\cdot)\) from (S.3) obtained when \(P_U\) is given by the approximating distribution with parameter \(\theta_{\hat{\Theta}}\). For some \(\underline{\phi}, \overline{\phi}\) such that \(0 < \underline{\phi} \leq \overline{\phi} < \infty\), define

\[
P_{U,3} = \left\{ P \in P_U : (i) \hat{\Theta} \Rightarrow \theta_{\hat{\Theta}} \text{ for some } \theta_{\hat{\Theta}} \in \overline{\Theta} \text{ such that } \phi_{\hat{\Theta}}(q) \in [\underline{\phi}, \overline{\phi}] \text{ for any } q, \right. \]

\[
(ii) \sup_{a \in [0, 1]} |a_T \Gamma_{P_{U},a}(k) a| \leq C_3 |k|^{-l} \text{ for } k = 0, \pm 1, \ldots, \text{ for some } C_3 < \infty, \]

for some \(l > \max\{2, (4q + 2)/(2 + q)\}\), for all \(a \in \mathbb{R}^p\) with \(||a|| = 1\),

\(q\) is as in \(K_3\) and satisfying \(8/q - 20q < 6\), and \(q < 11/2\),

\[
(iii) \sup_{k \geq 1} \sum_{k} \text{Var}_{P_{U}}(a^{(r)} \hat{J}_{k}(k) a) = O(1/Tb_{2,T}^{\text{opt}}), \text{ and}
\]

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PREWHITENED LRV ESTIMATION ROBUST TO NONSTATIONARITY

\( \left( \frac{1}{S_{\mathcal{P}, T}} \sum_{k=1}^{S_{\mathcal{P}, T}} \sqrt{T b_{2, T}^{opt}} \left| a' \hat{\Gamma} (k) a - a' T_{\mathcal{P}, T} (k) a \right| \right)^4 \leq C_4 \)

for some \( C_4 \leq \infty \) with \( S_{\mathcal{P}, T} = [ (b_{1, T}^{opt})^{-1} ] \) some \( r \in S (q, b, l) \),

where

\[ S (q, b, l) = \max \{ (b - 3/4 - q/2)/(b - 1), q/(l - 1) \} \]

\[ \min \{ (6 + 4q)/8, 15/16 + 3q/8 \} , \]

with \( b > 1 + 1/q \). The lower bound \( 0 < \phi \leq \phi_{\mathcal{P}} (q) \) eliminates any distribution for which \( \phi_{\mathcal{P}} (\cdot) = 0 \). For example, white noise sequences do not belong to \( \mathcal{P}_{U, 3} \) since then \( \phi (q) = 0 \). We discuss these cases at the end of the section. Let

\[ b_{1, \theta_{\mathcal{P}}, T} = \left[ 2 q K_{2, q}^{1, \theta_{\mathcal{P}}} (q) T b_{2, T}^{opt} / \int K_{1}^{2} (y) dy \int_{0}^{1} K_{2}^{2} (x) dx \right]^{-1/(2q+1)} \]

denote the optimal bandwidth for the case in which \( \mathcal{P}_{U} \) equals the approximating parametric model with parameter \( \theta_{\mathcal{P}} \). Let

\[ \hat{D}_{2, a} (u) \triangleq 2 \sum_{t=-[T^{4/25}]}^{[T^{4/25}]} a' \hat{c}_{T} (u_{0}, l) \hat{c}_{T} (u_{0}, l)' a, \]

where \( \hat{c}_{T} \) is defined as \( \hat{c}_{D, T} \) with \( \hat{V}_{i} \) and \( \hat{b}_{2, T} \) in place of \( \hat{V}_{D, i}^{*} \) and \( \hat{b}_{2, T}^{*} \), respectively,

\[ \hat{c}_{T} (rn_{T}/T, k) \triangleq \begin{cases} \left( T b_{2, T} \right)^{-1} \sum_{s=0}^{T} K_{2}^{*} \left( ((r+1)n_{T}-(s-k)/T) b_{2, T} \right) \hat{V}_{s} \hat{V}'_{s-k}, & k \geq 0 \\ \left( T b_{2, T} \right)^{-1} \sum_{s=-k}^{T} K_{2}^{*} \left( ((r+1)n_{T}-(s-k)/T) b_{2, T} \right) \hat{V}_{s+k} \hat{V}'_{s}, & k < 0 \end{cases}. \]

**Assumption S.B.5.** (i) We have \( \sup_{\mathcal{P} \in \mathcal{P}_{U, 3}} \mathbb{E} \left[ \inf \{ T/n_{3, T}, \sqrt{n_{2, T}/n_{3, T}} \} \left( \hat{\phi} (q)^{1/(2q+1)} - \phi_{\mathcal{P}}^{1/(2q+1)} \right) \right] \leq C_1 \) as \( T \to \infty \),

where \( q \) is as defined in \( K_{3} \), \( \hat{\phi} (q) \leq \bar{\phi} < \infty \), and \( n_{2, T}/T + n_{3, T}/T \to 0, n_{2, T}^{10/6}/T \to [c_{2}, \infty), n_{3, T}^{10/6}/T \to [c_{3}, \infty) \) with \( 0 < c_{2}, c_{3} < \infty \); (ii) \( \sqrt{T b_{2, T} (u)} (\hat{D}_{2, a} (u) - D_{2, U, a} (u)) = O_{\mathcal{P}} (1) \) for all \( u \in [0, 1] \); (iii) Assumption 3.5-(v, vii) hold.

Any estimator \( \hat{\phi} \) based on standard nonparametric estimators of \( \hat{a}_{1}^{(r)} (\cdot) \) and \( \hat{\sigma}^{(r)} (\cdot) \) satisfies Assumption S.B.5-(i). Assumption S.B.5-(ii) extends Assumption 3.5-(vi) to the distribution \( \mathcal{P} \). The following result shows that \( \hat{J}_{T} (b_{1, T}, \hat{b}_{2, T}) \) has the same asymptotic MSE properties under \( \mathcal{P} \) as the estimator \( \hat{J}_{T} (b_{1, \theta_{\mathcal{P}}, T}, \hat{b}_{2, T}^{opt}) \). Since the asymptotic MSE properties of the estimators with fixed bandwidth parameters have been determined in Section S.B.2, from this result follows the consistency of \( \hat{J}_{T} (b_{1, T}, \hat{b}_{2, T}) \) and its asymptotic optimality properties.
Theorem S.B.5. Consider any kernel $K_1 (\cdot) \in K_3$, $q$ as in $K_3$ and any $K_2 (\cdot) \in K_2$. Suppose Assumption S.B.1-S.B.5 hold. Then, for all $a \in \mathbb{R}^p$,

$$T^{8q/(2q+1)} \sup_{\varphi \in P_{U,3}} \left| \text{MSE}_{\varphi} (a' \hat{J}_T (b_{1,T}, \hat{b}_{2,T}) a) - \text{MSE}_{\varphi} (a' \hat{J}_T (b_{1,\theta \varphi}, \hat{b}_{2,T}) a) \right| \to 0.$$ 

Theorem S.B.5 combined with Theorem S.B.1 and Theorem S.B.2-(iii) establish upper and lower bounds on the asymptotic MSE. Results on asymptotic minimax optimality for data-dependent bandwidths parameters can be obtained using Theorem S.B.1, Theorem S.B.2-(iii) and Theorem S.B.4-S.B.5.

It remains to consider the case $\phi \neq 0$. When this occurs, $\hat{\phi}^{-1} (\cdot)$ is $O_{\varphi}((T/n_3,T)^2 + n_2,T)$. Under the additional condition $((T/n_3,T)^2 + n_2,T)/T^{4/5} \to O(1)$ in Assumption S.B.5-(i) we have $b_{1,T} = O_{\varphi} (1)$.

Thus, $\hat{J}_T (b_{1,T}, \hat{b}_{2,T}) - J_T \xrightarrow{p} 0$ also when the series is white noise. This is important in applied work because often researchers use robust standard errors even when they are not aware of whether any dependence is present at all.

S.C Proofs of the Results in Section 3

In some of the proofs below $\hat{\beta}$ is understood to be on the line segment joining $\hat{\beta}$ and $\beta_0$. We discard the degrees of freedom adjustment $T/(T-p)$ from the derivations since asymptotically it does not play any role. Similarly, we use $T/n_T$ in place of $(T-n_T)/n_T$ in the expression for $\hat{\Gamma}_D^k (k)$ and $\hat{\Gamma} (k)$. We collect the break dates in $\mathcal{T} \triangleq \{T_0^0, \ldots, T_{m_0}^0\}$.

S.C.1 Proof of Theorem 3.1

Let

$$\hat{J}_T^* = \hat{J}_T (b_{01,T}, b_{02,T}) \triangleq \sum_{k=-T+1}^{T-1} K_1 (b_{01,T} k) \hat{\Gamma}_k^*(k),$$

where $\hat{\Gamma}_k^* (k) \triangleq (n_T/T) \sum_{r=0}^{[T/n_T]} \hat{c}_T (rn_T/T, k)$ and

$$\hat{c}_T (rn_T/T, k) \triangleq \begin{cases} (Tb_{2,T})^{-1} \sum_{s=k+1}^{T} K_2^* \left( \frac{((r+1)n_T-(s-k/2))/T}{b_{02,T}} \right) \hat{V}_s \hat{V}_s' \hat{V}_{s-k} & k \geq 0 \\ (Tb_{2,T})^{-1} \sum_{s=-k+1}^{T} K_2^* \left( \frac{((r+1)n_T-(s+k/2))/T}{b_{02,T}} \right) \hat{V}_s \hat{V}_s' \hat{V}_{s+k} & k < 0 \end{cases} \quad (S.1)$$

with $\hat{V}_s^* = V_s^*(\hat{\beta})$ where $\hat{\beta}$ is elongated to include $\hat{A}_{j,k} (j = 1, \ldots, p_A)$. Define $\hat{J}_T^*$ as equal to $\hat{J}_T^*$ but with $V_{i,t}^* = V_i - \sum_{j=1}^{p_A} A_{r,j} V_{t-j}$ in place of $\hat{V}_s^*$ and define $J_{T,D}^*$ as equal to $J_T^*$ but with $V_{i,t}^*$ in place of $V_i$. The proof uses the following decomposition,

$$\hat{J}_{pw,T} - J_T = (\hat{J}_{pw,T} - J_{T,D}^*) + (J_{T,D}^* - J_{T,D}) + (J_{T,D} - J_T), \quad (S.2)$$

where $J_{T,D} = T^{-1} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} D_s E(V_s V_t') D_t'$, and $J_{T,D}^*$ is equal to $J_{T,D}$ but with $\hat{D}_s$ in place of $D_s$. 

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Lemma S.C.1. Under the assumptions of Theorem 3.1-(i), we have
\[ \hat{J}_T^* (b_{\theta_1,T}, b_{\theta_2,T}) - J_T^* = o_P (1). \] (S.3)

Proof. Under Assumption 3.1, \(|\int_0^1 f^{(0)}(u,0)|| < \infty\), where \(f^*\) is defined analogously to \(f^D\) but with \(D_s = 1\) for all \(s\). In view of \(K_{1,0} = 0\), Theorem 3.1-(i,ii) in Casini (2021b) [with \(q = 0\) in part (ii)] implies \(\hat{J}_T^* - J_T^* = o_P (1)\). Note that the assumptions of the aforementioned theorem are satisfied by \(\{V_t^*\}\) since they correspond to Assumption 2.1-3.1 here. Noting that \(\hat{J}_T^* - J_T^* = o_P (1)\) if and only if \(a'\hat{J}_T^*a - a'J_T^*a = o_P (1)\) for arbitrary \(a \in \mathbb{R}^p\). We shall provide the proof only for the scalar case. We show that \(\sqrt{n_T} b_{\theta_1,T} (J_T^* - J_T^*) = O_P (1)\). Let \(\hat{J}_T^* (\beta)\) denote the estimator that uses \(\{V_t^* (\beta)\}\) where \(\beta\) is elongated to include \(A_{-j} (j = 1, \ldots, p_A)\). A mean-value expansion of \(\hat{J}_T^* (\beta) = \hat{J}_T^*\) about \(\beta_0\) (elongated to include \(A_{-j} (j = 1, \ldots, p_A)\)) yields
\[ \sqrt{n_T} b_{\theta_1,T} (J_T^* - J_T^*) = b_{\theta_1,T} \frac{\partial}{\partial \beta'} \hat{J}_T^* (\beta) \sqrt{n_T} (\hat{\beta} - \beta_0) \]
\[ = b_{\theta_1,T} \sum_{k=-T+1}^{T-1} K_1 (b_{\theta_1,T} k) \frac{\partial}{\partial \beta'} \hat{\beta}^* (k) |_{\beta = \hat{\beta}} \sqrt{n_T} (\hat{\beta} - \beta_0), \] (S.4)
for some \(\hat{\beta}\) on the line segment joining \(\hat{\beta}\) and \(\beta_0\). Note also that \(\hat{\beta}^* (rn_T/T, k)\) depends on \(\beta\) although we have omitted it. We have for \(k \geq 0\) (the case \(k < 0\) is similar and omitted),
\[ \left\| \frac{\partial}{\partial \beta'} \hat{\beta}^* (rn_T/T, k) \right\|_{\beta = \hat{\beta}} \]
\[ = \left\| (Tb_{\theta_2,T})^{-1} \sum_{s=k+1}^{T} K_2^* (\frac{(r+1)n_T - (s+k/2)}{Tb_{\theta_2,T}}) \right. \]
\[ \times \left( V^*_s (\beta) \frac{\partial}{\partial \beta'} V^*_{s-k} (\beta) + \frac{\partial}{\partial \beta'} V^*_s (\beta) V^*_{s-k} (\beta) \right) \right\|_{\beta = \hat{\beta}} \]
\[ \leq 2 \left( (Tb_{\theta_2,T})^{-1} \sum_{s=k+1}^{T} K_2^* (\frac{(r+1)n_T - (s+k/2)}{Tb_{\theta_2,T}}) \right)^{1/2} \sup_{s \geq 1} \sup_{\beta} \left( V^*_s (\beta) \right)^{1/2} \]
\[ \times \left( (Tb_{\theta_2,T})^{-1} \sum_{s=k+1}^{T} K_2^* (\frac{(r+1)n_T - (s+k/2)}{Tb_{\theta_2,T}}) \right)^{1/2} \sup_{s \geq 1} \sup_{\beta} \left\| \frac{\partial}{\partial \beta'} V^*_s (\beta) \right\|^2 \]
\[ = O_P (1), \] (S.5)
where we have used the boundedness of the kernel \(K_2 (\text{and thus of } K_2^*)\), Assumption 3.2-(ii,iii) and Markov’s inequality to each term in parentheses; also \(\sup_{s \geq 1} \mathbb{E} \sup_{\beta} \left\| V^*_s (\beta) \right\|^2 < \infty\) under Assumption 3.2-(ii,iii) by a mean-value expansion and,
\[ (Tb_{\theta_2,T})^{-1} \sum_{s=k+1}^{T} K_2^* (\frac{(r+1)n_T - (s+k/2)}{Tb_{\theta_2,T}}) \rightarrow \int_0^1 K_2^2 (x) dx < \infty. \] (S.6)
Then, (S.4) is such that

\[ b_{\theta_1,T} \sum_{k=T+1}^{T-1} K_1 (b_{\theta_1,T}, k) \frac{\partial}{\partial \beta^r} \hat{\beta}^* (k) \mid_{\beta = \beta} \sqrt{n_T} (\beta - \beta_0) \]

\[ = b_{\theta_1,T} \sum_{k=-T+1}^{T-1} K_1 (b_{\theta_1,T} k) \frac{n_T}{T} \sum_{r=0}^{T/n_T} O_\varphi (1) O_\varphi (1) \]

\[ = O_\varphi (1), \]

where the last equality uses \( b_{\theta_1,T} \sum_{k=-T+1}^{T-1} |K_1 (b_{\theta_1,T} k)| \to \int |K_1 (x)| dx < \infty \). This concludes the proof of the lemma because \( \sqrt{n_T} b_{\theta_1,T} \to \infty \) by assumption. □

**Lemma S.C.2.** Under the assumptions of Theorem 3.1-(i), we have

\[ \hat{J}^*_T \left( b_{\theta_1,T}, b_{\theta_2,T} \right) - \hat{J}^*_T \left( \hat{b}^*_{1,T}, \hat{b}^*_{2,T} \right) = o_\varphi (1). \]  

(S.7)

**Proof.** Let \( S_T = \left[ b_{\theta_1,T}^{-r} \right] \) and

\[ r \in ( \max \left\{ (12b - 10q - 5) / 12 (b - 1), (b - 1 / 2 - q) / (b - 1), q / (l - 1) \right\} \]

\[ \min \left\{ (10q + 17) / 24, (3 + 2q) / 4, 5q / 6 + 5 / 12, 1 \right\} \).

We will use the following decomposition,

\[ \hat{J}^*_T \left( \hat{b}^*_{1,T}, \hat{b}^*_{2,T} \right) - \hat{J}^*_T \left( b_{\theta_1,T}, b_{\theta_2,T} \right) = \left( \hat{J}^*_T \left( \hat{b}^*_{1,T}, \hat{b}^*_{2,T} \right) - \hat{J}^*_T \left( b_{\theta_1,T}, \hat{b}^*_{2,T} \right) \right) \]

\[ + \left( \hat{J}^*_T \left( b_{\theta_1,T}, \hat{b}^*_{2,T} \right) - \hat{J}^*_T \left( b_{\theta_1,T}, b_{\theta_2,T} \right) \right) \]  

(S.8)

Let \( N_1 \equiv \{-S_T, -S_T + 1, \ldots, -1, 1, \ldots, S_T - 1, S_T\} \), and \( N_2 \equiv \{-T + 1, \ldots, -S_T - 1, S_T + 1, \ldots, T - 1\} \). Let us consider the first term above,

\[ \hat{J}^*_T \left( \hat{b}^*_{1,T}, \hat{b}^*_{2,T} \right) - \hat{J}^*_T \left( b_{\theta_1,T}, \hat{b}^*_{2,T} \right) \]

\[ = \sum_{k \in N_1} \left( K_1 \left( \hat{b}^*_{1,T} k \right) - K_1 \left( b_{\theta_1,T} k \right) \right) \hat{\Gamma}^* (k) \]

\[ + \sum_{k \in N_2} K_1 \left( \hat{b}^*_{1,T} k \right) \hat{\Gamma}^* (k) - \sum_{k \in N_2} K_1 \left( b_{\theta_1,T} k \right) \hat{\Gamma}^* (k) \]

\[ \triangleq A_{1,T} + A_{2,T} - A_{3,T}. \]

We first show that \( A_{1,T} \xrightarrow{P} 0 \). Let \( A_{1,1,T} \) denote \( A_{1,T} \) with the summation restricted over positive integers \( k \). Let \( \tilde{n}_T = \inf \left\{ T / n_3, T, \sqrt{n_2 T} \right\} \). We can use the Liptchitz condition on \( K_1 (\cdot) \in K_3 \) to yield,

\[ |A_{1,1,T}| \leq \sum_{k=1}^{S_T} C_2 \left| \hat{b}^*_{1,T} - b_{\theta_1,T} \right| k \left| \hat{\Gamma}^* (k) \right| \]  

(S.10)
\begin{equation}
\leq C \left| \hat{\phi}_D (q)^{1/(2q+1)} - \phi_{\beta^*}^{1/(2q+1)} \right| \left( \hat{\phi}_D (q, \phi_{\beta^*})^{-1/(2q+1)} \left( T_b^*_{b, T} \right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left| \hat{\Gamma}^* (k) \right| \right),
\end{equation}
for some $C < \infty$. By Assumption 3.5-(i),
\begin{equation}
\left| \hat{\phi}_D (q)^{1/(2q+1)} - \phi_{\beta^*}^{1/(2q+1)} \right| \left( \hat{\phi}_D (q, \phi_{\beta^*})^{-1/(2q+1)} = O_p (1). \right.
\end{equation}

Using the delta method, it suffices to show that $B_{1,T} + B_{2,T} + B_{3,T} \xrightarrow{p} 0$, where
\begin{equation}
B_{1,T} = \left( T_{b_{b, T}}^* \right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left| \hat{\Gamma}^* (k) - \Gamma^*_T (k) \right|
\end{equation}
\begin{equation}
B_{2,T} = \left( T_{b_{b, T}}^* \right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left| \hat{\Gamma}^* (k) - \Gamma^*_T (k) \right|
\end{equation}
\begin{equation}
B_{3,T} = \left( T_{b_{b, T}}^* \right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left| \Gamma^*_T (k) \right|
\end{equation}
with $\Gamma^*_T (k) \triangleq \left( n_T / T \right) \sum_{r=0}^{[T/n_T]} \hat{c}^* (rn_T / T, k)$. By a mean-value expansion, we have
\begin{equation}
B_{1,T} \leq \left( T_{b_{b, T}}^* \right)^{-1/(2q+1)} n_T^{-1/2} \sum_{k=1}^{S_T} k \left| \partial_{\beta} \hat{\Gamma}^* (k) \right| \sqrt{n_T} \left| \hat{\beta} - \beta_0 \right|
\end{equation}
\begin{equation}
\leq C \left( T_{b_{b, T}}^* \right)^{-1/(2q+1)} \left( T_{b_{b, T}}^* \right)^{2r/(2q+1)} n_T^{-1/2} \sup_{k \geq 1} \left| \partial_{\beta} \hat{\Gamma}^* (k) \right| \sqrt{n_T} \left| \hat{\beta} - \beta_0 \right|
\end{equation}
since $r < (10q + 17) / 24$, and $\sup_{k \geq 1} \left| \partial_{\beta} \hat{\Gamma}^* (k) \right| = O_p (1)$ using (S.5) and Assumption 3.2-(ii,iii) (the latter continues to hold for \{V^*_t \}). In addition,
\begin{equation}
\mathbb{E} \left( B^2_{2,T} \right) \leq \mathbb{E} \left( \left( T_{b_{b, T}}^* \right)^{-2/(2q+1)} \sum_{k=1}^{S_T} \sum_{j=1}^{S_T} k j \left| \hat{\Gamma}^* (k) - \Gamma^*_T (k) \right| \left| \hat{\Gamma}^* (j) - \Gamma^*_T (j) \right| \right)
\end{equation}
\begin{equation}
\leq \left( T_{b_{b, T}}^* \right)^{-2/(2q+1)} \sum_{k=1}^{S_T} \sup_{k \geq 1} T_{b_{b, T}}^* \text{Var} \left( \hat{\Gamma}^* (k) \right)
\end{equation}
\begin{equation}
\leq \left( T_{b_{b, T}}^* \right)^{-2/(2q+1)} \left( T_{b_{b, T}}^* \right)^{4r/(2q+1)} \sup_{k \geq 1} T_{b_{b, T}}^* \text{Var} \left( \hat{\Gamma}^* (k) \right)
\end{equation}
\begin{equation}
\leq \left( b_{b, T}^* \right)^{-2/(2q+1)} \left( T_{b_{b, T}}^* \right)^{-1/2} \left( T_{b_{b, T}}^* \right)^{16r/(2q+1)} \sup_{k \geq 1} T_{b_{b, T}}^* \text{Var} \left( \hat{\Gamma}^* (k) \right) \to 0,
\end{equation}
given that $r < (3 + 2q) / 4$ and $\sup_{k \geq 1} T_{b_{b, T}}^* \text{Var} \left( \hat{\Gamma}^* (k) \right) = O (1)$ by Lemma S.A.5 in Casini (2021b) that also holds with $\hat{\Gamma}^* (k)$ in place of $\Gamma (k)$. Next,
\begin{equation}
B_{3,T} \leq \left( T_{b_{b, T}}^* \right)^{-1/(2q+1)} S_T \sum_{k=1}^{\infty} \left| \Gamma^*_T (k) \right|
\end{equation}
\[ \left( T \hat{b}_{2,T}^s \right)^{(r-1)/(2q+1)} O_p(1) \to 0, \]

using Assumption 3.1-(i) since \( r < 1 \). This gives \( A_{1,T} \xrightarrow{p} 0 \). Next, we show that \( A_{2,T} \xrightarrow{p} 0 \). Let \( A_{2,1,T} = L_{1,T} + L_{2,T} + L_{3,T} \), where

\[
L_{1,T} = \sum_{k=S_T+1}^{T-1} K_1 \left( \hat{b}_{1,T}^s k \right) \left( \hat{\Gamma}^*(k) - \tilde{\Gamma}^*(k) \right), \\
L_{2,T} = \sum_{k=S_T+1}^{T-1} K_1 \left( \hat{b}_{1,T}^s k \right) \left( \hat{\Gamma}^*(k) - \Gamma_T^*(k) \right), \quad \text{and} \\
L_{3,T} = \sum_{k=S_T+1}^{T-1} K_1 \left( \hat{b}_{1,T}^s k \right) \Gamma_T^*(k).
\]

We apply a mean-value expansion and use \( \sqrt{n_T} (\hat{\beta} - \beta_0) = O_p(1) \) as well as (S.5) to obtain

\[
|L_{1,T}| = n_T^{-1/2} \sum_{k=S_T+1}^{T-1} C_1 \left( \hat{b}_{1,T}^s k \right)^{-b} \left| \left( \frac{\partial}{\partial \beta'} \hat{\Gamma}^*(k) \right) \bigg|_{\beta = \beta_0} \sqrt{n_T} \left( \hat{\beta} - \beta_0 \right) \right| \\
= T^{-1/3+4b/5(2q+1)} \sum_{k=S_T+1}^{T-1} C_1 k^{-b} \left| \left( \frac{\partial}{\partial \beta'} \hat{\Gamma}^*(k) \right) \bigg|_{\beta = \beta_0} \sqrt{n_T} \left( \hat{\beta} - \beta_0 \right) \right| \\
= T^{-1/3+4b/5(2q+1)+4r(1-b)/5(2q+1)} \left( \frac{\partial}{\partial \beta'} \hat{\Gamma}^*(k) \right) \bigg|_{\beta = \beta_0} \sqrt{n_T} \left( \hat{\beta} - \beta_0 \right) \right| \\
= T^{-1/3+4b/5(2q+1)+4r(1-b)/5(2q+1)} O(1) O_p(1),
\]

which converges to zero since \( r > (12b - 10q - 5)/(b - 1) \). Next,

\[
|L_{2,T}| = \sum_{k=S_T+1}^{T-1} C_1 \left( \hat{b}_{1,T}^s k \right)^{-b} \left| \hat{\Gamma}^*(k) - \Gamma_T^*(k) \right| \\
= C_1 \left( qK_{1,q} \phi_D(q) \right)^{b/(2q+1)} T^{b/(2q+1)-1/2} \left( \hat{b}_{2,T}^s \right)^{b/(2q+1)-1/2} \left( \sum_{k=S_T+1}^{T-1} k^{-b} \right) \sqrt{T \hat{b}_{2,T}^s} \left| \hat{\Gamma}^*(k) - \Gamma_T^*(k) \right|.
\]

Note that,

\[
\mathbb{E} \left( T^{b/(2q+1)-1/2} \left( \hat{b}_{2,T}^s \right)^{b/(2q+1)-1/2} \sum_{k=S_T}^{T-1} k^{-b} \sqrt{T \hat{b}_{2,T}^s} \left| \hat{\Gamma}^*(k) - \Gamma_T^*(k) \right| \right)^2 \leq T^{2b/(2q+1)-1} \left( \hat{b}_{2,T}^s \right)^{2b/(2q+1)-1} \left( \sum_{k=S_T}^{T-1} k^{-b} \right)^2 \mathcal{O}(1) \\
= T^{2b/(2q+1)-1} \left( \hat{b}_{2,T}^s \right)^{2b/(2q+1)-1} S_T^{2(1-b)} \mathcal{O}(1) \to 0,
\]

since \( r > (b - 1/2 - q)/(b - 1) \) and \( T \hat{b}_{2,T}^s \text{Var}(\hat{\Gamma}^*(k)) = \mathcal{O}(1) \), as above. Equations (S.17)-(S.18) combine
to yield $L_{2,T} \xrightarrow{p} 0$, since $\hat{\phi}_D (q) = O_P (1)$ by Assumption 3.5-(i). Let us turn to $L_{3,T}$. We have,

$$
\left | \sum_{k=S+1}^{T-1} K_1 \left ( \hat{b}_{1,T}^* k \right ) \Gamma^*_T (k) \right | \leq \sum_{k=S+1}^{T-1} \frac{n_T}{T} \sum_{r=0}^{T/n_T} |c^* (rn_T/T, k)| \tag{S.19}
$$

Equations (S.16)-(S.19) imply $A_{2,T} \xrightarrow{p} 0$. An analogous argument yields $A_{3,T} \xrightarrow{p} 0$. It remains to show that $(\hat{J}_T (b_{\theta_1,T}, \hat{b}_{2,T}^*) - \hat{J}_T (b_{\theta_1,T}, \tilde{b}_{2,T})) \xrightarrow{p} 0$. Its proof is the same as in Theorem 5.1-(i) in Casini (2021b) which can be repeated given the conditions $n_T^{-1/2}/(\hat{b}_{1,T}^*) \to 0$, $r < 5q/6+5/12$, and $r > (b-1/2-q)/(b-1)$.

**Lemma S.C.3.** Under the assumptions of Theorem 3.1-(ii), we have

$$
\sqrt{T b_{\theta_1,T} b_{\theta_2,T} \left ( \hat{J}_T^* (b_{\theta_1,T}, b_{\theta_2,T}) - J_T^* \right )} = O_P (1).
$$

**Proof.** Write

$$
\sqrt{T b_{\theta_1,T} b_{\theta_2,T} \left ( \hat{J}_T^* (b_{\theta_1,T}, b_{\theta_2,T}) - J_T^* \right )} = \sqrt{T b_{\theta_1,T} b_{\theta_2,T} \left ( \hat{J}_T^* (b_{\theta_1,T}, b_{\theta_2,T}) - \tilde{J}_T^* + \tilde{J}_T^* - J_T^* \right ).}
$$

Applying Theorem 3.1-(ii) in Casini (2021b) with $V^*_s$ in place of $V_s$, we have $\sqrt{T b_{\theta_1,T} b_{\theta_2,T} (\hat{J}_T^* - J_T^*)} = O_P (1)$. Thus, it is sufficient to show $\sqrt{T b_{\theta_1,T} b_{\theta_2,T} (\tilde{J}_T^* (b_{\theta_1,T}, b_{\theta_2,T}) - \tilde{J}_T^*)} = o_P (1)$. A second-order Taylor expansion gives

$$
\sqrt{T b_{\theta_1,T} b_{\theta_2,T} (\tilde{J}_T^* - J_T^*)} = \left [ \frac{\sqrt{T b_{\theta_2,T}}}{\sqrt{n_T}} \sqrt{b_{\theta_1,T}} \frac{\partial}{\partial \beta'} \tilde{J}_T^* (\beta_0) \right ] \sqrt{n_T} (\tilde{\beta} - \beta_0)
$$

$$
+ \frac{1}{2} \sqrt{n_T} (\tilde{\beta} - \beta_0)' \left [ \frac{\sqrt{T b_{\theta_2,T}}}{\sqrt{n_T}} \sqrt{b_{\theta_1,T}} \frac{\partial^2}{\partial \beta \partial \beta'} \tilde{J}_T^* (\beta) \right ] \sqrt{n_T} (\tilde{\beta} - \beta_0)
$$

$$
\triangleq G_T' \sqrt{n_T} (\tilde{\beta} - \beta_0) + \frac{1}{2} \sqrt{n_T} (\tilde{\beta} - \beta_0)' H_T \sqrt{n_T} (\tilde{\beta} - \beta_0).
$$

Using Assumption 3.3-(ii),

$$
\left | \left [ \frac{\partial^2}{\partial \beta \partial \beta'} \hat{c}^* (rn_T/T, k) \right ]_{\beta = \bar{\beta}} \right |
$$

$$
= \left | (T b_{\theta_2,T})^{-1} \sum_{s=k+1}^{T} K_2^* \left ( \frac{(r+1)n_T - (s+k)/2}{b_{\theta_2,T}} \right ) \left ( \frac{\partial^2}{\partial \beta \partial \beta'} V^*_s (\beta) V^*_{s-k} (\beta) \right ) \right |_{\beta = \bar{\beta}}
$$

$$
= O_P (1),
$$

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and thus,
\[ \|H_T\| \leq \left( \frac{T b_{\theta_2, T} b_{\theta_1, T}}{n_T^2} \right)^{1/2} \sum_{k=-T+1}^{T-1} |K_1(b_{\theta_1, T} k)| \sup_{\beta \in \Theta} \left| \frac{\partial^2}{\partial \beta \partial \beta'} \hat{G}^* (k) \right| \]
\[ \leq \left( \frac{T b_{\theta_2, T} b_{\theta_1, T}}{n_T^2} \right)^{1/2} \sum_{k=-T+1}^{T-1} |K_1(b_{\theta_1, T} k)| O_p(1) \]
\[ \leq \left( \frac{T b_{\theta_2, T}}{n_T^2} \right)^{1/2} b_{\theta_1, T} \sum_{k=-T+1}^{T-1} |K_1(b_{\theta_1, T} k)| O_p(1) = o_p(1), \]

since \( T b_{\theta_2, T} (n_T^2 b_{\theta_1, T}) \to 0 \). Next, we want to show that \( G_T = o_p(1) \). Following Andrews (1991) (cf. the last paragraph of p. 852), we apply the results of Theorem 3.1-(i,ii) in Casini (2021b) to \( \hat{J}_T^* \) where the latter is constructed using \( (V_t^*, \partial V_t^*/\partial \beta' - \mathbb{E}(\partial V_t^*/\partial \beta'))' \) rather than just with \( V_t^* \). The first row and column of the off-diagonal elements of this \( \hat{J}_T^* \) (written as column vectors) are now

\[ A_1 \triangleq \sum_{k=-T+1}^{T-1} K_1(b_{\theta_1, T} k) \frac{n_T^2}{T} \sum_{r=0}^{T/n_T} 1 T b_{\theta_2, T} \]
\[ \times \sum_{s=k+1}^{T} K_2^* \left( \frac{(r+1)n_T - (s+k/2)/T}{b_{\theta_2, T}} \right) V_s^* \left( \frac{\partial}{\partial \beta} V_{s-k}^* - \mathbb{E} \left( \frac{\partial}{\partial \beta} V_s^* \right) \right) \]
\[ A_2 \triangleq \sum_{k=-T+1}^{T-1} K_1(b_{\theta_1, T} k) \frac{n_T^2}{T} \sum_{r=0}^{T/n_T} 1 T b_{\theta_2, T} \]
\[ \times \sum_{s=k+1}^{T} K_2^* \left( \frac{(r+1)n_T - (s+k/2)/T}{b_{\theta_2, T}} \right) \left( \frac{\partial}{\partial \beta} V_s^* - \mathbb{E} \left( \frac{\partial}{\partial \beta} V_s^* \right) \right) V_{s-k}. \]

By Theorem 3.1-(i,ii) in Casini (2021b) each expression above is \( O_p(1) \). Given,

\[ G_T \leq \sqrt{\frac{T b_{\theta_2, T}}{n_T}} \sqrt{b_{\theta_1, T}} \left( A_1 + A_2 \right) + \sqrt{\frac{T b_{\theta_2, T}}{n_T}} \sqrt{b_{\theta_1, T}} \sum_{k=-T+1}^{T-1} K_1(b_{\theta_1, T} k) n_T^2 \sum_{r=0}^{T/n_T} 1 T b_{\theta_2, T} \]
\[ \times \sum_{s=k+1}^{T} K_2^* \left( \frac{(r+1)n_T - (s+k/2)/T}{b_{\theta_2, T}} \right) \left| (V_s^* + V_{s-k}^*) \mathbb{E} \left( \frac{\partial}{\partial \beta} V_s^* \right) \right| \]
\[ \triangleq \sqrt{\frac{T b_{\theta_2, T}}{n_T}} \sqrt{b_{\theta_1, T}} \left( A_1 + A_2 \right) + A_3 \sup_s \left| \mathbb{E} \left( \frac{\partial}{\partial \beta} V_s^* \right) \right|, \]

and the fact that \( T b_{\theta_2, T} b_{\theta_1, T}/n_T \to 0 \) it remains to show that \( A_3 = o_p(1) \). Note that

\[ \mathbb{E} \left( A_3^2 \right) \leq \frac{T b_{\theta_2, T} b_{\theta_1, T}}{n_T} \sum_{k=-T+1}^{T-1} \sum_{j=-T+1}^{T-1} |K_1(b_{\theta_1, T} k) K_1(b_{\theta_1, T} j)| \frac{\left( nt \right)^2}{T} \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T} \frac{1}{T b_{\theta_2, T}} \frac{1}{T b_{\theta_2, T}} \sum_{s=1}^{T} \sum_{l=1}^{T} K_2^* \left( \frac{(r+1)n_T - (s+k/2)/T}{b_{\theta_2, T}} \right), \]

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Let us consider the first term above, and that \( \mathbb{E}(V_s^* V_t^*) = c^*(u, h) + O(T^{-1}) \) uniformly in \( h = s - l \) and \( u = s/T \) by Lemma S.A.1 in Casini (2021b). Since \( \sum_{h=-\infty}^{\infty} \sup_{u \in [0, 1]} |c^*(u, h)| < \infty \),

\[
\mathbb{E}\left(A_2^2\right) \leq \frac{1}{n_T b_{\theta_1, T}} \left( b_{\theta_1, T} \sum_{k=-T+1}^{T-1} |K_1(b_{\theta_1, T} k)| \right)^2 \int_0^1 K_2^2(x) \, dx \int_0^1 \sum_{h=-\infty}^{\infty} |c^*(u, h)| \, du = o(1).
\]

This implies \( G_T = o_{\mathbb{P}}(1) \) which concludes the proof.

\[\square\]

**Lemma S.C.4.** Under the assumptions of Theorem 3.1-(ii), we have

\[
\sqrt{Tb_{\theta_1, T} b_{\theta_2, T}} \left( \hat{J}_T^* (\hat{b}_{1, T}, \hat{b}_{2, T}) - \hat{J}_T^* (b_{\theta_1, T}, b_{\theta_2, T}) \right) = o_{\mathbb{P}}(1).
\]

**Proof.** Let

\[
r \in (\max\{\{(−10 + 4q + 24b) / 24 (b − 1)\}, \{(b − 1)/2 \, / (b − 1)\} \text{ for } b > \max(1 + 1/q, 4)\},
\]

\[
\{(8q − 4) / (b − 1) (10q + 5)\}, \{(b − 2)/3 − q/3 \, / (b − 1)\} \text{ for } b > 1 + 1/q\}, q / (l − 1)\},
\]

\[
\min\{16q/48 + 44/48, 46/48 + 20q/48, 2/3 + q/3\},
\]

and \( S_T = [b_{\theta_1, T}^{-r}] \). We will use the following decomposition

\[
\hat{J}_T^* (\hat{b}_{1, T}, \hat{b}_{2, T}) - \hat{J}_T^* (b_{\theta_1, T}, b_{\theta_2, T}) = \left( \hat{J}_T^* (\hat{b}_{1, T}, \hat{b}_{2, T}) - \hat{J}_T^* (b_{\theta_1, T}, \hat{b}_{2, T}) \right)
\]

\[
+ \left( \hat{J}_T^* (\hat{b}_{1, T}, \hat{b}_{2, T}) - \hat{J}_T^* (b_{\theta_1, T}, b_{\theta_2, T}) \right).
\]

Let

\[
N_1 \triangleq \{-S_T, -S_T + 1, \ldots, -1, 1, \ldots, S_T - 1, S_T\}
\]

\[
N_2 \triangleq \{-T + 1, \ldots, -S_T - 1, S_T + 1, \ldots, T - 1\}.
\]

Let us consider the first term above,

\[
T^{8q/10(2q+1)} \left( \hat{J}_T^* (\hat{b}_{1, T}, \hat{b}_{2, T}) - \hat{J}_T^* (b_{\theta_1, T}, \hat{b}_{2, T}) \right)
\]

\[
= T^{8q/10(2q+1)} \sum_{k \in N_1} \left( K_1 (\hat{b}_{1, T}^* k) - K_1 (b_{\theta_1, T} k) \right) \hat{\Gamma}^*(k)
\]

\[
+ T^{8q/10(2q+1)} \sum_{k \in N_2} K_1 (\hat{b}_{1, T}^* k) \hat{\Gamma}^*(k)
\]

\[
- T^{8q/10(2q+1)} \sum_{k \in N_2} K_1 (b_{\theta_1, T} k) \hat{\Gamma}^*(k)
\]

\[
\triangleq A_{1, T} + A_{2, T} - A_{3, T}.
\]
We first show that $A_{1,T} \xrightarrow{P} 0$. Let $A_{1,T}$ denote $A_{1,T}$ with the summation restricted over positive integers $k$. Let $\tilde{n}_T = \inf\{T/n_3,T, \sqrt{n_2T}\}$. We can use the Lipschitz condition on $K_1(\cdot) \in K_3$ to yield,

$$|A_{1,T}| \leq T^{8q/10(2q+1)} \sum_{k=1}^{S_2} C_2 |\hat{b}_{1,T} - b_{\theta_1,T}| k |\tilde{\Gamma}^*(k)|$$ \hspace{1cm} (S.22)

$$\leq C\tilde{n}_T |\phi_D(q)|^{1/(2q+1)} - \phi_{\theta^*}^{1/(2q+1)}| (\phi_D(q) - \phi_{\theta^*})^{-1/(2q+1)}$$

$$\left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1) \tilde{n}_T^{-1} \sum_{k=1}^{S_2} k |\tilde{\Gamma}^*(k)|,$$

for some $C < \infty$. By Assumption 3.5-(ii), $(\tilde{n}_T |\phi_D(q) - \phi_{\theta^*}| = O_P(1))$ and using the delta method, it suffices to show that $B_{1,T} + B_{2,T} + B_{3,T} \xrightarrow{P} 0$, where

$$B_{1,T} = \left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1) \tilde{n}_T^{-1} \sum_{k=1}^{S_2} k |\tilde{\Gamma}^*(k) - \hat{\Gamma}^*(k)|,$$ \hspace{1cm} (S.23)

$$B_{2,T} = \left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1) \tilde{n}_T^{-1} \sum_{k=1}^{S_2} k |\hat{\Gamma}^*(k) - \Gamma_T^*(k)|,$$

and

$$B_{3,T} = \left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1) \tilde{n}_T^{-1} \sum_{k=1}^{S_2} k |\Gamma_T^*(k)|.$$

By a mean-value expansion, we have

$$B_{1,T} \leq \left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1) \tilde{n}_T^{-1} n_1^{-1} \sum_{k=1}^{S_2} k \left| \frac{\partial}{\partial \beta} \tilde{\Gamma}^*(k) \right| \sqrt{\tilde{n}_T} \left| \tilde{\beta} - \beta_0 \right|$$ \hspace{1cm} (S.24)

$$\leq C \left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1) (T b_{\theta_2,T})^{2r/(2q+1)} \tilde{n}_T^{-1} n_1^{-1} \sup_{k \geq 1} \left| \frac{\partial}{\partial \beta} \tilde{\Gamma}^*(k) \right| \sqrt{\tilde{n}_T} \left| \tilde{\beta} - \beta_0 \right|$$

$$\leq C \left(\tilde{b}_{2,T}\right)^{-1/(2q+1)} T(8q-10)/10(2q+1)+2r/(2q+1)+2r/(2q+1)+1/3 \tilde{n}_T^{-1} \sup_{k \geq 1} \left| \frac{\partial}{\partial \beta} \tilde{\Gamma}^*(k) \right| \sqrt{\tilde{n}_T} \left| \tilde{\beta} - \beta_0 \right| \xrightarrow{P} 0,$$

since $\tilde{n}_T / T^{1/3} \rightarrow \infty$, $r < 16q/48 + 44/48$, $\sqrt{\tilde{n}_T} \left| \tilde{\beta} - \beta_0 \right| = O_P(1)$, and $\sup_{k \geq 1} \left| (\partial/\partial \beta) \tilde{\Gamma}^*(k) \right| \beta_0 = O_P(1)$ using (S.5) and Assumption 3.2-(ii,iii). In addition,

$$E \left( \left(\tilde{b}_{2,T}\right)^{2} \right) \leq E \left( \left(\tilde{b}_{2,T}\right)^{-2/(2q+1)} T(8q-10)/5(2q+1) \tilde{n}_T^{-2} \sum_{k=1}^{S_2} \sum_{j=1}^{S_2} k \sqrt{\tilde{n}_T} \left| \tilde{\Gamma}^*(k) - \Gamma_T^*(k) \right| \left| \tilde{\Gamma}^*(j) - \Gamma_T^*(j) \right| \right)$$ \hspace{1cm} (S.25)

$$\leq \left(\tilde{b}_{2,T}\right)^{-2/(2q+1)-1} T(8q-10)/5(2q+1)-2/3-1 S_2^4 \sup_{k \geq 1} T b_{2,T} \text{Var} \left( \tilde{\Gamma}^*(k) \right)$$

$$\leq \left(\tilde{b}_{2,T}\right)^{-2/(2q+1)-1} T(8q-10)/5(2q+1)-2/3-1 \left( T b_{2,T}\right)^{4r/(2q+1)} \sup_{k \geq 1} T b_{2,T} \text{Var} \left( \tilde{\Gamma}^*(k) \right)$$

$$\leq T^{1/5} T^{2/(2q+1)} T(8q-10)/5(2q+1)-2/3-1 T^{4r/(2q+1)} \sup_{k \geq 1} T b_{2,T} \text{Var} \left( \tilde{\Gamma}^*(k) \right) \rightarrow 0,$$
given that $\sup_{k \geq 1} T_{k,T} \text{Var}(\tilde{\Gamma}^*(k)) = O(1)$ using Lemma S.A.5 in Casini (2021b) and $r < 46/48 + 20q/48$. Assumption 3.5-(iii) and $\sum_{k=1}^{\infty} k^{1-l} < \infty$ for $l > 2$ yield

$$B_{3,T} \leq \tilde{b}_{2,T}^{-1/(2q+1)} T^{(8q-10)/(2q+1)} n_T^{-1} C_3 \sum_{k=1}^{\infty} k^{1-l}$$

(S.26)

$$\leq T^{(-21-14q)/(2q+1)} C_3 \sum_{k=1}^{\infty} k^{1-l} \to 0,$$

where we have used the fact that $n_T / T^{1/3} \to \infty$. Combining (S.22)-(S.26) we deduce that $A_{1,1,T} \xrightarrow{P} 0$. The same argument applied to $A_{1,T}$ where the summation now also extends over negative integers $k$ gives $A_{1,T} \xrightarrow{P} 0$. Next, we show that $A_{2,T}$ (resp., $A_{2,2,T}$) to denote $A_{2,T}$ with the summation over positive (resp., negative) integers. Let $A_{2,1,T} = L_{1,T} + L_{2,T} + L_{3,T}$, where

$$L_{1,T} = L_{1,T}^A + L_{1,T}^B = T^{8q/10(2q+1)} \left( \sum_{k=1}^{D_y T^{1/2}} - \sum_{k=1}^{T-1} \right) K_1 \left( \tilde{b}_{1,T} k \right) \left( \tilde{\Gamma}^*(k) - \tilde{\Gamma}^*(k) \right),$$

(S.27)

$$L_{2,T} = L_{2,T}^A + L_{2,T}^B = T^{8q/10(2q+1)} \left( \sum_{k=1}^{D_y T^{1/2}} - \sum_{k=1}^{T-1} \right) K_1 \left( \tilde{b}_{1,T} k \right) \left( \tilde{\Gamma}^*(k) - \Gamma^*_T(k) \right),$$

and

$$L_{3,T} = T^{8q/10(2q+1)} \sum_{k=S_T+1}^{T-1} K_1 \left( \tilde{b}_{1,T} k \right) \Gamma^*_T(k).$$

We apply a mean-value expansion, use $\sqrt{n_T} (\tilde{\beta} - \beta_0) = O_P(1)$ as well as (S.5) to obtain

$$\left| L_{1,T}^A \right| = T^{8q/10(2q+1)-1/3} \sum_{k=S_T+1}^{D_y T^{1/2}} C_1 \left( \tilde{b}_{1,T} k \right)^{-b} \left| \frac{\partial}{\partial \beta'} \tilde{\Gamma}^*(k) \right|_{\beta=\tilde{\beta}} \sqrt{n_T} (\tilde{\beta} - \beta_0).$$

(S.28)

$$= T^{8q/10(2q+1)-1/3+4b/5(2q+1)} \sum_{k=S_T+1}^{D_y T^{1/2}} C_1 k^{-b} \left| \frac{\partial}{\partial \beta'} \tilde{\Gamma}^*(k) \right|_{\beta=\tilde{\beta}} \sqrt{n_T} (\tilde{\beta} - \beta_0)$$

$$= T^{8q/10(2q+1)-1/3+4b/5(2q+1)+4r(1-b)/5(2q+1)} \left| \frac{\partial}{\partial \beta'} \tilde{\Gamma}^*(k) \right|_{\beta=\tilde{\beta}} \sqrt{n_T} (\tilde{\beta} - \beta_0)$$

$$= T^{8q/10(2q+1)-1/3+4b/5(2q+1)+4r(1-b)/5(2q+1)} O_P(1) O_P(1),$$

which goes to zero since $r > (-10 + 4q + 24b) / 24(b - 1)$ with $b > \max\{1 + 1/q, 4\}$. We also have

$$\left| L_{1,T}^B \right| = T^{8q/10(2q+1)-1/3} \sum_{k=1}^{D_y T^{1/2}+1} C_1 \left( \tilde{b}_{1,T} k \right)^{-b} \left| \frac{\partial}{\partial \beta'} \tilde{\Gamma}^*(k) \right|_{\beta=\tilde{\beta}} \sqrt{n_T} (\tilde{\beta} - \beta_0)$$

$$= T^{8q/10(2q+1)-1/3+4b/5(2q+1)} \sum_{k=1}^{D_y T^{1/2}+1} C_1 k^{-b} \left| \frac{\partial}{\partial \beta'} \tilde{\Gamma}^*(k) \right|_{\beta=\tilde{\beta}} \sqrt{n_T} (\tilde{\beta} - \beta_0)$$

$$= T^{8q/10(2q+1)-1/3+4b/5(2q+1)} O_P(1) O_P(1).$$
\[ T^{8q/10(2q+1)-1/3+4b/5(2q+1)+(1-b)/2} \left( \frac{\partial}{\partial \beta'} \hat{\Gamma}^* (k) \right) |_{\beta' = \hat{\beta} \sqrt{n_T} (\hat{\beta} - \beta_0)} \]
\[ = T^{8q/10(2q+1)-1/3+4b/5(2q+1)+(1-b)/2} O_F (1) \overset{P}{\to} 0, \]
given that \( 1 - b < 0 \) and \( b > 1 + 1/q \). Let us now consider \( L_{2,T} \). We have
\[
|L_{2,T}^{A}| = T^{(8q-1)/10(2q+1)} \left[ \frac{D_T T^{1/2}}{b_{1,q} k} \right] C_1 \left( b_{1,T} k \right) |\hat{\Gamma}^* (k) - \Gamma_T^* (k)| \tag{S.29}
\]
\[ = C_1 \left( 2q K_2^2 \hat{\beta}_D (q) \right)^{b/(2q+1)} T^{8q/10(2q+1)+b/(2q+1)-1/2} \left( b_{2,T} \right) \left( b_{2,T} \right)^{b/(2q+1)-1/2} \sum_{k = S_T + 1}^{|D_T T^{1/2}} k^{-b} \times \sqrt{Tb_{2,T}^*} |\hat{\Gamma}^* (k) - \Gamma_T^* (k)|. \]

Note that
\[
\mathbb{E} \left( T^{8q/10(2q+1)+b/(2q+1)-1/2} \left( b_{2,T} \right)^{b/(2q+1)-1/2} \left( \sum_{k = S_T + 1}^{|D_T T^{1/2}} k^{-b} \sqrt{Tb_{2,T}^*} |\hat{\Gamma}^* (k) - \Gamma_T^* (k)| \right)^2 \right) \tag{S.30}
\]
\[ \leq T^{8q/5(2q+1)+2b/(2q+1)-1} \left( b_{2,T} \right)^{2b/(2q+1)-1} \left( \sum_{k = S_T + 1}^{|D_T T^{1/2}} k^{-b} \sqrt{Tb_{2,T}^*} \left( \text{Var} \left( \hat{\Gamma}^* (k) \right) \right)^{1/2} \right)^2 \]
\[ = T^{8q/5(2q+1)+2b/(2q+1)-1} \left( b_{2,T} \right)^{2b/(2q+1)-1} \sum_{k = S_T + 1}^{|D_T T^{1/2}} k^{-b} O (1) \]
\[ = T^{8q/5(2q+1)+2b/(2q+1)-1} \left( b_{2,T} \right)^{2b/(2q+1)-1} D_T^{2(1-b)} S_T^{2(1-b)} O (1) \to 0, \]
since \( r > (b - 1/2) / (b - 1) \) for \( b > 4 \) and \( \sqrt{Tb_{2,T}^*} \text{Var} \left( \hat{\Gamma}^* (k) \right) = O (1) \) as above. Further,
\[
|L_{2,T}^{B}| = T^{(8q-1)/10(2q+1)} \sum_{k = |D_T T^{1/2}| + 1}^{T-1} C_1 \left( b_{1,T} k \right) |\hat{\Gamma}^* (k) - \Gamma_T^* (k)| \tag{S.31}
\]
\[ = C_1 \left( 2q K_2^2 \hat{\beta}_D (q) \right)^{b/(2q+1)} T^{8q/10(2q+1)+b/(2q+1)-1/2} \left( b_{2,T} \right)^{b/(2q+1)-1/2} \]
\[ \times \sum_{k = |D_T T^{1/2}| + 1}^{T-1} k^{-b} \sqrt{Tb_{2,T}^*} |\hat{\Gamma}^* (k) - \Gamma_T^* (k)|. \]

Note that
\[
\mathbb{E} \left( T^{8q/10(2q+1)+b/(2q+1)-1/2} \left( b_{2,T} \right)^{b/(2q+1)-1/2} \sum_{k = |D_T T^{1/2}| + 1}^{T-1} k^{-b} \sqrt{Tb_{2,T}^*} |\hat{\Gamma}^* (k) - \Gamma_T^* (k)| \right)^2 \tag{S.32}
\]

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Proof of Theorem 3.1. We begin with part (i). Note that

\[ J_{T,D}^* = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} D_s E \left( V_s \left( V_t^* D_t \right)^t \right) \]

\[ = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} \left( I_p - \sum_{j=1}^{p_A} A_{D,s,j} \right)^{-1} E \left( V_s - \sum_{j=1}^{p_A} A_{D,s,j} V_{s-j} \right) \left( V_t^* \left( I_p - \sum_{j=1}^{p_A} A_{D,t,j} \right)^{-1} \right)^t \]

\[ = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} \left( I_p - \sum_{j=1}^{p_A} A_{D,s,j} \right)^{-1} \times E \left( \left( V_s - \sum_{j=1}^{p_A} A_{D,s,j} V_s + \sum_{j=1}^{p_A} A_{D,s,j} V_{s-j} \right) \left( I_p - \sum_{j=1}^{p_A} A_{D,t,j} \right)^{-1} \right)^t \]

since \( r > (b - 2)/3 - q/3 \) and \( \sqrt{Tb_{2,T}^* \text{Var} \left( \hat{\Gamma}^* (k) \right)} = O (1) \) as above. Combining (S.29)-(S.30) yields \( L_{2,T} \to 0 \). Let us turn to \( L_{3,T} \). By Assumption 3.5-(iii) and \| K_1 (\cdot) \| \leq 1 \), we have,

\[ |L_{3,T}| \leq T^{8q/10(2q+1)} \sum_{k=S_T}^{T-1} C_3 k^{-1} \leq T^{8q/10(2q+1)} C_3 S_T^{-1-l} \]

\[ \leq C_3 T^{8q/10(2q+1)} T^{-4r(l-1)/5(2q+1)} \to 0, \]

since \( r > q/(l-1) \). In view of (S.27)-(S.33) we deduce that \( A_{2,2,T} \to 0 \). Applying the same argument to \( A_{2,2,T} \), we have \( A_{2,2,T} \to 0 \). Using similar arguments, one has \( A_{3,2,T} \to 0 \). It remains to show that \( T^{8q/10(2q+1)} (J_{T}^* (b_{1,T}, \hat{b}_{2,T}^*, \hat{b}_2^*) - J_{T}^* (b_{1,T}, b_{2,T})) \to 0 \). The proof of the latter result follows from the proof of the corresponding result in Theorem 5.1-(ii) in Casini (2021b) with \( r < 2/3 + q/3 \) and \( r > (b - 2/3 - q/3) / (b - 1) \). \( \Box \)

Proof of Theorem 3.1. We begin with part (i). Note that

\[ J_{T,D}^* = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} D_s E \left( V_s \left( V_t^* D_t \right)^t \right) \]

By Lemma S.C.1-S.C.2 the right-hand side is \( o_p (1) \). It follows that the first term on the right-hand side of (S.2) is also \( o_p (1) \) because the presence of \( \hat{D}_s \) is irrelevant for the result to hold. We have,

\[ J_{T,D}^* = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} D_s E \left( V_s \left( V_t^* D_t \right)^t \right) \]

\[ = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} \left( I_p - \sum_{j=1}^{p_A} A_{D,s,j} \right)^{-1} E \left( V_s - \sum_{j=1}^{p_A} A_{D,s,j} V_{s-j} \right) \left( V_t^* \left( I_p - \sum_{j=1}^{p_A} A_{D,t,j} \right)^{-1} \right)^t \]

\[ = \frac{1}{T} \sum_{s=p_A+1}^{T} \sum_{t=p_A+1}^{T} \left( I_p - \sum_{j=1}^{p_A} A_{D,s,j} \right)^{-1} \times E \left( \left( V_s - \sum_{j=1}^{p_A} A_{D,s,j} V_s + \sum_{j=1}^{p_A} A_{D,s,j} V_{s-j} \right) \left( I_p - \sum_{j=1}^{p_A} A_{D,t,j} \right)^{-1} \right)^t \]

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For

Now note that the sum involving \( A_{D,s,j} \) has a telescopic form to a sum. Using the smoothness of \( A_{D,s,j} \), we have that the sum from any \( s \) to \( T \) is

\[
\left( I_p - \sum_{j=1}^{p_A} A_{D,s,j} \right)^{-1} \sum_{j=1}^{p_A} A_{D,s,j} (V_s - V_{s-j})
\]

\[
+ \left( I_p - \sum_{j=1}^{p_A} A_{D,s+1,j} \right)^{-1} \sum_{j=1}^{p_A} A_{D,s+1,j} (V_{s+1} - V_{s+1-j})
\]

\[
\ldots
\]

\[
+ \left( I_p - \sum_{j=1}^{p_A} A_{D,T,j} \right)^{-1} \sum_{j=1}^{p_A} A_{D,T,j} (V_T - V_{T-j}).
\]

For \( s \neq T_r^0 (r = 1, \ldots, m_0) \) local stationarity implies \( A_{D,s+1,j} = A_{D,s,j} + O(1/T) \). There are only a finite number of breaks \( T_r^0 (r = 1, \ldots, m_0) \) so that (S.36) is equal to

\[
\left( I_p - \sum_{j=1}^{p_A} A_{D,p_A+1,j} \right)^{-1} A_{D,p_A+1,p_A} V_1 + \left( I_p - \sum_{j=1}^{p_A} A_{D,T,j} \right) A_{D,T,p_A} V_T
\]

\[
+ \sum_{r=1}^{m_0} \left( I_p - \sum_{j=1}^{p_A} A_{D,T_r,j} \right)^{-1} \sum_{j=1}^{p_A} \left( A_{D,T_r,j} - A_{D,T_r+1,j} \right) V_{T_r}
\]

\( \triangleq C_{A,T} \).

It follows that

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(C_{A,T}) \left( V_t^* \left( I_p - \sum_{j=1}^{p_A} A_{D,t,j} \right)^{-1} \right) \rightarrow 0.
\]
Altogether, this implies \( J^*_{T, D} \xrightarrow{p} J_T \). Using Assumption 3.6 and simple manipulations, the second term on the right-hand side of (S.2) is \( o_p(1) \). Therefore,

\[
\hat{J}_{pw, T} - J_T = (\hat{J}_{pw, T} - J^*_{T, D}) + (J^*_{T, D} - J^*_{T, D}) + (J^*_{T, D} - J_T) = o_p(1),
\]

which concludes the proof of part (i).

Next, we move to part (ii). Given the decomposition (S.2), we have to show

\[
\sqrt{T} b_{\theta_1, T} b_{\theta_2, T} (J^*_{T, D} - J^*_{T, D}) = o_p(1),
\]

(3.38)

(3.39)

\[
\sqrt{T} b_{\theta_1, T} b_{\theta_2, T} (J^*_{T, D} - J_T) = o_p(1).
\]

(3.40)

Equation (3.38) follows from

\[
\sqrt{T} b_{\theta_1, T} b_{\theta_2, T} (\hat{J}_T (b_{\theta_1, T}, b_{\theta_2, T}) - J^*_{T}) = o_p(1),
\]

(3.41)

since the presence of \( \hat{D}_s \) in \( \hat{V}_{D, s} \) is irrelevant. Thus, Lemma S.C.3-S.C.4 yield (3.38). Given that \( \sqrt{T} b_{\theta_1, T} b_{\theta_2, T} / n_T \to 0 \), Assumption 3.6 and simple algebra yield (3.39). From the proof of part (i), it is easy to see that the multiplication by the factor \( \sqrt{T} b_{\theta_1, T} b_{\theta_2, T} \) in (3.40) does not change the fact that this term is \( o_p(1) \). Therefore, we conclude that \( T^{8q}/10(2q+1)(\hat{J}_{pw, T} - J_T) = O_p(1) \).

We now move to part (iii). The estimator \( \hat{J}_{T, pw} \) is actually a double kernel HAC estimator constructed using observations \( \{\hat{V}_{D, s}\} \), where the latter is SLS. Thus, using Theorem 3.2 and 5.1 in Casini (2021b) and Assumption 3.6, we deduce that

\[
\lim_{T \to \infty} \text{MSE} \left( T b_{\theta_1, T} b_{\theta_2, T}, \hat{J}_{pw, T}, J_T, W_T \right) = \lim_{T \to \infty} \text{MSE} \left( T b_{\theta_1, T} b_{\theta_2, T}, J^*_{T, D}, J_T, W_T \right).
\]

(3.43)

This implies that it is sufficient to determine the asymptotic MSE of \( J^*_{T, D} \). Note that \( J^*_{T, D} \) is simply a double kernel HAC estimator constructed using observations \( \{V^*_D\} \). It follows that \( \{V^*_D\} \) is SLS and thus it satisfies the conditions of Theorem 3.2 and 5.1 in Casini (2021b). The same argument in Casini (2021b) now with reference to Theorem 3.1-(i,ii) yields

\[
\lim_{T \to \infty} \text{MSE} \left( T b_{\theta_1, T} b_{\theta_2, T}, J^*_{T, D}, J_T, W_T \right)
\]

\[
= 4\pi^2 \left[ \gamma_0 K^2_{1,q} \text{vec} \left( \int_0^1 f^*_D (q) (u, 0) \, du \right) \text{vec} \left( \int_0^1 f^*_D (q) (u, 0) \, du \right) \right]
\]

\[
+ \int K^2_1 (y) \, dy \int K^2_2 (x) \, dx \text{tr} W \left( I_{p^*} - C_{pp} \right) \left( \int_0^1 f^*_D (u, 0) \, du \right) \otimes \left( \int_0^1 f^*_D (y, 0) \, dy \right) .
\]

The latter relation and (3.43) conclude the proof. □
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Supplemental Material not for Publication to

Prewhitened Long-Run Variance Estimation Robust to Nonstationarity

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Abstract

This supplemental material is not for publication and is structured as follows. Section N.A presents the proofs of the results of Section 4 and Section N.B presents the proof of the results of Section S.B.
N.A  Proof of the Results of Section 4

N.A.1  Proof of Theorem 4.1

Consider first the numerator of $t_{DM,i}$. We have

\[
T_n^{1/2} \delta_L = \delta^2 O_p \left( T_n^{1/2} n_n^{-1} \right) + O_p \left( T_n^{1/2} n_n^{-1} \right)^{1/2} N(0, J_{DM})
\]

for some $J_{DM} \in (0, \infty)$ where the factor $\delta^2$ follows from the quadratic loss.

Next, we focus on the expansion of the denominator of $t_{DM,i}$ which hinges on which LRV estimator is used. We begin with part (i). Under $b_T \to 0$ as $T \to \infty$, Theorem 3.1 in Casini, Deng, and Perron (2021) yields

\[
\hat{J}_{L,W87}^{DL} = \left[ \sum_{k=-[b_T^{-1}]}^{[b_T^{-1}]} (1 - |b_T k|) \tilde{P}(k) \right] \left[ \sum_{k=-[b_T^{-1}]}^{[b_T^{-1}]} (1 - |b_T k|) \int_0^1 c(u, k) du \right]
\]

\[
+ \sum_{k=-[b_T^{-1}]}^{[b_T^{-1}]} (1 - |b_T k|) \left( 2^{-1} \left( \frac{T_b - T_m - 1}{T_n} \right) \left( \frac{T_n - T_b - 2}{T_n} \right) \delta^4 + O_p(1) \right)
\]

\[
= CJ_{DM} + \sum_{k=-[b_T^{-1}]}^{[b_T^{-1}]} (1 - |b_T k|) \left( 2^{-1} \left( \frac{T_b - T_m - 1}{T_n} \right) \left( \frac{T_n - T_b - 2}{T_n} \right) \delta^4 + O_p(1) \right),
\]

for some $C > 0$ such that $C < \infty$. By Exercise 1.7.12 in Brillinger (1975),

\[
\sum_{k=-[b_T^{-1}]}^{[b_T^{-1}]} (1 - |b_T k|) \exp(-i\omega k) = b_T \left( \frac{\sin \frac{|b_T^{-1}| \omega}{2}}{\sin \frac{\omega}{2}} \right)^2.
\]

Evaluating the expression above at $\omega = 0$ and applying L’Hôpital’s rule we yield,

\[
\sum_{k=-[b_T^{-1}]}^{[b_T^{-1}]} (1 - |b_T k|) = b_T \left( \frac{|b_T^{-1}|}{2} \right)^2 = \left[ b_T^{-1} \right].
\]

Therefore, $\hat{J}_{L,W87}^{DL} = CJ_{DM} + \delta^4 O_p(b_T^{-1})$ and
where \( \hat{b} \) implies \( A^* \) and exhibits a break in the mean of magnitude \( T_{bT}^{\frac{1}{2}} \) after \( T \).

We now consider part (ii). We have \( b_T \to 0 \) as \( T \to \infty \). The whitening step for \( \hat{J}_{dL,pwNWS7,T} \) involves the following fitted least-squares regression,

\[
\hat{V}_t = \hat{A}_1 \hat{V}_{t-1} + \hat{V}_t^* \quad \text{for} \quad t = 2, \ldots, T_n - 1,
\]

where \( \hat{A}_1 \) is the least-squares estimate and \( \{\hat{V}_t^*\} \) is the corresponding least-squares residual. Under \( H_1 \), \( \hat{V}_t^* \) exhibits a break in the mean of magnitude \( \delta^2 \) because \( \hat{V}_t \) has a break in the mean of the same magnitude after \( T_b \). From Casini, Deng, and Perron (2021) it follows that \( \hat{A}_1 \overset{p}{\to} A_{1,B} \) with \( |1 - A_{1,B}| < |1 - A_1| \) where \( A_1 \) is such that \( \hat{A}_1 \overset{p}{\to} A_1 \) under the null hypothesis (i.e., under \( \delta = 0 \)). Let

\[
\hat{J}_T^* = \sum_{k=-\lfloor b_T^{-1} \rfloor}^{\lfloor b_T^{-1} \rfloor} K_1(b_T k) \hat{\Gamma}^*(k), \quad \hat{\Gamma}^*(k) = (T_n - 2)^{-1} \sum_{t=|k|+2}^{T_n-1} \hat{V}_t^* \hat{V}_{t-|k|}^*,
\]

and \( c^*(u, k) = \text{Cov}(\hat{V}_{T_n}^*, \hat{V}_{T_n-|k|}^*) \). Using Theorem 3.1 in Casini, Deng, and Perron (2021),

\[
\hat{J}_{dL,pwNWS7,T} = \left(1 - \hat{A}_1\right)^{-2} \hat{J}_T^* = \left(1 - \hat{A}_1\right)^{-2} \left\{ \sum_{k=-\lfloor b_T^{-1} \rfloor}^{\lfloor b_T^{-1} \rfloor} \left(1 - |b_T k|\right) \hat{\Gamma}^*(k) \right\}
\]

\[
= \left(1 - \hat{A}_1\right)^{-2} \sum_{k=-\lfloor b_T^{-1} \rfloor}^{\lfloor b_T^{-1} \rfloor} \left(1 - |b_T k|\right) \int_0^1 c^*(u, k) \, du
\]

\[
+ \left(1 - \hat{A}_1\right)^{-2} \sum_{k=-\lfloor b_T^{-1} \rfloor}^{\lfloor b_T^{-1} \rfloor} \left(1 - |b_T k|\right) \left(2^{-1} \left( \frac{T_b - T_m}{T_n} \right) \left( \frac{T_n - T_b}{T_n} \right) \delta^4 + o_{\Phi}(1) \right)
\]

\[
= C \left(1 - A_{1,B}\right)^{-2} J_{DM} + \left(1 - A_{1,B}\right)^{-2} \left\{ \sum_{k=-\lfloor b_T^{-1} \rfloor}^{\lfloor b_T^{-1} \rfloor} \left(1 - |b_T k|\right) \left(2^{-1} \left( \frac{T_b - T_m}{T_n} \right) \left( \frac{T_n - T_b}{T_n} \right) \delta^4 + o_{\Phi}(1) \right) \right\},
\]
for some $C > 0$ such that $C < \infty$. Therefore, $\tilde{J}_{d_{l,pwNW87},T} = C (1 - A_{1,B})^{-2} J_{DM} + (1 - A_{1,B})^{-2} \delta^4 O_p(b_T^{-1})$
and
\[
|t_{DM,pwNW87}| \leq \frac{\delta^2 O_p \left( T_n^{-1/2} n_\delta \right) + O_p(1)}{(1 - A_{1,B})^{-2} \delta^4 O \left( b_T^{-1} \right)}^{1/2} = \frac{(1 - A_{1,B}) \delta^2 O \left( T_n^{\zeta} \right)}{\delta^2 O \left( b_T^{-1/2} \right)} = O \left( T_n^{\zeta} b_T^{1/2} \right),
\]
which implies $P_\delta(|t_{pwDM,NW87}| > z_\alpha) \to 0$ since $T_n^{\zeta} b_1^{1/2} \to 0$.

In part (iii), $b_T = T^{-1}$. Proceeding as in (N.1) we have $|t_{DM,KVB}| = O(T_n^{-1})$ and $P_\delta(|t_{DM,KVB}| > z_\alpha) \to 0$ since $T_n^{\zeta - 1} \to 0$.

We consider part (iv). Using Theorem 3.3 in Casini, Deng, and Perron (2021), we have
\[
\hat{J}_{d_{l,DK},T} = \sum_{k=-T_n+2}^{T_n-2} K_1 \left( \tilde{b}_{1,T}^{*,k} \right) \frac{n_{T_n}}{T_n} \sum_{r=1}^{T_n/n_{T_n}} \tilde{c}_T \left( r n_{T_n}/T_n, k \right)
= \sum_{k=-T_n+2}^{T_n-2} K_1 \left( \tilde{b}_{1,T}^{*,k} \right) \frac{n_{T_n}}{T_n} \sum_{r=1}^{T_n/n_{T_n}} \left( c \left( r n_{T_n}/T_n, k \right) + \delta^2 1 \left\{ \left| r n_{T_n} + k/2 + T_n \tilde{b}_2^{*,T}/2 + 1 \right| - T_b \right\} \left( T_n \tilde{b}_2^{*,T} \right) \right) \right) + O_p(1)
= J_{DM} + \delta^2 O_p \left( \left( \tilde{b}_{1,T}^{*,T} \right)^{-1} \frac{n_{T_n}}{T_n} \right) + O_p(1).
\]
It follows that
\[
|t_{DM,DK}| = \frac{\delta^2 O_p \left( T_n^{-1/2} n_\delta \right) + O_p(1)}{J_{DM} + \delta^2 O_p \left( \left( \tilde{b}_{1,T}^{*,T} \right)^{-1} \frac{n_{T_n}}{T_n} \right)}^{1/2} = \delta^2 O \left( T_n^{\zeta} \right).
\]
Since $T_n^{\zeta} \to \infty$, we have $P_\delta(|t_{DM,DK}| > z_\alpha) \to 1$.

Finally, we consider part (v). The whitening step for $\hat{J}_{d_{l,pwDK},T}$ involves the following fitted least-squares regression,
\[
\hat{V}_t = \hat{A}_{r,1} \hat{V}_{t-1} + \hat{V}_t^* \quad \text{for} \quad t = r n_{T_n} + 1, \ldots, (r + 1) n_{T_n},
\]
(for the last block, $t = \lfloor n_{T_n}/n_{T_n} \rfloor n_{T_n} + 1, \ldots, T_n$) where $\hat{A}_{r,1}$ is the least-squares estimate and $\{\hat{V}_t^*\}$ is the corresponding least-squares residual. Under $H_1$, $\hat{V}_t^*$ exhibits a break in the mean of magnitude $\tilde{\delta}$ in the $r^*$th block such that $T_b \in \{r^* n_{T_n} + 1, \ldots, (r^* + 1) n_{T_n}\}$. This follows because $\hat{V}_t$ has a break in the mean of the same magnitude after $T_b$. Note that over the blocks $r \neq r^*$, $\hat{V}_t$ does not have a break in the mean.
Using Theorem 3.3 in Casini, Deng, and Perron (2021), we have
\[
\hat{J}_{d, pwDK, T} = \sum_{k=-T_n+1}^{T_n} K_1(\hat{b}_{1,T}k) \frac{n_{T_n}}{T_n} \sum_{r=1}^{[T_n/n_{T_n}]} \tilde{c}_{T,D} (r n_{T_n} / T_n, k)
\]
\[
= \sum_{k=-T_n+1}^{T_n} K_1(\hat{b}_{1,T}k) \frac{n_{T_n}}{T_n} \sum_{r=1}^{[T_n/n_{T_n}]} c^*_D (r n_{T_n} / T_n, k)
\]
\[
+ \tilde{\delta}^2 \mathbb{I} \left\{ \left( T_n / 2 + T_n \hat{b}_{2,T} / 2 + 1 - T_0 / (T_n \hat{b}_{2,T}) \right) \in (0, 1) \right\} + o_P(1)
\]
\[
= J_{DM} + \tilde{\delta}^2 O_P \left( \left( \hat{b}_{1,T} \right)^{-1} n_{T_n} / T_n \right) + o_P(1).
\]
It follows that
\[
|t_{DM, pwDK}| = \frac{\tilde{\delta}^2 O_P \left( T_n^{-1/2} n_\delta \right) + o_P(1)}{\left( J_{DM} + \tilde{\delta}^2 O_P \left( \left( \hat{b}_{1,T} \right)^{-1} n_{T_n} / T_n \right) \right)^{1/2}}
\]
\[
= \tilde{\delta}^2 O \left( T_n^\varsigma \right).
\]
Since \( T_n^\varsigma \to \infty \) we have \( \mathbb{P}_\delta(|t_{DM, pwDK}| > z_\alpha) \to 1. \) \( \Box \)

### N.B Proofs of the Results in Section S.B

In the proofs below involving \( \tilde{c}_T(u, k), \tilde{c}_T(u, k) \) and \( c(u, k), \) we assume \( k \geq 0 \) unless otherwise stated. The proofs for the case \( k < 0 \) are similar and omitted.

#### N.B.1 Proof of Theorem S.B.1

We first present upper and lower bounds on the asymptotic variance of \( \tilde{J}_T. \) Let \( \text{Var}_\mathcal{P}(\cdot) \) denote the variance of \( \cdot \) under \( \mathcal{P}. \)

**Lemma N.B.1.** Suppose that Assumption S.B.1 holds, \( K_2(\cdot) \in K_2, b_{1,T}, b_{2,T} \to 0, n_T \to \infty, n_T / T \to 0 \) and \( 1/ T b_{1,T} b_{2,T} \to 0. \) We have for all \( a \in \mathbb{R}^\delta : \)

(i) for any \( K_1(\cdot) \in K_1, \)

\[
\lim_{T \to \infty} \sup_{\mathcal{P} \in \mathcal{P}_U} T b_{1,T} b_{2,T} \text{Var}_\mathcal{P} \left( a' \tilde{J}_T a \right) = \lim_{T \to \infty} T b_{1,T} b_{2,T} \text{Var}_\mathcal{P} \left( a' \tilde{J}_T a \right)
\]
\[
= 8 \pi^2 \int_0^1 K_1^2(y) dy \int_0^1 K_2^2(x) dx \left( \int_0^1 f_{\mathcal{P}, a}(u, 0) du \right)^2;
\]

(ii) for any \( K_1(\cdot) \in K_{1, +}, \)

\[
\lim_{T \to \infty} \inf_{\mathcal{P} \in \mathcal{P}_L} T b_{1,T} b_{2,T} \text{Var}_\mathcal{P} \left( a' \tilde{J}_T a \right) = \lim_{T \to \infty} T b_{1,T} b_{2,T} \text{Var}_\mathcal{P} \left( a' \tilde{J}_T a \right)
\]
By definition of the fourth-order cumulant and by definition of $r$, \ldots, $\in P$,

\[ \begin{align*}
\sup_{\mathcal{P} \in \mathcal{P}_U} \left| \mathbb{E}_{\mathcal{P}} \left[ a' \left( \bar{c}_T (r n_T / T, k) - c_{\mathcal{P}, T} (r n_T / T, k) \right) a a' \left( \bar{c}_T (b n_T / T, j) - c_{\mathcal{P}, T} (b n_T / T, j) \right) a \right] \right| \\
= \left( T b_{2, T} \right)^{-2} \sum_{s=k+1}^T \sum_{l=j+1}^T K^*_2 \left( \frac{((r + 1) n_T - (s - k/2)) / T}{b_{2, T}} \right) K^*_2 \left( \frac{((b + 1) n_T - (l - j/2)) / T}{b_{2, T}} \right) \\
\times \left( \mathbb{E}_{\mathcal{P}} \left( Z_s Z_{s-k} Z_{l} Z_{l-j} \right) - \mathbb{E}_{\mathcal{P}} (Z_s Z_{s-k}) \mathbb{E}_{\mathcal{P}} (Z_l Z_{l-j}) \right).
\end{align*} \]

For any $k, j \geq 0$ and any $r, b = 0, \ldots, [T/n_T]$,

\[ \begin{align*}
\sup_{\mathcal{P} \in \mathcal{P}_U} \left| \mathbb{E}_{\mathcal{P}} \left[ a' \left( \bar{c}_T (r n_T / T, k) - c_{\mathcal{P}, T} (r n_T / T, k) \right) a a' \left( \bar{c}_T (b n_T / T, j) - c_{\mathcal{P}, T} (b n_T / T, j) \right) a \right] \right| \\
\leq \left( \frac{1}{T b_{2, T}} \right)^2 \sum_{s=k+1}^T \sum_{l=j+1}^T K^*_2 \left( \frac{((r + 1) n_T - (s - k/2)) / T}{b_{2, T}} \right) K^*_2 \left( \frac{((b + 1) n_T - (l - j/2)) / T}{b_{2, T}} \right) \\
\times \left( a' \Gamma_{\mathcal{P}_U, s/T} (s-l) a a' \Gamma_{\mathcal{P}_U, s-k} (s-k-l+j) a + a' \Gamma_{\mathcal{P}_U, s/T} (s-l+j) a a' \Gamma_{\mathcal{P}_U, s-k} (s-k-l) a \\
+ \kappa_s (-k, l-s, l-j-s) \right) \\
\leq \mathbb{E}_{\mathcal{P}_U} \left[ a' \left( \bar{c}_T (r n_T / T, k) - c_{\mathcal{P}_U, T} (r n_T / T, k) \right) a a' \left( \bar{c}_T (b n_T / T, j) - c_{\mathcal{P}_U, T} (b n_T / T, j) \right) a \right] \quad (N.1)
\end{align*} \]
where the last inequality holds by reversing the argument of the equality and the first inequality.

By a similar argument,

\[
\inf_{\varrho \in \mathcal{P}_L} |\mathbb{E}_{\varrho} (a' (\tilde{c}_T (rn_T/T, k) - c_{\varrho,T} (rn_T/T, k)) a)| \\
\geq \mathbb{E}_{\varrho_{L}} (a' (\tilde{c}_T (rn_T/T, k) - c_{\varrho_{L,T}} (rn_T/T, k)) a') \geq \mathbb{E}_{\varrho} (a' (\tilde{c}_T (bn_T/T, j) - c_{\varrho,T} (bn_T/T, j)) a')
\]

\[
+ 2 \left( \frac{1}{Tb_{2,T}} \right)^2 \sum_{s=k+1}^{T} \sum_{l=j+1}^{T} \kappa_s^* (-k, l - s, l - j - s). \tag{N.2}
\]

Let \(\tilde{J}_{T,K}^{*}\) be the same as \(\tilde{J}_T\) but with \(|K_1 (\cdot)|\) and \(|K_2 (\cdot)|\) in place of \(K_1 (\cdot)\) and \(K_2 (\cdot)\), respectively. Note that \(K_1 (\cdot) \in \mathcal{K}_1\) implies \(|K_1 (\cdot)| \in \mathcal{K}_1\) \(|K_2 (\cdot)| \in \mathcal{K}_2\). We have

\[
\lim_{T \to \infty} Tb_{1,Tb_{2,T}} \text{Var}_{\varrho_U} (a' \tilde{J}_Ta) \\
\leq \lim_{T \to \infty} \sup_{\varrho \in \mathcal{P}_U} \mathbb{E}_{\varrho} (a' \tilde{J}_Ta) \\
= \lim_{T \to \infty} \sup_{\varrho \in \mathcal{P}_U} Tb_{1,Tb_{2,T}} \sum_{k=-T+1}^{T-1} \sum_{j=-T+1}^{T-1} K_1 (b_{1,T}k) K_1 (b_{1,T}j) \\
\times \left( \frac{n_T}{T} \right) \sum_{r=0}^{T} \sum_{b=0}^{r} \left( \frac{1}{Tb_{2,T}} \right)^2 \sum_{s=k+1}^{T} \sum_{l=j+1}^{T} |K_2 (rn_T - (s - k/2))/Tb_{2,T}| \left| K_2 (bn_T - (l + j/2))/Tb_{2,T} \right| \]

\[
\times \mathbb{E}_{\varrho_U} (a' (\Gamma_{s/T} (k) - \mathbb{E}_{\varrho} (\Gamma_{s/T} (k))) a') \] 

\[
+ \lim_{T \to \infty} \mathbb{E}_{\varrho_U} (a' (\Gamma_{s/T} (k) - \mathbb{E}_{\varrho} (\Gamma_{s/T} (k))) a') \\
\times \kappa_s^* (-k, l - s, l - j - s) \\
= \lim_{T \to \infty} Tb_{1,Tb_{2,T}} \text{Var}_{\varrho_U} (a' \tilde{J}_{T,K} a), \tag{N.3}
\]

where the last inequality uses (N.1). For \(K_1 (\cdot) \in \mathcal{K}_{1,+}\), we can rely on an argument analogous to that of (N.3) using (N.2) in place of (N.1) to yield,

\[
\lim_{T \to \infty} Tb_{1,Tb_{2,T}} \text{Var}_{\varrho_L} (a' \tilde{J}_Ta) \geq \lim_{T \to \infty} \inf_{\varrho \in \mathcal{P}_L} Tb_{1,Tb_{2,T}} \text{Var}_{\varrho} (a' \tilde{J}_Ta)
\]
By Theorem 3.1 in Casini (2021b),
\[
\lim_{T \rightarrow \infty} Tb_{b_1,T}b_{b_2,T} \var_{\varphi_w} (a' \tilde{J}_T a) = 8\pi^2 \int K_1^2 (y) dy \int_0^1 K_2^2 (x) dx \left( \int_0^1 f_{\varphi_w,a} (u, 0) du \right)^2, \quad \text{and} \quad (N.5)
\]
\[
\lim_{T \rightarrow \infty} Tb_{b_1,T}b_{b_2,T} \var_{\varphi_w} (a' \tilde{J}_{K,T} a) = 8\pi^2 \int |K_1 (y)|^2 dy \int_0^1 |K_2 (x)|^2 dx \left( \int_0^1 f_{\varphi_w,a} (u, 0) du \right)^2, \quad (N.6)
\]
for \( w = L, U \). Equations (N.3), (N.5) and (N.6) combine to establish part (i) of the lemma:
\[
8\pi^2 \int K_1^2 (y) dy \int_0^1 K_2^2 (x) dx \left( \int_0^1 f_{\varphi_U,a} (u, 0) du \right)^2
\]
\[
= \lim_{T \rightarrow \infty} Tb_{b_1,T}b_{b_2,T} \var_{\varphi_U} (a' \tilde{J}_T a)
\]
\[
\leq \lim_{T \rightarrow \infty} \sup_{\varphi \in \mathcal{P}_k} Tb_{b_1,T}b_{b_2,T} \var_{\varphi} (a' \tilde{J}_T a)
\]
\[
\leq \lim_{T \rightarrow \infty} Tb_{b_1,T}b_{b_2,T} \var_{\varphi_U} (a' \tilde{J}_{K,T} a)
\]
\[
= 8\pi^2 \int K_1^2 (y) dy \int_0^1 K_2^2 (x) dx \left( \int_0^1 f_{\varphi_U,a} (u, 0) du \right)^2.
\]

By a similar reasoning, equations (N.4) and (N.5) yield part (ii). \( \Box \)

Upper and lower bounds on the asymptotic bias of \( \tilde{J}_T \) are given in the following lemma. Let \( J_{\varphi,U,T} \) be equal to \( J_{\varphi,T} \) but with the expectation \( \mathbb{E}_{\varphi} \) replaced by \( \mathbb{E}_{\varphi_w} \), \( w = U, L \).

**Lemma N.B.2.** Let Assumption S.B.1 hold, \( K_1 (\cdot) \in K_1, K_2 (\cdot) \in K_2, b_1, b_2, b_2 \to 0, n_T \to \infty, n_T/T \to 0, 1/Tb_1b_2 \to 0, 1/Tb_1b_2 \to 0, n_T/Tb_1 \to 0 \) and \( b_2^q, b_1^q \to 0 \) for some \( q \in [0, \infty) \) for which \( K_{1,q} \), \( \int f_{\varphi,w,a} (u, 0) du \in [0, \infty) \), \( w = U, L \). We have for all \( a \in \mathbb{R}^p \):

(i) \( \lim_{T \rightarrow \infty} \sup_{\varphi \in \mathcal{P}_U} b_{1,T}^{-q} \mathbb{E} \varphi a' \tilde{J}_T a - a' J_{\varphi,T} a = \lim_{T \rightarrow \infty} b_{1,T}^{-q} \mathbb{E} \varphi_U a' \tilde{J}_T a - a' J_{\varphi,U,T} a = 2\pi K_{1,q} f_{\varphi,U,a} \) and

(ii) \( \lim_{T \rightarrow \infty} \inf_{\varphi \in \mathcal{P}_L} b_{1,T}^{-q} \mathbb{E} \varphi a' \tilde{J}_T a - a' J_{\varphi,T} a = \lim_{T \rightarrow \infty} b_{1,T}^{-q} \mathbb{E} \varphi_L a' \tilde{J}_T a - a' J_{\varphi,L,T} a = 2\pi K_{1,q} f_{\varphi,L,a} \).

**Proof of Lemma N.B.2.** We begin with part (i). We have,
\[
\lim_{T \rightarrow \infty} \sup_{\varphi \in \mathcal{P}_U} b_{1,T}^{-q} \left| \mathbb{E} \varphi a' \tilde{J}_T a - a' J_{\varphi,T} a \right|
\]
\[
= \lim_{T \rightarrow \infty} \sup_{\varphi \in \mathcal{P}_U} b_{1,T}^{-q} \left| \sum_{k=-T+1}^{T-1} K_1 (b_1 k) a' \mathbb{E} \varphi \left( \tilde{\Gamma} (k) \right) a - \sum_{k=-T+1}^{T-1} a' \Gamma_{\varphi,T} (k) a \right|
\]
\[
= \lim_{T \rightarrow \infty} \sup_{\varphi \in \mathcal{P}_U} b_{1,T}^{-q} \left| \sum_{k=-T+1}^{T-1} K_1 (b_1 k) a' \mathbb{E} \varphi \left( \tilde{\Gamma} (k) \right) a - \sum_{k=-T+1}^{T-1} K_1 (b_1 k) a' \Gamma_{\varphi,T} (k) a \right|
\]
Thus, it follows that
\[
\Gamma (k) = \frac{1}{a} \left( \frac{(r + 1) n_T - (s + k/2)}{b_T} \right) - 1
\]

Let us first consider \( G_{1, \mathcal{F}, T} \). Note that for \( k \geq 0 \),
\[
a' \left( E_{\mathcal{F}} (\widetilde{\Gamma} (k)) - \Gamma (k) \right) a
\]
\[
= \left( \frac{n_T T}{T} \sum_{r=0}^{T} \sum_{s=0}^{T} T^{-1} \left( b_{2,T}^{-1} K_2 \left( \frac{(r + 1) n_T - (s + k/2)}{b_T} \right) - 1 \right) a'E_{\mathcal{F}} (V s'_{k-}) a \right).
\]

Thus,
\[
\sup_{\mathcal{F} \in \mathcal{P}_U} \left| a' \left( E_{\mathcal{F}} (\widetilde{\Gamma} (k)) - \Gamma (k) \right) a \right|
\]
\[
\leq \left| \frac{n_T T}{T} \sum_{r=0}^{T} \sum_{s=0}^{T} T^{-1} \left( b_{2,T}^{-1} K_2 \left( \frac{(r + 1) n_T - (s + k/2)}{b_T} \right) - 1 \right) a'E_{\mathcal{F}} (V s'_{k-}) a \right|
\]

By Lemma S.A.1 in Casini (2021b), \( E_{\mathcal{F}} (V s'_{k-}) = c(s/T, k) + O(T^{-1}) \) uniformly in \( s \) and \( k \). By the proof of Lemma S.A.6 in Casini (2021b),
\[
\sup_{\mathcal{F} \in \mathcal{P}_U} \left| a' \left( E_{\mathcal{F}} (\widetilde{\Gamma} (k)) - \Gamma (k) \right) a \right|
\]
\[
\leq \left| \frac{n_T T}{T} \sum_{r=0}^{T} \sum_{s=0}^{T} T^{-1} \left( b_{2,T}^{-1} K_2 \left( \frac{(r + 1) n_T - (s + k/2)}{b_T} \right) - 1 \right) a'E_{\mathcal{F}} (V s'_{k-}) a \right|
\]
\[
= O \left( \frac{n_T}{T} \right) + \left| \frac{1}{2} b_{2,T}^2 \int_0^1 x^2 K_2 (x) dx \int_0^1 a' \left( \frac{\partial^2}{\partial^2 u} c(u, k) \right) adu \right| + o \left( \frac{1}{T b_{2,T}} \right).
\]

It then follows that \( \lim_{T \to \infty} \sup_{\mathcal{F} \in \mathcal{P}_U} b_{1,T}^{-q} |G_{1, \mathcal{F}, T}| = 0 \) given the conditions \( n_T / T b_{1,T}^2 \to 0 \) and \( b_{2,T}^2 / b_{1,T}^q \to 0 \).

Next, given that \( 1 - K_1 (b_{1,T}k) \geq 0 \),
\[
\lim_{T \to \infty} \sup_{\mathcal{F} \in \mathcal{P}_U} b_{1,T}^{-q} |G_{2, \mathcal{F}, T}|
\]
\[
= \lim_{T \to \infty} \sup_{\mathcal{F} \in \mathcal{P}_U} b_{1,T}^{-q} \left| \sum_{k=-T+1}^{T-1} (K_1 (b_{1,T}k) - 1) a' \Gamma_{\mathcal{F}, T} (k) a \right|
\]
\[
= \lim_{T \to \infty} b_{1,T}^{-q} \sum_{k=-T+1}^{T-1} (1 - K_1 (b_{1,T}k)) a' \mathcal{F} \left( \Gamma (k) \right) a.
\]
Write the right-hand side above as,
\[
\lim_{T \to \infty} b_{1,T}^{-q} \sum_{k = -T+1}^{T-1} (1 - K_1 (b_1,T k)) a' \left( \mathbb{E}_{\mathcal{P}_U} \left( \hat{\Gamma} (k) \right) - \int_0^1 c_{\mathcal{P}_U} (u, k) \, du \right) a \\
+ \lim_{T \to \infty} b_{1,T}^{-q} \sum_{k = -T+1}^{T-1} (1 - K_1 (b_1,T k)) a' \left( \int_0^1 c_{\mathcal{P}_U} (u, k) \, du \right) a.
\] (N.7)

By Lemma S.A.1 in Casini (2021b), the first term above is less than,
\[
\lim_{T \to \infty} b_{1,T}^{-q} \sum_{k = -T+1}^{T-1} (1 - K_1 (b_1,T k)) O \left( T^{-1} \right) = 0.
\] (N.8)

Thus, it remains to consider the second term of (N.7). Let \( w(x) = (1 - K_1(x))/|x|^q \) for \( x \neq 0 \) and \( w(x) = K_1, q \) for \( x = 0 \). The following properties hold: \( w(x) \to K_1, q \) as \( x \to 0 \); \( w(\cdot) \) is non-negative and bounded. The latter property implies that there exists some constant \( C < \infty \) such that \( w(x) \leq C \) for all \( x \in \mathbb{R} \). Recall that \( \int_0^1 f_{\mathcal{P}_U}^{(q)} (u, 0) \, du \in [0, \infty) \), \( w = U, L \). Hence, given any \( \varepsilon > 0 \), we can choose a \( \hat{T} < \infty \) such that \( \int_{-\hat{T}}^{\hat{T}} \sum_{k = \hat{T}+1}^{\infty} |k|^q (a T, \mathcal{P}_U, a (k) du < \varepsilon / (4C) \). Then, using (N.8), we have
\[
\lim_{T \to \infty} \sup_{\mathcal{P} \in \mathcal{P}_U} b_{1,T}^{-q} \left| G_{2,T} - 2\pi K_1, q f_{\mathcal{P}_U}^{(q)} \right| \\
\leq \lim_{T \to \infty} \sup_{\mathcal{T} \rightarrow \infty} \sum_{k = -T}^{T} |w (b_1,T k) - K_1, q| |k|^q a' \left( \int_0^1 c (u, k) \, du \right) a \\
+ 2 \lim_{T \to \infty} \sup_{\mathcal{T} \rightarrow \infty} \sum_{k = -T+1}^{T} |w (b_1,T k) - K_1, q| |k|^q a' \left( \int_0^1 c (u, k) \, du \right) a \\
\leq \varepsilon.
\]

This concludes the proof of part (i). The proof of part (ii) is identical to that of part (i) except that \( \sup_{\mathcal{P} \in \mathcal{P}_U} , \Gamma_{\mathcal{P}_U, a} \) and \( f_{\mathcal{P}_U, a}^{(q)} \) are replaced by \( \inf_{\mathcal{P} \in \mathcal{P}_L} , \Gamma_{\mathcal{P}_L, a} \) and \( f_{\mathcal{P}_L, a}^{(q)} \). □

**Proof of Theorem S.B.1.** Parts (i) and (ii) of the theorem follow from Lemma N.B.1-(i) and Lemma N.B.2-(i), and Lemma N.B.1-(ii) and Lemma N.B.2-(ii), respectively. □

**N.B.2  Proof of Theorem S.B.2**

Lemma N.B.1-N.B.2 [with \( q = 0 \) in part (ii)] implies \( \tilde{J}_T - J_T = o_\mathcal{P} (1) \). Noting that \( \tilde{J}_T - \tilde{J}_T = o_\mathcal{P} (1) \) if and only if \( a', \tilde{J}_T a - a', \tilde{J}_T a = o_\mathcal{P} (1) \) for arbitrary \( a \in \mathbb{R}^p \) we shall provide the proof only for the scalar case. We first show that \( \sqrt{T} b_{1,T} (\tilde{J}_T - J_T) = O_\mathcal{P} (1) \) under Assumption 3.2. Let \( \tilde{J}_T (\beta) \) denote the estimator that uses \{\( \mathcal{V}_t (\beta) \)\}. A mean-value expansion of \( \tilde{J}_T (\beta) (\approx \tilde{J}_T) \) about \( \beta_0 \) yields,
\[
\sqrt{T} b_{1,T} \left( \tilde{J}_T - J_T \right) = b_{1,T} \sum_{k = -T+1}^{T-1} K_1 (b_1,T k) \frac{\partial}{\partial \beta'} \hat{\Gamma} (k) \Big|_{\beta = \beta_0} \sqrt{T} \left( \beta - \beta_0 \right),
\] (N.9)
for some $\bar{\beta}$ on the line segment joining $\hat{\beta}$ and $\beta_0$. We have for $k \geq 0$ (the case $k < 0$ is similar and omitted) (S.5)-(S.6). It follows that (N.9) is

$$b_{1,T} \sum_{k=-T+1}^{T-1} K_1(b_{1,T}k) \frac{\partial}{\partial \beta'} \tilde{\Gamma}(k) |_{\beta=\bar{\beta}} \sqrt{T} (\hat{\beta} - \beta_0)$$

$$\leq b_{1,T} \sum_{k=-T+1}^{T-1} K_1(b_{1,T}k) \frac{nT}{T} \sum_{r=0}^{T/nT} O_\varphi(1) O_\varphi(1)$$

$$= O_\varphi(1),$$

where we have used $b_{1,T} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| \to \int |K_1(x)| dx < \infty$. Given $\sqrt{T} b_{1,T} \to \infty$, this concludes the proof of Theorem S.B.2-(i).

Next, we show that $\sqrt{T} b_{1,T}(\tilde{J}_T - \bar{J}_T) = o_\varphi(1)$ under the assumptions of Theorem S.B.2-(ii). A second-order Taylor expansion yields

$$\sqrt{T} b_{1,T}(\tilde{J}_T - \bar{J}_T) = \left[ \sqrt{b_{1,T}} \frac{\partial}{\partial \beta'} \tilde{J}_T(\beta_0) \right] \sqrt{T} (\hat{\beta} - \beta_0)$$

$$+ \frac{1}{2} \sqrt{T} (\hat{\beta} - \beta_0)' \left[ \sqrt{b_{1,T}} \frac{\partial^2}{\partial \beta \partial \beta'} \tilde{J}_T(\beta) / \sqrt{T} \right] \sqrt{T} (\hat{\beta} - \beta_0)$$

$$\triangleq G_T \sqrt{T} (\hat{\beta} - \beta_0) + \frac{1}{2} \sqrt{T} (\hat{\beta} - \beta_0)' H_T \sqrt{T} (\hat{\beta} - \beta_0).$$

We can use the same argument as in (S.5) but now using Assumption S.B.3-(ii), sot that

$$\left\| \frac{\partial^2}{\partial \beta \partial \beta'} \tilde{c}(rt_{1,T}/T, k) \right\|_{\beta=\bar{\beta}}$$

$$= \left\| (Tb_{2,T})^{-1} \sum_{s=k+1}^{T} K_2^* \left( \frac{(r+1) nT - (s+k/2)}/b_{2,T} \right) \left( \frac{\partial^2}{\partial \beta \partial \beta'} V_s(\beta) V_{s-k}(\beta) \right) \right\|_{\beta=\bar{\beta}}$$

$$= O_\varphi(1),$$

and thus,

$$\|H_T\| \leq \left( \frac{b_{1,T}}{T} \right)^{1/2} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| \sup_{\beta \in \Theta} \left\| \frac{\partial^2}{\partial \beta \partial \beta'} \tilde{\Gamma}(k) \right\|$$

$$\leq \left( \frac{b_{1,T}}{T} \right)^{1/2} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| O_\varphi(1)$$

$$\leq \left( \frac{1}{T b_{1,T}} \right)^{1/2} b_{1,T} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| O_\varphi(1) = o_\varphi(1),$$

since $T b_{1,T} \to \infty$. Next, we show that $G_T = o_\varphi(1)$. We follow the argument in the last paragraph of p. 852 of Andrews (1991). We apply Theorem S.B.2-(i,ii) to $\tilde{J}_T$ where the latter is constructed using $(V_t', \partial V_t/\partial \beta' - E_{\varphi}(\partial V_t/\partial \beta'))'$ rather than just with $V_t$. The first row and column of the off-diagonal
elements of this $\tilde{J}_T$ are now

\[
A_1 \triangleq \sum_{k=-T+1}^{T-1} K_1(b_1,T,k) \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{T b_{2,T}} \times \sum_{s=k+1}^{T} K_2 \left( \frac{(r+1)n_T - (s+k/2)}{b_{2,T}} \right) V_s \left( \frac{\partial}{\partial \beta} V_{s-k} - \mathbb{E}_\mathcal{P} \left( \frac{\partial}{\partial \beta} V_s \right) \right)
\]

\[
A_2 \triangleq \sum_{k=-T+1}^{T-1} K_1(b_1,T,k) \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{T b_{2,T}} \times \sum_{s=k+1}^{T} K_2 \left( \frac{(r+1)n_T - (s+k/2)}{b_{2,T}} \right) \left( \frac{\partial}{\partial \beta} V_s - \mathbb{E}_\mathcal{P} \left( \frac{\partial}{\partial \beta} V_s \right) \right) V_{s-k},
\]

which are both $O_\mathcal{P}(1)$ by Theorem S.B.1. Note that

\[
G_T \leq \sqrt{b_{1,T}} (A_1 + A_2) + \sqrt{b_{1,T}} \sum_{k=-T+1}^{T-1} |K_1(b_1,T,k)| \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{T b_{2,T}} \times \sum_{s=k+1}^{T} K_2 \left( \frac{(r+1)n_T - (s+k/2)}{b_{2,T}} \right) \left| (V_s + V_{s-k}) \mathbb{E}_\mathcal{P} \left( \frac{\partial}{\partial \beta} V_s \right) \right|
\]

\[
\triangleq \sqrt{b_{1,T}} (A_1 + A_2) + A_3 \sup_{1 \leq s \leq T} \left| \mathbb{E}_\mathcal{P} \left( \frac{\partial}{\partial \beta} V_s \right) \right|.
\]

It remains to show that $A_3$ is $o_\mathcal{P}(1)$. We have,

\[
\mathbb{E}_\mathcal{P} \left( A_3^2 \right) \leq b_{1,T} \sum_{k=-T+1}^{T-1} \sum_{j=-T+1}^{T-1} |K_1(b_1,T,k) K_1(b_1,T,j)| 4 \left( \frac{n_T}{T} \right) \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T} \frac{1}{T b_{2,T} b_{2,T}} \sum_{s=1}^{T} \sum_{l=1}^{T} K_2 \left( \frac{(r+1)n_T - (s+k/2) + (l+j/2)}/b_{2,T} \right) \mathbb{E}_\mathcal{P} \left( V_{s,l} \right) \mathbb{E}_\mathcal{P} \left( V_{s,l} \right).
\]

Since $\mathcal{P} \in \mathcal{P}_U$, $|\mathbb{E}_\mathcal{P}(V_{s,l})| \leq |\mathcal{P}_{U,s,T}(l-s)|$. Given $\sum_{h=-\infty}^{\infty} \sup_{u \in [0,1]} |c_{\mathcal{P}_U}(u, h)| < \infty$, we have

\[
\mathbb{E}_\mathcal{P} \left( A_3^2 \right) \leq \frac{1}{b_{1,T} b_{2,T}} \left( b_{1,T} \sum_{k=-T+1}^{T-1} |K_1(b_1,T,k)| \right)^2 \int_0^1 K_2^2(x) \, dx \int_0^1 \sum_{h=-\infty}^{\infty} \mathbb{E}_\mathcal{P} \left( c_{\mathcal{P}_U}(u, h) \right) \, du = o(1),
\]

(N.10)

from which it follows that $G_T = o_\mathcal{P}(1)$ and so $\sqrt{Tb_{1,T}}(\tilde{J}_T - \tilde{J}_T) = o_\mathcal{P}(1)$. The latter concludes the proof of part (ii) because $\sqrt{Tb_{1,T}b_{2,T}}(\tilde{J}_T - J_T) = O_\mathcal{P}(1)$ by Theorem S.B.1.
Let us consider part (iii). Let \(\overline{G}_T = a'\hat{J}_T a - a'\tilde{J}_T a\). We have,

\[
\lim_{T \to \infty} \sup_{T \in \mathcal{P}_U} T b_1 b_2 T \left| \text{MSE}_{\mathcal{G}_T} \left( a'\hat{J}_T a \right) - \text{MSE}_{\mathcal{G}_T} \left( a'\tilde{J}_T a \right) \right| \\
= \lim_{T \to \infty} \sup_{T \in \mathcal{P}_U} T b_1 b_2 T \left| 2E_{\mathcal{G}_T} \left( a'\hat{J}_T a - a'\tilde{J}_T a \right) \overline{G}_T + \mathbb{E}_{\mathcal{G}_T} \left( \overline{G}_T^2 \right) \right| \\
\leq 2 \lim_{T \to \infty} \left( \sup_{T \in \mathcal{P}_U} T b_1 b_2 T \text{MSE}_{\mathcal{G}_T} \left( a'\hat{J}_T a \right) \right)^{1/2} \left( \sup_{T \in \mathcal{P}_U} T b_1 b_2 T \mathbb{E}_{\mathcal{G}_T} \left( \overline{G}_T^2 \right) \right)^{1/2} \\
+ \lim_{T \to \infty} \sup_{T \in \mathcal{P}_U} T b_1 b_2 T \mathbb{E}_{\mathcal{G}_T} \left( \overline{G}_T^2 \right).
\]

The right-hand side above equals zero if (a) \(\lim_{T \to \infty} \sup_{T \in \mathcal{P}_U} T b_1 b_2 T E_{\mathcal{G}_T} (\overline{G}_T^2) = 0\) and (b) \(\lim_{T \to \infty} \sup_{T \in \mathcal{P}_U} T b_1 b_2 T \text{MSE}_{\mathcal{G}_T} (a'\hat{J}_T a) < \infty\). Result (b) follows by Lemma N.B.1-(i). A second-order expansion yields,

\[
\overline{G}_T = \left[ \frac{\partial}{\partial \beta} a'\hat{J}_T (\beta_0) a \right] (\beta - \beta_0) + \frac{1}{2} \left( \beta - \beta_0 \right)' \left[ \frac{\partial^2}{\partial \beta \partial \beta} a'\hat{J}_T (\beta) a \right] (\beta - \beta_0) = \overline{G}_{1,T} + \overline{G}_{2,T},
\]

where \(\beta\) lies on the line segment joining \(\hat{\beta}\) and \(\beta_0\). Note that \(E_{\mathcal{G}_T} (\overline{G}_T^2) = E_{\mathcal{G}_T} (\overline{G}_{1,T}^2) + E_{\mathcal{G}_T} (\overline{G}_{2,T}^2) + 2E_{\mathcal{G}_T} (\overline{G}_{1,T} \overline{G}_{2,T})\). Thus, using Assumption S.B.4,

\[
\sup_{T \in \mathcal{P}_U} T b_1 b_2 T E_{\mathcal{G}_T} (\overline{G}_{1,T}^2)
\]

\[
\leq T b_1 b_2 T p^2 \max_{r \leq p} \sup_{T \in \mathcal{P}_U} E_{\mathcal{G}_T} \left( \frac{\partial}{\partial \beta(r)} a'\hat{J}_T (\beta_0) a \left( \beta(r) - \beta_0(r) \right) \right)^2
\leq \frac{1}{T b_1 b_2 T} \max_{r \leq p} \sup_{T \in \mathcal{P}_U} E_{\mathcal{G}_T} \left( H_{1,T} \sqrt{T} \left( \beta(r) - \beta_0(r) \right) \right)^2
\to 0,
\]

and

\[
\sup_{T \in \mathcal{P}_U} T b_1 b_2 T E_{\mathcal{G}_T} (\overline{G}_{2,T}^2)
\]

\[
\leq \frac{1}{4} T b_1 b_2 T p^2 \max_{r \leq p} \sup_{T \in \mathcal{P}_U} E_{\mathcal{G}_T} \left( \left| \frac{\partial^2}{\partial \beta(r) \partial \beta(r)} a'\hat{J}_T (\beta) a \left( \beta(r) - \beta_0(r) \right) \right| \right)^2
\leq \frac{b_2 T}{T b_1 b_2 T} \max_{r \leq p} \sup_{T \in \mathcal{P}_U} E_{\mathcal{G}_T} \left( \sqrt{T} \left| \beta(r) - \beta_0(r) \right| H_{2,T} \sqrt{T} \left| \beta(r) - \beta_0(r) \right| \right)^2
\to 0.
\]

Equations (N.12) to (N.14) and the Cauchy-Schwartz inequality yield result (a) and thus the desired result of the theorem. \(\square\)
N.B.3 Proof of Proposition S.B.1

For $K_2(\cdot) \in K_2$, using the definition of $\mathcal{P}_U$ and the arguments in (N.1),

$$\text{Var}_{\mathcal{P}_U} \left( a' \tilde{c}_T (u_0, k) a \right)$$

\[ \leq \sup_{\mathcal{P} \in \mathcal{P}_U} \text{Var}_{\mathcal{P}} \left( a' \tilde{c}_T (u_0, k) a \right) \]

\[ = \sup_{\mathcal{P} \in \mathcal{P}_U} \mathbb{E}_{\mathcal{P}} \left[ \left( T b_{2,T} \right)^{-1} \sum_{s=k+1}^{T} K_s (u_0 - (s + k/2) / T) b_{2,T} \left( \tilde{V}_{s-k} a' \left( \tilde{V}_{s-k} \right) a \right)^2 \right] \]

\[ = \sup_{\mathcal{P} \in \mathcal{P}_U} \mathbb{E}_{\mathcal{P}} \left[ \left( T b_{2,T} \right)^{-2} \sum_{s=k+1}^{T} \sum_{l=j+1}^{T} K_s \left( u_0 - (s + k/2) / T \right) K_s \left( u_0 - (l + j/2) / T \right) \right] \]

\[ \times a' \left( \tilde{V}_{s-k} \right) a a' \left( \tilde{V}_{l-j} \right) \]

\[ \leq \left( T b_{2,T} \right)^{-2} \sum_{s=k+1}^{T} \sum_{l=j+1}^{T} \left| K_s \left( u_0 - (s + k/2) / T \right) K_s \left( u_0 - (l + j/2) / T \right) \right| \]

\[ \times (\alpha \Gamma_{U,s/T} (s-l) a a \Gamma_{U,s-k} (s-k-l+j) a + \alpha \Gamma_{U,l/j} (s-l+j) a a \Gamma_{U,s-k} (s-k-l) a + \kappa_{\mathcal{P}_{U,aV}} (j, l-s, l-j-s)) \]

\[ \leq \mathbb{E}_{\mathcal{P}_U} \left( a' \left( \tilde{\sigma}_{T} (u_0, k) - \tau_{\mathcal{P}_U,T} (u_0, k) \right) a \right) \]

\[ + 2 \left( \frac{1}{T b_{2,T}} \right)^2 \sum_{s=k+1}^{T} \sum_{l=j+1}^{T} \left| K_s \left( \frac{(s-k) T - (s-k)}{T} \right) K_s \left( \frac{(l-j) T - (l-j)}{T} \right) \right| \]

\[ \times \kappa_{\mathcal{P}_{U,aV}} (j, l-s, l-j-s) \]

\[ = \text{Var}_{\mathcal{P}_U} \left( a' \tilde{c}_T (u_0, k) a \right), \quad (N.15) \]

where $\tilde{\sigma}_{T}(u_0, k)$ (resp. $\tau_{\mathcal{P}_U,T}(u_0, k)$) is equal to $\tilde{c}_T(u_0, k)$ (resp. $c_{\mathcal{P}_U,T}(u_0, k)$) but with $|K_2(\cdot)|$ in place of $K_2(\cdot)$. Since $K_2(\cdot) \geq 0$ by definition, Proposition 3.1 in Casini (2021b) implies

$$\text{Var}_{\mathcal{P}_U} \left( a' \tilde{c}_T (u_0, k) a \right)$$

\[ = \frac{1}{T b_{2,T}} \int_{0}^{1} K_2^2 (x) dx \sum_{l=\infty}^{\infty} a' \left( c_{\mathcal{P}_U} (u_0, l) \right) \left( c_{\mathcal{P}_U} (u_0, l) + c_{\mathcal{P}_U} (u_0, l + 2k) \right) a \]

\[ + \frac{1}{T b_{2,T}} \int_{0}^{1} K_2^2 (x) dx \sum_{h_1=\infty}^{\infty} \sum_{h_2=\infty}^{\infty} \kappa_{\mathcal{P}_{U,aV},U} (h_1, 0, h_2) \]

\[ + o \left( b_{2,T}^4 \right) + O \left( 1 / (b_{2,T} T) \right) \]

\[ = \text{Var}_{\mathcal{P}_U} \left( a' \tilde{\sigma}_{T} (u_0, k) a \right). \quad (N.16) \]

Next, we discuss the bias. We have,

$$\sup_{\mathcal{P} \in \mathcal{P}_U} \left| \mathbb{E}_{\mathcal{P}} \left( a' \tilde{c}_T (u_0, k) a - a' c_{\mathcal{P}_U} (u_0, k) a \right) \right|$$

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Lemma 2.1

\[ \lim_{T \to \infty} \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{2} \left( \frac{1}{b_{2,T}} \right)^{2} \right) \left( \left( (r+1)n_{T} - (s+k/2) \right) / b_{2,T} \right) a' \mathbb{E}_{\theta} (V_{s}V_{s-k}^{\theta}) a - a' c_{\theta}(u_{0}, k) a \right| \]

\[ \leq \frac{1}{2} b_{2,T}^{2} \int_{0}^{1} x^{2} \sup_{\theta \in \Theta} \left| a' \frac{\partial^2}{\partial^2 u} c_{\theta}(u_{0}, k) a \right| du + a \left( b_{2,T}^{2} \right) + O \left( \frac{1}{T b_{2,T}} \right), \quad (N.17) \]

where the inequality above follows from (S.1). Combining (N.16)-(N.17), we have that \( \sup_{\theta \in P_{U,2}} \text{MSE} \left( a' \tilde{c}_{T}(u_{0}, k,a) \right) \) is equal to the right-hand side of (S.2). The same result holds for \( \tilde{c}_{T}(u_{0}, k) \) since the proof of Theorem S.B.2 and \( P_{U,2} \subseteq P_{U} \) imply that \( \sup_{\theta \in P_{U,2}} \text{MSE} \left( a' \tilde{c}_{T}(u_{0}, k,a) \right) \) is asymptotically equivalent to \( \sup_{\theta \in P_{U,2}} \text{MSE} \left( a' \tilde{c}_{T}(u_{0}, k,a) \right) \). This gives (S.2). The form for the optimal \( b_{2,T} (\cdot) \) and \( K_{2} (\cdot) \) follow from the same argument as in Proposition 4.1 in Casini (2021b).

\[ \Box \]

N.B.4 Proof of Theorem S.B.3

If \( T b_{1,T}^{2q+1} b_{2,T} \to \gamma \in (0, \infty) \) for some \( q \in [0, \infty) \) for which \( K_{1,q}, \left| \int_{0}^{1} f_{U,a}^{(q)} (u, 0) du \right| \in [0, \infty) \), then by Lemma N.B.1-(i) and Lemma N.B.2-(i),

\[ \lim_{T \to \infty} T b_{1,T} b_{2,T} \sup_{\theta \in \Theta} \text{MSE} \left( a' \tilde{J}_{T} (b_{1,T}, K_{1}) a \right) \]

\[ = 4\pi^{2} \left[ \gamma K_{1,q}^{2} \left( \int_{0}^{1} f_{U,a}^{(q)} (u, 0) du \right)^{2} + \int K_{1}^{2} (y) dy \int_{0}^{1} (K_{2,0} (x))^{2} dx \left( \int_{0}^{1} f_{U,a} (u, 0) du \right)^{2} \right]. \]

Assume \( q = 2 \) so that \( T b_{1,T} b_{2,T} \to \gamma \). Then, \( T b_{1,T} b_{2,T} \to \gamma / (\int K_{1}^{2} (y) dy)^{5} \) and

\[ T b_{1,T} b_{2,T} = T b_{1,T} b_{2,T} \int K_{1}^{2} (y) dy. \]

Therefore, given \( K_{1,2} < \infty \),

\[ \lim_{T \to \infty} T b_{1,T} b_{2,T} \left( \sup_{\theta \in \Theta} \text{MSE} \left( a' \tilde{J}_{T} (b_{1,T}, K_{1}) a \right) - \sup_{\theta \in \Theta} \text{MSE} \left( a' \tilde{J}_{T}^{QS} (b_{1,T}) a \right) \right) \]

\[ = 4\gamma^{2} \left( \int_{0}^{1} f_{U,a}^{(q)} (u, 0) du \right)^{2} \int_{0}^{1} (K_{2} (x))^{2} dx \left[ K_{1,2}^{2} \left( \int K_{1}^{2} (y) dy \right)^{4} - (K_{1,2}^{QS})^{2} \right]. \]

The optimality of \( K_{1}^{QS} \) then follows from the same argument as in the proof of Theorem 4.1 in Casini (2021b).

\[ \Box \]

N.B.5 Proof of Theorem S.B.4

Suppose \( \gamma \in (0, \infty) \). Under the conditions of the theorem,

\[ (T b_{2,T})^{2q/(2q+1)} = (\gamma^{-1/(2q+1)} + o(1)) T b_{1,T} b_{2,T}. \]
By Theorem S.B.1-(i),
\[ \liminf_{T \to \infty} \left( Tb_{2,T} \right)^{2q/(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U}(\phi(q))} \mathbb{E}_{\mathcal{P}} L \left( \tilde{J}_{T} (b_{1,T}) , J_{\mathcal{P},T} \right) \]
\[ = \liminf_{T \to \infty} \left( \gamma^{-1/(2q+1)} + o(1) \right) Tb_{1,T} b_{2,T} \sup_{\mathcal{P} \in \mathcal{P}_{U}(\phi)} \sum_{r=1}^{p} \|w_{r}\| \text{MSE}_{\mathcal{P}} \left( a^{(r)} , \tilde{J}_{T} (b_{1,T}) a^{(r)} \right) \]
\[ = \gamma^{-1/(2q+1)} 4\pi^{-2} \sum_{r=1}^{p} \|w_{r}\| \left( \gamma K_{1,q}^{2} \int_{0}^{1} f_{\mathcal{P},a^{(r)}} (u, 0) du \right)^{2} \]
\[ + 2 \int K_{1}^{2} (x) dx \int_{0}^{1} K_{2}^{2} (y) dy \left( \int_{0}^{1} f_{\mathcal{P},a^{(r)}} (u, 0) du \right)^{2} \].

The right-hand side above is minimized at \( \gamma^{\text{opt}} = (2qK_{1,q}^{2} \phi(q))^{-1} (\int K_{1}^{2} (y) dy \int_{0}^{1} K_{2}^{2} (x) dx) \). Note that \( \gamma^{\text{opt}} > 0 \) provided that \( f_{\mathcal{P},a^{(r)}} (u, 0) > 0 \) and \( f_{\mathcal{P},a^{(r)}} (u, 0) > 0 \) for some \( u \in [0, 1] \) and some \( r \) for which \( w_{r} > 0 \). Hence, \( \{b_{1,T}\} \) is optimal in the sense that \( T_{b_{1,T}} b_{2,T} \to \gamma^{\text{opt}} \) if and only if \( b_{1,T} = b_{1,T}^{\text{opt}} + o((Tb_{2,T})^{-1/(2q+1)}) \). In virtue of Theorem S.B.2-(iii), eq. (N.18) holds also when \( \tilde{J}_{T}(b_{1,T}) \) is replaced by \( \tilde{J}_{T}(b_{1,T}) \). Thus, the final assertion of the theorem follows. \( \square \)

**N.B.6 Proof of Theorem S.B.5**

The proof of the theorem uses the following lemmas.

**Lemma N.B.3.** Let \( K_{1}(\cdot) , K_{2}(\cdot) , \{b_{1,\mathcal{P},T}\} , \{S_{\mathcal{P},T}\} , \hat{\phi}(\cdot) \) and \( q \) be as in Theorem S.B.5. Then, for all \( a \in \mathbb{R}^{p} \), (i)
\[ T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{T-1} K_{1} (b_{1,T} k) a' \hat{\Gamma} (k) a \right)^{2} \to 0; \]
(ii)
\[ T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=1}^{S_{\mathcal{P},T}} \left( K_{1} (b_{1,T} k) - K_{1} (b_{1,\mathcal{P},T} k) \right) a' \hat{\Gamma} (k) a \right)^{2} \to 0. \]

**Proof of Lemma N.B.3.** First we prove part (i). We have,
\[ \left( T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{T-1} K_{1} (b_{1,T} k) a' \hat{\Gamma} (k) a \right)^{2} \right)^{1/2} \]
\[ \leq \left( T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{T-1} K_{1} (b_{1,T} k) \left( a' \hat{\Gamma} (k) a - a' \Gamma_{\mathcal{P},T} a \right) \right)^{2} \right)^{1/2} \]

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\[
B_{2,T} \leq \left( T^{8q/10(2q+1)} \sup_{\mathcal{P} \in P_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{T-1} \frac{1}{k!} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{T-1} \left| K_1 \left( b_{1,T}k \right) a' \left( \int_0^1 \Gamma_{\mathcal{P},u}(k) \, du \right) a \right)^2 \right) \right)^{1/2}
\]

(N.20)

\[
\leq T^{8q/10(2q+1)} \sup_{\mathcal{P} \in P_{U,3}} \sum_{k=S_{\mathcal{P},T}+1}^{T-1} C_3 k^{-l}
\]

\[
\leq C_{3,1} T^{8q/10(2q+1)} \int_{S_{\mathcal{P},T}}^\infty k^{-l} dk
\]

\[
\leq C_{3,1} T^{8q/10(2q+1)} S_{T,\mathcal{P}}^{1-l}
\]

\[
= T^{8q/10(2q+1)+4r(1-l)/5(2q+1)} \to 0,
\]

for some constant \( C_{3,1} \in (0, \infty) \), using the fact that \( \inf_{\mathcal{P} \in P_{U,3}} \mathbb{P}_{\mathcal{P}}(\cdot) \geq \phi > 0 \) and \( q/(l-1) < r \). Let

\[
B_{1,1,T} = \left( T^{8q/5(2q+1)} \sup_{\mathcal{P} \in P_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{\lfloor D_T T^{1/2} \rfloor} K_1 \left( b_{1,T}k \right) a' \Gamma_{\mathcal{P},T} a \right)^2 \right)^{1/2}
\]

\[
B_{1,2,T} = \left( T^{8q/5(2q+1)} \sup_{\mathcal{P} \in P_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=\lfloor D_T T^{1/2} \rfloor+1}^{T} K_1 \left( b_{1,T}k \right) a' \Gamma_{\mathcal{P},T} a \right)^2 \right)^{1/2}
\]

We have

\[
B_{1,1,T}^2 \leq T^{8q/5(2q+1)-4/5} \sup_{\mathcal{P} \in P_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{\lfloor D_T T^{1/2} \rfloor} C_1 \left( b_{1,T}k \right)^{-b} \sqrt{T b_{\mathcal{P},T}^{opt}} a' \Gamma_{\mathcal{P},T} (k) a - a' \Gamma_{\mathcal{P},T} (k) a \right)^2
\]

(N.21)

\[
\leq T^{8q/5(2q+1)-4/5+8b/5(2q+1)} \sup_{\mathcal{P} \in P_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=S_{\mathcal{P},T}+1}^{\lfloor D_T T^{1/2} \rfloor} C_1 k^{-b} \sqrt{T b_{\mathcal{P},T}^{opt}} a' \Gamma_{\mathcal{P},T} (k) a - a' \Gamma_{\mathcal{P},T} (k) a \right)^2
\]

\[
\times \left( 2q K_1^2 \hat{\phi} (q) / ( \int K_1^2 (y) \, dy \int_0^1 K_2^2 (x) \, dx ) \right)^{2b/(2q+1)}
\]

\[
\leq C_{1,2} T^{8q/5(2q+1)-4/5+8b/5(2q+1)}
\]
for some constants $0 < C_{1,2}, C_{1,3} < \infty$, using the fact that $\hat{\phi}(q) \leq \overline{\phi} < \infty$, $\inf_{\mathcal{P} \in \mathcal{P}_{U,3}} \phi_{\mathcal{P}} \geq \underline{\phi} > 0$ and $r > 1.25$. Using similar manipulations,

\[
D_{1,2,T}^2 \leq C_{1,2} T^{8q/5(2q+1)-4/5} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=\lfloor D_{T} T^{1/2} \rfloor + 1}^{T} C_1 \left( \hat{b}_{1,T} k - b_{1,\mathcal{P}} \right)^{-b} T_{b_{1,T} T}^{\text{opt}} | a' \tilde{\Gamma} (k) a - a' \Gamma_{b_{1,T} T} (k) a | \right)^2 (N.22)
\]

\[
\leq C_{1,2} T^{8q/5(2q+1)-4/5+8b/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \left( \sum_{k=\lfloor D_{T} T^{1/2} \rfloor + 1}^{T} k^{-b} \right)^2 O(1)
\]

\[
\leq C_{1,3} T^{8q/5(2q+1)-4/5+8b/5(2q+1)-(b-1)} \to 0,
\]

for some constants $0 < C_{1,2}, C_{1,3} < \infty$ and with $q$ satisfying $8/q - 20q < 6$. Equations (N.19)-(N.22) combine to establish part (i). We now prove part (ii). Using the Lipschitz condition on $K_1 (\cdot)$, we get

\[
A_{1,T} = T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=1}^{S_{\mathcal{P}, T}} \left( K_1 \left( \hat{b}_{1,T} k - b_{1,\mathcal{P}} \right) a' \tilde{\Gamma} (k) a \right)^2 \right) (N.23)
\]

\[
\leq T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=1}^{S_{\mathcal{P}, T}} C_2 \left( \hat{b}_{1,T} - b_{1,\mathcal{P}} \right) k a' \tilde{\Gamma} (k) a \right)^2
\]

\[
\leq C_{2,1} T^{8q/5(2q+1)-8/5(2q+1)-n_{T}^{-1}} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=1}^{S_{\mathcal{P}, T}} \left( \sqrt{n_T} \left( \phi(q)^{1/(2q+1)} - \phi_{\mathcal{P}}(q)^{1/(2q+1)} \right) \right) k a' \tilde{\Gamma} (k) a \right)^2
\]

\[
\leq C_{2,1} T^{8q/5(2q+1)-8/5(2q+1)-6/10} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \sum_{k=1}^{S_{\mathcal{P}, T}} \left( \sqrt{n_T} \left( \phi(q)^{1/(2q+1)} - \phi_{\mathcal{P}}(q)^{1/(2q+1)} \right) \right) k a' \tilde{\Gamma} (k) a \right)^2
\]

for some constant $C_{2,1} \in (0, \infty)$, where $n_T = (\inf \{ n_{3,T}/T, \sqrt{n_{2,T}} \})^2$. Now decompose the right-hand
side above as follows,

\[
A_{1,1,T}^{1/2} \leq \left( C_{2,1} T^{8q/5(2q+1)-8/5(2q+1)-6/10} \sup_{\varphi \in \mathcal{F}_{q,3}} \mathbb{E}_{\varphi} \left( \frac{\sqrt{n_T} \left( \hat{\phi}(q) \right)^{1/(2q+1)} - \phi_{\varphi^*}(q)^{1/(2q+1)} }{\left( \hat{\phi}(q) \phi_{\varphi^*}(q) \right)^{1/(2q+1)}} \right)^2 \right)^{1/2} \tag{N.24}
\]

\[
\times \left( \sum_{k=1}^{S_{\varphi,T}} k \left( a' \hat{\Gamma} (k) a - a' \Gamma_{\varphi,T} (k) a \right) \right) \right)^{1/2} \]

\[
A_{1,1,T}^{2} \leq C_{2,1} T^{8q/5(2q+1)-8/5(2q+1)-3/5} S_{\varphi,T}^4 \sup_{\varphi \in \mathcal{F}_{q,3}} \mathbb{E}_{\varphi} \left( \frac{\sqrt{n_T} \left( \hat{\phi}(q) \right)^{1/(2q+1)} - \phi_{\varphi^*}(q)^{1/(2q+1)} }{\left( \hat{\phi}(q) \phi_{\varphi^*}(q) \right)^{1/(2q+1)}} \right)^2 \tag{N.25}
\]

\[
\times \left( \sum_{k=1}^{S_{\varphi,T}} \frac{k}{S_{\varphi,T}} \left( a' \hat{\Gamma} (k) a - a' \Gamma_{\varphi,T} (k) a \right) \right)^{2} \]

\[
\leq C_{2,1} T^{8q/5(2q+1)-8/5(2q+1)-3/5+16r/5(2q+1)-4/5} \]

\[
\times \left( \sup_{\varphi \in \mathcal{F}_{q,3}} \mathbb{E}_{\varphi} \left( \frac{\sqrt{n_T} \left( \hat{\phi}(q) \right)^{1/(2q+1)} - \phi_{\varphi^*}(q)^{1/(2q+1)} }{\left( \hat{\phi}(q) \phi_{\varphi^*}(q) \right)^{1/(2q+1)}} \right)^4 \right)^{1/2} \]

\[
\times \left( \sup_{\varphi \in \mathcal{F}_{q,3}} \mathbb{E}_{\varphi} \left( \frac{1}{S_{\varphi,T}} \sum_{k=1}^{S_{\varphi,T}} \sqrt{T \mathbb{E}_{q,T}^{opt} \left( a' \hat{\Gamma} (k) a - a' \Gamma_{\varphi,T} (k) a \right) } \right)^4 \right)^{1/2} \]

\[
\times \left( 2q K_{1,q}^2 \phi_{\varphi^*}(q) / \int K_{1}^2 (y) dy \int_{0}^{1} K_{2}^2 (x) dx \right)^{4r/(2q+1)} \]

\[
\to 0,
\]

for some constant \( C_{2,1} \in (0, \infty) \), since \( \sup_{\varphi \in \mathcal{F}_{q,3}} \phi_{\varphi^*} < \infty \) and \( r < 15/16 + 3q/8 \). In addition, we have

\[
A_{1,2,T}^{2} \leq C_{2,1} T^{8q/5(2q+1)-8/5(2q+1)-3/5} \sup_{\varphi \in \mathcal{F}_{q,3}} \mathbb{E}_{\varphi} \left( \frac{\sqrt{n_T} \left( \hat{\phi}(q) \right)^{1/(2q+1)} - \phi_{\varphi^*}(q)^{1/(2q+1)} }{\left( \hat{\phi}(q) \phi_{\varphi^*}(q) \right)^{1/(2q+1)}} \right)^2 \tag{N.26}
\]
We can apply Lemma \( \text{Hence, it suffices to show that} \)

\[
\sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \left( \sum_{k=1}^{N.29} k a T_{\mathcal{P}, T} (k) a \right)^2 \leq C_{2,1} T^{8q/5(2q+1) - 8/5(2q+1) - 3/5} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \mathbb{E}_{\mathcal{P}} \left( \frac{\sqrt{\pi_T} (\hat{\phi}(q)^{1/2(2q+1)} - \phi_{\mathcal{P}}^*(q)^{1/2(2q+1)})}{\hat{\phi}(q) \phi_{\mathcal{P}}^*(q)} \right)^2 \times \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \left( \sum_{k=1}^{N.29} k^{1-l} \right)^2 \rightarrow 0,
\]

where we have used the definition of \( \mathcal{P}_{U,3} \), \( q < 1/2 \) and \( l > 2 \) which implies that \( \sum_{k=1}^{\infty} k^{1-l} < \infty \). Equations (N.24)-(N.26) combine to establish part (ii) of the lemma. \( \square \)

**Proof of Theorem S.B.5.** Let \( \| \cdot \|_{\mathcal{P}} = (\mathbb{E}_{\mathcal{P}} (\cdot)^2)^{1/2} \). For any constant \( J \) and any random variables \( \hat{J}_1 \) and \( \hat{J}_2 \), the triangle inequality gives

\[
\| \hat{J}_1 - \hat{J}_2 \|_{\mathcal{P}} \geq \| \hat{J}_1 - J \|_{\mathcal{P}} - \| J - \hat{J}_2 \|_{\mathcal{P}}. \quad (\text{N.27})
\]

Hence, it suffices to show that

\[
T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \left\| a' \hat{J}_T \left( \hat{b}_{1,T}, \hat{b}_{2,T} \right) a - a' \hat{J}_T \left( b_{1,\theta, \mathcal{P}, T}, \hat{b}_{2,T}^\text{opt} \right) a \right\|_{\mathcal{P}}^2 \rightarrow 0. \quad (\text{N.28})
\]

The latter follows from

\[
T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \left\| a' \hat{J}_T \left( \hat{b}_{1,T}, \hat{b}_{2,T} \right) a - a' \hat{J}_T \left( b_{1,\theta, \mathcal{P}, T}, \hat{b}_{2,T} \right) a \right\|_{\mathcal{P}}^2 \quad (\text{N.29})
\]

\[
+ T^{8q/5(2q+1)} \sup_{\mathcal{P} \in \mathcal{P}_{U,3}} \left\| a' \hat{J}_T \left( b_{1,\theta, \mathcal{P}, T}, \hat{b}_{2,T} \right) a - a' \hat{J}_T \left( b_{1,\theta, \mathcal{P}, T}, \hat{b}_{2,T}^\text{opt} \right) a \right\|_{\mathcal{P}}^2 \rightarrow 0.
\]

Note that

\[
a' \hat{J}_T \left( \hat{b}_{1,T}, \hat{b}_{2,T} \right) a - a' \hat{J}_T \left( b_{1,\theta, \mathcal{P}, T}, \hat{b}_{2,T} \right) a \quad (\text{N.30})
\]

\[
= 2 \sum_{k=1}^{S_{\mathcal{P}, T} - 1} \left( K_1 \left( \hat{b}_{1,T} k \right) - K_1 \left( b_{1,\theta, \mathcal{P}, T} k \right) \right) a' \hat{\Gamma} (k) a
\]

\[
+ 2 \sum_{k=1}^{S_{\mathcal{P}, T}} K_1 \left( \hat{b}_{1,T} k \right) a' \hat{\Gamma} (k) a - 2 \sum_{k=1}^{S_{\mathcal{P}, T}} K_1 \left( b_{1,\theta, \mathcal{P}, T} k \right) a' \hat{\Gamma} (k) a.
\]

We can apply Lemma N.B.3-(ii) to the first term of (N.30) and Lemma N.B.3-(i) to second and third terms (with \( \{ b_{1,\theta, \mathcal{P}, T} \} \) in place of \( \{ \hat{b}_{1,T} \} \) for the third term). It remains to show that the second summand of (N.29) converges to zero. Let \( \hat{b}_{2,T} (r n_T / T, k) \) denote the estimator that uses \( b_{2,T}^\text{opt} (u) \) in place of \( \hat{b}_{2,T} (u) \).
We have for $k \geq 0$,
\[
\tilde{c}_T (rn_T / T, k) - \tilde{c}_{\theta_2,T} (rn_T / T, k) = \left( T_{b_{2,T}}^{\text{opt}} \right)^{-1} \sum_{s=k+1}^{T} \left( K_2^* \left( \frac{(r+1)n_T - (s - k/2)}{b_{2,T} ((r+1)n_T / T)} \right) \right) - K_2^* \left( \frac{(r+1)n_T - (s - k/2)}{b_{2,T}^{\text{opt}} ((r+1)n_T / T)} \right) \]
\[+ O_P \left( 1/T_{b_{2,T}}^{\text{opt}} \right). \]

Given Assumption 3.5-(v,vii) S.B.5-(ii,iii) and using the delta method, we have for $s \in \{ Tu - [Tb_{\theta_2,T}], \ldots , Tu + [Tb_{\theta_2,T}] \}$:
\[
K_2 \left( \frac{(Tu - (s - k/2)) / T}{b_{2,T} (u)} \right) - K_2 \left( \frac{(Tu - (s - k/2)) / T}{b_{2,T}^{\text{opt}} (u)} \right) \leq C_4 \left| \frac{Tu - (s - k/2)}{Tb_{2,T} (u)} - \frac{Tu - (s - k/2)}{Tb_{2,T}^{\text{opt}} (u)} \right| \leq CT^{-4/5 - 2/5} T^{2/5} \left| \left( \frac{D_2 (u)}{D_1 (u)} \right)^{-1/5} - \left( \frac{D_2 (u)}{D_{1,\theta} (u)} \right)^{-1/5} \right| |Tu - (s - k/2)| \leq CT^{-4/5 - 2/5} O_P (1) |Tu - (s - k/2)|.
\]

Therefore,
\[
T^{8q/10(2q+1)} \left( a' \tilde{J}_T (b_{\theta_1,T}, \hat{b}_{2,T}) a - a' \tilde{J}_T (b_{1,\theta_2,T}, b_{\theta_2,T}) a \right) \leq T^{8q/10(2q+1)} \sum_{k=-T+1}^{T-1} K_1 (b_{\theta_1,T} k) n_T / T \sum_{r=0}^{[T/n_T]} (a' \tilde{c} (rn_T / T, k) a - a' \tilde{c}_{\theta_2,T} (rn_T / T, k) a) \leq T^{8q/10(2q+1)} C \sum_{k=-T+1}^{T-1} |K_1 (b_{\theta_1,T} k)| n_T / T \sum_{r=0}^{[T/n_T]} \frac{1}{T_{b_{2,T}}^{\text{opt}}} \times \sum_{s=k+1}^{T} \left| K_2^* \left( \frac{(r+1)n_T - (s - k/2)}{b_{2,T} ((r+1)n_T / T)} \right) \right| - K_2^* \left( \frac{(r+1)n_T - (s - k/2)}{b_{2,T}^{\text{opt}} ((r+1)n_T / T)} \right) \times \left| (a' \hat{V}_s \hat{V}_{s-k}' a - E_{\theta_2} (a' \hat{V}_s \hat{V}_{s-k}' a)) + E_{\theta_2} (a' \hat{V}_s \hat{V}_{s-k}' a) \right| \leq H_{1,T} + H_{2,T}.
\]

We have to show that $H_{1,T} + H_{2,T} \xrightarrow{P} 0$. Let $H_{1,1,T}$ (resp. $H_{1,2,T}$) be defined as $H_{1,T}$ but with the sum over $k$ restricted to $k = 1, \ldots , S_T$ (resp. $k = S_T + 1, \ldots , T$). Let $H_{2,1,T}$ (resp. $H_{2,2,T}$) be defined as $H_{2,T}$ but with the sum over $k$ be restricted to $k = 1, \ldots , S_T$ (resp. $k = S_T + 1, \ldots , T$). Using the definition of $P_{U,3}$,
\[
\mathbb{E} \left( H_{1,1,T}^2 \right) \leq T^{8q/5(2q+1)} \sum_{k=1}^{S_T} \sum_{j=1}^{S_T} K_1 (b_{\theta_1,T} k) K_1 (b_{\theta_1,T} j) \left( \frac{n_T}{T} \right)^2 \sum_{r_1=0}^{[T/n_T]} \sum_{r_2=0}^{[T/n_T]} \frac{1}{(T_{b_{2,T}}^{\text{opt}})^2}.
\]

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This completes the proof of part (ii). \( \square \)
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