Rotationally invariant proof of Bell’s theorem without inequalities

Adán Cabello
Departamento de Física Aplicada II, Universidad de Sevilla, 41012 Sevilla, Spain
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The singlet state of two spin-$\frac{1}{2}$ particles allows a proof of Bell’s theorem without inequalities with two distinguishing features: any local observable can be regarded as an Einstein-Podolsky-Rosen element of reality, and the contradiction with local realism occurs not only for some specific local observables but for any rotation thereof.

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I. INTRODUCTION

Mermin’s version \[1, 2, 3\] of the Greenberger-Horne-Zeilinger (GHZ) proof \[4, 5, 6\] has been considered “the most simple, surprising, and convincing” \[7\]. The proof of Bell’s discovery \[8\] of the fact that Einstein-Podolsky-Rosen (EPR) “elements of reality” \[9\] are incompatible with quantum mechanics (QM). On the other hand, Hardy’s argument of “nonlocality without inequalities” \[10\] has been considered “the best version of Bell’s theorem” \[11\]. Besides their beauty and simplicity, however, both proofs lack of one of the distinguishing features of the original proof by Bell: rotational invariance. Bell’s proof is based on Bohm’s version \[12\] of the EPR experiment using the singlet state of two spin-$\frac{1}{2}$ particles. According to EPR, only “if, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity” \[8\]. Therefore, for the singlet state, and for any maximally entangled state of two spin-$s$ particles, any spin observable on each particle can be regarded as an element of reality. In contrast, for any GHZ state of three or more spin-$s$ particles \[13\] or for any Hardy state (i.e., an entangled but not maximally entangled pure state) of two spin-$s$ particles \[14\], only the results of some local observables can be predicted with certainty from spacelike separated measurements. Therefore, not all local observables can be regarded as elements of reality. Moreover, while in Bell’s proof the disagreement between elements of reality and QM occurs for a continuous range of local observables, both in the GHZ and Hardy’s proofs the algebraic contradictions between EPR elements of reality and QM appear only for a specific set of local observables, but vanishes for any other choice of observables.

A natural question is then, would it be possible to prove Bell’s theorem, without using inequalities, on a physical system in which (a) any local observable satisfies EPR’s criterion for elements of reality, and (b) the contradiction between QM and elements of reality appears not for some specific local observables but for a continuous range of them?

It can be proved that there does not exist a rotationally invariant GHZ state of three or more particles, and it is easy to see that Hardy states are not rotationally invariant. However, a proof of Bell’s theorem without inequalities which fulfills the above requirements is described in the following section.

II. PROOF WITHOUT INEQUALITIES

Let us consider two observers, Alice and Bob, in two distant regions. Each of them receives a spin-$\frac{1}{2}$ particle belonging to a pair initially prepared in the singlet state which, using the standard choice for the matrices $S_x$ (symmetric and real) and $S_y$ (antisymmetric and pure imaginary) \[15\], representing the spin along the $x$ and $y$ directions, can be expressed as

$$|\psi\rangle = \frac{1}{2} (|3/2, -3/2\rangle - |1/2, -1/2\rangle + |1/2, 1/2\rangle - |-3/2, 3/2\rangle).$$

The notation is the following: $|3/2, -3/2\rangle = |3/2\rangle_A \otimes | -3/2\rangle_B$, where $|3/2\rangle_A$ is the eigenstate with eigenvalue $3/2$ ($\hbar = 1$) of the spin along the $z$ direction of Alice’s particle. We shall choose $|3/2\rangle = (1, 0, 0, 0)$, $|1/2\rangle = (0, 1, 0, 0)$, $|-1/2\rangle = (0, 0, 1, 0)$, and $|-3/2\rangle = (0, 0, 0, 1)$. The singlet state $|\psi\rangle$ is rotationally invariant, which means that, if we act on both particles with the tensor product of two equal rotation operators, the result will be to reproduce the same state (within a possible phase factor). A method for preparing optical analogs of the singlet state of two $n$-dimensional systems for an arbitrary high $n$ has been recently described and has been experimentally implemented for low $n$ \[16, 17\].

It can be easily seen that, in the state $|\psi\rangle$, every local spin observable satisfies EPR’s criterion for elements of reality: its value can be predicted with certainty by a spacelike separated measurement on the other particle. Specifically, let us consider the local observables repre-
sented by the operators

\[ D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \]  
\[ d = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \]  
\[ U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]  
\[ u = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \]

and also the observables represented by the operators \( Dd, Du, Ud, Uu \).

As can be easily checked, in the singlet state (1), the result \( r_A(D) \), either \(-1\) or \(1\), of Alice’s measurement of the observable \( D \) on her particle and the result \( r_B(D) \) of Bob’s measurement of \( D \) on his particle are opposite. Moreover, it can be easily checked that, in the singlet state (1), the following correlations between Alice’s and Bob’s results would occur:

\[ r_A(D) = -r_B(D), \]  
\[ r_A(d) = -r_B(d), \]  
\[ r_A(U) = r_B(U), \]  
\[ r_A(u) = -r_B(u), \]  
\[ r_A(Dd) = r_B(D)d_B(d), \]  
\[ r_A(Uu) = -r_B(U)r_B(u), \]  
\[ r_A(D)r_A(u) = r_B(D)(Du), \]  
\[ r_A(U)r_A(d) = -r_B(U)d_B(Ud), \]  
\[ r_A(D)d_A(u) = r_B(D)(Du)r_B(Ud). \]

Let us show that any of the 12 local observables (six per particle) appearing in Eqs. (6)–(14) satisfies EPR’s criterion for elements of reality and thus possesses a preexisting result, either \(-1\) or \(1\), which is revealed when the corresponding measurement is performed, and is not altered when another compatible observable is measured. In particular, let us examine Eqs. (10)–(14), since each of them involves two elements of reality of the same particle. Let us take, for instance, Eq. (10). Following EPR and using Eq. (9), a measurement of \( D \) on Bob’s particle would reveal a preexisting element of reality \( r_B(D) \). Likewise, following EPR and using Eq. (7), a measurement of \( d \) on Bob’s particle would reveal a preexisting element of reality \( r_B(d) \). However, it could happen that the measurement of \( D \) could alter the preexisting element of reality \( r_B(d) \). How can we guarantee that a previous measurement of \( D \) will not affect the result of a subsequent measurement of (the compatible observable) \( d \)? We can guarantee it by invoking EPR’s criterion for elements of reality: since, by means of a spacelike separated measurement on his particle, Alice can predict with certainty \( r_B(d) \) using Eq. (7), regardless of whether or not Bob has measured \( D \) before measuring \( d \), then, following EPR, we conclude that Bob’s measurement of \( D \) does not change \( r_B(d) \). If \( d \) was an element of reality for Bob’s particle, a measurement of \( D \) on Bob’s particle does not alter the preexisting element of reality of \( d \). The same reasoning applies whenever a pair of compatible local observables is measured on the same particle, as in Eqs. (10)–(14).

The proof of Bell’s theorem of incompatibility between elements of reality and QM comes from the fact that it is impossible to assign preexisting results, either \(-1\) or \(1\), to the 12 local observables in such a way that satisfies the predictions of QM given by Eqs. (6)–(14). This can be checked in the following manner: if we take the product of Eqs. (6)–(14), each result (either \(-1\) or \(1\)) appears twice in each side, since each operator appears twice in each side. Therefore, the product of the left-hand sides must be \(1\), while the product of the right-hand sides must be \(-1\). We, therefore, conclude that any physical theory in which the notion of EPR elements of reality makes sense cannot reproduce the predictions of QM for the singlet state of two spin-\(\frac{3}{2}\) particles given by Eqs. (10)–(14).

### III. HOW TO MEASURE THE LOCAL OBSERVABLES

Let us now describe how to measure the local observables involved in the proof on a spin-\(\frac{3}{2}\) particle. Observable \( D \) can be measured by measuring the spin along the \(z\) direction using a Stern-Gerlach device: if the result of this measurement is \(S_z = 3/2\) or \(S_z = 1/2\), then \(r(D) = 1\); if the result is \(S_z = -1/2\) or \(S_z = -3/2\), then \(r(D) = -1\). Likewise, observable \(d\) can be measured by measuring the spin along the \(z\) direction; if the result is \(S_z = 3/2\) or \(S_z = 1/2\), then \(r(d) = 1\); if the result is \(S_z = -1/2\) or \(S_z = -3/2\), then \(r(d) = -1\). Therefore, a joint measurement of \(D\) and \(d\), like the one Bob needs to check Eq. (10), is equivalent to a measurement of the spin along the \(z\) direction on Bob’s particle; if the result is \(S_z = 3/2\) or \(S_z = 1/2\), then \(r(d) = 1\); if the result is \(S_z = -1/2\) or \(S_z = -3/2\), then \(r(d) = -1\). Any observable represented by a diagonal operator can be measured using this method. The recipe for measuring the other local observables involved in the proof is not as easy. In the case of a single spin-\(\frac{3}{2}\) particle, these observables require generalized Stern-Gerlach devices as described in Ref. [18]. Other possibility is to implement optical analogs of these observables by using the method described in Ref. [18].
IV. ROTATIONAL INVARIANCE

The most relevant feature of the proof introduced above is that it is rotationally invariant: the contradiction between elements of reality and QM occurs not only for the particular set of observables $\{D_A, D_B, d_A, \ldots\}$ used above, but for any set of observables $\{RD_A, RD_B, Rd_A, \ldots\}$, where $RD_A$ is the physical observable obtained by applying a rotation $R$ to the device for measuring the observable $D_A$. As can be easily checked, in the singlet state $|1\rangle$,

\begin{align*}
    r_A(RD) &= -r_B(RD), \\
r_A(Rd) &= -r_B(Rd), \\
r_A(RU) &= r_B(RU), \\
r_A(Ru) &= -r_B(Ru), \\
r_A(RDd) &= r_B(RD)r_B(Rd), \\
r_A(RUu) &= -r_B(RU)r_B(Ru), \\
r_A(RDd)r_A(RUu) &= r_B(RDu)r_B(RUd).
\end{align*}

Therefore, once Alice and Bob have found a set of observables leading to a contradiction between elements of reality and QM, then any common rotation of the local observables will lead to a similar contradiction, without rotating the source of entangled pairs.

V. CONCLUSIONS

To sum up, we have presented a proof of Bell’s theorem for the singlet state of two spin-$\frac{1}{2}$ particles which combines the simplicity of the GHZ proof with the symmetry of the original proof by Bell, in which any local spin observable can be regarded as an Einstein-Podolsky-Rosen element of reality, and in which the contradiction between elements of reality and QM occurs for a continuous range of settings.

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