Spin\(^c\) structures on Hantzsche-Wendt manifolds

R. Lutowski*, J. Popko*, and A. Szczepański*

*Institute of Mathematics, University of Gdańsk, Gdańsk, Poland

March 2, 2021

Abstract

Using a combinatorial description of Stiefel-Whitney classes of closed flat manifolds with diagonal holonomy representation, we show that no Hantzsche-Wendt manifold of dimension greater than three does not admit a spin\(^c\) structure.

1 Introduction

Hantzsche-Wendt manifolds are examples of flat manifolds, i.e. closed Riemannian manifolds with vanishing sectional curvature. They are generalizations of the three-dimensional flat orientable manifold defined in \[5\] and, following \[16\], we say that:

An orientable \(n\)-dimensional flat manifold is Hantzsche-Wendt if and only if its holonomy group is an elementary abelian \(2\)-group of rank \(n - 1\).

Every \(n\)-dimensional flat manifold \(X\) occurs as a quotient space of the action of \(\Gamma\) on the euclidean space \(\mathbb{R}^n\), where \(\Gamma\) is a Bieberbach group, i.e. a torsion-free, co-compact and discrete subgroup of the group \(\text{Isom}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n\) of isometries of \(\mathbb{R}^n\). \(X\) is an Eilenberg-MacLane space of type \(K(\Gamma, 1)\). By Bieberbach theorems (see \[18\]), \(\Gamma\) is defined by the following short exact sequence

\[
0 \longrightarrow \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{\pi} G \longrightarrow 1,
\]

where \(i(\mathbb{Z}^n)\) is the maximal abelian normal subgroup of \(\Gamma\), \(G\) is finite and coincides with the holonomy group of \(X\). Moreover, by conjugations in \(\Gamma\), \(G\) acts in a natural way on \(\mathbb{Z}^n\), giving it the structure of a \(G\)-module.

Taking into account the above definition we will say that a Bieberbach group \(\Gamma \subset \text{Isom}^+(\mathbb{R}^n) = \text{SO}(n) \ltimes \mathbb{R}^n\), defined by \((1.1)\), is a Hantzsche-Wendt group and \(X = \mathbb{R}^n/\Gamma\) is a Hantzsche-Wendt manifold (HW-group and HW-manifold for short) if \(G \simeq C_2^{n-1}\).

Among many properties of HW-manifolds which were objects of research one can list the following: they exist only in odd dimensions \([11]\), they are rational homology spheres \([19]\) and cohomologically rigid \([13]\). If \(\Gamma\) is a HW-group then it is an epimorphic image of a certain Fibonacci group \([10]\) and if its dimension is greater than or equal to 5, then its commutator and translation subgroups coincide \([14]\). One of the crucial – for the purposes of this paper – properties of
HW-manifolds (HW-groups) is the one described in [16]: they are diagonal, i.e. there exists a $\mathbb{Z}$-basis $B$ of the $G$-module $\mathbb{Z}^n$ such that

$$gb = \pm b$$

for every $b \in B$ and $g \in G$.

Now, let $n \geq 3$. The fundamental group $\pi_1(\text{SO}(n))$ of the special orthogonal group $\text{SO}(n)$ is of order 2. The spin group $\text{Spin}(n)$ is its double cover – and the universal cover in fact. Let $\lambda_n: \text{Spin}(n) \to \text{SO}(n)$ be the covering map. A spin structure on a smooth orientable manifold $X$ is an equivariant lift of its frame bundle via $\lambda_n$. Its existence is equivalent to the vanishing of the second Stiefel-Whitney class $w_2(X)$ of $X$, see [3, page 40]. In the case when $X$ is flat, it is closely connected to the Sylow 2-subgroup of its holonomy group [2] and can be determined by an algorithm [8]. The three-dimensional HW-manifold has a spin structure (see [7, Theorem VII.1]). But this is the only case – by [12, Example 4.6] no other HW-manifold admits any spin structure.

In the case when there are no spin structures, one can consider their complex analogue. We have that

$$\text{Spin}^c(n) := (\text{Spin}(n) \times S^1) / \langle (-1, -1) \rangle = \text{Spin}(n) \times_{C_2} S^1$$

is the double cover of $\text{SO}(n) \times S^1$ for which the spin$^c$ structure is defined – in analogy to the spin case – with the covering map $\bar{\lambda}_n: \text{Spin}^c(n) \to \text{SO}(n) \times S^1$ given by

$$\bar{\lambda}_n[x, z] := (\lambda_n(x), z^2).$$

The manifold $X$ has a spin$^c$ structure if and only if $w_2(X)$ is the mod 2 reduction of some integral cohomology class $z \in H^2(X, \mathbb{Z})$, see [3, page 49]. We immediately get that existence of spin structures determines existence of spin$^c$ structures – in fact the former induces the latter, but not the other way around. For example, by an unpublished work [20] all orientable 4-manifolds have some spin$^c$ structures, but by [15]: 3 of the 27 flat ones don’t have any.

In this paper we prove that every HW-manifold of dimension greater than or equal to 5 does not admit any spin$^c$ structure. Note that some examples of non-spin$^c$ HW-manifolds were given in [4].

The tools that we use have been introduced in [13] and used for example in [9]. They proved their effectiveness in cohomology-related properties of diagonal manifolds.

The structure of the paper is as follows. Sections 2 and 3 give a quick glance on a way of the encoding diagonal manifolds and their Stiefel-Whitney classes by certain matrices. This has been already presented in more detail in [13] and [9]. In Section 4 we give one of two theorems on conditions equivalent to the existence of spin$^c$ structures on HW-manifolds. For our further analysis we introduce HW-matrices. This description of HW-manifolds was introduced in [13] and is in fact one-to-one with the one given in [11]. Technical Section 6 gives us some properties and formulas for matrices that we work with. The second theorem on conditions equivalent to the existence of spin$^c$ structures on HW-manifolds is given in Section 7. After that we give a very specific form to a matrix which describes a (possible) spin$^c$ HW-manifold and at last we show that this form can never occur. This proves that no HW-manifold can admit a spin$^c$ structure.
2 Diagonal flat manifolds

In this section we give a combinatorial description of diagonal flat manifolds. This language is essential in the analysis of the Steifel-Whitney classes of such manifolds.

Remark 2.1. For any matrix $A$ by $A_{ij}, A_{i j}$ or $A_{i j}$ we shall denote the element in the $i$-th row and $j$-th column of $A$. By $A_i$ we shall understand the $i$-th row of $A$.

Remark 2.2. Let $k \in \mathbb{N}$. Cyclic groups of order $k$ with multiplicative and additive structure will be denoted by $C_k$ and $\mathbb{Z}_k := \mathbb{Z}/k$, respectively. Note that in the natural way $\mathbb{Z}_k$ is ring and possibly – a field.

Suppose $\Gamma$ is a Bieberbach group defined by the short exact sequence (1.1). As mentioned in the introduction, conjugations in $\Gamma$ define a $G$-module $\mathbb{Z}^n$. To be a bit more precise, corresponding representation $\rho: G \to \text{GL}_n(\mathbb{Z})$ is called an integral holonomy representation of $\Gamma$ and it is given by the formula

$$\rho_g(z) = \tau^{-1}(\gamma \epsilon(z) \gamma^{-1}),$$

where $z \in \mathbb{Z}^n, g \in G$ and $\gamma \in \Gamma$ is such that $\pi(\gamma) = g$. We will call $\Gamma$ diagonal or of diagonal type if the image of $\rho$ is a subgroup of the group

$$D = \{A \in \text{GL}(n, \mathbb{Z}) : A_{ij} = A_{ji} = 0 \text{ and } A_{ii} = \pm 1 \text{ for } 1 \leq i < j \leq n \} \cong \mathbb{Z}_2^n$$

of diagonal matrices of $\text{GL}(n, \mathbb{Z})$. It follows that $G = \mathbb{C}_2^k$ for some $1 \leq k \leq n - 1$.

Let $S^1 = \mathbb{R}/\mathbb{Z}$. As in [13] and [3], we consider the automorphisms $g_i : S^1 \to S^1$, given by

$$g_0([t]) = [t], \quad g_1([t]) = [t + \frac{1}{2}], \quad g_2([t]) = [-t], \quad g_3([t]) = [-t + \frac{1}{2}],$$

for $t \in \mathbb{R}$. Let $\mathcal{D} = \{g_i | i = 0, 1, 2, 3\}$. It is easy to see that $\mathcal{D} \cong \mathbb{C}_2 \times \mathbb{C}_2$ and $g_3 = g_1 g_2$. We define an action of $\mathcal{D}$ on $T^n$ by

$$(t_1, \ldots, t_n)(z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n),$$

for $(t_1, \ldots, t_n) \in \mathcal{D}^n$ and $(z_1, \ldots, z_n) \in T^n = S^1 \times \cdots \times S^1$.

Any minimal set of generators of a group $\mathbb{C}_2^d \subseteq \mathcal{D}^n$ defines a $(d \times n)$-matrix with entries in $\mathcal{D}$ which in turn defines a matrix $A$ with entries in the set $\mathcal{V} = \{0, 1, 2, 3\}$ under the identification $i \leftrightarrow g_i$, $0 \leq i \leq 3$. Note that elements of $\mathcal{V}$ are written in italic.

Definition 2.3. The structure of an additive group on $\mathcal{V}$ is given by

$$i + j = k \iff g_i g_j = g_k,$$

for $i, j, k \in \mathcal{V}$. This way $\mathcal{V} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is in the natural way a $\mathbb{Z}_2$-vector space.

Example 2.4. The three-dimensional HW-group has generators:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
hence the corresponding matrix $A \in \mathcal{V}^{2 \times 3}$ is of the form

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$  

**Remark 2.5.** Whenever our calculations involve $\mathbb{Z}_2 = \{0, 1\}$ and $\mathcal{V}$, it is done by identifying $\mathbb{Z}_2$ with the subgroup $\{0, 1\} < \mathcal{V}$.

We have the following characterization of the action of $C_d^2$ on $T^n$ and the associated orbit space $T^n/C_d^2$ via the matrix $A$.

**Lemma 2.6** ([13, page 1050]). Let $C_d^2 \subseteq D^n$ and define the matrix $A \in \mathcal{V}^{d \times n}$ as above. Then:

(i) the action of $C_d^2$ on $T^n$ is free if and only if there is 1 in the sum of any distinct collection of rows of $A$,

(ii) $C_d^2$ is the holonomy group of $T^n/C_d^2$ if and only if there is either 2 or 3 in the sum of any distinct collection of rows of $A$.

When the action of $C_d^2$ on $T^n$ defined by (2.2) is free, we will say that the associated matrix $A$ is free and we will call it the defining matrix of $T^n/C_d^2$. In addition, when $C_d^2$ is the holonomy group of $T^n/C_d^2$, we will say that $A$ is effective.

### 3 Stiefel-Whitney classes of diagonal flat manifolds

The goal of this section is to introduce a notation and some basic results on Stiefel-Whitney classes of diagonal flat manifolds. For more precise description see [9] and [13].

Let $n \in \mathbb{N}$ and $\Gamma$ be an $n$-dimensional diagonal Bieberbach group, given by the extension (1.1), with non-trivial holonomy group $G = C_d^2$ ($d > 0$). Let $A \in \mathcal{V}^{d \times n}$ be a defining matrix of the corresponding flat manifold $X = \mathbb{R}^n/\Gamma = T^n/C_d^2$.

It is well-known that

$$H^*(C_d^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_d],$$

where $\{x_1, \ldots, x_d\}$ is a basis of $H^1(C_d^2, \mathbb{Z}_2) = \text{Hom}(C_d^2, \mathbb{Z}_2)$ (see [1, Theorem 1.2]). Let

$$\pi^*: H^*(C_d^2, \mathbb{Z}_2) \to H^*(\Gamma, \mathbb{Z}_2)$$

be the induced cohomology ring homomorphism. By [3, Proposition 3.2] the total Stiefel-Whitney class is given by

$$w(X) = \pi^*(\text{sw}) \in H^*(\Gamma, \mathbb{Z}_2),$$

where

$$\text{sw} = \prod_{j=1}^{n}(1 + \alpha_j + \beta_j). \quad (3.1)$$
In the above formula for every $1 \leq j \leq n$, $\alpha_j, \beta_j \in H^1(C_d^1, \mathbb{Z}_2)$ are the cocycles defined by
\[
\alpha_j = \sum_{k=1}^{d} \alpha(A_{kj}) x_k, \quad \beta_j = \sum_{k=1}^{d} \beta(A_{kj}) x_k
\]
and the linear homomorphisms $\alpha, \beta \in \text{Hom}_{\mathbb{Z}_2}(\mathcal{V}, \mathbb{Z}_2)$ are uniquely defined by the following rules
\[
\alpha(2) = \beta(3) = 1 \quad \text{and} \quad \alpha(3) = \beta(2) = 0.
\]
Let
\[
\pi^*_i : H^i(C_d^{n-1}, \mathbb{Z}_2) \to H^i(\Gamma, \mathbb{Z}_2)
\]
be the induced group cohomology homomorphism (restriction of $\pi^*$ to the $i$-th gradation), for $0 \leq i \leq n$. Using again [9, Proposition 3.2] and the five-term exact sequence for the extension (1.1) (see [9, Formula (7)]) we get

**Lemma 3.1.** $\pi^*_i$ is injective and the kernel of $\pi^*_{i+1}$ is spanned by $\theta_j = \alpha_j \cup \beta_j$ for $1 \leq j \leq n$.

**Remark 3.2.** Note that the polynomials $sw, \alpha_j, \beta_j, \theta_j$, where $1 \leq j \leq n$, can be defined for any matrix $A \in \mathcal{V}^{d \times n}$. To emphasize this connection or in the case when it won’t be clear from the context, we will add the superscript $A$ to them and write $sw^A$ for example.

### 4 Bockstein maps and spin\(^c\) structures

We will keep the notation of the previous section and restrict our attention to the case of Hantzsche-Wendt manifolds of dimension greater than or equal to 5. Hence $n \geq 5$ is an odd integer and $d = n - 1$. Let $\tilde{\beta}_\Gamma$ and $\tilde{\beta}_\Gamma$ be the Bockstein homomorphisms of cohomology groups of $\Gamma$ associated to the short exact sequences
\[
0 \to \mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{\text{mod}_2} \mathbb{Z}_2 \to 0 \quad (4.1)
\]
and
\[
0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{mod}_2} \mathbb{Z}_2 \to 0 \quad (4.2)
\]
respectively. If $\rho : H^2(\Gamma, \mathbb{Z}) \to H^2(\Gamma, \mathbb{Z}_2)$ is the homomorphism induced by the mod 2 map, then we have the following commutative diagram
\[
\begin{array}{ccc}
H^1(\Gamma, \mathbb{Z}) & \xrightarrow{\tilde{\beta}_\Gamma} & H^2(\Gamma, \mathbb{Z}) \\
\downarrow{\beta_\Gamma} & & \downarrow{\rho} \\
H^1(\Gamma, \mathbb{Z}_2) & \xrightarrow{\tilde{\beta}_\Gamma} & H^2(\Gamma, \mathbb{Z}_2)
\end{array}
\]
with the row forming an exact sequence (see [8, Chapter 3.E]). By [14, Theorem 3.1] $H_1(\Gamma) \cong \mathbb{Z}_2^{n-1}$. By [18, Theorem 9.2] $H_2(\Gamma)$ is a finite group. Moreover from the universal coefficient theorem ([6, Theorem 3.2]),
\[
H^1(\Gamma, \mathbb{Z}) = 0 \quad \text{and} \quad H^1(\Gamma, \mathbb{Z}_2) \cong H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}_2^{n-1}.
\]
Hence $\tilde{\beta}_\Gamma$ is an isomorphism and $\text{Im}\, \tilde{\beta}_\Gamma = \text{Im}\, \rho$.

Let $\beta$ be the Bockstein homomorphism of cohomology groups of $C_2^{n-1}$ associated to the extension (4.1). The homomorphism $\pi$ induces the commutative diagram

$$
\begin{array}{ccc}
H^1(C_2^{n-1}, \mathbb{Z}_2) & \xrightarrow{\beta} & H^2(C_2^{n-1}, \mathbb{Z}_2) \\
\downarrow{\pi^*_{(1)}} & & \downarrow{\pi^*_{(2)}} \\
H^1(\Gamma, \mathbb{Z}_2) & \xrightarrow{\beta_{\Gamma}} & H^2(\Gamma, \mathbb{Z}_2)
\end{array}
$$

By Lemma 3.1, $\pi^*_{(1)}$ is a monomorphism of the elementary abelian 2-groups of rank $n-1$, hence it is an isomorphism and

$$
\text{Im}\, \rho = \text{Im}\, \beta_{\Gamma} = \text{Im}\, \beta_{\Gamma} \pi^*_{(1)} = \text{Im}\, \beta^* = \text{Im}\, \beta^* \beta.
$$

Let $sw_2$ be the sum of degree 2 terms of the polynomial $sw$. Then $w_2(X) = \pi^*(sw_2)$ and by definition the manifold $X = \mathbb{R}^n/\Gamma$ admits a spin$^c$ structure if and only if $\pi^*(sw_2) \in \text{Im}\, \pi^* \beta$. This condition is obviously equivalent to

$$(sw_2 + \ker \pi^*) \cap \text{Im} \beta \neq \emptyset.$$ 

In addition, one can easily show that for every $x \in H^1(C_2^{n-1}, \mathbb{Z}_2)$ and $a, b \in C_2^{n-1}$ we have

$$\beta(x)(a, b) = x(a)x(b) = x^2(a, b),$$

hence $\beta(x) = x^2$ and $\pi^*(\beta(x)) = \pi^*(x)^2$. Similarly, $\beta_{\Gamma}(f) = f^2$ for $f \in H^1(\Gamma, \mathbb{Z}_2)$.

Using Lemma 3.1 we get the following theorem:

**Theorem 4.1.** Assume that $n \geq 5$ is an odd integer and $X$ is an $n$-dimensional Hantzsche-Wendt manifold. Let $A \in V^{n-1 \times n}$ be a defining matrix of $X$. Then the following conditions are equivalent:

1. $X$ admits a spin$^c$ structure.
2. $w_2(X) \in H^*(\Gamma, \mathbb{Z}_2)$ is a square.
3. There exists $x \in H^1(\mathbb{Z}_2^{n-1}, \mathbb{Z}_2)$ such that $x^2 + sw_2^A \in \text{span}\{\theta_1^A, \ldots, \theta_n^A\}$.

## 5 HW matrices

Let $n \in \mathbb{N}$. Every $n$-dimensional HW-manifold $X$ defines some matrix $A \in V^{n-1 \times n}$. For the purpose of investigating spin$^c$ properties of $X$ it will be more convenient to work with a square matrix – a HW-matrix. HW-matrices were defined in [13].

Let $Z$ be a finite set. By $\mathcal{P}(Z)$ we denote the algebra (over the field $\mathbb{Z}_2$) of subsets of $Z$. Just recall that the addition and multiplication in $\mathcal{P}(Z)$ are defined by the symmetric difference and intersection respectively:

$$
\forall A, B \in \mathcal{P}(Z) \ A + B := (A \setminus B) \cup (B \setminus A) \text{ and } A \cdot B := A \cap B.
$$

Empty set and $Z$ are zero and one of this algebra, respectively. Let us note without a proof:

6
Lemma 5.1.

1. The map $|·|_2: \mathcal{P}(Z) \to \mathbb{Z}_2$, given by

$$U \mapsto |U| \mod 2,$$

is linear.

2. Every permutation of $Z$ is an algebra automorphism of $\mathcal{P}(Z)$.

Remark 5.2. We will use the notation $\mathcal{P}_d := \mathcal{P}([1, \ldots, d])$ for $d \in \mathbb{N}$.

Definition 5.3. Let $d, n \in \mathbb{N}$ and $A \in \mathcal{V}^{d \times n}$. For $S \in \mathcal{P}_n$ and $1 \leq i \leq d$ we have the sum of elements of the $i$-th row $A$ which lie in the columns from the set $S$:

$$\text{smr}^S_i(A) := \sum_{j \in S} A_{ij}$$

and we denote $\text{smr}^{\{1, \ldots, n\}}_i(A)$ simply by $\text{smr}_i(A)$. In a similar way we define the column sums $\text{smc}^S_j(A)$ (and $\text{smc}_j(A)$) for $S \in \mathcal{P}_d$ and $1 \leq j \leq n$. Moreover, we define a map $J_A: \mathcal{P}_d \to \mathcal{P}_n$ as follows

$$J_A(U) := \{ j : \text{smc}^U_j(A) = 1 \} .$$

Definition 5.4. The exists the unique $\mathbb{Z}_2$-linear involution $\cdot^$: $\mathcal{V} \to \mathcal{V}$ which maps 2 to 3. We call this map a conjugation. To be explicit, we have

$$\overline{0} = 0, \overline{1} = 1, \overline{2} = 3 \text{ and } \overline{3} = 2.$$

Definition 5.5. Let $A$ be a matrix with coefficients in $\mathcal{V}$. We call $A$:

- **self-conjugate** if $A^t = \overline{A}$, where $A^t$ is the transpose of $A$ and $\overline{A}$ is the element-wise conjugate of $A$;
- **distinguished** if it has 1 on the main diagonal and 2 or 3 everywhere else.

Remark 5.6. Recall that we speak about a principal submatrix of a given matrix if the sets of row and column indices which define it are the same (see [17, Definition 6.2.5] for example). We immediately get, that principal submatrices of self-conjugate and distinguished matrices are themselves self-conjugate and distinguished, respectively.

Lemma 5.7. Let $A \in \mathcal{V}^{k \times n}$ be distinguished, where $k \leq n$. Then the possible values for $\text{smc}_j(A)$, where $1 \leq j \leq n$ are given by the following table:

| $j \leq k$ | $j > k$ |
|------------|---------|
| $2 \mid k$ | 2 or 3  |
| $2 \not\mid k$ | 0 or 1 | 2 or 3 |

Proof. Simple calculation of the parity of the number of 2 and 3 in each column. $\Box$

Definition 5.8 ([13, Definition 2]),. Let $n \in \mathbb{N}$. We will call $A \in \mathcal{V}^{n \times n}$ a HW-matrix if:

1) $A$ is distinguished;
2) \( \text{smc}_j(A) = 0 \) for every \( 1 \leq j \leq n \);

3) \( J_A(U) \neq 0 \) for every \( U \in \mathcal{P}_n \setminus \{0,1\} \).

The set of HW-matrices of degree \( n \), or \( n \)-HW-matrices for short, will be denoted by \( \mathcal{H}_n \).

By Lemma 5.7 we immediately get:

**Corollary 5.9.** Every HW-matrix is of odd degree.

**Remark 5.10.** We can think of the above definition as coming from the encoding Hantzsche-Wendt groups presented in [11]. In connection to this description we note:

1. Any row of a HW-matrix may be removed and the corresponding torus quotient will remain the same. In other words, the removal will make the matrix a defining and effective one for the same HW-manifold.

2. Every HW-manifold defines some HW-matrix.

3. There is an action of the group \( G_n := C_2 \wr S_n \) on the set \( V_n \times n \). Namely, for every \( A \in V_n \times n \) we have that

   (a) \( c_k \) conjugates the \( k \)-th column of \( A \), where \( c_k \in C_2^n \) has non-trivial element of \( C_2 \) in the \( k \)-th coordinate only;

   (b) \( \sigma \cdot A := P_\sigma A P_\sigma^{-1} \), where \( P_\sigma \in \text{GL}_n(\mathbb{Z}) \) is the permutation matrix of \( \sigma \in S_n \).

Keeping the above remark in mind, we can reformulate [11, Proposition 1.5] as follows:

**Proposition 5.11.** The HW-manifolds \( X \) and \( X' \), with corresponding HW-matrices \( A, A' \in V_n \times n \), are affine equivalent if and only if \( A \) and \( A' \) are in the same orbit of the action of the group \( G_n \).

### 6 Square distinguished matrices

The following section is of a bit technical nature. Its purpose is to present some properties of square distinguished matrices. We start with a negative result:

**Lemma 6.1.** Let \( n > 1 \) be an integer. There does not exist a matrix \( M \in V_n \times n \) such that:

(A1) \( M \) is distinguished and self-conjugate;

(A2) the first row of \( M \) is of the form \( M_1 = [1,2,\ldots,2] \);

(A3) \( \text{smc}_i M = 1 \) for \( 1 \leq i \leq n \);

(A4) in every principal submatrix of \( M \) of odd degree there exists a column with sum of elements equal to 1.

**Proof.** Assume that such a matrix \( M \) exists. We will list some of its properties.
(P1) Action by permutations of the set \(\{2, 3, \ldots, n\}\) on \(M\), as in Remark 5.10, does not change its properties \([A1][A4]\).

(P2) \(\text{smr}_i(M) = 1\) for every \(1 \leq i \leq n\), since
\[
\text{smr}_i(M) = \sum_{j=1}^{n} M_{ij} = \sum_{j=1}^{n} M_{ji} = \sum_{j=1}^{n} \text{smc}_i(M) = T = 1.
\]

(P3) \(n\) is odd, by Lemma 5.7.

(P4) \(M_{2,1} = 3\) by self-conjugacy of \(M\).

(P5) The second row of \(M\) cannot be of the form \([3, 1, 2, \ldots, 2]\), otherwise
\[
\text{smr}_2(M) = 3 + 1 + (n - 2)2 = 2 + 2 = 0,
\]
which contradicts \(\text{(P2)}\).

(P6) The second row of \(M\) cannot be of the form \([3, 1, 3, \ldots, 3]\). Otherwise
\[
M = \begin{bmatrix} * & A \\ * & B \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 2 & \ldots & 2 \\ 3 & \ldots & 3 \end{bmatrix} \in \mathcal{V}^{2 \times n-2}
\]
Using \([A3]\), for every \(i > 2\) we get
\[
1 = \text{smc}_i(M) = 2 + 3 + \text{smc}_{i-2}(B) = 1 + \text{smc}_{i-2}(B),
\]
hence \(\text{smc}_{i-2}(B) = 0\) and this, together with \(\text{(P3)}\), contradicts \(\text{(A4)}\).

(P7) Using \(\text{(P1)}\), \(\text{(P5)}\) and \(\text{(P6)}\) we can assume that
\[
M_2 = [3, 1, 2, \ldots, 2, 3, \ldots, 3],
\]
where \(a, b > 0\). Moreover, \(a\) is even (and \(b = n - 2 - a\) is odd), since
\[
1 = \text{smr}_2(M) = 3 + 1 + a \cdot 2 + b \cdot 3 = 2 + a \cdot 2 + (n - 2 - a) \cdot 3
\]
\[
= (1 + a) \cdot 2 + (1 + a) \cdot 3 = (1 + a)(2 + 3) = (1 + a) \cdot 1 = 1 + a \cdot 1.
\]

(P8) Let \(M\) has the following block form
\[
M = \begin{bmatrix}
1 & 2 & 2 & 2 \\
3 & 1 & 2 & 3 \\
* & * & * & C \\
* & * & * & D
\end{bmatrix},
\]
where on the diagonal we have matrices of degree \(1, 1, a\) and \(b\). There exists an element of \(C\) equal to \(2\). Otherwise, for every \(i > a + 2\), we have
\[
1 = \text{smc}_i(M) = 2 + 3 + a \cdot 3 + \text{smc}_{i-a-2}(D) = 1 + \text{smc}_{i-a-2}(D)
\]
and since \(D\) is a principal submatrix of \(M\) of odd degree, we get a contradiction with \(\text{(A4)}\).
By [P8] there exist $i$ and $j$, such that $3 \leq i \leq a+2 < j \leq n$ and the principal submatrix $\Delta$ of $M$ given by indices $(2,i,j)$ is of the form

$$\Delta = \begin{bmatrix} 1 & 2 & 3 \\ * & 1 & 2 \\ * & * & 1 \end{bmatrix}.$$  

By self-conjugacy of $\Delta$ we immediately get

$$\Delta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix},$$

but this contradicts (A4).

**Remark 6.2.** To a logical sentence $\Theta$ we assign (in a natural way) an element $[\Theta] \in \mathbb{Z}_2$ as follows:

$$[\Theta] = 1 \iff \Theta \text{ is true}.$$  

**Remark 6.3.** Let $n \in \mathbb{N}, M \in V_{n \times n}$ and $U \in P_n$. By $M_U$ we denote the sum of the rows of $M$ from the set $U$:

$$M_U := \sum_{i \in U} M_i$$

and $M_U[j]$ – its $j$-th coordinate, for $1 \leq j \leq n$. We get

$$J_M(U) = \{ j : \text{smc}_j^U(M) = 1 \} = \{ j : M_U[j] = 1 \}.$$  

The following lemma, which describes map $J$ for distinguished matrices, extends [13, Proposition 3].

**Lemma 6.4.** Let $n \in \mathbb{N}, M \in V_{n \times n}$ be distinguished and $S, U \in P_n$. The following hold:

1. $J_M(U) = U$ if $|U| = 1$.
2. $J_M(U) \subset U$ if $|U|_2 = 1$.
3. $J_M(U) : U = 0$ if $|U|_2 = 0$.
4. $|J_M(U)|_2 = \sum_{i,j \in U} M_{ij}$ if $|U|_2 = 1$.
5. $|J_M(U)|_2 = \sum_{i,j \in U} M_{ij} + \sum_{i \in U} \text{smr}_i(M)$ if $|U|_2 = 0$.
6. $|J_M(U)S|_2 = \sum_{j \in U} |j \in S| M_U[j]$ if $|U|_2 = 1$.
7. $|J_M(U)S|_2 = \sum_{j \in U} |j \in S| M_U[j] + \sum_{i \in U} \text{smr}_i^S(M)$ if $|U|_2 = 0$.

**Proof.** Property 1 holds just because $M$ is distinguished – in fact, we have

$$\forall 1 \leq i \leq n, J_M(i) = \{ i \}. \quad (6.1)$$

Properties 2 and 3 hold by the same rule as in the proof of Lemma 5.7. This rule will be also used in the rest of the proof.
Note that 4. and 5. follow from 6. and 7. respectively, if one takes $S = \{1, \ldots, n\} = 1 \in \mathcal{P}_n$.

Recall Remark 2.5, by which $\mathbb{Z}_2$ is a subgroup of $\mathcal{V}$.

If $|U|$ is odd then $M_U[j] \in \{0, 1\}$ if and only if $j \in U$ and $[j \in J_M(U)] = M_U[j] \cdot [j \in U]$ for $1 \leq j \leq n$, hence

$$|J_M(U)S|_2 = \sum_{j=1}^{n} [j \in S][j \in J_M(U)] = \sum_{j=1}^{n} [j \in S]M_U[j] = \sum_{j \in U} [j \in S]M_U[j].$$

If $|U|$ is even on the other hand, we get that $M_U[j] \in \{0, 1\}$ if and only if $j \not\in U$ and $[j \in J_M(U)] = M_U[j] \cdot [j \not\in U]$. In a similar fashion as above we have

$$|J_M(U)S|_2 = \sum_{j=1}^{n} [j \not\in S][j \in J_M(U)] = \sum_{j=1}^{n} [j \not\in S]M_U[j]$$

$$= \sum_{j \in U} [j \in S]M_U[j] + \sum_{j=1}^{n} [j \not\in S]M_U[j]$$

$$= \sum_{j \in U} [j \in S]M_U[j] + \sum_{i \in U} [i \not\in S] \sum_{j \in U} M_{ij}$$

$$= \sum_{j \in U} [j \in S]M_U[j] + \sum_{i \in U} \sum_{j \in S} M_{ij} = \sum_{j \in U} [j \in S]M_U[j] + \sum_{i \in U} \text{smr}^S_i(M).$$

Directly from the definition of HW-matrices and the above lemma we get:

**Corollary 6.5 ([13, Proposition 3]).** Let $M$ be a HW-matrix. Then:

1) $J_M(1) = 0$;

2) $J_M(U) \neq 0$ for $U \in \mathcal{P}_n \setminus \{0, 1\}$;

3) $J_M(U) = J_M(1 + U)$.

7 Spin$^c$ structures and HW-matrices

In this section we give a necessary and sufficient condition for existence of a spin$^c$ structure on a manifold defined by a HW-matrix. Let us note an easy lemma.

**Lemma 7.1.** Let $d \in \mathbb{N}$. A map $\kappa_A : \mathbb{Z}_2[x_1, \ldots, x_d] \to \text{Map}(\mathcal{P}_d, \mathbb{Z}_2)$ defined by

$$\kappa_A(x_i)(U) = [i \in U],$$

where $1 \leq i \leq d$ and $U \in \mathcal{P}_d$, is an algebra homomorphism.

We will use the following properties of the map $\kappa_A$:

**Lemma 7.2.** Let $d, n \in \mathbb{N}$ and $A \in \mathcal{V}^{d \times n}$. Then:

1) $\kappa_A$ is a monomorphism in gradation 2;
2) $\kappa_A(\theta_A^j)(U) = [j \in J_A(U)]$.

Proof. Let $\kappa = \kappa_A$ and

$$x = \sum_{1 \leq i < j \leq d} \alpha_{ij} x_i x_j \in \ker \kappa,$$

where $\alpha_{ij} \in \mathbb{Z}_2$. For any $1 \leq k < l \leq d$ and $U = \{k, l\}$ we have

$$0 = \kappa \left( \sum_{1 \leq i < j \leq d} \alpha_{ij} x_i x_j \right)(U) = \sum_{1 \leq i < j \leq d} \alpha_{ij} \kappa(x_i)(U) \cdot \kappa(x_j)(U) = \alpha_{kl},$$

hence $x = 0$.

Now take $1 \leq j \leq n$. We have

$$\theta_j = \alpha_j \beta_j = \left( \sum_{i=1}^{d} \alpha(A_{ij}) x_i \right) \left( \sum_{k=1}^{d} \beta(A_{kj}) x_k \right)$$

and in the consequence, for any $U \in \mathcal{P}_d$,

$$\kappa(\theta_j)(U) = \left( \sum_{i \in U} \alpha(A_{ij}) \right) \left( \sum_{k \in U} \beta(A_{kj}) \right).$$

Denote by $a, b, c, d$ the number of $0, 1, 2, 3$ in the rows from the set $U$ of $j$-th column of $A$, respectively. We get $\kappa(\theta_j)(U) = (b+c)(b+d) \mod 2$, but

$$(b+c)(b+d) \mod 2 = 1 \Leftrightarrow (b+c) \mod 2 = (b+d) \mod 2 = 1.$$ 

Hence $\kappa(\theta_j)(U) = 1$ if and only if

$$1 = (b+c) \cdot 2 + (b+d) \cdot 3$$
$$= b \cdot (2 + 3) + c \cdot 2 + d \cdot 3$$
$$= a \cdot 0 + b \cdot 1 + c \cdot 2 + d \cdot 3 = \text{smc}_j^U(A),$$

which by definition means, that $j \in J_A(U)$. $\square$

**Proposition 7.3.** Let $n > 1$ be an odd integer and let $A \in \mathcal{V}^n_{\mathbb{Z}_2}$ be distinguished. The following conditions are equivalent:

1. There exists $x \in H^1(C^n_{\mathbb{Z}_2})$ such that $x^2 + \text{sw}_2^A \in \text{span}\{\theta_1^A, \ldots, \theta_n^A\}$.
2. $\sigma_2 \in V_3 := \text{span}\{\theta_1^A - x_1^2, \ldots, \theta_{n-1}^A - x_{n-1}^2, \theta_n^A\}$, where $\sigma_2$ is the elementary symmetric polynomial of degree 2 in variables $x_1, \ldots, x_n$.
3. There exists $S \in \mathcal{P}_n$, such that for every $U \in \mathcal{P}_{n-1}$ the equality (in $\mathbb{Z}_2$) holds

$$|(J_A(U) + U)S|_2 = \binom{|U|}{2}, \quad (7.1)$$

Proof. We will omit the super and subscript $A$ in the proof.

Denote by $V$ the subspace of $\mathbb{Z}_2[x_1, \ldots, x_{n-1}]$ of polynomials of degree 2. Let $V_s$ and $V_f$ be subspaces of $V$ generated by monomials which are and are
not squares, respectively. Let $p: V \to V_f$ be the projection coming from the decomposition $V = V_s \oplus V_f$. Note that
\[ p(\theta_j) = \theta_j - x_j^2 \quad \text{and} \quad p(\theta_n) = \theta_n \]
for $1 \leq j < n$, hence condition 1 is equivalent to
\[ p(\text{sw}_2) \in \text{span}\{p(\theta_1^4), \ldots, p(\theta_n^4)\} = V_\delta, \quad (7.2) \]
but directly from the formula (3.1), since $n$ is odd, we have that $p(\text{sw}_2) = \sigma_2$.
Assume $1 \leq j \leq n$ and let $\delta_j := p(\theta_j)$. For $U \in \mathcal{P}_{n-1}$ we have that
\[ \kappa(\delta_j)(U) = [j \in J(U) + U]. \quad (7.3) \]
Indeed, if $j < n$, using Lemma 7.2 we get
\[
\kappa(\delta_j)(U) = \kappa(\theta_j + x_j^2)(U) = \kappa(\theta_j)(U) + \kappa(x_j^2)(U) = \kappa(\theta_j)(U) + \kappa(x_j)(U) = [j \in J(U)] + [j \in U] = [j \in J(U) + U].
\]
Additionally, $\delta_n = \theta_n$ and $n \not\in U$, hence
\[ \kappa(\delta_n)(U) = [n \in J(U)] = [n \in J(U)] + [n \in U] = [n \in J(U) + U]. \]
Suppose that $\sigma_2 = \sum s_j \delta_j \in V_\delta$ and let $S := \{j : s_j = 1\} \in \mathcal{P}_n$. For every $U \in \mathcal{P}_{n-1}$ we have
\[ \kappa(\sigma_2)(U) = \sum_{j=1}^{n} s_j \kappa(\delta_j)(U). \]
Since
\[ \kappa(\sigma_2)(U) = \sum_{1 \leq k < l < n} [l \in U][l \in U] = \sum_{k, l \in U, k < l} 1 = \binom{|U|}{2} \]
and
\[ \sum_{j=1}^{n} s_j \kappa(\delta_j)(U) = \sum_{j=1}^{n} [j \in S][j \in J(U) + U] = \sum_{j=1}^{n} [j \in S \cdot (J(U) + U)] = |S \cdot (J(U) + U)|, \]
formula (7.1) follows.
Now assume that (7.1) holds for some $S \in \mathcal{P}_n$ and every $U \in \mathcal{P}_{n-1}$. By the above calculations it may be written as
\[ \sum_{j=1}^{n} [j \in S][j \in J(U) + U] = \kappa(\sigma_2)(U). \]
Put $s_j = [j \in S]$ and use (7.3). The above equation takes the form
\[ \sum_{j=1}^{n} s_j \kappa(\delta_j)(U) = \kappa(\sigma_2)(U). \]
Recall that $U$ is any element of $P_{n-1}$. Using this and the linearity of $\kappa$, we get

$$\kappa \left( \sum s_j \delta_j \right) = \kappa(\sigma_2).$$

By Lemma 7.2, $\sigma_2 = \sum s_j \delta_j \in V_\delta$.

**Definition 7.4.** Let $n \in \mathbb{N}, M \in \mathcal{H}_n \times n$ and $S \in P_n$.

1. We call $S$ a spin$^c$ set for $M$ if for every $U \in P_n$ the equation

$$|(J_M(U) + U)S|_2 = \binom{|U|}{2}$$

holds;

2. We call $S$ an almost spin$^c$ set for $M$ if for every $U \in P_{n-1}$ equation (7.4) holds.

If $S$ is a spin$^c$ set for $M$, we call $(M, S)$ a spin$^c$ pair.

**Lemma 7.5.** Let $n \in \mathbb{N}$ be odd, $M \in \mathcal{H}_n$ and $S \in P_n$.

1. If $S$ is an almost spin$^c$ set for $M$, then $|S|_2^2 = \frac{n-1}{2}$.

2. If $S$ is an almost spin$^c$ set set for $M$, then it is a spin$^c$ set for $M$.

**Proof.** Take $U = \{1, \ldots, n-1\}$. By Lemma 6.4 and Corollary 6.5, $J(U) = J(1+U) = J(\{n\}) = \{n\}$. Hence $J(U) + U = \{1, \ldots, n\} = 1$, $(J(U) + U)S = S$ and we get

$$|S|_2 = |(J(U) + U)S|_2 = \binom{|U|}{2} = \binom{n-1}{2} = \frac{n-1}{2}.$$  

Note again, that all equations above are in $\mathbb{Z}_2$. In particular the last one holds, because $n$ is odd.

Assume now that $S$ is an almost spin$^c$ set for $M$. Equation (7.4) holds for every $U \in P_{n-1}$. It is enough to show that it also holds whenever $n \in U$. In that case however $V = 1 + U \in P_{n-1}$, so we have

$$|(J(V) + V)S|_2 = \binom{|V|}{2}.$$  

By Corollary 6.3, $J(V) = J(1+U) = J(U)$, hence

$$(J(U) + U)S = (J(V) + V + 1)S = (J(V) + V)S + S$$

and by linearity of $|\cdot|_2$ we have

$$|(J(U) + U)S|_2 = |(J(V) + V)S|_2 + |S|_2 = \binom{|V|}{2} + \frac{n-1}{2}$$

$$= \left(\frac{n-|U|}{2}\right) + \frac{n-1}{2} = \binom{|U|}{2},$$

where in the last equality we again use the fact, that $n$ is odd. \qed
**Theorem 7.6.** Let \( n \in \mathbb{N}, n \geq 5, M \in \mathcal{H}_n \) and let \( X \) be the HW-manifold defined by \( M \). The following conditions are equivalent:

1. \( X \) admits a spin\(^c\) structure.
2. There exists a spin\(^c\) set for \( M \).

**Proof.** By Lemma 7.5 existence of a spin\(^c\) and an almost spin\(^c\) set are equivalent conditions. Let \( A \) be a matrix composed from the first \( n-1 \) rows of \( M \). Clearly it is distinguished and by Remark 5.10, \( A \) is defining and effective matrix for \( X \). In order to get the desired equivalence, notice that for every \( U \in \mathcal{P}_{n-1} \) the equality

\[
J_A(U) = J_M(U)
\]

holds, use Theorem 4.1 and Proposition 7.3.

---

**8 Standard forms of spin\(^c\) pairs**

Recall that in Remark 5.10 we have defined the action of the group \( G_n = C_2 \wr S_n \) on the space \( V^{n \times n} \), for every \( n \in \mathbb{N} \). We will show that in fact it can act on spin\(^c\) pairs.

**Lemma 8.1.** Let \( n \in \mathbb{N}, M \in V^{n \times n}, S \in \mathcal{P}_n \) be such that \((M, S)\) is a spin\(^c\) pair. Then for every \( \sigma \in S_n \), \((\sigma M, \sigma S)\) is also a spin\(^c\) pair.

**Proof.** Let \( U \in \mathcal{P}_n \) and \( \sigma \in S_n \). Using an easy observation that \( J_{\sigma M}(U) = J_M(\sigma^{-1}U) \) and Lemma 5.1 we get

\[
\begin{align*}
|J_{\sigma M}(U) + U| &= |(\sigma J_M(\sigma^{-1}U) + U)| \\
&= |\sigma ((J_M(\sigma^{-1}U) + \sigma^{-1}U))S| \\
&= |(J_M(\sigma^{-1}(U)) + \sigma^{-1}(U))S| \\
&= |(\sigma^{-1}(U)) + \sigma^{-1}(U)| \\
&= \left( \begin{array}{c} |U| \\ 2 \end{array} \right).
\end{align*}
\]

Note, with the assumptions of the above lemma, that \( G_n \) acts on \( \mathcal{P}_n \) by permutations, using the canonical epimorphism \( G_n \to S_n \). Moreover, if \( g \in G_n \) is an element which acts by conjugations of columns only, then \( J_gM = J_M \), since \( \mathcal{T} = 1 \). We immediately get

**Corollary 8.2.** Let \( n \in \mathbb{N}, M \in V^{n \times n}, S \in \mathcal{P}_n \) be such that \((M, S)\) is a spin\(^c\) pair. Then for every \( g \in G_n \), \((gM, gS)\) is also a spin\(^c\) pair.

**Lemma 8.3.** Let \( n \in \mathbb{N} \) and \( M \in V^{n \times n} \) be distinguished and such that

\[
|J_M(U)|_2 = 1
\]

for every two-element set \( U \in \mathcal{P}_n \). Then there exists an integer \( k \), such that \( 2k \geq n \) and in the orbit \( G_n M \) there exists a matrix \( M' \) in the following block form

\[
M' = \begin{bmatrix} A & C \\ C^t & B \end{bmatrix},
\]

where \( A \) and \( B \) are self-conjugate of degree \( k \) and \( n - k \), respectively. Moreover

\[
\text{smr}_1(M') = \ldots = \text{smr}_k(M') \neq \text{smr}_{k+1}(M') = \ldots = \text{smr}_n(M').
\]
Proof. Since the matrix $M'$ is distinguished, by Lemma 5.7 we get that the set 
$\{ \text{smr}_i(M) : 1 \leq i \leq n \}$ has at most two elements. Let $l = |\{i : \text{smr}_i(M) = \text{smr}_1(M)\}|$. If $2l \geq n$ take $k = l$ and $M'' = M$. Otherwise, construct $M''$ by conjugation of the first column of $M$. We have \( \text{smr}_1(M'') = \text{smr}_1(M) \) and \( \text{smr}_i(M'') = \text{smr}_i(M) + 1 \) for $i > 1$. Letting $k = |\{i : \text{smr}_i(M'') = \text{smr}_1(M'')\}|$ we have $2k \geq n$.

There exists a permutation $\sigma \in S_n$, which fixes 1 and such that $M' = \sigma M''$ is of the block form 
\[
\begin{bmatrix}
A & C \\
D & B
\end{bmatrix},
\]
where $A, B$ are of degrees $k, n - k$ respectively and the equation (8.1) holds.

Let $U = \{i, j\}$ for $1 \leq i < j \leq n$. By our assumptions and Lemma 6.4 we have
\[
1 = M'_{ii} + M'_{ij} + M'_{ji} + M'_{jj} + \text{smr}_i(M') + \text{smr}_j(M')
\]
and hence
\[
M'_{ij} + M'_{ji} = \text{smr}_i(M') + \text{smr}_j(M') + 1 \tag{8.2}
\]
Consider two cases:

1. $j \leq k$ or $i > k$. Equation (8.2) gives us $M'_{ij} + M'_{ji} = 1$ and since $M'$ is distinguished, $M'_{ij} = M'_{ji}$. Hence $A$ and $B$ are self-conjugate.

2. $i \leq k < j$ and hence $\text{smr}_i(M') = \text{smr}_j(M') + 1$. Equation (8.2) gives us $M'_{ij} = M'_{ji}$, hence $D = C'$.

Lemma 8.4. Let $n \in \mathbb{N}$ and $M \in V_{n \times n}$ be distinguished in the following block form
\[
M = \begin{bmatrix}
A & C \\
C^t & B
\end{bmatrix},
\]
where $A, B$ are of degrees $k, l$, respectively. Assume that $k > 0$ and:

1) $A$ is self-conjugate;

2) $M_1 = [1, 2, \ldots, 2]$;

3) $J_M(\{1, i, j\}) \neq 0$ for $1 \leq i \leq k < j \leq n$.

Then $C$ consists only of elements equal to 2.

Proof. If $l = 0$, there is nothing to prove. Assume that $l > 0$, take $i \leq k$ and $j > k$. The principal submatrix of $M$ defined by indices $1, i, j$ is of the form:
\[
\begin{bmatrix}
1 & 2 & 2 \\
3 & 1 & x \\
2 & x & 1
\end{bmatrix}
\]
If $x = 3$ then $J_M(\{1, i, j\}) = 0$, contrary to our assumptions, hence $M_{ij} = x = 2$. Together with the form of $M_1$, we get the desired result.

Definition 8.5. Let $n \in \mathbb{N}, M \in V_{n \times n}$ and $S$ be a spin$^c$ set for $M$. We will say that the spin$^c$ pair $(M, S)$ is in standard form if:
1) \( S = \{1, \ldots , |S|\} \);
2) \( M_1 = [1, 2, \ldots , 2] \);
3) \( M \) is distinguished and in the block form

\[
\begin{bmatrix}
A & 2 & \ast \\
2 & B & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\]

with elements on the diagonal of degrees \( k, l, r \);
4) \( k \geq l \) and \( k + l = |S| \);
5) \( A, B \) are self-conjugate;
6) \( \text{smr}_S(M) = \text{smr}_{k+1}^S(M) \neq \text{smr}_{k+2}^S(M) = \ldots = \text{smr}_{k+l}^S(M) \) (it is possible that \( l = 0 \)).

We can deduce some further restrictions on a standard form of a matrix.

**Lemma 8.6.** Keeping the notation from the above definition, let \( (M, S) \) be a spin\(^c\) pair in the standard form and \( k + l < m \leq n \). Then, in the block form

\[
M_m = \begin{bmatrix} a & \pi & \ast \end{bmatrix},
\]

where \( a \in \{2, 3\} \) is such that the equation

\[
ka + l\pi + (k - l - 1)2 = a
\]

(8.3)

holds.

**Proof.** Let \( i \leq k + l \). Using the fact that \( (M, S) \) is a spin\(^c\) pair in the standard form and Lemma 6.4, for \( U = \{i, m\} \) we get

\[
1 = \left( \frac{|U|}{2} \right) = (|J_M(U) + U|S|_2) = |J_M(U)S|_2 + |US|_2
\]

\[
= [i \in S](M_i + M_{mi}) + [m \in S](M_{si} + M_{mm})
\]

\[
= \text{smr}_i^S(M) + \text{smr}_m^S(M) + [i \in S] + [m \in S]
\]

\[
= M_{mi} + \text{smr}_i^S(M) + \text{smr}_m^S(M)
\]

and hence

\[
M_{mi} = \text{smr}_i^S(M) + \text{smr}_m^S(M) + 1 = \begin{cases}
\text{smr}_i^S(M) + \text{smr}_m^S(M) + 1 & \text{if } i \leq k \\
\text{smr}_i^S(M) + \text{smr}_m^S(M) & \text{if } i > k
\end{cases}
\]

Since \( M \) is distinguished and \( i < m \), setting \( a := M_{m1} \) gives us desired form of the \( m \)-th row of \( M \). The equation (8.3) follows from the fact that \( \text{smr}_1^S = 1 + (k + l - 1)2 \) and \( \text{smr}_m^S = ka + l\pi \).

By the following lemma, certain spin\(^c\) pairs can be transformed to standard forms.
Lemma 8.7. Let \( n \geq 3 \) be an odd integer and \( M \in \mathcal{V}^{n \times n} \) be distinguished. Let \( S \) be a spin\(^c\) set for \( M \). If
\[
J_M(U) \neq 0 \text{ for } U \subset S \text{ and } |U| = 3,
\]
then there exists \( g \in G_n \) such that \((gM, gS)\) is a spin\(^c\) pair in a standard form.

Proof. By Corollary 8.2 \((gM, gS)\) is a spin\(^c\) pair for any \( g \in G_n \). Our goal is to show that \((M, S)\) can be transformed to a pair in the standard form.

By permuting indices and conjugating columns, we can transform \((M, S)\) to a form where \( S = \{1, \ldots, |S|\} \) and \( M_1 = [1, 2, \ldots, 2] \).

Let \( N \) be the principal submatrix of \( M \) defined on the set \( S \). \( N \) is distinguished and for every \( U \in \mathcal{P}(S) = \mathcal{P}_{|S|} \) we have
\[
|J_N(U) + U|_2 = |(J_N(U) + U)S|_2 = |(J_M(U) + U)S|_2 = \binom{|U|}{2},
\]
In particular, \( |J_N(U)|_2 = 1 \) if \( |U| = 2 \). Using Lemma 8.3 we can act on \( M \) by an element of \( G_{|S|} \subset G_n \) such that \( N \) becomes
\[
N = \begin{bmatrix} A & C \\ C^t & B \end{bmatrix},
\]
where \( A \) and \( B \) are self-conjugate of degrees \( k, l \) respectively, such that \( k \geq l \) and
\[
\text{sm}_{r_1}(N) = \ldots = \text{sm}_{r_k}(N) \neq \text{sm}_{r_{k+1}}(N) = \ldots = \text{sm}_{r_{k+l}}(N).
\]
Note that \( \text{sm}_{r_1}(N) = \text{sm}_{r_1}^{S_1}(M) \) for \( 1 \leq i \leq |S| = k + l \).

By assumption and Lemma 6.4 we have
\[
J_N(U) = J_M(U)S = J_M(U) \neq 0
\]
for \( U \subset S \) and \( |U| = 3 \). By Lemma 8.4 we get that \( C = 2 \) and hence the spin\(^c\) pair \((M, S)\) was transformed to a standard form. \( \square \)

9 Spin\(^c\) structures on HW-manifolds

By the results of previous sections we know that the existence of a spin\(^c\) structure on a HW-manifold is equivalent to the existence of a spin\(^c\) set for its HW-matrix. We will show that this never happens in dimensions greater than 3.

Lemma 9.1. Let \( n \geq 5 \) be an odd integer and \( M \in \mathcal{H}_n \). There does not exist a spin\(^c\) set \( S \) for \( M \) such that \(|S| = n|\).

Proof. If such a set \( S \) exists, then by our assumptions \( J_M(U) \neq 0 \) for \(|U| = 3\) and by Lemma 8.7 we can assume that \((M, S)\) is in a standard form:
\[
M = \begin{bmatrix} A & 2 \\ 2 & B \end{bmatrix},
\]
where the degrees of \( A, B \) equal \( k, l \) respectively, \( k \geq l \) and:
\[
\text{sm}_{r_1}(M) = \begin{cases} 1 & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}
\]
By definition of HW-matrices we have
\[ 0 = \sum_{j=1}^{n} \text{smc}_j(M) = \sum_{i=1}^{n} \text{smr}_i(M) = k \cdot 1, \]
hence \( k \) is even and in particular \( k < n \).

Let \( U = \{1, \ldots, k\} \). Since it is of even size, \( J_M(U) = 0 \) by Lemma 6.4.
Moreover, for every \( j > k \) we have
\[ M_U[j] = \sum_{i \in U} M_{ij} = \sum_{i \in U} 2 = 0. \]

Hence \( J_M(U) = 0 \). Contradiction with the fact that \( M \in \mathcal{H}_n \).

Lemma 9.2. Let \( n \geq 5 \) be an odd integer and \( M \in \mathcal{H}_n \). There does not exist a spin\(^c\) set \( S \) for \( M \) such that \(|S| = n - 1\).

Proof. Similarly as in the proof of the previous lemma we can assume that
\[ M = \begin{bmatrix} A & 2 & \ast \\ 2 & B & \ast \\ \ast & \ast & 1 \end{bmatrix} \]
where the matrices on the diagonal are of degrees \( k, l, 1 \) respectively, \( k \geq l \) and \( M_1 = [1, 2, \ldots, 2] \).

Since \( k + l = n - 1 \) is even, \( k = l \mod 2 \). By Lemma 5.6 we get
\[ M_n = [a, \pi, 1] \text{ and } k \cdot 1 + 2 = a. \]

If \( k \) is odd, then \( l \) is odd and \( a = 3 \). By definition of a HW-matrix, we get \( \text{smc}_i(B) = 1 \) for some \( k + 1 \leq i \leq k + l \) and
\[ 0 = \text{smc}_{k+i}(M) = k \cdot 2 + \text{smc}_i(B) + 2 = \text{smc}_i(B) = 1, \]
a contradiction.

Assume that \( k \) is even. Then \( l \) is even and \( a = 2 \). If \( l = 0 \), then \( M_n = [2, \ldots, 2, 1] \) and \( J_M(\{1, n\}) = 0 \), which cannot happen. Suppose \( l > 0 \). Take \( U = \{1, \ldots, k\}, V = \{k+1, \ldots, l\} \). They are both sets of even size. By the form of \( M \) and Lemma 5.7 we have
\[ M_U[i] \in \{2, 3\} \text{ and } M_V[i] = l \cdot 2 = 0 \text{ if } i \leq k \]
and
\[ M_U[i] = k \cdot 2 = 0 \text{ and } M_V[i] \in \{2, 3\} \text{ if } k < i < n. \]
Since \( M \) is a HW-matrix, we get \( M_U[n] = M_V[n] = 1 \), but then
\[ 0 = \text{smc}_n(M) = M_U[n] + M_V[n] + 1 = 1, \]
a contradiction.

Lemma 9.3. Let \( n \geq 5 \) be an odd integer and \( M \in \mathcal{H}_n \). There does not exist a spin\(^c\) set \( S \) for \( M \) such that \(|S| = n - 2\).
Proof. Similarly as in the previous two cases, we may assume that
\[ M = \begin{bmatrix} A & 2 & * & * \\ 2 & B & * & * \\ * & * & I & * \\ * & * & * & I \end{bmatrix}, \]
where the blocks on the diagonal are of degrees \( k, l, 1, 1 \), respectively and \( k \geq l \).
Since \( k + l = n - 2 \) is odd, \( k = l + 1 \mod 2 \). By Lemma 8.6 we have
\[ M_{n-1} = [a, \eta, 1, *] \] and \( k \cdot 1 + \eta = a, \)
hence \( k \cdot 1 = 1 \), \( k \) is odd and \( l \) is even.
Assume that \( M_n = [b, \eta, *, I] \). We have \( a \neq b \), otherwise
\[ M_{n-1} + M_n = [0, \ldots, 0, c, d], \]
where \( c, d \in \{2, 3\} \), hence \( J_M(\{n - 1, n\}) = 0 \).
For every \( i \leq k \) we get
\[ 0 = \text{smc}_i(M) = \text{smc}_i(A) + l \cdot 2 + 2 + 3 = \text{smc}_i(A) + 1, \]
hence \( \text{smc}_i(A) = 1 \). But by Lemma 6.1 matrix \( A \) cannot exist, a contradiction.

**Proposition 9.4.** Let \( n \geq 5 \) be an odd integer and \( M \in \mathcal{H}_n \). There does not exist a spin\(_c\) set for \( M \).

**Proof.** Let \( S \) be a spin\(_c\) set for \( M \). By Lemmas 9.1, 9.2, and 9.3 we can assume that \( |S| \leq n - 3 \). In this case there exists a set \( U \in \mathcal{P}_n \) of size 3 such that \( US = 0 \). By Lemma 6.3 \( J_M(U) \subset U \), hence \( (J_M(U) + U)S = 0 \). Since \( S \) is a spin\(_c\) set for \( M \), we have
\[ 0 = |(J_M(U) + U)S|_2 = \binom{|U|}{2} - \binom{3}{2} = 1, \]
a contradiction.

Finally we are ready to state the main result of the paper:

**Theorem 9.5.** Let \( X \) be a Hantzsche-Wendt manifold of dimension \( n \geq 5 \). Then \( X \) does not admit a spin\(_c\)-structure.

**Proof.** This follows directly from Theorem 7.6 and Proposition 9.4.

**References**

[1] S. Console, R.J. Miatello, and J.P. Rossetti. “\( \mathbb{Z}_2 \)-cohomology and spectral properties of flat manifolds of diagonal type”. In: *J. Geom. Phys.* 60.5 (2010), pp. 760–781. issn: 0393-0440. doi:10.1016/j.geomphys.2010.01.006

[2] K. Dekimpe, M. Sadowski, and A. Szczepański. “Spin structures on flat manifolds”. In: *Monatsh. Math.* 148.4 (2006), pp. 283–296. issn: 0026-9255. doi:10.1007/s00605-005-0387-3

20
[19] Andrzej Szczepański. “Aspherical manifolds with the $\mathbb{Q}$-homology of a sphere”. In: *Mathematika* 30.2 (1983), 291–294 (1984). ISSN: 0025-5793. DOI: [10.1112/S0025579300010561](https://doi.org/10.1112/S0025579300010561).

[20] P. Teichner and E. Vogt. “All 4-manifolds have Spinc structures”. In: unpublished note, available from the authors’ webpage. URL: [https://math.berkeley.edu/~teichner/](https://math.berkeley.edu/~teichner/)