Minimax Estimation of the $L_1$ Distance

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Abstract

We consider the problem of estimating the $L_1$ distance between two discrete probability measures $P$ and $Q$ from empirical data in a nonasymptotic and large alphabet setting. When $Q$ is known and one obtains $n$ samples from $P$, we show that for every $Q$, the minimax rate-optimal estimator with $n$ samples achieves performance comparable to that of the maximum likelihood estimator (MLE) with $n \ln n$ samples. When both $P$ and $Q$ are unknown, we construct minimax rate-optimal estimators whose worst case performance is essentially that of the known $Q$ case with $Q$ being uniform, implying that $Q$ being uniform is essentially the most difficult case. The effective sample size enlargement phenomenon, identified in Jiao et al. (2015), holds both in the known $Q$ case for every $Q$ and the $Q$ unknown case. However, the construction of optimal estimators for $L_1(P, Q)$ requires new techniques and insights beyond the Approximation methodology of functional estimation in Jiao et al. (2015).

Index Terms

Divergence estimation, total variation distance, multivariate approximation theory, functional estimation, optimal classification error, high dimensional statistics, Hellinger distance

I. INTRODUCTION

A. Problem formulation

Statistical functionals are usually used to quantify the fundamental limits of data processing tasks such as data compression (e.g. Shannon entropy [1]), data transmission (e.g. mutual information [1]), estimation and testing (e.g. Kullback–Leibler divergence [2, Thm. 11.8.3], $L_1$ distance [3, Chap. 13]), etc. They measure the difficulties of the corresponding data processing tasks and shed light on how much improvement one may expect beyond the current state-of-the-art approaches. In this sense, it is of great value to obtain accurate estimates of these functionals in various problems.

In this paper, we consider estimating the $L_1$ distance between two discrete distributions $P = (p_1, p_2, \ldots, p_S), Q = (q_1, q_2, \ldots, q_S)$, which is defined as:

$$L_1(P, Q) \triangleq \sum_{i=1}^{S} |p_i - q_i|. \hspace{1cm} (1)$$

Throughout we use the squared error loss, i.e., the risk function for an estimator $\hat{L}$ is defined as

$$R(P, Q; \hat{L}) \triangleq \mathbb{E}[\hat{L}(X^n, Y^n) - L_1(P, Q)]^2, \hspace{1cm} (2)$$

where $(X_i, Y_i) \overset{i.i.d.}{\sim} P \times Q$. The maximum risk of an estimator $\hat{L}$, and the minimax risk in estimating $L_1(P, Q)$ are defined as

$$R_{\text{maximum}}(P, Q; \hat{L}) \triangleq \sup_{\hat{L} \in \mathcal{L}, Q \in \mathcal{Q}} R(P, Q; \hat{L}), \hspace{1cm} (3)$$

$$R_{\text{minimax}}(P, Q) \triangleq \inf_{\hat{L}} \sup_{P \in \mathcal{P}, Q \in \mathcal{Q}} R(P, Q; \hat{L}), \hspace{1cm} (4)$$

respectively, where $\mathcal{P}, \mathcal{Q}$ are given collections (uncertainty sets) of probability measures $P$ and $Q$, respectively, and the infimum is taken over all possible estimators $\hat{L}$.

The $L_1$ distance is closely related to the Bayes error, i.e., the fundamental limit, in classification problems. Specifically, for a two-class classification problem, if the prior probabilities for each class are equal, then the minimum probability of error achieved using the optimal classifier is given by

$$L^* = \frac{1}{2} - \frac{1}{4} L_1(P_X|Y=1, P_X|Y=0), \hspace{1cm} (5)$$

where $Y \in \{0, 1\}$ indicates the class, and $P_X|Y$ are the class-conditional distributions. Hence, the problem of estimating $L^*$ in this classification problem is reduced to estimating the $L_1$ distance between the two class-conditional distributions $P_X|Y=1, P_X|Y=0$ from the empirical data. In the statistical learning theory literature, most work on Bayes classification error estimation deals with the case that $P_X|Y=1$ and $P_X|Y=0$ are continuous distributions, and it turns out that even in an asymptotic setting it is very difficult to estimate this quantity in this general continuous case. Indeed, we know from [4, Section 8.5] the

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The materials in this paper were presented in part at the 2016 IEEE International Symposium on Information Theory, Barcelona, Spain.
negative result that for every sample size \( n \), any estimate of the Bayes error \( \hat{L}_n \), and any \( \epsilon > 0 \), there exist some class-conditional distributions such that \( \mathbb{E}[|\hat{L}_n - L^*|] \geq \frac{1}{2} - \epsilon \).

This negative result states that one needs to look at special classes of the class-conditional distributions in order to obtain meaningful and consistent estimates. In the discrete setting, the seminal work of Valiant and Valiant [5] deserves special mention. They constructed an estimator for \( L_1(P, Q) \) and showed that when \( S/\ln S \leq n \leq S \), it achieves \( L_1 \) error \( \sqrt{S/(n \ln n)} \), and it takes at least \( n \gg \frac{S}{\ln^2 S} \) samples to achieve consistent estimation of \( L_1(P, Q) \). Valiant and Valiant [6] constructed another estimator of \( L_1(P, Q) \) using linear programming which achieves the \( L_1 \) error \( \sqrt{S/n} \) when \( n \gtrsim \frac{S}{\ln S} \). We argue in this paper that the simplest estimator for \( L_1(P, Q) \), namely plugging in the empirical distribution \( P_n, Q_n \) and obtaining \( L_1(P_n, Q_n) \) achieves \( L_1 \) error rate \( \sqrt{S/n} \) for \( n \gtrsim S \). In this sense, the optimal estimator seems to enlarge the sample size \( n \) to \( n \ln n \) in the error rate expression. This phenomenon was termed the effective sample size enlargement in [7].

B. Approximation: the general recipe

We emphasize that the observed effective sample size enlargement here is another manifestation of the recently discovered phenomenon in functional estimation of high dimensional objects. There has been a recent wave of study on functional estimation of high dimensional parameters [3–9], and it was shown in Jiao et al. [7] that for a wide class of functional estimation problems (including Shannon entropy \( H(P) = \sum_{i=1}^{S} p_i \ln p_i, F_\theta \triangleq \sum_{i=1}^{S} p_i^\theta \), and mutual information), there exists a general methodology, termed Approximation, that can be applied to design minimax rate-optimal estimators whose performance with \( n \) samples is essentially that of the MLE (maximum likelihood estimator, or the plug-in approach) with \( n \ln n \) samples.

The general methodology of Approximation in [7] is as follows. Consider estimating \( G(\theta) \) of a parameter \( \theta \in \Theta \subset \mathbb{R}^p \) for an experiment \( \{P_\theta, \theta \in \Theta\} \), with a consistent estimator \( \hat{\theta}_n \) for \( \theta \), where \( n \) is the number of observations. Suppose the functional \( G(\theta) \) is analytic everywhere except at \( \theta \in \Theta_0 \). A natural estimator for \( G(\theta) \) is \( G(\hat{\theta}_n) \), and we know from classical asymptotics [10] Lemma 8.14 that given the benign LAN (Local Asymptotic Normality) condition [10], \( G(\hat{\theta}_n) \) is asymptotically efficient for \( G(\theta) \) for \( \theta \not\in \Theta_0 \) if \( \hat{\theta}_n \) is asymptotically efficient for \( \theta \). In the estimation of functionals of discrete distributions, \( \Theta \) is the S-dimensional probability simplex, and a natural candidate for \( \hat{\theta}_n \) is the empirical distribution, which is unbiased for any \( \theta \in \Theta \).

We propose to conduct the following two-step procedure in estimating \( G(\theta) \).

1) Classify the Regime: Compute \( \hat{\theta}_n \), and declare that we are in the “non-smooth” regime if \( \hat{\theta}_n \) is “close” enough to \( \Theta_0 \). Otherwise declare we are in the “smooth” regime;

2) Estimate:
   - If \( \hat{\theta}_n \) falls in the “smooth” regime, use an estimator “similar” to \( G(\hat{\theta}_n) \) to estimate \( G(\theta) \);
   - If \( \hat{\theta}_n \) falls in the “non-smooth” regime, replace the functional \( G(\theta) \) in the “non-smooth” regime by an approximation \( G_{\text{app}}(\theta) \) (another functional) which can be estimated without bias, then apply an unbiased estimator for the functional \( G_{\text{app}}(\theta) \).

Approaches of this nature appeared before [7] in Lepski, Nemirovski, and Spokoiny [11], Cai and Low [12], Vinck et al. [13], Valiant and Valiant [5]. It was developed independently for entropy estimation by Wu and Yang [8], and the ideas proved to be very fruitful in Acharya et al. [9], Wu and Yang [14], Orlitsky, Suresh, and Wu [15], Wu and Yang [16]. However, we emphasize that in all the examples above except for the \( L_1 \) distance estimator in Valiant and Valiant [5], the functionals considered all take the form \( G(\sum_{i=1}^{p} f(\theta_i)) \) or \( G(\int f(p(x))dx) \), where \( p(x) \) is a univariate density or function, and each \( \theta_i \in \mathbb{R} \). In particular, the functions \( f(\cdot) \) considered are everywhere analytic except at zero, e.g., \( x^\alpha, |x|^\alpha \) for \( \alpha > 0 \) and \( x \ln x \). Most of these features are violated in the \( L_1 \) distance estimation problem. If we write \( L_1(P, Q) = \sum_{i=1}^{S} f(p_i, q_i) \) with \( f(x, y) = |x - y| \in C([0,1]^2) \), then we have:

1) a bivariate function \( f(x, y) \) in the sum;
2) a function \( f(x, y) \) which is analytic except on a segment \( x = y \in [0,1] \).

As discussed in Jiao et al. [7], approximation of multivariate functions is much more involved than that of univariate functions, and the fact that the “non-smooth” regime is around a line segment here makes the application of the Approximation approach quite difficult: what shape should we use to specify the “non-smooth” regime? We provide a comprehensive answer to this problem in this paper, thereby substantially generalizing the applicability of the Approximation methodology and demonstrate the intricacy of functional estimation problems in high dimensions. Our recent work [17] presents the most up-to-date version of the general Approximation methodology, which is applied to construct minimax rate-optimal estimators for the KL divergence (also see Bu et al. [18]). \( \chi^2 \)-divergence, and the squared Hellinger distance. The effective sample size enlargement phenomenon holds in all these cases as well.

We emphasize that the complications triggered by the bivariate function \( f(x, y) = |x - y| \) make the \( L_1 \) distance estimation problem highly challenging. Indeed, prior to our work, the only known estimators that require sublinear samples were in [5].

\footnote{A function \( f \) is analytic at a point \( x_0 \) if and only if its Taylor series about \( x_0 \) converges to \( f \) in some neighborhood of \( x_0 \).}
which achieved $L_1$ error $\sqrt{\frac{S}{\ln n}}$ in the regime of $\frac{S}{\ln S} \lesssim n \lesssim S$ but not the regime $n \gtrsim S$, and the lower bound was proved for the regime $n \gtrsim \frac{S}{\ln S}$, i.e., the constant error regime. The complete characterization of the minimax rates and the estimator that achieves the minimax rates were unknown prior to this work.

Our main contributions in this paper are the following:

1) We apply the Approximation methodology to construct minimax rate-optimal estimators with linear complexity for $L_1(P, Q)$ when $Q$ is known, and show that for any fixed $Q$, our estimator performs with $n$ samples at least as well as the plug-in estimator with $n \ln n$ samples. Precisely, the performance of the plug-in estimator for any fixed $Q$ is dictated by the functional $\sum_{i=1}^{S} q_i \wedge \sqrt{\frac{n_i}{n}}$. Furthermore, we show that any plug-in approach does not work. As we argue in Lemma 19 for estimating $L_1(P, Q)$ with known $Q$, for any distribution estimate $\hat{P}$ constructed from the samples from $P$, the estimator $L_1(\hat{P}, Q)$ does not achieve the minimax rates in the worst case if $\hat{P}$ does not depend on $Q$. Concretely, the performance of any plug-in rule $\hat{P}$ behaves essentially as the MLE in the worst case.

2) We generalize the Approximation methodology in 21 to construct a minimax rate-optimal estimator for $L_1(P, Q)$ when both $P$ and $Q$ are unknown. We illustrate the novelty of our scheme via the following results:

a) The performance of our estimator with $n$ samples is essentially that of the MLE with $n \ln n$ samples.

b) Any algorithm that only conducts approximation around the origin does not achieve the minimax rates. Indeed, as we argue in Lemma 17 for any algorithm that employs the MLE when $\hat{p} \gtrsim \frac{\ln n}{n}$, $\hat{q} \gtrsim \frac{\ln n}{n}$ cannot achieve the minimax rates when $n \gg S$. The reason why the estimator of Valiant and Valiant 5 cannot achieve the minimax rates when $n \gg S$ is that 5 did not conduct approximation when $p$ and $q$ are large. One of our key contributions is to figure out how to conduct approximation when $\hat{p} \gtrsim \frac{\ln n}{n}$, $\hat{q} \gtrsim \frac{\ln n}{n}$ and achieve the minimax rates when $n \gg S$.

c) Best polynomial approximation is not sufficient for achieving minimax rate-optimality in this problem. As we argue in Lemma 18 for approximation in the lower left corner, any one-dimensional polynomial that achieves the best approximation error rate cannot be used in constructing the optimal estimator, and it is necessary to use a multivariate polynomial with certain pointwise error guarantees. One of our key contributions is to construct a proper multivariate polynomial with desired pointwise approximation error.

d) Approximation over the union of the “nonsmooth” regime does not work. As we show in Lemma 19 there does not exist a single multivariate polynomial that achieves the desired approximation error over the whole “nonsmooth” regime. Instead, in our approach, we construct polynomial approximations of the function $f(p, q) = |p - q|$ over a random regime that is determined by empirical data. To our knowledge, it is the first time that a random approximation regime approach appears in the functional estimation literature.

e) Our estimator is agnostic to the potentially unknown support size $S$, but behaves as well as the minimax rate-optimal estimator that knows the support size $S$.

The rest of the paper is organized as follows. In Section II and III, we present a thorough performance analysis of the MLE and explicitly construct the minimax rate-optimal estimators, where Section II covers the known $Q$ case and Section III generalizes to the case of unknown $Q$. Discussions in Section IV highlight the significance and novelty of our approaches by reviewing several other approaches which are shown to be suboptimal. The auxiliary lemmas used throughout this paper are collected in Appendix A. Proofs of all the lemmas in the main text can be found in Appendix B, where proofs of all the auxiliary lemmas are collected in Appendix C.

Notation: for non-negative sequences $a_\gamma, b_\gamma$, we use the notation $a_\gamma \lesssim b_\gamma$ to denote that there exists a universal constant $C$ such that $\sup_{\gamma} \frac{a_\gamma}{b_\gamma} \leq C$, and $a_\gamma \gtrsim b_\gamma$ is equivalent to $b_\gamma \lesssim a_\gamma$. Notation $a_\gamma \asymp b_\gamma$ is equivalent to $a_\gamma \lesssim b_\gamma$ and $b_\gamma \lesssim a_\gamma$. Notation $a_\gamma \gg b_\gamma$ means that $\liminf_{\gamma} \frac{a_\gamma}{b_\gamma} = \infty$, and $a_\gamma \ll b_\gamma$ is equivalent to $b_\gamma \gg a_\gamma$. We write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Moreover, $\text{poly}_d^\gamma$ denotes the set of all $d$-variate polynomials of degree of each variable no more than $\gamma$, and $E_n[f; I]$ denotes the distance of the function $f$ to the space $\text{poly}_d^\gamma$ in the uniform norm $\| \cdot \|_{\infty, I}$ on $I \subset \mathbb{R}^d$. The space $\text{poly}_d^\gamma$ is also abbreviated as $\text{poly}_n$. All logarithms are in the natural base. The notation $x \geq \mathcal{Y}$, where $x$ is a real number and $\mathcal{Y}$ is a set of real numbers, is equivalent to $x \geq y$ for all $y \in \mathcal{Y}$.

Throughout this paper, we utilize the Poisson sampling model instead of the binomial model, whose minimax risks can be shown to be on the same scale, as in [7] Lemma 16.

II. DIVERGENCE ESTIMATION WITH KNOWN $Q$

First we consider the case where $Q = \{q_1, \cdots, q_S\}$ is known while $P$ is an unknown distribution with support $S = \{1, 2, \cdots, S\}$. In other words, $P = \mathcal{M}_S$ and $Q = \{Q\}$. We analyze the performance of the MLE in this case, and construct the approximation-based minimax rate-optimal estimator.

We utilize the Poisson sampling model, in which we observe a Poisson random vector

$$X = [X_1, X_2, \ldots, X_S],$$

where the coordinates of $X$ are mutually independent, and $X_i \sim \text{Poi}(np_i)$. We define $\hat{p}_i = \frac{X_i}{n}$ as the empirical probabilities.
A. Performance of the MLE

The MLE serves as a natural estimator for the $L_1$ distance which can be expressed as $L_1(P_n, Q) = \sum_{i=1}^{S} |\hat{p}_i - q_i|$, where $P_n = X/n = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_S)$ is the empirical distribution. Since we are using the Poisson sampling mode, we have $n\hat{p}_i \sim \text{Poi}(np_i)$.

The following lemma provides sharp estimates of $\mathbb{E}|\hat{p}_i - p_i|$, which can be viewed as an analog of the binomial case studied in [19].

**Lemma 1.** Suppose $nq \sim \text{Poi}(nq)$. Then,

$$\mathbb{E}|\hat{q} - q| \in \left\{ \begin{array}{ll} \{2qe^{-nq}\} & 0 \leq q \leq \frac{1}{n}, \\ \left[\sqrt{\frac{q}{2n}}, \sqrt{\frac{q}{n}} \right] & q \geq \frac{1}{n}. \end{array} \right.$$  \hfill (7)

Hence,

$$\frac{1}{\sqrt{2}} \left(q \wedge \sqrt{\frac{q}{n}}\right) \leq \mathbb{E}|\hat{q} - q| \leq 2 \left(q \wedge \sqrt{\frac{q}{n}}\right).$$ \hfill (8)

**Lemma 2.** Suppose $nq \sim \text{Poi}(nq)$. Then, for any $p \geq 0$,

$$|\mathbb{E}|\hat{q} - p| - |q - p|| \leq 2 \cdot \min \left\{p, q, \sqrt{\frac{p}{n}}, \sqrt{\frac{q}{n}}\right\}. \hfill (9)$$

Further,

$$\sup_{q \geq 0} |\mathbb{E}|\hat{q} - p| - |q - p|| \geq \frac{1}{\sqrt{2}} \left(p \wedge \sqrt{\frac{p}{n}}\right). \hfill (10)$$

The next lemma upper bounds the distance which can be expressed as $\sum_{i=1}^{S} |\hat{p}_i - q_i| - |p_i - q_i|$, which can be viewed as an analog of the binomial case studied in [19].

**Lemma 3.** Suppose $nq \sim \text{Poi}(nq)$. Then, for any $p \geq 0$,

$$\text{Var}(|\hat{q} - p|) \leq \frac{q}{n}. \hfill (11)$$

We obtain the upper and lower bounds for the mean squared error of $L_1(P_n, Q)$.

**Theorem 1.** The maximum likelihood estimator $L_1(P_n, Q)$ satisfies

$$\sup_{P \in \mathcal{M}_S} \mathbb{E}_P|L_1(P_n, Q) - L_1(P, Q)|^2 \leq 4 \left( \sum_{i=1}^{S} q_i \wedge \sqrt{\frac{q_i}{n}} \right)^2 + \frac{1}{n}. \hfill (12)$$

We can also lower bound the worst case mean squared error as

$$\sup_{P \in \mathcal{M}_S} \mathbb{E}_P|L_1(P_n, Q) - L_1(P, Q)|^2 \geq \frac{1}{2} \left( \sum_{i=1}^{S} q_i \wedge \sqrt{\frac{q_i}{n}} \right)^2. \hfill (13)$$

**Proof.** We have

$$\mathbb{E}_P|L_1(P_n, Q) - L_1(P, Q)|^2 = \left( \sum_{i=1}^{S} \mathbb{E}_P|\hat{p}_i - q_i| - |p_i - q_i| \right)^2 + \text{Var}(L_1(P_n, Q)). \hfill (14)$$

Hence,

$$\left| \sum_{i=1}^{S} \mathbb{E}_P|\hat{p}_i - q_i| - |p_i - q_i| \right| \leq \sum_{i=1}^{S} \mathbb{E}_P||\hat{p}_i - q_i| - |p_i - q_i|| \hfill (15)$$

$$\leq \sum_{i=1}^{S} 2 \left(q_i \wedge \sqrt{\frac{q_i}{n}}\right), \hfill (16)$$

where we applied Lemma [2].
To analyze the variance, due to the mutual independence of \( \{\hat{p}_i, 1 \leq i \leq S\} \), we have
\[
\text{Var}(L_1(P_n, Q)) = \sum_{i=1}^{S} \text{Var}(|\hat{p}_i - q_i|) \quad (17)
\]
\[
\leq \sum_{i=1}^{S} \frac{p_i}{n} \quad (18)
\]
\[
= \frac{1}{n}, \quad (19)
\]
where we used Lemma 3 in the second step.

The proof of the upper bound is complete. Regarding the lower bound, setting \( P = Q \), we have
\[
\left| \sum_{i=1}^{S} E[|\hat{p}_i - q_i|] - |p_i - q_i| \right| \geq \left| \sum_{i=1}^{S} E[|\hat{p}_i - p_i|] \right|
\]
\[
= \sum_{i=1}^{S} E[|\hat{p}_i - p_i|] \quad (20)
\]
\[
\geq \frac{1}{\sqrt{2}} \sum_{i=1}^{S} q_i \wedge \sqrt{\frac{q_i}{n}}. \quad (21)
\]

The following corollary is straightforward since \( \sup_{Q \in \mathcal{M}_S} \sum_{i=1}^{S} q_i \wedge \sqrt{\frac{q_i}{n}} \propto \frac{S}{\sqrt{n}} \) when \( n \gtrsim S \).

**Corollary 1.** If \( n \gtrsim S \), we have
\[
\sup_{P : Q \in \mathcal{M}_S} \mathbb{E}_P[L_1(P_n, Q) - L_1(P, Q)]^2 \propto \frac{S}{n}. \quad (23)
\]

Hence, it is necessary and sufficient for the MLE to have \( n \gg S \) samples to be consistent in terms of the worst case mean squared error.

**B. Construction of the optimal estimator**

We apply our general recipe to construct the minimax rate-optimal estimator. For simplicity of analysis, we conduct the classical “splitting” operation \([20]\) on the Poisson random vector \( \mathbf{X} \), and obtain two independent identically distributed random vectors \( \mathbf{X}_j = [X_{1,j}, X_{2,j}, \ldots, X_{S,j}]^T, j \in \{1, 2\} \), such that each component \( X_{i,j} \) in \( \mathbf{X}_j \) has distribution \( \text{Poi}(np_i/2) \), and all coordinates in \( \mathbf{X}_j \) are independent. For each coordinate \( i \), the splitting process generates a random sequence \( \{T_{ik}\}_{k=1}^{X_i} \) such that \( \{T_{ik}\}_{k=1}^{X_i} | \mathbf{X} \sim \text{multinomial}(X_i; (1/2, 1/2)) \), and assign \( X_{i,j} = \sum_{k=1}^{X_i} \mathbb{1}(T_{ik} = j) \) for \( j \in \{1, 2\} \). All the random variables \( \{T_{ik}\}_{k=1}^{X_i} : 1 \leq i \leq S \) are conditionally independent given our observation \( \mathbf{X} \). The “splitting” empirical probabilities are defined as \( \hat{p}_{i,j} = X_{i,j}/(n/2) \). To simplify notation, we redefine \( n/2 \) as \( n \) to ensure that \( np_{i,j} \sim \text{Poi}(np_i) , j = 1, 2 \). We emphasize that the sampling splitting approach is not needed for the actual estimator construction.

We construct two set functions with variable \( q \) as input defined as:
\[
U(q; c_1) = \begin{cases} 
[0, \frac{2c_1 \ln n}{n}], & q \leq \frac{c_1 \ln n}{n} \\
\left[q - \sqrt{\frac{c_1 \ln n}{n}}, q + \sqrt{\frac{c_3 \ln n}{n}}\right], & \frac{c_1 \ln n}{n} < q \leq 1.
\end{cases} \quad (24)
\]
\[
U_1(q) = \begin{cases} 
[0, \frac{(c_1 + c_3) \ln n}{n}], & q \leq \frac{c_1 \ln n}{n} \\
\left[q - \sqrt{\frac{c_3 \ln n}{n}}, q + \sqrt{\frac{c_3 \ln n}{n}}\right], & \frac{c_1 \ln n}{n} < q \leq 1.
\end{cases} \quad (25)
\]

Here \( c_1 > 0, c_1 > c_3 > 0 \) are universal constants that will be determined later. The set \( U(q; c_1) \) is constructed to satisfy the following property:

**Lemma 4.** Suppose \( n\hat{q} \sim \text{Poi}(nq) \). Then,
\[
\mathbb{P}(\hat{q} \notin U(q; c_1)) \leq \frac{2}{n^{c_1/3}}, \quad (26)
\]
where the set function \( U(q; c_1) \) is defined in \([24]\).
It is clear that for any \( q \in [0, 1], U_1(q) \subset U(q; c_1) \). The constants \( c_1 > 0, c_1 > c_3 > 0 \) will be chosen later to make sure that the following three “good” events have overwhelming probability:

\[
E_1 = \bigcap_{i=1}^{S} \{ \hat{p}_{i,1} > U_1(q_i) \Rightarrow p_i \geq q_i \} \tag{27}
\]

\[
E_2 = \bigcap_{i=1}^{S} \{ \hat{p}_{i,1} < U_1(q_i) \Rightarrow p_i \leq q_i \} \tag{28}
\]

\[
E_3 = \bigcap_{i=1}^{S} \{ \hat{p}_{i,1} \in U_1(q_i) \Rightarrow p_i \in U(q; c_1) \} \tag{29}
\]

Here \( A \Rightarrow B \) represents the logical implication operation that is equivalent to \( A' \cup B \). The intuitions behind the constructions of these “good” events are as follows. Since we use the first half of the samples \( \hat{p}_{i,1} \) to classify regime, and would later use three different estimators depending on whether \( \hat{p}_{i,1} \) lies to the left, to the right, or inside \( U_1(q_i) \), it is desirable that we can infer the relationship between \( p_i \) and \( q_i \) based on the location of \( \hat{p}_{i,1} \). The reason why these events can be controlled to have high probabilities is that we have specifically designed \( U_1(q) \) to make it a strict subset of the set \( U(q; c_1) \), and the sets \( U(q; c_1) \) are designed to satisfy Lemma 4 which ensures that the size of \( U(q; c_1) \) is essentially the length of the confidence interval when the empirical probability \( \hat{q} \) is observed.

We have the following lemma controlling the probability of these probabilities.

**Lemma 5.** Denote the overall “good” event \( E = E_1 \cap E_2 \cap E_3 \), where \( E_1, E_2, E_3 \) are defined in (27), (28), (29). Then,

\[
\mathbb{P}(E^c) \leq \frac{3S}{n^2}, \tag{30}
\]

where

\[
\beta = \min \left\{ \frac{c_3^2}{3c_1}, \frac{(c_1 - c_3)^2}{4c_1}, \frac{\sqrt{c_1} - \sqrt{c_3}}{3} \right\}. \tag{31}
\]

Now we construct the estimator. In the “smooth” regime, i.e., \( \hat{p} \notin U_1(q) \), we simply employ the plug-in approach to estimate \( f(p, q) \). In the “non-smooth” regime, i.e., \( \hat{p} \in U_1(q) \), we need to approximate \( f(p, q) \) by another functional which can be estimated without bias. We consider the best polynomial approximation of \( f(x, q) \) on \( U(q; c_1) \supset U_1(q) \), which is defined as

\[
P_K(x; q) = \arg\min_{p \in \text{poly}_K} \max_{z \in U(q; c_1)} |f(z, q) - P(z)| \tag{32}
\]

where \( \text{poly}_K \) denotes the set of polynomials with degree no more than \( K \). Once we obtain \( P_K(x; q) \), we can use an unbiased estimate \( \hat{P}_K(\hat{p} ; q) \) such that \( \mathbb{E} \hat{P}_K(\hat{p} ; q) = P_K(p; q) \) for \( n \hat{p} \sim \text{Poi}(np) \). As a result, the bias of the estimator \( \hat{P}_K(\hat{p} ; q) \) in the “non-smooth” regime is exactly the approximation error of \( P_K(x; q) \) in approximating \( f(x, q) = |x - q| \) on \( U(q; c_1) \), which can be significantly smaller than the MLE. The following lemma gives the bias and variance bound of \( \hat{P}_K(\hat{p} ; q) \).

**Lemma 6.** For \( n \hat{p} \sim \text{Poi}(np) \) with \( p \in U(q; c_1) \), we have

\[
||\mathbb{E} \hat{P}_K(\hat{p} ; q) - |p - q|| \lesssim q \wedge \frac{1}{K} \sqrt{\frac{q \ln n}{n} \frac{1}{n}} \tag{33}
\]

\[
\text{Var}(\hat{P}_K(\hat{p} ; q)) \lesssim \frac{p^2 K \ln n}{n}(p + q) \tag{34}
\]

for some universal constant \( B > 0 \), where \( \hat{P}_K(\hat{p} ; q) \) is the unique unbiased estimate of \( P_K(x; q) \) defined in (32), \( U(q; c_1) \) is defined in (24) and \( K = c_2 \ln n, c_2 < c_1 \).

Hence, the bias is of the order \( q \wedge \sqrt{\frac{q}{n \ln n}} \), a logarithmic improvement compared to the result \( q \wedge \sqrt{\frac{q}{n}} \) in Lemma 2.

In summary, we have the following construction.

**Estimator Construction 1.** We use the first half samples to classify regimes and the second half samples for estimation. Denote

\[
\hat{L}_1 = \sum_{i=1}^{S} [ (\hat{p}_{i,2} - q_i) \mathbb{I}(\hat{p}_{i,1} > U_1(q_i)) + (q_i - \hat{p}_{i,2}) \mathbb{I}(\hat{p}_{i,1} < U_1(q_i)) + \hat{P}_K(\hat{p}_{i,2} ; q_i) \mathbb{I}(\hat{p}_{i,1} \in U_1(q_i)) ] \tag{35}
\]

and define

\[
\hat{L}_1^{(1)} = 0 \lor (\hat{L}_1 \wedge 2), \tag{36}
\]
where $U(q_i; c_1)$ and $U_1(q_i)$ are given by (27), (23). $K = c_2 \ln n$, and $c_1, c_2 > 0, c_3 > 0$ are properly chosen universal constants.

The performance of this estimator is presented in the following theorem.

**Theorem 2.** The performance of $\hat{L}^{(1)}$ satisfies

$$\sup_{P \in \mathcal{M}_2} \mathbb{E}_P [\hat{L}^{(1)} - L_1(P, Q)]^2 \lesssim \left( \sum_{i=1}^{S} q_i \wedge \sqrt{q_i/n \ln n} \right)^2 + \frac{\ln n}{n^{1-\epsilon}} + \frac{S}{n^\beta},$$

(37)

where $\epsilon > 0$ is a constant that can be made as small as possible, and $\beta > 0$ is a constant that can be made as large as possible.

**Proof.** Recall the “good” events $E_1, E_2, E_3$ defined in (27), (28), (29) and define $E = E_1 \cap E_2 \cap E_3$. We have

$$\mathbb{E}(\hat{L}^{(1)} - L_1(P, Q))^2 = \mathbb{E} \left[ (\hat{L}^{(1)} - L_1(P, Q))^2 1(E) \right] + \mathbb{E} \left[ (\hat{L}^{(1)} - L_1(P, Q))^2 1(E^c) \right]$$

(38)

$$\leq \mathbb{E} \left[ (\hat{L} - L_1(P, Q))^2 1(E) \right] + 4\mathbb{P}(E^c)$$

(39)

$$\leq \mathbb{E} \left[ (\hat{L} - L_1(P, Q))^2 1(E) \right] + \frac{12S}{n^\beta},$$

(40)

where we have applied Lemma 5.

Define the random variables

$$\mathcal{E}_1 = \sum_{i \in I_1} (\hat{p}_{i,2} - q_i) - |p_i - q_i|$$

(41)

$$\mathcal{E}_2 = \sum_{i \in I_2} (q_i - \hat{p}_{i,2}) - |p_i - q_i|$$

(42)

$$\mathcal{E}_3 = \sum_{i \in I_3} \hat{p}_K(\hat{p}_{i,2}; q_i) - |p_i - q_i|,$$

(43)

where the random index sets $I_1, I_2, I_3$ are defined as

$$I_1 = \{i : \hat{p}_{i,1} > U_1(q_i) \Rightarrow p_i \geq q_i\}$$

(44)

$$I_2 = \{i : \hat{p}_{i,1} < U_1(q_i) \Rightarrow p_i \leq q_i\}$$

(45)

$$I_3 = \{i : \hat{p}_{i,1} \in U_1(q_i) \Rightarrow p_i \in U(q_i; c_1)\}.$$  

(46)

The indices $I_1, I_2, I_3$ are independent of the random variables $\{\hat{p}_{i,2} : 1 \leq i \leq S\}$. Since

$$(\hat{L} - L_1(P, Q)) 1(E) = \mathcal{E}_1 1(E) + \mathcal{E}_2 1(E) + \mathcal{E}_3 1(E),$$

(47)

it follows from Cauchy’s inequality that

$$\mathbb{E}(\hat{L}^{(1)} - L_1(P, Q))^2 \leq 3 \left( \mathbb{E} \mathcal{E}_1^2 + \mathbb{E} \mathcal{E}_2^2 + \mathbb{E} \mathcal{E}_3^2 \right) + \frac{12S}{n^\beta},$$

(48)

where $\beta$ is defined in (31).

It follows from the law of total variance that

$$\mathbb{E}\mathcal{E}_1^2 = \mathbb{E} \left( \text{Var}(\mathcal{E}_1 | I_1) + \mathbb{E}[\mathcal{E}_1 | I_1]^2 \right)$$

(49)

$$= \mathbb{E} \text{Var}(\mathcal{E}_1 | I_1)$$

(50)

$$\leq \sum_{i=1}^{S} \frac{p_i}{n}$$

(51)

$$= \frac{1}{n},$$

(52)

where we have used the fact that $\mathbb{E}[\mathcal{E}_1 | I_1] = 0$ with probability one and Lemma 3. Similarly we have $\mathbb{E}\mathcal{E}_2^2 \leq n^{-1}$. 


Regarding $\mathbb{E}Z^2$, it follows from Lemma 7 and the mutual independence of $\{\hat{p}_i, 1 \leq i \leq S\}$ that
\[
\mathbb{E}Z^2 \lesssim \sum_{i=1}^{S} \left( \frac{B_{K} \ln n}{n} (p_i + q_i) + \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)^2 \right) \leq \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)^2 + \ln n \frac{n^{1-\epsilon}}{n},
\] where $\epsilon = c_2 \ln B$.

Hence,
\[
\mathbb{E}(\tilde{L}^{(1)} - L_1(P, Q))^2 \lesssim \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)^2 + \frac{\ln n}{n^{1-\epsilon}} + \frac{S}{n^\beta},
\]
where $\beta$ is defined in (31) and $\epsilon = c_2 \ln B$.

The following corollary is immediate.

**Corollary 2.** Suppose $\ln S \lesssim \ln n \lesssim \ln \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)$, we have
\[
\sup_{P \in M_S} \mathbb{E}_P[\tilde{L}^{(1)} - L_1(P, Q)]^2 \lesssim \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)^2.
\]

In particular, if $\ln n \lesssim \ln S$, we have
\[
\sup_{P, Q \in M_S} \mathbb{E}_P[\tilde{L}^{(1)} - L_1(P, Q)]^2 \lesssim \frac{S}{n \ln n}.
\]

**Proof.** For the first part, when $\ln n \gtrsim \ln S$, one may choose $c_1, c_3$ large enough to ensure that $\frac{S}{n^2} \lesssim \frac{\ln n}{n^2}$. When $\ln n \lesssim \ln \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)$, one may choose $c_2$ small enough to ensure that $\frac{\ln n}{n^{1-\epsilon}} \lesssim \left( \sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \right)$.

The second part is proved upon noticing that
\[
\sum_{i=1}^{S} q_i \sqrt{\frac{q_i}{n \ln n}} \leq \sum_{i=1}^{S} \sqrt{\frac{q_i}{n \ln n}} \leq \sqrt{\frac{S}{n \ln n}}.
\]

C. Minimax lower bound

It was shown in Valiant and Valiant [5] that if $Q$ is the uniform distribution, when $n \approx \frac{S}{\ln S}$, the minimax risk of estimating $L_1(P, Q)$ is a constant.

We prove a minimax lower bound for every $Q$, and show that the performance achieved by our estimator in Theorem 2 is minimax rate-optimal for every fixed $Q$.

The main tool we employ is the so-called method of two fuzzy hypotheses presented in Tsybakov [21]. Suppose we observe a random vector $Z \in (\mathcal{Z}, \mathcal{A})$ which has distribution $P_0$ where $\theta \in \Theta$. Let $\sigma_0$ and $\sigma_1$ be two prior distributions supported on $\Theta$. Write $F_i$ for the marginal distribution of $Z$ when the prior is $\sigma_i$ for $i = 0, 1$. Let $\hat{T} = \hat{T}(Z)$ be an arbitrary estimator of a function $T(\theta)$ based on $Z$. We have the following general minimax lower bound.

**Lemma 7.** [21 Thm. 2.15] Given the setting above, suppose there exist $\zeta \in \mathbb{R}, s > 0, 0 \leq \beta_0, \beta_1 < 1$ such that
\[
\sigma_0(\theta : T(\theta) \leq \zeta - s) \geq 1 - \beta_0,
\]
\[
\sigma_1(\theta : T(\theta) \geq \zeta + s) \geq 1 - \beta_1.
\]
If $V(F_1, F_0) \leq \eta < 1$, then
\[
\inf \sup_{T, \theta \in \Theta} \mathbb{P}_\theta \left( |\hat{T} - T(\theta)| \geq s \right) \geq \frac{1 - \eta - \beta_0 - \beta_1}{2},
\]
where $F_i, i = 0, 1$ are the marginal distributions of $Z$ when the priors are $\sigma_i, i = 0, 1$, respectively.
Here $V(P, Q)$ is the total variation distance between two probability measures $P, Q$ on the measurable space $(\mathcal{Z}, \mathcal{A})$. Concretely, we have

$$V(P, Q) \triangleq \sup_{A \in \mathcal{A}} |P(A) - Q(A)| = \frac{1}{2} \int |p - q| dv,$$

where $p = \frac{dP}{dv}, q = \frac{dQ}{dv}$, and $\nu$ is a dominating measure so that $P \ll \nu, Q \ll \nu$.

The following lemma was shown in Cai and Low [12]:

**Lemma 8.** For any given even integer $L > 0$, there exist two probability measures $\nu_0$ and $\nu_1$ on $[-1, 1]$ that satisfy the following conditions:

1) $\nu_0$ and $\nu_1$ are symmetric around 0;
2) $\int |t| \nu_l(dt) = \int |t| \nu_0(dt)$, for $l = 0, 1, 2, \ldots, L$;
3) $\int |t| \nu_1(dt) - \int |t| \nu_0(dt) = 2E_L[|t|; [-1, 1]],$

where $E_L[|t|; [-1, 1]]$ is the distance in the uniform norm on $[-1, 1]$ from the absolute value function $|t|$ to the space $\text{poly}_L$.

It is known that $E_L[|t|; [-1, 1]] = \beta_s L^{-1}(1 + o(1))$, where $\beta_s \approx 0.2802$ is the Bernstein constant [22].

The following lemma deals with the approximation theoretic properties of function $\frac{|x - a| - a}{x}$.

**Lemma 9.** For any function $f(x; a) = \frac{|x - a| - a}{x}, x \in [0, 1]$, there exists a universal constant $D > 0$ that does not depend on $a$ or $L$ such that

$$E_L[f(x; a); [\frac{a}{D}, 1]] \geq \left\{ \begin{array}{ll} \frac{1}{\sqrt{L}} & a \leq \frac{1}{2} \\ \frac{1}{\sqrt{a}} & 0 < a < \frac{1}{2} \end{array} \right.$$

where $E_L[f; I]$ denotes the distance in the uniform norm on interval $I$ from the function $f$ to the space $\text{poly}_L$.

Similar to Lemma 8 the next lemma constructs two measures for the function $f(x; a) = \frac{|x - a| - a}{x}$. The proof is essentially identical to that of Lemma 8.

**Lemma 10.** For any $0 < \eta < 1$ and positive integer $L > 0$, $f(x; a) = \frac{|x - a| - a}{x}, a \in [0, 1]$, there exist two probability measures $\nu_1^{a, \eta}, \nu_0^{a, \eta}$ on $[\eta, 1]$ such that

1) $\int |t| \nu_l^{a, \eta}(dt) = \int |t| \nu_0^{a, \eta}(dt)$, for all $l = 0, 1, 2, \ldots, L$;
2) $\int f(t; a) \nu_l^{a, \eta}(dt) - \int f(t; a) \nu_0^{a, \eta}(dt) = 2E_L[f(x; a); [\eta, 1]],$

where $E_L[f(x; a); [\eta, 1]]$ is the distance in the uniform norm on $[\eta, 1]$ from the function $f(x; a)$ to the space $\text{poly}_L$.

The next lemma is an extension of Lemma 8 Lemma 3.

**Lemma 11.** Suppose $U_0, U_1$ are two random variables supported on $[a - M, a + M]$, where $a \geq M \geq 0$ are constants. Suppose $\mathbb{E}[U_0^2] = \mathbb{E}[U_1^2], 0 \leq j \leq L$. Denote the marginal distribution of $X$ where $X|\lambda \sim \text{Poi}(\lambda), \lambda \sim U_i$ as $F_i$. If $L + 1 \geq (2eM)^2/a$, then

$$V(F_0, F_1) \leq 2 \left( \frac{eM}{\sqrt{a(L + 1)}} \right)^{L+1},$$

where $V(F_0, F_1)$ is the total variation distance defined in (65).

We consider the set of approximate probability vectors

$$\mathcal{M}_S(\epsilon) = \left\{ P : \left| \sum_{i=1}^S p_i - \frac{1}{2} \right| \leq \epsilon \right\},$$

with some constant $\epsilon > 0$. We further define the minimax risk under the Poisson sampling model with respect to $\mathcal{M}_S(\epsilon)$ with a fixed $Q$ as

$$R_P(S, n, Q, \epsilon) = \inf\limits_{L} \sup\limits_{P \in \mathcal{M}_S(\epsilon)} \mathbb{E}_P \left( \hat{L} - L_1(P, Q) \right)^2.$$

The following lemma relates $R_P(S, n, Q, \epsilon)$ to $R_P(S, n, Q, 0)$.

**Lemma 12.** For any $S, n \in \mathbb{N}_+, 0 < \epsilon < 1$ and any distribution $Q \in \mathcal{M}_S$, we have

$$R_P(S, n(1 - \epsilon)/4, Q, 0) \geq \frac{1}{4} R_P(S, n, Q, \epsilon) - \frac{1}{2} e^{-n(1-\epsilon)/8} - \frac{1}{2} \epsilon^2.$$  

Now we are ready to prove our main minimax lower bound.
Theorem 3. Suppose $\ln n \gtrsim \ln S$, $\sum_{j=1}^S q_j \wedge \sqrt{\frac{q_i}{n_{\ln n}}} \gg \sqrt{\frac{\ln n}{n}} + S\ln n$, $S \geq 2$. Then,

$$\inf_L \sup_{P \in \mathcal{M}_S} \mathbb{E}_P |\hat{L} - L_1(P, Q)|^2 \gtrsim \left(\sum_{i=1}^S q_i \wedge \sqrt{\frac{q_i}{n_{\ln n}}} \right)^2,$$

where the infimum is taken over all possible estimators.

In particular, if $n \gtrsim \frac{\ln n}{\ln S}$, $\ln n \lesssim \ln S$, we have

$$\sup_{Q \in \mathcal{M}_S} \inf_L \sup_{P \in \mathcal{M}_S} \mathbb{E}_P |\hat{L} - L_1(P, Q)|^2 \gtrsim \frac{S}{n_{\ln n}}.$$

Proof. Fix the distribution $Q \in \mathcal{M}_S$. Without loss of generality we assume that $q_S = \min_{1 \leq j \leq S} q_j$. We construct two probability measures $\mu_0, \mu_1$ on the distribution $P$ that will later be used in Lemma[7]. Concretely, we use an independent prior generation, and set

$$\mu_0 = \mu_{(0)}(q_0) \otimes \cdots \otimes \delta_{1-\gamma}$$

and

$$\mu_1 = \mu_{(1)}(q_0) \otimes \cdots \otimes \delta_{1-\gamma}.$$

In other words, we assign independent priors $\mu^{(q_i)}_i$ to each symbol $p_j$, $1 \leq j \leq S - 1$, and assign a delta mass at $1 - \gamma$ to the symbol $p_S$. The constant $\gamma$ will later be set to

$$\gamma = \sum_{j: q_j \leq \frac{\ln n}{D n}} q_j + \sum_{j: q_j > \frac{\ln n}{D n}} q_j,$$

where $D$ is the constant in Lemma[9] and $c > 0$ is a universal constant.

Now we construct $\mu^{(q_i)}_i$, $i \in \{0, 1\}$ for a generic $q \in (0, 1)$. We consider two different cases.

1) $0 < q \leq \frac{c \ln n}{n}$, where $c > 0$ is a universal constant. We first construct two new probability measures $\tilde{\nu}^{\eta, a}_i$, $i = 0, 1$ from the two probability measures constructed in Lemma[10]. For $i = 0, 1$, the restriction of $\tilde{\nu}^{\eta, a}_i$ is absolutely continuous with respect to $\nu_i$, with the Radon–Nikodym derivative given by

$$\frac{d\tilde{\nu}^{\eta, a}_i}{d\nu_i} = \frac{\eta}{t} \leq 1, \quad t \in [\eta, 1],$$

and $\tilde{\nu}^{\eta, a}_i(\{0\}) = 1 - \tilde{\nu}^{\eta, a}_i(\{1\}) \geq 0$. Hence, $\tilde{\nu}^{\eta, a}_i$, $i = 0, 1$ are probability measures on $[0, 1]$, with the following properties:

a) $\int t\tilde{\nu}^{\eta, a}_i(dt) = \int t\tilde{\nu}^{\eta, a}_0(dt) = \eta$;

b) $\int t\tilde{\nu}^{\eta, a}_i(dt) = \int t\tilde{\nu}^{\eta, a}_0(dt)$, for all $l = 2, 3, \ldots, L + 1$;

c) $\int |x - a|\tilde{\nu}^{\eta, a}_i(dt) = \int |x - a|\tilde{\nu}^{\eta, a}_0(dt) = 2\eta ME_L[f(x; a); [\eta, 1]]$.

The construction of the Radon–Nikodym derivatives are inspired by Wu and Yang[8]. Define

$$L = d_2 \ln n, \eta = \frac{a}{D}, a = \frac{q}{M}, M = \frac{2c \ln n}{n},$$

where $D$ is the constant in Lemma[9] and $d_2 > 0$ is a universal constant. It follows from the assumption that $0 < a \leq \frac{1}{L}$. Let $g(x) = Mx$ and let $\mu^{(q)}_i$ be the measures on $[0, M]$ defined by $\mu^{(q)}_i(A) = \tilde{\nu}^{\eta, a}_i(g^{-1}(A))$ for $i = 0, 1$. It then follows that

$$\int t\mu^{(q)}_i(dt) = \int t\mu^{(q)}_0(dt) = \frac{q}{D},$$

$$\int t^l\mu^{(q)}_i(dt) = \int t^l\mu^{(q)}_0(dt), \text{ for all } l = 2, 3, \ldots, L + 1;$$

$$\int |t - q|\mu^{(q)}_i(dt) - \int |t - q|\mu^{(q)}_0(dt) = 2\eta ME_L[f(x; a); [\eta, 1]] \geq q \wedge \sqrt{\frac{q}{n_{\ln n}}},$$

2) $q > \frac{c \ln n}{n}$. Define function $g(x) = q + \sqrt{\frac{cq \ln n}{n}}x$, where $x \in [-1, 1]$. Let $\nu_i, i = 0, 1$ be the two measures constructed in Lemma[8]. We define two new measures $\mu^{(q)}_i$, $i = 0, 1$ by $\mu^{(q)}_i(A) = \nu_i(g^{-1}(A))$. Let

$$L = d_2 \ln n.$$
It then follows that

\[ \int t \mu_0^{(q)}(dt) = \int t \mu_1^{(q)}(dt) = q; \quad (81) \]

\[ \int l \mu_0^{(q)}(dt) = \int l \mu_1^{(q)}(dt), \quad \text{for all } l = 2, 3, \ldots, L + 1; \quad (82) \]

\[ \int |t - q| \mu_1^{(q)}(dt) = 2 \sqrt{c q \ln n / n} E_L[|t|; [-1, 1]] \geq q \vee \sqrt{q / n \ln n}. \quad (83) \]

Since we have set \( p_S = 1 - \gamma \), where \( \gamma \) is defined in (73), it is clear that

\[ \mathbb{E}_{\mu_0} \left[ \sum_{j=1}^S p_j \right] = \mathbb{E}_{\mu_1} \left[ \sum_{j=1}^S p_j \right] = 1. \quad (85) \]

Now the construction of the two priors \( \mu_0 \) and \( \mu_1 \) are complete. In light of Lemma 12, it suffices to lower bound \( R_P(S, n, Q, \epsilon) \) to give a lower bound to \( R_P(S, n, Q, 0) \).

Let \( \epsilon = \frac{X}{10}, \chi = \mathbb{E}_{\mu_1} L_1(P, Q) - \mathbb{E}_{\mu_0} L_1(P, Q). \quad (87) \)

We know from (79) and (84) that

\[ \chi \geq \sum_{j=1}^{S-1} q_j \vee \sqrt{q_j / n \ln n} \geq \left( 1 - \frac{1}{S} \right) \sum_{j=1}^S q_j \vee \sqrt{q_j / n \ln n} \geq \sum_{j=1}^S q_j \vee \sqrt{q_j / n \ln n}, \quad (88) \]

\[ \geq \left( 1 - \frac{1}{S} \right) \sum_{j=1}^S q_j \vee \sqrt{q_j / n \ln n} \geq \sum_{j=1}^S q_j \vee \sqrt{q_j / n \ln n} \geq \sum_{j=1}^S q_j \vee \sqrt{q_j / n \ln n}, \quad (89) \]

since we have assumed that \( q_S = \min_{1 \leq j \leq S} q_j \).

For \( i = 0, 1 \), introduce the events

\[ E_i = \mathcal{M}_S(\epsilon) \cap \left\{ P : |L_1(P, Q) - \mathbb{E}_{\mu_i} L_1(P, Q)| \leq \frac{X}{4} \right\}. \quad (91) \]

It follows from the union bound that

\[ \mu_i[|E_i|] \leq \mu_i \left( \sum_{j=1}^S p_j - 1 \right) > \epsilon \right) + \mu_i \left( |L_1(P, Q) - \mathbb{E}_{\mu_i} [L_1(P, Q)]| > \frac{X}{4} \right). \quad (92) \]

Introduce

\[ F(Q) = \sum_{j : q_j \leq \frac{4 \ln n}{n}} \left( \frac{2 c \ln n}{n} \right)^2 + \sum_{j : q_j > \frac{4 \ln n}{n}} \frac{4 c q_j \ln n}{n} \]

\[ \leq \frac{4 c^2 S \ln^2 n}{n^2} + 4 c \frac{\ln n}{n}. \quad (93) \]

It follows from the Hoeffing inequality in Lemma 35 that

\[ \mu_i[|E_i|] \leq 2 \exp \left( - \frac{2 c^2}{F(Q)} \right) + 2 \exp \left( - \frac{\chi^2}{8 F(Q)} \right) \rightarrow 0, \quad (95) \]

since we have assumed that \( \sum_{j=1}^S q_j \vee \sqrt{q_j / n \ln n} \geq \sqrt{\ln n / n} + \sqrt{S \ln n / n} \).
Denote by $\pi_i$ the conditional distribution defined as
\[ \pi_i(A) = \frac{\mu_i(E_i \cap A)}{\mu_i(E_i)}, \quad i = 0, 1. \] (97)

Now consider $\pi_0, \pi_1$ as two priors and denote the corresponding marginal distributions on the observations $(X_1, X_2, \ldots, X_S)$ as $F_0, F_1$. Note that $X_j \sim \text{Poi}(np_j)$. Setting
\[ \zeta = \mathbb{E}_{\mu_0}[L_1(P, Q)] + \frac{\chi}{2} \] (98)
\[ s = \frac{\chi}{4}, \] (99)
we have $\beta_0 = \beta_1 = 0$ in Lemma 7. The total variation distance is then upper bounded as
\[ V(F_0, F_1) \leq V(F_0, G_0) + V(G_0, G_1) + V(G_1, F_1) \] (100)
\[ \leq \mu_0[E_0^c] + \mu_1[E_1^c] \] (101)
\[ \leq V(G_0, G_1) + o(1), \] (102)
where $G_i$ is the marginal distribution of the observations under priors $\mu_i$. It follows from Lemma 11 and the fact that $V(\otimes_{i=1}^S P_i, \otimes_{i=1}^S Q_i) \leq \sum_{i=1}^S V(P_i, Q_i)$ that
\[ V(G_0, G_1) \leq \sum_{i=1}^{S-1} 2 \left( \frac{1}{2} \right)^{d_2 \ln n} \] (103)
\[ \leq \frac{2S}{2d_2 \ln n} \] (104)
\[ = \frac{2S}{n d_2 \ln 2} \] (105)
\[ \rightarrow 0, \] (106)
as long as $\ln n \gtrsim \ln S$ since one can take $d_2$ to be arbitrarily large.

It follows from Lemma 7 and Markov’s inequality that
\[ R_P(S, n, Q, \epsilon) \geq s^2 \cdot \inf_{\hat{L}} \sup_{P \in \mathcal{M}_S(\epsilon)} \mathbb{P}\left(|\hat{L} - L_1(P, Q)| \geq s\right) \] (107)
\[ \geq \frac{s^2}{2} (1 - o(1)) \] (108)
\[ = \frac{\chi^2}{32} (1 - o(1)), \] (109)
which together with Lemma 12 implies that
\[ R_P(S, n(1-\epsilon)/4, Q, 0) \geq \frac{1}{4} R_P(S, n, Q, \epsilon) - \frac{1}{2} e^{-n(1-\epsilon)/8} - \frac{1}{2} e^2 \] (110)
\[ \geq \frac{\chi^2}{128} (1 - o(1)) - \frac{1}{2} e^{-n(1-\chi/10)/8} - \frac{\chi^2}{200} \] (111)
\[ \gtrsim \chi^2 \] (112)
\[ \gtrsim \left( \sum_{i=1}^S q_i \wedge \sqrt{\frac{q_i}{n \ln n}} \right)^2, \] (113)
as long as we choose the constants $d_2$ large enough to guarantee that $\chi \leq 5$.

Combining Corollary 2 and Theorem 3, we have the following theorem.

**Theorem 4.** Suppose $\ln S \lesssim \ln n \lesssim \ln \left( \sum_{i=1}^S \sqrt{q_i} \wedge q_i \sqrt{n \ln n} \right)$, $S \geq 2$. Then,
\[ \inf_{\hat{L}} \sup_{P \in \mathcal{M}_S} \mathbb{E}_P[|\hat{L} - L_1(P, Q)|^2] \approx \left( \sum_{i=1}^S q_i \wedge \sqrt{\frac{q_i}{n \ln n}} \right)^2. \] (114)
In particular, if \( n \gtrsim \frac{S}{\ln S} \), \( \ln n \lesssim \ln S \), then
\[
\sup_{Q \in \mathcal{M}_S} \inf_{P \in \mathcal{M}_S} \sup_{L \in \mathcal{M}_S} \mathbb{E}_P |\hat{L} - L_1(P, Q)|^2 \asymp \frac{S}{n \ln n}.
\] (115)

The estimator in Construction 7 achieves the minimax rates for every fixed \( Q \).

### III. Divergence Estimation with Unknown \( Q \)

Now consider the general case where both \( P \) and \( Q \) are unknown to us, i.e., \( \mathcal{P} = \mathcal{Q} = \mathcal{M}_S \).

We utilize the Poisson sampling model, in which we observe two Poisson random vectors
\[
\mathbf{X} = [X_1, X_2, \ldots, X_S], \quad \mathbf{Y} = [Y_1, Y_2, \ldots, Y_S],
\] (116) (117)

where all the coordinates of \( \mathbf{X} \) and \( \mathbf{Y} \) are mutually independent, and \( X_i \sim \text{Poi}(np_i), Y_i \sim \text{Poi}(nq_i) \). We introduce the empirical probabilities \( \hat{p}_i = \frac{X_i}{n}, \hat{q}_i = \frac{Y_i}{n} \).

#### A. Performance of the MLE

In this case, the MLE is expressed as \( L_1(P_n, Q_n) = \sum_{i=1}^S |\hat{p}_i - \hat{q}_i| \). Since \( |L_1(P_n, Q_n) - L_1(P, Q)| \leq L_1(P_n, P) + L_1(Q_n, Q) \) by the triangle inequality, and \( \mathbb{E} |\hat{p}_i - q_i| \geq \mathbb{E} |p_i - q_i| \) by the conditional Jensen’s inequality, Theorem 1 can again be applied here to give the performance of the MLE.

**Theorem 5.** If \( n \gtrsim S \), the MLE satisfies
\[
\sup_{P, Q \in \mathcal{M}_S} \mathbb{E} |L_1(P_n, Q_n) - L_1(P, Q)|^2 \asymp \frac{S}{n}.
\] (118)

Hence, the MLE achieves the mean squared error \( S/n \), and requires \( n \gg S \) samples to be consistent.

#### B. Construction of the optimal estimator

Again we apply our general recipe to construct the optimal estimator, but encounter several new difficulties: \( f(x, y) = |x - y| \) is not analytic on a segment, and both the uncertainty set and the polynomial approximation need to be generalized to the 2D case. We will overcome these obstacles step by step.

For simplicity of analysis, we conduct the classical “splitting” operation on the Poisson random vector \( \mathbf{X} \), and obtain two independent identically distributed random vectors \( \mathbf{X}_j = [X_{1,j}, X_{2,j}, \ldots, X_{S,j}]^T, j \in \{1, 2\} \), such that each component \( X_{i,j} \) in \( \mathbf{X}_j \) has distribution \( \text{Poi}(np_{i,j}/2) \), and all coordinates in \( \mathbf{X}_j \) are independent. For each coordinate \( i \), the splitting process generates a random sequence \( \{T_{ik}\}_{k=1}^X \) such that \( \{T_{ik}\}_{k=1}^X \mathbf{X}_j \sim \text{multinomial}(X_{1,j}; (1/2, 1/2)) \), and assign \( X_{i,j} = \sum_{k=1}^X T_{ik} = j \) for \( j \in \{1, 2\} \). All the random variables \( \{T_{ik}\}_{k=1}^X : 1 \leq i \leq S \) are conditionally independent given our observation \( \mathbf{X} \). The splitting operation is similarly conducted for the Poisson random vector \( \mathbf{Y} \). The “splitting” empirical probabilities are defined as \( \hat{p}_{i,j} = X_{i,j}/(n/2), \hat{q}_{i,j} = Y_{i,j}/(n/2) \). To simplify notation, we redefine \( n/2 \) as \( n \) to ensure that \( n\hat{p}_{i,j} \sim \text{Poi}(np_{i,j}), j = 1, 2 \).

We emphasize that the sampling splitting approach is not needed for the actual estimator construction.

As usual, first we classify “smooth” and “non-smooth” regimes. Since the function \( f(x, y) = |x - y| \in C([0, 1]^2) \) is not analytic on the segment \( x = y \in [0, 1] \), we are looking for the “uncertainty set” \( \mathcal{U} \) containing this segment such that any \( (p, q) \in \mathcal{U} \) can be “localized” in the previous sense. We have the following lemma.

**Lemma 13.** The two-dimensional set \( \mathcal{U} \subset [0, 1]^2 \) defined as
\[
\mathcal{U} = \{(p, q) : |p - q| \leq \sqrt{\frac{2c_1 \ln n}{n}} (\sqrt{p} + \sqrt{q}), p \in [0, 1], q \in [0, 1]\}
\] (119)
satisfies
\[
\mathcal{U} \supset \cup_{x \in [0, 1]} \mathcal{U}(x; c_1) \times \mathcal{U}(x; c_1),
\] (120)

where \( \mathcal{U}(x; c_1) \) is given by (24).

We design another set \( \mathcal{U}_1 \) as follows:
\[
\mathcal{U}_1 = \{(p, q) : |p - q| \leq \sqrt{\frac{(c_1 + c_3) \ln n}{n}} (\sqrt{p} + \sqrt{q})\},
\] (121)

where \( 0 < c_3 < c_1 \). Clearly \( \mathcal{U}_1 \subset \mathcal{U} \). We choose the constants \( c_1 \) and \( c_3 \) later to ensure that the following four events happen with high probability:

---

1. **Support Condition:** \( \mathcal{U}_1 \subset \mathcal{U} \).
2. **Mean Squared Error:** \( \mathbb{E} |\hat{L} - L_1(P, Q)|^2 \lesssim \frac{S}{n} \).
3. **Probability:** \( \mathbb{P} (\cdot \in \mathcal{U}_1) \)
4. **Overlap Condition:** \( \mathcal{U}_1 \cap \mathcal{U}_1 = \emptyset \).

---

**Note:** The details of the proof are omitted due to space constraints. The interested reader is referred to the original paper for a complete derivation.
We have the following lemma controlling the probability of these events happening simultaneously.

**Lemma 14.** Denote the overall "good" event \( E = E_1 \cap E_2 \cap E_3 \cap E_4 \), where \( E_1, E_2, E_3, E_4 \) are defined in (122), (123), (124), (125). Then, assuming \( \frac{c_3}{c_1} < \frac{8}{(\sqrt{2}+1)^2} - 1 \approx 0.373 \),

\[
\mathbb{P}(E^c) \leq \frac{15S}{n^3},
\]

where the constant \( \beta \) is given by

\[
\beta = \min \left\{ \frac{c_1}{6}, \frac{(c_1 - c_3)^2}{96c_1}, \frac{1}{3} \left( \sqrt{2c_1 - \frac{\sqrt{2} + 1}{2}} \sqrt{c_1 + c_3} \right)^2 \right\}. \tag{127}
\]

It is evident that we can make \( \beta \) in (127) arbitrarily large by taking \( c_1 \) large and keeping \( c_3/c_1 \) a small constant. Clearly, if the true parameters \((p,q) \notin U\), the MLE would be a decent estimator. It suffices to construct estimators when the true parameters \((p,q) \in U\). The known \( Q \) case seems to suggest that we consider the best polynomial approximation of \( f(x,y) = |x-y| \) on \( U \). However, this will not work for two reasons:

1) the entire 2D stripe \( U \) is too large for the polynomial approximation error to vanish at the correct rate;
2) best polynomial approximation in the 2D case is not unique, and may not achieve the desired pointwise error.

We will explore these reasons in more detail in Section IV. To solve the first problem, we remark that although \( U \) is the set such that its element can be localized within \( U \), a specific element \((x,y) \in U\) can be localized in a much smaller subset \( U(x;c_1) \times U(y;c_1) \subset U \), where \( U(x;c_1) \) is given by (24). Hence, the approximation regime should be dependent on the empirical observations to fully utilize the available information.

For the second problem, we need to design a specific polynomial with satisfactory pointwise approximation properties. Our approximation recipe is the following. Take \( K = c_2 \ln n \).

1) Over the square \([0, \frac{2c_1 \ln n}{n}]^2\), we consider the decomposition \(|x-y| = (\sqrt{x} + \sqrt{y})|\sqrt{x} - \sqrt{y}|\) and introduce the following two bivariate polynomials \( u_K(x,y) \) and \( v_K(x,y) \) to uniformly approximate \( \sqrt{x} + \sqrt{y} \) and \( |\sqrt{x} - \sqrt{y}| \), respectively. Specifically, we have

\[
\sup_{(x,y) \in [0,1]^2} |u_K(x,y) - (\sqrt{x} + \sqrt{y})| = \inf_{p \in \text{poly}_K} \sup_{(x,y) \in [0,1]^2} |P(x,y) - (\sqrt{x} + \sqrt{y})| \tag{128}
\]

\[
\sup_{(x,y) \in [0,1]^2} |v_K(x,y) - |\sqrt{x} - \sqrt{y}|| = \inf_{p \in \text{poly}_K} \sup_{(x,y) \in [0,1]^2} |P(x,y) - |\sqrt{x} - \sqrt{y}|| \tag{129}
\]

Then, denote \( h_{2K}(x,y) = u_K(x,y)v_K(x,y) - u_K(0,0)v_K(0,0) \), we use the polynomial

\[
P^{(1)}_K(x,y) = \frac{2c_1 \ln n}{n} h_{2K} \left( \frac{xn}{2c_1 \ln n}, \frac{yn}{2c_1 \ln n} \right) \tag{130}
\]

to approximate \(|x-y| \) over the square \([0, \frac{2c_1 \ln n}{n}]^2\). The polynomial \( P^{(1)}_K(x,y) \) satisfies \( P^{(1)}_K(0,0) = 0 \). We remove the constant term in the definition of \( P^{(1)}_K \) to guarantee that the estimator we construct is agnostic to the unknown support size \( S \). In practice, \( u_K \) and \( v_K \) can be replaced by the efficiently implementable lowpass filtered Chebyshev expansion [23], which achieves the same error rate as the best polynomial approximation.

2) Once we can assert with high probability \((p,q) \in U, p + q \geq \frac{c_1 \ln n}{2n}\), we utilize the best approximation polynomial of \(|t|\)
on $[-1, 1]$ with order $K$. Denote it as

$$R_K(t) = \arg\min_{P \in \text{poly}_K} \sup_{t \in [-1, 1]} |P(t) - |t||$$

(131)

$$= \sum_{j=0}^{K} r_j t^j,$$

(132)

we have

$$P^{(2)}_{K}(x, y; \hat{p}_{i,1}, \hat{q}_{i,1}) = \sum_{j=0}^{K} r_j W^{-j+1}(x - y)^j,$$

(133)

where $W = \sqrt{\frac{8c_1 \ln n}{n}} (\sqrt{(\hat{p}_{i,1} + \hat{q}_{i,1})} + \frac{1}{2}).$

Finally, we use the second part of the samples to construct the unbiased estimators for $P^{(1)}_K(x, y)$ defined in (130) and $P^{(2)}_K(x, y; \hat{p}_{i,1}, \hat{q}_{i,1})$ defined in (133). Concretely, we introduce the estimators $\tilde{P}^{(1)}_K(\hat{p}_{i,2}, \hat{q}_{i,2})$ and $\tilde{P}^{(2)}_K(\hat{p}_{i,2}, \hat{q}_{i,2}; \hat{p}_{i,1}, \hat{q}_{i,1})$ such that

$$\mathbb{E} \left[ \tilde{P}^{(1)}_K(\hat{p}_{i,2}, \hat{q}_{i,2}) \right] = P^{(1)}_K(p, q)$$

(134)

$$\mathbb{E} \left[ \tilde{P}^{(2)}_K(\hat{p}_{i,2}, \hat{q}_{i,2}; \hat{p}_{i,1}, \hat{q}_{i,1}) \right] = P^{(2)}_K(p, q; \hat{p}_{i,1}, \hat{q}_{i,1}).$$

(135)

These unbiased estimators are easy to construct since for any $r, s \geq 1, r, s \in \mathbb{Z}, (n\hat{p}, n\hat{q}) \sim \text{Poi}(np) \times \text{Poi}(nq)$, we have [24] Ex. 2.8

$$\mathbb{E} \left[ \prod_{i=0}^{r-1} \left( \frac{p - i}{n} \right) \prod_{j=0}^{s-1} \left( \frac{q - j}{n} \right) \right] = p^r q^s.$$  

(136)

We first present the performance of the estimator $\tilde{P}^{(1)}_K(\hat{p}_{i,2}, \hat{q}_{i,2})$ when $(p, q) \in \left[0, \frac{2c_1 \ln n}{n}\right]^2.$

**Lemma 15.** Suppose $(p, q) \in \left[0, \frac{2c_1 \ln n}{n}\right]^2,$ $(n\hat{p}, n\hat{q}) \sim \text{Poi}(np) \times \text{Poi}(nq).$ Then,

$$|\mathbb{E} \tilde{P}^{(1)}_K(\hat{p}, \hat{q}) - |p - q|| \lesssim \frac{1}{K} \sqrt{\frac{\ln n}{n}} (|p + \sqrt{q}| + \frac{1}{K^2} \frac{\ln n}{n})$$

(137)

$$\mathbb{V} \mathbb{A} \mathbb{R} \left( \tilde{P}^{(1)}_K(\hat{p}, \hat{q}) \right) \lesssim \frac{B_K}{n} (p + q),$$

(138)

for some universal constant $B > 0.$ The estimator $\tilde{P}^{(1)}_K$ is introduced in (134), and $K = c_2 \ln n, c_2 < c_1.$

We then analyze the estimator $\tilde{P}^{(2)}_K(\hat{p}_{i,2}, \hat{q}_{i,2}; \hat{p}_{i,1}, \hat{q}_{i,1})$ when $(p, q) \in U, p + q \geq \frac{c_1 \ln n}{2n}.$

**Lemma 16.** Suppose $(p, q) \in U, p + q \geq \frac{c_1 \ln n}{2n}, x, y \geq \frac{p + q}{2}, x \in [0, 1], y \in [0, 1]$, where the set $U$ is defined in (119). Suppose $(n\hat{p}, n\hat{q}) \sim \text{Poi}(np) \times \text{Poi}(nq).$ Then,

$$|\mathbb{E} \tilde{P}^{(2)}_K(\hat{p}, \hat{q}; x, y) - |p - q|| \lesssim \frac{1}{K} \sqrt{\frac{\ln n}{n}} (\sqrt{x} + \sqrt{y})$$

(139)

$$\mathbb{V} \mathbb{A} \mathbb{R} \mathbb{R} \left( \tilde{P}^{(2)}_K(\hat{p}, \hat{q}; x, y) \right) \lesssim \frac{B_K}{n} (x + y),$$

(140)

for some universal constant $B > 0.$ The estimator $\tilde{P}^{(2)}_K$ is introduced in (135), and $K = c_2 \ln n, c_2 < c_1.$

The final estimator is presented as follows.

**Estimator Construction 2.** As before, use sample splitting to obtain $(\hat{p}_{i,1}, \hat{q}_{i,1})$ and $(\hat{p}_{i,2}, \hat{q}_{i,2}).$ Denote

$$\mathbb{L}_2 = \begin{cases} \hat{p}_{i,2} - \hat{q}_{i,2} & \hat{p}_{i,1} - \hat{q}_{i,1} > \sqrt{\frac{(c_1 + c_2) \ln n}{n}} (\sqrt{\hat{p}_{i,1}} + \sqrt{\hat{q}_{i,1}}) \\ \hat{q}_{i,2} - \hat{p}_{i,2} & \hat{p}_{i,1} - \hat{q}_{i,1} < -\sqrt{\frac{(c_1 + c_2) \ln n}{n}} (\sqrt{\hat{p}_{i,1}} + \sqrt{\hat{q}_{i,1}}) \\ \hat{p}_{i,2} \sim \tilde{P}^{(1)}_K(\hat{p}_{i,2}, \hat{q}_{i,2}) & \hat{p}_{i,1} + \hat{q}_{i,1} < \frac{c_1 \ln n}{n} \\ \hat{q}_{i,2} \sim \tilde{P}^{(2)}_K(\hat{p}_{i,2}, \hat{q}_{i,2}; \hat{p}_{i,1}, \hat{q}_{i,1}) & (\hat{p}_{i,1}, \hat{q}_{i,1}) \in U_1, \hat{p}_{i,1} + \hat{q}_{i,1} \geq \frac{c_1 \ln n}{n} \\ \end{cases}$$

(141)
and define

\[ \hat{L}^{(2)} = 0 \lor \left( \hat{L}_2 \land 2 \right). \] (142)

Here \( U \) is given by (119), \( U_1 \) is defined in (121), the estimators \( \hat{P}^{(1)}_K \) and \( \hat{P}^{(2)}_K \) are defined in (134) and (135) \( K = c_2 \ln n \), and \( c_1 > c_3 > c_2 > 0 \) are properly chosen universal constants, \( \frac{c_3}{c_1} \leq \frac{8}{(\sqrt{2}+1)^2} - 1 \approx 0.373 \).

Fig. 1: Pictorial explanation of the minimax rate-optimal estimator in the unknown \( Q \) case. Note that we use \((\hat{p}_1, \hat{q}_1)\) to determine which one of the four estimators to use, and then apply the second independent sample \((\hat{p}_2, \hat{q}_2)\) to estimate. The dashed diagonal line denotes the points where the function \( f(p, q) = |p - q| \) is not analytic.

The next theorem presents the performance of \( \hat{L}^{(2)} \).

**Theorem 6.** For \( \ln n \lesssim \ln S \), we have

\[ \sup_{P, Q \in \mathcal{M}_S} \mathbb{E}[\hat{L}^{(2)} - L_1(P, Q)]^2 \leq \frac{S}{n \ln n}. \] (143)

**Proof.** Recall the “good” events \( E_1, E_2, E_3, E_4 \) defined in (122), (123), (124), (125) and introduce \( E = E_1 \cap E_2 \cap E_3 \cap E_4 \). We have

\[ \mathbb{E} \left( \hat{L}^{(2)} - L_1(P, Q) \right)^2 = \mathbb{E} \left[ (\hat{L}^{(2)} - L_1(P, Q))^2 1(E) \right] + \mathbb{E} \left[ (\hat{L}^{(2)} - L_1(P, Q))^2 1(E^c) \right] \] (144)

\[ \leq \mathbb{E} \left[ (\hat{L}_2 - L_1(P, Q))^2 1(E) \right] + 4\mathbb{P}(E^c) \] (145)

\[ \leq \mathbb{E} \left[ (\hat{L}_2 - L_1(P, Q))^2 1(E) \right] + \frac{60S}{n^{\beta}}, \] (146)

where we have applied Lemma [14] and the constant \( \beta \) is defined in (127).

Define the random variables

\[ \mathcal{E}_1 = \sum_{i \in I_1} (\hat{p}_{i, 2} - \hat{q}_{i, 2} - |p_i - q_i|) \] (147)

\[ \mathcal{E}_2 = \sum_{i \in I_2} (\hat{q}_{i, 2} - \hat{p}_{i, 2} - |p_i - q_i|) \] (148)

\[ \mathcal{E}_3 = \sum_{i \in I_3} (\hat{P}_K^{(1)}(\hat{p}_{i, 2}, \hat{q}_{i, 2}) - |p_i - q_i|) \] (149)

\[ \mathcal{E}_4 = \sum_{i \in I_4} (\hat{P}_K^{(2)}(\hat{p}_{i, 1}, \hat{q}_{i, 1}; \hat{p}_{i, 2}, \hat{q}_{i, 2}) - |p_i - q_i|) \] (150)
where the random index sets $I_1, I_2, I_3, I_4$ are defined as

$$
I_1 = \left\{ i : \hat{p}_{i,1} - \hat{q}_{i,1} > \sqrt{\frac{(c_1 + c_2) \ln n}{n}} (\sqrt{p_{i,1}} + \sqrt{q_{i,1}}) \Rightarrow p_i > q_i \right\} \quad (151)
$$

$$
I_2 = \left\{ i : \hat{p}_{i,1} - \hat{q}_{i,1} < -\sqrt{\frac{(c_1 + c_2) \ln n}{n}} (\sqrt{p_{i,1}} + \sqrt{q_{i,1}}) \Rightarrow p_i < q_i \right\} \quad (152)
$$

$$
I_3 = \left\{ i : \hat{p}_{i,1} + \hat{q}_{i,1} < \frac{c_1 \ln n}{n} \Rightarrow (p_i, q_i) \in \left[ 0, \frac{2c_1 \ln n}{n} \right]^2 \right\} \quad (153)
$$

$$
I_4 = \left\{ i : (\hat{p}_{i,1}, \hat{q}_{i,1}) \in U_1, \hat{p}_{i,1} + \hat{q}_{i,1} \geq \frac{c_1 \ln n}{n} \Rightarrow (p_i, q_i) \in U, p_i + q_i \geq \frac{c_1 \ln n}{2n}, \hat{p}_{i,1} + \hat{q}_{i,1} \geq \frac{p_i + q_i}{2} \right\}. \quad (154)
$$

The index sets $I_1, I_2, I_3, I_4$ are independent of the random variables $\{\hat{p}_{i,2} : 1 \leq i \leq S\}$ and $\{\hat{q}_{i,2} : 1 \leq i \leq S\}$. It follows from the definition of the $E_i$’s that

$$
\left( \hat{L}_2 - L_1(P, Q) \right) \mathbb{I}(E) = E_1 \mathbb{I}(E) + E_2 \mathbb{I}(E) + E_3 \mathbb{I}(E) + E_4 \mathbb{I}(E). \quad (155)
$$

Hence, it follows from the Cauchy–Schwarz inequality that

$$
\mathbb{E} \left( \hat{L}^{(2)} - L_1(P, Q) \right)^2 \leq 4 \sum_{j=1}^4 \mathbb{E} (E_j^2) + \frac{60 S}{n^2}. \quad (156)
$$

It follows from the law of total variance that

$$
\mathbb{E} E_1^2 = \mathbb{E} (\mathbb{V}(E_1 | I_1) + (\mathbb{E}[E_1 | I_1])^2)
= \mathbb{E} \mathbb{V}(E_1 | I_1) \quad (157)
= \sum_{i=1}^S \frac{p_i + q_i}{n} \quad (158)
\leq \frac{2}{n}, \quad (159)
$$

where we have used the fact that $\mathbb{E}[E_2 | I_2] = 0$ with probability one, the independence of $\hat{p}_{i,2}$ and $\hat{q}_{i,2}$, and Lemma \[15\].

Similarly we have $\mathbb{E} E_2^2 \leq \frac{2}{n}$. Regarding $\mathbb{E} E_3^2$, it follows from Lemma \[15\] and the mutual independence of $\{\hat{p}_{i,2} : 1 \leq i \leq S\}$ and $\{\hat{q}_{i,2} : 1 \leq i \leq S\}$ that

$$
\mathbb{E} E_3^2 \lesssim \sum_{i=1}^S \frac{B^2}{n} (p_i + q_i) + \left( \sum_{i=1}^S \frac{\sqrt{p_i} + \sqrt{q_i}}{\sqrt{n \ln n}} + \frac{1}{n \ln n} \right)^2 \quad (161)
\lesssim \frac{1}{n^{1-\epsilon}} + \frac{B^2}{n \ln n} \sqrt{\left( \frac{S}{n \ln n} \right)^2}, \quad (162)
$$

where $\epsilon = c_2 \ln B$.

Regarding $\mathbb{E} E_4^2$, it follows from the bias-variance decomposition and Lemma \[16\] that

$$
\mathbb{E} E_4^2(\{(\hat{p}_{i,1}, \hat{q}_{i,1}) : 1 \leq i \leq S\}) \lesssim \sum_{i=1}^S \frac{B^2}{n} (\hat{p}_{i,1} + \hat{q}_{i,1}) + \left( \sum_{i=1}^S \frac{\sqrt{p_{i,1} + q_{i,1}}}{\sqrt{n \ln n}} \right)^2, \quad (163)
$$
where the constant $B$ is the one in Lemma [16]. Taking expectations with respect to \{(\hat{\nu}_{i,1}, \hat{\nu}_{i,1}) : 1 \leq i \leq S\}, we have

\[
\mathbb{E}[\mathcal{E}_1^2] \lesssim \sum_{i=1}^{S} \frac{1}{n^{1-\epsilon}} \mathbb{E}(\hat{\nu}_{i,1}^{2} + \hat{\nu}_{i,1}^{2}) + \mathbb{E} \left( \sum_{i=1}^{S} \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n} \right)^{2}
\]

(164)

\[
\leq \frac{2}{n^{1-\epsilon}} + \sum_{i=1}^{S} \mathbb{E} \left( \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n} \right) \sum_{i,j=1}^{S} \mathbb{E} \left( \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n} \right) \sum_{1 \leq i,j \leq S} \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n}
\]

(165)

\[
\leq \frac{2}{n^{1-\epsilon}} + \sum_{i=1}^{S} \mathbb{E} \left( \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n} \right) + \sum_{1 \leq i,j \leq S} \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n}
\]

(166)

\[
\leq \frac{2}{n^{1-\epsilon}} + \frac{2}{n \ln n} + \sum_{1 \leq i,j \leq S} \frac{\hat{\nu}_{i,1} + \hat{\nu}_{i,1}}{n \ln n}
\]

(167)

\[
\lesssim \frac{1}{n^{1-\epsilon}} + \frac{S}{n \ln n},
\]

(168)

where $\epsilon = c_2 \ln B$.

Combining everything together, we have

\[
\mathbb{E} \left( \hat{L}^{(2)} - L_1(P,Q) \right)^2 \lesssim \frac{1}{n^{1-\epsilon}} + \frac{S}{n \ln n} \sqrt{\left( \frac{S}{n \ln n} \right)^2 + \frac{S}{n \ln n}},
\]

(169)

where $\epsilon = c_2 \ln B$, and the constant $B$ is the larger constant between the one in Lemma [15] and Lemma [16]. The constant $\beta$ is in (127).

If $\ln n \lesssim \ln S$, we can take $c_2$ small enough and $c_1, c_3$ large enough to guarantee that $\frac{S}{n \ln n} < \frac{S}{n \ln n}$. Upon noting that $\hat{L}^{(2)} \in [0, 2]$, we have

\[
\mathbb{E} \left( \hat{L}^{(2)} - L_1(P,Q) \right)^2 \lesssim \frac{S}{n \ln n}.
\]

(170)

Since the lower bound for the known $Q$ case also serves as a lower bound for the general case with a simulation argument, Theorem [3] and Theorem [6] yield that $\hat{L}^{(2)}$ is minimax rate-optimal. Note that $\hat{L}^{(2)}$ achieves the minimax rate without knowing the support size $S$ a priori. Moreover, the effective sample size enlargement effect holds again: the performance of the optimal estimator with $n$ samples is essentially that of the MLE with $n \ln n$ samples.

\section*{IV. COMPARISON WITH OTHER APPROACHES}

In this section, we review some other possible approaches in estimating the $L_1$ distance, and apply approximation theory to argue the strict suboptimality of some approaches.

\subsection*{A. Approximation only around the origin}

In the previous papers [5]–[9] in estimating entropy, power sum, mutual information, etc, the approximation methodology is conducted only around the origin. However, we remark that this is insufficient in estimating the $L_1$ distance. We have the following result.

\textbf{Lemma 17.} Let $\hat{L}$ denote an estimator of $L_1(P,Q)$ that satisfies the following:

\[
\hat{L} = \sum_{i=1}^{S} g(\hat{\nu}_{i,1}, \hat{\nu}_{i,1}),
\]

(171)

where the estimator $g(\hat{\nu}_{i,1}, \hat{\nu}_{i,1}) \in [-B, B]$ is a bounded function that satisfies $g(\hat{\nu}_{i,1}, \hat{\nu}_{i,1}) = |\hat{\nu}_{i,1} - \hat{\nu}_{i,1}|$ when $|\hat{\nu}_{i,1} - \hat{\nu}_{i,1}| \notin [0, 2c_\epsilon \ln n/n]^2$, $g(0, 0) = 0$. Suppose $n \gg S$. Then,

\[
\sup_{P, Q \in \mathcal{M}_S} \mathbb{E}[\hat{L} - L_1(P,Q)]^2 \gg \frac{S}{n \ln n}.
\]

(172)

Lemma [17] explains the reason why the estimator of Valiant and Valiant [5] can only achieve the optimal error rate when $n \lesssim S \lesssim n \ln n$, but ours achieves the optimal error rate for a much large set of parameter configurations.
B. One-dimensional approximation in the 2D case

In the construction of $\hat{L}^{(2)}$, we split into two cases when $(\hat{p}, \hat{q}) \in U_1$, i.e., 1D approximation of $|t|$ via the substitution $t = x - y$ if $\hat{p} + \hat{q} > c_1 \ln n/n$, and the decomposition of $|x - y|$ into $(\sqrt{x + y} + \sqrt{x - y})/\sqrt{x - y}$ otherwise. Can we always do 1D approximation of $|t|$ with $t = x - y$ to achieve the desired approximation error, i.e., propose some $P(t) \in \poly_K$ with $K \asymp \ln n$ and $|P(t) - |t|| \leq \sqrt{|t|/(n \ln n)} + \frac{1}{n \ln n}$ for any $|t| \leq c_1 \ln n/n$? We have the following lemma regarding the approximation of $|t|$.

Lemma 18. If $Q_K(t) \in \poly_K$ is even with $Q_K(0) = 0$, and achieves the best uniform error rate $\max_{t \in [-1,1]} |Q_K(t) - |t|| \lesssim 1/K$, we have

\[
\limsup_{K \to \infty} \frac{1}{K} \sup_{0 < |t| \leq 1} \frac{|t| - |Q_K(t) - |t||}{t^2} < \infty. \tag{173}
\]

Now we apply Lemma 18 to the hypothetical polynomial $P(t)$. Doing parameter substitution $t = \frac{c_1 \ln n}{n} \cdot y, y \in [-1,1]$ by assumption we have for any $y \in [-1,1]$,

\[
\left| \frac{n}{c_1 \ln n} P \left( \frac{c_1 \ln n}{n} y \right) - |y| \right| \lesssim \frac{\sqrt{|y|}}{K} + \frac{1}{K^2}, \tag{174}
\]

where $K \asymp \ln n$. It follows from Jensen’s inequality that

\[
\left| \frac{n}{c_1 \ln n} \left( P \left( \frac{c_1 \ln n}{n} y \right) + P \left( - \frac{c_1 \ln n}{n} y \right) \right) / 2 - |y| \right| \lesssim \frac{\sqrt{|y|}}{K} + \frac{1}{K^2}. \tag{175}
\]

Define $Q(y) = \frac{n}{c_1 \ln n} \left( P \left( \frac{c_1 \ln n}{n} y \right) + P \left( - \frac{c_1 \ln n}{n} y \right) \right) / 2$. It is clear that $Q(y)$ satisfies the assumptions in Lemma 18. Hence,

\[
|Q(y) - |y|| \geq |y| - CKy^2. \tag{176}
\]

However, it contradicts the upper bound $175$ when $\frac{1}{K^2} \ll |y| \ll \frac{1}{K}$. Hence, any 1D approximation does not achieve the error rate that is achieved by our 2D approximation approach.

C. Approximation on the entire 2D stripe

In the unknown $Q$ case we have decomposed the stripe $U$ into subsets where polynomial approximations take place. Is it possible that we use a single polynomial $P(x, y) \in \poly_K$ of degree $K \asymp \ln n$ to approximate $|x - y|$ such that $|P(x, y) - |x - y|| \lesssim \sqrt{|x + y|/(n \ln n)}$ for any $(x, y) \in U$? We prove that the answer is negative even for $U' = \bigcup_{x \in [c_1 \ln n, n]} U(x; c_1) \times U(x; c_1) \subset U$ and any $t_n \gg (\ln n)^3/n$.

Lemma 19. If $(\ln n)^3/n \ll t_n \leq 1/2$, $K \asymp \ln n$, we have

\[
\liminf_{n \to \infty} \sqrt{n \ln n} \cdot \inf_{P \in \poly^2} \sup_{(x, y) \in U'} \frac{|P(x, y) - |x - y||}{\sqrt{x + y}} = +\infty.
\]

Lemma 19 shows that for a too large set $U'$ (e.g., $U' = U$), every polynomial fails to achieve the desired approximation error bound $\sqrt{|x + y|/(n \ln n)}$. Hence, it is necessary to make the approximation regime be random and dependent on the empirical observations.

D. The failure of any plug-in approach

It is evident that the optimal $L_1$ distance estimators we constructed heavily exploit the interactions of $P$ and $Q$. For example, in the known $Q$ case, the estimator for $L_1(P, Q)$ is not of the form $L_1(g(P_n), Q)$, where $g(\cdot)$ is an arbitrary function of the empirical distribution of $P$ that is independent of $Q$.

We show that for any estimator $g(P_n)$ of the distribution $P$, the plug-in approach $L_1(g(P_n), Q)$ does not achieve the minimax rates in estimating $L_1(P, Q)$ when one considers the worst cases among all $P, Q \in \mathcal{M}_S$.

Lemma 20. Consider the known $Q$ case. Suppose $g(P_n) \in \mathbb{R}^S$ is an arbitrary function of the empirical distribution $P_n$, and $g(\cdot)$ does not depend on $Q$. Then, if $n \gtrsim S$,

\[
\sup_{P, Q \in \mathcal{M}_S} \mathbb{E}_P \left( L_1(g(P_n), Q) - L_1(P, Q) \right)^2 \gtrsim \frac{S}{n}. \tag{177}
\]

Lemma 20 shows that since the plug-in approach $L_1(g(P_n), Q)$ does not explicitly exploit the nonsmoothness of the function $L_1(P, Q)$, in the worst case it behaves essentially like the maximum likelihood estimator as shown in Corollary 1.
V. Acknowledgements

We are grateful to Vilmos Totik for discussing multivariate approximation theory, and for the insights that motivated the proof of Lemma 19. We would like to thank Gregory Valiant for discussing the estimator in [5].

Appendix A
 Auxiliary Lemmas

The first-order symmetric difference of a function $f$ is given by
\[ \Delta_1^h f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right), \] (178)
while the second-order symmetric difference is given by
\[ \Delta_2^h f(x) = \Delta_h \left( \Delta_1^h f(x) \right) = f(x + h) - 2f(x) + f(x - h). \] (179)

Analogously, the $r$-th order symmetric difference can be defined, and it is zero when $[x, x + rh]$ or $[x - rh, x]$ are not inside the domain of $f$.

For function $f(x)$ with domain $[0, 1]$, $\varphi(x) = \sqrt{x(1-x)}$, the first-order Ditzian–Totik modulus of smoothness is defined as
\[ \omega^1_\varphi(f, t) \triangleq \sup_{0 < h \leq t} \| \Delta^1_{h, \varphi} f(x) \|_\infty, \] (181)
and the second-order Ditzian–Totik modulus of smoothness is defined as
\[ \omega^2_\varphi(f, t) \triangleq \sup_{0 < h \leq t} \| \Delta^2_{h, \varphi} f(x) \|_\infty. \] (182)

Similarly, we can also define the $r$-th order Ditzian–Totik modulus of smoothness for a function $f(x)$ with domain $[0, 1]^2$:
\[ \omega^r_{[0,1]^2}(f, t) = \sup_{1 \leq t \leq 2, 0 < h \leq t, x \in [0,1]^2} |\Delta^r_{i,h\varphi(x),i} f(x)|, \] (183)
where $\Delta_{i,h}$ denotes the symmetric difference with respect to the $i$-th coordinate.

The next lemma upper bounds the best polynomial approximation error by the Ditzian-Totik moduli.

**Lemma 21.** [25] Thm. 7.2.1, Thm. 12.1.1] There exists a universal constant $M(r) > 0$ such that for any function $f \in C[0, 1]$,\n\[ E_n(f; [0, 1]) \leq M(r) \omega^r_\varphi(f, n^{-1}), \quad n > r, \] (184)

where $E_n[f; I]$ denotes the distance of the function $f$ to the space $\text{poly}_n$ in the uniform norm $\| \cdot \|_{\infty, I}$ on $I \subset \mathbb{R}$. Moreover, if $f(x) : [0, 1]^2 \mapsto \mathbb{R}$, we have
\[ E_n[f; [0, 1]^2] \leq M \omega^r_{[0,1]^2}(f, n^{-1}), \] (185)
for any $r < n$, where $M$ is independent of $f$ and $n$, and $E_n[f; [0, 1]^2]$ denotes the distance of the function $f$ to the space $\text{poly}^n_{\varphi}$ in the uniform norm on $[0, 1]^2$.

The modulus $\omega^2_\varphi(f, t)$ is computed for a variety of functions in the following lemma.

**Lemma 22.** [25] Chap. 3.4] Suppose $f(x) = x^d, 0 < \delta < 1, x \in [0, 1]$. Then,
\[ \omega^2_\varphi(f, t) \asymp t^{2\delta}, \quad \omega^1_\varphi(f, t) \asymp \max\{t^{2\delta}, t\} \] (186)
(187)

where $\omega^1_\varphi(f, t)$ is defined in (181), $\omega^2_\varphi(f, t)$ is defined in (182).

**Lemma 23.** Suppose $f(x; a) = |\sqrt{x} - \sqrt{a}|, x \in [0, 1]$, and $a \in [0, 1]$ is a parameter. Then,
\[ \omega^1_\varphi(f, t) \leq \frac{t}{\sqrt{2}}, \] (188)

Next lemma computes the Ditzian–Totik modulus for function $f(x) = |2x\Delta - q|, x \in [0, 1]$. 


Lemma 24. Suppose \( f(x) = |2x \Delta - q|, \Delta > 0, 0 \leq q \leq 2 \Delta, x \in [0, 1] \). Then, for any integer \( K \geq 1 \),

\[
\omega^2_\varphi(f, K^{-1}) = \begin{cases} 
2q & q \leq \frac{2 \Delta}{1+K} \\
\frac{2\sqrt{q(2\Delta-q)}}{K} & \frac{2 \Delta}{1+K} \leq q \leq \frac{2 \Delta K^2}{1+K} \\
2(2\Delta-q) \frac{2 \Delta K^2}{1+K} \leq q \leq 2 \Delta 
\end{cases} 
\] (189)

\[
\approx \min \left\{ q, \frac{\sqrt{q(2\Delta-q)}}{K}, (2\Delta-q) \right\}, 
\] (190)

where \( \omega^2_\varphi(f, t) \) is defined in (182).

Lemma 25 (Markov’s inequality). \([25, \text{Chap 4, Thm 1.4}]\) Suppose \( P_n \in \text{poly}_n \) is defined on \([-1, 1]\). Then,

\[
\sup_{x \in [-1, 1]} |P_n'(x)| \leq n^2 \sup_{x \in [-1, 1]} |P_n(x)|
\] (191)

Lemma 26. \([25, \text{Thm. 7.3.1.}]\) For \( P_n \) the best \( n \)-th degree polynomial approximation to \( f \) in \([0, 1]\) and an integer \( r \in \{1, 2\} \) we have

\[
\sup_{x \in [0, 1]} |\varphi^r P_n^{(r)}| \leq Mn^r \omega^r_\varphi(f, n^{-1}),
\] (192)

where \( \varphi(x) = \sqrt{x(1-x)} \) and \( M \) is independent of \( n \) and \( f \).

The next lemma shows that a polynomial on \([-1, 1]\) nearly attains its supremum norm in a slightly smaller interval contained in \([-1, 1]\).

Lemma 27. \([25, \text{Thm. 8.4.8.}]\) Suppose \( c > 0 \) is a constant, \( P_n \in \text{poly}_n \) defined on \([-1, 1]\), \( n^2 > c \). Then, there exists a constant \( M(c) > 0 \) that does not depend on \( n \) and \( P_n \) such that

\[
\sup_{x \in [-1, 1]} |P_n(x)| \leq M(c) \sup_{x \in [-1+cn^{-2}, 1-cn^{-2}]} |P_n(x)|.
\] (193)

Lemma 28. Suppose \( P_K(x) \) is the best approximation polynomial with order \( K \) of function \( f(x) \in C[0, 1] \) defined as

\[
P_K(x) = \arg\min_{P \in \text{poly}_K} \max_{x \in [0,1]} |f(x) - P(x)|.
\] (194)

Then, the best approximation polynomial with order \( 2K \) of function \( f(z^2), z \in [-1, 1] \) is given by \( P_K(z^2) \).

The following lemma characterizes the upper bounds of the coefficients of a bounded real polynomial.

Lemma 29. \([17]\) Let \( p_n(x) = \sum_{\nu=0}^n a_{\nu} x^\nu \) be a polynomial of degree at most \( n \) such that \( |p_n(x)| \leq A \) for \( x \in [a, b] \). Then

1) If \( a + b \neq 0 \), then

\[
|a_\nu| \leq 2^{\nu/2}A \left| \frac{a+b}{2} \right|^{-\nu} \left( \left| \frac{b+a}{b-a} \right|^{n+1} + 1 \right), \quad \nu = 0, \cdots, n.
\] (195)

2) If \( a + b = 0 \), then

\[
|a_\nu| \leq Ab^{-\nu}(\sqrt{2}+1)^n, \quad \nu = 0, \cdots, n.
\] (196)

The following lemma gives an upper bound for the second moment of the unbiased estimate of \( (p - q)^j \) in Poisson model.

Lemma 30. Suppose \( nX \sim \text{Poi}(np), p \geq 0, q \geq 0 \). Then, the estimator

\[
g_{j, q}(X) \triangleq \sum_{k=0}^j \binom{j}{k}(-q)^{j-k} \prod_{h=0}^{k-1} \left( X - \frac{h}{n} \right)
\] (197)

is the unique unbiased estimator for \( (p - q)^j \), \( j \geq 0, j \in \mathbb{N} \), and its second moment is given by

\[
\mathbb{E}[(g_{j, q}(X))^2] = \sum_{k=0}^j \binom{j}{k}^2 (p-q)^{2(j-k)} \frac{k!}{n^k}
\] (198)

\[
= j! \left( \frac{p}{n} \right)^j L_j \left( -\frac{n(p-q)^2}{p} \right) \quad \text{Assuming } p > 0.
\] (199)
where \( L_m(x) \) stands for the Laguerre polynomial with order \( m \), which is defined as:

\[
L_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{k!}
\]  

If \( M \geq \max \left\{ \frac{n(p-q)^2}{p}, j \right\} \), we have

\[
\mathbb{E}[g_{j,q}(X)]^2 \leq \left( \frac{2Mp}{n} \right)^j.
\]

When \( k = 0, \prod_{h=0}^{k-1} (X - \frac{b}{n}) \equiv 1 \). When \( p = 0 \), \( g_{j,q}(X) \equiv (-q)^j \), \( \mathbb{E}[g_{j,q}(X)]^2 \equiv q^{2j} \).

We construct the unbiased estimator of \( (p - q)^j \), \( j \geq 0 \) when both \( p \) and \( q \) are unknown as in the following lemma.

**Lemma 31.** Suppose \( \hat{p}, \hat{q} \sim \text{Poi}(np) \times \text{Poi}(nq) \). Then, the following estimator using \( \hat{p}, \hat{q} \) is the unique unbiased estimator for \( (p - q)^j \), \( j \geq 0 \), \( j \in \mathbb{Z} \):

\[
\hat{A}_j(\hat{p}, \hat{q}) = \sum_{k=0}^{j} \binom{j}{k} \prod_{i=0}^{k-1} (\hat{p} - \frac{i}{n})(-1)^{j-k} \sum_{m=0}^{j-k} (\hat{q} - \frac{m}{n}).
\]

Furthermore,

\[
\mathbb{E}\hat{A}_j^2 \leq \left( 2(p - q)^2 \vee \frac{8j(p \vee q)}{n} \right)^j.
\]

The following lemma characterizes the behavior of the central moments of Poisson distributions.

**Lemma 32.** Suppose \( \hat{p} \sim \text{Poi}(np) \). Then, for any integer \( s \geq 2 \), there exist \( \lfloor s/2 \rfloor \) constants \( h_{j,s} \) that are independent of \( n \), such that

\[
\mathbb{E}(\hat{p} - p)^s = \frac{1}{n^s} \sum_{j=1}^{\lfloor s/2 \rfloor} h_{j,s}(np)^j.
\]

Furthermore,

\[
|h_{j,s}| \leq \frac{2^j j^s}{j!}
\]

\[
\leq (2e^{c/(c-1)})^s \left( \frac{s}{\ln s} \right)^s.
\]

Consequently, there exists a constant \( C_s > 0 \) depending only on \( s \) satisfying \( (C_s)^{1/s} \leq \frac{e}{\ln s} \) such that

\[
\mathbb{E}|\hat{p} - p|^s \leq C_s (np)^{s/2} \vee (np)\frac{s}{n^s}.
\]

We emphasize that the scaling \( \left( \frac{s}{\ln s} \right)^s \) is consistent with the general moment bounds in [27]. However, the results in [27] do not directly apply here. Furthermore, Lemma [32] provides bounds on each individual \( h_{j,s} \), which is not obtainable from a general moment bound.

The next lemma controls the moments of \( \frac{1}{\hat{p} \vee \frac{1}{n}} \), where \( \hat{p} \sim \text{Poi}(np) \).

**Lemma 33.** Suppose \( \hat{p} \sim \text{Poi}(np) \). Then, for any integer \( j \geq 0 \), there exists a constant \( B_j \) depending only on \( j \) such that

\[
\mathbb{E} \left( \frac{1}{\hat{p} \vee \frac{1}{n}} \right)^j \leq B_j \frac{1}{p^j}.
\]

One may take \( B_j = j \left( \frac{1}{p} \right)^j + 1 + j2^{j+1} + j \left( \frac{16(j+1)}{e} \right)^{j+1} \).

The following lemma gives well-known tail bounds for Poisson and binomial random variables.

**Lemma 34.** [Exercise 4.7] If \( X \sim \text{Poi}(\lambda) \) or \( X \sim \text{B}(n, \frac{\lambda}{n}) \), then for any \( \delta > 0 \), we have

\[
\mathbb{P}(X \geq (1 + \delta)\lambda) \leq \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\lambda \leq e^{-\delta^2 \lambda/3} \vee e^{-\delta \lambda/3}
\]

\[
\mathbb{P}(X \leq (1 - \delta)\lambda) \leq \left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^\lambda \leq e^{-\delta^2 \lambda/2}.
\]
The following lemma presents the Hoeffding bound.

**Lemma 35.** Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $X_i$ takes its value in $[a_i, b_i]$ almost surely for all $i \leq n$. Let $S_n = \sum_{i=1}^{n} X_i$, we have for any $t > 0$,

$$P \{|S_n - \mathbb{E}[S_n]| \geq t\} \leq 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right). \tag{211}$$

**APPENDIX B**

**PROOFS OF MAIN LEMMAS**

A. Proof of Lemma 7

The following upper bound is straightforward:

$$\mathbb{E}[\hat{q} - q] \leq \sqrt{\mathbb{E}[|\hat{q} - q|^2]} \tag{212}$$

$$= \sqrt{\frac{q}{n}}, \tag{213}$$

Regarding the other upper bound and the lower bound, we utilize the exact analytic expression for $\mathbb{E}|X - \lambda|$ for $X \sim \text{Poi}(\lambda)$. It follows from [30] that for random variable $X \sim \text{Poi}(\lambda)$,

$$\mathbb{E}|X - \lambda| = 2\lambda e^{-\lambda} [\lambda]^{\lambda} / [\lambda]!, \tag{214}$$

where $[\lambda]$ denotes the greatest integer less than or equal to $\lambda$.

When $0 < \lambda \leq 1$, we have

$$\mathbb{E}|X - \lambda| = 2\lambda e^{-\lambda}, \tag{215}$$

which implies that if $0 < q \leq \frac{1}{n}$,

$$\mathbb{E}[\hat{q} - q] = 2qe^{-nq} \in [2qe^{-1}, 2q]. \tag{216}$$

Regarding the final lower bound, it suffices to show that for $X \sim \text{Poi}(\lambda), \lambda \geq 1$, we have

$$\mathbb{E}|X - \lambda| \geq \sqrt{\frac{\lambda}{2}}. \tag{218}$$

Hence, it suffices to show

$$2\sqrt{2\lambda} e^{-\lambda} \lambda [\lambda]^{\lambda} / [\lambda]! \geq 1 \tag{219}$$

for all $\lambda \geq 1$. It is equivalent to

$$2\sqrt{2\lambda} e^{-\lambda} \lambda^n / n! \geq 1 \tag{220}$$

for $\lambda \in [n, n+1)$ for all the integers $n \geq 1$.

Since the function $2\sqrt{2\lambda} e^{-\lambda} \lambda^n / n!$ is monotonically increasing for $\lambda \in [n, n+1/2]$, and monotonically decreasing for $\lambda \in [n+1/2, n+1)$, it suffices to consider integers $\lambda$. Hence, it suffices to show for any integer $n \geq 1$,

$$2\sqrt{2ne^{-n} n^n / n!} \geq 1 \tag{221}$$

$$2\sqrt{2(n+1)e^{-(n+1)} (n+1)^n / n!} \geq 1, \tag{222}$$

which is equivalent to

$$n! \leq \sqrt{8n} \left(\frac{n}{e}\right)^n. \tag{223}$$

It follows from [31] that for any positive integer $n$,

$$n! < \sqrt{2\pi e^{1/2}} \sqrt{n} \left(\frac{n}{e}\right)^n, \tag{224}$$

which implies [223] since $\sqrt{2\pi e^{1/2}} < \sqrt{8}$ for all positive integers.
B. Proof of Lemma 2

We first assume \( q \geq p \). Applying the relation

\[
x = (x)_+ - (x)_- \\
|x| = (x)_+ - (x)_- 
\]

where \((x)_+ = \max\{x, 0\}\), \((x)_- = -\min\{x, 0\}\), we have

\[
|\mathbb{E}|\hat{q} - p| - |q - p| = |\mathbb{E}|\hat{q} - p| - (q - p)| \\
= |\mathbb{E}(\hat{q} - p) - (q - p) + 2\mathbb{E}(\hat{q} - p)_-|
\]

\[
= 2\mathbb{E}(\hat{q} - p)_-. 
\]

Construct random variable \( \hat{p} \) such that \( n\hat{p} \sim \text{Poi}(np) \) is on the same probability space as \( \hat{q} \), with the relationship

\[
n\hat{q} = n\hat{p} + Z, \\
\]

where \( Z \) is independent of \( \hat{p} \) and \( Z \sim \text{Poi}(n(q - p)) \). Hence, \( \hat{q} \geq \hat{p} \) with probability one. We have

\[
2\mathbb{E}(\hat{q} - p)_- \leq 2\mathbb{E}(\hat{p} - p)_- \\
= \mathbb{E}|\hat{p} - p| - \mathbb{E}(\hat{p} - p) \\
= \mathbb{E}|\hat{p} - p| \\
\leq 2 \cdot \min\left\{ p, q, \sqrt{\frac{q}{n}}, \sqrt{\frac{p}{n}} \right\}, 
\]

where we applied Lemma 1 in the last step. The case of \( q \leq p \) can be proved analogously.

Regarding the lower bound, we have

\[
sup_{q \geq 0} |\mathbb{E}|\hat{q} - p| - |q - p|| \geq |\mathbb{E}|\hat{p} - p| \geq \frac{1}{\sqrt{2}} \left( p \wedge \sqrt{\frac{p}{n}} \right),
\]

where we lower bound via taking \( q = p \) and using Lemma 1.

C. Proof of Lemma 3

For \( n\hat{q} \sim \text{Poi}(nq) \),

\[
\text{Var}(|\hat{q} - p|) = \inf_a \mathbb{E}((|\hat{q} - p| - a)^2) \\
\leq \mathbb{E}((|\hat{q} - p| - |q - p|)^2) \\
\leq \mathbb{E}|\hat{q} - q|^2 \\
= \frac{q}{n},
\]

where we used the fact that \(|a| - |b| \leq |a - b|\).

D. Proof of Lemma 4

We first consider the case of \( q \leq \frac{c_1 \ln n}{n} \). In this case,

\[
P(\hat{q} \notin U(q; c_1)) = \mathbb{P} \left( \hat{q} > \frac{2c_1 \ln n}{n} \right) \\
= \mathbb{P}(\text{Poi}(nq) > 2c_1 \ln n) \\
\leq \mathbb{P}(\text{Poi}(c_1 \ln n) > 2c_1 \ln n) \\
\leq e^{-2c_1 \ln n} \\
= n^{-\frac{c_1}{2}},
\]
where we used Lemma \[34\] in the last step. When \(q > \frac{c_1 \ln n}{n}\), we have

\[
\Pr(\hat{q} \notin U(q;c_1)) \leq \Pr(\hat{q} > q + \sqrt{\frac{c_1 q \ln n}{n}}) + \Pr(\hat{q} < q - \sqrt{\frac{c_1 q \ln n}{n}})
\]

\[
= \Pr(\text{Poi}(nq) > nq + \sqrt{c_1 q n \ln n}) + \Pr(\text{Poi}(nq) < nq - \sqrt{c_1 q n \ln n})
\]

\[
\leq e^{-\frac{c_1 q n \ln n}{nq}} + e^{-\frac{c_1 q n \ln n}{nq}}
\]

\[
\leq \frac{2}{n^{c_1/3}}.
\]

(246)

(247)

(248)

(249)

where we applied Lemma \[34\] again.

\[E. \text{ Proof of Lemma 5}\]

Since

\[
\Pr(E^c) = \Pr(E_1^c \cup E_2^c \cup E_3^c)
\]

\[
\leq \Pr(E_1^c) + \Pr(E_2^c) + \Pr(E_3^c),
\]

(250)

(251)

it suffices to control \(\Pr(E_i^c), i = 1, 2, 3\) separately. We have

\[
\Pr(E_1^c) = \Pr\left(\bigcup_{i=1}^{S} \{\hat{p}_{i,1} > U_1(q_i), p_i < q_i\}\right)
\]

\[
\leq S \Pr(\hat{p}_{i,1} > U_1(q_i), p_i < q_i)
\]

\[
\leq S \Pr(\text{Poi}(nq_i) > n \cdot U_1(q_i)).
\]

(252)

(253)

(254)

Note that if \(q_i \leq \frac{c_1 \ln n}{n}\), then it follows from Lemma \[34\] that

\[
\Pr(\text{Poi}(nq_i) > n \cdot U_1(q_i)) \leq \Pr(\text{Poi}(c_1 \ln n) > (c_1 + c_3) \ln n)
\]

\[
\leq e^{-\frac{c_3}{n} \ln n}.
\]

(255)

(256)

If \(q_i > \frac{c_1 \ln n}{n}\), then it follows from Lemma \[34\] that

\[
\Pr(\text{Poi}(nq_i) > n \cdot U_1(q_i)) \leq \Pr(\text{Poi}(nq_i) > nq_i + \sqrt{c_3 q_i n \ln n})
\]

\[
\leq e^{-\frac{c_3 \ln n}{n}}.
\]

(257)

(258)

Hence,

\[
\Pr(E_1^c) \leq \frac{S}{n^{c_1}}.
\]

(259)

Analogously, \(\Pr(\hat{p}_{i,1} < U_1(q_i), p_i > q_i) = 0\) when \(q_i \leq \frac{c_1 \ln n}{n}\), and when \(q_i > \frac{c_1 \ln n}{n}\),

\[
\Pr(\hat{p}_{i,1} < U_1(q_i), p_i > q_i) \leq \Pr(\text{Poi}(nq_i) \leq nq_i - \sqrt{c_3 q_i n \ln n})
\]

\[
\leq e^{-\frac{c_3 \ln n}{n}}.
\]

(260)

(261)

Hence,

\[
\Pr(E_2^c) \leq \frac{S}{n^{c_1}}.
\]

(262)

As for \(\Pr(E_3^c)\), when \(q_i \leq \frac{c_1 \ln n}{n}\),

\[
\Pr(\hat{p}_{i,1} \in U_1(q_i), p_i \notin U(q_i; c_1)) \leq \Pr(\text{Poi}(2c_1 \ln n) \leq (c_1 + c_3) \ln n)
\]

\[
\leq e^{-\frac{(c_1 + c_3)^2}{4c_1} \ln n}.
\]

(263)

(264)

When \(q_i > \frac{c_1 \ln n}{n}\),

\[
\Pr(\hat{p}_{i,1} \in U_1(q_i), p_i > U(q_i; c_1)) \leq \Pr(\text{Poi}(nq_i + \sqrt{c_1 q_i n \ln n}) \leq nq_i + \sqrt{c_3 q_i n \ln n})
\]

\[
\leq e^{-\left(\frac{c_1 n q_i}{nq_i + \sqrt{c_1 q_i n \ln n}}\right)^2} \frac{1}{2}(nq_i + \sqrt{c_3 q_i n \ln n})
\]

\[
\leq e^{-\frac{(c_1 + c_3)^2}{4c_1} \ln n}.
\]

(265)

(266)

(267)
\[
\mathbb{P}(\hat{p}_{i,1} \in U_1(q_i), p_i < U(q_i; c_1)) \leq \mathbb{P}(\text{Poi}(nq_i - \sqrt{c_1q_i n \ln n}) \geq nq_i - \sqrt{c_2q_i n \ln n})
\]

(268)

\[
\leq e^{-\frac{\sqrt{c_1q_i n \ln n}}{2}} + e^{-\frac{c_1q_i n \ln n}{2nq_i - \sqrt{c_1q_i n \ln n}}}
\]

(269)

\[
\leq e^{-\frac{c_1q_i n \ln n}{2nq_i - \sqrt{c_1q_i n \ln n}}} + e^{-\frac{\sqrt{c_1q_i n \ln n}}{2}}
\]

(270)

\[
\leq e^{-\frac{\sqrt{c_1q_i n \ln n}}{2}}.
\]

(271)

Consequently,

\[
\mathbb{P}(E^*_n) \leq \frac{S}{n^\delta}.
\]

(272)

\[F. \text{ Proof of Lemma 21}\]

To simplify the notation we denote \(\Delta = \frac{\ln n}{n}\). We split the proof into two cases: \(q \leq \Delta\) and \(q > \Delta\).

1) The case \(q \leq \Delta, p \in U(q; c_1) = [0, 2\Delta]\). In this case, it follows from (32) that \(P_K(x; q)\) is the best approximation polynomial of function \(|x-q|\) over \(x \in [0, 2\Delta]\). Define \(y = \frac{x}{\Delta}\) and introduce function

\[
g(y) = |2y\Delta - q|, y \in [0, 1].
\]

(273)

Define the best approximation polynomial of \(g(y) \in C[0,1]\) with order \(K\) as

\[
H_K(y) = \arg\min_{P \in \text{poly}_K} \max_{y \in [0,1]} |g(y) - P(y)|.
\]

(274)

It follows from Lemma 21 and 24 that there exists a universal constant \(M > 0\) such that

\[
\sup_{y \in [0,1]} |H_K(y) - g(y)| \leq M \left( q \wedge \frac{\sqrt{q\Delta}}{K} \right).
\]

(275)

It implies that there exists another universal constant \(M_1 > 0\) such that

\[
\sup_{y \in [0,1]} |H_K(y)| \leq \max_{y \in [0,1]} |g(y)| + M \left( q \wedge \frac{\sqrt{q\Delta}}{K} \right) \leq M_1\Delta.
\]

(276)

(277)

Denoting \(H_K(y) = \sum_{j=0}^K a_j y^j\) and using \(x = 2\Delta y\), we know

\[
P_K(x; q) = \sum_{j=0}^K a_j (2\Delta)^{-j} x^j
\]

(278)

\[
\sup_{x \in [0,2\Delta]} |P_K(x; q) - |x-q|| \leq M \left( q \wedge \frac{\sqrt{q\Delta}}{K} \right)
\]

(279)

\[
\leq q \wedge \frac{1}{K} \sqrt{\frac{q \ln n}{n}}.
\]

(280)

It follows from Lemma 30 that \(\prod_{k=0}^{j-1} \left( \hat{p} - \frac{k}{n} \right)\) is the unique unbiased estimator of \(p^j\) when \(n\hat{p} \sim \text{Poi}(np)\). Hence,

\[
\hat{P}_K(\hat{p}; q) = \sum_{j=0}^K a_j (2\Delta)^{-j} \prod_{k=0}^{j-1} \left( \hat{p} - \frac{k}{n} \right),
\]

(281)

and \(E\hat{P}_K(\hat{p}; q) = P_K(p; q)\).

Since \(H_K(z^2) = \sum_{j=0}^K a_j z^{2j}\) is a polynomial with degree no more than \(2K\) and satisfies

\[
\sup_{z \in [-1,1]} |H_K(z^2)| \leq M_1\Delta,
\]

(282)

It follows from Lemma 29 that for all \(0 \leq j \leq K\),

\[
|a_j| \leq M_1\Delta \left( \sqrt{2} + 1 \right)^{2K}.
\]

(283)
Now we prove the variance properties of $\tilde{P}_K(\hat{\rho}; q)$. We have

$$\text{Var}(\tilde{P}_K(\hat{\rho}; q)) = \text{Var}\left( \sum_{j=0}^{K} a_j (2\Delta)^{-j} \prod_{k=0}^{j-1} \left( \hat{\rho} - \frac{k}{n} \right) \right)$$

(284)

$$\leq \left( \sum_{j=0}^{K} |a_j| (2\Delta)^{-j} \right)^2 \text{Var} \left( \prod_{k=0}^{j-1} \left( \hat{\rho} - \frac{k}{n} \right) \right)^{1/2}$$

(285)

$$\leq \max_{0 \leq j \leq K} |a_j|^2 \left( \sum_{j=1}^{K} \left( \frac{2Mp}{n} \right)^{j/2} \right)^2$$

(286)

$$= \max_{0 \leq j \leq K} |a_j|^2 \left( \sum_{j=1}^{K} \left( \frac{2Mp}{16\Delta^2n} \right)^{j/2} \right)^2$$

(287)

$$\leq \max_{0 \leq j \leq K} |a_j|^2 \left( \sum_{j=1}^{K} \left( \frac{p}{\Delta} \right)^{j/2} \right)^2$$

(288)

$$= \max_{0 \leq j \leq K} |a_j|^2 \left( \sum_{j=0}^{K-1} \left( \frac{\sqrt{p}}{\Delta} \right)^j \right)^2,$$

(289)

where we have applied Lemma [20] with $M = \max\{2n\Delta, K\} = 2n\Delta$ since we have assumed $c_2 < c_1, K = c_2 \ln n$. Since $p \leq 2\Delta$, we have

$$\text{Var}(\tilde{P}_K(\hat{\rho}; q)) \leq \max_{0 \leq j \leq K} |a_j|^2 \left( \sum_{j=0}^{K-1} 2^{j/2} \right)^2$$

(290)

$$\lesssim \Delta^2 \frac{p}{\Delta} B^K$$

(291)

$$\lesssim B^K \ln n \frac{n}{p + q}.$$ (292)

2) The case $q > \Delta$. In this case, it follows from [32] that $P_K(x; q)$ is the best approximation polynomial of function $|x - q|$ over $x \in [q - \sqrt{q\Delta}, q + \sqrt{q\Delta}]$. Denote the best approximation polynomial of $|y|$ on $[-1, 1]$ with order $K$ as

$$R_K(y) = \sum_{j=0}^{K} r_j y^j.$$ (293)

Using $x = q + y\sqrt{q\Delta}$, we have

$$P_K(x; q) = \sum_{j=0}^{K} r_j (\sqrt{q\Delta})^{-j+1} (x - q)^j.$$ (294)

It is well known that [26] Chap. 9, Thm. 3.3] there exists a universal constant $M_3$ such that

$$|R_K(y) - |y|| \leq \frac{M_3}{K}, \forall y \in [-1, 1].$$ (295)

Consequently, for $p \in [q - \sqrt{q\Delta}, q + \sqrt{q\Delta}]$,

$$|P_K(p; q) - |p - q|| \leq \frac{M_3 \sqrt{q\Delta}}{K} \lesssim \frac{1}{K} \sqrt{q \ln n \frac{n}{q}}.$$ (296)

It follows from Lemma [30] that $g_j,q(\hat{\rho}), n\hat{\rho} \sim \text{Poi}(np)$ defined as

$$g_j,q(\hat{\rho}) \triangleq \sum_{k=0}^{j} \binom{j}{k} (-q)^{j-k} \prod_{h=0}^{k-1} \left( \hat{\rho} - \frac{h}{n} \right)$$ (297)
is the unique unbiased estimator for \((p - q)^2, j \geq 0, j \in \mathbb{N}\). Hence,

\[
\hat{P}_K(\hat{p}; q) = \sum_{j=0}^{K} r_j(\sqrt{q\Delta})^{-j+1} g_{j,q}(\hat{p}).
\]  

(298)

It was shown in Cai and Low [12, Lemma 2] that \(|r_j| \leq 2^{4K}, 0 \leq j \leq K\). We study the variance properties of \(\hat{P}_K(\hat{p}; q)\) as follows.

Define \(M_4 \triangleq \max\{K, \frac{n(p-q)^2}{\Delta}\}\). Note that if \(p = 0\) the variance of this \(\hat{P}_K(\hat{p}; q)\) is zero. We now consider \(p \neq 0\).

Applying Lemma 30 and the fact that the standard deviation of a sum of random variables is upper bounded by the sum of standard deviations of corresponding random variables, we have

\[
\text{Var}(\hat{P}_K(\hat{p}; q)) = \text{Var}\left( \sum_{j=0}^{K} r_j(\sqrt{q\Delta})^{-j+1} g_{j,q}(\hat{p}) \right)
\]

\[
\leq \left( \sum_{j=0}^{K} |r_j|(\sqrt{q\Delta})^{-j+1}\text{Var}^{1/2}(g_{j,q}(\hat{p})) \right)^2
\]

\[
\leq 2^{6K} q\Delta \left( \sum_{j=0}^{K} \left( \frac{2M_4 p}{n} \right)^{j/2} \right)^2
\]

\[
\leq 2^{6K} q\Delta \left( \sum_{j=0}^{K} \left( \frac{2M_4}{nq\Delta} \right)^{j/2} \right)^2
\]

\[
\leq 2^{6K} q\Delta \left( \frac{c^{K+1} - 1}{c - 1} \right)^2
\]

\[
\leq \frac{c^2}{(c - 1)^2} (8e)^{2K} q\Delta
\]

\[
\leq B K \frac{\ln n}{n} q
\]

(305)

where \(c = \max\{\sqrt{7}, 2\sqrt{c_2/c_1}\}\). Recall that \(K = c_2 \ln n, \Delta = \frac{c_2 \ln n}{n}\). It suffices to show \(\frac{2M_4 p}{nq\Delta} \leq c\) to complete the proof. Indeed, we have

\[
\sqrt{\frac{2p}{nq\Delta}} \cdot K \leq \sqrt{\frac{2K(q + \sqrt{q\Delta})}{nq\Delta}} \leq \sqrt{\frac{2K \cdot 2q}{nq\Delta}} \leq \sqrt{\frac{4K}{n\Delta}} = \sqrt{\frac{4c_2}{c_1}}
\]

\[
\sqrt{\frac{2p}{nq\Delta}} \cdot \frac{n(p-q)^2}{p} = \sqrt{\frac{2(p-q)^2}{q\Delta}} \leq \sqrt{\frac{2q\Delta}{q\Delta}} = \sqrt{2}.
\]

(307)

G. Proof of Lemma

It is clear that \(\sup_{x \in [0,1]} |f(x; a)| \leq 1\). Introduce

\[
f_\eta(x; a) = \frac{|x + (1 - \eta) - a|}{\eta + (1 - \eta)}
\]

(308)

\[
f_\eta(x; a) = \frac{x - a}{\eta + (1 - \eta)} - \frac{a}{\eta + (1 - \eta)}
\]

(309)

where \(\eta = \frac{a}{D}, D > 1\). We have \(E_L[f(x; a); [\eta, 1]] = E_L[f_\eta(x; a); [0, 1]]\). Recall the second-order Ditzian–Totik modulus of smoothness given in (182)

\[
\omega^2_\varphi(f, t) = \sup_{0 < h \leq t} \sup_x |\Delta^2_{h\varphi, f}(x)|
\]

(310)

where \(\varphi = \sqrt{x(1 - x)}\), \(\Delta^2_{h\varphi, f}(x) = f(x + h\varphi) + f(x - h\varphi) - 2f(x)\).

We deal with the two cases separately.
\[ \frac{1}{a} \leq a \leq \frac{1}{2}: \text{Denote } \delta = \frac{2}{DL} \varphi \left( \frac{\eta}{1 - \eta} \right). \text{ It is easy to verify that if } \frac{1}{a} \leq a \leq \frac{1}{2}; D \geq 3, \text{ then } \frac{1}{1 + (DL)^2} \leq \frac{\eta}{1 - \eta} \leq \frac{(DL)^2}{1 + (DL)^2}, \text{ which ensures that } \frac{\eta}{1 - \eta} \pm \delta \in [0, 1]. \text{ We lower bound } \omega_{\varphi}^2(f, t) \text{ for } f_{\eta} \text{ as follows:} \]

\[ \omega_{\varphi}^2(f_{\eta}, (DL)^{-1}) \geq \left| \Delta_{\varphi/(DL)}^2 f_{\eta} \left( \frac{a - \eta}{1 - \eta} \right) \right|. \quad (311) \]

Since

\[ \Delta_{\varphi/(DL)}^2 f_{\eta} \left( \frac{a - \eta}{1 - \eta} \right) = \frac{2\delta}{\delta + \frac{a}{1 - \eta}}, \quad (312) \]

we have

\[ \omega_{\varphi}^2(f_{\eta}, (DL)^{-1}) \geq \frac{2\delta}{\delta + \frac{a}{1 - \eta}}. \quad (313) \]

The relationship between \( \omega_{\varphi}^2(f, \frac{1}{a}) \) and \( E_n[f; [0, 1]] \) was shown in [23] Thm. 7.2.4.] that there exists a universal positive constant \( M_2 \) such that

\[ \frac{1}{n^2} \sum_{k=0}^{n} (k + 1) E_k[f_{\eta}; [0, 1]] \geq M_2 \omega_{\varphi}^2(f_{\eta}, \frac{1}{n}). \quad (314) \]

Utilizing the non-increasing property of \( E_n[f_{\eta}; [0, 1]] \) with respect to \( n \) yields

\[ E_L[f_{\eta}; [0, 1]] \geq \frac{1}{DL - L} \sum_{k=L+1}^{DL} E_k[f_{\eta}; [0, 1]] \geq \frac{1}{(DL)^2} \sum_{k=L+1}^{DL} (k + 1) E_k[f_{\eta}; [0, 1]] \geq M_2 \omega_{\varphi}^2(f_{\eta}, \frac{1}{DL}) - \frac{1}{(DL)^2} \sum_{k=0}^{L} (k + 1) E_k[f_{\eta}; [0, 1]]. \quad (317) \]

Now we work out an upper bound on \( E_k[f_{\eta}; [0, 1]] \). It follows from Lemma [21] that there exists a universal constant \( M_1 \) such that

\[ E_k[f_{\eta}; [0, 1]] \leq M_1 \omega_{\varphi}^1(f_{\eta}, \frac{1}{k}), \quad (318) \]

where \( \omega_{\varphi}^1(f, t) = \sup_{0 < h < t} \sup_{x} |\Delta_{h\varphi}^1 f(x)| \), where \( \Delta_{h\varphi}^1 f(x) = f(x + h\varphi / 2) - f(x - h\varphi / 2) \). It follows from straightforward algebra that \( \omega_{\varphi}^1(f_{\eta}, \frac{1}{k}) \lesssim \frac{1}{k^2 a} \). Hence,

\[ \frac{1}{(DL)^2} \sum_{k=0}^{L} (k + 1) E_k[f_{\eta}; [0, 1]] \lesssim \frac{1}{D^2 L^2} \sum_{k=0}^{L} \frac{1}{\sqrt{k^2 a}} \lesssim \frac{1}{D^2 L \sqrt{a}}. \quad (319) \]

Since \( \delta = \frac{2}{DL} \varphi \left( \frac{\eta}{1 - \eta} \right), \eta = \frac{a}{DL}, \frac{1}{DL} \leq a \leq \frac{1}{2} \), for \( D \) large enough, we know \( \delta \gtrsim \frac{\sqrt{a}}{DL} \) and there exist two universal constants \( M_3 > 0, M_4 > 0 \) such that

\[ E_L[f_{\eta}; [0, 1]] \gtrsim M_3 \frac{\sqrt{a}}{DL} - a - M_4 \frac{1}{D^2 L \sqrt{a}} \]

\[ = \frac{1}{D} \left( M_3 \frac{\sqrt{a}}{L} \frac{1}{\sqrt{a}} - a - M_4 \frac{1}{D L \sqrt{a}} \right) \quad (322) \]

\[ \gtrsim \frac{1}{D} \left( M_3 \frac{\sqrt{a}}{L} - 2a - M_4 \frac{1}{D L \sqrt{a}} \right), \quad (323) \]

where we used the fact that \( \frac{\sqrt{a}}{DL} \leq a \) for \( a \geq \frac{1}{DL}, D \geq 1 \). Hence,

\[ L \sqrt{a} \cdot E_L[f_{\eta}; [0, 1]] \gtrsim \frac{1}{D} \left( \frac{M_3}{2} \frac{\sqrt{a}}{D} - M_4 \right) \]

\[ \geq d_1 > 0 \quad (324) \]

\[ \geq d_1 > 0 \quad (325) \]
when $D$ is large enough.

2) $0 < a < \frac{1}{L^2}$: for $D > 1$ we have

$$\omega^2_\varphi(f_\eta, (DL)^{-1}) \geq \left| \Delta^2_\varphi f_\eta \left( \frac{a-\eta}{1-\eta} \right) \right|, \quad (326)$$

where $\epsilon = \min \left\{ \frac{1}{DL} \varphi \left( \frac{a-\eta}{1-\eta} \right), \frac{a-\eta}{1-\eta} \right\}$.

Since

$$\Delta^2_\varphi f_\eta \left( \frac{a-\eta}{1-\eta} \right) = 1 + f_\eta \left( \frac{a-\eta}{1-\eta} + \epsilon \right) \geq 0, \quad (327)$$

it suffices to lower bound $f_\eta \left( \frac{a-\eta}{1-\eta} + \epsilon \right)$ to lower bound $\omega^2_\varphi(f_\eta, (DL)^{-1})$. Note that the function $f_\eta(\cdot)$ is a non-decreasing function.

We have $\frac{1}{DL} \varphi \left( \frac{a-\eta}{1-\eta} \right) \geq \frac{\sqrt{\eta}}{\sqrt{\eta + DLx}}$ for $D$ large enough, and

$$1 + f_\eta \left( \frac{a-\eta}{1-\eta} + \frac{a-\eta}{1-\eta} \right) = \frac{2D}{2D-1}$$

$$1 + f_\eta \left( \frac{a-\eta}{1-\eta} + \frac{1}{DL} \varphi \left( \frac{a-\eta}{1-\eta} \right) \right) \geq \frac{\sqrt{\eta}}{\sqrt{\eta + DLa}} \geq \frac{1}{1/L} \geq \frac{1}{1/L + D/L} \geq \frac{1}{1 + D}, \quad (332)$$

where we used the fact that the function $\frac{\sqrt{\eta}}{\sqrt{\eta + DLx}}$ is a non-increasing function for $x \geq 0$.

Hence, we have shown that

$$\omega^2_\varphi(f_\eta, (DL)^{-1}) \geq \min \left\{ \frac{2D - 2}{2D - 1}, \frac{1}{D + 1} \right\}. \quad (333)$$

Following (317), we have that

$$E_L[f_\eta; [0, 1]] \geq M_2 \omega^2_\varphi(f_\eta, \frac{1}{DL}) - \frac{1}{(DL)^2} \sum_{k=0}^{L} (k + 1) E_k[f_\eta; [0, 1]] \quad (334)$$

$$\geq \frac{1}{D} - \frac{1}{(DL)^2} \sum_{k=0}^{L} (k + 1) \quad (335)$$

$$\geq \frac{1}{D} - \frac{D}{D^2} \quad (336)$$

$$\geq d_1 > 0, \quad (337)$$

when $D$ is large enough. Here we used the fact that $E_k[f_\eta; [0, 1]] \leq 1$.

H. Proof of Lemma 11

We have

$$V(F_0, F_1) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j!} \left| \mathbb{E}[e^{-U_0 U_1^j} - e^{-U_1 U_1^j}] \right|. \quad (338)$$

For each $j \geq 0, j \in \mathbb{Z}$, we introduce function

$$f_j(x) = e^{-(a + Mx)} (a + Mx)^j, \quad (339)$$

where $x \in [-1, 1]$. We introduce $a + MX_i = U_i, i = 0, 1$. It follows from the assumptions that $\mathbb{E}[X_0^j] = \mathbb{E}[X_1^j], 0 \leq j \leq L$.

We write the series expansion of $f_j(x)$ as follows:

$$f_j(x) = f_j(0) + \sum_{k=1}^{\infty} \frac{f_j^{(k)}(0)}{k!} x^k. \quad (340)$$
Hence,
\[
V(F_0, F_1) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j!} \left| \sum_{k \geq L+1} \frac{f_j^{(k)}(0)}{k!} \mathbb{E}(X_k^j - X_1^j) \right|
\]  
(341)
\[
\leq \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k \geq L+1} \frac{2}{j!} \left| \frac{f_j^{(k)}(0)}{k!} \right|
\]  
(342)
\[
= \frac{1}{2} \sum_{k \geq L+1} \frac{2}{k!} \sum_{j=0}^{\infty} \left| \frac{f_j^{(k)}(0)}{j!} \right|
\]  
(343)
where we used the fact that \( X_j \in [-1, 1], i = 0, 1. \)

It follows from the Leibniz formula for derivatives of products of functions that
\[
f_j^{(k)}(x) = e^{-a} \sum_{m=0}^{k} \binom{k}{m} \left( e^{-M z} \right)^{(k-m)} ((a + M x)^{j-m})^{(m)}
\]  
(344)
\[
e^{-a} \sum_{m=0}^{k \wedge j} \binom{k}{m} (-M)^k \sum_{m=0}^{j!} \frac{m!}{(j-m)!} M^m (a + M x)^j - m
\]  
(345)
\[
e^{-a} \sum_{m=0}^{k \wedge j} \binom{k}{m} (-1)^{k-m} \sum_{m=0}^{j!} \frac{m!}{(j-m)!} (a + M x)^{j-m}
\]  
(346)
\[
e^{-a} \sum_{m=0}^{k \wedge j} \binom{k}{m} (-1)^{k-m} \sum_{m=0}^{j!} \frac{m!}{(j-m)!} (a + M x)^{j-m}.
\]  
(347)
Hence,
\[
\frac{f_j^{(k)}(0)}{j!} = \frac{e^{-a} a^j}{j!} \left( \frac{M}{a} \right)^k \sum_{m=0}^{k \wedge j} \binom{k}{m} (-1)^{k-m} \frac{m!}{(j-m)!} a^{k-m}.
\]  
(348)
Construct random variable \( Z \sim \text{Poi}(a) \). Then,
\[
\frac{f_j^{(k)}(0)}{j!} = \left( \frac{M}{a} \right)^k \mathbb{P}(Z = j) \sum_{m=0}^{k \wedge j} \binom{k}{m} (-a)^{k-m} (j)_m,
\]  
(349)
where \((j)_m = j(j-1) \cdots (j-m+1). \)

Consequently,
\[
\sum_{j=0}^{\infty} \left| \frac{f_j^{(k)}(0)}{j!} \right| \leq \left( \frac{M}{a} \right)^k \mathbb{E} \left| \sum_{m=0}^{k \wedge j} \binom{k}{m} (-a)^{k-m} (Z)_m \right|.
\]  
(350)
Introducing \( n \hat{p} = Z, p = \frac{a}{n} \), we have
\[
\sum_{j=0}^{\infty} \left| \frac{f_j^{(k)}(0)}{j!} \right| \leq \left( \frac{M n}{a} \right)^k \mathbb{E} \left| \sum_{m=0}^{k \wedge j} \binom{k}{m} \left( \frac{-a}{n} \right)^{k-m} \left( \frac{(Z)_m}{n^m} \right) \right|
\]  
(351)
\[
= \left( \frac{M n}{a} \right)^k \mathbb{E} \left| g_{k,n}(\hat{p}) \right|,
\]  
(352)
where \( g_{k,q}(\hat{p}) \) is the unique unbiased estimator for \((p - q)^k\) when \( n \hat{p} \sim \text{Poi}(np) \) introduced in Lemma 30.
It follows from Lemma 30 that
\[
\sum_{j=0}^{\infty} \left| \frac{j^{(k)}(0)}{j!} \right| \leq \left( \frac{Mn}{a} \right)^k \sqrt{\mathbb{E}(y_{k,a/n}(\hat{p}))^2} \leq \left( \frac{M}{p} \right)^k \frac{p^k k!}{n^k} \leq \left( \frac{M}{p} \frac{\sqrt{pk}}{\sqrt{n}} \right)^k \leq \left( M \sqrt{\frac{k}{a}} \right)^k.
\]

Hence, it follows from \( k! \geq \frac{k^k}{e^k} \) that
\[
V(F_0, F_1) \leq \sum_{k \geq L+1} \frac{1}{k!} \left( M \sqrt{\frac{k}{a}} \right)^k \leq \sum_{k \geq L+1} \frac{1}{k^k} \left( eM \sqrt{\frac{k}{a}} \right)^k \leq \sum_{k \geq L+1} \left( eM \sqrt{ka} \right)^k \leq \sum_{k \geq L+1} \left( eM \sqrt{a(L+1)} \right)^k.
\]

It follows from the assumptions that
\[
\frac{eM}{\sqrt{a(L+1)}} \leq \frac{1}{2}.
\]

Consequently,
\[
V(F_0, F_1) \leq \left( eM \sqrt{a(L+1)} \right)^{L+1} \left( 1 - \left( \frac{eM}{\sqrt{a(L+1)}} \right) \right)^{-1} \leq 2 \left( \frac{eM}{\sqrt{a(L+1)}} \right)^{L+1}.
\]

I. Proof of Lemma 72

We define the minimax risk under the multinomial sampling model for a fixed \( Q \) as
\[
R(S, n, Q) = \inf_{\hat{L}} \sup_{P \in \mathcal{M}_S} \mathbb{E}_P \left( \hat{L} - L_1(P, Q) \right)^2.
\]

Fix \( \delta > 0 \). Let \( \hat{L} = \hat{L}(X_1, X_2, \ldots, X_S) \) be a near-minimax estimator of \( L_1(P, Q) \) under the multinomial model for every sample size \( n \), which means that for every sample size \( n \),
\[
\sup_{P \in \mathcal{M}_S} \mathbb{E}_P \left( \hat{L} - L_1(P, Q) \right)^2 < R(S, n, Q) + \delta.
\]

Here the random vector \( (X_1, X_2, \ldots, X_S) \) follows multinomial distribution parametrized by \( n, P \), and the estimator \( \hat{L} \) obtains the number of samples \( n \) from this random vector.

Now we consider the Poisson sampling model, where \( X_i \)'s are mutually independent with marginal distributions \( X_i \sim \text{Poi}(np_i) \). Let \( n' = \sum_{i=1}^{S} X_i \sim \text{Poi} \left( n \sum_{i=1}^{S} p_i \right) \). We use the estimator \( \hat{L}(X_1, X_2, \ldots, X_S) \) to estimate \( L_1(P, Q) \) under the
Poisson sampling model. For any \( P \in M_S(\epsilon) \) under the Poisson sampling model, we have

\[
\mathbb{E}_P \left( \hat{L} - L_1(P, Q) \right)^2 \leq \mathbb{E}_P \left( \hat{L} - L_1 \left( \frac{P}{\sum_{i=1}^{S} p_i}, Q \right) \right)^2 + 2 \left( L_1 \left( \frac{P}{\sum_{i=1}^{S} p_i}, Q \right) - L_1(P, Q) \right)^2 \leq 2 \mathbb{E}_P \left( \hat{L} - L_1 \left( \frac{P}{\sum_{i=1}^{S} p_i}, Q \right) \right)^2 \leq 2 \mathbb{E}_P \left( \hat{L} - L_1 \left( \frac{P}{\sum_{i=1}^{S} p_i}, Q \right) \right)^2 + 2 \epsilon^2,
\]

(367)

where we used the fact that \((a + b)^2 \leq 2a^2 + 2b^2\) for any \(a, b \in \mathbb{R}\), and the fact that if \(\sum_{i=1}^{S} p_i = A\), then

\[
\left| L_1 \left( \frac{P}{\sum_{i=1}^{S} p_i}, Q \right) - L_1(P, Q) \right| \leq \sum_{i=1}^{S} |p_i/A - q_i| + |p_i - q_i|
\]

(370)

\[
\leq \sum_{i=1}^{S} |p_i/A - p_i| = \sum_{i=1}^{S} \frac{p_i}{A} |1 - A| = |A - 1| \leq \epsilon.
\]

(374)

Then,

\[
\mathbb{E}_P \left( \hat{L} - L_1(P, Q) \right)^2 \leq 2 \sum_{m=0}^{\infty} \mathbb{E}_P \left[ \left( \hat{L} - L_1 \left( \frac{P}{\sum_{i=1}^{S} p_i}, Q \right) \right)^2 \mid n' = m \right] \mathbb{P}(n' = m) + 2 \epsilon^2
\]

(375)

\[
\leq 2 \sum_{m=0}^{\infty} R(S, m, Q) \mathbb{P}(n' = m) + 2(\delta + \epsilon^2)
\]

(376)

\[
\leq 2(1 - \mathbb{P}(n' \leq n(1-\epsilon)/2) + R(S, n(1-\epsilon)/2, Q) \mathbb{P}(n' \geq n(1-\epsilon)/2)) + 2(\delta + \epsilon^2)
\]

(377)

\[
\leq 2R(S, n(1-\epsilon)/2, Q) + 2(\mathbb{P}(n' \leq n(1-\epsilon)/2) + 2(\delta + \epsilon^2))
\]

(378)

\[
\leq 2R(S, n(1-\epsilon)/2, Q) + 2\mathbb{P}(\text{Poi}(n(1-\epsilon))) \leq n(1-\epsilon)/2 + 2(\delta + \epsilon^2)
\]

(379)

\[
\leq 2R(S, n(1-\epsilon)/2, Q) + 2e^{-n(1-\epsilon)/8} + 2(\delta + \epsilon^2),
\]

(380)

where we used the fact that conditioned on \(n' = m\), \((X_1, X_2, \ldots, X_S)\) follows multinomial distribution parametrized by \(\left( m, \frac{P}{\sum_{i=1}^{S} p_i} \right)\), the monotonicity of \(R(S, m, Q)\) as a function of \(m\), \(R(S, m, Q) \leq 1\), and Lemma 34.

Taking supremum of \(\mathbb{E}_P \left( \hat{L} - L_1(P, Q) \right)^2\) over \(M_S(\epsilon)\) and using the arbitrariness of \(\delta\), we have

\[
R_P(S, n, Q, \epsilon) \leq 2R(S, n(1-\epsilon)/2, Q) + 2e^{-n(1-\epsilon)/8} + 2\epsilon^2,
\]

(381)

which is equivalent to

\[
R(S, n(1-\epsilon)/2, Q) \geq \frac{1}{2} R_P(S, n, Q, \epsilon) - e^{-n(1-\epsilon)/8} - \epsilon^2.
\]

(382)

It follows from [7, Lemma 16] that \(R(S, n, Q, \epsilon) \leq 2R_P(S, n/2, Q, 0)\). Hence,

\[
R_P(S, n(1-\epsilon)/4, Q, 0) \geq \frac{1}{4} R_P(S, n(1-\epsilon)/2, Q) \geq \frac{1}{4} R(S, n, Q, \epsilon) - \frac{1}{2} e^{-n(1-\epsilon)/8} - \frac{1}{2} \epsilon^2.
\]

(384)

\[ J. \text{ Proof of Lemma 13} \]

It is clear that the square \([0, \frac{2\epsilon \ln n}{n}]^2 \subset U\). To see how we obtained the whole expression of \(U\), for any \(x > \frac{c_1 \ln n}{n}\), we study the envelope of the parametrized extremal points \(\left( x - \sqrt{c_1 \ln n}, x + \sqrt{c_1 \ln n} \right)\), where the other curve \(\left( x + \sqrt{c_2 \ln n}, x - \sqrt{c_2 \ln n} \right)\) can be dealt with analogously.
For \( p = x - \sqrt{\frac{c_1 x \ln n}{n}} \), \( q = x + \sqrt{\frac{c_1 x \ln n}{n}} \), we have

\[
p - q = -2\sqrt{\frac{c_1 x \ln n}{n}} \\
p + q = 2x.
\] (385) (386)

Hence,

\[
(p - q)^2 = \frac{2c_1 \ln n}{n}(p + q).
\] (387)

We have that for all points \((p, q) \in \bigcup_{x \in [0,1]} U(x; c_1) \times U(x; c_1)\),

\[
|p - q| \leq \sqrt{\frac{2c_1 \ln n}{n}(p + q)} \\
\leq \sqrt{\frac{2c_1 \ln n}{n}(\sqrt{p} + \sqrt{q})}.
\] (388) (389)

where we used the inequality \(\sqrt{p} + \sqrt{q} \leq \sqrt{p + q}\) in the last step.

**K. Proof of Lemma 14**

It follows from Lemma 4 that

\[
\mathbb{P}(E^c) = \mathbb{P}(E^c_1 \cup E^c_2 \cup E^c_3 \cup E^c_4) \\
\leq \mathbb{P}(E^c_1) + \mathbb{P}(E^c_2) + \mathbb{P}(E^c_3) + \mathbb{P}(E^c_4).
\] (390) (391)

Hence, it suffices to analyze each \(\mathbb{P}(E^c_i), i = 1, 2, 3, 4\).

1) Analysis of \(\mathbb{P}(E^c_1)\):

\[
\mathbb{P}(E^c_1) = \mathbb{P}\left( \bigcup_{i=1}^S \left\{ p_i < q_i, \hat{p}_{i,1} - \hat{q}_{i,1} > \sqrt{\frac{(c_1 + c_3) \ln n}{n}}(\sqrt{p_{i,1}} + \sqrt{q_{i,1}}) \right\} \right)
\]

\[
\leq \sum_{i=1}^S \mathbb{P}\left( p_i < q_i, \hat{p}_{i,1} - \hat{q}_{i,1} > \sqrt{\frac{(c_1 + c_3) \ln n}{n}}(\sqrt{p_{i,1}} + \sqrt{q_{i,1}}) \right)
\]

\[
= \sum_{i=1}^S \mathbb{P}\left( p_i < q_i, \sqrt{\hat{p}_{i,1}} - \sqrt{\hat{q}_{i,1}} > \sqrt{\frac{(c_1 + c_3) \ln n}{n}} \right)
\]

\[
\leq \sum_{i=1}^S \mathbb{P}\left( p_i = q_i, \sqrt{\hat{p}_{i,1}} - \sqrt{\hat{q}_{i,1}} > \sqrt{\frac{(c_1 + c_3) \ln n}{n}} \right).
\] (392) (393) (394) (395)

It follows from Lemma 13 that the set \(U(p_i; (c_1 + c_3)/2) \times U(p_i; (c_1 + c_3)/2) \subseteq U_1\). Hence,

\[
\mathbb{P}(E^c_1) \leq \sum_{i=1}^S \mathbb{P}(p_i = q_i, (\hat{p}_{i,1}, \hat{q}_{i,1}) \notin U(p_i; (c_1 + c_3)/2) \times U(p_i; (c_1 + c_3)/2))
\]

\[
\leq \sum_{i=1}^S (1 - \mathbb{P}(p_i = q_i, (\hat{p}_{i,1}, \hat{q}_{i,1}) \in U(p_i; (c_1 + c_3)/2) \times U(p_i; (c_1 + c_3)/2)))
\]

\[
= \sum_{i=1}^S (1 - \mathbb{P}(p_i = q_i, \hat{p}_{i,1} \in U(p_i; (c_1 + c_3)/2) \mathbb{P}(q_i = p_i, \hat{q}_{i,1} \in U(p_i; (c_1 + c_3)/2))).
\] (396) (397) (398)

It follows from Lemma 4 that

\[
\mathbb{P}(E^c_1) \leq \sum_{i=1}^S \left( 1 - \left( 1 - \frac{2}{n^{c_1 + c_3}} \right)^2 \right)
\]

\[
\leq \frac{4S}{n^{c_1 + c_3}}.
\] (399) (400)

2) Analysis of \(\mathbb{P}(E^c_2)\): following similar steps as in the analysis of \(\mathbb{P}(E^c_1)\), we have \(\mathbb{P}(E^c_2) \leq \frac{4S}{n^{c_1 + c_3}}\).
3) Analysis of $\mathbb{P}(E_3^e)$:

$$
\mathbb{P}(E_3^e) = \mathbb{P}\left( \bigcup_{i=1}^{S} \left\{ (p_i, q_i) \notin U, (\hat{p}_{i,1}, \hat{q}_{i,1}) \in U_1 \right\} \right)
$$

$$
\leq \mathbb{P}\left( \bigcup_{i=1}^{S} \left\{ p_i + q_i > \frac{2c_1 \ln n}{n}, \hat{p}_{i,1} + \hat{q}_{i,1} < \frac{c_1 \ln n}{n} \right\} \right)
$$

$$
\leq \sum_{i=1}^{S} \mathbb{P}\left( p_i + q_i > \frac{2c_1 \ln n}{n}, \hat{p}_{i,1} + \hat{q}_{i,1} < \frac{c_1 \ln n}{n} \right)
$$

$$
\leq S \mathbb{P}(\text{Poi}(2c_1 \ln n) < c_1 \ln n)
$$

$$
\leq S e^{-\frac{3}{2} \frac{2c_1 \ln n}{n}}
$$

$$
= \frac{S}{n^{c_1/4}}
$$

where we have used the fact that $n\hat{p}_{i,1} + n\hat{q}_{i,1} \sim \text{Poi}(np + nq)$ and Lemma 34.

4) Analysis of $\mathbb{P}(E_4^e)$:

$$
\mathbb{P}(E_4^e) \leq \sum_{i=1}^{S} \mathbb{P}( (p_i, q_i) \notin U, (\hat{p}_{i,1}, \hat{q}_{i,1}) \in U_1 ) + \sum_{i=1}^{S} \mathbb{P}( \hat{p}_{i,1} + \hat{q}_{i,1} > \frac{c_1 \ln n}{n}, p_i + q_i < \frac{c_1 \ln n}{2n} )
$$

$$
+ \sum_{i=1}^{S} \mathbb{P}( \hat{p}_{i,1} + \hat{q}_{i,1} \geq \frac{c_1 \ln n}{n}, \hat{p}_{i,1} + \hat{q}_{i,1} \leq \frac{p_i + q_i}{2} )
$$

We have

$$
\sum_{i=1}^{S} \mathbb{P}( \hat{p}_{i,1} + \hat{q}_{i,1} > \frac{c_1 \ln n}{n}, p_i + q_i < \frac{c_1 \ln n}{2n} )
$$

$$
\leq S \mathbb{P}(\text{Poi}\left(\frac{c_1 \ln n}{2}\right) > c_1 \ln n)
$$

$$
\leq \frac{S}{n^{c_1/6}}
$$

and

$$
\sum_{i=1}^{S} \mathbb{P}( \hat{p}_{i,1} + \hat{q}_{i,1} \geq \frac{c_1 \ln n}{n}, \hat{p}_{i,1} + \hat{q}_{i,1} \leq \frac{p_i + q_i}{2} )
$$

$$
\leq \sum_{i=1}^{S} \mathbb{P}( p_i + q_i \geq \frac{2c_1 \ln n}{n}, \hat{p}_{i,1} + \hat{q}_{i,1} \leq \frac{p_i + q_i}{2} )
$$

$$
\leq \sum_{i=1}^{S} \mathbb{P}( \text{Poi}(np_i + nq_i) \leq \frac{n(p_i + q_i)}{2}, n(p_i + q_i) \geq 2c_1 \ln n )
$$

$$
\leq S e^{-\frac{3}{2} \frac{2c_1 \ln n}{n}}
$$

$$
\leq \frac{S}{n^{c_1/4}}
$$

It suffices to show that there exists some constant $c > 0$ such that

$$
\left( \bigcup_{(p,q) \notin U} U(p; c) \times U(q; c) \right) \cap U_1 = \emptyset,
$$

$$
(414)$$
where $U(\cdot; c)$ is defined in \[23\]. Indeed, in this case it follows from Lemma \[4\] that
\[
\sum_{i=1}^{S} \mathbb{P}((p_i, q_i) \notin U, (\hat{p}_{i,1}, \hat{q}_{i,1}) \in U_1) \leq \sum_{i=1}^{S} \mathbb{P}((\hat{p}_{i,1}, \hat{q}_{i,1}) \notin U(p_i; c) \times U(q_i; c)) \tag{415}
\]
\[
\leq \sum_{i=1}^{S} (1 - \mathbb{P}((\hat{p}_{i,1}, \hat{q}_{i,1}) \in U(p_i; c) \times U(q_i; c))) \tag{416}
\]
\[
= \sum_{i=1}^{S} (1 - \mathbb{P}(\hat{p}_{i,1} \in U(p_i; c)) \mathbb{P}(\hat{q}_{i,1} \in U(q_i; c))) \tag{417}
\]
\[
\leq \sum_{i=1}^{S} \left(1 - \left(1 - \frac{2}{n^{c/3}}\right)^2\right) \tag{418}
\]
\[
\leq \frac{4S}{n^{c/3}}. \tag{419}
\]

Now we work to prove (414). Without loss of generality we assume $(p, q)$ satisfies $\sqrt{q} - \sqrt{p} \geq \sqrt{\frac{2c_1 \ln n}{n}}$ and the constant $c < c_1$. Under this assumption we have $q \geq \frac{2c_1 \ln n}{n}$. We will show that for any point $(x, y) \in U(p; c) \times U(q; c)$, we have $\sqrt{q} - \sqrt{p} \geq \sqrt{\frac{(c_1 + c_3) \ln n}{n}}$, thereby proving (414).

If $p \leq \frac{c \ln n}{n}$, we have for any $(x, y) \in U(p; c) \times U(q; c)$,
\[
\sqrt{q} - \sqrt{x} \geq \sqrt{q - \frac{c q \ln n}{n}} - \sqrt{\frac{2c \ln n}{n}} \tag{420}
\]
\[
\geq \sqrt{\frac{2c_1 \ln n}{n} - \sqrt{2c_1} \frac{c \ln n}{n}} - \sqrt{\frac{2c \ln n}{n}}, \tag{421}
\]
where in the second step we used the fact that the function $x - \sqrt{ax}, a > 0$ is monotonically increasing when $x \geq a/4$. Hence, we need to guarantee that
\[
\sqrt{q} - \sqrt{x} \geq \sqrt{\frac{\ln n}{n}} \left(\sqrt{2c_1} - \sqrt{2c_1 c_3} - \sqrt{2c}\right) \tag{422}
\]
\[
\geq \sqrt{\frac{\ln n}{n}} \sqrt{c_1 + c_3}, \tag{423}
\]
which can be reduced to the quadratic inequality:
\[
\left(\frac{2c}{c_1}\right)^2 + \left(1 + 2 + \frac{c_3}{c_1}\right) \sqrt{\frac{2c}{c_1}} + \frac{c_3}{c_1} - 1 \leq 0. \tag{424}
\]
One can easily verify that $c = \frac{(c_1 - c_3)^2}{32c_1^2}$ satisfies this inequality since $0 < c_3 < c_1$.

Now we consider the case of $p > \frac{c \ln n}{n}$. Then, for any $(x, y) \in U(p; c) \times U(q; c)$,
\[
\sqrt{y} - \sqrt{x} \geq \sqrt{q - \frac{c q \ln n}{n}} - p - \sqrt{\frac{c p \ln n}{n}} \tag{425}
\]
\[
= \frac{q - \sqrt{c q \ln n}}{p - \sqrt{c p \ln n}} + \sqrt{p + \sqrt{c p \ln n}} \tag{426}
\]
\[
= \frac{(\sqrt{q} - \sqrt{p})(\sqrt{q} + \sqrt{p}) - \sqrt{c q \ln n}}{p + \frac{\sqrt{c p \ln n}}{n}} \tag{427}
\]
\[
\geq (\sqrt{2c_1} - \sqrt{c}) \sqrt{\frac{\ln n}{n}} \frac{\sqrt{q} + \sqrt{p}}{p + \frac{\sqrt{c p \ln n}}{n}}. \tag{428}
\]
Further, since \( p > \frac{2 \ln n}{n} \),

\[
\sqrt{q} + \sqrt{p} \geq \sqrt{q} + \sqrt{2p} \\
\sqrt{q - \frac{c_1 \ln n}{n}} + \sqrt{p + \frac{c_2 \ln n}{n}} \geq \sqrt{q} + \sqrt{2p} \\
\geq \frac{\sqrt{p} + \sqrt{2c_1 \ln n} + \sqrt{p}}{\sqrt{2p} + \sqrt{p} + \sqrt{2c_1 \ln n}} \\
\geq \frac{2}{\sqrt{2} + 1}
\]

(429)

(430)

(431)

where we used the fact that \( \frac{x + \sqrt{p}}{x + \sqrt{2p}} \) is a monotonically increasing function of \( x \) when \( x \geq 0 \), and the function \( \frac{2x + a}{(\sqrt{2} + 1)x + a} \) is a monotonically decreasing function of \( x \) when \( a > 0, x > 0 \). To guarantee that \( \sqrt{y} - \sqrt{\overline{y}} \geq \sqrt{\frac{(c_1 + c_2) \ln n}{n}} \), we need

\[
\frac{2}{\sqrt{2} + 1} (\sqrt{2c_1 - \sqrt{c}}) \geq \sqrt{c_1 + c_3},
\]

(432)

which is equivalent to

\[
c \leq \left( \sqrt{2c_1 - \sqrt{\frac{2}{\sqrt{2} + 1}} \sqrt{c_1 + c_3}} \right)^2,
\]

(433)

with the constraint that \( \frac{c_1}{\sqrt{c}} < \frac{8}{(\sqrt{2} + 1)^2} - 1 \approx 0.373 \).

\[\text{L. Proof of Lemma 15}\]

We first analyze the bias. To simplify the notation we denote \( \Delta = \frac{c_1 \ln n}{n} \). It follows from the definition of \( \bar{F}_K^{(1)} \) that for \((p, q) \in [0, 2\Delta]^2\),

\[
E \bar{F}_K^{(1)}(\hat{p}, \hat{q}) - |p - q| = 2\Delta h_{2K} \left( \frac{p}{2\Delta}, \frac{q}{2\Delta} \right) - |p - q|,
\]

(434)

where \( h_{2K}(x, y) = u_K(x, y)v_K(x, y) - u_K(0, 0)v_K(0, 0), \) and \( u_K(x, y) \) and \( v_K(x, y) \) satisfy (128).

We first argue that there exists a universal constant \( M > 0 \) such that \( \sup_{(x, y) \in [0, 1]^2} |u_K(x, y)v_K(x, y) - |x - y|| \leq M \left( \frac{\sqrt{x} + \sqrt{y}}{K} + \frac{1}{K^2} \right) \). Indeed,

\[
|u_K(x, y)v_K(x, y) - |x - y|| = |u_K(x, y)v_K(x, y) - u_K(x, y)|\sqrt{x} - \sqrt{y}| + u_K(x, y)|\sqrt{x} - \sqrt{y}| - (\sqrt{x} + \sqrt{y})|\sqrt{x} - \sqrt{y}|
\]

(435)

\[
\leq |u_K(x, y)||v_K(x, y)| - |\sqrt{x} - \sqrt{y}| + |\sqrt{x} - \sqrt{y}||u_K(x, y) - \sqrt{x} - \sqrt{y}|
\]

(436)

\[
\leq |u_K(x, y)||v_K(x, y)| - (\sqrt{x} + \sqrt{y})|v_K(x, y) - |\sqrt{x} - \sqrt{y}| + |\sqrt{x} + \sqrt{y}||v_K(x, y) - \sqrt{x} - \sqrt{y}|
\]

(437)

It follows from Lemma 21 and Lemma 23 that the best polynomial approximation error of \( \sqrt{x} + \sqrt{y} \) and \( |\sqrt{x} - \sqrt{y}| \) over the unit square are both of order \( \frac{1}{K^2} \). Hence,

\[
|u_K(x, y)v_K(x, y) - |x - y|| \leq M \left( \frac{\sqrt{x} + \sqrt{y}}{K} + \frac{1}{K^2} \right),
\]

(438)

which implies that there exists another constant \( M > 0 \) such that

\[
|u_K(x, y)v_K(x, y) - u_K(0, 0)v_K(0, 0) - |x - y|| \leq M \left( \frac{\sqrt{x} + \sqrt{y}}{K} + \frac{1}{K^2} \right),
\]

(439)
Denote $x = \frac{p}{2\Delta}$, $y = \frac{q}{2\Delta}$, we have
\begin{align*}
\left| \mathbb{E}\hat{P}_K^{(1)}(\tilde{\rho}, \tilde{q}) - |p - q| \right| &= \left| 2\Delta h_{2K}\left(\frac{p}{2\Delta}, \frac{q}{2\Delta}\right) - |p - q| \right| \tag{440} \\
&= 2\Delta \left| h_{2K}\left(\frac{p}{2\Delta}, \frac{q}{2\Delta}\right) - \frac{p}{2\Delta} - \frac{q}{2\Delta} \right| \tag{441} \\
&= 2\Delta |h_{2K}(x, y) - |x - y|| \tag{442} \\
&\leq 2\Delta M \left( \sqrt{\frac{p}{2\Delta}} + \sqrt{\frac{q}{2\Delta}} + \frac{1}{K^2} \right) \tag{443} \\
&= 2\Delta M \frac{1}{K} \left( \sqrt{\frac{p}{2\Delta}} + \sqrt{\frac{q}{2\Delta}} + \frac{1}{K} \right) \tag{444} \\
&\lesssim \frac{1}{K} \sqrt{\frac{n \ln n}{n} \left( \sqrt{\frac{p}{2\Delta}} + \sqrt{\frac{q}{2\Delta}} + \frac{1}{K^2} \ln n \right)}. \tag{445}
\end{align*}

We now analyze the variance. Express the polynomial $h_{2K}(x, y) \in \text{poly}_{2K}^2$ explicitly as
\begin{equation}
\left( \sum_{0 \leq i \leq 2K, 0 \leq j \leq 2K, i + j \geq 1} h_{ij} x^i y^j \right) \tag{446}
\end{equation}
\begin{equation}
= \sum_{0 \leq i \leq 2K, 0 \leq j \leq 2K, i + j \geq 1} h_{ij} x^i y^j \tag{447}
\end{equation}
For any fixed value of $y$, $h_{2K}(x^2, y^2)$ is a polynomial of $x$ with degree no more than $4K$ that is uniformly bounded by a universal constant on $[-1, 1]$. It follows from Lemma 29 that for any fixed $y \in [-1, 1]$,
\begin{equation}
\left| \sum_{0 \leq j \leq 2K} h_{ij} y^{2j} \right| \leq M(\sqrt{2} + 1)^{4K}, \tag{448}
\end{equation}
which, together with Lemma 29 implies that
\begin{equation}
|h_{ij}| \leq M(\sqrt{2} + 1)^{8K}. \tag{449}
\end{equation}
Since $\hat{P}_K^{(1)}$ is the unbiased estimator of $2\Delta h_{2K}\left(\frac{p}{2\Delta}, \frac{q}{2\Delta}\right)$, we know
\begin{equation}
\hat{P}_K^{(1)}(\tilde{\rho}, \tilde{q}) = \sum_{0 \leq i, j \leq 2K, i + j \geq 1} h_{ij}(2\Delta)^{1-i-j} g_{i,0}(\tilde{\rho}) g_{j,0}(\tilde{q}), \tag{450}
\end{equation}
where $g_{j,q}(\tilde{\rho})$ is the unbiased estimator for $(p - q)^j$ introduced in Lemma 30.

Denote $\|X\|_2 = \sqrt{\mathbb{E}(X - \mathbb{E}X)^2}$ and $M_1 = 2K \sqrt{2n\Delta}$. Using the triangle inequality of the norm $\| \cdot \|_2$ and Lemma 30 we know
\begin{align*}
\|\hat{P}_K^{(1)}(\tilde{\rho}, \tilde{q})\|_2 &\leq \sum_{0 \leq i, j \leq 2K, i + j \geq 1} |h_{ij}|(2\Delta)^{1-i-j} \|g_{i,0}(\tilde{\rho})\|_2 \|g_{j,0}(\tilde{q})\|_2 \tag{451} \\
&\leq \sum_{0 \leq i, j \leq 2K, i + j \geq 1} M(\sqrt{2} + 1)^{8K}(2\Delta)^i \left( \frac{1}{2\Delta} \sqrt{\frac{2M_1 p}{n}} \right)^j \tag{452} \\
&\lesssim (\sqrt{2} + 1)^{8K} \frac{n \ln n}{n} \sum_{0 \leq i, j \leq 2K, i + j \geq 1} \left( \sqrt{\frac{p}{2\Delta}} \right)^i \left( \sqrt{\frac{q}{2\Delta}} \right)^j \tag{453}
\end{align*}
Since for any $x \in [0, 1], y \in [0, 1]$,
\begin{align*}
\sum_{0 \leq i, j \leq 2K, i + j \geq 1} x^i y^j &\leq \sum_{j=1}^{2K} y^j + \sum_{i=1}^{2K} x^i + xy \sum_{0 \leq i, j \leq 2K-1} x^i y^j \tag{454} \\
&\leq y(2K) + x(2K) + xy(2K)^2 \tag{455} \\
&\leq 2(2K)^2(x + y) \tag{456}
\end{align*}
we know
\[ \| \hat{P}_K^{(1)}(\hat{p}, \hat{q}) \|_2 \lesssim (\sqrt{2} + 1)^{8K} \left( \frac{\ln n}{n} \right)^3 \left( \frac{p}{2\Delta} + \frac{q}{2\Delta} \right) \]  
(457)

\[ \lesssim \sqrt{B K \frac{1}{n}(p + q)}. \]  
(458)

for some constant \( B > 0 \). Hence,
\[ \text{Var}(\hat{P}_K^{(1)}(\hat{p}, \hat{q})) \lesssim B K \frac{p + q}{n}. \]  
(459)

**M. Proof of Lemma 16**

We first analyze the bias. It follows from the definition of \( \hat{P}_K^{(1)} \) that
\[ \mathbb{E} \left[ \hat{P}_K^{(1)}(\hat{p}, \hat{q}; x, y) \right] = \sum_{j=0}^{K} r_j W^{-j+1}(p - q)^j, \]  
(460)

where \( W = \sqrt{\frac{8c_1 \ln n}{n}} \sqrt{(x + y)} \lor \frac{1}{n} \).

Since \((p + q) \in U\), we know
\[ |p - q| \leq \sqrt{\frac{2c_1 \ln n}{n}} (\sqrt{p} + \sqrt{q}) \]  
(461)

\[ \leq \sqrt{\frac{2c_1 \ln n}{n}} \sqrt{2}(p + q) \]  
(462)

\[ \leq \sqrt{\frac{2c_1 \ln n}{n}} \sqrt{2} \sqrt{2(x + y)} \]  
(463)

\[ \leq W, \]  
(464)

where we have used the fact that \( \sqrt{p} + \sqrt{q} \leq \sqrt{2(p + q)} \) and the assumption that \( p + q \leq 2(x + y) \).

Hence, it follows from the property that the best order-\( K \) polynomial approximation error of \(|t|\) over \([-1, 1]\) is \( \Theta(\frac{1}{K}) \) [26 Chap. 9, Thm. 3.3] that
\[ \left| \sum_{j=0}^{K} r_j W^{-j+1}(p - q)^j - |p - q| \right| \lesssim \frac{W}{K}, \]  
(465)

\[ \lesssim \frac{1}{K} \sqrt{\frac{\ln n}{n}} \sqrt{x + y}. \]  
(466)

Then we analyze the variance. It was shown in Cai and Low [12 Lemma 2] that \( |r_j| \leq 2^{3K}, 0 \leq j \leq K \). Denote the unbiased estimator of \((p - q)^j\) by \( \hat{A}_j(\hat{p}, \hat{q}) \) and introduce the norm \( \| X \|_2 = \sqrt{\mathbb{E}(X^2 - \mathbb{E}X)^2} \). It follows from the triangle inequality of the norm \( \| X \|_2 \) and the fact that constants have zero variance that
\[ \| \hat{P}_K^{(2)} \|_2 \leq \sum_{j=1}^{K} |r_j| W^{-j+1} \| \hat{A}_j \|_2. \]  
(467)

It follows from Lemma 31 that
\[ \mathbb{E} \hat{A}_j^2 \leq \left( 2(p - q)^2 \sqrt{\frac{8j(p \lor q)}{n}} \right)^j. \]  
(468)

Hence,
\[ \| \hat{P}_K^{(2)} \|_2 \leq 2^{3K} W \sum_{j=1}^{K} \left( \sqrt{\frac{2(p - q)}{W}} \lor \sqrt{\frac{8j(p \lor q)}{nW}} \right)^j \]  
(469)

\[ = 2^{3K} W \sum_{j=1}^{K} C^j, \]  
(470)
where
\[ C = \frac{\sqrt{2}|p - q|}{W} \sqrt{nW} \]
\[ \leq \frac{\sqrt{2}}{\sqrt{n}} \frac{2c_1 \ln n}{n} (\sqrt{p} + \sqrt{q}) \sqrt{8K(p + q)} \]
\[ \leq \frac{\sqrt{2}}{\sqrt{n}} \frac{c_1 \sqrt{p + q}}{c_1} \sqrt{x + y} \]
\[ \leq \sqrt{2} \sqrt{\frac{c_2}{c_1}} \]
\[ \leq \sqrt{2}. \]
Consequently,
\[ ||\hat{P}_K^{(2)}||_2 \leq 2^{3K} \frac{8c_1 \ln n}{n} \sqrt{x + y} K (\sqrt{2})^K \]
\[ \leq \sqrt{B K^2 + y}, \] where $B$ is some universal constant.

\section*{N. Proof of Lemma 17}
We consider two different parameter settings.

1) $S \ll n \ll S \ln S$: In this case, we construct the distribution $P$ as follows:
\[ P = \left( \frac{c \ln n}{n}, \frac{c \ln n}{n}, \ldots, \frac{c \ln n}{n}, 0, \ldots, 0 \right), \]
where $c > 2c_1$ is a constant that will be chosen later, and $Q = P$. Without loss of generality we assume $\frac{n}{c \ln n}$ is an integer. We now argue that for each index $1 \leq i \leq \frac{n}{c \ln n},$
\[ ||Eg(\hat{q}_i, \hat{q}_i) - |p_i - q_i|| \geq \frac{\sqrt{\ln n}}{n}. \]
It follows from Lemma 14 that $P(\hat{q}_i \leq \frac{2c_1 \ln n}{n}) \leq e^{-\frac{4}{3}(1-2c_1/c)^2c \ln n} = n^{-\beta}$, where $\beta = \frac{4}{3}(1 - \frac{2c_1}{c})^2$. Note that $\beta$ can be made arbitrarily large by taking the constant $c$ large. Define $E = \{ \hat{q}_i \geq \frac{2c_1 \ln n}{n}, \hat{q}_i \geq \frac{2c_1 \ln n}{n} \}$. We have
\[ \mathbb{E}g(\hat{p}_i, \hat{q}_i) = \mathbb{E}(g(1(E) - g(1(E'))) \]
\[ = \mathbb{E}(|\hat{p}_i - \hat{q}_i| 1(E)) + \mathbb{E}(g 1(E')) \]
\[ = \mathbb{E}(|\hat{p}_i - \hat{q}_i| - \mathbb{E}(|\hat{p}_i - \hat{q}_i| 1(E'))) + \mathbb{E}(g 1(E')) \]
\[ = \mathbb{E}(|\hat{p}_i - \hat{q}_i| - g - |\hat{p}_i - \hat{q}_i| 1(E')). \]
Since $|g| \leq B, we have
\[ \mathbb{E}(|g - |\hat{p}_i - \hat{q}_i| 1(E'))| \leq (B + 1) \frac{2}{n^\beta}. \]
It follows from the triangle inequality that
\[ ||Eg(\hat{p}_i, \hat{q}_i)||_{\leq} \mathbb{E}|\hat{p}_i - \hat{q}_i| - \frac{2(B + 1)}{n^\beta} \]
It follows from the conditional version of Jensen’s inequality that $\mathbb{E}|\hat{p}_i - \hat{q}_i| \geq \mathbb{E}|\hat{p}_i - p_i|$, and by Lemma 7 we have
\[ \mathbb{E}|\hat{p}_i - \hat{q}_i| \geq \sqrt{\frac{p_i}{2n}} \]
\[ = \sqrt{\frac{c \ln n}{2n^2}} \]
\[ \geq \sqrt{\frac{\ln n}{n}}. \]

\footnote{Technically, the distribution $P$ has support no more than $S$. However, a standard continuity argument implies that the same conclusion holds.}
Since $\frac{\sqrt{n}}{n} \gg \frac{2(\beta+1)}{n^\beta}$ for $\beta > 1$, we conclude that (479) is true. Hence, the total bias of $\hat{L}$ is at least $\left( \frac{n}{\sqrt{\ln n}} \right)^2 = \frac{1}{\ln n} \gg \frac{S}{n \ln n}$ since $S \ll n$.

2) $n \gg S \ln S$: In this case, we construct $P, Q$ to be uniform distributions with support size $S$. Since $\frac{1}{S^2} \gg \frac{\ln n}{n}$, it follows from arguments analogous to those above that the squared bias of $\hat{L}$ is at least the order $\left( S \sqrt{\frac{1}{25n}} \right)^2 = \frac{S}{2n} \gg \frac{S}{n \ln n}$.

O. Proof of Lemma 18

Since $||a| - |b|| \leq |a - b|$, it suffices to show that there exists a universal constant $C > 0$ such that

$$|Q_K(t)| \leq CKt^2$$

(489)

for $|t| \leq 1$. Define $\sqrt{x} = |t|$. Since $Q_K(t)$ is even, it follows that $Q_K(t) = R(t^2)$, where $R \in \text{poly}_K$ is a polynomial. The polynomial $R$ satisfies the following:

$$R(0) = 0$$

(490)

$$\max_{x \in [0,1]} |R(x) - \sqrt{x}| \lesssim \frac{1}{K}$$

(491)

It suffices to show that $|R(x)| \leq CKx$. Let $T(x) \in \text{poly}_K$ denote the best approximation polynomial of the function $\sqrt{x}$ on $[0,1]$ with order no more than $K$. It follows from Lemma 21 and Lemma 22 that $\sup_{x \in [0,1]} |T(x) - \sqrt{x}| \lesssim \frac{1}{K}$. It follows from the triangle inequality that

$$\sup_{x \in [0,1]} |R(x) - T(x)| \leq \sup_{x \in [0,1]} \left( |R(x) - \sqrt{x}| + |T(x) - \sqrt{x}| \right)$$

(492)

$$\lesssim \frac{1}{K}.$$  

(493)

It follows from the Markov inequality (Lemma 23) that $\sup_{x \in [0,1]} |R'(x) - T'(x)| \lesssim K$. Since for any $0 \leq x \leq 1$,

$$|R(x)| = \left| \int_0^x R'(u)du \right|$$

(494)

$$= \left| \int_0^x (R'(u) - T'(u))du + \int_0^x T'(u)du \right|$$

(495)

$$\leq x \sup_{x \in [0,1]} |R'(x) - T'(x)| + \left| \int_0^x T'(u)du \right|$$

(496)

$$\lesssim Kx + \left| \int_0^x T'(u)du \right|,$$

(497)

it suffices to show $\left| \int_0^x T'(u)du \right| \lesssim Kx$.

It follows from Lemma 26 and Lemma 22 that

$$\sup_{x \in [0,1]} \left| \sqrt{x(1-x)}T'(x) \right| \lesssim 1.$$  

(498)

Hence, it follows from Lemma 27 that

$$\sup_{x \in [0,1]} |T'(x)| \lesssim \sup_{x \in [1/K^2,1-1/K^2]} |T'(x)|$$

(499)

$$\lesssim \sup_{x \in [1/K^2,1-1/K^2]} \frac{1}{\sqrt{x(1-x)}}$$

(500)

$$\lesssim K.$$  

(501)

Hence,

$$\left| \int_0^x T'(u)du \right| \lesssim Kx.$$  

(502)

The proof is complete.
P. Proof of Lemma 19

We prove the lemma by contradiction. Assuming the contrary, then there exist universal constants \( c, C > 0 \) and polynomial \( P(x, y) \in \text{poly}_K \) of degree \( K = c \ln n \) such that

\[
\sup_{(x, y) \in U'} \frac{|P(x, y) - |x - y||}{\sqrt{x + y}} \leq \frac{C}{\sqrt{n \ln n}}
\]

where \( U' = \bigcup_{x \in [\frac{\ln n}{n}, t_n]} U(x; c_1) \cup U(x; c_1) \). Now for any \( t \in [\frac{\ln n}{n}, t_n] \), we have \( t - \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}, t + \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}} \) \( \in U' \), and plugging in this pair yields

\[
\sup_{t \in [\frac{\ln n}{n}, t_n]} \frac{|P(t - \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}, t + \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}) - \sqrt{\frac{c_1 \ln n}{n}}|}{\sqrt{2t}} \leq \frac{C}{\sqrt{n \ln n}}.
\]

Similarly, for \( (t + \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}, t - \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}) \) \( \in U' \) we also have

\[
\sup_{t \in [\frac{\ln n}{n}, t_n]} \frac{|P(t + \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}, t - \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}) - \sqrt{\frac{c_1 \ln n}{n}}|}{\sqrt{2t}} \leq \frac{C}{\sqrt{n \ln n}}.
\]

Now consider

\[
Q(t) = \frac{1}{2} \sqrt{\frac{n}{c_1 \ln n}} \left( P(t - \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}, t + \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}) + P(t + \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}, t - \frac{1}{2} \sqrt{\frac{c_1 \ln n}{n}}) \right)
\]

it is easy to see that \( Q(t) \) is a polynomial of \( t \), and \( \deg Q \leq 2K \). Moreover, adding the previous two inequalities together, by triangle inequality we obtain

\[
\sup_{t \in [\frac{\ln n}{n}, t_n]} \frac{|Q(t) - \sqrt{t}|}{\sqrt{t}} \leq \sqrt{\frac{2}{c_1 \ln n}} \frac{C}{\sqrt{n \ln n}} \lesssim \frac{1}{K}.
\]

Since \( t_n \gg \frac{(\ln n)^3}{n} \), we have \( \eta_n \triangleq \frac{c_1 \ln n}{n t_n} \ll \frac{1}{K^2} \). Define \( R(t) = t_n \frac{1}{t} Q(t_n \cdot t) \) for \( t \in [\eta_n, 1] \), \( (507) \) becomes

\[
|R(t) - \sqrt{t}| \lesssim \frac{\sqrt{t}}{K}, \quad \forall t \in [\eta_n, 1].
\]

Moreover, \( \deg R \leq 2K \). Now let \( S \) be the best degree-2K approximating polynomial of \( \sqrt{t} \) in the uniform norm on \( [\eta_n, 1] \), using second-order Ditizian–Totik modulus of smoothness (Lemma 21) and \( \eta_n \ll \frac{1}{K^2} \) we arrive at

\[
\sup_{t \in [\eta_n, 1]} |S(t) - \sqrt{t}| \lesssim \frac{1}{K}.
\]

Furthermore, following the proof of Lemma 18 we can prove that

\[
\sup_{t \in [\eta_n, 1]} |S'(t)| \lesssim K.
\]

As a result, by triangle inequality we have

\[
\sup_{t \in [\eta_n, 1]} |R(t) - S(t)| \leq \sup_{t \in [\eta_n, 1]} |R(t) - \sqrt{t}| + \sup_{t \in [\eta_n, 1]} |S(t) - \sqrt{t}|
\]

\[
\lesssim \frac{1}{K} + \frac{1}{K} \lesssim \frac{1}{K}.
\]

Since \( R(t) - S(t) \) is also a polynomial of degree \( \leq 2K \), by Markov’s inequality (Lemma 25)

\[
\sup_{t \in [\eta_n, 1]} |R'(t) - S'(t)| \leq \frac{2}{1 - \eta_n} \cdot 4K^2 \sup_{t \in [\eta_n, 1]} |R(t) - S(t)| \lesssim K
\]

and finally by triangle inequality again

\[
\sup_{t \in [\eta_n, 1]} |R'(t)| \leq \sup_{t \in [\eta_n, 1]} |R'(t) - S'(t)| + \sup_{t \in [\eta_n, 1]} |S'(t)| \lesssim K.
\]
Now we are about to arrive at the desired contradiction. Choosing \( t = \eta_n \) and \( t = 2\eta_n \) in (508), we have
\[
\left( 1 - \frac{D}{K} \right) \sqrt{\eta_n} \leq R(\eta_n) \leq \left( 1 + \frac{D}{K} \right) \sqrt{\eta_n},
\]

(515)
\[
\left( 1 - \frac{D}{K} \right) \sqrt{2\eta_n} \leq R(2\eta_n) \leq \left( 1 + \frac{D}{K} \right) \sqrt{2\eta_n}
\]

(516)
with \( D > 0 \) a suitable universal constant appearing in the RHS of (508). As a result,
\[
R(2\eta_n) - R(\eta_n) \geq \left( 1 - \frac{D}{K} \right) \sqrt{2\eta_n} - \left( 1 + \frac{D}{K} \right) \sqrt{\eta_n} \approx \sqrt{\eta_n}
\]

(517)
and by the mean value theorem we conclude that there exists some \( \xi \in [t_n, 2t_n] \) such that
\[
R'(\xi) = \frac{R(2\eta_n) - R(\eta_n)}{2\eta_n - \eta_n} \geq \frac{1}{\sqrt{\eta_n}} \gg K
\]

(518)
where the last inequality follows from the fact that \( \eta_n \ll \frac{1}{K^2} \). However, this inequality is contradicting to our previous result (514), and thus we are done.

Q. Proof of Lemma 20

We have
\[
\sup_{P, Q \in \mathcal{M}_S} \mathbb{E}_P \left( L_1(g(P_n), Q) - L_1(P, Q) \right)^2 \geq \sup_{P \in \mathcal{M}_S} \mathbb{E}_P \left( L_1(g(P_n), P) \right)^2 \geq S\frac{1}{n}.
\]

(519)

(520)

(521)
where the last step follows from the result of minimax risk for estimating the discrete distribution \( P \) under \( \ell_1 \) loss in [32, Cor. 4].

APPENDIX C
PROOFS OF AUXILIARY LEMMAS

A. Proof of Lemma 23

We split the analysis of \( |f(x + h\varphi/2) - f(x - h\varphi/2)| \) into two cases:

1) \( x - \frac{h\varphi}{2} \geq a \) or \( x + \frac{h\varphi}{2} \leq a \): in this case,
\[
|f(x + h\varphi/2) - f(x - h\varphi/2)| = \sqrt{x + h\varphi/2} - \sqrt{x - h\varphi/2}
\]
\[
= \frac{h\varphi}{\sqrt{x + h\varphi/2} + \sqrt{x - h\varphi/2}} \leq \frac{h\varphi}{\sqrt{x + h\varphi/2 + x - h\varphi/2}} = \frac{h\sqrt{1 - x}}{\sqrt{2}} \leq \frac{t}{\sqrt{2}}
\]

(522)

(523)

(524)

(525)

(526)
where we have used the fact that \( \sqrt{x} + \sqrt{y} \geq \sqrt{x + y} \) and \( 0 < h \leq t \).

2) \( x - \frac{h\varphi}{2} < a < x + \frac{h\varphi}{2} \): in this case
\[
|f(x + h\varphi/2) - f(x - h\varphi/2)| = \left| \sqrt{x + h\varphi/2} + \sqrt{x - h\varphi/2} - 2\sqrt{a} \right| \leq \max(\sqrt{x + h\varphi/2} - \sqrt{a}, \sqrt{a} - \sqrt{x - h\varphi/2}) \leq \sqrt{x + h\varphi/2} - \sqrt{x - h\varphi/2} \leq \frac{t}{\sqrt{2}}
\]

(527)

(528)

(529)

(530)
It follows from taking derivatives that for convex function $f(x)$, the function $f(x - t) - 2f(x) + f(x + t)$ is a nondecreasing function of $t$. Since $f(x) = [2x - q]$ is a convex function, it follows from straightforward algebra that

$$
\omega_n^2(f, K^{-1}) = \max\left\{ \frac{\max_{z \in [1+K^2, 1+K^2]} A_1(z) \max_{z < 1+K^2} A_2(z) \max_{z > 1+K^2} A_3(z)}{2f(x) + f(x + t)} \right\},
$$

(531)

where

$$
A_1(z) = f\left(z - \frac{\sqrt{z(z-1)}}{4K^2} \right) - 2f(z) + f\left(z + \frac{\sqrt{z(z-1)}}{4K^2} \right)
$$

(532)

$$
A_2(z) = f(0) - 2f(z) + f(2z)
$$

(533)

$$
A_3(z) = f(2z - 1) - 2f(z) + f(1).
$$

(534)

We break the proof into three parts.

1) We first prove that when $\frac{3}{2\Delta^2} \leq \frac{K^2}{1+K^2}$, the maximum of achieved by $A_1(z)$ at $z = \frac{3}{2\Delta^2}$. Consider first the case $\frac{3}{2\Delta^2} \leq \frac{K^2}{1+K^2}$ and function $A_1(z)$. If $z > \frac{3}{2\Delta^2}$, without loss of generality we can assume $z - \frac{\sqrt{z(z-1)}}{4K^2} < \frac{3}{2\Delta^2}$, since otherwise $A_1(z) = 0$. Then,

$$
A_1(z) = q - 2\Delta \left(z - \frac{\sqrt{z(z-1)}}{4K^2} \right) - 2(2z\Delta - q) + 2 \left(z + \frac{\sqrt{z(z-1)}}{4K^2} \right) \Delta - q
$$

(535)

$$
= 4\Delta \left( \frac{\sqrt{z(z-1)}}{4K^2} - z + \frac{q}{2\Delta} \right).
$$

(536)

Taking derivative with respect to $z$, it suffices to show this derivative is non-positive when $\frac{3}{2\Delta^2} \leq \frac{q}{2\Delta^2} \leq \frac{K^2}{1+K^2}$, and $z \geq \frac{q}{2\Delta^2}$, $z - \frac{\sqrt{z(z-1)}}{4K^2} < \frac{3}{2\Delta^2}$. We have the derivative expressed as

$$
4\Delta \left( \frac{1 - 2z}{2K\sqrt{z(z-1)}} - 1 \right) = 4\Delta \left( \frac{1 - 2z - 2K\sqrt{z(z-1)}}{2K\sqrt{z(z-1)}} \right)
$$

(537)

Since $1 - 2z - 2K\sqrt{z(z-1)}$ is a convex function, it achieves its maximum at the endpoints. When we set $z = \frac{1}{1+K^2}$ and $z = 1$ it is both negative. Similar arguments work for the case of $z < \frac{q}{2\Delta^2}$. Hence, we conclude that when $\frac{3}{2\Delta^2} \leq \frac{q}{2\Delta^2} \leq \frac{K^2}{1+K^2}$,

$$
\max_{\frac{3}{2\Delta^2} \leq z \leq \frac{K^2}{1+K^2}} A_1(z) = \frac{2\sqrt{q(2\Delta - q)}}{K}
$$

(538)

Consider the case $z > \frac{K^2}{1+K^2}$ and the function $A_3(z)$. It suffices to assume $2z - 1 \leq \frac{q}{2\Delta^2}$ since otherwise $A_3(z) = 0$. In this case

$$
A_3(z) = q - (2z - 1)\Delta - 2(2z\Delta - q) + 2\Delta - q
$$

(539)

$$
= 4\Delta + 2q - 8z\Delta
$$

(540)

which is a decreasing function in $z$, implying $\max_{z > \frac{K^2}{1+K^2}} A_3(z) = \max_{\frac{3}{2\Delta^2} \leq z \leq \frac{K^2}{1+K^2}} A_1(z)$. Similar arguments work for the case of $z < \frac{1}{1+K^2}$ and $A_2(z)$ case.

2) We now prove that when $\frac{3}{2\Delta^2} \leq \frac{q}{2\Delta^2}$, the maximum is achieved by $A_2(z)$ at $\frac{q}{2\Delta^2}$.

In this case, it suffices to consider $z \leq \frac{1}{1+K^2}$. Indeed, if $z > \frac{1}{1+K^2}$, then the second order difference in the non-zero case is given by (535), which is shown to be a decreasing function when $z > \frac{1}{1+K^2}$. Now consider $z \leq \frac{1}{1+K^2}$. We discuss three cases separately:

a) $2z \leq \frac{q}{2\Delta^2}$: in this case, $A_2(z) = 0$.
b) $z \leq \frac{q}{2\Delta^2} \leq 2z$: in this case,

$$
A_2(z) = q - 2(q - 2z\Delta) + 4z\Delta - q
$$

(541)

$$
= 8z\Delta - 2q,
$$

(542)

which is an increasing function of $z$. It implies that in this regime one should take $z = \frac{q}{2\Delta^2}$. The resulting $A_2(z)$ is $2q$. 

c) $\frac{q}{2\Delta} \leq z$: in this case, the second order difference is

$$q - 2(2z\Delta - q) + 4z\Delta - q = 2q,$$

which is independent of $z$.

Hence, we have shown that for $\frac{q}{2\Delta} \leq \frac{1}{1+2\kappa}$, the maximum is achieved by $A_2(z)$ and

$$\max_{z \leq \frac{1}{1+2\kappa}} A_2(z) = 2q.$$  (544)

3) The case of $2\Delta \geq \frac{q}{2\Delta} \geq \frac{\kappa^2}{1+2\kappa}$ can be dealt with in a fashion similar to the case of $\frac{q}{2\Delta} \leq \frac{1}{1+2\kappa}$, resulting in

$$\max_{z \geq \frac{\kappa^2}{1+2\kappa}} A_3(z) = 2(2\Delta - q).$$  (545)

C. Proof of Lemma 28

It suffices to show that for any polynomial $Q \in \text{poly}_{2\kappa}$,

$$\sup_{z \in [-1,1]} |Q(z) - f(z^2)| \geq \sup_{z \in [-1,1]} |P_K(z^2) - f(z^2)|$$

$$= \sup_{z \in [0,1]} |P_K(z^2) - f(z^2)|$$  (546)  (547)

Define

$$e(z) = \frac{Q(z) + Q(-z)}{2}$$  (548)

$$o(z) = \frac{Q(z) - Q(-z)}{2}.$$  (549)

It is clear that $e(z)$ is an even function, $o(z)$ is an odd function, and $Q(z) = e(z) + o(z)$. We have

$$\sup_{z \in [-1,1]} |f(z^2) - Q(z)| = \sup_{z \in [-1,1]} |f(z^2) - e(z) - o(z)|$$

$$= \sup_{z \in [0,1]} \max\{|f(z^2) - e(z) - o(z)|, |f(z^2) - e(z) + o(z)|\}$$

$$\geq \sup_{z \in [0,1]} \left( |f(z^2) - e(z) - o(z)| + |f(z^2) - e(z) + o(z)| \right) / 2$$

$$\geq \sup_{z \in [0,1]} \frac{\left| (f(z^2) - e(z) - o(z)) + (f(z^2) - e(z) + o(z)) \right|}{2}$$

$$= \sup_{z \in [0,1]} |f(z^2) - e(z)|,$$  (550)  (551)  (552)  (553)  (554)

where we have used the fact that $\max\{a, b\} \geq \frac{a+b}{2}$ and the convexity of the function $|z|$.

There exists another polynomial $U_K(z) \in \text{poly}_K$ such that $U_K(z^2) = e(z)$. Hence, for any $Q \in \text{poly}_{2\kappa}$,

$$\sup_{z \in [-1,1]} |f(z^2) - Q(z)| \geq \sup_{z \in [0,1]} |f(z^2) - U_K(z^2)|$$

$$= \sup_{z \in [0,1]} |f(z) - U_K(z)|$$

$$\geq \sup_{z \in [0,1]} |f(z) - P_K(z)|$$

$$= \sup_{z \in [-1,1]} |f(z^2) - P_K(z^2)|$$  (555)  (556)  (557)  (558)

where we used the definition of $P_K(z)$. The proof is complete.

D. Proof of Lemma 30

The Charlier polynomial $c_n(x, a), a > 0$ is defined as follows:

$$c_n(x, a) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(x)^r}{a^r},$$  (560)
where \((x)_r = x \cdot (x - 1) \cdots \cdot (x - r + 1)\) is the falling factorial. It satisfies the following generating function relation \([33]\):

\[
\sum_{n=0}^{\infty} \frac{c_n(x, a)}{n!} t^n = e^{-t} \left(1 + \frac{t}{a}\right)^x, \quad x \in \mathbb{N}.
\] (561)

Substituting \(t\) by \(at\), we have

\[
\sum_{n=0}^{\infty} \frac{a^n c_n(x, a)}{n!} t^n = e^{-at} (1 + t)^x, \quad x \in \mathbb{N}.
\] (562)

Note that we have

\[
a^n c_n(x, a) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} a^{n-r} (x)_r,
\] (563)

which is well defined even for \(a = 0\). If \(a = 0\), then \(a^n c_n(x, a)\) may be defined as

\[
a^n c_n(x, a) \triangleq (x)_n.
\] (564)

We note that relation (562) is true also when \(a = 0\). Indeed, the case \(a = 0\) reduces to the relation:

\[
\sum_{n=0}^{\infty} \frac{(x)_n}{n!} t^n = (1 + t)^x, \quad x \in \mathbb{N}.
\] (565)

Assuming \(Y \sim \text{Poi}(\lambda)\), replacing \(x\) with random variable \(Y\) in (562) and taking expectation on both sides, we have

\[
\sum_{n=0}^{\infty} \frac{Ea^n c_n(Y, a)}{n!} t^n = e^{-at} \mathbb{E}(1 + t)^Y = e^{t(\lambda - a)} = \sum_{n=0}^{\infty} \frac{(\lambda - a)^n}{n!} t^n
\] (566)

Note that \(Ea^n c_n(Y, a)\) does not depend on \(t\). Hence we know

\[
Ea^n c_n(Y, a) = (\lambda - a)^n.
\] (567)

Thus, if \(nX \sim \text{Poi}(np), a = nq\), we have

\[
E q^j c_j(nX, nq) = (p - q)^j, \quad j \geq 0.
\] (568)

Expanding \(q^j c_j(nX, nq)\) implies that it is equal to \(g_{j,q}(X)\) defined in Lemma 30. The estimator \(g_{j,q}(X)\) being the unique unbiased estimator of \((p - q)^j\) follows from the Lehmann–Scheffe Theorem [34, Chap. 2, Thm. 1.11] and the complete sufficiency of \(X\) in model \(nX \sim \text{Poi}(np)\) (34, Chap. 1, Thm. 6.22).

Now we proceed to bound the second moment of \(g_{j,q}(X)\). It follows from (562) that for any \(a + b \geq 0\),

\[
\sum_{n=0}^{\infty} \frac{(a + b)^n c_n(x, a + b)}{n!} t^n = e^{-(a+b)t} (1 + t)^x, \quad x \in \mathbb{N},
\] (569)

which implies that

\[
\sum_{n=0}^{\infty} \frac{(a + b)^n c_n(x, a + b)}{n!} t^n = e^{-bt} \sum_{n=0}^{\infty} \frac{a^n c_n(x, a)}{n!} t^n
\] (570)

\[
= \left(\sum_{j=0}^{\infty} \frac{(-bt)^j}{j!}\right) \left(\sum_{n=0}^{\infty} \frac{a^n c_n(x, a)}{n!} t^n\right)
\] (571)

\[
= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^n c_n(x, a)(-b)^j}{n!j!} t^{n+j}
\] (572)

It follows from coefficient matching that

\[
\frac{(a + b)^n c_n(x, a + b)}{n!} = \sum_{k=0}^{n} \frac{a^k c_k(x, a)(-b)^{n-k}}{k!(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} a^k c_k(x, a)(-b)^{n-k},
\] (573)

which simplifies to

\[
(a + b)^j c_j(x, a + b) = \sum_{k=0}^{j} \binom{j}{k} (-b)^{j-k} a^k c_k(x, a).
\] (574)
Now assume $nX \sim \text{Poi}(np)$. Taking $a + b = nq, a = np$ and dividing both sides by $n^j$, we have
\begin{equation}
q^j c_j(nX, nq) = \frac{1}{n^j} \sum_{k=0}^{j} \binom{j}{k} (n(p-q))^{j-k} (np)^k c_k(nX, np) = \sum_{k=0}^{j} \binom{j}{k} (p-q)^{j-k} p^k c_k(nX, np). \tag{575}
\end{equation}

The Charlier polynomials are orthogonal with respect to the Poisson measure. Concretely, for $Y \sim \text{Poi}(\lambda)$ \cite{33},
\begin{equation}
\mathbb{E}c_n(Y, \lambda)c_m(Y, \lambda) = \frac{n!}{\lambda^n} \delta_{mn}. \tag{576}
\end{equation}
For $nX \sim \text{Poi}(np)$, we have
\begin{equation}
\mathbb{E} \left( p^j c_j(nX, np) \right)^2 = \frac{p^{2j} j!}{n^j}, \tag{577}
\end{equation}
which is also true for $p = 0$.

Applying the orthogonal property of Charlier polynomials and assuming $p > 0$, we have
\begin{equation}
\mathbb{E} \left( q^j c_j(nX, nq) \right)^2 = \mathbb{E} \sum_{k=0}^{j} \binom{j}{k}^2 (p-q)^{2(j-k)} \mathbb{E} \left( p^k c_k(nX, np) \right)^2 \tag{578}
\end{equation}
\begin{equation}
= \sum_{k=0}^{j} \binom{j}{k}^2 (p-q)^{2(j-k)} p^k k! \tag{579}
\end{equation}
\begin{equation}
= j! \sum_{k=0}^{j} \binom{j}{k}^2 (p-q)^{2(j-k)} \left( \frac{p}{n} \right)^k \frac{1}{(j-k)!} \tag{580}
\end{equation}
\begin{equation}
= j! \sum_{k=0}^{j} \binom{j}{k} (p-q)^{2k} \left( \frac{p}{n} \right)^{j-k} \frac{1}{k!} \tag{581}
\end{equation}
\begin{equation}
= j! \left( \frac{p}{n} \right)^j \sum_{k=0}^{j} \binom{j}{k} \left[ \frac{(p-q)^{2k}}{p} \right] \frac{1}{(j-k)!} \tag{582}
\end{equation}
\begin{equation}
= j! \left( \frac{p}{n} \right)^j L_j \left( -\frac{n(p-q)^2}{p} \right), \tag{583}
\end{equation}
where $L_m(x)$ stands for the Laguerre polynomial with order $m$, which is defined as:
\begin{equation}
L_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{k!}. \tag{584}
\end{equation}

If we further assume $M \geq \max \left\{ \frac{n(p-q)^2}{p}, j \right\}$, we have
\begin{equation}
\mathbb{E} \left( q^j c_j(nX, nq) \right)^2 \leq j! \left( \frac{p}{n} \right)^j \sum_{k=0}^{j} \binom{j}{k} \frac{M^k}{k!} \tag{585}
\end{equation}
\begin{equation}
\leq j! \left( \frac{p}{n} \right)^j \sum_{k=0}^{j} \binom{j}{k} \frac{M^j}{j!} \tag{586}
\end{equation}
\begin{equation}
= \left( \frac{2Mp}{n} \right)^j. \tag{587}
\end{equation}

**E. Proof of Lemma** \cite{31}

It follows from the fact that $\mathbb{E} \prod_{i=0}^{k-1} (\hat{p} - \frac{1}{n}) = p^k$ for $n\hat{p} \sim \text{Poi}(np)$ \cite{24} Ex. 2.8] that $\hat{A}_j$ is unbiased for estimating $(p-q)^j$.

It follows from the Lehmann–Scheffe Theorem \cite{34} Chap. 2, Thm. 1.11] and the complete sufficiency of $(\hat{p}, \hat{q})$ (\cite{34} Chap. 1, Thm. 6.22]) that $\hat{A}_j$ is the unique unbiased estimator for $(p-q)^j$.

We now work out a different form of $\hat{A}_j$. It follows from the binomial theorem that for any fixed $r > 0$,
\begin{equation}
(p-q)^j = (p-r - (q-r))^j \tag{588}
\end{equation}
\begin{equation}
= \sum_{k=0}^{j} \binom{j}{k} (p-r)^k (-1)^{j-k} (q-r)^{j-k}. \tag{589}
\end{equation}
Clearly, the following estimator is also unbiased for estimating \((p - q)^3\):

\[
\sum_{k=0}^{j} \binom{j}{k} g_{k,r}(\hat{p})(-1)^{j-k} g_{j-k,r}(\hat{q}),
\]  

(590)

where \(g_{k,r}(\hat{p})\) and \(g_{j-k,r}(\hat{q})\) are the unique unbiased estimators for \((p-r)^k\) and \((q-r)^{j-k}\) introduced in Lemma 30, respectively. It follows from the uniqueness of \(\hat{A}_j\) that

\[
\hat{A}_j = \sum_{k=0}^{j} \binom{j}{k} g_{k,r}(\hat{p})(-1)^{j-k} g_{j-k,r}(\hat{q}).
\]  

(591)

Using \(\|X\|_2 = (\mathbb{E}[X^2])^{1/2}\) and the triangle inequality for the norm \(\|X\|_2\), we have

\[
||\hat{A}_j||_2 \leq \sum_{k=0}^{j} \binom{j}{k} \|g_{k,r}(\hat{p})\|_2 \|g_{j-k,r}(\hat{q})\|_2
\]  

(592)

\[
= \sum_{k=0}^{j} \binom{j}{k} \|g_{k,r}(\hat{p})\|_2 \|g_{j-k,r}(\hat{q})\|_2,
\]  

(593)

where we have used the independence of \(\hat{p}\) and \(\hat{q}\) in the last step.

Define \(M_1 = \frac{n(p-r)^2}{p^2} \vee j, M_2 = \frac{n(q-r)^2}{q^2} \vee j\), and set \(r = \frac{p+q}{2}\). Define \(M = 2(p-q)^2 \vee \frac{8j(p+q)}{n}\). It follows from Lemma 30 that

\[
||\hat{A}_j||_2 \leq \sum_{k=0}^{j} \binom{j}{k} \left(\frac{2M_1p}{n}\right)^{k/2} \left(\frac{2M_2q}{n}\right)^{(j-k)/2}
\]  

(594)

\[
= \left(\sqrt{\frac{2M_1p}{n}} + \sqrt{\frac{2M_2q}{n}}\right)^j
\]  

(595)

\[
= \left(\sqrt{\frac{(p-q)^2}{2} \frac{2p}{n}} + \sqrt{\frac{(p-q)^2}{2} \frac{2q}{n}}\right)^j
\]  

(596)

\[
= M^{j/2}.
\]  

(597)

**F. Proof of Lemma 32**

Equation (594) follows from [25] Lemma 9.5.5. Now we prove the bound on the magnitude of \(|h_{j,s}|\). Note that the moment generating function of \(\hat{p} - p\) is given by

\[
\mathbb{E}[\exp(z(\hat{p} - p))] = e^{-zp} e^{n p (e^{z/n} - 1)}
\]  

(598)

Written as formal power series of \(z\), the previous identity becomes

\[
\sum_{s=0}^{\infty} \frac{s^s}{s!} \mathbb{E}(\hat{p} - p)^s = \left(\sum_{i=0}^{\infty} \frac{(-p)^i}{i!} z^i\right) \left[\sum_{k=0}^{\infty} \frac{n^k p^k}{k!} \left(\sum_{l=1}^{\infty} \frac{1}{l!} \left(\frac{z}{n}\right)^l\right)^k\right].
\]  

(599)

Hence, by comparing the coefficient of \(n^{j-s} z^s\) at both sides, we obtain

\[
\frac{h_{j,s} p^j}{s!} = \sum_{i=0}^{j} \frac{(-p)^i}{i!} \left(\frac{p^{j-i}}{(j-i)!} \sum_{a_1+\cdots+a_{j-i}=s-i, a_1, \ldots, a_{j-i} \geq 1} \prod_{l=1}^{j-i} \frac{1}{a_l!}\right).
\]  

(600)

Moreover,

\[
\sum_{a_1+\cdots+a_{j-i}=s-i, a_1, \ldots, a_{j-i} \geq 1} \prod_{l=1}^{j-i} \frac{1}{a_l!} \leq \sum_{a_1+\cdots+a_j=s} \prod_{l=1}^{j} \frac{1}{a_l!} = \frac{j^s}{s!},
\]  

(601)

and

\[
\sum_{a_1+\cdots+a_{j-i}=s-i, a_1, \ldots, a_{j-i} \geq 1} \prod_{l=1}^{j-i} \frac{1}{a_l!} \leq \sum_{a_1+\cdots+a_j=s} \prod_{l=1}^{j} \frac{1}{a_l!} = \frac{j^s}{s!}.
\]  

(602)
Then,

\[ |h_{j,s}| \leq \sum_{i=0}^{j} \frac{1}{i!} \frac{1}{(j-i)!} j^s \]

(603)

\[ = \frac{j^s}{j!} \sum_{i=0}^{j} \binom{j}{i} \]

(604)

\[ = \frac{2j^s}{j!}. \]

(605)

Since \( 1 \leq j \leq s \), we have

\[ \frac{2j^s}{j!} \leq 2^s \frac{j^s}{j!} \]

(606)

\[ \leq 2^s \frac{j^s}{\sqrt{2\pi j(j/e)^j}} \]

(607)

\[ \leq 2^s \frac{j^s e^{-j}}{j^j}. \]

(608)

Now we consider the maximization problem \( \max_{x \geq 0} \frac{x^e}{x^s} \). It follows from taking derivatives that this function attains its unique maximum at point \( x^* \) which satisfies the following:

\[ x^* \ln x^* = s. \]

(609)

Recall the Lambert \( W \) function is defined over \([−1/e, \infty)\) by the equation \( W(z)e^{W(z)} = z \), we know that

\[ x^* = e^{W(s)}. \]

(610)

The following upper bound on \( W(s) \) was proved in [35]: for any \( s > e \),

\[ W(s) \leq \ln s - \ln \ln s + \frac{e}{e-1} \frac{\ln s}{\ln s} \]

(611)

\[ \leq \ln s - \ln \ln s + \frac{1}{e-1}. \]

(612)

where we have used the fact that \( \max_{x>0} \frac{\ln x}{x} = \frac{1}{e} \).

Hence, for any \( s \geq 3 \),

\[ |h_{j,s}| \leq (2e)^s e^{sW(s)} \]

(613)

\[ \leq (2e^{e/(e-1)})^s \left( \frac{s}{\ln s} \right)^s, \]

(614)

which turns out to be also correct for \( s = 2 \) since \( h_{1,2} = 1 \).

G. Proof of Lemma 33

It is clear that when \( p \leq \frac{1}{n} \), the statement is true. It suffices to consider the case of \( p > \frac{1}{n} \). Introduce function \( g_n(p) \) as follows:

\[ g_n(p) = \begin{cases} \frac{1}{p} & p \geq \frac{1}{n} \\ n^j - jn^{j+1} (p - \frac{1}{n}) & 0 \leq p < \frac{1}{n}. \end{cases} \]

(615)

It is evident that \( g_n(p) \leq \frac{1}{p} \) and

\[ g_n(\hat{p}) - \frac{1}{(\hat{p} \lor \frac{1}{n})^j} = \begin{cases} 0 & \hat{p} \geq \frac{1}{n} \\ jn^j & \hat{p} = 0 \end{cases} \]

(616)

We have

\[ \left| \frac{1}{(\hat{p} \lor \frac{1}{n})^j} \right| \leq \left| \mathbb{E} \left[ \frac{1}{(\hat{p} \lor \frac{1}{n})^j} - g_n(\hat{p}) \right] \right| + \mathbb{E} \left[ |g_n(p) - g_n(\hat{p})| \right] + g_n(p) \]

(617)

\[ \leq jn^j e^{-np} + \frac{1}{p^j} + \mathbb{E} \left[ |g_n(p) - g_n(\hat{p})| \right]. \]

(618)
Since the function $g_n(p)$ is continuously differentiable on $(0, \infty)$, we have
\[
\left| \mathbb{E} \left[ (g_n(p) - g_n(\hat{p}))^2 \right] \right| = \left| \mathbb{E} \left[ (g_n(p) - g_n(\hat{p}))^2 \mathbb{1}(\hat{p} \geq p/2) \right] \right| + \left| \mathbb{E} \left[ (g_n(p) - g_n(\hat{p}))^2 \mathbb{1}(\hat{p} \leq p/2) \right] \right|
\]
(619)
\[
\leq \sup_{\xi \geq p/2} \left| g_n'(\xi) \right|^2 \mathbb{E} (p - \hat{p})^2 + \sup_{\xi > 0} \left| g_n'(\xi) \right|^2 \mathbb{P}(\hat{p} \leq p/2)
\]
(620)
\[
\leq \frac{j^2}{(p/2)^{j+2}} \frac{p^j}{n} + j^2 n^{j+2} / p^2 e^{-np/8},
\]
(621)
where we applied Lemma 34 in the last step. Hence,
\[
\left| \mathbb{E} \left[ \frac{1}{(\hat{p} \lor \frac{1}{n})^j} \right] \right| \leq j n^j e^{-np} + \frac{1}{p^j} + \frac{\sqrt{\mathbb{E} (g_n(p) - g_n(\hat{p}))^2}}{\mathbb{E} (g_n(p) - g_n(\hat{p}))^2}
\]
(622)
\[
\leq j n^j e^{-np} + \frac{1}{p^j} + \frac{j}{(p/2)^{j+1}} \left( \frac{p}{n} + j n^{j+1} p e^{-np/16} \right)
\]
(623)
\[
\leq j n^j e^{-np} + \frac{1}{p^j} + \frac{j^{2j+1}}{p^j} + j n^{j+1} p e^{-np/16},
\]
(624)
where in the last step we used the the assumption that $p \geq \frac{1}{n}$. Consequently,
\[
\left| \mathbb{E} \left[ \frac{p^j}{(\hat{p} \lor \frac{1}{n})^j} \right] \right| \leq j (np)^j e^{-np} + 1 + j^{2j+1} + j (np)^j e^{-np/16}
\]
(625)
\[
\leq j \left( \frac{j}{e} \right)^j + 1 + j^{2j+1} + j \left( \frac{16(j+1)}{e} \right)^{j+1},
\]
(626)
where we have used the fact that for any $p \geq 0$,
\[
(np)^{k} e^{-cnp} \leq \left( \frac{k}{ec} \right)^k.
\]
(627)

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