Some rigorous results on the Holstein-Hubbard model

Tadahiro Miyao
Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan
e-mail: miyao@math.sci.hokudai.ac.jp

Abstract

The Holstein model has been widely accepted as a model of electrons interacting with the phonons. Analysis of its ground states was accomplished decades ago. However obtained results did not completely take account of the repulsive Coulomb interactions. Recent progress has made it possible to treat such interactions rigorously. In this paper, we study the Holstein-Hubbard model with the repulsive Coulomb interactions. Ground state properties of the model are investigated. Especially, the ground state of the Hamiltonian is proven to be unique for an even number of electrons on bipartite connected lattice. In addition, an infrared bound on the two-point function is given. The effects of the repulsive Coulomb interaction induce several technical difficulties.

1 Introduction

The subtle interplay of electrons and phonons induces various physical phenomena. For instance, when electrons interact with phonons, the electrons have a tendency to pair. As a result, the ground state of such a system exhibits either superconducting or charge-density-wave order. Another example is high-temperature superconductivity. Since the discovery of it, studies of the coupled electron-phonon systems have become increasingly active. However unanimous mechanism of the origin of high-temperature superconductivity has not been established yet. These examples suggest that the coupled electron-phonon systems have provided a rich field of study so far. In this paper, we rigorously investigate ground state properties of the Holstein-Hubbard model which is a standard model of the electron-phonon interaction.

The importance of uniqueness of ground states for models of single electron interacting with a boson field was recognized through the rigorous studies of the quantum field theory \cite{3, 5, 8, 9}. In particular, Fröhlich’s method \cite{5} was applied to a model of one-electron which is positioned at discrete lattice system and interacts with the phonons of lattice by Löwen \cite{15}. Recently, this method was extended to a two-electron system interacting with the phonons \cite{16}. Remark that the Coulomb repulsion was considered in \cite{16}. Freericks and Lieb invented another crucial approach to show the uniqueness of many-body ground state of an electron-phonon Hamiltonian \cite{11}. Their method relies on the spin-reflection positivity discovered by Lieb \cite{14}. In \cite{17}, Lieb proved the uniqueness of ground states of the Hubbard model of interacting electrons. As a result, the ferrimagnetism in the ground state can be proven. The spin-reflection positivity originated from the quantum field theory \cite{19} and has various applications.
to the strongly correlated electron systems \[6, 21, 24\]. Freericks-Lieb’s method can be applicable to a general class of model, including the Holstein model \[10\], however it can not be directly applied to systems with the electron-electron interactions. To be precise, their results remain true when we add attractive electron-electron interactions, but if we add repulsive electron-electron interactions, their method does not work in general. In this paper, we will consider the Holstein-Hubbard model with repulsive electron-electron interactions. To prove the uniqueness of ground states, we employ the Lang-Firsov transformation \[13\]. The main difficulty we are faced is the following: Due to this transformation, elements of the hopping matrix of resulting Hamiltonian become complex-valued functions of the phonon coordinates. To overcome this difficulty, we extend a method proposed in \[17\]. In \[17\], ground state properties of the SSH model were investigated. The SSH model describes a many-electron system and its hopping matrix elements are real-valued functions of the phonon coordinates. Since elements of the hopping matrix become complex in our case, we establish more sophisticated analysis in the present paper.

Lieb’s results on the Hubbard model concern the ground state. On the other hand, Kubo and Kishi showed a finite temperature version of Lieb’s theorem \[12\]. Indeed they showed a uniform upper bound on the charge susceptibility of the Hubbard model at finite temperature. We extend their result to the Holstein-Hubbard model with repulsive electron-electron interactions. We remark that a difficulty similar to that in the proof of uniqueness of ground states occurs. Since we will not restrict ourselves to the on-site Coulomb interactions, general results can be obtained.

Our method requires a restriction on the electron-phonon coupling strength \(|g_0| \leq \sqrt{2U_0/\omega_0}\). We aware of no rigorous results when the electron-phonon coupling strength is large enough \(|g_0| > \sqrt{2U_0/\omega_0}\).

The organization of the paper is as follow: In Section 2, we introduce the Holstein-Hubbard model and display main results. Sections 3 through 6 are devoted to proving the main results in Section 2. In Section 3, we provide several expressions of the Hamiltonian. We will choose a suitable expression in each section below. The hole-particle transformation plays a key role in our study. In Section 4, we show Theorem 2.3 and Corollary 2.4. By choosing a suitable Hilbert cone, we prove that the heat semi-group generated by the Hamiltonian preserves the positivity. In Section 5, proofs of Theorem 2.7 and Corollary 2.8 are given. In fact, we show that the semi-group generated by the Hamiltonian improves the positivity with respect to the Hilbert cone constructed in Section 4. The uniqueness of ground states follows from Faris’ theorem which is a generalization of the Perron-Frobenius theorem. By applying this fact, some magnetic structures of the ground state are revealed. Section 6 is devoted to the proof of Theorem 2.9. We obtain an upper bound for the charge susceptibility by extending the method of Gaussian domination. In Appendices A-C, we give a list of basic facts which will be necessary in the main sections.

Acknowledgement: This work was supported by KAKENHI(20554421).
2 Model and results

2.1 The Holstein-Hubbard model

Let $G = (\Lambda, E)$ be a graph with vertex set $\Lambda$ and edge collection $E$. We suppose that $G$ is embedded in $\mathbb{R}^d$. An edge with endpoints $x$ and $y$ will be denoted by $\{x, y\}$. In this paper, we always assume that $\{x, x\} \notin E$ for any $x \in \Lambda$, i.e., any loops are excluded. $G$ is called connected if any of its vertices are linked by a path in $G$. $G$ is called bipartite if $\Lambda$ admits a partition into two classes such that every edge has its ends in different classes. In this paper, we always assume that the graph $G$ is bipartite.

The Hamiltonian of the Holstein-Hubbard model is given by

$$ H = - \sum_{\{x, y\} \in E, \sigma \in \{\uparrow, \downarrow\}} t(x - y) c^\dagger_{x\sigma} c_{y\sigma} + \sum_{x, y \in \Lambda} U(x - y) (n_x\uparrow - \frac{1}{2})(n_y\downarrow - \frac{1}{2}) + \sum_{x, y \in \Lambda, \sigma \in \{\uparrow, \downarrow\}} g(x - y) n_{x\sigma} (b^\dagger_{x\sigma} b_{y\sigma}) + \sum_{x \in \Lambda} \omega_b b^\dagger_x b_x, \quad (2.1) $$

where $c_{x\sigma}$ is the electron annihilation operator at vertex $x$ and $b_x$ is the phonon annihilation operator at vertex $x$. These operators satisfy the following relations:

$$ \{c_{x\sigma}, c^\dagger_{x'\sigma'}\} = \delta_{\sigma\sigma'}\delta_{xx'}, \quad [b_x, b^\dagger_{x'}] = \delta_{xx'}. \quad (2.2) $$

$n_{x\sigma} = c^\dagger_{x\sigma} c_{x\sigma}$ is the fermionic number operator at vertex $x \in \Lambda$. The hopping matrix is described by a real-valued function $t(x)$ on $\mathbb{R}^d$ such that $t(-x) = t(x)$ and $\max_x |t(x)| < \infty$. The electron-electron interaction is governed by a real-valued function $U(x)$ on $\mathbb{R}^d$ such that $\max_x |U(x)| < \infty$ and $U(-x) = U(x)$. $g(x)$ is a strength of electron-phonon interaction which is a real-valued function on $\mathbb{R}^d$ with $g(-x) = g(x)$ and $\max_x |g(x)| < \infty$. The phonons are assumed to be dispersionless with energy $\omega_0$. $H$ acts in the Hilbert space

$$ \mathcal{E} \otimes \mathcal{P}. \quad (2.3) $$

$\mathcal{E}$ is defined by $\mathfrak{F}_e \otimes \mathfrak{F}_e$. $\mathfrak{F}_e$ is the fermionic Fock space over $\ell^2(\Lambda)$ given by $\mathfrak{F}_e = \oplus_{n=0}^{\infty} \Lambda^n \ell^2(\Lambda)$, where $\Lambda^n \ell^2(\Lambda)$ is the $n$-fold anti-symmetric tensor product of $\ell^2(\Lambda)$. $\mathcal{P}$ is the bosonic Fock space over $\ell^2(\Lambda)$ defined by $\mathcal{P} = \oplus_{n=0}^{\infty} \Lambda^n \ell^2(\Lambda)$, where $\Lambda^n \ell^2(\Lambda)$ is the $n$-fold symmetric tensor product. By the Kato-Rellich theorem, $H$ is self-adjoint and bounded from below.

2.2 $M$-subspace

Let $N_e = \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{x \in \Lambda} n_{x\sigma}$, the fermionic number operator. We are interested in the ground state properties of $H$ at half filling. Thus we only consider the following subspace:

$$ \mathcal{H} = \mathcal{E}_{|\Lambda|} \otimes \mathcal{P}, \quad \mathcal{E}_{|\Lambda|} = \ker(N_e - |\Lambda|). \quad (2.4) $$

Let $S(z) = \frac{1}{2}(N_{e\uparrow} - N_{e\downarrow})$, where $N_{e\sigma} = \sum_{x \in \Lambda} n_{x\sigma}, \sigma \in \{\uparrow, \downarrow\}$. Since $S(z)$ commutes
with $H$, we have the following decompositions:

$$\mathcal{H} = \bigoplus_{M=-|\Lambda|/2}^{+|\Lambda|/2} \mathcal{H}_M, \quad \mathcal{H}_M = \left( \ker[S^{(z)} - M] \cap \mathcal{E}_{|\Lambda|} \right) \otimes \mathcal{P}, \quad (2.5)$$

$$H = \bigoplus_{M=-|\Lambda|/2}^{|\Lambda|/2} H_M, \quad H_M = H \upharpoonright \mathcal{H}_M. \quad (2.6)$$

$\mathcal{H}_M$ is called the $M$-subspace.

### 2.3 Ground state properties

#### 2.3.1 Positivity of a ground state

Before we state our results, we need to introduce some definitions.

**Definition 2.1** A function $F$ will be called positive semidefinite, if, for all $\{x\} x, y \in \mathcal{E}_{|\Lambda|}$,

$$\sum_{x,y \in \Lambda} \xi_x \xi_y F(x - y) \geq 0 \quad (2.7)$$

holds. ◦

**Example 1** (i) A typical example is the on-site interaction $F(x - y) = U_0 \delta(x - y)$, where $\delta(x) = 1$ if $x = o$, $\delta(x) = 0$ otherwise.

(ii) The smeared Coulomb interaction $F(x) = \left( \frac{1}{|x|} * \varrho \right)(x)$ is positive semidefinite if, for instance, $\varrho \in C_0^\infty(\mathbb{R}^3)$ and $\hat{\varrho}(k) \geq 0$, where $\hat{\varrho}$ is the Fourier transformation of $\varrho$. ◦

**Definition 2.2** Let $\mathfrak{X}$ be a complex Hilbert space. A closed cone $\mathfrak{X}_+$ in $\mathfrak{X}$ is called a Hilbert cone if it satisfies the following:

(i) $\mathfrak{X}_+ \cap (-\mathfrak{X}_+) = \{0\}$.

(ii) $x, y \in \mathfrak{X}_+ \Rightarrow \langle x, y \rangle \geq 0$.

(iii) Let $\mathfrak{X}_\mathbb{R}$ be a real subspace of $\mathfrak{X}$ generated by $\mathfrak{X}_+$. Then for all $x \in \mathfrak{X}_\mathbb{R}$, there exist $x_+, x_- \in \mathfrak{X}_+$ such that $x = x_+ - x_-$ and $\langle x_+, x_- \rangle = 0$.

(iv) $\mathfrak{X} = \mathfrak{X}_\mathbb{R} + i\mathfrak{X}_\mathbb{R} = \{x + iy \mid x, y \in \mathfrak{X}_\mathbb{R}\}$.

A vector $x$ is said to be positive w.r.t. $\mathfrak{X}_+$ if $x \in \mathfrak{X}_+$.

Let $A, B \in L^\infty(\mathfrak{X})$, where $L^\infty(\mathfrak{X})$ is the set of all bounded linear operators in $\mathfrak{X}$. If $(A - B)\mathfrak{X}_+ \subseteq \mathfrak{X}_+$, we write this as $A \succeq B$ w.r.t. $\mathfrak{X}_+$. In particular if $A \succeq 0$ w.r.t. $\mathfrak{X}_+$, then we say that $A$ preserves the positivity w.r.t. $\mathfrak{X}_+$. One of the most useful properties is that if $A \succeq B$ and $C \succeq D$, then $AC \succeq BD$ holds. ◦

\footnote{This symbol was introduced by Miura \cite{Miura}, see also \cite{Hiroshima}}
We introduce the effective Coulomb interaction by
\[ U_{\text{eff}}(x - y) = U(x - y) - \frac{2}{\omega_0} \sum_{z \in \Lambda} g(x - z)g(y - z). \tag{2.8} \]

In what follows, we assume the following.

(A. 1) For all \( y \in \Lambda \), \( \sum_{x \in \Lambda} g(x - y) \) is independent of \( y \).

Example 2  
(i) An example satisfying (A. 1) is \( g(x) = g_0 \delta(x) \).

(ii) Let us consider a linear chain of \( 2L \) atoms with periodic boundary conditions. We denote \( w_{i,j} \) the distance from atom \( i \) to atom \( j \). Assume \( w_{j,j+1} = \pi \) for all \( j \). If \( g(x) \) is a function of \( |x| \), i.e., \( g(x) = f(|x|) \), then (A. 1) is satisfied. Similarly if \( \Lambda \) has a symmetric structure, like C\(_{60}\) fullerene, then (A. 1) is fullfiled.

Theorem 2.3  
Assume that \( |\Lambda| \) is even. Assume (A. 1). Assume that \( U_{\text{eff}} \) is positive semidefinite. Then, for all \( M \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\} \), there exists a Hilbert cone \( H_{M,+} \) such that \( e^{-\beta H_M} \gg 0 \) w.r.t. \( H_{M,+} \) holds.

5
2.3.2 Uniqueness of ground states

To prove the uniqueness of ground states, we need the following assumption.

(A. 2) \( G \) is connected.

**Definition 2.5** A function \( F \) will be called positive definite, if, for all \( \{ \xi_x \}_{x \in \Lambda} \in \mathbb{C}^{|\Lambda|} \setminus \{0\}, \)
\[
\sum_{x,y \in \Lambda} \overline{\xi_x} \xi_y F(x - y) > 0 \tag{2.12}
\]
holds. ♦

It is clear that Example 1 (i) is positive definite. Example 1 (ii) is positive definite if \( \hat{\varrho}(k) > 0 \) for all \( k \in \mathbb{R}^3. \)

**Definition 2.6** In the following we will write \( x \geq 0 \) w.r.t. \( X_+ \) if \( x \in X_+. \) A vector \( y \geq 0 \) is called strictly positive w.r.t. \( X_+ \) whenever \( \langle x, y \rangle > 0 \) for all \( x \in X_+ \setminus \{0\} \). We write this as \( x > 0 \) w.r.t. \( X_+. \) ♦

**Theorem 2.7** Assume that \( |\Lambda| \) is even. Assume (A. 1) and (A. 2). Assume that \( U_{\text{eff}} \) is positive definite. Let \( H_M, X_+ \) be the Hilbert cone given by Theorem 2.3. Then one obtains \( e^{-\beta H_M} \geq 0 \) w.r.t. \( H_M, X_. \) In this case, we say that \( A \) improves the positivity w.r.t. \( X_+. \) ♦

Let us introduce the total spin operator by
\[
S^2_{\text{tot}} = S(z)^2 + \frac{1}{2} S^+_S S^- + \frac{1}{2} S^- S^+_S, \tag{2.13}
\]
where
\[
S_+ = \sum_{x \in \Lambda} c^*_x c_{x\uparrow}, \quad S_- = \sum_{x \in \Lambda} c^*_{x\downarrow} c_{x\uparrow}. \tag{2.14}
\]

**Corollary 2.8** Assume that \( |\Lambda| \) is even. Assume (A. 1) and (A. 2). Assume that \( U_{\text{eff}} \) is positive definite. For each \( M \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\}, \) the ground state of \( H_M \) is unique. Let \( \varphi_M \) be the unique ground state of \( H_M. \) Then we have the following:

(i) \( \varphi_M \) is strictly positive w.r.t. \( \mathcal{H}_{M, +}. \)

(ii) \( \tilde{P} \varphi_M \neq 0. \)

(iii) There exists a unique \( S \geq |M| \) such that \( S^2_{\text{tot}} \varphi_M = S(S + 1) \varphi_M. \)

(iv)
\[
\langle \varphi_M, S_+ S^- \varphi_M \rangle \begin{cases} > 0 & \text{if } x, y \in \Lambda_e \text{ or } x, y \in \Lambda_o \\ < 0 & \text{otherwise}. \end{cases} \tag{2.15}
\]

**Example 4** Consider the case where \( U(x) = U_0 \delta(x) \) and \( g(x) = g_0 \delta(x). \) Then \( U_{\text{eff}}(x) \) is positive definite if and only if \( |g_0| < \sqrt{\omega_0 U_0/2}. \) ♦

6
2.4 Infrared bound and its applications

2.4.1 Infrared bound

For each bounded operator $A$, we define a thermal average by

$$\langle A \rangle_{\beta, \Lambda} = \frac{\text{Tr}_{\beta M} [A e^{-\beta H_M}]}{Z_{\beta, \Lambda}}, \quad Z_{\beta, \Lambda} = \text{Tr}_{\beta M} [e^{-\beta H_M}].$$

(2.16)

Let $\delta n_x = n_x - 1$ with $n_x = n_x^{\uparrow} + n_x^{\downarrow}$. For each $h = \{h_x\}_{x \in \Lambda} \in \mathbb{C}^{\Lambda}$, we define

$$b(h) = \beta^{-1} \langle h, U_{\text{eff}} h \rangle,$$

(2.17)

$$c(h) = \beta \left\{ \left[ \langle \delta n, U_{\text{eff}} h \rangle^*, \left[ H_M, \langle \delta n, U_{\text{eff}} h \rangle \right] \right] \right\}_{\beta, \Lambda},$$

(2.18)

$$g(h) = \left\langle \left[ \langle \delta n, U_{\text{eff}} h \rangle^*, \langle \delta n, U_{\text{eff}} h \rangle \right] \right\rangle_{\beta, \Lambda},$$

(2.19)

where

$$\langle h, U_{\text{eff}} h \rangle = \sum_{x, y \in \Lambda} h_x^* U_{\text{eff}}(x - y) h_y,$$

(2.20)

$$\langle U_{\text{eff}} h \rangle_x = \sum_{y \in \Lambda} U_{\text{eff}}(x - y) h_y,$$

(2.21)

$$\langle \delta n, U_{\text{eff}} h \rangle = \sum_{x, y \in \Lambda} \delta n_x U_{\text{eff}}(x - y) h_y.$$  

(2.22)

We obtain the following infrared bound on the two-point function.

**Theorem 2.9** Suppose that $|\Lambda|$ is even. Assume (A. 1). In addition, assume $U_{\text{eff}}$ is positive semidefinite. Then, for each $M \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\}$ and $h \in \mathbb{C}^{\Lambda}$, we have

$$g(h) \leq \frac{1}{2} \left\{ c(h) b(h) \right\}^{1/2} \coth \sqrt{\frac{c(h)}{4b(h)}}.$$  

(2.23)

If there is no interaction between electrons and phonons, this type of bound is known [7, 12]. However if we take the interaction into consideration, it is far from trivial whether a bound of this kind is valid or not.

2.4.2 Application to a simple system

We will apply Theorem 2.9 to more concrete problems. We take $\Lambda = \mathbb{Z}^d \setminus [-\ell + 1, \ell]^d, \ell \in \mathbb{N}$. Note that our arguments below can be easily extended to general periodic lattices. Let $x = (x_1, \ldots, x_d) \in \Lambda$. We use the convention that if $x_j = \ell$, then $(x + \delta_j)_j = -\ell + 1$, where $\delta_j$ is the unit vector whose $j$-th component is 1. This means that $H_M$ has periodic boundary conditions. To be precise, $E$ is given by $E = \{ \{x, y\} \in \Lambda^2 \mid |x - y| = 1 \} \cup \partial$, where

$$\partial = \left\{ \{x, y\} \in \Lambda^2 \mid \exists i \in \{1, \ldots, d\} \text{ s.t. } |x_i - y_i| = 2\ell - 1 \right\}.$$  

(2.24)

and $|x_j - y_j| = 0 \forall j \in \{1, \ldots, d\} \setminus \{i\}$.  

(2.25)
For simplicity, we take \( t(x - y) = t \) if \( \{x, y\} \in E \).

For each \( f \in ell^{1}(\Lambda) \), the Fourier transformation of \( f \) is defined by

\[
\hat{f}(p) = \sum_{x \in \Lambda} e^{-ipx}f(x), \quad p \in \Lambda^{*},
\]

where \( \Lambda^{*} \) is the dual lattice defined by \( \Lambda^{*} = \{ p \in \mathbb{R}^{d} | p_{j} = n_{j}\pi/\ell, \quad n_{j} = -\ell + 1, \ldots, \ell, j = 1, \ldots, d \} \). Note that \( \hat{U}_{\text{eff}}(p) = \hat{U}(p) - 2(2\pi)^{d}\hat{g}(p)^{2}/\omega_{0} \).

By Theorem 2.9, we immediately obtain:

**Theorem 2.10** Assume that \( |\Lambda| \) is even. Assume (A.1). Assume that \( U_{\text{eff}} \) is positive semidefinite in a sense that

\[
\sum_{x,y \in \mathbb{Z}^{d}} \hat{\xi}_{x} \hat{\xi}_{y} U_{\text{eff}}(x - y) \geq 0, \quad \xi = \{ \xi_{x} \} \in ell^{2}(\mathbb{Z}^{d}).
\]

Let

\[
E(p) = |\Lambda|^{-1} \sum_{|x - y| = 1, \sigma \in \{\uparrow, \downarrow\}} t|e^{ipx} - e^{ipy}|^{2} \langle c_{x}^{*} c_{y} \rangle_{\beta\Lambda}.
\]

Let \( \hat{\delta}_{n} = |\Lambda|^{-1/2} \sum_{x \in \Lambda} e^{-ixp}\delta_{n,x} \). Then, for each \( M \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\} \), one obtains

\[
\langle \hat{\delta}_{n} \hat{\delta}_{m} \rangle_{\beta, \Lambda} \leq \frac{1}{2} (2\pi)^{-5d/4} \hat{U}_{\text{eff}}(p)^{-1/2} E(p)^{1/2} \cosh \left( \frac{(2\pi)^{d/2} E(p) \hat{U}_{\text{eff}}(p)}{4\beta^{2}} \right). \tag{2.28}
\]

**Remark 2.11** As \( \beta \to \infty \), we have

\[
\langle \hat{\delta}_{n} \hat{\delta}_{m} \rangle_{\Lambda} \leq \frac{1}{2} (2\pi)^{-5d/4} \hat{U}_{\text{eff}}(p)^{-1/2} E(p)^{1/2} |_{\beta = \infty}, \tag{2.29}
\]

where \( \langle A \rangle_{\Lambda} \) means the ground state expectation value of \( A \): \( \langle A \rangle_{\Lambda} = \langle \varphi_{\Lambda}, A \varphi_{\Lambda} \rangle \).

### 2.4.3 Absence of charge long range order

Suppose that there exists \( C_{0} > 0 \) such that

\[
\sum_{x,y \in \mathbb{Z}^{d}} \hat{\xi}_{x} \hat{\xi}_{y} U_{\text{eff}}(x - y) \geq C_{0}|\xi|^{2}, \quad \xi = \{ \xi_{x} \} \in ell^{2}(\mathbb{Z}^{d}). \tag{2.30}
\]

Then one has \( \hat{U}_{\text{eff}}(p) \geq C_{0} \) for all \( p \in [-\pi, \pi]^{d} \). Taking the fact \( \cosh x \leq x^{-1} + 1 \) into consideration, we arrive at

**Corollary 2.12** Assume that \( |\Lambda| \) is even. Assume (A.1) and (2.30). For all \( p \in \Lambda^{*} \), we have

\[
\langle \hat{\delta}_{n} \hat{\delta}_{m} \rangle_{\beta, \Lambda} \leq \beta^{-1}(2\pi)^{-3d/2} C_{0}^{-1} + \frac{1}{2} (2\pi)^{-5d/4} C_{0}^{-1/2} E_{0}(p)^{1/2}, \tag{2.31}
\]

where \( E_{0} = 2 \sum_{j=1}^{d} |t|(1 - \cos p_{j}) \).

**Example 5** Consider the case where \( U(x) = U_{0}\delta(x) \) and \( g(x) = g_{0}\delta(x) \). Then we can take \( C_{0} = U_{0} - 2g_{0}^{2}/\omega_{0} \) provided \( |g_{0}| < \sqrt{\omega_{0}U_{0}/2} \). \( \diamond \)
2.4.4 Possibility of charge long range order

The two-point function is

\[ G_{\beta,\Lambda}(x) = \langle \delta n_x \delta n_o \rangle_{\beta,\Lambda}. \] (2.32)

In what follows, we assume that the limit

\[ G_{\beta}(x) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} G_{\beta,\Lambda}(x) \] (2.33)

exists. Suppose that \( U_{\text{eff}} \) is positive semidefinite. In addition, assume that \( \hat{U}_{\text{eff}}(p) = 0 \) if and only if \( p = 0 \) and \( \hat{U}_{\text{eff}}(p) = O(|p|^\alpha) \) as \( p \rightarrow 0 \), \( \alpha \in (0, d) \). By applying the arguments in [7], we obtain

**Corollary 2.13** Assume (A. 1). Under the above conditions, \( G_{\beta}(x) \) can be expressed as

\[ G_{\beta}(x) = \int_{[-\pi,\pi]^d} dp \left\{ a\delta(p) + \rho(p) \right\} e^{-ixp}, \] (2.34)

where

\[ a = \lim_{|x| \rightarrow \infty} G_{\beta}(x) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle (\delta n_x)^2 \rangle_{\beta,\Lambda}, \] (2.35)

and \( \rho(k) \) is a non-negative function on \( [-\pi, \pi]^d \) satisfying the following inequality

\[ \rho(k) \leq (2\pi)^{-3d/2} \hat{U}_{\text{eff},\infty}(p)^{-1} + \frac{1}{2} (2\pi)^{-5d/4} \hat{U}_{\text{eff},\infty}(p)^{-1/2} E_0(p)^{1/2}, \quad p \neq 0 \] (2.36)

with \( \hat{U}_{\text{eff},\infty}(p) = \sum_{x \in \mathbb{Z}^d} U_{\text{eff}}(x) e^{-ixp} \).

**Example 6** Let \( d = 3 \). If \( \hat{U}_{\text{eff},\infty}(p) = C \sum_{j=1}^d (1 - \cos p_j) \) or \( \hat{U}_{\text{eff},\infty}(p) = C(1 - e^{-Bp^2}) \) with \( B, C > 0 \), then all assumptions of Corollary 2.13 are satisfied. In general, \( \hat{U}_{\text{eff},\infty}(p) = Cp^2 + O(|p|^3) \) as \( p \rightarrow 0 \), then \( G_{\beta}(p) \) could contain \( \delta \)-function. ♦

3 Rewriting the Hamiltonian

3.1 The Lang-Firsov transformation

Let

\[ q_x = \frac{1}{\sqrt{2\omega_0}}(b_x^* + b_x), \quad p_x = i \frac{1}{\sqrt{2\omega_0}}(b_x^* - b_x). \] (3.1)

Both operators are essentially self-adjoint. We denote their closures by the same symbols. Let

\[ L = -i\sqrt{2\omega_0}^{-3/2} \sum_{x,y \in \Lambda} g(x - y)n_x p_y. \] (3.2)
$L$ is essentially antiself-adjoint. We also denote its closure by the same symbol. Hence $e^L$ is a unitary operator. We see that

$$e^L c_{x\sigma} e^{-L} = \exp \left\{ i \sqrt{2} \omega_0^{-3/2} \sum_{y \in \Lambda} g(x - y) p_y \right\} c_{x\sigma}, \quad (3.3)$$

$$e^L b_x e^{-L} = b_x - \omega_0^{-1} \sum_{y \in \Lambda} g(y - x) n_y. \quad (3.4)$$

Let $V = (V_{xy})$ be a $|\Lambda| \times |\Lambda|$ matrix given by

$$V_{xy} = \sum_{z \in \Lambda} g(x - z) g(y - z). \quad (3.5)$$

We denote by $n_\sigma$ a family of operators $\{n_{x\sigma}\}_{x \in \Lambda}$. For notational simplicity, we set

$$\langle n_\sigma, V n_\sigma \rangle := \sum_{x,y \in \Lambda} V_{xy} n_{x\sigma} n_{y\sigma}. \quad (3.6)$$

Using the facts that

$$e^{-i \frac{\pi}{2} N_p} q_x e^{i \frac{\pi}{2} N_p} = \omega_0^{-1} p_x, \quad e^{-i \frac{\pi}{2} N_p} p_x e^{i \frac{\pi}{2} N_p} = \omega_0 q_x, \quad (3.7)$$

where $N_p = \sum_{x \in \Lambda} b_x^* b_x$, one arrives at the following.

**Proposition 3.1** Set $U = e^{-i \frac{\pi}{2} N_p} e^L$. Let

$$\hat{H}_M = U H_M U^* + \frac{1}{2} \sum_{x,y \in \Lambda} V(x - y) + \frac{g_2^2}{\omega_0} (|\Lambda| - 2M), \quad (3.8)$$

where $g_2 = \sum_{x \in \Lambda} g(x)$. Then one has

$$\hat{H}_M = -T_{g,\uparrow} - T_{g,\downarrow} + H_p + \mathbb{U}, \quad (3.9)$$

where

$$T_{x,y,\sigma} = \sum_{\{x,y\} \in E} t(x - y) c_{x\sigma}^* c_{y\sigma} \exp \left\{ \pm i \Phi_{\{x,y\}} \right\} + \frac{1}{2} \langle n_\sigma, V n_\sigma \rangle, \quad (3.10)$$

$$\Phi_{\{x,y\}} = \sum_{z \in \Lambda} \sqrt{2} \omega_0^{-1/2} \left( g(x - z) - g(y - z) \right) q_z, \quad (3.11)$$

$$H_p = \frac{1}{2} \sum_{x \in \Lambda} (p_x^2 + \omega_0^2 q_x^2), \quad (3.12)$$

$$\mathbb{U} = \sum_{x,y \in \Lambda} U_{\text{eff}}(x - y) (n_{x\uparrow} - \frac{1}{2})(n_{y\downarrow} - \frac{1}{2}), \quad (3.13)$$

$$U_{\text{eff}}(x - y) = U(x - y) - V(x - y). \quad (3.14)$$

**Proof.** We note the following:

$$\sum_{x,y \in \Lambda} V(x - y) n_{y\sigma} = \frac{2}{\omega_0} g_2^2 N_{\sigma\sigma} = \frac{1}{\omega_0} g_2^2 (|\Lambda| - 2M) \quad \text{on } \mathcal{H}_M. \quad (3.15)$$

Thus the formulas immediately follow from (3.3) and (3.4). \qed
3.2 Expression of the Hamiltonian in \((\mathcal{F}_e \otimes \mathcal{F}_e) \otimes \mathcal{P})\)

Note that
\[
c_{x \uparrow} = c_x \otimes 1, \quad c_{x \downarrow} = (-1)^{N_e} \otimes c_x,
\]
where \(c_x\) and \(c_x^*\) are the fermionic annihilation- and creation operators on \(\mathcal{F}_e\), \(N_e\) is the fermionic number operator given by \(N_e = \sum_{x \in \Lambda} n_x\) with \(n_x = c_x^* c_x\). Thus we have the following:

\[
T_{-g, \uparrow} = \sum_{\{x,y\} \in E} t(x - y)c_x^* c_y \otimes 1 \otimes \exp \left\{ -i \Phi_{\{x,y\}} \right\} - \mathbb{V} \otimes 1, \quad (3.17)
\]

\[
T_{-g, \downarrow} = \sum_{\{x,y\} \in E} t(x - y)1 \otimes c_x^* c_y \otimes \exp \left\{ -i \Phi_{\{x,y\}} \right\} - 1 \otimes \mathbb{V}, \quad (3.18)
\]

\[
\mathbb{V} = \frac{1}{2}(\mathbf{n}, \mathbb{V}\mathbf{n}) := \frac{1}{2} \sum_{x,y \in \Lambda} V(x - y)n_x n_y, \quad (3.19)
\]

\[
\mathbb{U} = \sum_{x,y \in \Lambda} U_{\text{eff}}(x - y)(n_x - \frac{1}{2}) \otimes (n_y - \frac{1}{2}). \quad (3.20)
\]

3.3 The hole-particle transformation

The **hole-particle transformation** is a unitary operator \(\mathcal{W}\) on \(E|_{\Lambda}\) such that

\[
\mathcal{W}c_x \otimes 1 \mathbb{W}^* = \gamma(x) c_x^* \otimes 1, \quad \mathcal{W}c_x^* \otimes 1 \mathbb{W}^* = \gamma(x) c_x \otimes 1, \quad \mathbb{W}1 \otimes c_x \mathbb{W}^* = 1 \otimes c_x.
\]

(3.21)

Observe that \(\mathcal{W}N_e \mathbb{W}^* = |\Lambda| - (N_e \otimes 1 - 1 \otimes N_e)\) and \(\mathcal{W}S^{(z)} \mathbb{W}^* = \frac{1}{2}|\Lambda| - \frac{1}{2}(N_e \otimes 1 + 1 \otimes N_e)\). Thus we have

\[
\mathcal{W}E|_{\Lambda} = \bigoplus_{n=0}^{|\Lambda|} \mathfrak{F}_{e,n} \otimes \mathfrak{F}_{e,n}, \quad \mathcal{W}N_M = \mathfrak{F}_{e,(|\Lambda| - 2M)/2} \otimes \mathfrak{F}_{e,(|\Lambda| - 2M)/2}, \quad (3.22)
\]

where \(\mathfrak{F}_{e,n} = \wedge^n \ell^2(\Lambda)\). In what follow, we put

\[
M^1 = \frac{1}{2}(|\Lambda| - 2M).
\]

**Lemma 3.2** We have the following.

(i) \(\mathcal{W}T_{-g, \uparrow} \mathbb{W}^* = T_{+g, \uparrow} - \frac{1}{2} \sum_{x,y \in \Lambda} V(x - y) + \frac{2g^2}{\omega_0} M^1\).

(ii) \(\mathcal{W}T_{-g, \downarrow} \mathbb{W}^* = T_{-g, \downarrow}\).

(iii) \(\mathcal{W}U \mathbb{W}^* = -U\).

**Proof.** (i) Remark that

\[
\mathcal{W}V \otimes 1 \mathbb{W}^* = \mathbb{V} \otimes 1 + \frac{1}{2} \sum_{x,y \in \Lambda} V(x - y) - \frac{2g^2}{\omega_0} M^1.
\]

(3.24)
By definition of $W$, we have
\[
W \sum_{\{x,y\} \in E} t(x-y)c_x^*c_y \otimes \mathbb{1} \otimes \exp \left\{ -i\Phi_{\{x,y\}} \right\} W^*
\]
\[
= \sum_{\{x,y\} \in E} t(x-y)\gamma(x)\gamma(y)c_x^*c_y \otimes \mathbb{1} \otimes \exp \left\{ -i\Phi_{\{x,y\}} \right\}.
\] (3.25)

Since $G$ is bipartite, $\gamma(x)\gamma(y) = -1$ holds for all $\{x, y\} \in E$. Thus \(R.H.S.\) of \(3.25\) = \[
\sum_{\{x,y\} \in E} t(x-y)c_x^*c_y \otimes \mathbb{1} \otimes \exp \left\{ -i\Phi_{\{x,y\}} \right\} (3.26)
\]
\[
= \sum_{\{y,x\} \in E} t(y-x)c_y^*c_x \otimes \mathbb{1} \otimes \exp \left\{ -i\Phi_{\{y,x\}} \right\}
\]
\[
= \sum_{\{x,y\} \in E} t(x-y)c_x^*c_y \otimes \mathbb{1} \otimes \exp \left\{ +i\Phi_{\{x,y\}} \right\}.
\] (3.27)

Here we used that $t(-x) = t(x)$ and $\Phi_{\{y,x\}} = -\Phi_{\{x,y\}}$. Thus we have (i). Similarly one obtains that $WT_{-g,\downarrow}W^* = T_{-g,\downarrow}$.

(iii) Since $Wn_x \otimes WW^* = (\mathbb{1} - n_x) \otimes \mathbb{1}$ and $W\mathbb{1} \otimes n_x W^* = \mathbb{1} \otimes n_x$, we see that
\[
WW^* = -\mathbb{1}.
\] (3.28)

**Corollary 3.3** Let $\mathbb{H}_M = W\hat{H}_M W^* + \frac{1}{2} \sum_{x,y \in \Lambda} V(x - y) - 2g^2 M^1/\omega_0$. Then one has
\[
\mathbb{H}_M = -T_{+g}\uparrow - T_{-g}\downarrow - \mathbb{1} + H_p.
\] (3.29)

### 3.4 Representation in $L^2(\mathcal{H}_{e,M}) \otimes L^2(\mathbb{Q})$

#### 3.4.1 A canonical cone in $L^2(\mathfrak{h})$

Before we proceed, we need some preliminaries.

Let $\mathfrak{h}$ be a complex Hilbert space. We denote the set of all Hilbert-Schmidt class operators on $\mathfrak{h}$ by $L^2(\mathfrak{h})$, i.e., $L^2(\mathfrak{h}) = \{ \xi \in L^\infty(\mathfrak{h}) \mid \text{Tr}[\xi^*\xi] < \infty \}$. Henceforth we regard $L^2(\mathfrak{h})$ as a Hilbert space equipped with the inner product $\langle \xi, \eta \rangle_{L^2} = \text{Tr}[\xi^*\eta]$, $\xi, \eta \in L^2(\mathfrak{h})$. For each $A \in L^\infty(\mathfrak{h})$, the left multiplication operator is defined by
\[
\mathcal{L}(A)\xi = A\xi, \quad \xi \in L^2(\mathfrak{h}).
\] (3.30)

Similarly the right multiplication operator is defined by
\[
\mathcal{R}(A)\xi = \xi A, \quad \xi \in L^2(\mathfrak{h}).
\] (3.31)

It is not so hard to check that
\[
\mathcal{L}(A)\mathcal{L}(B) = \mathcal{L}(AB), \quad \mathcal{R}(A)\mathcal{R}(B) = \mathcal{R}(BA), \quad A, B \in L^\infty(\mathfrak{h}).
\] (3.32)

A canonical Hilbert cone in $L^2(\mathfrak{h})$ is given by
\[
L^2(\mathfrak{h})_+ = \{ \xi \in L^2(\mathfrak{h}) \mid \xi \text{ is self-adjoint and } \xi \geq 0 \text{ as an operator on } \mathfrak{h} \}.
\] (3.33)

**Lemma 3.4** Let $A \in L^\infty(\mathfrak{h})$. Then $\mathcal{L}(A^*)\mathcal{R}(A) \geq 0$ w.r.t. $L^2(\mathfrak{h})_+$.

**Proof.** For each $\xi \in L^2(\mathfrak{h})_+$, we have $\mathcal{L}(A^*)\mathcal{R}(A)\xi = A^*\xi A \geq 0$. \(\square\)
3.4.2 Natural identification $\mathfrak{F}_{e,M'} \otimes \mathfrak{F}_{e,M'}$ with $L^2(\mathfrak{F}_{e,M'})$

Let $\vartheta$ be an involution on $\mathfrak{F}_{e,M'}$ defined by

$$\vartheta c_x = c_x, \quad \vartheta \Omega = \Omega,$$

where $\Omega$ is the Fock vacuum in $\mathfrak{F}_e$. We define an isometric isomorphism from $L^2(\mathfrak{F}_{e,M'})$ onto $\mathfrak{F}_{e,M'} \otimes \mathfrak{F}_{e,M'}$ by

$$\Phi_\vartheta (|\varphi\rangle\langle\psi|) = \varphi \otimes \vartheta \psi.$$  

(3.35)

Hence we can identify $L^2(\mathfrak{F}_{e,M'})$ with $\mathfrak{F}_{e,M'} \otimes \mathfrak{F}_{e,M'}$ by $\Phi_\vartheta$. Then one has

$$\Phi_\vartheta L (A) \Phi_\vartheta^{-1} = A \otimes \mathbb{1}, \quad \Phi_\vartheta R (\vartheta A^* \vartheta) \Phi_\vartheta^{-1} = \mathbb{1} \otimes A$$

(3.36)

for any bounded linear operator $A$ on $\mathfrak{F}_{e,M'}$.

3.4.3 The Schrödinger representation

Remark the following identification

$$P = L^2(Q, dq) = L^2(Q),$$

(3.37)

where $Q = \mathbb{R}^{|\Lambda|}$ and $dq = \prod_{x \in \Lambda} dq_x$ is the $|\Lambda|$-dimensional Lebesgue measure on $Q$. Under this identification, $q_x$ and $p_x$ can be viewed as multiplication and partial differential operators, respectively. Moreover the phonon energy term can be expressed as

$$H_p = \frac{1}{2} \sum_{x \in \Lambda} \left( -\nabla^2 q_x + \omega_0^2 q_x^2 \right) - \frac{|\Lambda|}{2}.$$  

(3.38)

Note the following identifications:

$$L^2(\mathfrak{F}_{e,M'}) \otimes P = L^2(\mathfrak{F}_{e,M'}) \otimes L^2(Q) = \int_{\mathfrak{F}} L^2(\mathfrak{F}_{e,M'}) dq.$$

(3.39)

**Definition 3.5** For each $\psi = \int_{\mathfrak{F}} \psi(q) dq \in L^2(\mathfrak{F}_{e,M'}) \otimes L^2(Q) = \int_{\mathfrak{F}} L^2(\mathfrak{F}_{e,M'}) dq$, let us define an isometric isomorphism $\Phi_\vartheta$ from $L^2(\mathfrak{F}_{e,M'}) \otimes L^2(Q)$ onto $[\mathfrak{F}_{e,M'} \otimes \mathfrak{F}_{e,M'}] \otimes L^2(Q)$ by

$$\Phi_\vartheta (\psi) = \int_{\mathfrak{F}} \Phi_\vartheta (\psi(q)) dq.$$  

(3.40)

Let $q \mapsto A(q)$ be an $L^\infty(\mathfrak{F}_{e,M'})$-valued measurable map such that $\sup_q \|A(q)\|_{L^\infty} < \infty$. Using (3.36), we see that

$$\Phi_\vartheta \int_{\mathfrak{F}} L(A(q)) dq \Phi_\vartheta^{-1} = \int_{\mathfrak{F}} A(q) \otimes \mathbb{1} dq,$$

(3.41)

$$\Phi_\vartheta \int_{\mathfrak{F}} R(\vartheta A(q)^* \vartheta) dq \Phi_\vartheta^{-1} = \int_{\mathfrak{F}} \mathbb{1} \otimes A(q) dq.$$  

(3.42)
Lemma 3.6 Under the identification (3.39), we have the following.

(i) \[ T_{+g,\uparrow} = \int_{Q}^{\oplus} L(T_{+g}(q)) dq, \quad T_{-g,\downarrow} = \int_{Q}^{\ominus} R(T_{+g}(q)) dq. \] (3.43)

where

\[ T_{\pm g}(q) = \sum_{\{x,y\} \in E} t(x-y)c_x^* c_y \exp \left\{ \pm i \Phi_{\{x,y\}}(q) \right\} + V, \] (3.44)

\[ \Phi_{\{x,y\}}(q) = \sum_{z \in \Lambda} \sqrt{2} \omega_0 \left( g(x-z) - g(y-z) \right) q_z, \] (3.45)

for each \( q = (q_x)_x \in Q \).

(ii) \[ U = \sum_{x,y \in \Lambda} U_{\text{eff}}(x-y) L(n_x - \frac{1}{2}) R(n_y - \frac{1}{2}). \]

Proof. (i) Since \( L(\cdot) \) is linear, i.e., \( L(aX + bY) = aL(X) + bL(Y) \), we have

\[ T_{+g,\uparrow} = \int_{Q}^{\oplus} \left( \sum_{\{x,y\} \in E} t(x-y) \exp \left\{ i \Phi_{\{x,y\}}(q) \right\} c_x^* c_y + V \right) \otimes \mathbb{1} dq \]

\[ = \int_{Q}^{\oplus} \left( \sum_{\{x,y\} \in E} t(x-y) \exp \left\{ i \Phi_{\{x,y\}}(q) \right\} L(c_x^* c_y) + L(\mathbb{1}) \right) dq \]

\[ = \int_{Q}^{\oplus} L(T_{+g}(q)) dq. \] (3.46)

Since \( R(\cdot) \) is linear and \( \partial c_x \partial = c_x \), we have

\[ T_{-g,\downarrow} = \int_{Q}^{\ominus} \mathbb{1} \otimes \left( \sum_{\{x,y\} \in E} t(x-y) \exp \left\{ -i \Phi_{\{x,y\}}(q) \right\} c_x^* c_y + V \right) dq \]

\[ = \int_{Q}^{\ominus} \left( \sum_{\{x,y\} \in E} t(x-y) \exp \left\{ -i \Phi_{\{x,y\}}(q) \right\} R(c_x^* c_y) + R(\mathbb{1}) \right) dq \]

\[ = \int_{Q}^{\ominus} \left( \sum_{\{x,y\} \in E} t(x-y) \exp \left\{ -i \Phi_{\{x,y\}}(q) \right\} R(c_x^* c_y) + R(\mathbb{1}) \right) dq \]

\[ = \int_{Q}^{\ominus} R(T_{+g}(q)) dq. \] (3.47)

Here we used that \( t(-x) = t(x) \) and \( \Phi_{\{y,x\}}(q) = -\Phi_{\{x,y\}}(q) \).

(ii) is immediate. ☐
Corollary 3.7  Under the identification (3.39), we have
\[ H_M = -T - U + H_p, \]
where
\[ T = \int_Q \mathcal{L}(T_+g(q))dq + \int_Q \mathcal{R}(T_+g(q))dq \]
and \( U \) is given by Lemma 3.6.

### 3.5 Kernel operators

Under the identification (3.39), each \( \psi \in L^2(\mathfrak{H}_{e,M}) \otimes L^2(Q) \) can be expressed as \( \psi = \int_Q \psi(q) dq \), where \( \psi(q) \in L^2(\mathfrak{H}_{e,M}) \) for a.e. \( q \).

**Definition 3.8** Let \( A \) be a bounded linear operator on \( L^2(\mathfrak{H}_{e,M}) \otimes L^2(Q) \). If there exists an \( L^\infty(L^2(\mathfrak{H}_{e,M})) \)-valued map \((q,q') \mapsto K(q,q')\) such that
\[ (A\psi)(q) = \int Q K(q,q')\psi(q')dq' \quad \text{a.e. } q, \forall \psi \in L^2(\mathfrak{H}_{e,M}) \otimes L^2(Q), \]
then we say that \( A \) has a kernel operator \( K \). We denote the kernel operator of \( A \) by \( A(q,q') \) if it exists. Trivially it holds that
\[ \langle \varphi, A\psi \rangle = \int_{Q \times Q} dq dq' \langle \varphi(q), A(q,q')\psi(q') \rangle_{L^2(\mathfrak{H}_{e,M})}. \]

In this subsection, we will clarify the kernel operator of \( \exp\{-\beta(-T + H_p)\} \).

In the remainder of this paper, we may assume \( \omega_0 = 1 \) without loss of generality.

Set \( A = C([0, \infty); Q) \), the set of all \( Q \)-valued continuous functions on \([0, \infty)\). Let \((A, \mathcal{B}(A), D\alpha)\) be the probability space for \( |A| \)-dimensional Brownian bridge \( \{\alpha(s) | 0 \leq s \leq 1\} = \{\{\alpha_x(s)\}_{x \in A} | 0 \leq s \leq 1\} \), i.e., the Gaussian process with covariance
\[ \int_A \alpha_x(s)\alpha_y(t) D\alpha = \delta_{xy} s(1 - t) \]
for \( 0 \leq s, t \leq 1 \) and \( x, y \in A \). Define, for each \( q,q' \in Q \),
\[ \omega(s) = (1 - \beta^{-1}s)q + \beta^{-1}s q' + \sqrt{\beta}(\beta^{-1}s). \]
The conditional Wiener measure \( d\mu_{q,q';\beta} \) is given by
\[ d\mu_{q,q';\beta} = P_{\beta}(q,q')D\alpha, \]
where \( P_{\beta}(q,q') = (2\pi\beta)^{-1/2} \exp\left(-\frac{1}{2\beta}|q - q'|^2\right) \).

For each \( \varphi \in A \), \( \omega(\varphi) \) indicates a function \( s \mapsto \omega(s)(\varphi) \), the sample path \( \omega(\cdot)(\varphi) \) associated with \( \varphi \). Let
\[ G_{\beta}(\omega(\varphi)) = \prod_{0}^{s} e^{\mathcal{T}_+g(\omega(s)(\varphi))} ds, \]
where the R.H.S. of (3.55) is the strong product integration, see Appendix B. Note that since \( \omega(s)(\varphi) \) is continuous in \( s \) for all \( \varphi \in A \), the the R.H.S. of (3.55) exists.
Proposition 3.9 Let

\[ K_M = -T + H_p. \]  

Then e\(^{-\beta K_M}\) has a kernel operator given by

\[ e^{-\beta K_M}(q, q') = \int d\mu_{q, q'; \beta} L\left[G_{\beta}(\omega)\right] R\left[G_{\beta}(\omega)^*\right] e^{-\int_0^\beta ds V(\omega(s))}, \]

where

\[ V(q) = \frac{1}{2} \sum_{x \in \Lambda} \omega_0^2 q_x^2 - \frac{1}{2} |\Lambda|. \]

Proof. First note that

\[ \left\langle f_0, e^{-\beta H_p/n} f_1 e^{-\beta H_p/n} f_2 \cdots f_n \right\rangle \]

\[ = \int_{\mathcal{Q} \times \mathcal{Q}} dq dq' \int d\mu_{q, q'; \beta} f_0(q) f_1(\omega^2_n) f_2(\omega^2_n) \cdots f_{n-1}(\omega^2_n) f_n(q') e^{-\int_0^\beta ds V(\omega(s))} \]

for \( f_0, f_n \in L^2(\mathcal{Q}) \) and \( f_1, \ldots, f_{n-1} \in L^\infty(\mathcal{Q}) \), see, e.g., [22]. Let \( T(q) = L(T_{+\beta}(q)) + R(T_{+\beta}(q)) \). By the Trotter-Kato product formula, we have

\[ \left\langle \varphi, e^{-\beta K_M} \psi \right\rangle = \lim_{n \to \infty} \left\langle \varphi, \left(e^{-\beta H_p/n} e^{\beta T/n}\right)^n \psi \right\rangle \]

\[ = \lim_{n \to \infty} \int_{\mathcal{Q} \times \mathcal{Q}} dq dq' \int d\mu_{q, q'; \beta} e^{-\int_0^\beta ds V(\omega(s))} \]

\[ \times \left\langle \varphi(q), e^{\beta T_{+\beta}(\omega(\frac{\beta}{n}))} \cdots e^{\beta T_{+\beta}(\omega(\frac{2\beta}{n}))} \psi(q') \right\rangle_{L^2(\mathcal{F}_e M^1)} \]

\[ = \lim_{n \to \infty} \int_{\mathcal{Q} \times \mathcal{Q}} dq dq' \int d\mu_{q, q'; \beta} e^{-\int_0^\beta ds V(\omega(s))} \]

\[ \times \left\langle \varphi(q), L\left[ e^{\beta T_{+\beta}(\omega(\frac{\beta}{n}))} \cdots e^{\beta T_{+\beta}(\omega(\frac{2\beta}{n}))} \right] \psi(q') \right\rangle_{L^2(\mathcal{F}_e M^1)} \]

(3.58)

By the dominated convergence theorem, we conclude (3.57). \( \square \)

4 Proofs of Theorem 2.3 and Corollary 2.4

4.1 Preliminaries

Let \( L^2(\mathcal{Q})_+ \) be a Hilbert cone in \( L^2(\mathcal{Q}) \) defined by \( L^2(\mathcal{Q})_+ = \{ F \in L^2(\mathcal{Q}) | F(q) \geq 0 \text{ a.e.} \} \). The canonical cone in \( L^2(\mathcal{F}_e M^1) \otimes L^2(\mathcal{Q}) \) is given by

\[ \mathcal{C}_M = \int_{\mathcal{Q}} L^2(\mathcal{F}_e M^1) d\mathcal{Q}. \]  

(4.1)
where the direct integral of $L^2(\mathfrak{H}_{e,M^1})_+$ over $\mathcal{Q}$ is defined by
\[
\int_\mathcal{Q} L^2(\mathfrak{H}_{e,M^1})_+ d\mathcal{q} = \left\{ \Psi \in L^2(\mathfrak{H}_{e,M^1}) \otimes L^2(\mathcal{Q}) \mid \Psi(\mathcal{q}) \geq 0 \text{ w.r.t. } L^2(\mathfrak{H}_{e,M^1})_+ \text{ for a.e. } \mathcal{q} \right\}.
\] (4.2)

**Lemma 4.1** Let $B : \mathcal{Q} \to L^2(\mathfrak{H}_{e,M^1}) : \mathcal{q} \mapsto B(\mathcal{q})$ be strongly continuous. Then we have
\[
\int_\mathcal{Q} \mathcal{L}(B(\mathcal{q})^*) \mathcal{R}(B(\mathcal{q})) d\mathcal{q} \geq 0 \text{ w.r.t. } \mathcal{C}_M.
\] (4.3)

In particular $\mathcal{L}(C^*) \mathcal{R}(C) \otimes \mathbb{I} \geq 0$ w.r.t. $\mathcal{C}_M$ for each $C \in L^\infty(L^2(\mathfrak{H}_{e,M^1}))$.

**Proof.** For a.e. $\mathcal{q}$, we obtain $\mathcal{L}(B(\mathcal{q})^*) \mathcal{R}(B(\mathcal{q})) \geq 0$ w.r.t. $L^2(\mathfrak{H}_{e,M^1})_+$ by Lemma 3.4.

Thus $\int_\mathcal{Q} \mathcal{L}(B(\mathcal{q})^*) \mathcal{R}(B(\mathcal{q})) d\mathcal{q}$ leaves $\mathcal{C}_M$ invariant. □

**4.2 Lower bounds for the effective Coulomb interaction**

**Proposition 4.2** One has the following.

(i) If $U_{\text{eff}}$ is positive semidefinite, then
\[
\sum_{x,y \in \Lambda} U_{\text{eff}}(x-y) \mathcal{L}(n_x - \frac{1}{2}) \mathcal{R}(n_y - \frac{1}{2}) \otimes \mathbb{I} \geq 0 \text{ w.r.t. } \mathcal{C}_M.
\] (4.4)

(ii) If $U_{\text{eff}}$ is positive definite, then there exists a $U_0 > 0$ such that
\[
\sum_{x,y \in \Lambda} U_{\text{eff}}(x-y) \mathcal{L}(n_x - \frac{1}{2}) \mathcal{R}(n_y - \frac{1}{2}) \otimes \mathbb{I} \geq U_0 \sum_{x \in \Lambda} \mathcal{L}(n_x) \mathcal{R}(n_x) \otimes \mathbb{I} - U_0 M^\dagger \mathbb{I} + \frac{1}{4} U_0 |\Lambda| \text{ w.r.t. } \mathcal{C}_M.
\] (4.5)

**Proof.** (i) Let $M = (M_{xy})$ be a $|\Lambda| \times |\Lambda|$ matrix defined by $M_{xy} = U_{\text{eff}}(x-y)$. By the assumption, $M$ is positive semidefinite. Thus there exists an orthogonal matrix $P$ such that $M = PDP^T$, where $D = \text{diag}(\lambda_x)$ is a diagonal matrix with $\lambda_x \geq 0$. Set $n = (n_x - \frac{1}{2})_{x \in \Lambda}$ and set $\tilde{n} = P^T n$. Writing $\tilde{n} = (\tilde{n}_x)_{x \in \Lambda}$, we have
\[
\sum_{x,y \in \Lambda} U_{\text{eff}}(x-y) \mathcal{L}(n_x - \frac{1}{2}) \mathcal{R}(n_y - \frac{1}{2}) = \langle \mathcal{L}(n), M \mathcal{R}(n) \rangle = \langle \mathcal{L}(\tilde{n}), D \mathcal{R}(\tilde{n}) \rangle = \sum_{x \in \Lambda} \lambda_x \mathcal{L}(\tilde{n}_x) \mathcal{R}(\tilde{n}_x).
\] (4.6)

Clearly the right hand side of (4.6) is positive w.r.t. $L^2(\mathfrak{H}_{e,M^1})_+$. 17
(ii) By the assumption, $M$ is positive definite. Thus the lowest eigenvalue of $M$ is strictly positive: $U_0 := \min_x \lambda_x > 0$. Thus by (4.6), one sees

$$\sum_{x,y \in \Lambda} U_{\text{eff}}(x - y) \mathcal{L}(n_x - \frac{1}{2}) \mathcal{R}(n_y - \frac{1}{2}) = \sum_{x \in \Lambda} \lambda_x \mathcal{L}(n_x) \mathcal{R}(n_x)$$

$$\geq U_0 \sum_{x \in \Lambda} \mathcal{L}(n_x) \mathcal{R}(n_x)$$

$$= U_0 \sum_{x \in \Lambda} \mathcal{L}(n_x - \frac{1}{2}) \mathcal{R}(n_x - \frac{1}{2})$$

$$= U_0 \sum_{x \in \Lambda} \mathcal{L}(n_x) \mathcal{R}(n_x) \otimes 1 - U_0 M^\dagger + \frac{1}{4} U_0 |\Lambda|.$$  

This completes the proof. □

4.3 Completion of proof of Theorem 2.3

**Proposition 4.3** Assume that $U_{\text{eff}}$ is positive semidefinite. For all $\beta \geq 0$ and $M^\dagger \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\}$, we have $e^{-\beta H_M} \geq 0$ w.r.t. $\mathcal{C}_M$.

**Proof.** Since $U \geq 0$ w.r.t. $\mathcal{C}_M$ by Proposition 4.2, we have

$$e^{\beta U} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} U^n \geq 0 \text{ w.r.t. } \mathcal{C}_M \text{ for all } \beta \geq 0.$$  

(4.8)

By (3.38), it holds that $e^{-\beta H_P} \geq 0$ w.r.t. $L^2(\mathcal{Q})_+$ for all $\beta \geq 0$. Let $K = H_P - U$. Thus it holds that $e^{-\beta K} = e^{-\beta H_P} e^{\beta U} \geq 0$ w.r.t. $\mathcal{C}_M$ for all $\beta \geq 0$.

By (3.49) and Lemma 4.1, we have

$$e^{\beta T} = \int_0^{\beta} \mathcal{L}(e^{\beta T + \gamma(q)}) \mathcal{R}(e^{\beta T + \gamma(q)}) dq \geq 0 \text{ w.r.t. } \mathcal{C}_M.$$  

(4.9)

Combining these properties, we have

$$\left( e^{\beta T/\sqrt{n}} e^{-\beta K/\sqrt{n}} \right)^n \geq 0 \text{ w.r.t. } \mathcal{C}_M \text{ for all } \beta \geq 0.$$  

(4.10)

Thus the proposition follows from the Trotter-Kato formula. □

4.4 Proof of Corollary 2.4

Let $J$ be a conjugation defined by $(J \Psi)(q) = \Psi^*(q)$ for each $\Psi \in L^2(\mathcal{F}_{e,M}) \otimes L^2(\mathcal{Q})$. Since $e^{-\beta H_M}$ preserves the positivity w.r.t. $\mathcal{C}_M$, $H_M$ commutes with $J$. Let $\lambda$ be an eigenvalue of $H_M$ and let $\Psi$ be a corresponding eigenvector. Put $\Psi_R = (\Psi + J \Psi)/2$ and $\Psi_I = (\Psi - J \Psi)/2i$. Then $\Psi_R(q)$ and $\Psi_I(q)$ are self-adjoint for a.e. $q$. In addition, they are eigenvectors of $H_M$ with the associated eigenvalue $\lambda$.

Let $\psi_M$ be a ground state of $H_M$. $\psi_M$ can then be written as $\psi_M = \int_0^{\beta T} \psi_M(q) dq$ under the identification (3.39). Without loss of generality, we may assume that $\psi_M(q)$ is self-adjoint for a.e. $q$. Let $\psi_{M,+}(q)$ (resp. $\psi_{M,-}(q)$) be the positive (resp. negative) part part
of $\psi_M(q)$. Hence it holds that $\psi_M = \psi_M^+ - \psi_M^-$, $\psi_M^\pm \in C_M$ and $\langle \psi_M^+, \psi_M^- \rangle = 0$. By Proposition 4.3 we have
\[
e^{-\beta E_M} = \langle \psi_M, e^{-\beta H_M} \psi_M \rangle \leq \langle |\psi_M|, e^{-\beta H_M} |\psi_M| \rangle, \tag{4.11}
\]
where $|\psi_M| = \psi_M^+ + \psi_M^-$. This means $|\psi_M|$ is a ground state of $H_M$ as well.

By using the notation in Subsection 5.1 we can express $|\psi_M|$ as
\[
|\psi_M| = \sum_{x, Y \in \Lambda^M |\Lambda \mu} \int Q |\psi_M|_{XY}(q)|e_X\rangle\langle e_Y|dq.	ag{4.12}
\]
Since $|\psi_M|$ is a non-zero vector, $|\psi_M|$ is non-zero as well. Thus there exists an $X_0 \in \Lambda^M |\Lambda \mu$ such that $|\psi_M|_{X_0 X_0}(q) \neq 0$. Observe that $S^2_{\text{tot}}|e_X\rangle\langle e_X| = 0$. From this it follows $P_{S=0}|\psi_M| \neq 0$, where $P_{S=0}$ is the orthogonal projection onto ker$[S^2_{\text{tot}}]$. By using the fact that $W^* S^2_{\text{tot}} W = S^2$, we obtain (i).

Note that $WS_{x+y} - W^* = \gamma(x) \gamma(y) L(c_x c_y) R((c_x c_y)^*)$. Hence
\[
\langle \varphi_M, S_{x+y} - \varphi_M \rangle = \gamma(x) \gamma(y) \langle |\psi_M|, L(c_x c_y) R((c_x c_y)^*) |\psi_M| \rangle. \tag{4.13}
\]
Since $|\psi_M|$ is positive and $L(c_x c_y) R((c_x c_y)^*) \geq 0$ w.r.t. $C_M$, we conclude (ii). □

5 Proofs of Theorem 2.7 and Corollary 2.8

5.1 Preliminaries

Let $G = (\Lambda, E)$ be a connected graph. For each $0 \leq n \leq |\Lambda|$, we set
\[
\Lambda^{(n)} = \{X = (x_1, \ldots, x_n) \in \Lambda^n \mid x_1 \neq \cdots \neq x_n\}. \tag{5.1}
\]
Let $S_n$ be the permutation group on the set $\{1, \ldots, n\}$. For $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Lambda^{(n)}$, if there exists a $\sigma \in S_n$ such that $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (y_1, \ldots, y_n)$, then we denote $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$. The binary relation $\sim^n$ on $\Lambda^{(n)}$ is an equivalence relation. We denote by $\Lambda^n \Lambda$ the quotient set $\Lambda^{(n)} / \sim$. For notational simplicity, let $(x_1, \ldots, x_n)$ denote the equivalence class $[x_1 \ldots, x_n)$ as well, if no confusion occurs. We say that $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \Lambda^n \Lambda$ are neighbors if there exists a unique $j$ such that $x_j$ and $y_j$ are neighbors in $G$ and $x_i = y_i$ holds for all $i \in \{1, \ldots, n\} \setminus \{j\}$. Now we define the fermionic graph by
\[
\wedge^n G = (\wedge^n \Lambda, \wedge^n E), \tag{5.2}
\]
\[
\wedge^n E = \{X, Y \in [\wedge^n \Lambda]^2 \mid X, Y \text{ are neighbors} \} \tag{5.3}
\]
with $\wedge^0 G = (\emptyset, \emptyset)$, the empty graph, and $\wedge^1 G = G$. Remark that since $|\wedge^{[\Lambda]} \Lambda| = 1$, $\wedge^{[\Lambda]} G$ is trivial.

The following proposition is often useful.

Proposition 5.1 If $G$ is connected, then $\wedge^n G$ is connected for all $0 < n < |\Lambda|$. 

19
Proof. See, e.g., [17].

A path in $\wedge^n G$ is a graph $P = (v, e) \subseteq \wedge^n G$ with $v = \{X_1, \ldots, X_N\}$ and $e = \{\{X_1, X_2\}, \{X_2, X_3\}, \ldots, \{X_{N-1}, X_N\}\}$, where all $X_j$ are distinct. In this paper, the path $P$ is simply denoted by $P = X_1X_2 \cdots X_N$. The number $N - 1$ is called the length of the path $P$ and denoted by $|P|$. For each $X, Y \in \wedge^n \Lambda$, we denote the set of all paths from $X$ to $Y$ by $P^{(n)}_{XY}$. For each $L \in \mathbb{N}$, we set

$$P^{(n)}_{XY}[L] = \left\{ P \in P^{(n)}_{XY} \mid |P| = L \right\}. \tag{5.4}$$

Clearly it holds that $P^{(n)}_{XY} = \bigcup_L P^{(n)}_{XY}[L].$

Let $e_x(y) = \delta(x - y)$. Then $\{e_x \mid x \in \Lambda\}$ is a complete orthonormal system (CONS) of $\ell^2(\Lambda)$. For each $X = (x_1, \ldots, x_n) \in \wedge^n \Lambda$, we define

$$e_X = e_{x_1} \wedge \cdots \wedge e_{x_n} \in \wedge^n \ell^2(\Lambda). \tag{5.5}$$

Then $\{e_X \mid X \in \wedge^n \Lambda\}$ is a CONS of $\wedge^n \ell^2(\Lambda)$ as well. Note that each $\psi \in L^2(\mathfrak{F}_{e,M}) \otimes L^2(Q)$ can be expressed as

$$\psi = \sum_{X,Y \in \wedge^M \Lambda} \int_Q \psi_{XY}(q)|e_X\rangle\langle e_Y|dq. \tag{5.6}$$

5.2 Positivity improving semigroup

Our main purpose in this section is to show the following:

**Theorem 5.2** Assume that $U_{\text{eff}}$ is positive definite. For all $\beta > 0$ and $M^1 \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\}$, one has $e^{-\beta H_M} > 0$ w.r.t. $\mathfrak{C}_M$.

As a corollary, we obtain the following result by Theorem A.2.

**Corollary 5.3** Assume that $U_{\text{eff}}$ is positive definite. Let $E_M$ be the ground state energy, i.e., the lowest eigenvalue of $H_M$. For each $M^1 \in \{-|\Lambda|/2, -(|\Lambda| - 1)/2, \ldots, |\Lambda|/2\}$, $E_M$ is nondegenerate and the corresponding eigenvector is strictly positive w.r.t. $\mathfrak{C}_M$.

Next we will explain how to prove Theorem 5.2.

**Proposition 5.4** Let $U_0$ be the strictly positive constant given by Proposition 4.2. Let

$$\mathbb{U}_0 = U_0 \sum_{x \in \Lambda} \mathcal{L}(n_x)/\mathcal{R}(n_x). \tag{5.7}$$

We define a new Hamiltonian $H_M^{(0)}$ by

$$H_M^{(0)} = K_M - \mathbb{U}_0. \tag{5.8}$$

If $e^{-\beta H_M^{(0)}} > 0$ w.r.t. $\mathfrak{C}_M$ for all $\beta > 0$, then $e^{-\beta H_M} > 0$ w.r.t. $\mathfrak{C}_M$ for all $\beta > 0$. 20
Proof. Put $C = -U_0M^\dagger + \frac{1}{4}U_0|\Lambda|$. Then by Proposition 4.2 it holds that $U \supset U_0 + C$ w.r.t. $\mathcal{C}_M$. Hence applying Proposition A.1, we have $e^{-\beta H_M} \supset e^{-\beta H_M^{(0)}} e^{-\beta C}$ w.r.t. $\mathcal{C}_M$. Thus if $e^{-\beta H_M^{(0)}} \supset 0$ w.r.t. $\mathcal{C}_M$, we conclude that $e^{-\beta H_M} \supset 0$ w.r.t. $\mathcal{C}_M$. □

By Proposition 5.3 it suffices to prove that $e^{-\beta H_M^{(0)}} \supset 0$ w.r.t. $\mathcal{C}_M$ for all $\beta > 0$. By the Duhamel formula, we have

$$e^{-\beta H_M^{(0)}} = \sum_{n \geq 0} \mathcal{D}_{n,\beta}, \quad (5.9)$$

$$\mathcal{D}_{n,\beta} = \int_{S_n(\beta)} e^{-s_1 K_M U_0} e^{-s_2 K_M U_0} \ldots e^{-s_n K_M U_0} e^{-(\beta - \sum_{j=1}^{n} s_j) K_M}, \quad (5.10)$$

where $\int_{S_n(\beta)} = \int_{0}^{\beta} dt_1 \int_{0}^{\beta - t_1} dt_2 \ldots \int_{0}^{\beta - \sum_{j=1}^{n-1} t_j} dt_n$ and $\mathcal{D}_{n,\beta} = e^{-\beta K_M}$. In Subsection 5.3 we will prove the following:

**Theorem 5.5 (Ergodicity)** \{\mathcal{D}_{n,\beta}\}_{n \in \mathbb{N}_0} is ergodic in a sense that, for each $\varphi, \psi \in \mathcal{C}_M \setminus \{0\}$, there are $\beta > 0$ and $n \in \mathbb{N}_0$ such that $\langle \varphi, \mathcal{D}_{n,\beta} \psi \rangle > 0$.

Assuming Theorem 5.5, we can prove Theorem 5.2.

**Proof of Theorem 5.2 given Theorem 5.5**

Basic idea comes from [5, 17]. Note that since $e^{\beta T} \supset 0$, $U_0 \supset 0$ w.r.t. $\mathcal{C}_M$, we see $\mathcal{D}_{n,\beta} \supset 0$ w.r.t. $\mathcal{C}_M$. Thus for each $n \in \mathbb{N}_0$, one has

$$e^{-\beta H_M^{(0)}} \supset \mathcal{D}_{n,\beta} \quad (5.11)$$

w.r.t. $\mathcal{C}_M$. Take $\varphi, \psi \in \mathcal{C}_M \setminus \{0\}$ arbitrarily. Then by Theorem 5.5 there exist $\beta > 0$ and $n \in \mathbb{N}_0$ such that $\langle \varphi, \mathcal{D}_{n,\beta} \psi \rangle > 0$. Hence using (5.11), we have $\langle \varphi, e^{-\beta H_M^{(0)}} \psi \rangle \geq \langle \varphi, \mathcal{D}_{n,\beta} \psi \rangle > 0$. To summarize, for each $\varphi, \psi \in \mathcal{C}_M \setminus \{0\}$, there exists a $\beta > 0$ such that $\langle \varphi, e^{-\beta H_M^{(0)}} \psi \rangle > 0$. This means $e^{-\beta H_M^{(0)}}$ improves the positivity w.r.t. $\mathcal{C}_M$ by Theorem A.2 □

**5.3 Proof of Theorem 5.5**

To prove Theorem 5.2 it suffices to show Theorem 5.5.

**Proposition 5.6** Let

$$\mathcal{C}_{n,\beta} = \left( U_0^M e^{-\beta K_M/(n-1)} \right)^n e^{\beta K_M/(n-1)} \supset U_0^M e^{\beta T/(n-1)} U_0^M \ldots e^{\beta T/(n-1)} U_0^M. \quad (5.12)$$

Suppose that \{\mathcal{C}_{n,\beta}\} is ergodic in a sense that for each $\varphi, \psi \in \mathcal{C}_M \setminus \{0\}$, there are $\beta > 0$ and $n \in \mathbb{N}_0$ such that $\langle \varphi, \mathcal{C}_{n,\beta} \psi \rangle > 0$. Then \{\mathcal{D}_{n,\beta}\} is ergodic.

**Proof.** Put $N(n) = nM^\dagger + (n-1)$. It suffices to show that $\{\mathcal{D}_{N(n),\beta}\}_{n,\beta}$ is ergodic. Let

$$F_n(s_1, \ldots, s_n) = e^{-s_1 K_M U_0} e^{-s_2 K_M U_0} \ldots e^{-s_n K_M U_0} e^{-(\beta - \sum_{j=1}^{n} s_j) K_M}. \quad (5.13)$$

21
By (5.10), it holds that
\[ \mathcal{D}_{N(n), \beta} = \int_{S_{N(n)}(\beta)} F_{N(n)}(s_1, \ldots, s_{N(n)}) \quad (5.14). \]

Notice that
\[ \mathcal{C}_{n, \beta} = F_{N(n)}(0, \ldots, 0, \beta/(n - 1), 0, \ldots, 0, \beta/(n - 1), 0, \ldots, 0) \quad (5.15). \]

In particular, \( \mathcal{C}_{n, \beta} \geq 0 \) w.r.t. \( \mathcal{C}_M \) for all \( n \in \mathbb{N}_0 \) and \( \beta \geq 0 \). Since \( \{\mathcal{C}_{n, \beta}\} \) is ergodic, for each \( \varphi, \psi \in \mathcal{C}_M \setminus \{0\} \), there are \( \beta > 0 \) and \( n \in \mathbb{N}_0 \) such that \( \langle \varphi, \mathcal{C}_{n, \beta} \psi \rangle > 0 \). Let \( f(s_1, \ldots, s_{N(n)}) = \langle \varphi, F_{N(n)}(s_1, \ldots, s_{N(n)}) \psi \rangle \). Then \( f \) is a non-zero positive function so that
\[ f(0, \ldots, 0, \beta/(n - 1), 0, \ldots, 0, \beta/(n - 1), 0, \ldots, 0) > 0. \quad (5.16) \]

Moreover, \( f \) is continuous in \( s_1, \ldots, s_{N(n)} \). Thus
\[ \langle \varphi, \mathcal{D}_{N(n), \beta} \psi \rangle = \int_{S_{N(n)}(\beta)} f(s_1, \ldots, s_{N(n)}) \quad (5.17). \]

This means that \( \{\mathcal{D}_{N(n), \beta}\}_{n, \beta} \) is ergodic. \( \square \)

Henceforth, we may assume
\[ U_0 = 1 \quad (5.18) \]

without loss of generality.

**Lemma 5.7** Let \( E_X = |e_X\rangle \langle e_X| \) for all \( X \in \Lambda^M \Lambda \). We have
\[ \mathbb{U}^M_{\Lambda} \ni \sum_{X \in \Lambda^M \Lambda} \mathcal{L}(E_X) \mathcal{R}(E_X) \quad \text{w.r.t.} \quad L^2(\mathcal{H}_{e,M^1}). \quad (5.19) \]

**Proof.** Since \( |\Lambda^M| \geq |\Lambda^M \Lambda| \) and \( E_X = n_{x_1} \cdots n_{x_{M^1}} \) for each \( X = (x_1, \ldots, x_{M^1}) \in \Lambda^M \Lambda \), we obtain
\[ \mathbb{U}^M_{\Lambda} \ni \sum_{X \in \Lambda^M \Lambda} \mathcal{L}(n_{x_1} \cdots n_{x_{M^1}}) \mathcal{R}(n_{x_1} \cdots n_{x_{M^1}}) \quad (5.20) \]

w.r.t. \( L^2(\mathcal{H}_{e,M^1}) \). \( \square \)

We introduce the following notation:
\[ \int d\nu_{q_1, \ldots, q_{n-1}}^{(n-1)} F(\omega_1, \ldots, \omega_{n-1}) \]
\[ := \int_{Q_{n-2}} \prod_{j=1}^{n-2} dq_j \int d\mu_{q_1, q_2; \beta} (\varphi_1) d\mu_{q_1, q_2; \beta} (\varphi_2) \cdots d\mu_{q_{n-2}, q_{n-1}; \beta} (\varphi_{n-1}) \times \exp \left[ \frac{1}{\beta} \int_0^\beta ds V(\omega_j(s) | \varphi_j) \right] F(\omega_1(\varphi_1), \ldots, \omega_{n-1}(\varphi_{n-1})). \quad (5.21) \]
Proposition 5.8 For each $P = X_1X_2 \cdots X_{|P|+1} \in \mathcal{S}^{(M^1)}_{XY}$ and $\varphi_1, \ldots, \varphi_{|P|} \in A$, let

$$
\mathcal{G}_\beta^{(M^1)}(P, \{\omega_j(\varphi_j)\}_{j=1}^{|P|}) = \prod_{j=1}^{|P|} E_{X_j} G_\beta(\omega_j(\varphi_j)) E_{X_{j+1}},
$$

(5.22)

where $\prod_{j=1}^n A_j := A_1 A_2 \cdots A_n$, the ordered product. Put $\bar{\beta} = \beta/(n - 1)$. The kernel operator of $\mathcal{K}_{n,\beta}$ satisfies the following operator inequality:

$$
\mathcal{C}_{n,\beta}(q, q') \geq \sum_{X_1, X_2, \ldots, X_n \in \Lambda^{M^1} \Lambda} d_{q, q'; \bar{\beta}}^{(n-1)} \int \mathcal{L} \left[ \mathcal{G}_\beta^{(M^1)}(P, \{\omega_j\}_{j=1}^{n-1}) \right] \mathcal{R} \left[ \mathcal{G}_\beta^{(M^1)}(P, \{\omega_j\}_{j=1}^{n-1}) \right]^* w.r.t. L^2(\bar{\mathfrak{S}}_{e,M^1}^+).
$$

(5.23)

Proof. By Proposition 3.9, $e^{-\beta K_{M^1}(q, q')} \geq 0$ w.r.t. $L^2(\bar{\mathfrak{S}}_{e,M^1}^+)$. Thus by Lemma 5.7 we have

$$
\mathcal{C}_{n,\beta}(q, q') \geq \sum_{X_1, \ldots, X_n \in \Lambda^{M^1} \Lambda} d_{q, q'; \bar{\beta}}^{(n-1)} \int \mathcal{L} \left[ E_{X_1} G_\beta(\omega_1) E_{X_2} G_\beta(\omega_2) \cdots G_\beta(\omega_{n-1}) E_{X_n} \right]

\times \mathcal{R} \left[ E_{X_1} G_\beta(\omega_1) E_{X_2} G_\beta(\omega_2) \cdots G_\beta(\omega_{n-1}) E_{X_1} \right]

\geq \sum_{X_1, X_2, \ldots, X_n \in \Lambda^{M^1} \Lambda} d_{q, q'; \bar{\beta}}^{(n-1)} \int \mathcal{L} \left[ E_{X_1} G_\beta(\omega_1) E_{X_2} G_\beta(\omega_2) \cdots G_\beta(\omega_{n-1}) E_{X_n} \right]

\times \mathcal{R} \left[ E_{X_1} G_\beta(\omega_1) E_{X_2} G_\beta(\omega_2) \cdots G_\beta(\omega_{n-1}) E_{X_1} \right]

\geq \sum_{X, Y \in \Lambda^{M^1} \Lambda} \int_{\mathcal{Q}} \psi_{XY}(q) e_X \langle e_Y | d q \int_{\mathcal{Q}} \varphi_{XY}(q) e_X \langle e_Y | d q.
$$

(5.24)

w.r.t. $L^2(\bar{\mathfrak{S}}_{e,M^1}^+)$. □

Let $\psi, \varphi \in \mathcal{C}_M \setminus \{0\}$. We can express as

$$
\psi = \sum_{X, Y \in \Lambda^{M^1} \Lambda} \int_{\mathcal{Q}} \psi_{XY}(q) e_X \langle e_Y | d q, \quad \varphi = \sum_{X, Y \in \Lambda^{M^1} \Lambda} \int_{\mathcal{Q}} \varphi_{XY}(q) e_X \langle e_Y | d q.
$$

Since $\psi \geq 0, \varphi \geq 0$ w.r.t. $\mathcal{C}_M$, one has $\langle e_X, \psi(q) e_X \rangle_{\bar{\mathfrak{S}}_{e,M^1}} \geq 0$ and $\langle e_X, \varphi(q) e_X \rangle_{\bar{\mathfrak{S}}_{e,M^1}} \geq 0$ for all $X \in \Lambda^{M^1} \Lambda$ which imply $\psi_{XX}(q) \geq 0$ and $\varphi_{XX}(q) \geq 0$ for all $X \in \Lambda^{M^1} \Lambda$ and
a.e. \( q \). Especially since both \( \psi \) and \( \varphi \) are non-zero, there exist \( X, Y \in \Lambda^{M^1} \Lambda \) and \( S_X, S_Y \subseteq \mathcal{Q} \) with non-vanishing Lebesgue measures such that \( \psi_{XX}(q) > 0 \) on \( S_X \) and \( \varphi_{YY}(q) > 0 \) on \( S_Y \). Then one obtains the following:

**Lemma 5.9** It holds that

\[
\langle \varphi, \mathcal{E}_{n, \beta} \psi \rangle \geq \sum_{q \in \mathcal{Q}} \sum_{P \in \mathcal{P}_{X, n}^{(M^1)\{n-1\}} \cap \mathcal{Q} \times \mathcal{Q}} dq \, dq' \int_{S_Y \times S_X} d\nu_{q, q'; \beta} \, \varphi_{YY}(q) \psi_{XX}(q') \int_{S_Y \times S_X} d\nu_{q, q'; \beta} \, \varphi_{YY}(q) \psi_{XX}(q') \\
\times \left| \left\langle e_Y, \mathcal{G}_{\beta}^{(M^1)}(P, \{\omega\}) \right\rangle \right|^2 \int_{\mathbb{R}^{M^1}} d\nu_{\gamma, \nu, \varphi, \psi, \xi}.
\] (5.25)

**Proof.** By (5.23), we have

\[
\langle \varphi, \mathcal{E}_{n, \beta} \psi \rangle \geq \sum_{X_1, X_n \in \Lambda^{M^1 \Lambda} \cap \mathcal{Q} \times \mathcal{Q}} \sum_{P \in \mathcal{P}_{X, n}^{(M^1)\{n-1\}} \cap \mathcal{Q} \times \mathcal{Q}} dq \, dq' \int_{S_Y \times S_X} d\nu_{q, q'; \beta} \, \varphi_{XX}(q) \psi_{XX}(q') \\
\times \left| \left\langle e_X, \mathcal{G}_{\beta}^{(M^1)}(P, \{\omega\}) \right\rangle \right|^2 \int_{\mathbb{R}^{M^1}} d\nu_{\gamma, \nu, \varphi, \psi, \xi}.
\] (5.26)

To show that \( \{\mathcal{E}_{n, \beta}\} \) is ergodic, it suffices to find some \( n \) and \( \beta \) such that the R.H.S. of (5.26) is strictly positive.

Before we proceed, we need some preparations. Set \( a_z = a_z(\{x, y\}) = \sqrt{2} \omega_0^{1/2} (g(x - z) - g(y - z)) \). Let

\[
\mathcal{Y} = \left\{ (q, q') \in \mathcal{Q} \times \mathcal{Q} \mid \exists \{x, y\} \in E \text{ s.t. } \sum_{z \in \Lambda} a_z(\{x, y\}) (q_z - q_z') \in 2\pi\mathbb{Z} \right\}.
\] (5.27)

Clearly \( \mathcal{Y} \) is a set of Lebesgue measure 0.

**Lemma 5.10** Let

\[
W_{q, q'} = \left\{ \varphi \in A \left| \lim_{\beta \to +0} \beta^{-1} \int_0^\beta ds \exp \left( \Phi_{\{x, y\}}(\omega(s)) \right) \neq 0 \forall \{x, y\} \in E \right\}.
\] (5.28)

For all \( \beta > 0 \) and \( (q, q') \in \mathcal{Y}^c \), the complement of \( \mathcal{Y} \), we have \( \mu_{q, q'; \beta}(W_{q, q'}) > 0 \).

**Proof.** It suffices to show \( \int_{W_{q, q'}} D\alpha > 0 \) for all \( (q, q') \in \mathcal{Y}^c \). Note that we can write \( \Phi_{\{x, y\}}(q) = \sum_{z \in \Lambda} a_z q_z \). Denote \( \int_A D\alpha F \) by \( \mathbb{E}[F] \). We observe that, by (3.52) and (3.53),

\[
\mathbb{E} \left[ \int_0^\beta ds \exp \left( \Phi_{\{x, y\}}(\omega(s)) \right) \right] = \int_0^\beta ds \mathbb{E} \left[ \exp \left( \Phi_{\{x, y\}}(\omega(s)) \right) \right] = \int_0^\beta ds \exp \left[ \sum_{z \in \Lambda} \left( ia_z \left( (1 - \beta^{-1}s)q_z + \beta^{-1}sq_z' \right) - s(1 - \beta^{-1}s)a_z^2 \right) \right] = \beta \int_0^1 dt \exp \left[ \sum_{z \in \Lambda} \left( ia_z \left( (1 - t)q_z + tq_z' \right) - \beta t(1 - t)a_z^2 \right) \right].
\] (5.29)
Thus, for all \((q, q') \in \mathcal{Y}\) and \(\{x, y\} \in E\), it holds that

\[
\lim_{\beta \to +0} \beta^{-1} E \left[ \int_0^\beta ds \exp \left\{ i\Phi_{\{x,y\}}(\omega(s)) \right\} \right] = \int_0^1 dt \exp \left[ \sum_{z \in \Lambda} \left\{ ia_z \left( (1-t)q_z + tq_z' \right) \right\} \right] \neq 0. \quad (5.30)
\]

Assume that \(\int_{W_{q,q'}} D\alpha = 0\) for some \((q, q') \in \mathcal{Y}\). Thus, by the dominated convergence theorem, we have

\[
\lim_{\beta \to +0} \beta^{-1} E \left[ \int_0^\beta ds \exp \left\{ i\Phi_{\{x,y\}}(\omega(s)) \right\} \right] = \int_{W_{q,q'}} D\alpha \lim_{\beta \to +0} \beta^{-1} \int_0^\beta ds \exp \left\{ i\Phi_{\{x,y\}}(\omega(s)(\varphi)) \right\} = 0 \quad (5.31)
\]

for some \(\{x, y\} \in E\). This contradicts \((5.30)\). \(\square\)

**Lemma 5.11** Let \((q, q') \in \mathcal{Y}\). For all \(\varphi \in W_{q,q'}\),

\[
\lim_{\beta \to +0} \beta^{-1} \left\langle e_X, \int_0^\beta ds \mathbb{T}_+(\omega(s)(\varphi))e_Y \right\rangle \neq 0 \quad (5.32)
\]

holds if and only if \(\{X, Y\} \in \wedge^M E\).

**Proof.** Note that, by Appendix C we can write

\[
\mathbb{T}_+(q) = d\Gamma(\mathbb{T}_+(q))_{\wedge^M},
\]

\[
\mathbb{T}_+(q) = \sum_{\{x,y\} \in E} t(x-y) \exp \left\{ i\Phi_{\{x,y\}}(q) \right\} |e_x\rangle \langle e_y| \quad (5.34)
\]

for all \(q \in Q\).

**STEP 1.** We claim that \(\lim_{\beta \to +0} \beta^{-1} \left\langle e_X, \int_0^\beta ds \mathbb{T}_+(\omega(s)(\varphi))e_Y \right\rangle \neq 0\) if and only if \(\{x, y\} \in E\) and \(\varphi \in W_{q,q'}\). To see this, we just remark that, if \(\{x, y\} \in E\),

\[
\left\langle e_X, \int_0^\beta ds \mathbb{T}_+(\omega(s)(\varphi))e_Y \right\rangle = \int_0^\beta ds \int_{\mathbb{T}_+(\omega(s)(\varphi))} e_x \langle e_x, e_y \rangle \quad (5.35)
\]

**STEP 2.** Let \(X = (x_1, \ldots, x_M), Y = (y_1, \ldots, y_M) \in \wedge^M \Lambda\). Assume that \(\{X, Y\} \in \wedge^M \Lambda\). Then there exists a unique \(j\) such that \(\{x_j, y_j\} \in E\) and \(x_i = y_i\) holds for all \(i \neq j\). By \((5.33)\), it holds that

\[
\left\langle e_X, \int_0^\beta ds \mathbb{T}_+(\omega(s)(\varphi))e_Y \right\rangle = \left\langle e_{X_j}, \int_0^\beta ds \mathbb{T}_+(\omega(s)(\varphi))e_{Y_j} \right\rangle. \quad (5.36)
\]

Thus if \(\varphi \in W_{q,q'}\) and \(\{X, Y\} \in \wedge^M E\), we have, by **STEP 1,**

\[
\lim_{\beta \to +0} \beta^{-1} \left\langle e_X, \int_0^\beta ds \mathbb{T}_+(\omega(s)(\varphi))e_Y \right\rangle \neq 0. \quad (5.37)
\]

On the other hand, if \(\{X, Y\} \notin E\) or \(\varphi \notin W_{q,q'}\), the L.H.S. of \((5.37)\) vanishes. \(\square\)
Proposition 5.12 (Connectivity) Let \( P \in \mathcal{P}_{XY}^{(M)}[L] \). Let \((q, q_1), (q_1, q_2), \ldots, (q_{L-1}, q') \in \mathcal{Y}^X\). Then, for all \( \varphi_1 \in W_{q_1}, \varphi_2 \in W_{q_1, q_2}, \ldots, \varphi_L \in W_{q_{L-1}, q'} \), there exists a \( \beta_\ast > 0 \) such that, for all \( \beta \in (0, \beta_\ast) \), one has

\[
\left\langle e_X, \mathcal{G}_\beta^{(M)} \left( P, \{ \omega_j(\varphi_j) \}_{j=1}^L \right) e_Y \right\rangle \mid_{\delta_{e,M}^L} \neq 0. \tag{5.38}
\]

Proof. For each \( P = X_1 X_2 \cdots X_{L+1} \in \mathcal{P}_{XY}^{(M)}[L] \), let

\[
\tau_\beta^{(M)} \left( P, \{ \omega_j(\varphi_j) \}_{j=1}^L \right) = \prod_{j=1}^L E_{X_j} \int_0^\beta ds T_{sg}(\omega_j(s)(\varphi_j)) E_{Y_j}. \tag{5.39}
\]

We have

\[
\mathcal{G}_\beta^{(M)} \left( P, \{ \omega_j(\varphi_j) \}_{j=1}^L \right) = \tau_\beta^{(M)} \left( P, \{ \omega_j(\varphi_j) \}_{j=1}^L \right) + \mathcal{O}(\beta^{L+1}). \tag{5.40}
\]

The error term \( \mathcal{O}(\beta^{L+1}) \) is uniform in \( \varphi_j \), i.e., \( \| \mathcal{O}(\beta^{L+1}) \| \leq C \beta^{L+1} \), where \( C \) is independent of \( \varphi_j \). To see this, we observe that

\[
\begin{align*}
\left\| E_{X_j} \left[ G_\beta(\omega_j(s_j)(\varphi_j)) - \int_0^\beta ds T_{sg}(\omega_j(s)(\varphi_j)) \right] E_{Y_j} \right\| & \leq C \left\| \int_0^\beta ds T_{sg}(\omega_j(s)(\varphi_j)) \right\|^2 \\
& \leq \beta^2 C(M)^2(\max_x |t(x)|)^2 \tag{5.41}
\end{align*}
\]

by Theorem B.1. Here we used the fact that \( E_{X_j} E_{Y_j} = 0 \). By Lemma 5.11, we have

\[
\lim_{\beta \to 0} \beta^{-L} \tau_\beta^{(M)} \left( P, \{ \omega_j(\varphi_j) \}_{j=1}^L \right) \neq 0. \tag{5.42}
\]

By combining this and (5.40), we obtain the desired result. \( \square \)

5.4 Completion of proof of Theorem 2.7

By Proposition 5.1, we can take \( n \in \mathbb{N} \) such that \( \mathcal{P}_{XY}^{(M)}[n-1] \neq \emptyset \). Let \((q, q_1), \ldots, (q_{n-1}, q') \in \mathcal{Y}^X\). For all \( P \in \mathcal{P}_{XY}^{(M)}[n-1] \), \( \beta \in (0, \beta_\ast) \) and \( \varphi_1 \in W_{q_1}, \varphi_2 \in W_{q_1, q_2}, \ldots, \varphi_{n-1} \in W_{q_{n-1}, q'} \), the term

\[
\left\langle e_Y, \mathcal{G}_\beta^{(M)} \left( P, \{ \omega_j(\varphi_j) \}_{j=1}^{n-1} \right) e_X \right\rangle \mid_{\delta_{e,M}^M}^2
\]

is strictly positive by Proposition 5.12.

Hence the R.H.S. of (5.25) is strictly positive provided that \( \beta \) is sufficiently small. Hence \( \{ \mathcal{C}_{n, \beta} \}_n \) is ergodic. \( \square \)

5.5 Proof of Corollary 2.8

By Theorems 2.7 and A.2, \( \mathbb{H}_M \) has a unique ground state \( \psi_M \). Moreover, by Corollary 5.3, \( \psi_M \) is strictly positive w.r.t. \( \mathcal{C}_M \). Hence we have (i).

(ii) immediately follows from (i) and Corollary 2.4.
Because $H_M$ commutes with $S_{tot}^2$ and because the ground state of $H_M$ is unique, we immediately obtain (iii).

By an argument similar to (4.13), we have

$$\langle \phi_M, S_x + S_y - \phi_M \rangle = \gamma(x)\gamma(y)\langle \psi_M, \mathcal{L}(c_x c_y^*) \mathcal{R}((c_x c_y^*)^*) \psi_M \rangle.$$  \hspace{1cm} (5.43)

Since $\psi_M$ is strictly positive and $\mathcal{L}(c_x c_y^*) \mathcal{R}((c_x c_y^*)^*) \succ 0$ w.r.t. $\mathcal{C}_M$, we conclude (iv). \hspace{1cm} $\blacksquare$

6 Proof of Theorem 2.9

6.1 Gaussian domination

For each $h = \{h_x\}_{x \in \Lambda} \in \mathbb{R}^{|

\Lambda|}$, let

$$\mathbb{W}(h) = \frac{1}{2} \sum_{x, y \in \Lambda} U_{\text{eff}}(x - y)(n_x \uparrow - n_x \downarrow + h_x)(n_y \uparrow - n_y \downarrow + h_y). \hspace{1cm} (6.1)$$

Let

$$\tilde{T}_{x, g, \sigma} = T_{x, g, \sigma} - \frac{1}{2}\langle (n_{\sigma} - \frac{1}{2}), U_{\text{eff}}(n_{\sigma} - \frac{1}{2}) \rangle, \quad \sigma \in \{\uparrow, \downarrow\}, \hspace{1cm} (6.2)$$

where $\langle (n_{\sigma} - \frac{1}{2}), U_{\text{eff}}(n_{\sigma} - \frac{1}{2}) \rangle = \sum_{x, y \in \Lambda} U_{\text{eff}}(x - y)(n_{x\sigma} - \frac{1}{2})(n_{y\sigma} - \frac{1}{2})$ and $T_{x, g, \sigma}$ is given by (3.10). We introduce a new Hamiltonian by

$$\mathbb{H}_M(h) = -\tilde{T}_{+g, \uparrow} - \tilde{T}_{-g, \downarrow} + \mathbb{W}(h) + H_p. \hspace{1cm} (6.3)$$

Note that

$$\mathbb{H}_M = \mathbb{H}_M(0). \hspace{1cm} (6.4)$$

The main purpose in this subsection is to show the following.

**Theorem 6.1** Let $Z_{\beta, \epsilon}(h) = \text{Tr}[e^{-\beta \mathbb{H}_M(h)} e^{-\epsilon H_p}]$. One has $Z_{\beta, \epsilon}(h) \leq Z_{\beta, \epsilon}(0)$ for all $h \in \mathbb{R}^{|

\Lambda|}$ and $\epsilon > 0$.

To see this, we need some preliminaries.

6.1.1 Auxiliary lemmas

Let $\tilde{T} = \tilde{T}_{+g, \uparrow} + \tilde{T}_{-g, \downarrow}$. Under the identification (3.39), we have

$$\tilde{T} = \int_{\mathcal{Q}} \tilde{T}(q) dq, \quad \tilde{T}(q) = \tilde{T}_{+g}(q) \otimes \mathbb{I} + \mathbb{I} \otimes \tilde{T}_{-g}(q), \hspace{1cm} (6.5)$$

where

$$\langle (n - \frac{1}{2}), U_{\text{eff}}(n - \frac{1}{2}) \rangle = \sum_{x, y \in \Lambda} U_{\text{eff}}(x - y)(n_x - \frac{1}{2})(n_y - \frac{1}{2}). \hspace{1cm} (6.7)$$

27
Lemma 6.2 Let $\tilde{K}_M = -\tilde{T} + H_p$. Let
\[
Z_{\beta, n, \varepsilon}(h) = \text{Tr} \left[ \left( e^{-\beta \tilde{K}_M \left( e^{-\beta \mathcal{W}(h)/n} \right)} n e^{-\varepsilon H_p} \right)^n \right], \quad n \in \mathbb{N}, \quad \varepsilon > 0.
\] (6.8)

Let us introduce the following notation:
\[
\int d\nu_{n, q, q'; \beta, \varepsilon} F(\omega_1, \ldots, \omega_{n+1})
\]
\[
= \int Q^n \prod_{j=1}^n dq_j \int d\mu_{q, q_1; \beta} \int d\mu_{q_1, q_2; \beta} \cdots \int d\mu_{q_{n-1}, q_n; \beta} \int d\nu_{q, q'; \beta}
\]
\[
\times \exp \left\{ - n \sum_{j=1}^n \int_0^\beta ds V(\omega_j(s)) - \int_0^\varepsilon ds V(\omega_{n+1}(s)) \right\} F(\omega_1, \ldots, \omega_{n+1}).
\] (6.9)

Then, putting $\tilde{\beta} = \beta/n$, one has
\[
Z_{\beta, n, \varepsilon}(h) = \frac{(4\pi)^{-n|\Lambda|/2}}{\prod_{|\Lambda|} \prod_{j=1}^n dk_j} \int Q^n d\nu_{n, q, q'; \beta, \varepsilon} e^{-i \sum_{j=1}^n k_j n^{-1} k_j^2 / 4}
\]
\[
\times \text{Tr} \delta_{n, M^1} \otimes \delta_{n, M^1} \left[ \prod_{j=1}^n \left( \prod_{0}^{\beta} e^{\tilde{T}(\omega_j(s))} ds \right) e^{-\tilde{\sum}_{\xi, y \in \Lambda} \tilde{\beta} k_j U_{\text{eff}}(x-y)(n_{\xi} \otimes 1 - 1 \otimes n_y)} \right].
\] (6.10)

Proof. By the Trotter-Kato product formula, we have
\[
e^{-\beta \tilde{K}_M} = \int d\mu_{q, q'; \beta} \left( \prod_{0}^{\beta} e^{\tilde{T}(\omega(s))} ds \right) e^{-\int_0^\beta ds V(\omega(s))}.
\] (6.11)

Let
\[
\mathcal{J}_{n, \beta, \varepsilon} = \left( e^{-\beta \tilde{K}_M \left( e^{-\beta \mathcal{W}(h)/n} \right)} n e^{-\varepsilon H_p} \right)^n e^{-\varepsilon H_p}.
\] (6.12)

By (6.11), the kernel operator of $\mathcal{J}_{n, \beta, \varepsilon}$ is obtained by the following observation:
\[
\mathcal{J}_{n, \beta, \varepsilon}(q_0, q_{n+1})
\]
\[
= \int Q^n \prod_{j=1}^n dq_j \left( \prod_{j=1}^n e^{-\beta \tilde{K}_M / n} (q_j - q_{j-1}) e^{-\beta \mathcal{W}(h)/n} \right) e^{-\varepsilon H_p (q_0, q_{n+1})}
\]
\[
= \int Q^n \prod_{j=1}^n dq_j \int d\mu_{q_0, q_1; \beta} \cdots \int d\mu_{q_{n-1}, q_n; \beta} e^{-\sum_{j=1}^n \int_0^\beta ds V(\omega_j(s)) - \int_0^\varepsilon ds V(\omega_{n+1}(s))}
\]
\[
\times \prod_{j=1}^n \left\{ \left( \prod_{0}^{\beta} e^{\tilde{T}(\omega_j(s))} ds \right) e^{-\beta \mathcal{W}(h)} \right\}
\]
\[
= \int d\nu_{n, q_0, q_{n+1}; \beta, \varepsilon} \left\{ \left( \prod_{0}^{\beta} e^{\tilde{T}(\omega_j(s))} ds \right) e^{-\beta \mathcal{W}(h)} \right\}.
\] (6.13)
Thus we have
\[
Z_{\beta,n,\varepsilon}(\hbar) = \text{Tr}[\mathcal{J}_{n,\beta,\varepsilon}] = \int_{\mathbb{Q}} dq \text{Tr}_{\mathcal{F}_{e,M}^1} \left[ \mathcal{J}_{n,\beta,\varepsilon}(q,q^*) \right] \\
= \int_{\mathbb{Q}} dq \int d\nu_{q,q^*} \text{Tr}_{\mathcal{F}_{e,M}^1} \left[ \prod_{j=1}^{n} \left( \prod_{0}^{\beta} e^{\mathcal{T}(\omega_j(s)) ds} e^{-\beta W(h)} \right) \right]. \quad (6.14)
\]

Finally applying the following identity
\[
e^{-\beta W(h)} = (4\pi)^{-n|\Lambda|/2} \int_{\mathbb{R}^{|\Lambda|}} dk e^{-i(k \cdot \omega)} e^{-k^2/4} e^{i\sum_{x,y} \beta h_{xy} (n_x-n_y)} , \quad (6.15)
\]
we obtain the assertion in the lemma. \(\square\)

**Lemma 6.3** One has
\[
Z_{\beta,n,\varepsilon}(\hbar) = (4\pi)^{-n|\Lambda|/2} \int_{\mathbb{R}^{|\Lambda|}} dk \int_{\mathbb{Q}} dq \int d\nu_{q,q^*} e^{-i\sum_{j=1}^{n} k_j \cdot \hbar e_{xy} - \sum_{j=1}^{n} k_j^2/4} \\
\times \left| \text{Tr}_{\mathcal{F}_{e,M}^1} \left[ \prod_{j=1}^{n} \left( \prod_{0}^{\beta} e^{\mathcal{T}_+(\omega_j(s)) ds} e^{|\sum_{x,y} \beta h_{xy} (n_x-n_y)} \right) \right] \right|^2. \quad (6.16)
\]

**Proof.** Note that \(\text{Tr}[A \otimes B] = \text{Tr}[A] \text{Tr}[B]\). By Lemma 6.2 we immediately have
\[
Z_{\beta,n,\varepsilon}(\hbar) = (4\pi)^{-n|\Lambda|/2} \int_{\mathbb{R}^{|\Lambda|}} dk \int_{\mathbb{Q}} dq \int d\nu_{q,q^*} e^{-i\sum_{j=1}^{n} k_j \cdot \hbar e_{xy} - \sum_{j=1}^{n} k_j^2/4} \\
\times \text{Tr}_{\mathcal{F}_{e,M}^1} \left[ \prod_{j=1}^{n} \left( \prod_{0}^{\beta} e^{\mathcal{T}_+(\omega_j(s)) ds} e^{|\sum_{x,y} \beta h_{xy} (n_x-n_y)} \right) \right] \\
\times \text{Tr}_{\mathcal{F}_{e,M}^1} \left[ \prod_{j=1}^{n} \left( \prod_{0}^{\beta} e^{\mathcal{T}_-(\omega_j(s)) ds} e^{-|\sum_{x,y} \beta h_{xy} (n_x-n_y)} \right) \right]. \quad (6.17)
\]

Let \(\Theta\) be a conjugation in \(\mathcal{F}_{e,M}^1\) defined by \(\Theta c_{x_1}^{*} \cdots c_{x_M}^{*} = c_{x_1}^{*} \cdots c_{x_M}^{*} \). Noting that \(\Theta c_x \Theta = c_x\), we have \(\Theta \mathcal{T}_-(\omega(s)) \Theta = \mathcal{T}_+(\omega(s))\) and \(\Theta n_x \Theta = n_x\). Thus it holds that
\[
\Theta \prod_{0}^{\beta} e^{\mathcal{T}_-(\omega(s)) ds} \Theta = \prod_{0}^{\beta} e^{\mathcal{T}_+(\omega(s)) ds}, \quad (6.18)
\]
\[
\Theta e^{-\sum_{x,y} \beta h_{xy} (n_x-n_y)} \Theta = e^{-\sum_{x,y} \beta h_{xy} (n_x-n_y)} . \quad (6.19)
\]
Hence using the fact \( \text{Tr}[A] = (\text{Tr}[\Theta A \Theta])^* \), one observes that

\[
\text{Tr}\left[ 3_{e,M} \frac{n}{j} \left( \prod_{j=1}^{n} e^{T_{-g}(\omega(s))ds} e^{-i \sum_{x,y \in A} \delta k_{jx} U_{\text{eff}}(x-y)n_y} \right) \right] \\
= \text{Tr}\left[ \Theta \prod_{j=1}^{n} \left( \prod_{0}^{\frac{\beta}{\delta}} e^{T_{-g}(\omega(s))ds} e^{-i \sum_{x,y \in A} \delta k_{jx} U_{\text{eff}}(x-y)n_y} \right) \right] \\
= \text{Tr}\left[ \Theta \prod_{j=1}^{n} \left( \prod_{0}^{\frac{\beta}{\delta}} e^{T_{-g}(\omega(s))ds} e^{-i \sum_{x,y \in A} \delta k_{jx} U_{\text{eff}}(x-y)n_y} \right) \right] \\
\text{This completes the proof.} \quad \blacksquare
\]

### 6.1.2 Proof of Theorem 6.1

Remark that except \( e^{-i \sum_{j=1}^{n} k_{j} \cdot h} \), every factors of the integrand in (6.10) are positive. Thus \( |Z_{\beta,n,\varepsilon}(h)| \leq Z_{\beta,n,\varepsilon}(0) \). As \( n \to \infty \), \( Z_{\beta,n,\varepsilon}(h) \) converges to \( Z_{\beta,\varepsilon}(h) = \text{Tr}[e^{-\beta H_M(h)}e^{-\varepsilon H_F}] \) by Lemma 6.4 below. Thus we have \( Z_{\beta,\varepsilon}(h) \leq Z_{\beta,n,\varepsilon}(0) \). \( \square \)

**Lemma 6.4** We denote by \( L^1(\mathcal{X}) \) the ideal of all trace class operators on \( \mathcal{X} \). Let \( A_n, A \in L^\infty(\mathcal{X}) \) and \( B_n, B \in L^1(\mathcal{X}) \) such that \( A_n \) converges to \( A \) strongly and \( \|B_n - B\|_1 \to 0 \) as \( n \to \infty \), where \( \| \cdot \|_1 \) is the trace norm. Then \( \|A_n B_n - AB\|_1 \to 0 \) as \( n \to \infty \).

*Proof. See [23, Chap. 2, Example 3]. \( \square \)

### 6.2 Upper bound for the susceptibility

Let \( Z_{\beta,\lambda} = \text{Tr}[e^{-\beta H_M}] \). We define the Duhamel two-point function by

\[
\langle A, B \rangle_{\beta,\lambda} = Z_{\beta,\lambda}^{-1} \int_{0}^{1} dx \text{Tr}[e^{-x\beta H_M} A e^{-(1-x)\beta H_M} B]. \quad (6.21)
\]

**Theorem 6.5** Let \( \delta n_x = n_{x^+} - n_{x^-} \). For all \( h \in \mathbb{C}^{[A]} \), one has

\[
\left( \delta n_x, U_{\text{eff}}(h) \right)_{\beta,\lambda} \leq \beta^{-1} \langle h, U_{\text{eff}}(h) \rangle,
\]

where \( \langle \delta n_x, U_{\text{eff}}(h) \rangle = \sum_{x,y \in A} U_{\text{eff}}(x-y) \delta n_x h_y \).

*Proof.* Let \( \lambda \in \mathbb{R} \). We note

\[
H_M(\lambda h) = H_M + \delta W(\lambda h), \quad (6.23)
\]

\[
\delta W(\lambda h) = W(\lambda h) - W(0) = \lambda \langle \delta n_x, U_{\text{eff}}(h) \rangle + \frac{\lambda^2}{2} \langle h, U_{\text{eff}}(h) \rangle. \quad (6.24)
\]
By the Duhamel formula, we have the norm convergent expansion:

\[
\frac{e^{-\beta H_M(h)}}{\sum_{n=0}^\infty D_n(\lambda)},
\]

\[
D_n(\lambda) = (-\beta)^n \int_{S_n(1)} e^{-s_1 \beta H_M} \delta W(\lambda h) \cdots e^{-s_n \beta H_M} \delta W(\lambda h) e^{-(1-s_1 \cdots s_n) \beta H_M}.
\]

By Lemma 6.4, we have

\[
Z_{\beta,\varepsilon}(\lambda h) = \sum_{n=0}^\infty \text{Tr} \left[ D_n(\lambda) e^{-\varepsilon H_p} \right].
\]

By Theorem 6.1, we have

\[
\text{Tr} \left[ D_1(\lambda) e^{-\varepsilon H_p} \right] = \frac{\lambda^2}{2} (h, U_{\text{eff}} h) \text{Tr} \left[ e^{-\beta H_M e^{-\varepsilon H_p}} \right]
\]

and

\[
\frac{Z_{\beta,\varepsilon}(0) - Z_{\beta,\varepsilon}(\lambda h)}{\lambda^2} \geq 0.
\]

Hence letting \( \lambda \to 0 \), it holds that

\[
\frac{\beta}{2} (h, U_{\text{eff}} h) \text{Tr} \left[ e^{-\beta H_M e^{-\varepsilon H_p}} \right] - \beta^2 \int_0^1 ds_1 \int_0^{1-s_1} ds_2 \text{Tr} \left[ e^{-s_1 \beta H_M (\delta \hat{n}, U_{\text{eff}} h)} e^{-s_2 \beta H_M (\delta \hat{n}, U_{\text{eff}} h)} e^{-(1-s_1-s_2) \beta H_M} \right] \geq 0.
\]

By applying Lemma 6.4, we have \( \lim_{\varepsilon \to +0} \text{Tr}[e^{-\beta H_M e^{-\varepsilon H_p}}] = Z_\beta \) and the second term in (6.30) becomes

\[
\frac{\beta^2}{2} \int_0^1 dx \text{Tr} \left[ (\delta \hat{n}, U_{\text{eff}} h) e^{-x \beta H_M (\delta \hat{n}, U_{\text{eff}} h)} e^{-(1-x) \beta H_M} \right], \quad \varepsilon \to +0.
\]

Thus we get (6.22) for \( h \) real-valued. To extend this to complex-valued \( h \)'s, we just note that, if \( A = A_R + iA_I \) with \( A_R^* = A_R, \ A_I^* = A_I \), we have \( (A^*, A)_{\beta,\Lambda} = (A_R, A_R)_{\beta,\Lambda} + (A_I, A_I)_{\beta,\Lambda} \).

6.3 Completion of proof of Theorem 2.9

Let

\[
b_0(h) = \left( \langle \delta \hat{n}, U_{\text{eff}} h \rangle, \langle \delta \hat{n}, U_{\text{eff}} h \rangle \right)_{\beta,\Lambda},
\]

where \( \langle (A, B) \rangle_{\beta,\Lambda} \) is the Duhamel two-point function associated with \( H_M \). Remark that \( \langle (\delta \hat{n}, U_{\text{eff}} h)^*, (\delta \hat{n}, U_{\text{eff}} h) \rangle_{\beta,\Lambda} = \langle (\delta \hat{n}, U_{\text{eff}} h)^*, (\delta \hat{n}, U_{\text{eff}} h) \rangle_{\beta,\Lambda} \). By [1] Theorem 3.2, we have

\[
g(h) \leq \frac{1}{2} \left( c_0(h) b_0(h) \right)^{1/2} \coth \sqrt{\frac{c_0(h)}{4b_0(h)}},
\]

where \( b_0(h) \leq b(h) \) by Theorem 6.5, we obtain the claim in the theorem. \( \square \)
A Hilbert cones and its related operator inequalities

Let \( X \) be a complex Hilbert space and \( X_+ \) be a Hilbert cone in \( X \).

**Proposition A.1** Let \( A, B \) be self-adjoint positive operators on \( X \). Suppose that

(i) \( e^{-\beta A} \succeq 0 \) w.r.t. \( X_+ \) for all \( \beta \geq 0 \);

(ii) \( A \succeq B \) w.r.t. \( X_+ \).

(iii) \( C = A - B \) is bounded.

Then we have \( e^{-\beta B} \succeq e^{-\beta A} \) w.r.t. \( X_+ \) for all \( \beta \geq 0 \).

**Proof.** By (ii), \( C \succeq 0 \) w.r.t. \( X_+ \) and \( B = A - C \). By the Duhamel formula, we have

\[
e^{-\beta B} = \sum_{n=0}^{\infty} D_n(\beta),
\]

where \( D_n(\beta) = \int_{S_n(\beta)} e^{-s_1 A} e^{-s_2 A} \cdots e^{-s_n A} e^{-(\sum_{j=1}^{n} s_j)A} \),

and \( S_n(\beta) = \int_0^{\beta} ds_1 \int_0^{\beta-s_1} ds_2 \cdots \int_0^{\beta-\sum_{j=1}^{n-1} s_j} ds_n \) and \( D_0(\beta) = e^{-\beta A} \). Since \( C \succeq 0 \) and \( e^{-tA} \succeq 0 \) w.r.t. \( X_+ \), it holds that \( D_n(\beta) \succeq 0 \) w.r.t. \( X_+ \) for all \( n \geq 0 \). Thus by (A.1), we have \( e^{-\beta B} \succeq D_0(\beta) = e^{-\beta A} \) w.r.t. \( X_+ \) for all \( \beta \geq 0 \). \( \blacksquare \)

The following theorem will play an important role.

**Theorem A.2** (Perron-Frobenius-Faris) Let \( A \) be a positive self-adjoint operator on \( X \). Suppose that \( 0 \leq e^{-tA} \) w.r.t. \( X_+ \) for all \( t \geq 0 \) and \( \inf \text{spec}(A) \) is an eigenvalue. Let \( P_A \) be the orthogonal projection onto the closed subspace spanned by eigenvectors associated with \( \inf \text{spec}(A) \). Then the following are equivalent:

(i) \( \dim \text{ran} P_A = 1 \) and \( P_A \succeq 0 \) w.r.t. \( X_+ \).

(ii) \( 0 \leq e^{-tA} \) w.r.t. \( X_+ \) for all \( t > 0 \).

(iii) For each \( x, y \in X_+ \setminus \{0\} \), there exists a \( t > 0 \) such that \( \langle x, e^{-tA} y \rangle > 0 \).

**Proof.** See, e.g., [3, 16, 20]. \( \blacksquare \)

B Strong product integration

Let \( \mathbb{C}_{n \times n} \) be the space of \( n \times n \) matrices with complex entries. Let \( A(\cdot) : [0, a] \to \mathbb{C}_{n \times n} \) be continuous. Let \( P = \{s_0, s_1, \ldots, s_n\} \) be a partition of \( [0, a] \) and \( \mu(P) = \max_j \{s_j - s_{j-1}\} \). The strong product integration of \( A \) is defined by

\[
\prod_{s_0}^{s_n} e^{A(s)ds} := \lim_{\mu(P) \to 0} e^{A(s_1-s_0)} e^{A(s_2-s_1)} \cdots e^{A(s_n-s_{n-1})}, \tag{B.1}
\]

Note that the limit is independent of any partition \( P \).
Theorem B.1 It holds that
\[ \left\| \prod_{0}^{a} e^{A(s)ds} - \mathbb{I} - \int_{0}^{a} ds A(s) \right\| \leq e^{\int_{0}^{a} ds \| A(s) \|} - 1 - \int_{0}^{a} ds \| A(s) \|. \] (B.2)

Proof. See [2]. □

C Second quantization

Let \( A \) be a bounded linear operator on \( \ell^2(\Lambda) \). The second quantization of \( A \) is defined by
\[ d \Gamma(A)_N = \sum_{j=1}^{N} \mathbb{I} \otimes \cdots \otimes A_{jth} \otimes \cdots \otimes \mathbb{I}. \] (C.1)
d\( \Gamma(A)_N \) acts in \( \wedge^N \ell^2(\Lambda) \). Set \( a_{xy} = \langle e_x, Ae_y \rangle \). Then \( d \Gamma(A)_N \) can be expressed as
\[ d \Gamma(A)_N = \sum_{x,y \in \Lambda} a_{xy} c_x^* c_y. \] (C.2)

References

[1] F. J. Dyson, E. H. Lieb, B. Simon, Phase transitions in quantum spin systems with isotropic and nonisotropic interactions, J. Stat. Phys. 18 (1978), 335-383.
[2] J. D. Dollard, C. N. Friedman, Product integration with application to differential equations, Encyclopedia of mathematics and its applications Volume 10, 1979, Addison-Wesley Publishing Company.
[3] W. G. Faris, Invariant cones and uniqueness of the ground state for fermion systems. J. Math. Phys. 13 (1972), 1285-1290.
[4] J. K. Freericks, E. H. Lieb, Ground state of a general electron-phonon Hamiltonian is a spin singlet, Phys. Rev. B 51 (1995), 2812-2821.
[5] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons. Ann. Inst. H. Poincaré Sect. A (N.S.) 19 (1973), 1-103.
[6] J. Fröhlich, R. Israel, E. H. Lieb, B. Simon, Phase transitions and reflection positivity. I. General theory and long range lattice models, Comm. Math. Phys., 62 (1978), 1-34.
[7] J. Fröhlich, B. Simon, T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking, Comm. Math. Phys. 50 (1976), 79-95.
[8] J. Glimm, A. Jaffe, Quantum physics. A functional integral point of view. Second edition. Springer-Verlag, New York, 1987.
[9] L. Gross, Existence and uniqueness of physical ground states. J. Funct. Anal. (1972), 52-109.
[10] T. Holstein, Studies of polaron motion: Part I. The molecular-crystal model, Ann. Phys. 8 (1959), 325-342.

[11] A. Kishimoto, D. W. Robinson, Positivity and monotonicity properties of $C_0$-semigroups. I. Comm. Math. Phys. 75 (1980), 67-84. Positivity and monotonicity properties of $C_0$-semigroups. II. Comm. Math. Phys. 75 (1980), 85-101.

[12] K. Kubo, T. Kishi, Rigorous bounds on the susceptibilities of the Hubbard model, Phys. Rev. B 41 (1990), 4866-4868.

[13] I. G. Lang, Y. A. Firsov, Kinetic theory of semiconductors with low mobility, Sov. Phys. JETP, 16 (1963), 1301.

[14] E. H. Lieb, Two theorems on the Hubbard model, Phys. Rev. Lett. 62 (1989), 1201-1204.

[15] H. Löwen, Absence of phase transitions in Holstein systems, Phys. Rev. B 37 (1988), 8661-8667.

[16] T. Miyao, Nondegeneracy of ground states in nonrelativistic quantum field theory, J. Operator Theory, 64 (2010), 207-241.

[17] T. Miyao, Ground state properties of the SSH model, J. Stat. Phys. 149 (2012), 519-550.

[18] Y. Miura, On order of operators preserving selfdual cones in standard forms. Far East J. Math. Sci. (FJMS) 8 (2003), no. 1, 1–9.

[19] K. Osterwalder, R. Schrader, Axioms for Euclidean Green’s functions. Comm. Math. Phys. (1973), 83-112.

[20] M. Reed, B. Simon, Methods of Modern Mathematical Physics Vol. IV, Academic Press, New York, 1978.

[21] S. H. Shen, Strongly correlated electron systems: Spin-reflection positivity and some rigorous results, Int. J. Mod. Phys. B 12 (1998), 709.

[22] B. Simon, Functional integration and quantum physics, Academic Press, New York, San Francisco, London, 1979.

[23] B. Simon, “Trace ideals and their applications” Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.

[24] G-S. Tian, Lieb’s spin-reflection positivity methods and its applications to strongly correlated electron systems, J. Stat. Phys. 116 (2004), 629-680.