On ground fields of arithmetic hyperbolic reflection groups. III

Viacheslav V. Nikulin

Dedicated to I. R. Shafarevich for his 85th birthday

Abstract

The paper continues from the work of Nikulin. Using our methods of 1980 and 1981, we define some explicit finite sets of number fields containing all ground fields of arithmetic hyperbolic reflection groups in dimensions at least 3, and we give good upper bounds for their degrees (over \( \mathbb{Q} \)). This extends the earlier results of Nikulin for dimensions at least 4. This finally delivers a possibility, in principle, of effective finite classification of maximal arithmetic hyperbolic reflection groups (more generally, of reflective hyperbolic lattices) in all dimensions. Our results also give another proof of finiteness in dimension 3. In fact, using our methods, we show that finiteness in dimension 3 follows from finiteness in dimension 2.

1. Introduction

Finiteness of the number of maximal arithmetic hyperbolic reflection groups and, more generally, of reflective hyperbolic lattices was recently (2006) established in full generality in works by Agol [1], Long et al. [6], Nikulin [7–10], and Vinberg [18–21]. See the introductions of [10, 13] for history, definitions, and results concerning the subject. For the enumeration of these finite sets, perhaps the first and the most important problem is to enumerate the ground fields of these groups and to give the upper bounds for their degrees (over \( \mathbb{Q} \)). Indeed, by [7, 8], there exists an effective finite algorithm for the enumeration of these lattices and groups for fixed dimension \( n \) of hyperbolic space and degree \( N \) of ground fields. By Vinberg [18–21], the dimension \( n \) is less than 30.

Using our methods of [7, 8], and results by Borel [3] (in dimension 2) and Takeuchi [17], in [12, 13] we defined some explicit finite sets of totally real algebraic number fields containing all ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 4 \). We gave good explicit upper bounds for their degrees. In particular, we showed in [13] that \( N \leq 56 \) for the degrees \( N \) of the ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 6 \). We showed in [12] that \( N \leq 138 \) in dimensions \( n \geq 4 \).

Here we continue this study for the dimension \( n = 3 \). Using similar methods, we define some explicit finite sets of totally real algebraic number fields containing all ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 3 \). We show that \( N \leq 909 \) for degrees \( N \) of fields from these sets. Here we also use our upper bound (at most 44) from [13] for the degrees of the ground fields of arithmetic Fuchsian groups of genus 0. This follows from results by Long et al. [6] where the finiteness of the number of maximal such groups had been established, and results by Borel [3] (in dimension 2) and Takeuchi [17] (see also [14–16]).

It is also important that all of these fields are clearly determined by some small parts of fundamental chambers of arithmetic hyperbolic reflection groups, and we hope that they can be further geometrically investigated and restricted by considering some larger parts of the fundamental chambers.
Thus, the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 3 \) are bounded by 909.

We note that finiteness of the number of maximal arithmetic hyperbolic reflection groups in dimension 3 was obtained by Agol [1]. It follows a theoretical existence of some upper bound for the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimension 3. Our result here is different in that we give an explicit upper bound (at most 909) for the degrees, which is important for effective finite classification. This also gives another proof of finiteness in dimension 3. In fact, using our methods, we show that finiteness in dimension 3 follows from finiteness in dimension 2.

We have mentioned above that the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimension \( n = 2 \) are bounded by 44.

Thus, now, an explicit upper bound for the degrees of the ground fields of arithmetic hyperbolic reflection groups is known in all dimensions \( n \geq 2 \). As we discussed above, this gives the possibility, in principle, of an effective finite classification of maximal arithmetic hyperbolic reflection groups in all dimensions together. More generally, this gives a possibility, in principle, of an effective finite classification of similarity classes of reflective hyperbolic lattices \( S \). Their full reflection groups \( W(S) \) contain all arithmetic hyperbolic reflection groups as subgroups of finite index.

Shortly after a preliminary version of this paper had been published in [11], Belolipetsky [2] showed that the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimension 3 are bounded by 35. His result was based on the results by Agol [1], and it also used some deep arithmetic results by Borel [3] (in dimension 3) and Chinburg and Friedman [5]. One can consider our worse upper bound 909 as the price that we have to pay to obtain finiteness in dimension 3 by Agol [1] from finiteness in dimension 2 by Long et al. [6], using our methods.

2. A reminder of some basic facts about hyperbolic fundamental polyhedra

Here we remind the reader of some basic definitions and results about fundamental chambers (always for discrete reflection groups) in hyperbolic spaces and their Gram matrices; see [7, 8, 18].

We consider a Klein model of the hyperbolic space \( \mathcal{L} \) associated to a hyperbolic form \( \Phi \) over the field \( \mathbb{R} \) of real numbers with signature \((1, n)\), where \( n = \dim \mathcal{L} \). Let \( V = \{ x \in \Phi \mid x^2 > 0 \} \) be the positive cone defined by \( \Phi \), and \( V^+ \) be one of the two halves of the cone. The hyperbolic space \( \mathcal{L} \) is the set \( V^+/\mathbb{R}^+ \) of rays in \( V^+ \); we let \([x]\) denote the element of \( \mathcal{L} \) determined by the ray \( \mathbb{R}^+x \), where \( x \in V^+ \) and \( \mathbb{R}^+ \) is the set of all positive real numbers. The hyperbolic distance is given by the formula

\[
\text{ch}(\rho([x],[y])) = \frac{(x \cdot y)}{\sqrt{x^2 y^2}}, \quad [x],[y] \in \mathcal{L}.
\]

Then the curvature of \( \mathcal{L} \) is equal to \(-1\).

Every half-space \( \mathcal{H}^+ \) in \( \mathcal{L} \) determines, and is determined by, the orthogonal element \( e \in \Phi \) with square \( e^2 = -2 \):

\[
\mathcal{H}^+ = \mathcal{H}^+_e = \{ [x] \in \mathcal{L} \mid x \cdot e \geq 0 \}.
\]

It is bounded by the hyperplane

\[
\mathcal{H} = \mathcal{H}_e = \{ [x] \in \mathcal{L} \mid x \cdot e = 0 \}
\]

orthogonal to \( e \). If two half-spaces \( \mathcal{H}^+_e \) and \( \mathcal{H}^+_f \), where \( e_1^2 = e_2^2 = -2 \), have a common non-empty open subset in \( \mathcal{L} \), then \( \mathcal{H}^+_e \cap \mathcal{H}^+_f \) is an angle of the value \( \phi \), where \( 2 \cos \phi = e_1 \cdot e_2 \) if \(-2 < e_1 \cdot e_2 \leq 2 \), and the distance between the hyperplanes \( \mathcal{H}_e \) and \( \mathcal{H}_f \) is equal to \( \rho \), where \( 2 \text{ch} \rho = e_1 \cdot e_2 \) if \( e_1 \cdot e_2 > 2 \).
A convex polyhedron $\mathcal{M}$ in $\mathcal{L}$ is the intersection of a finite number of half-spaces $\mathcal{H}_e^+$, for $e \in P(\mathcal{M})$, where $P(\mathcal{M})$ is the set of vectors with square $-2$ that are orthogonal to codimension one faces of $\mathcal{M}$ and directed outward. The Gram matrix

$$A = (a_{ij}) = (e_i \cdot e_j), \quad e_i, e_j \in P(\mathcal{M}), \quad (1)$$

of $P(\mathcal{M})$ is called the Gram matrix $\Gamma(\mathcal{M}) = \Gamma(P(\mathcal{M}))$ of $\mathcal{M}$. It determines $\mathcal{M}$ uniquely up to motions of $\mathcal{L}$. If $\mathcal{M}$ is sufficiently general, then $P(\mathcal{M})$ generates $\Phi$, and the form $\Phi$ is equal to

$$\Phi = \sum_{e_i, e_j \in P(\mathcal{M})} a_{ij} X_i Y_j \mod \text{Kernel.} \quad (2)$$

The set $P(\mathcal{M})$ is naturally identified with a subset of $\Phi$, and it determines $\mathcal{M}$.

A convex polyhedron $\mathcal{M}$ is a fundamental chamber of a discrete reflection group $W$ in $\mathcal{L}$ if and only if, for all $i \neq j$, we have $a_{ij} \geq 0$, and $a_{ij} = 2 \cos(\pi/m_{ij})$, where $m_{ij} \geq 2$ is an integer if $a_{ij} < 2$. Symmetric real matrices $A$ satisfying these conditions and having all of their diagonal elements equal to $-2$ are called fundamental (then the set $P(\mathcal{M})$ formally corresponds to indices of the matrix $A$). As usual, we further identify fundamental matrices with fundamental graphs (or diagrams) $\Gamma$. Their vertices correspond to $P(\mathcal{M})$. Two different vertices $e_i \neq e_j \in P(\mathcal{M})$ are connected by a thin edge of integer weight $m_{ij} \geq 3$ if $0 < a_{ij} = 2 \cos(\pi/m_{ij}) < 2$, by a thick edge if $a_{ij} = 2$, and by a broken edge of weight $a_{ij}$ if $a_{ij} > 2$. In particular, vertices $e_i$ and $e_j$ are disjoint if and only if $e_i \cdot e_j = a_{ij} = 2 \cos(\pi/2) = 0$. Equivalently, $e_i$ and $e_j$ (or the corresponding hyperplanes $\mathcal{H}_{e_i}$ and $\mathcal{H}_{e_j}$) are perpendicular. See some examples of such graphs in Figures 1–5.
For a real \( t > 0 \), we say that a fundamental matrix \( A = (a_{ij}) \) (and the corresponding fundamental chamber \( M \)) has minimality \( t \) if \( a_{ij} < t \) for all \( a_{ij} \). Here we follow \([7, 8]\). In our further considerations, minimality \( t = 14 \) will be especially important.

It is known that fundamental domains of arithmetic hyperbolic groups must have finite volume. Let us assume that this is valid for a fundamental chamber \( M \) of a hyperbolic discrete reflection group. Vinberg \([18]\) showed that, in order for \( M \) to be a fundamental chamber of an arithmetic reflection group \( W \) in \( L \), it is necessary and sufficient that all of the cyclic products

\[
b_{i_1 \ldots i_m} = a_{i_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_{m-1} i_m} \cdot a_{i_m i_1}
\]  

\( (3) \)
be algebraic integers, that the field \( \mathbb{K} = \mathbb{Q}(\{a_{ij}\}) \) be totally real, and that, for any embedding \( \mathbb{K} \rightarrow \mathbb{R} \) not the identity on the ground field \( \mathbb{K} = \mathbb{Q}(\{b_{1},...,b_{n}\}) \) generated by all of the cyclic products (3), the form (2) be negative definite.

A fundamental real matrix \( A = (a_{ij}) \), such that \( a_{ij} = e_{i} \cdot e_{j} \) for \( e_{i}, e_{j} \in P(\mathcal{M}) \) (or the corresponding graph), is called V-arithmetic if \( A \) satisfies these Vinberg conditions and the corresponding form \( \Phi \) in (2) is hyperbolic (here we do not require that the corresponding hyperbolic polyhedron \( \mathcal{M} \) has finite volume). It is well known and easy to see (for example, look at arguments in Subsections 4.3 and 4.4) that any subset \( P \subset P(\mathcal{M}) \) also defines a V-arithmetic matrix \( (e_{i} \cdot e_{j}) \), where \( e_{i}, e_{j} \in P \), with the same ground field \( \mathbb{K} \) if the subset \( P \) is hyperbolic, that is, the form (2) corresponding to \( P \) is hyperbolic.

Below, we assume (it was first observed and used in [7]) that the set of ground fields \( \mathbb{K} \) of V-arithmetic matrices (or graphs) of fixed minimality \( t > 0 \) and fixed degree \( N = [\mathbb{K} : \mathbb{Q}] \) is finite. Indeed, \( \mathbb{K} \subset \mathbb{Q}(\{a_{ij}\}) \). All numbers \( a_{ij} \) are algebraic integers from \( \mathbb{K} \); for each embedding \( \sigma : \mathbb{K} \rightarrow \mathbb{R} \) different from the geometric (identity) embedding \( \sigma^{(+)} : \mathbb{K} \rightarrow \mathbb{R} \), we have \( 0 \leq \sigma(a_{ij}^{2}) \leq 4 \); moreover, \( 0 \leq \sigma^{(+)}(a_{ij}^{2}) < t^{2} \). It follows that the set of possible algebraic integers \( a_{ij} \) is finite. Then the set of fields \( \mathbb{K} \) is also finite.

3. Ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 3 \)

This paper is a direct continuation of [12, 13], and we use the notation, definitions, and results of these papers.

In [13, Sections 3 and 4], we defined the explicit finite sets \( \mathcal{F}L^{4}, \mathcal{F}T, \mathcal{FT}_{1}^{(4)}(14) \), where \( 1 \leq i \leq 5 \), and \( \mathcal{FT}_{2,4}(14) \) of totally real algebraic number fields. The set \( \mathcal{F}L^{4} \) consists of all ground fields of arithmetic Lannér diagrams with at least 4 vertices and has three fields of degrees at most 2. The set \( \mathcal{F}T \) consists of all ground fields of arithmetic triangles (plane) and has thirteen fields of degrees at most 5 (it includes \( \mathcal{F}L^{4} \)). The set \( \mathcal{FT}_{1}^{(4)}(14) \), where \( 1 \leq i \leq 5 \), consists of all ground fields of V-arithmetic finite edge polyhedra of minimality 14 with connected Gram graph having 4 vertices. They are determined by five types of graphs \( \mathcal{F}T_{1}^{(4)}(14) \), where \( i = 1, 2, 3, 4, 5 \). The degrees of fields from these sets are bounded by 22, 39, 53, 56, and 54, respectively. The set \( \mathcal{FT}_{2,4}(14) \) consists of all ground fields of arithmetic quadrangles (plane) of minimality 14. Their degrees are bounded by 11.

Using methods of [7, 8], and results by Borel [3] (in dimension 2 only) and Takeuchi [17], in [13, Theorem 9], we obtained the following result.

**Theorem 3.1** [13]. In dimensions \( n \geq 6 \), the ground field of any arithmetic hyperbolic reflection group belongs to one of the finite sets of fields \( \mathcal{F}L^{4}, \mathcal{F}T, \mathcal{FT}_{1}^{(4)}(14) \), where \( 1 \leq i \leq 5 \), and \( \mathcal{FT}_{2,4}(14) \). In particular, its degree is bounded by 56.

In [12, Sections 2 and 3], we defined further explicit finite sets \( \mathcal{FT}_{1}^{(6)}(14), \mathcal{FT}_{2}^{(6)}(14), \mathcal{FT}_{3}^{(6)}(14), \mathcal{FT}_{1}^{(7)}(14), \mathcal{FT}_{2}^{(7)}(14), \mathcal{FT}_{2,5}(14) \) of totally real algebraic number fields. The sets \( \mathcal{FT}_{1}^{(6)}(14), \mathcal{FT}_{2}^{(6)}(14), \mathcal{FT}_{3}^{(6)}(14), \mathcal{FT}_{1}^{(7)}(14), \mathcal{FT}_{2}^{(7)}(14) \) consist of all ground fields of some V-arithmetic pentagon graphs \( \mathcal{G}_{1}^{(6)}(14), \mathcal{G}_{2}^{(6)}(14), \mathcal{G}_{3}^{(6)}(14), \mathcal{G}_{1}^{(7)}(14), \mathcal{G}_{2}^{(7)}(14) \), respectively, of minimality 14. They correspond to some fundamental pentagons of minimality 14 on a hyperbolic plane. The degrees of fields from these sets are bounded by 56, 75, 138, 38, and 138, respectively. The set \( \mathcal{FT}_{2,5}(14) \) consists of all ground fields of arithmetic pentagons (plane) of minimality 14. Their degrees are bounded by 12.

Using similar but much more complicated considerations, we proved in [12] the following result.
Theorem 3.2. In dimensions \( n \geq 4 \), the ground field of any arithmetic hyperbolic reflection group belongs to one of the finite sets of fields \( \mathcal{F} \mathcal{L}^4, \mathcal{F} \mathcal{T}, \mathcal{F} \Gamma_i^{(i)}(14) \), where \( 1 \leq i \leq 5 \), \( \mathcal{F} \Gamma_2,4(14), \mathcal{F} \Gamma_1^{(6)}(14), \mathcal{F} \Gamma_2^{(6)}(14), \mathcal{F} \Gamma_3^{(6)}(14), \mathcal{F} \Gamma_1^{(7)}(14), \mathcal{F} \Gamma_2^{(7)}(14), \) and \( \mathcal{F} \Gamma_2,5(14) \).

In particular, its degree is bounded by 138.

Applying the same methods, here we extend these results to \( n \geq 3 \), also considering \( n = 3 \).

First, we introduce some other explicit finite sets of fields. All of them are related to fundamental polygons on a hyperbolic plane.

Let us consider arithmetic reflection groups on a hyperbolic plane with fundamental polygons \( \mathcal{M}_2 \) of minimality 14. This means (see Section 2) that \( \delta_1 \cdot \delta_2 < 14 \) for all \( \delta_1, \delta_2 \in P(\mathcal{M}_2) \). The corresponding polygons \( \mathcal{M}_2 \) are also called arithmetic polygons of minimality 14.

Definition 3.3. We denote by \( \Gamma_2^{(14)} \) the set of Gram graphs \( \Gamma(P(\mathcal{M}_2)) \) of all arithmetic polygons \( \mathcal{M}_2 \) of minimality 14. The set \( \mathcal{F} \Gamma_2^{(14)} \) consists of all of their ground fields.

In [13, Subsection 4.5], we showed that the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimension 2 are bounded by 44. This follows from results by Long et al. [6], Borel [3] (in dimension 2 only), and Takeuchi [17]. Thus, the degrees of the fields from \( \mathcal{F} \Gamma_2^{(14)} \) are also bounded by 44.

Next, let us consider V-arithmetic 3-dimensional chambers \( \mathcal{M} \) that are defined by Gram graphs \( \Gamma_6^{(4)} \) with four vertices \( \delta_1, \delta_2, \delta_3, \) and \( e \) as shown in Figure 1. The chambers \( \mathcal{M} \) satisfy the following condition: the 2-dimensional face \( \mathcal{M}_e \) of \( \mathcal{M} \) that is perpendicular to \( e \) is an open fundamental triangle \( \mathcal{M}_2 \) bounded by three lines that are perpendicular to

\[
P(\mathcal{M}_2) = \{\delta_1, \delta_2, \tilde{\delta}_3\},
\]

where

\[
\tilde{\delta}_3 = \frac{\delta_3 + \cos(\pi/m)e}{\sin(\pi/m)}
\]

(see Figure 1). The open fundamental triangle \( \mathcal{M}_2 \) has one angle \( \pi/k \), for \( k \geq 3 \), defined by \( \delta_1 \) and \( \delta_2 \). All of its other sides do not intersect. The planes \( \mathcal{H}_\delta_1 \) and \( \mathcal{H}_\delta_2 \) are perpendicular to \( \mathcal{H}_e \), and the plane \( \mathcal{H}_\tilde{\delta}_3 \) has angle \( \pi/m \), for \( m \geq 3 \), with \( \mathcal{H}_e \).

Definition 3.4. We denote by \( \Gamma_6^{(4)}(14) \) the set of all V-arithmetic 3-dimensional graphs \( \Gamma_6^{(4)} \) (or the corresponding 3-dimensional V-arithmetic chambers) of minimality 14; see Figure 1. Thus, the inequalities \( 2 < a_{ij} < 14 \) are satisfied. We denote by \( \mathcal{F} \Gamma_6^{(4)}(14) \) the set of all of their ground fields.

Next, let us consider V-arithmetic 3-dimensional chambers \( \mathcal{M} \) that are defined by Gram graphs \( \Gamma_1^{(5)} \) with the five vertices \( \delta_1, \delta_2, \delta_3, \delta_4, \) and \( e \) as shown in Figure 2. The chambers \( \mathcal{M} \) satisfy the following condition: the 2-dimensional face \( \mathcal{M}_e \) of \( \mathcal{M} \) that is perpendicular to \( e \) is an open fundamental quadrangle \( \mathcal{M}_2 \) bounded by four lines that are perpendicular to

\[
P(\mathcal{M}_2) = \{\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3, \tilde{\delta}_4\},
\]

where

\[
\tilde{\delta}_1 = \frac{\delta_1 + \cos(\pi/m_1)e}{\sin(\pi/m_1)}, \quad \tilde{\delta}_3 = \frac{\delta_3 + \cos(\pi/m_3)e}{\sin(\pi/m_3)}
\]
(see Figure 2). The open fundamental quadrangle $M_2$ has two right angles defined by $\tilde{\delta}_1$ and $\delta_2$, and $\tilde{\delta}_3$ and $\delta_4$. All of its other sides do not intersect. The planes $H_{\delta_2}$ and $H_{\delta_4}$ are perpendicular to $H_e$, and the planes $H_{\delta_1}$ and $H_{\delta_3}$ have angles $\pi/m_1$ for $m_1 \geq 3$, and $\pi/m_3$ for $m_3 \geq 3$, with $H_e$, respectively.

**Definition 3.5.** We denote by $\Gamma^{(5)}_1(14)$ the set of all $V$-arithmetic 3-dimensional graphs $\Gamma^{(5)}_i$ (or the corresponding 3-dimensional $V$-arithmetic chambers) of minimality 14; see Figure 2. Thus, the inequalities $2 < a_{ij} < 14$ are satisfied. We denote by $\mathcal{F} \Gamma^{(5)}_1(14)$ the set of all their ground fields.

Now, let us consider $V$-arithmetic 3-dimensional chambers $M$ that are defined by Gram graphs $\Gamma^{(6)}_4$ with the six vertices $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$, and $e$, as shown in Figure 3. The chambers $M$ satisfy the following condition: the 2-dimensional face $M_e$ of $M$ that is perpendicular to $e$ is an open fundamental pentagon $M_2$ bounded by five lines that are perpendicular to $P(M_2) = \{\delta_1, \delta_2, \delta_3, \tilde{\delta}_4, \delta_5\}$, where

$$\tilde{\delta}_4 = \frac{\delta_4 + \cos(\pi/m)e}{\sin(\pi/m)}$$

(see Figure 3). It has four consecutive right angles defined by $\delta_2, \delta_3, \tilde{\delta}_4$, $\delta_5$, and $\delta_1$. Its two consecutive sides that are perpendicular to $\delta_1$ and $\delta_2$ do not intersect. All of the planes $H_{\delta_1}$ and $H_{\delta_2}$ are perpendicular to $H_e$, except the plane $H_{\delta_4}$, which has angle $\pi/m$ with $H_e$.

**Definition 3.6.** We denote by $\Gamma^{(6)}_4(14)$ the set of all $V$-arithmetic 3-dimensional graphs $\Gamma^{(6)}_4$ (or the corresponding 3-dimensional $V$-arithmetic chambers) of minimality 14; see Figure 3. Thus, the inequalities $2 < a_{ij} < 14$ are satisfied. We denote by $\mathcal{F} \Gamma^{(6)}_4(14)$ the set of all of their ground fields.

We have the following result.

**Theorem 3.7.** The sets of $V$-arithmetic graphs $\Gamma^{(4)}_6(14), \Gamma^{(5)}_1(14)$, and $\Gamma^{(6)}_4(14)$ are finite.

(i) The degrees of the fields from $\mathcal{F} \Gamma^{(4)}_6(14)$ are bounded by 56.

(ii) The degrees of the fields from $\mathcal{F} \Gamma^{(5)}_1(14)$ are bounded by 909.

(iii) The degrees of the fields from $\mathcal{F} \Gamma^{(6)}_4(14)$ are bounded by 99.

The proof of Theorem 3.7 requires long considerations and calculations. It will be given separately in Section 4.

We have the following main result of the paper.

**Theorem 3.8.** In dimensions $n \geq 3$, the ground field of any arithmetic hyperbolic reflection group belongs to one of the finite sets of fields $\mathcal{F} L^4, \mathcal{F} T, \mathcal{F} \Gamma^{(4)}_1(14), \mathcal{F} \Gamma^{(4)}_2(14)$ (fields in dimensions at least 6), and $\mathcal{F} \Gamma^{(6)}_1(14), \mathcal{F} \Gamma^{(6)}_2(14), \mathcal{F} \Gamma^{(6)}_3(14), \mathcal{F} \Gamma^{(7)}_1(14), \mathcal{F} \Gamma^{(7)}_2(14), \mathcal{F} \Gamma^{(7)}_3(14), \mathcal{F} \Gamma_4(14), \mathcal{F} \Gamma_5(14), \mathcal{F} \Gamma_6(14), \mathcal{F} \Gamma_7(14)$ (additional fields in dimension 3).

In particular, its degree is bounded by 909.
Proof. By [12], if \( n \geq 4 \), then the ground field \( \mathbb{K} \) belongs to one of the sets \( \mathcal{F}L^4, \mathcal{F}T, \mathcal{F}T_i(4)(14) \), where \( 1 \leq i \leq 5 \), \( \mathcal{F}T_2,4(14), \mathcal{F}T_1(6)(14), \mathcal{F}T_2(6)(14), \mathcal{F}T_3(6)(14), \mathcal{F}T_1(7)(14), \mathcal{F}T_2(7)(14), \) or \( \mathcal{F}T_2,5(14) \). Thus, we can further assume that the ground field \( \mathbb{K} \) does not belong to these sets, and that the dimension is \( n = 3 \).

Let \( W \) be an arithmetic hyperbolic reflection group in dimension \( n = 3 \), and let \( M \) be its fundamental chamber. By [7, Section 3 and Appendix], there exists \( e \in P(M) \) that defines a narrow face \( M_e \) of \( M \) of minimality 14. It means that \( \delta_1 \cdot \delta_2 < 14 \) for all \( \delta_1, \delta_2 \in P(M, e) \subseteq P(M) \), where

\[
P(M, e) = \{ \delta \in P(M) \mid \mathcal{H}_\delta \cap \mathcal{H}_e \neq \emptyset \}
\]
corresponds to all faces neighbouring \( M_e \).

Let us assume that \( e \cdot \delta = 0 \) for any \( \delta \in P(M, e) - \{ e \} \), or, equivalently, all neighbouring 2-dimensional faces of \( M \) to the polygon \( M_e \) are perpendicular to \( M_e \). Then \( M_e \) satisfies \( P(M_e) = P(M, e) - \{ e \} \), and \( M_e \) is a fundamental polygon for an arithmetic hyperbolic plane reflection group with the same ground field \( \mathbb{K} \) as for \( W \) (see the remarks at the end of Section 2). Since \( P(M_e) \subset P(M, e) \), it has minimality 14. Then \( \mathbb{K} \in \mathcal{F}T_2(14) \) as required.

Thus, we further assume that \( e \cdot \delta = 2 \cos(\pi/m) > 0 \) for one of \( \delta \in P(M, e) - \{ e \} \); equivalently, one neighbouring 2-dimensional face \( M_\delta \) of \( M \) to the polygon \( M_e \) is not perpendicular to \( M_e \).

All cases when the polygon \( M_e \) has less than six sides were considered in [12, 13]. It was shown that the ground field \( \mathbb{K} \) then belongs to one of the sets of fields \( \mathcal{F}L^4, \mathcal{F}T, \mathcal{F}T_i(4)(14) \), where \( 1 \leq i \leq 5 \), \( \mathcal{F}T_2,4(14), \mathcal{F}T_1(6)(14), \mathcal{F}T_2(6)(14), \mathcal{F}T_3(6)(14), \mathcal{F}T_1(7)(14), \mathcal{F}T_2(7)(14), \) or \( \mathcal{F}T_2,5(14) \). Thus, we further additionally assume that \( M_e \) has at least six sides, and the ground field \( \mathbb{K} \) does not belong to any of these sets of fields.

By [13, Lemma 7], if \( \mathbb{K} \) does not belong to \( \mathcal{F}L^4, \mathcal{F}T, \) or \( \mathcal{F}T_i(4)(14) \), where \( 1 \leq i \leq 4 \), then the Coxeter graph \( C(v) \) of any vertex \( v \in M_e \) has all connected components having only one or two vertices. If, additionally, \( \mathbb{K} \) does not belong to \( \mathcal{F}T_5(4)(14) \), then the hyperbolic connected component of the edge graph \( \Gamma(v) \) defined by any edge \( v = v_1v_2 \subset M_e \) has less than four vertices. Further, we mark these facts as (*)

By (*), both angles of \( M_e \) at the edge \( M_{e,0} \) perpendicular to \( \delta \) are right angles. Moreover, if \( f_1, f_2 \in P(M, e) \) define two neighbouring edges of the edge \( M_{e,0} \) of \( M_e \), then \( e \cdot f_1 = e \cdot f_2 = \delta \cdot f_1 = \delta \cdot f_2 = 0 \).

Assume that the polygon \( M_e \) has a non-right angle with edges perpendicular to \( \delta_1, \delta_2 \in P(M, e) \). By (*), then \( \delta_1 \cdot \delta_2 = 2 \cos(\pi/k) > 0 \), and \( \{ \delta_1, \delta_2 \} \) are perpendicular to \( \{ e, \delta \} \). Then \( \delta_1, \delta_2, \delta_3 = \delta \) and \( e \) have Gram graph \( \Gamma_6(4)(14) \), and the ground field \( \mathbb{K} \) belongs to \( \mathcal{F}T_6(4)(14) \), as required.

Now assume that all of the angles of the polygon \( M_e \) are right angles and that there exist two elements \( \delta_1, \delta_3 \in P(M, e) - \{ e \} \) such that \( e \cdot \delta_1 = 2 \cos(\pi/m_1) > 0 \) and \( e \cdot \delta_3 = 2 \cos(\pi/m_3) > 0 \). By (*), then \( \delta_1 \cdot \delta_3 > 0 \), and \( \delta_1 \) and \( \delta_3 \) are perpendicular to two non-consecutive sides of \( M_e \). Since \( M_e \) has more than five vertices, we can find two of their neighbouring edges that are perpendicular to \( \delta_2, \delta_4 \in P(M, e) \) such that the Gram graph of \( \delta_1, \delta_2, \delta_3, \delta_4 \), and \( e \) is \( \Gamma_4^{(5)} \). Then the ground field belongs to \( \mathcal{F}T_4^{(5)}(14) \), as required.

Now assume that all of the angles of the polygon \( M_e \) are right angles and that there exists only one element \( \delta_2 \in P(M, e) - \{ e \} \) such that \( \delta_2 \cdot e \neq 0 \). Then \( \delta_1 \cdot e = 2 \cos(\pi/m) > 0 \). Since \( M_e \) has at least six sides, we can find five consecutive sides of \( M_e \) that are perpendicular to \( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \in P(M, e) \) such that the Gram graph of \( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \), and \( e \) is \( \Gamma_4^{(6)} \). Then the ground field \( \mathbb{K} \) belongs to \( \mathcal{F}T_4^{(6)}(14) \), as required.

This completes the proof of the theorem. \( \square \)
Here we prove Theorem 3.7.

4.1. Some general results

We use the following general results from [8].

**Theorem 4.1** [8, Theorem 1.2.1]. Let \( \mathbb{F} \) be a totally real algebraic number field, and let each embedding \( \sigma : \mathbb{F} \to \mathbb{R} \) correspond to an interval \( [a_{\sigma}, b_{\sigma}] \) in \( \mathbb{R} \), where
\[
\prod_{\sigma} b_{\sigma} - a_{\sigma} < 1.
\]
In addition, let the natural number \( m \) and the intervals \( [s_1, t_1], \ldots, [s_m, t_m] \) in \( \mathbb{R} \) be fixed. Then there exists a constant \( N(s_i, t_i) \) such that, if \( \alpha \) is a totally real algebraic integer and if the following inequalities hold for the embeddings \( \tau : \mathbb{F}(\alpha) \to \mathbb{R} \):
\[
\begin{align*}
s_i &\leq \tau(\alpha) \leq t_i \quad \text{for } \tau = \tau_1, \ldots, \tau_m, \\
a_{\tau|\mathbb{F}} &\leq \tau(\alpha) \leq b_{\tau|\mathbb{F}} \quad \text{for } \tau \neq \tau_1, \ldots, \tau_m,
\end{align*}
\]
then
\[
[\mathbb{F}(\alpha) : \mathbb{F}] \leq N(s_i, t_i).
\]

**Theorem 4.2** [8, Theorem 1.2.2]. Under the conditions of Theorem 4.1, the constant \( N(s_i, t_i) \) can be taken to be \( N(s_i, t_i) = N_0 \), where \( N_0 \) is the least natural number solution of the inequality
\[
N_0 M \ln \frac{1}{R} - M \ln(N_0 + 1) - \ln B \geq \ln S.
\]
Here
\[
M = [\mathbb{F} : \mathbb{Q}], \quad B = 2\sqrt{\text{discr} \mathbb{F}},
\]
\[
R = \sqrt{\prod_{\sigma} b_{\sigma} - a_{\sigma}}, \quad S = \prod_{i=1}^{m} \frac{2r_i}{b_{\sigma_i} - a_{\sigma_i}},
\]
where
\[
\sigma_i = \tau_i|\mathbb{F}, \quad r_i = \max\{|t_i - a_{\sigma_i}|, |b_{\sigma_i} - s_i|\}.
\]

We note that the proofs of Theorems 4.1 and 4.2 use a variant of Fekete’s theorem (1923) about the existence of non-zero integer polynomials of bounded degree that differ only slightly from zero on appropriate intervals; see [8, Theorem 1.1.1].

Below, we apply these results to two cases that are very similar to those used in [8; 13, Subsection 5.5]. Cases 1 and 2 below are natural generalizations of the cases (1) and (2) that were considered in [12, Subsection 3.1].

**Case 1.** For a natural \( l \geq 3 \), we define \( \mathbb{F}_l = \mathbb{Q}(\cos(2\pi/l)) \). We consider a totally real algebraic number field \( \mathbb{K} \), where \( \mathbb{F}_l \subset \mathbb{K} = \mathbb{Q}(\alpha) \), and the algebraic integer \( \alpha \) satisfies
\[
-a_1 \sigma \left( \sin^2 \left( \frac{\pi}{l} \right) \right) < \sigma(\alpha) < a_2 \sigma \left( \sin^2 \left( \frac{\pi}{l} \right) \right)
\]
for all \( \sigma : \mathbb{K} \to \mathbb{R} \) such that \( \sigma \neq \sigma^{(+)} \), and
\[
b_1 < \sigma^{(+)}(\alpha) < b_2,
\]
where $\sigma(+) : \mathbb{K} \to \mathbb{R}$ is the identity. We assume that $a_1 \geq 0$, $a_2 \geq 0$, and $0 < a = \max\{a_1, a_2\} < 4$. We assume that $b_1 < b_2$ and define $b = \max\{|b_1|, |b_2|\}$. We also assume that $a \leq b$. We want to estimate $[\mathbb{K} : F_{\ell}] = N_0$ and $N = [\mathbb{K} : \mathbb{Q}] = N_0 \cdot [F_{\ell} : \mathbb{Q}]$ from above.

For $l \geq 3$, we have $[F_{\ell} : \mathbb{Q}] = \varphi(l)/2$, where $\varphi(l)$ is the Euler function, and $N_{F_{\ell}/\mathbb{Q}}(\sin^2(\pi/l)) = \gamma(l)/4^{\varphi(l)/2}$, where

$$N_{F_{\ell}/\mathbb{Q}}\left(4\sin^2\left(\frac{\pi}{l}\right)\right) = \gamma(l) = \begin{cases} p & \text{if } l = p^t > 2, \text{ where } p \text{ is prime}, \\ 1 & \text{otherwise}. \end{cases}$$

We have

$$\frac{b^a N_{\mathbb{K}/\mathbb{Q}}(\sin^2(\pi/l))}{a \sin^2(\pi/l)} > |N_{\mathbb{K}/\mathbb{Q}}(\alpha)| \geq 1$$

and

$$\frac{b(a/4)^N \gamma(l) 2^N \varphi(l)}{a \sin^2(\pi/l)} > 1.$$  

Equivalently, we have

$$N \left(\ln \frac{2}{\sqrt{a}} - \ln \frac{\gamma(l)}{\varphi(l)}\right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{l} \quad \text{and} \quad \frac{\varphi(l)}{2} \bigg| N. \quad (11)$$

Since $\gamma(l) \leq l$ and $\varphi(l) \geq 6l/\ln(l)$ for $l \geq 6$, where $C = \varphi(6) \ln(\ln 6)/6 \geq 0.194399$ and $\sin(\pi/l) \leq \pi/l$ for $l \geq 3$, there exist only a finite number of $l \geq 3$ such that (11) has solutions $N \in \mathbb{N}$.

More exactly, there exist only a finite number of exceptional $l \geq 3$ such that

$$\ln \frac{2}{\sqrt{a}} - \ln \frac{\gamma(l)}{\varphi(l)} \leq 0. \quad (12)$$

All non-exceptional $l$ satisfy the inequality

$$\frac{\varphi(l)}{2} \left(\ln \frac{2}{\sqrt{a}} - \ln \frac{\gamma(l)}{\varphi(l)}\right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{l}. \quad (13)$$

Note that exceptional $l$ also satisfy this inequality.

If $\gamma(l) = 1$, then (13) implies that $l$ satisfies the inequality

$$\frac{C}{2} \ln \left(\frac{2}{\sqrt{a}}\right) l < \left(\ln l + \ln \left(\frac{\sqrt{b/a}}{\pi}\right)\right) \ln \ln l. \quad (14)$$

It follows that

$$l < L_0, \quad (15)$$

where $L_0 > 3$ satisfies

$$\frac{C}{2} \ln \left(\frac{2}{\sqrt{a}}\right) L_0 \geq \left(\ln L_0 + \ln \left(\frac{\sqrt{b/a}}{\pi}\right)\right) \ln \ln L_0. \quad (16)$$

If $l = p^t$, where $p$ is prime, then (13) implies that $l$ satisfies the inequality

$$\frac{C}{2} \Delta(a) l < \left(\ln l + \ln \left(\frac{\sqrt{b/a}}{\pi}\right)\right) \ln \ln l, \quad (17)$$

where

$$\Delta(a) = \min_{l = p^t \geq L_0} \left\{\ln \frac{2}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} > 0\right\}. \quad (18)$$

It follows that

$$l < L_1, \quad (19)$$
where \( L_1 \geq L_0 \) is a solution of the inequality
\[
\frac{C}{2} \Delta(a) L_1 \geq \left( \ln L_1 + \ln \left( \frac{\sqrt{b/a}}{\pi} \right) \right) \ln \ln L_1. \tag{20}
\]
Thus, to find all non-exceptional \( l \) satisfying (13), we should check (13) for all \( l \) such that \( 3 \leq l < L_1 \); moreover, if \( L_0 \leq l < L_1 \), then we can assume that \( l = p^j \). Their number is finite, and all of them can be effectively found.

For non-exceptional \( l \) satisfying (13), we obtain the bounds
\[
N_0 = [K : F_l] \leq \left[ \frac{\ln \sqrt{b/a} - \ln \sin(\pi/l)}{(\varphi(l)/2)(\ln(2/\sqrt{a}) - (\ln \gamma(l))/\varphi(l))} \right], \tag{21}
\]
and
\[
N = [K : Q] \leq \left[ \frac{\ln \sqrt{b/a} - \ln \sin(\pi/l)}{(\varphi(l)/2)(\ln(2/\sqrt{a}) - (\ln \gamma(l))/\varphi(l))} \right] \cdot \frac{\varphi(l)}{2}. \tag{22}
\]
We call this use of the norm Method B (as in [13, Subsection 5.5]).

On the other hand, for fixed \( l \), we obtain a bound for \( N_0 \) using Theorems 4.1 and 4.2 applied to \( F = F_l \) and \( \alpha \). We can take
\[
R = \sqrt{N_{F_l/Q} \left( \sin^2 \left( \frac{\pi}{7} \right) \right) \left( \frac{a_1 + a_2}{4} \right)^{\varphi(l)/2}} = \left( \frac{(\gamma(l)^{1/\varphi(l)}(a_1 + a_2)^{1/2})}{4} \right)^{\varphi(l)/2}, \tag{23}
\]
where
\[
R < 1 \quad \text{if and only if} \quad \ln \frac{4}{\sqrt{a_1 + a_2}} - \frac{\ln \gamma(l)}{\varphi(l)} > 0, \tag{24}
\]
\[
M = [F_l : Q] = \frac{\varphi(l)}{2}, \quad B = 2\sqrt{|\text{disc } F_l|} \tag{25}
\]
where the discriminant \( |\text{disc } F_l| \) is given in (A.4), and
\[
S = \frac{2e \max \{a_2, b_2, a_2 - b_1, a_1, -b_1, b_2 + a_1\}}{(a_1 + a_2) \sin^2(\pi/l)} \tag{26}
\]
Then \([K : F_l] \leq n_0 \) and \([K : Q] \leq n_0 \varphi(l)/2\), where \( n_0 \) is the least natural solution of the inequality (4):
\[
n_0 M \ln \frac{1}{R} - M \ln(n_0 + 1) - \ln B \geq \ln S. \tag{27}
\]
In particular, this gives a bound for \([K : Q]\) for exceptional \( l \) satisfying (24) and improves the bound (21) for \( N_0 \) when it is poor, which also improves the bound for \([K : Q]\). We call this use of Theorems 4.1 and 4.2 Method A (as in [13, Subsection 5.5]).

We shall apply Methods A and B to \( \Gamma^{(6)}_1(14) \) in Subsection 4.4.

Case 2. For natural \( k \geq s \geq 3 \), we define \( F_{k,s} = Q(\cos(2\pi/k), \cos(2\pi/s)) \). We consider a totally real algebraic number field \( K \), where \( F_{k,s} \subset K = Q(\alpha) \), and the algebraic integer \( \alpha \) satisfies
\[
- a_1 \alpha \left( \sin^2 \left( \frac{\pi}{k} \right) \sin^2 \left( \frac{\pi}{s} \right) < \alpha \left( \sin^2 \left( \frac{\pi}{k} \right) \sin^2 \left( \frac{\pi}{s} \right) \right) \right) < a_2 \alpha \left( \sin^2 \left( \frac{\pi}{k} \right) \sin^2 \left( \frac{\pi}{s} \right) \right) \tag{28}
\]
for all \( \sigma : K \to \mathbb{R} \) such that \( \sigma \neq \sigma^{(+)} \), and
\[
b_1 < \sigma^{(+)}(\alpha) < b_2, \tag{29}
\]
where \( \sigma^{(+)} : K \to \mathbb{R} \) is the identity. We assume that \( a_1 \geq 0, a_2 \geq 0 \), and \( 0 < a = \max \{a_1, a_2\} \leq 16 \). We assume that \( b_1 < b_2 \) and define \( b = \max \{b_1, b_2\} \). Also, we assume that \( a \leq b \). We want
to estimate \([K : \mathbb{F}_{k,s}] = N_0\) and \(N = [K : \mathbb{Q}] = N_0[\mathbb{F}_{k,s} : \mathbb{Q}]\) for non-exceptional \(k\) and \(s\), where \(l \geq 3\) is called exceptional if
\[
\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} \leq 0.
\] (30)

We also assume that \(k \geq s \geq s_0 \geq 3\), where \(s_0 \geq 3\) is fixed.

We have \([\mathbb{F}_{k,s} : \mathbb{Q}] = \varphi([k, s])/2\rho(k, s)\), where \(\rho(k, s) = 1\) or \(2\) is given in (A.5), and \(N_{\mathbb{F}_{k,s}/\mathbb{Q}}(\sin^2(\pi/l)) = \gamma(l)/4\varphi(l)^2\), where \(\gamma(l)\) is given in (10). We have
\[
\frac{b a^N}{\sin^2(\pi/k)\sin^2(\pi/s)} |N_{\mathbb{F}_{k,s}/\mathbb{Q}}(\sin^2(\pi/k))| |N_{\mathbb{F}_{k,s}/\mathbb{Q}}(\sin^2(\pi/s))| > |N_{\mathbb{F}_{k,s}/\mathbb{Q}}(\alpha)| \geq 1
\]
and
\[
\frac{b(a/16)^N\gamma(k)^{2\varphi(k)/\varphi(s)}\gamma(s)^{2N/\varphi(s)}}{a\sin^2(\pi/k)\sin^2(\pi/s)} > 1.
\]

Equivalently, we obtain
\[
N \left(\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)}\right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s} \quad \text{and} \quad \frac{\varphi([k, s])}{2\rho(k, s)} \geq N.
\] (31)

Since \(\gamma(l) \leq l\) and \(\varphi(l) \geq C l/\ln(\ln l)\) for \(l \geq 6\), where \(C = \varphi(6)\ln(\ln 6)/6\) and \(\sin(\pi/l) \leq \pi/l\) for \(l \geq 3\), there exist only a finite number of pairs \((k, s)\) such that (31) has solutions \(N \in \mathbb{N}\) where both \(k\) and \(s\) are non-exceptional.

More exactly, there exist only a finite number of exceptional pairs \((k, s)\), where a pair \((k, s)\) (consisting of non-exceptional \(k\) and \(s\)) is called exceptional if
\[
\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \leq 0.
\] (32)

All non-exceptional pairs \((k, s)\) satisfying (31) satisfy the inequality
\[
\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)}\right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}.
\] (33)

Note that exceptional pairs \((k, s)\) also satisfy this inequality.

If \(\gamma(k) = \gamma(s) = 1\) and \(k \geq s\), then (33) implies that
\[
\frac{C}{2} \ln \left(\frac{4}{\sqrt{a}}\right) k < \left(2 \ln k + \ln \left(\frac{\sqrt{b/a}}{\pi^2}\right)\right) \ln \ln k.
\] (34)

It follows that
\[
s_0 \leq s \leq k < K_0,
\] (35)

where \(K_0 > 3\) satisfies
\[
\frac{C}{2} \ln \left(\frac{4}{\sqrt{a}}\right) K_0 \geq \left(2 \ln K_0 + \ln \left(\frac{\sqrt{b/a}}{\pi^2}\right)\right) \ln \ln K_0.
\] (36)

If one of \(\gamma(k)\) and \(\gamma(s)\) is not equal to 1, then (33) implies, for non-exceptional pairs \((k, s)\), that
\[
\frac{C}{2} \Delta_1(a) k < \left(2 \ln k + \ln \left(\frac{\sqrt{b/a}}{\pi^2}\right)\right) \ln \ln k,
\] (37)

where
\[
\Delta_1(a) = \min_{k \geq s \geq s_0, k \geq K_0} \left\{ \ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(s)}{\varphi(s)} - \frac{\ln \gamma(k)}{\varphi(k)} > 0 \right\}.
\] (38)
It follows that

\[ s_0 \leq s \leq k < K_1, \quad (39) \]

where \( K_1 \geq K_0 \) is a solution of the inequality

\[ \frac{C}{2} \Delta_1(a) K_1 \geq \left( 2 \ln K_1 + \ln \left( \frac{\sqrt{b/a}}{\pi^2} \right) \right) \ln \ln K_1. \quad (40) \]

Thus, to find all non-exceptional pairs \((k, s)\) satisfying (33), we should check (33) for all \( s_0 \leq s \leq k < K_1 \); moreover, if \( K_0 \leq k \leq K_1 \), then we can assume that one of \( k \) and \( s \) is equal to \( p^e \), where \( p \) is prime. The number of such pairs is finite, and all of them can be effectively found.

For such non-exceptional pairs \((k, s)\) satisfying (33), we obtain the bounds

\[ N_0 = [K : \mathbb{F}_{k,s}] \leq \left[ \frac{\ln \sqrt{b/a} - \ln \sin(\pi/k) - \ln \sin(\pi/s)}{\varphi([k, s])/2\rho(k, s) \cdot \left( \ln \left( 4/\sqrt{a} \right) - \ln \gamma(k)/\varphi(k) - \ln \gamma(s)/\varphi(s) \right)} \right] \quad (41) \]

and

\[ N = [K : \mathbb{Q}] \leq \left[ \frac{\ln \sqrt{b/a} - \ln \sin(\pi/k) - \ln \sin(\pi/s)}{\varphi([k, s])/2\rho(k, s) \cdot \left( \ln \left( 4/\sqrt{a} \right) - \ln \gamma(k)/\varphi(k) - \ln \gamma(s)/\varphi(s) \right)} \right] \frac{\varphi([k, s])}{2\rho(k, s)}. \quad (42) \]

We call this use of the norm Method \( B \) (as in [13, Subsection 5.5]).

On the other hand, for a fixed pair \((k, s)\), we can obtain a bound for \( N_0 \) using Theorems 4.1 and 4.2 applied to \( F = \mathbb{F}_{k,s} \) and \( \alpha \). We can take

\[ R = \sqrt{|N_{\mathbb{F}_{k,s}/\mathbb{Q}} \left( \sin^2 \frac{\pi}{K} \sin^2 \frac{\pi}{s} \right) \left( \frac{a_1 + a_2}{4} \right)^{\varphi([k, s])/2\rho(k, s)}} \]

\[ = \left( \frac{\gamma(k)^{1/\varphi(k)} \gamma(s)^{1/\varphi(s)} (a_1 + a_2)^{1/2}}{8} \right)^{\varphi([k, s])/2\rho(k, s)}, \quad (43) \]

where

\[ R < 1 \quad \text{if and only if} \quad \ln \frac{8}{\sqrt{a_1 + a_2}} < \ln \frac{\gamma(k)}{\varphi(k)} - \ln \frac{\gamma(s)}{\varphi(s)} > 0, \quad (44) \]

\[ M = [\mathbb{F}_{k,s} : \mathbb{Q}] = \frac{\varphi([k, s])}{2\rho(k, s)}, \quad B = 2\sqrt{\text{disc} \mathbb{F}_{k,s}}, \quad (45) \]

where the discriminant \(|\text{disc} \mathbb{F}_{k,s}|\) is given in (A.7) and (A.8), and

\[ S = \frac{2e \max \{a_2, b_2, a_2 - b_1, a_1, -b_1, b_2 + a_1\}}{(a_1 + a_2) \sin^2 (\pi/s) \sin^2 (\pi/k)}. \quad (46) \]

For all pairs \((k, s)\) satisfying (44), we obtain the bounds \([K : \mathbb{F}_{k,s}] \leq n_0 \) and \([K : \mathbb{Q}] \leq n_0 \varphi([k, s])/(2\rho(k, s))\), where \( n_0 \) is the least natural solution of the inequality (4):

\[ n_0 M \ln \frac{1}{R} - M \ln(n_0 + 1) - \ln B \geq \ln S. \]

For \( a < 16 \) and \( k, s \geq 3 \), all pairs \((k, s)\), except for a finite number, satisfy (44), and we can apply this method to all of these pairs. In particular, this gives a bound for \([K : \mathbb{Q}]\) for all exceptional pairs \((k, s)\) satisfying (44), and it improves the bound (41) for \( N_0 \) when it is poor, which also improves the bound for \([K : \mathbb{Q}]\). We call this use of Theorems 4.1 and 4.2 Method \( A \) (as in [13, Subsection 5.5]).

We apply Methods \( A \) and \( B \) to \( \Gamma_6^{(4)}(14) \) in Subsection 4.2 and to \( \Gamma_1^{(5)}(14) \) in Subsection 4.3.
4.2. V-arithmetic 3-graphs $\Gamma^{(4)}_6(14)$ and their fields

Here we consider V-arithmetic 3-dimensional graphs $\Gamma^{(4)}_6(14)$ (see Figure 1) and their fields.

This case was considered in [12, Subsection 3.2], where a bound for the degrees of fields $K$ from $\mathcal{F}T^{(6)}_1(14)$ was obtained. To get this bound, in [12, Subsection 3.2] a bound for the degrees of fields defined by a subgraph $\Gamma^{(4)}_6(14)$ of this graph was obtained. This uses Methods A and B of Case 2 in Subsection 4.1 applied to $a_1 = 0$, $a_2 = 4$, $b_1 = 12$, and $b_2 = 28^2$. The bound is $[K : \mathbb{Q}] \leq 56$.

4.3. V-arithmetic 3-graphs $\Gamma^{(5)}_1(14)$ and their fields

Here we consider V-arithmetic 3-dimensional graphs $\Gamma^{(5)}_1(14)$ (see Figure 2) and their fields.

First, let us consider the corresponding plane graph defined by $\beta_1 = \delta_1$, $\beta_2 = \delta_2$, $\beta_3 = \delta_3$, and $\beta_4 = \delta_4$ that give $P(M_2)$. We denote $b_{ij} = \beta_i \cdot \beta_j$ when it is not 0. This graph is given in Figure 4.

Any three elements from $\beta_1, \ldots, \beta_4$ generate the form defining the hyperbolic plane. Thus the determinant of their Gram matrix must be positive for the geometric embedding $\sigma^{(+)}$ and must be negative for $\sigma \neq \sigma^{(+)}$. For example, for $\beta_1, \beta_2$, and $\beta_3$ it is equal to $-8 + 2b_{13}^2 + 2b_{23}^2$. Thus, for $\sigma$ we obtain the inequalities $b_{13}^2 + b_{23}^2 < 4$. Moreover, the determinant

$$16 + b_{13}^2 b_{24}^2 + b_{14}^2 b_{23}^2 - 4b_{13}^2 - 4b_{14}^2 - 4b_{23}^2 - 4b_{24}^2 - 2b_{13} b_{14} b_{23} b_{24}$$

of the Gram matrix of all four elements $\beta_1, \ldots, \beta_4$ is 0. Combining all of these conditions, we obtain the following conditions on $M_2$ for $\sigma \neq \sigma^{(+)}$:

$$b_{13} b_{14} b_{23} b_{24} = 8 - 2b_{13}^2 - 2b_{14}^2 - 2b_{23}^2 - 2b_{24}^2 + \frac{(b_{13}^2 b_{24}^2 + b_{14}^2 b_{23}^2)}{2}. 
\begin{align*}
&b_{13}^2 + b_{23}^2 < 4, \\
&b_{23}^2 + b_{24}^2 < 4, \\
&b_{24}^2 + b_{14}^2 < 4, \\
&b_{14}^2 + b_{13}^2 < 4.
\end{align*}
(47)

One can find the minimum and maximum of $b_{13} b_{14} b_{23} b_{24}$ under the closure of these conditions, which shows that

$$-4 < \sigma(b_{13} b_{14} b_{23} b_{24}) \leq 1. 
(48)$$

Here, the minimum is achieved for $b_{ij} = \pm \sqrt{2}$ where the number of $(-)$ is odd, and the maximum is achieved for $b_{ij} = \pm 1$ where the number of $(-)$ is even. From expressions for $\beta_i$ using $\delta_i$ and $e$, we obtain

$$b_{13} b_{14} b_{23} b_{24} = \frac{a_{13} a_{14} a_{23} a_{24} + 2 \cos(\pi/m_1) \cos(\pi/m_3) a_{14} a_{23} a_{24}}{\sin^2(\pi/m_1) \sin^2(\pi/m_3)}. 
(49)$$

We consider the algebraic integer $\alpha \in K$, which is

$$\alpha = 2a_{13} a_{14} a_{23} a_{24} + 4 \cos \frac{\pi}{m_1} \cos \frac{\pi}{m_3} a_{14} a_{23} a_{24}. 
(50)$$

From (48) and (49), we obtain

$$-8\sigma \left( \sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_3} \right) < \sigma(\alpha) \leq 2\sigma \left( \sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_3} \right). 
(51)$$

For the geometric embedding $\sigma^{(+)}$, we have

$$2 \cdot 2^4 + 2^3 = 5 \cdot 2^3 < \sigma^{(+)}(\alpha) < 2 \cdot 14^4 + 4 \cdot 14^3 = 32 \cdot 14^3. 
(52)$$
We can apply this method to any pair $(a, b)$ (Case 2) of pairs $(s_0, K)$ that satisfy (41), which is:

$$\frac{\ln \sqrt{b} - \ln \gamma(l)}{\varphi(l)} \leq 0.$$  

(53)

It follows that $l = 3, 4, 5$ are the only exceptional values.

All exceptional pairs $(k, s)$, where $k \geq s \geq 6$, that is, when (32), namely

$$\frac{\ln \sqrt{2} - \ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \leq 0,$$

(54)

is satisfied, are as follows: $(k, s = 7)$ where either $7 \leq k \leq 241$ is prime, or $k = 8, 9, 16, 25, 27, 32, 49$; $(k, s = 8)$ where $k = 8, 9, 11, 13, 17$; $(k, s = 9)$ where $k = 9, 11, 13, 17, 19$; $(k, s = 11)$ where $11 \leq k \leq 31$ is prime; $(k, s = 13)$ where $13 \leq k \leq 23$ is prime; and $(k = 17, s = 17)$.

We can take $K_0 = 911$ in (36). Then (here we take $s_0 = 6$)

$$\Delta_1(s) = \ln \sqrt{2} - \frac{\ln 7}{6} - \frac{\ln 911}{910} \geq 0.0147667,$$

and $K_1 = 38563$ can be taken in (40). Checking (33) for $6 \leq s \leq k \leq 38563$, we obtain that $6 \leq s \leq 330$ and $6 \leq s \leq k \leq 5460$. Moreover, $s \leq k \leq 330$ for $20 \leq s \leq 330$. For all of these pairs $(k, s)$ satisfying (33), which is

$$\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \sqrt{2} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)}\right) < \ln \sqrt{4 \cdot 14^3} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s},$$

(55)

we obtain

$$[F_{k,s} : \mathbb{Q}] = \frac{\varphi([k, s])}{2\rho(k, s)} \leq 909,$$

(56)

where 909 is achieved for $(k, s) = (607, 7)$. Moreover, for all of these non-exceptional pairs $(k, s)$ we obtain the bound (41), which is

$$N_0 = [K : \mathbb{Q}] \leq \left[ \frac{\ln \sqrt{4 \cdot 14^3} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\varphi([k, s])/2\rho(k, s) \cdot (\ln \sqrt{2} - \ln \gamma(k)/\varphi(k) - \ln \gamma(s)/\varphi(s))} \right],$$

(57)

and finally we obtain the bound (42), which is

$$N = [K : \mathbb{Q}] \leq \left[ \frac{\ln \sqrt{4 \cdot 14^3} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\varphi([k, s])/2\rho(k, s) \cdot (\ln \sqrt{2} - \ln \gamma(k)/\varphi(k) - \ln \gamma(s)/\varphi(s))} \right] \cdot \frac{\varphi([k, s])}{2\rho(k, s)}.$$

(58)

If a pair $(k, s)$ is exceptional or if the right-hand side of (58) is more than 909 (these are possible only for pairs $(k, s)$ with $6 \leq s \leq 17$ and $s \leq k \leq 421$), then we also apply Method A of Case 2 to the pair $(k, s)$. This improves the poor bound (57) for the non-exceptional $(k, s)$. We can apply this method to any pair $(k, s)$ with $k \geq s \geq 6$ since (44) is valid if $a_1 + a_2 = 10$. We obtain that $[K : \mathbb{Q}] \leq 909$ for all $k \geq s \geq 6$.

Let us assume that $s = 3, 4, 5$ is exceptional. It means that either $m_1 = 3, 4, 5$ or $m_3 = 3, 4, 5$ for $\Gamma(3)/14$. For example, let $m_1 = 3, 4, 5$. Let us consider the V-arithmetic graph defined by $\alpha$, $\delta_1$, and $\delta_4$. We define $\alpha = a_{14}^2$, where the algebraic integer $a_{14} \cdot \delta_1 \cdot \delta_4$. The determinant of the Gram matrix of $\alpha$, $\delta_1$, and $\delta_4$ is equal to $2\alpha - 8\sin^2(\pi/m_1)$. It follows that $0 < \sigma(\alpha) < 4\sigma(\sin^2(\pi/m_1))$ for $\sigma \neq \sigma^{(s)}$ and $4 < \sigma^{(s)}(\alpha) < 4\cdot \pi^2$. Then $K = \mathbb{Q}(\alpha)$ and $F_{m_1} \subset K$. Thus, we can apply Method A of Case 1 in Subsection 4.1 to $a_1 = 0$, $a_2 = 4$, $b_1 = 4$, $b_2 = 14^2$, and $l := m_1$, where $m_1 = 3, 4, 5$. We obtain that $[K : \mathbb{Q}] \leq 76$ for $m_1 = 3$, $[K : \mathbb{Q}] \leq 31$ for $m_1 = 4$, and $[K : \mathbb{Q}] \leq 24$ for $m_1 = 5$. 

It follows that $K = \mathbb{Q}(\alpha)$. Since $8 < 16$, this case is similar to that considered in [13, Subsection 5.5].
Thus, finally, \( |\mathbb{K} : \mathbb{Q}| \leqslant 909 \) for all graphs \( \Gamma_4^{(5)}(14) \).

4.4. V-arithmetic 3-graphs \( \Gamma_4^{(6)}(14) \) and their ground fields

Here we consider V-arithmetic 3-dimensional graphs \( \Gamma_4^{(6)}(14) \) and their fields.

First, let us consider the corresponding plane graph defined by \( \beta_1 = \delta_1, \beta_2 = \delta_2, \beta_3 = \delta_3, \beta_4 = \delta_4, \) and \( \beta_5 = \delta_5 \) that give \( P(\mathcal{M}_2) \). We define \( b_{ij} = \beta_i \cdot \beta_j \) when it is not 0. We also define \( c = b_{12} \). One can consider it as a kind of angle between the corresponding lines. This graph is given in Figure 5.

Considering the determinants of Gram matrices of subsets of \( \beta_1, \ldots, \beta_5 \), we obtain all the equations of \( \mathcal{M}_2 \) as follows:

\[
\begin{align*}
4b_{13}^2 &= (4 - b_{14}^2)(4 - b_{35}^2), \\
4b_{25}^2 &= (4 - b_{24}^2)(4 - b_{35}^2), \\
4b_{14}^2 + 4c^2 + 4cb_{14}b_{24} &= (4 - b_{13}^2)(4 - b_{24}^2), \\
4b_{24}^2 + 4c^2 + 4cb_{14}b_{24} &= (4 - b_{14}^2)(4 - b_{25}^2), \\
b_{35}^2(4 - c^2) + 2cb_{13}b_{25}b_{35} + 4c^2 &= (4 - b_{13}^2)(4 - b_{25}^2).
\end{align*}
\]

(59)

For \( \sigma \neq \sigma^{(+)} \), they also satisfy the inequalities \( b_{ij}^2 < 4 \) for all \( b_{ij} \), and \( c^2 < 4 \). By a direct calculation of the minimum and maximum of \( b_{13}b_{14}b_{24}b_{25}b_{35} \) for \( 0 \leqslant b_{ij}^2 \leqslant 4 \) and \( 0 \leqslant c^2 \leqslant 4 \) satisfying equations (59), we obtain that

\[
-3.1 < \sigma(b_{13}b_{14}b_{24}b_{25}b_{35}) < 3.1.
\]

(60)

Here the minimum \(-3.07\ldots\) is achieved for \( c = -0.39\ldots, b_{14} = b_{24} = \pm 1.4\ldots, b_{13} = \pm 1.166\ldots, b_{25} = \pm 1.166\ldots, b_{35} = \pm 1.1549\ldots \) and \( c = 0.39\ldots, b_{14} = -b_{24} = \pm 1.4\ldots, b_{13} = \pm 1.166\ldots, b_{25} = \pm 1.166\ldots, b_{35} = \pm 1.1549\ldots \) Here the maximum \( 3.07\ldots \) is achieved for \( c = -1.569\ldots, b_{14} = b_{24} = \pm 1.4\ldots, b_{13} = \pm 1.166\ldots, b_{25} = \pm 1.166\ldots, b_{35} = \pm 1.1549\ldots \) and \( c = 1.569\ldots, b_{14} = -b_{24} = \pm 1.4\ldots, b_{13} = \pm 1.166\ldots, b_{25} = \pm 1.166\ldots, b_{35} = \pm 1.1549\ldots \)

From expressions for \( \beta_i \) using \( \delta_i \) and \( \epsilon \), we obtain

\[
b_{13}b_{14}b_{24}b_{25}b_{35} = \frac{a_{13}a_{14}a_{24}a_{25}a_{35}}{\sin^2(\pi/m)}.
\]

(61)

We consider the algebraic integer \( \alpha \in \mathbb{K} \) given by

\[
\alpha = a_{13}a_{14}a_{24}a_{25}a_{35}.
\]

(62)

From (60), we obtain

\[
-3.1 \cdot \sigma\left(\frac{\sin^2 \pi \alpha}{m} \right) < \sigma(\alpha) < 3.1 \cdot \sigma\left(\frac{\sin^2 \pi \alpha}{m} \right).
\]

(63)

For the geometric embedding \( \sigma^{(+)} \), we have

\[
2^5 < \sigma^{(+)}(\alpha) < 14^5.
\]

(64)

It follows that \( \mathbb{K} = \mathbb{Q}(\alpha) \).

We can apply Methods A and B of Case 1 in Subsection 4.1 to \( a_1 = 3.1, a_2 = 3.1 \) (then \( a = 3.1 \), \( b_1 = 2^5, b_2 = 14^5 \) (then \( b = 14^5 \)), and \( l := m \).

First, we apply Method B. All exceptional \( l \geqslant 3 \), that is, when (12), which is

\[
\ln \frac{2}{\sqrt{3.1}} - \frac{\ln \gamma(l)}{\varphi(l)} \leqslant 0,
\]

(65)

is satisfied are \( l = 3, 4, 5, 7, 8, 9, 11, 13, 17, 19, 23 \).

We can take \( L_0 = 2053 \) in (16). Then

\[
\Delta_1(3.1) = \ln \frac{2}{\sqrt{3.1}} - \frac{\ln 2053}{2052} > 0.1237,
\]
and $L_1 = 2125$ can be taken in (20). Checking (13) for $3 \leq l < 2125$, we obtain that $3 \leq l \leq 510$. For all of these $l$ such that (13), which is
\[
\frac{\varphi(l)}{2} \cdot \left( \ln \frac{2}{\sqrt{3.1}} - \ln \frac{\gamma(l)}{\varphi(l)} \right) < \ln \sqrt{\frac{14^5}{3.1} - \ln \sin \frac{\pi}{l}},
\]
holds, we obtain
\[
[F_l : \mathbb{Q}] = \frac{\varphi(l)}{2} \leq 99,
\]
where 99 is achieved for $l = 199$. Moreover, for all of these non-exceptional $l$ we obtain the bound (21), which is
\[
N_0 = [K : F_l] \leq \left[ \frac{\ln \sqrt{\frac{14^5}{3.1} - \ln \sin (\pi/l)}}{\varphi(l)/2 \cdot (\ln (2/\sqrt{3.1}) - \ln \gamma(l)/\varphi(l))} \right],
\]
and finally we obtain the bound (22), which is
\[
N = [K : \mathbb{Q}] \leq \left[ \frac{\ln \sqrt{\frac{14^5}{3.1} - \ln \sin (\pi/l)}}{\varphi(l)/2 \cdot (\ln (2/\sqrt{3.1}) - \ln \gamma(l)/\varphi(l))} \right] \cdot \frac{\varphi(l)}{2}.
\]

If either $l$ is exceptional, or if the right-hand side of (69) is more than 99 (this is possible for $3 \leq l \leq 113$ only), then we also apply Method A of Case 1 to $l$. This improves the poor bound (68) for the non-exceptional $l$. We can apply this method to any $l \geq 4$ since (24) is valid for all $l \geq 4$ if $a_1 + a_2 = 6.2$. For $l \geq 6$, this method gives what we want: $[K : \mathbb{Q}] \leq 99$. For $l = 4$, it only gives $[K : \mathbb{Q}] \leq 120$; for $l = 5$, it only gives $[K : \mathbb{Q}] \leq 172$.

If $l = 3, 4, 5$, or, equivalently, $m = 3, 4, 5$, then, considering the subgraph of $e, \delta_4$, and $\delta_2$, exactly the same consideration as for the graph $\Gamma^{(5)}_1(14)$ above for $m_1 = 3, 4, 5$ gives that $[K : \mathbb{Q}] \leq 76$ for $m = 3$, $[K : \mathbb{Q}] \leq 31$ for $m = 4$, and $[K : \mathbb{Q}] \leq 24$ for $m = 5$.

Thus, $[K : \mathbb{Q}] \leq 99$ for all graphs $\Gamma^{(6)}_4(14)$. This completes the proof of Theorem 3.7.

**Appendix. Some results about cyclotomic fields**

Here we formulate some results about cyclotomic fields. Proofs can be found in [12, Appendix]; of course, they are standard (see [4]).

We consider the cyclotomic field $\mathbb{Q}(\sqrt[l]{T})$ and its totally real subfield $F_l = \mathbb{Q}(\cos(2\pi/l))$. We have $[\mathbb{Q}(\sqrt[l]{T}) : \mathbb{Q}] = \varphi(l)$. We have $F_l = \mathbb{Q}(\sqrt[l]{T}) = \mathbb{Q}$ for $l = 1, 2$, and $[F_l : \mathbb{Q}] = \varphi(l)/2$ for $l \geq 3$.

It is known (see, for example, [4]) that the discriminant of the field $\mathbb{Q}(\sqrt[l]{T})$ is equal to (where $p$ is prime)
\[
|\text{discr} \mathbb{Q}(\sqrt[l]{T})| = \frac{l^{\varphi(l)}}{\prod_{p|l} p^{e(l)/(p-1)}}.
\]

We have, for $l \geq 3$, that
\[
N_{F_l/\mathbb{Q}} \left( 4 \sin^2 \left( \frac{\pi}{l} \right) \right) = \gamma(l) = \begin{cases} p & \text{if } l = p^t > 2, \text{ where } p \text{ is prime}, \\ 1 & \text{otherwise}, \end{cases}
\]
and
\[
N_{F_l/\mathbb{Q}} \left( 4 \sin^2 \left( \frac{2\pi}{l} \right) \right) = \tilde{\gamma}(l) = \begin{cases} \gamma(l) & \text{if } l \geq 3 \text{ is odd}, \\ \gamma(l/2) & \text{if } l/2 \geq 3 \text{ is odd}, \\ \gamma(l/2)^2 & \text{if } l/2 \geq 4 \text{ is even}, \\ 4 & \text{if } l = 4. \end{cases}
\]
Moreover, we have the following formula for the discriminant:

$$|\text{discr} F_l| = \left(\frac{|\text{discr} \mathbb{Q}(\sqrt{l})|}{\gamma(l)}\right)^{1/2} \quad \text{for } l \geq 3,$$

(A.4)

where $|\text{discr} \mathbb{Q}(\sqrt{l})|$ is given by (A.1), and $\gamma(l)$ is given by (A.3).

We define $F_{k,s} = \mathbb{Q}(\cos(2\pi/k), \cos(2\pi/s))$. We further assume that $k, s \geq 3$. Let $m = [k, s]$ be the least common multiple of $k$ and $s$, and

$$\rho(k, s) = \begin{cases} 2 & \text{if } (k, s) \mid 2, \\ 1 & \text{otherwise}. \end{cases}$$

(A.5)

We have

$$[F_{k,s} : \mathbb{Q}] = \frac{\varphi(m)}{2\rho(k, s)}.$$

(A.6)

Moreover, we have

$$|\text{discr} F_{k,s}| = |\text{discr} F_m| \quad \text{if } (k, s) \nmid 2,$$

(A.7)

where $|\text{discr} F_m|$ is given by (A.4), and

$$|\text{discr} F_{k,s}| = |\text{discr} F_k|^{(\varphi(s)/2)}|\text{discr} F_s|^{(\varphi(k)/2)} \quad \text{if } (k, s) \mid 2,$$

(A.8)

where the discriminants $|\text{discr} F_k|$ and $|\text{discr} F_s|$ are given by (A.4).

References

1. I. Agol, ‘Finiteness of arithmetic Kleinian reflection groups’, Proceedings of the International Congress of Mathematicians, Madrid, 2005, vol. 2 (American Mathematical Society, Providence, RI, 2007) 951–960 (see also math.GT/0512560).
2. M. Belolipetsky, ‘On fields of definition of arithmetic Kleinian reflection groups’, Preprint, 2007, arXiv:0710.5108 [math.GT].
3. A. Borel, ‘Commensurability classes and volumes of hyperbolic 3-manifolds’, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) 8 (1981) 1–33.
4. J. W. S. Cassels and A. Fröhlich (eds), ‘Algebraic number theory’, Proceedings of an Instructional Conference Organized by LMS (Academic Press, London, 1967) xviii+366.
5. T. Chinburg and E. Friedman, ‘The smallest arithmetic hyperbolic three-orbifold’, Invent. Math. 86 (1986) 507–527.
6. D. D. Long, C. Maclachlan and A. W. Reid, ‘Arithmetic Fuchsian groups of genus zero’, Pure Appl. Math. Q. 2 (2006) 1–31.
7. V. V. Nikulin, ‘On arithmetic groups generated by reflections in Lobachevsky spaces’, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980) 637–699 (Russian), Math. USSR Izv. 16 (1981) 573–601 (English).
8. V. V. Nikulin, ‘On the classification of arithmetic groups generated by reflections in Lobachevsky spaces’, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981) 113–142 (Russian), Math. USSR Izv. 18 (1982) 99–123 (English).
9. V. V. Nikulin, ‘Discrete reflection groups in Lobachevsky spaces and algebraic surfaces’, Proceedings of the International Congress of Mathematicians, Berkeley, CA, 1986, vol. 1 (American Mathematical Society, Providence, RI, 1987) 654–671.
10. V. V. Nikulin, ‘Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces’, Izv. Ross. Akad. Nauk Ser. Mat. 71 (2007) 55–60 (Russian), Izv. Math. 71 (2007) 53–56 (English) (see also arXiv.org:math.AG/0609256).
11. V. V. Nikulin, ‘On ground fields of arithmetic hyperbolic reflection groups. III’, Preprint, 2007, arXiv:0710.2340 [math.AG].
12. V. V. Nikulin, ‘On ground fields of arithmetic hyperbolic reflection groups. II’, Moscow Math. J. 8 (2008) 789–812 (see also Preprint arXiv:0710.0162 [math.AG]).
13. V. V. Nikulin, ‘On ground fields of arithmetic hyperbolic reflection groups’, CRM Proceedings & Lecture Notes (American Mathematical Society, Providence, RI; volume dedicated to J. McKay), to appear (see also Preprint arXiv:0708.3991 [math.AG]).
14. K. Takeuchi, ‘A characterization of arithmetic Fuchsian groups’, J. Math. Soc. Japan 27 (1975) 600–612.
15. K. Takeuchi, ‘Arithmetic triangle groups’, J. Math. Soc. Japan 29 (1977) 91–106.
16. K. Takeuchi, ‘Commensurability classes of arithmetic triangle groups’, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. 24 (1977) 201–212.
17. K. Takeuchi, ‘Arithmetic Fuchsian groups with signature (1, e)’, J. Math. Soc. Japan 35 (1983) 381–407.
18. E. B. Vinberg, ‘Discrete groups generated by reflections in Lobačevskii spaces’, Mat. Sb. (N.S.) 72 (1967) 471–488 (Russian), Math. USSR Sb. 1 (1967) 429–444 (English).

19. E. B. Vinberg, ‘The nonexistence of crystallographic reflection groups in Lobačevskii spaces of large dimension’, Funkts. Anal. i Prilozhen. 15 (1981) 67–68 (Russian), Funct. Anal. Appl. 15 (1981) 216–217 (English).

20. E. B. Vinberg, ‘Absence of crystallographic reflection groups in Lobačevskii spaces of large dimension’, Trudy Moskov. Mat. Obshch. 47 (1984) 68–102 (Russian), Trans. Moscow Math. Soc. 47 (1985) 75–112 (English).

21. E. B. Vinberg, ‘Discrete reflection groups in Lobachevsky spaces’, Proceedings of the International Congress of Mathematicians, Warsaw, 1983, vol. 1 (American Mathematical Society, Providence, RI, 1984) 593–601.

Viacheslav V. Nikulin
Department of Pure Mathematics
The University of Liverpool
Liverpool
L69 3BX
United Kingdom

Steklov Mathematical Institute of Russian
Academy of Sciences
ul. Gubkina 8
GSP-1
Moscow 117966
Russia

vnikulin@liv.ac.uk
vvnikulin@list.ru