Domain walls of D=5 supergravity and fixed points of $N=1$ Super Yang Mills

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Abstract

Employing the AdS/CFT correspondence, we give an explicit supergravity picture for the renormalization group flow of couplings 4-d super Yang Mills with four supercharges. The solution represents a domain wall of 5-d, $N=2$ supergravity, that interpolates between two (different) $AdS_5$ vacua and is obtained by gauging a $U(1)$ subgroup of the $SU(2)$ R-symmetry. On the supergravity side the domain wall couples only to scalar fields from vector multiplets, but not to scalars from hyper multiplets. We discuss the $c$-theorem, the $\beta$-functions and consider two examples: one is the sugra solution related to $Z_k$ orbifolds (corresponding to $N=2$ SYM) and the other is an orientifold construction for an elliptically fibered CY with $F_1$ basis (corresponding to $N=1$ SYM).

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1 Introduction

Many field theories flow under the renormalization group (RG) towards fixed points, where they become finite and universal (scheme independent). However, the fixed point values of the couplings are not necessarily small and generically perturbation theory breaks down. On the other hand, employing the AdS/CFT correspondence \cite{1,2,3} we can formulate these interacting field theories in terms of AdS gravity. Super Yang-Mills in 4 dimensions e.g., becomes conformal near fixed points and is expected to be dual to AdS$_5$ gravity – at least in appropriate limits. The flow between two different fixed points corresponds in supergravity to domain walls (DW), which are kink solutions connecting two vacua which appear as extrema of the potential (typically AdS spaces). For a review on domain walls see \cite{4} and different aspects of the AdS/CFT correspondence (or more general DW/QFT correspondence) are discussed in \cite{5,6}.

Domain walls are supported by at least one scalar field and appear naturally in supergravity where a subgroup of the R-symmetry has been gauged. As a consequence of the gauging the sugra Lagrangian contains a potential and the flow of the field theory is encoded in non-trivial supergravity scalars that run from one extremum of the potential to another. From these fixed scalars it is straightforward to obtain the corresponding fixed couplings in the field theory. If the potential allows only for one extremum the field theory has either an IR or UV fixed point, but not both.

For a concrete supergravity potential, it is straightforward to expand a solution around a given fixed point. However, in order to get a global picture, we need an explicit solution that interpolates between two (different) AdS$_5$ vacua on both sides of the wall. Having this solution it should be straightforward to calculate field theory quantities like $\beta$-functions or anomalous dimensions.

The qualitative picture depends very much on the amount of unbroken supersymmetry and only in field theory with at most four supercharges we can expect a flow between two non-trivial fixed points. One way to break partly supersymmetry has been discussed in \cite{7,8,9} and represents a non-abelian gauging of D=5, N=8 supergravity. Another possibility is to gauge a generic N=2 supergravity which has in the ungauged case only eight supercharges and the domain wall will break at least one half of them. In these models the scalar fields enter two different multiplets: the vector multiplets or hypermultiplets, and for many purposes it is reasonable to consider one sector only. This truncation is always possible as long as the scalars are not charged. An interesting domain wall solution for this non-abelian gauging has been discussed in \cite{10} and it contains an arbitrary number of vector multiplets which couple to the universal hypermultiplet; this formulation has been extended to include non-universal hypermultiplets \cite{11}.

In this paper we will explore the situation where only a $U(1)$ subgroup of the R-symmetry has been gauged and the hypermultiplets remain uncharged and we can consistently decouple this sector. As we will see, these domain walls provide a su-
pergravity solution for a non-trivial RG flow connecting two fixed points. Since this solution holds for any prepotential, we obtain an exact expression in the two-derivative approximation for the $\beta$-functions.

We have organized our paper as follows. We start with a discussion of the domain wall solution and show that it represents a BPS configuration. In section 3 we will describe the supersymmetric flow between extrema of the superpotential and we formulate the $\mathcal{C}$-theorem, where the $\mathcal{C}$-function will determine the $\beta$-functions. The domain wall solution is quite general and is not related to a specific example or field theory. To be concrete we discuss in section 4 two examples, one related to $\mathcal{N}=2$ and the other to $\mathcal{N}=1$ super Yang Mills, where the scalars run from the boundary of the moduli space towards an $SU(2)$ enhancement point. The central charges and $\beta$-functions are calculated. In the conclusion we will summarize our results.

2 Abelian gauged supergravity in 5 dimensions

2.1 General remarks

Let us start with some general remarks about gauged $\mathcal{N}=2$ supergravity in 5 dimensions, see [12, 13] for more details. As mentioned in the introduction we are interested in an abelian gauging and can truncate the model to the vector multiplets sector only. The bosonic Lagrangian is then given by

$$S_5 = \int \left[ \frac{1}{2} R + g^2 P - \frac{1}{4} G_{IJ} F_{\mu \nu}^I F^{\mu \nu J} - \frac{1}{2} g_{AB} \partial_\mu \Phi^A \partial^\mu \Phi^B \right] + \frac{1}{48} \int C_{IJK} F^I \wedge F^J \wedge A^K \quad (1)$$

where $(\mu, \nu) = 0, 1, \ldots, 4$ are space-time indices; $A, B = 1, \ldots, n$ count the number of vector multiplets (each containing one real scalar $\Phi^A$ and one gauge field), in addition there is one gauge field in the gravity multiplet so that $I, J = 0, \ldots, n$. Both the gauge fields and the scalar part have non-trivial couplings $G_{IJ}$ and $g_{AB}$ which depends on the scalar fields. The potential $P$ arises due to the gauging of an $U(1)$ subgroup of the $R$-symmetry and the corresponding gauge field is a linear combination of all other abelian gauge fields

$$A_\mu^I = V_I A^I_\mu \quad (2)$$

where $V_I$ are constants (Fayet-Illiopoulos terms). Notice, by this abelian gauging fermions become charged, but all scalars from the vector as well as hypermultiplets remain neutral. In contrast, under a non-abelian gauging the hypermultiplets become charged and one can not ignore them.

We are especially interested in a solution, which describes a 3-brane or domain wall living in a 5-d asymptotic anti-de Sitter space. We assume that it is flat, static and isotropic which yields as ansatz for the metric

$$ds^2 = e^{2U} g^2 r^2 \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + e^{-4U} \frac{dr^2}{g^2 r^2} \quad (3)$$
where $U$ is a function of the radius $r$ and becomes constant in the asymptotic AdS vacuum.

The gauging can be viewed in different ways. From the M- or F-theory perspective, one compactifies on a Calabi-Yau, but with *non-trivial* (=non-zero mode) internal components of the 4-form field strength. As discussed in [14] this is required for a consistent reduction of the Horava-Witten model [15] and yields naturally 3-branes in 5 dimensions. On the other hand from the 10-d perspective [16, 17], the space transverse to the 3-branes is given by a cone, $ds_{\perp} \sim dr^2 + r^2 ds_X^2$, over the horizon manifold $ds_X^2$ which is generically not spherical symmetric. This is the case at least as long as one is in a region which we call AdS vacuum, where the scalars become constant and which correspond to fixed points. In general however, one cannot expect that the 10-d metric factorizes, e.g., non-extreme AdS black holes can be interpreted as rotating 3-branes [18, 19, 20]. To get a better understanding of our solution it may help to adopt the M- or even F-theory approach.

Before we come to the generic N=2 case, we will discuss the solution that can naturally be embedded into $N=4,8$ supergravity.

### 2.2 A simple example

In the past years we have learned a lot about 5-d black holes and string-type solutions, which are the natural objects that are charged. Much less is known about domain wall solutions. The main difference is that black holes are well defined in an asymptotic flat spacetime whereas domain walls are asymptotically AdS, if we impose that the asymptotic space is maximal supersymmetric. But it is straightforward to truncate, or if one likes to promote, any (static) black hole into a domain wall. What one has to do is:

(i) embed it into an AdS space,
(ii) replace the $S_3$-horizon by $S_{3,k}$ (3-sphere with constant curvature $k$) and
(iii) take the limit $k \to 0$.

The first step has been discussed in [21], the second one in [22] and the last one is trivial, see also [23, 24, 25, 26], where other dimensions have also been discussed.

Let us explain it for the simple example of the 3-charge solution. This is an interesting example, because due to the solutions generating technique [25, 26], any black hole that can be embedded into ungauged $N=4,8$ supergravity can be obtained from this solution.

\footnote{However the $U$-duality group is broken by the gauging and we will not discuss here to which extend the solution generating technique is applicable.}
After step (i) and (ii) the extreme solution reads [22]

\[
ds^2 = -\frac{f}{(H_1 H_2 H_3)^{2/3}} \, dt^2 + (H_1 H_2 H_3)^{1/3} \left[ \frac{dr^2}{f} + r^2 d\Omega_3 \right], \quad A^I_0 = \sqrt{k/H_1},
\]

\[
\Phi^1 = \left( \frac{H_1 H_2 H_3}{H_1} \right)^{1/3}, \quad \Phi^2 = \left( \frac{H_1 H_2 H_3}{H_2} \right)^{1/3}, \quad f = k + g^2 r^2 H_1 H_2 H_3
\]

where for \( k = 1 \) the horizon is a sphere and for \( k = -1 \) it becomes a hyperboloid. In the limit \( k \to 0 \) this solution becomes a 3-brane compatible with our ansatz [3]

\[
ds^2 = (H_1 H_2 H_3)^{1/3} g^2 r^2 \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + \frac{dr^2}{g^2 r^2 (H_1 H_2 H_3)^{2/3}},
\]

\[
\Phi^1 = \left( \frac{H_1 H_2 H_3}{H_2} \right)^{1/3}, \quad \Phi^2 = \left( \frac{H_1 H_2 H_3}{H_2} \right)^{1/3}, \quad F^I_{\mu \nu} = 0
\]

where the harmonic functions \( H_I \) are given by

\[
H_{1,2,3} = 1 + \frac{q_{1,2,3}}{r^2}.
\]

For this case the supergravity potential is (see below)

\[
P = 2 \left( \frac{1}{\Phi^1} + \frac{1}{\Phi^2} + \Phi^1 \Phi^2 \right),
\]

with the minimum at \( \Phi^1 = \Phi^2 = 1 \). Let us stress, that in order to perform the truncation to a flat \( (k = 0) \) 3-brane, it is crucial to consider gauged supergravity with an asymptotic AdS vacuum – the extreme ungauged solution (with no potential) cannot be truncated to a flat 3-brane! Let us also mention, that in the non-extreme case the gauge fields survive the domain wall limit \( k \to 0 \), see [18, 19, 20].

This solution contains three classes corresponding to: one, two or three non-trivial harmonic functions. Asymptotically all solutions become \( AdS_5 \), but they differ near the core. To discuss the different cases in more detail we may equalize all non-vanishing harmonics, which means that we replace \( H_1 H_2 H_3 = H^n \) \( (n = 1, 2, 3) \). Choosing a coordinate system where the metric becomes

\[
ds^2 = e^{2A} \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + d\rho^2
\]

we find near \( r \simeq 0 \)

\[
e^{2A} \sim \begin{cases} 
q g^4 (\rho - a)^2, & n = 1 \\
q g^{5/2} \sqrt{|\rho - a|}, & n = 2 \\
q g^2, & n = 3.
\end{cases}
\]

where \( a \) is an arbitrary parameter. Only the case \( n = 3 \) behaves smooth near \( \rho = a \) (or at \( r = 0 \), the \( n = 2 \) case has a curvature singularity and for \( n = 1 \) the metric exhibits a conical singularity (the case \( n = 2 \) has been addressed also in [27, 28]) In
addition, after equalizing the harmonic functions we can have at most one scalar field, which either vanishes or diverges near the origin and plays the role of the dilaton. This singularity indicates that for these two cases the corresponding Yang-Mills coupling runs either to a strongly or weakly coupled regime (electric or magnetic picture cp. [29]).

Perhaps more interesting than these singular cases is the case \( n = 3 \) where the spacetime is regular at \( r = 0 \) (\( \rho = a \)). By equalizing all three harmonics all scalars are trivial and the metric reads

\[
\begin{align*}
\text{ds}^2 &= H g^2 r^2 \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + \frac{dr^2}{g^2 r^2 H^2}, \\
&= g^2 \rho^2 \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + \frac{dr^2}{g^2 \rho^2},
\end{align*}
\]

which is nothing other than AdS_5 \( (r^2 H \equiv r^2 + q = \rho^2) \). So, in order to obtain a non-trivial solution we need different \( H' \)s, or equivalently, we have to turn on scalar fields. But still, the metric will remain regular at \( r = 0 \) where \( \rho = \sqrt{q} \) and spacetime does not end there. This is very similar to the extreme Reissner-Norström solution, but with the difference that we do not have a horizon in the case at hand. In fact near \( r = 0 \) the metric (5) becomes

\[
\text{ds}^2 = (q_1 q_2 q_3)^{1/3} g^2 \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + \frac{r^2 dr^2}{g^2 (q_1 q_2 q_3)^{2/3}}.
\]

Since this metric, as well as the scalars, are invariant under the \( Z_2 \) symmetry: \( r \to -r \) we can continue the solution beyond the point \( r = 0 \). If we do not break this reflection symmetry, we effectively identify the two asymptotic regions, which is equivalent to a compact transverse space with a radius given by the cosmological constant. Note, the infinite radial part of an AdS space can always be mapped on a finite interval. This case corresponds to the situation described in [30, 31], where the radial coordinate is an angle and the identification \( r \simeq -r \) yields an \( S/Z_2 \) orbifold as transversal space. For earlier work on reflection symmetric domain walls see [32]. But a generic domain wall breaks the discrete symmetry and thus we may treat both sides differently. One possibility would be to connect the solution to flat space, which means that the potential has to vanish. This supersymmetry breaking setup has been explored in [33, 34].

In summary, the 3-brane as given in (5) represents a domain wall interpolating between two AdS spaces or an AdS space and flat spacetime. In order to match the scalars and the metric, we have to equalize the \( q' \)s on both side and if one wants to describe a flat space on one side, the constant parts of the harmonics have to vanish on that side (see also below). When discussing genuine \( N=2 \) solutions in the next section, we will see that we can glue together topological distinct vacua at this point, where each of them may have a different cosmological constant. In the dual field theory this setup will correspond to the incorporation of additional perturbations and we expect a flow between different fixed points.
2.3 Definitions and conventions of 5-d supergravity

A generic solution has of course more scalars and is in the ungauged case not duality equivalent to a 3-charge solution, which we discussed in the last section. The \( n \) physical scalars \( \Phi^A \) of 5-d supergravity parameterize a hypersurface in an \( (n + 1) \)-dimensional space parameterized by the coordinates \( X^I \). This space will be called scalar manifold or occasionally moduli space, but we have to keep in mind that the gauging breaks the \( U \)-duality group. Let us summarize some basic features; we will mainly follow here \[35, 34, 12\], but in slightly modified notation. To be concrete the scalar manifold is defined by the constraint

\[
\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K = 1 ,
\]

where in the case of a Calabi-Yau compactification the constants \( C_{IJK} \) are the topological intersection numbers. For many physical interesting cases this space is given by a coset like \[12\]

\[
\mathcal{M} = \frac{SO(n - 1, 1)}{SO(n - 1)} \times SO(1, 1) .
\]

In general there is no restriction in the number of scalar fields, the case discussed in the last section corresponds to \( n = 2 \) and \( C_{123} = 1 \).

The coupling matrices entering the Lagrangian \([1]\) are defined by

\[
G_{IJ} = -\frac{1}{2} (\partial_I \partial_J \log \mathcal{V})_{\mathcal{V}=1} , \quad g_{AB} = \left( \partial_A X^I \partial_B X^J G_{IJ} \right)_{\mathcal{V}=1} \]

where \( \partial_A \equiv \frac{\partial}{\partial \Phi^A} \) and \( \partial_I \equiv \frac{\partial}{\partial X^I} \). It follows that

\[
G_{IJ} = -\frac{1}{2} \left[ C_{IJ} - \frac{1}{4} C_I C_J \right] ,
\]

\[
\partial_K G_{IJ} = -\frac{1}{2} \left[ C_{IJK} - \frac{1}{2} (C_{IJ} C_K + \text{cycl.}) + \frac{1}{4} C_I C_J C_K \right] ,
\]

with \( C_I = C_{IJK} X^J X^K \) and \( C_{IJ} = C_{IJK} X^K \). Moreover, the normal vector on the scalar manifold is given by\(^3\)

\[
X_I \equiv \frac{2}{3} G_{IJ} X^J = \frac{1}{6} C_{IJK} X^J X^K = \frac{1}{3} \partial_I \mathcal{V}
\]

(normalized as \( X_I X^I = 1 \)). It follows that

\[
\partial_A X_I = -\frac{2}{3} G_{IJ} \partial_A X^J ,
\]

\(^3\)In proper coordinates, the defining equation of a surface \( F(X^I) = 1 \) depends only on the transversal coordinates, at least locally. Therefore, \( \partial_I F \) is a normal vector and \( \partial_A F \equiv (\partial_A X^I) \partial_I F(X) = 0 \).
and since \( \partial_A X^I \) gives the tangent vectors
\[
X_I \partial_A X^I = -X^I \partial_A X_I = 0 .
\]
Finally, the potential \( P \) in the Lagrangian (1) reads
\[
P = 6 V_I V_J \left( X^I X^J - \frac{3}{4} g^{AB} \partial_A X^I \partial_B X^J \right)
\]
\[
= 6 \left( W^2 - \frac{3}{4} g^{AB} \partial_A W \partial_B W \right) ,
\]
where \( V_I \) are constants (FI terms) and the superpotential \( W \) is
\[
W = V_I X^I .
\]
Notice, \( W \) is subject to the constraint (12) which makes it non-linear in the physical scalars \( \Phi^A \).

### 2.4 BPS domain walls in gauged \( N=2 \) supergravity

We are interested in a 3-brane, that couples to \( n \) scalars of the vector multiplets and for which the gauge fields are trivial. This domain wall solution allows for unbroken supersymmetries if the gaugino and gravitino variations
\[
\delta \lambda_A = \left( - \frac{i}{2} g_{AB} \Gamma^\mu \partial_\mu \Phi^B + i \frac{3}{2} g \partial_A W \right) \epsilon ,
\]
\[
\delta \psi_\mu = \left( \partial_\mu + \frac{1}{4} \omega_{\mu ab} \Gamma_{ab} + \frac{1}{2} g \Gamma_\mu W \right) \epsilon ,
\]
have non-trivial zeros. Solutions of these equations are BPS configurations of \( N=2 \) gauged supergravity and basically there are two dual cases related to the possibilities to express the scalar fields in terms of \( X_I \) or its dual \( \tilde{X}_I \). Both coordinates parameterize dual cycles of the internal manifold, \( X_I \) is related to 4-cycles whereas \( \tilde{X}_I \) to 2-cycles. Taking into account non-trivial gauge fields, the solution expressed by \( X_I \) corresponds to electric whereas the other to magnetic solutions. In this paper we will explore only the electric solution. Most likely the magnetic solution is not supersymmetric, at least as long as one keeps the ansatz for \( U(1) \) gauge fields \( A_\mu = V_I A_\mu^I \), where \( V_I \) is a constant “electric” vector which by supersymmetry will be related to the electric moduli (see below). For a discussion of electric-magnetic duality in gauged supergravity see [33].

Our (electric) solution reads
\[
ds^2 = e^{2U} g^{2} r^2 \left[ -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right] + e^{-4U} \frac{dy_4^2}{g^{2r^2}} ,
\]
\[
X_I = \frac{1}{3} e^{-2U} H_I , \quad H_I = h_I + \frac{a_I}{r^2}
\]
\[\text{In our case of abelian gauging, the hypprino variation are trivially solved for constant hyper scalars.}\]
where all gauge fields vanish. As it has been shown in \[22\] this configuration solves the equations of motion coming from the Lagrangian \[1\] for any prepotential. What remains to be shown is that this domain wall allows for unbroken supersymmetry. Let us start with the gaugino variation. Using the definition \[14\] and \[17\] we find

\[
g_{AB} \partial_\mu \Phi^B = G_{IJ} \partial_A X^I \partial_\mu X^J = -\frac{3}{2} \partial_A X^I \partial_\mu X_I
\]

and therefore the gaugino variation becomes

\[
\delta \lambda_A = \frac{3i}{4} \partial_A X^I (\Gamma^\mu \partial_\mu X_I + 2g V_I) \epsilon .
\]

It follows from \[18\] and our solution \[22\] that

\[
(\partial_A X^I) X_I = \frac{1}{3} e^{-2U} (\partial_A X^I) H_I = 0
\]

and thus

\[
\partial_A X^I \Gamma^\mu \partial_\mu X_I = e^{-2U} \frac{2}{3r} \partial_A X^I h_I \Gamma^r .
\]

After transforming \(\Gamma^r\) into the tangent space, the gaugino variation becomes

\[
\delta \lambda_i \sim \partial_A X^I (\Gamma^r h_I + 3V_I) \epsilon .
\]

Taking the projector

\[
(1 + \Gamma^r) \epsilon = 0
\]

the constants in the harmonic functions are fixed by the \(V_I\) vector

\[
h_I = 3V_I .
\]

Notice, treating \(\Gamma^r\) as \(\Gamma^5\), we project out one chirality from the 4-d perspective. The chirality discussed in this paper yields a Killing spinor (see eq. \(37\)) which is located near \(r = 0\); whereas the Killing spinor with respect to the other chirality would be located at \(r = \infty\); the location is given by an exponentially increase or fall-off in adapted coordinates.

Taking the normalization that \(e^{2U} \rightarrow 1\) for \(r \rightarrow +\infty\), we find that

\[
X_I |_{+\infty} = V_I
\]

i.e., the moduli fix the constant vector \(V_I\).

Next we turn to the gravitino variation and find for the non-vanishing spin connections

\[
\omega^{0r} = g^2 r e^{2U} (r e^U)' dt , \quad \omega^{mr} = g^2 r e^{2U} (r e^U)' dx^m .
\]

As a consequence of the scalar constraint \(X_I X^I = 1, e^{2U}\) becomes

\[
e^{2U} = \frac{1}{3} X^I H_I
\]
and therefore (recall $H_I \partial X^I = 0$)

$$(e^{2U})' = \frac{2}{r} \left( \frac{1}{3} X^I h_I - e^{2U} \right), \quad (33)$$

or

$$(r e^U)' = \frac{1}{3} (X^I h_I) e^{-U} = W e^{-U}, \quad (34)$$

with $W$ defined in (20). Using this relation we find that the worldvolume components of the gravitino variation

$$\Gamma_\alpha \left[ \frac{1}{2} g^2 r e^{2U} (r e^U)' \Gamma_r + \frac{1}{2} g^2 r e^U W \right] \epsilon = 0 \quad (35)$$

vanish ($\alpha = 0 \ldots 3$). In order to determine the Killing spinor $\epsilon$ we have to solve $\delta \psi_r = 0$, which becomes

$$\delta \psi_r = \left[ \partial_r + \frac{1}{2r} e^{-2U} W \Gamma_r \right] \epsilon = \left[ \partial_r + \frac{1}{2} \partial_r \left( U + \log r \right) \Gamma_r \right] \epsilon = 0 . \quad (36)$$

Using the projection (28) it is straightforward to solve this differential equation and the Killing spinor reads

$$\epsilon = e^{-\frac{1}{2}(U + \log r)} \left( 1 - \Gamma_r \right) \epsilon_0 \quad (37)$$

where $\epsilon_0$ is an arbitrary constant spinor.

This completes the discussion of supersymmetry. We have shown that the solution (22) represents a BPS domain wall of gauged $N=2$ supergravity.

One may of course ask whether this is just a special solution or to which extent it is general. Let us add some comments. First, that the projector (28) depends only on $\Gamma^r$ is dictated by the geometry (isotropic 3-brane) and employing eq. (18) the gaugino variation becomes

$$\partial_A X^I \left( -r e^{2U} \partial_r X_I + 2V_I \right) \epsilon \equiv \partial_A X^I \left( -r \partial_r (e^{2U} X_I) + 2V_I \right) \epsilon = 0 . \quad (38)$$

with the solution given by

$$-r \partial_r (e^{2U} X_I) + 2V_I = A X_I \quad (39)$$

for some function $A$. To solve the equation explicitly we contract it with $X^I$ and get: $r \partial_r e^{2U} - 2W = -A$. In order to fix $A$ we use the gravitino variation, which yields

$$W = e^U \partial_r (r e^U) \quad (40)$$

and hence $A = 2 e^{2U}$. Inserting this back in (39) we get the unique solution

$$e^{2U} X_I = \frac{1}{3} \left( 3V_I + q_I r^2 \right) = \frac{1}{3} H_I , \quad (41)$$

(the $\frac{1}{3}$ is a convenient normalization). Thus, eq. (22) is not only a special ansatz, it represents the generic isotropic 3-brane solution of 5-dimensional gauged supergravity with trivial hypermultiplets and AdS boundary condition. These equations are also known as stabilization equations and have been first discussed for the black hole entropy (36) and later for general stationary BPS solutions N=2 sugra in (37, 38).
The AdS/CFT correspondence tells us that the renormalization group flow of Yang-Mills couplings translate into a running (radial dependent) scalar fields. In this setup fixed points correspond to regions where the scalars become constant and extremize the potential; the metric is then anti de Sitter. In this section we will discuss some general features of this flow and afterwards we will present examples.

Let us start by rewriting the metric in a different way. Defining

\[ r e^U = \mu \quad \quad \quad (42) \]

and using (34) we get \( \mu d\mu = W r dr \) and thus

\[ ds^2 = g^2 \mu^2 \left( -dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right) + \frac{d\mu^2}{(gW)^2 \mu^2}. \quad \quad \quad (43) \]

Therefore, whenever we approach an extrema of the superpotential where \( \partial_A W = 0 \) and thus \( W \) becomes constant, we obtain an AdS\(_5\) space with a radius given by \( l = 1/(gW_0) \) and fixed scalars. In fact, from the gaugino variation it follows, that if the scalars are constant the supergravity potential is extremal

\[ \partial_A W = 0 \quad \quad \quad (44) \]

which implies that also the supergravity potential is extremal \( (\partial_A P = 0) \). In the dual Yang-Mills theory we have reached a conformal fixed point. The scalars are generically ratios of harmonic functions and therefore only \( r = \infty \) or \( r = 0 \) could correspond to fixed points. But as we will see below, the point \( r = 0 \) is not a fixed point but a phase transition point.

But let us first expand the potential around a fixed point. The first derivative vanishes and the second derivative gives

\[ \partial_A \partial_B W = \frac{1}{3} h_{IJ} \partial_A \partial_B X^I. \quad \quad \quad (45) \]

From the definition of the scalar metric we get

\[ g_{AB} = \partial_A X^I \partial_B X^J G_{IJ} = -\frac{3}{2} \partial_A X_I \partial_B X^I = -\frac{3}{2} \partial_A \left( X_I \partial_B X^I \right) + \frac{3}{2} X_I \partial_A \partial_B X^I \quad \quad \quad (46) \]

where the first term on the rhs vanishes identically, see eq. (25). Inverting this equation yields (recall \( X_I X^I = 1 \))

\[ \partial_A \partial_B X^I = \frac{2}{3} g_{AB} X^I + T_{ABC} \partial_D X^I \quad \quad \quad (47) \]

where \( T_{ABC} \) is defined as the projection of \( C_{IJK} \) on the scalar manifold \( 35 \)

\[ T_{ABC} = \partial_A X^I \partial_B X^J \partial_C X^K C_{IJK}. \quad \quad \quad (48) \]
Contracting (47) with $V_I$ yields

$$\partial_A \partial_B W = \frac{2}{3} g_{AB} W + T_{ABC}^C \partial_C W$$

and thus we get at the fixed point ($\partial C W = 0$) the well-known relation

$$\partial_A \partial_B W \bigg|_0 = \frac{2}{3} g_{AB} W_0 .$$

Hence, supersymmetric fixed points (extrema) remain stable as long as the scalar metric $g_{AB}$ is positive definite. Notice, it is $W$ that drives the supersymmetric flow and not the sugra potential $P$. The expansion can be continued and we find for the third derivative near the extremum

$$\partial_A \partial_B \partial_C W \bigg|_0 = \frac{8}{3} W_0 T_{ABC} .$$

Putting the terms together, the superpotential becomes

$$W = W_0 \left( 1 + \frac{1}{3} g_{AB} \phi^A \phi^B + \frac{4}{9} T_{ABC} \phi^A \phi^B \phi^C + \ldots \right) ,$$

where $\phi^A = \Phi^A - \Phi^A_0$ are small fluctuations around the fixed point value. It is straightforward to continue this expansion if one uses the fact that for many physical interesting cases $T_{ABC}$ is covariantly constant, which implies that the scalar manifold is symmetric and homogeneous [35].

The mass terms of the scalars are extracted from the expansion of the potential

$$P = 6 W_0^2 (1 + \frac{1}{3} g_{AB} \phi^A \phi^B - \frac{32}{3} T_{ABC} \phi^A \phi^B \phi^C + \ldots)$$

where $6 g^2 W_0^2$ is the (negative) cosmological constant and the mass of $\Phi^A$ is

$$m_A^2 = 4 W_0^2 \lambda_A ,$$

with $\lambda_A$ as eigenvalue of $g_{AB}$ calculated at the fixed point. Following the standard procedure [3] these masses translate into conformal dimensions of the corresponding operator in the field theory $\Delta_A = 2 \left( 1 + \sqrt{1 + m_A^2 / 4} \right) = 2 \left( 1 + \sqrt{1 + W_0^2 \lambda_A} \right)$. In addition to the scalar masses, the gauging yields also mass terms for the fermions [12]

$$-i W \bar{\psi}_\mu \Gamma^{\mu
u} \psi_\nu + i \frac{3}{2} (g_{AB} W + 4 \sqrt{2} T_{ABC} \partial_C W) \bar{\lambda}^A \lambda^B .$$

At the fixed point the gravitino mass term is therefore given by the cosmological constant and, as required by supersymmetry, the masses of the gauginos coincide with the scalar masses. Notice, the scalar masses come from the sugra potential $P$ whereas the fermionic masses from the superpotential $W$. Let us also mention that the supergravity potential can vanish identically, $P \equiv 0$, but $W \neq 0$ [33] (assuming that $T_{ABC}$
is covariantly constant). Obviously, in this case the scalars remain massless, but the gauginos feel a potential and supersymmetry is broken, see also [39]. Furthermore, let us also mention that all masses are suppressed by the cosmological constant, which has to be small for a reliable supergravity picture.

Next, following the discussion of [7, 9] we can formulate a $c$-theorem for the supersymmetric flow. By investigating the Einstein equations (see eq. (24) in [22]) we find

$$-R_0^0 + R^r_r = -3g^2 r^2 e^{4U} [U'' + 2U'r + U'] = g^2 [ ||h||^2 - \frac{2}{3} (X \cdot h)^2 ] = -3g^2 W \mu \frac{d}{d\mu} W = g_{AB} \partial A \Phi^A \cdot \partial B \Phi^B \geq 0 .$$  (56)

Using the projector $g^{AB} \partial A X^I \partial B X^J = g^{IJ} - \frac{2}{3} X^I X^J$ which becomes $g^{AB} \partial A W \partial B W = \frac{1}{9} ||h||^2 - \frac{2}{3} W^2$ this equation can be written as

$$W \mu \frac{d}{d\mu} W + 3g^{AB} \partial A W \partial B W = 0 .$$  (57)

Hence, $W^2$ is a monotonic decreasing function in $\mu$. On the other hand the $c$-theorem conjectures a $C$-function which is monotonic under the RG flow and becomes the central charge of the CFT at their extremum. Generically it interpolates between two different CFT with different central charges. These central charges, are basically given by the radius of the AdS space [10]: $c \sim l^2$, up to universal constants. Recalling that at the fixed point $l = (g||W||)^{-1}$, a natural ansatz for the $C$-function is

$$C(\mu, \Phi^A) \sim \frac{1}{(g||W||)^{\frac{3}{2}}} .$$  (58)

Moreover, expressed in terms of $\mu$ (recall $\mu d\mu = Wr dr$) the gaugino variation gives us an expression for the $\beta^A$-function for the coupling $\Phi^A$

$$\beta^A \equiv \mu \frac{d}{d\mu} \Phi^A = g^{AB} \partial B \log C .$$  (59)

Using the definition of $W$ together with the constraint (12) this represents an exact expression for the $\beta$-function for any prepotential. Inserting the expansion for $W$ this $\beta$-function can be expanded as

$$\beta^A = -3 g^{AB} \partial A W W = -2 \phi^A - 4 T^A_{BC} \phi^B \phi^C \pm \ldots .$$  (60)

Using this we can write eq. (57) as the RG equation for the $C$-function

$$(\mu \frac{d}{d\mu} - \beta^A \partial A) C(\mu, \Phi^B) = 0 .$$  (61)

Since $W^2$ is a strictly decreasing function and for $W > 0$, the $C$-function is strictly increasing under the RG flow ($c$-theorem)

$$\mu \frac{d}{d\mu} C \geq 0 .$$  (62)
It also means that the superpotential behaves monotonic during the flow.

Like the entropy in thermodynamics, the c-theorem reflects the irreversibility of a QFT under the RG flow, where the RG parameter \( \mu \) represents the scale below which (massive) modes have been integrated out, for a recent discussion see [41]. In fact, there seems to be a close relationship between the gravitational entropy and the central charge of the field theory. Both quantities can be obtained by an extremization with respect to the moduli: the central charge from the \( C \)-function as described before and the Bekenstein-Hawking entropy from the BPS mass [12, 13]. As a consequence, both are moduli independent and depend only on universal quantities like topological data. On the other hand the BPS mass itself as well as the \( C \)-function are not universal, the mass depends on the moduli and the \( C \)-function is scheme dependent. To be concrete, the Bekenstein-Hawking entropy of black holes in 5 dimensions is given by

\[ S = M_{\text{extr}}^{3/2}, \]

where \( M_{\text{extr}} \) is extremum of the BPS mass as a function of the moduli \( \sim q_I X^I \) while keeping fix the charges \( q_I \). This definition of the entropy does not require a horizon or a black hole as it can directly be derived from the first law of thermodynamics [14], but see also [15]. However, this approach has a subtle point - the temperature. E.g., extremal black holes have a non-vanishing entropy, whereas there is no natural definition of temperature in an asymptotic flat spacetime. On the other hand, our situation is different, because anti de Sitter space has an intrinsic temperature. Black holes, e.g., need a minimal temperature \( T_0 \sim \sqrt{-\Lambda} \) to be in a thermodynamical equilibrium with the AdS space [46] and at this critical point also the mass is directly fixed by the cosmological constant \( M \sim 1/(-\Lambda) \). Recalling that the cosmological constant is \( \Lambda \sim W_0^2 \), the first law gives for the entropy

\[ S = \left( \frac{M}{T_0} \right)_{\text{extr}} \sim W_0^{-3/2}, \]

where the extremum is taken with respect to the moduli while keeping fixed the parameter \( V_I \) (F-I terms). As mentioned in [15] the extremization with respect to the moduli yields an extremum for the temperature as well, \( T_0 \) is therefore a reasonable candidate for the temperature. Since this extremization is exactly equivalent to the equation (41) calculated at the fixed point [43], this analogy suggests that

\[ c_{FT} = \frac{1}{(gW_0)^3} \sim S_{\text{grav}}^{-2}. \]  

Note, the field theory central charge decreases in the flow towards the IR, while the gravitational entropy increases towards the UV and recall, the gravitational UV regime translates into the field theory IR and vice versa! One could object, that the equations (41) allow in general for more solutions, but the entropy should be a unique quantity for a given system. On the other hand, different solutions will correspond to different fixed scalar fields and therefore translate into different fixed points and thus there is no puzzle (since the field theory is expected to be very different at different fixed points).

As next step, let us describe the supersymmetric flow in detail. As one may have realized, the crucial part of the solution is given by equation (41), which states that

\[ c_{FT} = \frac{1}{(gW_0)^3} \sim S_{\text{grav}}^{-2}. \]  

\footnote{Of course these arguments have to be justified by a microscopic or statistical analysis.}
Figure 1: The scalar fields are not constant and therefore every point on the scalar manifold $M$ represents a different point in spacetime. A solution of the scalar field equation defines a trajectory $\Phi = \Phi(r)$ connecting two fixed point values. Supersymmetry is preserved, if along this trajectory the normal vector of $M$ remains parallel the harmonic function $H_I = h_I + \frac{q_I}{r^2}$ which fixes the solution. Although $M$ is smooth and differentiable the trajectory may make turns related to source terms at the domain wall.

the normal vector $X_I$ on the scalar manifold has to be parallel to $H_I$ or in other words: for a given radius $r$ a supersymmetric solution is given by the point(s) on the scalar manifold where $X_I \parallel H_I$. Moreover, since $H_I$ is a vector that interpolates between $h_I$ at $r = \infty$ and $q_I$ near $r \simeq 0$, every point on a supersymmetric path (flow) corresponds to a different radius (energy) and along this path the normal vector has to remain parallel to the (with $r$ changing) vector $H_I$. Our solution shows that there is always a path departing from a fixed point, with the normal vector given by $h_I$ towards any normal vector $q_I$, see figure 1.

But the point $r = 0$ is not a new fixed point. A fixed point would imply that the metric becomes anti de Sitter, but instead near $r = 0$ the spacetime becomes flat. In order to see this, let us note that as long as all $q_I$ are non-vanishing, none of the $X_I$ vanishes or diverges. Therefore near this point $e^{2U}$ has to scale like $1/r^2$ and the metric (22) becomes flat. Moreover, if it would be a fixed point, the first term in the gaugino variation had to vanish, but the first term in (27) vanishes only if $h_I \sim H_I$ (see (25)). This is the case only if $r \rightarrow \infty$ or for $h_I \parallel q_I$, the latter case would mean that we stay at a given fixed point. Hence, in order to reach a second fixed point we have to continue our solution through this point and at $r = -\infty$ we reach again an anti de Sitter space, i.e., we are again at a fixed point.

The continuation can be done in a symmetric way, i.e. by identifying the solution under $r \rightarrow -r$, but for us more interesting is the non-symmetric case, where the asymptotic AdS spaces differ. In order to have a well-defined transition we have to require that the scalars and the metric join smoothly which is ensured if the $q_I$ vector is the same on both sides. Also, we do not want to change the gauging while running from one fixed point.

\footnote{To verify this, take arbitrary ratios of eq. (41).}
point to another and therefore we impose that $h_I \sim V_I$ is the same on both sides. Hence, we use the same set of harmonic functions on both sides which implies that the normal vector $X_I$ behaves smoothly at $r = 0$. As a consequence of all this, the scalars and $W$ are regular. In order to ensure that we nevertheless move to another fixed point and not to come back to our starting point, we have to change the topology of the internal manifold at $r = 0$. There has to be a phase transition, which changes the intersection form (12). An example is a flop transition, which implies that the intersection form gets an additional term $\mathcal{V} \to \mathcal{V} - \frac{1}{6}t_2^3$ (if we assume that the cycle parameterized by $t_2$ vanishes at the transition point). For this type of transition, not only the scalar manifold but also the first and second order derivatives pass the transition point at $t_2 = 0$ smoothly, but higher derivatives, as typical for phase transitions, may jump; see also the discussion in [47]. As we have shown earlier, the second derivative of $W$ is positive definite at the fixed points (scalar mass terms), suggesting that $W$ has only minima and no saddle points and extrema (for $W > 0$). But the information about the first and second order derivative is not enough to get a global picture of $W$, instead we have to take into account that the scalar manifold (moduli space) has boundaries and typically at these boundaries massless states appear and second order derivatives of $W$ vanishes, see $W_+$ in (83). For a strictly non-degenerate Kähler metric, the number of minima should be related to the number of boundaries and if there is only one boundary like for torus compactification (see figure 2 in [12]) we get an unique extremum for $W$. The smoothness of the scalar manifold does not mean that the trajectory of the flow given by $\Phi = \Phi(r)$ is differentiable at $r = 0$. In the symmetric case where we identify both sides it is rather like a reflection, where the scalars have maximal velocity at the wall. At this point the velocity ($\sim \partial_r \Phi$) changes its sign, which produces a $\delta$-function in the second derivative and indicates source terms located at the wall [13]. In the non-symmetric case, there is still a sign change in the velocity, but it is rather a non-complete reflection or a refraction; see also the discussion in [32, 4]. Nevertheless, the underlying manifold should be smooth, only the flow trajectory is making a turn at $r = 0$. Let us also mention, that the vector $q_I$ determines the trajectory or equivalently the $q$’s are allowed deformations of a given trajectory. But not for all choices of $q_I$ we reach a phase transition point, i.e., for those cases the flow has to come back to the starting point.

Also, one may ask whether the superpotential has more extrema for a given model with a given intersection form (recall, that we still assume trivial hyper multiplets). Any extremum of the superpotential implies an AdS vacuum and allows for some unbroken supersymmetries and therefore, has to be part of our solution. As we discussed earlier, supersymmetry requires that the normal vector of the scalar manifold (i.e., $X_I$) becomes at the fixed point parallel to the constant vector $h_I$ or $V_I$, i.e., $h_I \partial_A X^I = 0$. Therefore, extrema of the superpotential coincide with extrema of the scalar manifold with respect $h_I$, see figure [4]. However, the scalar manifold as defined by the cubic equation $\mathcal{V} = 1$, is generically not connected and the different branches are separated by singular lines where the prepotential vanishes $\mathcal{V} = 0$. For many physical interesting cases this
Figure 2: Case (a) shows a coupling that runs towards a non-vanishing IR of UV fixed point value $\Phi_0$ (the arrow indicates the IR flow). On the other side, case (b) shows a running coupling with two fixed points. This case is typical for $N=1$ super Yang-Mills model.

4 Examples

Let us start with a simple example that can be solved completely. It is given by the prepotential

$$\mathcal{V} = X^0 X^A \eta_{AB} X^B$$

and corresponds in the ungauged case to a compactification on $K3 \times T_2$ with an appropriate signature for $\eta_{AB}$. Therefore it will give us a supergravity picture of branes at $Z_k$ orbifold singularities, as e.g., described in [49]. First we have to solve the equations (64), which become

$$H_0 = e^{2U} X^A \eta_{AB} X^B, \quad H_A = 2 X^0 e^{2U} \eta_{AB} X^B.$$  \hfill (65)

Fixing $X^0$ by the requirement $\mathcal{V} = 1$ and taking $X^A = \Phi^A$ as physical scalars we find as solution

$$X^A = e^{-4U} \eta^{AB} H_B H_0, \quad e^{3U} = \frac{1}{2} \sqrt{H_0 H_A \eta^{AB} H_B}.$$  \hfill (66)

The superpotential becomes

$$W = \frac{V_0}{\Phi_A \Phi^A} + V_A \Phi^A$$  \hfill (67)
which allows for a minimum \((\partial_t W = 0)\) where
\[
W_0^{3/2} = \frac{1}{2} \sqrt{27 V_0 (V_A \eta^{AB} V_B)} , \quad \Phi^A = 2^{1/3} \frac{V_0 V_A}{(V_0 V_A \eta^{AB} V_B)^{2/3}}
\] (68)
and \(g^2 W_0^2 = -\Lambda\) is the cosmological constant which defines also the central charge of the CFT. We can also calculate the \(\beta\)-function as a function of the scalars and find (see eq. (60))
\[
\beta^A = -3 g^{AB} \frac{\partial \phi^A}{W} = -\frac{1}{V_0 + (\Phi \cdot \Phi)(\Phi \cdot V)} \left[ 4 \Phi^A (\Phi \cdot \Phi)(\Phi \cdot V) - 2 V_0 \Phi^A - 3 (\Phi \cdot \Phi)^2 V^A \right].
\] (69)

But not only the supergravity side, also the field theory can be explored more explicitly. E.g., this model provides a sugra picture for branes at \(Z_k\) orbifolds as described in [49], where a subclass of our scalars corresponds to blow-up modes of 2-cycles and the charge parameter \(q_I\) entering the harmonic functions corresponds to the number of branes which are on top of each other. As consequence of the orbifold, the gauge group factories and the sugra scalars (moduli) parameterize the space of gauge couplings for the different \(U(q_I)\) gauge groups. The fixed point values of these gauge couplings are inversely related to the fixed scalars in (68) and the central charge is \(c \sim W_0^{-3}\). The non-vanishing \(\beta\)-function means the conformal symmetry is generically broken, but for special values of \(V_I\) the sugra scalars are constant everywhere. This is exactly the case if \(q_I \parallel V_I\) where the radial dependence of the scalars drops out (no running) and the domain wall becomes exactly \(AdS_5\). For black holes, this case is known as double extreme solution (an extremal black hole with extremal mass).

As a second example we will discuss a topological non-trivial case, which has two fixed points. The dashed line indicates the supergravity phase transition at \(r = 0\) between the distinct vacua and in our example we consider a transition related to an additional cubic term in the intersection form, as it has been discussed for black holes with constant scalars in [47] and later on for non-trivial scalar in [50]. The two phases correspond to different triangulations of an elliptically fibered Calabi-Yau with the base \(F_1\); for more information about this flop transition we refer to [51, 52] and for a general discussion about Calabi-Yau phase transitions to [53]. For this Calabi-Yau exists an orientifold limit [54] and the type I dual description (compactified on K3) has been discussed in [55]. We expect that in the dual field theory the space of gauge couplings can again be parameterized by our supergravity scalars.

Let us come to the concrete model. One phase, which corresponds to the second Chern class \(c_2 = (92, 102, 36)\) is described by the prepotential [52]
\[
\mathcal{V}_+ = \frac{3}{8} (X^2)^3 + \frac{1}{2} X^2 (X^1)^2 - \frac{1}{6} (X^3)^3
\] (70)

\[\text{Notice, on the supergravity side the dilaton sits in the universal hypermultiplet which is fixed in our setup (not running) and therefore also on the field theory side we have to keep fix the over-all norm of the gauge couplings as well. In this respect it would be interesting to explore the domain wall solution of [10] which has a non-trivial dilaton.}\]
and as it will turn out at the end, this model yields the bigger central charge. The coordinates that we are using are related to the Calabi-Yau moduli by

\[ t_1 = X^3, \quad t_2 = X^2 - X^3, \quad t_3 = X^1 - \frac{3}{2}X^2. \quad (71) \]

In order to stay inside the Kähler cone all \( t \)'s have to be positive or equivalently:

\[ \frac{2}{3}X^1 > X^2 > X^3 > 0. \]

We reach the boundaries of the moduli space at \( t_1 = X^3 = 0 \) (elliptic fibration over \( P_2 \)) and at \( t_3 = X^1 - \frac{3}{2}X^2 = 0 \) where tensionless strings emerge. As a side remark, these \( t \)-moduli correspond to different 2-cycles and from the F-theory perspective our domain wall can be seen as 7-branes with four coordinates wrapped in the internal space. Then, the tensionless strings can either be understood as 3-branes wrapping a vanishing 2-cycle which is not part of the 7-brane, or as zero-size instantons on the world volume, see [51] for more details. Let us recall, we consider gauged supergravity which, as mentioned before, corresponds to non-trivial (non-zero mode) fluxes for some cycles. Continuing our discussion, we pass the flop transition at

\[ t_2 = X^2 - X^3 = 0. \quad (72) \]

After this transition we enter a different CY with \( c_2 = (92, 36, 24) \), which corresponds to the prepotential [52]

\[ \mathcal{V}_- = \frac{5}{24}(X^2)^3 + \frac{1}{2}X^2(X^1)^2 - \frac{1}{2}X^2(X^3)^2 + \frac{1}{2}(X^2)^2X^3 \quad (73) \]

where the Calabi-Yau coordinates are now given by

\[ \tilde{t}_1 = X^2, \quad \tilde{t}_2 = X^3 - X^2, \quad \tilde{t}_3 = X^1 - \frac{1}{2}X^2 - X^3. \quad (74) \]

The Kähler cone is again defined by the domain of positive \( \tilde{t} \)s; the moduli space ends at \( \tilde{t}_1 = 0 \) (again tensionless strings emerge) and at \( \tilde{t}_3 = 0 \) where an \( SU(2) \) symmetry enhancement occurs. The flop transition is at \( \tilde{t}_2 = 0 \).

Let us discuss the two phases separately.

**1) The “+” phase.** In order to find the scalars we have to solve (11) which gives for the prepotential \( \mathcal{V}_+ \)

\[ H_1 = e^{2U}X^1X^2, \]
\[ H_2 = e^{2U}\left[ \frac{9}{8}(X^2)^2 + \frac{1}{2}(X^1)^2 \right], \]
\[ H_3 = e^{2U}\left[ -\frac{1}{8}(X^3)^2 \right]. \quad (75) \]

These equations have more solutions, but only one fulfills the conditions \( \frac{2}{3}X^1 > X^2 > \)
\(X^3 > 0\) (i.e., lies inside the Kähler cone) \[47\], \[50\]

\[
\begin{align*}
X^1 &= e^{-U} \sqrt{H_2 + \sqrt{H_2^2 - \frac{9}{4}H_1^2}}, \\
X^2 &= \frac{2}{3} e^{-U} \sqrt{H_2 - \sqrt{H_2^2 - \frac{9}{4}H_1^2}}, \\
X^3 &= e^{-U} \sqrt{-2H_3}, \\
e^{2U} &= \frac{1}{3} \left[ H_1X^1 + H_2X^2 + H_3X^3 \right]
\end{align*}
\]

where the harmonic functions have to satisfy

\[
H_2 \geq \frac{3}{2} H_1, \quad H_2 + \sqrt{H_2^2 - \frac{9}{4}H_1^2} \geq \frac{9}{2} |H_3|.
\]

At the transition point where \(t_2 = \bar{t}_2 = 0\) we have to ensure that \(X^2 = X^3\), which translates into a condition for the charges

\[
\frac{2}{9} \left( q_2 - \sqrt{q_2^2 - \frac{9}{4}q_1^2} \right) = |q_3|,
\]

(recall: \(q_3 < 0\)). Recall, the \(q\)’s are deformation parameter for the flow.

**2) The “-“ phase.** Here we have to consider the prepotential \(\mathcal{V}_-\) and find

\[
\begin{align*}
H_1 &= e^{2U} X^2 X^1 , \\
H_2 &= e^{2U} \left[ \frac{5}{8} (X^2)^2 + \frac{1}{2} (X^1)^2 - \frac{1}{2} (X^3)^2 + X^2 X^3 \right], \\
H_3 &= e^{2U} \left[ -X^2 X^3 + \frac{1}{2} (X^2)^2 \right].
\end{align*}
\]

Again these equations have more solutions, but only one of them can be connected at \(r = 0\) to the solution \(76\). It is given by \[47\], \[50\]

\[
\begin{align*}
X^1 &= e^{-U} \sqrt{\frac{\sqrt{2}H_1}{H_2 + \frac{1}{2}H_3 + \sqrt{(H_2 + \frac{1}{2}H_3)^2 - 2H_1^2 - H_3^2}}}, \\
X^2 &= \frac{1}{\sqrt{2}} e^{-U} \sqrt{H_2 + \frac{1}{2}H_3 + \sqrt{(H_2 + \frac{1}{2}H_3)^2 - 2(H_1^2 - H_3^2)}}, \\
X^3 &= \left( \frac{1}{2} X^2 - \frac{H_3}{H_2} e^{-2U} \right) \\
e^{2U} &= \frac{1}{3} \left[ H_1X^1 + H_2X^2 + H_3X^3 \right].
\end{align*}
\]

Notice, at the transition point we did not change the harmonic functions, neither \(q_I\) nor \(h_I\), we changed only the prepotential.
In order to simplify the situation further, we can go to the symmetry enhancement line, which means for the “+” phase: \( X^1 = \frac{3}{2} X^2 \) and for the “-” phase: \( X^3 = X^1 - \frac{1}{2} X^2 \). In both cases this translates into one condition for the harmonic functions

\[
H_2 = \frac{3}{2} H_1 .
\]  

(81)

After imposing this constraint, we have only one physical scalar and we can describe the situation as shown in figure (b). For this we regard \( X^3 = \Phi \) as the physical scalar and we start our flow in the “+” phase at the boundary of the \( X^3 \) modulus, i.e., at \( X^3 = t^3 = 0 \). It means, we have to set \( h_3 = 0 \) and find

\[
\Phi \to \begin{cases} 
0 , & \text{in the “+” phase } (r \to +\infty) \\
(48/5)^{1/3} , & \text{in the “-” phase } (r \to -\infty) .
\end{cases}
\]  

(82)

The superpotentials read \((V_1 = 2/3 V_2 = V; V_3 = 0)\)

\[
W_+ = V \left( \frac{72}{7} \right)^{1/3} \left[ 1 + \frac{1}{6} \Phi^2 \right]^{1/3} ,
\]

\[
W_- = V \left[ \frac{5}{3} \sqrt{f(\Phi)} + \frac{1}{f(\Phi)} \right] ,
\]

where \( f(\Phi) = \sqrt{9/4 \Phi^2 + 3 - 3/2 \Phi} \). Obviously, the extremum in the “+” phase is at \( \Phi = 0 \) and the second derivative of \( W \) vanishes at the extremum, which means that the Kähler metric degenerates. This confirms our expectations, because this extrema lies on the boundary of the moduli space where tensionless strings appear. In the “-” phase the situation is different, the fixed scalars as well as the second derivative is non-vanishing. The ratio of the extrema gives the ratio of the central charges

\[
\frac{c_-}{c_+} = \left( \frac{W_+}{W_-} \right)_{\text{extr.}}^3 = \frac{(e^{6U})_+}{(e^{6U})_-} = \frac{24}{25} .
\]

(84)

5 Conclusion

In this paper we employed domain wall solutions to give a supergravity description of the RG flow of 4-d super Yang-Mills. On the supergravity side the domain wall interpolates between two asymptotic AdS spaces on both sides and is a solution of 5-d gauged supergravity, where a \( U(1) \) group of the R-symmetry has been gauged. In this setup the running of the supergravity scalars between two fixed point values corresponds to the running of couplings of operators in super Yang-Mills. These fixed point values are reached in the two asymptotic AdS spaces on both sides of the domain wall. Identifying both sides yields an \( S/Z_2 \) orbifold in the radial direction, but it does not describe a flow between different fixed points. Instead, to reach a second fixed
point one has to assume that a phase transition takes place while passing the domain wall. The natural framework to discuss this transition is $N=2$ supergravity and we have discussed an explicit example. This phase transition happens at a boundary of the Kähler cone and therefore, while passing the domain wall in spacetime one leaves a given Kähler cone as well.

Exploring the equations of motion and the supersymmetry variations we discuss the $c$-theorem and give a supergravity expression for the field theory $\beta$-functions. These expressions are non-perturbative in the sense that they hold for any prepotential of $N=2$ supergravity but they are restricted to the case of trivial hypermultiplets. Expanding the supergravity potential as well as the superpotential around a given fixed point we obtain the cosmological constant and the masses for the scalar fields. The cosmological constant, which suppresses all mass terms, is obtained by an extremization of the superpotential with respect to the moduli which suggests a relation to the 5-d gravitational entropy.

At the end we discussed two examples in more detail. One is a gauging of the K3 compactification of IIB string theory which should be dual $N=2$ super Yang-Mills. As expected this has only one fixed point. In the second example we discuss a gauged model of elliptically fibered Calabi-Yau with $F_1$ basis, which allows for an orientifold limit and is dual to type I on K3. This second example exhibits a phase transition and passing this transition corresponds to a running coupling between different fixed points of the dual $N=1$ super Yang Mills. In both cases the field theory gauge group is expected to be a direct product of different $U(q_I)$, at least in the orbifold/orientifold limit, and we argue that deformations of the relative gauge couplings are parameterized by the supergravity scalars.

There are a couple of interesting directions that are worthwhile to explore. One is to take into account a non-trivial dilaton, e.g., by exploiting the solution given in [10]. A further direction is to investigate the electric-magnetic duality. In general by gauging we break the duality and therefore we cannot expect dyonic solutions, but it may happen that two different models flow to the same fixed point and at this fixed point they are dual to each other. In fact for non-supersymmetric vacua (with vanishing potential) generalized electro-magnetic dualities have been discussed in [34]. Of course, to investigate non-supersymmetric vacua, would be interesting in its own. Finally, let us also mention that having the explicit supergravity solutions, it is straightforward to calculate the effective supergravity action. As they solve the equations of motion only surface terms will survive, which should generate the amplitudes of super Yang Mills.

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