On the number of eigenvalues of a model operator related to a system of three particles on lattices

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Abstract

We consider a quantum mechanical system on a lattice $\mathbb{Z}^3$ in which three particles, two of them being identical, interact through a zero-range potential. We admit a very general form for the ’kinetic’ part $H_0^\gamma$ of the Hamiltonian, which contains a parameter $\gamma$ to distinguish the two identical particles from the third one (in the continuum case this parameter would be the inverse of the mass). We prove that there is a value $\gamma^*$ of the parameter such that only for $\gamma < \gamma^*$ the Efimov effect (infinite number of bound states if the two-body interactions have a resonance) is absent for the sector of the Hilbert space which contains functions which are antisymmetric with respect to the two identical particles, while it is present for all values of $\gamma$ on the symmetric sector. We comment briefly on the relation of this result with previous investigations on the Thomas effect. We also establish the following asymptotics for the number $N(z)$ of eigenvalues $z$ below $E_{\text{min}}$, the lower limit of the essential spectrum of $H_0$. In the symmetric subspace
\[ \lim_{z \to E_{\text{min}}} \frac{N^s(z)}{\log |E_{\text{min}} - z|} = \mathcal{U}^s_0(\gamma), \quad \forall \gamma, \]
whereas in the antisymmetric subspace
\[ \lim_{z \to E_{\text{min}}} \frac{N^{as}(z)}{\log |E_{\text{min}} - z|} = \mathcal{U}^{as}_0(\gamma), \quad \forall \gamma > \gamma^*, \]
where $\mathcal{U}^s_0(\gamma), \mathcal{U}^{as}_0(\gamma)$ are written explicitly as a function of the integral kernel of operators acting on $L^2((0, r) \times (L^2(S^2) \otimes L^2(S^2)))$ ($S^2$ is the unit sphere in $\mathbb{R}^3$).

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Introduction

The spectral theory of continuous and lattice three-particle Schrödinger operators in $\mathbb{R}^3$ shows the remarkable phenomenon known as the ‘Efimov effect’. If all Hamiltonians of all the two-body subsystems are positive and if at least two of them have a zero-energy resonance, then the three-body system has an infinite number of negative eigenvalues accumulating at zero.

This remarkable spectral property was discovered by Efimov [14] and has since become the subject of many papers [3, 4, 10, 13, 19, 30, 36–38, 42]. The first mathematical proof of the existence of the Efimov effect was given in [42] and Sobolev established [36] the asymptotics of the number of eigenvalues near the threshold of the essential spectrum of the three-body problem. See also [38].

Recently, Wang [41] has proved the existence of the Efimov effect in the system with $N \geq 4$ particles in $\mathbb{R}^3$ but in this case the properties of the spectrum have not been fully comprehended yet.

A similar effect (infinitely many eigenvalues below a threshold) is known in nuclear physics, and recent years it is also called Efimov effect [15, 21, 40] suggesting a relation which is not proved so far.

Nuclear forces are of short range, and a mathematical model for them consists in taking them to be of zero range [16]. In this limit, not only there are infinitely many bound states for a three-body system, but the spectrum is unbounded below with eigenfunctions which tend to have support in a region whose volume tends to zero (‘collapse to a point’ effect). This effect was discovered by Thomas [39] and is therefore called the ‘Thomas effect’ (fall to the center effect). The Thomas effect consists therefore in the fact that for a three-particle system, when all particles interact pairwise through ‘zero-range interactions’, the spectrum is unbounded below, the positive real axis is the (absolutely) continuous spectrum and there are infinitely many negative eigenvalues (bound states) accumulating at $-\infty$.

The first theoretical analysis of the Thomas effect was given in [35]. Danilov [12] obtained the condition providing self-adjointness of the underlying three-body Hamiltonian with zero-range pair interactions. The first rigorous proof was given by Faddeev and Minlos [20] who also gave a precise meaning to the ‘zero-range interaction’ through the theory of self-adjoint extensions.

Later, the presence of this effect was also proved with another technique [13] in which the construction of $N$-body zero-range interactions is made using quadratic (energy) forms.

In nuclear physics, the Thomas effect (often called the Efimov effect) is described as the appearance of infinitely many bound states in the three-body system with a large two-body scattering length, with an accumulation point at zero energy in the limit of infinite scattering length (unitary limit) [11, 21]. In this limit, the ratio of two successive eigenvalues tends to a constant.

This effect has recently received significant attention in nuclear physics as a consequence of major advances in experimental techniques.

There are striking similarities between the Efimov and Thomas effects, apart from the occurrence of infinitely many bound states.

In the case of the Efimov effect on the lattice and with a special form of the ‘kinetic energy’, it was proved in [1] by Albeverio et al that if two of the particles are identical fermions and the ‘kinetic energy’ of the other particle is sufficiently large, the Efimov effect is not present.

Similarly, in the continuum case of nuclear theory, if two of the particles are identical fermions (so that there is no zero-range interaction between them), the Thomas effect is present.
only if the mass of the third particle is sufficiently small (the mass of the identical fermions is set to be equal to 1); this was proved by Minlos [28] with later improvements by [27] and Shermatov [34].

This striking similarity between these two effects has led in the physics literature to call both ‘Efimov effect’ and in the mathematical literature to a suggestion (see e.g. [2] and [24]) that there is some relation between the two formalisms, in spite of the fact that one (Efimov) is a ‘long-range effect’ (infrared) and the other (Thomas) is a ‘collapse to a point’ (ultraviolet) effect (the ‘collapse’ is due to the fact that the eigenfunctions have a singularity in the region where the coordinates of the three particles tend to coincide).

There are other similarities: in order to have the Efimov effect there must be a zero-energy resonance in at least two of the three channels, i.e. a distinguished element which does not belong to the Hilbert space of the square integrable functions of the channel.

Similarly in the Thomas effect, in at least two of the channels, there is a distinguished element in the domain of the Hamiltonian operator; this element is the Green function of the Laplacian which is locally square integrable but does not belong to the Sobolev space $H^2_{|\text{loc}}$ as the remaining part of the domain.

Remark that in the case of the Thomas effect the space $H_{\text{Thomas}}$ is related to a fractional Sobolev space of functions supported on the coincidence hyperplanes (coincidence of the positions of particle pairs). In the case of the Efimov effect, the space $H_{\text{Efimov}}$ is $L^2(\mathbb{R}^9)$.

It is difficult at the present stage to say whether or not the similarity of the two effects can be given a precise mathematical explanation.

Recall that the point interaction Hamiltonian $H$ is defined as a self-adjoint extension of the symmetric operator $-\Delta$ acting on functions that are in the Sobolev space $H_2$ and vanish in a neighborhood of the origin in $\mathbb{R}^3$. Its domain are the function $\phi(x)$ which can be written as the sum of a term $c|x|^{-1}$ and a term $\psi(x)$ which is in the Sobolev space $H_2$, supplemented by a relation between $c$ and $\psi(0)$. The unitary limit (infinite scattering length) corresponds to taking $\psi(0) = 0$.

The operator $H$ acts on its domain as $(H\phi)(x) = -\Delta \phi$.

This relation can be written formally as

$$(H\psi)(x) = -(\Delta \psi)(x) - c\delta(x)\psi(x),$$

suggesting that $H$ represents the interaction with a potential concentrated at the origin.

This formulation is mathematically inconsistent, but may suggest, from the physical point of view, that an approximation procedure, including perhaps a ‘renormalization’, may lead to a correct answer [13].

An approximation could consist in starting with a model on a lattice and then passing to the continuum while choosing the parameters in such a way that the limit exists. This requires passing to the limit both in the kinetic part of the Hamiltonian and in the part which refers to the interaction.

In models of solid state physics [26, 29] and also in lattice quantum field theory [25], discrete lattice operators are considered which are lattice analogs of the three-particle Schrödinger operator in the continuum.

From the formal analysis we have given above, it is reasonable to take the interaction potential as concentrated in a point of the lattice, with strength that increases without bound when one takes the continuum limit. We defer for the moment the problem of the choice of the potential, with the proviso that in order to achieve the correct limit one should make sure that the resonance of the lattice model goes over to the Green function of the model in the continuum.
As for the kinetic energy, note that while in the continuum model each particle has kinetic energy \( \frac{1}{2m} p^2 \), the lattice kinetic energy of a particle is a bounded function defined on a torus of side \( 2\pi \frac{\alpha}{L} \), where \( L \) is the lattice size (for simplicity we take the lattice to be cubical) and \( \alpha \) is a parameter (with dimension those of an action) which fixes the units of measure.

In order to achieve the desired limit, one must therefore require that the particle kinetic energy, that we denote by \( e_0(p) \), is approximated for a small value of \( |p| \) by \( e_0(p) \approx \frac{1}{2m} p^2 \) for some value of \( m \).

The common choice for the lattice Laplacian \( e_0(p) = 1 - \cos \frac{|p|}{\sqrt{m}} \) satisfies this requirement for small values of \( p \).

For the purpose of studying the continuum limit of the Lattice model for the Efimov effect we can choose \( e_0(p) \) for larger values of \( |p| \) in the way which is more convenient for us.

It is therefore useful to be able to give theorems for the lattice case under the weakest possible assumption on \( e_0(p) \). The only assumption to be retained is that it has an absolute minimum at \( p = 0 \) and that the coefficient of the quadratic term at zero be considered as the inverse of twice the mass in statements such as if the mass is greater than \( m^* \).

This is the motivation for considering the Efimov effect under a weaker assumption on \( e_0(p) \) than that previously considered.

The presence of the Efimov effect for the three-particle discrete Schrödinger operators has been proved in [6, 22, 23].

For the system of three particles on a three-dimensional lattice, the presence of the Efimov effect was found in [26, 29], and it was rigorously proved in [22, 23] and then in [6]; in [6] moreover, an asymptotic form of the three-body discrete spectrum was found to be similar to the continuous case.

The existence the Efimov effect for model three-particle Schrödinger operators associated with a system of three particles on the three-dimensional lattice has been obtained in [9, 32].

In [9], the availability of the Efimov effect for the three-body model operator, interacting via pair non-local potentials, is shown, when all corresponding Frīdrihīs models have a resonance at the threshold.

A model operator in [32] is studied in the case of a special form of the function \( E \) that parameterizes the non-perturbed operator.

In this paper, we investigate a model operator \( H_\gamma \) associated with the three-particle discrete Schrödinger operator on the three-dimensional cubic lattice, with zero-range pair potentials, where the role of the two-particle discrete Schrödinger operators is taken by a family of Frīdrihīs models with parameters \( h_\alpha(k), \alpha = 1, 2, k \in \mathbb{T}^3 \).

Under some natural conditions on the parameter kinetic energies \( e_\alpha(\cdot), \alpha = 1, 2, 3 \), we obtain the following results.

We prove that there is a value \( \gamma^* \) of the parameter such that only for \( \gamma < \gamma^* \) the Efimov effect (infinite number of bound states if the two-body interaction have a resonance) is absent for the sector of the Hilbert space which contains functions which are antisymmetric with respect to the two identical particles, while it is present for all values of \( \gamma \) on the symmetric sector.

We comment briefly on the relation of this result with previous investigations on the Thomas effect.

We also establish the following asymptotics for the number \( N(z) \) of eigenvalues \( z \) below \( E_{\min} \), the lower limit of the essential spectrum of \( H_0 \):

In the symmetric subspace

\[
\lim_{z \to E_{\min}} \frac{N^s(z)}{\log |E_{\min} - z|} = \mathcal{U}_s(\gamma), \quad \forall \gamma,
\]
whereas in the antisymmetric subspace
\[
\lim_{z \to E_{mn}} \frac{N^a(z)}{\log |E_{mn} - z|} = U^a_0(\gamma), \quad \forall \gamma > \gamma^*,
\]
where \(U^a_0(\gamma), U^a_0(\gamma^*)\) are written explicitly as functions of the integral kernels of operators acting on \(L^2((0, r) \times (L^2(S^2) \otimes L^2(S^2)))\) (\(S^2\) is the unit sphere in \(\mathbb{R}^3\)).

We remark that in the case of an antisymmetric wavefunction, the number \(\gamma^*\) is a critical value for the mass ratio, where the Efimov effect is present or absent. Interestingly, the case of three fermions, two identical and different from the third one, was also considered from a more physical point of view by Petrov [31] and he also found a critical value for the mass ratio that allows or forbids the Efimov effect.

Throughout this paper we adopt the following conventions. Denote by \(L^2(\Omega)\) the Hilbert space of square-integrable functions defined on a measurable set \(\Omega \subseteq \mathbb{R}^d\), and by \(L^2(\Omega)^n\) the Hilbert space of \(n\)-component vector functions \(f = (f_1, f_2, \ldots, f_n)\), \(f_\alpha \in L^2(\Omega), \alpha = 1, 2, \ldots, n\). We denote by \(\text{diag}\{B_1, B_2\}\) the \(2 \times 2\) diagonal matrix with operators \(B_1, B_2\) as diagonal entries. Let \(\mathbb{Z}^d\) be the \(d\)-dimensional lattice and \(\mathbb{T}^d\) be the \(d\)-dimensional torus, the cube \((-\pi, \pi]^d\) with appropriately identified sides, \(d \geq 1\). We remark that the torus \(\mathbb{T}^d\) will always be considered as an Abelian group with respect to the addition and multiplication by the real numbers regarded as operations on \(\mathbb{R}^d\) modulo \((2\pi \mathbb{Z})^d\). Denote by \(L^2_d((\mathbb{T}^d)^2)\) respectively \(L^2_d((\mathbb{T}^d)^3)\) the subspace of antisymmetric respectively symmetric functions of the Hilbert space \(L^2((\mathbb{T}^d)^2)\).

For each \(\delta > 0\) the notation \(U_\delta(0) = \{K \in \mathbb{T}^3 : |K| < \delta\}\) stands for a \(\delta\)-neighborhood of the origin and \(U^0_\delta(0) = U_\delta(0) \setminus \{0\}\) for a punctured \(\delta\)-neighborhood.

The plan of this work is as follows. In section 1, we give some results for the Friedrichs model. In section 2, we introduce the investigating operator \(H_\gamma\) and its restrictions \(H^{a}\) and \(H^{\prime}\) to the subspaces \(L^2_d((\mathbb{T}^d)^2)\) and \(L^2_d((\mathbb{T}^d)^3)\), respectively. Introducing the channel operators in section 3 we describe the essential spectra \(H^{a}\) and \(H^{\prime}\) by the spectrum of the Friedrichs models. In section 4, we formulate the main results of this paper. After formulating the main theorem in section 5 the corresponding Birman–Schwinger principles for the operators \(H^{a}\) and \(H^{\prime}\) are reviewed. Section 6 is devoted to proving the main result.

Some technical materials are collected in appendices A and B.

In the proof we follow closely the Sobolev method to derive the asymptotics for the number of eigenvalues of \(H_\gamma\) (theorems 4.2 and 4.3).

1. Threshold analysis of the family of Friedrichs models \(h(k), k \in \mathbb{T}^3\)

In the study of the spectral properties of the two-particle discrete Schrödinger Hamiltonians the problem may be reduced to the study of the family of self-adjoint Friedrichs models with local interaction operators.

In this section, we study some spectral properties of the families of Friedrichs models \(h(k), k \in \mathbb{T}^3\), acting in \(L^2(\mathbb{T}^3)\) and given by
\[
h(k) = h^0(k) - \mu v,
\]
where \(h^0(k)\) is the multiplication operator by the function \(\mathcal{E}_x(\cdot)\):
\[
(h^0(k)f)(q) = \mathcal{E}_x(q)f(q), \quad f \in L^2(\mathbb{T}^3),
\]
\[
\mathcal{E}_x(q) = a_1\mathcal{E}(q) + a_2\mathcal{E}(k - q),
\]
and \(v\) is a zero-range interaction operator
\[
(vf)(q) = \int_{\mathbb{T}^3} f(t) \, dt, \quad f \in L^2(\mathbb{T}^3).
\]
Here, $\epsilon(\cdot)$ is a real-valued continuous function on $\mathbb{T}^3$ and $a_1, a_2, \mu$ are positive real numbers.

**Remark 1.1.** The spectrum and resonances of the Friedrichs model are studied in [8, 17, 18, 43]. The operators $h(k), k \in \mathbb{T}^3$, are the two-particle Schrödinger operators on the lattice $\mathbb{Z}^3$ which describe a two-particle system interacting through zero-range attractive potentials and $E_k(\cdot)$ is the two-particle dispersion relation of normal modes associated with a two-particle ‘free’ discrete Hamiltonian.

The perturbation $v$ of the operator $h_0(k), k \in \mathbb{T}^3$, is a Hilbert–Schmidt operator and, therefore, in accordance with Weyl’s theorem the essential spectrum of the operator $h(k)$ coincides with the spectrum of $h_0(k)$:

$$\sigma_{\text{ess}}(h(k)) = [E_{\text{min}}(k), E_{\text{max}}(k)],$$

where

$$E_{\text{min}}(k) = \min_{q \in \mathbb{T}^3} E_k(q), \quad E_{\text{max}}(k) = \max_{q \in \mathbb{T}^3} E_k(q). \quad (1.2)$$

**Assumption 1.2.** The function $\epsilon(\cdot)$ is a conditionally negative definite function and three times differentiable function on $\mathbb{T}^3$ with a unique non-degenerate minimum at the origin.

Recall (see, e.g., [33]) that a complex-valued bounded function $\epsilon : \mathbb{T}^m \to \mathbb{C}$ is called conditionally negative definite if $\epsilon(p) = \overline{\epsilon(-p)}$ and

$$\sum_{i,j=1}^n \epsilon(p_i - p_j)z_i \overline{z_j} \leq 0$$

for all $p_1, p_2, \ldots, p_n \in \mathbb{T}^m$ and all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ satisfying $\sum_{i=1}^n z_i = 0$.

**Remark 1.3.** The proof of the Efimov effect on a lattice for a generic choice of the dispersion relation for the free Hamiltonian of the particles may give a clue for the problem of finding a mathematical relation between the Efimov and Thomas effects.

Indeed, taking the limit when the lattice spacing goes to zero, one can consider two cases.

One can consider a sequence of potentials, satisfying the hypothesis that leads to the Efimov effect, that are regular in the limit and use the standard lattice approximation of the Laplacian to prove that the Efimov effect for the three-particle system is present also in the limit.

Or one can consider a sequence of potentials which are supported by a given point of the lattice and whose strength diverges in the limit, and try to prove that for an appropriate choice of the dispersion relations the resolvent of the two-body potential problem converges (may be weakly) to the resolvent of the two-body point interaction, and that in this limit the Efimov for the three-particle system turns into the Thomas effect.

**Definition 1.4.** Let assumption 1.2 be fulfilled. The operator $h(0)$ is said to have a resonance at the threshold of its essential spectrum if $1$ is an eigenvalue of the compact operator acting in $C(\mathbb{T}^3)$ and given by

$$(G\psi)(q) = \mu \int_{\mathbb{T}^3} \frac{\psi(t) \, dt}{\tilde{E}_0(t) - \tilde{\epsilon}_{\text{min}}(0)}, \quad \psi \in C(\mathbb{T}^3),$$

and the associated eigenfunction $\psi$ (up to a constant factor) satisfies the condition $\psi(0) \neq 0$. 

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Let the operator \( h(0) \) has a resonance at the threshold of its essential spectrum. Then, the function
\[
\phi(q) = \frac{1}{E_0(t) - E_{\min}(0)}
\]
is a solution (up to a constant factor) of the Schrödinger equation \( h(0) f = E_{\min}(0) f \) and \( \phi \) belongs to \( L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3) \).

**Remark 1.5.** This requirement of the existence of a zero-energy resonance (an element that does not belong to the domain of the Schrödinger Hamiltonian) in order to have the Efimov effect is to be compared with the formulation of the Thomas effect in the continuum case. In that case, the role of ‘resonance’ is taken by the element one is forced to add to the domain of the symmetric operator \(-\Delta\), defined initially on smooth functions with support not containing the origin, to extend it to a self-adjoint operator.

Note that in the case of the lattice the resonance belongs to \( L^1(\mathbb{T}^3) \) but not to \( L^2(\mathbb{T}^3) \) which is the natural domain of the (bounded) operator \( e(p) \).

Set
\[
\mu^0 = \left( \int_{\mathbb{T}^3} \frac{dt}{E_0(t) - E_{\min}(0)} \right)^{-1}.
\]

Let \( \mathbb{C} \) be the field of complex numbers. For any \( k \in \mathbb{T}^3 \), we define an analytic function \( \Delta(k, \cdot) \) (the Fredholm determinant associated with the operator \( h(k) \)) on \( \mathbb{C} \setminus [E_{\min}(k), E_{\max}(k)] \) by
\[
\Delta(k, z) = 1 - \mu \int_{\mathbb{T}^3} \frac{dq}{E_k(q) - z}.
\]

Note that the function \( \Delta(k, z) \) is real analytic in \( \mathbb{T}^3 \times (\mathbb{C} \setminus [E_{\min}(k), E_{\max}(k)] \).

Set
\[
\Delta(0, E_{\min}(0)) = 1 - \mu \int_{\mathbb{T}^3} \frac{dq}{E_0(q) - E_{\min}(0)}.
\]

The following lemma is a simple consequence of the Birman–Schwinger principle and the Fredholm theorem and it describes whether the bottom of the essential spectrum of \( h(0) \) is a resonance.

**Lemma 1.6.** Under assumption 1.2.

(i) For any \( \mu > 0 \) and \( k \in \mathbb{T}^3 \), the operator \( h(k) \) has an eigenvalue \( z \in \mathbb{C} \setminus [E_{\min}(k), E_{\max}(k)] \) if and only if \( \Delta(k, z) = 0 \).

(ii) The operator \( h(0) \) has a a resonance at the threshold of its essential spectrum if and only if
\[
\mu = \mu^0 \text{ (or } \Delta(0, E_{\min}(0)) = 0),
\]
where \( \mu^0 \) is defined by (1.3).

As a consequence we have

**Lemma 1.7.** (i) For any \( \mu > 0 \) and \( k \in \mathbb{T}^3 \), the operator \( h(k) \) has no eigenvalue lying above the essential spectrum.

(ii) The operator \( h(k) \) has a (unique) eigenvalue lying below the essential spectrum if and only if for some \( z_0 \leq E_{\min}(k) \) the inequality \( \Delta(k, z_0) < 0 \) holds.
Proof.

(i) The proof of this assertion follows from the inequality \( \Delta(k, z) > 0, \ z > \mathcal{E}_{\text{max}}(k) \) and part (i) of lemma 1.6.

(ii) For each \( \mu > 0 \) and \( k \in \mathbb{T}^3 \), the function \( \Delta(k, \cdot) \) is strictly monotone decreasing on \( (-\infty, \mathcal{E}_{\text{min}}(k)) \) and \( \Delta(k, z) \to 1 \) as \( z \to -\infty \). Since \( \Delta(k, z_0) < 0 \), there exists a unique number \( z \in (-\infty, z_0) \) such that \( \Delta(k, z) = 0 \), that is by lemma 1.6 the number \( z \) is a (unique) eigenvalue of the operator \( h(k) \).

Let \( z \) be a (unique) eigenvalue of the operator \( h(k) \) that is \( \Delta(k, z) = 0 \). Therefore, from (i) the eigenvalue \( z \) belongs in \( (-\infty, \mathcal{E}_{\text{min}}(k)) \).

Since \( \Delta(k, \cdot) \) is strictly monotone decreasing on \( (-\infty, \mathcal{E}_{\text{min}}(k)) \) for any number \( z_0, z < z_0 \leq \mathcal{E}_{\text{min}}(k) \), the inequality \( \Delta(k, z_0) < 0 \) holds.

We need a simple inequality [5] which will play a crucial role in the proof of theorem 1.10.

For the reader's convenience we recall some results concerning the two-particle dispersion relations of normal modes associated with two-particle free discrete Hamiltonians.

Lemma 1.8. Assume that the function (dispersion relation) \( \varepsilon(\cdot) \) is a real-valued continuous conditionally negative definite function on \( \mathbb{T}^3 \). Assume, in addition, that \( \varepsilon(0) \) is the unique minimum of the function \( \varepsilon(p) \). Then, for all \( q \in \mathbb{T}^3 \setminus \{0\} \) the inequality

\[
\varepsilon(p) + \varepsilon(q) > \frac{\varepsilon(p + q) + \varepsilon(p - q)}{2} + \varepsilon(0), \quad \text{a.e.} \quad p \in \mathbb{T}^3.
\]

holds.

Lemma 1.9. Let the assumptions of lemma 1.8 be fulfilled. Then, for any (fixed) \( k, q \in \mathbb{T}^3 \) such that either \( k \neq q \) or \( q \neq 0 \)

\[
\mathcal{E}_0(p) - \mathcal{E}_0(0) + \mathcal{E}_0(q) - \frac{\mathcal{E}_0(p + q) + \mathcal{E}_0(q - p)}{2} > 0, \quad \text{a.e.} \quad p \in \mathbb{T}^3.
\]

In particular, if \( k \neq 0 \), and \( p(k) \) is a point where the function \( \mathcal{E}_k(\cdot) \) attains its minimal value, that is,

\[
\mathcal{E}_{\text{min}}(k) = \mathcal{E}_k(p(k)),
\]

the following inequality holds:

\[
\mathcal{E}_0(p) - \mathcal{E}_{\text{min}}(0) + \mathcal{E}_{\text{min}}(k) - \frac{\mathcal{E}_0(p + p(k)) + \mathcal{E}_0(p(k) - p)}{2} > 0, \quad \text{a.e.} \quad p \in \mathbb{T}^3.
\]

The following theorem gives the existence of the eigenvalues of \( h(k) \), \( k \in \mathbb{T}^3 \), below the essential spectrum.

Theorem 1.10. Let the positive operator \( h(0) \) have a resonance at the threshold of its essential spectrum. Then, for all \( k \in \mathbb{T}^3 \), \( k \neq 0 \), the operator \( h(k) \) has a unique eigenvalue \( z(k) \) below the bottom \( \mathcal{E}_{\text{min}}(k) \) of its essential spectrum. Moreover, \( z(k) \) as a function of \( k \), \( k \in \mathbb{T}^3 \setminus \{0\} \), is even and \( z(k) > \mathcal{E}_{\text{min}}(0) \) for \( k \neq 0 \).

Proof. Since \( \Delta(0, \mathcal{E}_{\text{min}}(0)) = 0 \) (see lemma 1.6) and a strict monotonicity of \( \Delta(k, \cdot) \) in \( (-\infty, \mathcal{E}_{\text{min}}(k)) \), the operator \( h(0) \) is positive.

Let \( p(k) \) be a (any) point in \( \mathbb{T}^3 \) where the function \( \mathcal{E}_k(\cdot) \) attains its minimal value, that is,

\[
\mathcal{E}_{\text{min}}(k) = \mathcal{E}_k(p(k)).
\]
Since
\[ E_k(p) = E_{-k}(-p) \quad \text{and} \quad E_{\min}(k) = E_{\min}(-k), \]
the point \(-\bar{p}(k) \in T^3\) is again a minimum point of \(E_{-k}(\cdot)\).

Therefore, without loss of generality, we can assume that
\[ p(-k) = -p(k), \quad k \in T^3. \quad (1.4) \]

For any \(k \in T^3\) we set
\[ \hat{E}(k, p) = E_k(p + p(k)) - E_{\min}(k), \quad k \in G. \quad (1.5) \]
It is easily seen that for any fixed \(k \in T^3\) respectively \(p \in T^3\) the function \(\hat{E}(k, p)\) as a function of the argument \(p \in T^3\) respectively \(k \in T^3\) is even.

Since for any \(k \in T^3\) the function \(\Delta(k, \cdot)\) is monotone in \((-\infty, E_{\min}(k))\), the following (finite or infinite) limit
\[ \lim_{z \to E_{\min}(k)} \Delta(k, z) := \Delta(k, E_{\min}(k)) \]
extists.

Let us denote by \(G\) the set of \(k \in T^3 \setminus \{0\}\) which \(|\Delta(k, E_{\min}(k))| < \infty\).

In accordance with equality \((1.4)\), if \(k \in G\), then \(-k \in G\).

On account of the definition of \(G\) for any \(k \in G\) by virtue of the Levi theorem we obtain
\[ \Delta(k, E_{\min}(k)) = 1 - \mu^0 \int_{T^3} \frac{dq}{E_x(k, q) - E_{\min}(k)}. \quad (1.6) \]

Making the change of variable \(q := q + p(k)\) on the right-hand side of \((1.6)\) and using equality \((1.3)\) and notation \((1.5)\) one obtains the representation
\[ \Delta(k, E_{\min}(k)) = \mu^0 \int_{T^3} \frac{G(k, q)}{\hat{E}(k, q) E(0, q)} dq + \frac{\mu^0}{2} \int_{T^3} \frac{\hat{E}(k, q) - \hat{E}(-k, q)}{\hat{E}(k, q) E(0, q)} dq, \quad (1.7) \]
where
\[ G(k, q) = \frac{E(k, q) + \hat{E}(-k, q)}{2} - E_0(q) + E_{\min}(0) \]
or
\[ G(k, q) = \frac{E_k(p + p(k)) + E_k(p(k) - p)}{2} + E_{\min}(0) - E_0(p) - E_{\min}(k). \]

Since the function \(\Delta(k, \cdot)\) and \(E_{\min}(k)\) are even in \(G\), the following holds:
\[ \Delta(k, \cdot) = \frac{\Delta(k, \cdot) + \Delta(-k, \cdot)}{2} \]
and henceforth from \((1.7)\) we receive
\[ \Delta(k, E_{\min}(k)) = \mu^0 \int_{T^3} \frac{G(k, q)}{\hat{E}(k, q) E(0, q)} dq - \frac{\mu^0}{4} \int_{T^3} \frac{(\hat{E}(k, q) - \hat{E}(-k, q))^2}{\hat{E}(k, q) E(-k, q) E(0, q)} dq. \quad (1.8) \]

By lemma \(1.8\) the inequality \(G(k, q) < 0\) holds, a.e. \(q \in T^3\) for any fixed \(k \in G\), \(k \neq 0\).
Then, from \((1.8)\) follows
\[ \Delta(k, E_{\min}(k)) < \Delta(0, E_{\min}(0)) = 0, \quad k \in G, \quad k \neq 0. \]

If \(k \in T^3 \setminus G\), then by the definition of the set \(G\) we have
\[ \lim_{z \to E_{\min}(k)} \Delta(k, z) = -\infty < \Delta(0, 0) = 0, \quad k \in T^3 \setminus G, \quad k \neq 0. \]

Therefore by lemma \(1.7\) for \(k \in T^3, k \neq 0\), there is a unique eigenvalue \(z(k)\) of \(h(k)\) below the bottom of the essential spectrum.
Now we will prove that the number \( z(k), k \neq 0 \) lies on the right of \( \varepsilon_{\min}(0) \).

By equality (1.3) the function has the form
\[
\Delta(k, 0) = \mu \int_{\mathbb{R}^3} \frac{a_2(\varepsilon(k - p) - \varepsilon(p))}{(E_0(p) - \varepsilon_{\min}(0))(E_1(p) - \varepsilon_{\min}(0))} \, dp.
\]
Making the change of variable \( q = \frac{1}{2} - p \) on the right-hand side of (1.9) and using the equality \( \Delta(k, 0) = \Delta(-k, 0) \), we obtain
\[
\Delta(k, 0) = \mu^0 \int_{\mathbb{R}^3} a_1 a_2 \left( \varepsilon \left( \frac{k}{2} + p \right) - \varepsilon \left( \frac{k}{2} - p \right) \right)^2 F(k, p) \, dp,
\]
where
\[
F(k, p) = \frac{E_0(\frac{k}{2} + p) + E_0(\frac{k}{2} - p) - 2E_{\min}(0)}{(E_0(\frac{k}{2} + p) - \varepsilon_{\min}(0))(E_0(\frac{k}{2} - p) - \varepsilon_{\min}(0))(E_1(\frac{k}{2} + p) - \varepsilon_{\min}(0))(E_1(\frac{k}{2} - p) - \varepsilon_{\min}(0))}.
\]
Since for any \( k \in \mathbb{T}^3 \) the inequality \( F(k, p) > 0 \) a.e. \( p \in \mathbb{T}^3 \), we have
\[
\Delta(k, \varepsilon_{\min}(0)) > 0, \quad k \in 0 \neq \mathbb{T}^3.
\]
So the function \( \Delta(k, \cdot) \) is strictly monotonous in \((-\infty, \varepsilon_{\min}(k))\), the eigenvalue \( z(k) \) of \( h(k) \) belongs to \((\varepsilon_{\min}(0), \varepsilon_{\min}(k))\).

Since \( \Delta(\cdot, z) \) is even as a function of \( k, k \in \mathbb{T}^3 \), so is \( z(k) \) in \( \mathbb{T}^3 \setminus \{0\} \).

The proof is complete. \( \Box \)

A result corresponding to theorem 1.10 has been proven [5] by a variational method.

Let \( W \) be a \( 3 \times 3 \)-matrix of the second-order partial derivatives of the function \( \varepsilon(\cdot, \cdot) \) at the point \( p = 0 \).

The following lemma 1.11 plays a crucial role in the proof of the infiniteness (respectively finiteness) of the number of eigenvalues lying below the bottom of the essential spectrum for a model operator \( H \). The proof may be handled in much the same way as in [6].

**Lemma 1.11.** Assume that hypothesis 1.2 is fulfilled and let the operator \( h(0) \) have a resonance at the threshold.

(i) Then, for all \( k \in U_3(0) \) and \( z \leq \varepsilon_{\min}(k) \) the following expansion holds:
\[
\Delta(k, z) = \frac{4\sqrt{2} \pi^2 \mu^0}{(a_1 + a_2)^2 \det(W)^2} \sqrt{\varepsilon_{\min}(k) - z} + \Delta^{\text{res}}(\varepsilon_{\min}(k) - z) + \Delta^{\text{res}}(k, z),
\]
where \( \Delta(\varepsilon_{\min}(k) - z) = O((\varepsilon_{\min}(k) - z)^{\frac{3}{4}}) \) as \( \varepsilon_{\min}(k) - z \to 0, z < \varepsilon_{\min}(k) \), and \( \Delta^{\text{res}}(k, z) = O(k^2) \) as \( k \to 0 \) uniformly in \( z \leq \varepsilon_{\min}(k) \);
(ii) for some \( c_1, c_2 > 0 \) the following inequalities hold:
\[
c_1 |k| \leq \Delta(k, \varepsilon_{\min}(0)) \leq c_2 |k|, \quad k \in U_3(0),
\]
\[
\Delta(k, \varepsilon_{\min}(0)) \geq c, \quad k \in \mathbb{T}^3 \setminus U_3(0).
\]

**Remark 1.12.** Lemma 1.11 gives threshold energy expansions for the Fredholm determinant, leading to behaviors of the threshold energy resonance.
2. Three-particle model operator

Let us consider the bounded and self-adjoint operator \( H_\gamma \) acting in the Hilbert space \( L^2(\mathbb{T}^3)^2 \) by

\[
H_\gamma = H_\gamma^0 - V, \quad V = V_1 + V_2 + V_3,
\]

where \( H_\gamma^0 \) is the multiplication operator by the function \( E(p, q) \):

\[
(H_\gamma^0 f)(p, q) = E(p, q) f(p, q), \quad f \in L^2(\mathbb{T}^3)^2,
\]

and \( V_\alpha, \alpha = 1, 2, 3, \) are the ‘zero-range’ interaction operators

\[
(V_\alpha f)(p, q) = \mu_\alpha E(p + q) + \varepsilon(p) + \varepsilon(q), \quad p, q \in \mathbb{T}^3,
\]

where \( \varepsilon(\cdot) \) is satisfied by assumption 1.2 on \( \mathbb{T}^3 \) and \( \gamma, \mu_\alpha, \alpha = 1, 2, \) are positive real numbers.

The subspaces

\[
L^2(\mathbb{T}^3)^2, \quad \text{as} (\text{subspaces} \ L^2(\mathbb{T}^3)^2) \text{are not changed, while}
\]

\[
V_\alpha \text{ to these subspaces are not changed, while}
\]

\[
E(p, q) f)(p, q), f \in L^2(\mathbb{T}^3)^2.
\]

Here \( \varepsilon(\cdot) \) is satisfied by assumption 1.2 on \( \mathbb{T}^3 \) and \( \gamma, \mu_\alpha, \alpha = 1, 2, \) are positive real numbers.

To study the spectral properties of the operator \( H_\gamma \), we introduce the following two families of bounded self-adjoint operators (the Friedrichs model) \( h_\alpha(k), k \in \mathbb{T}^3, \alpha = 1, 2, \) acting in \( L^2(\mathbb{T}^3) \) by

\[
h_\alpha(k) = h_\alpha^0(k) - \mu_\alpha v,
\]

where

\[
l_1 = 1, \quad l_2 = \gamma.
\]

For the model operators we use the notations introduced in section 2 but endow them with the subscript \( \alpha \). For example, \( E_{\min}^{(\alpha)}(k) \) means the minimal value (1.2) for the multiplication function \( E_k^{(\alpha)}(\cdot) \).

Under assumption 1.2 the following relations hold:

\[
E_{\min}^{(\alpha)}(0) < E_{\min}^{(\alpha)}(k), \quad k \neq 0 \in \mathbb{T}^3, \quad \text{and} \quad E_{\min} = E_{\min}^{(\alpha)}(0) + l_\alpha \varepsilon(0).
\]

It is easy to check that the subspaces \( L^2_{\alpha}(\mathbb{T}^3)^2 \) respectively \( L^2_{\gamma}(\mathbb{T}^3)^2 \) invariant by the action of \( H_\gamma \) and restrictions of the operators \( V_2, V_3 \) to these subspaces are not changed, while the restriction \( V_1 |_{L^2_{\alpha}(\mathbb{T}^3)^2} \) of the operator \( V_1 \) on \( L^2_{\alpha}(\mathbb{T}^3)^2 \) is equal to zero operator, that is

\[
V_1 |_{L^2_{\alpha}(\mathbb{T}^3)^2} = 0.
\]

Denote by \( H_\gamma^{\alpha} \) respectively \( H_\gamma^{\gamma} \) the restriction of the model operator \( H_\gamma \) to \( L^2_{\alpha}(\mathbb{T}^3)^2 \) respectively \( L^2_{\gamma}(\mathbb{T}^3)^2 \). These restrictions have the form

\[
H_\gamma^{\alpha} = H_\gamma^0 - V^{\alpha}, \quad V^{\alpha} = V_2 + V_3,
\]

and

\[
H_\gamma^{\gamma} = H_\gamma^0 - V^{\gamma}, \quad V^{\gamma} = V_1 + V_2 + V_3.
\]
3. The essential spectrum of the operator $H_γ$

3.1. Channel operators

In this section, to give the representation for the essential spectra of $H_γ$, $H_γ^s$ and $H_γ^{ss}$ we introduce the channel operators $H_α$ acting on the Hilbert space $L^2((T^3)^2)$ as

$$H_α = H_0^γ - V_α, \quad α = 1, 2.$$  

The decomposition of the space $L^2((T^3)^2)$ into the direct integral

$$L^2((T^3)^2) = \int_{k ∈ T^3} \oplus ⊕ L^2(T^3) \, dk$$

yields for the operator $H_α$ the decomposition into the direct integral

$$H_α = \int_{k ∈ T^3} \oplus (h_α(k) + m_αε(k)I) \, dp,$$

where $h_α(k)$ is defined by (2.4) and $m_α$ being the number such that $m_1 = γ$ and $m_2 = 1$.

For any $K$, $k ∈ T^3$ we set

$$E_{min}^{(α)}(k) = E_{min}^{(α)}(k) + m_αε(k), \quad E_{max}^{(α)}(k) = E_{max}^{(α)}(k) + m_αε(k),$$

where $E_{min}^{(α)}(k)$ and $E_{max}^{(α)}(k)$ are defined in (1.2).

**Lemma 3.1.** For any $k ∈ T^3$ the number $z ∈ \mathbb{C} \setminus [E_{min}(k), E_{max}(k)]$ is an eigenvalue of the operator $h_α(k) + m_αε(k)I$ if and only if

$$\Delta_α(k, z - m_αε(k)) = 0, \quad α = 1, 2.$$ 

The proof of lemma 3.1 is similar to that of lemma 1.6.

Set

$$E_{min} = \min_{p,q ∈ T^3} E(p,q), \quad E_{max} = \max_{p,q ∈ T^3} E(p,q).$$

**Lemma 3.2.** For any $k ∈ T^3$ the equality

$$σ(H_α) = σ_{two}(H_α) ∪ [E_{min}, E_{max}]$$

holds, where

$$σ_{two}(H_α) = \bigcup_{k ∈ T^3} [σ_d(h_α(k)) + m_αε(k)], \quad α = 1, 2.$$ 

**Proof.** The theorem on the spectrum of decomposable operators (see, e.g., [33]) gives the proof of the theorem.

Set

$$\mu_{min}^{(α)} = \min_{k ∈ T^3} \left( \int_{T^3} \frac{dq}{\xi_k^{(α)}(q) + m_αε(k) - E_{min}} \right)^{-1},$$

$$\mu_{max}^{(α)} = \max_{k ∈ T^3} \left( \int_{T^3} \frac{dq}{\xi_k^{(α)}(q) + m_αε(k) - E_{min}} \right)^{-1}.$$ 

**Remark 3.3.** The relations are valid $μ_{min}^{(α)} = μ_0^γ$ and from theorem 1.10 we have

$$σ_{two}(H_α) = \text{Im}\{z_α(·) + m_αε(·)\},$$
where $z\alpha(0) = 0$ and $z\alpha(k), k \neq 0 \in \mathbb{T}^3$, is the eigenvalue of the operator $h\alpha(k), k \neq 0 \in \mathbb{T}^3$, (see theorem 1.10).

Set
\[ m_{\alpha} = \inf \sigma_{\text{two}}(H_{\alpha}), \quad M_{\alpha} = \sup \sigma_{\text{two}}(H_{\alpha}), \quad \alpha = 1, 2. \]

The following lemma describes the location of the essential spectrum of the operator $H_{\alpha}$.

**Lemma 3.4.**

(i) Let $0 < \mu_{\alpha} \leq \mu_{\text{min}}^{(\alpha)}$; then, the spectrum $\sigma(H_{\alpha})$ of $H_{\alpha}$ coincides with the three-particle branch
\[ \sigma(H_{\alpha}) = [E_{\min}, E_{\max}]. \]

(ii) Let $\mu_{\text{min}}^{(\alpha)} < \mu_{\alpha} \leq \mu_{\text{max}}^{(\alpha)}$; then, the spectrum $\sigma(H_{\alpha})$ consists of the interval
\[ \sigma(H_{\alpha}) = [m_{\alpha}, E_{\max}] \quad \text{and} \quad m_{\alpha} < E_{\min}. \]

(iii) Let $\mu_{\text{max}}^{(\alpha)} < \mu_{\alpha}$; then, $\sigma(H_{\alpha})$ consists of union of the two (closed) intervals
\[ \sigma(H_{\alpha}) = [m_{\alpha}, M_{\alpha}] \cup [E_{\min}, E_{\max}] \quad \text{and} \quad M_{\alpha} < E_{\min}. \]

**Proof.** Comparing lemmas 1.7, 3.1 and 3.2 one can obtain the proof. \qed

### 3.2. Essential spectrum of the operators $H_{\gamma}, H_{\hat{\gamma}}$ and $H_{\gamma}^{\text{ess}}$

In this subsection, we exhibit the location of the essential spectrum of the model operator $H_{\gamma}$. The following theorem may be proved in much the same way as in [7].

**Theorem 3.5.** For the essential spectrum $\sigma_{\text{ess}}(H_{\gamma}^{\text{opt}})$ of $H_{\gamma}^{\text{opt}}$ respectively $\sigma_{\text{ess}}(H_{\gamma})$ of $H_{\gamma}$ the equality
\[ \sigma_{\text{ess}}(H_{\gamma}^{\text{opt}}) = \sigma_{\text{two}}(H_{\gamma}^{(2)}) \cup [E_{\min}, E_{\max}] \]
respectively
\[ \sigma_{\text{ess}}(H_{\gamma}) = \sigma_{\text{two}}(H_{\gamma}^{(1)}) \cup \sigma_{\text{two}}(H_{\gamma}^{(2)}) \cup [E_{\min}, E_{\max}], \]
holds.

Since the domain and ranges of $H_{\gamma}^{(1)}$ and $H_{\gamma}^{\text{opt}}$ are orthogonal spaces, we have
\[ \sigma_{\text{ess}}(H_{\gamma}) = \sigma_{\text{ess}}(H_{\gamma}^{(1)}) \cup \sigma_{\text{ess}}(H_{\gamma}^{\text{opt}}). \]

The following lemma describes the location of the essential spectrum of the operator $H_{\gamma}$ and it is a simple consequence of lemmas 3.2 and 3.4.

**Lemma 3.6.**

(i) Let $0 < \mu_{1} \leq \mu_{\text{min}}^{(1)}$ and $0 < \mu_{2} \leq \mu_{\text{min}}^{(2)}$; then, the essential spectrum $\sigma_{\text{ess}}(H_{\gamma})$ of $H_{\gamma}$ coincides with the three-particle branch of the essential spectrum
\[ \sigma_{\text{ess}}(H_{\gamma}) = [E_{\min}, E_{\max}]. \]

(ii) Let $\mu_{\text{min}}^{(1)} < \mu_{1} \leq \mu_{\text{max}}^{(1)}$ and $0 < \mu_{2} \leq \mu_{\text{min}}^{(2)}$; then, the essential spectrum $\sigma_{\text{ess}}(H_{\gamma})$ of $H_{\gamma}$ consists of the interval
\[ \sigma_{\text{ess}}(H_{\gamma}) = [m_{1}, E_{\max}] \quad \text{and} \quad m_{1} < E_{\min}. \]

(iii) Let $\mu_{\text{max}}^{(1)} < \mu_{1}$ and $0 < \mu_{2} \leq \mu_{\text{min}}^{(2)}$; then, the essential spectrum $\sigma_{\text{ess}}(H_{\gamma})$ of $H_{\gamma}$ consists of union of the two (closed) intervals
\[ \sigma_{\text{ess}}(H_{\gamma}) = [m_{1}, M_{1}] \cup [E_{\min}, E_{\max}] \quad \text{and} \quad M_{1} < E_{\min}. \]
4. Statement of the main results

Let \( \tau_{\text{ess}}(H_\gamma), \tau_{\text{ess}}(H_\gamma^{au}) \) and \( \tau_{\text{ess}}(H'_\gamma) \) be the bottom of the essential spectrum of the operators \( H_\gamma, H_\gamma^{au} \) and \( H'_\gamma \), respectively.

By virtue of theorem 3.5 the following relations hold:

\[
\tau_{\text{ess}}(H_\gamma^{au}) \leq \tau_{\text{ess}}(H_\gamma) \quad \text{and} \quad \tau_{\text{ess}}(H'_\gamma) = \tau_{\text{ess}}(H_\gamma).
\]

Let \( N(z), N^{au}(z) \) and \( N^s(z) \) be the number of eigenvalues (and counted according their multiplicities) of \( H_\gamma, H_\gamma^{au} \) and \( H'_\gamma \) lying below \( z \leq \tau_{\text{ess}}(H_\gamma), z \leq \tau_{\text{ess}}(H'_\gamma) \) and \( z \leq \tau_{\text{ess}}(H_\gamma^{au}) \), respectively.

Since \( H_\gamma^{au} \perp H'_\gamma \), we have

\[
N(z) = N^{au}(z) + N^s(z), \quad z \leq \tau_{\text{ess}}(H'_\gamma).
\]

Let \( \gamma^* \) be a solution of the equation

\[
\frac{2(1 + \gamma)^2}{\pi \gamma \sqrt{Y + 2\gamma}} - \frac{2(1 + \gamma)^2}{\pi \gamma^2} \arcsin \frac{\gamma}{1 + \gamma} = 1, \quad \gamma > 0.
\]

Remark 4.1. Note that this equation has a unique positive solution (see proof of (ii) of appendix A).

In the following theorems we precisely describe the dependence of the number of eigenvalues of \( H_\gamma^{au} \) on the parameters \( \gamma > 0 \).

**Theorem 4.2.** Let \( \mu_2 = \mu_1^0 \) and assumption 1.2 be fulfilled.

(i) For any \( 0 < \gamma < \gamma_0 \), the operator \( H_\gamma^{au} \) has a finite number of eigenvalues lying below the bottom \( E_{\text{min}} \) of the essential spectrum.

(ii) For any \( \gamma > \gamma^* \), the operator \( H_\gamma^{au} \) has infinitely many eigenvalues lying below the bottom of the essential spectrum. The function \( N^{au}(z) \) obeys the relation

\[
\lim_{z \to E_{\text{ess}}^-} \frac{N^{au}(z)}{\log |E_{\text{min}} - z|} = \mu_0^{au}(\gamma) \quad (\mu_0^{au}(\gamma) > 0). \tag{4.1}
\]

The following theorem is proved closely related to that in the work [6].

**Theorem 4.3.** Let \( 0 < \mu_1 \leq \mu_1^0, \mu_2 = \mu_2^0 \) and assumption 1.2 be fulfilled. Then, the operator \( H'_\gamma \) has infinitely many eigenvalues lying below the bottom of the essential spectrum and the function \( N^s(z) \) obeys the relation

\[
\lim_{z \to E_{\text{ess}}^-} \frac{N^s(z)}{\log |E_{\text{min}} - z|} = \mu_0^s (\mu_0^s > 0). \tag{4.2}
\]

Remark 4.4. The constant \( \mu_0^s \) is given as a positive function depending only on the variable \( \gamma, \gamma > 0 \).

Remark 4.5. In [6] a result analogous to theorem 4.3, has been proven for the three-particle Schrödinger operators on the lattice \( \mathbb{Z}^3 \) in the case, where the function \( E(\cdot, \cdot) \) is of the form

\[
E(p, q) = a_1 \varepsilon(p) + a_2 \varepsilon(p - q) + a_3 \varepsilon(q),
\]

where \( a_1, a_2, a_3 > 0 \) and

\[
\varepsilon(q) = 3 - \cos q_1 - \cos q_2 - \cos q_3, \quad q = (q_1, q_2, q_3) \in \mathbb{T}^3.
\]
It follows from the positivity of limits (4.1) and (4.2) that the discrete spectrum of the operators $H^0_\gamma$ and $H^s_\gamma$ are infinite.

Lemma 3.6 and theorems 4.3, 4.2 give us the main result of this paper.

**Theorem 4.6.** Let assumption 1.2 and $\mu_2 = \mu_2^0$ be fulfilled.

(i) Let $\mu_1^0 < \mu_1 \leq \mu_1^{(1)}$. For any $\gamma > \gamma^*$, the operator $H_\gamma$ has infinitely many eigenvalues of finite multiplicity in the continuous spectrum accumulating at the bottom of the three-particle branch of the essential spectrum.

(ii) Let $\mu(\gamma^*) < \mu_1$. For any $\gamma > \gamma^*$, the operator $H_\gamma$ has infinitely many eigenvalues of finite multiplicity in the gap, which accumulate at the bottom of the three-particle branch of the essential spectrum.

5. The Birman–Schwinger principle

We define the unitary operator $U$ on $L^2((\mathbb{T}^3)^2)$ to be

$$(Uf)(p, q) = f(p, q + p)$$

and its adjoint will be denoted by $U^*$. Let $\Phi_\alpha : L^2((\mathbb{T}^3)^2) \to L^2(\mathbb{T}^3)$, $\alpha = 1, 2, 3$, be the operator defined by

$$(\Phi_2 f)(q) = \int_{\mathbb{T}^3} f(t, q) \, dt,$$

$$(\Phi_3 f)(p) = \int_{\mathbb{T}^3} f(p, t) \, dt,$$

$\Phi_1 = \Phi_3 U^*$, $f \in L^2((\mathbb{T}^3)^2)$

and denote by $\Phi_\alpha^*$ its adjoint.

With these notations we have

$$V_\alpha = \mu_\alpha \Phi_\alpha^* \Phi_\alpha.$$

Set

$$D_\alpha(z) = I - \mu_\alpha \Phi_\alpha R_0(z) \Phi_\alpha^*, \quad z \in \mathbb{C} \setminus [E_{\text{min}}, E_{\text{max}}], \quad \alpha = 1, 2, 3, \quad (5.1)$$

where $R_0(z) = (H_0^0 - z I)^{-1}$ is the resolvent of $H_0^0$ and $I$ respectively $I$ is the identity operator on $L^2(\mathbb{T}^3)$ respectively $L^2((\mathbb{T}^3)^2)$.

One can check that

$$D_2(z) = D_3(z) \quad (5.2)$$

and $D_\alpha(z)$, $\alpha = 1, 2$, is the multiplication operator by the function $\Delta_\alpha(q, z - m_\alpha \varepsilon(q))$ on $L^2(\mathbb{T}^3)$.

By lemma 3.1, for any $z \in \mathbb{C} \setminus \sigma(H_\alpha)$, $\alpha = 1, 2$, the inequality $\Delta_\alpha(p, z + I_\alpha \varepsilon(q)) \neq 0$, $\alpha = 1, 2$, holds. Hence, the operator $D_\alpha(z)$, $\alpha = 1, 2$, $z \in \mathbb{C} \setminus \Sigma$, is invertible. Let $D_\alpha^{-1}(z)$, $\alpha = 1, 2$, be its inverse. In the case $z \leq \inf \sigma(H_\alpha)$, $\alpha = 1, 2$, according to lemma 1.7 the operator $D_\alpha(z)$, $\alpha = 1, 2$, is positive and we denote its square root by $D_{\alpha}^{1/2}(z)$, $\alpha = 1, 2$.

5.1. The Birman–Schwinger principle

For a bounded self-adjoint operator $B_\alpha$, we define $n(\lambda, B)$ by

$$n(\lambda, B) = \sup \{ \dim F : (Bu, u) > \lambda, u \in F, ||u|| = 1 \}.$$
\( n(\lambda, B) \) is equal to infinity if \( \lambda \) is in the essential spectrum of \( B \) and if \( n(\lambda, B) \) is finite, it is equal to the number of the eigenvalues of \( B \) larger than \( \lambda \). By the definition of \( N(z) \) we have
\[
N(z) = n(-z, -H), \quad -z > -\tau_{\text{ess}}(H).
\]

In our analysis of the spectrum of \( H \) the crucial role is played by the self-adjoint compact Faddeev–Newton-type integral operator
\[
T(z), z < \tau_{\text{ess}}(H)
\]
in \( L^2(T^3) \) with the entries
\[
T_{\alpha\alpha}(z) = 0, \quad T_{\alpha\beta}(z) = \sqrt{C_{\alpha\beta}}D_{\alpha\beta}(z), \quad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta.
\]

The following lemma follows from the well-known Birman–Schwinger principle for the operator \( H \) (see [6, 36, 38]). In the discrete case, their associated model operators was obtained in [6, 9]. We refer to these papers for the proof.

**Lemma 5.1.** The operator \( T(z) \) is compact and continuous in \( z < \tau_{\text{ess}}(H) \) and
\[
N(z) = n(1, T(z)).
\]

For the operators \( H^s \) and \( H^a \) are also defined the corresponding Faddeev–Newton-type integral operators.

We decompose the Hilbert space \( L^2(T^3) \) as
\[
L^2(T^3) = \{0\} \oplus \mathcal{H}^a \oplus (L^2(T^3) \oplus \mathcal{H}_s),
\]
where
\[
\mathcal{H}^a = \{ (f, -f) \in L^2(T^3) : f \in L^2(T^3) \}, \quad \mathcal{H}_s = \{ (f, f) \in L^2(T^3) : f \in L^2(T^3) \}.
\]

One can check that the subspaces \( \{0\} \oplus \mathcal{H}^a \) and \( L^2(T^3) \oplus \mathcal{H}_s \) are invariant with \( T(z) \). Let \( T^a(z) \) and \( T^s(z) \) be restrictions of this on these spaces, respectively.

It is easily seen that the operator \( T^a(z) \) respectively \( T^s(z) \) acting in \( L^2(T^3) \) respectively \( L^2(T^3) \) as
\[
T^a(z) = -\mu D_2^{-1}(z)\Phi_2 R_0(z)\Phi_2 D_2^{-1}(z),
\]
respectively
\[
T^s(z) = T_{22}(z), \quad \alpha, \beta = 1, 2, \quad T^s_2 = -T^a(z).
\]

The unitarity equivalence is given by the unitary operator \( P^a : \{0\} \oplus \mathcal{H}^a \rightarrow L^2(T^3) \) as
\[
P^a = \frac{1}{\sqrt{2}}(I, -I),
\]
respectively \( P^s : L^2(T^3) \oplus \mathcal{H}_s \rightarrow L^2(T^3) \) as
\[
P^s = \begin{pmatrix} I & 0 & 0 \\ \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \end{pmatrix},
\]
where \( I \) is the identity operator on \( L^2(T^3) \).

The next lemma is the analog of lemma 5.1. For completeness we give the proof.

**Lemma 5.2.** The operator \( T^a(z) \) is compact and continuous in \( z < \tau_{\text{ess}}(H^a) \) and
\[
N^a(z) = n(1, T^a(z)), \quad (N^a(z) = n(1, T^a(z))).
\]
Proof. This lemma is deduced by the same arguments as in [6, 36]. We give the proof only for the operator \( H_p^{\alpha} \).

Since for any \( z < \tau_{\text{ess}}(H_p^{\alpha}) \) the following relation holds,

\[
f \in L_{\alpha}^2((T^3)^2), \quad (H_p^{\alpha}, f, f) < z(f, f) \iff \left( R_0^\frac{1}{2}(z)(V_2 + V_3)R_0^\frac{1}{2}(z)g, g \right) > (g, g),
\]

we have

\[
N^{as}(z) = n(1, R_0^\frac{1}{2}(z)(V_2 + V_3)R_0^\frac{1}{2}(z)).
\]

We decompose \( R_0^\frac{1}{2}(z)V R_0^\frac{1}{2}(z) = B^*B \), with the vector operator \( B : L_{as}^2((T^3)^2) \rightarrow \mathcal{H}^{as} \) defined by

\[
B = \left( \sqrt{\mu_2} \Phi_2 R_0^\frac{1}{2}(z), \sqrt{\mu_3} \Phi_3 R_0^\frac{1}{2}(z) \right).
\]

One can see that the operator \( M(z) = BB^* \) acts in \( \mathcal{H}^{as} \) with the entries

\[
M_{\alpha\beta}(z) = \mu_2 \Phi_2 R_0(z) \Phi^*_\beta, \quad \alpha, \beta = 2, 3.
\]

Both operators \( B^*B \) and \( M(z) = BB^* \) have the same nonzero eigenvalues with the same multiplicities and the essential spectrum. We use this argument to obtain the equality

\[
N^{as}(z) = n(1, M(z)). \quad (5.3)
\]

We decompose \( M(z) \) into the sum \( M(z) = M_0(z) + K(z) \), where

\[
M_0(z) = \text{diag}(\mu_2 \Phi_2 R_0(z) \Phi^*_2, \mu_2 \Phi_3 R_0(z) \Phi^*_3), \quad K(z) = \begin{pmatrix} 0 & \mu \Phi_1 R_0(z) \Phi^*_2 \\ \mu \Phi_2 R_0(z) \Phi^*_1 \end{pmatrix}.
\]

Since equality (5.1) and \( \Delta_3(p, z + \epsilon(p)) > 0 \), \( p \in T^3, \ z < \tau_{\text{ess}}(H_p^{\alpha}) \), the operator \( \mathcal{I} - M_0(z), \ z < \tau_{\text{ess}}(H_p^{\alpha}) \), is invertible and we denote by \( (\mathcal{I} - M_0(z))^{-\frac{1}{2}} \) its positive root.

Direct calculations show that

\[
n(1, M(z)) = n(1, (\mathcal{I} - M_0(z))^{-\frac{1}{2}} K(z)(\mathcal{I} - M_0(z))^{-\frac{1}{2}}).
\]

Then, to finish the proof, it suffices to use the equality \( T^{as}(z) = P^{as}(\mathcal{I} - M_0(z))^{-\frac{1}{2}} K(z)(\mathcal{I} - M_0(z))^{-\frac{1}{2}} (P^{as})^* \) and (5.3).

□

5.2. The Birman–Schwinger principle at the threshold

Throughout this subsection, we assume that \( \mu_2 = \mu_3^0 \) i.e the operator \( h_2(0) \) has a a resonance at the threshold \( E_{\text{min}}^{(2)}(0) \) of its essential spectrum.

The proof of theorem 4.2 is divided into several steps.

(1) It should be noted that this operator \( T^{as}(z) \) can be defined as a bounded operator even for the point \( z = E_{\text{min}} \) by

\[
T^{as}(E_{\text{min}}) = -\mu D_2^{-\frac{1}{2}}(E_{\text{min}}) \Phi_\alpha (H_p^{\alpha})^{-1} \Phi^*_\beta D_2^{-\frac{1}{2}}(E_{\text{min}}),
\]

\[
\alpha, \beta = 2, 3, \quad \alpha \neq \beta.
\]

Remark 5.3. The operator \( T^{as}(z) \) converges strongly (but not uniformly) as \( z \to E_{\text{min}} = 0 \) to \( T^{as}(E_{\text{min}}) \). Here, we do not give the proof of this convergence. The convergence of these types of operators was shown in [23, 42].

The next lemma is an analog of lemma 5.2. For completeness we give its proof.
Lemma 5.4. For any $z, z \leq E_{\text{min}}$, the inequality
\[ N_{\text{as}}(z) \leq n(1, T^{\alpha}(E_{\text{min}})) \]
occurs.

Proof. Let the operator $M(E_{\text{min}})$ act in $\mathcal{H}_{\alpha}$ with the entries
\[ M_{\alpha \beta}(E_{\text{min}}) = \frac{\mu_2}{\Phi_1^2} \left( H_0 \gamma - E_{\text{min}} I \right)^{-1} \Phi_1^\beta, \quad \alpha, \beta = 2, 3. \]

Since the operator $M(z)$ is increasing in $(-\infty, E_{\text{min}})$ and equality (5.3), we have
\[ N_{\text{as}}(z) \leq n(1, M(E_{\text{min}})), \quad z \leq E_{\text{min}}. \quad (5.4) \]

We decompose $M(E_{\text{min}})$ as the proof of lemma 5.2 into the sum
\[ M(E_{\text{min}}) = M_0(E_{\text{min}}) + K(E_{\text{min}}), \]
where
\[ M_0(E_{\text{min}}) = \text{diag}\{ M_{22}(E_{\text{min}}), M_{33}(E_{\text{min}}) \}, \quad K(E_{\text{min}}) = \begin{pmatrix} 0 & M_{23}(E_{\text{min}}) \\ M_{32}(E_{\text{min}}) & 0 \end{pmatrix}. \]

Since $\text{Ker}(I - M_{\alpha \alpha}(E_{\text{min}})) = \{0\}, \alpha = 2, 3$, the operator $I - M_{\alpha \alpha}(E_{\text{min}}) = D_{\alpha}(E_{\text{min}})$ is invertible and a direct calculation shows that
\[ n(1, M(E_{\text{min}})) = n(1, (I - M_0(E_{\text{min}}))^{-\frac{1}{2}} K(E_{\text{min}})(I - M_0(E_{\text{min}}))^{-\frac{1}{2}}). \]

Then, to finish the proof, it suffices to coincide the equality $T^{\alpha}(E_{\text{min}}) = P^{\alpha}(I - M_0(E_{\text{min}}))^{-\frac{1}{2}} K(E_{\text{min}})(I - M_0(E_{\text{min}}))^{-\frac{1}{2}} (P^{\alpha})^*$ and (5.4). □

6. The proof of the main results

The proof of the theorems are long and are divided into several subsections.

We note that the main part of the operator $T^{\alpha}(z)$ is unitarily equivalent to a Toeplitz-type operator $S^{\alpha}$. By assumption 1.2, we obtain
\[ E_{\alpha}(p, q) = E^{(\alpha)}_p(q) + m_{\alpha} \epsilon(q) \]
\[ = E_{\text{min}} + \frac{1}{2} ((1 + l_{\alpha})(Wp, p) + 2l_{\alpha}(Wp, q) + (l_{\alpha} + m_{\alpha})(Wq, q)) \]
\[ + O(|p|^{3+\theta} + |q|^{3+\theta}), \quad (6.1) \]
as $p, q \to 0$ and
\[ \epsilon^{(\alpha)}_{\text{min}}(k) = \epsilon^{(2\gamma)}_{\text{min}}(0) + \frac{l_{\alpha}}{2} \frac{1}{1 + l_{\alpha}} (Wk, k) + O(|k|^{3+\theta}) \quad \text{as} \quad k \to 0, \]
where $W$ is the $3 \times 3$-matrix of the second-order partial derivatives of the function $\epsilon(\cdot)$ at the point $p = 0$ and $l_1 = 1, m_1 = \gamma$ and $l_2 = \gamma, m_2 = 1$.

Applying the asymptotics for $\epsilon^{(2\gamma)}_{\text{min}}(k)$ and using lemma 1.11, we have
\[ \Delta_{\alpha}(k, z - m_{\alpha} \epsilon(k)) = \frac{4\pi^2 \mu_0}{(1 + l_{\alpha})^{3/2} \det(W)^{1/2}} [n_{\alpha}(Wk, k) + 2(E_{\text{min}} - z)]^{1/2} \]
\[ + O\left(|p|^{12} + |E_{\text{min}} - z|^{1/2} \right) \quad \text{as} \quad |E_{\text{min}} - z| \to 0, \quad (6.2) \]
where
\[ n_{\alpha} = \frac{1 + 2\gamma}{1 + l_{\alpha}}. \]
6.1. The asymptotics for the number of eigenvalues of the operator $H_\delta^{as}$

In this subsection, we derive the asymptotics (4.1) for the number of eigenvalues of $H_\delta^{as}$.

We recall that in the subsection we closely follow Sobolev's method (see [36]) to derive the asymptotics for the number of eigenvalues of the operator $H_\delta^{as}$ (see theorem 4.2).

Let $T(\delta; |E_{\min} - z|)$ be the integral operator in $L^2(\mathbb{T}^3)$ with the kernel

$$T^{as}(\delta, |E_{\min} - z|; p, q) = -d_0 \frac{\chi_\delta(p) \chi_\delta(q) (\alpha(Wp, p) + 2|E_{\min} - z|^{-1})}{(1 + \gamma)(Wp, p) + 2\gamma(Wp, q) + (1 + \gamma)(Wq, q) + 2|E_{\min} - z|^{-1}}$$

where $\chi_\delta(\cdot)$ is the characteristic function of the region $\hat{U}_\delta(0) = \{ p \in \mathbb{T}^3 : |W^{1/2} p| < \delta \}$ and

$$d_0 = \frac{\det \Gamma}{2\pi^2 (1 + \gamma)^2}.$$

Lemma 6.1. Let the conditions of theorem 4.2 be fulfilled. The operator $T^{as}(z) - T^{as}(\delta; |E_{\min} - z|)$ belongs to the Hilbert–Schmidt class and is continuous in $z \leq E_{\min}$.

Proof. Applying asymptotics (6.1) and (6.2) one can estimate the kernel of the operator

$$T^{as}(z) - T^{as}(\delta; |E_{\min} - z|), \ z \leq E_{\min},$$

by the square-integrable function

$$C \left( \frac{|p|^{1 + \theta} + |q|^{1 + \theta}}{|p|^2 (|p|^2 + |q|^2)|q|^2} + \frac{|E_{\min} - z|^2 (p^2 + q^2)^{-1}}{(|p|^2 + |E_{\min} - z|)^2 (|q|^2 + |E_{\min} - z|)^2} + 1 \right).$$

Hence, the operator $T^{as}(z) - T^{as}(\delta; |E_{\min} - z|)$ belongs to the Hilbert–Schmidt class for all $z \leq E_{\min}$. In combination with the continuity of the kernel of the operator in $z < E_{\min}$, this gives the continuity of $T^{as}(z) - T^{as}(\delta; |E_{\min} - z|)$ in $z \leq E_{\min}$.

Remark 6.2. The content of lemma 6.1 is the reduction of the estimates on the number of eigenvalues below the threshold of the continuum to a corresponding estimate in the case when the kinetic energy is approximated with its quadratic part. This allows a straightforward adaptation of the techniques used in [36] and makes evident that any form of the kinetic energy of the particles can be used in the lattice case, provided that it has an absolute minimum and that it is three times differentiable at the minimum.

Moreover, it suggests that the same procedure of comparison with an operator which has a simpler kernel but under a different scaling (more appropriate to letting the lattice size and the strength of the potential diverge) may lead to study the spectrum on the three-body problem in the Thomas effect.

Note that the procedure followed by Faddeev and Minlos was also to compare the kernel of the extension (of the symmetric operator) with a simpler kernel chosen in such a way that the asymptotics of the eigenvalues is not altered by this substitution.

Let $\tilde{T}_0^{as}(\delta; |E_{\min} - z|)$ be the restriction of the integral operator $T^{as}(\delta; |E_{\min} - z|)$ to the subspace $L^2(\hat{U}(0))$. One verifies that the operator $\tilde{T}_0^{as}(\delta; |E_{\min} - z|)$ is unitarily equivalent to the integral operator $T^{as}_1(r)$ acting in $L^2(U(0))$, where $r = |E_{\min} - z|^{-1}$ and $U(0) = \{ p \in \mathbb{R}^3 : |p| < r \}$, with the kernel

$$T^{as}_1(r; p, q) = -d_1 \frac{(np^3 + 2r)^{-1/4} (nq^2 + 2r)^{-1/4}}{(1 + \gamma)p^2 + 2\gamma(p, q) + (1 + \gamma)q^2 + 2},$$

where

$$d_1 = \frac{(1 + \gamma)^{3/2}}{2\pi^2}.$$
Lemma 6.3. Let 

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The equivalence is given by the unitary dilation

$$B : L^2(\hat{U}_1(0)) \to L^2(U_1(0)), \quad (B, f)(p) = \left( \frac{r}{\delta} \right)^{-3/2} f \left( \frac{\delta}{r} W^f p \right).$$

Further, we may replace

$$(np^2 + 2)^{-1/4}, (nq^2 + 2)^{-1/4} \quad \text{and} \quad (1 + \gamma p^2 + 2 \gamma (p, q) + (1 + \gamma)q^2 + 2$$

by

$$(np^2)^{-1/4}(1 - \chi_1(p)), \quad (nq^2)^{-1/4}(1 - \chi_1(q)) \quad \text{and} \quad (1 + \gamma p^2 + 2 \gamma (p, q) + (1 + \gamma)q^2,$$

respectively, since the error will be a Hilbert–Schmidt operator continuous up to $z = E_{\min}$.

We have denoted by $\chi_1(\cdot)$ the characteristic function of the ball $U_1(0)$. By the replacement we obtain the integral operator $T_2^{\alpha r}(r)$ in $L^2(U_1(0) \setminus U_1(0))$ with a kernel

$$T_2^{\alpha r}(r; p, q) = -\frac{d_1}{n^2} |p|^{-1/2} |q|^{-1/2} (1 + \gamma p^2 + 2 \gamma (p, q) + (1 + \gamma)q^2).$$

By the dilation

$$M : L^2(U_1(0) \setminus U_1(0)) \to L^2((0, r), S^2), \quad r = 1/2 \log |E_{\min} - z|,$$

where $S^2$ is the unit sphere in $\mathbb{R}^3, (M f)(x, w) = e^{x/2} f(xw), x \in (0, r), w \in S^2,$ one sees that the operator $T_2^{\alpha r}(r)$ is unitarily equivalent to the integral operator $S_\gamma^ux$ with the kernel

$$S_\gamma^ux(x - x';<\xi,\eta>), \quad \xi, \eta \in S^2, x, x' \in \mathbb{R}^2,$$

where

$$S_\gamma^ux(x; t) = -(2\pi)^{-2} \frac{u}{\cosh x + st}, \quad u = \frac{1 + \gamma}{\sqrt{1 + 2\gamma}}, \quad s = \frac{\gamma}{1 + \gamma}, \quad t = (\xi, \eta).$$

Recall the lemma in [36].

**Lemma 6.3.** Let $A(z) = A_0(z) + A_1(z)$, where $A_0 (A_1)$ is compact and continuous in $z < 0$ ($z \leq 0$). Assume that for some function $f(\cdot), f(z) \to 0, z \to -0$ the limit

$$\lim_{z \to -0} f(z)n(\lambda, A_0(z)) = l(\lambda)$$

exists and is continuous in $\lambda > 0$. Then, the same limit exists for $A(z)$ and

$$\lim_{z \to -0} f(z)n(\lambda, A(z)) = l(\lambda).$$

The following theorem is important for the proof of the asymptotics (4.1) and can be proved in a similar way as theorem 7.4 in [6].

**Theorem 6.4.** Let the conditions of part (ii) of theorem 4.2 be fulfilled. The following equalities hold:

$$\lim_{E_{\min} - z \to 0} \frac{n(1, T^{\alpha r}(z))}{|\log |E_{\min} - z||} = \lim_{r \to \infty} \frac{1}{2} r^{-1} n(1, S_\gamma^ux) = u^\alpha_0(\gamma), \quad u^\alpha_0(\gamma) > 0. \quad (6.4)$$

**Proof.** The coefficient $u^\alpha_0(\gamma)$ on the rhs of the asymptotics (6.4) will be expressed by means of the self-adjoint integral operator $\hat{S}_\gamma(\gamma), \gamma \in \mathbb{R}$, in $L^2(S^2)$, whose kernel depends on the scalar product $t = \langle \xi, \eta \rangle >$ of the arguments $\xi, \eta \in S^2$ and has the form

$$\hat{S}_\gamma(r; \gamma) = -(2\pi)^{-1} u \frac{\sinh \gamma \arccos r}{\sqrt{1 - s^2 t^2} \sinh (\pi \gamma)}. \quad (6.5)$$

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For $\nu > 0$, define
\[
U^{\nu}(\nu; \gamma) = (4\pi)^{-1} \int_{-\infty}^{\infty} n(\nu, \hat{S}_\nu(y)) \, dy, \quad \nu > 0. \tag{6.6}
\]

The function $U^{\nu}(\nu; \gamma)$ is very important for the proof of the existence of the Efimov effect. Denote $U_0^{\nu}(\gamma) = U^{\nu}(1; \gamma)$.

Similar to [36], we can derive that
\[
U_0^{\nu}(\gamma) = \frac{1}{4\pi} \int_{-\infty}^{\infty} n(1, \hat{S}_\nu(y)) \, dy = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1) \int_{-\infty}^{\infty} n(1, \hat{S}_\nu^{(l)}(y)) \, dy, \tag{6.7}
\]
where $\hat{S}_\nu^{(l)}(y)$ is the multiplication operator by number
\[
\hat{S}_\nu^{(l)}(y) = 2\pi \int_{-1}^{1} P_l(t) \hat{S}_\nu(y) \, dt \tag{6.8}
\]
in $G_l$, the subspace of the harmonics of degree $l$, and $P_l(t)$ are Legendre polynomials. It follows from (6.7) and lemma 6.5 that
\[
U_0^{\nu}(\gamma) \geq \frac{1}{4\pi} \int_{-\infty}^{+\infty} n(1, \hat{S}_\nu^{(1)}(y)) \, dy = \frac{3}{4\pi} \text{mes} \{ x : \hat{S}_\nu^{(1)}(x) > 1 \}.
\]

For the proof this theorem the following lemma is important.

**Lemma 6.5.** The following assertions are true:

(a) $\hat{S}_\nu^{(0)}(y) < 0$.

(b) $\sup_y \hat{S}_\nu^{(1)}(y) > 1, \gamma > \gamma^*,$ and $\sup_y \hat{S}_\nu^{(1)}(y) < 1, \gamma < \gamma^*$.

(c) $\hat{S}_\nu^{(2)}(y) < 0, \gamma > 0$, and for any $\gamma, \gamma < \gamma^*$ there exists a positive number $\varepsilon = \varepsilon_\gamma$ such that $\sup_y \hat{S}_\nu^{(l)}(y) < 1 - \varepsilon_\gamma, l \geq 3$.

**Proof.** For the proof see appendix A. 

The positivity of $U_0^{\nu}(\gamma)$ follows from the fact that $\text{mes} \{ x : \hat{S}_\nu^{(1)}(x) > 1 \} > 0$, if $\gamma > \gamma^*$, which is proved in lemma 6.5.

We remark that for all $\gamma > \gamma_0$, the number $U_0^{\nu}(\gamma)$ is finite and
\[
U_0(\gamma) \leq \frac{3}{\pi} \frac{\log \left( \frac{1+\gamma}{\sqrt{1+2\gamma}} \right)}{\pi - 2 \arcsin \frac{1}{\sqrt{1+2\gamma}}} \left[ \frac{\log \left( \frac{16(1+\gamma)}{\pi + 3\pi \sqrt{1+2\gamma}} \right)}{\log \left( \frac{1+\gamma}{\sqrt{1+2\gamma}} \right) \sqrt{1+2\gamma}} + 1 \right]^2.
\]

The difference of the operators $S^{as}_\nu$ and $T^{as}(z)$ is compact (up to unitarity equivalence) and, hence, taking into account that $r = 1/2 \log |E_{\min} - z|$ and lemma 6.3, we obtain
\[
\lim_{|z - E_{\min}| \to 0} |\log |z - E_{\min}|^{-1} n(1, T^{as}(z)) = U_0^{\nu}(1; \gamma).
\]

theorem 6.4 is proved.
6.2. The finiteness of the discrete spectrum of $H_γ^{as}$

In this section, we discuss the case $γ < γ^*$ and prove the finiteness of the negative discrete spectrum.

We start the proof with the following assertion.

**Theorem 6.6.** The operator $H_γ$ has no eigenvalues lying on the right-hand side of the essential spectrum $σ_{ess}(H_γ)$.

**Proof.** Since $V = V_1 + V_2 + V_3$ is a positive operator and $\sup(σ_{ess}(H_γ)) = \sup(σ(H_0)) = E_{max}$, we have that the operator $H_γ = H_0 - V$ has no eigenvalues larger than $E_{max}$. □

The following lemma and lemma 5.4 complete the proof of part (i) of theorem 4.2.

**Lemma 6.7.** Let $γ < γ^*$ and the hypothesis of part (i) of theorem 4.2 be fulfilled. Then, there exists the number $\epsilon = \epsilon_γ$ depending on $γ$ so that

$$\sup σ_{ess}(T_γ^{as}(E_{min})) < 1 - \epsilon_γ.$$ 

**Proof.** See appendix B. □

6.3. The discrete spectrum of the operator $H_γ^s$

We recall that in the section we closely follow Sobolev’s method to derive the asymptotics for the number of eigenvalues of the operator $H_γ^s$ (see theorem 4.3).

As we will see, the discrete spectrum asymptotics of the operator $T_γ(z)$ as $z \to -E_{min}$ is determined by the integral operator

$$S_γ^r, r = 1/2| \log(E_{min} - z)|$$

in

$$L^2((0, r) \times L^2(S^2))$$

with the kernel $S_γ^r(x - x'; (ξ, η))$, $ξ, η ∈ S^2$; $S^2$ is the unit sphere in $\mathbb{R}^3$, where

$$S_γ^1(x; t) = 0, \quad S_γ^2(x; t) = S_γ^{as}(x; t),$$

$$S_γ^α(x; t) = (2π)^{-2} \frac{u_αβ}{\cosh(x + r_αβ) + s_αβt}, \quad α, β = 1, 2, \quad α ≠ β,$$

where $S_γ(x; t)$ is defined by (6.3) and

$$u_{12} = u_{21} = k_{αβ} 2^{1 + γ} \frac{1}{\sqrt{1 + 2γ}}, \quad s_{12} = s_{21} = \frac{1}{\sqrt{2(1 + γ)}},$$

$$r_{12} = \frac{1}{2} \log \frac{2}{1 + γ}, \quad r_{21} = \frac{1}{2} \log \frac{1 + γ}{2},$$

$k_{αβ}$ being such that $k_{12} = 1$ if both operators $h_1(0)$ and $h_2(0)$ have a resonance at the threshold (or $μ_1 = μ_0^1$ and $μ_2 = μ_0^2$); otherwise $k_{αβ} = 0$.

The next theorem concludes the proof of theorem 4.3.

Since the difference of the operators $S_γ^r$ and $T_γ(z)$ is compact, the following theorem can be proved in the same way as theorem 7.4 in [6] (or theorem 6.4 in section 6).

**Theorem 6.8.** Let the conditions of theorem 4.3 be fulfilled. The following equality holds:

$$\lim \frac{n(1, T_γ(z))}{|E_{min} - z|} = \lim \frac{1}{r^{1 − 1/2}} n(1, S_γ^r) = U_0^0, \quad U_0^0 > 0.$$
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Appendix A

Proof of lemma 6.5. Case $l = 0$. By (6.7) we first calculate $\hat{S}_y^{(0)}(y)$:

$$\hat{S}_y^{(0)}(y) = -\frac{u}{\sinh(\pi y)} \int_{-1}^{1} \sinh[y(\arccos(st))] \sqrt{1 - s^2 t^2} \, dt. \quad (A.1)$$

In [36] it is proven that for any $y \in \mathbb{R}$,

$$\frac{u \sinh[y(\arcsin s)]}{s y \cosh(\pi y)} = \frac{u}{\sinh(\pi y)} \int_{-1}^{1} \sinh[y(\arccos(st))] \sqrt{1 - s^2 t^2} \, dt. \quad (A.2)$$

Since this number is positive, by (A.1) we have $\hat{S}_y^{(0)}(y) < 0$ for all $y \in \mathbb{R}$.

Case $l = 1$. For any $y \in \mathbb{R}^1$, the $\hat{S}_y^{(1)}(y)$ can be written in the form

$$\hat{S}_y^{(1)}(y) = \frac{u}{\sinh(\pi y)} \int_{0}^{1} t \sinh[y(\arcsin(st))] \sqrt{1 - s^2 t^2} \, dt. \quad (A.2)$$

Since the integrand in (A.2) is positive, we obtain that the function $\hat{S}_y^{(1)}(y)$ is also positive and even as $y \in \mathbb{R}^1$. The function $\frac{\sinh(\pi y)}{\sinh(\pi y)}$ strictly decreases on $[0, \infty)$ for $\alpha > \beta$ and hence the function $\hat{S}_y^{(1)}(y)$ strictly decreases on $[0, \infty)$.

Let $b_1(y) = \sup_y \hat{S}_y^{(1)}(y)$. Then, we have

$$b_1(y) = \frac{2(1 + y)^2}{\pi y \sqrt{1 + 2y}} = \frac{2(1 + y)^2}{\pi y^2} \arcsin \frac{y}{1 + y}. \quad (A.3)$$

Since $\hat{S}_y^{(1)}(y) > 0$ for all $y \in \mathbb{R}^1$ and $\lim_{y \to \infty} \hat{S}_y^{(1)}(y) = 0$, it follows from (A.2) and (A.3) that the range of the function $\hat{S}_y^{(1)}(y)$ coincides with the set $(0, b_1(y)]$.

The function $b_1(y)$ defined on $(0, \infty)$ is continuous and strictly increases. Moreover,

$$\lim_{y \to 0} b_1(y) = 0 \quad \text{and} \quad \lim_{y \to \infty} b_1(y) = \infty.$$ 

Thus, the equation $b_1(y) = 1$ has a unique simple solution $y_0 > 0$ and $b_1(y) < 1$ (respectively $b_1(y) > 1$) for $y < y^*$ (respectively $y > y^*$).

Since $b_1(16) > \frac{3.20}{\pi} > 1$,

the solution $y^*$ of the equation $b_1(y) = 1$ satisfies the inequality $y^* < 16$.

Case $l = 2$. The function $\hat{S}_y^{(2)}(y)$ is calculated as

$$\hat{S}_y^{(2)}(y) = -\frac{u}{2} \int_{0}^{1} \frac{3t^2 - 1}{\sqrt{1 - s^2 t^2}} \cosh[y(\arcsin st)] \cosh(\frac{\pi y}{2}) \, dt.$$
It is easy to see that the inequalities
\[
\frac{\cosh[y(\arcsin st)]}{\sqrt{1-s^2t^2}} < \frac{\cosh[y(\arcsin(1/\sqrt{3})s)]}{\sqrt{1-(1/3)s^2}}, \quad t \in \left[0, \frac{1}{\sqrt{3}}\right), \quad (A.4)
\]
and
\[
\frac{\cosh[y(\arcsin st)]}{\sqrt{1-s^2t^2}} > \frac{\cosh[y(\arcsin(1/\sqrt{3})s)]}{\sqrt{1-(1/3)s^2}}, \quad t \in \left(\frac{1}{\sqrt{3}}, 1\right], \quad (A.5)
\]
hold.

Then, using inequalities (A.4) and (A.5) we obtain the following inequality:
\[
\hat{S}^{(2)}(y) < -\frac{u}{2} \frac{\cosh[y(\arcsin(1/\sqrt{3})s)]}{\sqrt{1-(1/3)s^2} \cosh\left(\frac{\pi y}{2}\right)} \int_0^1 (3t^2 - 1) \, dt = 0.
\]
From here it follows that for all \( y > 0 \)
\[
\sup \sigma(\hat{S}^{(2)}(y)) \, dy \leq 0.
\]

**Case I \( I \geq 3 \).** By lemma 3.2 of [36] we obtain that
\[
|\hat{S}^{(l)}(y)| < F(l; y) = \frac{16}{\pi^2 [3(2l+1)]^2} \frac{1+y}{\sqrt{1+2y}} \left(\frac{y}{1+y+\sqrt{1+2y}}\right)^{l},
\]
for all \( l \geq 0 \) and \( y > 0 \).

Since the function \( F(l; \cdot) \) is increasing and \( F(\cdot; y) \) is decreasing, we have
\[
\sup_y |\hat{S}^{(l)}(y)| < F(l; y^*) < F(3; y^*) < F(3; 16) < 1
\]
for any \( y \leq y^* \) and \( l \geq 3 \).

Thus, for any \( y \geq 0 \) and \( y \in \mathbb{R}^1 \), we have \( \hat{S}^{(l)}(y) \leq 0, \ l = 0, 2 \), and for any \( y < y^* \) (respectively for any \( y \geq 0 \) and \( l \geq 3 \)) we have \( |\hat{S}^{(l)}(y)| < 1 \) (respectively \( |\hat{S}^{(l)}(y)| < 1 \)).

These facts complete the proof of the lemma.

**Appendix B**

**Proof of lemma 6.7.** Since the operator \( T^{as}(E_{\min}) - T^{as}(\delta; 0) \) is Hilbert–Schmidt (see lemma 6.1), Weyl’s theorem implies that
\[
\sigma_{ess}(T^{as}(E_{\min})) = \sigma_{ess}(T^{as}(\delta; 0)).
\]
The space \( L^2(\hat{U}_\delta(0)) \) of all functions \( w(p) \) having support in \( \hat{U}_\delta(0) = \{ p \in T^3 : |W^{1/2} p| < \delta \} \) is an invariant subspace of the operator \( T^{as}_\eta(\delta; 0) \).

Let \( \hat{T}^{as}_\eta(\delta; 0) \) be the restriction of the operator \( T^{as}_\eta(\delta; 0) \) to the invariant subspace \( L^2(\hat{U}_\delta(0)) \).

The operator \( \hat{T}^{as}_\eta(\delta; 0) \) is unitarily equivalent to the integral operator \( S_\eta \) acting in \( L^2(0, +\infty) \otimes L^2(S^2) \), with the kernel \( S_\eta(x - x'; \langle \xi, \eta \rangle) \), \( \xi, \eta \in S^2 \), where
\[
S_\eta(x; t) = -\frac{2}{(2\pi)^2} \frac{u}{\cosh x \, st}, \quad u = \frac{1+y}{\sqrt{1+2y}}, \quad s = \frac{\gamma}{1+y}, \quad (B.1)
\]
The equivalence is given by the unitary operator \( M : L^2(U_\delta(0)) \rightarrow L^2((0, \infty) \otimes L^2(S^2)) \), where
\[
(Mf)(x, w) = \delta^{1/2} e^{-\frac{u}{2}} f(\delta e^{-\frac{u}{2}} W^{1/2} w), \quad w \in S^2.
\]
Therefore, the essential spectra of $T^{(\gamma)}(\delta)$ and $S_{\gamma}$ coincide.

Similar to [36], we have

$$S_{\gamma} = \sum_{l=0}^{\infty} \oplus \left( S_{\gamma}^{(l)} \otimes \mathcal{P}_l \right),$$  \hspace{1cm} (B.2)

where $\mathcal{P}_l : L^2(0, \infty) \otimes L^2(S^2) \to \mathcal{G}$ is the orthogonal projector onto $\mathcal{G}$; $S_{\gamma}^{(l)}$ are operators in $L^2(0, \infty)$ with the kernel

$$S_{\gamma}^{(l)}(x - x') = 2\pi \int_{-1}^{1} P_l(t) S_{\gamma}(x - x'; t) \, dt.$$  \hspace{1cm} (B.3)

Now, comparing definitions (6.5) and (B.1) and applying the equality

$$\frac{\sinh(y\theta)}{\sin \theta \sinh(\pi y)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda y} \frac{dx}{\cosh x + \cos \theta}, \quad 0 < \theta < \pi,$$

we see that

$$\tilde{S}_{\gamma}(t; y) = \int_{-\infty}^{\infty} e^{-i\lambda y} S_{\gamma}(x; t) \, dx.$$  \hspace{1cm} (B.4)

By (B.1) and (B.3) this yields

$$\tilde{S}_{\gamma}^{(l)}(y) = \int_{-\infty}^{\infty} e^{-i\lambda y} S_{\gamma}^{(l)}(x) \, dx.$$  \hspace{1cm} (B.4)

One can check that the functions in (6.8) and (B.4) coincide.

Since the operator $S_{\gamma}^{(l)}$ is of convolution type we conclude that it is unitarily equivalent to the operators $\tilde{S}_{\gamma}^{(l)}$ of multiplication by the functions $\tilde{S}_{\gamma}^{(l)}(y)$ in the space $L^2(\mathbb{R}^1)$.

Therefore,

$$\sigma(\tilde{S}_{\gamma}^{(l)}) = \sigma(\tilde{S}_{\gamma}^{(l)}) = \text{Ran}(\tilde{S}_{\gamma}^{(l)}(y)).$$  \hspace{1cm} (B.5)

Recall that the number $\gamma^*$ is the unique solution of the equation $b_1(\gamma) = 1$.

As it was mentioned in lemma 6.5, for any $\gamma < \gamma^*$ there exists the number $\epsilon_{\gamma} > 0$ such that for all $l \geq 0$

$$\sup_{y} \left| \tilde{S}_{\gamma}^{(l)}(y) \right| < 1 - \epsilon_{\gamma}.$$  \hspace{1cm} (B.6)

Equalities (B.2), (B.5) and (B.6) proved the inequality in the lemma.

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