Oscillation Numbers for Continuous Lagrangian Paths and Maslov Index

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Abstract
In this paper we present the theory of oscillation numbers and dual oscillation numbers for continuous Lagrangian paths in $\mathbb{R}^{2n}$. Our main results include a connection of the oscillation numbers of the given Lagrangian path with the Lidskii angles of a special symplectic orthogonal matrix. We also present Sturmian type comparison and separation theorems for the difference of the oscillation numbers of two continuous Lagrangian paths. These results, as well as the definition of the oscillation number itself, are based on the comparative index theory (Elyseeva, 2009). The applications of these results are directed to the theory of Maslov index of two continuous Lagrangian paths. We derive a formula for the Maslov index via the Lidskii angles of a special symplectic orthogonal matrix, and hence we express the Maslov index as the oscillation number of a certain transformed Lagrangian path. The results and methods are based on a generalization of the recently introduced oscillation numbers and dual oscillation numbers for conjoined bases of linear Hamiltonian systems (Elyseeva, 2019 and 2020) and on the connection between the comparative index and Lidskii angles of symplectic matrices (Šepitka and Šimon Hilscher, 2021).

Keywords Oscillation number · Lagrangian path · Lidskii angle · Symplectic matrix · Comparative index · Maslov index

Mathematics Subject Classification Primary 34C10; Secondary 53D12
1 Introduction

Let $n \in \mathbb{N}$ be a given dimension and $[a, b] \subseteq \mathbb{R}$ a given interval. In this paper we develop the theory of oscillation numbers for arbitrary continuous Lagrangian paths on $[a, b]$. A continuous matrix-valued function $Y : [a, b] \rightarrow \mathbb{R}^{2n \times n}$ is a Lagrangian path if

$$Y^T(t) \mathcal{J} Y(t) = 0, \quad \text{rank} Y(t) = n, \quad t \in [a, b],$$

where $\mathcal{J} \in \mathbb{R}^{2n \times 2n}$ is the canonical skew-symmetric matrix. When a Lagrangian path is constant on $[a, b]$, then we call it a Lagrangian plane. Lagrangian paths arise, among others, as particular solutions (called conjoined or isotropic bases) of the linear Hamiltonian differential system

$$y' = \mathcal{J} \mathcal{H}(t) y, \quad t \in [a, b],$$

(H)

where the coefficient matrix $\mathcal{H} : [a, b] \rightarrow \mathbb{R}^{2n \times 2n}$ is symmetric and piecewise continuous. In this case the function $Y$ is piecewise continuously differentiable on $[a, b]$ and satisfies (1.1). We partition the matrices $\mathcal{J}, Y(t)$, and $\mathcal{H}(t)$ into $n \times n$ blocks as

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}, \quad \mathcal{H}(t) = \begin{pmatrix} -C(t) & A(t)^T \\ A(t) & B(t) \end{pmatrix}.$$ (1.2)

The motivation for the present study comes from the qualitative theory of canonical systems of the form (H) in [9, 13, 18, 19, 40, 45, 46].

The oscillation number $\mathcal{N}(Y, [a, b])$ and the dual oscillation number $\mathcal{N}^*(Y, [a, b])$ of a conjoined basis $Y$ of (H) on $[a, b]$ were defined in [17, 19, 21] as quantities describing the oscillations of the first component $X$ of the conjoined basis $Y$. They are based on the notions of the comparative index and the dual comparative index, see [11, 12] or [10, Section 3], and on piecewise constant symplectic transformations of solutions of system (H) from [1, 2].

According to the definitions in (2.1) below, for two (constant) Lagrangian planes $Y$ and $\hat{Y}$ the comparative index $\mu(Y, \hat{Y})$ and the dual comparative index $\mu^*(Y, \hat{Y})$ are integers between 0 and $n$, which are defined in algebraic way from the rank and the index of certain $n \times n$ matrices constructed from the blocks of the matrices $Y$ and $\hat{Y}$. The utility of the comparative index for the oscillation and spectral theory of system (H), as well as for the parallel theory of symplectic difference systems, is documented e.g. in [10, 13–15, 17, 20, 40, 42–44]. If system (H) satisfies the Legendre condition

$$B(t) \geq 0 \quad \text{for all} \quad t \in [a, b],$$

then the oscillation number $\mathcal{N}(Y, [a, b])$ reduces to the total number of left proper focal points of the conjoined basis $Y$ in the interval $(a, b)$, while the dual oscillation number $\mathcal{N}^*(Y, [a, b])$ reduces to the total number of right proper focal points of $Y$ in the interval $[a, b)$. These notions were introduced in [35, 47]. The main results in [19, 21] provide, without assuming (1.3) and without any majorant condition on the involved coefficient matrices, the Sturmian type comparison and separation theorems for conjoined bases of two possibly uncontrollable linear Hamiltonian systems of the form (H) in terms of the oscillation numbers.

The main purpose of this paper is to extend the concepts of an oscillation number $\mathcal{N}(Y, [a, b])$ and a dual oscillation number $\mathcal{N}^*(Y, [a, b])$, following the definitions presented in [17, 19, 21] via the comparative index, to an arbitrary continuous Lagrangian path $Y$ on $[a, b]$ and to connect these notions with the Maslov index. As the main tools we employ the traditional theory of Lidskii angles (or arguments) for symplectic matrices introduced in [37, 48, 49] and a new connection of the Lidskii angles with the comparative index obtained...
recently in [46]. More precisely, with a given continuous Lagrangian path \( Y \) on \([a, b]\) we associate the special continuous symplectic and orthogonal matrix \( Z_Y(t) \) defined by

\[
Z_Y(t) := (J Y(t) K_Y(t) Y(t) K_Y(t)^T), \quad K_Y(t) := [Y^T(t) Y(t)]^{-1/2}, \quad t \in [a, b], (1.4)
\]

where \( K_Y(t) > 0 \) represents a normalization factor, and consider its continuous Lidskii angles \( \varphi_j(t) \) on \([a, b]\) for \( j \in \{1, \ldots, n\} \). We then express (Theorem 3.8) the oscillation number \( \mathcal{N}(Y, [a, b]) \) in terms of the cumulative changes of specific integers \( q_j(t) \) at the endpoints of \([a, b]\). For each \( j \in \{1, \ldots, n\} \) the integer \( q_j(t) \) has the property that the Lidskii angle \( \varphi_j(t) \) belongs to the half-open interval \([2\pi q_j(t), 2\pi (q_j(t) + 1)]\). In a similar way we express the dual oscillation number \( \mathcal{N}^* (Y, [a, b]) \) in terms of the integer quantities \( q_j^*(t) \) corresponding to the Lidskii angles \( \varphi_j(t) \), which are located in the half-open interval \((2\pi q_j^*(t), 2\pi (q_j^*(t) + 1)]\). Combining the latter two results yields a formula (Theorem 3.13) relating the oscillation number and the dual oscillation number of one Lagrangian path on \([a, b]\). We also prove a special invariance property of the oscillation numbers under a symplectic orthogonal transformation (Theorem 3.12). We also present (under some monotonicity assumptions) formulas for the oscillation and dual oscillation numbers of the Lagrangian path \( Y \) in terms of the changes in the rank of the first component \( X \) of \( Y \) (Theorem 4.6).

Using the Lidskii angles in our analysis of the oscillation numbers leads in a natural way to their connection with the Maslov index of two continuous Lagrangian paths \( Y \) and \( \hat{Y} \) on \([a, b]\), denoted by \( \text{Mas}(Y, \hat{Y}, [a, b]) \). Here we use the analytic definition of the Maslov index from [5], see also [27–29]. We show (Theorem 4.2) that the Maslov index of \( Y \) and \( \hat{Y} \) on \([a, b]\) can be calculated from the changes at the endpoints of \([a, b]\) of the Lidskii angles of the symplectic matrix \( Z_Y^{-1}(t) Z_{\hat{Y}}(t) \). This approach is known in [50, Definition 2.2], [6, Eqs. (2.4)–(2.5)], or in the special case in [38, 39]. Consequently, the Maslov index of \( Y \) and \( \hat{Y} \) on \([a, b]\) is equal to the oscillation number of the transformed Lagrangian path \( Z_Y^{-1} \hat{Y} \) on \([a, b]\), i.e.,

\[
\text{Mas}(Y, \hat{Y}, [a, b]) = \mathcal{N}(Z_Y^{-1} \hat{Y}, [a, b]), (1.5)
\]

where the symplectic and orthogonal matrices \( Z_Y(t) \) and \( Z_{\hat{Y}}(t) \) are defined according to (1.4). Equivalently, the number in (1.5) is equal to the oscillation number \( \mathcal{N}(Z^{-1} \hat{Y}, [a, b]) \), where \( Z(t) \) is any continuous symplectic matrix, whose second block column is equal to \( Y(t) \). In particular, the oscillation number of \( Y \) on \([a, b]\) can be expressed via the Maslov index as

\[
\mathcal{N}(Y, [a, b]) = \text{Mas}(E, Y, [a, b]), (1.6)
\]

where the matrix \( E = (0 I)^T \) represents the vertical Lagrangian plane. We also discuss (Remark 4.5) a notion of the dual Maslov index, which is related in analogous way to the dual oscillation number and to the dual comparative index, and a new monotonicity result for calculating the Maslov index (Theorem 4.8). Thus, we contribute in the spirit of [27–29] to a detailed investigation of the Maslov index for Lagrangian paths in \( \mathbb{R}^{2n} \).

The main properties of the oscillation numbers obtained via the Lidskii angles allow us to derive Sturmian type comparison theorems for the oscillation numbers (Theorem 5.1) of two continuous Lagrangian paths on \([a, b]\), essentially extending the results in [19, Theorem 4.4] and [21, Theorem 4.3] to the context of continuous Lagrangian paths. These formulas involve the comparative index or the dual comparative index of \( Y(t) \) and \( \hat{Y}(t) \) evaluated at the endpoints of the interval \([a, b]\). In addition, we derive Sturmian type separation theorems (Theorems 5.7 and 5.9) for specific continuous Lagrangian paths, which belong to the set of paths determined by a given continuous symplectic matrix \( \Phi(t) \) on \([a, b]\). In this way we
generalize the result in [44, Theorem 1.1] to the context of continuous Lagrangian paths. The above mentioned comparison theorem for the oscillation numbers also yields (Corollary 5.4) the expression of the Maslov index \( \text{Mas}(Y, \hat{Y}, [a, b]) \) in terms of the two reference Maslov indices \( \text{Mas}(E, \hat{Y}, [a, b]) \) and \( \text{Mas}(E, Y, [a, b]) \) and in terms of the comparative index of \( \hat{Y} \) and \( Y \) evaluated at the endpoints of the interval \([a, b]\).

We are convinced that the results in this paper contribute to the understanding of the role of the comparative index in the oscillation theory of continuous Lagrangian paths. We believe that further investigations in this direction will lead to substantial advancements in several areas of theoretical mathematics, such as in the spectral theory of linear Hamiltonian systems, where the monotonicity assumption on the spectral parameter is dropped, in the oscillation theory on discrete time domains, or in the theory of continuous symplectic matrices in general. In particular, the presented methods are instrumental for future development of the oscillation numbers and the Maslov index on unbounded intervals (including the properties of the rotation number) or for the investigations of the connections between the Hörmander index and the comparative index.

The paper is organized as follows. In Sect. 2 we recall the definitions of the comparative index, the dual comparative index, and the Lidskii angles of a symplectic matrix. We also present their relationship based on special argument functions derived in [46]. In Sect. 3 we define the oscillation number and the dual oscillation number for a continuous Lagrangian path on \([a, b]\) and study their relationship with the Lidskii angles. In Sect. 4 we investigate connections of the oscillation number and the dual oscillation number with the Maslov index. In Sect. 5 we present comparison and separation theorems for the oscillation numbers and the dual comparative index.

In Sect. 6 we comment about the results of this paper and their future development.

2 Comparative Index and Lidskii Angles

In this section we recall the definitions of the comparative index, the dual comparative index, and the Lidskii angles for a symplectic matrix. We also present formulas for the calculation of the comparative index and the dual comparative index by means of the Lidskii angles. We also present formulas for the calculation of the comparative index and the dual comparative index by means of the Lidskii angles.

where the \( n \times n \) matrices \( \mathcal{M} \) and \( \mathcal{P} \) are given by

\[
\mathcal{M} := (I - X^\dagger X) W(Y, \hat{Y}), \quad \mathcal{P} := V [W(Y, \hat{Y})]^T X^\dagger \hat{X} V, \quad V := I - \mathcal{M}^\dagger \mathcal{M},
\]

and where \( W(Y, \hat{Y}) := Y^T \mathcal{J} \hat{Y} \) is the Wronskian of \( Y = (X^T, U^T)^T \) and \( \hat{Y} = (\hat{X}^T, \hat{U}^T)^T \), following the notation in (1.2). The dagger in (2.2) denotes the Moore–Penrose pseudoinverse, see e.g. [4, 7]. The notation \( \text{ind} \mathcal{P} \) means the number of negative eigenvalues of the symmetric matrix \( \mathcal{P} \). Note that \( \text{ind} \mathcal{P} + \text{ind}(-\mathcal{P}) = \text{rank} \mathcal{P} \). For convenience we also define the constant \( 2n \times n \) matrix \( E \) representing the vertical Lagrangian plane, i.e.,

\[
E := \begin{pmatrix} 0 & 1 \end{pmatrix}^T.
\]

We will use the following basic invariant properties of the comparative indices defined in (2.1), see [12, Properties 1–2, p. 448] or [10, Theorem 3.5(i)–(iii)]. For any nonsingular
matrices $C_1, C_2 \in \mathbb{R}^{n \times n}$, for any symplectic lower block triangular matrix $L \in \mathbb{R}^{2n \times 2n}$, and for any symplectic matrix $Z \in \mathbb{R}^{2n \times 2n}$ with $ZE = Y$ we have the properties
\[
\mu(YC_1, \hat{Y}C_2) = \mu(Y, \hat{Y}), \quad \mu^*(YC_1, \hat{Y}C_2) = \mu^*(Y, \hat{Y}), \quad \det C_1 \neq 0, \quad \det C_2 \neq 0.
\] (2.4)
\[
\mu(LY, L\hat{Y}) = \mu(Y, \hat{Y}), \quad \mu^*(LY, L\hat{Y}) = \mu^*(Y, \hat{Y}), \quad L = \begin{pmatrix} P & 0 \\ K & p^{T-1} \end{pmatrix}, \quad L^T J L = \mathcal{J},
\] (2.5)
\[
\mu(Y, \hat{Y}) = \mu^*(Z^{-1}E, Z^{-1}\hat{Y}), \quad \mu^*(Y, \hat{Y}) = \mu(Z^{-1}E, Z^{-1}\hat{Y}), \quad Z^T J Z = \mathcal{J}, \quad ZE = Y. \] (2.6)

Next we recall the definition of Lidskii angles. Consider a real symplectic matrix $S$ which belong to the interval $I$, for all $j \in \{1, \ldots, n\}$. Then we set
\[
\mathcal{J} := \left( \begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S_{22} \end{array} \right), \quad S_{ij} \in \mathbb{R}^{n \times n}, \quad S^T J S = \mathcal{J}.
\] (2.7)

We define, according to [3, 37, 49] or [30, Section 3.1], the complex matrix $W_S := (S_{11} - iS_{12})^{-1}(S_{11} + iS_{12})^{-1}$, where $i$ is the imaginary unit ($i^2 = -1$). The matrix $W_S$ is well-defined, symmetric, and unitary, and in particular its eigenvalues $w_j$ lie on the unit circle $U$ in the complex plane. Hence, $w_j = \exp(i\varphi_j)$ with the real arguments $\varphi_j$. The numbers $\varphi_j$ for $j \in \{1, \ldots, n\}$ are called the Lidskii angles corresponding to the symplectic matrix $S$. The angles $\varphi_j$ are thus defined uniquely up to an additive term $2\pi m_j$ for $m_j \in \mathbb{Z}$ and
\[
\sum_{j=1}^n \varphi_j = \arg \det W_S = \left( 2 \arg \det (S_{11} + iS_{12}) \right) \mod 2\pi.
\] (2.9)

In view of (2.9) the Lidskii angles $\varphi_j$ satisfy
\[
\frac{1}{2} \sum_{j=1}^n \varphi_j = \text{Arg}_3(S) \mod \pi,
\] (2.10)
where $\text{Arg}_3(S)$ is one of the arguments of the symplectic matrix $S$ considered by Yakubovich in [48, p. 263], see also [31, Lemma 5.6], namely
\[
\text{Arg}_3(S) := \arg \det (S_{11} + iS_{12}) = \arg \det \left( I \begin{pmatrix} 0 \\ I \end{pmatrix} S \begin{pmatrix} I \\ iI \end{pmatrix} \right).
\] (2.11)

In [46] we introduced the following argument function, which is motivated by (2.10). Given a real interval $I$ and a real symplectic matrix $S$, we consider the Lidskii angles $\varphi_j$ of $S$, which belong to the interval $I$ for all $j \in \{1, \ldots, n\}$. Then we set
\[
\text{Arg}_I(S) := \frac{1}{2} \sum_{j=1}^n \varphi_j \quad \text{with } \varphi_j \in I \text{ for all } j \in \{1, \ldots, n\}.
\] (2.12)

We will use the above argument function $\text{Arg}_I$ with the particular choice of the half-open intervals $I = [2\pi q, 2\pi (q + 1))$ and $\hat{I} = (2\pi q, 2\pi (q + 1)]$ for $q \in \mathbb{Z}$.

\textbf{Remark 2.1} It is known that the number of the Lidskii angles $\varphi_j$ of $S$, which are integer multiples of $2\pi$, is equal to the defect of the block $S_{12}$ in (2.7), see e.g. [22, Proposition 2.5] or [45, Eq. (4.10)].

The main result in [46] provides a connection between the comparative index (or the dual comparative index) and the Lidskii angles. The connection is based on the symplectic and
orthogonal matrices of the form (1.4). More precisely, for a real constant $2n \times n$ matrix $Y$ satisfying (1.1) we define the constant $2n \times 2n$ matrix

$$Z_Y := (J Y K_Y Y K_Y), \quad K_Y := (Y^T Y)^{-1/2},$$

where the symmetric matrix $K_Y$ obeys the condition $K_Y > 0$. Then the matrix $Z_Y$ is symplectic and orthogonal, i.e., $Z_Y^T J Z_Y = J$ and $Z_Y^T Z_Y = I$. This implies that the matrices $Z_Y$ and $J$ commute. Note also that $Z_E = I$.

**Remark 2.2** Observe that for an arbitrary symplectic matrix $S$ we have

$$S = Z_{SE} L,$$

where $L = Z_{SE}^{-1} S = Z_{SE}^T S$ is a symplectic lower block triangular matrix in the form given by (2.5). This fact follows from equation (2.13) with $Y := SE$. Indeed, the right upper block of the matrix $L$ is equal to $(I_0) LE = (I_0) Z_{SE}^T SE = -K_Y(Y^T J Y) = 0$. Moreover, if $S$ is a symplectic and orthogonal matrix, then in (2.14) we have $L = I$ and hence by (2.13) we get $Z_{SY} = S Z_Y$ and $Z_{SE} = S$.

The following result is derived in [46, Theorem 1.1].

**Proposition 2.3** (Comparative index and Lidskii angles) Let $Y$ and $\hat{Y}$ be real constant $2n \times n$ matrices satisfying (1.1) and define the symplectic matrices $Z_Y$ and $Z_{\hat{Y}}$ by (2.13). Then the comparative index and the dual comparative index defined in (2.1) with (2.2) satisfy

$$\mu(Y, \hat{Y}) = \frac{1}{\pi} \left\{ \text{Arg}_{[0,2\pi]}(Z_{\hat{Y}}) - \text{Arg}_{[0,2\pi]}(Z_Y) + \text{Arg}_{[0,2\pi]}(Z_{\hat{Y}}^{-1} Z_Y) \right\},$$

$$\mu^*(Y, \hat{Y}) = n - \frac{1}{\pi} \left\{ \text{Arg}_{[0,2\pi]}(Z_{\hat{Y}}) - \text{Arg}_{[0,2\pi]}(Z_Y) + \text{Arg}_{[0,2\pi]}(Z_{\hat{Y}}^{-1} Z_Y) \right\},$$

where $\text{Arg}_{\mathcal{T}}$ is the special argument function defined in (2.12).

In the last comment in this section we discuss the continuity property of the Lidskii angles.

**Remark 2.4** Consider a continuous Lagrangian path $Y$ on the interval $[a, b]$ and define the symplectic and orthogonal matrix $Z_Y(t)$ on $[a, b]$ by (1.4). Note that this matrix corresponds to the matrix $Z_Y(t)$ in (2.13). Then the matrix $Z_Y(t)$ is continuous on $[a, b]$, which implies that the corresponding matrix $W_{Z_Y(t)}$ defined through (2.8) is also continuous on $[a, b]$. Therefore, the Lidskii angles $\varphi_j(t)$ of the symplectic matrix $Z_Y(t)$ can be chosen to be continuous functions on $[a, b]$, see also the proof of [30, Theorem 3.6]. This fact is utilized in Definition 3.4 below. In the next sections we will always work with such continuous Lidskii angles $\varphi_j(t)$ on $[a, b]$ for all $j \in \{1, \ldots, n\}$.

### 3 Oscillation Numbers for Lagrangian Paths

In this section we generalize the notion of the oscillation number and the dual oscillation number from [17, pp. 17–18], [19, pp. 311–312], and [21, Definition 3.1], which are defined through the comparative index and the dual comparative index, in the context of continuous Lagrangian paths on $[a, b]$ and study their properties in terms of the Lidskii angles. For a given continuous Lagrangian path $Y$ on $[a, b]$, let $D := \{a = t_0 < t_1 < \cdots < t_{p-1} < t_p = b\}$ be a finite partition of the interval $[a, b]$ and let $\{R_k(t)\}_{k=0}^{p-1}$ be a finite system of real symplectic matrices on $[a, b]$ such that for every $k \in \{0, \ldots, p-1\}$ the matrix $R_k(t)$ is continuous.
on the interval $[t_k, t_{k+1}]$ and the transformed Lagrangian path $\tilde{Y}_k = (\tilde{X}_k^T, \tilde{U}_k^T)^T := R_k^{-1}Y$ satisfies

$$\text{rank}\ X_k(t) = \text{rank}\ W(R_k(t)E, Y(t)) = \text{const}, \quad \text{rank}\ ((I 0) R_k(t)E) = \text{const},$$

\[ t \in [t_k, t_{k+1}], \quad k \in \{0, 1, \ldots, p - 1\}. \tag{3.1} \]

For a continuous Lagrangian path $Y$ on $[a, b]$ and a partition $D = \{t_k\}_{k=0}^p$ of $[a, b]$ with the system of symplectic matrices $R := \{R_k(t)\}_{k=0}^{p-1}$ satisfying (3.1) we define the following quantities, compare with [17, Eq. (2.6)], [19, Eq. (3.4)], and [21, Eq. (3.1)],

$$\mathcal{N}(Y, D, R, [a, b]) := \sum_{k=0}^{p-1} \mu(Y(t), R_k(t)E) |t_{k+1}^{t_k}|, \tag{3.2}$$

$$\mathcal{N}^*(Y, D, R, [a, b]) := \sum_{k=0}^{p-1} \mu^*(Y(t), R_k(t)E) |t_{k+1}^{t_k}|, \tag{3.3}$$

where we use the comparative index in (3.2) and the dual comparative index in (3.3). Here for a function $f : [a, b] \to \mathbb{R}$ we use the standard notation

$$f(t)|_{\tau_1}^{\tau_2} := f(\tau_2) - f(\tau_1), \quad \tau_1, \tau_2 \in [a, b].$$

Based on (3.1) and property (2.6) of the comparative index we derive the connections

$$\mathcal{N}(Y, D, R, [a, b]) = -\mathcal{N}^*(Z^{-1}E, D, \tilde{R}, [a, b]), \tag{3.4}$$

$$\mathcal{N}^*(Y, D, R, [a, b]) = -\mathcal{N}(Z^{-1}E, D, \tilde{R}, [a, b]).$$

Here $Z(t)$ is any continuous symplectic matrix with $Z(t) E = Y(t)$ on $[a, b]$ and $\tilde{R} := \{\tilde{R}_k(t)\}_{k=0}^{p-1}$ is the transformed system of symplectic matrix-valued functions $\tilde{R}_k(t) := Z^{-1}(t) R_k(t)$ obtained from the original system $R := \{R_k(t)\}_{k=0}^{p-1}$, compare with [21, Proposition 3.3(vi)]. Indeed, by applying (2.6) we rewrite equations (3.2) and (3.3) as

$$\mathcal{N}(Y, D, R, [a, b]) = \sum_{k=0}^{p-1} \mu^*(Z^{-1}(t) E, Z^{-1}(t) R_k(t)E) |t_{k+1}^{t_k}|, \tag{3.5}$$

$$\mathcal{N}^*(Y, D, R, [a, b]) = \sum_{k=0}^{p-1} \mu(Z^{-1}(t) E, Z^{-1}(t) R_k(t)E) |t_{k+1}^{t_k}|, \tag{3.6}$$

where the pair of matrices $Z^{-1}(t) E$ and $\tilde{R}_k(t)$ satisfies the properties in assumption (3.1), replacing respectively the matrices $Y(t)$ and $R_k(t)$ therein. Incorporating the order of the substitutions in (3.2) and (3.3), we derive the result in (3.4) from equations (3.5) and (3.6).

**Remark 3.1** (i) In [17] we considered a partition $D$ and a system $R = \{R_k\}_{k=0}^{p-1}$ of constant symplectic matrices such that instead of (3.1) we have that

$$\tilde{X}_k(t)$$

is invertible on the interval $[t_k, t_{k+1}]$. \tag{3.7}

For the special case of

$$R_k := R_{\alpha_k} = \begin{pmatrix} (\cos \alpha_k) I & (\sin \alpha_k) I \\ -(\sin \alpha_k) I & (\cos \alpha_k) I \end{pmatrix}$$


For an arbitrary continuous Lagrangian path 

\[ Y \]

(ii) In [19, Eq. (3.13)] we introduced an equivalent definition of the oscillation numbers from [17] for conjoined bases \( Y \) of system (H). Namely, consider a partition \( a = s_0 < s_1 < \cdots < s_{r-1} < s_r = b \) such that for any \( \ell \in \{0, \ldots, r-1\} \) there exists a conjoined basis \( Y_\ell \) of system (H) with the nonsingular upper block \( X_\ell(t) \) on \([s_\ell, s_{\ell+1}]\), i.e.,

\[ \det X_\ell(t) \neq 0, \quad t \in [s_\ell, s_{\ell+1}], \quad \ell \in \{0, \ldots, r-1\}. \]  

(3.9)

The oscillation numbers in [19, Eq. (3.13)] are defined by (3.2) with \( R_\ell(t) := Z_\ell(t) \), where \( Z_\ell(t) \) is a symplectic fundamental matrix of system (H) such that \( Z_\ell(t) E = Y_\ell(t) \) on \([s_\ell, s_{\ell+1}]\). Note that for this choice of \( R_\ell(t) \) all conditions in (3.1) are satisfied, in particular \( Y \) and \( R_\ell(t) E = Y_\ell(t) \) are conjoined bases of (H) and then the first condition in (3.1) automatically holds. Hence, we can see that condition (3.1) is a proper generalization of conditions (3.7) and (3.9).

(iii) For any arbitrary continuous Lagrangian path \( Y \) on \([a, b] \) one can present an analog of the definition from part (ii) by putting \( R_\ell(t) := Z_{Y(t)} C_\ell \), where \( Z_{Y(t)} \) are symplectic and orthogonal matrices given by (1.4) and \( C_\ell \) for \( k \in \{0, 1, \ldots, p - 1\} \) are constant symplectic matrices such that the upper block of \( R_k(t) E \) is nonsingular for all \( t \in [t_k, t_{k+1}] \) or, more generally, it has constant rank on \([t_k, t_{k+1}]\).

(iv) For the subsequent proofs it is important that condition (3.1) remains valid for the matrices \( Z_{R_k E} E = R_k E K_{R_k E} \) and \( Z_{Y(t)} E = Y(t) K_{Y(t)} \), which are associated with \( R_k E \) and \( Y(t) \) via (2.13). Indeed, according to the definition in (1.4) the matrices \( K_{R_k E} \) and \( K_{Y(t)} \) are nonsingular and the values rank \( W(Z_{R_k E} E, Z_{Y(t)} E) = \) rank \( W(R_k E, Y(t)) \) and rank((I 0) \( R_k(t) E K_{R_k E} \)) are then constant on \([t_k, t_{k+1}]\).

For the cases described in Remark 3.1(i), (ii) the numbers defined in (3.2) and (3.3) are invariant with respect to the partition \( D \) of \([a, b] \) and with respect to the special choice of the system \( R \) of symplectic matrices satisfying condition (3.7) or (3.9), see [19, Lemma 2.2] and [21, Proposition 3.3(ii)]. One can prove the same invariant properties for the general case (3.1), based only on the properties of the comparative index and the dual comparative index. Note that the same will follow from our Theorem 3.8 below (see Remark 3.9). The invariant properties of the numbers in (3.2) and (3.3) then justify the following definition.

**Definition 3.2** (Oscillation numbers) Let \( Y : [a, b] \rightarrow \mathbb{R}^{2n \times n} \) be a continuous Lagrangian path. The quantity defined in (3.2) is called the oscillation number of \( Y \) on \([a, b] \) and it is denoted by \( \mathcal{N}(Y, [a, b]) \). The quantity defined in (3.3) is called the dual oscillation number of \( Y \) on \([a, b] \) and it is denoted by \( \mathcal{N}^*(Y, [a, b]) \).

**Remark 3.3** (i) Since by (2.4) the comparative index and the dual comparative index are invariant under the multiplication of its arguments by invertible \( n \times n \) matrices from the right, it follows that the same property holds for the oscillation number and the dual oscillation number. More precisely, if \( Y' \) is a continuous Lagrangian path on \([a, b] \) and \( C : [a, b] \rightarrow \mathbb{R}^{n \times n} \) is a continuous invertible matrix function on \([a, b] \), then \( Y'C \) is also a continuous Lagrangian path on \([a, b] \) with

\[ \mathcal{N}(Y'C, [a, b]) = \mathcal{N}(Y', [a, b]), \quad \mathcal{N}^*(Y'C, [a, b]) = \mathcal{N}^*(Y', [a, b]). \]  

(3.10)
(ii) The definition of the oscillation number and the dual oscillation number through (3.2) and (3.3) yields their additivity with respect to the base interval, see also [19, Remark 3.7]. Namely, for any point \( c \in (a, b) \) we have

\[
\mathcal{N}(Y, [a, b]) = \mathcal{N}(Y, [a, c]) + \mathcal{N}(Y, [c, b]), \\
\mathcal{N}^*(Y, [a, b]) = \mathcal{N}^*(Y, [a, c]) + \mathcal{N}^*(Y, [c, b]).
\] (3.11)

We will now present the main results of this section, which use the following terminology based on Remark 2.4.

**Definition 3.4 (Lidskii angles of Lagrangian path)** Let \( Y \) be a continuous Lagrangian path on \([a, b]\) and let \( Z_Y(t) \) be the associated symplectic (and orthogonal) matrix defined in (1.4). Then the continuous Lidskii angles \( \varphi_j(t) \) for \( j \in \{1, \ldots, n\} \) and \( t \in [a, b] \) of \( Z_Y(t) \) are called the Lidskii angles of the Lagrangian path \( Y \).

With the above terminology, for fixed continuous branches of the Lidskii angles \( \varphi_j(t) \) we consider the uniquely defined integers \( q_j(t) \) and \( q^*_j(t) \), which satisfy the properties

\[
\varphi_j(t) \in [2\pi q_j(t), 2\pi (q_j(t) + 1)), \quad t \in [a, b], \quad j \in \{1, \ldots, n\},
\] (3.12)

\[
\varphi_j(t) \in (2\pi q^*_j(t), 2\pi (q^*_j(t) + 1]], \quad t \in [a, b], \quad j \in \{1, \ldots, n\},
\] (3.13)

that is, the Lidskii angle \( \varphi_j(t) \) of \( Y \) belongs to the indicated half-open interval of the length \( 2\pi \). This means that the integers \( q_j(t) \) and \( q^*_j(t) \) are given by

\[
q_j(t) = \left\lfloor \frac{\varphi_j(t)}{2\pi} \right\rfloor, \quad q^*_j(t) = \left\lceil \frac{\varphi_j(t)}{2\pi} \right\rceil - 1, \quad t \in [a, b],
\] (3.14)

where for \( x \in \mathbb{R} \) the notation \( \lfloor x \rfloor \) and \( \lceil x \rceil \) stand for the greatest integer which is smaller or equal to \( x \) (the floor function) and for the smallest integer which is greater or equal to \( x \) (the ceiling function). It follows from this definition that \( q_j(t) = q^*_j(t) \) if and only if the angle \( \varphi_j(t) \) belongs to the interior of the intervals in (3.12) or (3.13), while \( q_j(t) = q^*_j(t) + 1 \) holds when the angle \( \varphi_j(t) \) is an integer multiple of \( 2\pi \). This means in view of Remark 2.1 that

\[
\sum_{j=1}^n q_j(t) = \text{def } X(t) + \sum_{j=1}^n q^*_j(t) \quad \text{for all } t \in [a, b].
\] (3.15)

Observe that for the argument function in (2.10) we then have

\[
\text{Arg}_3(Z_Y(t))\big|^{\tau_2}_{\tau_1} = \frac{1}{2} \sum_{j=1}^n \varphi_j(t)\big|^{\tau_2}_{\tau_1} = \text{Arg}_{[0,2\pi]}(Z_Y(t))\big|^{\tau_2}_{\tau_1} + \pi \sum_{j=1}^n q_j(t)\big|^{\tau_2}_{\tau_1},
\] (3.16)

\[
\text{Arg}_3(Z_Y(t))\big|^{\tau_2}_{\tau_1} = \frac{1}{2} \sum_{j=1}^n \varphi_j(t)\big|^{\tau_2}_{\tau_1} = \text{Arg}_{[0,2\pi]}(Z_Y(t))\big|^{\tau_2}_{\tau_1} + \pi \sum_{j=1}^n q^*_j(t)\big|^{\tau_2}_{\tau_1}
\] (3.17)

for all \( \tau_1, \tau_2 \in [a, b] \) with \( \tau_1 < \tau_2 \). We have the following important property of the integers \( q_j(t) \) and \( q^*_j(t) \) in (3.12) and (3.13).

**Lemma 3.5** Let \( Y \) be a continuous Lagrangian path on \([a, b]\) with the associated Lidskii angles \( \varphi_j(t) \) on \([a, b]\) for \( j \in \{1, \ldots, n\} \) according to Definition 3.4. If the upper block \( X(t) \) of \( Y(t) \) has constant rank on \([\tau_1, \tau_2] \subseteq [a, b] \), then

\[
q_j(t) = \text{def } q_j \quad \text{and } q^*_j(t) = \text{def } q^*_j \quad \text{are constant on } [\tau_1, \tau_2] \text{ for all } j \in \{1, \ldots, n\},
\] (3.18)

where \( q_j(t) \) and \( q^*_j(t) \) are given by (3.12) and (3.13).
Proof According to Remark 2.1 and using the continuity of \( \varphi_j(t) \) on \([a, b]\) it follows that each angle \( \varphi_j(t) \) either remains in the open interval \( (2\pi q_j, 2\pi (q_j + 1)) = (2\pi q^*_j, 2\pi (q^*_j + 1)) \) on \([a, b]\), or it is constant on the interval \([\tau_1, \tau_2]\) with the value \( \varphi_j(t) \equiv 2\pi q_j = 2\pi (q^*_j + 1) \). Hence, the property in (3.18) holds.

Combining Proposition 2.3 with equations (3.16) and (3.17) we derive the following main property of the comparative index for two continuous Lagrangian paths.

Proposition 3.6 (Comparative index for continuous Lagrangian paths) Let \( Y \) and \( \hat{Y} \) be continuous Lagrangian paths on \([a, b]\) with the associated Lidskii angles \( \varphi_j(t) \) and \( \hat{\varphi}_j(t) \) on \([a, b]\) for \( j \in \{1, \ldots, n\} \) according to Definition 3.4. Consider the continuous symplectic matrix

\[
\tilde{S}(t) := Z_{\hat{Y}}^{-1}(t) Z_Y(t) = Z_{\hat{Y}}^T(t) Z_Y(t), \quad t \in [a, b],
\]

and its continuous Lidskii angles \( \tilde{\varphi}_j(t) \), where \( Z_Y(t) \) and \( Z_{\hat{Y}}(t) \) are the symplectic and orthogonal matrices associated with \( Y \) and \( \hat{Y} \) through (1.4). Given the integers \( q_j(t) \), \( \hat{q}_j(t) \), \( \tilde{q}_j(t) \) and \( \tilde{q}^*_j(t) \), \( \tilde{q}^*_j(t) \), \( \tilde{q}^*_j(t) \) associated through (3.12) and (3.13) with the angles \( \varphi_j(t) \), \( \hat{\varphi}_j(t) \), \( \tilde{\varphi}_j(t) \), then for any two points \( t_1, t_2 \in [a, b] \) the comparative index and the dual comparative index defined in (2.1) with (2.2) satisfy

\[
\begin{align}
\mu(Y(t), \hat{Y}(t))|_{t_1}^{t_2} &= \sum_{j=1}^n (q_j(t) - \hat{q}_j(t) - \tilde{q}_j(t))|_{t_1}^{t_2}, \\
\mu^*(Y(t), \hat{Y}(t))|_{t_1}^{t_2} &= -\sum_{j=1}^n (q^*_j(t) - \hat{q}^*_j(t) - \tilde{q}^*_j(t))|_{t_1}^{t_2}.
\end{align}
\]

Proof According to [24, pg. 163], for any symplectic and orthogonal matrices \( Z(t) \) and \( \hat{Z}(t) \) we have the multiplicative property

\[
(I \ 0) \hat{Z}^{-1}(t) Z(t) \left( \begin{array}{c}
I \\
iI
\end{array} \right) = (I \ 0) \hat{Z}^{-1}(t) \left( \begin{array}{c}
I \\
iI
\end{array} \right) (I \ 0) Z(t) \left( \begin{array}{c}
I \\
iI
\end{array} \right).
\]

Then the argument function \( \text{Arg}_3 \) defined by (2.11) for the matrix \( \hat{Z}^{-1}(t) Z(t) \) is the sum of the argument functions of \( \hat{Z}^{-1}(t) \) and \( Z(t) \), and hence

\[
\text{Arg}_3(\hat{Z}(t))|_{t_1}^{t_2} - \text{Arg}_3(Z(t))|_{t_1}^{t_2} + \text{Arg}_3(\hat{Z}^{-1}(t) Z(t))|_{t_1}^{t_2} = 0.
\]

For any continuous Lagrangian paths \( Y \) and \( \hat{Y} \) on \([a, b]\) we then have by (2.15) and (3.16) that

\[
\begin{align}
\pi \mu(Y(t), \hat{Y}(t))|_{t_1}^{t_2} \overset{(2.15)}{=} & \text{Arg}_{[0,2\pi)}(Z_{\hat{Y}}(t))|_{t_1}^{t_2} - \text{Arg}_{[0,2\pi)}(Z_Y(t))|_{t_1}^{t_2} \\
& + \text{Arg}_{[0,2\pi)}(Z_{\hat{Y}}^{-1}(t) Z_Y(t))|_{t_1}^{t_2} \\
\overset{(3.16)}{=} & \text{Arg}_3(Z_{\hat{Y}}(t))|_{t_1}^{t_2} - \text{Arg}_3(Z_Y(t))|_{t_1}^{t_2} + \text{Arg}_3(Z_{\hat{Y}}^{-1}(t) Z_Y(t))|_{t_1}^{t_2} \\
& + \pi \sum_{j=1}^n (q_j(t) - \hat{q}_j(t) - \tilde{q}_j(t))|_{t_1}^{t_2}.
\end{align}
\]

Then by (3.22) we derive the equality in (3.20). The proof of equality (3.21) is similar and it follows from (2.16), (3.17), and (3.22).
Corollary 3.7  Under the assumptions and the notation of Proposition 3.6 we have the following implications. If rank $\hat{X}(t)$ is constant on $[\tau_1, \tau_2]$, then
\[
\mu(Y(t), \hat{Y}(t))|_{\tau_1}^{\tau_2} = \sum_{j=1}^{n} (q_j(t) - \hat{q}_j(t))|_{\tau_1}^{\tau_2},
\]
while if rank $W(\hat{Y}(t), Y(t))$ is constant on $[\tau_1, \tau_2]$, then
\[
\mu(Y(t), \hat{Y}(t))|_{\tau_1}^{\tau_2} = \sum_{j=1}^{n} (q_j(t) - \hat{q}_j(t))|_{\tau_1}^{\tau_2},
\]
\[
\mu^*(Y(t), \hat{Y}(t))|_{\tau_1}^{\tau_2} = \sum_{j=1}^{n} (q_j^*(t) - \hat{q}_j^*(t))|_{\tau_1}^{\tau_2}.
\]

In particular, if both rank $\hat{X}(t)$ and rank $W(\hat{Y}(t), Y(t))$ are constant on $[\tau_1, \tau_2]$, then
\[
\mu(Y(t), \hat{Y}(t))|_{\tau_1}^{\tau_2} = \sum_{j=1}^{n} q_j(t)|_{\tau_1}^{\tau_2}, \quad \mu^*(Y(t), \hat{Y}(t))|_{\tau_1}^{\tau_2} = \sum_{j=1}^{n} q_j^*(t)|_{\tau_1}^{\tau_2}. \tag{3.25}
\]

Proof  Under the stated assumptions, the results in (3.23), (3.24), and (3.25) follow from Proposition 3.6 and Lemma 3.5.

Our main result presented below shows that the oscillation number and the dual oscillation number of $Y$ on $[a, b]$ count the cumulative change in the differences of the integers $q_j(t)$ and of the integers $q_j^*(t)$ at the endpoints of the interval $[a, b]$.

Theorem 3.8  Let $Y$ be a continuous Lagrangian path on $[a, b]$ with the associated Lidskii angles $\varphi_j(t)$ for $j \in \{1, \ldots, n\}$ on $[a, b]$ according to Definition 3.4. Given the integers $q_j(t)$ and $q_j^*(t)$ satisfying conditions (3.12) and (3.13), then the oscillation number of $Y$ on $[a, b]$ and the dual oscillation number of $Y$ on $[a, b]$ satisfy
\[
N(Y, [a, b]) = \sum_{j=1}^{n} (q_j(b) - q_j(a)), \tag{3.26}
\]
\[
N^*(Y, [a, b]) = \sum_{j=1}^{n} (q_j^*(b) - q_j^*(a)). \tag{3.27}
\]

Proof  According to Definition 3.2 we choose a partition $D = \{t_k\}_{k=0}^{p}$ of the interval $[a, b]$ and a system $R = \{R_k(t)\}_{k=0}^{p-1}$ of symplectic matrix-valued functions such that (3.1) holds. By (3.25) in Corollary 3.7 applied to $\hat{Y}(t) := R_k(t)E$ and $\tau_1 := t_k$, $\tau_2 := t_{k+1}$ for $k \in \{0, 1, \ldots, p - 1\}$ we have for the variations of the comparative indices in (3.2) and (3.3) that
\[
\mu(Y(t), R_k(t)E)|_{t_k}^{t_{k+1}} = \sum_{j=1}^{n} q_j(t)|_{t_k}^{t_{k+1}},
\]
\[
\mu^*(Y(t), R_k(t)E)|_{t_k}^{t_{k+1}} = \sum_{j=1}^{n} q_j^*(t)|_{t_k}^{t_{k+1}}, \tag{3.28}
\]

$\square$
where we used the facts that the rank of the upper block of \( R_k(t)E \) and rank of the Wronskian \( W(R_k(t)E, Y(t)) \) are constant for \( t \in [t_k, t_{k+1}] \) according to (3.1). By the first equation in (3.28) we obtain, with the telescope summation (and using \( t_0 = a \) and \( t_p = b \)),

\[
\mathcal{N}(Y, [a, b]) \stackrel{(3.2)}{=} \sum_{k=0}^{p-1} \mu(Y(t), R_k(t)E)|_{t_k}^{t_{k+1}} (3.28) \equiv \sum_{j=1}^{n} \sum_{k=1}^{p-1} q_j(t)|_{t_k}^{t_{k+1}} = n \sum_{j=1}^{n} (q_j(b) - q_j(a)),
\]

which completes the proof of equation (3.26). Analogously, by the second equality in (3.28) and by (3.3) we obtain with the telescope summation that

\[
\mathcal{N}^*(Y, [a, b]) \stackrel{(3.3)}{=} \sum_{k=0}^{p-1} \mu^*(Y(t), R_k(t)E)|_{t_k}^{t_{k+1}} (3.28) \equiv \sum_{j=1}^{n} \sum_{k=1}^{p-1} q_j^*(t)|_{t_k}^{t_{k+1}} = n \sum_{j=1}^{n} (q_j^*(b) - q_j^*(a)),
\]

which completes the proof of equation (3.27).

\[ \Box \]

**Remark 3.9** The results in Theorem 3.8 confirm that the values in (3.2) and (3.3) indeed do not depend on the chosen partition \( D = \{t_k\}_{k=0}^{p} \) of \([a, b]\) as well as on the chosen system \( R = \{R_k(t)\}_{k=0}^{p-1} \) of symplectic matrix-valued functions satisfying (3.7).

As a consequence of Theorem 3.8 we obtain the value of the oscillation number and the dual oscillation number of \( Y \), when the upper block \( X(t) \) of \( Y(t) \) has constant rank on \([a, b]\), see also [19, Proposition 3.2(ii)] and [21, Proposition 3.3(iii)].

**Corollary 3.10** Assume that \( Y \) is a continuous Lagrangian path on \([a, b]\) such that its upper block \( X(t) \), according to the notation in (1.2), has constant rank on \([a, b]\). Then we have

\[
\mathcal{N}(Y, [a, b]) = 0, \quad \mathcal{N}^*(Y, [a, b]) = 0. \tag{3.29}
\]

**Proof** The result follows from Theorem 3.8 and from Lemma 3.5 (with the choice of \( \tau_1 := a \) and \( \tau_2 := b \)). \( \Box \)

The following result presents the invariance of the oscillation number and the dual oscillation number under a special continuous symplectic transformation.

**Corollary 3.11** Let \( Y \) be a continuous Lagrangian path on \([a, b]\) and let \( L : [a, b] \rightarrow \mathbb{R}^{2n \times 2n} \) be a continuous symplectic lower block triangular matrix-valued function, i.e., the matrix \( L(t) \) has the form as in (2.5). Then we have

\[
\mathcal{N}(Y, [a, b]) = \mathcal{N}(LY, [a, b]), \quad \tag{3.30}
\]

\[
\mathcal{N}^*(Y, [a, b]) = \mathcal{N}^*(LY, [a, b]). \quad \tag{3.31}
\]

**Proof** The main role in the proof is played by the invariant property of the comparative index with respect to block lower triangular symplectic transformations, see (2.5). Obviously, \( LY \) is a continuous Lagrangian path on \([a, b]\). Under the assumptions and the notation of Theorem 3.8, we show that equations (3.26) and (3.27) are also valid for \( \mathcal{N}(LY, [a, b]) \) and
Applying Definition 3.2 to the path $LY$ we choose a partition $D = \{t_k\}_{k=0}^p$ of $[a, b]$ and a system $\tilde{R} := \{\tilde{R}_k(t)\}_{k=0}^{p-1}$ of symplectic matrix-valued functions $\tilde{R}_k(t)$ such that

$$\text{rank } W(\tilde{R}_k(t), E, L(t)Y(t)) \text{ and rank } ((I 0) \tilde{R}_k(t), E) \text{ are constant on } [t_k, t_{k+1}] (3.32)$$

for every $k \in \{0, \ldots, p - 1\}$. By using the assumption that the matrix $L(t)$ is symplectic block lower triangular one can rewrite (3.32) in the equivalent form

$$W(L^{-1}(t) \tilde{R}_k(t), E, Y(t)) \text{ and rank } ((I 0) L^{-1}(t) \tilde{R}_k(t), E) \text{ are constant on } [t_k, t_{k+1}], (3.33)$$

where we used that $\mathcal{J}L(t) = L^{-1}(t)\mathcal{J}$.

It follows from (3.33) that one can use the same partition for $L(t)Y(t)$ and $Y(t)$ with the corresponding transformation matrices $\tilde{R}_k(t)$ and $L^{-1}(t) \tilde{R}_k(t)$. Finally, by (2.5) we have, instead of (3.28), the equalities

$$\mu(L(t)Y(t), \tilde{R}_k(t)E)|_{t_k}^{t_{k+1}} (2.5) = \mu(Y(t), L^{-1}(t) \tilde{R}_k(t)E)|_{t_k}^{t_{k+1}} = \sum_{j=1}^n \mu_j(t)|_{t_k}^{t_{k+1}},$$

$$\mu^*(L(t)Y(t), \tilde{R}_k(t)E)|_{t_k}^{t_{k+1}} (2.5) = \mu^*(Y(t), L^{-1}(t) \tilde{R}_k(t)E)|_{t_k}^{t_{k+1}} = \sum_{j=1}^n \mu^*_j(t)|_{t_k}^{t_{k+1}},$$

where by analogy with the proof of Theorem 3.8 we applied Corollary 3.7 with the matrix $\hat{Y}(t) := L^{-1}(t) \tilde{R}_k E$. Summing the equalities derived above for all $k \in \{0, \ldots, p - 1\}$ we complete the proof of (3.30) and (3.31).

Based on Corollary 3.11 we are able to prove the invariance of the oscillation number and the dual oscillation number in the sense that every continuous symplectic transformation of a continuous Lagrangian path can be realized with a special continuous symplectic and orthogonal transformation.

**Theorem 3.12** Let $Y$ be a continuous Lagrangian path on $[a, b]$ and let $S(t)$ be a continuous symplectic matrix on $[a, b]$. Then we have the equalities

$$\mathcal{N}(S^{-1} Y, [a, b]) = \mathcal{N}(Z_{SE}^{-1} Y, [a, b]), (3.34)$$

$$\mathcal{N}^*(S^{-1} Y, [a, b]) = \mathcal{N}^*(Z_{SE}^{-1} Y, [a, b]), (3.35)$$

where the symplectic and orthogonal matrix $Z_{SE}(t) = Z_{S(t)E}$ is defined according to (1.4).

**Proof** By property (2.14) in Remark 2.2 we have $S(t) = Z_{S(t)E} L(t)$, where $L(t)$ is a continuous symplectic block lower triangular matrix. Then in view of Corollary 3.11 we have

$$\mathcal{N}(S^{-1} Y, [a, b]) = \mathcal{N}(L^{-1} Z_{SE}^{-1} Y, [a, b]) (3.30) = \mathcal{N}(Z_{SE}^{-1} Y, [a, b]),$$

$$\mathcal{N}^*(S^{-1} Y, [a, b]) = \mathcal{N}^*(L^{-1} Z_{SE}^{-1} Y, [a, b]) (3.31) = \mathcal{N}^*(Z_{SE}^{-1} Y, [a, b]),$$

which completes the proof.

Next we present a formula relating the oscillation number $\mathcal{N}(Y, [a, b])$ and the dual oscillation number $\mathcal{N}^*(Y, [a, b])$, see also [21, Proposition 3.3(iv)] and [40, Theorem 5.1].

**Theorem 3.13** Let $Y$ be a continuous Lagrangian path on $[a, b]$. Then the oscillation number and the dual oscillation number of $Y$ on $[a, b]$ are related by the formula

$$\mathcal{N}(Y, [a, b]) + \text{rank } X(b) = \mathcal{N}^*(Y, [a, b]) + \text{rank } X(a), (3.36)$$

where $X(t)$ is the upper block of $Y(t)$ as in (1.2).
Proof The result follows from Theorem 3.8 and from the relationship between the integers \( q_j(t) \) and \( q_j^*(t) \) in (3.15). Namely, we have

\[
N(Y, [a, b]) + \text{rank } X(b) \overset{(3.26)}{=} \text{rank } X(b) + \sum_{j=1}^{n} (q_j(b) - q_j(a))
\]

\[
\overset{(3.15)}{=} \text{rank } X(b) + \text{def } X(b) - \text{def } X(a) + \sum_{j=1}^{n} (q_j^*(b) - q_j^*(a))
\]

\[
\overset{(3.27)}{=} N^*(Y, [a, b]) + \text{rank } X(a),
\]

which proves the result in (3.36).

Corollary 3.14 Let \( S(t) \) be a continuous symplectic matrix on \([a, b]\), which is partitioned into \( n \times n \) blocks as in (2.7). Then we have the equalities

\[
N(SE, [a, b]) + N(S^{-1}E, [a, b]) = \text{rank } S_{12}(a) - \text{rank } S_{12}(b), \tag{3.37}
\]

\[
N^*(SE, [a, b]) + N^*(S^{-1}E, [a, b]) = \text{rank } S_{12}(b) - \text{rank } S_{12}(a). \tag{3.38}
\]

Proof According to (3.4), for the continuous Lagrangian path \( Y(t) := S(t)E \) on \([a, b]\) we have

\[
N(SE, [a, b]) = -N^*(S^{-1}E, [a, b]), \quad N^*(SE, [a, b]) = -N(S^{-1}E, [a, b]). \tag{3.39}
\]

Equations (3.37) and (3.38) now follow directly from formula (3.36) (with \( Y(t) := S(t)E \) and \( X(t) := S_{12}(t) \)) and from the relations in (3.39). The proof is complete.

In the final part of this section we will discuss some additional properties of the oscillation number and the dual oscillation number, which are based on Theorem 3.8. At the first place we obtain the following representations of the changes in the corresponding special arguments \( \text{Arg}_{[0,2\pi]} \) and \( \text{Arg}_{(0,2\pi]} \) of the matrix \( Z_Y(t) \).

Corollary 3.15 Let \( Y \) be a continuous Lagrangian path on \([a, b]\) and let \( Z_Y(t) \) be the continuous symplectic and orthogonal matrix defined in (1.4). Then we have the representations

\[
\text{Arg}_{[0,2\pi]}(Z_Y(b)) - \text{Arg}_{[0,2\pi]}(Z_Y(a)) = \text{Arg}_3(Z_Y(b)) - \text{Arg}_3(Z_Y(a)) - \pi N(Y, [a, b]), \tag{3.40}
\]

\[
\text{Arg}_{(0,2\pi]}(Z_Y(b)) - \text{Arg}_{(0,2\pi]}(Z_Y(a)) = \text{Arg}_3(Z_Y(b)) - \text{Arg}_3(Z_Y(a)) - \pi N^*(Y, [a, b]). \tag{3.41}
\]

Proof The result in (3.40) follows by combining formula (3.16) at \( t = a \) and \( t = b \) with (3.26) in Theorem 3.8, while the result in (3.41) follows by combining formula (3.17) at \( t = a \) and \( t = b \) with (3.27) in Theorem 3.8.

The representations of the oscillation number and the dual oscillation number in Theorem 3.8 yield their additive property with respect to a block diagonal structure of the components \( X \) and \( U \) of the Lagrangian path \( Y \), also called a direct symplectic sum. More precisely, consider the dimensions \( n_1, n_2 \in \mathbb{N} \) and the permutation matrix

\[
\Pi := \begin{pmatrix}
I_{n_1} & 0 & 0 \\
0 & I_{n_2} & 0 \\
0 & I_{n_1} & 0 \\
0 & 0 & I_{n_2}
\end{pmatrix}, \tag{3.42}
\]
of dimension $2(n_1 + n_2) \times 2(n_1 + n_2)$. Here $I_k$ denotes the $k \times k$ identity matrix. The following result will be useful in particular for the construction of higher dimensional examples. We recall the convention that $\text{diag}(A, B)$ denotes the block diagonal matrix with the matrices $A$ and $B$ on the diagonal.

**Theorem 3.16** Assume that $Y_1 = (X_1^T, U_1^T)^T$ and $Y_2 = (X_2^T, U_2^T)^T$ are continuous Lagrangian paths on $[a, b]$ with values in $\mathbb{R}^{2n_1 \times n_1}$ and $\mathbb{R}^{2n_2 \times n_2}$, respectively. Consider the continuous Lagrangian path on $[a, b]$ defined by

$$ Y := \Pi \text{diag}(Y_1, Y_2) = \left(\begin{array}{c} \text{diag}(X_1, X_2) \\ \text{diag}(U_1, U_2) \end{array}\right) $$

(3.43)

with values in $\mathbb{R}^{2n \times n}$, where $n := n_1 + n_2$ and where the matrix $\Pi$ is given by (3.42). Then

$$ N(Y, [a, b]) = N(Y_1, [a, b]) + N(Y_2, [a, b]), \quad (3.44) $$

$$ N^*(Y, [a, b]) = N^*(Y_1, [a, b]) + N^*(Y_2, [a, b]). \quad (3.45) $$

**Proof** According to (1.4) applied to the Lagrangian paths $Y_1, Y_2$, and to the Lagrangian path $Y$ defined in (3.43), for the corresponding symplectic and orthogonal matrices $S_1(t) := Z_{Y_1}(t), S_2(t) := Z_{Y_2}(t)$, and $S(t) := Z_Y(t)$ we have

$$ S(t) = \Pi \text{diag}\{S_1(t), S_2(t)\} \Pi, \quad t \in [a, b]. $$

This implies that the matrices $W_{S_1}(t)$, $W_{S_2}(t)$, and $W_S(t)$ defined through equation (2.8) satisfy

$$ W_S(t) = \text{diag}\{W_{S_1}(t), W_{S_2}(t)\}, \quad t \in [a, b]. \quad (3.46) $$

Formula (3.46) shows that the Lidskii angles $\varphi_j(t)$ for $j \in \{1, \ldots, n\}$ of the Lagrangian path $Y$ according to Definition 3.4 consists exactly of the Lidskii angles $\varphi^{[1]}_j(t)$ for $j \in \{1, \ldots, n_1\}$ and $\varphi^{[2]}_j(t)$ for $j \in \{1, \ldots, n_2\}$ of the Lagrangian paths $Y_1$ and $Y_2$. Therefore, equalities (3.44) and (3.45) follow respectively from formula (3.26) for the oscillation numbers $N(Y, [a, b]), N(Y_1, [a, b]), N(Y_2, [a, b])$ and from formula (3.27) for the dual oscillation numbers $N^*(Y, [a, b]), N^*(Y_1, [a, b]), N^*(Y_2, [a, b]).$ \hfill $\square$

## 4 Oscillation Numbers and Maslov Index

In this section we make a connection of the oscillation number and the dual oscillation number of two continuous Lagrangian paths $Y$ and $\hat{Y}$ on $[a, b]$ with the Maslov index. Here we use the definition of the Maslov index $\text{Mas}(Y, \hat{Y}, [a, b])$ from [5, Definition 1.5], which we recall.

Let $Y$ and $\hat{Y}$ be continuous Lagrangian paths on $[a, b]$ with their partitions into $n \times n$ blocks as in (1.2). According to [28, Section 1], we consider the complex $n \times n$ matrix

$$ \Gamma(t) := -[X(t) + iU(t)][X(t) - iU(t)]^{-1} [\hat{X}(t) - i\hat{U}(t)] \times $$

$$ \times [\hat{X}(t) + i\hat{U}(t)]^{-1}, \quad t \in [a, b]. $$

(4.1)

Then the matrix $\Gamma(t)$ is well-defined, continuous, and unitary on $[a, b]$. The last property also follows from the proof of Lemma 4.1 below, see equation (4.7). Let $\gamma_j(t)$ for $j \in \{1, \ldots, n\}$ be the eigenvalues of the matrix $\Gamma(t)$, which are continuous on $[a, b]$ and lie on the unit circle $\mathbb{U}$ in the complex plane. Let us fix a point $\tau_0 \in [a, b]$. Since the matrix $\Gamma(\tau_0)$ has $n$ (i.e., finitely
many) eigenvalues \( \gamma_j(\tau_0) \), there exists a neighborhood \( \mathcal{O}(\tau_0) \) of \( \tau_0 \) and a number \( \varepsilon \in (0, \pi) \) such that the matrix \( \Gamma(t) - \exp \{ i(\pi \pm \varepsilon) \} I \) is invertible for all \( t \in \mathcal{O}(\tau_0) \). This implies by the compactness of \([a, b]\) that there exists a finite partition \( D = \{a = t_0 < t_1 \cdots < t_p = b\} \) of \([a, b]\) along with numbers \( \varepsilon_k \in (0, \pi) \) such that for every \( k \in \{1, \ldots, p\} \) the matrix \( \Gamma(t) - \exp \{ i(\pi \pm \varepsilon_k) \} I \) is invertible for all \( t \in (t_{k-1}, t_k) \), i.e., \( \exp \{ i(\pi \pm \varepsilon_k) \} \) is not an eigenvalue of \( \Gamma(t) \) for \( t \in (t_{k-1}, t_k) \). Moreover, for each \( t \in (t_{k-1}, t_k) \) there are at most \( n \) angles \( \theta \in [0, \varepsilon_k] \subseteq [0, \pi) \) such that \( \exp \{ i(\pi + \theta) \} \) is an eigenvalue of \( \Gamma(t) \). This allows to define the number

\[
\ell(t, \varepsilon_k) := \sum_{\theta \in [0, \varepsilon_k]} \text{def} \left( \Gamma(t) - \exp \{ i(\pi + \theta) \} I \right), \quad t \in [t_{k-1}, t_k]. \tag{4.2}
\]

Hence, \( \ell(t, \varepsilon_k) \) is equal to the number of the eigenvalues of \( \Gamma(t) \), which lie on the arc

\[
A_k := \{ \exp \{ i \theta \}, \; \theta \in [\pi, \pi + \varepsilon_k) \}
\]

of the unit circle \( \mathbb{U} \). By [5, Definition 1.5] or [28, Definition 1.4] the Maslov index of the continuous Lagrangian paths \( Y \) and \( \hat{Y} \) is defined as the integer number

\[
\text{Mas}(Y, \hat{Y}, [a, b]) := \sum_{k=1}^{p} \left( \ell(t_k, \varepsilon_k) - \ell(t_{k-1}, \varepsilon_k) \right). \tag{4.3}
\]

Note that this definition does not depend on the choice of the partition \( D = \{t_k\}_{k=0}^{p} \) of \([a, b]\) and on the choice of the numbers \( \varepsilon_k \), as long as they satisfy the above properties, see e.g. [5, pg. 10]. This property will also follow from our main result below (see Theorem 4.2 and Remark 4.3).

**Lemma 4.1** Let \( Y \) and \( \hat{Y} \) be continuous Lagrangian paths on \([a, b]\) and define the matrix \( \Gamma(t) \) by (4.1). Consider the symplectic matrix

\[
S(t) := Z_Y^{-1}(t) Z_{\hat{Y}}(t) = Z_Y^T(t) Z_{\hat{Y}}(t), \quad t \in [a, b], \tag{4.4}
\]

where \( Z_Y(t) \) and \( Z_{\hat{Y}}(t) \) are the symplectic and orthogonal matrices associated with \( Y(t) \) and \( \hat{Y}(t) \) through (1.4). Moreover, consider the unitary and symmetric matrix \( W_S(t) := W_{S(t)} \) on \([a, b]\) defined in (2.8) which is associated with the above matrix \( S(t) \), i.e.,

\[
W_S(t) = K_{\hat{Y}}^{-1}(t) \left[ Y^T(t) \hat{Y}(t) + i W(Y(t), \hat{Y}(t)) \right]^{-1} \times
\]

\[
\times \left[ Y^T(t) \hat{Y}(t) - i W(Y(t), \hat{Y}(t)) \right] K_{\hat{Y}}(t). \tag{4.5}
\]

Then for each \( t \in [a, b] \) the matrices \( W_S(t) \) and \( -\Gamma(t) \) are similar.

**Proof** Let us fix any \( t \in [a, b] \). For brevity we will suppress the argument \( t \) in the following calculations. Consider the auxiliary Lagrangian paths

\[
Y_* := YK_Y = \begin{pmatrix} X_* \\ U_* \end{pmatrix}, \quad \hat{Y}_* := \hat{Y}K_{\hat{Y}} = \begin{pmatrix} \hat{X}_* \\ \hat{U}_* \end{pmatrix}, \tag{4.6}
\]

which form the second block columns of the matrices \( Z_Y \) and \( Z_{\hat{Y}} \), i.e., \( Y_* = Z_Y E \) and \( \hat{Y}_* = Z_{\hat{Y}} E \). Since \( X_* = XK_Y, U_* = UK_Y \) and \( \hat{X}_* = \hat{X}K_{\hat{Y}}, \hat{U}_* = \hat{U}K_{\hat{Y}} \), where the matrices \( K_Y \) and \( K_{\hat{Y}} \) are invertible, it follows from the definition of \( \Gamma \) in (4.1) that

\[
\Gamma = -(X_* + i U_*)(X_* - i U_*)^{-1} (\hat{X}_* - i \hat{U}_*) (\hat{X}_* + i \hat{U}_*)^{-1}. \tag{4.7}
\]
Note that the orthogonality of the matrices $Z_Y$ and $Z_{\hat{Y}}$ yields that
\[
(X_* + i U_*)^{-1} = X_*^T \mp i U_*^T, \quad (\hat{X}_* + i \hat{U}_*)^{-1} = \hat{X}_*^T \mp i \hat{U}_*^T.
\] (4.8)

We express the matrix $S$ in the form
\[
S = (4.4) \begin{pmatrix} K_Y Y^T \hat{Y} K_{\hat{Y}} & -K_Y W(Y, \hat{Y}) K_{\hat{Y}} \\ K_Y W(Y, \hat{Y}) K_{\hat{Y}} & K_Y Y^T \hat{Y} K_{\hat{Y}} \end{pmatrix} = (4.6) \begin{pmatrix} Y_*^T \hat{Y}_* & -W(Y_*, \hat{Y}_*) \\ W(Y_*, \hat{Y}_*) & Y_*^T \hat{Y}_* \end{pmatrix}. \] (4.9)

Then according to the form of the matrix $W_S$ in (4.5) we have
\[
W_S = (4.9) \begin{pmatrix} Y_*^T \hat{Y}_* + i W(Y_*, \hat{Y}_*) \end{pmatrix}^{-1} \begin{pmatrix} Y_*^T \hat{Y}_* - i W(Y_*, \hat{Y}_*) \end{pmatrix} = L^{-1} \tilde{L}^T,
\] (4.10)

where the $n \times n$ matrix $L$ is defined by
\[
L := \hat{Y}_*^T Y_* - i W(Y_*, \hat{Y}_*) = [Y_*^T \hat{Y}_* + i W(Y_*, \hat{Y}_*)]^T, \quad L^{-1} = \tilde{L}^T. \] (4.11)

This means that $L$ is a unitary matrix. Then we obtain that
\[
-L = (4.7) (X_* + i U_*) (X_* - i U_*)^{-1} (\hat{X}_* - i \hat{U}_*) (\hat{X}_* + i \hat{U}_*)^{-1}
\]
\[
= (X_* + i U_*) \left( Y_*^T \hat{Y}_* - i W(Y_*, \hat{Y}_*) \right) \left[ \hat{Y}_*^T Y_* + i W(Y_*, Y_*) \right] (X_* + i U_*)^{-1}
\]
\[
= (X_* + i U_*) L^{-1} \tilde{L} (X_* + i U_*)^{-1} \equiv (X_* + i U_*) L^{-1} W_S L (X_* + i U_*)^{-1}, \] (4.10)

where in the last step we used the symmetry of the matrix $W_S$. Therefore, we showed that the matrices $-\Gamma$ and $W_S$ are similar, which completes the proof of this lemma.

Based on the above preliminary considerations we can now prove the main result of this section, which connects the Maslov index with the Lidskii angles and hence with the oscillation number through equation (1.5). More precisely, the Maslov index of $Y$ and $\hat{Y}$ over the interval $[a, b]$ can be calculated as the total change in the interval $[a, b]$ of the integers $q_j(t)$, which are associated through (3.12) or (3.14) with the continuous Lidskii angles $\varphi_j(t)$ of the symplectic matrix $S(t)$ defined in (4.4), compare with [50, Definition 2.2] and [6, Eqs. (2.4)–(2.5)]. Consequently, the Maslov index is equal to the oscillation number of the transformed Lagrangian path $Z_{\hat{Y}}$ on $[a, b]$.

**Theorem 4.2** Let $Y$ and $\hat{Y}$ be continuous Lagrangian paths on $[a, b]$ and let the symplectic and orthogonal matrices $Z_Y(t)$, $Z_{\hat{Y}}(t)$ together with the invertible matrices $K_Y(t)$, $K_{\hat{Y}}(t)$ be defined according to (1.4) on $[a, b]$. Then
\[
\text{Mas}(Y, \hat{Y}, [a, b]) = \sum_{j=1}^n (q_j(b) - q_j(a)).
\] (4.12)

where $q_j(t)$ are the integers associated through (3.12) with the continuous Lidskii angles $\varphi_j(t)$ of the symplectic matrix $S(t)$ defined in (4.4). Consequently, for any continuous symplectic matrix $Z(t)$ such that $Y(t) = Z(t) E$ on $[a, b]$ we have
\[
\text{Mas}(Y, \hat{Y}, [a, b]) = \mathcal{N}(Z^{-1} \hat{Y}, [a, b]) = \mathcal{N}(\hat{Y}, [a, b]),
\]
\[
\hat{Y} := Z_{\hat{Y}}^{-1} \hat{Y} = \begin{pmatrix} -K_Y W(Y, \hat{Y}) \\ K_Y Y^T \hat{Y} \end{pmatrix}. \] (4.13)
Proof Let the matrices $Z_Y(t)$, $Z_{\hat{Y}}(t)$ and $K_Y(t)$, $K_{\hat{Y}}(t)$ be as in the theorem. Since the matrix $Z_Y(t)$ is orthogonal, the form of the transformed Lagrangian path $\hat{Y}$ defined in (4.13) follows from the same calculation as in (4.9). Consider the matrix $\Gamma(t)$ defined in (4.1) on $[a, b]$ and a partition $D = \{t_k\}_{k=0}^p$ of $[a, b]$ with the numbers $\varepsilon_k \in (0, \pi)$, which are used in (4.2) in order to define the numbers $\ell(t, \varepsilon_k)$. With the symplectic matrix $S(t)$ in (4.4) we consider the corresponding matrix $W_S(t)$ in (4.5) on $[a, b]$. Denote by $\varphi_j(t)$ for $j \in \{1, \ldots, n\}$ the continuous Lidskii angles of the symplectic matrix $S(t)$ on $[a, b]$ with the corresponding integers $q_j(t)$ satisfying (3.12). Since by Lemma 4.1 the matrices $\Gamma(t)$ and $-W_S(t)$ are similar for all $t \in [a, b]$, then these matrices have the same eigenvalues. And since the arguments of the eigenvalues of the matrices $-W_S(t)$ and $W_S(t)$ differ by $\pi$, we obtain from (4.2) that

$$\ell(t, \varepsilon_k) = \sum_{\theta \in [0, \varepsilon_k]} \text{def} \left( W_S(t) - \exp(i \theta) I \right), \quad t \in [t_{k-1}, t_k].$$  \hfill (4.14)

Equation (4.14) shows that $\ell(t, \varepsilon_k)$ is equal to the number of the eigenvalues of the matrix $W_S(t)$, which lie on the arc $B_k := \{ \exp(i \theta), \; \theta \in [0, \varepsilon_k] \}$ of the unit circle $U$. Hence, the changes (i.e., incrementing or decrementing) of the integers $\ell(t, \varepsilon_k)$ when the eigenvalues of $\Gamma(t)$ pass through $-1$, as it is commented in [28, pp. 796–797], can be calculated by the same changes when the eigenvalues of $W_S(t)$ pass through 1. More precisely, the number $\ell(t, \varepsilon_k)$ increases by one if and only if there is a corresponding Lidskii angle $\varphi_j(t)$ of the matrix $S(t)$, which leaves an integer multiple of 2$\pi$ in the downward direction. Consequently, the sum of the numbers $q_j(t)$ increases or decreases in the interval $[t_{k-1}, t_k]$ by the same amount as $\ell(t, \varepsilon_k)$, and thus

$$\ell(t_k, \varepsilon_k) - \ell(t_{k-1}, \varepsilon_k) = \sum_{j=1}^{n} (q_j(t_k) - q_j(t_{k-1})).$$ \hfill (4.15)

By the definition of the Maslov index of $Y$ and $\hat{Y}$ on $[a, b]$ in (4.3) we then obtain

$$\text{Mas}(Y, \hat{Y}, [a, b]) \stackrel{(4.15)}{=} \sum_{k=1}^{p} \sum_{j=1}^{n} (q_j(t_k) - q_j(t_{k-1})) = \sum_{j=1}^{n} (q_j(b) - q_j(a)), \quad \text{Mas}(\hat{Y}, [a, b]) \stackrel{(3.10)}{=} \mathcal{N}(\hat{Y}, [a, b]),$$ \hfill (4.16)

which shows the result in (4.12). By combining equality (4.16) and (3.26) in Theorem 3.8 (with the continuous Lagrangian path $Y := SE = \hat{Y}K_{\hat{Y}}$) we conclude that

$$\text{Mas}(Y, \hat{Y}, [a, b]) = \sum_{j=1}^{n} (q_j(b) - q_j(a)) = \mathcal{N}(\hat{Y}K_{\hat{Y}}, [a, b]) = \mathcal{N}(\hat{Y}, [a, b]),$$

where in the last step we used the invariance of the oscillation number with respect to the multiplicity of $\hat{Y}$ by a continuous invertible $n \times n$ matrix function from the right (see Remark 3.3(i)). Therefore, the second equality in (4.13) is proved. Next we apply Theorem 3.12 with $S(t) := Z(t)$ to prove the first equality in (4.13) for an arbitrary continuous symplectic matrix $Z(t)$ such that $Z(t) E = Y(t)$. The proof is complete. \hfill \square

Remark 4.3 (i) Equation (4.12) also proves the independence of the value \text{Mas}(Y, \hat{Y}, [a, b]) defined in (4.3) on the choice of the therein used partition $D = \{t_k\}_{k=0}^p$ and numbers $\varepsilon_k$.\hfill \s

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(ii) According to the definition in (4.1) the Maslov index is invariant under the multiplication of its arguments $Y$ and $\hat{Y}$ by invertible $n \times n$ matrices from the right, i.e.,

$$\text{Mas}(YC, \hat{Y}\hat{C}, [a, b]) = \text{Mas}(Y, \hat{Y}, [a, b])$$

for arbitrary continuous nonsingular matrices $C(t)$ and $\hat{C}(t)$ on $[a, b]$. This fact also follows from formula (4.13) and from the invariant properties of the oscillation numbers according to Remark 3.3(i) and Corollary 3.11. In particular, the symplectic matrix $Z(t)$ in Theorem 4.2 can be chosen to be in a more general form $Z(t) E = Y(t) C(t)$ on $[a, b]$.

The result in Theorem 4.2 leads to the evaluation of the oscillation number in terms of the Maslov index, as we announced in equation (1.6). It shows that the oscillation number $N(Y, [a, b])$ is precisely the Maslov index detecting the intersections between the Lagrangian subspaces associated with $Y(t)$ on $[a, b]$ and the Dirichlet target space.

**Corollary 4.4** Let $Y$ be a continuous Lagrangian path on $[a, b]$. Then

$$N(Y, [a, b]) = \text{Mas}(E, Y, [a, b]), \quad (4.17)$$

where the matrix $E$ defined in (2.3) represents the constant vertical Lagrangian path.

**Proof** Formula (4.17) follows from equation (4.13) in Theorem 4.2 (with $Y := E$ and $\hat{Y} := Y$), since in this case the matrix $Z_Y(t) = Z_E(t) \equiv I$ on $[a, b]$. \hfill $\Box$

Next we discuss a dual notion to the Maslov index $\text{Mas}(Y, \hat{Y}, [a, b])$, which reflects the results regarding the dual oscillation numbers obtained in Sects. 3 and 5.

**Remark 4.5** Let $Y$ and $\hat{Y}$ be two given continuous Lagrangian paths on $[a, b]$. Assume that $D = \{a = t_0 < t_1 < \cdots < t_p = b\}$ is a partition of $[a, b]$, which is used along with numbers $\varepsilon_k \in (0, \pi)$ in the definition of the Maslov index $\text{Mas}(Y, \hat{Y}, [a, b])$ in (4.3). Then instead of the numbers $\ell(t, \varepsilon_k)$ defined in (4.2) we consider the numbers

$$\ell^* (t, \varepsilon_k) := \sum_{\theta \in [0, \varepsilon_k]} \text{def} \left( \Gamma(t) - \exp \{i (\pi - \theta) \} I \right), \quad t \in [t_{k-1}, t_k]. \quad (4.18)$$

This means that $\ell^* (t, \varepsilon_k)$ is equal to the number of the eigenvalues of $\Gamma(t)$, which lie on the arc $A^*_{\varepsilon_k} := \{ \exp (i \theta), \theta \in (\pi - \varepsilon_k, \pi) \}$ of the unit circle $U$. By analogy with (4.3) we define the dual Maslov index of the continuous Lagrangian paths $Y$ and $\hat{Y}$ as the integer

$$\text{Mas}^*(Y, \hat{Y}, [a, b]) := \sum_{k=1}^{p} \left( \ell^*(t_{k-1}, \varepsilon_k) - \ell^*(t_k, \varepsilon_k) \right). \quad (4.19)$$

Then similarly as in Theorem 4.2 we obtain that for an arbitrary continuous symplectic matrix $Z(t)$ associated with $Y(t)$ via the condition $Y(t) = Z(t) E$

$$\text{Mas}^*(Y, \hat{Y}, [a, b]) = N^*(Z^{-1} \hat{Y}, [a, b]) = N^*(Z^{-1} \hat{Y}, [a, b]). \quad (4.20)$$

In view of (3.4), the representation formulas (4.13) and (4.20) for the Maslov index and the dual Maslov index imply that

$$\text{Mas}^*(Y, \hat{Y}, [a, b]) = N^*(Z^{-1} \hat{Y}, [a, b]) = - \text{Mas}(\hat{Y}, Y, [a, b]), \quad (4.21)$$

where the continuous symplectic matrix $\hat{Z}(t)$ is such that $\hat{Y}(t) = \hat{Z}(t) E$. This yields, in view of Theorem 3.8, the geometric interpretation of the dual Maslov index $\text{Mas}^*(Y, \hat{Y}, [a, b])$ as...
The main idea of the proof is to construct a linear Hamiltonian system in form (H)

\[ \text{Mas}^\ast(Y, \hat{Y}, [a, b]) = \sum_{j=1}^{n} (q_j^\ast(b) - q_j^\ast(a)). \] (4.22)

From (4.20) we then obtain the dual version of Corollary 4.4 in the form

\[ \mathcal{N}^\ast(Y, [a, b]) = \text{Mas}^\ast(E, Y, [a, b]). \] (4.23)

Finally, from Theorem 3.13 we obtain the equality

\[ \text{Mas}^\ast(Y, \hat{Y}, [a, b]) = \text{Mas}(Y, \hat{Y}, [a, b]) + \text{rank } W(Y(b), \hat{Y}(b)) - \text{rank } W(Y(a), \hat{Y}(a)), \] (4.24)

or in combination with (4.21) the equalities

\[ \text{Mas}(\hat{Y}, Y, [a, b]) = -\text{Mas}(Y, \hat{Y}, [a, b]) - \text{rank } W(Y(b), \hat{Y}(b)) + \text{rank } W(Y(a), \hat{Y}(a)), \] (4.25)

\[ \text{Mas}^\ast(\hat{Y}, Y, [a, b]) = -\text{Mas}^\ast(Y, \hat{Y}, [a, b]) + \text{rank } W(Y(b), \hat{Y}(b)) - \text{rank } W(Y(a), \hat{Y}(a)). \] (4.26)

In view of formulas (4.17) and (4.23), the results in (3.29) and (4.21), (4.24)–(4.26) correspond to the vanishing property and to the flipping property (or the symmetry) of the Maslov index and the dual Maslov index, see [6, Proposition 2.3.1(e), (f)] and [8, Property XI, pg. 130].

In [17, Theorem 2.3], [19, Propositions 3.2 and 3.4], and [21, Proposition 3.3(v)] we proved that the oscillation number and the dual oscillation number count the left and right proper focal points of a conjoined basis $Y$ of system (H) satisfying the Legendre condition (1.3). Below we extend these properties to the case when $Y$ is an arbitrary piecewise continuously differentiable Lagrangian path on $[a, b]$. For this purpose we recall that if $A(t)$ is a given matrix defined on the interval $[a, b]$, then $\text{rank } A(t)$ denote the left-hand and the right-hand limits of the quantity $\text{rank } A(t)$ at the point $t_0 \in [a, b]$.

**Theorem 4.6** Let $Y$ be a piecewise continuously differentiable Lagrangian path on $[a, b]$ with the partition as in (1.2) and assume that

\[ [Y'(t)]^T Y(t) = -Y^T(t) \mathcal{J} Y'(t) \geq 0, \quad t \in [a, b]. \] (4.27)

Then the subspace $\text{Im } X(t)$ is piecewise constant on $[a, b]$ and

\[ \mathcal{N}(Y, [a, b]) = \sum_{t_0 \in [a, b]} (\text{rank } X(t_0^+) - \text{rank } X(t_0^-)) \geq 0, \] (4.28)

\[ \mathcal{N}^\ast(Y, [a, b]) = \sum_{t_0 \in [a, b]} (\text{rank } X(t_0^+) - \text{rank } X(t_0^-)) \geq 0, \] (4.29)

where the sums in equations (4.28) and (4.29) are finite.

**Proof** The main idea of the proof is to construct a linear Hamiltonian system in form (H) with the condition $\mathcal{H}(t) \geq 0$ on $[a, b]$ for its coefficient matrix $\mathcal{H}(t)$ and such that $YQ$ is a conjoined basis of this system for a suitably chosen nonsingular matrix-valued function.
\( Q : [a, b] \rightarrow \mathbb{R}^{n \times n}. \) Then by a classical result for such systems, see e.g. [34, Theorem 3], [23, Proof of Lemma 3.6(a)], or [16, Theorem 2.4], [10, Theorems 1.79 and 1.81], the condition \( \mathcal{H}(t) \geq 0 \) will imply that the sets Ker \( X(t) \) \( Q(t) \) and \( \text{Im } X(t) \) are piecewise constant on \([a, b] \). Hence, the quantity rank \( X(t) \) is also piecewise constant on \([a, b] \), giving a correct meaning to the right-hand sides of equations (4.28) and (4.29).

A construction of such Hamiltonian system is given in [16], see also [10, Section 5.2.1], where the authors considered symplectic spectral problems with self-adjoint boundary conditions depending on spectral parameter \( \lambda \in \mathbb{R} \). In the notation of this paper and according to the proof of [16, Lemma 4.1], we consider the solution \( Q(t) \) of the linear differential system

\[
Q' = -K_Y(t)Y^T(t)Y(t)Q, \quad t \in [a, b], \quad Q(a) = I.
\]

Since the coefficient matrix in (4.30) is piecewise continuous on \([a, b] \), it follows that the matrix \( Q(t) \) is piecewise continuously differentiable and invertible on \([a, b] \). Consider the invertible matrix \( M(t) := K_Y(t)Q^{T-1}(t) \) and modify the symplectic matrix \( Z_Y(t) \) in (1.4) to become

\[
\tilde{Z}_Y(t) := Z_Y(t)\text{diag}(M(t), M^{-1}(t))
\]

(4.31)

so that the adjoint system to (4.30), which is the linear Hamiltonian system

\[
\tilde{Z}_Y(t) := \tilde{Z}_Y(t)\text{diag}(M(t), M^{-1}(t))
\]

(4.32)

is symmetric and satisfies with the above defined \( \tilde{Z}_Y(t) \) the linear Hamiltonian system

\[
[\tilde{Z}_Y(t)]' = \mathcal{H}(t)\tilde{Z}_Y(t), \quad t \in [a, b].
\]

(4.33)

Note that \( \mathcal{H}(t) = \Psi(\tilde{Z}_Y(t)) \) with the notation from [16, Propositions 2.2 and 2.3]. Upon calculating the derivative of the matrix \( \tilde{Z}_Y(t) \) in (4.31), we get (suppressing the argument \( t \))

\[
(\tilde{Z}_Y)' = (\mathcal{J}Y'K_Y^2Q^{T-1} + \mathcal{J}Y(K_Y^2)'Q^{T-1} + \mathcal{J}YK_Y^2(Q^{T-1})', \ Y'Q + YQ').
\]

(4.34)

Then by combining equations (4.30)–(4.34) with (1.1) and assumption (4.27) we conclude that

\[
\tilde{Z}_Y^{T}(t)\mathcal{H}(t)\tilde{Z}_Y(t) = -V^T(t)\text{diag}\{Y^T(t)\mathcal{J}Y'(t), Y^T(t)\mathcal{J}Y'(t)\}V(t) \geq 0
\]

(4.35)

on \([a, b] \), where the matrix \( V(t) := \text{diag}(K_Y^2(t)Q^{T-1}(t), Q(t)) \) is invertible. Note that we also used that the matrix \( Q^{T-1}(t) \) solves on \([a, b] \) the adjoint system to (4.30), which is the linear differential system with the coefficient matrix equal to \([Y'(t)]^T\mathcal{J}Y(t)K_Y^2(t) \). Thus, according to (4.31) and (4.35) the function \( YQ \) is a conjoined basis of the linear Hamiltonian system (4.33) with \( \mathcal{H}(t) \geq 0 \) on \([a, b] \), so that the Legendre condition (1.3) for this system holds. Applying [17, Theorem 2.3] we obtain that the oscillation number for the piecewise continuously differentiable Lagrangian path \( YQ \) is equal to the total number of left proper focal points of \( YQ \) in the interval \([a, b] \), i.e.,

\[
\mathcal{N}(YQ, [a, b]) = \sum_{t_0 \in (a, b)} \{\text{rank}(X(t_0^-)Q(t_0^-)) - \text{rank}(X(t_0^-)Q(t_0))\}
\]

\[
= \sum_{t_0 \in (a, b)} \{\text{rank}(X(t_0^-)) - \text{rank}(X(t_0))\} \geq 0.
\]
In a similar way, by applying [21, Proposition 3.3(v)] we obtain that the dual oscillation number for \( YQ \) is equal to the total number of right proper focal points of \( YQ \) in the interval \([a, b]\), i.e.,

\[
\mathcal{N}^*(YQ, [a, b]) = \sum_{t_0 \in [a,b]} \{ \text{rank} (X(t_0^+) Q(t_0^+)) - \text{rank} (X(t_0) Q_0(t_0)) \}
\]

\[
= \sum_{t_0 \in [a,b]} (\text{rank} X(t_0^+) - \text{rank} X(t_0)) \geq 0.
\]

However, since the matrix \( Q(t) \) is nonsingular on \([a, b]\), it follows from Remark 3.3(i) that \( \mathcal{N}(YQ, [a, b]) = \mathcal{N}(Y, [a, b]) \) and \( \mathcal{N}^*(YQ, [a, b]) = \mathcal{N}^*(Y, [a, b]) \), which completes the proof of this theorem. \( \square \)

**Remark 4.7** The proof of Theorem 4.6 shows that the conclusion of the piecewise constant image of \( X(t) \) on \([a, b]\) in the statement of this theorem follows from the stronger property that the matrix \( X(t) Q(t) \) has piecewise constant kernel on \([a, b]\), where \( Q(t) \) is the solution of (4.30).

By combining Theorems 4.2 and 4.6 we derive the following monotonicity property of the Maslov index of two continuously differentiable Lagrangian paths on \([a, b]\). Note that here we do not make any strict monotonicity assumption.

**Theorem 4.8** Let \( Y \) and \( \hat{Y} \) be piecewise continuously differentiable Lagrangian paths \([a, b]\). Assume that there exist a piecewise continuously differentiable symplectic matrix \( Z(t) \) and a nonsingular piecewise continuously differentiable \( n \times n \) matrix \( P(t) \) such that

\[
[\hat{Y}'(t)]^T J \hat{Y}(t) \geq 0, \quad \hat{Y}(t) := Z^{-1}(t) \hat{Y}(t), \quad Z(t) E = Y(t) P(t), \quad t \in [a, b]. \tag{4.36}
\]

Then the subspace \( \text{Im} W(Y(t), \hat{Y}(t)) \) is piecewise constant on \([a, b]\) and

\[
\text{Mas}(Y, \hat{Y}, [a, b]) = \sum_{t_0 \in [a,b]} \{ \text{rank} W(Y(t_0^-), \hat{Y}(t_0^-)) - \text{rank} W(Y(t_0), \hat{Y}(t_0)) \} \geq 0. \tag{4.37}
\]

\[
\text{Mas}^*(Y, \hat{Y}, [a, b]) = \sum_{t_0 \in [a,b]} \{ \text{rank} W(Y(t_0^+), \hat{Y}(t_0^+)) - \text{rank} W(Y(t_0), \hat{Y}(t_0)) \} \geq 0. \tag{4.38}
\]

In particular, assumption (4.36) is satisfied under the conditions

\[
[Y'(t)]^T J Y(t) \leq 0, \quad [\hat{Y}'(t)]^T J \hat{Y}(t) \geq 0, \quad t \in [a, b]. \tag{4.39}
\]

**Proof** The function \( \hat{Y} \) defined in (4.36) is a piecewise continuously differentiable Lagrangian path on \([a, b]\). Then by applying formula (4.13) in Theorem 4.2 and Remark 4.3(ii) we get

\[
\text{Mas}(Y, \hat{Y}, [a, b]) = \text{Mas}(YP, \hat{Y}, [a, b]) = \mathcal{N}(\hat{Y}, [a, b]),
\]

\[
\text{Mas}^*(Y, \hat{Y}, [a, b]) = \text{Mas}^*(YP, \hat{Y}, [a, b]) = \mathcal{N}^*(\hat{Y}, [a, b]).
\]

For the calculation of these oscillation numbers we apply Theorem 4.6 (with \( Y := \hat{Y} \)). Note that the upper block \( \hat{X}(t) \) of \( \hat{Y}(t) \) has the form \( \hat{X}(t) = -P^T(t) W(Y(t), \hat{Y}(t)) \). Then, by the nonsingularity of the matrix \( P(t) \), the results in (4.37) and (4.38) follow from equations (4.28) and (4.29). Finally, we assume that (4.39) holds and consider the matrices \( Q(t), \hat{Z}_Y(t), \) and \( \mathcal{H}(t) \) given by (4.30), (4.31), and (4.32). Then we have (repeating the proof of Theorem 4.6)
from the first condition in \((4.39)\) that \(\mathcal{H}(t) \leq 0\) (pay attention to the sign change) on \([a, b]\).

Next we put \(P(t) := \hat{Q}(t)\) and \(Z(t) := \hat{Z}_Y(t)\) in \((4.36)\), so that

\[
\begin{align*}
\begin{bmatrix}
\hat{Y}'(t)
\end{bmatrix}^T \mathcal{J} \hat{Y}(t) &= (-\tilde{Z}^{-1}_Y(t)[\tilde{Z}_Y(t)]' \tilde{Z}^{-1}_Y(t) \hat{Y}(t) + \tilde{Z}^{-1}_Y(t) \hat{Y}'(t)) \mathcal{J} \tilde{Z}^{-1}_Y(t) \hat{Y}(t) \\
&= [\hat{Y}'(t)]^T \mathcal{J} \hat{Y}(t) - \hat{Y}^T(t) \mathcal{H}(t) \hat{Y}(t) \geq 0
\end{align*}
\]

on \([a, b]\), where we used that the matrix \(\tilde{Z}^{-1}_Y(t)\) is symplectic and that the matrix \(\mathcal{H}(t)\) is symmetric. The last inequality then follows from \(\mathcal{H}(t) \leq 0\) and the second condition in \((4.39)\). The proof is complete. \(\square\)

**Remark 4.9** In [28, Section 4] the authors consider monotonicity properties of the eigenvalues of the matrix \(\Gamma(t)\) in \((4.1)\) in the following sense. As the parameter \(t \in [a, b]\) varies in a fixed direction, the eigenvalues of the matrix \(\Gamma(t)\) move monotonically around unit circle \(\mathbb{U}\). These results demand strict monotonicity assumptions for the derivatives in \((4.36)\) and \((4.27)\), see [28, Lemma 4.2] in particular. In this case we would have the strict inequality \(\mathcal{H}(t) > 0\) on \([a, b]\) for the Hamiltonian of the differential system \((4.33)\) associated with \(Y(t)\) in the proof of Theorem 4.6. Therefore, in this case the system \((4.33)\) would be completely controllable on \([a, b]\). The approach without a strict monotonicity assumption and hence without the complete controllability condition, as presented in \((4.39)\) above, is known in [45] in the analysis of proper focal points of conjoined bases of system (H).

### 5 Comparison Theorems for Lagrangian Paths

In this section we derive Sturmian type comparison theorems for two continuous Lagrangian paths \(Y\) and \(\hat{Y}\) on \([a, b]\). Thus we extend the results in [19, Theorem 4.4] and [21, Theorem 4.3] to the context of arbitrary continuous Lagrangian paths on \([a, b]\). At the same time we utilize the more general definition of the oscillation number and the dual oscillation number as presented in Sect. 3. Compared with the results in the latter two references, which were proven by using the properties of the comparative index, we employ the results from Sect. 3 based on the theory of Lidskii angles.

**Theorem 5.1** (Comparison theorem for oscillation numbers) Let \(Y\) and \(\hat{Y}\) be continuous Lagrangian paths on \([a, b]\). Then

\[
\begin{align*}
\mathcal{N}(Y, [a, b]) - \mathcal{N}(\hat{Y}, [a, b]) &= \mu(Y(b), \hat{Y}(b)) - \mu(Y(a), \hat{Y}(a)) + \mathcal{N}(Z^{-1}_Y, [a, b]) \\
\mathcal{N}^*(Y, [a, b]) - \mathcal{N}^*(\hat{Y}, [a, b]) &= \mu^*(Y(a), \hat{Y}(a)) - \mu^*(Y(b), \hat{Y}(b)) + \mathcal{N}^*(Z^{-1}_Y, [a, b]),
\end{align*}
\]

(5.1) (5.2)

where \(Z^{-1}_{\hat{Y}}(t) = Z^{-1}_{\hat{Y}(t)}\) is the symplectic and orthogonal matrix defined in \((1.4)\), which is associated with the Lagrangian path \(\hat{Y}(t)\). Moreover, the matrix \(Z_{\hat{Y}}(t)\) in \((5.1)\) and \((5.2)\) can be replaced by an arbitrary continuous symplectic matrix \(\hat{Z}(t)\) with \(\hat{Z}(t) E = \hat{Y}(t)\) on \([a, b]\).
Proof Applying Proposition 3.6 (with the special case of $\tau_1 := a$ and $\tau_2 := b$) we derive by (3.20) and (3.21) that
\[
\mu(Y(t), \hat{Y}(t))_a^b = \sum_{j=1}^n q_j(t)|_a^b - \sum_{j=1}^n \hat{q}_j(t)|_a^b - \sum_{j=1}^n \hat{\hat{q}}_j(t)|_a^b,
\]
(5.3)
\[
\mu^*(Y(t), \hat{Y}(t))_a^b = -\sum_{j=1}^n q_j^*(t)|_a^b + \sum_{j=1}^n \hat{q}_j^*(t)|_a^b + \sum_{j=1}^n \hat{\hat{q}}_j^*(t)|_a^b,
\]
(5.4)
where according to Theorem 3.8 we have
\[
N(Y, [a, b]) = \sum_{j=1}^n q_j(t)|_a^b, \quad N(\hat{Y}, [a, b]) = \sum_{j=1}^n \hat{q}_j(t)|_a^b,
\]
\[
N(Z^{-1}_{\hat{\gamma}} Y, [a, b]) = \sum_{j=1}^n \hat{\hat{q}}_j(t)|_a^b,
\]
\[
N^*(Y, [a, b]) = \sum_{j=1}^n q_j^*(t)|_a^b, \quad N^*(\hat{Y}, [a, b]) = \sum_{j=1}^n \hat{q}_j^*(t)|_a^b,
\]
\[
N^*(Z^{-1}_{\hat{\gamma}} Y, [a, b]) = \sum_{j=1}^n \hat{\hat{q}}_j^*(t)|_a^b.
\]
By substituting the last representations into formulas (5.3) and (5.4) we complete the proofs of the results in (5.1) and (5.2). Moreover, by using (3.34) and (3.35) in Theorem 3.12 (with $S(t) := \hat{Z}(t)$ and $S(t) E = \hat{Y}(t)$ on $[a, b]$) we see that the matrix $Z_{\hat{\gamma}}(t)$ in (5.1) and (5.2) can be replaced by an arbitrary continuous symplectic matrix $\hat{Z}(t)$ with $\hat{Z}(t) E = \hat{Y}(t)$ on $[a, b]$.

If the Lagrangian paths $Y$ and $\hat{Y}$ have the same values at the endpoints of the interval $[a, b]$, then we obtain from Theorem 5.1 the following.

Corollary 5.2 Let $Y$ and $\hat{Y}$ be continuous Lagrangian paths on $[a, b]$ such that $Y(a) = \hat{Y}(a)$ and $Y(b) = \hat{Y}(b)$, then
\[
N(Y, [a, b]) - N(\hat{Y}, [a, b]) = N(Z^{-1}_{\hat{\gamma}} Y, [a, b]),
\]
(5.5)
\[
N^*(Y, [a, b]) - N^*(\hat{Y}, [a, b]) = N^*(Z^{-1}_{\hat{\gamma}} Y, [a, b]),
\]
(5.6)
where $Z_{\hat{\gamma}}(t) = Z_{\hat{\gamma}}(t)$ is the symplectic and orthogonal matrix defined in (1.4), which is associated with the Lagrangian path $\hat{Y}(t)$. Moreover, the matrix $Z_{\hat{\gamma}}(t)$ in (5.5) and (5.6) can be replaced by an arbitrary continuous symplectic matrix $\hat{Z}(t)$ with $\hat{Z}(t) E = \hat{Y}(t)$ on $[a, b]$.

Proof Under the assumptions $Y(a) = \hat{Y}(a)$ and $Y(b) = \hat{Y}(b)$ the comparative indices and the dual comparative indices appearing in (5.1) and (5.2) are zero. Hence, the results in (5.5) and (5.6), as well as the last statement of this corollary, follow from Theorem 5.1.

Equations (5.1) and (5.2) can be simplified for special Lagrangian paths on $[a, b]$, for which the involved comparative indices vanish. For this purpose we introduce the notation $Y_a$ and $Y_b$ for continuous Lagrangian paths, which satisfy the initial conditions
\[
Y_a(a) = E = Y_b(b).
\]
(5.7)
When $Y_a$ and $Y_b$ correspond to conjoined bases of system (H), then they are uniquely determined by (5.7) as solutions of (H). In this case $Y_a$ and $Y_b$ are called the principal solutions of system (H) at the points $a$ and $b$. In the general situation of continuous Lagrangian paths on $[a, b]$ we can still derive some important properties of the paths $Y_a$ and $Y_b$. Then we have the following extension of [21, Corollary 4.6].

**Corollary 5.3** Let $Y_a$ and $Y_b$ be continuous Lagrangian paths on $[a, b]$ satisfying condition (5.7). Then their oscillation numbers and dual oscillation numbers satisfy the relations

$$\mathcal{N}(Y_b, [a, b]) - \mathcal{N}^*(Y_a, [a, b]) = \mathcal{N}(Z_{Y_b}^{-1}Y_b, [a, b]),$$  \hspace{1cm} (5.8)

$$\mathcal{N}^*(Y_b, [a, b]) - \mathcal{N}(Y_a, [a, b]) = \mathcal{N}^*(Z_{Y_a}^{-1}Y_b, [a, b]).$$  \hspace{1cm} (5.9)

**Proof** We partition the Lagrangian paths $Y_a$ and $Y_b$ on $[a, b]$ according to (1.2). We apply equality (5.1) with $Y := Y_b$ and $\hat{Y} := Y_a$. Then $\mu(Y_b(b), Y_a(b)) = \text{rank } X_a(b)$ and $\mu(Y_b(a), Y_a(a)) = 0$, so that by Theorem 3.13 (with $Y := Y_a$) we obtain

$$\mathcal{N}(Y_b, [a, b]) = \mathcal{N}(Y_a, [a, b]) + \text{rank } X_a(b) + \mathcal{N}(Z_{Y_a}^{-1}Y_b, [a, b]).$$  \hspace{1cm} (3.36)

This shows (5.8). Next we apply (5.2) with $Y := Y_b$ and $\hat{Y} := Y_a$. Then $\mu^*(Y_b(a), Y_a(a)) = 0$ and $\mu^*(Y_b(b), Y_a(b)) = \text{rank } X_a(b)$, so that by Theorem 3.13 (with $Y := Y_a$) we get

$$\mathcal{N}^*(Y_b, [a, b]) = \mathcal{N}^*(Y_a, [a, b]) - \text{rank } X_a(b) + \mathcal{N}^*(Z_{Y_a}^{-1}Y_b, [a, b]).$$  \hspace{1cm} (3.36)

This shows (5.9) and the proof is complete. □

Based on the connections of the oscillation number and the Maslov index derived in Sect. 4 we can now reformulate the comparison theorem for the oscillation numbers (Theorem 5.1) in terms of the Maslov index. More precisely, we obtain a formula calculating the Maslov index $\text{Mas}(Y, \hat{Y}, [a, b])$ in terms of the two reference Maslov indices $\text{Mas}(E, \hat{Y}, [a, b])$ and $\text{Mas}(E, Y, [a, b])$ and in terms of the comparative index of $\hat{Y}$ and $Y$ evaluated at the endpoints of $[a, b]$.

**Corollary 5.4** Let $Y$ and $\hat{Y}$ be continuous Lagrangian paths on $[a, b]$. Then we have

$$\text{Mas}(Y, \hat{Y}, [a, b]) = \text{Mas}(E, \hat{Y}, [a, b]) - \text{Mas}(E, Y, [a, b]) + \mu(\hat{Y}(a), Y(a)) - \mu(\hat{Y}(b), Y(b)).$$  \hspace{1cm} (5.10)

If in addition $Y(a) = \hat{Y}(a)$ and $Y(b) = \hat{Y}(b)$ hold, then

$$\text{Mas}(Y, \hat{Y}, [a, b]) = \text{Mas}(E, \hat{Y}, [a, b]) - \text{Mas}(E, Y, [a, b]).$$  \hspace{1cm} (5.11)

**Proof** Let $Y$ and $\hat{Y}$ be continuous Lagrangian paths on $[a, b]$. We apply formula (5.1) in Theorem 5.1, in which we interchange the roles of $Y$ and $\hat{Y}$. Then we get

$$\mathcal{N}(\hat{Y}, [a, b]) = \mathcal{N}(Y, [a, b]) + \mu(\hat{Y}(a), Y(a)) - \mu(\hat{Y}(b), Y(b)) + \mathcal{N}(Z_{\hat{Y}}^{-1}\hat{Y}, [a, b]).$$  \hspace{1cm} (5.12)

If we now replace the oscillation numbers appearing in (5.12) by the corresponding Maslov indices from (4.17) and (4.13), then we obtain the result in (5.10). Finally, equation (5.11)
follows from identity (5.10), in which the comparative indices vanish under the assumptions that \( Y(a) = \hat{Y}(a) \) and \( Y(b) = \hat{Y}(b) \).

\( \square \)

**Remark 5.5** Combining equations (4.20) and (4.23) with the comparison theorem for the dual oscillation numbers in (5.2) we derive a dual version of Corollary 5.4 in the form

\[
\begin{align*}
\Mas^*(Y, \hat{Y}, [a, b]) &= \Mas^*(E, \hat{Y}, [a, b]) - \Mas^*(E, Y, [a, b]) \\
+ \mu^*(\hat{Y}(b), Y(b)) - \mu^*(\hat{Y}(a), Y(a)).
\end{align*}
\]

(5.13)

In addition, if \( Y(a) = \hat{Y}(a) \) and \( Y(b) = \hat{Y}(b) \) hold, then (5.13) reduces to

\[
\Mas^*(Y, \hat{Y}, [a, b]) = \Mas^*(E, \hat{Y}, [a, b]) - \Mas^*(E, Y, [a, b]).
\]

(5.14)

Comparison formulas (5.10) and (5.13) appear to be new in the context of the Maslov index.

**Remark 5.6** From Corollary 5.4 and Remark 5.5 we obtain a precise statement about the validity of equations (5.11) and (5.14). In particular, the equality in (5.11) holds if and only if \( \mu^*(\hat{Y}(a), Y(a)) = \mu^*(\hat{Y}(b), Y(b)) \), while the equality in (5.14) is satisfied if and only if \( \mu^*(\hat{Y}(b), Y(b)) = \mu^*(\hat{Y}(a), Y(a)) \). The coincidence of the initial and terminal values of \( Y \) and \( \hat{Y} \) at the endpoints of \([a, b]\) or the closedness of the paths \( Y \) and \( \hat{Y} \) of course guarantee that the involved comparative indices are the same.

Next we consider the corresponding Sturmian type separation theorems for the oscillation numbers and the dual oscillation numbers. These results extend the special situation, when both Lagrangian paths \( Y \) and \( \hat{Y} \) are conjoined bases of one linear Hamiltonian system \( (H) \). We refer to [13, Theorems 2.2 and 2.3] and [40, Theorem 4.1] for the case when the Legendre condition (1.3) holds, and to [18, Theorem 4.1] and [21, Theorem 4.4] for the case without assumption (1.3).

Let us fix a continuous symplectic matrix \( \Phi(t) \) on \([a, b]\). Consider the set

\[
\mathcal{F}(\Phi) := \{ \Phi(\cdot)C, \text{ where } C \text{ is a Lagrangian plane} \}.
\]

(5.15)

Then the elements \( Y \in \mathcal{F}(\Phi) \) are continuous Lagrangian paths on \([a, b]\), which are constant multiples of the given symplectic matrix \( \Phi(t) \) on \([a, b]\), that is, \( Y(t) = \Phi(t)C \) on \([a, b]\) for some matrix \( C \in \mathbb{R}^{2n \times n} \) with \( C^TJC = 0 \) and rank \( C = n \). In the context of system \( (H) \) the set \( \mathcal{F}(\Phi) \) corresponds to the set of all conjoined bases of \( (H) \). With the notation in (5.15) we can formulate the following.

**Theorem 5.7** (Separation theorem) Let \( \Phi(t) \) be a continuous symplectic matrix on \([a, b]\). For any continuous Lagrangian paths \( Y \) and \( \hat{Y} \) belonging to the set \( \mathcal{F}(\Phi) \) defined in (5.15) we have

\[
\mathcal{N}(Y, [a, b]) - \mathcal{N}(\hat{Y}, [a, b]) = \mu(Y(b), \hat{Y}(b)) - \mu(Y(a), \hat{Y}(a)),
\]

(5.16)

\[
\mathcal{N}^*(Y, [a, b]) - \mathcal{N}^*(\hat{Y}, [a, b]) = \mu^*(Y(a), \hat{Y}(a)) - \mu^*(Y(b), \hat{Y}(b)).
\]

(5.17)

**Proof** Formulas (5.16) and (5.17) follow from the comparison theorem (Theorem 5.1) and from Corollary 3.10. Indeed, considering the continuous Lagrangian path \( \hat{Y} := \hat{Z}^{-1}Y \) on \([a, b]\), then its upper block \( \hat{X} \) has the form \( \hat{X}(t) = -W(\hat{Y}(t), \hat{Y}(t)) \), which is a constant matrix on \([a, b]\) in the setting of this theorem. Hence, we have the equalities \( \mathcal{N}(\hat{Y}, [a, b]) = 0 \) and \( \mathcal{N}^*(\hat{Y}, [a, b]) = 0 \) by Corollary 3.10 and then equations (5.1) and (5.2) yield the results in (5.16) and (5.17).
The results in Theorem 5.7 show that for a given continuous Lagrangian path \( Y \in \mathcal{F}(\Phi) \) it is possible to calculate the value \( \mathcal{N}(Y, [a, b]) \) from the oscillation number of a suitable reference Lagrangian path from the set \( \mathcal{F}(\Phi) \) in (5.15).

**Remark 5.8** Consider the continuous Lagrangian paths \( Y_a, Y_b \in \mathcal{F}(\Phi) \), which are associated in (5.15) with the matrices \( C_a := \Phi^{-1}(a)E \) and \( C_b := \Phi^{-1}(b)E \), i.e.,

\[
Y_a(t) = \Phi(t)C_a, \quad Y_b(t) = \Phi(t)C_b, \quad t \in [a, b], \quad Y_a(a) = E = Y_b(b). \tag{5.18}
\]

Since for any Lagrangian path \( Y \in \mathcal{F}(\Phi) \) we have \( \mu(Y(a), E) = 0 \) and \( \mu^*(Y(b), E) = 0 \) by (2.1), it follows from (5.16) with \( \tilde{Y} := Y_a \) and from (5.17) with \( \tilde{Y} := Y_b \) that

\[
\mathcal{N}(Y, [a, b]) = \mathcal{N}(Y_a, [a, b]) + \mu(Y(b), Y_a(b)), \quad \mathcal{N}^*(Y, [a, b]) = \mathcal{N}^*(Y_b, [a, b]) + \mu^*(Y(a), Y_b(a)). \tag{5.19}
\]

In addition, in the same spirit as in [21, Theorem 4.5] we obtain from (5.19) and (5.20) for any Lagrangian path \( Y \in \mathcal{F}(\Phi) \) the estimates

\[
\mathcal{N}(Y_a, [a, b]) \leq \mathcal{N}(Y, [a, b]) \leq \mathcal{N}(Y_b, [a, b]), \quad \mathcal{N}^*(Y_a, [a, b]) \leq \mathcal{N}^*(Y, [a, b]) \leq \mathcal{N}^*(Y_b, [a, b]). \tag{5.21}
\]

Moreover, with the special choices of \( Y := Y_b \) in (5.19) and \( Y := Y_a \) in (5.20) we deduce that

\[
\mathcal{N}(Y_a, [a, b]) = \mathcal{N}(Y_b, [a, b]) = \mathcal{N}^*(Y_a, [a, b]) = \mathcal{N}^*(Y_b, [a, b]), \tag{5.23}
\]

\[
\mathcal{N}(Y_b, [a, b]) - \mathcal{N}(Y_a, [a, b]) = \text{rank} W(Y_a, Y_b),
\]

\[
= \mathcal{N}^*(Y_a, [a, b]) - \mathcal{N}^*(Y_b, [a, b]), \tag{5.24}
\]

compare with [40, Corollary 5.4 and Theorem 5.6]. Note that in order to derive (5.23) and (5.24) from (5.19) and (5.20) we used that \( \mu(E, Y_a(b)) \) and \( \mu^*(E, Y_b(a)) \) have the same value, since the Wronskian \( W(Y_a(t), Y_b(t)) \equiv W(Y_a, Y_b) \) is in this case constant on \([a, b]\).

Based on the above separation theorem we are able to answer the question about the existence of a continuous Lagrangian path \( Y \) in the given set \( \mathcal{F}(\Phi) \), whose oscillation number and dual oscillation number attain prescribed values satisfying estimates (5.21) and (5.22). This result generalizes [44, Theorem 1.1] to arbitrary continuous Lagrangian paths or even to conjoined bases of system (H) without assuming the Legendre condition (1.3).

**Theorem 5.9** Let \( \Phi(t) \) be a continuous symplectic matrix on \([a, b]\) and let \( Y_a, Y_b \in \mathcal{F}(\Phi) \) be the unique continuous Lagrangian paths determined by formula (5.18) in Remark 5.8. Then for any integers \( \ell \) and \( r \) satisfying

\[
\mathcal{N}(Y_a, [a, b]) \leq \ell \leq \mathcal{N}(Y_b, [a, b]) \quad \text{and} \quad \mathcal{N}^*(Y_a, [a, b]) \leq r \leq \mathcal{N}^*(Y_b, [a, b]) \tag{5.25}
\]

there exists a continuous Lagrangian path \( Y \in \mathcal{F}(\Phi) \) such that

\[
\mathcal{N}(Y, [a, b]) = \ell \quad \text{and} \quad \mathcal{N}^*(Y, [a, b]) = r. \tag{5.26}
\]

Moreover, if \( \ell \geq r \), then the Lagrangian path \( Y \) can be chosen with \( X(a) = I \), and if \( \ell \leq r \), then the Lagrangian path \( Y \) can be chosen with \( X(b) = I \). In particular, when \( \ell = r \) the Lagrangian path \( Y \) may be chosen with both \( X(a) \) and \( X(b) \) invertible.
Proof Let $\ell$ and $r$ be given integers satisfying (5.25). Define the integers
\[
p := \ell - \mathcal{N}(Y_a, [a, b]), \quad q := r - \mathcal{N}^*(Y_b, [a, b]).
\]
Then from (5.25) and (5.24) it follows that $\max\{p, q\} \leq w := \text{rank } W(Y_a, Y_b)$ holds, where the Wronskian is constant on $[a, b]$ by the last part of Remark 5.8. We will apply the ideas of [44, Theorem 2.1], which we adopt to the setting of continuous Lagrangian paths in the set $\mathcal{F}(\Phi)$. We partition the Lagrangian paths $Y_a = (X_a^T, U_a^T)^T$ and $Y_b = (X_b^T, U_b^T)^T$ according to notation (1.2). First we assume that $\ell \geq r$ holds. In view of (5.23) we then obtain that $p \geq q$. In the spirit of the proof of [44, Theorem 2.1] we construct the Lagrangian path $Y \in \mathcal{F}(\Phi)$ by
\[
Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix} := \Phi(t) C, \quad t \in [a, b],
\]
\[
C := \Phi^{-1}(a) \begin{pmatrix} I \\ D + R_b(a) U_b(a) X_b^\dagger(a) \end{pmatrix},
\]
where $R_b(a) := X_b(a) X_b^\dagger(a)$ is the orthogonal projector onto $\text{Im } X_b(a)$ and where the symmetric $n \times n$ matrix $D$ has $q$ negative eigenvalues $\lambda_j = -1$ and $w - p$ positive eigenvalues $\lambda_j = 1$. More precisely, see [44, Eqs. (2.29)–(2.31)], we take
\[
D := L \text{ diag}[-I_q, I_{w-p}, 0_{n-w+p-q}] L^T,
\]
where $L$ is an orthogonal matrix satisfying
\[
R_b(a) = L \text{ diag}\{I_w, 0_{n-w}\} L^T, \quad R_b(a) D R_b(a) = D.
\]
Equations (5.29) and (5.30) imply, compare with [44, Lemma 2.2], that
\[
\mu(Y(b), Y_a(b)) = w - \text{ind}[-R_b(a) D R_b(a)] \tag{5.30}
\]
\[
\mu^*(Y(a), Y_b(a)) = \text{ind}[R_b(a) D R_b(a)] \tag{5.30}
\]
\[
\mu^*(Y(b), Y_a(b)) = \text{ind}[R_b(a) D R_b(a)] \tag{5.30}
\]
Then by (5.19) and (5.20) we obtain that
\[
\mathcal{N}(Y, [a, b]) \tag{5.19}
\]
\[
\mathcal{N}(Y_a, [a, b]) + \mu(Y(b), Y_a(b)) \tag{5.31}
\]
\[
\mathcal{N}(Y_a, [a, b]) + p \tag{5.27}
\]
\[
\mathcal{N}^*(Y, [a, b]) \tag{5.20}
\]
\[
\mathcal{N}^*(Y_b, [a, b]) + \mu^*(Y(b), Y_a(b)) \tag{5.32}
\]
\[
\mathcal{N}^*(Y_b, [a, b]) + q \tag{5.27}
\]
This completes the proof of (5.26) for the case of $\ell \geq r$. Moreover, from (5.28) with $t = a$ we can see that $X(a) = I$ holds. Next we suppose that $\ell \leq r$, so that $p \leq q$ in view of (5.23). We follow the second part of the proof of [44, Theorem 2.1]. Hence, we construct the Lagrangian path $Y \in \mathcal{F}(\Phi)$ by the formula
\[
Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix} := \Phi(t) C, \quad t \in [a, b],
\]
\[
C := \Phi^{-1}(b) \begin{pmatrix} I \\ D + R_a(b) U_a(b) X_a^\dagger(b) \end{pmatrix},
\]
where $R_a(b) := X_a(b) X_a^\dagger(b)$ is the orthogonal projector onto $\text{Im } X_a(b)$ and where as in [44, Eq. (2.33)] the symmetric $n \times n$ matrix $D$ has the form
\[
D := L \text{ diag}[-I_{w-q}, I_p, 0_{n-w+q-p}] L^T
\]
with an orthogonal matrix $L$ satisfying
\[ R_a(b) = L \text{diag}(I_w, 0_{n-w}) L^T, \quad R_a(b) D R_a(b) = D. \] (5.35)
Equations (5.34) and (5.35) imply, compare with [44, Lemma 2.3], that
\[ \mu(Y(b), Y_a(b)) = \text{ind}[-R_a(b) D R_a(b)] \overset{(5.35)}{=} \text{ind}(-D) \overset{(5.34)}{=} p, \]
\[ \mu^*(Y(a), Y_b(a)) = w - \text{ind}[R_a(b) D R_a(b)] \overset{(5.35)}{=} w - \text{ind} D \overset{(5.34)}{=} w - (w - q) = q. \] (5.36)
Then by (5.19) and (5.20) we obtain that
\[ \mathcal{N}(Y, [a, b]) \overset{(5.19)}{=} \mathcal{N}(Y_a, [a, b]) + \mu(Y(b), Y_a(b)) \overset{(5.36)}{=} \mathcal{N}(Y_a, [a, b]) + p \overset{(5.27)}{=} \ell, \]
\[ \mathcal{N}^*(Y, [a, b]) \overset{(5.20)}{=} \mathcal{N}^*(Y_b, [a, b]) + \mu^*(Y(a), Y_b(a)) \overset{(5.37)}{=} \mathcal{N}^*(Y_b, [a, b]) + q \overset{(5.27)}{=} r. \]
This completes the proof of (5.26) for the case of $\ell \leq r$. Moreover, from (5.33) with $t = b$ we can see that $X(b) = I$ holds. Finally, if $\ell = r$, then from (5.26) we have $\mathcal{N}(Y, [a, b]) = \mathcal{N}^*(Y, [a, b])$. By formula (3.36) in Theorem 3.13 it follows that rank $X(a) = \text{rank} X(b)$. This means that if $X(a) = I$, then $X(b)$ is nonsingular, while if $X(b) = I$, then $X(a)$ is nonsingular. In conclusion, if $\ell = r$, then the Lagrangian path $Y$ can be chosen with both $X(a)$ and $X(b)$ invertible. The proof is complete. \hfill \Box

**Remark 5.10** The proof of Theorem 5.9 shows that the Lagrangian path $Y \in \mathcal{F}(\Phi)$ satisfying condition (5.26) is constructed on $[a, b]$ as a constant multiple of the matrix $\Phi(t)$ by prescribing its initial condition at $a$ in (5.28) if $\ell \geq r$, or at $b$ in (5.33) if $\ell \leq r$.

### 6 Conclusions

In this section we make comments about the main results of this paper and their relationship with some related mathematical problems. The main purpose of this paper was to study the oscillation number $\mathcal{N}(Y, [a, b])$ and the dual oscillation number $\mathcal{N}^*(Y, [a, b])$ for a continuous Lagrangian path $Y$ on $[a, b]$. These are integer quantities defined in an algebraic way through the comparative index and the dual comparative index by using a certain partition of the interval $[a, b]$. Here we use a more general definition than in [17, 21] in the sense that we use nonconstant symplectic matrices in the partition. As the main results (Theorem 3.8 and the subsequent results in Sect. 5) we express the quantities $\mathcal{N}(Y, [a, b])$ and $\mathcal{N}^*(Y, [a, b])$ in terms of the total changes in the interval $[a, b]$ of the integers $q_j(t)$ and $q_j^*(t)$, which are associated through (3.12) and (3.13) with the continuous Lidskii angles $\varphi_j(t)$ of the symplectic and orthogonal matrix $Z(t)$ on $[a, b]$ defined in (1.4). The methods, which were used for the above analysis, are based on the Lidskii angles of symplectic matrices and their relationship with the comparative index obtained recently in [46]. This approach allowed us to connect the oscillation numbers with the Maslov index (Theorems 4.2 and 4.6, Corollary 4.4, and Remarks 4.5 and 5.5). In addition, we derive a general comparison theorem (Theorem 5.1) for the oscillation numbers and the dual oscillation numbers of two arbitrary continuous Lagrangian paths $Y$ and $\hat{Y}$ on $[a, b]$, as well as general separation theorems for the case when $Y$ and $\hat{Y}$ are constant multiples of a given continuous symplectic matrix (Theorems 5.7 and 5.9).

The theory presented in this paper is based on detailed matrix analysis with motivations coming from the theory of differential equations, resp. from the oscillation and spectral
theory of linear Hamiltonian systems (H). It can be further developed in the direction of the oscillation and spectral theory on discrete time domains [10, 16, 20, 41] or in the direction of singular comparison theorems for the oscillation numbers, as we recently presented in [42, 43] for conjoined bases of nonoscillatory linear Hamiltonian systems. Such results may have fundamental applications in the theory of Maslov index on unbounded intervals, such as in [26], or in the study of the rotation number of a family of linear Hamiltonian systems, such as in [30, 31]. We expect related research activity in the theory of matrices in general, for example in the limit theorems for symmetric matrix valued functions generalizing the results in [32, 36] or [33, Theorem 3.3.7]. The obtained results on Maslov index also open the door for connecting the Hörmander index [25, 50] with the comparative index. We will investigate this topic in our subsequent work.

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