Proposal of a gauge-invariant treatment of $l=0$, 1-mode perturbations on Schwarzschild background spacetime

Kouji Nakamura
Gravitational-Wave Science Project, National Astronomical Observatory of Japan,
2-21-1, Osawa, Mitaka, Tokyo 181-8588, Japan
E-mail: kouji.nakamura@nao.ac.jp

Abstract. A gauge-invariant treatment of the monopole- ($l=0$) and dipole ($l=1$) modes in linear perturbations of the Schwarzschild background spacetime is proposed. Through this gauge-invariant treatment, we derived the solutions to the linearized Einstein equation for these modes with a generic matter field. In the vacuum case, these solutions include the Kerr parameter perturbations in the $l=1$ odd modes and the additional mass parameter perturbations of the Schwarzschild mass in the $l=0$ even modes. The linearized version of Birkhoff’s theorem is also confirmed in a gauge-invariant manner. In this sense, our proposal is reasonable.

Keywords: gauge-invariant perturbations, Schwarzschild spacetime, $l=0$, 1 modes

1. Introduction

In 2015, the direct observation of gravitational waves was finally accomplished by the Laser Interferometer Gravitational-wave Observatory [1]. This event was the beginning of the gravitational-wave astronomy and multi-messenger astronomy including gravitational waves [2]. One future direction of gravitational-wave astronomy is the development as a precise science by the detailed studies of source science and the tests of general-relativity through the developments of the global network of gravitational-wave detectors [2, 3]. In addition to these ground-based detectors, some projects of space gravitational-wave antenna are also progressing [4, 5]. Among them, the Extreme-Mass-Ratio-Inspiral (EMRI) is one of the targets of the Laser Interferometer Space Antenna [4]. The EMRI is a source of gravitational waves, which is the motion of a stellar mass object around a supermassive black hole, and black hole perturbation theories are used to describe the EMRI. Therefore, theoretical sophistications of black hole perturbation theories and their higher-order extensions are required to support the development of precise experimental science. This paper is motivated by such theoretical sophistication of black hole perturbation theories toward higher-order perturbations.

Although realistic black holes have angular momentum and we must consider the perturbation theory of a Kerr black hole for direct application to the EMRI, further sophistication is possible even in perturbation theories on the Schwarzschild background spacetime. Based on the pioneering works by Regge and Wheeler [6] and Zerilli [7], there have been many studies on the perturbations in the Schwarzschild background spacetime [8, 9, 10, 11]. Because the Schwarzschild spacetime has a spherical symmetry, we may decompose the perturbations on this spacetime using the spherical harmonics $Y_{lm}$ and classify them into odd- and even-modes based on their parity. However, the current consensus is that $l=0$ and
l = 1 modes should be separately treated \cite{10}, and “gauge-invariant” treatments for l = 0 and l = 1 even-modes remain unknown.

However, toward unambiguous sophisticated nonlinear general-relativistic perturbation theories, we have been developing the general formulation of a higher-order gauge-invariant perturbation theory on a generic background spacetime \cite{12, 13, 14, 15} and have applied it to cosmological perturbations \cite{16, 17, 18}. We briefly review our framework of the gauge-invariant perturbation theory \cite{12, 13} in Sec. 2. This framework is based on a conjecture (Conjecture 2.1 below), which roughly states that we already know the procedure to find gauge-invariant variables for linear-order metric perturbations. A proof of Conjecture 2.1 was discussed in \cite{14}. In this proof, we assumed the existence of the Green functions for some elliptic derivative operators and ignored the kernel modes of these elliptic derivative operators. We call these kernel modes “zero modes,” and the treatment of these zero modes remains unclear. We also called the problem of finding a treatment to these zero modes as the “zero-mode problem.”

In the case of the gauge-invariant perturbation theory on the Schwarzschild background spacetime, the zero modes are just above the l = 0, 1 modes. The special treatments of these modes become an obstacle when we develop nonlinear perturbation theory because mode couplings owing to the nonlinear effects produce higher-order l = 0, 1 modes \cite{11}. Therefore, the finding of a gauge-invariant treatment of l = 0, 1 modes in the perturbations on Schwarzschild background spacetime is a resolution of the above zero-mode problem in a specific background spacetime. Furthermore, this resolution is an important step of the development of the higher-order gauge-invariant perturbation theory on the Schwarzschild background spacetime. In addition to the perturbation theory on a specific background spacetime, this resolution will become a clue to the perturbation theory on a generic background spacetime.

In this paper, we propose a gauge-invariant treatment of l = 0, 1, perturbations on the Schwarzschild background spacetime and show that Conjecture 2.1 is true even for these modes. We also derive the solutions to the linearized Einstein equation for these modes.

The organization of this paper is as follows: In Sec. 2, we briefly review the framework of the general-relativistic gauge-invariant perturbation theory within the linear perturbation theory, although this framework can be extended to higher-order perturbations \cite{12, 13, 14, 15}. In Sec. 3, we propose a strategy for gauge-invariant treatments of l = 0, 1 modes after the explanation of the situation, which in many studies requires the special treatments of l = 0, 1 modes. In Sec. 4, we construct gauge-invariant variables including l = 0, 1 modes through the proposal described in Sec. 3. This is a complete proof of Conjecture 2.1 for perturbations on the background spacetimes with spherical symmetry. In Sec. 5, we derive the solutions to the Einstein equations for l = 0, 1 modes. Finally, in Sec. 6, we provide some concluding remarks and discussions regarding this research.

Throughout this paper, we use the unit G = c = 1, where G is Newton’s constant of gravitation, and c is the velocity of light.

2. Brief review of general-relativistic gauge-invariant perturbation theory

Herein, we briefly review the framework of the gauge-invariant perturbation theory \cite{12, 13}. In this review, we concentrate only on the linear perturbations, because we treat only the linear perturbations within this paper.

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime (\(M_{ph}, g_{ab}\)), which is identified with our nature itself, and we want to describe this spacetime (\(M_{ph}, g_{ab}\)) by perturbations. The other is the background spacetime (\(M, g_{ab}\)),
which is prepared as a reference by hand. Note that these two spacetimes are distinct. Furthermore, in any perturbation theory, we always write equations for the perturbation of the variable $Q$ as follows:

$$Q(\text{"p"}) = Q_0(p) + \delta Q(p). \quad (2.1)$$

Equation (2.1) gives a relation between variables on different manifolds. Actually, $Q(\text{"p"})$ in Eq. (2.1) is a variable on $\mathcal{M}_{\text{ph}}$, whereas $Q_0(p)$ and $\delta Q(p)$ are variables on $\mathcal{M}$. Because we regard Eq. (2.1) as a field equation, Eq. (2.1) includes an implicit assumption of the existence of a point identification map $\mathcal{M} \to \mathcal{M}_{\text{ph}} : p \in \mathcal{M} \mapsto "p" \in \mathcal{M}_{\text{ph}}$. This identification map is called a gauge choice in general-relativistic perturbation theories. This is the notion of the second-kind gauge pointed out by Sachs [19]. Note that this second-kind gauge is a different notion from the degree of freedom of the coordinate transformation on a single manifold, which is called the first-kind gauge [13].

To develop this understanding of the “gauge,” we introduce an infinitesimal parameter $\lambda$ for perturbations and $4+1$-dimensional manifold $\mathcal{N} = \mathcal{M}_{\text{ph}} \times \mathbb{R}$ ($4 = \text{dim} \mathcal{M}$) such that $\mathcal{M} = \mathcal{N}\mid_{\lambda=0}$ and $\mathcal{M}_{\text{ph}} = \mathcal{M}\mid_{\lambda=\lambda}$. On $\mathcal{N}$, the gauge choice is regarded as a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M} \to \mathcal{M}_\lambda$. This gauge choice is a point-identification. Furthermore, we introduce a gauge choice $\mathcal{X}_\lambda$ as an exponential map with a generator $\mathcal{X}_a\eta^a$, which is chosen such that its integral curve in $\mathcal{N}$ is transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$. Points lying on the same integral curve are regarded as the “same” by the gauge choice $\mathcal{X}_\lambda$.

The first-order perturbation of the variable $Q$ on $\mathcal{M}_\lambda$ is defined as the pulled-back $\mathcal{X}_\lambda^*Q$ on $\mathcal{M}$, which is induced by $\mathcal{X}_\lambda$, and is expanded as

$$\mathcal{X}_\lambda^*Q = Q_0 + \lambda \mathcal{L}_{\mathcal{X}_\lambda}Q + O(\lambda^2), \quad (2.2)$$

where $Q_0 = Q\mid_{\lambda=0}$ is the background value of $Q$ and all terms in Eq. (2.2) are evaluated on the background spacetime $\mathcal{M}$. Because Eq. (2.2) is the perturbative expansion of $\mathcal{X}_\lambda^*Q_\lambda$, the first-order perturbation of $Q$ is given by $\mathcal{Q} \equiv \mathcal{Q}_\lambda := \mathcal{X}_\lambda^*Q\mid_{\lambda=0}$.

When we have two gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ with the generators $\mathcal{X}_a\eta^a$ and $\mathcal{Y}_a\eta^a$, respectively, and when these generators have different tangential components to each $\mathcal{M}_\lambda$, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are regarded as different gauge choices. A gauge-transformation is regarded as the change of the point-identification $\mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, which is given by the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda : \mathcal{M} \to \mathcal{M}$. The diffeomorphism $\Phi_\lambda$ does change the point-identification.

Here, $\Phi_\lambda$ induces a pull-back from the representation $\mathcal{X}_\lambda^*Q_\lambda$ to the representation $\mathcal{Y}_\lambda^*Q_\lambda$ as $\mathcal{Y}_\lambda^*Q_\lambda = \Phi_\lambda^* \mathcal{X}_\lambda^*Q_\lambda$. From general arguments of the Taylor expansion [20], the pull-back $\Phi_\lambda^*$ is expanded as

$$\Phi_\lambda^*Q_\lambda = \mathcal{X}_\lambda^*Q_\lambda + \lambda \mathcal{L}_{\xi(1)} \mathcal{X}_\lambda^*Q_\lambda + O(\lambda^2), \quad (2.3)$$

where $\xi(1)$ is the generator of $\Phi_\lambda$. From Eqs. (2.2) and (2.3), each order gauge-transformation is given as

$$\mathcal{Q}_\lambda - \mathcal{Q}_0 = \lambda \mathcal{L}_{\mathcal{X}_\lambda} \mathcal{Q}_0. \quad (2.4)$$

We also employ the order by order gauge invariance as a concept of gauge invariance [17]. We call the $k$th-order perturbation $\mathcal{Q}_k$ as gauge invariant if and only if $\mathcal{Q}_k = (k)\mathcal{Q}$ for any gauge choice $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$.

Based on the above setup, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations [12] [13]. In this paper, we concentrate only on the explanations
of the linear perturbations. First, we expand the metric on the physical spacetime $\mathcal{M}_\lambda$, which was pulled back to the background spacetime $\mathcal{M}$ through a gauge choice $\mathcal{X}_\lambda$ as

$$\mathcal{X}_\lambda^{-1} g_{ab} = g_{ab} + \lambda \, \mathcal{F}_{ab} + O(\lambda^2).$$

Although the expression (2.5) depends entirely on the gauge choice $\mathcal{X}_\lambda$, henceforth, we do not explicitly express the index of the gauge choice $\mathcal{X}$ in the expression if there is no possibility of confusion. The important premise of our proposal was the following conjecture \cite{12, 13} for the linear metric perturbation $h_{ab}$:

**Conjecture 2.1.** If the gauge-transformation rule for a tensor field $h_{ab}$ is given by $\delta h_{ab} = \mathcal{F}_{ab} + \mathcal{X}_{ab}$, with the background metric $g_{ab}$, there then exist a tensor field $\mathcal{X}_{ab}$ and a vector field $Y^a$ such that $h_{ab}$ is decomposed as $h_{ab} = \mathcal{F}_{ab} + \mathcal{X}_{ab}$, where $\mathcal{X}_{ab}$ and $Y^a$ are transformed into $\mathcal{X}_{ab} - \mathcal{F}_{ab} = 0$ and $Y^a - \mathcal{F}_{ab} = \xi^a \lambda(1)$ under the gauge transformation, respectively.

We call $\mathcal{X}_{ab}$ and $Y^a$ as the gauge-invariant and gauge-variant parts of $h_{ab}$, respectively.

The proof of Conjecture 2.1 is highly nontrivial \cite{14}, and it was found that the gauge-invariant variables are essentially non-local, as mentioned in Sec. 1. Despite this non-triviality, once we accept Conjecture 2.1 we can construct gauge-invariant variables for the linear perturbation of an arbitrary tensor field $(1) \mathcal{Q}$, whose gauge-transformation is given by Eq. (2.4), through the gauge-variant part of the metric perturbation $Y_a$ in Conjecture 2.1 as

$$(1) \mathcal{Q} := (1) \mathcal{Q} - \xi \, \mathcal{F}^a Y_a. \tag{2.6}$$

This definition implies that the linear perturbation $(1) \mathcal{Q}$ of an arbitrary tensor field $\mathcal{X}_{ab}$ is always decomposed into its gauge-invariant part $(1) \mathcal{Q}$ and gauge-variant part $\xi \, \mathcal{F}^a Y_a$ as

$$(1) \mathcal{Q} = (1) \mathcal{Q} + \xi \, \mathcal{F}^a Y_a. \tag{2.7}$$

As an example, the linearized Einstein tensor $(1) \mathcal{G}_{ab}$ and the linear perturbation of the energy-momentum tensor $(1) T_a^b$ are also decomposed as

$$(1) \mathcal{G}_{ab} = (1) \, \mathcal{G}_{ab} - \xi \, \mathcal{F}^c Y_a, \quad (1) T_a^b = (1) \, \mathcal{F}^c T_a^b + \xi \, \mathcal{F}^c Y_a, \tag{2.8}$$

where $G_{ab}$ and $T_{ab}$ are the background values of the Einstein tensor and the energy-momentum tensor, respectively. The gauge-invariant part $(1) \mathcal{G}_{ab}$ of the linear-order perturbation of the Einstein tensor is given by

$$(1) \mathcal{G}_{ab} = (1) \mathcal{G}_{ab} - \frac{1}{2} \delta_{a}^{b} \sum_{c} \mathcal{G}_{c} A, \tag{2.9}$$

$$(1) \mathcal{G}_{ab} = (1) \mathcal{G}_{ab} - \frac{1}{2} \delta_{a}^{b} \sum_{c} \mathcal{G}_{c} A, \tag{2.10}$$

where $A_{ab}$ is an arbitrary tensor field of the second rank and $\delta_{a}^{b} \sum_{c} \mathcal{G}_{c} A$ corresponds to the gauge-invariant part of the linear perturbation of the Ricci tensor $R_a^b$ \cite{13, 18}. Then, using the background Einstein equation $G_{ab} = 8\pi T_{ab}$, the linearized Einstein equation $(1) G_{ab} = 8\pi (1) T_{ab}$ is automatically given in the gauge-invariant form

$$(1) \mathcal{G}_{ab} = 8\pi (1) \mathcal{F}_{ab} \tag{2.11}$$

even if the background Einstein equation $G_{ab} = 8\pi T_{ab}$ is nontrivial. We also note that, in the case of a vacuum background case, i.e., $8\pi T_{ab} = G_{ab} = 0$, Eq. (2.8) shows that the linear perturbations of the Einstein tensor and the energy-momentum tensor is automatically gauge-invariant.
We can also derive the perturbation of the divergence of $\nabla_a \bar{T}_b^a$ of the second-rank tensor $\bar{T}_b^a$ on $(\mathcal{M}_{ph}, \bar{g}_{ab})$. Through the gauge choice $\mathcal{F}_\lambda$, $\bar{T}_b^a$ is pulled back to $\mathcal{F}_\lambda^* \bar{T}_b^a$ on the background spacetime $(\mathcal{M}, g_{ab})$, and the covariant derivative operator $\nabla_a$ on $(\mathcal{M}_{ph}, \bar{g}_{ab})$ is pulled back to a derivative operator $\nabla_a = \mathcal{F}_\lambda^* \nabla_a (\mathcal{F}_\lambda^{-1})^*$ on $(\mathcal{M}, g_{ab})$. Note that the derivative $\nabla_a$ is the covariant derivative associated with the metric $\mathcal{F}_\lambda \bar{g}_{ab}$, whereas the derivative $\nabla_a$ on the background spacetime $(\mathcal{M}, g_{ab})$ is the covariant derivative associated with the background metric $g_{ab}$. Bearing in mind the difference in these derivative, the first-order perturbation of $\nabla_a \bar{T}_b^a$ is given by

$$\nabla_a \bar{T}_b^a = 0$$

The derivation of the formula (2.12) is given in Ref. [13]. If the tensor field $\bar{T}_b^a$ is the Einstein tensor $\bar{G}_b^a$, Eq. (2.12) yields the linear-order perturbation of the Bianchi identity

$$\nabla_a \bar{G}_b^a [\mathcal{F}] + H_{ca} [\mathcal{F}] T_b^c = 0$$

and if the background Einstein tensor vanishes $G_a^b = 0$, we obtain the identity

$$\nabla_a \bar{G}_b^a [\mathcal{F}] = 0$$

By contrast, if the tensor field $\bar{T}_b^a$ is the energy-momentum tensor, Eq. (2.12) yields the continuity equation of the energy-momentum tensor

$$\nabla_a \bar{T}_b^a + H_{ca} [\mathcal{F}] T_b^c = 0$$

where we used the background continuity equation $\nabla_a T_b^a = 0$. If the background spacetime is vacuum $T_{ab} = 0$, Eq. (2.15) yields a linear perturbation of the energy-momentum tensor given by

$$\nabla_a \bar{G}_b^a = 0$$

Thus, starting from the Conjecture 2.1, we can develop the gauge-invariant perturbation theory through the above framework. Furthermore, this formulation can be extended to any order perturbations [12, 13, 14, 15] from Conjecture 2.1. In this sense, the proof of the Conjecture 2.1 is crucial to this framework. We should note that the decomposition of the metric perturbation $h_{ab}$ into its gauge-invariant part $\mathcal{F}_{ab}$ and into its gauge-variant part $\psi^a_a$ is not unique [17, 18]. For example, The gauge-invariant part $\mathcal{F}_{ab}$ has six components and we can create the gauge-invariant vector field $Z^a$ through the component $\mathcal{F}_{ab}$ such that the gauge-transformation of the vector field $Z^a$ is given by $\mathcal{F} Z^a - \mathcal{F} Z^a = 0$. Using this gauge-invariant vector field $Z^a$, the original metric perturbation can be expressed as follows:

$$h_{ab} = \mathcal{F}_{ab} - \mathcal{F} Z_{ab} + \mathcal{F} Y Z_{ab} =: \mathcal{H}_{ab} + \mathcal{F} X g_{ab}$$

The tensor field $\mathcal{H}_{ab} := \mathcal{F}_{ab} - \mathcal{F} Z_{ab}$ is also regarded as the gauge-invariant part of the perturbation $h_{ab}$ because $\not\mathcal{H}_{ab} = \not\mathcal{H}_{ab} = 0$. Similarly, the vector field $X^a := Z^a + Y^a$ is also regarded as the gauge-variant part of the perturbation $h_{ab}$ because $\not X^a = \not X^a = \not Y^a$. This non-uniqueness appears in the solutions derived in Sec. 5.

Finally, we comment on the relation between the gauge-transformation $\Phi_\psi$ and the coordinate transformation $\psi$. As mentioned above, the notion of the second-kind gauges above is different from the notion of the degree of freedom of the coordinate transformation on a single manifold. However, the gauge-transformation $\Phi_\psi$ of the second kind induces the coordinate transformations. To see this, we introduce the coordinate system $\{ O_\alpha, \psi_\alpha \}$ on the background spacetime $\mathcal{M}$, where $O_\alpha$ are open sets on the background spacetime and $\psi_\alpha$ are diffeomorphisms from $O_\alpha$ to $\mathbb{R}^4$ (4 = dim $\mathcal{M}$). The coordinate system $\{ O_\alpha, \psi_\alpha \}$ is the set of collections of the pair of open sets $O_\alpha$ and diffeomorphism $O_\alpha \mapsto \mathbb{R}^4$. If we
employ a gauge choice \( \mathcal{Y}_\lambda \) of the second kind, we have the correspondence of the physical spacetime \( \mathcal{M}_\lambda = \mathcal{M}_\psi \) and the background spacetime \( \mathcal{M} \). Together with the coordinate system \( \psi_\alpha \) on \( \mathcal{M} \), this correspondence between \( \mathcal{M}_\lambda \) and \( \mathcal{M} \) induces the coordinate system on \( \mathcal{M}_\lambda \). Actually, \( \mathcal{Y}_\lambda (O_\alpha) \) for each \( \alpha \) is an open set of \( \mathcal{M}_\lambda \). Then, \( \psi_\alpha \circ \mathcal{Y}_\lambda^{-1} \) becomes a diffeomorphism from an open set \( \mathcal{Y}_\lambda (O_\alpha) \subset \mathcal{M}_\lambda \) to \( \mathbb{R}^4 \). This diffeomorphism \( \psi_\alpha \circ \mathcal{Y}_\lambda^{-1} \) induces a coordinate system of an open set on \( \mathcal{M}_\lambda \). When we have two different gauge choices \( \mathcal{Y}_\lambda \) and \( \mathcal{Y}_\mu \) of the second kind, \( \psi_\alpha \circ \mathcal{Y}_\lambda^{-1} : \mathcal{M}_\lambda \rightarrow \mathbb{R}^4 \{ \{ x^\mu \} \} \) and \( \psi_\alpha \circ \mathcal{Y}_\mu^{-1} : \mathcal{M}_\mu \rightarrow \mathbb{R}^4 \{ \{ x^\mu \} \} \) become different coordinate systems on \( \mathcal{M}_\lambda \). We can also consider the coordinate transformation from the coordinate system \( \psi_\alpha \circ \mathcal{Y}_\lambda^{-1} \) to another coordinate system \( \psi_\alpha \circ \mathcal{Y}_\mu^{-1} \). Because the gauge transformation \( \mathcal{Y}_\lambda \rightarrow \mathcal{Y}_\mu \) is induced by the diffeomorphism \( \Phi_\lambda := (\mathcal{Y}_\lambda)^{-1} \circ \mathcal{Y}_\mu \), this diffeomorphism \( \Phi_\lambda \) induces the coordinate transformation as

\[
y^\mu (q) := x^\mu (p) = ((\Phi_\lambda^{-1})^\ast x^\mu) (q) \quad (2.18)
\]

in the passive point of view \([12, 20]\), where \( p, q \in \mathcal{M} \) are identified to the same point “\( p \)” \( \in \mathcal{M}_\lambda \) by the gauge choices \( \mathcal{Y}_\lambda \) and \( \mathcal{Y}_\mu \), respectively. If we represent this coordinate transformation in terms of the Taylor expansion \((2.3)\), we have the coordinate transformation

\[
y^\mu (q) = x^\mu (q) - \lambda \varepsilon^\mu (q) + O(\lambda^2). \quad (2.19)
\]

We should emphasize that the coordinate transformation \((2.19)\) is not the starting point of the gauge-transformation but a result of the above framework. Because our above framework of the gauge-invariant perturbation theory is constructed without a coordinate transformation \((2.19)\), we avoid the use of the coordinate transformation \((2.19)\) as much as possible.

3. Linear perturbations on spherically symmetric background

Here, we use the 2+2 formulation of the perturbations on spherically symmetric background spacetimes, which was originally proposed by Gerlach and Sengupta \([9]\). The topological space of spherically symmetric spacetimes is the direct product \( \mathcal{M} = \mathcal{M}_1 \times S^2 \), and the metric on this spacetime is

\[
g_{ab} = \gamma_{ab} + \varepsilon_{ab}, \quad \gamma_{ab} = \gamma_{AB} (dx^A)_a (dx^B)_b, \quad \gamma_{ab} = \gamma_{pq} (dx^p)_a (dx^q)_b, \quad (3.1)
\]

where \( x^A = (t, r) \) and \( x^p = (\theta, \phi) \). In addition, \( \gamma_{pq} \) is a metric of the unit sphere. In the Schwarzschild spacetime, the metric \((3.1)\) is given by

\[
\gamma_{ab} = - f (dt)_a (dt)_b + f^{-1} (dr)_a (dr)_b, \quad f = 1 - \frac{2M}{r}, \quad (3.2)
\]

\[
\gamma_{ab} = (d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b. \quad (3.3)
\]

On this background spacetime \((\mathcal{M}, g_{ab})\), we consider the components of the metric perturbation as

\[
h_{ab} = h_{AB} (dx^A)_a (dx^B)_b + 2h_{Ap} (dx^A)_a (dx^p)_b + h_{pq} (dx^p)_a (dx^q)_b. \quad (3.4)
\]

In \([9]\), these components of the metric perturbation are decomposed through the decomposition \((2.11)\) using the spherical harmonics \( S = Y_{lm} \) as follows:

\[
h_{AB} = \sum_{l,m} \tilde{h}_{AB} S, \quad h_{Ap} = r \sum_{l,m} \left[ \tilde{h}_{(l+1)A} \hat{D}_p S + \tilde{h}_{(l+1)A} \varepsilon_{pq} \hat{D}_q S \right], \quad (3.5)
\]

\[
h_{pq} = r^2 \sum_{l,m} \left[ \frac{1}{2} \gamma_{pq} \tilde{h}_{l(0)S} + \tilde{h}_{l(2)} \left( \hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{D} S \right) + 2 \tilde{h}_{l(2)} \varepsilon_{pq} \hat{D}_q S \right], \quad (3.6)
\]

where \( \hat{D}_p \) is the covariant derivative associated with the metric \( \gamma_{pq} \) on \( S^2 \), \( \hat{D} := \gamma_{pq} \hat{D}_q \), and \( \varepsilon_{pq} = \varepsilon_{[pq]} \) is the totally antisymmetric tensor on \( S^2 \). Although the matrix representations
the independent harmonic functions are used in pioneer works \cite{6,7}. This decomposition formulae (3.5)-(3.6) is the starting point of the perturbations on spherically symmetric spacetimes proposed by Gerlach and Sengupta \cite{9}. Gerlach and Sengupta showed the constructions of gauge-invariant variables for $l \geq 2$ and derived the linearized Einstein equations. However, separate treatments are required for $l = 0, 1$ modes \cite{10}, which is the main target of this paper.

Herein, we describe the situation of $l = 0, 1$ modes in the $2 + 2$ formulation. Note that the decompositions (3.5)-(3.6) implicitly state that the Green function of the derivative operators $\hat{\Delta} := \hat{D}^r \hat{D}_r$ and $\hat{\Delta} + 2 := \hat{D}^r \hat{D}_r + 2$ should exist if the one-to-one correspondence between $\{ \hat{h}_{lp}, \hat{h}_{pq} \}$ and $\{ \tilde{h}_{(l+1)A}, \tilde{h}_{r(0)A}, \tilde{h}_{r(2)}, \tilde{h}_{r(2)} \}$ is guaranteed. Because the eigenvalue of the derivative operator $\hat{\Delta}$ on $S^2$ is $-l(l+1)$, the kernels of the operators $\hat{\Delta}$ and $\hat{\Delta} + 2$ are $l = 0$ and $l = 1$ modes, respectively. Thus, the one-to-one correspondence between $\{ \hat{h}_{lp}, \hat{h}_{pq} \}$ and $\{ \tilde{h}_{(l+1)A}, \tilde{h}_{r(0)A}, \tilde{h}_{r(2)}, \tilde{h}_{r(2)} \}$ is lost for $l = 0, 1$ modes in decomposition formulae (3.5)-(3.6) with $S = Y_{lm}$. If we choose the decomposition formulae (3.5)-(3.6) as the starting point of the metric perturbations even for $l = 0, 1$ modes, the gauge-invariance of $l = 0, 1$-mode perturbations becomes unclear owing to the loss of this one-to-one correspondence. Therefore, we should regard that the second decomposition formula in Eq. (3.5) with $S = Y_{lm}$ does not include the $l = 0$ mode and the formula (3.6) with $S = Y_{lm}$ does not include $l = 0, 1$ modes. This situation is also seen from the harmonics $D_p Y_{lm}, \epsilon_{pr} \hat{D}^r Y_{lm}, (\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{\Delta}) Y_{lm}$, and $2\epsilon_{(p} D_{q)} \hat{D}^r Y_{lm}$ for the $l = 0, 1$ modes as $D_p Y_{00} = \epsilon_{pr} \hat{D}^r Y_{00} = 0$, $(\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{\Delta}) Y_{00} = 2\epsilon_{(p} D_{q)} \hat{D}^r Y_{00} = 0$, and $(\hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{\Delta}) Y_{lm} = 2\epsilon_{(p} D_{q)} \hat{D}^r Y_{lm} = 0$. Then, for these kernel modes, the one-to-one correspondence between $\{ \hat{h}_{lp}, \hat{h}_{pq} \}$ and $\{ \tilde{h}_{(l+1)A}, \tilde{h}_{r(0)A}, \tilde{h}_{r(2)}, \tilde{h}_{r(2)} \}$ is not guaranteed. For this reason, separate treatments for $l = 0, 1$ modes are required.

To resolve this situation, in this paper, we introduce the mode functions $k_{(\Delta)}$ and $k_{(\Delta+2)lm}$ instead of $Y_{00}$ and $Y_{lm}$, respectively. These mode functions satisfy the equations

\[ \hat{\Delta} k_{(\Delta)} = 0, \quad [\hat{\Delta} + 2] k_{(\Delta+2)lm} = 0. \]  

(3.7)

Although the tensor decomposition formulae (3.5)-(3.6) with the harmonic function $Y_{lm}$ does not have an inverse relation for the $l = 0, 1$ modes, these equations may have an inverse relation even for the $l = 0, 1$ modes if we choose the harmonic function $S$ such that

\[ S_\delta = \begin{cases} Y_{lm} & \text{for } l \geq 2; \\ k_{(\Delta+2)lm} & \text{for } l = 1; \\ k_{(\Delta)} & \text{for } l = 0. \end{cases} \]  

(3.8)

Using Eq. (3.8) instead of $Y_{lm}$, we expand the metric perturbation through the decomposition formulae (3.5)-(3.6). To derive the inverse relation of this new decomposition formula, we first use the fact that the operators $\hat{\Delta}^{-1}\hat{\Delta}$ and $[\hat{\Delta} + 2]^{-1}[\hat{\Delta} + 2]$ are projection operators, which excludes the functions belonging to the kernels $\mathcal{H}_\Delta := \{ f | \hat{\Delta} f = 0 \}$ and $\mathcal{H}_{\Delta+2} := \{ f | [\hat{\Delta} + 2] f = 0 \}$, respectively. For $l \geq 2$ modes, we use the orthogonality of spherical harmonics $Y_{lm}$:

\[ \int_{S^2} d\Omega_{lm} Y_{lm}^* Y_{lm'} = \delta_{ll'} \delta_{mm'}. \]  

(3.9)

Furthermore, we can show that the set of the harmonic functions

\[ \left\{ S_\delta, D_p S_\delta, \epsilon_{pq} \hat{D}^p S_\delta, \left( \hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{\Delta} \right) S_\delta, 2\epsilon_{(p} D_{q)} \hat{D}^r S_\delta \right\} \]  

(3.10)

are linear-independent if the conditions

\[ k_{(\Delta)} \in \mathcal{H}_\Delta, \quad D_p k_{(\Delta)} \neq 0, \quad D_p D_q k_{(\Delta)} \neq 0, \]  

(3.11)

\[ k_{(\Delta+2)lm} \in \mathcal{H}_{\Delta+2}, \quad D_p k_{(\Delta+2)lm} \neq 0, \quad k_{(\Delta+2)lm} = 0. \]  

(3.12)
Here, we consider the gauge-transformation rule with the metric (3.1)–(3.3), through Eqs. (3.5)–(3.6) with the harmonic functions $S^z$ for the linear-order perturbations on a spherically symmetric background with the metric (3.1). We rewrite this gauge-transformation rule through the decomposition of the generator by Eq. (3.8). Then, Eqs. (3.5)–(3.6) become invertible with the inclusion of $l = 0, 1$ modes. We decompose the metric perturbations $h_{ab}$ on the background spacetime $\delta$ into the mode-by-mode analyses including $l = 0, 1$ modes. The $\phi$-dependence of $k_{(\delta+2)m}$ in Eq. (3.12) is used to resolve the $m = 0, \pm 1$ mode-degeneracy through the formula

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i(m-m')\phi} = \delta_{mm'}.$$  

(3.14)

As the explicit functions of $k_{(\delta)}$ and $k_{(\delta+2)m}$, we employ the function

$$k_{(\delta)} = 1 + \delta \left( \ln \left( \frac{1-z}{1+z} \right) \right)^{1/2} \delta \in \mathbb{R},$$  

(3.15)

$$k_{(\delta+2)m} = z \left\{ 1 + \delta \left( \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right) \right) \right\},$$  

(3.16)

$$k_{(\delta+2)m} = (1-z^2)^{1/2} \left\{ 1 + \delta \left( \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right) + \frac{z}{1-z^2} \right) \right\} e^{\pm i\phi},$$

(3.17)

where $z = \cos \theta$. This choice satisfies the conditions (3.11)–(3.13), but is singular if $\delta \neq 0$. When $\delta = 0$, we have $k_{(\delta)} \sim Y_{00}$ and $k_{(\delta+2)m} \sim Y_{1m}$.

Using the above harmonics functions $S^z$ in Eq. (3.5), we propose the following strategy:

**Proposal 3.1.** We decompose the metric perturbations $h_{ab}$ on the background spacetime $\delta$ into the mode-by-mode analyses including $l = 0, 1$ modes. After deriving the field equations such as linearized Einstein equations using the harmonic function $S^z$, we choose $\delta = 0$ when we solve these field equations as the regularity of the solutions.

Through this strategy, we can construct gauge-invariant variables and evaluate the field equations through the mode-by-mode analyses including $l = 0, 1$ modes.

### 4. Construction of gauge-invariant variables for linear perturbations

Here, we consider the gauge-transformation rule

$$\delta h_{ab} - \delta h_{ab} = \xi_{(ab)} + 2\nabla_a \xi_b,$$  

(4.1)

for the linear-order perturbations on a spherically symmetric background with the metric (3.1).

We rewrite this gauge-transformation rule by the mode-decomposition of the generator

$$\xi_{(a)} = \xi_{(a)} (dx^1)_a + \xi_{(a)} (dx^r)_a,$$

(4.2)

$$\xi_{(a)} = \sum_{l,m} \xi_{(a)} S^z_{lm},$$

(4.3)

From the mode-decomposition (3.5)–(3.6), the mode-by-mode components of the gauge-transformation rule (4.1) including $l = 0, 1$ modes are summarized as follows:

$$\delta h_{(a)} = \delta h_{(a)} = r D_a \left( \frac{1}{r} \xi_{(a)} S^z_{lm} \right), \quad \delta h_{(a)} = - \frac{1}{r} \xi_{(a)} S^z_{lm},$$

(4.4)

$$\delta h_{AB} = \delta h_{AB} = 2 D_{(A} \xi_{B)},$$

(4.5)

$$\delta h_{(a)} = \delta h_{(a)} = \frac{1}{r} \xi_{(a)} + r D_a \left( \frac{1}{r} \xi_{(a)} S^z_{lm} \right), \quad \delta h_{(a)} = - \frac{2}{r} \xi_{(a)} S^z_{lm},$$

(4.6)

$$\delta h_{(a)} = - \delta h_{(a)} = - \frac{2}{r} (l+1) \xi_{(a)} + \frac{4}{r} D^A r \xi_{(a)}.$$  

(4.7)
where $\bar{D}_A$ is the covariant derivative associated with the metric $\gamma_{AB}$ on $\mathcal{M}_1$. The perturbations $\bar{h}_{AB}$, $\bar{h}_{(e_1)A}$, $\bar{h}_{(e_0)}$, and $\bar{h}_{(e_2)}$ and the generator $\zeta_A$ and $\zeta_{(1)}$ are called even modes, and the perturbations $\bar{h}_{(o_1)A}$, $\bar{h}_{(o_2)}$, and $\zeta_{(o_1)}$ are called odd modes. These even- and odd-mode perturbations are independent of each other, and we may treat them separately.

Inspecting gauge-transformation rules (4.4), for the odd mode, gauge-invariant variable $\tilde{F}_A$ and gauge-variant variable $\tilde{Y}_{(o_2)}$ are defined by

$$\tilde{F}_A := \bar{h}_{(o_1)A} + r\bar{D}_A\bar{h}_{(o_2)} \quad \tilde{Y}_{(o_2)} := -r^2\bar{h}_{(o_2)} \quad \forall \tilde{Y}_{(o_2)} - \mathcal{A} \tilde{Y}_{(o_2)} = r\zeta_{(o_1)}.$$  (4.8)

For even-mode perturbations, we first define the gauge-invariant variable $\tilde{F}_A$ by

$$\tilde{Y}_{(e_2)} := \frac{r^2}{2}\bar{h}_{(e_2)}, \quad \forall \tilde{Y}_{(e_2)} - \mathcal{A} \tilde{Y}_{(e_2)} = r\zeta_{(e_1)}$$  (4.9)

from the second equation in Eq. (4.6). Further, Eq. (4.6) leads to the following definition of $\tilde{Y}_A$:

$$\tilde{Y}_A := r\bar{h}_{(e_1)A} - \frac{r^2}{2}\bar{D}_A\bar{h}_{(e_2)}, \quad \forall \tilde{Y}_A - \mathcal{A} \tilde{Y}_A = \zeta_A.$$  (4.10)

From the gauge-transformation rules (4.5), (4.7), (4.9), and (4.10), we define the two gauge-invariant variables $\tilde{F}_{AB}$ and $F$ as

$$\tilde{F}_{AB} := \bar{h}_{AB} - 2\bar{D}_A\bar{Y}_{B}, \quad \tilde{F} := \bar{h}_{(e_0)} - \frac{4}{r}\bar{Y}_{A}\bar{D}^A r + \frac{2}{r^2}\bar{Y}_{(e_2)} l(l + 1).$$  (4.11)

From the variables $\tilde{Y}_{(o_2)}, \tilde{Y}_{(e_2)}$, and $\tilde{Y}_A$, which are defined by Eqs. (4.8), (4.9), and (4.10), respectively, we introduce the vector field $Y_a$ through

$$Y_a := Y_a dx^A A_a, \quad Y_A := \sum_{l,m} \tilde{Y}_A S_A,$$

$$Y_p := \sum_{l,m} \tilde{Y}_{(e_2)} D_p S_A + \tilde{Y}_{(e_2)} E_{pr} \mathcal{D}^r S_A.$$  (4.12)

Here, the gauge-transformation rule of the vector field $Y_a$ is given by

$$\forall Y_a - \mathcal{A} Y_a = \sum_{l,m} \zeta_A S_A (dx^A) A_a + r \sum_{l,m} \zeta_{(1)} D_p S_A + \zeta_{(o_1)} E_{pr} \mathcal{D}^r S_A (dx^A) A_a = \xi_a.$$  (4.14)

We also introduce the gauge-invariant variables $F_{AB}, F_{AP},$ and $F$ by

$$\sum_{l,m} \tilde{F}_{AB} S_A, \quad \sum_{l,m} \tilde{F}_{AP} E_{pq} \mathcal{D}^q S_A, \quad F := \sum_{l,m} \tilde{F} S_A.$$  (4.15)

In terms of the variables (4.13) and (4.15), the original components (3.5–3.6) of the metric perturbations are given by

$$h_{AB} = F_{AB} + 2\bar{D}_A Y_B, \quad h_{AP} = rF_{AP} + \bar{D}_p Y_A + \bar{D}_A Y_P - \frac{2}{r}(\bar{D}_A r) Y_P,$$  (4.16)

$$h_{pq} = \frac{1}{2} \gamma_{pq} F + 2r(\bar{D}^A r) Y_P A + 2\bar{D}_p (r Y_q).$$  (4.17)

The components of the gauge-invariant metric perturbation $F_{ab}$ are identified as

$$\mathcal{F}_{AB} := F_{AB}, \quad \mathcal{F}_{AP} := rF_{AP}, \quad \mathcal{F}_{pq} := \frac{1}{2} \gamma_{pq} F.$$  (4.18)

The expressions (4.10–4.18) are summarized as

$$h_{ab} = \mathcal{F}_{ab} + \xi_{Y_{ab}}.$$  (4.19)

Note that the above arguments include not only the $l \geq 2$ mode but also $l = 0, 1$ modes of metric perturbations. Equation (4.19) is a complete proof of the Conjecture (2.1) for the perturbations on the spherically symmetric background spacetime. Therefore, the general arguments in our gauge-invariant perturbation theory are applicable to perturbations on the Schwarzschild background spacetime without special treatment of $l = 0, 1$ modes. We have therefore resolved the zero-mode problem in the perturbations on the Schwarzschild background spacetime.
5. \( l = 0, 1 \) solutions to the linearized Einstein equations

As shown in Sec. 2, the linearized Einstein tensor \( (1)G_{ab} \) for the linear metric perturbation in the form (4.19) with the background Einstein equation \( G_{ab} = 0 \) is given by \( (1)G_{ab} = (1)g_{ab} [\mathcal{F}] \) and Eqs. (2.9)–(2.10). Herein, we consider the linearized non-vacuum Einstein equation (2.11). Because the background spacetime is the vacuum solution, the first-order perturbations \( (1)T_{ac} \) and \( (1)\mathcal{F}_a \) of the energy-momentum tensor are automatically gauge-invariant as shown in Eq. (2.8). We then have the gauge-invariant part of the first-order perturbation of the energy-momentum tensor by \( (1)\mathcal{F}_{ac} := (1)T_{ac} \) and \( (1)\mathcal{F}_a := (1)T_{a0} = \mathcal{F}^{b(1)} \mathcal{F}_{ab} \). Furthermore, because we only consider the perturbations on the vacuum background solution based on the conventional general relativity, the linear-metric perturbation \( h_{ab} \) is not included in \( (1)\mathcal{F}_{ac} \) or \( (1)\mathcal{F}_a \).

The total energy-momentum tensor satisfies the continuity equation on the physical spacetime, which is pulled back to the background spacetime. Herein, we note that our background spacetime is the vacuum solution, and the first-order perturbation of this continuity equation is given by Eq. (2.16). We decompose the linear-order perturbation of the energy-momentum tensor \( (1)\mathcal{F}_{ac} \) and the components of this tensor is given by

\[
(1)\mathcal{F}_{ac} = \sum_{l,m} \tilde{T}_{ac}^l S_{d}^l (dx^A)_{a} (dx^C)_{c} + 2r \sum_{l,m} \left\{ \tilde{T}_{(e1)}^l \tilde{D}_p \tilde{S}_{d}^l + \tilde{T}_{(o1)}^l \epsilon_{pq} \tilde{D}_q \tilde{S}_{d}^l \right\} (dx^A)^{(a)} (dx^p)^{(c)}
\]

\[
+ r^2 \sum_{l,m} \left\{ \tilde{T}_{(e0)}^l \frac{1}{2} \gamma_{pq} \tilde{S}_{d}^l + \tilde{T}_{(e2)}^l \left( \tilde{D}_p \tilde{D}_q - \frac{1}{2} \gamma_{pq} \tilde{D}_r \tilde{D}^r \right) \tilde{S}_{d}^l \right\} (dx^A)^{(a)} (dx^p)^{(c)}.
\]

In terms of these mode-decomposition, the components of the linearized continuity equation (2.16) is summarized as follows:

\[
\tilde{D}^C \tilde{T}_{e1}^B + \frac{2}{r} (\tilde{D}^D r) \tilde{T}_{e1}^B - \frac{1}{r} (l + 1) \tilde{T}_{(e1)}^B - \frac{1}{r} (\tilde{D}^B r) \tilde{T}_{(e0)}^B = 0,
\]

\[
\tilde{D}^C \tilde{T}_{(e1)}^C + \frac{3}{r} (\tilde{D}^C r) \tilde{T}_{(e1)}^C + \frac{1}{2r} \tilde{T}_{(e0)}^C - \frac{1}{2r} (l - 1)(l + 2) \tilde{T}_{(e2)}^C = 0,
\]

\[
\tilde{D}^C \tilde{T}_{(o1)}^C + \frac{3}{r} (\tilde{D}^D r) \tilde{T}_{(o1)}^D + \frac{1}{r} (l - 1)(l + 2) \tilde{T}_{(o2)}^D = 0.
\]

Now, we consider the solutions to the Einstein equation for \( l = 0, 1 \) mode imposing the regularity of the harmonics \( S_{\delta} \) through the choice \( \delta = 0 \). We should note that the harmonics \( \tilde{D}_p \tilde{D}_q - \frac{1}{2} \gamma_{pq} \tilde{D}_r \tilde{D}^r \) \( \tilde{S}_{\delta} \) vanish for \( l = 0, 1 \) modes when \( \delta = 0 \). This indicates that the components \( \tilde{T}_{(e2)} \) and \( \tilde{T}_{(o2)} \) in Eq. (5.1) do not appear in the \( l = 0, 1 \) modes. We may therefore safely choose \( \tilde{T}_{(e2)} = 0 \) and \( \tilde{T}_{(o2)} = 0 \) for \( l = 1 \), 0 modes. In addition, we also note that harmonics \( \tilde{D}_p S_{\delta} \) and \( \epsilon_{pq} \tilde{D}_q S_{\delta} \) vanish for \( l = 0 \) mode when \( \delta = 0 \). This indicates that the components \( \tilde{T}_{(e1)}^A \) and \( \tilde{T}_{(o1)}^A \) in Eq. (5.1) do not appear in \( l = 0 \) mode, and we may also choose \( \tilde{T}_{(e1)}^A = 0 \) and \( \tilde{T}_{(o1)}^A = 0 \) for \( l = 0 \) mode. Through this choice and Eq. (5.3), we should regard \( \tilde{T}_{(e0)} = 0 \) for \( l = 0 \) mode.

5.1. \( l = 1 \) odd mode perturbations

If we impose the regularity on the harmonics \( S_{\delta} \) by choosing \( \delta = 0 \), there is no \( l = 0 \) mode in the odd-mode perturbation. We therefore can concentrate only on \( l = 1 \) mode. Furthermore, we only consider \( m = 0 \) mode because the generalizations to \( m = \pm 1 \) modes are straightforward. To evaluate the \( l = 1 \) odd-mode solutions, we introduce the components
Eqs. (5.7) are integrated as
\[ rF_A(dx^A)_a = X_{(o)}(dt)_a + r^2 \partial_r W_{(o)}(dr)_a. \] (5.5)

Furthermore, it is convenient to introduce the functions \( a_1(t, r) \) by
\[ \frac{6M}{r^2} a_1(t, r) := \partial_t \left( \frac{1}{r^2} X_{(o)} \right) - \partial_t \partial_r W_{(o)}. \] (5.6)

Using the variables \( a_1(t, r) \) and \( r f \partial_r W_{(o)} \), the Einstein equations for the \( l = 1 \) odd-mode perturbations are summarized as
\[ \partial_a a_1(t, r) = -\frac{16\pi}{3M} r^3 f \tilde{T}_{(o)tr}, \quad \partial_t a_1(t, r) = -\frac{16\pi}{3M} r^3 f \tilde{T}_{(o)tr}, \] (5.7)
\[ + \partial_t^2 (rf \partial_r W_{(o)}) - f \partial_t (f \partial_r (rf \partial_r W_{(o)})) + \frac{1}{r^2} f [2 - 3 (1 - f)] (rf \partial_r W_{(o)}) = 16\pi f \left( + f \tilde{T}_{(o)tr} + r f \tilde{D}_r \tilde{T}_{(o)tr} + (1 - f) \tilde{T}_{(o)tr} \right). \] (5.8)

The integrability condition of Eqs. (5.7) is guaranteed by Eq. (5.4) with \( \tilde{T}_{(o)tr} = 0 \). Therefore, Eqs. (5.7) are integrated as
\[ a_1(t, r) = -\frac{16\pi}{3M} r^3 f \int dt \tilde{T}_{(o)tr} + a_{10} = -\frac{16\pi}{3M} \int dr r^3 f \tilde{T}_{(o)tr} + a_{10}. \] (5.9)

where \( a_{10} \) is a constant. By contrast, Eq. (5.8) has the form of the Regge-Wheeler equation [6] with \( l = 1 \) for the variable \( r f \partial_r W_{(o)} \), although the original Regge-Wheeler equation is valid only in the case of \( l \geq 2 \). From Eq. (5.9) and the solution to Eq. (5.8), the component \( X_{(o)} \) of \( r \tilde{F}_A \) is obtained by the integration of Eq. (5.6), and the explicit odd-mode solution is given by
\[ 2 \mathcal{F}_A (dx^A)_{(o)(dx^B)_b} = \left( 6Mr^2 \int dr \frac{1}{r^4} a_1(t, r) \right) \sin^2 \theta (dt)_{(a)(d\phi)_b} + r \mathcal{V}_{(o)a} g_{ab}, \] (5.10)
\[ V_{(o)a} = (\beta(t) + W_{(o)}(t, r)) r^2 \sin^2 \theta (d\phi)_a, \] (5.11)
where \( \beta(t) \) is an arbitrary function of \( t \). In the vacuum case where \( \tilde{T}_{(o)tr} = \tilde{T}_{(o)tr} = 0 \), the function \( a_1(t, r) \) becomes constant \( a_{10} \). In this case, (5.10) is the linearized Kerr solution with the Kerr parameter \( a = a_{10} \) on the Schwarzschild background spacetime, where \( a \) is the total angular momentum of the spacetime per mass [22]. Also note that the vector field \( V_{(o)a} \) and \( \mathcal{V}_{(o)a} g_{ab} \) are gauge invariant.

5.2. \( l = 0, 1 \) even mode perturbations

Because the component \( \tilde{T}_{(e)tt} \) of the energy-momentum tensor vanishes in both \( l = 0, 1 \) modes, one of the Einstein equations yields
\[ \tilde{F}_D^{\parallel} = 0. \] (5.12)

We may then regard that the tensor \( \tilde{F}_{AB} \) is traceless in \( l = 0, 1 \) modes. Furthermore, we introduce the components of \( \tilde{F}_D \) by
\[ \tilde{F}_A (dx^A)_a (dx^B)_b : = X_{(e)} \left\{- f (dt)_a (dt)_b - f^{-1} (dr)_a (dr)_b \right\} + 2Y_{(e)} (dt)_a (dr)_b. \] (5.13)

In terms of the components \( X_{(e)} \) and \( Y_{(e)} \), the linearized Einstein equations yield the initial value constraints
\[ - \partial_t X_{(e)} + f \partial_r Y_{(e)} + \frac{1 - f}{r} Y_{(e)} - \frac{1}{2} \partial_r F = 16\pi r \tilde{T}_{(e)tr}, \] (5.14)
\[ - \partial_t Y_{(e)} - f \partial_r X_{(e)} + \frac{1 - f}{r} X_{(e)} - \frac{1}{2} \partial_r F = 16\pi r f \tilde{T}_{(e)tr}. \] (5.15)
The other Einstein equations are three evolution equations for the variables \(X(e),Y(e),\) and \(\tilde{F}.\)

To evaluate these evolution equations, it is convenient to introduce the Moncrief variable \(\Phi(e)\) using

\[
\Phi(e) := \frac{r}{\Lambda} \left[ fX(e) - \frac{1}{4} \Lambda \tilde{F} + \frac{1}{2} rf \partial_r \tilde{F} \right], \quad \Lambda := (l - 1)(l + 2) + 3(1 - f). \tag{5.16}
\]

Furthermore, with these evolution equations and the above constraints \(5.14\) and \(5.15\), we have

\[
l(l + 1)Y(e) = \frac{2\Lambda}{f} \partial_t \Phi(e) + \frac{\Lambda + 3f - 1}{2f} r \partial_r \tilde{F} + 16\pi r^2 T_{tr}, \tag{5.17}
\]

\[
l(l + 1)\tilde{F} = -8f \Lambda \partial_t \Phi(e) + \frac{4}{r} \left[ 6f(1 - f) - l(l + 1)\Lambda \right] \Phi(e) - 64\pi r^2 T_{tt}, \tag{5.18}
\]

and the evolution equations

\[
-\frac{1}{f} \partial_t^2 \Phi(e) + \partial_r \left[ f \partial_r \Phi(e) \right] - V_{\text{even}} \Phi(e) = +16\pi \frac{r}{\Lambda} S(\Phi(e)), \tag{5.19}
\]

\[
V_{\text{even}} := \frac{1}{r^2 \Lambda^2} \left\{ \Lambda^2 \left[ \Lambda - 2(2 - 3f) \right] + 6(1 - f) \left[ (1 - 3f) \Lambda + 3f(1 - f) \right] \right\}, \tag{5.20}
\]

\[
S(\Phi(e)) := \left( \frac{\Lambda}{4f} + \frac{1}{2} \right) \tilde{T}_{tt} - \frac{1}{2} r \partial_r \tilde{T}_{tt} - \frac{3(1 - f)}{\Lambda} \tilde{F} + \left( \frac{2f - \frac{1}{2}}{4} \right) f \tilde{T}_{tr} + \frac{1}{2} f^2 r \partial_r \tilde{T}_{tr} - \frac{f}{2} \tilde{T}_{(t)(e) r} - l(l + 1)f \tilde{T}_{(t)(e) r}. \tag{5.21}
\]

Equation \((5.19)\) is the Zerilli equation. Furthermore, the evolution equation for \(\tilde{F}\) given by

\[
-\frac{1}{f} \partial_t^2 \tilde{F} + \partial_r (f \partial_r \tilde{F}) = \frac{1}{r^2} 3(1 - f) \tilde{F} + \frac{4}{r^3} \Lambda \Phi(e) - \left[ \frac{1}{f} \tilde{T}_{tt} + f \tilde{T}_{tr} + 4f \tilde{T}_{(t)(e) r} \right] \tag{5.22}
\]

is also useful. We can check the consistency of Eqs. \((5.14)-(5.22)\) based on the continuity equations \((5.2)\), \((5.3)\), and \(\tilde{T}_{(t)(e) r} = 0\) of the energy-momentum tensor.

### 5.2. 1. \(l = 0\) mode solutions

In the \(l = 0\) case, Eqs. \((5.17)\) and \((5.18)\) do not yield the variables \(Y(e)\) or \(\tilde{F}\) as solutions. Instead, introducing the variable

\[
m_1(t,r) := -\frac{1}{2} (1 - 3f) \Phi(e), \tag{5.23}
\]

which corresponds to the mass perturbations, these equations are given by

\[
\partial_t m_1(t,r) = 4\pi r^2 f \tilde{T}_{tr}, \quad \partial_r m_1(t,r) = 4\pi r^2 \frac{1}{f} \tilde{T}_{tt}. \tag{5.24}
\]

The integrability of Eqs. \((5.24)\) is guaranteed through the \(t\)-component of the continuity equation \((5.2)\). Furthermore, the \(l = 0\) mode version of Eq. \((5.19)\) with the potential \((5.20)\) and the source term \((5.21)\) is trivial. Actually, Eqs. \((5.24)\) are integrated as

\[
m_1(t,r) = 4\pi \int dr \left( \frac{r^2}{f} \tilde{T}_{tt} \right) + M_1 = 4\pi \int dt \left( r^2 f \tilde{T}_{tr} \right) + M_1, \quad M_1 \in \mathbb{R}. \tag{5.25}
\]

The solution \((5.23)\) to Eqs. \((5.24)\) gives the variable \(\Phi(e)\) as a solution to the Einstein equation through Eq. \((5.23)\). The variable \(\tilde{F}\) is obtained as a solution to Eq. \((5.22)\) with the solution \(\Phi(e)\). From \((\tilde{F},\Phi(e))\), the variable \(X(e)\) is given as a solution to the Einstein equation through Eq. \((5.16)\). Through the solution \((\tilde{F},X(e))\), the variable \(Y(e)\) is obtained from the constraints \((5.14)\) and \((5.15)\).
Eqs. (5.19) and (5.22), and Eqs. (5.14) and (5.15) are integrated as follows:

Herein, we note that the vector field

The integrability condition of Eqs. (5.14) and (5.15) for the

Using this solution (5.26), we obtain

where we defined the vector field

Note that Eq. (5.22) has the same form as the inhomogeneous version of the Regge-Wheeler equation with \( l = 0 \), although the original Regge-Wheeler equation is valid only in the case \( l \geq 2 \). We denote the solution to Eq. (5.22) as

\[ F = \partial_\tau Y. \]  \hfill (5.26)

Using this solution (5.26), we obtain

The integrability condition of Eqs. (5.14) and (5.15) for the \( l = 0 \) mode is guaranteed through Eqs. (5.19) and (5.22), and Eqs. (5.14) and (5.15) are integrated as follows:

\[ Y_{(e)} = -\frac{2}{r^2} \int dt m_1(t, r) + \frac{3}{4r} (1 - f) \partial_t Y - \frac{1}{4} (1 - 3f) \partial_r Y + \frac{1}{2} r \partial_\tau (f \partial_r Y) \]

\[ + 8\pi r \int dt \tilde{T}_t + \frac{1}{2} r \zeta(r) - \frac{1}{4f} \int dr (1 - 3f) \zeta(r) + \frac{\xi}{T}, \]  \hfill (5.28)

where \( \zeta(r) \) is an arbitrary function of \( r \), and \( \xi \) is an arbitrary constant. Substituting Eqs. (5.27) and (5.28) into (5.13) using (4.15), and (4.18), we obtain

\[ \mathcal{F}_{ab} = \frac{2}{r} \left( M_1 + 4\pi \int dr \left[ \frac{r^2}{T} \tilde{T}_t \right] \right) \left( (dt)_a (dt)_b + \frac{1}{T^2} (dr)_a (dr)_b \right) \]

\[ + 2 \left[ 4\pi r \int dt \left( \frac{1}{T} \tilde{T}_t + f \tilde{T}_r \right) \right] (dt)_a (dr)_b \]

\[ + \xi \gamma_{(e)} g_{ab}, \]  \hfill (5.29)

where we defined the vector field \( V_{(e)ab} \) and an arbitrary function \( \gamma(r) \) by

\[ V_{(e)ab} := \left( \frac{1}{4} f Y + \frac{1}{4} r f \partial_r Y + \gamma(r) \right) (dt)_a + \frac{1}{4f} r \partial_\tau (dr)_a, \]

\[ \gamma(r) := f \int dr \left[ \frac{1}{2} r \zeta(r) - \frac{1}{4f^2} \int dr (1 - 3f) \zeta(r) + \frac{\xi}{T} \right]. \]  \hfill (5.31)

Herein, we note that the vector field \( V_a \) defined by Eq. (5.30) is gauge invariant.

5.2.2. \( l = 1 \) mode solutions \hspace{1cm} In the \( l = 1 \) even-mode case, we obtain the components \( \tilde{F}, Y_{(e)}, \) and \( X_{(e)} \) through Eqs. (5.18), (5.17), and (5.16) if we obtain the variable \( \Phi_{(e)} \) as a solution to the linearized Einstein equations. The variable \( \Phi_{(e)} \) is determined by the master equation (5.19) with the appropriate boundary conditions. From Eqs. (5.12), (5.13), (4.15), and (4.18), we obtain the gauge-invariant metric perturbation \( \mathcal{F}_{ab} \). Herein, we only consider the \( m = 0 \) mode in the \( l = 1 \) mode perturbations. We then obtain

\[ \mathcal{F}_{ab} = -\frac{16\pi r^2 f^2}{3(1-f)} \left[ \frac{1}{2} \tilde{T}_{rr} + r f \partial_r \tilde{T}_t - \tilde{\Phi}_{(e)1} - 4\tilde{T}_{(e)1} \right] \cos \theta (dt)_a (dt)_b \]

\[ + 16\pi^2 \left\{ \tilde{T}_{rr} - \frac{2r}{3f(1-f)} \partial_\tau \tilde{T}_t \right\} \cos \theta (dt)_a (dr)_b \]

\[ + 8\pi^2 \left[ \int (1 - 3f) \right] \left[ \tilde{T}_t - \frac{2rf}{3f(1-f)} \partial_\tau \tilde{T}_t \right] \cos \theta (dr)_a (dr)_b \]

\[ - \frac{16\pi^4}{3f(1-f)} \tilde{T}_t \cos \theta \gamma_{ab} + \xi \gamma_{(e)} g_{ab}, \]  \hfill (5.32)
where
\begin{equation}
V_{(e1)a} := -r\partial_t\Phi_{(e)}\cos\theta (dt)_a + (\Phi_{(e)} - r\partial_r\Phi_{(e)}) \cos\theta (dr)_a \\
- r\Phi_{(e)} \sin\theta (d\theta)_a.
\end{equation}
\tag{5.33}

Here, the vector field \(V_{(e1)a}\) is also gauge invariant. The components of the energy-momentum tensor in Eq. (5.32) satisfy the linear perturbations (5.2) and (5.3). We also note that there may exist an additional gauge-invariant term that has the form of the Lie derivative of the background metric in addition to the term \(\varepsilon V_{(e1)} g_{ab}\) in Eq. (5.32). This depends on the equation of state of the perturbation of the energy-momentum tensor.

6. Summary and Discussions

To summarize, we proposed a gauge-invariant treatment of the \(l = 0, 1\)-mode perturbations on the Schwarzschild background spacetime. Instead of the spherical harmonics \(Y_{lm}\) with \(l = 0, 1\), we used the mode functions \(k_{(\delta)}\) and \(k_{(\delta+2)m}\) with the parameter \(\delta\). These functions are the kernel mode of the derivative operators \(\hat{\Delta}\) and \(\hat{\Delta} + 2\), respectively. We choose the parameter \(\delta\) such that the choice \(\delta = 0\) realizes the usual spherical harmonics \(Y_{lm}\), and examined the linear independence of the scalar harmonic functions \(S_\delta\) defined by (3.8), the vector harmonics \(d_\alpha S_\delta\) and \(e_\alpha S_\delta\), and the tensor harmonics \(\frac{1}{2}d_\alpha d_\beta S_\delta\), \(\frac{1}{2}e_\alpha e_\beta S_\delta\), \(\frac{1}{2}d_\alpha e_\beta S_\delta\), \(\frac{1}{2}e_\alpha d_\beta S_\delta\), \(\frac{1}{2}e_\alpha e_\beta S_\delta\), \(\frac{1}{2}d_\alpha d_\beta S_\delta\), \(\frac{1}{2}e_\alpha e_\beta S_\delta\), \(\frac{1}{2}d_\alpha e_\beta S_\delta\), \(\frac{1}{2}e_\alpha d_\beta S_\delta\), \(\frac{1}{2}e_\alpha e_\beta S_\delta\), \(\frac{1}{2}d_\alpha d_\beta S_\delta\), and \(\frac{1}{2}e_\alpha e_\beta S_\delta\). We thus proposed Proposal 3.1 as a strategy of a gauge-invariant treatment of the \(l = 0, 1\) perturbations on the Schwarzschild background spacetime. Following this proposal, we derived the \(l = 0, 1\) mode solutions to the Einstein equations with the general linear perturbations of the energy-momentum tensor in the gauge-invariant manner. Herein, it is assumed that these general linear perturbations of the energy-momentum tensor satisfy the linear perturbations of the divergence of the energy-momentum tensor.

The derived solution in the \(l = 1\) odd mode actually realizes the linearized Kerr solution in the vacuum case. Apart from the term described as the Lie derivative of the background metric, the unique solution in the odd-mode vacuum perturbation case is the Kerr parameter perturbation. Furthermore, we also derived the \(l = 0, 1\) even-mode solutions to the Einstein equations. In the vacuum case, in which all components of \((1)\mathcal{F}_{ab}\) vanish, a \(l = 0\) even-mode solution realizes the only the additional mass parameter perturbation of the Schwarzschild spacetime, apart from the terms described by the Lie derivative of the background metric. This is the realization of the linearized gauge-invariant version of Birkhoff’s theorem. Owing to this realization, we can state that our proposal is physically reasonable. Herein, we note that the terms described by the Lie derivative of the background spacetime in Eq. (5.29) is necessary if we include the Schwarzschild mass perturbation \(M_1\) as the solution to the linearized Einstein equations. Actually, if we choose \(\Upsilon = 0\), the mass parameter \(M_1\) must vanish from Eq. (5.29).

Also note that all \(l = 1\) odd-mode and \(l = 0, 1\) even-mode solutions include the term which is described by the Lie derivative of the background metric. Because the definitions of gauge-invariant and gauge-variant variables are not unique, as explained through Eq. (2.17) in Sec. 2, such terms may appear in our perturbation theory. Furthermore, although these terms can be eliminated through the gauge-fixing method at any time, they are gauge-invariant, i.e., they are invariant under the change of point-identifications between the physical spacetime \(\mathcal{M}_{ph}\) and the background spacetime \(\mathcal{M}\). The gauge-invariance of these terms implies that the point-identification between the physical spacetime \(\mathcal{M}_{ph}\) and the background spacetime \(\mathcal{M}\) are already fixed. Therefore, these terms may have a physical meaning. As an example, the function \(\beta(t)\) in Eq. (5.11) can be produced by the infinitesimal coordinate transformation.
\phi \rightarrow \phi + \omega(t) t \) in the background metric (3.1) with Eqs. (3.2) and (3.3). Because this function \( \beta(t) \) is gauge-invariant, the coordinate transformation \( \phi \rightarrow \phi + \omega(t) t \) should not be regarded as the coordinate transformation (2.18) induced by the gauge-transformation \( \Phi_\lambda \) described in Sec. 2 but should be regarded as the coordinate transformation within the background spacetime, i.e., the first-kind gauge transformation on the background spacetime. Because this point-identification is already fixed owing to the gauge-invariance of this term, the coordinate transformation on the physical spacetime \( \mathcal{M}_{\text{ph}} \) is also regarded as the coordinate transformation on the physical spacetime \( \mathcal{M}_{\text{ph}} \). Because the coordinate transformation \( \phi \rightarrow \phi + \omega(t) t \) describes the rotation of the Universe, it is regarded as the coordinate transformation into a non-inertia frame and the function \( \beta(t) \) represents an inertia force which appears as a property of the physical spacetime \( \mathcal{M}_{\text{ph}} \). This will be a physical meaning of the function \( \beta(t) \) in Eq. (5.11). Of course, it is highly non-trivial that all terms in the \( l = 1 \) odd-mode and \( l = 0, 1 \) even-mode solutions that have the form of the Lie derivative of the background metric can be interpreted in a similar manner as this function \( \beta(t) \), which remains an open question.

Finally, we should emphasize that we confirmed Conjecture 2.1 for the linear-metric perturbations in the Schwarzschild background case, including the \( l = 0, 1 \) modes. Because these are zero modes in Refs. [14], we resolved the zero-mode problem for the perturbations on the Schwarzschild background spacetime. Conjecture 2.1 is important and is the only non-trivial premise of our general framework of the gauge-invariant higher-order perturbation theory. For this reason, in principle, the extension to any-order perturbations through our gauge-invariant formulation [15] is possible, at least in the Schwarzschild background case. Although a short discussion was already given in our companion paper [23], we will discuss this extension to the higher-order perturbation elsewhere.

Acknowledgements

The author would like to thank Prof. Shuhei Mano for the valuable comments and discussions. The author also deeply acknowledges Prof. Hiroyuki Nakano for various discussions and suggestions during the past 20 years.

References

[1] B. P. Abbot et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116 (2016), 061102.
[2] LIGO Scientific Collaboration home page: https://ligo.org
[3] Virgo home page: https://www.virgo-gw.eu
KAGRA home page: https://gwcenter.icrr.u-tokyo.ac.jp
LIGO INDIA home page: https://www.ligo-india.in
[4] LISA home page: https://lisa.nasa.gov
[5] DECIGO home page: https://decigo.jp
[6] T. Regge and J. A. Wheeler, Phys. Rev. 108 (1957), 1063.
[7] F. Zerilli, Phys. Rev. Lett. 24 (1970), 737;
F. Zerilli, Phys. Rev. D a2 (1970), 2141;
H. Nakano, Private note on “Regge-Wheeler-Zerilli formalism” (2019).
[8] V. Moncrief, Ann. Phys. (N.Y.) 88 (1974), 323;
V. Moncrief, Ann. Phys. (N.Y.) 88 (1974), 343;
C. T. Cunningham, R. H. Price, and V. Moncrief, Astrophys. J. 224 (1978), 643;
S. Chandrasekhar, The mathematical theory of black holes (Oxford: Clarendon Press, 1983).
[9] U.H. Gerlach and U.K. Sengupta, Phys. Rev. D 19 (1979), 2268;
U.H. Gerlach and U.K. Sengupta, Phys. Rev. D 20 (1979), 3009;
U.H. Gerlach and U.K. Sengupta, J. Math. Phys. 20 (1979), 2540;
U.H. Gerlach and U.K. Sengupta, Phys. Rev. D 22 (1980), 1300.
[10] C. Gundlach and J.M. Martín-García, Phys. Rev. D61 (2000), 084024.
J.M. Martín-García and C. Gundlach, Phys. Rev. D64 (2001), 024012.
A. Nagar and L. Rezzolla, Class. Quantum Grav. 22 (2005), R167, Erratum ibid. 23 (2006), 4297;
K. Martel and E. Poisson, Phys. Rev. D 71 (2005), 104003.

[11] D. Brizuela, J. M. Martín-García, and G. A. Mena Marugán, Phys. Rev. D 76 (2007), 024004.

[12] K. Nakamura, Prog. Theor. Phys. 110 (2003), 723.

[13] K. Nakamura, Prog. Theor. Phys. 113 (2005), 481.

[14] K. Nakamura, Class. Quantum Grav. 28 (2011), 122001;
K. Nakamura, Int. J. Mod. Phys. D 21 (2012), 124004;
K. Nakamura, Prog. Theor. Exp. Phys. 2013 (2013), 043E02.

[15] K. Nakamura, Class. quantum Grav. 31, (2014), 135013.

[16] K. Nakamura, Phys. Rev. D 74 (2006), 101301(R);
K. Nakamura, Prog. Theor. Phys. 117 (2007), 17;
K. Nakamura, Bulgarian Journal of Physics 35 (2008), 489;
K. Nakamura, Prog. Theor. Phys. 121 (2009), 1321;
A. J. Christopherson, K. A. Malik, D. R. -Matravers, K. Nakamura, Class. Quantum Grav. 28 (2011), 225024.

[17] K. Nakamura, Phys. Rev. D 80 (2009), 124021.

[18] K. Nakamura, Advances in Astronomy, 2010 (2010), 576273;
K. Nakamura et al., “Theory and Applications of Physical Science vol.3,” (Book Publisher International, 2020),
DOI:10.9734/bpitas/v3. (Preprint arXiv:1912.12805).

[19] R. K. Sachs, “Gravitational radiation,” in Relativity, Groups and Topology, C. DeWitt and B. DeWitt, Eds.,
Gordon and Breach, New York, NY, USA, 1964;
J. M. Stewart and M. Walker, Proc. R. Soc. London A 311 (1974), 49;
J. M. Stewart, Class. Quantum Grav. 7 (1990), 1169;
J. M. Stewart, Advanced General Relativity (Cambridge University Press, Cambridge, 1991).

[20] M. Bruni, S. Matarrese, S. Mollerach and S. Sonego, Class. Quantum Grav. 14 (1997), 2585;
M. Bruni, S. Sonego, Class. Quantum Grav. 16 (1999), L29;
S. Matarrese, S. Mollerach and M. Bruni, Phys. Rev. D 58 (1998), 043504;
M. Bruni, L. Gualtieri and C. F. Sopuerta, Class. Quantum Grav. 20 (2003), 535;
S. Sonego and M. Bruni, Commun. Math. Phys. 193 (1998), 209;
C. F. Sopuerta, M. Bruni and L. Gualtieri, Phys. Rev. D 70 (2004), 064002.

[21] J. W. York, Jr., J. Math. Phys. 14 (1973), 456;
J. W. York, Jr., Ann. Inst. H. Poincaré 21 (1974), 319;
S. Deser, Ann. Inst. H. Poincaré 7 (1967), 149.

[22] R.M Wald, General Relativity (Chicago, IL: University of Chicago Press, 1984).

[23] K. Nakamura, Preprint arXiv:2102.10650.