Assigning a classifying space to a fusion system up to F-isomorphism

Nora Seeliger

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Abstract

We give an abstract criterion for the cohomology of the classifying space of a group model of a given fusion system to be $F$–isomorphic in the sense of Quillen to the stable elements.

1 Introduction

In the topological theory of $p$–local finite groups introduced by Broto, Levi, and Oliver one tries to approximate the classifying space of a finite group via the $p$–local structure of the group, at least up to $\mathbb{F}_p$–cohomology. Every finite group gives canonically rise to a saturated fusion system for every prime dividing its order however not every fusion system is the fusion system of a finite group. However there exist infinite groups realizing arbitrary fusion systems. In this article we give an abstract criterion for a classifying space of a group model to be $F$–isomorphic in the sense of Quillen to the stable elements.

2 Preliminaries

2.1 Fusion Systems

We review the basic definitions of fusion systems and centric linking systems and establish our notations. Our main references are [6], [7], and [13]. Let $S$ be a finite $p$-group. A fusion system $\mathcal{F}$ on $S$ is a category whose objects are all the subgroups of $S$, and which satisfies the following two properties for all $P,Q \leq S$: The set $\text{Hom}_\mathcal{F}(P,Q)$ contains injective group homomorphisms and amongst them all morphisms induced by conjugation of elements in $S$ and each element is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion. Two subgroups $P,Q \leq S$ will be called $\mathcal{F}$–conjugate if they are isomorphic in $\mathcal{F}$. Define $\text{Out}_\mathcal{F}(P) = \text{Aut}_\mathcal{F}(P)/\text{Inn}(P)$ for all $P \leq S$. A subgroup $P \leq S$ will be called $\mathcal{F}$–centric if $C_S(P') \leq P'$ for all $P'$ which are $\mathcal{F}$–conjugate to $P$. $\mathcal{F}$ is called saturated if for all $P \leq S$ which is fully normalized in $\mathcal{F}$, $P$ is fully centralized in $\mathcal{F}$ and $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(P))$ and moreover if $P \leq S$ and $\phi \in \text{Hom}_\mathcal{F}(P,S)$ are such that $\phi(P)$ is fully centralized, and if we set $N_\phi = \{g \in N_S(P)|\phi g \phi^{-1} \in \text{Aut}_S(\phi(P))\}$, then there is $\phi \in \text{Hom}_\mathcal{F}(N_\phi,S)$ such that $\phi|_P = \phi$. A subgroup $P \leq S$ will be called $\mathcal{F}$–centric if $C_S(P') \leq P'$ for all $P'$ which are $\mathcal{F}$–conjugate to $P$. Denote $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ with objects the $\mathcal{F}$–centric subgroups of $S$. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$, and ”distinguished” monomorphisms $\delta_P : P \rightarrow \text{Aut}_\mathcal{L}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$ such that the following conditions are satisfied: $\pi$ is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P,Q \in \mathcal{L}$, $Z(P)$ acts freely on...
Let $S$ be a finite $p$–group and let $P_1, ..., P_r, Q_1, ..., Q_t$ be subgroups of $S$. Let $\phi_1, ..., \phi_t$ be injective group homomorphisms $\phi_i : P_i \to Q_i \forall i$. The fusion system generated by $\phi_1, ..., \phi_t$ is the minimal fusion system $\mathcal{F}$ over $S$ containing $\phi_1, ..., \phi_t$. Let $\mathcal{F}$ be a fusion system on a finite $p$–group $S$. Fix any pair $S \leq G$, where $G$ is a (possibly infinite) group and $S$ is a finite $p$–group. Define $\mathcal{F}_S(G)$ to be the category whose objects are the subgroups of $S$, and where $\text{Mor}_{\mathcal{F}_S(G)}(P,Q) = \text{Hom}_G(P,Q) = \{c_g \in \text{Hom}(P,Q) \mid g \in G, g Pg^{-1} \leq Q \}$, $N_G(P,Q)/C_G(P)$. Here $c_g$ denotes the homomorphism conjugation by $g$, and $N_G(P,Q) = \{g \in G \mid g Pg^{-1} \leq Q \}$ (the transporter set). For each $P \leq S$, let $C_G(P)$ be the maximal $p$–perfect subgroup of $C_G(P)$. For each $\mathcal{F}_S(G)$–centric subgroup $P \leq G$, let $\delta_P : P \to \text{Out}_{\mathcal{F}_S(G)}(P)$ be the monomorphism induced by the inclusion $P \leq N_G(P)$. A triple $(S, \mathcal{F}, \mathcal{L})$ where $S$ is a finite $p$–group, $\mathcal{F}$ is a saturated fusion system on $S$, and $\mathcal{L}$ is an associated centric linking system to $\mathcal{F}$, is called a $p$–local finite group. It’s classifying space is $|\mathcal{L}|_p^\wedge$ where $(-)^\wedge_p$ denotes the $p$–completion functor in the sense of Bousfield and Kan. This is partly motivated by the fact that every finite group $G$ gives canonically rise to a $p$–local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ and we have $BG_p^\wedge \simeq |\mathcal{L}|_p^\wedge$, see [6]. In particular, all fusion systems coming from finite groups are saturated.

Let $\mathcal{F}$ be a fusion system on the the finite $p$–group $S$. The fusion system $\mathcal{F}$ is called an Alperin fusion system if there are subgroups $P_1 = S, P_2, \cdots, P_r$ of $S$ and finite groups $L_1, \ldots, L_r$ such that: $P_i$ is the largest normal $p$–subgroup of $L_i$, $C_{L_i}(P_i) = Z(P_i)$, and $L_i/P_i \simeq \text{Out}_{\mathcal{F}}(P_i)$ for each $i$. Moreover $N_S(P_i)$ is a Sylow $p$–subgroup of $L_i$, and $\mathcal{F}_{N_S(P_i)}(L_i)$ is contained in $\mathcal{F}$ for each $i$ such that $\mathcal{F}$ is generated by all the $\mathcal{F}_{N_S(P_i)}(L_i)$. The groups $L_i$ are isomorphic to $\text{Aut}_{\mathcal{L}}(P_i)$ for all $i = 1, \cdots, r$. Recall that every saturated fusion system is Alperin, as proven in [4, Section 4].

### 2.2 Groups Realizing a Given Fusion System

Given a fusion system $\mathcal{F}$ on a finite $p$–group $S$ it is not always true that there exists a finite group $G$ such that $\mathcal{F}_S = \mathcal{F}_S(G)$, (see [6], chapter 9 for example). However for every fusion system $\mathcal{F}$ there exists an infinite group $\mathcal{G}$ such that $\mathcal{F}_S(G) = \mathcal{F}$. We now describe the classical constructions by G. Robinson [18], and I. Leary and R. Stancu [10] as well as our generalisation of Robinson’s construction [19] and our work with Libman, see [12].

The groups of Robinson type are iterated amalgams of automorphism groups in the linking system over the $S$–normalizers of the respective $\mathcal{F}$–centric subgroups of $S$.

**Theorem 2.1** ([18, Theorem 2.]) Let $\mathcal{F}$ be an Alperin fusion system on a finite $p$–group $S$ and associated groups $L_1, ..., L_n$ as in the definition. Then there is a finitely generated group $\mathcal{G}$ which has $S$ as a Sylow $p$–subgroup such that the fusion system $\mathcal{F}$ is realized by $\mathcal{G}$.

The group $\mathcal{G}$ is given explicitly by $\mathcal{G} = L_1 \ast_{N_S(P_2)} L_2 \ast_{N_S(P_3)} \cdots \ast_{N_S(P_n)} L_n$ with $L_i, P_i$ as in the definition.

Corresponding to the various versions of Alperin’s fusion theorem (essential subgroups, centric subgroups, centric radical subgroups) there exist canonical choices for the groups generating $\mathcal{F}$. The group constructed by I. Leary and R. Stancu is an iterated HNN-construction.
Theorem 2.2 ([10], Theorem 2.) Suppose that \( \mathcal{F} \) is the fusion system on \( S \) generated by \( \Phi = \{ \phi_1, \ldots, \phi_r \} \). Let \( T \) be a free group with free generators \( t_1, \ldots, t_r \), and define \( G \) as the quotient of the free product \( S \ast T \) by the relations \( t_i^{-1}u_i = \phi_i(u) \) for all \( i \) and for all \( u \in P_i \). Then \( S \) embeds as a \( p \)-Sylow subgroup of \( G \) and \( \mathcal{F}_S(G) = \mathcal{F} \).

In [19] we generalize Robinson’s construction to an amalgam of certain subgroups of the \( L \)'s, and the group model for \( \mathcal{F} \) we construct in [12] is a quotient group of Robinson’s construction. All these models have the property that their classifying spaces can be described as homotopy colimits of a certain functor over a 1-dimensional category, and that they come with a map to \( r : B\mathcal{G} \to |\mathcal{L}| \). The classifying spaces of all but Leary-Stancu’s models are \( p \)-good and the \( \mathbb{F}_p \)-cohomology is noetherian.

2.3 A map to the stable elements

In [20] we prove that for every group which realizes a prescribed fusion system we have a map in \( \mathbb{F}_p \)-cohomology into the stable elements.

Theorem 2.3 Let \( \mathcal{F} \) be a fusion system over the finite \( p \)-group \( S \) and \( \mathcal{G} \) a group such that \( S \in Syl_p(\mathcal{G}) \) and \( \mathcal{F} = \mathcal{F}_S(\mathcal{G}) \). Then there exist a natural map of unstable algebras over the Steenrod algebra \( H^*(BG) \xrightarrow{\delta} H^*(\mathcal{F}) \) making \( H^*(\mathcal{F}) \) a finitely generated module over \( H^*(BG) \).

Moreover we have the following splitting result in \( \mathbb{F}_p \)-cohomology over the Steenrod algebra.

Theorem 2.4 Let \( (S, \mathcal{F}, \mathcal{L}) \) be a \( p \)-local finite group and \( \mathcal{G} \) be one of the group models for a saturated fusion system \( \mathcal{F} \) presented above. Then there exist natural maps of algebras over the Steenrod algebra \( H^*(BG) \xrightarrow{\delta} H^*(|\mathcal{L}|) \) and \( H^*(|\mathcal{L}|) \xrightarrow{i^*} H^*(BG) \) such that we obtain a split short exact sequence of unstable modules over the Steenrod algebra

\[
0 \to W \xrightarrow{\text{incl}} H^*(BG) \xrightarrow{\delta} H^*(|\mathcal{L}|) \xrightarrow{i^*} H^*(BG) \to 0,
\]

where \( W \cong \text{Ker}(Res^G_S) \) and \( q \) is the map from the previous theorem.

3 The main theorem

The following theorem is a special case of the theorem we prove at the end of this section. The techniques of the proof are quite different and, as we believe, interesting in themselves. This is why, in addition, we would like to include this special case separately.

Theorem 3.1: Let \( S \) be a finite \( p \)-group and \( \mathcal{F} \) a fusion system over \( S \). Let \( \mathcal{G} \) be a group with finite virtual dimension such that \( S \in Syl_p(\mathcal{G}) \), \( \mathcal{F}_S(\mathcal{G}) = \mathcal{F} \) and \( H^*(BG) \) is noetherian. Then \( H^*(BG; \mathbb{F}_p) \) is \( \mathcal{F} \)-monomorphic in the sense of Quillen to \( H^*(\mathcal{F}) \).

Proof: Note that from the theorem above we have a map \( f : H^*(BG) \to H^*(\mathcal{F}) \). We want to show that all elements in the kernel of \( f \) are nilpotent. We will show that the nilpotent elements in \( H^*(BG; \mathbb{F}_p) \) are precisely the ones who lie in the kernel of all the restriction maps to all the elementary abelian subgroups \( V \leq \mathcal{G} \).

Let \( A \) be a commutative graded connected algebra over the field \( \mathbb{F} \) and \( I \subset A \) an ideal. The set \( Ass(I) \) of associated prime ideals of \( I \) is finite and \( \sqrt{I} = \bigcap \{ p | p \in Ass(I) \} \).
Since ideals in a Noetherian algebra are finitely generated an ideal in which all elements are nilpotent is a nilpotent ideal. We will show that the kernel of the restriction map is contained in the nilradical of $H^*(BG;\mathbb{F}_p)$. We will show that an element $x \in \ker(\text{res}^*)$ belongs to every minimal prime ideal.

Let $\mathcal{P}^*$ denote the Steenrod algebra of the Galois field $\mathbb{F}_p$ and recall that for any reasonable topological space $X$ that $H^*(X;\mathbb{F}_p)$ is an unstable algebra over $\mathcal{P}^*$. For any unstable algebra $H^*$ over $\mathcal{P}^*$ one says that an ideal $I \subset H^*$ is $\mathcal{P}^*$-invariant if $\theta(I) \subseteq I$ for all $\theta \in \mathcal{P}^*$. There is a $\mathcal{P}^*$-version of the Lasker-Noether-Theorem, [14]. Moreover we have that if $H^*$ is a Noetherian unstable algebra over the Steenrod algebra $\mathcal{P}^*$ and $I \subseteq H^*$ a $\mathcal{P}^*$-invariant ideal, then the associated prime ideals of $I$ are all $\mathcal{P}^*$-invariant. This implies that the minimal primes of a Noetherian unstable $\mathcal{P}^*$-algebra are $\mathcal{P}^*$-invariant.

Let $H^*$ be a Noetherian unstable algebra. Denote by $\text{min}(H^*)$ the set of minimal prime ideals of $H^*$. For $p \in \text{min}(H^*)$ the quotient algebra $H^*/p$ is therefore an unstable Noetherian integral domain over $\mathcal{P}^*$. By [1] we can therefore find an embedding $H^*/p \to \mathbb{F}_p[V_p]$ which is a finite ring extension, where $V_p$ is a finite dimensional vector space over $\mathbb{F}_p$ and $\mathbb{F}_p[V_p]$ is the polynomial algebra on $V_p$ regarded as an unstable $\mathcal{P}^*$-algebra in the canonical way. Moreover, if $\phi : H^* \to \mathbb{F}_p[V]$ is a map of unstable algebras then the kernel of $\phi$ is an invariant prime ideal, and if $\mathbb{F}_p[V]$ is finitely generated as an $H^*$-module even a minimal prime of $H^*$. For a Noetherian cohomology algebra $H^*(BG;\mathbb{F}_p)$ this means the nil-radical can be described in the following way.

**Theorem 3.2** Let $H^*$ be a Noetherian unstable algebra over the Steenrod algebra. If $p \subset H^*$ is a minimal prime ideal then there is a homomorphism $\phi : H^* \to \mathbb{F}_p[V_0]$ into a finitely generated module over $H^*$. Therefore $h \in H^*$ is nilpotent if and only if $h$ belongs to the kernel of every homomorphism $\phi : H^* \to \mathbb{F}_p[V_0]$, where $V_0$ is a finite dimensional vector space over $\mathbb{F}_p$.

At this point we want to note there is a difference between the cases $p = 2$ and $p \neq 2$. For $p = 2$ one has $\mathbb{F}_2[V] \cong H^*(BG;\mathbb{F}_2)$, where as for $p \neq 2$ one only has $\mathbb{F}_p[V] \cong H^*(BG;\mathbb{F}_p)/\sqrt{0}$.

We would like to specialize the algebra $H^*$ in the previous Theorem to $H^*(BG;\mathbb{F}_p)$ where $G$ is some kind of reasonable discrete group and try to describe homomorphisms $\phi : H^*(BG;\mathbb{F}_p) \to \mathbb{F}_p[V_0]$ in terms of $p$--elementary abelian subgroups $V_0 \leq G$ and the homomorphism induced by the inclusion. This leads to a number of technical problems that were nicely dealt with in [15] and [22]. First of all we need to review some results of [15] and [16]. For this reason we are forced at first to introduce the full Steenrod algebra $A^*$ of the Galois field $\mathbb{F}_p$, namely the algebra generated by not just the Steenrod reduced powers, but in addition to them also the Bockstein. From [15] and [16] we need the following fact (see [9] Observation 2 for a short proof).

**Theorem 3.3 (Quillen)** If $H^*(BG;\mathbb{F}_p)$ (is Noetherian and) $\phi : H^*(BG;\mathbb{F}_p) \to H^*(BG;\mathbb{F}_p)$ is a homomorphism of algebras over the Steenrod algebra $A^*$ making $H^*(BG;\mathbb{F}_p)$ into a finitely generated $H^*(BG;\mathbb{F}_p)$--module, then there is an inclusion $V \leq G$ such that the compositions

\[
H^*(BG;\mathbb{F}_p) \xrightarrow{\phi} H^*(BG;\mathbb{F}_p) \xrightarrow{\pi} H^*(BG;\mathbb{F}_p)/\sqrt{0}
\]

are the same, where $\pi$ is the canonical quotient map.

Duflot, Landweber, and Stong point out (see [9] the note following the corollary on page 74) that a group homomorphism $\rho : V \to G$ for which $H^*(BV;\mathbb{F}_p)$ becomes a finitely generated $H^*(BG;\mathbb{F}_p)$--module has to be a monomorphism. They pose the question: If $H^*(BG;\mathbb{F}_p)$ (is Noetherian and) $\phi, \psi : H^*(BG;\mathbb{F}_p) \to H^*(BV;\mathbb{F}_p)$ are homomorphisms of algebras over the Steenrod algebra $A^*$ making $H^*(BG;\mathbb{F}_p)$ into a finitely generated $H^*(BG;\mathbb{F}_p)$--module, such that the compositions

\[
H^*(BG;\mathbb{F}_p) \xrightarrow{\phi} H^*(BG;\mathbb{F}_p) \xrightarrow{\pi} H^*(BG;\mathbb{F}_p)/\sqrt{0}
\]

are the same then is $\phi = \psi$? This is answered and more in [22] in the affirmative. One has.
Theorem 3.4 (Zarati) If $H^*(BG; \mathbb{F}_p)$ is Noetherian and $\phi, \psi : H^*(BG; \mathbb{F}_p) \to H^*(BV; \mathbb{F}_p)$ are homomorphisms of algebras over the Steenrod algebra $A^*$ making $H^*(BV; \mathbb{F}_p)$ into a finitely generated $H^*(BG; \mathbb{F}_p)$–module, such that the compositions

$$H^*(BG; \mathbb{F}_p) \xrightarrow{\phi} H^*(BV; \mathbb{F}_p) \xrightarrow{\psi} H^*(BV; \mathbb{F}_p)/\sqrt{0}$$

are the same, then $\phi = \psi$.

The upshot is that for $H^*(BG; \mathbb{F}_p)$ a Noetherian algebra we know that the minimal prime ideals $p \subset H^*(BG; \mathbb{F}_p)$ are precisely the ideals that arise as the kernels of an induced map $\text{Res}^* : H^*(BG; \mathbb{F}_p) \to H^*(BV; \mathbb{F}_p)$, where $V$ ranges over the $p$–elementary abelian subgroups (or in any case their conjugacy classes) $V$ of $G$. This gives the following criterion for nilpotence.

Theorem 3.5 Suppose that $H^*(BG; \mathbb{F}_p)$ is Noetherian and $u \in H^*(BG; \mathbb{F}_p)$. Then $u$ is nilpotent if and only if for every $p$–elementary abelian subgroup $V$ of $G$ the element $u$ belongs to the kernel of the induced map $\text{Res}^* : H^*(BG; \mathbb{F}_p) \to H^*(BV; \mathbb{F}_p)$. If $G$ has a $p$–Sylow subgroup $S \leq G$ then it suffices to know that $u$ is in the kernel of the induced map $\text{Res}^* : H^*(BG; \mathbb{F}_p) \to H^*(BS; \mathbb{F}_p)$.

Theorem 3.6 Let $\mathcal{G}$ be a group of finite virtual cohomological dimension with a Sylow $p$–subgroup $S$ and $\mathcal{F}_S(\mathcal{G})$ be a saturated fusion system. Then $H^*(BG; \mathbb{F}_p)$ is $F$–isomorphic to the stable elements.

**Proof**: Recall that $\mathcal{G}$ is a group of finite virtual cohomological dimension. It is proved in [16] that $H^*(BG; \mathbb{F}_p)$ is $F$–isomorphic to the inverse limit of $H^*(-; \mathbb{F}_p)$ over the Quillen category which will be denoted by $H^*(Q)$. Moreover Broto, Levi, and Oliver show in [3] that the ring of stable elements $H^*(\mathcal{F})$ is $F$–isomorphic to the ring $H^*(Q)$. We therefore obtain a commutative diagram: $H^*(BG) \xrightarrow{f} H^*(\mathcal{F})$ where $f$ is the natural map in $\mathbb{F}_p$–cohomology of Theorem 2.3, and $g, h$ are the natural restriction maps to $H^*(Q)$ respectively. It follows from diagram chase that the map $f$ is an $F$–isomorphism. Since $\ker(f) \subseteq \ker(h)$ and $h$ is an $F$–monomorphism it follows that all the elements in $\ker(f)$ are nilpotent and therefore $f$ is an $F$–monomorphism as well. It remains to show that $f$ is an $F$–epimorphism, which means that for every $x$ in $H^*(\mathcal{F})$ there is $n \geq 0$ and $y$ in $H^*(BG)$ such that $f(y) = x^n$. Since $h$ is an $F$–epimorphism we have that for every $x$ in $H^*(\mathcal{F})$ there exist $m \geq 0$ and $y'$ in $H^*(BG)$ such that $h(y') = g(x)^m = g(x^n)$. Moreover we have because of the commutativity that $g(f(y') - x^n) = 0$. This implies that there is $k \geq 0$ with $(f(y') - x^n)^k = 0$ and therefore we have $(f(y') - x^n)^p = 0$ and since we are in characteristic $p$ this implies $f(y'^p) = f(y')^p = x^mp^\circ$. □

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Max-Planck-Institut für Mathematik Bonn, Vivatsgasse 11, 53111 Bonn, Germany.
email: seeliger@mpim-bonn.mpg.de