The Least $H$-eigenvalue of Generalized Power Hypergraphs*

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Abstract: The generalized power of a simple graph $G$, denoted by $G^{k,s}$, which is obtained from $G$ by blowing up each vertex into an $s$-set and each edge into a $k$-set, where $1 \leq s \leq k/2$. When $1 \leq s < k/2$, $G^{k,s}$ is always odd-bipartite. It is known that $G^{k,1}$ is non-odd-bipartite if and only if $G$ is non-bipartite, and $G^{k,\frac{k}{2}}$ has the same adjacency (respectively, signless Laplacian) spectral radius as $G$. In this paper, we prove that $G^{k,\frac{k}{2}}$ has the same least adjacency or signless Laplacian $H$-eigenvalue as $G$. Furthermore, all adjacency or signless Laplacian eigenvalues of $G$ are contained in the adjacency or signless Laplacian spectrum of $G^{k,\frac{k}{2}}$. By the above relationship, we minimize the least adjacency or signless Laplacian $H$-eigenvalues among all (or all non-odd-bipartite) hypergraphs $G^{k,s}$ of fixed order. We also discuss the limit points of the least adjacency $H$-eigenvalues of hypergraphs, and construct a sequence of non-odd-bipartite hypergraphs whose least adjacency $H$-eigenvalues converge to $-\sqrt{2 + \sqrt{5}}$.

Keywords: Hypergraph; adjacency tensor; signless Laplacian tensor; least eigenvalue; limit point

1 Introduction

A hypergraph $G = (V(G), E(G))$ consists of a set of vertices, say $V(G) = \{v_1, v_2, \ldots, v_n\}$, and a set of edges, say $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $e_j \subseteq V(G)$. If $|e_j| = k$ for each $j = 1, 2, \ldots, m$, then $G$ is called a $k$-uniform hypergraph. In particular, the 2-uniform hypergraphs are exactly the classical simple graphs. The degree $d_v$ of a vertex $v \in V(G)$ is defined as $d_v = |\{e_j : v \in e_j \in E(G)\}|$. A walk $W$ of length $l$ in $G$ is a sequence of alternate vertices and edges: $v_0, e_1, v_1, e_2, \ldots, v_l$, where $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, l-1$. The hypergraph $G$ is connected if every two vertices are connected by a walk.

In recent years spectral hypergraph theory has emerged as an important field in algebraic graph theory. Let $G$ be a $k$-uniform hypergraph. The adjacency tensor $A = A(G) = (a_{i_1i_2\ldots i_k})$ of $G$ is a $k$th order $n$-dimensional symmetric tensor, where

$$a_{i_1i_2\ldots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = D(G)$ be a $k$th order $n$-dimensional diagonal tensor, where $d_{v_i} = d_v$, for all $i \in [n] := \{1, 2, \ldots, n\}$. Then $L = L(G) = D(G) - A(G)$ is the Laplacian tensor of the hypergraph $G$, and $Q = Q(G) = D(G) + A(G)$ is the signless Laplacian tensor of $G$. If $k = 2$, the above tensors are the classical matrices of simple graphs. The spectral radius (or the least $H$-eigenvalue) of the adjacency, Laplacian and signless Laplacian tensor of $G$ are denoted respectively by $\rho^A(G), \rho^L(G), \rho^Q(G)$ (or respectively by $\lambda^A_{\min}(G), \lambda^L_{\min}(G), \lambda^Q_{\min}(G)$).

*Supported by the National Natural Science Foundation of China (11371028), Scientific Research Fund for Fostering Distinguished Young Scholars of Anhui University(KJJQ1001), Academic Innovation Team of Anhui University Project (KJTD001B).
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The spectral radius (or the largest $H$-eigenvalue) of the adjacency or signless Laplacian tensor of a hypergraph has enjoyed a lot of research exposure; see \( [3, 11, 12, 14, 16, 19, 21] \). However, the least $H$-eigenvalue receives little attention. Nikiforov \( [15] \) gave a lower bound of $\lambda_{\text{min}}^A(G)$ for an even uniform hypergraph $G$ in terms of order and size. In fact he reduced the problem to discussing an odd-bipartite hypergraph; see \( [15] \) Theorem 8.1. Here an even uniform hypergraph $G$ is called odd-bipartite if $V(G)$ has a bipartition $V(G) = V_1 \cup V_2$ such that each edge has an odd number of vertices in both $V_1$ and $V_2$. Shao et al. \( [21] \) proved that the adjacency $H$-spectrum (or the adjacency spectrum) of $G$ is symmetric with respect to the origin if and only if $k$ is even and $G$ is odd-bipartite. So, if $G$ is an odd-bipartite even uniform hypergraph, then $\lambda_{\text{min}}^A(G) = -\rho^A(G)$.

Qi \( [19] \) showed that $\rho^L(G) \leq \rho^Q(G)$, and posed a question of identifying the conditions under which the equality holds. Hu et al. \( [11] \) proved that if $G$ is connected, then the equality holds if and only if $k$ is even and $G$ is odd-bipartite. Shao et al. \( [21] \) proved a stronger result that the Laplacian $H$-spectrum (respectively, Laplacian spectrum) and the signless Laplacian $H$-spectrum (respectively, signless Laplacian spectrum) of a connected $k$-uniform hypergraph $G$ are equal if and only if $k$ is even and $G$ is odd-bipartite. So, for an even $k$ and a connected $k$-uniform hypergraph $G$, if $G$ is odd-bipartite, then $\lambda_{\text{min}}^Q(G) = \lambda_{\text{min}}^L(G) = 0$.

So, if we discuss the least $H$-eigenvalue of the adjacency or signless Laplacian tensor of a connected even uniform hypergraph, it suffices to consider non-odd-bipartite hypergraphs. Up to now, most known examples of hypergraphs are odd-bipartite. Hu, Qi, Shao \( [12] \) introduced the cored hypergraphs and the power hypergraphs, where the cored hypergraph is one such that each edge contains at least one vertex of degree 1, and the $k$th power of a simple graph $G$, denoted by $G^k$, is obtained by replacing each edge (a 2-set) with a $k$-set by adding $k - 2$ new vertices. These two kinds of hypergraphs are both odd-bipartite. Peng \( [17] \) introduced $s$-paths and $s$-cycles, which are both $k$-uniform hypergraphs. An $s$-path is always odd-bipartite \( [13] \). But this does not hold for $s$-cycles. When $1 \leq s < k/2$, an $s$-cycle is odd-bipartite; and when $s = k/2$, it is odd-bipartite if and only if it has an even length.

We \( [13] \) introduced a generalized power hypergraph $G^{k,s}$ from a simple graph $G$, where $1 \leq s \leq k/2$. If $s < k/2$, then $G^{k,s}$ is odd-bipartite; and $G^{k,k/2}$ ($k$ being even) is non-odd-bipartite if and only if $G$ is non-bipartite \( [13] \). So we can construct non-odd-bipartite hypergraphs from non-bipartite simple graphs. In the paper \( [18] \), we proved that $\rho^A(G) = \rho^A(G^{k,s})$ and $\rho^Q(G) = \rho^Q(G^{k,s})$. We wonder whether the equalities also hold for the $\lambda_{\text{min}}^A(G)$ and $\lambda_{\text{min}}^Q(G)$. In Section 3 we give a confirmative answer to the problem. Furthermore, we show that all eigenvalues of $A(G)$ or $Q(G)$ are contained in the spectrum of $A(G^{k,s})$ or $Q(G^{k,s})$. Thus, by the results on minimizing the least adjacency or signless Laplacian eigenvalue of simple graphs $G$, we get some corresponding results of hypergraphs $G^{k,s}$. We also discuss the limit points of the least adjacency $H$-eigenvalues of hypergraphs, and construct a sequence of non-odd-bipartite hypergraphs whose least adjacency $H$-eigenvalues converge to $-\sqrt{2 + \sqrt{5}}$.

## 2 Preliminaries

For integers $k \geq 3$ and $n \geq 2$, a real tensor (also called hypermatrix) $T = (t_{i_1\ldots i_k})$ of order $k$ and dimension $n$ refers to a multidimensional array with entries $t_{i_1\ldots i_k}$ such that $t_{i_1\ldots i_k} \in \mathbb{R}$ for all $i_j \in [n]$ and $j \in [k]$. The tensor $T$ is called symmetric if its entries are invariant under any permutation of their indices. A subtensor of $T$ is a multidimensional array with entries $t_{i_1\ldots i_k}$ such that $i_j \in S_j \subseteq [n]$ for some $S_j$ and $j \in [k]$, denoted by $T|S_1|S_2|\cdots|S_k$. If $S_1 = S_2 = \cdots = S_k =: S$, then we simply write $T|S_1|S_2|\cdots|S_k$ as $T[S]$, which is called the principal subtensor of $T$. Given a vector $x \in \mathbb{R}^n$, $Tx^k$ is a real number, and
\(T x^{k-1}\) is an \(n\)-dimensional vector, which are defined as follows:

\[
T x^k = \sum_{i_1, i_2, \ldots, i_k \in [n]} t_{i_1i_2\ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad (Tx^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} t_{i_2\ldots i_k} x_{i_2} \cdots x_{i_k} \text{ for } i \in [n].
\]

Let \(I\) be the identity tensor of order \(k\) and dimension \(n\), that is, \(i_1i_2\ldots i_k = 1\) if and only if \(i_1 = i_2 = \cdots = i_k \in [n]\) and \(i_1i_2\ldots i_k = 0\) otherwise.

**Definition 2.1** [20] Let \(T\) be a \(k\)th order \(n\)-dimensional real tensor. For some \(\lambda \in \mathbb{C}\), if the polynomial system \((\lambda I - T)x^{k-1} = 0\), or equivalently \(Tx^{k-1} = \lambda x^{k-1}\), has a solution \(x \in \mathbb{C}^n \setminus \{0\}\), then \(\lambda\) is called an eigenvalue of \(T\) and \(x\) is an eigenvector of \(T\) associated with \(\lambda\), where \(x^{k-1} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \in \mathbb{C}^n\).

If \(x\) is a real eigenvector of \(T\), surely the corresponding eigenvalue \(\lambda\) is real. In this case, \(x\) is called an \(H\)-eigenvector and \(\lambda\) is called an \(H\)-eigenvalue. Furthermore, if \(x \in \mathbb{R}^n_+\) (the set of nonnegative vectors of dimension \(n\)), then \(\lambda\) is called an \(H^+\)-eigenvalue of \(T\); if \(x \in \mathbb{R}^n_{++}\) (the set of positive vectors of dimension \(n\)), then \(\lambda\) is said to be an \(H^{++}\)-eigenvalue of \(T\). The smallest \(H\)-eigenvalue and the largest \(H\)-eigenvalue of \(T\) are denoted by \(\lambda_{\min}(T)\) and \(\lambda_{\max}(T)\), respectively. The spectral radius of \(T\) is defined as

\[
\rho(T) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\}.
\]

Surely, by the following Theorem 2.2 if \(T\) is nonnegative, then \(\rho(T) = \lambda_{\max}(T)\).

To generalize the classical Perron-Frobenius Theorem from nonnegative matrices to nonnegative tensors, we need the definition of the irreducibility of tensor. Chang et al. [11] introduced the irreducibility of tensor. A tensor \(T = (t_{i_1\ldots i_k})\) of order \(k\) and dimension \(n\) is called reducible if there exists a nonempty proper subset \(I \subset [n]\) such that \(t_{i_1i_2\ldots i_k} = 0\) for any \(i_1 \in I\) and any \(i_2, \ldots, i_k \notin I\). If \(T\) is not reducible, then it is called irreducible.

Friedland et al. [9] proposed a weak version of irreducibility. The graph associated with \(T\), denoted by \(G(T)\), is the directed graph with vertices \(1, \ldots, n\) and an edge from \(i\) to \(j\) if and only if \(t_{ij\ldots i_k} > 0\) for some \(i_l = j, \ l \in \{2, 3, \ldots, k\}\). The tensor \(T\) is called weakly irreducible if \(G(T)\) is strongly connected. Surely, an irreducible tensor is always weakly irreducible. Pearson and Zhang [10] proved that the adjacency tensor of a uniform hypergraph \(G\) is weakly irreducible if and only if \(G\) is connected. Clearly, this shows that if \(G\) is connected, then \(A(G), L(G)\) and \(Q(G)\) are all weakly irreducible.

**Theorem 2.2 (The Perron-Frobenius Theorem for nonnegative tensors)**

1. (Yang and Yang 2010 [23]) If \(T\) is a nonnegative tensor of order \(k\) and dimension \(n\), then \(\rho(T)\) is an \(H^+\)-eigenvalue of \(T\).

2. (Friedland, Gaubert and Han 2013 [9]) If furthermore \(T\) is weakly irreducible, then \(\rho(T)\) is the unique \(H^{++}\)-eigenvalue of \(T\), with the unique eigenvector \(x \in \mathbb{R}^n_+\), up to a positive scaling coefficient.

3. (Chang, Pearson and Zhang 2008 [11]) If moreover \(T\) is irreducible, then \(\rho(T)\) is the unique \(H^+\)-eigenvalue of \(T\), with the unique eigenvector \(x \in \mathbb{R}^n_+\), up to a positive scaling coefficient.

For a connected hypergraph \(G\), by Theorem 2.2 there exists a unique positive eigenvector of \(A(G)\) (respectively, \(Q(G)\)) up to scales corresponding to its spectral radius, which is called the Perron vector of \(A(G)\) (respectively, \(Q(G)\)). The product \(A(G)x^k\) or \(Q(G)x^k\) has a graph interpretation as following:

\[
A(G)x^k = \sum_{\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G)} kx_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_k}}, \quad (2.1)
\]
\[ Q(G)x^k = \sum_{v \in V(G)} d_v x_v^k + \sum_{\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G)} k x_{v_{i_1}} x_{v_{i_2}} \cdots x_{v_{i_k}}. \]  

(2.2)

The eigenvector equation \( A(G)x^{k-1} = \lambda x^{k-1} \) could be interpreted as

\[ \lambda x_i^{k-1} = \sum_{\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G)} x_{v_{i_2}} x_{v_{i_3}} \cdots x_{v_{i_k}}, \text{ for each } v \in V(G). \]  

(2.3)

The eigenvector equation \( Q(G)x^{k-1} = \lambda x^{k-1} \) could be interpreted as

\[ [\lambda - d_v] x_i^{k-1} = \sum_{\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G)} x_{v_{i_2}} x_{v_{i_3}} \cdots x_{v_{i_k}}, \text{ for each } v \in V(G). \]  

(2.4)

It is known that

\[ \lambda^A_{\min}(G) \leq \min_{\|x\|_k = 1} A(G)x^k, \lambda^Q_{\min}(G) \leq \min_{\|x\|_k = 1} Q(G)x^k, \]

with equality if and only if \( x \) is an eigenvector corresponding to \( \lambda^A_{\min}(G) \) (respectively, \( \lambda^Q_{\min}(G) \)).

Denote \([n] := \{1, 2, \ldots, n\}\); and denote by \( \Delta(G) \) (respectively, \( \delta(G) \)) the maximum degree (respectively, the minimum degree) of a hypergraph \( G \).

**Lemma 2.3** Let \( T \) be a tensor of order \( k \) and dimension \( n \), and let \( T[S] \) be a principle subtensor of \( T \) with \( S \subseteq [n] \). Then

\[ \lambda_{\min}(T) \leq \lambda_{\min}(T[S]), \lambda_{\max}(T[S]) \leq \lambda_{\max}(T); \]

if, in addition \( T \) is nonnegative and weakly irreducible, then \( \rho(T[S]) < \rho(T) \).

**Proof.** Let \( x \) (respectively, \( y \)) be an eigenvector of \( T[S] \) corresponding to \( \lambda_{\min}(T[S]) \) (respectively, \( \lambda_{\max}(T[S]) \)) with \( \|x\|_k = 1 \) (respectively, \( \|y\|_k = 1 \)). Define a vector \( \bar{x} \) (respectively, \( \bar{y} \)) such that \( \bar{x}_i = x_i \) (respectively, \( \bar{y}_i = y_i \)) if \( i \in S \) and \( \bar{x}_i = \bar{y}_i = 0 \) otherwise. Then

\[ \lambda_{\min}(T) = \min_{\|x\|_k = 1} Tz^k \leq T \bar{x}^k = T[S]\bar{x}^k = \lambda_{\min}(T[S]), \]

and

\[ \lambda_{\max}(T) = \max_{\|x\|_k = 1} Tz^k \geq T \bar{y}^k = T[S]\bar{y}^k = \lambda_{\max}(T[S]). \]

If \( T \) is nonnegative and weakly irreducible, then by Theorem 2.2 there exists a positive eigenvector \( z \) of \( T \) corresponding to \( \rho(T) \), i.e. \( Tz^{k-1} = \rho(T)z^{k-1} \). Also by the irreducibility of \( T \), there exists at least one \( i \in S \) such that \( T_{iix_{i_2}x_{i_k}} > 0 \) for some \( i \notin S \), where \( t \in \{2, 3, \ldots, k\} \). So, \( T[S]z[S]^{k-1} \leq \rho(T)z[S]^{k-1} \), where \( z[S] \) denotes the subvector of \( z \) indexed by the elements of \( S \). Now by [13, Corollary 3.4], we have \( \rho(T[S]) < \rho(T) \).

By Lemma 2.3, for a hypergraph \( G \), if taking a vertex \( u \) with \( d_u = \delta(G) \), then \( Q(G)[u] = \delta(G) \), and hence \( \lambda^Q_{\min}(G) \leq \delta(G) \). Similarly, if taking a vertex \( u \) with \( d_u = \Delta(G) \), then \( \rho^Q(G) \geq \Delta(G) \). Furthermore, considering a component \( H \) of \( G \) which contains the vertex \( u \), then \( \rho^Q(G) \geq \rho^Q(H) > \Delta(H) = \Delta(G) \) as \( Q(H) \) is irreducible. The latter result has been shown in [11] with a more accurate bound. Here we use a unified method to deal with the bounds of \( \lambda^Q_{\min}(G) \) and \( \rho^Q(G) \).

**Corollary 2.4** Let \( G \) be a \( k \)-uniform hypergraph. Then \( \rho^Q(G) > \Delta(G) \). If furthermore \( k \) is even, then \( \lambda^Q_{\min}(G) < \delta(G) \).

**Proof.** Without loss of generality, let \( e = \{1, 2, \ldots, k\} \) be an edge of \( G \). Considering the \( H \)-eigenvalues of the principal subtensor \( Q(G)[e] \), by the eigenvector equation \( Q(G)[e]x^{k-1} = \lambda x^{k-1} \), where \( x \in \mathbb{R}^k \), we have

\[ (\lambda - d_i)x_i^{k-1} = \Pi_{j \in [k] \setminus \{i\}} x_j, \text{ for } i = 1, 2, \ldots, k. \]
If there exists some \( i \) such that \( x_i = 0 \), letting \( j \) be such that \( x_j \neq 0 \), by the \( j \)th equation we have \( \lambda = d_i \). Otherwise, all \( x_i \)'s are nonzero; and multiplying both sides of the above equations over all \( i \)'s, we get that

\[
f(\lambda) := (\lambda - d_1)(\lambda - d_2) \cdots (\lambda - d_k) - 1 = 0.
\]

If \( e \) contains the vertex with maximum degree, then \( f(\Delta(G)) < 0 \). Noting that \( f(\lambda) \to +\infty \) when \( \lambda \to +\infty \), so we have \( \rho(Q(G)[e]) > \Delta(G) \). Similarly, if \( k \) is even and \( e \) contains the vertex with minimum degree, then \( f(\lambda) \to +\infty \) when \( \lambda \to -\infty \), and \( f(\delta(G)) < 0 \), which implies that \( \lambda_{\min}(Q(G)[e]) < \delta(G) \). The result now follows by Lemma 2.3.

**Remark:** If \( k \) is odd, then the second result of Corollary 2.4 does not hold. As \( Q(G)a^k \) is an odd function in this case, \( \lambda_{\min}^O(G) = -\rho^O(G) < -\Delta(G) \).

Finally we introduce the generalized power hypergraphs defined in [13].

**Definition 2.5** [13] Let \( G = (V, E) \) be a simple graph. For any \( k \geq 3 \) and \( 1 \leq s \leq k/2 \), the generalized power of \( G \), denoted by \( G^{k,s} \), is defined as the \( k \)-uniform hypergraph with the vertex set \( \{v : v \in V\} \cup \{e : e \in E\} \), and the edge set \( \{u \cup v \cup e : e = \{u, v\} \in E\} \), where \( v \) is an \( s \)-set containing \( v \) and \( e \) is a \((k - 2s)\)-set corresponding to \( e \).

\[\text{Fig. 2.1 (c.f. [13]) Constructing power hypergraphs } G^6 \text{ (right upper), } G^{6,2} \text{ (left below) and } G^{6,3} \text{ (right below)} \text{ from a simple graph } G \text{ (left upper), where a closed curve represents an edge.}\]

Intuitively, \( G^{k,s} \) is obtained from \( G \) by replacing each vertex \( v \) by an \( s \)-subset \( v \) and replacing each edge \( \{u, v\} \) by a \( k \)-set obtained from \( u \cup v \) by adding \((k - 2s)\) new vertices. If \( s = 1 \), then \( G^{k,s} \) is exactly the \( k \)th power hypergraph of \( G \). When \( G \) is a path or a cycle, then \( G^{k,s} \) is an \( s \)-path or \( s \)-cycle for \( s \leq k/2 \).

Note that if \( s < k/2 \), then \( G^{k,s} \) is a cored hypergraphs and hence is odd-bipartite. If \( s = k/2 \) (\( k \) being even), then \( G^{k,s} \) is obtained from \( G \) by only blowing up its vertices. In this case, \( k \) is always assumed to be even; \( \{u, v\} \) is an edge of \( G \) if and only if \( u \cup v \) is an edge of \( G^{k/2} \), where we use the bold \( v \) to denote the blowing-up of the vertex \( v \) in \( G \). For simplicity, we write \( uv \) rather than \( u \cup v \), and call \( u \) a half edge of \( G^{k/2} \). Denote by \( d_u \) the common degree of the vertices in \( u \). For a nonempty subset \( S \subseteq V(G^{k/2}) \), denote \( x^S := \Pi_{v \in S} x_v \), where \( x \) is a vector defined on the vertices of \( G^{k/2} \).

**Lemma 2.6** [13] Let \( G \) be a simple graph and let \( k \) be an even positive integer. The hypergraph \( G^{k/2} \) is non-odd-bipartite if and only if \( G \) is non-bipartite.
3 Relationship between the eigenvalues of $G$ and $G^{k, \frac{1}{2}}$

Let $G$ be a simple graph on $n$ vertices. We first list some properties of the eigenvalues and eigenvectors of $\mathcal{A}(G^{k, \frac{1}{2}})$ and $Q(G^{k, \frac{1}{2}})$. First $\mathcal{A}(G^{k, \frac{1}{2}})$ has zero eigenvalues with multiplicity at least $\frac{k}{2}$ (the number of vertices of $G^{k, \frac{1}{2}}$). Let $v$ be an arbitrary fixed vertex of $G^{k, \frac{1}{2}}$. Define a vector $x$ on $G^{k, \frac{1}{2}}$ such that $x_v = 1$ and $x_u = 0$ for any other vertices $u \neq v$. It is easy to verify by (2.3) that 0 is an eigenvalue of $\mathcal{A}(G^{k, \frac{1}{2}})$ with $x$ as an eigenvector. So the geometric multiplicity of the eigenvalue 0 is at least $\frac{k}{2}$ (the number of vertices in the half edge $v$).

From the above facts, we find that the vertices in the same half edge of $G^{k, \frac{1}{2}}$ may have different values given by eigenvectors of $\mathcal{A}(G^{k, \frac{1}{2}})$ or $Q(G^{k, \frac{1}{2}})$. However, if $\lambda \neq 0$ (respectively, $\lambda \neq d_v$ for some vertex $v$) as an eigenvalue of $\mathcal{A}(G^{k, \frac{1}{2}})$ (respectively, an eigenvalue of $Q(G^{k, \frac{1}{2}})$), we will have the following property on the eigenvectors of $\lambda$.

**Lemma 3.1** Let $G$ be a simple graph. Let $u$ and $\bar{u}$ be two vertices in the same half edge $u$ of a hypergraph $G^{k, \frac{1}{2}}$. If $x$ is an eigenvector of $\mathcal{A}(G^{k, \frac{1}{2}})$ corresponding to an eigenvalue $\lambda \neq 0$, or an eigenvector of $Q(G^{k, \frac{1}{2}})$ corresponding an eigenvalue $\lambda \neq d_v$, then $|x_u| = |x_{\bar{u}}|$.

**Proof.** If $x$ is an eigenvector of $\mathcal{A}(G^{k, \frac{1}{2}})$, by the eigenvector equation (2.1),

$$\lambda x_u^{k-1} = \sum_{v \in E(G^{k, \frac{1}{2}})} x^{uv}(u), \lambda x_{\bar{u}}^{k-1} = \sum_{v \in E(G^{k, \frac{1}{2}})} x^{uv}(\bar{u}).$$

So we have $\lambda x_u^{k} = \lambda x_{\bar{u}}^{k}$, which implies the result. Similarly, if $x$ is an eigenvector of $Q(G^{k, \frac{1}{2}})$, by (2.2) we have $(\lambda - d_v)x_u^{k} = (\lambda - d_{\bar{u}})x_{\bar{u}}^{k}$. As $\lambda \neq d_v$, the result also follows. 

Suppose $G$ is a $k$-uniform hypergraph on $n$ vertices which contains at least one edge. By Theorem 2.22 and Corollary 2.4, $\rho^A(G) > 0$, $\rho^Q(G) > \Delta(G)$, $\lambda_{\min}^A(G) < \delta(G)$. Note that the sum of all eigenvalues of $\mathcal{A}(G)$ is $(k-1)^{n-1}\text{tr}(\mathcal{A}(G)) = 0$ ([20]), which implies $\lambda_{\min}^A(G) < 0$, where $\text{tr}(\mathcal{A})$ is the trace of $\mathcal{A}$. Now, if $G$ is a simple graph containing at least one edge, then by Lemma 3.1 all vertices in a half edge $u$ of $G^{k, \frac{1}{2}}$ have the same modular given by the eigenvector of $\lambda_{\min}^A(G)$ or $\lambda_{\min}^Q(G)$; the common modular is denoted by $|x_u|$.

**Theorem 3.2** Let $G$ be a simple graph. Each eigenvalue of $\mathcal{A}(G)$ (respectively, $Q(G)$) is contained in the spectrum of $\mathcal{A}(G^{k, \frac{1}{2}})$ (respectively, $Q(G^{k, \frac{1}{2}})$).

**Proof.** For each vertex $v$ of $G$, we assume that $v$ is also contained in the corresponding half edge $v$ in $G^{k, \frac{1}{2}}$. Let $x$ be an eigenvector of $\mathcal{A}(G)$ corresponding to an eigenvalue $\lambda$. Let $x$ be a vector defined on $G^{k, \frac{1}{2}}$ as follows:

$$x_v = \text{sgn}(x_v)|x_v|^{2/k}, x_{\bar{v}} = |x_v|^{2/k}, \text{ for each vertex } \bar{v} \in v \setminus \{v\} \text{ and each } v \in V(G).$$

Then

$$x^\top = x_v, \text{ for each } v \in V(G)$$

and

$$\|x\|_2^2 = \frac{k}{2}\|x\|_2^2.$$

Noting that $\lambda x_v = \sum_{uv \in E(G)} x_u$, it is easy to verify that

$$\lambda x_v^{k-1} = \sum_{uv \in E(G^{k, \frac{1}{2}})} x^{uv}(v) \text{ and } \lambda x_{\bar{v}}^{k-1} = \sum_{uv \in E(G^{k, \frac{1}{2}})} x^{uv}(\bar{v}) \text{ for each } \bar{v} \in v \setminus \{v\}.$$
So, \( \lambda \) is also an eigenvalue of \( A(G^{k, \frac{1}{2}}) \).

Similarly, if \( x \) is an eigenvector of \( Q(G) \) corresponding to an eigenvalue \( \lambda \), then \((\lambda - d_v)x_v = \sum_{uv \in E(G)}^{} x_u \) for each vertex \( v \) of \( G \). Defining a vector \( x \) as in (3.1), we get \((\lambda - d_v)x_v = \sum_{uv \in E(G^{k, \frac{1}{2}})}^{} x_{uv} \) for each vertex \( v \) of \( G^{k, \frac{1}{2}} \).

**Remark:** We give a note on Theorem 3.2 by two examples. Let \( G = P_2 \), a simple path on 2 vertices. Then \( P_2^{k, \frac{1}{2}} \) is a hypergraph on vertices 1, 2, \( \ldots \), \( k \) with only one edge \( \{1, 2, \ldots, k\} \). It is known that the spectrum of \( A(P_2) \) are \((1, -1)\), and the spectrum of \( Q(P_2) \) are \((0, 2)\). We now compute the eigenvalues of \( A(P_2^{k, \frac{1}{2}}) \) and \( Q(P_2^{k, \frac{1}{2}}) \). Let \( \lambda \) be an eigenvalue of \( A(P_2^{k, \frac{1}{2}}) \) corresponding to an eigenvector \( x \). Then by (2.3) we have

\[
\lambda x_v^{k-1} = \prod_{j \in [k], j \neq i}^{} x_j, \quad \text{for } i = 1, 2, \ldots, k.
\]

Just like the discussion in Corollary 2.4, if \( x_i = 0 \) for some \( i \), then \( \lambda = 0 \); otherwise, \( \lambda^k = 1 \). Each \( \lambda_j = e^{\frac{2\pi i}{k}} (j \in [k], i^2 = -1) \) is an eigenvalue of \( A(P_2^{k, \frac{1}{2}}) \) with the eigenvector \( x^{(j)} \) defined as following: for any chosen \( j \)-subset \( S \) of \( [k] \), \( x^{(j)}(i) = e^{\frac{2\pi i}{k}} \) if \( i \in S \), and \( x^{(j)}(i) = 1 \) otherwise. So the eigenvalues of \( A(P_2^{k, \frac{1}{2}}) \) are \( 0, \lambda_k = 1, \lambda_{k/2} = -1 \) and \( \lambda_j (j \in [k] \setminus \{k/2, k\}) \). As \( Q(P_2^{k, \frac{1}{2}}) = I + A(P_2^{k, \frac{1}{2}}) \), so the eigenvalues of \( Q(P_2^{k, \frac{1}{2}}) \) are 1 (the degree), 2, 0 and \( 1 + \lambda_j (j \in [k] \setminus \{k/2, k\}) \).

From this example, we find that the eigenvalues of \( A(G^{k, \frac{1}{2}}) \) (respectively, \( Q(G^{k, \frac{1}{2}}) \)) may not contained in the spectrum of \( A(G) \) (respectively, \( Q(G) \)) due to the following reasons. Firstly, \( A(G) \) (respectively, \( Q(G) \)) may not contains zero eigenvalue (respectively, the degrees as eigenvalues) but \( A(G^{k, \frac{1}{2}}) \) (respectively, \( Q(G^{k, \frac{1}{2}}) \)) must contain zero eigenvalues (respectively, the degree of each vertex as an eigenvalue). Secondly, \( A(G^{k, \frac{1}{2}}) \) or \( Q(G^{k, \frac{1}{2}}) \) may have complex eigenvalues while \( A(G) \) or \( Q(G) \) cannot have such eigenvalues. Thirdly, \( A(G^{k, \frac{1}{2}}) \) or \( Q(G^{k, \frac{1}{2}}) \) may have some \( H \)-eigenvalues which are not contained in the spectrum of \( A(G) \) or \( Q(G) \); see the following example.

Let \( P_3 \) be a path on 3 vertices \( v_1, v_2, v_3 \) with edges \( \{v_1, v_2\}, \{v_2, v_3\} \). It is known that the spectrum of \( A(P_3) \) are \( \{\sqrt{2}, 0, -\sqrt{2}\} \), and the spectrum of \( Q(P_3) \) are \( \{0, 1, 3\} \). Define a vector \( x \) on \( P_3^{k, \frac{1}{2}} \) such that \( x_u = 0 \) if \( u \in v_1 \), and \( x_u = 1 \) otherwise. Then \( x \) is an eigenvector of \( A(P_3^{k, \frac{1}{2}}) \) corresponding an \( H \)-eigenvalue 1, which is not an eigenvalue of \( A(P_3) \). Define a vector \( y \) on \( P_3^{k, \frac{1}{2}} \) such that \( y_u = 0 \) if \( u \in v_1 \), \( y_u = 1 \) if \( u \in v_2 \), and \( y_u = (\frac{\sqrt{2} - 1}{2})^\frac{k}{2} \) if \( u \in v_3 \). It is easy to verify that \( y \) is an eigenvector of \( Q(P_3^{k, \frac{1}{2}}) \) corresponding an \( H \)-eigenvalue \( \frac{3 + \sqrt{5}}{2} \), which is not an eigenvalue of \( Q(P_3) \).

By Theorem 3.2, we know that \( \lambda_{\min}^A(G^{k, \frac{1}{2}}) \geq \lambda_{\min}^A(G) \) and \( \lambda_{\min}^Q(G^{k, \frac{1}{2}}) \geq \lambda_{\min}^Q(G) \). In fact, the above two inequalities hold as equalities.

**Theorem 3.3** Let \( G \) be a simple graph. Then \( \lambda_{\min}^A(G) = \lambda_{\min}^A(G^{k, \frac{1}{2}}) \) and \( \lambda_{\min}^Q(G) = \lambda_{\min}^Q(G^{k, \frac{1}{2}}) \).

**Proof.** For each vertex \( v \) of \( G \), we assume that \( v \) is also contained in the corresponding half edge \( v \) in \( G^{k, \frac{1}{2}} \). Let \( x \) be an eigenvector of \( A(G) \) corresponding to the least eigenvalue of \( \lambda_{\min}^A(G) \). Let \( x \) be a vector defined on \( G^{k, \frac{1}{2}} \) as in (3.1). Now by (3.2) and (3.3)

\[
\lambda_{\min}^A(G^{k, \frac{1}{2}}) \leq \frac{A(G^{k, \frac{1}{2}})x_k}{\|x\|_k^2} = \sum_{uv \in E(G^{k, \frac{1}{2}})}^{k} x_{uv} \|x\|_k^2 = \sum_{uv \in E(G)}^{2} x_{uv} \|x\|_2^2 = \lambda_{\min}^A(G).
\]

If \( x \) is an eigenvector of \( Q(G) \) corresponding to the least eigenvalue of \( \lambda_{\min}^Q(G) \), also defining \( x \) as in (3.1),
then 

\[
\lambda_{\min}^{Q}(G^{k, \frac{1}{2}}) \leq \frac{Q(G^{k, \frac{1}{2}})x^k}{\|x\|_{k}^2} = \sum_{u \in V(G)} \frac{d_u x_u^k}{\|x\|_{k}^2} + \sum_{uv \in E(G)} \frac{kx^u x^v}{\|x\|_{k}^2} = \sum_{u \in V(G)} \frac{d_u x_u^2}{\|x\|_{2}^2} + \sum_{uv \in E(G)} \frac{2x_u x_v}{\|x\|_{2}^2} = \lambda_{\min}^{Q}(G).
\]

On the other hand, let \( x \) be an eigenvector of \( A(G^{k, \frac{1}{2}}) \) corresponding to \( \lambda_{\min}^{A}(G^{k, \frac{1}{2}}) \). Let us define a vector \( x \) on \( G \) such as

\[
x_v = x^v \quad \text{for each} \quad v \in V(G),
\]

By Lemma 3.1 \(|x_v| = |x_v|^{\frac{k}{2}} \), and hence \( \|x\|_{2}^2 = \frac{2}{k} \|x\|_{k}^2 \). So

\[
\lambda_{\min}^{A}(G) \leq \sum_{uv \in E(G)} \frac{2x_u x_v}{\|x\|_{2}^2} = \sum_{uv \in E(G)} \frac{kx^u x^v}{\|x\|_{k}^2} = \frac{A(G^{k, \frac{1}{2}})x^k}{\|x\|_{k}^2} = \lambda_{\min}^{A}(G^{k, \frac{1}{2}}).
\]

If \( x \) is an eigenvector of \( Q(G^{k, \frac{1}{2}}) \) corresponding to \( \lambda_{\min}^{Q}(G^{k, \frac{1}{2}}) \), then

\[
\lambda_{\min}^{Q}(G) \leq \sum_{u \in V(G)} \frac{d_u x_u^2}{\|x\|_{k}^2} + \sum_{uv \in E(G)} \frac{2x_u x_v}{\|x\|_{2}^2} = \sum_{u \in V(G)} \frac{d_u x_u^k}{\|x\|_{k}^2} + \sum_{uv \in E(G)} \frac{kx^u x^v}{\|x\|_{k}^2} = \frac{Q(G^{k, \frac{1}{2}})x^k}{\|x\|_{k}^2} = \lambda_{\min}^{Q}(G^{k, \frac{1}{2}}).
\]

So we get the desired equality.

The proof in Theorem 3.3 also gives a construction of eigenvectors of the least \( H \)-eigenvalue of \( A(G^{k, \frac{1}{2}}) \) or \( Q(G^{k, \frac{1}{2}}) \) from the eigenvectors of the least eigenvalue of \( A(G) \) or \( Q(G) \) and vice versa. Denote \( G_n \) (respectively, \( \text{nb} G_n \)) the class of simple connected graphs (respectively, non-bipartite graphs) of order \( n \). Denote \( G_n^{k, \frac{1}{2}} = \{ G^{k, \frac{1}{2}} : G \in G_n \} \) and \( \text{nb} G_n^{k, \frac{1}{2}} = \{ G^{k, \frac{1}{2}} : G \in \text{nb} G_n \} \). By Theorem 3.3 we easily get the following results on the minimum of the least eigenvalue of the adjacency or signless Laplacian tensor of the hypergraphs in \( G_n^{k, \frac{1}{2}} \) and \( \text{nb} G_n^{k, \frac{1}{2}} \).

**Corollary 3.4** Among all hypergraphs in \( G_n^{k, \frac{1}{2}} \), the minimum least \( H \)-eigenvalue of the adjacency tensor is achieved uniquely by \( K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \), where \( K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \) is the complete bipartite graph with two parts having \( \lfloor n/2 \rfloor \) and \( \lceil n/2 \rceil \) vertices respectively.

**Proof.** Use Theorem 3.3 and the result in [2, 8] for the least adjacency eigenvalue of simple graphs.

**Corollary 3.5** Among all hypergraphs in \( \text{nb} G_n^{k, \frac{1}{2}} \), the minimum least \( H \)-eigenvalue of the adjacency tensor is achieved uniquely by \( (K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} + e)^{k, \frac{1}{2}} \), where \( K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} + e \) is the graph obtained from \( K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \) by adding an edge within the part with \( \lceil n/2 \rceil \) vertices.

**Proof.** Use Theorem 3.3 and the result in [6] for the least adjacency eigenvalue of non-bipartite simple graphs.

**Corollary 3.6** Among all hypergraphs in \( \text{nb} G_n^{k, \frac{1}{2}} \), the minimum least \( H \)-eigenvalue of the signless Laplacian tensor is achieved uniquely by \( E_{3,n-3}^{k, \frac{1}{2}} \), where \( E_{3,n-3} \) is the graph obtained from a triangle by appending a path of order \( n - 3 \) at some vertex.

**Proof.** Use Theorem 3.3 and the result in [4, 7] for the least signless Laplacian eigenvalue of non-bipartite simple graphs.
4 Limit points of the least Eigenvalues

Hoffman [10] observed if a simple graph G properly contains a cycle, then \( \rho(A(G)) > \tau^{1/2} + \tau^{-1/2} = \tau^{3/2} = \sqrt{2 + \sqrt{5}} \), where \( \tau = (\sqrt{5} + 1)/2 \) is the golden mean. He proved that \( \tau^{3/2} \) is a limit point, and found all limit points of the adjacency spectral radii less than \( \tau^{3/2} \). The work of Hoffman was extended by Shearer [18] to show that every real number \( r \geq \tau^{3/2} \) is the limit point of the adjacency spectral radii of simple graphs. Furthermore, Doob [5] proved that for each \( r \geq \tau^{3/2} \) (respectively, \( r \leq -\tau^{3/2} \)) and for any \( k \), there exists a sequence of graphs whose \( k \)th largest eigenvalues (respectively, \( k \)th smallest eigenvalues) converge to \( r \).

The smallest limit point of the adjacency spectral radii of simple graphs is 2, which is realized by a sequence of paths. If \( r < \tau^{3/2} \) is a limit point, it suffices to consider the trees by Hoffman’s observation. The construction of graphs whose adjacency spectral radii converge to \( r \geq \tau^{3/2} \) in [5, 18] are trees \( T(n_1, n_2, \ldots, n_k) \) called caterpillars, which is obtained from a path on vertices \( v_1, v_2, \ldots, v_k \) by attaching \( n_j \) pendant edges at the vertex \( v_j \) for each \( j = 1, 2, \ldots, k \).

As the spectrum of a tree is symmetric with respect to the origin, the minus of the limit points of the spectral radius are the limit points the least eigenvalue. Since \( -\rho^A(T) = \lambda^{A}_{\min}(T) = \lambda^{A}_{\min}(T^k) \) by Theorem 3.3, we get the following result on the limit points of the eigenvalues of \( k \)-uniform hypergraphs.

**Theorem 4.1** For \( n = 1, 2, \ldots \), let \( \beta_n \) be the positive root of \( P_n(x) = x^{n+1} - (1 + x + x^2 + \cdots + x^{n-1}) \). Let \( \alpha_n = \beta_n^{1/2} + \beta_n^{-1/2} \). Then \( -2 = -\alpha_1 > -\alpha_2 > \cdots \) are all limit points of the least \( H \)-eigenvalues of the adjacency tensor of hypergraphs greater than \( -(\tau^{1/2} + \tau^{-1/2}) = -\lim_n \alpha_n \).

**Theorem 4.2** Each real number \( r \leq -\tau^{3/2} \) is a limit point of the least \( H \)-eigenvalue of the adjacency tensor of hypergraphs.

Finally we will construct a sequence of non-bipartite graphs \( G \) whose least adjacency eigenvalues converge to \( -\tau^{3/2} \). Consequently we get sequence of non-odd-bipartite hypergraphs \( G^{k,k/2} \) whose least adjacency eigenvalues converge to \( -\tau^{3/2} \). Denote by \( C_n + e \) the simple graph obtained from a cycle \( C_n \) on \( n \) vertices by appending a pendant edge \( e \) at some vertex.

**Lemma 4.3** \( \lim_{n \to \infty} \lambda^{A}_{\min}(C_{2n+1} + e) = -\tau^2 \)

**Proof.** Label the vertices of \( C_{2n+1} + e \) as follows: the pendant vertex is labeled by \( v_0 \), starting from the vertex of degree 3 the vertices of the cycle are labeled by \( v_1, v_2, \ldots, v_{2n+1} \) clockwise. Note that now \( e = v_0v_1 \). Denote \( T_{2n+1} := C_{2n+1} + e - v_{n+1}v_{n+2} \). Let \( x \) be a unit vector corresponding to the least eigenvalue of \( C_{2n+1} + e \). By symmetry, \( x_v = x_{v_{2n+3-k}} \) for \( k = 2, 3, \ldots, n+1 \); in particular \( x_{v_{n+1}} = x_{v_{n+2}} \). So

\[
\lambda^{A}_{\min}(C_{2n+1} + e) = \sum_{uv \in E(C_{2n+1} + e)} 2x_u x_v = x^T A(T_{2n+1}) x + 2x_{v_{n+1}} x_{v_{n+2}} > \lambda^{A}_{\min}(T_{2n+1}).
\]

On the other hand, let \( y \) be a unit vector corresponding to the least eigenvalue of \( T_{2n+1} \). Also by symmetry, \( y_{v_{n+1}} = y_{v_{n+2}} \). As \( T_{2n+1} \) is bipartite, \( \rho^A(T_{2n+1}) = -\lambda^{A}_{\min}(T_{2n+1}) \) and \( |y| \) is the Perron vector of \( A(T_{2n+1}) \). In addition, as shown by Hoffman [10], \( \rho^A(T_{2n+1}) \) increasingly converges to \( \tau^{3/2} > 2 \). So, for sufficiently large \( n \), \( \rho^A(T_{2n+1}) > 2 \). By a discussion similar in [18] Lemma 4.7, we get that \( |y_v| > |y_{v_1}| > \cdots > |y_{v_{n+1}}| \). Note that

\[
1 = \sum_{i=0}^{2n+1} y_{v_i}^2 > y_{v_1}^2 + 2(y_{v_2}^2 + \cdots + y_{v_{n+1}}^2) > (2n+1)y_{v_{n+1}}^2.
\]
So $2y_{v_{n+1}} < \frac{2}{2n+1}$. As $y_{v_{n+1}} = y_{v_{n+2}}$, we have

$$\lambda^A_{\min}(C_{2n+1} + e) = \sum_{uv \in E(C_{2n+1} + e)} 2y_uy_v = y^T A(T_{2n+1})y + 2y_{v_{n+1}}y_{v_{n+2}} < \lambda^A_{\min}(T_{2n+1}) + \frac{2}{2n+1}.$$ 

By the above discussion, for sufficiently large $n$,

$$\lambda^A_{\min}(T_{2n+1}) < \lambda^A_{\min}(C_{2n+1} + e) < \lambda^A_{\min}(T_{2n+1}) + \frac{2}{2n+1}.$$ 

So

$$\lim_{n \to \infty} \lambda^A_{\min}(C_{2n+1} + e) = \lim_{n \to \infty} \lambda^A_{\min}(T_{2n+1}) = -\tau^{3/2}.$$ 

**Corollary 4.4** $-\tau^{3/2}$ is a limit point of the least $H$-eigenvalue of adjacency tensor of non-odd-bipartite hypergraphs.

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