Random Switching near Bifurcations

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Abstract

The interplay between bifurcations and random switching processes of vector fields is studied. More precisely, we provide a classification of piecewise deterministic Markov processes arising from stochastic switching dynamics near fold, Hopf, transcritical and pitchfork bifurcations. We prove the existence of invariant measures for different switching rates. We also study, when the invariant measures are unique, when multiple measures occur, when measures have smooth densities, and under which conditions finite-time blow-up occurs. We demonstrate the applicability of our results for three nonlinear models arising in applications.

1 Introduction

In this work we study the dynamics of randomly switched ordinary differential equations (ODEs) of the form

\[ \frac{dx}{dt} = f(x,p), \quad x = x(t) \in \mathbb{R}^d, \quad x(0) = x_0, \]  

(1.1)
near bifurcation points. More precisely, we select two parameters \( p = p_\pm \in \mathbb{R} \) so that the ODE (1.1) has non-equivalent dynamics \cite{35}, which are separated by a distinguished bifurcation point \( p_* \in (p_-, p_+) \). Then we look at the piecewise-deterministic Markov process (PDMP) generated by switching between the vector fields \( f(x, p_-) \) and \( f(x, p_+) \). This idea is motivated by several observations. Here we just name a few:

(M1) In parametric families of vector fields, bifurcations occur generically. Therefore, they are immediately relevant for the study of PDMPs as well. In addition, the interplay between random switching and bifurcation points is not studied well enough yet.

(M2) From the perspective of PDMPs, this setting provides natural examples to test and extend the general theory of invariant measures.

(M3) Stochastic bifurcation theory is a very active area, where still many questions remain open. Hence, studying a well-defined set of standard cases involving bifurcations and stochasticity is highly desirable.

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Parameters in many models are usually only known via a possible distribution and not exactly. Therefore, our work contributes to the uncertainty quantification for nonlinear systems arising in applications.

Before describing our main results, we briefly review some of the background from PDMPs and from nonlinear dynamics to provide a broader perspective.

The study of randomly switched deterministic vector fields goes back at least to the works of Goldstein [24] and Kac [30]. The set-up can informally be described as follows: Given a starting point \( x_0 \in \mathbb{R}^d \) and an initial vector field \( f_i \) taken from a finite collection \( \{f_j\} \) of smooth vector fields on \( \mathbb{R}^d \), we follow the flow along \( f_i \) starting at \( x_0 \) for an exponentially distributed random time. Then a switch occurs, meaning that \( f_i \) is replaced with a new vector field \( f_j, j \neq i \). We flow along \( f_j \) for another exponential time and switch again. This yields a continuous and piecewise smooth trajectory in \( \mathbb{R}^d \) that is, however, not the trajectory of a Markov process. To obtain a Markov process, one needs to supplement the switching process on \( \mathbb{R}^d \) with a second stochastic process that keeps track of the driving vector field. The resulting two-component process belongs to the class of piecewise deterministic Markov processes (PDMPs).

PDMPs were first introduced by Davis [18] in an even more general setting. For instance, PDMPs may involve jumps not only on the collection of vector fields but also on \( \mathbb{R}^d \) [19, 39]. The class of PDMPs considered in this article is also known under the names of hybrid systems [50] and random evolutions [28], [21, Chapter 12]. Randomly switched vector fields have applications to areas such as ecology [12], gene regulation [15], molecular motors [23], epidemiology [37], queueing theory [1], and climate science [38], to name just a few.

Aside from their uses in modeling, randomly switched vector fields have intriguing theoretical properties. For example, switching between stable vector fields can result in an unstable situation, and vice versa. Recently, examples of randomly switched vector fields were found that exhibit such a reversal of stability for almost all realizations of switching times [11, 36]. Another interesting phenomenon is the regularizing effect random switching can have on a dynamical system. For example, random switching between two Lorenz vector fields with just slightly different parameter values induces an invariant probability measure that is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R}^3 \), whereas the dynamics associated to each individual vector field concentrate on attractors of Lebesgue measure zero [47, 3]. Another recent topic is the ergodic theory for randomly switched vector fields. Important contributions to the question whether a switching system on a noncompact state space admits an invariant probability measure were made in [10, 7, 8]. In [3, 9], it was shown that a Hörmander-type hypoellipticity condition on the vector fields at an accessible point yields uniqueness and absolute continuity of the invariant probability measure.

In this work we focus on the invariant probability measure aspect and relate it to bifurcation points. Bifurcation theory [26, 35] has become one of the most widely used techniques to study nonlinear systems [48]. Informally, the main idea is to study vector fields under parameter variation and to determine at which points the dynamics changes fundamentally, i.e., to detect the points where the phase portraits of the vector fields are not topologically equivalent upon small parameter variation. Almost full classification results exist for bifurcations with relatively few parameters, i.e., codimension one or two. These results provide suitable unfoldings, which are basically partitions of parameter space into non-equivalent phase portraits [35].

Recently, substantial interest has been focused on understanding the interplay between stochasticity and bifurcations. Yet, the setting in almost all of these works is focused on either stochastic
differential equations (SDEs) involving (space-)time stochastic forcing processes \[2\] [13], or less frequently on random differential equations (RDEs) with a fixed random parameter distribution \[14\] [42]. Particularly interesting dynamics seems to appear for SDEs in oscillatory situations \[6\] [20] [46]. Recently numerical and semi-analytical work shows that interesting effects also occur for switched systems near bifurcations \[34\]. Therefore, it is very natural that one should try to link PDMPs with bifurcation theory.

In this paper, we provide a full mathematical classification of the PDMPs associated to (1.1) switched near local bifurcations for codimension one bifurcations. We not only include the generic fold and Hopf bifurcations but also study the frequently occurring one-parameter transcritical and pitchfork bifurcations. We prove under which conditions on the switching rates invariant measures occur, when they are unique, when their densities are smooth, and we also provide explicit formulas for these densities in certain cases. In addition, we prove finite-time blow-up results for certain parameter regimes. In summary, our theorems provide building blocks, which can be employed in various PDMPs. In addition, we demonstrate that we may also derive insights from our results in three nonlinear models arising respectively in ecology, nonlinear oscillations, and collective motion.

The paper is structured as follows: In Section 2 we provide more technical background from local bifurcation theory and PDMPs. In Section 3 we focus on all cases where below and above the bifurcation point there are non-trivial trapping regions. In these cases we characterize the occurring invariant probability measures completely. In Section 4 we consider the cases with only one non-trivial trapping region. We again study the invariant measures in full detail but now also finite-time blow-up can appear. In Section 5 we indicate how our results can be used in three models arising from applications.

## 2 Background

We briefly recall the technical background needed from the two main areas we consider in this work. Hence, this section mainly serves as a reference and to fix the notation. Readers familiar with local bifurcation theory \[26\] [35] and PDMPs can skip ahead to Section 3.

### 2.1 Local Bifurcation Theory

Consider an ordinary differential equation (ODE) given by

\[
\frac{dx}{dt} = x' = f(x, p), \quad x = x(t) \in \mathbb{R}^d, \; x(0) =: x_0,
\]

\[ (2.1) \]

where \( p \in \mathbb{R} \) is the (main) bifurcation parameter, and we assume that the vector field \( f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) is sufficiently smooth; in particular, in what follows \( f \in C^3(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d) \) is going to suffice. We also refer to \( \mathbb{R}^d \) as the phase space of (2.1). The phase space together with the foliation by trajectories \( x(t) \) is called phase portrait. Suppose \( x_* \) is an equilibrium point (or steady state) of (2.1) for the parameter value \( p_* \) so that \( f(x_*, p_*) = 0 \). Without loss of generality, upon translating coordinates in the phase space \( \mathbb{R}^d \) and the parameter space \( \mathbb{R} \), we may assume that \( (x_*, p_*) = (0, 0, \ldots, 0) =: 0 \). Consider the linearized problem near the steady state

\[
X' = D_xf(0)X = AX, \quad X = X(t) \in \mathbb{R}^d.
\]

\[ (2.2) \]
Then $x_*$ is called hyperbolic if the matrix $A \in \mathbb{R}^{d \times d}$ has no spectrum on the imaginary axis. In the hyperbolic case, the Hartman-Grobman Theorem (see e.g. [49]) implies that the systems (2.1) and (2.2) are locally topologically equivalent, i.e., small parameter variations for $p \in (-p_0, p_0)$, $p_0 > 0$, do not qualitatively alter the phase portrait as the hyperbolic structure of $A$ is robust under small parameter perturbations. More precisely, for any two parameter values $p_1, p_2 \in (-p_0, p_0)$, there exists a homeomorphism $h : \mathbb{R}^d \to \mathbb{R}^d$ such that the phase portraits of $f(x, p_1)$ and $f(x, p_2)$ are mapped to each other by $h$ preserving the direction of time on trajectories.

Suppose $A$ is not hyperbolic so that $\text{spec}(A) \cap i\mathbb{R} \neq \emptyset$. A local bifurcation occurs at $p_* = 0$ if for any $p_0 > 0$ and any open neighbourhood $U = U(0)$ of $x_* = 0$, there exist two locally (wrt $U$) non-homeomorphic phase portraits of (2.1) for two values $p_1, p_2 \in (-p_0, p_0)$. In particular, a bifurcation just corresponds to the appearance of a topologically non-equivalent phase portrait under parameter variation.

The general strategy to analyze bifurcation problems [35, 26] proceeds as follows: (I) the system is reduced to the dimension $d_c$ of $\ker(A)$ using a center manifold $W_{\text{loc}}^c(0)$, (II) on $W_{\text{loc}}^c(0)$ one uses smoothness to Taylor-expand the vector field and then simplifies it using coordinate changes, and (III) one proves that a finite number of polynomial terms is locally sufficient to determine the topological equivalence class so a truncation yields a finite-degree polynomial vector field. The steps (I)-(III) lead to different classes/families of polynomial vector fields, also called normal forms, depending upon degeneracy of $\text{spec}(A)$ and depending upon a finite number of partial derivatives of $f$.  

In this work we shall focus on the four most common bifurcations used in practical applications for $d_c = 1$ and $d_c = 2$, which just require a single bifurcation parameter $p$, and where the system has already been reduced to normal form. These cases will be the fold, Hopf, transcritical, and pitchfork bifurcations. As a motivating example, consider the supercritical pitchfork normal form

$$x' = px - x^3, \quad x \in \mathbb{R}, \; p \in \mathbb{R}. \tag{2.3}$$

Clearly, the equilibrium $x_* = 0$ undergoes a bifurcation at $p_* = 0$ as the phase portrait for $p < 0$ has one globally stable equilibrium, while the phase portrait for $p > 0$ has three equilibria. For $p > 0$, we find that $x_* = 0$ is unstable while the equilibria $x_{\pm} = \pm \sqrt{p}$ are both locally stable; see also Figure [1]

![Figure 1: Sketch of the bifurcation diagram for the supercritical pitchfork bifurcation normal form (2.3). There are two classes of topologically non-equivalent phase portraits here denoted by $S_{p \leq 0}$ and $S_{p > 0}$.](image)
However, note that from the viewpoint of applications, treating \( p \) just as a static parameter is not always realistic. This approach presumes \( p \) is changed infinitely slowly to bring the system to and across the bifurcation point. One option is to consider the case when \( p \) is just switched across the bifurcation point (e.g., consider shot noise effects, control action, activation of external interfaces of the system, etc). As an example, consider the problem of switching between \( p < 0 \) and \( p > 0 \) in the context of the pitchfork normal form (2.3). This leads us naturally to consider piecewise deterministic Markov processes as introduced in the next section.

### 2.2 Piecewise Deterministic Markov Processes

In this subsection we introduce a class of PDMPs characterized by Poissonian random switching between a finite number of deterministic vector fields. Let \( I \) be a finite index set, and let \((f_i)_{i \in I}\) be a collection of vector fields on \( \mathbb{R}^d \) with some degree of smoothness. To introduce the basic framework, we just assume that \((f_i)_{i \in I}\) are in \( C^1(\mathbb{R}^d, \mathbb{R}^d) \), but for some of the results stated below higher degrees of smoothness are required. To be able to associate flows to the vector fields, we assume in addition that \((f_i)_{i \in I}\) are forward complete, i.e. for any \( x_0 \in \mathbb{R}^d \) the initial-value problem

\[
x'(t) = f_i(x), \quad x(0) = x_0
\]

has a unique solution \( t \mapsto \Phi^t_i(x_0) \) that is defined for all \( t \geq 0 \). Given a starting point \( x_0 \in \mathbb{R}^d \) and an initial vector field \( f_i \), the random dynamical system we consider follows the flow associated to \( x_0 \) and \( f_i \) for a random time. Then a switch occurs, which means that the driving vector field \( f_i \) is replaced by a new vector field \( f_j \) chosen at random from \( \{f_k : k \in I \setminus \{i\}\} \). Again, the system flows along \( f_j \) for a random time until another switch occurs, etc. The stochastic process \( X = (X_t)_{t \geq 0} \) that records the position of the switching trajectory on \( \mathbb{R}^d \) is not Markov because knowing \((X_s)_{s \leq t}\) lets us infer the driving vector field at time \( t \). If the times between consecutive switches are exponentially distributed and independent conditioned on the sequence of driving vector fields, and if the vector fields are chosen according to a Markov chain on \( I \), then the two-component process \((X, E)\) is already Markov, where \( E_t \in I \) gives the index of the driving vector field at time \( t \). For more general distributions of switching times, one needs to adjoin a third component that keeps track of the time elapsed since the latest switch. It is possible to consider the situation where the rate of switching depends continuously on the location of the switching trajectory on \( \mathbb{R}^d \) [9], [22]. For simplicity we assume that the switching rates do not depend on the process \( X \). We can then give the following rigorous description of \((X, E)\). Let \( E = (E_t)_{t \geq 0} \) be an irreducible continuous-time Markov chain on the state space \( I \). Let \( X = (X_t)_{t \geq 0} \) be the solution to the control problem

\[
X_t = x + \int_0^t f_{E_s}(X_s) \, ds.
\]

The Markov process \((X, E)\) has infinitesimal generator \( L \) acting on functions \( g : \mathbb{R}^d \times I \to \mathbb{R} \) that are smooth in \( x \) according to

\[
Lg(x, i) = \langle f_i(x), \nabla_x g(x, i) \rangle + \sum_{j \neq i} \lambda_{i,j}(g(x, j) - g(x, i)),
\]

where \( \lambda_{i,j} \) is the rate at which \( E \) transitions from state \( i \) to state \( j \). We denote the Markov semigroup of \((X, E)\) by \((P^t)_{t \geq 0}\) or just by \((P^t)\). An invariant probability measure (IPM) of \((P^t)\) is
a probability measure \( \mu \) on \( \mathbb{R}^d \times I \) such that \( \mu = \mu P_t \) for all \( t \geq 0 \). Below, we collect some results on existence, uniqueness and absolute continuity for IPM of \( (P_t) \) that have been established in the literature.

We call a set \( M \subset \mathbb{R}^d \) positive invariant if \( M \) is positive invariant under the flows \( (\Phi_i)_{i \in I} \) associated with the vector fields \( (f_i)_{i \in I} \), i.e. if for any \( x \in M \), \( i \in I \) and \( t \geq 0 \), we have \( \Phi_i^t(x) \in M \). Thus, trajectories of \( X \) starting in a positive invariant set \( M \) or entering \( M \) at some time remain in \( M \) for all future times. If there is a compact positive invariant set \( M \), existence of an IPM is guaranteed by the Krylov–Bogoliubov method [17, Theorem 3.1.1], which applies because \( (X, E) \) is Feller [9, Proposition 2.1]. In the noncompact situation, an IPM is guaranteed to exist, for instance, if \( (X, E) \) is on average contracting [10, Corollary 1.11]. By Harris’s ergodic theorem, existence also holds if the semigroup \( (P_t) \) admits a Lyapunov function as well as a minorizing measure \( \nu_K \) for every compact set \( K \subset \mathbb{R}^d \).

Recall that the support of a Borel measure \( \mu \) on \( \mathbb{R}^d \times I \) is the set of points \( (x, i) \in \mathbb{R}^d \times I \) such that \( \mu(U \times \{i\}) > 0 \) for every open neighborhood \( U \subset \mathbb{R}^d \) of \( x \). If \( x^* \in \mathbb{R}^d \) is an equilibrium for each of the vector fields \( f_i \), then the product of the Dirac measure at \( x^* \) and the unique IPM \( \nu \) of the continuous-time Markov chain \( E \) is a trivial IPM for \( (P_t) \). If \( M \subset \mathbb{R}^d \) is a compact positive invariant set containing such a common equilibrium \( x^* \), the Krylov–Bogoliubov method is not sufficient to decide whether there are any additional IPM whose support is contained in \( M \times I \). This more subtle existence question can often be addressed using the theory of stochastic persistence as developed by Benaïm [7], and as applied to the case of a common equilibrium by Benaïm and Strickler [47].

We now outline an existence result from [47] that will be needed later on. Let \( M \subset \mathbb{R}^d \) be a compact positive invariant set containing the point \( x^* = 0 \), which we assume to be an equilibrium for all vector fields \( f_i \). As a technical condition, we also require that there is \( \delta > 0 \) such that whenever \( x \in M \) and \( \|x\| \leq \delta \), then the entire line segment from 0 to \( x \) is contained in \( M \). For \( i \in I \), let

\[
A_i = Df_i(0)
\]

be the Jacobian matrix of \( f_i \) at 0. Then, the cone

\[
C_M = \{tx : t \geq 0, x \in M, \|x\| \leq \delta\} \subset \mathbb{R}^d
\]

is positive invariant with respect to the flows of the linear vector fields given by \( (A_i)_{i \in I} \). On \( C_M \times I \), we define the PDMP \( (Y, E) \), which is obtained from \( (X, E) \) by replacing each vector field \( f_i \) with its linearization \( A_i \). Whenever \( Y_t \neq 0 \), we define the angular process

\[
\Theta_t = \frac{Y_t}{\|Y_t\|},
\]

which evolves on the compact set \( S^{d-1} \cap C_M \). By Krylov–Bogoliubov, \( (\Theta, E) \) admits at least one IPM. For any IPM \( \nu \) of \( (\Theta, E) \), define the average growth rate as

\[
\Lambda(\nu) = \sum_{i \in I} \int_{S^{d-1} \cap C_M} \theta^\top A_i \theta \nu(d\theta \times \{i\}).
\]

Since \( \|Y_t\| \) satisfies

\[
\frac{d}{dt}\|Y_t\| = \Theta_t^\top A_{E_t} \Theta_t \|Y_t\|,
\]

we have
Birkhoff’s ergodic theorem implies that for almost every realization of \((\Theta, E)\) with initial distribution \(\nu\), we have
\[
\lim_{t \to \infty} \frac{\ln(\|Y_t\|)}{t} = \Lambda(\nu).
\]
Recall that an IPM \(\nu\) of a Markov process with Markov semigroup \((P^t)\) and state space \(X\) is called ergodic if \(\nu(A) \in \{0, 1\}\) for every measurable \(A \subset X\) such that for all \(t \geq 0\), \(P^t_y(A) = 1\) for \(\nu\)-almost every \(x \in A\). Let \(\Lambda^-\) denote the infimum and \(\Lambda^+\) the supremum of \(\Lambda(\nu)\) over all ergodic IPM \(\nu\) of \((\Theta, E)\). In many situations of interest, \((\Theta, E)\) has exactly one IPM, so \(\Lambda^- = \Lambda^+\).

**Definition 2.1.** We call a point \(x \in \mathbb{R}^d\)

1. **reachable from** \(y \in \mathbb{R}^d\) if there is a finite sequence of indices \(i_1, \ldots, i_n \in I\) and a corresponding sequence of positive real numbers \(t_1, \ldots, t_n\) such that
   \[
   \Phi^{t_n}_{i_n} \circ \cdots \circ \Phi^{t_1}_{i_1}(y) = x;
   \]
2. **accessible from** \(y\) if for any neighborhood \(U\) of \(x\) there is \(z \in U\) such that \(z\) is reachable from \(y\);
3. **accessible from** \(S \subset \mathbb{R}^d\) if it is accessible from any \(y \in S\). If \(x\) is accessible from \(S \subset \mathbb{R}^d\), we simply say that \(x\) is accessible.

A point \(x \in \mathbb{R}^d\) is accessible if and only if for every neighborhood \(U\) of \(x\), for every \(y \in \mathbb{R}^d\) and for every \(i, j \in I\) there is \(t > 0\) such that \(P^t_{y,(i)}(U \times \{j\}) > 0\). If \(x\) is accessible, then the points \((x, i), i \in I\), are contained in the support of any IPM for \((P^t)\).

**Theorem 2.2** (Benaîm, Strickler, [8]). Let \(M_+ = M \setminus \{0\}\). The following statements hold.

1. If \(\Lambda^- > 0\), then there exists an IPM \(\mu\) of \((X, E)\) such that \(\mu(M_+ \times I) = 1\). In addition, for any starting point \(x \in M_+\), \(X_t\) almost surely does not converge to 0 as \(t \to \infty\).
2. If \(\Lambda^+ < 0\) and if the point 0 is accessible, then for any starting point \(x \in M\), \(X_t\) converges almost surely to 0 as \(t \to \infty\). In particular, there is no IPM that assigns positive mass to \(M_+ \times I\).

Now, we review sufficient conditions for uniqueness and absolute continuity of the IPM. Recall that the Lie bracket of \(C^1\) vector fields \(f_0\) and \(f_1\) on \(\mathbb{R}^d\) is defined as
\[
[f_0, f_1](x) = Df_1(x)f_0(x) - Df_0(x)f_1(x), \quad x \in \mathbb{R}^d.
\]
Let \(\mathcal{L}\) denote the Lie algebra generated by \((f_i)_{i \in I}\), i.e. \(\mathcal{L}\) is the smallest collection of \(C^\infty\) vector fields on \(\mathbb{R}^d\) that contains \((f_i)_{i \in I}\), and is closed under linear combinations and the Lie bracket operation.

**Definition 2.3.** We say that the weak bracket condition is satisfied at a point \(x \in \mathbb{R}^d\) if
\[
\{f(x) : f \in \mathcal{L}\} = \mathbb{R}^d.
\]

The weak bracket condition is essentially Hörmander’s condition for smoothness of transition densities for a diffusion process with the noise acting along \((f_i)_{i \in I}\), see [43, Section 2.3].
Theorem 2.4 (Benaim, Le Borgne, Malrieu, Zitt, [9]; Bakhtin, Hurth, [3]). Let $U \subset \mathbb{R}^d$ be an open positive invariant set. Suppose $(P^t)$ admits an IPM $\mu$ such that $\mu(U \times I) = 1$. Assume in addition that there exists $x \in U$ such that (i) $x$ is accessible from $U$ and (ii) the weak bracket condition holds at $x$. Then, $\mu$ is the unique IPM assigning full measure to $U \times I$, and $\mu$ is absolutely continuous with respect to the product of Lebesgue measure on $\mathbb{R}^d$ and counting measure on $I$.

If $d = 1$, the weak bracket condition holds at any point that is not an equilibrium of all $(f_i)_{i \in I}$. The interesting condition is then existence of an accessible point.

Suppose now that $(P^t)$ admits an absolutely continuous IPM with probability density function $\rho(x,i)$. We refer to the projections $\rho_i = \rho(\cdot, i), i \in I$, as invariant densities. For some simple PDMPs on $\mathbb{R} \times I$, we can give explicit formulas for invariant densities. Besides, we have the following regularity result.

Theorem 2.5 (Bakhtin, Hurth, Mattingly, [4]). Assume that $(f_i)_{i \in I}$ are $C^\infty$ vector fields on $\mathbb{R}$ with locally finite sets of critical points each. Let $x \in \mathbb{R}$ such that $f_i(x) \neq 0$ for every $i \in I$. Then, the invariant densities $(\rho_i)_{i \in I}$ of an absolutely continuous IPM are $C^\infty$ smooth at $x$.

3 Two Nontrivial Trapping Regions

Given a vector field $f$ on $\mathbb{R}^d$ with flow function $\Phi$ and a set $\mathcal{V} \subset \mathbb{R}^d$, we call $\mathcal{V}$ a trapping region with respect to $f$ if for any $x \in \mathcal{V}$ and any $t > 0$ we have $\Phi^t(x) \in \mathcal{V}$. We split our analysis into two cases, which can occur for our normal forms in different parameter regimes. Either, there exists a trapping region $\mathcal{V} \subset \mathbb{R}^d$ of finite positive Lebesgue measure. Or, trajectories leave any bounded set except for a set of measure zero, which is going to consist of unstable equilibria in our case. In this section, we cover the case when such a trapping region exists both below and above the bifurcation value. The case when a trapping region exists only below or only above the bifurcation value is covered in Section 4.

3.1 Supercritical Pitchfork Bifurcation

Consider the ODE (2.1) for $d = 1$ and assume the existence of a trivial branch of equilibria $f(x_*, p) = 0$ for all $p$. Assume that the following conditions hold at $(x_*, p)$:

$$\partial_x f(x_*, p) = 0, \quad \partial_{xx} f(x_*, p) = 0, \quad \partial_{xxx} f(x_*, p) < 0, \quad \partial_x p f(x_*, p) \neq 0. \quad (3.1)$$

Then a bifurcation occurs at $(x_*, p)$, which can be proven to be locally topologically equivalent to the supercritical pitchfork bifurcation normal form

$$x' = px - x^3. \quad (3.2)$$

The dynamics of (3.2) is easy to analyze. For $p < 0$, there is a unique globally stable equilibrium point $x_* = 0$. For $p > 0$, $x_* = 0$ is unstable while the equilibria $x_{\pm} = \pm \sqrt{p}$ are locally stable. For any $p \in \mathbb{R}$, all trajectories remain bounded so trapping regions of positive measure are easy to find. We now analyze the normal form (3.2) from the viewpoint of PDMPs by switching the parameter $p$. For fixed parameters $p_- < 0$ and $p_+ > 0$, we switch between the vector fields

$$f_{-1}(x) = p_- x - x^3, \quad f_1(x) = p_+ x - x^3.$$
We denote the rate of switching from $f_{-1}$ to $f_1$ by $\lambda_-$ and the rate of switching from $f_1$ to $f_{-1}$ by $\lambda_+$. Since 0 is an equilibrium for both vector fields, the semigroup $(P^t)$ associated with the PDMP $(X, E)$ admits at least one IPM, namely the product of the Dirac measure at 0 and the measure on $I = \{-1, 1\}$ that assigns probability $\frac{\lambda_+}{\lambda_+ + \lambda_-}$ to $-1$ and $\frac{\lambda_-}{\lambda_+ + \lambda_-}$ to 1. The latter is precisely the IPM of the continuous-time Markov chain $E$ on the state space $I$. For ease of reference, we call this trivial IPM $\delta$.

**Theorem 3.1.** The following statements hold.

1. If $\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}$, the semigroup $(P^t)$ admits exactly three ergodic IPM: the trivial measure $\delta$, a measure $\mu$ such that $\mu((0, \infty) \times I) = 1$, and a measure $\pi$ such that $\pi((-\infty, 0) \times I) = 1$.

2. If $\frac{\lambda_+}{p_+} \geq -\frac{\lambda_-}{p_-}$, then $\delta$ is the unique IPM for $(P^t)$.

**Theorem 3.2.** Suppose that $\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}$. Then, the ergodic IPM $\mu$ and $\pi$ assigning measure 1 to $(0, \infty) \times I$ and $(-\infty, 0) \times I$, respectively, are absolutely continuous. Moreover, the corresponding invariant densities $\rho^\mu$ and $\rho^\pi$ are given by

$$
\rho^\mu_i(x) = \rho^\pi_i(-x) = Cx^{\frac{\lambda_-}{p_-} - \frac{\lambda_+}{p_+} - 1}(-p_- + x^2)^{\frac{\lambda_-}{2p_-} - \frac{1}{2}(1-i)}(p_+ - x^2)^{\frac{\lambda_+}{2p_+} - \frac{1}{2}(1+i)}1_{(0, \sqrt{p_+})}(x), \quad i \in I.
$$

Here, $C$ is a normalizing constant.

**Proof of Theorem 3.1.** Let $M = [0, \sqrt{p_+})$ and $M_+ = (0, \sqrt{p_+})$. Then, $M$ is a compact positive invariant set containing the common equilibrium 0. Moreover, as 0 is globally asymptotically stable for $f_{-1}$, 0 is accessible from $M$. If we linearize $f_{-1}$ and $f_1$ at $x = 0$, we obtain

$$
A_{-1} = \frac{d}{dx}f_{-1}(x)|_{x=0} = p_-, \quad A_1 = \frac{d}{dx}f_1(x)|_{x=0} = p_+.
$$

We have $C_M = [0, \infty)$ and $C_M \cap S^0 = \{1\}$. The angular process $(\Theta, E)$ has a unique IPM $\nu$ that assigns probability $\frac{\lambda_+}{\lambda_+ + \lambda_-}$ to $\{1\} \times \{-1\}$ and probability $\frac{\lambda_-}{\lambda_+ + \lambda_-}$ to $\{1\} \times \{1\}$. Thus,

$$
\Lambda^+ = \Lambda^- = \Lambda(\nu) = \frac{p_- \lambda_+ + p_+ \lambda_-}{\lambda_+ + \lambda_-},
$$

which is positive if $\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}$ and negative if $\frac{\lambda_+}{p_+} > -\frac{\lambda_-}{p_-}$. By Theorem 2.2, if $\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}$, there exists an IPM $\mu$ such that $\mu(M_+ \times I) = 1$; and if $\frac{\lambda_+}{p_+} > -\frac{\lambda_-}{p_-}$, there is no IPM assigning positive mass to $M_+ \times I$. Suppose now that $\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}$, and consider the open positive invariant set $(0, \infty)$. Let $x \in (0, \sqrt{p_+})$ and observe that $x$ is accessible from $(0, \infty)$. Since $f_{-1}$ and $f_1$ do not vanish at $x$, the weak bracket condition is satisfied. By Theorem 2.4, there is exactly one IPM $\mu$ assigning full measure to $(0, \infty) \times I$. This measure is ergodic: By the ergodic decomposition theorem (see, e.g., [27, Theorem 5.7]), there exists an ergodic IPM $\pi$ assigning positive mass to $(0, \infty) \times I$. Since $(0, \infty)$ is positive invariant, we have $\pi((0, \infty) \times I) = 1$ and hence $\pi = \mu$. This argument also shows that $\mu$ is the only ergodic IPM that assigns positive mass to $(0, \infty) \times I$. A completely analogous reasoning applies to the positive invariant set $(-\infty, 0)$.

It remains to consider the critical case $\frac{\lambda_+}{p_+} = -\frac{\lambda_-}{p_-}$, where Theorem 2.2 does not apply. To obtain a contradiction, we assume that there is an ergodic IPM $\mu$ that, without loss of generality, assigns measure 1 to $(0, \infty) \times I$. By Theorem 2.4, $\mu$ has a density $\rho$, and by Theorem 2.5, $\rho_{-1}$ and
\( \rho_1 \) are smooth in \((0, \sqrt{p_+})\). Therefore, they satisfy the formula in Theorem 3.2 As \( \frac{\lambda_-}{p_-} + \frac{\lambda_+}{p_+} = 0 \), \( \rho_- \) and \( \rho_1 \) behave asymptotically as \( x^{-1} \) as \( x \downarrow 0 \). Since \( x^{-1} \) is not integrable in a neighborhood of 0, we arrive at a contradiction. \( \square \)

**Proof of Theorem 3.2.** Absolute continuity of \( \mu \) and \( \pi \) follows from Theorem 2.4 As shown in the proof of Theorem 3.1 \( \mu((0, \sqrt{p_+}) \times I) = 1 \), so the invariant densities \( (\rho^\mu_i)_{i \in I} \) vanish outside of \([0, \sqrt{p_+}]\). By Theorem 2.5 \( (\rho^\mu_i)_{i \in I} \) are \( C^\infty \) on \((0, \sqrt{p_+})\) and thus satisfy the Fokker–Planck equations, see for instance [22]. Written in terms of probability fluxes \( \varphi_i = \rho^\mu_i f_i, i \in I \), the Fokker–Planck equations read for \( x \in (0, \sqrt{p_+}) \)

\[
\begin{align*}
\varphi'_{-1}(x) &= -\frac{\lambda_-}{f_{-1}(x)} \varphi_{-1}(x) + \frac{\lambda_+}{f_1(x)} \varphi_1(x), \\
\varphi'_1(x) &= -\frac{\lambda_+}{f_1(x)} \varphi_1(x) + \frac{\lambda_-}{f_{-1}(x)} \varphi_{-1}(x).
\end{align*}
\]

Then,

\[ \varphi'_{-1} + \varphi'_1 \equiv 0, \]

so \( \varphi_{-1} + \varphi_1 \) is constant. We even have \( \varphi_{-1} + \varphi_1 \equiv 0 \) [5]. The ODE in (3.3) becomes

\[ \varphi'_{-1}(x) = \left(-\frac{\lambda_-}{f_{-1}(x)} + \frac{\lambda_+}{f_1(x)}\right) \varphi_{-1}(x), \]

which is solved by

\[ \varphi_{-1}(x) = C \exp \left(-\lambda_- \int \frac{dx}{f_{-1}(x)} - \lambda_+ \int \frac{dx}{f_1(x)}\right) = C x^{-\frac{\lambda_-}{p_-} - \frac{\lambda_+}{p_+}} (-p_- + x^2)^{\frac{\lambda_-}{2p_-}} \left( p_+ - x^2 \right)^{\frac{\lambda_+}{2p_+}}. \]

We obtain the desired formula for \( \rho^\mu \) with \( \rho^\mu_{-1} = \varphi_{-1}/f_{-1} \) and \( \rho^\mu_1 = -\varphi_{-1}/f_1 \). The formula for \( \rho^\pi \) follows from the fact that both \( f_{-1} \) and \( f_1 \) are odd. \( \square \)

### 3.2 Supercritical Hopf Bifurcation

Consider the ODE (2.1) for \( d = 2 \). Assume that \( x_* = x_*(p) \) is a family of equilibrium points for all \( p \) in a parameter-space neighbourhood of \( p_* \). Let \( A = D_x f(x_*(p), p) \) and assume that

\[ \text{spec}(A) = \{ \alpha(p) \pm i \omega(p) \}, \quad \alpha(p_*) = 0, \quad \alpha'(p_*) \neq 0, \quad \omega(p_*) \neq 0. \]  

(3.5)

Furthermore, consider the first Lyapunov coefficient \( l_1 = l_1(p) \), which is computable from \( f \) using partial derivatives up to and including third order; see the formulas in [20, 35]. Assume that \( l_1(p_*) < 0 \). Then a bifurcation occurs at \((x_*, p_*)\), which can be proven to be locally topologically equivalent to the supercritical Hopf bifurcation normal form

\[
\begin{align*}
x'_1 &= px_1 - x_2 - x_1(x_1^2 + x_2^2), \\
x'_2 &= x_1 + px_2 - x_2(x_1^2 + x_2^2).
\end{align*}
\]

(3.6)
The dynamics of (3.6) can be analyzed a lot easier upon changing to polar coordinates \((x_1, x_2) = (r \cos \theta, r \sin \theta)\), which gives
\[
\begin{align*}
\theta' &= 1, \\
r' &= pr - r^3.
\end{align*}
\] (3.7)

Analyzing the simple vector field (3.7) and returning to Euclidean coordinates, one finds that for \(p < 0\), there is a unique globally stable equilibrium point \(x_* = 0\). For \(p > 0\), \(x_* = 0\) is unstable while there exists a family of stable periodic orbits \(\{\|\|_2 = \sqrt{p}\}\). For any \(p \in \mathbb{R}\), all trajectories remain bounded so trapping regions of positive measure always exist. We now analyze the normal form (3.6) from the viewpoint of PDMPs by switching the parameter \(p\), again working in polar coordinates. For fixed \(p_- < 0\) and \(p_+ > 0\), we switch between the vector fields
\[
g_{-1}(\theta, r) = (1, p_- r - r^3)^	op, \quad g_1(\theta, r) = (1, p_+ r - r^3)^	op.
\]

In analogy to the case of the supercritical pitchfork bifurcation, we denote the rate of switching from \(g_{-1}\) to \(g_1\) by \(\lambda_-\), and the rate of switching from \(g_1\) to \(g_{-1}\) by \(\lambda_+\). As before, the origin is an equilibrium for both vector fields, so \(\delta\), defined as the product of the Dirac measure at the origin and the discrete measure assigning probability \(\frac{\lambda_+}{\lambda_+ + \lambda_-}\) to \(-1\) and \(\frac{\lambda_-}{\lambda_+ + \lambda_-}\) to \(1\), is an IPM. Let \(\nu\) denote the unique IPM for the PDMP induced by switching between the one-dimensional vector fields
\[
f_{-1}(r) = p_- r - r^3, \quad f_1(r) = p_+ r - r^3, \quad r > 0
\]
at rates \(\lambda_-\) and \(\lambda_+\), whose existence is guaranteed by Theorem 3.1.

**Theorem 3.3.** The following statements hold.

1. If \(\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}\), \((P^t)\) admits exactly two ergodic IPM: the measure \(\delta\) and a measure \(\mu\) that is the product of Lebesgue measure on the unit circle \(S^1\), normalized by the factor \(\frac{1}{2\pi}\), and the IPM \(\nu\).

2. If \(\frac{\lambda_+}{p_+} \geq -\frac{\lambda_-}{p_-}\), then \(\delta\) is the unique IPM for \((P^t)\).

**Proof:** Suppose first that \(\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}\). Let \(\mu\) denote the product of Lebesgue measure on \(S^1\), normalized by the factor \(\frac{1}{2\pi}\), and the IPM \(\nu\). Then, for \(t > 0\), \(i, j \in I\), \(\theta \in S^1\), \(r > 0\) and measurable sets \(A \subset S^1\), \(B \subset (0, \infty)\), we have
\[
\mathcal{P}_{\theta, r, j}^t(A \times B \times \{i\}) = I_A(\theta + t) \hat{\mathcal{P}}_{r, j}^t(B \times \{i\}),
\]
where \(\theta + t\) should be understood modulo \(2\pi\), and where \(\hat{\mathcal{P}}\) denotes the semigroup associated with the PDMP induced by \(f_{-1}\) and \(f_1\). This form of independence for \(\theta\) and \(r\) holds because the evolution of the angular component \(\theta\) is entirely deterministic and in particular not affected by the switching times. Thus,
\[
\mu^t(A \times B \times \{i\}) = \frac{1}{2\pi} \int_{S^1} I_A(\theta + t) \, d\theta \sum_{j \in \{-1, 1\}} \int_0^\infty \hat{\mathcal{P}}_{r, j}^t(B \times \{i\}) \, \nu(dr \times \{j\})
\]
\[
= \frac{\text{Leb}(A)}{2\pi} \nu(B \times \{i\}) = \frac{\text{Leb}(A)}{2\pi} \nu(B \times \{i\}) = \mu(A \times B \times \{i\}).
\]

Hence, \(\mu\) is an IPM for \((P^t)\). Next, we show that \(\mu\) is the only IPM such that \(\mu(S^1 \times (0, \infty) \times I) = 1\). First we show that any point in \(S^1 \times (0, \sqrt{p_+})\) is accessible from \(S^1 \times (0, \infty)\). Fix two points
$(\alpha, p) \in S^1 \times (0, \sqrt{p_+})$ and $(\beta, q) \in S^1 \times (0, \infty)$. For $i \in I$, we denote the flow associated with the vector field $g_i$ by $\Phi_i$. As $s \to \infty$, the radial component of $\Phi^s_{-1}(\beta, q)$ tends to 0. Let $s > 0$ such that $\Phi^s_{-1}(\beta, q)$ has angular component $\alpha$ and radial component $u < p$. As $p \in (0, \sqrt{p_+})$, a short computation shows that $\Phi^s(\alpha, p)$ has radial component

$$\left(\frac{e^{2p+t}p+C}{1+e^{2p+t}C}\right)^{\frac{1}{2}},$$

where $C = \frac{p^2+p^2}{p^2-p^2} > 0$. This shows that the vector field $g_1$ is both forward and backward complete on the punctured disk $S^1 \times (0, \sqrt{p_+})$, with $\lim_{t \to -\infty} \Phi^t_1(\alpha, p) = 0$. Hence, there is $t < 0$ such that $\Phi^t(\alpha, p)$ has angular component $\alpha$ and radial component $v < u$. For $T \geq 0$, let $h(T)$ denote the difference of the radial components of $\Phi^T_{-1}(\alpha, u)$ and $\Phi^T_1(\alpha, v)$. Then, $h(0) = u - v > 0$ and $h(-t) \leq u - p < 0$. As $h$ is continuous, there is $\tau \in (0, -t)$ such that $h(\tau) = 0$. As the points $\Phi^T_{-1}(\alpha, u)$ and $\Phi^T_1(\alpha, v)$ have the same angular component for any $T \geq 0$, we have

$$\Phi^{T}_{-1}(\alpha, u) = \Phi^T_1(\alpha, v).$$

Thus, we can reach the point $(\alpha, p)$ from $(\beta, q)$ as follows: First, flow along the vector field $g_{-1}$ for time $s + \tau$, then make a switch and flow along $g_1$ for time $-t - \tau$.

For $(\alpha, p) \in S^1 \times (0, \sqrt{p_+})$, the vectors $g_{-1}(\alpha, p)$ and $g_1(\alpha, p)$ are clearly transversal, so the weak bracket condition holds as well. By Theorem 2.4, $\mu$ is indeed the only IPM assigning mass 1 to $S^1 \times (0, \infty)$. The fact that $\delta$ and $\mu$ are the only ergodic IPM follows along the same lines as in the proof of Theorem 3.1.

Now, we consider the case $\frac{\lambda_+}{p_+} \geq -\frac{\lambda_+}{p_-}$. To obtain a contradiction, suppose that there is an IPM $\pi$ for $(P^t)$ such that $\pi(S^1 \times (0, \infty) \times I) > 0$. By the ergodic decomposition theorem, we may assume without loss of generality that $\pi(S^1 \times (0, \infty) \times I) = 1$. Consider the marginal $\pi(\cdot) = \pi(S^1 \times \cdot)$, which is a probability measure on $(0, \infty) \times I$. For $t > 0$ and with $\hat{P}$ defined as above, we have for measurable $B \subset (0, \infty)$ and $i \in I$

$$\hat{\pi}(B \times \{i\}) = \sum_{j \in I} \int_0^\infty \hat{P}^t_{r,j}(B \times \{i\}) \pi(dr \times \{j\})$$

$$= \sum_{j \in I} \int_{S^1} \int_0^\infty \hat{P}^t_{r,j}(B \times \{i\}) \pi(d\theta \times dr \times \{j\})$$

$$= \sum_{j \in I} \int_{S^1} \int_0^\infty \mathbb{I}_{S^1}(\theta + t) \hat{P}^t_{r,j}(B \times \{i\}) \pi(d\theta \times dr \times \{j\})$$

$$= \sum_{j \in I} \int_{S^1} \int_0^\infty P^t_{\theta,x,j}(S^1 \times B \times \{i\}) \pi(d\theta \times dr \times \{j\}) = \pi(S^1 \times B \times \{i\}) = \pi(B \times \{i\}).$$

This computation shows that $\hat{\pi}$ is an IPM for $(\hat{P}^t)$. But Theorem 3.1 implies that $(\hat{P}^t)$ has no IPM if $\frac{\lambda_+}{p_+} \geq -\frac{\lambda_+}{p_-}$, a contradiction.

### 3.3 Transcritical Bifurcation

Consider the ODE (2.1) for $d = 1$ and assume the existence of a trivial branch of equilibria $f(x_*, p) = 0$ for all $p$. Assume that the following conditions hold at $(x, p) = (x_*, p_*)$:

$$\partial_x f(x_*, p_*) = 0, \quad \partial_x f(x_*, p_*) \neq 0, \quad \partial_x^2 f(x_*, p_*) = 0. \quad (3.8)$$
Then a bifurcation occurs at \((x_*, p_*)\), which can be proven to be locally topologically equivalent to the transcritical bifurcation normal form

\[
x' = px - x^2.
\]  

\(\text{(3.9)}\)

The dynamics of \(\text{(3.9)}\) works as follows. There are two families of equilibrium points \(x_*=0\) and \(x_{ss}=p\). For \(p<0\), \(x_*\) is locally stable, while \(x_{ss}\) is unstable. For \(p>0\), the stabilities switch. There are bounded trapping regions of positive measure given in the different parameter regimes by

\[
V_{p<0} = [x_{ss}, 0] \quad \text{and} \quad V_{p>0} = [0, x_{ss}]
\]

with the special case \(V_{p=0} = [0, K]\) for any \(K > 0\). For fixed \(p_- < 0\) and \(p_+ > 0\), consider the vector fields

\[
f_{-1}(x) = p_-x - x^2, \quad f_1(x) = p_+x - x^2.
\]

These vector fields are not forward complete: trajectories for \(f_{-1}\) that start to the left of \(p_-\) and trajectories for \(f_1\) that start to the left of 0 move off to \(-\infty\) in finite time. To obtain a well-defined PDMP, we therefore restrict ourselves to the positive invariant set \([0, \infty)\), where both \(f_{-1}\) and \(f_1\) have bounded trajectories and are in particular forward complete. We switch from \(f_{-1}\) to \(f_1\) at rate \(\lambda_-\) and from \(f_1\) to \(f_{-1}\) at rate \(\lambda_+\), and we let \(\delta\) denote the product of the Dirac measure at 0 and the IPM of \(E\).

**Remark:** It is possible to define a PDMP that involves switching between \(f_{-1}\) and \(f_1\) on the larger interval \((p_-, \infty)\). Since \((p_-, \infty)\) is not a trapping region for \(f_1\), one needs to ensure that we switch away from \(f_1\) before reaching the point \(p_-\). This can be achieved by letting the switching rate \(\lambda_+\) depend on the location \(x\) of the switching trajectory, with \(\lambda_+(x)\) blowing up as \(x \to p_-\) from the right.

**Theorem 3.4.** The following statements hold.

1. If \(\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}\), there are exactly two ergodic IPM: the trivial measure \(\delta\) and a measure \(\mu\) such that \(\mu((0, \infty) \times I) = 1\).

2. If \(\frac{\lambda_+}{p_+} \geq -\frac{\lambda_-}{p_-}\), then \(\delta\) is the only IPM.

This statement can be shown along the same lines as Theorem 3.1. We therefore omit the proof.

**Theorem 3.5.** Suppose that \(\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}\). Then, the ergodic IPM \(\mu\) assigning mass 1 to \((0, \infty) \times I\) is absolutely continuous. Moreover, the corresponding invariant density \(\rho\) is given by

\[
\rho_i(x) = C x^{-\frac{\lambda_-}{p_-} - \frac{\lambda_+}{p_+} - 1} (-p_- + x)^{\frac{\lambda_-}{p_-} - \frac{1}{2}(1+i)} (p_+ - x)^{\frac{\lambda_+}{p_+} - \frac{1}{2}(1+i)} \mathbb{1}(0, p_+)(x), \quad i \in I.
\]

**Proof:** Absolute continuity of \(\mu\) follows from Theorem 2.4. As \(\mu((0, p_+) \times I) = 1\), the invariant densities \((\rho_i)_{i \in I}\) vanish outside of \([0, p_+]\). By Theorem 2.5, \((\rho_i)_{i \in I}\) are \(C^\infty\) on \((0, p_+)\) and thus satisfy the Fokker – Planck equations. For the probability flux \(\varphi_{-1}\), we have

\[
\varphi_{-1}(x) = C \exp \left( -\lambda_- \int \frac{dx}{f_{-1}(x)} - \lambda_+ \int \frac{dx}{f_1(x)} \right) = C x^{-\frac{\lambda_-}{p_-} - \frac{\lambda_+}{p_+} - 1} (-p_- + x)^{\frac{\lambda_-}{p_-} - \frac{1}{2}(1+i)} (p_+ - x)^{\frac{\lambda_+}{p_+} - \frac{1}{2}(1+i)}.
\]

As in the case of the supercritical pitchfork bifurcation, we obtain the desired formula with \(\rho_{-1} = \varphi_{-1}/f_{-1}\) and \(\rho_1 = -\varphi_{-1}/f_1\). \(\square\)
Proposition 3.6. Let \( \nu \) be a probability measure on \( \mathbb{R} \times I \) such that \( \nu((-\infty, 0) \times I) = 1 \), and let \( \nu \mathbb{P}^a \) denote the law of \((X, E)\) with initial distribution \( \nu \) and stopped at time \( \tau_a \). There is a nonincreasing function \( g : (-\infty, p_- - 2] \to (0, \infty) \) such that \( \int_{-\infty}^{p_- - 2} g(a) \, da < \infty \) and
\[
\nu \mathbb{P}^a (\tau_{p_- - 1} < \infty) > 0, \quad \nu \mathbb{P}^a (\tau_a - \tau_{a+1} < g(a) \mid \tau_{p_- - 1} < \infty) = 1, \quad a \leq p_- - 2.
\]

Proposition 3.6 essentially says that \( X_t \) goes off to \(-\infty\) in finite time with positive probability if the initial distribution assigns full measure to \((-\infty, 0) \times I\): There is a positive probability that \( X \) reaches the interval \((-\infty, p_- - 1]\) in finite time. And once \( X \) is in \((-\infty, p_- - 1]\), it blows up to \(-\infty\) with probability 1 in time less than
\[
(\tau_{p_- - 2} - \tau_{p_- - 1}) + (\tau_{p_- - 3} - \tau_{p_- - 2}) + \ldots \leq \sum_{k=2}^{\infty} g(p_- - k) < \infty.
\]

**Proof of Proposition 3.6.** Fix \( a \leq p_- - 2 \). Let us first show that \( \nu \mathbb{P}^a (\tau_{p_- - 1} < \infty) > 0 \). Let \( \delta > 0 \) be so small that \( \nu((-\infty, -\delta] \times I) > 0 \), and let \( r, s > 0 \) such that
\[
\Phi^r_{-1}(-\delta) = -\frac{\delta}{2}, \quad \Phi^s_{-1}(-\frac{\delta}{2}) = p_- - 1.
\]
If \( \nu \mathbb{P}^a (s + r > \tau_a) > 0 \), we also have
\[
\nu \mathbb{P}^a (\tau_{p_- - 1} < \infty) > 0.
\]
If \( \nu \mathbb{P}^a (s + r \leq \tau_a) = 1 \), we use the estimate
\[
\nu \mathbb{P}^a (\tau_{p_- - 1} < \infty) \geq \nu \mathbb{P}^a (\tau_{p_- - 1} < \infty, X_0 \leq -\delta, E_t = 1 \ \forall t \in [r, r + s]),
\]
Suppose that \( s + r \leq \tau_a \) and \( X_0 \leq -\delta \). Then, we have \( X_r \leq -\frac{\delta}{2} \). If in addition \( E_t = 1 \) for all \( t \in [r, r + s] \), it follows that \( \tau_{p_- - 1} < \infty \). Hence, the term on the right side of (3.10) equals
\[
\nu \mathbb{P}^a (X_0 \leq -\delta, E_t = 1 \ \forall t \in [r, r + s]) > 0.
\]
Now, we come to the second statement. We will specify the function \( g \) later in the proof. Since there is \( c > 0 \) such that \( f_1(x) \leq f_{-1}(x) \leq -c \) for all \( x \in (-\infty, p_- - 1] \), we have \( \tau_a < \infty \) for all \( a \leq p_- - 2 \) whenever \( \tau_{p_- - 1} < \infty \). By the strong Markov property,
\[
\nu \mathbb{P}^a (\tau_a - \tau_{a+1} < g(a) \mid \tau_{p_- - 1} < \infty) = \pi \mathbb{P}^a (\tau_a < g(a)),
\]
where \( \pi \) is the distribution of \((X, E)_{\tau_{a+1}}\) under \( \nu \mathbb{P}^a (\cdot \mid \tau_{p_- - 1} < \infty) \), and thus satisfies
\[
\pi((\infty, a + 1] \times I) = 1.
\]
In light of \( f_1 \leq f_{-1} \leq -c \), we have under \( \pi \mathbb{P}^a \)
\[
X_t \leq \Phi^1_{-1} (a + 1), \quad t \geq 0.
\]
As a result, if we let \( g(a) \) be defined by the relation \( \Phi_{a}^{\hat{g}(a)}(a + 1) = a \), we have \( \tau_a < g(a) \) under \( \pi P^{\tau_a} \). Since trajectories of \( f_{-1} \) starting in \((\neg \infty, p_{-} - 1]\) tend to \( \neg \infty \) in finite time, we also have \( \int_{\neg \infty}^{\tau_a} g(a) \, da < \infty \).

Proposition \(3.6\) and the ergodic decomposition theorem imply that there is no IPM assigning positive mass to \((\neg \infty, 0) \times I\). Looking at Proposition \(3.6\), it is natural to ask under which conditions a blow-up of \( X_t \) to \( \neg \infty \) in finite time happens almost surely. The answer follows from Theorem \(3.7\) below.

**Theorem 3.7.** Let \( a \leq p_{-} - 2 \) and let \( \nu \) be a probability measure on \( \mathbb{R} \times I \) such that \( \nu((-\infty, 0) \times I) = 1 \).

1. If \( \frac{\lambda_{+}}{p_{+}} < -\frac{\lambda_{-}}{p_{-}} \), we have \( \nu P^{\tau_a} (\tau_{p_{-} - 1} < \infty) = 1 \).

2. If \( \frac{\lambda_{+}}{p_{+}} > -\frac{\lambda_{-}}{p_{-}} \) and if \( \nu((p_{-}, 0) \times I) > 0 \), we have \( \nu P^{\tau_a} (\tau_{p_{-} - 1} < \infty) < 1 \) and

\[
\nu P^{\tau_a} \left( \{\tau_{p_{-} - 1} < \infty\} \cup \left\{\lim_{t \to \infty} X_t = 0\right\} \right) = 1.
\]

As stated in the following lemma, with probability 1, \( X_t \) either diverges to \( \neg \infty \) in finite time or converges to 0 as \( t \to \infty \).

**Lemma 3.8.** If \( a \leq p_{-} - 2 \) and if \( \nu \) is a probability measure on \( \mathbb{R} \times I \) such that \( \nu((-\infty, 0) \times I) = 1 \), we have

\[
\nu P^{\tau_a} \left( \{\tau_{p_{-} - 1} < \infty\} \cup \left\{\lim_{t \to \infty} X_t = 0\right\} \right) = 1.
\]

**Proof of Lemma 3.8** Under \( \nu P^{\tau_a} \), the complement of \( \{\tau_{p_{-} - 1} < \infty\} \cup \left\{\lim_{t \to \infty} X_t = 0\right\} \) is

\[
\{\tau_{p_{-} - 1} = \infty\} \cap \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{t \geq k} X_t \leq \frac{1}{n}.
\]

For fixed \( n \in \mathbb{N} \), consider the event

\[
\bigcap_{k=1}^{\infty} \bigcup_{t \geq k} X_t \leq \frac{1}{n}.
\]

On this event, there is \( T > 0 \) such that \( X_t \leq \frac{1}{n} \) for every \( t \geq T \), or there is a sequence of times \( t_{j} \to \infty \) such that \((X, E)_{t_{j}} = (-\frac{1}{n}, 1)\) for every \( j \). In the former case, let \( s > 0 \) such that

\[
\Phi_{1}^{*}(\frac{1}{n}) = p_{-} - 1.
\]

Then, \( \tau_{p_{-} - 1} < \infty \) or, \( \nu P^{\tau_a} \)-almost surely, there is \( r > T \) such that \( E_t = 1 \) for \( r \leq t \leq r + s \), which also yields \( \tau_{p_{-} - 1} < \infty \). In the latter case, observe that the first return time to state \((\neg \frac{1}{n}, 1)\) is finite with probability strictly less than 1, so by the strong Markov property the event \( \{(X, E)_{t_{j}} = (-\frac{1}{n}, 1) \ \forall j\} \) has probability 0.

**Proof of Theorem 3.7.** Assume first that \( \frac{\lambda_{+}}{p_{+}} < -\frac{\lambda_{-}}{p_{-}} \). By Lemma 3.8 it suffices to show that

\[
\nu P^{\tau_a} \left( \lim_{t \to \infty} X_t = 0 \right) = 0.
\]
Let \( M = [\frac{p_{-}}{4}, 0] \) and \( M_+ = [\frac{p_{+}}{4}, 0] \). Let \( \tilde{f}_1 \) be a smooth vector field that coincides with \( f_1 \) on the interval \([\frac{p_{-}}{4}, 0] \), is strictly negative on \( (\frac{p_{-}}{4}, 0) \), and has \( \frac{p_{+}}{4} \) as an equilibrium point. In addition, we assume that \( \tilde{f}_1(x) \geq f_1(x) \) for all \( x \in \mathbb{R} \). Then, \( M \) is positive invariant for the vector fields \( f_{-1} \) and \( \tilde{f}_1 \), and 0 is accessible from \( M \). Let \( (\tilde{X}, \tilde{E}) \) denote the PDMP with vector fields \( f_{-1}, \tilde{f}_1 \) and switching rates \( \lambda_-, \lambda_+ \). Following the proof of Theorem 3.1 and applying Theorem 2.2, we see that for any starting point \( x \in M_+, \tilde{X}_t \) almost surely does not converge to 0 as \( t \to \infty \). The Markov property and the fact that any switching trajectory starting in \((-\infty, 0)\) and converging to 0 has to visit points in \( M_+ \) imply that this result extends to starting points \( x \in (-\infty, 0) \). Since \( \tilde{f}_1 \geq f_1 \), we finally infer (3.11).

Now, we consider the case \( \lambda \frac{p_{-}}{p_{+}} > -\frac{\lambda}{p_{-}} \), assuming that \( \nu((p_-, 0) \times I) > 0 \). Defining \( \tilde{f}_1 \) and \((\tilde{X}, \tilde{E})\) as above, we obtain with Theorem 2.2 that for any starting point \( x \in M = [\frac{p_{-}}{4}, 0] \), \( \tilde{X}_t \) converges almost surely to 0 as \( t \to \infty \). Fix \( x \in [\frac{p_{-}}{4}, 0] \). Then,

\[
\delta_{x,-1} \tilde{P} \left( \lim_{t \to \infty} \tilde{X}_t = 0 \right) = 1,
\]

where \( \delta_{x,-1} \tilde{P} \) is the distribution of \((\tilde{X}, \tilde{E})\) with initial distribution \( \delta_{x,-1} \). The first return time for state \((x, -1)\) must then be infinite with positive probability. In other words, there is a positive probability that the PDMP \((\tilde{X}, \tilde{E})\) starting in \((x, -1)\) stays in \([\frac{p_{-}}{4}, 0] \times I \) for all \( t \geq 0 \) and thus coincides with \((X, E)\) starting in \((x, -1)\). In particular, \( P_{\tau_{x,-1}(\lim_{t \to \infty} X_t = 0)} > 0 \) for all \( x \in (\frac{p_{-}}{4}, 0) \). Let \( \epsilon > 0 \) be so small that \( \nu((p_-, \epsilon, 0) \times I) > 0 \), and let \( s > 0 \) such that \( \Phi_{s-1}((p_-, \epsilon) = \frac{p_{-}}{4} \). Then, for any \((y, i) \in (p_-, \epsilon) \), \((P_{y,i}^s)_{\tau_{y,i}}((\frac{p_{-}}{4}, 0) \times \{-1\}) > 0 \). It follows that

\[
\nu P_{\tau_{x,-1}(\lim_{t \to \infty} X_t = 0)} \geq \sum_{i \in I} \int_{p_- + \epsilon}^{p_-} \int_{\frac{p_{-}}{4}}^{p_-} \left( \lim_{t \to \infty} X_t = 0 \right) (P_{y,i}^s)_{\tau_{y,i}}(dx \times \{-1\}) \nu(dy \times \{i\}) > 0.
\]

The claim made in part 2 of Theorem 3.7 then follows from Lemma 3.8. \( \square \)

4 One Nontrivial Trapping Region

4.1 Subcritical Pitchfork Bifurcation

Consider the ODE (2.1) for \( d = 1 \) and assume the existence of a trivial branch of equilibria \( f(x, p) = 0 \) for all \( p \). Assume that the following conditions hold at \((x, p) = (x_*, p_*)\):

\[
\partial_x f(x_*, p_*) = 0, \quad \partial_{xx} f(x_*, p_*) = 0, \quad \partial_{xxx} f(x_*, p_*) > 0, \quad \partial_{xp} f(x_*, p_*) \neq 0. \quad (4.1)
\]

Then a bifurcation occurs at \((x_*, p_*)\), which can be proven to be locally topologically equivalent to the subcritical pitchfork bifurcation normal form

\[
x' = px + x^3. \quad (4.2)
\]

The dynamics of (4.2) works as follows. For \( p < 0 \), there are three equilibrium points \( x_0 = 0 \) and \( x_{\pm} = \pm \sqrt{-p} \). \( x_0 \) is locally stable, while \( x_{\pm} \) are unstable. For \( p > 0 \), \( x_0 = 0 \) is the only equilibrium point and it is unstable. For \( p \geq 0 \) there is no trapping region of positive measure. However, \([x_-, x_+] = V\) is a trapping region for the dynamics when \( p < 0 \). For fixed \( p_- < 0 \) and \( p_+ > 0 \), we switch between

\[
f_{-1}(x) = p_- x + x^3, \quad f_1(x) = p_+ x + x^3
\]
at rates $\lambda_-$ and $\lambda_+$. The trivial measure $\delta$ is defined exactly as for the supercritical pitchfork bifurcation. As for the transcritical bifurcation, the vector fields $f_{-1}$ and $f_1$ are not forward complete. E.g., any trajectory of $f_1$ not starting at 0 blows up in finite time. If one wishes to define a PDMP outside of the common equilibrium point 0, one can either let the rate $\lambda_+$ of switching from $f_1$ to $f_{-1}$ depend on the location $x$, with $\lambda_+(x)$ diverging to $\infty$ as $x$ approaches $-\sqrt{-p_-}$ from the right and $\sqrt{-p_-}$ from the left. Or one can stop the PDMP with constant switching rates once it reaches certain thresholds. For the latter model, $\delta$ is the unique IPM. Besides, we have the following result that is reminiscent of Proposition 3.6 and Theorem 3.7. Since $f_{-1}$ and $f_1$ are odd functions, we may restrict ourselves to the interval $(0, \infty)$, with the understanding that there are completely analogous statements about $(-\infty, 0)$.

**Theorem 4.1.** Let $\nu$ be a probability measure on $\mathbb{R} \times I$ such that $\nu((0, \infty) \times I) = 1$, and let
\[ \tau_a = \inf\{t \geq 0 : X_t \geq a\} \]
for $a > \sqrt{-p_-}$. Let $\nu P^{\tau_a}$ denote the law of $(X, E)$ with initial distribution $\nu$ and stopped at time $\tau_a$.

1. There is a nonincreasing function $g : [\sqrt{-p_-}+2, \infty) \to (0, \infty)$ such that $\int_{\sqrt{-p_-}+2}^{\infty} g(a) \, da < \infty$ and
\[ \nu P^{\tau_a}(\tau_{\sqrt{-p_-}+1} < \infty) > 0, \quad \nu P^{\tau_a}(\tau_a - \tau_{a-1} < g(a) \, | \, \tau_{\sqrt{-p_-}+1} < \infty) = 1, \quad a \geq \sqrt{-p_-} + 2. \]

2. If $\frac{\lambda_+}{p_+} < -\frac{\lambda_-}{p_-}$, we have $\nu P^{\tau_a}(\tau_{\sqrt{-p_-}+1} < \infty) = 1$ for $a \geq \sqrt{-p_-} + 2$.

3. If $\frac{\lambda_+}{p_+} > -\frac{\lambda_-}{p_-}$ and $\nu((0, \sqrt{-p_-}) \times I) > 0$, we have $\nu P^{\tau_a}(\tau_{\sqrt{-p_-}+1} < \infty) < 1$ and
\[ \nu P^{\tau_a} \left( \{\tau_{\sqrt{-p_-}+1} < \infty\} \cup \left\{ \lim_{t \to \infty} X_t = 0 \right\} \right) = 1. \]

The proof is analogous to the ones of Proposition 3.6 and Theorem 3.7, and we omit it.

### 4.2 Subcritical Hopf Bifurcation

Consider the same setting as in Section 3.2 except that we now assume that the first Lyapunov coefficient satisfies $l_1(p_*) > 0$. This leads to a subcritical Hopf bifurcation normal form
\[
\begin{align*}
x'_1 &= px_1 - x_2 + x_1(x_1^2 + x_2^2), \\
x'_2 &= x_1 + px_2 + x_2(x_1^2 + x_2^2).
\end{align*}
\]
(4.3)

Here the unstable bifurcating family of periodic orbits $\{\|x\|_2 = \sqrt{-p}\}$ exists for $p < 0$ and in this case $x_* = 0$ is locally stable. $x_*$ is unstable for $p \geq 0$. For $p < 0$, there is a trapping region of positive measure $\mathcal{V}_{p < 0} = \{x \in \mathbb{R}^2 : \|x\|_2 \leq \sqrt{-p}\}$. After a change of variables to polar coordinates, the system in (4.3) becomes
\[
\begin{align*}
\theta' &= 1, \\
r' &= pr + r^3.
\end{align*}
\]
For fixed \( p_- < 0 \) and \( p_+ > 0 \), we then switch between 
\[
g_{-1}(\theta, r) = (1, p_- r + r^3)\top, \quad g_1(\theta, r) = (1, p_+ r + r^3)\top
\]
at rates \( \lambda_- \) and \( \lambda_+ \). Here, we encounter the same issue as for the subcritical pitchfork bifurcation. Then, Theorem 4.1 applies to the PDMP induced by the vector fields 
\[
f_{-1}(r) = p_- r + r^3, \quad f_1(r) = p_+ r + r^3,
\]
and thus to the evolution of the radial component of the PDMP induced by \( g_{-1} \) and \( g_1 \).

### 4.3 Fold Bifurcation

Consider the ODE \((2.1)\) for \( d = 1 \). Assume that the following conditions hold at \((x, p) = (x_*, p_*)\):
\[
f(x_*, p_*) = 0 = \partial_x f(x_*, p_*) = 0, \quad \partial_{xx} f(x_*, p_*) \neq 0, \quad \partial_p f(x_*, p_*) \neq 0.
\]
(4.4)

Then a bifurcation occurs at \((x_*, p_*)\), which can be proven to be locally topologically equivalent to the fold (or saddle-node) bifurcation normal form
\[
x' = p - x^2.
\]
(4.5)

For \( p > 0 \), there are two equilibrium points \( x_\pm = \pm \sqrt{p} \). \( x_+ \) is locally stable, while \( x_- \) is unstable. For \( p < 0 \), there are no equilibria. Only for \( p > 0 \), there is a trapping region given by \( \mathcal{V}_{p>0} = [x_-, x_+] \). For \( p_- < 0, p_+ > 0 \), we switch between 
\[
f_{-1}(x) = p_- - x^2, \quad f_1(x) = p_+ - x^2
\]
at rates \( \lambda_- \) and \( \lambda_+ \).

#### Theorem 4.2

Let \( \nu \) be a probability measure on \( \mathbb{R} \times I \), and let
\[
\tau_a = \inf\{ t \geq 0 : X_t \leq a \}
\]
for \( a < -\sqrt{p_+} \). Let \( \nu P_{\tau_a} \) be the law of \((X, E)\) with initial distribution \( \nu \) and stopped at time \( \tau_a \).

Then, there is a nonincreasing function \( g : (-\infty, -\sqrt{p_+} - 2] \to (0, \infty) \) such that \( \int_{-\infty}^{0} g(a) da < \infty \) and 
\[
\nu P_{\tau_a}(\tau_{-\sqrt{p_+} - 1} < \infty) = 1, \quad \nu P_{\tau_a}(\tau_a - \tau_{a+1} < g(a) \mid \tau_{-\sqrt{p_+} - 1} < \infty) = 1, \quad a \leq -\sqrt{p_+} - 2.
\]

In words, \( X_t \) diverges to \( -\infty \) in finite time almost surely.

**Proof of Theorem 4.2**. Let us show that \( \nu P_{\tau_a}(\tau_{-\sqrt{p_+} - 1} < \infty) = 1 \). The rest is analogous to the proof of Proposition 4.3. For fixed \( x \geq -\sqrt{p_+} - 1 \), let \( s_1 \geq 0 \) such that \( \Phi_{s_1}^a(x) = -\sqrt{p_+} - 1 \), and let \( s_2 > 0 \) such that \( \Phi_{-1}^s(\sqrt{p_+}) = -\sqrt{p_+} - 1 \). Set \( s = \max\{s_1, s_2\} \) and let \( i \in I \). With \( \delta_{x,i} P_{\tau_a} \)-probability 1, we have \( \tau_{-\sqrt{p_+} - 1} < \infty \) or there is \( r \geq 0 \) such that \( E_{r+} = -1 \) for all \( t \in [r, r+s] \). But the latter case also implies \( \tau_{-\sqrt{p_+} - 1} < \infty \) because any switching trajectory starting from \( x \) cannot move to the right of \( \max\{x, \sqrt{p_+}\} \). As a result,
\[
\nu P_{\tau_a}(\tau_{-\sqrt{p_+} - 1} < \infty) = \sum_{i \in I} \int_{-\infty}^{\infty} \delta_{x,i} P_{\tau_a}(\tau_{-\sqrt{p_+} - 1} < \infty) \nu(dx \times \{i\}) = 1.
\]
\(\square\)
5 Applications

In this section, we provide several very brief examples of systems where the switching viewpoint near bifurcations via PDMPs can yield insight into concrete dynamical systems arising in applications. In particular, the normal form results can be used sufficiently close to bifurcation points after a normal form transformation. Furthermore, they can also be used directly to form conjectures about the dynamics of the applications.

5.1 The Paradox of Enrichment

The paradox of enrichment is a classical topic in ecology. One simple variant can be found in classical predator-prey systems, such as the Rosenzweig-MacArthur [45] model

\[ \begin{align*}
    x' &= x \left(1 - \frac{x}{p}\right) - \frac{xy}{1+x}, \\
    y' &= \beta \frac{xy}{1+x} - my,
\end{align*} \tag{5.1} \]

where \( x, y \in [0, \infty) \) are population densities of prey and predator, \( \beta > 0 \) is a parameter representing a conversion factor, \( m > 0 \) is the mortality of the predator, while \( p > 0 \) is the carrying capacity for the prey. The basic concept of the paradox of enrichment [44] is that increasing the carrying capacity \( p > 0 \) can actually lead to more likely extinction events triggered by additional stochastic effects, which can be supported by a classical bifurcation analysis of (5.1) as follows:

Let us fix \( m = 1 \) and \( \beta = 3 \) while varying \( p \) as the main bifurcation parameter. Besides the two boundary equilibrium points \( (x, y) = (0, 0) \) and \( (x, y) = (p, 0) \), we find the nontrivial co-existence equilibrium point

\[ (x_*, y_*) = (x_*, y_*(p)) = \left( \frac{1}{2}, \frac{3(2p - 1)}{4p} \right) \]

which is in the relevant domain given by the non-negative quadrant for \( p > 1/2 \). Linearizing (5.1) around \( (x_*, y_*) \) shows that the coexistence equilibrium is locally asymptotically stable for \( p \in (0, 2) \). Another direct calculation shows that a supercritical Hopf bifurcation occurs at \( p_* = 2 \). The resulting locally asymptotically stable limit cycle generated in the Hopf bifurcation for \( p > 2 \) grows in phase space. Hence, solutions can get closer to the two coordinate axes \( \{x = 0, y \geq 0\} \) and \( \{y = 0, x \geq 0\} \), which could make it more likely that a stochastic effect triggers an extinction event of a species. Therefore, enrichment may lead to a potential increase in extinction events. Of course, it is important to mention that there is still a debate in the literature on the mechanisms and possible variations of the paradox of enrichment [31, 32, 40]. We do not provide here a full discussion of the various arguments made in favor or against the paradox but instead point towards the effect of randomness in the parameter \( p \).

From an ecological perspective, it can be plausible to view \( p \) as a parameter, which switches randomly between different carrying capacities since the environment might be driven by external random events such as droughts, floods, storms, earthquakes, sudden human intervention, or even just different seasonal climate conditions. Suppose we switch \( p \) randomly in a range near \( p_* = 2 \) with rates \( \lambda_\pm \) and values \( p_\pm \) as defined in Section 3.2 where the parameters \( p_\pm \) are chosen so that the supercritical Hopf normal form (3.6) is a good local approximation of (5.1) near
Then Theorem 3.3 suggests an interesting dichotomy of the ergodic IPM. Either, we have only the invariant measure $\delta$, which is concentrated on the equilibrium branch $(x_*, y_*(p))$ with probabilities determined by the switching rates, or we have two probability measures given by $\delta$ and $\nu$, where $\nu$ is a non-trivial product measure also supported on the periodic orbit. One natural ecological interpretation of this effect is that we can actually avoid the paradox of enrichment from the viewpoint of measures if we restrict to those switching rates, which only lead to the IPM $\delta$, i.e., that we switch frequently enough from the periodic stable regime above the bifurcation to the stationary stable regime below the bifurcation.

5.2 Relaxation Oscillations

Consider the van der Pol (vdP) / FitzHugh-Nagumo (FHN) system

$$
x' = p - \frac{1}{3}x^3 + x,
$$

$$
p' = -\varepsilon x,
$$

which is a classical model used for bistable systems with relaxation oscillations and excitability. The parameter $\varepsilon$ is usually assumed to be small, so that in the singular limit $\varepsilon \to 0$, we obtain the fast subsystem ODE

$$
x' = p - \frac{1}{3}x^3 + x,
$$

(5.2)

where $p \in \mathbb{R}$ becomes a parameter. We can also view $p$ as a random parameter for the dynamics. Observe that there are several branches of equilibrium points for (5.2) given by solving

$$
p = \frac{1}{3}x^3 - x.
$$

If $p \in (-2/3, 2/3)$, then there are three equilibria, two locally asymptotically stable and one unstable. At $p_* = -2/3$ and $p_* = 2/3$, there are non-degenerate fold bifurcations. While for $|p| > 2/3$, there is always only one globally stable equilibrium point $x_*$. Let us focus on the case of switching $p$ near $p_* = 2/3$; the case $p_* = -2/3$ simply follows by a symmetry argument. If we switch the dynamics randomly above and below the fold bifurcation, Theorem 4.2 suggests that with probability one, we are going to diverge away from the region of the fold, i.e., we are going to obtain a point measure eventually concentrated on single remaining equilibrium $x_*$ existing for $p_* > 2/3$. Hence, if we have random switching across both folds, then it is possible to obtain the classical structure of a relaxation oscillations [41]. This confirms similar observations made already numerically for a similar class of randomly switched van der Pol oscillators in [34].

5.3 Adaptive Swarming

The next ODE model we are going to discuss is motivated by the swarming motion of locusts in a ring-shaped arena [16]. An adaptive network [25] model for this situation was proposed in [29]. The network model views locusts in clockwise-moving and anti-clockwise-moving nodes and keeps track of interactions between different locusts/nodes via the links of the network. The model is reduced to a low-dimensional ODE via moment closure [33] and we focus here only on
the essential features of the following low-dimensional ODE model

\[\begin{align*}
    x_1' &= q(x_1 - x_2) + w_3(y_3^3/(2x_2) - y_3^2/(2x_1)), \\
    x_2' &= q(x_2 - x_1) + w_3(y_3^3/(2x_1) - y_3^2/(2x_2)), \\
    y_1' &= q(y_3 - 2y_1) + w_3(y_3 + y_3^2/L - 2y_3y_1/x_1), \\
    &\quad + w_3(y_3^2/x_2 + y_3^2/(2x_2^2) - y_3^2y_1/x_1^2) + a_e x_1^2 - d_e y_1, \\
    y_2' &= q(y_3 - 2y_2) + w_2(y_2 + y_3^2/x_3 - 2y_3y_1/x_1), \\
    &\quad + w_3(y_3^2/x_1 + y_3^2/(2x_1^2) - y_3^2y_2/x_2^2) + a_e x_2^2 - d_e y_2.
\end{align*}\]  

and the conservation equation

\[\begin{align*}
    (y_1 + y_2 + y_3)' &= a_0 x_1 x_2 - d_0 y_3 + a_e (x_1^2 + x_2^2) - d_e (y_1 + y_2),
\end{align*}\]  

where \(q, w_2, w_3, a_e, d_e, a_0, d_0\) are positive parameters. Basically, \(x_1\) and \(x_2\) correspond to proportions of clockwise \(R\) (“right”) and anti-clockwise \(L\) (“left”) moving nodes, while \(y_{1,2,3}\) capture the link densities \(RR, LL\) and \(RL\) between the two classes of nodes respectively. One checks that for \(a_e = 0 = d_e\), one can solve the steady-state problem, which provides a branch of solutions given by

\[x_1 = \frac{1}{2} = x_2.\]

This state corresponds to an equal number of left and right moving nodes. This disordered state is locally asymptotically stable up to a supercritical pitchfork bifurcation at \(a_0^* = 2d_0 \sqrt{2q/w_3}\). The parameter \(a_0\) controls the rate at which new connections between left and right moving nodes form. Therefore, for a high connectivity between different groups, the system can move into an ordered phase given by the steady state

\[(x_1)_\pm := \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 8qd_0/w_3a_0^*}\]

and similarly for \(x_2\) with reversed signs. This corresponds to a classical symmetry-breaking and above the supercritical pitchfork, the two majority states are locally asymptotically stable. Clearly, we can also view \(a_0 =: p\) as our randomly switched parameter across the pitchfork bifurcation. Then Theorem 3.1 provides us with the case of either one or three IPM if we switch near \(a_0^*\). The interpretation for swarming is that we effectively can allow for a certain percentage of disordered motion as long as the switching rate back into the ordered phase is large enough to get an effective ordered phase. Furthermore, if we have the case of three IPM, then we are bound to observe not only a pure ordered state but intermittent phases of disordered motion as the non-trivial measures are supported also near the locally unstable state above the bifurcation point.

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