GENERALIZED INJECTIVITY OF BANACH MODULES

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Abstract. In this paper, we study the notion of φ-injectivity in the special case that φ = 0. For an arbitrary locally compact group G, we characterize the 0-injectivity of $L^1(G)$ as a left $L^1(G)$ module. Also, we show that $L^1(G)^{**}$ and $L^p(G)$ for $1 < p < \infty$ are 0-injective Banach $L^1(G)$ modules.

1. Introduction

The homological properties of Banach modules such as injectivity, projectivity, and flatness were first introduced and investigated by Helemskii; see [5, 6]. White in [11] gave a quantitative version of these concepts, i.e., he introduced the concepts of C-injective, C-projective, and C-flat Banach modules for a positive real number C. Recently Nasr-Isfahani and Soltani Renani introduced a version of these homological concepts based on a character of a Banach algebra A and they showed that every injective (projective, flat) Banach module is a character injective (character projective, character flat respectively) module but that the converse is not valid in general. With the use of these new homological concepts, they gave a new characterization of φ-amenability of a Banach algebra A such that $\phi \in \Delta(A)$ and a necessary condition for φ-contractibility of A; see [8].

2. Preliminaries

Let A be a Banach algebra and $\Delta(A)$ denote the character space of A, i.e., the space of all non-zero homomorphisms from A onto C. We denote by A-mod and mod-A the category of all Banach left A-modules and all Banach right A-modules, respectively. In the case that A has an identity we denote by A-unmod the category of all Banach left unital modules. For $E, F \in$ A-mod, let $A_B(E, F)$ be the space of all bounded linear left A-module morphisms from $E$ into $F$. 

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For each Banach space $E$, $B(A,E)$; the Banach space consisting of all bounded linear operator from $A$ into $E$, is in $\textbf{A-mod}$ with the following module action:

$$(a \cdot T)(b) = T(ba) \quad (T \in B(A,E), a, b \in A).$$

**Definition 2.1.** Let $A$ be a Banach algebra and $J \in \textbf{A-mod}$. We say that $J$ is injective if for each $F, E \in \textbf{A-mod}$ and admissible monomorphism $T : F \to E$ the induced map $T_J : \_\_AB(E,J) \to \_\_AB(F,J)$ defined by $T_J(R) = R \circ T$ is onto.

Suppose that $\phi \in \Delta(A)$. For $E \in \textbf{A-mod}$, put

$I(\phi, E) = \text{span}\{a \cdot \xi - \phi(\xi)a : a \in A, \xi \in E\}$,

$\phi B(A^2, E) = \{T \in B(A^2, E) : T(ab-\phi(b)a) = a \cdot T(b-\phi(b)e^2), (a, b \in A)\}$. It is clear that $I(\phi, E) = \{0\}$ if and only if the module action of $E$ is given by $a \cdot x = \phi(a)x$ for all $a \in A$ and $x \in E$.

Obviously, $\phi B(A^2, E)$ is a Banach subspace of $B(A^2, E)$. On the other hand, for each $b \in \ker(\phi)$, if $T \in \phi B(A^2, E)$, then $T(ab) = a \cdot T(b)$ for all $a \in A$. Therefore, we conclude that $\phi B(A^2, E)$ is a Banach left $A$-submodule of $B(A^2, E)$.

Note that if $E, F \in \textbf{A-mod}$ and $\rho : E \to F$ is a left $A$-module homomorphism, we can extend the module actions of $E$ and $F$ from $A$ into $A^2$ and $\rho$ to a left $A^2$-module homomorphism in the following way:

$$(a, \lambda) \cdot e = a \cdot e + \lambda e \quad (a \in A, \lambda \in \mathbb{C}, e \in E)$$

$$(a, \lambda) \cdot f = a \cdot f + \lambda f \quad (a \in A, \lambda \in \mathbb{C}, f \in F).$$

So, $\rho((a, \lambda) \cdot e) = a \cdot \rho(e) + \lambda \rho(e) = (a, \lambda) \cdot \rho(e)$.

For Banach spaces $E$ and $F$, $T \in B(E,F)$ is admissible if and only if there exists $S \in B(F,E)$ such that $T \circ S \circ T = T$.

The following definition of a $\phi$-injective Banach module, was introduced by Nasr-Isfahani and Soltani Renani in [8].

**Definition 2.2.** Let $A$ be a Banach algebra, $\phi \in \Delta(A)$ and $J \in \textbf{A-mod}$. We say that $J$ is $\phi$-injective if for each $F, E \in \textbf{A-mod}$ and admissible monomorphism $T : F \to E$ with $I(\phi, E) \subseteq \text{Im}T$, the induced map $T_J$ is onto.

By Definitions 2.1 and 2.2, one can easily check that each injective module is $\phi$-injective, although by [8, Example 2.5], the converse is not valid. In [4], the authors with the use of the semigroup algebras, gave two good examples of $\phi$-injective Banach modules which are not injective.

Let $E, F$ be in $\textbf{A-mod}$. An operator $T \in \_\_AB(E,F)$ is called a retraction if there exists an $S \in \_\_AB(F,E)$ such that $T \circ S = \text{Id}_F$. In this case
$F$ is called a retract of $E$. Also, an operator $T \in _A B(E, F)$ is called a coretraction if there exists an $S \in _A B(F, E)$ such that $S \circ T = Id_E$.

For $E \in \text{A-mod}$, let $\phi \Pi^2 : E \to _A B(A^\sharp, E)$ be defined by $\phi \Pi^2(x)(a) = a \cdot x$ for all $a \in A^\sharp$ and $x \in E$.

**Theorem 2.3.** [8, Theorem 2.4] Let $A$ be a Banach algebra and $\phi \in \Delta(A)$.

For $J \in \text{A-mod}$ the following statements are equivalent.

1. $J$ is $\phi$-injective.
2. $\phi \Pi^2 \in _A B(J, _A B(A^\sharp, J))$ is a coretraction.

### 3. 0-injectivity of Banach modules

In this section, we give the definition of a 0-injective Banach left $A$-module and show that this class of Banach modules are strictly larger than the class of injective Banach modules.

For each $E \in \text{A-mod}$ define

$$0_B(A^\sharp, E) = \{ T \in B(A^\sharp, E) : T(ab) = a \cdot T(b) \text{ for all } a, b \in A \}.$$ 

Clearly, $0_B(A^\sharp, E)$ is a Banach left $A$-submodule of $B(A^\sharp, E)$. It is well-known that $E^\ast$ is in mod-$A$ with the following module action:

$$(f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in E, f \in E^\ast).$$

**Definition 3.1.** Let $A$ be a Banach algebra and $E \in \text{A-mod}$. We say that $E$ is a (left) 0-injective if for each $F, K \in \text{A-mod}$ and admissible monomorphism $T : F \to K$ for which $A \cdot K = \text{span}\{ a \cdot k : a \in A, k \in K \} \subseteq \text{Im}T$, the induced map $T_J$ is onto.

Similarly, one can define the concept of (right) 0-injective $A$-module. We say that $E \in \text{A-mod}$ is 0-flat if $E^\ast \in \text{mod-A}$ is (right) 0-injective.

Clearly, each injective module is 0-injective.

We use the following characterization of 0-injectivity in the sequel without giving a reference.

**Proposition 3.2.** Let $A$ be a Banach algebra and $E \in \text{A-mod}$. Then $E$ is 0-injective if and only if $0_B \Pi^2$ is a coretraction.

**Proof.** Suppose $E \in \text{A-mod}$ is 0-injective. Take $F = E$, $K = 0_B(A^\sharp, E)$ and $T = 0_B$. Then $A \cdot K \subseteq \text{Im}(0_B)$ and $a \cdot T = 0_B(T(a))$ for each $a \in A$ and $T \in K$. Hence, for the identity map $I_E \in _A B(F, E) = _A B(E, E)$, there exists $\rho \in _A B(K, E) = _A B(B(A^\sharp, E), E)$ such that $\rho \circ 0_B = \rho \circ T = I_E$.

Conversely, let $0_B \rho : 0_B(A^\sharp, E) \to E$ be a left $A$-module morphism and a left inverse for the canonical morphism $0_B$. Suppose that $F, K \in \text{A-mod}$ and $T : F \to K$ is an admissible monomorphism such that $A \cdot K \subseteq \text{Im}T$. Let $W \in _A B(F, E)$ and define the map $R : K \to 0_B(A^\sharp, E)$ by

$$R(k)(a) = W \circ T'(a \cdot k) \quad (k \in K, a \in A^\sharp),$$
where \( T' \in B(K, F) \) satisfies \( T \circ T' \circ T = T \). We show that \( R \) is well defined, i.e., \( R(k) \in \omega B(A^\sharp, E) \) for each \( k \in K \). So, we will show that \( R(k)(ab) = a \cdot R(k)(b) \) for each \( a, b \in A \). By assumption \( A \cdot K \subseteq \text{Im}T \) and so there exist \( f \in F \) such that \( b \cdot k = T(f) \). Therefore

\[
a \cdot R(k)(b) = a \cdot W \circ T'(b \cdot k) = a \cdot W \circ T'(T(f)) = a \cdot W(f) = W(a \cdot f) = W \circ T'(T(a \cdot f)) = W \circ T'(ab \cdot k) = R(k)(ab).
\]

Moreover, for each \( b \in A^\sharp \) we have

\[
R(a \cdot k)(b) = W \circ T'(b \cdot (a \cdot k)) = W \circ T'(ba \cdot k) = R(k)(ba) = (a \cdot R(K))(b).
\]

It follows that \( R(a \cdot k) = a \cdot R(k) \). Now, take \( S = \omega \rho \circ R \in \omega B(K, E) \). Since \( R \circ T = \omega \Pi \circ W \), we conclude that \( S \circ T = W \), which completes the proof. \( \square \)

Now, we give a sufficient condition for 0-injectivity which provide us a large class of Banach algebras \( A \) such that they are 0-injective in \( A\text{-mod} \).

Recall that by [10, Corollary 2.2.8(i)], if \( A \in A\text{-mod} \) is injective, then \( A \) has a right identity. Moreover, the converse is not valid in general even in the case that \( A \) has an identity; see Example 3.4.

**Proposition 3.3.** Let \( A \) be a Banach algebra. If \( A \) has an identity, then \( A \in A\text{-mod} \) is 0-injective.

**Proof.** Let \( e \) be the identity of \( A \). Define \( \rho : \omega B(A^\sharp, A) \to A \) by \( \rho(T) = T(e) \) for all \( T \in \omega B(A^\sharp, A) \). It is obvious that \( \rho \) is a left inverse for \( \omega \Pi^\sharp \), because for each \( a \in A \), we have

\[
\rho \circ \omega \Pi^\sharp(a) = (\omega \Pi^\sharp(a))(e) = ea = a.
\]

Also, \( \rho \) is a left \( A \)-module morphism, because for each \( a \in A \) and \( T \in \omega B(A^\sharp, A) \) we have

\[
\rho(a \cdot T) = (a \cdot T)(e) = T(ea) = T(a)
\]

\[
a \cdot \rho(T) = a \cdot T(e) = T(ae) = T(a).
\]

Therefore, \( A \in A\text{-mod} \) is 0-injective. \( \square \)

For each locally compact group \( G \), let \( M(G) \) be the Banach algebra consisting of all complex regular Borel measure of \( G \) and let \( L^\infty(G) \) be the space of all measurable complex-valued functions on \( G \) which are essentially bounded; see [1] for more details.
The group $G$ is said to be amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$, where $L_x f(y) = f(x^{-1}y)$.

Regarding the last proposition we give the following example which shows the difference between 0-injectivity and injectivity.

**Example 3.4.** Let $G$ be a non-amenable locally compact group. Then by [10, Theorem 3.1.2], $M(G) \in M(G)$-mod is not injective, but it is 0-injective.

By [6, Proposition VII.1.35], if $E \in A$-unmod, each retract of $E$ is injective. For 0-injective Banach modules we have the following proposition.

**Proposition 3.5.** Let $A$ be a Banach algebra and let $E \in A$-mod be 0-injective. Then each retract of $E$ is also 0-injective.

**Proof.** Let $F \in A$-mod be a retract of $E$. Also, let $T \in \overset{\bigcirc}{A}B(E,F)$ and $S \in \overset{\bigcirc}{A}B(F,E)$ be such that $T \circ S = I_F$.

Since $E \in A$-mod is 0-injective, there exists $E\rho^\sharp \in \overset{\bigcirc}{A}B(\overset{\bigcirc}{A}B(A^\sharp, E), E)$ for which $E\rho^\sharp \circ E\Pi^\sharp(x) = x$ for all $x \in E$.

Now, define the map $F\rho^\sharp : \overset{\bigcirc}{A}B(A^\sharp, F) \to F$ by $$F\rho^\sharp(W) = T \circ E\rho^\sharp(S \circ W) \quad (W \in \overset{\bigcirc}{A}B(A^\sharp, F)).$$ It is straightforward to check that $F\rho^\sharp$ is a left $A$-module morphism. On the other hand, for each $y \in F$ we have

$$F\rho^\sharp \circ F\Pi^\sharp(y) = F\rho^\sharp(F\Pi^\sharp(y))$$
$$= T \circ E\rho^\sharp(S \circ F\Pi^\sharp(y))$$
$$= T \circ E\rho^\sharp(E\Pi^\sharp(S(y)))$$
$$= T \circ S(y) = y.$$

Therefore, $F \in A$-mod is 0-injective. \hfill \Box

Now, we try to characterize 0-injectivity of $L^1(G)$ in $L^1(G)$-mod. First we give the following lemma.

**Lemma 3.6.** Let $A$ be a Banach algebra and $E \in A$-mod. If $E$ is 0-injective, then

$$\overset{\bigcirc}{A}B(A^\sharp, E) = \{T : T = R_x \text{ on } A \text{ for some } x \in E\},$$

where $R_x a = a \cdot x$ for all $a \in A$.

**Proof.** Let $E \in A$-mod be 0-injective. So, there exists $0\rho^\sharp \in \overset{\bigcirc}{A}B(\overset{\bigcirc}{A}B(A^\sharp, E), E)$ with $0\rho^\sharp \circ 0\Pi^\sharp(x) = x$ for all $x \in E$.

Let $T$ be an element of $\overset{\bigcirc}{A}B(A^\sharp, E)$. Hence

$$T(b) = 0\rho^\sharp \circ 0\Pi^\sharp(T(b)) = 0\rho^\sharp(0\Pi^\sharp(T(b))) = 0\rho^\sharp(b \cdot T) = b \cdot 0\rho^\sharp(T).$$
Take $x_0 = 0 \rho^*(T)$. So, $T = R_{x_0}$ on $A$ and this completes the proof. \hfill \Box

Recall that $E \in \mathbf{A}\text{-}mod$ is faithful in $A$, if for each $x \in E$, the relation $a \cdot x = 0$ for all $a \in A$, implies $x = 0$.

**Theorem 3.7.** Let $G$ be a locally compact group. Then $L^1(G) \in \mathbf{L}^1(G)\text{-}mod$ is 0-injective if and only if $G$ is discrete.

**Proof.** Let $G$ be a discrete group. Then $L^1(G)$ is unital and so the result follows from Proposition 3.3.

Conversely, let $G$ be non-discrete. So, $L^1(G) \neq M(G)$. Suppose that $\mu \in M(G) \setminus L^1(G)$. Since $L^1(G)$ is an ideal of $M(G)$, the operator $T_\mu$ defined by

$$T_\mu((f, \lambda)) = f \cdot \mu \quad ((f, \lambda) \in L^1(G)^\sharp),$$

is in $\rho(B(L^1(G)^\sharp, L^1(G)))$, but it is not of the form $R_x$ for some $x \in L^1(G)$, because $M(G)$ is faithful in $L^1(G)$. Therefore, by Lemma 3.6, $L^1(G)$ in $\mathbf{L}^1(G)\text{-}mod$ is not 0-injective. \hfill \Box

Recall that a Banach algebra $A$ is left 0-amenable if for every Banach $A$-bimodule $X$ with $a \cdot x = 0$ for all $a \in A$ and $x \in X$, every continuous derivation $D : A \to X^*$ is inner, or equivalently, $H^1(A, X^*) = 0$ where $H^1(A, X^*)$ denotes the first cohomology group of $A$ with coefficients in $X^*$; see [7] for more details.

Now, we investigate the relation between 0-injectivity and 0-amenability.

Let $E, F \in \mathbf{A}\text{-}mod$. Suppose that $Z^1(A \times E, F)$ denotes the Banach space of all continuous bilinear maps $B : A \times E \to F$ satisfying

$$a \cdot B(b, \xi) - B(ab, \xi) + B(a, b \cdot \xi) = 0 \quad (a, b \in A, \xi \in E).$$

Define $\delta_0 : B(E, F) \to Z^1(A \times E, F)$ by $(\delta_0 T)(a, \xi) = a \cdot T(\xi) - T(a \cdot \xi)$ for all $a \in A$ and $\xi \in E$. Then we have

$$\text{Ext}^1_A(E, F) = Z^1(A \times E, F)/\text{Im} \delta_0.$$

By [5, Proposition VII.3.19], we know that $\text{Ext}^1_A(E, F)$ is topologically isomorphic to $H^1(A, B(E, F))$ where $B(E, F)$ is a Banach $A$-bimodule with the following module actions:

$$(a \cdot T)(\xi) = a \cdot T(\xi), \quad (T \cdot a)(\xi) = T(a \cdot \xi) \quad (a \in A, \xi \in E, T \in B(E, F)).$$

To see further details about $\text{Ext}^1_A(E, F)$; see [6].

**Lemma 3.8.** Let $E \in \mathbf{A}\text{-}mod$. If $\text{Ext}^1_A(F, E) = \{0\}$ for all $F \in \mathbf{A}\text{-}mod$ with $A \cdot F = 0$, then $E \in \mathbf{A}\text{-}mod$ is 0-injective.

**Proof.** To show this, let $K, W \in \mathbf{A}\text{-}mod$ and $T : K \to W$ be an admissible monomorphism with $A \cdot W \subseteq \text{Im}T$. We claim that the induced map $T_E$ is onto.
We know that the short complex $0 \to K \xrightarrow{T} W \xrightarrow{q} W_{\operatorname{Im}T} \to 0$ is admissible where $q$ is the quotient map. But for all $a \in A$ and $x \in W$, $a \cdot (x + \operatorname{Im}T) = \operatorname{Im}T$, because $A \cdot W \subseteq \operatorname{Im}T$. Therefore, by assumption $\operatorname{Ext}^1_A(W_{\operatorname{Im}T}, E) = \{0\}$.

Now, by [6, III Theorem 4.4], the complex

$$0 \to A \otimes_B A(W_{\operatorname{Im}T}, E) \xrightarrow{T_E} A(W, E) \xrightarrow{T_E} A(K, E) \to \operatorname{Ext}^1_A(W_{\operatorname{Im}T}, E) \to \cdots,$$

is exact. Therefore, $T_E$ is onto.

Recall that if $E, F$ be two Banach spaces and $E \hat{\otimes} F$ denotes the projective tensor product space, then $(E \hat{\otimes} F)^*$ is isomorphic to $B(E, F^*)$ as two Banach spaces with the pairing

$$<Tx, y> = T(x \otimes y) \quad (x \in E, y \in F, T \in (E \hat{\otimes} F)^*).$$

Also, note that $E \hat{\otimes} F$ is isometrically isomorphic to $F \hat{\otimes} E$ as two Banach spaces.

**Theorem 3.9.** Let $A$ be a Banach algebra. Then $A$ is left 0-amenable if and only if each $J \in \text{mod-A}$ is 0-flat.

**Proof.** Suppose that $A$ is left 0-amenable. We show that $\operatorname{Ext}^1_A(E, J^*) = \{0\}$ for all $E \in \text{A-mod}$ with $A \cdot E = 0$. We have

$$\operatorname{Ext}^1_A(E, J^*) = H^1(A, B(E, J^*)) = H^1(A, (E \hat{\otimes} J)^*) = \{0\},$$

because $E \hat{\otimes} J \in \text{mod-A}$ has the module action, $a \cdot z = 0$ for all $z \in E \hat{\otimes} J$.

Therefore, by Lemma 3.8, $J^* \in \text{A-mod}$ is 0-injective.

Conversely, let $J \in \text{mod-A}$ be 0-flat. So, for Banach right $A$-module $C$ with module action $\lambda \cdot a = 0$ for all $a \in A$ and $\lambda \in C$ we have

$$H^1(A, J^*) = H^1(A, B(J, C)) = H^1(A, B(J, C^*))$$

$$= H^1(A, (J \hat{\otimes} C)^*)$$

$$= H^1(A, (C \hat{\otimes} J)^*)$$

$$= H^1(A, B(C, J^*))$$

$$= \operatorname{Ext}^1_A(C, J^*)$$

$$= 0.$$

Hence, if we take $J$ a left $A$ module with module action $a \cdot x = 0$ for all $a \in A$ and $x \in J$, then the above relation implies that $A$ is 0-amenable.

By [2, Corollary 4.7], we know that $L^1(G)^{**} \in L^1(G)-\text{mod}$ is injective if and only if $G$ is an amenable group. Also, if $1 < p < \infty$ by [3, Theorem 9.6], $L^p(G) \in L^1(G)-\text{mod}$ is injective if and only if $G$ is an amenable group.
Corollary 3.10. Let $G$ be a locally compact group, $1 < p < \infty$ and $E \in L^1(G)\text{-mod}$ be $L^p(G)$ or $L^1(G)^{**}$. Then $E \in L^1(G)\text{-mod}$ is 0-injective.

Proof. Since $L^1(G)$ has a bounded approximate identity by [7, Proposition 3.4 (i)], we know that $L^1(G)$ is 0-amenable. So, by Theorem 3.9 we conclude the result. The second part follows similarly, because for each $1 < p < \infty$ we know that $L^q(G)^* = L^p(G)$ where $q$ satisfies the relation $q^{-1} + p^{-1} = 1$. □

Remark 3.11. In general, by [7, Proposition 3.4 (i)], if $A$ is a Banach algebra with a bounded approximate identity, then each $E \in \text{mod-A}$ is 0-flat.

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