On the Ambarzumyan’s theorem for the Quasi-periodic Problem

Alp Arslan Kiraç

Department of Mathematics, Faculty of Arts and Sciences, Pamukkale University, 20070, Denizli, Turkey

Abstract
We obtain the classical Ambarzumyan’s theorem for the Sturm-Liouville operators \( L_t(q) \) with \( q \in L^1[0, 1] \) and quasi-periodic boundary conditions, \( t \in [0, 2\pi) \), when there is not any additional condition on the potential \( q \).

Keywords: Ambarzumyan theorem; inverse spectral theory; Hill operator

1. Introduction

In this study we consider the Sturm-Liouville operator \( L_t(q) \) generated in the space \( L^2[0, 1] \) by the expression

\[-y'' + q(x)y\]  (1)

and the quasi-periodic boundary conditions

\[y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0),\]  (2)

where \( q \in L^1[0, 1] \) is a real-valued function and \( t \) is a fixed real number in \([0, 2\pi)\). Note that the operator \( L_t(q) \) is self-adjoint and the cases \( t = 0 \) and \( t = \pi \) correspond to the periodic and antiperiodic problems, respectively. Since the spectrum \( S(L(q)) \) of Hill operator \( L(q) \) generated in the space \( L^2(-\infty, \infty) \) by expression (1) with periodic potential \( q \) is the union of the spectra \( S(L_t(q)) \) of the operators \( L_t(q) \) for \( t \in [0, 2\pi) \) (e.g., see [1]), the operators \( L_t(q) \) have a fundamental role in the spectral theory of the operator.

Email address: aakirac@pau.edu.tr

Preprint submitted to Elsevier

March 9, 2015
In 1929, Ambarzumyan [2] obtained the following theorem considered as the first theorem in inverse spectral theory:

If \( \{ n^2 : n = 0, 1, \ldots \} \) is the spectrum of the Sturm-Liouville operator with Neumann boundary condition, then \( q = 0 \) a.e.

In [3], Chern and Shen proved Ambarzumyan’s theorem for the Sturm-Liouville differential systems with Neumann boundary conditions. Later, in [4], by imposing an additional condition on the potential they extended the classical Ambarzumyan’s theorem for the Sturm-Liouville equation to the general separated boundary conditions. See basics and further references in [5, 6].

At this point we refer in particular to [7, 8]. In [7], for the vectorial Sturm-Liouville problem under periodic or antiperiodic boundary conditions, Yang-Huang-Yang found two analogs of Ambarzumyan’s theorem. Their result supplements the Pöschel-Trubowitz inverse spectral theory [9]. More recently, Cheng-Wang-Wu [8] proved the following theorem:

\( (a) \) If all eigenvalues of the operator \( L_0(q) \) are nonnegative and they include \( \{ (2n\pi)^2 : n \in \mathbb{N} \} \), then \( q = 0 \) a.e.

\( (b) \) If all eigenvalues of the operator \( L_\pi(q) \) are not less than \( \pi^2 \) and they include \( \{ (2n\pi - \pi)^2 : n \in \mathbb{N} \} \), and

\[
\int_0^1 q(x) \cos(2\pi x) \, dx \geq 0,
\]

then \( q = 0 \) a.e.

The present work was stimulated by the papers [4, 8]. For the first time, we obtain Ambarzumyan’s theorem for the operator \( L_t(q) \) with \( t \in [0, 2\pi) \), generated by quasi-periodic boundary conditions [2]. The result established below show that the potential \( q \) can be determined from one spectrum and there is not any additional condition on \( q \) such as (3) for the operator \( L_t(q) \) with \( t = \pi \) (see also [4, 7]). The result of this paper is the following.

**Theorem 1.** If first eigenvalue of the operator \( L_t(q) \) for any fixed number \( t \) in \( [0, 2\pi) \) is not less than the value of \( \min \{ t^2, (2\pi - t)^2 \} \) and the spectrum \( S(L_t(q)) \) contains the set \( \{ (2n\pi - t)^2 : n \in \mathbb{N} \} \), then \( q = 0 \) a.e.

2. Preliminaries and Proof of the result

We now introduce some preliminary facts. In [10] (see also [? ]), without using the assumption \( q_0 = 0 \), they proved the following result:
The eigenvalues $\lambda_n(t)$ of the operator $L_t(q)$ for $q \in L^1[0,1]$ and $t \neq 0, \pi$, satisfy the following asymptotic formula

$$\lambda_n(t) = (2\pi n + t)^2 + q_0 + O\left(n^{-1} \ln|n|\right) \quad \text{as } |n| \to \infty,$$

(4)

where $q_n = (q, e^{i2\pi nx})$ for $n \in \mathbb{Z}$ and $(.,.)$ is the inner product in $L^2[0,1]$.

Note that when $q = 0$, $(2\pi n + t)^2$ for $n \in \mathbb{Z}$ is the eigenvalue of the operator $L_t(0)$ for any fixed $t \in [0,2\pi)$ corresponding to the eigenfunction $e^{i(2\pi n + t)x}$.

**Proof of Theorem 1.** Using the assumption that, for any $n \in \mathbb{N}$, $(2n\pi - t)^2$ belongs to the spectrum $S(L_t(q))$ and taking into account that, for sufficiently large $|n|$, the asymptotic formulas (4) for $t \neq 0, \pi$, and, in [8], (1.2)-(1.3) for $t = 0, \pi$ (see Theorem 1.1. of [8]), we obtain

$$q_0 = \int_0^1 q(x) \, dx = 0.$$  

(5)

Let us show that, for fixed $t \in [0,2\pi)$, the first eigenvalue of the operator $L_t(q)$ is either $t^2$ or $(2\pi - t)^2$ corresponding to the eigenfunctions $y = e^{itx}$ or $y = e^{i(-2\pi + t)x}$, respectively. First, suppose that the value of $\min\{t^2, (2\pi - t)^2\}$ is $t^2$. By the variational principle and (5), we have for $y = e^{itx}$

$$t^2 \leq \lambda_0(t) \leq \frac{\int_0^1 -yy'' \, dx + \int_0^1 q(x)|y|^2 \, dx}{(y, y)} = t^2 + q_0 = t^2.$$  

(6)

This implies that the first eigenvalue of the operator $L_t(q)$ is $\lambda_0(t) = t^2$ and the test function $y = e^{itx}$ is the first eigenfunction of the operator. Thus, Substituting the expressions $y = e^{itx}$ and $\lambda_0(t) = t^2$ into the equation

$$-y'' + q(x)y = \lambda y,$$

we get $q = 0$ in $L^1[0,1]$. Similarly, one can readily show that if the value of $\min\{t^2, (2\pi - t)^2\}$ is $(2\pi - t)^2$, then the function $y = e^{i(-2\pi + t)x}$ is the first eigenfunction corresponding to the first eigenvalue $(2\pi - t)^2$ and $q = 0$ in $L^1[0,1]$. \qed

**Remark 1.** Note that instead of the subset $\{(2n\pi - t)^2 : n \in \mathbb{N}\}$ of the spectrum $S(L_t(q))$ in Theorem 1 if we use either of the subsets

$\{(2n\pi + t)^2 : n \in \mathbb{N}\}$, $\{m^2 : m \text{ is either } (2n\pi - t) \text{ or } (2n\pi + t) \text{ for all } n \in \mathbb{N}\}$,

then the assertion of Theorem 1 remains valid.
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