Evolution of a qubit under the influence of a succession of unsharp measurements

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Abstract

We investigate the evolution of a single qubit subject to a continuous unitary dynamics and an additional interrupting influence which occurs periodically. One may imagine a dynamically evolving closed quantum system which becomes open at certain times. The interrupting influence is represented by an operation, which is assumed to equivalently describe a non-selective unsharp measurement. It may be decomposed into a positive operator, which in case of a measurement represents the pure measurement part, followed by an unitary back-action operator. Equations of motion for the state evolution are derived in the form of difference equations. It is shown that the “free” Hamiltonian is completed by an averaged Hamiltonian, which goes back to the back-action. The positive operator specifies a decoherence rate and results in a decoherence term. The continuum limit to a master equation is performed. The selective evolution is discussed and correcting higher order terms are worked out in an Appendix.

1 Introduction

Experimental and theoretical studies of the dynamics of single two-level systems have become very important in the context of quantum computation and quantum information. In this article we investigate the evolution of a single qubit subject to a continuously acting unitary dynamics (undisturbed or “free” dynamics) with operator $U$ as well as affected by an interrupting additional influence, which is non-unitary and acts periodically at times $t_n = t_0 + n\tau$, $n = 1, 2,\ldots$. The duration $\delta\tau$ of this influence is assumed to
be much shorter than $\tau$, so that it can be neglected. One may imagine a dynamically evolving closed quantum system which becomes open at times $t_n$. The corresponding single influence is represented by an operation $\mathcal{E}$ which transforms the state of the qubit given by its reduced density operator $\rho$ according to

$$\rho \rightarrow \mathcal{E}(\rho) = \sum_{k=\pm} M_k \rho M_k^+$$

(1)

with operation elements $M_k$, which are sometimes also called Kraus operators. $\mathcal{E}$ is assumed to be trace preserving. The representation (1) of the single operation is called the operator-sum representation or Kraus representation.

Such a periodically occurring, nearly instantaneous change can be caused by a recurring interaction with a second system provided that this system does not "remember" the influence it may have experienced from the qubit at former times (Markov process). Typically the second system could be an environment or it could consist of a number of systems of the same kind which interact only once with the qubit, as it is the case in a sequence of scattering processes.

According to eqn. (1) we are not dealing with the most general form of such an influence on a qubit, which would correspond to an operation with four operation elements, but we restrict to interactions which may be represented by only two elements $M_+$ and $M_-$. Pairs of operation elements can describe such important operations as for example amplitude damping and phase damping, bit flips and phase flips as well as projection measurements and unsharp measurements. The concept of an unsharp measurement will be explained below. We will restrict to an operation which is equivalent, as far as the map $\mathcal{E}$ is concerned, to a non-selective unsharp measurement. Note that this may still comprise many physical processes which at first glance do not look like an unsharp measurement. An example is the experiment of Brune, Haroche et al. [1] to measure the number of photons in a cavity, which is discussed under the aspect of an unsharp measurement in [2].

According to the polar decomposition theorem each operation element $M_\pm$ may be written as a product of a unitary operator and a positive operator

$$M_\pm = U_\pm |M_\pm|.$$ 

(2)

We are dealing with a class of generalized measurements with two outcomes + and −. The unitary part $U_\pm$ extracts no information from the qubit. It is in a different context often called the feedback part of the quantum operation [3, 4]. The set $\{|M_-|^2, |M_+|^2\}$ represents a positive-operator valued measure (POVM). The demand of unsharpness of the measurement is introduced below in specifying these $|M_\pm|$ further. A sequence of such generalized measurements are of practical importance because they can be employed to explore the original dynamics of the system [3, 4] or to control its dynamics.
by means of a specific feedback \[3, 6\].

In the non-selective case of eqn. (1), when the measurement results are not read off, our total physical setup defined by \(U, \tau, U_\pm\) and \(M_\pm\) may also be regarded as a particular noisy channel. Below the unitary operators \(U_\pm\) will not be neglected because they form an important part of the sequential operations in realistic situations. In general it would need a non-trivial feedback procedure to eliminate their influence, compare \[2\] for an example.

It is our goal to derive equations of motions for the state of the system subject to the periodical influence. We will do this in form of difference equations and – on a coarse grained time scale – in form of master equations. Difference equations take into account the discrete nature of the influence due to the finite time \(\tau\) between its occurrences and grant therefore a more exact description. It may reveal specific physical influences which cannot be seen in the continuum limit furthermore.

We also look at the limit \(\tau \to 0\) of continuous measurements and compute the corresponding master equation (non-selective description). The master equation can be understood as a means to compute approximately the dynamics of a system which is subject to a sequence of operations of the type (1). Since there are a lot more mathematical methods to solve differential equations than difference equations, it is often useful to approximate sequential measurements by continuous measurements.

Master equations for special cases of measurements with non-minimal disturbance of the state, i.e. with a non-vanishing unitary part of the operation elements have been considered in the literature. A master equation for general feedback was derived by Wiseman \[7\]. In the Markovian limit, if the time delay between feedback and measurement vanishes and the feedback depends only on the outcome of the last measurement (instantaneous feedback), the action of the feedback can be represented as unitary part of the operation of the measurement. However \[7\] does not comprise our results since it deals with a special kind of continuous measurements. They have poissonian statistics and allow finite state changes during infinitesimal time intervals. More precisely this means that only very seldom a certain measurement result occurs which is then connected to a finite state change during a infinitesimal time while for other measurement results the state changes only infinitesimally. We are excluding Poissonian statistics and require the state change in the continuum limit to be infinitesimal during infinitesimal times. Thus Wiseman’s and our studies do not overlap.

In a later paper Wiseman \[4\] employed the operation formalism to analyze a homodyne measurement in quantum optics to apply instantaneous feedback in order to minimize disturbance, i.e., compensate the unitary part of the operation. Korotkov investigated a measurement with non-minimal disturbance in the context of continuous measurement of a qubit by means of a single electron transistor \[8, 9\]. He noted that this non-minimal disturbance acts in the master equation like a change of the distance between the
two energy levels of the qubit. We find this effect as a special case of our
studies ($[U_\pm, H] = 0$). Korotkov also derives a modification of the stochastic
master equation (selective regime) due to the non-minimal disturbance [8],
which is averaged out in the non-selective case.

Many examples of the application of instantaneous feedback in continuous
measurements by means of external changes of the Hamiltonian of
the system can be found in the literature of quantum control (e.g. [3, 6]).
They correspond to special choices of the unitary part of the measurement
operation and obey the equations of motions derived here, provided the
measurements do not inflict finite changes during infinitesimal times.

In the context of quantum dissipation the influence of a heat bath on an
infinite dimensional quantum system has been investigated by Caldeira and
Leggett [10]. The coupling was such that in terms of the operation formalism
the corresponding operation of the system had a unitary part additional to
the one stemming from its free original evolution. They derived a master
equation for high temperatures which was later modified to also describe
medium temperatures [11]. Although we are looking at a qubit we find in
the difference equation among others similar terms like there, but only one
of them survives in the continuum limit.

In order to discuss the problem in an illustrative way but without re-
striction of generality, we treat the periodical influence in terms of a se-
quential measurement. We proceed as follows. We first consider the sin-
gle measurements of the sequence. Then we bundle the whole sequence in
subsequences of N measurements (“N-series“). The resulting operation has
Gaussian shape. Afterwards we integrate over all outcomes to obtain the av-
erage state change due to a N-series (non-selective regime). We then discuss
how to find the right continuum limit which conserves the physical charac-
teristics of the N-series and derive the master equation. Finally we deal with
the selective regime of measurement and write down the stochastic master
equation. The Appendices serve to derive the difference equation for the
non-selective regime up to second order in the occurring small parameters.

2 The single quantum operation

An example for a physical realisation of the operation $E$ from (1) is given
by a qubit which interacts at times $t_n$ unitarily for a short duration $\delta \tau$ with
an environment and thus becomes an open system. The resulting change
of its reduced density operator may formally be expressed with the help of
operation elements $M_\pm$ as in eqn. (1). Just the same change of the reduced
density operator results if i) a projection measurement on the environment
is performed with outcome + or − transferring the qubit to the states

\[ \rho \longrightarrow \rho_\pm = M_\pm \rho M_\pm. \] (3)
respectively and ii) the outcome is not read off (non-selective case). In
the generic case, eqn. (3) describes thereby a generalized measurement
of the qubit. For simplicity reasons the terminology we are going to use
will refer to measurements, but the results apply equally to any operation
with operator-sum representation (1) if the same specifications of $M_\pm$
are made. This is independent of how the operation is experimentally realized.

Because of the polar decomposition theorem, the operation elements $M_\pm$
may be written as products of a unitary operator and a positive operator

\[ M_\pm = U_\pm |M_\pm| . \]  

(4)

We introduce the POVM effects

\[ E_\pm = |M_\pm|^2 , \]  

(5)

which obey the completeness relation

\[ E_+ + E_- = 1 . \]  

(6)

The probability of the outcome $+$ or $-$ is given by

\[ p_\pm = \langle E_\pm \rangle_\rho \]  

(7)

with $p_+ + p_- = 1$.

Eqn. (4) represents a decomposition of the operation into a pure measure-
ment part described by $|M_\pm|$, followed by a unitary back-action given
by $U_\pm$ depending on the result $+$ or $-$. These denominations are justi-
fied for the following reasons: All the information which can be read
off from the meter is related to $|M_\pm|$ which therefore represents the unavoid-
able minimal disturbance. The unitary operators leave the von Neumann
entropy unchanged and therefore do not allow to export information to an
observer. Because they depend on the result $+$ or $-$, they may be inter-
preted as a specific back-action of the measuring apparatus inducing an
additional Hamiltonian evolution of the qubit. We formally introduce the
corresponding Hamiltonians $H_\pm$ according to

\[ U_\pm =: \exp \left( -\frac{i}{\hbar} H_\pm \tau \right) . \]  

(8)

This unitary back-action represents an important part of the quantum op-
eration and appears naturally in the generic situation.

According to eqns. (5) and (6), $|M_+|^2$ and $|M_-|^2$ commute. Therefore
we can find orthonormal basisvectors $|1\rangle$ and $|2\rangle$ of the qubit Hilberts-
pace with respect to which $|M_\pm|^2$ are diagonal. We introduce the eigenvalues $p_1$
and $p_2$ of $|M_+|^2$, which are positive and because of eqn. (6) obey $0 \leq p_{1,2} \leq$
1. Without restriction of generality we choose \( p_2 \geq p_1 \). Reading off the eigenvalues of \(|M_-|^2\) from (8) and taking the square root we find

\[
|M_+| := \sqrt{p_1} |1\rangle\langle 1| + \sqrt{p_2} |2\rangle\langle 2|
\]

\[
|M_-| := \sqrt{1 - p_1} |1\rangle\langle 1| + \sqrt{1 - p_2} |2\rangle\langle 2|
\]

The elements \(|M_+|\) and \(|M_-|\) commute. We will characterize the operation later on by the parameters

\[
p_0 := \frac{1}{2}(p_1 + p_2) , \quad \Delta p := p_2 - p_1
\]

with \( 0 \leq \Delta p \leq 1 \). Introducing

\[
\sigma_z := |1\rangle\langle 1| - |2\rangle\langle 2|
\]

the effects \( E_{\pm} \) of eqn. (5) are rewritten in the form

\[
E_+ = p_0 \mathbf{1} - \frac{1}{2} \Delta p \sigma_z , \quad E_- = (1 - p_0) \mathbf{1} + \frac{1}{2} \Delta p \sigma_z .
\]

In the limiting case \( \Delta p = 1 \), the pure part of the measurement (8) results in a projection on \(|1\rangle\) or \(|2\rangle\) depending on the measurement outcome + or −. We call this a sharp measurement of an observable with eigenvectors \(|1\rangle\) and \(|2\rangle\), for example \(\sigma_z\). Note that also for a sharp measurement the result of the quantum operation (7) will in general not be the state \(|1\rangle\) or \(|2\rangle\) because of the remaining influence of the unitary back-action.

In the contrary limit \( \Delta p \ll 1 \) the \(|M_+|\) and \(|M_-|\) are nearly proportional to the identity operator. The probability \( p_+ \) (or \( p_- \)) to obtain the measurement result + (or −) is then nearly independent of the initial state of the qubit. There is almost no state discrimination. Because of this low sensitivity we call this an unsharp measurement. Note that in this limit the parameters \( p_0 \) and \( 1 - p_0 \) become, because of eqns. (7) and (12), approximately the mean probabilities to obtain the measurement results + or − respectively. “Unsharpness” does of course not originate from a measurement apparatus which is “broken”.

We are now able to further specify the particular class of quantum operations (1) which we are going to discuss. We will restrict to the case that the pure measurement part represents an unsharp measurement: \( \Delta p \ll 1 \).

We will mainly be interested in the non-selective case where an information about the results \(+\) or the corresponding states of the environment is not available. The influence on the qubit at times \( t_n \) may then be written in the operator-sum representation as in eqn. (1).

\[
\rho \rightarrow \mathcal{E}(\rho) = \sum_{k=\pm} M_k \rho M_k^+, \quad (13)
\]

whereby

\[
\sum_{k=\pm} M_k^+ M_k = \mathbf{1} .
\]

because of (8). The quantum operations at times \( t_n \) are trace preserving.
3 N-series and related operation

The time between measurements is $\tau$. We assume that the duration $\delta \tau$ of a measurement is much shorter than $\tau$. The undisturbed or “free” dynamics of the system between the measurements is given by the Hamiltonian $H$. We bundle $N$ consecutive measurements to a $N$-series of duration $\Delta t = N\tau$ as we have done in [2] (cp. also [12]). This procedure has several advantages. We will obtain effects of a Gaussian structure. This enables us to work out the operator sum explicitly. A comparison with the results in the literature regarding continuous measurements becomes more evident. And finally the discussion of the selective case is simpler.

We require

$$N \gg 1. \tag{15}$$

We relate $N$ to the sharpness $\Delta p \ll 1$ of the measurement by

$$N \cdot \Delta p \ll 1 \tag{16}$$

and demand in addition

$$\Delta t \| H \| \ll \hbar \tag{17}$$

and

$$\Delta t \| H_\pm \| \ll \hbar. \tag{18}$$

This means that the influence of the undisturbed dynamics of the qubit and the unitary back-action dynamics due to the measurements are both small over the duration $\Delta t$ of a $N$-series. With $\Delta t = N\tau$ we have obtained above restrictions for $N, \tau, H$ and $H_\pm$.

The density operator resulting at the end of a $N$-series of measurements with results $m_1, ... m_N$, each of which can assume the values “+” and “−”, read

$$\rho(t + \Delta t) = M_{m_N} U ... M_{m_2} U \rho(t) U^+ M_{m_1}^+ ... U^+ M_{m_N}^+ \tag{19}$$

with

$$U := \exp \{-i \frac{\hbar}{\tau} H \}. \tag{20}$$

The influences of the systems dynamics and the measurement will in general not commute. The following relation is derived in Appendix A:

$$M_{m_N} U M_{m_{N-1}} U ... M_{m_1} U = M_{m_N} M_{m_{N-1}} ... M_{m_1} U^N (1 + C_1) \tag{21}$$

with

$$\|C_1\| \leq O(N \Delta p \Delta t \| H \| / \hbar) + O(\Delta t^2 \| H \| \max \{\| H_\pm \| \} / \hbar^2). \tag{22}$$

Also the operations $M_\pm$ will not commute. Based on the decomposition [4] we show in Appendix A that

$$U_{m_N} | M_{m_N} \rangle ... U_{m_1} | M_{m_1} \rangle U^N = U_+^{N_+} U_-^{N-N_+} | M_+ \rangle^{N_+} | M_- \rangle^{N-N_+} U^N (1 + C_2). \tag{23}$$
\[
\|C_2\| \leq \mathcal{O}(N\Delta p \Delta t) \max\{\|H_{\pm}\|/\hbar\} + \mathcal{O}(\Delta t^2\|H_{\pm}\|\|H_{\mp}\|/\hbar^2).
\] (24)

\(N_+\) and \(N - N_+\) are the total numbers of measurement results “+” and “−” in the N-series respectively. Because of the assumptions (16), (17) and (18) we may neglect \(C_1\) and \(C_2\). The calculation which takes into account \(C_1\) and \(C_2\), is given in the Appendix C.

In our approximation the influence (19) of the N-series becomes a function of \(N_+\) only, independent of the \((\binom{N}{N_+})\) different orderings of the “+” and “−” results (cp. eqn. (23)). Therefore the total N-series of duration \(\Delta t\) including the “free” dynamics corresponds to a quantum operation with operation elements

\[
M(N_+, N) = U_+^{N_+} U_-^{N - N_+} |M(N_+, N)| \exp \left\{ -\frac{i}{\hbar} H \Delta t \right\}
\] (25)

with

\[
|M(N_+, N)| = \sqrt{\binom{N}{N_+}} |M_+|^{N_+} |M_-|^{N - N_+} \quad (26)
\]

\[
= \sqrt{\binom{N}{N_+}} \left( p_1^{N_+} (1 - p_1)^{N - N_+} |1\rangle\langle 1| + p_2^{N_+} (1 - p_2)^{N - N_+} |2\rangle\langle 2| \right).
\]

The square root in front ensures the completeness relation of the effects

\[
\sum_{N_+=0}^N M(N_+, N)^\dagger M(N_+, N) = 1.
\] (27)

The unitary back-action part caused by the N-series measurements can be expressed as

\[
U(N_+, N) := U_+^{N_+} U_-^{N - N_+} = \exp \left\{ -\frac{i}{\hbar} (N_+ H_+ + (N - N_+) H_-) \tau \right\}.
\] (28)

We now make use of the condition that \(N\) is large, so that \(|M(N_+, N)|\) of eqn. (26) may approximately be written in form of a Gaussian

\[
|M(N_+, N)| = \frac{1}{\sqrt{2\pi NE_+ E_-}} \exp \left\{ -\frac{(NE_+ - N_+)^2}{4NE_+ E_-} \right\},
\] (29)

which contains the operators \(E_\pm\) of (12). Because we assumed that the measurements are unsharp and therefore \(\Delta p \ll 1\), the “spread” of the Gaussian becomes in lowest order a c-number

\[
E_+ E_- = \left( p_0(1 - p_0) - \frac{1}{4}(\Delta p)^2 \right) I + \Delta p(p_0 - \frac{1}{2}) \sigma_z = p_0(1 - p_0),
\] (30)
where we have ignored terms of order \( \Delta p \) and higher on the right-hand side of (30). The error thus committed in the Gaussian in (29) is of order \( N \Delta p^3 \), which can be seen by inserting \( E_+ \) from (12) and \( N_+/N \) from (31) and expanding Gaussian (29) in powers of \( \Delta p \). A more detailed calculation can be found in Appendix C.

We introduce a new variable \( s \) to replace the readout \( N_+/N \) according to
\[
\frac{N_+}{N} =: p_0 - \frac{1}{2} \Delta ps. \tag{31}
\]
Because \( N \) is large we may approximately regard \( s \) to be continuous. Its range is limited by:
\[
0 \leq p_0 - \frac{1}{2} \Delta ps \leq 1. \tag{32}
\]

In addition we introduce the new quantity
\[
\gamma := \frac{(\Delta p)^2}{4p_0(1-p_0)\tau}, \tag{33}
\]
which will turn out to be the decoherence rate. It contains apart from \( \Delta p \) and \( p_0 \) also the time interval \( \tau \) between two measurements. These three parameters characterize completely the influence of the sequence of pure measurements. \( \gamma \) increases when the measurements become sharper and accordingly have a stronger influence on the qubit. A decreasing time difference \( \tau \) between two measurements results as well in an increase of \( \gamma \). This reflects a Zeno type effect which also happens for unsharp measurements.

Installing \( \gamma \) we get the ultimate form of the operations, valid for \( \Delta p \ll \tau/\Delta t \ll 1 \):
\[
|M_s| = \frac{1}{\sqrt{2\pi/(\gamma \Delta t)}} \exp \left\{ -\gamma \left( \sigma_z - s \right)^2 \frac{4}{\Delta t} \right\}. \tag{34}
\]
The resulting effects \( E_s = |M_s|^2 \) have Gaussian form. They show the characteristics which are known for instance from an unsharp position measurement as investigated e.g. in [13]. Instead of a continuous observable however, we are dealing here with a discrete observable.

With reference to \( s \), the complete operation element including the back-action and the “free” evolution is given by
\[
M_s = U_s|M_s| \exp \left\{ -\frac{i}{\hbar} H \Delta t \right\}, \tag{35}
\]
where, using (28) and (31), we obtain for \( U_s \) which replaces \( U(N_+, N) \):
\[
U_s = \exp \left\{ -\frac{i}{\hbar} H_{AV} \Delta t - \frac{i}{2\hbar} \Delta H s \Delta p \Delta t \right\}. \tag{36}
\]
We have thereby introduced the averaged back-action Hamiltonian \( H_{AV} \) and the difference \( \Delta H \) of the back-action Hamiltonians, respectively:
\[
H_{AV} := p_0 H_+ + (1-p_0) H_- \tag{37}
\]
\[
\Delta H := (H_- - H_+). \tag{38}
\]
\( M_s \) of (35) replaces \( M(N_+, N) \) of eqn. (25) for the continuous variable \( s \).

The N-series operation elements above correspond to a continuous set of effects with the Gaussian distribution function

\[
p_s = \langle M_s^+ M_s \rangle \rho.
\]

The completeness relation is satisfied if we extend the range of \( s \) to the whole real axis:

\[
\int_{-\infty}^{\infty} M_s^† M_s ds = 1
\]

The statistical weight of the unphysical values of \( s \) will be negligible provided that \( p_0 \not\approx 0, 1 \) and this justifies the formal extension of the values of \( s \) beyond their physical range (32).

4 Non-selective evolution

In the non-selective case the state change during a N-series can be expressed in the operator-sum representation as

\[
\rho(t + \Delta t) = \int_{-\infty}^{\infty} M_s \rho M_s^† ds.
\]

We are going to expand the r.h.s. up to linear terms in \( \Delta t \).

The unitary parts of the operation \( M_s \) which are generated by \( H \), \( H_{AV} \) and \( \Delta H \) lead to

\[
\Delta \rho := \rho(t + \Delta t) - \rho(t) = -\frac{i}{\hbar} [H + H_{AV}, \rho(t)] \Delta t + D(\rho(t)) - \rho(t).
\]

The integral

\[
D(\rho) := \int_{-\infty}^{\infty} \exp \left\{ -\frac{i}{2\hbar} \Delta H \sigma_L \Delta p \Delta t \right\} |M_s|\rho|M_s| \exp \left\{ \frac{i}{2\hbar} \Delta H s \Delta p \Delta t \right\} ds
\]

over the parts which depend on \( s \) will first be calculated and then expanded.

Introducing operators which act from the left and are denoted with \( L \) (e.g. \( \sigma^L_z \rho := \sigma_z \rho \)) as well as operators which act from the right and are denoted with \( R \) (e.g. \( \sigma^R_z \rho := \rho \sigma_z \)), we rewrite the integrand of \( D(\rho) \)

\[
D(\rho) = \frac{1}{\sqrt{2\pi\gamma/\Delta t}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{i}{2\hbar} \left( \Delta H^L - \Delta H^R \right) s \Delta p \Delta t \right\} \times \exp \left\{ -\frac{\gamma}{4} \left( (\sigma^L_z - s)^2 + (\sigma^R_z - s)^2 \right) \Delta t \right\} ds \rho.
\]

It is important to take into account a further operator ordering for the integrand, namely that \( \Delta H^L, \Delta H^R \) should remain leftmost and rightmost,
respectively. The resulting integral is Gaussian in $s$. It may be solved in a closed form:

$$
D(\rho) = \exp\left\{ -\frac{\gamma}{8}(\sigma_L^z - \sigma_R^z)^2 + \frac{\Delta p^2}{8\gamma\hbar^2} (\Delta H_L - \Delta H_R)^2 + i\frac{\Delta p}{4\hbar} \left( \sigma_L^z + \sigma_R^z \right) \left( \Delta H_L - \Delta H_R \right) \right\} \Delta t \rho.
$$

(45)

Now we expand it up to the leading linear term in $\Delta t$ and restore the usual operator formalism according to the rules for $L$ and $R$. This leads for example to

$$
(\sigma_L^z - \sigma_R^z)^2 \rho = (\sigma_L^z - \sigma_R^z)(\sigma_L^z - \sigma_R^z)\rho = (\sigma_L^z - \sigma_R^z)[\sigma_z, \rho] = [\sigma_z, [\sigma_z, \rho]]
$$

(46)

and all together to

$$
D(\rho) - \rho = \left( -\frac{\gamma}{8}[\sigma_z, [\sigma_z, \rho]] - \frac{(\Delta p)^2}{8\gamma\hbar^2}[\Delta H, [\Delta H, \rho]] - i\frac{\Delta p}{4\hbar}[\Delta H, \{\sigma_z, \rho]\} \right) \Delta t.
$$

(47)

While the second and third term are proportional to small quantities (cp. (16) – (18)), the first contains the ratio of the two small quantities $\Delta p^2$ and $\tau$ (cp. (33)). We assume $\gamma$ not to be small. We thus obtain as final result for the state evolution during one N-series

$$
\Delta \rho = \left( -\frac{i}{\hbar} [H + H_{AV}, \rho] - \frac{\gamma}{8}[\sigma_z, [\sigma_z, \rho]] \right) \Delta t.
$$

(48)

The first term on the r.h.s. represents the unitary dynamical evolution related to the “free” Hamiltonian $H$ and to the averaged Hamiltonian $H_{AV} = p_0 H_+ + (1 - p_0) H_-$. $p_0$ and $1 - p_0$ are approximately the probabilities that the back-action causes a Hamiltonian development with $H_+$ or $H_-$ respectively (cp. section 2). The second term on the r.h.s. reflects the decoherence induced by the pure measurement part $|M_\pm|$ of the operation. The structure of both terms is clearly what one would expect on physical grounds.

The second and third term of eqn. (47) indicate additional physical effects, which are to be expected in a higher order approximation. The second term corresponds to further decoherence induced by unitary back-action. The third term goes back to friction caused also by the back-action. A complete list of additional terms is given in Appendix C.

Eqn. (48) has been derived on the basis of the following approximations:

We have changed the order of $U$, $U_\pm$ and $|M_\pm|$ in the operation of the N-series, neglecting the commutators between them, cp. (21) and (23), the estimated error is smaller than $O(\Delta p \Delta t ||H||/\hbar) + O(\Delta t^2 ||H|| \max\{||H_\pm||\}/\hbar^2) + O(N\Delta p \Delta t ||\max\{||H_\pm||\}/\hbar) + O(\Delta t^2 ||H_\pm||||H_-||/\hbar^2)$. We also have approximated the q-number denominator of the Gaussian operation elements.
by a c-number, cp. (30), which leads to an error of order $O(\Delta p \Delta t \gamma)$. We further expanded the operation in powers of $\Delta t$ up to the first order, cf. (42) and (44), which results in errors of order $O(\Delta t^2 \| H + H_{AV} \|^2 / \hbar^2)$ and $O(\Delta t^2 \gamma \| H + H_{AV} \| / \hbar)$. In the continuum limit the errors all vanish but they can play an important role for discrete sequences of measurements if $\Delta p \ll 1$ is not fulfilled. Appendix C contains a calculation of the state change up to higher orders. This more accurate calculation confirms the order of the errors estimated here. In case of the neglected commutators in (21) and (23) the error turns out to be actually smaller (cp. (68)).

5 Continuum limit

In Section 4 we have worked out the operation given by the discrete state transformation between an initial state $\rho(t)$ and the final state $\rho(t + \Delta t)$ after a N-series of instantaneous interactions of a qubit with an environment, which are of the type of an unsharp measurement. It can be applied to a truly sequential measurement by dividing the sequence of elementary measurements into a succession of N-series. Since the r.h.s. of the equation (45) is proportional to $\Delta t$, the given approximation is not sensitive to the division as long as $N$ is large. The rate of the change of $\rho$ is invariant under this the division. This discrete-time analysis is the most natural approach to sequential measurements. For example see [5]. Eqn. (48) above reveals the underlying physics as represented by the decoherence rate $\gamma$ and the measurement induced unitary development given by $H_{AV}$.

There are elaborated schemes for the treatment of permanently open quantum systems by continuous-time descriptions. Master equations are an example. One may profit from these schemes as approximations in the sequentially open case too, if a physically reasonable continuum limit $\tau \to 0$ is carried out. The corresponding demand for such a limit is that the physical characteristics of the sequential situation have to be taken over. We proceed as follows:

The quantity $p_0$ is the mean probability to obtain the measurement result $+$. We leave the value of $p_0$ unchanged in the continuum limit. In order to not change the decoherence behavior in the continuum limit we secondly demand for the decoherence rate:

$$\lim_{\tau \to 0} \frac{(\Delta p)^2}{4p_0(1 - p_0)\tau} = \gamma = \text{const.} \quad (49)$$

The smaller $\Delta p$ the weaker the single measurement. With $\tau \to 0$ and the strength $\Delta p$ of the single measurement unchanged, a Zeno effect would be obtained. This is prevented by appropriately diminishing the strength $\Delta p$ of the measurement according to (49). This demand can also be found in the literature [14].
If in a given sequential physical situation the $H_{AV}$ is non-vanishing, then the total Hamiltonian dynamics is according to (48) governed by the Hamiltonian $H + H_{AV}$. We want to keep this dynamics in the continuum limit on physical grounds and demand therefore that $H_{AV}$ remains unchanged.

Performing the limit $\tau \to 0$ as specified above results in the master equation

$$\dot{\rho} = -\frac{i}{\hbar} [H + H_{AV}, \rho] - \frac{\gamma}{8} [\sigma_z, [\sigma_z, \rho]],$$

(50)

which describes approximately the discontinuous situation in the noisy channel characterized above.

We note that the master equation (50) could have been obtained in the limit (49) directly from the elementary measurements. The intervening formulation in terms of N-series will be exploited by the forthcoming equations of selective evolution.

6 Selective evolution

In Sec. 3 we calculated the Gaussian form (34) of effective operation (35) valid for a N-series. In Sec. 5 we derived the master equation (50) valid exactly in the continuous limit (49). As a matter of fact, the master equation describes the non-selective evolution. Selective evolution is, on the contrary, conditioned on the random measurement results (readout) and described by stochastic equations. In our case, the readout is $s$. It is the continuously measured unsharp value of the observable $\sigma_z$ obtained in the N-series in the limit (49).

The theory of the selective evolution has been available since long ago [15]. From the Gaussian operations (34) in the limit $\Delta t \to 0$, it has been proved that the selective evolution of the quantum state, conditioned on the measurement result $s$, satisfies the conditional master equation:

$$\dot{\rho} = -\frac{i}{\hbar} [H + H_{AV}, \rho] - \frac{\gamma}{8} [\sigma_z, [\sigma_z, \rho]] + w \sqrt{\gamma} \left\{ \sigma_z - \langle \sigma_z \rangle, \rho \right\}$$

(51)

The function $w(t)$ is the standard white-noise and the equation should be understood in the Itô-stochastic sense. The state evolution couples to the readout $s$ by:

$$s = \langle \sigma_z \rangle + \frac{1}{\sqrt{\gamma}} w$$

(52)

Obviously the stochastic mean of the conditional master equation (51) reduces to the unconditional master equation (50) as it should. Of course eqn. (51) applies to pure initial states as well. Then the pure state property $\rho^2 = \rho$ is preserved. The derivation may completely be identical to that in Ref. [15]. In the continuum limit (49) the value of $\Delta p$ must vanish and the feed-back $U_s$ is thus deterministic, given by $H_{AV}$ alone.
The above equations of selective evolution are exact in the following sense. Elementary operations are being applied with frequency growing to infinity and strength decreasing to zero as given by (49), i.e., at fixed $\gamma$. We read out the rate $N_+/N$ averaged over time $\Delta t$ which should go to zero in such a way that $N = \Delta t/\tau$ still goes to infinity. The elementary time $\tau$ goes “faster” to zero than the time $\Delta t$ to calculate the rate $N_+/N$. The calculated current rate $N_+/N$ is related to $s$ by (31):

$$s = \frac{p_0 - N_+/N}{\sqrt{\gamma p_0(1-p_0)\tau}}$$

(53)

The continuous limit (49) of $s$ exists. As follows from (52), it is centered around a state dependent part $\langle \sigma_z \rangle$ and superposed by the white-noise of constant intensity $1/\gamma$.

Appendix A

In the appendices we sketch the calculation of the change of state in the non-selective regime including all terms up to order $O(\Delta p^2)$ and $O(\Delta t^2)$, where $\Delta t$ occurs in products with either $H/\hbar$, $H_\pm/\hbar$ or $\gamma$.

We start with the exact operation element $\Omega$ for a N-series with unitary development $U$ between consecutive measurements.

$$\Omega_{(m)} := U_{m_N}...U_{m_1}|M_{m_N}|...|M_{m_1}|U^{N-1} + R_1 + R_2,$$

(54)

where $R_1$ and $R_2$ are the terms which arise from commuting out the evolution operators $U$ and the feedback operators $U_\pm$, respectively, cp. equations (21) and (23). The relation of $R_1$ and $R_2$ to $C_1$ and $C_2$ and an estimation of the order of magnitude of $C_1$ and $C_2$ is described below.

Since the commutators $K_{m_1} := [U, M_{m_1}]$ occurring in $R_1$ are of order $O(\Delta p^2/h) + O(\Delta t^2)\max\{\|H_\pm\|/\hbar^2\}$, we can neglect terms containing products of two such commutators.

$$R_1 = M_{m_N}K_{m_{N-1}}...M_{m_2}U^{N-1} + 2M_{m_N}M_{m_{N-1}}K_{m_{N-2}}...M_{m_2}U^{N-1}$$

$$+ ... + (N-1)M_{m_N}...K_{m_1}U^{N-1}$$

(55)

$$= \Delta p\sigma_z U_{m_N}...U_{m_1} \sum_{n=1}^{N-1} \frac{n}{a_{m_{N-n}}} |M_{m_N}|...|M_{m_1}|U^{N-1}$$

$$- \frac{\tau^2}{\hbar^2} \sum_{n=1}^{N-1} n [H, H_{m_{N-n}}] U_{m_N}...U_{m_1} |M_{m_N}|...|M_{m_{N-n}}|...|M_{m_1}|U^{N-1},$$

where $a_+ := -4\sqrt{p_0}$ and $a_- := 4\sqrt{1-p_0}$. Please note, that in (55) behind the sum sign the products of $|M|$'s and $U$'s are meant to not contain $M_{m_{N-n}}$ and $U_{m_{N-n}}$, except if explicitly mentioned. In the last two lines we have commuted out the feedback $U_\pm$. The resulting error is of higher order and can be neglected.
Also in $R_2$ we only take into account the terms containing one commutator $K_{k,l} := [M_{m_k}, U_{m_l}]$, which is of order $O(\Delta p \tau \max\{\|H_0\|\}/\hbar)$.

\[ R_2 = U_{m_N} \cdots U_{m_2} K_{2,1} |M_{m_N}| \cdots |M_{m_2}| |M_{m_1}| U^N + \ldots \] (56)

\[ + \left( U_{m_N} \cdots U_{m_2} K_{N,1} + U_{m_N} \cdots U_{m_3} U_{m_1} K_{N,2} + \ldots \right) + U_{m_N} U_{m_{N-2}} \cdots U_{m_1} K_{N,N-1} \right) |M_{m_{N-1}}| \cdots |M_{m_1}| U^N \]

\[ = -\frac{i \Delta p \tau}{\hbar} \sum_{k=2}^{N} \sum_{l=1}^{k-1} \frac{1}{a_{mk}} [\sigma_z, H_{m_l}] U_{m_N} \cdots U_{m_{l+1}} U_{m_{l-1}} \cdots U_{m_1} \times \]

\[ \times |M_{m_N}| \cdots |M_{m_{k-1}}| |M_{m_{k-1}}| \cdots |M_{m_1}| U^{N-1} \]

\[ -\frac{\tau^2}{\hbar^2} \sum_{k=2}^{N} \sum_{l=1}^{k-1} [H_{m_k}, H_{m_l}] U_{m_N} \cdots U_{m_{k+1}} U_{m_{k-1}} \cdots U_{m_{l+1}} \times \]

\[ \times U_{m_{l-1}} \cdots U_{m_1} |M_{m_N}| \cdots |M_{m_1}| U^{N-1}. \]

Let us shortly motivate the estimation of the order of magnitude of $C_1$ and $C_2$ given in (22) and (24). First we observe that $R_1 = M_{m_N} \cdots M_{m_1} U^N C_1$ and $R_2 = U_{N+1} U_{N^{-N+1}} |M_{N^{-N+1}}| \cdots U_{N} C_2$. A moment’s thought shows that the order of magnitude of the summands contained in $C_i$ is equal to the order of the commutators $K_{m_i}$ and $K_{k,l}$ in $R_i$. Since there are approximately $N^2$ such summands in $C_i$, the norm of $C_i$ can be estimated to be less or equal to $N^2$ times the order of the commutators in $R_i$ which leads to the claims (22) and (24).

Appendix B

The state change due to a N-series in the non-selective regime reads

\[ \rho(t + \Delta t) = \sum_{\Omega(m_i)} \rho \Omega^+_{(m_i)} \]

\[ = \sum_{m_1 \cdots m_N} U_{m_N} \cdots U_{m_1} |M_{m_N}| \cdots |M_{m_1}| U(\Delta t) \times \]

\[ \times \rho U^+ (\Delta t)|M_{m_1}| \cdots |M_{m_N}| U^+_{m_1} \cdots U^+_{m_N} \]

\[ + \tilde{R}_1 + \tilde{R}_2 + O(R_1^2) + O(R_2^2) + O(R_1 R_2) \] (57)

with

\[ \tilde{R}_i := \sum_{m_1 \cdots m_N} \left\{ R_i \rho U^+ (\Delta t)|M_{m_1}| \cdots |M_{m_N}| U^+_{m_1} \cdots U^+_{m_N} + h.c. \right\} \]

\[ i = 1, 2. \] (58)

In order to carry out the summation in (57) its terms can be expressed by means of binomial distributions. The latter can be approximated by
integrals over Gaussians. For the first term in (57) this recipe has already been demonstrated in section 3 and 4. $\tilde{R}_1$ can be written in binomial form by observing that:

$$
\sum_{m_1 \ldots m_N} \sum_{n_1 = -1}^{N-1} n b_{mN-n} U_m \ldots U_m \left| M_{m_N} \right| \ldots \left| M_{m_N-n+1} \right| \left| M_{m_N-n-1} \right| \ldots \left| M_{m_1} \right| \times \\
\times \rho \left| M_{m_1} \right| \ldots \left| M_{m_N} \right| U^+_m \ldots U^+_m \\
= \frac{(N - 1)(N - 2)}{2} \left( \sum_m b_m U^L_m \left| M_m \right| R(U^+_m)^R \right) \sum_{N_+ = 0}^{N-1} \left( \begin{array}{c}
N - 1 \\
N_+
\end{array} \right) U^N_+ U^{N-N_+-1}_- \times \\
\times \left| M_+ \right|^{N_+} \left| M_- \right|^{-N-N_+-1} \rho \left| M_+ \right|^{N_+} \left| M_- \right|^{-N-N_+-1} (U^+_+)^{N_+} (U^+_-)^{N-N_+-1},
$$

(59)

where we have again used the notation that operators with upper script $L$ and $R$ act from the left and from the right respectively. A similar formula is obtained when instead of $\left| M_{m_N-n} \right|$ in the first line $U_{m_N-n+1}$ is missing. Then only $U^L_m$ has to be replaced by $\left| M_m \right|^L$.

$\tilde{R}_2$ can be simplified employing

$$
\sum_{m_1 \ldots m_N} \sum_{k=2}^{k-1} C_{m_k,m_l} U_{m_N} \ldots U_{m_k+1} U_{m_k-1} \ldots U_{m_1} \times \\
\times \left| M_{m_N} \right| \ldots \left| M_{m_k+l} \right| \left| M_{m_k-l} \right| \ldots \left| M_{m_1} \right| \rho \left| M_{m_1} \right| \ldots \left| M_{m_N} \right| U^+_m \ldots U^+_m \\
= \frac{(N - 1)(N - 2)}{2} \left( \sum_{m,n=0}^{N,2} C_{m,n} U^L_m \left| M_m \right| R(U^+_m)^R \right) \times \\
\times \sum_{N_+ = 0}^{N-2} \left( \begin{array}{c}
N - 2 \\
N_+
\end{array} \right) U^N_+ U^{N-N_+-2}_- \left| M_+ \right|^{N_+} \left| M_- \right|^{-N-N_+-2} \times \\
\times \rho \left| M_+ \right|^{N_+} \left| M_- \right|^{-N-N_+-2} (U^+_+)^{N_+} (U^+_-)^{N-N_+-2},
$$

(60)

Formulae (59) and (61) neglect commutators between the operators they contain. In our case corrections containing these commutators would be of higher order and therefore too small.

Applying formulae (59), (60) to $\tilde{R}_1$, $\tilde{R}_2$ respectively and expressing $N_+/N$ in terms of variable $s$ according to (31) we obtain:

$$
\tilde{R}_1 = -\frac{iN \Delta p \Delta t}{\hbar} \left\{ \left[ H, \sigma_z \right] \sum_m \frac{U_m}{a_m} \tilde{D}(\rho) \left| M_m \right| U^+_m + h.c. \right\} \\
- \frac{\Delta t^2}{\hbar^2} \sum_m \left\{ \left[ H, H_m \right] \left| M_m \right| \tilde{D}(\rho) \left| M_m \right| U^+_m + h.c. \right\},
$$

(61)

where $\tilde{D}(\rho) = \int_{-\infty}^{\infty} M_s \rho M^*_s ds$ with $M_s$ as given by (33) with $\Delta t$ replaced by $(N - 1)\tau$. $\tilde{R}_2$ now reads

$$
\tilde{R}_2 = -\frac{iN \Delta p \Delta t}{\hbar} \sum_{m,m\tilde{m}} \left\{ \left[ \sigma_z, \frac{H_m}{a_m} \right] U_m \left| M_m \right| \tilde{D}(\rho) \left| M_{m\tilde{m}} \right| U^+_m \left| M_m \right| U^+_m + h.c. \right\}
$$
\[- \frac{\Delta t^2}{2\hbar^2} \sum_{m \neq \hat{m}} \left\{ [H_m, \hat{H}_{\hat{m}}] |M_m||M_{\hat{m}}||\hat{D}(\rho)||U_{\hat{m}}^+||M_m||U_m^+ + h.c. \right\} \quad (62)\]

In $\tilde{R}_2$, when inserting $M_\pm$ from (35) in $\tilde{D}(\rho)$, $\Delta t$ has to be replaced by $(N - 2)\tau$. The second sum in (62) vanishes since the summand with $m = +, \hat{m} = -$ and the summand with $m = -, \hat{m} = +$ add to zero. Inserting the lowest order of $\tilde{D}(\rho)$ namely $\tilde{D}(\rho) \approx \rho$, it is easy to see that $\tilde{R}_1$ and $\tilde{R}_2$ contribute the terms in the sixth and the seventh line of equation (68) respectively to the change of state.

Appendix C

Having calculated $\tilde{R}_1$ and $\tilde{R}_2$, we want to sketch how to process the main contribution to the state change, which is represented by the first term in equation (57). As mentioned above this part of the operation can be written by means of a binomial distribution and then be expressed with operation elements whose modulus $|M(N_+, N)|$ are the square root of Gaussians, cf. equation (29). In contrast to section 3 and 4, we now take into account the full q-number denominators of the modulus $|M(N_+, N)|$ in (29). Expressing the operation elements in terms of variables (cp. (31)) we obtain for their modulus

$$|M_s| = \frac{1}{\sqrt{2\pi/(\hat{\gamma}\Delta t)}} \exp \left\{ - \frac{\hat{\gamma} (\sigma_z - s)^2}{4} \Delta t \right\}, \quad (63)$$

with

$$\hat{\gamma} := \frac{(\Delta p)^2}{4E_+ E_- \tau} \quad (64)$$

Expanding the unitary part of the operation up to order $\Delta t^2$ leads to the state change (without the contribution from $R_1$ and $R_2$)

$$\rho(t + \Delta t) = D(\rho) - \frac{i\Delta t}{\hbar} [H + H_{AV}, D(\rho)] - \frac{\Delta t^2}{2\hbar^2} \left\{ (H + H_{AV})^2, \rho \right\}$$

$$+ \frac{\Delta t^2}{\hbar^2} (H + H_{AV}) \rho (H + H_{AV}) \quad (65)$$

with

$$D(\rho) = \int_{-\infty}^{\infty} \exp \left\{ \frac{i}{2\hbar} \left( \Delta H^L - \Delta H^R \right) s \Delta p \Delta t \right\} \exp \left\{ - \frac{\hat{\gamma}^L}{4} (\sigma^L_z - s)^2 \Delta t \right\} \times$$

$$\times \exp \left\{ - \frac{\hat{\gamma}^R}{4} (\sigma^R_z - s)^2 \Delta t \right\} ds \frac{\sqrt{\hat{\gamma}^L \hat{\gamma}^R}}{\sqrt{2\pi/\Delta t}} \rho. \quad (66)$$

We note that in eqn. (65) $H$ is meant to act in operator products directly on $\rho$. This is due to the order of operators in the operation elements (cp.
The integral $D(\rho)$ has a closed form solution which can be expanded in powers of $\Delta t$ and $\Delta p$. $\gamma$ without hat is given by (33).

$$D(\rho) = \left\{ 1 - \Delta t \left[ \frac{\gamma}{8} \left( \sigma_z - \sigma_z^2 \right)^2 \left( 1 - \frac{1}{2} \left( \frac{\Delta p(p_0 - 1/2)}{p_0p_0} \right)^2 \times \left( 1 - \sqrt{p_0p_0(p_0p_0 + 3 - 2^{-1/4})} \right) + \frac{\Delta p^2}{8\gamma\hbar^2} \left( \Delta H^L - \Delta H^R \right)^2 + i\frac{\Delta p}{4\hbar} \left( \Delta H^L - \Delta H^R \left( \sigma^L + \sigma^R \right) \right) + \Delta t^2 \frac{\gamma^2}{32} \left( \sigma^L - \sigma^R \right)^2 + O(\Delta p\Delta t^2) + O(\Delta p^2\Delta t^2) + O(\Delta t^3) \right) \right\} \rho.$$  

Collecting all terms up to order $O(\Delta t^2)$ and $O(\Delta t\Delta p^2)$ we obtain the following difference equation:

$$\Delta \rho = \Delta t \left( -\frac{i}{\hbar} [H + H_{AV}, \rho] \right)$$

$$- \frac{\gamma}{8} \left( 1 - \frac{1}{2} \left( \frac{\Delta p(p_0 - 1/2)}{p_0p_0} \right)^2 \left( 1 - \sqrt{p_0p_0(p_0p_0 + 3 - 2^{-1/4})} \right) \right) [\sigma_z, [\sigma_z, \rho]]$$

$$- \frac{\Delta p^2}{8\gamma\hbar^2} [\Delta H, [\Delta H, \rho]] - \frac{i\Delta p}{4\hbar} \left( \Delta H, \{\sigma_z, \rho\} \right)$$

$$+ \Delta t^2 \left( -\frac{i\gamma}{8\hbar} \left( [\sigma_z, [\sigma_z, [H, \rho]]] + [H_{AV}, [\sigma_z, [\sigma_z, \rho]]] \right) \right)$$

$$- \frac{1}{2\hbar^2} \left( (H + H_{AV})^2, \rho \right) + \frac{1}{\hbar^2} (H + H_{AV}) \rho (H + H_{AV}) + \frac{\gamma^2}{32} [\sigma_z, [\sigma_z, \rho]] \right)$$

$$\left( -\frac{i\Delta t^2}{2\hbar} + \frac{3i\Delta t\Delta p^2}{8\hbar p_0p_0} \right) \left( \left[ [H, \sigma_z] \rho \sigma_z + h.c. \right] - \frac{i}{\hbar} \left[ \left[ [H, H_{AV}], \rho \right] \right] \right)$$

$$+ \frac{1}{4} \left( [\sigma_z, \tilde{p}_0 H_+ + p_0 H_-] \rho \sigma_z + h.c. \right) \right)$$

$O(\Delta p\Delta t^2) + O(\Delta p^2\Delta t^2) + O(\Delta t^3)$

with $\tilde{p}_0 := 1 - p_0$. In the order terms $\Delta t$ occurs in products with one of the three: $H/\hbar, H_+/\hbar$ or $\gamma$.

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