Fractal Scaling of Population Counts Over Time Spans
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Abstract

Attributes which are infrequently expressed in a population can require weeks or months of counting to reach statistical significance. But replacement in a stable population increases long-term counts to a degree determined by the probability distribution of lifetimes.

If the lifetimes are in a Pareto distribution with shape factor $1 - r$ between 0 and 1, then the expected counts for a stable population are proportional to time raised to the $r$ power. Thus $r$ is the fractal dimension of counts versus time for this population.

Furthermore, the counts from a series of consecutive measurement intervals can be combined using the $L^p$-norm where $p = 1/r$ to approximate the population count over the combined time span.

Data from digital advertising support these assertions and find that fractal scaling is useful for early estimates of reach, and that the largest reachable fraction of an audience over a long time span is about $1 - r$.

Keywords: population statistics; fractal dimension; digital advertising

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1. **Introduction**

In online (digital) advertising, “third-party” data vendors provide streams of anonymized unique user identifiers (UUID) along with their alleged attributes for use in deciding which users to buy advertisements for.

The primary assumption here is that one UUID (in a population) being associated with a feature is persistent and independent of other UUIDs being associated with the feature. This makes it a stationary Bernoulli process (coin-toss) of $N$ trials, with expected value $N \cdot P$, where $P$ is the probability that a UUID has a particular attribute.

It is natural to ask what is the size of a pool of users, those with a particular attribute or combination of attributes, and also the total population. But the counts of such pools depend on the time span over which the unique identifiers are counted.

Popular web browsers offer “incognito” modes whose UUIDs (in the form of cookies) are forgotten at the end of the session. Despite the efforts of third-party vendors to filter out ephemeral UUIDs, short-lived UUIDs comprise the majority of the UUIDs seen week-to-week.

Digital advertisers want to evaluate the effectiveness of targeting any of the hundreds or thousands of third-party attributes in driving sales. With typical success rates of only 0.1%, hundreds of advertisements must be bought per sale. In order to achieve statistically significant measurements, the UUID counts for attributes and combinations of attributes must span weeks.

Counting unique cookies over a 6 month span can be expensive in computing and storage costs. This investigation began as a study of the relationship between weekly counts and counts over multiple weeks.

2. **The $L^p$-norm**

Consider the population counts for two consecutive weeks and a two-week count for the same time period. Every individual counted in the weekly counts must also appear in the two-week count and every individual counted in the two-week count must appear in at least one of the weekly counts. So these three counts must obey the triangle inequality.

The triangle inequality suggests that the weekly counts might be treated as dimensions, and the two-week count as the result of a norm applied to their vector sum. If the population is very long-lived, then few individuals get replaced, and the population count will be nearly constant with time. If the individuals in a stable population are short-lived, then the population count will grow nearly linearly with the duration of the count. Experimentation with the graphs quickly converged to the distinct $p$ exponents in Section 4, which worked so well that it prompted this exploration of the mathematics.

The $L^p$-norm is:

$$\|C_1, \ldots, C_n\|_p = (|C_1|^p + \ldots + |C_n|^p)^{1/p}$$

All population counts $C_j$ are non-negative, so the absolute values are superfluous to this application. With $p = 1$ the weekly counts add linearly. As $p$ approaches $\infty$, the $L^\infty$-norm returns the maximum of its inputs. These limits satisfy the earlier reasoning.

The $L^p$-norm definition implies a scaling law. If all the $C_j$ have the same value $C$, then:

$$\|C_1, C_2\|_p, C_3, \ldots, C_n\|_p = \left(\left(|C_1|^p + |C_2|^p\right)^{1/p} + |C_3|^p + \ldots + |C_n|^p\right)^{1/p}$$

$$= (|C_1|^p + |C_2|^p + |C_3|^p + \ldots + |C_n|^p)^{1/p}$$

With the assumption that $p$ remains constant over time, the graphs in Section 4 show that we can estimate the population over a span of $n$ weeks from population counts of each of the constituent weeks using the $L^p$-norm.

The $L^p$-norm definition implies a scaling law. If all the $C_j$ have the same value $C$, then:
\[ \|C_1, \ldots, C_n\|_p = \left( \sum_{1}^{n} |C|^p \right)^{1/p} = (n \cdot |C|^p)^{1/p} = C \cdot n^{1/p} \]  

(1)

The norm for the analogous \(L^p\) space is:

\[ \|C\|_p \equiv \left( \int_{0}^{t} |C(t)|^p \, dt \right)^{1/p} \]

(2)

When \(C\) is constant, norm (2) obeys the same scaling law as the \(L^p\)-norm (1).

3. Fractal Dimension

On viewing scaling law (1), the authors realized that \(r = 1/p\) is a fractal dimension (as described by Mandelbrot[1]). Similarly to the length of a coastline growing as the measurement resolution is increased, the count of a population increases as the time span of counting increases.

This implied scaling law (1) is plotted along with the true and \(L^p\)-estimated counts in the graphs in Section 4. It is in rough agreement with the multiple-week counts and \(L^p\)-estimates, even though there is some variation in the weekly counts.
4. Digital Advertising Data

The figures show the UUIDs per week, the $L^p$-norm of $n$ weekly counts, the UUIDs counted over a span of $n$ weeks, and the scaling law with its coefficient being the geometric mean of the weekly counts.

Figure 1 shows a 7-week span starting in March 2016 of all UUIDs seen by a large third-party vendor. The variation in the number of UUIDs per week is tracked well by the $L^{1.65}$-norm; less so by the scaling law with its assumption of identical weekly counts.

![Accumulation of Unique Users in Third-Party Sample B](image1)

Figure 1

Figure 2 shows a 14-week span starting 2017-11-20 of all UUIDs seen by another third-party vendor. This vendor provides some attributes which depend on how many times a UUID clicks, which violates the assumption of a stationary Bernoulli process.

![Accumulation of Unique Users in Third-Party Sample M](image2)

Figure 2
Figure 3 shows a 14-week span starting 2017-11-20 of all UUIDs seen by Digilant advertisers’ pixels. Being unfiltered, this data-set has a fractal dimension larger than 0.92.

Figure 4 shows a 31-day span starting 2017-11-20 of the same pixels hits. The cumulative counts are in close agreement with $L^{1.085}$-norm counts.

5. **Asymptotics**

Assume a stable population of size $n$ and time $t \ll n$.

In order to uniquely count the population over time $t$, the storage required is $O(t^n \log n)$; and the running time is $O(t n \log n)$. If cumulative counts are to be computed every time period, then the storage is $O(t n)$ and running time is $O(t^2 n \log n)$.

If instead, counts are made every time unit (to be combined with the $L_p$-norm), then the running time is $O(t n \log n)$, the short-term storage is $O(n)$ and the long-term storage is $O(t)$.

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1 In digital advertising a pixel is a tiny image used to learn the UUIDs of visitors to a web-page containing that pixel.
6. Pareto Distribution

Is there a probability distribution for lifetimes which produces \( L^p \)-norm and fractal scaling of population counts? Going through McLaughlin’s “A compendium of common probability distributions” [2], it was found that the Pareto probability distribution, which is used for modeling income and longevity distributions, has the fractal scaling properties.

Let \( X \) be a Pareto random variable for UUID lifetime with positive scale factor \( A \leq X \) and positive shape factor \( B < 1 \).

\[
p_X(x) = P(X = x) = \frac{B A^B}{x^{B+1}} \quad P(X < x) = 1 - \left( \frac{A}{x} \right)^B
\]

With a stable population, each individual is replaced when its lifetime \( X \) expires. The count of replacements over time \( t \) is:

\[
R(t) = \frac{N t}{X}
\]

For \( t \geq 1 \) let \( A = 1/t \). Ignoring cohorts with average lifetimes shorter than the unit time interval, the expected replacement count is:

\[
E[R(t)] = \int_1^\infty \frac{N t \cdot p_X(x)}{x} \, dx = N t \int_1^\infty \frac{B A^B}{x^{B+2}} \, dx = N \frac{B}{B+1} t^{1-B}
\]

With \( r = 1 - B \), the expected count (3) comes into the same fractal scaling form as equation (1):

\[
E[R(t)] = N \frac{B}{B+1} t^{1-B} = N \frac{1-r}{2-r} t^r = E[R(1)] t^r
\]

The expected population count \( E[C(t)] = N t^r \) is proportional to the expected replacement count:

\[
E[C(t)] = \frac{2-r}{1-r} E[R(t)] \quad E[C(t)] = E[C(1)] t^r \tag{4}
\]

Given the fractal scaling of expected count (4), what can be inferred about splitting \( E[C(t)] \) into \( t \) equal size counts \( C = C(1) \)? From equation (4) the unknown function \( f(C, \ldots, C) = C \cdot t^r \). Raising both sides to the \( 1/r \) power:

\[
f(C, \ldots, C)^{1/r} = t \cdot C^{1/r} = C^{1/r} + \cdots + C^{1/r}
\]

Raising both sides to the \( r \) power:

\[
f(C, \ldots, C) = \left( C^{1/r} + \cdots + C^{1/r} \right)^r \tag{5}
\]

The right side of equation (5) is the formula for the \( L^p \)-norm \( \| C_1, \ldots, C_t \|_{1/r} \) for non-negative \( C_j \). Thus the Pareto lifetime distribution implies a scaling law, which in turn implies the \( L^p \)-norm with \( p = 1/r \) for successive non-overlapping counts.

7. Changing Population Size

So far we have assumed that population sizes did not experience much increase or decrease during the measurement interval. While the triangle inequality holds when the constituent counts are very different in magnitude, a population cannot drop to zero in one time period without invalidating the longer lifetimes in the probability distribution of the previous period.

However, the \( L^p \)-norm estimates of sample B (Figure 1) and pixel (Figures 3 and 4) populations track the cumulative counts well through variations in the weekly and daily counts (over 3.5:1 in the daily pixel case).
8. Measuring the Fractal Dimension

Given positive daily population counts $C_m, \ldots, C_n$ and corresponding (monotonically increasing) cumulative counts $Q_m, \ldots, Q_n$, for $0 < m \leq j \leq n$ we would like to find the optimal $p$ to minimize the difference between $\|C_1, \ldots, C_j\|_p$ and $Q_j$. Equivalently, we wish to minimize the difference between $Q_j^p - Q_{j-1}^p$ and $C_j^p$ (where $Q_0 = 0$).

Suppose we have an initial value for $p$ which does not extinguish the difference between $Q_j^p - Q_{j-1}^p$ and $C_j^p$. Let $\delta$ be the change in exponent $p$ which makes them equal:

$$Q_j^{p+\delta} - Q_{j-1}^{p+\delta} = C_j^{p+\delta}$$

$$Q_j^p Q_{j-1}^p = C_j^p$$

If $\delta$ is near zero and $Q_j^p$ and $Q_{j-1}^p$ are close in value, then they can be approximated by their average $Q_{j'} = (Q_j + Q_{j-1})/2$.

$$Q_{j'}^p (Q_j^p - Q_{j-1}^p) \approx C_j^p$$

$$Q_j^p - Q_{j-1}^p \approx C_j^p$$

$$\log \left( \frac{Q_j^p - Q_{j-1}^p}{C_j} \right) \approx \delta_j \log \left( \frac{C_j}{Q_{j'}} \right)$$

$$\delta_j \approx \log \left( \frac{[Q_j^p - Q_{j-1}^p]/C_j^p}{log (2 C_j / [Q_j + Q_{j-1}])} \right)$$

By averaging $\delta$ over $j$, $p$ can be improved for the dataset as a whole:

$$\delta = \frac{1}{n - m + 1} \sum_{j=m}^{n} \log \left( \frac{[Q_j^p - Q_{j-1}^p]/C_j^p}{log (2 C_j / [Q_j + Q_{j-1}])} \right)$$

$$p \leftarrow \delta + p$$

Overshoot from $p \leftarrow \delta + p$ leads to slow oscillatory convergence. $p \leftarrow 0.632 \delta + p$ converges about one decimal digit per iteration. Once $p$ has settled, its standard-deviation can be calculated:

$$\sigma = \sqrt{\frac{1}{n - m + 1} \sum_{j=m}^{n} \left[ \log \left( \frac{[Q_j^p - Q_{j-1}^p]/C_j^p}{log (2 C_j / [Q_j + Q_{j-1}])} \right) \right]^2}$$

In practice, the contribution from step $j$ should only be included when $0 < 1.6 C_j < (Q_j + Q_{j-1})/2$; the count $n - m + 1$ is reduced by the number of excluded steps.

9. Reach and Saturation

Reach is the total number of individuals who received or viewed an advertisement during the campaign; reach goals are often part of advertising contracts. Fractal scaling provides good early estimates of large reaches, as can be seen from the graphs in Section 4.

The $p$ for daily reach from more than fifty Digilant managed campaigns in the month of June 2018 were between 1.002 and 1.24 with a mean of 1.074; the weekly $p$ for the same time period were between 1.007 and 1.20 with a mean of 1.077.2 That these averages (1.074 and 1.077) are so close to the $p$ for pixel hits (1.085 and 1.08), is evidence that the fractal dimension is an intrinsic property of the user population.

Saturation is the ratio of the reach to the number of UUIDs with the targeted attributes. For UUID populations with long lifetimes, this ratio can approach 1. The ratio is small for populations with short lifetimes because the UUIDs tend not to be online long enough to see many advertisements.

Looking at 6 months of Digilant advertising campaigns which targeted attributes from samples B and M, the largest saturation achieved (for sample B and for sample M) was roughly $1 - r$.

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2. The only campaign which was purely retargeting (repeatedly showing advertisements to the same users) during June had a $p$ of 1.79 and was not included in the averages.
10. **Conclusion**

For populations having a Pareto distribution of lifetimes with shape factor $0 < B < 1$, counts made over successive time intervals can be combined using the $L^p$-norm to closely approximate the count which would result from counting over the combined time span. The norm’s exponent $p = 1/r$ where $r = 1 - B$ is the fractal dimension of the population counts over time. Fractal scaling allows counts collected over very different time spans to be effectively compared.

Digital advertising UUIDs are an example of such a population. Collecting daily or weekly counts, then aggregating using the $L^p$-norm, allows longer term studies with better confidence to be conducted without straining resources.

Fractal scaling laws imply aggregation using the $L^p$-norm. Regions where the norm doesn’t scale with the expected exponent might be used to locate anomalies in large temporal or spacial data-sets.

11. **References**

[1] Benoît B. Mandelbrot. *Fractals: form, chance, and dimension*. W. H. Freeman and Company, New York, NY, USA, revised edition, 1977. Translated from the French.

[2] Michael P. McLaughlin. A compendium of common probability distributions. URL https://www.causascientia.org/math_stat/Dists/Compendium.pdf, 2016.