Local induction approximation in the theory of superfluid turbulence. Numerical consideration

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Abstract

The local induction approximation (LIA) of the Biot-Savart law is often used for numerical and analytical investigations of vortex dynamics in the theory of superfluid turbulence. In this paper, using numerical simulation, some features of the LIA is considered. The temporal evolution of vortex loop spectrum is studied numerically.

PACS: 67.40.Vs, 47.32.Cc, 07.05.Tp

1 Introduction.

It is well known that chaotic vortex structures, appearing in volume of superfluid helium under the particular conditions influence on hydrodynamic and thermodynamic properties of HeII. At this time, any explicit macroscopic equations to consider this influence are absent. In hydrodynamic approaches usually the Vinen equation is used. The system of hydrodynamic equations for superfluid turbulence, where the Vinen equation is incorporated, was obtained by some authors. (The details can be found for example in [1][2].) In turn, the Vinen equation is a phenomenological relation and it’s validity is, in principle, unknown. From this point of view, for an adequate description of various phenomena in HeII, it is desirable to consider exact microscopic equations of vortex dynamics, based on the Biot-Savart law. But any analytical progress in this approach is rather difficult (practically zero at this time). Numerical investigations also have some problems. The Biot-Savart law implies the nonlocal interaction and therefore large computer resources are demanded. Since the works [3][4] to the present moment, the local induction approximation (LIA) of the Biot-Savart law is often used. But the validity of the LIA to describe vortex dynamics is the object of discussions until now.

This report devoted to the numerical investigation of the LIA and present the continuation of the early analytical work [5]. In [5] was shown that the LIA can not describe a flux of amplitudes of harmonics across the spectrum in the one-dimesional Fourier-representation. This result seems rather unusual, taking into account that the LIA is a nonlinear differential equation. The aim of the paper is to check this with help of numerical modeling.
2 The problem statement.

The dynamical equation of the evolution of a vortex filament has the following LIA-form [3][4]:

\[
\frac{d\vec{s}(\xi, t)}{dt} = \beta \vec{s}' \times \vec{s}'' + \nu \vec{s}''
\] (1)

Here \( \vec{s}(\xi, t) \) is a radius vector of a vortex line point labeled by the variable \( \xi \); \( \vec{s}' \) is the derivative on the parameter \( \xi \); \( t \) is time; the quantity \( \beta \) is the coefficient of the nonlinearity \( \beta = \frac{\kappa^2}{8\pi} \log \frac{R}{r_0} \), with the circulation \( \kappa \) and the cutting parameters \( R \) (the external size, i.e. the averaged radius of curvature) and \( r_0 \) (the vortex core size). The coefficient of dissipation \( \nu \) appears in the eq.(1) when external counterflow is absent [3][4]. Below one supposed that \( \xi \) is the arclength.

Amplitudes of one-dimensional Fourier harmonics are defined as:

\[
\vec{s}_k(t) \simeq \int d\xi \ s(t, \xi) e^{-ik\xi}
\] (2)

The numerical calculations was executed for two cases. Firstly, the spectral evolution of some initial configuration was investigated. At that it was supposed \( \nu = 0 \) in the eq.(1) and thus only nonlinear term was presented in the dynamical equation. The evolution of harmonic amplitudes was defined by this term only. Secondly, a random force term of the Gaussian type was added to the right hand side of eq.(1). Thus eq.(1) was modified as:

\[
\frac{d\vec{s}(\xi, t)}{dt} = \beta \vec{s}' \times \vec{s}'' + \nu \vec{s}'' + \vec{f}(t, \xi)
\] (3)

Here \( \vec{f}(t, \xi) \) is defined by the correlator which in the one-dimensional \( k \)-space looks like:

\[
< \vec{f}_k(t_1) \vec{f}_{-k}(t_2) > = \frac{D}{k^y} \delta(t_1 - t_2)
\] (4)

3 The details of the numerical procedure

The technical detail: the figures, mentioned below are available in the individual files. For example the file Fig.1.gif corresponds to Fig.1 etc.

In this chapter the full details of the numerical procedures are described. A reader, not interesting this theme, may omit this section.

A vortex line was presented as a number of points in the coordinate space \( (x, y, z) \). For example, the ring with radius 1 contains 1200-5000 points, situated on the equal distance \( (\Delta s) \) from each other. The fourth order Runge-Kutta method was used for modeling the temporal evolution. The time step was defined as \( \Delta t = (\Delta s)^2/C \). Where \( \Delta s \) is the minimal for the entire configuration space step, \( C \) is a constant. In the work [6] one maintain that this method is stable when \( C > \sqrt{2} \). The experimental testing in our case gives the result \( C \approx 500-1000 \) if the vortex line is strongly intricated. The criterion of the explicity was the conservation of the full length and the integral of the curvature square. (This values are some of the LIA invariants.) For example, this integral was equal 38222.2167961644 and it was conserved with the explicity \( \pm 0.0000001 \) during the program ran, the total length was conserved down to 12 decimal signs. Besides to test the explicity, the iterative Crank-Nicolson type numerical scheme was
developed. This scheme was similar to the first one, described in the work [7]. But the distinction was the following: the equation for \( \mathbf{s} \) was used, instead of the equation for a tangent vector \( \mathbf{l} \) as in [7]. (There method is really unusable for the required explicity.) The criterion of the iteration convergence was about \( 10^{-18} \). Let’s note that the explicity of the floating number representation in a standard computer is about 19 characters (the format ”double”). For example, the same initial configurations (it is shown at the Fig.1 on the (x,y) plane) were calculated with these methods till 192547 and 1174140 time steps, but for the approximately same physical time. One can see from Fig.2 and Fig.3 (the configurations on the \((x, y)\)-plane, corresponding to these two calculations), the results are absolutely identical. The spectral distributions are also identical.

To create a strongly intricated initial configuration, the following procedure was used. Usual ring, with radius 1 was created at first. After that, sinusoidal disturbances were added to this configuration. The disturbances were added step by step, and after each step the procedure of the rescaling of the length was applied, besides the coordinates of points were corrected to place them uniformly. In this way (step by step) some harmonics were excited.

When the randomized model was studied, the additional \( f(t, \xi) \) was incorporated into the numerical method. At that this value was modeled in the following way:

\[
 f^\alpha(t, \xi) = \sum_{n=n_{low}}^{n=n_{high}} c_\alpha^n \sin \left( \frac{2\pi n \xi}{L_t} + \phi_\alpha^n \right)
\]

Here \( \left( \frac{2\pi n}{L} \right) \) is a one-dimensional wave vector \( k_n \) with number \( n \), \( L_t \) is a total length of a vortex line (recalculated at each time step), \( \xi \) is a current distance along the line, to clarify y see the rel.(4), \( \alpha \) denotes a spatial component, \( c_\alpha^n \) is a random amplitude, defined by a random-number generator, \( \phi_\alpha^n \) is a random phase. The last two values were generated at each time step.

Besides, a random disturbance also was used to create an initial configuration. At that a bunch of harmonics was excited, and then the random force was switched off.

During the was program running, one was possible to observe the changing of various parameters and the spectral picture on the monitor. To develop the program, Borland C++Builder 5 tool was used.

4 The results, discussion and conclusions

Thus, in the previous paper [5] was shown that the LIA don’t describe a flux of harmonic amplitudes across the spectrum. The aim of the numerical experiment was to confirm this result. The first part of the experiment looks as following. One was created an initial strongly intricated configuration with some spectral structure. After that, the vortex filament evolved accordingly to the eq.(1) (under the condition \( \nu = 0 \)). The reconnecting processes was omitted from the consideration, i.e. parts of a vortex line freely passed across each other when they intersected. The changing of the spectral picture was studed. The initial spectral distribution ploted on the Fig.4 from \( n = 1 \) till \( n = 150 \) (there and below a red line correspond to an initial state, black line denotes a current distribution). A wave vector \( k_n = 2\pi n/L_t \), where \( L_t \) is the total length of the filament at the moment \( t \). In coordinate spase this configuration
plotted on the (x,y)-plane. The successive steps of the spectral evolution are shown at the Fig.5-Fig.8. Finally, the spectral distribution looks as at the Fig.9. At later time a stationary state was formed and any changes were only inside the area, marked by blue line. In the stationary state, the average size of the vortex tangle ceased to grow and only slight pulses occurred. Inside of the marked area some peaks appeared, which disappeared after a time. The general spectral distribution was not changed. The harmonic excitation in the low region (about the first harmonic) was absent just as the excitation in the high area. Thus, after the stationary state was formed, an spectrum changes was ceased. The initial spectral distribution was changed cardinally. One can seem that this point is not in agreement with the result of [5]. But the procedure, used in [5] implicitly implies just stationary state. As well known, a standard situation look as following. A wave packet spreads to infinity when a nonlinear, nondissipative equation is used.

The stability of the harmonic distribution, perhaps connected with the infinite number of the invariants, produced by the LIA. For example, total length $L = \text{const}$, $\int d\xi s^{\frac{3}{2}} = \text{const}$, etc. High harmonics can not excite because it means the growth of the curvature and, hence, of the curvature square, this is impossible. Increasing of low harmonics lead to the growth of a loop size, this is also impossible because $L$ is a constant and so the total curvature must decrease in this case. Thus some transfer across the spectrum is certainly absent if a vortex configuration achieved a stationary state.

In the second part of the numerical experiment the temporary evolution of a detached narrow harmonic bunch was studied. The harmonics in the environment of this bunch was equal to zero. The initial configuration presented a ring. Some scores of time steps this ring undergo the random disturbances of the kind: $f^\alpha(t, \xi) = \sum_{i=n_{\text{low}}}^{i=n_{\text{high}}} c_{i}^{\alpha}(2\pi i \xi L_{t}) + \phi_{i}^{\alpha}$. The amplitudes $c_{i}^{\alpha}$ and the phases $\phi_{i}^{\alpha}$ possessed random values at each time steps. (Their values was defined by the random number generator.) As the result the initial spectral configuration was obtained (Fig.10). (At the Fig.10 the stripe of the spectrum is plotted from harmonic number $n = 70$ up to number $n = 120$.) Later the external force was switched off and the evolution under the LIA nonlinearity only was investigated.

The result is following. There are no any additional harmonics exciting in the system, on condition that they equal to zero initially. This result was checked thoroughly. The boundaries of the excited area was thoroughly examined after about 800 thousands time steps. They were examined at the enlarge scale factor $10^{8}$ (in the vertical direction) relative to the amplitude of the stripe. As it shown at the Fig.11 where the boundaries of the excited harmonic bunch plotted (scale factor $10^{8}$), there are no any disturbances beyong the initial stripe. At the same time the spectral distribution inside of the perturbed area was varied. This change represented the periodical process (the period is about 20 thousands time steps). The steps of the changes is shown on the Fig.12-Fig.18. Obviously, the spectral distribution strictly return to it’s initial state. The period of the return depends on the width of an excited strip and also on current numbers of the harmonics. Note again, in a standard nonlinear task we have a wave packet is smearing and tending to it’s stationary spectral distribution. Note, however, after some millions time steps, distortion of the spectral stripe was observed. Apparently it is produced not strictly zero initial background. It was tested for the smooth ring. Obviously, in the later case only the first harmonic must
be present. But other harmonics had amplitudes of order $10^{-7}$ (relative to the amplitudes, plotted on the Fig.12 - Fig.18). Because, any numerical method has inadvertent errors, background amplitudes was conserved within about 1 percent. At the Fig.19 (scale factor $5 \times 10^9$) shown the divergence from zero for the initial (red) and current (black) backgrounds. Besides, the graphic on the Fig.11 is wider the on the Fig.12 - Fig.18 because the greater magnification in the vertical direction allows to see small amplitudes which are lost details.

Also a case when the external force not switched off was considered. The force was during all time when the program was running. At that, of course, $\nu \neq 0$ in the eq.(3). The result is the same. After the quasi-equilibrium between pumping and dissipation was formed (i.e. when the tendency for changing the total length and the total curvature vanished), any new harmonics not appeared in the system.

Thus two conclusions for sure following for the LIA.

1. There are no transfer across the spectrum when a vortex system is in the stationary state.
2. There are only initially excited harmonics remain in a vortex system. New harmonics never excite. All redistributions of harmonic amplitudes take place between excited harmonics.

And the last remark. It can be suppose that the question: "whether can LIA approximately describe the superfluid turbulence" is rather make no sense until we don’t know exactly the role of the processes of reconnections in dynamics of vortex loops. But this problem is too much difficult.

The work was carried out under support of INTAS (grant N 2001-0618) and RFBR (grant N 03-02-16179).

References

1. S. K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. 67, 37 (1995).
2. Donnelly R. J., Quantized Vortices in Helium II, Cambridge University, Cambridge, England (1991).
3. K. W. Schwarz, Phys. Rev. B 18, 245 (1978).
4. K. W. Schwarz, Phys. Rev. B 38, 2398 (1988).
5. M. V. Nedoboiko, cond-mat/0302144.
6. R.G.M. Aarts, A numerical study of quantized vortices in HeII, Tech. Univer. Eindhoven,(1993).
7. T. F. Buttke Journal of Computational Physics 76, pp. 301-326, (1988).
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