POINTS IN GENERIC POSITION AND CONDUCTOR
OF VARIETIES WITH ORDINARY MULTIPLE
SUBVARIETIES OF CODIMENSION ONE

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Abstract. We extend results of our previous papers, on ordinary multiple points
of curves [9], and on the computation of their conductor [8], to ordinary multiple
subvarieties of codimension one.

1. Introduction.

Let $A$ be the local ring, at a multiple (that is singular) point $x$, of an algebraic
reduced curve $C$ with embedding dimension $emdim(A) = r + 1$ and multiplicity
$e(A) = e$. Let $\mathfrak{m}$ be the maximal ideal of $A$. $Spec(G(A))$ is the tangent cone and
$Proj(G(A))$ the projectivized tangent cone to $C$ at $x$.

The scheme $Proj(G(A))$ is reduced if and only if it consists of $e$ points of $\mathbb{P}^r$, that
is, the tangent cone considered as a set consists of $e$ lines of $\mathbb{A}^{r+1}$ through $x$ (the
tangents to the curve at $x$). In this case we say that $x$ is an ordinary multiple point
(or an ordinary singularity) of $C$ [9, Lemma-Definition 2.1]. Clearly if $Spec(G(A))$
is reduced then $Proj(G(A))$ is reduced. The converse, in general, doesn’t hold (see
[9, Example 1] or [7, Section 4]), but if $Proj(G(A))$ is reduced and consists of points
in generic $e$ position, then $Spec(G(A))$ is reduced [9, Theorem 3.3]. Furthermore,
if the points of $Proj(G(A))$ are in generic $e - 1$, $e$ position, then the conductor of $A$
in its normalization $\overline{A}$ is $\mathfrak{m}^\nu$ where $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$ [8, Theorem 4.4].

In this paper first we extend these results to any one dimensional reduced local
ring $B$ with finite normalization $\overline{B}$. Let $\mathfrak{m}$ be the maximal ideal of $B$ and $K$ be
the algebraic closure of the residue field $k(\mathfrak{m})$ of $B$. Set $e = e(B)$, $emdim(B) = r + 1$.
Considering the ring $G(B) \otimes_{k(\mathfrak{m})} K$, instead of $G(A)$, we prove that, if
$Proj(G(B) \otimes_{k(\mathfrak{m})} K)$ is reduced and consists of points in generic $e$ position, then
$Spec(G(B) \otimes_{k(\mathfrak{m})} K)$ and $Spec(G(B))$ are reduced [Theorem 5]. If in addition the
points of $Proj(G(B) \otimes_{k(\mathfrak{m})} K)$ are in generic $e - 1$ position then the conductor of $B$
in $\overline{B}$ is $\mathfrak{m}^\nu$ where $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$ [Theorem 11].

Then we apply the previous results to the ring $B = A_p$, where $p$ is a prime ideal
of codimension one of a local reduced ring $A$, with finite normalization $\overline{A}$.

Using also the properties of normal flatness we get the following result on the
conductor $b$ of $A$ (in $\overline{A}$).

Let $e(A_p) = e > 1$, $emdim(A_p) = r + 1$ and $K$ be the algebraic closure of
the residue field $k(p) = A_p/pA_p$ of $A$ at $p$. Assume $A/p$ regular, $\sqrt{b} = p$ and

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Let \( p \) be a prime ideal of codimension one of a reduced Cohen-Macaulay ring \( A \) with finite normalization \( \overline{A} \). Assume \( A/p \) regular, \( \sqrt{\mathfrak{b}} = p \) and \( \text{Proj}(G(A_p) \otimes_{k(p)} K) \) reduced. If \( \text{emdim}(A) = \dim(A)+1 \) and \( e(A) = e(A_p) = e \) then \( \mathfrak{b} = p^{e-1} \) [Corollary 16].

The previous results have the following geometrical consequences.

Let \( X = \text{Spec}(R) \) be an algebraic variety and \( Y = \text{Spec}(R/q) \) be an irreducible codimension one subvariety of \( X \). Suppose that \( e(R_q) = e > 1 \) that is \( Y \) is a multiple subvariety of \( X \), of multiplicity \( e \). Let \( \text{emdim}(R_q) = r + 1 \) and \( K \) be the algebraic closure of the residue field \( k(q) \) of \( R \) in \( q \). If \( \text{Proj}(G(R_q) \otimes_{k(q)} K) \) is reduced and its points are in generic \( e-1, e \) position, then there exists an open nonempty subset \( U \) of \( Y \) such that for every closed point \( x \) of \( U \) the conductor of the local ring \( A \) of \( X \) at \( x \), is \( p^\nu \), where \( p \) is the prime ideal in \( A \) defining \( Y \) and \( \nu = \text{Min}\{n \in \mathbb{N} \mid e \leq (\frac{n+r}{r})\} \) [Theorem 19].

Note that \( \text{Proj}(G(R_q) \otimes_{k(p)} K) \) is reduced if and only if there exists an open nonempty subset \( U \) of \( Y \) such that, for every closed point \( x \) of \( U \), the tangent cone to \( X \) at \( x \) is the union, as a set, of \( e \) distinct linear spaces of \( A^{r+1} \) [Theorem 21]. Then (extending the definition given for curves) it is natural to say that in this case \( Y \) is an ordinary multiple subvariety of \( X \).

Let \( X = \text{Spec}R \) be a reduced non-normal \( S_2 \) variety. Assume that the irreducible components of the non-normal locus of \( X \) are ordinary multiple subvarieties \( Y_i = \text{Spec}(R/q_i) \) of multiplicity \( e_i = e(R_{q_i}) > 1 \) \( (1 \leq i \leq t) \). Set \( \text{emdim}(R_{q_i}) = r_i + 1 \) and let \( K_i \) be the algebraic closure of the residue field \( k(q_i) \). Suppose that the varieties \( Y_i \) are nonsingular and that the points of \( \text{Proj}(G(A_{q_i}) \otimes_{k(q_i)} K_i) \) are in generic \( e_i - 1, e_i \) position, for any \( i \). Considering the following conditions:

(a) \( X = \text{Spec}(R) \) is Cohen-Macaulay, equimultiple of multiplicity \( e_i \) along \( Y_i \) and has constant embedding dimension along \( Y_i \), for any \( i \);

(b) \( X \) is normally flat along \( Y_i \), for any \( i \);

(c) \( \mathfrak{b} = p_1^{e_1} \cap \ldots \cap p_t^{e_t} \);

then (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) [Theorem 24].

In particular if, in (a), \( X \) is a hypersurface, then the conductor of \( R \) is \( \mathfrak{b} = p_1^{e_1-1} \cap \ldots \cap p_t^{e_t-1} \) [Corollary 26].
Throughout the paper all ring are supposed to be commutative, with identity and noetherian.

Let $A$ be a semilocal ring. If $\mathfrak{p}$ is an ideal of $A$, $G_{\mathfrak{p}}(A) = \bigoplus_{n \geq 0}(\mathfrak{p}^n/\mathfrak{p}^{n+1})$ is the associated graded ring with respect to $\mathfrak{p}$. By $G(A)$ we denote the associated graded ring with respect to the Jacobson radical $\mathfrak{J}$ of $A$. If $x \in A, x \neq 0, x \in \mathfrak{J}^n - \mathfrak{J}^{n+1}, n \in \mathbb{N}$, we say that $x$ has degree $n$ and the image $x^* \in \mathfrak{J}^n/\mathfrak{J}^{n+1}$ of $x$ in $G(A)$ is said to be the initial form of $x$. If $\mathfrak{a}$ is an ideal of $A$, by $G(\mathfrak{a})$ we denote the ideal of $G(A)$ generated by all the initial forms of the elements of $\mathfrak{a}$.

If $A$ is local with maximal ideal $\mathfrak{m}$, $H^0(A, n) = \text{dim}_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}), n \in \mathbb{N}$, denotes the Hilbert function of $A$ and $e(A)$ the multiplicity of $A$ at $\mathfrak{m}$. The embedding dimension $\text{emdim}(A)$ of $A$ is given by $H^0(A, 1)$. If $i \in \mathbb{N}$, the functions $H^i(A, n)$ are given by the relations $H^i(A, n) = \sum_{j=0}^{n}H^{i-1}(A, j)$.

If $S = \bigoplus_{n \geq 0}S_n$ is a standard graded finitely generated algebra over a field $k$, of maximal homogeneous ideal $\mathfrak{n}$, $H^0(S, n) = \text{dim}_kS_n = H^0(S_n, n)$ denotes the Hilbert function of $S$ and $\text{emdim}(S) = H^0(S, 1) = \text{emdim}(S_n)$ the embedding dimension of $S$. The multiplicity of $S$ is $e(S) = e(S_n)$. One has $e(A) = e(G(A))$ and $\text{emdim}(A) = \text{emdim}(G(A))$.

If $B$ is any ring $\text{dim}(B)$ denotes the dimension of $B$.

2. The one dimensional case.

Let $B$ be a local ring or a standard finitely generated graded $k$-algebra over a field $k$. Suppose $B$ has dimension one and is Cohen-Macaulay. Set $\text{emdim}(B) = r + 1$ and $e(B) = e$. It is well known that, for any $n \in \mathbb{N}$, $H^0(B, n) \leq \text{Min}\{e, (\binom{n+r}{r})\}$ and if $H^0(B, m) = e$ then $H^0(B, n) = e$, for any $n \geq m$ [14].

1. Definition. The ring $B$ has maximal Hilbert function if, for every $n \in \mathbb{N}$,

$$H^0(B, n) = \text{Min}\{e, \binom{n+r}{r}\}$$

2. Definition. A set of points $X = \{P_1, ..., P_e\} \subset \mathbb{P}^r$ is in generic position (or the points $P_1, ..., P_e$ are in generic position) [8, Definition 3.1] if the Hilbert function of its homogeneous coordinate ring $R$ is maximal. The set $X$ is in generic $t$-position, $t \leq e$, if every $t$-subset of $X$ is in generic position (then generic $e$-position is generic position).

Remark. It is proved in [2, Theorem 4] that, for any $e$ and $r$, “generic position” is an open nonempty condition.

3. Example. It is easily seen that any set of points of $\mathbb{P}^1$ is in generic $t$-position, for any $t$.

4. Example. A set of $\binom{n+r}{r}$ points in $\mathbb{P}^r (n > 0, r > 0)$ is in generic position if and only if they do not lie on a hypersurface of degree $n$ [8, Corollary 3.4], in particular six points in $\mathbb{P}^2$ are in generic position if and only if they do not lie on a conic.
5. Theorem. Let $B$ be a one dimensional reduced local ring of maximal ideal $\mathfrak{m}$. Let $K$ be the algebraic closure of the residue field $k(\mathfrak{m}) = B/\mathfrak{m}$. Then:

(a) the Hilbert functions of $B$, $G(B)$ and $G(B) \otimes_{k(\mathfrak{m})} K$ are the same, $e(B) = e(G(B)) = e(G(B) \otimes_{k(\mathfrak{m})} K)$ and $\emdim(B) = \emdim(G(B)) = \emdim(G(B) \otimes_{k(\mathfrak{m})} K) = r + 1$;

(b) if $\text{Proj}(G(B) \otimes_{k(\mathfrak{m})} K)$ is reduced and consists of points in generic position in $\mathbb{P}^r$ then $B$ has maximal Hilbert function and the rings $G(B) \otimes_{k(\mathfrak{m})} K$ and $G(B)$ are reduced.

Proof. (a) If $G(B) \otimes K$ is the $K$-vector space obtained extending to $K$ the field of scalars of the $k(\mathfrak{m})$-vector space $G(B)$, then the Hilbert functions of $G(B)$ (that is of $B$) and of $G(B) \otimes K$ are the same. This implies the equalities of the multiplicities and of the embedding dimensions.

(b) If the points of $\text{Proj}(G(B) \otimes K)$ are in generic position the Hilbert function of $G(B) \otimes K$ is maximal [Definitions 1 and 2]. Then by (a), the Hilbert function of $B$ is maximal. This implies that $G(B)$ is Cohen-Macaulay [11, Theorem 3.2], that is, there exists $y^* \in \mathfrak{m}/\mathfrak{m}^2$ which is a non zero divisor of $G(B)$ (this is easy to prove if $k(\mathfrak{m})$ is infinite and if $k(\mathfrak{m})$ is finite one can pass to the ring $R[U]_{mR[U]}$, $U$ indeterminate). Then, since a field extension is flat, $y^* \otimes 1$ is a non zero divisor of $G(B) \otimes K$ which then is Cohen-Macaulay and reduced, since $\text{Proj}(G(B) \otimes K)$ is reduced. Finally, by the flatness of $G(B)$ over $k(\mathfrak{m})$, we have $G(B) \subset G(B) \otimes K$ and $G(B)$ is reduced. \(\square\)

Remark. If the ring $B$ of Theorem 5 is the local ring $A$ at a multiple point of a curve over an algebraically closed field $k$, then $K = k(\mathfrak{m}) = k$ and $G(B) \otimes_{k(\mathfrak{m})} K = G(A)$. Hence in this case Theorem 5,(b) proves that, if $\text{Proj}(G(A))$ is reduced and consists of points in generic position, then $G(A)$ is reduced which is Theorem 3.3 of [9]. In general the condition $\text{Proj}(G(A))$ reduced doesn’t imply that $G(A)$ is reduced as has been shown with various examples in [9, Section 3, Example 1] and [7, Section 4]. Another example (pointed out by A. De Paris), which answers also to a question posed in [3, Example 13], is the following.

6. Example. Let $B = \mathbb{C}[g, tg, fg]$, $f = t^5 - 1$, $g = tf$ and $A$ be the local ring of the curve $\text{Spec}(B)$ at $\mathfrak{m} = (g, tg, fg)$. We have $e(A) = 6$. Let $a_i$, $i = 1, \ldots, 5$, be the fifth roots of the unity. It is proved in [7,Section 4] that $\text{Proj}(G(A))$ consists of the points $P_i = (1, a_i, 0)$, $i = 1, \ldots, 5, P_6 = (1, 0, -1)$ which lie on the conic $yz = 0$ and then they are not in generic position [Example 4]. But $G(A)$ is not reduced as we are going to show. Let $\mathfrak{n}$ be the maximal homogeneous ideal of $G(A)_{\text{red}} = G(A)/\text{nil}(G(A)) = \mathbb{C}[X, Y, Z]/\mathfrak{a} = \mathbb{C}[x, y, z]$, where $\mathfrak{a} = (Z, Y^5 - X^5) \cap (Y, X + Z)$. There is a natural surjective homomorphism $\phi: \mathfrak{m}^2/\mathfrak{m}^3 \rightarrow \mathfrak{n}^2/\mathfrak{n}^3$ given by $\phi(H(g, tg, fg)) = H(x, y, z)$, $H(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ homogeneous of degree 2. Let $F(X, Y, Z) = Z^2 + XZ$. Since $z^2 + xz = 0$, if we show that $F(g, tg, fg) = fg(fg + g) \notin \mathfrak{m}^3$ we have that $\phi$ is not injective and $G(A)$ is not reduced. Now if $fg(fg + g) \in \mathfrak{m}^3$ one has

\begin{equation}
(*) \quad fg(fg + g) = P(1, t, f)g^3 + g^4h
\end{equation}

where $h \in \mathbb{C}[t]$ and $P(X, Y, Z)$ is a homogeneous polynomial of degree 3. But $fg(fg + g) = f(f + 1)g^2 + t\delta^2$. Substituting in $(*)$ and dividing by $g^3$ we have...
$t^4 - P(1, t, f) = gh$. But $t^4 - P(1, t, f) = q + fr$ where $q, r \in \mathbb{C}[t]$ and $q$ is monic of degree 4. Thus $q = gh - fr = f(th - r)$ which is impossible because the degree of $f$ is 5.

We recall that the conductor $b$ of a ring $B$ in its normalization $\overline{B}$ is the ideal (of $B$ and $\overline{B}$) $b = \{b \in B | \overline{B}b \subset B\}$.

7. Theorem. Let $B$ be a one dimensional reduced local ring with finite normalization $\overline{B}$. Let $k(m) = B/m$ be the residue field of $B$ and $K$ be the algebraic closure of $k(m)$, If $G(B) \otimes_{k(m)} K$ is reduced then:

(a) $G(B)$ is reduced, there is a natural immersion $G(B) \subset G(\overline{B})$ and $G(\overline{B})$ is the normalization of $G(B)$;

(b) there is a natural immersion $G(B) \otimes_{k(m)} K \subset G(\overline{B}) \otimes_{k(m)} K$ and $G(\overline{B}) \otimes_{k(m)} K$ is the normalization of $G(B) \otimes_{k(m)} K$;

(c) If $\nu = \text{Min}\{n \in \mathbb{N} | e = H(B, n)\}$, the ideal $(m)^\nu$ is contained in the conductor of $G(B)$ in $G(\overline{B})$, and if $B$ (that is $G(B)$) has maximal Hilbert function, then $\nu = \text{Min}\{n \in \mathbb{N} | e \leq (n + r)\}$, where $e = e(B)$ and $r + 1 = \text{emedim}(B)$.

Proof. (a) We have $G(B) \subset G(B) \otimes K$ so if $G(B) \otimes K$ is reduced then so is $G(B)$. Then there is a natural immersion $G(B) \subset G(\overline{B})$ [8, Proposition 2.2] and $G(\overline{B})$ is the normalization of $G(B)$ [10, Proposition 1.7].

(b) By (a) and by the flatness of $K$ over $k(m)$ we have the immersion $G(B) \otimes K \subset G(\overline{B}) \otimes K$. Furthermore $G(\overline{B})$ is the normalization of $G(B)$. From this we want to deduce that $G(\overline{B}) \otimes K$ is the normalization of $G(B) \otimes K$. It is easily seen that $G(\overline{B}) \otimes K$ is integral over $G(B) \otimes K$ and that $G(\overline{B}) \otimes K$ is contained in the total ring of quotients of $G(B) \otimes K$. Since by assumption $G(B) \otimes K$ is reduced, $G(\overline{B}) \otimes K$ is reduced. But, if $D = \overline{B}/m\overline{B}$, $G(\overline{B}) \cong D[T]$ and $G(\overline{B}) \otimes K \cong (D \otimes K)[T]$. Thus $D \otimes K$ is an artinian reduced ring, that is a direct sum of fields. Hence $G(\overline{B}) \otimes K$ is normal.

(c) If $b$ is the conductor of $B$ in $\overline{B}$ it is well known ([7, Lemma 2.12] and [12, Theorem 1.3]) that $m^\nu \subset b$. Then $G(m)^\nu \subset G(b)$. But, $G(b)$ is contained in the conductor of $G(B)$ in $G(\overline{B})$. In fact if $b^* \in G(\overline{B})$ and $x^* \in G(b)$ have degree respectively $s$ and $t$ and are initial forms of elements in $\mathfrak{J}^s - \mathfrak{J}^{s+1}$ and $x \in \mathfrak{J}^t \cap b - \mathfrak{J}^{t+1}$ (where $\mathfrak{J}$ is the Jacobson radical of $\overline{B}$), then $bx \in \mathfrak{J}^{t+s} \cap A = m^{s+t}$ [8, Proposition 2.2] i.e. $b^* x^* \in G(A)$.

Finally the last statement is an easy consequence of the definition of maximal Hilbert function [Definition 1].

8. Theorem. Let $S$ be a standard graded finitely generated $k$-algebra, over an algebraically closed field $k$, with maximal homogeneous ideal $n$. Assume $S$ one dimensional and reduced. Let $e = e(S)$ and $r + 1 = \text{emedim}(S)$. Then $\text{Proj}(S)$ consists of $e$ points of $\mathbb{P}^r$. If these points are in generic $e - 1$, $e$ position, then the conductor of $S$ in its normalization $\overline{S}$ is $n^\nu$, where $\nu = \text{Min}\{n | e \leq (\binom{n + r}{r})\}$.

Proof. [8, Proposition 3.5 and Theorem 4.3].
9. Theorem. Let \( A \) be the local ring of a curve at a singular point with reduced tangent cone. Let \( \mathfrak{m} \) be the maximal ideal of \( A \). If, for some integer \( n \), \( G(\mathfrak{m}^n) = G(\mathfrak{m})^n \) is the conductor of \( G(A) \) in \( G(\overline{A}) \), then \( \mathfrak{m}^n \) is the conductor of \( A \) in \( \overline{A} \).

Proof. Let \( \mathfrak{J} \) be the Jacobson radical of \( G(\overline{A}) \). Since by assumption \( G(\mathfrak{m}^n) \) is the conductor we have

\[
G(\mathfrak{m}^n) = G(\mathfrak{m}^n)G(\overline{A}) = G(\mathfrak{m}^n\overline{A}) = G(\mathfrak{J}^n)
\]

and then the result follows from [8, Theorem 2.3]. \( \square \)

The following is the main result of [8] (see Theorem 4.4).

10. Theorem. Let \( A \) be the local ring of a curve at a singular point, with maximal ideal \( \mathfrak{m} \). Let \( e = e(A) \) and \( \text{emdim}A = r + 1 \). Assume that \( \text{Proj}(G(A)) \) is reduced and consists of points in generic \( e-1,e \) position. Then \( \mathfrak{m}^\nu \) is the conductor of \( A \), where \( \nu = \text{Min}\{n \mid e \leq \binom{n+r}{r}\} \).

Proof. By Theorem 5, (b) (see also the following Remark) \( G(A) \) is reduced. Then, by Theorem 8, the conductor of \( G(A) \) in its normalization \( G(\overline{A}) \) is \( G(\mathfrak{m}^\nu) = G(\mathfrak{m}^\nu) \) (note that \( G(\mathfrak{m}) \) is the maximal homogeneous ideal of \( G(A) \)). Hence, by Theorem 9, \( \mathfrak{m}^\nu \) is the conductor of \( A \) in \( \overline{A} \). \( \square \)

Theorem 10 can be extended to any one dimensional local ring in the following way:

11. Theorem. Let \( B \) be a one dimensional reduced local ring with finite normalization \( \overline{B} \) and maximal ideal \( \mathfrak{m} \). Let \( k(\mathfrak{m}) = B/\mathfrak{m} \) be the residue field of \( B \) and \( K \) be the algebraic closure of \( k(\mathfrak{m}) \). Set \( e = e(B) \) and \( \text{emdim}(B) = r + 1 \). If \( \text{Proj}(G(B) \otimes_{k(\mathfrak{m})} K) \) is reduced and consists of points in generic \( e-1,e \) position then the conductor of \( B \) in \( \overline{B} \) is \( \mathfrak{m}^\nu \) where \( \nu = \text{Min}\{n \mid e \leq \binom{n+r}{r}\} \).

Proof. If \( \text{Proj}(G(B) \otimes K) \) is reduced and consists of points in generic \( e-1,e \) position then \( G(B) \otimes K \) is reduced [Theorem 5, (b)], its normalization is \( G(\overline{B}) \otimes K \) [Theorem 7,(b)], and its conductor is \( (G(\mathfrak{m})\otimes K)^\nu = G(\mathfrak{m})^\nu \otimes K \) [Theorem 8 and Theorem 5, (a)]. Furthermore, if \( \mathfrak{b} \) is the conductor of \( G(B) \), then \( \mathfrak{b} \otimes K \) is contained in the conductor of \( G(B) \otimes K \). In fact if \( y \in \mathfrak{b} \), \( b \in G(\overline{B}) \) and \( c,c' \in K \), \( (b\otimes c)(y\otimes c') = by\otimes cc' \in G(\overline{B}) \otimes K \). But, by Theorem 7,(c), \( G(\mathfrak{m})^\nu \subset \mathfrak{b} \) and then, by flatness \( G(\mathfrak{m})^\nu \otimes K \subset \mathfrak{b} \otimes K \). Thus

\[
G(\mathfrak{m})^\nu \otimes K = \mathfrak{b} \otimes K
\]

But, again by flatness,

\[
dim_k(\mathfrak{b}/G(\mathfrak{m})^\nu) \otimes K = \dim_k((\mathfrak{b} \otimes K)/(G(\mathfrak{m})^\nu \otimes K)) = 0
\]

that is \( \mathfrak{b} = G(\mathfrak{m})^\nu \) and, by Theorem 9, the conductor of \( B \) is \( \mathfrak{m}^\nu \). \( \square \)

Remark. Theorem 10 is the main result of a more general statement on the conductor of a curve at an ordinary singularity [8, Theorem 4.4]. This statement can be easily extended to one dimensional rings in the same way of Theorem 11.
3. The codimension one case.

We need some preliminaries.

12. Definition. Let $p$ be a prime ideal of a local ring $A$. $A$ is normally flat along $p$ if $p^n/p^{n+1}$ is flat over $A/p$, for all $n \geq 0$. Note that $A$ is normally flat along $p$ if and only if $G_p(A)$ is free over $A/p$.

13. Theorem. Let $p$ be a prime ideal of a local ring $A$ such that $A/p$ is regular and $\dim(A/p) = d \geq 1$. Then:

$$H^0(A, n) \geq H^d(A_p, n),$$

for any $n$ and equality holds if and only if $A$ is normally flat along $p$.

In particular

$$\text{emdim}(A) \geq \text{emdim}(A_p) + \dim(A/p), e(A) \geq e(A_p)$$

and if $A$ is normally flat along $p$, then

$$\text{emdim}(A) = \text{emdim}(A_p) + \dim(A/p), e(A) = e(A_p)$$

Proof. See [5, Theorem (22.24) and Proposition (30.2)].

14. Theorem. Let $p$ be a prime ideal of codimension one of a Cohen-Macaulay local ring $A$ such that $A/p$ is regular. Assume $e(A) = e(A_p) = e$ and $\text{emdim}(A) = \text{emdim}(A_p) + \dim(A/p)$. If $H^0(A_p, n)$ is maximal then $A$ is normally flat along $p$ and $G(A)$ is Cohen-Macaulay.

Proof. Set $\dim(A/p) = d$, $e(A_p) = e$ and $\text{emdim}(A_p) = r + 1$. Since $A$ is Cohen-Macaulay there is an $A$-sequence $x_1, ..., x_d$ of elements of degree 1 (this can always be arranged if the residue field of $A$ is infinite and if it is finite, by passing, if necessary, to the ring $A[U]_{\text{mA}[U]}$, $U$ indeterminate) and such that the ring $B = A/(x_1, ..., x_d)$ is Cohen-Macaulay. Then, by the assumption, we have $e(B) = e(A) = e(A_p) = e$, $\text{emdim}(B) = e\text{emdim}(A) - d = e\text{emdim}(A_p) = r + 1$. Furthermore $B$ and $A_p$ are one dimensional and then $H^0(B, n) \leq \text{Min} \{e,(n+e)\} = H^0(A_p, n)$. [Definition 1 and preceding comments] Hence $H^d(B, n) \leq H^d(A_p, n)$. But $H^d(B, n) \geq H^0(A, n)$ [11, Theorem 3.1 (note that in the statement there is a misprint, namely the exponent $s$ of the Hilbert function $H^s$ should be $s + 1$)]. Furthermore $H^0(A, n) \geq H^d(A_p, n)$ by Theorem 13. We have then $H^d(B, n) = H^0(A, n) = H^d(A_p, n)$. Hence $A$ is normally flat along $p$ [Theorem 13] and the initial forms $x_1^*, ..., x_d^*$ form a regular sequence of $G(A)$ [11, Theorem 3.1]. Furthermore, since $H^0(B, n) = H^0(A_p, n)$, the one dimensional ring $B$ has maximal Hilbert function and then $G(B)$ is Cohen-Macaulay [11, Theorem 3.2]. But $G(B) \cong G(A)/(x_1^*, ..., x_d^*)$ [14, Chapter 2, Lemma 3.2] hence $G(A)$ is Cohen-Macaulay.

15. Theorem. Let $p$ be a prime ideal of codimension one of a reduced local ring $A$ with finite normalization $\overline{A}$. Let $e(A_p) = e > 1$, $\text{emdim}(A_p) = r + 1$ and denote by $K$ the algebraic closure of the residue field $k(p) = A_p/pA_p$ of $A$ at $p$. Let $b$ be the conductor of $A$ in $\overline{A}$. Assume $A/p$ regular, $\sqrt{b} = p$ and $\text{Proj}(G(A_p) \otimes_{k(p)} K)$ reduced with points in generic $e - 1, e$ position.
Given the following conditions:

(a) $A$ is Cohen-Macaulay and $\text{emdim}(A) = \text{emdim}(A_p) + \dim(A/p)$, $e(A) = e(A_p)$;

(b) $A$ is $S_2$ and normally flat along $p$;

(c) $b$ is primary and $A$ is normally flat along $p$;

(d) $b = p^\nu$, where $\nu = \text{Min}\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$;

then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ A Cohen-Macaulay ring is $S_2$ [6, 17.I]. By Theorem 5, (b) if $\text{Proj}(G(A_p) \otimes_{k(p)} K)$ is reduced and consists of points in generic position, the Hilbert function of $A_p$ is maximal and, by Theorem 14, $A$ is normally flat along $p$.

$(b) \Rightarrow (c)$ If $A$ is $S_2$ then its conductor $b$ is unmixed [4, Lemma 7.4] and then primary, since $\sqrt{b} = p$.

$(c) \Rightarrow (d)$ $bA_p$ is the conductor of $A_p$ in its normalization $\overline{A}$ and then, by Theorem 11, $bA_p = (pA_p)^\nu = p^\nu A_p$. Furthermore, if $A$ is normally flat along $p$ then $p^\nu$ is primary [13, Proposition 1.1] and then

$$b = A \cap bA_p = A \cap p^\nu A_p = p^\nu$$

□

16. Corollary. Suppose $p$ is a prime ideal of codimension one of a Cohen-Macaulay reduced local ring $A$ with finite normalization $\overline{A}$. Denote by $K$ the algebraic closure of the residue field $k(p) = A_p/pA_p$. Let $b$ be the conductor of $A$ in $\overline{A}$. Assume $A/p$ regular, $\sqrt{b} = p$ and $\text{Proj}(G(A_p) \otimes_{k(p)} K)$ reduced. If $\text{emdim}(A) = \dim(A) + 1$ and $e(A) = e(A_p) = e$ then $b = p^{e-1}$.

Proof. Let $\text{emdim}(A) = \dim(A) + 1$. By assumption $\dim(A) = \dim(A/p) + 1$ and, by Theorem 13, $\text{emdim}(A) \geq \text{emdim}(A_p) + \dim(A/p)$. Then $\text{emdim}(A_p) = 2$ since $e(A_p) > 1$ and $\text{emdim}(A) = \text{emdim}(A_p) + \dim(A/p)$. Hence $\text{emdim}(G(A_p) \otimes_{k(p)} K) = 2$ [Theorem 5, a)] that is $\text{Proj}(G(A_p) \otimes_{k(p)} K) \subset \mathbb{P}^1$ and then its points are always in generic $e - 1$, $e$ position [Example 3]. Hence the claim follows from Theorem 15, $(a) \Rightarrow (d)$ (if $r = 1$ one has $\nu = e - 1$). □

In the following we want to apply the previous results to the geometric case.

Standing notation. From now on $X = \text{Spec}(R)$ is a reduced variety over an algebraically closed field $k$ and $Y = \text{Spec}(R/q)$ is an irreducible codimension one subvariety of $X$ (i.e. $q$ is a prime ideal of codimension one in $R$). By $K$ we denote the algebraic closure of the residue field $k(q)$ of $R$ at $q$. We set $\text{emdim}(R_q) = r + 1$ and $e(R_q) = e$.

17. Definition. Let $x$ be any closed point of $Y$ and $A$ be the local ring of $X$ at $x$. Let $p = qA$ be the prime ideal in $A$ defining the subvariety $Y$. Then:

(i) $Y$ is a multiple subvariety of $X$ of multiplicity $e$ if $e(R_q) = e > 1$;

(ii) $X$ is normally flat along $Y$ at $x$ if $A$ is normally flat along $p$;

(iii) $X$ is normally flat along $Y$ if it is so at every point of $Y$;

(iv) $Y$ is nonsingular at $x$ if $A/p$ is regular.

Remark. With the notations of Definition 17 we have $A_p = R_q$ for any closed point $q$ of $X$. 

18. **Theorem.** Let $Y$ be a multiple subvariety of $X$. There exists an open nonempty subset $U$ of $Y$ such that, for every closed point $x$ of $U$, $Y$ is nonsingular at $x$ and $X$ is normally flat along $Y$ at $x$.

**Proof.** It is well known that the nonsingular points of $Y$ form an open nonempty set and that $X$ is normally flat along $Y$ at the points of an open nonempty subset of $Y$ [5, Corollary (24.5)]. □

19. **Theorem.** Set $\text{emdim}(R_q) = r + 1$. Suppose that $Y$ is a multiple subvariety of $X$ of multiplicity $e$. If $\text{Proj}(G(R_q) \otimes_{k(q)} K)$ is reduced and consists of points in generic $e - 1$ position then there exists an open nonempty subset $U$ of $Y$ such that, for every closed point $x$ of $U$, the conductor $b$ of the local ring $A$ of $X$ at $x$ is primary and equal to $p^\nu$ where $p$ is the prime ideal in $A$ defining $Y$ and $\nu = \text{Min}\{n \mid e \leq \binom{n+r}{r}\}$.

**Proof.** By the Remark to Definition 17, if $x$ is any point of $X$ and $A$ is the corresponding local ring, one has $e(A_p) = e(R_q)$, hence, by assumption, $e(A_p) > 1$ that is, since $\dim(A_p) = 1$, $A_p$ is not normal. Then it is well known that $b$ is contained in $p$ . But the conductor contains a nonzero divisor. Thus the codimension one ideal $p$ is a minimal prime of $b$ . Now it is easily shown that there exists an open nonempty subset $U_1$ of $Y$ such that, if $x$ is a closed point of $U_1$, $p$ is the unique prime ideal of $A$ associated to $b$. Then $b$ is primary and $\sqrt{b} = p$. Furthermore, by Theorem 18, there exists an open nonempty subset $U_2$ of $Y$ such that for every closed point $x$ of $U_2$, $Y$ is nonsingular at $x$ and $X$ is normally flat along $Y$ at $x$. Then, if we apply Theorem 15, $(c) \Rightarrow (d)$ to the local rings of the points of $U = U_1 \cap U_2$, we have the claim. □

We want now to characterize geometrically the condition: $\text{Proj}(G(R_q) \otimes_{k(q)} K)$ is reduced. We need the following preliminary general result.

20. **Theorem.** Let $A$ be the local ring of $X$ at a closed point $x$ of $Y$ and $p$ be the prime ideal defining $Y$ in $A$. Set $\dim(A/p) = d$. Suppose $Y$ is nonsingular at $x$ and $X$ is normally flat along $Y$ at $x$. Then there is an isomorphism of graded $k$-algebras

$$G(A) \cong (G_p(A) \otimes_{A/p} k)[T_1, \ldots, T_d]$$

**Proof.** See [5, Corollary (21.11)]. □

Assume that $Y$ is a multiple subvariety of $X$ and $A$ is the local ring of $X$ at a closed point $x$ of $Y$. $A/p$ is the local ring of $Y$ at $x$. Then $\text{Spec}(G(A))$ is the tangent cone to $X$ at $x$ and $\text{Spec}(G(A/p))$ is the tangent cone to $Y$ at $x$. Furthermore, there is a natural surjective homomorphism $G(A) \rightarrow G(A/p)$ and then $\text{Spec}(G(A/p))$ naturally embeds in $\text{Spec}(G(A))$. In this setting we have the following result:

21. **Theorem.** Let $K$ be the algebraic closure of the residue field $k(q)$ of $R$ in $q$. Then $\text{Proj}(G(R_q) \otimes_{k(q)} K)$ is reduced (that is consists of $e$ points) if and only if there exists an open nonempty subset $U$ of $Y$ such that, for every closed point $x$ of $U$, the tangent cone to $X$ at $x$ is the union, as a set, of $e$ distinct linear varieties, whose intersection is the tangent cone to $Y$ at $x$.

**Proof.** If $\text{Proj}(G(R_q) \otimes_{k(q)} K)$ has $e$ points by [1, Theorem 1.1] there exists an open nonempty subset $U$ of $Y$ such that, for every closed point $x$ of $U$, the tangent cone
23. **Definition.** Let $\text{Spec}(G(A))$ to $x$ at $x$, has $e$ irreducible components, that is $G(A)$ has $e$ minimal primes. Furthermore, by Theorems 18 and 20, there exists an open nonempty subset $U_2$ of $Y$ such that, for every closed point $x$ of $U_2$, $G(A)$ is a polynomial ring over $G_p(A) \otimes_{A/p} k$. Then if $x$ is a point of $U_2$ the minimal primes of $G(A)$ are extensions of the minimal primes of the one dimensional finitely generated graded $k$-algebra $G_p(A) \otimes_{A/p} k$, hence they are generated by linear forms. Since, at every closed point $x \in U = U_1 \cap U_2$ the tangent cone to $Y$ is contained in the tangent cone to $X$, we have the claim. Vice versa if, at every point of an open nonempty subset of $Y$, the tangent cone to $X$ has $e$ irreducible components, again by [1, Theorem 1.1], also the zero dimensional scheme $\text{Proj}(G(A_p) \otimes_{k(p_i)} K)$ has $e$ irreducible components, that is $e$ points, and then, since by Theorem 5,(a) $e(G(A_p) \otimes_{k(p_i)} K) = e(G(A_p)) = e(A_p) = e$, is reduced. \quad \square

**Remark.** Theorem 21 extends Corollary 1.4 of [1] which states the only if part in

24. **Theorem.** Let $X = \text{Spec}R$ be a reduced non-normal $S_2$ variety. Assume that the irreducible components of the non-normal locus of $X$ are ordinary multiple subvarieties $Y_i = \text{Spec}(R/q_i)$, $(1 \leq i \leq t)$ of multiplicity $e_i = e(R_{q_i}) > 1$. Set $\text{emdim}(R_{q_i}) = r_i + 1$ and let $K_i$ be the algebraic closure of the residue field $k(q_i)$. If $\text{Proj}(G(R_{q_i}) \otimes_{k(q_i)} K_i)$ has points in generic $e_i - 1, e_i$ position, for any $i$, then the conductor of $R$ is $J = 2^{(\nu_1)} \circ \cdots \circ 2^{(\nu_t)}$, where $\nu_i = \text{Min}\{n | e_i \leq (n+r_i)\}$, and...
$q_i^{(\nu_i)}$ denotes the $\nu_i$-th symbolic power of $q_i$. Furthermore assume that the $Y_i$ are nonsingular varieties, for any $i$, and consider the following conditions:

(a) $X = \text{Spec}(R)$ is Cohen-Macaulay, equimultiple along $Y_i$ and has constant embedding dimension along $Y_i$, for any $i$;

(b) $X$ is normally flat along $Y_i$, for any $i$;

(c) $b = q_i^{\nu_1} \cap ... \cap q_i^{\nu_t}$;

then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

**Proof.** Let $b = a_1 \cap ... \cap a_t$ be a minimal primary decomposition, $\sqrt{a_i} = q_i$ ($1 \leq i \leq t$). By Theorem 11, for any $i = 1, ..., t$, $a_i R_{q_i} = b R_{q_i} = q_i^{\nu_i} R_{q_i} = q_i^{(\nu_i)} R_{q_i}$. Then

$$a_i = R \cap a_i R_{q_i} = R \cap q_i^{\nu_i} R_{q_i} = q_i^{(\nu_i)}$$

which is the first claim. The implications of the second claim follow easily from the analogous local implications of Theorem 15. $\Box$

If we apply Theorem 24 to the particular case $t = 1$ we get that, under the assumption (a), Theorem 19 holds for any point $x$ of the subvariety $Y$ (not only on an open subset of $Y$).

**25. Corollary.** Let $X = \text{Spec}(R)$ be a reduced non-normal Cohen-Macaulay variety. Assume that the non-normal locus of $X$ is an ordinary multiple nonsingular subvariety $Y = \text{Spec}(R/q)$ of multiplicity $e = e(R_q) > 1$ and that $X$ is equimultiple and has constant embedding dimension along $Y$. Then, for every point $x$ of $Y$, the conductor $b$ of the local ring $A$ of $X$ at $x$ is primary and equal to $p^\nu$ where $p$ is the prime ideal in $A$ defining $Y$ and $\nu = \text{Min}\{n \mid e \leq (n+\nu)\}$

**Proof.** Setting $t = 1$ in Theorem 24 we have that $b R_{q_i} = q_i^{\nu_i} R_{q_i} = (q R_q)^{\nu_i} = p^\nu$ is the conductor of $A$. $\Box$

**26. Corollary.** Let $H$ be a non-normal hypersurface. Assume that the irreducible components of the non-normal locus of $X$ are ordinary multiple subvarieties $Y_i = \text{Spec}(R/q_i)$ of multiplicity $e_i = e(R_{q_i}) > 1$ ($1 \leq i \leq t$). If the $Y_i$ are nonsingular varieties and $X$ is equimultiple along $Y_i$, for any $i$, then the conductor of $R$ is $b = q_i^{e_i - 1} \cap ... \cap q_i^{e_i - 1}$.

**Proof.** If $H = \text{Spec}(R)$ is a hypersurface, then $R$ is Cohen-Macaulay and its embedding dimension at any point is constant. Furthermore $\text{Proj}(G(R_{q_i}) \otimes_{k(q_i)} K_i) \subset \mathbb{P}^1$ has points in generic $e_i - 1, e_i$ position [Example 3]. Then the claim follows by Theorem 24, (a) $\Rightarrow$ (c), in which $r_i = 1$. $\Box$

**27. Example.** Let $H = \text{Spec}(R)$, $R = \mathbb{C}[T_0, ..., T_r]/(L_1...L_n)$ ($r \geq 2, n \geq 2$) be the union of $n$ hyperplanes of $\mathbb{P}^r$. Let $q_i$ ($1 \leq i \leq t$) denote the codimension one primes of $R$ of the form $\langle L_p, L_q \rangle$, $p, q \in \{1, ..., n\}, p \neq q$, and $Y_1, ..., Y_t$ be the corresponding linear varieties (that is the irreducible components of the non-normal locus of $X$). Then $e(R_{q_i}) = e_i$ is the number of hyperplanes passing through $Y_i$. Furthermore since $Y_i$ are linear it is easily shown that $q_i^{e_i} = q_i^{(e_i)}$, for any $m$. Hence from Theorems 21 and 24, it follows that the conductor of $R$ is $b = q_i^{e_i - 1} \cap ... \cap q_i^{e_i - 1}$.
Example. Let $R = \mathbb{C}[X, Y, Z]/(XY^n - Z^n) = \mathbb{C}[x, y, z]$ $(n \geq 2)$. The non-normal locus of the hypersurface $H = \text{Spec} R$ is the line $L : y = 0, z = 0$ of multiplicity $n$, that is $e(R_q) = n$, where $q = (y, z)$. Moreover it is easily seen that $H$ is equimultiple along $L$. Furthermore, if $a \in \mathbb{C}, a \neq 0$, the tangent cone, at the point $(a, 0, 0)$ of $L$, consists, as a set, of the $n$ distinct planes $z = by$, where $b^n = a$, and then, by Theorem 21 and Corollary 26 and, the conductor of $R$ in $\overline{R}$ is $(y, z)^{n-1}$.

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