EXTREMAL K-CONTACT METRICS

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Abstract. Extending a result of He to the non-integrable case of K-contact manifolds, it is shown that transverse Hermitian scalar curvature may be interpreted as a moment map for the strict contactomorphism group. As a consequence, we may generalize the Sasaki-Futaki invariant to K-contact geometry and establish a number of elementary properties.

Moreover, we prove that in dimension 5 certain deformation-theoretic results can be established also under weaker integrability conditions by exploiting the relationship between J-anti-invariant and self-dual 2-forms.

1. Introduction

On a symplectic manifold \((M, \omega)\), consider the space \(\mathcal{AC}(\omega)\) of all \(\omega\)-compatible almost-complex structures \(J\) and the subspace \(\mathcal{C}(\omega)\) of integrable ones. A crucial observation due to Fujiki \cite{11} is that \(\mathcal{C}(\omega)\) may be viewed as an infinite-dimensional Kähler manifold and that the natural action of the group of Hamiltonian symplectomorphisms admits a moment map, associating to a complex structure \(J\) the scalar curvature of the metric \(g = \omega(\cdot, J\cdot)\). An important generalization of this result to \(\mathcal{AC}(\omega)\), the non-integrable case, was established by Donaldson \cite{10}.

The critical points of the square-norm of this moment map give canonical representatives of almost-complex structures \(J\) (corresponding to metrics) called extremal almost-Kähler metrics \cite{2, 20}. These metrics are a natural extension of Calabi’s extremal Kähler metrics \cite{7, 8}.

Recently, He \cite{21} introduced a similar moment map picture to Sasakian geometry, which may be viewed as an odd-dimensional counterpart of Kähler geometry. The first goal of this paper is to generalize in Theorem 16 the result of He to the non-integrable case (so-called K-contact structures), as conjectured in \cite[Remark 4.3]{21}. The moment map now takes a K-contact structure to its transverse Hermitian scalar curvature.

We define extremal K-contact metrics again as critical points. Theorem \cite{16} has a number of consequences (such as a K-contact Futaki invariant \cite{5, 6, 13}), which we investigate in Sections \cite{3, 6}. These metrics appear as natural extensions of extremal Sasakian metrics, introduced by Boyer–Galicki–Simanca \cite{5, 6} and motivated by the examples of irregular Sasaki-Einstein metrics (see for instance \cite{19}).

In Sections \cite{3, 6} we consider the deformation-theoretic behaviour of extremal K-contact metrics, leading to the notion of a semi-Sasakian structure. As opposed to the integrable case, our considerations are limited to dimension 5, as we exploit the relationship between J-anti-invariant and self-dual 2-forms. We also generalize the transverse \(\partial\bar{\partial}\)-Lemma \cite{22} to the K-contact case.

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2. Preliminaries

2.1. K-contact structures. Let $(M, \eta)$ be a contact manifold of dimension $2n+1$, where $\eta$ is the contact 1-form satisfying $\eta \wedge (d\eta)^n \neq 0$ at every point of $M$. The Reeb vector field $\xi \in \frak{x}(M)$ for $\eta$ is uniquely determined by the requirements

$$\eta(\xi) = 1, \quad \iota_\xi(d\eta) = 0.$$ 

The corresponding distribution $\mathcal{F}_\xi = \mathbb{R}\xi \subset TM$ defines the characteristic foliation.

Denote by $\frak{Con}(M, \eta)$ the strict contactomorphism group of all diffeomorphisms $f$ satisfying $f^* \eta = \eta$. Its Lie algebra are all vector fields $X$ with $\frak{L}_X \eta = 0$. For $M$ compact, the contact Hamiltonian of $X$ is the unique basic function $f \in C^\infty_B(M)$ satisfying $\eta(X) = f$, $d\eta(X, \cdot) = -df$. This gives an identification

$$\text{Lie} \frak{Con}(M, \eta) \cong C^\infty_B(M), \quad f \leftrightarrow X_f.$$ 

We use it to transport the metric on $C^\infty_B(M) \subset L^2(M, dv_\eta)$ to $\text{Lie} \frak{Con}(M, \eta)$.

Definition 1. A $K$-contact structure $(\eta, \xi, \Phi)$ consists of a contact form $\eta$ on $M$ with Reeb field $\xi$ together with an endomorphism $\Phi : TM \to TM$ satisfying

$$\Phi^2 = -\text{id}_{TM} \pm \xi \otimes \eta, \quad \mathcal{L}_\xi \Phi = 0.$$ 

We require also the following compatibility conditions with $\eta$:

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \quad d\eta(Z, \Phi Z) > 0 \quad \forall X, Y \in TM, Z \in \ker(\eta) \setminus \{0\}$$

Definition 2. Fixing $\eta$, the set of $K$-contact structures $\Phi$ on $(M, \eta)$ is denoted $K_\eta$.

From Definition 1 one may deduce $\Phi(\xi) = 0$, $\eta \circ \Phi = 0$. Moreover, to any $K$-contact structure there belongs a metric $g = g_\Phi$ given by

$$g_\Phi(X, Y) = d\eta(X, \Phi Y) + \eta(X)\eta(Y).$$

The leaves of $\mathcal{F}_\xi$ are geodesics with respect to $g$ and the foliation is Riemannian (see [4, Section 2]). In particular, we have a transverse Levi-Civita connection $D^\Gamma$ on the normal bundle $\nu = TM/\mathbb{R}\xi$. This is the unique metric, torsion-free connection on $\nu$ (i.e. $D^\Gamma_X(\pi Y) - D^\Gamma_Y(\pi X) = \pi[X, Y]$ for the projection $\pi : TM \to \nu$).

Remark 3. A Sasakian structure is a $K$-contact structure $(\eta, \xi, \Phi, g)$ satisfying the integrability condition $D^\nu_X \Phi = \xi \otimes X^s - X \otimes \eta$ ($\forall X \in \frak{x}(M)$) for the Levi-Civita connection $D^g$ on $(M, g)$ and where $X^s = g(\cdot, X)$. It is well-known that this is equivalent to the almost-Kähler cone $\left(\mathbb{R}_{>0} \times M, dr^2 + r^2 g, d\left(\frac{r^2}{2} \eta\right)\right)$ being Kähler.

2.2. Basic Forms and Transverse Structure. Let $\mathcal{F}$ be a foliation on $M$ given by an integrable subbundle $\mathcal{T}\mathcal{F}$ of $TM$. A $p$-form $\alpha$ on $M$ is called basic if

$$\iota_\xi \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0 \quad \forall \xi \in \Gamma(M, \mathcal{T}\mathcal{F}).$$

Let $(\Omega^*_B(M), d_B)$ denote the subcomplex of basic forms of the de Rham complex and let $C^\infty_B(M) = \Omega^*_B(M)$. The basic cohomology is $H^*_\mathcal{F}(M) = H^*(\Omega_B(M), d_B)$.

A transverse symplectic, almost-complex, or Riemannian structure is a corresponding structure on the normal bundle $\nu$ whose Lie derivative in direction of vectors tangent to the leaves vanishes (see [33]). For example, a $K$-contact structure $(\eta, \xi, \Phi, g)$ gives a transverse almost complex structure $\Phi^\mathcal{T} = \Phi|_{\nu} \in \text{End}(\nu)$, metric $g^\mathcal{T} = g_\Phi|_{\nu}$, and symplectic form $\omega^\mathcal{T} = d\eta|_{\nu} \in \Gamma(M, \Lambda^2\nu^*)$ for $\mathcal{F} = \mathcal{F}_\xi$. 

Definition 4. Fixing \( \xi \) and \( J \in \text{End}(\nu) \), let \( K(\xi, J) \) be the space of all \( K \)-contact structures \((\eta, \xi, \Phi)\) with Reeb field \( \xi \) and induced transverse structure \( \Phi^T = J \).

We briefly describe a \( K \)-contact structure in local coordinates (see [21] and also \cite{15} in the Sasakian case). We may pick contact Darboux coordinates \cite{18} Theorem 2.5.1 which means the contact form may be written as

\[
\eta = dx^n + \sum_{i=1}^{n} x^{2i-1} dx^{2i}, \quad \xi = \partial/\partial x^n.
\]

Then \( \omega^T = \sum_{i=1}^{n} dx^{2i-1} \wedge dx^{2i} \). The subspace \( \ker \eta \), which identifies via \( X \mapsto X - \eta(X)X \) with the normal bundle \( TM/\mathbb{R}\xi = \nu \), is spanned by

\[
e_{2i-1} = \partial/\partial x^{2i-1}, \quad e_{2i} = \partial/\partial x^{2i} - x^{2i-1}\partial/\partial x^n \quad (1 \leq i \leq n)
\]

Using \( \ker(\eta) \perp \xi \), the metric \cite{2} has \( g_{ij} = g(e_i, e_j) = g(\partial/\partial x^i, \partial/\partial x^j) \) for \( i, j \geq 1 \) and \( g_{00} = 1 \). \( \Phi \) is described by the basic functions \( \Phi(e_i) = \Phi^j_i e_j \). We have

\[
\Phi^k_i \Phi^j_k = -\delta^k_i, \quad g_{jk} \Phi^j_k = g_{ij} \Phi^k_i.
\]

Hermitian curvature. The (transverse) Hermitian connection \( \nabla^T \) on \( \nu \) may be defined using the transverse Levi-Civita connection \( D^T \) via

\[
\nabla^T_X Y = D^T_X Y - \frac{1}{2} \Phi^T (D^T_X \Phi^T) Y.
\]

(see \cite{17} \cite{29} and \cite{16} Sections 9.2, 9.3 for details on Hermitian connections.) This gives the unique connection on \( \nu \) with \( \nabla^T h = 0 \), where \( 2h = \omega^T - i\omega \), and whose torsion is the transverse Nijenhuis tensor \cite{14}. Let \( \tilde{R}^T \) be the curvature of \( \nabla^T \) and

\[
\tilde{R}^T(X, Y) = -\Lambda_{\omega}(\Phi^T \circ \tilde{R}^T_{X, Y}),
\]

where \( \Lambda_{\omega} \) denotes the adjoint of \( \omega^T \wedge - \) on basic forms, see \cite{16}. The (transverse) Hermitian scalar curvature is defined via the Hermitian Ricci 2-form \( \tilde{\rho}^T \) as

\[
s^T = 2\Lambda_{\omega}(\tilde{\rho}^T).
\]

If \( \eta \) is fixed we shall emphasize the dependence on \( \Phi \) by writing \( s^T_\Phi \).

3. Basic cohomology of \( K \)-contact structures

Throughout this section, fix a compact \( K \)-contact manifold \((M, \eta, \xi, \Phi, g)\) of dimension \( 2n + 1 \) with transverse almost complex structure \( J = \Phi^T \). Our first goal is to describe in Theorem \cite{15} the space \( K(\xi, J) \) when \( 2n + 1 = 5 \). This requires the development of some machinery of ‘almost Kähler geometry in the transverse,’ for example the transverse \( \partial\bar{\partial}\)-Lemma (generalizing a result of El-Kacimi-Alaoui \cite{22} to the \( K \)-contact case).

3.1. Transverse Almost Kähler Geometry. The endomorphism \( \Phi \) induces an action on basic \( p \)-forms via

\[
(\Phi \alpha)(X_1, \cdots, X_p) = (-1)^p \alpha(\Phi X_1, \cdots, \Phi X_p).
\]

For instance \( \Phi \eta = 0 \). This action preserves basic forms \( \alpha \in \Omega^*_p(M) \) since

\[
i_\xi(\Phi \alpha)(X_2, \cdots, X_p) = (-1)^p \alpha(\Phi \xi, \Phi X_2, \cdots, \Phi X_p) = 0,
\]

\[
\mathcal{L}_\xi(\Phi \alpha) = (\mathcal{L}_\xi \Phi) \alpha = 0.
\]

On \( p \)-forms we have \( \Phi^2|_{\Omega^*_p} = (-1)^p \text{id} \). This action coincides for all \( \Phi \in K(\xi, J) \) and accordingly we may speak just of \( J \)-invariant basic forms.
The twisted exterior derivative on $p$-forms is $d^c = (-1)^p \Phi d \Phi$ and preserves basic forms. We write $d^c_B = d^c|_{\Omega^p_B}$.

**Remark 5.** For a basic function $f$ we have

$$[(d^c_B d_B + d_B d^c_B) f](X,Y) = d^c_B f(N\Phi(X,Y)),$$

using the transverse Nijenhuis tensor

$$(4) \quad N\Phi(X,Y) = [\Phi X, \Phi Y] + \Phi^2 [X,Y] - \Phi [\Phi X, Y] - \Phi [X, \Phi Y].$$

Furthermore, when the $K$-contact structure $(\eta, \xi, \Phi, g)$ is Sasakian, $N\Phi = -2\xi \otimes d\eta$ (see for instance [4, p. 204]) and thus $d^c_B d_B + d_B d^c_B = 0$.

The transverse Hodge star operator $\tilde{*}$ (see [11, p. 215] or [33]) is defined in terms of the usual Riemannian Hodge operator $*$ by setting

$$\tilde{*}\alpha = * (\eta \wedge \alpha) = (-1)^p \iota_\xi (*\alpha), \quad \alpha \in \Omega^p_B.$$ 

In particular, it maps basic forms to themselves and on $p$-forms we have

$$(\tilde{*})^2|_{\Omega^p_B} = (-1)^p.$$

As in almost Kähler geometry, define also adjoint differentials by $\delta_B = -\tilde{*} d_B \tilde{*}$ and $\tilde{\delta}^c_B = -\tilde{*} d^c_B \tilde{*}$. The basic (twisted) Laplacian is then given by

$$\Delta_B = d_B \delta_B + \delta_B d_B, \quad \Delta^c_B = d^c_B \delta_B + \delta_B d^c_B.$$

One can easily check the following (similarly to the almost-Kähler case, see [16]):

**Lemma 6.** Let $\nabla$ be a torsion free connection on $M$ with $\nabla(d\eta) = 0$. For a local, positively oriented, $g$-orthonormal basis $\xi, e_1, \ldots, e_{2n}$ of $TM$ we have

$$\delta^c_B \alpha = -\sum_{i=1}^{2n} \iota_{\Phi e_i} (\nabla_{e_i} \alpha), \quad \alpha \in \Omega^p_B.$$ 

(such a connection $\nabla$ can be constructed similarly to the Levi-Civita connection.)

We introduce also the adjoint operators $L: \Omega^p_B \to \Omega^{p+2}_B$ and $\Lambda: \Omega^{p+2}_B \to \Omega^p_B$ by

$$(6) \quad L(\alpha) = \alpha \wedge d\eta, \quad \Lambda = -\tilde{*} L \tilde{*}.$$

**Proposition 7.** We have the Kähler identities on basic forms:

$$\ell [L, \delta^c_B] = -d_B, \quad [L, \delta_B] = d^c_B, \quad [\Lambda, \delta^c_B] = \delta_B, \quad [\Lambda, d_B] = -\delta^c_B.$$ 

**Proof.** Using $\Phi^2 \alpha = (\tilde{*})^2 \alpha = (-1)^p \alpha$, we reduce to proving only the first identity. Choose an oriented, local, $g$-orthonormal basis $\{\xi, e_i\}$. For a basic $p$-form $\alpha$ we
then compute, using Lemma 8 and a torsion free connection ∇ preserving dη:

\[
[\delta^c_{\infty}, L] \alpha = \delta^c_B (\alpha \wedge d\eta) - d\eta \wedge \delta^c_B \alpha \\
= \sum_{i=1}^{2n} -\iota_{\Phi_{\alpha}} (\nabla_{e_i} (\alpha \wedge d\eta)) + d\eta \wedge \iota_{\Phi_{\alpha}} (\nabla_{e_i} \alpha) \\
= \sum_{i=1}^{2n} -\iota_{\Phi_{\alpha}} (\nabla_{e_i} \alpha \wedge d\eta) + d\eta \wedge \iota_{\Phi_{\alpha}} (\nabla_{e_i} \alpha) \\
= \sum_{i=1}^{2n} -\iota_{\Phi_{\alpha}} (d\eta) \wedge \nabla_{e_i} \alpha \\
= \sum_{i=1}^{2n} e^c_i \wedge \nabla_{e_i} \alpha = d_B \alpha.
\]

Note here that the expression for dα in terms of covariant derivatives reduces on basic forms to the last equation since ∇ξ α = 0 for basic α. To see this, note that ∇(dη) = 0 implies dη(∇Xξ, Z) = 0 for any X, Z and the Reeb field ξ. Therefore ∇Xξ is a multiple of ξ, so for basic forms i∇Xξ α = 0. Using that ∇ is torsion free,

\[0 = \mathcal{L}_\xi \alpha (X_1, \ldots, X_p) = \nabla_\xi \alpha (X_1, \ldots, X_p) + \sum_{k=1}^{n} (-1)^k \alpha (\nabla X_k \xi, X_1, \ldots, \hat{X}_k, \ldots, X_p),\]

for arbitrary tangent vectors X_k, and ∇ξ α = 0 follows. □

Remark 8. It follows from Remark 6 and the Kähler identities (7) that Δ_B = Δ^c_B whenever the K-contact structure (η, ξ, Φ, g) is Sasakian.

The Laplacians (8) are basic transversely elliptic operators (see [22]). Hence they are Fredholm operators, so we get basic Green operators G_B and G^c_B with

\[G_B \Delta_B \alpha = \Delta_B G_B \alpha = \alpha + (\alpha)_H, \quad G^c_B \Delta_B \alpha = \Delta^c_B G^c_B \alpha = \alpha + (\alpha)_H,\]

where (α)_H, (α)_H^c are orthogonal projections onto Δ_B and Δ^c_B harmonic forms. Recall that G_B, Δ_B commute with d_B, δ_B, while G^c_B, Δ^c_B commute with d^c_B, δ^c_B.

Lemma 9. For any d_B-exact, d^c_B-closed α ∈ Ω^p_B there exists ψ ∈ Ω^{p-2}_B with

\[\alpha = G_B d_B d^c_B \psi = d_B G^c_B d_B d^c_B \psi.\]

Proof. This is an analogue of [22] Lemma 3.1. Decomposing α with respect to Δ_B using (8) gives α = d_B δ_B G_B α. With respect to Δ^c_B we have α = (α)_H + d^c_B δ^c_B G^c_B α. Now we apply the Kähler identities (7) to deduce

\[\alpha = d_B \delta_B G_B \alpha = d_B \delta_B G_B [(\alpha)_H + d^c_B \delta^c_B G^c_B \alpha] \\
= d_B G_B \delta_B (\alpha)_H + d^c_B G_B \delta_B d_B \delta^c_B G^c_B \alpha \\
= d_B G_B [\Lambda, d_B] (\alpha)_H - d_B G_B d_B \delta^c_B \delta^c_B G^c_B \alpha \\
= -d_B G_B d_B \delta^c_B \Lambda (\alpha)_H - d_B G_B d_B \delta^c_B \delta^c_B G^c_B \alpha \\
= d_B G_B d_B \delta^c_B (-(\Lambda) (\alpha)_H), \quad \square\]

Corollary 10. Suppose ω_1, ω_2 ∈ Ω^2_B are d_B-closed, J-invariant, and basic cohomologous. Then there exists a smooth basic function f with ω_2 = ω_1 + d_B G_B d^c_B f.

Remark 11. Corollary 10 generalizes the transverse ∂̅-Lemma [22]. Indeed, Remark 8 implies that in the Sasakian case ω_2 = ω_1 + d_B G_B d^c_B f.
3.2. The Space $\mathcal{K}(\xi, J)$ in Dimension 5. In dimension $2n + 1$ the bundle $\Lambda^2_B$ of basic 2-forms decomposes into $\pm 1$-eigenspaces of the transverse Hodge operator

$$\Lambda^2_B = \Lambda^+ B \oplus \Lambda^- B.$$ 

Similarly $\Lambda^2_B = \Lambda^+_B \oplus \Lambda^+ B$ into the $\pm 1$-eigenspaces of $J$. In dimension 5 we have

$$\Lambda^+_B = \mathbb{R}. \, d\eta \oplus \Lambda^+ J, \quad \Lambda^- B = \Lambda^+ B,$$

where $\Lambda^+ J$ is the subbundle of $J$-invariant 2-forms pointwise orthogonal to $d\eta$.

We denote by $b^+_B$ (resp. $b^-_B$) the dimension of the space of $\Delta_B$-harmonic $\bar{\partial}$-self-dual (resp. $\bar{\partial}$-anti-self-dual) basic 2-forms. Since $\Delta_B$-harmonic basic 2-forms are preserved by the $\bar{\partial}$-operator, the dimension of the basic cohomology is

$$b^2_B := \dim H^2_B(M) = b^+_B + b^-_B.$$ 

Let $h^+_B$ be the dimension of the $\Delta_B$-harmonic $J$-anti-invariant basic 2-forms. It is easy to see that this definition agrees with that in [9].

**Proposition 12.** Let $(M, \eta, \xi, \Phi, g)$ be a 5-dimensional compact $K$-contact manifold. If $h^+_B = b^+_B - 1$, then for any basic function $f$, $d_B \mathbb{G}_B d^-_B f$ is $J$-invariant.

**Proof.** The proof is similar to that of [28, Proposition 2], so we only sketch the argument. Beginning with (10), a computation using the Kähler identities (7) shows that the $J$-anti-invariant part of $d_B \mathbb{G}_B d^-_B f$ is

$$(d_B \mathbb{G}_B d^-_B f)^+_J = \frac{1}{2}(f_0 d\eta)_H - \frac{1}{4} g((f_0 d\eta)_H, d\eta) d\eta,$$

for the orthogonal projection $f_0$ of $f$ onto the complement of the constants and the $\Delta_B$ harmonic part $(f_0 d\eta)_H$. The condition $h^+_B = b^+_B - 1$ implies $(f_0 d\eta)_H = 0$. \hfill \Box

Using Remark [8] we see $h^+_B = b^+_B - 1$ when the $K$-contact structure is Sasakian. It is well-known that an almost complex structure is integrable precisely when $(dd^c + d^c d)f$ is $J$-invariant for every function $f$. The condition $h^+_B = b^+_B - 1$ appears therefore as a semi-integrability condition.

**Definition 13.** A $K$-contact structure is semi-Sasakian if $h^+_B = b^+_B - 1$ holds.

**Remark 14.** Under a smooth variation of the transverse almost-complex structure $J_t$, the dimension $h^+_B$ is a semi-continuous function of $t$. To see this, consider the family of basic transversally strongly elliptic differential operators

$$P_t : \Omega^+ B \rightarrow \Omega^+ B, \quad \alpha \mapsto (d_B^t \alpha)^+ B.$$ 

Here, $\Omega^+ B$ are basic $J_t$-anti-invariant 2-forms, $(\cdot)^+ B$ is the projection, and $d^t B$ is the adjoint of $d_B$ with respect to the $K$-contact metric induced by $J_t$. Then $h^+_B$ is the kernel of $P_t$, whose dimension is an upper semi-continuous function (using [23, Theorem 6.1], an adaptation of [24, Theorem 4.3], see also [9]).

We may now generalize [1] Proposition 7.5.7] to the semi-Sasakian case:

**Theorem 15.** Let $(M, \eta, \xi, \Phi, g)$ be a 5-dimensional compact semi-Sasakian manifold with transverse structure $J = \Phi^\top$. Then we have a diffeomorphism

$$\mathcal{K}(\xi, J) \simeq \mathcal{H} \times C^\infty_{B,0}(M) \times \mathcal{H}^1(M, \mathbb{R}),$$

for the basic functions with zero integral $C^\infty_{B,0}(M)$ and where

$$\mathcal{H} = \{ f \in C^\infty_{B,0}(M) \mid (d\eta + d_B \mathbb{G}_B d^-_B f)(X, \Phi X) > 0 \ \forall X \in \ker (\eta + \mathbb{G}_B d^-_B f) \}.$$
Theorem 16. (see [21, Remark 4.3]) The action of $\mathfrak{Con}(M, \eta)$ on $K_\eta$ is Hamiltonian. The moment map $\mu: K_\eta \rightarrow \text{Lie} \mathfrak{Con}(M, \eta)^* \cong C^\infty_B(M)^*$ is

$$
\mu(\Phi)(f) = - \int_M s^\Phi(f) \, d\eta,
$$

for the transverse Hermitian scalar curvature $s^\Phi$ of $\Phi$ and where we use [11].
Proof. The infinitesimal action of $X \in \mathrm{LieCon}(M, \eta)$ at a point $\Phi \in \mathcal{K}_\eta$ is given by $\mathcal{L}_X(\Phi) = -s_X(\Phi)$. For a tangent vector $A \in T_{\Phi} \mathcal{K}_\eta$ let $Q(A) \in C_m^\infty$ be the derivative of the map $\Phi \mapsto \Phi^A$ in direction $A$. We must show

$$\Omega_{\Phi}(\mathcal{L}_{X^f} \Phi, A) = -\int_M f \cdot Q(A) dv_\eta.$$ \hfill (12)

We do local computations as in [21]. Pick local coordinates as in Subsection 2.2. We shall write $f_k = f_{,k} = \partial f / \partial x^k$ and $f_{,k}$ for covariant differentiation.

From $d\eta(X_f, \cdot) = -df$ we have (we adopt the summation convention that roman indices, if appearing twice, range over $1, \ldots, n$, i.e. excluding zero)

$$X_f = -\frac{1}{2} \Phi^i_k g^{k i} f_j \frac{\partial}{\partial x^j} + f \frac{\partial}{\partial x^i}$$ \hfill (13)

Since $\Phi$ is basic, we may locally write

$$\mathcal{L}_{X_f} \Phi = B^i_j \frac{\partial}{\partial x^j} \otimes dx^i.$$

The local coordinate formula for the Lie derivative combined with (13) gives

$$B^i_j = -\frac{1}{2} \Phi^i_k g^{k i} f_j + \frac{1}{2} \Phi^i_p \left( \Phi^k_l g^{k i} f_1 \right)_,p - \frac{1}{2} \Phi^i_l \left( \Phi^k_j g^{k p} f_1 \right)_,l.$$

From (3) we see then

$$\Phi^i_k B^i_j = -\frac{1}{2} \Phi^i_k \Phi^j_p \Phi^p_l g^{k j} f_1 + \frac{1}{2} \Phi^i_k \Phi^j_p \left( \Phi^k_l g^{k i} f_1 \right)_,p - \frac{1}{2} \Phi^i_k \left( \Phi^k_j g^{k p} f_1 \right)_,l.$$

Therefore we have

$$\Omega_{\Phi}(\mathcal{L}_{X_f} \Phi, A) = \int_M \text{trace}(\Phi \circ \mathcal{L}_{X_f} \Phi \circ A) dv_\eta$$

$$= \int_M \left( -\frac{1}{2} \Phi^i_k g^{k j} f_1 A^j_k + \frac{1}{2} \Phi^i_p \left( \Phi^k_l g^{k i} f_1 \right)_,p A^j_k + \frac{1}{2} \left( \Phi^i_k g^{k j} f_1 \right)_,l A^j_k \right) dv_\eta.$$ \hfill (13)

Let $C^i_j = \Phi^i_k \Phi^j_p \Phi^p_l g^{k j} f_1$. Using (3) and its derivative one checks $g_{s} C^s_j = -g_{sj} C^s_i$, so $C$ is $g^T$-anti-symmetric. On the other hand, $A^j_k$ is $g^T$-symmetric and so the trace $C^j_k A^j_k$, the first summand in the bracket, vanishes.

The second and third summand are equal (from $\Phi^j_k A^j_k = -\Phi^j_k A^j_k$) so

$$\Omega_{\Phi}(\mathcal{L}_{X_f} \Phi, A) = \int_M \left( \Phi^i_k g^{k j} f_1 \right)_,l A^j_k dv_\eta.$$ \hfill (14)

The variation of the transverse Hermitian scalar curvature (see [14]) is given in terms of the variation of $\Phi$ along $A$ by

$$\hat{s}^T = Q(A) = -(g^{k s} \Phi^i_k (A^j_s)_,j)_,l.$$

Using integration by parts twice (justified as in [21]), we conclude the proof of (12):

$$\int_M \left( \Phi^i_k g^{k j} f_1 \right)_,l A^j_k dv_\eta = -\int_M \Phi^i_k g^{k j} f_1 (A^j_k)_,j dv_\eta = \int_M f \left( \Phi^i_k g^{k j} (A^j_s)_,j \right)_,l dv_\eta.$$

Much of the above works in greater generality; instead of a contact manifold, begin with a closed manifold $M$ and closed 2-form $\omega$ with $\omega^q$ never zero and $\omega^{q+1} = 0$. This amounts to a codimension 2 foliation $\mathcal{T}F = \text{ker} \omega$ with transverse symplectic structure. The argument of Proposition 7 gives Kähler identities on basic forms. One may then define a Fréchet space $\mathcal{AC}(\omega)$ of $\omega$-compatible transverse almost complex structures $J$, which has a Kähler structure. Combining $J$ with $\omega$ yields
a bundle-like metric $g$, so again we obtain a transverse Levi-Civita connection $D^T$ and Hermitian connection and corresponding scalar curvatures.

Introducing the variation of connection $\delta A$ in direction $A \in T_A\mathcal{AC}(\omega)$ in the standard way (see [10] Section 9.5) defines, using that the connection is basic (see [33] Proposition 3.6), a basic 1-form. The Mohsen formula $2\delta A = g(\delta A, \cdot)$ for $\delta A = -\sum_k e_k D^T_{e_k} A$, summing over an orthonormal frame $e_k$ of the normal bundle, can be established using the argument of [10] Proposition 9.5.1. The variation $\delta A$ being basic, the Kähler identities then give $s^2_{\mathcal{C}} = -\delta J(\delta A)^2$.

It is not hard to define a transverse Hamiltonian group that acts on $\mathcal{AC}(\omega)$. As in Theorem [10] this action is Hamiltonian with moment map $J \mapsto s^2_{\mathcal{C}}$.

**Definition 17.** The square-norm of the moment map defines a functional
\[
\mathcal{C}: \mathcal{K}_\eta \to \mathbb{R}, \quad \mathcal{C}(\Phi) = ||\mu(\Phi)||^2 = \int_M (s^2_{\mathcal{C}})^2 \, dv_\eta.
\]

The critical points of this functional are called *extremal K-contact metrics*.

Given a K-contact structure $(\eta, \xi, \Phi, g)$, we denote by $X_{\mathcal{C}} \in \text{Lie}\text{Conn}(M, \eta)$ the vector field belonging via [1] to the scalar curvature $s^2_{\mathcal{C}}$.

**Proposition 18.** $\Phi$ is extremal if and only if $X_{\mathcal{C}}$ is a Killing vector field with respect to the metric $g_\Phi$ induced by $\Phi$ (equivalently, when $\mathcal{L}_{X_{\mathcal{C}}} \Phi = 0$).

**Proof.** This follows from the moment map set-up (see [2] [26] in the case of extremal almost-Kähler metrics). For $A \in T_\Phi \mathcal{K}_\eta$ the differential of (14) in direction $A$ is
\[
\mathcal{C}_{\bullet, \Phi}(A) = 2(\mu_{\bullet, \Phi}(A), \mu(\Phi)) = 2d||\mu||^2(\Phi) = 2\Omega_{\Phi}(\mathcal{L}_{X_{\mathcal{C}}} \Phi, A),
\]
where we write $\mu_f = \mu(\cdot)(f)$. The last equality is by Theorem [10].

**Example 19.** Calabi’s extremal problem [7] [8] was extended to Sasaki geometry by Boyer–Galicki–Simanca in [5] [6] where they introduce the notion of *extremal Sasakian metrics*. This notion generalizes *Sasaki-Einstein metrics* (more generally the so-called $\eta$-Einstein metrics, see for instance [32]) and constant scalar curvature Sasaki metrics. From Proposition [13] extremal Sasaki metrics $\Phi$ are extremal K-contact metrics. Indeed, when $\Phi$ is Sasakian the Riemannian scalar curvature coincides with the Hermitian scalar curvature $s^2$.

**Remark 20.** Extremal K-contact metrics are a natural extension of extremal Sasaki metrics [5] [6] to K-contact geometry. Given a background Sasaki structure $(\eta, \xi, \Phi, g)$, Boyer–Galicki–Simanca consider the space $\mathcal{S}(\xi, J)$, of Sasaki structures with common Reeb vector field $\xi$ and transverse *integrable* almost-complex structure $J = \Phi^T$, arising from deforming the contact form $\eta$ by $\eta \mapsto \eta_t = \eta + t\alpha$, where $\alpha$ is a basic 1-form with respect to the characteristic foliation $\mathcal{F}_\xi$. Hence, by *Gray’s Stability Theorem* (see [21] or for instance [21] p. 190), there exist a *diffeomorphism* $\gamma$ such that $\gamma^* \eta_t = \eta_t$ for any $(\eta', \xi', \Phi', g') \in \mathcal{S}(\xi, J)$. This gives a new Sasaki structure $(\eta, \xi, \gamma^{-1}_* \Phi', \gamma^* g')$ with $\gamma^{-1}_* \Phi' \gamma \in \mathcal{K}_\eta$.

5. A K-contact Futaki invariant

We continue to draw consequences of Theorem [10]. In this section, we generalize the Futaki invariant from [5] [13] to the non-integrable K-contact setting. Fix throughout a $(2n + 1)$-dimensional compact contact manifold $(M, \eta)$ with volume form $dv_\eta = (2n)!^{-1} \eta \wedge (d\eta)^{2n}$. Moreover, we shall assume $\mathcal{K}_\eta \neq \emptyset$. In this case
where different maximal tori are not necessarily conjugate.

Since $G$ is compact, an averaging argument shows that the subspace $\mathcal{K}^G_\eta \subset \mathcal{K}_\eta$ of $G$-invariant $K$-contact structures is contractible as well.

Let $\Pi^G$ be the orthogonal projection from $C_B^\infty(M)$, the space of basic functions, onto $\mathfrak{g}_\eta$ the contact Hamiltonians of $\text{Lie}(G)$, recalling the identification \cite{11}.

As a generalization of \cite{11} in the Kähler case we find:

**Proposition 21.** For every smooth curve $\Phi_t \in \mathcal{K}^G_\eta$ the projection of the Hermitian scalar curvature $\Pi^G(s^T_\Phi) \in \mathfrak{g}_\eta$ is independent of $t$.

*Proof.* We may equivalently show that $\mu|_{\mathcal{K}^G_\eta}(\Phi_t)(X)$ is constant for any $X \in \text{Lie}(G)$:

$$\left. \frac{d}{dt} \right|_{t_0} \mu(\Phi_t)(X) = d\mu(\Phi_t)(X) = \Omega_{\Phi_{t_0}}(\dot{\Phi}(0), \dot{X}(\Phi_{t_0})) = 0,$$

using that the infinitesimal action $\dot{X} = \left. \frac{d}{dt} \right|_{t_0} X(\Phi_{t_0})$ vanishes ($\Phi_{t_0}$ being $G$-invariant). $\square$

**Definition 22.** For fixed $G \subset \text{Con}(M, \eta)$, we define the vector field $Z^G_\eta \in \text{Lie}(G)$ corresponding to the contact Hamiltonian $z^G_\eta = \Pi^G s^T_\Phi \in \mathfrak{g}_\eta$, via \cite{11}, using an arbitrary $K$-contact structure $\Phi \in \mathcal{K}^G_\eta$.

By Proposition 21 and $\mathcal{K}^G_\eta \simeq \{ \text{pt} \}$, the extremal vector field $Z^G_\eta$ is well-defined (see \cite{14} in the Kähler case).

**Proposition 23.** A $K$-contact structure $\Phi \in \mathcal{K}^G_\eta$ is extremal precisely when

$$s^T_\Phi = z^G_\eta. \quad (15)$$

*Proof.* Suppose $(\eta, \xi, \Phi, g)$ is extremal. By Proposition 18 $X_{\mathfrak{x}\mathfrak{t}}$ is a $G$-invariant Killing field. $\Phi$ being $G$-invariant, $G$ is a subgroup of the isometry group for $(M, g_\Phi)$. Consider the connected Lie subgroup $H \subset \text{Isom}(M, g)$ belonging to the abelian subalgebra $\text{Lie}(G) + \mathbb{R} \cdot X_{\mathfrak{x}\mathfrak{t}}$. The closure $\overline{H}$ is a torus in $\text{Isom}(M, g) \cap \text{Con}(M, \eta)$ (and also in $\text{Con}(M, \eta)$), containing $G$. By maximality, $G \subset H = \overline{H} \subset G$, so $X_{\mathfrak{x}\mathfrak{t}} \in \text{Lie}(G)$ and \cite{15} follows. Conversely, from \cite{15} we have $s^T_\Phi \in \text{Lie}(G) \subset \text{Lie}(\text{Isom}(M, g))$ so $X_{\mathfrak{x}\mathfrak{t}}$ is a Killing field and $\Phi$ is extremal by Proposition 18. $\square$

Consider the ‘angle’ map $(\cdot, Z^G_\eta)$ on $\text{Lie}(G)$. If $(M, \eta)$ admits an extremal metric $\Phi$, then by the previous proposition the angle map completely determines its scalar curvature $s^T_\Phi$. A more explicit definition of this map is as follows:

**Definition 24.** The $K$-contact Futaki invariant relative to the group $G$ is the map

$$\mathfrak{F}_{K^G} : \text{Lie}(G) \to \mathbb{R}, \quad X \mapsto \int_M \eta(X)s^T_\Phi dv_\eta,$$

where $s^T_\Phi = s^T_\Phi - \int_M s^T_\Phi dv_\eta$ is the zero integral part (for $\Phi \in \mathcal{K}^G_\eta$ arbitrary).

The previous discussion implies (see \cite{3} Proposition 5.2) in the Sasakian case:

**Proposition 25.** If $(M, \eta)$ admits an extremal $K$-contact metric, the following are equivalent:

1. Every (some) extremal metric has constant Hermitian scalar curvature.
2. $\mathfrak{F}_{K^G} \equiv 0.$
Proposition 26. The vector field $Z^G_\eta$ is invariant under $G$-invariant strict contact isotopy of $\eta$.

Proof. Suppose that we have a smooth $G$-invariant family of contact forms $\eta_t$ with the same Reeb vector field $\xi$ (such that $\eta_0 = \eta$). Then, by Gray’s Stability Theorem, there exists a smooth family of diffeomorphisms $\gamma_t$ such that $\gamma_0 = \text{id}$ and $\gamma_t^* \eta_t = \eta$. Then, $\gamma_t^* (Z^G_\eta) = Z^G_{\eta_t}$. Moreover, using the $G$-invariance of the vector field generating $\gamma_t^*$, we have $\gamma_t^* (Z^G_\eta) = Z^G_{\eta_t}$. □

On a compact contact manifold $(M, \eta)$ consider the space $K^G(\xi)$ of all $G$-invariant $K$-contact structures with contact forms strictly isotopic to $\eta$ and common Reeb field $\xi$. One easily deduces the following: if $K^G(\xi)$ contains a $K$-contact metric with constant transverse Hermitian scalar curvature then $Z^G_\eta = 0$. Conversely, if $Z^G_\eta = 0$, any extremal $K$-contact metric in $K^G(\xi)$ is of constant transverse Hermitian scalar curvature.

Since the Reeb field $\xi$ lies in $\text{Lie}(G)$ it follows that

$$\int_M s^\nabla^T dv_\eta = \int_M Z^G_\eta dv_\eta$$

so that

$$\mathfrak{F}_{K^G_\eta}(Z^G_\eta) = \int_M \eta(Z^G_\eta)s^\nabla^T dv_\eta = \int_M z^G_\eta s^\nabla^T dv_\eta = \int_M (z^G_\eta)^2 dv_\eta - \frac{(\int_M s^\nabla^T dv_\eta)^2}{\int_M dv_\eta}.$$ 

We obtain a lower bound for the functional (14):

**Proposition 27.** Let $S_\eta = \int_M s^\nabla^T dv_\eta$ and $V_\eta = \int_M dv_\eta$. For all $\Phi \in K^G_\eta$ we have

$$\int_M (s^\nabla^T)^2 dv_\eta \geq \mathfrak{F}_{K^G_\eta}(Z^G_\eta) + \frac{S_\eta^2}{V_\eta}.$$ 

Equality holds if and only if $\Phi \in K^G_\eta$ induces an extremal metric.

Proof. The inequality follows from the above discussion. Moreover, equality holds if and only if $s^\nabla^T = z^G_\eta$, i.e. by Proposition 23 when $\Phi$ is extremal. □

### 6. Deformations of Extremal $K$-contact Metrics in Dimension 5

In the Sasakian setting, Boyer-Galicki-Simanca developed the notion of Sasakian cone [5, 6] and proved in [5] that the existence of extremal Sasakian metrics is an open condition in the Sasakian cone, as in the Kähler set-up [25, 30, 12].

In this section, we show that a similar result holds in the semi-Sasakian case. Let $K^G_{\eta, \text{semi}}$ be the subspace of $K^G_{\eta}$ of those $\Phi$ that are semi-Sasakian (see Definition 13).

**Theorem 28.** Let $(M, \eta)$ be a 5-dimensional compact contact manifold and $G$ be a maximal torus in $\text{Con}(M, \eta)$. Let $\Phi_t$ be a smooth curve in $K^G_{\eta, \text{semi}}$ with $\Phi_0$ an extremal Sasakian metric. Then there exists a smooth curve $\Phi_t$ of $G$-invariant extremal $K$-contact metrics with $\Phi_0 = \Phi_0$ and $\Phi_t$ diffeomorphic to $\Phi_t$.

Proof. We follow mainly Boyer–Galicki–Simanca proof [5]. However, in our case, $J_t = \Phi_t^T$ may vary. Let $g_\eta = \{\eta(X) \mid X \in \text{Lie}(G)\}$ be the space of contact Hamiltonian functions associated to $\text{Lie}(G)$. Using Theorem 15, we consider the deformations...
of \((\eta, \xi, \Phi_0, g_0)\) defined by

\[
\begin{align*}
\eta_{t,\phi} &= \eta + \mathcal{G}_t d^*_t \Delta_{B,t} \phi, \\
\Phi_{t,\phi} &= \Phi_t - (\xi \otimes (\eta_{t,\phi} - \eta)) \circ \Phi_t, \\
g_{t,\phi} &= d\eta_{t,\phi} \circ (\text{id} \otimes \Phi_{t,\phi}) + \eta_{t,\phi} \otimes \eta_{t,\phi},
\end{align*}
\]

where \(\mathcal{G}_t\) is the Green’s operator associated to the basic Laplacian \(\Delta_{B,t}\), with respect to the \(K\)-contact metric \((\eta, \xi, \Phi_t, g_t)\), \(\phi\) is an element of the space \(C^\infty_{G,\perp}\) of smooth \(G\)-invariant basic functions which are \(L^2\)-orthogonal (with respect to \(dv_\eta\)) to \(g_\eta\). Here, \(d^*_t\) stands for \(\Phi_t d_B\).

Denote by \(\Pi_{\eta,\phi}\) the \(L^2\)-orthogonal projection of basic functions on the space \(g_{\eta,\phi} = \{\eta_t(X) | X \in \text{Lie}(G)\}\) with respect to the volume form \(dv_{\eta,\phi}\). Let \(\mathcal{W}^{p,k}\) be the Sobolev completion of \(C^\infty_{G,\perp}\) involving derivatives up to order \(k\).

Let \(U \subset \mathbb{R} \times \mathcal{W}^{p,k}\) be a neighborhood of \((0,0)\) such that \((\eta_{t,\phi}, \xi, \Phi_{t,\phi}, g_{t,\phi})\) is a \(K\)-contact structure for any \((t, \phi) \in U\) and that \(\ker(\text{id} - \Pi_\eta) \circ (\text{id} - \Pi_{\eta,\phi}) = \ker(\text{id} - \Pi_{\eta,\phi})\) (by possibly shrinking \(U\)). Consider then the map (defined by extension)

\[
\Psi : U \subset \mathbb{R} \times \mathcal{W}^{p,k+4} \longrightarrow \mathbb{R} \times \mathcal{W}^{p,k}
\]

\[
(t, \phi) \longmapsto (t, (\text{id} - \Pi_\eta) \circ (\text{id} - \Pi_{\eta,\phi}) \tilde{s}^T_t(\eta, \xi, \Phi_{t,\phi}, g_{t,\phi}),
\]

where \(\tilde{s}^T_t(\eta, \xi, \Phi_{t,\phi}, g_{t,\phi})\) is the transverse Hermitian scalar curvature of \((\eta_{t,\phi}, \xi, \Phi_{t,\phi}, g_{t,\phi})\). The map is well defined for \(pk > 5\).

By Proposition \[23\] \(\Psi(t, \phi) = (t, 0)\) if and only if \((\eta_{t,\phi}, \xi, \Phi_{t,\phi}, g_{t,\phi})\) is an extremal \(K\)-contact structure. Hence, by hypothesis, \(\Psi(0,0) = (0,0)\).

\(\Psi\) is a \(C^1\) map. Indeed, the dimension of the kernel of the basic Laplacian, with respect to the metric \((\eta, \xi, \Phi_t, g_t)\), applied on 1-forms, is equal to the dimension of \(H^1(M, \mathbb{R})\) (see \[4\] Proposition 7.2.3) and so the dimension of the kernel of \(\Delta_{B,t}\) is independent of \(t\). Thus, \(\mathcal{G}_t\) is a \(C^1\) map (see \[23\] Theorem 6.1) and consequently \(\Psi\) is. The linearization of \(\Psi\) at \((0,0)\) is given by (see \[5\] Proposition 7.3)

\[
(D\Psi)_{(0,0)}(t, \phi) = (t, t(\ast) - 2(\text{id} - \Pi_\eta)L_B^0(\phi)),
\]

where \(L_B^0(\phi) = \delta_B^0 \delta_B^0 \left(D_B^0 d_B^0\right) J_0^{-1}\) is a basic self-adjoint transversally elliptic differential operator of order 4 (here, \((\ast)\) denotes some expression depending on \(\frac{d}{dt}|_{t=0} \Phi_t\)). By the standard arguments (see \[5\] Proposition 7.5 and \[1\] Lemma 4) and the main result of \[22\], \((D\Psi)_{(0,0)}\) is an isomorphism.

It follows from the Inverse Function Theorem for Banach spaces that there exists a neighborhood \(V \subset \mathbb{R} \times \mathcal{W}^{p,k+4}\) of \((0,0)\) and \(\epsilon > 0\) such that, for \(|t| < \epsilon\), \((\eta_{\Psi^{-1}_{\beta}(t,0)}, \xi, \Phi_{\Psi^{-1}_{\beta}(t,0)}, g_{\Psi^{-1}_{\beta}(t,0)})\) is an extremal \(K\)-contact metric.

By a standard bootstrapping argument (used for instance in \[27\]), we get then a smooth family of \(G\)-invariant extremal \(K\)-contact metrics defined for a sufficiently small \(t\). Theorem \[23\] now follows from Gray’s Stability Theorem.

\[\square\]

**Remark 29.** Suppose that at time \(t = 0\) we have \(b_B^+ = 1\). Then, using \[23\] Theorem 6.1, \(b_B^+ = 1\) for small \(|t| < \epsilon\). Remark \[14\] now implies that \(\Phi_t\) is automatically semi-Sasakian for small values of \(|t|\).
References

[1] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. W. Tønnesen-Friedman, Extremal Kähler metrics on projective bundles over a curve, Adv. Math. 227 (2011), no. 6, 2085–2424.

[2] V. Apostolov and T. Drăghici, The curvature and the integrability of almost-Kähler manifolds: a survey, Fields Inst. Commun. Series 35, AMS (2003), 25–53.

[3] C. P. Boyer, Maximal Tori in Contactomorphism Groups, Differential Geom. Appl. 31 (2013), no. 2, 190–216.

[4] C. P. Boyer and K. Galicki, Sasakian geometry, Oxford Mathematical Monographs. Oxford University Press, Oxford (2008).

[5] C. P. Boyer, K. Galicki and S. R. Simanca, Canonical Sasakian metrics, Comm. Math. Phys. 279 (2008), no. 3, 705–733.

[6] E. Calabi, Extremal Kähler metrics, in Seminar of Differential Geometry, S. T. Yau (eds), Annals of Mathematics Studies 102, Princeton University Press (1982), 259–290.

[7] A. Fujiki and G. Schumacher, The moduli space of extremal compact Kähler manifolds and generalized Weil–Petersson metrics, Publ. Res. Inst. Math. Sci. 26 (1990), no. 1, 101–183.

[8] A. Futaki, An obstruction to the existence of Einstein Kähler metrics, Invent. Math., 73 (1983), 437–443.

[9] A. Futaki and T. Mabushi, Bilinear forms and extremal Kähler vector fields associated with Kähler classes, Math. Ann. 301 (1995), 199–210.

[10] K. Kodaira and J. Morrow, Complex manifolds, Holt, Rinehart and Winston (1971).

[11] C. LeBrun and S. R. Simanca, On the Kähler Classes of Extremal Metrics, Geometry and Global Analysis (Sendai, Japan 1993), FirstMath. Soc. Japan Intern. Res. Inst. Eds. Kotake, Nishikawa and Schoen.

[12] M. Lejmi, Extremal almost-Kähler metrics, Internat. J. Math. 21 (2010), no. 12, 1639–1662.

[13] Stability under deformations of extremal almost-Kähler metrics in dimension 4, Math. Res. Lett. 17 (2010), no. 4, 601–612.
[28] _____, Stability under deformations of Hermite-Einstein almost-Kähler metrics, To appear at Annales de l’institut Fourier, 64 (2014).

[29] P. Libermann, Sur les connexions hermitiennes, C. R. Acad. Sci. Paris 239 (1954), 1579–1581.

[30] S. R. Simanca, Canonical metrics on compact almost complex manifolds, Publicações Matemáticas do IMPA, IMPA, Rio de Janeiro (2004), 97 pp.

[31] _____, Heat Flows for Extremal Kähler Metrics, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 4 (2005), 187–217.

[32] J. Sparks, Sasakian-Einstein manifolds, arXiv:1004.2461.

[33] P. Tondeur, Geometry of Riemannian foliations, volume 20 of Seminar on Mathematical Sciences. Keio University, Department of Mathematics, Yokohama (1994).