Entropy Function and Universality of Entropy-Area Relation for Small Black Holes

Rong-Gen Cai1, Chiang-Mei Chen2, Kei-ichi Maeda3,4, Nobuyoshi Ohta5 and Da-Wei Pang2,6

1Institute of Theoretical Physics, Chinese Academy of Sciences, P.O.Box 2735, Beijing 100080, China
2Department of Physics and Center for Mathematics and Theoretical Physics, National Central University, Chungli 320, Taiwan
3Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan
4Advanced Research Institute for Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan
5Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan and
6Graduate University of the Chinese Academy of Sciences, YuQuan Road 19A, Beijing 100049, China

(Dated: March 4, 2008)

We discuss the entropy-area relation for the small black holes with higher curvature corrections by using the entropy function formalism and field redefinition method. We show that the entropy $S_{BH}$ of small black hole is proportional to its horizon area $A$. In particular we find a universal result that $S_{BH} = A/2G$, the ratio is two times of Bekenstein-Hawking entropy-area formula in many cases of physical interest. In four dimensions, the universal relation is always true irrespective of the coefficients of the higher-order terms if the dilaton couplings are the same, which is the case for string effective theory, while in five dimensions, the relation again holds irrespective of the overall coefficient if the higher-order corrections are in the GB combination. We also discuss how this result generalizes to known physically interesting cases with Lovelock correction terms in various dimensions, and possible implications of the universal relation.

I. INTRODUCTION

Black hole is a fascinating object in gravity which opens a window to shed light to some quantum effects of gravity. In particular, black hole behaves like a thermal system which can be described by macroscopic quantities such as temperature and entropy. The existence of Hawking temperature indicates that the black holes emit thermal radiation due to the quantum effect, just like the usual thermodynamic objects. In Einstein’s general relativity, the black hole entropy is given by quarter of the area of the event horizon, $S_{BH} = A/4G$, known as the Bekenstein-Hawking entropy-area formula, which inspires the proposal of holographic principle of gravity [1,2].

Extremal charged black holes with degenerate horizon are particularly simple and crucial in studying many aspects of gravity. In string theory, extremal black holes are some configurations which generally preserve partial supersymmetry. For the thermal property, extremal black holes have zero temperature and therefore are thermodynamically stable. For some extremal black holes, generically with the dilaton filed, the degenerate horizon shrinks to a point (thus the horizon area vanishes) and the singularity is not protected by a regular horizon to asymptotic observer. This type of classical solutions is called “small black hole” [3]. If we naively apply the Bekenstein-Hawking entropy-area formula, the entropy of small black holes vanishes and the expected quantum degrees of freedom seem to totally “disappear”. This discrepancy comes from the fact that the general relativity is only a classical effective theory of quantum gravity. It has been pointed out that for such kind of black holes, higher curvature corrections inspired by the low-energy effective action of quantum theory of gravity, such as Gauss-Bonnet (GB) and Lovelock terms etc, are expected to stretch the horizon and reproduce correct entropy corresponding to the microstate degrees of freedom [3]. In this case, the Bekenstein-Hawking entropy-area relation, $S_{BH} = A/4G$, breaks and should be significantly revised by the higher curvature terms and could be obtained by Wald’s entropy formula [4, 5, 6].

The near-horizon geometry of small black holes is $AdS_2 \times S^{D-2}$ after “stringy cloaking” and, for this kind of geometry, recently Sen has developed an elegant approach, the so-called entropy function formalism, to calculate the entropy coming from higher curvature corrections [7, 8]. (See [9] for a review on this topic.)

Following Sen’s entropy function approach, various extremal black holes were investigated, including the two-charge small black holes from heterotic string compactified on $S^1 \times T^{9-D}$ with momentum $n$ and winding $w$ on $S^1$ in which...
the statistical entropy can be explicitly computed as \( S_{\text{statistic}} = 4\pi \sqrt{aw} \). Among numerous investigation of small black holes, there is an interesting general property first observed in [13] and later in [14, 16, 17]: the black hole entropy and the area of stretched horizon are related as

\[
S_{BH} = \frac{1}{2G} A. \tag{1}
\]

This adjusted relation indicates that the higher curvature terms contribute equal amount of entropy as Hilbert-Einstein action (scalar curvature). This is an important observation and it is natural to ask how general or universal this revised relation (1) is between the entropy and horizon area and what the physical implications behind the relation are, if any. This is the basic motivation of this paper.

As a simple example, in Sec. II we first investigate the single-charge extremal black hole in four-dimensional dilaton gravity with general quadratic curvature corrections. We find that the relation (1) is indeed universal; it is always true irrespective of the coefficients of the higher-order terms if the dilaton couplings are the same, which is the case for string effective theory. In Sec. III we generalize the analysis to arbitrary dimensions and find a constraint on the coefficients of Ricci and Riemann square terms for the universal ratio (1). In five dimensions, we find that the relation is again universal irrespective of the overall coefficient if the higher-order corrections are in the GB combination. This may be interpreted as another evidence why the higher-order corrections in string theory should be in this GB combination, in addition to the known argument of no-ghost condition [18]. These results obtained in Secs. II and III are reproduced by using the field redefinition method in Appendix A and there we also show why the entropy of small black hole with near-horizon geometry \( AdS_2 \times S^{D-2} \) is always proportional to its horizon area.

We then generalize our study to the Lovelock gravity with one gauge field in Sec. IV. In four and five dimensions, there are only GB terms with common dilaton couplings, and we are uniquely lead to the universal relation (1). It is then natural to examine how this result may be extended to higher dimensions with more Lovelock terms. It turns out that this demands relations between the coefficients for higher-order terms, as given in [19]. We also discuss more general solutions in this type of theories. Two-charge case is discussed in Appendix B.

It is worth pointing out that for the particular classes of black holes in which the microstate counting is available, we find that the macroscopic entropy satisfies the universal ratio (1). This suggests that the ratio (1) may be an important physical principle. How about the inverse statement: can the requirement that entropy-area ratio (1) be universal lead to the macroscopic entropy matching the statistical entropy? In Sec. V we examine this issue by looking at two-charge extremal black holes with only quadratic curvature corrections. If we assume the ratio (1) as a priori requirement, then the corresponding macroscopic entropy almost matches the degrees of freedom from microstate counting up to an overall numerical factor. It is interesting that this normalization factor is universal independent of spacetime dimensions. However our result so far is based only on one example. It would be interesting to find more evidence for this conjecture. Sec. VI is devoted to conclusion and discussion.

### II. UNIVERSALITY IN \( D = 4 \) DILATON GRAVITY WITH QUADRATIC CURVATURE TERMS

As a first example, let us consider the four-dimensional dilatonic Einstein-Maxwell gravity including most general quadratic curvature corrections, scalar curvature square, Ricci tensor square and Riemann tensor square, with arbitrary coefficients \( a, b, c \) and dilaton couplings \( \alpha_1, \alpha_2 \) and \( \alpha_3 \). The action reads

\[
I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{\alpha_1 \phi} F^2 + ae^{\alpha_1 \phi} R^2 - be^{\alpha_2 \phi} R_{AB}^2 + ce^{\alpha_3 \phi} R_{ABCD}^2 \right]. \tag{2}
\]

We should note that \( \alpha = \alpha_1 = \alpha_2 = \alpha_3 \) is valid for the low-energy effective theories of heterotic strings.

The near-horizon geometry of a small black hole is supposed to have \( AdS_2 \times S^2 \) geometry with constant dilaton and gauge fields (only electric here)

\[
ds^2 = v_1 \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\Omega_2^2, \quad F_{lr} = e, \quad \phi = \phi_s. \tag{3}
\]

1 In Ref. [19], the author chooses a special set of coefficients of Lovelock terms so that the gravity entropy of the black hole gives the microstate degrees of freedom in any dimension. In Appendix B we give a simple proof that with that set of coefficients, one has \( S_{BH} = A/2G \).
The solution to the set of equations (10) is to be distinguished. The remaining part of the solution depends on the values of parameters in the Lagrangian and there are three cases for $v_1, v_2$.

The following geometrical and physical quantities can be computed straightforwardly. The Riemann curvature tensors for $AdS_2$ and $S^2$ are

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{v_1}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \quad R_{mnpq} = \frac{1}{v_2}(g_{mp}g_{nq} - g_{mq}g_{np}),$$

and the Ricci tensor and scalar curvature are

$$R_{\alpha\beta} = -\frac{1}{v_1}g_{\alpha\beta}, \quad R_{mn} = \frac{1}{v_2}g_{mn}, \quad R = -\frac{2}{v_1} + \frac{2}{v_2}. \quad (5)$$

The associated quadratic combinations of curvature and the field strength square $F^2$ are

$$R_{AB}^2 = \frac{2}{v_1^2} + \frac{2}{v_2^2}, \quad R_{ABCD}^2 = \frac{4}{v_1^4} + \frac{4}{v_2^4}, \quad F^2 = -\frac{2e^2}{v_1^2}. \quad (6)$$

Here we have used the notation that the upper case indices $A, B, \ldots$ run over the whole spacetime, while $\alpha, \beta, \ldots$ over the AdS space and $m, n, \ldots$ over the sphere. This notation is used throughout this paper.

The entropy of small black holes can be derived from the entropy function by a Legendre transformation with respect to $q$.

The physical charge $q$ corresponding to the field strength $e$ at the horizon is given by

$$q = \frac{\partial f}{\partial e} = \frac{e^\alpha \phi_s}{4G} v_2 \frac{v_2}{v_1} e. \quad (9)$$

The values of $v_1, v_2$ and $\phi_s$ are determined by extremizing the entropy function, i.e. by the following equations

$$G \frac{\partial f}{\partial v_1} = \frac{1}{2} e^2 v_2 v_1 e^\alpha \phi_s - \left(\frac{v_2}{v_1} - \frac{1}{v_2}\right) \left(a e^{\alpha_1 \phi_s} - \frac{b}{2} e^{\alpha_2 \phi_s} + c e^{\alpha_3 \phi_s}\right) = 0, \quad (10)$$

$$G \frac{\partial f}{\partial v_2} = -\frac{1}{2} e^2 v_1 e^\alpha \phi_s - \left(\frac{v_1}{v_2} - \frac{1}{v_1}\right) \left(a e^{\alpha_1 \phi_s} - \frac{b}{2} e^{\alpha_2 \phi_s} + c e^{\alpha_3 \phi_s}\right) = 0,$$

$$G \frac{\partial f}{\partial \phi_s} = v_1 v_2 \left[\frac{a e^2}{8v_1^2} e^{\alpha_2 \phi_s} + a_1 e^{\alpha_1 \phi_s} \left(\frac{1}{v_1} - \frac{1}{v_2}\right)^2 - \left(\frac{b_2}{2} e^{\alpha_2 \phi_s} - c_2 e^{\alpha_3 \phi_s}\right) \left(\frac{1}{v_1^2} + \frac{1}{v_2^2}\right)\right] = 0,$$

and the black hole entropy can be obtained from the entropy function by a Legendre transformation with respect to the physical charge:

$$S_{BH} = 2\pi(eq - f). \quad (11)$$

The solution to the set of equations (10) is

$$v_1 = v_2 = 4G^2 q^2 e^{-\alpha \phi_s}. \quad (12)$$

The remaining part of the solution depends on the values of parameters in the Lagrangian and there are three cases to be distinguished.

---

2 This is only a local solution near horizon and its regular extension globally to asymptotic flat infinity is not always guaranteed. See the examples discussed in [13].
1. \( \alpha_2 \neq 0 \):
In this case, we can solve the last equation in (10) as

\[
b = \frac{2G^2\alpha q^2}{\alpha_2} e^{-(\alpha + \alpha_2)\phi_s} + \frac{2\alpha_3}{\alpha_2} e^{(\alpha_3 - \alpha_2)\phi_s},
\]

(13)

which should be understood as an equation for determining \( \phi_s \) as a function of the physical charge \( q \) but it is generally difficult to give its explicit form. The area is given by

\[
A = 4\pi v_2 = 16\pi G^2 q^2 e^{-\alpha \phi_s},
\]

(14)

and the entropy is

\[
S_{BH} = 2\pi(eq - f) = \frac{4\pi}{G} \left[ \frac{G^2(\alpha + \alpha_2)q^2}{\alpha_2} e^{-\alpha \phi_s} + \frac{c(\alpha_3 - \alpha_2)}{\alpha_2} e^{\alpha_3 \phi_s} \right].
\]

(15)

Therefore, the entropy-area relation is

\[
S_{BH} = \frac{1}{4G} \left[ \frac{\alpha + \alpha_2}{\alpha_2} + \frac{c(\alpha_3 - \alpha_2)}{G^2\alpha_2 q^2} e^{(\alpha + \alpha_3)\phi_s} \right] A.
\]

(16)

For fixed coefficients \( a, b, c \) and exponents \( \alpha, \alpha_1, \alpha_2 \) and \( \alpha_3, \phi_s \) is a function of the physical charge \( q \). It can be checked by using Eq. (13) that the square bracket in Eq. (16) is constant only for \( \alpha_2 = \alpha_3 \). Then the entropy is always proportional to the area irrespective of the coefficients \( a, b, c \) and exponents \( \alpha \)'s, a manifestation of holographic principle. Note that the relation (16) is independent of the value of \( \alpha_1 \). This is because the contribution of the scalar curvature vanishes in the solution.

The fact that the entropy is proportional to the area may appear somewhat trivial for the spherically symmetric small black holes. However we emphasize that Eq. (16) does not mean the proportionality. Given the entropy and the area expressed in terms of solutions, we can formally write their relation like Eq. (16), but it has to be checked whether the coefficient is a constant or not. Indeed, in our above example, we have shown that we must have \( \alpha_2 = \alpha_3 \); otherwise the coefficient changes when we change the physical charge. We cannot say that they are proportional for such a general case.

In the string effective theory, \( \alpha = \alpha_2 = \alpha_3 \) and the relation (16) reduces to the universal one (1). The relation (1) is universally true irrespective of the precise values of the coefficients \( a, b, c \) and exponents \( \alpha \)'s, and only the relative magnitudes of the exponents \( \alpha \)'s are important. In particular, it is not necessary that the higher-order corrections are in the GB combination. Thus the dilaton coupling appears to automatically adjust these values to produce the universal result (1) in string theories. Here we can solve Eq. (13) for \( \phi_s \) to obtain

\[
\phi_s = \frac{1}{2\alpha} \ln \left( \frac{2G^2 q^2}{b - 2c} \right).
\]

(17)

2. \( \alpha_3 \neq 0 \):
We have basically the same equations as in item 1 with \( (\alpha_2, b) \) and \( (\alpha_3, -2c) \) interchanged, and get the universal relation (1) for \( \alpha = \alpha_2 = \alpha_3 \).

3. \( \alpha_2 = \alpha_3 = 0 \):
In this case, the last equation of (10) gives constraint \( v_1v_2\frac{2G^2}{b-2c} e^{\alpha \phi_s} = 0 \), namely

\[
\alpha v_2 = 0.
\]

(18)

For the case \( v_2 = 0 \) and \( \alpha \neq 0 \), which amounts to \( \phi_s \rightarrow \alpha \times \infty \) from (12) (the exceptional case is neutral solution \( q = 0 \) which is not of our interest), both area and entropy vanish. The near-horizon geometry (3) is not valid in this case, and we expect that further higher curvature corrections are necessary.

For the other case \( \alpha = 0 \), the dilaton is completely decoupled and the system is not of our interest. Nevertheless, let us see what we get. The horizon area and the entropy are given by (this is the only case in which \( f \neq 0 \))

\[
A = 4\pi v_2 = 16\pi G^2 q^2, \quad S_{BH} = 2\pi(eq - f) = 4\pi G q^2 + \frac{2\pi(b - 2c)}{G},
\]

(19)
independently of the value of $\alpha_1$. We find the entropy-area relation

$$S_{BH} = \frac{1}{4G} \left( 1 + \frac{b - 2e}{2G^2 q^2} \right) A.$$  \tag{20}

When the higher-order corrections are absent, this gives

$$S_{BH} = \frac{1}{4G} A,$$  \tag{21}

which is nothing but the result for Einstein gravity.

Because the scalar curvature vanishes in the solution, their corrections do not play any role in the above evaluation. So the above results are always valid even if other higher-order corrections in the scalar curvature are added to the action \cite{2}

$$\sum_{n=2} c_n e^{\alpha_n \phi} R^n.$$  \tag{22}

We have thus found that small black holes in four dimensions are very special in that they have the universal relation \cite{1} for arbitrary combination of the curvature square terms, not just for the GB combination as usually supposed to be. We will see that this is no longer true in higher dimensions with higher curvature corrections. However if we consider Lovelock type corrections, we get the relation \cite{1}.

\section{III. ARBITRARY DIMENSIONS}

In this section, we examine the entropy-area relation for the general theory in arbitrary $D$ dimensions with the action

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{\alpha \phi} F^2 + ae^{\alpha_1 \phi} R^2 - be^{\alpha_2 \phi} R_{AB} + ce^{\alpha_3 \phi} R_{ABCD} \right].$$  \tag{23}

The extremal black holes in $D$ dimensions are assumed to have near-horizon geometry $AdS_2 \times S^{D-2}$ and the relevant geometric quantities are

$$R_{\alpha\beta} = -\frac{1}{v_1} g_{\alpha\beta}, \quad R_{mn} = \frac{D-3}{v_2} g_{mn}, \quad R = -\frac{2}{v_1} + \frac{(D-2)(D-3)}{v_2},$$

$$R_{AB} = \frac{2}{v_1^2} + \frac{(D-2)(D-3)}{v_2^2}, \quad R_{ABCD} = \frac{4}{v_1^2} + \frac{2(D-2)(D-3)}{v_2^2}, \quad F^2 = -\frac{2e^2}{v_1^2}.$$  \tag{24}

It is straightforward to compute the entropy function

$$f \equiv \frac{\Omega_{D-2} v_1 v_2^{\frac{D-2}{2}}}{16\pi G} \left[ \frac{(D-2)(D-3)}{v_2} - \frac{2}{v_1} + \frac{1}{2} e^{\alpha \phi} \frac{e^2}{v_1^2} + ae^{\alpha_1 \phi} \left( \frac{2}{v_1} - \frac{(D-2)(D-3)}{v_2} \right)^2 \right.$$

$$-be^{\alpha_2 \phi} \left( \frac{2}{v_1^2} + \frac{(D-2)(D-3)}{v_2^2} \right) + 2ae^{\alpha_3 \phi} \left( \frac{2}{v_1^2} + \frac{(D-2)(D-3)}{v_2^2} \right) \left. \right],$$  \tag{26}

where $\Omega_{D-2} = 2\pi \frac{D-1}{\Gamma\left(\frac{D-1}{2}\right)}$ is surface area of unit sphere $S^{D-2}$. The relation of physical charge $q$ and the field strength $e$ is

$$e = \frac{16\pi G v_1}{\Omega_{D-2} e^{\alpha \phi} v_2^{\frac{D-2}{2}}} q.$$  \tag{27}

In general, it is complicated to find the extremal value of $f$ for arbitrary dilaton couplings. For simplicity we focus on the special case of string effective theory in which $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ (all dilaton couplings are equal). The conditions for $f$ to have extremal value give the relation

$$v_2 = \frac{1}{2} (D-2)(D-3)v_1,$$  \tag{28}
which implies vanishing scalar curvature, \( R = 0 \), and the constants \( v_1 \) and \( \phi_s \) must satisfy the equations

\[
v_1 = \frac{(D - 3)e^2 + 8c(D - 4)}{D(D - 3)} e^{\alpha \phi_s},
\]

\[
b = \frac{1}{4} \frac{(D - 2)(D - 3)e^2 + 8c(D^2 - 5D + 8)}{D(D - 3)}. \tag{29}
\]

Note that \( \partial f / \partial \phi_s = 0 \) implies either \( \alpha = 0 \) in which dilaton is totally decoupled (and we are not interested in this) or \( f = 0 \). The explicit solutions for \( v_1 \) and \( \phi_s \) are complicated. However, the explicit forms are not essential for deriving the entropy-area ratio.

The area of event horizon is

\[
A = \Omega_D^{-2} \frac{b_2}{v_2} = \Omega_D^{-2} \left[ (2Db - 6b - 4c) e^{\alpha \phi_s} \right] \frac{b_2}{v_2}, \tag{31}
\]

and the entropy is \((f = 0)\)

\[
S_{BH} = 2\pi(eq - f) = \frac{\Omega_D^{-2} \frac{b_2}{v_2} \left[ D(D - 3)b - 2(D^2 - 5D + 8)c \right]}{8G(Db - 3b - 2c)}. \tag{32}
\]

Thus the entropy-area relation is

\[
S_{BH} = \frac{D(D - 3)b - 2(D^2 - 5D + 8)c}{8G(Db - 3b - 2c)} A. \tag{33}
\]

We thus again find that the entropy is always proportional to the area irrespective of the coefficients \( a, b, c \) and exponents \( \alpha \)'s. We can derive this relation from the Bekenstein-Hawking formula by field redefinition (see Appendix A).

Let us also check when we get the relation \( S_{BH} = A/2G \). It turns out that we must have either \( D = 4 \) or

\[
(D - 3)b = 2(D - 1)c. \tag{34}
\]

This means that the corrections are in the GB combination for \( D = 5 \). In this case, only the relative magnitudes of the coefficient are determined and the universal relation \([\text{I}]\) is true irrespective of the overall factor. If we regard the universal relation as a physically important principle, this might be another evidence why the higher-order corrections must come in the GB combination in string effective theories in addition to the ghost-free condition \([18]\).

For theories in \( D > 5 \), \([\text{II}]\) does not give the GB combination. Does this mean that the relation \([\text{I}]\) is not valid in dimensions higher than five? Considering that GB term is the leading correction in heterotic string theory, this problem gets physical interest. Rather than hastily jumping to such a conclusion, we suggest that this is an indication that we should consider more higher curvature corrections, such as the Lovelock terms. We are now going to consider this possibility and see that indeed we can obtain the relation \([\text{I}]\) in higher dimensions.

### IV. LOVELOCK GRAVITY

In this section, let us consider the dilatonic Einstein-Maxwell theory with Lovelock higher curvature corrections. In the string frame, the action is

\[
I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} S \sum_{m=1}^{\alpha} \alpha^{m-1} L_m, \tag{35}
\]

where \( S \) on the rhs is the dilaton field. The leading term in the Lagrangian is

\[
L_1 = R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} F^2, \tag{36}
\]

and the higher-order terms are

\[
L_m = \frac{\lambda_m}{2m} \delta_{\rho_1 \sigma_1} \cdots \delta_{\rho_m \sigma_m} R_{\rho_1 \nu_1 \cdots \rho_m \nu_m \rho_1 \sigma_1} \cdots R_{\mu_1 \nu_1 \cdots \mu_m \nu_m \rho_1 \sigma_1}, \quad m = 2, \ldots, [D/2], \tag{37}
\]
where $\lambda_m$ are dimensionless parameters. This action, written in the string frame, is a generalization of those actions considered in the previous sections to higher dimensions with higher curvature terms for equal dilaton coupling.

The entropy function for the black hole with $AdS_2 \times S^{D-2}$ near-horizon geometry is

$$f = \frac{\Omega_{D-2} u_S v_1 v_2^{D-2}}{16\pi G} \left\{ \frac{e^2}{2v_1^2} + \sum_{m=1}^{[D/2]} \frac{(D-2)!}{(D-2m)!} \alpha^{m-1} \lambda_m \left[ (D-2m)(D-2m-1) - \frac{2mv_2}{v_1} \right] \right\}, \quad (38)$$

where $u_S$ is the near-horizon value of dilaton field $S$. Here the Hilbert-Einstein term (scalar curvature) is included as the $m = 1$ term by defining $\lambda_1 = 1$. The summation in the entropy function can be rearranged in terms of power of $v_2$ as

$$\sum_{m=1}^{[D/2]} (\cdots) = \sum_{m=1}^{[D/2]} K_m v_2^{1-m}, \quad (39)$$

where the coefficients $K_m$, functions of $v_1$, are

$$K_1 = -\frac{2}{v_1}, \quad K_m = \alpha^{m-2} \frac{(D-2)!}{(D-2m)!} \left( \lambda_{m-1} - \frac{2m\alpha'}{v_1} \lambda_m \right), \quad m \geq 2. \quad (40)$$

The equation $\partial f/\partial u_S = 0$ for extremal value of $f$ requires $f = 0$,

$$\sum_{m=2}^{[D/2]} K_m v_2^{1-m} + \frac{e^2}{2v_1^2} - \frac{2}{v_1} = 0, \quad (41)$$

and the equation $\partial f/\partial v_2 = 0$ gives

$$\sum_{m=2}^{[D/2]} (m-1)K_m v_2^{-m} = 0. \quad (42)$$

The last equation $\partial f/\partial v_1 = 0$ is more complicated

$$\sum_{m=2}^{[D/2]} \frac{(D-2)!}{(D-2m)!} m\alpha^{m-1} \lambda_m v_2^{1-m} - \frac{e^2}{2v_1} + 1 = 0. \quad (43)$$

Finally, the value of $u_S$ is determined from the physical charge defined by

$$q = \frac{\partial f}{\partial e} = \frac{\Omega_{D-2} u_S v_1 v_2^{D-2}}{16\pi G v_1} e. \quad (44)$$

Now our goal is to solve equations (41)–(44) to find the values of $u_S, v_1, v_2$ and $e$ in terms of the physical charge $q$ and the parameters $\lambda_m$’s in the theory. Formally the horizon area and the black hole entropy ($f = 0$) can be expressed as

$$A = \Omega_{D-2} u_S v_2^{D-2} = 16\pi G \frac{v_1 q}{e}, \quad S_{BH} = 2\pi(eq - f) = 2\pi eq, \quad (45)$$

and the ratio is

$$S_{BH} = \frac{e^2}{8Gv_1} A. \quad (46)$$

Here again the entropy is proportional to the area. Note that here the universal relation (11) is equivalent to

$$v_1 = \frac{e^2}{4}. \quad (47)$$
A. Universal relation in $D = 4, 5$ and generalization to higher dimensions

In $D = 4, 5$, there are only GB terms. Let us first consider $D = 4$ though this is a special case of those discussed in Sec. III. From Eqs. (42) and (41), we find

$$v_1 = 4\alpha'\lambda_2, \quad e^2 = 4v_1 = 16\alpha'\lambda_2, \quad (48)$$

which already assures that the universal relation (1) holds in this case (see (43)). The remaining Eq. (48) then gives $v_2$ as

$$v_2 = 4\alpha'\lambda_2 = v_1. \quad (49)$$

The dilaton field can be determined as $u_1 = Gq/(4\alpha'\lambda_2)$ through (44). Thus all moduli fields are given in terms of the physical charge $q$ and the parameter $\lambda_2$. Note that here the relation $S = A/2G$ is independent of the value of $\lambda_2$, namely, $\lambda_2$ can be arbitrary, in agreement with the result in Sec. III.

Next for the case $D = 5$, we obtain

$$v_1 = 4\alpha'\lambda_2, \quad e^2 = 4v_1 = 16\alpha'\lambda_2, \quad (50)$$

and $v_2 = 12\alpha'\lambda_2$. Thus, once again, we have the universal relation (1) irrespective of the value of $\lambda_2$, in agreement with the results in Sec. III. Of course, if we want to match the result with the microstate counting, the value of $\lambda_2$ should be fixed in some way (see for example, [19] or [22] where $\lambda_2$ is determined to be $\frac{1}{2}$). Note that in the case of $D = 5$, one has also $K_2 = 0$.

When we go up to $D = 6$ and $D = 7$, the next Lovelock term contributes. We know from Eq. (45) that if one requires that the entropy should be independent of the spacetime dimension, $e$ must be independent of the dimension (note that $q$ is a physical charge, which is an input quantity). On the other hand, since we already know that $K_2 = 0$ from the case of $D = 4$ and $5$, we conclude that $K_3 = 0$ from Eq. (42). The only solution of Eq. (41) is then $v_1 = 4e^2$, which is independent of the spacetime dimension. We thus come back to the universal result (1), again whatever the value of $\lambda_2$ is. We see that this is the value that new higher-order correction does not modify the solution for the lower-dimensional solutions. The condition $K_3 = 0$ gives

$$\lambda_3 = \frac{v_1}{6\alpha'\lambda_2}. \quad (51)$$

Combined with Eq. (43), Eq. (51) leads to

$$\lambda_3 = \frac{1}{3!}(2\lambda_2)^2, \quad (52)$$

so that we can see that only $\lambda_2$ is a free parameter.

As we have observed, the solution $v_2$ depends on dimension $D$, but $v_1$ does not. It is natural to take the value of $v_1$ as it is in all dimensions [19]. Under this condition, Eqs. (41) and (42) determine the solution$^5$ by $K_m = 0$ for $m \geq 2$, which completely fix all other $\lambda_m$ by $\lambda_2$ via

$$\lambda_m = \frac{v_1}{2m\alpha'}\lambda_{m-1} = \frac{1}{m!}\left(\frac{v_1}{2\alpha'}\right)^{m-1} = \frac{1}{m!}(2\lambda_2)^{m-1}. \quad (53)$$

where we also used the result (43). Obviously the universal relation $S_{BH} = A/2G$ still holds by (46) and (47). Note that $v_2$ is determined by Eq. (43) and it is easy to show that it has at least one positive solution. This point will be discussed later in more general context.

The relation (43) also indicates that $e$ is not a free parameter but its value is fixed by parameters $\alpha'$ and $\lambda_2$ in the Lagrangian. Thus the value of $e$ does not appear to contain any information on the magnitude of charge. This seems peculiar since $e$ is the value of $F_1$ at horizon. This strange result is due to the fact that the dependence of field strength on charge parameter is hidden in the string frame by the coupling of the dilaton. From Eq. (44), one can see that the value $u_1$ of the dilaton at horizon actually carries the charge information. The relation between the field strength and charge becomes more transparent if the solution is presented in the Einstein frame.

---

$^3$ Note that in this case, the unique solution of (43) is $K_2 = 0$.

$^4$ Similar argument based on the microstate counting is given in Ref. [19].

$^5$ Note that each $K_i$ is different by a numerical factor in different dimensions.
Finally we note that there may be other possibility for achieving the relation $S_{BH} = A/2G$ if we do not require that the coefficients be determined step by step from lower dimensions to higher dimensions as above. For example, consider $D = 10$ with free parameters $\lambda_2, \lambda_3$ and $\lambda_4$. Let us take $\lambda_2 = \frac{1}{8}$ and $\lambda_3 = 0$, which are the known values for these in string theory \[22\]. To have the relation $S_{BH} = A/2G$, we must have $v_1 = e^2 / 4$ and the extremal conditions of the entropy function give

$$v_1 = \frac{7(19823 - 31\sqrt{89761})}{165243} \alpha'^{0} > 0, \quad v_2 = \frac{7}{223}(195 + \sqrt{89761})\alpha' > 0, \quad \lambda_4 = \frac{49(366563 + 3737\sqrt{89761})}{15968976480}.$$  \hspace{1cm} (54)

Thus the relation (11) is possible in arbitrary dimensions by adjusting $\lambda$‘s even if we do not require $\lambda$‘s are determined in lower dimensions. These values need not be taken serious because it is known that there are other corrections in the order of $R^4$ in the heterotic strings \[22\]. (The string case in higher dimensions is beyond the scope of this paper and may be a subject of future study.) We now discuss this kind of general solutions in more detail.

### B. General solution

In this section, we show that we can formally solve Eqs. (11)–(14) for $u_S, v_1, v_2$ and $e$, and obtain positive $v_1$ and $v_2$. Eliminating $e^2$, which contains $u_S$, from Eqs. (11) and (13), we find

$$\sum_{m=2}^{\lfloor D/2 \rfloor} \frac{(D-2)!}{(D-2m)!} \alpha^{m-2} (\lambda_{m-1} v_1 - m\alpha' \lambda_m) v_2^{1-m} = 0.$$  \hspace{1cm} (55)

This equation is rewritten as

$$A_1 v_1 = B_1,$$  \hspace{1cm} (56)

where

$$A_1 = \sum_{m=2}^{\lfloor D/2 \rfloor} \frac{(D-2)!}{(D-2m)!} \alpha^{m-2} \lambda_{m-1} v_1^{1-m},$$

$$B_1 = \sum_{m=2}^{\lfloor D/2 \rfloor} m \frac{(D-2)!}{(D-2m)!} \alpha^{m-1} \lambda_m v_2^{1-m} + 1.$$  \hspace{1cm} (57)

Now Eq. (13) has a simple expression

$$e^2 = 2v_1 B_1,$$  \hspace{1cm} (58)

and Eq. (12) can also be rewritten in the form of

$$A_2 v_1 = B_2,$$  \hspace{1cm} (59)

where

$$A_2 = \sum_{m=2}^{\lfloor D/2 \rfloor} \frac{(m-1)(D-2)!}{(D-2m)!} \alpha^{m-2} \lambda_{m-1} v_1^{1-m},$$

$$B_2 = 2 \sum_{m=2}^{\lfloor D/2 \rfloor} \frac{m(m-1)(D-2)!}{(D-2m)!} \alpha^{m-1} \lambda_m v_2^{1-m}.$$  \hspace{1cm} (60)

Eliminating $v_1$ from Eqs. (56) and (59), we find the equation for $v_2$ as

$$\mathcal{F}(v_2) \equiv \left(v_2^{\lfloor D/2 \rfloor - 2} [A_2 B_1 - A_1 B_2]ight) = 0,$$  \hspace{1cm} (61)

whose explicit form is

$$\mathcal{F}(v_2) = \left(\sum_{m=2}^{\lfloor D/2 \rfloor} \frac{(m-1)(D-2)!}{(D-2m)!} \alpha^{m-2} \lambda_{m-1} v_2^{1-m} \right) \left(v_2^{\lfloor D/2 \rfloor - 1} + \sum_{m=2}^{\lfloor D/2 \rfloor} \frac{m(D-2)!}{(D-2m)!} \alpha^{m-1} \lambda_m v_2^{1-m} \right)$$

$$-2 \left(\sum_{m=2}^{\lfloor D/2 \rfloor} \frac{(D-2)!}{(D-2m)!} \alpha^{m-2} \lambda_{m-1} v_2^{1-m} \right) \left(\sum_{m=2}^{\lfloor D/2 \rfloor} \frac{m(m-1)(D-2)!}{(D-2m)!} \alpha^{m-1} \lambda_m v_2^{1-m} \right).$$  \hspace{1cm} (62)
The equation $F(v_2) = 0$ is the $(2[D/2] - 3)$ order algebraic equation for $v_2$. Once we find the solution for $v_2$, we obtain the solutions for $v_1$ and $u_S$ as

$$v_1 = \frac{B_1}{A_1} = \frac{B_2}{A_2}, \quad (63)$$

$$u_S^2 = \frac{1}{2} \left( \frac{16\pi G q}{\Omega_{D-2}} \right)^2 \frac{1}{A_1v_2^{D-2}}. \quad (64)$$

Hence if there exists a solution of $F(v_2) = 0$ for $v_2$, we obtain the solution for any coupling constant $\lambda_m$ and any charge $q$.

Here let us give two simple examples.

1. Case of $[D/2] = 2$ ($D = 4$ or 5):
   The equation for $v_2$ is very simple and the solution is $v_2 = 2(D - 2)(D - 3)\alpha'\lambda_2$. We find $v_1 = 4\alpha'\lambda_2$, that is $v_2 = (D - 2)(D - 3)v_1/2$. (Note that $\lambda_1 = 1$.)

2. Case of $[D/2] = 3$ ($D = 6$ or 7):
   $F(v_2) = 0$ gives the cubic equation for $v_2$ as

$$v_2^3 - 4(2D - 7)\alpha'\lambda_2v_2^2 - 9(D - 2)(D - 3)(D - 4)(D - 5)\alpha'^2\lambda_3v_2$$

$$-6(D - 2)(D - 3)(D - 4)^2(D - 5)^2\alpha'^3\lambda_2\lambda_3 = 0. \quad (65)$$

In general, it is difficult to obtain an explicit solution of $v_2$ from $F(v_2) = 0$. Now, for this solution to be acceptable, we must show that there exists at least one positive solution for $v_2$. Since $F(v_2) = (D - 2)(D - 3)v_2^{2[D/2]-3} + \cdots$, $F(v_2) \rightarrow +\infty$ as $v_2 \rightarrow \infty$. Also

$$F(v_2)|_{v_2=0} = -\frac{\Omega_{[D/2]}(\Omega_{[D/2]}-1)}{\Omega_{[D/2]}(\Omega_{[D/2]}-1)!}2\alpha'^2\lambda_2\lambda_1\lambda_3 < 0, \quad (66)$$

if $\lambda_{[D/2]-1}\lambda_{[D/2]} > 0$. Because the continuous function $F(v_2)$ changes from negative value at $v_2 = 0$ to infinity at $v_2 = \infty$, we have at least one positive solution for $v_2$ when this last condition is satisfied.

As for the entropy, inserting our solution into the definition $S = 2\pi eq$, we have

$$S_{BH} = B_1 \times \frac{A}{4G}. \quad (67)$$

We can reproduce our earlier results using these formulae. We see from Eq. [57] that $B_1 > 1$ if $\lambda_m$’s are all positive.

### V. Matching Microstate Counting

In this section we elaborate on the connection between the entropy-area relation and the microstate counting. We consider a theory with two gauge fields in any dimension. The statistical entropy of this theory is known and it has been verified that the microstate entropy can be reproduced from gravity side, for example by including Lovelock corrections with an appropriate tuning of coefficients [19]. Here we would like to change our viewpoint to a different side. We include only general quadratic curvature corrections and check whether the entropy-area relation leads to the statistical entropy. The action under consideration in the string frame is

$$I = \frac{1}{16\pi G} \int d^Dx \sqrt{-g}S \left[ R + S^{-2}(\partial S)^2 - T^{-2}(\partial T)^2 - T^2 \left( F^{(1)}_{[2]} \right)^2 - T^{-2} \left( F^{(2)}_{[2]} \right)^2 + \alpha' (a R^2 - b R_{AB}^2 + c R_{ABCD}^2) \right], \quad (68)$$

and the near-horizon data are exactly the ones in the previous sections. The entropy function is

$$f = \frac{\Omega_{D-2}u_Sv_1v_2^{D-2}}{16\pi G} \left[ \frac{(D - 2)(D - 3)}{v_2} - \frac{2}{v_1} + \frac{2e_1^2u_T^2}{v_1^2} + \frac{2e_2^2}{u_T^2u_T^2} + \alpha' \left( \frac{2}{v_1} - \frac{(D - 2)(D - 3)}{v_2} \right)^2 - \alpha' \left( \frac{2}{v_1^2} + \frac{(D - 2)(D - 3)}{v_2^2} \right) + 2\alpha' \left( \frac{2}{v_1^2} + \frac{(D - 2)(D - 3)}{v_2^2} \right) \right]. \quad (69)$$
The physical charges are

\[ q_i = \frac{\partial f}{\partial e_i}, \quad q_1 = \frac{\Omega_{D-2}u_Su_2^2v_2^{\frac{D-2}{2}}}{4\pi Gv_1} e_1, \quad q_2 = \frac{\Omega_{D-2}u_Sv_2^{\frac{D-2}{2}}}{4\pi Gv_1^2 v_1} e_2, \]

and the solution for an extremal \( f \) is, for \( D = 4 \),

\[ u_1^2 = \frac{e_2}{e_1}, \quad v_2 = v_1, \quad v_1 = 2e_1e_2 = 2(b - 2c)\alpha', \quad e_1e_2 = (b - 2c)\alpha', \]

and for \( D > 4 \)

\[ u_1^2 = \frac{e_2}{e_1}, \quad v_2 = \frac{(D - 2)(D - 3)v_1}{2}, \quad v_1 = \frac{4[2e_1e_2 + \alpha'b(D - 4)]}{D^2 - 5D + 8}, \]

\[ e_1e_2 = \frac{[D(D - 3)b - 2(D^2 - 5D + 8)c]\alpha'}{2(D - 2)(D - 3)} \]

Since \( f = 0 \), the entropy and area (in the Einstein frame) are

\[ S_{BH} = 2\pi(e_1q_1 + e_1q_2) = \frac{\Omega_{D-2}u_Sv_2^{\frac{D-2}{2}}}{2Gv_1}(2e_1e_2), \quad A = u_S\Omega_{D-2}v_2^{\frac{D-2}{2}}, \]

and they are related by

\[ S_{BH} = \frac{A}{2G} \left( \frac{2e_1e_2}{v_1} \right). \]

For \( D = 4 \), we have \( v_1 = 2e_1e_2 \), therefore the relation \( S_{BH} = A/2G \) holds independently of the values of \( a, b, c \) and the value of entropy \( S_{BH} = 4\pi u_S e_1e_2/G \). Moreover, from the relation \[19\]

\[ q_1 = \frac{2n}{\sqrt{\alpha'}}, \quad q_2 = \frac{2w}{\sqrt{\alpha'}}, \quad \frac{q_1}{q_2} = \frac{e_2}{e_1} \frac{n}{w}, \]

we have

\[ e_1 = \sqrt{(b - 2c)\alpha' \frac{w}{n}}, \quad e_2 = \sqrt{(b - 2c)\alpha' \frac{n}{w}}, \quad u_S = \frac{2G}{\sqrt{b - 2c} \sqrt{\alpha'}} \sqrt{nw}. \]

The entropy can be expressed in terms of the momentum \( n \) and winding number \( w \) as

\[ S_{BH} = 8\pi \sqrt{b - 2c \sqrt{nw}}. \]

From this check, we conclude that the entropy-area relation \[11\] can lead to the black hole entropy to match the statistical entropy up to a numerical factor. In order to fix this numerical factor we should require \( b - 2c = 1/4 \) to match the microstate counting \( S = 4\pi \sqrt{nw} \). It is interesting, as we will see, this same additional requirement is also necessary in any dimension.

For a general dimension, we have

\[ e_1 = \sqrt{\alpha' \Delta \frac{w}{n}}, \quad e_2 = \sqrt{\alpha' \Delta \frac{n}{w}}, \quad u_S = \frac{8\pi Gv_1}{\Omega_{D-2}v_2^{\frac{D-2}{2}} \sqrt{\Delta \alpha'}} \sqrt{nw}, \]

where

\[ \Delta = \frac{D(D - 3)b - 2(D^2 - 5D + 8)c}{2(D - 2)(D - 3)}. \]

The entropy is

\[ S_{BH} = \frac{\Omega_{D-2}u_Sv_2^{\frac{D-2}{2}}}{2Gv_1}(2e_1e_2) = 8\pi \sqrt{\Delta \sqrt{nw}}. \]
In order to match the microstate counting, we should have \( \Delta = 1/4 \). Moreover, if we first require the condition \( S_{BH} = A/2G \), then we have relation \( v_1 = 2c_1c_2 \), which gives \((D - 3)b = 2(D - 1)c\), this is identical to the result (44) for the single charge case. But note that it is not sufficient to make \( \Delta = 1/4 \). Therefore, it seems that the condition \( S_{BH} = A/2G \) and matching to microstate counting in general are two independent requirements when we consider quadratic correction. When \( D = 4 \) or 5, however, if

\[
b = 4c = 1/2, \quad (81)
\]

these quadratic curvature terms in the action can be written in the GB combination. In that case, we have \( S = A/2G \) for both \( D = 4 \) and 5. On the other hand, \( \Delta = 1/4 \). Thus \( S_{BH} = A/2G \) matches the microstate counting. Note that \( c = 1/8 \) from (51) just gives \( \lambda_2 = 1/8 \) in the previous section, while the latter is predicted by heterotic string theory. Finally, let us note that an interesting general feature appears once we demand the ratio of entropy and area \( (51) \). Under this condition, we have \((D - 3)b = 2(D - 1)c\) and then \( \Delta = b - 2c \), which is independent of spacetime dimension. Thus the matching condition, \( b - 2c = 1/4 \), yields

\[
b = D - 1 \quad 8, \quad c = D - 3 \quad 16. \quad (82)
\]

Although Eq. (82) gives the GB combination only when \( D = 5 \), we expect this argument to reproduce statistical entropy can apply to other cases. On the other hand, if we include higher-order Lovelock terms as in \( [19] \), the universal entropy-area relation \( (1) \), \( S_{BH} = A/2G \), can be matched to the microstate counting, as is shown in Appendix \( [B] \).

\section{VI. Conclusion}

In the Einstein gravity, the entropy of black holes is universally given by the so-called area formula \( S_{BH} = A/4G \). However it is well known that the area formula no longer holds in general if one considers higher-order curvature correction terms. Still one can calculate black hole entropy by employing the Wald’s entropy formula in the higher-order derivative gravity theories. To calculate black hole entropy using Wald’s entropy formula, one has to know the black hole solution. In general, however, it is difficult to find analytical black hole solutions in higher-order derivative gravity theories. The entropy function method proposed by Sen is a powerful approach to get the entropy of black hole with higher-curvature correction corrections.

In this paper we showed that the entropy of small black holes with near-horizon geometry \( AdS_2 \times S^{D-2} \) is always proportional to its horizon area by employing the entropy function method and field redefinition approach. In particular we found a universal result that the ratio is two times of Bekenstein-Hawking entropy-area formula in many cases of physical interest, namely \( S_{BH} = A/2G \). In four dimensions, the universal relation always holds irrespective of the coefficients of the higher-order terms if the dilaton couplings are the same, which is the case for string effective theory, while in five dimensions, the relation is again universal irrespective of the overall coefficient if the higher-order corrections are in the GB combination. We also discussed how this result generalizes to known physically interesting cases with Lovelock correction terms in various dimensions. In the Lovelock gravity with two gauge fields, the requirement to match the microstate counting of black holes is consistent with the universal entropy-area relation, \( S_{BH} = A/2G \). Based on the results derived in the present paper and those in the literature, one expects that the relation \( S_{BH} = A/2G \) might be a guidance to match the microstate counting of small black holes. Of course, to confirm this conjecture, more evidence needs to be accumulated.

It would also be interesting to generalize our results to more general black hole solutions like those with deformed horizons.

\section*{Acknowledgement}

RGC and CMC are grateful to Kinki University for hospitality in August 2007 when this work was initiated. KM and NO thank KITPC at Beijing for hospitality during their stay when part of this work was carried out. RGC and DWP were supported by a grant from Chinese Academy of Sciences, grants from NSFC with No. 10325525 and No. 90403029. The work of CMC was supported by the National Science Council under the grant NSC 96-2112-M-008-006-MY3, and in part by the National Center for Theoretical Sciences. The work of NO was supported in part by Grants-in-Aid for Scientific Research Fund of the JSPS Nos. 16540250 and 060402. The work of KM was partially supported by the Grant-in-Aid for Scientific Research Fund of the JSPS (No.19540308) and for the Japan-U.K. Research Cooperative Program, and by the Waseda University Grants for Special Research Projects and for the 21st-Century COE Program (Holistic Research and Education Center for Physics Self-Organization Systems) at Waseda University.
APPENDIX A: AREA OF HORIZON AND FIELD REDEFINITION

Here we derive the entropy-area relation including higher curvature corrections from the Bekenstein-Hawking formula by a field redefinition. Suppose we have a model with Lagrangian given by an arbitrary function $F$ of Ricci tensor $R_{AB}$, i.e.

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} F(g^{AB}, R_{AB}, \psi),$$  

(A1)

where $\psi$ denotes matter field. Introducing the new metric, which is a kind of field redefinition, as

$$\sqrt{-q} g^{AB} = \sqrt{-g} \frac{\partial F}{\partial R_{AB}},$$  

(A2)

we can rewrite our system as the Einstein equations with respect to $q_{AB}$, and $g^{AB}$ behaves as a spin 2 tensor field [20]. The explicit form of $q^{AB}$ is not important in calculation of black hole entropy. In this Einstein frame (we call it $q$-frame), we have the Bekenstein-Hawking formula, that is, $S_{BH} = A_q/4G$ [21], where $A_q$ is the area of a black hole in $q$-frame. Using Eq. (A2), we find the entropy-area relation in the original $g$-frame. Note that black hole entropy is invariant under the frame transformation.

Let us discuss a concrete example. We first consider the model in Sec. IIII with $c = 0$. In this case, we find

$$\sqrt{-q} g^{\alpha\beta} = \sqrt{-g} \left[ 1 + 2ae^{\alpha\psi} R \right] g^{\alpha\beta} - 2be^{\alpha\psi} R^{AB}.$$  

(A3)

Using the near-horizon solution (24) and (28), we find

$$\sqrt{-q} g^{\alpha\beta} = \sqrt{-g} \left( 1 + \frac{2b}{v_1} e^{\alpha\psi} \right) g^{\alpha\beta},$$  

$$\sqrt{-q} g^{\mu\nu} = \sqrt{-g} \left( 1 - \frac{4b}{(D-2)v_1} e^{\alpha\psi} \right) g^{\mu\nu}.$$  

(A4)

These relations can be rewritten as

$$q_{\alpha\beta} = \left( 1 + \frac{2b}{v_1} e^{\alpha\psi} \right)^{\frac{D-2}{2}} \left( 1 - \frac{4b}{(D-2)v_1} e^{\alpha\psi} \right) g_{\alpha\beta},$$

$$q_{\mu\nu} = \left( 1 + \frac{2b}{v_1} e^{\alpha\psi} \right)^{\frac{D-2}{2}} g_{\mu\nu}.$$  

(A5)

The area is given by $\sqrt{\det(g_{mn})}$ in $g$-frame and by $\sqrt{\det(q_{mn})}$ in $q$-frame. Hence we obtain the relation between two areas $A_q$ and $A_g$ as

$$A_q = \left( 1 + \frac{2b}{v_1} e^{\alpha\psi} \right) A_g.$$  

(A6)

From Eqs. (29) and (30), we find

$$v_1 = \frac{e^2}{D} e^{\alpha\psi}, \quad b = \frac{(D-2)e^2}{4D}.$$  

(A7)

Then we have

$$A_q = \frac{D}{2} A_g.$$  

(A8)

Since we have the Bekenstein-Hawking entropy formula in $q$-frame, we therefore obtain

$$S_{BH} = \frac{A_q}{4G} = \frac{D}{8G} A_g.$$  

(A9)

We recover (33) for $c = 0$.

If we have Riemann tensor in the action, we cannot use this method to calculate black hole entropy in general. However, if we restrict our spacetime to the present metric form, i.e. $AdS_2 \times S^{D-2}$ near horizon, we have only two
metric constants $v_1$ and $v_2$, and then we can write the Riemann curvature in terms of the Ricci and scalar curvatures. Hence we find the effective action only with the Ricci and scalar curvatures, which is equivalent to any model with Lagrangian given by an arbitrary function $F$ of Riemann tensor $R_{ABCD}$, i.e.

$$I = \frac{1}{16\pi G} \int d^Dx \sqrt{-g} F(g^{AB}, R_{ABCD}, \psi).$$  \hspace{1cm} (A10)$$

Here we give a simple example, which has been discussed in Sec. III. The scalar curvature $R$ and the Ricci curvature square $R_{AB}^2$ are given by Eqs. (24) and (25). We express $v_1$ and $v_2$ in terms of $R$ and $R_{AB}^2$ as

$$\frac{1}{v_1} = \frac{1}{2D} \left[ -2R + \sqrt{2(D-2)(DR_{AB}^2 - R^2)} \right],$$

$$\frac{1}{v_2} = \frac{1}{D(D-2)(D-3)} \left[ (D-2)R + \sqrt{2(D-2)(DR_{AB}^2 - R^2)} \right],$$  \hspace{1cm} (A11)$$

where we have chosen the plus sign when we solve the second order algebraic equation in order to guarantee both $v_1$ and $v_2$ are positive.

Inserting the expression (A11) into the Riemann curvature square (25), we find

$$R_{ABCD}^2 = \frac{2}{D^2(D-3)} \left[ -(D-4)^2 R^2 + D(D^2 - 5D + 8)R_{AB}^2 - 2(D - 4)R\sqrt{2(D-2)(DR_{AB}^2 - R^2)} \right].$$  \hspace{1cm} (A12)$$

Plugging this expression into the original action (23), we obtain the equivalent action only with $R$ and $R_{AB}^2$ as

$$I \simeq \frac{1}{16\pi G} \int d^Dx \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{\alpha \phi_s} F^2 + e^{\alpha \phi} \left( \tilde{a} R^2 - \tilde{b} R_{AB}^2 - \tilde{d} R\sqrt{2(D-2)(DR_{AB}^2 - R^2)} \right) \right] ,$$  \hspace{1cm} (A13)$$

where

$$\tilde{a} = a - \frac{2(D-4)^2}{D^2(D-3)} c, \quad \tilde{b} = b - \frac{2(D^2 - 5D + 8)}{D(D-3)} c, \quad \tilde{d} = \frac{4(D-4)}{D^2(D-3)} c.$$  \hspace{1cm} (A14)$$

Since this action is of the form of Eq. (A1), we can apply the method of field redefinition.

The redefined metric (A2) is now

$$\sqrt{-g} q^{AB} = \sqrt{-g} \left[ 1 - \tilde{a} e^{\alpha \phi_s} \sqrt{2D(D-2)R_{CD}^2} \right] g^{AB} - 2\tilde{b} e^{\alpha \phi_s} R^{AB}.$$  \hspace{1cm} (A15)$$

Here we have used the fact that the scalar curvature $R$ vanishes for our black hole solutions. Rewriting Eq. (A15), we obtain

$$q_{mn} = \left[ 1 + 2(\tilde{b} - D\tilde{d}) e^{\alpha \phi_s} \right] \frac{v_1}{v_1^2} g_{mn},$$  \hspace{1cm} (A16)$$

which implies that

$$A_q = \left[ 1 + 2(\tilde{b} - D\tilde{d}) \frac{e^{\alpha \phi_s}}{v_1} \right] A_g.$$  \hspace{1cm} (A17)$$

From Eqs. (29) and (30), we find that

$$\frac{e^{\alpha \phi_s}}{v_1} = \frac{(D-2)(D-3)}{4[(D-3)b - 2c]}.$$  \hspace{1cm} (A18)$$

Inserting this solution into Eq. (A17) and using the relations (A14), we find

$$A_q = \frac{D(D-3)b - 2(D^2 - 5D + 8)c}{2[(D-3)b - 2c]} A_g.$$  \hspace{1cm} (A19)$$

Note that we have $S_{BH} = A_q/4G$ in $g$-frame, and thus we reproduce the entropy-area relation (33) in $g$-frame.

In principle we can use the present method of field redefinition to small black holes with near-horizon geometry $AdS_2 \times S^{D-2}$ in any gravity theory with curvature scalar, Ricci tensor and Riemann tensor, e.g., the Lovelock gravity discussed in the section IV but the equations turn out to be very complicated in that case.
APPENDIX B: MATCHING THE MICROSTATE COUNTING WITH THE RELATION $S_{BH} = A/2G$ FOR TWO-CHARGE SYSTEM

In [19], Prester considered small black holes in Lovelock gravity with two gauge fields. The action is

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} S \sum_{m=1}^{\alpha' m} \mathcal{L}_m.$$  \hfill (B1)

where $S$ is the dilaton. The leading term in $\alpha'$ given by [3]

$$L_1 = R + S^{-2} (\partial S)^2 - T^{-2} (\partial T)^2 - T^2 \left( F^{(1)} \right)^2 - T^{-2} \left( F^{(2)} \right)^2$$  \hfill (B2)

and the higher-order terms are

$$L_m = \frac{\lambda_m}{2m} \delta_{\mu_1 \nu_1 \cdots \mu_m \nu_m} R^\mu_1 \nu_1 \cdots R^\mu_m \nu_m, \quad m = 2, \cdots, [D/2],$$  \hfill (B3)

where $\lambda_m$ are dimensionless parameters. By entropy function method, it turns out that if one chooses the following set of parameters

$$\lambda_m = \frac{4}{4^{m-1} m!},$$  \hfill (B4)

the microstate entropy $S = 4\pi \sqrt{nw}$ of the small black hole with two charges can be reproduced by the gravity entropy of the black hole in the action (B1) in any dimension. Here we would like to give a simple proof that the choice (B4) is consistent with the entropy-area relation (1), $S_{BH} = A/2G$.

With the data of the near-horizon geometry $AdS_2 \times S^{D-2}$, the entropy function is easy to calculate as [19]

$$f = \frac{\Omega_{D-2}}{16\pi G} u_s v_1^{1/2} v_2^{(D-2)/2} \left[ \frac{2 u_1^2 e_1^2}{v_1} + \frac{2 e_1^2}{v_1^2} - \frac{2}{v_1} - \left( \frac{1}{v_1} - \frac{2}{\alpha'} \right) A \right]$$

$$= \frac{\Omega_{D-2}}{16\pi G} u_s v_1^{1/2} v_2^{(D-2)/2} g.$$  \hfill (B5)

where

$$A = A(v_2) = \sum_{m=1}^{[D/2]} \alpha' m \frac{2 m (D-2)!}{(D-2m-2)!} \frac{1}{v_2^m}.$$  \hfill (B6)

Note that $\partial f/\partial u_S = 0$ and $\partial f/\partial u_T = 0$ give us

$$g = 0, \quad u_T = (e_2/e_1)^{1/2},$$  \hfill (B7)

$\partial f/\partial v_2 = 0$ leads to $\partial g/\partial v_2 = 0$, and the latter gives

$$v_1 = \alpha'/2.$$  \hfill (B8)

By definition, we have the physical charges $q_1$ and $q_2$ as

$$q_1 = \frac{\Omega_{D-2}}{4\pi G} u_s v_1^{1/2} v_2^{(D-2)/2} e_2, \quad q_2 = \frac{\Omega_{D-2}}{4\pi G} u_s v_1^{1/2} v_2^{(D-2)/2} e_1,$$  \hfill (B9)

where we have used the relation $u_T^2 = e_2/e_1$. Furthermore, combining (B7) with (B8) yields

$$e_1 e_2 = v_1^2/2 = \alpha'/4.$$  \hfill (B9)

Thus we can obtain the value of dilaton on the horizon through (B8) as

$$u_S = \frac{8\pi G}{\Omega_{D-2} v_2^{(D-2)/2} \sqrt{nw}}.$$  \hfill (B10)
where we have used the relations, $q_1 = 2n/\sqrt{\alpha'}$ and $q_2 = 2w/\sqrt{\alpha'}$. The horizon area in the Einstein frame is

$$A = u S \Omega_{D-2} v_2^{(D-2)/2} = 8\pi G \sqrt{nw},$$

while the gravity entropy of the black hole turns out to be

$$S_{BH} = 2\pi (e_1 q_1 + e_2 q_2) = 4\pi \sqrt{nw} = A/2G.$$  \hfill (B10)

Thus without knowing $v_2$, we show $S_{BH} = A/2G$. Indeed, the parameters given in (B4) just corresponds to the choice with $\lambda_2 = 1/8$ in (52).

[1] G. ’t Hooft, “The black hole interpretation of string theory,” Nucl. Phys. B 335 (1990) 138.
[2] L. Susskind, “Some speculations about black hole entropy in string theory,” arXiv:hep-th/9309145.
[3] A. Sen, “Extremal black holes and elementary string states,” Mod. Phys. Lett. A 10, 2081 (1995) [arXiv:hep-th/9504147];
“How does a fundamental string stretch its horizon?”, JHEP 0505, 059 (2005) [arXiv:hep-th/0411255];
“How stretching the horizon of a higher dimensional small black hole”, JHEP 0507, 073 (2005) [arXiv:hep-th/0505122].
[4] R. M. Wald, “Black hole entropy in the Noether charge,” Phys. Rev. D 48, 3427 (1993) [arXiv:gr-qc/9307038].
[5] T. Jacobson, G. Kang and R. C. Myers, “On Black Hole Entropy,” Phys. Rev. D 49, 6587 (1994) [arXiv:gr-qc/9312023].
[6] T. Jacobson, G. Kang and R. C. Myers, “Black hole entropy in higher curvature gravity,” arXiv:gr-qc/9502009.
[7] A. Sen, “Black hole entropy function and the attractor mechanism in higher derivative gravity,” JHEP 0509, 038 (2005) [arXiv:hep-th/0506177].
[8] A. Sen, “Entropy function for heterotic black holes,” JHEP 0603, 008 (2006) [arXiv:hep-th/0508042].
[9] A. Sen, “Black Hole Entropy Function. Attractors and Precision Counting of Microstates,” [arXiv:0708.1270 [hep-th]].
[10] A. Dabholkar, “Exact counting of black hole microstates,” Phys. Rev. Lett. 94, 241301 (2005) [arXiv:hep-th/0409148].
[11] A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, “Exact and asymptotic degeneracies of small black holes,” JHEP 0508, 021 (2005) [arXiv:hep-th/0502157].
[12] A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, “Precision counting of small black holes,” JHEP 0510, 096 (2005) [arXiv:hep-th/0507014].
[13] A. Dabholkar, R. Kallosh and A. Maloney, “A stringy cloak for a classical singularity,” JHEP 0412, 059 (2004) [arXiv:hep-th/0410076].
[14] Y. Hubeny, A. Maloney and M. Rangamani, “String-corrected black holes,” JHEP 0505, 035 (2005) [arXiv:hep-th/0411272].
[15] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, “Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy,” JHEP 0412, 075 (2004) [arXiv:hep-th/0412287].
[16] D. Bak, S. Kim and S. J. Rey, “Exactly soluble BPS black holes in higher curvature N = 2 supergravity,” arXiv:hep-th/0501014.
[17] C. M. Chen, D. V. Gal’tsov and D. G. Orlov, “Extremal black holes in D = 4 Gauss-Bonnet gravity,” Phys. Rev. D 75, 084030 (2007) [arXiv:hep-th/0701004].
[18] B. Zwiebach, “Curvature Squared Terms And String Theories,” Phys. Lett. B 156, 315 (1985).
[19] P. Prester, “Lovelock type gravity and small black holes in heterotic string theory,” JHEP 0602, 039 (2006) [arXiv:hep-th/0511306].
[20] A. Jakubiec and J. Kijowski, Gen. Rel. Grav. 19, 719 (1987); Phys. Rev. D 37, 1406 (1988); G. Magnano, M. Ferraris, and M. Francaviglia, Gen. Rel. Grav. 19, 465 (1987); M. Ferraris, M. Francaviglia, and G. Magnano, Class. Quantum Grav. 5, L95 (1988); Class. Quantum Grav. 7, 261 (1990); K. Maeda, Phys. Rev. D 39, 3159 (1989).
[21] J. Koga and K. Maeda, “Equivalence of black hole thermodynamics between a generalized theory of gravity and the Einstein theory,” Phys. Rev. D 58, 064020 (1998) [arXiv:gr-qc/9803086].
[22] D. J. Gross and J. H. Sloan, “The Quartic Effective Action for the Heterotic String,” Nucl. Phys. B 291, 41 (1987).