Modal Reasoning = Metric Reasoning
via Lawvere
(Extended Version)

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Abstract—Graded modal types systems and coeffects are becoming a standard formalism to deal with context-dependent computations where code usage plays a central role. The theory of program equivalence for modal and coeffectful languages, however, is considerably underdeveloped if compared to the denotational and operational semantics of such languages. This raises the question of how much of the theory of ordinary program equivalence can be given in a modal scenario. In this work, we show that coinductive equivalences can be extended to a modal setting, and we do so by generalising Abramsky's applicative bisimilarity to coeffectful behaviours. To achieve this goal, we develop a general theory of ternary program relations based on the novel notion of a comonadic lax extension, on top of which we define a modal extension of Abramsky's applicative bisimilarity (which we dub modal applicative bisimilarity). We prove such a relation to be a congruence, this way obtaining a compositional technique for reasoning about modal and coeffectful behaviours. To this end, we establish a correspondence between modal program relations and program distances. This correspondence shows that modal applicative bisimilarity and (a properly extended) applicative bisimilarity distance coincide, this way revealing that modal program equivalences and program distances are just two sides of the same coin.

I. INTRODUCTION

Program equivalence, the study of notions of equality between programs, is a central topic in programming language semantics since the very birth of the discipline. For a fixed programming language, a notion of program equivalence is usually given in terms of an equivalence relation between program phrases relating pairs of programs exhibiting the same behaviour. Obviously, if the programming language at hand is equipped with a denotational semantics, a notion of program equivalence is given by mathematical equality in the semantic model. However, interesting notions of program equivalence can be given even without a denotational semantics, but relying on the operational behaviour of programs, only. Examples of such equivalences are Morris' contextual testability of program equivalence for modal and coeffectful languages, whereby equivalent programs can be safely re-placed for one another inside larger software systems. More precisely, compositionality states that program equality is compatible with the syntactical constructs of the language, so that putting equivalent programs in arbitrary contexts always produces equivalent programs. If we think about contexts of the language as tests an external observer can perform on programs, then compositionality states that no test can distinguish between two equivalent programs. Therefore, program equivalence captures the idea of program indistinguishability in a black-box testing scenario. Compositionality then entails that program equality is context insensitive: to be equal, two programs must exhibit the same behaviour in any possible context, i.e. under any possible test.

Oftentimes, however, we would like to reason about programs precisely in virtue of the environment in which such programs are used. This ability is crucial in scenarios where, e.g., resource consumption, data security, and information leakage play a central role, as it often happens in today's software systems. Let us think, for instance, to a software manipulating sensitive data. In such a scenario, we would like users with unprivileged permissions to identify all programs critically relying on classified data (such as medical data or passwords), even if an idealised observer could tell them apart. Why? Because if a user with unprivileged permissions can discriminate between programs that differ only for the classified data they manipulate, then the user can infer some information about such data, meaning that the program causes an information leakage. This is precisely the idea behind non-interference, one of the main properties studied in the field of information flow. A similar story can be told for, e.g., resource consumption, where one may want to identify programs that can only be discriminated by means of expensive computations, or only on certain hardwares, which may not be available to an external observer.

What does the literature offers to cope with these scenarios? From a programming language perspective, several new type systems disciplining code usage have been recently developed. All these systems share two common features: (i) they are resource-sensitive, relying on some
sort of linearity [14]; (ii) they have modal type constructors indexed by grades, resources, or capabilities that specify how code can be manipulated. Such type constructors generalise bounded exponential modalities [15] and can be instantiated to recover modalities for resource analysis [9], program sensitivity [16], and information flow [6], [17]. For these reasons, we generically speak of graded modal types or coeffects.]

But what about program equivalence? Having good notions of program equivalence is crucial when dealing with modal types, as context-sensitive reasoning is usually modelled as a form of program equivalence. This is witnessed by a number of important theorems — such as non-interference [6], metric preservation, and proof irrelevance [18] — which are cornerstones of programming language techniques in fields like information flow, program security, and differential privacy. Non-interference, for instance, relies on a notion of program equality parametrised by users’ permissions, which are cornerstones of programming language behaviours related to code usage and context-dependency.

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In this work, we answer this question in the affirmative by extending Abramsky’s applicative bisimilarity [5] to coeffectful behaviours. Such an extension is obtained through the development of a general relational theory of modal program equivalence based on monoidal Kripke relations and lax extensions of comonads. Let us expand on that.

a) Ternary Relations: As witnessed by, e.g., non-interference, in a modal setting equivalences should relate programs with respect to objects reflecting usage constraints of (modal) programs. Following standard practice in logic [19]–[22], we model this feature by working in a category of ternary relations, which we call monoidal Kripke relations, relating programs with respect to possible worlds living in monoidal preorders, i.e. preorders endowed with a monoid structure.

b) Comonadic Lax Extensions: The main novelty of our approach is the interpretation of modal types in terms of monoidal Kripke relations: to define such an interpretation we introduce the novel notion of a lax extension of a comonad. Lax extensions of monads [23]–[25] are maps extending the action of monads on functions to (binary) relations, which play a central role in topology [24]–[27], coalgebra [28]–[30], and programming language semantics [31]–[35]. Here, we show that modal types and coeffectful behaviour can be uniformly understood in terms of lax extensions of comonads to the category of monoidal Kripke relations. Actually, we do not even need to extend arbitrary comonads: we can consider lax extension of the identity comonad (which we call comonadic lax extensions), only. This reflects the intuition that the action of a modal type on a program does not modify the program, but its usage.

Comonadic lax extensions give an abstract axiomatisation of the action of changing possible worlds, the consequence being that comparing programs via a comonadic lax extension forces one to compare the very same programs, but in a different possible world (where, for instance, the observer has more or less resources at her disposal). Compared with other relational interpretations of modal types [13], [36], comonadic lax extensions build upon the rich theory of relation lifting [23], [25]: it is precisely this level of abstraction that allows us to relate (monadic) effectful and (comonadic) coeffectful behaviours relationally, and thus to account for both infinitary (and, as we will see, generic monadic) and coeffectful phenomena at the same time, something not readily possible in other relational frameworks.

c) Modal Applicative Bisimilarity: On top of this relational framework, we define modal applicative bisimilarity

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1The word coeffect is usually employed to denote the family of program behaviours related to code usage and context-dependency.
— the extension of applicative bisimilarity to a modal setting
— for a call-by-value λ-calculus with (graded) modal and recursive types. Our first main result (Theorem 1) states that modal applicative bisimilarity is a congruence, from which a general compositionality theorem subsuming the aforementioned non-interference, proof irrelevance, and metric preservation theorems follows. Proving such a congruence result, however, is nontrivial. In fact, proving applicative bisimilarity to be a congruence is difficult already in a non-modal setting, the proof being based on a non-elementary relational construction known as Howe’s method [37], [38]. In a modal setting, the situation is even worse, as possible worlds have to be taken into account, with the consequence that routine lemmas (notably, substitutivity) now require nontrivial proofs based on the axiomatics of a comonadic lax extension.

d) Modalities as Metrics: But this is not the end of the story. Our second main result (Theorem 2 and Theorem 3) states that modal program equivalence and program distance are one and the same. More precisely, our notion of a comonadic lax extension can be used to improve Gavazzo’s theory of abstract program distance [39] (the latter being a theory of coinductively-defined distances based on Lawvere analysis of metric spaces as enriched categories [40]) with the consequent result that modal applicative bisimilarity and applicative bisimilarity distance are just two sides of the same coin. This equivalence (that can be easily extended to other notions of equivalence and metric) allows the modal and the metric worlds to improve one another. For instance, it is an easy exercise to show that the many results on program distance for languages with monadic effects can be now imported into the modal world, this way obtaining a large family of operationally-based techniques for effectful and coeffectful languages that, to the best of the authors’ knowledge, are not available in any of the operational theories of coeffects present in the literature.

Summary and Outline Summing up, our contributions are: (i) the definition of a relational theory of modal program equivalence based on the novel notion of a comonadic lax extension (Section III) and Section IV; (ii) the definition of modal applicative bisimilarity and a compositionality theorem for it based on a nontrivial extension of Howe’s method (Section VI); (iii) a correspondence between modal equivalence and program distance (Section VII).

II. MODAL CALCULI

The vehicle calculus of this work is a call-by-value affine λ-calculus with modal necessity types graded in an algebra \( \mathcal{J} \), which we are going to formally introduce. We call our calculus \( \Lambda \mathcal{J} \). The raw syntax of \( \Lambda \mathcal{J} \) is given by the following grammars, respectively for types, values, and terms (where \( j \) ranges over elements in \( \mathcal{J} \)).

\[
\begin{align*}
\tau & ::= a \mid \tau \to \tau \mid \mu a. \tau \mid \Box_j \tau \\
v & ::= x \mid \lambda x.t \mid \text{fold } v \mid [v] \\
t & ::= v \mid vv \mid \text{unfold } v \mid \text{let } x = t \text{ in } t \mid \text{let } [x] = v \text{ in } t.
\end{align*}
\]

For ease of exposition, we consider a minimal set of types consisting of recursive types \( \mu a. \tau \), affine arrow types \( \tau \to \sigma \), and graded modal necessity types \( \Box_j \tau \): this is enough to study modal, higher-order, and infinitary properties of programming languages. Nonetheless, our results have been developed for an extension of \( \Lambda \mathcal{J} \) including product, tensor, and finitary sum types.

We adopt standard notational and terminological conventions [41]. In particular, we work with types and expressions modulo α-conversion and write \( t[s/x] \) for the capture-avoiding substitution of \( s \) for all the free occurrences of the variable \( x \) in \( t \). We employ a similar notation for types. Before going any further, it is convenient to formally introduce grade algebras.

Definition 1. A grade algebra \( \mathcal{J} = (\mathcal{J}, \leq, +, 0, 1, \infty) \) is a preordered semiring with top element, i.e a semiring \( (\mathcal{J}, +, 0, 1) \) together with a preorder \( \leq \) for which both + and * are monotone, and \( \infty \) is the top element.

Elements of a grade algebra \( \mathcal{J} \) are called grades (but also resources or capabilities [9]) and are used to constrain the way code can be manipulated. The following examples will clarify the concept.

Example 1. 1. The one element semiring \( \{ \infty \} \) is used to model the exponential modality of linear logic [14], [42]. An expression of type \( \Box \infty \tau \) represents a piece of code that can be freely duplicated and discharged. This property comes from idempotency of the semiring addition, which gives \( \infty + \infty = \infty \). Notice that here the zero, unit, and top elements of the semiring coincide.

2. The semiring of natural numbers extended with infinity \( (\mathbb{N} \cup \{ \infty \}, =, +, 0, 1, \infty) \) is used to model the exact usage modality from bounded linear logic [15]. Accordingly, a term of type \( \Box \infty \tau \) represents a piece of code that has to be used exactly \( n \) times. Notice that here the semiring addition is not idempotent.

3. The semiring of extended non-negative real numbers \( \mathbb{R} \) \( ([0, \infty], \leq, +, 0, 1, \infty) \) is used to model type systems for program sensitivity [16]. Accordingly, we think about expressions of type \( \Box \tau \to \sigma \) as representing functions that are \( j \)-Lipschitz continuous.

4. Distributive lattices \( (\mathcal{L}, \geq, \wedge, \lor, \top, \bot) \) are used to model information flow modalities, such as modalities describing security levels [6], [17], [43]. As a running example,
we consider the two-element lattice \{low \leq high\}. A
term of type \square_{\text{high}} \tau represents a piece of code accessible
only by users with high confidentiality level. Dually, a
term of type \square_{\text{low}} \tau can be used by users with at least
low confidentiality level, and thus by any user.

Notice that grade algebras are not required to satisfy \(0 \neq 1\).
However, it is convenient to require \(0\) to be the bottom
element of the algebra (i.e. \(0 \leq j\), for any \(j\)). Moreover, we
also require the join of two elements to exist, so that for all
\(j, i \in J\), the element \(j \lor i\) exists and belongs to
\(J\)\(^3\). Such a condition is necessary to guarantee soundness
of our type system \(^4\).

A. Statics and Dynamics

Let us now fix a grade algebra \(J = (J, \leq, \lor, \cdot, 0, 1, \infty)\).
We endow \(\Lambda_J\) with a typing system relying on the
judgments \(\Gamma \vdash^\cdot v : \tau\) and \(\Gamma \vdash^\land t : \sigma\), where \(\sigma\) is a closed
type and \(\Gamma\) is an environment. Environments are sets of
graded variables, i.e. expressions of the form \(x :_j \sigma\) acting as
placeholders for code of type \(\tau\) that has to be manipulated
according to \(j\). For instance, if \(j\) belongs to the algebra
of natural numbers extended with infinity, then \(x :_j \sigma\) is a
placeholder for a piece of code that has to be used \(j\) times.
We write \(\cdot\) for the empty environment, and extend
the semiring structure of \(J\) to environments in the natural
way.

We use judgments to distinguish between arbitrary terms
and values \(^5\), the former representing programs to be evaluated,
and the latter representing the result of the evaluation
of such programs \(^6\). Accordingly, a judgment of the form
\(\Gamma \vdash^\cdot v : \tau\) asserts that \(v\) is a value of type \(\tau\) in environment \(\Gamma\), whereas judgments of the form
\(\Gamma \vdash^\land t : \sigma\) assert that \(t\) is a
term of type \(\tau\) in environment \(\Gamma\).

Finally, we endow \(\Lambda_J\) with the type system defined in
Figure 1\(^8\). We write \(\Lambda^\cdot_J\) and \(\Lambda^\land_J\) for the collections of terms
and values having type \(\tau\) in the environment \(\Gamma\), simply
writing \(\Lambda^\cdot\) and \(\Lambda^\land\) if the environment is empty.

Let us now comment on the rules in Figure 1\(^9\) starting
with the role played by semiring operations on environments.
Following the intuition that terms of type \(\square\tau\) are pieces of
code of type \(\tau\) that can be manipulated according to the
modal label \(j\), we see that whenever we have a value \(v\)
with free variables in \(\Gamma\) and we want to use \(v\) in place of
a variable used by another program according to \(j\), then we
need variables in \(\Gamma\) to be themselves usable according \(j\):
this is formalised by the environment \(j \star \Gamma\), where we use
semiring multiplication to give conditions on code to be used
inside other code. For instance, if a variable \(x\) is used by \(v\) twice and we want to replace \(v\) for a variable \(y\) used 3
times in a term \(t\), then \(x\) will be used 6 times in \(t[v/y]\).

\(^3\) Actually, we do not need \(j \lor i\) to exist for any \(i\), but just for \(i = 1\).
\(^4\) Notice that any value can be regarded as a term that simply evaluates
to itself.

We can now look at some technical features of the rules
in Figure 1. Most of such rules are standard in the context
of graded calculi, although there are minor differences with
other presentations. For instance, in the first rule in Figure 1
it is often required \(j = 1\) and the environment \(0 \star \Gamma\) is
used in place of \(\Gamma\), this way staying closer to linear calculi.
Our choice of allowing \(j\) to be greater than or equal to 1 is
in line with examples coming from differential privacy and
information flow \([6, 16, 45]\). Nonetheless, all our results
hold for calculi where one requires \(j = 1\).

Another important difference is given by the typing rule
for sequencing, which comes from type systems for abstract
program metrics \([39]\) and that has been used in the context
of modal types \([13]\) more recently. The rationale behind such
a rule can be easily understood in terms of resource usage.
Suppose to have a term \(s\) using a variable \(x\) zero times,
and a term \(t\) using a variable \(x\) two times. How many times
\(\text{let } x = t \text{ in } s\) uses \(y\)? One may be tempted to say that \(y\)
is not used in \(\text{let } x = t \text{ in } s\) at all, as \(t\) is simply thrown
away in \(s\). However, in a call-by-value scenario the term
\(\text{let } x = t \text{ in } s\) first evaluates \(t\), and then it throws it away.
As a consequence, the variable \(y\) is still used twice in \(\text{let } x = t \text{ in } s\).
If we were to replace \(j \lor 1\) with \(j\) in the conclusion
of the sequencing rule, then we would obtain that \(y\) is not
used in \(\text{let } x = t \text{ in } s\), which is just unsound. Our choice
of using \((j \lor 1) \star \Gamma\) in place \(j \star \Gamma\) models the fact that \(t\) is
evaluated in \(\text{let } x = t \text{ in } s\), and thus it is used at least once.

Example 2 (Modal Calculi). We now instantiate \(\Lambda_J\)
with suitable grade algebras to recover several examples of modal
calculi that have been studied in the literature both in
isolation \([6, 15, 16, 42, 45–47]\) and in the context of
general modal, quantitative, and coeffectful calculi \([7–11, 13, 43, 49]\).

1) Let us instantiate \(J\) as the one-element semiring \(\{\infty\}\).
Obviously, addition and multiplication on \(\{\infty\}\) are
idempotent operations so that in a judgment of the form
\(\Gamma \vdash t : \tau\), all variables in \(\Gamma\) can be freely erased
and duplicated. Therefore, \(\Lambda_J\) is nothing but a standard,
non-modal call-by-value \(\lambda\)-calculus.
2) Instantiating \(J\) as the three-element chain \(\{0 \leq 1 \leq \infty\}\),
we recover a standard affine call-by-value \(\lambda\)-calculus.
3) Instantiating \(J\) as \(\langle \mathbb{N}^\infty, \leq, +, \cdot, 0, 1, \infty\rangle\),
we obtain a type system for approximate usage analysis.
Accordingly, a term of \(\square_n \tau\) can be used at most \(n\) times
and a judgment of the form \(x :_n \tau \vdash^\land t\) states that to produce
(a single unit) \(t\) we need to use at most \(n\) copies of \(x\).
4) Instantiating \(J\) as \(\langle [0, \infty], \leq, +, \cdot, 0, 1, \infty\rangle\),
we obtain a variation of Fuzz \([16]\), a call-by-value \(\lambda\)-calculus with
a type system tracking program sensitivity \([16, 43, 50, 51]\).
We read judgments of the form \(x :_j \tau \vdash^\land t : \sigma\) as
stating that \(t\) is a \(j\)-continuous function, in
the sense of Lipschitz-continuity, and refer to \(j\) as
the sensitivity of \(t\) in \(t\).
5) Instantiating $J$ as the opposite of a security lattice ([43]), we obtain a call-by-value $\lambda$-calculus for information flow ([6], [17]). Here, we regard elements in $L$ as security levels and read $i_1 \leq i_2$ as stating that data labelled with $i_2$ are more secure (or confidential) than those labelled with $i_1$. We take the two-element lattice $\{\text{high}, \text{low}\}$ as a running example. A term of type $\Box \tau$ is a piece of code that can be used in any security level below $i$, so that if $t$ is a high confidential term (so that it has type $\Box_{\text{high}} \tau$), then it can be used also as a low confidential term, but the vice versa does not hold. More generally, we regard the opposite $(L, \leq, \land, \lor, \top, \bot)$ of any security lattice as a grade algebra.

We conclude this section with a remark on our design choices.

**Remark 1** (On Language Extensions and Comparison). $\Lambda_J$ is a minimal calculus whose type system supports higher-order, modal, and infinitary features. However, all our results are robust with respect to language extensions, such as the addition of finitary product, tensor, and sum types. Several calculi for graded modal types have been recently proposed in the literature, most of them differing (one another) for small details, such as side conditions on typing rules or the axioms of grade algebras. $\Lambda_J$ is meant to be a simple vehicle calculus to illustrate our relational techniques, rather than as a calculus giving a comprehensive account of modal types. In fact, it is easy to realise that our techniques and results can be adapted to all the main (graded) modal calculi in the literature.

Last but not least, we define the dynamic semantics of $\Lambda_J$ as an ordinary operational semantics for a call-by-value $\lambda$-calculus with recursive types extended with the reduction rule `let $[x] = v$ in $t \rightarrow t[v/x]$. Notice that modal types essentially play no role here. We write $t \Downarrow v$ if $t$ converges. The relation $\Downarrow$ being deterministic, we obtain a (type-indexed family of) map(s) $[\cdot] : \Lambda_J \rightarrow (V_L)_{\bot}$ defined by $[t] \Downarrow v$ if $t \Downarrow v$, and $\bot$ otherwise.

### III. RELATIONAL REASONING

In the previous section, we introduced the syntax and semantics of $\Lambda_J$, a call-by-value $\lambda$-calculus parametric with respect to grade algebras. Now we want to define notions of program equivalence for such a calculus. As a first observation, we notice that according to our operational semantics, modal types do not influence the operational behaviour of programs, as the presence of modalities does not affect program execution. According to our information reading, what modalities do is to act on the external observer by modifying the way she can use and test (and thus ultimately discriminate between) programs. We formalise this intuition by defining notions of program equivalence not as relations, but as relations indexed by possible worlds (or, equivalently, as ternary relations), the latter capturing suitable observer’s features.

Formally, instead of associating to any type $\tau$ a relation between terms of type $\tau$, we associate to it a function $\sim_\tau : W \rightarrow P(\Lambda_\tau \times \Lambda_\tau)$ from possible worlds to relations mapping each world $w$ to a relation $\sim_\tau^w$ meant to capture operational indistinguishability for programs of type $\tau$ at world $w$. We thus use possible worlds to represent possible states in which an external observer compares programs.

**Definition 2.** A monoidal Kripke frame ($\text{MKF}$, for short) is a symmetric monoidal preorder $\mathcal{W} = (\{w\}, \leq, \cdot, \varepsilon)$ such that $\varepsilon$ is the bottom element. A $\mathcal{W}$-relation is a monotone map $R : (\mathcal{W}, \leq) \rightarrow (P(X \times Y), \subseteq)$.

We call elements $w, v, \ldots$ of a MKF possible worlds and refer to generic $\mathcal{W}$-relations as monoidal Kripke relations. Fixed a MKP $\mathcal{W}$, a $\mathcal{W}$-relation is thus a monotone map $R : \mathcal{W} \rightarrow P(\mathcal{W} \times \mathcal{W})$— equivalently described as a monotone ternary relation $R \subseteq X \times Y \times \mathcal{W}$— in the sense that $w \leq v$ implies $R(w, v) \subseteq R(v)$. Following the resource-semantics reading ([19]–[22]), we think about possible worlds as consumable resources, this way regarding program comparison as a resource consuming process. $\mathcal{W}$-relations share most of the algebra of the usual (binary) relations, so that we can transfer several notions used in relational reasoning (such as the notion of an equivalence or of a preorder) to the realm $\mathcal{W}$-relations. We now recall the basic algebraic construction needed in this work.

We denote by $\mathcal{W}$-Rel$(X, Y)$ the collection of $\mathcal{W}$-relations over sets $X$ and $Y$, and write $R : X \rightarrow Y$ in place of $R \in \mathcal{W}$-Rel$(X, Y)$, provided that $\mathcal{W}$ is clear from the context. Given a $\mathcal{W}$-relation $R$, we use the notations $X R(w) \mathcal{W} Y$, $R(x, y, w)$, and $(x, y, w) \in R$ interchangeably. $\mathcal{W}$-relations form a category, denoted by $\mathcal{W}$-Rel, whose objects are sets and whose arrows are $\mathcal{W}$-relations. For
any set $X$, the identity $\mathcal{R}$-relation $I$ over $X$ is defined as $I = \{(x, x, w) \mid x \in X, w \in W\}$, whereas the composition $R; S$ of $\mathcal{R}$-relations $R, S$ (of the appropriate type), is defined as:

$$x \in (R; S)(w) \iff \exists y, v, u \quad w \geq v \cdot u \land x \in R(v) \land y \in S(u) \land z.$$

Notice that $I$ and $R; S$ are monotone (provided that $R$ and $S$ are), and thus indeed $\mathcal{R}$-relations. We define the converse of a $\mathcal{R}$-relation $R : X \to Y$ as the $\mathcal{R}$-relation $R^\circ : Y \to X$ defined by $y \in R^\circ(w) \iff x \in R(w) \land y$. As a consequence, we say that a $\mathcal{R}$-relation $R \in \mathcal{R}-\text{Rel}(X, X)$ is reflexive if $I \subseteq R$, symmetric if $R^\circ \subseteq R$, and transitive if $R; R \subseteq R$. Altogether, we obtain the notions of a $\mathcal{R}$-preorder and $\mathcal{R}$-equivalence. Additionally, each set $\mathcal{R}-\text{Rel}(X, Y)$ forms a complete lattice when endowed with (the pointwise extension of) subset inclusion, so that we can define $\mathcal{R}$-relations both inductively and coinductively. Finally, we observe that any function $f : X \to Y$ can be regarded as an arrow in $\mathcal{R}-\text{Rel}$ via the $\mathcal{R}$-relation $\{(x, f(x), w) \mid x \in X, w \in W\}$. For simplicity, we use the notation $f : X \to Y$ even when regarding $f$ as an arrow in $\mathcal{R}-\text{Rel}$.

An important construction on $\mathcal{R}$-relations we will extensively use in this work, is the tensor product of $\mathcal{R}$-relations: given $\mathcal{R}$-relations $R : X \to Y, S : X' \to Y'$, define

$$R \otimes S : X \times X' \to Y \times Y'$$

by $(x, x')(R \otimes S)(w, y') \iff \exists y, u \text{ such that } w \geq v \cdot u \land x \in R(v) \land y \in S(u) \land y'$. Notice that $R \otimes S$ is the monoidal counterpart of the usual (cartesian) product $R \times S$, defined as

$$(x, x')(R \times S)(w, y') \iff x \in R(w) \land y \in S(u) \land y'.$$

Now that we have introduced some basic notions on $\mathcal{R}$-relations, we give some examples of MKFs and their corresponding $\mathcal{R}$-relations one finds in the literature on programming language semantics.

**Example 3.** 1. For the grade algebra $([0, \infty], \leq, +, \cdot, 0, 1)$, let $\mathcal{R}$ be $([0, \infty], \leq, +, 0)$. A $\mathcal{R}$-relation $R$ relate terms with respect to non-negative extended real numbers with the intended meaning that if $t R(a)$ holds, then the $R$-distance between $t$ and $s$ is bounded by $a$. Obviously, if $a \leq b$ and $t R(a)$, we also have $t R(b)$ (if $t$ and $s$ are at most $a$ far, then they also are at most $b$ far). Reed and Pierce [16] use $\mathcal{R}$-relations to define a logical relation $\sim_\mathcal{R} : [0, \infty] \to \Lambda_\mathcal{R} \times \Lambda_\mathcal{R}$ to characterise program distance.

In fact, it is easy to see that $\mathcal{R}$-relations over sets $X$ and $Y$ correspond to metric-like functions $X \times Y \to [0, \infty]$ through the maps $\Phi : \mathcal{R}-\text{Rel}(X, Y) \to [0, \infty]^{X \times Y}$ and $\Psi : [0, \infty]^{X \times Y} \to \mathcal{R}-\text{Rel}(X, Y)$ thus defined:

$$\Phi(R)(x, y) \equiv \inf\{a \mid t R(a)\}$$

$$\Psi(\delta)(a) = \{(x, y) \mid \delta(x, y) \leq a\}.$$  

2. Consider the grade algebra $\mathcal{L}$ of security levels, and let $\mathcal{R}$ be $(\mathcal{L}, \geq, \wedge, \perp)$. A $\mathcal{R}$-relation relates terms at specific confidentiality levels, and we read $t R(\ell)$ as stating that a user with confidentiality level $\ell$ regards the terms $t$ and $s$ as $R$-related. In particular, for an equivalence $\mathcal{R}$-relation $\sim_\mathcal{R}$ we read $t \sim_\mathcal{R} s$ as stating that $t$ and $s$ are indistinguishable for a user with permission level $\ell$.

3. Given a MKF $\mathcal{W}$, the set $\text{End}(\mathcal{W})$ of $\mathcal{W}$-endomorphisms forms a grade algebra with semiring multiplication given by function composition, unit element given by the identity function, and all other operations defined pointwise. In particular, if $\mathcal{W} = (\{0, \infty\}, \leq, +, 0)$, then elements in $\text{End}(\mathcal{W})$ give the so-called $f$-sensitivities [52].
the novel notion of a lax extension of a graded (monoidal) comonad, which is to comonads what a lax extension of a monad is to a monad. The fact that lax extensions of a comonad work on \( \mathcal{J} \)-relations (and not just on relations) is of paramount importance: in fact, equivalence at modal types is actually defined by means of lax extension of the identity comonad. Such lax extensions, which we dub comonadic lax extensions, essentially leave programs unchanged but modify possible worlds, this way reflecting our intuition that modalities do not act on the computational behaviour of programs (hence the identity comonad), but on the external observer.

**Definition 3.** Let \( F \) be a functor on \( \text{Set} \). A lax extension of \( F \) is \( (\text{family of}) \) maps \( \Gamma : \mathcal{J} \text{-Rel}(X, Y) \rightarrow \mathcal{J} \text{-Rel}(F(X), F(Y)) \) satisfying the following laws:

\[
I \subseteq \Gamma(I) \quad \Gamma(R) : \Gamma(S) \subseteq \Gamma(R; S) \quad F(f) \subseteq \Gamma(f) \\
R \subseteq S \implies \Gamma(R) \subseteq \Gamma(S) \quad F(f)^- \subseteq \Gamma(f^-)
\]

Definition 3 is nothing but the usual definition of a lax extension of a functor \([26]\) properly modified to the setting of \( \mathcal{J} \)-relations: it states that the mapping \( X \rightarrow F(X) \), \( R \rightarrow \Gamma(R) \) is a \( \mathcal{J} \)-2-functor on \( \mathcal{J} \text{-Rel} \) that agrees with \( F \) on functions. We now introduce the notion of a comonadic lax extension (i.e. of a lax extension of the identity comonad). It is straightforward to generalise Definition 4 to arbitrary comonads.

**Definition 4.** Given a grade algebra \( (\mathcal{J}, \leq, +, 0, 1, \infty) \), a comonadic lax extension \( \Delta \) associates to any \( \mathcal{J} \)-relation \( R : X \rightarrow Y \) a \( \mathcal{J} \)-indexed family of \( \mathcal{J} \)-relations \( \Delta_j R : X \rightarrow Y \) in such a way that each \( \Delta_j R \) is a lax extension of the identity functor and that the following hold, where \( d : X \rightarrow X \times X \) denotes the duplication (or contraction) map sending \( x \) to \( (x, x) \):

\[
\Delta_0(R) \subseteq R \quad (\text{Com}_1) \\
\Delta_{j+1}(R) \subseteq \Delta_j(\Delta_j(R)) \quad (\text{Com}_2) \\
\Delta_j(R) \otimes \Delta_j(S) \subseteq \Delta_j(R \otimes S) \quad (\text{Mon}_1) \\
\Delta_{j+1}(R) \subseteq d^-(\Delta_j(R) \otimes \Delta_j(R)); d^- \quad (\text{Com}_3) \\
j \leq i \implies \Delta_j(R) \subseteq \Delta_i(R) \quad (\text{Contra})
\]

Let us comment on Definition 4. Requiring each maps \( \Delta_j \) to be a lax extension for the identity functor amounts to require each mapping \( X \rightarrow X, R \rightarrow \Delta_j(R) \) to be a lax 2-functor on \( \mathcal{J} \text{-Rel} \). The real novelty of Definition 4 is that it requires the counit and comultiplication of the (identity) comonad to be lax natural transformations on \( \mathcal{J} \text{-Rel} \). Since both counit and comultiplication are the identity function, they are not visible in laws \( (\text{Com}_1) \) and \( (\text{Com}_2) \). A similar reading explains laws \( (\text{Mon}_1) \) and \( (\text{Com}_3) \), where one exploits the fact that the identity comonad is monoidal. All of this, is adapted to a graded setting. The only rule in Definition 4 that involves the presence of functions (although regarded as \( \mathcal{J} \)-relations) is rule \( (\text{Mon}_1) \). Diagrammatically, we express this rule as follows:

\[
\begin{array}{c}
X \xrightarrow{d} X \times X \\
\Delta_{j+1}(R) \subseteq \Delta_j(R) \otimes \Delta_j(R) \\
Y \xrightarrow{d} Y \times Y
\end{array}
\]

Finally, law \( (\text{Contra}) \) states that what we have is actually a \( \mathcal{J} \)-graded monoidal comonad. Operationally, we can read antitionicity as stating that if two expressions are equivalent when used according to \( j \), then they are also equivalent when used “less” than \( j \) (e.g. if two expressions are equivalent when used an arbitrary number of times, then they must be so when used at most \( n \) times). Notice that \( (\text{Contra}) \) and \( (\text{Com}_3) \) imply \( \Delta_j(R) \subseteq R \), for any \( j \geq 1 \).

**A. Examples**

In this section, we give some examples of comonadic lax extensions that apply to the modal calculi seen so far. We leave for future research further applications of lax extensions of comonads.

Our first example is an abstract extension which we call action extension. In order to define it, we need an action \( \hat{-} : J \times \mathcal{W} \rightarrow \mathcal{W} \) making \( \mathcal{W} \) a \( \mathcal{J} \)-module.

**Proposition 1.** Recall that a lax action is a monotone map \( \hat{-} : J \times \mathcal{W} \rightarrow \mathcal{W} \) satisfying the following laws, where we write \( \hat{j}(w) \) for the action of \( \hat{j} \) on \( j \cdot w \):

\[
\hat{j}(w) \leq w \\
\hat{j}(w \cdot v) \leq \hat{j}(w) \cdot \hat{j}(v) \\
\hat{j}(\hat{i}(w)) \leq \hat{j}(j \cdot i)(w) \\
\hat{i}(w) \leq w \\
\hat{j}(w) \cdot \hat{i}(w) \leq (j + i)(w)
\]

Define the action extension \( ! \) as follows:

\[
x ! j(R)(w) y \equiv \exists v. w \geq \hat{j}(v) \& x R(v) y.
\]

Then, \( ! \) is a comonadic lax extension.

A standard example of a lax action is obtained by taking the MKF \( (J, \leq, +, 0) \) and defining \( \hat{j}(i) \) as \( j \cdot i \). This is precisely the structure one considers when studying program metrics \([16, 45, 50]\). Moreover, as the notation suggests, these action extensions are extensively used to deal with linear-like calculi, where one has \( x ! j R(i)(y) \) if and only if there exists \( g \) such that \( i \geq j \cdot g \) and \( x R(g)(y) \). Notice that any modal calculus comes with this “canonical” comonadic lax extension, to which we refer to as the canonical extension.

**Example 4.** When instantiated to the grade algebra \( ([0, \infty], \leq, +, 0, 1, \infty) \), the canonical extension gives \( x ! j R(i)(y) \) if and only if \( \exists g. i \geq j \cdot g \) and \( x R(g)(y) \). In particular, if \( x R(i)(y) \), then \( x ! j R(j \cdot i)(y) \). This is essentially the modal type clause used by Reed and Piece to define metric logical relations \([16]\); intuitively, it states that if \( x \) and \( y \) are at most \( i \) far when measured by \( R \), then they also are at most \( j \cdot i \) far when measured by \( ! j R \).
Recall that the set $\End(W)$ of $\wedge$-endomorphisms on a MKF $\mathcal{W}$ forms a grade algebra with semiring multiplication given by function composition. In this case, a lax action is given by function application $\hat{f}(w) = f(w)$, so that the canonical extension gives $x !_h R(w) y$ if and only if $\exists v. w \geq h(v)$ and $x R(v) y$. In particular, by taking $\wedge$ as $[0, \infty] = \leq, +, 0)$ and looking at $\wedge$-relations as defining function spaces, we obtain the so-called $f$-sensitivity 52.

Our second example of a comonadic lax extension comes from modal logic, Kripke semantics of intuitionistic logic, and Kripke logical relations 55.

**Proposition 2.** Let $J$ be the one-element grade algebra and consider the (cartesian) MKF $(\mathcal{W}, \leq, \lor, \top)$. Define the Kripke extension (we do not write the unique grade subscript) $\square$ by:

$$x \square R(w) y \equiv \forall v \geq w. x R(v) y.$$  

Then, $\square$ gives a comonadic lax extension.

The map $\square$ is nothing but the relational counterpart of the propositional construction used in the Kripke semantics of the necessity modality. Notice also that $\Box$ can be used to recover the Kripke logical relation semantics of (intuitionistic) arrow types (where we encode $\Box$-relations as defining $\square$-relations).

Our last example of a comonadic lax extension deals with information flow. Let us fix a (op-)lattice of security levels $\langle L, \geq, \wedge, \lor, \top \rangle$.

**Proposition 3.** Define the masking extension $\uparrow$ thus:

$$x \uparrow_j R(i) y \equiv j \not\leq i \text{ or } x R(i) y.$$  

Then, $\uparrow$ is a comonadic lax extension.

The action of $\uparrow_j$ is to make code invisible to users with permission below $j$. Recall that a judgment of the form $\Gamma \vdash t R(i) s$ has the intended meaning that terms $t$ and $s$ are $R$-indistinguishable to users with security permission $i$. For instance, two classified terms $t, s$ are indistinguishable to a user with low confidentiality permission, even if the two terms are actually different.

V. TERM RELATIONS AND THEIR ALGEBRA

In previous sections, we have introduced $\wedge$-relations and their algebra, in the abstract. However, studying $\Lambda_J$ we are interested not in general $\wedge$-relations but in relations between $\Lambda_J$-terms. Following standard practice, we refer to such relations as term relations 38, 50.

**Definition 5.** A term relation is a $\wedge$-relation $R$ between judgments belonging to the same syntactic class (that is, relating values with values, and terms with terms) satisfying the following properties:

$$R(\Gamma \vdash^\wedge t : \tau_1, \Gamma \vdash^\wedge s : \tau_2, w) \implies \Gamma_1 = \Gamma_2 \text{ and } \tau_1 = \tau_2$$

$$R(\Gamma \vdash^\nu v : \tau_1, \Gamma \vdash^\nu w : \tau_2, w) \implies \Gamma_1 = \Gamma_2 \text{ and } \tau_1 = \tau_2.$$  

We employ the notation $\Gamma \vdash^\wedge t R(w) s : \tau$ in place of $R(\Gamma \vdash^\wedge t : \tau, \Gamma \vdash^\wedge s : \tau, w)$ (and similarly, for values) and write $\Gamma \vdash t R(w) s : \tau$ if the distinction between values and terms is not relevant. We also require term relations to be closed under weakening, meaning that $\Gamma \vdash t R(w) s : \tau$ implies $\Gamma + \Delta \vdash t R(w) s : \tau$.

**Definition 6.** A closed term relation is a term relation relating judgments of the form $\vdash t : \tau$ (we write $t R(w) s : \tau$ in place of $\vdash t R(w) s : \tau$). The open extension of a closed term relation $R$ is the term relation $R^o$ thus defined

$$\Gamma \vdash t R^o(w) s : \tau \equiv \forall \gamma \in \Subst(\Gamma) \cdot t \gamma R(w) s \gamma : \tau,$$  

where $\Subst(\Gamma)$ denotes the collection of maps $\gamma$ sending variables $\langle x : \tau \rangle \in \Gamma$ to closed values of type $\tau$. Dually, the closed term projection of a term relation $R$ is the closed term relation $R^c$ obtained by restricting $R$ to closed terms.

Term relations being specific $\wedge$-relations, all the constructions seen in previous sections apply to term relations as well. Additionally, we can extend the algebra of term relations relying on specific features of $\Lambda_J$. We already did that with the open extension of a term relation, and now we do it again by introducing the central notion a compatible refinement.

**Definition 7.** The compatible refinement of a term relation $R$ is the term relation $\hat{R}$ inductively defined by the rules in Figure 2.

Intuitively, the compatible refinement of a term relation $R$ is the relation obtained from $R$ by closing $R$-related expressions under syntactic constructors. Notice that $\hat{R}$ is indeed a $\wedge$-relation (viz. it is monotone). A natural way to understand the defining rules of $\hat{R}$ is to look at those rules as diagrams in $\wedge$-Rel. We consider the case of sequencing as an illustrative example. Let the function

$$\text{seq}_\wedge : \Lambda_\tau^x \times \Lambda_\sigma^\Delta x : \tau \rightarrow \Lambda_\sigma^j \tau + \Delta$$  

map $(t, s)$ to let $x = t \cdot s$. Then, we see that the clause in Figure 2 for sequencing is nothing but the pointwise version of the following lax commutative diagram in $\wedge$-Rel.

$$\begin{array}{c}
\Lambda_\tau^x \times \Lambda_\sigma^\Delta x : \tau \\
\text{seq}_\wedge \\
\Lambda_\tau^x \times \Lambda_\sigma^\Delta x : \tau
\end{array} \xrightarrow{\text{seq}_\wedge} \begin{array}{c}
\Lambda_\tau^j \tau + \Delta \\
\hat{R} \\
\Lambda_\tau^j \tau + \Delta
\end{array}$$

Notice how we use the tensor product to account for multiple premises of the rule, as well as the comonadic lax extension $\Delta_j$ to account for graded variables. For instance, if we take the canonical extension, then we see that, e.g., the compatible refinement rule for the box introduction specialises to
the usual rule one finds in (graded) linear-like type systems \cite{DBLP:journals/tcs/Bozzelli10, DBLP:journals/ml/Pfenning12}:

\[
\Gamma \vdash^v v R(g) w : \tau \quad i \geq j * g
\]

\[
j * \Gamma \vdash^v [v] \bar{R}(i) [w] : \Box_j \tau
\]

Definition 7 induces a monotone map \( R \mapsto \bar{R} \) on the collection of term relations that allows us to formalise the notion of a compatible term relation, i.e. of a term relation closed under syntactical constructors of \( \Lambda_\tau \).

Definition 8. We say that a term relation is compatible if \( \bar{R} \subseteq R \).

In particular, a term relation is compatible if and only if it is a pre-fixed point of \( R \mapsto \bar{R} \). It is not hard to prove that the identity term is such a pre-fixed point, and it actually is the least such. As a consequence, any compatible term relation is reflexive. Another important operation on term relations is the one of a substitutive refinement.

Definition 9. The substitutive refinement of a term relation \( R \) is the term relation \( R^{\text{sub}} \) inductively defined by the following rule:

\[
\Gamma, x : \tau \vdash t R(w) s : \sigma \quad v \Delta_j(R)(v) w : \tau \quad u \geq w \cdot v
\]

\[
\Gamma \vdash t\bar{v}/x R^{\text{mon}}(u) s\bar{w}/x] : \sigma
\]

We say that \( R \) is substitutive if \( R^{\text{sub}} \subseteq R \).

In non-modal calculi, substitutivity and compatibility are important properties that tell us that equivalent expressions can be safely replaced for one another inside a more complex expression, this way giving the following compositionality law: if \( t \simeq s \), then \( C[t] \simeq C[s] \), where \( C \) is an arbitrary context that we regard, for the sake of the arguments, as a term with a free variable \( x \) (so that \( C[t] \) is \( C[t/x] \)).

Compositionality as it is does not hold for modal calculi, as one should also account for possible worlds. In fact, even if two terms \( t, s \) are equivalent at a given world \( w \), then \( C[t] \) and \( C[s] \) may differ at \( w \). Why? Because \( C \) may use \( t \) (resp. \( s \)) in a modal context, so that what we will observe is the behaviour of \( t \) (resp. \( s \)) not at \( w \) but at a different world \( v \), where \( t \) and \( s \) may differ. Formally, this means that the ‘naive’ compositionality law \( t \simeq (w) s \implies C[t] \simeq (w) C[s] \) is unsound.

Comonadic lax extensions offer a solution to this problem by replacing the premise \( t \simeq (w) s \) with \( t \Delta_j(\simeq)(w) s \), this way letting \( \Delta_j(\simeq) \) to give information on the equality of \( t \) and \( s \) at world \( w \) but when used as prescribed by \( j \). This is summarised by the compositionality law:

\[
t \Delta_j(\simeq)(w) s \& x : \tau \vdash C : \sigma \implies C[t] \simeq (w) C[s],
\]

which follows from substitutivity by noticing that \( x : \tau \vdash C R(\varepsilon) C : \sigma \). In light of all these considerations, what we look for is a substitutive and compatible equivalence term relation. Among the many options available, we choose Abramsky’s applicative bisimilarity \cite{DBLP:conf/lics/Abramsky90}.

VI. MODAL APPLICATIVE (B1)SIMILARITY

In this section, we introduce modal applicative (b1)similarity, the extension of Abramsky’s applicative (b1)similarity \cite{DBLP:conf/lics/Abramsky90} to a modal and coeffectful setting, and prove that what we obtain is indeed a compatible and substitutive term relation.

Definition 10. Given \( R : X \rightarrow Y \), define \( R_{\bot} : X_{\bot} \rightarrow Y_{\bot} \) by \( xR_{\bot}(w)y \iff x \neq \bot \implies y \neq \bot \) and \( xR(w)y \). Let \( \Delta \) be a comonadic lax extension. We define the map \( R \mapsto [R] \) on closed term relations as follows:

\[
t [R]^\wedge(w) s : \tau \equiv [t] R^\wedge(\bar{w})[s]
\]

\[
v [R]^\vee(w) w : \tau \equiv \forall u \in V_{\tau,vu} R^\wedge(w) wu : \sigma
\]

\[
\text{fold } v [R]^\vee(w) \text{ fold } w : \mu a. \tau \equiv v R^\wedge(\bar{w}) w : \tau[\mu a. \tau/a]
\]

\[
[v] [R]^\vee(w) [w] : \Box_j \tau \equiv v \Delta_j(R)^\vee(w) w : \tau
\]

We say that a closed term relation \( R \) is an applicative simulation if \( R \subseteq [R] \), and that \( R \) is an applicative bisimulation if both \( R \) and \( R^\wedge \) are applicative simulations.
The first three clauses in Definition 10 are the possible-world counterparts of the usual defining clause of a (non-modal) applicative simulation. The real novelty of Definition 10 is the last clause, where modal values are related relying on comonadic lax extensions.

Since lax extensions are monotone, the mapping \( R \mapsto [R] \) is monotone too and thus it has a greatest fixed point which we call applicative similarity and denote by \( \preceq \). Applicative bisimilarity is defined as \( \preceq \cap \preceq^\circ \) and denoted by \( \simeq \). Notice that \( \preceq \) being defined coinductively (i.e. as the greatest fixed point of a monoton function) it obeys the coinduction proof principle, whereby to prove \( t \preceq (w) \) \( s : \tau \) it is enough to find an applicative simulation \( R \) such that \( t R(w) s : \tau \).

Example 5. 1. For the canonical extension, we see that we have \( [v] \preceq^0 (i) [w] \) \( : \Box_j \tau \) iff there exists \( g \) such that \( i \geq j + g \) and \( v \preceq^0 (g) w : \tau \). Taking non-negative real numbers both as grade algebra and as MKF, we see that modal applicative (bi)similarity gives a relational presentation of applicative bisimulation metrics [39], [57] and a coinductive counterpart of the modal logical relations by Reed and Pierce [16].

2. For the Kripke extension, we have \( [v] \preceq^0 (w) [w] : \Box \tau \) iff \( v \preceq^0 (v) w : \tau \), for any \( v \geq w \). Therefore, we see that modal applicative bisimilarity gives a coinductive counterpart of the well-known Kripke logical relations [58].

3. For the masking extension \( \uparrow \) we have \( [v] \preceq^0 (i) [w] : \Box_j \tau \) iff \( v \preceq^0 (i) w : \tau \). In particular, if two \( j \)-masked values \( [v] \) and \( [w] \) are equivalent at security level \( i \), then either \( i \) is a too low security level \( (i \not\geq j) \) to access \( v \) and \( w \), or the non-masked values \( v \) and \( w \) are actually equivalent at \( i \). Modal applicative bisimilarity thus gives the coinductive counterpart of the usual logical relations used in the field of information flow [59], [60].

Finally, we extend applicative (bi)similarity to open expressions by means of its open extension (and write \( \preceq \) in place of \( \preceq^0 \)). What remains to be done is to prove that applicative bisimilarity is indeed a notion of program equivalence, in the sense that it is compatible and substitutive \( \forall \)-equivalence.

Proposition 4. Applicative similarity \( \preceq \) is a \( \forall \)-preorder, and applicative bisimilarity \( \simeq \) is a \( \forall \)-equivalence.

Proof. By coinduction, showing, e.g., that \( \preceq ; \preceq \) is an applicative simulation.

A. Compositionality and Howe’s Method

Proving that applicative (bi)similarity is compatible and substitutive is highly nontrivial. In this section, we prove such results by extending Howe’s method [37], [38] to a modal setting. Accordingly, we construct a substitutive and compatible relation \( \preceq^u \) out of \( \preceq \) and prove that \( \preceq^u \) and \( \simeq \) coincide (from which it follows that both \( \preceq \) and \( \simeq \) are substitutive and compatible). In non-modal calculi, the proof of substitutivity of \( \preceq^u \) consists of a routine induction, whereas proving that it coincides with applicative similarity is more challenging and requires a mixed induction-coinduction argument. Modal calculi present an additional difficulty, as in such calculi also proving substitutivity of \( \preceq^u \) is nontrivial; and in fact, the defining axioms of a comonadic lax extension turned out to be precisely what is needed to ensure such a property.

Definition 11. Given a closed term relation \( R \), define its Howe extension \( R^u \) as the least fixed point of the mapping \( X \mapsto X^o ; R^o \). That is, \( R^u \) is the least term relation satisfying the following inference rule [67].

\[
\begin{align*}
\Gamma, t &\vdash R^u(v) p : \tau \\
\Gamma &\vdash p R^o(u) s : \tau \quad w \geq v \cdot u \\
\Gamma &\vdash t R^u(w) s : \tau
\end{align*}
\]

The Howe extension of a term relation enjoys several nice properties, which are summarised by the following result.

Lemma 1. Let \( R \) be a reflexive and transitive closed term relation. Then \( R^u \) is a compatible term relation such that \( R^o \subseteq R^u \) and \( R^u \subseteq R \).

In particular, \( \preceq^u \) is a compatible and reflexive term relation that extends \( \preceq \). Our goal now is to prove that \( \preceq^u \) is substitutive and equal to \( \simeq \) (which entails that \( \preceq \) itself is compatible and substitutive). In order to prove substitutivity of \( \preceq^u \), we need the auxiliary notion of a value substitutive term relation.

Definition 12. A term relation \( R \) is value substitutive if the following rule holds (notice that \( v \) is a closed value).

\[
\begin{align*}
\Gamma, x : j &\vdash t R(w) s : \sigma \quad \vdash v \cdot x : \tau \\
\Gamma &\vdash t[v/x] R(w) s[v/x] : \sigma
\end{align*}
\]

Obviously, any substitutive relation is value substitutive. Moreover, since the defining rule of value substitutivity involves closed values only (so that sequential and simultaneous substitution coincide), we see that the open extension of a term relation is always value substitutive. We are now ready to prove our substitutivity lemma.

Lemma 2 (Substitutivity). Let \( R \) be a reflexive and transitive term relation. Then, \( R^u \) is substitutive.

Proof sketch. Define the map \( \text{subst} : \Lambda_\circ^\Gamma \times \mathcal{V}_\tau \rightarrow \Lambda_\circ^\Gamma \) by \( \text{subst}(t, v) = t[v/x] \). We have to prove \( R^u \otimes \Delta_j(R^u) \subseteq \text{subst}; R^u \); \( \text{subst}^- \). Denoting by \( R^u_n \) the \( n \)-th approximation of \( R^u \), so that \( R^u = \bigcup_{n \geq 0} R^u_n \), it is sufficient to show \( \bigcup_{n \geq 0} R^u_n \otimes \Delta_j(R^u) \subseteq \text{subst}; R^u; \text{subst}^- \) which itself follows from \( \bigcup_{n \geq 0} (R^u_n \otimes \Delta_j(R^u)) \subseteq \text{subst}; R^u; \text{subst}^- \). We prove \( \forall n \geq 0 \). \( R^u_n \otimes \Delta_j(R^u) \subseteq \text{subst}; R^u; \text{subst}^- \) by induction on \( n \) relying on the properties of \( \Delta_j(R^u) \).
prove that it coincides with applicative (bi)similarity itself. This is the content of the so-called Key Lemma, which states that if $R$ is a reflexive and transitive applicative simulation, then $R^\mathsf{it}$ (restricted to closed terms) is an applicative simulation too (and thus, by coinduction, it is included in $\preceq$).

Our proof of the key lemma follows the abstract Howe’s method by Dal Lago et al. [31]. Let us sketch how it goes. The crux of the argument is showing that $R^\mathsf{it}$ satisfies the first clause in Definition 10. Assuming $t \, R^\mathsf{it} \, w$, we proceed by induction on the evaluation of $t$ with a case analysis on the shape of $t$. The difficult case is given by sequencing, where one sees that for the proof to go through, the underlying comonadic lax extensions need to satisfy the (lax distributive) law $\Delta_j \triangleright (R_{\perp}) \subseteq (\Delta_j \triangleright (R))_{\perp}$, which now becomes part of the hypothesis of the Key Lemma. Notice that $\bot$, $\top$, and $\triangleright$ all satisfy the desired law.

**Lemma 3 (Key Lemma).** Assume the law $\Delta_j \triangleright (R_{\perp}) \subseteq (\Delta_j \triangleright (R))_{\perp}$, for any $R$. Then, for any reflexive and transitive applicative simulation $R$, $R^\mathsf{it}$ (restricted to closed terms) is an applicative simulation.

**Theorem 1.** Both applicative similarity and applicative bisimilarity are substitutive and compatible term relations.

We conclude this section by noticing that we obtain Metric Preservation [16], Non-interference [6], as well as other similar results (such as Proof Irrelevance [18]) as immediate corollaries of Theorem 1.

**Corollary 1.** 1. For any term $x : \Delta \tau$, $\forall \alpha \vdash t : \rho$ and all values $v$, $w$, $w'$ (of the right type), we have $t[v, w, x, y] \simeq (\Delta \tau) t[v, w', x, y] : \rho$. That is, users with low security permissions cannot observe changes in classified values.

2. For any term $x_1 : \Delta_1 \tau_1$, $\ldots$, $x_n : \Delta_n \tau_n \vdash t : \tau$ and values $v = v_1, \ldots, v_n$, $w = w_1, \ldots, w_n$ (of the right type), if $v_i \simeq (g_i) w_i (\forall i \leq n)$, then $t[v/x] \simeq (\sum_{i \leq n} j_i : g_i) t[w/x]$. That is, terms behave as Lipschitz-continuous functions, with Lipschitz constant given by grades and determined by typing.

**VII. MODAL REASONING = METRIC REASONING**

In Example 3 we have seen how metric reasoning is a specific example of modal reasoning where possible worlds give upper bounds to distances between programs. In this section, we show that modal reasoning and metric reasoning are actually one and the same, provided that the latter is formulated in the general setting of quantale-valued distances [24], as pioneered by Lawvere in his seminal work on generalised metric spaces as enriched categories [40]. Accordingly, the distance between two objects is not a number, but an element of a quantale [62] representing an abstract difference.

**Definition 13.** A commutative quantale $V = (\mathcal{V}, \leq, \otimes, k)$ is a complete lattice $(\mathcal{V}, \leq)$ equipped with a binary commutative multiplication $\otimes$ such that: (i) $\otimes$ has a unit $k$; (ii) join distributes over multiplication, i.e. $a \otimes (\bigvee_i b_i) = \bigvee_i (a \otimes b_i)$.

Given a quantale $V = (\mathcal{V}, \leq, \otimes, k)$, a $\mathcal{V}$-matrix $\alpha : X \rightarrow Y$ over sets $X, Y$ is a map $\alpha : X \times Y \rightarrow V$.

When the quantale $V$ is left unspecified, we generically refer to quantale-valued matrices (or, sometimes, quantale-valued distances). Before giving examples of quantales and quantale-valued matrices, we observe that fixed a quantale $V$, we have a category — called $\mathcal{V}$-$\text{Mat}$ — with sets as objects and $\mathcal{V}$-matrices as arrows. The identity arrow $1 : X \rightarrow X$ maps a pair $(x, y)$ to $k$ if $x = y$, and to $\perp$ (the bottom element of $\mathcal{V}$) otherwise. Given $\mathcal{V}$-matrices $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$, their composition $\alpha \otimes \beta : X \rightarrow Z$ is given by the so-called matrix multiplication formula [24]:

$$(\alpha \otimes \beta)(x, z) \triangleq \bigvee_y \alpha(x, y) \otimes \beta(y, z).$$

Moreover, the complete lattice structure of $\mathcal{V}$ extends to $\mathcal{V}$-matrices pointwise, so that we can say that a $\mathcal{V}$-matrix $\alpha : X \rightarrow X$ is reflexive if $1 \leq \alpha$, transitive if $\alpha \circ \alpha \leq \alpha$, and symmetric if $\alpha \leq \alpha^\ast$ (here, the transpose of a $\mathcal{V}$-matrix $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$, their composition $\alpha \otimes \beta : X \rightarrow Z$ is given by the so-called matrix multiplication formula [24]:

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$$(\alpha \otimes \beta)(x, z) \triangleq \bigvee_y \alpha(x, y) \otimes \beta(y, z).$$
6. The set $\Delta \triangleq \{ f \in [0,1]^{[0,\infty]} \mid f$ monotone and $f(a) = \bigvee_{b \leq a} f(b)\}$. Equivalence $\Delta$-matrices gives probabilistic metric spaces [69] (the informal reading of a $\Delta$-relation $\alpha$ is that $\alpha(x, y)(a)$ gives the probability that $x$ and $y$ are at most $a$-far).

But what does quantales and quantale-valued matrices have to do with Kripke monoidal relations? First, we notice that for any MKF $\mathcal{W}$ we can extend the notion of an applicative bisimulation distance to forms of modality. We can thus rely on Proposition 5 to define the language it builds upon has only a single kind of algebraic effects (and it is thus more general than ours, in this case). Although such a theory deals with $\lambda$-relations as quantale-valued matrices. In fact, the set $\{ \varphi \in \{\text{false, true}\}^\mathcal{W} \mid \varphi$ monotone $\}$ carries a quantale structure with the complete lattice structure defined pointwise and quantale multiplication defined thus:

$$(\varphi \otimes \psi)(w) \equiv \exists v, u. w \geq v \cdot u \text{ and } \varphi(v) \text{ and } \psi(u).$$

We denote the resulting quantale as $\mathbb{B}^\mathcal{W}$. As a consequence, we see that $\mathcal{W}$-relations are a special case of quantale-valued matrices. But this is only half of the result. In fact, $\mathcal{V}$-matrices can be regarded as monoidal Kripke relations of a special kind, namely as meet-preserving monoidal Kripke relations on the MKF $\mathcal{V}$ (meaning that $R(\bigvee_i a_i) = \bigwedge_i R(a_i)$). We write $\text{Inf}(\mathcal{V}^{\text{op}}, \text{Rel}(X, Y))$ for the collection of such relations. Altogether, we obtain the following correspondence (cf. [24]).

**Proposition 5.** We have $\mathcal{W}$-Rel$(X, Y) \cong \mathbb{B}^{\mathcal{W}}$-Mat$(X, Y)$ and $\mathcal{V}$-Mat$(X, Y) \cong \text{Inf}(\mathcal{V}^{\text{op}}, \text{Rel}(X, Y))$ via the maps

$x \Phi(a)(y) \equiv a \leq \alpha(x, y) \quad \Psi(R)(x, y) \equiv x R(y)$

Proposition 5 has a clear mathematical meaning. But what is its pragmatic relevance? Quantale-valued matrices have been extensively studied as abstract notions of distances [24], [27], [67], [70], meaning that there is a large body of results that, thanks to Proposition 5, we can rely on to improve our theory of modal program equivalence. In particular, Gavazzo [39], [53] developed a theory of quantale-based applicative (bi)simulation distances for higher-order languages with algebraic effects [71]–[73]. Although such a theory deals with algebraic effects (and it is thus more general than ours, in this respect), the language it builds upon has only a single kind of grade algebras. Additionally, lacking the general notion of a comonadic lax extension, applicative bisimilarity distance is defined with respect to the analogue of our lax actions only, and thus it cannot capture the behaviour of more general forms of modality. We can thus rely on Proposition 5 to extend the notion of an applicative bisimulation distance to the general setting of modal types. First, observe that by Proposition 5 any notion we have defined in terms of $\mathcal{W}$-relations has a $\mathcal{V}$-Mat counterpart. Thus, for instance, we have a notion term matrix (cf. term relation), as well as a notion of a comonadic lax extension to $\mathcal{V}$-Mat. Finally, for a $\mathcal{V}$-matrix $\alpha : X \rightarrow Y$, we define $\alpha_\bot : X_\bot \rightarrow Y_\bot$ by $\alpha_\bot(x, y) = k$ if $x = \bot$, $\alpha_\bot(x, y) = \alpha(x, y)$ if $x, y \neq \bot$, and to $\alpha_\bot(x, y) = \bot$ otherwise. We refer to the original paper on applicative bisimilarity distance(s) [39] for details.

**Definition 14.** Fixed a quantale $\mathcal{V}$ and comonadic lax extension $\Delta$ to $\mathcal{V}$-Mat, we define applicative similarity distance $\delta$ as the largest term matrix $\alpha$ such that

$$\alpha_\bot^\mathcal{V}(t, s) \leq \alpha_\bot^\mathcal{V}([t], [s])$$

$$\alpha_\bot^\mathcal{V}(\lambda x. t, \lambda x. s) \equiv \bigwedge_{v \in \mathcal{V}} \alpha_\bot^\mathcal{V}(t[v/x], s[v/x])$$

$$\alpha_{\mu_a, \tau}^\mathcal{V}(\text{fold } v, \text{fold } w) \leq \alpha_{\mu_a, \tau}^\mathcal{V}(v, w) \leq \alpha_{\mu_a, \tau}^\mathcal{V}([v], [w]) \leq \Delta_\bot^\mathcal{V}(v, w),$$

Applicative bisimilarity distance is defined as $\delta \land \Delta_\bot^\mathcal{V}$.

The following result states that modal reasoning is indeed equal to metric reasoning, via Lawvere.

**Theorem 2.** Let $\mathcal{V}$ and $\mathcal{W}$ be a quantale and a monoidal Kripke frame, respectively. Modulo the isomorphisms of Proposition 5 we have that $\Phi(\delta) = \leq$, where $\leq$ is defined on $\mathcal{V}^{\text{op}}$, and that $\psi(\leq) = \delta$, where $\delta$ is defined on $\mathbb{B}^{\mathcal{W}}$.

Since modal applicative (bi)similarity is compatible and substitutive, by Theorem 2 we obtain that applicative (bi)similarity distance is compatible and substitutive, too. From this result, it also follows an abstract metric preservation theorem stating that we can reason compositionality about program distances.

**Theorem 3.** For any term $x_1 : j_1, \ldots, x_n : j_n \vdash t : \tau$, and for all values $v \triangleq v_1, \ldots, v_n, w \triangleq w_1, \ldots, w_n$ of the appropriate type, we have:

$$\bigwedge_{i \leq n} \Delta_\bot^\mathcal{V}(\delta_{\tau_i})(v_i, w_i) \leq \delta_\bot^\mathcal{V}(t[v/x], t[w/x]).$$

**Remark 2 (On Effects).** Theorem 3 and Theorem 2 relate modal and metric reasoning. Although mathematically pleasant, the reader may wonder what one really gains from such a relationship (after all, one could ignore program distance and work directly with $\mathcal{W}$-relations). The advantage of the correspondence between program distances and modal equivalences is that the former comes with a collection of results and techniques that are not readily available in a modal setting. For instance, since applicative bisimilarity distance has been originally defined on languages with arbitrary algebraic effects [72], [73] (such as pure and probabilistic nondeterminism, imperative stores, exceptions, etc), Theorem 3 can be easily generalised to extensions of $\Lambda_\tau$ with algebraic operations $\delta$ la Plotkin and Power. As a consequence, we obtain a collection of relational and metric-like techniques for reasoning about programs exhibiting both effectful and coeffectful behaviours.
VIII. CONCLUSION

In this work, we have developed a relational theory of program equality for higher-order languages with graded modal types and coeffects. Such a theory builds upon some nontrivial and abstract notions, notably the one of a comonadic lax extension, which make the theory a robust and unifying framework for the operational analysis of coeffectful languages. Even if new, we have showed that our relational theory is de facto equivalent to a general theory of program distance, the latter being built on the category of quantale-valued matrices using suitable notions of lax extensions. This correspondence allows us to improve both theories at once giving, for instance, relational techniques for the analysis of languages with both (algebraic) effects and coeffects.

a) Future Work: We have only touched the results obtainable from the aforementioned correspondence between modal relational reasoning and metric reasoning. In the future, the authors would like to rely on this correspondence to develop Böhm tree-like distances by means of a notion of modal Böhm tree equivalence. The latter syntactically compares Böhm trees of programs with respect to a possible world determining the granularity of the inspection. The action of a comonadic lax extension is then to change this granularity making, e.g., parts of the tree visible or invisible.

The authors would also like to investigate whether abstract metric semantics can be used to give a uniform denotational semantics to languages with modal types. In fact, denotational semantics of such languages have been given in terms of general categories and (graded) monads and comonads [7], [48]. The specific categories such semantics instantiate to, however, considerably change from case to case (giving, e.g., a presheaves semantics in the case of information flow and a metric semantics in the case of program sensitivity). It is thus desirable to have a more uniform, albeit more concrete semantics. This has been done by Breuvart and Pagni [36], who gave a denotational semantics to coeffectful calculi where programs are interpreted as suitable relations. We believe that it is also possible to give modal calculi a uniform (abstract) metric semantics by interpreting types as categories enriched over a quantale and programs as enriched functors (which, in such a setting, generalise non-expansive maps).

Finally, the authors would like to explore further applications of comonadic lax extensions. For instance, although in this work the focus was on lax extensions of the identity comonad, it is a straightforward exercise to generalise our notions to arbitrary comonads. That allows us to develop relational techniques for truly comonadic calculi, such as those based on Uustalu and Vene’s comonadic notions of computation [73].

b) Related Work: In recent years, there has been a growing interest for typing disciplines regulating how code can be manipulated. Specific examples of such disciplines date back at least to the 90s, originating from (bounded) linear logic [14], [15], [42]. programming languages-based approaches to information flow [6], [17], and investigations into the Curry-Howard correspondence for modal logic(s) [46], [75], [76]. More recently, researchers started to design calculi with types governing more general notions of resource consumptions [8], [48], quantitative aspects of code usage [9], [10], [16], and environmental requirements [11], [13], this way obtaining general modal-like type systems [7], [9], [13]. From a semantical perspective, such systems have been investigated by means of (comonadic) denotational semantics [7], [55], [48] and (mostly heap-based) resource sensitive operational semantics [8], [9], [12], [13], [78].

Concerning (operationally-based) program equivalence, the work closest to ours is the one by Abel and Bernardy [13], where logical relations for a (call-by-name) $\lambda$-calculus with modal and polymorphic types is introduced. As we do in this work, Abel and Bernardy define logical relations as monoidal Kripke relations. Their treatment of modalities, however, is different from ours, as they lack the notion of a comonadic lax extension. Moreover, the language of Abel and Bernardy includes polymorphism (which we do not have) and is pure and strongly normalising (the calculus does not have neither general recursion nor effects), whereas $\Lambda_j$ has recursive types and, as argued in Remark 2, we can safely add algebraic effects to it.

REFERENCES

[1] J. Morris, “Lambda calculus models of programming languages,” Ph.D. dissertation, MIT, 1969.
[2] I. A. Mason and C. L. Talcott, “Equivalence in functional languages with effects,” J. Funct. Program., vol. 1, no. 3, pp. 287–327, 1991.
[3] G. Plotkin, “Lambda-definability and logical relations,” 1973, technical Report SAI-RM-4, School of A.I., University of Edinburgh.
[4] J. Reynolds, “Types, abstraction and parametric polymorphism,” in IFIP Congress, 1983, pp. 513–523.
[5] S. Abramsky, “The lazy lambda calculus,” in Research Topics in Functional Programming, D. Turner, Ed. Addison Wesley, 1990, pp. 65–117.
[6] M. Abadi, A. Banerjee, N. Heintze, and J. G. Riecke, “A core calculus of dependency,” in POPL ’99. Proceedings of the 26th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, San Antonio, TX, USA, January 20-22, 1999, pp. 147–160.
[7] M. Gaboardi, S. Katsumata, D. A. Orchard, F. Breuvart, and T. Uustalu, “Combining effects and coeffects via grading,” in Proc. of ICFP 2016, 2016, pp. 476–489.
[8] A. Brunel, M. Gaboardi, D. Mazza, and S. Zdancewic, “A core quantitative coeffect calculus,” in Proc. of ESOP 2014, 2014, pp. 351–370.
[9] D. Orchard, V.-B. Liepelt, and H. Eades III, “Quantitative program reasoning with graded modal types,” Proc. ACM Program. Lang., vol. 3, no. ICFP, pp. 110:1–110:30, 2019.
[10] R. Atkey, “Syntax and semantics of quantitative type theory,” in Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018, 2018, pp. 56–65.
[11] T. Petricek, D. A. Orchard, and A. Mycroft, “Coefficients: a calculus of context-dependent computation,” in Proc. of ICFP 2014, 2014, pp. 123–135.
APPENDIX A
MODAL CALCULI

In this Section, we provide further details on the syntax and (operational) semantics of \( \Lambda_\mathcal{J} \). First, we extend the semiring structure of \( \mathcal{J} \) to environments pointwise. More precisely, we first require two environments \( \Gamma, \Delta \) to be compatible, meaning that if \( x \) appears both in \( \Gamma \) and \( \Delta \), then it has the same type. We tacitly assume environments to be pairwise compatible.

**Definition 15.** For an operation \( \circ \in \{+, \ast \} \), we then define \( \Gamma \circ \Delta \) as follows, where the last clause handles the case for \( x \) not appearing among variables in \( \Delta \).

\[
\Gamma \circ \Delta \triangleq \begin{cases} 
\Gamma & \text{if } x \text{ does not appear in } \Delta, \\
(x : j, \Gamma) \circ (x : i, \Delta) \triangleq x : j \circ i, \Gamma, \Delta & \text{if } x \text{ appears in } \Delta.
\end{cases}
\]

We endow \( \Lambda_\mathcal{J} \) with an inductive call-by-value big step semantics given by the rules Figure 3, where in a judgment \( t \downarrow v \), \( t \) is a closed term and \( v \) is closed values (of the same type of \( t \)).

![Fig. 3. Call-by-Value Operational Semantics](image)

To simplify the meta-theory of \( \Lambda_\mathcal{J} \), we also endow \( \Lambda_\mathcal{J} \) with a more abstract *monadic* operational semantics [31], [72], [79]: this allows us to rely on the abstract Howe’s method of Dal Lago et al. [31] to prove congruence properties of applicative bisimilarity as well as to smoothly extend \( \Lambda_\mathcal{J} \) with effectful primitives.

**Definition 16.** The **maybe** or partiality monad \( \mathbb{M} \) (on \( \text{Set} \)) is the triple \( \mathbb{M} = (\mathcal{M}, \eta^\mathbb{M}, -^\mathbb{M}) \), where \( T(X) = X_\bot = X + \{ \bot \} \) and (where \( f : X \to Y_\bot \)):

\[
\eta^\mathbb{M}(x) = x \quad f^\mathbb{M}(x) = \begin{cases} 
\bot & \text{if } x = \bot; \\
n(x) & \text{otherwise}.
\end{cases}
\]

Sets of the form \( X_\bot \) can always be endowed with a \( \omega \)-complete pointed partial order (\( \omega \)-cppo, for short) structure [81] by considering the flat order \( \sqsubseteq \). The bottom element of \( X_\bot \) is \( \bot \). Moreover, any \( \omega \)-chain \( x_0 \sqsubseteq x_1 \sqsubseteq \cdots \) in \( X_\bot \) has a least upper bound which we denote by \( \bigsqcup_{n \geq 0} x_n \). The monad and \( \omega \)-cppo structure of the construction \( X_\bot \) properly interact, in the sense that the following strictness and continuity laws hold, where function spaces of the form \( T(X) \to T(Y) \) are endowed with the \( \omega \)-cppo structure inherited from \( T(Y) \) pointwise.

\[
f^\mathbb{M}(\bot) = \bot; \quad f^\mathbb{M}(\bigsqcup_{n \geq 0} x_n) = \bigsqcup_{n \geq 0} f^\mathbb{M}(x_n); \quad (\bigsqcup_{n \geq 0} f^\mathbb{M}(x_n))(\cdot) = \bigsqcup_{n \geq 0} f^\mathbb{M}(x_n).
\]

We can now define an evaluation map mapping each closed term \( t \) to an element \( [t] \in \mathcal{V}_\bot \). Notice that the map \( \llbracket - \rrbracket : \Lambda \to \mathcal{V}_\bot \) is thus equivalent to a deterministic relation.

12 Recall that a monad is a triple \( \mathbb{T} = (T, \eta, \mu) \), with \( T \) an endofunctor (we consider the case of \( \text{Set} \) monads only), and \( \eta_X : X \to T(X) \) and \( \mu_X : T(T(X)) \to T(X) \) natural transformations, subject to suitable coherence conditions. Oftentimes, we do not work with monads directly but with the equivalent notion of a Kleisli triple \( (T, \eta, -) \) [80].

13 Recall that \( x \sqsubseteq y \) if \( x \neq \bot \) implies \( x = y \).
Definition 17. Define the \( \mathbb{N} \)-indexed family of (type-indexed) evaluation maps\(^{14} \) \( [-]_n : \Lambda \rightarrow \mathbb{V}_\bot \) recursively as follows:

\[
\begin{align*}
[l]_0 & \triangleq \bot \\
[l]_{n+1} & \triangleq \eta^{[l]}(v) \\
[\lambda x.t]_{n+1} & \triangleq [[t[v/x]]_n] \\
[\text{unfold} \ (\text{fold} \ v)]_{n+1} & \triangleq [[t[v/x]]_n] \\
[\text{let} \ [x] = [v] \ \text{in} \ t]_{n+1} & \triangleq (v \mapsto [[s(v/x)]_n]^{[l]}([l]_n)).
\end{align*}
\]

The function \( [-]_n \) maps each (closed) computation of type \( \tau \) either to a (closed) value of type \( \tau \) or to the divergence symbol \( \bot \). Moreover, it is straightforward to see that for any (closed) computation \( t \) we have an \( \omega \)-chain \( \llbracket t \rrbracket_0 \sqsubseteq \llbracket t \rrbracket_1 \sqsubseteq \cdots \) so that we can define \( \llbracket t \rrbracket = \bigsqcup_{n \geq 0} \llbracket t \rrbracket_n \).

Lemma 4. Let \( t \) be an arbitrary closed computation. Then:

\( t \Downarrow v \iff \llbracket t \rrbracket = v \) and \( t \Downarrow \not\iff \llbracket t \rrbracket = \bot \).

Now that we have endowed \( \Lambda_J \) with a typing system and an operational semantics, we move to the main topic of this work: relational reasoning and program equivalence.

Appendix B

Relational Reasoning

We give some preliminaries on relational reasoning. Oftentimes, it will be helpful to reason about monoidal Kripke relations pointfree style. It is thus useful to keep in mind the pointwise reading of relations of the form \( f; S; g^{-} \), for a relation \( S : Z \rightarrow W \) and functions \( f : X \rightarrow Z, \ g : Y \rightarrow W \):

\[ x (f; S; g^{-})(w) y \iff f(x) S(w) g(y). \]

Given \( R : X \rightarrow Y \) we can thus express a generalised monotonicity condition in pointfree fashion as:

\[ R \subseteq f; S; g^{-}. \]

Indeed, taking \( f = g \), we obtain standard monotonicity of \( f \). We will make extensively use of the following adjunction rules \(^{24} \) (also knowns as shunting \(^{82} \)), for \( f : X \rightarrow Y, \ g : Y \rightarrow Z, \ R : X \rightarrow Y, \ S : Y \rightarrow Z, \) and \( Q : X \rightarrow Z \):

\[ R; g \subseteq Q \iff R \subseteq Q; g^{-} \quad \text{(adj 1)} \]
\[ f^{-}; Q \subseteq S \iff Q \subseteq f; S. \quad \text{(adj 2)} \]

Using (adj 1) and (adj 2) we see that generalised monotonicity \( R \subseteq f; S; g^{-} \) can be equivalently expressed via the following lax commutative diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow R & & \downarrow S \\
Y & \xrightarrow{g} & W
\end{array} \]

The diagram acts as a graphical representation of the expression \( R; g \subseteq f; S \), which, by (adj 1), is equivalent to \( R \subseteq f; S; g^{-} \).

\(^{14}\) We omit type subscripts.
A. Lax Extensions

Lemma 5. Lax commutative diagrams in \( \mathcal{W} \)-Rel are preserved by the mapping \( X \mapsto F(X), R \mapsto \Gamma(R) \). That is:

\[
\begin{array}{c}
X \xrightarrow{f} Z \\
R \xrightarrow{\subseteq} S \quad \Rightarrow \quad \Gamma(R) \xrightarrow{\subseteq} \Gamma(S)
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{f} F(Z) \\
R \xrightarrow{\subseteq} S \quad \Rightarrow \quad \Gamma(R) \xrightarrow{\subseteq} \Gamma(S)
\end{array}
\]

Proof. Let us consider a lax commutative diagram in \( \mathcal{W} \)-Rel expressed in linear notation: \( R; g \subseteq f; S \). By shunting, the latter is equivalent to \( f^{-}; R; g \subseteq S \). By monotonicity of \( \Gamma \), we thus obtain \( \Gamma(f^{-}; R; g) \subseteq \Gamma(S) \), and thus \( \Gamma(f^{-}); \Gamma(R); \Gamma(g) \subseteq \Gamma(S) \), by lax functoriality. We now apply stability on \( \Gamma(g) \) and \( \Gamma(f^{-}) \), this way obtaining (by monotonicity) \( F(f^{-}); \Gamma(R); F(g) \subseteq \Gamma(S) \), and thus the desired thesis, by shunting. \( \square \)

Although \( \Lambda_\mathcal{J} \) is a pure calculus, we handled divergence by giving it a monadic (operational) semantics based on the partiality monad. We thus follow Dal Lago et al. \cite{31} and rely on lax extensions of monads to define applicative bisimilarity.

Definition 18. A lax extension of a monad \( \mathbb{T} = (T, \eta, \mu) \) is a lax extension of \( T \) satisfying the following laws:

\[
R \subseteq \eta; \Gamma(R); \eta^{-} \\
\Gamma(\Gamma(R)) \subseteq \mu; \Gamma(R); \mu^{-}
\]

(lax monad 1)

(lax monad 2)

As before, we can conveniently express laws \((\text{lax monad 1})\) and \((\text{lax monad 2})\) as diagrams:

\[
\begin{array}{c}
X \xrightarrow{T(X)} T(Y) \\
R \xrightarrow{\subseteq} \Gamma(R) \quad \Rightarrow \quad \Gamma(T(R)) \xrightarrow{\subseteq} \Gamma(T(Y))
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{f} Z \\
R \xrightarrow{\subseteq} S \quad \Rightarrow \quad \Gamma(R) \xrightarrow{\subseteq} \Gamma(S)
\end{array}
\]

Notice that any lax extension \( \Gamma \) of a monad \( \mathbb{T} = (T, \eta, \mu) \) satisfies the following law:

\[
R \subseteq f^{-}; \Gamma(S); g \quad \Rightarrow \quad \Gamma(R) \subseteq (f^{-})^\dagger; \Gamma(S); g^\dagger
\]

(lax monad bind)

which can be can conveniently expressed diagrammatically as follows:

\[
\begin{array}{c}
X \xrightarrow{f} T(Z) \\
R \xrightarrow{\subseteq} \Gamma(S) \quad \Rightarrow \quad \Gamma(R) \xrightarrow{\subseteq} \Gamma(S)
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{f} T(Z) \\
R \xrightarrow{\subseteq} \Gamma(S) \quad \Rightarrow \quad \Gamma(R) \xrightarrow{\subseteq} \Gamma(S)
\end{array}
\]

Actually, in presence of law \((\text{lax monad 1})\) the laws \((\text{lax monad 2})\) and \((\text{lax monad bind})\) are equivalent \cite{53}.

Proposition 6. Let \( R : X \to Y \) be a \( \mathcal{W} \)-relation. Define \( R_\perp : X_\perp \to Y_\perp \) as follows:

\[
x R_\perp(w) y \iff x \neq \perp \& y \neq \perp \& x \mathcal{W} R(w) y.
\]

Then, \( (-)_\perp \) is a lax extension of the maybe/partiality monad.

Intuitively, \( R_\perp \) gives a generalisation of the usual clause used to define operational preorders between programs. Accordingly, a term \( t \) approximates the behaviour of a term \( s \) at world \( w \) if either \( t \) diverges or
both \( t \) and \( s \) converge and the resulting values are related at \( w \). If we take \([0, \infty], \leq_1, +, 0\) as possible worlds structure and read \( t R(j) s \) as stating that the \( R \)-distance between \( t \) and \( s \) is at most \( j \), then \( t R(j) s \) tells us that if \( t \) diverges, then the \( R_1 \)-distance between \( t \) and \( s \) is bounded by any \( j \) — and thus it is bounded by 0. Otherwise, \( t \) converges to value \( v \), and thus \( s \) converges to a value \( w \) such that the \( R_1 \)-distance between \( t \) and \( s \) is the \( R \)-distance between \( v \) and \( w \).

**Appendix C**

**Howe’s Method**

**A. Applicative Bisimilarity**

First, recall the definition of a modal applicative (bi)simulation.

**Definition 19.** Recall the definition of the relator \( \Gamma^\perp \) for the partiality monad given in Proposition 6. Define the mapping \( R \mapsto [R] \) on closed term relations as follows:

\[
\begin{align*}
t &= [R]^\wedge (w) \iff [t] \; R_1^\wedge (w) \; [s] & \text{(App eval)}
\end{align*}
\]

\[
\begin{align*}
\lambda x. t &= [R]^\vee (w) \lambda x. s \to \sigma \iff \forall v \in V, \; t[v/x] \; R^\wedge (w) \; s / x : \sigma & \text{(App abs)}
\end{align*}
\]

\[
\begin{align*}
\text{fold} \; v &= [R]^\vee (w) \text{ fold} \; w : \mu a. \tau \iff v \; R^\vee (w) \; w : \tau / \mu a. \tau / a & \text{(App fold)}
\end{align*}
\]

\[
\begin{align*}
[v] \; [R]^\vee (w) \; [w] : \sqcup \tau & \iff v \; \Delta_j (R)^\vee (w) \; w : \tau & \text{(App box)}
\end{align*}
\]

We say that a closed term relation \( R \) is an applicative simulation if \( R \subseteq [R] \), and that \( R \) is an applicative bisimulation if both \( R \) and \( R^- \) are applicative simulations.

Equivalently, \( R \) is an applicative simulation if the following hold:

\[
\begin{align*}
t \; R^\wedge (w) \; s & \mapsto [t] \; R_1^\wedge (w) \; [s] \\
v \; R^\vee (w) \; w & \to \sigma \mapsto \forall v \in V, \; v[w/u] \; R^\wedge (w) \; w u : \sigma \\
\text{fold} \; v \; R^\vee (w) \; \text{fold} \; w : \mu a. \tau & \mapsto v \; R^\vee (w) \; w : \tau / \mu a. \tau / a \\
[v] \; R^\vee (w) \; [w] : \sqcup \tau & \mapsto v \; \Delta_j (R)^\vee (w) \; w : \tau
\end{align*}
\]

**Proposition 7.** Applicatively similarly \( \preceq \) is a preorder term relation, and applicative bisimilarity \( \simeq \) is an equivalence term relation.

**Proof.** By coinduction. For instance, we show that \( \simeq \) is an applicative simulation, and thus it is included in \( \preceq \). We show one case as a paradigmatic example. Suppose \( t \preceq (\preceq)^\wedge (w) \) \( p : \tau \), so that we have \( w \succeq_v u \) with \( t \preceq^\wedge (v) \) \( s : \tau \) and \( s \preceq^\wedge (u) \) \( p : \tau \), for suitable worlds \( v, u \), and an expression \( s \). Say \( t \Downarrow v \). Then, \( t \preceq^\wedge (v) \) \( s : \tau \) implies \( s \Downarrow w \) for some value \( w \) such that \( v \preceq^\wedge (w) \) \( w : \tau \). From \( s \Downarrow w \) and \( s \preceq^\wedge (u) \) \( p : \tau \) we obtain the existence of a value \( u \) such that \( p \Downarrow u \) and \( w \preceq^\wedge (u) \) \( u : \tau \). We thus conclude \( v \preceq^\wedge (w) \) \( u : \tau \), and thus we are done.

**B. Howe’s Method**

It is convenient to give an explicit, syntax-oriented characterisation of the Howe extension of a term relation \( R \). We do so by means of judgments of the form \( \Gamma \vdash t \; R^\wedge (w) s : \tau \) (declined, as usual into value and computation judgments) and of the inference rules in Figure 4. Then, two open terms \( \Gamma \vdash t, s : \tau \) are related by \( R^\wedge \) at world \( w \) if and only if \( \Gamma \vdash t \; R^\wedge (w) s : \tau \) is derivable. We also define the relation \( R^\wedge_n \), for \( n \in \mathbb{N} \), by saying that \( \Gamma \vdash t, s : \tau \) are related by \( R^\wedge_n \) at world \( w \) if and only if \( \Gamma \vdash t \; R^\wedge_n (w) s : \tau \) is derivable with a derivation of depth at most \( n \). As a consequence, we see that \( R^\wedge = \bigcup_{n \geq 0} R^\wedge_n \).

**Lemma 2** (Substitutivity). Let \( R \) be a reflexive and transitive term relation. Then, \( R^\wedge \) is substitutive.
Since $\vdash \vdash \Gamma$

Therefore, to prove substitutivity it is sufficient to show that $\vdash \vdash w \geq v \cdot u$

We have to prove $\vdash \vdash R^\Gamma(w) : \tau$

\[
\Gamma \vdash \vdash R^\Gamma(w) : \tau
\]

\[
\Gamma, x :_1 \vdash \vdash R^\Gamma(w) : \tau \quad \Gamma \vdash \vdash \Lambda \vdash w \geq v \cdot u
\]

\[
\Gamma \vdash \vdash R^\Gamma(w) : \tau
\]

\[
\Gamma, x :_1 \vdash \vdash R^\Gamma(w) : \tau \quad \Gamma \vdash \vdash \Lambda \vdash w \geq v \cdot u
\]

\[
\Gamma, x :_1 \vdash \vdash R^\Gamma(w) : \tau
\]

\[
\Gamma \vdash \vdash v \cdot R^\Gamma(v) : \tau
\]
We proceed by induction on \( n \). The case for \( n = 0 \) is trivial. We prove \( R_n^{*+1} \cap \Delta_j(R^n) \subseteq \text{subst}; R^n; \text{subst}^\ast \), assuming \( R_m^n \cap \Delta_j(R^n) \subseteq \text{subst}; R^n; \text{subst}^\ast \) for all \( m \leq n \). Notice that, formally, we are universally quantifying over \( j \).

- Suppose to be in the following case:

\[
\Gamma, x : j \vdash \sigma \vdash y R^o(w) u : \sigma (j \geq 1) \quad v \quad \Delta_j(R^n)^\gamma(v) w : \tau \quad u \geq w \otimes v
\]

\[
\Gamma, x : j \vdash \sigma \vdash y R^o(u) u[w/x] : \sigma
\]

Then, since \( R^o \) is value substitutive, from \( \Gamma, x : j \vdash \sigma \vdash y R^o(w) u : \sigma \) we infer \( \Gamma \vdash v \quad R^o(w) u[w/x] : \tau \). Moreover, since \( j \geq 1 \) (and thus \( \Delta_j(R^n) \subseteq \Delta_1(R^n) \subseteq R^n \)), from \( v \quad \Delta_j(R^n)^\gamma(v) w : \tau \) we infer \( \Gamma \vdash v \quad R^o(w) u[w/x] : \tau \). Putting things together, we obtain \( \Gamma \vdash v \quad (R^o; R^n)(u) u[w/x] : \tau \), which gives \( \Gamma \vdash v \quad R^o(u) u[w/x] : \tau \), thanks to quasi-transitivity.

- Suppose to be in the following case:

\[
\Gamma, x : j \vdash \sigma \vdash y R^o(w) u : \sigma (i \geq 1) \quad v \quad \Delta_j(R^n)^\gamma(v) w : \tau \quad u \geq w \otimes v
\]

\[
\Gamma, y : \sigma \vdash \sigma \vdash y R^o(u) u[w/x] : \sigma
\]

Then, since \( R^o \) is value substitutive, from \( \Gamma, x : j \vdash \sigma \vdash y R^o(w) u : \sigma \) we infer \( \Gamma, x : j \vdash \sigma \vdash y R^o(w) u[w/x] : \sigma \) and thus \( \Gamma, x : j \vdash \sigma \vdash y R^o(w) u[w/x] : \sigma \). We conclude the thesis, since \( w \leq w \otimes u \).

- Suppose to be in the following case:

\[
(i \lor 1) \ast \Gamma + \Delta, x : (\iota_{(\lor 1)} \vdash \gamma + g) \quad \vdash \gamma \quad \Delta(x) \vdash y = t \quad \text{lin} \quad s R^o_{n+1}(w) p : \rho \quad v \quad \Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z) w : \tau \quad y \geq w \otimes z
\]

\[
(i \lor 1) \ast \Gamma + \Delta \vdash \gamma \quad \Delta(y) \vdash y = t[v/x] \quad \text{lin} \quad s[v/x] R^o(y) p[w/x]
\]

where \( \mathcal{D} \) is the following derivation

\[
\Gamma, x : j \vdash \sigma \vdash t \Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(t') : \sigma
\]

\[
\Delta, x : g \vdash \sigma \vdash s R^o_{n}(u) s' : \rho
\]

\[
(i \lor 1) \ast (\Gamma, x : j) \vdash (\iota_{(\lor 1)} \vdash \gamma + g) \quad \Delta(x) \vdash y = t' \quad \text{lin} \quad s'[v/x] R^o(z) p : \rho \quad w \geq v \otimes u \otimes z
\]

By law \( \text{Com}_2 \), from \( v \quad \Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z) w : \tau \) we obtain:

\[
\begin{align*}
\Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z_1) w : \tau \\
\Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z_2) w : \tau \\
z \geq z_1 \otimes z_2.
\end{align*}
\]

Moreover, from \( \text{I} \) we infer \( v \quad \Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z_1) w : \tau \) by law \( \text{Com}_2 \). We next apply the induction hypothesis, obtaining

\[
R^o_{n} \otimes \Delta_j(R^n) \subseteq \text{subst}; R^n; \text{subst}^\ast
\]

which in turn gives

\[
\Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z_1) w : \tau
\]

by stability (Lemma \( \text{S} \)). Finally, we use law \( \text{Mon}_1 \) and obtain

\[
\Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z_1) w : \tau
\]

\[
\Delta_{(\iota_{(\lor 1)} \vdash \gamma + g)}\gamma(v)(z_2) w : \tau
\]

\[
z \geq z_1 \otimes z_2.
\]
From the above inclusion, \( \Gamma, x; \tau \vdash^{\Delta} t \Delta_{i_1}(R^o_{n}(v)) t' : \sigma \), and \( v \Delta_{(i_1)}(\Delta_j(R^o))(z_1) w : \tau \) we infer
\[
\Gamma \vdash^{\Delta} t[v/x] \Delta_{i_1}(R^o)(v \cdot z_1) t'[w/x] : \sigma.
\]
We then apply the induction hypothesis on \( \Box \) and \( \Delta, x; \tau, y; \sigma \vdash^{\Delta} s R^o_{n}(u) s' : \rho \), hence inferring
\[
\Delta, y; \sigma \vdash^{\Delta} s[v/x] R^o(u \cdot z_2) s'[w/x] : \rho.
\]
Finally, since \( R^o \) is value substitutive \( (i \vee 1) \Gamma + \Delta, x; (i_1)_{\vee 1} + g \tau \vdash^{\Delta} y = t' \in s' R^o(z) \rho \) implies \( (i \vee 1) \Gamma + \Delta \vdash^{\Delta} y = t[v/x] \in s[v/x] R^o(z) p[u/x] : \rho \), so that we can conclude the thesis by the very definition of Howe extension of a relation as follows:
\[
(i \vee 1) \Gamma + \Delta \vdash^{\Delta} y = t'[v/x] \in s'[v/x] R^o(z) p[u/x] : \rho \quad \text{for the closed projection of}
\]
\[
(i \vee 1) \Gamma + \Delta \vdash^{\Delta} y = t[v/x] \in s[v/x] R^o(z) p[u/x] : \rho.
\]
• Suppose to be in the following case:
\[
\Gamma, x; \tau \vdash^{\Delta} u \Delta_i(R^o_{n}(u)) u' : \sigma
\]
\[
i \Gamma, x; \tau \vdash^{\Delta} u' [v] R^o(v) z : \Box \sigma \quad z \geq v \cdot u
\]
\[
i \Gamma, x; i, \sigma \vdash^{\Delta} v \Delta_{i}(R^o_{n+1}(z)) w : \tau \quad w \geq y \cdot z
\]
\[
i \Gamma, x; i, \sigma \vdash^{\Delta} u[v/x] R^o(w) z[w/x] : \Box_1 \sigma
\]
We proceed as in previous case. The main passages are summarised in the following chain of implication:
\[
IH \implies R^o_{n} \otimes \Delta_j(R^o) \subseteq \text{subst}; R^o; \text{subst}^{-}
\]
\[
\implies \Delta_i(R^o_{n} \otimes \Delta_j(R^o)) \subseteq \text{subst}; \Delta_i(R^o); \text{subst}^{-}
\] (By Lemma 5)
\[
\implies \Delta_i(R^o_{n} \otimes \Delta_i(R^o_{n})) \subseteq \text{subst}; \Delta_i(R^o_{n}); \text{subst}^{-}
\] (By law (Mon))
\[
\implies \Delta_i(R^o_{n}) \otimes \Delta_i(R^o_{n}) \subseteq \text{subst}; \Delta_i(R^o_{n}); \text{subst}^{-}
\] (By law (Com))
As a consequence, \( \Gamma, x; \tau \vdash^{\Delta} u \Delta_i(R^o_{n}(u)) u' : \sigma \) and \( v \Delta_{i_1}(R^o)(v) w : \tau \) implies
\[
\Gamma \vdash^{\Delta} u[v/x] \Delta_i(R^o_{n}(u) y) u'[w/x] : \sigma.
\]
Moreover, since \( R^o \) is value substitutive, \( i \Gamma, x: i, \tau \vdash^{\Delta} [u'] R^o(v) z : \Box \_1 \sigma \) entails
\[
i \Gamma \vdash^{\Delta} [u'[w/x]] R^o(v) z[w/x] : \Box_1 \sigma,
\]
from which the thesis follows by very definition of \( R^o \).

The remaining cases follow the same pattern of the one seen so far, or are even easier. \( \square \)

We now prove that applicative (bi)similarity is compatible and substitutive.

**Lemma 6.** If the closed projection of \( \preceq^H \) is an applicative simulation, then \( \preceq^H \) coincides with \( \preceq^o \).

**Proof.** We already know that \( \preceq^o \subseteq \preceq^H \), so that it is enough to prove the converse inclusion. First, notice that since \( (\preceq^H)^c \) is an applicative simulation, \( (\preceq^O)^c \) is contained in \( \preceq \), and thus \( (\preceq^H)^c \) is contained in \( (\preceq^o)^c \). We are done since \( \preceq^H \subseteq ((\preceq^H)^c)^o \).

Therefore, since \( \preceq^H \) is compatible and substitutive, it is enough to show that the closed projection of \( \preceq^H \) is an applicative simulation. First, let us observe that the value clauses of Definition 10 are satisfied \( (\preceq^H)^c \). In the remaining part of this section, to improve readability we will write \( R^o_{n} \) for the closed projection of the Howe extension of \( R \).
Lemma 7. Let \( R \) be a reflexive and transitive applicative simulation. Then, \((R^H)^c\) satisfies clauses \((\text{App abs}), \text{App box})\), and \((\text{App fold})\).

Proof. The proof is straightforward, and thus we just show the case of clause \((\text{App box})\) as an illustrative example. Notice that since we deal with closed relations, if \( vR^H_0(w): \tau \), then \( v \) and \( w \) have the same syntactic structure, which is determined by \( \tau \) (for instance, if \( \tau = \square_j \sigma \), then \( v \) and \( w \) must be two boxed values). So suppose to have the following derivation:

\[
\quad v \bigtriangleup_j ((R^H_0)^\forall(v)) w : \tau \quad [w]R^\forall(u)[u] : \square_j \tau \quad w \geq v \cdot u
\]

Since \( R \) is an applicative simulation, \([w]R^\forall(u)[u] : \square_j \tau \) implies \( w \bigtriangleup_j (R)^\forall(u) u : \tau \), which, together with \( v \bigtriangleup_j ((R^H_0)^\forall(v)) w : \tau \) and \( w \geq v \cdot u \) gives \( v \bigtriangleup_j (R); \bigtriangleup_j ((R^H_0)^\forall(v)) \) \( u : \tau \) and thus the desired thesis by quasi-transitivity.

It thus remains to prove that \( R^H \) satisfies clause \((\text{App eval})\), where \( R \) is a reflexive and transitive applicative simulation (we write \( R^H \) for the closed restriction of the Howe extension of \( R \)). That essentially amounts to show the inclusion

\[
R^H \subseteq [\square]; R^H_0; [-]^\perp.
\]

We will do that by case analysis on \( R^H \). However, since \([\square]\) is defined as \( \bigsqcup_{n \geq 0} [\square]^n \), we would also like to reason inductively in terms of \([\square]^n \). We can do so by observing that \((-)_{\perp} \) supports the following reasoning principles:

\[
\perp R_{\perp}(w) y \quad (\forall n \geq 0. \; x_n R_{\perp}(w) y) \implies \bigsqcup_{n \geq 0} x_n R_{\perp}(w) y.
\]

As a consequence, to prove that \( R^H \) satisfies clause \((\text{App eval})\) it is enough to show the following statement:

\[
\forall n \geq 0. \; R^H \subseteq [\square]^n; R^H_0; [-]^\perp.
\]

We proceed by induction on \( n \). The case for \( n = 0 \) directly follows from law \((\text{Induction 1})\). For the inductive step, we proceed by cases on the definition of \( R^H \). Most cases are standard, but we encounter a further difficulty in the case of sequencing. Suppose to have:

\[
\begin{align*}
\text{let } x = t \in s & \lim_{n+1} R^H_\perp(w)[p] : \sigma \\
\text{let } x = t \in s & \lim_{n+1} R^H_\perp(w)[p] : \sigma
\end{align*}
\]

We have to prove \([\text{let } x = t \in s]_{n+1} R^H_\perp(w)[p] : \sigma \). Since

\[
[\text{let } x = t \in s]_{n+1} = (v \mapsto [s[v/x]]_n)^{\dagger}([t]_n)
\]

we already see that we may want to rely one lax \((\text{lax monad bind})\). But let us proceed by step by step. First, by induction hypothesis we have \( R^H \subseteq [\square]^n; R^H_0; [-]^\perp \) and thus, by stability (Lemma 5),

\[
\bigtriangleup_j \forall_1(R^H) \subseteq [\square]^n; \bigtriangleup_j \forall_1(R^H_0); [-]^\perp.
\]

As a consequence, from \( t \bigtriangleup_j \forall_1(R^H)^\forall(v) t' : \tau \) we infer \([t]_n \bigtriangleup_j \forall_1(R^H_0)^\forall(v) [t'] \). Let us now move \( x : \tau \vdash s; \bigtriangleup_j \forall_1(R^H_u) s' : \sigma \). Let write \( \hat{s}, \hat{s}' \) for the maps mapping a closed value \( v \) of type \( \tau \) to \( s[v/x] \) and \( s'[v/x] \),
We have thus proved the following result.

**Lemma 3 (Key Lemma).** Assume the law \( \Delta_{\vee 1}(R_{\perp}) \subseteq (\Delta_{\vee 1}(R))_{\perp} \), for any \( R \). Then, for any reflexive and transitive applicative simulation \( R \), \( R^H \) (restricted to closed terms) is an applicative simulation.

It then follows that applicative similarity is a preorder and that applicative bisimilarity is a congruence.

\(^{15}\)

Actually, we would conclude \([\text{let } x = t \text{ in } \sigma]_{\perp, +1} \) \( R^{H}(v \cdot u) \) \( \text{let } x = t' \text{ in } s' \) from which we can then infer the thesis, since we have \( \text{let } x = t' \text{ in } s' R(z) p : \sigma \) (which, \( R \) being an applicative simulation, entails \( [\text{let } x = t' \text{ in } s'] R_{\perp} (z) [p] \)) and by quasi-transitivity \( R_{\perp} ; R^{H}_{\perp} \subseteq R^{H}_{\perp} \).
APPENDIX D
ON EFFECTFUL EXTENSIONS

We informally sketch how to extend the main results in this paper to an effectful setting. We follow the same structure given by Gavazzo [39], to which we refer for details.

The ingredients are:
1) A collection \( \Sigma \) of effect-triggering operation symbols (such as probabilistic choice operations or primitives for input-output).
2) A monad \( \mathbb{T} \) to interpret such operations (such as the distribution monad). Formally, that means to any \( n \)-ary operation symbol in \( \Sigma \) is associated an \( n \)-ary algebraic operation on \( \mathbb{T} \) [71]–[73].
3) A lax extension \( \Gamma \) of \( \mathbb{T} \) to \( \mathbb{V} \)-Mat.

First, we extend \( \Lambda_J \) with operations in \( \Sigma \): for any \( n \)-ary operation \( \text{op} \in \Sigma \), we stipulate that \( \text{op}(t_1, \ldots, t_n) \) is a term of the calculus, whenever \( t_1, \ldots, t_n \) are. The operational semantics of the calculus thus obtained, which we denote by \( \Lambda_J, \Sigma \), is defined as for \( \Lambda_J \): there, in fact, we rely on the monad structure of the partiality monad, rather than on partiality itself. However, we also rely on the domain structure of \( X \bot \) and some strictness and continuity properties of Kleisli extension: we require such properties to hold for \( \mathbb{T} \) too, this way restricting our analysis to continuous monads [31], [83]. A monad \( \mathbb{T} = (T, \eta, \varepsilon) \) is continuous if \( TX \) is an \( \omega \)-cppo for any set \( X \), and satisfy the laws

\[
f^I(\perp) = \perp; \quad f^I\left( \bigsqcup_{n \geq 0} x_n \right) = \bigsqcup_{n \geq 0} f^I(x_n); \quad \left( \bigsqcup_{n \geq 0} f^I \right)(x) = \bigsqcup_{n \geq 0} f^I(x).
\]

Next, we move to the definition of applicative similarity distance, which is defined as in Definition 14 except for the clause on terms which now involves the lax extension \( \Gamma \) of \( \mathbb{T} \):

\[
\delta^\Lambda(t, s) \leq \Gamma(\delta^\mathbb{V})([[t]], [[s]]).
\]

To prove compatibility of applicative similarity distance, we also need the lax extension \( \Gamma \) of \( \mathbb{T} \) to satisfy the properties we relied on in our proof of the key lemma (Lemma 3), namely distributivity of comonadic lax extensions over \( \Gamma \), and a suitable induction principle based on continuity of \( \mathbb{T} \). We summarise these properties as follows:

1) \( \Gamma \) must be a lax extension of \( \mathbb{T} \) (on \( \mathbb{V} \)-Mat).
2) \( \Gamma \) must be inductive, meaning that the following laws hold:

\[
k \leq \Gamma(\alpha)(\perp, \chi) \quad \bigwedge_n \Gamma(\alpha)(\chi_n, \phi) \leq \Gamma(\alpha)(\bigsqcup_n \chi_n, \phi)
\]

3) \( \Delta \) must distributes over \( \Gamma \):

\[
\Delta_J(\Gamma(\alpha)) \subseteq \Gamma(\Delta_J(\alpha)).
\]

If these conditions are satisfied, then applicative similarity distance is substitutive and compatible, and the abstract metric preservation theorem holds.

We conclude this overview by mentioning a couple of concrete examples of effectful extensions of \( \Lambda_J \) (see [39] for further examples).

**Example 7.**

1) The discrete subdistribution monad \( \mathcal{D} \) is continuous and probabilistic choice operations are algebraic on it. A lax extension of \( \mathcal{D} \) to \( \mathbb{L} \)-Mat is given by the so-called Wasserstein-Kantorovich metric lifting [85]. The duality theorem for countable transportation problems [86] stating that the Wasserstein and Kantorovich metric coincide gives us two equivalent formulations of such a lifting, which allow one to prove that we indeed have a lax extension of \( \mathcal{D} \). Such extension satisfies the required induction

\[\text{Notice that continuous monads is just an } \omega \text{-cppo enriched monad [84].}\]
principle [87] and comonadic lax extension given by multiplication by a constant in $[0, \infty)$ distributes over it. The resulting notion of (probabilistic) applicative similarity distance is thus compatible and substitutive.

2) The powerset monad $\mathcal{P}$ is countable and set-theoretic union is algebraic on it. A lax extension of $\mathcal{P}$ is given the so-called Hausdorff lifting [88]. The Hausdorff lifting has been extended to arbitrary quantales [89] (and thus to any possible worlds structure), so that it is always possible to inject pure nondeterminism in modal calculi.

3) More generally, canonical lax extensions of monads $T$ to $\mathcal{V}$-$\text{Mat}$ can be obtained by algebra maps $\xi : T(\mathcal{V}) \to \mathcal{V}$ respecting the quantale structure of $\mathcal{V}$ [90].