A Semantics for Approximate Program Transformations

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Abstract

An approximate program transformation is a transformation that can change the semantics of a program within a specified empirical error bound. Such transformations have wide applications: they can decrease computation time, power consumption, and memory usage, and can, in some cases, allow implementations of incomputable operations. Correctness proofs of approximate program transformations are by definition quantitative. Unfortunately, unlike with standard program transformations, there is as of yet no modular way to prove correctness of an approximate transformation itself. Error bounds must be proved for each transformed program individually, and must be re-proved each time a program is modified or a different set of approximations are applied.

In this paper, we give a semantics that enables quantitative reasoning about a large class of approximate program transformations in a local, composable way. Our semantics is based on a notion of distance between programs that defines what it means for an approximate transformation to be correct up to an error bound. The key insight is that distances between programs cannot in general be formulated in terms of metric spaces and real numbers. Instead, our semantics admits natural notions of distance for each type construct; for example, numbers are used as distances for numerical data, functions are used as distances for functional data, and polymorphic lambda-terms are used as distances for polymorphic data. We then show how our semantics applies to two example approximations: replacing reals with floating-point numbers, and loop perforation.

1. Introduction

Approximation is a fundamental concept in engineering and computer science, including notions such as floating-point numbers, lossy compression, and approximation algorithms for NP-hard problems. Such techniques are often used to trade off accuracy of the result for reduced resource usage, for resources such as computation time, power, and memory. In addition, some approximation techniques are also used to ensure computability. For example, true representations of real numbers (e.g., [7], [1]), require some operations, such as comparison, to be incomputable; floating-point comparison, in contrast, is efficiently decidable on modern computers.

Recently, there has been a growing interest in language-based approximations, where approximate program transformations are performed by the programming language environment [21], [12], [19], [18], [4], [3], [16]. Such approaches allow the user to give an exact program as a specification, and then apply some set of transformations to this specification, yielding an approximate program. The goal is for approximations to be performed on behalf of the programmer, either fully automatically or with only high-level input from the user, while still maintaining a given error bound; i.e., the goals are automation and correctness. These goals can in turn increase programmer productivity while helping to remove programmer errors, where the latter can be especially important, for example, in safety-critical systems.

This leads us to a fundamental question: what does it mean for an approximate transformation to be correct? A good answer to this question must surely be quantitative, since approximate transformations should not change the output by too much; i.e., they must respect user-specified error bounds. Correctness should also be modular, meaning that, in settings where approximate transformations $T_1, \ldots, T_k$ are applied together to a program $P$, it should be possible to reduce the proof of correctness of $\{T_1, \ldots, T_k\}$ to individual proof obligations for the $T_i$s. Current formal approaches to approximate program transformations, however, do not permit such modular reasoning about approximations.
Typically, they are tailored to specific forms of approximation — for example, the use of floating-point numbers or loop perforation (skipping certain iterations in long-running loops). Even when multiple approximations can be combined, reasoning about them is monolithic; in the above example, if $T_1$ is changed slightly, we would need to re-prove the correctness of $P$ with respect to not just $T_1$ but $\{T_1, \ldots, T_k\}$. In addition, current approaches to approximate computation are essentially incomputable operations, such as floating-point numbers instead of reals, uses inexact but computable versions of potentially incomputable operations, such as $f(n)$ for finite $n$ in place of $\lim_{x \to \infty} f(x)$; and approximate optimizations: $a$ performs a cheaper, less precise version of a computation, such as the identity function in place of $\sin(x)$.

The remainder of the paper is organized as follows. Section 2 motivates our semantics with a high-level overview. Section 3 defines our input language, $F^{ADT+}$, which is System F with algebraic datatypes and built-in operations. Section 4 defines our semantics by giving a general, composable semantics for program approximation, for higher-order programs with polymorphic types. In our semantics, individual approximate transformations are proved to be quantifiably correct, i.e., to induce a given local error expression. Error expressions are then combined compositionally, yielding a top-level error expression for the whole program that is built up from the errors of the individual approximate transformations being used. This approach has a number of benefits. First, it allows for more portable proofs: an approximate transformation $T$ can now be proved correct once, and the resulting error expression can be used many times in many different contexts. Second, it is mechanical and opens up opportunities for automation: approximation errors for a whole program and a set of disparate approximate transformations are generated simply by composing the errors of the individual transformations. Finally, our approach reduces the correctness of approximately transformed programs to the much easier question of whether a generated error expression is less than or equal to a given error bound.

The key technical insight that makes our semantics possible is that, despite past work on using metric spaces and real numbers in program semantics (e.g., [20], [15], [11]), we argue that real numbers cannot in general capture how, for example, the output error of a function depends on the input and its error. Instead, our approach allows arbitrary System F types for errors. We show that this can accurately capture errors, for example, by using functions as errors for functional data and polymorphic lambdas as errors for polymorphic data. To allow this, our semantics is based around a novel notion of an approximation type, which is a ternary logical relation between exact expressions $e$, approximate expressions $a$, and error expressions $q$. In addition to the above benefits, this approach can also handle changes of type between exact and approximate expressions, e.g., approximating real numbers by floating-point.

2. Approximating Programs

The goal of this work is to give a semantics for approximate transformations, which convert an exact program $e$ into an approximate program $a$ that, although not identical to the $e$, is within some quantifiable error bound $q$. Our notion of approximate transformation is very general, but it includes at least:

- **Data Approximations:** $a$ uses a less exact datatype, such as floating-point numbers instead of reals;
- **Approximations of Incomputable Operations:** $a$ uses inexact but computable versions of potentially incomputable operations, such as $f(n)$ for finite $n$ in place of $\lim_{x \to \infty} f(x)$; and
- **Approximate Optimizations:** $a$ performs a cheaper, less precise version of a computation, such as the identity function in place of $\sin(x)$.

The point of language-based approximations is that these transformations become part of the language semantics, which precisely captures the relationship between exact programs, their approximations, and the associated error bounds.

As a simple yet illustrative example, consider an approximate transformation that replaces the $\sin$ operator with the identity function $\lambda x. x$. Such an approximation can greatly reduce computation, since $\sin$ (for floating-point numbers) can be a costly operation, while the identity function is a no-op. Intuitively, we know that this change does not greatly affect the output when $x$ is close to 0. If, however, the output of a call to $\sin$ is then passed to a numerically sensitive operation, such as a reciprocal, then the small change resulting from replacing $\sin$ by the identity could lead to a large change in the final result. Thus, our goal is to quantify the error introduced by this approximate...
transformation in a local, compositional way, allowing this error to be propagated through the rest of the program.

Figure 1 allows us to visualize the error resulting from replacing \( \sin \) with the identity function. We assume that the input \( in \) has already been approximated, yielding an approximate input \( in' \) with some approximation error \( err \). The exact output, \( \sin(in) \), gets replaced by an approximate result equal to \( in' \).

The error for this expression can then be calculated as \( err + in - \sin(in) \), as described in the figure.

This example leads to the following observations:

1) **Errors are programs:** Although the standard approach to quantification and errors is to use metric spaces and real numbers, the error expression \( err + in - \sin(in) \) in our example depends on both the input \( in \) and the input error \( err \); i.e., it is a function, which cannot be represented by a single real number.

2) **Errors are compositional:** Approximation errors in the input to \( \sin \) are substituted into the approximation error for \( \sin \) itself, and this error is in turn passed to any further approximations that occur at a later point in the computation.

In order to formalize the relationship between programs, their approximations, and the resulting errors, we define a semantic notion below called an approximation type. An approximation type is a form of logical relation (see, e.g., \[6\], \[17\]), satisfying certain properties discussed below, that relates exact expressions \( e \), approximate expressions \( a \), and error expressions \( q \). If \( A \) is an approximation type, we write \( \{e\}_A^q \) for the set of all exact expressions \( e \) related by \( A \) to approximate expression \( a \) and error expressions \( q \). Thus, \( e \in \{a\}_A^q \) means that \( e \) can be approximated by \( a \) with approximation error no greater than \( q \) when using approximation type \( A \).

As a first example, we define the approximation type \( F1 \) that relates real number expressions \( e \), floating-point expressions \( a \), and non-negative real number expressions \( q \) iff \( q \) is the distance between the \( e \) and the real number corresponding to \( a \). For instance, the real number \( \pi \) is within error 0.1415926... of the floating-point number 3, meaning that \( \pi \in \{0.3\}_F^1 \). If the only difference between \( a \) and \( e \) is that the former uses floating-point numbers in place of reals, then showing \( e \in \{a\}_F^q \) is essentially a form of floating-point error analysis. Using \( F1 \) is more general, however, because it allows the possibility that \( a \) performs further approximations, e.g., using the identity in place of \( \sin \). Taking this comparison further, \( F1 \) is specifically like an interval-based error analysis; we examine this more closely in Section 5.2.

For functions, following Figure 1 we use an approximation type where the error of approximating a function is itself a function, that maps exact inputs and their approximation errors to output approximation errors. More formally, if \( A_1 \) and \( A_2 \) are approximation types for the input and output of a function, respectively, then the functional approximation type \( A_1 \Rightarrow A_2 \) is defined such that \( e \in \{a\}_A^q \Rightarrow A_2 \) iff \( e_{c1} \in \{a_{c1}\}_A^q c_{e1} q_1 \) for all \( c_{e1} \), \( a_{c1} \), and \( q_1 \) such that \( a_{c1} \in \{a\}_A^q \).

These notions can then be used to prove correctness of approximate transformations, as follows. First, the designer of an approximate transformation gives an approximation rule for the judgment \( \vdash e \Rightarrow a \leq q : A \), stating that expressions matching \( e \) can be transformed into \( a \) with error at most \( q \) using approximation type \( A \). Approximation rules are explained in more detail below, but, as an example, our sin approximation above can be captured as:

\[ \vdash \sin \Rightarrow \lambda x. x \leq \lambda x. q q + x - \sin(x) : F1 \Rightarrow F1 \]

After formulating this rule, it must then be proved correct, by proving that \( e \in \{a\}_A^q \) for all \( e \), \( a \), and \( q \) that match the rule. As argued above, a significant benefit of this approach is that it allows approximate transformations to be analyzed and proved correct on their own, without reference to the programs in which they are being used.

3. System F as a Logical Language

In the remainder of this paper, we work in System F with algebraic datatypes (ADTs) (see, e.g., Pierce \[13\]), extended to allow uncountability and incomputability. Specifically, ADTs can have uncountably many constructors, while built-in operations can perform potentially incomputable functions. The former is useful for modeling the real numbers, while the latter is useful for modeling operations on real numbers such as comparison that are incomputable in general (e.g., \[17\], \[11\]). We refer to this language as \( F^{ADT+} \).
The only additional technical machinery needed for these extensions to System F is to require that all built-in functions are definable in the meta-language (set theory or type theory). More specifically, we assume as given some set of built-in operation symbols \( f \), each with a given type \( \tau_f \) and meta-language relation \( R_f \) relating allowed inputs for the function defined by \( f \) to their corresponding outputs. Further, we assume that each \( R_f \) obeys \( \tau_f \) and relates only one output to any given set of inputs. The small-step evaluation relation of System F is then be extended to allow \( f \) applied to any values to evaluate to any output related to those inputs by \( R_f \). The details are straightforward but tedious, and so are omitted here.

We use \( \rightarrow \) to denote the small-step evaluation relation of \( F^{ADT+} \); i.e., \( e \rightarrow e' \) means that \( e \) evaluates in one step to \( e' \). Typing contexts \( \Gamma \) are lists of type variables \( X \) that are considered in scope, along with pairs \( x : \tau \) of expression variables \( x \) along with their types. Substitutions for expressions \( e_1 \) through \( e_n \) for variables \( x_1 \) through \( x_n \) are written \([e_1/x_1, \ldots, e_n/x_n]\). These are represented with the letter \( \sigma \), and we define capture-avoiding substitution \( \sigma e \) for expressions and \( \sigma \tau \) for types in the usual manner. Well-formedness of types and expressions is given respectively by the kinding judgment \( \Gamma \vdash \tau : \ast \) and typing judgment \( \Gamma \vdash e : \tau \), defined in the standard manner. We write \([\Gamma \vdash \tau] \) for the set of all expressions \( e \) such that \( \Gamma \vdash e : \tau \). Expressions or types with no free variables are ground. Typing is extended to typing contexts \( \Gamma \) and substitutions in the usual manner: \( \vdash \Gamma \) indicates that all types in \( \Gamma \) are well-formed, while \( \Gamma \vdash \sigma : \Gamma' \) indicates that \( \text{Dom}(\sigma) = \text{Dom}(\Gamma') \), \( \Gamma \vdash \sigma(X) : \ast \) for all \( X \in \text{Dom}(\sigma) \), and \( \Gamma \vdash \sigma(x) : \Gamma'(x) \) for all \( x \in \text{Dom}(\sigma) \). We assume all types are well-formed and all expressions, contexts, and substitutions are well-typed below.

We write \( e \downarrow \) to denote that \( e \) is terminating. An expression context \( C \) an expression with exactly one occurrence of a “hole” \( \{ \} \), and \( C\{ e \} \) denotes the (non-capture-avoiding) replacement of \( \{ \} \) with \( e \). Contextual equivalence \( e_1 \simeq e_2 \) is then defined to hold iff, for all \( C \) and \( \tau' \) such that \( \cdot \vdash C\{e_1\} : \tau' \) for \( i \in \{1, 2\} \), we have that \( C\{e_i\} \downarrow \) iff \( C\{e_2\} \downarrow \). We use \( \downarrow \) to denote an arbitrary non-terminating expression. \( \text{fix}(\lambda x. x) \) of a given type \( \tau \). We assume below type \( R \) of real numbers, represented with an uncountable number of constructors, and the type \( \mathbb{R}^{+\infty} \) of the non-negative reals with infinity. We use \( 0_R \) for the \( R \)-constructor corresponding to the real number 0, and \( \leq_R, +_R \), and \( \cdot_R \) for the function symbols whose functions perform real number comparison, addition, and absolute difference, both on \( R \) and, abusing notation slightly, on \( \mathbb{R}^{+\infty} \).

Since \( F^{ADT+} \) can contain incomputable built-in operations, we can additionally use it as a meta-language by embedding any relations of a given meta-language, such as set theory or type theory, as built-in operations. We assume an ADT Prop with the sole constructor \( \top : \text{Prop} \), which will intuitively be used as the type of propositions; a “true” proposition terminates to \( \top \) and a “false” one is non-terminating, i.e., is contextually equivalent to \( \bot \). A meta-language relation \( R \) can then be added to \( F^{ADT+} \) as a function symbol \( f_R \) of type \( \forall \tau. \tau \rightarrow \text{Prop} \) iff \( R \) is a set of tuples \( \langle \tau', e \rangle \) such that \( \cdot \vdash f_R[\tau'] e : \text{Prop} \) that is closed under contextual equivalence and contains only terminating expressions \( e \). Note that this latter restriction is not too significant because we can always use a “thunkified” relation \( R' \) of type \( \forall \tau. (\text{Unit} \rightarrow \tau_1) \rightarrow \ldots \rightarrow (\text{Unit} \rightarrow \tau_n) \rightarrow \text{Prop} \) such that \( \langle \tau', e \rangle \in R \) iff \( \langle \tau', \lambda x.e_1, \ldots, \lambda x.e_n \rangle \in R' \). As an example, we can immediately see that the thunkified version of contextual equivalence itself is a relation of type \( \forall X. (\text{Unit} \rightarrow X) \rightarrow (\text{Unit} \rightarrow X) \rightarrow \text{Prop} \).

Lemma 1 (Consistency): For any class of built-in operations \( f \) with functions \( F_f \), \( \top \equiv \bot \) does not hold.

In the below, \( \phi \) refers to expressions of type \( \forall X. \tau \rightarrow \text{Prop} \). A constrained context is a pair \( \langle \Gamma; \phi \rangle \) of a context \( \Gamma \) and an expression \( \phi \) such that \( \Gamma \vdash \phi : \text{Prop} \). A substitution \( \sigma \) satisfies \( \langle \Gamma; \phi \rangle \), written \( \sigma \vdash \langle \Gamma; \phi \rangle \), iff \( \cdot \vdash \sigma : \Gamma \) and \( \sigma(\phi) \equiv \top \). We say that \( \langle \Gamma; \phi \rangle \) entails \( \phi', \) written \( \Gamma ; \phi \vdash \phi \) true, iff \( \forall \sigma \vdash \langle \Gamma; \phi \rangle, \sigma(\phi) \equiv \top \) holds. Finally, we say that \( \sigma \) is a substitution from \( \langle \Gamma_1; \phi_1 \rangle \) to \( \langle \Gamma_2; \phi_2 \rangle \), written \( \langle \Gamma_2; \phi_2 \rangle \vdash \sigma : \langle \Gamma_1; \phi_1 \rangle \), iff \( \Gamma_2 \vdash \sigma : \Gamma_1 \) and \( \langle \Gamma_2; \phi_2 \rangle \vdash \sigma(\phi_1) \) true.

4. A Semantics of Program Approximation

In this section, we define approximation types and illustrate them with examples. The main technical difficulty is that the straightforward way to handle free (expression and type) variables involves a circular definition. Specifically, to fit with the logical relations approach, \( e \in \{a\}_A \) for \( e, a, \) and \( q \) with free variables should only hold iff \( \sigma e \in \{a\}'_\mathcal{A}_q \) for all substitutions \( \sigma \) for exact variables \( x^a \), approximate variables \( x^a \), and error variables \( x^a \) such that \( \sigma x^a \in \{a\}_\mathcal{A}_q \vDash x^a \) for some approximation \( \mathcal{A}_q \). This defines approximation types in terms of approximation types! To remove this circularity, we first define approximation types that handle a given set of free variables arbitrarily, in Section 4.1. We then show in Section 4.2 how to lift ground approximation types into approximation functions, which uniformly handle any given context of variables in the “right” way.
4.1. Approximation Types

Definition 1 (Expression-Preorder): Let $\Gamma \vdash \tau : \ast$. A relation $\leq$ is called a $(\Gamma \vdash \tau)$-expression preorder iff it is a preorder (reflexive and transitive) over the equivalence classes of $[\Gamma \vdash \tau]$.

Definition 2 (Quantification Type): A Quantification type is a tuple $Q = (\Gamma, Q, \leq, +, 0)$ of: a typing context $\Gamma$ and a type $Q$ such that $\Gamma \vdash Q : \ast$; a $(\Gamma \vdash Q)$-expression preorder $\leq$; and two expressions $+$ (sometimes written in infix notation) and $0$ of types $Q \rightarrow Q \rightarrow Q$ and $Q$ (relative to $\Gamma$), respectively, such that $0$ is a least element for $\leq$, $+$ is monotone with respect to $\leq$ and $(\| \Gamma \vdash Q \|, +, 0)$ forms a monoid with respect to the equivalence relation $\equiv = (\leq \cap \geq)$.

Stated differently, the following must hold:
- **Closedness:** $e_1, e_2 \in [\Gamma \vdash Q]$ implies $e_1 + e_2 \in [\Gamma \vdash Q]$.
- **Monotonicity:** $e_1 \leq e_1'$ and $e_2 \leq e_2'$ implies $e_1 + e_2 \leq e_1' + e_2'$.
- **Leastness of 0:** $0 \leq e$ for all $e \in [\Gamma \vdash Q]$.
- **Identity:** $e + 0 \leq e$ for all $e \in [\Gamma \vdash Q]$.
- **Commutativity:** $e_1 + e_2 \leq e_2 + e_1$ for $e_1, e_2 \in [\Gamma \vdash Q]$.
- **Associativity:** $e_1 + (e_2 + e_3) \leq (e_1 + e_2) + e_3$ for $e_1 \in [\Gamma \vdash Q]$.

Such a tuple is sometimes called a $Q$- or $(\Gamma \vdash Q)$-Quantification type. If $\Gamma = \cdot$ then $Q$ is said to be ground.

In the below, we often omit $\Gamma$ when it is clear from context, writing $(Q, \leq, +, 0)$ etc. We also write $Q, \leq, +, 0$ for the corresponding elements of $Q$. We sometimes omit the $Q$ subscript where $Q$ can be inferred from context. The following lemma gives a final, implied constraint, that $\bot$ is always infinity:

Lemma 2: For any quantification type $Q$, $e \leq \bot$ for all $e \in [\Gamma \vdash Q]$.

Example 1 (Non-Negative Reals): $Q_R = \langle R^{\geq 0}, \leq_R, +_R, 0_R \rangle$ is a ground quantification type that corresponds to the standard ordering and addition on the non-negative reals.

Example 2 (Non-Negative Real Functions): For any ground $\tau$, $\langle \tau \rightarrow R^{\geq 0}, \leq_{\tau \rightarrow R^{\geq 0}}, \lambda x_1. \lambda x_2. \lambda y. x_1 y + R x_2 y, \lambda y. 0_R \rangle$ is a ground quantification type, where addition is performed pointwise on functions and $e_1 \leq_{\tau \rightarrow R^{\geq 0}} e_2$ iff $e_1 e \leq_R e_2$ for all ground $e$ of type $\tau$; i.e., if the output of $e_1$ is always no greater than that of $e_2$.

Definition 3 (Approximation Equality): Let $\Gamma^0$ and $\Gamma^e$ be any contexts with disjoint domains, let $Q$ be a $(\Gamma^0, \Gamma^e \vdash Q)$-quantification type, and let $E$ be any type such that $\Gamma^e \vdash E : \ast$. A $(Q, \Gamma^e, E)$-approximate equality relation is a ternary relation $e_1 \approx^q e_2 \subseteq [\Gamma^0, \Gamma^e \vdash Q] \times [\Gamma^e \vdash E] \times [\Gamma^e \vdash E]$, where $q \in [\Gamma^0, \Gamma^e \vdash Q]$ and each $e_i \in [\Gamma^e \vdash E]$, that satisfies the following:

**Upward Closedness:** $e_1 \approx^q e_2$ and $q \leq q'$ implies $e_1 \approx^{q'} e_2$.

**Reflexivity:** $e_1 \approx e_2$ implies $e_1 \approx^{0, e_2}$.

**Symmetry:** $e_1 \approx^{q, e_2}$ implies $e_2 \approx^{q, e_1}$.

**Triangle Inequality:** $e_1 \approx^{q_1} e_2$ and $e_2 \approx^{q_2} e_3$ implies $e_1 \approx^{q_1 + q_2, e_3}$.

**Completeness:** $e_1 \approx^q e_2$ for all $e_1, e_2 \in [\Gamma \vdash E]$.

An approximation equality relation is ground iff $\Gamma^e = \Gamma^0 = \cdot$. In the below, we often omit $\Gamma^0$ and $\Gamma^e$ when clear from context. We use a subscript to denote which approximate equality relation is intended, as in $e_1 \approx^q_{\ast=\cdot} e_2$, when it is not clear from context.

Example 3 (Reals): The relation $e_1 \approx^q_R e_2$ that holds iff $q \leq \bot$ or $e_1 \cong e_2$, or $e_1 - e_2 \leq_R q$ is a ground $(R^{\geq 0}, R)$-approximate equality relation that corresponds to the standard distance over the reals.

Example 4 (Real Functions): The relation $e_1 \approx^q_{\ast=\cdot} e_2$ that holds iff $q \leq \bot$, or $e_1 \cong e_2$, or $|e_1 - e_2| \leq_R (q + 0^R)$ for all $e \in R$ and $r^R \in R^{\geq 0}$, is a ground $(R \rightarrow R^{\geq 0} \rightarrow R^{\geq 0}, R \rightarrow R)$-approximate equality relation, where, intuitively, the distance between two functions $e_1$ and $e_2$ is given by a function $q$ that bounds the distance between their outputs for each input.

Definition 4 (Approximation Type): Let $\Gamma^0$, $\Gamma^e$, and $\Gamma^a$ be three domain-disjoint contexts. A $(\Gamma^0, \Gamma^e, \Gamma^a)$-approximation type is a tuple $\langle Q, E, A, \sim, \{, \rangle \rangle$ of: a $(\Gamma^0, \Gamma^e \vdash Q)$-quantification type $Q$; types $E$ and $A$, called respectively the exact and approximate types, such that $\Gamma^e \vdash E : \ast$ and $\Gamma^a \vdash A : \ast$; a $(Q, \Gamma^0, E)$-approximate equality relation $\sim$; and a mapping $\langle [a] \rangle^q : ([\Gamma^0, \Gamma^e \vdash Q] \times ([\Gamma^a \vdash A]) \rightarrow [P([\Gamma^e \vdash E])]$, where $q \in [\Gamma^0, \Gamma^e \vdash Q]$ and $a \in [\Gamma^a \vdash A]$ that satisfies:

**Error Weakening:** $q_1 \leq q_2$ implies $[a]^{q_1} \subseteq [a]^{q_2}$.

**Error Addition:** $e_1 \in [a]^{q_1}$ and $e_2 \equiv e'$ implies $e_2 \in [a]^{q_1 + q'}$.

**Equivalence:** If $q \equiv^q q'$, $a \equiv^a a'$, and $e \equiv^e e'$ then $e \in [a]^{q_1}$ implies $e' \in [a']^{q_1}$.

**Approximate Equality:** $e_1, e_2 \in [a]^{q_1}$ implies $e_1 \equiv^{q_1 + q_2} e_2$.

A ground approximation type is one where $\Gamma^e = \Gamma^0 = \Gamma^a = \cdot$. In the below, we write $A$ for approximation types, writing $\langle [a] \rangle_{\sim, A}$ for the set to which $A$ maps $q$ and $a$, $\sim_{\cdot, A}$ for the approximate equality relation, $E_A$ and $A_A$ for the exact and approximate types, $Q_A$ for the quantification type, and $Q_{A_A} \leq_{\cdot, A}$, and $+_{\cdot, A}$ for the elements of $Q_A$. Again, we often omit the subscript $A$ when it can be inferred from context.

Example 5 (Floating-Point Numbers): Let
$e \in \{0\}^q_{\mathcal{F}_1}$ iff $q \cong \perp$, or $e \equiv \perp$ and $\alpha \equiv \perp$, or $|e - \text{real}(a)| \leq R q$ where \text{real} maps (the value of) $a$ to its corresponding real number. We then have that $\mathcal{F}_1 = (Q_R, R, \mathcal{F}_1, \approx_R, \{\cdot\}^q_{\mathcal{F}_1})$ is a ground approximation type where a real can be approximated by a floating-point number with an error given by the real-number distance between the two.

**Example 6 (Floating-Point Functions):** Let $e \in \{0\}^q_{\mathcal{F}_1 \Rightarrow \mathcal{F}_1}$ iff $q \cong \perp$, or $e \equiv \perp$ and $\alpha \equiv \perp$, or $e \in \{0\}^q_{\mathcal{F}_1}$ for all $e \in \{0\}^q_{\mathcal{F}_1}$. We then have that $\mathcal{F}_1 \Rightarrow \mathcal{F}_1 = (\mathcal{R} \Rightarrow \mathcal{R}^+ \Rightarrow Q_R, \mathcal{R} \Rightarrow \mathcal{R}, \mathcal{F}_1 \Rightarrow \mathcal{F}_1, \approx_{\mathcal{F}_1 \Rightarrow \mathcal{F}_1}, \{\cdot\}^q_{\mathcal{F}_1 = \mathcal{F}_1})$ is a ground approximation type, where $\mathcal{R} \Rightarrow \mathcal{R}^+ \Rightarrow Q_R$ is the quantification type obtained from $Q_R$ by applying the construction of Example 2 twice. Intuitively, this approximation type allows a real function $f$ to be approximated by a floating-point function $f'$ with an error $q$ whenever $Q_R$ bounds the error between calling $f$ on exact real number $r$ and calling $f'$ on a floating-point number with at most distance $q_r$ from $r$.

### 4.2. Approximation Families

**Definition 5 (Approximation Families):** Let $\Gamma = \Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$ for four domain-disjoint typing contexts and $\phi$ be an expression such that $\Gamma \vdash \phi : \text{Prop}$. We say that $\mathcal{F} = (E, A, Q, 0, +, F)$ is a $(\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4, \phi)$-approximation familyiff $\Gamma^0 \vdash E : *$, $\Gamma^0 \vdash A : *$, $\Gamma^0, \Gamma^1 \vdash Q : *$, $\Gamma^0, \Gamma^1 \vdash 0 : Q$, $\Gamma^0, \Gamma^1 \vdash 0 : Q \Rightarrow Q \Rightarrow Q$, and $F$ is a meta-language function from substitutions $\sigma$ such that $\Gamma \vdash (\Gamma; \phi)$ to ground approximation types $\mathcal{A}$ such that $E_A = \sigma(E), A_A = \sigma(A), Q_A = \sigma(Q), 0_A = \sigma(0)$, and $+_A = \sigma(+)$. We use a subscript $\mathcal{F}$ to denote the elements of $\mathcal{F}$, e.g., $E_\mathcal{F}$ denotes the exact type $E$ of $\mathcal{F}$. We write $F_\mathcal{F}(\sigma)$ for the approximation resulting from applying the $F$ component to $\sigma$. $\Lambda\langle\cdot,\cdot,\cdot,\cdot,\cdot\rangle$-approximation family is called ground. Note that the ground $\mathcal{F}$ are isomorphic to the ground approximation types $\mathcal{A}$, since, if $\mathcal{F}$ is ground, then the domain of $F_\mathcal{F}$ consists of the sole pair $(\cdot; \cdot)$.

We define approximation contexts $\Xi$ with grammar:

- $\Xi, (x^e, x^a, x^\phi) : \mathcal{F} \mid \Xi, (X^e, X^a, X^\phi) : \mathcal{F} \mid \Xi, (\Gamma, \phi)$

The form $(x^e, x^a, x^\phi) : \mathcal{F}$ introduces variables $x^e$, $x^a$, and $x^\phi$ such that $x^e$ is an approximation of $x^a$ with error $x^\phi$ in some returned approximation by approximation family $\mathcal{F}$. The form $(X^e, X^a, X^\phi) : \mathcal{F}$ introduces type variables $X^e$, $X^a$, and $X^\phi$, along with a variable $\xi$ that quantifies over approximations of $X^e$ by $X^a$ with error $X^\phi$. Finally, the form $\Gamma, (\phi)$ introduces additional variables in $\Gamma$ and constraint $\phi$.

More formally, let $|\Xi|^e, |\Xi|^a, |\Xi|^q, |\Xi|^A$ be typing contexts that contain, respectively: all $x^e$ and $X^e$; all $x^a$ and $X^a$; all $x^\phi$ and $X^\phi$, and contexts $\Gamma$ in a $\Gamma, (\phi)$ form; and all $\xi$ variables in $\Xi$. Further, let $|\Xi|^p$ be the conjunction of the following formulas: $x^e \in \{x^e\}^a_\Xi^q$ for each $(x^e, x^a, x^\phi) : \mathcal{F}$; the formula is $\text{approx} \subseteq (X^e, X^a, X^\phi)$ stating that $\xi$ is an approximation of $X^e$ by $X^a$ with error $X^\phi$ for each $(X^e, X^a, X^\phi) : \Xi$; and $\phi$ for each $\Gamma, (\phi)$.

We use the abbreviations $|\Xi|^eq = |\Xi|^e, |\Xi|^q$ and $|\Xi|^eq\phi = |\Xi|^e, |\Xi|^a, |\Xi|^q, |\Xi|^A$. The approximation context $\Xi$ is well-formed, written $\Xi :: \text{Prop}$, iff $\Xi \vdash |\Xi|^eq\phi$ and $|\Xi|^eq\phi \vdash : \text{Prop}$. We say $\mathcal{F}$ is a $\Xi$-approximation family, written $\Xi \vdash \mathcal{F}$, iff $\mathcal{F}$ is a $(|\Xi|^eq, |\Xi|^a, |\Xi|^q, |\Xi|^A; |\Xi|^p)$-approximation family. A $\sigma$ is a substitution from $\Xi$ to $\Xi'$, written $\Xi \vdash \sigma : \Xi'$, iff $\Xi \vdash (|\Xi|^eq\sigma; |\Xi'|)^p \vdash \sigma : (|\Xi|^eq\sigma; |\Xi'|)^p$.

Although the functions $F$ in approximation families return only ground approximations, we can create non-ground approximation types from $\mathcal{F}$ as follows. Let $\Xi$ be any approximation context and $\mathcal{F}$ be any $\Xi$-approximation family. The notation $\Xi \vdash \mathcal{F}$ then denotes $\langle (|\Xi|^eq, |\Xi|^a, |\Xi|^q), \text{approx} \rangle \vdash Q, E, A, \approx_\mathcal{F} \vdash \mathcal{F}$ where $Q = |\Xi|^eq_\mathcal{F}, Q_\mathcal{F} \Rightarrow 0, +_\mathcal{F} \Rightarrow Q$. We write $\text{approx}$ for $\mathcal{F}$.

**Theorem 1:** If $\Xi \vdash \mathcal{F}$ and $\Xi \vdash F$ then $\Xi \vdash \mathcal{F}$ is a valid approximation type.

**Lemma 3 (Approximation Weakening):** Let $\Xi \vdash \mathcal{F}$ and $\Xi \vdash F \vdash \mathcal{F}$. Then we have that: $\Xi \vdash \mathcal{F}; q_1 \leq_{\Xi, \Xi, \Xi, \Xi, \Xi} q_2$ implies $q_1 \leq_{\Xi, \Xi, \Xi, \Xi, \Xi} q_2$; $e_1 \approx_{\Xi, \Xi, \Xi, \Xi, \Xi} e_2$ implies $e_1 \approx_{\Xi, \Xi, \Xi, \Xi, \Xi} e_2$; $e \in \{0\}^q_{\Xi, \Xi, \Xi, \Xi, \Xi}$ implies $e \in \{0\}^q_{\Xi, \Xi, \Xi, \Xi, \Xi}$.

We define substitution $\sigma(F)$ into approximation families as yielding the approximation family $\langle \sigma(E), \sigma(A), \sigma(Q), 0, +, \lambda \sigma(F) \rangle$.

**Lemma 4 (Approximation Substitution):** If $\Xi \vdash \mathcal{F}$ and $\Xi \vdash \sigma : \mathcal{F}$ then: $\Xi \vdash \sigma(F); q_1 \leq_{\Xi, \Xi, \Xi, \Xi, \Xi} q_2$ implies $q_1 \leq_{\Xi, \Xi, \Xi, \Xi, \Xi} q_2$; $e_1 \approx_{\Xi, \Xi, \Xi, \Xi, \Xi} e_2$ implies $e_1 \approx_{\Xi, \Xi, \Xi, \Xi, \Xi} e_2$; and $e \in \{0\}^q_{\Xi, \Xi, \Xi, \Xi, \Xi}$ implies $e \in \{0\}^q_{\Xi, \Xi, \Xi, \Xi, \Xi}$.

We define $p_{\sigma(F)}$ and $p_{\mathcal{F}}$ such that $p_{\mathcal{F}}(\sigma)$ is defined such that: $q_1 \leq_{\mathcal{F}(\sigma)} q_2$ iff $q_1 \leq_{\mathcal{F}(\sigma)} q_2$.

**Definition 6 (II-Approximations):** If $\mathcal{F}$ is a $\Xi, \Xi$-approximation family, then $\Pi(\mathcal{F})$ is the $\Xi$-approximation family $\langle |\Xi|^eq \Rightarrow E_\mathcal{F}, |\Xi|^a \Rightarrow A_\mathcal{F}, |\Xi|^q \Rightarrow Q_\mathcal{F}, \lambda |\Xi|^eq, 0_{\mathcal{F}}, \lambda |\Xi|^eq, +_{\mathcal{F}}, F' \rangle$ where $F'(\sigma)$ for $\sigma \vdash (|\Xi|^eq\phi; |\Xi'|)$ is defined such that: $q_1 \leq_{F'(\sigma)} q_2$ iff $q_1 \leq_{F'(\sigma)} q_2$. $|\Xi|^eq, e_1 \approx_{F'(\sigma)} e_2$ iff
To approximate the polymorphic type $\forall X.E$ we use

$$\Pi((X^e, X^a, X^q) : \xi, z^0 : X^q, z^+ : X^q \rightarrow X^q, (z^0 \equiv 0_\xi \land z^+ \equiv +\xi)).F$$

This approximation family quantifies over the type variables $X^e, X^a,$ and $X^q$ for the exact, approximate, and error types, as well as over the variables $z^0$ and $z^+$ for the zero error and error addition of the approximation type $\xi.$ The latter variables are explicitly abstracted in order to allow error expressions to refer to them: recall that $\xi$ is only bound in $[\xi^p], \not [\xi^a]$; i.e., error terms refer only to $X^e$ and $X^q$, not to $\xi$. Abusing notation slightly, we abbreviate the above approximation family as $\Pi(X, z : \xi).F$.

5. Verifying an Approximating Compiler

As discussed in the Introduction, the long-term goal of this work is to enable language-based approximations, where a compiler or other tool performs approximate transformations in a correct and automated manner. In this section, we show how to verify such a tool, the goal of the current work, with the semantics given in the previous section. Specifically, we consider a tool that performs two transformations: it compiles real numbers into floating-point implementations; and it optionally performs loop perforation [12], [19]. For the current work, we assume only that our tool can be specified with an approximate compilation judgment $\Xi \vdash e \rightsquigarrow a \leq q : A$ such that this judgment can be derived whenever the tool might approximate exact expression $e$ by $a$ with error bound by $q$ using approximation type $A$ and assumptions $\Xi$. Intuitively, each rule of this judgment corresponds to an approximate transformation that the tool might perform; for example, the approximation rule of Section 2 for approximating sin by $\lambda x. x$, might be included. We ignore the specifics of how the tool chooses which approximations to use where, as long as all possible choices are contained in the approximate compilation judgment.

To verify such a tool, we then prove soundness of its approximate compilation judgment. Soundness here means that $\Xi \vdash e \rightsquigarrow a \leq q : A$ implies $e \in \{a\}_{F(\xi)}^\gamma_{E_X}$.

This can be proved in a local, modular fashion, by verifying each approximation rule individually; more specifically, if an approximation rule derives $\Xi \vdash e \rightsquigarrow a \leq q : A$ from assumptions $\Xi \vdash e_i \rightsquigarrow a_i \leq q_i : A_i$ for $1 \leq i \leq n$ and side conditions $\phi_i$ for $1 \leq j \leq m$, then the rule is correct iff $e \in \{a\}_{\Xi^\gamma_{E_X}}$ holds whenever $e_i \in \{a_i\}_{\Xi^\gamma_{E_X}}$, for all $i$ and $j$. This also allows extensibility, since additional rules can always be added as long as they are proved correct. In the remainder of this section, we consider rules that would be used in our example tool, including: compositionality rules (Section 5.1), rules for replacing real numbers by floating-point implementations (Section 5.2); and a rule for performing loop perforation (Section 5.3).

5.1. Compositionality Rules

In order to combine errors from individual approximate transforms into a single, whole-program error, we now introduce the compositionality rules. These rules, given in Figure 2 are essentially the identity, stating that each expression construct can be approximated by itself; however, they show how to build up and combine error expressions for different constructs.

The first rule, A-Weak, allows the error bound to be weakened from $q$ to any greater error $q'$. The A-Var rule approximates variable $x^e$ by $x^a$ with error $x^q$ when these variables are associated in $\Xi$. The rule A-Lam approximates a lambda-abstraction $\lambda x^e.e$ with a lambda-abstraction $\lambda x^a.a$ by approximating the body $e$ by $a$, using the error function $\lambda x^e.\lambda x^q.a$ that abstracts over the input $x^e$ and its approximation error $x^q$. The A-App rule approximates applications $e_1.e_2$ by applying the approximation of $e_1$ to that of $e_2$ and applying the error for $e_1$ to both $e_2$ and its error. The A-Tlam rule approximates polymorphic lambdas $\Lambda X^e.e$ by approximating the body $e$ in the extended approximation context $\Xi, X, z : \xi$, recalling the abbreviation $X, z : \xi$ from Section 4.2 that abstracts the various components of approximation families. A-Tapp approximates type applications $e[E_X]$ where the type involved is the exact type of some approximation family $F$. This is accomplished by first approximating $e$ to some $a$ with error $q$ in the polymorphic approximation $\Pi((X, z : \xi).F)$ introduced in Definition 3 and then applying $a$ to the approximate type $A_X$ of $F$. The error $q$ is applied to the necessary components of $F$, and the resulting approximation is $F'$ with $F$ substituted for $\xi$ and all the appropriate
components of $F$ substituted for the $X$ and $Z$ variables. This is abbreviated as $[F/\xi, \ldots, F/\xi']$.

Fixed-points $\text{fix}(e)$ are approximated using $\text{A-Fix}$. This approximates $e$ to $a$ with error $q$, applying $\text{fix}$ to the results in the conclusion. The approximation family used for $e$ is $\mathcal{F} \Rightarrow \mathcal{F}$, augmented with the assumption that the inputs $x^e$, $x^a$, and $x^q$ are equal to the exact, approximate, and error fix-expressions in the conclusion of the rule.

Finally, if-expressions are approximated with $\text{A-If}$. First, each component of the if-expression is approximated; the condition can use an arbitrary approximation family $\mathcal{F}'$, while the then and else branches must use the same family as the whole expression. The final condition requires that the output error $q$ bounds the error between the branches taken in the exact and approximate expressions, even if one takes the then branch and the other takes the else branch. This is stated by quantifying over all combinations of True or False in the exact and approximate conditions that meet the error $q'$ computed for approximating the if-condition, and then requiring that the corresponding then or else branches are within error $q$.

Lemma 5: Each rule of Figure 2 is sound.

5.2. Floating-Point Approximation

To approximate real-number programs with their floating-point equivalents, we use the $\mathcal{F}_l$ approximation type formalized in Example 5. We then add rules

$$\Xi \vdash e \leadsto a \leq q : F \quad \Xi \vdash e \leadsto a \leq q' : F$$

$$\Xi \vdash e \leadsto a \leq q : F'$$

$$\Xi \vdash \lambda x^e. e \leadsto \lambda x^a. a \leq \lambda x^q. q : \Pi((x^e, x^a, x^q) : \mathcal{F}, \mathcal{F}')$$

$$\Xi \vdash e \leadsto a \leq q : \Pi((x^e, x^a, x^q) : \mathcal{F}), \Xi \vdash e \leadsto a \leq q : F$$

$$\Xi \vdash e \leadsto a \leq q : \Xi, x : F \vdash e \leadsto a \leq q : F'$$

for each real number $r$ and each built-in binary operation (such as $+$, $\ast$, etc.) $\text{op}$, where $\text{op}^R$ and $\text{op}^\text{Float}$ are the version of $\text{op}$ for reals and floating-points, respectively, and $\text{op}^q$ is the error function calculating the size of the interval error in the output from those of the inputs. The error functions $\text{op}^q$ to use for the various operations can be derived in a straightforward manner by considering the smallest error that bounds the difference between the real result of $\text{op}^r$ and any potential result of $\text{op}^\text{Float}$ on floating-point numbers in the input interval. The infinite error $\perp$ is returned in case of overflow or Not-A-Number results. For instance, the error $+^q$ can defined (in pseudocode) as:

$$+^q x^e x^a y^e y^q =$$

let $I = \text{round}(([x^e - x^a] + [y^e - y^q])$,

$$(x^e + x^a) + (y^e + y^q))$)$ \text{in}$

if $\exists r \in I, |r| \geq \text{MAXFLOAT}$ then $\perp$ else $\max_{r', r} |r' - r|$ 

where $\text{round}$ rounds all reals in an interval to floating-point numbers (using the current rounding mode) and $\text{MAXFLOAT}$ is the maximum absolute value of the floating-point representation being used. The errors for other operations can be defined similarly.

Using these rules with the compositionality rules of Figure 2 yields an interval-based floating-point error analysis that works for higher-order and even polymorphic terms. Although recent work has given more precise floating-point error analyses than intervals [10], [5], we anticipate, as future work, that such approaches can also be incorporated into our framework, allowing them to be used on higher-order, polymorphic programs.

Theorem 2: The rules listed above for compiling reals to floating-points are sound.
repeating the values that are computed

mate transformation that takes a loop which combines
5.3. Loop Perforation
operating on reductions
formalize a simplified version of loop perforation as
function $f$ values of
the function that reduces, using
by first approximating each
$q$

numbers are used for the exact, approximate, and error
types. Next, the rule finds an error function

the relationship between expressions of different types,
such as real-valued functions and functions on floating-
point numbers used to implement them.

Approximate bisimulation [9] is a new technique for
relating discrete, continuous and hybrid systems in a
manner that can tolerate errors in a system. We are
still investigating the relationships between our system
and approximate bisimulation, but one key difference
with our work is that this approach considers only
transition systems, and so cannot be applied to higher-
order programs, programs with recursive data, etc.; however, such systems still encompass the large and practical class of control systems.

Chaudhuri et al. [4], [3] have investigated a static
analysis for proving robustness of programs, which
they then argue is a useful precondition for approx-
imate transformations. They define robustness as the
$K$-Lipschitz condition, which, again, can be captured
in our system as the error $\lambda x^e, \lambda x^q, K * x^q$ in
an approximation of a function.

Loop perforation [12], [19] transforms certain map-
reduce programs, written as for-loops in C, to perform
only a subset of their iterations. Section 5.3
shows how to capture a variant of this transformation in
our system. More recent work [21] has extended
loop perforation to sampling, which takes a random
(instead of a controlled) sample of the iterations of a
loop. This work also adds substitution transformations,
where different implementations of the same basic
operations (such as sin or log) are substituted to try
to trade off accuracy for performance.

Emerj [18] allows the programmer to specify, with
type modifiers, whether data is exact or inexact. Inexact
data means the program can tolerate errors in this data.
Such data is then stored in low-power memory, which
is cheaper but is susceptible to random bit flips.

Carbin et al. [2] present a programming model with
relaxation, where certain values in a program are
allowed to vary in a pre-defined way, to model errors.
The authors then show how to verify properties of
relaxed programs despite these errors. Although this
work addresses some of the same questions as ours,
one significant drawback is that it cannot represent
the relationship between expressions of different types,
such as real-valued functions and functions on floating-
point numbers used to implement them.

A number of recent papers relate to program ap-
proximations. Possibly the closest to this work is the
work of Reed and Pierce [16], because they consider a
higher-order (though not polymorphic) input language.
They show how to perform an approximate program
transformation that adds noise to a database query to
ensure differential privacy, i.e., that a query cannot
violate the privacy of a single individual recorded in
the database. In order to ensure that adding noise does
not change the query results too much, a type system
is used to capture functions that are $K$-Lipschitz,
meaning that a change of $\delta$ in the input yields a change
of at most $K * \delta$ in the output. This condition can be
captured in our system by the error $\lambda x^e, \lambda x^q, K * x^q$ in
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type modifiers, whether data is exact or inexact. Inexact
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is cheaper but is susceptible to random bit flips.
cally equivalent real-number program. We showed how to accomplish this in our setting in Section 5.2, yielding what we believe is the first floating-point error analysis to work for higher-order and polymorphic programs. Although state-of-the-art analyses use affine expressions, instead of intervals as we used above, we anticipate that we will be able to accommodate these approaches in our semantics as well, thereby adapting them to higher-order and polymorphic programs.

On a technical note, our quantification types and approximate equality relations were inspired by Flagg and Kopperman’s continuity spaces [8], one of the few works that considers a more general notion of distance than metric spaces. Specifically, the distributive lattice completion of a quantification type corresponds exactly to Flagg and Kopperman’s abstract notion of distance, called a quantale. A similar transformation turns an approximate equality relation into a continuity space, their generalization of metric spaces.

7. Conclusion

We have introduced a semantics for approximate program transformations. The semantics relates an exact program to an approximation of it, and quantifies this relationship with an error expression. Rather than specifying errors solely with real numbers and metric spaces, our approach is based on approximation types, an extension of logical relations that allows us to use, for example, functions as the errors for approximations of functions, and polymorphic types as the errors for polymorphic types. We then show how approximation types can be used to verify approximate transforms in a modular, composable fashion, by proving soundness of each transform individually and by including a set of compositionality rules, also proved correct, that combine errors from individual approximate transforms into a whole-program error.

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