ESTIMATING MONOTONE DENSITIES
BY CELLULAR BINARY TREES

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Abstract. We propose a novel, simple density estimation algorithm for bounded monotone densities with compact support under a cellular restriction. We show that its expected error ($L_1$ distance) converges at a rate of $n^{-1/3}$, that its expected runtime is sublinear and, in doing so, find a connection to the theory of Galton–Watson processes.

Keywords. Density estimation, cellular computation, binary trees, Galton–Watson trees.

MSC2020 subject classification. Primary 62G07, 68Q87, 68W40; secondary 60C05, 60J85.

1. INTRODUCTION

In this note, we consider density estimation on $[0,1]$ from an i.i.d. sample $X_1,...,X_n$ drawn from an unknown density $f$ that is monotonically decreasing on its support.

The idea of estimating a density using a recursive partition of the space into intervals (or rectangles in $\mathbb{R}^d$) is quite old. In fact, histogram estimates based on a regular grid can be viewed as estimates that use a complete tree to partition the space. We take the view that the creation of the partition is the main problem: given a partition, we estimate the density by the standard histogram estimate. The data mining community (Schmidberger, 2009 [20]; Ram and Gray, 2011 [19]; Anderlini, 2016 [2]) refers to this general method as “density estimation trees”.

We take a view similar to Biau and Devroye (2013) [7] in their cellular approach to classification. The recursive partition of the space is achieved by a simple immediate decision taken independently for each node in the tree, assuming each node corresponds to a region of the space, and the data in it. If the decision is not to split, then the node is a leaf in the tree, and the region is a set in the final partition. If the decision is to split, then there are two child nodes in the tree, corresponding to two disjoint regions whose union is the parent region. All decisions must be restricted to the locally available data—the region and the points in it—so, no decision can use the original data size, “$n$”, for example.

To illustrate this principle, we look at the benchmark problem of the estimation of a monotonically decreasing density $f$ on $[0,1]$.

Imagine that an interval $C$ is given, and a sample drawn from $f$ restricted to that interval. Let $C'$ and $C''$ be the left and right halves of $C$, and let $N'$ and $N''$ be the number of sample points falling in $C'$ and $C''$, respectively. We are asked to decide whether it is best to use a histogram estimate on $C$ (estimating $f$ on it by $(N'+N'')/(n \cdot \lambda(C))$ where $\lambda(C)$ is the length of $C$), or to split $C$ into $C'$ and $C''$. That decision is only allowed to
depend on \(N'\) and \(N''.\) This design restriction is at once an advantage—as it leads to a simple and easy-to-understand estimator—and a curse—since we do not know if one can even get consistency or good rates of convergence.

The process can be viewed as a binary tree of intervals (or cells), with the leaves corresponding to the intervals on which we apply the time-honored histogram estimate. As each decision to split is made locally, the process of partitioning \([0, 1]\) into a set of intervals is perfectly geared to distributive computing.

The rule we propose is extremely simple: split when \(N' - N'' > \gamma \sqrt{N' + N''}\) for fixed universal design constant \(\gamma.\) We will show that

\[
E \left\{ \int_0^1 |f_n(x) - f(x)| \, dx \right\} = O\left(\frac{B^{2/3}}{n^{1/3}}\right)
\]

where \(B = f(0)\) is the value at the mode, and \(f_n\) is the histogram estimate on the partition induced by the leaves. Also, when \(B = \infty,\) we still have consistency, i.e., \(E\{\int |f_n - f|\} \to 0\) as \(n \to \infty.\)

It is noteworthy that if \(\mathcal{M}_B\) denotes the class of all monotone densities on \([0, 1]\) bounded by \(B,\) then

\[
\inf_{\text{all estimators } f_n} \sup_{f \in \mathcal{M}_B} E \left\{ \int_0^1 |f_n(x) - f(x)| \, dx \right\} \geq \alpha \left(\frac{\log(1 + B)}{n}\right)^{1/3}
\]

for a universal constant \(\alpha > 0\) (Birgé, 1983 [5]; see also Devroye and Györfi, 1985 [8]). Our estimate achieves the minimax rate of convergence, albeit with a worse constant multiplicative factor. Several estimates achieve the minimax rate with the correct multiplicative factor, notably Grenander’s histogram estimate (Grenander, 1956 [11]) (which uses a partition based on the smallest concave majorant of the distribution function) and Birgé’s histogram estimate (Birgé, 1983 [5]) (which uses a partition of exponentially increasing widths). Standard histogram estimates of equal bin widths can at best achieve a rate of \((B/n)^{1/3}.\)

The design principle used here can be extended and generalized to densities with special structures—convex or concave densities, log-concave or log-convex densities, and unimodal densities spring to mind. More generalizations will be discussed in the conclusion of the paper (section 7). We start with our main theorem (section 2) and discuss its computational complexity (section 3). Then we note the importance of Galton–Watson trees in the analysis (section 4): as every density is locally nearly uniform, the performance of our splitting rule on a uniform density \(f\) matters a lot to explain the behaviour near the bottom of the tree. We will show in section 4 that for uniform \(f,\) the binary tree is essentially an extinct Galton–Watson tree of constant expected size.

In sections 5 and 6, we develop the partly combinatorial and partly probabilistic proof of Theorem [11].
2. MAIN THEOREM

Throughout this section and the paper, \( f_n \) is the histogram estimate, i.e.

\[
f_n(x) = \frac{N(C)}{n\lambda(C)}, \quad x \in C
\]

where \( C \) is the unique interval to which \( x \) belongs, \( \lambda(C) \) is the length (or Lebesgue measure) of \( C \), and \( N(C) \) is the number of \( X_i \)'s falling in \( C \). For our \( f_n \) we will show:

**Theorem 1.** Let \( f \) be a bounded decreasing probability density function on \([0,1]\) with \( B = f(0) \). Then

\[
\sup_{f \in M_B} E\left\{ \int_0^1 |f_n(x) - f(x)| dx \right\} \leq \beta \left( \frac{B^{2/3}}{n^{1/3}} \right)
\]

for a universal constant \( \beta \).

Remark 1. Note that if we partition \([0,1]\) into \( k \) equal intervals, then

\[
E\left\{ \int_0^1 |f_n - f| \right\} \leq E\left\{ \int_0^1 |f_n - Ef_n| \right\} + \int_0^1 |Ef_n - f|,
\]

where \( E\{f_n(x)\} = p(C)/\lambda(C), \ x \in C, \) and \( p(C) = \int_C f \). Thus, \( E\{f_n(x)\} = kp(C) \).

A simple shifting argument shows that

\[
\int_0^1 |Ef_n - f| \leq \frac{B}{k},
\]

Also,

\[
E\left\{ \int_0^1 |f_n - Ef_n| \right\} = \sum_C E\left\{ \left| \frac{N(C)}{n} - p(C) \right| \right\}
\leq \sum_C 1 \cdot \sum_C E\left\{ \left| \frac{N(C)}{n} - p(C) \right|^2 \right\}
\leq \sqrt{\frac{k}{n} \sum_C p(C)} = \sqrt{\frac{k}{n}},
\]

and therefore, taking \( k = \lceil (2B)^{2/3}n^{1/3} \rceil \) to optimize the sum, we obtain

\[
\sup_{f \in M_B} E\left\{ \int_0^1 |f_n - f| \right\} \leq \left( \frac{1}{2^{2/3}} + 2^{1/3} + o(1) \right) \left( \frac{B}{n} \right)^{1/3}.
\]

Note that without knowledge of \( B \), the histogram estimate does not have a better convergence rate than our estimate, despite the significant restrictions under which the latter operates.

3. ALGORITHM AND TIME COMPLEXITY

From an algorithmic standpoint, the splitting described above amounts to a branching process that constructs a binary tree. For any \( x \in \mathbb{R} \) and sorted list of numbers \( L \), let
\(i(L, x)\) denote the index of \(x\) if it were inserted into \(L\). Given a sorted list of size \(n\) whose elements are an i.i.d. sample \(X_1, ..., X_n\) drawn from an unknown density \(f\) on \([a, b] \subseteq \mathbb{R}\), the following recursive algorithm constructs a partition tree of \([a, b]\) according to our splitting rule.

**Algorithm 1** Interval partitioning using a binary tree

1: function \(\text{BuildTree}(r, [X_1, ..., X_n], [a, b])\) \(\triangleright r\) is a tree node
2: \(L \leftarrow i([X_1, ..., X_n], (a + b)/2)\) \(\triangleright L\) is the number of data points in the left half of \([a, b]\)
3: \(R \leftarrow n - L\) \(\triangleright R\) is the number of data points on the right half of \([a, b]\)
4: if \(L - R > \gamma \sqrt{n}\) then \(\triangleright \gamma\) is a parameter in \((0, \infty)\)
5: \(\text{BuildTree}(r.\text{left}, [X_1, ..., X_L], [a, (a + b)/2])\)
6: \(\text{BuildTree}(r.\text{right}, [X_{L+1}, ..., X_n], [(a + b)/2, b])\)
7: else
8: \(r.\text{value} \leftarrow [a, b]\)
9: end if
10: end function
11: Initialize new tree node \(r\)
12: \(\text{BuildTree}(r, [X_1, ..., X_n], [a, b])\)
13: return \(r\)

While not strictly necessary, the assumption that \([X_1, ..., X_n]\) is sorted allows us to decide whether or not to split in logarithmic time by using binary search to compute the number of points on the left and right halves of \([a, b]\). It also greatly simplifies the algorithm’s pseudocode to construct left and right sublists in the case where we perform a recursive call.

As a corollary to the results shown later in the paper, we can derive the following sublinear upper bound on the expected runtime of our algorithm.

**Corollary 2.** This algorithm’s expected runtime is \(O(n^{1/3} \log_2(n))\) if the input data are sorted.

*Proof.* See appendix. \(\blacksquare\)

### 4. GALTON–WATSON TREES AND THE UNIFORM CASE

We recall the definition of a Galton–Watson tree (see, e.g., Athreya and Ney, 1972 [3]): every node in the tree has a number of children distributed as \(Z\), where \(Z \geq 0\) has a fixed distribution. All realizations of \(Z\) are independent. If \(\mathbb{E}Z = m < 1\), then the expected size of the tree is \(1/(1 - m)\). See, e.g., Lyons and Peres, 2016 [17].

As previously discussed, our splitting procedure can be viewed as a (randomly generated) binary tree of intervals which we will henceforth denote by \(T_n\). An elegant connection to the theory of branching processes can be established in the case where our data are sampled
from a uniform distribution. More specifically, one can show that the resulting tree would closely resemble a Galton–Watson tree whose nodes have two children with probability \( p_2 = \Pr \{N(0,1) > \gamma \} := \Phi(\gamma) \) and no children with probability \( p_0 = 1 - p_2 \) (where \( N(0,1) \) is a standard normal and \( \gamma \) is the parameter chosen in the algorithm, which is assumed to be \( \leq 1 \) in this section).

Let \( C \) be an arbitrary subinterval of \([0,1]\) and assume that \( C \) contains \( N \) data points. The number of points in the left and right halves of \( C \), which we denote by \( N' \) and \( N'' \) respectively, are both binomial random variables with parameters \( N \) and \( 1/2 \). Noting that \( 2N' - N = N' - N'' \), the probability of splitting the interval is

\[
\Pr \{N' - N'' > \gamma \sqrt{N} \} = \Pr \{\frac{N' - N/2}{\sqrt{N/4}} > \gamma \}
\]

which by the Berry-Esseen theorem (Berry, 1941 [4], see also Petrov, 1975 [18]) is equal to \( \Phi(\gamma) + \theta/\sqrt{N} =: p_2 \) for some \( |\theta| \leq 1 \).

Let \( \epsilon > 0 \) be arbitrary, and let \( T_n' \) be the subtree of \( T_n \) in which all nodes \( C \) (we refer to \( C \) as a node as well as an interval associated with that node) contain at least \( N_\epsilon := \lceil 1/(\epsilon \cdot \Phi(\gamma))^2 \rceil \) points. Then for these nodes, the probability \( p_2 \) of splitting is smaller than

\[
(1 + \theta \cdot \epsilon) \Phi(\gamma).
\]

We infer that for \( \epsilon \) small enough,

\[
E\{|T_n'|\} \leq \frac{1}{1 - 2(1 + \epsilon)\Phi(\gamma)}.
\]

Furthermore, every leaf of \( T_n' \) is either a leaf of \( T_n \) or an internal node containing less than \( N_\epsilon \) points. In the latter case, we can derive a uniform upper bound for the expected size of subtrees that hang from such leaves as a function of \( \epsilon \).
Our splitting criterion is such that any interval with a single point is never split. By analyzing the expected minimum distance between any two points in an interval, we can determine an upper bound for the expected height (and in turn size) of a tree.

Consider \( N \) uniformly distributed points on an interval (without loss of generality, \([0, 1]\)). Let \( D \) be an integer random variable taking value \( i \) when the minimum distance between two points of the interval lies in \( (2^{-i-1}, 2^{-i}] \).

We split \([0, 1]\) dyadically until each interval has 0 or 1 point (as depicted in figure 1). The expected number of internal nodes of this tree is

\[
\sum_{\ell=0}^{\infty} 2^\ell \cdot \Pr\left\{ \left[ 0, \frac{1}{2^\ell} \right] \text{ contains at least 2 points} \right\}
\leq \sum_{\ell=0}^{\infty} 2^\ell \cdot \left( \frac{N - 1}{2} \right) \frac{1}{2^{2\ell}} \leq \frac{(N - 1)^2}{2} \sum_{\ell=0}^{\infty} \frac{1}{2^\ell} = (N - 1)^2
\]

where \( \ell \) is the level number in the tree.

Thus, the expected size is \( \leq 2(N - 1)^2 + 1 < 2N^2 \) since the number of leaves equals the number of internal nodes plus one.

We conclude that the expected tree size is finite and uniformly bounded over all values of \( n \):

\[
\mathbb{E}\{|T_n|\} \leq \inf_{\epsilon > 0} \frac{1}{1 - 2(1 + \epsilon)\Phi(\gamma)} \cdot 2N^2
\]

\[
\leq \inf_{\epsilon > 0} \frac{2}{1 - 2(1 + \epsilon)\Phi(\gamma)} \cdot \left( \frac{1}{(\epsilon \cdot \Phi(\gamma))^2} + 1 \right)^2 \overset{\text{def}}{=} \varphi(\gamma) < \infty.
\]

For this example, we have \( \mathbb{E}|f_n - f| = O(1/\sqrt{n}) \) by elementary calculations.

5. THE DETERMINISTIC INFINITE TREE

5.1. Notation, setup and main proposition

Towards our goal of proving Theorem 1, we begin with the analysis of the infinite full binary tree depicted in figure 3 and denoted by \( T_\infty \). It is analogous to \( T_n \) in that each node of \( T_\infty \) is associated with a subinterval of \([0, 1]\): more specifically, if \( C_1, \ldots, C_{2^\ell} \) are \( T_\infty \)’s level \( \ell \) nodes labelled left to right, then \( C_i \) corresponds to the interval \([i - 1)/2^\ell, i/2^\ell] \) for any \( 1 \leq i \leq 2^\ell \). When we later analyze the random tree \( T_n \), it will be useful to view it as a subset of this infinite deterministic tree.

As in the introduction, the left and right halves of a node \( C \in T_\infty \) are denoted by \( C' \) and \( C'' \), respectively. It is said to be balanced (and is uncoloured in figure 3) if it satisfies

\[
p(C') - p(C'') \leq \gamma \sqrt{\frac{p(C)}{n}},
\]

where \( \gamma \) is the parameter previously defined for the algorithm [1] and, as above, \( p(C) = \int_C f \). The set of all such nodes is denoted by \( \mathcal{B} \). All other (coloured) nodes are said to be
unbalanced and belong to \( B^c \), the complement of \( B \). Similarly, for any positive real number \( \alpha \), we denote by \( B^{(\alpha)} \) the set of nodes satisfying
\[
p(C') - p(C'') \leq (\alpha \gamma) \sqrt{\frac{p(C)}{n}},
\]
noting that \( B = B^{(1)} \). The integer \( \ell^* \) is defined as
\[
\ell^* = \min \left\{ \ell \in \mathbb{Z}_{>0} : \frac{B}{2^{\ell+1}} \leq \gamma \cdot 2^{\ell/2} \sqrt{\frac{B}{n}} \right\}.
\]
We denote by \( P_j(T_\infty) = P_j \) the set of nodes of \( T_\infty \) with exactly \( j \) balanced ancestors.

Lastly, for any node \( C \), the average value of \( f \) on \( C \) is denoted by \( f(C) \).

Note that if we were to truncate \( T_\infty \) by deleting nodes that fall below those belonging to \( B \cap P_0 \), the resulting tree (with leaf set \( B \cap P_0 \)) would be the tree generated by the algorithm \( 0 \) if every interval \( C \) contained its expected number of data points, \( np(C) \) (in which case our splitting rule becomes the negation of \( 1 \)). If this were the case, the density estimate extracted from this tree would therefore, on each leaf \( C \), be equal to \( f(C) \). We begin by showing that Theorem \( 1 \) holds for this function, as stated in the following proposition.

**Proposition 3.** Let \( f \) be a bounded decreasing probability density function on \([0, 1]\) with \( B = f(0) < \infty \), and let \( F_n \) be the function that takes the value \( f(C) \) on every \( C \in P_0 \cap B \). Then the \( L_1 \) distance between these two functions does not exceed \( c_0 \cdot (B^{2/3}/n^{1/3}) \) for some constant \( c_0 \in \mathbb{R}_{>0} \) that does not depend on \( B \) or \( n \).

### 5.2. Preliminary results and lemmas

The following three lemmas are needed to prove Proposition \( 3 \).

**Lemma 4.** For any \( C \in T_\infty \),
\[
p(C') - p(C'') \leq \int_C |f - f(C)| \leq 2(p(C') - p(C'')).
\]
Proof. Let \( x_0 := \sup \{ x \in C : f(x) \geq f(C) \} \). Without loss of generality, assume \( C = [0,1] \) and \( p(C) > 0 \). Our result is clear when \( f \) is constant on \( C \), so we assume otherwise. Assume first that \( x_0 < 1/2 \), and define

\[
A := \int_0^{x_0} (f - f(C)), \quad B_1 := \int_{x_0}^{1/2} (f(C) - f), \quad B_2 := \int_{1/2}^1 (f(C) - f).
\]

Our assumption on \( f \) guarantees that \( A, B_1 \) and \( B_2 \) are all positive. It is clear that

\[
\int_C |f - f(C)| = A + B_1 + B_2 \text{ and } p(C') - p(C'') = A + (B_2 - B_1),
\]

which shows the leftmost inequality. Note that \( B_2 \geq B_1 \), since otherwise we would have

\[
p(C') - p(C'') = A + (B_2 - B_1) < A,
\]

which would only be possible \(|x_0 - 1/2| \geq 1/2\), forcing \( x_0 = 0, A = 0 \) and \( f \) to be constant.

Using the fact that \( A = B_1 + B_2 \) (by definition of \( x_0 \)),

\[
2(p(C') - p(C'')) = A + B_1 + B_2 + 2(B_2 - B_1) \geq \int_C |f - f(C)|.
\]

The case \( x_0 > 1/2 \) can be taken care of similarly. \( \square \)

Lemma 5. Let \( \ell \in \mathbb{Z}^+ \) be fixed, and let \( \mathcal{A}_\ell \) be the set of nodes in \( \mathcal{T}_\infty \) of depth \( \ell \). Then

\[
\sum_{C \in \mathcal{A}_\ell} (p(C') - p(C'')) \leq \frac{B}{2^{\ell+1}}.
\]

Proof. Let \( \{C_i\}_{i=1}^{2^\ell} \) be an enumeration of \( \mathcal{A}_\ell \) from left to right (where the leftmost node has 0 as one of its interval endpoints). We have

\[
\sum_{C \in \mathcal{A}_\ell} (p(C') - p(C'')) = \sum_{i=1}^{2^\ell} (p(C_i') - p(C_i'')) \\
\leq p(C_1') - p(C_1'') \leq p(C_1') \leq \frac{B}{2^{\ell+1}}.
\]

\( \square \)

Lemma 6. Let \( \ell \in \mathbb{Z}^+ \) be fixed, and let \( \mathcal{A}_\ell \) be the set of nodes in \( \mathcal{T}_\infty \) of depth \( \ell \). Then

\[
\sum_{C \in \mathcal{A}_\ell} \sqrt{\frac{p(C)}{n}} \leq 2^{\ell/2} \sqrt{\frac{B}{n}}.
\]

Proof. By Jensen’s inequality, \( \sqrt{f(C)} \leq \int_C \sqrt{f}/\lambda(C) \) and

\[
\sqrt{p(C)} = \sqrt{\lambda(C)f(C)} \leq \frac{\int_C \sqrt{f}}{\sqrt{\lambda(C)}}.
\]

It follows that

\[
\sum_{C \in \mathcal{A}_\ell} \sqrt{\frac{p(C)}{n}} \leq 2^{\ell/2} \frac{1}{\sqrt{n}} \int_C \sqrt{f} \leq 2^{\ell/2} \sqrt{\frac{B}{n}}.
\]

\( \square \)
5.3. Proof of proposition 3

Armed with these lemmas, we may now prove Proposition 3.

Proof. The $L_1$ distance between $f$ and $f_n$ on the whole of $[0, 1]$ can be computed by summing the error over the leaf set $B \cap P_0$, and is thus equal to

$$\sum_{C \in B \cap P_0} \int_C |f - f(C)|.$$

Using lemma 4 and the definition $B \cap P_0$, we can upper bound this quantity and write

$$\sum_{C \in B \cap P_0} \int_C |f - f(C)| \leq 2 \cdot \sum_{C \in B \cap P_0} (p(C') - p(C'')) \leq 2 \cdot \sum_{C \in B \cap P_0} \gamma \sqrt{\frac{p(C)}{n}}.$$  

By lemmas 5 and 6,

$$\sum_{C \in B \cap P_0} \int_C |f - f(C)| \leq 2 \cdot \sum_{C \in B \cap P_0} |A_{\ell} \setminus B| \leq 2 \cdot \sum_{\ell=0}^{\ell^*} \min \left( \frac{B}{2^{\ell+1}}, \gamma \cdot 2^{\ell/2} \sqrt{\frac{B}{n}} \right).$$  

Recall that $\ell^* = \min\{\ell \in \mathbb{Z}_+ : B/2^{\ell+1} \leq \gamma \cdot 2^{\ell/2} \sqrt{B/n}\}$ and note that $\ell^*$ is within 1 of

$$\log_2 \left( \frac{B}{4} \right)^{1/3} \left( \frac{1}{\gamma} \right)^{2/3},$$

and that the summation in (3) is bounded above by

$$2 \left( \sum_{\ell=0}^{\ell^*-1} \gamma \sqrt{\frac{B}{n} \cdot 2^{\ell/2}} + \sum_{\ell=\ell^*}^{\infty} \frac{B}{2^{\ell+1}} \right) \leq 2\gamma \sqrt{\frac{B}{n}} 2^{(\ell^*-1)/2} \left( \frac{1}{1 - 1/\sqrt{2}} \right) + \frac{2B}{2^{\ell^*}}$$

$$\leq \frac{\gamma^{2/3} B^{2/3}}{n^{1/3}} \left( \frac{27/6}{(\sqrt{2} - 1)} + 2^{5/3} \right).$$

Note that this a nonasymptotic bound that is uniform over all bounded monotone densities $f$.  

5.4. Additional results regarding the infinite tree

We conclude this section by stating a few properties of $T_\infty$ that we will make use of in the proof of Theorem 1. The following two lemmas are proved in subsections 1 and 2 of the appendix, respectively.

Lemma 7. If $\ell \geq \ell^*$,

$$|A_{\ell} \setminus B| \leq \frac{2\sqrt{2} \sqrt{Bn}}{\gamma 2^{\ell/2}},$$

and it follows, using the definition of $\ell^*$, that

$$|B^c| \leq 14 \cdot \frac{B^{1/3} n^{1/3}}{\gamma^{2/3}}.$$  

Furthermore, for any $\alpha > 0$,

$$|(B^{(\alpha)})^c| \leq \left( 2 + \frac{13}{\alpha} \right) \frac{B^{1/3} n^{1/3}}{\gamma^{2/3}}.$$  

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Lemma 8. Recall that $B$ is the subset of balanced nodes and $B^c$ the subset of unbalanced nodes of $T_\infty$. For any $j \in \mathbb{Z}_{>0}$, define $P_j = P_j(T_\infty)$ to be the set of nodes of $T_\infty$ with exactly $j$ ancestors in $B$, then

$$|P_j| \leq (|B^c| + 1) \cdot 2^j.$$ 

6. PROOF OF THEOREM 1

Making use of the results above, we now turn our attention to the proof of our main result, Theorem 1. Our approach will be similar in that the expected $L_1$ distance between $f$ and $f_n$ (as defined previously) will be computed by summing over $T_n$’s leaf set, denoted by $L$. By Scheffé’s identity (see Devroye and Györfi, 1985 [8]), we have

$$\mathbb{E}\left\{\int_0^1 |f - f_n| \right\} = 2 \cdot \mathbb{E}\left\{\int_0^1 (f - f_n)^+ \right\}$$

(4)

where $(x)^+ := \max(x, 0)$. Now, (I) is bounded from above by

$$2 \mathbb{E}\left\{ \sum_{C \in L} \int_C (f - f(C))^+ \right\} + 2 \mathbb{E}\left\{ \sum_{C \in L} \int_C \left( f(C) - \frac{N(C)/n}{\lambda(C)} \right)^+ \right\}.$$ 

(II)

where $N(C)$ is defined to be the number of data points in $C$. Our strategy will be to bound each of these terms separately.

Before we proceed, we make the following remarks regarding notation. As mentioned previously, we view $T_n$ as a subtree of $T_\infty$. This allows us to recycle most of the notation introduced above. For instance, leaves of $T_n$ with depth $\ell$ are the elements of $L \cap A_\ell$, while leaves that are balanced are elements of $L \cap B$.

6.1. Upper bound for (I)

We begin with a few preliminary results.

Lemma 9. Let $C$ be any non-leaf node of $T_n$ with depth $\ell$ and let $D \subseteq L$ be the set of leaves of the subtree rooted at $C$, then

$$\sum_{C^* \in D} \int_{C^*} (f - f(C^*))^+ \leq \int_C (f - f(C))^+.$$ 

Proof. See appendix. ■

Lemma 10. Let $C \in T_n \setminus B(\sqrt{2})$, and $\xi(C) := p(C') - p(C'') - \gamma \sqrt{2p(C)/n} > 0$. Then for such $C$, we have

$$\mathbb{P}\{C \in L\} \leq \frac{2p(C)}{2p(C) + n\xi(C)^2} + \frac{4}{np(C)}.$$ 

Proof. See appendix. ■

Using these lemmas, we prove the following proposition.
Figure 4. Definitions used in the proof of lemma 9.

Proposition 11.

\[ \sup_{f \in \mathcal{M}_B} E \left\{ \sum_{C \in L} \int_C (f - f(C))_+ \right\} \leq \frac{B^{2/3}}{n^{1/3}} c_1(\gamma) + o(n^{-1/3}) \]

where

\[ c_1(\gamma) := \left( 4\gamma^{2/3} + \frac{6(\gamma + \sqrt{\gamma^2 + 1})}{\gamma^{1/3}} \right) \]

is a strictly positive constant depending only upon \( \gamma \).

Proof. The term we are trying to bound can be viewed as the expected \( L_1 \) distance between \( f \) and the estimator obtained by taking the (random) partition of \([0,1]\) given by \( T_n \), and estimating \( f \) by its average value on each interval in said partition. Informally, one notices that if the branching process that generated \( T_n \) behaved “as expected”, this estimator would be more or less equal to \( \mathcal{F}_n \) from Proposition 3.

Deeper leaves in \( T_n \) yield a finer partition of \([0,1]\). Taking intuition from the Riemann integral, one would guess that since we approximate \( f \) by its average value on each interval of this partition, a finer partition would help us minimize \( L_1 \) distance. Conversely, we can use a coarser partition to upper bound said distance, as shown by lemma 9.

It follows that we can use the partition given by \( T_n \) truncated below level \( \ell^* \) to derive our upper bound. By lemmas 3 and 9 we have

\[ (I) \leq E \left\{ \sum_{\ell=0}^{\ell^*} \sum_{C \in L \cap A_\ell} \int_C (f - f(C))_+ \right\} + 2 \cdot E \left\{ \sum_{C \in A_{\ell^*}} (p(C') - p(C'')) \right\}, \tag{5} \]

and an application of lemma 9 yields

\[ E \left\{ \sum_{C \in A_{\ell^*}} (p(C') - p(C'')) \right\} \leq \frac{B}{2^{\ell^*+1}} \leq \frac{B^{2/3} \gamma^{2/3}}{n^{1/3}} \cdot 4^{1/3}. \tag{6} \]

Next, we recall that \( \mathcal{B}(\sqrt{3}) \) is the set of nodes of \( T_\infty \) satisfying

\[ p(C') - p(C'') \leq \gamma \sqrt{\frac{2p(C)}{n}}, \]
as defined earlier. Any node \( C \) belonging to the complement of \( B(\sqrt{2}) \) satisfies

\[
p(C') - p(C'') = \gamma \sqrt{\frac{2p(C)}{n}} + \xi(C)
\]

where \( \xi(C) := p(C') - p(C'') - \gamma \sqrt{2p(C)/n} \) is a strictly positive real number.

We use lemma 4 once more to bound the leftmost term in (5), writing

\[
E\left\{ \sum_{\ell=0}^{\ell^*} \sum_{C \in L \cap A} \int_C (f - f(C))_+ \right\}
= E\left\{ \sum_{\ell=0}^{\ell^*} \sum_{C \in A_\ell} \int_C (f - f(C))_+ 1_{[C \in L]} \right\}
\leq \sum_{\ell=0}^{\ell^*} \left( \sum_{C \in B(\sqrt{2}) \cap A_\ell} \gamma \sqrt{\frac{2p(C)}{n}} \right.
+ \left. \sum_{C \in A_\ell \setminus B(\sqrt{2})} \min \left( \gamma \sqrt{\frac{2p(C)}{n}} + \xi(C), p(C') - p(C'') \right) P\{C \in L\} \right).
\tag{7}
\]

Applying lemma 6 we find that the first of the two inner summations in (7) is bounded above by

\[
\gamma \sqrt{\frac{Bn}{n^{\ell+1}}}. \tag{8}
\]

To bound the second summation, we use lemma 10 as well as the fact that \((p(C') - p(C''))/p(C) \leq 1\) to write

\[
\sum_{C \in A_\ell \setminus B(\sqrt{2})} \min \left( \gamma \sqrt{\frac{2p(C)}{n}} + \xi(C), p(C') - p(C'') \right) P\{C \in L\}
\leq \sum_{C \in A_\ell \setminus B(\sqrt{2})} \left( \gamma \sqrt{\frac{2p(C)}{n}} + \xi(C) \right) \frac{1}{1 + \frac{\xi(C)^2}{2p(C)/n}} + \frac{4}{n}. \tag{9}
\]

For any positive real numbers \( a \) and \( b \), the following identity holds:

\[
\frac{a + b}{1 + b^2} \leq \sqrt{a^2 + 1}.
\]

Using it on \( a = \gamma \) and \( b = \xi(C)/\sqrt{2p(C)/n} \) inside the summation in (9), we have

\[
\sum_{C \in A_\ell \setminus B(\sqrt{2})} \left( \sqrt{\frac{2p(C)}{n}} \left( \frac{a + b}{1 + b^2} \right) + \frac{4}{n} \right) \leq \sum_{C \in A_\ell \setminus B(\sqrt{2})} \left( \sqrt{\frac{2p(C)}{n}} \sqrt{\gamma^2 + 1} + \frac{4}{n} \right). \tag{10}
\]

Since we are only bounding the quantity above for values of \( \ell \) that are smaller than \( \ell^* \), we have

\[
|A_\ell \setminus B(\sqrt{2})| \leq |A_\ell| \leq |A_{\ell^*}| \leq 2^\ell^*.
\]

By definition of \( \ell^* \), this yields

\[
|A_\ell \setminus B(\sqrt{2})| \leq 2^\ell^* \leq \frac{2(Bn)^{1/3}}{\gamma^{2/3}}.
\]
It follows by lemma 6 that for any $\ell \leq \ell^*$,

$$
\sum_{C \in A \setminus B(\sqrt{2})} \left( \frac{2p(C)}{n} \sqrt{\gamma^2 + 1} + \frac{4}{n} \right) \leq \left( \sqrt{\gamma^2 + 1} \right) \sqrt{\frac{B}{n} 2^{(\ell+1)/2}} + \frac{2B^{1/3}}{(\gamma n)^{2/3}}.
$$

Invoking equations 6 and 8, our bound on (I) in 5 becomes

$$(I) \leq \frac{4B^{2/3} \gamma^{2/3}}{n^{1/3}} + \sum_{\ell=0}^{\ell^*} \left( \gamma + 2\sqrt{\gamma^2 + 1} \right) \sqrt{\frac{B}{n} 2^{(\ell+1)/2}} + \frac{2B^{1/3}}{(\gamma n)^{2/3}}$$

$$\leq \frac{B^{2/3}}{n^{1/3}} c_1(\gamma) + \frac{2(\ell^* + 1)B^{1/3}}{(\gamma n)^{2/3}}.
$$

Note that $\ell^*/n^{2/3} = O(\log_2(n)/n^{2/3})$ uniformly over all monotone densities bounded by $B$.
This completes the proof of Proposition 11. ■

6.2. Upper bound for (II)

Our final step towards proving Theorem 1 is deriving an upper bound for (II), as stated in Proposition 13 below. The latter proof relies on the following preliminary result.

Lemma 12. Assume that $\gamma > 2$, then if $C \in B^{(1/2)}$,

$$E \left\{ \left( p(C) - \frac{N(C)}{n} \right) + 1_{[C \in L]} \right\} \leq c_2(\gamma)^{3/2} \sqrt{\frac{p(C)}{n}}
$$

where

$$c_2(\gamma) = \frac{1}{1 + \gamma^2/4} < \frac{1}{2}.
$$

Proof. See appendix. ■

Proposition 13. Assume that $\gamma > 2$, then

$$E \left\{ \sum_{C \in L} \int_C \left( f(C) - \frac{N(C)/n}{\lambda(C)} \right)_+ \right\} \leq c_3(\gamma) \frac{B^{1/6}}{n^{1/3}}
$$

where

$$c_3(\gamma) = \gamma^{-1/3} \left( 4 + \frac{5}{1 - \sqrt{2}c_2(\gamma)} \right).
$$

Proof. We have

$$E \left\{ \sum_{C \in L} \int_C \left( f(C) - \frac{N(C)/n}{\lambda(C)} \right)_+ \right\} = E \left\{ \sum_{C \in L} \left( p(C) - \frac{N(C)}{n} \right)_+ \right\}.
$$

Our strategy will be to partition nodes $C \in L$ according to which $\mathcal{P}_j$ they belong to, as well as whether or not they belong to $B^{(1/2)}$, seeing that lemmas 7 and 8 provide upper bounds to the number of elements in these sets. We write

$$E \left\{ \sum_{C \in L} \left( p(C) - \frac{N(C)}{n} \right)_+ \right\}
\leq \sum_{j=0}^{\infty} \sum_{C \in B^{(1/2)} \cap \mathcal{P}_j} E \left\{ \left( p(C) - \frac{N(C)}{n} \right)_+ 1_{[C \in L]} \right\} + \sum_{C \notin B^{(1/2)}} \sqrt{\frac{p(C)}{n}}.
$$
then use the Cauchy-Schwarz inequality to obtain
\[
\sum_{C \in B^{(1/2)}} \sqrt{\frac{p(C)}{n}} \leq \sqrt{\left( \sum_{C \in B^{(1/2)}} \frac{1}{n} \right) \left( \sum_{C \in B^{(1/2)}} \frac{p(C)}{n} \right)} \\
\leq \frac{1}{\sqrt{n}} \sqrt{\left| \left( B^{(1/2)} \right)^c \right| \cdot \sum_{C \in B^{(1/2)}} p(C)} \\
\leq \frac{1}{\sqrt{n}} \sqrt{\left| \left( B^{(1/2)} \right)^c \right|}.
\]

By lemma 7, the latter is dominated by
\[
6 \cdot \frac{B^{1/6}}{\gamma^{1/3} n^{1/3}}.
\]
Lastly, we use lemma 12 to write
\[
\sum_{j=0}^{\infty} \sum_{C \in B^{(1/2)} \cap P_j} \mathbb{E} \left\{ \left( p(C) - \frac{N(C)}{n} \right)_+ 1_{\{C \in L\}} \right\} \\
\leq \sum_{j=0}^{\infty} \sum_{C \in B^{(1/2)} \cap P_j} c_2(\gamma)^{j/2} \sqrt{\frac{p(C)}{n}} \\
\leq \frac{1}{\sqrt{n}} \sum_{j=0}^{\infty} c_2(\gamma)^{j/2} \sqrt{|P_j|} \cdot \sum_{C \in P_j} \frac{p(C)}{n} \quad \text{(by Jensen’s inequality)} \\
\leq \frac{1}{\sqrt{n}} \sum_{j=0}^{\infty} c_2(\gamma)^{j/2} \sqrt{\left( 14 \cdot \gamma^{-2/3} B^{1/3} n^{1/3} + 1 \right) \cdot 2^j} \quad \text{(by lemmas 7 and 8)} \\
\leq \sqrt{\frac{14 \cdot B^{1/3} n^{1/3} + 1}{\gamma^{2/3} n}} \cdot \frac{1}{1 - \sqrt{2c_2(\gamma)}} \\
\leq \left( \frac{1}{1 - \sqrt{2c_2(\gamma)}} \right) \left( \frac{4B^{1/6} + 1}{\gamma^{1/3} n^{1/3}} \right) \\
\leq c_3(\gamma) \frac{B^{1/6}}{n^{1/3}}.
\]

Theorem 1 is a direct consequence of propositions 11 and 13.

7. CONCLUSION

Within the same framework, we can replace the histogram on each set of the partition by a linear estimate with parameter (slope and intercept at the center point of an interval) only depending upon $N', N'', \text{ and } \lambda(C)$. Such estimates should adapt better to the smoothness of the density, and should be studied for the larger class of bounded monotone densities with bounded first derivative.

Simple extensions include more complex local rules that depend upon the cardinalities $N(C_1), \ldots, N(C_k)$, where $C$ is split into $k$ equal-size intervals. The tree would still be binary
but the decision to split or not is allowed to depend upon the given cardinalities. Such rules could lead to nice and simple estimates of convex, concave, log-convex and log-concave densities.

Unimodal densities on $[0, 1]$ with unknown location of the mode offer a particular challenge. In fact, under our restrictions, it should be possible to estimate the mode itself using a $k = 4$ rule (referring to the $k$ of the previous paragraph) since the subinterval containing the mode is always the maximal probability interval (among the $k$ intervals under consideration) or one of its neighbours).

A natural rule for splitting whenever $|N' - N''| \geq \gamma \sqrt{N' + N''}$ could be of use for a large class of densities.

Finally, one can easily picture extensions to $[0, 1]^d$ for monotone densities (e.g., monotone in each coordinate when all others are held fixed). Splitting decisions would then depend upon the $2^d$ cardinalities of all equal quadrants that partition a cell $C$. The splits themselves could either be binary (along a preferred dimension) or $d$-ary. In the latter case, one would obtain random quadtrees.

8. APPENDIX

8.1. Proof of lemma 7

List the unbalanced nodes of $A_\ell$ in order from right to left, where the leftmost node is that for which the left interval endpoint is the smallest. Denote this list $\{C_i\}_{i=1}^k$, where $k = |A_\ell \setminus B|$.

By monotonicity of $f$, we have $p(C_0) \leq p(C_1) \leq \cdots \leq p(C_k)$, and we can therefore write $p(C_i) = \sum_{j=0}^i q_j$ for every $i$, where $q_1, \ldots, q_k$ are nonnegative. Since every $C_i$ is unbalanced, we have $p(C'_i) - p(C''_i) > \gamma \sqrt{p(C_i)/n}$ which, combined with the fact that $p(C_i) = p(C'_i) + p(C''_i)$, yields $2p(C'_i) \geq \gamma \sqrt{p(C_i)/n} + p(C_i)$ and in turn

$$p(C_{i+1}) \geq \gamma \sqrt{\frac{p(C_i)}{n}} + p(C_i)$$

for any $1 \leq i \leq k$. We use this fact to prove that for any $1 \leq i \leq k$, $q_i \geq (\gamma^2/4n)(i+1)$.

If $i = 0$, we have $q_0 = p(C_0) \geq \gamma^2/n \geq \gamma^2/(4n)$. Now assume that the claim regarding $q_i$ holds for some $i$, then

$$q_{i+1} = p(C_{i+1}) - p(C_i)$$

$$\geq \gamma \sqrt{\frac{p(C_i)}{n}}$$

$$\geq \frac{\gamma \sqrt{n}}{4n} \left( \frac{\gamma^2(i+1)(i+2)}{2} \right)^{1/2}$$

$$\geq \frac{\gamma^2}{4n}(i+2)$$

and it follows by induction that this claim holds for any $i$. We therefore have

$$p(C_k) = \sum_{i=1}^k q_k \geq \frac{\gamma^2}{4n} \sum_{i=1}^k (i+1) \geq \frac{\gamma^2 k^2}{8n}.$$
and the first part of the lemma follows since \( p(C_k) \leq B/2^k \). The upper bound on \( |B^c| \) follows from the fact that it is no larger than \( 2^\ell + \sum_{\ell \geq k} |A_\ell \setminus B| \). Lastly, an identical argument yields the upper bound for \( |(B^{(\alpha)})^c| \).

8.2. Proof of lemma 8

We begin by noticing that all but finitely many nodes of \( T_\infty \) are in \( B \). It follows that for any \( i \in \mathbb{Z}_{>0} \), \( |P_i| \leq |P_{i+1}| \) since any node in \( P_i \) is the root of a tree that contains at least one balanced node in \( P_{i+1} \).

Next, we examine the effect that switching a balanced node with its parent has on the various \( |P_j| \)'s. Let \( C \) be an arbitrary balanced node of \( T_\infty \), \( D \) be its parent and \( a \) be the number of balanced ancestors of \( D \). We may assume that \( D \) is unbalanced, since otherwise switching \( D \) and \( C \) would leave the tree unaffected. Our operation is depicted in figure 5 . If we let \( T \) be the subtree of \( T_\infty \) rooted at \( C \)'s sibling, then switching \( C \) and \( D \) applies the map

\[
|P_j| \mapsto \begin{cases} 
|P_j| & \text{if } j \geq a \\
|P_j| - |P_j(T)| + |P_{j+1}(T)| & \text{if } j < a
\end{cases}
\]

to every \( |P_j| \).

Since \( |P_j(T)| \leq |P_{j+1}(T)| \) for any \( j \), this map’s output is always greater than or equal to \( |P_j| \). In other words, the node configuration that maximizes \( |P_j| \) for every \( j \) is that in which all the unbalanced nodes are pushed to the top. This forms a tree of unbalanced nodes whose leaves are the roots of full, infinite binary trees where all nodes are balanced.

In this configuration, it is clear that \( |P_0| \leq (|B^c| + 1) \) and that \( |P_{j+1}| \leq 2|P_j| \), from which the lemma follows.

8.3. Proof of lemma 9

It suffices to show that for all pairs of disjoint intervals \( C_1, C_2 \subseteq C \) such that \( C_1 \cup C_2 = C \)

\[
\int_{C_1} (f - f(C_1))_+ + \int_{C_2} (f - f(C_2))_+ \leq \int_C (f - f(C))_+.
\]

The lemma then follows by induction. Without loss of generality, assume that \( C = [0, 1] \).

Let \( C_1, C_2 \) be an arbitrary such pair and note that \( f(C_2) \leq f(C) \leq f(C_1) \) since \( f \) is decreasing. Let \( x_1 := \sup \{ x : f(x) \geq f(C_1) \} \), \( x_2 := \sup \{ x : f(x) \geq f(C_2) \} \), and let
A simple application of Chebyshev’s inequality gives

\[ A := \int_0^{x_1} (f - f(C_1)) \text{ and } B := \int_{x_2}^1 (f - f(C_2)) \]

as depicted in figure [1]. Then it is obvious that

\[ A + B = \int_{C_1} (f - f(C_1))_{+} + \int_{C_2} (f - f(C_2))_{+} \leq \int_C (f - f(C))_{+}. \]

8.4. Proof of lemma 10

Let \( C \in T_n \) be arbitrary. Recall that \( N(C) \) is the number of data points lying in \( C \) (similarly, \( N(C') \) and \( N(C'') \) are the number of points lying in the left/right halves of \( C \) respectively). Notice that \( N(C) \overset{D}{=} \text{Bin}(n, p(C)) \) (where \( \overset{D}{=} \) denotes equivalence in law and \( \text{Bin}(n, p) \) a binomial \( n, p \)). For \( C \) to be a leaf, we must have decided not to split its node. Therefore

\[
\mathbb{P}\{C \in L\} \leq \mathbb{P}\left\{ N(C') - N(C'') < \gamma \sqrt{N(C)} \right\}
\leq \sup_{m \geq np(C)/2} \mathbb{P}\left\{ N(C') - N(C'') < \gamma \sqrt{N(C)} \mid N(C) = m \right\}
+ \mathbb{P}\{ |N(C) - np(C)| > np(C)/2 \}. \tag{13}
\]

A simple application of Chebyshev’s inequality gives

\[
\mathbb{P}\{ |N(C) - np(C)| > np(C)/2 \} \leq \frac{4(1 - p(C))}{np(C)} \leq \frac{4}{np(C)}
\]

Next, given \( N(C) = m, N(C') \overset{D}{=} \text{Bin}(m, p(C')/p(C)) \). Note that

\[ p(C') + p(C'') = p(C) \]
\[ p(C') - p(C'') = \gamma \sqrt{2p(C)/n} + \xi(C) \]

so that

\[
\frac{p(C')}{p(C)} = \frac{1}{2} + \frac{1}{2} \gamma \sqrt{\frac{2}{np(C)} + \frac{1}{2} \frac{\xi(C)}{p(C)}}.
\]

Using these observations and the fact that \( N(C') - N(C'') = 2N(C') - N(C) \), we find that

\[
\mathbb{P}\left\{ N(C') - N(C'') < \gamma \sqrt{N(C)} \mid N(C) = m \right\}
= \mathbb{P}\left\{ \text{Bin} \left( m, \frac{p(C')}{p(C)} \right) - m \frac{p(C')}{p(C)} < \gamma \sqrt{\frac{m}{2}} - \gamma \frac{m}{2} \sqrt{\frac{2}{np(C)} - \frac{m \xi(C)}{2 p(C)}} \right\}
\leq \mathbb{P}\left\{ \text{Bin} \left( m, \frac{p(C')}{p(C)} \right) - m \frac{p(C')}{p(C)} < - \frac{m \xi(C)}{2 p(C)} \right\}
\]

if \( m \geq np(C)/2 \), which is the case in the supremum taken in [13]. Using the Chebyshev-Cantelli inequality (see Lugosi, Massart and Boucheron, 2013 [16]) and the fact that the variance of a binomial \( n, p \) is at most \( n/4 \), we have

\[
\mathbb{P}\left\{ \text{Bin} \left( m, \frac{p(C')}{p(C)} \right) - m \frac{p(C')}{p(C)} < - \frac{m \xi(C)}{2 p(C)} \right\} \leq \frac{m/4}{m/4 + \frac{m^2 \xi(C)^2}{4(p(C))^2}}
\leq \frac{2p(C)}{2p(C) + n\xi(C)^2} \left( \text{since } m \geq \frac{np(C)}{2} \right)
\]
8.5. Proof of lemma 12

Using Cauchy-Schwarz, we write

\[ E \left\{ 1_{C \in L} \left( p(C) - \frac{N(C)}{n} \right) \right\} \leq \sqrt{P\{C \in L, N(C) \leq np(C)\}} \cdot \sqrt{\frac{p(C)}{n}}. \]

We claim that

\[ P\{C \in L, N(C) \leq np(C)\} \leq c_2(\gamma)^2, \]

where \( c_2(\gamma) \) is defined in the lemma’s statement. Note that this bound does not depend on the level \( \ell \) at which the node is located. To prove this, it suffices to show that for any balanced node \( C \in B(1/2) \), if \( N(C) \leq np(C) \),

\[ P\{N(C') - N(C'') > \gamma \sqrt{N(C)} \mid N(C)\} \leq c_2(\gamma), \]

since consecutive splits are independent given the values of \( N(C) \). For simplicity, temporarily denote \( N(C), N(C') \) and \( N(C'') \) by \( N, N' \) and \( N'' \) respectively, and similarly for \( p, p', p'' \). Then observe that

\[ P\left\{ N' - N'' > \gamma \sqrt{N} \mid N \right\} \leq P\left\{ N' - N'' - N\left( \frac{p' - p''}{p} \right) > \gamma \sqrt{N} - N\left( \frac{p' - p''}{p} \right) \mid N \right\} \]

\[ \leq 1_{(p' - p'')/p \geq \gamma \sqrt{n}/2} + P\left\{ N' - N'' - N\left( \frac{p' - p''}{p} \right) > \gamma \sqrt{N} \mid N \right\} \]

\[ \leq 1_{\sqrt{N} > (\gamma/2)(p/(p' - p''))} + P\left\{ \frac{N' - N'' - N\cdot \frac{p' - p''}{p}}{\sqrt{4N \cdot \frac{p' - p''}{p}}} > \frac{\gamma \cdot p/2}{2 \sqrt{p'p''}} \mid N \right\}. \]

By definition of \( B(1/2) \), we have \( p' - p'' < (\gamma/2)\sqrt{p/n} \). If the indicator in the expression above were one, this would imply that \( \sqrt{N} > \sqrt{np} \) which cannot be true since \( N \leq np \), hence the indicator is equal to 0. As for the probability term, notice that

\[ \sqrt{4N \cdot \frac{p' - p''}{p}} \]

is the conditional variance of

\[ N' - N'' - N\cdot \frac{p' - p''}{p}, \]

which is a random variable with mean 0 and unit variance. Noting that \( p/2 \geq \sqrt{p'p''} \) and that \( \gamma > 2 \) by assumption, we can apply the Chebyshev-Cantelli inequality to get

\[ P\left\{ \frac{N' - N'' - N\left( \frac{p' - p''}{p} \right)}{\sqrt{4N \cdot \frac{p' - p''}{p}}} > \frac{\gamma \cdot p/2}{2 \sqrt{p'p''}} \mid N \right\} \leq \frac{1}{1 + (\gamma/2)^2} \overset{\text{def}}{=} c_2(\gamma) < 1/2. \]
8.6. Proof of corollary 2

The decision whether to split an interval containing $k$ points can be made in order $\log_2(k)$ time, which is the time taken to determine which points lie on the left and right halves of the interval, respectively, via binary search. It therefore suffices to show that the expected number of leaves of the tree generated by the algorithm is of the order of $n^{1/3}$. Letting $L$ denote the tree’s leaf set, we have

$$E(|L|) \leq E\{|(B^{(1/2)})^c|\} + E\{|L \cap B^{(1/2)}|\} = E\{|(B^{(1/2)})^c|\} + \sum_{j=0}^{\infty} \sum_{C \in B^{(1/2)} \cap P_j} E\{1_{[C \in L]}\}.$$ 

The first term is $O(n^{1/3})$ by lemma 7 and the second is $O(1)$ by the proof of lemma 12.

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