On a new type of orbifold equivalence and M-theoretic
$AdS_4/CFT_3$ duality

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abstract

We consider the large-$N$ limit of $N = 6 \ U(N) \times U(N)$ superconformal Chern-Simons (ABJM) theory with fixed level $k$, which is conjectured to be dual to M-theory on $AdS_4 \times (S^7/Z_k)$ background. We point out that the so-called orbifold equivalence on the gravity side, combined with the $AdS_4/CFT_3$ duality, predicts a hitherto unknown type of duality on the gauge theory side. It establishes the equivalence between a class of observables, which are not necessarily protected by supersymmetry, in strongly coupled ABJM theories away from the planar approximation, with different values of $k$ and $N$ but sharing common $kN$. This limit is vastly different from the planar limit, and hence from the gauge theory point of view the duality is more difficult to explain compared to the previously known analogous equivalence between planar gauge theories, where one can explicitly prove the equivalence diagrammatically using the dominance of the planar diagrams.

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Introduction and main result

Three-dimensional gauge theories with a Chern-Simons term have been a subject of interest in theoretical physics ever since their first appearance \[1, 2\]. They have direct applications in condensed matter physics and exhibit rich mathematical structure. In recent years Chern-Simons theories coupled to matter fields, with various amount of supersymmetry, have gained renewed interest because they capture some part of the dynamics of the elusive M-theory, the eleven-dimensional theory defined by the strong coupling limit of type IIA superstring theory. In particular Aharony, Bergman, Jafferis and Maldacena constructed an \(\mathcal{N} = 6\) \(U(N_1) \times U(N_2)\) superconformal Chern-Simons theory (ABJM theory) with a certain coupling constant parametrised by an integer (the level) \(k\) \[3\], following important earlier works \[4, 5\]. The model has been proposed to be the low energy effective theory of \(N\) M2 branes (which are important degrees of freedom in M-theory) put on an orbifolded (transverse) space \(\mathbb{R}^8/\mathbb{Z}_k\). It has been conjectured that the low energy effective theory of \(N\) M2 branes in flat space \(\mathbb{R}^8\) should be dual to M-theory in \(AdS_4 \times S^7\) when \(N\) is large \[6\], and in \[3\], accordingly, it was conjectured that the ABJM theory is dual to M-theory on \(AdS_4 \times S^7/\mathbb{Z}_k\).

Similar orbifold geometries have been considered in the context of the \(AdS_5/CFT_4\) duality in \[7\], where the duality between type IIB string theory in \(AdS_5 \times S^5/\mathbb{Z}_k\) and the corresponding Yang-Mills theories in four dimensions was discussed. It was found that, from the gravity side, one can predict an equivalence between these field theories for observables which are invariant under the \(\mathbb{Z}_k\) projection (up to a calculable factor depending on \(k\)). The equivalence holds only at tree level on the gravity side. This is because for tree level processes, all intermediate states are also invariant under \(\mathbb{Z}_k\), whereas \(\mathbb{Z}_k\) breaking intermediate states can appear in loop corrections.

In this note we will discuss the application of this idea to the \(AdS_4/CFT_3\) duality. We will show that similar arguments lead to predictions for the dual gauge theory which are more surprising and difficult to explain using field theoretical methods. Let us consider two ABJM theories with levels \(k_1, k_2\) and gauge groups \(U(N_1) \times U(N_2)\), \(U(N_2) \times U(N_2)\). The curvature radius of the dual background in eleven-dimensional Planck units is given by \[3\]

\[
\frac{R}{l_p} = (2^5 \pi^2 N_i')^{\frac{1}{6}},
\]

where \(N_i' \equiv N_i k_i\). \[4\] Hence, for theories with the same value of \(N'\),

\[
N_1 k_1 = N' = N_2 k_2,
\]

any observables which are blind both to \(\mathbb{Z}_{k_1}\) and \(\mathbb{Z}_{k_2}\) transformations should be equal in the two theories, as long as loop corrections on the gravity side are negligible. \[5\] There are many observables which are blind to the \(\mathbb{Z}_{k_1}\) and \(\mathbb{Z}_{k_2}\) transformations, both supersymmetric and non-supersymmetric. \[6\] We discuss some examples later.

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\[4\] In addition to the metric there is also four-form flux, but it depends on the parameters of the theory only through \(R\), so the full supergravity solution depends only on \(N'\).

\[5\] We notice that in general there are observables, say in the first theory dual to the \(\mathbb{Z}_{k_1}\) orbifold, that are charged under the \(\mathbb{Z}_{k_2}\) transformation associated to the second orbifold. They are absent in the second ABJM theory, and hence there are no orbifold equivalence for these operators.

\[6\] For BPS operators, the orbifold equivalence may be considered as a simple consequence of the usual prescription to compute correlation functions in \(AdS/CFT\) \[9, 10\]. In the \(AdS_5/CFT_4\) case, an extension to near-BPS operators is discussed in \[11\]. Orbifold equivalence for non-BPS operators might be understood along this line.
Let us discuss the conditions under which quantum corrections are negligible. In general this depends strongly on the observable (or the process) we consider. However, one can give necessary conditions: if length scales involved in the model (on the gravity side) are comparable to the eleven-dimensional Planck scale, one cannot expect quantum gravity corrections to be small. There are two length scales in the geometry, the curvature radius of the background $R$ given in (1) and the radius of the M-theory circle $R/k$. Requiring both of them to be much larger than the Planck scale, one obtains the conditions $N' = Nk \gg 1$ and $N \gg k^5$, where the latter implies the former (because $k, N$ are integers).

The argument given so far, which is on the gravity side, is completely parallel to that in the $AdS_5/CFT_4$ duality. However, from the gauge theory point of view we find a big difference. In the context of $AdS_5/CFT_4$ duality, it is possible to prove the equivalence directly in the gauge theory [3]; the essential point in the proof is that loop effects on the gravity side correspond to $1/N$ corrections on the gauge theory side. Hence one should focus on planar diagrams, and one can indeed prove the equivalence using purely field theoretic arguments. However, in our case the same type of argument is not applicable. The reason is as follows. In ABJM theories, the 't Hooft expansion is the double expansion in terms of $1/N$ and the 't Hooft coupling $\lambda = N/k$. The dominance of planar diagrams holds in the large-$N$ limit with fixed $\lambda$, which is dual to type IIA superstring in $AdS_5 \times CP^3$ [3]. However, in the M-theory regime, $k$ is typically of order one and $N$ is still large, so that neglecting contributions from non-planar diagrams cannot be justified. Thus our conclusion is that the $AdS_4/CFT_3$ duality, combined with the orbifold equivalence, predicts non-trivial relations for very strongly coupled (in the sense that the planar approximation cannot be used) ABJM theories. This new equivalence is much more difficult to explain in field theoretic terms than the analogous equivalence relations found in the $AdS_5/CFT_4$ context (and their extensions such as those considered in [12]).

### Observables

Because in the region of parameters that we are considering the gauge theory is strongly coupled, it is not easy to explicitly compute observables. However, recent developments using the localization technique provide us with some exact results in ABJM theories. In particular, the “free energy” $\log Z$ of the theory in a compact three-sphere $S^3$ in the planar limit ($N, k \to \infty$ with $N/k$ fixed), and some sub-leading terms in the $1/N$ expansion were obtained in [14, 15]. Let us discuss the free energy using an expression proposed in [16] which sums up all $1/N$ corrections

$$\log Z = \log \left(2\pi C_1 Ai\left(\frac{\pi}{\sqrt{2}} \left(\frac{N}{\lambda}\right)^2 \lambda_{ren}^{3/2}\right)^{2/3}\right), \quad (3)$$

where $Ai(x)$ is the Airy function,

$$C_1 = \frac{1}{\sqrt{2}} \left(\frac{2\pi}{k}\right)^{-1/3} \quad (4)$$

\footnote{Of course, one can consider the more familiar type of the orbifold equivalence associated with planar diagrams in the $AdS_4/CFT_3$ context as well, but in the limit where $N/k$ is fixed, not in the M-theoretic limit we are interested in. For these planar orbifold equivalence, the internal space $CP^3$ is orbifolded on the gravity side. On the gauge theory side, the gauge group changes as in the $AdS_5/CFT_4$ case; see e.g. [13].}
and the ‘renormalized ’t Hooft coupling’ $\lambda_{\text{ren}}$ is given by
\[
\lambda_{\text{ren}} = \lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}.
\] (5)

Whether this formula is applicable in the M-theory regime is not very clear, as the limit $N \to \infty$ with $k$ fixed is very different from the planar limit. However, a direct extrapolation of this expression to the M-theory region provides us with a result which is consistent with the orbifold equivalence. The asymptotic behavior of the Airy function at a large positive value of $x$
\[
Ai(x) \sim \frac{1}{2\sqrt{\pi}} \left(\frac{1}{x}\right)^{1/4} \exp\left(-\frac{2}{3} x^{3/2}\right)
\] (6)
yields, to leading order,
\[
\log Z \sim -\frac{\sqrt{2\pi}}{3} k^{1/2} N^{3/2} = -\frac{\sqrt{2\pi}}{3} \frac{N^{3/2}}{k}.
\] (7)
The $1/k$ scaling of the last expression is what is expected from the orbifold equivalence – the free energy is proportional to the volume of $S^7/Z_k$ on the gravity side. The leading order term in the asymptotic form of the Airy function in the M-theory regime ($k$ finite, $N \to \infty$) is actually the same as the type IIA supergravity limit, where $\lambda \to \infty$ is taken after the $N \to \infty$ limit with fixed $\lambda = \frac{N}{k}$. This is an expected result from the gravity side [3] and is first derived on the field theory side in [15] by rewriting the ’t Hooft expansion in terms of M-theory variables. One may also read off the sub-leading correction in this regime (which is first obtained in [15]) from (3),
\[
\log Z = -\frac{\sqrt{2\pi}}{3} \frac{N^{3/2}}{k} \left(1 - \frac{1}{2N^\gamma} - \frac{k}{16N^\gamma}\right).
\] (8)

We note that the last term which does not satisfy the expected $1/k$ behaviour should be interpreted as coming from quantum correlations in $S^7/Z_k$.

Another class of observables comprises an infinite family of local BPS operators [3]
\[
\text{Tr} \left(C_{I_1} C_{J_1}^\dagger \cdots C_{I_l} C_{J_l}^\dagger\right),
\] (9)
where $I_1, \cdots, I_l$ and $J_1, \cdots, J_l$ are symmetrized. Here, $C_I$ ($I = 1, 2, 3, 4$) denote four complex bi-fundamental scalar fields in ABJM theories which describe the collective coordinates of M2-branes [10], on which the $Z_k$ symmetry acts as $C_I \to e^{2\pi i/k} C_I$. Operators [4] are manifestly $Z_k$ invariant. That their scaling dimensions are protected is of course consistent with the orbifold equivalence. There are other classes of gauge invariant BPS operators written in terms of monopole operators [3]. One can choose the monopole charges $m_1$ and $m_2$ for the two theories such that $k_1 m_1 = k_2 m_2$ holds. These operators are invariant under both $Z_{k_1}$ and $Z_{k_2}$ symmetries. Our argument predicts that the equivalence should also hold for correlation functions of these operators [11], at tree level on the gravity side.

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8 Note added: After the first version of this paper appeared in the arXiv, numerical results are obtained which support the validity of [3] in the M-theory region [17]. See also [18].

9 In [16], a discrepancy between the sub-leading correction in this formula and a calculation in string theory [19] was pointed out. These terms do not affect the leading order expression [7].

10 Our notation is that of the original paper [3].

11 We thank O. Aharony for very useful comments on this issue.
The observables we have discussed so far involve operators which preserve some supersymmetry. There are also non-supersymmetric, or non-BPS, observables which are $\mathbb{Z}_k$ invariant. The argument on the gravity side predicts equivalence of non-supersymmetric observables as well, as far as the tree level approximation on the gravity (M-theory) side is justified. One class of examples is provided by various observables at finite temperature. In this case, the gravity counterpart is calculated via a non-BPS solution to eleven-dimensional supergravity. Another example is the loop operator discussed in [21], which is analogous to the Wilson-loop in the $AdS_5/CFT_4$ correspondence [22, 23]. The M-theory counterpart is the volume of the membrane minimal hyper-surface, with appropriate boundary conditions. For generic boundary conditions, these loop operators are non-BPS (although locally BPS). In these cases, it is clear that as far as the length scale involved in the solutions (of the supergravity equation in the finite-temperature physics, and of the minimal hyper-surface equation for the loop operator) is much larger than the Planck scale, the tree level approximation in M-theory should apply. In general, the dictionary of the AdS/CFT correspondence for non-BPS operators is not well established, compared to BPS operators. Hence for some non-BPS observables, the validity of the tree level approximation is not immediately clear.

Conclusion and Discussion

The most crucial point in our observation is that a very simple equivalence on the gravity (M-theory) side translates into a highly non-trivial equivalence on the gauge theory side. This is because we have considered the M-theory regime where $k$ is kept finite while $N$ is taken to be large. It is remarkable that one can predict many relations in this limit, which is considered to be less under control than the 't Hooft limit, or equivalently the type IIA regime, where $N/k$ is fixed and both $k$ and $N$ are large.

Our argument brings in the following crucial question, “how should one distinguish, on the gauge theory side, the tree and loop processes in the dual gravity theory?” or “what is the loop expansion parameter in the gravity theory, when expressed in the language of the gauge theory?” In the standard $AdS_5/CFT_4$ duality and the $AdS_4/CFT_3$ in the type IIA regime, the answers are of course provided by the 't Hooft expansion [24]: loop level processes (on the gravity side) correspond to non-planar diagrams and the expansion parameter is $1/N$. It is natural that these answers fail in the M-theory regime, as the 't Hooft expansion implies that the dual theory is a theory of strings, whereas the fundamental degrees of freedom of eleven-dimensional M-theory are not strings. It is clearly important to answer these questions in the M-theory regime; the orbifold equivalence may well be useful here as it provides a way to discriminate the tree level and the loop level effects in M-theory.

It is also important to find a way to understand the equivalence in purely field theoretic

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12 Non-BPS operators are not protected by supersymmetry and hence equivalence between them would be more difficult to show in the gauge theory. We note however that, in ABJM theories (as opposed to the four-dimensional $\mathcal{N} = 4$ Yang-Mills theory), overall coefficients of even three-point correlation functions of BPS operators are believed not to be protected, though scaling dimensions are believed to be protected; see e.g. [20].

13 It may be pointed out that we are implicitly assuming that M-theory obeys usual principles of quantum mechanics; in particular we are assuming the existence of a semi-classical expansion, or the distinction between tree-level and loop-level processes in M-theory.

14 We thank D. Young for a very useful discussion on this point.
terms, which is available for the usual planar orbifold equivalence [8]. Once one understands the equivalence without using the gravitational dual, it may be possible to generate many examples of orbifold equivalence between very strongly coupled Chern-Simons theories including inherently non-supersymmetric ones. These equivalences, in particular those between non-supersymmetric theories, may have applications in condensed matter. Although supersymmetry is often considered to be crucial for the gauge/gravity duality, purely field theoretical orbifold equivalence (in the planar limit) holds without relying on supersymmetry. Indeed, this idea was pursued in [12] as a viable strategy to study finite density QCD theories, circumventing the infamous sign problem.

The orbifold equivalence considered in this paper would provide a nontrivial test of the $AdS_4/CFT_3$ duality conjecture in M-theory, although checking it directly is not an easy problem. The formulation of M-theory is not yet established, and the dynamics of the best candidate for such a formulation in flat spacetime, the matrix model [25, 26], is far from being understood. M-theory dynamics on curved spacetime, including $AdS_4 \times S^7$, is even less explored. On the gauge theory side, for the ABJM theory, not even a non-perturbative formulation suitable for computer simulation is known, although in the ’t Hooft limit, a matrix model formulation [28] following the large-$N$ reduction technique introduced in [29] is available. However, experience with planar orbifold equivalences [8] suggests that it is not necessary to understand the full dynamics in order to prove the equivalence; in that case the proof was rather kinematical. Thus, orbifold equivalence may well be a good starting point to study the M-theoretic aspects of $AdS_4/CFT_3$ duality. It seems to be possible to relate our non-planar orbifold equivalence to a conjectured mirror symmetry in three dimensional gauge theories [30, 31]: the non-planar orbifold equivalence in this paper may be understood as a planar orbifold equivalence between the mirror theories [32]. We hope to report progress in this direction in the near future.

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