ON THE FI-HOMOLOGY OF THE INJECTIVE COGENERATORS

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Abstract. The purpose of this paper is to give information on the FI-homology of the standard injective cogenerators of the category of FI-modules, where FI is the category of finite sets and injections.

Working over a field $k$ of characteristic zero, a full calculation is given in homological degree zero and a conjectural description in higher homological degree.

The proof of the main theorem reduces to a calculation in representation theory of the symmetric groups, exploiting the Young orthonormal basis.

1. Introduction

The category $\text{FI}$ of finite sets and injections is a fundamental tool for relating representations of the symmetric groups, which appear as the automorphism groups of the objects of $\text{FI}$. In particular, for any commutative, unital ring, one can consider $\mathcal{F}(\text{FI})$, the category of functors from $\text{FI}$ to $k$-modules, often known as the category of $\text{FI}$-modules. This provides a framework for studying phenomena such as representation stability, for example. In this paper, $k$ is taken to be a field of characteristic zero, since the proof of the main results exploits the representation theory of the symmetric groups in characteristic zero.

The category $\mathcal{F}(\text{FI})$ has enough projectives and enough injectives, which are provided by Yoneda’s lemma. For $b \in \mathbb{N}$, writing $b := \{1, \ldots, b\}$, the associated standard projective generator is $k\text{Hom}_{\text{FI}}(b, -)$, the $k$-linearization of the set-valued functor $\text{Hom}_{\text{FI}}(b, -)$. Correspondingly, one has the standard injective cogenerator given by $k\text{Hom}_{\text{FI}}(-, b)^{\text{tr}}$, formed by taking set maps with values in $k$. Contrary to the standard projectives, these are torsion $\text{FI}$-modules; in particular, for $a \in \mathbb{N}$, and $a := \{1, \ldots, a\}$, $k^{\text{Hom}_{\text{FI}}(-, b)}(a) = k^{\text{Hom}_{\text{FI}}(a, b)}$, which is zero for $a > b$.

For the purposes of this paper (with its application in [Pow22] in view), it is convenient to describe the above injective cogenerator as

$$k\text{Hom}_{\text{FI}}(-, b)^{\text{tr}} : \text{FI} \to k-\text{mod},$$

where $^{\text{tr}}$ indicates the following ‘transpose’ covariant structure: $k\text{Hom}_{\text{FI}}(-, b)^{\text{tr}}(a)$ is the $k$-vector space with basis $\{[f] \mid f \in \text{Hom}_{\text{FI}}(a, b)\}$. Then for $i : a \to a'$ a map in $\text{FI}$:

$$k\text{Hom}_{\text{FI}}(i, b)[f] = \sum_{f' \in \text{Hom}_{\text{FI}}(a', b) \text{ with } f' \circ i = f} [f'].$$

There is more structure, namely one can take into account the action of $\mathfrak{S}_b := \text{Aut}(b)$, so that the above is a functor from $\text{FI}$ to $k\mathfrak{S}_b$-modules.

The category $\Sigma$ of finite sets and bijections identifies as the maximal subgroupoid of $\text{FI}$. Functors from $\Sigma$ to $k$-vector spaces are denoted by $\mathcal{F}(\Sigma)$ and there is an extension by zero functor $\mathcal{F}(\Sigma) \to \mathcal{F}(\text{FI})$. This has left adjoint $H_0^\text{FI} : \mathcal{F}(\text{FI}) \to \mathcal{F}(\Sigma)$ and, by definition, $\text{FI}$-homology $H_*^\text{FI}$ is given by forming the left derived functors.

Behaviour of $\text{FI}$-homology on the standard projective is easily understood:

$$(H_0^\text{FI}k\text{Hom}_{\text{FI}}(b, -))(a) = \begin{cases} k\mathfrak{S}_b \quad &a = b \\ 0 &\text{otherwise.} \end{cases}$$

Clearly higher $\text{FI}$-homology of $k\text{Hom}_{\text{FI}}(b, -)$ vanishes, since these are defined as left derived functors.

A contrario, the case of the standard injective cogenerators $k\text{Hom}_{\text{FI}}(-, b)^{\text{tr}}$, for $b \in \mathbb{N}$, is much more complicated. In particular, $H_0^\text{FI}k\text{Hom}_{\text{FI}}(-, b)^{\text{tr}}$ is not in general supported on $b$.

The main result of this paper, Theorem 3.1, calculates this $H_0^\text{FI}$, in the statement, for a partition $\lambda$, $S^\lambda$ is the simple $\mathfrak{S}_b$-module indexed by $\lambda$ (see Section 3 for some recollections on the representation theory of the symmetric groups and references).

Theorem 1. For $1 \leq a \leq b$, there is an isomorphism of $\mathfrak{S}_b \times \mathfrak{S}_a$-modules:

$$(H_0^\text{FI}k\text{Hom}_{\text{FI}}(-, b)^{\text{tr}})(a) \cong \bigoplus_{\lambda \vdash b} S^\lambda \boxtimes S^\lambda_{\lambda\vdash b-a},$$

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This work was partially supported by the ANR Project ChroK, ANR-16-CE40-0003.
where \( \hat{\lambda} \) is the partition obtained from \( \lambda \) by removing \( \lambda_1 \).

In particular, this is multiplicity-free.

This result has a striking aspect that can be paraphrased as stating that the above \( \text{FI} \)-homology in homological degree zero is as small as it can be, given the structure of the representations involved (cf. Lemma \( [4,8] \)). That such a result should be true was suggested by calculations carried out by the author using Sage \( [\text{Sage}22] \) in January 2022.

The starting point is the identification as representations

\[
\kappa\text{Hom}_{\text{FI}}(a,b)^{\text{tr}} \cong \bigoplus_{\lambda+b,a\vdash\lambda, \lambda \leq \nu} S^\lambda \boxtimes S^\nu.
\]

Here the condition \( \hat{\lambda} \leq \nu \leq \lambda \) is equivalent to the skew partition \( \lambda/\nu \) being a ‘horizontal strip’ (i.e., having skew Young diagram with at most one box in each column).

The proof of Theorem \( [4] \) boils down to a property of the skew representation \( S^{\lambda/\nu} \) when \( \nu \neq \hat{\lambda} \), as is explained in Section \( [5.1] \). Since \( \lambda/\nu \) is a horizontal strip, \( S^{\lambda/\nu} \) is a permutation representation; the difficulty in proving the Theorem stems from the fact that the explicit element which intervenes is defined in terms of the Young permutation basis, which is not a permutation basis.

This proof takes up most of Section \( [5] \) and exploits elementary, brute force techniques. The author believes that there should be a better proof of this result and suspects that the result should already occur in some form in the literature.

Emboldened by Theorem \( [4] \) it is natural to consider the \( \text{FI} \)-homology in higher homological degree. Proposition \( [6.2] \) gives the following lower bound for this (the notation used in the statement is introduced in Section \( [6] \):

**Proposition 2.** For \( b \geq a \in \mathbb{N} \) and homological degree \( n \), there is an inclusion of \( \Sigma_b \times \Sigma_a \)-modules:

\[
\bigoplus_{(\lambda,\mu) \in \text{Crit}(a,b)} S^\lambda \boxtimes S^\mu \subset \mathcal{H}_{\text{FI}}^n(\kappa\text{Hom}_{\text{FI}}(-,b))(a).
\]

For \( n = 0 \) this is an isomorphism.

An optimistic conjecture is that this Proposition actually gives the full \( \text{FI} \)-homology. Since this is not the main thrust of the paper, the proposed strategy for establishing the conjecture is only outlined here.

The appendix, Section \( [A] \) revisits these structures using the Schur-Weyl correspondence, i.e., by passing to Schur \((bi)\)functors. This allows the Koszul complex that calculates the \( \text{FI} \)-homology to be made completely explicit (see Theorem \( [A.6] \)) and also introduces further structure (see Corollary \( [A.8] \)). This is significant in the application in \( \text{Pow}22 \), where it is encoded in the wall categories appearing there.

1.1. **Acknowledgement.** The author is grateful to Christine Vespa for comments which lead to the natural description of the main players.

2. **Ingredients**

The category of finite sets and injective maps is denoted \( \text{FI} \) and its maximal subgroupoid by \( \Sigma \), which identifies with the category of finite sets and bijections.

**Notation 2.1.**

1. For \( t \in \mathbb{N} \), \( t \) denotes the finite set \( \{1, \ldots, t\} \), which can be considered as an object of \( \Sigma \) (and hence of \( \text{FI} \)). By convention, \( 0 = \emptyset \).
2. For \( s \leq t \), \( t_s, t \in \text{Hom}_{\text{FI}}(s,t) \) denotes the canonical inclusion \( \{1, \ldots, s\} \subset \{1, \ldots, t\} \).

Morphisms in \( \text{FI} \) yield a functor

\[
\text{Hom}_{\text{FI}}(-,-) : \text{FI}^{\text{op}} \times \text{FI} \to \text{Set}^f,
\]

where \( \text{Set}^f \) is the category of finite sets. The \( \kappa \)-linearization gives a functor \( \kappa\text{Hom}_{\text{FI}}(-,-) \) to \( \mathcal{V}^f_\kappa \subset \mathcal{V}_\kappa \), the full subcategory of finite-dimensional \( \kappa \)-vector spaces.

In addition, if one restricts the second variable to \( \Sigma \subset \text{FI} \), \( \kappa\text{Hom}_{\text{FI}}(-,-) \) has a covariant functoriality with respect to the first variable:

**Notation 2.2.** Denote by \( \kappa\text{Hom}_{\text{FI}}(-,-)^{\text{tr}} \) the functor \( \text{FI} \times \Sigma \to \mathcal{V}^f_\kappa \) given by

\[
\kappa\text{Hom}_{\text{FI}}(-,-)^{\text{tr}} : (a,b) \mapsto \kappa\text{Hom}_{\text{FI}}(a,b)
\]

where, for \( i : a \hookrightarrow a' \) in \( \text{FI} \) and \( f \in \text{Hom}_{\text{FI}}(a,b) \), \( \kappa\text{Hom}_{\text{FI}}(i,b)^{\text{tr}}[f] \) is the sum \( \sum [f'] \), where \( f' \in \text{Hom}_{\text{FI}}(a',b) \) runs over the set of injective maps such that \( f' \circ i = f \).
Denote by $\mathcal{F}(\mathcal{F})$ the category of functors from $\mathcal{F}$ to $\mathbb{k}$-vector spaces. Then, for each $b \in \mathbb{N}$, the functor $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}$ is injective in $\mathcal{F}(\mathcal{F})$, by Yoneda’s lemma. More explicitly, it corepresents the functor $F \mapsto F(b)^{\dagger}$, for $F \in \text{Ob} \mathcal{F}(\mathcal{F})$, where $\dagger$ denotes vector space duality. It follows that $\{\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)} \mid b \in \mathbb{N}\}$ is a set of injective cogenerators of $\mathcal{F}(\mathcal{F})$.

**Proposition 2.3.** For $b \in \mathbb{N}$,

1. $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}$ is isomorphic to the injective cogenerator $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}$ of $\mathcal{F}(\mathcal{F})$;
2. $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}$ yields a functor $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}^{\dagger}: \mathcal{F} \to \mathbb{k}\mathcal{S}_b\text{-mod.}$

**Proof.** The first statement follows by using the isomorphism of $\mathbb{k}$-vector spaces

$$\mathbb{k}X \cong \mathbb{k}^X,$$

for $X$ a finite set. This corresponds to the pairing $\mathbb{k}X \otimes \mathbb{k}X \to \mathbb{k}$ sending $[x] \otimes [x']$ to 1 if $x = x' \in X$ and zero otherwise; this is equivariant with respect to automorphisms of $X$, for the trivial action on $\mathbb{k}$. The structure of $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}$ corresponds to that of $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}^{\dagger}$ across these isomorphisms.

The second statement is clear. $\square$

**Remark 2.4.** The description $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}^{\dagger}$ is preferred here, as opposed to $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}$, since it is generalized in [Pow22] to a functor on a larger category (in which $b$ is allowed to vary), for which this formulation is more natural.

For $0 < a \leq b \in \mathbb{N}$, applying $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}(-, b)}^{\dagger}$ to $\iota_{a-1, a} \in \operatorname{Hom}_{\mathcal{F}}(a - 1, a)$ gives a $\mathcal{S}_a^\text{op} \times \mathcal{S}_b$-equivariant map

$$\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a - 1, b) \to \mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a, b) \downarrow_{\mathcal{S}_a},$$

where $\mathcal{S}_{a-1} \subset \mathcal{S}_a$ is induced by $a - 1 \subset a$.

As a $\mathcal{S}_a^\text{op} \times \mathcal{S}_b$-module, $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a - 1, b)$ is generated by $[\iota_{a-1, b}]$. The above map is thus determined by the image of $[\iota_{a-1, b}]$. From the definition of $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a - 1, b)$, one checks that the following holds:

**Lemma 2.5.** For $0 < a \leq b \in \mathbb{N}$, the morphism $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(\iota_{a-1, a}, b)$ is determined as a morphism of $\mathcal{S}_a^\text{op} \times \mathcal{S}_b$-modules by:

$$[\iota_{a-1, b}] \mapsto \sum_{z \in b \backslash a-1} (a, z)[\iota_{a, b}],$$

where the transposition $(a, z) \in \mathcal{S}_b$ acts via the $\mathcal{S}_b$-action on $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a, b)$.

**Notation 2.6.** For $0 < a \leq b \in \mathbb{N}$, denote by

$$\mathbb{S}_{a,b} : \mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a - 1, b) \uparrow_{\mathcal{S}_{a-1}} \to \mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a, b).$$

the $\mathcal{S}_a^\text{op} \times \mathcal{S}_b$-equivariant map adjoint to (2.1). For $a = 0$, $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a - 1, b)$ (and hence the map $\mathbb{S}_{0,b}$ also) is taken to be 0.

**Remark 2.7.** For $0 < a \leq b$, clearly $[\iota_{a-1, b}]$ generates $\mathbb{k}^{\operatorname{Hom}_{\mathcal{F}}}(a - 1, b)$ as a $\mathcal{S}_a^\text{op} \times \mathcal{S}_b$-module, hence $\mathbb{S}_{a,b}$ is also determined by the image of $[\iota_{a-1, b}]$ given in Lemma 2.5.

3. Representation theory of the symmetric groups

This Section reviews elements of the representation theory of the symmetric groups that are required later. References are given to the book [CSST10] so as to have a convenient single reference for the reader. In particular, notation and conventions follow this reference. Here $\mathbb{k}$ is taken to be a field of characteristic zero.

3.1. Background.

**Notation 3.1.** For $n \in \mathbb{N}$,

1. $\mathcal{S}_n$ denotes the symmetric group on $n$ letters, which identifies with the group of automorphisms of $\mathfrak{n}$;
2. for $\lambda \vdash n$ a partition of $n$, $S^\lambda$ denotes the associated simple representation, indexed so that $S^{(n)}$ is the trivial representation and $S^{(1^n)}$ is the signature representation.

The conventions used here follow [CSST10]. In particular, a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ has associated Young diagram in which the $i$th row has length $\lambda_i$. For instance, the partition $(4, 2, 1)$ has Young diagram:

Young diagrams are given coordinates via (rows, columns). For example, in the above case:
Notation 3.2. Write $\preceq$ for the partial order on the set of partitions defined by $\mu \preceq \lambda$ if and only if $\mu_i \leq \lambda_i$ for all $i \in \mathbb{N}$ (unspecified entries are understood to be zero). Equivalently, $\mu \preceq \lambda$ if and only if the Young diagram of $\mu$ is contained in that of $\lambda$.

If $\mu \preceq \lambda$, $\lambda/\mu$ denotes the associated skew diagram; this is viewed as the complement of the Young diagram of $\mu$ in that of $\lambda$.

Example 3.3. If $\mu = (1,1)$ and $\lambda = (4,2,1)$ then $\mu \preceq \lambda$, which can be represented by the diagram

\[
\begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\end{array}
\]

in which the Young diagram of $\mu$ is shaded.

The skew diagram $\lambda/\mu$ is represented by:

\[
\begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\end{array}
\]

Remark 3.4. For a fixed partition $\lambda$, the partial order $\preceq$ induces a partial order on the skew partitions such that $\lambda/\mu_1 \preceq \lambda/\mu_2$ if and only if $\mu_2 \preceq \mu_1$ (note the reversal of the order). This corresponds to the inclusion of skew diagrams.

Example 3.5. For $\lambda = (4,2,1)$ and $\mu = (1,1), \mu' = (2,1)$, clearly one has $\mu \preceq \mu'$, hence $\lambda/\mu' \preceq \lambda/\mu$, which corresponds at the level of the skew diagrams to the inclusion of a sub-diagram: $\begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\end{array} \preceq \begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\end{array}$.

Definition 3.6. For partitions $\mu \preceq \lambda$, the skew partition $\lambda/\mu$ is a horizontal strip if each column of the skew diagram contains at most one box.

This can be reformulated using the following.

Notation 3.7. For a partition $\lambda$, let $\hat{\lambda}$ denote the partition obtained by forgetting $\lambda_1$. (In terms of the associated Young diagrams, this corresponds to removing the first row.)

Clearly $\hat{\lambda} \preceq \lambda$ and $\lambda/\hat{\lambda}$ is a horizontal strip with Young diagram that contains precisely one box in each non-empty column of $\lambda$. Moreover, one has:

Lemma 3.8. For $\mu \preceq \lambda$, $\lambda/\mu$ is a horizontal strip if and only if $\hat{\lambda} \preceq \mu$. In particular, $\lambda/\hat{\lambda}$ is the maximal element of the poset of horizontal strips for $\lambda$.

Example 3.9. In Example 3.5, $\hat{\lambda} = \mu'$. Visibly $\lambda/\mu'$ is a horizontal strip, whereas $\lambda/\mu$ is not, which corresponds to the obvious fact that $\mu' \not\preceq \mu$.

To each skew partition $\lambda/\mu$, one associates a skew representation as follows. Here, for $a, b \in \mathbb{N}$, one uses $\iota_{a,b} : a \subset b$ to define the Young subgroup $\mathfrak{S}_a \times \mathfrak{S}_{b-a} \subset \mathfrak{S}_b$, as usual. Recall the following:

Definition 3.10. For $\lambda \vdash b$ and $\mu \vdash a$ with $\mu \preceq \lambda$, the skew representation $S^{\lambda/\mu}$ is the $\mathfrak{S}_{b-a}$-module

$S^{\lambda/\mu} := \text{Hom}_{\mathfrak{S}_a}(S^\mu, S^\lambda)$,

where the $\mathfrak{S}_{b-a}$ action is induced by the restriction along $\mathfrak{S}_{b-a} \subset \mathfrak{S}_b$ of the action on $S^\lambda$.

The importance of the skew representations is illustrated by the following (see [CSST10] Proposition 3.5.5), for example.):

Proposition 3.11. For $a \leq b \in \mathbb{N}$ and $\lambda \vdash b$, the restriction of $S^\lambda$ to $\mathfrak{S}_a \times \mathfrak{S}_{b-a}$ identifies as:

$S^\lambda \downarrow_{\mathfrak{S}_a \times \mathfrak{S}_{b-a}} \cong \bigoplus_{\mu \preceq \lambda} S^\mu \boxtimes S^{\lambda/\mu}$,

where $\boxtimes$ denotes the exterior tensor product.

3.2. Tableaux and axial distance. Recall that, for $\lambda \vdash b$, a Young tableau of shape $\lambda$ is a bijection between the boxes of the Young diagram of $\lambda$ and the set $b$.

A Young tableau $T$ of shape $\lambda$ is standard if the entries increase both along the rows and down the columns of $T$. The set of standard tableaux of shape $\lambda$ is denoted by $\text{Tab}(\lambda)$.

These notions carry over to skew diagrams of the form $\lambda/\mu$: if $\lambda \vdash a$ and $\mu \vdash b$ with $\mu \preceq \lambda$, then a tableau of shape $\lambda/\mu$ is a bijection between the boxes of the Young diagram and $b - a$. A tableau of shape $\lambda/\mu$ is standard if entries increase both along the rows and down the columns. The set of standard skew tableaux of shape $\lambda/\mu$ is denote $\text{Tab}(\lambda/\mu)$. (Thus, on taking $\mu = (0)$, one recovers $\text{Tab}(\lambda)$.)

For $\lambda/\mu$ a horizontal strip, one has the distinguished standard skew-tableau:
For $\lambda/\mu$ a horizontal strip, let $T^{rev} \in \text{Tab}(\lambda/\mu)$ be the standard Young tableau of shape $\lambda/\mu$ such that numbers increase from left to right.

**Example 3.13.** Take $\lambda = (5,2,1,1)$ and $\mu = (3,1,1)$, so that $\lambda \preceq \mu \preceq \lambda$; then for $\lambda/\mu$, the tableau $T^{rev}$ is:

\[
\begin{array}{cccccc}
\text{\#} & \text{\#} & \text{\#} & \text{\#} & \text{\#} \\
2 & 3 & 4 & 5 & 6
\end{array}
\]

We introduce the following:

**Notation 3.14.** For $i \in b$ and a Young tableau of shape $\lambda$, denote by $(u_i, v_i)$ the coordinates of $i$.

Then, given a tableau $T$ of shape $\lambda/\mu$ one has the notion of axial distance between entries of the tableau:

**Definition 3.15.** For $T$ a tableau of shape $\lambda/\mu$, where $\lambda \vdash b$ and $\mu \vdash a$ with $\mu \preceq \lambda$:

1. for $i, j \in b - a$, the axial distance $a(j, i)(T)$ from $j$ to $i$ in $T$ is $(v_j - v_i) - (u_j - u_i)$.
2. for $j \in b - a - 1$, $r_j = r_j(T) := a(j + 1, j)(T)$, the axial distance from $j + 1$ to $j$ in $T$.

Axial distance satisfies the following evident additivity property:

**Lemma 3.16.** For $T$ a tableau of shape $\lambda/\mu$, where $\lambda \vdash b$ and $\mu \vdash a$ with $\mu \preceq \lambda$, and for $i, j, k \in b - a$:

\[a(k, i)(T) = a(k, j)(T) + a(j, i)(T)\]

In particular, $a(3,1)(T) = r_1(T) + r_2(T)$.

Axial distance gives the following criterion for a skew diagram $\lambda/\mu$ to be a horizontal strip:

**Lemma 3.17.** Suppose $\lambda \vdash b$ and $\mu \vdash a$ with $\mu \preceq \lambda$. Then $\lambda/\mu$ is a horizontal strip if and only if there exists $T \in \text{Tab}(\lambda/\mu)$ such that, for all $1 \leq i < j \leq b - a$, the axial distance $a(j, i)(T)$ from $j$ to $i$ is positive.

Moreover, such a standard Young tableau is unique, namely is $T^{rev}$.

**Proof.** If $\lambda/\mu$ is not a horizontal strip, there exists a column containing at least two boxes. For $T \in \text{Tab}(\lambda/\mu)$, consider such adjacent boxes with labels $\Box$. Since $T$ is standard, $i < j$ and $a(j, i)(T) = -1$.

If $\lambda/\mu$ is a horizontal strip, then one checks that $T^{rev}$ is the unique element of $\text{Tab}(\lambda/\mu)$ with the required property.

3.3. **Young’s orthogonal form.** Since, for $\lambda \vdash b$, a tableau of shape $\lambda$ is an isomorphism from the set of boxes of the Young diagram of $\lambda$ to $b$, the symmetric group $S_b$ acts by post-composition. There is a subtlety, if $T \in \text{Tab}(\lambda)$ and $\sigma \in S_b$, $\sigma T$ is not necessarily standard.

We focus upon the Coxeter generators of the symmetric groups:

**Notation 3.18.** For $n \in \mathbb{N}$, the set of Coxeter generators of $S_n$ is the set of transpositions $s_i := (i, i + 1)$, for $1 \leq i < n$. (The value of $n$ is usually known, so is not included in the notation.)

If $T \in \text{Tab}(\lambda)$ and $1 \leq j < b$, then $s_jT$ is standard unless the entries $j, j + 1$ occur in adjacent boxes or equivalently $|r_j| = |r_j(T)| = 1$.

In order to use Young’s orthogonal form, we work over $k = \mathbb{R}$. For $\lambda \vdash b$ (see [CSST10, Theorem 3.4.4]) Young’s orthonormal basis of $S^\lambda$ is

\[
\{w_T | T \in \text{Tab}(\lambda)\}
\]

with the $S_b$-action determined by the following action of the Coxeter generators:

\[
s_jw_T = \frac{1}{r_j}w_T + \sqrt{1 - \frac{1}{r_j^2}}w_{s_jT}, \quad s_jw_{s_jT} = -\frac{1}{r_j}w_{s_jT} + \sqrt{1 - \frac{1}{r_j^2}}w_T,
\]

(assuming that $s_jT$ is standard for the second equation). As above, $s_jT$ is non-standard if and only if $|r_j| = 1$ (when $r_{jT} = r_j$), in which case the first equation reduces to

\[
s_jw_T = r_jw_T.
\]

The Young orthonormal basis adapts to working with skew diagrams (see [CSST10, Theorem 3.5.5]). For $\lambda \vdash b$ and $\mu \vdash a$ with $\mu \preceq \lambda$, the skew representation $S^{\lambda/\mu}$ has orthonormal basis

\[
\{\Phi_T | T \in \text{Tab}(\lambda/\mu)\}
\]
where the Coxeter generators of $\mathfrak{S}_{b-a}$ act by
\[ s_j \Phi_T = \frac{1}{r_j} \Phi_T + \sqrt{1 - \frac{1}{r_j}} \Phi_{s_j T}. \]

**Remark 3.19.** If $\lambda/\mu$ is a horizontal strip, then $s_j T$ is not standard if and only if $r_j = 1$, in which case the above action reduces to $s_j \Phi_T = \Phi_T$.

### 3.4. First applications

For partitions $\lambda, \lambda'$ of $b$, since the contragredient of $S^\lambda$ is isomorphic to $S^\lambda$, one has
\[ (S^\lambda \otimes S^{\lambda'})^{\mathfrak{S}_b} \cong \left\{ \begin{array}{ll} k & \lambda = \lambda' \\
0 & \text{otherwise} \end{array} \right. \]
where $\mathfrak{S}_b$ acts diagonally. The Young orthonormal basis allows one to identify an explicit generator of these invariants in the case $\lambda = \lambda'$:

**Lemma 3.20.** For $\lambda \vdash b$, the element $\sum_{T \in \text{Tab}(\lambda)} w_T \otimes w_T \in S^\lambda \otimes S^\lambda$ is a generator of $(S^\lambda \otimes S^\lambda)^{\mathfrak{S}_b} = k$.

**Proof.** It is sufficient to check that this element is invariant under the (diagonal) action of the Coxeter generators. This is shown using the explicit action given above. \qed

Likewise, if $\mu \vdash a$ such that $\mu \preceq \lambda$, then
\[ (S^{\lambda/\mu})^{\mathfrak{S}_{b-a}} \cong \left\{ \begin{array}{ll} k & \lambda/\mu \text{ is a horizontal strip} \\
0 & \text{otherwise}, \end{array} \right. \]
(see [CSST10] Proposition 3.5.12, for example) and, moreover, if $\lambda/\mu$ is a horizontal strip, then $S^{\lambda/\mu}$ is a permutation module. More precisely (see [CSST10] Proposition 3.5.8], for example), if $\lambda/\mu$ is a horizontal strip with rows of lengths $b_1, \ldots, b_t$ (so that $\sum_i b_i = b - a$), then
\[ S^{\lambda/\mu} \cong k \otimes_{\prod_i \mathfrak{S}_{b_i}}. \]

This can be made explicit using the standard tableau $T^{rev} \in \text{Tab}(\lambda/\mu)$ (see Notation 3.12):

**Proposition 3.21.** For $\lambda \vdash b$, $\mu \vdash a$ such that $\lambda \preceq \mu \preceq \lambda$, the map $1 \mapsto \Phi_{T^{rev}}$ induces an isomorphism of $\mathfrak{S}_{b-a}$-modules
\[ \mathfrak{S}_{b-a} \otimes_{\prod_i \mathfrak{S}_{b_i}} k \xrightarrow{\cong} S^{\lambda/\mu}. \]

**Proof.** The hypothesis implies that $\lambda/\mu$ is a horizontal strip.

Using the action of the Coxeter generators, one sees that the Young subgroup $\prod_i \mathfrak{S}_{b_i} \subset \mathfrak{S}_{b-a}$ stabilizes $\Phi_{T^{rev}}$, hence one obtains the given morphism of $\mathfrak{S}_{b}$-modules. This is surjective, as is seen for example by applying [CSST10] Proposition 3.5.6]. Hence it is an isomorphism, for dimension reasons. \qed

Moreover, the Young orthonormal basis allows an explicit generator for the invariants $(S^{\lambda/\mu})^{\mathfrak{S}_{b-a}}$ to be exhibited:

**Proposition 3.22.** If $\lambda \preceq \mu \preceq \lambda$, then $(S^{\lambda/\mu})^{\mathfrak{S}_{b-a}} \cong k$ is generated by the element of the form
\[ \sum_{T \in \text{Tab}(\lambda/\mu)} \beta_T \Phi_T \]
that is uniquely determined by $\beta_{T^{rev}} = 1$.

The coefficients are determined by the following: if $T', T'' \in \text{Tab}(\lambda/\mu)$ such that $T'' = s_i T'$, then
\[ \frac{\beta_{T''}}{\beta_{T'}} = \sqrt{\frac{r_i - 1}{r_i + 1}}, \]
where $r_i = r_i(T')$, which satisfies $r_i \not\in \{-1, 0, 1\}$, by the hypothesis on $T', T''$. In particular, $\beta_T \in k^\times$ for all $T \in \text{Tab}(\lambda)$.

**Proof.** Under the hypothesis upon $\lambda/\mu$, $(S^{\lambda/\mu})^{\mathfrak{S}_{b-a}} = k$, hence there exists an invariant element of the form $\sum_{T \in \text{Tab}(\lambda/\mu)} \beta_T \Phi_T$ with not all coefficients zero. Using the action of the Coxeter elements, as explained below, one deduces that all coefficients must be non-zero. The expression can then be normalized by imposing $\beta_{T^{rev}} = 1$, so that the $\beta_T$ are uniquely determined.

To establish the explicit relation between $\beta_{T'}$ and $\beta_{T^{rev}}$, we use that invariance requires that the coefficient of $\Phi_{T'}$ in $s_i \left( \sum_{T \in \text{Tab}(\lambda/\mu)} \beta_T \Phi_T \right)$ must be equal to $\beta_T$. This coefficient is equal to that of $\Phi_{T'}$ in $s_i (\beta_T \Phi_T + \beta_{T^{rev}} \Phi_{T^{rev}})$ which is $\beta_T r_i + \beta_{T^{rev}} \sqrt{1 - \frac{1}{r_i}}$. The stated relation follows from the equality $\beta_T = \beta_T r_i + \beta_{T^{rev}} \sqrt{1 - \frac{1}{r_i}}$.

The rational number $\frac{r_i - 1}{r_i + 1}$ is defined and strictly positive, since the integer $r_i$ does not belong to $\{-1, 0, 1\}$.

This gives the calculation of all of the coefficients $\beta_T$ recursively, starting from $\beta_{T^{rev}} = 1$ since the equivalence relation on $\text{Tab}(\lambda/\mu)$ generated by $T' \sim T''$ if there exists a Coxeter generator $s_i$ such that $T'' = s_i T'$ has a
single equivalence class (see [CSST10, Corollary 3.1.6], for example). In particular, one sees that all of the coefficients $\beta_T$ are non-zero.

\[ \text{Remark 3.23.} \] Comparing Propositions 3.21 and 3.22 highlights one of the subtleties here. When $\lambda/\mu$ is a horizontal strip, $S^{\lambda/\mu}$ is a permutation module with permutation basis $\sigma \Phi_{T^{\mu_{\lambda}}} \text{ as } \sigma$ ranges over a set of coset representatives for $\mathcal{G}_{b-a}/\prod_i \mathcal{G}_i$.

However, the Young orthonormal basis of $S^{\lambda/\mu}$ is clearly not a permutation basis (except in degenerate cases). This is witnessed by the appearance of coefficients $\beta_T \neq 1$ in Proposition 3.22.

4. A first analysis of $k\operatorname{Hom}_{FI}(a, b)$

In this Section, $k$ is a field of characteristic zero. In particular, the categories of representations that intervene are all semisimple.

4.1. The permutation representation. For $a, b \in \mathbb{N}$, $k\operatorname{Hom}_{FI}(a, b)$ is a $\mathcal{G}^{op}_a \times \mathcal{G}_b$-module. It is the permutation module associated to the transitive $\mathcal{G}_a^{op}$-set $\operatorname{Hom}_{FI}(a, b)$; in particular, it is non-zero if and only if $a \leq b$. If $a \leq b$, a generator is given by the canonical inclusion $i_{a:b} : a = \{1, \ldots, a\} \subset b = \{1, \ldots, b\}$.

\[ \text{Remark 4.1.} \] The inverse $g \mapsto g^{-1}$ gives the isomorphism of groups $\mathcal{G}_a \cong \mathcal{G}^{op}_a$, so that $\operatorname{Hom}_{FI}(a, b)$ can be considered as a $\mathcal{G}_a \times \mathcal{G}_b$-set and $k\operatorname{Hom}_{FI}(a, b)$ as a $\mathcal{G}_a \times \mathcal{G}_b$-module.

The inclusion $i_{a:b}$ determines $\mathcal{G}_a \subset \mathcal{G}_b$ and hence the Young subgroup $\mathcal{G}_a \times \mathcal{G}_{b-a} \subset \mathcal{G}_b$.

\[ \text{Lemma 4.2.} \] For $a \leq b \in \mathbb{N}$, considered as a $\mathcal{G}_b \times \mathcal{G}_a$-set, there is an isomorphism

\[ k\operatorname{Hom}_{FI}(a, b) \cong (\mathcal{G}_b \times \mathcal{G}_a)/(\mathcal{G}_{b-a} \times \Delta \mathcal{G}_a), \]

where $\Delta \mathcal{G}_a \subset \mathcal{G}_b \times \mathcal{G}_a$ is the diagonal inclusion.

Hence there is an isomorphism of $\mathcal{G}^{op}_a \times \mathcal{G}_b$-sets

\[ k\operatorname{Hom}_{FI}(a, b) \cong \mathcal{G}_b/\mathcal{G}_{b-a} \]

with the regular left $\mathcal{G}_b$-action and $\mathcal{G}_a$ acting via right multiplication on $\mathcal{G}_b$; i.e., for $g, g' \in \mathcal{G}_b$ and $h \in \mathcal{G}_a$, $g[g']h = [gg'h]$.

\[ \text{Proof.} \] The stabilizer of $i_{a:b}$ for the left $\mathcal{G}_b \times \mathcal{G}_a$-action on $k\operatorname{Hom}_{FI}(a, b)$ is the subgroup $(\mathcal{G}_{b-a} \times \Delta \mathcal{G}_a)$, which gives the first statement.

As a left $\mathcal{G}_b$-set, the quotient by the action of the diagonal subgroup $\Delta \mathcal{G}_a$ clearly identifies as $\mathcal{G}_b/\mathcal{G}_{b-a}$. It remains to consider the $\mathcal{G}_a$-action. Considered now as a right action, one checks that this is as given. (Note that this is well-defined, since the right action of $\mathcal{G}_a$ on $\mathcal{G}_b$ commutes with that of $\mathcal{G}_{b-a}$, due to the inclusion $\mathcal{G}_a \times \mathcal{G}_{b-a} \subset \mathcal{G}_b$.)

We next identify the composition factors of the permutation module $k\operatorname{Hom}_{FI}(a, b)$. For this, recall that the set of isomorphism classes of simple $\mathcal{G}_b \times \mathcal{G}_a$-modules is indexed by pairs of partitions $(\lambda \vdash b, \nu \vdash a)$, corresponding to $S^\lambda \boxtimes S^\nu$.

Recall the notation $\hat{\lambda}$ introduced in Notation 3.7 which has the property that, for $\mu \subseteq \lambda$, $\lambda/\mu$ is a horizontal strip if and only if $\hat{\lambda} \leq \mu$.

\[ \text{Proposition 4.3.} \] For $a \leq b \in \mathbb{N}$, there is an isomorphism of $\mathcal{G}_b \times \mathcal{G}_a$-modules

\[ k\operatorname{Hom}_{FI}(a, b) \cong \bigoplus_{\lambda \vdash b, \nu \vdash a, \hat{\lambda} \leq \nu \leq \lambda} S^\lambda \boxtimes S^\nu. \]

In particular, the $\mathcal{G}_b \times \mathcal{G}_a$-module $k\operatorname{Hom}_{FI}(a, b)$ is multiplicity-free.

\[ \text{Proof.} \] To prove the result, it suffices to show that

\[ k\operatorname{Hom}_{\mathcal{G}_a \times \mathcal{G}_b} (k\operatorname{Hom}_{FI}(a, b), S^\lambda \boxtimes S^\nu) = \begin{cases} 1 & \lambda \vdash b, \nu \vdash a, \hat{\lambda} \leq \nu \leq \lambda \\ 0 & \text{otherwise.} \end{cases} \]

By Lemma 4.2, $k\operatorname{Hom}_{\mathcal{G}_a \times \mathcal{G}_b} (k\operatorname{Hom}_{FI}(a, b), S^\lambda \boxtimes S^\nu) \cong (S^\lambda \boxtimes S^\nu)_{\mathcal{G}_{b-a} \times \Delta \mathcal{G}_a}$. The right hand side only depends on the restriction of the $\mathcal{G}_b$-action on $S^\lambda$ to $\mathcal{G}_{b-a} \times \mathcal{G}_a$ and, by Proposition 3.11, identifies with

\[ \bigoplus_{\lambda \vdash b, \nu \vdash a, \hat{\lambda} \leq \nu \leq \lambda} (S^\lambda \boxtimes S^\nu)_{\mathcal{G}_{b-a} \times \mathcal{G}_a}, \]

where $\mathcal{G}_{b-a}$ acts on $S^\lambda/\mu$ and $\mathcal{G}_a$ acts diagonally on $S^\nu$. This identifies as:

\[ \bigoplus_{\lambda \vdash b, \nu \vdash a, \hat{\lambda} \leq \nu \leq \lambda} (S^\lambda \boxtimes S^\nu)_{\mathcal{G}_a}. \]
Now, as recalled in Section 3.4,
\[(S^\lambda/\mu)_{\Theta_{a-\nu}} \cong \begin{cases} k \quad \tilde{\lambda} \preceq \mu \preceq \lambda \\ 0 \quad \text{otherwise}, \end{cases}\]
and, as in equation (3.1),
\[(S^\mu \otimes S^\nu)_{\Theta_{a}} = \begin{cases} k \quad \mu = \nu \\ 0 \quad \text{otherwise}. \end{cases}\]

The result follows. \hfill \Box

4.2. An explicit map. In this subsection, Proposition 4.3 is refined by exhibiting an explicit non-trivial map from \(k\text{Hom}_{FI}(a, b)\) to \(S^\lambda \otimes S^\nu\).

Notation 4.4. For \(\nu \preceq \lambda\), let \(\text{Tab}(\lambda; \nu)\) be the set of standard tableaux \(T\) for which the restriction \(T|_{\nu}\) to the diagram \(\nu\) belongs to \(\text{Tab}(\nu)\) (i.e., contains only the numbers \(\{1, \ldots, |\nu|\}\)).

The following is clear:

Lemma 4.5. For \(\nu \preceq \lambda\), the map
\[\text{Tab}(\lambda; \nu) \to \text{Tab}(\nu) \times \text{Tab}(\lambda/\nu)\]
\[T \mapsto (T|_{\nu}, T|_{\lambda/\nu})\]
is a bijection, where the restriction \(T|_{\lambda/\nu}\) to the skew diagram \(\lambda/\nu\) is considered as an element of \(\text{Tab}(\lambda/\nu)\) by relabelling (i.e., subtracting \(|\nu|\) from the entries).

Proposition 4.6. For \(\lambda \vdash b\), \(\nu \vdash a\) such that \(\tilde{\lambda} \preceq \nu \preceq \lambda\), the element
\[X_{\lambda, \nu} = \sum_{T \in \text{Tab}(\lambda; \nu)} \beta_{T|_{\lambda/\nu}} w_T \otimes w_{T|_{\nu}} \in S^\lambda \otimes S^\nu\]
is a generator of \((S^\lambda \otimes S^\nu)_{\Theta_{b-a}, \Delta_{\Theta_a}} \cong k\), where the coefficients \(\beta_a\) are given by Proposition 3.2.

Hence a generator of \(\text{Hom}_{\Theta_b \times \Theta_a}(k\text{Hom}_{FI}(a, b), S^\lambda \otimes S^\nu)\) is given by \(a, b \mapsto X_{\lambda, \nu}\).

Proof. Suppose that \(T \in \text{Tab}(\lambda; \nu)\); then restricting to \(\Theta_{b-a} \times \Theta_a\) and exploiting the bijection of Lemma 4.5, the action of the relevant Coxeter generators on \(w_T\) corresponds to that on
\[\Phi_{T|_{\lambda/\nu}} \otimes w_{T|_{\nu}}\]
where \(\Theta_{b-a}\) acts on the first factor and \(\Theta_a\) on the second. The first statement then follows by putting together Lemma 3.20 and Proposition 3.22.

The second statement then follows from Proposition 4.3 and its proof. \hfill \Box

4.3. Comparing representations. Our ultimate goal is to understand the cokernel of
\[\text{Tr}_{a, b} : k\text{Hom}_{FI}(a - 1, b) \uparrow_{\Theta_a} \to k\text{Hom}_{FI}(a, b).\]
In this subsection, we only compare these representations.

If \(1 \leq a \leq b\), then Proposition 4.3 applies also to \(k\text{Hom}_{FI}(a - 1, b)\), considered as a \(\Theta_b \times \Theta_{a-1}\)-module. One can then induce up to a \(\Theta_b \times \Theta_a\)-module, giving the following Corollary, in which the second isomorphism is given by Pieri’s rule:

Corollary 4.7. For \(1 \leq a \leq b \in \mathbb{N}\), there is an isomorphism of \(\Theta_b \times \Theta_a\)-modules
\[k\text{Hom}_{FI}(a - 1, b) \uparrow_{\Theta_a} \cong \bigoplus_{\lambda \vdash b, \nu \vdash a} \bigoplus_{\lambda \geq \nu \geq 1} S^\lambda \otimes (S^\nu \uparrow_{\Theta_a}).\]

As an immediate consequence, one has:

Lemma 4.8. For \(\lambda \vdash b \in \mathbb{N}\), a composition factor \(S^\lambda \otimes S^\nu\) of \(k\text{Hom}_{FI}(a, b)\) does not occur in \(k\text{Hom}_{FI}(a - 1, b) \uparrow_{\Theta_a}\) if and only if \(\nu = \tilde{\lambda}\) and \(\lambda_1 = b - a\).

Hence, there is a bijection between the set of such composition factors and both of the following:

(1) \(\{\lambda \vdash b \mid \lambda_1 = b - a\}\);
(2) \(\{\nu \vdash a \mid \nu_1 \leq b - a\}\).

Proof. Suppose that \(S^\lambda \otimes S^\nu\) is a composition factor of \(k\text{Hom}_{FI}(a, b)\) and that \(\lambda/\nu\) is not the maximal horizontal strip (i.e., \(\tilde{\lambda} \preceq \nu\) is not an equality). Then consider the rightmost non-empty column of \(\lambda\) in which \(\lambda/\nu\) is empty; such a column exists by the hypothesis. Let \(\kappa \vdash a - 1\) be the unique partition obtained from \(\nu\) by decreasing the length of this column by one (the verification that \(\kappa\) is a partition is left to the reader).

By construction, \(\lambda/\kappa\) is a horizontal strip and \(S^\kappa \uparrow_{\Theta_a}\) contains \(S^\nu\) as a composition factor, by Pieri’s rule.

It follows from Corollary 4.7 that \(S^\lambda \otimes S^\nu\) is a composition factor of \(k\text{Hom}_{FI}(a - 1, b) \uparrow_{\Theta_a}\).

If \(\nu = \tilde{\lambda}\), then the composition factor \(S^\lambda \otimes S^\nu\) cannot arise in \(k\text{Hom}_{FI}(a - 1, b) \uparrow_{\Theta_a} \).

\hfill \Box
5. Calculating the cokernel of $\text{Tr}_{a,b}$

Fix $1 \leq a \leq b$; our aim is to determine the cokernel of

$$\text{Tr}_{a,b} : k\text{Hom}_{\text{FI}}(a - 1, b) \uparrow_{\mathfrak{S}_{a-1}} \to k\text{Hom}_{\text{FI}}(a, b)$$

as a $\mathfrak{S}^\mathfrak{op}_a \times \mathfrak{S}_b$-module.

Recall that, for $\nu \preceq \lambda$, $\lambda/\nu$ is a horizontal strip if and only if $\hat{\lambda} \preceq \nu$. The following strengthens Lemma 4.8 to describe $\text{Tr}_{a,b}$:

**Theorem 5.1.** For $1 \leq a \leq b$, there is an isomorphism of $\mathfrak{S}_b \times \mathfrak{S}_a$-modules:

$$\text{Coker} \text{Tr}_{a,b} \cong \bigoplus_{\lambda_1=b-a} S^\lambda \boxtimes \bar{S}^\lambda.$$

The proof of this result occupies the whole of this Section.

5.1. A first reduction. By Lemma 4.5 and Proposition 4.6 to prove the Theorem it is sufficient to prove the following statement:

**Proposition 5.2.** Suppose that $\lambda \vdash b$ with $\lambda_1 > b - a$ and $\nu \vdash a$ such that $\hat{\lambda} \preceq \nu \preceq \lambda$. Then

$$\sum_{i \in b \setminus a - 1} (a, i)X_{\lambda,\nu} \neq 0,$$

where $(a, i) \in \mathfrak{S}_b$.

**Remark 5.3.** The hypothesis $\lambda_1 > b - a$ is equivalent to the assertion that $\nu \not\sim \lambda$.

Reasoning as in the proof of Lemma 4.8 one has:

**Lemma 5.4.** Suppose that $\lambda \vdash b$ with $\lambda_1 > b - a$ and $\nu \vdash a$ such that $\hat{\lambda} \preceq \nu \preceq \lambda$. Then there exists $\kappa \vdash a - 1$ such that $\hat{\lambda} \preceq \kappa \preceq \nu \preceq \lambda$. In particular, the skew diagram $\lambda/\kappa$ is a horizontal strip.

**Hypothesis 5.5.** Henceforth in this Section, we fix $\kappa, \nu, \lambda$ as in Lemma 5.4 so that $\lambda \vdash b$ such that $\lambda_1 > b - a$, $\nu \vdash a$, $\kappa \vdash a - 1$ with $\hat{\lambda} \preceq \kappa \preceq \nu \preceq \lambda$.

**Notation 5.6.** Denote by $\text{Tab}(\lambda/\kappa; \nu/\kappa) \subset \text{Tab}(\lambda/\kappa)$ the set of standard skew tableaux such that the restriction to $\nu/\kappa$ is standard (i.e., the single box is labelled by 1).

The following is clear (cf. Lemma 4.5):

**Lemma 5.7.** Restriction and relabelling yield a bijection:

$$\text{Tab}(\lambda/\kappa; \nu/\kappa) \xrightarrow{\cong} \text{Tab}(\lambda/\nu)$$

$$T \mapsto T|_{\lambda/\nu}.$$

**Notation 5.8.**

1. Let $T \in \text{Tab}(\lambda/\kappa; \nu/\kappa)$ be the unique standard tableau such that $T|_{\lambda/\nu} = T^{\text{rev}} \in \text{Tab}(\lambda/\nu)$.

2. Set

$$Y_{\lambda,\nu,\kappa} := \sum_{T \in \text{Tab}(\lambda/\kappa; \nu/\kappa)} \beta_T|_{\lambda/\nu} \Phi_T \in S^{\lambda/\kappa}.$$

The following is a consequence of Proposition 3.22; it explains the significance of $Y_{\lambda,\nu,\kappa}$.

**Lemma 5.9.** The element $Y_{\lambda,\nu,\kappa}$ is invariant under the action of $\mathfrak{S}_{b-a} \subset \mathfrak{S}_{b-a+1}$ corresponding to the inclusion $\{2, \ldots, b - a + 1\} \subset b - a + 1$.

Below we consider $S^{\lambda/\kappa}$ as a representation of $\text{Aut}(b - a + 1)$ rather than of the subgroup $\mathfrak{S}_{b-a+1} \subset \mathfrak{S}_b$ used above. Under this convention, the element $\sum_{i \in b \setminus a - 1} (a, i) \in \mathfrak{S}_b$ appearing in Proposition 4.6 is reindexed to give $\sum_{i \in b \setminus a + 1} (1, i) \in \mathfrak{S}_{b-a+1}$.

The following reduces the proof of Theorem 5.1 to establishing a property of the skew representation $S^{\lambda/\nu}$:

**Proposition 5.10.** Under Hypothesis 5.5

$$\sum_{i \in b - a + 1} (1, i)Y_{\lambda,\nu,\kappa} \neq 0.$$
Proof of Theorem 5.1 assuming Proposition 5.10. It suffices to show that Proposition 5.10 implies Proposition 5.2. Consider
\[ \sum_{i \in b \backslash a-1} (a, i)X_{\lambda, \nu} = \sum_{T \in \text{Tab}(\lambda, \mu)} \left( \sum_{i \in b \backslash a-1} \beta_{T|\lambda/\mu}(a, i)w_T \right) \otimes w_{T|\nu}. \]

The hypothesis on $\kappa$ ensures that there exists a tableau $\tilde{T} \in \text{Tab}(\nu)$ such that the box of $\nu/\kappa$ is labelled by $a$. Now we may restrict to considering the $T \in \text{Tab}(\lambda; \nu)$ such that $T|\nu = \tilde{T}$; such a tableau is determined by the standard skew tableau $T|_{\lambda/\nu}$.

To prove Proposition 5.2, it suffices to show that the corresponding term
\[ \sum_{T \in \text{Tab}(\lambda; \nu)} \sum_{T|\nu = \tilde{T}} \beta_{T|\lambda/\nu}(a, i)w_T = \sum_{i \in b \backslash a-1} (a, i) \left( \sum_{T \in \text{Tab}(\lambda; \nu)} \beta_{T|\lambda/\nu}w_T \right) \]
is non-zero.

The summation over $T \in \text{Tab}(\lambda; \nu)$ such that $T|\nu = \tilde{T}$ is equivalent to summing over $T' \in \text{Tab}(\lambda; \nu/\kappa)$ and one has the correspondence $w_T \leftrightarrow \Phi_T'$, compatibly with the respective actions of the symmetric groups. Thus, after reindexing and by definition of $Y_{\lambda, \nu, \kappa}$, Proposition 5.10 gives that the right hand side is non-zero, as required.

Remark 5.11.

1. A straightforward calculation using Lemma 5.9 shows that
\[ \sum_{i \in b \backslash a+1} (1, i)Y_{\lambda, \nu, \kappa} \]
lies in $(S^{\lambda/\kappa})^{\otimes b-a+1}$, i.e., is fully invariant. Since $(S^{\lambda/\kappa})^{\otimes b-a+1} = k$, establishing non-triviality of this element is equivalent to comparing it with the generator given in Proposition 5.22.

2. As in Proposition 3.21, $S^{\lambda/\kappa}$ is a permutation representation. The difficulty in proving Proposition 5.10 is that the element $Y_{\lambda, \nu, \kappa}$ is not defined in terms of the permutation basis. (Cf. Remark 3.23.)

5.2. A further reduction. Recall the skew tableau $T \in \text{Tab}(\lambda; \nu) \subset \text{Tab}(\lambda/\kappa)$ introduced in Notation 5.5. We work with the Young orthonormal basis $\{ \Phi_T | T \in \text{Tab}(\lambda/\kappa) \}$ for $S^{\lambda/\kappa}$.

Notation 5.12. Let $\theta(\lambda, \nu, \kappa) \in \mathbb{R}$ denote the coefficient of $\Phi_T$ in $\sum_{i \in b \backslash a+1} (1, i)Y_{\lambda, \nu, \kappa}$.

Proposition 5.10 clearly follows from:

Proposition 5.13. Under Hypothesis 5.5, $\theta(\lambda, \nu, \kappa) > 0$.

The proof of this result (and hence of Theorem 5.1) will be given in the next section.

Remark 5.14. The following example shows that $\theta(\lambda, \nu, \kappa)$ can be an arbitrarily small positive real number.

Take $\lambda$ to be the partition $(n, n-1)$ and $\kappa \leq \nu \leq \lambda$ to be given by $\kappa = (n-1)$ and $\nu = (n)$. Hence the tableau $T \in \text{Tab}(\lambda; \nu/\kappa)$ is the following (in which the position of 1 is given by the single box of $\nu/\kappa$):

\[
\begin{array}{ccccccccccc}
\text{1} & 2 & \ldots & b & \text{1} & \text{2} & \ldots & \text{b-1} & \text{1} & \text{2} & \ldots & \text{b-a+1} \\
\end{array}
\]

A straightforward calculation gives $\theta((n, n-1), (n), (n-1)) = \frac{1}{n}$.

5.3. Proof of Proposition 5.13. Throughout this section, Hypothesis 5.5 is in force.

In order to exploit the explicit action of the symmetric group given by the action of the Coxeter generators, the following elementary result is used:

Lemma 5.15. For $n \in \mathbb{N}$ and $1 \leq j < n$, in $\mathfrak{S}_n$ one has $(1, j+1) = s_j(1, j)s_j$.

Notation 5.16. Set $\rho_2 = \text{id}$ and, recursively for $j > 2$, $\rho_j = s_{j-1}\rho_{j-1}$, so that $\rho_j$ is the cycle $(j, j-1, \ldots, 2)$ given as an explicit product of Coxeter generators.

Lemma 5.17.
\[ \sum_{i \in b \backslash a+1} (1, i)Y_{\lambda, \nu, \kappa} = Y_{\lambda, \nu, \kappa} + \sum_{j=2}^{b-a+1} \rho_j(12)Y_{\lambda, \nu, \kappa}. \]

Proof. Lemma 5.13 implies that, for $j \geq 2$, $(1, j) = \rho_j(1, 2)\rho_j^{-1}$. Now $\rho_j^{-1}$ lies in the subgroup $\text{Aut}((2, \ldots, b-a+1)) \cong \mathfrak{S}_{b-a} \subset \mathfrak{S}_{b-a+1} \cong \text{Aut}(b-a)$, hence acts trivially upon $Y_{\lambda, \nu, \kappa}$, by Lemma 5.9. The result follows.

Recall that, for $T \in \text{Tab}(\lambda/\kappa)$ and a Coxeter generator $s_i$, one has the two, mutually exclusive possibilities:

1. $i, i+1$ are in the same row of $T$ and $s_i\Phi_T = \Phi_T$;
2. $s_iT \in \text{Tab}(\lambda/\kappa)$ (i.e., $s_iT$ is standard) and $s_i\Phi_T$ is a linear combination of $\Phi_T$ and $\Phi_{s_iT}$.

This places a strong restriction on the tableaux in $\text{Tab}(\lambda; \nu/\kappa)$ for which the corresponding term in $Y_{\lambda, \nu, \kappa}$ can contribute to $\theta(\lambda, \nu, \kappa)$, motivating the following definition:
Definition 5.18.

(1) If \( b > a \), set \( \langle T \rangle = \{ T \} \). If \( b > a \), let \( \langle T \rangle \subset \text{Tab}(\lambda/\kappa; \nu/\kappa) \) be the set of standard tableaux of the following form:

\[
s^{a}_{2} s^{a}_{3} \ldots s^{b-a}_{b} T
\]

where \( \epsilon_{i} \in \{0, 1\} \) such that \( \epsilon_{i} = 0 \) if \( i, i + 1 \) lie in the same row of the standard tableau \( s^{a}_{i+1} \ldots s^{b-a}_{b} T \).

By convention, \( \epsilon_{1} := 0 \); in particular, if \( b - a = 1 \), then \( \langle T \rangle = \{ T \} \).

(2) Set

\[
\lambda, \nu, \kappa := \sum_{T \in \langle T \rangle} \beta_{T} \Phi_{T} \in S^{\lambda/\kappa}.
\]

Remark 5.19. The ‘admissibility’ condition imposed on the \( \epsilon_{i} \) in the definition of \( \langle T \rangle \) ensures that, given a tableau \( s^{a}_{2} s^{a}_{3} \ldots s^{b-a}_{b} T \) in \( \langle T \rangle \), for all \( 1 \leq k \leq b-a \), the tableau:

\[
s^{a}_{k} \ldots s^{b-a}_{b} T
\]

is standard.

Lemma 5.20. \( \theta(\lambda, \nu, \kappa) \) is equal to the coefficient of \( \Phi_{T} \) in

\[
\lambda, \nu, \kappa := \sum_{j=2}^{b-a+1} \rho_{j}(12) \lambda, \nu, \kappa.
\]

Proof. By Lemma 5.17, \( \theta(\lambda, \nu, \kappa) \) is equal to the coefficient of \( \Phi_{T} \) in \( Y_{\lambda, \nu, \kappa} + \sum_{j=2}^{b-a+1} \rho_{j}(12) Y_{\lambda, \nu, \kappa} \).

By construction, \( \lambda, \nu, \kappa \) is the sum of the terms appearing in \( Y_{\lambda, \nu, \kappa} \) that can contribute non-trivially to \( \theta(\lambda, \nu, \kappa) \).

The cases of small \( b-a \) illustrate the basic behaviour. For \( b-a \in \{0, 1\} \), one has:

Example 5.21.

(1) If \( b-a = 0 \), then \( \lambda, \nu, \kappa = \Phi_{T} \) where \( T \) has a single box, labelled by 1. Clearly \( \theta(\lambda, \nu, \kappa) = 1 \) in this case.

(2) If \( b-a = 1 \), then again one has \( \lambda, \nu, \kappa = \Phi_{T} \), where now \( T \) has two boxes, labelled 1 and 2. In this case, \( \theta(\lambda, \nu, \kappa) \) is the coefficient of \( \Phi_{T} \) in \( \Phi_{T} + s_{1} \Phi_{T} \). This is equal to \( 1 + \frac{1}{r_{1}(T)} \). Since \( r_{1}(T) \in \mathbb{Z}\{0, 1\} \), this is well-defined and positive.

The next case exhibits crucial new ingredients:

Example 5.22. Suppose that \( b-a = 2 \), thus \( T \) has three boxes, labelled 1, 2 and 3, and \( r_{2}(T) \geq 1 \), with equality if and only if 2 and 3 occur in the same row. In this case:

\[
\lambda, \nu, \kappa = \Phi_{T} + \beta \Phi_{s_{2}T}
\]

where \( \beta = \frac{1}{r_{1}(T)} \) for \( r = r_{2}(T) \). (Here, if \( s_{2}T \) is not standard, \( \Phi_{s_{2}T} \) should be understood to be zero; one also has \( \beta = 0 \).)

By definition, \( \theta(\lambda, \nu, \kappa) \) is the coefficient of \( \Phi_{T} \) in

\[
\lambda, \nu, \kappa + s_{1} \lambda, \nu, \kappa + s_{2} s_{1} \lambda, \nu, \kappa.
\]

Only \( \Phi_{T} \) contributes to the first two expressions, giving \( 1 + \frac{1}{r_{1}(T)} \), as in the previous example.

For the final term:

(1) \( s_{2} s_{1} \Phi_{T} \) contributes \( \frac{1}{r_{1}(s_{2}T)} \); and

(2) suppose that \( \beta \neq 0 \), then \( s_{2} s_{1} \beta \Phi_{s_{2}T} \) contributes

\[
\frac{1}{r_{1}(s_{2}T)} \beta \left( 1 - \frac{1}{r_{2}(s_{2}T)} \right) = \frac{1}{r_{1}(s_{2}T)} \left( r_{2}(T) \right) - 1 \left( r_{2}(T) \right) - 1 \left( r_{1}(T) \right) = \frac{1}{r_{1}(s_{2}T)} \left( r_{2}(T) \right) - 1 \left( r_{1}(T) \right)
\]

by using the action of the Coxeter generators on the Young orthonormal basis, the fact that \( r_{2}(s_{2}T) = -r_{2}(T) \) and that \( r = r_{2}(T) \). \( \geq 1 \).

Summing these two contributions gives:

\[
\frac{1}{r_{1}(s_{2}T)} \left( r_{1}(s_{2}T) \right) + r_{2}(T) - 1 \left( r_{1}(T) \right) = \frac{1}{r_{1}(s_{2}T)} \left( 1 + \frac{1}{r_{1}(T)} \right),
\]

where the equality is obtained by using the identity (see Lemma 5.17):

\[
a(3, 1)(s_{2}T) = r_{2}(s_{2}T) + r_{1}(s_{2}T),
\]

which, since \( a(3, 1)(s_{2}T) = r_{1}(T) \), gives \( r_{1}(s_{2}T) = r_{1}(T) + r_{2}(T) \).
Putting these facts together, one gets:

$$\theta(\lambda, \nu, \kappa) = \begin{cases} 
(1 + \frac{1}{r_1(T)}) (1 + \frac{1}{r_2(T)}) s_2 T & \text{standard} \\
1 + 2 \frac{1}{r_1(T)} & \text{otherwise}.
\end{cases}$$

A straightforward verification (using that $r_1(T) \neq 1$) shows that these expressions are equal if one uses the identity $r_1(s_2 T) = r_1(T) + r_2(T)$ to define $r_1(s_2 T)$ when $s_2 T$ is not standard, since $r_2(T) = 1$ in this case. Equivalently, one can replace $r_1(s_2 T)$ by $a(3, 1)(T)$, giving the unified expression:

$$\theta(\lambda, \nu, \kappa) = \left(1 + \frac{1}{r_1(T)}\right) \left(1 + \frac{1}{a(3, 1)(T)}\right).$$

The factor $\left(1 + \frac{1}{a(3, 1)(T)}\right)$ should be interpreted as being the value of $\theta$ obtained when replacing $\lambda/\kappa$ by the skew-diagram given by omitting the box labelled by 2 and reindexing. This fits into a general inductive scheme, as below.

We next give an explicit expression for $\theta(\lambda, \nu, \kappa)$ (see Proposition 5.27), using the ingredients used in Example 5.22.

**Notation 5.23.** For $T$ in $\langle T \rangle$, let $\theta(\lambda, \nu, \kappa)(T)$ be the coefficient of $\Phi_T$ in

$$\beta_{T|\lambda,\nu}(\Phi_T + \sum_{j=2}^{b-a+1} \rho_j(12)\Phi_T).$$

Lemma 5.20 implies:

**Lemma 5.24.** $\theta(\lambda, \nu, \kappa) = \sum_{T \in \langle T \rangle} \theta(\lambda, \nu, \kappa)(T)$.

To state Proposition 5.27, we require the following notation:

**Notation 5.25.** Suppose that $b - a \geq 1$ and $T \in \langle T \rangle$ given by the sequence $(\epsilon_j)_{2 \leq j \leq b-a}$.

1. for $1 \leq i \leq b-a$, denote by $r_i^j(T)$ the axial distance from $i+1$ to $i$ in $s_i^{b-a} T$ (thus $r_i^1(T) = r_1(T)$);
2. denote by $J(T) := \inf\{i \mid \epsilon_j = 0 \forall j > i\}$.

We note the following important property of the axial distances $r_i^j(T)$:

**Lemma 5.26.** For $T \in \langle T \rangle$ given by the sequence $(\epsilon_j)_{2 \leq j \leq b-a}$ and $2 \leq j \leq b-a$:

$$r_j^j(T) \geq 1$$

with equality if and only if $j, j+1$ lie in the same row of $s_i^{b-a} T$.

If $T = T^{rev}$ (i.e., the box labelled by 1 is the leftmost box of the skew tableau $T$), then $r_j^j(T) \geq 1$.

**Proof.** The first statement is proved by increasing induction upon $b-a$ by using Lemma 8.17.

The box labelled 1 does not intervene, hence (up to relabelling) one may restrict to $\lambda/\nu$. Restricted to $\lambda/\nu$, by construction one has $T \equiv T^{rev}$. In particular, this gives $r_j^i(T) = r_i(T) \geq 1$ by Lemma 8.17. The statement on the equality is clear.

For the inductive step, consider $s_i^{b-a} T$. Forgetting the box labelled by $b-a + 1$ (this is the maximal label, after the relabelling), this gives a sub skew diagram $\lambda'/\kappa$ and the corresponding restricted tableau is $T \in \text{Tab}(\lambda'/\kappa; \nu/\kappa)$. Hence the inductive hypothesis applies.

Finally, if $T = T^{rev}$, it is clear that $r_1(T) \geq 1$ also.

**Proposition 5.27.** Suppose that $b - a \geq 1$. For $T \in \langle T \rangle$ given by the sequence $(\epsilon_j)_{2 \leq j \leq b-a}$,

$$\theta(\lambda, \nu, \kappa)(T) = \delta_{T,T} + \sum_{k=J(T)+1}^{b-a} \prod_{i=1}^{k} \left(\frac{1}{r_i^j(T)}\right)^{1-\epsilon_j} \left(r_j^j(T) - 1\right)^{\epsilon_j},$$

where $\delta_{T,T}$ is 1 if $T = T$ and 0 otherwise.

**Proof.** Firstly, Proposition 5.22 allows the calculation of $\beta_{T|\lambda,\nu}$ by a straightforward induction on $b-a$. Namely, one shows that

$$\beta_{T|\lambda,\nu} = \prod_{j=2}^{b-a} \left(\frac{r_j^j - 1}{r_j^j + 1}\right)^{\epsilon_j}$$

(using $0^0 = 1$ in the case $r_j^j = 1$, for which $\epsilon_j = 0$).

Now, consider $\Phi_T + \sum_{j=2}^{b-a} \rho_j(12)\Phi_T$. Clearly $\Phi_T$ has a non-trivial coefficient of $\Phi_T$ if and only if $T = T$, which is equivalent to $\epsilon_j = 0$ for $1 \leq j \leq b-a$. In this case, $\beta_{T|\lambda,\nu} = 1$, so this accounts for the term $\delta_{T,T}$ in the statement.
Now consider $\rho_k(12) \Phi_T$ for $2 \leq k \leq b-a+1$. A straightforward calculation using the action of the Coxeter generators on the Young orthonormal basis shows that the coefficient of $\Phi_T$ is non-zero if and only if $\epsilon_i = 0$ for all $i > k$. This condition is equivalent to $k \geq J(T)$, by definition of the latter.

Moreover, in this case, arguing as in Example 5.22, the coefficient is equal to

$$\prod_{1 \leq j \leq k} \left( \frac{1}{r_j^T} \right)^{1-\epsilon_j} \left( \sqrt{\frac{r_j^T}{r_j^T-1}} \right)^{\epsilon_j}.$$

Again as in the Example, using the identity

$$\sqrt{\frac{r_j^T}{r_j^T-1}} \frac{r_j^T - r_j^T}{r_j^T} = \frac{r_j^T - 1}{r_j^T},$$

under the above hypothesis that $\epsilon_j = 0$ for all $i > k$, one has the equality

$$\beta_{\lambda/\nu} \prod_{1 \leq j \leq k} \left( \frac{1}{r_j^T} \right)^{1-\epsilon_j} \left( \sqrt{\frac{r_j^T}{r_j^T-1}} \right)^{\epsilon_j} = \prod_{1 \leq j \leq k} \left( \frac{1}{r_j^T} \right)^{1-\epsilon_j} \left( \frac{r_j^T - 1}{r_j^T} \right)^{\epsilon_j}.$$

Summing the terms over $k \geq J(T)$ gives the required result. \hfill \qed

Remark 5.28.

1. The expression in the statement of Proposition 5.27 is defined over $\mathbb{Q}$.
2. If $b-a \geq 1$, it can be rewritten as

$$\theta(\lambda, \nu, \kappa)(T) = \delta_{T,T} + \frac{1}{r_1^T} \prod_{k=J(T)}^{b-a} \prod_{2 \leq j \leq k} \left( \frac{1}{r_j^T} \right)^{1-\epsilon_j} \left( \frac{r_j^T - 1}{r_j^T} \right)^{\epsilon_j}.$$

Proposition 5.27 leads to the easy case of Proposition 5.13.

**Proposition 5.29.** Suppose that $\lambda/\nu$ is non-empty and that $\mathbb{T} = \mathbb{T}^{rev}$. Then $\theta(\lambda, \nu, \kappa) > 1$.

**Proof.** The hypothesis together with Lemma 5.24 ensure that $r_1^T > 0$ for all $1 \leq i \leq b - a$. Hence, by Proposition 5.27, all contributions to $\theta(\lambda, \nu, \kappa)$ are non-negative and the contribution from $\mathbb{T}$, $1 + r_1^{-1}$, is strictly greater than 1. \hfill \qed

It remains to treat the cases not covered by Proposition 5.29 namely when $\lambda/\nu$ is non-empty and $\mathbb{T} \neq \mathbb{T}^{rev}$. The difficulty in this case stems from the fact that, for $T \in \mathbb{T}$, $r_1^T$ need not be positive.

Under the above hypothesis, the left-most box in $\mathbb{T}$ is labelled by 2, with the box labelled 1 occurring as a (different) inner corner. Then, by ‘removing’ the box labelled 2 from the skew tableau and relabelling the remaining boxes of the skew tableau (other than 1) by $n \mapsto n - 1$, one obtains $\mathbb{T}$ associated to the sub skew diagram without the left-most box. This is the basis of an inductive strategy for understanding $\theta(\lambda, \nu, \kappa)$.

**Notation 5.30.** Suppose that $\lambda/\nu$ is non-empty and that $\mathbb{T} \neq \mathbb{T}^{rev}$. Let $\nu^+$ (respectively $\kappa^+$) be the partition obtained from $\nu$ (resp. $\kappa$) by increasing the length of the first column by one. (At the level of the Young tableaux of $\nu$ and $\kappa$, this corresponds to adding the box labelled 2 to the respective Young tableaux; respectively, for the skew tableaux $\lambda/\nu$ and $\lambda/\kappa$, it corresponds to removing it.)

**Remark 5.31.** The hypothesis upon $\mathbb{T}$ serves here only to eliminate the possibility that $\mathbb{T}$ contains the configuration $\equiv \equiv$, in which case it is not possible to ‘remove’ the box labelled by 2 and retain a skew diagram.

By construction, one has the following:

**Lemma 5.32.** Suppose that $\lambda/\nu$ is non-empty and that $\mathbb{T} \neq \mathbb{T}^{rev}$, then $\nu^+ \vdash a + 1$, $\kappa^+ \vdash a$ and

$$\hat{\lambda} \preceq \kappa^+ \preceq \nu^+ \preceq \lambda.$$

Moreover, the skew diagram $\lambda/\kappa^+$ is obtained from that of $\lambda/\kappa$ by removing the left-most box.

There is a bijection between $\text{Tab}(\lambda/\kappa^+; \nu^+/\kappa^+)$ and the subset of $\text{Tab}(\lambda/\kappa; \nu/\kappa)$ of tableaux such that the left-most box is labelled by 2.

**Notation 5.33.** Suppose that $\lambda/\nu$ is non-empty and that $\mathbb{T} \neq \mathbb{T}^{rev}$. For $T \in \text{Tab}(\lambda/\kappa^+; \nu^+/\kappa^+)$, denote by $T^{[2]} \in \text{Tab}(\lambda/\kappa^+; \nu^+/\kappa^+)$ the tableau obtained by the bijection of Lemma 5.32.

Denote by $\mathbb{T} \equiv \mathbb{Tab}(\lambda/\kappa^+; \nu^+/\kappa^+)$ the tableau associated to the triple $(\lambda, \nu^+, \kappa^+)$ as in Notation 5.8 (This is obtained from $\mathbb{T}$ by forgetting the box labelled 2 and renumbering.)

The definition of $(\mathbb{T})$ given in Definition 5.18 leads to the following, which is the basis for an inductive approach.

**Lemma 5.34.** Suppose that $|\lambda/\nu| \geq 2$ and that $\mathbb{T} \neq \mathbb{T}^{rev}$. The set $(\mathbb{T})$ decomposes as $(\mathbb{T}) = (\mathbb{T})_0 \sqcup (\mathbb{T})_1$, where $(\mathbb{T})_0 = \{T^{[2]}: T \in (\mathbb{T})\}$ and $(\mathbb{T})_1 = \{s_2(T^{[2]}): T \in (\mathbb{T})\}$ such that $s_2(T^{[2]})$ is standard.
Proof. By Definition 5.18 an element of \((T)\) is determined by an expression
\[ s_2^t s_3^x \ldots s_{n-1}^y T \]
for a unique sequence \((\epsilon_t)\). The decomposition into the two components corresponds to the two possibilities \(\epsilon_t \in \{0, 1\}\).

\[ \theta \]

Proposition 5.35. Suppose that \(|\lambda/\nu| \geq 2\) with \(T \neq T^{rev}\). Then
\[ \theta(\lambda, \nu, \kappa) = \left(1 + \frac{1}{r_1(T)}\right) \theta(\lambda, \nu^+, \kappa^+). \]

Remark 5.36. This result generalizes the case \(|\lambda/\nu| = 2\) that is treated in Example 5.22. In particular, in the example, it was shown that
\[ \theta(\lambda, \nu, \kappa) = \frac{1}{a(3, 1)(T)} \theta(\lambda, \nu^+, \kappa^+) \]
To conclude in this case, it suffices to observe that \(\theta(\lambda, \nu^+, \kappa^+) = \left(1 + \frac{1}{a(3, 1)(T)}\right)\), which follows from Example 5.21 using the fact that \(r_1(T) = a(3, 1)(T)\).

Proof of Proposition 5.35. For the purposes of this proof, we introduce the following notation:

1. For \(T' \in (T)\), define \(\theta_{\geq 2}(T')\) by
\[ \theta(\lambda, \nu, \kappa)(T') = \delta_{T', \lambda} \left(1 + \frac{1}{r_1(T')}\right) + \theta_{\geq 2}(T') \]

2. For \(T \in (T)\), define \(\theta_1(T)\) and \(\theta_1(T)\) by
\[ \theta(\lambda, \nu^+, \kappa^+)(T) = \delta_{T, \lambda} + \theta_1(T) = \delta_{T, \lambda} + \frac{1}{r_1(T)} \theta_1(T) \]
The key to the proof is to relate \(\theta_{\geq 2}(T^{[2]})\) and \(\theta_{\geq 2}(s_2 T^{[2]})\) (when \(s_2 T^{[2]}\) is standard) to \(\theta_1(T)\) and \(\theta_1(T)\).

First one checks (for example, using the explicit description given in Proposition 5.27) that
\[ \theta_{\geq 2}(T^{[2]}) = \frac{1}{r_1(T^{[2]}) r_2(T^{[2]})} \theta_1(T) \]
where the second equality follows from the relationship between \(\theta_1(T)\) and \(\theta_1(T)\).

Now, \(s_2 T^{[2]}\) is not standard if and only if the configuration \(s_3 T^{[2]}\) occurs in \(T^{[2]}\). In this case the additivity of axial distances given by Lemma 3.10 implies the equality
\[ r_1(T^{[2]}) = 1 + r_1(T^{[2]}), \]
using \(r_2(T^{[2]}) = 1\). Thus, in this case, one has the equality:
\[ \theta_{\geq 2}(T^{[2]}) = \left(1 + \frac{1}{r_1(T^{[2]})}\right) \theta_1(T), \]
using that \(r_1(T^{[2]}) = r_1(T)\), by construction of \(T^{[2]}\).

Now consider the case where \(s_2 T^{[2]}\) is standard. One has the identity \(r_1(T) = r_1(s_2 T^{[2]})\) and, reasoning as for \(\theta_{\geq 2}(T^{[2]}), one deduces the equality:
\[ \theta_{\geq 2}(s_2 T^{[2]}) = \frac{r_2(T^{[2]})}{r_2(T^{[2]})} \theta_1(T), \]
(come also Example 5.22).

Thus,
\[ \theta_{\geq 2}(T^{[2]}) + \theta_{\geq 2}(s_2 T^{[2]}) = \frac{1}{r_2(T^{[2]})} \left(\frac{r_1(T)}{r_1(T^{[2]})} + r_2(T^{[2]}) - 1\right) \theta_1(T). \]
Again using Lemma 3.10 one has \(r_1(T) = r_1(T^{[2]}) + r_2(T^{[2]}), so that
\[ \frac{r_1(T)}{r_1(T^{[2]})} + r_2(T^{[2]}) - 1 = r_2(T^{[2]})(1 + \frac{1}{r_1(T^{[2]})}). \]
This gives:
\[ \theta_{\geq 2}(T^{[2]}) + \theta_{\geq 2}(s_2 T^{[2]}) = \left(1 + \frac{1}{r_1(T)}\right) \theta_1(T). \]
Summing over \(T \in (T)\) gives the result, by Lemma 5.34 together with Lemma 5.24. \qed
Proof of Proposition 5.13. By inspection, the result holds in the case $|\lambda/\nu| \leq 1$ (for example, use Example 5.21). Proposition 5.29 establishes the result in the case $\mathbb{T} = T^{rev}$. These form the initial cases for an inductive proof.

The inductive step treats the case $|\lambda/\nu| \geq 2$ and $\mathbb{T} \neq T^{rev}$. Proposition 5.35 gives the equality:

$$\theta(\lambda, \nu, \kappa) = \left(1 + \frac{1}{r_1(\lambda)}\right) \theta(\lambda, \nu^+, \kappa^+).$$

The inductive hypothesis implies that $\theta(\lambda, \nu^+, \kappa^+) > 0$ and $(1 + \frac{1}{r_1(\lambda)}) \geq \frac{1}{2}$, since $\frac{1}{r_1(\lambda)} \geq -\frac{1}{2}$. The result follows.

6. Higher FI-homology of $k\text{Hom}_{FI}(-, b)^{tr}$

The purpose of this Section is to place Theorem 5.1 in the more general context of studying FI-homology. Recall that $\Sigma \subset \text{FI}$ is the maximal subgroupoid; the category of functors from $\text{FI}$ to $k$-modules is denoted $\mathcal{F}(\text{FI})$ (respectively $\mathcal{F}(\Sigma)$ for functors on $\Sigma$) and sometimes referred to as FI-modules (resp. $\Sigma$-modules).

There is an exact extension functor $\mathcal{F}(\Sigma) \rightarrow \mathcal{F}(\text{FI})$ that sends a $\Sigma$-module $G$ to the $\text{FI}$-module taking the same values and on which morphisms of $\text{FI}$ that are not bijections act via zero. This has a left adjoint $H^\text{FI}_0 : \mathcal{F}(\text{FI}) \rightarrow \mathcal{F}(\Sigma)$.

This identifies explicitly as follows: for $F$ a $\text{FI}$-module, $(H^\text{FI}_0 F)(a)$ is the cokernel of the map

$$F(a - 1) \overset{\theta_a}{\rightarrow} F(a)$$

induced by $\iota_a$. Using this, one has:

**Proposition 6.1.** For $a, b \in \mathbb{N}$, there is a natural isomorphism

$$\text{CokerTr}_{a,b} \cong (H^\text{FI}_0 k\text{Hom}_{FI}(-, b)^{tr})(a).$$

By definition, FI-homology is given by the left derived functors of $H^\text{FI}_0$. It is natural to seek to identify the functors

$$(a, b) \mapsto (H^\text{FI}_0 k\text{Hom}_{FI}(-, b)^{tr})(a)$$

for $n \in \mathbb{N}$, generalizing the case $n = 0$ given by Theorem 5.1.

The main result of this Section, Proposition 6.21 gives a lower bound for this FI-homology which, conjecturally, coincides with the FI-homology.

6.1. FI-homology - some recollections. The FI-homology of $F \in \text{Ob} \mathcal{F}(\text{FI})$ can be calculated as the homology of an explicit Koszul complex $\mathcal{R}^\text{FI}_{a,b} F$ in $\mathcal{F}(\Sigma)$ such that $\mathcal{R}^\text{FI}_{a,b}$ is an exact functor from $\mathcal{F}(\text{FI})$ to chain complexes in $\mathcal{F}(\Sigma)$.

To describe the part of the structure that is required below, first recall the convolution product on $\mathcal{F}(\Sigma)$. This is the symmetric monoidal structure on $\mathcal{F}(\Sigma)$.

$$G_1 \otimes G_2(X) = \bigoplus_{X = X_1 \sqcup X_2} G_1(X_1) \otimes G_2(X_2),$$

where the sum is taken over the set of ordered decompositions of $X$ into two subsets. (The $\Sigma$-module $k(0)$ is $k$ evaluated on $0$ and zero evaluated on a non-empty finite set.)

Next, recall the orientation $\Sigma$-module $\text{Or}$ that is given on $X \in \text{Ob} \Sigma$ by

$$\text{Or}(X) = A^{|X|}(kX),$$

the top exterior power of the $k$-linearization of $X$. Thus $\text{Or}(n)$ is the signature representation of $\Sigma_n$, for $n \in \mathbb{N}$. This is understood to have homological degree $n$.

For $F$ a $\text{FI}$-module, by restricting to $\Sigma$ one can form the convolution product $\text{Or} \circ F$ in $\mathcal{F}(\Sigma)$. The Koszul complex of $F$ in $\mathcal{F}(\Sigma)$ has the form

$$(\mathcal{R}^\text{FI}_{a,b} F, d) = (\text{Or} \circ F, d),$$

in which the differential takes into account the FI-module structure of $F$.

**Remark 6.2.** In Section 5.2, the associated complex given by passage to Schur bifunctors is described explicitly in the case of interest, i.e., when $F$ is $k\text{Hom}_{FI}(-, b)^{tr}$.

Evaluating on $a$, for $a \in \mathbb{N}$, in homological degree $n$ with $0 \leq n \leq a$, one has:

$$\mathcal{R}^\text{FI}_{n} F(a) \cong (\text{sgn} \otimes F(a - n)) 1_{\Sigma_n \times \Sigma_{a-n}}$$

as $\Sigma_{a}$-modules. For $n > a$, $\mathcal{R}^\text{FI}_{n} F(a) = 0$. 


6.2. The case \( F = k\text{Hom}_{\text{FI}}(-, b)^{tr} \). Henceforth \( k \) is taken to be a field of characteristic zero.

We fix \( b \in \mathbb{N} \) and consider the \( \text{FI} \)-module \( F = k\text{Hom}_{\text{FI}}(-, b)^{tr} \). Proposition 6.3 identified the composition factors of the underlying \( \Sigma \)-module in \( \mathfrak{S}_b \)-modules. Thus we can identify the composition factors occurring in the Koszul complex \( \mathfrak{S}_b \text{FI} F \), as below.

The following is the counterpart of Definition \( 6.6 \).

**Definition 6.3.** For partitions \( \lambda, \mu \) such that \( \mu \subseteq \lambda \), the skew partition \( \lambda/\mu \) is a vertical strip if it contains at most one box in each row.

The following gives the counterpart of \( \lambda \) of Notation \( 6.7 \).

**Notation 6.4.** For a partition \( \lambda \), denote by \( \tilde{\lambda} \) the partition \( \tilde{\lambda}_i = \lambda_i - 1 \) if \( \lambda_i > 0 \). (In terms of Young diagrams, \( \lambda \) is obtained from \( \tilde{\lambda} \) by removing the first column.)

**Remark 6.5.** Recall that the transpose of a partition \( \lambda \) is the partition \( \lambda^\dagger \) what has Young diagram given by reflecting that of \( \lambda \) in the diagonal, thus interchanging rows and columns. For instance, the transpose of \( (1^n) \) is \( (1^n) \) and vice versa.

If \( \mu \subseteq \lambda \) then \( \mu^\dagger \subseteq \lambda^\dagger \) and \( \lambda/\mu \) is a vertical strip if and only if \( \lambda^\dagger/\mu^\dagger \) is a horizontal strip. In particular, the ‘transpose’ to Lemma 6.9 gives that \( \lambda/\mu \) is a vertical strip if and only if \( \lambda \subseteq \mu \subseteq \lambda \).

Pieri’s rule gives:

**Lemma 6.6.** For \( \nu \vdash t \) and \( n \in \mathbb{N} \),

\[
(S^\nu \boxtimes \text{sgn}_n) \uparrow^{\mathfrak{S}_t \times \mathfrak{S}_n}_{\mathfrak{S}_t \times \mathfrak{S}_n} \cong \bigoplus_{\mu \vdash t+n} S^\mu.
\]

Equivalently, the sum is indexed over partitions \( \mu \) such that \( \nu \subseteq \mu \) and \( \lambda/\mu \) is a vertical strip.

**Hypothesis 6.7.** Henceforth the bidegree \( (a, b) \in \mathbb{N}^2 \) is fixed. (Note that we do not suppose that \( a \leq b \).)

Consider the \( n \)th homological degree term of the Koszul complex:

\[
(\text{sgn}_n \boxtimes k\text{Hom}_{\text{FI}}(a - n, b)) \uparrow^{\mathfrak{S}_a \times \mathfrak{S}_n}_{\mathfrak{S}_a \times \mathfrak{S}_n} \cong \bigoplus_{\nu \vdash a+n} S^\nu.
\]

This is zero if either \( a > b + n \) or \( a < n \).

**Notation 6.8.** For \( (a, b) \) and \( n \) as above and \( \nu \vdash a - n \), denote by \( \mathfrak{P}(a, b; \nu) \) the set of pairs of partitions \( (\lambda \vdash b, \mu \vdash a) \) such that the following conditions are satisfied:

1. \( \lambda \subseteq \nu \subseteq \lambda \);
2. \( \mu \subseteq \nu \subseteq \mu \).

(Thus \( \lambda/\nu \) is a horizontal strip and \( \mu/\nu \) is a vertical strip.)

Proposition 6.3, Lemma 6.6 together with the definition of \( \mathfrak{P}(a, b; \nu) \) imply the following:

**Lemma 6.9.** For \( (a, b) \) and \( n \) as above, there is an isomorphism of \( \mathfrak{S}_b \times \mathfrak{S}_a \)-modules:

\[
(\mathfrak{R}_a \text{FI} k\text{Hom}_{\text{FI}}(-, b))((a)) \cong \bigoplus_{\nu \vdash a - n} \bigoplus_{(\lambda, \mu) \in \mathfrak{P}(a, b; \nu)} S^\lambda \boxtimes S^\mu.
\]

**Remark 6.10.** Given \( (\lambda, \mu) \in \mathfrak{P}(a, b; \nu) \), the homological degree \( n \) can be recovered as \(|\mu/\nu|\).

6.3. Reindexing by \( \mathfrak{M} \). So as to focus upon the simple composition factors \( S^\lambda \boxtimes S^\mu \), we introduce the following:

**Notation 6.11.** For a fixed pair \( (\lambda \vdash b, \mu \vdash a) \) let \( \mathfrak{M}(\lambda, \mu) \) denote the set of partitions \( \nu \) such that both the following conditions hold:

1. \( \lambda \subseteq \nu \subseteq \lambda \);
2. \( \mu \subseteq \nu \subseteq \mu \).

**Notation 6.12.** For partitions \( \lambda, \mu \), let \( \lambda \cap \mu \) and \( \lambda \cup \mu \) be respectively the infimum and supremum with respect to \( \subseteq \). (With respect to the associated Young diagrams, these correspond to the intersection and the union, in the obvious sense.)

By construction, Lemma 6.9 can be reformulated as:

**Lemma 6.13.** For \( (a, b) \) and \( n \) as above, there is an isomorphism of \( \mathfrak{S}_b \times \mathfrak{S}_a \)-modules:

\[
(\mathfrak{R}_a \text{FI} k\text{Hom}_{\text{FI}}(-, b))((a)) \cong \bigoplus_{\lambda \vdash b, \mu \vdash a} \bigoplus_{\nu \in \mathfrak{M}(\lambda, \mu)} S^\lambda \boxtimes S^\mu.
\]

In particular, \( S^\lambda \boxtimes S^\mu \) occurs in the Koszul complex if and only if the following equivalent conditions are satisfied:
(1) \( \lambda \cap \mu \in \mathcal{M}(\lambda, \mu) \);
(2) \( \mathcal{M}(\lambda, \mu) \neq \emptyset \).

Moreover, \( \mathcal{M}(\lambda, \mu) \) is the total multiplicity of \( S^\lambda \otimes S^\mu \) in the Koszul complex (i.e., allowing arbitrary homological degree).

Clearly one has:

**Lemma 6.14.**

(1) Suppose that \( \nu \in \mathcal{M}(\lambda, \mu) \), then \( \nu \preceq (\lambda \cap \mu) \).
(2) Moreover, if \( \nu \preceq \gamma \preceq (\lambda \cap \mu) \) and \( \nu \in \mathcal{M}(\lambda, \mu) \), then \( \gamma \in \mathcal{M}(\lambda, \mu) \).

Recall that, if \( \nu' \preceq \nu \) with \( |\nu'| = |\nu| - 1 \), then \( \nu' \) is obtained from \( \nu \) by removing an outer corner of the Young diagram representing \( \nu \). (An outer corner of a Young diagram is a box without neighbours either to the right or below.) Then, if \( \nu \in \mathcal{M}(\lambda, \mu) \), it is natural to ask under what condition \( \nu' \) lies in \( \mathcal{M}(\lambda, \mu) \).

Given an outer corner, by hypothesis this belongs both to the diagram of \( \lambda \) and that of \( \mu \). Consider the boxes (potentially) to the right and below this outer corner:

Here the unlabelled box is an outer corner of \( \nu \), by hypothesis, so the boxes \( \square \) and \( \blacksquare \) do not belong to \( \nu \); they may possibly belong to either the diagram \( \lambda \) or to that of \( \mu \), but this is not necessarily the case.

**Lemma 6.15.** In the above situation, \( \nu' \in \mathcal{M}(\lambda, \mu) \) if and only if both the following conditions are satisfied:

1. the box \( \square \) does not belong to \( \mu/\nu \);
2. the box \( \blacksquare \) does not belong to \( \nu/\nu' \).

Proof. One checks that \( \mu/\nu' \) is a vertical strip if and only if \( \square \) does not belong to \( \mu/\nu \). The ‘transpose’ gives the second condition.

One also has:

**Lemma 6.16.** If \( \nu \in \mathcal{M}(\lambda, \mu) \), then \( (\lambda \cap \mu)/\nu \) is a skew diagram consisting of outer corners of \( \lambda \cap \mu \).

Proof. The condition on the skew diagram is equivalent to the following:

- the skew diagram \( (\lambda \cap \mu)/\nu \) does not contain a sub diagram of the form \( \square \) or \( \blacksquare \).

Suppose that \( (\lambda \cap \mu)/\nu \) contains \( \blacksquare \) then \( \mu/\nu \) does also, in particular it is not a vertical strip, contradicting the hypothesis \( \nu \in \mathcal{M}(\lambda, \mu) \).

The other case is treated by the ‘transpose’ argument.

Putting these points together, one arrives at the following concrete description of the set \( \mathcal{M}(\lambda, \mu) \):

**Proposition 6.17.** Suppose that \( (\lambda \cap \mu) \in \mathcal{M}(\lambda, \mu) \) (equivalently, that \( \mathcal{M}(\lambda, \mu) \neq \emptyset \)) and let \( \nu(\lambda, \mu) \preceq (\lambda \cap \mu) \) be the diagram obtained by removing all the outer corners of \( (\lambda \cap \mu) \) that satisfy the criterion of Lemma 6.15 (taking \( \nu = (\lambda \cap \mu) \)).

Then \( \mathcal{M}(\lambda, \mu) = \{ \nu'' | \nu(\lambda, \mu) \preceq \nu'' \preceq (\lambda \cap \mu) \} \). In particular, this has cardinal \( 2^{(\lambda \cap \mu)/\nu(\lambda, \mu)} \).

Proof. Most of the statement follows directly from Lemma 6.14 and Lemma 6.16. The only point that remains to be established is that \( \nu(\lambda, \mu) \) belongs to \( \mathcal{M}(\lambda, \mu) \). This is established by using the criterion of Lemma 6.15 and the observation that removing outer corners of \( \lambda \cap \mu \) does not affect the right and below neighbours of the remaining outer corners.

6.4. **Criticality.** One distinguishes the critical cases where \( \mathcal{M}(\lambda, \mu) = \{ \lambda \cap \mu \} \). By Lemma 6.13 (see also Proposition 6.17), for such cases \( S^\lambda \otimes S^\mu \) occurs with multiplicity one in the Koszul complex, in homological degree \( |\mu/(\lambda \cap \mu)| \).

**Notation 6.18.** Let \( \text{Crit}(a, b) \) denote the set of pairs of partitions \( (\lambda \vdash b, \mu \vdash a) \) such that \( \mathcal{M}(\lambda, \mu) = \{ \lambda \cap \mu \} \).

**Example 6.19.** The following examples given pairs \( (\lambda, \mu) \in \text{Crit}(a, b) \) for various values of \( a \) and \( b \). In the diagrams, \( \mu/(\lambda \cap \mu) \) is represented in gray and \( \lambda/(\lambda \cap \nu) \) in light gray; \( \lambda \cap \mu \) is in white. Hence the homological degree of the element of \( \mathcal{M}(\lambda, \mu) \) is the number of dark gray boxes.

In the following, the reader should check for themselves that the outer corners of \( \lambda \cap \mu \) cannot be removed, using the criterion of Lemma 6.15:

1. \( a = 1, b = 2, \) with \( \lambda = (1, 1), \) \( \mu = (1) \):

\[ \square \]

2. \( a = 4, b = 3, \) with \( \lambda = (3), \) \( \mu = (4) \):

\[ \square \square \square \square \]
Lemma 6.20. Suppose that \( \mu \preceq \lambda \). Then \( (\lambda, \mu) \in \text{Crit}(a, b) \) if and only if \( \mu = \lambda' \).

Proof. Since \( \mu \preceq \lambda \) by hypothesis, \( \mu = (\lambda \cap \mu) \).

Suppose that \( \mathcal{M}(\lambda, \mu) \) is non-empty, so that \( \mu = (\lambda \cap \mu) \in \mathcal{M}(\lambda, \mu) \). Since \( \lambda/\mu \) is a horizontal strip, one has \( \lambda \preceq \mu \). Suppose that this is not an equality, then there exists \( \mu' \) obtained by removing an outer corner from \( \mu \) such that \( \lambda \preceq \mu' \preceq \mu \). One checks that \( \mu' \in \mathcal{M}(\lambda, \mu) \), hence \( (\lambda, \mu) \notin \text{Crit}(a, b) \), a contradiction.

Conversely, if \( \mu = \lambda' \), one checks easily that \( (\lambda, \mu) \in \text{Crit}(a, b) \). \( \square \)

Proposition 6.21. The set \( \text{Crit}(a, b) \) is in bijection with the set of pairs of partitions \( (\gamma, \delta) \) that satisfy the following conditions:

1. \( \gamma \cap \delta = \gamma \cap \delta \);
2. \( |\gamma \cup \delta| = b \);
3. \( |\gamma \cup \delta| = a \).

The bijection is defined by setting:

\[
\lambda := \gamma \cup \tilde{\delta}, \\
\mu := \tilde{\gamma} \cup \delta.
\]

In particular, \( \mu \preceq \lambda \) if and only if \( \delta = (0) \) and \( \lambda \preceq \mu \) if and only if \( \gamma = (0) \).

Proof. One first checks that, given a pair \( (\gamma, \delta) \), the associated \( (\lambda, \mu) \) lies in \( \text{Crit}(a, b) \). The case \( \delta = (0) \) corresponds to Lemma 6.20 and the case \( \gamma = (0) \) is proved by the ‘transpose’ argument.

Hence suppose that both \( \gamma \) and \( \delta \) are non-trivial. Consider the case where \( \delta \) has one row and \( \gamma \) one column. The condition \( \gamma \cap \delta \preceq \gamma \cap \delta \) implies that \( \gamma \cup \delta \) has the form of a hook:

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

and this corresponds to a critical pair, by the criterion of Lemma 6.15. (Note that the light gray box, corresponding to \( \lambda/(\lambda \cap \mu) \) is to the left and below the dark gray box, which corresponds to \( \mu/(\lambda \cap \mu) \).)

The general case follows by applying the argument of Lemma 6.20 (and its transpose) to the parts of the diagram arising from \( \gamma \) and \( \delta \) respectively.

To show that this defines a bijection, given \( \lambda \) and \( \mu \) representing an element of \( \text{Crit}(a, b) \), we require to exhibit the appropriate \( \gamma \) and \( \delta \). The key observation is that (again using the colouring of Example 6.19), all light gray boxes must lie to the left and below the dark gray boxes, as explained below.

Suppose otherwise, then the edge of \( \lambda \cup \mu \) would contain a sub-hook of the form:

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

By the criterion of Lemma 6.15, one derives a contradiction to criticality (both white outer corners can be removed - and this applies even in the case \( \mu = \lambda' \)).

Hence define \( \delta \preceq \mu \) to be the smallest partition of the form \( (\mu_1, \ldots, \mu_s) \) with Young diagram containing all the dark gray boxes; \( \gamma \preceq \lambda \) is defined likewise using the transpose construction (thus columns) for the light gray boxes. It remains to check that \( (\gamma, \delta) \) satisfies the given conditions. This is left as an exercise for the reader. \( \square \)

There is a pleasing symmetry in the above construction, using the passage to the transpose partition \( \lambda \mapsto \lambda' \).
Corollary 6.22. There is a bijection \( \text{Crit}(a, b) \cong \text{Crit}(b, a) \) given by \( (\lambda, \mu) \mapsto (\mu^\dagger, \lambda^\dagger) \).

Under the bijection of Proposition 6.21, this corresponds to \( (\gamma, \delta) \mapsto (\delta^\dagger, \gamma^\dagger) \).

6.5. Homological consequences. Recall that the Koszul complex \( \mathcal{R}_a^{FI}(k\text{Hom}_{FI}(-, b))(a) \) is zero in homological degree \( > a \) and each of the terms is a direct sum of finitely many simple \( \mathcal{S}_b \times \mathcal{S}_a \)-modules. This allows one to reason in terms of the Grothendieck group of such modules and to consider the Euler-Poincaré characteristic of the complex, which coincides with that of its homology.

Notation 6.23. The class of a finite \( \mathcal{S}_b \times \mathcal{S}_a \)-module \( X \) in the Grothendieck group is denoted by \([X]\).

Using the notion of criticality, one deduces the following:

Proposition 6.24. For \( a, b \in \mathbb{N} \) and homological degree \( n \), there is an inclusion of \( \mathcal{S}_b \times \mathcal{S}_a \)-modules:

\[
\bigoplus_{(\lambda, \mu) \in \text{Crit}(a, b)} S^\lambda \otimes S^\mu \subset H^\text{FI}_n(k\text{Hom}_{FI}(-, b))(a).
\]

For \( n = 0 \) this is an isomorphism.

Moreover, there is an equality in the Grothendieck group of \( \mathcal{S}_b \times \mathcal{S}_a \)-modules

\[
\sum_{(\lambda, \mu) \in \text{Crit}(a, b)} (-1)^{\mu/(\lambda \cap \mu)} [S^\lambda \otimes S^\mu] = \sum_{n \in \mathbb{N}} (-1)^n [H^\text{FI}_n(k\text{Hom}_{FI}(-, b))(a)].
\]

Proof. By Lemma 6.13 for \( (\lambda, \mu) \in \text{Crit}(a, b) \), \( S^\lambda \otimes S^\mu \) occurs with total multiplicity one in the Koszul complex calculating the \( \text{FI} \)-homology \( H^\text{FI}_n(k\text{Hom}_{FI}(-, b))(a) \) and this factor is in the given homological degree. This gives the inclusion.

The case \( n = 0 \) follows from Lemma 6.20 and Theorem 5.1.

The final statement follows by analyzing Proposition 6.17 which gives

\[ \mathfrak{M}(\lambda, \mu) = \{ \nu'' | \nu''(\lambda, \mu) \leq \nu'' \leq (\lambda \cap \mu) \}, \]

where \( \nu'' \) represents an element in homological degree \( \mu/\nu'' \leq (\lambda \cap \mu) / \nu'' \).

Now \( (\lambda, \mu) \) is critical if and only if \( (\lambda \cap \mu) / \nu''(\lambda, \mu) > 0 \). If \( (\lambda, \mu) \) is not critical, then \( 2^{(\lambda \cap \mu)/\nu''(\lambda, \mu)} \) is even and one checks that the occurrences of \( S^\lambda \otimes S^\mu \) in the Koszul complex sum to zero on forming the Euler-Poincaré characteristic. Since this is equal to the Euler-Poincaré characteristic of the homology, the result follows. \( \square \)

On the basis of Proposition 6.23 one can optimistically conjecture the following:

Conjecture 6.25. For \( a, b \in \mathbb{N} \) and homological degree \( n \), there is an isomorphism of \( \mathcal{S}_b \times \mathcal{S}_a \)-modules:

\[ H^\text{FI}_n(k\text{Hom}_{FI}(-, b))(a) \cong \bigoplus_{(\lambda, \mu) \in \text{Crit}(a, b)} S^\lambda \otimes S^\mu. \]

6.6. A possible approach. We outline in this subsection a possible strategy for attacking the conjecture, based on the dévissage provided by Proposition 6.27 below.

Write \( k(1) \) for the \( \text{FI} \)-module that is zero on \( \mathfrak{n} \) unless \( n = 1 \), when it takes value \( k \) (with the only possible action of \( \mathcal{S}_1 \)). Then the following is standard:

Lemma 6.26. For \( F \in \text{Ob} \mathcal{F}(\mathcal{F}_1) \), \( k(1) \circ F \) carries a canonical \( \mathcal{F}_1 \)-module structure induced by that of \( F \).

Proof. For a finite set \( U \), \( k(1) \circ F(U) = \bigoplus_{U' \subseteq U} F(U') \). Given an injection of finite sets \( i : U \hookrightarrow V \), and \( U' = U \setminus \{x\} \), take \( V' := V \setminus \{i(x)\} \), so that \( i \) induces \( i' : U' \hookrightarrow X' \). This construction gives rise to the required \( k(1) \circ F(U) \Rightarrow k(1) \circ F(V) \).

\( \square \)

This allows the statement of the following result, which is the basis for an inductive analysis of the functors \( k\text{Hom}_{FI}(-, b) \):

Proposition 6.27. For \( b \in \mathbb{N} \), there is a short exact sequence of \( \mathcal{F}_1 \)-modules with values in \( k[\mathcal{S}_{b-1}] \)-modules:

\[ 0 \to k(1) \circ k\text{Hom}_{FI}(-, b - 1) \to k\text{Hom}_{FI}(-, b) \xrightarrow{\pi} k\text{Hom}_{FI}(-, b - 1) \to 0 \]

where the surjection \( \pi \) is the retract of the canonical inclusion \( k\text{Hom}_{FI}(-, b - 1) \subset k\text{Hom}_{FI}(-, b) \) that sends a generator \([f] \), where image\((f) \not\in b \), to zero.
proof. The key point is that $\pi$ is a morphism of $\mathbf{FI}$-modules. This is an elementary, but important verification. That it is $\mathfrak{S}_{n-1}$-equivariant is clear.

To complete the proof, it remains to identify the kernel; this follows from the observation that, given $W \subset a$ with $|W| = a - 1$, there is an isomorphism

$$\text{Hom}_{\mathbf{FI}}(W, b - 1) \cong \text{Hom}_{\mathbf{FI}}(a, b)$$

given by sending the element of $a/W$ to $b \in b$. Conversely, given $f \in \text{Hom}_{\mathbf{FI}}(a, b)$ such that $f \not\subseteq b - 1$, taking $W := a \setminus f^{-1}(b)$, $f$ is the image of $f|_W$.

Putting these points together, one obtains the result. \hfill \square

Applying $H^\text{FI}_{\mathbf{FI}}$ to the short exact sequence of Proposition 6.27 gives a long exact sequence:

$$\cdots \to \mathbb{k}(1) \odot H_n^\text{FI}_k \text{Hom}_\text{FI}(\bullet, b - 1) \to H_n^\text{FI}_k \text{Hom}_\text{FI}(\bullet, b) \odot \mathfrak{S}_{n-1} \to H_{n-1}^\text{FI}_k \text{Hom}_\text{FI}(\bullet, b - 1) \to \cdots$$

using that $\mathbb{k}(1) \odot -$ commutes with the formation of $H^\text{FI}_n$.

The above long exact sequence can be used to analyse $H^\text{FI}_n k \text{Hom}_\text{FI}(\bullet, b)$, by increasing induction on $b$. The difficulty is that, in order to prove the conjecture, one requires to show that the connecting morphisms are all as non-trivial as is possible.

It is possible to quantify exactly what is required (this is based on the explicit description of the $\mathfrak{C} \text{rit}(\bullet, -)$ given by Proposition 6.28. However, this requires non-trivial input which should be expected to be as difficult as proving Theorem 5.1.

Remark 6.28. The above also provides an alternative strategy for proving Theorem 5.1 reducing to establishing the non-triviality of the first connecting morphism. Taking into account the addition structure introduced in [Pow22] could facilitate this approach.

Appendix A. The Koszul Complex and Schur Functors

The purpose of this Section is to identify the complex of Schur (bi)functors that is associated to the Koszul complex considered in Section B in the specific cases of interest.

The passage to Schur functors is part of the Schur-Weyl correspondence (see [GW09, Chapter 4] for example), which is a powerful tool when considering representations of the symmetric groups, especially when working over $k$ a field of characteristic zero (see [SS12], for example).

A.1. From $\mathcal{F}(\Sigma)$ to Schur functors. For $G \in \text{Ob} \mathcal{F}(\Sigma)$, the associated Schur functor (from $V_k^f$ to $V_k$, where $V_k$ is the category of $k$-vector spaces and $V_k^f$ the full subcategory of finite-dimensional spaces) is given by

$$G(V) := \bigoplus_{n \in \mathbb{N}} V^\otimes n \otimes \mathfrak{S}_n \text{G}(n),$$

where $\mathfrak{S}_n$ acts by place permutations of tensor factors on $V^\otimes n$.

Example A.1. If $\lambda \vdash n$ and $S^\lambda$ is the associated simple representation of $\mathfrak{S}_n$, since $k$ is a field of characteristic zero, $S_n^\lambda(V)$ is the usual Schur functor associated to the partition $\lambda$. It is a simple functor.

As is well-known (see [SS12], for example), this construction is symmetric monoidal with respect to the convolution product on $\mathcal{F}(\Sigma)$ and the pointwise tensor product of functors:

Proposition A.2. For $G_1, G_2 \in \text{Ob} \mathcal{F}(\Sigma)$, there is a natural isomorphism of functors:

$$(G_1 \circ G_2)(V) \cong G_1(V) \otimes G_2(V).$$

The Schur functor construction clearly passes to the bivariant case, namely functors from $\Sigma^{op} \times \Sigma$ (or, equivalently, $\Sigma \times \Sigma$) to $V_k$. For example, for a left $\mathfrak{S}_n^{op} \times \mathfrak{S}_n$-module, $M$, the Schur bifunctor is

$$(V, W) \mapsto W^\otimes a \otimes \mathfrak{S}_n M \otimes \mathfrak{S}_n V^\otimes a$$

for $(V, W) \in (V_k^f)^{\times 2}$.

Example A.3. Take $b = a$ and consider $k \mathfrak{S}_n$ as a $\mathfrak{S}_n$-bimodule with respect to the regular structures. Then the associated Schur bifunctor is

$$W^\otimes a \otimes \mathfrak{S}_n k \mathfrak{S}_n \otimes \mathfrak{S}_n V^\otimes a.$$
Proof. By Lemma 12 the $\mathbb{S}_a^o \times \mathbb{S}_b$-module $\mathfrak{k}\text{Hom}_{\mathbb{F}1}(a, b)$ is isomorphic to the permutation bimodule on $\mathbb{S}_b/S_{b-a}$. The result follows by a straightforward extension of the argument outlined in Example A.3.

A.2. Revisiting the Koszul complex. The Schur bifunctor construction applies to the Koszul complex $\mathfrak{S}_b^{\mathbb{F}1}$ of an $\mathbb{F}1$-bimodule $\mathfrak{k}\text{Hom}_{\mathbb{F}1}(a, b)$ since, for fixed $a, b \in \mathbb{N}$, this gives a complex of $\mathbb{S}_a^o \times \mathbb{S}_b$-modules and hence a natural complex in bifunctors $(V, W) \mapsto \mathfrak{S}_b^{\mathbb{F}1}(a, b)(V, W)$, considering all $a, b$ at once.

Remark A.5. The natural numbers $a$ and $b$ can be recovered respectively as the polynomial degree with respect to $V$ and the polynomial degree with respect to $b$ (compare Proposition A.4).

Theorem A.6. There is an isomorphism of complexes

$$\mathfrak{S}_b^{\mathbb{F}1}(a, b)(V, W), d) \cong (S^*(W) \otimes \Lambda^*(V) \otimes S^*(W \otimes V), d)$$

with Koszul-type differential from homological degree $n + 1$ to $n$ given in ‘polynomial bidegree’ $(a, b)$ as the composite

$$S^{b-a+n+1}(W) \otimes \Lambda^{n+1}(V) \otimes S^{a-n-1}(W \otimes V) \rightarrow S^{b-a+n}(W) \otimes \Lambda^n(V) \otimes (W \otimes V) \otimes S^{a-n-1}(W \otimes V)$$

where the first map is induced by the coproducts

$$S^*(W) \rightarrow S^{*+1}(W) \otimes W$$

$$\Lambda^*(V) \rightarrow \Lambda^{*+1}(V) \otimes V$$

and the second by the product of the symmetric algebra $S^*(W \otimes V)$.

In particular, this is a complex in the category of $S^*(W \otimes V)$-modules.

Proof. The explicit identification of the terms in $\mathfrak{S}_b^{\mathbb{F}1}(a, b)(V, W)$ follows from the definition of $\mathfrak{S}_b^{\mathbb{F}1}(a, b)$, the identification of the Schur functor associated with the orientation module Or as the functor $V \mapsto \Lambda^*(V)$, the exterior algebra on $V$, together with Proposition A.2 to treat the convolution product.

The Koszul-type differential is simply a translation of the explicit differential in $(\mathfrak{S}_b^{\mathbb{F}1}(a, b), d)$ in terms of the Schur bifunctors.

Remark A.7. Working over a field of characteristic zero, the Koszul complex $(\mathfrak{S}_b^{\mathbb{F}1}(a, b), d)$ is determined by the description of the Schur bifunctor given in Theorem A.6.

Theorem A.6 contains further important information on the $\mathbb{F}1$-homology of $\mathfrak{k}\text{Hom}_{\mathbb{F}1}(a, b)$ that has been omitted in the body of the text:

Corollary A.8. The homology of $(\mathfrak{S}_b^{\mathbb{F}1}(a, b), d)$ takes values in the category of $S^*(W \otimes V)$-modules.

Remark A.9. One of the contributions of [Pow22] is to present a natural categorical framework for this structure that does not require passage to Schur bifunctors.

A.3. Flipping the complex. Corollary [Pow22] highlighted a duality of $\text{Crit}(-, -)$ under $\dagger$. A form of this already holds at the level of the Koszul complexes.

To see this it is useful to rewrite the Koszul complex in the following way:

$$(S^*(W) \otimes S^*(sV) \otimes S^*(W \otimes V), d),$$

where $sV$ denotes $V$ concentrated in homological degree one and $S^*(-)$ is defined using Koszul signs in the category of graded vector spaces.

Now, consider the following substitution:

$$W \ := \ sX$$

$$V \ := \ s^{-1}Y,$$

where $X, Y$ are $k$-vector spaces, so that $W$ is in homological degree one and $V$ in homological degree $-1$.

This yields the complex:

$$(S^*(sX) \otimes S^*(Y) \otimes S^*(X \otimes Y), d),$$

which identifies with the original Koszul complex, with the roles of $X, Y$ reversed.
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