The structure constants of geometric structures on a local Lie group

Ercüment H. Ortaçgil

May 12, 2022

Abstract

We define the structure constants of the canonical almost complex, almost symplectic and Riemannian structures on a local Lie group.

1 A short review of absolute parallelizm

We will recall some formulas from Part I of [O1] that will be needed below and refer to [O1], [O2] for further details.

Let $M$ be a smooth manifold with the principal frame bundle $F(M) \to M$. Elements of $F(M)$ are 1-jets of local diffeomorphisms (called 1-arrows in [O1]) with source at the origin of $o$ of $\mathbb{R}^n$, $n = \dim M$, and target at some $x \in M$. $M$ is parallelizable if and only if the structure group $GL(n, \mathbb{R})$ of $F(M) \to M$ can be reduced to the identity. Such a reduction $w$, which we fix once and for all, is called the structure object in [O1] and called a framing by topologists. For each $x \in M$, $w$ assigns a unique 1-arrow from $o$ to $x$. In terms of coordinates around $x$, $w$ is of the form $w^i_{(j)}(x)$. The parenthesis around the index $j$ indicates that $w^i_{(j)}(x)$ is not subject to any transformation upon a coordinate change $(x) \to (y)$ on $M$, since this index refers to the standard coordinates in $\mathbb{R}^n$. We define $z(x) = w(x)^{-1}$, which is a 1-arrow from $x \in M$ to $o \in \mathbb{R}^n$, so that

$$w^i_{(a)}(x)z^a_{(j)}(x) = \delta^i_j \quad z^a_{(i)}(x)w^a_{(j)}(x) = \delta^a_{(j)}$$

with the standard summation convention. Note that the linear map $\delta^i_j$ is the identity map at the tangent space $T_x(M)$ whereas $\delta^a_{(j)}$ is the identity map at $T_o(\mathbb{R}^n) = \mathbb{R}^n$. The structure object $w$ induces a transitive groupoid $\varepsilon$ on $M$ whose 1-arrows on $M$ are given by

$$\varepsilon^i_j(x, y) \overset{\text{def}}{=} w^i_{(a)}(y)z^a_{(j)}(x)$$

We define
\[
\Gamma_{jk}^i(x) \overset{def}{=} \begin{bmatrix}
\frac{\partial \varepsilon_{ji}(x,y)}{\partial y^k} \\
\frac{\partial \varepsilon_{ji}(x,y)}{\partial x^k}
\end{bmatrix}_{y=x} = \begin{bmatrix}
z_j^{(a)}(x) \frac{\partial w_{ai}^i(x)}{\partial x^k} \\
- \frac{\partial \varepsilon_{ji}(x)}{\partial x^k} w_i^{(a)}(x)
\end{bmatrix} = y = x = z \overset{def}{=} \begin{bmatrix}
w_{ai}^i(x) \\
-w_i^{(a)}(x)
\end{bmatrix}
\]

\[
T_{jk}^i(x) \overset{def}{=} \Gamma_{jk}^i(x) - \Gamma_{kj}^i(x)
\]

Now \( T = (T_{jk}^i) \) is called the integrability object of \( T \) in [O2]. \( \Gamma = (\Gamma_{jk}^i) \) defines a first order linear operator \( \bar{\nabla} \) which may be interpreted as linear connection on the tangent bundle of \( M \). With this interpretation, \( T \) turns out to be the torsion of this connection and is called the torsion of \((M, w)\) in [O1], a misleading name modified in [O2]. The linear curvature \( R \) is defined by

\[
R_{ijk} = \Gamma_{jk}^i(x) - \Gamma_{kj}^i(x)
\]

Substituting (2) into (7) gives

\[
w_{ai}^i(y)z_j^{(a)}(x)T_{cd}^b(x)w_{c}^e(x)z_k^{(e)}(y)w_i^{(f)}(x)z_k^{(f)}(y) = T_{jk}(y)
\]

Substituting (2) into (7) gives

\[
w_{ai}^i(y)z_j^{(a)}(x)T_{cd}^b(x)w_{c}^e(x)z_k^{(e)}(y)w_i^{(f)}(x)z_k^{(f)}(y) = T_{jk}(y)
\]

Seperating the variables in (8), (8) is equivalent to

\[
T_{jk}(y) = T_{jk}(y) = T_{jk}(y)
\]

From (9) we conclude that the expression on the left hand side of (9) is independent of the variable \( x \). Therefore, this expression is constant on \( M \) and there exist some constants \( C_{(j)(k)} \) such that

\[
z_{a}^{(i)}(x)T_{bc}^a(x)w_{(j)}^b(x)w_{(k)}^c(x) = z_{a}^{(i)}(y)T_{bc}^a(y)w_{(j)}^b(y)w_{(k)}^c(y)
\]

It is desirable to the express the important formula (10) in a coordinate free language. Given the parallelizable manifold \((M, w)\), we choose some \( p \in M \) and define a bilinear form \( T(p) : T_p(M) \times T_p(M) \rightarrow T_p(M) \) by the formula

\[
T_{jk}(x) \overset{def}{=} T_{jk}(x) - \Gamma_{kj}^i(x)
\]
\[ [T(p)(\xi, \mu)]^i \stackrel{\text{def}}{=} T^{ia}_{ob}(\overline{p}) \xi^a \mu^b \]  

where \( \overline{p} = (\overline{p}) \) are the coordinates of \( p \) in the coordinate system used in (11). Clearly \( T(p) \) is skew-symmetric by (4) but does not necessarily satisfy the Jacobi identity, i.e., it does not define a Lie algebra. Now \( z(p) = (z_i^{(j)}(\overline{p})) \) defines an isomorphism \( \sharp z_p : T_p(M) \to T_o(\mathbb{R}^n) \) by the formula \( \sharp z_p(\xi) \stackrel{\text{def}}{=} z_a^{(i)}(\overline{p}) \xi^a \). This isomorphism extends to the tensor algebra \( \sharp z_p : T^{\tau, \sigma}_p(M) \to T^{\tau, \sigma}_p(M) \). Let \( A(o) \) be the image of \( T(p) \) under this isomorphism. Doing this construction at some \( q \in M \), let \( B(o) \) be the image of \( T(q) \). Now we need not have \( A(o) = B(o) \) because the isomorphism \( \sharp z(p, q) : T_p(M) \to T_o(\mathbb{R}^n) \) (which extends to the tensor algebra in the same way) need not map \( T(p) \) to \( T(q) \). Indeed, we have \( \sharp z_q \circ \sharp z(p, q) = \sharp z_p \) by (2) and therefore \( \sharp z_q \circ \sharp z(p, q) T_p(M) = \sharp z_p T_p(M) \). Thus \( \sharp z(p, q) T_p(M) = T_o(M) \) implies \( A(o) = B(o) \). However, \( \Re = 0 \) implies \( \varepsilon \)-invariance of \( T \) and therefore the independence of \( \sharp z_p(T(p)) = C \in T^{1,2}_o(\mathbb{R}^n) \) on \( p \). Now we can easily show that (11) becomes a Lie algebra whose structure constants with respect to the coordinate basis at \( p \) used in (11) are the constants \( C^{(i)}_{(j)(k)} \) defined by (10). Furthermore, (11) is the localization at \( p \) of some \( \varepsilon \)-invariant Lie algebra of vector fields on \( M \) under the usual bracket.

This coordinate free argument indicates a very simple general principle underlying the above constancy condition.

2 A general principle on a LLG

We observe that \( w_{(j)}^{(i)}, 1 \leq j \leq n = \dim M, \) define \( n \) independent vector fields \( w_{(j)} \) on \( M \) and the tensor \( T = (T_{i}^{jk}) \) is defined in terms of the first order derivatives of these vector fields, i.e., \( T \) is a first order joint tensorial differential invariant (tdi) of \( w_{(j)} \). Now given a smooth manifold \( M \), let \( \alpha \) be a tensor field on \( M \) and \( \beta \) another tensor field defined using the \( k \)’th order derivatives of \( \alpha \), i.e., \( \beta \) is a tdi of \( \alpha \) of order \( k \). Here are three well known examples.

1) If \( \alpha \) is a \( m \)-form, its exterior derivative \( d\alpha = \beta \) is a tdi of \( \alpha \) of order one.

2) If \( \alpha \) is a Riemannian metric, its Riemann curvature tensor, Ricci tensor and scalar curvature are tdi’s of \( \alpha \) of order two.

3) If \( \alpha \) is an almost complex structure, then its Nijenhuis tensor \( \beta = N(\alpha) \) is a tdi of order one.

In fact, we can replace \( \alpha \) with \( r \) tensor fields \( \alpha_1, \ldots, \alpha_r \) and assume that \( \beta \) is a joint tdi of \( \alpha_i \)’s as in our main example of \( (M, w) \). Our principle applies also in this case, but for simplicity, we will assume \( r = 1 \) henceforth.

Now we assume that \( \alpha \) is an \( \varepsilon \)-invariant \( (r, s) \)-tensor field on some parallelizable \( (M, w) \). By the above argument \( \sharp z_p(\alpha(p)) \in T^{\tau, \sigma}_o(\mathbb{R}^n) \) does not depend on \( p \in M \). We call the components of this tensor the structure constants of \( \alpha \). As in (10), we can express these components in local coordinates around \( p \). In this setting, our general principle is

\textbf{GP}: If \( \Re = 0 \), then any tdi \( \beta \) of \( \alpha \) of any order is also \( \varepsilon \)-invariant.
Indeed, \( \mathcal{R} = 0 \) implies that the groupoid \( \Upsilon \) of 1-arrows integrates to the pseudogroup \( \mathcal{G} \) whose local diffeomorphisms leave \( \alpha \) invariant since \( \alpha \) is \( \varepsilon \)-invariant. Differentiation of this invariance condition up to order \( k \) shows that \( \mathcal{G} \) leaves \( k \)-jets of \( \alpha \) invariant for any \( k \). It follows that \( \mathcal{G} \) leaves also \( \beta \) invariant. However 1-arrows induced by \( \mathcal{G} \) belong to \( \Upsilon \) which means that \( \beta \) is \( \varepsilon \)-invariant. Consequently, the tensor \( \tilde{z}_p(\beta(p)) \) is constant whose components we call the structure constants of \( \beta \).

So far so good, but now comes a nontrivial fact.

**Proposition 1** Let \((M, w, \mathcal{G})\) be a LLG, i.e., some parallelizable \((M, w)\) with \( \mathcal{R} = 0 \). Let \( \alpha \) be an \( \varepsilon \)-invariant tensor field and \( \beta \) a tdi of \( \alpha \) (which is also \( \varepsilon \)-invariant by GP). Then the structure constants of \( \beta \) can be expressed in terms of the structure constants of \( \alpha \) and the structure constants of the Lie algebra defined by (11).

Proposition 1 holds trivially for \( a = w \) and \( \beta = T \) by (10). Rather than attempting a rigorous statement and proof of Proposition 1, we will illustrate it with three well known examples.

### 3 Three examples

1) Let \( \hat{J} \) be the \( 2n \times 2n \) matrix whose diagonal consists of \( n \) blocks of \( \hat{J} = 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \). Now \( \hat{J} \) defines a linear map of \( \mathbb{R}^{2n} \) with the property \( \hat{J}^2 = -I_{2n} \).

Given some parallelizable \((M, w)\), we can carry \( \hat{J} \) from \( \mathbb{R}^{2n} \) onto \( M \) by defining

\[
J^i_j(x) \overset{def}{=} w^i_{(a)}(x) \hat{J}^{(a)}_{(b)} \tilde{z}^j_{(b)}(x)
\] (12)

Clearly \( J = (J^i_j) \) is a 1-1 tensor field on \( M \), that is, a linear map at the tangent space \( T_p(M) \) for all \( p \in M \) and we easily check that \( J^2 = -I \), i.e., \( J \) is an almost complex structure on \( M \). Clearly \( \tilde{\nabla} J = 0 \) or equivalently \( J \) is \( \varepsilon \)-invariant. The structure constants of \( J \) are the components of \( \hat{J} \). We call \( J \) the canonical almost complex structure of \((M, w)\).

We recall that the Nijenhuis tensor \( N(J) \) of \( J \) is defined by

\[
N(J)^i_{jk} = \left[ J^a_j \frac{\partial J^i_k}{\partial x^a} + J^a_i \frac{\partial J^j_k}{\partial x^a} \right]_{[jk]} = J^a_j \frac{\partial J^i_k}{\partial x^a} + J^a_i \frac{\partial J^j_k}{\partial x^a} - J^a_k \frac{\partial J^i_j}{\partial x^a} - J^a_i \frac{\partial J^j_k}{\partial x^a} \quad (13)
\]

It is easy to deduce the expression in (13) in coordinates: Let \( \varpi \in (U, x^i) \), \( J = (J^i_j(x)) \) be a matrix defined on \((U, x^i)\) satisfying \( J^2 = -I \) on \( U \). We can always find a coordinate change \((x) \to (y)\) such that \( J(\varpi) = \tilde{J} \), that is, we can always normalize the value \( J(\varpi) \). Can we also normalize its derivative as zero, i.e., can we find a coordinate change \((x) \to (y)\) with the property \( \frac{\partial J_j^i}{\partial y^a} = 0? \)
Some elementary computations now show that a necessary condition is $N(J) = 0$ at $\pi$. The local converse of this statement is the Newlander-Nirenberg theorem: If $N(J) = 0$ on $(U, x^i)$, then there exists a coordinate change $(x) \to (y)$ such that $J(y) = \tilde{J}$ on $(V, y^i)$.

Now we substitute (12) into (13). After some straightforward computation, we find

$$N(J)_{jk}^i = [X_{jk}^i]_{jk} = X_{jk}^i - X_{kj}^j$$

where $X_{jk}^i$ is the expression

$$X_{jk}^i \overset{df}{=} \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial w^i}{\partial x^k} \partial_{z^j} \frac{\partial z^j}{\partial x^k} = \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial w^i}{\partial x^k} \partial_{z^j} \frac{\partial z^j}{\partial x^k}$$

We handle the first term on the right hand side of (15) as follows:

$$w^a_{(b)} \tilde{\gamma}^b_{(c)} z_j \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k} = w^a_{(b)} \tilde{\gamma}^b_{(c)} z_j \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k}$$

$$= w^a_{(b)} \tilde{\gamma}^b_{(c)} z_j \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k}$$

$$= \frac{\partial w^a_{(b)}}{\partial x^{\alpha}} \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k}$$

where we substituted from (1) and (3). Similar computation for the second and third terms give

$$= \frac{\partial w^a_{(b)}}{\partial x^{\alpha}} \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k}$$

For the fourth term we have

$$= \frac{\partial w^a_{(b)}}{\partial x^{\alpha}} \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k} = \frac{\partial w^a_{(b)}}{\partial x^{\alpha}} \frac{\partial w^i}{\partial x^{\alpha}} \partial_{x^j} \frac{\partial z^j}{\partial x^k}$$
From (15), (16), (17), (18) and (19) we conclude
\[ w^i_{(b)} \tilde{J}^{(c)}_{(e)} z^a_{(d)} w^a_{(d)} \frac{\partial}{\partial x^k} = w^i_{(b)} \tilde{J}^{(c)}_{(e)} \gamma_d^{(e)} \frac{\partial}{\partial x^k} \]
\[ = w^i_{(b)} \tilde{J}^{(c)}_{(e)} \gamma_d^{(e)} \frac{\partial}{\partial x^k} \]
\[ = -w^i_{(b)} \gamma_d^{(e)} \frac{\partial}{\partial x^k} \]
\[ = -w^i_{(a)} \frac{\partial}{\partial x^k} \]
\[ = -\Gamma^i_{jk} \]

(19)

From (15), (16), (17), (18) and (19) we conclude
\[ X^i_{jk} = J^b_{k} J^a_{j} \Gamma^i_{ba} = J^a_{j} J^b_{ka} + J^a_{a} J^b_{bk} J^i_{j} - \Gamma^i_{jk} \]

(20)

and therefore
\[ N(J)^i_{jk} = X^i_{jk} - X^i_{kj} \]
\[ = T^i_{ab} J^a_{k} J^b_{j} + J^a_{a} T^b_{bk} J^i_{j} - J^a_{a} T^b_{bj} J^i_{k} - T^i_{jk} \]

(21)

We observe that (21) holds on any parallelizable \((M, \omega)\). If \(\mathfrak{R} = 0\), note that \(\varepsilon\)-invariance of \(N(J)\), a consequence of GP, can be seen directly from (21) by applying \(\nabla\) to both sides of (21) in view of (5). Using (11), we can write (21) in a coordinate free form as
\[ N(J)(u,v) = [J(u), J(v)] + J[J(u), v] - J[J(v), u] - [u, v] \]

(22)

Note that \([ , ]\) is the usual bracket of vector fields in (22) which localizes at any point according to (11). Generalizing this bracket to Frölicher-Nijenhuis bracket, we recall that (22) can be taken as the modern definition of the Nijenhuis tensor on any almost complex manifold. We observe that \(N(J)^a_{ak} = -2T^a_{ak}\) by (21) so that \(N(J)\) defines the same secondary characteristic classes as \(T\) if \(\mathfrak{R} = 0\) (see Chapter 12 of [O1]).

Now if \(\mathfrak{R} = 0\), then \(N(J)\) is \(\varepsilon\)-invariant by GP and therefore \(\natural z_p (N(J)(p)) \in T^1_a \mathbb{R}^{2n}\) is constant. In coordinates, this means
\[ z_{(i)}^a N(J)^a_{bc} u^b_{(j)} w^c_{(k)} = N(J)^{(i)}_{(j)(k)} \]

(23)
in the same way as (10). Evaluating (21) at \(p\) and translating (21) to \(o \in \mathbb{R}^{2n}\) by the isomorphism \(\natural z_p\) amounts to contracting both sides of (21) with the components of \(z\) and \(w\). The result is
\[ N(J)^{(i)}_{(j)(k)} = C^{(i)}_{(a)(b)} \tilde{J}^{(a)}_{(k)} \tilde{J}^{(b)}_{(j)} + C^{(a)}_{(b)(k)} \tilde{J}^{(i)}_{(a)} \tilde{J}^{(b)}_{(j)} - C^{(a)}_{(b)(j)} \tilde{J}^{(i)}_{(a)} \tilde{J}^{(b)}_{(k)} - C^{(i)}_{(j)(k)} \]

(24)
which expresses the structure constants of $N(\gamma)$ in terms of the structure constants of $J$ and the Lie algebra (11). We observe that (21) and (24) are essentially the same formulas but (21) holds on $(M, w)$ whereas (24) needs $R = 0$.

2) Let $\widehat{\omega}$ be the $2n \times 2n$ matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We define the skew-symmetric and nondegenerate 2-form $\omega = (\omega_{ij})$ on $(M, w)$ by

$$\widehat{\omega}(a)(b) z_i^{(a)} z_j^{(b)} = \omega_{ij}(x)$$  \hspace{1cm} (25)

We call $\omega$ the canonical almost symplectic structure of $(M, w)$ which is $\varepsilon$-invariant with structure constants as the components of $\widehat{\omega}$. An easy computation shows that a necessary condition to normalize the 1-jet of $\omega$ as zero is the vanishing of the exterior derivative

$$ (d\omega)_{kij} = \frac{\partial \omega_{kj}}{\partial x^i} - \frac{\partial \omega_{ik}}{\partial x^j} $$ \hspace{1cm} (26)

(omitting the factor $\frac{1}{3}$) at that point and the local converse of this statement is the Darboux theorem. We substitute (25) into (26) and what we find is

$$ (d\omega)_{kij} = T^a_k \omega_{ja} - T^a_k \omega_{ia} - T^a_k \omega_{ka} $$ \hspace{1cm} (27)

or

$$ d\omega(\xi, \gamma, \rho) = \omega(\rho, [\xi, \gamma]) - \omega(\gamma, [\xi, \rho]) - \omega(\xi, [\rho, \gamma]) $$ \hspace{1cm} (28)

which holds on $(M, w)$. Note that (28) is "half" of the well known formula for the exterior derivative (the other half vanishes since $\omega$ is $\nabla$-parallel and $\nabla$ is "formal Lie derivative" ([O2]). If $R = 0$, (27) gives

$$ (d\omega)_{(i)(j)(k)} = T^a_{(k)(i)} \omega^{(j)(a)} - T^a_{(k)(j)} \omega^{(i)(a)} - T^a_{(j)(i)} \omega^{(k)(a)} $$ \hspace{1cm} (29)

3) This is by far the most important example and is studied to some extent in [O3] (we warn the reader that some formulas in [O3] are not correct as they stand). The local computations are quite more involved compared to the above examples since Riemann curvature tensor is a second order tdi of the canonical metric defined by

$$ g_{ij}(x) \equiv g_{(a)(b)} z_i^{(a)} z_j^{(b)} $$ \hspace{1cm} (30)

where $g_{ij} \equiv \delta_{ij}$ and therefore $g_{ij} = z_i^{(a)} z_j^{(a)}$ with a summation over $a$. We will leave the details of these computations to the interested reader. However, there is an important issue here which has been lingering for a long time and needs to be settled: Since $g$ is $\varepsilon$-invariant, the scalar curvature $c(x)$ is also $\varepsilon$-invariant by GP if we assume $R = 0$. However a function is $\varepsilon$-invariant if and only if it is constant and there the $c(x)$ is constant. On the other hand, $c$ can be expressed in terms of the structure constants of the Lie algebra defined by (11) according to Proposition 1. We invite the interested reader to derive this expression. Therefore, Pommaret’s reasoning that "a single constant can not be related to
any Lie algebra” which he uses to support his conceptual framework in many of his works (see, for instance [P], pg.29), is not justified. He also claims in various places (see again [P], pg.29) that the modern concept of curvature must be revisited due to a confusion between the Janet and Spencer sequences. We believe that there are two concepts of curvatures in geometry. The first one is well known and is widely accepted by the geometers: It originates from Riemannian geometry where the central concept is a connection and curvature is an invariant of a connection together with a very intriguing invariant called torsion. As a peculiarity among others, only connections on “certain” principal bundles have torsion! The second one emerges from Lie theory as a ”local obstruction to the integration of the groupoid of k-arrows to a pseudogroup” or equivalently, as a local obstruction to the formal integrability of certain geometric PDE’s where a ”connection” appears as part of the definition of these PDE’s. This second curvature is formulated, in its simplest incarnation $k = 0$, in Part I of [O1] and studied further in [O1], [O2] where torsion becomes the measure of the difference between the ”left-right actions”. Even though the distinction between these two curvatures may seem to be only a matter of interpretation at first sight, it has some quite unexpected consequences.

Given some parallelizable $(M, w)$ we can now use the notation $(M, w, J, \varpi, g)$ where the last three structures are canonically defined as above and any two of them determines the other. This raises a natural question: What are the relations between the tdi’s of these structures? Another question is how these arguments generalize to prehomogeneous geometries in [O1], [O2].

We will conclude this note by expressing our conviction that a parallelizable manifold and more generally a prehomogeneous geometry is an extremely rich geometric structure encompassing most (if not all) of differential geometry originating from Klein’s Erlangen Program.

References

[O1] E.H.Ortaçgil: An Alternative Approach to Lie Groups and Geometric Structures, OUP, 2018
[O2] E.H.Ortaçgil: Curvature without connection, arXiv: 2003.06593, 2020
[O3] E.H.Ortaçgil: The canonical geometry of a local Lie group, arXiv: 2004.04029, 2020
[P] J.F.Pommaret: Partial Differential Equations and Group Theory, New Perspectives for Applications, Kluwer Academic Publishers, 1994

Ercüment Ortaçgil
ortacgile@gmail.com