Free vibrations of cylindrical laminated shells – introduction to optimal design in divergence and flutter problems

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Abstract. Aeroelastic properties, such as free vibrations of multilayered composite cylindrical shells, are investigated in the present paper. It is well known that the eigenmode (eigenvalue) analysis is a crucial point in the estimations/calculations of the divergence and flutter speeds characterizing dynamic instability. In this study the maximization problem of eigenfrequencies and eigenmodes is discussed, especially from the point of view of stacking sequences of composite panels. A special attention is focused on two types of the shell wall construction: a) angle-ply ±θ and b) symmetrical discrete laminates built of layers oriented at 0°, ±45° and 90°.

The analytical formulae is derived with the use of the Rayleigh-Ritz method for simply supported cylinders. It allows us to solve the optimum problem easily for angle-ply fibre orientations. For symmetrical laminates with discrete fibre orientations the optimal solutions are searched for using finite element analysis combined with genetic or evolutionary algorithms. The influence of geometrical parameters describing the cylindrical panel is also demonstrated herein.

1. Introduction

A flutter phenomenon is defined as a dynamic aeroelastic instability. Aeroelasticity denotes the combination of aerodynamic, inertial and elastic forces in such a way that the structure and the flow around it interact with each other. As a result, an energy transfer between the fluid and the structure takes place and, according to the nature of this transfer, will lead to either stable or unstable motion. Actually flutter phenomenon strongly depends on the flexibility of the structure and appears when the mechanical work is lower than the aerodynamic work, i.e. when the mechanical damping is too small to overcome the aerodynamic excitations. In external flows, flutter appears when two vibrating modes interact together at distinct frequencies. In general the flutter phenomenon can lead and result in the appearance of structural failure such as fatigue damage or progressive fracture of structural components. A detailed knowledge in this area might have prevented dangerous accidents, particularly in the case of aircrafts.

The number of aeroelastic problems and the complexity of analysis depend on the type of structure considered (plates, shells – laminated composites, sandwiches, functionally graded materials - FGM), speed regime (the value of the Mach number – subsonic and/or supersonic speed) and the stages of design. The solution of the flutter problems actually starts with the finite element modeling and free vibration analysis.

For free vibration analysis, damping matrix is assumed to be zero and there no external force acting on the system. This is a generalized eigenvalue problem; the solution can be obtained using eigenvalue extraction routine since the solutions are always real and positive if the aerodynamic pressure forces are equal to zero. For increasing dynamic pressure forces the eigenfrequencies are not constant some
can decrease or increase in value. If the critical pressure parameter is achieved two of the eigenvalues can become complex (not necessarily neighbourhood modes, e.g. the first and the second). The merging point is called as frequency coalescence – see Figure 1.

**Figure 1.** Variation of dimensionless natural frequencies with the aerodynamic pressure

An important class of problems in the flutter analysis concerns the free vibrations of systems. In the literature vibrations of composite multilayered cylindrical shells has been of great interest of numerous authors and most of their work has been referenced e.g. by Leissa [1], Bert *et al.* [2] Kalnins [3] and Cohen [4]. The analysis of natural vibrations of doubly-curved composite and sandwich structures was conducted by Tornabene *et al.* [5]. Similarly as previously optimization of natural frequencies has also became one of a central concept in the appropriate design of cylindrical composite structures. In the classical manner design parameters (denoted by the vector s) such as layer thicknesses and ply angles were usually employed to achieve an optimized structures.

Structures with some frequency constraints can be optimized using computationally different approaches – see Grandhi [6], Szyszkowski, King [7]. To avoid vibrational resonance in aerospace and naval structures, laminated panels are usually designed for maximum fundamental frequency constraints \( \omega \), i.e.:

\[
\text{Max } \omega(s)
\]  

Bert [8] and Reiss, Ramachandran [9] maximised the fundamental frequency of thin symmetric laminates by treating the fibre orientation angle of each layer as a continuous design variable. The frequency analysis was performed using a closed form formula or numerical approach.

### 2. Governing Relations

Consider a circular cylindrical shell of mean radius \( R = D/2 \), wall thickness \( t \), and length \( L \). To formulate the fundamental equations of the composite shells, we will use the Hamilton principle:

\[
\delta H = \delta \int \left( K - \Pi \right) \mathrm{d} \tau
\]  

and the basic relations for laminates consisting of \( N \) layers:

\[
\{ \rho^{(0)}, \rho^{(1)}, \rho^{(2)} \} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_{k}} \rho_{k}(l,z,z^2) \mathrm{d}z
\]  

where \( \rho_{k}(l) \) denotes the density of \( k \)-th layer in the laminate. Usually, they are identical. \( K \) is the kinetic energy, and \( \Pi \) the total strain energy. Derivation of the fundamental relations from the Hamilton principle for various global, global-local and local formulation is presented in details in Refs [10, 11]. For the multilayered cylindrical shells with the circular cross-section, the motion equations are expressed by the relations (4). Assuming radii of curvature in the \( x \) (longitudinal), \( y \) (circumferential)
directions: \( R_1 \rightarrow \infty, R_2 = R_3 \); Lame parameters: \( A_i = 1, A_j = R \), we obtain finally (\( L_{ij} \) are differential operators, \( i, j = 1, 2, 3 \)):

\[
\begin{bmatrix} r^{L-K} \end{bmatrix}^T = [u, v, w], \quad L_{ij} r_j^{L-K} = \rho^{(0)} r_j^{L-K} - \rho^{(1)} \frac{\partial r_j^{L-K}}{\partial x}, \quad L_{ij} r_j^{L-K} = \rho^{(0)} r_j^{L-K} - \rho^{(1)} \frac{\partial r_j^{L-K}}{\partial y},
\]

\[
L_{33} r_j^{L-K} = \rho^{(0)} r_j^{L-K} + \rho^{(1)} \left( \frac{\partial r_j^{L-K}}{\partial x} + \frac{\partial r_j^{L-K}}{\partial y} \right) - \rho^{(2)} \nabla^2 r_j^{L-K}
\]  

(4)

3. Optimization of laminate configuration for cylindrical shell

Various optimization problems of natural frequencies of the multi-layered composite cylindrical shells were discussed in Refs [12-15].

Assuming the mid-plane symmetric case (\( B_{ij} = 0 \)) the value of natural frequencies can be expressed as the sum of two terms (see Muc et al. [16, 17]):

\[
\rho \omega^2 \alpha_m^2 = D_{11} \alpha_m^4 + 2(D_{12} + 2D_{66}) \alpha_m^2 \beta_n^2 + D_{22} \beta_n^4 + p(A_{ij}) \alpha_m = \frac{m\pi}{L} \beta_n^2 = \frac{n}{R}
\]

\[
p(A_{ij}) = \frac{\alpha_m^2}{R^2} \alpha_m^2 A_{11}, A_{66} + \alpha_m^2 \beta_n^2 (A_{11} A_{22} - A_{12}^2 - 2A_{12} A_{66}) + \beta_n^4 A_{12} A_{66}
\]

where the first term is analogous to the formulae valid for the natural frequencies of plates, whereas the second term, expressed symbolically as the function \( p \), is a polynomial function of the membrane stiffnesses \( A_{ij} \). The detailed analysis of the influence of stacking sequences on the optimal solutions of natural frequencies is discussed by Muc [18].

A variation of fibre orientation in the individual layers results in an increase (decrease) of the fundamental critical frequency. An example of numerical solutions for angle-ply laminates \( \pm \theta \) is shown in Figure 2. The NKTP 32 elements (NISA II Family of Programs) were used – quadrilateral taking into account the first-order transverse shear deformation theory. An angle \( \theta = 0^\circ \) means the direction parallel to the longitudinal shell direction in the shell. As it may be seen the location of the optimal orientation of the fibers (Eq (1)) depends on the form of boundary conditions and the geometrical parameters characterizing the cylinder.

![Figure 2](image-url)  

**Figure 2.** Natural frequencies for the cross-ply laminates \((L/D=1, t/R=2/15, E_z/E_l = 0.1)\).  

For the prescribed form of boundary conditions at the edges of the shell and using continuous design variables, the search for the optimal fiber orientation in individual layers (the objective function is defined by relationship (5)) can be carried out with the use of the optimization package in the FEM program, e.g. NISAOPT. During solving this problem numerically a fundamental role plays the appropriate modeling of the problem. The basic issue is to avoid the so-called twin forms of the critical vibration (Figure 3a, b - first mode). Figure 3c shows a second mode shape. For this purpose,
only one half of the cylinder should be considered in the circumferential direction with the symmetry (anti-symmetry) conditions at the edges. The form of the critical vibration depends on the orientation of the laminate, as shown in Figure 3d. The change in the form of the basic vibrations can also be seen in Figure 1. This corresponds to the approach of the curves characterizing the first two forms of the vibrations. Therefore, the solution to the optimization problem (1) always exists, although the application of the specific numerical values requires the use of optimization algorithms.

Figure 3. The natural frequencies for clamped cylindrical shells (the analyzed shell in the longitudinal direction): (a) orientation $\theta = 0^\circ$ - 1 mode shape (frequency = 1.12254E+03 Hz); (b) orientation $\theta = 0^\circ$ - twin 1 mode shape (frequency = 1.12254E+03 Hz), (c) orientation $\theta = 0^\circ$ - 2 mode shape (frequency = 1.34058E+03 Hz), (d) orientation $\theta = 90^\circ$ - 1 mode shape (frequency = 1.25436E+03 Hz).

Searching for the optimal laminate configuration as a given discrete set of orientation of the individual layers ($0^\circ$, $\pm 45^\circ$, $90^\circ$) is considered the optimization problem cannot be solved using standard FEM software. It is necessary to introduce the external optimization procedures (e.g. genetic or evolutionary algorithms). The objective function is changed here, because we are currently looking for the stacking sequences of the laminate that corresponds to the minimum natural frequencies. In the analysis it is assumed that the laminate is composed of 64 individual layers, so the variable chain (chromosome) contains 16 elements (design variables).

The problem was solved by looking for the minimum frequency for specific (prescribed in advance) modes of vibrations $m$, $n$. The numerical computations of the objective function (FEM) was used. In the calculations, 1/8 cylinder was discretized using the four node quadrilateral shell elements (NKTP 32).

Figure 4. The optimal values of the natural frequencies ($L/D=1$, $t/R=0.02$, $E_*/E_1=1/6$).

The distributions presented in Figure 4 demonstrate, that for the value of $m=n=1$ the minimum is achieved and it corresponds to the string in the form: $[3333222222221111111]$, $[1-0^0, 2-\pm 45^0, 3-90^0]$. The form of the optimal chromosome clearly shows, that in this case the junction between the flexural/bending effects (terms of the $[D]$ matrix) and the membrane effects (terms of the $[A]$ matrix) occurs. It can be verified easily that in the case of only flexural effects, the problem of searching for an
optimum is reduced to the analysis of the plate free vibrations. In the case of the domination of flexural effects (high value \( n \)) for cylindrical shells, the optimal configuration is to made of the fibers oriented at the angle 0° – the chain of ones.

4. Concluding Remarks

For angle-ply laminated cylindrical shells the optimal fibre configurations can be found easily plotting the relation (5) versus fibre orientations for the prescribed geometrical parameters and the number of modes in buckling.

Using the discrete fibre orientations (0°, ±45°, 90°) the numerical FE analysis is required connected with the application of genetic or evolutionary algorithms described in Refs [19-21]. It is necessary to emphasize that the use of them does not allow to obtain the global optimum but allows us to obtain better solution than starting ones. The set of the optimization problems allows comparing the effectiveness of a developed version of the optimization algorithms. As the assessment criterion, the number of population generations required to complete the optimization process was adopted (for simplification in the drawings marked as iterations). It should be emphasized, that in each of the numerical examples analyzed in the section 3, the convergence of the process was achieved. However, it was always conditioned by the adoption of the appropriate values of the indicators regulating the convergence process. It was noted, that:

- in the genetic algorithms (GA) the values of the probabilities must be low, close to zero and cross points may not be too large (the best results were obtained for values 0.4-0.6),
- in the simulated annealing (SA) algorithms it is advantageous to adopt a moderate cooling rate,
- in the genetic algorithms, increasing the number of the cross points (above one) does not improve the convergence of the process, but even makes it worse,
- both in the genetic algorithms and the simulated annealing algorithms, in-creased population size improves process convergence, but in the case of using FEM packets, this requires extending the calculation time (in the examples studied the maximum number of individuals was equal to 150).

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