On the Computational Complexity of Minimal Cumulative Cost Graph Pebbling

Jeremiah Blocki∗ Samson Zhou†

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Abstract

We consider the computational complexity of finding a legal black pebbling of a DAG $G = (V, E)$ with minimum cumulative cost. A black pebbling is a sequence $P_0, \ldots, P_t \subseteq V$ of sets of nodes which must satisfy the following properties: $P_0 = \emptyset$ (we start off with no pebbles on $G$), $\text{sinks}(G) \subseteq \bigcup_{j \leq t} P_j$ (every sink node was pebbled at some point) and $\text{parents}(P_{i+1} \setminus P_i) \subseteq P_i$ (we can only place a new pebble on a node $v$ if all of $v$’s parents had a pebble during the last round). The cumulative cost of a pebbling $P_0, P_1, \ldots, P_t \subseteq V$ is $cc(P) = |P_1| + \ldots + |P_t|$. The cumulative pebbling cost is an especially important security metric for data-independent memory hard functions, an important primitive for password hashing. Thus, an efficient (approximation) algorithm would be an invaluable tool for the cryptanalysis of password hash functions as it would provide an automated tool to establish tight bounds on the amortized space-time cost of computing the function. We show that such a tool is unlikely to exist. In particular, we prove the following results.

• It is $\text{NP-Hard}$ to find a pebbling minimizing cumulative cost.
• The natural linear program relaxation for the problem has integrality gap $\tilde{O}(n)$, where $n$ is the number of nodes in $G$. We conjecture that the problem is hard to approximate.
• We show that a related problem, find the minimum size subset $S \subseteq V$ such that $\text{depth}(G - S) \leq d$, is also $\text{NP-Hard}$. In fact, under the unique games conjecture there is no $(2 - \epsilon)$-approximation algorithm.

1 Introduction

Given a directed acyclic graph (DAG) $G = (V, E)$ the goal of the (parallel) black pebbling game is to place pebbles on all sink nodes of $G$ (not necessarily simultaneously). The game is played in rounds and we use $P_i \subseteq V$ to denote the set of currently pebbled nodes on round $i$. Initially all nodes are unpebbled, $P_0 = \emptyset$, and in each round $i \geq 1$ we may only include $v \in P_i$ if all of $v$’s parents were pebbled in the previous configuration ($\text{parents}(v) \subseteq P_{i-1}$) or if $v$ was already pebbled in the last round ($v \in P_{i-1}$). In the sequential pebbling game we can place at most one new pebble on the graph in any round (i.e., $|P_i \setminus P_{i-1}| \leq 1$), but in the parallel pebbling game no such restriction applies.

∗Department of Computer Science, Purdue University, West Lafayette, IN. Email: jblocki@purdue.edu.
†Department of Computer Science, Purdue University, West Lafayette, IN. Email: samsonzhou@gmail.com. Research supported by NSF CCF-1649515.
We define the cumulative cost (respectively space-time cost) of a pebbling $P = (P_1, \ldots, P_t) \in \mathcal{P}_G$ to be $cc(P) = |P_1| + \ldots + |P_t|$ (resp. $st(P) = t \times \max_{1 \leq i \leq t} |P_i|$), that is, the sum of the number of pebbles on the graph during every round. Here, $\mathcal{P}_G^\parallel$ (resp. $\mathcal{P}_G$) denotes the set of all valid parallel (resp. sequential) pebblings of $G$. The parallel cumulative pebbling cost of $G$, denoted $\Pi_{cc}^\parallel(G)$ (resp. $\Pi_{st}(G) = \min_{P \in \mathcal{P}_G} st(P)$), is the cumulative cost of the best legal pebbling of $G$. Formally,

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\Pi_{cc}^\parallel(G) = \min_{P \in \mathcal{P}_G^\parallel} cc(P), \quad \Pi_{st}(G) = \min_{P \in \mathcal{P}_G} st(P).
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In this paper, we consider the computational complexity of $\Pi_{cc}^\parallel(G)$, showing that the value is $\text{NP-Hard}$ to compute. We also provide evidence that $\Pi_{cc}^\parallel(G)$ is hard to approximate by demonstrating that the natural linear programming relaxation for the problem has a large integrality gap.

### 1.1 Motivation

The pebbling cost of a DAG $G$ is closely related to the cryptanalysis of data-independent memory hard functions (iMHFs) [AS15], a particularly useful primitive for password hashing [PHC, BDK16].

In particular, an efficient algorithm for (approximately) computing $\Pi_{cc}^\parallel(G)$ would enable us to automate the cryptanalysis of candidate iMHFs. The question is particularly timely as the Internet Research Task Force considers standardizing Argon2i [BDK16], the winner of the password hashing competition [PHC], despite recent attacks [CGBS16, AB16a, AB16b] on the construction. Despite recent progress [AB16b, ABP16] the precise security of Argon2i and alternative constructions is poorly understood.

An iMHF is defined by a DAG $G$ (modeling data-dependencies) on $n$ nodes $V = \{1, \ldots, n\}$ and a compression function $H$ (usually modeled as a random oracle in theoretical analysis). The label $\ell_1$ of the first node in the graph $G$ is simply the hash $H(x)$ of the input $x$. A vertex $i > 1$ with parents $i_1 < i_2 < \cdots < i_k$ has label $\ell_i(x) = H(i, \ell_{i_1}(x), \ldots, \ell_{i_k}(x))$. The output value is the label $\ell_n$ of the last node in $G$. It is easy to see that any legal pebbling of $G$ corresponds to an algorithm computing the corresponding iMHF. Placing a new pebble on node $i$ corresponds to computing the label $\ell_i$ and keeping (resp. discarding) a pebble on node $i$ corresponds to storing the label in memory (resp. freeing memory). Alwen and Serbinenko proved that in the parallel random oracle model (pROM) of computation, any algorithm evaluating such an iMHF could be reduced to a pebbling strategy with (approximately) the same cumulative memory cost.

It should be noted that any graph $G$ on $n$ nodes has a sequential pebbling strategy $P \in \mathcal{P}_G$ that finishes in $n$ rounds and has cost $cc(P) \leq st(P) \leq n^2$. Ideally, a good iMHF construction provides the guarantee that the amortized cost of computing the iMHF remains high (i.e., $\tilde{\Omega}(n^2)$) even if the adversary evaluates many instances (e.g., different password guesses) of the iMHF. Unfortunately, neither large $\Pi_{st}(G)$ nor large $\min_{P \in \mathcal{P}_G} st(P)$, are sufficient to guarantee that $\Pi_{cc}^\parallel(G)$ is large [AS15]. More recently Alwen and Blocki [AB16a] showed that Argon2i [BDK16], the winner of the recently completed password hashing competition [PHC], has much lower than desired amortized space-time complexity. In particular, $\Pi_{cc}^\parallel(G) \leq \tilde{O}(n^{1.75})$.

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1 Because the data-dependencies in an iMHF are specified by a static graph, the induced memory access pattern does not depend on the secret input (e.g., password). This makes iMHFs resistant to side-channel attacks. Data-dependent memory hard functions (MHFs) like scrypt [Per09] are potentially easier to construct, but they are potentially vulnerable to cache-timing attacks.
In the context of iMHFs, it is important to study $\Pi_{cc}(G)$, the cumulative pebbling cost of a graph $G$, in addition to $\Pi_{st}(G)$. Traditionally, pebbling strategies have been analyzed using space-time complexity or simply space complexity. While sequential space-time complexity may be a good model for the cost of computing a single-instance of an iMHF on a standard single-core machine (i.e., the costs incurred by the honest party during password authentication), it does not model the amortized costs of a (parallel) offline adversary who obtains a password hash value and would like to evaluate the hash function on many different inputs (e.g., password guesses) to crack the user’s password [AS15, AB16a]. Unlike $\Pi_{st}(G)$, $\Pi_{cc}(G)$ models the amortized cost of evaluating a data-independent memory hard function on many instances [AS15, AB16a].

An efficient algorithm to (approximately) compute $\Pi_{cc}(G)$ would be an incredible asset when developing and evaluating iMHFs. For example, the Argon2i designers argued that the Alwen-Blocki attack [AB16a] was not particularly effective for practical values of $n$ (e.g., $n \leq 2^{20}$) because the constant overhead was too high [BDK16]. However, they could not rule out the possibility that more efficient attacks might exist [2]. An efficient algorithm to (approximately) compute $\Pi_{cc}(G)$ would have allowed us to immediately resolve such debates by automatically generating upper/lower bounds on the cost of computing Argon2i for each running time parameters ($n$) that one might select in practice. Alwen et al. [ABP16] showed how to construct graphs $G$ with $\Pi_{cc}(G) = \Omega\left(\frac{n^2}{\log n}\right)$. This construction is essentially optimal in theory as results of Alwen and Blocki [AB16a] imply that any constant indegree graph has $\Pi_{cc}(G) = O\left(\frac{n^2 \log \log n}{\log n}\right)$. However, the exact constants one could obtain through a theoretical analysis are most-likely small. A proof that $\Pi_{cc}(G) \geq \frac{10^{-6} \times n^2}{\log n}$ would be an underwhelming theoretical analysis, where we may have $n \approx 10^6$. An efficient algorithm to compute $\Pi_{cc}(G)$ would allow us to immediately determine whether these new construction provide meaningful security guarantees in practice.

1.2 Results

Our primary contribution is a proof that the decision problem “is $\Pi_{cc}(G) \leq k$ for a positive integer $k \leq \frac{n(n+1)}{2}$ is NP-Complete” [3]. In fact, our result holds even if the DAG $G$ has constant indeg [3]. We also provide evidence that $\Pi_{cc}(G)$ is hard to approximate. Thus, it is unlikely that it will be possible to automate the cryptanalysis process for iMHF candidates. In particular, we define a natural integer program to compute $\Pi_{cc}(G)$ and consider its linear programming relaxation. We then show that the integrality gap is at least $\Omega\left(\frac{n}{\log n}\right)$ leading us to conjecture that it is hard to approximate $\Pi_{cc}(G)$. We also give an example of a DAG $G$ on $n$ nodes with the property that any optimal pebbling (minimizing $\Pi_{cc}$) requires more than $n$ pebbling rounds.

The computational complexity of several graph pebbling problems has been explored previously in various settings [GLT80, HP10]. However, we stress that the structure of a pebbling minimizing cumulative cost can be very different from the structure of a pebbling minimizing space-time cost or

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[2] Indeed, Alwen and Blocki [AB16a] subsequently introduced heuristics to improve their attack and demonstrated that their attacks were effective even for smaller (practical) values of $n$ by simulating their attack against real Argon2i instances.

[3] Note that for any $G$ with $n$ nodes we have $\Pi_{cc}(G) \leq 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ since we can always pebble $G$ in topological order in $n$ steps if we never remove pebbles.

[4] For practical reasons most iMHF candidates are based on a DAG $G$ with constant indegree.
space. Thus, we need fundamentally new ideas to construct appropriate gadgets for our reduction. We first introduce a new problem which we call Bounded 2-Linear Covering (B2LC) and show that it is NP-Complete by reduction from 3-PARTITION. We then show that we can encode a B2LC instance as a graph pebbling problem.

In Appendix B we also investigate the computational complexity of determining how “depth-reducible” a DAG \( G \) is showing that the problem is NP-Complete even if \( G \) has constant indegree. A DAG \( G \) is \((e, d)\)-reducible if there exists a subset \( S \subseteq V \) of size \( |S| \leq e \) such that \( \text{depth}(G - S) < d \). That is, after removing nodes in the set \( S \) from \( G \), any remaining directed path has length less than \( d \). If \( G \) is not \((e, d)\)-reducible, we say that it is \((e, d)\)-depth robust. It is known that a graph has high cumulative cost (e.g., \( \tilde{\Omega}(n^2) \)) if and only if the graph is highly depth robust (e.g., \( e, d = \tilde{\Omega}(n) \) \([AB16a, ABP16]\). Our reduction from Vertex Cover preserves approximation hardness. Thus, assuming that \( P \neq \text{NP} \) it is hard to 1.3-approximate \( e \), the minimum size of a set \( S \subseteq V \) such that \( \text{depth}(G - S) < d \) \([DS05]\). Under the Unique Games Conjecture \([Kho02]\), it is hard to \((2 - \varepsilon)\)-approximate \( e \) for any fixed \( \varepsilon > 0 \) \([KR08]\).

### 2 Preliminaries

Given a directed acyclic graph (DAG) \( G = (V, E) \) and a node \( v \in V \) we use \( \text{parents}(v) = \{u : (u, v) \in E\} \) to denote the set of nodes \( u \) with directed edges into node \( v \) and we use \( \text{indeg}(v) = |\text{parents}(v)| \) to denote the number of directed edges into node \( v \). We use \( \text{indeg}(G) = \max_{v \in V} \text{indeg}(v) \) to denote the maximum indegree of any node in \( G \). For convenience, we use \( \text{indeg} \) instead of \( \text{indeg}(G) \) when \( G \) is clear from context. We say that a node \( v \in V \) with \( \text{indeg}(v) = 0 \) is a source node and a node with no outgoing edges is a sink node. We use \( \text{sinks}(G) \) (resp. \( \text{sources}(G) \)) to denote the set of all sink nodes (resp. source nodes) in \( G \). We will use \( n = |V| \) to denote the number of nodes in a graph, and for convenience we will assume that the nodes \( V = \{1, 2, 3, \ldots, n\} \) are given in topological order (i.e., \( 1 \leq j < i \leq n \) implies that \( (i, j) \notin E \)). We use \( \text{depth}(G) \) to denote the length of the longest directed path in \( G \). Given a positive integer \( k \geq 1 \) we will use \( [k] = \{1, 2, \ldots, k\} \) to denote the set of all integers \( 1 \) to \( k \) (inclusive).

**Definition 1** Given a DAG \( G = (V, E) \) on \( n \) nodes a legal pebbling of \( G \) is a sequence of sets \( P = (P_0, \ldots, P_t) \) such that: (1) \( P_0 = \emptyset \), (2) \( \forall i > 0, v \in P_i \setminus P_{i-1} \) we have \( \text{parents}(v) \subseteq P_{i-1} \), and (3) \( \forall v \in \text{sinks}(G) \exists 0 < j \leq t \) such that \( v \in P_j \). The cumulative cost of the pebbling \( P \) is \( \text{cc}(P) = \sum_{i=1}^{t} |P_i| \), and the space-time cost is \( \text{st}(P) = t \times \max_{0 \leq i \leq t} |P_i| \).

The first condition states that we start with no pebbles on the graph. The second condition states that we can only add a new pebble on node \( v \) during round \( i \) if we already had pebbles on all of \( v \)'s parents during round \( i - 1 \). Finally, the last condition states that every sink node must have been pebbling during some round.

We use \( P_G^\parallel \) to denote the set of all legal pebblings, and we use \( P_G \subset P_G^\parallel \) to denote the set of all sequential pebblings with the additional requirement that \( |P_i \setminus P_{i-1}| \leq 1 \) (i.e., we place at most one new pebble on the graph during ever round \( i \)). We use \( \Pi^\parallel_{\text{cc}}(G) = \min_{P \in P_G^\parallel} \text{cc}(P) \) to denote the cumulative cost of the best legal pebbling.

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\(^5\)See additional discussion in Section 3.2.

\(^6\)Note that when \( d = 0 \) testing whether a graph \( G \) is \((e, d)\) reducible is equivalent to asking whether \( G \) has a vertex cover of size \( e \). Our reduction establishes hardness for \( d \gg 1 \).
Definition 2 We say that a directed acyclic graph (DAG) \( G = (V,E) \) is \((e,d)\)-depth robust if \( \forall S \subseteq V \) of size \( |S| \leq e \) we have \( \text{depth}(G - S) \geq d \). If \( G \) contains a set \( S \subseteq V \) of size \( |S| \leq e \) such that \( \text{depth}(G - S) \leq d \) then we say that \( G \) is \((e,d)\)-reducible.

2.1 Decision Problems

The decision problem \( \text{minCC} \) is defined as follows:

**Input:** a DAG \( G \) on \( n \) nodes and an integer \( k < n(n + 1)/2 \).

**Output:** Yes, if \( \Pi^\text{cc}_{k}(G) \leq k \); otherwise No.

Given a constant \( \delta \geq 1 \) we use \( \text{minCC}_\delta \) to denote the above decision problem with the additional constraint that \( \text{indeg}(G) \leq \delta \). It is clear that \( \text{minCC} \in \text{NP} \) and \( \text{minCC}_\delta \in \text{NP} \) since it is easy to verify that a candidate pebbling \( P \) is legal and that \( CC(P) \leq k \). One of our primary results is to show that the decision problems \( \text{minCC} \) and \( \text{minCC}_2 \) are \( \text{NP-Complete} \). In fact, these results hold even if we require that the DAG \( G \) has a single sink node.

The decision problem \( \text{REDCIBLE}_d \) is defined as follows:

**Input:** a DAG \( G \) on \( n \) nodes and positive integers \( e, d \leq n \).

**Output:** Yes, if \( G \) is \((e,d)\)-reducible; otherwise No.

We show that the decision problem \( \text{REDCIBLE}_d \) is \( \text{NP-Complete} \) for all \( d > 0 \) by a reduction from Cubic Vertex Cover, defined below. Note that when \( d = 0 \) \( \text{REDCIBLE}_d \) is Vertex Cover.

The decision problem \( VC \) (resp. \( \text{CubicVC} \)) is defined as follows:

**Input:** a graph \( G \) on \( n \) vertices (\( \text{CubicVC} \): each with degree 3) and a positive integer \( k \leq \frac{n}{2} \).

**Output:** Yes, if \( G \) has a vertex cover of size at most \( k \); otherwise No.

To show that \( \text{minCC} \) is \( \text{NP-Complete} \) we introduce a new decision problem \( \text{B2LC} \). We will show that the decision problem \( \text{B2LC} \) is \( \text{NP-Complete} \) and we will give a reduction from \( \text{B2LC} \) to \( \text{minCC} \).

The decision problem \( \text{Bounded 2-Linear Covering} \) (\( \text{B2LC} \)) is defined as follows:

**Input:** \( n \) variables \( x_1, \ldots, x_n \), \( k \) positive constants \( 0 \leq c_1, \ldots, c_k \), an integer \( m \leq k \) and \( k \) equations of the form \( x_{\alpha_i} + c_i = x_{\beta_i} \), where \( \alpha_i, \beta_i \in [n] \) and \( i \in [k] \). We require that \( \sum_{i=1}^{k} c_i \leq p(n) \) for some fixed polynomial \( n \).

**Output:** Yes, if we can find \( mn \) values \( x_{y,z} \) (for each \( 1 \leq y \leq m \) and \( 1 \leq z \leq n \)) such that for each \( i \in [k] \) there exists \( 1 \leq y \leq m \) such that \( x_{y,\alpha_i} + c_i = x_{y,\beta_i} \) (that is the assignment \( x_1, \ldots, x_n = x_{y,1}, \ldots, x_{y,n} \) satisfies the \( i \)th equation); otherwise No.

To show that \( \text{B2LC} \) is \( \text{NP-Complete} \) we will reduce from the problem \( \text{3-PARTITION} \), which is known to be \( \text{NP-Complete} \). The decision problem \( \text{3-PARTITION} \) is defined as follows:

**Input:** A multi-set \( S \) of \( m = 3n \) positive integers \( x_1, \ldots, x_m \geq 1 \) such that (1) we have \( \frac{1}{3n} < x_i < \frac{1}{2n} \) for each \( 1 \leq i \leq m \), where \( T = \sum x_i \), and (2) we require that \( T \leq \frac{p(n)}{2} \) for a fixed polynomial \( p \).

**Output:** Yes, if there is a partition of \( m \) into \( n \) subsets \( S_1, \ldots, S_n \) such that \( \sum_{j \in S_i} x_j = \frac{T}{n} \) for each \( 1 \leq i \leq n \); otherwise No.

\footnote{We may assume \( \frac{1}{3n} < x_i < \frac{1}{2n} \) by taking any set of positive integers and adding a large fixed constant to all terms, as described in \cite{Dem14}.}
3 Related Work

The sequential black pebbling game was introduced by Hewitt and Paterson [HP70], and by Cook [Coo73]. It has been particularly useful in exploring space/time trade-offs for various problems like matrix multiplication [Tom78], fast fourier transformations [SS78, Tom78], integer multiplication [SS79b] and many others [Cha73, SS79a]. In cryptography it has been used to construct/analyze proofs of space [DFKP15, RD16], proofs of work [DNW05, MMV13] and memory-bound functions [DGN03] (functions that incur many expensive cache-misses [ABW03]). More recently, the black pebbling game has been used to analyze memory hard functions [AS15, AB16a].

3.1 Password Hashing and Memory Hard Functions

Users often select low-entropy passwords which are vulnerable to offline attacks if an adversary obtains the cryptographic hash of the user’s password. Thus, it is desirable for a password hashing algorithm to involve a function $f(\cdot)$ which is moderately expensive to compute. The goal is to ensure that, even if an adversary obtains the value $(\text{username}, f(\text{pwd}, \text{salt}), \text{salt})$ (where salt is some randomly chosen value), it is prohibitively expensive to evaluate $f(\cdot, \text{salt})$ for millions (billions) of different password guesses. PBKDF2 (Password Based Key Derivation Function 2) [Kal00] is a popular moderately hard function which iterates the underlying cryptographic hash function many times (e.g., $2^{10}$). Unfortunately, PBKDF2 is insufficient to protect against an adversary who can build customized hardware to evaluate the underlying hash function. The cost computing a hash function $H$ like SHA256 or MD5 on an Application Specific Integrated Circuit (ASIC) is dramatically smaller than the cost of computing $H$ on traditional hardware [NB+15].

[ABW03], observing that cache-misses are more egalitarian than computation, proposed the use of “memory-bound” functions for password hashing — a function which maximizes the number of expensive cache-misses. Percival [Per09] observed that memory costs tend to be stable across different architectures and proposed the use of memory-hard functions (MHFs) for password hashing. Presently, there seems to be a consensus that memory hard functions are the ‘right tool’ for constructing moderately expensive functions. Indeed, all entrants in the password hashing competition claimed some form of memory hardness [PHC]. As the name suggests, the cost of computing a memory hard function is primarily memory related (storing/retrieving data values). Thus, the cost of computing the function cannot be significantly reduced by constructing an ASIC. Percival [Per09] introduced a candidate memory hard function called scrypt, but scrypt is potentially vulnerable to side-channel attacks since its computation yields a memory access pattern that is data-dependent (i.e., depends on the secret input/password). Due to the recently completed password hashing competition [PHC] we have many candidate data-dependent memory hard functions such as Catena [FLW13] and the winning contestant Argon2i-A [BDK15].

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3 The PARTITION problem is called $P[3,1]$ in [GJ75].

9 The specification of Argon2i has changed several times. We use Argon2i-A to refer to the version of Argon2i from the password hashing competition, and we use Argon2i-B to refer to the version that is currently being considered for standardization by the Cryptography Form Research Group (CFRG) of the IRTF [BDKJ16].
3.1.1 iMHFs and Graph Pebbling

All known candidate iMHFs can be described using a DAG $G$ and a hash function $H$. Graph pebbling is particularly useful as a tool to analyze the security of an iMHF [AS15, CGBS16, FLW13]. A pebbling of $G$ naturally corresponds to an algorithm to compute the iMHF. Alwen and Serbinenko [AS15] showed that in the pROM model of computation, any algorithm to compute the iMHF corresponds to a pebbling of $G$.

3.1.2 Measuring Pebbling Costs

In the past, MHF analysis has focused on space-time complexity [Per09, FLW13, CGBS16]. For example, the designers of Catena [FLW13] showed that their DAG $G$ had high sequential space-time pebbling cost $\Pi^{st}(G)$ and Corigan-Gibbs et al. [CGBS16] showed that Argon2i-A and their own iMHF candidate iBH (“balloon hash”) have (parallel) space-time cost $\Omega(n^2)$. Alwen and Serbinenko [AS15] observed that these guarantees are insufficient for two reasons: (1) the adversary may be parallel, and (2) the adversary might amortize his costs over multiple iMHF instances (e.g., multiple password guesses). Indeed, there are now multiple known attacks on Catena [BK15, AS15, AB16a]. Alwen and Blocki [AB16a, AB16b] gave attacks on Argon2i-A, Argon2i-B, iBH, and Catena with lower than desired amortized space-time cost — $\Pi^{cc}(G) \leq O(n^{1.6})$ for Argon2i-B, $\Pi^{cc}(G) \leq \tilde{O}(n^{1.75})$ for Argon2i-A and iBH and $\Pi^{cc}(G) \leq O(n^{5/3})$ for Catena. This motivates the need to study cumulative cost $\Pi^{cc}$ instead of space-time cost since amortized space-time complexity approaches $\Pi^{cc}$ as the number of iMHF instances being computed increases.

Alwen et al. [ABP16] recently constructed a constant indegree graph $G$ with $\Pi^{cc}(G) = \Omega(n^{2.5\log n})$. From a theoretical standpoint, this is essentially optimal as any constant indeg DAG has $\Pi^{cc} = O\left(\frac{n^2 \log \log n}{\log n}\right)$ [AB16a], but from a practical standpoint the critically important constants terms in the lower bound are not well understood.

3.2 Computational Complexity of Pebbling

The computational complexity of various graph pebbling has been explored previously in different settings [GLT80, HIP10]. Gilbert et al. [GLTS0] focused on space-complexity of the black-pebbling game. Here, the goal is to find a pebbling which minimizes the total number of pebbles on the graph at any point in time (intuitively this corresponds to minimizing the maximum space required during computation of the associated function). Gilbert et al. [GLTS0] showed that this problem is PSPACE complete by reducing from the truly quantified boolean formula (TQBF) problem.

We need different tools to analyze the computational complexity of the problem of finding a pebbling with low cumulative cost. By contrast, observe that $\text{mincc} \in \text{NP}$ because any DAG $G$ with $n$ nodes this algorithm has a pebbling $P$ with $\text{cc}(P) \leq \text{st}(P) \leq n^2$. Thus, if we are minimizing cc or st cost, the optimal pebbling of $G$ will trivially never require more than $n^2$ steps. By contrast, the optimal (space-minimizing) pebbling of the graphs from the reduction of Gilbert et al. [GLTS0] often require exponential time.

In Appendix D we show that the optimal pebbling from [GLTS0] does take polynomial time if the TQBF formula only uses existential quantifiers (i.e., if we reduce from 3SAT). Thus, the reduction of Gilbert et al. [GLTS0] can also be extended to show that it is NP-Complete to check whether a DAG $G$ admits a pebbling $P$ with $\text{st}(P) \leq k$ for some parameter $k$. The reduction,
which simply appends a long chain to the original graph, exploits the fact that that if we increase space-usage even temporarily we will dramatically increase st cost. However, this reduction does not extend to cumulative cost because the penalty for temporarily placing large number of pebbles can be quite small as we do not keep these pebbles on the graph for a long time.

4 NP-Hardness of minCC

In this section we prove that minCC is NP-Complete by reduction from B2LC. Suppose we are given an instance of B2LC. That is, we are given n variables $x_1, \ldots, x_n$ and k equations of the form $x_\alpha + c_i = x_\beta$, where $\alpha, \beta \in \{1, \ldots, n\}$ and $i \in [k]$, where $c_i$ are positive integers bounded by some polynomial in $n$. Our goal is to to decide if there exist a set of $m < k$ variable assignments: $x_{ij} 1 \leq i \leq m$ and $1 \leq j \leq n$ so that each equation $x_\alpha + c_i = x_\beta$ is satisfied by some assignment $x_{i1}, \ldots, x_{in}$.

We shall construct a graph $G_{B2LC}$ so that pebbling nodes corresponds to synchronized variables which are “offset” by the summand in satisfied equations. Namely, our graph consists of several components. Our first gadget is a chain of length $c = \sum c_i$ so that each node is connected to the previous node, and can only be pebbled if there exists a pebble on the previous node in the previous step, such as in Figure 1. We will use another gadget to ensure that, in any legal pebbling, we need to “walk” a pebble down each chain $m$ different times. Intuitively, each chain represents a variable $x_i$ and the time at which we start walking the $j^{th}$ pebble down the chain will correspond to the value of the variable $x_i$ in the $j^{th}$ assignment. In any “non-cheating” pebbling strategy we will keep at most one pebble on each chain at any point in time. We could “cheat” by placing multiple pebbles along a chain representing a specific variable to satisfy the equation nodes. To discourage cheating, we create $\tau$ copies of the path gadget representing each variable. Let $C_1^l, \ldots, C_\tau^l$ denote the $\tau$ chains representing variable $x_i$, and $v_i^{j1}, \ldots, v_i^{jc}$ denote the nodes in chain $C_i^l$.

![Fig. 1: A chain of length 5 for variable $x_i$.](image)

For the $j^{th}$ equation $x_\alpha + c_i = x_\beta$, create a chain $E_j$ of length $c - c_i$, so that the $\alpha^{th}$ node in chain $E_j$ has incoming edges from vertices $v_\alpha^{j1}$ and $v_\beta^{j1, a + c_i}$ for all $1 \leq l \leq \tau$, as demonstrated in Figure 2. Intuitively, if the equation $x_\alpha + c_i = x_\beta$ is satisfied by the $j^{th}$ assignment then on the $j^{th}$ time we walk pebbles down the chain $x_\alpha$ and $x_\beta$ the pebbles on each chain will be synchronized (i.e., when we have a pebble on $v_\alpha^{j1}$, the $\alpha^{th}$ link in the chain representing $x_\alpha$ we will have a pebble on the node $v_\beta^{j1, a + c_i}$ during the same round) so that we can pebble all of these new nodes. We create a single sink node linked from each of the $k$ equation chains, which can only be pebbled if all equation nodes are pebbled.

Our final gadget is a path of length $cm$ so that each node is connected to the previous node. We create a path $M_i$ of length $cm$ for each variable $x_i$, and for every node $v_i^{lp + qc}$ in the path with position $p + qc > 1$, where $1 \leq p \leq c$ and $0 \leq q < m - 1$, we create an edge to the node from node $v_i^{lp}$, $1 \leq l \leq \tau$ (that is, the $p^{th}$ in all $\tau$ paths representing the variable). We connect the final node in each of the $n$ paths to the final sink node. This forces any valid pebbling to walk along each path of length $cm$, in the process walking along each chain $C_i^l$, $1 \leq j \leq \tau$, at least $m$ times.
Lemma 4 If the B2LC instance has a valid solution, then \( \Pi_{cc}(G_{B2LC}) \leq \tau cmn + 2cmn + 2ckm + 1 \).

Lemma 5 If the B2LC instance does not have a valid solution, then \( \Pi_{cc}(G_{B2LC}) \geq \tau cmn + \tau \).

We refer to the appendix for the formal proofs of Lemma 4 and Lemma 5. Intuitively, any solution to B2LC corresponds to \( m \) walks across the \( \tau n \) chains \( C_i^j, 1 \leq i \leq n, 1 \leq j \leq \tau \) of length \( c \). If the B2LC instance is satisfiable then we can synchronize each of these walks so that we can pebble every equation chain \( E_j \) and path \( M_j \) along the way. Thus, the total cost is \( \tau cmn \) plus the cost to pebble the \( k \) equation chains \( E_j \) \( (\leq 2ckm) \), the cost to pebble the \( n \) paths \( M_j \) \( (\leq 2cmn) \) plus the cost to pebble the sink node 1. If the B2LC instance is not satisfied then we must adopt a “cheating” pebbling strategy. This means that for some \( i \leq n \) the pebbling either (1) keeps at least 2 pebbles on each of the chains \( C_i^1, \ldots, C_i^\tau \) during some pebbling round, or (2) delay walking the pebble down the chains \( C_i^1, \ldots, C_i^\tau \) at some point (mid-walk) so that there are \( mc + 1 \) pebbling rounds with 1 pebble on each of these chains. In either case we pay at least \( \tau (mc + 1) \) to pebble the chains \( C_i^1, \ldots, C_i^\tau \) and at least \( \tau mc(n - 1) \) to pebble the other chains \( C_i^j, \) with \( i' \neq i \) and \( 1 \leq j \leq \tau \).

Thus, the cumulative cost is at least \( \tau cmn + \tau \).

Theorem 6 \( \text{minCC} \) is NP-Complete.

Proof: It suffices to show that there is a polynomial time reduction from B2LC to \( \text{minCC} \) since B2LC is NP-Complete (see Theorem 8). Given an instance \( \mathcal{P} \) of B2LC, we create the corresponding graph \( G \) as described above. This is clearly achieved in polynomial time. By Lemma 4 if \( \mathcal{P} \) has a valid solution, then \( \Pi_{cc}(G) \leq \tau cmn + 2cmn + 2ckm + 1 \). On the other hand, by Lemma 5 if \( \mathcal{P} \) does not have a valid solution, then \( \Pi_{cc}(G) \geq \tau cmn + \tau \). Therefore, setting \( \tau > 2cmn + 2ckm + 1 \) (such as \( \tau = 2cmn + 2ckm + 2 \)) allows one to solve B2LC given an algorithm which outputs \( \Pi_{cc}(G) \). \( \square \)

Theorem 7 \( \text{minCC}_\delta \) is NP-Complete.
Note that the only possible nodes with indegree greater than two are the nodes in the equation gadgets $E_1, \ldots, E_m$, and the final sink node. The equation gadgets can have indegree up to $2\tau + 1$, while the final sink node has indegree $n + m$. We replace the incoming edges to each of these nodes with a binary tree, so that all vertices have indegree at most two. By changing $\tau$ appropriately, we can still distinguish between instances of $B2LC$ using the output of $\minCC$. We refer to the appendix for a sketch of the proof of Theorem 7.

**Theorem 8** $B2LC$ is NP-Complete.

We refer to the appendix for the proof of Theorem 8, where we show that there is a polynomial time reduction from $3$-PARTITION to $B2LC$.

## 5 LP Relaxation has Large Integrality Gap

Let $G = (V,E)$ be a DAG with maximum indegree $\delta$ and with $V = \{1, \ldots, n\}$, where $1, 2, 3, \ldots, n$ is topological ordering of $V$. We start with an integer program for DAG pebbling, taking Step 1a in Figure 5.

Intuitively, $x_v^t = 1$ if we have a pebble on node $v$ during round $t$. Thus, Constraint 1 says that we must have a pebble on the final node at some point. Constraint 2 says that we do not start with any pebbles on $G$ and Constraint 3 enforces the validity of the pebbling. That is, if $v$ has parents we can only have a pebble on $v$ in round $t + 1$ if either (1) $v$ already had a pebble during round $t$, or (2) all of $v$’s parents had pebbles in round $t$. It is clear that the above Integer Program yields the optimal pebbling solution. If we want the optimal pebbling solution that terminates in $\text{exactly } t^* \geq \text{depth}(G)$ steps then we can simply change Constraint 1 to say that $x_v^{t^*} \geq 1$.

We would like to convert our Integer Program to a Linear Program. The natural relaxation is simply to allow $0 \leq x_v^t \leq 1$. However, we show that this LP has a large integrality GAP $\tilde{\Omega}\left(\frac{n}{\log n}\right)$ even for DAGs $G$ with constant indegree. In particular, there exist DAGs with constant indegree $\delta$ for which the optimal pebbling has $\Pi^{\text{opt}}_\text{LC}(G) = \tilde{\Omega}\left(\frac{n^{\delta+1}}{\log n}\right)$ [ABP16], but we will provide a fractional solution to the LP relaxation with cost $O(n)$. 

Fig. 4: An example of a complete reduction
(1) Variables: For $1 \leq v \leq n$ and $0 \leq t \leq n^2$,
   
   (a) Integer Program: $x_v^t \in \{0, 1\}$
   
   (b) Relaxed Linear Program: $0 \leq x_v^t \leq 1$

(2) Goal (minimize cumulative pebbling cost): $\min \sum_{v \in V} \sum_{t=0}^{n^2} x_v^t$.

(3) Constraint 1 (Must Finish): $\sum_{t=0}^{n^2} x_v^t \geq 1$.

(4) Constraint 2 (No Pebbles At Start): $\sum_{v>0} x_0^v \leq 0$.

(5) Constraint 3 (Pebbling Is Valid): For all $v$ s.t $|\text{Parents}(v)| \geq 1$ and $0 \leq t \leq n^2 - 1$ we have

$$x_v^{t+1} \leq x_v^t + \sum_{v' \in \text{Parents}(v)} x_{v'}^t / |\text{Parents}(v)|.$$

Fig. 5: Integer Program for Pebbling.

**Theorem 9** Let $G$ be with constant indegree $\delta$. Then there is a fractional solution to our LP Relaxation (of the Integer Program in Figure 5) with cost at most $3n$.

The proof of Theorem 9 is in Appendix A, we sketch the proof here. We first “fractionally” pebble $G$ in topological order using fractional pebbles with weight $\frac{1}{n}$. Since there are $n$ pebbles, this process requires $n$ time steps. Once this process is complete, we may increase $x_n^i$ by $\frac{1}{n}$ for each subsequent time step $x_v^{t-1}$, since all of its parents $v$ must have value $x_v^{t-1} = \frac{1}{n}$. Thus, after $n - 1$ more time steps, we have $x_n = 1$. We refer to the appendix for a formal proof.

One tempting way to “fix” the linear program is to require that the pebbling take at most $n$ steps since the fractional assignment used to establish the integrality gap takes $2n$ steps. There are two issues with this approach: (1) It is not true in general that the optimal pebbling of $G$ takes at most $n$ steps. See Appendix C for a counter example and discussion. (2) For constant indegree DAGs we can give a fractional assignment that takes exactly $n$ steps and costs $O(n \log n)$. Thus, the integrality gap is still $\tilde{\Omega}(n)$. Briefly, in this assignment we set $x_i^1 = 1$ for $i \leq n$ and for $i < t$ we set $x_i^t = \max \left\{ \frac{1}{n}, (2^{-\text{dist}(i,t+j)}-j+2 : j \geq 1) \right\}$, where $\text{dist}(x,y)$ is the length of the shortest path from $x$ to $y$. In particular, if $j = 1$ (we want to place a ‘whole’ pebble on node $j + t$ in the next round by setting $x_j^{t+1} = 1$) and node $i$ is a parent of node $t + j$ then we have $\text{dist}(i, t + j) = 1$ so we will have $x_i^{t+1} = 1$ (e.g., a whole pebble on node $i$ during the previous round).

6 Conclusions

We initiate the study of the computational complexity of cumulative cost minimizing pebbling in the parallel black pebbling model. This problem is motivated by the urgent need to develop and analyze secure data-independent memory hard functions for password hashing. We show that it is NP-Hard to find a parallel black pebbling minimizing cumulative cost, and we provide evidence that the problem is hard to approximate. Thus, it seems unlikely that we will be able to develop tools to automate the task of a cryptanalyst to obtain strong upper/lower bounds on the security
of a candidate iMHF. However, we cannot absolutely rule out the possibility that an efficient approximation algorithm exists. The primary remaining challenge is to either give an efficient $\alpha$-approximation algorithm to find a pebbling $P \in P^\parallel$ with $\text{cc}(P) \leq \alpha \Pi^\parallel_{\text{cc}}(G)$ or show that $\Pi^\parallel_{\text{cc}}(G)$ is hard to approximate. We believe that the problem offers many interesting theoretical challenges and a solution could have very practical consequences for secure password hashing. It is our hope that this work encourages others in the TCS community to explore these questions.

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A Missing Proofs

Reminder of Lemma 4
If the B2LC instance has a valid solution, then \( \Pi_{cc}^\parallel(G_{B2LC}) \leq \tau cmn + 2cmn + 2ckm + 1 \).

Proof of Lemma 4
Suppose the given instance of B2LC has a valid solution, \( \{x_{i,j}\} \). Recall that a pebble must pass \( m \) times through each of the \( \tau \) chains of length \( c \) representing each variable. For a set \( k \), let \( x_{k,i_1} \leq x_{k,i_2} \leq \ldots \leq x_{k,i_n} \). We start the \( k^{th} \) pass through the \( \tau \) chains by placing a pebble
on $C^i_{j_n}, \ldots, C^t_{j_n}$, the $\tau$ chains representing variable $j_n$. At each subsequent time step, we move the existing pebbles to the next node along the chain. When the pebbles on chains $C^i_{j_n}, \ldots, C^t_{j_n}$ reach the $(x_{k_{j_n}} - x_{k_{j_n-1}} + 1)^{\text{th}}$ nodes, we place a pebble on $C^i_{j_n-1}, \ldots, C^t_{j_n-1}$, the $\tau$ chains representing variable $j_{n-1}$. We continue this process by moving existing pebbles to the next node along each chain for each subsequent time step.

When pebbles on chains $C^i_{j_1}, \ldots, C^t_{j_1}$ reach the $(x_{k_{j_1}} - x_{k_{j_1-1}} + 1)^{\text{th}}$ nodes, we place a pebble on $C^i_{j_1-1}, \ldots, C^t_{j_1-1}$, the $\tau$ chains representing variable $j_{1-1}$. Thus, the gadgets $E_1, \ldots, E_m$ which are satisfied by $x_{k_1}, \ldots, x_{k_m+1}$ can be pebbled during this pass, since by construction, we offset the positions of the pebbles by the appropriate distances. When a pebble reaches the end of its chain, we remove the pebble. Hence, we see that each chain has $c$ time steps with pebbles, and each of these steps needs only one pebble. There are $n$ variables, $\tau$ chains representing each variable, and $m$ passes through each chain. Across all variable chains, the cumulative complexity is $\tau c m n$ since there are $n$ variables and $\tau$ chains representing each variable.

Similarly, making $m$ walks through the paths $C^i_{j_1}, \ldots, C^t_{j_1}$ allows us to pebble the path $M_1$ ($j \leq n$). These paths each have length $cm$ and we keep at most one pebble on $M_1$ at any point in time. There may be a delay of up to $c$ time steps between consecutive walks through the paths $C^i_{j_1}, \ldots, C^t_{j_1}$ during which we cannot progress our pebble through the path $M_1$. However, there are at most $2cm$ total steps in which we have a pebble on $M_1$. Thus, the cumulative complexity across all paths $M_1, \ldots, M_n$ is at most $2cmn$.

Likewise, since each equation gadget $E_j$ is represented by a chain, one pebble at each time step suffices for each of these paths. We may have to keep a pebble on the gadget $E_j$ while we make $m$ walks through the paths, but we keep this pebble on $E_j$ for at most $2cm$ steps (each walk takes $c$ steps and the delay between consecutive walks is at most $c$ steps). Since there are $k$ equations, and there is a chain for each equation, the cumulative complexity across all equation chains is at most $2ckm$.

There is one final sink node, so the cumulative complexity for an instance of B2LC with a valid solution is at most $\tau c m n + 2cmn + 2ckm + 1$.

**Reminder of Lemma 5.** *If the B2LC instance does not have a valid solution, then $\Pi_{cc}^\parallel (G_{B2LC}) \geq \tau c m n + \tau$.*

**Proof of Lemma 5.** Suppose the given instance of B2LC does not have a valid solution. We first observe that in any legal pebbling $P = (P_0, \ldots, P_i) \in \mathcal{P}^\parallel$ we must walk a pebble down each of the paths $M_1, \ldots, M_n$ of length $cm$. Let $t^z_i$ denote the first time step in which we place a pebble on $v^z_i$ — the $z$’th node on path $M_i$. Clearly, $t^z_i < t^{z+1}_i$ for $z < cm$ and at time $t^z_i - 1$ we must have a pebble on all nodes in parents($v^z_i$). In particular, $v^y_i, \ldots, v^t_i \in P_{t^z_i-1}$ where $y - 1 = z \mod c$ so that the pebbling configuration at time $t^z_i - 1$ includes at least one pebble on each of the chains $C^i_1, \ldots, C^t_i$. There are $n$ variables $x_i$, $\tau$ chains $C^i_1$ of length $c$ representing each variable $x_i$, and $m$ passes through each chain. Thus, across all variable chains, the cumulative complexity is at least $\tau c m n$, and any “cheating” pebbling has cumulative complexity at least $\tau c m n + \tau$. Recall that a “cheating” pebbling either has (1) one step in which we have at least 2 pebbles on each of the chains $C^i_1, \ldots, C^t_i$ for some $i \leq n$, or (2) $cm + 1$ time steps in which we have a pebble on each of the chains $C^i_1, \ldots, C^t_i$ for some $i \leq n$. Any non-cheating pebbling corresponds to a valid B2LC solution. If an equation $E_j$ is not satisfied by the solution then there is no legal way to pebble the equation chain $E_j$ without cheating because at no point during the $m$ walks are both variables in the equation offset by the correct amount. Thus, any instance of B2LC which does not have a valid solution requires at least $\tau c m n + \tau$ cumulative complexity.
Reminder of Theorem 7 \( \text{minCC}_S \) is NP-Complete.

Proof of Theorem 7 (sketch) We sketch the construction for \( \delta = 2 \). Although we maintain \( \tau \) chains representing each variable, we can no longer maintain the gadgets for the equation, \( E_1, \ldots, E_m \), for which each vertex has indegree \( 2\tau + 1 \), one edge from its predecessor in the gadget, and \( 2\tau \) edges from the chains representing the variables involved in the equation. Instead, in place of the \( 2\tau \) edges, we construct a binary tree, where the bottom layer of the tree has at least \( \binom{2}{2} \) nodes. Each of the \( \binom{2}{2} \) nodes connects to a separate instance of the chains representing the variables involved in the equation. Thus, if the equation involves variables \( x_i \) and \( x_j \), then each of the \( \binom{2}{2} \) nodes will have an edge from one of the \( \tau \) chains representing \( x_i \), and an edge from one of the \( \tau \) chains representing \( x_j \), offset by an appropriate amount. This construction ensures that the root of the tree is pebbled only if all equations are satisfied, but any “dishonest” walk among the chains will require at least \( \tau \) additional pebbles. Similarly, we replace the \( 2\tau \) edges in the paths \( P_1, \ldots, P_n \) of length \( cm \) with binary trees with base \( \binom{2}{2} \). Finally, we replace the \( m + n \) incident edges to the sink node with a binary tree with base \( \binom{m+n}{2} \). Since each of these terms are polynomial in \( c, m, n \), there exists a \( \tau \) that is also polynomial in \( c, m, n \) which allows us to distinguish between honest walks and dishonest walks. As a result, we can decide between instances of B2LC.

\( \square \)

Reminder of Theorem 8 B2LC is NP-Complete.

Proof of Theorem 8 First, sort \( S \) so that \( S = \{s_1, s_2, \ldots, s_m\} \) and \( s_i \leq s_j \) for any \( i < j \). Let \( T = \sum_{i=1}^{m} s_i \). We then create \( mn = 3n^2 \) equations:

\[
\begin{align*}
x_1 + s_1 &= x_2, & x_1 + s_2 &= x_3, & \ldots, & x_m + s_m &= x_{m+1}, \\
x_1 + 0 &= x_2, & x_2 + 0 &= x_3, & \ldots, & x_m + 0 &= x_{m+1}, \\
x_1 + T &= x_2, & x_2 + T &= x_3, & \ldots, & x_m + T &= x_{m+1}, \\
x_1 + 2T &= x_2, & x_2 + 2T &= x_3, & \ldots, & x_m + 2T &= x_{m+1}, \\
x_1 + (n-2)T &= x_2, & x_2 + (n-2)T &= x_3, & \ldots, & x_m + (n-2)T &= x_{m+1},
\end{align*}
\]

Finally, we create the additional \( n \) equations:

\[
x_1 + \frac{T}{n} + 3(i-1)(n-2)T = x_{m+1},
\]

for \( i \in [n] \). This gives a total of \( 3n^2 + n \) equations so that the reduction is clearly achieved in polynomial time.

Recall that B2LC is true if and only if there exist \( (a_{i,j}) \), \( i \in [n], j \in [m] \) so that each equation \( x_i + c_{i,j} = x_j \) is satisfied by the assigning \( x_i = a_{i,k} \) and \( x_j = a_{j,k} \) for some \( k \). We show that there exists a solution to 3-PARTITION if and only if there exists a solution to B2LC.

Suppose there exists a partition of \( S \) into \( n \) triplets \( S_1, S_2, \ldots, S_n \) so that the sum of the integers in each triplet is the same, and equals \( \frac{T}{n} \). We set \( a_{1,1} = 0 \) and for each \( s_i \) that appears in \( S_1 \), we take the equation, \( a_{1,i} + s_i = a_{1,i+1} \). Otherwise, if \( s_i \) does not appear in \( S_1 \), we take the equation \( a_{1,i} + 0 = a_{1,i+1} \).

Suppose that \( a_{i,j} \) are defined for all \( i < k \). Then we set \( a_{k,1} = 0 \) and for each \( s_i \) that appears in \( S_k \), we take the equation, \( a_{k,i} + s_i = a_{k,i+1} \). If \( s_i \) appears in \( S_1 \) for some \( i < k \), then we take the equation \( a_{k,i} + (i-2)T = a_{k,i+1} \). Otherwise, if \( s_i \) does not appear in \( S_1, \ldots, S_k \), we take the equation \( a_{k,i} + (i-1)T = a_{k,i+1} \).

Thus, to get \( a_{i,m+1} \) from \( a_{i,1} \), we add in three elements whose sum is \( \frac{T}{n} \). We then add in \( 3(i-1) \) instances of \( (i-2)T \), for each of the elements which appear in \( S_1, \ldots, S_{i-1} \). Finally, we add in
(3n - 3i) instances of (i - 1)T. Hence, it follows that

\[ a_{i,1} + \frac{T}{n} + 3(i - 1)(i - 2)T + (3n - 3i)(i - 1)T = a_{i,m+1} \]
\[ a_{i,1} + \frac{T}{n} + (3in - 3n - 6i + 6)T = a_{i,m+1} \]
\[ a_{i,1} + \frac{T}{n} + 3(i - 1)(n - 2)T = a_{i,m+1}, \]

so that all equations in \( i \) are satisfied. Thus, a solution for 3-PARTITION yields a solution for B2LC.

Suppose there exists a solution for the above instance of B2LC. Without loss of generality, assume each equation in \( i \) \( x_i + \frac{T}{n} + 3(i - 1)(n - 2)T = x_{m+1} \) holds for \( a_{i,1}, a_{i,m+1} \), since two different equations clearly cannot hold for the same \( a_{i,1}, a_{i,m+1} \). We observe that \( a_{i,1} + \frac{T}{n} \equiv a_{i,m+1} \) (mod T) for all i. Hence, to obtain \( a_{i,m+1} \) from \( a_{i,1} \), we must take three equations of the form \( x_i + s_i = x_{i+1} \) since the summing less than three elements in S is less than \( \frac{T}{n} \), while the summing more than three elements in S is more than \( \frac{T}{n} \) but less than \( T + \frac{T}{n} \) (as each element of S is greater than \( \frac{T}{n} \) and less than \( \frac{T}{n} \)) and the sum of all elements in S is T). Say that these three equations use the terms \( s_{i_1}, s_{i_2}, s_{i_3} \). Then we let \( S_1 = \{s_{i_1}, s_{i_2}, s_{i_3}\} \), which indeed sums to \( \frac{T}{n} \).

Similarly, to obtain \( a_{k,m+1} \) from \( a_{k,1} \), we must take three equations of the form \( x_i + s_i = x_{i+1} \). Since there are 3n equations of this form which must be satisfied and each of the n assignments of the form \( a_{i,1}, \ldots, a_{i,m+1} \) (where \( 1 \leq i \leq n \)) uses exactly three of these equations, then each assignment of \( a_{i,1}, \ldots, a_{i,m+1} \) must satisfy a disjoint triplet of the 3n equations. Say that the three satisfied equations for \( a_{i,1}, \ldots, a_{i,m+1} \) are \( S_k = \{s_{k_1}, s_{k_2}, s_{k_3}\} \). Then we let \( S_k = \{s_{k_1}, s_{k_2}, s_{k_3}\} \), which indeed sums to \( \frac{T}{n} \).

Therefore, we have a partition of S into triplets, which each sum to \( \frac{T}{n} \), as desired. Thus, a solution for B2LC yields a solution for 3-PARTITION. \( \square \)

**Reminder of Theorem 9** Let G be with constant indegree \( \delta \). Then there is a fractional solution to our LP Relaxation (of the Integer Program in Figure 5) with cost at most 3n.

**Proof of Theorem 9** In particular, for all time steps \( t \leq n \), we set \( x^t_i = \frac{1}{n} \) for all \( i \leq t \), and \( x^t_i = 0 \) for \( i > t \). For time steps \( n < t \leq 2n \), we set \( x^t_i = \frac{t-n+1}{n} \) and \( x^t_i = \frac{1}{n} \) for \( v \neq n \). We first argue that this is a feasible solution. Trivially, Constraints 1 and 2 are satisfied. Note that if \( x^t_v = \frac{1}{n} \), then \( x^t_u = \frac{1}{n} \) for all \( u < v \), so certainly \( \sum_{v \in \text{Parents}(v)} \frac{x^t_v}{|\text{Parents}(v)|} = \frac{1}{n} \). Moreover, \( x^{t-1}_n = \frac{t-n}{n} \) for \( n < t \leq 2n \) implies that setting

\[ x^t_n = x^{t-1}_n + \sum_{v \in \text{Parents}(v)} \frac{x^{t-1}_v}{|\text{Parents}(v)|} = t - n + \frac{1}{n} = \frac{t - n + 1}{n} \]

is valid. Therefore, Constraint 3 is satisfied.

Finally, we claim that \( \sum_{v \in V} \sum_{t \leq 2n} x^t_v \leq 3n \). To see this, note that for every round \( t \leq n \), we have \( x_v^t \leq \frac{1}{n} \) for all \( v \in V \). Thus,

\[ \sum_{v \in V} \sum_{t \leq n} x^t_v \leq \sum_{v \in V} \sum_{t \leq n} \frac{1}{n} \leq \sum_{v \in V} 1 = n. \]
For time steps $n < t \leq 2n$, note that $x^t_n \leq 1$ and $x^t_v = \frac{1}{n}$ for all $v \neq n$. Thus,
\[
\sum_{v \in V} \sum_{n < t \leq 2n} x^t_v \leq \sum_{t \leq n} \left(1 + \frac{n-1}{n}\right) \leq 2n.
\]
Therefore,
\[
\sum_{v \in V} \sum_{t \leq 2n} x^t_v \leq 3n.
\]

B NP-Hardness of REDUCIBLE\textsubscript{d}

The attacks of Alwen and Blocki \cite{AB16a, AB16b} exploited the fact that the Argon2i-A, Argon2i-B, iBH and Catena DAGs are not depth-robust. In general, Alwen and Blocki \cite{AB16a} showed that any $(e, d)$-reducible DAG $G$ can be pebbled with cumulative cost $O(ne + n\sqrt{nd})$. Thus, depth-robustness is necessary condition for a secure iMHF. Recently, Alwen \textit{et al.} \cite{ABP16} showed that depth-robustness is sufficient for a secure iMHF. In particular, they showed that an $(e, d)$-depth reducible graph has $\Pi_{cc}^{\parallel}(G) \geq ed$.\footnote{Alwen \textit{et al.} \cite{ABP16} also gave tighter upper and lower bounds on $\Pi_{cc}^{\parallel}(G)$ for the Argon2i-A, iBH and Catena iMHFs. For example, $\Pi_{cc}^{\parallel}(G) = \Omega\left(n^{1.06}\right)$ and $\Pi_{cc}^{\parallel}(G) = O\left(n^{1.71}\right)$ for a random Argon2i-A DAG $G$ (whp).} Thus, to cryptanalyze a candidate iMHF it would be useful to have an algorithm to test for depth-robustness of an input graph $G$. To the best of our knowledge no one has explored the computational complexity of testing whether a given DAG $G$ is $(e, d)$-depth robust.

We have many constructions of depth-robust graphs \cite{EGS75, PR80, Sch82, Sch83, MMV13}, but the constant terms in these constructions are typically not well understood. For example, Erdős, Graham and Szemerédi \cite{EGS75} constructed an $(\Omega(n), \Omega(n))$-depth robust graph with indeg = $O\left(\log n\right)$. Alwen \textit{et al.} \cite{ABP16} showed how to transform an $n$ node $(e, d)$-depth robust graph with maximum indegree indeg to a $(e, d \times \text{indeg})$-depth robust graph with maximum indeg = 2 on $n \times \text{indeg}$ nodes. Applying this transform to the Erdős, Graham and Szemerédi \cite{EGS75} construction yields a constant-indegree graph on $n$ nodes such that $G$ is $(\Omega(n / \log(n)), \Omega(n))$-depth robust — implying that $\Pi_{cc}^{\parallel}(G) = \Omega\left(\frac{n^2 \log n}{\log n}\right)$. From a theoretical standpoint, this is essentially optimal as any constant indeg DAG has $\Pi_{cc}^{\parallel} = O\left(\frac{n^2 \log \log n}{\log n}\right)$ \cite{AB16a}. From a practical standpoint it is important to understand the exact values of $e$ and $d$ for specific parameters $n$ in each construction.

B.1 Results

We first produce a reduction from Vertex Cover which preserves approximation hardness. Thus, it is NP-Hard to 1.3-approximate $e$, the minimum size of a set $S \subseteq$ such that depth$(G - S) < d$ \cite{DS05}. Moreover, it is NP-Hard to $(2 - \epsilon)$-approximate $e$ for any fixed $\epsilon > 0$ \cite{KR08}, under the Unique Games Conjecture \cite{Kho02}. We also produce a reduction from Cubic Vertex Cover to show REDUCIBLE\textsubscript{d} is NP-Complete even when the input graph has bounded indegree.

\textbf{Theorem 10} REDUCIBLE\textsubscript{d} is NP-Complete.
Proof: We provide a reduction from \( VC \) to \( REDUCIBLE_d \). Given an instance \( G = (V, E) \) of \( VC \), arbitrarily label the vertices \( 1, \ldots, n \), where \( n = |V| \), and direct each edge of \( E \) so that \( 1, \ldots, n \) is a topological ordering of the nodes. For each node \( i \) we add two directed paths: (1) a path of length \( i - 1 \) with an edge from the last node on the path to node \( i \), (2) a path of length \( n - i \) with an edge from node \( i \) to the first node of the path. To avoid abuse of notation, let \( U \) represent the original \( n \) vertices (from the given instance of \( VC \)) in the modified graph.

We claim \( VC \) has a vertex cover of size at most \( k \) if and only if the resulting graph is \((k, n)\)-reducible. Indeed, if there exists a vertex cover of size \( k \), we remove the corresponding \( k \) vertices in the resulting construction. Then there are no edges connecting vertices of \( U \). By construction, any path of length \( n \) must contain at least two vertices of \( U \). Hence, the resulting graph is \((k, n)\)-reducible.

On the other hand, suppose that there is no vertex cover of size \( k \). Given a set \( S \) of \( k = |S| \) vertices we say that a node \( u \in U \) is “untouched” by \( S \) if \( u \notin S \) and \( S \) does not contain any vertex from the chain(s) we connected to node \( u \). If there is no vertex cover of size \( k \), then removing any set \( S \) of \( k = |S| \) vertices from the graph leaves an edge \((u, v)\) with the following properties: (1) \( u, v \in U \), (2) \( u \) and \( v \) are both untouched by \( S \). Suppose \( u \) has label \( i \). Then the (untouched) directed path which ends at \( u \) has length \( i - 1 \). Similarly, there still exists some directed path of length \( \geq n \) which begins at \( v \). Thus, there exists a path of length at least \( n \), so the resulting graph is not \((k, n)\)-reducible.
size \( k \), then removing any set \( S \) of \( k = |S| \) vertices from the graph leaves an edge \((u, v)\) with the following properties: (1) \( u, v \in V \) and have different color classes, (2) \( u \) and \( v \) are both untouched by \( S \). Suppose \( u \) has color \( i \). Then the (untouched) directed path which ends at \( u \) has length \( \frac{n}{k} \). Similarly, there still exists some directed path of length \( \geq n - \frac{k}{n} \) which begins at \( v \). Thus, there exists a path of length at least \( n \), so the resulting graph is not \((k, n)\)-reducible.

\[\square\]

C On Pebbling Time and Cumulative Cost

In this section we address the following question: is it necessarily case that an optimal pebbling of \( G \) (minimizing cumulative cost) takes \( n \) steps? For example, the pebbling attacks of Alwen and Blocki [AB16a, AB16b] on depth-reducible graphs such as Argon2i-A, Argon2i-B, Catena all take exactly \( n \) steps. Thus, it seems natural to conjecture that the optimal pebbling of \( G \) always finishes in \( \text{depth}(G) \) steps. In this section we provide a concrete example of an DAG \( G \) (with \( n \) nodes and \( \text{depth}(G) = n \)) with the property that any optimal pebbling of \( G \) must take more than \( n \) steps. More formally, let \( \Pi^i(G, t) \) denote the set of all legal pebblings of \( G \) that take at most \( t \) pebbling rounds. We prove that \( \min_{P \in \Pi^i(G, t)} cc(P) > \min_{P \in \Pi^i(G)} cc(P) = \Pi^i_{cc}(G) \).

**Theorem 12** There exists a graph \( G \) with \( n = 16 \) nodes and \( \text{depth}(G) = n \) s.t. \( \frac{\min_{P \in \Pi^i(G, t)} cc(P)}{\Pi^i_{cc}(G)} \geq \frac{28}{27} \).

**Proof:** Consider the following DAG \( G \) on 16 nodes \( \{1, \ldots, 16\} \) with the following directed edges (1) \((i, i+1)\) for \( 1 \leq i < 16 \), (2) \((i, i+9)\) for \( 1 \leq i < 5 \) and (3) \((i, i+7)\) for \( 8 \leq i \leq 9 \). We first show in Claim 13 that there is a pebbling \( P \) with cost 27 that takes 18 rounds. Theorem 12 then follows from Claim 14 where we show that any \( P \in \Pi^i(G, 16) \) has cumulative cost at least 28. Intuitively, any legal pebbling must delay either by pebbling nodes 15 and 16 or during rounds 15 – i (for \( 0 \leq i \leq 5 \)) we must have at least one pebble on some nodes in the set \( \{9, 8, \ldots, 9 - i\} \).

**Claim 13** For the above DAG \( G \) we have \( \Pi^i_{cc}(G) \leq 27 \).

**Proof:** Consider the following pebbling: \( P_0 = \emptyset, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\}, P_5 = \{5\}, P_6 = \{6\}, P_7 = \{7\}, P_8 = \{8\}, P_9 = \{1, 9\}, P_{10} = \{2, 10\}, P_{11} = \{3, 11\}, P_{12} = \{4, 12\}, P_{13} = \{5, 13\}, P_{14} = \{6, 14\}, P_{15} = \{7, 14\}, P_{16} = \{8, 14\}, P_{17} = \{9, 15\}, P_{18} = \{16\} \). It is easy to verify that \( cc(P) = 9 + 2 \times 9 = 27 \) since there are 9 steps in which we have one pebble on \( G \) and 9 steps in which we have two pebbles on \( G \).

**Claim 14** For the above DAG \( G \) we have \( \min_{P \in \Pi^i(G, 16)} cc(P) \geq 28 \).

**Proof:** Let \( P = (P_1, \ldots, P_{16}) \in \Pi^i(G, 16) \) be given. Clearly, \( i \in P_i \) for each round \( 1 \leq i \leq 16 \). To pebble nodes 15 and 16 on steps 15 and 16 we must have \( 9 \in P_{15} \) and \( 8 \in P_{14} \). By induction, this means that \( P_{14-i} \cap \{8, \ldots, 8-i\} \neq \emptyset \) for each \( i > 0 \). To pebble node \( 9+i \) at time \( 9+i \) we require that \( i \in P_{8+i} \) for each \( 1 \leq i \leq 5 \). These observations imply that \( |P_9| \geq 3, |P_{10}| \geq 3, \ldots, |P_{13}| \geq 3 \). We also have \( |P_{15}| \geq 2 \) and \( |P_{14}| \geq 2 \). We also have \( |P_i| \geq 1 \) for all \( 1 \leq i \leq 16 \). The cost of rounds 1–8 and round 16 is at least 9. The cost of rounds 9–13 is at least 15 and the cost of rounds 14 and 15 is at least 4. Thus, \( cc(P) \geq 28 \).

\[\square\]
Open Questions: Theorem 12 shows that $\Pi_{\parallel cc}(G)$ can be smaller than $\min_{P \in \Pi_{\parallel cc}(G, t)} cc(P)$, but how large can this gap be in general? Can we prove upper/lower bounds on the ratio: $\frac{\min_{P \in \Pi_{\parallel cc}(G, t)} cc(P)}{\Pi_{\parallel cc}(G)}$ for any $n$ node DAG $G$? Is it true that $\min_{P \in \Pi_{\parallel cc}(G, t)} cc(P) \leq c$ for some constant $c$? If not does this hold for $n$ node DAGs $G$ with constant indegree? Is it true that the optimal pebbling of $G$ always takes at most $cn$ steps for some constant $c$?

D NP-Hardness of $\min ST$

Recall that the space-time complexity of a graph pebbling is defined as $st(P) = t \times \max_{1 \leq i \leq t} |P_i|$. We define $\min ST$ and $\min SST$ based on whether the graph pebbling is parallel or sequential. Formally, the decision problem $\min ST$ is defined as follows:

**Input:** a DAG $G$ on $n$ nodes and an integer $k < n(n + 1)/2$.

**Output:** Yes, if $\min_{P \in P_{\parallel cc}(G)} st(P) \leq k$; otherwise No.

The decision problem $\min SST$ is defined as follows:

**Input:** a DAG $G$ on $n$ nodes and an integer $k < n(n + 1)/2$.

**Output:** Yes, if $\Pi_{st}(G) \leq k$; otherwise No.

Gilbert et al. [GLT80] provide a construction from any instance of TQBF to a DAG $G_{TQBF}$ with pebbling number $3n + 3$ if and only if the instance is satisfiable. Here, the pebbling number of a DAG $G$ is $\min_{P = (P_1, \ldots, P_t) \in P} \max_{1 \leq i \leq t} |P_i|$ is the number of pebbles necessary to pebble $G$. An important gadget in their construction is the so-called pyramid DAG. We use a triangle with the number $k$ inside to denote a $k$-pyramid (see Figure 6 for an example of a 3-pyramid). The key property of these DAGs is that 

**Remark:** We note that [GLT80] focused on sequential pebblings ($P \in P$) in their analysis, but their analysis extends to parallel pebblings ($P \in P_{\parallel}$) as well.

We observe that any instance of TQBF where each quantifier is an existential quantifier requires at most a quadratic number of pebbling moves. Specifically, we look at instances of 3-SAT, such
Fig. 7: An existential quantifier, with $x_i$ set to true in the left figure and $x_i$ set to false in the right figure.

as in Figure 8. In such a graph representing an instance of 3-SAT, the sink node to be pebbled is $q_n$. By design of the construction, any true statement requires exactly three pebbles for each pyramid representing a clause. On the other hand, a false clause requires four pebbles, so that false statements require more pebbles. Thus, by providing extraneous additions to the construction which force the number of pebbling moves to be a known constant, we can extract the pebbling number, given the space-time complexity. For more details, see the full description in [GLT80].

Lemma 15 [GLT80] The quantified Boolean formula $Q_1x_1Q_2x_2 \cdots Q_nx_nF_n$ is true if and only if the corresponding DAG $G_{\text{TQBF}}$ has pebbling number $3n + 3$.

Lemma 16 Suppose that we have a satisfiable TQBF formula $Q_1x_1Q_2x_2 \cdots Q_nx_nF_n$ with $Q_i = \exists$ for all $i \leq n$. Then there is a legal sequential pebbling $P = (P_0, \ldots, P_t) \in \mathcal{P}(G_{\text{TQBF}})$ of the corresponding DAG $G_{\text{TQBF}}$ from [GLT80] with $t \leq 6n^2 + 33n$ pebbling moves and $\max_{i \leq t} |P_i| \leq 3n + 3$.

Proof: We describe the pebbling strategy of Gilbert et al. [GLT80], and analyze the pebbling time of their strategy. Let $T(i)$ be the time it takes to pebble $q_i$ in the proposed construction for any instance with $i$ variables, $i$ clauses, and only existential quantifiers.

Suppose that $x_i$ is allowed to be true for the existential quantifier $Q_i = \exists$. Then vertex $x'_i$ is pebbled using $s_i$ moves, where $s_i = 3n - 3i + 6$. Similarly, vertices $d_i$ and $f_i$ are pebbled using $s_i - 1$ and $s_i - 2$ moves respectively. Additionally, $f_i$ is moved to $x'_i$ and then $x_i$ is moved to $x'_i$ in the following step, for a total of two more moves. We then pebble $q_{i+1}$ using $T(i+1)$ moves and finish by placing a pebble on $x'_i$ and moving it to $c_i$, $b_i$, $a_i$, and $q_i$, for five more moves. Finally, we use six more moves to pebble an additional clause. Thus, in this case,

$$T_{\text{true}}(i) = s_i + (s_i - 1) + (s_i - 2) + 13 + T(i + 1).$$

On the other hand, if $x_i$ is allowed to be false for the existential quantifier $Q_i = \exists$, then first we pebble $x'_i$, $d_i$, and $f_i$ sequentially, using $s_i$, $s_i - 1$, and $s_i - 2$ moves respectively. We then move
the pebble from $f_1$ to $x'_i$ and then to $x_i$, for a total of two more moves. We then pebble $q_{i+1}$ using $T(i+1)$ moves. The pebble on $q_{i+1}$ is subsequently moved to $c_i$ and then $b_i$, using two more moves. Picking up all pebbles except those on $b_i$ and $x'_i$, and using them to pebble $f_i$ takes $s_i - 2$ more moves. Additionally, the pebble on $f_i$ is moved to $x'_i$ and then $a_i$, while the pebble on $x'_i$ is moved to $x_i$ and then $q_i$, for four more moves. Finally, we use six more moves to pebble an additional clause. In total,

$$T_{\text{false}}(i) = s_i + (s_i - 1) + (s_i - 2) + (s_i - 2) + 14 + T(i+1).$$

Therefore,

$$T(i) \leq 4s_i + 10 + T(i + 1),$$

where $s_i = 3n - 3i + 6$. Thus,

$$T(i) \leq 12(n - i) + 34 + T(i + 1).$$

Writing $R(i) = T(n - i)$ then gives

$$R(i) \leq 12i + 34 + R(i - 1),$$

so that $R(n) \leq \sum_{i=1}^{n}(12i + 34) = 6n^2 + 40n$. Hence, it takes at most $6n^2 + 40n$ moves to pebble the given construction for any instance of TQBF which only includes existential quantifiers. 

\[ \square \]

**Theorem 17** \textit{minST} is NP-Complete.

**Proof:** We provide a reduction from \textit{3-SAT} to \textit{minST}. Given an instance $I$ of \textit{3-SAT} with at most $n$ clauses or variables, we create the corresponding graph from [GLT80]. Additionally, we append a chain of length $300n^3 + 6n^2 + 40n + 100$ to the graph with an edge from the sink node $q_n$ from [GLT80] to the first node in our chain. By adding a chain of length $300n^3 + 6n^2 + 40n + 100$ we can ensure that for any legal pebbling $P = (P_0, \ldots, P_t) \in \mathcal{P}(G)$ of our graph $G$ we have $t \geq 300n^3 + 6n^2 + 40n + 100$. By Lemma 16 at most $6n^2 + 40n$ moves are necessary to pebble the $3$-$\text{SAT}$ portion of the graph, while the chain requires exactly $300n^3 + 6n^2 + 40n + 100$ moves. Thus, if $I$ is satisfiable then we can find a legal pebbling $P = (P_0, \ldots, P_t)$ with space $\max_{1 \leq i \leq t} |P_i| \leq 3n + 3$ and $t \leq 300n^3 + 12n^2 + 80n + 100$ moves. First, pebble the sink $q_n$ in $t' = 6n^2 + 40n$ steps and $\max_{1 \leq t' \leq t} |P_{t'}| \leq 3n + 3$ space by Lemma 16 and then walk single pebble down the chain in $300n^3 + 6n^2 + 40n + 10$ steps keeping at most one pebble on the DAG in each step. The space time cost is at most $st(P) \leq 900n^4 + 936n^3 + 276n^2 + 540n + 300$. If $I$ is not satisfiable then, by Lemma 15 for any legal pebbling $P = (P_0, \ldots, P_t) \in \mathcal{P}(G)$ of our graph we have $\max_{1 \leq i \leq t} |P_i| \geq 3n + 4$ and $t \geq 300n^3 + 6n^2 + 40n + 100$. Thus, $st(P) \geq 900n^4 + 1218n^3 + 144n^2 + 460n + 400$ we have

$$900n^4 + 1218n^3 + 144n^2 + 460n + 400 - (900n^4 + 936n^3 + 276n^2 + 540n + 300) > 0.$$ 

for all $n > 0$. Thus, $I$ is satisfiable if and only if there exists a legal pebbling with $st(P) \leq 900n^4 + 936n^3 + 276n^2 + 540n + 300$. Clearly, this reduction can be done in polynomial time, and so there is indeed a polynomial time reduction from \textit{3-SAT} to \textit{minST}. 

We note that the same reduction from \textit{TQBF} to \textit{minST} fails, since there exist instances of \textit{TQBF} where the pebbling time is $2^{\Omega(n)}$. However, the same relationships do hold for sequential pebbling.

The proof of Theorem 18 is the same as the proof of Theorem 17 because we can exploit the fact that the pebbling of $G_{\text{TQBF}}$ from Lemma 16 is sequential.

**Theorem 18** \textit{minSST} is NP-Complete.
Why Doesn’t the Gilbert et al. Reduction Work for Cumulative Cost?  The construction from [GLT80] is designed to minimize the number of pebbles simultaneously on the graph, at the expense of a larger number of necessary time steps and/or a larger average number of pebbles on the graph during each pebbling round. As a result, one can bypass several time steps by simply adding additional pebbles during some time step. For example, it may be beneficial to temporarily keep pebbles on all three nodes $x_i$, $\overline{x}_i$, and $\overline{x}_i'$ at times so that we can avoid repebbling the $s_i-2$-pyramid later. Also if we do not need the pebble on node $x_i$ for the next $\left(\frac{s_i}{2}\right)$ steps then it is better to discard any pebbles on $x'_i$ and $x_i$ entirely to reduce cumulative cost. Because we would need to repebble the $s_i$-pyramid later our maximum space usage will increase, but our cumulative cost would decrease. Our reduction to space-time cost works because $st$ cost is highly sensitive to an increase in the number of pebbles on the graph even if this increase is temporary. Cumulative, unlike space cost or space-time cost, is not very sensitive to such temporary increases in the number of pebbles on the graph.
Fig. 8: Graph $G_{TQBF}$ for $\exists x_1, x_2, x_3, x_4$ s.t. $(x_1 \lor x_2 \lor x_4) \land (x_2 \lor x_3 \lor \overline{x}_4)$. 