The Nernst heat theorem for an atom interacting with graphene: Dirac model with nonzero energy gap and chemical potential

G. L. Klimchitskaya\(^1,2\) and V. M. Mostepanenko\(^1,2,3\)

\(^1\)Central Astronomical Observatory at Pulkovo of the Russian Academy of Sciences, Saint Petersburg, 196140, Russia
\(^2\)Institute of Physics, Nanotechnology and Telecommunications, Peter the Great Saint Petersburg Polytechnic University, Saint Petersburg, 195251, Russia
\(^3\)Kazan Federal University, Kazan, 420008, Russia

Abstract

We derive the low-temperature behavior of the Casimir-Polder free energy for a polarizable atom interacting with graphene sheet which possesses the nonzero energy gap \(\Delta\) and chemical potential \(\mu\). The response of graphene to the electromagnetic field is described by means of the polarization tensor in the framework of Dirac model on the basis of first principles of thermal quantum field theory in the Matsubara formulation. It is shown that the thermal correction to the Casimir-Polder energy consists of three contributions. The first of them is determined by the Matsubara summation using the polarization tensor defined at zero temperature, whereas the second and third contributions are caused by an explicit temperature dependence of the polarization tensor and originate from the zero-frequency Matsubara term and the sum of all Matsubara terms with nonzero frequencies, respectively. The asymptotic behavior for each of the three contributions at low temperature is found analytically for any value of the energy gap and chemical potential. According to our results, the Nernst heat theorem for the Casimir-Polder free energy and entropy is satisfied for both \(\Delta > 2\mu\) and \(\Delta < 2\mu\). We also reveal an entropic anomaly arising in the case \(\Delta = 2\mu\). The obtained results are discussed in connection with the long-standing fundamental problem in Casimir physics regarding the proper description of the dielectric response of matter to the electromagnetic field.
I. INTRODUCTION

At the moment there is a strong interest to theoretical and experimental investigations of the Casimir \[1\] and Casimir-Polder \[2\] forces which act between electrically neutral bodies spaced at short separations one from the other. These forces are of entirely quantum nature and are caused by the zero-point and thermal fluctuations of the electromagnetic field. At separations exceeding several nanometers the Casimir force, which acts between two macroscopic bodies, and the Casimir-Polder force, acting between an atom and a material surface, are of relativistic character by depending on both the Planck constant \(\hbar\) and the speed of light \(c\). In fact these forces present the relativistic generalization of the familiar van der Waals forces \[3\], but take on greater significance due to multidisciplinary applications not only in atomic physics \[4–12\] and condensed matter physics \[13–15\], but also in quantum field theory \[16–18\], gravitation and cosmology \[19–22\], and for constraining predictions of high energy physics, supersymmetry and supergravity \[23–28\].

Theoretical description of the Casimir and Casimir-Polder forces is based on the semi-classical Lifshitz theory \[29, 30\], which treats the electromagnetic field in the framework of quantum field theory, but uses the classical description of matter by means of some phenomenological response functions. In the framework of this theory, the Casimir-Polder atom-plate interaction is expressed via the frequency dependent atomic polarizability and the dielectric permittivity of a plate material. Although the Lifshitz theory was successfully used over a period of several decades, modern precise experiments performed during the last few years revealed serious contradictions between experiment and theory. Specifically, for two metallic bodies the theoretical predictions obtained with taken into account relaxation properties of free (conduction) electrons were found to be in a irreconcilable contradiction with the measurement data (see Refs. \[31–39\] and reviews in Refs. \[40–42\]). The contradiction arises if the available optical data of a metal are extrapolated down to zero frequency by the well tested Drude model taking the proper account of the relaxation properties of free electrons and dies away if the lossless plasma model is used which should be applicable only at high frequencies.

By an intriguing coincidence, the Casimir entropy, calculated using the Lifshitz theory combined with the Drude model, does not vanish with vanishing temperature for metals with perfect crystal lattices and depends on the volume and other parameters of a system \[43–\]}
Thus, the Nernst heat theorem, which demands that for a physical system in thermal equilibrium the entropy at zero temperature must either vanish or be equal to the universal constant independent on the system parameters [48, 49], is violated in this case. In doing so, the Nernst heat theorem is satisfied if the plasma model is used [43–47] which is consistent with measurements of the Casimir force, but is in conflict with all our knowledge about the electric phenomena occurring at low frequencies. It was noticed also [50–52] that the Casimir entropy jumps to zero at a very low temperature starting from the negative value if the Drude model is used for metals with an imperfect crystal lattice containing some fraction of impurities. This observation, however, does not help to bring the Drude-based theory in agreement with the measurement results for the Casimir force.

Somewhat similar situation was discovered for the Casimir force between two dielectric bodies and for the Casimir-Polder force between a polarizable atom in close proximity to a dielectric plate. It was found that theoretical predictions of the Lifshitz theory obtained with taken into account conductivity at a constant current (dc conductivity) of a dielectric material are in contradictions with the measurement data of Casimir experiments [53–56]. To bring the theoretical predictions in agreement with the measurement data, one needs to omit in computations the really observable dc conductivity of a material [11, 53–56]. It seems meaningful that the calculated values of both the Casimir and Casimir-Polder entropies at zero temperature were found to violate the Nernst heat theorem if the dc conductivity of a dielectric body is included in calculations and in agreement with this theorem otherwise [57–61]. Thus, the theoretical approach consistent with the results of Casimir experiments, in spite of its inconsistency with clearly established facts in other fields of physics, was again found in accordance with the requirements of thermodynamics. The above contradictions have been called the Casimir puzzle [62] and the Casimir conundrum [63] for metallic and dielectric materials, respectively, which still remain unresolved.

From the above reasoning it may be suggested that the Nernst heat theorem plays an important role as a test for different approaches to a description of the dielectric response of matter. The weak point of existing approaches is the use of phenomenological local dielectric permittivities given by the Drude and plasma models. It is the matter of fact that real dielectrics and metals are too complicated systems, so that their response to the electromagnetic field cannot be found exactly on the basis of first principles of thermal quantum field theory. In this regard, much attention is currently attracted to graphene...
which is a 2D-sheet of carbon atoms packed in a hexagonal lattice. The remarkable feature of graphene is that at energies below 1–2 eV it is described by the Dirac model where the speed of light is replaced with the Fermi velocity \( v_F \approx c/300 \)\(^{64–66}\). This opens opportunities for a full description of the nonlocal dielectric properties of graphene in the framework of thermal quantum field theory in the Matsubara formulation. It should be noted also that the Casimir-Polder interaction of different atoms with graphene and graphene-coated substrates attracts much recent attention\(^{67–76}\). This raises a question on whether or not the Casimir and Casimir-Polder entropy in graphene systems is consistent with the Nernst heat theorem.

This question can be investigated by describing the dielectric response of graphene in terms of its polarization tensor. The exact expressions for the polarization tensor of graphene with a nonzero energy gap \( \Delta \) at zero temperature have been found in Ref.\(^{77}\). In Ref.\(^{78}\) they were generalized for the case of nonzero temperature, but only at the pure imaginary Matsubara frequencies. In Ref.\(^{79}\) another representation for the polarization tensor of graphene was obtained valid over the entire plane of complex frequencies. In Ref.\(^{80}\) it was generalized to the case of nonzero chemical potential \( \mu \). A validity of the Kramers-Kronig relations for the obtained dielectric response has been demonstrated in Ref.\(^{81}\). Thus, it was proven that the dielectric response of graphene satisfies the causality condition. Using the results of Ref.\(^{79}\), it was shown that the Casimir entropy of two parallel sheets of pristine graphene, possessing the zero energy gap and chemical potential, as well as the Casimir-Polder entropy for an atom interacting with a pristine graphene sheet, satisfy the Nernst heat theorem\(^{82, 83}\). The low-temperature expansion of the Casimir-Polder free energy for an atom interacting with real graphene sheet possessing any values of \( \Delta \) and \( \mu \) was considered in Ref.\(^{84}\), and several main terms under different relationships between \( \Delta \) and \( \mu \) have been found. Some of them, however, turned out to be in disagreement with the results of Ref.\(^{85}\) obtained only in the special case \( \Delta > 2\mu \). Thus, the issue on a validity of the Nernst heat theorem for an atom interacting with real graphene sheet remained open.

In this paper, we investigate the analytic behavior of the Casimir-Polder free energy and entropy at low temperature for an atom interacting with real graphene sheet for any relationships between the energy gap \( \Delta \) and chemical potential \( \mu \) basing on first principles of thermal quantum field theory in the Matsubara formulation. For this purpose, the thermal correction to the Casimir-Polder energy is presented as a sum of three contributions. The first of them is obtained using the polarization tensor of graphene at zero temperature. In
this case the temperature dependence arises only due to a summation over the Matsubara frequencies. The second and third contributions originate from an explicit dependence of the polarization tensor on temperature as a parameter in the Matsubara term with zero frequency and in the sum of terms with all nonzero Matsubara frequencies, respectively. It is shown that for $\Delta > 2\mu$ the Casimir-Polder free energy at sufficiently low temperature behaves as $\sim (k_B T)^5$ where $k_B$ is the Boltzmann constant whereas for $\Delta < 2\mu$ as $\sim (k_B T)^2$. These behaviors are determined by the first contribution to the thermal correction. The conclusion is made that for $\Delta > 2\mu$ and $\Delta < 2\mu$ the Casimir-Polder free energy and entropy for an atom interacting with graphene sheet are in agreement with the Nernst heat theorem. The main terms of the second and third contributions in the thermal correction to the Casimir-Polder energy are also found. According to the obtained results, for $\Delta = 2\mu$ the Casimir-Polder free energy at low temperature is of the order of $k_B T$ and is determined by the third contribution to the thermal correction. The physical meaning of the resulting entropic anomaly is discussed.

The paper is organized as follows. In Sec. II, the Casimir-Polder free energy for an atom interacting with real graphene sheet is conveniently expressed via the polarization tensor. In Sec. III, the low-temperature behavior of the first contribution to the thermal correction arising due to the Matsubara summation is found using the polarization tensor at zero temperature. Section IV considers the second contribution to the thermal correction arising from an explicit temperature dependence of the polarization tensor in the zero-frequency Matsubara term. In Sec. V, the third contribution to the thermal correction is found at low temperature which arises in a similar manner from the sum of all terms with nonzero Matsubara frequencies. In Sec. VI, the reader will find our conclusions and a discussion. Appendixes A and B contain some details of the used asymptotic expansions.

II. THE CASIMIR-POLDER FREE ENERGY FOR AN ATOM INTERACTING WITH REAL GRAPHENE SHEET DESCRIBED BY THE POLARIZATION TENSOR

The free energy of an atom spaced at a distance $a$ from real graphene sheet kept at temperature $T$ in thermal equilibrium with the environment has the form following from the Lifshitz theory for an atom interacting with any plate or some planar structure [42]. For
our purposes, it is convenient to present this equation in terms of dimensionless Matsubara
frequencies \( \zeta_l = \xi/\omega_c \), where \( l = 0, 1, 2, \ldots \), \( \xi = 2\pi k_B T/\hbar \) are the standard dimensional
Matsubara frequencies, and \( \omega_c = c/(2a) \) is the characteristic frequency. We also use the
dimensionless integration variable \( y \) which is connected with the magnitude of the wave
vector projection on the plane of a plate \( k_\perp \) by \( y = 2a(k_\perp^2 + \xi_l^2/c^2)^{1/2} \). Then the Casimir-
Polder free energy is expressed as

\[
F(a, T) = -\frac{k_B T}{8a^3} \sum_{l=0}^{\infty} \alpha_l \int_{\zeta_l}^{\infty} dy e^{-y} \left[(2y^2 - \zeta_l^2)r_{TM}(i\zeta_l, y, T) - \zeta_l^2 r_{TE}(i\zeta_l, y, T)\right],
\]

where \( \alpha_l = \alpha(i\omega_c \zeta_l) \) is the atomic electric polarizability, the prime on the summation sign
divides the term with \( l = 0 \) by two, and \( r_{TM}, r_{TE} \) are the reflection coefficients of electromagnetic waves with the transverse magnetic (TM) and transverse electric (TE) polarizations
on the plate (planar structure).

For a dielectric plate described by some phenomenological dielectric permittivity, \( r_{TM} \)
are \( r_{TE} \) are the standard Fresnel reflection coefficients. However, for graphene the reflection
coefficients are expressed via the polarization tensor of graphene found on the basis of
first principles of thermal quantum field theory [77–80]. For us it is convenient to use the
dimensionless polarization tensor \( \tilde{\Pi}_{mn,l}(y, T, \Delta, \mu) \equiv \tilde{\Pi}_{mn}(i\zeta_l, y, T, \Delta, \mu) \), \( m, n = 0, 1, 2, \)
which is expressed via the dimensional one by \( \tilde{\Pi}_{mn,l} = 2a\Pi_{mn,l}/\hbar \). Then the reflection
coefficients on a graphene sheet take the form [77–80]

\[
\begin{align*}
  r_{TM}(i\zeta_l, y, T) &= \frac{y\tilde{\Pi}_{00,l}(y, T, \Delta, \mu)}{y\tilde{\Pi}_{00,l}(y, T, \Delta, \mu) + 2(y^2 - \zeta_l^2)}, \\
  r_{TE}(i\zeta_l, y, T) &= -\frac{\tilde{\Pi}_l(y, T, \Delta, \mu)}{\tilde{\Pi}_l(y, T, \Delta, \mu) + 2y(y^2 - \zeta_l^2)},
\end{align*}
\]

where the following notation is introduced

\[
\tilde{\Pi}_l \equiv (y^2 - \zeta_l^2)tr\tilde{\Pi}_l - y^2\tilde{\Pi}_{00,l}
\]

and \( tr\tilde{\Pi}_l = \tilde{\Pi}_{mn}^m(i\zeta_l, y, T, \Delta, \mu) \).

For real graphene sheet the quantities \( \tilde{\Pi}_{00,l} \) and \( \tilde{\Pi}_l \) depend on the energy gap \( \Delta \) and
chemical potential \( \mu \). (Thus, the reflection coefficients also depend on \( \Delta \) and \( \mu \), but we
do not explicitly indicate this dependence for the sake of brevity.) Note that a nonzero
energy gap in the spectrum of electronic excitations arises under the influence of electron-electron interaction, defects of the crystal structure, for graphene deposited on a substrate
whereas the value of the chemical potential is connected with the doping concentration. Explicit expressions for $\tilde{\Pi}_{00,t}$ and $\tilde{\Pi}_t$ can be conveniently presented as the sums of independent and dependent on $\mu$ parts:

$$
\tilde{\Pi}_{00,t}(y, T, \Delta, \mu) = \tilde{\Pi}_{00,t}^{(0)}(y, \Delta) + \tilde{\Pi}_{00,t}^{(1)}(y, T, \Delta, \mu),
$$

$$
\tilde{\Pi}_t(y, T, \Delta, \mu) = \tilde{\Pi}_t^{(0)}(y, \Delta) + \tilde{\Pi}_t^{(1)}(y, T, \Delta, \mu).
$$

The independent on $\mu$ parts on the right-hand side of Eq. (4) take the following simple form:

$$
\tilde{\Pi}_{00,t}^{(0)}(y, \Delta) = \alpha y^2 - \zeta_l^2 p_l \Psi(D),
$$

$$
\tilde{\Pi}_t^{(0)}(y, \Delta) = \alpha (y^2 - \zeta_l^2) p_l \Psi(D).
$$

Here, $D \equiv \Delta/(h \omega_c)$, the function $\Psi(x)$ is defined as

$$
\Psi(x) = 2 \left[x + (1 - x^2) \arctan(x^{-1})\right],
$$

$\alpha = e^2/(hc)$ and $\bar{v}_F = v_F/c \approx 1/300$ are the fine structure constant and the Fermi velocity normalized to the speed of light, and

$$
p_l = \left[\bar{v}_F^2 y^2 + (1 - \bar{v}_F^2)\zeta_l^2\right]^{1/2}.
$$

The $\mu$-dependent parts on the right-hand side of Eq. (4) are more complicated. They are given by Eqs. (13) and (14) in Ref. 76 where it is convenient to replace the integration variable $u$ with

$$
t = \frac{h c p_l u}{2 \alpha \Delta}
$$

$$
\tilde{\Pi}_{00,t}^{(1)}(y, T, \Delta, \mu) = \frac{4 \alpha D}{\bar{v}_F^2} \int_1^{\infty} dw(t, T, \Delta, \mu) X_{00,t}(t, y, D),
$$

$$
\tilde{\Pi}_t^{(1)}(y, T, \Delta, \mu) = -\frac{4 \alpha D}{\bar{v}_F^2} \int_1^{\infty} dw(t, T, \Delta, \mu) X_t(t, y, D),
$$

where $w$ is defined as

$$
w(t, T, \Delta, \mu) = \left(e^{\frac{\Delta - 2\mu}{\bar{v}_F^2}} + 1\right)^{-1} + \left(e^{\frac{\Delta - 2\mu}{\bar{v}_F^2}} + 1\right)^{-1}
$$

and the quantities $X_{00,t}$ and $X_t$ are given by

$$
X_{00,t}(t, y, D) = 1 - \text{Re}\left[\frac{p_l^2 - \bar{v}_F^2 (y^2 - \zeta_l^2) D^2 + 2i\zeta_l p_l D t}{[p_l^4 - \bar{v}_F^2 D^2 t^2 + \bar{v}_F^2 (y^2 - \zeta_l^2) D^2 + 2i\zeta_l p_l D t]}\right]^{1/2},
$$

$$
X_t(t, y, D) = \zeta_l^2 - \text{Re}\left[\frac{\zeta_l^2 p_l^2 - \bar{v}_F^2 (y^2 - \zeta_l^2) D^2 + 2i\zeta_l p_l D t}{[p_l^4 - \bar{v}_F^2 D^2 t^2 + \bar{v}_F^2 (y^2 - \zeta_l^2) D^2 + 2i\zeta_l p_l D t]}\right]^{1/2}.
$$
As noted in Sec. I, we are interested to investigate the thermal correction to the Casimir-Polder energy as a function of temperature. For this purpose the Casimir-Polder free energy is presented in the form

\[ F(a, T) = E(a) + \delta_T F(a, T), \]  

(11)

where the Casimir-Polder energy is given by

\[ E(a) = -\frac{\hbar c}{32\pi a^4} \int_0^\infty d\zeta \alpha(i\omega_c\zeta) \int_\zeta^\infty dy e^{-y} \left[ (2y^2 - \zeta^2) r_{TM}(i\zeta, y, 0) - \zeta^2 r_{TE}(i\zeta, y, 0) \right], \]  

(12)

and the thermal correction vanishes with vanishing temperature

\[ \lim_{T \to 0} \delta_T F(a, T) = 0. \]  

(13)

The reflection coefficients in Eq. (12) are given by Eq. (2) taken at \( T = 0 \). They are expressed via the polarization tensor of graphene calculated at zero temperature and contain a continuous parameter \( \zeta \) in place of the discrete Matsubara frequencies \( \zeta_l \).

An important point is that in the limiting case of zero chemical potential, \( \mu \to 0 \), the quantities \( \tilde{\Pi}_{00,l}^{(0)} \) and \( \tilde{\Pi}_{l}^{(0)} \) defined in Eqs. (11) and (13) just have the meaning of the 00 component of the polarization tensor at zero temperature and the combination of its components defined in Eq. (13):

\[ \tilde{\Pi}_{00,l}^{(0)}(y, \Delta) = \tilde{\Pi}_{00,l}(y, 0, \Delta, 0), \quad \tilde{\Pi}_{l}^{(0)}(y, \Delta) = \tilde{\Pi}_{l}(y, 0, \Delta, 0). \]  

(14)

In this case the quantities \( \tilde{\Pi}_{00,l}^{(1)} \) and \( \tilde{\Pi}_{l}^{(1)} \) have the meaning of the thermal corrections to the zero-temperature polarization tensor:

\[ \tilde{\Pi}_{00,l}^{(1)}(y, T, \Delta, 0) = \delta_T \tilde{\Pi}_{00,l}(y, T, \Delta, 0), \quad \tilde{\Pi}_{l}^{(1)}(y, T, \Delta, 0) = \delta_T \tilde{\Pi}_{l}(y, T, \Delta, 0), \]  

(15)

which goes to zero with vanishing \( T \).

According to results of Ref. [76], similar situation holds for \( \mu \neq 0 \) satisfying the condition \( \Delta > 2\mu \). Under this condition the polarization tensor at zero temperature does not depend on \( \mu \), so that, once again, we have

\[ \tilde{\Pi}_{00,l}^{(0)}(y, \Delta) = \tilde{\Pi}_{00,l}(y, 0, \Delta, \mu), \quad \tilde{\Pi}_{l}^{(0)}(y, \Delta) = \tilde{\Pi}_{l}(y, 0, \Delta, \mu), \]  

(16)

and

\[ \tilde{\Pi}_{00,l}^{(1)}(y, T, \Delta, \mu) = \delta_T \tilde{\Pi}_{00,l}(y, T, \Delta, \mu), \quad \tilde{\Pi}_{l}^{(1)}(y, T, \Delta, \mu) = \delta_T \tilde{\Pi}_{l}(y, T, \Delta, \mu), \]  

(17)
where the quantities in Eq. (17) go to zero when \( T \) goes to zero.

Another situation takes place for \( \mu \neq 0 \) satisfying the condition \( \Delta < 2\mu \). In this case the quantities \( \bar{\Pi}_{00,I}^{(0)} \) and \( \bar{\Pi}_{l}^{(0)} \) are not equal to the 00 component of the polarization tensor at zero temperature and to the combination of its components defined in Eq. (3). In fact under the condition \( \Delta < 2\mu \) the polarization tensor at \( T = 0 \) depends on \( \mu \). The precise expressions for the quantities \( \bar{\Pi}_{00,I}(y,0,\Delta,\mu) \) and \( \bar{\Pi}_{l}(y,0,\Delta,\mu) \) in this case have been obtained in Eqs. (21) and (24) of Ref. [89] by direct calculation using Eqs. (4)–(10). In terms of the dimensionless variables used above they are given by

\[
\bar{\Pi}_{00,I}(y,0,\Delta,\mu) = \frac{8\alpha\mu}{\bar{v}_F^2\hbar\omega_c} - \frac{2\alpha(y^2 - \zeta_l^2)}{p_l^3} \left\{ (p_l^2 + D^2) \text{Im} \left( z_l \sqrt{1 + z_l^2} \right) \right. \\
+ (p_l^2 - D^2) \left[ \text{Im} \ln \left( z_l + \sqrt{1 + z_l^2} \right) - \frac{\pi}{2} \right] \right\},
\]

\[
\bar{\Pi}_{l}(y,0,\Delta,\mu) = \frac{8\alpha\mu\zeta_l^2}{\bar{v}_F^2\hbar\omega_c} + \frac{2\alpha(y^2 - \zeta_l^2)}{p_l} \left\{ (p_l^2 + D^2) \text{Im} \left( z_l \sqrt{1 + z_l^2} \right) \right. \\
- (p_l^2 - D^2) \left[ \text{Im} \ln \left( z_l + \sqrt{1 + z_l^2} \right) - \frac{\pi}{2} \right] \right\},
\]

where

\[
z_l \equiv z_l(y,\Delta,\mu) = \frac{p_l}{\bar{v}_F \sqrt{p_l^2 + D^2} \sqrt{y^2 - \zeta_l^2}} \left( \zeta_l + i \frac{2\mu}{\hbar\omega_c} \right). \tag{19}
\]

It is easily seen that for \( \mu = \Delta = 0 \) these equations reduce to the result given by Eq. (5) with \( \Delta = 0 \).

Now we are in a position to present the reflection coefficients (2) in the form

\[
r_{TM(TE)}(i\zeta_l, y, T) = r_{TM(TE)}(i\zeta_l, y, 0) + \delta_T r_{TM(TE)}(i\zeta_l, y, T), \tag{20}
\]

where the first contributions on the right-hand side are determined by the polarization tensor at \( T = 0 \)

\[
r_{TM}(i\zeta_l, y, 0) = \frac{y\bar{\Pi}_{00,I}(y,0,\Delta,\mu)}{y\bar{\Pi}_{00,I}(y,0,\Delta,\mu) + 2(y^2 - \zeta_l^2)},
\]

\[
r_{TE}(i\zeta_l, y, 0) = -\frac{\bar{\Pi}_{l}(y,0,\Delta,\mu)}{\bar{\Pi}_{l}(y,0,\Delta,\mu) + 2y(y^2 - \zeta_l^2)}, \tag{21}
\]

whereas the second contribution has the meaning of the thermal correction and goes to zero with vanishing \( T \). This equation, however, is valid in both cases \( \Delta > 2\mu \) [here, in accordance
to Eq. (16), the polarization tensor at \( T = 0 \) is presented in Eq. (5) and \( \Delta < 2\mu \) [here it is given by Eq. ??eq18]. As to the case \( \Delta = 2\mu \), it is discussed in the next sections, as well as the explicit approximate expressions for thermal corrections to the reflection coefficients on the right-hand side of Eq. (20).

Now we substitute Eq. (20) in Eq. (1) and, using Eq. (11), present the thermal correction to the Casimir-Polder energy as a sum of three contributions

\[
\delta_T \mathcal{F}(a, T) = \mathcal{F}(a, T) - E(a) = \delta_T^{(1)} \mathcal{F}(a, T) + \delta_T^{(2)} \mathcal{F}(a, T) + \delta_T^{(3)} \mathcal{F}(a, T). \tag{22}
\]

Here, the first term on the right-hand side is given by the difference of Eqs. (1) and (12) where in Eq. (1) the reflection coefficients at zero temperature \( (21) \) are substituted in place of the respective reflection coefficients \( (2) \) defined at temperature \( T \). From this it follows that the thermal correction \( \delta_T^{(1)} \mathcal{F} \) is completely determined by a summation over the Matsubara frequencies. The sum of the thermal corrections \( \delta_T^{(2)} \mathcal{F} \) and \( \delta_T^{(3)} \mathcal{F} \) is given by Eq. (1) where the thermal corrections to the reflection coefficients \( \delta_T r_{TM} \) and \( \delta_T r_{TE} \) from Eq. (20) are substituted in place of the reflection coefficients \( r_{TM} \) and \( r_{TE} \), respectively, defined in Eq. (2).

In doing so, \( \delta_T^{(2)} \mathcal{F} \) is equal to the zero-frequency term of the obtained sum alone, whereas \( \delta_T^{(3)} \mathcal{F} \) is the sum over all nonzero Matsubara frequencies. The corrections \( \delta_T^{(2)} \mathcal{F} \) and \( \delta_T^{(3)} \mathcal{F} \) are nonzero due to an explicit dependence of the polarization tensor on temperature as a parameter.

In the next sections, the low-temperature behaviors of the thermal corrections \( \delta_T^{(1)} \mathcal{F} \), \( \delta_T^{(2)} \mathcal{F} \) and \( \delta_T^{(3)} \mathcal{F} \) are investigated one after another.

### III. THERMAL CORRECTION TO THE CASIMIR-POLDER ENERGY DUE TO MATSUBARA SUMMATION USING THE ZERO-TEMPERATURE REFLECTION COEFFICIENTS

In this section, we find the behavior of the first contribution to the thermal correction, \( \delta_T^{(1)} \mathcal{F} \), at low temperature under different relationships between the energy gap and chemical potential. As defined in Sec. II, \( \delta_T^{(1)} \mathcal{F} \) is given by the difference of the sum \( (11) \) over the discrete Matsubara frequencies, where the reflection coefficients \( r_{TM(TE)}(i\zeta_l, y, T) \) are replaced with \( r_{TM(TE)}(i\zeta_l, y, 0) \), and the integral \( (12) \) with respect to continuous imaginary frequency containing the reflection coefficients \( r_{TM(TE)}(i\zeta, y, 0) \). Using the Abel-Plana for-
mula, this difference can be written in the form

$$\delta_T^{(1)} \mathcal{F}(a, T) = -i \alpha_0 k_B T \int_0^\infty dt \frac{\Phi(it\tau) - \Phi(-it\tau)}{e^{2\pi t} - 1}$$

(23)

where \(\Phi(x) = \Phi_1(x) + \Phi_2(x)\),

$$\Phi_1(x) = 2 \int_x^\infty dy y^2 e^{-y} r_{TM}(ix, y, 0),$$

$$\Phi_2(x) = -x^2 \int_x^\infty dy e^{-y}[r_{TM}(ix, y, 0) + r_{TE}(ix, y, 0)]$$

and the dimensionless temperature parameter is defined as \(\tau = 4\pi a k_B T/(\hbar c) = 2\pi k_B T/(\hbar \omega_c)\).

In Eq. (23), we have preserved only the static atomic polarizability \(\alpha_0 = \alpha(0)\) in the expansion of \(\alpha(it\tau) = \alpha(ix)\) in the powers of \(x\). This is because we are looking for the main term in the expansion of the Casimir-Polder free energy \(\mathcal{F}\) at low \(T\) (\(\tau \ll 1\)). Note also that care must be exercised when expanding the functions \(\Phi_1\) and \(\Phi_2\) in the powers of \(x\). It may happen that an expansion of the reflection coefficients in the powers of \(x\) with subsequent integration leads to incorrect results because common powers of \(x\) arise from different expansion orders of the reflection coefficients (see below).

We begin with the case of a slightly doped graphene \(\Delta > 2\mu\). In this case the polarization tensor at zero temperature does not depend on \(\mu\) [see Eq. (16)] and is given by Eq. (5). The reflection coefficients entering the thermal correction \(\delta_T^{(1)} \mathcal{F}(a, T)\) are given by Eq. (21). For the function \(\Phi_1\), defined in Eq. (24), it is not productive to expand the reflection coefficient \(r_{TM}\) in powers of \(x\) with subsequent integration as noted above. Instead, an expansion of \(\Phi_1\) in the Taylor series in powers of \(x\) using Eqs. (5) and (21) results in \(\Phi_1(0) = \Phi_1^{(3)}(0) = \Phi_1^{(5)}(0) = 0\). Then we conclude that the leading contribution of \(\Phi_1\) to \(\delta_T^{(1)} \mathcal{F}(a, T)\) is of higher order than \(T^6\) because the even powers in \(x\) do not contribute to Eq. (23).

An expansion of the function \(\Phi_2\) defined in Eq. (24) in powers of \(x\) can be found by expanding the sum of the reflection coefficients \(r_{TM}\) and \(r_{TE}\) in powers of \(x\) with subsequent integration with respect to \(y\). This is done under an assumption \(D > 1\) which is valid at sufficiently large separations \(a > 1\ \mu m\). Taking into account that the main contributions to the integrals in Eq. (24) are given by \(y \sim 1\) and that \(\zeta = \tau l\), at low temperature the quantity \(p_l\) defined in Eq. (7) satisfies the inequality \(p_l \ll 1\), so that \(D/p_l \gg 1\). Then the main contribution to the function \(\Psi\) in Eqs. (5) and (6) is given by

$$\Psi \left( \frac{D}{p_l} \right) \approx \frac{8}{3} \frac{p_l}{D}. \quad (25)$$
With account of this equation one obtains

$$\Phi_2(x) = \frac{\hbar c\alpha(1 + \tilde{\nu}_F^2)}{3\tilde{\nu}_F^2a\Delta}x^4\text{Ei}(-x) + Cx^4 + O(x^5), \quad (26)$$

where \(\text{Ei}(z)\) is the exponential integral and \(C\) is a constant which does not contribute to Eq. (23).

Substituting Eq. (26) in Eq. (23), one finds

$$\delta T^{(1)} F(a, T) = -\frac{\alpha_0(k_B T)^5}{(\hbar c)^2 \Delta} \frac{8\alpha(1 + \tilde{\nu}_F^2)}{\tilde{\nu}_F^2}. \quad (27)$$

Thus, under a condition \(\Delta > 2\mu\) the thermal correction \(\delta T^{(1)} F\) vanishes with temperature faster than for a pristine graphene where it is of the order of \((k_B T)^4\) [83].

We are coming now to the case of \(\Delta < 2\mu\). In this case the thermal correction \(\delta T^{(1)} F\) is again given by the difference of the sum Eq. (1) with the zero-temperature reflection coefficients \(r_{TM,TE}(i\zeta, y, 0)\) and the integral [12] resulting in Eq. (23). It is convenient, however, to present the function \(\Phi(x)\) in an equivalent form \(\Phi(x) = \chi_1(x) + \chi_2(x)\), where

$$\begin{align*}
\chi_1(x) &= \int_{-\infty}^{\infty} dy e^{-y(2y^2 - x^2)}r_{TM}(ix, y, 0), \\
\chi_2(x) &= -x^2 \int_{-\infty}^{\infty} dy e^{-y}r_{TE}(ix, y, 0). \quad (28)
\end{align*}$$

The reflection coefficients are again given by Eq. (23), but the polarization tensor is now presented in Eqs. (18) and (19) where the discrete Matsubara frequencies \(\zeta_l = \tau l\) are replaced with \(x\). Then the polarization tensor in Eq. (21) is replaced with \(\tilde{\Pi}_{00}(x, y, 0, \Delta, \mu)\).

The low-temperature expansion of the quantity \(\chi_1(x)\) can be performed in the same way as of \(\Phi_1(x)\), i.e., by expanding \(\chi_1(x)\) in the Taylor series in powers of \(x\). Using Eqs. (28) and (21), one obtains

$$\chi_1'(0) = 2 \int_{0}^{\infty} dy e^{-y^2} \frac{\partial}{\partial x} \left. \frac{y\tilde{\Pi}_{00}(x, y, 0, \Delta, \mu)}{y\tilde{\Pi}_{00}(x, y, 0, \Delta, \mu) + 2(y^2 - x^2)} \right|_{x=0}$$

$$= 4 \int_{0}^{\infty} dy e^{-y^2} \frac{\partial}{\partial x} \left. \frac{\tilde{\Pi}_{00}(x, y, 0, \Delta, \mu)}{[\tilde{\Pi}_{00}(0, y, 0, \Delta, \mu) + 2y]^2} \right|_{x=0} \quad (29)$$

In what follows we use the condition

$$\sqrt{4\mu^2 - \Delta^2} > \hbar \omega_c, \quad (30)$$
which is valid at sufficiently large separations. Then from the first formula in Eq. (18) we have
\[ \tilde{\Pi}_{00}(0, y, 0, \Delta, \mu) = \frac{8\alpha}{\tilde{v}_F} \frac{\mu}{\hbar \omega_c} \equiv Q_0. \] (31)

By calculating the derivative of the first formula in Eq. (18) at \( x = 0 \), we obtain
\[ \frac{\partial}{\partial x} \tilde{\Pi}_{00}(x, y, 0, \Delta, \mu) \bigg|_{x=0} = \frac{4\alpha}{\tilde{v}_F^3 y} \frac{4\mu^2 - (\hbar \omega_c \tilde{v}_F y)^2}{\sqrt{4\mu^2 - (\hbar \omega_c \tilde{v}_F y)^2 - \Delta^2}}. \] (32)

Taking into account the condition (30), the inequality \( \tilde{v}_F \ll 1 \) and the fact that the main contribution to Eq. (29) is given by \( y \sim 1 \), Eq. (32) can be simplified to
\[ \frac{\partial}{\partial x} \tilde{\Pi}_{00}(x, y, 0, \Delta, \mu) \bigg|_{x=0} = \frac{16\alpha}{\tilde{v}_F^3 y} \frac{\mu^2}{\hbar \omega_c \sqrt{4\mu^2 - \Delta^2}}. \] (33)

Substituting Eqs. (31) and (33) in Eq. (29), one finds
\[ \chi_1'(0) = \frac{16\alpha \mu^2}{\tilde{v}_F^3 \hbar \omega_c \sqrt{4\mu^2 - \Delta^2}} \int_0^\infty dy y^2 e^{-y} \frac{4y^2}{(2y + Q_0)^2} \]
\[ = -\frac{16\alpha \mu^2}{\tilde{v}_F^3 \hbar \omega_c \sqrt{4\mu^2 - \Delta^2}} \left[ \frac{2 + Q_0}{2} + \frac{Q_0(4Q_0 + 4)}{4} e^{Q_0/2} \text{Ei} \left( -\frac{Q_0}{2} \right) \right]. \] (34)

Now we have the desired result
\[ \chi_1(x) = \chi_1(0) + \chi_1'(0)x + O(x^2), \] (35)
where \( \chi_1(0) \) does not contribute to Eq. (23) and the value of the first derivative at \( x = 0 \) is presented in Eq. (34). From Eq. (28) it is easily seen that \( \chi_2(0) = \chi_2'(0) = 0 \) and similar expansion for the function \( \chi_2(x) \) takes the form
\[ \chi_2(x) = Cx^2 + O(x^3), \] (36)
where \( C \) is a constant which does not contribute to Eq. (23). Thus, from Eq. (35)
\[ \Phi(\tau t) - \Phi(-\tau t) = 2i\chi_1'(0)\tau t \] (37)
and, after substitution to Eq. (23) with account of Eq. (34), one obtains
\[ \delta_{T}^{(1)} \mathcal{F}(a, T) = -\frac{\alpha_0 \mu^2 (k_B T)^2}{(\hbar c)^2 a \sqrt{4\mu^2 - \Delta^2}} \frac{16\alpha}{\tilde{v}_F} \left[ 2 + Q_0 + \frac{Q_0(4Q_0 + 4)}{2} e^{Q_0/2} \text{Ei} \left( -\frac{Q_0}{2} \right) \right]. \] (38)
It is seen that in the case $\Delta < 2\mu$ the behavior of the thermal correction $\delta T^{(1)} F$ at low temperature is different from the case of graphene with $\Delta > 2\mu$ [see Eq. (27)] and from the case of pristine graphene.

Now we consider the low-temperature behavior of $\delta T^{(1)} F$ for the case $\Delta = 2\mu$. This case cannot be considered by the limiting transition $\Delta \rightarrow 2\mu$ from our result (38) obtained for $\Delta < 2\mu$ because it was derived under the condition (30).

Below we show that in the case $\Delta = 2\mu$ the low-temperature behavior of $\delta T^{(1)} F$ is again given by Eq. (27) derived in the case $\Delta > 2\mu$. We start from the polarization tensor at zero temperature (5) where $\zeta_l = \tau l$ is replaced with $x$. To be specific, we consider

$$\tilde{\Pi}_{00}^{(0)}(x, y, \Delta) = \alpha y^2 - x^2 \frac{\Psi}{p(x)},$$

where $\Psi$ is defined in Eq. (6) and $p(x) = [\tilde{v}_F^2 y^2 + (1 - \tilde{v}_F^2)x^2]^{1/2}$. Under the condition $D > 1$, we consider the value of $\tilde{\Pi}_{00}^{(0)}$ at zero

$$\tilde{\Pi}_{00}^{(0)}(0, y, \Delta) = \frac{\alpha y}{\tilde{v}_F} \Psi \left( \frac{D}{\tilde{v}_F y} \right).$$

Expanding this quantity in powers of the small parameter $\tilde{v}_F y / D$, one arrives at

$$\tilde{\Pi}_{00}^{(0)}(0, y, \Delta) = \frac{8\alpha y}{\tilde{v}_F} \sum_{k=0}^{\infty} (-1)^k \frac{k + 1}{(2k + 1)(2k + 3)} \left( \frac{\tilde{v}_F y^2}{D} \right)^{2k+1}.$$  \hfill (41)

This equation is also valid at $\Delta = 2\mu$. To make sure that this is the case, we consider the first formula in Eq. (18) expressing the zero-temperature polarization tensor in the case $\Delta < 2\mu$, replace there $\zeta_l$ with $x$, put $\mu = \Delta / 2$, $x = 0$ and obtain

$$\tilde{\Pi}_{00}(0, y, 0, \Delta, \Delta/2) = \frac{4\alpha D}{\tilde{v}_F^2} - \frac{2\alpha}{\tilde{v}_F^3 y} \left\{ \tilde{v}_F yD + (\tilde{v}_F^2 y^2 - D^2) \left[ \text{Im} \ln(iD + \tilde{v}_F y) - \frac{\pi}{2} \right] \right\}.$$  \hfill (42)

Expanding this equation in powers of $\tilde{v}_F y / D$, one again obtains the right-hand side of Eq. (41) with a conclusion that

$$\tilde{\Pi}_{00}^{(0)}(0, y, \Delta) = \tilde{\Pi}_{00}(0, y, 0, \Delta, \Delta/2).$$  \hfill (43)

In a similar way, it is easy to show that

$$\tilde{\Pi}^{(0)}(0, y, \Delta) = \tilde{\Pi}(0, y, 0, \Delta, \Delta/2)$$  \hfill (44)
and also
\[ \left. \frac{\partial \tilde{\Pi}^{(0)}_{00}}{\partial x} \right|_{x=0} = \left. \frac{\partial \Pi_{00}}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial \tilde{\Pi}^{(0)}_{\ell}}{\partial x} \right|_{x=0} = \left. \frac{\partial \Pi_{\ell}}{\partial x} \right|_{x=0} = 0. \] (45)

We conclude that the polarization tensor at zero temperature (5) and (18) is continuous at the point \( \Delta = 2\mu \), and the thermal correction \( \delta_T^{(1)} F \) at this point is really given by Eq. (27).

IV. THE ROLE OF EXPPLICIT TEMPERATURE DEPENDENCE OF REFLECTION COEFFICIENTS: ZERO-FREQUENCY CONTRIBUTION

In this section we consider the low-temperature behavior of the second contribution \( \delta_T^{(2)} F \) to the thermal correction defined in Eq. (22). It is given by
\[ \delta_T^{(2)} F(a, T) = -\frac{\alpha_0 k_B T}{8a^3} \int_0^\infty dy e^{-y^2} \delta_T r_{TM}(0, y, T). \] (46)

To find \( \delta_T r_{TM} \), we substitute the representation for the polarization tensor 
\[ \tilde{\Pi}_{00,\ell}(y, T, \Delta, \mu) = \tilde{\Pi}_{00,\ell}(y, 0, \Delta, \mu) + \delta_T \tilde{\Pi}_{00,\ell}(y, T, \Delta, \mu) \] (47)
in the first formula in Eq. (2) and expand the obtained expression up to the first power in small parameter
\[ \frac{\delta_T \tilde{\Pi}_{00,\ell}(y, T, \Delta, \mu)}{\tilde{\Pi}_{00,\ell}(y, 0, \Delta, \mu)}. \] (48)

The result is
\[ \delta_T r_{TM}(\imath \zeta, y, T) = \frac{2y(y^2 - \zeta^2) \delta_T \tilde{\Pi}_{00,\ell}(y, T, \Delta, \mu)}{[y \tilde{\Pi}_{00,\ell}(y, 0, \Delta, \mu) + 2(y^2 - \zeta^2)]^2}. \] (49)

In this section we use Eq. (49) at \( l = 0 \), but in Sec. V below it is used at all \( l \geq 1 \).

We start with the case \( \Delta > 2\mu \) where, according to Eq. (17), \( \delta_T \tilde{\Pi}_{00,0} = \tilde{\Pi}_{00,0}^{(1)} \). The latter quantity is contained in Eqs. (8)–(10) taken at \( l = 0 \). We restrict ourselves by only the second contribution to the right-hand side of Eq. (9) (below it is shown that the first one leads to an additional exponentially small factor in the result). Thus, the thermal correction to the polarization tensor has the form
\[ \delta_T \tilde{\Pi}_{00,0}(y, T, \Delta, \mu) = \frac{4\alpha D}{\tilde{v}_F^2} \int_1^\infty dt \left( e^{\frac{i\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} \left\{ 1 - \Re \frac{\tilde{v}_F^2 y^2 - D^2 t^2}{\tilde{v}_F y \sqrt{\tilde{v}_F^2 y^2 - D^2 (t^2 - 1)^2}} \right\}^{1/2}. \] (50)
The integral of the first term on the right-hand side of this equation is given by
\[\frac{4\alpha D}{\tilde{v}_F^2} \int_1^\infty dt \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} = \frac{8\alpha k_B T}{\tilde{v}_F^2} \ln \left( 1 + e^{\frac{\Delta - 2\mu}{2k_B T}} \right) \approx \frac{8\alpha k_B T}{\tilde{v}_F^2} \frac{e^{-\frac{\Delta - 2\mu}{2k_B T}}}{\hbar \omega_c} \quad (51)\]

at \(k_B T \ll \Delta - 2\mu\).

As shown in Appendix A, for the integral of the second term on the right-hand side of Eq. (50) one has
\[\frac{4\alpha D}{y\tilde{v}_F^3} \int_1^f(y) dt \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} < \frac{4\alpha \sqrt{D^2 + \tilde{v}_F^2 y^2} e^{-\frac{\Delta - 2\mu}{2k_B T} + \frac{2\omega c F y^2}{4k_B T}}} {\frac{2\tilde{v}_F^2}{D^2} - \tilde{v}_F^2 e^{-\frac{\Delta - 2\mu}{2k_B T} + \frac{2\omega c F y^2}{4k_B T}}}, \quad (52)\]

where
\[f(y) = \sqrt{1 + \frac{\tilde{v}_F^2 y^2}{D^2}}. \quad (53)\]

The quantity in Eq. (52) contains an additional factor exponentially small at \(T \to 0\), as compared to Eq. (51), and, thus, can be neglected. As a result, we have
\[\delta_T \tilde{\Pi}_{00,0}(y, T, \Delta, \mu) = \frac{8\alpha k_B T}{\tilde{v}_F^2} \frac{e^{-\frac{\Delta - 2\mu}{2k_B T}}}{\hbar \omega_c} \quad (54)\]

Note that the first contribution on the right-hand side of Eq. (9) omitted above would lead to an additional exponentially small factor of the order of \(e^{-2\mu/(k_B T)}\).

From Eqs. (5) and (25) one also obtains
\[\tilde{\Pi}_{00,0}(y, 0, \Delta, \mu) = \frac{8\alpha \hbar \omega_c y^2}{3\Delta}. \quad (55)\]

Substituting Eqs. (54) and (55) in Eq. (49) taken at \(l = 0\), we find
\[\delta_T \tilde{\Pi}_{00,0}(0, y, T) = \frac{4\alpha \frac{k_B T}{\tilde{v}_F^2} e^{-\frac{\Delta - 2\mu}{2k_B T}}}{y(1 + qy)^2}, \quad (56)\]

where \(q = 4\alpha \hbar \omega_c/(3\Delta)\).

Substituting this equation in Eq. (46) and calculating the integral, one obtains
\[\delta_T \tilde{F}(a, T) = -\alpha_0 \frac{\alpha (k_B T)^2}{a^2 \tilde{v}_F^2 \hbar c} e^{-\frac{\Delta - 2\mu}{2k_B T}} \int_0^\infty dy e^{-y} \frac{y}{(1 + qy)^2} \approx -\alpha_0 \frac{\alpha (k_B T)^2}{a^2 \tilde{v}_F^2 \hbar c} e^{-\frac{\Delta - 2\mu}{2k_B T}} \left[ 1 + \left( 1 + \frac{1}{q} \right) e^{1/q} \text{Ei} \left( -\frac{1}{q} \right) \right]. \quad (57)\]

Then under the condition \(D > 1 (\Delta > \hbar \omega_c)\) we arrive at
\[\delta_T \tilde{F}(a, T) \approx -\alpha_0 \frac{\alpha (k_B T)^2}{a^2 \tilde{v}_F^2 \hbar c} e^{-\frac{\Delta - 2\mu}{2k_B T}} (1 - 4q) \approx -\alpha_0 \frac{\alpha (k_B T)^2}{a^2 \tilde{v}_F^2 \hbar c} e^{-\frac{\Delta - 2\mu}{2k_B T}}. \quad (58)\]
This is quite different behavior at low $T$ than that obtained in Eq. (27) for the thermal correction $\delta_T^{(1)} F$ under the condition $\Delta > 2\mu$.

Now we turn to the case $\Delta < 2\mu$ for the thermal correction $\delta_T^{(2)} F$. In this case the thermal correction $\delta_T \Pi_{00,0}$ is given by

$$\delta_T \Pi_{00,0}(y, T, \Delta, \mu) = \Pi_{00,0}^{(1)}(y, T, \Delta, \mu) - \Pi_{00,0}^{(1)}(y, 0, \Delta, \mu),$$

(59)

where $\Pi_{00,0}^{(1)}$ is defined in Eqs. (8)–(10). For $l = 0$ one has

$$\Pi_{00,0}^{(1)}(y, T, \Delta, \mu) = \frac{4\alpha D}{v_F^2} (I_1 + I_2),$$

(60)

where

$$I_1 = \int_1^\infty dt \left( e^{\frac{\Delta}{2k_B T} + 1} \right)^{-1},$$

(61)

$$I_2 = \frac{1}{v_F y} \Re \int_1^\infty dt \left( e^{\frac{\Delta}{2k_B T} + 1} \right)^{-1} \frac{D^2 t^2 - \tilde{v}_F^2 y^2}{[\tilde{v}_F^2 y^2 - D^2 (t^2 - 1)]^{1/2}}.$$

Similar to the case of $\Delta > 2\mu$, the first exponential term on the right-hand side of Eq. (9) and the integral $I_2$ in Eq. (61) lead to additional exponentially decreasing factors when the temperature vanishes. For this reason, we do not consider them below. Thus, according to Eqs. (59)–(61), the thermal correction to the polarization tensor takes the form

$$\delta_T \Pi_{00,0}(y, T, \Delta, \mu) = \frac{4\alpha D}{v_F^2} \left[ \int_1^\infty dt \left( e^{\frac{\Delta}{2k_B T} + 1} \right)^{-1} - \int_1^{2\mu/\Delta} dt \right]$$

$$= \frac{4\alpha D}{v_F^2} \left[ \frac{2k_B T}{\Delta} \ln \left( e^{\frac{\Delta}{2k_B T} + 1} - \frac{2\mu}{\Delta} + 1 \right) \right]$$

$$= \frac{8\alpha k_B T}{\tilde{v}_F^2 \hbar \omega_c} \ln \left( 1 + e^{-\frac{2\mu}{2k_B T}} \right) \approx \frac{8\alpha k_B T}{\tilde{v}_F^2 \hbar \omega_c} e^{-\frac{2\mu}{2k_B T}},$$

(62)

at sufficiently low temperature.

Substituting Eq. (62) in Eq. (49) with $l = 0$ and taking into account that $\Pi_{00,0}(y, 0, \Delta, \mu) = Q_0$, where $Q_0$ is defined in Eq. (31), one finds

$$\delta_T r_{TM}(0, y, T) = \frac{16\alpha}{v_F^2} \frac{k_B T}{\hbar \omega_c} e^{-\frac{2\mu}{2k_B T}} \frac{y}{(Q_0 + 2y)^2}.$$

(63)

Finally from Eqs. (49) and (63) we obtain

$$\delta_T^{(2)} F(a, T) = -\alpha_0 \frac{\alpha (k_B T)^2}{a^2 \tilde{v}_F^2 \hbar c} e^{-\frac{2\mu}{2k_B T}} \int_0^\infty dy e^{-y} \frac{4y^3}{(2y + Q_0)^2}$$

$$= -\alpha_0 \frac{\alpha (k_B T)^2}{a^2 \tilde{v}_F^2 \hbar c} e^{-\frac{2\mu}{2k_B T}} \left[ 1 - \frac{Q_0}{4} (4 + Q_0) - \frac{Q_0^2}{8} (Q_0 + 6) e^{Q_0/2} \text{Ei}\left(-\frac{Q_0}{2}\right) \right].$$

(64)
By comparing Eqs. (58) and (64), one can conclude that in the case $\Delta < 2\mu$ the thermal correction $\delta_T^{(2)}F$ at low temperatures again decreases with $T$ exponentially fast.

Let us now consider the last case $\Delta = \mu$. Similar in Sec. III, it can be considered starting from the results obtained for $\Delta > 2\mu$. Now, however, the last transformation in Eq. (51) is not allowed because $\exp\left[\left(-\Delta + 2\mu\right)/(2k_B T)\right] = 1$. As a result, Eq. (54) should be replaced with

$$\delta_T \Pi_{00,0}(y, T, \Delta, \mu) = \frac{8\alpha}{\bar{v}_F^2} \frac{k_B T}{\hbar \omega_c} \ln 2.$$  (65)

Substituting Eqs. (55) and (65) in Eq. (49), we have

$$\delta_T r_{TM}(0, y, T) = \frac{4\alpha \ln 2}{\bar{v}_F^2} \frac{k_B T}{\hbar \omega_c} \frac{1}{y(1 + qy)^2}. \quad (66)$$

Then, from Eq. (46), in place of Eq. (58) we finally obtain

$$\delta_T^{(2)}F(a, T) = -\alpha_0 \frac{\alpha \ln 2 (k_B T)^2}{a^2 \bar{v}_F^2 \hbar c}, \quad (67)$$
i.e., the same behavior with $T$ as was found for $\delta_T^{(1)}F$ in the case $\Delta < 2\mu$ [see Eq. (58)].

V. EXPLICIT TEMPERATURE DEPENDENCE OF REFLECTION COEFFICIENTS: SUMMATION OVER THE NONZERO MATSUBARA FREQUENCIES

Here we consider the low-temperature behavior of the last, third, contribution $\delta_T^{(3)}F$ to the thermal correction in Eq. (22) which is determined by an explicit dependence of the polarization tensor on $T$ in all Matsubara terms with $l \neq 0$. It is given by

$$\delta_T^{(3)}F(a, T) = -\alpha_0 \frac{k_B T}{8a^3} \sum_{l=1}^{\infty} \int_{i\zeta_l}^{\infty} dy e^{-y} G(\zeta_l^2, y, T, \Delta, \mu),$$

$$G(\zeta_l^2, y, T, \Delta, \mu) = (2y^2 - \zeta_l^2) \delta_T r_{TM}(i\zeta_l, y, T) - \zeta_l^2 \delta_T r_{TE}(i\zeta_l, y, T). \quad (68)$$

An expression for the thermal correction $\delta_T r_{TM}$ is already given in Eq. (49). To derive similar expression for $\delta_T r_{TE}$, we substitute the representation

$$\tilde{\Pi}_l(y, T, \Delta, \mu) = \tilde{\Pi}_l(y, 0, \Delta, \mu) + \delta_T \tilde{\Pi}_l(y, T, \Delta, \mu) \quad (69)$$
in the second formula in Eq. (2) and expand the obtained expression up to the first power in small parameter

$$\frac{\delta_T \tilde{\Pi}_l(y, T, \Delta, \mu)}{\tilde{\Pi}_l(y, T, \Delta, \mu)}. \quad (70)$$
The desired result is given by

\[
\delta_T r_{\text{TE}}(i\zeta_l, y, T) = -\frac{2y(y^2 - \zeta_l^2)\delta_T \Pi_l(y, T, \Delta, \mu)}{\Pi_l(y, 0, \Delta, \mu) + 2y(y^2 - \zeta_l^2)^2}. \tag{71}
\]

We start with the case \(\Delta > 2\mu\) where, according to Eq. (17), \(\delta_T \Pi_{00,l} = \Pi_{00,l}^{(1)}\) and \(\delta_T \Pi_l = \Pi_{l}^{(1)}\) with \(\Pi_{00,l}^{(1)}\) and \(\Pi_{l}^{(1)}\) are defined in Eqs. (8)–(10). According to Eq. (16), \(\Pi_{00,l}(y, 0, \Delta, \mu) = \Pi_{00,l}^{(0)}\) and \(\Pi_l(y, 0, \Delta, \mu) = \Pi_l^{(0)}\), where the right-hand sides of these equations are given by Eqs. (5)–(7), and under the condition \(D > 1\) the quantity \(\Psi(D/p_l)\) can be replaced with \(8p_l/(3D)\) [see Eq. (25)].

Then, in the lowest order of the small parameter \(p_l/D\), Eqs. (49) and (71) take the form

\[
\delta_T r_{\text{TM}}(i\zeta_l, y, T) = \frac{2y\delta_T \Pi_{00,l}(y, T, \Delta, \mu)}{(y^2 - \zeta_l^2)\left(\frac{8\alpha y}{3D} + 2\right)} \approx \frac{y\delta_T \Pi_{00,l}(y, T, \Delta, \mu)}{2(y^2 - \zeta_l^2)},
\]

\[
\delta_T r_{\text{TE}}(i\zeta_l, y, T) = -\frac{2y\delta_T \Pi_l(y, T, \Delta, \mu)}{(y^2 - \zeta_l^2)\left(\frac{8\alpha y^2}{3D} + 2y\right)} \approx -\frac{\delta_T \Pi_l(y, T, \Delta, \mu)}{2y(y^2 - \zeta_l^2)}. \tag{72}
\]

Note that we have omitted two small terms, \(8\alpha y/(3D)\) and \(8\alpha p_l^2/(3D)\), in the denominators because, similar to Eqs. (56)–(58), they lead to the thermal corrections of higher orders which can be neglected in the result. The function \(G\) in Eq. (68), which depends on \(\zeta_l^2\), can be expanded in the powers of \(\zeta_l^2 = (\tau l)^2\)

\[
G(\zeta_l^2, y, T, \Delta, \mu) = 2y^2\delta_T r_{\text{TM}}(0, y, T) + \zeta_l^2 \frac{\partial G}{\partial \zeta_l^2} \bigg|_{\zeta_l=0} + \ldots. \tag{73}
\]

Substituting this expansion in Eq. (68), one obtains

\[
\delta_T^{(3)} F(a, T) = -\alpha_0 \frac{k_B T}{8a^3} (J_1 + J_2), \tag{74}
\]

where

\[
J_1 = 2 \sum_{l=1}^{\infty} \int_{\zeta_l}^{\infty} dy e^{-y} y^2 \delta_T r_{\text{TM}}(0, y, T), \tag{75}
\]

\[
J_2 = \sum_{l=1}^{\infty} \zeta_l^2 \int_{\zeta_l}^{\infty} dy e^{-y} \frac{\partial G}{\partial \zeta_l^2} \bigg|_{\zeta_l=0} + \ldots. \tag{76}
\]

From Eqs. (74) and (72) we find

\[
\delta_T r_{\text{TM}}(0, y, T) = \frac{8\alpha k_B Ta}{v_F h c y} e^{-\frac{\Delta - 2\mu}{2k_B T}}. \tag{76}
\]
Taking this into account, we rewrite the quantity $J_1$ in Eq. (75) as

$$J_1 = \frac{16\alpha k_B Ta}{v_F^2 \hbar c} e^{-\frac{2\Delta - 2\mu}{2k_B T}} \sum_{l=1}^{\infty} \int_{\tilde{\xi}}^{\infty} dy e^{-y^3} = \frac{16\alpha k_B Ta}{v_F^2 \hbar c} e^{-\frac{2\Delta - 2\mu}{2k_B T}} \left[ \frac{1}{e^{\tau} - 1} + \frac{\tau e^{\tau}}{(e^{\tau} - 1)^2} \right] \sum_{l=1}^{\infty} \int_{\tilde{\xi}}^{\infty} dy e^{-y^3}.$$

As shown in Appendix B, the integral $J_2$ contains the same exponentially fast decreasing with $T$ factor and differs from Eq. (77) only by the pre-exponent coefficient. Because of this, using Eq. (77), we obtain from Eq. (74)

$$\delta_T^{(3)} F(a, T) \sim \frac{k_B T}{a^3} e^{-\frac{2\mu - \Delta}{2k_B T}}.$$

It is seen that here the factor in front of the exponent decreases slower than in $\delta_T^{(2)} F$ [see Eq. (58)].

The case $\Delta < 2\mu$ can be considered in a similar manner. Using Eqs. (31) and (49), we have

$$\delta_T r_{TM}(i\zeta, y, T) = \frac{2y(y^2 - \xi^2)\delta_T \tilde{\Pi}_{00,T}(y, T, \Delta, \mu)}{[y Q_0 + 2(y^2 - \xi^2)]^2}.$$

Substituting here Eq. (82) and taking into account that under the condition (30) the inequality $Q_0 \gg 1$ holds, one can neglect by $2y$ as compared to $Q_0$ and obtain

$$\delta_T r_{TM}(0, y, T) = \frac{32\alpha k_B Ta}{v_F^2 \hbar c Q_0^2} e^{-\frac{2\mu - \Delta}{2k_B T}} y.$$

Then the quantity $J_1$ defined in Eq. (75) is

$$J_1 = \frac{64\alpha k_B Ta}{v_F^2 \hbar c Q_0^2} e^{-\frac{2\mu - \Delta}{2k_B T}} \sum_{l=1}^{\infty} \int_{\tilde{\xi}}^{\infty} dy e^{-y^3} \left[ \frac{3e^{\tau}(1 + 4e^{\tau} + e^{2\tau})}{(e^{\tau} - 1)^4} + 3\tau e^{\tau}(1 + e^{\tau}) + 6\tau e^{\tau}(e^{\tau} - 1)\tau + \frac{6}{e^{\tau} - 1} \right].$$

Similar to Appendix B, it can be shown that the integral $J_2$ leads to the same, up to a factor, dependence on $T$, as in Eq. (81). Thus, from Eq. (74) one finds

$$\delta_T^{(3)} F(a, T) \sim \frac{k_B T}{a^3} e^{-\frac{2\mu - \Delta}{2k_B T}}.$$

This is a dependence like that given by Eq. (78) for the case $\Delta > 2\mu$.

Now we consider the behavior of $\delta_T^{(3)} F$ at low temperature in the case $\Delta = 2\mu$. Similar to the correction $\delta_T^{(2)} F$ in Sec. IV, this behavior can be investigated using the results obtained
for $\Delta > 2\mu$. For this purpose, we take into account that $\Delta = 2\mu$ and from the next to last transformation in Eq. (62) obtain

$$\delta T \tilde{\Pi}_{00,0}(y, T, \Delta, \mu) = \frac{8\alpha k_B T}{\bar{v}_F^2 h \omega_c} \ln 2.$$  

(83)

Then from Eq. (72) we have

$$\delta T r_{TM}(0, y, T) = \frac{8\alpha k_B T \ln 2}{\bar{v}_F^2 h_c y}.$$  

(84)

Repeating the same derivations as in the case $\Delta > 2\mu$, one arrives at

$$J_1 = \frac{16\alpha k_B T a \ln 2}{2} \frac{2}{\tau} = \frac{8\alpha \ln 2}{\bar{v}_F^2 \pi}.$$  

(85)

and for the thermal correction $\delta T^{(3)} F$ for $\Delta = 2\mu$ finally finds

$$\delta T^{(3)} F(a, T) \sim \alpha_0 \frac{k_B T}{a^3}.$$  

(86)

This expression differs fundamentally from the behaviors of all thermal corrections considered above. According to the obtained results, in the cases $\Delta > 2\mu$ and $\Delta < 2\mu$ the Casimir-Polder entropy

$$S(a, T) = -\frac{\partial F(a, T)}{\partial T},$$  

(87)

where the Casimir-Polder free energy $F(a, T)$ is defined in Eqs. (1) and (11), vanishes with vanishing $T$. The same holds in the case $\Delta = 2\mu$ for the contribution to the free energy determined by the thermal corrections $\delta T^{(1)} F$ and $\delta T^{(2)} F$

$$\lim_{T \to 0} S^{(1,2)}(a, T) = -\lim_{T \to 0} \frac{\partial}{\partial T} \left[ \delta T^{(1)} F + \delta T^{(2)} F \right] = 0.$$  

(88)

However, according to Eq. (86), for the thermal correction $\delta T^{(3)} F$ in the case $\Delta = 2\mu$ there is some kind of entropic anomaly

$$\lim_{T \to 0} S^{(3)}(a, T) = -\lim_{T \to 0} \frac{\partial}{\partial T} \delta T^{(3)} F(a, T) \neq 0.$$  

(89)

Here, the entropy at zero temperature is not equal to zero and depends on the parameters of the system which means a violation of the Nernst heat theorem. These results are discussed in Sec. VI in connection with similar problems of the Casimir physics arising for metallic and dielectric materials.
VI. CONCLUSIONS AND DISCUSSION

In the foregoing, we have found the behavior of the Casimir-Polder free energy and entropy at low temperature for a polarizable atom interacting with real graphene sheet possessing nonzero energy gap and chemical potential. As discussed in Sec. I, this subject is of much fundamental interest in connection with problems arising in Casimir physics when using the commonly accepted local models of the dielectric response for both metallic and dielectric materials. The distinctive feature of graphene is that its nonlocal dielectric response, described by the polarization tensor, is found exactly on the basis of first principles of thermal quantum field theory. At the same time, the dielectric responses of conventional materials, described, e.g., by the Drude or plasma models, are partially the phenomenological ones. They are well confirmed experimentally only for real electromagnetic fields on a mass-shell, although in the Lifshitz theory the integration is made over all momenta both on and off a mass-shell.

According to our results, the contribution $\delta_1^{(1)}(T)F$ to the thermal correction to the Casimir-Polder energy, originating from a summation over the Matsubara frequencies using the zero-temperature polarization tensor, behaves as $\sim (k_B T)^5$ and $\sim (k_B T)^2$ at low temperature under the conditions $\Delta > 2\mu$ and $\Delta < 2\mu$, respectively. The contribution $\delta_2^{(2)}(T)F$ to the Casimir-Polder energy, which is caused by an explicit temperature dependence of the polarization tensor in the zero-frequency Matsubara term, behaves as

$$
\delta_2^{(2)}(T)F \sim \begin{cases} (k_B T)^2 e^{-\Delta/2k_B T}, & \Delta > 2\mu, \\ (k_B T)^2 e^{-2\mu/2k_B T}, & \Delta < 2\mu. \end{cases}
$$

In the case $\Delta = 2\mu$, these dependences reduce to $\delta_2^{(2)}(T)F \sim (k_B T)^2$.

The most interesting situation arises for the thermal correction $\delta_3^{(3)}(T)F$ originating from an explicit temperature dependence of the polarization tensor in the sum of all Matsubara terms with nonzero frequencies. As shown in this paper, a summation over all nonzero Matsubara frequencies reduces by one the power of the leading temperature dependence in each of the cases $\Delta > 2\mu$, $\Delta < 2\mu$, and $\Delta = 2\mu$. As a result, one obtains that

$$
\delta_3^{(3)}(T)F \sim \begin{cases} k_B T e^{-\Delta/2k_B T}, & \Delta > 2\mu, \\ k_B T e^{-2\mu/2k_B T}, & \Delta < 2\mu, \end{cases}
$$

and $\delta_3^{(3)}(T)F \sim k_B T$ for $\Delta = 2\mu$.  

22
The above results for all three contributions to the thermal correction combined together lead us to a conclusion that in both cases $\Delta > 2\mu$ and $\Delta < 2\mu$ the Casimir-Polder free energy and entropy satisfy the Nernst heat theorem. In doing so, the leading terms in the Casimir-Polder free energy at low temperature behave as $\sim (k_B T)^5$ and $(k_B T)^2$ for $\Delta > 2\mu$ and $\Delta < 2\mu$, respectively. Thus, our results do not support the statement of Ref. [84] that “the first order correction is quadratic over temperature $\sim T^2$.” This is true for the case $\Delta < 2\mu$ but not for $\Delta > 2\mu$ where the total free energy $F \sim (k_B T)^5$. Also, if the exact equality $\Delta = 2\mu$ is valid, the Casimir-Polder free energy is linear in temperature $F \sim k_B T$. In this case the Casimir-Polder entropy at zero temperature is equal to a nonzero constant depending on the parameters of a system and, thus, the Nernst heat theorem is violated. Note for a pristine graphene where $F \sim (k_B T)^3$ the Nernst heat theorem is satisfied [83].

As discussed in Sec. I, for dielectrics and metals the models of dielectric response leading to a violation of the Nernst heat theorem also result in contradictions between the theoretical predictions and the experimental data for the Casimir and Casimir-Polder forces. Up to date there is a single experiment on measuring the Casimir interaction between a Au-coated sphere and a graphene sheet deposited on a substrate [90], and its data are in good agreement with theoretical results obtained using the polarization tensor of graphene [91]. In fact the values of $\Delta$ and $\mu$ for a graphene sample used in the experiment are not known precisely so that from the practical standpoint the equality $\Delta = 2\mu$ cannot be satisfied exactly. For comparison purposes, the character of the real part of conductivity of graphene as a function of frequency also changes qualitatively depending on whether $\Delta > 2\mu$ or $\Delta < 2\mu$ [92], so that the condition $\Delta = 2\mu$ defines a singular point.

One can conclude that with the only exception of a physically unrealizable case $\Delta = 2\mu$ the Casimir-Polder free energy and entropy for an atom interacting with real graphene sheet characterized by nonzero energy gap and chemical potential satisfy the Nernst heat theorem. This result provides further support to the assumption that the widely known problems in Casimir physics discussed in Sec. I may be connected with the phenomenological character of local response functions used for both metallic and dielectric materials.
Acknowledgments

The work of G.L.K. and V.M.M. was partially supported by the Peter the Great Saint Petersburg Polytechnic University in the framework of the Program “5–100–2020”. The work of V.M.M. was partially funded by the Russian Foundation for Basic Research, Grant No. 19-02-00453 A. His work was also partially supported by the Russian Government Program of Competitive Growth of Kazan Federal University.

Appendix A: Bound for the contribution to $\delta_T \tilde{\Pi}_{00,0}$ in the case $\Delta > 2\mu$

In this Appendix, we consider the integral used in Eq. (52) and for $\Delta > 2\mu$ and restrict it as follows:

$$\int_1^\infty f(y) \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} \left[ \frac{D^2 t^2 - \tilde{v}_F^2 y^2}{D^2(t^2 - 1)} \right]^{1/2} < D^2 \int_1^\infty f(y) \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} \left[ \frac{D^2 t^2}{D^2(t^2 - 1)} \right]^{1/2} \equiv I,$$

where $f(y)$ is defined in Eq. (53).

The integral on the right-hand side of Eq. (A1) can be integrated by parts

$$I = -\int_1^\infty f(y) t \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} d\left[ \frac{\tilde{v}_F^2 y^2 - D^2(t^2 - 1)}{D^2(t^2 - 1)} \right]^{1/2}$$

$$= \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} \tilde{v}_F y + \int_1^\infty d \left[ t \left( e^{\frac{\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} \right] \frac{\tilde{v}_F^2 y^2 - D^2(t^2 - 1)}{D^2(t^2 - 1)} \right]^{1/2}.

The square root on the right-hand side of Eq. (A2) only increases if to replace it with $\tilde{v}_F y$.

Then Eq. (A2) transforms to

$$I < \tilde{v}_F y f(y) \left( e^{\frac{f(y)\Delta - 2\mu}{2k_B T}} + 1 \right)^{-1} \cdot (A3)$$

Now we take into account that, according to Eq. (53),

$$f(y) = \sqrt{1 + \frac{\tilde{v}_F^2 y^2}{D^2}} \approx 1 + \frac{\tilde{v}_F^2 y^2}{2D^2} = 1 + \frac{\tilde{v}_F^2 y^2(\hbar \omega_c)^2}{2\Delta^2}$$

and that for sufficiently low $T$ the inequality $\Delta - 2\mu \gg 2k_B T$ holds. Then one can neglect by the unity in Eq. (A3) as compared to the exponent and, substituting Eq. (A4) to its power, obtain

$$I < \frac{\tilde{v}_F y}{D} \sqrt{D^2 + \tilde{v}_F^2 y^2} e^{\frac{\Delta - 2\mu}{2k_B T}} e^{-\frac{\hbar \omega_c \tilde{v}_F y^2}{4k_B T}}.$$

(A5)

Multiplying Eqs. (A1) and (A5) by the factor $4\alpha D/(y\tilde{v}_F^3)$, we arrive at Eq. (52).
Appendix B: Estimation for the contribution to $\delta_T^{(3)}F$ in the case $\Delta > 2\mu$

Here, we estimate the contribution $J_2$ to the thermal correction to the Casimir-Polder free energy (74) defined by the second expression in Eq. (75).

According to Eq. (68), the value of the first derivative of $G$, entering Eq. (75), is given by

$$\frac{\partial G}{\partial \zeta^2} \bigg|_{\zeta=0} = -\delta_T r_{TM}(0, y, T) + 2y^2 \frac{\partial}{\partial \zeta^2} \delta_T r_{TM}(i\zeta, y, T) \bigg|_{\zeta=0} - \delta_T r_{TE}(0, y, T), \quad (B1)$$

where expressions for the thermal corrections to the reflection coefficients are contained in Eq. (72).

The derivative to the thermal correction $\delta_T r_{TM}$ is calculated using the first expression in Eq. (72)

$$\frac{\partial}{\partial \zeta^2} \delta_T r_{TM}(i\zeta, y, T) \bigg|_{\zeta=0} = \frac{1}{2y^3} \delta_T \tilde{\Pi}_{00,0}(y, T, \Delta, \mu) + \frac{1}{2y} \frac{\partial}{\partial \zeta^2} \delta_T \tilde{\Pi}_{00,1}(y, T, \Delta, \mu) \bigg|_{\zeta=0}. \quad (B2)$$

Using Eq. (17) and Eqs. (8)–(10), where only the second term contributes in Eq. (9), one obtains at sufficiently low $T$

$$\frac{\partial}{\partial \zeta^2} \delta_T \tilde{\Pi}_{00,1}(y, T, \Delta, \mu) \bigg|_{\zeta=0} = \frac{4\alpha D}{\hbar^2} \int_{1}^{\infty} dt \left( e^{\Delta - \frac{2\mu}{\hbar\omega}} + 1 \right)^{-1} \frac{\partial \chi_{00,1}}{\partial \zeta^2} \bigg|_{\zeta=0} \approx \frac{b_1}{y^4} \frac{k_B T}{\hbar\omega_c} e^{-\frac{\Delta - 2\mu}{2k_B T}}, \quad (B3)$$

where the numerical value of the constant $b_1$ is of no concern for us now.

As to the thermal correction $\delta_T \tilde{\Pi}_{00,0}$ in Eq. (B2), at low $T$ it is contained in Eq. (54) and can be written in the form

$$\delta_T \tilde{\Pi}_{00,0}(y, T, \Delta, \mu) = b_2 \frac{k_B T}{\hbar\omega_c} e^{-\frac{\Delta - 2\mu}{2k_B T}}. \quad (B4)$$

Substituting Eqs. (B3) and (B4) in Eq. (B2), one obtains

$$\frac{\partial}{\partial \zeta^2} \delta_T r_{TM}(i\zeta, y, T) \bigg|_{\zeta=0} = \frac{k_B T}{2\hbar\omega_c} \left( \frac{b_1}{y} + \frac{b_2}{y^3} \right) e^{-\frac{\Delta - 2\mu}{2k_B T}}. \quad (B5)$$

Now we return to Eq. (B1) where the first term on the right-hand side is found from Eqs. (72) and (B4)

$$\delta_T r_{TM}(0, y, T) = \frac{b_2}{2y} \frac{k_B T}{\hbar\omega_c} e^{-\frac{\Delta - 2\mu}{2k_B T}}. \quad (B6)$$
The remaining term $\delta_T r_{TE}(0, y, T)$ in Eq. (B1) is given by the second expression in Eq. (72) where, in accordance to Eq. (17), the quantity $\delta_T \tilde{\Pi}_0$ is expressed by Eq. (5). The following result is found with the help of Eqs. (5)–(10):

$$
\delta_T r_{TE}(0, y, T) = \frac{4\alpha D^3 y}{\tilde{v}_F} \int_1^{f(y)} dt \left( e^{\frac{\Delta - 2\mu}{2k_BT}} + 1 \right)^{-1} \frac{t^2 - 1}{\tilde{v}_F^2 y^2 - D^2(t^2 - 1)}^{1/2},
$$

(B7)

where $f(y)$ is defined in Eq. (53). This quantity is similar to that considered in Appendix A. By repeating the derivations of Appendix A, it is easy to see that it contains an exponentially decreasing with $T$ factor in addition to that one contained in Eq. (B6). Thus, we can neglect by the quantity (B7) in Eq. (B1) as compared to other terms.

Using Eqs. (B5) and (B6), one obtains from Eq. B1)

$$
\frac{\partial G}{\partial \zeta_l} \bigg|_{\zeta_l=0} = \frac{k_B T}{2 \hbar \omega_c} e^{-\frac{\Delta - 2\mu}{2k_BT}} \left( \frac{b_2}{y} + \frac{2b_1}{y^3} \right).
$$

(B8)

Substituting this equation to the second expression in Eq. (75), we find

$$
J_2 = \frac{k_B T}{2 \hbar \omega_c} e^{-\frac{\Delta - 2\mu}{2k_BT}} \sum_{l=1}^{\infty} \int_{\zeta_l}^{\infty} dy e^{-y} \left( \frac{b_2}{y} + \frac{2b_1}{y^3} \right) \zeta_l^2 + \ldots.
$$

(B9)

Introducing the integration variable $v = y/\zeta_l$, one obtains from Eq. (B9)

$$
J_2 = \frac{k_B T}{2 \hbar \omega_c} e^{-\frac{\Delta - 2\mu}{2k_BT}} \int_1^{\infty} dv \left[ \frac{b_2}{v} \sum_{l=1}^{\infty} \zeta_l^2 e^{-\zeta_l v} + \frac{2b_1}{v^3} \sum_{l=1}^{\infty} e^{-\zeta_l v} \right] + \ldots
$$

(B10)

$$
= \frac{2k_B T}{\hbar \omega_c} e^{-\frac{\Delta - 2\mu}{2k_BT}} \left[ b_2 T^2 \int_1^{\infty} \frac{dv e^{\tau v}(e^{\tau v} + 1)}{v(e^{\tau v} - 1)^3} + 2b_1 \int_1^{\infty} \frac{dv}{v^3(e^{\tau v} - 1)} \right] + \ldots
$$

$$
= \frac{k_B T}{\hbar \omega_c} e^{-\frac{\Delta - 2\mu}{2k_BT}} \frac{b_1 + b_2}{\tau} \int_1^{\infty} \frac{dv}{v^4} + \ldots = \frac{k_B T}{\hbar \omega_c} e^{-\frac{\Delta - 2\mu}{2k_BT}} \frac{b_1 + b_2}{3\tau} + \ldots \sim e^{-\frac{\Delta - 2\mu}{2k_BT}} + \ldots.
$$

Thus, a summation in nonzero $l$ again results in the additional factor $\sim 1/\tau$. The same holds for all expansion terms in the higher powers of $\zeta_l$ notated by dots in Eq. (B10). Thus, $J_2$ contains the same exponentially decreasing with $T$ factor as $J_1$, and the result (78) remains valid with account of $J_2$.

[1] H. B. G. Casimir, On the attraction between two perfectly conducting plates, Proc. Kon. Ned. Akad. Wet. B 51, 793 (1848).
[2] H. B. G. Casimir and D. Polder, The influence of retardation on the London–van der Waals forces, Phys. Rev. **73**, 360 (1948).

[3] V. A. Parsegian, *Van der Waals Forces: A Handbook for Biologists, Chemists, Engineers, and Physicists* (Cambridge University Press, Cambridge, 2005).

[4] F. Shimizu, Specular reflection of very slow metastable neon atoms from a solid surface, Phys. Rev. Lett. **86**, 987 (2001).

[5] H. Friedrich, G. Jacoby, and C. G. Meister, Quantum reflection by Casimir-van der Waals potential tails, Phys. Rev. A **65**, 032902 (2002).

[6] V. Druzhinina and M. DeKieviet, Experimental observation of quantum reflection far from threshold, Phys. Rev. Lett. **91**, 193202 (2004).

[7] T. E. Judd, R. G. Scott, A. M. Martin, B. Kaczmarek, and T. M. Fromhold, Quantum reflection of ultracold atoms from thin films, graphene and semiconductor heterostructures, New J. Phys. **13**, 083020 (2011).

[8] G. Rojas-Rorenzo, J. Rubayo-Soneira, S. Miret-Artés, and E. Pollak, Quantum reflection of rare-gas atoms and clusters from a grating, Phys. Rev. A **98**, 063604 (2018).

[9] A. E. Leanhardt, Y. Shin, A. P. Chikkatur, D. Kielpinski, W. Ketterle, and D. E. Pritchard, Bose-Einstein Condensates near a Microfabricated Surface, Phys. Rev. Lett. **90**, 100404 (2003).

[10] Y. Lin, I. Teper, C. Chin, and V. Vuletić, Impact of the Casimir-Polder Potential and Johnson Noise on Bose-Einstein Condensate Stability Near Surfaces, Phys. Rev. Lett. **92**, 050404 (2004).

[11] J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari, and E. A. Cornell, Measurement of the temperature dependence of the Casimir-Polder force, Phys. Rev. Lett. **98**, 063201 (2007).

[12] A. Tononi and L. Salasnich, Bose-Einstein Condensation on the Surface of a Sphere, Phys. Rev. Lett. **123**, 160403 (2019).

[13] M. Krech, The Casimir Effect in Critical Systems (World Scientific, Singapore, 1994).

[14] M. Kardar and R. Golestanian, The "friction" of vacuum and other fluctuation-induced forces, Rev. Mod. Phys. **71**, 1233 (1999).

[15] S. Y. Buhmann, D.-G. Welsch, and T. Kampf, Ground-state van der Waals forces in planar multilayer magnetodielectrics, Phys. Rev. A **72** 032112 (2005).
[16] M. Bordag, E. Elizalde, K. Kirsten, and S. Leseduarte, Casimir energies for massive scalar fields in a spherical geometry, Phys. Rev. D 56, 4896 (1997).

[17] K. A. Milton, The Casimir Effect: Physical Manifestations of Zero-Point Energy (World Scientific, Singapore, 2001).

[18] Quantum Field Theory under the Influence of External Conditions, edited by K. A. Milton and M. Bordag (World Scientific, Singapore, 2010).

[19] E. Elizalde, The vacuum energy density for spherical and cylindrical universes, J. Math. Phys. 35, 3308 (1994).

[20] E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Ogushi, Casimir effect in de Sitter and anti-de Sitter braneworlds, Phys. Rev. D 67, 063515 (2003).

[21] A. A. Saharian, Surface Casimir densities and induced cosmological constant on parallel branes in AdS spacetime, Phys. Rev. D 70, 064026 (2004).

[22] A. A. Saharian, Surface Casimir densities and induced cosmological constant in higher dimensional braneworlds, Phys. Rev. D 74, 124009 (2006).

[23] P. Brax, C. van de Bruck, A.-C. Davis, D. F. Mota, and D. Show, Detecting chameleons through Casimir force measurements, Phys. Rev. D 76, 124034 (2007).

[24] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Constraints on the parameters of an axion from measurements of the thermal Casimir-Polder force, Phys. Rev. D 89, 035010 (2014).

[25] A. Almasi, P. Brax, D. Iannuzzi, and R. I. P. Sedmik, Force sensor for chameleon and Casimir force experiments, Phys. Rev. D 91, 102002 (2015).

[26] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Constraining axion coupling constants from measuring the Casimir interaction between polarized test bodies, Phys. Rev. D 94, 035011 (2016).

[27] G. L. Klimchitskaya and V. M. Mostepanenko, Constraints on axionlike particles and non-Newtonian gravity from measuring the difference in Casimir forces, Phys. Rev. D 95, 123013 (2017).

[28] G. L. Klimchitskaya, P. Kuusk, and V. M. Mostepanenko, Constraints on non-Newtonian gravity and axionlike particles from measuring the Casimir force in nanometer separation range, Phys. Rev. D 101, 056013 (2020).

[29] I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, General theory of van der Waals
forces, Usp. Fiz. Nauk **73**, 381 (1961) [Sov. Phys. Usp. **4**, 153 (1961)].

[30] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part II (Pergamon, Oxford, 1980).

[31] R. S. Decca, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, D. López, and V. M. Mostepanenko, Improved tests of extra-dimensional physics and thermal quantum field theory from new Casimir force measurements, Phys. Rev. D **68**, 116003 (2003).

[32] R. S. Decca, D. López, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Tests of new physics from precise measurements of the Casimir pressure between two gold-coated plates, Phys. Rev. D **75**, 077101 (2007).

[33] C.-C. Chang, A. A. Banishev, R. Castillo-Garza, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Gradient of the Casimir force between Au surfaces of a sphere and a plate measured using an atomic force microscope in a frequency-shift technique, Phys. Rev. B **85**, 165443 (2012).

[34] A. A. Banishev, C.-C. Chang, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Measurement of the gradient of the Casimir force between a nonmagnetic gold sphere and a magnetic nickel plate, Phys. Rev. B **85**, 195422 (2012).

[35] A. A. Banishev, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Demonstration of the Casimir force between ferromagnetic surfaces of a Ni-coated sphere and a Ni-coated plate, Phys. Rev. Lett. **110**, 137401 (2013).

[36] G. Bimonte, D. López, and R. S. Decca, Isoelectronic determination of the thermal Casimir force, Phys. Rev. B **93**, 184434 (2016).

[37] J. Xu, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Reducing detrimental electrostatic effects in Casimir-force measurements and Casimir-force-based microdevices, Phys. Rev. A **97**, 032501 (2018).

[38] M. Liu, J. Xu, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Examining the Casimir puzzle with an upgraded AFM-based technique and advanced surface cleaning, Phys. Rev. B **100**, 081406(R) (2019).

[39] M. Liu, J. Xu, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Precision measurements of the gradient of the Casimir force between ultraclean metallic surfaces at larger separations, Phys. Rev. A **100**, 052511 (2019).

[40] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, The Casimir force between real materials: Experiment and theory, Rev. Mod. Phys. **81**, 1827 (2009).
[41] L. M. Woods, D. A. R. Dalvit, A. Tkatchenko, P. Rodriguez-Lopez, A. W. Rodriguez, and R. Podgornik, Materials perspective on Casimir and van der Waals interactions, Rev. Mod. Phys. 88, 045003 (2016).

[42] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Advances in the Casimir Effect (Oxford University Press, Oxford, 2015).

[43] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Violation of the Nernst heat theorem in the theory of thermal Casimir force between Drude metals, Phys. Rev. A 69, 022119 (2004).

[44] M. Bordag and I. Pirozhenko, Casimir entropy for a ball in front of a plane, Phys. Rev. D 82, 125016 (2010).

[45] M. Bordag, Low Temperature Expansion in the Lifshitz Formula, Adv. Math. Phys. 2014, 981586 (2014).

[46] G. L. Klimchitskaya and V. M. Mostepanenko, Low-temperature behavior of the Casimir free energy and entropy of metallic films, Phys. Rev. A 95, 012130 (2017).

[47] G. L. Klimchitskaya and C. C. Korikov, Analytic results for the Casimir free energy between ferromagnetic metals, Phys. Rev. A 91, 032119 (2015).

[48] L. D. Landau and E. M. Lifshitz, Statistical Physics, Part I (Pergamon, Oxford, 1980).

[49] Yu. B. Rumer and M. S. Ryvkin, Thermodynamics, Statistical Physics, and Kinetics (Mir, Moscow, 1980).

[50] S. Boström and Bo E. Sernelius, Entropy of the Casimir effect between real metal plates, Physica A 339, 53 (2004).

[51] I. Brevik, J. B. Aarseth, J. S. Høye, and K. A. Milton, Temperature dependence of the Casimir effect, Phys. Rev. E 71, 056101 (2005).

[52] J. S. Høye, I. Brevik, S. A. Ellingsen, and J. B. Aarseth, Analytical and numerical verification of the Nernst theorem for metals, Phys. Rev. E 75, 051127 (2007).

[53] F. Chen, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Control of the Casimir force by the modification of dielectric properties with light, Phys. Rev. B 76, 035338 (2007).

[54] G. L. Klimchitskaya and V. M. Mostepanenko, Conductivity of dielectric and thermal atom-wall interaction, J. Phys. A: Math. Theor. 41, 312002 (2008).

[55] C.-C. Chang, A. A. Banishev, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Reduction of the Casimir Force from Indium Tin Oxide Film by UV Treatment, Phys. Rev.
[56] A. A. Banishev, C.-C. Chang, R. Castillo-Garza, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Modifying the Casimir force between indium tin oxide film and Au sphere, Phys. Rev. B 85, 045436 (2012).

[57] B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, Thermal quantum field theory and the Casimir interaction between dielectrics, Phys. Rev. D 72, 085009 (2005).

[58] G. L. Klimchitskaya, U. Mohideen and V. M. Mostepanenko, Casimir-Polder interaction between an atom and a dielectric plate: Thermodynamics and experiment, J. Phys. A: Math. Theor. 41, 432001 (2008).

[59] G. L. Klimchitskaya and C. C. Korikov, Casimir entropy for magnetodielectrics, J. Phys.: Condens. Matter 27, 214007 (2015).

[60] G. L. Klimchitskaya and V. M. Mostepanenko, Casimir free energy of dielectric films: classical limit, low-temperature behavior and control, J. Phys.: Condens. Matter 29, 275701 (2017).

[61] C. C. Korikov and V. M. Mostepanenko, Nernst heat theorem for the Casimir-Polder interaction between a magnetizable atom and ferromagnetic dielectric plate, Mod. Phys. Lett. A 35, 2040010 (2020).

[62] G. L. Klimchitskaya and V. M. Mostepanenko, Experiment and theory in the Casimir effect, Contemp. Phys. 47, 131 (2006).

[63] G. L. Klimchitskaya and V. M. Mostepanenko, Graphene may help to solve the Casimir conundrum in indium tin oxide systems, Phys. Rev. B 98, 035307 (2018).

[64] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81, 109 (2009).

[65] M. I. Katsnelson, Graphene: Carbon in Two Dimensions (Cambridge University Press, Cambridge, 2012).

[66] Physics of Graphene, edited by H. Aoki and M. S. Dresselhaus (Springer, Cham, 2014).

[67] M. Chaichian, G. L. Klimchitskaya, V. M. Mostepanenko, and A. Tureanu, Thermal Casimir-Polder interaction of different atoms with graphene, Phys. Rev. A 86, 012515 (2012).

[68] S. Ribeiro and S. Scheel, Shielding vacuum fluctuations with graphene, Phys. Rev. A 88, 042519 (2013); 89 039904(E) (2014).

[69] T. Cysne, W. J. M. Kort-Kamp, D. Oliver, F. A. Pinheiro, F. S. S. Rosa, and C. Farina, Tuning the Casimir-Polder interaction via magneto-optical effects in graphene, Phys. Rev. A
90, 052511 (2014).

[70] G. L. Klimchitskaya and V. M. Mostepanenko, Classical Casimir-Polder force between polarizable microparticles and thin films including graphene, Phys. Rev. A 89, 012516 (2014).

[71] K. Kaur, J. Kaur, B. Arora, and B. K. Sahoo, Emending thermal dispersion interaction of Li, Na, K and Rb alkali-metal atoms with graphene in the Dirac model, Phys. Rev. B 90, 245405 (2014).

[72] G. L. Klimchitskaya and V. M. Mostepanenko, Impact of graphene coating on the atom-plate interaction, Phys. Rev. A 89, 062508 (2014).

[73] K. Kaur, B. Arora, and B. K. Sahoo, Dispersion coefficients for the interactions of the alkali-metal and alkaline-earth-metal ions and inert-gas atoms with a graphene layer, Phys. Rev. A 92, 032704 (2015).

[74] N. Knusnutdinov, R. Kashapov, and L. M. Woods, Casimir-Polder effect for a stack of conductive planes, Phys. Rev. A 94, 012513 (2016).

[75] N. Knusnutdinov, R. Kashapov, and L. M. Woods, Thermal Casimir and Casimir-Polder interactions in N parallel 2D Dirac materials, 2D Materials 5, 035032 (2018).

[76] C. Henkel, G. L. Klimchitskaya, and V. M. Mostepanenko, Influence of chemical potential on the Casimir-Polder interaction between an atom and gapped graphene or graphene-coated substrate, Phys. Rev. A 97, 032504 (2018).

[77] M. Bordag, I. V. Fialkovsky, D. M. Gitman, and D. V. Vassilevich, Casimir interaction between a perfect conductor and graphene described by the Dirac model, Phys. Rev. B 80, 245406 (2009).

[78] I. V. Fialkovsky, V. N. Marachevsky, and D. V. Vassilevich, Finite-temperature Casimir effect for graphene, Phys. Rev. B 84, 035446 (2011).

[79] M. Bordag, G. L. Klimchitskaya, V. M. Mostepanenko, and V. M. Petrov, Quantum field theoretical description for the reflectivity of graphene, Phys. Rev. D 91, 045037 (2015); 93, 089907(E) (2016).

[80] M. Bordag, I. Fialkovskiy, and D. Vassilevich, Enhanced Casimir effect for doped graphene, Phys. Rev. B 93, 075414 (2016); 95, 119905(E) (2017).

[81] G. L. Klimchitskaya and V. M. Mostepanenko, Kramers-Kronig relations and causality conditions for graphene in the framework of Dirac model, Phys. Rev. D 97, 085001 (2018).

[82] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Nernst heat theorem
for the thermal Casimir interaction between two graphene sheets, Phys. Rev. A 94, 042501 (2016).

[83] G. L. Klimchitskaya and V. M. Mostepanenko, Low-temperature behavior of the Casimir-Polder free energy and entropy for an atom interacting with graphene, Phys. Rev. A 98, 032506 (2018).

[84] N. Khusnutdinov and N. Emelianova, Low-temperature expansion of the Casimir-Polder free energy for an atom interacting with a conducting plane, Int J. Mod. Phys. A 34, 1950008 (2019).

[85] G. L. Klimchitskaya, The Casimir-Polder interaction of an atom and real graphene sheet: Verification of the Nernst heat theorem, Mod. Phys. Lett. A 35, 2040004 (2020).

[86] P. K. Pyatkovsky, Dynamical polarization, screening, and plasmons in gapped graphene, J. Phys.: Condens. Matter 21, 025506 (2009).

[87] V. P. Gusynin, S. G. Sharapov, and J. P. Carbotte, Heat conduction in graphene: experimental study and theoretical interpretation, New J. Phys. 11, 095013 (2009)

[88] L. A. Falkovsky, Optical properties of graphene, J. Phys.: Conf. Series 129, 012004 (2008).

[89] G. Bimonte, G. L. Klimchitskaya, and V. M. Mostepanenko, Thermal effect in the Casimir force for graphene and graphene-coated substrates: Impact of nonzero mass gap and chemical potential, Phys. Rev. B 96, 115430 (2017).

[90] A. A. Banishev, H. Wen, J. Xu, R. K. Kawakami, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Measuring the Casimir force gradient from graphene on a SiO₂ substrate Phys. Rev. B 87, 205433 (2013).

[91] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Theory of the Casimir interaction for graphene-coated substrates using the polarization tensor and comparison with experiment, Phys. Rev. B 89, 115419 (2014).

[92] G. L. Klimchitskaya, V. M. Mostepanenko, and V. M. Petrov, Conductivity of graphene in the framework of Dirac model: Interplay between nonzero mass gap and chemical potential, Phys. Rev. B 96, 235432 (2017).