Up- and down-quark masses from QCD sum rules

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Abstract

The QCD up- and down-quark masses are determined from an optimized QCD Finite Energy Sum Rule (FESR) involving the correlator of axial-vector current divergences. In the QCD sector this correlator is known to five loop order in perturbative QCD (PQCD), together with non-perturbative corrections from the quark and gluon condensates. This FESR is designed to reduce considerably the systematic uncertainties arising from the hadronic spectral function. The determination is done in the framework of both fixed order and contour improved perturbation theory. Results from the latter, involving far less systematic uncertainties, are: $\bar{m}_u(2\text{ GeV}) = (2.5 \pm 0.4)\text{ MeV}$, $\bar{m}_d(2\text{ GeV}) = (5.2 \pm 0.4)\text{ MeV}$, and the sum $\bar{m}_{ud} \equiv (\bar{m}_u + \bar{m}_d)/2$, is $\bar{m}_{ud}(2\text{ GeV}) = (3.9 \pm 0.2)\text{ MeV}$. 

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1 Introduction

Quark masses together with the strong coupling are the fundamental parameters of Quantum Chromodynamics (QCD). Their values at some given scale can be determined numerically from Lattice QCD (LQCD), as well as analytically from QCD sum rules (QCDSR) [1]-[3]. Historically, QCDSR were first formulated in the framework of Laplace transforms [2]-[5]. As precision determinations became necessary, in order to compare results with those from LQCD, current QCDSR are formulated in the complex squared energy, s-plane, as first proposed in [6]. In this plane the only singularities in current correlators are along the real positive semi-axis. They correspond to hadronic bound states on this axis, as well as resonances in the second Riemann sheet.

Cauchy’s theorem applied to current correlators relates QCD information on the circle to hadronic physics on the real axis (quark-hadron duality). For the determination of the light-quark masses, $m_{u,d}$, the appropriate correlator is that involving the axial-vector current divergences.

Figure 1: Integration contour in the complex s-plane.
\[ \psi_5(s \equiv -q^2) = i \int d^4x \ e^{iqx} \langle 0 \left| T(j_5(x) j_5(0)) \right| 0 \rangle, \]  \hspace{1cm} (1) \]

where

\[ j_5(x) \equiv \partial \mu A_\mu(x) = (m_d + m_u) : \bar{d}(x) i \gamma_5 u(x) :. \]  \hspace{1cm} (2) \]

Cauchy’s theorem for this correlation function becomes

\[ \frac{1}{2\pi i} \oint_{C(|s_0|)} ds \psi_5(s) \big|_{\text{QCD}} P_5(s) + \int_{s_{th}}^{s_0} ds \frac{1}{\pi} \text{Im} \psi_5(s) \big|_{\text{HAD}} P_5(s) = \sum_i R_i, \]  \hspace{1cm} (3) \]

where \( P_5(s) \) is some meromorphic function, and \( R_i \) the residues at the pole(s).

The purpose of the function \( P_5(s) \) is to quench the hadronic resonance contribution to the FESR. For the case of the pseudoscalar correlator, Eq.(1), the hadronic spectral function involves the pion pole followed by at least two radial excitations. While the mass and width of these resonances is known, this information is hardly enough to reconstruct the hadronic spectral function. Non-resonant background, inelasticity, resonance interference, etc. are realistically impossible to model. For these reasons the kernel \( P_5(s) \) was introduced in previous quark-mass determinations \[7\]-\[9\] in order to quench the contribution of the resonance region. The choice of \( P_5(s) \) in the present determination will be an analytic function. Hence, there will be no residue contribution to the right-hand-side of Eq.(3).

The contour integral in Eq.(3) is usually performed in two ways, i.e. fixed order perturbation theory (FOPT), and contour improved perturbation theory (CIPT). In FOPT the strong coupling, \( \alpha_s(s) \), is frozen on the integration contour, and the renormalization group (RG) is implemented after integration. Conversely, in CIPT the strong coupling is running and the RG improvement is used before integration. In a variety of applications either both methods give similar results, or CIPT leads to more accurate predictions. The latter will turn out to be the case in this determination.
Previous FESR results for the up- and down-quark masses, in the framework of CIPT [9], made use of the kernel $P_5(s) = 1 - a_0 s - a_1 s^2$, with $a_0 = 0.897 \text{ GeV}^{-2}$ and $a_1 = -0.1806 \text{ GeV}^{-4}$. In the QCD sector, CIPT was chosen, with the perturbative QCD (PQCD) expansion of $\psi_5(q^2)$, Eq. [1], up to order $O(\alpha_s^4)$. The strong coupling was expressed in terms of the QCD scale $\Lambda_{\text{QCD}}$, as in $\alpha_s(s) \propto 1/\ln(s/\Lambda_{\text{QCD}}^2)$, a procedure that will not be followed here as it leads to unnecessary larger uncertainties. Instead, the renormalization group equation for the strong coupling will be used in order to express the coupling in terms of some well known value at a given scale, e.g. at the tau-lepton mass scale. Finally, the issue of the convergence of the perturbative QCD expansion was not addressed in [9], but it will be considered in the present determination.

2 Pseudoscalar current correlator in QCD

The pseudoscalar current correlator, Eq.(1), in QCD is given by

$$\psi_5(q^2) = (\bar{m}_u + \bar{m}_d)^2 \left\{-q^2 \Pi_0(q^2) + O(m_{u,d}^2) \right. $$

$$- \frac{C_q}{-q^2} (\bar{m}_u + \bar{m}_d) \langle \bar{q} q \rangle + \frac{C_4 \langle O_4 \rangle}{-q^2} + O \left(\frac{1}{q^4}\right) \right\}, \quad (4)$$

where $\bar{m}_q$ stands for the running quark mass in the MS-bar renormalization scheme. The perturbative QCD function, $\Pi_0(q^2)$, can be obtained from [10]-[12], whilst the $O(\alpha_s^4)$ result can be found in [13]. To $O(\alpha_s^4)$ it is given by

$$\Pi_0(q^2) = \frac{1}{16 \pi^2} \left[-12 + 6 L + a_s A_1(q^2) + a_s^2 A_2(q^2) + a_s^3 A_3(q^2) + a_s^4 A_4(q^2) \right], \quad (5)$$

where $L \equiv \ln(-q^2/\mu^2)$, $a_s \equiv \alpha_s(-q^2)/\pi$, and the $A_i(q^2)$ are
\[ A_1(q^2) = -\frac{131}{2} + 34 L - 6 L^2 + 24 \zeta(3), \]  
(6)

\[ A_2(q^2) = \left(4 n_F \zeta(3) - \frac{65}{4} n_F - 117 \zeta(3) + \frac{10801}{24}\right)L + \left(\frac{11}{3} n_F - 106\right)L^2 \]
\[ + \left(-\frac{n_F}{3} + \frac{19}{2}\right)L^3 + \text{constants}, \]  
(7)

\[ A_3(q^2) = C_1 L - 6 \left(\frac{4781}{18} - \frac{475}{8} \zeta(3)\right)L^2 + 229 L^3 - \frac{221}{16} L^4, \]  
(8)

\[ C_1 = \frac{4748953}{864} - \frac{\pi^4}{6} - \frac{91519}{36} \zeta(3) + \frac{715}{2} \zeta(5), \]  
(9)

and

\[ A_4(q^2) = \sum_{i=1}^{5} H_i L^i, \]  
(10)

with \(\zeta(n)\) the Riemann zeta-function, \(n_F = 3\) in the light quark sector, and the coefficients \(H_i\) involving long expressions [13], numerically reducing to \(H_1 = 33532.3\), \(H_2 = -15230.645111\), \(H_3 = 3962.454926\), \(H_4 = -534.0520833\), and \(H_5 = 24.17187500\). Next, the non-perturbative terms are

\[ C_q = \frac{1}{2} + \frac{7}{3} a_s, \]  
(11)

\[ C_4(O_4) = -\frac{1}{8} a_s \left\langle G_{\mu\nu} G_{\mu\nu} \right\rangle \left[1 + \frac{11}{2} a_s\right]. \]  
(12)
In FOPT one can either use the correlator $\psi_5(q^2)$, Eq. (4), or its (convergent) second derivative. However, this is not the case in CIPT, which requires the use of the second derivative, $\psi_5''(q^2)$. The PQCD result for $\psi_5''(q^2)$ was obtained in [13], which for three flavours leads to the simplified (renormalization group improved) expression [3], [9]

$$
\psi_5''(s) \big|_{\text{RGI}}^{\text{PQCD}} = -\bar{m}_{ud}(s) \frac{1}{4 \pi^2} \frac{1}{s} \sum_{m=0}^{4} K_m \left( \frac{\alpha_s(s)}{\pi} \right)^m
$$

where $s \equiv -q^2$, and

$$
\bar{m}_{ud}(s) \equiv \frac{\bar{m}_u(s) + \bar{m}_d(s)}{2},
$$

and the coefficients $K_m$ are: $K_0 = 6$, $K_1 = 22$, $K_2 = 5071/24 - 105 \zeta(3)$, $K_3 = 5985291/2592 - \pi^4/6 - 65869 \zeta(3)/36$, and $K_4 = 3070.9698$.

The leading order non-perturbative terms in $\psi_5''(q^2)$ are:

$$
\psi_5''(q^2) |_{\langle G^2 \rangle} = -\frac{1}{4} \bar{m}^2_{ud} \frac{1}{(q^2)^3} \left( \frac{\alpha_s}{\pi} G^2 \right) \left( 1 + \frac{11}{2} \alpha_s(q^2) \right),
$$

$$
\psi_5''(q^2) |_{\langle \bar{q} q \rangle} = \frac{\bar{m}_{ud}}{(q^2)^3} \bar{m}_{ud} \langle \bar{q} q \rangle \left( 1 + \mathcal{O}(\alpha_s) \right).
$$

Unlike the case of the correlator determining the strange-quark mass, it is safe to ignore here the quark-condensate contribution. This ensures that the gluon condensate is renormalization-group invariant [17].

As mentioned in the Introduction, the strong coupling is expressed in terms of a given scale $s = s^*$ where its value is known with high precision. Using
the renormalization group equation for \( a_s(s) \equiv \alpha_s(s)/\pi \) one can perform a Taylor expansion at some given reference scale \( s = s^* \), leading to\(^{18-19} \)

\[
a_s(s) \equiv \frac{\alpha_s(s)}{\pi} = a_s(s^*) + [a_s(s^*)]^2 (-\beta_0 \eta) + [a_s(s^*)]^3 (-\beta_1 \eta + \beta_0^2 \eta^2)
+ [a_s(s^*)]^4 \left(-\beta_2 \eta + \frac{5}{2} \beta_0 \beta_1 \eta^2 - \beta_0^3 \eta^3\right)
+ [a_s(s^*)]^5 \left(-\beta_3 \eta + \frac{3}{2} \beta_1^2 \eta^2 + 3 \beta_0 \beta_2 \eta^2 - \frac{13}{3} \beta_0^2 \beta_1 \eta^3 + \beta_0^4 \eta^4\right)
+ [a_s(s^*)]^6 \left(-\beta_4 \eta + \frac{7}{2} \beta_0 \beta_1 \eta^2 + \frac{7}{2} \beta_0 \beta_3 \eta^2 - \frac{35}{6} \beta_0 \beta_1^2 \eta^3 - 6 \beta_0^2 \beta_2 \eta^3
+ \frac{77}{12} \beta_0^3 \beta_1 \eta^4 - \beta_0^5 \eta^5\right),
\]

where \( \eta \equiv \ln(s/s^*) \). The beta function is

\[
\beta(a_s) = -a_s^2 \left(\beta_0 + a_s \beta_1 + a_s^2 \beta_2 + a_s^3 \beta_3 + a_s^4 \beta_4\right),
\]

which is known up to \( \mathcal{O}(a_s^6) \). Our convention for the coefficients of the \( \beta \)-function for three flavours is such that \( \beta_0 = 9/4, \beta_1 = 4, \) etc. We shall choose the scale \( s^* = M_T^2 \), with \(^{20} \)

\[
\alpha_s(s^* \equiv M_T^2) = 0.328 \pm 0.013.
\]

Another important advantage of this procedure is that one gains an additional term in the perturbative expansion of the strong coupling, i.e. the sixth-order term above.

Similarly, by solving the renormalization group equation for \( \bar{m}(s) \), the quark
mass can also be expressed in terms of its value at some scale $s = s^*$ \[18\], \[21\].

\[
\bar{m}(s) = \bar{m}(s^*) \left\{ 1 - a(s^*) \gamma_0 \eta + \frac{1}{2} a^2(s^*) \eta \left[ -2 \gamma_1 + \gamma_0 (\beta_0 + \gamma_0) \eta \right] \right. \\
- \frac{1}{6} a^3(s^*) \eta \left[ 6 \gamma_2 - 3 \left( \beta_1 \gamma_0 + 2 (\beta_0 + \gamma_0) \right) \eta + \gamma_0 (2 \beta_0^2 + 3 \beta_0 \gamma_0 + \gamma_0^2) \eta^2 \right] \\
+ \frac{1}{24} a^4(s^*) \eta \left[ 24 \gamma_3 + 12 (\beta_2 \gamma_0 + 2 \beta_1 \gamma_1 + \gamma_1^2 + 3 \beta_0 \gamma_2 + 2 \gamma_0 \gamma_2) \eta \right] \\
- 4 \left( 6 \beta_0^2 \gamma_1 + 3 \gamma_0^2 (\beta_1 + \gamma_1) + \beta_0 \gamma_0 (5 \beta_1 + 9 \gamma_1) \right) \eta^2 + \gamma_0 (6 \beta_0^3 + 11 \beta_0^2 \gamma_0 \\
+ 6 \beta_0 \gamma_0^2 + \gamma_0^3 \eta^3 \left\{ \right. \\
+ \frac{1}{120} a^5(s^*) \eta \left[ -120 \gamma_4 + \frac{1}{\beta_0} \left( 60 \beta_1 \beta_2 \gamma_0 + 4 \beta_0^2 \gamma_3 + \beta_0 \left( 7 \beta_1 \gamma_0 + \beta_3 \gamma_0 \right) \\
- 2 \beta_2 \gamma_3 + 3 \beta_1 \gamma_2 + 2 \gamma_1 \gamma_2 + 2 \gamma_0 \gamma_3 \right) \eta - 20 \left( 3 \beta_1 \gamma_0 + \beta_1 \left( 14 \beta_0 + 9 \gamma_0 \right) \gamma_1 \right) \\
+ 3 (2 \beta_0 + \gamma_0) \left( \beta_2 \gamma_0 + \gamma_1^2 + 2 \beta_0 \gamma_2 + \gamma_0 \gamma_2 \right) \eta^2 + 10 \left( 12 \beta_0^3 \gamma_1 + \gamma_0^3 (3 \beta_1 + 2 \gamma_1) \right) \\
+ \beta_0 \gamma_0^2 \left( 13 \beta_1 + 12 \gamma_1 \right) + \beta_0^2 \gamma_0 (13 \beta_1 + 22 \gamma_1) \right) \eta^3 - \gamma_0 \left( 24 \beta_0^4 + 50 \beta_0^3 \gamma_0 \right) \\
+ 35 \beta_0^2 \gamma_0^2 + 10 \beta_0 \gamma_0^3 + \gamma_0^4 \eta^4 \right\} + \mathcal{O}(a^6(s^*)) \right\}, \tag{20}
\]

where the $\gamma(a_s)$ function is \[15\]

\[
\gamma(a_s) = -a_s (\gamma_0 + a_s \gamma_1 + a_s^2 \gamma_2 + a_s^3 \gamma_3 + a_s^4 \gamma_4) \tag{21}
\]

with the convention such that e.g. $\gamma_0 = 1$, $\gamma_1 = 91/24$, etc., for three flavours.
3 Hadronic pseudoscalar current correlator

In the hadronic sector, the spectral function of the current correlator $\psi_5(q^2)$, Eq.(1), involves the pion pole followed by the three-pion resonance contribution

$$\frac{1}{\pi} \text{Im} \psi_5|_{\text{HAD}}(s) = 2 f_{\pi}^2 M_{\pi}^4 \delta(s - M_{\pi}^2) + \frac{1}{\pi} \text{Im} \psi_5|_{\text{RES}}(s) \quad (22)$$

where $f_{\pi} = (92.07 \pm 1.20) \text{ MeV} \ [22]$, $M_{\pi} = (134.9770 \pm 0.0005) \text{ MeV} \ [22]$, and the three-pion resonance contribution is due to the $\pi(1300)$ followed by the $\pi(1800) \ [22]$. In the chiral limit the threshold behaviour of the three-pion state, first obtained in [23], is

$$\frac{1}{\pi} \text{Im} \psi_5(s)|_{\pi\pi\pi} = \theta(s) \frac{1}{3} \frac{M_{\pi}^4}{f_{\pi}^2} \frac{1}{2^8 \pi^4} s \ . \quad (23)$$

Beyond the chiral limit the threshold behaviour, first obtained in [24] and later corrected for misprints in [25], is given by

$$\frac{1}{\pi} \text{Im} \psi_5(s)|_{\pi\pi\pi} = \theta(s - 9 M_{\pi}^2) \frac{1}{9} \frac{M_{\pi}^4}{f_{\pi}^2} \frac{1}{2^8 \pi^4} I_{PS}(s) \ . \quad (24)$$

where the phase-space integral $I_{PS}(s)$ is
\[ I_{PS}(s) = \int_{4M_\pi^2}^{(\sqrt{s} - M_\pi)^2} du \sqrt{1 - \frac{4M_\pi^2}{u}} \lambda^{1/2}(1, u/s, M_\pi^2/s) \left\{ 5 + \frac{1}{2} \frac{1}{(s - M_\pi^2)^2} \right. \]
\[ \times \left[ (s - 3u + 3M_\pi^2)^2 + 3 \lambda(s, u, M_\pi^2) \left( 1 - \frac{4M_\pi^2}{u} \right) + 20 M_\pi^4 \right] \]
\[ + \frac{1}{(s - M_\pi^2)} \left[ 3(u - M_\pi^2) - s + 9M_\pi^2 \right] \right\}, \quad (25) \]

where

\[ \lambda(1, u/s, M_\pi^2/s) \equiv \left[ 1 - \frac{(\sqrt{s} + M_\pi)^2}{s} \right] \left[ 1 - \frac{(\sqrt{u} - M_\pi)^2}{s} \right], \quad (26) \]

\[ \lambda(s, u, M_\pi^2) \equiv \left[ s - (\sqrt{s} + M_\pi)^2 \right] \left[ s - (\sqrt{u} - M_\pi)^2 \right], \quad (27) \]

which in the chiral limit it reduces to \( I_{PS} = 3 \).

This threshold expression normalizes the hadronic resonance spectral function, modelled as a combination of Breit-Wigner forms \( BW_i(s) \)
\[
\frac{1}{\pi} \text{Im} \psi_5(s)|_{\text{RES}} = \text{Im} \psi_5(s)|_{\pi\pi} \frac{[BW_1(s) + \kappa BW_2(s)]}{(1 + \kappa)}, \quad (28)
\]

where \( BW_1(s_{th}) = BW_2(s_{th}) = 1 \), with

\[ BW_i(s) = \frac{(M_i^2 - s_{th})^2 + M_i^2 \Gamma_i^2}{(s - M_i^2)^2 + M_i^2 \Gamma_i^2} \quad (i = 1, 2), \quad (29) \]
and \( \kappa \) is a free parameter controlling the relative weight of the resonances. The value \( \kappa = 0.1 \) results in a smaller contribution of the second resonance compared to the first, and it will be used in the sequel. The widths of these radial excitations of the pion are affected by large uncertainties [22]. For the first resonance, \( \pi (1300) \) we shall use the determination from the two-photon process \( \gamma \gamma \rightarrow \pi^+ \pi^- \pi^0 \), as it is the most reliable [26]. The width is \( \Gamma_1 = (260 \pm 36) \) MeV. The second resonance is the \( \pi(1800) \), with a width \( \Gamma_2 = (208 \pm 12) \) MeV [22].

![Figure 2: Hadronic spectral function in the resonance region, Eqs. (28)-(29) with \( \kappa = 0.1 \), and involving two radial excitations of the pion, \( \pi(1300) \) and \( \pi(1800) \).](image)

4 QCD sum rules and results

The starting point is the analysis of the convergence of the PQCD expansion. In order to analyse it one needs to use FOPT. The reason being that the strong coupling is fixed for a given radius \( s_0 \) in the complex \( s \)-plane, unlike
the case of CIPT. The quark mass, \( \bar{m}_{ud}(s_0) \), is determined in FOPT from the FESR, Eq.(3), as

\[
(m_u + m_d)^2 = \frac{\delta_5(s_0)|_{\text{HAD}}}{\delta_5(s_0)|_{\text{QCD}}},
\]

\( \delta_5(s_0)|_{\text{HAD}} = \int_{s_{th}}^{s_0} ds \frac{1}{\pi} \text{Im} \hat{\psi}_5(s)|_{\text{HAD}} P_5(s), \)

\( \delta_5(s_0)|_{\text{QCD}} = -\frac{1}{2\pi i} \oint_{C(|s_0|)} ds \hat{\psi}_5(s)|_{\text{QCD}} P_5(s), \)

where \( \hat{\psi}_5(s)|_{\text{QCD}} \) stands for the correlator, Eq.(4), with the overall quark-mass squared factor removed, and \( P_5(s) \) is an analytic integration kernel designed to quench the hadronic contribution to the sum rule. Notice that the dimension, \( d \), of \( \delta_5(s_0)|_{\text{HAD}} \) is \( d = 6 \), while that of \( \delta_5(s_0)|_{\text{QCD}} \) is \( d = 4 \).

Substituting the PQCD result, as given in Eqs.(4)-(5), at a typical scale of \( s_0 = 3.3 \text{ GeV}^2 \), leads to

\[
[\delta_5(s_0)|_{\text{PQCD}}]^{-1/2} = 2.04 \left( 1 + 1.90 \alpha_s + 6.12 \alpha_s^2 + 18.26 \alpha_s^3 + 50.77 \alpha_s^4 \right)^{-1/2},
\]

in units of GeV\(^{-2} \), and \( \alpha_s \equiv \alpha_s(s_0) \). Using Eqs.(17)-(19) to obtain \( \alpha_s(s_0) \) shows that all terms beyond the leading order are roughly of the same size.

\[
[\delta_5(s_0)|_{\text{PQCD}}]^{-1/2} = 2.04 \left( 1 + 0.62 + 0.64 + 0.62 + 0.56 \right)^{-1/2}.
\]

Since the quark mass actually depends on the square-root of \( \delta_5 \), the relevant power series expansion is instead
\[
[\delta_5(s_0)|_{\text{PQCD}}]^{-1/2} = 2.04 (1 - 0.95 \alpha_s - 1.70 \alpha_s^2 - 2.55 \alpha_s^3 - 2.46 \alpha_s^4). \quad (35)
\]

Substituting in \(\alpha_s(s_0)\) (found from Eqs. (17)-(19), Eqs. (35) becomes

\[
[\delta_5(s_0)|_{\text{PQCD}}]^{-1/2} = 2.04 (1 - 0.31 - 0.18 - 0.09 - 0.03), \quad (36)
\]

which shows a much improved convergence. Interestingly, the expansion, Eq. (35), is an example of a Padé approximant; in this case a \([4/0]\) approximant. As a consequence of this we have tried other types of Padé approximants, but this simple one provides the optimal expansion in this application. While this Padé improvement is unquestionably a positive feature, there remain other unwelcome issues with FOPT. These include a large negative impact on the results for \(\bar{m}_{ud}\) from (i) the uncertainties in the value of \(s_0\), (ii) the estimate of the unknown six-loop contribution, and (iii) the uncertainties in \(\alpha_s\) when using Padé approximants. These issues are under much better control in CIPT, which is described next.

In the framework of CIPT the QCD sum rule is given by

\[
- \frac{1}{2\pi i} \oint_{C(|s_0|)} ds \psi_5''(s)|_{\text{QCD}} [F(s) - F(s_0)]
\]

\[
= 2 f_\pi^2 M_\pi^4 P_5(M_\pi^2) + \frac{1}{\pi} \int_{s_{th}}^{s_0} ds \text{Im} \psi_5(s)|_{\text{RES}} P_5(s), \quad (37)
\]

where \(F(s)\) depends on the explicit form of the kernel \(P_5(s)\). Several functional forms for the hadronic quenching integration kernel, \(P_5(s)\), were considered, with the optimal being

\[
P_5(s) = (s - c)(s - s_0), \quad (38)
\]
where $c = 2.4 \text{GeV}^{-2}$ lies halfway between the two resonances. This kernel quenches the hadronic contribution to the FESR at $s = s_0$, as well as in the region between the two resonances. It also leads to the most stable result for the quark masses in the wide region $s_0 \simeq (1.5 - 4.0) \text{GeV}^2$. The function $F(s)$ corresponding to this integration kernel is given by

$$F(s) = \frac{1}{12} s^4 - \frac{1}{6} (c + s_0) s^3 + \frac{1}{2} c s_0 s^2 + \left( \frac{s_0^3}{6} - \frac{1}{2} c s_0^2 \right) s,$$  

(39)

and $F(s_0)$ becomes

$$F(s_0) = \frac{s_0^3}{12} (-2c + s_0).$$  

(40)

After substituting Eqs. (13) in Eq. (37) the left-hand-side of the FESR, Eq. (37), becomes (after renormalization group improvement)

$$\delta_{5}(s_0)_{\text{PQCD}}^{\text{RGI}} = \frac{\bar{m}_u^2}{16 \pi^2} \sum_{n=0}^{4} K_n \frac{1}{2 \pi i} \int_{C(|s_0|)} \frac{ds}{s} \left[ F(s) - F(s_0) \right] \left( \frac{\bar{\alpha}_s(s)}{\pi} \right)^n,$$  

(41)

with the coefficients $K_n$ defined in Eq. (13). After substituting Eqs. (39) and (40) into Eq. (41), there are two types of integrals involved, to be computed numerically,

$$I_{NM}^{a}(s_0) \equiv \frac{1}{2 \pi i} \int \frac{ds}{s} s^N \left( \frac{\bar{\alpha}_s}{\pi} \right)^M,$$  

(42)

and

$$I_{NM}^{b}(s_0) \equiv \frac{1}{2 \pi i} \int \frac{ds}{s} \left( \frac{\bar{\alpha}_s}{\pi} \right)^M,$$  

(43)
where $N$ and $M$ are positive integers.

Finally, the running of the quark mass must be taken into account. This is achieved by starting from the RG equation for the mass

$$\frac{dm}{m} = \frac{\gamma(\alpha_s)}{\beta(\alpha_s)} d\alpha_s$$  \hspace{1cm} (44)

where $\gamma(\alpha_s)$ and $\beta(\alpha_s)$ were defined in Eqs. (18),(21).

After the change of variables $d\alpha_s(x) = i\beta(\alpha_s) dx$ in Eq. (44), followed by integration, Eq. (44) becomes

$$m(x) = m(x_0) \exp \left[ i \int_{x_0}^x \gamma[\alpha_s(x')] dx' \right]$$  \hspace{1cm} (45)

and the running quark mass entering the FESR is given by

$$\bar{m}_{ud}(x) = \bar{m}_{ud}(s_0) \exp \left[ -i \int_0^x dx' \sum J \frac{\gamma_J}{\alpha_s(x')} \right]$$  \hspace{1cm} (46)

such that the FESR determines $\bar{m}_{ud}(s_0)$. The initial value of the strong coupling is obtained from Eqs. (17)-(19).

In the non-perturbative sector we use the value of the gluon condensate in Eq. (15) from a recent precision determination [27] (earlier determinations are discussed in detail in [3])

$$\left\langle \frac{\alpha_s G^2}{\pi} \right\rangle = (0.037 \pm 0.015) \text{GeV}^4.$$

In the hadronic sector the spectral function is parametrized as in Eqs. (28)-(29). The parameter $\kappa$ can be varied in the wide range $\kappa = 0.1 - 0.2$, subject
to the requirement that the first resonance should be leading. Such a variation produces only a 1% change in $\bar{m}_{ud}$.

The result for $\bar{m}_{ud}$ as a function of $s_0$ from the sum rule, Eq.(37), in CIPT is shown in Fig.3.

![Figure 3: The quark mass $\bar{m}_{ud}(2 \text{ GeV})$ as a function of $s_0$ in CIPT from the FESR, Eq.(37).](image)

The error bar is the total uncertainty due to the various sources shown in Table 1. These are: (i) the uncertainty in the strong coupling, $\alpha_s$, Eq.(19), (ii) the value of the gluon condensate, Eq.(47), (iii) the range $s_0 = (1.5 - 4.0) \text{ GeV}^2$, (iv) the uncertainty in the resonance widths and the parameter $\kappa$.

| $\bar{m}_{ud}(4 \text{ GeV}^2)$ (MeV) | $\Delta_{\alpha_s}$ | $\Delta_{\langle G^2 \rangle}$ | $\Delta_{s_0}$ | $\Delta_{\text{HAD}}$ | $\Delta_{6}\text{-loop}$ | $\Delta_T$ |
|--------------------------------------|---------------------|---------------------|-----------------|-----------------|------------------|-----------------|
| CIPT                                 | 3.860               | 0.144               | 0.049           | 0.016           | 0.083            | 0.137           | 0.221           |

Table 1: Results for the various uncertainties from CIPT, together with the total uncertainty added in quadrature, $\Delta_T$. 

The result for $\bar{m}_{ud}$ as a function of $s_0$ from the sum rule, Eq.(37), in CIPT is shown in Fig.3.
in the hadronic spectral function, and (v) the assumption that the unknown PQCD six-loop contribution is equal to the five-loop one. This leads to

\[ \bar{m}_{ud}^{\text{CIPT}}(2 \text{ GeV}) = (3.86 \pm 0.22) \text{ MeV}, \quad (48) \]

to compare with the PDG value \[ \bar{m}_{ud}^{\text{PDG}}(2 \text{ GeV}) = (3.5 \pm 0.6) \text{ MeV}. \] In order to disentangle the individual mass values one requires as external input the quark mass ratio \( m_u/m_d \). Using the recent PDG value \[ 22 \]

\[ \frac{m_u}{m_d} = 0.48 \pm 0.08, \quad (49) \]

results in

\[ \bar{m}_u(2 \text{ GeV}) = (2.5 \pm 0.4) \text{ MeV}, \quad (50) \]

\[ \bar{m}_d(2 \text{ GeV}) = (5.2 \pm 0.4) \text{ MeV}, \quad (51) \]

to be compared with the PDG values: \( \bar{m}_u(2 \text{ GeV}) = (2.2 \pm 0.5) \text{ MeV}, \) and \( \bar{m}_d(2 \text{ GeV}) = (4.7 \pm 0.6) \text{ MeV}. \)

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**Notice:** A Mathematica® code used in the numerical evaluations is attached as a supplementary resource.
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