ON TWO APPROACHES TO
FRACTIONAL SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract

Two complementary approaches of $\mathcal{N} = 2$ fractional supersymmetric quantum mechanics of order $k$ are studied in this article. The first one, based on a generalized Weyl-Heisenberg algebra $W_k$ (that comprizes the affine quantum algebra $U_q(sl_2)$ with $q^k = 1$ as a special case), apparently contains solely one bosonic degree of freedom. The second one uses generalized bosonic and $k$-fermionic degrees of freedom. As an illustration, a particular emphasis is put on the fractional supersymmetric oscillator of order $k$.

1 Introduction

Since more than three decades, supersymmetry is a concept largely used in Physics. It was first introduced in elementary particle physics on the basis of the unification of internal and external symmetries. Curiously enough, this kind of unification has never been used directly in Physics but its corollary according to which the Poincaré group has to be replaced by an extended Poincaré group proved to be fruitful. In this respect, the fact that fermions and bosons can be accommodated in a given irreducible representation of the $\mathbb{Z}_2$-graded Poincaré group is at the origin of the idea of a superparticle.

The experimental evidence for supersymmetry is not yet firmly established. Some arguments in favour of supersymmetry come from: (i) condensed matter physics with the fractional quantum Hall effect and high temperature superconductivity; (ii) nuclear physics where supersymmetry could connect the complex structure of odd-odd nuclei to much simpler even-even and odd-$A$ systems; and (iii) high energy physics (especially in the search of supersymmetric particles and the lighter Higgs boson) where the recently observed signal at 115 GeV/c$^2$ for a neutral Higgs boson is compatible with the hypotheses of supersymmetry.

In spite of the absence of a decisive evidence for supersymmetry, supersymmetric quantum mechanics (SSQM), a supersymmetric quantum field theory in $D = 1+0$ dimension [6], has received a great deal of attention in the last twenty years and is still in a state of development. In recent years, the investigation of quantum groups, with

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deformation parameters taken as roots of unity, has been a catalyst for the study
of fractional supersymmetric quantum mechanics (FSSQM) which is an extension
of ordinary SSQM. This extension takes its motivation in the so-called intermediate
or exotic statistics like: (i) anyonic statistics in $D = 1 + 2$ dimensions connected to
braid groups,\textsuperscript{3,7–9} (ii) para-bosonic and para-fermionic statistics in $D = 1 + 3$
dimensions connected to permutation groups,\textsuperscript{10–13} and (iii) $q$-deformed statistics (see,
for instance,\textsuperscript{14,15} arising from $q$-deformed oscillator algebras.\textsuperscript{16–20} Along this vein,
intermediate statistics constitute a useful tool for the study of physical phenomena
in condensed matter physics (e.g., fractional quantum Hall effect and supraconduc-
tivity at high critical temperature).

Supersymmetric quantum mechanics needs two degrees of freedom: one bosonic
degree (described by a complex variable) and one fermionic degree (described by a
Grassmann variable). From a mathematical point of view, we then have a $Z_2$-grading
of the Hilbert space of physical states (involving bosonic and fermionic states).
Fractional supersymmetric quantum mechanics of order $k$ is an extension of ordinary
SSQM for which the $Z_2$-grading is replaced by a $Z_k$-grading with $k \in \mathbb{N} \setminus \{0, 1\}.$
The $Z_k$-grading corresponds to a bosonic degree of freedom (described again by a
complex variable) and a para-fermionic or $k$-fermionic degree of freedom (described
by a generalized Grassmann variable of order $k$). In other words, to pass from
ordinary supersymmetry or SSQM to fractional supersymmetry or FSSQM of order
$k$, we retain the bosonic variable and replace the fermionic variable by a para-
fermionic or $k$-fermionic variable (which can be represented by a $k \times k$ matrix).

A possible approach to FSSQM of order $k$ thus amounts to replace fermions by
para-fermions of order $k - 1$. This yields para-supersymmetric quantum mechanics
as first developed, with one boson and one para-fermion of order 2, by Rubakov
and Spiridonov\textsuperscript{21} and extended by various authors.\textsuperscript{22–27} An alternative approach to
FSSQM of order $k$ consists in replacing fermions by $k$-fermions which are objects
interpolating between bosons (for $k \rightarrow \infty$) and fermions (for $k = 2$) and which
satisfy a generalized Pauli exclusion principle according to which one cannot put
more than $k - 1$ particles on a given quantum state.\textsuperscript{28} The $k$-fermions proved to
be useful in describing Bose-Einstein condensation in low dimensions\textsuperscript{14} (see also
Ref. [15]). They take their origin in a pair of $q$- and $\bar{q}$-oscillator algebras (or $q$- and
$\bar{q}$-uon algebras) with

$$q = \frac{1}{\bar{q}} := \exp \left( \frac{2\pi i}{k} \right),$$

where $k \in \mathbb{N} \setminus \{0, 1\}$. Along this line, a fractional supersymmetric oscillator was
derived in terms of boson and $k$-fermion operators in Ref. [29].

Fractional supersymmetric quantum mechanics was also developed without an
explicit introduction of $k$-fermionic degrees of freedom.\textsuperscript{30,31} In this respect, FSSQM
of order $k = 3$ was worked out by Quesne and Vansteenkiste\textsuperscript{31} owing to the intro-
duction of the the $C_{3k}$-extended oscillator algebra. Their work is an extension of
the construction by Plyushchay\textsuperscript{30} of SSQM, viz., FSSQM of order $k = 2$, with one
bosonic degree of freedom only.
The connection between FSSQM (and thus SSQM) and quantum groups has been worked out by several authors\textsuperscript{32−40} mainly with applications to exotic statistics in mind. In particular, LeClair and Vafa\textsuperscript{32} studied the isomorphism between the affine quantum algebra $U_q(sl_2)$ and $\mathcal{N} = 2$ FSSQM in $D = 1 + 1$ dimensions when $q^2$ goes to a root of unity ($\mathcal{N}$ is the number of supercharges); in the special case where $q^2 \to -1$, they recovered ordinary SSQM.

It is the aim of this paper to approach $\mathcal{N} = 2$ FSSQM of order $k$ from different routes: (i) first, from a generalized Weyl-Heisenberg algebra $W_k$ (defined in Sec. 2) and (ii) second, in terms of generalized bosonic and $k$-fermionic operators (Secs. 4 and 5). In Sec. 3, a fractional supersymmetric Hamiltonian is derived from the generators of $W_k$ and specialized to the case of a fractional supersymmetric oscillator. In Sec. 5, this fractional supersymmetric oscillator is further investigated on the basis of a $Q$-uon approach to the algebra $W_k$, with $Q$ going to a $k$-th root of unity. Finally, differential realizations, involving bosonic and generalized Grassmannian variables, of FSSQM are given in Sec. 6 for some particular cases of $W_k$. Three appendices complete this paper: The quantum algebra $U_q(sl_2)$ with $q^k = 1$ is connected to $W_k$ in Appendix A, a boson + $k$-fermion decomposition of a $Q$-uon for $Q \to q$ is derived in Appendix B and the action of deformed-boson and $k$-fermion operators on the $Z_k$-graded Hilbert-Fock space is given in Appendix C.

In the present paper, we use the notation $[A, B]_Q := AB - QBA$ for any complex number $Q$ and any pair of operators $A$ and $B$. As particular cases, we have $[A, B]_1$ or $[A, B]_- := [A, B]_1$ and $[A, B]_+ := [A, B]_{-1}$ for the commutator and the anti-commutator, respectively, of $A$ and $B$.

2 A generalized Weyl-Heisenberg algebra $W_k$

2.1 The algebra

For fixed $k$, with $k \in \mathbb{N} \setminus \{0, 1\}$, we define a generalized Weyl-Heisenberg algebra, denoted as $W_k$, as an algebra spanned by four linear operators $X_-$ (annihilation operator), $X_+$ (creation operator), $N$ (number operator) and $K$ (grading operator) acting on some Hilbert space and satisfying the following relations:

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N) \Pi_s, \quad (2a)$$

$$[N, X_-] = -X_-, \quad [N, X_+] = +X_+, \quad (2b)$$

$$[K, X_+]_q = [K, X_-]_{\bar{q}} = 0, \quad (2c)$$

$$[K, N] = 0, \quad (2d)$$

$$K^k = 1, \quad (2e)$$
where \( q \) is the \( k \)-th root of unity given by (1). In Eq. (2a), the \( f_s \) are reasonable functions (see below) and the operators \( \Pi_s \) are polynomials in \( K \) defined by

\[
\Pi_s := \frac{1}{k} \sum_{t=0}^{k-1} q^{-st} K^t
\]

for \( s = 0, 1, \ldots, k - 1 \). Furthermore, we suppose that the operator \( K \) is unitary \((K^\dagger = K^{-1})\), the operator \( N \) is self-adjoint \((N^\dagger = N)\), and the operators \( X_- \) and \( X_+ \) are connected via Hermitean conjugation \((X_+^\dagger = X_-)\). The functions \( f_s : N \mapsto f_s(N) \) must satisfy the constrain relation

\[
f_s(N)^\dagger = f_s(N)
\]

(with \( s = 0, 1, \ldots, k - 1 \)) in order that \( X_+ = X_+^\dagger \) be verified.

### 2.2 Projection operators

It is clear that we have the resolution of the identity operator

\[
\sum_{s=0}^{k-1} \Pi_s = 1
\]

and the idempotency relation

\[
\Pi_s \Pi_t = \delta(s, t) \Pi_s
\]

where \( \delta \) is the Kronecker symbol. Consequently, the \( k \) self-adjoint operators \( \Pi_s \) are projection operators for the cyclic group \( Z_k = \{1, K, \ldots, K^{k-1}\} \) of order \( k \) spanned by the generator \( K \). In addition, these projection operators satisfy

\[
\Pi_s X_+ = X_+ \Pi_{s-1} \iff X_- \Pi_s = \Pi_{s-1} X_-
\]

with the convention \( \Pi_{-1} \equiv \Pi_{k-1} \) and \( \Pi_k \equiv \Pi_0 \) (more generally, \( \Pi_{s+kn} \equiv \Pi_s \) for \( n \in \mathbb{Z} \)). Note that Eq. (3) can be reversed in the form

\[
K^t = \sum_{s=0}^{k-1} q^{ts} \Pi_s
\]

with \( t = 0, 1, \ldots, k - 1 \).

### 2.3 Representation

We now consider an Hilbertean representation of the algebra \( W_k \). Let \( \mathcal{F} \) be the Hilbert-Fock space on which the generators of \( W_k \) act. Since \( K \) obeys the cyclicity condition \( K^k = 1 \), the operator \( K \) admits the set \( \{1, q, \ldots, q^{k-1}\} \) of eigenvalues. It
thus makes it possible to graduate, via a $Z_k$-grading, the representation space $\mathcal{F}$ of the algebra $W_k$ as

$$\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s$$  \hspace{1cm} (5a)

where

$$\mathcal{F}_s := \{|kn+s\rangle : n \in \mathbb{N}\}$$  \hspace{1cm} (5b)

with

$$K|kn+s\rangle = q^s|kn+s\rangle.$$

Therefore, to each eigenvalue $q^s$ (with $s = 0, 1, \cdots, k-1$) we associate a subspace $\mathcal{F}_s$ of $\mathcal{F}$. It is evident that

$$\Pi_s|kn+t\rangle = \delta(s,t)|kn+s\rangle$$

and, thus, the application $\Pi_s: \mathcal{F} \to \mathcal{F}_s$ yields a projection of $\mathcal{F}$ onto its subspace $\mathcal{F}_s$.

The action of $X_\pm$ and $N$ on $\mathcal{F}$ can be taken to be

$$N|kn+s\rangle = n|kn+s\rangle$$

and

$$X_-|kn+s\rangle = \sqrt{F(n)} |k(n-1)+s-1\rangle, \quad s \neq 0, \hspace{1cm} (6a)$$

$$X_-|kn\rangle = \sqrt{F(n)} |k(n-1)+k-1\rangle, \quad s = 0, \hspace{1cm} (6b)$$

$$X_+|kn+s\rangle = \sqrt{F(n+1)} |k(n+1)+s+1\rangle, \quad s \neq k-1, \hspace{1cm} (6c)$$

$$X_+|kn+k-1\rangle = \sqrt{F(n+1)} |k(n+1)\rangle, \quad s = k-1. \hspace{1cm} (6d)$$

The function $F$ is a structure function that fulfills the initial condition $F(0) = 0$ (see Refs. [41,42]). Furthermore, it satisfies

$$X_-X_+ = F(N+1), \quad X_+X_- = F(N)$$

and

$$F(N+1) - F(N) = \sum_{s=0}^{k-1} f_s(N) \Pi_s$$

which admits the classical solution $F(N) = N$ for $f_s = 1$ ($s = 0, 1, \cdots, k-1$).
2.4 Particular cases

The algebra $W_k$ covers a great number of situations encountered in the literature.\cite{29,30,31,43,44}

These situations differ by the form given to the right-hand side of (2a) and can be classified as follows.

(i) As a particular case, the algebra $W_2$ for $k = 2$ with

$$[X_-, X_+] = 1 + c K, \quad [N, X_+] = \pm X_+,$$

$$[K, X_+] = 0, \quad [K, N] = 0, \quad K^2 = 1,$$

where $c$ is a real constant ($f_0 = 1 + c, f_1 = 1 - c$), corresponds to the Calogero-Vasiliev\cite{43} algebra considered by Gazeau\cite{44} for describing a system of two anyons, with an $\text{Sl}(2, \mathbb{R})$ dynamical symmetry, subjected to an intense magnetic field and by Plyushchay\cite{30} for constructing SSQM without fermions. Of course, for $k = 2$ and $c = 0$ we recover the algebra describing the ordinary or $Z_2$-graded supersymmetric oscillator.

If we define

$$c_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-ts} f_t(N),$$

with the functions $f_t$ chosen in such a way that $c_s$ is independent of $N$ (for $s = 0, 1, \ldots, k - 1$), the algebra $W_k$ defined by

$$[X_-, X_+] = \sum_{s=0}^{k-1} c_s K^s,$$

(7)

(8)

together with Eqs. (2b)-(2e), corresponds to the $C_\lambda$-extented harmonic oscillator algebra introduced by Quesne and Vansteenkiste\cite{31} for formulating FSSQM of order 3. The latter algebra was explored in great detail in the case $k = 3$.\cite{31}

(ii) Going back to the general case where $k \in \mathbb{N} \setminus \{0, 1\}$, if we assume in Eq. (2a) that $f_s = G$ is independent of $s$ with $G(N) = G(N)$, we get

$$[X_-, X_+] = G(N).$$

(9)

We refer the algebra $W_k$ defined by Eq. (9) together with Eqs. (2b)-(2e) to as a nonlinear Weyl-Heisenberg algebra (see also Ref. [13]). The latter algebra was considered by the authors as a generalization of the $Z_k$-graded Weyl-Heisenberg algebra describing a generalized fractional supersymmetric oscillator.\cite{29}

(iii) As a particular case, for $G = 1$ we have

$$[X_-, X_+] = 1$$

(10)

and here we can take

$$N := X_+ X_-.$$
The algebra $W_k$ defined by Eqs. (10) and (11) together with Eqs. (2b)-(2e) corresponds to the $\mathbb{Z}_k$-graded Weyl-Heisenberg algebra connected to the fractional supersymmetric oscillator studied in Ref. [29].

(iv) Finally, it is to be noted that the affine quantum algebra $U_q(sl_2)$ with $q^k = 1$ can be considered as a special case of the generalized Weyl-Heisenberg algebra $W_k$ (see Appendix A). This result is valid for all the representations (studied in Ref. [45]) of the algebra $U_q(sl_2)$.

3 A general supersymmetric Hamiltonian

3.1 Supercharges

We are now in a position to introduce supercharges which are basic operators for the formulation of FSSQM. We define the supercharge operators $Q_-$ and $Q_+$ by

\[
Q_- := X_-(1 - \Pi_1),
\]

\[
Q_+ := X_+(1 - \Pi_0),
\]

or alternatively

\[
Q_- := X_-(\Pi_2 + \cdots + \Pi_{k-2} + \Pi_{k-1} + \Pi_0),
\]

\[
Q_+ := X_+(\Pi_1 + \Pi_2 + \cdots + \Pi_{k-2} + \Pi_{k-1}).
\]

Indeed, we have here one of $k$, with $k \in \mathbb{N} \setminus \{0, 1\}$, possible equivalent definitions of the supercharges $Q_-$ and $Q_+$ corresponding to the $k$ circular permutations of the indices $0, 1, \cdots, k - 1$. Obviously, we have the Hermitean conjugation relation

\[
Q_-^* = Q_+.
\]

Thus, our approach corresponds to a $\mathcal{N} = 2$ formulation of FSSQM of order $k$ ($\frac{\mathcal{N}}{2}$ is the number of independent supercharges). By making use of the commutation relations between the projection operators $\Pi_s$ and the shift operators $X_-$ and $X_+$ [see Eqs. (4)], we easily get

\[
Q_-^m = X_-^m(\Pi_0 + \Pi_{m+1} + \Pi_{m+2} + \cdots + \Pi_{k-1})
\]

\[
Q_+^m = X_+^m(\Pi_1 + \Pi_2 + \cdots + \Pi_{k-m-1} + \Pi_{k-m})
\]

for $m = 0, 1, \cdots, k - 1$. By combining Eqs. (12) or (13) and (14), we obtain

\[
Q_-^k = Q_+^k = 0.
\]

Hence, the supercharge operators $Q_-$ and $Q_+$ are nilpotent operators of order $k$.

We continue with some relations at the basis of the derivation of a supersymmetric Hamiltonian. The central relations are

\[
Q_+Q_-^m = X_+X_-^m(1 - \Pi_m)(\Pi_0 + \Pi_{m+1} + \cdots + \Pi_{k-1})
\]
\[ Q^m Q_+ = X^m X_+ (1 - \Pi_0)(\Pi_m + \Pi_{m+1} + \cdots + \Pi_{k-1}) \] (15b)

with \( m = 0, 1, \ldots, k - 1 \). From Eqs. (15), we can derive the following identities giving \( Q^m Q_+ Q_\ell \) with \( m + \ell = k - 1 \).

(i) We have
\[ Q_+ Q^{k-1}_- = X_+ X_-^{k-1} \Pi_0 \] (16a)
\[ Q^{k-1}_- Q_+ = X_-^{k-1} X_+ \Pi_{k-1} \] (16b)
in the limiting cases corresponding to \((m = 0, \ell = k - 1)\) and \((m = k - 1, \ell = 0)\).

(ii) Furthermore, we have
\[ Q^m Q_+ Q_\ell = X^m X_+ X_\ell (\Pi_0 + \Pi_{k-1}) \] (16c)
with the conditions \((m \neq 0, \ell \neq k - 1)\) and \((m \neq k - 1, \ell \neq 0)\).

### 3.2 The Hamiltonian

Following Rubakov and Spiridonov,\(^2\) we consider the multilinear relation
\[ Q^{k-1}_- Q_+ + Q^{k-2}_- Q_+ Q_- + \cdots + Q_+ Q^{k-1}_- = Q^{k-2}_- H, \]
where \( H \) is an operator that depends on the algebra \( W_k \). The operator \( H \) defines the Hamiltonian for a supersymmetric system associated to \( W_k \). This dynamical system, that we shall refer to a fractional or \( Z_k \)-graded supersymmetric system, depends on the functions \( f_s \) occurring in the definition (1) of \( W_k \). By repeated use of Eqs. (1) and (16), we find that the most general expression of \( H \) is
\[
H = (k - 1)X_+ X_- - \sum_{s=k-2}^{k-1} \sum_{t=2}^{s-1} (t - 1) f_t(N - s + t) \Pi_s
\]
\[
- \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t - k) f_t(N - s + t) \Pi_s
\] (17)
in terms of the product \( X_+ X_- \), the operators \( \Pi_s \) and the functions \( f_s \). In the general case, we can check that
\[ H^\dagger = H \] (18)
and
\[ [H, Q_-] = [H, Q_+] = 0. \] (19)

Equations (18) and (19) show that the two supercharge operators \( Q_- \) and \( Q_+ \) are two (non independent) constants of the motion for the Hamiltonian system described by the self-adjoint operator \( H \). From Eqs. (17-19), it can be seen that the Hamiltonian \( H \) is a linear combination of the projection operators \( \Pi_s \) with coefficients corresponding to isospectral Hamiltonians (or supersymmetric partners) associated to the various subspaces \( \mathcal{F}_s \) with \( s = 0, 1, \ldots, k - 1 \).
3.3 Particular cases

The general expression (17) for the Hamiltonian $H$ can be particularized to some interesting cases. These cases correspond to the above-mentioned forms of the generalized Weyl-Heisenberg algebra $W_k$.

(i) In the particular case $k = 2$, by taking $f_0 = 1 + c$ and $f_1 = 1 - c$, where $c$ is a real constant, the Hamiltonian (17) gives back the one derived by Plyushchay\textsuperscript{30} for SSQM.

More generally, by restricting the functions $f_t$ in Eq. (17) to constants (independent of $N$) defined by

$$f_t = \sum_{s=0}^{k-1} q^{ts} c_s$$

in terms of the constants $c_s$ (cf. Eq. (7)), the so-obtained Hamiltonian $H$ corresponds to the $C_{\lambda}$-oscillator fully investigated for $k = 3$ in Ref. [31].

(ii) In the case $f_s = G$ (independent of $s = 0, 1, \cdots, k-1$), i.e., for a generalized Weyl-Heisenberg algebra $W_k$ defined by (2b)-(2e) and (9), the Hamiltonian $H$ can be written as

$$H = (k-1)X_+ X_- - \sum_{s=2}^{k-1} \sum_{t=1}^{s-1} G(N-t)(1 - \Pi_1 - \Pi_2 - \cdots - \Pi_s)$$

$$+ \sum_{s=1}^{k-1} \sum_{t=0}^{k-s-1} (k - s - t)G(N+t) \Pi_s.$$  

The latter expression was derived in Ref. [29].

(iii) If $G = 1$, i.e., for a Weyl-Heisenberg algebra defined by (2b)-(2e) and (10), Eq. (20) leads to the Hamiltonian

$$H = (k-1)X_+ X_- + (k-1) \sum_{s=0}^{k-1} (s + 1 - \frac{1}{2}k)\Pi_{k-s}$$

for a fractional supersymmetric oscillator. The energy spectrum of $H$ is made of equally spaced levels with a ground state (singlet), a first excited state (doublet), a second excited state (triplet), \cdots, a $(k-2)$-th excited state ($(k-1)$-plet) followed by an infinite sequence of further excited states (all $k$-plets).

(iv) In the case where the algebra $W_k$ is restricted to $U_q(sl_2)$, see Appendix A, the corresponding Hamiltonian $H$ is given by Eq. (17) where the $f_t$ are given in Appendix A. This yields

$$H = (k-1)J_+ J_- + \frac{1}{\sin \frac{2\pi}{k}} \sum_{s=k-2}^{k} \sum_{t=2}^{s-1} (t - 1) \sin \frac{4\pi t}{k} \Pi_s$$

$$+ \frac{1}{\sin \frac{2\pi}{k}} \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t - k) \sin \frac{4\pi t}{k} \Pi_s.$$
Alternatively, Eq. (22) can be rewritten in the form (20) where \(X_\pm \equiv J_\pm\) and \(N \equiv J_3\) and where the function \(G\) is defined by

\[
G(X) := -[2X]_q,
\]

where the symbol \([\ ]_q\) is defined by

\[
[X]_q := \frac{q^{2X} - q^{-2X}}{q - q^{-1}}
\]

with \(X\) an arbitrary operator or number. The quadratic term \(J_+J_-\) can be expressed in term of the Casimir operator \(J^2\) of \(U_q(sl_2)\), see Appendix A. Thus, the so-obtained expression for the Hamiltonian \(H\) is a simple function of \(J^2\) and \(J_3\).

4 A deformed-boson + \(k\)-fermion approach to fractional supersymmetry

4.1 A deformed-boson + \(k\)-fermion realization of \(W_k\)

In this section, the main tools consist of \(k\) pairs \((b(s)_-, b(s)_+)\) with \(s = 0, 1, \cdots, k-1\) of deformed bosons and a pair \((f_-, f_+)\) of \(k\)-fermions. The operators \(f_\pm\) satisfy (see Appendix B)

\[
[f_-, f_+]_q = 1, \quad f^k_- = f^k_+ = 0,
\]

and the operators \(b(s)_\pm\) the commutation relation

\[
[b(s)_-, b(s)_+] = f_s(N), \quad (23)
\]

where the functions \(f_s\) with \(s = 0, 1, \cdots, k-1\) and the operator \(N\) occur in Eq. (2). In addition, the pairs \((f_-, f_+)\) and \((b(s)_-, b(s)_+)\) are two pairs of commuting operators and the operators \(b(s)_\pm\) commute with the projection operators \(\Pi_s\) with \(s, t = 0, 1, \cdots, k-1\). We also introduce the linear combinations

\[
b_- := \sum_{s=0}^{k-1} b(s)_- \Pi_s, \quad b_+ := \sum_{s=0}^{k-1} b(s)_+ \Pi_s.
\]

It is immediate to verify that we have the commutation relation

\[
[b_-, b_+] = \sum_{s=0}^{k-1} f_s(N) \Pi_s, \quad (24)
\]

a companion of Eq. (23).

We are now in a situation to find a realization of the generators \(X_-, X_+\) and \(K\) of the algebra \(W_k\) in terms of the \(b\)'s and \(f\)'s. Let us define the shift operators \(X_-\) and \(X_+\) by

\[
X_- := b_- \left( f_- + \frac{f^k_-}{[[k-1]]_q!} \right), \quad (25)
\]
\[ X_+ := b_+ \left( f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^{k-1}, \] 

(26)

where the new symbol \([[ \ ]]]_q \) is defined by

\[ [[X]]_q := \frac{1 - q^X}{1 - q} \]

with \( X \) an arbitrary operator or number and where the \( q \)-deformed factorial is given by

\[ [[n]]_q! := [[1]]_q [[2]]_q \cdots [[n]]_q \]

for \( n \in \mathbb{N}^* \) (and \([ [0]]_q ! := 1\)). It is also always possible to find a representation for which the relation \( X_+^\dagger = X_+ \) holds (see Appendix C). Furthermore, we define the grading operator \( K \) by

\[ K := [f_-, f_+]. \] 

(27)

In view of the remarkable property

\[ \left( f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^k = 1, \]

we obtain

\[ [X_-, X_+] = [b_-, b_+]. \] 

(28)

Equations (24) and (28) show that Eq. (2a) is satisfied. It can be checked also that the operators \( X_-, X_+ \) and \( K \) satisfy Eqs. (2c) and (2e). Of course, Eqs. (2b) and (2d) have to be considered as postulates. However, note that the operator \( N \) is formally given in terms of the \( b \)'s by

\[ F(N + 1) = b_- b_+ = \sum_{s=0}^{k-1} b(s)_- b(s)_+ \Pi_s, \]

\[ F(N) = b_+ b_- = \sum_{s=0}^{k-1} b(s)_+ b(s)_- \Pi_s, \]

with the help of the structure function \( F \) introduced in Sec. 2. We thus have a realization of the generalized Weyl-Heisenberg algebra \( W_k \) by multilinear forms involving \( k \) pairs \((b(s)_-, b(s)_+)\) of deformed-boson operators \((s = 0, 1, \ldots, k - 1)\) and one pair \((f_-, f_+)\) of \( k \)-fermion operators.

### 4.2 The resulting Hamiltonian

The supercharges \( Q_- \) and \( Q_+ \) can be expressed by means of the deformed-bosons and \( k \)-fermions. By using the identity

\[ \Pi_s \left( f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^n = \left( f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^n \Pi_{s+n}, \]
with \( s = 0, 1, \cdots, k - 1 \) and \( n \in \mathbb{N} \), Eqs. (12) can be rewritten as

\[
Q_- = \left( f_- + \frac{f_+^{k-1}}{[[k-1]]q!} \right) \sum_{s=1}^{k-1} b(s)_- \Pi_{s+1},
\]

\[
Q_+ = \left( f_- + \frac{f_+^{k-1}}{[[k-1]]q!} \right) \sum_{s=1}^{k-1} b(s + 1)_+ \Pi_s,
\]

with the convention \( b(k)_+ = b(0)_+ \). Then, the supersymmetric Hamiltonian \( H \) given by Eq. (17) assumes the form

\[
H = (k - 1) \sum_{s=0}^{k-1} F_s(N) \Pi_s - \sum_{s=k-2}^{k} \sum_{t=2}^{s-1} (t - 1) f_t(N - s + t) \Pi_s
\]

\[
- \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t - k) f_t(N - s + t) \Pi_s,
\]

in terms of the operators \( b(s)_{\pm} \), the projection operators \( \Pi_s \) (that may be written with \( k \)-fermion operators), the structure functions \( F_s \) and the structure constants \( f_s \) with \( s = 0, 1, \cdots, k - 1 \).

5 The fractional supersymmetric oscillator

5.1 A special case of \( W_k \)

In this section, we deal with the particular case where \( f_s = 1 \) and the deformed bosons \( b(s)_{\pm} \equiv b_{\pm} \) are independent of \( s \) with \( s = 0, 1, \cdots, k - 1 \). We thus end up with a pair \((b_-, b_+)\) of ordinary bosons, satisfying \([b_-, b_+] = 1\), and a pair \((f_-, f_+)\) of \( k \)-fermions. The ordinary bosons \( b_{\pm} \) and the \( k \)-fermions \( f_{\pm} \) may be considered as originating from the decomposition of a pair of \( Q \)-uons when \( Q \) goes to the root of unity \( q \) (see Appendix B).

Here, the two operators \( X_- \) and \( X_+ \) are given by Eqs. (25) and (26), where now \( b_{\pm} \) are ordinary boson operators. They satisfy the commutation relation \([X_-, X_+] = 1\).

Then, the number operator \( N \) may be defined by

\[
N := X_+ X_-,
\]

which is amenable to the form

\[
N = b_+ b_-.
\]

Finally, the grading operator \( K \) is still defined by Eq. (27). We can check that the operators \( X_-, X_+, N \) and \( K \) so-defined generate the generalized Weyl-Heisenberg algebra \( W_k \) defined by Eq. (2) with \( f_s = 1 \) for \( s = 0, 1, \cdots, k - 1 \). The latter algebra \( W_k \) can thus be realized with multilinear forms involving ordinary boson operators \( b_{\pm} \) and \( k \)-fermion operators \( f_{\pm} \).
5.2 The resulting fractional supersymmetric oscillator

The supercharge operators $Q_-$ and $Q_+$ as well as the Hamiltonian $H$ associated to
the algebra $W_k$ introduced in Sec. 4.2 (in terms of the operators $b_-, b_+, f_-$ and $f_+$)
can be constructed according to the prescriptions given in Sec. 3. This leads to the
expression

$$ H = (k - 1)b_+b_- + (k - 1) \sum_{s=0}^{k-1} (s + 1 - \frac{1}{2}k)\Pi_{k-s} $$

to be compared with Eq. (21).

Most of the properties of the Hamiltonian $H$ are essentially the same as the ones
given above for the Hamiltonian (21). In particular, we can write

$$ H = \sum_{m=1}^{k} H_m \Pi_m, \quad H_m := (k - 1) \left( b_+b_- + \frac{1}{2}k + 1 - m \right) $$

and thus $H$ is a linear combination of projection operators with coefficients $H_m$
corresponding to isospectral Hamiltonians (remember that $\Pi_k := \Pi_0$).

To close this section, let us mention that the fractional supercoherent state $|z, \theta\rangle$
defined in Appendix B is a coherent state corresponding to the Hamiltonian $H$. As
a point of fact, we can check that the action of the evolution operator $\exp(-iHt)$
on the state $|z, \theta\rangle$ gives

$$ \exp(-iHt) |z, \theta\rangle = \exp \left[ -\frac{i}{2}(k-1)(k+2)t \right] e^{-i(k-1)t}z, e^{+i(k-1)t}\theta) $$

i.e., another fractional supercoherent state.

5.3 Examples

Example 1. As a first example, we take $k = 2$, i.e., $q = -1$. Then, the operators

$$ X_{\pm} := b_{\pm} (f_- + f_+) $$

and the operators $K$ and $N$, see Eqs. (27) and (29), are defined in terms of bilinear
forms of ordinary bosons $(b_-, b_+)$ and ordinary fermions $(f_-, f_+)$. The operators
$X_-, X_+, N$ and $K$ satisfy

$$ [X_-, X_+] = 1, \quad [N, X_{\pm}] = \pm X_{\pm}, $$

$$ [K, X_{\pm}] = 0, \quad [K, N] = 0, \quad K^2 = 1, $$

which reflect bosonic and fermionic degrees of freedom, the bosonic degree corre-
sponding to the triplet $(X_-, X_+, N)$ and the fermionic degree to the Klein involution
operator $K$. The projection operators

$$ \Pi_0 = 1 - f_+f_-, \quad \Pi_1 = f_+f_- $$
are here simple chirality operators and the supercharges
\[ Q_- = b_- f_+, \quad Q_+ = b_+ f_- \]
have the property
\[ Q_-^2 = Q_+^2 = 0. \]
The Hamiltonian \( H \) assumes the form
\[ H = [Q_-, Q_+] \]
which can be rewritten as
\[ H = b_+ b_- \Pi_0 + b_- b_+ \Pi_1. \]
It is clear that the self-adjoint operator \( H \) commutes with \( Q_- \) and \( Q_+ \) and we recover that \( Q_- \), \( Q_+ \) and \( H \) span the Lie superalgebra \( \mathfrak{s}\mathfrak{l}(1/1) \). We have
\[ H = b_+ b_- + f_+ f_- \]
so that \( H \) acts on the \( Z_2 \)-graded space \( \mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \). The operator \( H \) corresponds to the ordinary or \( Z_2 \)-graded supersymmetric oscillator whose energy spectrum \( E \) is (in a symbolic way)
\[ E = 1 \oplus 2 \oplus 2 \oplus \cdots \]
with equally spaced levels, the ground state being a singlet (denoted by 1) and all the excited states (viz., an infinite sequence) being doublets (denoted by 2). Finally, note that the fractional supercoherent state \( |z, \theta \rangle \) of Appendix B with \( k = 2 \) is a coherent state for the Hamiltonian \( H \) (see also Ref. [46]).

Example 2. We continue with \( k = 3 \), i.e.,
\[ q = \exp \left( \frac{2\pi i}{3} \right). \]
In this case, we have
\[ X_- = b_- \left( f_- - q f_+^2 \right), \]
\[ X_+ = b_+ \left( f_+ + f_-^2 + q^2 f_+^2 f_- \right), \]
and \( K \) and \( N \) as given by (27) and (29), where here \((b_-, b_+)\) are ordinary bosons and \((f_-, f_+)\) are 3-fermions. We hence have
\[ [X_-, X_+] = 1, \quad [N, X_\pm] = \pm X_\pm, \]
\[ [K, X_+]_q = [K, X_-]_q = 0, \quad [K, N] = 0, \quad K^3 = 1. \]
Our general definitions can be specialized to
\[ \Pi_0 = 1 + (q - 1) f_+ f_- - q f_+ f_- f_+ f_- \]
\[ \Pi_1 = -q f_+ f_- + (1 + q) f_+ f_- f_+ f_- \]
\[ \Pi_2 = f_+ f_- - f_+ f_- f_+ f_- \]
for the projection operators and to

\[ Q_- = b_- f_+ (f_+^2 - q f_+) \]

\[ Q_+ = b_+ (f_- - q f_+^2) f_- \]

for the supercharges with the property

\[ Q_-^3 = Q_+^3 = 0. \]

By introducing the Hamiltonian \( H \) via

\[ Q^2_+ Q_+ + Q_- Q_+ Q_- + Q_+ Q_-^2 = Q_- H \]

we obtain

\[ H = (2b_+ b_- - 1) \Pi_0 + (2b_+ b_- + 3) \Pi_1 + (2b_+ b_- + 1) \Pi_2 \]

which acts on the \( \mathbb{Z}_3 \)-graded space \( \mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \) and can be rewritten as

\[ H = 2b_+ b_- - 1 + 2(1 - 2q) f_+ f_- + 2(1 + 2q) f_+ f_- f_+ f_- \]

in terms of boson and 3-fermion operators. We can check that the self-adjoint operator \( H \) commutes with \( Q_- \) and \( Q_+ \). The energy spectrum of \( H \) reads

\[ E = 1 \oplus 2 \oplus 3 \oplus 3 \oplus \cdots. \]

It contains equally spaced levels with a nondegenerate ground state (denoted as 1), a doubly degenerate first excited state (denoted as 2) and an infinite sequence of triply degenerate excited states (denoted as 3).

### 6 Differential realizations

In this section, we consider the case of the algebra \( W_k \) defined by Eqs. (2b)-(2e) and Eq. (8) with \( c_0 = 1 \) and \( c_s = c\delta(s,1) \), \( c \in \mathbb{R} \), for \( s = 1, 2, \cdots, k - 1 \). In other words, we have

\[ [X_-, X_+] = 1 + cK, \quad K^k = 1, \quad (30a) \]

\[ [K, X_+]_q = [K, X_-]_q = 0, \quad (30b) \]

which corresponds to the \( C_\lambda \)-extended oscillator. The operators \( X_-, X_+ \) and \( K \) can be realized in terms of a bosonic variable \( x \) and its derivative \( \frac{d}{dx} \) satisfying

\[ \left[ \frac{d}{dx}, x \right] = 1 \]
and a $k$-fermionic variable (or generalized Grassmann variable) $\theta$ and its derivative $\frac{d}{d\theta}$ satisfying\textsuperscript{21,32} (see also Refs. [22-28])

\[
[\frac{d}{d\theta}, \theta]_q = 1, \quad \theta^k = \left(\frac{d}{d\theta}\right)^k = 0.
\]

Of course, the sets \{x, $\frac{d}{dx}$\} and \{\theta, $\frac{d}{d\theta}$\} commute. It is a simple matter of calculation to derive the two following identities

\[
\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_q!}\right)^k = 1
\]

and

\[
\left(\frac{d}{d\theta} - \theta \frac{d}{d\theta}\right)^k = 1,
\]

which are essential for the realizations given below.

As a first realization, we can show that the shift operators

\[
X_- = \frac{d}{dx} \left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_q!}\right)^{k-1} - \frac{c}{x} \theta,
\]

\[
X_+ = x \left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_q!}\right),
\]

and the Witten grading operator

\[
K = \left[\frac{d}{d\theta}, \theta\right]
\]

satisfy Eqs. (30). This realization of $X_-$, $X_+$ and $K$ clearly exhibits the bosonic and $k$-fermionic degrees of liberty via the sets \{x, $\frac{d}{dx}$\} and \{\theta, $\frac{d}{d\theta}$\}, respectively. In the particular case $k = 2$, the $k$-fermionic variable $\theta$ is an ordinary Grassmann variable and the supercharge operators $Q_-$ and $Q_+$ take the simple form

\[
Q_- = \left(\frac{d}{dx} - \frac{c}{x}\right) \theta
\]

\[
Q_+ = x \frac{d}{d\theta}.
\]

(Note that the latter realization for $Q_-$ and $Q_+$ is valid for $k = 3$ too.)

Another possible realization of $X_-$ and $X_+$ for arbitrary $k$ is

\[
X_- = P \left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_q!}\right)^{k-1} - \frac{c}{x} \theta,
\]

\[
X_+ = X \left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_q!}\right),
\]
where $P$ and $X$ are the two canonically conjugated quantities

$$P := \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} - \frac{c}{2x}K \right)$$

and

$$X := \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} + \frac{c}{2x}K \right).$$

This realization is more convenient for a Schrödinger type approach to the supersymmetric Hamiltonian $H$. According to Eq. (17), we can derive an Hamiltonian $H$ involving bosonic and $k$-fermionic degrees of freedom. To illustrate this point, let us continue with the particular case $k = 2$. It can be seen that the supercharge operators (31) and (32) must be replaced by

$$Q_- = \left( P - \frac{c}{X} \right) \theta$$

$$Q_+ = X \frac{d}{d\theta}.$$  

(Note the formal character of $Q_-$ since the definition of $Q_-$ lies on the existence of an inverse for the operator $X$.) Then, we obtain the following Hamiltonian

$$H = -\frac{1}{2} \left[ \left( \frac{d}{dx} - \frac{c}{2x}K \right)^2 - x^2 + K + c(1 + K) \right].$$

For $c = 0$, we have (cf. Ref. [6])

$$H = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2} K$$

that is the Hamiltonian for an ordinary super-oscillator, i.e., a $\mathbb{Z}_2$-graded supersymmetric oscillator. Here, the bosonic character arises from the bosonic variable $x$ and the fermionic character from the ordinary Grassmann variable $\theta$ in $K$.

7 Concluding remarks

A first facet of this work concerns an approach of $\mathcal{N} = 2$ FSSQM of order $k$ ($k \in \mathbb{N} \setminus \{0,1\}$) in $D = 1 + 1$ dimensions through a generalized Weyl-Heisenberg algebra $W_k$ which is an extension of the Calogero-Vasiliev algebra. We have shown how the algebra $W_k$ is connected to the quantum algebra $U_q(sl_2)$ with $q^k = 1$. This approach of FSSQM, in the spirit of the pioneer works in Refs. [30,31], differs from the one developed in Refs. [21-27] via the introduction of two degrees of freedom, a bosonic one and a para-fermionic one. At first glance, our approach seems to be of an entirely bosonic character. However, the para-fermionic or $k$-fermionic character is hidden
behind the (Klein-Witten) operator $K$. This operator ensures a $Z_k$-grading of the Hilbert space $\mathcal{F}$ of the physical states according to the decomposition $\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s$. The generators of $W_k$ (and consequently of $U_q(sl_2)$) have been used for constructing a general fractional supersymmetric Hamiltonian $H$ which is a linear combination of projection operators on the subspaces $\mathcal{F}_s$ ($s = 0, 1, \cdots, k - 1$), the coefficients of which being isospectral Hamiltonians. The general Hamiltonian $H$ covers the particular case of the fractional supersymmetric oscillator.

A second facet of this paper is devoted to a $Q$-uon approach of $\mathcal{N} = 2$ FSSQM of order $k$ with $Q$ going to $q = \exp(2\pi i / k)$. The bosonic and $k$-fermionic degrees of freedom are present since the very beginning, a situation which parallels the à la Rubakov and Spiridonov\textsuperscript{21,22} construction of para-supersymmetric quantum mechanics. Indeed, the $Q$-uon $\rightarrow$ boson + $k$-fermion decomposition obtained when $Q \sim q$ has been exploited for building a realization of $W_k$ corresponding to the fractional supersymmetric oscillator. This approach of FSSQM is especially appropriate for deriving the fractional supercoherent states associated to this fractional supersymmetric oscillator. In addition, it is appropriate to the writing of supercharges and fractional supersymmetric Hamiltonians in terms of ordinary bosonic variables and generalized Grassmann variables, as shown with the specific differential realizations of Sec. 6.

The two approaches of FSSQM developed in this paper are obviously complementary. In this direction, it is to be emphasized that this work might be useful for generating isospectral Hamiltonians for exactly integrable potentials and for constructing their coherent states.

Finally, two comments of a group-theoretical nature are in order. First, we have shown here that supercharges and fractional supersymmetric Hamiltonians can be expressed from the generators of $U_q(sl_2)$, with $q$ a $k$-th root of unity, in a way independent of the representations (i), (ii) and (iii) of Appendix A chosen for the quantum algebra $U_q(sl_2)$. This approach is different from the one in Refs. [32,35] where nilpotent and cyclic representations of $U_q(sl_2)$, with $q^2$ being a root of unity, are separately considered for an investigation of $\mathcal{N} = 2$ FSSQM in $D = 1 + 1$ dimensions. Second, the algebra $U_q(sl_2)$ has not to be confused with the algebra spanned by the supercharges $Q_-$ and $Q_+$ and the Hamiltonian $H$. The latter algebra coincides with the $Z_2$-graded Lie algebra $sl(1/1)$ for $q = -1$, i.e., $k = 2$, in the case of $\mathcal{N} = 2$ SSQM. An open question is to find the algebra spanned by $Q_-$, $Q_+$ and $H$ for $k \geq 3$ in the case of $\mathcal{N} = 2$ FSSQM. In an other terminology, can $\mathcal{N} = 2$ FSSQM of order $k$ be described by a $q$-deformed algebra (with $q^k = 1$) that gives back $sl(1/1)$ for $q = -1$ ? It is hoped that the results in this paper shall shed light on this question.
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Appendix A: Connection between $W_k$ and $U_q(sl_2)$

Let us now show that the quantum algebra $U_q(sl_2)$, with $q$ being the $k$-th root of unity given by (1), turns out to be a particular form of $W_k$. The algebra $U_q(sl_2)$ is spanned by the generators $J_-, J_+, q^{J_3}$ and $q^{-J_3}$ that satisfy the relationships

\[ [J_+, J_-] = [2J_3]_q, \]

\[ q^{J_3} J_+ q^{-J_3} = q J_+, \quad q^{J_3} J_- q^{-J_3} = q J_-, \]

\[ q^{J_3} q^{-J_3} = q^{-J_3} q^{J_3} = 1. \]

It is straightforward to prove that the operator

\[ J^2 := J_- J_+ + \frac{q^{+1} q^{2J_3} + q^{-1} q^{-2J_3}}{(q - q^{-1})^2} \]

or

\[ J^2 := J_+ J_- + \frac{q^{-1} q^{2J_3} + q^{+1} q^{-2J_3}}{(q - q^{-1})^2} \]

is an invariant of $U_q(sl_2)$. In view of Eq. (1), the operators $J_-^k$, $J_+^k$, $(q^{J_3})^k$, and $(q^{-J_3})^k$ belong, likewise $J^2$, to the center of $U_q(sl_2)$.

In the case where the deformation parameter $q$ is a root of unity, the representation theory of $U_q(sl_2)$ is richer than the one for $q$ generic. The algebra $U_q(sl_2)$ admits finite-dimensional representations of dimension $k$ such that

\[ J_-^k = A, \quad J_+^k = B, \]

where $A$ and $B$ are constant matrices. Three types of representations have been studied in the literature:\textsuperscript{45}

(i) $A = B = 0$ (nilpotent representations),

(ii) $A = B = 1$ (cyclic or periodic representations),

(iii) $A = 0$ and $B = 1$ or $A = 1$ and $B = 0$ (semi-periodic representations).

Indeed, the realization of FSSQM based on $U_q(sl_2)$ does not depend of the choice (i), (ii) or (iii) in contrast with the work in Ref. [32] where nilpotent representations corresponding to the choice (i) were considered. The only important ingredient is to take

\[ (q^{J_3})^k = 1 \]

that ensures a $Z_k$-grading of the Hilbertean representation space of $U_q(sl_2)$.
The contact with the algebra $W_k$ is established by putting
\[ X_\pm := J_\pm, \quad N := J_3, \quad K := q^{J_3}, \]
and by using the definition (3) of $\Pi_s$ as function of $K$. Here, the operator $\Pi_s$ is a projection operator on the subspace, of the representation space of $U_q(sl_2)$, corresponding to a given eigenvalue of $J_3$. It is easy to check that the operators $X_-, X_+, N$ and $K$ satisfy Eqs. (2) with
\[ f_s(N) = -[2s]_q = -\frac{\sin \frac{4\pi s}{k}}{\sin \frac{2\pi}{k}} \]
for $s = 0, 1, \cdots, k - 1$. The quantum algebra $U_q(sl_2)$, with $q$ given by (1), then appears as a further particular case of the generalized Weyl-Heisenberg algebra $W_k$.

**Appendix B: The $Q$-uon → boson + $k$-fermion decomposition**

We shall limit ourselves to give an outline of this decomposition (see Dunne et al.\(^{47}\) and Mansour et al.\(^{48}\) for an alternative and more rigorous mathematical presentation based on the isomorphism between the braided $Z$-line and the $(z, \theta)$-superspace).

We start from a $Q$-uon algebra spanned by three operators $a_-, a_+$ and $N_a$ satisfying the relationships\(^{16}\) (see also Refs. [17-20])
\[ [a_-, a_+]_Q = 1, \quad [N_a, a_\pm] = \pm a_\pm, \]
where $Q$ is generic (a real number different from zero). The action of the operators $a_-, a_+$ and $N_a$ on a Fock space $F := \{|n\rangle : n \in \mathbb{N}\}$ is given by
\[ N_a|n\rangle = n|n\rangle, \]
and
\[ a_-|n\rangle = (\left[[n + \sigma - \frac{1}{2}]_Q\right]^{\alpha} |n - 1\rangle, \]
\[ a_+|n\rangle = (\left[[n + \sigma + \frac{1}{2}]_Q\right]^{\beta} |n + 1\rangle, \]
where $\sigma = \frac{1}{2}$ and $\alpha + \beta = 1$ with $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. For $\alpha = \beta = \frac{1}{2}$, let us consider the $Q$-deformed Glauber coherent state\(^{16}\) (see also Ref. [49])
\[ |Z\rangle := \sum_{n=0}^{\infty} \frac{(Za_+)^n}{[[n]]_Q^{1/2}} |0\rangle = \sum_{n=0}^{\infty} \frac{Z^n}{[[n]]_Q^{1/2}} |n\rangle \]
(with $Z \in \mathbb{C}$). If we do the replacement
\[ Q \sim q := \exp \left(\frac{2\pi i}{k}\right), \quad k \in \mathbb{N} \setminus \{0, 1\}, \]
we have \([[[k]]]q! \rightarrow [[[k]]]q! = 0\). Therefore, in order to give a sense to \(|Z\) for \(Q \sim q\), we have to do the replacement

\[ a_+ \sim f_+ \quad \text{with} \quad f^k_+ = 0 \]

and, for the sake of symmetry, a similar replacement for \(a_-\). We thus end up with what we call a \(k\)-fermionic algebra spanned by the operators \(f_-, f_+\) and \(N_f \equiv N_a\) completed by the adjoints \(f^\dagger_+\) and \(f^\dagger_-\) of \(f_+\) and \(f_-\), respectively. The defining relations for the \(k\)-fermionic algebra are

\[ [f_-, f_+] = 1, \quad [N_f, f_\pm] = \pm f_\pm, \quad f^\pm_+ = f_\pm^k = 0, \]

and similar relations for \(f^\dagger_+\) and \(f^\dagger_-\). The case \(k = 2\) corresponds to ordinary fermion operators and the case \(k \rightarrow \infty\) to ordinary boson operators. The \(k\)-fermions are objects interpolating between fermions and bosons. They share some properties with the para-fermions\(^{21,22,24}\) and the anyons as introduced by Goldin et al.\(^{[8]}\) (see also Ref.\([7]\)). If we define

\[ b_\pm := \lim_{Q \sim q} \frac{a_\pm^k}{([[k]]_q)!^{1/2}} \]

we obtain

\[ [b_-, b_+] = 1 \]

so that the operators \(b_-\) and \(b_+\) can be considered as ordinary boson operators. This is at the root of the two following results.\(^{28}\)

As a first result, the set \(\{a_-, a_+\}\) gives rise, for \(Q \sim q\), to two commuting sets: The set \(\{b_-, b_+\}\) of boson operators and the set of \(k\)-fermion operators \(\{f_-, f_+\}\). As a second result, this decomposition leads to the replacement of the \(Q\)-deformed coherent state \(|Z\) by the so-called fractional supercoherent state

\[ |z, \theta\rangle := \sum_{n=0}^{\infty} \sum_{s=0}^{k-1} \frac{\theta^s}{([s]_q)! \sqrt{n!}} z^n |kn + s\rangle, \]

where \(z\) is a (bosonic) complex variable and \(\theta\) a \((k\)-fermionic\) generalized Grassmann variable\(^{21,24,32,50}\) with \(\theta^k = 0\). The fractional supercoherent state \(|z, \theta\rangle\) is an eigenvector of the product \(f_- b_-\) with the eigenvalue \(z\theta\). The state \(|z^k, \theta\rangle\) can be seen to be a linear combination of the coherent states introduced by Vourdas\(^{51}\) with coefficients in the generalized Grassmann algebra spanned by \(\theta\) and the derivative \(\frac{d}{d\theta}\).

In the case \(k = 2\), the fractional supercoherent state \(|z, \theta\rangle\) turns out to be a coherent state for the ordinary (or \(Z_2\)-graded) supersymmetric oscillator.\(^{46}\) For \(k \geq 3\), the state \(|z, \theta\rangle\) is a coherent state for the \(Z_k\)-graded supersymmetric oscillator (see Sec. 5).
Appendix C: Actions on the space $\mathcal{F}$

Equation (23) is satisfied by

$$b(s)_-b(s)_+ = F_s(N + 1), \quad b(s)_+b(s)_- = F_s(N),$$

where the structure functions $F_s$ are connected to the structure constants $f_s$ via

$$F_s(N + 1) - F_s(N) = f_s(N)$$

and to the structure function $F$ via

$$F(N) = \sum_{s=0}^{k-1} F_s(N) \Pi_s$$

(see Sec. 2 for the definition of $f_s$ and $F$).

Let us consider the operators $X_-$ and $X_+$ defined by Eqs. (25) and (26) and acting on the Hilbert-Fock space $\mathcal{F}$ (see Eqs. (5)). We choose the action of the constituent operators $b_\pm$ and $f_\pm$ on the state $|kn + s\rangle$ to be given by

$$b_-|kn + s\rangle = b(s)_-|kn + s\rangle = \sqrt{F_s(n + \sigma - \frac{1}{2})} |k(n - 1) + s\rangle,$$

$$b_+|kn + s\rangle = b(s)_+|kn + s\rangle = \sqrt{F_s(n + \sigma + \frac{1}{2})} |k(n + 1) + s\rangle,$$

and

$$f_-|kn + s\rangle = |kn + s - 1\rangle, \quad f_-|kn\rangle = 0,$$

$$f_+|kn + s\rangle = [[s + 1]]_q |kn + s + 1\rangle, \quad f_+|kn + k - 1\rangle = 0,$$

where $n \in \mathbb{N}$ and $s = 0, 1, \ldots, k - 1$. The action of $b_\pm$ is standard and the action of $f_\pm$ corresponds to $\alpha = 0$ and $\beta = 1$ (see Appendix B). Then, we can show that the relationships (6) are satisfied. In this representation, it is easy to prove that the Hermitian conjugation relation $X_+^\dagger = X_-$ is true.

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