First Integrals of holonomic systems without Noether symmetries

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Abstract

A theorem is proved which determines the first integrals of the form

\[ I = K_{ab}(t, q)\dot{q}^a\dot{q}^b + K_a(t, q)\dot{q}^a + K(t, q) \]

of autonomous holonomic systems using only the collinearizations of the kinetic metric which is defined by the kinetic energy or the Lagrangian of the system. It is shown how these first integrals can be associated via the inverse Noether theorem to a gauged weak Noether symmetry which admits the given first integral as a Noether integral. It is shown also that the associated Noether symmetry is possible to satisfy the conditions for a Hojman or a form-invariance symmetry therefore the so-called non-Noetherian first integrals are gauged weak Noether integrals. The application of the theorem requires a certain algorithm due to the complexity of the special conditions involved. We demonstrate this algorithm by a number of solved examples. We choose examples from published works in order to show that our approach produces new first integrals not found before with the standard methods.

1 Introduction

The symmetries of dynamical systems, described by differential equations, are described/understood in two different ways.

\begin{itemize}
  \item[a.] The geometric method. In this method one identifies the differential equation with the set of all the solution curves. Then a symmetry of the differential equation is understood as a point transformation in the appropriate space of the variables which preserves the set of solution curves in the sense, that takes one point of a solution curve to a point on another solution curve. The point transformation is defined by means of a vector field which is called the generator of the transformation. If the point transformation depends only on the variables and not on their derivatives it is just called a point transformation and the generator is a vector field in the (base) space defined by the variables of the equation. If the point transformation contains derivatives of the dependent variables it is called a generalized transformation (contact transformations, B"acklund transformations etc.) whereas the generator of the transformation is a vector field on the appropriate jet space over the base space. Finally if the solution curves are parameterized then the point transformation is possible to preserve or not the parameter and it is characterized accordingly.
  \item[b.] The algebraic method. In this method the symmetry of the equation is a change of coordinates in the space of the variables so that in the new variables the resulting differential equation has the same form as the original, therefore it has the same set of solutions. It is possible that in the new variables the symmetries of the differential equation are evident or they can be understood. If this is the case, they are computed in the new variables and then by the inverse transformation one gets them in the original variables.
\end{itemize}

The geometric method is the prevailing one in practice. It was initiated and systematized by Sophus Lie\textsuperscript{1,\textit{2,3}} who used the theory of continuous transformation groups to define the Lie symmetries. Using the Lie symmetries one may define appropriate variables in which the differential equation is reduced and in general it is simplified. Furthermore it is possible that a Lie symmetry leads to a first integral (FI).

In the present work we consider the autonomous holonomic dynamical system of the form

\[ \ddot{q}^a = \omega^a(q) \]
where $\omega^a = -\Gamma^a_{bc} \dot{q}^b \dot{q}^c - V^a(q) + F^a(q, \dot{q})$. In (1) $\Gamma^a_{bc}$ are the Riemann connection coefficients of the metric $\gamma_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}$ defined by the Lagrangian (or the kinetic energy), $V$ stands for all conservative forces, $F^a$ for the non-conservative generalized forces and the Einstein summation convention is used. A dot indicates total derivative wrt the parameter $t$. Every such equation defines the Hamiltonian vector field

$$\Gamma \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^a \frac{\partial}{\partial q^a} + \omega^a \frac{\partial}{\partial \dot{q}^a},$$

The condition that the vector field

$$X = \xi(t, q, \dot{q}) \partial_t + \eta^a(t, q, \dot{q}) \partial_{q^a}$$

is a Lie symmetry of (1) is that there exists a function $\lambda(t, q, \dot{q})$ such that

$$[X^{[1]}, \Gamma] = \lambda(t, q, \dot{q}) \Gamma$$

where

$$X^{[1]} = \xi(t, q, \dot{q}) \partial_t + \eta^a(t, q, \dot{q}) \partial_{q^a} + \left(\dot{\eta}^a - \dot{\xi} \dot{q}^a\right) \partial_{\dot{q}^a}$$

is the first prolongation of $X$ in the first jet space $J^1(t, q, \dot{q})$. It can be shown that the Lie symmetry condition (3) can be written

$$X^{[2]} H^a = 0 \implies X^{[1]} H^a + \eta^{a[2]} = 0$$

where $H^a \equiv \ddot{q}^a - \omega^a, \eta^{a[2]} = \dot{\eta} - 2\dot{\xi} \dot{\dot{q}}^a - \dot{\dot{\xi}} \dot{q}^a$ and $X^{[2]} = X^{[1]} + \eta^{a[2]} \partial_{\dot{q}^a}$ is the second prolongation of $X$ in $J^2\{t, q^a, \dot{q}^a, \ddot{q}^a\}$.

As it is well-known a velocity-dependent Lie symmetry has an extra degree of freedom which allows the introduction of a scalar condition, in which case one speaks for a gauged Lie symmetry. The standard gauge condition is $\xi = 0$ which simplifies the Lie symmetry condition (5) as follows

$$X^{[1]}(\omega^a) = \Gamma (\Gamma(\eta^a)) = \ddot{\eta}^a.$$  

1.1 The case of First Integrals (FIs)

FIs are the most useful tool in the study of the dynamical equations. The first major contribution on this topic is the work of Noether who introduced the Noether symmetries. A Noether symmetry of a Lagrangian system is a Lie symmetry which satisfies in addition the Noether condition. Geometrically a Noether symmetry is a point transformation in $J^1(t, q, \dot{q})$ under which the action integral is invariant up to a perfect differential with zero endpoint variation, so that the resulting Euler-Lagrange equations (hence the set of solutions) remain the same. The important result is that to each Noether symmetry there corresponds a concomitant first integral, called a Noether integral.

The Noether integrals are usually autonomous but they can also be time-dependent. The time-dependent FIs are equally important as the autonomous ones. Indeed time-dependent FIs can be used to test the (Liouville) integrability and also the superintegrability of a Hamiltonian system. Specifically, the Liouville theorem on integrability requires $n$ functionally independent FIs (i.e. their gradients on the phase space are linearly independent) in involution of the form $I(q, p)$. However, one can also use time-dependent FIs of the form $H(q, p, t)$. It is to be noticed that both Theorems in Refs. 9 and 10 refer to a time-dependent Hamiltonian $H(q, p, t)$.

Perhaps the next most systematic important approach after the Noetherian one is the direct approach of Katzin and Levine, originated by Darboux, which is discussed in section 3.

An additional different approach has been developed by Hojman who showed that under certain conditions a Lie symmetry leads to a FI, called a Hojman integral. These FIs are coordinate-dependent therefore they are not useful (at least in Physics where the Covariance Principle requires that the physical quantities must be covariant wrt the fundamental group of the theory). Finally it has also been shown that under certain conditions a form-invariance symmetry is also possible to give a FI.

In conclusion, there are only two important and systematic approaches for the computation of the FIs: The Noether approach and the direct approach of Katzin and Levine.
2 The conditions for a weak Noether symmetry

For holonomic dynamical systems defined by a Lagrangian \( L(t, q, \dot{q}) \) and generalized non-conservative forces \( F^a \) one has to use the weak Noether condition\(^{15} \)

\[
X^W (L) + L\dot{\xi} = X^{[1]} (L) + \phi^a \frac{\partial L}{\partial \dot{q}^a} + L\dot{\xi} = \dot{f}
\]  

(7)

where the function \( f(t, q^a, \dot{q}^a) \) is called the Noether or the gauge function and to the vector field (2) corresponds the weak first prolongation

\[
X^W \equiv X^{[1]} + \phi^a (t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^a}.
\]  

(8)

The weak Noether condition leads to the FI

\[
I = f - L\xi - \frac{\partial L}{\partial \dot{q}^a} (\eta^a - \xi \dot{q}^a)
\]  

(9)

provided that the functions \( \phi^a (t, q, \dot{q}) \) are defined by the condition

\[
F_a (\eta^a - \xi \dot{q}^a) = \phi^a \frac{\partial L}{\partial \dot{q}^a}.
\]  

(10)

Substituting (10) in (7) we obtain the so-called Noether-Bessel-Hagen (NBH) equation

\[
X^{[1]} (L) + L\dot{\xi} + F_a (\eta^a - \xi \dot{q}^a) = \dot{f}.
\]  

(11)

For velocity-dependent Noether symmetries in the gauge \( \xi = 0 \) the latter conditions simplify as follows

\[
I = f - \frac{\partial L}{\partial \dot{q}^a} \eta^a
\]  

(12)

\[
F_a \eta^a = \phi^a \frac{\partial L}{\partial \dot{q}^a}.
\]  

(13)

The FIs of Hojman and the ones defined by the form-invariance symmetry have been called non-Noetherian FIs because the generators of the corresponding point transformations do not satisfy the weak Noether condition. However there is an alternative approach to look at the non-Noetherian and the Noetherian FIs. Indeed according to the Inverse Noether Theorem to every FI one may associate (in general) a velocity-dependent gauged Noether symmetry whose generator is not necessarily the same with the one deriving the non-Noetherian FI. Therefore, in a sense, all FIs are or can be Noether integrals. The Inverse Noether theorem for velocity-dependent Noether symmetries has as follows (see Ref. 15).

**Theorem 1 (Inverse Noether theorem)** Suppose \( \Lambda \) is a FI of a holonomic dynamical system with regular Lagrangian \( L(t, q^a, \dot{q}^a) \) and generalized non-conservative forces \( F^a (t, q, \dot{q}) \). Then the vector \( X = \xi (t, q, \dot{q}) \frac{\partial}{\partial t} + \eta^a (t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^a} \) with a weak first prolongation

\[
X^W = X^{[1]} + \phi^a (t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^a} = \xi \partial_t + \eta^a \partial_{\dot{q}^a} + \left( \dot{\eta}^a - \dot{q}^a \dot{\xi} + \phi^a \right) \partial_{\dot{q}^a}
\]  

(14)

is the generator of a weak Noether symmetry with gauge function \( f(t, q, \dot{q}) \) provided that

\[
\eta^a = -\gamma^{ab} \frac{\partial \Lambda}{\partial \dot{q}^b} + \xi \dot{q}^a
\]  

(15)

\[
\phi^a \frac{\partial L}{\partial \dot{q}^a} = -F^a \frac{\partial \Lambda}{\partial \dot{q}^a}
\]  

(16)

\[
\xi = \frac{1}{L} \left( f - \Lambda + \gamma^{ab} \frac{\partial L}{\partial \dot{q}^b} \frac{\partial \Lambda}{\partial \dot{q}^a} \right).
\]  

(17)

This weak Noether symmetry produces the given FI \( \Lambda \). Therefore any FI for such systems can be associated to a weak Noether symmetry.
Proof.

We write the Euler-Lagrange equations as follows

\[ \ddot{q}^a = \gamma^{ab} \left( F_b + \frac{\partial L}{\partial \dot{q}^b} - \frac{\partial^2 L}{\partial t \partial \dot{q}^b} - \frac{\partial^2 L}{\partial q^b \partial \dot{q}^c} \dot{q}^c \right) \]  

(18)

where \( \gamma_{ab} = \frac{\partial^2 F}{\partial q^a \partial q^b} \). Using (18) the FI condition \( \frac{\partial \Lambda}{\partial t} = 0 \) gives

\[ \frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial q^a} \dot{q}^a + \gamma_{ab} \frac{\partial \Lambda}{\partial q^b} \left( F_b + \frac{\partial L}{\partial \dot{q}^b} - \frac{\partial^2 L}{\partial t \partial \dot{q}^b} - \frac{\partial^2 L}{\partial q^b \partial \dot{q}^c} \dot{q}^c \right) = 0. \]  

(19)

Taking into account the above results it is sufficient to show that the weak Noether condition (7) is satisfied for the set \( (\xi, \eta^a, \phi^a, f) \) defined by the conditions (15) - (17). We compute

\[
\dot{\xi} = -L^{-2} \left( \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^c} \dot{q}^c + \frac{\partial L}{\partial q^b} \ddot{q}^b \right) \left( f - \Lambda + \gamma_{ab} \frac{\partial L}{\partial q^a} \frac{\partial \Lambda}{\partial q^b} \right) +
\]
\[+ L^{-1} \left[ \dot{f} + \gamma_{ab} \frac{\partial L}{\partial q^b} \frac{\partial \Lambda}{\partial q^a} + \gamma_{bc} \frac{\partial L}{\partial q^c} \frac{\partial \Lambda}{\partial q^b} \right]
\]
\[= -L^{-2} \left( \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^c} \dot{q}^c + \frac{\partial L}{\partial q^b} \ddot{q}^b \right) \left( f - \Lambda + \gamma_{ab} \frac{\partial L}{\partial q^a} \frac{\partial \Lambda}{\partial q^b} \right) +
\]
\[+ L^{-1} \left[ \dot{f} + \gamma_{ab} \frac{\partial L}{\partial q^b} \frac{\partial \Lambda}{\partial q^a} + F_a \frac{\partial \Lambda}{\partial q^a} + \gamma_{ab} \frac{\partial L}{\partial q^a} \frac{\partial \Lambda}{\partial q^b} + \gamma_{ab} \frac{\partial L}{\partial q^a} \frac{\partial \Lambda}{\partial q^b} \right].
\]

and

\[ \dot{\eta}^a = -\gamma_{ab} \frac{\partial \Lambda}{\partial q^b} - \gamma_{ab} \left( \frac{\partial^2 L}{\partial q^c \partial q^b} \ddot{q}^c + \frac{\partial^2 L}{\partial q^c \partial q^b} \dot{q}^c \right) + \xi \dot{q}^a + \eta^b \frac{\partial \Lambda}{\partial q^b}. \]

Substituting \( \xi, \eta^a \) from (15) - (17) and the total derivatives computed above in (17) we find that the weak Noether condition is trivially satisfied. Therefore the set \( (\xi, \eta^a, \phi^a, f) \) generates a weak Noether symmetry whose FI is the FI \( I = \Lambda \). This completes the proof.

In the case of the gauge \( \xi = 0 \) the conditions defining a gauged weak Noether (generalized) symmetry are reduced as follows

\[ \eta^a = -\gamma_{ab} \frac{\partial \Lambda}{\partial q^b} \]  

(20)

\[ \phi^a \frac{\partial L}{\partial q^a} = -F_a \frac{\partial \Lambda}{\partial q^a} \]  

(21)

\[ f = \Lambda + \eta^a \frac{\partial L}{\partial q^a}. \]  

(22)

Moreover, by applying the Inverse Noether Theorem to a general QFI of the form

\[ \Lambda = K_{ab}(t, q) \dot{q}^a \dot{q}^b + K_a(t, q) \dot{q}^a + K(t, q) \]

we deduce from the conditions (20) - (22) that \( \Lambda \) is associated to the gauged weak Noether symmetry as follows

\[ \eta^a = -\gamma_{ab} (2K_{bc} \dot{q}^c + K_b) \]  

(23)

\[ \phi^a \frac{\partial L}{\partial q^a} = -F_a (2K_{ab} \dot{q}^b + K_a) \]  

(24)

\[ f = K_{ab} \dot{q}^a \dot{q}^b + K_a \dot{q}^a + K - \gamma_{ab} (2K_{bc} \dot{q}^c + K_b) \frac{\partial L}{\partial q^a} \]  

(25)

where \( \frac{\partial \Lambda}{\partial q^a} = 2K_{ab} \dot{q}^b + K_a \).
For $L = \frac{1}{2}\gamma_{ab}\dot{q}^a\dot{q}^b - V(q)$ the conditions (23) - (25) become

$$\eta^a = -\gamma^{ab}(2K_a\dot{q}^b + K)$$

$$\left(\phi_a + 2K_{ab}\dot{F}^b\right)\dot{q}^a + K_aF^a = 0$$

$$f = -K_{ab}\dot{q}^a\dot{q}^b + K.$$  

It is easy to check that the gauged weak Noether symmetry

$$(\xi = 0, \eta_a = -2K_{ab}\dot{q}^b - K_a, \phi_a, f = -K_{ab}\dot{q}^a\dot{q}^b + K)$$

such that $\left(\phi_a + 2K_{ab}\dot{F}^b\right)\dot{q}^a + K_aF^a = 0$ does produce the Noether FI $\Lambda$. Another gauged weak Noether symmetry which generates the same result is the

$$(\xi = 0, \eta_a = -K_{ab}\dot{q}^b - K_a, \phi_a, f = K)$$

such that $\left(\phi_a + K_{ab}\dot{F}^b\right)\dot{q}^a + K_aF^a = 0.$

3 Lie symmetries, FIs and collineations

Over the years various works have appeared with the view to put the Lie and the Noether approach in geometric terms. The reason for this is that if one manages to relate the Lie and the Noether symmetries with the symmetries (collineations) of the kinetic metric, which is defined by the dynamical system itself, then one may use the vast results of Differential Geometry to compute the generators of the Lie/Noether symmetries admitted by the dynamical system and consequently the FIs. It appears that the first clear approach in this direction was done in the 80s mainly by Katzin and Levine.\textsuperscript{16,17,18,19,20,21}

These authors considered a holonomic conservative system and showed that the generators of the Lie point symmetries are the projective collineations (PCs) of the kinetic metric and the generators of the Noether point symmetries are the elements of the homothetic algebra of the kinetic metric. Perhaps these results were not emphasized enough and due to the special $\delta$-derivative approach they used, it appears that they passed rather unnoticed. As a result other works followed using the same approach\textsuperscript{22,23} concerning mainly special types of autonomous conservative systems (see also Refs 3 and 15). The results of these latter works follow as special cases of the general results of Katzin and Levine. A clear, complete and systematic presentation\textsuperscript{24} of this approach has been given recently in Ref. 24 which has lead to a number of interesting applications.\textsuperscript{25}

The approach of Katzin and Levine can be summarized as follows. Instead they to consider the Lie/Noether symmetries, which are the intermediate steps in the determination of the FIs, they focused directly on the computation of the FIs $I$ from the condition $\frac{dI}{dt} = 0$. To do that they considered the quadratic FIs (QFIs) $I$ of the form

$$I = K_{ab}(t, q)\dot{q}^a\dot{q}^b + K_a(t, q)\dot{q}^a + K(t, q)$$

where $K_{ab}$ is a symmetric tensor, $K_a$ is a vector and $K$ is an invariant and required that $\Gamma(I) = \frac{dI}{dt} = 0$. This requirement leads to a system of conditions for the coefficients $K_{ab}, K_a, K$ whose solution gives all second (and first) order autonomous and time-dependent QFIs admitted by the dynamical equations. In this approach the problem of determining the QFIs of dynamical equations is reduced to the solution of the resulting system of equations for the coefficients $K_{ab}, K_a, K$. Obviously these equations depend on the form of the general quantity $\omega$ (i.e. the particular dynamical system) whereas the solution of the resulting system is a formidable task.

In a recent paper\textsuperscript{26} (henceforth referred as paper A) the complete systematic solution of the system of these equations for the case of autonomous conservative dynamical systems has been given under the assumption that the tensors involved have the general form $K_{ab}(t, q) = g(t)C_{ab}(q)$, $K_a(t, q) = f(t)L_a(q) + B_a(q)$, where $g(t), f(t)$ are analytic functions. This solution (Theorem 1 of paper A) provided all the autonomous and the time-dependent QFIs of the dynamical equations $I$ for $F^a = 0$ in terms of the collineations (including the Killing tensors-KTs) of the kinetic metric.

The purpose of the present work is to generalize the results of paper A to the case of autonomous holonomic dynamical systems which move in a Riemannian space under the action of generalized forces of the form $F^a = -P^a(q) + A^a_0(q)\dot{q}^b$. The dynamical equations for these systems are

$$\ddot{q}^a = -\Gamma^a_{bc}\dot{q}^b\dot{q}^c - Q^a(q) + A^a_0(q)\dot{q}^b$$

where the generalized forces $Q^a = V^a + P^a$ contain all the forces conservative and non-conservative. We assume again that $I$ has the general form $\Gamma^a_{bc}$ and determine the system of equations resulting from the condition
$dI/dt = 0$ together with the integrability conditions for the scalar $K$. We solve this system of equations in terms of the collineations of the kinetic metric (including the KTs) and the result is stated in Theorem 2.

The structure of the paper is as follows. In section 4 we derive the system of equations which result from the condition $dI/dt = 0$ for the dynamical system (32). These equations reduce to the corresponding equations of paper A for $F^a = 0$. In section 5 we give the general solution of the system of equations as Theorem 2. In section 6 we study the behavior of the case Integral 1 of Theorem 2. In section 7 we give a brief theory concerning the determination of KTs in a Riemannian space in terms of the collineations of the kinetic metric. In sections 8, 9 we present some useful results concerning the KTs of the kinetic metric (including the KTs) and the result is stated in Theorem 2. In section 10 we demonstrate the significance of Theorem 2 by considering various examples. It is shown that Theorem 2 besides the Noether Fls also computes the Hojman integrals and finally in section 11 we draw our conclusions. In Appendix we sketch the proof of Theorem 2.

4 The conditions for a QFI

Equations (1) may be considered as the Euler-Lagrange equations for the Lagrangian $L(q, \dot{q}) = \frac{1}{2} \gamma_{ab} \dot{q}^a \dot{q}^b - V(q)$ with generalized forces $F_a$. The Lagrangian is assumed to be regular, that is $det \frac{\partial^2 L}{\partial q^a \partial \dot{q}^b} \neq 0$, and defines the new non-degenerate kinetic metric $\gamma_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}$. The kinetic metric is not the metric of the space where motion occurs except in the case of a free system (that is $V = 0, F^a = 0$) in which case equations (1) are the geodesic equations. In the following the covariant derivatives and the rising/lowering of indices are done with the kinetic metric $\gamma_{ab}$.

We consider the function $I$ given by (31) and using the dynamical equations (32) to replace the terms $\ddot{q}^a$ we write the condition $dI/dt = 0$ as

$$0 = K_{(ab)c}\dot{q}^a \dot{q}^b \dot{q}^c + \left(K_{ab} + K_{ab} + 2K_{c(ab)\dot{A}_b}^c\right) \dot{q}^a \dot{q}^b + \left(K_{a,t} + K_{a} - 2K_{ab}Q^b + K_{b} A_a^b\right) \dot{q}^a + K_{a} Q^a$$

from which follows the system of equations

$$K_{(ab)c} = 0$$
$$K_{ab,t} + K_{(ab)} + 2K_{a(t)\dot{A}_b}^c = 0$$
$$-2K_{ab}Q^b + K_{a,t} + K_{a} + K_{b} A_a^b = 0$$
$$K_{a} - K_{a} Q^a = 0.$$ (33) (34) (35) (36)

Condition (33) implies that $K_{ab}$ is a Killing tensor (KT) of order 2 (possibly zero) of the kinetic metric $\gamma_{ab}$.

The most general choice for the KT $K_{ab}$ in the case of an autonomous system is

$$K_{ab}(t, q) = C_{(0)ab}(q) + \sum_{N=1}^{n} C_{(N)ab}(q) t^N$$ (37)

where $C_{(N)ab}, N = 0, 1, ..., n$ is a sequence of arbitrary KTs of order 2 of the kinetic metric $\gamma_{ab}$.

This choice of $K_{ab}$ and equation (34) indicate that we set

$$K_a(t, q) = \sum_{M=0}^{m} L_{(M)a}(q) t^M$$ (38)

where $L_{(M)a}(q)$ are arbitrary vectors.

We note that both powers $n, m$ in the above polynomial expressions may be infinite.

Substituting (37), (38) in the system of equations (33) - (36) (equation (33) is identically zero since $C_{(N)ab}$ are assumed to be KTs) we obtain
0 = C_{(1)ab}t + C_{(2)ab}t^n + \cdots + C_{(n)ab}t^{n-1} + L_{(0)(a;b)} + L_{(1)(a;b)}t + \cdots + L_{(m)(a;b)}t^m + 2C_{(0)c(a)}A_b^c + 2C_{(1)c(a)}A_b^c t + \cdots + 2C_{(0)c(a)}A_b^c t^n \tag{39}

0 = -2C_{(1)ab}Q^b - 2C_{(2)ab}Q^b t - \cdots - 2C_{(n)ab}Q^b t^{n-1} + 2L_{(2)a} + 6L_{(3)a}t + \cdots + mL_{(m)a}t^{m-2} + L_{(0)b}Q^b t + \cdots + L_{(m)b}A_b^a t + mL_{(m)b}A_b^a t^m \tag{40}

0 = K_{,t} - L_{(0)a}Q^a - L_{(1)a}Q^a t - \cdots - L_{(m)a}Q^a t^m \tag{41}

Conditions 39 - 41 must be supplemented with the integrability conditions $K_{,at} = K_{,ta}$ and $K_{,[ab]} = 0$ for the scalar function $K$. The integrability condition $K_{,at} = K_{,ta}$ gives - if we make use of 40 and 41 - the equation

0 = -2C_{(1)ab}Q^b - 2C_{(2)ab}Q^b t - \cdots - 2C_{(n)ab}Q^b t^{n-1} + 2L_{(2)a} + 6L_{(3)a}t + \cdots + mL_{(m)a}t^{m-2} + (L_{(0)b}Q^b)_{,a} + \cdots + (L_{(m)b}A_b^a)_{,a} t + \cdots + (L_{(m)b}A_b^a)_{,a} t^m + L_{(1)b}A_b^a + 2L_{(2)b}A_b^a t + \cdots + mL_{(m)b}A_b^a t^m \tag{42}

Condition $K_{,[ab]} = 0$ gives the equation

0 = \left(2C_{(1)[a][c]}Q^c\right)_{,b} + 2\left(C_{(1)[a][c]}Q^c\right)_{,b} t + \cdots + 2\left(C_{(n)[a][c]}Q^c\right)_{,b} t^n - L_{(1)[a][b]} - 2L_{(2)[a][b]} t - \cdots - mL_{(m)[a][b]} t^{m-1} - \left(L_{(0)[c][b]}A_b^a\right)_{,a} - L_{(1)[c][b]}A_b^a t - \cdots - L_{(m)[c][b]}A_b^a t^{m} - L_{(0)\n}[a][b]}A_b^a t^{m} - L_{(1)\n}[a][b]}A_b^a t^{m} - L_{(m)\n}[a][b]}A_b^a t^{m} \tag{43}

which for 2d systems with $F^a = 0$ is known as the second order Bertrand-Darboux equation (see Ref. 11).

Equations 39 - 43 constitute the system of equations we have to solve.

5 The Theorem

The solution of the system of equations 39 - 43 can be found in the Appendix and it is stated in Theorem 2 below.

Theorem 2 The independent QFIs of a dynamical system 32 are the following28:

\textbf{Integral 1.}

\[ J_1 = \left(\frac{t^n}{n} C_{(n)ab} + \cdots + \frac{t^2}{2} C_{(2)ab} + tC_{(1)ab} + C_{(0)ab}\right) + t^n L_{(n)ab} Q^a + \cdots + t^2 L_{(2)ab} Q^a + tL_{(1)ab} Q^a + L_{(0)ab} Q^a + \cdots + \frac{t^{n+1}}{n+1} L_{(n)a} Q^a + \cdots + \frac{t^2}{2} L_{(1)a} Q^a + tL_{(0)a} Q^a + G(q) \]

where $C_{(N)ab}$ for $N = 0, 1, \cdots, n$ are KTs, $C_{(1)ab} = -L_{(0)(a;b)} - 2C_{(0)c(a)}A_b^c$, $C_{(k+1)ab} = -L_{(k)(a;b)} - \frac{2}{k} C_{(k)c(a)}A_b^c$ for $k = 1, \cdots, n-1$, $L_{(n)(a;b)} = -\frac{2}{n} C_{(n)c(a)}A_b^c$, $(L_{(k-1)b}Q^b)_{,a} = 2C_{(k)b}Q^b - k(k+1)L_{(k+1)a} - kL_{(k)b}A_b^a$ for $k = 1, \cdots, n-1$, $(L_{(n-1)b}Q^b)_{,a} = 2C_{(n)b}Q^b - nL_{(n)b}A_b^a$, $L_{(n)a}Q^a = s$ and $G_{,a} = 2C_{(0)b}Q^b - L_{(1)a} - L_{(0)b}A_b^a$.

\textbf{Integral 2.}

\[ J_2 = e^\lambda \left(\lambda C_{ab} Q^a + \lambda L_{ab} Q^a + L_{ab} Q^a\right) \]

where $\lambda \neq 0$, $C_{ab}$ is a KT, $\lambda C_{ab} = -L_{(a;b)} - 2C_{(0)c(a)}A_b^c$ and $(L_{b}Q^b)_{,a} = 2\lambda C_{ab}Q^b - \lambda^2 L_a - \lambda L_{ab}A_b^a$.

We note that in all cases $C_{(N)ab}$ are KTs of order two whereas in many cases the vector $Q^a$ is a KV. This emphasizes the already known result from previous studies (see Refs. 5, 22 and 28) of the important role played by the KTs and the KVs of the kinetic metric in the determination of the FIs of 32.

In the case $A_b^a = 0$ Theorem 3 takes the following form.

Theorem 3 The independent QFIs of the dynamical system 32 for $A_b^a = 0$ are the following:
\[ I_{(1)} = \left( -\frac{t^{2\ell}}{2\ell} L_{(2\ell-1)(a;b)} - \ldots - \frac{t^{4}}{4} L_{(3)(a;b)} - \frac{t^{2}}{2} L_{(1)(a;b)} + C_{ab} \right) \dot{q}^a \dot{q}^b + \frac{t^{2\ell}}{2\ell} L_{(2\ell-1)a} \dot{q}^a + \ldots + t^{3} L_{(3)a} \dot{q}^a + +tL_{(1)a} \dot{q}^a + \frac{t^{2\ell}}{2\ell} L_{(2\ell-1)a} Q^a + \ldots + \frac{t^{4}}{4} L_{(3)a} Q^a + \frac{t^{2}}{2} L_{(1)a} Q^a + G(q) \]

where \( C_{ab} \), \( L_{(M)(a;b)} \) for \( M = 1, 3, \ldots, 2\ell - 1 \) are KTs, \( L_{(2\ell-1)b} Q^b \), \( L_{(k-1)b} Q^b \) are KTs, \( (k + 1) L_{(k+1)a} \) for \( k = 2, 4, \ldots, 2\ell - 2 \) and \( G_{a} = 2C_{ab} Q^b - L_{(1)a} \).

\[ I_{(2)} = \left( -\frac{t^{2\ell+1}}{2\ell + 1} L_{(2\ell)(a;b)} - \ldots - \frac{t^{4}}{3} L_{(2)(a;b)} - tL_{(0)(a;b)} \right) \dot{q}^a \dot{q}^b + \frac{t^{2\ell+1}}{2\ell + 1} L_{(2\ell)a} \dot{q}^a + \ldots + t^{3} L_{(3)a} \dot{q}^a + +tL_{(1)a} \dot{q}^a + \frac{t^{2\ell+1}}{2\ell + 1} L_{(2\ell)a} Q^a + \ldots + \frac{t^{4}}{3} L_{(2)a} Q^a + tL_{(0)a} Q^a \]

where \( L_{M(a;b)} \) for \( M = 0, 2, \ldots, 2\ell \) are KTs, \( L_{(2\ell)b} Q^b \) and \( L_{(k-1)b} Q^b \) are KTs, \( k(k + 1) L_{(k+1)a} \) for \( k = 1, 3, \ldots, 2\ell - 1 \).

\[ I_{(3)} = e^{\lambda t} \left( -L_{(a;b)} \dot{q}^b + \lambda L_{a} \dot{q}^a + L_{a} Q^a \right) \]

where \( L_{a} \) is such that \( L_{(a;b)} \) is a KT and \( L_{a} Q^a = -2L_{(a;b)} Q^b - \lambda^2 L_{a} \).

We observe that for \( A_{\mathfrak{s}}^a = 0 \) the QFI \( J_1 \) breaks into two independent QFIs the \( I_{(1)} \) and \( I_{(2)} \) corresponding to even and odd powers of \( t \). The case of autonomous conservative dynamical systems is obtained if one sets \( Q^a = V^a \). Theorem \( \text{[38]} \) is a generalized version of Theorem 1 of paper \( \Lambda \) because the assumption \( \text{[37]} \) is more general than the one made in paper \( \Lambda \).

It is apparent that before one attempts to compute the QFIs of a given dynamical system of the form \( \text{[32]} \) using Theorem \( \text{[2]} \) one has to know the collineations of the kinetic metric including the second order KTs. This is not a trivial requirement. However because the kinetic metric is non-degenerate (the Lagrangian is assumed to be regular) it is always possible to bring it to its canonical form by means of a proper change of the coordinates and then use existing results of Differential Geometry to compute the collineations and its KTs. A particular important and fairly general case is that of spaces of constant curvature where these quantities are known.\( \text{[32]} \).

In general one has to use special methods to compute the KTs.\( \text{[33,34,35,36,37,38]} \)

In section \( \text{[4]} \) we consider briefly the determination of KTs from the projective collineations (PCs) in a Riemannian space and state the results for the case of spaces of constant curvature.

\section*{6 Computing \( J_1 \equiv I_n \) in terms of the \( I_0 \)}

We prove that all QFIs \( I_N \) where \( N = 1, 2, \ldots, n \) of the case \textbf{Integral 1} of Theorem \( \text{[2]} \) can be constructed from the QFI \( I_0 \) by using the following systematic algorithm.

1) Write the QFI \( I_0 \).

2) Introduce a new KT \( C_{(1)ab} \) and a new vector \( L_{(1)a} \).

3) Construct \( I_1 \) by adding to the expression \( I_0 \) the time-dependent terms \( tC_{(1)ab} \dot{q}^a \dot{q}^b \), \( tL_{(1)a} \dot{q}^a \) and \( \frac{t^2}{2} L_{(1)a} Q^a \).

4) Expand the conditions for \( I_0 \) so as to satisfy the requirement \( \frac{dI_0}{dt} = 0 \).

5) Continue in a similar manner with the construction of \( I_2 \) by using \( I_1 \).

6) After some steps use the \( I_{n-1} \) to construct \( I_n \) by adding the terms \( \frac{t^n}{n!} C_{(n)ab} \dot{q}^a \dot{q}^b \), \( t^n L_{(n)a} \dot{q}^a \) and \( \frac{t^{n+1}}{n+1} L_{(n)a} Q^a \).
We illustrate the above procedure for the small values of \(n\).

- For \(n = 0\):
  We have the QFI
  \[
  I_0 = C_{(0)ab}q^a q^b + L_{(0)a} q^a + st + G(q)
  \]
  where \(C_{(0)ab}\) is a KT and \(L_{(0)a}, G\) are computed from the expressions
  \[
  L_{(0)(a;b)} = -2C_{(0)c(a} A^c_{b)}, \quad L_{(0)b}Q^b = s, \quad G_{,a} = 2C_{(0)ab}Q^b - L_{(0)b}A^b_a.
  \]

- For \(n = 1\):
  We have the QFI
  \[
  I_1 = (tC_{(1)ab} + C_{(0)ab}) q^a q^b + tL_{(1)a} q^a + L_{(0)a} q^a + \frac{t^2}{2}s + tL_{(0)a}Q^a + G(q)
  \]
  where \(C_{(1)ab}\) is a KT computed from the relation
  \[
  C_{(1)ab} = -L_{(0)(a;b)} - 2C_{(0)c(a} A^c_{b)}
  \]
  and the vector \(L_{(1)a}\) and the quantity \(G\) are computed from the relations
  \[
  L_{(1)(a;b)} = -2C_{(1)c(a} A^c_{b)}, \quad L_{(1)a} Q^a = s, \quad (L_{(0)b}Q^b)_{,a} = 2C_{(1)ab}Q^b - L_{(1)b}A^b_a
  \]
  \[
  L_{(1)a} = 2C_{(0)ab}Q^b - L_{(0)b}A^b_a - G_{,a}.
  \]

- For \(n = 2\):
  We have the QFI
  \[
  I_2 = \left(\frac{t^2}{2} C_{(2)ab} + tC_{(1)ab} + C_{(0)ab}\right) q^a q^b + t^2 L_{(2)ab} q^a + tL_{(1)ab} q^a + L_{(0)ab} q^a + \frac{t^3}{3}s + \frac{t^2}{2}L_{(1)ab}Q^a +
  \]
  \[
  + tL_{(0)ab}Q^a + G(q)
  \]
  where \(C_{(2)ab}\) is a KT computed from the relation
  \[
  C_{(1)ab} = -L_{(0)(a;b)} - 2C_{(0)c(a} A^c_{b)}, \quad C_{(2)ab} = -L_{(1)(a;b)} - 2C_{(1)c(a} A^c_{b)}
  \]
  whereas the vector \(L_{(2)a}\) and the quantity \(G\) are computed from the relations
  \[
  L_{(2)(a;b)} = -C_{(2)c(a} A^c_{b)}, \quad L_{(2)ab} Q^a = s, \quad (L_{(1)b}Q^b)_{,a} = 2C_{(2)ab}Q^b - 2L_{(2)b}A^b_a
  \]
  \[
  L_{(1)a} = 2C_{(0)ab}Q^b - L_{(0)b}A^b_a - G_{,a}, \quad L_{(2)a} = C_{(1)ab}Q^b - \frac{1}{2}L_{(1)b}A^b_a - \frac{1}{2} (L_{(0)b}Q^b)_{,a}.
  \]

In a similar manner we continue for higher values of \(n\).

We observe that for all values of \(n\) the KTs \(C_{(N)ab}\), the vectors \(L_{(N)a}\) and hence the conditions for \(I_n\) can be written in terms of the triplet \(\{G(q), L_{(0)a}, C_{(0)ab} = KT\}\).

7 Killing Tensors and collineations

A symmetry of a geometric object \(A\) generated by the vector field \(X\) is an equation of the form
\[
\mathcal{L}_X A = B
\]
where \(B\) is a tensor with the same number of indices and the same symmetries of indices as \(A\). In a Riemannian space the symmetries of geometric objects which are defined in terms of the metric and its derivatives are called collineations. The basic collineation is
\[
\mathcal{L}_X g_{ab} = 2 X_{(a;b)} = 2\psi(x) g_{ab}
\]
where \( g_{ab} \) is the metric tensor of the space and the vector \( X \) is called a conformal Killing vector (CKV) with conformal factor \( \psi \). If \( \psi_{ij} = 0 \) the CKV is said to be a special CKV (SCKV). If \( \psi_i = 0 \) and \( \psi \neq 0 \), \( X \) is called a homothetic vector (HV); and if \( \psi = 0 \) a KV. The next set of collineations are of the form

\[
\mathcal{L}_X \Gamma_{bc}^a = 2\delta_{(i}^a \phi_{c)}.
\]

(46)

In that case the vector \( X \) is called a projective collineation (PC) with projective factor \( \phi \). If \( \phi_{ab} = 0 \) (i.e. \( \phi_a \) is a gradient KV) is called a special PC (SPC). If \( \phi_{c} = 0 \) is called an affine collineation (AC).

In general it holds the identity\(^39\)

\[
\mathcal{L}_X \Gamma_{bc}^a = X^a_{;bc} - R^a_{bcde}X^d
\]

(47)

which when \( X \) is a PC takes the form

\[
X^a_{;bc} - 2\delta_{(i}^a \phi_{c)} = R^a_{bcde}X^d.
\]

(48)

A well-known result is that if \( f, S \) define gradient KVs then the vector \( fS_a \) is an AC (since for any gradient KV \( R_{abcd}S^{cd} = 0 \)).

The HV and the KVs are ACs. An AC which is not generated from neither KVs nor the HV is called a proper AC (PCA).

A direct consequence of (48) is that a PC \( X \) defines the KT

\[
C_{ab} = X_{(a;b)} - 2\phi_{ab}.
\]

(49)

If \( X \) is an AC, then \( X_{(a;b)} \) is a KT of order two. Therefore from the \( s \) (say) proper ACs \( \eta^a \) of a space one constructs the \( s \) KTs \( C_{ab} = \eta_{(a;b)} \).

Concerning the KTs (19) they can be written in terms of a vector field \( L^a \) in the reducible form \( C_{ab} = L_{(a;b)} \).

In that case \( L_a = X_a + M_a \) where \( X^a \) is a PC with projective factor \( \phi \) and \( M_a \) is a CKV with conformal factor \( -2\phi \) (because \( M_{(a;b)} = -2\phi_{ab} \) and from (15) \( M_{(a;b)} = \psi_{ab} \)).

Therefore if a Riemannian space admits \( m \) CKVs \( M^a \) and \( m \) PCs \( X^a \) such that \( \psi(M) = -2\phi(X) \), we construct \( m \) KTs of order two of the form \( C_{ab} = L_{(a;b)} \) where \( L_a = M_a + X_a \).

The maximum number of linearly independent KTs\(^{40,41} \) of order 2 in a Riemannian (or pseudo-Riemannian) manifold of dimension \( n \) is \( \binom{n+1}{2}(n+2) \) and this is the necessary and sufficient condition for the space to be maximal symmetric, or of constant curvature (see Refs. 32, 38, 40, and 41).

It is important to note that not all KTs of order 2 in a maximal symmetric space are reducible, that is of the form \( C_{ab} = L_{(a;b)} \). For example in the cases of \( E^2 \) and \( E^3 \) (see sections 8 and 9) the reducible KTs are subcases of more general non-reducible KTs.

Concerning the KTs of the form \( C_{ab} = L_{(a;b)} \) defined on \( V^n \) we have the following proposition.

**Proposition 4** In a space \( V^n \) the vector fields of the form

\[
L_a = c_{11}S_{I,a} + c_{22}M_{Ia} + c_3HV_a + c_4AC_a + c_5S_I S_{I,a} + 2c_6S_I M_{Ia} + c_7(K(PC_ka + CKV_{Ka})
\]

(50)

where \( S_{I,a} \) are the gradient KVs, \( M_{Ia} \) are the non-gradient KVs, \( HV_a \) is the homothetic vector, \( AC_a \) are the proper ACs, \( S_I S_{I,a} \) are non-proper ACs, \( PC_ka \) are proper PCs with a projective factor \( \phi_K \) and \( CKV_{Ka} \) are conformal KVs with conformal factor \( -2\phi_K \), produce the KTs of order 2 of the form \( C_{ab} = L_{(a;b)} \). In the case of maximally symmetric spaces there do not exist proper PCs and proper ACs\(^42 \) therefore only the vectors generated by the KVs are necessary, that is in these spaces

\[
L_a = c_{11}S_{I,a} + c_{22}M_{Ia} + c_3HV_a + c_5S_I S_{I,a} + 2c_6S_I M_{Ia}.
\]

(51)

The KVs alone give the solution \( C_{ab} = 0 \) and the HV generates the trivial KT \( g_{ab} \).

The special projective Lie algebra of a maximally symmetric space consists of the vector fields of Table 1 (\( I, J = 1, 2, ..., n \)).
Table 1: Collineations of Euclidean space $E^n$.

| Collineation                            | Gradient                      | Non-gradient                  |
|----------------------------------------|-------------------------------|-------------------------------|
| Killing vectors (KV)                   | $K_I = \delta_I^j \partial_j$ | $X_{IJ} = \delta^j_I \delta^j_J x_j \partial_i$ |
| Homothetic vector (HV)                 | $H = x^i \partial_i$         |                               |
| Affine Collineations (AC)              | $A_{IJ} = x_I \delta_j^I \partial_i$ | $A_{IJ} = x_I \delta_j^I \partial_i$, $I \neq J$ |
| Special Projective collineations (SPC) | $P_I = x_I H$                 |                               |

Therefore a maximally symmetric space of dimension $n$ admits
- $n$ gradient KVs and $\binom{n(n-1)}{2}$ non-gradient KVs
- 1 gradient HV
- $n^2$ non-proper ACs
- $n$ PCs which are special (that is the partial derivative of the projective function is a gradient KV).

8 The KTs of $V^2$

The KTs of the Euclidean space $E^2$ are well-known (see Ref. [26]). However for the convenience of the reader we refer briefly the KTs of $V^2$ with metric $g_{ab} = (\varepsilon, 1)$ where $\varepsilon = \pm 1$ in order to include the 2d Minkowski space $L^2$.

- $V^2$ admits two gradient KVs $\partial_x, \partial_y$ whose generating functions are $x, y$ respectively and one non-gradient KV (the rotation) $y \partial_x - \varepsilon x \partial_y$. These vectors can be written collectively

$$L^a = \begin{pmatrix} b_1 + b_2 y \\ b_2 - \varepsilon b_3 x \end{pmatrix}$$

where $b_1, b_2, b_3$ are arbitrary constants, possibly zero.

- The general KT of order 2 in $V^2$ is

$$C_{ab} = \begin{pmatrix} \gamma y^2 + 2ay + A & -\gamma xy - ax - \beta y + C \\ -\gamma xy - ax - \beta y + C & \gamma x^2 + 2\beta x + B \end{pmatrix}.$$  

(53)

- The vector $L^a$ generating KTs of $V^2$ of the form $C_{ab} = L_{(a,b)}$ is

$$L_a = \begin{pmatrix} -2\beta y^2 + 2axy + Ax + a_8 y + a_{11} \\ -2ax^2 + 2\beta xy + a_{10} x + B y + a_9 \end{pmatrix}.$$  

(54)

- The KTs $C_{ab} = L_{(a;b)}$ in $V^2$ generated from the vector (54) are

$$C_{ab} = L_{(a;b)} = \begin{pmatrix} \frac{1}{2} L_{x,x} & \frac{1}{2} (L_{x,y} + L_{y,x}) \\ \frac{1}{2} (L_{x,y} + L_{y,x}) & L_{y,y} \end{pmatrix} = \begin{pmatrix} 2ay + A & -ax - \beta y + C \\ -ax - \beta y + C & 2\beta x + B \end{pmatrix}.$$  

(55)

where $43 2C = a_8 + a_{10}$. Observe that these KTs are special cases of the general KTs (53) for $\gamma = 0$.

9 The geometric quantities of $E^3$

In $E^3$ the general KT of order 2 has independent components

$$C_{11} = \frac{a_6}{2} y^2 + \frac{a_1}{2} x^2 + a_4 y z + a_5 y + a_2 z + a_3$$

$$C_{12} = \frac{a_{10}}{2} x^2 - \frac{a_6}{2} x y - \frac{a_4}{2} x z - \frac{a_1}{2} y z - \frac{a_5}{2} x - \frac{a_{15}}{2} y + a_{16} z + a_{17}$$

$$C_{13} = \frac{a_{14}}{2} y^2 - \frac{a_4}{2} x y - \frac{a_1}{2} x z - \frac{a_{10}}{2} y z - \frac{a_2}{2} x + a_{18} y - \frac{a_{11}}{2} z + a_{19}$$

$$C_{22} = \frac{a_8}{2} x^2 + \frac{a_7}{2} y^2 + a_{14} x z + a_{15} x + a_{12} z + a_{13}.$$  

(56)
\[ C_{33} = \frac{a_4}{2} x^2 - \frac{a_{14}}{2} x y - \frac{a_{10}}{2} x z - \frac{a_7}{2} y z - (a_{16} + a_{18}) x - \frac{a_{12}}{2} y - \frac{a_8}{2} z + a_2 \]
\[ C_{33} = \frac{a_1}{2} x^2 + \frac{a_7}{2} y^2 + a_{10} x y + a_{11} x + a_8 y + a_9 \]

where \(a_i\) with \(I = 1, 2, ..., 20\) are arbitrary real constants.

The vector \(L^a\) generating the KT \(C_{ab} = L_{(a,b)}\) is

\[
L_a = \begin{pmatrix}
-a_{15} y^2 - a_{11} z^2 + a_3 x y + a_2 x z + 2(a_{16} + a_{18}) y z + a_3 x + 2a_4 y + 2a_1 z + a_6 \\
-a_{5} x^2 - a_{8} z^2 + a_{15} x y - 2a_{18} x z + a_{12} y z + 2(a_{17} - a_4) x + a_{13} y + 2a_7 z + a_{14} \\
-a_2 x^2 - a_3 y^2 - 2a_{16} x y + a_{11} x + a_8 y z + 2(a_{19} - a_1) x + 2(a_{20} - a_7) y + a_9 z + a_{10}
\end{pmatrix}
\]

(57)

and the generated KT is

\[
C_{ab} = \begin{pmatrix}
a_{5} y + a_2 z + a_3 & -\frac{a_9}{2} x - \frac{a_{11}}{2} y + a_{16} z + a_{17} & -\frac{a_9}{2} x + a_1 y - \frac{a_{11}}{2} y - \frac{a_{17}}{2} y + a_{19} \\
-\frac{a_9}{2} x - a_2 y + a_{16} z + a_{17} & a_{15} x + a_{12} z + a_{13} & -(a_{16} + a_{18}) x - \frac{a_{13}}{2} y - \frac{a_{15}}{2} y + a_{20} \\
-\frac{a_9}{2} x + a_1 y + \frac{a_{11}}{2} z + a_{19} & -(a_{16} + a_{18}) x - \frac{a_{13}}{2} y + \frac{a_{15}}{2} y + a_{20} & a_{11} x + a_8 y + a_9
\end{pmatrix}
\]

(58)

which is a subcase of the general KT (56) for \(a_1 = a_4 = a_6 = a_7 = a_{10} = a_{14} = 0\).

We note that the covariant expression of the most general KT \(\Lambda_{ij}\) of order 2 of \(E^3\) is \(A_{m^m}^{n^m}B_{i^i}^{j^j}D_{ij}\)

\[
(\varepsilon_{ikm}\varepsilon_{jln} + \varepsilon_{jkm}\varepsilon_{iln})A_{m^m}^{n^m}q^k q^l + (B_{i^i}^{j^j}k^l + \lambda_i\delta_{j}k - \delta_{ij}\lambda_k)q^k + D_{ij}
\]

(59)

where \(A_{m^m}, B_{i^i}, D_{ij}\) are constant tensors all being symmetric and \(B_{i^i}\) also being traceless; \(\lambda^k\) is a constant vector. This result is obtained from the solution of the Killing tensor equation in Euclidean space.

Observe that \(A_{m^m}, D_{ij}\) have each 6 independent components; \(B_{i^i}\) has 5 independent components; and \(\lambda^k\) has 3 independent components. Therefore \(\Lambda_{ij}\) depends on \(6 + 6 + 5 + 3 = 20\) arbitrary real constants, a result which is in accordance with the one found earlier in (56).

### 10 Applications

In this section we discuss various applications of Theorem 2

#### 10.1 Case of geodesics

We apply Theorem 2 to the geodesic equations in order to recover the results of Ref. 20 in a simple and straightforward manner. In that case \(Q^a = 0\) and \(A^a_i = 0\); and the conditions of the FI Integral 1 imply that \(I_{n>2} = 0\). Therefore the only QFI which survives is the

\[
I_2 = \left(\frac{t^2}{2}G_{ab} - tL_{(a;b)} + C_{(0)ab}\right)q^a q^b - tG_{,a}q^a + L_{(0)a}q^a + G(q)
\]

where \(C_{(0)ab}, G_{ab}\) and \(L_{(0)(a;b)}\) are KTs.

The FI \(I_2\) consists of the three independent FIs46 (see Ref. 20)

\[
I_{2a} = C_{ab}q^aq^b, \quad I_{2b} = \frac{t^2}{2}G_{ab}q^aq^b - tG_{,a}q^a + G(q), \quad I_{2c} = -tL_{(a;b)}q^aq^b + L_{a}q^a.
\]

The time-dependent QFIs \(I_{2b}, I_{2c}\) are the ones found in Ref. 20 The QFI \(I_{2a}\) is not found because the authors were looking only for time-dependent FIs.

| QFI          | Condition                      |
|--------------|--------------------------------|
| \(I_{2a}\)   | \(C_{ab} = \)KT               |
| \(I_{2b}\)   | \(G_{ab} = \)KT               |
| \(I_{2c}\)   | \(L_{(a;b)} = \)KT            |

Table 2: The QFIs of geodesic equations.
As an application of the above general results let us compute the QFIs of the geodesic equations of the 3d metric.  

\[ ds^2 = z^2 \left( dx^2 + dy^2 \right) + dz^2. \]  

(60)

In this case the kinetic metric is \( g_{ab} = \text{diag}(z^2, z^2, 1) \). The Ricci Scalar \( R = -\frac{2}{z^2} \). Therefor this metric is not of constant curvature and consequently the number of KTs is less than 20.

The geodesic equations are

\[
\begin{align*}
\ddot{x} &= -\frac{2}{z} \dot{x} \dot{z} \\
\ddot{y} &= -\frac{2}{z} \dot{y} \dot{z} \\
\ddot{z} &= z (\dot{x}^2 + \dot{y}^2).
\end{align*}
\]

(61)  

(62)  

(63)

Solving the condition \( C_{(ab,c)} = 0 \) we find that the metric (60) admits the following KTs

\[
C_{ab} = \begin{pmatrix}
\left( \frac{c_1}{z^2} + \frac{c_2}{2} y^2 + c_3 y + c_4 \right) z^4 & -\frac{1}{2} \left( c_2 xy + c_3 x + c_5 y - 2 c_7 \right) z^4 & 0 \\
-\frac{1}{2} \left( c_2 xy + c_3 x + c_5 y - 2 c_7 \right) z^4 & \left( \frac{c_1}{z^2} + \frac{c_2}{2} x^2 + c_5 x + c_6 \right) z^4 & 0 \\
0 & 0 & c_1
\end{pmatrix}
\]

where \( c_\kappa, \kappa = 1, 2, ..., 7 \) are arbitrary constants. Therefore there exist 7 linearly independent KTs as many as the free parameters involved.

In order to find the reducible KTs we solve the constraint \( C_{ab} = L_{(a;b)} \) for a vector \( L_a \). We have the following system of equations:

\[
\begin{align*}
L_{1,1} + z L_{3} &= c_1 z^2 + \frac{c_2}{2} y^2 z^4 + c_3 y z^4 + c_4 z^4 \\
L_{1,2} + L_{2,1} &= -c_2 x y z^4 - c_3 x z^4 - c_5 y z^4 + 2 c_7 z^4 \\
z L_{1,3} + z L_{3,1} - 2 L_1 &= 0 \\
z L_{2,2} + L_{3,3} &= c_1 z^2 + \frac{c_2}{2} x^2 z^4 + c_5 x z^4 + c_6 z^4 \\
z L_{2,3} + z L_{3,2} - 2 L_2 &= 0 \\
L_{3,3} &= c_1.
\end{align*}
\]

The solution of the above system is the vector

\[
L_a = \begin{pmatrix}
\frac{1}{z^2} (b_1 y + b_2) \\
-\frac{1}{z^2} (b_1 x + b_3)
\end{pmatrix}, \quad L_{(a;b)} = c_1 \begin{pmatrix}
z^2 & 0 & 0 \\
0 & z^2 & 0 \\
0 & 0 & 1
\end{pmatrix} = c_1 g_{ab}
\]

that is \( L_a \) is a homothetic vector.

In the case that the generating vector \( L_a = G.a \) we find that \( b_1 = b_2 = b_3 = 0 \) and \( G = \frac{1}{z} z^2 \). Then we have

\[
G.a = \begin{pmatrix}
0 \\
0 \\
c_1 z
\end{pmatrix}, \quad G_{ab} = c_1 \begin{pmatrix}
z^2 & 0 & 0 \\
0 & z^2 & 0 \\
0 & 0 & 1
\end{pmatrix} = c_1 g_{ab}.
\]

In order to compute the QFIs for the geodesic equations of (60) we apply the results of Table 2. We have:

1) The QFI \( I_{2a} \).

\[
I_{2a} = C_{ab} q^a \dot{q}^b
\]

\[
= \left( \frac{c_1}{z^2} + \frac{c_2}{2} y^2 + c_3 y + c_4 \right) z^4 \dot{x}^2 - (c_2 xy + c_3 x + c_5 y - 2 c_7) z^4 \dot{y} \dot{x} + \left( \frac{c_1}{z^2} + \frac{c_2}{2} x^2 + c_5 x + c_6 \right) z^4 \dot{y}^2 + c_1 z^2
\]

\[
= 2c_1 \frac{1}{2} \left( z^2 \dot{x}^2 + z^2 \dot{y}^2 + z^2 \right) - \frac{c_2}{2} z^4 (\dot{x} \dot{y} - \dot{y} \dot{x})^2 + c_3 z^4 \dot{x} (\dot{y} - \dot{y}) + c_4 z^4 \dot{x}^2 - c_5 z^4 \dot{y} (\dot{x} - \dot{y}) +
\]

= kinetic energy
\[ + c_6 z^4 \dot{y}^2 + 2c_7 z^4 \dot{x} \dot{y}. \]

This expression contains the independent FIs
\[ T = \frac{1}{2} \left( z^2 \dot{x}^2 + z^2 \dot{y}^2 + z^2 \right), \quad I_{2a1} = z^2 \dot{x}, \quad I_{2a2} = z^2 \dot{y}, \quad I_{2a3} = z^2 (x \dot{y} - y \dot{x}). \]

We note that
\[ T = \frac{1}{2} \left( \dot{x} I_{2a1} + \dot{y} I_{2a2} + \dot{z}^2 \right), \quad I_{2a3} = x I_{2a2} - y I_{2a1}. \]

2) The QFI \( I_{2b} \).

\[
I_{2b} = \frac{t^2}{2} G_{ab} \dot{q}^a \dot{q}^b - tG_{a} \dot{q}^a + G(q) \\
= c_1 \frac{t^2}{2} \left( z^2 \dot{x}^2 + z^2 \dot{y}^2 + \dot{z}^2 \right) - c_1 t \dot{z} \dot{z} + c_1 \frac{z^2}{2}.
\]

Therefore
\[ I_{2b} = -t^2 T + t \dot{z} \dot{z} - \frac{z^2}{2}. \]

3) The QFI \( I_{2c} \).

\[
I_{2c} = -t L_{(a,b)} \dot{q}^a \dot{q}^b + L_a \dot{q}^a \\
= -c_1 t \left( z^2 \dot{x}^2 + z^2 \dot{y}^2 + \dot{z}^2 \right) + z^2 (b_1 y + b_2) \dot{x} - z^2 (b_1 x + b_1) \dot{y} + c_1 \dot{z} \\
\]

which contains the new irreducible FI
\[ I_{2c1} = -tT + \frac{\dot{z} \dot{z}}{2} = \frac{1}{2} \frac{d}{dt} \left( -t^2 T + \frac{z^2}{2} \right). \]

We note that
\[ I_{2b} = t I_{2c1} + \frac{t \dot{z} \dot{z}}{2} - \frac{z^2}{2}. \]

We collect the results in the following table.

| \( T \) | \( I_{2a1} \) | \( I_{2a2} \) | \( I_{2a3} \) |
| --- | --- | --- | --- |
| \[ \frac{1}{2} \left( z^2 \dot{x}^2 + z^2 \dot{y}^2 + z^2 \right) \] | \[ z^2 \dot{x} \] | \[ z^2 \dot{y} \] | \[ z^2 (x \dot{y} - y \dot{x}) \] |
| \[ I_{2c} = -tT + \frac{\dot{z} \dot{z}}{2} \] | \[ I_{2b} = -t^2 T + t \dot{z} \dot{z} - \frac{z^2}{2} \] |

Table 3: The LFIs/QFIs of geodesics of (60).

Since the metric \( g_{ab} \) is not flat the conjugate momenta \( p_a \) of the Hamiltonian formalism are not equal to the velocities \( \dot{q}^a \). Hence to compute the Poisson brackets (PBs) of the FIs we have to make the required transformation.

The conjugate momenta are
\[ p_a = \frac{\partial T}{\partial \dot{q}^a} = g_{ab} \dot{q}^b = \begin{pmatrix} z^2 \dot{x} \\ z^2 \dot{y} \\ \dot{z} \end{pmatrix}. \]

Then the FIs become

| \( T \) | \( I_{2a1} \) | \( I_{2a2} \) | \( I_{2a3} \) |
| --- | --- | --- | --- |
| \[ \frac{1}{2} \left( \dot{q}_1^2 \dot{q}_2^2 + \dot{q}_2^2 \dot{q}_3^2 + \dot{q}_3^2 \right) \] | \[ p_1 \] | \[ p_2 \] | \[ x p_2 - y p_1 \] |
| \[ I_{2c} = -tT + \frac{\dot{z} \dot{z}}{2} \] | \[ I_{2b} = -t^2 T + t \dot{z} \dot{z} - \frac{z^2}{2} \] |

Table 4: The LFIs/QFIs of geodesics of (60) in the phase space \( (q^a, p_a) \).
The system is (Liouville) integrable because the three FIs $T, I_{2a1}, I_{2a2}$ are linearly independent and in involution.

Therefore we can find the solution of the system by quadrature using $T, I_{2a1}, I_{2a2}$. However, it is simpler to use instead of $T$ the time-dependent FI $I_{2c}$. Indeed we have

$$\begin{align*}
&\begin{cases} 
  z^2 \dot{x} = k_1 \\
  z^2 \dot{y} = k_2 \\
  2z \dot{z} = 2k_4 t + k_3 
\end{cases} \implies 
  \frac{dz^2}{dt} = 2k_4 t + k_3 \implies 
  z(t) = \pm (k_4 t^2 + k_3 t + k_0)^{1/2}
\end{align*}$$

where $k_0, k_1 \equiv I_{2a1}, k_2 \equiv I_{2a2}, k_3 \equiv 4I_{2c}, k_4 \equiv 2T$ are arbitrary constants.

Substituting in the remaining FIs we find

$$\dot{x} = \frac{k_1}{z^2} \implies x(t) = \frac{2k_1}{(4k_0 k_4 - k_3^2)^{1/2}} \tan^{-1}\left[ \frac{2k_4 t + k_3}{(4k_0 k_4 - k_3^2)^{1/2}} \right] + c$$

and

$$\dot{y} = \frac{k_2}{z^2} \implies y(t) = \frac{2k_2}{(4k_0 k_4 - k_3^2)^{1/2}} \tan^{-1}\left[ \frac{2k_4 t + k_3}{(4k_0 k_4 - k_3^2)^{1/2}} \right] + c'$$

where $c, c'$ are arbitrary constants.

The solution is

$$q^a(t) = \begin{pmatrix} x(t) \\
  y(t) \\
  z(t) 
\end{pmatrix} = \begin{pmatrix} \frac{2k_1}{(4k_0 k_4 - k_3^2)^{1/2}} \tan^{-1}\left[ \frac{2k_4 t + k_3}{(4k_0 k_4 - k_3^2)^{1/2}} \right] + c \\
  \frac{2k_2}{(4k_0 k_4 - k_3^2)^{1/2}} \tan^{-1}\left[ \frac{2k_4 t + k_3}{(4k_0 k_4 - k_3^2)^{1/2}} \right] + c' \\
  \pm (k_4 t^2 + k_3 t + k_0)^{1/2} \end{pmatrix}.$$  

### 10.2 The Whittaker dynamical system

The Whittaker dynamical system is a 2d Newtonian system with equations

$$\ddot{x} = x, \quad \ddot{y} = \dot{x}.$$  

For that system the kinetic metric is the Euclidean metric $\delta_{ab}$ of $E^2$.

In the notation of the Theorem 2 we have $A^a_b = \delta^a_2 \delta^b_2 = \begin{pmatrix} 0 & 0 \\
  1 & 0 \end{pmatrix}$ and $Q^a = V^a = -x \delta^a = \begin{pmatrix} -x \\
  0 \end{pmatrix}$ where $V = -\frac{1}{2} \dot{y}^2$.

We apply the Theorem 2 to determine the QFIs.

**Integral 1.**

$$I_n = \left( \begin{array}{c} \frac{t^n}{n} C_{(n)ab} + \ldots + \frac{t^2}{2} C_{(2)ab} + t C_{(1)ab} + C_{(0)ab} \end{array} \right) \hat{q}^a \hat{q}^b + t^n L_{(n)a} \hat{q}^a + \ldots + t^2 L_{(2)a} \hat{q}^a + t L_{(1)a} \hat{q}^a + L_{(0)a} \hat{q}^a +$$

$$+ \frac{t^{n+1}}{n+1} L_{(n)a} Q^a + \ldots + \frac{t^2}{2} L_{(1)a} Q^a + t L_{(0)a} Q^a + G(q)$$

where $C_{(N)ab}$ are KTs. Taking into consideration the quantities mentioned above we find

$$C_{(1)11} = -L_{(0)(1;1)} - 2C_{(0)12}, \quad C_{(1)12} = -L_{(0)(1;2)} - C_{(0)22}, \quad C_{(1)22} = -L_{(0)(2;2)}$$

$$C_{(k+1)11} = -L_{(k)(1;1)} - \frac{1}{k} C_{(k)12}, \quad C_{(k+1)12} = -L_{(k)(1;2)} - \frac{1}{k} C_{(k)22}, \quad C_{(k+1)22} = -L_{(k)(2;2)}, \quad k = 1, \ldots, n - 1$$

$$-x L_{(k-1)1}, \quad -x L_{(k-1)1} = -2x C_{(k)11} + k(k+1) L_{(k+1)1} - k L_{(k)2}, \quad -x L_{(k-1)1} = -2x C_{(k)12} - k(k+1) L_{(k+1)2}, \quad k = 1, \ldots, n - 1$$
\[ L_{(n)(1;1)} = -\frac{2}{n} C_{(n)12}, \quad L_{(n)(1;2)} = -\frac{1}{n} C_{(n)22}, \quad L_{(n)(2;2)} = 0 \]
\[-xL_{(n)} = \mathbf{s}, \quad (-xL_{(n-1)})_1 = -2xL_{(n)11} - nL_{(n)2}, \quad (-xL_{(n-1)})_2 = -2xL_{(n)12} \]
\[ G_{,1} = -2xC_{(0)11} - L_{(1)1} - L_{(0)2}, \quad G_{,2} = -2xC_{(0)12} - L_{(1)2}. \]

We note that all the QFIs \( I_n (n > 1) \) reduce to the QFI \( I_1 \). Therefore we continue only with the case \( n = 1 \).

We have
\[ I_1 = (tD_{ab} + C_{ab} \hat{q}^a \hat{q}^b + tLa \hat{q}^a + B_a \hat{q}^a + \frac{1^2}{2} s + tB_a Q^a + G(q) \]
where \( C_{ab}, D_{ab} \) are KTIs and
\[ D_{11} = -B_{(1;1)} - 2C_{12}, \quad D_{12} = -B_{(1;2)} - C_{22}, \quad D_{22} = -B_{(2;2)} \]
\[ L_{(1;1)} = -2D_{12}, \quad L_{(1;2)} = -D_{22}, \quad L_{(2;2)} = 0 \]
\[-xL_1 = s \]
\[ (-xB_1)_1 = -2xD_{11} - L_2, \quad (-xB_1)_2 = -2xD_{12} \]
\[ G_{,1} = -2xC_{11} - L_1 - B_2, \quad G_{,2} = -2xC_{12} - L_2. \]

The KTIs \( C_{ab}, D_{ab} \) are of the form (see section 8)
\[ C_{ab} = \begin{pmatrix} \gamma_0y^2 + 2a_0y + A_0 & -\gamma_0xy - a_0x - \beta_0y + C_0 \\ -\gamma_0xy - a_0x - \beta_0y + C_0 & \gamma_0x^2 + 2\beta_0x + E_0 \end{pmatrix} \]
and
\[ D_{ab} = \begin{pmatrix} \gamma_1y^2 + 2a_1y + A_1 & -\gamma_1xy - a_1x - \beta_1y + C_1 \\ -\gamma_1xy - a_1x - \beta_1y + C_1 & \gamma_1x^2 + 2\beta_1x + E_1 \end{pmatrix}. \]

Solving the conditions \( 65 \) we find
\[ L_a = \begin{pmatrix} \gamma_1x^2 + a_1x^2 + 2\beta_1x - 2C_1x + k_1 \\ -\gamma_1x^3 - 3\beta_1x^2 - 2E_1x + k_2 \end{pmatrix}. \]

Substituting in \( 66 \) we get \( a_1 = \beta_1 = \gamma_1 = C = k_1 = 0 \) and \( s = 0 \). Therefore
\[ L_a = \begin{pmatrix} 0 \\ -2E_1x + k_2 \end{pmatrix}, \quad D_{ab} = \begin{pmatrix} A_1 & 0 \\ 0 & E_1 \end{pmatrix}. \]

Solving the conditions \( 64 \) we find
\[ B_a = \begin{pmatrix} \gamma_0x^2 + 2\beta_0xy + a_0x^2 - (A_1 + 2C_0)x + k_3 \\ -\gamma_0x^3 - 3\beta_0x^2 - 2E_0x - E_1y + k_4 \end{pmatrix} \]

which when replaced in \( 67 \) gives \( a_0 = \beta_0 = \gamma_0 = 0, k_2 = k_3 \) and \( E_1 = 2(A_1 + C_0) \). It follows:
\[ B_a = \begin{pmatrix} -(A_1 + 2C_0)x + k_2 \\ -2E_0x - 2(C_0 + A_1)y + k_4 \end{pmatrix}, \quad C_{ab} = \begin{pmatrix} A_0 & 0 \\ C_0 & E_0 \end{pmatrix}. \]

Substituting in the integrability condition of \( 68 \) we find \( A_1 = 0 \implies E_1 = 2C_0 \). Therefore
\[ L_a = \begin{pmatrix} 0 \\ -4C_0x + k_2 \end{pmatrix}, \quad D_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2C_0 \end{pmatrix}, \quad B_a = \begin{pmatrix} -2C_0x + k_2 \\ -2E_0x - 2C_0y + k_4 \end{pmatrix}, \quad C_{ab} = \begin{pmatrix} A_0 & 0 \\ C_0 & E_0 \end{pmatrix}. \]

Finally integrating the conditions \( 68 \) we have
\[ G(x, y) = (E_0 - A_0)x^2 + 2C_0xy - k_4x - k_2y. \]

The FI is
\[ J_1 = 2tC_0y^2 + A_0x^2 + 2C_0\dot{x}y + E_0y^2 - 4tC_0xy + tk_2\dot{y} - 2C_0\dot{x} + k_2\dot{x} - 2E_0x\dot{y} - 2C_0y\dot{y} + k_4\dot{y} + \]
which consists of the FIs
\[ J_{1a} = (\dot{y} - x) [t(\dot{y} - x) + \dot{x} - y] \]
\[ J_{1b} = \dot{x}^2 - x^2, \quad J_{1c} = (\dot{y} - x)^2, \quad J_{1d} = t(\dot{y} - x) + \dot{x} - y, \quad J_{1e} = \dot{y} - x. \]

The independent FIs are the following:
\[ J_{11} = \dot{x}^2 - x^2, \quad J_{12} = \dot{y} - x, \quad J_{13} = t(\dot{y} - x) + \dot{x} - y. \]

**Integral 2.**
\[ J_2 = e^{\lambda t} (\lambda C_{ab} \dot{q}_a \dot{q}_b + \lambda L_a \ddot{q}_a + L_a Q^a) \]
where \( \lambda \neq 0, \) \( C_{ab} \) is a KT, \( \lambda C_{11} = -L_{(1;1)} - 2C_{12}, \) \( \lambda C_{12} = -L_{(1;2)} - C_{22}, \) \( \lambda C_{22} = -L_{(2;2)} \) and \( (-xL_1)_{,a} = -2\lambda xC_{1a} - \lambda^2 L_a - \lambda L_2 \delta_{a1}. \)

We have the conditions
\[ L_{1,1} = -\lambda C_{11} - 2C_{12} \quad \text{(69)} \]
\[ L_{1,2} + L_{2,1} = -2\lambda C_{12} - 2C_{22} \quad \text{(70)} \]
\[ L_{2,2} = -\lambda C_{22} \quad \text{(71)} \]
\[ (-xL_1)_{,a} = -2\lambda xC_{1a} - \lambda^2 L_a - \lambda L_2 \delta_{a1}. \quad \text{(72)} \]

Solving the system of PDEs \((69)-(71)\) we find that
\[ L_a = \begin{pmatrix} ax^2 + 2\lambda \beta y^2 + 2(\beta - \lambda a)xy + k_1 y - (\lambda A + 2C)x + k_2 \\ 2(\lambda a - 3\beta)x^2 - 2\lambda \beta xy - \lambda By - (2\lambda C + 2B + k_1)x + k_3 \end{pmatrix} \]
and
\[ C_{ab} = \begin{pmatrix} 2ay + A & -ax - \beta y + C \\ -ax - \beta y + C & 2\beta x + B \end{pmatrix}. \]

Substituting in the last condition \((72)\) we get \( a = \beta = B = k_3 = 0. \) We consider the following subcases:

i) Case \( \lambda = \pm 1. \)
We find \( A = k_1 = 0. \)
Then
\[ L_a = \begin{pmatrix} -2Cx + k_2 \\ \mp 2Cx \end{pmatrix}, \quad C_{ab} = C \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The FI
\[ J_{2a} = e^{\pm t} \left[ \pm 2C \dot{x} \dot{y} \mp (2Cx + k_2) \dot{x} - 2C \dot{y} + 2C (C^2 - k_2) \right] \]
which contains the FIs
\[ J_{21} = e^{\pm t} (\dot{x} \mp x), \quad J_{21b} = e^{\pm t} (\dot{y} - x)(\dot{x} \mp x) = J_{21} J_{12}. \]

We note that the FI \( J_{11} \) can be derived from \( J_{21+} \) and \( J_{21-} \) as follows
\[ J_{21+} + J_{21-} = e^t (\dot{x} - x) e^{-t} (\dot{x} + x) = \dot{x}^2 - x^2 = J_{11}. \]

Therefore \( J_{11} \) is not an independent FI.

ii) Case \( \lambda = \pm 2. \)
We find \( C = k_1 = k_2 = 0. \)
Then
\[ L_a = \begin{pmatrix} \mp 2Ax \\ 0 \end{pmatrix}, \quad C_{ab} = A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

We collect the results in the Table 5 below.
10.2.1 Table of FIs

| J_{12} = y - x | J_{13} = \dot{y} - \dot{x} - y = tJ_{12} + \dot{x} - y | J_{21\pm} = e^{\pm t}(\dot{x} \mp x) |

Table 5: FIs of Whittaker system.

In order to study the integrability of the Whittaker system we compute the PBs of the independent FIs. We have

\{J_{12}, J_{13}\} = 0, \quad \{J_{12}, J_{21\pm}\} = -e^{\pm t}, \quad \{J_{13}, J_{21\pm}\} = -e^{\pm t}(t \mp 1), \quad \{J_{21\pm}, J_{21\mp}\} = -2.

Therefore the 2d Whittaker system is integrable because the FIs $J_{12}, J_{13}$ are (functionally) independent and in involution.

However, the solution of the system can be found immediately by using $J_{12}$ and, instead of $J_{13}$, the time-dependent FIs $J_{21\pm}$. It follows that

\begin{align*}
x(t) &= \frac{1}{2}(c_-e^t - c_+e^{-t}), \quad y(t) = c_0t + \frac{1}{2}(c_-e^t + c_+e^{-t}) + c_1
\end{align*}

where $c_\pm, c_0, c_1$ are arbitrary constants.

10.3 The autonomous linearly coupled 2d damped harmonic oscillator

This is the two-dimensional dynamical system with equations of motion

\begin{align*}
\ddot{x} + kx &= py - 2m\dot{x} \\
\ddot{y} + ky &= -px - 2m\dot{y}
\end{align*}

where $m, p, k$ are (real or imaginary, non-zero) constants and $q^1 = x, q^2 = y$. The determination of the QFIs of this example have been discussed before (see example 6.5 in Ref. 15) where it has been found one new time-dependent QFI by giving arbitrary values to the quantities involved in the weak Noether condition (equivalently the NBH equation). Using Theorem 2 we shall recover this QFI plus a number of new QFIs not found before.

A Lagrangian that describes this system is the Lagrangian of a 2d simple harmonic oscillator

\begin{equation}
L = T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2)
\end{equation}

with the generalized external forces

\begin{equation}
F^a = -P^a + A^a_b q^b
\end{equation}

where $P^a = \begin{pmatrix} -py \\ px \end{pmatrix}$ and $A^a_b = -2m\delta^a_b$. The equations of motion are written

\begin{equation}
\ddot{q}^a = -Q^a + A^a_b \dot{q}^b
\end{equation}

where $Q^a = V^a + P^a = \begin{pmatrix} kx - py \\ ky + px \end{pmatrix}$. The kinetic metric is the Euclidean metric $\delta_{ab}$ of the plane $E^2$.

We apply Theorem 2 to determine the QFIs of that system.

**Integral 1.**

The conditions of the QFI $I_n$ become

\begin{align*}
C_{(1)ab} &= -L_{(0)(a;b)} + 4mC_{(0)ab} \\
C_{(k+1)ab} &= -L_{(k)(a;b)} + \frac{4m}{k}C_{(k)ab}, \quad k = 1, ..., n-1 \\
(L_{(k-1)b}Q^b)_a &= 2C_{(k)ab}Q^b - k(k+1)L_{(k+1)a} + 2mkL_{(k)a}, \quad k = 1, ..., n-1 \\
L_{(n)(a;b)} &= \frac{4m}{n}C_{(n)ab}
\end{align*}
\[ L_{(n)a}Q^a = s \]  
\[ (L_{(n-1)b}Q^b)_{,a} = 2C_{(n)ab}Q^b + 2mnL_{(n)a} \]  
\[ G_{,a} = 2C_{(0)ab}Q^b - L_{(1)a} + 2mL_{(0)a}. \]

Since \( C_{(N)ab} = 0 \), \( L_{(N)a} = 0 \) for \( N = 2, 3, ..., n \) it follows that only the QFI \( I_1 \) survives. Therefore:

\[ I_1 = (tD_{ab} + C_{ab})q^{a,b} + tL_aq^{a} + B_aq^{a} + \frac{t^2}{2}s + tB_aQ^a + G(q) \]

where \( C_{ab} = \frac{1}{4m}B_{(a;b)} + \frac{1}{10m}L_{(a)b}; D_{ab} = \frac{1}{4m}L_{(a)b} \) are KTs and

\[ L_aq^{a} = s \]  
\[ (B_{a}Q^{b})_{,a} = 2D_{ab}Q^b + 2mL_a \]  
\[ G_{,a} = 2C_{ab}Q^b + 2mB_a - L_a. \]

Since \( C_{ab}, D_{ab} \) are KTs we have that \( B_{(a;b)}, L_{(a)b} \) are reducible KTs. Therefore from section \( 8 \) we have

\[ B_a = \begin{pmatrix} -2\beta_1 y^2 + 2a_1 xy + A_1 x + n_8 y + n_{11} \\ -2a_1 x^2 + 2\beta_1 y + n_{10} x + B_1 y + n_9 \end{pmatrix}, \quad B_{(a;b)} = \begin{pmatrix} 2a_1 y + A_1 & -a_1 x - \beta_1 y + C_1 \\ -a_1 x - \beta_1 y + C_1 & 2\beta_1 x + B_1 \end{pmatrix} \]

\[ L_a = \begin{pmatrix} -2\beta_2 y^2 + 2a_2 xy + A_2 x + w_8 y + w_{11} \\ -2a_2 x^2 + 2\beta_2 y + w_{10} x + B_2 y + w_9 \end{pmatrix}, \quad L_{(a;b)} = \begin{pmatrix} 2a_2 y + A_2 & -a_2 x - \beta_2 y + C_2 \\ -a_2 x - \beta_2 y + C_2 & 2\beta_2 x + B_2 \end{pmatrix} \]

where \( 2C_1 = n_8 + n_{10} \) and \( 2C_2 = w_8 + w_{10} \).

Substituting \( L_a \) in the condition \( 96 \) we obtain \( a_2 = \beta_2 = s = 0 \) and there remain the following two cases:

1) \( k = \pm ip \) with \( w_9 = \mp iw_{11}, \ w_8 = \pm iB_2, \ w_{10} = \mp iA_2 \); and
2) \( w_9 = w_{11} = 0, \ A_2 = B_2, \ w_8 = -w_{10} = \frac{i}{p}A_2 \).

We continue the consideration of the remaining conditions for these two cases.

1) Case \( k = \pm ip \) with \( w_9 = \mp iw_{11}, \ w_8 = \pm iB_2 \) and \( w_{10} = \mp iA_2 \).

1.1. The subcase \( k = ip \).

Then

\[ L_a = \begin{pmatrix} A_2 x + iB_2 y + w_{11} \\ -iA_2 x + B_2 y - iw_{11} \end{pmatrix}, \quad L_{(a;b)} = \begin{pmatrix} A_2 \pm \frac{i}{2}(B_2 - A_2) \\ \frac{i}{2}(B_2 - A_2) \end{pmatrix} \]

and the condition \( 87 \) gives

\[ a_1 = \beta_1 = 0, \ (p - 4im^2)A_2 = 0, \ A_2 = B_2, \ w_{11} = 0, \ n_8 = iB_1, \ n_{10} = -iA_1, \ n_9 = -iw_{11}. \]

Therefore

\[ B_a = \begin{pmatrix} A_1 x + iB_1 y + n_{11} \\ -iA_1 x + B_1 y - in_{11} \end{pmatrix}, \quad B_{(a;b)} = \begin{pmatrix} A_1 \pm \frac{i}{2}(B_1 - A_1) \\ \frac{i}{2}(B_1 - A_1) \end{pmatrix}. \]

From \( (p - 4im^2)A_2 = 0 \) we have

1.1.1. Subcase \( A_2 = B_2 = 0 \Rightarrow L_a = 0, \ D_{ab} = 0 \) and \( C_{ab} = \frac{1}{4m}B_{(a;b)}. \)

From the integrability condition of \( 88 \) we find \( (p - 4im^2)(A_1 + B_1) = 0 \) which gives:

1.1.1.1. \( A_1 = -B_1. \)

We have

\[ B_a = \begin{pmatrix} -B_1 x + iB_1 y + n_{11} \\ iB_1 x + B_1 y - in_{11} \end{pmatrix}, \quad B_{(a;b)} = B_1 \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \quad C_{ab} = \frac{B_1}{4m} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} \]

and integrating \( 88 \) we find

\[ G(x, y) = mB_1(-x^2 + y^2) + 2imB_1xy + 2mn_{11}(x - iy). \]
The FI is
\[ J_1(11a) = -\frac{B_1}{4m}(\dot{x}^2 - 2ix\dot{y} - \dot{y}^2) + (-B_1x + iB_1y + n_{11})\dot{x} + (iB_1x + B_1y - in_{11})\dot{y} + mB_1(-x^2 + y^2) + 2imB_1xy + 2mn_{11}(x - iy) \]

which consists of the independent FIs

\[ J_{1a}(11a) = -\frac{1}{4m}(\dot{x} - iy)^2 + (-x + iy)\dot{x} + (ix + y)\dot{y} - m(x - iy)^2 \equiv J_{11+} \]
\[ J_{1b}(11a) = i\dot{x} + \dot{y} + 2m(ix + y) \equiv J_{12}. \]

1.1.1.B. For \( p = 4im^2 \).
Integrating condition \( SS \) we find
\[ G(x, y) = -miA_1xy + miB_1xy - \frac{m}{2}(B_1 - A_1)(x^2 - y^2) + 2mn_{11}(x - iy). \]

The FI is
\[ J_1(11b) = \frac{A_1}{4m}(\dot{x}^2 - \frac{i}{4m}(A_1 - B_1)\dot{y} + \frac{B_1}{4m}\dot{y}^2 + (A_1x + iB_1y)\dot{x} + in_9\dot{x} + (-iA_1x + B_1y)\dot{y} + n_9\dot{y} + \frac{1}{2}m(A_1 - B_1)(x^2 - y^2) - im(A_1 - B_1)xy + 2imn_9(x - iy) \]

which consists of the irreducible FIs

\[ J_{1a}(11b) = \frac{1}{4m}\dot{x}^2 - \frac{i}{4m}\dot{y} + x\dot{x} - ix\dot{y} + \frac{1}{2}m(x^2 - y^2) - imxy \]
\[ J_{1b}(11b) = \frac{1}{4m}\dot{y}^2 + \frac{i}{4m}\dot{y} + iy\dot{x} + y\dot{y} - \frac{1}{2}m(x^2 - y^2) + imxy \]
\[ J_{1c}(11b) = i\dot{x} + \dot{y} + 2imx + 2my. \]

1.1.2. Subcase \( p = 4im^2 \). We have
\[ L_a = A_2 \begin{pmatrix} x + iy \\ -ix + y \end{pmatrix}, \quad L_{(a;b)} = A_2\delta_{ab}, \quad C_{ab} = \frac{1}{4m}\begin{pmatrix} A_1 + \frac{m}{4m} & \frac{i}{4m}(B_1 - A_1) \\ \frac{i}{2m}(B_1 - A_1) & A_1 + \frac{m}{4m} \end{pmatrix}. \]

From the integrability condition of \( SS \) we find \( A_2 = 0 \). Therefore \( L_a = 0 \) and \( C_{ab} = \frac{1}{4m}B_{(a;b)} \).

We retrieve the FI \( J_1(11b) \).

1.2. The subcase \( k = -ip \).

Working similarly for the case \( k = -ip \) we find the FIs:

1.2.1. \( A_1 = -B_1 \).
\[ J_1(k = -ip) = -\frac{B_1}{4m}\dot{x}^2 - \frac{iB_1}{2m}\dot{y} + \frac{B_1}{4m}\dot{y}^2 - B_1(x + iy)\dot{x} - in_9\dot{x} + B_1(-ix + y)\dot{y} + n_9\dot{y} - mB_1(x^2 - y^2) - 2imB_1xy - 2imn_9x + 2mn_9y \]

which gives the irreducible FIs

\[ J_{11} = -\frac{1}{4m}\dot{x}^2 - \frac{i}{2m}\dot{y} + \frac{1}{4m}\dot{y}^2 - (x + iy)\dot{x} + (-ix + y)\dot{y} - m(x^2 - y^2) - 2imxy \]
\[ J_{12} = -i\dot{x} + \dot{y} - 2imx + 2my. \]

1.2.2. \( p = -4im^2 \).
\[ J_1(k = -ip = -4m^2) = \frac{A_1}{4m}\dot{x}^2 + \frac{i}{4m}(A_1 - B_1)\dot{y} + \frac{B_1}{4m}\dot{y}^2 + (A_1x - iB_1y)\dot{x} - in_9\dot{x} + (iA_1x + B_1y)\dot{y} + n_9\dot{y} + \frac{1}{2}m(A_1 - B_1)(x^2 - y^2) + im(A_1 - B_1)xy - 2imn_9x + 2mn_9y \]
which consists of the irreducible FIs

\[ J_{1a} = \frac{1}{4m} x^2 + \frac{i}{4m} x\dot{y} + x\dot{x} + i\dot{x}\dot{y} + \frac{1}{2} m(x^2 - y^2) + imxy \]
\[ J_{1b} = \frac{1}{4m} y^2 - \frac{i}{4m} x\dot{y} - iy\dot{x} + y\dot{y} - \frac{1}{2} m(x^2 - y^2) - imxy \]
\[ J_{1c} = -i\dot{x} + \dot{y} - 2imx + 2my. \]

We note that collectively the FIs \( J_{11\pm} \) and \( J_{12\pm} \) are written

\[ J_{11\pm} = -\frac{1}{4m}(\dot{x} \mp iy)^2 + (-x \mp iy)\dot{x} + (\pm ix + y)\dot{y} - m(x^2 - y^2) \pm 2imxy \]
\[ J_{12\pm} = \dot{x} \mp iy + 2m(x \mp iy). \]

We observe that \( J_{11\pm} = -\frac{1}{4m}(J_{12\pm})^2 \) therefore the FIs \( J_{11\pm} \) are not irreducible.

2) Case \( w_5 = w_{11} = 0, A_2 = B_2 \) and \( w_8 = -w_{10} = \frac{k}{p}B_2 \Rightarrow C_2 = 0. \)

We have

\[ L_a = B_2 \left( \frac{x + \frac{k}{p}y}{-\frac{k}{p}x + y} \right), \quad L_{(a;b)} = B_2 \delta_{ab}. \]

Then

\[ 4mC_{ab} = B_{(a;b)} + \frac{1}{4m}L_{(a;b)} \]
\[ = \begin{pmatrix} 2a_1y + A_1 + \frac{p}{4m} & -a_1x - \beta_1y + C_1 \\ -a_1x - \beta_1y + C_1 & 2\beta_1x + B_1 + \frac{B_2}{4m} \end{pmatrix} \]

which in \( \text{(S7)} \) gives \( a_1 = \beta_1 = 0 \) and the following subcases:

2.1. \( k = \pm ip, n_{11} = \pm in_9, n_8 = \pm iB_1, n_{10} = \mp iA_1 \) and \( B_2(p \mp 4im^2) = 0. \)

This subcase gives again the FIs found in the case 1.

2.2. \( n_9 = n_{11} = 0, n_8 = -n_{10} = \frac{k}{p}B_1 - \frac{B_2}{4mp}(k + 4m^2), A_1 = B_1 \) and \( B_2(p^2 - 4m^2k) = 0. \)

We have

\[ B_a = \begin{pmatrix} B_1x + \frac{k}{p}B_1y - \frac{B_2}{4mp}(k + 4m^2) & y \\ \frac{k}{p}B_1x + \frac{B_2}{4mp}(k + 4m^2)x + B_1y \end{pmatrix}, \quad B_{(a;b)} = B_1\delta_{ab}, \quad 4mC_{ab} = \left( B_1 + \frac{B_2}{4m} \right) \delta_{ab}. \]

2.2.A. \( B_2 = 0. \)

We have \( L_a = 0, \)

\[ B_a = \begin{pmatrix} B_1x + \frac{k}{p}B_1y \\ -\frac{k}{p}B_1x + B_1y \end{pmatrix}, \quad B_{(a;b)} = B_1\delta_{ab}, \quad 4mC_{ab} = B_1\delta_{ab}. \]

The integrability condition of \( \text{(S8)} \) implies that \( p^2 = 4m^2k \) for non-trivial FIs and we compute

\[ G(x, y) = B_1 \left( \frac{k}{4m} + m \right) (x^2 + y^2). \]

The FI is

\[ J_1 = \frac{1}{4m}(\dot{x}^2 + \dot{y}^2) + (x + \frac{k}{p}y) \dot{x} + (y - \frac{k}{p}x) \dot{y} + \left( \frac{k}{4m} + m \right) (x^2 + y^2) \]
\[ \Rightarrow \quad \bar{J}_1 = \frac{p}{k}J_1 = \frac{1}{4mk} \left( \dot{x}^2 + \dot{y}^2 \right) + \left( \frac{p}{k}x + y \right) \dot{x} + \left( \frac{p}{k}y - x \right) \dot{y} + p \left( \frac{1}{4m} + \frac{m}{k} \right) (x^2 + y^2). \]

2.2.B. \( p^2 = 4m^2k. \)

The integrability condition of \( \text{(S8)} \) implies \( p^2 = -4m^4 \Rightarrow k = -m^2, p = \pm 2im^2 \) and by integration we compute

\[ G(x, y) = \frac{3m}{4}B_1(x^2 + y^2) - \frac{9}{16}B_2(x^2 + y^2). \]
The FI is written

\[
J_1(2.2) = \frac{B_2}{4m} (\dot{x}^2 + \dot{y}^2) + \frac{B_1}{4m} (x^2 + y^2) + \frac{B_2}{16m^3} (x^2 + y^2) + \frac{t}{8m} B_2 \left( x \pm \frac{i}{2} y \right) \dot{x} + \\
+ \frac{3m}{4} B_2 (x^2 + y^2) + \frac{3m}{4} B_1 (x^2 + y^2) - \frac{9}{16} B_2 (x^2 + y^2)
\]

which consists of the irreducible FIs

\[
J_{1a}(2.2) = \frac{1}{4m} (\dot{x}^2 + \dot{y}^2) + \frac{1}{16m^2} (x^2 + y^2) + t \left( x \pm \frac{i}{2} y \right) \dot{x} + t \left( \mp \frac{i}{2} x + y \right) \dot{y} + \\
\pm \frac{3i}{8m} (y \dot{x} - x \dot{y}) + \frac{3m}{4} (x^2 + y^2) - \frac{9}{16} (x^2 + y^2)
\]

\[
J_{1b}(2.2) = \frac{1}{4m} (\dot{x}^2 + \dot{y}^2) + \left( x \pm \frac{i}{2} y \right) \dot{x} + \left( y \mp \frac{i}{2} x \right) \dot{y} + \frac{3m}{4} (x^2 + y^2).
\]

Integral 2.

\[
J_2 = e^{\lambda t} \left( \lambda C_{ab} \dot{q}^a \dot{q}^b + \lambda L_a \dot{q}^a + L_a Q^a \right)
\]

where \( \lambda \neq 0 \) and

\[
L_{a(ab)} = (4m - \lambda) C_{ab}
\]

\[
( L_b Q^b )_{,a} = 2 \lambda C_{ab} Q^b + \lambda (2m - \lambda) L_a.
\]

Since \( C_{ab} \) is a KT condition implies that \( L_{a(ab)} \) is a KT as well. Consider the following cases:

1) Case \( \lambda = 4m \).

From we find that \( L_a \) is a KV, i.e. \( L_a = (b_1 + b_3 y) \partial_x + (b_2 - b_3 x) \partial_y \).

Then condition becomes

\[
( L_b Q^b )_{,a} = 8m C_{ab} Q^b - 8m^2 L_a.
\]

Substituting the KV \( L_a \) and the KT in we have the following six subcases:

1.1. \( k = \frac{p^2}{4m^2} \), \( C = 0 \), \( A = B \), \( a = \beta = \gamma = 0 \), \( b_1 = b_2 = 0 \) and \( b_3 = - \frac{p}{m} A \).

The FI is

\[
J_2 \left( k = \frac{p^2}{4m^2} \right) = e^{4mt} \left[ \dot{x}^2 + \dot{y}^2 - \frac{p}{m} (y \dot{x} - x \dot{y}) + 4m (x^2 + y^2) \right].
\]

1.2. \( k = \frac{p^2}{4m^2} \), \( C = 0 \), \( A = B \), \( a = \beta = \gamma = 0 \), \( b_3 = - \frac{p}{m} A \), \( p = \pm i (k + 8m^2) \) and \( b_1 = \mp ib_2 \).

Substituting \( k = \frac{p^2}{4m^2} \) in \( p = \pm i (k + 8m^2) \) we get a second order equation wrt \( p \) with solutions

\[
p_1 = \mp 8im^2 \implies k_1 = -16m^2, \quad p_2 = \pm 4im^2 \implies k_2 = -4m^2.
\]

We have the following FI (all satisfy \( k = \frac{p^2}{4m^2} \)):

\[
J_2 = e^{4mt} \left[ 4m A (x^2 + y^2) + 4m (\mp ib_2 - \frac{p}{m} Ay) \dot{x} + 4m \left( b_2 + \frac{p}{m} Ax \right) \dot{y} + \\
\quad + \left( \mp ib_2 - \frac{p}{m} Ay \right) \left( kx - py \right) + \left( b_2 + \frac{p}{m} Ax \right) \left( ky + px \right) \right]
\]

\[
e^{4mt} \left[ 4m A (x^2 + y^2) + 4m (\mp ib_2 - \frac{p}{m} Ay) \dot{x} + 4m \left( b_2 + \frac{p}{m} Ax \right) \dot{y} + \\
\quad + \left( b_2 + \frac{p}{m} Ax \right) \left( px + ky \mp ikx \pm ipy \right) \right]
\]

\[22\]
which consists of the FIs $J_2 \left( k = \frac{p^2}{4m^2} \right)$ found earlier in the subcase 1.1 and the

$$ J_{21} = e^{4mt} \left( \mp 4mi\dot{x} + 4m\dot{y} + px + ky \mp ikx \pm ipy \right). $$

Specifically we have

$$ J_{21}(k = -16m^2, p = \mp 8im^2) = J_{21}(k = -4m^2, p = \pm 4im^2) = e^{4mt} \left( \mp i\dot{x} + \dot{y} \mp 2imx - 2my \right). $$

1.3. $C = 0$, $A = B = 0$, $a = \beta = \gamma = 0$, $b_3 = 0$, $b_1 = \mp b_2$ and $p = \pm i(k + 8m^2)$.

We find again the FI $J_{21}$ of the case 1.2. that is

$$ J_2(p = \pm i(k + 8m^2)) = e^{4mt} \left( \mp 4mi\dot{x} + 4m\dot{y} + px + ky \mp ikx \pm ipy \right). $$

1.4. $k = \pm ip$: $L_a = 0$ and $C_{ab} = B \begin{pmatrix} -1 & \pm i \\ \pm i & 1 \end{pmatrix}$.

We have the FI

$$ J_2(k = \pm ip) = e^{4mt}(\dot{x}^2 - \dot{y}^2 \mp 2i\dot{x}\dot{y}). $$

1.5. $k = \pm ip$: $L_a = \mp ib_2\partial_x + b_2\partial_y$, $C_{ab} = B \begin{pmatrix} -1 & \pm i \\ \pm i & 1 \end{pmatrix}$ and $p = \pm i(k + 8m^2) = \pm 4im^2$ which implies $k = -4m^2$.

We have the irreducible FIs $J_2(k = \pm ip)$ found in the subcase 1.4 and the FI (already found)

$$ J_2(1.5) = e^{4mt} \left( \mp 4mi\dot{x} + 4m\dot{y} + px \mp ikx \pm ipy \right). $$

1.6. $k = \pm ip$: $A = B \pm 2iC$, $b_3 = 4m(C \mp iB)$, $b_1 = \mp ib_2$ and $p = \pm i(k + 8m^2) = \pm 4im^2$ which implies $k = -4m^2$.

Then we write the FI

$$ J_2(1.6) = e^{4mt} \left[ (B \pm 2iC)\dot{x}^2 + B\dot{y}^2 + 2C\dot{x}\dot{y} \mp ib_2\dot{x} + 4m(C \mp iB)y\dot{x} + b_2\dot{y} - 4m(C \mp iB)x\dot{y} \mp 2imb_2x - 2mb_2y \mp 4m^2i(C \mp iB)(x^2 + y^2) \right] $$

which consists of the FIs

$$ J_{21}(1.6) = e^{4mt} \left[ \dot{x}^2 + \dot{y}^2 \mp 4mi(y\dot{x} - x\dot{y}) - 4m^2(x^2 + y^2) \right] $$

$$ J_{22}(1.6) = e^{4mt} \left[ \pm i\dot{x}^2 + \dot{x}\dot{y} + 2m(y\dot{x} - x\dot{y}) \mp 2m^2i(x^2 + y^2) \right] $$

$$ J_{23}(1.6) = e^{4mt} \left( \mp i\dot{x} + \dot{y} \pm 2m\dot{x} - 2my \right). $$

2) Case $\lambda \neq 4m$.

Condition (89) gives the KT

$$ C_{ab} = \frac{1}{4m - \lambda} L_{(a;b)} \tag{92} $$

where the vector

$$ L_a = \begin{pmatrix} -2\beta y^2 + 2axy + A\dot{x} + a_{11} \\ -2ax^2 + 2\beta xy + a_{10}\dot{x} + By + a_9 \end{pmatrix} $$

generates the reducible KT

$$ L_{(a;b)} = \begin{pmatrix} 2ay + A & -ax - \beta y + C \\ -ax - \beta y + C & 2\beta x + B \end{pmatrix} $$

where $2C = a_8 + a_{10}$.

Substituting (92) in (91) we get

$$ (L_0 Q^b)_a = \frac{2\lambda}{4m - \lambda} L_{(a;b)} Q^b + \lambda(2m - \lambda) L_a \tag{93} $$
which implies that

\[
\begin{align*}
    a(\lambda - 3m) &= 0 \quad (94) \\
    \beta(\lambda - 3m) &= 0 \quad (95) \\
    pa - \frac{3}{5}(m^2 - k)\beta &= 0 \quad (96) \\
    \frac{3}{5}(m^2 - k)a + p\beta &= 0 \quad (97) \\
    A + B - \frac{\lambda^2}{2p}(a_s - a_{10}) &= 0 \quad (98)
\end{align*}
\]

\[
4p(\lambda - 2m)B + \frac{\lambda^3}{2}(3a_{10} - a_s) - 2m\lambda^2(a_{10} + 2a_s) + 2k\lambda(a_s + a_{10}) + 8m^2\lambda a_s - 4km(a_s + a_{10}) = 0 \quad (99)
\]

\[
\begin{align*}
    [\lambda^3 - 6m\lambda^2 + 4(2m^2 + k)\lambda - 8km] A + p\lambda a_s + p(3\lambda - 8m)a_{10} &= 0 \quad (100) \\
    [\lambda^3 - 6m\lambda^2 + 4(2m^2 + k)\lambda - 8km] B - p(3\lambda - 8m)a_s - p\lambda a_{10} &= 0 \quad (101) \\
    pa_9 + \frac{\lambda^3 - 6m\lambda^2 + (8m^2 + k)\lambda - 4km}{\lambda - 4m}a_{11} &= 0 \quad (102) \\
    \frac{\lambda^3 - 6m\lambda^2 + (8m^2 + k)\lambda - 4km}{\lambda - 4m}a_9 - pa_{11} &= 0 \quad (103)
\end{align*}
\]

The set of the above conditions leads to three distinct QFIs because conditions (94) - (97) concern only the parameters \(a, \beta\); conditions (98) - (101) the parameters \(A, B, a_s, a_{10}\) and conditions (102) - (103) only the parameters \(a_9, a_{11}\). Therefore when we write the final form of the QFI this will consist of three independent FIs one for each set of parameters.

The crucial parameter is the \(\lambda\). We consider two cases \(\lambda \neq 2m\) and \(\lambda = 2m\) (where in both cases it is assumed that \(\lambda \neq 4m\)).

2.1. The subcase \(\lambda \neq 2m\).

2.1.1. Non-vanishing parameters \(a, \beta\) and the rest equal to zero (i.e. \(A = B = a_s = a_{10} = 0, a_9 = a_{11} = 0\)).

In this case we have only the conditions (94) - (97).

Only for \(\lambda = 3m\) the parameters \(a, \beta\) can be non-zero and thus give a non-trivial FI, because then (91), (95) vanish identically.

In that case the linear system of (96) - (97) has the non-zero solution \(\beta = \pm ia\) when \(p = \pm i\lambda^2(m^2 - k)\) and we have for this set of parameters the QFI

\[
J_{2}(2.1.1) = e^{3 \lambda t} \left[ 3y\dot{x}^2 \pm 3ixy^2 - 3(x \pm iy)\dot{x}\dot{y} + 3m(\mp iy^2 + xy)\dot{x} + 3m(-x^2 \pm ix)\dot{y} + (\mp iy^2 + xy)(kx - py) + (-x^2 \pm ix)(ky + px) \right].
\]

2.1.2. Non-vanishing parameters \(A, B, a_s, a_{10}\) and all remaining zero.

In this case we have the conditions (98) - (101).

From (98), (99) the parameters \(A, B\) are expressed as linear combinations of \(a_s, a_{10}\) since \(\lambda \neq 2m\).

These expressions \(A(a_s, a_{10}), B(a_s, a_{10})\) when replaced in (100) and (101) respectively give a homogeneous linear system. This system has non-vanishing solution of the form \(a_s = Da_{10}\) only when \(D = 1\). In that case \(a_s = a_{10}\) with \(p = \pm \frac{\lambda}{4}(\lambda^2 - 4m\lambda + 4k)\) which implies that \(A = -B = \mp ia_s\).

Then

\[
C_{ab} = \frac{a_s}{4m - \lambda} \begin{pmatrix} \mp i & 1 \\ 1 & \pm i \end{pmatrix}, \quad L_a = a_s \begin{pmatrix} \mp ix + y \\ x \pm iy \end{pmatrix}
\]

and the QFI for this set of parameters is

\[
J_{2}(2.1.2) = e^{\lambda t} \left[ \frac{\lambda}{4m - \lambda}(\mp ix^2 \pm iy^2 + 2\dot{x}\dot{y}) + \lambda(\mp ix + y)\dot{x} + \lambda(x \pm iy)\dot{y} + (p \mp ik)(x^2 - y^2) + 2(k \pm ip)xy \right].
\]

2.1.3. Non-vanishing parameters \(a_9, a_{11}\) and all remaining parameters zero.
In this case we have the homogeneous linear system \([102] - [103]\) which has the non-zero solution \(a_{11} = \mp ia_{9}\) for \(p = \pm i\frac{\lambda^{3} - 6m\lambda^{2} + (8m^{2} + k)\lambda - 4km}{\lambda - 4m}\). Therefore \(C_{ab} = 0\) and \(L_{a} = a_{9}(\mp i\partial_{x} + \partial_{y})\). Since

\[
p = \pm i\frac{\lambda^{3} - 6m\lambda^{2} + (8m^{2} + k)\lambda - 4km}{\lambda - 4m} = \pm i\frac{(\lambda - 4m)(\lambda^{2} - 2\lambda m + k)}{\lambda - 4m} = \pm i(\lambda^{2} - 2\lambda m + k)
\]

we end up with the FI which we find in case 3) below.

2.2. The subcase \(\lambda = 2m\).

In this case condition \([\text{93}]\) becomes (since \(\lambda \neq 4m\))

\[
(L_{b}Q_{b})_{,a} = 2L_{(a;b)}Q_{b}.
\]

From \([104]\) we find that \(a = \beta = 0, A = -B, a_{8} = a_{10} \implies C = a_{8}\) and we end up with the system

\[
\begin{align*}
pa_{9} + ka_{11} &= 0 \\
k a_{9} - pa_{11} &= 0
\end{align*}
\]

which leads to two subcases: 2.2.1) \(a_{9} = a_{11} = 0\) and 2.2.2) \(k = \pm ip\) and \(a_{9} = \mp ia_{11}\).

2.2.1. Subcase \(a_{9} = a_{11} = 0\).

We have

\[
C_{ab} = \frac{1}{2m} \begin{pmatrix} A & a_{8} \\ a_{8} & -A \end{pmatrix}, \quad L_{a} = \begin{pmatrix} Ax + a_{8}y \\ a_{8}x - Ay \end{pmatrix}.
\]

The FI is

\[
J_{2}(\lambda = 2m) = e^{2mt} \left[ A\hat{x}^{2} - Ay^{2} + 2a_{8}\hat{x}\hat{y} + 2mAx\hat{x} + 2ma_{8}\hat{x}\hat{y} + 2ma_{8}\hat{y}\hat{x} - 2mAyy + Ak(x^{2} - y^{2}) - Apxy + a_{8}kxy - a_{8}y^{2} + a_{8}kxy + a_{8}px^{2} - Apxy \right]
\]

which consists of the irreducible FIs

\[
\begin{align*}
J_{2a}(2.2.1) &= e^{2mt} \left[ \hat{x}^{2} - \hat{y}^{2} + 2m(x\hat{x} - y\hat{y}) + k(x^{2} - y^{2}) - 2pxy \right] \\
J_{2b}(2.2.1) &= e^{2mt} \left[ i\hat{y} + m(y\hat{x} + x\hat{y}) + \frac{p}{2}(x^{2} - y^{2}) + kxy \right].
\end{align*}
\]

The FI \(J_{2b}(2.2.1)\) is the one found in Ref. \([\text{15}]\) (see equation (38) in Ref. \([\text{15}]\)).

2.2.2. Subcase \(a_{9} = \mp ia_{11}\).

In that case \(k = \pm ip\),

\[
C_{ab} = \frac{1}{2m} \begin{pmatrix} A & a_{8} \\ a_{8} & -A \end{pmatrix}, \quad L_{a} = \begin{pmatrix} Ax + a_{8}y + a_{11} \\ a_{8}x - Ay + ia_{11} \end{pmatrix}.
\]

We find again the two FIs of the case 3.1 and the additional FI

\[
J_{2}(2.2.2) = e^{2mt}(\hat{x} \mp i\hat{y}).
\]

We note that the FI of the case 1.4 can be derived from the above FI as follows

\[
[J_{2}(2.2.2)]^{2} = e^{4mt}(\hat{x} \mp i\hat{y})^{2} = e^{4mt}(\hat{x}^{2} - \hat{y}^{2} \mp 2i\hat{x}\hat{y}) = J_{2}(1.4).
\]

3) Case \(C_{ab} = 0\).

Condition \([\text{88}]\) implies that \(L_{a} = (b_{1} + b_{3}y)\partial_{x} + (b_{2} - b_{3}x)\partial_{y}\) is a KV and the remaining condition \([\text{90}]\) is written

\[
(L_{b}Q_{b})_{,a} = \lambda(2m - \lambda)L_{a}.
\]

(105)
Substituting the KV $L_a$ in \[105\] we get $b_3 = 0$ and non-vanishing values for $b_1, b_2$ only when

$$p = \pm i(\lambda^2 - 2\lambda m + k) \implies \begin{cases} 
\lambda_1 = m + \sqrt{m^2 - k \pm ip} \\
\lambda_2 = m - \sqrt{m^2 - k \pm ip}.
\end{cases}$$

Then $b_1 = \mp ib_2$.

Observe that since $p \neq 0$ also $\lambda^2 - 2\lambda m + k \neq 0$.

The FI is

$$J_{2(3)t} = e^{4t}(\mp i\lambda \dot{x} + \lambda \dot{y} \mp ikx \pm ipy + px + ky), \quad p = \pm i(\lambda^2 - 2\lambda m + k).$$

We collect the above results in the following Table (see section\[10.3.1\]).

### 10.3.1 Table of FIs of example \[10.3\]

| Condition | LFI, QFI |
|-----------|----------|
| $p = \pm 4im^2$, $k = -4m^2$ | $J_{29} = e^{2mt} \left[ (x^2 - y^2) + 2m(x \dot{x} - y \dot{y}) + k(x^2 - y^2) - 2pxy \right]$ |
| $k = \pm ip$ | $J_{29} = e^{2mt}(\dot{x} \pm iy)$ |
| $p = \pm 2im^2$, $k = -m^2$ | $J_{16} = \frac{4m}{3m} \left[ (x^2 + y^2) + t \left( x \pm \frac{1}{2}y \right) \dot{x} + t \left( \mp \frac{1}{2}x + y \right) \dot{y} \pm \frac{8m}{2m} \left( x \dot{y} - y \dot{x} \right) \pm \frac{4m}{2m} \left( x^2 + y^2 \right) \right]$ |
| $k = \frac{p^2}{4m^2}$ | $J_{21} = e^{4mt} \left[ \dot{x}^2 + y^2 - \frac{m}{2m} (y \dot{x} - x \dot{y}) + \frac{4m^2}{2m} (x^2 + y^2) \right]$ |
| $p = \pm i(k + 8m^2)$ | $J_{29} = e^{2mt} \left[ \mp 4im \dot{x} + 4m \dot{y} + px + ky \mp ikx \mp ipy \right]$ |
| $p = \pm i(\lambda^2 - 4\lambda m + 4k)$, $\lambda \neq 2m, 4m$ | $J_{27} = e^{\lambda t} \left[ \frac{\lambda}{4m-\lambda} (\mp 4imx \dot{y}^2 \pm iy^2 \dot{y} + 3m \dot{x}y^2 + x \dot{y} \lambda (\mp iy^2 + x \dot{y}) + 3m(-x^2 \pm ipx) \dot{y} \right]$ |
| $p = \mp i(\lambda^2 - 2\lambda m + k)$ | $J_{28} = e^{\lambda t} \left[ \pm iy \dot{x}^2 - 3m \dot{x}y - 3m(\mp iy^2 + x \dot{y}) \dot{x} + 3m(-x^2 \pm ipx) \dot{y} + (\mp iy^2 + x \dot{y}) (ky - px) \right]$ |
| $p = \pm i\frac{m^2}{4}(m^2 - k)$ | $J_{26} = e^{\lambda t} \left[ 3m \dot{x}^2 \mp 3m x \dot{y} \dot{x} \dot{y} + 3m(\mp iy^2 + x \dot{y}) \dot{x} + 3m(-x^2 \pm ipx) \dot{y} + (\mp iy^2 + x \dot{y}) (ky - px) \right]$ |

Table 6: FIs of the 2d harmonic oscillator with external forces.

In the above Table some sets of conditions are a subset of other more general conditions. In that case the FIs corresponding to that more general conditions are also FIs for the special subset of these conditions; however the opposite does not hold. For example:

- The $J_{2a}$, $J_{2b}$ are FIs for all values of $k,p,m$.
- The set of conditions $(k = -4m^2, p = \pm 4im^2)$ gives $p = \mp ik \implies k = \pm ip$ which means that the $J_{12}$, $J_{29}$ are FIs of that set in addition to $J_{13}$, $J_{14}$, $J_{26}$. Observe that for that special set of conditions $J_{13} - J_{14} = \frac{4m}{2m} (J_{12})^2$ and $J_{11} = J_{13} + J_{14}$.

**NOTE 1:** The set of conditions $k = \pm ip$ with $p_+ = i(k + 8m^2)$ and $p_- = -i(k + 8m^2)$ implies that $k = -4m^2$ and $p = \pm 4im^2$. 


NOTE 2: For $k = \frac{p^2}{4m^2}$, $p = \pm i(k + 8m^2)$ we find that $p_+ = 4im^2, -8im^2$ and $p_- = -4im^2, 8im^2$. In that case the corresponding FI $J_{23}$ reduces to

$$J_{23}(k = p^2/4m^2) = e^{4imt}(\mp i\dot{x} + \dot{y} \pm 2imx - 2my).$$

10.3.2 Discussing integrability

- For arbitrary values of $k, p, m$.

For arbitrary values of $k, p, m$ the dynamical system admits two time-dependent QFIs the $J_{2a}, J_{2b}$. These FIs are not in involution, that is, their PB does not vanish. Therefore an arbitrary 2d harmonic oscillator with arbitrary external forces is not in general integrable; in order to achieve integrability we have to look for special values of $k, p, m$ where more FIs are admitted.

- $k = \pm ip$.

The FIs $J_{12}, J_{29}$ are functionally independent and in involution since $\{J_{12\pm}, J_{29\pm}\} = 0$. Therefore the system in that special case is Liouville integrable and can be integrated by quadratures.

Indeed we have

$$\begin{cases} \dot{x} \mp i\dot{y} + 2m(x \mp iy) = c_{1\pm} \\
e^{2mt}(\dot{x} \mp i\dot{y}) = c_{2\pm} \end{cases} \implies z(t) = \frac{-c_{2\pm}}{2m}e^{-2mt} + \frac{c_{1\pm}}{2m}$$

(106)

where $z(t) \equiv x(t) \mp iy(t)$ and $c_{1\pm}, c_{2\pm}$ are arbitrary complex constants.

- $p = \pm 4im^2, k = -4m^2$.

These values satisfy the conditions $k = \pm ip, k = \frac{p^2}{4m^2}$ and $p = \pm i(k + 8m^2)$. Since $k = \pm ip$ it is straightforward that the system is integrable (previous case).

11 Conclusions

We discussed the relation between the collineations of the kinetic metric and the existence of QFIs of the form (31) for autonomous holonomic dynamical systems of the form (32). We reviewed previous results in the literature and derived a system of equations whose solution derives all the QFIs admitted. In particular a given dynamical system defines its own geometry via the kinetic metric and Theorem 2 shows that the collineations of the kinetic metric are sufficient for the computation of the FIs. In a sense then a dynamical system is ‘constrained’ by the geometry it contains, because it is the collineations of this geometry which provide the FIs for the dynamical system and consequently specify its evolution.

The direct method of computing the QFIs from the condition $dI/dt = 0$ is complementary to the standard method of Noether which uses the standard formal calculations required by the Lie approach. It appears that in cases of low dimension the method discussed in this work would be more convenient due to the strong results of Differential Geometry in the field of collineations. However it cannot be compared with the generality of the Noether approach. Moreover it has been shown that the characterization of some FIs as non-Noetherian is meaningless because all FIs may be associated with a gauged weak Noether symmetry.

It would be useful one to extend the geometric analysis on the symmetry conditions (39) - (43) in order to understand in detail how geometry is related to the force-term $F^a$. Such an analysis will provide important information for the geometric properties of integrable models and also a possible geometric classification for the separable and non-separable Hamiltonian systems. An equally important step is the general solution of the system of equations (39) - (43) which will provide all QFIs of (32) of the form (31).

12 Data Availability

The data that supports the findings of this study are available within the article.
A Proof of Theorem 2

Recall that

\[ K_{ab}(t, q) = C(0)_{ab}(q) + C(1)_{ab}(q)t + C(2)_{ab}(q)\frac{t^2}{2} + \ldots + C(n)_{ab}(q)\frac{t^n}{n} \]

and

\[ K_a(t, q) = L(0)_{a}(q) + L(1)_{a}(q)t + L(2)_{a}(q)t^2 + \ldots + L(m)_{a}(q)t^m. \]

We consider various cases\(^{49}\).

I. Case \(n = m\) (both \(n, m\) finite)

From equation (39) we obtain

\[ C_{(1)ab} = -L_{(0)ab} - 2C_{(0)c(a \cdot A^b_c)}, \quad C_{(k)ab} = -L_{(k-1)ab} - \frac{2}{k-1}C_{(k-1)c(a \cdot A^b_c)}, \quad k = 2, \ldots, n, \quad L_{(n)ab} = -\frac{2}{n}C_{(n)c(a \cdot A^b_c)}. \]

The condition (40) implies

\[ L_{(n)a}Q^a = s, \quad (L_{(n-1)b}Q^b)_a = 2C_{(n)ab}Q^b - nL_{(n)b}A^b_a, \]

\[ (L_{(k-2)b}Q^b)_a = 2C_{(k-1)ab}Q^b - k(k-1)L_{(k-1)a} - (k-1)L_{(k-1)b}A^b_a, \quad k = 2, \ldots, n. \]

The solution of (41) is

\[ K = L_{(0)a}Q^a t + L_{(1)a}Q^a \frac{t^2}{2} + \ldots + L_{(n)a}Q^a \frac{t^{n+1}}{n+1} + G(q). \]

Substituting in (41) we find

\[ G_a = 2C_{(0)ab}Q^b - L_{(1)a} - L_{(0)b}A^b_a. \]

The FI is

\[ I_n = \left( \frac{t^n}{n} C_{(n)ab} + \ldots + \frac{t^2}{2} C_{(2)ab} + tC_{(1)ab} + C_{(0)ab} \right) q^a q^b + t^n L_{(n)a} q^a + \ldots + t^2 L_{(2)a} q^a + tL_{(1)a} q^a + L_{(0)a} q^a + G(q) \]

where \(C_{(N)ab}\) are KT, \(C_{(1)ab} = -L_{(0)ab} - 2C_{(0)c(a \cdot A^b_c)}, \quad C_{(k+1)ab} = -L_{(k)ab} - \frac{2}{k}C_{(k)c(a \cdot A^b_c)}\) for \(k = 1, \ldots, n-1, \quad L_{(n)ab} = -\frac{2}{n}C_{(n)c(a \cdot A^b_c)}, \quad L_{(n)a}Q^a = s, \quad (L_{(n-1)b}Q^b)_a = 2C_{(n)ab}Q^b - nL_{(n)b}A^b_a, \quad (L_{(k-1)b}Q^b)_a = 2C_{(k)ab}Q^b - k(k+1)L_{(k+1)a} - kL_{(k)b}A^b_a \)

for \(k = 1, \ldots, n-1\) and \(C_{(n)ab}Q^b - L_{(1)a} - L_{(0)b}A^b_a\).\n
We note that \(I_0 < I_1 < I_2 < I_3 < I_4 < \ldots\), that is each QFI \(I_k\) is a subcase of the next QFI \(I_{k+1}\) for all \(k \in \mathbb{N}\). Therefore we have only one independent QFI the \(I_n\). The value of \(n\) is determined by the symmetries of the kinetic metric and the dynamics of each specific system.

Observe that for \(A^b_a = 0\) the FI \(I_n\) reduces to

\[ I_{ns} = \left( \frac{t^n}{n} L_{(n-1)ab} - \ldots - \frac{t^2}{2} L_{(2)ab} + L_{(0)ab} \right) q^a q^b + t^n L_{(n)a} q^a + \ldots + t^2 L_{(2)a} q^a + tL_{(1)a} q^a + L_{(0)a} q^a + G(q) \]

where \(C_{(0)ab}, L_{(N)(ab)}\) are KT, \(L_{(n)a}Q^a = s, \quad (L_{(n-1)b}Q^b)_a = -2L_{(n-1)(ab)}Q^b, \quad (L_{(k)b}Q^b)_a = -2L_{(k)(ab)}Q^b - k(k+1)L_{(k+1)a} - kL_{(k)b}A^b_a \)

for \(k = 1, \ldots, n-1\) and \(G_a = 2C_{(0)ab}Q^b - L_{(1)a} - L_{(0)b}A^b_a\).\n
We shall prove that \(I_n(A^b_a = 0)\) consists of two independent FIs.

In the case that \(A^b_a = 0\) we have the following:

- For \(n = 0\),

\[ I_0 = C_{(0)ab}q^a q^b + L_{(0)a}q^a + st + G(q) \]
where \( C_{(0)ab} \) is a KT, \( L_{(0)a} \) is a KV, \( L_{(0)a}Q^a = s \) and \( G, a = 2C_{(0)ab}Q^b \).

This FI consists of the independent FIs

\[
\begin{align*}
I_{01} &= C_{(0)ab} \dot{q}^a \dot{q}^b + G(q) \\
I_{02} &= L_{(0)a} \dot{q}^a + st.
\end{align*}
\]

- For \( n = 1 \).

\[
I_1 = \left( -tL_{(0)(a;b)} + C_{(0)ab} \right) \dot{q}^a \dot{q}^b + tL_{(1)a} \dot{q}^a + L_{(0)a} \dot{q}^a + \frac{t^2}{2} s + tL_{(0)a}Q^a + G(q)
\]

where \( C_{(0)ab} , L_{(0)(a;b)} \) are KT, \( L_{(1)a} \) is a KV, \( L_{(1)a}Q^a = s \), \( (L_{(0)b}Q^b)_{,a} = -2L_{(0)(a;b)}Q^b \) and \( G, a = 2C_{(0)ab}Q^b - L_{(1)a} \).

This FI consists of

\[
\begin{align*}
I_{11} &= C_{(0)ab} \dot{q}^a \dot{q}^b + tL_{(1)a} \dot{q}^a + \frac{t^2}{2} s + G(q) \\
I_{12} &= -tL_{(0)(a;b)} \dot{q}^a \dot{q}^b + L_{(0)a} \dot{q}^a + tL_{(0)a}Q^a.
\end{align*}
\]

- For \( n = 2 \).

\[
I_2 = \left( -\frac{t^2}{2} L_{(1)(a;b)} - tL_{(0)(a;b)} + C_{(0)ab} \right) \dot{q}^a \dot{q}^b + t^2 L_{(2)a} \dot{q}^a + tL_{(1)a} \dot{q}^a + L_{(0)a} \dot{q}^a + \frac{t^3}{3} s + tL_{(0)a}Q^a + G(q)
\]

where \( C_{(0)ab} , L_{(M)(a;b)} \) for \( M = 0, 1 \) are KT, \( L_{(2)a} \) is a KV, \( L_{(2)a}Q^a = s \), \( (L_{(1)b}Q^b)_{,a} = -2L_{(1)(a;b)}Q^b \), \( (L_{(0)b}Q^b)_{,a} = -2L_{(0)(a;b)}Q^b - 6L_{(2)a} \) and \( G, a = 2C_{(0)ab}Q^b - L_{(1)a} \).

This FI consists of

\[
\begin{align*}
I_{21} &= \left( -\frac{t^2}{2} L_{(1)(a;b)} + C_{(0)ab} \right) \dot{q}^a \dot{q}^b + tL_{(1)a} \dot{q}^a + \frac{t^2}{2} L_{(1)a}Q^a + G(q) \\
I_{22} &= -tL_{(0)(a;b)} \dot{q}^a \dot{q}^b + t^2 L_{(2)a} \dot{q}^a + L_{(0)a} \dot{q}^a + \frac{t^3}{3} s + tL_{(0)a}Q^a.
\end{align*}
\]

- For \( n = 3 \).

\[
I_3 = \left( -\frac{t^3}{3} L_{(2)(a;b)} - \frac{t^2}{2} L_{(1)(a;b)} - tL_{(0)(a;b)} + C_{(0)ab} \right) \dot{q}^a \dot{q}^b + t^3 L_{(3)a} \dot{q}^a + t^2 L_{(2)a} \dot{q}^a + tL_{(1)a} \dot{q}^a + L_{(0)a} \dot{q}^a + \frac{t^4}{4} s + \frac{t^3}{3} L_{(2)a}Q^a + tL_{(0)a}Q^a + G(q)
\]

where \( C_{(0)ab} , L_{(M)(a;b)} \) for \( M = 0, 1, 2 \) are KT, \( L_{(3)a} \) is a KV, \( L_{(3)a}Q^a = s \), \( (L_{(2)b}Q^b)_{,a} = -2L_{(2)(a;b)}Q^b \), \( (L_{(1)b}Q^b)_{,a} = -2L_{(1)(a;b)}Q^b - 6L_{(3)a} \), \( (L_{(0)b}Q^b)_{,a} = -2L_{(0)(a;b)}Q^b - 2L_{(2)a} \) and \( G, a = 2C_{(0)ab}Q^b - L_{(1)a} \).

This FI consists of

\[
\begin{align*}
I_{31} &= \left( -\frac{t^2}{2} L_{(1)(a;b)} + C_{(0)ab} \right) \dot{q}^a \dot{q}^b + t^3 L_{(3)a} \dot{q}^a + tL_{(1)a} \dot{q}^a + \frac{t^4}{4} s + \frac{t^2}{2} L_{(1)a}Q^a + G(q) \\
I_{32} &= \left( -\frac{t^3}{3} L_{(2)(a;b)} - tL_{(0)(a;b)} \right) \dot{q}^a \dot{q}^b + t^2 L_{(2)a} \dot{q}^a + L_{(0)a} \dot{q}^a + \frac{t^3}{3} L_{(2)a}Q^a + tL_{(0)a}Q^a.
\end{align*}
\]

- For \( n = 4 \).

\[
I_4 = \left( -\frac{t^4}{4} L_{(3)(a;b)} - \frac{t^3}{3} L_{(2)(a;b)} - \frac{t^2}{2} L_{(1)(a;b)} - tL_{(0)(a;b)} + C_{(0)ab} \right) \dot{q}^a \dot{q}^b + t^4 L_{(4)a} \dot{q}^a + t^3 L_{(3)a} \dot{q}^a + t^2 L_{(2)a} \dot{q}^a + \frac{t^5}{5} s + \frac{t^4}{4} L_{(3)a}Q^a + \frac{t^3}{3} L_{(2)a}Q^a + \frac{t^2}{2} L_{(1)a}Q^a + tL_{(0)a}Q^a + G(q)
\]
where \( C_{(0)ab}, L_{(M)(a;b)} \) for \( M = 0, \ldots, n - 1 \) are KTIs, \( L_{(4)a}Q^a = s, (L_{(3)b}Q^b)_{,a} = -2L_{(3)(a;b)}Q^b \), 
\((L_{(2)b}Q^b)_{,a} = -2L_{(2)(a;b)}Q^b - 2L_{(4)a}, (L_{(1)b}Q^b)_{,a} = -2L_{(1)(a;b)}Q^b - 6L_{(3)a}, (L_{0)b}Q^b)_{,a} = -2L_{0)(a;b)Q^b - 2L_{(2)a} \)
and \( G_{,a} = 2C_{(0)ab}Q^b - L_{(1)a} \).

This FI consists of

\[
I_{41} = \left( -\frac{t^4}{4}L_{(3)(a;b)} - \frac{t^2}{2}L_{(1)(a;b)} + C_{(0)ab} \right) q^a q^b + t^3 L_{(3)a}q^a + t L_{(1)a}q^a + \frac{t^4}{4}L_{(3)a}q^a + \frac{t^2}{2}L_{(1)a}q^a + G(q)
\]

\[
I_{42} = \left( -\frac{t^3}{3}L_{(2)(a;b)} - t L_{(0)(a;b)} \right) q^a q^b + t^4 L_{(4)a}q^a + t^2 L_{(2)a}q^a + L_{(0)a}q^a + \frac{t^5}{5}s + \frac{t^3}{3}L_{(2)a}q^a + t L_{(0)a}q^a.
\]

For \( n = 5 \),

\[
I_5 = \left( -\frac{t^5}{5}L_{(4)(a;b)} - \frac{t^4}{4}L_{(3)(a;b)} - \frac{t^3}{3}L_{(2)(a;b)} - \frac{t^2}{2}L_{(1)(a;b)} - t L_{(0)(a;b)} + C_{(0)ab} \right) q^a q^b + t^5 L_{(5)a}q^a + t^4 L_{(4)a}q^a + \frac{t^6}{6}s + \frac{t^5}{5}L_{(4)a}q^a + \frac{t^4}{4}L_{(3)a}q^a + \frac{t^3}{3}L_{(2)a}q^a + \frac{t^2}{2}L_{(1)a}q^a + t L_{(0)a}q^a + G(q)
\]

where \( C_{(0)ab}, L_{(M)(a;b)} \) for \( M = 0, \ldots, n - 1 \) are KTIs, \( L_{(5)a}Q^a = s, (L_{(4)b}Q^b)_{,a} = -2L_{(4)(a;b)}Q^b \), 
\((L_{(3)b}Q^b)_{,a} = -2L_{(3)(a;b)}Q^b - 20L_{(5)a}, (L_{(2)b}Q^b)_{,a} = -2L_{(2)(a;b)}Q^b - 12L_{(4)a}, (L_{(1)b}Q^b)_{,a} = -2L_{(1)(a;b)}Q^b - 6L_{(3)a}, (L_{0)b}Q^b)_{,a} = -2L_{0)(a;b)Q^b - 2L_{(2)a} \)
and \( G_{,a} = 2C_{(0)ab}Q^b - L_{(1)a} \).

The FI consists of

\[
I_{51} = \left( -\frac{t^4}{4}L_{(3)(a;b)} - \frac{t^2}{2}L_{(1)(a;b)} + C_{(0)ab} \right) q^a q^b + t^5 L_{(5)a}q^a + t^3 L_{(3)a}q^a + t L_{(1)a}q^a + \frac{t^6}{6}s + \frac{t^4}{4}L_{(3)a}q^a + \frac{t^3}{3}L_{(2)a}q^a + t L_{(0)a}q^a + G(q)
\]

\[
I_{52} = \left( -\frac{t^3}{3}L_{(4)(a;b)} - \frac{t^2}{2}L_{(2)(a;b)} - t L_{(0)(a;b)} \right) q^a q^b + t^4 L_{(4)a}q^a + t^2 L_{(2)a}q^a + L_{(0)a}q^a + \frac{t^5}{5}L_{(4)a}q^a + \frac{t^3}{3}L_{(2)a}q^a + t L_{(0)a}q^a.
\]

If we continue in the same way, we prove that for \( A_5^b = 0 \) the FI \( I_n \) consists of the independent FIs:

\[
I_{11} = \left( -\frac{t^{2\ell}}{2\ell}L_{(2\ell-1)(a;b)} - \ldots - \frac{t^4}{4}L_{(3)(a;b)} - \frac{t^2}{2}L_{(1)(a;b)} + C_{(0)ab} \right) q^a q^b + t^{2\ell-1}L_{(2\ell-1)a}q^a + \ldots + t^3 L_{(3)a}q^a + t L_{(1)a}q^a + \frac{t^{2\ell}}{2\ell}L_{(2\ell-1)a}Q^a + \ldots + \frac{t^4}{4}L_{(3)a}Q^a + \frac{t^2}{2}L_{(1)a}Q^a + G(q)
\]

where \( C_{(0)ab}, L_{(M)(a;b)} \) for \( M = 1, 3, \ldots, 2\ell - 1 \) are KTIs, \( (L_{(2\ell-1)a}Q^b)_{,a} = -2L_{(2\ell-1)(a;b)}Q^b \), \( (L_{(k-1)b}Q^b)_{,a} = -2L_{(k-1)(a;b)}Q^b - k(k+1)L_{(k+1)a} \) for \( k = 2, 4, \ldots, 2\ell - 2 \) and \( G_{,a} = 2C_{(0)ab}Q^b - L_{(1)a} \).

\[
I_{22} = \left( -\frac{t^{2\ell+1}}{2\ell+1}L_{(2\ell)(a;b)} - \ldots - \frac{t^3}{3}L_{(2)(a;b)} - t L_{(0)(a;b)} \right) q^a q^b + t^{2\ell}L_{(2\ell)a}q^a + \ldots + t^2 L_{(2)a}q^a + \frac{t^{2\ell+1}}{2\ell+1}L_{(2\ell)a}Q^a + \ldots + \frac{t^4}{4}L_{(2)a}Q^a + t L_{(0)a}Q^a.
\]

where \( L_{(M)(a;b)} \) for \( M = 0, 2, \ldots, 2\ell \) are KTIs, \( (L_{(2\ell)a}Q^b)_{,a} = -2L_{(2\ell)(a;b)}Q^b \) and \( (L_{(k-1)b}Q^b)_{,a} = -2L_{(k-1)(a;b)}Q^b - k(k+1)L_{(k+1)a} \) for \( k = 1, 3, \ldots, 2\ell - 1 \).

Observe that the set of the constraints of the FI \( I_n(A_5^b = 0) \) is divided into one set involving the odd vectors \( L_{(2\ell+1)a} \), the KT \( C_{0ab} \) and the function \( G(q) \); and a second set involving only the even vectors \( L_{(2k)a} \). This explains why in that case \( I_n \) consists of two independent FIs.
II. Case \( n \neq m \). (\( n \) or \( m \) may be infinite)

We find QFIs that are subcases of those found in Case I and Case III.

III. Both \( n, m \) are infinite.

In this case we consider the solution to have the form\(^5\)

\[
K_{ab}(t, q) = g(t)C_{ab}(q), \quad K_{a}(t, q) = f(t)Q_{a}(q)
\]

where the functions \( g(t), f(t) \) are analytic so that they may be represented by polynomial functions as follows

\[
g(t) = \sum_{k=0}^{n} c_k t^k = c_0 + c_1 t + \ldots + c_n t^n
\]

\[
f(t) = \sum_{k=0}^{m} d_k t^k = d_0 + d_1 t + \ldots + d_m t^m.
\]

Only the following subcase give a new independent FI (this is the \( J_2 \) of the Theorem 2). All the other subcases give trivial results already analyzed in the previous cases.

\textbf{Subcase} (\( g = e^{\lambda t}, f = e^{\mu t} \)). \( \lambda \mu \neq 0 \).

\[
\begin{align*}
\begin{cases}
\text{(39)} & \Rightarrow \lambda e^{\lambda t}C_{ab} + e^{\mu t}L_{(a;b)} + 2e^{\lambda t}C_{c(a;A^c_{b})} = 0 \\
\text{(40)} & \Rightarrow -2e^{\lambda t}C_{ab}Q_{b} + \mu e^{\mu t}L_{a} + K_{a} + e^{\mu t}L_{b}A_{a}^b = 0 \\
\text{(41)} & \Rightarrow K_{t} = e^{\mu t}L_{a}Q_{a} \\
\text{(42)} & \Rightarrow \mu^2 e^{\mu t}L_{a} + \mu e^{\mu t}L_{b}A_{a}^b + e^{\mu t}L_{b}Q_{b})_a - 2\lambda e^{\lambda t}C_{ab}Q_{b} = 0.
\end{cases}
\end{align*}
\]

We consider the following subcases.

a) For \( \lambda \neq \mu \):

From (39) we have that \( C_{ab} = -\frac{2}{\lambda} C_{c(a;A^c_{b})} \) and \( L_{a} \) is a KV.

From (42) we find that \( C_{ab}Q_{b} = 0 \) and \( \mu^2 L_{a} + \mu L_{b}A_{a}^b + (L_{b}Q_{b})_a = 0 \).

The solution of (41) is

\[
K = \frac{1}{\mu} e^{\mu t}L_{a}Q_{a} + G(q)
\]

which when replaced in (40) gives \( G_{a} = 0 \), that is \( G(q) = \text{const} \equiv 0 \).

The FI is

\[
I_{a}(\lambda \neq \mu) = e^{\lambda t}C_{ab}q^{a}q_{b} + e^{\mu t}L_{a}q^{a} + \frac{1}{\mu} e^{\mu t}L_{a}Q_{a}
\]

where \( C_{ab} = -\frac{2}{\lambda} C_{c(a;A^c_{b})} \) is a KT such that \( C_{ab}Q_{b} = 0 \) and \( L_{a} = -\frac{1}{\mu^2} (L_{b}Q_{b})_a - \frac{1}{\mu} L_{b}A_{a}^b \) is a KV.

We note that \( I_{a}(\lambda \neq \mu) \) consists of the two independent FIs

\[
J_{2a} = e^{\lambda t}C_{ab}q^{a}q_{b}, \quad J_{2b} = e^{\mu t}L_{a}q^{a} + \frac{1}{\mu} e^{\mu t}L_{a}Q_{a}.
\]

The FIs \( J_{2a}, J_{2b} \) are new.

b) For \( \lambda = \mu \):

From (39) we have that \( C_{ab} = -\frac{1}{\lambda} L_{(a;b)} - \frac{2}{\lambda} C_{c(a;A^c_{b})} \).

From (42) we find that \( \lambda^2 L_{a} + \lambda L_{b}A_{a}^b + (L_{b}Q_{b})_a - 2\lambda C_{ab}Q_{b} = 0 \).

The solution of (41) is

\[
K = \frac{1}{\chi} e^{\lambda t}L_{a}Q_{a} + G(q)
\]
which when replaced in \((40)\) gives \(G(q) = \text{const} \equiv 0\).

The FI is
\[
I_\epsilon(\lambda = \mu) = e^{\lambda t} C_{ab} \dot{q}^a \dot{q}^b + e^{\lambda t} L_a \dot{q}^a + \frac{1}{\lambda} e^{\lambda t} L_a Q^a \equiv J_2
\]
where \(C_{ab} = -\frac{1}{\lambda} L_{(a;b)} - \frac{2}{\lambda} C_{(a} A_{b)}^b\) is a KT and the vector \(L_a = -\frac{1}{\lambda} (L_a Q)^a\) is a KT and the vector \(L_a = -\frac{1}{\lambda} L_a A^b + \frac{2}{\lambda} C_{ab} Q^b\).

We note that the FIs \(J_{2a}, J_{2b}\) found previously are subcases of the new FI \(J_2\). Specifically, \(J_{2a} = J_2(L_a = 0)\) and \(J_{2b} = J_2(C_{ab} = 0)\). Therefore, the Case III leads to only one independent FI the \(J_2\).

The above complete the proof of Theorem 2

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\[
E_a(\Delta L(t, q, \dot{q})) = \Delta F_a(t, q, \dot{q}) \implies E_a \left( X^{[1]}(L) \right) = X^{[1]}(F_a).
\]
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\[
f_N(t) = \sum_{M=0}^{n} d_{(N)M} t^M = d_{(N)0} + d_{(N)1} t + \ldots + d_{(N)n} t^n
\]
then
\[
K_{ab} = \sum_{N=1}^{n} \sum_{M=0}^{n} d_{(N)M} t^M D_{(N)ab}(q) = \sum_{M=0}^{n} \left( \sum_{N=1}^{n} d_{(N)M} D_{(N)ab}(q) \right) t^M = \sum_{M=0}^{n} D_{(M)ab}(q) t^M.
\]
28 We note that the FI \(J_1\) is for \(n\) finite whereas \(J_2\) is for \(n\) infinite hence the term \(e^{\lambda t}\).
29 We note that for \(n = 0\) the conditions for the QFI \(J_1(n = 0)\) can be derived if we set equal to zero the quantities \(C_{(N)ab}\) and \(L_{(N)ab}\) for \(N \neq 0\).
30 If in addition \(F^a = 0\) the \(Q^a = V^a\) the case is reduced to that of the autonomous conservative systems.
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Note that $L^a$ in (54) is the sum of the non-proper ACs of $E^2$ and not of its KVs which give $C_{ab} = 0$.

If we set $z = i t$, the line element (60) takes the form $d s^2 = - d t^2 - t^2 (d x^2 + d y^2)$ which is a conformally flat spacetime.

Equation (43) is not necessary, because the integrability condition $K_{[ab]} = 0$ does not intervene in the calculations. However, it has been checked that equation (43) is always satisfied identically from the solutions of the other equations of the system.

To find a solution we consider $C_{(0)ab} = c_0 C_{ab}$, $C_{(1)ab} = c_1 C_{ab}$, ..., $C_{(n)ab} = n c_n C_{ab}$, $L_{(0)a} = d_0 L_a$, $L_{(1)a} = d_1 L_a$, ..., $L_{(m)a} = d_m L_a$. 

33