Cosmology of Galileons with non-minimal Maxwell coupling

Shahab Shahidi
School of Physics, Damghan University, Damghan, 41167-36716, Iran
E-mail: s.shahidi@du.ac.ir

Abstract: Galileon theory in curved space-time coupled non-minimally to the Maxwell field is considered. We will show that the theory admits two independent exact de Sitter solutions in the FRW background, one driven by the cosmological constant and the other by the Galileon field. The dynamical system analysis of the theory shows that these two exact solutions are stable fixed points. Also, cosmological perturbations over these solutions shows that the cosmological constant based solution is healthy at linear level but the Galileon based solution suffers from a gradient instability in the scalar sector. This proves that the cosmological constant is needed in the Galileon-Maxwell system in order to have a healthy de Sitter solution.
1 Introduction

Modifying Einstein’s general relativity has a long history. Perhaps the first modification, can be attributed to the addition of the cosmological constant to the gravitational field equation by Einstein himself [1]. From then, infinite number of modifications have come out, concentrating on both ultraviolet/infrared limits of the Einstein’s field equations [2]. Cosmology however, suffers from many problems, one of the most important is the accelerated expansion of the universe at late times. This can be explained by introducing some light degree of freedom (dof) to the Einstein’s field equations, which can be responsible for the IR modification of gravity. Many proposals have been suggested so far in the literature, including the addition of some extra field to the Einstein’s theory, which can be a scalar/vector/tensor field [3], or enriching the gravitational action itself like higher order derivative theories [4], Weyl-Cartan theories [5] or massive gravity theories [6]. Also one can assume some non-trivial matter-geometry coupling to explain the accelerated expansion of the universe.

Among all, addition of a scalar field may be the minimal modification of the theory. This adds one additional dof to the Einstein’s theory (with two dof) if the Lagrangian for the scalar field is healthy. In order for the scalar interactions to becomes healthy, the scalar field should not have more than two time derivatives at the level of equations of motion, and the interaction terms should have a form which avoid gradient/tachyonic instabilities. The scalar field theories is then divided into two major classes; those which
produce accelerated expansion from the Kinetic interactions [7], and those which do that from non-trivial potential terms [3].

One the most interesting scalar field theories for the above goal, is the so-called Galileon theory [8]. Galileons are scalar fields which has more than second order time derivatives in the action but due to the special form of the interactions, it has at most second order time derivatives in the equations of motion. This makes the theory free from Ostrogradski instability. Galileon terms has an internal symmetry under which the interaction terms remain invariant if one shift the scalars as

\[ \phi \rightarrow \phi + b_\mu x^\mu + c, \]

where \( \phi \) is the Galileon scalar and \( b_\mu \) and \( c \) are constants. Many works has been done in the literature, considering cosmological [9], balck holes [10], quantum nature [11] and some generalizations of the Galileon scalars [12]. However, one of most interesting facts about the Galileons is that they can be interpreted as a position of the 4D brane world embedded in the 5D flat space [13]. This suggests that the Galileons interaction terms can not have an arbitrary form and as a result we have a finite number of Galileon interaction terms in any dimension [8].

Upon generalizing the Galileon interactions to curved space time, one immediately find out that higher order time derivatives come back to the equations of motion [14]. This is due to the fact that in curved space time, partial derivatives do not commute. This problem can be solved by adding to the action some higher order derivative terms which compensate the higher order time derivatives in the equations of motion. However, these terms breaks the Galileon invariance [14]. The most general scalar-tensor interactions in curved space time which has the property that the equations of motion are healthy is called the Horndeski theory [15]. Among all the Horndeski terms, four terms bring more attention in the sense that any combination of these terms have a consistent self-tuning mechanism on FRW background [16]. These terms can be written as:

\[
\begin{align*}
L_{\text{john}} &= \sqrt{-g}V_{\text{john}}(\phi)G^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi, \\
L_{\text{george}} &= \sqrt{-g}V_{\text{george}}(\phi)R, \\
L_{\text{paul}} &= \sqrt{-g}V_{\text{paul}}(\phi)P^{\mu\nu\alpha\beta}\nabla_\mu \phi \nabla_\alpha \phi \nabla_\nu \nabla_\beta \phi, \\
L_{\text{ringo}} &= \sqrt{-g}V_{\text{ringo}}(\phi)\mathcal{G},
\end{align*}
\]

where \( P^{\mu\nu\alpha\beta} \) is the double dual of the Riemann tensor and \( \mathcal{G} \) is the Gauss-Bonnet invariant.

In this paper, we will investigate cosmological consequences of a scalar field theory coupled to a Maxwell field. The procedure of defining the action is that we write an Einstein-Maxwell system in the presence of the cosmological constant, and then couple the energy momentum tensor of this theory with the kinetic term of the scalar field [17]. This will construct the John term of the Horndeski theory coupled non-minimally to the Maxwell field. We will see that the theory allow us to have two different exact de Sitter solutions which we will separately investigate the cosmological implications in this paper.
2 The action

In this section we will introduce the model and construct the action. This was first done in [17]. Let us begin with a gravitational action minimally coupled to the Maxwell field

\[ S = \int d^4x \sqrt{-g} \left[ \kappa^2 R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{2.1} \]

where we have introduced the cosmological constant \( \Lambda \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the strength tensor related to the electromagnetic potential \( A_\mu \). Now variation of each term in (2.1) with respect to the metric tensor gives the Einstein’s tensor \( G_{\mu\nu} \), the metric tensor \( g_{\mu\nu} \) and the energy momentum tensor of the Maxwell field \( T_{\mu\nu} \) defined as

\[ T_{\mu\nu} = \frac{1}{2} F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{8} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}, \tag{2.2} \]

respectively. In this level we can couple a scalar field with the theory (2.1) by multiplying \( \partial_\mu \phi \partial_\nu \phi \) with the terms obtained from variation of the action (2.1). The resulting action becomes

\[ S = \int d^4x \sqrt{-g} \left[ \kappa^2 R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha \partial_\mu \phi \partial_\mu \phi + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \gamma T^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right], \tag{2.3} \]

where \( \alpha, \beta \) and \( \gamma \) are arbitrary constants. In the above action, the \( \alpha \)-term is the kinetic term of the scalar field, the \( \beta \)-term is a special case of the “John” term of the Horndeski theory [18] which is the non-minimal interaction between scalar field and geometry, and the \( \gamma \)-term is the non-minimal interaction between scalar field and the Maxwell field. Note that in order to have a canonical kinetic term for the scalar field, one should set \( \alpha = -1/2 \). However, we will keep it arbitrary since there is a non-trivial background cosmological solution for \( \alpha \neq -1/2 \).

We note that the \( \beta \)-term in the action (2.3) contains higher order derivatives. However due to the galileon invariance \( \partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu \), the field equation obtained from this term contains at most second time derivatives. Also the theory has a \( U(1) \) symmetry on the Maxwell field \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda \) with \( \lambda \) is an arbitrary function. In this sense, the field equations corresponding to the scalar field \( \phi \) and the Maxwell field \( A_\mu \) can be written as a conservation of the corresponding Noether charge. The metric field equation can be written
\begin{equation}
G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu} - 2\alpha \nabla_\mu \phi \nabla_\nu \phi + \alpha g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \\
+ \beta \left( \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R - 2 \nabla_\alpha \phi \nabla_{(\mu} \phi R_{\nu)}^\alpha - \nabla^\alpha \phi \nabla_\beta \phi R_{\mu\alpha\nu\beta} - (\nabla_\mu \phi \nabla_\alpha \phi)(\nabla_\nu \nabla_\alpha \phi) \right) \\
+ (\nabla_\mu \nabla_\nu \phi) \Box \phi + \frac{1}{2} G_{\mu\nu}(\nabla^2 \phi) - g_{\mu\nu} \left[ \frac{1}{2}(\Box \phi)^2 - \frac{1}{2}(\nabla^\alpha \nabla_\beta \phi)(\nabla_\alpha \nabla_\beta \phi) - \frac{\alpha g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi}{2} \right] \\
- \frac{1}{2} \gamma \left( F_{\mu\nu} F_{\nu\beta} \nabla^\alpha \phi \nabla_\beta \phi + 2 \nabla_{(\mu} \phi F_{\nu)\alpha} F^\beta \alpha \nabla_\beta \phi - \frac{1}{2} g_{\mu\nu} F_{\beta \alpha} \nabla^\beta \phi \nabla^\gamma \phi \\
+ \frac{1}{2} g_{\mu\nu} \nabla_\beta \phi \nabla_\gamma \phi F_{\tau\mu} F^{\tau\rho} - \frac{1}{2} F_{\mu\nu} F_\sigma \nabla^\sigma \phi \nabla_\beta \phi - \frac{1}{4} \nabla_\mu \phi \nabla_\nu \phi F_{\tau\beta} F^{\tau\beta} \right). \tag{2.4}
\end{equation}

The scalar and vector field equations can be written respectively as

\begin{equation}
\nabla_\mu \left[ \left( \alpha g^{\mu\nu} + \beta G^{\mu\nu} + \gamma T^{\mu\nu} \right) \nabla_\nu \phi \right] = 0, \tag{2.5}
\end{equation}

\begin{equation}
\nabla_\mu \left[ \left( 1 + \frac{1}{2} \gamma \nabla_\alpha \phi \nabla^\alpha \phi \right) F^{\mu\nu} + 2 \gamma F_{\sigma}^{(\mu} \nabla^{\nu)} \phi \nabla^\sigma \phi \right] = 0. \tag{2.6}
\end{equation}

As we have discussed the last two equations of motion can be written in the form \( \partial_\mu (\sqrt{-g} J^\mu) = 0 \) which is the the conservation equations related to Galileon and \( U(1) \) symmetries.

### 3 Background cosmology

Let us now consider the cosmological consequences of the model (2.4)-(2.6). Let us assume that the universe can be described by the FRW ansatz with line element

\begin{equation}
ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2), \tag{3.1}
\end{equation}

where \( a = a(t) \) is the scalar factor. In the case of isotropic and homogeneous space-time, the vector field \( A_\mu \) should have the form

\begin{equation}
A_\mu = (A_0(t), 0, 0, 0), \tag{3.2}
\end{equation}

and the scalar field can be written as \( \phi = \phi(t) \). The field equations then reduces to

\begin{equation}
-3\kappa^2 H^2 + \Lambda - \frac{1}{2} \alpha \dot{\phi}^2 + \frac{9}{2} \beta H^2 \dot{\phi}^2 = 0, \tag{3.3}
\end{equation}

\begin{equation}
-\kappa^2 (2\dot{H} + 3H^2) + \Lambda + \beta \ddot{\phi}^2 + \frac{3}{2} \beta H^2 \dot{\phi}^2 + 2 \beta H \dot{\phi} \ddot{\phi} + \frac{1}{2} \alpha \dot{\phi}^2 = 0, \tag{3.4}
\end{equation}

\begin{equation}
2 \alpha \dddot{\phi} + 6 \alpha H \dot{\phi} - 6 \beta H^2 \ddot{\phi} - 6 \beta H^3 \dot{\phi} - 12 \beta H \left( \dot{H} + H^2 \right) \dot{\phi} = 0, \tag{3.5}
\end{equation}

where \( H \) is the Hubble parameter. Note that the vector field equation of motion satisfied identically in the case of homogeneous and isotropic universe since our theory is \( U(1) \)
invariant. Also note that in the above field equations, the scalar field appears at most with two time derivatives. This is the consequence of the Galileon invariance of the $\beta$-term in the action.

The above set of equations has two Exact solutions corresponding to an accelerated expanding universe. The first one has non-vanishing cosmological constant, with

$$\phi = \phi_0, \quad H = \sqrt{\frac{\Lambda}{3\kappa^2}},$$  \hspace{1cm} (3.6)

where $\phi_0$ is an arbitrary constant. We refer to this solution as $\Lambda$-based solution. This solution is nothing but the standard dS solution of the Einstein-Hilbert theory with non-vanishing cosmological constant. This happens actually because we have assumed that the scalar field is constant and the field equation contains at least first order time derivative on the scalar field. So, the scalar field will be disappeared from the equations. Despite the fact that the background solution is the same as in the standard Einstein’s theory we will see that at the level of perturbations the physics becomes different from that of Einstein’s theory.

The theory has another dS solution with vanishing cosmological constant $\Lambda = 0$ and

$$\phi = \frac{\kappa}{\sqrt{\beta}} t, \quad H = \sqrt{\frac{\alpha}{3\beta}},$$ \hspace{1cm} (3.7)

which we will refer as the Galileon-based solution. Note that in this case, $\alpha$ and $\beta$ should be positive constants. This is actually the non-trivial solution of the Galileon-Maxwell system and the accelerated expansion comes from the Galileon field. Note that the Maxwell field does not contribute to the background solutions since as noted above, we have assumed isotropic and homogeneous universe. In order to investigate the effects of the Maxwell field, one should consider for example anisotropic space-times.

### 3.1 Dynamical system analysis

Let us write the Friedman equation (3.3) as

$$-\frac{\alpha\dot{\phi}^2}{6\kappa^2 H^2} + \frac{\kappa^2 \bar{\Lambda}}{3H^2} + \frac{3\bar{\beta}\dot{\phi}^2}{2\kappa^2} = 1,$$ \hspace{1cm} (3.8)

where we have defined dimensionless constants $\bar{\beta} = \kappa^2 \beta$ and $\bar{\Lambda} = \kappa^4 \Lambda$. From the above equation one can define two dynamical variables as

$$\Omega_\Lambda = \frac{\kappa^2}{3H^2}, \quad \Omega_\phi = \frac{\dot{\phi}^2}{\kappa^4}.$$  

Equation (3.8) shows that $\Omega_\phi$ can be obtained as a function of $\Omega_\Lambda$ so the system has only one dynamical degree of freedom. Using equations (3.4) and (3.5) one can write the autonomous equation of this degree of freedom as

$$\frac{d\Omega_\Lambda}{d\ln a} = \frac{6\Omega_\Lambda (\bar{\Lambda} \Omega_\Lambda - 1) (-4\alpha \bar{\beta} \Omega_\Lambda + 3\bar{\beta}^2 + \alpha^2 \Omega_\Lambda^2)}{\alpha \beta \Omega_\Lambda (\bar{\Lambda} \Omega_\Lambda + 3) + 3\bar{\beta}^2 (\bar{\Lambda} \Omega_\Lambda - 2) - \alpha^2 \Omega_\Lambda^2}.$$ \hspace{1cm} (3.9)
Table 1. The fixed points of the dynamical system (3.9).

![Table](https://example.com/table1.png)

Also, the effective equation of state parameter \( \omega_{eff} = -1 - 2 \dot{H}/3H^2 \) can be obtained as

\[
\omega_{eff} = \frac{2 (\Lambda \Omega - 1) (\beta - \alpha \Omega) (3 \beta - \alpha \Omega)}{\alpha \beta \Omega (\Lambda \Omega + 3) + 3 \beta^2 (\Lambda \Omega - 2) - \alpha^2 \Omega^2} - 1.
\] (3.10)

The above system has four fixed points as have been written in table (3.1). One can see from the table that there are three de Sitter fixed points associated with the dynamical system (3.9). The fixed points (2) and (4) are stable and are related to the Galileon-based and \( \Lambda \)-based solution we have obtained in the previous subsection respectively. There is however an unstable fixed point (1) associated with a matter dominated universe. In the next section we will prove that the \( \Lambda \)-based solution is healthy over linear perturbations but the Galileon-based solution suffers from instability. Since the matter dominated fixed point placed at the origin, one can deduce that for the values \( \alpha < \beta \Lambda \) the \( \Lambda \)-based de Sitter fixed point (4) is closer to the matter dominated fixed point (1) and the dynamical behavior will becomes healthy; starting from an unstable matter dominated fixed point (1) and end at a stable healthy de Sitter fixed point (4).

3.2 General solutions

Before considering the cosmological perturbations of the above exact solutions, let us solve the system (3.3)-(3.5) numerically. Defining the dimensionless parameters

\[
H = H_0 h, \quad \tau = H_0 t, \quad \phi = \kappa \psi, \quad \Lambda = \lambda \kappa^2 H_0^2, \quad \beta = H_0^2 \beta,
\] (3.11)

one can rewrite the equations of motion as

\[
h^2 \left( 9 \tilde{\beta} \psi'^2 - 6 \right) - \alpha \psi'^2 + 2 \lambda h = 0,
\]

\[
3 h \psi'' \left( 2 \tilde{\beta} h' - \alpha \right) + 3 \tilde{\beta} h^2 \psi'' + 9 \tilde{\beta} h^3 \psi' - \alpha \psi'' = 0,
\]

\[
(2 h' + 3 h^2) \left( \tilde{\beta} \psi'^2 - 2 \right) + 4 \tilde{\beta} h \psi' \psi'' + \alpha \psi'^2 + 2 \lambda = 0,
\] (3.12)

where prime denotes derivative with respect to \( \tau \). Figure (3.2) shows the behavior of the Hubble parameter, the scalar field and the deceleration parameter defined as \( q = -1 - \dot{H}/H^2 \) as a function of \( \tau \). The value of the parameter \( \alpha \) is \( \alpha = -1.1, 1.1, -1 \) for the dotted, dashed and solid lines respectively. One can see from the figures that the positive values of the parameter \( \alpha \) can produce late time acceleration but the value of the scalar field diverges at late times. One should note that the Galileon-based solution is a subset of this case. We will show in the next section that this solution has a gradient instability in the scalar sector.
Figure 1. Plot of the Hubble parameter, the scalar field and the deceleration parameter as a function of the dimensionless time parameter \( \tau \). The values of the parameters are \( \alpha = -1, 1, 1, -1 \), \( \beta = 1, 7, 2, 3, 4 \) and \( \lambda = 5, 2, 1, 5, 4, 9 \) for the dotted/dashed/solid lines respectively.

4 Perturbations

In this section we will investigate the cosmological perturbations around the background solutions introduced in section 3. The metric perturbations around FRW background can be written as

\[
\mathbf{ds}^2 = -(1 + 2\varphi)\,dt^2 + 2a(S_i + \partial_i B)dx^i\,dt + a^2\left((1 + 2\psi)\delta_{ij} + \partial_i\partial_j E + \partial_i F_{ij} + h_{ij}\right)dx^i\,dx^j,
\]

(4.1)

where \( \varphi, \psi, E \) and \( B \) are the scalar perturbations, \( S_i \) and \( F_i \) are the vector perturbations with vanishing divergence \( \partial_i S_i = 0 = \partial_i F_i \), and \( h_{ij} \) is the traceless and transverse tensor perturbation, \( h_{ii} = 0 = \partial_i h_{ij} \). Note that in our notation the spatial indices are raised and lowered by the flat-space metric \( \delta_{ij} \). The Maxwell field can be decomposed as

\[
A_\mu = (A_0 + \delta A_0, \xi_i + \partial_i \delta A),
\]

(4.2)

where \( A_0 \) is the background value of the Maxwell field. Note that due to \( U(1) \) symmetry of the action (2.3), the Maxwell field did not appear in the background field equations and \( A_0 \) remains an arbitrary function. In this section for simplicity we will assume that \( A_0 \) is a constant. In the decomposition of the Maxwell field (4.2), \( \delta Q_0 \) and \( \delta Q \) are the scalar
perturbations and $\xi_i$ is a transverse vector perturbation $\partial_i \xi_i = 0$. The Galileon field can also be decomposed as

$$\phi = \phi_0 + \delta \phi. \quad (4.3)$$

Note that $\phi_0$ is not constant in the Galileon-based solution.

Now, let us define the gauge invariant perturbation quantities. Under the infinitesimal coordinate transformation of the form $x^\mu \rightarrow x^\mu + \delta x^\mu$, the scalar perturbations transform as

$$\varphi \rightarrow \varphi - \partial_t \delta x^0, \quad B \rightarrow B + \frac{1}{a} \delta x^0 - a \partial_t \delta x, \quad \psi \rightarrow \psi - H \delta x^0, \quad E \rightarrow E - 2 \delta x,$$

$$\delta A \rightarrow \delta A - A_0 \delta x^0, \quad \delta A_0 \rightarrow \delta A_0 - A_0 \partial_t \delta x^0, \quad \delta \phi \rightarrow \delta \phi - \dot{\phi}_0 \delta x^0. \quad (4.4)$$

We can construct five gauge invariant scalar perturbations as

$$\Phi = \varphi + \partial_t \left( a B - \frac{a^2}{2} \partial_t E \right), \quad \Psi = \psi + H \left( a B - \frac{a^2}{2} \partial_t E \right),$$

$$X = \delta A_0 + A_0 \partial_t \left( a B - \frac{a^2}{2} \partial_t E \right), \quad Y = \delta A_0 \left( a B - \frac{a^2}{2} \partial_t E \right),$$

$$Z = \delta \phi + \dot{\phi}_0 \left( a B - \frac{a^2}{2} \partial_t E \right). \quad (4.5)$$

Note that for the $\Lambda$-based solution the scalar perturbation $\delta \phi$ is gauge invariant and we have $Z = \delta \phi$.

For the vector perturbation we have

$$S_i \rightarrow S_i - a \partial_t \eta_i, \quad F_i \rightarrow F_i - 2 \eta_i, \quad \xi_i \rightarrow \xi_i, \quad (4.6)$$

and we can construct two gauge invariant vector perturbations of the form

$$\rho_i = S_i - \frac{1}{2} a \partial_t F_i, \quad \xi_i \rightarrow \xi_i. \quad (4.7)$$

The tensor perturbation $h_{ij}$ does not transform under the infinitesimal coordinate transformation and so it is gauge invariant.

### 4.1 Tensor perturbation

Let us consider the tensor perturbation of the theory (2.3). The tensor perturbation $h_{ij}$ has two polarizations which we will denote by $h_x$ and $h_+$. After expanding the action up to second order in $h_{ij}$ and Fourier transforming the resulting action one obtains

$$S_{(2)}^{\text{tensor}} = \frac{1}{2} \sum_{+,\times} \int d^3k dt k^2 a^3 \left[ a_1 h_{ij} \dot{h}_{ij} - a_2 \frac{k^2}{a^2} h_{ij} \dot{h}_{ij} \right], \quad (4.8)$$

where $a_1 = 1 = a_2$ for $\Lambda$-based solution and $a_1 = 1/2$, $a_2 = 3/2$ for Galileon-based solution. In both cases the tensor perturbation is healthy. This is in fact expectable. The scalar and vector interaction terms does not contribute to the tensor perturbation in the $\Lambda$-based solution since the background values $\phi_0$ and $A_0$ are constant. However, for the Galileon-based solution where the background value of the scalar field depends on time one has a tensor contribution from the $\beta$-term in the action (2.3).
4.2 Vector perturbation

For the vector perturbation we have two gauge invariant quantities. After Fourier transformation, one can obtain the vector part of the second order perturbed action as

\[
S^{(2)}_{\text{vector}} = \int d^3k dt a_1 \left( \frac{\dot{\xi}_i^2}{a^2} - \frac{\vec{k}^2}{a^2} \xi_i^2 \right) + a_2 \kappa^2 \vec{k}^2 \rho_i^2,
\]

where \( a_1 = 1 = a_2 \) for \( \Lambda \)-based solution and \( a_1 = 1 - \kappa^2 \gamma / 2 \beta, \) \( a_2 = 1 / 2 \) for Galileon-based solution. Note that \( \rho_i \) is nondynamical with equation of motion \( \rho_i = 0 \), so the third term in (4.9) vanishes and one obtains the vector perturbation action as

\[
S^{(2)}_{\text{vector}} = \int d^3k dt a_1 a \left( \frac{\dot{\xi}_i^2}{a^2} - \frac{\vec{k}^2}{a^2} \xi_i^2 \right).
\]

One can see from the above relation that the \( \Lambda \)-based solution is always healthy. However, in order to have a stable vector perturbation in the case of Galileon-based solution, one should impose \( \gamma < 2 \beta / \kappa^2 \).

4.3 Scalar perturbation

For the scalar perturbation, there are five gauge invariant scalar quantities. In what follows we will consider the scalar perturbations over two background solutions separately.

4.3.1 \( \Lambda \)-based solution

After Fourier transformation of the second order action, one obtains

\[
S^{(2)}_{\text{scalar}} = \int d^3k dt \left[ \Phi \left( 8\sqrt{3}a^3 \kappa \sqrt{\Lambda} \dot{\Psi} + 8 a \kappa^2 k^2 \Psi \right) - 2 a \lambda' k^2 \dot{Y} - 4 a^3 \Lambda \Phi^2 + 4 a \kappa^2 k^2 \Psi^2 \\
+ a \left( k^2 \dot{Y}^2 + 2 a^2 \left( \frac{\beta \Lambda}{\kappa^2} - \alpha \right) \dot{Z}^2 - 12 a^2 \kappa^2 \dot{\Psi}^2 \right) + a \left( 2 a k^2 - \frac{2 \beta \kappa^2 \Lambda}{\kappa^2} \right) Z^2 + a k^2 X^2 \right].
\]

One can see from the above action that \( \Phi \) and \( \lambda' \) are non-dynamical with equations of motion

\[
\lambda' = \dot{Y}, \quad \Phi = \sqrt{\frac{3 \kappa^2}{\Lambda} \dot{\Psi} + \frac{\kappa^2 k^2}{\Lambda a^2} \Psi}.
\]

Substituting back the solutions (4.12) to the action (4.11) one can see that \( \dot{Y} \) vanishes from the action and also \( \Psi \) becomes non-dynamical with equation of motion \( \Psi = 0 \). At the end we have left with an action with one scalar dynamical degree of freedom

\[
S^{(2)}_{\text{scalar}} = \int d^3k dt 2 a^3 \left( \frac{\beta \Lambda}{\kappa^2} - \alpha \right) \left[ \dot{Z}^2 - \frac{\vec{k}^2}{a^2} Z^2 \right].
\]

In order to have a healthy scalar perturbation on top of the \( \Lambda \)-based solution one should have \( \beta > \alpha \kappa^2 / \Lambda \).
4.3.2 Galileon-based solution

In this subsection, we will concentrate on the scalar perturbation over the de Sitter background of the Galileon-based solution (3.7). After Fourier transforming the second order action, one obtains

\[
S^{(2)}_{\text{scalar}} = \int d^3k dt a^3 \left[ \frac{6\alpha k^2 \Phi^2}{\beta} + \frac{\chi^2 k^2 (2\beta + \gamma \kappa^2)}{2a^2 \beta} + \frac{6\kappa^2 k^2 \Psi^2}{a^2} - \frac{\chi k^2 (2\beta + \gamma \kappa^2)}{a^2 \beta} \right] \\
+ \frac{3k^2 (2\beta + \gamma \kappa^2) \dot{\psi}^2}{6a^2 \beta} - \frac{36a^2 \beta k^2 \dot{\psi}^2 + 48\sqrt{3}\sqrt{\alpha} a^2 \beta \kappa \dot{\psi} \dot{Z}}{6a^2 \beta} - \frac{8\sqrt{\alpha} kk^2 \Psi Z}{\sqrt{3}a^2} \\
+ \Phi \left( \frac{-24\sqrt{3} \alpha a^2 \sqrt{\alpha} \beta \dot{\Psi} - 48\alpha a^2 \sqrt{\beta} k \dot{Z}}{6a^2 \beta} + \frac{4\kappa^2 k^2 \Psi}{a^2} - \frac{8\sqrt{\alpha} kk^2 Z}{\sqrt{3}a^2} \right). \tag{4.14}
\]

It is evident that \(\Phi\) and \(\chi\) are non-dynamical variables with equations of motion

\[
\Phi = -\frac{\beta k^2 \Psi}{3\alpha a^2} + \frac{2\beta k^2 \dot{Z}}{3\sqrt{3}\alpha \kappa a^2} + \frac{\sqrt{\alpha} \beta \dot{\dot{Z}}}{\sqrt{3}a} + \frac{2\sqrt{\beta} \dot{Z}}{3\kappa}, \tag{4.15}
\]

and \(\chi = \dot{\psi}\). Substituting the above equations back into the action (4.14), one obtains

\[
S^{(2)}_{\text{scalar}} = \int d^3k dt \left[ \frac{8\sqrt{3} a \kappa k^2 \dot{\Psi} \dot{Z}}{3a} - \frac{8\alpha \kappa k^2 Z \dot{\Psi} - 8a^2 \kappa^2 \dot{\psi}^2 + \frac{16a^3 \sqrt{\alpha} \kappa \dot{Z}}{\sqrt{3}}} \right] \\
- \frac{8}{3} a^3 \dot{\dot{\Psi}} - \frac{2\beta k^2 k^4 \Psi^2}{3a\alpha} - \frac{8\beta k^4 \dot{Z}^2}{9a} + \frac{8\beta k^4 \Psi Z}{3\sqrt{3}a \sqrt{\alpha}} \\
+ \frac{16}{3} a \kappa^2 k^2 \dot{\psi}^2 + \frac{8}{9} a^2 k^2 \dot{Z}^2 - \frac{8a \sqrt{\alpha} \kappa k^2 \Psi Z}{\sqrt{3}} \right]. \tag{4.16}
\]

Upon transforming the perturbation variables \(\Psi\) and \(Z\) as

\[
\mathcal{N} = \Psi + \frac{\sqrt{3} Z \kappa}{\sqrt{\alpha}}, \quad \mathcal{M} = \Psi - \frac{Z \sqrt{\alpha}}{\sqrt{3} \kappa}, \tag{4.17}
\]

One can see that the variable \(\mathcal{N}\) becomes non-dynamical. After obtaining the equation of motion for \(\mathcal{N}\) and substituting back to the action (4.16), one obtains

\[
S^{(2)}_{\text{scalar}} = \int d^3k dt 8\kappa^2 a^3 \left( 3\mathcal{M}^2 + \frac{k^2}{a^2} \mathcal{M}^2 \right), \tag{4.18}
\]

showing that the remaining scalar perturbation suffers from gradient instability. This can be traced back to the fact the sign of the kinetic term for the Galileon field is positive.

5 Conclusions

In this paper we have considered the cosmological implications of a theory consists of a Galileon field in curved space-time coupled non-minimally to the Maxwell field. The Galileon term has a “John” self interaction form and the non-minimal coupling between
the scalar and the vector field is the interaction between the kinetic term of the scalar field with the energy-momentum tensor of the Maxwell field. The theory has two internal symmetries; the Galileon symmetry associated with the scalar field and the $U(1)$ symmetry associated with the Maxwell field. One should note that the terms appearing in the action (2.3) can also be found in the Stueckelberg transformation of the generalized Proca theory [19]. However, our theory is not a special case of the generalized Proca theory since there is no combination of generalized Proca interactions that gives the action (2.3). Cosmological consequence of the generalized Proca theory is considered in [20].

The theory has two independent exact de Sitter solutions; one is driven by the cosmological constant and is equivalent to the de Sitter solution of the Einstein-Hilbert action. The other is driven by a non-constant, time dependent Galileon field. This solution does not need a cosmological constant but the coupling constant $\alpha$ for the canonical kinetic term of the scalar field should be positive. We have shown that the positive values for $\alpha$ makes the scalar field to diverge at late times and hence the Galileon-based solution should have some difficulties. The dynamical system analysis of the theory shows that the system is one dimensional and has four fixed points. The $\Lambda$-based and Galileon-based solutions coincides with two stable dS fixed points of the theory. There is also an unstable matter dominated fixed point in which the dynamical evolution of the universe can start, and end at the stable $\Lambda$-based dS fixed point at late times provided that $\alpha < \beta \Lambda$. However, there is an unstable dS fixed point which can not be used for the late time accelerated expansion of the universe. Since this fixed point is unstable one can think that the inflation era can be described by this unstable fixed point. In should be noted that the unstable fixed point placed after the Galileon-based dS fixed point and then if the universe start from this unstable fixed point, it should end at an unhealthy Galileon-based fixed point. As a result the system can not be used for inflation.

The cosmological perturbations over these solutions shows that the $\Lambda$-based solution is healthy at linear level for all perturbations. The only difference between this solution and the standard Einstein-Hilbert theory is that the scalar sector put a lower bound on the values of $\beta$.

The Galileon-based solution has a healthy tensor perturbations. Also the vector sector, put an upper bound on the values of $\gamma$. However, the scalar sector shows a gradient instability which can be traced back to the sign of $\alpha$. In fact, the presence of the Galileon interaction can not compensate the $\alpha$-term in the action and wrong sign of $\alpha$ affect the perturbations at linear level. As a result, one can see that the Galileon-Maxwell system does not have a healthy dS solution without the cosmological constant.

References

[1] S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.
[2] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rep. 513 (2012) 1; S. Nojiri, S.D. Odintsov and V.K. Oikonomou, Phys. Rep. 692 (2017) 1; R. Maartens and K. Koyama, Living Rev. Rel. 13 (2010) 5.
[3] S. Tsujikawa, Class. Quant. Grav. 30 (2013) 214003.
[4] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010) 451; A. De Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010) 3; Z. Haghani, T. Harko, F. S. N. Lobo, H. R. Sepangi, and S. Shahidi, Phys. Rev. D 88 (2013) 044023.

[5] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48 (1976) 393; Z. Haghani, N. Khosravi, and S. Shahidi, Classical Quantum Gravity 32 (2015) 215016; J. B. Jimenez and T. S. Koivisto, Classical Quantum Gravity 31 (2014) 135002.

[6] C. de Rham, Living Rev. Rel. 17 (2014) 7.

[7] C. Armendariz-Picon, T. Damour and V. Mukhanov, Phys. Lett. B 458 (1999) 209.

[8] A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D 79 (2009) 064036.

[9] A. De Felice and S. Tsujikawa, Phys. Rev. Lett. 105 (2010) 111301; T. Kobayashi, Phys. Rev. D 81 (2010) 103533; F. P. Silva and K. Koyama, Phys. Rev. D 80 (2009) 121301; N. Chow and J. Khoury, Phys. Rev. D 80 (2009) 024037; A. De Felice, R. Kase and S. Tsujikawa, Phys. Rev. D 83 (2011) 043515; C. Burrage, C. de Rham, D. Seery and A. J. Tolley, JCAP 01 (2011) 014; R. Banerjee, S. Chakraborty, A. Mitra and P. Mukherjee, Phys. Rev. D 96 (2017) 064023.

[10] E. Bellini, N. Bartolo and S. Matarrese, JCAP 06 (2012) 019; A. De Felice, R. Kase, and S. Tsujikawa, Phys. Rev. D 85 (2012) 044059; E. Babichev, Phys. Rev. D 86 (2012) 084003.

[11] G. G. G. Goon, K. Hinterbichler, A. Joyce, and M. Trodden, J. High Energy Phys. 11 (2016) 100; A. Nicolis and R. Rattazzi, J. High Energy Phys. 06 (2004) 059; K. Hinterbichler, A. Nicolis, and M. Porrati, J. High Energy Phys. 09 (2009) 089; F. Charmchi, Z. Haghani, S. Shahidi, and L. Shahkarami, Phys. Rev. D 93 (2016) 124044; A. Amado, Z. Haghani, A. Mohammadi, and S. Shahidi, Phys. Lett. B 772 (2017) 141; I. Saltas and V. Vitagliano, JCAP 1705 (2017) 020.

[12] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D 80 (2009) 064015; A. Padilla, P. M. Saffin and S. Zhou, JHEP 12 (2010) 031; C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D 82 (2010) 061501; R. Banerjee and P. Mukherjee, Class. Quantum Grav. 34 (2017) 235005.

[13] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B 485 (2000) 208; A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D 79 (2009) 064036.

[14] C. Deffayet, G. Esposito-Farese, and A. Vikman, Phys. Rev. D 79 (2009) 084003.

[15] G. W. Horndeski, Int. J. Theor. Phys. 10 (1974) 363.

[16] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, Phys. Rev. Lett. 108 (2012) 051101.

[17] E. Babichev, C. Charmousis, M. Hassaine, JCAP 05 (2015) 031.

[18] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, Phys. Rev. D 85, 104040 (2012) [arXiv:1112.4866 [hep-th]]; C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, Phys. Rev. Lett. 108, 051101 (2012) [arXiv:1106.2000 [hep-th]].

[19] L. Heisenberg, JCAP 05 (2014) 015.

[20] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa and Y.-li Zhang, JCAP 06 (2016) 048; A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa and Y.-li Zhang, Phys. Rev. D 94 (2016) 044024.