Generic Stability and Modes of Convergence

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Abstract
We study generically stable types/measures in both classical and continuous logics, and their connection with randomization and modes of convergence of types/measures.

1 Introduction

Pillay and Tanović [PT11] introduced the notion of generically stable type, for arbitrary theories. The notion abstracts/expresses/preserves the crucial properties of definable types in stable theories, and definable and finitely satisfiable types in NIP theories. In [HPS13], this notion was correctly generalized to Keisler measures, for NIP theories. Assuming NIP, there are also equivalences on the notion, namely dfs, fam, and fim. The important question that remains is what is the “correct” notion of generic stability for measures in arbitrary theories. This seems to be more complicated than what is expected in the first encounter.

Recently [Kha22], it is shown that a global type $p$ in a countable theory is generically stable over a set $A$ if and only if $p$ is definable over $A$, AND there is a sequence $(c_i : i < \omega)$ inside $A$ such that $(tp(c_i/U) : i < \omega)$ converges to $p$, in a strong sense. The significance of this result is that

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1These are abbreviations for definable and finitely satisfiable measure, finitely approximated measure, and frequency interpretation measure, respectively. Cf. [CGH21] for definitions.
2DBSC-convergence. Cf. [Kha22, Def. 3.4] or Remark 2.10 below.
this notion is characterized by only a specific sequence in the model/set, and the information of all Morley sequences is encoded in that sequence so that we can call it the Morley sequence. On the other hand, it gives an analytical characterization of the notion which links it to other fields including topology/functional analysis/descriptive set theory. This suggests that a similar result holds for measures in arbitrary (countable) theories.

On the other hand, a line of research is randomization of first-order structures/theories. The randomization of a structure/theory was introduced by Keisler [Kei99] and formalized as a metric structure/theory by Ben Yaacov and Keisler [BK09]. Intuitively, for a (complete) classical theory the randomization $T^R$ of $T$ is a (complete) continuous theory such that elements of models of $T^R$ are random elements of models of $T$. A default theme is that model-theoretic properties are preserved in randomization. As an example $T$ is stable/NIP if and only if $T^R$ is so. It is known that “measures” in classical logic correspond to “types” in randomization. With these ideas in mind, and since the notion of “generic stability for types” is practically known even for continuous logic, we are looking for some properties of measures in classical logic which are close to generic stability of types in randomization.

The present paper aims to investigate ‘generic stability’ for measures in arbitrary theories. (We focus more on measures in classical logic which are not type, and types in continuous logic/randomization.) For this, we first generalize/adapt some results of [Kha22] on generically stable types to continuous logic, and then we transfer these results to measures in classical logic using randomization. We define the notion of $R$-generic stability, as a property of measures in classical logic, and show that a measure has this property if and only if a canonical random-type behaves like generically stable types in the sense of continuous logic. We will see that, in countable languages, every $R$-generically stable measure is a fam measure. On the other hand, we show that every fim measure is $R$-generically stable, and assuming a local variant of NIP, a global measure is fim if and only if its corresponding random-type is generically stable (over a model of the form $M \otimes A$). Finally, we refine some of the previous results/observations. In particular, we prove that a measure is $R$-generically stable iff its corresponding random-type is generically stable (over a model of the form $M \otimes A$).

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3 Although, there are counterexamples that this is not always the case. For example, randomization of an unstable simple theory is not simple [Ben13]. The present paper leads to other counterexamples.

4 Cf. Section 3.2, for definition of models of the form $M \otimes A$. 

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Let us give our motivation and background. First, we believe that model theory has a deep analytical nature that is not yet fully studied. In [KP18], [Kha20], [Kha19b], [Kha21a], [Kha21] and [Kha22], some analytical aspects of model theory/classification theory were studied. Roughly, it is shown that some model-theoretic notions appeared independently in topology/functional analysis/descriptive set theory, and moreover various characterizations yield the characterization of $NOP/NIP/NSOP$ in a model $M$ or set $A$, and some important theorems in model theory have twins there. Also, there are connections between classification in model theory and classification of Baire class 1 functions which lead to a better understanding of both of these topics [Kha19b]. The key idea is that the study of the model-theoretic properties of formulas in ‘models’ instead of only these properties in ‘theories’ develops a sharper stability theory and establishes important links between model theory and other areas of mathematics. In the present paper, we continue this approach and complete/generalize some results of [Kha21] and [Kha22]. Another work that has influenced this article is [G21], which an analytic aspect of generically stable types in (countable) theories is given (cf. [G21, Thm 4.8]). And of course, the note [Ben14] on finite satisfiability in randomization and [Ben09] on continuous VC-theory have influenced our work.

**Convention 1.1.** (1): Although many of our results can be generalized to uncountable case as well, in this paper, we study countable theories, classic or continuous. When we say $T$ is a theory we mean a countable classical theory, and in otherwise we say $T$ is a continuous theory; of course it is still countable. Also, all theories are complete.

(2): The monster model for a classical theory is denoted by $\mathcal{U}$ and for a continuous theory is denoted by $\tilde{\mathcal{U}}$. This distinct is important, especially when we pointed out that any model of the form $\mathcal{U} \otimes A$ is a strict substructure of $\mathcal{U}$.

(3): In this paper, when we say that $(a_i) \subset \mathcal{U}$ (or $(a_i) \subset \tilde{\mathcal{U}}$) is a sequence, we mean the usual notion in the sense of analysis. That is, every sequence is indexed by $\omega$. Similarly, we consider Morley sequences indexed by $\omega$.

(4): In this paper, a variable $x$ is a tuple of length $n$ (for $n < \omega$). Sometimes we write $\bar{x}$ or $x_1, \ldots, x_n$ instead of $x$. All types are $n$-type (for $n < \omega$) unless the difference between models of the form $M \otimes A$ and the other models are important in this paper. (Cf. Remark 3.13.)

Although all arguments are true for infinite variables, to make the proofs readable, we consider finite tuples.
explicitly stated otherwise. Similarly, a sequence \((a_i) \subset U\) (or \((a_i) \subset U\)) is a sequence of tuples of length \(n\) (for \(n < \omega\)).

This paper is organized as follows. In Section 2, we generalize/adapt some results of [Kha22] on generically stable types to continuous logic. We give characterizations of generic stability of continuous types in the terms of convergence of (Morley) sequences of types. We also show that generically stable continuous types are \(fim\). In Section 3, we introduce the notion \(R\)-generic stability for measures in classical logic. We show that a measure (in classical logic) is \(R\)-generically stable if and only if there is a canonical random-type which behaves like generically stable types (in the sense of continuous logic). We show the equivalence of \(fim\) measures and their corresponding random-types, assuming a local version of \(NIP\). In Section 4, we refine some of the previous results and prove that a measure is \(R\)-generically stable iff its corresponding random-type is generically stable.

2 Continuous logic and generic stability

In this section we introduce the notion of generic stability for types in continuous logic and give characterizations of this notion which will be used later. The proofs are adaptations of the arguments in classical logic [Kha22].

If the reader thinks that it is not necessary to state, repeat and review the results of classical case (i.e. [Kha22]) for continuous logic, we must give the following warnings: (a) Not all results of classical logic can be directly translated into continuous logic. For example, Shelah’s theorem (i.e. stable=\(NIP+NSOP\)) can be translated into continuous logic (cf. [Kha20a]). (b) It is important to have correct definitions in continuous logic and to check that the proofs works in this case, and why these proofs still work or do not. (c) The results in continuous case sometimes lead to new results even in classical logic, as we will see in Section 3 of the present paper and it was seen in [Kha20a].

Therefore, we introduce the notions in details but we will refer the proofs to the classical case and explain why arguments in classical case work here as well. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU08].

\footnote{This section can’t be read without firm grasp of [Kha22].}
In this section, we fix a continuous first order language $L$ a complete countable (continuous) $L$-theory $T$ (not necessarily NIP), and a small set $A$ of the monster model $U$. The space of global types in the variable $x$ is denoted by $S_x(U)$ or $S(U)$.

In the following, $\phi(\bar{x})$ is a formula, $r, s$ are real numbers in $[0, 1]$, and $\phi(\bar{x}) \leq r$, $\phi(\bar{x}) \geq r$, and $\phi(\bar{x}) = r$ are $L$-statements (or $L$-conditions) in continuous logic.

Recall that, in continuous logic, we use $\sup$, $\inf$, $\min$, $\max$, $\leq$ instead of $\forall$, $\exists$, $\land$, $\lor$, $\neg$, respectively. However, we will sometimes continue to use classic symbols to make our article more readable.

**Remark 2.1.** Notice that $L$ is countable and we assume that $M$ is a separable model (i.e., there is a countable dense subset $M_0 \subseteq M$). We work with a countable system of connectives containing $0, 1, \min, \max, \cdot, /, ^{-}$ (where $x^{-}y = \max(x - y, 0)$). Recall from [BBHU08, Thm. 6.3] that this does not impose any restrictions. In this paper, we can work with $L(M_0)$-statements of the form $\phi(\bar{x}) \leq r$ and $\phi(\bar{x}) \geq s$ where $r, s$ are rational numbers in $[0, 1]$. Therefore, the set of all statements are countable and all diagonal arguments in [Kha22] work well in the present paper.

**Definition 2.2.** Let $A \subset U$ and $\phi(x_1, \ldots, x_n) \in L(A)$. We say that $\phi(x_1, \ldots, x_n)$ is symmetric if for any permutation $\sigma$ of $\{1, \ldots, n\}$,

$$\sup_{\bar{x}} \left| \phi(x_1, \ldots, x_n) - \phi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \right| = 0.$$  

**Example 2.3.** Let $\phi(x, y)$ be a formula, and $r < s$. The following formula $\theta_{\phi,n}^{r,s}(x_1, \ldots, x_n)$ is symmetric:

$$\forall F \subseteq \{1, \ldots, n\} \exists y_F \left( \bigwedge_{i \in F} \phi(x_i, y_F) \leq r \land \bigwedge_{i \notin F} \phi(x_i, y_F) \geq s \right).$$

This formula should play a key role in the arguments in the present paper. See Remark 2.4 and Theorem 2.7 below, and also [Kha22, Thm. 2.8].

**Remark 2.4.** Let $(a_i : i < \omega)$ be an indiscernible sequence of $x$-types and $\phi(x, y)$ a formula. Then, it is easy to check that the following are equivalent:

As mentioned earlier, in this paper, the language is countable, however we can consider separable languages. Also, we can study separable fragments of the language. To make the proofs more readable, we assume that the theory is countable.
(i) For any parameter $b \in U$, the truth value of sequence $(\phi(a_i, b) : i < \omega)$ is eventually constant.

(ii) For any $r < s$ there is a natural number $n$ such that $\models \theta_{\phi,n}^r (a_1, \ldots, a_n)$.

The following is an adaptation/generalization of [Kha22, Def. 2.3].

**Definition 2.5.** (i) Let $(b_i)$ be a sequence of $U$ and $A \subset U$ a set. The eventual Ehrenfeucht-Mostovski type (abbreviated EEM-type) of $(b_i)$ over $A$, which is denoted by $EEM((b_i)/A)$, is the following (partial) type in $S_\omega(A)$:

$$(\phi(x_1, \ldots, x_k) = r) \in EEM((b_i)/A) \iff \lim_{i_1 < \cdots < i_k, i_1 \to \infty} \phi(b_{i_1}, \ldots, b_{i_k}) = r.$$  

(ii) Let $(b_i)$ be a sequence of $U$ and $A \subset U$ a set. The symmetric eventual Ehrenfeucht-Mostovski type (abbreviated SEEM-type) of $(b_i)$ over $A$, which is denoted by $SEEM((b_i)/A)$, is the following partial type in $S_\omega(A)$:

$$\{ \phi(x_1, \ldots, x_k) = r : (\phi(\bar{x}) = r) \in EEM((b_i)/A) \text{ and } \phi \text{ is symmetric} \}.$$  

Whenever $(b_i)$ is $A$-indiscernible, we sometimes write $SEM((b_i)/A)$ instead of $SEEM((b_i)/A)$.

(iii) Let $p(x)$ be a type in $S_\omega(A)$ (or $S_\omega(U)$). The symmetric type of $p$, denoted by $\text{Sym}(p)$, is the following partial type:

$$\{ (\phi(x) = r) \in p : \phi \text{ is symmetric} \}.$$  

The sequence $(b_i)$ is called eventual indiscernible over $A$ if $EEM((b_i)/A)$ is a complete type. In this case, for any $L(A)$-formula $\phi(x)$, the limit $\lim_{i \to \infty} \phi(b_i)$ is well-defined.

Let $p(x)$ be a global $A$-invariant type. The Morley type (or sequence) of $p(x)$ can be easily defined similar to classical logic. The Morley type (or sequence) of $p(x)$ is denoted by $p^{(\omega)}$. The restriction of $p(x)$ to $A$ is denoted by $p|_A$. A realisation $(d_i : i < \omega)$ of $p^{(\omega)}|_A$ is called a Morley sequence of/in $p$ over $A$.

**Lemma 2.6.** Let $p(x) \in S(U)$ be finitely satisfiable in $M$ where $M$ is separable model. Then there is a sequence $(c_i)$ in $M$ such that $SEEM((c_i)/M) = \text{Sym}(p^{(\omega)}|_M)$.

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9Cf. [S15], subsection 2.2.1 for definition in classical case.

10We can assume that $M$ is a separable set in some model.
Proof. Let $I = (d_i)$ be a Morley sequence in $p$ over $M$. Since $T$ is countable and $M$ is separable, there is a sequence $(c_i)$ in $M$ such that $\lim tp(c_i/M|I) = p|MI$.\footnote{Notice that, in this case, the space $S(MI)$ is metrizable.} We can assume that $(c_i)$ is eventual indescirnable over $MI$. That is, the type $EEM((c_i)/MI)$ is complete. (Notice that, as $MI$ is separable and $T$ is countable, using Ramsey’s theorem and a diagonal argument, there is a subsequence of $(c_i)$ which is eventual indescirnable over $MI$. See also Remark 2.7 above.) We claim that $(c_i)$ is the desirable sequence. That is, $SEEM((c_i)/M) = SEM((d_i)/M) = \text{Sym}(p^{\omega}|M)$. The proof is by induction on symmetric formulas similar to \cite{Kha22} Lemma 2.4]. \hfill \Box

We say that a sequence $(d_i) \in U$ of $x$-tuples converges (or is convergent) if the sequence $(tp(d_i/U) : i < \omega)$ converges in the logic topology. Equivalently, for any $L(U)$-formula $\phi(x)$, the sequence $(\phi(d_i) : i < \omega)$ is convergent, i.e. for any $\epsilon > 0$ there is an natural number $n$ such that $|\phi(d_i) - \phi(d_j)| < \epsilon$ for all $i, j \geq n$. If $(tp(d_i/U) : i < \omega)$ converges to a type $p$, then we write $\lim tp(d_i/U) = p$ or $tp(d_i/U) \to p$. Notice that $tp(d_i/U) \to p$ iff for any $L(U)$-formula $\phi(x)$,

$$ (\phi(x) = r) \in p \iff \lim \phi(d_i) = r. $$

**Theorem 2.7.** Let $T$ be a separable continuous theory, $M$ a separable model, and $p(x) \in S(U)$ a global type which is finitely satisfiable in $M$. Let $(d_i)$ be a Morley sequence of $p$ over $M$. If $(d_i)$ converges then there is a sequence $(c_i)$ in $M$ such that $tp(c_i/U) \to p$.

**Proof.** The proof is an adaptation of the argument of Theorem 2.8 of \cite{Kha22}. Indeed, by the argument of Lemma 2.6, we can assume that there is a sequence $(c_i)$ in $M$ such that $tp(c_i/M\cup(d_i)) \to p|MI(d_i)$ and $SEEM((c_i)/M) = \text{Sym}(p^{\omega}|M)$. We show that $tp(c_i/U) \to p$. Let $q$ be an accumulation point of $\{tp(c_i/U) : i \in \omega\}$. Then $q|MI(d_i) = p|MI(d_i)$. Therefore, by an easy induction, we can show that the Morley types (sequences) $p^{\omega}|M$ and $q^{\omega}|M$ are the same. Now, as $(d_i)$ converges, it is easy to show that $p = q$. (If not, using an standard argument, one can build a Morley sequence $(a_i)$ such that for a parameter $b \in U$, a formula $\phi(x, y)$, and for $r < s$, we have $\phi(a_i, b) < r$ if $i$ is even, and $\phi(a_i, b) > s$ in otherwise. This contradicts the convergence of $(d_i)$. Cf. the Claim 1 in \cite{Kha22}. Also, using Rosenthal’s lemma (cf. \cite{Kha22}, Fact 2.7)\footnote{Notice that Rosenthal’s lemma works in real-valued case too.}, similar to the Claim 2 in \cite{Kha22} Thm. 2.8, we can show that
the sequence \((tp(c_i/U) : i < \omega)\) converges. If not, there are a formula \(\phi(x, y)\), \(r < s\), and a Morley sequence \((a_i : i < \omega)\) such that for any \(n\) the formula \(\theta_{\phi,n}^{r,s}\) holds in this Morley sequence. (Cf. Example 2.3.) This means that \((a_i : i < \omega)\) diverges, a contradiction. These prove the theorem.

The following is the counterpart of Definition 3.1 to continuous logic.

**Definition 2.8.** Let \(M\) be a model and \((p_n(x) : n < \omega)\) be a sequence of types over \(M\). We say that \((p_n : n < \omega)\) Baire-1/2-converges (or is Baire-1/2-convergent) if the independence property is semi-uniformly blocked on \((p_n : n < \omega)\), that is, for any formula \(\phi(x, y)\), and for each \(r < s\), there is a natural number \(N = N_{r,s}^\phi\) and a set \(E \subset \{1, \ldots, N\}\) such that for each \(i_1 < \cdots < i_N < \omega\), and any parameter \(b \in M\), the following does not hold

\[
\bigwedge_{j \in E} (\phi(x, b) \leq r) \in p_{i_j} \land \bigwedge_{j \notin N \setminus E} (\phi(x, b) \geq s) \in p_{i_j}.
\]

In the above, we can assume that \(M\) is the monster model.

**Remark 2.9.** Notice that in Theorem 2.7, the sequence \((tp(c_i/U) : i < \omega)\) is Baire-1/2-convergent. In fact, as \(SEEM((c_i)/M) = Sym(p^{(\omega)}|_M)\), any/some Morley sequence is convergent if and only if \((tp(c_i/U) : i < \omega)\) is Baire-1/2-convergent. In particular, if the Morley sequence of \(p\) is totally indiscernible (e.g. \(p\) is generically stable),

\[
EEM((c_i)/M) = SEEM((c_i)/M) = Sym(p^{(\omega)}|_M) = p^{(\omega)}|_M.
\]

This means that Morley sequences of \(p\) are controlled by the sequence \((c_i) \in M\), and vice versa. We will shortly see that this fact leads to a new and useful characterization of the following notion, namely generic stability.

**Remark 2.10.** Recall that, in classical logic, the types are \(\{0, 1\}\)-valued. In this case, we say that the sequence \((p_n : n < \omega)\) of types is DBSC-convergent (or DBSC-converges) if it is Baire-1/2-convergent as in Definition 2.8. When \((p_n : n < \omega)\) is DBSC-convergent, we say the independence property is uniformly blocked on \((p_n : n < \omega)\). (Notice that, in this case, for any \(r < s\) and formula \(\phi(x, y)\), the natural number \(N\) (in Definition 2.8) depend just on \(\phi\).) Recall from [Kha21a] that these notions of convergence are related to different subclasses of Baire-1 functions. In fact, the limit of a DBSC-convergent (resp. Baire-1/2-convergent) sequence is a DBSC (resp. Baire-1/2) function, and the class of DBSC functions is a proper subclass of Baire-1/2 functions.
The following is the natural/correct adaptation of generic stability from [PT11] to continuous logic (cf. Remark 2.12(v) below).

**Definition 2.11** (Generic stability). Let $T$ be a continuous theory and $A$ a small set of the monster model. A global type $p(x)$ is generically stable over $A$ if $p$ is $A$-invariant and any Morley sequence $(a_i : i < \omega)$ of $p$ over $A$ is totally indiscernible AND it has no order; that is, there is no a sequence $(b_i : i < \omega)$, a formula $\phi(x,y)$, and $r < s$ such that $\phi(a_i, b_j) \leq r$ if $i < j$ and $\phi(a_i, b_j) \geq s$ in otherwise.

**Remark 2.12.** Let $p(x)$ be a global $A$-invariant type. The following are equivalent.

(i) $p$ is generically stable over $A$.

(ii) $p$ is $A$-invariant and for any Morley sequence $(a_i : i < \omega)$ of $p$ over $A$, we have $\lim tp(a_i/\mathcal{U}) = p$.

(iii) The condition (ii) holds and furthermore, the sequences $(tp(a_i/\mathcal{U}) : i < \omega)$ are Baire-1/2-convergent.

(iv) Any Morley sequence of $p$ is totally indiscernible AND convergent.

(v) For any Morley sequence $(a_i : i < \alpha)$ (any $\alpha$, not only $\omega$) of $p$ over $A$, any formula $\phi(x)$ (with parameters from $\mathcal{U}$) and $r < s$, at least one of \{i : \models \phi(a_i) \leq r\} or \{i : \models \phi(a_i) \geq s\} is finite.

**Proof.** (i) $\iff$ (ii) is a straightforward generalization of Proposition 3.2 of [CG20].

(iii) follows from (ii) and the indiscernibility of Morley sequences. Indeed, if some Morley sequence is not Baire-1/2-convergent, then there is a Morley sequence which is NOT convergent, a contradiction. (Cf. [Kha22] for classical case.)

(i) $\iff$ (iv) is a straightforward generalization of Lemma 4.2 of [Kha22].

(iv) $\iff$ (v): Easy. 

The following theorem gives a characterizations of generically stable types for countable theories. The important one to note immediately is (ii).

**Theorem 2.13.** Let $T$ be a continuous theory, $M$ a small model of $T$, and $p(x) \in S(\mathcal{U})$ a global $M$-invariant type. The following are equivalent:

(i) $p$ is generically stable over $M$.

(ii) $p$ is definable over a small model, AND there is a sequence $(c_i)$ in $M$ such that $(tp(c_i/\mathcal{U}) : i < \omega)$ Baire-1/2-converges to $p$.

(iii) $p$ is definable over and finitely satisfiable in some small model, AND there is a convergent Morley sequence of $p$ over $M$. 


Proof. The proof is an adaptation of the argument of Theorem 4.4 of [Kha22]. First, notice that, similar to classical case, we can assume that $M$ is separable.

(i) $\implies$ (ii) follows from Theorem 2.7. (See also Remark 2.9.)

(ii) $\implies$ (i): As $p$ definable and finitely satisfiable, any Morley sequence is totally indiscernible. Let $(d_i)$ be a Morley sequence. Similar to Lemma 2.6, it is easy to see that $\text{SEEM}((c_i)/M) = \text{Sym}(tp((d_i)/M))$. Therefore, as $(c_i)$ is Baire-1/2-convergent, the Morley sequence $(d_i)$ converges. By Remark 2.12(iv), $p$ is generically stable.

(i) $\implies$ (iii) follows from Remark 2.12(iv) and the fact that the Morley sequences of definable and finitely satisfiable types are totally indiscernible.

(i) $\implies$ (iii) is evident. \qed

The following is a consequence of the previous results, although it has not been stated anywhere before, even for classical logic.

**Corollary 2.14.** Let $T$ be a continuous theory, $M$ a small model of $T$, and $p(x) \in S(U)$ a global $M$-invariant type. The following are equivalent:

(i) $p$ is generically stable over $M$.

(ii) There is a sequence $(c_i)$ in $M$ such that $(tp(c_i/U) : i < \omega)$ Baire-1/2-converges to $p$, AND $(c_i)$ has no order, that is, there is no $(b_i) \in U$, $r < s$, and formula $\phi(x,y)$ such that $\phi(a_i,b_j) \leq r$ if $i < j$ and $\phi(a_i,b_j) \geq s$ in otherwise.

Proof. This follows from Theorem 2.13 AND Grothendieck’s double limit theorem (cf. [Kha19b, Fact 2.2]). \qed

**Continuous types and $fim$**

In this section we introduce the notion $fim$ to continuous types and show that any generically stable type in continuous logic is $fim$.

Let $\phi(x,y)$ be a formula, and $\bar{a} = (a_1, \ldots, a_n)$ and $b$ parameters. Then we define $Av(\bar{a})(\phi(x,b)) := \frac{1}{n} \sum_{i=1}^{n} \phi(a_i,b)$.

**Definition 2.15 (Continuous $fim$).** Let $p(x)$ be a continuous type. We say that $p$ is $fim$ if for any formula $\phi(x,y)$ and $\epsilon > 0$, there is a statement $\theta(x_1, \ldots, x_n)$ such that:

(i) $\theta(x_1, \ldots, x_n) \in p^{(n)}$, and

(ii) for all $\bar{a} \models \theta(x_1, \ldots, x_n)$ we have

$$\sup_{b \in U} |p((\phi(x,b))) - Av(\bar{a})(\phi(x,b))| \leq \epsilon.$$
Lemma 2.16. Let $p(x)$ be a continuous type. Suppose that its Morley type/sequence $p^\omega$ is totally indiscernible AND convergent. Then for any formula $\phi(x, y)$ and $\epsilon > 0$ there is a natural number $n_{\phi, \epsilon}$ such that for any Morley sequence $(a_i : i < \omega) \models p^\omega$ we have:

for any $b \in U$, the number of $i$ such that $|\phi(a_i, b) - \lim_i \phi(a_i, b)| > \epsilon$ is $\leq n_{\phi, \epsilon}$.

Proof. If not, using total indiscernibility and compactness, one can find a divergent Morley sequence, a contradiction. □

Proposition 2.17. Let $p(x)$ be a continuous type. If $p$ is generically stable, then it is $\text{fim}$. 

Proof. The proof is an adaptation of Proposition 3.2 of [CG20]. As $p$ is definable, for any formula $\phi(x, y)$ and any $\epsilon > 0$, there is a formula $\psi_{\phi, \epsilon} = \psi(y)$ (with parameters), which is a finite continuous combination of the instances of $\phi(a, y)$, such that $\sup_{b \in U} |p((\phi(x, b))) - \psi(b))| \leq \epsilon$. On the other hand, as $p$ is generically stable, its Morley sequence is totally indiscernible AND convergent. By Lemma 2.16 there is a natural number $n_{\phi, \epsilon} = N_{r, s}^\phi$ (with $|r - s| \leq \epsilon$) as in Definition 2.8 such that for any Morley sequence $(a_n : n < \omega)$ of $p$, there is a set $E \subset \{1, \ldots, n_{\phi, \epsilon}\}$ such that for each $i_1 < \cdots < i_{n_{\phi, \epsilon}} < \omega$, and any parameter $b$, the following does not hold

$\bigwedge_{j \in E} (\phi(a_{i_j}, b) \leq r) \land \bigwedge_{j \in n_{\phi, \epsilon} \setminus E} (\phi(a_{i_j}, b) \geq s).$

(Notice that $n_{\phi, \epsilon} = N_{r, s}^\phi$ for all $r, s$ with $|r - s| \leq \epsilon$.) Recall that any Morley sequence $(a_n : n < \omega)$ is totally indiscernible, and $\lim tp(a_n/U) = p$.

Define the statement $\theta(x_1, \ldots, x_n)$ as follows:

$\forall y \bigwedge_{I \subset n, |I| > n_{\phi, \epsilon}} \left( \exists J < I, |J| \geq |I| - n_{\phi, \epsilon} \left( \bigwedge_{i \in J} |\phi(x_i, y) - \psi(y)| \leq \epsilon \right) \right).$

It is easy to verify that for big enough $n \gg n_{\phi, \epsilon}$, $\theta(x_1, \ldots, x_n) \in p^{(n)}$, AND the condition (ii) of Definition 2.15 holds. □

3 Measures and random types

In this section we first introduce the notion of $R$-generic stability, as a property of a measure in classical logic, and then using the results of Section 2 we
study this notion and related random-types (in the sense of continuous logic) and their connections. We study two variants of random-types related to a measure, namely the natural extension (Section 3.2) and the corresponding random-type (Section 3.3). We study their relationship with fam and fim measures.

3.1 R-generic stability

In this section we introduce the notion of $R$-generically stable measure in classical logic, and show that this notion and its corresponding notion in NIP theories [HPS13] coincide. We show that every $R$-generically stable measure is a fam measure.

In this section, we fix a (classical) first-order language $L$, a complete countable $L$-theory $T$ (not necessarily NIP), and a small set $A$ of the monster model $U$. The space of global types in the variable $x$ is denoted by $S_x(U)$ or $S(U)$.

We first give a notion/notation. Let $\mu(x)$ be a global measure, $r_1, \ldots, r_k \in [0,1]$ such that $\sum r_i = 1$. The measure $\mu^{\sum r_i}$ is defined as follows: for any formula $\phi(x,y)$, and any parameters $b_1, \ldots, b_k$,

$$\mu^{\sum r_i}(\phi; (b_i)^k) := \sum_1^k r_i \cdot \mu(\phi(x, b_i)).$$

The following notion is technical but we will shortly see that it is the natural generalization of the corresponding notion in the case of type.

**Definition 3.1.** Let $(\mu_n(x) : n < \omega)$ be a sequence of global $A$-invariant measures. We say that $(\mu_n : n < \omega)$ is randomly convergent (or randomly converges) if for any formula $\phi(x, y)$ and for each $r < s$, there are a natural number $N = N_{r,s}$ and a set $E \subset \{1, \ldots, N\}$ such that for any $r_1, \ldots, r_k \in [0,1]$ with $\sum r_t = 1$ and any parameters $b_1, \ldots, b_k$ and for each $i_1 < \cdots < i_N < \omega$, the following does not hold

$$\bigwedge_{j \in E} \mu^{\sum r_i}(\phi; (b_i)^k) \leq r \land \bigwedge_{j \in N \setminus E} \mu^{\sum r_i}(\phi; (b_i)^k) \geq s. \quad (*)$$

**Remark 3.2.** (i): In the above definition, if $(*)$ holds just for $k = 1$, then we say that $(\mu_n : n < \omega)$ is Baire-1/2-convergent (or Baire-1/2 converges).
A question arise. Is every Baire-1/2-convergent sequence randomly convergent? We will return to this question in Section 4. (Cf. Remark 4.8)

(ii): It is easy to verify that:

\[ \mu_n: n < \omega \]

randomly converges if and only for any formula \( \phi(x, y) \) and for each \( r < s \), there are a natural number \( N = N_{r,s}^\phi \) and a set \( E \subset \{1, \ldots, N\} \) such that for any \( r_1, \ldots, r_k \in [0, 1] \) with \( \sum r_i = 1 \) the sequence \( (\mu_n^{\sum r_i} : n < \omega) \) Baire-1/2 converges (with respect to \( N, E \) fixed above). This is important in the proof of Theorem 3.12 below.

Definition 3.3. (i) Let \( \mu(x) \) be a global measure. We say that \( \mu \) is \( R \)-generically stable over \( A \) if \( \mu \) is definable over \( A \), AND there is a sequence \( (\mu_n : n < \omega) \) of measures such that:
- \( \mu_n = \frac{1}{k_n} \sum_{i=1}^{k_n} p_{n,i} \) where \( a_{n,i} \models p_{n,i} \) and \( a_{n,i} \in A \) (for all \( n \)), and
- \( (\mu_n : n < \omega) \) randomly converges to \( \mu \).

(ii) Let \( p(x) \) be a global type. We say that \( p \) is \( R^{\text{type}} \)-generically stable over \( A \) if \( p \) is definable over \( A \), AND there is a sequence \( (p_n : n < \omega) \) of types such that:
- \( a_n \models p_n \) and \( a_n \in A \) (for all \( n \)), and
- \( (p_n : n < \omega) \) randomly converges to \( p \).

Remark 3.4. (1): Notice that for a type \( p \), the conditions in (ii) implies the conditions in (i), but there is no reason for the converse to be true.

(2): On the other hand, we can make a more subtle distinction. Indeed, for any finite set \( F \) of real numbers in \( [0, 1] \), we can give a generalization of Definition 3.3(ii) as follows. Let \( \mu(x) \) be a global \( F \)-valued measure. We say that \( \mu \) is \( R^{\text{type}} \)-generically stable over \( A \) if \( p \) is definable over \( A \), AND there is a sequence \( (\mu_n : n < \omega) \) of \( F \)-valued measures such that:
- \( \mu_n \) is a \( F \)-valued measures of a convex combination of types realized in \( A \) (for all \( n \)), and
- \( (\mu_n : n < \omega) \) randomly converges to \( \mu \).

Although we will not study it in this article, we believe that for a fixed \( F \), the latter notion and the usual notion of generically stable type are of the same nature and complexity.

First we show that, for types, \( R^{\text{type}} \)-generic stability and the usual notion of generic stability coincide. The following is a new characterization of generic stability which is interesting in itself.

Fact 3.5. Let \( p(x) \) be a global \( A \)-invariant type. The following are equivalent:
(i) $p$ is generically stable over $A$ (as in [PT11]).
(ii) $p$ is $R^{type}$-generically stable over $A$ (as in Definition 3.3).

Proof. We use a result from [Kha22], namely Theorem 4.4. Using this theorem, $p$ is generically stable (in the usual sense) iff it is definable AND there is a sequence $(a_n) \in A$ such that $(tp(a_n/\mathcal{U}) : n < \omega)$ DBSC-converges to $p$.
(Cf. Remark 2.10, for definition of DBSC-convergent. Notice that for types Baire-1/2 convergent (as in Remark 3.2) and DBSC-convergent coincide.) It is easy to check that if $(tp(a_n/\mathcal{U}) : n < \omega)$ randomly converges, then it DBSC-converges. Conversely, if $(tp(a_n/\mathcal{U}) : n < \omega)$ DBSC-converges, then by Lemma 2.8 in [Kha19b], for any formula $\phi(x,y)$ there is a natural number $m$ such that for any parameters $b_i$,

$$\sum_{n=1}^{\infty} |\phi(a_n, b) - \phi(a_{n+1}, b)| \leq m.$$ 

Now, by the triangle inequality, it is easy to check that for any $r_1, \ldots, r_k \in [0,1]$ with $\sum r_i = 1$ and parameters $(b_i)_1^k$, we have

$$\sum_{n=1}^{\infty} \left| \sum_{i=1}^{k} r_i \cdot \phi(a_n, b_i) - \sum_{k=1}^{k} r_i \cdot \phi(a_{n+1}, b_i) \right| \leq m. \quad (\dagger)$$

This implies that $(tp(a_n/\mathcal{U}) : n < \omega)$ randomly converges. Indeed, suppose for a contradiction that there are parameters $(b_i)_1^k$, natural number $N > m/|r-s|$, $r_1, \ldots, r_k \in [0,1]$ with $\sum r_i = 1$, and $j_1 < \ldots < j_N < \omega$ such that $\mu^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) \leq r}$ if $t$ is odd, and $\mu^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) \geq s}$ if $t$ is even. (Here $\mu^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) \leq t} = \sum_{i=1}^{r_i} r_i \cdot \phi(a_n, b_i).$) Then

$$\sum_{n=1}^{\infty} \left| \mu_{n+1}^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) - \mu_{n}^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) \leq s} \right| \geq \sum_{i=1}^{N} \left| \mu_{j_t}^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) - \mu_{j_t}^{\sum_{i=1}^{r_i}(\phi; (b_i)_1^k) \geq s} \right| \geq N \cdot |r-s| > m.$$ 

This contradicts $(\dagger)$. (The inequality $(\ast)$ can be clearly verified and confirmed.)

Recall that the notion of generically stable measures for NIP theories was introduced and studied in [HPS13]. The next observation show that the new and usual notions coincide, in NIP theories.
Proposition 3.6. (Assuming \( T \) is NIP.) Let \( \mu \) be a global measure. Then \( \mu \) is \( R \)-generically stable over \( A \) (as in Definition 3.3(i)) iff it is generically stable over \( A \) (as in [HPS13]).

Proof. Recall that, in NIP theories, any definable and finitely satisfiable measure is generically stable. Therefore, this follows from the definition. (In particular, for types, \( R \)-generic stability and \( R^{\text{type}} \)-generic stability are the same.)

We need to recall from [G21 Definition 3.1] the notion of sequential approximation of measures. Let \( \mu(x) \) be a global measure. We say that \( \mu \) is sequentially approximated over \( A \) if there is a sequence \( (\mu_n : n < \omega) \) of measures such that:

- \( \mu_n = \frac{1}{k_n} \sum_{i=1}^{k_n} p_i \) where \( a_i \models p_i \) and \( a_i \in A \) (for all \( n \)), and
- \( (\mu_n : n < \omega) \) converges to \( \mu \).

Notice that convergence is in the logic topology or equivalently the topology of pointwise convergence. (Cf. [G21], Definition 2.1.)

Proposition 3.7. Let \( \mu(x) \) be a global measure. If \( \mu \) is \( R \)-generically stable over \( A \), then \( \mu \) is \( \text{fam} \) over \( A \).

Proof. This follows from Proposition 3.4 of [G21]. First notice that, as \( T \) is countable and \( \mu \) is definable, we can assume that \( A \) is countable. Clearly, by definition, \( \mu \) is sequentially approximated over \( A \). As \( \mu \) is definable and sequentially approximated, Proposition 3.4 of [G21] implies that \( \mu \) is \( \text{fam} \).

The above observation shows that the notion of \( R \)-generic stability is strong enough for the purposes of the present paper. In the next section we study connections between \( R \)-generic stability and random-types in randomization.

Remark 3.8. We believe that \( R \)-generic stability is strictly stronger than \( \text{fam} \). In [Kha22], we make a similar claim: generic stability (for types) is strictly stronger than \( \text{sad} \). (Recall from [G21] that a type is called \( \text{sad} \) if it is both sequentially approximated and definable. Notice that sequential approximation for types and measures are differently defined in his paper.) Although, in [Kha22] we suggested an example of a non-generic and \( \text{sad} \) type,

\[13\] We believe that \( \text{sad} \) types are sad because they are not generically stable.
but we have not found clear examples yet. On the other hand, for measures, there is still no consensus on a specific notion for being generic stability. However, the following question remains: Does there is a global measure $\mu$ which is $R$-generically stable but is not fam?

3.2 Random types and $R$-generic stability

Randomization of a first-order structure $M$, as formalized by Ben Yaacov and Keisler [BK09], is a new continuous structure whose elements are random elements of $M$. In this section we show that a measure $\mu$ (in classical logic) is $R$-generically stable (as in Definition 3.3) if and only if a canonical random-type (i.e. the natural extension) behaves like generically stable types in continuous logic (as in Theorem 2.13). Although, in Section 4, we show that $R$-generic stability is equivalent to a perfect analog of generic stability in the sense of continuous logic.

We assume familiarity with the basic notions about randomization of classical structures/theories as developed in [BK09] and [Ben14]. Although we recall some notion and results from [Ben14]. In the following $T$ is a classical theory and $T^R$ its randomization, as a continuous theory.

Convention 3.9. In this section, the symbol $\otimes$ is not used for the Morley product of types/measures, but will be used in another sense.

Convention 3.10. In the rest of the article, whenever necessary, we write the parameters in continuous logic (i.e. in $\mathbb{U}$) in bold letters $a, b, c, \ldots$. Otherwise, we use $a, b, c, \ldots$.

Let $M$ be a classical $L$-structure and $A$ an atomless measure algebra. The $L^R$-pre-structure $(M \otimes A)_0$ is defined as follows. The domain consists of all formal finite sums $\sum_{i<k} m_i \otimes e_i$, also written $\bar{m}\bar{e}$, where $m \in M$ and $\bar{e} = (e_i)_{i<k} \subseteq A$ is a partition of the identity. If $e'$ is any other event then one can easily refine the partition and we identify members of $(M \otimes A)_0$ with other members obtained by refinement of partition. In this case, it is easy to check that:

$$f(\bar{a} \otimes \bar{e}, \bar{b} \otimes \bar{e}, \ldots) = (f(a_i, b_i, \ldots)) \otimes \bar{e},$$

$$[P(\bar{a} \otimes \bar{e}, \bar{b} \otimes \bar{e}, \ldots)] = \bigvee \{e_i : P(a_i, b_i, \ldots) \in A\} \in A.$$

As the distance symbol interprets a metric on $(M \otimes A)_0$, its completion is denoted by $M \otimes A$. Notice that if $M \models T$ then $M \otimes A \models T^R$.  

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Let \( \mu(\bar{x}) \) be a measure over \( M \). Then, it is a random type over \( M \), but not over \( M \otimes A \). Although there is a natural extension \( \mu \otimes A \) of \( \mu \) over \( M \otimes A \). For every \( L \)-formula \( \phi \) we define

\[
P \left[ \phi \left( \bar{x}, \sum m_i e_i \right) \right]^{\mu \otimes A} = \sum P[e_i]P^\mu[\phi(\bar{x}, m_i)].
\]

As this is only defined for formulae over the parameter set \((M \otimes A)_0\), it can be extended by continuity to the whole structure. The type \( \mu \otimes A \) is called the natural extension of \( \mu \).

Fact 3.11 ([Ben14], Proposition 1.1). Let \( \mathcal{U} \) be the monster model of \( T \), \( \mu \) a global measure, and \( A \) an atomless measure algebra. Suppose that \( M \) is a separable model of \( T \) and \( A_0 \leq A \) is separable.

(i) \( M \otimes A_0 \preceq \mathcal{U} \otimes A \).

(ii) \( \mu \) is definable over \( M \) if and only if the type \( \mu \otimes A \) is definable over \( M \).

(iii) \( \mu \) is finitely satisfied in \( M \) if and only if the type \( \mu \otimes A \) is finitely satisfied in \( M \otimes A_0 \).

We are now ready to give the main result/observation of this subsection.

Theorem 3.12. Let \( T \) be a (countable) classical theory, \( M \) a small model of \( T \), and \( \mu(x) \) a global \( M \)-invariant measure. Let \( A \) be an atomless measure algebra such that \([0,1] \preceq A\). The following are equivalent:

(i) \( \mu \) is \( R \)-generically stable over \( M \) (as in Definition 3.3).

(ii) There is a sequence \((a_n) \in M \otimes [0,1]\) such that \((tp(a_n/\mathcal{U} \otimes A) : n < \omega)\) Baire-1/2-converges to \( \mu \otimes A \).

Proof. We can assume that \( M \) is separable. Let \( M_0 \subseteq M \) be dense and countable, and \( A_0 \subseteq [0,1] \) be dense and countable. By Fact 3.11, \( \mu \) is finitely satisfiable in, and definable over \( M \) if and only if \( \mu \otimes A \) is finitely satisfiable in, and definable over \( M \otimes [0,1] \). We will show that the following are equivalent:

(1) There is a sequence \((\mu_n) = \frac{1}{k_n} \sum_{i=1}^{k_n} p_{n,i} : n < \omega\), where \( a_{n,i} \models p_{n,i} \) and \( a_{n,i} \in M_0 \) (for all \( n \) and \( i \)), such that \((\mu_n : n < \omega)\) randomly converges to \( \mu \).

(2) There is a sequence \((a_n) \in M_0 \otimes A_0\) such that \((tp(a_n/\mathcal{U} \otimes A) : n < \omega)\) Baire-1/2-converges to \( \mu \otimes A \).

Let \( a = \sum_{i<k} m_i \otimes e_i \) where \( m_i \in M_0 \) and \( e_i \in A_0 \). Suppose that \( a \) realizes \( \mu \otimes A \) where \( \mu = \mu(x) \) is a measure over \( \mathcal{U} \). Then, for any \( L \)-formula \( \phi(x,y) \)
and $b = \sum_{k < k'} m'_j \otimes e'_j \in (U \otimes A)_0$ we have

$$
P[\phi(x, \sum_j m'_je'_j)]^\mu \otimes A = \sum_j P[e'_j]P[\phi(x, m'_j)]
$$

$$
= \sum_j P[e'_j] \left( \sum_i P[e_i] \cdot \phi(m_i, m'_j) \right)
$$

$$
= \sum_i P[e_i] \left( \sum_j P[e'_j] \cdot \phi(m_i, m'_j) \right)
$$

$$
= \sum_i P[e_i] \cdot \left[ \phi(x, \sum_j m'_je'_j) \right]^{m_i \otimes A}
$$

Here $m_i \otimes A$ is the natural extension of type $tp(m_i/U)$. (Notice that $\mu(\phi(x, m'_j)) = \sum_i P[e_i] \cdot \phi(m_i, m'_j)$ for any $m'_j$.)

For any $n \in \mathbb{N}$, let $\mu_n := \frac{1}{k_0} \sum_j tp(a_{i,n}/U)$ where $a_{i,n} \in M_0$. With the above observations, it is not hard to check that the sequence $(\mu_n : n < \omega)$ is randomly convergent if and only if the sequence $(\mu_n \otimes A : n < \omega)$ is Baire-1/2-convergent (as in Definition 2.8). (See also Remark 3.2 above.) Therefore, as $T^R$ admits quantifiers elimination (cf. [BK09, Thm. 2.9]), (1),(2) above are equivalent. This proves the theorem, by density of $A_0$ and $M_0$.

\[ \square \]

**Remark 3.13.** (i) (Warning) One can NOT expect that a combination of Theorem 3.12 and Theorem 2.13 implies that a measure $\mu$ (in classical logic) is $R$-generically stable if and only if “its randomization” is generically stable (in continuous logic). The reasons are that: (1) There is no guarantee that $U \otimes A$ is the monster model of $T^R$, and $\mu \otimes A$ is not the first candidate for randomization of $\mu$, unlike $\mu |^U$ defined in Subsection 3.3 below. (See the following example on (non-)saturation of models of the form $M \otimes A$.)

(2) The randomization $\mu |^U$ of $\mu$, as defined below, is an extension of $\mu \otimes A$, AND there is no reason that $\mu |^U$ will be finitely satisfible assuming that $\mu \otimes A$ is finitely satisfible. This is the reason why in Proposition 2.1 of [Ben14] the author made another assumption (i.e. NIP). Although in the following we prove, with a weaker assumption than NIP, that a measure $\mu$ is fin if and only if $\mu |^U$ is generically stable over a model of the form $M \otimes A$. (Cf. Theorem 3.21).

(ii) Despite the above, surprisingly we can refine Theorem 3.12. Although for this we need more powerful tools from continuous VC-theory which we present in Section 4. (Cf. Theorem 4.7 below.) In fact, we will see that
R-generic stability of $\mu$ implies generic stability of $\mu \upharpoonright U$.  

(iii) (Example) James Hanson pointed out to us that randomizations of the form $M \otimes A$ are not sufficiently saturated in general. For an easy example, let $T$ be DLO with constants added for $\mathbb{Q}$. Consider the Lebesgue measure on $[0,1]$ and let $p$ be the corresponding type in $T^R$. No model of $T^R$ of the form $M \otimes A$ can realize $p$, because for any element $a$ of such a model, $tp(a)$ corresponds to a measure in $T$ with atoms. (And this is in the completion, not just the pre-model. Any type corresponding to a measure with finite support has distance 1 from $p$ in $T^R$’s type space.) He also suggested a characterization of a theory $T$ such that models of $T^R$ of the form $M \otimes A$ can be arbitrarily saturated: a proper subclasses of stable theories. 

(iv) It is known by Conant, Gannon and Hanson (in an unavailable paper) that there are definable and finitely satisfiable measures which their randomizations are not finitely satisfiable. Although, Using the dominated convergence theorem, it is easy to see that randomizations of $fam$ (and so $R$-generically stable) measures are finitely satisfiable in models of the form $M \otimes A$. (Cf. the argument of 3.23.) The converse is problematic for generically stable types in continuous logic, because we do NOT know whether if randomization of $\mu$ is finitely satisfiable (in a model not necessarily of the form $M \otimes A$) then $\mu$ will be finitely satisfiable or not. One of the reasons that we think the notion of $R$-generic stability is suitable for our purpose is that this problem does not appear here. (Notice that we will shortly see randomization of $fim$ measures are generically stable (cf. Fact 3.20), but the converse is problematic.)

### 3.3 Local NIP and $fim$

In this section we first introduce the notion of corresponding random-type $\mu \upharpoonright U$ of a global measure $\mu$ (in classical logic), and then prove that, assuming a local version of NIP, $\mu$ is generically stable iff $\mu \upharpoonright U$ is generically stable (over a model of the form $M \otimes A$).

Let $T$ be a classical theory, $M \models T$ and $A \subseteq M$. Let $\mu(x)$ be a measure
over \( M \) such that \( \mu(x) \) is definable over \( A \). For any formula \( \phi(x, y) \), there is a continuous function \( f^\phi_M : S_y(A) \to [0, 1] \) such that \( \mathbb{P}^\mu[\phi(x, b)] = f^\phi_M(q) \) where \( q = tp(b/A) \). If \( M \) is a model of \( T^R \) containing \( M \) (e.g. \( M \models M \otimes A \) or \( M = U \) the monster model of \( T^R \)), there is an extension \( \mu \upharpoonright M \) of \( \mu \) over \( M \):

\[
\mathbb{P}[\phi(x, b)]^\mu_M = \int f^\phi_M(q) \, d\, tp(b/A),
\]

where \( tp(b/A) \) is a Borel probability measure on \( S_y(A) \). The type \( \mu \upharpoonright M \) is called the corresponding random-type of \( \mu \).

**Remark 3.14.** (i) Notice that definability guarantees that \( f^\phi_M \) is continuous (and so measurable), and is uniquely determined by \( \mu \).

(ii) In addition, if \( \mu \) is finitely satisfiable in \( A \), then \( \mu \upharpoonright M \) can be defined as follows:

\[
\mathbb{P}[\phi(x, b)]^\mu_M = \int f^\phi_M(q) \, d\, tp_\phi(b/A),
\]

where \( f^\phi_M : q \mapsto \mathbb{P}_\mu[\phi(x,b)] \) with \( q = tp_\phi(b/A) \), is the defining function of \( \mu \) on \( S_\phi(A) \), and \( tp_\phi(b/A) \) is a Borel probability measure on \( S_\phi(A) \). (In this case, we say that the definition of \( \mu \) is factors via \( S_\phi(A) \). That is, \( \mathbb{P}_\mu[\phi(x,b)] = \mathbb{P}_\mu[\phi(x,b')] \iff tp_\phi(b/A) = tp_\phi(b'/A) \).

(iii) Notice that if \( M = M \otimes A \), then \( \mu \upharpoonright M = \mu \otimes A \) (as in the previous section).

(iv) Ben Yaacov [Ben14] provided a wider definition, namely without definability of measures. Although, as we study very nice measures in the present paper, this assumption is not limitation.

Let \( M, M^* \) be models of a classical/continuous theory. We say that \( M^* \) is a **good extension of** \( M \), if it is an elementary extension of \( M \) such that any \( n \)-type \( (n \in \mathbb{N}) \) over \( M \) is realized in \( M^* \).

**Definition 3.15.** Let \( M \) be a model (classic or continuous) and \( \phi(x, y) \) a formula. We say that \( \phi(x, y) \) is uniformly NIP in \( M \) if for any real numbers \( r < s \) there is a natural number \( n = n^{r,s}_{\phi,M} \) such that there is no \( a_1, \ldots, a_n \in M \) such that for any \( I \subseteq \{1, \ldots, n\} \), \( M \models \exists y \bigwedge_{i \in I} \phi(a_i, y) \leq r \land \bigwedge_{i \notin I} \phi(a_i, y) \geq s \).

We say that \( M \) is uniformly NIP if every formula \( \phi(x, y) \) is uniformly NIP in \( M \).

---

\[\text{In the notion, ‘uniformly’ emphasizes that, in contrast to ‘NIP in a model’ in [KP18], there is a natural number } n^{r,s}_{\phi,M} \text{ for any formula } \phi \text{ and } r, s.\]

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It is easy to see that \( \phi \) has NIP for the theory \( T \) iff it is uniformly NIP in the monster model of \( T \) iff it is uniformly NIP in some model of \( T \) in which all types over the empty set in countably many variables are realised.

**Convention 3.16.** In the rest of the article, if the reader is not comfortable with the statement “\( M^* \) is a good extension of \( M \) which is uniformly NIP”, he/she can simply assume that: \( T \) is NIP.

**Fact 3.17.** Let \( \mu \) be global type and definable over \( M_0 \models T \). Let \( \mathcal{A}_0 \prec [0,1] \) be any atomless probability algebra. Suppose that \( M \) is a good extension of \( M_0 \) such that \( M \) is uniformly NIP. Let \( \mu \) be a measure over \( M \) such that it is finitely satisfiable in \( M_0 \). Then, \( \mu \restriction U \) is approximately finitely satisfiable in \( M_0 \otimes \mathcal{A}_0 \).

**Proof.** This is an adaptation of Proposition 1.2 of [Ben14]. (Notice that, as \( M \) is a good extension and uniformly NIP, it is easy to check that the measure \( \nu = \text{tp}_{\phi^*}(b/M_0) \) in the argument of the proof of [Ben14, Pro. 2.1] can be approximated by an average of types in \( S_{\phi^*}(M_0) \). Indeed, see Remarks 4.13 and 5.2 in [Kha21]. See also the argument of Theorem 3.19 below.) \( \square \)

In [Kha21a], using a crucial result due to Bourgain, Fremlin, and Talagrand, we gave an alternative argument of [G21, Thm. 5.10]. (See Theorem A.1 in [Kha21a].) In fact, [Kha21a, Thm. A.1] is a refinement of [G21, Thm. 5.10]. The following is another refinement/argument.

**Theorem 3.18.** Let \( M \models T \) be countable, and \( M^* \) a good extension of \( M \otimes [0,1] \) such that it is uniformly NIP. Then, every global measure \( \mu \) which is finitely satisfiable in \( M \) is the limit of a sequence of average types realised in \( M \), that is, there is a sequence \( (\bar{a}_n) \in M^{<\omega} \) such that \( \lim_n \text{Av}(\bar{a}_n) = \mu \).

**Proof.** The corresponding random type \( \mu \restriction U \) is finitely satisfiable in \( M \otimes \mathcal{A}_0 \) by Fact 3.17 (Here \( \mathcal{A}_0 \prec [0,1] \) is countable and dense. Notice that, as \( M^* \) is uniformly NIP, there is \( N \succeq M \) which is a good extension and uniformly NIP.) Now, as \( M^* \) is good and uniformly NIP, some/any Morley sequence of \( \mu \restriction U \) (over \( M \otimes \mathcal{A}_0 \)) is convergent. (Notice that we can consider a Morley sequence in \( M^* \).) By Theorem 2.7 there is a sequence in \( M \otimes \mathcal{A}_0 \) which is convergent to \( \mu \restriction U \), and so there is a sequence of averages measures of realized types in \( M \) which converges to \( \mu \). This is enough. \( \square \)

The following is a kind of complement to Theorem 3.12. (Although we will refine it in Section 4, its proof is interesting and instructive.)
Theorem 3.19. Let $T$ be a (countable) classical theory, $M, M^*$ a small models of $T$, and $\mu(x)$ a global measure. Let $\mathcal{A}$ be an atomless measure algebra such that $[0, 1] \preceq \mathcal{A}$. If $M^*$ is a good extension of $M$ which is uniformly NIP, then (i) $\implies$ (ii).

(i) $\mu$ is $R$-generically stable over $M$.

(ii) There is a sequence $(a_n) \in M \otimes [0, 1]$ such that $(tp(a_n/U : n < \omega)$ Baire-1/2-converges to $\mu \upharpoonright U$. (Equivalently, $\mu \upharpoonright U$ is generically stable over $M \otimes [0, 1]$.)

Proof. Suppose that $\mu$ is $R$-generically stable over $M$. By Theorem 3.12, there is a sequence $(a_n) \in M \otimes [0, 1]$ such that $(tp(a_n/U \otimes \mathcal{A}) : n < \omega)$ Baire-1/2-converges to $\mu \otimes \mathcal{A}$. Let $U \supseteq U \otimes \mathcal{A}$ be the monster model of $(T_M)^R$ (i.e. the randomization of $T_M$, where $T_M$ is the extension of $T$ by parameters of $M$). Set $\mu_n = tp(a_n/U)$ for all $n$.

Similar to the argument of Proposition 2.1 of [Ben14], it is easy to verify that the sequence $(\mu_n : n < \omega)$ Baire-1/2 converges to $\mu \upharpoonright U$, and so by Theorem 2.13 $\mu \upharpoonright U$ is generically stable. Indeed, fix a formula $\phi(x, y)$ and a parameter $b \in U$. Let $\nu = tp_{\tilde{\mathcal{A}}}(b/M)$ viewed as a $\tilde{\mathcal{A}}$-measure over $M$. By [HP11] Lemma 4.8, as $M^*$ is good and uniformly NIP (or simply $T$ is NIP), $\nu$ can be approximated up to $\epsilon$ by an average $\frac{1}{k} \sum_i p_i$, where $p_i \in S_{\tilde{\mathcal{A}}}(M)$ are types. (The same can be done for a finite family of formulas.) Let $a_i \models p_i$ where $a_i \in M^*$ (for $i \leq k$). Let $\mathcal{A}$ be an extension of $[0, 1]$ contain an equal $k$-partition $\bar{e}$ of 1. Set $b' = a \otimes \bar{e}$. Then $|\phi(x, b')^\mu_{\mathcal{A}} - \phi(x, b)^\mu_{\mathcal{A}}| < \epsilon$. (Notice that this make sense because $\mu$ is definable over $M$.)

On the other hand, as $(a_n) \in M \otimes [0, 1]$, it is easy to verify that $|\phi(x, b')^{\mu_n} - \phi(x, b)^{\mu_n}| < 3 \cdot \epsilon$ for all $n$. Indeed, for each $a_n$, there is a measure of the form $\sum r_i \cdot m_i$ with $m_i \in M$ such that $d(a_n, \sum r_i \cdot m_i) < \epsilon$. (This means that the distance between the global types of $a$ and $\sum r_i \cdot m_i$ (in the sense of continuous logic) is $< \epsilon$.) In particular, $|\phi(x, b)^{\mu_n} - \phi(x, b)^{\sum r_i \cdot m_i}| < \epsilon$ for all $b \in U$. Notice that the measure $\sum r_i \cdot m_i$ is definable over $M$. Therefore, by definition, $|\phi(x, b')^{\sum r_i \cdot m_i} - \phi(x, b)^{\sum r_i \cdot m_i}| < \epsilon$. Using the triangle inequality, we are done.

So putting everything together we can verify that $(\mu_n : n < \omega)$ Baire-1/2 converges to $\mu \upharpoonright U$. By Theorem 2.13 this means that $\mu \upharpoonright U$ is generically stable.

Fact 3.20. Let $M$ be models of $T$ and $\mu(x)$ be a global measure (in $T$) which is definable over $M$. Then (i) $\implies$ (ii).

(i) $\mu$ is $\text{fim}$ (over $M$).
(ii) $\mu \upharpoonright U$ is generically stable (as in Definition 2.11).

Proof. It seems that this was first proved by Conant, Gannon and Hanson. (See https://james-hanson.github.io/conferences/Gen-Stab-Rand.pdf)

[We can give an sketch of an alternative: As $\mu$ is $\text{fim}$, by Theorem 5.5 of [Kha21], $\mu$ commutes with itself: $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$. (Here, the notation $\otimes$ is used for Morley products and is different from its original use in the present paper.) This implies that $\mu \upharpoonright U$ commutes with itself, and so the Morley sequence of $\mu \upharpoonright U$ is totally indiscernible. On the other hand, since $\mu$ is $\text{fim}$, it is easy to see that for some Morley sequence $(a_i)$ of $\mu \upharpoonright U$ we have $(\ast) \sum_n \phi(a_i, b) - \phi(a_i, b)^{\mu \upharpoonright U} \to 0$ as $n \to \infty$, for any formula $\phi$ and parameter $b$. As $(a_i)$ is indiscernible, $(\ast)$ holds for any subsequence of $(a_i)$, and so $\phi(a_i, b) \to \phi(a_i, b)^{\mu \downharpoonright U}$. Therefore, the Morley sequence of $\mu \upharpoonright U$ is convergent. By Remark 2.12(iv), $\mu \upharpoonright U$ is generically stable.] \hfill \Box

In Section 4, we prove a stronger result. (Cf. Theorem 4.9)

The following is in a way the reverse of Fact 3.20 and Theorem 3.19.

**Theorem 3.21.** Let $M$ be model of $T$, and $\mu(x)$ a global measure (in $T$) which is definable over $M$. Suppose that $\mu \upharpoonright U$ is generically stable over $M \otimes A$. Then,

(i) $\mu$ is $R$-generically stable (over $M$).

Furthermore, suppose that $M$ has a good extension $M^*$ which is uniformly $\text{NIP}$, then

(ii) $\mu$ is $\text{fim}$ (over $M$).

**Proof.** (i): By Theorem 2.13, there is a sequence $(a_n) \in M \otimes [0, 1]$ such that $(\text{tp}(a_n/U : n < \omega)$ Baire-1/2-converges to $\mu \upharpoonright U$. In particular, $(\text{tp}(a_n/U \otimes A) : n < \omega$) Baire-1/2-converges to $\mu \otimes A$. By Theorem 3.12, (i) holds.

(ii): Furthermore, suppose that $M^*$ is a good extension of $M$ which is uniformly $\text{NIP}$. Then, (ii) follows from Theorem 5.3 of [Kha21]. Indeed, we have to check that $\mu$ is finitely satisfiable in a small model. Notice that as $\mu \upharpoonright U$ is finitely satisfiable in $M \otimes A$, by Fact 3.11, $\mu$ is finitely satisfiable in $M$. Now, all the assumptions of [Kha21, Thm. 5.3] hold. \hfill \Box

**Remark 3.22.** In Theorem 3.21, the assumption that $\mu \upharpoonright U$ is generically stable over a model of the form $M \otimes A$ is essential. The problem here is finite satisfiability of $\mu$. Is it true without this assumption? (Cf. Remark 3.13.)
At the end this section we compare \textit{fim} and \textit{R}-generically stable measures.

\textbf{Proposition 3.23.} Let \( M \) be model of \( T \), and \( \mu(x) \) a global measure (in \( T \)) which is definable over \( M \). Then (i) \( \implies \) (ii). Furthermore, suppose that \( M \) has a good extension \( M^* \) which is uniformly NIP, then (ii) \( \implies \) (i).

(i) \( \mu \) is fim.
(ii) \( \mu \) is \( R \)-generically stable.

\textit{Proof.} (ii) \( \implies \) (i) follows from Theorem 3.19 and Theorem 3.21. (In fact, as \( \mu \) is definable over, and finitely satisfiable in \( M \), this follows from Theorem 5.3 of \cite{Kha21}.)

(i) \( \implies \) (ii): As \( \mu \) is fim over \( M \), \( \mu \) is \( \text{fam} \) over \( M \). By Proposition 3.4 of \cite{G21}, \( \mu \) is is sequentially approximated over \( M \). This means that there is a sequence \((\mu_n : n < \omega)\) of measures of the form \( \mu_n := \frac{1}{k_n} \sum_{i=1}^{k_n} a_{n,i} \), where \( a_{n,i} \in M \), such that the sequence \( \mu_n \rightarrow \mu \). Equivalently, for any formula \( \phi \), we have \( f_{\mu_n}^\phi \rightarrow f_{\mu}^\phi \). By the Dominated Convergence Theorem, for any \( b \in U \), \( \int f_{\mu_n}^\phi \ d \text{tp}(b/M) \rightarrow \int f_{\mu}^\phi \ d \text{tp}(b/M) \). As all the \( \mu_n \)'s are in a model of the form \( M \otimes A \), this implies that \( \mu \upharpoonright U \) is finitely satisfiable in \( M \otimes A \).

On the other hand, by Fact 3.20, \( \mu \upharpoonright U \) is generically stable. As this measure is definable over \( M \), and finitely satisfiable in \( M \otimes A \), \( \mu \upharpoonright U \) is generically stable over \( M \otimes A \). By Theorem 2.13, there is a sequence \((c_i) \) in \( M \otimes A \) such that \( (\text{tp}(c_i/U) : i < \omega) \) Baire-1/2-converges to \( \mu \upharpoonright U \). By Theorem 3.12, \( \mu \) is \( R \)-generically stable over \( M \).

\textit{Question 3.24.} Without additional assumptions, is every \( R \)-generically stable measure a fim measure?

\section{Continuous VC-theory and generic stability}

In this section, using powerful tools provided in \cite{Ben09}, we can refine some of the previous results. We give an argument of the fact that the generic stability for \textbf{types} is preserved in randomization (cf. 4.6), and we then generalizes it to \textbf{measures} (cf. 4.9). For two reasons we first give the \textbf{types} case, and then the \textbf{measures} case: (1) The type case is easier and is a bridge to the measure case. (2) The type case has a point that is not in the measure case and it is useful to express it separately (cf. 4.10). 24
We first recall some notion and notation from [Ben09]. Let $J$ be a set, and $Q \subseteq [0, 1]^J$ a collection of functions from $J$ to $[0, 1]$. For any $r < s$, there is a collection $Q_{r,s} = \{q_{r,s} : q \in Q\}$ of fuzzy sets, where the $q_{r,s}$'s are defined as follows: $j \in q_{r,s}$ if $q(j) \leq r$ and $j \notin q_{r,s}$ if $q(j) \geq s$; and it is NOT known/important that $j$ belongs to $q_{r,s}$ or not if $r < g(j) < s$. The VC-index for $Q_{r,s}$, denoted by $VC(Q_{r,s})$, is defined similar to classical case. (We refer to [Ben09, page 316] for a precise definition of fuzzy sets and the VC-indexes of $Q$.) $Q$ is called a VC-class if for any $r < s$, $VC(Q_{r,s}) < \infty$. If $Q$ is a VC-class, one can easily show that for every $\epsilon > 0$ there is an upper bound $d_\epsilon < \infty$ for the VC-indexes of classes $Q_{r, r+\epsilon}$ where $r \in [0, 1-\epsilon]$.

To summarize, the notion of a VC-class is such that we have the following important observations/connections:

**Observation 4.1.** Let $J$ be a set, and $\varphi : \mathbb{N} \times J \to [0, 1]$ a function. Then,

(i) $\iff$ (ii) $\implies$ (iii).

(i) $\varphi' = \{\varphi(\cdot, j) : J \to [0, 1] | j \in J\}$ is a VC-class.

(ii) $\varphi_\mathbb{N} = \{\varphi(i, \cdot) : J \to [0, 1] | i \in \mathbb{N}\}$ is dependent. That is, for every $\epsilon > 0$ there is a natural number $N = N_\epsilon$ such that for every $r < s$ with $s - r = \epsilon$ the following does not hold

\[ \exists F \subseteq \mathbb{N}, |F| = N \forall E \subseteq F \exists j \in J \left( \bigwedge_{i \in E} \varphi(i,j) \leq r \wedge \bigwedge_{i \in F \setminus E} \varphi(i,j) \geq s \right). \]

(iii) There is a subsequence $(\varphi_n : n < \omega)$ of $\varphi_\mathbb{N}$ such that $(\varphi_n : n < \omega)$ Baire-$1/2$-converges. That is, for each $r < s$, there is a natural number $N = N_{r,s}$ and a set $E \subseteq \{1, \ldots, N\}$ such that for each $i_1 < \cdots < i_N < \omega$, the following does not hold

\[ \exists j \in J \left( \bigwedge_{k \in E} \varphi(i_k,j) \leq r \wedge \bigwedge_{k \in N \setminus E} \varphi(i_k,j) \geq s \right). \]

**Proof.** The equivalence (i) $\iff$ (ii) is folklore in classical logic (cf. [van98]). The argument is a straightforward adaptation of the classical case.

(ii) $\implies$ (iii) is similar to the argument of the direction (i) $\implies$ (iii) of Proposition 2.14 in [Kha19b]. Indeed, suppose for a contradiction that there is no Baire-$1/2$-convergent subsequence. Using Ramsey theorem, let $(\varphi_n : n < \omega)$ be a $\varphi$-$N$-$A$-indiscernible sequence as in Definition 3.1 of [Kha20a]. (Here $N$ is a natural number and $\{r, s\} = A$.) For suitable $N$ and $A$, the condition $(*)$ above holds for $(\varphi_n : n < \omega)$, because this sequence is NOT Baire-$1/2$-convergent. As $N$ is arbitrary, this contradicts (ii).
Observation 4.2. With the above notation, every Baire-1/2-convergent sequence \( \{ \varphi(n, \cdot) : n \in \mathbb{N} \} \) is dependent (as in (ii) in 4.1).

Proof. Immediate.

\( \square \)

Compare Definitions 2.8, 3.1, and Remark 3.2.

Fact 4.3 ([Ben09], Pro. 2.15). With the above notation, \( \varphi_I \) is a \( \text{VC} \)-class iff \( \varphi_J \) is so.

Fix a probability space \((\Omega, \mathcal{B}, \mu)\). Given sets \( I \) and \( J \), a family of \([0,1]\)-valued functions on \( I \times J \) is given as \( \varphi : \Omega \rightarrow [0,1]^{I \times J} \) by \( \omega \mapsto \varphi_\omega(\cdot, \cdot) \). For every \( \omega \), \( \varphi_\omega = \{ (\varphi_\omega)_i : i \in I \} \) and similarly \( \varphi_\omega^J = \{ (\varphi_\omega)_j : j \in J \} \). The family \( \varphi = \{ \varphi_\omega : \omega \in \Omega \} \) is called uniformly dependent if for every \( \epsilon > 0 \) there is \( d = d_{\varphi, \epsilon} \) such that \( \text{VC}((\varphi_\omega)^J_{r,r+\epsilon}) \leq d \) for every \( r \in [0,1] - \epsilon \) and \( \omega \in \Omega \).

We say that \( \varphi = \{ \varphi_\omega : \omega \in \Omega \} \) is a measurable family if for any \((i,j) \in I \times J\), the function \( \omega \mapsto \varphi_\omega(i,j) \) is measurable. Then, we define function \( \mathbb{P}[\varphi] : I \times J \rightarrow [0,1] \) by \( \mathbb{P}[\varphi](i,j) = \mathbb{P}[\varphi(i,j)] \).

We say that \( \mathbb{P}[\varphi] : I \times J \rightarrow [0,1] \) is dependent, if for every \( r < s \), the collection \( (\mathbb{P}[\varphi]^J)_{r,s} \) is a \( \text{VC} \)-class. (Cf. [Ben09], page 316 and Proposition 2.15.)

The key result is the following.

Fact 4.4 ([Ben09], Corollary 4.2). If \( \varphi = \{ \varphi_\omega : \omega \in \Omega \} \) is a measurable family of uniformly dependent functions, then \( \mathbb{P}[\varphi] : I \times J \rightarrow [0,1] \) is dependent.

4.1 The case of types

The following localizes Theorem 5.3 of [Ben09] in two way: for a formula AND a sequence. (We strongly suggest that the proof of [Ben09], Thm. 5.3 should be read before reading the rest of this section.)

Theorem 4.5. Let \( M \) be a model of \( T \), \( \psi(\bar{x}, \bar{y}) \) a formula, and \( \langle a_i \rangle \) a sequence in \( M \) of \( \bar{x} \)-tuples. If the sequence \( \langle \psi(a_i, \bar{y}) : i < \omega \rangle \) DBSC-converges, then the sequence \( \langle \mathbb{P}[\psi(a_i, \bar{y})] : i < \omega \rangle \) Baire-1/2-converges. (Here \( \mathbb{P}[\psi(\bar{x}, \bar{y})] \) is the corresponding formula in \( T^R \).)

Proof. The proof is an adaptation of the argument of [Ben09], Thm. 5.3. We write \( \varphi(\bar{x}, \bar{y}) = \mathbb{P}[\psi(\bar{x}, \bar{y})] \). We will show that \( \varphi^U \) is dependent on \( \{ a_i : i < \)
Let $\omega \times \mathbb{U}^m \ni \{ b_j : j \in J \}$, and set $A = \{ a_i : i < \omega \}$. Let $p = tp(A, \mathbb{U}^m/\emptyset)$. We may write it as $p(x_i, y_j)_{i \in \omega, j \in J} \in S_{\omega \times J}(T^R)$, and identify it with a probability measure $\mu$ on $\Omega = S(\omega \times J \times \mathbb{U}^m)(T)$ such that for every formula $\rho(z)$ of the theory $T$, $z \subseteq \{ x_i, y_j \}_{i \in \omega, j \in J}$:

$$P[\rho(z)]^p = \mu(\{ q \in \omega : \rho(z) \in q \}) = \int_\Omega \rho(z)^q d\mu(q).$$

We can replace $\Omega$ by $\Omega' := \{ q \in \Omega : (c_i) \models tp(A/\emptyset) \text{ for any } (c_i)_{i < \omega} \cup (d_j)_{j \in J} \models q \}$, and assume that $\mu$ is concentrated on $\Omega'$. (In fact, we can assume that $P[\rho(z)]^p = \int_{S_1 I \times J}(T) \rho(a_i, y_j)^q d\mu(q_j) = \int_{\Omega'} \rho(x_i, y_j)^q d\mu(q)$.)

For $i \in \omega$, $j \in J$ and $q \in \Omega'$ define: $\chi'_q(i, j) = \psi(x_i, y_j)^q$. Then $\chi = \{ \chi'_q : q \in \Omega' \}$ is a measurable family of $\{0, 1\}$-valued functions on $I \times J$ and $\varphi(a_i, b_j) = \varphi(x_i, y_j)^p = P[\chi](i, j)$ where expectation is with respect to $\mu$. (Notice that, as $\psi(x, y)$ is a formula, for $(i, j) \in I \times J$, the functions $q \mapsto \chi'_q(i, j)$ are measurable.) Then, by the assumption of DBSC-convergence, the family $\{ \chi'_q : q \in \Omega' \}$ is uniformly dependent. (That is, for any $\epsilon > 0$, there is $d = d_{\varphi, \epsilon}$ such that $VC(\chi'_q)[r, r + \epsilon] \leq d$ for all $r \in [0, 1 - \epsilon]$ and $q \in \omega'$.)

By Fact 4.4, $P[\chi] : \omega \times J \to [0, 1]$ is dependent. This means that $\varphi : A \times \mathbb{U}^m \to [0, 1]$ is dependent. Equivalently, the sequence $(P[\psi(a_i, y)]) : i < \omega)$ Baire-1/2-converges. (Cf. Observations 4.1 4.2 and Fact 4.3 above.)

The direction $(i) \implies (ii)$ of the following is a consequence of Fact 3.20 but its argument can be enlightening.

**Corollary 4.6.** Let $p$ be a global type (in $T$) which is definable. Then the following are equivalent:

(i) $p$ is generically stable.

(ii) There is a sequence $(a_i) \in U$ such that $(tp(a_i/U) : i < \omega)$ Baire-1/2-converges to $p \upharpoonright U$. (Therefore, $p \upharpoonright U$ is generically stable.)

**Proof.** $(i) \implies (ii)$ follows from Theorem 4.5. Indeed, by Theorem 4.4 of [Kha22], there is a sequence $(a_i) \in U$ such that $(tp(a_i/U) : i < \omega)$ DBSC-converges to $p$. By Theorem 4.5 $(tp(a_i/U) : i < \omega)$ Baire-1/2-converges to $p \upharpoonright U$. Indeed, notice that, by Theorem 4.5 for every atomic formula $\phi(x, y)$, $(\phi(a_i, y) : i < \omega)$ Baire-1/2-converges to $\phi(x, y)^p$. By Lemma 3.6 of [Ben09], this holds for every quantifier formula. By quantifier elimination and Lemma 2.16 of [Ben09], this holds for every formula. (Notice that, as
Theorem 4.7. Let $M$ be a model of $T$, $\psi(\bar{x}, \bar{y})$ a formula, and $((a_{i,1}, \ldots, a_{i,k_i}) : i < \omega)$ a sequence in $M$ of $\bar{x}$-tuples. If the sequence $\left(\frac{1}{k_i}\sum_{j < k_i} \psi(a_{i,j}, \bar{y}) : i < \omega\right)$ Baire-1/2-converges, then the sequence $\left(\mathbb{P}[\psi(\sum_{j} a_{i,j} \otimes e_{i,j}, \bar{y})] : i < \omega\right)$ Baire-1/2-converges. (Here, $\mathbb{P}[\psi(\bar{x}, \bar{y})]$ is the corresponding formula in $T^R$, and for all $i$, $e_{i,1}, \ldots, e_{i,k_i}$ is a partition of 1 in the measure algebra.)

Proof. We write $\varphi(\bar{x}, \bar{y}) = \mathbb{P}[\psi(\bar{x}, \bar{y})]$. We will show that $\varphi^\mathbb{U}$ is dependent on $\{a_i : i < \omega\} \times \mathbb{U}$. Let us enumerate $\mathbb{U} = \{b_j : j \in J\}$, and set $A = \{a_i : i < \omega\}$. Let $p = tp(A, \mathbb{U}/\emptyset)$. (In the following, $\mathfrak{R}(X)$ is the space of all regular Borel probability measures on $X$.) We may write it as $p(\bar{x}_i, \bar{y}_j)_{i \in \omega, j \in J} \in S_{\omega \cup J}(T^R)$, and identify it with a Dirac measure $\mu = \delta_p$ on $\Omega' = \mathfrak{R}(S_{(\omega \times n) \cup (J \times m)}(T))$ such that for every formula $\rho(\bar{z})$ of the theory $T$, $\bar{z} = \bar{z}_i \bar{z}_j \in \{\bar{x}_i, \bar{y}_j\}_{i \in \omega, j \in J}$:

$$\mathbb{P}[\rho(\bar{z})]^p = \int_{\Omega'} \rho(\bar{z}_i, \bar{z}_j)^{\nu_{i,j}} d\mu(\nu_{i,j}) = \int_{S(\omega \times m)(T)} \rho(a_i, \bar{z}_j)^{\eta_j} d\mu(\eta_j) = \int_{S(\omega \times m)(T)} \sum_{j=1}^{k_i} \mathbb{P}[e_{i,j}] \rho(a_{i,j}, \bar{z}_j)^{\eta_j} d\mu(\eta_j) = \int_{\Omega''} \sum_{j=1}^{k_i} \mathbb{P}[e_{i,j}] \rho(\bar{z}_i, \bar{z}_j)^{\eta_j} d\mu(\eta_j),$$

where $\Omega' := \{\nu_{i,j} \in \Omega : (c_i) \models tp(A/\emptyset)\}$ for any $(c_i)_{i < \omega} \cup (d_j)_{j \in J} \models \nu_{i,j}$, and we assume that $\mu$ is concentrated on $\Omega'$. (Here, $\mu$ is defined similar to $\mu$. It is the corresponding measure of $p$.) And $\Omega'' := \{\nu_{i,j} \in \Omega' : (d_j) \in U$ for any $(c_i)_{i < \omega} \cup (d_j)_{j \in J} \models \nu_{i,j}\}$. We can also assume that $\mu$ is concentrated on $\Omega''$.
For $i \in \omega$, $j \in J$ and $\nu \in \Omega''$ define: $\chi_\nu(i,j) = \psi(\bar{x}_i, \bar{y}_j)^\nu$. Then $\chi = \{\chi_\nu : \nu \in \Omega''\}$ is a measurable family of $[0, 1]$-valued functions on $I \times J$ and

$\varphi(\bar{a}_i, \bar{b}_j) = \varphi(\bar{x}_i, \bar{y}_j)^\nu = P[\chi](i,j)$ where expectation is with respect to $\mu$. Then, by the assumption of Baire-1/2-convergence, the family $\{\chi_\nu : \nu \in \Omega''\}$ is uniformly dependent. By Fact 4.4, $P[\chi] : \omega \times J \rightarrow [0, 1]$ is dependent. Equivalently, the sequence $(P[\psi(a_i, b_j)] : i < \omega)$ Baire-1/2-converges.

Remark 4.8. Theorem 4.7 gives a positive answer to the question in Remark 3.2(i).

The following generalizes 4.6.

Corollary 4.9. Let $\mu$ be a global type (in $T$) which is definable over a model $M$. Then the following are equivalent:

(i) There is a sequence $((a_{i,j})_{i<j}^k : i < \omega) \in U$ such that $(\frac{1}{k_i} \sum_{j<k_i} tp(a_{i,j}/U) : i < \omega)$ Baire-1/2-converges to $\mu$.

(ii) There is a sequence $(a_i) \in M \otimes [0, 1]$ such that $(tp(a_i/U) : i < \omega)$ Baire-1/2-converges to $\mu |^U$. (Therefore, $\mu |^U$ is generically stable.)

Proof. This follows from 4.7, similar to 4.6.

Question 4.10. (i): In the condition (ii) of 4.6, we assumed that $(a_i) \in U$. This is a strong assumption, because one can assume the $a_i$'s are measures for generic stability of $p |^U$. There are two problems: Suppose that $(\mu_n)$ are measure which are finitely satisfiable in a model $N$ of $T^R$ and $\mu_n \rightarrow p |^U$ Baire-1/2. (1) Is $N$ of the form $M \otimes A$ (for $M \models T$)? (2) If the answer to (1) is positive, is there a sequence $(a_i) \in M$ such that $(a_n) DBSC$-converges to $p$? To summarize, does generic stability of $p |^U$ imply generic stability of $p$?

(ii): Notice that, for measures and the notion $R$-generic stability, the problem (1) remains. (Let $\mu |^U$ be generically stable over a model $N$ of $U$. Suppose that $(\mu_n)$ are measure which are finitely satisfiable in $N$ and $\mu_n \rightarrow \mu |^U$ Baire-1/2. (1) Is $N$ of the form $M \otimes A$ (for $M \models T$)? Does generic stability of $\mu |^U$ imply $R$-generic stability of $\mu$?)

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References

[Ben09] I. Ben Yaacov. Continuous and Random Vapnik-Chervonenkis Classes. Israel Journal of Mathematics 173 (2009), 309-333

[Ben13] I. Ben-Yaacov, On theories of random variables, Israel J. Math. 194 (2013), no. 2, 957-1012

[Ben14] I. Ben-Yaacov, Transfer of properties between measures and random types, unpublished note, 2008. http://math.univ-lyon1.fr/~begnac/articles/MsrPrps.pdf

[BBHU08] I. Ben-Yaacov, A. Berenstein, C. W. Henson, A. Usvyatsov, Model theory for metric structures, Model theory with Applications to Algebra and Analysis, vol. 2 (Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008.

[BK09] I. Ben Yaacov, and H.J. Keisler, Randomizations of models as metric structures. Confluentes Mathematici, 1:197–223 (2009).

[CG20] G. Conant, K. Gannon, Remarks on generic stability in independent theories, Ann. Pure Appl. Logic 171 (2020), no. 2, 102736, 20. MR 4033642

[CGH21] G. Conant, K. Gannon, J. Hanson, Keisler measures in the wild, arxiv 2021

[G21] K. Gannon, Sequential approximations for types and Keisler measures, preprint: 2021 https://arxiv.org/abs/2103.09946v2

[G52] A. Grothendieck, Criteres de Compacite dans les Espaces Fonctionnels Generaux, Am. J. Math, 74 (1952), 168-186.

[HP11] E. Hrushovski, A. Pillay, On NIP and invariant measures, Journal of the European Mathematical Society, 13 (2011), 1005-1061.

[HPS13] E. Hrushovski, A. Pillay, and P. Simon. Generically stable and smooth measures in NIP theories. Transactions of the American Mathematical Society 365.5 (2013): 2341-2366.
[Kei99] H. Jerome Keisler. Randomizing a Model. Advances in Math 143 (1999), 124-158.

[Kha21] K. Khanaki, Dependent measures in independent theories, submitted, arXiv:2109.11973, 2021.

[Kha19b] K. Khanaki, Dividing lines in unstable theories and subclasses of Baire 1 functions, Archive for Mathematical Logic (2022), https://doi.org/10.1007/s00153-022-00816-8

[Kha21a] K. Khanaki, Glivenko-Cantelli classes and NIP formulas, submitted, arXiv:2103.10788v4, 2021.

[Kha22] K. Khanaki, Remarks on convergence of Morley sequences, arXiv:1703.08731, 2017.

[Kha20a] K. Khanaki, On classification of continuous first order theories, submitted, 2020.

[Kha20] K. Khanaki, Stability, the NIP, and the NSOP; Model Theoretic Properties of Formulas via Topological Properties of Function Spaces, Math. Log. Quart. 66, No. 2, 136-149 (2020) / DOI 10.1002/malq.201500059

[KP18] K. Khanaki, A. Pillay, Remarks on NIP in a model, Math. Log. Quart. 64, No. 6, 429-434 (2018) / DOI 10.1002/malq.201700070

[PT11] A. Pillay and P. Tanović, Generic stability, regularity, and quasiminimality, Models, logics, and higher-dimensional categories, CRM Proc. Lecture Notes, vol. 53, Amer. Math. Soc., Providence, RI, 2011, pp. 189–211. MR 2867971

[Ros74] H. P. Rosenthal, A characterization of Banach spaces containing $l^1$, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411-2413.

[S15] P. Simon. A guide to NIP theories. Cambridge University Press, 2015.

[van98] Lou van den Dries, Tame topology and o-minimal structures, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998