Existence of Hyperbolic Motions to a Class of Hamiltonians and Generalized N-Body System via a Geometric Approach

JIAYIN LIU, DUOKUI YAN & YUAN ZHOU

Communicated by P. Rabinowitz

Abstract

For the classical $N$-body problem in $\mathbb{R}^d$ with $d \geq 2$, Maderna–Venturelli in their remarkable paper (Ann Math 192:499–550, 2020) proved the existence of hyperbolic motions with any positive energy constant, starting from any configuration and along any non-collision configuration. Their original proof relies on the long time behavior of solutions by Chazy 1922 and Marchal-Saari 1976, on the Hölder estimate for Mañé’s potential by Maderna 2012, and on the weak KAM theory. We give a new and completely different proof for the above existence of hyperbolic motions. The central idea is that, via some geometric observation, we build up uniform estimates for Euclidean length and angle of geodesics of Mañé’s potential starting from a given configuration and ending at the ray along a given non-collision configuration. Moreover, our geometric approach works for Hamiltonians $\frac{1}{2} \| p \|^2 - F(x)$, where $F(x) \geq 0$ is lower semicontinuous and decreases very slowly to 0 faraway from collisions. We therefore obtain the existence of hyperbolic motions to such Hamiltonians with any positive energy constant, starting from any admissible configuration and along any non-collision configuration. Consequently, for several important potentials $F \in C^2(\Omega)$, we get similar existence of hyperbolic motions to the generalized $N$-body system $\ddot{x} = \nabla_x F(x)$, which is an extension of Maderna–Venturelli [Ann Math 2020].
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1. Introduction

The classical \(N\)-body problem in \(\mathbb{R}^d\) with \(d \geq 2\) consists in studying the second order ordinary system

\[
\ddot{x} = \nabla_x U(x),
\]

where \(x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{dN}\), each body \(x_i\) has mass \(m_i > 0\), \(U(x)\) denotes the Newtonian potential, and \(\nabla_x U = (\nabla_{x_i} U)_{1 \leq i \leq N}\) with \(\nabla_{x_i} = \frac{1}{m_i} \frac{\partial}{\partial x_i}\) being the gradient with respect to the mass scalar product. In other words,

\[
U(x) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|}, \quad \text{and} \quad \nabla_{x_i} U(x) = \sum_{j=1, j \neq i}^{N} \frac{m_j}{|x_j - x_i|^{3}}(x_j - x_i).
\]

The Hamiltonian corresponding to the system (1.1) is \(\frac{1}{2} \|p\|^2 - U(x)\), where the weighted norm \(\|p\|\) is defined by \(\|p\| := (\sum_{i=1}^{N} m_i |p_i|^2)^{1/2}\) for \(p = (p_1, \ldots, p_N) \in \mathbb{R}^{dN}\). We denote by \(\Omega\) the set of non-collision configurations and by \(\Sigma\) the set of configurations including some collision, that is,

\[
\Omega := \{x \in \mathbb{R}^{dN} : x_i \neq x_j, \forall 1 \leq i < j \leq N\} \quad \text{and} \quad \Sigma := \mathbb{R}^{dN} \setminus \Omega.
\]

It is well known that \(\Omega\) is a path connected open set (that is, a domain) in \(\mathbb{R}^{dN}\).

The long-time behavior of solutions to the system (1.1) has been insistently concerned after the fundamental works by Chazy in 1918; see [5,12,13,17,20,22,28,31] and the references therein. Indeed, Chazy classified different asymptotic types of solutions for the three-body problem, and moreover, established the continuity of the limit shape; see [12,13]. Later, these results were generalized to the \(N\)-body problem for any \(N \geq 4\); see [22,35–37,39,43]. Following Chazy [13], if \(x : [0, \infty) \to \mathbb{R}^{dN}\) is a solution to (1.1) and satisfies \(x(t) = ta + o(t)\) as \(t \to +\infty\) for some non-collision configuration \(a \in \Omega\), we call it a hyperbolic motion to (1.1) along \(a\). The following was proved in [12,13,22]:

**Theorem 1.1.** Let \(x : [0, \infty) \to \mathbb{R}^{dN}\) be a hyperbolic motion to (1.1) along \(a \in \Omega\). Then we have the following:

(i) The solution \(x\) has asymptotic velocity \(a\), that is, \(\lim_{t \to \infty} \dot{x}(t) = a\);
(ii) Given any $\varepsilon > 0$, there are constants $t^* > 0$ and $\delta > 0$ such that, for any maximal solution $y : [0, T) \to \Omega$ satisfying $|y(0) - x(0)| < \delta$ and $|\dot{y}(0) - \dot{x}(0)| < \delta$, we have

(i) $T = +\infty$, $|y(t) - ta| < t\varepsilon$ for all $t > t^*$ and
(ii) there is $b \in \Omega$ with $|b - a| < \varepsilon$ such that $y = tb + o(t)$.

In 1976, Marchal-Saari provided the following classification of long-time behavior for the $N$-body problem; see Theorem 1 and Section 10 in [31]:

**Theorem 1.2.** Let $x : [0, +\infty) \to \mathbb{R}^{dN}$ be a solution to (1.1). Then, as $t \to +\infty$, either

$$\frac{R(t)}{t} \to +\infty,$$

where $R(t) := \max\{|x_i(t) - x_j(t)| : 1 \leq i < j \leq N\}, \quad (1.2)$

or

there exists $A \in \mathbb{R}^{dN}$ such that $x(t) = At + o(t). \quad (1.3)$

Note that if $x$ satisfies (1.3) with $A \in \Omega$, then $x(t)$ is exactly a hyperbolic motion. In general, (1.3) may not be true; and even when it is true, the condition $A \in \Omega$ does not necessarily hold. Thus Theorem 1.2 does not affirm the existence of hyperbolic motions.

In 2020, Maderna–Venturelli showed the existence of hyperbolic motions to (1.1) starting from any configuration and along any given non-collision configuration; see Theorem 1.1 in [28].

**Theorem 1.3.** Given any initial configuration $x_0 \in \mathbb{R}^{dN}$, any non-collision configuration $a \in \Omega$ with $\|a\| = 1$, and any choice of the energy constant $\lambda > 0$, there always exists a hyperbolic motion $x : [0, \infty) \to \mathbb{R}^{dN}$ to (1.1) starting from $x_0$ and along $\sqrt{2\lambda}a$, that is,

$x(0) = x_0$ and $x(t) = \sqrt{2\lambda}at + o(t)$ as $t \to +\infty.$

The original proof of Theorem 1.3 by Maderna–Venturelli relies on Theorems 1.1 and 1.2, and also uses the weak KAM theory and Hölder estimate for Mañé’s potential. Indeed, starting from given $x_0$ and along given $a$ as in Theorem 1.3, Maderna–Venturelli [28] constructed unbounded calibrating curves of viscosity solutions to Hamilton-Jacobi equation involving the Newtonian potential, where the possibility of bounded calibrating curves was ruled out by exploiting a classical result by Von Zeipel [44]. Next, thanks to Marchal-Saari’s classification result in Theorem 1.2, such unbounded calibrating curves satisfy either (1.2) or (1.3). Via some Hölder estimate for Mañé’s potential $m_\lambda$ by Maderna [27] (see also [28, Theorem 2.11]), the possibility of (1.2) is excluded. Thus such unbounded calibrating curves must satisfy (1.3) for some $A \in \mathbb{R}^{dN}$. Finally, by using the asymptotic behavior in Theorem 1.1 by Chazy [13] and Gingold-Solomon [22], they showed that $A$ must be $\sqrt{2\lambda}a$ as desired.

In this paper, we give a new and completely different proof for Theorem 1.3, which is based on some geometric observation for geodesics of Mañé’s potential. The central point is that: we consider a solution $\gamma^{(n)} : [0, \sigma^{(n)}] \to \mathbb{R}^{dN}$ to the
system (1.1) joining \( x_0 \) and \( x_0 + s^*a + 2^n a \) for all sufficiently large \( n \), where \( s^* \) is a constant depending on \( x_0 \) and \( a \); indeed, \( \gamma^{(n)} \) is a geodesic of Mañé’s potential \( m_\lambda \). Via some geometric observation and an induction argument, we are able to bound, uniformly in \( n \), the Euclidean length of \( \gamma^{(n)}|_{[0,1]} \) and also the angle between \( \gamma^{(n)}(t) - x_0 - s^*a \) and \( a \) whenever \( t \in [\sigma_0, \sigma^{(n)}] \) for some constant \( \sigma_0 \) depending on \( x_0, \lambda \) and \( a \). The desired hyperbolic motion is then given by the limit of \( \gamma^{(n)} \) as \( n \to \infty \) (up to some subsequence). See Section 2 for more detail.

Moreover, our geometric approach works for Hamiltonians \( \frac{1}{2} \| p \|^2 - F(x) \), where \( F(x) \geq 0 \) is lower semicontinuous and decreases very slowly to 0 faraway from collisions. We therefore obtain the existence of hyperbolic motions to such Hamiltonian with any positive energy constant, starting from any admissible configuration, and along any non-collision configuration. Consequently, we get a similar Hamiltonian \( \frac{1}{2} \| p \|^2 - F(x) \) defined by\

\[
F(x) := \sum_{1 \leq i < j \leq N} F_{ij}(x_i, x_j) \quad \forall \, x = (x_1, \ldots, x_N) \in \mathbb{R}^dN, \quad \text{where}
\]

\[
F_{ij} : \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta \to [0, \infty) \text{ is locally bounded,}
\]

\[
\text{and is also lower semicontinuous, that is,}
\]

\[
\text{for any } \delta \in \mathbb{R} \text{ the level set } \{ (x_i, x_j) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta : F_{ij}(x_i, x_j) > \delta \} \text{ is open.}
\]

Here and below we write \( \Delta = \{ (z, z) : z \in \mathbb{R}^d \} \). In a natural way, one may extend \( F_{ij} \) to be a lower semicontinuous function in \( \mathbb{R}^d \times \mathbb{R}^d \) with values in \( [0, \infty) \), and hence extend \( F \) to be a lower semicontinuous function in \( \mathbb{R}^{dN} \) with values in \( [0, \infty) \). See Appendix for details.

For any energy constant \( \lambda > 0 \), Mañé’s potential or the action potential \( m_\lambda \) is defined by\

\[
m_\lambda(x, y) := \inf \{ A_\lambda(\gamma) : \gamma \in \mathcal{AC}(x, y; [0, \sigma]; \mathbb{R}^{dN}) \text{ for some } \sigma > 0 \}, \quad \forall x, y \in \mathbb{R}^{dN},
\]

where\

\[
A_\lambda(\gamma) := \int_0^\sigma \left[ \frac{1}{2} |\dot{\gamma}(s)|^2 + F \circ \gamma(s) + \lambda \right] ds,
\]

and for any set \( W \subset \mathbb{R}^{dN} \),\

\[
\mathcal{AC}(x, y; [0, \sigma]; W) := \{ \gamma : [0, \sigma] \to W \text{ is absolutely continuous, and } \gamma(0) = x, \gamma(\sigma) = y \}.
\]

Since \( \Omega \) is path-connected, \( (\Omega, m_\lambda) \) is a metric space. Denote by \( \tilde{\Omega} \) the set\

\[
\{ y \in \mathbb{R}^{dN} : \text{there exists } \lambda > 0 \text{ and } x \in \Omega \text{ such that } m_\lambda(x, y) < +\infty \}. \quad (1.6)
\]
We remark that the set $\tilde{\Omega}$ is independent of the choice of $\lambda > 0$ and $x \in \Omega$; see Appendix. Configurations $x \in \tilde{\Omega}$ are also said to be admissible. It turns out that $(\tilde{\Omega}, m_\lambda)$ is a geodesic space, and geodesics joining any pair of distinct configurations $x, y \in \tilde{\Omega}$ are given by the arc-length parametrization of minimizers of $A_\lambda$ in the class $\bigcup_{\sigma > 0} AC(x, y; [0, \sigma]; \tilde{\Omega})$; see Appendix for more details. Below, to clarify the geometric nature, we call a minimizer of $A_\lambda$ as an $m_\lambda$-geodesic with canonical parameter. We also call a continuous curve $\gamma : [0, \infty) \to \tilde{\Omega}$ as an $m_\lambda$-geodesic ray with canonical parameter if, for any $\sigma > 0$, the restriction $\gamma|_{[0, \sigma]}$ is an $m_\lambda$-geodesic with canonical parameter. Moreover, in the spirit of Chazy, if $\gamma$ is an $m_\lambda$-geodesic ray with canonical parameter for some $\lambda > 0$ and satisfies $x(t) = at + o(t)$ as $t \to +\infty$ for some non-collision configuration $a$, we call $\gamma$ as a hyperbolic motion to the Hamiltonian $\frac{1}{2} \| p \|^2 - F(x)$ with energy constant $\lambda > 0$ and along $a$, or for simplicity, hyperbolic motion to $m_\lambda$ along $a$.

Next, assuming in addition that $F_{ij} \in C^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta; \mathbb{R})$ for all $1 \leq i < j \leq N$ (for simplicity we write $F \in C^2(\Omega)$ below), we consider the generalized $N$-body system

$$\ddot{x} = \nabla_x F(x), \quad (1.7)$$

which corresponds to the Hamiltonian $\frac{1}{2} \| p \|^2 - F(x)$. Following Chazy, if a solution $x : [0, \infty) \to \mathbb{R}^{dN}$ to (1.7) satisfies $x(t) = at + o(t)$ as $t \to +\infty$ for some non-collision configuration $a \in \Omega$, then we call it as a hyperbolic motion to (1.7) along $a$. Recall that if $\gamma : [0, \sigma) \to \tilde{\Omega}$ is an $m_\lambda$-geodesic with canonical parameter and if $\gamma$ is collision free interiorly, that is, $\gamma|_{(0, \sigma)} \subset \Omega$, then it is a solution to the system (1.7) with energy constant $\lambda$, see Lemma A.3 and Lemma A.4 in Appendix. Consequently, a hyperbolic motion to $m_\lambda$ without collision in $(0, \infty)$ is a hyperbolic motion to (1.7).

Below, we build up an existence result of hyperbolic motions to the Hamiltonian $\frac{1}{2} \| p \|^2 - F(x)$, and hence to the corresponding N-body system $\ddot{x} = \nabla_x F(x)$ when, additionally $F \in C^2(\Omega)$, provided that $F$ satisfies the following growth assumption faraway from collisions:

\[
\begin{cases}
F_{ij}(x_i, x_j) \leq f(|x_i - x_j|), \quad \forall (x_i, x_j) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta_2 \text{ for all } 1 \leq i < j \leq N, \\
\text{where } f \in C^0(\mathbb{R}_+, \mathbb{R}_+), \quad \sum_{k=1}^{\infty} \left(2^{-k} \int_{2^k}^{2^{k+1}} f(s) \, ds \right)^{1/2} < \infty.
\end{cases}
\]

(1.8)

Here and below $\mathbb{R}_+ := (0, \infty)$, and $\Delta_\delta := \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |z - w| < \delta\}$.

Note that, if $f$ is decreasing, then

$$\int_4^\infty \frac{\sqrt{f(s)} \, ds}{s} \leq \sum_{k=2}^{\infty} \left(2^{-k} \int_{2^k}^{2^{k+1}} f(s) \, ds \right)^{1/2} \leq \sum_{k=2}^{\infty} \frac{2 \int_2^\infty \sqrt{f(s)} \, ds}{s}.$$

Moreover, if $f(s) \leq C(\log s)^{-\beta}, \forall s \geq 2$, for some $\beta > 2$, then

$$\sum_{k=1}^{\infty} \left(2^{-k} \int_{2^k}^{2^{k+1}} f(s) \, ds \right)^{1/2} \leq \frac{C^{1/2}}{(\log 2)^{\beta/2}} \sum_{k=1}^{\infty} k^{-\beta/2} < \infty.$$
Theorem 1.4. Let $F$ be as in (1.4) and satisfying (1.8). Given any energy constant $\lambda > 0$, any initial configuration $x \in \Omega$ and any non-collision configuration $a \in \Omega$ with $\|a\| = 1$, we have the following:

(i) There exists a hyperbolic motion $\gamma : [0, +\infty) \to \tilde{\Omega}$ to the Hamiltonian $\frac{1}{2}\|p\|^2 - F(x)$ with the energy constant $\lambda > 0$, along $\sqrt{2\lambda}a$ and starting from $x$, that is,

$$\gamma(0) = x \text{ and } \gamma(t) = \sqrt{2\lambda}at + o(t) \text{ as } t \to +\infty.$$  

(ii) There exists $t_0 = t_0(x, a, \lambda) \geq 0$ such that the restriction $\gamma|_{[t_0, \infty)}$ is collision-free, that is, $\gamma|_{[t_0, \infty)} \subset \Omega$.

(iii) If $F \in C^2(\Omega)$, then the restriction $\gamma|_{[t_0, \infty)}$ is a hyperbolic motion to (1.7) along $\sqrt{2\lambda}a$.

(iv) If $F \in C^2(\Omega)$, and suppose in addition $\gamma|_{(0, t_0)}$ is collision free, then $\gamma$ is a hyperbolic motion to (1.7) starting from $x$ and along $\sqrt{2\lambda}a$.

The study of (non-)collisions is one of the main issues in this field. To some extent, (non-)collisions are determined by the behavior of $F$ near the set $\Sigma$ of collisions. There do exist potentials $F$, for which one can find some $m_\lambda$-geodesic including interior collisions, see [7,8,33] for constructions. Therefore, in Theorem 1.4 (iv), since no other assumption is made on $F$ around $\Sigma$, we can not expect in general that the restriction $\gamma|_{(0, t_0)}$ is collision-free.

On the other hand, it is well known (see [3,4,21,29,32]) that all $m_\lambda$-geodesics are collision-free interiorly, and hence are solutions to the system (1.7), provided that $F(x)$ satisfies any one of the following conditions near $\Sigma$:

(E1) For all $1 \leq i < j \leq N$, $F_{ij}(x_i, x_j) \geq C m_i m_j |x_i - x_j|^2$ in $\Delta_{1/2}$, where $C > 0$.

(E2) For all $1 \leq i < j \leq N$, $F_{ij}$ coincides with the homogeneous potential $m_i m_j |x_i - x_j|^{-\alpha}$ in $\Delta_{1/2}$, where $0 < \alpha < \infty$. When $\alpha = 2$, it was also called as the Jacobi-Banachiewitz potential; see [2,14,42].

(E3) For all $1 \leq i < j \leq N$, $F_{ij}$ is given by the quasi-homogeneous potential $m_i m_j (|x_i - x_j|^{-\alpha} + \delta|x_i - x_j|^{-\beta})$ in $\Delta_{1/2}$, where $0 < \beta < \alpha < \infty$ and $\delta > 0$.

It was also called as the Manev potential when $\alpha = 2$ and $\beta = 1$ (see [16,26,30]), and as the Schwarzschild potential when $\alpha = 3$ and $\beta = 1$ (see [1]).

(E4) For all $1 \leq i < j \leq N$, $F_{ij}$ is given by the Lennard-Jones potential (see [6,24,25]), that is, $A|x_i - x_j|^{-12} - B|x_i - x_j|^{-6}$ in $\Delta_\delta$ for some $\delta = \delta(A, B) > 0$, where $A, B > 0$.

(E5) For all $1 \leq i < j \leq N$, $F_{ij}$ is given by the Seeliger-Yukawa potential (see [40,41]), that is, $A e^{-B|x_i - x_j|}|x_i - x_j|^{-1}$ in $\Delta_{1/2}$, where $A, B > 0$.

(E6) For all $1 \leq i < j \leq N$, $F_{ij}$ is given by the Mücket-Treder potential (see [34,38]), that is, $|x_i - x_j|^{-1}(A - B \log |x_i - x_j|)$ in $\Delta_{1/2}$, where $A, B > 0$.

(E7) For all $1 \leq i < j \leq N$, $F_{ij}$ coincides with the logarithmic potential $m_i m_j \log |x_i - x_j|^{-1}$ in $\Delta_{1/2}$. 
Thus, as a direct consequence of Theorem 1.4 (iv), we have the following.

**Corollary 1.5.** Let $F \in C^2(\Omega)$ be as in (1.4) and satisfy (1.8). Suppose that $F$ satisfies any one of (E1)–(E7) near $\Sigma$.

Given any energy constant $\lambda > 0$, any initial configuration $x \in \widetilde{\Omega}$ and any non-collision configuration $a \in \Omega$ with $\|a\| = 1$, there is a hyperbolic motion to the system (1.7) with the energy constant $\lambda$, along $\sqrt{2\lambda}a$ and starting from $x$.

In Sections 3–8, we prove Theorem 1.4 and hence Corollary 1.5; in particular, when $F = U$ with $f(s) = s^{-1}$, we reprove Theorem 1.3. For reader’s convenience, in Section 2 we sketch our geometric proof to Theorem 1.4, and also introduce some necessary notations used in Sections 3–8 and Appendix.

**Remark 1.6.** We remark that it seems possible to extend our geometric approach to general Hamiltonians $H(x, p)$, and also to the setting of (sub)Riemannian manifolds. We would like to see such extension.

**2. Sketch of the Proof to Theorem 1.4**

Note that we only prove Theorem 1.4 under the assumption that

the mass $m_i = 1$ for each $1 \leq i \leq N$. \hspace{1cm}(2.1)

For general masses, by some necessary and standard modifications, our argument also works; we omit the details.

Before sketching the proof, we introduce several notations which are needed below. In the sequel of this paper, we always assume that (2.1) holds and hence the corresponding weighted norm $\| \cdot \|$ (resp. weighted inner product) is then the Euclidean norm (resp. inner product), that is,

$$\|x\| = \left( \sum_{i=1}^{N} |x_i|^2 \right)^{1/2} = |x| \quad \text{and hence}$$

$$\langle \langle x, y \rangle \rangle = \sum_{i=1}^{N} \langle x_i, y_i \rangle \quad \text{for } x, y \in \mathbb{R}^{dN}. \hspace{1cm}(2.2)$$

To measure the distance of any configuration $y \in \mathbb{R}^{dN}$ to the set of collisions, we always set

$$y^b := \min\{|y_i - y_j| : 1 \leq i < j \leq N\}. \hspace{1cm}(2.3)$$

Note that $y^b > 0$ if and only if $y \in \Omega$, that is,

$$\Omega = \{y \in \mathbb{R}^{dN} : y^b > 0\}.$$  

For any $x, y \in \mathbb{R}^{dN}$, write $[x, y]$ (resp. $(x, y)$) as the closed (resp. open) line-segment between them, that is,

$$[x, y] = \{tx + (1 - t)y \}_{t \in [0, 1]} \quad \text{and}$$
\( (x, y) = [x, y] \setminus \{x, y\} = \{tx + (1-t)y \}_{t \in (0,1)}. \) \hfill (2.4)

Write \( \mathbb{R}_+ = (0, \infty) \), the open half ray \( x + \mathbb{R}_+ y = \{x + ty : t \in \mathbb{R}_+\} \) and \( \overline{x + \mathbb{R}_+ y} \) as the closure of \( x + \mathbb{R}_+ y \). Given \( A \subset \mathbb{R}^{dN} \) and \( z \in \mathbb{R}^{dN} \), let us denote \( d_E(z, A) \) the Euclidean distance from \( z \) to \( A \). In particular, \( d_E(z, [x, y]) \) is the Euclidean distance from \( z \) to the line-segment \( [x, y] \), and \( d_E(z, x + \mathbb{R}_+ y) \) is the Euclidean distance from \( z \) to the open half ray \( x + \mathbb{R}_+ y \).

For any absolutely continuous curve \( \gamma : [0, \sigma] \to \mathbb{R}^{dN} \), denote by \( l(\gamma) \) its Euclidean length, that is,

\[ l(\gamma) = \int_0^\sigma |\dot{\gamma}(s)| \, ds. \] \hfill (2.5)

For any two configurations \( a, b \in \mathbb{R}^{dN} \) with \( a, b \neq 0 \), denote by \( \angle(a, b) \) the angle between them, that is the unique element of \( [0, \pi] \) satisfying

\[ \cos \angle(a, b) = \frac{\langle a, b \rangle}{|a||b|}. \]

Denote by \( S^{dN-1}_{\mathbb{R}} \) the set of all configuration \( a \in \mathbb{R}^{dN} \) satisfying \( |a| = 1 \). Note that, for any two configurations \( a, b \in S^{dN-1}_{\mathbb{R}} \), the smallness of \( |b - a| \) is equivalent to that of the angle \( \angle(a, b) \); indeed,

\[ \frac{1}{2} |a - b| = \sin \frac{1}{2} \angle(a, b). \] \hfill (2.6)

Now we are ready to sketch the proof to Theorem 1.4 and hence Corollary 1.5, in particular, when \( \hat{F} = U \) and \( f(s) = s^{-1} \), we sketch a new proof of Theorem 1.3.

**Sketch of the proof to Theorem 1.4.** Given any energy constant \( \lambda > 0 \), any initial configuration \( x \in \tilde{\Omega} \) and any non-collision configuration \( a \in \Omega \) with \( |a| = 1 \), we obtain the desired hyperbolic motion in Theorem 1.4 in 4 steps.

**Step 1.** Writing \( x^* := x + 50(1 + |x|)a/a^b \), we have

\[ (x^* + tb)^b \geq 2 \] and hence \( x^* + tb \in \Omega \) whenever \( t \geq 0 \) and \( b \in S^{dN-1}_{\mathbb{R}} \) with

\[ |b - a| \leq \frac{1}{20} a^b. \] \hfill (2.7)

For any \( t, b \) as in (2.7), and for any \( m_{\lambda} \)-geodesic \( \gamma \) with canonical parameter joining \( x \) and \( x^* + tb \), we have the upper bounds

\[ t \leq l(\gamma) \leq \frac{1}{\sqrt{2\lambda}} m_{\lambda}(x, x^* + tb) \leq t + \Psi(t), \] \hfill (2.8)

where

\[ \Psi(t) := \frac{1}{\sqrt{2\lambda}} m_{\lambda}(x, x^*) + \frac{1}{\lambda} N^2 \frac{1}{a^b} \int_{(x^*)^b}^{2t + 2|x|} f(s) \, ds. \]
Both (2.7) and (2.8) are stated in Section 5. Their proofs are given in Sections 3 and 4 based on the basic geometry of non-collision configurations established in Section 3 and some careful analysis. The choice of the configuration $x^*$ is necessary in general to get (2.8) since $F$ is only assumed to be bounded by $f$ faraway from collisions.

We remark that the growth assumption (1.8) guarantees that

$$
\tilde{\Psi}(t) = \sum_{j \geq [\log_2 t]} \sqrt{2^{-j} \Psi(2^j)} \to 0 \quad \text{as } t \to \infty; \quad (2.9)
$$

see Sect. 5. Below we fix $n_0 \in \mathbb{N}$ large enough, in particular, to guarantee that $\tilde{\Psi}(2n_0) \leq 2^{-10} a^b$.

**Step 2.** For any integer $n \gg n_0$, denote by $\gamma^{(n)} : [0, \sigma^{(n)}] \to \Omega$ an $m_\lambda$-geodesic with canonical parameter joining $x$ and $x^* + 2^n a$. In Sect. 5, we prove the following crucial geometric observation: if $x^* + Sb = \gamma^{(n)}(\tau)$ for some $b \in S^{dN-1}$ with $|a - b| \leq \frac{1}{20} a^b$, and $S \geq 2^{n_0}$ and $\tau \in (0, \sigma^{(n)}]$, then one has

$$
|\gamma^{(n)}(\tau_{1/2}) - (x^* + \frac{S}{2} b)| \leq 2\sqrt{S\Psi(S)} \quad \text{with}
$$

$$
\tau_{1/2} := \max \left\{ 0 < t < \tau : |x^* - \gamma^{(n)}(t)| = \frac{S}{2} \right\}, \quad (2.10)
$$

and

$$
d_E(\gamma^{(n)}(t), x^* + \mathbb{R}^+ b) \leq 4\sqrt{S\Psi(S)}, \quad \forall t \in [\tau_{1/2}, \tau]. \quad (2.11)
$$

To see this, first note that the assumptions in $S$ and $\tau$ allow us to conclude $l(\gamma^{(n)}|[0,\tau]) \leq S + \Psi(S)$ from (2.8). Next, we construct some suitable triangles as illustrated by Figs. 1, 2, 3, 4 and 5 in Section 5, and then bound the lengths of the longest sides via the Euclidean lengths of certain restrictions of $\gamma^{(n)}$, which will be estimated with the aid of $l(\gamma^{(n)}|[0,\tau])$. Using the geometry of triangles, we get the desired (2.10) and (2.11) through careful calculation.

**Step 3.** Thanks to (2.8), (2.10) and (2.11), by an induction argument as illustrated in Fig. 6 of Section 6, we obtain the following upper bound for the Euclidean length

$$
l(\gamma^{(n)}|[0,\tau]) \leq \frac{1}{\sqrt{2\lambda}} m_\lambda(x, \gamma^{(n)}(t)) \leq |\gamma^{(n)}(t) - x^*| (1 + \tilde{\Psi}^2(|\gamma^{(n)}(t) - x^*|)) \quad \text{whenever}
$$

$$
\sigma^{(n)}_{n_0} \leq t < \sigma^{(n)} \quad (2.12)
$$

with

$$
\sigma^{(n)}_{n_0} := \max \{ s \in [0, \sigma^{(n)}] : |\gamma^{(n)}(s) - x^*| = 2^{n_0} \},
$$
and also obtain an upper bound for the angle $\angle(\gamma^{(n)}(t) - x^*, a)$ between $\gamma^{(n)}(t) - x^*$ and $a$, equivalently,

$$|\frac{\gamma^{(n)}(t) - x^*}{|\gamma^{(n)}(t) - x^*|} - a| \leq 2^4 \tilde{\Psi}(|\gamma^{(n)}(t) - x^*|) \quad \text{whenever } \sigma_{n_0}^{(n)} \leq t < \sigma^{(n)}. \quad (2.13)$$

See Section 6 for more details.

**Step 4.** Recall that $\gamma^{(n)}$ has energy constant $\lambda$, that is,

$$|\dot{\gamma}^{(n)}(s)|^2 = 2F \circ \gamma^{(n)}(s) + 2\lambda \quad \text{for almost all } s \in [0, \sigma^{(n)}];$$

see Lemma A.3 in Appendix. Combining this, (2.12) and (2.13), we prove in Section 7 that

$$|\frac{\gamma(t) - x^* - \sqrt{2\lambda}a t}{\sqrt{2\lambda}t}| \leq 2^5 \tilde{\Psi} \left( \frac{1}{2}\sqrt{2\lambda}t \right) \quad \text{whenever } \sigma^{(n)} \geq t \geq \frac{1}{\sqrt{2\lambda}} 2^{n_0+1}. \quad (3.1)$$

Next, sending $n \to \infty$ (up to some subsequence), $\gamma^{(n)}$ converges to some $m_\lambda$-geodesic ray $\gamma$ with canonical parameter, which satisfies

$$|\frac{\gamma(t) - x^* - \sqrt{2\lambda}a t}{\sqrt{2\lambda}t}| \leq 2^5 \tilde{\Psi} \left( \frac{1}{2}\sqrt{2\lambda}t \right) \quad \text{whenever } t \geq \frac{1}{\sqrt{2\lambda}} 2^{n_0+1}. \quad (3.2)$$

Recall (2.9), that is, $\tilde{\Psi}(s) \to 0$ as $s \to \infty$. We know that $\gamma$ is the desired hyperbolic motion starting from $x$ and along $\sqrt{2\lambda}a$. For details see Section 8. \( \square \)

### 3. Some Basic Geometric Properties of Non-collision Configurations

In what follows, we always assume the mass $m_i = 1$ for each $1 \leq i \leq N$. Recall that $\Omega$ is the set of non-collision configurations, that is, all $x \in \mathbb{R}^{dN}$ with $x^b > 0$. We refer to (2.2)–(2.5) and therein for several notations which will be used below.

In this section, we prove several basic geometric properties of non-collision configurations (see Lemma 2.1–Lemma 2.4), which will be used later.

**Lemma 3.1.** Let $b \in \mathbb{R}^{dN}$ and $a \in \mathbb{S}^{dN-1}$ with $a^b > 0$. If $|a - b| \leq \delta a^b$ for some $0 < \delta < \frac{1}{2}$, then

$$b^b > a^b - 2|b - a| \geq (1 - 2\delta)a^b, \quad (3.1)$$

and

$$\cos \angle(a_i - a_j, b_i - b_j) = \frac{\langle a_i - a_j, b_i - b_j \rangle}{|a_i - a_j||b_i - b_j|} \geq 1 - 4\delta, \quad \forall 1 \leq i < j \leq N. \quad (3.2)$$

If we further assume $b \in \mathbb{S}^{dN-1}$, then

$$\cos \angle(a, b) = \langle a, b \rangle \geq 1 - 2\delta^2. \quad (3.3)$$
**Proof.** By the triangle inequality,
\[ |b_i - b_j| \geq |a_i - a_j| - |b_i - a_i| - |b_j - a_j| > |a_i - a_j| - 2|b - a|,\]
\[ \forall 1 \leq i < j \leq N.\]

From this, we conclude
\[ b^b > a^b - 2|b - a|.\]

Thus, by \(|a - b| \leq \delta a^b\), it holds that
\[ b^b \geq (1 - 2\delta)a^b.\]

Moreover, for all \(1 \leq i < j \leq N\), by the Cauchy-Schwartz inequality and the triangle inequality,
\[ \langle a_i - a_j, b_i - b_j \rangle = \langle a_i - a_j, a_i - a_j \rangle + \langle a_i - a_j, b_i - a_i - b_j + a_j \rangle \geq |a_i - a_j|(|a_i - a_j| - 2|b - a|).\]

Note that
\[ |b_i - b_j| \leq |a_i - a_j| + |b_i - a_i| + |b_j - a_j| \leq |a_i - a_j| + 2|b - a|.\]

We then have
\[ \langle a_i - a_j, b_i - b_j \rangle \geq \frac{|a_i - a_j| - 2|b - a|}{|a_i - a_j| + 2|b - a|} \geq 1 - \frac{2|b - a|}{a^b},\]
and hence, by \(|a - b| \leq \delta a^b\), it holds that
\[ \cos \angle(a_i - a_j, b_i - b_j) \geq 1 - 4\delta.\]

For the last inequality (3.2), note that \(a^b < 2\) and
\[ 2 - 2\langle a, b \rangle = |a - b|^2 \leq \delta^2(a^b)^2.\]

This implies
\[ \langle a, b \rangle = 1 - \frac{1}{2}|a - b|^2 \geq 1 - \frac{1}{2}\delta^2(a^b)^2 \geq 1 - 2\delta^2,\]
as desired. \(\Box\)

**Lemma 3.2.** Let \(x \in \mathbb{R}^{dN}\) and \(a \in S^{dN-1}\) with \(a^b > 0\). For \(s \geq 50 \frac{1 + |x|}{a_0}\), we have
\[ \left| \frac{x + sa}{|x + sa|} - a \right| \leq \frac{a^b}{20} \]
and
\[ (x + sa)^b \geq \frac{24}{25} a^b > 2.\]
Proof. For $s > |x|$, write
\[ \left| \frac{x + sa}{|x + sa|} - a \right| = \frac{1}{|x + sa|} |x + sa - |x + sa|a| . \]

By the triangle inequality, one has
\[ |x + sa - |x + sa|a| \leq |x| + |s - |x + sa|| = |x| + ||sa| - |x + sa|| \leq 2|x| , \]
and hence
\[ \left| \frac{x + sa}{|x + sa|} - a \right| \leq \frac{1}{|x + sa|} 2|x| \leq \frac{1}{s - |x|} 2|x| . \]

By $s \geq 50 \frac{1 + |x|}{a^2}$ and $a^b < 2$, we have $s - |x| \geq 40 \frac{|x|}{a^2}$ and hence
\[ \frac{1}{s - |x|} 2|x| \leq \frac{a^b}{20} \]
as desired.

Moreover, it is easy to see that
\[ (x + sa)^b \geq sa^b - 2|x| \geq \frac{24}{25} sa^b . \]
The proof is complete. \[ \square \]

Lemma 3.3. If $z, b \in \Omega$ satisfy
\[ \angle (b_i - b_j, z_i - z_j) \leq \frac{\pi}{2} \text{ or equivalently } \cos \angle (b_i - b_j, z_i - z_j) \geq 0 \ \forall 1 \leq i < j \leq N , \]
then $(z + tb)^b > z^b > 0$, and hence $z + tb \subset \Omega$, for all $t \in [0, \infty)$.

Proof. It suffices to prove that for any $1 \leq i < j \leq N$,
\[ |z_i + tb_i - (z_j + tb_j)| \geq |z_i - z_j| \text{ for all } t > 0 . \]

To see this, write
\[ q = z_i - z_j \text{ and } e = b_i - b_j , \text{ and hence, } |z_i + tb_i - (z_j + tb_j)| = |q + te| . \]

Then $|e| \geq b^b > 0$ and $|q| \geq z^b > 0$. The assumption $\angle (q, e) \leq \pi/2$ implies $\langle q, e \rangle \geq 0$. Thus
\[ |q + et| = \sqrt{\langle q + et, q + et \rangle} = \sqrt{|q|^2 + 2\langle q, e \rangle t + |e|^2 t^2} \geq \sqrt{|q|^2 + |e|^2 t^2} \geq |q| \geq z^b > 0 , \ \forall t \geq 0 \]
as desired. \[ \square \]
Lemma 3.4. Let \( x \in \mathbb{R}^{dN} \) and \( a \in S^{dN-1} \) with \( a^b > 0 \). If \( b \in S^{dN-1} \) with \( |a - b| \leq \frac{a^b}{20} \), then for all \( s \geq 50 \frac{1 + |x|}{a^b} \) and \( t \geq 0 \), one has

\[
(x + sa + tb)^b \geq (x + sa)^b \geq \frac{24}{25} s a^b \geq 2.
\] (3.3)

Therefore, \( x + sa + tb \in \Omega \).

Proof. For any fixed \( s \geq 50 \frac{1 + |x|}{a^b} \), write \( c = x + sa \). By Lemma 3.2, \( |c - a| \leq \frac{a^b}{20} \).

Thus \( |c - b| \leq \frac{b^b}{9} \). By Lemma 3.1, it implies that \( \angle(c, b) \leq \arccos \frac{79}{81} < \pi / 2 \). Then for all \( \delta \geq 0 \), it holds that \( \angle(c + \delta b, b) \leq \angle(c, b) < \pi / 2 \). Hence,

\[
\left| \frac{c + \delta b}{|c + \delta b|} - b \right| = 2 \sin \frac{1}{2} \angle(c + \delta b, b) \\
\leq 2 \sin \frac{1}{2} \angle(c, b) = |c - b| \leq \frac{b^b}{9}, \quad \forall \delta \geq 0.
\] (3.4)

Note that for any \( t \geq 0 \),

\[
\frac{x + sa + tb}{|x + sa + tb|} = \frac{c + \frac{t}{|x + sa|} b}{|c + \frac{t}{|x + sa|} b|}.
\]

By (3.4), it follows that

\[
\left| \frac{x + sa + tb}{|x + sa + tb|} - b \right| = \left| \frac{c + \frac{t}{|x + sa|} b}{|c + \frac{t}{|x + sa|} b|} - b \right| \leq \frac{b^b}{9}.
\]

By Lemma 3.1, it holds that

\[
\cos \angle(x_i + sa_i + tb_i - x_j - sa_j - tb_j, b_i - b_j) \geq \frac{5}{9} > \frac{1}{2}, \quad \forall 1 \leq i < j \leq N.
\]

From Lemma 3.3 and Lemma 3.2, it follows that

\[
(x + sa + tb)^b \geq (x + sa)^b \geq \frac{24}{25} s a^b.
\]

We have completed the proof. \( \square \)
4. An Upper Bound of Euclidean Length of $m_\lambda$-Geodesics

In this section, we build up the following upper bound of the Euclidean length of $m_\lambda$-geodesics with canonical parameter:

**Lemma 4.1.** Let $x \in \tilde{\Omega}$, $a \in \mathbb{S}^{dN-1}$ with $a^b > 0$ and $\lambda > 0$. For any $s \geq 50 \frac{1+|x|}{a^b}$, $b \in \mathbb{S}^{dN-1}$ with $|a-b| \leq \frac{a^b}{100}$, and any $T \geq 0$, if $\gamma \in \mathcal{AC}(x, x + sa + Tb; \emptyset; \tilde{\Omega})$ is an $m_\lambda$-geodesic with canonical parameter, then we have

$$l(\gamma) \leq \frac{1}{\sqrt{2\lambda}} m_\lambda(x, x + sa + Tb) \leq T + \frac{1}{\sqrt{2\lambda}} m_\lambda(x, x + sa) + \frac{1}{\lambda} N^2 \frac{1}{a^b} \int_{(x+sa)^b} f(\tau) d\tau.$$

(4.1)

To prove this we begin with the following trivial upper bound.

**Lemma 4.2.** If $\gamma \in \mathcal{AC}(x, y; [0, \sigma]; \tilde{\Omega})$ is an $m_\lambda$-geodesic with canonical parameter for some $\lambda > 0$, then one has

$$l(\gamma) \leq \frac{1}{\sqrt{2\lambda}} m_\lambda(x, y) \leq |y - x| + \frac{1}{2\lambda} \int_0^{|y-x|} F\left(x + s \frac{y-x}{|y-x|}\right) ds. \quad (4.2)$$

**Proof.** Concerning the the first inequality in (4.2), by the Cauchy-Schwarz inequality one has

$$\frac{1}{2} |\dot{\gamma}(s)|^2 + F(\gamma(s)) + \lambda \geq \frac{1}{2} |\dot{\gamma}(s)|^2 + \lambda \geq 2\sqrt{\lambda} \sqrt{\frac{1}{2} |\dot{\gamma}(s)|^2} = 2\sqrt{\lambda} |\dot{\gamma}(s)|, \quad a.e.$$

Thus

$$m_\lambda(x, y) = A_\lambda(\gamma) \geq 2\sqrt{\lambda} l(\gamma).$$

To see the second inequality in (4.2), we set $a = \frac{y-x}{|y-x|}$ and $\eta(s) = x + \sqrt{2\lambda} sa$ for $s \in [0, \frac{|y-x|}{\sqrt{2\lambda}}]$. By definition, $m_\lambda(x, y) \leq A_\lambda(\eta)$. Since

$$A_\lambda(\eta) = \int_0^{\frac{|y-x|}{\sqrt{2\lambda}}} \left[\lambda + F(x + \sqrt{2\lambda} sa)\right] ds + \lambda \frac{|y-x|}{\sqrt{2\lambda}}$$

$$= \sqrt{2\lambda} |y-x| + \frac{1}{\sqrt{2\lambda}} \int_0^{\frac{|y-x|}{\sqrt{2\lambda}}} F(x + sa) ds,$$

we obtain the second inequality in (4.2), as desired. \hfill \Box

**Remark 4.3.** Note that above if $[x, y] \subset \Omega$, then $\int_0^{\frac{|y-x|}{\sqrt{2\lambda}}} F(x + sa) ds < \infty$; otherwise, it may happen that $\int_0^{\frac{|y-x|}{\sqrt{2\lambda}}} F(x + sa) ds = \infty$. Recall that no assumption is made on the behavior of $F$ in $\Sigma$ essentially. Below via some necessary geometric argument in Section 3, we could only meet the the integral of type $\int_0^A F(y + sb) ds$ where $A > 0$, $y \in \Omega$, $b \in \mathbb{S}^{dN-1} \cap \Omega$ and the line-segment $[y, y + Ab] \subset \Omega$. 

Next we bound \( \int_0^T F(z + sa) \, ds \) for some \( z \) and \( a \) with \( \bar{z} + \mathbb{R}_+ a \subset \Omega \).

**Lemma 4.4.** Let \( \theta \in (0, \pi/2) \), \( z \in \Omega \) with \( z^b \geq 2 \) and \( a \in \mathbb{S}^{d-1} \) with \( a^b > 0 \). If
\[
\sup_{1 \leq i < j \leq N} \angle (a_i - a_j, z_i - z_j) \leq \theta,
\]
then
\[
\int_0^T F(z + sa) \, ds \leq \frac{N^2}{2 \cos \theta} \frac{1}{a^b} \int_{z^b}^{2T + 2|z|} f(s) \, ds
\]
(4.3)
holds for any \( T > 0 \).

**Proof.** By Lemma 3.3, one has \( (z + sa)^b \geq z^b \geq 2 \) for each \( s \geq 0 \). By (1.4) and (1.8), we then obtain
\[
\int_0^T F(z + sa) \, ds \leq \sum_{1 \leq i < j \leq N} \int_0^T F_{ij}(z_i + sai, z_j + sa_j) \, ds
\]
\[
\leq \sum_{1 \leq i < j \leq N} \int_0^T f(|(z_i - z_j) + s(a_i - a_j)|) \, ds.
\]
We claim that, for any \( 1 \leq i < j \leq N \),
\[
\int_0^T f(|(z_i - z_j) + s(a_i - a_j)|) \, ds
\]
\[
\leq \frac{1}{\cos \theta} \frac{1}{|a_i - a_j|} \int_{|z_i - z_j|}^{|z_i - z_j + T(a_i - a_j)|} f(t) \, dt.
\]
(4.4)
Assume this claim holds for the moment. Note that
\[
|z_i - z_j| \geq z^b \geq 2, \quad |a_i - a_j| \geq a^b > 0
\]
and
\[
|z_i - z_j + T(a_i - a_j)| \leq 2|z| + T|a_i - a_j| \leq 2|z| + 2T.
\]
One has
\[
\int_0^T F(z + sa) \, ds \leq \frac{N^2}{2 \cos \theta} \frac{1}{a^b} \int_{z^b}^{2T + 2|z|} f(t) \, dt,
\]
that is, (4.3) holds.

Finally, we prove the above claim (4.4). For any fixed \( 1 \leq i < j \leq N \), write \( q = z_i - z_j \) and \( e = a_i - a_j \), and hence
\[
\int_0^T f(|(z_i - s_j) + s(a_i - a_j)|) \, dt = \int_0^T f(|q + se|) \, dt.
\]
Write \( t(s) = |q + se| \) for \( s \geq 0 \). Since the assumption \( \angle(q, e) \leq \theta < \pi/2 \) implies \( \langle q, e \rangle \geq |q||e| \cos \theta > 0 \), we have

\[
t(s) = \sqrt{\langle q + se, q + se \rangle} = \sqrt{|q|^2 + 2\langle q, e \rangle s + |e|^2 s^2} \geq |q| \geq 2, \quad \forall s \geq 0,
\]

and

\[
\frac{dt}{ds} = \frac{\langle q + se, e \rangle}{|q + se|} \geq \frac{|e||q| \cos \theta + |e|^2 s}{|q + se|} \geq |e| \cos \theta \frac{|q| + |e|s}{|q + se|} \geq |e| \cos \theta > 0, \quad \forall s \geq 0.
\]

Thus \( t = t(s) : [0, \infty) \rightarrow [|q|, \infty) \) is a strictly increasing \( C^1 \) function. Obviously,

\[
\frac{ds}{dt} \leq \frac{1}{|e| \cos \theta}.
\]

Hence, by the change of variable, we have

\[
\int_0^T f(|q + se|) \, ds \leq \frac{1}{\cos \theta} \frac{1}{|e|} \int_{|q|}^{|q + Te|} f(t) \, dt,
\]

as desired. \( \Box \)

As a consequence of Lemma 4.4 and Lemma 3.1, we have the following:

**Corollary 4.5.** Let \( z \in \mathbb{R}^{dN} \) with \( z^b \geq 2 \) and \( a \in S^{dN-1} \) with \( a^b > 0 \). If

\[
\left| \frac{z}{|z|} - a \right| \leq \frac{a^b}{20},
\]

then for any \( b \in S^{dN-1} \) with \( |a - b| \leq \frac{a^b}{20} \), we have

\[
\int_0^T F(z + sb) \, ds \leq N^2 \frac{2}{a^b} \int_{z^b}^{2T+2|z|} f(s) \, ds, \quad \forall T > 0.
\]

**Proof.** Since \( b \in S^{dN-1} \) and \( |a - b| \leq \frac{a^b}{20} \), by Lemma 3.1, one has \( b^b > \frac{9}{10} a^b \). It follows that

\[
\left| \frac{z}{|z|} - b \right| \leq \left| \frac{z}{|z|} - a \right| + |a - b| \leq \frac{a^b}{10} \leq \frac{b^b}{9}.
\]

By Lemma 3.1, it holds that

\[
\cos \angle(b_i - b_j, z_i - z_j) = \cos \angle \left( b_i - b_j, \frac{z_i}{|z|} - \frac{z_j}{|z|} \right) \geq 1 - \frac{4}{9} \geq \frac{1}{2}.
\]

Thus by applying Lemma 4.4 with \( \theta = \arccos \frac{1}{2} \) and \( b^b > \frac{9}{10} a^b \), we get the desired estimate. \( \Box \)

As a consequence of Corollary 4.5 and Lemma 3.2, the following result holds:
Corollary 4.6. Let $x \in \mathbb{R}^{dN}$ and $a \in \mathbb{S}^{dN-1}$ with $a^b > 0$. If $s \geq 50 \frac{1+|x|}{a^b}$, then for any $b \in \mathbb{S}^{dN-1}$ with $|a - b| \leq \frac{a^b}{20}$, we have

$$\int_{0}^{T} F(x + sa + t)b \ dt \leq N^2 \frac{2}{a^b} \int_{(x+sa)^b}^{2^{s+2}|x+sa|} f(t) \ dt, \ \forall T > 0. \ (4.5)$$

Proof. Given any $s \geq 50 \frac{1+|x|}{a^b}$, let $z = x + sa$. By Lemma 3.2, we know that $z^b \geq 2$ and $|z| - a| \leq \frac{a^b}{20}$. Then Corollary 4.5 implies inequality (4.5). \qed

We now apply Corollary 4.6 to prove Lemma 4.1.

Proof of Lemma 4.1. By Lemma 3.4, for any $T \geq 0$, one has

$$(x + sa + Tb)^b \geq (x + sa)^b \geq sa^b - 2|x| > 48|x| \geq 0.$$ 

It follows that $x + sa + Tb \in \Omega$. Thus $m_{\lambda}(x, x + sa + Tb) < \infty$.

For any $m_{\lambda}$-geodesic $\gamma : [0, \sigma] \rightarrow \mathbb{G}$ joining $x$ and $x + sa + Tb$, it holds that

$$l(\gamma) \leq \frac{1}{\sqrt{2}\lambda} m_{\lambda}(x, x + sa + Tb).$$

Note that

$$m_{\lambda}(x, x + sa + Tb) \leq m_{\lambda}(x, x + sa) + m_{\lambda}(x + sa, x + sa + Tb).$$

By Lemma 4.2, one has

$$m_{\lambda}(x + sa, x + sa + Tb) \leq \sqrt{2\lambda} T + \frac{1}{\sqrt{2\lambda}} \int_{0}^{T} F(x + sa + t b) \ dt.$$ 

By Corollary 4.6, it holds that

$$\int_{0}^{T} F(x + sa + t b) \ dt \leq 2N^2 \frac{1}{a^b} \int_{(x+sa)^b}^{2^{s+2}|x+sa|} f(t) \ dt,$$

and hence

$$m_{\lambda}(x + sa, x + sa + Tb) \leq \sqrt{2\lambda} T + \sqrt{\frac{2}{\lambda}} N^2 \frac{1}{a^b} \int_{(x+sa)^b}^{2^{s+2}|x+sa|} f(t) \ dt.$$ 

We therefore obtain (4.1). \qed
5. A Key Geometric Observation for $m_\lambda$-Geodesics

To state our geometric observation, we first introduce some conventions. Given $x \in \bar{\Omega}$ and $a \in S^{dN-1}$ with $a^\flat > 0$, we always write

$$x^* := x + (50 \frac{1 + |x|}{a^\flat})a.$$  

By Lemma 3.4, one has

$$(x^* + Tb)^\flat \geq 2, \text{ and hence } x^* + Tb \in \bar{\Omega},$$

$$\forall T \geq 0 \text{ and } b \in S^{dN-1} \text{ with } |a - b| \leq \frac{a^\flat}{20}.$$  

Given any $\lambda > 0$, we always set

$$\Psi(T) = \Psi_{\lambda,x,a}(T) := \frac{1}{\sqrt{2\lambda}} m_\lambda(x, x^*) + \frac{1}{\lambda} N^2 \frac{1}{a^\flat} \int_{(x^*)^\flat}^{2T+2|x^*|} f(t) \, dt, \quad \forall T \geq 0. \quad (5.1)$$

Note that the assumption (1.8) guarantees

$$\sum_{j=1}^{\infty} \sqrt{2^{-j} \Psi(2^j)} < \infty. \quad (5.2)$$

Indeed, let $\xi \in \mathbb{N}$ such that $2^{\xi-1} \leq |x^*| < 2^\xi$.

By (5.1), we have

$$\sum_{j=1}^{\infty} \sqrt{2^{-j} \Psi(2^j)} \leq \frac{1}{(2\lambda)^{1/4}} \sum_{j \geq 1} \sqrt{2^{-j} \Psi(2^j)}$$

$$+ \frac{N}{(\lambda a^\flat)^{1/2}} \sum_{j \geq 1} \sqrt{2^{-j} \int_{(x^*)^\flat}^{2j+1+2|x^*|} f(t) \, dt}$$

$$\leq \frac{3}{(2\lambda)^{1/4}} \sqrt{m_\lambda(x, x^*)} + \frac{N}{(\lambda a^\flat)^{1/2}} \sum_{j \geq 1} \sqrt{2^{-j} \int_{1}^{2j+1+2j+2|x^*|} f(t) \, dt}$$

where in the last inequality we use the facts that $|x^*| < 2^\xi$ and $2^a + 2^b \leq 2^{a+b}$ whenever $a \geq 1$ and $b \geq 1$. Note that

$$\sum_{j \geq 1} \sqrt{2^{-j} \int_{1}^{2j+1+2j+2} f(t) \, dt} \leq \sum_{j \geq 1} \sum_{k=0}^{j+\xi+1} 2^{-j/2} \sqrt{\int_{2^k}^{2^{k+1}} f(t) \, dt}$$

$$\leq C(x^*) \sum_{k \geq 0} \sqrt{2^{-k} \int_{2^k}^{2^{k+1}} f(t) \, dt}.$$
for some positive constant $C(x^*)$. Thanks to (1.8), the right-hand side of the last inequality is finite. Thus (5.2) holds.

We further write
\[
\tilde{\Psi}(t) := \sum_{j \geq \lfloor \log_2 t \rfloor + 1} \sqrt{2^{-j} \Psi(2^j)} < \infty, \quad \forall \, t \geq 1.
\]

Then $\tilde{\Psi}(t)$ is decreasing to 0 as $t \to \infty$. Moreover,
\[
\sqrt{t^{-1} \Psi(t)} \leq \tilde{\Psi}(t), \quad \forall \, t \geq 1.
\]

To see this, letting $2^k \leq t < 2^{k+1}$ for some $k \in \mathbb{N}$, we have
\[
\tilde{\Psi}(t) = \sum_{j \geq k+1} \sqrt{\frac{\Psi(2^j)}{2^j}} \geq \sqrt{\Psi(2^{k+1})} \sum_{j \geq k+1} \frac{1}{2^j}
\]
\[
\geq \sqrt{\Psi(2^{k+1})} \frac{2^{-(k+1)}}{1 - 2^{-1}} \geq \sqrt{\frac{\Psi(2^{k+1})}{2^k}} \geq \sqrt{\frac{\Psi(t)}{t}},
\]
where in the first and last "$\geq$" we use the fact that $\Psi(t)$ is increasing with respect to $t$, and in the second last "$\geq$" we use the fact $(1 - 2^{-1})^{-1} \geq \sqrt{2}$.

From (5.3), it follows that
\[
\tilde{\Psi}(t) = \sum_{j = n}^\infty \sqrt{2^{-j} \Psi(2^j)} = 2^{-10} a^b < \infty.
\]

Obviously, one has
\[
\Psi(T) \leq 2^{-20} (a^b)^2 T, \quad \text{for all } T \geq 2^{n_0}.
\]

We remark that with the aid of $x^*$ and $\Psi$, we restate Lemma 4.1 with $s = 50 \frac{1 + |x|}{a^b}$ as follows:

**Lemma 4.1**. Let $b \in S^{dN-1}$ with $|a - b| \leq \frac{a^b}{20}$. For any $T > 0$, and for any $m_\lambda$-geodesic $\gamma$ with canonical parameter joining $x$ and $x^* + Tb$, we have
\[
l(\gamma) \leq \frac{1}{\sqrt{2^\lambda}} m_\lambda(x, x^* + Tb) \leq T + \Psi(T).
\]

We are ready to state our geometric observation as below, which plays a key role in the proof of Theorem 1.4.

**Lemma 5.1.** Let $\gamma \in AC(x, x^* + 2^n a; [0, \sigma]; \tilde{\Omega})$ be an $m_\lambda$-geodesic with canonical parameter, where $n > n_0$. Let $b \in S^{dN-1}$ with $|a - b| \leq \frac{a^b}{10}$. Suppose that $x^* + Sb = \gamma(\tau)$ for some $S \geq 2^{n_0}$ and $\tau \in (0, \sigma]$. Then
\[
|\gamma(\tau_{1/2}) - (x^* + \frac{S}{2} b)| \leq 2\sqrt{S}\Psi(S),
\]
where
\[
\tau_{1/2} := \max \left\{ 0 < t < \tau : |x^* - \gamma(t)| = \frac{S}{2} \right\};
\]
and moreover,
\[
d_E(\gamma(t), x^* + \mathbb{R}_+ b) \leq 4\sqrt{S\Psi(S)}, \quad \forall t \in [\tau_{1/2}, \tau]. \tag{5.8}
\]
We first prove (5.7) in Lemma 5.1.

**Proof of (5.7) in Lemma 5.1.** Let
\[
\hat{\tau}_{1/2} := \max\{0 < t < \tau : |x^* - \gamma(t)| = |\gamma(t) - (x^* + bS)|\}.
\]
Observing \(|\gamma(\hat{\tau}_{1/2}) - x^*| \geq \frac{S}{2}\), we have \(\tau_{1/2} \leq \hat{\tau}_{1/2}\) (See Fig. 1 for an illustration).

By (5.5),
\[
\Psi(S) \leq 2^{-20}(a^b)^2 S \leq 2^{-18} S. \tag{5.9}
\]
To get (5.7), it suffices to prove that
\[
|\gamma(\hat{\tau}_{1/2}) - (x^* + b\frac{S}{2})| \leq (1 + 2^{-19})\sqrt{S\Psi(S)} \tag{5.10}
\]
and
\[
|\gamma(\hat{\tau}_{1/2}) - \gamma(\tau_{1/2})| \leq 2\Psi(S) \leq 2^{-8} \sqrt{S\Psi(S)}. \tag{5.11}
\]
Now, we prove the above two inequalities: (5.10) and (5.11). As in Fig. 1, we write
\[
y = x^* + Sb, \quad w = x^* + \frac{S}{2}b, \quad p = \gamma(\hat{\tau}_{1/2}) \quad \text{and} \quad q = \gamma(\tau_{1/2}).
\]
Note that \(w\) is the middle point of the straight line segment \([x^*, y]\). One has
\[
|x^* - p| = |p - y| > \frac{S}{2} \quad \text{and} \quad |x^* - w| = |w - y| = \frac{1}{2} |x^* - y| = \frac{S}{2}.
\]
Obviously,
\[
\langle p - w, x^* - y \rangle = \langle p - w, b \rangle = 0.
\]

**Proof of (5.10).** Note that
\[
|x^* - p| = \frac{1}{2} \left[ |x^* - p| + |p - y| \right] \leq \frac{1}{2} \left[ |x - p| + |p - y| + |x - x^*| \right].
\]
Since \(|x - p| \leq l(\gamma|_{[0, \hat{\tau}_{1/2}]})\) and \(|p - y| \leq l(\gamma|_{[\hat{\tau}_{1/2}, \tau]})\), we obtain
\[
|x^* - p| \leq \frac{1}{2} \left[ l(\gamma|_{[0, \hat{\tau}_{1/2}]}) + l(\gamma|_{[\hat{\tau}_{1/2}, \tau]}) + |x - x^*| \right] = \frac{1}{2} l(\gamma|_{[0, \tau]}) + \frac{1}{2} |x - x^*|.
By (5.6),

\[ I(y|_{[0, \tau_1]} \leq S + \Psi(S), \]

we therefore obtain

\[ |x^* - p| \leq \frac{1}{2} S + \frac{1}{2} \Psi(S) + \frac{1}{2} |x - x^*|. \]

Note that \( S \geq 2^{n_0}, \Psi(S) \geq \Psi(2^{n_0}) \) and \( \Psi(S) \leq 2^{-18} S \). By the definition of \( n_0 \) in (5.4) and Lemma 4.2, we know that

\[ |x - x^*| \leq 2^{n_0 - 20} \leq 2^{-20} S \quad \text{and} \quad |x - x^*| \leq \frac{1}{\sqrt{2\lambda}} m_\lambda(x, x^*) < \Psi(S). \]

Thus,

\[ |p - w| = \sqrt{|x^* - p|^2 - |x^* - w|^2} \]

\[ \leq \sqrt{\left[ \frac{S}{2} + \frac{1}{2} \Psi(S) + \frac{1}{2} |x - x^*|^2 \right] - \left[ \frac{S}{2} \right]^2} \]

\[ = \sqrt{\frac{1}{2} \Psi(S)[S + \frac{1}{2} \Psi(S)] + \frac{1}{2} |x - x^*|\left| \frac{1}{2} |x - x^*| + S + \Psi(S) \right|} \]

\[ \leq \sqrt{S\Psi(S)} \left( 1 + 2^{-19} + 2^{-20} + 2^{-22} \right)^{1/2} \]

\[ \leq \sqrt{S\Psi(S)} (1 + 2^{-18})^{1/2} \]

\[ \leq \sqrt{S\Psi(S)} (1 + 2^{-19}) \]

where in the last inequality we applied \((1 + c)^{1/2} \leq 1 + \frac{c}{2}\) for any \( c \geq 0 \). Therefore, (5.10) holds.

**Proof of (5.11).** Recalling \( p = \gamma(\hat{\tau}_{1/2}) \) and \( \langle p - w, w - y \rangle = 0 \) one has

\[ I(y|_{[\hat{\tau}_{1/2}, \tau]} \geq |p - y| \geq |w - y| = \frac{S}{2}, \]

and

\[ I(y|_{[0, \hat{\tau}_{1/2}]}) \geq |x - p| \geq |x^* - p| - |x^* - x| \geq \frac{S}{2} - |x^* - x|. \]

From this and (5.6), it follows that

\[ I(y|_{[\hat{\tau}_{1/2}, \tau]} = I(y|_{[0, \tau]} - I(y|_{[0, \hat{\tau}_{1/2}]}) \leq S + \Psi(S) - \frac{S}{2} + |x^* - x| \]

\[ = \frac{S}{2} + \Psi(S) + |x^* - x|. \]

Moreover, by recalling \(|x^* - q| = S/2 \) and Lemma 4.2, we have

\[ I(y|_{[0, \hat{\tau}_{1/2}]}) \geq |x - q| \geq |q - x^*| - |x^* - x| \geq \frac{S}{2} - \frac{1}{\sqrt{2\lambda}} m_\lambda(x, x^*) \geq \frac{S}{2} - \Psi(S). \]
From this and (5.6), one deduces that

\[
\begin{align*}
  l(\gamma|_{[\tau_1/2, \hat{\tau}_1/2]}) &= l(\gamma|_{[0, \tau_1]}) - l(\gamma|_{[0, \tau_1/2]}) \\
  &\leq S + \Psi(S) - \frac{S}{2} + \Psi(S) \\
  &\leq \frac{S}{2} + 2\Psi(S). 
\end{align*}
\]  

(5.12)

Since \(\tau_1/2 \leq \hat{\tau}_1/2\), one has

\[
|p - q| \leq l(\gamma|_{[\tau_1/2, \hat{\tau}_1/2]}) = l(\gamma|_{[\tau_1/2, \tau_1]}) - l(\gamma|_{[\hat{\tau}_1/2, \tau_1]}) \leq \frac{S}{2} + 2\Psi(S)
\]

\[
-\frac{S}{2} \leq 2\Psi(S) \leq 2^{-8}\sqrt{S\Psi(S)}.
\]

Thus,

\[
|p - q| \leq 2\Psi(S) \leq 2^{-8}\sqrt{S\Psi(S)},
\]

which shows (5.11).

Next we prove (5.8) in Lemma 5.1.
Proof of (5.8) in Lemma 5.1. Recall that $y = \gamma(\tau)$ and $q = \gamma(\tau_{1/2})$. For any $t \in (\tau_{1/2}, \tau)$, the definition of $\tau_{1/2}$ implies that $|\gamma(t) - x^*| \geq \frac{S}{2}$, moreover there is a unique $z \in [q, y]$ such that

$$d_E(\gamma(t), [q, y]) = |\gamma(t) - z|.$$ 

It then suffices to prove

$$|\gamma(t) - z| \leq 2\sqrt{S\Psi(S)}. \quad (5.13)$$

Indeed, considering $y = \gamma(\tau)$, inequalities (5.10) and (5.11) imply that

$$d_E(z, x^* + \mathbb{R}_+b) \leq d_E(z, [x^*, y]) \leq d_E(q, [x^*, y]) \leq |q - p| + |p - w| \leq 2\sqrt{S\Psi(S)}.$$ 

Thus

$$d_E(\gamma(t), x^* + \mathbb{R}_+b) \leq |\gamma(t) - z| + d_E(z, x^* + \mathbb{R}_+b) \leq 4\sqrt{S\Psi(S)},$$

which shows (5.8). See Fig. 3 for illustration.

Finally, we prove (5.13) by considering three cases: Case $z \in (q, y) = (\gamma(\tau_{1/2}), \gamma(\tau))$, Case $z = y = \gamma(\tau)$ and Case $z = q = \gamma(\tau_{1/2})$. Recall that $(q, y) := \{sq + (1-s)y : s \in (0, 1)\}$. Moreover, since $\gamma|_{[\tau_{1/2}, \tau]}$ joins $q$ to $\gamma(t)$ and also $\gamma|_{[0, \tau]}$ joins $\gamma(t)$ to $y$, we always have

$$|q - \gamma(t)| + |\gamma(t) - y| \leq l(\gamma|_{[\tau_{1/2}, \tau]}) + l(\gamma|_{[0, \tau]}) = l(\gamma|_{[\tau_{1/2}, \tau]}). \quad (5.14)$$

Case $z \in (q, y)$. The proof in this case is illustrated by Fig. 4.

In this case, we first observe that

$$\langle \gamma(t) - z, q - y \rangle = 0. \quad (5.15)$$

Geometrically, (5.15) is obvious. A standard proof to (5.15) goes as follows. If $|\delta| < \delta_0$ for some $\delta_0 > 0$, one has $z + \delta(q - y) \in (q, y)$. By the definition of $z$, the function

$$h(\delta) := |\gamma(t) - z - \delta(q - y)|^2$$

reaches minimal in $(-\delta_0, \delta_0)$ at $\delta = 0$. Thus $0 = h'(0) = -2\langle \gamma(t) - z, q - y \rangle$, which gives (5.15).

By (5.15), we have

$$|\gamma(t) - z|^2 = \frac{1}{2} \left[ |q - \gamma(t)|^2 - |q - z|^2 + |\gamma(t) - y|^2 - |y - z|^2 \right]$$

$$= \frac{1}{2} \left[ (|q - \gamma(t)| + |\gamma(t) - y|)^2 - 2|q - \gamma(t)||y - \gamma(t)| \right.$$

$$- (|q - z| + |y - z|)^2 + 2|q - z||y - z|].$$

Note that $z \in (q, y)$ implies

$$|q - z| + |y - z| = |q - y|. $$
By \( (\gamma(t) - z, q - y) = 0 \), one has
\[
|q - z| \leq |q - \gamma(t)|, \quad |y - z| \leq |y - \gamma(t)|,
\]
and hence
\[
|q - \gamma(t)||y - \gamma(t)| \geq |q - z||y - z|.
\]
These together with (5.14) yield
\[
|\gamma(t)-z|^2 \leq \frac{1}{2} \left[ l(\gamma|_{\tau_1/2, \tau})^2 - |q - y|^2 \right]. \tag{5.16}
\]
Since \( |q - x^*| = \frac{S}{2} \), one has
\[
|q - y| \geq |x^* - y| - |x^* - q| = \frac{S}{2}. \tag{5.17}
\]
By (5.12), (5.16) and (5.17), we further obtain
\[
|\gamma(t)-z|^2 \leq \frac{1}{2} \left( \frac{S}{2} + 2\Psi(S))^2 - \left(\frac{S}{2}\right)^2 \right) = S\Psi(S) + 2\Psi(S)^2.
\]
Since \( \Psi(S) < 2^{-18}S \) by (5.9), we have \( |\gamma(t) - z| < 2\sqrt{S\Psi(S)} \) as desired.

\textit{Case } \( z = y \). The proof in this case is illustrated by Fig. 5 below.

In this case, note that
\[
\angle(\gamma(t) - y, q - y) \geq \frac{\pi}{2}, \tag{5.18}
\]
Geometrically, (5.18) is obvious. We give a proof. If \( 0 \leq \delta \leq 1 \), one has
\[
z + \delta(q - y) = y + \delta(q - y) \in [q, y].
\]
By the definition of \( z \), the function
\[
h(\delta) := |\gamma(t) - z - \delta(q - y)|^2
\]
reaches minimal in \([0, 1]\) at \( \delta = 0 \). Thus
\[
0 \leq \lim_{\delta \to 0+} \frac{1}{\delta} [h(\delta) - h(0)] = -2(\gamma(t) - y, q - y)
\]
which gives (5.18) as desired.

Considering the triangle \( \triangle(\gamma(t), \gamma(\tau), \gamma(\tau_{1/2})) \). (5.17) and (5.18) imply that
\[
|\gamma(t) - q| \geq |q - y| \geq \frac{S}{2}.
\]
Applying (5.14) and (5.12), by \( z = y \) we therefore obtain
\[
|\gamma(t) - z| = \left[ |\gamma(t) - q| + |\gamma(t) - y| \right] - |\gamma(t) - q|
\leq \frac{S}{2} + 2\Psi(S) - \frac{S}{2} = 2\Psi(S) < 2\sqrt{S\Psi(S)}.
\]

\textit{Case } \( z = q \). The proof in this case is much similar to Case \( z = y \). We omit the details.

This ends the proof. \( \square \)
Fig. 3. An illustration of showing (5.8)

Fig. 4. An illustration of Case $z \in (q, y)$

Fig. 5. An illustration of Case $z \notin [\gamma(\tau_{1/2}), \gamma(\tau)]$

6. Estimates of Euclidean Length and Angle of $m_{\lambda}$-Geodesics

Given any $x \in \tilde{\Omega}$, any $a \in S^{dN-1}$ with $a^b > 0$, and any $\lambda > 0$, let $x^*$, $\Psi$, $\tilde{\Psi}$ and $n_0$ be as in Section 5.

Lemma 6.1. Let $\gamma \in AC(x, x^* + 2^n a; [0, \sigma]; \tilde{\Omega})$ be an $m_{\lambda}$-geodesic with canonical parameter, where $n > n_0$. Set

$$\sigma_{n_0} := \max\{s \in [0, \sigma] : |\gamma(s) - x^*| = 2^n\}.$$
For $t \in [\sigma_{n_0}, \sigma]$, we have
\begin{equation}
|\gamma(t) - x^*| - a \leq 2\tilde{\psi}(|\gamma(t) - x^*|) 
\end{equation}
and
\begin{equation}
\ell(\gamma|_{[0,t]}) \leq \frac{1}{\sqrt{2\lambda}}m_\lambda(x, \gamma(t)) \leq |\gamma(t) - x^*|(1 + \tilde{\psi}^2(|\gamma(t) - x^*|)).
\end{equation}

In particular,
\begin{equation}
\gamma(t) \in \Omega \quad \text{with} \quad (\gamma(t))^b \geq (x^*)^b > 0 \quad \text{for all} \quad t \in [\sigma_{n_0}, \sigma].
\end{equation}

**Proof.** For $n_0 \leq i \leq n$, set
\[
\sigma_i := \max\{t \in (0, \sigma) : |x^* - \gamma(t)| = 2^i\}.
\]
Write
\[
a^{(n)} = a, \quad \text{and} \quad a^{(i)} := 2^{-i}(\gamma(\sigma_i) - x^*) \quad \text{for} \quad n_0 \leq i \leq n - 1.
\]
Obviously,
\[
a^{(i)} = \frac{\gamma(\sigma_i) - x^*}{|\gamma(\sigma_i) - x^*|} \in S^{dN-1}.
\]

**Step 1.** We claim that
\begin{equation}
|a^{(i)} - a^{(i+1)}| \leq 4\sqrt{2^{-i+1}\psi(2^{i+1})}, \quad \text{for any} \quad n_0 \leq i \leq n - 1.
\end{equation}

We prove this by induction; see Fig. 6 for an illustration.

First, we see that (6.4) holds for $i = n - 1$. Indeed, applying Lemma 5.1 with $b = a = a^{(n)}$, $\tau = \sigma$ and $\tau_{i/2} = \sigma_{n-1}$, we know that
\[
|\gamma(\sigma_{n-1}) - (x^* + 2^{n-1}a)| \leq 2\sqrt{2^n\psi(2^n)}.
\]

Hence dividing both sides by $2^{n-1}$ in the above inequality, we have
\[
|a^{(n-1)} - a^{(n)}| = \frac{|\gamma(\sigma_{n-1}) - x^*|}{2^{n-1}} - a \leq 4\sqrt{2^{-n}\psi(2^n)} \leq 2^{-8}a^b,
\]
where in the last inequality we use (5.5). Then by Lemma 3.1, we know $a^{(n-1)} \in \Omega$.

Next, given any $j \geq n_0$, assume that (6.4) holds for all $j + 1 \leq i \leq n - 1$. By the definition of $n_0$ in (5.4), it holds that $\tilde{\psi}(2^{n_0}) \leq 2^{-10}a^b$. This implies
\[
|a^{(j+1)} - a^{(n)}| \leq \sum_{i=j+1}^{n-1} |a^{(i)} - a^{(i+1)}| \leq \sum_{i=j+2}^{\infty} 4\sqrt{2^{-i}\psi(2^i)} = 4\tilde{\psi}(2^{j+2}) \leq 4\tilde{\psi}(2^{n_0}) \leq 2^{-8}a^b.
\]
Here, again, in the last inequality we use (5.5) and \( a^{(j)} \in \Omega \). By this, applying Lemma 5.1 with \( b = a^{(j+1)} \), \( \tau = \sigma_{j+1} \) and \( \tau_{1/2} = \sigma_j \), we have

\[
|\gamma(\sigma_j) - (x^* + 2^ja^{(j+1)})| \leq 2\sqrt{2^{j+1}\Psi(2^{j+1})}
\]

Since \( \gamma(\sigma_j) = x^* + 2^ja^{(j)} \), we have

\[
|a^{(j)} - a^{(j+1)}| \leq 4\sqrt{2^{-(j+1)}\Psi(2^{j+1})}.
\]

That is, (6.4) holds for \( i = j \).

Thus (6.4) holds for \( n_0 \leq i \leq n - 1 \) as desired.

**Step 2.** We show (6.1), that is,

\[
|\gamma(\sigma_i) - x^*| - a \leq 4\tilde{\Psi}(|\gamma(\sigma_i) - x^*|), \quad \text{for } t \in [\sigma_{n_0}, \sigma].
\]

By Step 1, if \( n_0 \leq i \leq n - 1 \), we have

\[
|a^{(i)} - a^{(n)}| \leq \sum_{j=i+1}^{n} |a^{(j-1)} - a^{(j)}| \leq \sum_{j=i}^{\infty} 4\sqrt{2^{-j}\Psi(2^j)} = 4\tilde{\Psi}(2^i).
\]

That is,

\[
|\gamma(\sigma_i) - x^*| - a \leq 4\tilde{\Psi}(|\gamma(\sigma_i) - x^*|).
\]

For any \( t \in [\sigma_i, \sigma_{i+1}] \), by the definition of \( \sigma_i \) and \( \sigma_{i+1} \), we know

\[
|\gamma(t) - x^*| \geq |\gamma(\sigma_i) - x^*| = 2^i.
\]

Applying Lemma 5.1 with \( b = a^{(i+1)} \) in (5.8), \( \tau = \sigma_{i+1} \) and \( \tau_{1/2} = \sigma_i \), by the increasing property of \( \Psi \) we have

\[
d_E(\gamma(t), x^* + \mathbb{R}^+a^{(i+1)}) \leq 4\sqrt{2^{i+1}\Psi(2^{i+1})} \leq 4\sqrt{2|\gamma(t) - x^*|\Psi(2|\gamma(t) - x^*|)}.
\]

Then

\[
\sin \angle(\gamma(t) - x^*, a^{(i+1)}) = \frac{d_E(\gamma(t), x^* + \mathbb{R}^+a^{(i+1)})}{|\gamma(t) - x^*|} \leq \frac{4\sqrt{2^{i+1}\Psi(2^{i+1})}}{2^i}.
\]

By (5.9), we know that \( 2^{-(i+1)}\Psi(2^{i+1}) \leq 2^{-18} \) and hence

\[
\sin \angle(\gamma(t) - x^*, a^{(i+1)}) \leq 2^{-6}.
\]

This also gives

\[
\cos \angle(\gamma(t) - x^*, a^{(i+1)}) \geq \sqrt{1 - 2^{-12}} \geq \frac{9}{10}.
\]
It follows that
\[
|\gamma(t) - (x^* + |\gamma(t) - x^*|a^{(i+1)})| \\
= d_E(\gamma(t), x^* + \mathbb{R}_+a^{(i+1)}) \cdot \frac{1}{\cos(\frac{1}{2} \angle (\gamma(t) - x^*, a^{(i+1)}))} \\
\leq d_E(\gamma(t), x^* + \mathbb{R}_+a^{(i+1)}) \cdot \frac{1}{\cos \angle (\gamma(t) - x^*, a^{(i+1)}))} \\
\leq \frac{10}{9} d_E(\gamma(t), x^* + \mathbb{R}_+a^{(i+1)}).
\]

We conclude that
\[
|\frac{\gamma(t) - x^*}{|\gamma(t) - x^*|} - a^{(i+1)}| = \frac{|\gamma(t) - (x^* + |\gamma(t) - x^*|a^{(i+1)})|}{|\gamma(t) - x^*|} \\
\leq \frac{10}{9} \frac{d_E(\gamma(t), x^* + \mathbb{R}_+a^{(i+1)})}{|\gamma(t) - x^*|} \\
\leq \frac{80}{9} \sqrt{\Psi(2|\gamma(t) - x^*|)} \\
\leq \frac{80}{9} \tilde{\Psi}(|\gamma(t) - x^*|).
\]

Thus for any \( t \in [\sigma_i, \sigma_{i+1}] \), it holds that
\[
|\frac{\gamma(t) - x^*}{|\gamma(t) - x^*|} - a| \leq \frac{|\gamma(t) - x^*|}{|\gamma(t) - x^*|} - a^{(i+1)}| + |a^{(i+1)} - a| \\
\leq \frac{80}{9} \tilde{\Psi}(|\gamma(t) - x^*|) + 4\tilde{\Psi}(2^{i+1}) \\
\leq 16\tilde{\Psi}(|\gamma(t) - x^*|).
\]

**Step 3.** For \( t \geq \sigma_{n_0} \), we have
\[
16\tilde{\Psi}(|\gamma(t) - x^*|) \leq 16\tilde{\Psi}(2^{n_0}) \leq 2^{-6}a^b,
\]
and hence
\[
|\frac{\gamma(t) - x^*}{|\gamma(t) - x^*|} - a| \leq 2^{-6}a^b.
\]

This shows \( \frac{\gamma(t) - x^*}{|\gamma(t) - x^*|} \in \Omega \) by Lemma 3.1. Write
\[
\gamma(t) = x^* + Tb = x + sa + Tb \quad \text{with} \quad b = \frac{\gamma(t) - x^*}{|\gamma(t) - x^*|} \quad \text{and} \quad T = |\gamma(t) - x^*|.
\]

By (3.3), we see that \( (\gamma(t))^b \geq (x^*)^b > 0 \) and \( \gamma(t) \in \Omega \), that is, (6.3) holds.

Recalling (5.3), that is \( \sqrt{i-1}\Psi(t) \leq \tilde{\Psi}(t) \), we conclude (6.2) from (5.6). \( \square \)
7. Large Time Behavior of $m_\lambda$-Geodesics

Given any $x \in \tilde{\Omega}$, any $a \in \mathbb{S}^{dN-1}$ with $a^b > 0$, and any $\lambda > 0$, let $x^*, \Psi, \tilde{\Psi}$ and $n_0$ be as in Section 5.

**Lemma 7.1.** Let $\gamma \in AC(x, x^* + 2^n a; [0, \sigma]; \tilde{\Omega})$ be an $m_\lambda$-geodesic with canonical parameter, where $n > n_0$. Then for any $t \geq \sigma n_0$, we have

$$\int_0^t |\dot{\gamma}(s)|^2 \ ds \leq \sqrt{2\lambda} \gamma(t) - x^* |(1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)) \leq 8\lambda t, \quad (7.1)$$

$$\frac{1}{2} \leq \frac{1}{1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)} \leq \frac{|\gamma(t) - x^*|}{\sqrt{2\lambda} t} \leq 1 + \tilde{\Psi}^2(|\gamma(t) - x^*|) \leq 2 \quad (7.2)$$

and

$$\frac{|\gamma(t) - x^* - \sqrt{2\lambda} at|^2}{(\sqrt{2\lambda} t)^2} \leq 2^n \tilde{\Psi}^2(|\gamma(t) - x^*|) \leq 2^n \tilde{\Psi}^2(\frac{1}{2} \sqrt{2\lambda} t). \quad (7.3)$$

**Proof.** Lemma A.3 implies that

$$|\dot{\gamma}(s)| = \sqrt{2(F \circ \gamma(s) + \lambda)}, \quad \text{for almost all } s \geq 0, \quad (7.4)$$

and hence

$$|\dot{\gamma}(s)|^2 = \frac{1}{2} |\dot{\gamma}(s)|^2 + F \circ \gamma(s) + \lambda, \quad \text{for almost all } s \geq 0. \quad (7.5)$$

We first show (7.2). Let $t \geq \sigma n_0$. Then

$$|\gamma(t) - x^*| \geq 2^{n_0} \text{ and hence } \tilde{\Psi}(|\gamma(t) - x^*|) \leq \tilde{\Psi}(2^{n_0}) \leq 2^{-10} a^b \leq 2^{-9}. \quad (7.6)$$
By (7.5) and (6.2), one has
\[ \int_0^t |\dot{\gamma}(s)|^2 \, ds = A_\lambda(\gamma_{[0,t]}) = m_\lambda(x, \gamma(t)) \leq \sqrt{2\lambda}|\gamma(t) - x^*|(1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)). \] (7.7)

Since (7.4) gives \(|\dot{\gamma}(t)| \geq \sqrt{2\lambda}t\), by (7.7) one has
\[ |\gamma(t) - x^*|[1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)] \geq \sqrt{2\lambda}t. \]

Note that \(\tilde{\Psi}(|\gamma(t) - x^*|) \leq 1\). By (7.6), one has
\[ \frac{|\gamma(t) - x^*|}{\sqrt{2\lambda}t} \geq [1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)]^{-1} \geq \frac{1}{2}. \]

On the other hand, since
\[ \frac{1}{t}|\gamma(t) - x|^2 \leq \frac{1}{t} \left( \int_0^t |\dot{\gamma}(s)| \, ds \right)^2 \leq \int_0^t |\dot{\gamma}(s)|^2 \, ds, \]
by (7.7), one has
\[ \frac{|\gamma(t) - x|^2}{\sqrt{2\lambda}t|\gamma(t) - x^*|} \leq 1 + \tilde{\Psi}^2(|\gamma(t) - x^*|). \]

By (6.1) and (7.6),
\[ \frac{|\gamma(t) - x^*|}{|\gamma(t) - x^*| - a|} \leq 2^4 \tilde{\Psi}(|\gamma(t) - x^*|) \leq 2^{-5}. \]

It follows that \(|\gamma(t) - x| \geq |\gamma(t) - x^*|\) for all \(t \geq \sigma_{n_0}\) (See Figure 6 for illustration). Since \(\tilde{\Psi}(|\gamma(t) - x^*|) \leq 1\) by (7.6),
\[ \frac{|\gamma(t) - x^*|}{\sqrt{2\lambda}t} \leq \frac{|\gamma(t) - x|^2}{\sqrt{2\lambda}t|\gamma(t) - x^*|} \leq 1 + \tilde{\Psi}^2(|\gamma(t) - x^*|) \leq 2. \]

We therefore obtain (7.2).

By (7.7), (7.2) and \(\tilde{\Psi}(|\gamma(t) - x^*|) \leq 1\), one has
\[ \int_0^t |\dot{\gamma}(s)|^2 \, ds \leq \sqrt{2\lambda}|\gamma(t) - x^*|(1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)) \leq 8\lambda t, \]
which gives (7.1).

Finally, we prove (7.3). For \(t \geq \sigma_{n_0}\), by (2.6) one has
\[ \langle \frac{\gamma(t) - x^*}{|\gamma(t) - x^*|}, a \rangle = \cos \left( \frac{\gamma(t) - x^*}{|\gamma(t) - x^*|}, a \right) = 1 \]
\[ -2\sin^2 \frac{1}{2} \langle \frac{\gamma(t) - x^*}{|\gamma(t) - x^*|}, a \rangle = 1 - \frac{1}{2} |\gamma(t) - x^* - a|^2. \]
By (6.1) and \( \Psi(|\gamma(t) - x^*|) \leq 2^{-9} \) as given by (7.6), one gets
\[
\langle \gamma(t) - x^* \mid \gamma(t) - x^* \rangle, a \rangle \geq 1 - 2^{7} \bar{\Psi}^2(|\gamma(t) - x^*|) > 0.
\]
From this and (7.2), we deduce that
\[
\frac{|\gamma(t) - x^* - \sqrt{2} \lambda at|^2}{(\sqrt{2} \lambda t)^2} = \frac{|\gamma(t) - x^*|^2}{(\sqrt{2} \lambda t)^2} + 1 - 2 \frac{|\gamma(t) - x^*| \langle \gamma(t) - x^*, a \rangle}{\sqrt{2} \lambda t} |\gamma(t) - x^*|
\]
\[
\leq [1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)]^2 + 1 - 2[1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)]^{-1}[1 - 2^{7} \bar{\Psi}^2(|\gamma(t) - x^*|)]
\]
\[
\leq [1 + \tilde{\Psi}^2(|\gamma(t) - x^*|)]^2 + 1 - 2[1 - \tilde{\Psi}^2(|\gamma(t) - x^*|)][1 - 2^{7} \bar{\Psi}^2(|\gamma(t) - x^*|)]
\]
\[
\leq 2^9 \bar{\Psi}^2(|\gamma(t) - x^*|).
\]
Since \( \Psi(|\gamma(t) - x^*|) \leq 2^{-9} \) and \( |\gamma(t) - x^*| \geq \frac{1}{2} \sqrt{2} \lambda t \), one has
\[
\frac{|\gamma(t) - x^* - \sqrt{2} \lambda at|^2}{(\sqrt{2} \lambda t)^2} \leq 2^9 \bar{\Psi}^2(|\gamma(t) - x^*|) \leq 2^9 \bar{\Psi}^2\left(\frac{1}{2} \sqrt{2} \lambda t\right).
\]
Thus (7.3) holds. The proof is complete. \( \Box \)

8. Proof of Theorem 1.4

Below we prove Theorem 1.4 and hence Corollary 1.5; in particular, when \( F = U \) with \( f(s) = s^{-1} \), we reprove Theorem 1.3.

Proof of Theorem 1.4. Given any \( x \in \widetilde{\Omega}, \) any \( a \in \mathbb{S}^{dN - 1} \) with \( a^b > 0 \), and any \( \lambda > 0, \) let \( x^*, \Psi, \tilde{\Psi} \) and \( n_0 \) be as in Section 5. The proofs consists of 5 steps.

Step 1. According to Lemma A.2, for any \( n \in \mathbb{N} \), let \( \gamma^{(n)} \in \mathcal{AC}(x, x^* + 2^n a; [0, \sigma^{(n)}]; \widetilde{\Omega}) \) be an \( m_\lambda \)-geodesic with canonical parameter. For \( n \geq j \geq n_0, \) set
\[
\sigma_j^{(n)} := \max\{t \in [0, \sigma^{(n)}] : |\gamma^{(n)}(t) - x^*| = 2^j\}.
\]
By (7.2), we have
\[
\frac{1}{2} \leq \frac{|\gamma^{(n)}(\sigma_j^{(n)} - x^*|}{\sqrt{2} \lambda \sigma_j^{(n)}} \leq \frac{2^j}{\sqrt{2} \lambda \sigma_j^{(n)}} \leq 2.
\]
Thus
\[
\frac{1}{\sqrt{2} \lambda} 2^{j-1} \leq \sigma_j^{(n)} \leq \frac{1}{\sqrt{2} \lambda} 2^{j+1}.
\]
Moreover, \( t \geq \frac{1}{\sqrt{2} \lambda} 2^{n_0 + 1} \) implies that \( t \geq \sigma_j^{(n)} \) for all possible \( n \geq n_0. \) Thus, by (6.3) from Lemma 6.1, we have
\[
(\gamma^{(n)}(t))^b \geq (x^*)^b > 0 \text{ whenever } \frac{1}{\sqrt{2} \lambda} 2^{n_0 + 1} \leq t \leq \sigma^{(n)} \text{ for all possible } n \geq n_0.
Step 2. We show that for each $j \geq n_0$, the family \( \{ \gamma^{(n)}|_{[0, \frac{1}{\sqrt{2\lambda}}j]} \}_{n \geq j+1} \) is uniformly bounded and equi-continuous. Indeed, by (5.6), (6.1) and $\Psi(2^j) \leq 1$, one has

\[
I(\gamma^{(n)}|_{[0, \sigma_j^{(n)}]}) \leq \frac{1}{\sqrt{2\lambda}} m(x, \gamma^{(n)}(\sigma_j^{(n)})) \leq 2^j + \Psi(2^j) \leq 2^{j+1}, \quad \forall \ j \geq n_0.
\]

By (7.1),

\[
\int_0^{\sigma_j^{(n)}} |\dot{\gamma}^{(n)}(s)|^2 \, ds \leq 8\lambda \sigma_j^{(n)} \leq 4\sqrt{2\lambda} 2^{j+1}.
\]

Thus one has

\[
I(\gamma^{(n)}|_{[0, \sqrt{2\lambda}2^j]}) \leq I(\gamma^{(n)}|_{[0, \sigma_j^{(n)}]}) \leq 2^{j+2} \quad \text{and hence} \quad \gamma^{(n)}|_{[0, \frac{1}{\sqrt{2\lambda}}2^j]} \subset B(x, 2^{j+2}).
\]

While for any $0 \leq s < t \leq \frac{1}{\sqrt{2\lambda}}2^j$, we have

\[
|\gamma^{(n)}(t) - \gamma^{(n)}(s)| \leq \int_s^t |\dot{\gamma}^{(n)}(\delta)| \, d\delta \leq |t - s|^{1/2} \left( \int_s^{\sigma_j^{(n)}} |\dot{\gamma}^{(n)}(\delta)|^2 \, d\delta \right)^{1/2} \leq 4 \cdot 2^{j/2} (2\lambda)^{1/4}|t - s|^{1/2}.
\]

Step 3. For $j = n_0$, by Arzela-Ascoli theorem, there is some infinite subset $N_j \subset N_{j-1}$ such that the subsequence

\[
\{ \gamma^{(n)}|_{[0, \frac{1}{\sqrt{2\lambda}}2^{n_0}]} \}_{n \in N_{n_0}} \text{ converges uniformly to some curve} \eta_0 \in C^0([0, \frac{1}{\sqrt{2\lambda}}2^{n_0}]; \mathbb{R}^dN)
\]

For any $j > n_0$, we can find some infinite subset $N_j \subset N_{j-1}$ such that the subsequence

\[
\{ \gamma^{(n)}|_{[0, \frac{1}{\sqrt{2\lambda}}2^j]} \}_{n \in N_j} \text{ converges uniformly to some curve} \eta_j \in C^0([0, \frac{1}{\sqrt{2\lambda}}2^j]; \mathbb{R}^dN).
\]

Since $N_j \subset N_{j-1}$ we know that

\[
\eta_j|_{[0, \frac{1}{\sqrt{2\lambda}}2^{j-1}]} = \eta_{j-1}.
\]

This allows us to define a ray $\gamma : [0, \infty) \to \Omega$ via

\[
\gamma|_{[0, \frac{1}{\sqrt{2\lambda}}2^j]} := \eta_j \quad \forall \ j \geq n_0.
\]
We also observe that for any \( t \geq \frac{1}{\sqrt{2\lambda}} 2^{n_0+1} \), if \( j \) is large enough such that \( t \leq \frac{1}{\sqrt{2\lambda}} 2^{j-1} \), then
\[
(\gamma(t))^b = (\eta_j(t))^b = \left( \lim_{N_j \ni n \to \infty} \gamma^{(n)}(t) \right)^b = \lim_{N_j \ni n \to \infty} (\gamma^{(n)}(t))^b \geq (x^*)^b > 0.
\]

Thus
\[
\gamma(t) \in \Omega \quad \text{for all} \quad t \geq \frac{1}{\sqrt{2\lambda}} 2^{n_0+1}. \tag{8.1}
\]

Step 4. We show that \( \gamma \) is an \( m_\lambda \)-geodesic ray with canonical parameter. To see this, by Lemma A.2, it suffices to prove that for all \( j \leq \eta_j \) is an \( m_\lambda \)-geodesic with canonical parameter, that is, \( \eta_j \in \mathcal{AC} \left( x, \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right); \Omega_1 \right) \) and
\[
A_\lambda(\eta_j) = m_\lambda \left( x, \eta_j \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right).
\]

Indeed, since \( \frac{1}{\sqrt{2\lambda}} 2^j \leq \lambda^{(n)}_{j+1} \), we can apply Tonelli’s Theorem for convex Lagrangians to get
\[
\frac{1}{2} \int_0^{\frac{1}{\sqrt{2\lambda}} 2^j} |\dot{\eta}_j(s)|^2 ds \leq \frac{1}{2} \liminf_{n \to \infty} \int_0^{\frac{1}{\sqrt{2\lambda}} 2^j} |\gamma^{(n)}(s)|^2 ds
\]
and Fatou’s Lemma to obtain that
\[
\int_0^{\frac{1}{\sqrt{2\lambda}} 2^j} F(\eta_j(s)) ds \leq \liminf_{n \to \infty} \int_0^{\frac{1}{\sqrt{2\lambda}} 2^j} F(\gamma^{(n)}(s)) ds.
\]
Therefore, we conclude that
\[
A_\lambda(\eta_j) \leq \liminf_{n \in \mathbb{N}, n \to \infty} A_\lambda \left( \gamma^{(n)}_{[-1/\sqrt{2\lambda} 2^j]} \right) \\
\leq \liminf_{n \in \mathbb{N}, n \to \infty} m_\lambda \left( x, \gamma^{(n)}_{[\sigma_{j+1}^n]} \right) \leq \sqrt{2\lambda} 2^{j+2}. \tag{8.2}
\]

Thus \( \eta_j \in \mathcal{AC} \left( x, \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right); \Omega_1 \right) \). Obviously, \( m_\lambda( x, \eta_j(\frac{1}{\sqrt{2\lambda}} 2^j)) \leq A_\lambda(\eta_j) \).

Below we show that \( A_\lambda(\eta_j) \leq m_\lambda \left( x, \eta_j \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right) \). Since \( \gamma^{(n)} \) is an \( m_\lambda \)-geodesic with canonical parameter, one has
\[
A_\lambda \left( \gamma^{(n)}_{[0, \frac{1}{\sqrt{2\lambda}} 2^j]} \right) = m_\lambda \left( x, \gamma^{(n)} \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right) \leq m_\lambda \left( x, \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right) + m_\lambda \left( \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right), \gamma^{(n)} \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right).
\]

By Lemma 4.2, write \( z_{n,j} = \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) - \gamma^{(n)} \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \) one has
\[
m_\lambda \left( \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right), \gamma^{(n)} \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right) \leq \sqrt{2\lambda} |z_{n,j}|.
\]
\[
+ \frac{1}{\sqrt{2\lambda}} \int_0^{|z_{n,j}|} F \left( \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right), s \frac{z_{n,j}}{|z_{n,j}|} \right) ds.
\]

Since \(F\) is bounded in any compact subset of \(\Omega\), noting \(|z_{n,j}| \to 0\) as \(N_j \ni n \to \infty\) we know that

\[
\lim_{n \in N_j, n \to \infty} m_\lambda \left( \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right), \gamma^{(n)} \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right) = 0.
\]

From this and (8.2), we conclude that

\[
A_\lambda(\eta_j) \leq \limsup_{n \in N_j, n \to \infty} A_\lambda \left( \gamma^{(n)}|_{[0, \frac{1}{\sqrt{2\lambda}} 2^j]} \right) \leq m_\lambda \left( x, \gamma \left( \frac{1}{\sqrt{2\lambda}} 2^j \right) \right)
\]

as desired.

**Step 5.** For any \(t > \frac{1}{\sqrt{2\lambda}} 2^{n_0}\), letting \(j > n_0\) such that \(t < \frac{1}{\sqrt{2\lambda}} 2^j\), by (7.3) one has

\[
\frac{|\gamma(t) - x^* - \sqrt{2\lambda}at|}{\sqrt{2\lambda}t} = \frac{|\eta_j(t) - x^* - \sqrt{2\lambda}at|}{\sqrt{2\lambda}t} = \lim_{n \in N_j, n \to \infty} \frac{|\gamma^{(n)}(t) - x^* - \sqrt{2\lambda}at|}{\sqrt{2\lambda}t} \leq 2^{5} \Psi_0 \left( \frac{1}{2} \sqrt{2\lambda}t \right).
\]

Hence

\[
\lim_{t \to \infty} \frac{|\gamma(t) - \sqrt{2\lambda}at|^2}{(\sqrt{2\lambda}t)^2} = \lim_{t \to \infty} \frac{|\gamma(t) - x^* - \sqrt{2\lambda}at|^2}{(\sqrt{2\lambda}t)^2} = 0,
\]

That is, Theorem 1.4 (i) holds.

Theorem 1.4 (iv) follows from Theorem 1.4 (i) and Lemma A.4. Moreover, Theorem 1.4 (ii) follows from (8.1) with \(t_0 = \frac{1}{\sqrt{2\lambda}} 2^{n_0+1}\). Theorem 1.4 (iii) follows from Theorem 1.4 (i), Theorem 1.4 (ii) and Lemma A.4.

We end the proof of Theorem 1.4.

**Acknowledgements.** The authors would like to thank the anonymous referee for the careful reading, several valuable suggestions/comments, and many important and detailed corrections, which significantly improve the final presentation of the paper. In particular, the authors thank the referee for pointing out a wrong statement in the original version about the relationship between \((\tilde{\Omega}, m_\lambda)\) and the completion of \((\Omega, m_\lambda)\). In this revision, we correct this wrong statement in Remark A.1 (ii) and (iii). In Remark A.1 (ii), under the assumption that the potential \(F = +\infty\) in \(\Sigma\), we do show that \((\tilde{\Omega}, m_\lambda)\) is the completion of \((\Omega, m_\lambda)\). However, in Remark A.1 (iii), we construct a potential \(F\), for which the set \(\{F = +\infty\}\) is strictly contained in \(\Sigma\), so that the completion of \((\Omega, m_\lambda)\) is strictly contained in \((\tilde{\Omega}, m_\lambda)\). We are really in debt to the anonymous referee.
Appendix A Properties of Hamiltonians and Mañé’s potentials

In the appendix we always assume that $m_i = 1$ for all $1 \leq i \leq N$. For general masses, all of the following conclusions still hold, up to some obvious modifications. We omit the details.

Let $F$ be as in (1.4). Apriori, $F$ is only defined in the set $/Omega_1$ since $F_{ij}$ is only defined in $\mathbb{R}^d \times \mathbb{R}^d \setminus /Delta_1$ for all $1 \leq i < j \leq \infty$. Now we extend the definition of $F$ to the whole $\mathbb{R}^{dN}$ by defining the value of $F_{ij}$ for all $1 \leq i < j \leq \infty$ as follows:

$$F_{ij}(z, z) = \lim \inf_{(x_i, x_j) \to (z, z)} F_{ij}(x_i, x_j), \quad \forall (z, z) \in /Delta_1.$$ 

It may happen that $F_{ij}(z, z) = \infty$ for some $z \in \mathbb{R}^d$, and hence $F(x) = \infty$ for some $x \in \Sigma$. But one may directly check that that $F_{ij}$ is lower semicontinuous in $\mathbb{R}^d \times \mathbb{R}^d$, that is, the set $\{(x_i, x_j) \in \mathbb{R}^d \times \mathbb{R}^d : F_{ij}(x_i, x_j) > \delta\}$ is open for all $\delta > 0$. Thus $F$ is also lower semicontinuous in whole $\mathbb{R}^{dN}$.

For any $\lambda > 0$, Mañé’s potential $m_\lambda$ is defined in (1.5). Observe that

$$m_\lambda(x, y) \geq \sqrt{2 \lambda |x - y|} \text{ for all } x, y \in \mathbb{R}^{dN}. \quad (A.1)$$

Indeed, for any $\gamma \in AC(x, y; [0, \sigma], \mathbb{R}^{dN})$, by the Cauchy-Schwartz inequality, its Euclidean length

$$l(\gamma) \leq (2\sigma)^{1/2} \left( \int_0^\sigma \frac{1}{2} |\dot{y}(s)|^2 \, ds \right)^{1/2} \leq (2\sigma)^{1/2} (A_\lambda - \lambda \sigma)^{1/2},$$

and hence by $|x - y| \leq l(\gamma)$, one has

$$A_\lambda(\gamma) \geq \frac{1}{2\sigma} l(\gamma)^2 + \lambda \sigma \geq \frac{1}{2\sigma} |x - y|^2 + \lambda \sigma.$$ 

Since $\frac{1}{2\sigma} |x - y|^2 + \lambda \sigma$ reaches its minimal value at $\sigma = |x - y|/\sqrt{2\lambda}$, we have $A_\lambda(\gamma) \geq \sqrt{2\lambda} |x - y|$ as desired.

On the other hand, we have

$$m_\lambda(x, y) < \infty, \quad \forall x, y \in \Omega.$$ 

Indeed, since $\Omega$ is path connected, one can always find a smooth curve $\gamma_{x,y} \in AC(x, y; [0, \sigma]; \Omega)$ for some $\sigma > 0$. By the semicontinuity of $F$ and $F < \infty$ in $\Omega$,
we know that $F$ is bounded in the compact set $\gamma([0, \sigma])$ and hence we know that $A_\lambda(\gamma) < \infty$. But for $x \in \Omega$ and $y \notin \Omega$, it is not clear whether $m_\lambda(x, y)$ is finite or not. This is indeed determined by the behaviour of $F$ around $\Sigma$.

Set

$$\Omega_\lambda := \{ y \in \mathbb{R}^{dN} : m_\lambda(x, y) < \infty, \forall x \in \Omega \}.$$ 

Obviously, $\Omega \subset \Omega_\lambda \subset \mathbb{R}^{dN}$. We observe that $\Omega_\lambda = \Omega_\mu$ for all $0 < \lambda < \mu < \infty$. To see this, it suffices to show that

$$m_\lambda(x, y) < \infty \text{ if and only if } m_\mu(x, y) < \infty \text{ for any } x \in \Omega, y \in \mathbb{R}^{dN}.$$ 

Since $A_\lambda(\gamma) \leq A_\mu(\gamma)$ for all possible $\gamma$, by definition, one always has $m_\lambda(x, y) \leq m_\mu(x, y)$. On the other hand, if $m_\lambda(x, y) < \infty$, we have $A_\lambda(\gamma) < \infty$ for some $\gamma \in \mathcal{AC}(x, y; [0, t]; \mathbb{R}^{dN})$. Thus

$$A_\mu(\gamma) \leq A_\lambda(\gamma) + (\mu - \lambda)t < \infty,$$

which implies $m_\mu(x, y) < \infty$ as desired. Recalling (1.6), we write $\tilde{\Omega} = \Omega_\lambda$ for any $\lambda > 0$.

One may directly check that $m_\lambda$ is a distance in $\tilde{\Omega}$, and hence $(\tilde{\Omega}, m_\lambda)$ is a metric space. Observe that $(\tilde{\Omega}, m_\lambda)$ is always complete. Indeed, let $\{y_n\}_{n \in \mathbb{N}} \subset \tilde{\Omega}$ be a Cauchy sequence with respect to $m_\lambda$. Recall that (A.1) gives $|z - w| \leq \frac{1}{\sqrt{2}} m_\lambda(z, w)$ for all $z, w \in \mathbb{R}^{dN}$. It follows that $\{y_n\}_{n \in \mathbb{N}}$ is also a Cauchy sequence with respect to the Euclidean distance, and hence $|y_n - y| \to 0$ as $n \to \infty$ for some $y \in \mathbb{R}^{dN}$. It then suffices to show that $y \in \tilde{\Omega}$ and $m_\lambda(y_n, y) \to 0$ as $n \to \infty$. If there exists $m \in \mathbb{N}$ such that $y_n = y$ for all $n \geq m$, then $y_m \in \tilde{\Omega}$ implies $y \in \tilde{\Omega}$ and $m_\lambda(y, y) = 0$ for all $n \geq m$ as desired. Now we assume that there are infinitely many $k$ such that $y_k \neq y$. Thanks to this, it is standard to find a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ of $\{y_n\}_{n \in \mathbb{N}}$ such that $0 < m_\lambda(y_{n_k}, y_{n_k+1}) \leq 2^{-k}$. For each $k \in \mathbb{N}$, there is $y_k \in \mathcal{AC}(y_{n_k}, y_{n_k+1}; [0, \sigma_k]; \mathbb{R}^{dN})$ such that $A_\lambda(y_k) \leq 2m_\lambda(y_{n_k}, y_{n_k+1}) \leq 2^{-k+1}$. Fix an arbitrary $x \in \Omega$. There is $y_0 \in \mathcal{AC}(x, y_1; [0, \sigma_0]; \mathbb{R}^{dN})$ such that $A_\lambda(y_0) \leq 2m_\lambda(x, y_1) < \infty$. Noting that $\sigma_k \leq \frac{1}{\lambda} A_\lambda(y_k)$ for all $k \in \mathbb{N} \cup \{0\}$, it implies that

$$\sigma := \sum_{k=0}^{\infty} \sigma_k \leq \frac{1}{\lambda} \sum_{k=0}^{\infty} A_\lambda(y_k) \leq \frac{1}{\lambda} \sum_{k=0}^{\infty} 2^{-k+1} < \infty.$$ 

Then the concatenation $\gamma$ of these curves $y_k$ (that is, $\gamma = \ast_k y_k$) belongs to $\mathcal{AC}(x, y; [0, \sigma]; \mathbb{R}^{dN})$ and satisfies $A_\lambda(\gamma) \leq \sum_{k=0}^{\infty} A_\lambda(y_k) < \infty$. Thus $m_\lambda(x, y) < \infty$, that is, $y \in \tilde{\Omega}$, and moreover

$$m_\lambda(y_{n_k}, y) \leq \sum_{j=k}^{\infty} A_\lambda(y_j) \leq \sum_{j=k}^{\infty} 2^{-j+2} = 2^{-k+3},$$

which implies $m_\lambda(y_{n_k}, y) \to 0$ as $k \to \infty$. Since $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $m_\lambda$, by a standard argument, one further gets $m_\lambda(y_n, y) \to 0$ as $n \to \infty$ as desired.
Recall that $\Omega \subset \tilde{\Omega}$, and $\Omega$ is an open subset of $\tilde{\Omega}$ with respect to $m_\lambda$. Thus $(\Omega, m_\lambda)$ is also a metric space as a subspace of $(\tilde{\Omega}, m_\lambda)$. Since there is no any other assumption made on the behaviour of $F$ around $\Sigma$ (except the lower semicontinuity), $\Omega$ is not necessarily closed or complete in general. It is interesting to determine the relation between $(\Omega, m_\lambda)$, or its completion/closure, and $(\tilde{\Omega}, m_\lambda)$. In the following remark, we have some discussion about this issue.

**Remark A.1.** (i) It is easy to see that if

$$F_{ij}(x_i, x_j) \geq C|x_i - x_j|^{-2} \text{ when } |x_i - x_j| \leq \frac{1}{2} \text{ for all } 1 \leq i < j \leq N,$$

then $\tilde{\Omega} = \Omega$, and hence, $(\Omega, m_\lambda)$ is complete. One may also check that if

$$|F_{ij}(x_i, x_j)| \leq C|x_i - x_j|^{-\alpha} \text{ when } |x_i - x_j| \leq \frac{1}{2} \text{ for all } 1 \leq i < j \leq N \text{ and } \alpha \in (0, 2),$$

then $\Omega \subset \tilde{\Omega} = \mathbb{R}^n$. But in general, it is difficult to determine the set $\tilde{\Omega}$. For example, see [4,7,10,11,15,21,29,33] and references therein.

(ii) If $F = +\infty$ in $\Sigma = \mathbb{R}^{dN} \setminus \Omega$, then $\Omega$ is dense in $(\tilde{\Omega}, m_\lambda)$, equivalently, the completion of $(\Omega, m_\lambda)$ is $(\tilde{\Omega}, m_\lambda)$. Indeed, if $\Omega \subsetneq \tilde{\Omega}$, given any $y \in \tilde{\Omega} \setminus \Omega$ and $x \in \Omega$, there is $\gamma \in AC(x, y; [0, \sigma]; \mathbb{R}^{dN})$ with $A_\lambda(\gamma) < \infty$. Since $F = \infty$ in $\Sigma$, we know that $\gamma(t) \subset \Omega$ for almost all $t \in [0, \sigma]$. Therefore there exists a sequence $\{t_n\} \subset [0, \sigma]$ so that $t_n \to \sigma$ with $\gamma(t_n) \in \Omega$, and $A_\lambda(\gamma|\{t_n, \sigma\}) \to 0$, which implies $m_\lambda(\gamma(t_n), y) \to 0$ as $n \to \infty$, that is, $y$ is the limit of the Cauchy sequence $\{\gamma(t_n)\} \subset \Omega$ with respect to $m_\lambda$. This shows the closure $\bar{\Omega}^{m_\lambda}$ of $\Omega$ with respect to $m_\lambda$. On the other hand, since $\Omega \subset \tilde{\Omega}$ and $\tilde{\Omega}$ is complete with respect to $m_\lambda$, it follows that $\bar{\Omega}^{m_\lambda} \subset \tilde{\Omega}$. We conclude that $\bar{\Omega}^{m_\lambda} = \tilde{\Omega}$.

(iii) Without the assumption $F = +\infty$ in $\Sigma$, in general one can not expect that $\Omega$ is dense in $(\tilde{\Omega}, m_\lambda)$, equivalently, the completion of $(\Omega, m_\lambda)$ is $(\tilde{\Omega}, m_\lambda)$. Indeed, we construct a potential $F : \mathbb{R}^{dN} \to (0, \infty)$ so that $\Omega$ is not dense in $(\tilde{\Omega}, m_\lambda)$; see the following 3 steps.

**Step 1.** Constructions of $h_\pm : [0, \infty) \to (0, \infty)$ by modifying $r^{-2}$.

Let $h_\pm : [0, \infty) \to (0, \infty)$ be two functions defined by

$$h_+(r) = \begin{cases} 1 & \text{when } 0 \leq r \leq 1; \\ \frac{1}{r^2} & \text{when } 1 < r < \infty \end{cases}$$

and

$$h_-(r) = \begin{cases} 1 & \text{when } r \in \Lambda := [0] \cup \left( \bigcup_{k \geq 3} [2^{-k^2 - 1}, 2^{-k^2 + 1}] \right); \\ \frac{1}{r^2} & \text{when } r \in (0, \infty) \setminus \Lambda. \end{cases}$$

Obviously, $h_+ \in C^0([0, \infty))$, and $h_-$ is lower semicontinuous in $[0, \infty)$ with

$$h_-(r) = \liminf_{t \to r} h_-(t) \quad \forall \ 0 \leq r < \infty.$$
moreover, \( h_+(r) \leq h_-(r) \leq r^{-2} \) for all 0 < \( r < \infty \), and \( h_+ = h_- \) in \( \Lambda \cup [1, +\infty) \).

**Step 2.** Construction of a potential \( F : \mathbb{R}^{dN} \to (0, \infty) \) via \( h_\pm \).

For any nonempty subset \( E \subset \mathbb{R}^d \), recall the standard characteristic function \( \chi_E : \mathbb{R}^d \to \{0, 1\} \), which maps the elements of \( E \) to 1, and all other elements to 0. Define

\[
F(x) := F_{12}(x_1, x_2) \quad \text{for all } x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}
\]

with

\[
F_{12}(x_1, x_2) := h_-(|x_1 - x_2|)\chi_{\Gamma \times \Gamma} + h_+(|x_1 - x_2|)\chi_{\mathbb{R}^d \times \mathbb{R}^d \setminus [\Gamma \times \Gamma]},
\]

where \( x_i = (x_i^1, \ldots, x_i^d) \in \mathbb{R}^d \) and \( \Gamma := \{x_i \in \mathbb{R}^d : x_i^j < 0, \forall 1 \leq j \leq d\} \).

Obviously, \( F_{12} \) is locally bounded in \( \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d : x_1 = x_2, x_1 \in \Gamma\} \), and \( F_{12}(x_1, x_2) = 1 \) whenever \( |x_1 - x_2| \in \Lambda \). Moreover, \( F_{12} \) is lower semicontinuous in \( \mathbb{R}^d \times \mathbb{R}^d \) with

\[
F_{12}(x_1, x_2) = \liminf_{(z_1, z_2) \to (x_1, x_2)} F_{12}(z_1, z_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^{2d}.
\]

In particular, \( F \) satisfies the condition (1.4).

Note that

\[
\limsup_{(z_1, z_2) \to (x_1, x_2)} F_{12}(z_1, z_2) = +\infty \quad \text{whenever } x_1 = x_2 \text{ and } x_1 \in \Gamma.
\]

Below we write

\[
\Sigma_{12}^- := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : x_1 = x_2 \in \Gamma\} \subset \Gamma \times \Gamma \times \mathbb{R}^{d(N-2)}
\]

and

\[
\Sigma_{12} := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : x_1 = x_2\}.
\]

Notice that the local boundedness of \( F_{12} \) in \( \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d : x_1 = x_2, x_1 \in \Gamma\} \) implies the local boundedness of \( F \) in \( \mathbb{R}^{dN} \setminus \Sigma_{12} \). In particular, \( F \) is locally bounded in \( \Sigma \setminus \Sigma_{12} \).

**Step 3.** We show that \( \widetilde{\Omega} = \mathbb{R}^{dN} \) and \( \Sigma_{12}^- \subset \widetilde{\Omega} \setminus \Omega^{m_\lambda} \).

We first show that \( \widetilde{\Omega} = \mathbb{R}^{dN} \). It suffices to show that \( \Sigma \subset \widetilde{\Omega} \). We consider three cases: \( x \in \Sigma \setminus \Sigma_{12} \), \( x \in \Sigma_{12} \setminus \Sigma_{12}^- \) and \( x \in \Sigma_{12}^- \).

**Case** \( x \in \Sigma \setminus \Sigma_{12} \). By Step 2, we know that \( F \) is locally bounded at \( x \). Since \( \mathbb{R}^{dN} \setminus \Sigma_{12} \) is a connected open subset, there exists an Euclidean ball \( B(x, r_x) \subset \mathbb{R}^{dN} \setminus \Sigma_{12} \) for some \( r_x > 0 \) such that \( F|_{B(x, r_x)} \leq M_x \) for some \( M_x > 0 \). Noting that \( \Sigma = \mathbb{R}^{dN} \setminus \Omega \) is closed and does not have interior points, we can find a curve \( \gamma_x \subset C([0, 1], B(x, r_x)) \) with end point \( x \) such that \( \gamma_x \setminus \{x\} \subset \Omega \) and \( F|_{\gamma_x} \leq M_x \). Therefore, we see that \( m_\lambda(\gamma_x(0), x) \leq A_\lambda(\gamma_x) < \infty \), that is, \( x \in \widetilde{\Omega} \).

**Case** \( x \in \Sigma_{12} \setminus \Sigma_{12}^- \). By the definition of \( F \), we know \( F(x) = 1 \). Note that \( \mathbb{R}^{dN} \setminus \Sigma_{12}^- \) is a connected open subset of \( \mathbb{R}^{dN} \). Similar to the first case, there is an Euclidean ball
Given Lemma A.2.

Next we prove the following result:

Case \( x \in \Sigma_{12} \). Letting \( y = (y_i)_{i=1}^N \in \Sigma_{12} \setminus \Sigma_{12}^- \) such that \( y_1 = y_2 = (1, \ldots, 1) \in \mathbb{R}^d \) and \( y_i = x_i \in \mathbb{R}^d \) for \( i = 3, \ldots, N \), taking \( \gamma \) as the line-segment joining \( y \) and \( x \), then \( \gamma \subset \Sigma_{12} \) and hence \( F|_{\gamma} = 1 \).

Since \( y \in \Sigma_{12} \setminus \Sigma_{12}^- \subset \tilde{\Omega} \), there exists some \( x_0 \in \Omega \), such that \( m_{\lambda}(x, x_0) \leq m_{\lambda}(x, y) + m_{\lambda}(y, x_0) < \infty \). Hence, \( x \in \tilde{\Omega} \).

Finally, we show that \( \Sigma_{12}^- \subset \tilde{\Omega} \setminus \tilde{\Omega}^{m_{\lambda}} \). Let \( x \in \Sigma_{12}^- \) be an arbitrary point. It then suffices to show that

\[
\inf_{z \in \Omega} m_{\lambda}(z, x) > 0. \tag{A.2}
\]

To prove this, we need some properties of \( m_{\lambda} \)-geodesics joining \( z, x \) which will be introduced/proved below. So we postpone the proof of (A.2) to the end of this appendix.

Moreover, we know that \( (\tilde{\Omega}, m_{\lambda}) \) is a geodesic space, that is, for any \( x, y \in \tilde{\Omega} \) with \( x \neq y \), there exists a \( \eta \in \mathcal{AC}(x, y; [0, m_{\lambda}(x, y)]; \mathbb{R}^d) \) such that

\[
\eta([0, m_{\lambda}(x, y)]) \subset \tilde{\Omega}, \quad \text{and} \quad m_{\lambda}(\eta(s), \eta(t)) = t - s \text{ for all } 0 \leq s < t \leq m_{\lambda}(x, y).
\]

Indeed, thanks to Lemma A.2 below, the desired geodesic \( \gamma \) comes from the arc length (with respect to \( m_{\lambda} \)) parametrisation of the following minimizer of \( A_{\lambda} \) in the class of \( \cup_{\sigma > 0} \mathcal{AC}(x, y; [0, \sigma]; \tilde{\Omega}) \).

**Lemma A.2.** Given \( \lambda > 0 \) and \( x, y \in \tilde{\Omega} \) with \( x \neq y \), there is a curve \( \gamma \in \mathcal{AC}(x, y; [0, \sigma]; \tilde{\Omega}) \) such that \( A_{\lambda}(\gamma) = m_{\lambda}(x, y) \). Moreover, for any \( \gamma \in \mathcal{AC}(x, y; [0, \sigma]; \tilde{\Omega}) \) satisfying \( A_{\lambda}(\gamma) = m_{\lambda}(x, y) \), we have

\[
m_{\lambda}(\gamma(s), \gamma(t)) = A_{\lambda}(\gamma|_{[s, t]}), \quad \forall 0 \leq s < t \leq \sigma.
\]

The proof of Lemma A.2 is standard, for readers’ convenience we give it later.

Considering the geodesic nature, as in the introduction we call a minimizer \( \gamma : [0, \sigma] \to \tilde{\Omega} \) of \( A_{\lambda} \) with given endpoints \( x, y \) as an \( m_{\lambda} \)-geodesic with the canonical parameter joining \( x, y \). We also call a ray \( \gamma : [0, \infty) \to \tilde{\Omega} \) as an \( m_{\lambda} \)-geodesic ray with the canonical parameter if \( \gamma|_{[0, \sigma]} \) is an \( m_{\lambda} \)-geodesic with the canonical parameter for any \( \sigma > 0 \).

Next we prove the following result:

**Lemma A.3.** Given any \( x, y \in \tilde{\Omega} \) and \( \lambda > 0 \), let \( \gamma \in \mathcal{AC}(x, y; [0, \sigma], \mathbb{R}^d) \) be any \( m_{\lambda} \)-geodesic with canonical parameter. Then \( \gamma \) has energy constant \( \lambda \), that is,

\[
|\dot{\gamma}(s)| = \sqrt{2(F(\gamma(s)) + \lambda)} \text{ almost all } s \in [0, \sigma].
\]

Moreover,

\[
m_{\lambda}(x, y) = \tilde{m}_{\lambda}(x, y) = \tilde{A}_{\lambda}(y),
\]

The proof of Lemma A.3 is straightforward.
where
\[ \tilde{m}_\lambda(x, y) := \inf \{ \tilde{A}_\lambda(\eta) : \eta \in \bigcup_{\tau > 0} \mathcal{AC}(x, y; [0, \tau], \mathbb{R}^d) \} \quad \forall x, y \in \mathbb{R}^d, \]
and
\[ \tilde{A}_\lambda(\eta) := \int_0^\tau |\dot{\eta}(s)| \sqrt{2F \circ \eta + 2\lambda} \, ds. \]

Here in the above inequality, we use the convention that
\[ |\dot{\eta}(s)| \sqrt{2F \circ \eta + 2\lambda} = 0 \]
when \( |\dot{\eta}(s)| = 0 \) and \( F \circ \eta(s) = \infty \).

Consequently, we have

**Lemma A.4.** Under additional assumption \( F \in C^2(\Omega) \), for any \( \lambda > 0 \), if \( \gamma \in \mathcal{AC}(x, y; [0, \sigma], \mathbb{R}^d) \) is an \( m_\lambda \)-geodesic with canonical parameter joining \( x, y \) and also is interiorly collision-free (that is, \( \gamma|_{(0, \sigma)} \subset \Omega \)), then \( \gamma \) is a solution to \( \ddot{x} = \nabla F \) in \( (0, \sigma) \) starting from \( x \) and ending at \( y \).

Consequently, if \( \gamma : [0, \infty) \to \mathbb{R}^d \) is an \( m_\lambda \)-geodesic ray with canonical parameter starting from \( x \) and also is interiorly collision-free (that is, \( \gamma|_{(0, \infty)} \subset \Omega \)), then \( \gamma \) is a solution to \( \ddot{x} = \nabla F \) in \( (0, \infty) \) starting from \( x \).

Below we prove Lemma A.2-Lemma A.4.

**Proof of Lemma A.2.** Let \( x, y \in \tilde{\Omega} \) be two given configurations, with \( x \neq y \). Since \( 0 < m_\lambda(x, y) < \infty \), there exist a sequence of curves \( \{ \delta_n \in \mathcal{AC}(x, y; [0, \sigma_n]; \mathbb{R}^d) \}_{n \in \mathbb{N}} \) such that
\[ \lim_{n \to \infty} A_\lambda(\delta_n) = m_\lambda(x, y). \]
Thus for \( n \) large enough we have \( A_\lambda(\delta_n) \leq m_\lambda(x, y) + 1 \). Without loss of generality, we may assume that for all \( n \), this holds. This also implies that \( \delta_n \in \mathcal{AC}(x, y; [0, \sigma_n]; \tilde{\Omega}) \); otherwise \( \delta_n(t) \notin \tilde{\Omega} \) for some \( 0 < t < \sigma_n \) and hence
\[ m_\lambda(x, \delta_n(t)) < A_\lambda(\delta_n|_{[0,t]}) < A_\lambda(\delta_n) < \infty, \]
which is a contradiction with the definition of \( \tilde{\Omega} \).

Next, for every \( n \), since
\[ \frac{1}{2\sigma_n} |x - y|^2 + \lambda \sigma_n \leq A_\lambda(\delta_n) \leq 1 + m_\lambda(x, y), \]
we know
\[ 0 < \liminf_{n \to \infty} \sigma_n \leq \limsup_{n \to \infty} \sigma_n < +\infty. \]
and therefore up to considering subsequence, we may assume that \( \sigma_n \to \sigma_0 \) as \( n \to \infty \).

Now, for each \( n > 0 \), we parametrize \( \delta_n \) linearly as below. Define \( \gamma^{(n)}(t) = \delta_n(\sigma_n^{-1}\sigma) \) for \( t \in [0, \sigma_0] \). A direct calculation leads to
\[ \lim_{n \to \infty} A_\lambda(\gamma^{(n)}) = \lim_{n \to \infty} A_\lambda(\delta_n) = m_\lambda(x, y). \]
We may also assume that
\[ A_\lambda (\gamma^{(n)}) \leq m_\lambda (x, y) + 1, \quad \forall n \geq 1. \]

It is easy to see that \( \{\gamma^{(n)}\} \) is uniform bounded and equicontinuous. Indeed, for any \( s, t \in [0, \sigma_0] \) with \( s < t \), one has
\[
|\gamma^{(n)}(s) - \gamma^{(n)}(t)| \leq \int_s^t |\dot{\gamma}^{(n)}(\tau)| \, d\tau \leq \left( \int_0^{\sigma_0} |\dot{\gamma}^{(n)}(\tau)|^2 \, d\tau \right)^{1/2} |s - t|^{1/2} \\
\leq [2(m_\lambda(x, y) + 1)]^{1/2} |s - t|^{1/2}.
\]

In particular,
\[
|\gamma^{(n)}(0) - \gamma^{(n)}(t)| \leq [2(m_\lambda(x, y) + 1)]^{1/2} |t|^{1/2} \leq [2(m_\lambda(x, y) + 1)]^{1/2} |\sigma_0|^{1/2}.
\]

By Arzela-Ascoli Theorem, up to some subsequence, we may assume that the sequence \( \{\gamma^{(n)}\} \) converges uniformly to a curve \( \gamma \in C^0([0, \sigma_0]; \mathbb{R}^{dN}) \) with \( \gamma(0) = x, \gamma(\sigma_0) = y \). Note that \( \gamma \) is absolutely continuous. Indeed, for any \( \epsilon > 0 \) and for any family \( \{[s_i, t_i]\}_{1 \leq i \leq k} \) of mutually disjoint intervals such that \( \sum_{i=1}^k |s_i - t_i| < \epsilon \), applying the Cauchy-Schwarz inequality, we have
\[
\sum_{i=1}^k |\gamma^{(n)}(s_i) - \gamma^{(n)}(t_i)| \leq \int_{\bigcup_{1 \leq i \leq k}[s_i, t_i]} |\dot{\gamma}^{(n)}(\tau)| \, d\tau \\
\leq \left[ \int_{\bigcup_{1 \leq i \leq k}[s_i, t_i]} |\dot{\gamma}^{(n)}(\tau)|^2 \, d\tau \right]^{1/2} \left[ \sum_{i=1}^k |s_i - t_i| \right]^{1/2} \\
\leq 2^{1/2}[m_\lambda(x, y) + 1]^{1/2} \left[ \sum_{i=1}^k |s_i - t_i| \right]^{1/2} \leq 2^{1/2}[m_\lambda(x, y) + 1]^{1/2} \epsilon^{1/2}.
\]

Thus \( \gamma \in AC(x, y; [0, \sigma_0]; \mathbb{R}^{dN}) \). Apply Tonelli’s Theorem for convex Lagrangians to get
\[
\frac{1}{2} \int_0^{\sigma_0} |\dot{\gamma}(s)|^2 \, ds \leq \frac{1}{2} \liminf_{n \to \infty} \int_0^{\sigma_0} |\dot{\gamma}^{(n)}(s)|^2 \, ds
\]
and Fatou’s Lemma to obtain that
\[
\int_0^{\sigma_0} F(\gamma) \, ds \leq \liminf_{n \to \infty} \int_0^{\sigma_0} F(\gamma^{(n)}) \, ds.
\]

Therefore \( A_\lambda(\gamma) \leq m_\lambda(x, y) \), which is only possible if the equality holds. Note that \( A_\lambda(\gamma) < \infty \) also implies that \( \gamma \in AC(x, y; [0, \sigma_0]; \widehat{\Omega}) \).

Next for any \( \gamma \in AC(x, y; [0, \sigma]; \widehat{\Omega}) \) such that \( A_\lambda(\gamma) = m_\lambda(x, y) \) and for any \( 0 \leq s < t \leq \sigma \), we claim that
\[
m_\lambda(\gamma(s), \gamma(t)) \leq A_\lambda(\gamma|_{[s, t]}).
\]
We show this by contradiction. If 
\[ m_\lambda(\gamma(s), \gamma(t)) < A_\lambda(\gamma|_{s,t}], \]
we find \( \gamma_0 \in AC(\gamma(s), \gamma(t); [0, \sigma_0]; \tilde{\Omega}) \) such that
\[ A_\lambda(\gamma_0) = m_\lambda(\gamma(s), \gamma(t)) = m_\lambda(\gamma(s), \gamma(t)) < A_\lambda(\gamma|_{s,t}], \]
The concatenation of \( \gamma|_{[0,s]}, \gamma_0 \) and \( \gamma|_{t,\sigma} \) gives a curve \( \eta \in AC(x, y; [0, \sigma_0 + \sigma - (t-s)]; \tilde{\Omega}) \) such that
\[ m_\lambda(x, y) \leq A_\lambda(\eta) = A_\lambda(\gamma|_{0,s}) + A_\lambda(\gamma_0) + A_\lambda(\gamma|_{t,\sigma}) < A_\lambda(\gamma|_{0,s}) + A_\lambda(\gamma|_{s,t}) + A_\lambda(\gamma|_{t,\sigma}) = A_\lambda(\gamma') = m_\lambda(x, y), \]
which is a contradiction. \( \square \)

To prove lemma A.3, we need the following auxiliary Lemma A.5 - Lemma A.8.

**Lemma A.5.** Given any \( \lambda > 0 \) and \( x, y \in \Omega \), let \( \gamma \in AC(x, y; [0, \sigma]; \tilde{\Omega}) \) satisfying \( m_\lambda(x, y) = A_\lambda(\gamma) \). Then \( F \circ \gamma < \infty \) and \( |\dot{\gamma}| > 0 \) almost everywhere.

To prove Lemma A.5 and for later use, we recall the following two change of variable formulas. We refer to for example [18, Section 3.3.3, Theorem 2] for the first one, which comes from the area formula and works for Lipschitz maps; and refer to [9, Proposition 2.2.18] for the second one, which works for absolute continuous maps.

**Lemma A.6.** (i) Let \( f : \mathbb{R} \to \mathbb{R} \) be Lipschitz. For any \( g \in L^1(\mathbb{R}) \), one has
\[ \left| \sum_{s \in f^{-1}(t)} g(s) \right| \leq \sum_{s \in f^{-1}(t)} |g(s)| \in L^1(\mathbb{R}), \]
where \( f^{-1}(t) \) is at most countable for almost all \( t \in \mathbb{R} \), and
\[ \int_{\mathbb{R}} g(s) |f'(s)| \, ds = \int_{\mathbb{R}} \left[ \sum_{s \in f^{-1}(t)} g(s) \right] \, dt. \]
If \( f \) is injective in addition, then \( g \circ f^{-1} \in L^1(\mathbb{R}) \) and
\[ \int_{\mathbb{R}} g(s) |f'(s)| \, ds = \int_{\mathbb{R}} g \circ f^{-1}(t) \, dt. \]
(ii) Let \( f : [\alpha, \beta] \to [a, b] \) be absolutely continuous and increasing, and satisfy \( a = f(\alpha) \) and \( b = f(\beta) \). Then for any \( g \in L^1([a, b]) \), one has \( (g \circ f) f' \in L^1([\alpha, \beta]) \) and
\[ \int_{a}^{b} g(t) \, dt = \int_{\alpha}^{\beta} g(f(s)) f'(s) \, ds. \]
Proof of Lemma A.5. Note that $A_2(\gamma) < \infty$ implies the integrability of $F \circ \gamma$ in $[0, \sigma]$, and hence $F \circ \gamma(s) < \infty$ for almost all $s \in [0, \sigma]$. Moreover, denote by $E$ the set of $t \in [0, \sigma]$ such that $|\dot{\gamma}(t)| = 0$. We prove $|E| = 0$ by contradiction. Assuming that $|E| > 0$ below.

First, we show that there is no interval $[s_0, s_1] \subset [0, \sigma]$ such that almost all points in $[s_0, s_1]$ are in $E$. Otherwise, assume that $|\dot{\gamma}| = 0$ in the interval $[s_0, s_1] \subset [0, \sigma]$ with $s_0 < s_1$. Then $\gamma(s) = \gamma(s_0)$ for $s \in [s_0, s_1]$. Let $\eta(s) = \gamma(s)$ if $0 \leq s \leq s_0$, and $\eta(s) = \gamma(s + (s_1 - s_0))$ if $\sigma - (s_1 - s_0) \geq s \geq s_0$.

Obviously, $\eta \in AC(x,y; [0,\sigma - (s_1 - s_0)]; \tilde{\Omega})$ and $A(\eta) < A(\gamma)$, which is a contradiction.

Next define

$$\phi(s) := \int_0^s \chi_{[0,\sigma]\setminus E}(\delta) \, d\delta.$$ 

It is obvious that $\phi$ is Lipschitz, and hence absolutely continuous, $\phi'(s) = \chi_{[0,\sigma]\setminus E}(s)$ for almost all $s \in [0, \sigma]$. Moreover, $|\phi(E)| = 0$ and $\phi([0, \sigma]) = [0, \sigma - |E|]$. Since $E$ does not contain any interval, we know that $\phi$ is strictly increasing in $[0, \sigma]$, and hence injective. Thus the inverse $\phi^{-1} : [0, \sigma - |E|] \to [0, \sigma]$ is well-defined.

For any $g \in L^1([0, \sigma])$, we claim that

$$g \circ \phi^{-1} \in L^1([0, \sigma - |E|]) \text{ and } \int_0^\sigma g \chi_{[0,\sigma]\setminus E} \, ds = \int_0^{\sigma - |E|} g \circ \phi^{-1} \, ds \quad (A.3)$$

Indeed, let $\tilde{\phi}(t) = \int_0^t \chi_E \, dc$ for $t \in \mathbb{R}$. Then $\tilde{\phi}$ is Lipschitz and strictly increasing in whole $\mathbb{R}$. $\tilde{\phi}|_{[0,\sigma]} = \phi$ and $(\tilde{\phi})^{-1}|_{[0,\sigma - |E|]} = \phi^{-1}$. Applying the change of variable formula in Lemma A.6(i) to $g \chi_{[0,\sigma]}$ and $\bar{\phi}$, noting that $\phi[0, \sigma] = [0, \sigma - |E|]$ and $\phi' = \chi_{[0,\sigma]\setminus E}$ almost everywhere, one has

$$\int_0^\sigma g \chi_{[0,\sigma]\setminus E} \, ds = \int_{\mathbb{R}} (g \chi_{[0,\sigma]}) \, |(\tilde{\phi})'| \, ds$$

$$= \int_{\mathbb{R}} (g \chi_{[0,\sigma]}) \circ (\tilde{\phi})^{-1} \, ds = \int_0^{\sigma - |E|} g \circ \phi^{-1} \, ds$$

as desired.

Write $\eta(t) = \gamma(\phi^{-1}(t))$ for $t \in [0, \sigma - |E|]$. Since $\gamma$ is absolutely continuous (hence $\dot{\gamma} \in L^1([0, \sigma])$) and $\dot{\gamma} = 0$ in $E$, for any $t \in [0, \sigma - |E|]$ we have

$$\eta(t) - \eta(0) = \gamma(\phi^{-1}(t)) - \gamma(0) = \int_0^{\phi^{-1}(t)} \dot{\gamma}(s) \, ds$$

$$= \int_0^\sigma (\dot{\gamma} \chi_{[0,\phi^{-1}(t)]}) \chi_{[0,\sigma]\setminus E}) (s) \, ds.$$ 

Applying (A.3) to $\dot{\gamma} \chi_{[0,\phi^{-1}(t)]}$ for all $t \in [0, \sigma - |E|]$, one has $\dot{\gamma} \circ \phi^{-1} \in L^1([0, \sigma - |E|])$ and

$$\eta(t) - \eta(0) = \int_0^{\sigma - |E|} (\dot{\gamma} \chi_{[0,\phi^{-1}(t)]}) \circ \phi^{-1}(s) \, ds.$$
Thanks to $\chi_{[0,\phi^{-1}(t)]} \circ \phi^{-1} = \chi_{[0,t]}$, we obtain

$$
\eta(t) - \eta(0) = \int_0^{\sigma - |E|} \dot{\eta} \circ \phi^{-1} \chi_{[0,t]}(\delta) d\delta = \int_0^t \dot{\eta} \circ \phi^{-1}(\delta) d\delta.
$$

Thus $\eta \in AC(x,y; [0,\sigma - |E|], \tilde{\Omega})$ and hence differentiable almost everywhere in $[0,\sigma - |E|]$ with

$$
\dot{\eta} = \dot{\gamma} \circ \phi^{-1} \in L^1([0,\sigma - |E|]). \tag{A.4}
$$

By (A.4) and $|\dot{\gamma}|^2 \in L^1([0,\sigma])$, applying (A.3) to $|\dot{\gamma}|^2$ and $F \circ \gamma \in L^1([0,\sigma])$ one has

$$
A_{\lambda}(\eta) = \int_0^{\sigma - |E|} \left[\frac{1}{2} |\dot{\eta}(t)|^2 + F \circ \eta(t) + \lambda \right] dt
= \int_0^{\sigma - |E|} \left[\frac{1}{2} |\dot{\gamma} \circ \phi^{-1}(t)|^2 + (F \circ \gamma) \circ \phi^{-1}(t) + \lambda \right] dt
= \int_0^\sigma \left[\frac{1}{2} |\dot{\gamma}(t)|^2 \chi_{[0,\sigma \setminus |E|]}(t) + (F \circ \gamma)(t) \chi_{[0,\sigma \setminus |E|]}(t) \right] dt + \lambda(\sigma - |E|)
$$

Since $|[0,\sigma] \setminus |E| = \sigma - |E|$, one has

$$
A_{\lambda}(\eta) = \int_0^\sigma \frac{1}{2} |\dot{\gamma}(s)|^2 + F \circ \gamma(s) + \lambda \chi_{[0,\sigma \setminus |E|]}(s) ds < A_{\lambda}(\gamma).
$$

Hence $A_{\lambda}(\eta) < m_{\lambda}(x,y)$, which contradicts to the definition of $m_{\lambda}(x,y)$. \hfill \Box

**Lemma A.7.** Let $\gamma \in AC(x,y; [0,\sigma]; \mathbb{R}^d)$ with $\tilde{A}_{\lambda}(\gamma) < \infty$. We can reparameterize $\gamma$ to get a new curve $\tilde{\xi} \in AC(x,y; [0,\tau]; \mathbb{R}^d)$ so that $|\tilde{\xi}| > 0$ almost everywhere and $\tilde{A}_{\lambda}(\tilde{\xi}) = \tilde{A}_{\lambda}(\gamma)$.

**Proof.** The proof is much similar to that of Lemma A.5. \hfill \Box

**Lemma A.8.** Let $\gamma \in AC(x,y; [0,\sigma]; \mathbb{R}^d)$ with $\tilde{A}_{\lambda}(\gamma) < \infty$, and $|\dot{\gamma}| > 0$ almost everywhere. Then $\gamma \in AC(x,y; [0,\sigma]; \tilde{\Omega})$, and we can reparameterize $\gamma$ to get a new curve $\eta \in AC(x,y; [0,\tau]; \tilde{\Omega})$ such that

$$
|\dot{\eta}(t)| = \sqrt{2(F \circ \eta(t) + \lambda)} \text{ almost everywhere.}
$$

Moreover,

$$
A_{\lambda}(\eta) = \tilde{A}_{\lambda}(\eta) = \tilde{A}_{\lambda}(\gamma) \leq A_{\lambda}(\gamma).
$$

**Proof.** Write

$$
\psi(s) = \int_0^s \frac{|\dot{\gamma}(\delta)|}{\sqrt{2F \circ \gamma(\delta) + 2\lambda}} d\delta \quad \forall s \in [0,\sigma].
$$

Note that $\psi(0) = 0$ and $\psi(\sigma) < \infty$. Obviously, $\psi$ is absolutely continuous.

$$
\psi'(s) = \frac{|\dot{\gamma}(s)|}{\sqrt{2F \circ \gamma(s) + 2\lambda}} \text{ for almost all } s \in [0,\sigma]. \tag{A.5}
$$
Note that $|\dot{\gamma}(s)|\sqrt{2F \circ \gamma(s) + 2\lambda} \in L^1([0, \sigma])$ and $|\dot{\gamma}| > 0$ almost everywhere, hence $F \circ \gamma < \infty$ almost everywhere. Since $|\dot{\gamma}| > 0$ almost everywhere, we have $\psi' > 0$ almost everywhere. Thus $\psi$ is continuous and strictly increasing. Therefore $\psi([0, \sigma]) = [0, \psi(\sigma)]$, and $\psi^{-1} : [0, \psi(\sigma)] \to [0, \sigma]$ is also continuous, strictly increasing.

Next we show that $\psi^{-1}$ is absolutely continuous, that is, $|\psi^{-1}(E)| = 0$ whenever $E \subset [0, \psi(\sigma)]$ has measure $|E| = 0$. We only need to prove that for any set $E \subset [0, \sigma]$, if $\psi(E)$ has measure 0, then $E$ has measure 0. Indeed, $\psi(E)$ must be contained in a $G_\delta$-set $H$ which has measure 0, and hence $E \subset \psi^{-1}(H)$. So if $\psi^{-1}(H)$ has measure 0, then $E$ has measure 0. Since $H$ is a $G_\delta$-set, we know that $\psi^{-1}(H)$ is also a $G_\delta$-set, one can see that

$$0 = |H| = \int_{\psi^{-1}(H)} \psi'(s) \, ds.$$  

Since $\psi' > 0$ almost everywhere, we have $|\psi^{-1}(H)| = 0$ as desired.

Let $\eta(t) = \gamma(\psi^{-1}(t))$ for $t \in [0, \psi(\sigma)]$. Thus, for any $t \in [0, \psi(\sigma)]$, one has

$$\eta(t) - \eta(0) = \int_0^t \psi^{-1}(r) \dot{\gamma}(\delta) \, d\delta = \int_0^\sigma (\dot{\gamma} \chi_{[0, \psi^{-1}(t)]})(\delta) \, d\delta.$$  

Applying the change of variable formula given in Lemma A.6(ii) to $\dot{\gamma} \chi_{[0, \psi^{-1}]} \in L^1([0, \sigma])$ and $\psi^{-1}$, noting $\chi_{[0, \psi^{-1}(t)]} \circ \psi^{-1} = \chi_{[0, t]}$, one has $\dot{\gamma} \circ \psi^{-1}(\psi^{-1})' \in L^1([0, t])$ and

$$\eta(t) - \eta(0) = \int_0^\sigma (\dot{\gamma} \chi_{[0, \psi^{-1}(t)]})(\psi^{-1}(s))(\psi^{-1})'(s) \, ds = \int_0^\sigma \dot{\gamma}(\psi^{-1}(s))(\psi^{-1})'(s) \, ds.$$  

Thus $\eta \in AC(x, y; [0, \psi(\sigma)]; \tilde{\Omega})$ with $\dot{\eta} = \dot{\gamma} \circ \psi^{-1}(\psi^{-1})'$ almost everywhere.

Since $t = \psi \circ \psi^{-1}(t)$ for all $t \in [0, \psi(\sigma)]$, by the chain rule we have $1 = \psi' \circ \psi^{-1}(\psi^{-1})'$ almost everywhere in $[0, \psi(\sigma)]$. Denote by $E$ the set where $\psi$ is differentiable and $\psi' > 0$. Since $|[0, \psi(\sigma)] \setminus E| = 0$, by the absolute continuity of $\psi$, we have $|[0, \psi(\sigma)] \setminus \psi(E)| = |\psi([0, \sigma] \setminus E)| = 0$. Thus $\psi' \circ \psi^{-1} > 0$ in $\psi(E)$, and hence, almost everywhere in $[0, \psi(\sigma)]$. Recalling (A.5), we obtain

$$(\psi^{-1})'(t) = \frac{1}{\psi' \circ \psi^{-1}(t)} = \frac{\sqrt{2F \circ \gamma \circ \psi^{-1}(t) + 2\lambda}}{|\dot{\gamma} \circ \psi^{-1}(t)|}$$  

for almost all $t \in [0, \psi(\sigma)]$, and hence

$$|\dot{\eta}(t)| = \sqrt{2F \circ \gamma(\psi^{-1}(t)) + 2\lambda} = \sqrt{2F \circ \eta(t) + 2\lambda}$$  

for almost all $t \in [0, \psi(\sigma)]$.

Thus

$$\frac{1}{2} |\dot{\eta}(t)|^2 + F \circ \eta(t) + \lambda = |\dot{\eta}(t)|\sqrt{2F \circ \eta(t) + 2\lambda}$$  

for almost all $t \in [0, \psi(\sigma)]$. 

We therefore obtain that
\[ A_{\lambda}(\eta) = \tilde{A}_{\lambda}(\eta) = \tilde{A}_{\lambda}(\gamma) \]
as desired. \( \square \)

We are now in a position to show

**Proof of Lemma A.3.** Note that for any \( \gamma \in AC(x, y; [0, \sigma]; \mathbb{R}^d) \), by Cauchy-Schwartz inequality, one has
\[
\frac{1}{2} |\dot{\gamma}(s)|^2 + F(\gamma(s)) + \lambda \geq 2\sqrt{F(\gamma(s)) + \lambda |\dot{\gamma}(s)|},
\]
which gives \( A_{\lambda}(\gamma) \geq \tilde{A}_{\lambda}(\gamma) \). Thus \( m_{\lambda}(x, y) \geq \tilde{m}_{\lambda}(x, y) \).

We prove by contradiction that \( |\dot{\gamma}(s)| = \sqrt{2(F \circ \gamma(t) + \lambda)} \) almost everywhere. Suppose that this is not correct. Write
\[ E = \{ s \in [0, \tau] : |\dot{\gamma}(s)| \neq \sqrt{2(F(\gamma(s)) + \lambda)}, F(\gamma(s)) < \infty \}. \]

Then \( |E| > 0 \). Moreover for \( s \in E \), one has
\[
\frac{1}{2} |\dot{\gamma}(s)|^2 + F(\gamma(s)) + \lambda > 2\sqrt{F(\gamma(s)) + \lambda |\dot{\gamma}(s)|} = \sqrt{2(F(\gamma(s)) + \lambda)|\dot{\gamma}(s)|}.
\]
Thus
\[ m_{\lambda}(x, y) = A_{\lambda}(\gamma) > \tilde{A}_{\lambda}(\gamma). \]

Reparameterize \( \gamma \) to get \( \eta \) as in Lemma A.5 we know that
\[ A_{\lambda}(\eta) = \tilde{A}_{\lambda}(\eta) = \tilde{A}_{\lambda}(\gamma). \]

Note that \( m_{\lambda}(x, y) \leq A_{\lambda}(\eta) \). This is a contradiction.

It is easy to see that \( m_{\lambda}(x, y) \geq \tilde{m}_{\lambda}(x, y) \). We prove \( \tilde{m}_{\lambda}(x, y) = m_{\lambda}(x, y) \) by contradiction. Suppose this is not correct. Then \( \tilde{m}_{\lambda}(x, y) < m_{\lambda}(x, y) \). There exists a curve \( \gamma \) such that
\[ \tilde{m}_{\lambda}(x, y) \leq \tilde{A}_{\lambda}(\gamma) < m_{\lambda}(x, y). \]

Reparameterizing \( \gamma \) like in Lemma A.5, we find a curve \( \eta \) such that
\[ A_{\lambda}(\eta) = \tilde{A}_{\lambda}(\eta) = \tilde{A}_{\lambda}(\gamma). \]

Since \( m_{\lambda}(x, y) \leq A_{\lambda}(\eta) \), this is a contradiction. \( \square \)
Proof of Lemma A.4. We only need to prove Lemma A.4 for \( m_\lambda \)-geodesics with canonical parameter. Assume that \( \gamma \in (x, y; [0, \sigma]; \tilde{\Omega}) \) is an \( m_\lambda \)-geodesic with canonical parameter joining \( x, y \), and moreover \( \gamma|_{(0,\sigma)} \subset \Omega \). Up to considering \( \gamma|_{[\delta,\sigma-\delta]} \subset \Omega \), we may assume that \( \gamma \subset \Omega \). Recall that 
\[
m_\lambda(x, y) = A_\lambda(\gamma) \leq A_\lambda(\gamma + \varepsilon \xi), \quad \forall \xi \in C^1(0, 0; [0, \sigma]; \mathbb{R}^{dN}).
\]
Since \( \gamma \subset \Omega \), there exist \( \varepsilon_0 \) depending on \( \gamma \) and \( \xi \) such that \( A_\lambda(\gamma + \varepsilon \xi) < \infty \) for any \( \varepsilon < \varepsilon_0 \). Thus
\[
0 = \frac{d}{d\varepsilon}|_{\varepsilon=0}A_\lambda(\gamma + \varepsilon \xi),
\]
that is,
\[
0 = \frac{d}{d\varepsilon}|_{\varepsilon=0}\int_0^\sigma \left( \frac{1}{2}|(\dot{\gamma} + \varepsilon \dot{\xi})(s)|^2 + F(\gamma + \varepsilon \xi)(s) + \lambda \right) ds.
\]
A direct calculation gives
\[
\frac{d}{d\varepsilon}|_{\varepsilon=0}\left( \frac{1}{2}|(\dot{\gamma} + \varepsilon \dot{\xi})(s)|^2 + F(\gamma + \varepsilon \xi)(s) + \lambda \right) = \left( \langle (\dot{\gamma}(s), \dot{\xi}(s)) \rangle + \langle \nabla F(\gamma(s)), \xi(s) \rangle \right).
\]
Therefore we obtain
\[
\int_0^\sigma \left( \langle (\dot{\gamma}(s), \dot{\xi}(s)) \rangle + \langle \nabla F(\gamma(s)), \xi(s) \rangle \right) ds = 0, \quad \forall \xi \in C^1(0, 0; [0, \sigma]; \mathbb{R}^{dN}).
\]
Note that \( \gamma \subset \Omega \) is free of collision. Since \( F \in C^2(\Omega) \), it follows from [19, Chapter 3] that \( \gamma \in C^2 \), and hence,
\[
\int_0^\sigma \langle [-\ddot{\gamma}(s) + \nabla F(\gamma(s))], \xi(s) \rangle ds = 0.
\]
By the arbitrariness of \( \xi \), we have \( \ddot{\gamma}(s) = \nabla F(\gamma(s)) \) for all \( s \in (0, \sigma) \) as desired.

Finally, we prove (A.2) in the Step 3 in Remark A.1(iii).

Proof of (A.2). Let all notations and notions be as in Remark A.1(iii). In particular, recall
\[
\Sigma_{12} = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : x_1 = x_2 \in \Gamma \} \text{ and } \Gamma = \{ x_i \in \mathbb{R}^d : x_i^j < 0, \ \forall 1 \leq j \leq d \}.
\]
Let \( x \in \Sigma_{12} \) be an arbitrary point. We have \( x_2 = x_1 \in \Gamma \), that is, \( x_2^j = x_1^j < 0 \) for all \( 1 \leq j \leq d \). Set
\[
c_\ast(x) := \min\{1, |x_1^1|, \ldots, |x_1^d|\} = \min\{1, -x_1^1, \ldots, -x_1^d\}.
\]
Then $c_+(x) > 0$. To get (A.2), that is, $\inf_{z \in \Omega} m_\lambda(z, x) > 0$, obviously it suffices to prove

$$\inf_{z \in \Omega} m_\lambda(z, x) \geq \sqrt{2\lambda} \frac{1}{2} c_+(x). \quad (A.6)$$

To see (A.6), given any $z \in \Omega$, let $\gamma \in \mathcal{AC}(z, x; [0, \sigma]; \tilde{\Omega})$ for some $\sigma > 0$ be an $m_\lambda$-geodesic with canonical parameter so that $A_\lambda(\gamma) = m_\lambda(z, x) < \infty$; see Lemma A.2 for its existence. From Lemma A.2 and (A.1) it follows that

$$m_\lambda(z, x) = A_\lambda(\gamma) \geq A_\lambda(\gamma|_{[s, \sigma]}) = m_\lambda(\gamma(s), x) \geq \sqrt{2\lambda} |\gamma(s) - x| \quad \forall s \in [0, \sigma). \quad (A.7)$$

We then claim that

there exists an $s \in [0, \sigma)$ such that $|\gamma(s) - x| \geq \frac{1}{2} c_+(x)$. \quad (A.8)

Obviously, from (A.7) and the claim (A.8) it follows that $m_\lambda(z, x) \geq \frac{1}{2} \sqrt{2\lambda} c_+(x)$. Thus (A.6) holds as desired.

Below we prove the above claim (A.8) by contradiction. Assume that the claim (A.8) is not correct, in other words, assume that

$$|\gamma(s) - x| < \frac{1}{2} c_+(x) \quad \text{for all } s \in [0, \sigma]. \quad (A.9)$$

Write $\gamma = (\gamma_1, \ldots, \gamma_N)$ and $\gamma_i = (\gamma_i^1, \ldots, \gamma_i^d)$. Given any $i = 1, 2$ and $1 \leq j \leq d$, by the hypothesis (A.9) one has

$$|\gamma_i^j(s) - x_i^j| \leq |\gamma(s) - x| < \frac{1}{2} c_+(x) \quad \text{and hence} \quad \gamma_i^j(s) \leq x_i^j + \frac{1}{2} c_+(x).$$

Noting $0 < c_+(x) \leq -x_i^j$, we get

$$\gamma_i^j(s) \leq \frac{1}{2} x_i^j < 0.$$
Set
\[ \kappa := \min \{ s \in [0, \sigma] \mid v(s) = 0 \}. \]
Then \( 0 < \kappa \leq \sigma \). Below we will prove that
\[ \int_0^\kappa |\dot{\gamma}| \sqrt{h_-(\circ v)} \, ds = \infty. \] (A.10)
Before we give the proof of (A.10), we show that (A.10) leads to a contradiction. Indeed, Lemma A.2 and Lemma A.3 yield
\[ m_\lambda(z, x) \geq m_\lambda(z, \gamma(\kappa)) \geq \tilde{A}_\lambda(\gamma|_{[0, \kappa]}). \]
Write
\[ \tilde{A}_\lambda(\gamma|_{[0, \kappa]}) = \int_0^\kappa |\dot{\gamma}| \sqrt{2F \circ \gamma + 2\lambda} \, ds = \int_0^\kappa |\dot{\gamma}| \sqrt{2h_-(\circ v) + 2\lambda} \, ds \geq \sqrt{2} \int_0^\kappa |\dot{\gamma}| \sqrt{h_-(\circ v)} \, ds. \]
We see that (A.10) implies \( \tilde{A}_\lambda(\gamma|_{[0, \kappa]}) = \infty \) and hence \( m_\lambda(z, x) = \infty \). However, since \( z \in \Omega \), we have \( m_\lambda(z, x) < \infty \). This is a contradiction. Therefore, we conclude that the hypothesis (A.9) is not correct, and hence, the claim (A.8) must hold as desired.
Finally, we prove (A.10) via the following 4 substeps.
Substep 1. From the definition of \( \kappa \) it follows that
\[ v(\kappa) = 0, \ v(t) > 0 \text{ for } t \in [0, \kappa), \ \gamma(\kappa) \in \Sigma_{12}^- \text{ and } \gamma|_{[0, \kappa)} \subset \mathbb{R}^d \setminus \Sigma_{12}^- . \]
Set
\[ \tau = \sup_{s \in [0, \kappa]} v(s). \]
The choice of \( \kappa \) and the continuity of \( v \) give that \( v([0, \kappa]) = [0, \tau] \). Moreover, we observe that \( \tau \leq c_+(x) \leq 1 \). Indeed, by \( x_1 = x_2 \) and the triangle inequality, for any \( s \in [0, \kappa] \) we have
\[ v(s) = |\gamma_1(s) - \gamma_2(s)| \leq |\gamma_1(s) - x_1(s)| + |x_2(s) - \gamma_2(s)| \leq 2|\gamma - x| \]
and hence, by the hypothesis (A.9), \( v(s) \leq c_+(x) \).
For any \( s \in [0, \kappa] \), recall that
\[ h_-(\circ v)(s) = \begin{cases} 1 & \text{ if } v(s) \in \Lambda = \{0\} \cup \bigcup_{k \geq 3}[2^{-k-1}, 2^{-k+1}]; \\ 1/v(s)^2 & \text{ if } v(s) \in [0, \tau] \setminus \Lambda. \end{cases} \]
Write \( E := \{ s \in [0, \kappa] | v(s) \in [0, \tau] \setminus \Lambda \} \). Then
\[ h_-(v(s)) = \frac{1}{v(s)^2} \text{ for } s \in E. \]
and

\[ \chi_E(s) = \chi_{[0, \tau] \setminus A}(v(s)) = \chi_{[0, \tau] \setminus A} \circ v(s) \quad \forall s \in [0, \kappa]. \]

Thus

\[
\int_0^\kappa |\dot{\gamma}| \sqrt{h_\omega \circ v} \, ds \geq \int_0^\kappa \chi_E(s)|\dot{\gamma}(s)| \frac{1}{v(s)} \, ds \\
= \int_0^\kappa \chi_{[0, \tau] \setminus A} \circ v(s)|\dot{\gamma}(s)| \frac{1}{v(s)} \, ds. \quad (A.11)
\]

**Substep 2.** Since both of \( \gamma_1 \) and \( \gamma_2 \) are absolutely continuous in \( [0, \kappa] \), we know that \( v \) is also absolutely continuous in \( [0, \kappa] \), and hence \( \dot{v} \in L^1([0, \kappa]) \). One further has

\[
|\dot{v}(t)| = |\langle (\dot{\gamma}_1 - \dot{\gamma}_2)(t), \gamma_1(t) - \gamma_2(t) \rangle | \\
\leq |(\dot{\gamma}_1 - \dot{\gamma}_2)(t)| \leq 2|\dot{\gamma}(t)| \quad \text{for almost all } t \in [0, \kappa]. \quad (A.12)
\]

Applying this in (A.11), we obtain

\[
\int_0^\kappa |\dot{\gamma}| \sqrt{h_\omega \circ v} \, ds \geq \frac{1}{2} \int_0^\kappa \chi_{[0, \tau] \setminus A} \circ v(s)|\dot{\gamma}(s)| \frac{1}{v(s)} \, ds. \quad (A.13)
\]

**Substep 3.** We claim that

\[
\int_0^\kappa \frac{1}{v} |\dot{v}| \chi_{[0, \tau] \setminus A} \circ v \, dt \geq \int_{[0, \tau] \setminus A} \frac{1}{t} \, dt, \quad (A.14)
\]

whose proof will be given in Substep 4. Assume that the claim (A.14) holds for the moment. By (A.13) and the claim (A.14), we obtain

\[
\int_0^\kappa |\dot{\gamma}| \sqrt{h_\omega \circ v} \, ds > \int_{[0, \tau] \setminus A} \frac{1}{t} \, dt.
\]

Let \( k_\tau \geq 3 \) such that \( 2^{-k_\tau^2+1} \leq \tau \). Then

\[
\int_{[0, \tau] \setminus A} \frac{1}{t} \, dt \geq \sum_{k \geq k_\tau} \int_{2^{-k_\tau^2+1}} 2^{-(k-1)^2+1} \frac{1}{t} \, dt = \sum_{k \geq k_\tau} \ln \frac{2^{-(k-1)^2+1}}{2^{-k_\tau^2+1}} \\
\geq \sum_{k \geq k_\tau} \ln 2^{2k-3} \geq (\ln 2) \sum_{k \geq k_\tau} k = \infty.
\]

Thus

\[
\int_0^\kappa |\dot{\gamma}| \sqrt{h_\omega \circ v} \, ds = \infty,
\]

which gives (A.10).

**Substep 4.** We prove the claim (A.14) via Lemma A.6(i). Since \( v \) is only defined in \( [0, \sigma] \) and also not necessarily Lipschitz in \( [0, \kappa] \), we can not use Lemma A.6(i) directly.
To overcome this difficulty, we show that the restriction of \( v \) in subintervals \([0, \kappa - \varepsilon]\) is Lipschitz for all sufficiently small \( \varepsilon > 0 \), and extend them to \( \mathbb{R} \) via the McShane’s extension. To be precise, by Lemma A.3 and \( F \circ \gamma = h_\cdot \circ v \), one has

\[
|\dot{\gamma}| = \sqrt{2(F \circ \gamma + \lambda)} = \sqrt{2(h_\cdot \circ v + \lambda)}
\]

almost everywhere in \([0, \kappa]\).

Recall that \( 0 < v \leq 1 \) in \([0, \kappa)\), \( \lambda > 0 \), and \( h_\cdot(t) \leq \frac{1}{\tau^2} \) by definition, we have

\[
h_\cdot \circ v + \lambda \leq (1 + \lambda) \frac{1}{v^2}
\]

in \([0, \kappa]\).

Thus, by (A.12),

\[
|\dot{v}| \leq 2|\dot{\gamma}| \leq 2\sqrt{2(1 + \lambda)} \frac{1}{v} \quad \text{almost everywhere in } [0, \kappa].
\]

For \( 0 < \varepsilon < \kappa \), set

\[
\delta_\varepsilon := \min\{v(t), t \in [0, \kappa - \varepsilon]\}.
\]

Since \( v \) is continuous in \([0, \kappa]\) and \( v > 0 \) in \([0, \kappa)\), we know that \( \delta_\varepsilon > 0 \). Thus

\[
|\dot{v}| \leq 2\sqrt{2(1 + \lambda)} \frac{1}{\delta_\varepsilon} \quad \text{almost everywhere in } [0, \kappa - \varepsilon].
\]

From this and the absolute continuity of \( v \) in \([0, \kappa]\), we know that \( v \) is Lipschitz in \([0, \kappa - \varepsilon]\) with

\[
|v(s) - v(t)| = |\int_s^t \dot{v}(\xi) d\xi| \leq 2\sqrt{2(1 + \lambda)} \frac{1}{\delta_\varepsilon} |s - t|, \quad \forall s, t \in [0, \kappa - \varepsilon].
\]

Denote by \( \tilde{v}_\varepsilon \) the McShane’s extension of \( v|_{[0, \kappa - \varepsilon]} \) into \( \mathbb{R} \), that is,

\[
\tilde{v}_\varepsilon(t) = \inf \left\{ v(s) + 2\sqrt{2(1 + \lambda)} \frac{1}{\delta_\varepsilon} |t - s| \left| s \in [0, \kappa - \varepsilon] \right. \right\}, \quad \forall t \in \mathbb{R}. \quad \text{(A.15)}
\]

Then

\[
|\tilde{v}_\varepsilon(s) - \tilde{v}_\varepsilon(t)| \leq 2\sqrt{2(1 + \lambda)} \frac{1}{\delta_\varepsilon} |s - t|, \quad \forall s, t \in \mathbb{R},
\]

that is, \( \tilde{v}_\varepsilon \) is Lipschitz in \( \mathbb{R} \), and moreover, \( \tilde{v}_\varepsilon|_{[0, \kappa - \varepsilon]} = v|_{[0, \kappa - \varepsilon]} \); see for example [23, Chapter 6].

Moreover, write

\[
\int_0^\kappa \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \wedge v \ dt = \lim_{\varepsilon \to 0} \int_0^{\kappa - \varepsilon} \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \wedge v \ ds.
\]

Since \( v \) is continuous in \([0, \kappa]\), \( v > 0 \) in \([0, \sigma)\) and \( \varphi(\kappa) = 0 \), we know that \( \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). When \( \varepsilon > 0 \) is sufficiently small, one also has \( v([0, \kappa - \varepsilon]) = [\delta_\varepsilon, \tau] \), that is,

\[
\chi_{[0, \tau]} \wedge v(s) = \chi_{[\delta_\varepsilon, \tau]} \wedge v(s) \quad \text{for any } s \in [0, \kappa - \varepsilon].
\]
Therefore, for all sufficiently small $\varepsilon > 0$, one has
\[
\int_0^{\kappa - \varepsilon} \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \Lambda \circ v \, ds = \int_0^{\kappa - \varepsilon} \frac{1}{v} |\dot{v}| \chi_{[\delta_\varepsilon, \tau]} \Lambda \circ v \, ds.
\]
Denote by $\tilde{v}^\varepsilon$ the McShane’s extension of $v([0, \kappa - \varepsilon])$ into $\mathbb{R}$ as in (A.15). Since $\tilde{v}^\varepsilon([0, \kappa - \varepsilon]) = v([0, \kappa - \varepsilon])$, it follows that
\[
\int_0^{\kappa - \varepsilon} \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \Lambda \circ v \, ds = \int_\mathbb{R} \chi_{[0, \kappa - \varepsilon]}(s) \chi_{[\delta_\varepsilon, \tau]} \Lambda \circ \tilde{v}^\varepsilon(s) \frac{1}{v^\varepsilon(s)} |(\tilde{v}^\varepsilon)'(s)| \, ds.
\]
Since $\tilde{v}^\varepsilon$ is Lipschitz in $\mathbb{R}$ and $\chi_{[0, \kappa - \varepsilon]} \frac{1}{v^\varepsilon} \chi_{[\delta_\varepsilon, \tau]} \Lambda \circ \tilde{v}^\varepsilon \in L^1(\mathbb{R})$, we are able to apply the change of variable formula in Lemma A.6(i) with $f = \tilde{v}^\varepsilon$ in $\mathbb{R}$ and $g = \chi_{[0, \kappa - \varepsilon]} \frac{1}{v^\varepsilon} \chi_{[\delta_\varepsilon, \tau]} \Lambda \circ \tilde{v}^\varepsilon$ therein to get
\[
\int_0^{\kappa - \varepsilon} \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \Lambda \circ v \, ds = \int_\mathbb{R} \sum_{s \in (s')^{-1}([t])} \left[ \chi_{[0, \kappa - \varepsilon]}(s) \chi_{[\delta_\varepsilon, \tau]} \Lambda \circ \tilde{v}^\varepsilon(s) \frac{1}{v^\varepsilon(s)} \right] \, dt.
\]
(A.16)

For any sufficiently small $\varepsilon > 0$, since $v([0, \kappa - \varepsilon]) = [\delta_\varepsilon, \tau]$, given any $t \in [\delta_\varepsilon, \tau] \Lambda$, one can find at least one $s \in [0, \kappa - \varepsilon]$ such that $\tilde{v}^\varepsilon(s) = v(s) = t$, and hence,
\[
\sum_{s \in (s')^{-1}([t])} \left[ \chi_{[0, \kappa - \varepsilon]}(s) \chi_{[\delta_\varepsilon, \tau]} \Lambda \circ \tilde{v}^\varepsilon(s) \frac{1}{v^\varepsilon(s)} \right] \geq \frac{1}{t}.
\]
From this and (A.16) one deduces that
\[
\int_0^{\kappa - \varepsilon} \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \Lambda \circ v \, ds \geq \int_{[\delta_\varepsilon, \tau] \Lambda} \frac{1}{t} \, dt.
\]
Sending $\varepsilon \to 0$, by $\delta_\varepsilon \to 0$ one gets
\[
\int_0^{\kappa} \frac{1}{v} |\dot{v}| \chi_{[0, \tau]} \Lambda \circ v \, dt \geq \int_{[0, \tau] \Lambda} \frac{1}{t} \, dt,
\]
which gives the claim (A.14) as desired. The proof is complete. \qed

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J. Liu · D. Yan
School of Mathematical Science,
Beihang University,
Changping District Shahe Higher Education Park South Third Street No. 9,
Beijing
102206 People’s Republic of China.
e-mail: duokuiyan@buaa.edu.cn

and

J. Liu
Department of Mathematics and Statistics,
University of Jyväskylä,
Jyvaskyla
Finland.
e-mail: jiayin.mat.liu@jyu.fi

and

Y. Zhou
School of Mathematical Sciences,
Beijing Normal University,
Haidian District Xinejikou Waidajie No.19,
Beijing
100875 People’s Republic of China.
e-mail: yuan.zhou@bnu.edu.cn

(Received January 25, 2022 / Accepted May 31, 2023)
Published online June 12, 2023
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