THE SECOND STIEFEL-WHITNEY CLASS OF $\ell$-ADIC COHOMOLOGY

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Abstract. For a proper smooth variety of even dimension over a field of characteristic different from 2 or $\ell$, the second Stiefel-Whitney class of the $\ell$-adic cohomology and the second Hasse-Witt class of the de Rham cohomology are both defined in the second Galois cohomology. We state a conjecture on their relation and give several evidences.

The cohomology of middle degree of a proper smooth variety of even dimension carries a non-degenerate symmetric bilinear form. This defines various Stiefel-Whitney classes depending on the cohomology theory we consider. The second Stiefel-Whitney class of $\ell$-adic cohomology allows us to deduce that the $L$-function has the positive sign in the functional equation, from a reasonable hypothesis [16]. For the de Rham cohomology, we also call the Stiefel-Whitney class the Hasse-Witt class since it is defined by a quadratic form.

In this paper, we propose a conjecture Conjecture 2.3 comparing the Stiefel-Whitney classes of $\ell$-adic cohomology and of de Rham cohomology and prove several evidences. In the case where the variety is of dimension 0, it amounts to a formula of Serre on the Hasse-Witt invariant of the trace form for a finite separable extension [19]. Conjecture 2.3 may also be regarded as a version in degree 2 of the formula for the determinant of cohomology proved in [15].

We state the main Conjecture 2.3 in Section 2 after some preliminaries on the Stiefel-Whitney classes of orthogonal Galois representations and of quadratic vector spaces in Section 1. We give an apparently simpler reformulation in Corollary 2.11 by using a graded variant.

As evidences for Conjecture 2.3, we prove the following cases in Theorem 2.5 under some additional conditions: 1. The base field is a finite extension of $\mathbb{Q}_p$ for $p \neq \ell$ and the variety has a certain mild degeneration. 2. The base field is a finite unramified extension of $\mathbb{Q}_\ell$ and the variety has a good reduction. 3. The base field is $\mathbb{R}$. 4. The base field is an extension of $\mathbb{Q}$. 5. The base field is arbitrary and the variety is a smooth hypersurface in a projective space.

Curiously, Theorem 2.5 implies that the second Stiefel-Whitney class of $\ell$-adic cohomology in fact may depend on $\ell$, as in Example 2.6, contrary to the first Stiefel-Whitney class which is independent of $\ell$ as a consequence of the Weil conjecture (see Lemma 2.1).

We formulate a generalization Conjecture 7.2 of Conjecture 2.3 for families in Section 7 after introducing the Stiefel-Whitney classes for symmetric complexes in Section 6. A similar construction is studied by Balmer in [2]. Contrary to there, we are more interested in the invariants of individual complexes rather than the
invariants of the categories. The author learned from \cite{5} that a similar construction is also studied in \cite{22}. The rest of the article is devoted to the proof of Theorem 2.5.

In Section 3, we prove the assertion 1 of Theorem 2.5 using nearby cycles and the sheaves of differential forms with logarithmic poles. We observe in Lemma 3.7 that the mysterious appearance of the term involving 2 in Serre’s formula arises from the multiplicity of the exceptional divisor of the blow-up at an ordinary double point.

We verify that the assertion 2 is essentially proved in \cite{16} in Section 4. In Section 5, we prove the assertion 3 using Hodge structures. We prove the assertion 4 by partly proving the generalization Conjecture 7.2 for families by a transcendental argument in Section 8. In Section 9, using the universal family of hypersurfaces, we prove the assertion 5 by a global arithmetic argument. In the proof of 5., we combine all the results obtained in the other parts of the article.

The author thanks Paul Balmer for discussion on symmetric complexes. The author also thanks Asher Auel for his interest which encouraged him to complete the article. The author thanks Luc Illusie for informing an unpublished preprint \cite{21}.

The research is partly supported by Grants-in-aid for Scientific Research S-19104001.

1. Stiefel-Whitney classes and Hasse-Witt classes

We recall some basic definitions to formulate a conjecture. In order to distinguish the Stiefel-Whitney classes of orthogonal representations from those of quadratic forms, we will write $sw$ for the former and $hw$ for the latter. We call the latter the Hasse-Witt class.

Let $\pi$ be a group and $L$ be a field of characteristic 0. An orthogonal $L$-representation of $\pi$ is a triple $(V, b, \rho)$ consisting of an $L$-vector space $V$ of finite dimension, a non-degenerate symmetric bilinear form $b : V \otimes V \to L$ and a representation $\rho : \pi \to O(V, b)$ to the orthogonal group $O(V, b)$. If $\pi$ is a profinite group, we assume that $L$ is a finite extension of the $\ell$-adic field $\mathbb{Q}_\ell$ and that $\rho$ is continuous.

Let $V$ be an orthogonal representation of $\pi$. The first Stiefel-Whitney class $sw_1(V) \in H^1(\pi, \mathbb{Z}/2\mathbb{Z})$ is the determinant $\det \rho : \pi \to \{\pm 1\} \subset L^\times$ regarded as an element of $H^1(\pi, \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(\pi, \{\pm 1\})$. The second Stiefel-Whitney class $sw_2(V) \in H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ is defined as follows. Let

$$(1.0.1) \quad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{O}(V) \longrightarrow O(V) \longrightarrow 1$$

be the central extension of the algebraic group $O(V) = O(V, b)$ by $\mathbb{Z}/2\mathbb{Z}$ defined by using the Clifford algebra $\text{Cl}(V)$ as in \cite{9, 7, 16}. The canonical map $\tilde{O}(V) \to O(V)$ sends $x \in \tilde{O}(V) \subset \text{Cl}(V)$ to the automorphism $v \mapsto I(x)vx^{-1}$ where $I$ denotes the automorphism of the Clifford algebra defined by the multiplication by $-1$ on the odd part. By pulling back the central extension \eqref{1.0.1} by $\rho$, we obtain a central extension of $\pi$ by $\mathbb{Z}/2\mathbb{Z}$. The Stiefel-Whitney class $sw_2(V) \in H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ is defined as the class of this central extension \cite[CHAP. XIV Theorem 4.2]{4}.

For the orthogonal sum of orthogonal representations, we have $sw_1(V \oplus V') = sw_1(V) + sw_1(V')$ and $sw_2(V \oplus V') = sw_2(V) + sw_1(V)sw_1(V') + sw_2(V')$. If we
introduce the notation $sw(V) = 1 + sw_1(V) + sw_2(V)$, the equalities are rewritten as $sw(V \oplus V') = sw(V) \cdot sw(V')$ [16, Lemma 2.1].

If an orthogonal representation $V$ of $\pi$ admits a direct sum decomposition $V = W \oplus W'$ by $\pi$-stable and isotropic subspaces, the Stiefel-Whitney classes are computed as follows. For a character $\chi: \pi \to L^\times$, we define $\tilde{c}_1(\chi)$ in $H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ to be the class of the pull-back by $\chi$ of the central extension
\begin{equation}
(1.0.2) \quad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1.
\end{equation}
For a finite dimensional $L$-representation $V$ of $\pi$, we put $\tilde{c}_1(V) = \tilde{c}_1(\det V)$.

**Lemma 1.1.** Let $W$ be an $L$-vector space of finite dimension and we define a symmetric non-degenerate bilinear form on the direct sum $V = W \oplus W^\vee$ with the dual by the canonical pairing. We regard $GL(W)$ as a subgroup of $O(V)$ by $g \mapsto g \oplus g^{\vee\!-\!1}$. Then, the pull-back of (1.0.1) by the inclusion $GL(W) \to O(V)$ is isomorphic to the pull-back of (1.0.2) by $det: GL(W) \to \mathbb{G}_m$.

**Proof.** Since the algebraic group $SL(W)$ is connected and simply connected, the pull-back of (1.0.1) by the inclusion $GL(W) \to O(V)$ is isomorphic to the pull-back of a central extension of $\mathbb{G}_m$ by $det: GL(W) \to \mathbb{G}_m$. We define a section $\mathbb{G}_m \to GL(W)$ by taking a line in $W$. Then, it is reduced to the case where $W$ is a line.

We assume dim $W = 1$ and identify $GL(W) = \mathbb{G}_m$. Let $e \in W$ be a basis and $f \in W^\vee$ be the linear form defined by $f(e) = 1/2$. Then, the map $\mathbb{G}_m \to \tilde{O}(V)$ defined by $a \mapsto a + (1/a - a)f \cdot e$ is a group homomorphism since $f \cdot e$ is an idempotent. Further, it makes the diagram

$$
\begin{array}{ccc}
\mathbb{G}_m & \longrightarrow & \mathbb{G}_m = GL(W) \\
\downarrow & & \downarrow \\
\tilde{O}(V) & \longrightarrow & O(V)
\end{array}
$$

commutative since $e \cdot e = 0$ and $f \cdot e \cdot f = f$. Hence the assertion follows. $\square$

**Corollary 1.2.** Let $V$ be an orthogonal $L$-representation of $\pi$ and $V = W \oplus W'$ be a decomposition by $\pi$-stable isotropic subspaces. Then, we have $sw_2(V) = \tilde{c}_1(W)$ in $H^2(\pi, \mathbb{Z}/2\mathbb{Z})$.

**Proof.** By the assumption, we may identify $W'$ with the dual space of $W$. Then, it follows from Lemma 1.1 $\square$

We compute the Stiefel-Whitney class of the twist by a character of order 2. We prepare lemmas on central extensions.

**Lemma 1.3.** Let $n > 1$ be an integer. Let $1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{G} \to G \to 1$ and $1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{G}' \to G' \to 1$ be central extensions and $\chi: G \to \mathbb{Z}/n\mathbb{Z}$ and $\chi': G' \to \mathbb{Z}/n\mathbb{Z}$ be characters. We define a new group structure on the quotient
\begin{equation}
(1.3.1) \quad E = (\tilde{G} \times \tilde{G}')/\text{Ker}(+: \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z})
\end{equation}
by \((g, g') \cdot (h, h') = (\chi(h) \cdot \chi'(g'))(gh, g'h')\) where \(\cdot\) in the right hand side denotes the multiplication of the ring \(\mathbb{Z}/n\mathbb{Z}\).

Then the class \([E] \in H^2(G \times G', \mathbb{Z}/n\mathbb{Z})\) of the central extension \(1 \to \mathbb{Z}/n\mathbb{Z} \to E \to G \times G' \to 1\) is equal to the sum \(\text{pr}_1^*[	ilde{G}] + \text{pr}_2^*[	ilde{G}']\).

**Proof.** Let \(U(\mathbb{Z}/n\mathbb{Z}) \subset GL_2(\mathbb{Z}/n\mathbb{Z})\) be the subgroup consisting of unipotent upper triangular matrices. If we put \(G = G' = \mathbb{Z}/n\mathbb{Z}, \tilde{G} = \tilde{G}' = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}\) and \(\chi = \chi' = \text{id}\), then \(U(\mathbb{Z}/n\mathbb{Z})\) is obtained as \(E\) defined in (1.3.1). It is a central extension of \(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\) by \(\mathbb{Z}/n\mathbb{Z}\) and its class in \(H^2(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})\) is \(\text{pr}_1 \cup \text{pr}_2\).

The class of the central extension \(E_0\) with the same underlying set as \(E\) and with the group structure without modification is equal to the sum \(\text{pr}_1^*[\tilde{G}] + \text{pr}_2^*[\tilde{G}']\). Since \(E\) is obtained by modifying \(E_0\) by the pull-back of \(U(\mathbb{Z}/n\mathbb{Z})\) by \(\chi \times \chi',\) the assertion follows. \(\square\)

**Lemma 1.4.** Let \(b\) be a non-degenerate symmetric bilinear form on an \(L\)-vector space \(V\) of finite dimension \(n \geq 1\).

1. Let \(1 \to \mu_2 \to A \to \mu_2 \to 1\) be the central extension defined as the pull-back of (1.0.1) by the inclusion \(\mu_2 \to O(V)\). Then, the class \([A] \in H^2(\mu_2, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}\) is \((n\choose 2)\).

2. We apply the construction (1.3.1) to the central extensions \(1 \to \mu_2 \to A \to \mu_2 \to 1\) and \(1 \to \mu_2 \to \tilde{O}(V) \to O(V) \to 1\) and the characters \(\text{id}^{n-1}: \mu_2 \to \mu_2\) and \(\text{det}: O(V) \to \mu_2\) to define a central extension \(E\) of \(O(V) \times \mu_2\) by \(\mu_2\). Then, the inclusions induce a morphism \(E \to \tilde{O}(V)\) of groups.

**Proof.** After extending \(L\) if necessary, we take an orthonormal basis \(x_1, \ldots, x_n\) of \(V\).

1. Since the subgroup \(A\) is generated by \(x_1 \cdots x_n\) and \(-1\), the assertion follows from \((x_1 \cdots x_n)^2 = (-1)^{\binom{n}{2}}\).

2. For \(x \in V\), we have an equality \(x_1 \cdots x_n \cdot x = (-1)^{n-1}x \cdot x_1 \cdots x_n\) in the Clifford algebra \(\text{Cl}(V)\). Since the algebraic group \(\tilde{O}(V)\) is generated by \(V \cap \tilde{O}(V) \subset \text{Cl}(V)\), the assertion follows. \(\square\)

**Corollary 1.5.** Let \(\rho: \pi \to O(V)\) be an orthogonal representation of degree \(n\) and \(\chi: \pi \to \mu_2\) be a character of order 2. We regard \(\text{det} \rho\) and \(\chi\) as elements in \(H^1(\pi, \mathbb{Z}/2\mathbb{Z})\). Then, we have

\[
(1.5.1) \quad \text{sw}_2(\rho \otimes \chi) = \text{sw}_2(\rho) + (n - 1) \text{det} \rho \cup \chi + \binom{n}{2} \chi \cup \chi
\]

in \(H^2(\pi, \mathbb{Z}/2\mathbb{Z})\) and \(\text{det}(\rho \otimes \chi) = \text{det} \rho + n \chi\) in \(H^1(\pi, \mathbb{Z}/2\mathbb{Z})\).

**Proof.** By Lemma 1.4, the Stiefel-Whitney class \(\text{sw}_2(\rho \otimes \chi)\) is the class of the central extension defined as the pull-back of \(E\) in Lemma 1.4 by \(\rho \times \chi: \pi \to O(V) \times \mu_2\). Hence the equation (1.5.1) follows from Lemmas 1.3 and 1.4.1. The assertion for \(\text{det}\) is clear. \(\square\)
We generalize the definitions to graded case. Let $V^\bullet = \bigoplus_{q \in \mathbb{Z}} V^q$ be a graded $L$-representation. For each integer $q \in \mathbb{Z}$, we assume $V^q$ is a finite dimensional $L$-vector space equipped with a continuous representation of $\pi$ and $V^q = 0$ except for finitely many $q$. We assume that $V^0$ is an orthogonal representation and that, for each $q > 0$, $V^q \oplus V^{-q}$ is equipped with a $\pi$-invariant $(-1)^q$-symmetric non-degenerate form such that $V^q$ and $V^{-q}$ are totally isotropic. Then, we define $sw_1(V^\bullet) \in H^1(\pi, \mathbb{Z}/2\mathbb{Z})$ to be $\det(V^0)$ and $sw_2(V^\bullet) \in H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ by

\begin{equation}
sw_2(V^\bullet) = sw_2(V^0) + \sum_{q < 0} \tilde{c}_1(V^q).
\end{equation}

The definition is equivalent to the equality

\begin{equation}
1 + sw_1(V^\bullet) + sw_2(V^\bullet) = (1 + \det V^0 + sw_2(V^0)) \cdot \prod_{q < 0} (1 + \tilde{c}_1(V^q))^{(-1)^q}
\end{equation}

in $1 + H^1(\pi, \mathbb{Z}/2\mathbb{Z}) + H^2(\pi, \mathbb{Z}/2\mathbb{Z})$.

If $K$ is a field and $\pi$ is the absolute Galois group $G_K = \text{Gal}(\bar{K}/K)$, the Stiefel-Whitney classes $sw_i(V)$ are defined in the Galois cohomology $H^i(K, \mathbb{Z}/2\mathbb{Z}) = H^i(G_K, \mathbb{Z}/2\mathbb{Z})$ for $i = 1, 2$. If the characteristic of $K$ is not 2, we identify $H^1(K, \mathbb{Z}/2\mathbb{Z})$ with $K^\times/(K^\times)^2$. For $a \in K^\times$, let $\{a\} \in H^1(K, \mathbb{Z}/2\mathbb{Z})$ denote its class. For $a, a' \in K^\times$, let $\{a, a'\} \in H^2(K, \mathbb{Z}/2\mathbb{Z})$ denote the cup-product $\{a\} \cup \{a'\}$.

Lemma 1.6. 1. Let $\ell$ be a prime number and $\chi_\ell: \pi_1(\text{Spec } \mathbb{Z}[\frac{1}{\ell}])^{ab} \to \mathbb{Q}_\ell^\times$ be the $\ell$-adic cyclotomic character of the abelianized algebraic fundamental group. Then, $c_\ell = \tilde{c}_1(\chi_\ell)$ is the generator of the cyclic group $H^2(\pi_1(\text{Spec } \mathbb{Z}[\frac{1}{\ell}])^{ab}, \mathbb{Z}/2\mathbb{Z})$ of order 2.

2. Let $K$ be a field of characteristic $\neq 2$ and $\chi: G_K \to \{\pm 1\} \subset L^\times$ be a character. Then, we have $\tilde{c}_1(\chi) = \chi \cup \chi$.

Proof. 1. Since $\chi_\ell$ defines an isomorphism $\pi_1(\text{Spec } \mathbb{Z}[\frac{1}{\ell}])^{ab} \to \mathbb{Z}_\ell^\times$, the profinite group $\pi_1(\text{Spec } \mathbb{Z}[\frac{1}{\ell}])^{ab}$ is isomorphic to the product of $\mathbb{Z}_\ell$ with a cyclic group $C$ of even order. Hence, $H^2(\pi_1(\text{Spec } \mathbb{Z}[\frac{1}{\ell}])^{ab}, \mathbb{Z}/2\mathbb{Z})$ is of order 2 and is generated by the pull-back of the central group of $\text{Spec } \mathbb{Z}[\frac{1}{\ell}]$ by $\mathbb{Z}/2\mathbb{Z}$.

2. Since $\tilde{c}_1(\text{id})$, id $\cup$ id $\in H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ are the unique non-trivial element, we have $\tilde{c}_1(\text{id}) = \text{id} \cup \text{id}$. Hence, we obtain $\tilde{c}_1(\chi) = \chi \cup \chi$. Since $\{a, a\} = \{a, -1\}$ for $a \in K^\times$ corresponding to $\chi \in H^1(K, \mathbb{Z}/2\mathbb{Z})$, the assertion follows.

For a field $K$ of characteristic different from $\ell$, let

\begin{equation}
c_\ell = \tilde{c}_1(\chi_\ell) \in H^2(K, \mathbb{Z}/2\mathbb{Z})
\end{equation}

also denote the pull-back by the canonical map $G_K \to \pi_1(\text{Spec } \mathbb{Z}[\frac{1}{\ell}])^{ab}$. If $K = \mathbb{Q}$, then $c_\ell \in H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ is the unique element ramifying exactly at $\ell$ and $\infty$. We have $c_2 = \{-1, -1\}$ for example. If $K$ is of positive characteristic, we have $c_\ell = 0$.

Let $K$ be a field of characteristic different from 2. We call a pair $(D, b)$ of a $K$-vector space $D$ of finite dimension and a non-degenerate symmetric bilinear form $b: D \otimes_K D \to K$ a quadratic $K$-vector space. The first and the second Hasse-Witt classes $hw_1(D, b) \in H^1(K, \mathbb{Z}/2\mathbb{Z})$ and $hw_2(D, b) \in H^2(K, \mathbb{Z}/2\mathbb{Z})$ are defined as follows [13]. Let $x_1, \ldots, x_r$ be an orthogonal basis of $D$ and put $a_i = b(x_i, x_i)$. The
discriminant of $D$ is defined by \( \text{disc } D = \prod_{i=1}^{r} a_i \in K^x/K^{x2} \). We define the classes by
\[
\text{hw}(D) = \{\text{disc } D\} = \sum_{i=1}^{r} \{a_i\} \quad \text{and} \quad \text{hw}(D) = \sum_{1 \leq i < j \leq r} \{a_i, a_j\}.
\]
They are independent of the choice of an orthogonal basis.

For the orthogonal sum of quadratic vector spaces, we have \( \text{hw}(D \oplus D') = \text{hw}(D) + \text{hw}(D') \) and \( \text{hw}(D \oplus D'') = \text{hw}(D) + \text{hw}(D') \text{hw}(D'') + \text{hw}(D'') \). If we introduce the notation \( \text{hw}(D) = 1 + \text{hw}(D) + \text{hw}(D) \), the equalities are rewritten as \( \text{hw}(D \oplus D') = \text{hw}(D) \cdot \text{hw}(D') \).

**Lemma 1.7.** Let \((D, b)\) be a quadratic \(K\)-vector space.

1. Let \(D^-\) be a totally isotropic subspace of dimension \(r\). Then the restriction of \(b\) to \(D^+ = (D^-)^\perp\) induces a non-degenerate symmetric bilinear form \(b^0\) on \(D^0 = D^+/D^-\) and we have
\[
\text{hw}(D) = \text{hw}(D^0) + r\{-1, \text{disc } D^0\} + \left(\begin{array}{c} r \\ 2 \end{array}\right)\{-1, -1\}.
\]

2. For \(a \in K^x\), we have
\[
\text{hw}(D, a \cdot b) = 1 + \dim D \cdot \{a\} + \left(\begin{array}{c} \dim D \\ 2 \end{array}\right) \cdot \{a, a\} + \text{disc } D + (\dim D - 1)\{a, \text{disc } D\} + \text{hw}(D, b).
\]

**Proof.**
1. The quadratic space \(D\) is isomorphic to the orthogonal direct sum of \(D^0\) with \(r\)-copies of the hyperbolic plane. Hence we have \(\text{hw}(D) = \text{hw}(D^0)(1 + \{-1\})^r\).

2. Clear from the definition. \(\square\)

We generalize the definitions to graded case. Let \(D^* = \bigoplus_{q \in \mathbb{Z}} D^q\) be a graded \(K\)-vector space. For each integer \(q \in \mathbb{Z}\), we assume \(D^q\) is a finite dimensional \(K\)-vector space and \(D^q = 0\) except for finitely many \(q\). We assume that \(D^0\) is a quadratic vector space and that, for each \(q > 0\), \(D^q \oplus D^{-q}\) is equipped with a \((-1)^q\)-symmetric non-degenerate form such that \(D^q\) and \(D^{-q}\) are totally isotropic. We put \(r = \sum_{q < 0} (-1)^q \dim D^q\). Then, we define \(\text{hw}(D^*) \in H^1(K, \mathbb{Z}/2\mathbb{Z})\) and \(\text{hw}(D^*) \in H^2(K, \mathbb{Z}/2\mathbb{Z})\) by
\[
(1.7.1) \quad \text{hw}(D^*) = \text{disc } (D^0) + r \cdot \{-1\}
\]
\[
(1.7.2) \quad \text{hw}(D^*) = \text{hw}(D^0) + r \cdot \{-1, \text{disc } D\} + \left(\begin{array}{c} r \\ 2 \end{array}\right) \cdot \{-1, -1\}.
\]

The definition is equivalent to the equality
\[
1 + \text{hw}(D^*) + \text{hw}(D^*) = (1 + \text{hw}(D^0) + \text{hw}(D^0)) \cdot \prod_{q < 0} (1 + \{-1\})^{(-1)^q \dim D^q}
\]
in \(1 + H^1(K, \mathbb{Z}/2\mathbb{Z}) + H^2(K, \mathbb{Z}/2\mathbb{Z})\).
2. Conjecture

To formulate the conjecture, we prove a preliminary result on the determinant of \( \ell \)-adic cohomology.

**Lemma 2.1.** Let \( S \) be a connected normal scheme, \( f: X \to S \) be a proper smooth morphism and \( q \geq 0 \) be an integer. For a prime number \( \ell \) invertible on \( S \), let \( \chi_\ell : \pi_1(S)^{ab} \to \mathbb{Q}_\ell^\times \) denote the \( \ell \)-adic cyclotomic character.

1. ([21 Corollary 2.2.3]) The rank \( b_{et,q} \) of the smooth \( \mathbb{Q}_\ell \)-sheaf \( R^q f_* \mathbb{Q}_\ell \) is independent of \( \ell \) invertible on \( S \).

2. ([21 Corollary 3.3.5]) The character \( e_q : \pi_1(S)^{ab} \to \mathbb{Q}_\ell^\times \) defined by

\[
e_q = \det R^q f_* \mathbb{Q}_\ell : \chi_\ell^{q-b_{et,q}/2}
\]

is independent of \( \ell \) invertible on \( S \) and takes values in \( \{ \pm 1 \} \). Further if \( q \) is odd, the character \( e_q \) is trivial.

**Proof.**

1. By a standard limit argument, we may assume \( S \) is of finite type over \( \mathbb{Z} \). Then, it follows from the proper base change theorem, the Weil conjecture and [21 Corollary 2.2.3].

2. First, we consider the case where \( S = \text{Spec } k \) for a finite field \( k \). Then by the Weil conjecture proved by Deligne, \( \det(F_r k: H^q(X_k, \mathbb{Q}_\ell)) \) is a rational integer independent of \( \ell \) and is of absolute value \( q^{b_{et,q}/2} \)-th power of Card \( k \). Hence we have \( \det(F_r k: H^q(X_k, \mathbb{Q}_\ell)) = \pm(\text{Card } k)^{q^{b_{et,q}/2}} \). Further by [21 Corollary 3.3.5]), the sign is positive if \( q \) is odd. Thus the assertion follows in this case.

We prove the general case. By replacing \( S \) by a dense open, we may assume \( S \) is affine. By a standard limit argument, we may assume \( S \) is of finite type over \( \mathbb{Z} \). By the Chebotarev density theorem [18 Theorem 7], [20 Theorem 9.11], the reciprocity map \( \bigoplus_{s \in S_0} \mathbb{Z} \to \pi_1(S)^{ab} \) has dense image, where \( S_0 \) denotes the set of closed points in \( S \). Since it is proved for the spectrum of a finite field, it follows that the character \( \det R^q f_* \mathbb{Q}_\ell \) is independent of \( \ell \) and its square is equal to \( \chi_\ell^{-q^{b_{et,q}}} \). Further if \( q \) is odd, the character \( \det R^q f_* \mathbb{Q}_\ell \) itself is equal to \( \chi_\ell^{-q^{b_{et,q}}/2} \). \( \square \)

Let \( X \) be a proper smooth scheme of even dimension \( n \) over a field \( K \). Let \( \ell \) be a prime number different from the characteristic of \( K \). The cup-product defines a non-degenerate symmetric bilinear form on \( V = H^n(X_K, \mathbb{Q}_\ell)(\frac{1}{2}) \). We consider \( V \) as an orthogonal representation of the absolute Galois group \( G_K = \text{Gal}(\bar{K}/K) \) and define

\[
sw_2(H^n_\ell(X)) \in H^2(K, \mathbb{Z}/2\mathbb{Z})
\]

to be its second Stiefel-Whitney class.

Assume the characteristic of \( K \) is not 2. The cup-product defines a non-degenerate symmetric bilinear form also on \( D = H^n_{dR}(X/K) \). We consider \( D \) as a quadratic \( K \)-vector space and define

\[
hw_2(H^n_{dR}(X)) \in H^2(K, \mathbb{Z}/2\mathbb{Z})
\]

to be its second Hasse-Witt class. We also put

\[
d_X = \text{disc } H^n_{dR}(X/K) = hw_1(H^n_{dR}(X)) \in H^1(K, \mathbb{Z}/2\mathbb{Z}).
\]
To state a conjecture on the relation between $sw_2(H^n_ℓ(X))$ and $hw_2(H^n_{dR}(X))$, we introduce auxiliary invariants. For an integer $q$, we put

\begin{equation}
2.2.4 \quad b_{ℓ,q} = \dim H^q(X_ℓ, Q_ℓ) \quad \text{and} \quad b_{dR,q} = \dim H^q_{dR}(X/K).
\end{equation}

If $K$ is of characteristic 0, we have $b_{ℓ,q} = b_{dR,q}$ and we simply write them $b_q$.

**Conjecture 2.3.** Let $X$ be a proper and smooth scheme of even dimension $n$ over a field $K$ of characteristic $\neq 2, ℓ$. For an integer $q \geq 0$, we regard the character $e_q : G_K \to \{±1\}$ \eqref{2.1.1} as an element of $\text{Hom}(G_K, \{±1\}) = H^1(K, \mathbb{Z}/2\mathbb{Z})$ and put $e = \sum_{q<n} e_q$.

\begin{equation}
2.3.1 \quad r = \sum_{q<n} (-1)^q b_{dR,q}, \quad \eta = \sum_{q<n} (-1)^q \left(\frac{n}{2} - q\right) \chi(X, Ω^n_{X/K}).
\end{equation}

If the characteristic of $K$ is 0, we further define

\begin{equation}
2.3.2 \quad \beta = \frac{1}{2} \sum_{q<n} (-1)^q(n - q)b_q.
\end{equation}

Then we have an equality

\begin{equation}
2.3.3 \quad sw_2(H^n_ℓ(X)) + \{e, -1\} + \beta \cdot c_ℓ = hw_2(H^n_{dR}(X))
\end{equation}

in $H^2(K, \mathbb{Z}/2\mathbb{Z})$.

An apparently simpler reformulation of the conjecture will be given at Corollary 2.1. A generalization for a family is stated in Conjecture 7.2 after defining the Stiefel-Whitney class for a symmetric perfect complex in Section 6.

**Remark 2.4.** Let $X$ be a projective smooth scheme of even dimension $n$ over a field $K$ of characteristic $\neq 2, ℓ$. We regard the character $e_n : G_K \to \{±1\}$ defined by $e_n(σ) = \det(σ : H^n(X_ℓ, Q_ℓ) \to H^n(X_ℓ, Q_ℓ))$ as an element of $H^1(K, \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(G_K, \{±1\})$. Then the equality

\begin{equation}
2.4.1 \quad e_n = \{d_X\} + \begin{cases} r \cdot \{-1\} & \text{if } n \equiv 0 \mod 4, \\ (r + b_{dR,n}) \cdot \{-1\} & \text{if } n \equiv 2 \mod 4 \end{cases}
\end{equation}

in $H^1(K, \mathbb{Z}/2\mathbb{Z})$ is proved in [13, Theorem 2]. Conjecture 2.3 is a degree 2 version of the equality \eqref{2.4.1}.

In this paper, we prove the following evidence for Conjecture 2.3.
\textbf{Theorem 2.5.} Let $X$ be a proper and smooth scheme of even dimension $n$ over a field $K$ of characteristic $\neq 2, \ell$. Conjecture 2.3 is true in the following cases.

1. $K$ is a finite extension of $\mathbb{Q}_p$, $p \neq 2, \ell$ and there exists a projective regular flat model $X_{\mathcal{O}_K}$ over the integer ring $\mathcal{O}_K$ such that the closed fiber has at most ordinary double points as singularities.

2. $K$ is a finite extension of $\mathbb{Q}_p$, $p = \ell > n + 1$ and there exists a proper smooth model $X_{\mathcal{O}_K}$ over the integer ring $\mathcal{O}_K$.

3. $K = \mathbb{R}$ and $X$ is projective.

4. $K \supset \mathbb{Q}$.

5. $X$ is a smooth hypersurface in $\mathbb{P}_K^{n+1}$ and $l > n + 1$.

Theorem 2.5 implies that the Stiefel-Whitney class $sw_2(H^n_\ell(X))$ may depend on $\ell$.

\textbf{Example 2.6.} Let $p$ be a prime and $X$ be a proper smooth variety of even dimension $n$ over $K = \mathbb{Q}_p$ with good reduction. Then for any prime $\ell \neq p$, we have $sw_2(H^n_\ell(X)) = hw_2(H^d_{dR}(X)) = 0$ and $d, e \in H^1(\mathbb{Q}_p, \mathbb{Z}/2\mathbb{Z})$ are unramified. Further if $p > 3$ and $A$ is an abelian surface, Theorem 2.5 implies $sw_2(H^2_\ell(A)) = c_p \neq 0$ since $\beta = \frac{1}{2}(2 - 4) \equiv 1 \text{ mod } 2$ and $\eta = 0$.

Similarly, for an abelian surface $A$ over $\mathbb{R}$, we have

$$sw_2(H^2_\ell(A)) \neq hw_2(H^2_{dR}(A)) = 0.$$

If $n = 0$, the assertion 5 in Theorem 2.5 is nothing but the following theorem of Serre since $e = 1$ and $r, \beta, \eta = 0$ in this case. The proof of the assertion 5 gives a new proof of the formula of Serre.

\textbf{Theorem 2.7 ([19, Theorem 1])}. Let $K$ be a field of characteristic $\neq 2$ and $L$ be a finite separable extension of $K$. We consider $V = \mathbb{Q}^{\mathrm{Mor}_K(L, \bar{K})}$ as an orthogonal representation of $G_K$ and $D = L$ as a quadratic $K$-vector space with a non-degenerate symmetric bilinear form $(x, y) \mapsto \mathrm{Tr}_{L/K}(xy)$. Let $d_{L/K} \in H^1(K, \mathbb{Z}/2\mathbb{Z})$ be the discriminant of $D = L$. Then we have

$$sw_2(\mathbb{Q}^{\mathrm{Mor}_K(L, \bar{K})}) = hw_2(L, \mathrm{Tr}_{L/K}(xy)) + \{2, d_{L/K}\}.$$

\textbf{Proof.} We put $X = \text{Spec } L$. Then the $\ell$-adic representation $H^0(X_K, \mathbb{Q}_\ell)$ is the extension of scalars of the continuous representation $\mathbb{Q}^{\mathrm{Mor}_K(L, \bar{K})}$. Hence, the left hand side is equal to $sw_2(H^0_\ell(X))$. Since the cup-product on $L = H^0_{dR}(X/K)$ is the multiplication of $L$ and the trace map $\mathrm{Tr}: H^0_{dR}(X/K) = L \to K$ is the usual trace map for the extension $L$ over $K$, the first term of the right hand side equals $hw_2(H^0_{dR}(X))$. Hence the assertion follows from the case where $n = 0$ of the assertion 5 in Theorem 2.5.

We give some simplified versions of the formula (2.3.3).

\textbf{Lemma 2.8.} Let $hw_2(H^n_{dR}(X)^\prime)$ denote the second Hasse-Witt class of $H^n_{dR}(X)$ with the symmetric bilinear form multiplied by $(-1)^{n/2}$ and $d_X' = hw_1(H^n_{dR}(X)^\prime)$ denote
the discriminant. Then, if \( n \equiv 2 \mod 4 \), we have

\[
(2.8.1) \quad h w_2(H_{dR}^n(X)) = h w_2(H_{dR}^n(X)) + (b_{dR,n} - 1) \cdot \{d_x, -1\} + \left(\frac{b_{dR,n}}{2}\right) \{-1, -1\},
\]

\[
d'_x = d_x + b_{dR,n} \cdot \{-1\}.
\]

**Proof.** It suffices to apply Lemma 1.7.2. □

**Corollary 2.9.** The equality (2.3.3) is equivalent to the following:

\[
(2.9.1) \quad sw_2(H_{dR}^n(X)) + \{e, -1\} + \beta \cdot c_\ell
\]

\[
= hw_2(H_{dR}^n(X)) + r \cdot \{d'_x, -1\} + \left(\frac{r}{2}\right) \{-1, -1\} + \{2, d_x\} + \eta \cdot (c_\ell - c_2)
\]

in \( H^2(K, \mathbb{Z}/2\mathbb{Z}) \).

The equality (2.3.3) is further simplified if we use generalizations of the classes for graded orthogonal representations and graded quadratic forms. Following the definition (1.4.2), we put

\[
H_{q,\ell}(X) = H_q(X, \Omega^\ell_{X/K}),
\]

(2.9.2)

\[
(2.9.2) \quad sw_2(H_{dR}^n(X)) = sw_2(H_{dR}^n(X)) + \sum_{0 \leq q < n} \tilde{e}_1(H_q^\ell(X)).
\]

Similarly, following (1.7.2), we put

\[
(2.9.3) \quad hw_2(H_{dR}^n(X)) = hw_2(H_{dR}^n(X)) + r \cdot \{-1, -1\}
\]

where \( r = \sum_{q<n} (-1)^q b_{dR,q} \) (2.3.1).

**Lemma 2.10.** We have the following equality:

\[
(2.10.1) \quad sw_2(H_{dR}^n(X)) = sw_2(H_{dR}^n(X)) + \{e, -1\} + \beta \cdot c_\ell.
\]

**Proof.** By the definition of \( e_q \), we have \( \det H^q_{\ell}(X) = e_q \cdot \chi_{\ell}^{(n-q)b_{dR,q}/2} \). Hence, we obtain \( \sum_{0 \leq q < n} \tilde{e}_1(H_q^\ell(X)) = \tilde{e}_1(e) + \beta \tilde{e}_1(\chi_{\ell}) \) by the definitions \( e = \sum_{q<n} e_q \) and \( \beta = \frac{1}{2} \sum_{q<n} (-1)^q (n-q) b_{\ell,q} \). Thus, the equality (2.10.1) follows from Lemma 1.6. □

**Corollary 2.11.** The equality (2.3.3) is equivalent to the following:

\[
(2.11.1) \quad sw_2(H_{dR}^n(X)) = hw_2(H_{dR}^n(X)) + \{2, d_x\} + \eta \cdot (c_\ell - c_2).
\]

We compare the de Rham cohomology with the Hodge cohomology. We put \( n = 2m \) and we regard \( H^m(X, \Omega^{2m}_{X/K}) \) as a quadratic vector space by symmetric bilinear form defined by the composition

\[
H^m(X, \Omega^{2m}_{X/K}) \times H^m(X, \Omega^{2m}_{X/K}) \overset{\cup}{\longrightarrow} H^{2m}(X, \Omega^{2m}_{X/K}) \overset{\text{Tr}}{\longrightarrow} K.
\]
Lemma 2.12. The dimensions \( b_{dR,n} = H^n_{dR}(X/K) \) and \( h^{m,m} = H^m(X, \Omega^m_{X/K}) \) have the same parity. We put \( b_{dR,n} = h^{m,m} + 2s \). Then, we have

\[
(2.13.1) \quad hw_2(H^m_{dR}(X/K)) = hw_2(H^m(X, \Omega^m_{X/K})) + s \cdot \{ -1, hw_1(H^m(X, \Omega^m_{X/K})) \} + \left( \frac{s}{2} \right) \cdot \{-1, -1\}
\]

and \( hw_1(H^m_{dR}(X/K)) = hw_1(H^m(X, \Omega^m_{X/K})) + s \cdot \{-1\} \).

Proof. The Hodge to de Rham spectral sequence \( E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^r_{dR}(X/K) \) defines decreasing filtrations \( F^* \) on \( E_1^{m,m} = H^m(X, \Omega^m_{X/K}) \) and on \( H^m_{dR}(X/K) \). We choose numbering of the filtration so that we have \( F^0/F^1 = E^{m,m}_\infty \) and put \( s_1 = \dim F^1 H^m_{dR}(X/K) \) and \( s_2 = \dim F^2 H^m(X, \Omega^m_{X/K}) \). Then, since the spectral sequence is compatible with the Serre duality, we have \( F^0 = (F^1)^\perp \). Hence we have \( s = s_1 - s_2 \) and \( hw(H^m_{dR}(X/K)) = hw(E^{m,m}_\infty)(1 + \{-1\})^{s_1} \) and \( hw(H^m(X, \Omega^m_{X/K})) = hw(E^{m,m}_\infty)(1 + \{-1\})^{s_2} \) by Lemma 2.11. Thus, the assertion follows. \( \square \)

If the characteristic is 0, Conjecture 2.3 may be restated as follows.

Lemma 2.13. Let \( X \) be a proper and smooth scheme of even dimension \( n \) over a field \( K \) of characteristic 0.

1. We put

\[
(2.13.1) \quad r' = \begin{cases} b_0 - b_2 + b_4 - \cdots - b_{n-2} & \text{if } n \equiv 0 \text{ mod } 4 \\ -b_0 + b_2 - b_4 + \cdots - b_{n-2} + b_n & \text{if } n \equiv 2 \text{ mod } 4 \end{cases}
\]

and

\[
(2.13.2) \quad h = \sum_{q<\frac{n}{2}} \left( \frac{n}{2} - q \right) \dim H^{n-q}(X, \Omega^q_{X/K}).
\]

Then in this case, the equality \((2.3.3)\) in Conjecture 2.3 is equivalent to

\[
(2.13.3) \quad sw_2(H^m(X)) + \{ e, -1 \} = hw_2(H^m_{dR}(X)) + \{ 2, d_X \} + h \cdot (c_2 - c_2) + (r' - \frac{n}{2}) \{ d_X, -1 \} + \left( \frac{r'}{2} \right) \{-1, -1\}.
\]

2. Assume further \( X \) is projective. Let \( H^n(X, \mathbb{Q}_\ell)(\frac{n}{2}) = \bigoplus_{q \leq n, \text{even}} P^q \) be the Lefschetz decomposition into primitive parts by an ample invertible sheaf \( L \) and put \( P^+ = \bigoplus_{q<n, q \equiv 0 \text{ mod } 4} P^q \) and \( P^- = \bigoplus_{q<n, q \equiv 2 \text{ mod } 4} P^q \). Then, we have a congruence

\[
(2.13.4) \quad r + 2\beta \equiv -\begin{cases} \dim P^- & \text{if } n \equiv 0 \text{ mod } 4 \\ \dim P^+ & \text{if } n \equiv 2 \text{ mod } 4, \end{cases}
\]

modulo 4 and an equality

\[
(2.13.5) \quad e = \begin{cases} \det P^- & \text{if } n \equiv 0 \text{ mod } 4 \\ \det P^+ & \text{if } n \equiv 2 \text{ mod } 4 \end{cases}
\]
in $H^1(K,\mathbb{Z}/2\mathbb{Z})$.

**Proof.**

1. Since $c_2 = \{-1, -1\}$, it is sufficient to show the congruences

$\beta \equiv \eta + h$  \hspace{1cm} (2.13.6)

$r' \equiv \begin{cases} r & \text{if } n \equiv 0 \mod 4, \\ r + b_n & \text{if } n \equiv 2 \mod 4 \end{cases}$  \hspace{1cm} (2.13.7)

\[
\left( \frac{r'}{2} \right) \equiv \beta + \begin{cases} \left( \frac{r}{2} \right) & \text{if } n \equiv 0 \mod 4, \\ \left( \frac{r+b_n}{2} \right) & \text{if } n \equiv 2 \mod 4 \end{cases}
\]  \hspace{1cm} (2.13.8)

modulo 2. By the Lefschetz principle, we may assume $K = \mathbb{C}$.

We prove (2.13.6). We put $h^{q,p} = \dim H^p(X, \Omega^q_X)$. By the Hodge symmetry and the Serre duality, we have $h^{q,p} = h^{p,q}$ and $h^{n-q,n-p}$ respectively. Since the Hodge to de Rham spectral sequence degenerates, we have $b_q = \sum_{i+j=q} h^{i,j}$. By the definition $\beta = \frac{1}{2} \sum_{q<n} (-1)^q (n - q) b_q$, we have

$$2\beta = \sum_{p+q<n} (1)^{p+q} (n - (p + q)) h^{p,q}$$

$$= \sum_{p+q<n} (1)^{p+q} \left( \frac{n}{2} - p \right) h^{p,q} + \sum_{p+q<n} (1)^{p+q} \left( \frac{n}{2} - q \right) h^{p,q}.$$ 

Hence by the Hodge symmetry and the Serre duality, we obtain

$$\beta = \sum_{p+q<n} (1)^{p+q} \left( \frac{n}{2} - q \right) h^{p,q}$$

$$= \sum_{p+q<n,q<\frac{n}{2}} (1)^{p+q} \left( \frac{n}{2} - q \right) h^{p,q} - \sum_{p+q>n,q<\frac{n}{2}} (1)^{p+q} \left( \frac{n}{2} - q \right) h^{p,q}.$$ 

Thus, we obtain $\beta + h \equiv \sum_{q<n} (1)^p q \left( \frac{n}{2} - q \right) h^{p,q} = \eta \mod 2$.

We prove (2.13.7) and (2.13.8). By Hodge symmetry, it follows that the Betti number $b_q$ is even for odd $q$. Hence by the definition $r = \sum_{q<n} (-1)^q b_q$ and the definition of $\beta$ recalled above, we have

$$r + 2\beta = \sum_{q<n} (-1)^q (n - q + 1) b_q$$  \hspace{1cm} (2.13.9)

$$= \sum_{q<n, even} (n - q + 1) b_q \equiv r' - \begin{cases} 0 & \text{if } n \equiv 0 \mod 4, \\ b_n & \text{if } n \equiv 2 \mod 4 \end{cases}$$ 

modulo 4. Hence the congruence (2.13.7) follows. Since $\left( \frac{a+2b}{2} \right) \equiv \left( \frac{a}{2} \right) + b \mod 2$, the congruence (2.13.8) also follows.

2. By (2.13.9), we have

$$r + 2\beta \equiv \begin{cases} (b_0 - b_2) + \cdots + (b_{n-4} - b_{n-2}) & \text{if } n \equiv 0 \mod 4, \\ -b_0 + (b_2 - b_4) + \cdots + (b_{n-4} - b_{n-2}) & \text{if } n \equiv 2 \mod 4 \end{cases}$$
modulo 4. Since $b_q - b_{q-2} = \dim P^q$ for $2 \leq q \leq n-2$ and $b_0 = \dim P^0$, the congruence (2.13.4) follows.

For odd $q$, $H^q(X_{\bar{K}}, \mathbb{Q}_\ell)$ carries a non-degenerate alternating form by hard Lefschetz and hence we have $e_q = 1$. For even $2 \leq q \leq n-2$, we have $e_q - e_{q-2} = \det P^q$ and $e_0 = \det P^0$. Since

$$e = \begin{cases} (e_0 - e_2) + \cdots + (e_{n-4} - e_{n-2}) & \text{if } n \equiv 0 \mod 4, \\ -e_0 + (e_2 - e_4) + \cdots + (e_{n-4} - e_{n-2}) & \text{if } n \equiv 2 \mod 4, \end{cases}$$

the equality (2.13.5) is also proved. \hfill \Box

3. Degenerations

In this section, we assume that $K$ is a complete discrete valuation field with residue field $F$ of characteristic $p \neq 2$. Let $I = \Gal(\bar{K}/K^ur) \subset G_K = \Gal(\bar{K}/K)$ denote the inertia subgroup and let $P$ denote the kernel of the canonical surjection $I \to \prod_{p' \neq p} \mathbb{Z}_{p'}(1)$. Since $P$ is a pro-$p$ group, the canonical map $H^q(I, \mathbb{Z}/2\mathbb{Z}) \to H^q(I, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism and they are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $q = 0, 1$ and is 0 for $q > 1$. Since the extension $1 \to I/P \to G_K/P \to \tilde{G}_F \to 1$ splits, we have an exact sequence

$$(3.0.1) \quad 0 \longrightarrow H^2(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(K, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial} H^1(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0.$$

In this section, we prove that the both sides of (2.3.3) in Conjecture 2.3 have the same images by the map $\partial$ in some cases. In particular, if $H^2(F, \mathbb{Z}/2\mathbb{Z}) = 0$ for example if the residue field $F$ is finite, this will imply Conjecture 2.3 in these cases.

First, we consider a consequence of the following elementary lemma.

Lemma 3.1. Let $K$ be a complete discrete valuation field and assume that the characteristic $p$ of the residue field $F$ is not 2.

1. Let $L$ be a finite extension of $\mathbb{Q}_\ell$ and $V$ be an orthogonal $L$-representation of $G_K = \Gal(\bar{K}/K)$. If the inverse image $I' \subset I$ of the pro-$2$ part of $\mathbb{Z}_2(1) \subset \prod_{p' \neq p} \mathbb{Z}_{p'}(1)$ acts trivially on $V$, then we have $\partial sw_2(V) = 0$.

2. Let $D$ be a quadratic $K$-vector space. If $D$ has a non-degenerate $\mathcal{O}_K$-lattice, then we have $\partial hw_2(D) = 0$.

Proof. 1. The class $sw_2(V)$ is in the image of $H^2(G_K/I', \mathbb{Z}/2\mathbb{Z})$. Since $I/I'$ has no pro-$2$ part, the canonical map $H^2(F, \mathbb{Z}/2\mathbb{Z}) = H^2(G_K/I, \mathbb{Z}/2\mathbb{Z}) \to H^2(G_K/I', \mathbb{Z}/2\mathbb{Z})$ is an isomorphism. Hence the assertion follows from the exact sequence (3.0.1).

2. The class $hw_2(D)$ lies in the image $H^2(F, \mathbb{Z}/2\mathbb{Z}) \subset H^2(K, \mathbb{Z}/2\mathbb{Z})$ of $\{\mathcal{O}_K^\times, \mathcal{O}_K^\times\}$. \hfill \Box

We say a scheme $X$ over $S = \Spec \mathcal{O}_K$ is generalized semi-stable if étale locally on $X$, it is étale over $\Spec \mathcal{O}_K[T_0, \ldots, T_n]/(T_0^{m_0} \cdots T_r^{m_r} - \pi)$ for some integer $0 \leq r \leq n$, a prime element $\pi$ of $K$ and integers $m_0, \ldots, m_r \geq 1$ invertible in $\mathcal{O}_K$. A scheme $X$ is semi-stable if and only if it is generalized semi-stable and if the closed fiber $X_s$ is reduced.

Let $X$ be a generalized semi-stable scheme over $S$. Let $\ell$ be a prime number invertible on $S$ and $R^q\psi\mathbb{Z}_\ell$ be the sheaf of nearby cycles. Then, for each geometric
point $\bar{x}$ of the geometric closed fiber $X_s$, the action of $I$ on the stalk $R^q\psi\mathcal{Z}_{d,\bar{x}}$ is tamely ramified for every $q \geq 0$. More precisely, if $m_1, \ldots, m_r$ are the multiplicities of irreducible components of the closed fiber $X_{\bar{x}} \times_S \text{Spec } F = \sum_i m_i D_i$ of the strict henselization and if $d_\ell$ denotes the greatest common divisor, the inertia $I$ acts on $R^q\psi\mathcal{Z}_{d,\bar{x}}$ through the quotient $\mu_{d_\ell}$ [14 Proposition 6].

The sheaf $\Omega^1_{X/S}(\log / \log)$ of logarithmic differential 1-forms is étale locally defined by patching $(\bigoplus_{i=0}^r \mathcal{O}_X d\log T_i \oplus \bigoplus_{r+1}^n \mathcal{O}_X dT_i)/(\sum_i m_i d\log T_i)$. In the following, we write

$$A^1_{X/S} = \Omega^1_{X/S}(\log / \log)$$

and

$$A^q_{X/S} = \wedge^q_{\mathcal{O}_X} A^1_{X/S}$$

for short. The $\mathcal{O}_X$-module $A^1_{X/S}$ is locally free of rank $n = \dim X_K$. The canonical map $\Omega^1_{X/S} \to A^1_{X/S}$ induces an isomorphism on the generic fiber $X_K$. It also induces an isomorphism $\omega_{X/S}(X_{s,\text{red}} - X_s) \to A^n_{X/S}$ where $\omega_{X/S} = \det \Omega^1_{X/S}$ denotes the relative dualizing sheaf.

**Lemma 3.2.** Let $X$ be a proper generalized semi-stable scheme over $S = \text{Spec } \mathcal{O}_K$. Assume that every irreducible component of the closed fiber $X_s = X \times_S \text{Spec } F$ has odd multiplicity in $X_s$.

1. The inverse image $I' \subset I$ of the pro-$2$ part of $\mathbb{Z}_2(1) \subset \prod_{p' \neq p} \mathbb{Z}_{p'}(1)$ acts trivially on $H^q(X_K, \mathbb{Q}_\ell)$ for every $q$.

2. We define an effective Cartier divisor $D$ by $2D = X_s - X_{s,\text{red}}$. Then, the image of $H^m(X, A^n_{X/S}(D))$ defines a non-degenerate $\mathcal{O}_K$-lattice of $H^m(X_K, \Omega^m_{X_K/K})$.

**Proof.** 1. By the assumption, the inverse image $I' \subset I$ of the pro-$2$ part acts trivially on $R^q\psi\mathcal{Z}_{d,\bar{x}}$ for every $q$. Hence, it follows from the spectral sequence $E^p,q_2 = H^p(X_{\bar{x}}, R^q\psi\mathcal{Z}_{d,\bar{x}}) \Rightarrow H^{p+q}(X_K, \mathbb{Q}_\ell)$.

2. By the assumption $X_s - X_{s,\text{red}} = 2D$, we have an isomorphism $\omega_{X/S}(-2D) \to A^n_{X/S}$. It induces an isomorphism $A^n_{X/S}(D) \to \text{Hom}(A^n_{X/S}(D), \omega_{X/S})$. By Grothendieck duality, it induces an isomorphism

$$H^m(X, A^n_{X/S}(D))/(\text{torsion part}) \to \text{Hom}_{\mathcal{O}_K}(H^m(X, A^n_{X/S}(D)), \mathcal{O}_K).$$

Hence the assertion follows. \hfill $\square$

**Corollary 3.3.** Let $X$ be a proper generalized semi-stable scheme over $S = \text{Spec } \mathcal{O}_K$ satisfying the condition in Lemma 3.2. Then, for a prime number $\ell$ invertible in $\mathcal{O}_K$, we have

$$\partial \omega_2(H^m_l(X_K/K)) = \partial w_2(H^m_{dR}(X_K/K)) = 0$$

in $H^1(F, \mathbb{Z}/2\mathbb{Z})$.

In particular, further if the residue field $F$ is finite, Conjecture 2.3 is true.

**Proof.** By Lemmas 3.2.1 and 3.1.1, we have $\partial \omega_2(H^m_l(X_K/K)) = 0$. By Lemmas 3.2.2 and 3.1.2, we have $\partial w_2(H^m(X_K, \Omega^m_{X_K/K})) = 0$. Further by Lemma 2.1.2, we obtain $\partial w_2(H^m_{dR}(X_K/K)) = 0$.

Assume $F$ is finite. Then, the map $\partial : H^2(K, \mathbb{Z}/2\mathbb{Z}) \to H^1(F, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism. Since $2, d, -1$ are units in $\mathcal{O}_K$, the terms $\{2, d\}, \{d, -1\}, \{-1, -1\}$ in (2.3.3) are $0$. Hence the equality (3.3.1) implies Conjecture 2.3. \hfill $\square$
In the rest of this section, we prove the assertion 1 in Theorem 2.5.

**Lemma 3.4.** Let $K$ be a complete discrete valuation field with residue field $F$ of characteristic different from 2. Let $K'$ be a totally ramified extension of $K$ of degree 2 and $I' \subset G_{K'}$ be the inertia subgroup. Let $\chi_{K'}/K' \in H^1(K, \mathbb{Z}/2\mathbb{Z}) = K^*/K^{*2}$ be the quadratic character of $G_K$ corresponding to $K'$ or equivalently the discriminant of the quadratic extension $K'$.

1. Let $V$ be an orthogonal representation of $G_{K}/I' = G_F \times I/I'$ and let $V = V_0 \oplus (V_1 \otimes \chi_{K'}/K)$ be the decomposition into the unramified part $V_0$ and the ramified part $V_1 \otimes \chi_{K'}/K$. Let $r$ be the degree of the representation $V_1$ of $G_F = G_{K}/I$.

Then we have

$$\partial sw_2(V) = \begin{pmatrix} r \\ 2 \end{pmatrix} \{-1\} + \det V_1 \begin{pmatrix} 0 \\ \det V/\chi_{K'}/K \end{pmatrix} \text{ if } r \text{ is even}$$

$$\text{if } r \text{ is odd.}$$

2. Let $(D, b)$ be a quadratic $K$-vector space and $L$ be an $O_K$-lattice of $D$ satisfying $m_K L^* \subset L \subset L^* = \{ x \in D \mid b(x, y) \in O_K \text{ for } y \in L \}$. Then, there exists a unique non-degenerate $O_{K'}$-lattice $L'$ of $D_{K'} = D \otimes_K K'$ satisfying $m_{K'} L' \subset O_{K'} L \subset L'$. The bilinear form $b$ induces a non-degenerate form on the $F$-vector space $L_1 = L'/O_{K'} L$.

We put $r = \dim L_1$.

Then we have

$$\partial hw_2(D) = \begin{pmatrix} r \\ 2 \end{pmatrix} \{-1\} + \text{disc } \bar{L}_1 + \begin{pmatrix} 0 \\ \text{disc } D/d_{K'}/K \end{pmatrix} \text{ if } r \text{ is even}$$

$$\text{if } r \text{ is odd.}$$

**Proof.** 1. Since $sw(V) = sw(V_0) \cdot sw(V_1 \otimes \chi_{K'}/K)$, we have

$$\partial sw_2(V) = \partial sw_1(V_1 \otimes \chi_{K'}/K) \cdot sw_1(V_0) + \partial sw_2(V_1 \otimes \chi_{K'}/K).$$

We have $\partial sw_1(V_1 \otimes \chi_{K'}/K) = r \in H^0(F, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. By Corollary 1.5, we have $\partial sw_2(V_1 \otimes \chi_{K'}/K) = (r - 1) \det(V_1) + \binom{r}{2} \{-1\}$ since $\partial \chi_{K'/K} = 1$ in $H^0(F, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $\chi_{K'/K} \cup \chi_{K'/K} = \chi_{K'/K} \cup \{-1\}$ in $H^2(K, \mathbb{Z}/2\mathbb{Z})$. If $r$ is odd, we have $sw_1(V_0) = \det V/\det V_1 \chi_{K'/K}$. Hence the assertion follows.

2. Since the canonical map $\bar{L}_0 = L/m_K L^* \rightarrow L/m_K L^* = \text{Hom}_F(L/m_K L, F)$ is injective, $b$ induces a non-degenerate form on $\bar{L}_0$. Take a direct summand $L_0$ of $L$ such that $L_0/m_K L_0 = \bar{L}_0$ and put $L_1 = L_0^\perp = \{ x \in L \mid b(x, y) = 0 \text{ for } y \in L_0 \}$. Then, $L_0$ is non-degenerate and we have $L = L_0 \oplus L_1$ and $L^* = L_0 \oplus m_K^{-1} L_1$.

The conditions $m_K L' \subset O_{K'} L \subset L'$ and that $L'$ is non-degenerate imply $L' \subset O_{K'} L^* \subset m_K L^*$ and hence $O_{K'} L + m_K L^* \subset L' \subset O_{K'} L^* \cap m_K L$. Since $O_{K'} L + m_K L^*$ and $O_{K'} L' \cap m_K L$ are both equal to $O_{K'} L_0 \oplus m_K^{-1} L_1$, the uniqueness of $L'$ follows. It is clear that $L' = O_{K'} L_0 \oplus m_K^{-1} L_1$ is non-degenerate. Since $L'/m_K L'$ is the orthogonal direct sum $\bar{L}_0 \oplus \bar{L}_1$, the bilinear form $b$ induces non-degenerate forms on $\bar{L}_0$ and $\bar{L}_1$.

We put $D_0 = KL_0$ and $D_1 = KL_1$. Since $hw(D) = hw(D_0) \cdot hw(D_1)$, we have

$$\partial hw_2(D) = \partial hw_2(D_0) \cdot hw_2(D_1) + \partial hw_2(D_1).$$

We have $\partial hw_1(D_1) = r \in H^0(F, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Since $L_1$ is non-degenerate with respect to the restriction of $\pi^{-1} b$ and since $\{\pi, \pi\} = \{\pi, -1\}$, we have $\partial hw_2(D_1) = \partial hw_1(D_1) = 0$.
The following Lemma is inspired by [11].

**Lemma 3.5.** Let $X$ be a proper generalized semi-stable scheme over $S = \text{Spec } \mathcal{O}_K$. Assume that the divisor $D = X_s - X_{s,\text{red}}$ is smooth over $F$ and that the complement $X \setminus D$ is smooth over $S$. Let $K'$ be a totally ramified extension of degree 2 and put $S' = \text{Spec } \mathcal{O}_{K'}$.

1. The normalization $X'$ of the base change $X \times_S S'$ is semi-stable. The reduced inverse image $D' = (D \times_X X')_{\text{red}}$ is smooth over $F$. The double covering $\varphi : D' \to D$ is étale outside $C = D \cap (X_s - 2D)$ and is totally ramified along $C$.

2. Let $\ell$ be a prime number invertible in $\mathcal{O}_K$ and $I' \subset G_{K'} \subset G_K$ be the inertia subgroup. Then, the Galois representation on $H^q(X_{K'}, \mathbb{Q}_\ell)$ factors through the quotient $G_K/I'$.

Let $H^q(X_{K'}, \mathbb{Q}_\ell)^- \subset H^q(X_{K'}, \mathbb{Q}_\ell)$ and $H^q(D'_F, \mathbb{Q}_\ell)^- \subset H^q(D'_F, \mathbb{Q}_\ell)$ be the minus-parts with respect to the actions of $I/I'$. Then, the cospecialization map $H^q(X_{K'}, \mathbb{Q}_\ell) \to H^q(X_{K'}, \mathbb{Q}_\ell)^-$ induces an isomorphism $H^q(D'_F, \mathbb{Q}_\ell)^- \to H^q(X_{K'}, \mathbb{Q}_\ell)^-.$

3. Assume that $X_K$ is of even dimension $n = 2m$. Then, the canonical map $H^m(X', A^m_{X'/S'}) \otimes \mathcal{O}_{K'}$ $\to H^m(X'_F, A^m_{X'_F/F})$ is injective and the image is the orthogonal of the image of the torsion part $H^m(X', A^m_{X'/S'})^\text{tors}.$

Let $H^m(X'_F, A^m_{X'_F/F}) = H^m(X'_F, A^m_{X'_F/F})^+ \oplus H^m(X'_F, A^m_{X'_F/F})^-$ be the decomposition into plus and minus part with respect to the action of $I/I' = \text{Gal}(K'/K)$. Then, the image of $H^m(X, A^m_{X/S}) \otimes \mathcal{O}_K$ in the plus-part $H^m(X'_F, A^m_{X'_F/F})^+$ is equal to the image of $H^m(X', A^m_{X'/S'}) \otimes \mathcal{O}_{K'}$ $\subset F$ and the image of $H^m(X, A^m_{X/S}) \otimes \mathcal{O}_K$ in the minus-part $H^m(X'_F, A^m_{X'_F/F})^-$ is $H^m(D'_F, \Omega^m_{D'/F})^{-}$.

**Proof.** 1. Since the assertion is étale local on $X$, we may assume $X$ is smooth over $A = \mathcal{O}_K[x, y]/(xy^2 - \pi)$ and $K' = K(\sqrt{\pi})$ for a prime element $\pi$ of $K$. Then, the normalization of $A \otimes \mathcal{O}_K \mathcal{O}_{K'}$ is $\mathcal{O}_{K'}[x', y]/(x'y - \sqrt{\pi})$ where $x = x^2$ and the assertion follows.

2. We consider the spectral sequence $E_2^{p, q} = H^p(X_F, R^q\psi\mathbb{Q}_\ell) \Rightarrow H^{p+q}(X_{K'}, \mathbb{Q}_\ell).$ We recall computation of $R^0\psi\mathbb{Q}_\ell$ from [14, Proposition 6]. We have $R^q\psi\mathbb{Z}_\ell = 0$ for $q > 1.$ Further, the restriction $R^0\psi\mathbb{Z}_\ell|_D$ is canonically isomorphic to $\varphi_*\mathbb{Z}_\ell|_{D'}$, and the restriction $R^0\psi\mathbb{Z}_\ell|_{X_s - 2D}$ is canonically isomorphic to $\mathbb{Z}_\ell|_{(X_s - 2D)}$. The isomorphism is compatible with the actions of $I$. The sheaf $R^1\psi\mathbb{Z}_\ell$ is the direct image of a locally constant sheaf of rank 1 on the intersection $C = D \cap (X_s - 2D)$ with the trivial action of $I$.

Hence, the inertia action on $H^p(X_F, R^0\psi\mathbb{Q}_\ell)$ is trivial for $q \neq 0$. For $q = 0$, the action of $I'$ on $H^p(X_F, R^0\psi\mathbb{Q}_\ell)$ is trivial. Further, the minus part of $H^p(X_F, R^0\psi\mathbb{Q}_\ell)$ is isomorphic to $H^p(D'_F, \mathbb{Q}_\ell)^-$. 
3. We consider the commutative diagram

\[
0 \to H^m(X', A^m_{X'/S'}) \otimes \mathcal{O}_{K'} \to H^m(X'_F, A^m_{X'_F/F}) \to \text{Tor}_1^{\mathcal{O}_{K'}}(H^{m+1}(X', A^m_{X'/S'}), F) \to 0
\]

\[
\downarrow \hspace{1cm} \downarrow
\]

\[
H^m(X'_F, A^m_{X'_F/F})^\vee \to (H^m(X', A^m_{X'/S'})_{\text{tors}} \otimes \mathcal{O}_{K'}, F)^\vee.
\]

The upper line is exact and $\vee$ denote the $F$-linear dual. By Grothendieck duality, the vertical arrows are isomorphisms. Thus, the first paragraph is proved.

We consider the decomposition $\mathcal{O}_{D'} = \mathcal{O}_D \oplus \mathcal{L}$ by the action of the Galois group $\text{Gal}(K'/K) = \text{Gal}(D'/D)$. The computation in the proof of 1. shows that we also have a decomposition $\mathcal{O}_{X'_F} = \mathcal{O}_{X_{F,\text{red}}} \oplus \mathcal{L}$. We put $A^m_{D'/F} = \Omega^m_{D/F}(\log C)$ and $A^m_{D'/F} = \Omega^m_{D'/F}(\log C)$. The canonical map $A^m_{D'/F} \otimes_{\mathcal{O}_D} \mathcal{O}_{D'} \to A^m_{D'/F} = A^m_{D/F} \oplus (A^m_{D'/F} \otimes_{\mathcal{O}_D} \mathcal{L})$ is an isomorphism. Since the sequences $0 \to \Omega^m_{D'/F} \to A^m_{D'/F} \to \Omega^m_{C'/F} \to 0$ and $0 \to \Omega^m_{D'/F} \to A^m_{D'/F} \to \Omega^m_{C'/F} \to 0$ are exact, the inclusion $\Omega^m_{D'/F} \to A^m_{D'/F}$ induces an isomorphism $\Omega^m_{D'/F} \to A^m_{D'/F} = A^m_{D/F} \otimes_{\mathcal{O}_D} \mathcal{L}$ on the minus parts. Since the canonical map $\frac{A^m_{X'/S'}}{\mathcal{O}_{X'}} \otimes_{\mathcal{O}_D} \mathcal{O}_{D'} \to A^m_{D'/F}$ is an isomorphism, it is also identified with $A^m_{X'/F}$. Thus, we have $H^m(X'_F, A^m_{X'_F/F})^\vee = H^m(D, \Omega_{D'/F})^\vee$.

Further the computation in the proof of 1. shows that the canonical map $\text{Coker}(\mathcal{O}_{X \times S'} \to \mathcal{O}_{X'}) \to \text{Coker}(\mathcal{O}_D \to \mathcal{O}_{D'}) = \mathcal{L}$ is an isomorphism. Hence, we have exact sequences

\[
0 \to A^m_{X'/S'} \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times S'} \to A^m_{X'/S'} \otimes_{\mathcal{O}_{D'/F}} \to 0
\]

\[
\text{and} \quad H^m(X, A^m_{X'/S}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \to H^m(X', A^m_{X'/S'}) \to H^m(D', \Omega_{D'/F}) \to \ldots
\]

Thus, we obtain an exact sequence

\[
H^m(X, A^m_{X'/S}) \otimes_{\mathcal{O}_K} F \to H^m(X', A^m_{X'/S'}) \otimes_{\mathcal{O}_K}, F \to H^m(D', \Omega_{D'/F}) \to \ldots
\]

and the assertion in the second paragraph follows. 

\[\square\]

**Proposition 3.6.** Let the notation be as in Lemma 3.5.

1. The following conditions are equivalent:
   
   (1) The determinant det $H^n(X_K, \mathbb{Q}_\ell)$ is unramified.
   
   (2) The discriminant $d_X = \text{disc } H^n_{\text{dR}}(X/K) \in K^\times/K^\times$ is in the image of $F^\times/F^\times$.
   
   (3) The Euler number $\chi(D_F, \mathbb{Q}_\ell) = \chi_{\text{dR}}(D/F)$ is even.

2. Let $\chi^-(D')$ denote the Euler number $\sum_q (-1)^q \dim H^q(D'_F, \mathbb{Q}_\ell)^-$ of the minus-part and let $\chi_{K'/K} \in H^1(K, \mathbb{Z}/2\mathbb{Z})$ denote the character of order 2 corresponding to the quadratic extension $K'$ over $K$. Then, the image of the left hand side $s_{w_2}(H^n_{\ell}(X)) + \{e, -1\} + \beta \cdot c_\ell$ of (2.3.3) by $\partial$ is equal to

\[
\left(\frac{\chi^-(D')}{2}\right) \{-1\} + \det \left( H^n(D'_F, \mathbb{Q}_\ell)^- \left( \frac{n}{2} \right) \right)
\]

\[
+ \begin{cases} 
0 & \text{if } \det H^n(X_K, \mathbb{Q}_\ell) \text{ is unramified,} \\
\det H^n(X_K, \mathbb{Q}_\ell)/\chi_{K'/K} & \text{if } \det H^n(X_K, \mathbb{Q}_\ell) \text{ is ramified}
\end{cases}
\]

in $H^1(F, \mathbb{Z}/2\mathbb{Z})$. 


3. Let \( h^{m,m-}(D') \) denote the dimension dim \( H^m(D'_F, \Omega^m_{D'/F})^- \) of the minus-part and let \( d_{K'/K} \in H^1(K, \mathbb{Z}/2\mathbb{Z}) \) denote the discriminant of the quadratic extension \( K' \) over \( K \). Let \( H^n_{dR}(D'/F)' \) denote \( H^n_{dR}(D'/F) \) with the symmetric bilinear form multiplied by \((-1)^{n/2}\) and let \( H^n_{dR}(D'/F)'^- \) denote its minus part with respect to the action of Gal\((D'/D)\). Let \( r \) be as in (2.3.1).

Then, the image of the right hand side of (2.3.3) by \( \partial \) is equal to

\[
\begin{align*}
(3.6.2) \quad & \left( \frac{b^-_{n,dR}(D')}{2} \right) \{ -1 \} + \text{disc } H^n_{dR}(D'/F)'^- + \chi_{dR}(D/F) \cdot \{ 2 \} \\
& + \begin{cases} 
0 & \text{if } d_X \in F^x/F^{x^2}, \\
(d'_X - d_{K'/K}) + r\{ -1 \} & \text{if } d_X \notin F^x/F^{x^2}
\end{cases}
\end{align*}
\]

in \( H^1(F, \mathbb{Z}/2\mathbb{Z}) \).

**Proof.** 1. By Lemma 3.5.2, the condition (1) is equivalent to that dim \( H^n(D'_F, \mathbb{Q}_\ell^-) \) is even. Hence, it is equivalent to that the Euler number \( \chi(D'_F, \mathbb{Q}_\ell^-) \) is even. Since \( \chi(D'_F, \mathbb{Q}_\ell^-) = \chi(D_F, \mathbb{Q}_\ell^-) - \chi(C_F, \mathbb{Q}_\ell) \) and since the Euler number \( \chi(C_F, \mathbb{Q}_\ell) \) is even for proper smooth scheme \( C_F \) of odd dimension, it is equivalent to the condition (3).

By Lemma 3.5.3, the condition (2) is equivalent to that dim \( H^n(D', \Omega^m_{D'/F})^- \) is even. By Serre duality and the Hodge to de Rham spectral sequence, it is equivalent to that the Euler number \( \chi_{dR}(D'/F)^- \) is even. Since \( \chi_{dR}(D'/F)' = \chi_{dR}(D/F) - \chi_{dR}(C/F) \) and since the Euler number \( \chi_{dR}(C/F) \) is even for proper smooth scheme \( C \) of odd dimension, it is also equivalent to the condition (3).

2. We put \( b^-_n = \dim H^n(D'_F, \mathbb{Q}_\ell^-) \). By Lemmas 3.4.1 and 3.5.2, we have

\[
\partial(sw_2(H^n_{\ell}(X))) = \left( \frac{b_n}{2} \right) \{ -1 \} + \text{det } H^n(D'_F, \mathbb{Q}_\ell^-) \left( \frac{n}{2} \right) \\
& + \begin{cases} 
0 & \text{if } b^-_n \text{ is even}, \\
\text{det } H^n(X_K, \mathbb{Q}_\ell^-) \left( \frac{n}{2} \right) /d_{K'/K} & \text{if } b^-_n \text{ is odd}.
\end{cases}
\]

Further by Lemma 3.5.2, the character \( \text{det } H^n(X_K, \mathbb{Q}_\ell^-) \) is unramified if and only if \( b^-_n \) is even. We put \( r^- = \sum_{q<n}(-1)^q \dim H^q(D'_F, \mathbb{Q}_\ell^-) \). Then, further by Lemma 3.5.2, the character \( e = \prod_{q<n} \left( \text{det } H^q(X_K, \mathbb{Q}_\ell) \right)^{-1} \) is unramified if and only if \( r^- \) is even. Hence, we have \( \partial\{ e, -1 \} = r^- \{ -1 \} \). Since \( \chi^-(D') = b^-_n + 2r^- \), we have \( \left( \chi^-(D') \right) = \left( \frac{b^-_n}{2} \right) + r^- \). Since \( \partial(c_\ell) = 0 \), the assertion follows.

3. By (2.9.3), the right hand side of (2.3.3) is equal to \( hw_2(H^*_{dR}(X_K/K)) + \{ 2, d_X \} + \eta(c_\ell - c_2) \) as in Corollary 2.11. Let \( H^m(X_K, \Omega^m_{X/K})' \) denote the symmetric \( K \)-vector space \( H^m(X_K, \Omega^m_{X/K}) \) with the symmetric bilinear form multiplied by \((-1)^m\). We put \( h^{m,m} = H^m(X_K, \Omega^m_{X/K}) \) and define \( r' \) by \( \chi_{dR}(X/K) = h^{m,m} + 2r' \).
Then, by Lemma 2.12, we have

\[ \partial(hw_2(H_{dR}^*(X_K/K))) = \partial(hw_2(H^m(X_K, \Omega^m_{X/K}'))) + \begin{cases} 0 & \text{if } \disc H^m(X_K, \Omega^m_{X/K}) \in F^\times/F^\times 2 \\ r' \cdot \{ -1 \} & \text{if } H^m(X_K, \Omega^m_{X/K}) \notin F^\times/F^\times 2. \end{cases} \]

The condition \( \disc H^m(X_K, \Omega^m_{X/K}) \in F^\times/F^\times 2 \) is equivalent to \( d_X = \disc H^m_{dR}(X/K) \in F^\times/F^\times 2 \). It is further equivalent to that \( h_{m,m}^{-}(D') \) is even.

We apply Lemma 3.4.2. to \( H^m(X_K, \Omega^m_{X/K})' \) and define \( \tilde{L}_1 \) as in Lemma 3.4.2. Then, we obtain

\[ \partial(hw_2(H^m(X_K, \Omega^m_{X/K}'))) = \left( \dim \tilde{L}_1 \right) \{ -1 \} + \disc \tilde{L}_1 \]

\[ + \begin{cases} 0 & \text{if } h_{m,m}^{-}(D') \text{ is even} \\ \disc H^m(X_K, \Omega^m_{X/K}')/dK'/K & \text{if } h_{m,m}^{-}(D') \text{ is odd}. \end{cases} \]

We put \( s = \dim H^m(X', A_{X'/S}^{\text{tors}} \otimes K, F) \). Then, we have \( h_{m,m}^{-}(D') = \dim \tilde{L}_1 + 2 \cdot s \) and \( \disc H^m(D', \Omega_{D'/F}^{-}) = \disc \tilde{L}_1 + s \cdot \{ -1 \} \) by Lemma 3.5.3. Hence, for the right hand side, we have

\[ \left( \dim \tilde{L}_1 \right) \{ -1 \} + \disc \tilde{L}_1 = \left( \frac{h_{m,m}^{-}(D')}{2} \right) \{ -1 \} + \disc H^m(D', \Omega_{D'/F}^{-}). \]

It is further equal to \( \left( h_{m,m}^{-}(D') \right) \{ -1 \} + \disc H^m_{dR}(D'/F)' \) by Lemma 2.12.

We have \( \partial(\ell, 2) = h_{m,m}^{-}(D') \cdot \{ 2 \} \) and \( h_{m,m}^{-}(D') \equiv \chi^{-}(D') \mod 2 \). Further, as in the end of the proof of 1., we have \( \chi^{-}(D') \equiv \chi_{dR}(D/F) \mod 2 \). We also have \( \partial(c_1) = \partial(c_2) = 0 \). Since \( \disc H^m(D_K, \Omega^m_{X/K})' - \disc H^m_{dR}(X_K/K)' = (r' - r') \{ -1 \} \), the assertion follows.

**Lemma 3.7.** Let \( D \) be a projective smooth scheme of even dimension \( n \) over a field \( F \) of characteristic \( \neq 2 \). Let \( D' \to D \) be a double covering ramified along a smooth divisor \( C \subset D \). Let \( H^m_{dR}(D'/F)' \) be the quadratic \( F \)-vector space \( H^m_{dR}(D'/F) \) with symmetric bilinear form multiplied by \(( -1 )^{n/2} \) and let \( H^m_{dR}(D'/F)' \) denote the minus part with respect to the action of \( \text{Gal}(D'/D) = \{ \pm 1 \} \). We put \( r_{D'}^{-} = \sum_{q \leq n} (-1)^q \dim H^q_{dR}(D'/F) \).

Then, we have

\[ (3.7.1) \quad \det H^n(D'_F, \mathbb{Q}_\ell)^- = \disc H^n_{dR}(D'/F)' + r_{D'}^{-} \{ -1 \} + \chi_{dR}(D/F) \cdot \{ 2 \} \]

in \( H^1(F, \mathbb{Z}/2\mathbb{Z}) \).

**Proof.** We put \( r_D = \sum_{q \leq n} (-1)^q \dim H^q_{dR}(D/F) \) and \( r_{D'} = \sum_{q \leq n} (-1)^q \dim H^q_{dR}(D'/F) \).

Let \( H^n_{dR}(D/F)' \) denote the quadratic \( F \)-vector space \( H^n_{dR}(D/F) \) with symmetric bilinear form multiplied by \(( -1 )^{n/2} \). Then, by [13 Theorem 2], we have \( \det H^n(D_F, \mathbb{Q}_\ell) = \disc H^n_{dR}(D/F)' + r_D \{ -1 \} \) and \( \det H^n(D'_F, \mathbb{Q}_\ell) = \disc H^n_{dR}(D'/F)' + r_{D'} \{ -1 \} \). On the left hand sides, we have

\[ \det H^n(D'_F, \mathbb{Q}_\ell) = \det H^n(D_F, \mathbb{Q}_\ell) \cdot \det H^n(D'_F, \mathbb{Q}_\ell)^- \]
since \( H^n(D'_F, \mathbb{Q}_\ell) = H^n(D_F, \mathbb{Q}_\ell) \oplus H^n(D'_F, \mathbb{Q}_\ell) \). On the other hand, since \( H^\text{dR}_n(D'/F) = H^\text{dR}_n(D/F) \oplus H^\text{dR}_n(D'/F) \), we have \( r_{D'} = r_D + r_{D'} \). Further since the restriction of the symmetric bilinear form of \( H^\text{dR}_n(D'/F) \) on \( H^\text{dR}_n(D/F) \subset H^\text{dR}_n(D'/F) \) is 2-times that of \( H^\text{dR}_n(D/F) \), we have

\[
\text{disc } H^\text{dR}_n(D'/F) = 2^{\chi(D)} \cdot \text{disc } H^\text{dR}_n(D/F) \cdot \text{disc } H^\text{dR}_n(D'/F).
\]

Hence the assertion follows.

Corollary 3.8. Let the notation be as in Lemma 3.5. Assume \( D \) and \( X_K \) are projective. Then, the both sides of (2.3.3) have the same images by \( \partial \). In particular, further if the residue field \( F \) is finite, Conjecture 2.3 is true in this case.

Proof. We compare the sums of the terms in the first lines in (3.6.1) and (3.6.2). Since \( \chi^-(D') = b_{n,dR}(D') + 2r_{D'} \), they are equal to each other by Lemma 3.7. In the case where the equivalent conditions in Proposition 3.6.1 do not hold, the remaining terms are also equal to each other by [15, Theorem 2].

Corollary 3.9. Let \( K \) be a complete discrete valuation field with residue field \( F \) of characteristic \( \neq 2, \ell \). Let \( X_{O_K} \) be a proper regular flat scheme over a discrete valuation ring \( O_K \). Assume that the generic fiber \( X_K \) is projective and smooth of even dimension and that the closed fiber \( X_s \) has at most ordinary double points as singularities.

Then, the both sides of (2.3.3) have the same images by \( \partial \). In particular, further if the residue field \( F \) is finite, Conjecture 2.3 is true in this case.

Proof. The blow-up of \( X_{O_K} \) at the singular points of the closed fibers satisfies the assumption of Corollary 3.8.

4. Cristalline representations

In this short section, we derive the assertion 2 in Theorem 2.5 from the following computation of the second Stiefel-Whitney class of an orthogonal cristalline representation.

Lemma 4.1 ([16, Theorem 2.3]). Let \( K \) be a complete discrete valuation field of characteristic 0 with perfect residue field \( F \) of characteristic \( p > 2 \). We assume that \( p \) is a prime element of \( K \). Let \( V \) be an orthogonal cristalline \( \mathbb{Q}_p \)-representation of the absolute Galois group \( G_K = \text{Gal}(\overline{K}/K) \) and \( D = D_{\text{crys}}(V) \) be the associated quadratic \( K \)-vector space. Assume that the Hodge filtration \( F^\bullet \) on \( D \) satisfies \( F^{p-1} D = 0 \).

Then we have

\[
\partial \text{sw}_2(V) = \sum_{q > 0} q \cdot \dim_K \text{Gr}^q F \cdot \partial c_p
\]

in \( H^1(F, \mathbb{Z}/2) \). In particular, further if the residue field \( F \) is finite, we have \( \text{sw}_2(V) = \sum_{q > 0} q \cdot \dim_K \text{Gr}^q F \cdot c_p \) in \( H^2(K, \mathbb{Z}/2) \).

Applying Lemma 4.1, we compute the second Stiefel-Whitney class in a good reduction case.
**Proposition 4.2.** Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $F$ of characteristic $p \neq 0$. We assume that $p$ is a prime element of $K$. Let $X$ be a proper smooth scheme of even dimension $n$ over $K$ and assume that $X$ has a proper smooth model $X_{\mathcal{O}_K}$ over the integer ring $\mathcal{O}_K$. We further assume that $H^q(X, \Omega^{n-q}_{X/K}) = 0$ for $|q - \frac{n}{2}| \geq \frac{p-1}{2}$.

We put $h = \sum_{q<\frac{n}{2}} \left(\frac{n}{2} - q\right) \dim H^{n-q}(X, \Omega^q_{X/K})$ (2.13.2), as in Lemma 2.13. Then we have

$$\partial h^2_h(H^p_p(X)) = h \cdot \partial c_p$$

in $H^1(F, \mathbb{Z}/2)$. In particular, further if $K$ is a finite extension of $\mathbb{Q}_p$, we have $\partial h^2_h(H^p_p(X)) = h \cdot c_p$ in $H^2(K, \mathbb{Z}/2)$.

**Proof.** Since $H^n(X_{\mathcal{O}_K}, \mathbb{Q}_p)$ is a cristalline representation by [8], it is enough to apply Lemma 2.1 if $p > 2$. If $p = 2$, the assumption implies that $H^n(X_{\mathcal{O}_K}, \mathbb{Q}_p)\left(\frac{n}{2}\right)$ is unramified and $h = 0$. Hence the assertion follows.

**Corollary 4.3.** We keep the assumption in Proposition 4.2. The both sides of (2.3.3) have the same images by $\partial$. In particular, further if $K$ is a finite unramified extension of $\mathbb{Q}_p$, Conjecture 2.3 is true in this case.

The assumption $H^q(X, \Omega^{n-q}_{X/K}) = 0$ for $|q - \frac{n}{2}| \geq \frac{p-1}{2}$ of Proposition 4.2 is satisfied if $n+1 < p = \ell$. Hence, Corollary 4.3 implies the assertion 2 in Theorem 2.5 for $p = \ell \neq 2$. The case $p = \ell = 2$ will be proved as a consequence of Proposition 9.4.

**Proof.** By Corollary 3.3, we have $\partial h\sigma(H^2_{dR}(X/K)) = 0$. By Lemma 3.2, we also have $\partial dX = 0$ and $\partial e = 0$. Thus, the image by $\partial$ of the left hand side of (2.3.3) is $\partial h^2_h(H^p_p(X))$ and that of the right hand side is $h \cdot \partial e_{\ell}$. Hence, it follows from Proposition 4.2.

### 5. Hodge structures

In this section, we prove the assertion 3 of Theorem 2.5 for a projective smooth variety over $\mathbb{R}$, using polarizations of Hodge structures.

Before starting proof, we recall some terminology on Hodge structures. An $\mathbb{R}$-Hodge structure of weight 0 over $\mathbb{R}$ is an $\mathbb{R}$-vector space $V$ of finite dimension endowed with the following structures: A representation $\text{Gal}(\mathbb{C}/\mathbb{R}) \to GL_V(V)$ and a decreasing filtration $F^\bullet$ on the $\mathbb{R}$-vector space $D = (V \otimes_{\mathbb{R}} \mathbb{C})^{\text{Gal}(\mathbb{C}/\mathbb{R})}$ giving a Hodge decomposition $V_C = V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_p V^{p,-p}$ where $V^{p,q} = F^p_C \cap F^q_C$. Here $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $V_C$ as $\sigma \otimes 1$ and $-\sigma$ denotes $\sigma \otimes -1$. We say an $\mathbb{R}$-Hodge structure of weight 0 over $\mathbb{R}$ is polarized if it is equipped with a non-degenerate symmetric bilinear form $b_V : V \times V \to \mathbb{R}$ satisfying the following condition. Let $C$ be an automorphism of $V_C$ which is defined to be the multiplication by $i^{-p-q} = (-1)^p$ on $V^{p,q} = V^{p,-q}$. Then the condition is that the bilinear form $(x, y) \mapsto b_V(x, Cy)$ on $V$ is symmetric and positive definite. Let $b_D$ denote the symmetric bilinear form on $D$ defined as the restriction of the symmetric bilinear form on $V_C$ induced by $b_V$. We put $h^{p,q} = \dim_C V^{p,q}$ and let $h^{0,0;\pm}$ be the multiplicity of eigenvalue $\pm 1$ of the complex conjugation on $V_{\mathbb{R}}^{0,0} = V \cap V^{0,0}$. 


For an orthogonal representation $V$ of $\text{Gal}(\mathbb{C}/\mathbb{R})$, let $v^- = \dim V^-$ be the dimension of the $(-1)$-eigenspace $V^- = \{ x \in V \mid \sigma(x) = -x \}$ where $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$ denotes the complex conjugate. Then we have $sw_1(V) = v^-\{-1\}$ and $sw_2(V) = (\nu^-_2)\{-1,-1\}$. For a quadratic $\mathbb{R}$-vector space $(D, b_D)$, let $d^- = \dim D^-$ be the dimension of a maximal subspace $D^-$ of $D$ where the symmetric bilinear form $b_D|_{D^-}$ is negative definite. Then we have $hw_1(D) = d^-\{-1\}$ and $hw_2(D) = (d^-_2)\{-1,-1\}$.

**Lemma 5.1.** Let $V$ be a polarized $\mathbb{R}$-Hodge structure of weight 0 over $\mathbb{R}$. Then we have

$$v^- = d^- = \sum_{p>0} h^{p,-p} + h^{0,0,-}.$$  

**Proof.** Since $V = V^{0,0}_R \oplus (V \cap \bigoplus_{p>0} V^{p,-p})$, it suffices to consider the cases where $V = V^{0,0}_R$ and $V^{0,0}_R = 0$ respectively. If $V = V^{0,0}_R$, the symmetric bilinear form $b_V$ is positive definite and we obtain $v^- = d^- = h^{0,0,-}$ and $h^{p,-p} = 0$ for $p > 0$. If $V^{0,0}_R = 0$, we have $h^{0,0,-} = 0$ and $v^-, d^-$ and $\sum_{p>0} h^{p,-p}$ are equal to $\dim V/2$. \hfill $\square$

By the comparison theorem of singular cohomology and étale cohomology, we have $sw_1(H^n_d(X/\mathbb{R})) = sw_1(H^n(X(\mathbb{C}), \mathbb{Q}))$. Now we are ready to prove the following main result of this section.

**Proposition 5.2.** Let $X$ be a projective smooth scheme over $\mathbb{R}$ of even dimension $n$. Let $\beta$ be as in Lemma 2.8, $\kappa$ be as in Conjecture 2.3 and $e = \sum_{q<n} e_q \in H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ as in Conjecture 2.3. Let $d_X = H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ and $d_X' = H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ be as in Lemma 2.8. Then we have

$$(5.2.1) \quad sw_2(H^n_d(X/\mathbb{R})) + \{ e, -1 \} + \beta \cdot \{-1,-1\} = hw_2(H^n_d(X/\mathbb{R}')) + r\{ d_X', -1 \} + \binom{r}{2} \{-1,-1\}$$

in $H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$.

Since $\{2, d\}, c_\ell = c_2 \in H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ in the formula (2.9.1) are 0, Proposition 5.2 implies the assertion 3 in Theorem 2.5.

**Proof.** We take an ample invertible sheaf and consider the associated Lefschetz decomposition $H^n(X(\mathbb{C}), \mathbb{R})\left(\frac{1}{2}\right) = \bigoplus_{q \leq n, q \text{ even}} P^q$ by the primitive parts. Then each $P^q$ is an $\mathbb{R}$-Hodge structure of weight 0 over $\mathbb{R}$. Further $(-1)^{\frac{n}{2}}$-times the restriction of $b_\mathbb{C}$ to $V^q$ defines a polarization on it. Let $D = H^1_d(X/\mathbb{R}) = \bigoplus_{q \leq n, q \text{ even}} D^q$ be the corresponding Lefschetz decomposition.

For an even integer $0 \leq q \leq n$, let $e^-_q$ be the multiplicity of the eigenvalue $-1$ of the action of the complex conjugate on $H^n(X(\mathbb{C}), \mathbb{Q})\left(\frac{1}{2}\right)$ and we put $e^- = \sum_{q \leq n, q \text{ even}} e^-_q$. Let $(d^+, d^-)$ be the signature of the quadratic vector space $H^1_d(X/\mathbb{R})'$ with the symmetric bilinear form multiplied by $(-1)^{n/2}$. We put $P^+ = \bigoplus_{q < n, q \equiv 0 \bmod 4} P^q$ and $P^- = \bigoplus_{q < n, q \equiv 2 \bmod 4} P^q$ as in Lemma 2.13 and prove the congruence

$$(5.2.2) \quad e^-_n - 2e^- = d^- - \left\{ \begin{array}{ll} \dim P^- & \text{if } n \equiv 0 \bmod 4, \\ \dim P^+ & \text{if } n \equiv 2 \bmod 4 \end{array} \right.$$
Let \( v^q \) be the multiplicity of the eigenvalue \(-1\) of the action of the complex conjugate on \( P^q \) and let \((d^q_+, d^q_-)\) be the signature of the quadratic vector space \( D^q \). Then, we have \( e_q - e_{q-2} = v^q \) for \( 2 \leq q \leq n \) even and \( e_0 = v^0 \). Since \( P^q \) is \((-1)^{q/2}\)-definite, we have

\[
v^q = \begin{cases} d^q_- & \text{if } q \equiv 0 \mod 4, \\ d^q_+ & \text{if } q \equiv 2 \mod 4. \end{cases}
\]

If \( n \equiv 0 \mod 4 \), we have

\[
e^-_n - 2e^- \equiv \sum_{q \leq n, \text{even}} v^-_q - 2 \sum_{q \leq n, q \equiv 0} (4) v^-_q = \sum_{q \leq n, q \equiv 0} v^-_q - \sum_{q \leq n, q \equiv 2} v^-_q = \sum_{q \leq n, q \equiv 0} d^-_q - \sum_{q \leq n, q \equiv 2} d^+_q = d^- - \dim P^-.
\]

If \( n \equiv 2 \mod 4 \), we have

\[
e^-_n - 2e^- \equiv \sum_{q \leq n, \text{even}} v^-_q - 2 \sum_{q \leq n, q \equiv 0} (4) v^-_q = \sum_{q \leq n, q \equiv 2} v^-_q - \sum_{q \leq n, q \equiv 0} v^-_q = \sum_{q \leq n, q \equiv 2} d^+_q - \sum_{q \leq n, q \equiv 0} d^-_q = d^+ - \dim P^+.
\]

Thus (5.2.2) is proved.

We have \( sw(H^n(X)) = (1 + \{ -1 \})e^-_n \) and \( hw(H^dR(X))' = (1 + \{ -1 \})d^- \). Hence by (5.2.2), we obtain

\[
sw(H^n(X)) \cdot (1 + e^- \{ -1, -1 \}) = hw(H^dR(X))' \cdot \begin{cases} (1 + \{ -1 \})^{-\dim P^-} & \text{if } n \equiv 0 \mod 4, \\ (1 + \{ -1 \})^{-\dim P^+} & \text{if } n \equiv 2 \mod 4. \end{cases}
\]

Further, by (2.13.4), we have

\[
sw(H^n(X)) \cdot (1 + (e^- + \beta) \{ -1, -1 \}) = hw(H^dR(X))' \cdot (1 + \{ -1 \})^r.
\]

Since \( e = e^- \{ -1 \} \), the assertion is proved. \( \Box \)

6. Stiefel-Whitney classes of symmetric complexes

In this section, we define and study the Stiefel-Whitney classes of symmetric complexes. A similar framework is studied in [2]. After recalling some elementary constructions on complexes, we prepare basic properties on symmetric strict perfect complexes in (6.2)–(6.8) and we recall fundamental properties of the Stiefel-Whitney classes of symmetric bundles in (6.9)–(6.10). We define the class for symmetric strict perfect complexes in Definition 6.11 and establish fundamental properties in Proposition 6.12. Finally, we generalize the definition to the derived category \( D_{\text{perf}}(X) \) in Corollary 6.16.

We will follow the sign convention on complexes in [3]. Let \( \mathcal{C} \) be an abelian category. The mapping cone \( \text{Cone}(f) \) of a morphism \( f : K \to L \) of complexes of objects of \( \mathcal{C} \) is defined to be the simple complex associated to the double complex

\[
\begin{array}{c}
\ldots \\
\begin{array}{c}
\text{Cone}(f) = \begin{cases} K & \text{if } q \equiv 0 \mod 4, \\ L & \text{if } q \equiv 2 \mod 4. \end{cases}
\end{array} \\
\end{array}
\]
\[ \cdots \to 0 \to K \to L \to 0 \to \cdots \] where \( L \) is put on the first degree 0. More concretely, the \( i \)-th component is \( K^{i+1} \oplus L^i \) and the map is given by the left multiplication by the matrix \( \begin{pmatrix} -d^{i+1} & 0 \\ f^{i+1} & d^i \end{pmatrix} \). Canonical maps \( L \to \text{Cone}(f) \to K[1] \) are defined as \( L = \text{Cone}(0 \to L) \to \text{Cone}(f) \to \text{Cone}(K \to 0) = K[1] \).

Similarly, the mapping fiber \( \text{Fib}(f) \) is defined to be the simple complex associated to the double complex \( \cdots \to 0 \to K \to L \to 0 \to \cdots \) where \( K \) is put on the first degree 0. The \( i \)-th component is \( K^i \oplus L^{i-1} \) and the map is given by the left multiplication by the matrix \( \begin{pmatrix} d^i & 0 \\ f^i & -d^{i-1} \end{pmatrix} \). Canonical maps \( L[-1] \to \text{Fib}(f) \to K \) are defined as \( L[-1] = \text{Fib}(0 \to L) \to \text{Fib}(f) \to \text{Fib}(K \to 0) = K \). We identify \( \text{Fib}(f)[1] \) with \( \text{Cone}(-f) \) by the identity.

Let \( f: K \to L \) and \( g: L \to M \) be morphisms of complexes such that the composition \( g \circ f \) is homotope to 0. Then, a homotopy \( t \) connecting \( g \circ f \) to 0 defines a map \( (g, t): \text{Cone}(f) \to M \). Recall that a homotopy \( t \) consists of maps \( t^i: K^i \to M^{i-1} \) satisfying \( g^i \circ f^i = t^{i+1} \circ d^i + d^{i-1} \circ t^i \). The \( i \)-th component of the map \( (g, t) \) is given by \( t^{i+1} \oplus g^i: K^{i+1} \oplus L^i \to M^i \). The composition \( L \xrightarrow{\text{can}} \text{Cone}(f) \xrightarrow{(g, t)} M \) is \( g \). A bijection of the set of homotopies connecting \( g \circ f \) to 0 to the set of maps of complexes \( \text{Cone}(f) \to M \) such that the composition with \( L \to \text{Cone}(f) \) is \( g \) is defined by sending \( t \) to \( (g, t) \). A homotopy \( t \) also corresponds to a map \( (f, t): K \to \text{Fib}(g) \) such that the composition with the canonical map \( \text{Fib}(g) \to L \) is equal to \( f \).

For a homotopy \( t \) connecting \( g \circ f \) to 0, we define a complex

\[
(6.1.1) \quad C = C(K \overset{f}{\to} L \overset{g}{\to} M)_t
\]

to be \( \text{Fib}((g, t): \text{Cone}(f) \to M) = \text{Cone}((f, t): K \to \text{Fib}(g)) \). For an integer \( i \), the \( i \)-th component \( C^i \) is given by \( C^i = K^{i+1} \oplus L^i \oplus M^{i-1} \) and the map \( d^i: C^i \to C^{i+1} \) is given by the matrix \( \begin{pmatrix} -d^{i+1} & 0 & 0 \\ f^{i+1} & d^i & 0 \\ t^{i+1} & g^i & -d^{i-1} \end{pmatrix} \). For a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L & \xrightarrow{g} & M \\
| a & \downarrow & b & \downarrow & c \\
K' & \xrightarrow{f'} & L' & \xrightarrow{g'} & M'
\end{array}
\]

of morphisms of complexes and homotopies \( t \) and \( t' \) connecting \( g \circ f \) and \( g' \circ f' \) to 0 respectively satisfying \( c \circ t = t' \circ a \), the maps \( a, b \) and \( c \) induce a map

\[
(6.1.2) \quad C = C(K \overset{f}{\to} L \overset{g}{\to} M)_t \longrightarrow C' = C(K' \overset{f'}{\to} L' \overset{g'}{\to} M')_{t'}
\]

of complexes.
Further we consider a commutative diagram

\[
\begin{array}{ccc}
K' & \xrightarrow{f'} & L' & \xrightarrow{g'} & M' \\
\downarrow{a'} & & \downarrow{b'} & & \downarrow{c'} \\
K'' & \xrightarrow{f''} & L'' & \xrightarrow{g''} & M''
\end{array}
\]

of morphisms of complexes such that \(a' \circ a, b' \circ b\) and \(c' \circ c\) are 0 and a homotopy \(t''\) connecting \(g'' \circ f''\) to 0 satisfying \(c' \circ t'' = t'' \circ a'\). Then, we obtain maps \(C \to C' \to C'' = C(K'' \xrightarrow{f''} L'' \xrightarrow{g''} M'')\) and the composition is 0. The complex \(C^\circ = C(C \to C' \to C'')\) is the simple complex associated to the double complex \(\cdots \to 0 \to C \to C' \to C'' \to 0 \to \cdots\) where \(C'\) is put on the first degree 0. We put \(K^\circ = C(K \xrightarrow{a} K' \xrightarrow{a'} K'')\), \(L^\circ = C(L \xrightarrow{b} L' \xrightarrow{b'} L'')\), \(M^\circ = C(M \xrightarrow{c} M' \xrightarrow{c'} M'')\) and let \(f^\circ\): \(K^\circ \to L^\circ\) and \(g^\circ\): \(L^\circ \to M^\circ\) be the induced maps. The homotopies \(t, t'\) and \(t''\) induce a homotopy \(t^\circ\) connecting \(g^\circ \circ f^\circ\) to 0. A canonical isomorphism

\[
(6.1.3) \quad C^\circ = C(C \to C' \to C'') \xrightarrow{\sim} C(K^\circ \xrightarrow{f^\circ} L^\circ \xrightarrow{g^\circ} M^\circ)_{t^\circ}
\]

is defined by 1 on \(L, K', L', M', L''\) and \(-1\) on \(K, M, K'', M''\).

Let \(D: \mathcal{C} \to \mathcal{C}\) be a contravariant additive functor and \(c: \text{id} \to DD\) be a morphism of functors. In practice, the category \(\mathcal{C}\) will be that of \(\mathcal{O}_X\)-modules on a ringed space \((X, \mathcal{O}_X)\), the functor \(D\) will be defined by \(DF = \text{Hom}(F, \mathcal{O}_X)\) and \(c\) will be defined by the canonical morphism \(F \to DD\). For a complex \(K\) of objects of \(\mathcal{C}\), let \(DK\) denote the dual complex. The \(i\)-th component \((DK)^i\) is equal to \(D(K^{-i})\) and \(d^i: D(K^{-i}) \to D(K^{-i-1})\) is given by \((-1)^{i+1}D(d^{-i-1})\). The \(i\)-th component of the bidual complex \(DDK\) is equal to \(DD(K^i)\) and \(d^i\) is given by \(-DD(d^i)\). The canonical map \(c_K: K \to DDK\) is defined by \((-1)^{i}\) times the canonical map \(K^i \to DDK(K^i)\). The composition \(Dc_K \circ c_{DK}: D(K) \to DDDK \to DK\) is the identity.

We define a canonical isomorphism \(D\text{Cone}(f) \to \text{Fib}(Df)\) by \((1, (-1)^{-i})\) on the \(i\)-th component \(D(L^{-i}) \oplus D(K^{-i+1})\). In the case where \(L = 0\), this gives a canonical isomorphism \(D(K[1]) \to (DK)[-1]\). Consequently, for an even integer \(n\) the canonical isomorphism \(D(K[n]) \to (DK)[-n]\) is defined by the multiplication by \((-1)^{n/2}\). Similarly, we define a canonical isomorphism \(D\text{Fib}(f) \to \text{Cone}(Df)\) by \((1, (-1)^{-i+1})\) on the \(i\)-th component \(D(K^{-i}) \oplus D(L^{-i-1})\). The diagram

\[
\begin{array}{ccc}
\text{Cone}(f) & \xrightarrow{\text{Cone}(c_L)} & \text{Cone}(DDf) \\
\downarrow{c_{\text{Cone}(f)}} & & \uparrow{} \\
DD\text{Cone}(f) & \xleftarrow{} & D\text{Fib}(Df)
\end{array}
\]

is commutative. A similar diagram obtained by switching \(\text{Cone}\) and \(\text{Fib}\) is also commutative.

The dual homotopy \(Dt\) connecting \(Df \circ Dg\) to 0 consists of \((Dt)^i: D(M^{-i}) \to D(K^{-i})\) defined by \((Dt)^i = (-1)^iD(t^{1-i})\). We define a canonical isomorphism

\[
(6.1.4) \quad D(C(K \xrightarrow{f} L \xrightarrow{g} M)_t) \xrightarrow{\sim} C(DM \xrightarrow{Dg} DL \xrightarrow{Df} DK)_{Dt}
\]
to be the composition of
\[ D(\text{Fib}((g, t): \text{Cone}(f) \to M)) \longrightarrow \text{Cone}(D(g, t): DM \to D\text{Cone}(f)) \]
\[ \longrightarrow \text{Cone}((Dg, Dt): DM \to \text{Fib}(Df)). \]
It is defined by \((-1)^i \oplus 1 \oplus (-1)^{i+1}\) on the \(i\)-th component \(D(M^{-1-i}) \oplus D(L^{-i}) \oplus D(K^{1-i})\). The diagram
\[
\begin{array}{c}
\xymatrix{ C(K \xrightarrow{f} L \xrightarrow{g} M)_t \ar[r] \ar[d] & C(DDK \xrightarrow{DDf} DDL \xrightarrow{DDg} DDM)_{DDt} \ar[u] \\
DDC(K \xrightarrow{f} L \xrightarrow{g} M)_t & DC(DM \xrightarrow{Dg} DL \xrightarrow{Df} DK)_{Dt} }
\end{array}
\]
(6.1.5)
is commutative.

Let \((X, \mathcal{O}_X)\) be a local ringed space in the rest of this section. Recall that a complex \(K = (K^i, d^i)\) of \(\mathcal{O}_X\)-modules is called a strict perfect complex if \(K^i\) is locally free of finite rank for each \(i \in \mathbb{Z}\) and if \(K^i\) are 0 except for finitely many \(i \in \mathbb{Z}\). A strictly perfect complex is acyclic if and only if it is locally homotope to 0. A map \(f\) of strict perfect complexes is a quasi-isomorphism if and only if the mapping cone \(\text{Cone}(f)\) is acyclic. If \(K\) is a strict perfect complex, the canonical map \(c_K: K \to DDK\) is an isomorphism.

**Definition 6.2.** Let \(K\) be a strict perfect complex of \(\mathcal{O}_X\)-modules.

1. We say a morphism \(q: K \to DK\) of complexes is symmetric if the composition \(Dq \circ c_K: K \to DDK \to DK\) is equal to \(q\).
2. If a symmetric morphism \(q: K \to DK\) is a quasi-isomorphism, we call the pair \((K, q)\) a symmetric strict perfect complex.
3. Let \(f: L \to DL\) be a symmetric morphism. We say a homotopy \(t\) connecting \(f\) to 0 is symmetric if we have \(Dt \circ c_L = t\).

Let \((K, q)\) be a symmetric strict perfect complex. Let \(f: L \to K\) be a morphism of strict perfect complexes and let \(t\) be a symmetric homotopy connecting the symmetric morphism \(Df \circ q \circ f: L \to K \to DK \to DL\) to 0. We define a complex
\[
(6.2.1) \quad M = M(L \xrightarrow{f} K)_{q,t}
\]
to be \(C(L \xrightarrow{f} K \xrightarrow{Df \circ q} DL)_t\) (6.1.1) defined by the homotopy \(t\). We define a map
\[
(6.2.2) \quad q_M: M \longrightarrow DM
\]
to be the composition of the map
\[
\bar{q}: M = C(L \to K \to DL)_t \to C(DDL \to DK \to DL)_{Dt} \quad (6.1.2)
\]
defined by the commutative diagram
\[
(6.2.3) \quad \xymatrix{ L \ar[r]^f & K \ar[r]^{Df \circ q} & DL \\
& K \ar[r]^{Df} & DL } \quad \xymatrix{ DDL \ar[r]^{Dq \circ Df} & DK \ar[r]^{Df} & DL. }
\]
with the inverse of the isomorphism
\[ DM = DC(L \to K \to DL)_t \to C(DDL \to DK \to DL)_{Dt} \] (6.1.4).

**Lemma 6.3.** Let \((K, q)\) be a symmetric strict perfect complex. Let \(f : L \to K\) be a morphism of strict perfect complexes and let \(t\) be a symmetric homotopy connecting the symmetric morphism \(Df \circ q \circ f : L \to K \to DK \to DL\) to 0.

Then, the pair \((M, q_M)\) is a symmetric strict perfect complex.

**Proof.** The morphism \(q_M\) is a quasi-isomorphism since the vertical arrows in (6.2.3) are quasi-isomorphisms. We show \(q_M = Dq_M \circ c_M\). By (6.1.5), the upper square of the diagram

\[
\begin{array}{ccc}
M = C(L \to K \to DL)_t & \xrightarrow{\tilde{c}} & C(DDL \to DDK \to DDDL)_{DDt} \\
\downarrow c_M & & \uparrow \\
DDM = DDC(L \to K \to DL)_t & \leftarrow & DC(DDL \to DK \to DL)_{Dt} \\
\downarrow D\tilde{q} & & \downarrow Dc_M \\
DM = DC(L \to K \to DL)_t & & 
\end{array}
\] (6.3.1)

is commutative. The composition \(DDM \to DM\) is \(Dq_M\). Let
\[
\tilde{D}q : C(DDL \to DDK \to DDDL)_{DDt} \to C(DDL \to DK \to DL)_{Dt}
\]
be the map defined by the dual diagram

\[
\begin{array}{ccc}
DDL & \xrightarrow{DDf} & DDK & \xrightarrow{DDDf \circ DDq} & DDDL \\
\uparrow Dq & & \downarrow Dc_M & & \downarrow Dc_M \\
DDL & \xrightarrow{Dq \circ DDf} & DK & \xrightarrow{Df} & DL
\end{array}
\] (6.3.2)

of (6.2.2). Then, the diagram

\[
\begin{array}{ccc}
DC(DDL \to DK \to DL)_{Dt} & \to & C(DDL \to DDK \to DDDL)_{DDt} \\
\downarrow D\tilde{q} & & \downarrow D\tilde{q} \\
DM = DC(L \to K \to DL)_t & \to & C(DDL \to DK \to DL)_{Dt}
\end{array}
\]

is commutative. Since the composition \(DL \to DDDL \to DL\) is the identity, we have \(\tilde{q} = \tilde{D}q \circ \tilde{c}\) where \(\tilde{c}\) is the upper horizontal arrow in (6.3.1). Thus, we obtain \(q_M = Dq_M \circ c_M\). \(\Box\)

In the notation \(M(L \to K)_{q,t}\), if the homotopy \(t\) is 0, we drop it from the notation to make \(M(L \to K)_q\).

**Corollary 6.4.** Let \((K, q)\) be a symmetric strict perfect complex. Let \(K^{>0} \subset K\) denote the subcomplex defined by \((K^{>0})^i = K^i\) for \(i > 0\) and \((K^{>0})^i = 0\) for \(i \leq 0\) and define a symmetric strict perfect complex \((K^2, q_{K^2})\) to be \(M(K^{>0} \to K)_q\) defined in (6.2.1) and (6.2.2).
Then, the complex $K^2$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{E} = \mathcal{H}^0(K^2)$ is locally free of finite rank. The map $q_{K^2}: K^2 \to DK^2$ induces a non-degenerate symmetric bilinear form $q_\mathcal{E}: \mathcal{E} \to D\mathcal{E}$.

**Proof.** The complex $K^2$ is quasi-isomorphic to the strict perfect complex $\text{Cone}(K^{>0} \to (DK)^{>0})$ supported on degree $\geq 0$. Since it is also quasi-isomorphic to the complex $\text{Fib}(K^{\leq 0} \to D(K^{>0}))$, it is acyclic at degree $> 0$. Hence the assertion follows. \[\square\]

By abuse of terminology, we call a locally free $\mathcal{O}_X$-module of finite rank a bundle on $X$. For a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank, we call a locally free $\mathcal{O}_X$-submodule $\mathcal{F}$ a subbundle if the quotient $\mathcal{E}/\mathcal{F}$ is locally free, or equivalently, if $\mathcal{F}$ is locally a direct summand of $\mathcal{E}$.

**Definition 6.5.** Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank.

1. We say an $\mathcal{O}_X$-linear map $q: \mathcal{E} \to D\mathcal{E} = \text{Hom}(\mathcal{E}, \mathcal{O}_X)$ is symmetric if $q$ is equal to the composition $Dq \circ c_\mathcal{E}: \mathcal{E} \to D\mathcal{E} \to D\mathcal{E}$.
2. If a symmetric map $q: \mathcal{E} \to D\mathcal{E}$ is an isomorphism, we say that $q$ is non-degenerate and we call the pair $(\mathcal{E}, q)$ a symmetric bundle.
3. Let $(\mathcal{E}, q)$ be a symmetric bundle. We say a subbundle $\mathcal{F}$ is isotropic if the composition $\mathcal{F} \to \mathcal{E} \to D\mathcal{E} \to D\mathcal{F}$ is the 0-map. Further, if the sequence $\mathcal{F} \to \mathcal{E} \to D\mathcal{F}$ is exact, we say $\mathcal{F}$ is Lagrangean.

If $\mathcal{F}$ is an isotropic subbundle of a symmetric bundle $(\mathcal{E}, q)$, the subquotient

\[(6.5.1) \quad \mathcal{E}' = \mathcal{H}(\mathcal{F} \to \mathcal{E} \to D\mathcal{F}) = \text{Ker}(\mathcal{E} \to D\mathcal{F})/\mathcal{F}\]

is locally free and $q$ induces a non-degenerate symmetric bilinear form $q': \mathcal{E}' \to D\mathcal{E}'$.

**Proposition 6.6.** Let $(K, q)$ be a symmetric strict perfect complex. Let $f: L \to K$ be a morphism of strict perfect complexes and let $t$ be a symmetric homotopy connecting $Df \circ q \circ f$ to 0. Let $(M, q_M)$ be the symmetric strict perfect complex $M = M(L \xrightarrow{f} K)_{q,t}$ defined by (6.2.1) and (6.2.2). Further we define $M^2 = M(M^{>0} \to M)_{q_M}$ as in Corollary 6.3.

Assume that $K$ is acyclic except at degree 0 and that the cohomology sheaf $\mathcal{H}^0(K) = \mathcal{E}$ is locally free. Then there exists an isotropic subbundle $\mathcal{F}$ of $\mathcal{H}^0(M^2) = \mathcal{E}'$ satisfying the following properties:

(1) There exists an isomorphism $\mathcal{H}(\mathcal{F} \to \mathcal{E}' \to D\mathcal{F}) \to \mathcal{E}$ of symmetric bundles.

(2) There exists an exact sequence $0 \to DL^1 \to \mathcal{F} \to D\text{Ker}(K^1 \to K^2) \to 0$ of locally free $\mathcal{O}_X$-modules.

**Proof.** First, we prove the case where $L = 0$. In this case, we have $M = K$ and $M^2 = C(K^{>0} \to K \to DK^{>0})$. By the assumption that $K$ is acyclic except at degree 0, the subcomplex $K^{>0}$ is also acyclic except at degree 1 and $\mathcal{H}^1(K^{>0}) = \text{Ker}(K^1 \to K^2)$ is locally free of finite rank. Hence $M^2$ is also acyclic except at degree 0. Further, the cohomology sheaf $\mathcal{E}' = \mathcal{H}^0(M^2)$ is locally free and has an increasing 3 step filtration $F^\bullet$ by subbundle such that $\text{Gr}_{F}^1\mathcal{E}' = \mathcal{H}^1(K^{>0})$, $\text{Gr}_{F}^0\mathcal{E}' = \mathcal{E} = \mathcal{H}^0(K)$ and $\text{Gr}_{F}^{-1}\mathcal{E}' = \mathcal{H}^{-1}(DK^{>0}) = D\mathcal{H}^1(K^{>0})$. Since the composition $(DK^{>0})[1] \to M^2 = C(K^{>0} \to K \to DK^{>0}) \to K^{>0}[1]$ is the zero map, the subbundle $\mathcal{F} = \mathcal{E}'$. 
The second Stiefel-Whitney class of $\ell$-adic cohomology

Gr$^{-1}$E' = H$^{-1}(DK^{>0})$ is isotropic and the assertion (1) follows. The assertion (2) follows from $H^1(K^{>0}) = \text{Ker}(K^1 \to K^2)$.

We prove the general case. The complex $L^o = C(L^1 \to L \to D(L^1))$ is acyclic except at degree 1 and we have $H^1(L^o) = L^1$. By (6.1.3), we have a canonical isomorphism $M^2 \to C(L^o \to K^2 \to DL^o)$. Hence $\mathcal{E}' = H^0(M^2)$ is locally free and has an increasing 3 step filtration $F^\bullet$ by subbundle such that $Gr^1 F' = H^1(L^o) = L^1$, $Gr^0 F' = H^0(K^2)$ and $Gr^{-1} F' = H^{-1}(D(L^o)) = DL^1$. Similarly as in the special case proved above, the subbundle $Gr^{-1} E' = DL^1$ is isotropic. Further since it has been already proved for $0 \to K^3$, there exist an isotropic subbundle $F' = H^1(K^{>0}) = \text{Ker}(K^1 \to K^2)$ of $Gr^0 F' = H^0(K^2)$ an isomorphism $\mathcal{H}(F' \to Gr^0 F' \to DF') \to E$ of symmetric bundles. Thus, the inverse image $\mathcal{F} \subset \mathcal{E}'$ of $\mathcal{F}'$ is isotropic and is an extension of $\mathcal{F}' = \text{Ker}(K^1 \to K^2)$ by $Gr^{-1} E' = DL^1$. Thus the assertion follows. □

Definition 6.7. Let $(K, q)$ be a symmetric strict perfect complex. Let $f : L \to K$ be a morphism of strict perfect complexes.

If there exists a symmetric homotopy $t$ connecting $Df \circ q \circ f$ to 0 such that the complex $M = M(L \to K)_{q,t}$ (6.2.1) is acyclic, we say $f : L \to K$ is Lagrangean.

The condition $M = M(L \to K)_{q,t}$ is acyclic is equivalent to that the map $(Df \circ q, t) : \text{Cone}(f) \to DL$ is a quasi-isomorphism.

Lemma 6.8. Let $(K, q)$ be a symmetric strict perfect complex. Let $f : L \to K$ be a morphism of strict perfect complexes and let $t$ be a symmetric homotopy connecting $Df \circ q \circ f$ to 0. We put $M = M(L \to K)_{q,t}$ and $N = \text{Fib}(K \to DL)$. Then the direct sum $N \to K \oplus M$ of the maps defined by

\[
\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow & & \downarrow \\
0 & \longrightarrow & K \\
\end{array}
\quad
\begin{array}{ccc}
0 & \longrightarrow & DL \\
\downarrow & & \downarrow \\
L & \longrightarrow & K \\
\downarrow & & \downarrow \\
0 & \longrightarrow & DL \\
\end{array}
\]

is Lagrangean with respect to the symmetric bilinear form $q \oplus -qM$ on $K \oplus M$.

Proof. It suffices to show that the complex $C(N \to K \oplus M \to DN)_0$ is acyclic. Let $K^o$ denote the acyclic complex $C(K \to K \oplus K \to DK)_0$ where $K \to K \oplus K$ is the diagonal map and $K \oplus K \to DK$ is the difference $(Dq, -Dq)$. By (6.1.3), we have an isomorphism $C(N \to K \oplus M \to DN)_0 \to C(\text{Fib}(id_L) \to K^o \to \text{Cone}(id_{DL}))_0$ and the assertion follows. □

In the following, we assume that $X$ is a scheme over $\mathbb{Z}/4\mathbb{Z}$. For a symmetric bundle $(\mathcal{E}, q)$ on $X$, the Stiefel-Whitney class $w(\mathcal{E}) = 1 + w_1(\mathcal{E}) + w_2(\mathcal{E}) + \cdots$ is defined in $H^*(X, \mathbb{Z}/2\mathbb{Z}) = \prod_{i=0}^\infty H^*(X, \mathbb{Z}/2\mathbb{Z})$, see [7, §1]. For a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank $r$, we put

\[(6.8.1) \quad c(\mathcal{F}) = \sum_{i=0}^r (1 + \{-1\})^{r-i}c_i(\mathcal{F})\]
in $H^*(X, \mathbb{Z}/2\mathbb{Z})$.

The Stiefel-Whitney class is characterized by the following properties:

1. For a morphism $f: X \to Y$ of schemes and a symmetric bundle $(E, q)$ on $Y$, we have $f^*w(E) = w(f^*E)$.
2. For the direct sum $(E \oplus E', q \oplus q')$ of symmetric bundles $(E, q)$ and $(E', q')$ on $X$, we have $w(E \oplus E') = w(E) \cdot w(E')$.
3. Assume $E$ is of rank 1 and let $w_1(E)$ be the class of $E$ as an element of $H^1(X, \mathcal{O}(1)) = H^1(X, \mathbb{Z}/2\mathbb{Z})$. Then, we have $w(E) = 1 + w_1(E)$.

**Proposition 6.9.** Let $(E, q)$ be a symmetric bundle.

1. We have

$$w(E, q) \cdot w(E, -q) = \overline{c}(E).$$

2. Let $\mathcal{F}$ be an isotropic subbundle of $E$ and regard $\mathcal{E}' = \mathcal{H}(\mathcal{F} \to E \to D\mathcal{F})$ (6.5.1) as a symmetric bundle by the symmetric bilinear form $q'$ induced by $q$. Then, we have

$$w(E) = w(E') \cdot \overline{c}(\mathcal{F}).$$

**Proof.** We define a map $\mathcal{F}' = \ker(\mathcal{E} \to D\mathcal{F}) \to E \oplus \mathcal{E}'$ to be the sum of the inclusion and the projection. Then, it is Lagrangean with respect to the symmetric bilinear form $q \oplus -q'$. Hence, by (6.9.2) and [2] Proposition 5.5, we obtain $w(E, q) \cdot w(E', -q') = \overline{c}(\mathcal{F}')$. By considering the case where $\mathcal{F} = 0$, we obtain the equality (6.9.4). Further, we obtain $w(E, q) \cdot w(E', q')^{-1} = \overline{c}(\mathcal{F}') \cdot \overline{c}(\mathcal{E})^{-1} = \overline{c}(\mathcal{F})$. \qed

For a strict perfect complex $K$, we put

$$\overline{c}(K) = \prod_i \overline{c}(K^i)^{(-1)^i}$$

in $H^*(X, \mathbb{Z}/2\mathbb{Z})$. For a quasi-isomorphism $K_1 \to K_2$, we have $\overline{c}(K_1) = \overline{c}(K_2)$. For the dual complex $DK$, we have $\overline{c}(DK) = \overline{c}(K)$.

**Corollary 6.10.** Let $(K, q)$ be an acyclic symmetric strict perfect complex and put $E = \mathcal{H}^0(K^2)$ for $K^2 = M(K^>0 \to K)_q$. Then, we have

$$w(E) \cdot \overline{c}(K^>0) = 1.$$

**Proof.** We apply Proposition 6.6 to $L = 0$. Then $\mathcal{F} = D\ker(K^1 \to K^2)$ is a Lagrangean subbundle of $E$. Hence, by Proposition 6.9, we obtain $w(E) = \overline{c}(\ker(K^1 \to K^2))$. Since $K$ is acyclic, we have $\overline{c}(K^>0) = \overline{c}(\ker(K^1 \to K^2))^{-1}$. \qed

**Definition 6.11.** Let $(K, q)$ be a symmetric strict perfect complex and put $E = \mathcal{H}^0(K^2)$ for $K^2 = M(K^>0 \to K)_q$. Then, we define the total Stiefel-Whitney class $w(K) \in H^*(X, \mathbb{Z}/2\mathbb{Z})$ by

$$w(K) = w(E) \cdot \overline{c}(K^>0).$$

**Proposition 6.12.** The Stiefel-Whitney classes satisfy the following properties:

1. For a morphism $f: X \to Y$ of schemes and a symmetric strict perfect complex $(K, q)$ on $Y$, we have $f^*w(K) = w(f^*K)$. 

For a symmetric strict perfect complex $(K_1, q_1)$ and $(K_2, q_2)$, we have $w(K_1 \oplus K_2) = w(K_1) \cdot w(K_2)$.

For a symmetric bundle $(\mathcal{E}, q)$ on $X$ and $K = \mathcal{E}[0]$, we have $w(K) = w(\mathcal{E})$.

For symmetric strict perfect complexes $(K_1, q_1)$ and $(K_2, q_2)$ of $\mathcal{O}_X$-modules and a quasi-isomorphism $f : K_1 \to K_2$ such that the diagram

$$
\begin{array}{ccc}
K_1 & \xrightarrow{f} & K_2 \\
\downarrow q_1 & & \downarrow q_2 \\
\text{DK}_1 & \xleftarrow{Df} & \text{DK}_2
\end{array}
$$

is commutative up to homotopy, we have $w(K_1) = w(K_2)$.

For a symmetric strict perfect complex $(K, q)$ and a Lagrangean morphism $L \to K$ of strict perfect complexes, we have $w(K) = \bar{c}(L)$.

For a symmetric strict perfect complex $(K, q)$, we have $w(K) \cdot w(K, -q) = \bar{c}(K)$.

**Proof.** The properties (6.12.1–3) are clear from the definition. We show (6.12.6).

We put $\mathcal{E} = H^0(K^\natural)$. Then, by Proposition 6.9.1, we obtain

$$w(K, q) \cdot w(K, -q) = w(\mathcal{E}, q_\mathcal{E}) \cdot w(\mathcal{E}, -q_\mathcal{E}) \cdot \bar{c}(K^{>0})^2 = \bar{c}(\mathcal{E}) \cdot \bar{c}(K^{>0})^2.$$

Since $\bar{c}(\mathcal{E}) = \bar{c}(K^\natural) = \bar{c}(K) \cdot \bar{c}(K^{>0})^{-2}$, we obtain (6.12.6). To show (6.12.5), we first prove the following.

**Lemma 6.13.** Assume that $K$ is acyclic except at degree 0 and that $\mathcal{E} = H^0(K)$ is locally free of finite rank. Let $f : L \to K$ be a Lagrangean morphism of strict perfect complexes. Then, we have $w(\mathcal{E}) = \bar{c}(L)$.

**Proof.** Let $t$ be a symmetric homotopy connecting $Df \circ q \circ f$ to 0. We apply Proposition 6.6 to $L \to K$. Then, by Proposition 6.9.2, we have $w(\mathcal{E}') = w(\mathcal{E}) \cdot \bar{c}(L^1) \cdot \bar{c}(\text{Ker}(K^1 \to K^2))$. By the assumption that $L$ is Lagrangean, the complex $M = M(L \to K)'_\natural$ is acyclic. Hence, by Corollary 6.10.1, we have $w(\mathcal{E}') \cdot \bar{c}(M^{>0}) = 1$. Since $K$ is acyclic except at degree 0, we have $\bar{c}(K^{>0}) = \bar{c}(\text{Ker}(K^1 \to K^2))^{-1}$. Thus, we obtain $\bar{c}(K^{>0}) = \bar{c}(\mathcal{E}) \cdot \bar{c}(L^1) \cdot \bar{c}(M^{>0})$. Since $\bar{c}(M^{>0}) = \bar{c}(K^{>0}) \cdot \bar{c}(L^{>1})^{-1} \cdot \bar{c}(D(L^{<0}))^{-1}$ and $\bar{c}(L) = \bar{c}(L^1)^{-1} \cdot \bar{c}(L^{>1}) \cdot \bar{c}(D(L^{<0}))$, the assertion follows. \qed

We prove (6.12.5). Let $K^\natural$ denote the strict perfect complex $M(K^{>0} \to K)_q = C(K^{>0} \to K \to DK^{>0})_q$. Then the map $L' = C(0 \to L \to DK^{>0}) \to K^\natural$ induced by $f : L \to K$ is Lagrangean. By applying Lemma 6.13 to $L' \to K^\natural$, we obtain $w(\mathcal{E}) = \bar{c}(L') = \bar{c}(L) \cdot \bar{c}(K^{>0})^{-1}$. Thus, we obtain $w(K) = w(\mathcal{E}) \cdot \bar{c}(K^{>0}) = \bar{c}(L)$.

Finally, we deduce (6.12.4) from (6.12.2), (6.12.5) and (6.12.6). We consider the direct sum $(K_1 \oplus K_2, q_1 \oplus -q_2)$ as a symmetric strict perfect complex. Let $t$ be a homotopy connecting $Df \circ q_2 \circ f$ to $q_1$. By replacing $t$ by the homotopy $(t + Dt \circ c_{K_2})/2$, we may assume $t$ is symmetric. Then, the map $(\text{id}_{K_1}, f) : K_1 \oplus K_2$ is Lagrangean with respect to $-q_1 \oplus q_2$. Hence, by (6.12.2) and (6.12.5), we obtain $w(K_1, -q_1) \cdot w(K_2, q_2) = \bar{c}(K_2)$. Thus, we obtain $w(K_1, q_1) = w(K_2, q_2)$ by (6.12.6). \qed
The properties (6.12.2–5) characterize the Stiefel-Whitney classes. We deduce the uniqueness from Lemma 6.8 applied to $L = K > 0$. Since the symmetric strict perfect complex $K^3 = M(K > 0 \to K)_q$ is quasi-isomorphic to $E = \mathcal{H}^0(K^2)$, we obtain
\[ w(K, q) \cdot w(E, -q_E) = w(K, q) \cdot w(K^2, -q_{K^2}) = w(K \oplus K^2, q \oplus -q_{K^2}) = \bar{c}(K > 0). \]

**Corollary 6.14.** Let $(K, q)$ be a symmetric strict perfect complex.

1. Let $f: L \to K$ be a morphism of strict perfect complexes and let $t$ be a symmetric homotopy connecting $Df \circ q \circ f$ to $0$. We put $M = M(L \to K)_q t$. Then, we have
\[ w(K) = w(M) \cdot \bar{c}(L). \]

2. Assume that $\mathcal{H}^i(K)$ is locally free of finite rank for every $i \in \mathbb{Z}$ and regard $\mathcal{H}^0(K)$ as a symmetric bundle by the symmetric form induced by $q$. Then, we have
\[ w(K) = w(\mathcal{H}^0(K)) \cdot \prod_{i < 0} \bar{c}(\mathcal{H}^i(K))^{(-1)^i}. \]

**Proof.** 1. We put $N = \text{Fib}(K \to DL)$ as in Lemma 6.8. Then, by (6.12.2), (6.12.5), and by Lemma 6.8, we have $w(K, q) \cdot w(M, -q_M) = \bar{c}(N)$. Further by (6.12.6), we obtain $w(K, q) \cdot \bar{c}(M) = w(M, q_M) \cdot \bar{c}(N)$. By $\bar{c}(M) = \bar{c}(L)^{-1} \cdot \bar{c}(N)$, the assertion follows.

2. We define a subcomplex $L \subset K$ by $L_i = K_i$ if $i < 0$, $L^0 = \text{Im}(K^{-1} \to K^0)$ and $L^i = 0$ if $i > 0$. Then, the inclusion $i: L \to K$ satisfies $Di \circ q \circ i = 0$. The complex $M = M(L \to K)_q$ is acyclic except at degree 0 and we have $\mathcal{H}^0(M) = \mathcal{H}^0(K)$. Thus, the assertion follows from 1. \qed

In the rest of this section, we assume that the scheme $X$ is divisorial [10, Definition 2.2.5] and either separated or noetherian. Recall from [10, Proposition 2.2.9 b)] that the natural functor $K(X) \to D_{\text{perf}}(X)$ from the homotopy category of strict perfect complexes of $O_X$-modules to the derived category of perfect complexes of $O_X$-modules induces an equivalence of categories from the quotient category divided by quasi-isomorphisms. For an object $K$ of $D_{\text{perf}}(X)$ and an isomorphism $q: K \to DK$ satisfying $q = Dq \circ c_K$, we call the pair $(K, q)$ a symmetric perfect complex on $X$.

**Lemma 6.15.** Let $X$ be a divisorial scheme over $\mathbb{Z}[\frac{1}{2}]$ either separated or noetherian. Let $K$ be an object of the derived category $D_{\text{perf}}(X)$ of perfect complexes of $O_X$-modules and $q: K \to DK$ an isomorphism of $D_{\text{perf}}(X)$ satisfying $q = Dq \circ c_K$.

1. There exist a symmetric strict perfect complex $(K_0, q_0)$ and a quasi-isomorphism $f_0: K_0 \to K$ of $O_X$-modules such that the diagram
\[
\begin{array}{ccc}
K_0 & \xrightarrow{f_0} & K \\
q_0 \downarrow & & \downarrow q \\
DK_0 & \xleftarrow{Df_0} & DK
\end{array}
\]

in $D_{\text{perf}}(X)$ is commutative.

2. Let $(K_1, q_1)$ be another symmetric strict perfect complex and $f_1: K_1 \to K$ be a quasi-isomorphism such that the diagram (6.15.1) with suffix 0 replaced by 1 is...
commutative. Then, there exist a strict perfect complex $K_2$ and quasi-isomorphisms $g_0: K_2 \rightarrow K_0$ and $g_1: K_2 \rightarrow K_1$ such that the diagram

\[
\begin{array}{cccc}
K_2 & \xrightarrow{g_0} & K_0 & \xrightarrow{q_0} & DK_0 \\
g_1 & & & \downarrow Dg_0 & \\
K_1 & \xrightarrow{q_1} & DK_1 & \xrightarrow{Dg_1} & DK_2
\end{array}
\]

(6.15.2)

in commutative up to homotopy.

**Proof.**

1. There exist strict perfect complexes $K_1$ and $K_2$, quasi-isomorphisms $f_1: K_1 \rightarrow K$ and $f_2: K_2 \rightarrow K$ and a morphism $q_1: K_1 \rightarrow DK_2$ such that the diagram

\[
\begin{array}{cccc}
K_1 & \xrightarrow{f_1} & K & \\
q_1 & & & \downarrow q \\
DK_2 & \xrightarrow{Df_2} & DK
\end{array}
\]

(6.15.3)

in $D_{\text{perf}}(X)$ is commutative. Further, there exist a strict perfect complex $K_0$ and quasi-isomorphisms $g_1: K_0 \rightarrow K_1$ and $g_2: K_0 \rightarrow K_2$ such that the diagram

\[
\begin{array}{cccc}
K_0 & \xrightarrow{g_1} & K_1 & \xrightarrow{f_1} & K \\
g_2 & & & \downarrow f_1 \\
K_2 & \xrightarrow{f_2} & K
\end{array}
\]

(6.15.4)

in $D_{\text{perf}}(X)$ is commutative. The composition $q_0 = Dg_0 \circ q_1 \circ g_1: K_0 \rightarrow DK_0$ makes the diagram (6.15.1) commutative and hence is a quasi-isomorphism. Since it is homotope to $Dq_0 \circ c_{K_0}$, the map $q_0 = (q_0 + Dg_0 \circ c_{K_0})/2: K_0 \rightarrow DK_0$ is symmetric and is homotope to $q_0^\prime$.

2. We consider a commutative diagram (6.15.4) with suffix 1, 2, 0 replaced by 0, 1, 2. Then, we obtain a diagram (6.15.2) commutative up to homotopy. $\square$

**Corollary 6.16.** Let $X$ and $(K, q)$ be as in Lemma 6.15. Then, for a symmetric strict perfect complex $(K_1, q_1)$ and an isomorphism $f_1: K_1 \rightarrow K$ as in Lemma 6.15.1, the Stiefel-Whitney class $w(K_1)$ is independent of $(K_1, q_1)$.

**Proof.** Let $(K_2, q_2)$ be another symmetric strict perfect complex and $f_2: K_2 \rightarrow K$ be an isomorphism as in Lemma 6.15.1. We consider a strict perfect complex $K_0$, quasi-isomorphisms $g_1: K_0 \rightarrow K_1$ and $g_2: K_0 \rightarrow K_2$ and a symmetric homotopy $t$ as in Lemma 6.15.2. Then the composition $q_0 = Dg_1 \circ q_1 \circ g_1: K_0 \rightarrow DK_0$ defines a symmetric strict perfect complex. By (6.12.1), we obtain $w(K_1) = w(K_0) = w(K_2)$. $\square$

For a symmetric perfect complex $(K, q)$ on $X$, the Stiefel-Whitney class $w(K)$ is defined as $w(K_0)$ by taking a quasi-isomorphism $f_0: K_0 \rightarrow K$ as in Lemma 6.15.1. It is well-defined by Corollary 6.16.
We give a slight generalization. Let $K$ be a perfect complex and $n$ be an even integer. We say that an isomorphism $q: K \rightarrow DK[-2n]$ in the derived category $D_{\text{perf}}(X)$ is symmetric if $q$ is equal to the composition

$$
(6.16.1) \quad K \xrightarrow{c_{K}} DDK \xrightarrow{\text{can}} D(DK[-2n])[-2n] \xrightarrow{Dq[-2n]} DK[-2n].
$$

Let $q: K \rightarrow DK[-2n]$ be a symmetric isomorphism. We put $K' = K[n]$ and define $q': K' \rightarrow DK'$ to be the composition

$$
K' = K[n] \xrightarrow{q[n]} DK[-n] \xrightarrow{\text{can}} D(K[n]) = DK'.
$$

Then, $(K', q')$ is a symmetric perfect complex and the Stiefel-Whitney class $w(K')$ is defined.

**Corollary 6.17.** Let $K$ be a perfect complex, $n$ be an even integer and $q: K \rightarrow DK[-2n]$ be a symmetric isomorphism in the derived category $D_{\text{perf}}(X)$. Let $(K[n], q')$ be the symmetric perfect complex defined above.

Assume that $\mathcal{H}(K)$ is locally free of finite rank for every $i \in \mathbb{Z}$ and regard $\mathcal{E} = \mathcal{H}^{0}(K)$ as a symmetric bundle by the symmetric form $q_{\mathcal{E}}$ induced by $q$. Then, we have

$$
w(K) = w(\mathcal{E}, (-1)^{n/2}q_{\mathcal{E}}) \cdot \prod_{i<0} c(\mathcal{H}^{i}(K))^{(-1)^{i}}.
$$

**Proof.** For $K' = K[n]$, we have $\mathcal{H}^{0}(K') = \mathcal{H}^{0}(K) = \mathcal{E}$. By the sign convention on the canonical isomorphism $D(K[n]) \rightarrow DK[-n]$, the symmetric bilinear form on $\mathcal{E}$ induced by $q'$ is equal to $(-1)^{n/2}q_{\mathcal{E}}$. Hence the assertion follows from Corollary 6.14.2. 

7. Families

We generalize definitions in Section 2. Let $S$ be a normal scheme over $\mathbb{Z}[\frac{1}{2}]$ and $L$ be a finite extension of $\mathbb{Q}_{\ell}$. We say that a smooth $L$-sheaf $V$ on $S$ is orthogonal if it is endowed with a non-degenerate symmetric bilinear form $b : V \otimes V \rightarrow L$. We define the Stiefel-Whitney class

$$
w_{2}(V) \in H^{2}(S, \mathbb{Z}/2\mathbb{Z})
$$

in étale cohomology as follows. It suffices to consider the case where $S$ is connected. In this case, an orthogonal $L$-sheaf $(V, b)$ is defined by an orthogonal representation $\rho_{x} : \pi_{1}(S, \bar{x}) \rightarrow O(V_{x}, b_{x})$ for a base point $\bar{x}$. Then we define the Stiefel-Whitney class $w_{2}(V, b) \in H^{2}(S, \mathbb{Z}/2\mathbb{Z})$ to be the image of $w_{2}(\rho_{x}) \in H^{2}(\pi_{1}(S, \bar{x}), \mathbb{Z}/2\mathbb{Z})$ by the canonical map $H^{2}(\pi_{1}(S, \bar{x}), \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{2}(S, \mathbb{Z}/2\mathbb{Z})$. It is independent of the choice of base point $\bar{x}$. When there is no fear of confusion, we drop $b$ and write simply $w_{2}(V)$. Similarly as (1.5.2), for a graded smooth $L$-sheaf $V^{\bullet}$ on $S$, we define the Stiefel-Whitney classes $w_{1}(V^{\bullet}) \in H^{1}(S, \mathbb{Z}/2\mathbb{Z})$ and $w_{2}(V^{\bullet}) \in H^{2}(S, \mathbb{Z}/2\mathbb{Z})$.

The definition of the Stiefel-Whitney class commutes with base change. When $S$ is Spec $K$ for a field $K$ and the smooth orthogonal $L$-sheaf $V$ on $S$ is defined by an orthogonal $L$-representation $\rho$ of $\text{Gal}(\bar{K}/K)$, we have $w_{2}(V) = w_{2}(\rho)$ in $H^{2}(S, \mathbb{Z}/2\mathbb{Z}) = H^{2}(\text{Gal}(\bar{K}/K), \mathbb{Z}/2\mathbb{Z})$. 


The first Stiefel-Whitney class \( sw_1(V, b) \in H^1(S, \mathbb{Z}/2\mathbb{Z}) \) of an orthogonal smooth \( L \)-sheaf \( V \) on \( S \) is similarly defined as follows. We may assume \( S \) is connected and consider the corresponding representation \( \rho : \pi_1(S, \bar{x}) \to O(V) \). Then, the class \( sw_1(V, b) \) is the homomorphism \( \det \rho : \pi_1(S, \bar{x}) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z} \) regarded as an element of \( H^1(S, \mathbb{Z}/2\mathbb{Z}) = Hom(\pi_1(S, \bar{x}), \{\pm 1\}) \).

Let \( S \) be a connected normal scheme over \( \mathbb{Z}[(1/2)] \). Let \( f : X \to S \) be a proper smooth morphism of relative even dimension \( n \). Let \( V^i \) denote the smooth \( \mathbb{Q}_\ell \)-sheaf \( R^{i+n}f_!\mathbb{Q}_\ell(\frac{q}{2}) \) on \( X \). The cup-product defines a non-degenerate \((-1)^i\)-symmetric bilinear form \( V^i \times V^{-i} \to R^{2n}f_!\mathbb{Q}_\ell(n) \to \mathbb{Q}_\ell \) by sending \((x, y)\) to \( \text{Tr}(x \cup y) \). We define the second Stiefel-Whitney classes by

\[
sw_2(H^a_\ell(X/S)) = sw_2(V^0),
sw_2(H^\bullet_\ell(X/S)) = sw_2(V^\bullet) = sw_2(V^0) + \sum_{q<0} \bar{c}_1(V^q).
\]

as elements in \( H^2(S, \mathbb{Z}/2\mathbb{Z}) \), similarly as in (1.5.2).

For an integer \( q \geq 0 \), we consider the character \( e_q : \pi_1(S)^{ab} \to \{\pm 1\} \) as an element of \( H^1(S, \mathbb{Z}/2\mathbb{Z}) \) and let \( b_{\ell, q} \) be the rank of the smooth \( \mathbb{Q}_\ell \)-sheaf \( R^qf_!\mathbb{Q}_\ell \).

We put \( e = \sum_{q<n} e_q \) as in Conjecture 2.3. We also put \( \beta = \frac{1}{2} \sum_{q<n} (-1)^q(n - q)b_{\ell, q} \).

By the definition (1.5.2) and Lemma 1.6.1, they are related by the equality

\[
sw_2(H^\bullet_\ell(X/S)) = sw_2(H^a_\ell(X/S)) + \{e, -1\} + \beta \cdot c_\ell.
\]

The first Stiefel-Whitney class \( sw_1(H^a_\ell(X/S)) = sw_1(H^\bullet_\ell(X/S)) \in H^1(S, \mathbb{Z}/2\mathbb{Z}) \) is defined as \( \det R^nf_!\mathbb{Q}_\ell(\frac{\ell}{2}) \).

We define the Hasse-Witt class for the de Rham cohomology. Let \( S \) be a divisorial scheme [10, Definition 2.2.5] over \( \mathbb{Z}[(1/2)] \) either separated or noetherian and let \( f : X \to S \) be a proper smooth morphism of relative even dimension \( n \). We put \( H^\bullet_{dR}(X/S) = Rf_*\Omega^\bullet_{X/S} \). The product \( \Omega^\bullet_{X/S} \otimes \Omega^\bullet_{X/S} \to \Omega^\bullet_{X/S} \) and the trace map \( R^nf_*\Omega^\bullet_{X/S} \to \mathcal{O}_S \) induce a symmetric isomorphism \( H^\bullet_{dR}(X/S) \to D H^\bullet_{dR}(X/S)[-2n] \) by Poincaré duality. Hence, the Hasse-Witt classes \( hw_i(H^\bullet_{dR}(X/S)) \in H^i(S, \mathbb{Z}/2\mathbb{Z}) \) are defined as at the end of Section 6.

**Lemma 7.1.** Assume that \( H^a_{dR}(X/S) = R^af_*\Omega^a_{X/S} \) are locally free for all \( q \in \mathbb{Z} \) and we put \( r = \sum_{q<n} (-1)^q \text{rank} H^q_{dR}(X/S) \). Then \( H^a_{dR}(X/S) \) is a symmetric bundle and we have

\[
\begin{align*}
\hw_2(H^\bullet_{dR}(X/S)) &= \hw_2(H^a_{dR}(X/S)) \\
&= \hw_2(H^\bullet_{dR}(X/S)) + \left\{ \begin{array}{ll}
\frac{r}{2} d_X, & d_X \equiv 1 \pmod{4}, \\
\frac{r}{2} d_X - 1, & d_X \equiv 0 \pmod{4}
\end{array} \right. + \frac{1}{2} d_X - 1
\end{align*}
\]

\[
\sum_{0 \leq q < n} c_1(H^q_{dR}(X/S))
\]
in $H^2(S, \mathbb{Z}/2\mathbb{Z})$ and

$$hw_1(H^n_{dR}(X/S)) = hw_1(H^n_{dR}(X/S)) + \frac{n}{2} \cdot b_{dR,n} \cdot \{-1\}$$

in $H^1(S, \mathbb{Z}/2\mathbb{Z})$.

The condition of Lemma 7.1 is satisfied if $S$ is the spectrum of a field or $S$ is a reduced scheme over $\mathbb{Q}$.

**Proof.** Similarly as Corollary 2.9, it follows from Corollary 6.17 and the definition of $\overline{c}(\mathcal{F})$ in (6.8.1).

For the relation between $sw_2(H^n_{\ell}(X/S))$ and $hw_2(H^n_{dR}(X/S))$, we state the following conjecture.

**Conjecture 7.2.** Let $S$ be a normal divisorial scheme over $\mathbb{Z}[\frac{1}{2}]$ either separated or noetherian and let $f : X \to S$ be a proper smooth morphism of relative even dimension $n$. We put

$$\eta = \sum_{q < \frac{n}{2}} (-1)^q \left(\frac{n}{2} - q\right) \text{rank} R^q f_* \Omega^q_{X/S}$$

as in (2.3.1) and define $c_2, c_\ell \in H^2(S, \mathbb{Z}/2\mathbb{Z})$ as in (1.6.1).

Then, we have an equality

$$(7.2.1) \quad sw_2(H^n_{\ell}(X/S)) = hw_2(H^n_{dR}(X/S)) + \{2, hw_1(H^n_{dR}(X/S))\} + \eta \cdot (c_\ell - c_2)$$

in $H^2(S, \mathbb{Z}/2\mathbb{Z})$.

If the condition of Lemma 7.1 is satisfied, the equality (7.2.1) is equivalent to

$$(7.2.2) \quad sw_2(H^n_{\ell}(X/S)) + \{e, -1\} + \beta \cdot c_\ell = hw_2(H^n_{dR}(X/S)) \quad \text{if } n \equiv 0 \text{ mod } 4,$$

$$(r + b_{dR,n} - 1)\{d_X, -1\} + \left(\frac{r + b_{dR,n}}{2}\right)\{-1, -1\} \quad \text{if } n \equiv 2 \text{ mod } 4,$$

$$(2, d_X) + \eta \cdot (c_\ell - c_2) + \sum_{0 \leq q < n} c_1(H^q_{dR}(X/S)).$$

Thus Conjecture 7.2 is a generalization of Conjecture 2.3.

The following weak evidence on Conjecture 7.2 will be used in the proof of the assertion 5 of Theorem 2.5 in the final section.

**Lemma 7.3.** Let $S$ be a smooth scheme over $\mathbb{Z}[\frac{1}{2}]$ and $f : X \to S$ be a proper smooth morphism of even relative dimension $n$. Let $\ell$ be a prime number. Assume that the following condition is satisfied:
(P) For every finite unramified extension $K$ of $\mathbb{Q}_\ell$ and every morphism $f: \text{Spec } \mathcal{O}_K \to S$, the pull-back $X \times_S K$ by the restriction $f|_K: \text{Spec } K \to S$ satisfies Conjecture 2.3.

Then, the difference $\delta \in H^2(S[\frac{1}{\ell}], \mathbb{Z}/2\mathbb{Z})$ of the both sides of (7.2.1) for $X[\frac{1}{\ell}] \to S[\frac{1}{\ell}]$ is in the image of the restriction map $H^2(S, \mathbb{Z}/2\mathbb{Z}) \to H^2(S[\frac{1}{\ell}], \mathbb{Z}/2\mathbb{Z})$.

Proof. Since $S$ is assumed smooth over $\mathbb{Z}[\frac{1}{\ell}]$, we have an exact sequence

$$
H^2(S, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(S[\frac{1}{\ell}], \mathbb{Z}/2\mathbb{Z}) \quad \longrightarrow \quad H^1(S, \mathbb{Z}/2\mathbb{Z})
$$

by local acyclicity of smooth morphism. By a generalization of the Chebotarev density theorem [18, Theorem 7], [20, Theorem 9.11], it suffices to show that the image of $\delta$ by the composition

$$
H^2(S[\frac{1}{\ell}], \mathbb{Z}/2\mathbb{Z}) \quad \longrightarrow \quad H^1(S, \mathbb{Z}/2\mathbb{Z}) 
$$

is 0 for every closed point $s$ of $S[\frac{1}{\ell}]$.

Let $K$ be an unramified extension of $\mathbb{Q}_\ell$ with residue field $\kappa(s)$. Since $S$ is smooth, the closed immersion $s \to S$ is lifted to a morphism $f: \text{Spec } \mathcal{O}_K \to S$. Then, we have a commutative diagram

$$
\begin{array}{ccc}
H^2(S[\frac{1}{\ell}], \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^1(S, \mathbb{Z}/2\mathbb{Z}) \\
\downarrow f|_K & & \downarrow \\
H^2(K, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^1(s, \mathbb{Z}/2\mathbb{Z}).
\end{array}
$$

By the assumption (P), the pull-back $X_K$ satisfies Conjecture 2.3. By the remark preceding Lemma 7.3, it means $f|_K^*(\delta) = 0$ and the assertion follows. \hfill \Box

If $n = 0$, Conjecture 7.2 is nothing but the following.

Theorem 7.4 ([7 Theorem 2.3]). For a finite étale morphism $f: X \to S$, we have

$$
\text{sw}_2(f_*\mathbb{Q}) = h\text{w}_2(f_*\mathcal{O}_X, \text{Tr}_{X/S}(x^2)) + \{2, d_X\}.
$$

8. TRANSCENDENTAL ARGUMENT

In this section, we prove the assertion 4 of Theorem 2.5 by a transcendental argument. Since it will be reduced to proving Conjecture 7.2 for schemes over $\mathbb{C}$, we study them first.

We introduce some terminology. Let $S$ be a connected normal scheme of finite type over $\mathbb{C}$ and let $S^{an}$ denote the associated analytic space. We say a local system $V$ of $\mathbb{C}$-vector spaces on $S^{an}$ is orthogonal if it is equipped with a non-degenerate symmetric bilinear form $b: V \otimes V \to \mathbb{C}$. An orthogonal local system $V$ corresponds to an orthogonal representation $\rho_{\bar{x}}: \pi_1(S^{an}, \bar{x}) \to O(V_{\bar{x}}, b_{\bar{x}})$ of the fundamental group for a geometric point $\bar{x}$. The Stiefel-Whitney class $\text{sw}_2(V, b) \in H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$ is defined as the image of the Stiefel-Whitney class $\text{sw}_2(\rho_{\bar{x}}) \in H^2(\pi_1(S^{an}, \bar{x}), \mathbb{Z}/2\mathbb{Z})$ by the canonical map $H^2(\pi_1(S^{an}, \bar{x}), \mathbb{Z}/2\mathbb{Z}) \to H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$. It is independent of the choice of base point $\bar{x}$. 

A locally free $\mathcal{O}_{S^{an}}$-module $D$ is called quadratic if it is equipped with a non-degenerate symmetric bilinear form $\beta : D \otimes D \to \mathcal{O}_{S^{an}}$. For a quadratic locally free $\mathcal{O}_{S^{an}}$-module $(D, \beta)$, the Hasse-Witt class $hw_2(D, \beta) \in H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$ is defined, see for example [7, §1]. If $D$ is of rank $r$, it is the image of the class of $D$ in $H^1(S^{an}, \mathcal{O}(r)^{an})$ by the boundary map $H^1(S^{an}, \mathcal{O}(r)^{an}) \to H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$

**Lemma 8.1.** Let $S$ be a normal scheme of finite type over $\mathbb{C}$. Let $V$ be an orthogonal local system of $\mathbb{C}$-vector spaces on $S^{an}$ and $D = V \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}}$ be the corresponding quadratic locally free $\mathcal{O}_{S^{an}}$-module. Then we have $sw_2(V) = hw_2(D)$ in $H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$.

**Proof.** We may assume $S$ is connected and let $\pi_1(S^{an}, \bar{b})$ be the topological fundamental group defined by a base point $\bar{b}$. The isomorphism class of the orthogonal local system $V$ is defined as an element of $H^1(S^{an}, O(r, \mathbb{C})) = \pi_1(S^{an}, \bar{b}), O(r, \mathbb{C}))$. By the definition of the Stiefel-Whitney class $sw_2(V)$ and the functoriality of the map from group cohomology to singular cohomology, it is equal to its image by the boundary map $H^1(S^{an}, O(r, \mathbb{C})) \to H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$ defined by the central extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{O}(r, \mathbb{C}) \longrightarrow O(r, \mathbb{C}) \longrightarrow 1.$$ 

Hence the assertion follows from the commutative diagram

$$
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \tilde{O}(r, \mathbb{C}) & \longrightarrow & O(r, \mathbb{C}) & \longrightarrow & 1 \\
\| & & \| & & \| & & \| \\
1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \tilde{O}(r)^{an} & \longrightarrow & O(r)^{an} & \longrightarrow & 1
\end{array}
$$

of central extensions of sheaves on $S^{an}$.

We prove the main result of this section.

**Proposition 8.2.** Let $S$ be a normal scheme of finite type over a noetherian ring over an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Let $f : X \to S$ be a proper smooth morphism of relative even dimension $n$. Then, for every prime number $\ell$, we have

$$sw_2(H^r_f(X/S)) = hw_2(H^r_{dR}(X/S))$$

in $H^2(S, \mathbb{Z}/2\mathbb{Z})$.

If $K$ is an extension of $\bar{\mathbb{Q}}$, the remaining terms in (2.3.3) are 0. Hence Proposition 8.2 implies the assertion 4 in Theorem 2.3.

**Proof.** By a standard argument, we may assume $S$ is of finite type over $\bar{\mathbb{Q}}$. By Lefschetz principle $H^2(S, \mathbb{Z}/2\mathbb{Z}) \simeq H^2(S_{\mathbb{C}}, \mathbb{Z}/2\mathbb{Z})$, it is reduced to the case where $S$ is of finite type over $\mathbb{C}$.

We identify $H^2(S, \mathbb{Z}/2\mathbb{Z}) = H^2(S^{an}, \mathbb{Z}/2\mathbb{Z})$ by the canonical isomorphism. Let $V$ be the orthogonal local system $R^nf_{\ast}^{an}\mathbb{C}$ and $D$ be the symmetric bundle $R^nf_{\ast}^{an}\Omega^{an}_{X^{an}/S^{an}}$ on $X^{an}$. Then, we have $sw_2(H^r_f(X/S)) = sw_2(V)$ and $hw_2(H^r_{dR}(X/S)) = hw_2(D)$. Since $D = V \otimes_{\mathbb{C}} \mathcal{O}_{X^{an}}$, the assertion follows from Lemma 8.1.

□
9. Arithmetic argument

We prove the assertion 5 of Theorem 2.5 by a global arithmetic argument. We will apply the following to the moduli space of hypersurfaces.

Lemma 9.1. Let $S$ be a connected normal noetherian scheme, $K$ be the fraction field of $S$ and $K$ be an algebraic closure of $K$. Let $p : P \to S$ be a proper smooth and geometrically connected scheme over $S$ and $U \subset P$ be an open subscheme. Let $m$ be an integer invertible on $S$. We assume that the following conditions (1)–(3) are satisfied:

1. The open subscheme $U \subset P$ is the complement of a divisor $D \subset P$ flat over $S$. Let $D^0 \subset D$ be the largest open subscheme smooth over $S$. Then, for every $s \in S$, the fiber $D^0_s$ is dense in $D_s$.

2. For every irreducible component $C$ of the geometric generic fiber $D_{K_s}$, the divisor class $[C] \in \text{Pic}(P_K)$ is in the image of the multiplication $m \times : \text{Pic}(P_K) \to \text{Pic}(P_K)$.

3. $R^1p_*\mu_m = 0$.

Let $D_1, \ldots, D_r$ be the irreducible components of $D$ with the reduced closed subscheme structures, $F_i$ be the fraction field of $D_i$ and $K_i$ be the local field at the generic point of $D_i$ for $1 \leq i \leq r$. Then, the compositions $H^2(U, \mu_m) \to H^2(K_i, \mu_m) \to H^1(F_i, \mu_m)$ with the boundary map define an exact sequence

$$H^2(S, \mu_m) \longrightarrow H^2(U, \mu_m) \longrightarrow H^2(U_K, \mu_m) \oplus \bigoplus_{i=1}^{\tau} H^1(F_i, \mathbb{Z}/m\mathbb{Z}).$$

Proof. We consider the commutative diagram

$$
\begin{array}{ccc}
H^2(S, \mu_m) & \longrightarrow & H^2(U, \mu_m) \\
\downarrow & & \downarrow \\
H^2(P, \mu_m) & \longrightarrow & H^2(U, \mu_m) \\
\downarrow & & \downarrow \\
H^2(P_K, \mu_m) & \longrightarrow & H^2(U_K, \mu_m).
\end{array}
$$

We show that the conditions (1)–(3) respectively imply the exactness of the middle row, the injectivity of the bottom horizontal arrow and the exactness of the left column. The assertion follows from this by diagram chasing.

We show the exactness of the middle row assuming the condition (1). The union $V = U \cup D^0$ is open in $P$ and we regard it as an open subscheme of $P$. Since $D^0$ is a smooth divisor of $V$, we have an exact sequence $H^2(V, \mu_m) \to H^2(U, \mu_m) \to H^1(D^0, \mathbb{Z}/m\mathbb{Z})$ by relative purity. Since $H^1(D^0, \mathbb{Z}/m\mathbb{Z}) \to \bigoplus_{i=1}^{\tau} H^1(F_i, \mathbb{Z}/m\mathbb{Z})$ is injective, it is sufficient to show that the restriction map $H^q(P, \mu_m) \to H^q(V, \mu_m)$ is an isomorphism for $q \leq 2$. Let $v : V \to S$ be the restriction of $p : P \to S$. By the assumption (1), the codimension of the complement $P_s \setminus V_s$ in each fiber $P_s$ is at least 2. Hence the map $R^qv_*\mathbb{Z}/m\mathbb{Z} \to R^qp_*\mathbb{Z}/m\mathbb{Z}$ is an isomorphism for $q \geq 2N - 2$ where $N$ is the relative dimension of $P$ over $S$. By Poincaré duality, the map $R^q p_* \mathbb{Z}/m\mathbb{Z} \to R^q v_* \mathbb{Z}/m\mathbb{Z}$ is an isomorphism for $q \leq 2$. Hence the assertion
follows by comparing the Leray spectral sequences $H^p(S, R^q\nu_\ast \mu_m) \Rightarrow H^{p+q}(P, \mu_m)$ and $H^p(S, R^q\nu_\ast \mu_m) \Rightarrow H^{p+q}(V, \mu_m)$.

Next we show that the restriction map $H^2(P, \mu_m) \rightarrow H^2(U, \mu_m)$ is injective assuming the condition (2). Let $C_1, \ldots, C_{r'}$ be the irreducible components of $D_K$. By the same argument as above, we obtain an exact sequence $\bigoplus_{j=1}^r H^0(C_j, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^2(P, \mu_m) \rightarrow H^2(U, \mu_m)$. By the exact sequence $\text{Pic}(P) \xrightarrow{\mu} \text{Pic}(P) \rightarrow H^2(P, \mu_m)$, the injectivity is in fact equivalent to the condition (2).

Finally, the condition (3) implies the exactness of the left column by the Leray spectral sequence. Thus the assertion is proved. \qed

Let $n \geq 0$ be an integer and $d \geq 1$ be an integer. Let $P = \mathbb{P}(\Gamma(\mathbb{P}^{n+1}_\mathbb{Z}, \mathcal{O}(d)))$ be the moduli space of hypersurfaces of degree $d$ and $f : X \rightarrow P$ be the universal family of hypersurfaces. More explicitly, it is described as follows. Let $T_0, \ldots, T_{n+1} \in \Gamma(\mathbb{P}^{n+1}_\mathbb{Z}, \mathcal{O}(1))$ be the homogeneous coordinate of $\mathbb{P}^{n+1}_\mathbb{Z}$. The monomials $T_0^{i_0} \cdots T_{n+1}^{i_{n+1}}$ of degree $i_0 + \cdots + i_{n+1} = d$ form a basis of $\Gamma(\mathbb{P}^{n+1}_\mathbb{Z}, \mathcal{O}(d))$. Let $(a_{i_0, \ldots, i_{n+1}})$ for $i_0 + \cdots + i_{n+1} = d$ be the dual basis of $\Gamma(\mathbb{P}^{n+1}_\mathbb{Z}, \mathcal{O}(d))$. They form a basis of $\Gamma(P, \mathcal{O}(1))$ and $X$ is defined in $\mathbb{P}^{n+1}_\mathbb{Z} \times P$ by the equation $F = 0$ where

$$F = \sum_{i_0 + \cdots + i_{n+1} = d} a_{i_0, \ldots, i_{n+1}} T_0^{i_0} \cdots T_{n+1}^{i_{n+1}} \in \Gamma(\mathbb{P}^{n+1}_\mathbb{Z} \times P, \mathcal{O}(d, 1))$$

is the universal polynomial.

Let $U \subset P$ be the open subscheme where the universal hypersurface $f : X \rightarrow P$ is smooth. If $X_K \subset \mathbb{P}^{n+1}_K$ is a smooth hypersurface defined by a polynomial $\sum a_{i_0, \ldots, i_{n+1}} = d$ for $i_0 + \cdots + i_{n+1} = d$, the pull-back of the universal family $X \rightarrow P$ by the map Spec $K \rightarrow U$ defined by the coefficients $(a_{i_0, \ldots, i_{n+1}})$.

**Lemma 9.2.** Let $\Delta \subset X$ be the complement of the largest open subscheme of $X$ smooth over $P$. We regard $\Delta$ as a closed subscheme of $X$ defined by the equations $\frac{\partial F}{\partial T_0} = \cdots = \frac{\partial F}{\partial T_{n+1}} = 0$ and we put $N + 1 = \text{rank} \Gamma(\mathbb{P}^{n+1}_\mathbb{Z}, \mathcal{O}(d))$.

1. The scheme $X$ is a $\mathbb{P}^{N-1}$-bundle over $\mathbb{P}^{n+1}_\mathbb{Z}$ with respect to the first projection. Consequently, it is regular and irreducible.

2. The scheme $\Delta$ is a $\mathbb{P}^{N-(n+2)}$-bundle over $\mathbb{P}^{n+1}_\mathbb{Z}$ with respect to the first projection. Consequently, it is regular and irreducible. The immersion $\Delta \rightarrow X$ is a regular immersion of codimension $n+1$.

3. Locally on $X$, the coherent $\mathcal{O}_X$-module $\mathcal{O}_X^{\mu_1}$ is isomorphic to the cokernel of the map $\mathcal{O}_X \rightarrow \mathcal{O}_X^{\mu_1}$ defined by a system of generators $a_1, \ldots, a_{n+1}$ of the ideal $\mathcal{I}_\Delta \subset \mathcal{O}_X$.

**Proof.** It suffices to consider over the open subscheme $D(T_i) \subset \mathbb{P}^{n+1}_\mathbb{Z}$ for each $i = 0, \ldots, n+1$.

1. Since $X$ is defined by the linear form $F$ on $A_{i_0, \ldots, i_{n+1}}$, the assertion follows.

2. This is proved in [17, Proposition 2.8] using the fact that the closed scheme $\Delta \subset P \times \mathbb{P}^{n+1}_\mathbb{Z} D(T_i)$ is defined by $F = \frac{\partial F}{\partial T_0} = \cdots = \frac{\partial F}{\partial T_{n+1}} = 0$ that follows from $d \cdot F = \sum_j T_j \cdot \frac{\partial F}{\partial T_j}$. \qed
3. We define a regular function $f_i$ on $X \times_{\mathbb{Z}[\frac{1}{d}]} D(T_i)$ by $F = f_i \cdot T_i^d$ and put $t_j = T_j/T_i$. Then, on $X \times_{\mathbb{Z}[\frac{1}{d}]} D(T_i)$, the cokernel of the map $O_X \to O_X^{n+1}$ defined by \( \frac{\partial f_i}{\partial \sigma_0}, \ldots, \frac{\partial f_i}{\partial \sigma_{i-1}}, \frac{\partial f_i}{\partial \sigma_i}, \ldots, \frac{\partial f_i}{\partial \sigma_{n+1}} \). It follows from the proof of 2. that they form a system of generators of the ideal $I_{\Delta} \subset O_X$.

We regard the image $D \subset P$ of $\Delta \subset X$ as a reduced closed subscheme. On the scheme $D$, the following is proved in \cite{SAGA4} Lemma 2.10, Proposition 2.12). The degree is computed in \cite{n5}.

**Lemma 9.3.** The scheme $D$ is a divisor of degree $(n+2)(d-1)^{n+1}$ flat over $\text{Spec} \mathbb{Z}$ and has geometrically irreducible fibers. The largest open subscheme of $D$ smooth over $\text{Spec} \mathbb{Z}$ has non-empty intersection with every fiber.

**Proposition 9.4.** For a prime number $\ell \geq n+1$, Conjecture 7.2 is true for the universal family $f_{U_{\ell}[\frac{1}{d}]} : X_{U_{\ell}[\frac{1}{d}]} \to U_{\ell}[\frac{1}{d}]$ of smooth hypersurfaces.

As Conjecture 7.2 implies Conjecture 2.3 Proposition 9.4 implies the assertion 5 of Theorem 2.5. It also implies the assertion 2 of Theorem 2.5 in the case $p = \ell = 2$ since in this case we have $n = 0$.

**Proof.** Let $\delta \in H^2(U_{\ell}[\frac{1}{d}], \mathbb{Z}/2\mathbb{Z})$ be the difference of the both sides of (7.2.1). If $\ell \neq 2$, the difference $\delta$ lies in the image of $H^2(U_{\ell}[\frac{1}{d}], \mathbb{Z}/2\mathbb{Z}) \to H^2(U_{\ell}[\frac{1}{d}], \mathbb{Z}/2\mathbb{Z})$, by Lemma 7.3 and Corollary 4.3. For $\ell = 2$, it is obvious. In order to show that $\delta$ lies in the image of $H^2(Z[\frac{1}{2}], \mathbb{Z}/2\mathbb{Z}) \to H^2(U_{\ell}[\frac{1}{d}], \mathbb{Z}/2\mathbb{Z})$, we verify that $U_{\ell}[\frac{1}{2}] \subset P[\frac{1}{2}] \to \text{Spec} \mathbb{Z}[\frac{1}{2}]$ and $m = 2$ satisfy the conditions (1)–(3) of Lemma 9.1.

Let $\xi$ denote the generic point of the irreducible closed divisor $D \subset P$. The conditions (1) and (2) follow from Lemma 9.3. Since $P[\frac{1}{2}]$ is a projective space over $\mathbb{Z}[\frac{1}{2}]$, the condition (3) is also satisfied. Thus, by Lemma 9.4, the sequence (9.4.1)

\[
H^2(\text{Spec} \mathbb{Z}[\frac{1}{2}], \mathbb{Z}/2\mathbb{Z}) \to H^2(U_{\ell}[\frac{1}{2}], \mathbb{Z}/2\mathbb{Z}) \to H^2(U_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) \oplus H^1(\kappa(\xi), \mathbb{Z}/2\mathbb{Z})
\]

is exact.

Let $K$ be the completion of the fraction field of the discrete valuation ring $O_{P, \ell}$. Then, since $X$ is regular, the base change of $X \to P$ to the valuation ring $O_K$ is also regular. Further its closed fiber has at most an ordinary double point as singularity by \cite{12} Proposition 3.2 (cf. Proof of 12 Theorem 3.5). Hence by Corollary 3.9 the image of $\delta$ in $H^1(\kappa(\xi), \mathbb{Z}/2\mathbb{Z})$ is 0. By Proposition 8.2, its image in $H^2(U_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$ is also 0. Thus, by the exact sequence (9.4.1), it is in the image of $H^2(Z[\frac{1}{2}], \mathbb{Z}/2\mathbb{Z})$.

Let $x \in U(\mathbb{R})$ be the point corresponding to the Fermat hypersurface. By Proposition 5.2 we also have $x^*\delta = 0$ in $H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$. Since the composite map

\[
H^2(Z[\frac{1}{2}], \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(U_{\ell}[\frac{1}{d}], \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mathbb{R}^*} H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})
\]

is an isomorphism, we have $\delta = 0$. □

**References**

[1] M. Artin, *Théorème de changement de base par un morphisme lisse et applications*, SGA 4. III Exposé XVI, Springer Lecture Notes in Math. 305, 209-252 (1973).
[2] P. Balmer, *Triangular Witt groups. Part II: from usual to derived*, Math. Z. **236** (2001) 351-382.

[3] P. Berthelot, L. Breen, W. Messing, *Théorie de Dieudonné cristalline II*, Springer Lecture Notes in Mathematics, Volume 930, 1982.

[4] H. Cartan, S. Eilenberg, *HOMOLOGICAL ALGEBRA*, Princeton UP, 1953.

[5] Ph. Cassou-Noguès, B. Erez, M. J. Taylor, *Hasse-Witt invariants of symmetric complexes: an example from geometry*, C. R. Acad. Sci. Paris, Ser. I, **334** (2002), no. 10, 839-842.

[6] M. Demazure, *Résiduant, Discriminant*, unpublished Bourbaki manuscript dated July 1969; to appear in L’Enseignement Mathématique.

[7] H. Esnault, B. Kahn and E. Viehweg, *Coverings with odd ramification and Stiefel-Whitney classes*, J. für Reine und Angew. Math. **441** (1993) pp. 145-188

[8] J.-M. Fontaine, W. Messing, *p-adic periods and p-adic etale cohomology*, Current trends in arithmetical algebraic geometry, 179-207, Contemp. Math., **67**, AMS, Providence, RI, 1987.

[9] A. Fröhlich, *Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants*, J. für Reine und Angew. Math. **360** (1985) pp. 84-123

[10] L. Illusie, *Existence de résolutions globales*, Exp. II, SGA 6, Springer LNM 225, p. 160-221, Springer-Verlag, Berlin-New York, 1971.

[11] ——, *Sur la formule de Picard-Lefschetz*, Adv. St. in Pure Math., Alg. Geom. 2000 Azumino, **36**, 249-268, 2002.

[12] N. M. Katz, *Pinceaux de Lefschetz: Théorème d’existence*, Exp. XVII, SGA 7 II, Springer LNM 340, p. 212-253, Springer-Verlag, Berlin-New York, 1973.

[13] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1970) 318-344.

[14] T. Saito, *ε-factor of a tamely ramified sheaf on a variety*, Invent. Math. **113** (1993) 389-417.

[15] ——, *Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies*, J. of Alg. Geom., **3** (1994) pp. 411-434

[16] ——, *The sign of the functional equation of the L-function of an orthogonal motive*, Invent. Math. **120** (1995) pp. 119-142

[17] ——, *The discriminant and the determinant of a hypersurface of even dimension*, preprint.

[18] J-P. Serre, *Zeta and L-functions*, Arithmetical Algebraic Geometry, Harper and Row, New York (1965), 82-92, (Oe. 64, volume 2, 249-259).

[19] ——, *L’invariant de Witt de la forme Tr(x²)*, Comm. Math. Helv., **59** (1984) pp. 651-676.

[20] ——, *Lectures on N_X(p)*. (Research Notes in Mathematics) A K Peters/CRC Press, (2012).

[21] J. Suh, *Symmetry and parity in slopes of Frobenius*, preprint.

[22] C. Walter, *Obstructions to the existence of symmetric resolutions*, preprint (2001).

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