Velocity selection of ultra-cold atoms with Fabry–Perot laser devices: improvements and limits

A Ruschhaupt, F Delgado and J G Muga
Departamento de Química-Física, UPV-EHU, Apartado 644, 48080 Bilbao, Spain

Received 6 May 2005, in final form 21 June 2005
Published 13 July 2005
Online at stacks.iop.org/JPhysB/38/2665

Abstract
We discuss a method to select the velocities of ultra-cold atoms with a modified Fabry–Perot type of device made of two effective barriers and a well, created, respectively, by blue- and red-detuned lasers. The laser parameters may be used to select the peak and width of the transmitted velocity window. In particular, lowering the central well provides a peak arbitrarily close to zero velocity having a minimum but finite width. The low-energy atomic scattering off this laser device is parametrized and approximate formulae are found to describe and explain its behaviour and limits. We illustrate the velocity-selection method with an atomic wave packet prepared as a Bose–Einstein condensate.

1. Introduction
Velocity selection is one of the basic operations in quantum optics and atomic physics for a plethora of applications such as cooling, interferometry, atom lithography, atom lasers and lenses and measurements of collision cross sections, momentum distributions or other kinetic-energy-dependent quantities. There are mechanical (slotted discs) [1] and non-mechanical (optical) techniques available, useful for different experimental circumstances, species and energies. The low energies and large wavelengths achieved with laser cooling have made the traditional methods no longer effective because of the increasing importance of gravity and the need to take into account the quantum nature of translational motion. For example, the standard classical-mechanical analysis of mechanical velocity-selection methods becomes invalid for small-time temporal slits since they produce momentum spread in agreement with a time–energy uncertainty principle [2]. Among the new methods, the velocity selection using Doppler sensitive stimulated Raman transitions [3, 4], and coherent population trapping into a dark state [5], provide selectivity in the ‘transverse direction’ parallel to the lasers, and rely on specific internal level configurations.

Fabry–Perot (FP) cavities have also been proposed to provide coherent velocity selection or trapping for longitudinal motion, using detuned lasers perpendicular to the incident atoms [6] to implement the partially reflecting mirrors. In addition, resonant microwave cavities have been used for the velocity selection of a two-level atom [7] and a three-level atom [8].
Non-resonant microwave cavities have also been studied [9]. Moreover, the transmission behaviour of a Bose–Einstein condensate [10, 11] through a double barrier in a waveguide has been described in [12], and through an optical lattice in [13].

The velocity selection with a Fabry–Perot cavity is produced by the filtering effect of resonance peaks in the transmission probability. The potential of FP cavities as trapping devices also stems from characteristic resonance features, namely, high densities and large life times in the interaction region.

The aim of this work is to discuss an improvement of the FP cavities, provide formulae to describe their behaviour and study the fundamental limitations to lower the peak width and velocity of the transmitted wave packet. The basic idea is to add a well with controllable depth between the two external barriers, see figure 1. Effective barriers and well can be implemented with blue- and red-detuned lasers perpendicular to the atomic ('x'-'directional) motion, which do not excite the impinging ground state atom and cause a mechanical effect only. We will use a one-dimensional description, which is accurate in the large detuning regime since the excited component is negligible and no momentum transfer in the laser direction occurs. Formally, the effective potential depends only on the longitudinal ('x'-) coordinate [14].

The depth of the well and the barrier height can be varied with the intensities of the lasers. Making the well deeper, rather than wider, displaces the resonance peaks in the transmittance to lower energies without diminishing the inter-resonance spacing. Thus, deepening the well is the ideal way to achieve a sharp, low-energy velocity selection. The displacement of the resonance peak of the transmittance towards lower velocities when increasing the well depth can be seen in figure 2. The resonance velocity shift is accompanied by a peak width reduction until a minimum, non-zero width is attained when the peak reaches zero velocity at a critical ‘threshold depth’ associated with the onset of a new bound state. Beyond that depth the peak broadens, moves back to higher velocities and its maximum decays, as shown in the inset of figure 2.

In section 2 we will describe the models used, and in section 3 the parameter regime useful for velocity selection, which comprises well depths smaller than the threshold depth. Section 4 is devoted to the fundamental limitations of the velocity-selection process at or near the threshold depth. In section 5 the velocity selection will be applied to a wave packet formed as a Bose–Einstein condensate. The paper ends with a summary.

2. Models

We shall use both a realistic model based on three Gaussians and a simplified version with two square barriers and a well. They are depicted in figure 1. The scattering off the two potential
models is very similar but the square model enables us to obtain analytical exact results and approximate but physically illuminating expressions.

Let us consider, for a single ultra-cold atom, the Hamiltonian
\[
H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_b(x + d) - V_w(x) + V_b(x - d),
\]
where \( V_{b,w}(x) = \hat{V}_{b,w}/\Pi_1(x) \), and \( \Pi_1(x) \) can take the forms
\[
\Pi_G(x) = \exp \left( -\frac{x^2}{2\sigma^2} \right), \quad \Pi_s(x) = \begin{cases} 
1 & \text{if } -d/2 < x < d/2 \\
0 & \text{otherwise}
\end{cases}
\]
for the Gaussian and square models respectively. For simplicity we have set all Gaussians with the same width \( \sigma \), and the square segments with the same length \( d \).

We assume that the atom impinges from the left, so only positive incident velocities are considered. We are interested in the transmission amplitude \( T \) and the transmission probability (or ‘transmittance’) \( |T|^2 \) of the scattering solutions of \( H\phi_v(x) = E_v\phi_v(x) \), where \( E_v = \frac{mv^2}{2} = \frac{\hbar^2 k^2}{2m} \). Both velocity, \( v \), and wavenumber, \( k \), will be used; the latter being more appropriate for complex plane analysis and the former for presenting the physical results. For the Gaussian model, the transmission probability \( |T(k)|^2 \) is calculated numerically by using the invariant imbedding method [15, 16] whereas, for the square model, \( T(k) \) is known analytically,
\[
T(k) = -4e^{-2ikb^k}k_b^k k_w^k \times \left\{ e^{i(k + k_w^k)} \left[ ik_b(k + k_w)C + (kk_w - k_b^k)S \right]^2 - e^{i(k + k_w^k)} \left[ -ik_b(k - k_w)C + (kk_w + k_b^k)S \right]^2 \right\}^{-1},
\]
for all \( k \), where \( C = \cosh(dk_b), S = \sinh(dk_b), k_w = (k^2 + K_w^2)^{1/2}, k_b = (K_b^2 - k^2)^{1/2}, K_b = (2mV_b)^{1/2}/\hbar \) and \( K_w = (2mV_w)^{1/2}/\hbar \).

As is well known, resonance scattering can be described by using poles. Even though the calculations can be made exactly, it is therefore useful and physically illuminating to find approximate expressions and dominant dependences relating the poles of \( T \) in the complex momentum plane to the well depth or other potential parameters, and to visible features such
as position and width of the resonance peaks. To this end, we shall provide a minimal but sufficient theory of the analytical structure of $T(k)$ in the following subsection.

### 2.1. Two-pole approximation

Since the barrier is symmetrical with respect to parity, the (Gamow) resonance states have well-defined parity and can be classified as symmetrical and antisymmetrical. For increasing resonance energies the corresponding poles appear alternatively in one of the two eigenvalues of the $2 \times 2$ $S$ matrix, corresponding to symmetrical, $S_0$, or antisymmetrical scattering, $S_1$. The transmission amplitude is given by

$$T = (S_0 + S_1)/2.$$  \hspace{1cm} (3)

We shall centre our attention on the first, symmetrical, lowest-energy resonance at zero well depth, and its transformation as the well is made deeper. The first symmetrical resonance is the one that will become the ground state for deep enough wells. We assume that the first (antisymmetrical) pole of $S_1$ is far from the origin so that a two-pole approximation suffices.

Retaining only two poles in the canonical pole expansion for cut-off potentials [17],

$$S_0 = -e^{-2ikr}(k - k_1^*)(k - k_2^*)/(k - k_1)(k - k_2), \quad S_1 = e^{-2ikr},$$  \hspace{1cm} (4)

where, for resonance scattering, $k_1$ is the resonance pole in the fourth quadrant. Due to the hermiticity of the Hamiltonian, it is accompanied by an antiresonance pole at $k_2 = -k_1^*$.

In (4), $r = 3d/2$ for the square model. For the Gaussian model, we could truncate the potential at a large $|x|$ value and apply equation (4). In any case the phase factor does not play any role to calculate the filtering function $|T|^2$:

$$|T(k)|^2 = 1/4 |S_0 + S_1|^2 = 1/4 \left|1 - (k - k_1^*)(k - k_2^*)/(k - k_1)(k - k_2)\right|^2.$$  \hspace{1cm} (5)

Expressions for the two important poles can be obtained with the square model under some approximations, as we shall see later on.

When the transmission peak is isolated and far from the origin, it admits a simple, one-pole, Breit–Wigner (BW) parametrization (see section 3). However, in both models, at some critical, ‘threshold’ well depth $\hat{V}_w$, a new bound state is formed. For well depths close to these thresholds, the description of the transmission peak is not as simple as for isolated BW resonances and requires a two-pole description (see section 4).

### 3. Velocity-selection control by well deepening

The Breit–Wigner, ‘BW’, regime, associated with a clear dominance of a single BW resonance pole $k_1$ in the fourth quadrant of the $k$-complex plane, is the most important one for velocity selection. The antiresonance pole at $k_2 = -k_1^*$ may normally be ignored if the resonance is sharp (i.e. $k_1$ is close to the real axis) and far from the origin. An increase in the well depth displaces $k_1$ to the left and upwards in the complex plane, so that the transmittance curve decreases both its peak velocity and width. By inspection of the $S$ matrix (4), it is clear that in this regime the transmittance reaches the unitary limit $|T(k)|^2 = 1$ for $k \approx \text{Re}(k_1)$.

We define $E_R = m v_R^2/2$ and $v_R$ as the energy and velocity of the transmittance maximum. In the BW regime $E_R \approx \hbar^2 \text{Re}(k_1)^2/2m$. For the two models and different barrier heights $\hat{V}_b$, the dependence of the maximum of the resonance $v_R$ on the well depth $\hat{V}_w$, is shown in figure 3(a). The width is also an important feature of the resonance. We define a velocity
velocity selection of ultra-cold atoms with fabry–perot laser devices

\[
\begin{align*}
V^w [h^{-1} s] & \quad v_R [\text{cm/s}] \\
0 & \quad 0.02 \\n0 & \quad 0.04 \\n0 & \quad 0.06 \\n0 & \quad 0.08 \\n0 & \quad 0.1 \\
50 & \quad 100 \\n150 & \quad 200 \\n250 & \quad 0
\end{align*}
\]

\[
\begin{align*}
\delta v_R [\text{cm/s}] & \quad V^w [h^{-1} s] \\
0 & \quad 0.001 \\n0 & \quad 0.002 \\n0 & \quad 0.003 \\n0 & \quad 0.004 \\
50 & \quad 100 \\n150 & \quad 200 \\n250 & \quad 0
\end{align*}
\]

(a) Resonance velocity \(v_R\) versus \(\hat{V}_w\) (symbols connected with dotted line). Filled symbols indicate the case \(|T(v_R)|^2 > 0.995\), and empty symbols otherwise. The solid lines show the approximation of equation (6). Gaussian functions, see figure 1(a): \(\hat{V}_b = 300\bar{h}/s\), \(\alpha = 0.65\) (diamonds), \(\hat{V}_b = 500\bar{h}/s\), \(\alpha = 0.70\) (squares); square functions, see figure 1(b): \(\hat{V}_b = 300\bar{h}/s\), \(\alpha = 0.79\) (triangles up), \(\hat{V}_b = 500\bar{h}/s\), \(\alpha = 0.85\) (triangles down). (b) Velocity width \(\Delta v_R\) of the resonance versus \(\hat{V}_w\); meaning of symbols as in (a). The solid lines show the approximation of equation (7). The parameter \(\beta\), in the same order of potentials used for (a) is \(0.00585 \text{ s}/\bar{h}\), \(0.00610 \text{ s}/\bar{h}\), \(0.00424 \text{ s}/\bar{h}\), and \(0.00418 \text{ s}/\bar{h}\).

width \(\Delta v_R\) as the width of the transmittance peak at half height. Its dependence with the well depth is shown in figure 3(b).

Simple parametrizations of the ‘Breit–Wigner’ regime are provided by perturbation theory or semiclassical formulae. Let \(E_{R0}\) and \(v_{R0}\) be the real energy and velocity of the resonance peak ‘without well’, i.e., when \(\hat{V}_w = 0\). Then the energy of the resonance with a non-zero well can be approximated within a perturbation theory for resonance functions [18]. Up to first order in \(\hat{V}_w\),

\[
E_R = E_{R0} - \alpha \hat{V}_w, \quad v_R = \sqrt{v_{R0}^2 - 2\alpha \hat{V}_w/m}.
\]

A semiclassical treatment [19] for opaque barriers gives \(\alpha = 1\), but keeping \(\alpha\) as a fitting parameter the dependence of equation (6) is valid even beyond very opaque barriers or very small depths, as can be seen in figure 3(a).

In figure 3(b) the BW regime corresponds to the slow decrease of \(\Delta v_R\) with well depth ending with an abrupt, almost vertical increase. As before, we are looking for a simple dependence with \(\hat{V}_w\). A semiclassical estimate for the energy width of the resonance is given by a well frequency factor times the WKB probability to escape through a barrier from the

\[
\begin{align*}
\text{Figure 3.} \quad (a) & \quad \text{Resonance velocity } v_R \text{ versus } \hat{V}_w \text{ (symbols connected with dotted line). Filled symbols indicate the case } |T(v_R)|^2 > 0.995, \text{ and empty symbols otherwise. The solid lines show the approximation of equation (6). Gaussian functions, see figure 1(a): } \\
\text{ } & \quad \hat{V}_b = 300\bar{h}/s, \alpha = 0.65 \text{ (diamonds), } \hat{V}_b = 500\bar{h}/s, \alpha = 0.70 \text{ (squares); square functions, see figure 1(b): } \\
\text{ } & \quad \hat{V}_b = 300\bar{h}/s, \alpha = 0.79 \text{ (triangles up), } \hat{V}_b = 500\bar{h}/s, \alpha = 0.85 \text{ (triangles down). (b) Velocity width } \Delta v_R \text{ of the resonance versus } \hat{V}_w; \text{ meaning of symbols as in (a). The solid lines show the approximation of equation (7). The parameter } \beta, \text{ in the same order of potentials used for (a) is } \text{0.00585 s}/\bar{h}, \text{0.00610 s}/\bar{h}, \text{0.00424 s}/\bar{h}, \text{and 0.00418 s}/\bar{h}.}
\end{align*}
\]
well (see, e.g., [19]). Retaining dominant dependences in the opaque barrier and shallow well limit,
\[ \Delta v_R = \Delta v_{R0} \exp(-\beta \hat{V}_w), \] (7)
which, again, by using \( \beta \) as an effective fitting parameter, describes the correct behaviour in the whole BW regime, up to well depths very near the threshold, see figure 3(b).

4. Fundamental limits of a Fabry–Perot device for velocity selection

Near the threshold depth (for which \( v_R = 0 \) and a new bound state appears) the velocity and width of the peak are affected more and more by the nearby antiresonance in the third quadrant, \( k_2 = -k_1^* \), see the ‘crosses’ in figure 4. The examination of the pole motion near threshold depths is worthwhile since it establishes the ultimate physical lower limit of the peak velocity and width using a Fabry–Perot filtering device. At a critical ‘collision’ depth \( \hat{V}_{w,\text{coll}} \) both poles meet at \(-i\kappa_{\text{coll}}, \kappa_{\text{coll}} > 0\), on the negative imaginary axis, see the ‘square’ in figure 4. In one-dimensional scattering, as in s-wave scattering, the pole collision is not at the origin because the bound states are not degenerate. Note also that, in spite of the zero real part of the poles at the collision, the velocity peak is not at \( v = 0 \). As the well becomes deeper the two poles move in opposite directions, now along the imaginary axis as ‘virtual’ poles until the upper one arrives at the origin at the threshold depth \( \hat{V}_{w,\text{thres}} \), with the lower pole at \(-i\kappa_{\text{thres}}\), see the ‘unfilled circles’ in figure 4. From (5) with \( k_1 = 0 \), and \( k_2 \equiv -i\kappa_{\text{thres}} \), it follows that, for the threshold depth,
\[ |T(k)|^2 = \frac{\kappa_{\text{thres}}^2}{k^2 + \kappa_{\text{thres}}^2}. \]
Thus, the width at half height at ‘threshold’, considering only positive momenta, is just \( \kappa_{\text{thres}} \) in wavenumber units because \( |T(0)|^2 = 1 \) and \( |T(\kappa_{\text{thres}})|^2 = 1/2 \). The threshold is a singular, abnormal point in which the transmission peak reaches the origin, \( T(0) = 1 \), whereas \( T(0) = 0 \) for any other well depth.

The two poles continue moving in opposite directions for \( \hat{V}_w > \hat{V}_{w,\text{thres}} \) with one of them being a bound state, see the ‘filled circles’ in figure 4. The transmittance peak broadens dramatically and moves to higher positive velocities; also the peak maximum becomes smaller than one, so this regime is no longer useful for velocity filtering, see figures 2 and 3.
The motion of the two poles just before the collision and even beyond threshold is well described by expanding the denominator of $S_0$ in powers of $(\hat{V}_w - \hat{V}_{w,\text{coll}})$ and retaining the first term. This gives

$$k_{1,2} = -\i \kappa_{\text{coll}} \pm \i \gamma (\hat{V}_w - \hat{V}_{w,\text{coll}})^{1/2},$$

with $\gamma$ real. A consequence is that, at threshold, $k_1 = 0$, $k_2 = -\i \kappa_{\text{thres}} \approx -2 \i \kappa_{\text{coll}}$.

Alternative approximate expressions for $k_1$ and $k_2$ may be obtained from (2). We have to find zeros of the denominator of $T$ or, equivalently, the $\chi \equiv \frac{K_w}{K_b}$ that satisfies

$$\frac{K_b}{\sqrt{K_b^2 + K_w^2 \chi^2}} \cot \left( d \sqrt{K_b^2 + K_w^2 \chi^2 / 2} \right) = \frac{\sqrt{1 - \chi^2} - \i \chi \tanh (d K_b \sqrt{1 - \chi^2})}{\sqrt{1 - \chi^2} (\sqrt{1 - \chi^2} \tanh (d K_b \sqrt{1 - \chi^2}) - \i \chi)}.$$ 

We assume $\hat{V}_b, \hat{V}_w > 0$ and $\chi \ll 1$. Neglecting $\mathcal{O}(\chi^3)$ we arrive at the quadratic equation

$$\alpha_2 \chi^2 + 2 \i \alpha_1 \chi - \alpha_0 = 0$$

with

$$\alpha_0 = \cot(d K_w / 2) - K_w \coth(d K_b) K_b, \quad \alpha_1 = \frac{K_w}{2 K_b \sinh^2(d K_b)},$$

$$\alpha_2 = \frac{K_b^2}{4 K_w^2} \sin^2(d K_w / 2) + K_b \coth(d K_b) \left( \cosh^2(d K_b) - 3 \right) + d K_b \frac{K_w}{2 K_b \sinh^2(d K_b)}.$$ 

The two solutions of (9) are given by

$$\chi_{1/2} = -\i \alpha_1 / \alpha_2 \pm \left( \sqrt{\alpha_0 \alpha_2 - \alpha_1^2} \right) / \alpha_2.$$  

At $\hat{V}_{w,\text{thres}} \equiv \hbar^2 K_{w,\text{thres}} / 2m$, a zero of the denominator of $T$ is at $k = 0$, so we find the threshold well depth $\hat{V}_{w,\text{thres}}$ by solving

$$\alpha_0(K_{w,\text{thres}}) = 0.$$  

At threshold depth the other pole is at $-\i \kappa_{\text{thres}} = -2\i K_b \alpha_1 / \alpha_2$ and it determines the minimal velocity width

$$\Delta v_{R,\text{thres}} = \frac{\hbar}{m} \kappa_{\text{thres}} = \frac{\hbar}{m} 2 K_b \alpha_1 / \alpha_2$$

$$= \frac{\hbar}{m} 4 K_b^2 K_{w,\text{thres}}^2 \left[ (d K_b + 2 \coth(d K_b))(K_b^2 + K_{w,\text{thres}}^2 \coth(d K_b)^2) \right. $$

$$+ 2 K_{w,\text{thres}}^2 (d K_b - 3 \coth(d K_b)) / \sinh^2(d K_b)^2 \left. \right]^{-1}.$$  

Figure 5 shows also the resonance velocity $v_R$ and velocity width $\Delta v_R$ using the two-pole expression (5) with the poles calculated by solving (9) versus $\lambda$ with

$$\lambda = \begin{cases} -|k_{1,2} - (-\i \kappa_{\text{coll}})| & \text{before the collision} \\ |k_{1,2} - (-\i \kappa_{\text{coll}})| & \text{after the collision}. \end{cases}$$ 

The result is indistinguishable from the exact one. Moreover, figure 5 shows the approximation using only one pole. This approximation differs significantly from the exact result which, once more, underlines that it is important to consider both poles near ‘threshold’.

If $d K_b \gg 1$, i.e., for a very opaque barrier, $\coth(d K_b) \approx 1$ and an approximate solution for $K_{w,\text{thres}}$ can be found. Because of the condition (11), $d K_{w,\text{thres}} / 2 \approx \arccot(K_{w,\text{thres}} / K_b) \approx \pi / 2 - K_{w,\text{thres}} / K_b$, so we get

$$K_{w,\text{thres}} \approx \frac{\pi}{d} \frac{1}{1 + \frac{1}{d R_k}} \ll K_b.$$
As an application example we shall compute the transmitted velocity distributions resulting from the FP filtering of an atomic wave packet prepared as a Bose–Einstein condensate \[10, 11\] in a trap. The wavefunction of the condensate can be described by the Gross–Pitaevskii equation \[20\].

This is the modelled experiment: the condensate is prepared in a trap while the trap is moved with a certain velocity with respect to the laboratory frame and turned off suddenly at \( t = 0 \); finally, the condensate expands until the nonlinear interaction between the atoms can be neglected and encounters the FP cavity. (Alternatively the FP potential can be moved with the trap at rest.) We shall now provide the corresponding theoretical treatment.

A harmonic trap is assumed with frequency \( \omega_x \) in \( x \) direction and \( \omega_{yz} \) in \( y \) and \( z \) directions and its (normalized to 1) ground state \( \Psi_0(x, y, z) \) can be found by solving the stationary Gross–Pitaevskii equation

\[
\mu \Psi_0 = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m \omega_x^2}{2} x^2 + \frac{m \omega_{yz}^2}{2} (y^2 + z^2) + \frac{4\pi \hbar^2 Na}{m} |\Psi_0(x, y, z)|^2 \right] \Psi_0, \tag{13}
\]

where \( \mu \) is the chemical potential, \( m = \text{mass}^{23}\text{Na}, N \) is the number of atoms in the condensate and \( a \) is the three-dimensional s-wave scattering length. We take \( a = 2.93 \times 10^{-9}m \) \[21\].

We will use a one-dimensional approximation of the situation which we are going to motivate now. Assuming that \( \omega_{yz} \gg \omega_x \), the nonlinearity in the \( y-z \)-plane can be neglected.
Velocity selection of ultra-cold atoms with Fabry–Perot laser devices

Figure 6. $|\psi(v,t)|^2$ for $v_0 = 0.0336$ cm s$^{-1}$, $v_{\text{TRAP}} = -600$ µm; Gaussian functions, see figure 1(a), $V_b = 300 \hbar/\ell$, $\omega_0 = 5$ s$^{-1}$, $\omega_{yz} = 100$ s$^{-1}$, $N = 5 \times 10^5$; $t = 0$ (dotted line); $t = 0.8$ s; solid line; $t = 8$ s: $V_w = 1400$ s (thick dotted line), $V_w = 1500$ s (dashed line), $V_w = 1600$ s (thick solid line); the circles mark the corresponding resonance velocities $v_R$.

We use therefore the ansatz $\Psi_0(x, y, z) = \Phi_0(y, z)\bar{\psi}_0(x)$ for the ground state where $\Phi_0(y, z)$ is the (normalized to 1) Gaussian ground state of the Hamiltonian

$$H_{yz} = -\hbar^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{m\omega_{yz}^2}{2}(y^2 + z^2),$$

namely,

$$\Phi_0(y, z) = \sqrt{\frac{m\omega_{yz}}{\hbar\pi}} \exp\left( -\frac{m\omega_{yz}}{2\hbar}(y^2 + z^2) \right).$$

(14)

By multiplying (13) with $\Phi_0^*(y, z)$ and integrating over $y$ and $z$ we arrive at

$$(\mu - \hbar\omega_{yz})\bar{\psi}_0(x) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega_{yz}^2}{2} x^2 + 2\hbar N a_0 \omega_{yz} |\bar{\psi}_0(x)|^2 \right) \bar{\psi}_0(x).$$

This equation is also derived in [22]. We compute the ground state $\bar{\psi}_0$ numerically and then we change to the lab frame where the trap moves with velocity $v_0$. The ground state $\bar{\psi}_0$ in this reference frame is $\bar{\psi}_0(x) = e^{i v_0 t} \bar{\psi}_0(x)$.

If at $t = 0$ the trap, at position $x_{\text{TRAP}}$, is turned off, and the velocity-selection potentials are switched on, the subsequent time evolution is given by

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x, y, z) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_b(x + d) - V_b(x) \\
+ V_b(x - d) + \frac{4\pi\hbar^2 N a}{m} |\Psi(t, x, y, z)|^2 \right] \Psi(t, x, y, z).$$

(15)

We consistently assume that the three-dimensional wavefunction has the form $\Psi(t, x, y, z) = \Phi(t, y, z)\psi(t, x)$ with $i\hbar \frac{\partial}{\partial t} \Phi(t, y, z) = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(t, y, z)$ and $\Phi(0, y, z) = \Phi_0(y, z)$. By multiplying (15) with $\Phi^*(t, y, z)$ and integrating over $y$ and $z$, we arrive at

$$i\hbar \frac{\partial}{\partial t} \psi(t, x) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_b(x + d) - V_b(x) + V_b(x - d) + 4\hbar N a_0 \omega_{yz} \frac{|\psi(t, x)|^2}{1 + \omega_{yz}^2 t^2} \right] \psi(t, x).$$

(16)

Note that the nonlinearity decays because of the spreading in $y$ and $z$ directions, but not exponentially as in [23]. Figure 6 shows the momentum distribution of the ground state of
the trap at $t = 0$. At $t = 0.8$ s the nonlinearity has practically vanished, so the momentum distribution stays stable until the velocity selection. The filtered distributions at $t = 8$ s for several resonance velocities obtained with different well depths are also shown. They demonstrate the applicability of the proposed method for velocity selection.

6. Summary

In summary, we have proposed an improvement of Fabry–Perot cavities to select the velocity of ultra-cold atoms by using a red-detuned potential well between the partially reflecting blue-detuned mirrors. Making the well deeper is the ideal way to achieve a sharp low-energy velocity selection because the resonance peaks are displaced to lower energies without diminishing the inter-resonance spacing. At a critical ‘threshold’ depth, the peak reaches zero velocity but still with a non-zero width. We have shown that the device behaviour can be understood and quantified using the S matrix formalism within a two-pole approximation which provides simple formulae for the critical ‘threshold’ depth and the minimal velocity width. Finally, we have exemplified the velocity selection by well deepening with an atomic wave packet prepared as a Bose–Einstein condensate.

Acknowledgments

This work is supported by ‘Ministerio de Ciencia y Tecnología-FEDER’ (BFM2003-01003), and UPV-EHU (grant 15968/2004). AR acknowledges the support of the German Academic Exchange Service (DAAD) and Ministerio de Educación y Ciencia.

References

[1] van den Meijdenberg C J 1988 Atomic and Molecular Beam Methods ed G Scoles (Oxford, New York: Oxford University Press) chapter 13
[2] Szriftgiser P, Guéry-Odelin D, Arndt M and Dalibard J 1996 Phys. Rev. Lett. 77 4
[3] Kasevich M, Weiss D S, Riis E, Moler K, Kasapi S and Chu S 1991 Phys. Rev. Lett. 66 2297
[4] Moler K, Weiss D S, Kasevich M and Chu S 1992 Phys. Rev. A 45 342
[5] Aspect A, Arimondo E, Kaiser R, Vansteenkiste N and Cohen-Tannoudji C 1988 Phys. Rev. Lett. 61 826
[6] Wilkens M, Goldestein E, Taylor B and Meystre P 1993 Phys. Rev. A 47 2366
[7] Löffler M, Meyer G M and Walther H 1998 Europhys. Lett. 41 593
[8] Zhang Z M, Xie S W, Chen Y L, Xia Y X and Zhou S K 1999 Phys. Rev. A 60 3321
[9] Martin J and Bastin T 2004 Eur. Phys. J. D 29 133
[10] Anglin J R and Ketterle W 2002 Nature 416 211
[11] Bongs K and Sengstock K 2004 Rep. Prog. Phys. 67 907
[12] Paul T, Richter K and Schlagheck P 2005 Phys. Rev. Lett. 94 020404
[13] Carusotto I and La Rocca G C 2000 Phys. Rev. Lett. 84 399
[14] Ruschhaupt A, Damborenea J A, Navarro B, Muga J G and Hegerfeldt G C 2004 Europhys. Lett. 67 1
[15] Singer S, freed K F and Band Y B 1982 J. Chem. Phys. 77 1942
[16] Band Y B and Tuı̈ 1994 J. Chem. Phys. 100 8869
[17] Nussenzveig H M 1972 Causality and Dispersion Relations (New York: Academic)
[18] Kukulin V I, Krasnopol’sky V M and Horáček J 1989 Theory of Resonances (Dordrecht: Kluwer)
[19] Bohm D 1951 Quantum Theory (New York: Dover)
[20] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Rev. Mod. Phys. 71 463
[21] van Abeelen F A and Verhaar B J 1999 Phys. Rev. A 59 578
[22] Bao W, Jaksh D and Markowich P A 2003 J. Comp. Phys. 187 318
[23] Bongs K, Burger S, Birkel G, Sengstock K, Ertmer W, Rzgulski K, Sanpera A and Lewenstein M 1999 Phys. Rev. Lett. 83 3577