Bogomolov Instability and Kawamata-Viehweg Vanishing

GUILLERMO FERNÁNDEZ DEL BUSTO

Introduction

The purpose of this note is to show how the Kawamata-Viehweg vanishing theorem for fractional divisors leads to a quick new proof of Bogomolov’s instability theorem for rank two vector bundles on an algebraic surface.

Let $X$ be a smooth complex projective surface and let $E$ be a rank two holomorphic vector bundle on $X$. Bogomolov’s theorem states that if $c_1(E)^e > \Delta |_E(\mathcal{E})$, then $E$ satisfies a strong instability condition, which roughly speaking means that $E$ contains an exceptionally positive rank one subbundle. Bogomolov’s original proof $[\text{B}], \text{Rd}$ revolved around a beautiful argument with geometric invariant theory. Another proof—using characteristic $p$ techniques—was given by Gieseker $[\text{G}]$, and Miyaoka $[\text{M}]$ subsequently found a simple way to reduce the question to some restriction theorems of Mumford, Mehta and Ramanathan $[\text{MR}]$. Each of these proofs also yields an analogous assertion for higher rank bundles.

It is well understood that the rank two case of Bogomolov’s result can be used to prove various sorts of vanishing theorems on the surface $X$. For example, Mumford $[\text{Rd}]$ used this approach to give a quick proof of Ramanujam’s form of the Kodaira vanishing theorem. Somewhat later, Reider $[\text{Rdr}]$ realized that similar techniques yield criteria for the vanishing of groups of the form $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z)$, where $\mathcal{I}_Z$ is the ideal sheaf of a set of points $Z \subseteq X$. He deduces thereby his celebrated theorem on freeness and very ampleness of adjoint linear series on $X$. A cohomological approach to these questions, based on Miyaoka’s vanishing theorem for Zariski decompositions, was given by Sakai $[\text{S}]$.

More recently, in connection with three dimensional analogues of Reider’s results, Ein and Lazarsfeld $[\text{EL}]$ found that one can prove (special cases of) Reider’s results using the Kawamata-Viehweg vanishing theorem. The argument draws on the cohomological techniques pioneered by Kawamata, Kollár, Reid and Shokurov in connection with the minimal model program (c.f. $[\text{CKM}]$ or $[\text{KMM}]$). The question then arises whether similar techniques can be used to deduce the full theorem of Bogomolov. We complete this circle of ideas by showing that this is indeed the case, and obtain a transparent new proof of Bogomolov’s theorem.

The idea of the proof is very simple. Given the rank two bundle $E$, after twisting by a sufficiently ample line bundle we may assume that $E$ has lots of sections. The Koszul complex associated to a general section $s$ then expresses $E$ as sitting in an exact sequence

\[ 0 \to \mathcal{O}_X \overset{f}{\to} E \to \mathcal{O}_X(L) \otimes \mathcal{I}_Z \to 0. \]

\[ ^{1}\text{Partially supported by UNAM} \]
where $L$ is ample, and $Z$ consists of distinct points. As in the proof of Reider’s theorem in [ET], the positivity of $c_1^2(\mathcal{E}) - \Delta|_\mathcal{E}(\mathcal{E})$ implies the existence of a divisor $D$ in $|nL|$ with high multiplicity on $Z$. If the singular points of $D$ are (close to being) isolated, then one would obtain a vanishing which contradicts the local freeness of $\mathcal{E}$. Consequently, $D$ must have some special components appearing with high multiplicity. We use these distinguished components to construct a divisor $\Gamma \subseteq D$ through $Z$. The Kawamata-Viehweg vanishing theorem implies then that the inclusion $\mathcal{O}_X(L - \Gamma) \to \mathcal{O}_X(L) \otimes \mathcal{I}_Z$ lifts to an embedding of $\mathcal{O}_X(L - \Gamma)$ into $\mathcal{E}$, and we argue that (the saturation of) $\mathcal{O}_X(L - \Gamma) \subseteq \mathcal{E}$ is a destabilizing subsheaf. A new feature in this approach is that we need to be somewhat careful in our choice of $Z$ and $D$. We use a monodromy argument to show that for a general choice of $s$ and $D$, $D$ will have the same multiplicity at each point of $Z = Z(s)$.

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§1. The Uniform Multiplicity Property and Beginning of Proof.

Consider a rank 2 vector bundle $\mathcal{E}$ on a projective and nonsingular surface $X$; recall that $\mathcal{E}$ is said to be (Bogomolov)-unstable if there exists an exact sequence

$$0 \to \mathcal{O}_X(M) \to \mathcal{E} \to \mathcal{O}_X(N) \otimes \mathcal{I}_\chi \to 1,$$

with $M$ and $N$ divisors, and $\chi \subseteq X$ a 0-dimensional subscheme with sheaf of ideals $\mathcal{I}_\chi$, such that $M - N$ is in the positive cone of the Néron-Severi group, i.e. $(M - N)^2 > 0$ and $(M - N) \cdot H > 0$ for any ample divisor $H$ on $X$. Bogomolov’s theorem then asserts that an equivalent condition for $\mathcal{E}$ to be unstable is that $c_1(\mathcal{E})^\xi > \Delta|_\mathcal{E}(\mathcal{E})$.

Since Bogomolov’s theorem is invariant under tensor product by line bundles, we may assume that $\mathcal{E}$ is globally generated. Let $s \in H^0(X, \mathcal{E})$ be a general section; the Koszul resolution associated to $s$ defines $\mathcal{E}$ as an extension

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{E} \to \mathcal{O}_X(L) \otimes \mathcal{I}_Z \to 1$$

with $Z = Z(s)$ the 0-scheme of $s$; we may suppose that $L = c_1(\mathcal{E})$ is ample and that $Z$ consists of $c_2(\mathcal{E})$ distinct points.

The local freeness of $\mathcal{E}$ imposes some conditions on the finite set $Z$ (c.f. [GH] or [OSS]). In fact, there is an element $e \in \text{Ext}^1(\mathcal{O}_X(L) \otimes \mathcal{I}_Z, \mathcal{O}_X)$ corresponding to the extension class $[\mathcal{E}]$. If $Z' \subseteq Z$ is any proper subset (possibly empty) the local freeness of $\mathcal{E}$ then implies that $e$ is not in the image of $\text{Ext}^1(\mathcal{O}_X(L) \otimes \mathcal{I}_{Z'}, \mathcal{O}_X) \to \text{Ext}^\infty(\mathcal{O}_X(L) \otimes \mathcal{I}_Z, \mathcal{O}_X)$. Since $\text{Ext}^1(\mathcal{O}_X(L) \otimes \mathcal{I}_Z, \mathcal{O}_X)$ is Grothendieck dual to $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z)$, this is equivalent of the non-surjectivity of the evaluation map

$$H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z'}) \to H^1(Z - Z', \mathcal{O}_{Z - Z'}(K_X + L)),$$  

(2)
i.e. \(Z - Z'\) cannot impose independent conditions on \(|\mathcal{O}_X(\mathcal{K}_X + \mathcal{L}) \otimes \mathcal{I}_Z|\).

Suppose now that \(c_1(\mathcal{E}) \in \Delta_\mathcal{E}(\mathcal{E})\). As in the proof of Reider’s theorem in [EL], for \(n \gg 0\) there is a divisor \(D\) in \(|n\mathcal{L}|\) with \(\text{mult}_x(D) \geq 2n + 1\) for any \(z \in Z\). (In brief, by Riemann-Roch \(h^0(X, \mathcal{O}_X(\mathcal{L})) \sim \sum_\mathcal{E} L_\mathcal{E}\), whereas it is \(\left(\frac{2n+2}{2}\right) \deg(Z) \sim \frac{n^2}{2} 4 \deg(Z)\) conditions on a divisor to have multiplicity at least \(2n + 1\) at each point of \(Z\)).

The proof will involve analysis of \(D\) and \(s\), but we will need to have some control on the geometry of \(D\). In fact, we need to show that if the section \(s \in H^0(X, \mathcal{E})\) and the divisor \(D \in |n\mathcal{L}|\) are sufficiently general, then \(D\) and \(Z = Z(s)\) satisfy the following

**Uniform Multiplicity Property (UMP).** For any rational number \(\delta > 0\), the multiplicity of \([\delta D]\) is the same at every point of \(Z\).

For the proof, let \(\mathbb{P} (= \mathbb{P} H^0(\mathcal{E}))\) be the projective space parametrizing sections of \(\mathcal{E}\). Let

\[
X \times \mathbb{P} \supseteq Z = \{(\xi, f) : f(\xi) = t\}
\]

be the universal 0-scheme, with projections \(p\) and \(q\). Since \(\mathcal{E}\) is globally generated, \(Z \to X\) is a projective bundle, hence \(Z\) is irreducible. Note that \(Z\) is generically finite over \(\mathbb{P}\) and that \(\dim Z = \dim \mathbb{P}\). Now let \(D \subseteq X \times \mathbb{P}\) be an effective divisor flat over \(\mathbb{P}\): we’ll write \(D_s\) for the fibre of \(D\) over \(s\). The situation is as follows:

\[
\begin{array}{c}
D \\
\cap \\
\downarrow \\
\mathbb{P} \\
\nearrow \\
\searrow \\
Z \subset X \times \mathbb{P}
\end{array}
\]

The main point is the following

**Lemma.** Let \(k\) be a positive integer, and suppose that for a general \(s \in \mathbb{P}\), there is a point \(x \in Z(s)\) such that \(\text{mult}_x(D_s) \geq k\). Then for a general \(s \in \mathbb{P}\), \(\text{mult}_x(D_s) \geq k\) for every \(x \in Z(s)\).

**Proof.** Let \(Z \supseteq Z_{\parallel} = \{(\xi, f) : f \cap \cup_{s}(D_s) \geq \parallel\}\). By hypothesis \(Z_{\parallel}\) dominates \(\mathbb{P}\), hence

\[
\dim Z_{\parallel} = \dim \mathbb{P} = \dim Z
\]

and since \(Z\) is irreducible, we have that \(Z_{\parallel} = Z\). □

For the (UMP), consider the sheaf \(\mathcal{F} = \sqrt[\mathcal{E}_1(\mathcal{L}) \otimes I_{Z}^{\infty}}\). Since \(c_1(\mathcal{E}) \in \Delta_\mathcal{E}(\mathcal{E})\), \(q_\ast \mathcal{F}\) has positive rank. Hence \(H^0(X \times \mathbb{P}, q^\ast \mathcal{O}_\mathbb{P}(\mathcal{H}) \otimes \mathcal{F}) \neq t\) for sufficiently positive \(H\). Let \(D\) be the corresponding divisor of a section of \(q^\ast \mathcal{O}_\mathbb{P}(\mathcal{H}) \otimes \mathcal{F}\). Now apply the previous lemma to the divisor \([\delta D]\).
The basic tool for our cohomological approach to Bogomolov instability is the vanishing theorem of Kawamata-Viehweg. Sakai noticed (c.f. [EL, 1.1]) that on a surface \( X \), the vanishing of \( \mathbb{Q} \)-divisors holds without any normal crossing assumption:

**Theorem (Kawamata-Viehweg).** Let \( X \) be a nonsingular projective surface, and let \( M \) be any big and nef \( \mathbb{Q} \)-divisor on \( X \). Then

\[
H^i(X, \mathcal{O}_X(K_X + \lceil M \rceil)) = 0 \text{ for } i > 1.
\] (3)

§2. End of Proof.

With notation as in §1 now we fix a general section \( s \in \mathbb{P} \), and write \( Z = Z(s) \) and \( D = D_s \). We assume that the (UMP) holds for \( Z \) and \( D \). The (UMP) implies that \( D \) has the same multiplicity at every point \( z \in Z \); denote by \( m \) such multiplicity. Let

\[
D = \sum d_j D_j + F,
\]
where \( D_j \) are the components of \( D \) intersecting \( Z \), and \( F \) consists of the components of \( D \) disjoint from \( Z \). Let \( d = \max \{d_j\} \) and consider the divisor

\[
D_0 = \lceil \sum \frac{d_j}{d} D_j \rceil.
\]

Then \( D_0 \subseteq X \) is reduced and contains (by construction) at least one point of \( Z \), hence \( Z \subseteq D_0 \) because of the (UMP).

Notice next that \( 2d > m \). In fact, if \( 2d \leq m \) then as in the proof of Reider’s theorem in [EL], let \( f: Y \to X \) be the blow-up of \( X \) at \( Z \) and let \( E_i \subseteq Y \) be the exceptional divisors. The \( \mathbb{Q} \)-divisor \( f^*(L - \frac{2d}{m}D) - \delta_2 \sum E_i \equiv \delta_1 f^*(L - \frac{2}{m}D) + (1 - \delta_1) f^*L - \delta_2 \sum E_i \), with \( 0 < \delta_2 < \delta_1 < 1 \) and \( \delta_2 \geq 2(1 - \delta_1) \), is nef and big, and applying the Kawamata-Viehweg vanishing theorem, it follows that

\[
H^1(X, \mathcal{O}_X(K_X + L - \lceil \frac{\epsilon}{d} F \rceil) \otimes \mathcal{I}_Z) = 0,
\]
which is absurd by (2).

Assume henceforth that \( 2d > m \). In particular, \( D_0 \) is nonsingular at each point of \( Z \) and each point of \( Z \) is in exactly one component of \( D_0 \). The \( \mathbb{Q} \)-divisor \( L - \frac{1}{d} D \) is ample, so \( H^1(X, \mathcal{O}_X(K_X + L - \lceil \frac{\epsilon}{d} D \rceil)) = 0 \). Therefore there exists a minimal subdivisor \( \Delta \subseteq \lceil \frac{\epsilon}{d} F \rceil \) such that

\[
H^1(X, \mathcal{O}_X(K_X + L - D_0 - \Delta)) = 0,
\] (4)
in the sense that if (4) holds for \( \Delta \), and if \( \Delta' \) is a component of \( \Delta \) (whenever \( \Delta \neq 0 \) then \( H^1(X, \mathcal{O}_X(K_X + L - D_0 - \Delta + \Delta')) \neq 0 \). Let

\[
\Gamma = D_0 + \Delta.
\]
Then (4) implies that Ext\(^1\)(\(O_\mathcal{X}(\mathcal{L} - \Gamma), O_\mathcal{X}\)) = \(\iota\), and in consequence there is an injection \(O_\mathcal{X}(\mathcal{L} - \Gamma) \hookrightarrow \mathcal{E}\) with

\[
O_\mathcal{X}(\mathcal{L} - \Gamma) \\
\downarrow \\
0 \rightarrow O_\mathcal{X} \rightarrow \mathcal{E} \rightarrow O_\mathcal{X}(\mathcal{L}) \otimes \mathcal{I}_Z \rightarrow 0.
\]

The purpose now is to show that (the saturation of) \(O_\mathcal{X}(\mathcal{L} - \Gamma)\) is the destabilizing subsheaf of \(\mathcal{E}\). For this end, we need to establish some inequalities.

Note first that if \(\Delta'\) is a component of \(\Delta\), then

\[
(L - \Gamma) \cdot \Delta' \leq 0.
\]

In fact, the exact sequence

\[
0 \rightarrow O_\mathcal{X}(K_\mathcal{X} + \mathcal{L} - \Gamma) \rightarrow O_\mathcal{X}(K_\mathcal{X} + \mathcal{L} - \Gamma + \Delta') \rightarrow O_{\Delta'}(K_{\Delta'} + \mathcal{L} - \Gamma) \rightarrow \iota,
\]

and \(H^1(X, O_\mathcal{X}(K_\mathcal{X} + \mathcal{L} - \Gamma + \Delta')) \neq \iota\) imply that \(H^1(\Delta', O_{\Delta'}(K_{\Delta'} + (\mathcal{L} - \Gamma))) \neq \iota\), whence (\(\mathfrak{f}\)).

We claim next that

\[
(L - \Gamma) \cdot D_0 \leq \deg(Z).
\]

In fact, since each point of \(Z\) is in exactly one component of \(D_0\) and each component of \(D_0\) contains at least one point of \(Z\), then (\(\mathfrak{f}\)) will follow if we show that if \(D'\) is a component of \(D_0\) containing a subset \(Z' \subseteq Z\), then \((L - \Gamma) \cdot D' \leq \deg(Z')\). Suppose to the contrary that \((L - \Gamma) \cdot D' > \deg(Z')\). Then any deg\((Z')\) nonsingular points on \(D'\) impose independent conditions on \(|O_{D'}(K_{D'} + \mathcal{L} - \Gamma)|\). On the other hand, it follows from (4) that

\[
H^0(X, O_\mathcal{X}(K_\mathcal{X} + \mathcal{L} - \Gamma + D')) \rightarrow \mathcal{H}'(D', O_{D'}(K_{D'} + \mathcal{L} - \Gamma))
\]

is surjective. Since \(Z - Z' \subseteq \text{Supp}(\Gamma - D')\) we have then that \(Z'\) imposes independent conditions on \(H^0(X, O_\mathcal{X}(K_\mathcal{X} + \mathcal{L}) \otimes \mathcal{I}_{Z-Z'})\). But this contradicts (\(\mathfrak{f}\)), so (\(\mathfrak{f}\)) is established. Combining (\(\mathfrak{f}\)) and (\(\mathfrak{f}\)) yields

\[
(L - \Gamma) \cdot \Gamma \leq \deg(Z).
\]

We now assert that

\[
\deg(Z) > \frac{1}{2} L \cdot \Gamma.
\]

For this, note that

\[
L - \Gamma \equiv (1 - \frac{n}{d})L + (\sum \frac{d_j}{d} D_j - D_0) + (\frac{1}{d} F - \Delta).
\]

We assert first that

\[
(\frac{1}{d} F - \Delta) \cdot \Gamma \geq 0.
\]
In fact, let \( \frac{1}{d} F - \Delta = \Delta_1 + \Delta_2 \), with \( \Delta_1 \) an effective \( \mathbb{Q} \)-divisor all of whose components are components of \( \Delta \), and with \( \Delta_2 \) and \( \Delta \) having no common components. Then \( \Delta_2 \cdot \Gamma \geq 0 \), hence \( (\frac{1}{d} F - \Delta) \cdot \Gamma \geq \Delta_1 \cdot \Gamma \) and \( \Delta_1 \cdot \Gamma \geq \Delta_1 \cdot L \geq 0 \) because of (5). Recalling that \( \sum \frac{d_j}{d} D_j - D_0 \) and \( \Delta \) have no common components, observe next:

\[
(\sum \frac{d_j}{d} D_j - D_0) \cdot \Gamma \geq (\sum \frac{d_j}{d} D_j - D_0) \cdot D_0 \\
\geq \frac{1}{d} \sum_{z \in Z} (\text{mult}_z(D) - d) \\
\geq \deg(Z)(\frac{m}{d} - 1).
\]

Combining (7) and (9) we then conclude that

\[
\deg(Z) \geq (L - \Delta) \cdot \Gamma \geq (1 - \frac{n}{d})L \cdot \Gamma + \deg(Z)(\frac{m}{d} - 1)
\]

and since \( 2d > m > 2n \), (8) follows.

Now consider the subsheaf \( \mathcal{O}_X(L - \Gamma) \hookrightarrow \mathcal{E} \). The saturation of this subsheaf is of the form \( \mathcal{O}_X(L - \bar{\Gamma}) \), for a subdivisor \( \bar{\Gamma} \subseteq \Gamma \). We claim that \( \mathcal{O}_X(L - \bar{\Gamma}) \) is a destabilizing subsheaf of \( \mathcal{E} \); this is equivalent to the inequalities

\[
(L - 2\bar{\Gamma})^2 > 0 \\
(L - 2\bar{\Gamma}) \cdot L > 0.
\]

Now the first of this inequalities is a consequence of the hypothesis \( c_1^2(\mathcal{E}) > \Delta_1 \cdot (\mathcal{E}) \) by computing \( c_2(\mathcal{E}) \) from the exact sequence \( 0 \rightarrow \mathcal{O}_X(L - \bar{\Gamma}) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(\bar{\Gamma}) \otimes I_{\mathcal{Z}'} \rightarrow 0 \).

As for the second inequality, it is a consequence of (8):

\[
L^2 > 4 \deg(Z) > 2L \cdot \Gamma \geq 2L \cdot \bar{\Gamma}.
\]

This completes the proof of Bogomolov’s theorem.

**Remark.** A similar argument as in the proof of Bogomolov’s theorem completes the proof of Reider’s theorem in [EL, 1.4].

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA 90024-1555
e-mail: gonzalez@math.ucla.edu